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SELMER COMPLEXES

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Abstract. **—** This book builds new foundations of Iwasawa theory, based on a systematic study of cohomological invariants of big Galois representations in the framework of derived categories. A new duality formalism is developed, which leads to generalized Cassels-Tate pairings and generalized p-adic height pairings. One of the applications is a parity result for Selmer groups associated to Hilbert modular forms.

Résumé **(Complexes de Selmer). —** Ce livre construit de nouvelles fondations pour la théorie d'Iwasawa, basées sur une étude systématique d'invariants cohomologiques (vivant dans des catégories dérivées) pour les grosses représentations galoisiennes. On développe un nouveau formalisme de dualité dont on déduit des accouplements de Cassels-Tate généralisés et des hauteurs p-adiques généralisées. Une des applications est un résultat de parité pour les groupes de Selmer attachés aux formes modulaires de Hilbert.

CONTENTS

CHAPTER 0

INTRODUCTION

0.0. Big Galois representations

In this work we study cohomological invariants of "big Galois representations"

$$\rho : G \longrightarrow \mathrm{Aut}_R(T),$$

where

(i) G is a suitable Galois group.

(ii) R is a complete local Noetherian ring, with a finite residue field of characteristic p.

(iii) T is an R-module of finite type.

(iv) ρ is a continuous homomorphism of pro-finite groups.

We develop a general machinery that covers duality theory, Iwasawa theory, generalized Cassels-Tate pairings and generalized height pairings.

0.1. Examples

0.1.0. An archetypal example of a big Galois representation arises as follows. Let K be a field of characteristic $\mathrm{char}(K) \neq p$. For every K-scheme $X \to \mathrm{Spec}(K)$ put $\overline{X} = X \otimes_K K^{\mathrm{sep}}$, where K^{sep} is a fixed separable closure of K. Given a projective system $X_\infty = (X_\alpha)_{\alpha \in I}$ (indexed by some directed set I) of separated K-schemes of finite type with *finite* transition morphisms $X_\beta \to X_\alpha$, put

$$H^i(X_\infty) := \varprojlim_{\alpha} H^i_{\mathrm{et}}(\overline{X}_\alpha, \mathbf{Z}_p) = \varprojlim_{\alpha} \varprojlim_{n} H^i_{\mathrm{et}}(\overline{X}_\alpha, \mathbf{Z}/p^n\mathbf{Z}),$$

where the transition morphisms are given by trace maps. This is a representation of $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$, linear over the $\mathbf{Z}_p$-algebra generated by "endomorphisms" of the tower X_∞. In practice, $H^i(X_\infty)$ is often too big and must be first decomposed into smaller constituents. One can also use more general coefficient sheaves, not just $\mathbf{Z}/p^n\mathbf{Z}$.

0.1.1. Iwasawa theory. — Let K_∞/K be a Galois extension (contained in K^{sep}) with $\Gamma = \mathrm{Gal}(K_\infty/K)$ isomorphic to $\mathbf{Z}_p^r$ for some $r \geqslant 1$. Write $K_\infty = \bigcup K_\alpha$ as a union of finite extensions of K. For a fixed separated K-scheme of finite type $X \to \mathrm{Spec}(K)$, consider the projective system $X_\alpha = X \otimes_K K_\alpha$. In this case

$$H^i(X_\infty) \xrightarrow{\sim} H^i_{\mathrm{et}}(\overline{X}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \Lambda,$$

where

$$\Lambda = \mathbf{Z}_p[\![\Gamma]\!] \xrightarrow{\sim} \mathbf{Z}_p[\![X_1, \ldots, X_r]\!]$$

is the Iwasawa algebra of Γ and G_K acts on Λ by the tautological character

$$\chi_\Gamma : G_K \twoheadrightarrow \Gamma \hookrightarrow \Lambda^*$$

(or its inverse, depending on the sign conventions). Thus $T = H^i(X_\infty)$ is a big Galois representation of $G = G_K$ over $R = \Lambda$.

0.1.2. Hida theory. — Let $N \geqslant 1$ be an integer not divisible by p (and such that $Np > 4$). Let X_∞ be the projective system of modular curves[(1)]

$$X_1(Np) \longleftarrow \cdots \longleftarrow X_1(Np^r) \longleftarrow X_1(Np^{r+1}) \longleftarrow \cdots$$

over $K = \mathbf{Q}$. The tower X_∞ has many endomorphisms, namely Hecke correspondences.

The Galois module $H^1(X_\infty)$ is too big, but its *ordinary part* $H^1(X_\infty)^{\mathrm{ord}}$, defined as the maximal $\mathbf{Z}_p$-submodule on which the Hecke operator[(2)] $T(p)$ is invertible, is of finite type over the *ordinary Hecke algebra* $\mathfrak{h}^{\mathrm{ord}}$, defined as the projective limit of the ordinary parts of the $\mathbf{Z}_p$-Hecke algebras acting on $S_2(\Gamma_1(Np^r))$ (for variable r).

The ring $\mathfrak{h}^{\mathrm{ord}}$ is semilocal, in fact finite and free over $\Lambda = \mathbf{Z}_p[\![\Gamma]\!] \xrightarrow{\sim} \mathbf{Z}_p[\![X]\!]$, where $\Gamma = 1 + p\mathbf{Z}_p$ (resp., $\Gamma = 1 + 4\mathbf{Z}_2$, if $p = 2$) acts on X_∞ by diamond operators. If we fix a maximal ideal $\mathfrak{m} \subset \mathfrak{h}^{\mathrm{ord}}$, then $T = H^1(X_\infty)^{\mathrm{ord}}_{\mathfrak{m}}$ is a big Galois representation of $G = G_{\mathbf{Q}}$ over $R = \mathfrak{h}^{\mathrm{ord}}_{\mathfrak{m}}$.

0.1.3. One can, of course, combine the constructions in 0.1.1–0.1.2.

0.2. Selmer groups

In the case when K is a number field and the projective system X_∞ has good reduction (*i.e.*, each X_α has) outside a finite set S of places of K containing all places above p, then $H^i(X_\infty)$ is a representation of the Galois group with restricted ramification $G_{K,S} = \mathrm{Gal}(K_S/K)$, where K_S is the maximal extension of K which is unramified outside S.

(1) There are two choices of transition morphisms; see *e.g.* [**N-P**] for more details.

(2) Again, there are two choices of $T(p)$; which one is correct depends on the choice made in (1).

In general, given a big Galois representation T of $G_{K,S}$, the main objects of interest are the following:

(i) (continuous) Galois cohomology groups $H^i_{\mathrm{cont}}(G_{K,S}, T)$.

(ii) Selmer groups

$$\mathrm{Sel}(G_{K,S}, T) \subset H^1_{\mathrm{cont}}(G_{K,S}, T),$$

consisting of elements satisfying suitable local conditions in $H^1_{\mathrm{cont}}(G_v, T)$ for $v \in S$ (where $G_v = \mathrm{Gal}(\overline{K}_v/K_v)$).

Similar objects were first considered by R. Greenberg [**Gre4**] as a natural generalization of Iwasawa theory.

Greenberg expressed hope that there should be a variant of the Main Conjecture of Iwasawa theory in this context, *i.e.*, a relation between the "characteristic power series" of a big Selmer group and an appropriate p-adic L-function.

A big Galois representation ρ can be viewed as a family of "usual" Galois representations $\rho_\lambda : G_{K,S} \to \mathrm{GL}_n(\mathbf{Z}_p)$, which depends analytically on the parameter λ. One of the main motivations of the present work was to develop a homological machinery that would control the variation of the Selmer groups associated to the individual ρ_λ's as a function of λ. A statement such as the Main Conjecture for T should then imply a relation between the Selmer group of ρ_λ and the special value at λ of the p-adic L-function in question.

0.3. Big vs. finite Galois representations

Every big Galois representation $\rho : G \to \mathrm{Aut}_R(T)$ is the projective limit of Galois representations $\rho_n : G \to \mathrm{Aut}_R(T/\mathfrak{m}^n T)$ with finite targets. Using known properties of ρ_n one can sometimes pass to the limit and deduce results valid for ρ.

Consider, for example, a representation $\rho : G_K \to \mathrm{Aut}_R(T)$ of $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ for a local field K (with finite residue field) of characteristic $\mathrm{char}(K) \neq p$. Writing D for the Pontrjagin dual functor

$$D(-) = \mathrm{Hom}_{\mathrm{cont}}(-, \mathbf{R}/\mathbf{Z}),$$

Tate's local duality states that the (finite) cohomology groups

$$H^i(G_K, T/\mathfrak{m}^n T) \xleftarrow{\;D\;}\!\!\!\rightarrow H^{2-i}(G_K, D(T/\mathfrak{m}^n T)(1))$$

are Pontrjagin duals of each other. Taking projective limit one obtains Pontrjagin duality between a compact and a discrete R-module

$$H^i(G_K, T) \xleftarrow{\;D\;}\!\!\!\rightarrow H^{2-i}(G_K, D(T)(1)),$$

where

$$H^i(G_K, T) = H^i_{\mathrm{cont}}(G_K, T) = \varprojlim_n H^i(G_K, T/\mathfrak{m}^n T).$$

0.4. Compact vs. discrete modules

Attentive readers will have noticed that Greenberg [**Gre2, Gre3, Gre4**] considers Selmer groups for discrete Galois modules, while our T is compact. Let us investigate the relationship between discrete and compact Galois representations more closely. In fact, understanding the interplay between discrete and compact modules is at the basis of the whole theory developed in this work.

Let us first consider the "classical" case of $R = \mathbf{Z}_p$. Given a representation

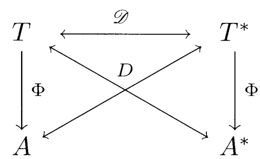
$$\rho : G \longrightarrow \mathrm{Aut}_{\mathbf{Z}_p}(T),$$

where T is free of finite rank over $\mathbf{Z}_p$, there are three more representations of G associated to T, namely

$$\begin{aligned} A &= T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p, \\ T^* &= \mathrm{Hom}_{\mathbf{Z}_p}(T, \mathbf{Z}_p) = D(A), \qquad (0.4.1) \\ A^* &= T^* \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p = D(T). \end{aligned}$$

They can be arranged into the following diagram:

(0.4.2)



Here $\mathscr{D}(-) = \mathrm{Hom}_{\mathbf{Z}_p}(-, \mathbf{Z}_p)$ and $\Phi(-) = (-) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p$.

What is the analogue of this construction for general R (or even for $R = \mathbf{Z}_p$ if T is not free over $\mathbf{Z}_p$)? Let us temporarily ignore the action of G and consider this question only for R-modules. For $R = \mathbf{Z}_p$, the tensor product

$$T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p$$

loses any information about the torsion submodule $T_{\mathrm{tors}} \subset T$. On the other hand,

$$T_{\mathrm{tors}} \xrightarrow{\sim} \mathrm{Tor}_1^{\mathbf{Z}_p}(T, \mathbf{Q}_p/\mathbf{Z}_p).$$

This suggests that one should consider the derived tensor product

$$A = T \overset{\mathbf{L}}{\otimes}_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p$$

as a correct version of (0.4.1). In concrete terms, $\mathbf{Q}_p/\mathbf{Z}_p$ has a natural flat resolution

$$[\mathbf{Z}_p \longrightarrow \mathbf{Q}_p] \qquad (0.4.3)$$

(in degrees $-1, 0$) and A is represented by the complex

$$T \otimes_{\mathbf{Z}_p} [\mathbf{Z}_p \longrightarrow \mathbf{Q}_p] = \left[T \longrightarrow T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p\right],$$

again in degrees $-1, 0$.

What is the analogue of (0.4.3) for general R? Fix a system of parameters of R, *i.e.*, elements $x_1, \dots, x_d \in \mathfrak{m}$ (where $d = \dim(R)$) such that $\dim(R/(x_1, \dots, x_d)) = 0$. An analogue of (0.4.3) is then given by the complex

$$C^\bullet = C^\bullet(R, (x_i)) = \left[R \longrightarrow \bigoplus_i R_{x_i} \longrightarrow \bigoplus_{i<j} R_{x_i x_j} \longrightarrow \cdots \longrightarrow R_{x_1 \cdots x_d} \right]$$

in degrees $[-d, 0]$, with standard "Čech differentials". This complex depends on the chosen system of parameters. In order to remove this ambiguity it is necessary to consider $C^\bullet$ as an object of the (bounded) derived category $D^b({}_R\mathrm{Mod})$. In more rigorous terms, if (y_j) is another system of parameters of R, then there is a canonical isomorphism

$$R \xrightarrow{\sim} \mathrm{Hom}_{D^b({}_R\mathrm{Mod})}\big(C^\bullet(R, (x_i)), C^\bullet(R, (y_j))\big).$$

Given $T \in D_{ft}({}_R\mathrm{Mod})$, we then define

$$A = \Phi(T) = T \overset{\mathbf{L}}{\otimes}_R C^\bullet = T \otimes_R C^\bullet$$

(for the last equality note that $C^\bullet$ is a complex of flat R-modules).

The Pontrjagin dual of R

$$I = D(R)$$

is an injective hull of the (finite) residue field $k = R/\mathfrak{m}$; we abandon our earlier convention about D and instead define

$$D(M) = \mathrm{Hom}_R(M, I)$$

for every R-module M. This functor coincides with Pontrjagin dual for Noetherian (hence compact) or Artinian (hence discrete) R-modules.

We have, so far, defined analogues of the vertical and diagonal arrows in the diagram (0.4.2). What about the horizontal arrow? A derived version of the adjunction isomorphism

$$\mathrm{adj} : \mathbf{R}\mathrm{Hom}_R(X \overset{\mathbf{L}}{\otimes}_R Y, Z) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\mathrm{Hom}_R(Y, Z)),$$

applied to $X = T$, $Y = C^\bullet$, $Z = I$, shows that

$$D \circ \Phi(-) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_R(-, D(C^\bullet)).$$

The object of $D^b({}_R\mathrm{Mod})$ represented by the complex

$$D(C^\bullet)$$

is known as the *dualizing complex* $\omega \in D^b_{ft}({}_R\mathrm{Mod})$ and the functor

$$\mathscr{D}(-) = \mathbf{R}\mathrm{Hom}_R(-, \omega)$$

as Grothendieck's dual (if $R = \mathbf{Z}_p$, then $\omega = \mathbf{Z}_p$).

To sum up, a general version of (0.4.2) is given by the following "duality diagram"

$$\begin{array}{ccc} T & \overset{\mathscr{D}}{\longleftrightarrow} & T^* \\ \Big\downarrow \Phi & \overset{D}{\times} & \Big\downarrow \Phi \\ A & & A^* \end{array} \tag{0.4.4}$$

with $T, T^* \in D_{ft}({}_R\mathrm{Mod})$, $A, A^* \in D_{coft}({}_R\mathrm{Mod})$,

$$D(-) = \mathrm{Hom}_R(-, I)$$
$$\mathscr{D}(-) = \mathbf{R}\mathrm{Hom}_R(-, \omega)$$
$$\Phi(-) = (-)\overset{\mathbf{L}}{\otimes}_R D(\omega)$$

Commutativity of this diagram (up to canonical isomorphisms) is equivalent to three duality theorems: Matlis duality ($\mathrm{id} \xrightarrow{\sim} D \circ D$), Grothendieck duality ($\mathrm{id} \xrightarrow{\sim} \mathscr{D} \circ \mathscr{D}$) and local duality ($\mathscr{D} \xrightarrow{\sim} D \circ \Phi$ together with the isomorphism $\Phi[-d] \xrightarrow{\sim} \mathbf{R}\Gamma_{\{\mathfrak{m}\}}$).

The diagram (0.4.4) gives rise to a spectral sequence

$$E_2^{i,j} = \mathbb{E}\mathrm{xt}_R^i(D(H^j(A)), \omega) = \mathbb{E}\mathrm{xt}_R^i(H^{-j}(T^*), \omega) \Longrightarrow H^{i+j}(\mathscr{D}(T^*)) = H^{i+j}(T). \tag{0.4.5}$$

0.5. (Ind-)admissible $R[G]$-modules

In order to incorporate the Galois action into the diagram (0.4.4), it is necessary to enlarge the category of Galois modules we consider. For example, $T \otimes_R C^\bullet$ has components of the form

$$T \otimes_R R_x = \varinjlim_n \left[T \xrightarrow{x} T \xrightarrow{x} T \xrightarrow{x} \cdots\right] = \bigcup_{n \geqslant 1} T \otimes_R \left\{\frac{r}{x^n} \mid r \in R\right\}.$$

This suggests that we should consider $R[G]$-modules M satisfying the following condition (which makes sense for any topological group G):

Axiom 1. — *$M = \bigcup M_\alpha$, where $M_\alpha \subset M$ are $R[G]$-submodules of M, which are of finite type over R and such that the map $G \to \mathrm{Aut}_R(M_\alpha)$ is continuous (with respect to the pro-finite topology on the target).*

As

$$D\left(\bigoplus_\alpha M_\alpha\right) = \prod_\alpha D(M_\alpha),$$

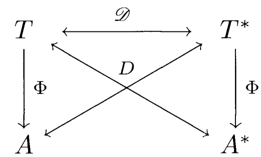
there are cases when Axiom 1 is satisfied by M, but not by $D(M)$. For this reason we impose an additional, purely algebraic, condition:

Axiom 2. — $\mathrm{Im}(R[G] \to \mathrm{End}_R(M))$ *is an R-module of finite type.*

An $R[G]$-module satisfying Axiom 1 and Axiom 2 (resp., only Axiom 1) will be called *admissible* (resp., *ind-admissible*). Admissible modules form a full subcategory $\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$ of $\big({}_{R[G]}\mathrm{Mod}\big)$, which is stable under subquotients, finite direct sums, tensor products and internal Hom's. In particular, if T is admissible, so is $T \otimes_R R_x$, for every $x \in R$.

The duality diagram

(0.5.1)



then makes sense for $T, T^* \in D_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$, $A, A^* \in D_{R\text{-}coft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$.

0.6. Continuous cohomology

For an admissible $R[G]$-module M we define the complex of continuous (non-homogeneous) cochains of G with values in M as

$$C^\bullet_{\mathrm{cont}}(G, M) = \varinjlim_{\alpha} C^\bullet_{\mathrm{cont}}(G, M_\alpha) = \varinjlim_{\alpha} \varprojlim_{n} C^\bullet_{\mathrm{cont}}(G, M_\alpha/\mathfrak{m}^n M_\alpha),$$

where each $M_\alpha/\mathfrak{m}^n M_\alpha$ has discrete topology. The functor $M \mapsto C^\bullet_{\mathrm{cont}}(G, M)$ gives rise to an exact functor

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G, -) : D^*\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big) \longrightarrow D^*({}_R\mathrm{Mod})$$

for $* = +$ (resp., for $* = +, b$, if G is a pro-finite group satisfying $\mathrm{cd}_p(G) < \infty$). In fact, this construction requires only Axiom 1, and so it makes sense for ind-admissible modules.

In the situation of (0.5.1), the functor $\mathbf{R}\Gamma_{\mathrm{cont}}(G, -)$ commutes with Φ (up to a canonical isomorphism). For $R = \mathbf{Z}_p$ this statement boils down to the fact that there exists a long cohomology sequence of continuous G-cohomology associated to

$$0 \longrightarrow T \longrightarrow V \longrightarrow A \longrightarrow 0,$$

where $V = T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

0.7. Duality for Galois cohomology

The machinery behind the duality diagram (0.5.1) makes the passage from finite to big Galois representations very easy. As we have seen in 0.3, classical duality results for finite Galois modules imply a duality with respect to D, while compatibility of $\mathbf{R}\Gamma_{\mathrm{cont}}(G, -)$ with Φ is automatic; combining the two facts we obtain a duality with respect to $\mathscr{D}$. The final outcome (cf. Chapter 5) is the following:

0.7.0. Duality over local fields (Tate). — Let K be a local field (with finite residue field) of characteristic $\mathrm{char}(K) \neq p$ and $G = G_K$. If $T, T^* \in D^b_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G_K]}\mathrm{Mod}\big)$ and $A, A^* \in D^b_{R\text{-}coft}\big({}^{\mathrm{ad}}_{R[G_K]}\mathrm{Mod}\big)$ are related as in (0.5.1), then the four objects of $D^b_{(co)ft}({}_R\mathrm{Mod})$ in the diagram

$$\begin{array}{ccc} \mathbf{R}\Gamma_{\mathrm{cont}}(G_K, T) & \overset{\mathscr{D}}{\longleftrightarrow} & \mathbf{R}\Gamma_{\mathrm{cont}}(G_K, T^*(1))\,[2] \\ \downarrow{\scriptstyle \Phi} & \overset{D}{\times} & \downarrow{\scriptstyle \Phi} \\ \mathbf{R}\Gamma_{\mathrm{cont}}(G_K, A) & & \mathbf{R}\Gamma_{\mathrm{cont}}(G_K, A^*(1))\,[2] \end{array}$$

are related as in (0.4.4).

0.7.1. Duality over global fields (Poitou - Tate). — Let K be a global field of characteristic $\mathrm{char}(K) \neq p$ and S a finite set of primes of K containing all primes above p and all archimedean primes of K. If $p = 2$, assume for simplicity that K has no real prime (otherwise one would have to consider also Tate cohomology groups at real primes). Denote by S_f the set of all non-archimedean primes in S; for $v \in S_f$ put $G_v = G_{K_v}$ and fix an embedding $\overline{K} \hookrightarrow \overline{K}_v$. Set $G_{K,S} = \mathrm{Gal}(K_S/K)$, where K_S the maximal extension of K unramified outside S. For every admissible $G_{K,S}$-module M define the complex of continuous cochains with compact support by[(3)]

$$C^\bullet_{c,\mathrm{cont}}(G_{K,S}, M) = \mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(G_{K,S}, M) \longrightarrow \bigoplus_{v \in S_f} C^\bullet_{\mathrm{cont}}(G_v, M) \right)[-1].$$

This defines an exact functor

$$\mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, -) : D^*\big({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\big) \longrightarrow D^*({}_R\mathrm{Mod})$$

$(* = +, b)$. If $T, T^* \in D^b_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)$ and $A, A^* \in D^b_{R\text{-}coft}\big({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)$ are related as in (0.5.1) (for $G = G_{K,S}$), then the objects of $D^b_{(co)ft}({}_R\mathrm{Mod})$ in the diagram

$$\begin{array}{ccc} \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, T) & \overset{\mathscr{D}}{\longleftrightarrow} & \mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, T^*(1))\,[3] \\ \downarrow{\scriptstyle \Phi} & \overset{D}{\times} & \downarrow{\scriptstyle \Phi} \\ \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, A) & & \mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, A^*(1))\,[3] \end{array}$$

are related as in (0.4.4).

(3) This differs from the cochains with compact support as defined by Kato [**Ka1**]. Our definition makes the duality theorem work, while Kato's definition, which incorporates cochains at all infinite places, gives rise to objects naturally related to "zeta elements".

0.7.2. As in (0.4.5), the previous diagram gives rise to the following spectral sequences:

$$E_2^{i,j} = \mathbb{E}\mathrm{xt}^i_R(H^{3-j}_{c,\mathrm{cont}}(G_{K,S}, T^*(1)), \omega) = \mathbb{E}\mathrm{xt}^i_R(D(H^j_{\mathrm{cont}}(G_{K,S}, A)), \omega) \Longrightarrow H^{i+j}_{\mathrm{cont}}(G_{K,S}, T)$$

$$'E_2^{i,j} = \mathbb{E}\mathrm{xt}^i_R(H^{3-j}_{\mathrm{cont}}(G_{K,S}, T^*(1)), \omega) = \mathbb{E}\mathrm{xt}^i_R(D(H^j_{c,\mathrm{cont}}(G_{K,S}, A)), \omega) \Longrightarrow H^{i+j}_{c,\mathrm{cont}}(G_{K,S}, T).$$

0.8. Selmer complexes

Let us keep the notation of 0.7.1. Selmer groups have been traditionally defined as subgroups of elements of $H^1(G_{K,S}, -)$ satisfying suitable local conditions in $H^1(G_v, -)$ (for $v \in S$). In our approach we have no choice but to impose local conditions on the level of complexes, rather than cohomology.

0.8.0. Let T, T^*, A, A^* be bounded complexes of admissible $R[G_{K,S}]$-modules, which are related in the derived category as in (0.5.1). *Local conditions* for any $X \in \{T, A, T^*(1), A^*(1)\}$ are given by a collection $\Delta(X) = (\Delta_v(X))_{v\in S_f}$, where each $\Delta_v(X)$ is a morphism of complexes of R-modules

$$i_v^+(X) : U_v^+(X) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v, X),$$

with $U_v^+(X)$ satisfying appropriate finiteness conditions.

The *Selmer complex* associated to the local conditions $\Delta(X)$ is defined as the total complex

$$\mathrm{Tot}\begin{pmatrix} C^\bullet_{\mathrm{cont}}(G_{K,S}, X) & \to & \bigoplus_{v\in S_f} C^\bullet_{\mathrm{cont}}(G_v, X) \\ & & \uparrow \\ & & \bigoplus_{v\in S_f} U_v^+(X) \end{pmatrix}.$$

The corresponding object of the derived category will be denoted by

$$\widetilde{\mathbf{R}\Gamma}_f(X) = \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X; \Delta(X)) \in D_*({}_R\mathrm{Mod}), \qquad * = \begin{cases} \mathit{ft}, & X = T, T^*(1) \\ \mathit{coft}, & X = A, A^*(1) \end{cases}$$

and its cohomology by $\widetilde{H}^i_f(X) = \widetilde{H}^i_f(G_{K,S}, X; \Delta(X))$. If we put

$$U_v^-(X) = \mathrm{Cone}(U_v^+(X) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v, X)),$$

then the exact triangle

$$\widetilde{\mathbf{R}\Gamma}_f(X) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, X) \longrightarrow \bigoplus_{v\in S_f} U_v^-(X)$$

gives rise to a long exact sequence

$$(0.8.0.1)\quad \cdots \longrightarrow \bigoplus_{v\in S_f} H^{i-1}(U_v^-(X)) \longrightarrow \widetilde{H}^i_f(X) \longrightarrow H^i_{\mathrm{cont}}(G_{K,S},X) \longrightarrow \bigoplus_{v\in S_f} H^i(U_v^-(X)) \longrightarrow \cdots$$

In particular, the canonical map

$$\widetilde{H}^1_f(X) \longrightarrow H^1_{\mathrm{cont}}(G_{K,S},X)$$

need not be injective.

0.8.1. In the present work we consider only *Greenberg's local conditions*, defined as follows (these are the only local conditions that can be handled by elementary methods; the general case would require a heavy dose of crystalline machinery, which is not yet available). Fix a subset $\Sigma \subset S_f$ containing all primes above p and put $\Sigma' = S_f - \Sigma$.

(i) For $v \in \Sigma$, assume that we are given a morphism of complexes of admissible $R[G_v]$-modules $X_v^+ \to X$; put

$$U_v^+(X) = C^\bullet_{\mathrm{cont}}(G_v, X_v^+).$$

(ii) For $v \in \Sigma'$ we take the "unramified local conditions"

$$U_v^+(X) = C^\bullet_{\mathrm{ur}}(G_v, X).$$

Morally, these should defined as

$$\text{“}C^\bullet_{\mathrm{cont}}(G_v/I_v, \tau_{\leqslant 0}\, C^\bullet_{\mathrm{cont}}(I_v,X)) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v/I_v, C^\bullet_{\mathrm{cont}}(I_v,X)) \xrightarrow{\sim} C^\bullet_{\mathrm{cont}}(G_v,X)\text{”},$$

where $I_v \subset G_v$ is the inertia subgroup. Unfortunately, there does not seem to be a satisfactory general formalism of the Hochschild-Serre spectral sequence for continuous cohomology. As a result, we define $C^\bullet_{\mathrm{ur}}$ by explicit formulas. For example, if $X = X^0$ is concentrated in degree zero, then

$$C^\bullet_{\mathrm{ur}}(G_v, X) = C^\bullet_{\mathrm{cont}}\big(G_v/I_v, (X^0)^{I_v}\big)$$

is quasi-isomorphic to the complex

$$\Big[(X^0)^{I_v} \xrightarrow{f_v - 1} (X^0)^{I_v}\Big]$$

in degrees $0, 1$; here $f_v \in G_v/I_v$ denotes the geometric Frobenius element.

0.9. Duality for Selmer complexes

In order to obtain a duality result similar to 0.7.1 for various $\widetilde{\mathbf{R}\Gamma}_f(X)$, it is necessary to impose suitable "orthogonality constraints" on the local conditions. For example, we require, for all $v \in S_f$, the composite morphism of complexes

$$U_v^-(X) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v, X) \xrightarrow{\alpha} D\big(C^\bullet_{\mathrm{cont}}(G_v, D(X)(1))\big)[-2] \longrightarrow D\big(U_v^+(D(X)(1))\big)[-2]$$

(in which the map α underlies Tate's local duality) to be a quasi-isomorphism.

For Greenberg's local conditions 0.8.1, this follows from a suitable orthogonality of X_v^+ and $D(X)(1)_v^+$.

This implies that the following pairs of Selmer complexes are related by Pontrjagin duality:

$$\begin{array}{ccc} \widetilde{\mathbf{R}\Gamma}_f(T) & \overset{D}{\longleftrightarrow} & \widetilde{\mathbf{R}\Gamma}_f(A^*(1))\,[3] \\ \widetilde{\mathbf{R}\Gamma}_f(A) & \overset{D}{\longleftrightarrow} & \widetilde{\mathbf{R}\Gamma}_f(T^*(1))\,[3] \end{array} \tag{0.9.1}$$

In general, (0.9.1) cannot be completed to a full duality diagram

$$\begin{array}{ccc} \widetilde{\mathbf{R}\Gamma}_f(T) & \overset{\mathscr{D}}{\longleftrightarrow} & \widetilde{\mathbf{R}\Gamma}_f(T^*(1))\,[3] \\ \Big\downarrow \Phi & \overset{D}{\times} & \Big\downarrow \Phi \\ \widetilde{\mathbf{R}\Gamma}_f(A) & & \widetilde{\mathbf{R}\Gamma}_f(A^*(1))\,[3], \end{array} \tag{0.9.2}$$

as the local conditions need not be compatible with respect to Φ. In other words, there is an exact (= distinguished) triangle

$$\Phi(U_v^+(T)) \longrightarrow U_v^+(A) \longrightarrow \mathrm{Err}_v(\Phi, T)$$

involving an "error term" $\mathrm{Err}_v(\Phi, T)$, which leads to another exact triangle

$$\Phi\big(\widetilde{\mathbf{R}\Gamma}_f(T)\big) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(A) \longrightarrow \bigoplus_{v \in S_f} \mathrm{Err}_v(\Phi, T).$$

For Greenberg's local conditions 0.8.1 we have $\mathrm{Err}_v(\Phi, T) = 0$ for $v \in \Sigma$, but not for $v \in \Sigma'$, in general. For example, assume that $R = \mathbf{Z}_p$ and T is a free $\mathbf{Z}_p$-module of finite rank, concentrated in degree zero. As before, $A = V/T$ for $V = T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

The unramified local conditions at $v \in \Sigma'$ are quasi-isomorphic to

$$U_v^+(X) = \Big[X^{I_v} \xrightarrow{f_v - 1} X^{I_v}\Big] \qquad (X = T, A),$$

hence $\Phi(U_v^+(T))$ is quasi-isomorphic to

$$\Big[V^{I_v}/T^{I_v} \xrightarrow{f_v - 1} V^{I_v}/T^{I_v}\Big] = \Big[(A^{I_v})_{\mathrm{div}} \xrightarrow{f_v - 1} (A^{I_v})_{\mathrm{div}}\Big].$$

It follows that $\mathrm{Err}_v(\Phi, T)$ is quasi-isomorphic to

$$\Big[(A^{I_v})/(A^{I_v})_{\mathrm{div}} \xrightarrow{f_v - 1} (A^{I_v})/(A^{I_v})_{\mathrm{div}}\Big] = \Big[H^1(I_v, T)_{\mathrm{tors}} \xrightarrow{f_v - 1} H^1(I_v, T)_{\mathrm{tors}}\Big].$$

The cohomology groups of $\mathrm{Err}_v(\Phi, T)$ are finite groups of common order

$$|H^0(\mathrm{Err}_v(\Phi, T))| = |H^1(\mathrm{Err}_v(\Phi, T))| = \Big|H^1(I_v, T)_{\mathrm{tors}}^{f_v = 1}\Big|,$$

equal to the "local Tamagawa factor" of T at $v \in \Sigma'$, which appears in the formulation of the Bloch-Kato conjecture in the language of [**Fo-PR**] (this is a generalization of the "fudge factors" in the conjecture of Birch and Swinnerton-Dyer).

Similarly, there is an error term for the horizontal arrow $\mathscr{D}$ in (0.9.2). Under a suitable boundedness hypothesis, the arrow corresponds to a cup product

$$\widetilde{\mathbf{R}\Gamma}_f(T)\overset{\mathbf{L}}{\otimes}_R \widetilde{\mathbf{R}\Gamma}_f(T^*(1)) \longrightarrow \omega[-3]. \tag{0.9.3}$$

0.10. Comparison with classical Selmer groups

Let E be an elliptic curve over $\mathbf{Q}$, with good reduction outside a finite set of primes $S \supset \{p, \infty\}$. The classical p-power descent on E gives rise to Selmer groups

$$\mathrm{Sel}(G_{\mathbf{Q},S}, -) \subset H^1(G_{\mathbf{Q},S}, -)$$

(for $- = T = T_p(E)$ or $- = A = E[p^\infty]$), sitting in exact sequences

$$0 \longrightarrow E(\mathbf{Q}) \otimes \mathbf{Z}_p \longrightarrow \mathrm{Sel}(G_{\mathbf{Q},S}, T) \longrightarrow T_p\text{Ш}(E/\mathbf{Q}) \longrightarrow 0$$

$$0 \longrightarrow E(\mathbf{Q}) \otimes \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow \mathrm{Sel}(G_{\mathbf{Q},S}, A) \longrightarrow \text{Ш}(E/\mathbf{Q})[p^\infty] \longrightarrow 0.$$

We also use the notation

$$\mathrm{Sel}(G_{\mathbf{Q},S}, V) = \mathrm{Sel}(G_{\mathbf{Q},S}, T) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

for $V = V_p(E) = T_p(E) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

Assume that E has *ordinary reduction*[(4)] at p and, for simplicity, that $p \neq 2$. As a representation of $G_p = G_{\mathbf{Q}_p}$, the Tate module T is reducible. There is an exact sequence of $\mathbf{Z}_p[G_p]$-modules

$$0 \longrightarrow T_p^+ \longrightarrow T \longrightarrow T_p^- \longrightarrow 0,$$

in which each $T_p^\pm$ is free of rank 1 over $\mathbf{Z}_p$ and I_p acts trivially on T_p^- (*i.e.*, T_p^- is unramified). Consider Greenberg's local conditions for T, given by

$$U_v^+(T) = \begin{cases} C^\bullet_{\mathrm{cont}}(G_p, T_p^+), & v = p \\ C^\bullet_{\mathrm{cont}}(G_v/I_v, T^{I_v}), & v \in S_f,\ v \neq p. \end{cases}$$

One deduces from (0.8.0.1) an exact sequence (cf. 9.6.3, 9.6.7)

$$0 \longrightarrow H^0(G_p, T_p^-) \longrightarrow \widetilde{H}^1_f(T) \longrightarrow \mathrm{Sel}(G_{\mathbf{Q},S}, T) \longrightarrow C \longrightarrow 0,$$

in which C is a finite group and

$$H^0(G_p, T_p^-) = \begin{cases} \mathbf{Z}_p, & \text{if } E \text{ has split multiplicative reduction at } p \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $\widetilde{H}^1_f(T)$ is an "extended Selmer group" in the sense of Mazur, Tate and Teitelbaum [**M-T-T**] and the term $H^0(G_p, T_p^-)$ detects the presence of a "trivial zero" of the p-adic L-function of E. This is one of the simplest instances of the following general principle: classical Selmer groups correspond to complex L-functions, while Selmer complexes to p-adic L-functions.

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(4) *i.e.*, either good ordinary or multiplicative reduction.

0.11. Iwasawa theory

The formalism of big Galois representations greatly simplifies Iwasawa theory. Let K be a number field and S as in 0.7.1. Assume that we are given an intermediate Galois extension $K \subset K_\infty \subset K_S$ with $\Gamma = \mathrm{Gal}(K_\infty/K) \xrightarrow{\sim} \mathbf{Z}_p^r$ for some $r \geqslant 1$ (in fact, one can treat in the same way also "tame Iwasawa theory", when $\Gamma \xrightarrow{\sim} \mathbf{Z}_p^r \times \Delta$, for a finite abelian group Δ). Put $G = G_{K,S}$.

Writing $K_\infty = \bigcup K_\alpha$ as a union of finite extensions of K, we define the Iwasawa algebra of Γ over R as

$$\overline{R} = R[\![\Gamma]\!] = \varprojlim_\alpha R[\mathrm{Gal}(K_\alpha/K)].$$

As in the classical case ($R = \mathbf{Z}_p$), any choice of an isomorphism $\Gamma \xrightarrow{\sim} \mathbf{Z}_p^r$ gives an isomorphism of R-algebras

$$\overline{R} \xrightarrow{\sim} R[\![X_1, \ldots, X_r]\!].$$

We denote by

$$\chi_\Gamma : G \twoheadrightarrow \Gamma \hookrightarrow R[\Gamma]^*$$

the tautological character of G and by

$$\iota : \overline{R} \longrightarrow \overline{R}$$

the R-linear involution satisfying $\iota(\gamma) = \gamma^{-1}$ for all $\gamma \in \Gamma$.

If M is an $\overline{R}[G]$-module and $n \in \mathbf{Z}$, we define $\overline{R}[G]$-modules $M < n >$ and M^ι by requiring that

(i) $M < n >= M$ as an $\overline{R}$-module; $g \in G$ acts by

$$g_{M<n>} = \chi_\Gamma(g)^n g_M.$$

(ii) $M^\iota = M$ as an $R[G]$-module; $\gamma \in \Gamma$ acts by

$$\gamma_{M^\iota} = \iota(\gamma)_M = (\gamma^{-1})_M.$$

0.12. Duality for Galois cohomology in Iwasawa theory

The main point is that cohomology of R-representations over K_∞ can be expressed in terms of cohomology of $\overline{R}$-representations over K (cf. [**Gre4**, Prop. 3.2]; [**Col**, Prop. 2]).

Let T, T^* (resp., A, A^*) be bounded complexes of admissible $R[G]$-modules, of finite (resp., co-finite) type over R. In Iwasawa theory one is often interested in the cohomology groups

$$H^i_{\mathrm{Iw}}(K_\infty/K, T) = \varprojlim_\alpha H^i_{\mathrm{cont}}(\mathrm{Gal}(K_S/K_\alpha), T)$$

$$H^i(K_S/K_\infty, A) = \varinjlim_\alpha H^i_{\mathrm{cont}}(\mathrm{Gal}(K_S/K_\alpha), A)$$

(and their counterparts with compact support). An easy application of Shapiro's Lemma (cf. Sect. 8.3 and 8.4) shows that these are the cohomology groups of the following objects of $D({}_{\overline{R}}\mathrm{Mod})$:

$$\begin{aligned}\mathbf{R}\Gamma_{\mathrm{Iw}}(K_\infty/K,T) &= \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S},\mathscr{F}_\Gamma(T))\\ \mathbf{R}\Gamma(K_S/K_\infty,A) &= \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S},F_\Gamma(A)),\end{aligned}$$

where

$$\begin{aligned}\mathscr{F}_\Gamma(T) &= (T\otimes_R \overline{R}) < -1 >\\ F_\Gamma(A) &= \mathrm{Hom}_{R,\mathrm{cont}}(\overline{R},A) < -1 >\end{aligned}$$

(and similarly for cohomology with compact support).

The crucial observation (cf. 8.4.6.6) is the following: if T, T^*, A, A^* are related (over R) as in (0.5.1), then $\mathscr{F}_\Gamma(T)$, $\mathscr{F}_\Gamma(T^*)$, $F_\Gamma(A)$, $F_\Gamma(A^*)$ are related by the duality diagram

(0.12.1)
$$\begin{array}{ccc}\mathscr{F}_\Gamma(T) & \overset{\overline{\mathscr{D}}}{\longleftrightarrow} & \mathscr{F}_\Gamma(T^*)^\iota\\ \downarrow{\scriptstyle\overline{\Phi}} & \overset{\overline{D}}{\times} & \downarrow{\scriptstyle\overline{\Phi}}\\ F_\Gamma(A) & & F_\Gamma(A^*)^\iota\end{array}$$

over $\overline{R}$ (here we use the notation $\overline{\mathscr{D}}(-) = \mathbf{R}\mathrm{Hom}_{\overline{R}}(-,\omega_{\overline{R}})$, and similarly for $\overline{\Phi}$ and $\overline{D}$).

Applying the Poitou-Tate duality 0.7.1 (over $\overline{R}$) to (0.12.1), we obtain a duality diagram in $D({}_{\overline{R}}\mathrm{Mod})$

(0.12.2)
$$\begin{array}{ccc}\mathbf{R}\Gamma_{\mathrm{Iw}}(K_\infty/K,T) & \overset{\overline{\mathscr{D}}}{\longleftrightarrow} & \mathbf{R}\Gamma_{c,\mathrm{Iw}}(K_\infty/K,T^*(1))^\iota[3]\\ \downarrow{\scriptstyle\overline{\Phi}} & \overset{\overline{D}}{\times} & \downarrow{\scriptstyle\overline{\Phi}}\\ \mathbf{R}\Gamma(K_S/K_\infty,A) & & \mathbf{R}\Gamma_c(K_S/K_\infty,A^*(1))^\iota[3]\end{array}$$

and spectral sequences

(0.12.3)
$$\begin{aligned}E_2^{i,j} &= \mathbb{E}\mathrm{xt}^i_{\overline{R}}(\overline{D}(H^j(K_S/K_\infty,A)),\omega_{\overline{R}}) = \mathbb{E}\mathrm{xt}^i_{\overline{R}}(H^{3-j}_{c,\mathrm{Iw}}(K_\infty/K,T^*(1)),\omega_{\overline{R}})^\iota\\ &\qquad\qquad \Longrightarrow H^{i+j}_{\mathrm{Iw}}(K_\infty/K,T)\\ {}'E_2^{i,j} &= \mathbb{E}\mathrm{xt}^i_{\overline{R}}(\overline{D}(H^j_c(K_S/K_\infty,A)),\omega_{\overline{R}}) = \mathbb{E}\mathrm{xt}^i_{\overline{R}}(H^{3-j}_{\mathrm{Iw}}(K_\infty/K,T^*(1)),\omega_{\overline{R}})^\iota\\ &\qquad\qquad \Longrightarrow H^{i+j}_{c,\mathrm{Iw}}(K_\infty/K,T)\end{aligned}$$

In the classical case $R = \mathbf{Z}_p$ the ring $\overline{R} = \mathbf{Z}_p[\![\Gamma]\!] = \Lambda$ is the usual Iwasawa algebra. The spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_{\overline{R}}(\overline{D}(H^j(K_S/K_\infty,A)),\Lambda) \Longrightarrow H^{i+j}_{\mathrm{Iw}}(K_\infty/K,T)$$

was in this case constructed in an unpublished note of Jannsen [**Ja3**] (who also considered the case of non-commutative Γ).

Back to the general case, recall that an $\overline{R}$-module M is *pseudo-null* if it is finitely generated and its support $\mathrm{supp}(M)$ has codimension $\geqslant 2$ in $\mathrm{Spec}(\overline{R})$. As $\mathrm{codim}_{\overline{R}}(\mathrm{supp}(E_2^{i,j})) \geqslant i$, the spectral sequence E_r degenerates in the quotient category $\big({}_{\overline{R}}\mathrm{Mod}\big)/(\text{pseudo-null})$ into short exact sequences

$$0 \longrightarrow E_2^{1,n-1} \longrightarrow H^n_{\mathrm{Iw}}(K_\infty/K, T) \longrightarrow E_2^{0,n} \longrightarrow 0,$$

in which $E_2^{0,n}$ has no $\overline{R}$-torsion and $E_2^{1,n-1}$ has support in codimension $\geqslant 1$.

0.13. Duality for Selmer complexes in Iwasawa theory

Given suitably compatible systems of local conditions $U_v^+(X)$ along the tower of fields $\{K_\alpha\}$, one can define Selmer complexes

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, X),\ \widetilde{\mathbf{R}\Gamma}_f(K_S/K_\infty, Y) \qquad (X = T, T^*(1),\ Y = A, A^*(1)).$$

Although over each finite layer K_α the diagram (0.9.2) may involve non-zero error terms, the limit of these error terms over K_∞ is very often pseudo-null (or co-pseudo-null).

For example, Greenberg's local conditions 0.8.1 induce similar local conditions over each K_α. If we assume that no prime $v \in \Sigma'$ splits completely in K_∞/K, then (cf. 8.9.9)

(0.13.1)
$$\begin{array}{ccc} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, T) & \xleftrightarrow{\ \mathscr{D}\ } & \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1))^\iota[3] \\ \big\downarrow \overline{\Phi} & \overline{D}\ (\text{crossed arrows}) & \big\downarrow \overline{\Phi} \\ \widetilde{\mathbf{R}\Gamma}_f(K_S/K_\infty, A) & & \widetilde{\mathbf{R}\Gamma}_f(K_S/K_\infty, A^*(1))^\iota[3] \end{array}$$

becomes a duality diagram without any error terms, if we consider the top (resp., bottom) two objects in $D_{ft}(\big({}_{\overline{R}}\mathrm{Mod}\big)/(\text{pseudo-null}))$ (resp., in $D_{coft}({}_{\overline{R}}\mathrm{Mod}/(\text{co-pseudo-null})))$. Equivalently, for every prime ideal $\overline{\mathfrak{p}} \in \mathrm{Spec}(\overline{R})$ of height $\mathrm{ht}(\overline{\mathfrak{p}}) = 1$, the localization of (0.13.1) at $\overline{\mathfrak{p}}$ is a duality diagram in $D({}_{\overline{R}_{\overline{\mathfrak{p}}}}\mathrm{Mod})$. As in 0.12, this leads to exact sequences of $\overline{R}_{\overline{\mathfrak{p}}}$-modules

(0.13.2)
$$0 \longrightarrow \mathrm{Ext}^1_{\overline{R}}\big(\widetilde{H}^{4-q}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1)), \omega_{\overline{R}}\big)^\iota_{\overline{\mathfrak{p}}} \longrightarrow \widetilde{H}^q_{f,\mathrm{Iw}}(K_\infty/K, T)_{\overline{\mathfrak{p}}}$$
$$\longrightarrow \mathrm{Hom}_{\overline{R}}\big(\widetilde{H}^{3-q}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1)), \omega_{\overline{R}}\big)^\iota_{\overline{\mathfrak{p}}} \longrightarrow 0,$$

in which $X^\iota_{\overline{\mathfrak{p}}}$ is a shorthand for $(X^\iota)_{\overline{\mathfrak{p}}}$ and

$$\widetilde{H}^q_{f,\mathrm{Iw}}(K_\infty/K, T^*(1)) \xrightarrow{\sim} \overline{D}\big(\widetilde{H}^{3-q}_f(K_S/K_\infty, A)^\iota\big)$$

(and similarly for T and $A^*(1)$). If R has no embedded primes, then we obtain isomorphisms

$$\widetilde{H}^q_{f,\mathrm{Iw}}(K_\infty/K,T)_{\overline{R}\text{-tors}} \xrightarrow{\sim} \mathrm{Ext}^1_{\overline{R}}\big(\widetilde{H}^{4-q}_{f,\mathrm{Iw}}(K_\infty/K,T^*(1)),\omega_{\overline{R}}\big)^\iota$$

in $\big({}_{\overline{R}}\mathrm{Mod}\big)/(\text{pseudo-null})$.

If there is a prime $v \in \Sigma'$ that splits completely in K_∞/K, then the above statements hold for those prime ideals $\overline{\mathfrak{p}} \in \mathrm{Spec}(\overline{R})$ of height $\mathrm{ht}(\overline{\mathfrak{p}}) = 1$ which are not of the form $\overline{\mathfrak{p}} = \mathfrak{p}\overline{R}$, where $\mathfrak{p} \in \mathrm{Spec}(R)$ has $\mathrm{ht}(\mathfrak{p}) = 1$ (cf. 8.9.8).

In particular, if R is regular and no prime $v \in \Sigma'$ splits completely in K_∞/K, then the $\overline{R}$-modules

$$\big(\overline{D}(\widetilde{H}^1_f(K_S/K_\infty,A))\big)_{\mathrm{tors}} \text{ and } \big(\overline{D}\big(\widetilde{H}^1_f(K_S/K_\infty,A^*(1))\big)\big)^\iota_{\mathrm{tors}}$$

are pseudo-isomorphic. This is a generalization of Greenberg's results ([**Gre2**, Thm. 2]; [**Gre3**, Thm. 1]), according to which for $R = \mathbf{Z}_p$ and K_∞/K the cyclotomic $\mathbf{Z}_p$-extension(5) the two Λ-modules in question have the same characteristic power series (more precisely, Greenberg works with his "strict Selmer groups" $S^{\mathrm{str}}_A(K_\infty)$ and $S^{\mathrm{str}}_{A^*(1)}(K_\infty)$; their relation to our $\widetilde{H}^1_f$ is explained in 9.6). A similar result for Selmer groups of abelian varieties defined in terms of flat cohomology was proved by Wingberg [**Win**].

If the complex $T = \sigma_{\leqslant 0}T$ is concentrated in non-positive degrees, then one can say much more: the horizontal arrow in (0.13.1) becomes an isomorphism after tensoring with $R_{\mathfrak{q}}$, for any minimal prime $\mathfrak{q}$ of R (Sect. 8.9.11–Sect. 8.9.12).

Greenberg [**Gre2**] also defined "non-strict" Selmer groups $S_A \supseteq S^{\mathrm{str}}_A$. One of their interesting features is the fact that a trivial zero over K can sometimes be detected by the Λ-module $S_A(K_\infty)$ (but *not* by the Selmer group $S_A(K)$ over K). Although the Pontrjagin duals of $S_A(K_\infty)$ and $\widetilde{H}^1_f(K_S/K_\infty,A)$ may often have the same characteristic power series, they need not be isomorphic as Λ-modules, as $S^{\mathrm{str}}_A(K_\infty)$ is a subgroup of $S_A(K_\infty)$, but a quotient of $\widetilde{H}^1_f(K_S/K_\infty,A)$ (cf. 9.6.2–9.6.6). It seems that in the presence of a trivial zero $\widetilde{H}^1_f(K_S/K_\infty,A)$ has better semi-simplicity properties than $S_A(K_\infty)$.

0.14. Classical Iwasawa theory

0.14.0. Traditionally, the main objects of interest in Iwasawa theory have been the following:

(i) The Galois group $\mathrm{Gal}(M_\infty/K_\infty)$ of the maximal abelian pro-p-extension of K_∞, unramified outside primes above S.

(5) And assuming, in addition, that both $\overline{D}\big(\widetilde{H}^1_f(K_S/K_\infty,A)\big)$ and $\overline{D}\big(\widetilde{H}^1_f(K_S/K_\infty,A^*(1))\big)$ are torsion over $\Lambda = \mathbf{Z}_p[\![\Gamma]\!]$.

(ii) The projective (resp., inductive) limit X_∞ (resp., A_∞) of the p-primary parts of the ideal class groups of $\mathcal{O}_{K_\alpha}$.

(iii) The projective (resp., inductive) limit X'_∞ (resp., A'_∞) of the p-primary parts of the ideal class groups of $\mathcal{O}_{K_\alpha,S_\alpha}$, where S_α is the set of primes of K_α above S.

These are closely related to $\mathbf{R}\Gamma_{\mathrm{Iw}}(K_\infty/K,T)$ and $\mathbf{R}\Gamma_{c,\mathrm{Iw}}(K_\infty/K,T)$ for $T = \mathbf{Z}_p, \mathbf{Z}_p(1)$ (with $R = \mathbf{Z}_p$, $\overline{R} = \Lambda = \mathbf{Z}_p[\![\Gamma]\!]$). Can one obtain anything interesting from the general machinery (Sect. 0.12–Sect. 0.13) in this classical setup?

0.14.1. First of all, the spectral sequence ${}'E_r$ in (0.12.3) for $T = \mathbf{Z}_p$ and $T = \mathbf{Z}_p(1)$ gives very short proofs of the following well-known results (cf. 9.3):

(i) The Pontrjagin dual of A'_∞ contains no non-zero pseudo-null Λ-submodules.

(ii) If the weak Leopoldt conjecture holds for K_∞ (*i.e.*, $H^2(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p) = 0$), then $\mathrm{Gal}(M_\infty/K_\infty)$ contains no non-zero pseudo-null Λ-submodules.

0.14.2. For $\Gamma \xrightarrow{\sim} \mathbf{Z}_p$, Iwasawa [**Iw**] constructed canonical isomorphisms in $({}_\Lambda\mathrm{Mod})/(\text{pseudo-null})$

$$\mathrm{Ext}^1_\Lambda(X_\infty,\Lambda) \xrightarrow{\sim} \overline{D}(A_\infty), \quad \mathrm{Ext}^1_\Lambda(X'_\infty,\Lambda) \xrightarrow{\sim} \overline{D}(A'_\infty).$$

One expects analogous statements to hold for arbitrary $\Gamma \xrightarrow{\sim} \mathbf{Z}_p^r$ (cf. [**McCa2**]).

Our machinery gives only partial results in this direction, such as the following (cf. 9.4–9.5):

(i) There is a canonical morphism of Λ-modules

$$\alpha' : X'_\infty \longrightarrow \mathrm{Ext}^1_\Lambda(\overline{D}(A'_\infty),\Lambda).$$

(ii) $\mathrm{Coker}(\alpha')$ is almost pseudo-null in the sense that there is an explicit finite set P of height one prime ideals $\mathfrak{p} \in \mathrm{Spec}(\Lambda)$ such that $\mathrm{Coker}(\alpha')_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(\Lambda)$, $\mathrm{ht}(\mathfrak{p}) = 1$, $\mathfrak{p} \notin P$.

(iii) The characteristic power series of $\overline{D}(A'_\infty)$ divides that of X'_∞.

Slightly weaker results can be proved for X_∞ and A_∞.

In 1998 the author announced a proof of the fact that $\mathrm{Coker}(\alpha')$ is pseudo-null. Unfortunately, the argument for exceptional $\mathfrak{p} \in P$ turned out to be flawed, which means that the claim has to be retracted.

0.14.3. Greenberg's local conditions have built into them a fundamental base change property (cf. 8.10.1)

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K,T)\overset{\mathbf{L}}{\otimes}_{\overline{R}}R \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_f(T)$$

with respect to the augmentation map $\overline{R} \to R$. This can be interpreted as a derived version of Mazur's "control theorem" for Selmer groups, according to which the canonical map

$$\mathrm{Sel}(G_{K,S}, E[p^\infty]) \longrightarrow \mathrm{Sel}(G_{K_\infty,S}, E[p^\infty])^\Gamma$$

has finite kernel and cokernel (assuming that $\Gamma \xrightarrow{\sim} \mathbf{Z}_p$ and E has good ordinary reduction at all primes dividing p).

More generally, there are canonical isomorphisms

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K,T)\overset{\mathbf{L}}{\otimes}_{\overline{R}} R[\![\mathrm{Gal}(L/K)]\!] \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T) \tag{0.14.3.1}$$

for arbitrary intermediate fields $K \subset L \subset K_\infty$.

0.14.4. A typical situation in which Mazur's control theorem fails for the classical Selmer groups but holds for the extended Selmer groups $\widetilde{H}^1_f(-, E[p^\infty])$ is the following: E is an elliptic curve defined over $\mathbf{Q}$ with multiplicative reduction at p and K_∞/K is the anti-cyclotomic $\mathbf{Z}_p$-extension of an imaginary quadratic field K in which p is inert. Note that a trivial zero is again lurking behind this example.

0.14.5. Our duality results also show that, in the classical case when $R = \mathbf{Z}_p$ and $T = T^0$ is concentrated in degree zero, the objects $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K,T)$ can often be represented by relatively simple complexes (see 9.7), which then control $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T)$ and the corresponding height pairings for *all* subextensions L/K of K_∞/K, thanks to (0.14.3.1). These results were used, in the context of classical Selmer groups, in the work of Mazur and Rubin [**M-R2**] on "organizing complexes" in Iwasawa theory of elliptic curves.

0.15. Generalized Cassels-Tate pairings

One of the main applications of the duality theory for Selmer complexes is a construction of higher-dimensional generalizations of Cassels-Tate pairings (cf. Chapter 10). These pairings are used in 10.7 to prove several versions of the following general principle (for Greenberg's local conditions): the parity of ranks of *extended* Selmer groups $\widetilde{H}^1_f(T_\lambda)$ associated to a one-parameter family T_λ of self-dual Galois representations (with respect to a family of skew-symmetric isomorphisms $T_\lambda \xrightarrow{\sim} T^*_\lambda(1)$ respecting the local conditions) is constant. In [**N-P, Ne3, Ne5**] and in Chapter 12 we deduce from this principle parity results for ranks of Selmer groups associated to modular forms, elliptic curves and Hilbert modular forms, respectively.

On should keep in mind the following topological analogue (see 10.1): if X is a compact oriented topological manifold of (real) dimension 3, then Poincaré duality with finite coefficients induces a non-degenerate symmetric pairing

$$H^2(X,\mathbf{Z})_{\mathrm{tors}} \times H^2(X,\mathbf{Z})_{\mathrm{tors}} \longrightarrow \mathbf{Q}/\mathbf{Z}.$$

0.15.0. Let us return to the exact sequence (0.13.2), assuming in addition that $\mathrm{depth}(\overline{R}_{\overline{\mathfrak{p}}}) = \dim(\overline{R}_{\overline{\mathfrak{p}}}) = 1$. Under this assumption the first term in (0.13.2) is canonically isomorphic to

$$\mathrm{Hom}_{\overline{R}_{\overline{\mathfrak{p}}}}\Big(\big(\widetilde{H}^{4-q}_{f,\mathrm{Iw}}(K_\infty/K,T^*(1))^{\iota}_{\overline{\mathfrak{p}}}\big)_{(\overline{R}_{\overline{\mathfrak{p}}})\text{-tors}}, I_{\overline{R}_{\overline{\mathfrak{p}}}}\Big),$$

where $I_{\overline{R}_{\overline{\mathfrak{p}}}} \xrightarrow{\sim} H^0(\omega_{\overline{R}})_{\overline{\mathfrak{p}}} \otimes_{\overline{R}_{\overline{\mathfrak{p}}}} \big(\mathrm{Frac}(\overline{R}_{\overline{\mathfrak{p}}})/\overline{R}_{\overline{\mathfrak{p}}}\big)$ denotes the injective hull of the $(\overline{R}_{\overline{\mathfrak{p}}})$-module $\overline{R}_{\overline{\mathfrak{p}}}/\overline{\mathfrak{p}}\overline{R}_{\overline{\mathfrak{p}}}$ (*e.g.* $I_{\overline{R}_{\overline{\mathfrak{p}}}} = \mathrm{Frac}(\overline{R}_{\overline{\mathfrak{p}}})/\overline{R}_{\overline{\mathfrak{p}}}$, if $\overline{R}_{\overline{\mathfrak{p}}}$ is a discrete valuation ring). As a result, one obtains a non-degenerate bilinear form (cf. 10.3.3, 10.5.5)

$$\big(\widetilde{H}^q_{f,\mathrm{Iw}}(K_\infty/K,T)_{\overline{\mathfrak{p}}}\big)_{(\overline{R}_{\overline{\mathfrak{p}}})\text{-tors}} \times \big(\widetilde{H}^{4-q}_{f,\mathrm{Iw}}(K_\infty/K,T^*(1))^{\iota}_{\overline{\mathfrak{p}}}\big)_{(\overline{R}_{\overline{\mathfrak{p}}})\text{-tors}} \longrightarrow I_{\overline{R}_{\overline{\mathfrak{p}}}}.$$

This pairing is of particular interest for $q=2$. In the **self-dual case**, *i.e.*, when there is a skew-symmetric isomorphism $T_{\mathfrak{p}} \xrightarrow{\sim} T^*(1)_{\mathfrak{p}}$ compatible with isomorphisms $(T_v^+)_{\mathfrak{p}} \xrightarrow{\sim} ((T^*(1))_v^+)_{\mathfrak{p}}$ for all $v \in \Sigma$, then the induced pairing

$$(0.15.0.1)\quad \langle\ ,\ \rangle : \Big(\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K,T)_{\overline{\mathfrak{p}}}\Big)_{(\overline{R}_{\overline{\mathfrak{p}}})\text{-tors}} \times \Big(\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K,T)^{\iota}_{\overline{\mathfrak{p}}}\Big)_{(\overline{R}_{\overline{\mathfrak{p}}})\text{-tors}} \longrightarrow I_{\overline{R}_{\overline{\mathfrak{p}}}}$$

is skew-Hermitian (cf. 10.3.4.2).

0.15.1. All of the above makes sense in the absence of Γ (*i.e.*, for $\overline{R}=R$ and $\overline{\mathfrak{p}}=\mathfrak{p}$), when we obtain bilinear forms (cf. 10.3.2, 10.5.3)

$$\Big(\widetilde{H}^q_f(T)_{\mathfrak{p}}\Big)_{R_{\mathfrak{p}}\text{-tors}} \times \Big(\widetilde{H}^{4-q}_f(T^*(1))_{\mathfrak{p}}\Big)_{R_{\mathfrak{p}}\text{-tors}} \longrightarrow I_{R_{\mathfrak{p}}},$$

which can be degenerate (because of the presence of error terms in (0.9.2)).

In the self-dual case, the induced pairing

$$\Big(\widetilde{H}^2_f(T)_{\mathfrak{p}}\Big)_{R_{\mathfrak{p}}\text{-tors}} \times \Big(\widetilde{H}^2_f(T)_{\mathfrak{p}}\Big)_{R_{\mathfrak{p}}\text{-tors}} \longrightarrow I_{R_{\mathfrak{p}}}$$

is *skew-symmetric* (cf. 10.2.5).

For $R=\mathbf{Z}_p$, $\mathfrak{p}=(p)$ and $T=T_p(E)$ (where E is an elliptic curve with ordinary reduction at all primes above p), we recover essentially the classical Cassels-Tate pairing on the quotient of $\mathrm{Sel}(G_{K,S},E[p^\infty])$ by its maximal divisible subgroup (combining 10.8.7 with 9.6.7.3 and 9.6.3).

Perhaps the simplest non-classical example comes from Hida theory (cf. [**N-P**]). For simplicity, let us begin with an elliptic curve E over $\mathbf{Q}$, with good ordinary reduction at p (if $p=2$, then the following discussion has to be slightly modified). It is known that E is modular [**B-C-D-T**], hence $L(E,s)=L(f_E,s)$ for a newform $f_E \in S_2(\Gamma_0(N),\mathbf{Z})$, where N is the conductor of E.

Our assumptions imply that $p \nmid N$ and $f_E = f_2$ is a member of a Hida family of ordinary eigenforms[(6)] $f_k \in S_k(\Gamma_0(N),\mathbf{Z}_p)$, for weights $k \in \mathbf{Z}_{\geqslant 2}$ sufficiently close to 2 in the p-adic space of weights $\mathbf{Z}_p \times \mathbf{Z}/(p-1)\mathbf{Z}$.

The Galois representations associated to various f_k can be interpolated by a big Galois representation

$$\rho : G_{\mathbf{Q},S} \longrightarrow \mathrm{Aut}_R(T),$$

[(6)] In this introduction we ignore the phenomenon of p-stabilization.

where S consists of all primes dividing Np (and infinity) and R, T are as in (0.1.2). Replacing T by a suitable twist ([**N-P**, §3.2.3]), one obtains a representation (also denoted by T) with the following properties:

(i) There is a prime ideal $\mathcal{P} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathcal{P}) = 1$ such that

$$(T/\mathcal{P}T) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \xrightarrow{\sim} V_p(E).$$

Moreover, $R_{\mathcal{P}}$ is a discrete valuation ring, unramified over $\Lambda_{\mathcal{P}\cap\Lambda}$.

(ii) T is self-dual, *i.e.*, there is a skew-symmetric isomorphism

$$T \xrightarrow{\sim} T^*(1) = \mathrm{Hom}_{\Lambda}(T, \Lambda)(1).$$

(iii) There is a self-dual exact sequence of $R_{\mathcal{P}}[G_{\mathbf{Q}_p}]$-modules

$$0 \longrightarrow T^+_{\mathcal{P}} \longrightarrow T_{\mathcal{P}} \longrightarrow T^-_{\mathcal{P}} \longrightarrow 0,$$

in which $T^{\pm}_{\mathcal{P}}$ is free of rank one over $R_{\mathcal{P}}$ and there is an isomorphism $T^+_{\mathcal{P}}/\mathcal{P}T^+_{\mathcal{P}} \xrightarrow{\sim} V_p(E)^+$ (compatible with that in (i)).

The corresponding big Selmer complex $\widetilde{\mathbf{R}\Gamma}_f(G_{\mathbf{Q},S}, T_{\mathcal{P}})$ satisfies the base change property 12.7.13.4 (i)

$$\widetilde{\mathbf{R}\Gamma}_f(T_{\mathcal{P}}) \overset{\mathbf{L}}{\otimes}_{R_{\mathcal{P}}} R_{\mathcal{P}}/\mathcal{P} \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_f(V_p(E)), \tag{0.15.1.1}$$

which gives an exact cohomology sequence

$$0 \longrightarrow \widetilde{H}^1_f(T_{\mathcal{P}})/\mathcal{P} \longrightarrow \widetilde{H}^1_f(V_p(E)) \longrightarrow \widetilde{H}^2_f(T_{\mathcal{P}})[P] \longrightarrow 0,$$

in which the middle term is equal to the classical Selmer group $\mathrm{Sel}(G_{\mathbf{Q},S}, V_p(E))$.

The existence of a canonical non-degenerate skew-symmetric pairing

$$\widetilde{H}^2_f(T_{\mathcal{P}})_{R_{\mathcal{P}}\text{-tors}} \times \widetilde{H}^2_f(T_{\mathcal{P}})_{R_{\mathcal{P}}\text{-tors}} \longrightarrow \mathrm{Frac}(R_{\mathcal{P}})/R_{\mathcal{P}}$$

then implies the following result (see 12.7.13.5).

0.15.2. Let E be an elliptic curve over $\mathbf{Q}$ with a good ordinary reduction at p. Then:

(i) There exists a canonical decreasing filtration by $\mathbf{Q}_p$-vector spaces on $S = \mathrm{Sel}(G_{\mathbf{Q},S}, V_p(E))$:

$$S = S^1 \supseteq S^2 \supseteq \cdots .$$

(ii) There exist non-degenerate skew-symmetric pairings

$$S^i/S^{i+1} \times S^i/S^{i+1} \longrightarrow \mathbf{Q}_p \quad (i \geqslant 1)$$

depending on the choice of an isomorphism $\Gamma = 1+q\mathbf{Z}_p \xrightarrow{\sim} \mathbf{Z}_p$, and otherwise canonical (where $q = p$ (resp., $q = 4$) if $p \neq 2$ (resp., $p = 2$)).

(iii) The common kernel

$$S^{\infty} = \bigcap_{i\geqslant 1} S^i$$

is equal to the "generic subspace of S"

$$S^{\mathrm{gen}} = \mathrm{Im}\big(\widetilde{H}^1_f(T_{\mathcal{P}}) \longrightarrow \widetilde{H}^1_f(V_p(E))\big).$$

In particular,

$$\dim_{\mathbf{Q}_p}(S) \equiv \dim_{\mathbf{Q}_p}(S^{\mathrm{gen}}) \pmod 2.$$

Above, $\dim_{\mathbf{Q}_p}(S^{\mathrm{gen}}) = \dim_{\mathscr{L}} \widetilde{H}^1_f(\mathscr{V})$, where $\mathscr{L} = \mathrm{Frac}(R_{\mathcal{P}})$, $\mathscr{V} = T \otimes_R \mathscr{L}$ and $\widetilde{H}^1_f(\mathscr{V}) = \widetilde{H}^1_f(T_{\mathcal{P}}) \otimes_{R_{\mathcal{P}}} \mathscr{L}$ is the Selmer group associated to the whole Hida family passing through f_E.

A similar result holds for extended Selmer groups $\widetilde{H}^1_f$ of self-dual Galois representations associated to Hilbert modular forms (see 12.7.13.5).

The results of [**N-P**] show that certain non-vanishing conjectures for the two-variable p-adic L-function of E would imply (at least for $p > 3$) that

$$\dim_{\mathbf{Q}_p}(S^{\mathrm{gen}}) = \begin{cases} 0, & \text{if } 2 \mid \mathrm{ord}_{s=1} L(E,s) \\ 1, & \text{if } 2 \nmid \mathrm{ord}_{s=1} L(E,s) \end{cases}$$

(in the language of 12.7.13.5, $\widetilde{h}^1_f(\mathbf{Q}, V') = 0$ (resp., $= 1$) for infinitely many $\mathcal{P}'$, hence $\dim_{\mathbf{Q}_p}(S^{\mathrm{gen}}) = \dim F^{\infty}$ is also equal to 0 (resp., to 1)).

0.15.3. Self-duality in Iwasawa theory is more complicated; because of the presence of the involution ι, we obtain skew-Hermitian pairings (Sect. 10.3.4.2 (ii)). In an important *dihedral case* it is possible to get rid of the involution and obtain, as in 0.15.2, skew-symmetric pairings.

Consider, for example, the following situation. Let $K = \mathbf{Q}(\sqrt{D})$, $D < 0$, be an imaginary quadratic field and K_∞/K the *anti-cyclotomic* $\mathbf{Z}_p$-extension of K. This is a dihedral extension of $\mathbf{Q}$, *i.e.*,

$$\Gamma^+ = \mathrm{Gal}(K_\infty/\mathbf{Q}) = \Gamma \rtimes \{1, \tau\},$$

where

$$\tau^2 = 1, \ \tau\gamma\tau^{-1} = \gamma^{-1} \qquad (\gamma \in \Gamma \xrightarrow{\sim} \mathbf{Z}_p).$$

This implies that, for every $R[\Gamma^+]$-module M, the action of $\tau \in \Gamma^+$ induces an isomorphism of $R[\Gamma]$-modules

$$\tau : M \xrightarrow{\sim} M^{\iota}.$$

Applying this remark to $M = \widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, T_p(E))$, where E is an elliptic curve over $\mathbf{Q}$ with good ordinary reduction at p, the non-degenerate skew-Hermitian pairing

$$\langle\ ,\ \rangle : \big(\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, T_p(E))_{\mathfrak{p}}\big)_{\Lambda_{\mathfrak{p}}\text{-tors}} \times \big(\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, T_p(E))_{\mathfrak{p}}\big)_{\Lambda_{\mathfrak{p}}\text{-tors}} \longrightarrow \mathrm{Frac}(\Lambda_{\mathfrak{p}})/\Lambda_{\mathfrak{p}}$$

from (0.15.0.1) (where $R = \mathbf{Z}_p$, $\overline{R} = \Lambda = \mathbf{Z}_p[\![\Gamma]\!]$, $T = T_p(E)$, $\mathfrak{p} \in \mathrm{Spec}(\Lambda) = \mathrm{Spec}(\mathbf{Z}_p[\![\Gamma]\!])$, $\mathrm{ht}(\mathfrak{p}) = 1$, $\mathfrak{p} \neq (p)$) induces a non-degenerate skew-symmetric pairing

$$(\ ,\ ) : \big(\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, T_p(E))_{\mathfrak{p}}\big)_{\Lambda_{\mathfrak{p}}\text{-tors}} \times \big(\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, T_p(E))_{\mathfrak{p}}\big)_{\Lambda_{\mathfrak{p}}\text{-tors}} \longrightarrow \mathrm{Frac}(\Lambda_{\mathfrak{p}})/\Lambda_{\mathfrak{p}},$$

$$(x, y) = \langle x, \tau y \rangle.$$

This implies that, for each prime ideal $\mathfrak{p}$ as above, we have

$$\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, T_p(E))^\iota_{\mathfrak{p}} \xrightarrow{\sim} \left(\overline{D}\left(\mathrm{Sel}(G_{K_\infty,S}, E[p^\infty])\right)\right)_{\mathfrak{p}},$$

with

$$\left(\overline{D}(\mathrm{Sel}(G_{K_\infty,S}, E[p^\infty]))_{\mathfrak{p}}\right)_{\Lambda_{\mathfrak{p}}\text{-tors}} \xrightarrow{\sim} Y \oplus Y$$

for some $\Lambda_{\mathfrak{p}}$-module Y of finite length. The control theorem 0.14.3 for Selmer groups then gives a congruence analogous to that in 0.15.2 (iii)

$$\dim_{\mathbf{Q}_p}\left(\mathrm{Sel}(G_{K,S}, V_p(E))\right) \equiv \mathrm{rk}_\Lambda \overline{D}(\mathrm{Sel}(G_{K_\infty,S}, E[p^\infty])) \pmod 2 \tag{0.15.3.1}$$

(which holds in a general "dihedral" context; see 10.7.19).

If $p \neq 2$ and K satisfies the following "Heegner condition"

(Heeg) Every prime dividing N_E splits in K,

then everything works even for $\mathfrak{p} = (p)$ (see 10.7.18). A recently proved ([**Cor, Va**]) conjecture of Mazur [**Maz2**] implies that, assuming (Heeg), the R.H.S. in (0.15.3.1) is equal to 1. As shown in [**Ne3**], the congruence (0.15.3.1) for suitably chosen K implies that

$$\dim_{\mathbf{Q}_p}\left(\mathrm{Sel}(G_{\mathbf{Q},S}, V_p(E))\right) \equiv \mathrm{ord}_{s=1} L(E,s) \pmod 2$$

(still assuming that E has good ordinary reduction at p).

A generalization of this parity result to Hilbert modular forms is proved in Chapter 12.

If

$$\dim_{\mathbf{Q}_p}\left(\mathrm{Sel}(G_{K,S}, V_p(E))\right) \equiv 1 \pmod 2,$$

the congruence (0.15.3.1) together with the control theorem 0.14.3 imply that

$$\dim_{\mathbf{Q}_p}\left(\mathrm{Sel}(G_{K',S}, V_p(E))\right) \geqslant [K' : K],$$

for all finite subextensions K'/K of K_∞/K (cf. 10.7.19). The phenomenon of systematic growth of Selmer groups in dihedral extensions was systematically investigated by Mazur and Rubin [**M-R3, M-R4**] (cf. 12.12).

0.16. Generalized height pairings

0.16.0. Our formalism also gives a new approach to p-adic (in fact, R-valued) height pairings, which greatly simplifies all previous constructions (due to many people, including Zarhin, Schneider, Perrin-Riou, Mazur and Tate, Rubin, and the author; see the references in [**Ne2**] and in Sect. 11.3 below). Let

$$J = \mathrm{Ker}(\overline{R} \longrightarrow R)$$

be the augmentation ideal of $\overline{R}$; there is a canonical isomorphism

$$J/J^2 \xrightarrow{\sim} \Gamma_R = \Gamma \otimes_{\mathbf{Z}_p} R$$

$$\gamma - 1 \pmod{J^2} \longmapsto \gamma \otimes 1 \qquad (\gamma \in \Gamma).$$

Assume that Γ_R is flat over R. For T as in (0.12.1), denote $\mathscr{F}_\Gamma(T)$ by $\overline{T}$. The exact triangle

$$T \otimes_R J/J^2 \longrightarrow \overline{T}/J^2\overline{T} \longrightarrow \overline{T}/J\overline{T} \longrightarrow T \otimes_R J/J^2[1]$$

is canonically isomorphic to

$$T \otimes_R \Gamma_R \longrightarrow \overline{T}/J^2\overline{T} \longrightarrow T \longrightarrow T \otimes_R \Gamma_R[1]. \tag{0.16.0.1}$$

Greenberg's local conditions for T induce similar local conditions for each term in (0.16.0.1); the corresponding Selmer complexes also form an exact triangle in $D^b_{ft}({}_R\mathrm{Mod})$

$$\widetilde{\mathbf{R}\Gamma}_f(T) \otimes_R \Gamma_R \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(\overline{T}/J^2\overline{T}) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(T) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(T) \otimes_R \Gamma_R\,[1],$$

which can also be obtained by applying $\widetilde{\mathbf{R}\Gamma}_f(\overline{T})\overset{\mathbf{L}}{\otimes}_{\overline{R}}(-)$ to the exact triangle

$$J/J^2 \longrightarrow \overline{R}/J^2 \longrightarrow R \longrightarrow J/J^2\,[1].$$

The cup product (0.9.3) and the "Bockstein map"

$$\beta : \widetilde{\mathbf{R}\Gamma}_f(T) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(T) \otimes_R \Gamma_R\,[1]$$

induce a morphism in $D^b_{ft}({}_R\mathrm{Mod})$

$$\widetilde{\mathbf{R}\Gamma}_f(T)\overset{\mathbf{L}}{\otimes}_R\widetilde{\mathbf{R}\Gamma}_f(T^*(1)) \longrightarrow \omega \otimes_R \Gamma_R\,[-2],$$

which is a derived version of the height pairing. In practice, the only interesting component of this pairing is given by

$$\widetilde{h} : \widetilde{H}^1_f(T) \otimes_R \widetilde{H}^1_f(T^*(1)) \longrightarrow H^0(\omega) \otimes_R \Gamma_R,$$

which can be written as

$$\widetilde{h}(x \otimes y) = \mathrm{Tr}(\beta(x) \cup y).$$

This construction makes sense also in the case when Γ is a finite abelian group of exponent p^m and R is an $\mathbf{Z}/p^m\mathbf{Z}$-algebra. It is very likely that there is a similar cohomological formalism behind real-valued heights.

There is also a more general version of this construction, which yields pairings

$$\begin{aligned}\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T)\overset{\mathbf{L}}{\otimes}_{R[\![\mathrm{Gal}(L/K)]\!]}\,&\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T^*(1))^\iota\\ &\longrightarrow \omega \otimes_R R[\![\mathrm{Gal}(L/K)]\!] \otimes_R \mathrm{Gal}(K_\infty/L)_R\,[-2]\end{aligned}$$

for arbitrary subextensions L/K of K_∞/K (assuming that $\mathrm{Gal}(K_\infty/L)_R$ is flat over R).

0.16.1. This approach to height pairings has many advantages over traditional treatments even in the simplest case when $R = \mathbf{Z}_p$ and $T = T_p(E)$, where E is an elliptic curve over $\mathbf{Q}$ with ordinary reduction at p. For example,

(i) The pairing

$$\widetilde{h} : \widetilde{H}^1_f(T_p(E)) \otimes_R \widetilde{H}^1_f(T_p(E)) \longrightarrow \mathbf{Z}_p$$

has values in $\mathbf{Z}_p$; there are *no denominators* involved.

(ii) If E has split multiplicative reduction at p, then $\widetilde{h}$ is a *natural* height pairing on the *extended Selmer group* (Mazur, Tate and Teitelbaum [**M-T-T**] and their followers used an ad-hoc definition).

(iii) Universal norms in $\widetilde{H}^1_f(T_p(E))$ (*i.e.*, the image of $\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K, T_p(E))$) lie in $\mathrm{Ker}(\beta)$, hence are *automatically* contained in the kernel of the height pairing.

0.16.2. The definition of $\widetilde{h}$ in terms of the Bockstein map β also sheds new light on the formulas of the Birch and Swinnerton-Dyer type proved by Perrin-Riou [**PR2, PR3, PR4, PR5**] and Schneider [**Sch2, Sch3**]. These formulas express (for $\Gamma \xrightarrow{\sim} \mathbf{Z}_p$) the leading term of the "arithmetic p-adic L-function" (*i.e.*, of the characteristic power series of $\det_{\overline{R}} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, T)$) as a product of the determinant of the height pairing $\widetilde{h}$ with, essentially, the p-part of the various rational terms appearing in the conjecture of Birch and Swinnerton-Dyer.

In our approach, such formulas boil down to the additivity of Euler characteristics in a suitable exact triangle (see 11.7.11). For example, in the classical case $R = \mathbf{Z}_p$, the leading term in question is equal, up to a p-adic unit, to the product of

$$\det(\widetilde{h}) \prod_{i=1}^{3} \left| \widetilde{H}^i_f(T)_{\mathrm{tors}} \right|^{(-1)^i}$$

with a certain fudge factor. As in the classical case, all this works under the following assumptions:

(i) The $\overline{R}$-modules $\widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K, X)$ $(X = T, T^*(1))$ satisfy suitable finiteness properties.

(ii) $\widetilde{h}$ is non-degenerate.

If (i) holds but (ii) fails, then it is necessary to consider also higher order terms E_r in the Bockstein spectral sequence and suitable higher order height pairings that generalize the "derived heights" of Bertolini and Darmon [**B-D1, B-D2**].

In [**PR3**], Perrin-Riou considered the case of anti-cyclotomic $\mathbf{Z}_p$-extensions and proved a suitable Λ-valued version of the formulas alluded to above. This result is also covered by our machinery.

Burns and Venjakob [**Bu-Ve**] combined our approach to the formulas of the Birch and Swinnerton-Dyer type with the formalism of non-commutative Iwasawa theory.

0.17. Parity results

The symplectic pairings constructed in Chapter 10 can be used to generalize the parity results proved in [**Ne3**] to Hilbert modular forms and abelian varieties of GL(2)-type over totally real number fields. We refer the reader to Sections 12.1–12.2 for a detailed description of our results, which include, for example, the following generalization of [**Ne3**, Thm. A] (see Corollary 12.2.10 below):

Theorem. — *Let F be a totally real number field, F_0/F an abelian 2-extension, E an elliptic curve over F which is potentially modular in the sense of 12.11.3(1) below*[(7)] *and p a prime number such that E has potentially ordinary (= potentially good ordinary or potentially multiplicative) reduction at each prime of F above p. Assume that at least one of the following conditions holds:*

(1) *$j(E) \notin \mathcal{O}_F$.*

(2) *E is modular over F and $2 \nmid [F : \mathbf{Q}]$.*

(3) *$j(E) \in \mathcal{O}_F$, E has good ordinary reduction at each prime of F above p, the prime number p is unramified in $F_0/\mathbf{Q}$ and $p > 3$. If E does not have CM, assume, in addition, that $\mathrm{Im}\big(G_F \to \mathrm{Aut}(E[p])\big) \supseteq \mathrm{SL}_2(\mathbf{F}_p)$.*

Then: for each finite Galois extension of odd degree F_1/F_0,

$$\mathrm{rk}_{\mathbf{Z}} E(F_1) + \mathrm{cork}_{\mathbf{Z}_p} Ш(E/F_1)[p^\infty] \equiv \mathrm{ord}_{s=1} L(E/F_1, s) \pmod 2.$$

These parity results can be combined with (generalizations of) (0.15.3.1), giving rise to many situations in which Selmer groups "grow systematically" in the sense of [**M-R3**]. See Sect. 12.12 for more details.

0.18. Contents

Let us give a brief description of the contents of each chapter of this work. In Chapter 1 we collect the necessary background material from homological algebra. We pay particular attention to signs, as one of our main goals is to construct higher-dimensional generalizations of the Cassels-Tate pairing, and verify that they are skew-symmetric. The reader is strongly advised to skip this chapter and return to it only when necessary. In Chapter 2 we recall the formalism of Grothendieck's duality theory over (complete) local rings. In Chapter 3 we develop from scratch the formalism of continuous cohomology for what we call (ind)-admissible $R[G]$-modules. Chapter 4 deals with finiteness results for continuous cohomology of pro-finite groups. In Chapter 5 we deduce from the classical duality results for Galois cohomology of finite Galois modules over local and global fields (due to Tate and Poitou) the corresponding results for big Galois representations. In Chapter 6 we introduce Selmer complexes in

(7) Potential modularity of E seems to be well-known to the experts [**Tay5**]; a proof is expected to appear in a forthcoming thesis of a student of R. Taylor.

an axiomatic setting and prove a duality theorem for them (as a consequence of the Poitou-Tate duality in our formalism). In Chapter 7 we investigate a generalization of unramified cohomology (over local fields) in our set-up. In Chapter 8 we apply Shapiro's Lemma to deduce duality results in Iwasawa theory from those over number fields. Chapter 9 is devoted to applications to classical Iwasawa theory, namely to p-parts of ideal class groups (resp., of S-ideal class groups). It also includes comparison results between the extended Selmer groups $\widetilde{H}^1_f$, Greenberg's (strict) Selmer groups and classical Selmer groups for abelian varieties. In Chapter 10 we construct and study various incarnations of generalized Cassels-Tate pairings. We pay particular attention to the self-dual case, which is important for arithmetic applications. In Chapter 11 we construct generalized p-adic height pairings and relate them to the formulas of the Birch and Swinnerton-Dyer type. In Chapter 12, we apply the results from Chapter 10 to big Galois representations arising from Hida families of Hilbert modular forms of parallel weight, and also to anticyclotomic Iwasawa theory of CM points on Shimura curves. This allows us to deduce a far-reaching generalization of the parity results from [**Ne3**].

0.19. Directions for further research

The fact that Selmer complexes 'see' trivial zeros of p-adic L-functions and satisfy the base change properties (Sect. 0.14.3) and (0.15.1.1) indicates that they – and not the usual Selmer groups – are the correct algebraic counterparts of p-adic L-functions.

It would be of some interest, therefore, to reformulate all aspects of Iwasawa theory from this perspective.

0.19.1. Non-commutative Iwasawa theory. — It seems very likely that the results discussed in 0.11–0.13 can be generalized to a fairly large class of non-commutative p-adic Lie groups Γ. One would expect the duality diagram (0.12.1) to hold over $\overline{R} = R[\![\Gamma]\!]$ again with $\omega_{\overline{R}} = \omega_R \otimes_R \overline{R}$ (this time as a complex of $\overline{R}$-bimodules) and $\mathscr{F}_\Gamma$, F_Γ defined as in 8.3.1. Note that both $\mathscr{F}_\Gamma(M)$ and $F_\Gamma(M)$ are $\overline{R}$-bimodules equipped with an involution compatible with the bimodule structure, and the action of $G = G_{K,S}$ commutes with one of the $\overline{R}$-module structures. It seems that this extra structure can be used to generalize the cohomological theory of admissible $\overline{R}[G]$-modules to the non-commutative setting.

0.19.2. Local Iwasawa theory. — It would be highly desirable to develop a theory of local conditions at primes dividing p that would go beyond Greenberg's local conditions. One should view Perrin-Riou's theory [**PR6**] interpolating the Bloch-Kato exponential in the local cyclotomic $\mathbf{Z}_p$-extension as a first step in this direction. Another challenge is posed by the families of Galois representations arising from Coleman's theory of rigid analytic families of modular forms. It is clear that in this generality one would have to work with more general coefficient rings R. In fact, there should be a common generalization of 0.19.1 and 0.19.2, perhaps in the context of "Fréchet-Stein algebras" introduced by Schneider-Teitelbaum [**Sch-Te**]. One can also envisage a version of the theory involving directly étale cohomology of towers of varieties, rather than Galois cohomology.

0.19.3. Euler systems. — The machinery of Euler systems is a powerful tool for obtaining upper bounds for the size of (dual) Selmer groups. It would seem natural to incorporate Selmer complexes into this theory, which would allow for the treatment of trivial zeros.

In practice, elements of an Euler system are obtained from suitable elements of motivic cohomology to which one applies the p-adic regulator or the p-adic Abel-Jacobi map. It is a natural question whether one can, in the presence of a trivial zero, canonically lift an Euler system from the Selmer group to its extended version $\widetilde{H}^1_f$. This can be done, for example, for the Euler system of Heegner points in the presence of an "anticyclotomic trivial zero" ([**B-D3**, §2.6]).

0.20. Miscellaneous

0.20.1. An embryonic version of Selmer complexes appears in [**Fo1**] (following a suggestion of Deligne). The first consistent use of derived categories in Iwasawa theory is due to Kato [**Ka1**]; his approach has been incorporated into the general formalism of Equivariant Tamagawa Number Conjecture [**Bu-Fl1, Bu-Fl2**]. Recent articles of Burns-Greither [**Bu-Gr**], Burns-Venjakob [**Bu-Ve**], Fukaya-Kato [**Fu-Ka**] and Mazur-Rubin [**M-R1, M-R2, M-R3**] are also closely related to our framework.

0.20.2. To our great embarrassment, it has proved impossible to keep the length of this work under control, even though much of what we do is just an exercise in linear algebra. This is a consequence of our early decision not to use any homotopical machinery (such as infinite hierarchies of higher-order homotopies) in our treatment of Selmer complexes, only brute force.

0.20.3. The idea of a 'Selmer complex' occurred to the author while he was staying at Institut Henri Poincaré in Paris in spring 1997. It was further developed during stays at the Isaac Newton Institute in Cambridge in spring 1998 and (again) at Institut Henri Poincaré in spring 2000. During the visits at IHP the author was partially supported by a grant from EPSRC and by the EU research network "Arithmetic Algebraic Geometry". Main results of this theory were presented in a series of lectures at University of Tokyo in spring 2001; this visit was supported by a fellowship from JSPS. The first version of this work was completed during the author's visit at Institut de Mathématiques de Jussieu in Paris in October 2001. The author is grateful to all these institutions for their support. He would also like to thank D. Blasius, D. Burns,

O. Gabber, R. Greenberg, H. Hida, U. Jannsen, B. Mazur, K. Rubin, P. Schneider, A.J. Scholl, C. Skinner, R. Taylor and A. Wiles for helpful discussions and inspiring questions, and to C. Cornut, D. Mauger, J. Oesterlé, L. Orton, J. Pottharst and the referee for pointing out several inaccuracies in the text.

0.20.4. To sum up, this work gives a unified treatment of much of (commutative) Iwasawa theory[(8)] organized around a small number of simple, but sufficiently general principles. We hope that our attempt to Grothendieckify the subject will help integrate it into a wider landscape of arithmetic geometry.

Selmer groups are dead.
Long live Selmer complexes!

[(8)] We consider only the algebraic side of the subject. The relation of Selmer complexes to p-adic L-functions remains to be explored, but it is natural to expect that $\det_{\overline{R}} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, T)$, whenever defined, should be closely related to a suitable analytic p-adic L-function.

CHAPTER 1

HOMOLOGICAL ALGEBRA: PRODUCTS AND SIGNS

This chapter should be skipped at first reading. Sect. 1.1 and 1.2 collect basic conventions involving signs, tensor products and Hom's in derived categories. In Sect. 1.3 we define and study abstract pairings between cones. Such pairings will be used in Chapter 6 as a fundamental tool for developing duality theory for Selmer complexes.

1.1. Standard notation and conventions

We follow the sign conventions of [**B-B-M**] (with one important correction; see 1.2.8 below). We fix an abelian category $\mathcal{C}$ and work with the corresponding category of complexes $C(\mathcal{C})$.

1.1.1. Translations (= shifts). — For $n \in \mathbf{Z}$, the translation by n of a complex X (resp., of a morphism of complexes $f : X \to Y$) is given by

$$X[n]^i = X^{n+i}, \quad d^i_{X[n]} = (-1)^n d_X^{i+n}, \quad f[n]^i = f^{i+n}.$$

1.1.2. Cones. — The cone of a morphism of complexes $f : X \to Y$ is equal to

$$\mathrm{Cone}(f) = Y \oplus X[1]$$

with differential

$$d^i_{\mathrm{Cone}(f)} = \begin{pmatrix} d^i_Y & f^{i+1} \\ 0 & -d^{i+1}_X \end{pmatrix} : Y^i \oplus X^{i+1} \longrightarrow Y^{i+1} \oplus X^{i+2}.$$

There is an exact sequence of complexes

$$0 \longrightarrow Y \xrightarrow{j} \mathrm{Cone}(f) \xrightarrow{p} X[1] \longrightarrow 0,$$

in which j and p are the canonical inclusion and projection, respectively; the corresponding boundary map

$$\partial : H^i(X[1]) = H^{i+1}(X) \longrightarrow H^{i+1}(Y)$$

is induced by f^{i+1}.

1.1.3. Exact (= distinguished) triangles. — These are isomorphic (in the derived category $D(\mathcal{C})$) to triangles of the form

$$X \xrightarrow{f} Y \xrightarrow{j} \mathrm{Cone}(f) \xrightarrow{-p} X[1],$$

or, equivalently, to

$$\mathrm{Cone}(f)[-1] \xrightarrow{p[-1]} X \xrightarrow{f} Y \xrightarrow{j} \mathrm{Cone}(f).$$

The translation of an exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is equal to

$$X[1] \xrightarrow{f[1]} Y[1] \xrightarrow{g[1]} Z[1] \xrightarrow{-h[1]} X[2].$$

1.1.4. Exact sequences. — For every exact sequence of complexes

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0, \tag{1.1.4.1}$$

the morphism of complexes

$$q : \mathrm{Cone}(f) \longrightarrow Z$$

equal to g (resp., to zero) on Y (resp., on $X[1]$) is a quasi-isomorphism (Qis). The corresponding map in the derived category

$$h : Z \xleftarrow{q} \mathrm{Cone}(f) \xrightarrow{-p} X[1]$$

defines an exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

such that the map $H^i(h) : H^i(Z) \to H^i(X[1]) = H^{i+1}(X)$ is equal to the coboundary map arising from the original exact sequence (1.1.4.1).

Assume that, for each $i \in \mathbf{Z}$, the epimorphism in $\mathcal{C}$

$$g_i : Y^i \longrightarrow Z^i$$

admits a section

$$s_i : Z^i \longrightarrow Y^i, \quad g_i s_i = \mathrm{id}.$$

Then there is a unique collection of morphisms in $\mathcal{C}$

$$\beta_i : Z^i \longrightarrow X^{i+1}$$

characterized by

$$d_Y^i s_i - s_i d_Z^i = f_{i+1}\beta_i.$$

As

$$d_X^{i+1}\beta_i = -\beta_{i+1} d_Z^i,$$

the collection of maps $\beta = (\beta_i)$ is a morphism of complexes

$$\beta : Z \longrightarrow X[1].$$

The morphism of complexes

$$r = (s, -\beta) : Z \longrightarrow \mathrm{Cone}(f)$$

is a section of q, which implies that the morphism of complexes

$$\beta = -p \circ r : Z \longrightarrow X[1]$$

represents the 'boundary' map h in the derived category $D(\mathcal{C})$.

1.1.5. Homotopies. — A homotopy a between morphisms of complexes $f, g : X \to Y$ (*i.e.*, a collection of maps $a = (a^i : X^{i+1} \to Y^i)$ such that $da + ad = g - f$) will be denoted by $a : f \rightsquigarrow g$. If $u : X' \to X$ (resp., $v : Y \to Y'$) is a morphism of complexes, then $a \star u = (a^i \circ u^{i+1} : (X')^{i+1} \to Y^i)$ (resp., $v \star a = (v^i \circ a^i : X^{i+1} \to (Y')^i)$) is a homotopy $a \star u : fu \rightsquigarrow gu$ (resp., $v \star a : vf \rightsquigarrow vg$). A second order homotopy α between homotopies $a, b : f \rightsquigarrow g$ (*i.e.*, a collection of maps $\alpha = (\alpha^i : X^{i+2} \to Y^i)$ such that $\alpha d - d\alpha = b - a$) will be denoted by $\alpha : a \rightsquigarrow b$.

Assume that we are given complexes $X^\bullet, Y^\bullet, Z^\bullet$ and collections of maps $h = (h^i : X^{i+1} \to Y^i)$, $h' = ((h')^i : Y^{i+1} \to Z^i)$. Then

$$dh + hd : X^\bullet \longrightarrow Y^\bullet, \quad dh' + h'd : Y^\bullet \longrightarrow Z^\bullet$$

are morphisms of complexes,

$$(dh' + h'd) \star h, h' \star (dh + hd) : 0 \rightsquigarrow (dh' + h'd) \circ (dh + hd)$$

are homotopies and

$$H = h'h : (dh' + h'd) \star h \rightsquigarrow h' \star (dh + hd)$$

is a 2-homotopy between these homotopies.

1.1.6. Functoriality of cones ([Ve2, §3.1]). — Let $\mathrm{tr}_1(\mathcal{C})$ be the category with objects $f : X \to Y$ (morphisms of complexes in $\mathcal{C}$) and morphisms (g, h, a)

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & \overset{a}{\Nearrow} & \downarrow{\scriptstyle h} \\ X' & \xrightarrow[f']{} & Y', \end{array}$$

where $g : X \to X'$, $h : Y \to Y'$ are morphisms of complexes and $a : f'g \rightsquigarrow hf$ is a homotopy. The composition of

$$(X \xrightarrow{f} Y) \xrightarrow{(g,h,a)} (X' \xrightarrow{f'} Y') \xrightarrow{(g',h',a')} (X'' \xrightarrow{f''} Y'')$$

is defined as $(g'g, h'h, a' \star g + h' \star a)$, where $h' \star a : h'f'g \rightsquigarrow h'hf$, $a' \star g : f''g'g \rightsquigarrow h'f'g$.

A morphism

$$(g, h, a) : (X \xrightarrow{f} Y) \longrightarrow (X' \xrightarrow{f'} Y')$$

in $\mathrm{tr}_1(\mathcal{C})$ defines a morphism of complexes

$$\mathrm{Cone}(g, h, a) : \mathrm{Cone}(f) \longrightarrow \mathrm{Cone}(f')$$

given by

$$\mathrm{Cone}(g, h, a)^i = \begin{pmatrix} h^i & a^i \\ 0 & g^{i+1} \end{pmatrix} : Y^i \oplus X^{i+1} \longrightarrow Y'^i \oplus X'^{i+1}.$$

In other words, "Cone" is a functor

$$\mathrm{Cone} : \mathrm{tr}_1(\mathcal{C}) \longrightarrow C(\mathcal{C}).$$

1.1.7. Homotopies in $\mathrm{tr}_1(\mathcal{C})$**.** — By definition, a homotopy

$$(b, b', \alpha) : (g, h, a) \rightsquigarrow (g', h', a')$$

between two morphisms

$$(g, h, a), (g', h', a') : (X \xrightarrow{f} Y) \rightrightarrows (X' \xrightarrow{f'} Y')$$

in $\mathrm{tr}_1(\mathcal{C})$ consists of homotopies

$$b : g \rightsquigarrow g', \quad b' : h \rightsquigarrow h'$$

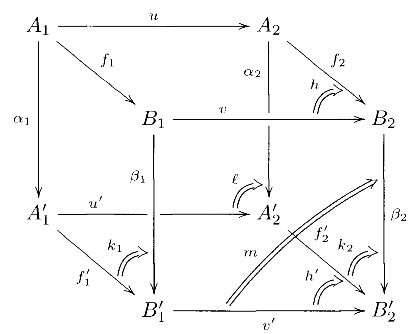
and a second order homotopy

$$\alpha : f' \star b + a' \rightsquigarrow b' \star f + a.$$

Such a homotopy in $\mathrm{tr}_1(\mathcal{C})$ induces a homotopy

$$\begin{pmatrix} b' & \alpha \\ 0 & -b \end{pmatrix} : \mathrm{Cone}(g, h, a) \rightsquigarrow \mathrm{Cone}(g', h', a').$$

1.1.8. Assume that we are given the following cubic diagram of complexes:



In other words, $A_1, \ldots, B'_2$ are complexes in $\mathcal{C}$; $u, \ldots, \beta_2$ are morphisms of complexes and $h : v \circ f_1 \rightsquigarrow f_2 \circ u, \ldots, k_2 : f'_2 \circ \alpha_2 \rightsquigarrow \beta_2 \circ f_2$ are homotopies.

Assume, in addition, that the boundary of the cube is trivialized by a 2-homotopy $H = (H^i : A_1^{i+2} \to (B'_2)^i)$, *i.e.*,

$$H : v' \star k_1 + m \star f_1 + \beta_2 \star h \rightsquigarrow k_2 \star u + h' \star \alpha_1 + f'_2 \star \ell.$$

Then the triple (k_1, k_2, H) defines a homotopy

$$(k_1, k_2, H) : (f'_1, f'_2, h') \circ (\alpha_1, \alpha_2, \ell) = (f'_1\alpha_1, f'_2\alpha_2, h' \star \alpha_1 + f'_2 \star \ell)$$
$$\rightsquigarrow (\beta_1 f_1, \beta_2 f_2, m \star f_1 + \beta_2 \star h) = (\beta_1, \beta_2, m) \circ (f_1, f_2, h),$$

i.e., the diagram

$$\begin{array}{ccc} \mathrm{Cone}(u) & \xrightarrow{(f_1,f_2,h)} & \mathrm{Cone}(v) \\ \big\downarrow{\scriptstyle (\alpha_1,\alpha_2,\ell)} & & \big\downarrow{\scriptstyle (\beta_1,\beta_2,m)} \\ \mathrm{Cone}(u') & \xrightarrow{(f_1',f_2',h')} & \mathrm{Cone}(v') \end{array}$$

is commutative up to homotopy.

1.1.9. A *covariant* additive functor $F : \mathcal{C} \to \mathcal{C}'$ induces a functor on complexes $F : C(\mathcal{C}) \to C(\mathcal{C}')$ given by $d^i_{F(X)} = F(d^i_X)$. The identity morphisms define (for all $n \in \mathbf{Z}$) canonical isomorphisms of complexes

$$F(X[n]) \xrightarrow{\sim} F(X)[n].$$

For a *contravariant* additive functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}'$ we define $F : C(\mathcal{C})^{\mathrm{op}} \to C(\mathcal{C}')$ by

$$d^i_{F(X)} = (-1)^{i+1} F(d_X^{-i-1}).$$

1.1.10. If $G : (\mathcal{C}')^{\mathrm{op}} \to \mathcal{C}''$ is another contravariant additive functor, then

$$d_{G(F(X))} = -G(F(d_X)).$$

We define an isomorphism of complexes

$$G(F(X)) \xrightarrow{\sim} (G \circ F)(X)$$

to be equal to $(-1)^i$ times the identity morphism in degree i.

1.1.11. Truncations. — If X is a complex, we use the usual notation for the truncations

$$\begin{array}{rlccccccccccl}
\sigma_{\leqslant i} X &= [\cdots & X^{i-2} & \longrightarrow & X^{i-1} & \longrightarrow & X^i & \longrightarrow & 0 & \longrightarrow & 0 & \cdots] \\
\tau_{\leqslant i} X &= [\cdots & X^{i-2} & \longrightarrow & X^{i-1} & \longrightarrow & \mathrm{Ker}(d^i_X) & \longrightarrow & 0 & \longrightarrow & 0 & \cdots] \\
\sigma_{\geqslant i} X &= [\cdots & 0 & \longrightarrow & 0 & \longrightarrow & X^i & \to & X^{i+1} & \longrightarrow & X^{i+2} & \cdots] \\
\tau_{\geqslant i} X &= [\cdots & 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Coker}(d^{i-1}_X) & \longrightarrow & X^{i+1} & \longrightarrow & X^{i+2} & \cdots]
\end{array}$$

1.2. Tensor products and Hom

In the rest of Chapter 1, $\mathcal{C} = ({}_R\mathrm{Mod})$ will be the category of modules over a commutative ring R. If $X^\bullet$ is a complex and $x \in X^i$, we denote the degree of x by $\overline{x} = i$.

1.2.1. For complexes $X = X^\bullet, Y = Y^\bullet$ we define the complexes $\mathrm{Hom}^\bullet_R(X,Y)$ and $X \otimes_R Y$ by

$$\mathrm{Hom}^n_R(X,Y) = \prod_{i\in\mathbf{Z}} \mathrm{Hom}_R(X^i, Y^{i+n})$$
$$(X\otimes_R Y)^n = \bigoplus_{i\in\mathbf{Z}} (X^i \otimes_R Y^{n-i}),$$

with differentials

$$df = d\circ f + (-1)^{\overline{f}-1} f\circ d$$
$$d(x\otimes y) = dx\otimes y + (-1)^{\overline{x}} x\otimes dy.$$

If $Y = Y^0$ is concentrated in degree zero, then $\mathrm{Hom}^\bullet_R(X,Y) = F(X)$ for $F(-) = \mathrm{Hom}_R(-, Y^0)$ (with the sign conventions of 1.1.9).

If Y is a bounded (resp., bounded below) complex of injective R-modules and X is any (resp., bounded above) complex of R-modules, then $\mathrm{Hom}^\bullet_R(X,Y)$ represents $\mathbf{R}\mathrm{Hom}_R(X,Y)$.

An element $f \in \mathrm{Hom}^0_R(X,Y)$ satisfies

(1.2.1.1) $$df = 0 \iff f \text{ is a morphism of complexes } f : X \longrightarrow Y$$
$$f = dg \iff g \text{ is a homotopy } g : 0 \rightsquigarrow f.$$

There is also a "naive" version $\mathrm{Hom}^{\bullet,\mathrm{naive}}_R(X,Y)$ of $\mathrm{Hom}^\bullet_R(X,Y)$, in which the differential of $f : X^i \to Y^j$ is equal to

$$d^{\mathrm{naive}} f = d\circ f + (-1)^j f\circ d.$$

1.2.2. Morphisms of complexes $u : X \to X'$, $v : Y \to Y'$ induce morphisms

$$\begin{array}{rccc} \mathrm{Hom}^\bullet(u,v) : & \mathrm{Hom}^\bullet_R(X',Y) & \longrightarrow & \mathrm{Hom}^\bullet_R(X,Y') \\ & f & \longmapsto & v\circ f\circ u \end{array}$$

and

$$\begin{array}{rccc} u\otimes v : & X\otimes_R Y & \longrightarrow & X'\otimes_R Y' \\ & x\otimes y & \longmapsto & u(x)\otimes v(y) \end{array}$$

1.2.3. Tensor products of complexes admit various symmetries, such as

Associativity isomorphism:

$$\begin{array}{ccc} (X\otimes_R Y)\otimes_R Z & \xrightarrow{\sim} & X\otimes_R (Y\otimes_R Z) \\ (x\otimes y)\otimes z & \longmapsto & x\otimes(y\otimes z) \end{array}$$

Transposition isomorphism:

$$\begin{array}{ccc} s_{12} : X\otimes_R Y & \xrightarrow{\sim} & Y\otimes_R X \\ x\otimes y & \longmapsto & (-1)^{\overline{x}\,\overline{y}}\, y\otimes x \end{array}$$

Another transposition isomorphism:

$$\begin{array}{rcl} s_{23} : (X \otimes_R A) \otimes_R (Y \otimes_R B) & \xrightarrow{\sim} & (X \otimes_R Y) \otimes_R (A \otimes_R B) \\ (x \otimes a) \otimes (y \otimes b) & \longmapsto & (-1)^{\overline{a}\overline{y}} (x \otimes y) \otimes (a \otimes b) \end{array}$$

1.2.4. Lemma. — *The following diagram is commutative:*

$$\begin{array}{ccc} (X \otimes_R A) \otimes_R (Y \otimes_R B) & \xrightarrow{s_{23}} & (X \otimes_R Y) \otimes_R (A \otimes_R B) \\ \Big\downarrow s_{12} & & \Big\downarrow s_{12} \otimes s_{12} \\ (Y \otimes_R B) \otimes_R (X \otimes_R A) & \xrightarrow{s_{23}} & (Y \otimes_R X) \otimes_R (B \otimes_R A) \end{array}$$

Proof. — $(-1)^{(\overline{x}+\overline{a})(\overline{y}+\overline{b})}(-1)^{\overline{x}\overline{b}} = (-1)^{\overline{a}\overline{y}}(-1)^{\overline{x}\overline{y}}(-1)^{\overline{a}\overline{b}}$. □

1.2.5. With the sign conventions of 1.2.1, the canonical isomorphism

$$\mathrm{Hom}^\bullet_R(X, Y)[n] \xrightarrow{\sim} \mathrm{Hom}^\bullet_R(X, Y[n])$$

does not involve any signs, *i.e.*, it is given in all degrees by the identity maps.

1.2.6. The adjunction morphism on the level of complexes

$$\begin{array}{rcl} \mathrm{adj} : \mathrm{Hom}^\bullet_R(X \otimes_R Y, Z) & \longrightarrow & \mathrm{Hom}^\bullet_R(X, \mathrm{Hom}^\bullet_R(Y, Z)) \\ f & \longmapsto & \big(x \mapsto (y \mapsto f(x \otimes y))\big) \end{array}$$

induces a morphism of R-modules

$$\mathrm{Hom}_{C({}_R\mathrm{Mod})}(X \otimes_R Y, Z) \longrightarrow \mathrm{Hom}_{C({}_R\mathrm{Mod})}(X, \mathrm{Hom}^\bullet_R(Y, Z)),$$

which preserves homotopy classes (by (1.2.1.1)). Both of these maps are monomorphisms; they are isomorphisms, provided X and Y are bounded above and Z is bounded below.

1.2.7. The evaluation maps

$$\begin{array}{rcl} \mathrm{ev}_1 : \mathrm{Hom}^\bullet_R(X, Y) \otimes_R X & \longrightarrow & Y \\ f \otimes x & \longmapsto & f(x) \end{array}$$

and

$$\begin{array}{rcl} \mathrm{ev}_2 : X \otimes_R \mathrm{Hom}^\bullet_R(X, Y) & \longrightarrow & Y \\ x \otimes f & \longmapsto & (-1)^{\overline{x}\overline{f}} f(x) \end{array}$$

are morphisms of complexes making the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Hom}^\bullet_R(X, Y) \otimes_R X & \xrightarrow{\mathrm{ev}_1} & Y \\ \Big\downarrow s_{12} & & \| \\ X \otimes_R \mathrm{Hom}^\bullet_R(X, Y) & \xrightarrow{\mathrm{ev}_2} & Y \end{array} \tag{1.2.7.1}$$

We have $\mathrm{adj}(\mathrm{ev}_1) = \mathrm{id}$ under the adjunction morphism

$$\mathrm{adj} : \mathrm{Hom}_{C({}_R\mathrm{Mod})}(\mathrm{Hom}^\bullet_R(X, Y) \otimes_R X, Y) \longrightarrow \mathrm{Hom}_{C({}_R\mathrm{Mod})}(\mathrm{Hom}^\bullet_R(X, Y), \mathrm{Hom}^\bullet_R(X, Y)).$$

More generally, there are evaluation morphisms

$$\begin{array}{rcl} \mathrm{ev}_1 : \mathrm{Hom}^\bullet_R(X,Y) \otimes_R \mathrm{Hom}^\bullet_R(W,X) & \longrightarrow & \mathrm{Hom}^\bullet_R(W,Y) \\ g \otimes f & \longmapsto & g \circ f \\ \mathrm{ev}_2 : \mathrm{Hom}^\bullet_R(W,X) \otimes_R \mathrm{Hom}^\bullet_R(X,Y) & \longrightarrow & \mathrm{Hom}^\bullet_R(W,Y) \\ f \otimes g & \longmapsto & (-1)^{\overline{f}\overline{g}} g \circ f \end{array}$$

satisfying $\mathrm{ev}_1 = \mathrm{ev}_2 \circ s_{12}$. Another generalization of ev_1 is given by the morphism

$$\begin{array}{rcl} \mathrm{Hom}^\bullet_R(X,Y) \otimes_R (X \otimes_R Z) & \longrightarrow & Y \otimes_R Z \\ f \otimes (x \otimes z) & \longmapsto & f(x) \otimes z, \end{array}$$

which corresponds to

$$\begin{gathered} \mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_R(X,Y), \mathrm{Hom}^\bullet_R(X \otimes_R Z, Y \otimes_R Z)) \\ f \longmapsto f \otimes \mathrm{id}_Z \end{gathered}$$

under the adjunction map.

1.2.8. The biduality morphism

$$\varepsilon_Y : X \longrightarrow \mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_R(X,Y), Y)$$

is given on $x \in X^i$ by

$$x \longmapsto \big((-1)^{ik} x_k^{**}\big)_{k \in \mathbf{Z}},$$

where

$$\begin{array}{rcl} X^i & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(X^i, Y^{i+k}), Y^{i+k}) \\ x & \longmapsto & (x_k^{**} : f \longmapsto f(x)) \end{array}$$

is the usual biduality map. This corrects a sign error in [**B-B-M**, §0.3.4.2] – also discovered by B. Conrad [**Con**] – where the authors give an erroneous sign $(-1)^i$ instead of $(-1)^{ik}$; that would not make ε_Y a morphism of complexes.

We have $\mathrm{adj}(\mathrm{ev}_2) = \varepsilon_Y$ under the adjunction morphism

$$\begin{aligned} \mathrm{adj} : \mathrm{Hom}_{C({}_R\mathrm{Mod})}(X \otimes_R \mathrm{Hom}^\bullet_R(X,Y), Y) & \\ \longrightarrow \mathrm{Hom}_{C({}_R\mathrm{Mod})}(X, \mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_R(X,Y),Y)). \end{aligned}$$

The statements of Lemmas 1.2.9–1.2.13, 1.2.16 below follow immediately from the definitions; we leave the details to the reader (for the homotopy versions of 1.2.11–1.2.13 it is sufficient to recall that adj preserves homotopy classes).

1.2.9. Lemma. — *The following diagram (and a symmetric diagram, in which the roles of* ev_1 *and* ev_2 *are interchanged) is commutative:*

$$\begin{array}{ccc} X \otimes_R \mathrm{Hom}^\bullet_R(X,Y) & \xrightarrow{\mathrm{ev}_2} & Y \\ \big\downarrow{\scriptstyle \varepsilon_Y \otimes \mathrm{id}} & & \| \\ \mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_R(X,Y),Y) \otimes_R \mathrm{Hom}^\bullet_R(X,Y) & \xrightarrow{\mathrm{ev}_1} & Y. \end{array}$$

1.2.10. Lemma. — *Assume we are given a morphism of complexes $f : X \to \mathrm{Hom}^\bullet_R(X,Y)$; denote the composite morphism of complexes*

$$X \xrightarrow{\varepsilon_Y} \mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_R(X,Y),Y) \xrightarrow{\mathrm{Hom}^\bullet(f,\mathrm{id}_Y)} \mathrm{Hom}^\bullet_R(X,Y)$$

by g. Then the following diagram is commutative:

$$\begin{array}{ccc} X \otimes_R X & \xrightarrow{\mathrm{id}_X \otimes g} & X \otimes_R \mathrm{Hom}^\bullet_R(X,Y) \\ \big\downarrow{\scriptstyle f\otimes \mathrm{id}_Y} & & \big\downarrow{\scriptstyle \mathrm{ev}_2} \\ \mathrm{Hom}^\bullet_R(X,Y) \otimes_R X & \xrightarrow{\mathrm{ev}_1} & Y. \end{array}$$

1.2.11. Lemma. — *If the following diagram of morphisms of complexes is commutative (resp., commutative up to homotopy)*

$$\begin{array}{ccc} X \otimes_R Y' & \xrightarrow{\mathrm{id}\otimes f'} & X \otimes_R X' \\ \big\downarrow{\scriptstyle f\otimes \mathrm{id}} & & \big\downarrow{\scriptstyle g} \\ Y \otimes_R Y' & \xrightarrow{h} & Z, \end{array}$$

so is

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{adj}(g)} & \mathrm{Hom}^\bullet_R(X',Z) \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \mathrm{Hom}^\bullet(f',\mathrm{id})} \\ Y & \xrightarrow{\mathrm{adj}(h)} & \mathrm{Hom}^\bullet_R(Y',Z). \end{array}$$

1.2.12. Lemma. — *If the following diagram of morphisms of complexes is commutative (resp., commutative up to homotopy)*

$$\begin{array}{ccc} X \otimes_R Y & \xrightarrow{u} & Z \\ \big\downarrow{\scriptstyle f\otimes g} & & \big\downarrow{\scriptstyle h} \\ X' \otimes_R Y' & \xrightarrow{u'} & Z', \end{array}$$

so is

$$\begin{array}{ccccc} X & \xrightarrow{\mathrm{adj}(u)} & \mathrm{Hom}^\bullet_R(Y,Z) & \xrightarrow{\mathrm{Hom}^\bullet(\mathrm{id},h)} & \mathrm{Hom}^\bullet_R(Y,Z') \\ \big\downarrow{\scriptstyle f} & & & & \| \\ X' & \xrightarrow{\mathrm{adj}(u')} & \mathrm{Hom}^\bullet_R(Y',Z') & \xrightarrow{\mathrm{Hom}^\bullet(g,\mathrm{id})} & \mathrm{Hom}^\bullet_R(Y,Z'). \end{array}$$

1.2.13. Lemma. — *Let A, B, B', U, U', C be complexes of R-modules and*

$$A \otimes_R B \xrightarrow{f} U, \quad A \otimes_R B' \xrightarrow{f'} U', \quad B' \otimes_R C \xrightarrow{b} B, \quad U' \otimes_R C \xrightarrow{u} U$$

morphisms of complexes. If the diagram

$$\begin{array}{ccccc} (A \otimes_R B') \otimes_R C & \xrightarrow{\sim} & A \otimes_R (B' \otimes_R C) & \xrightarrow{\mathrm{id}\otimes b} & A \otimes_R B \\ \big\downarrow{\scriptstyle f'\otimes \mathrm{id}} & & & & \big\downarrow{\scriptstyle f} \\ U' \otimes_R C & & \xrightarrow{\qquad\qquad u \qquad\qquad} & & U \end{array}$$

is commutative (resp., commutative up to homotopy), so is

$$\begin{array}{ccc} A & \xrightarrow{\operatorname{adj}(f)} & \operatorname{Hom}^\bullet_R(B,U) \\ \downarrow{\scriptstyle \operatorname{adj}(f')} & & \\ \operatorname{Hom}^\bullet_R(B',U') & & \downarrow{\scriptstyle \operatorname{Hom}^\bullet_R(b,\mathrm{id})} \\ \downarrow & & \\ \operatorname{Hom}^\bullet_R(B'\otimes_R C, U'\otimes_R C) & \xrightarrow{\operatorname{Hom}^\bullet_R(\mathrm{id},u)} & \operatorname{Hom}^\bullet_R(B'\otimes_R C, U) \end{array}$$

1.2.14. Lemma. — *If*

$$\begin{array}{ccc} X\otimes_R Y & \xrightarrow{\lambda} & Z \\ \downarrow{\scriptstyle s_{12}\circ(f\otimes g)} & & \downarrow{\scriptstyle h} \\ Y\otimes_R X & \xrightarrow{\mu} & Z \end{array}$$

is a commutative diagram of morphisms of complexes, so are

$$\begin{array}{ccc} Y & \xrightarrow{\operatorname{adj}(\mu)\circ g} & \operatorname{Hom}^\bullet_R(X,Z) \\ \downarrow{\scriptstyle \varepsilon_Z} & & \downarrow{\scriptstyle \operatorname{Hom}^\bullet(f,\mathrm{id})} \\ \operatorname{Hom}^\bullet_R(\operatorname{Hom}^\bullet_R(Y,Z),Z) & \xrightarrow{\operatorname{Hom}^\bullet(\operatorname{adj}(\lambda),h)} & \operatorname{Hom}^\bullet_R(X,Z). \end{array}$$

and

$$\begin{array}{ccc} X & \xrightarrow{\operatorname{adj}(\lambda)} & \operatorname{Hom}^\bullet_R(Y,Z) \\ \downarrow{\scriptstyle \varepsilon_Z\circ f} & & \downarrow{\scriptstyle \operatorname{Hom}^\bullet(\mathrm{id},h)} \\ \operatorname{Hom}^\bullet_R(\operatorname{Hom}^\bullet_R(X,Z),Z) & \xrightarrow{\operatorname{Hom}^\bullet(\operatorname{adj}(\mu)\circ g,\mathrm{id})} & \operatorname{Hom}^\bullet_R(Y,Z). \end{array}$$

Proof. — Let $y\in Y^j$, $x\in X^i$. Then $\operatorname{Hom}^\bullet(f,\mathrm{id})\circ\operatorname{adj}(\mu)\circ g(y)$ sends x to $\mu(g(y)\otimes f(x)) = (-1)^{ij}h(\lambda(x\otimes y))$. In the notation of 1.2.8, the only component of $\varepsilon_Z(y) = ((-1)^{jk}y_k^{**})_{k\in\mathbf{Z}}$ contributing to $\operatorname{Hom}^\bullet(\operatorname{adj}(\lambda),h)\circ\varepsilon_Z(y)(x)$ is $(-1)^{ji}y_i^{**}$; its image in $\operatorname{Hom}^\bullet_R(X,Z)$ also sends x to $(-1)^{ij}h(\lambda(x\otimes y))$. A similar argument works for the second diagram: $\operatorname{Hom}^\bullet(\mathrm{id},h)\circ\operatorname{adj}(\lambda)(x)$ sends y to $h(\lambda(x\otimes y)) = (-1)^{ij}\mu(g(y)\otimes f(x))$, while the only component of $\varepsilon_Z\circ f(x) = ((-1)^{ik}f(x)_k^{**})_{k\in\mathbf{Z}}$ contributing to $\operatorname{Hom}^\bullet(\operatorname{adj}(\mu)\circ g,\mathrm{id})\circ\varepsilon_Z(f(x))$ is $(-1)^{ij}f(x)_j^{**}$, the image of which in $\operatorname{Hom}^\bullet_R(Y,Z)$ sends y to $(-1)^{ij}\mu(g(y)\otimes f(x))$. □

1.2.15. For each $n \in \mathbf{Z}$, the following formulas define isomorphisms of complexes:

$$s_n : X^\bullet[n] \otimes_R Y^\bullet \xrightarrow{\sim} (X^\bullet \otimes_R Y^\bullet)[n]$$
$$x \otimes y \longmapsto x \otimes y$$
$$s'_n : X^\bullet \otimes_R (Y^\bullet[n]) \xrightarrow{\sim} (X^\bullet \otimes_R Y^\bullet)[n]$$
$$x \otimes y \longmapsto (-1)^{n\overline{x}} x \otimes y$$
$$t_n : \mathrm{Hom}^\bullet_R(X^\bullet, Y^\bullet) \xrightarrow{\sim} \mathrm{Hom}^\bullet_R(X^\bullet[n], Y^\bullet[n])$$
$$f \longmapsto (-1)^{n\overline{f}} f.$$

1.2.16. Lemma. — *Given a morphism of complexes $u : A^\bullet \otimes_R B^\bullet \to C^\bullet$ and $n \in \mathbf{Z}$, put*

$$v = u[n] \circ s'_n : A^\bullet \otimes_R (B^\bullet[n]) \xrightarrow{s'_n} (A^\bullet \otimes_R B^\bullet)[n] \xrightarrow{u[n]} C^\bullet[n].$$

Then $\mathrm{adj}(v)$ *is equal to the composite map*

$$A^\bullet \xrightarrow{\mathrm{adj}(u)} \mathrm{Hom}^\bullet_R(B^\bullet, C^\bullet) \xrightarrow{t_n} \mathrm{Hom}^\bullet_R(B^\bullet[n], C^\bullet[n]).$$

1.2.17. Tensor product of homotopies. — Assume that

$$f_i : X \longrightarrow X', \quad g_i : Y \longrightarrow Y' \qquad (i = 1, 2)$$

are morphisms of complexes and

$$u : f_1 \rightsquigarrow f_2, \quad v : g_1 \rightsquigarrow g_2$$

homotopies between them. Then the formulas

$$(u \otimes v)_1(x \otimes y) = u(x) \otimes g_1(y) + (-1)^{\overline{x}} f_2(x) \otimes v(y)$$
$$(u \otimes v)_2(x \otimes y) = u(x) \otimes g_2(y) + (-1)^{\overline{x}} f_1(x) \otimes v(y)$$

define two homotopies

$$(u \otimes v)_j : f_1 \otimes g_1 \rightsquigarrow f_2 \otimes g_2 \qquad (j = 1, 2)$$

between the morphisms

$$f_i \otimes g_i : X \otimes_R Y \longrightarrow X' \otimes_R Y'.$$

The formula

$$\alpha(x \otimes y) = (-1)^{\overline{x}} u(x) \otimes v(y)$$

defines a second order homotopy

$$\alpha : (u \otimes v)_1 \rightsquigarrow (u \otimes v)_2.$$

1.2.18. If X, Y are complexes of R-modules, then the maps

$$x \otimes y \longmapsto (-1)^{\overline{x}}\, x \otimes y, \quad y \otimes x \longmapsto y \otimes x$$

define isomorphisms of complexes

$$X \otimes_R (Y[1]) \xrightarrow{\sim} (X \otimes_R Y)[1], \quad (Y[1]) \otimes_R X \xrightarrow{\sim} (Y \otimes_R X)[1],$$

which make the following diagram commutative:

$$\begin{array}{ccc}
X \otimes_R (Y[1]) & \xrightarrow{\sim} & (X \otimes_R Y)[1] \\
\big\downarrow{\scriptstyle s_{12}} & & \big\downarrow{\scriptstyle s_{12}[1]} \\
(Y[1]) \otimes_R X & \xrightarrow{\sim} & (Y \otimes_R X)[1].
\end{array}$$

1.2.19. Lemma. — *Assume that the following commutative diagram of morphisms of complexes has exact rows and columns:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & A'' & \xrightarrow{\rho_A} & A' & \xrightarrow{\sigma_A} & A & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle i''} & & \big\downarrow{\scriptstyle i'} & & \big\downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & B'' & \xrightarrow{\rho_B} & B' & \xrightarrow{\sigma_B} & B & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle j''} & & \big\downarrow{\scriptstyle j'} & & \big\downarrow{\scriptstyle j} & & \\
0 & \longrightarrow & C'' & \xrightarrow{\rho_C} & C' & \xrightarrow{\sigma_C} & C & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then the diagram

$$\begin{array}{ccccccccc}
 & & & & & & & & H^{q-1}(C) \\
 & & & & & & & & \big\downarrow{\scriptstyle \partial} \\
 & & & & & & & & H^q(A) \\
 & & & & & & & & \big\downarrow{\scriptstyle i} \\
 & & & & & & H^q(B') & \xrightarrow{\sigma_B} & H^q(B) \\
 & & & & & & \big\downarrow{\scriptstyle j'} & & \big\downarrow{\scriptstyle j} \\
 & & H^{q-1}(C) & \xrightarrow{\partial_C} & H^q(C'') & \xrightarrow{\rho_C} & H^q(C') & \xrightarrow{\sigma_C} & H^q(C) \\
 & & \big\downarrow{\scriptstyle -\partial} & & \big\downarrow{\scriptstyle \partial''} & & \big\downarrow{\scriptstyle \partial'} & & \big\downarrow{\scriptstyle \partial} \\
H^q(A') & \xrightarrow{\sigma_A} & H^q(A) & \xrightarrow{\partial_A} & H^{q+1}(A'') & \xrightarrow{\rho_A} & H^{q+1}(A') & \xrightarrow{\sigma_A} & H^{q+1}(A)
\end{array}$$

is also commutative, and if $[c''] \in H^q(C'')$, $[b'] \in H^q(B')$ *are cohomology classes satisfying* $\rho_C[c''] = j'[b']$, *then there is a unique coset* $[a] + \mathrm{Im}(\partial) \in H^q(A) + \mathrm{Im}(\partial)$ *such that* $i[a] = \sigma_B[b']$. *This coset satisfies*

$$\partial_A[a] + \partial''[c''] \in \mathrm{Im}(\partial_A \partial) = \mathrm{Im}(\partial'' \partial_C).$$

Above, ∂, ∂', ∂'', ∂_A and ∂_C denote coboundary maps associated to the original diagram.

Proof. — This is a well-known fact, which can be verified by an explicit calculation. □

1.3. Products

In this section we construct products in a slightly more general context than considered by Nizioł ([**Ni**, Prop. 3.1]). The main difference is that we allow certain diagrams to commute only up to homotopy. In what follows, R is a commutative ring and all complexes are complexes of R-modules.

1.3.1. Assume we are given the following data:

1.3.1.1. Complexes A_j, B_j, C_j $(j = 1, 2, 3)$.

1.3.1.2. Morphisms of complexes

$$A_j \xrightarrow{f_j} C_j \xleftarrow{g_j} B_j \qquad (j = 1, 2, 3).$$

1.3.1.3. Morphisms of complexes

$$\begin{aligned} \cup_A &: A_1 \otimes_R A_2 \longrightarrow A_3 \\ \cup_B &: B_1 \otimes_R B_2 \longrightarrow B_3 \\ \cup_C &: C_1 \otimes_R C_2 \longrightarrow C_3 \end{aligned}$$

1.3.1.4. A pair $h = (h_f, h_g)$ of homotopies

$$\begin{aligned} h_f &: \cup_C \circ (f_1 \otimes f_2) \rightsquigarrow f_3 \circ \cup_A \\ h_g &: \cup_C \circ (g_1 \otimes g_2) \rightsquigarrow g_3 \circ \cup_B \end{aligned}$$

We define new complexes E_j by

$$E_j = \mathrm{Cone}\Big(A_j \oplus B_j \xrightarrow{f_j - g_j} C_j\Big)[-1], \qquad (j = 1, 2, 3)$$

i.e.,

$$\begin{aligned} E_j^n &= A_j^n \oplus B_j^n \oplus C_j^{n-1}, \\ d(a_j, b_j, c_j) &= (da_j, db_j, -f_j(a_j) + g_j(b_j) - dc_j). \end{aligned}$$

An element $e_j = (a_j, b_j, c_j)$ has degree $\overline{e}_j = \overline{a}_j = \overline{b}_j = 1 + \overline{c}_j$.

1.3.2. Proposition. — *Given the data 1.3.1.1–1.3.1.4, then*

(i) *For every $r \in R$ the formula*

$$\begin{aligned} (a_1, b_1, c_1) \cup_{r,h} (a_2, b_2, c_2) = \big(&a_1 \cup_A a_2, b_1 \cup_B b_2, c_1 \cup_C (r f_2(a_2) + (1 - r) g_2(b_2)) \\ &+ (-1)^{\overline{a}_1}((1 - r) f_1(a_1) + r g_1(b_1)) \cup_C c_2 - (h_f(a_1 \otimes a_2) - h_g(b_1 \otimes b_2))\big) \end{aligned}$$

defines a morphism of complexes

$$\cup_{r,h} : E_1 \otimes_R E_2 \longrightarrow E_3.$$

(ii) *For $r_1, r_2 \in R$, the formula*

$$k((a_1, b_1, c_1) \otimes (a_2, b_2, c_2)) = (0, 0, (-1)^{\overline{a}_1}(r_1 - r_2)\, c_1 \cup_C c_2)$$

defines a homotopy $k : \cup_{r_1,h} \rightsquigarrow \cup_{r_2,h}$.

(iii) *If $h' = (h'_f, h'_g)$ is another pair of homotopies as in 1.3.1.4, then*

$$\cup_{r,h'} - \cup_{r,h} : (a_1, b_1, c_1) \otimes (a_2, b_2, c_2) \mapsto (0, 0, (h_f - h'_f)(a_1 \otimes a_2) - (h_g - h'_g)(b_1 \otimes b_2)).$$

If $\alpha : h_f \rightsquigarrow h'_f$, $\beta : h_g \rightsquigarrow h'_g$ is a pair of second order homotopies, then the formula

$$k((a_1, b_1, c_1) \otimes (a_2, b_2, c_2)) = (0, 0, \alpha(a_1 \otimes a_2) - \beta(b_1 \otimes b_2))$$

defines a homotopy $k : \cup_{r,h} \rightsquigarrow \cup_{r,h'}$.

Proof. — Explicit calculation (cf. [**Ni**, Prop. 3.1]). □

1.3.3. Functoriality of products. — Assume we are given another piece of data as in 1.3.1.1–1.3.1.4, namely morphisms of complexes

$$\widetilde{A}_j \xrightarrow{\widetilde{f}_j} \widetilde{C}_j \xleftarrow{\widetilde{g}_j} \widetilde{B}_j,$$

products $\widetilde{\cup}_*$ (for $* = A, B, C$) and homotopies $\widetilde{h} = \left(\widetilde{h}_{\widetilde{f}}, \widetilde{h}_{\widetilde{g}}\right)$, yielding complexes $\widetilde{E}_j$. A morphism between the data

$$(A_j, B_j, C_j, f_j, g_j, \cup_*, h) \longrightarrow (\widetilde{A}_j, \widetilde{B}_j, \widetilde{C}_j, \widetilde{f}_j, \widetilde{g}_j, \widetilde{\cup}_*, \widetilde{h})$$

consists of the following:

1.3.3.1. Morphisms of complexes

$$\begin{aligned} \alpha_j &: A_j \longrightarrow \widetilde{A}_j \\ \beta_j &: B_j \longrightarrow \widetilde{B}_j \qquad (j = 1, 2, 3) \\ \gamma_j &: C_j \longrightarrow \widetilde{C}_j \end{aligned}$$

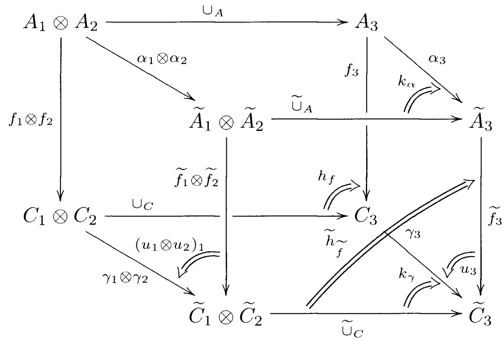
1.3.3.2. Homotopies

$$\begin{aligned} u_j &: \widetilde{f}_j \circ \alpha_j \rightsquigarrow \gamma_j \circ f_j \\ v_j &: \widetilde{g}_j \circ \beta_j \rightsquigarrow \gamma_j \circ g_j \end{aligned} \qquad (j = 1, 2, 3)$$

1.3.3.3. Homotopies

$$\begin{aligned} k_\alpha &: \widetilde{\cup}_A \circ (\alpha_1 \otimes \alpha_2) \rightsquigarrow \alpha_3 \circ \cup_A \\ k_\beta &: \widetilde{\cup}_B \circ (\beta_1 \otimes \beta_2) \rightsquigarrow \beta_3 \circ \cup_B \\ k_\gamma &: \widetilde{\cup}_C \circ (\gamma_1 \otimes \gamma_2) \rightsquigarrow \gamma_3 \circ \cup_C \end{aligned}$$

1.3.3.4. A second order homotopy K_f between 0 and the composition of the six homotopies associated to the faces of the following cubic diagram:



In other words, K_f is a collection of maps

$$K_f = \big(K_f^i : (A_1 \otimes_R A_2)^i \longrightarrow (\widetilde{C}_3)^{i-2}\big)$$

satisfying

$$\begin{aligned} dK_f - K_f d = -k_\gamma \star (f_1 \otimes f_2) - \gamma_3 \star h_f + u_3 \star \cup_A \\ + \widetilde{f}_3 \star k_\alpha + \widetilde{h}_{\widetilde{f}} \star (\alpha_1 \otimes \alpha_2) - \widetilde{\cup}_C \star (u_1 \otimes u_2)_1, \end{aligned}$$

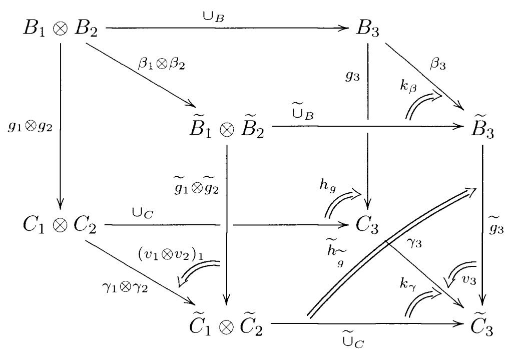
i.e., trivializing the boundary of the cube (the homotopy $(u_1 \otimes u_2)_1$ was defined in 1.2.17).

1.3.3.5. A second order homotopy

$$K_g = \big(K_g^i : (B_1 \otimes_R B_2)^i \longrightarrow (\widetilde{C}_3)^{i-2}\big)$$

satisfying an analogous condition, with (A, f, α, u) being replaced by (B, g, β, v):

$$\begin{aligned} dK_g - K_g d = -k_\gamma \star (g_1 \otimes g_2) - \gamma_3 \star h_g + v_3 \star \cup_B \\ + \widetilde{g}_3 \star k_\beta + \widetilde{h}_{\widetilde{g}} \star (\beta_1 \otimes \beta_2) - \widetilde{\cup}_C \star (v_1 \otimes v_2)_1, \end{aligned}$$



1.3.4. Proposition

(i) *Given the data 1.3.3.1–1.3.3.2, then the formula*

$$\varphi_j(a_j, b_j, c_j) = (\alpha_j(a_j), \beta_j(b_j), \gamma_j(c_j) + u_j(a_j) - v_j(b_j))$$

defines a morphism of complexes

$$\varphi_j : E_j \longrightarrow \widetilde{E}_j \qquad (j = 1, 2, 3).$$

(ii) *Given the data 1.3.3.1–1.3.3.5 and $r \in R$, the formula*

$$\begin{aligned} H((a_1, b_1, c_1) \otimes (a_2, b_2, c_2)) = \\ \big(k_\alpha(a_1 \otimes a_2), k_\beta(b_1 \otimes b_2), -k_\gamma\big(c_1 \otimes (rf_2(a_2) + (1-r)g_2(b_2)) \\ + (-1)^{\overline{a}_1}((1-r)f_1(a_1) + rg_1(b_1)) \otimes c_2\big) \\ + (-1)^{\overline{a}_1}\gamma_1(c_1)\widetilde{\cup}_C(ru_2(a_2) + (1-r)v_2(b_2)) \\ - (-1)^{\overline{a}_1}\big((1-r)u_1(a_1) + rv_1(b_1)\big)\widetilde{\cup}_C\big(\gamma_2(c_2) + u_2(a_2) - v_2(b_2)\big) \\ - K_f(a_1 \otimes a_2) + K_g(b_1 \otimes b_2)\big) \end{aligned}$$

defines a homotopy

$$H : \widetilde{\cup}_{r,\widetilde{h}} \circ (\varphi_1 \otimes \varphi_2) \rightsquigarrow \varphi_3 \circ \cup_{r,h},$$

i.e., *the diagram*

$$\begin{array}{ccc} E_1 \otimes_R E_2 & \xrightarrow{\cup_{r,h}} & E_3 \\ \big\downarrow \varphi_1\otimes\varphi_2 & & \big\downarrow \varphi_3 \\ \widetilde{E}_1 \otimes_R \widetilde{E}_2 & \xrightarrow{\widetilde{\cup}_{r,\widetilde{h}}} & \widetilde{E}_3 \end{array}$$

is commutative up to homotopy.

Proof. — The first part is a special case of 1.1.6, while (ii) can be proved by a tedious, but routine calculation. □

1.3.5. Transpositions. — Assume that, in addition to the data 1.3.1.1–1.3.1.4, we are given the following objects:

1.3.5.1. Morphisms of complexes

$$\begin{aligned} \mathcal{T}_A &: A_j \longrightarrow A_j \\ \mathcal{T}_B &: B_j \longrightarrow B_j \qquad (j = 1, 2, 3) \\ \mathcal{T}_C &: C_j \longrightarrow C_j \end{aligned}$$

1.3.5.2. Morphisms of complexes

$$\begin{aligned} \cup'_A &: A_2 \otimes_R A_1 \longrightarrow A_3 \\ \cup'_B &: B_2 \otimes_R B_1 \longrightarrow B_3 \\ \cup'_C &: C_2 \otimes_R C_1 \longrightarrow C_3 \end{aligned}$$

1.3.5.3. A pair $h' = (h'_f, h'_g)$ of homotopies

$$h'_f : \cup'_C \circ (f_2 \otimes f_1) \rightsquigarrow f_3 \circ \cup'_A$$
$$h'_g : \cup'_C \circ (g_2 \otimes g_1) \rightsquigarrow g_3 \circ \cup'_B$$

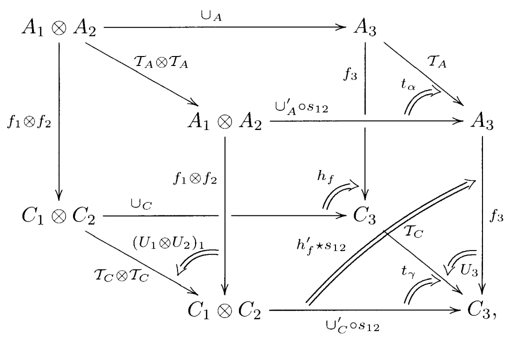
1.3.5.4. Homotopies

$$\begin{matrix} U_j : f_j \circ \mathcal{T}_A \rightsquigarrow \mathcal{T}_C \circ f_j \\ V_j : g_j \circ \mathcal{T}_B \rightsquigarrow \mathcal{T}_C \circ g_j \end{matrix} \qquad (j = 1, 2, 3)$$

1.3.5.5. Homotopies

$$t_\alpha : \cup'_A \circ s_{12} \circ (\mathcal{T}_A \otimes \mathcal{T}_A) \rightsquigarrow \mathcal{T}_A \circ \cup_A$$
$$t_\beta : \cup'_B \circ s_{12} \circ (\mathcal{T}_B \otimes \mathcal{T}_B) \rightsquigarrow \mathcal{T}_B \circ \cup_B$$
$$t_\gamma : \cup'_C \circ s_{12} \circ (\mathcal{T}_C \otimes \mathcal{T}_C) \rightsquigarrow \mathcal{T}_C \circ \cup_C$$

1.3.5.6. A second order homotopy H_f trivializing the boundary of the following cube:



i.e., satisfying

$$dH_f - H_f d = -t_\gamma \star (f_1 \otimes f_2) - \mathcal{T}_C \star h_f + U_3 \star \cup_A + f_3 \star t_\alpha$$
$$+ h'_f \star (s_{12} \circ (\mathcal{T}_A \otimes \mathcal{T}_A)) - (\cup'_C \circ s_{12}) \star (U_1 \otimes U_2)_1.$$

1.3.5.7. A second order homotopy H_g trivializing the boundary of an analogous cube in which (A, α, f) are replaced by (B, β, g).

With these data, the formula

$$(a_2, b_2, c_2) \cup'_{r,h'} (a_1, b_1, c_1) = \big(a_2 \cup'_A a_1, b_2 \cup'_B b_1, c_2 \cup'_C (r f_1(a_1) + (1-r) g_1(b_1))$$
$$+ (-1)^{\overline{a}_2}((1-r) f_2(a_2) + r g_2(b_2)) \cup'_C c_1 - (h'_f(a_2 \otimes a_1) - h'_g(b_2 \otimes b_1))\big)$$

(for a fixed $r \in R$) defines a morphism of complexes

$$\cup'_{r,h'} : E_2 \otimes_R E_1 \longrightarrow E_3.$$

The complexes

$$\widetilde{A}_j = A_j, \widetilde{B}_j = B_j, \widetilde{C}_j = C_j,$$

morphisms of complexes

$$\widetilde{f}_j = f_j, \widetilde{g}_j = g_j, \alpha_j = \mathcal{T}_A, \beta_j = \mathcal{T}_B, \gamma_j = \mathcal{T}_C,$$
$$u_j = U_j, v_j = V_j, \widetilde{\cup}_A = \cup'_A \circ s_{12}, \widetilde{\cup}_B = \cup'_B \circ s_{12}, \widetilde{\cup}_C = \cup'_C \circ s_{12},$$

homotopies

$$k_\alpha = t_\alpha, k_\beta = t_\beta, k_\gamma = t_\gamma, \widetilde{h}_f = h'_f \star s_{12}, \widetilde{h}_g = h'_g \star s_{12}$$

and second order homotopies

$$K_f = H_f, \quad K_g = H_g$$

satisfy the conditions 1.3.3.1–1.3.3.5. Applying Proposition 1.3.4 and observing that the following diagram of morphisms of complexes

$$\begin{array}{ccc} E_1 \otimes_R E_2 & \xrightarrow{\widetilde{\cup}_{r,\widetilde{h}}} & E_3 \\ \Big\downarrow s_{12} & & \| \\ E_2 \otimes_R E_1 & \xrightarrow{\cup'_{1-r,h'}} & E_3 \end{array}$$

is commutative, we obtain the following statement.

1.3.6. Proposition

(i) *Given the data 1.3.5.1 and 1.3.5.4, then the formula*

$$\mathcal{T}_j(a_j, b_j, c_j) = (\mathcal{T}_A(a_j), \mathcal{T}_B(b_j), \mathcal{T}_C(c_j) + U_j(a_j) - V_j(b_j))$$

defines a morphism of complexes

$$\mathcal{T}_j : E_j \longrightarrow E_j.$$

(ii) *Given the data 1.3.5.1–1.3.5.7 and $r \in R$, the diagram*

$$\begin{array}{ccc} E_1 \otimes_R E_2 & \xrightarrow{\cup_{r,h}} & E_3 \\ \Big\downarrow s_{12}\circ(\mathcal{T}_1 \otimes \mathcal{T}_2) & & \Big\downarrow \mathcal{T}_3 \\ E_2 \otimes_R E_1 & \xrightarrow{\cup'_{1-r,h'}} & E_3 \end{array}$$

is commutative up to homotopy.

1.3.7. Corollary. — *Under the assumptions of Proposition 1.3.6(ii), the following diagrams are commutative up to homotopy:*

$$\begin{array}{ccc} E_2 & \xrightarrow{\mathrm{adj}(\cup'_{1-r,h'})\circ\mathcal{T}_2} & \mathrm{Hom}^\bullet_R(E_1, E_3) \\ \Big\downarrow \varepsilon_{E_3} & & \Big\downarrow \mathrm{Hom}^\bullet(\mathcal{T}_1, \mathrm{id}) \\ \mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_R(E_2, E_3), E_3) & \xrightarrow{\mathrm{Hom}^\bullet(\mathrm{adj}(\cup_{r,h}),\mathcal{T}_3)} & \mathrm{Hom}^\bullet_R(E_1, E_3) \end{array}$$

$$\begin{array}{ccc} E_1 & \xrightarrow{\mathrm{adj}(\cup_{r,h})} & \mathrm{Hom}^\bullet_R(E_2, E_3) \\ \Big\downarrow{\scriptstyle \varepsilon_{E_3}\circ \mathcal{T}_1} & & \Big\downarrow{\scriptstyle \mathrm{Hom}^\bullet(\mathrm{id},\mathcal{T}_3)} \\ \mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_R(E_1, E_3), E_3) & \xrightarrow{\mathrm{Hom}^\bullet(\mathrm{adj}(\cup'_{1-r,h'})\circ \mathcal{T}_2,\mathrm{id})} & \mathrm{Hom}^\bullet_R(E_2, E_3). \end{array}$$

Proof. — Apply a homotopy version of Lemma 1.2.14 to the diagram in Proposition 1.3.6 (ii). □

1.3.8. Bockstein maps. — Assume that, in addition to the data 1.3.1–1.3.4, we are given an R-module Γ_R and the following objects:

1.3.8.1. Morphisms of complexes

$$\beta_{j,Z} : Z_j \longrightarrow Z_j\,[1] \otimes_R \Gamma_R \qquad (j = 1, 2;\ Z = A, B, C)$$

1.3.8.2. Homotopies

$$\begin{aligned} u_j &: f_j\,[1] \circ \beta_{j,A} \rightsquigarrow \beta_{j,C} \circ f_j \\ v_j &: g_j\,[1] \circ \beta_{j,B} \rightsquigarrow \beta_{j,C} \circ g_j \end{aligned} \qquad (j = 1, 2).$$

Above, the map

$$f_j\,[1] \otimes \mathrm{id} : A_j\,[1] \otimes_R \Gamma_R \longrightarrow C_j\,[1] \otimes_R \Gamma_R$$

is abbreviated as $f_j\,[1]$ (and similarly for $g_j\,[1]$).

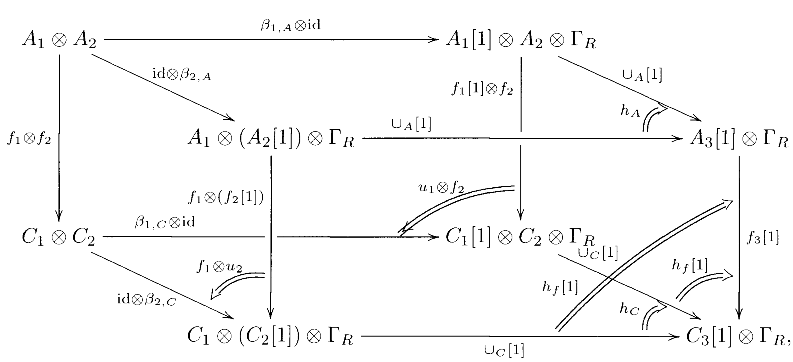
1.3.8.3. Homotopies

$$h_Z : \cup_Z\,[1] \circ (\mathrm{id} \otimes \beta_{2,Z}) \rightsquigarrow \cup_Z\,[1] \circ (\beta_{1,Z} \otimes \mathrm{id}) \qquad (Z = A, B, C).$$

Again, we write $\cup_Z\,[1]$ instead of

$$\cup_Z\,[1] \otimes \mathrm{id} : (Z_1 \otimes_R (Z_2\,[1])) \otimes_R \Gamma_R \longrightarrow Z_3\,[1] \otimes_R \Gamma_R.$$

1.3.8.4. A second order homotopy H_f trivializing the boundary of the following cubic diagram:



i.e., such that

$$dH_f - H_f d = \cup_C[1] \star (u_1 \otimes f_2) + f_3[1] \star h_A + h_f[1] \star (\mathrm{id} \otimes \beta_{2,A}) \\ - \cup_C[1] \star (f_1 \otimes u_2) - h_C \star (f_1 \otimes f_2) - h_f[1] \star (\beta_{1,A} \otimes \mathrm{id}).$$

Above, we implicitly use the canonical isomorphisms from 1.2.18.

1.3.8.5. A second order homotopy H_g trivializing the boundary of an analogous cube in which (A, f, u) are replaced by (B, g, v).

1.3.9. Proposition

(i) *Given the data 1.3.8.1–1.3.8.2, the formula*

$$\beta_{j,E}(a_j, b_j, c_j) = (\beta_{j,A}(a_j), \beta_{j,B}(b_j), -\beta_{j,C}(c_j) - u_j(a_j) + v_j(b_j))$$

defines a morphism of complexes

$$\beta_{j,E} : E_j \longrightarrow E_j[1] \otimes_R \Gamma_R \qquad (j = 1, 2).$$

(ii) *Given the data 1.3.8.1–1.3.8.5 and $r \in R$, the diagram*

$$\begin{array}{ccc} E_1 \otimes_R E_2 & \xrightarrow{\beta_{E,1} \otimes \mathrm{id}} & E_1[1] \otimes_R E_2 \otimes_R \Gamma_R \\ \big\downarrow{\scriptstyle \mathrm{id} \otimes \beta_{E,2}} & & \big\downarrow{\scriptstyle \cup_{r,h}[1]} \\ E_1 \otimes_R (E_2[1]) \otimes_R \Gamma_R & \xrightarrow{\cup_{r,h}[1]} & E_3[1] \otimes_R \Gamma_R \end{array}$$

is commutative up to homotopy.

Proof. — The proof is analogous to that of Proposition 1.3.4. In (ii), we again use the canonical isomorphisms 1.2.18. □

1.3.10. Proposition. — *Given the data 1.3.5.1–1.3.5.7, 1.3.8.1–1.3.8.5 and $r \in R$, the diagram*

$$\begin{array}{ccccccc} E_1 \otimes_R E_2 & \xrightarrow{\beta_{E,1} \otimes \mathrm{id}} & E_1[1] \otimes_R E_2 \otimes_R \Gamma_R & \xrightarrow{\cup_{r,h}[1]} & E_3[1] \otimes_R \Gamma_R & \xrightarrow{\mathcal{T}_3[1]} & E_3[1] \otimes_R \Gamma_R \\ \big\downarrow{\scriptstyle s_{12}} & & & & & & \| \\ E_2 \otimes_R E_1 & \xrightarrow{\beta_{E,2} \otimes \mathrm{id}} & E_2[1] \otimes_R E_1 \otimes_R \Gamma_R & \xrightarrow{\mathcal{T}_2[1] \otimes \mathcal{T}_1} & E_2[1] \otimes_R E_1 \otimes_R \Gamma_R & \xrightarrow{\cup'_{1-r,h'}[1]} & E_3[1] \otimes_R \Gamma_R \end{array}$$

is commutative up to homotopy (above, $\mathcal{T}_j$ are as in Proposition 1.3.6 (i)).

Proof. — Combine Proposition 1.3.6 (ii) and 1.3.9 (ii) (using the canonical isomorphisms 1.2.18). □

1.3.11. The morphisms $\beta_{j,X}$ arise naturally in the following context (for simplicity, we suppress the index j from the notation). Assume that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A'' & \xrightarrow{\rho_A} & A' & \xrightarrow{\sigma_A} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & C'' & \xrightarrow{\rho_C} & C' & \xrightarrow{\sigma_C} & C & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle g''} & & \uparrow{\scriptstyle g'} & & \uparrow{\scriptstyle g} & & \\ 0 & \longrightarrow & B'' & \xrightarrow{\rho_B} & B' & \xrightarrow{\sigma_B} & B & \longrightarrow & 0 \end{array}$$

is a commutative diagram of morphisms of complexes with exact rows. Assume, in addition, that in each degree $i \in \mathbf{Z}$ the epimorphism

$$\sigma_X^i : (X')^i \longrightarrow X^i \qquad (X = A, B, C)$$

admits a section

$$s_X^i : X^i \longrightarrow (X')^i \qquad (X = A, B, C).$$

Writing

$$E = \mathrm{Cone}\Big(A \oplus B \xrightarrow{f-g} C\Big)[-1]$$

(and similarly for E', E''), then the maps

$$\rho_E^i = (\rho_A^i, \rho_B^i, \rho_C^{i-1}), \quad \sigma_E^i = (\sigma_A^i, \sigma_B^i, \sigma_C^{i-1})$$

define an exact sequence of complexes

$$0 \longrightarrow E'' \xrightarrow{\rho_E} E' \xrightarrow{\sigma_E} E \longrightarrow 0$$

and

$$s_E^i = (s_A^i, s_B^i, s_C^{i-1})$$

is a section of σ_E^i $(i \in \mathbf{Z})$.

The recipe from 1.1.4 yields morphisms of complexes

$$\beta_X : X \longrightarrow X''[1] \qquad (X = A, B, C, E)$$

characterized by

(1.3.11.1) $$\rho_X[1] \circ \beta_X = d \circ s_X - s_X \circ d \qquad (X = A, B, C, E)$$

and such that

$$X'' \xrightarrow{\rho_X} X' \xrightarrow{\sigma_X} X \xrightarrow{\beta_X} X''[1]$$

is an exact triangle in the derived category. In order to express β_E in terms of β_A, β_B and β_C, we compute

$$\begin{aligned} &d(s_E(a,b,c)) - s_E(d(a,b,c)) \\ &= d(s_A(a), s_B(b), s_C(c)) - \big(s_A(da), s_B(db), s_C(-dc - f(a) + g(b))\big) \\ &= ((ds_A - s_A d)(a), (ds_B - s_B d)(b), -(ds_C - s_C d)(c) - (f' s_A - s_C f)(a) + (g' s_B - s_C g)(b)), \end{aligned}$$

hence

$$\beta_E(a,b,c) = (\beta_A(a), \beta_B(b), -\beta_C(c) - u(a) + v(b)), \tag{1.3.11.2}$$

where

$$A \xrightarrow{u} C'' \xleftarrow{v} B$$

are morphisms of complexes characterized by

$$\begin{aligned} \rho_C \circ u &= f' \circ s_A - s_C \circ f \\ \rho_C \circ v &= g' \circ s_B - s_C \circ g. \end{aligned}$$

As

$$\begin{aligned} - d_{C'} \circ (f' \circ s_A - s_C \circ f) + (f' \circ s_A - s_C \circ f) \circ d_A \\ = (d \circ s_C - s_C \circ d) \circ f - f' \circ (d \circ s_A - s_A \circ d) \end{aligned}$$

(and similarly for (B, g) instead of (A, f)), the morphisms u and v are, in fact, homotopies

$$\begin{aligned} u &: f''[1] \circ \beta_A \rightsquigarrow \beta_C \circ f \\ v &: g''[1] \circ \beta_B \rightsquigarrow \beta_C \circ g. \end{aligned}$$

In the special case when $X'' = X \otimes_R \Gamma_R$ ($X = A, B, C$) we thus obtain the data 1.3.8.1–1.3.8.2, as well as the formula for β_E from Proposition 1.3.9 (i).

CHAPTER 2

LOCAL DUALITY

In this chapter we recall the formalism of Grothendieck's duality for R-modules ([**LC, RD**]). At first reading there is no need to continue beyond 2.10.4 (the subsequent sections are used in the construction of generalized Cassels-Tate pairings in Chapter 10).

2.1. Notation

Throughout Chapters 2–11 (with the exception of Sect. 2.10), R will be a complete Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. The dimension of R will be denoted by d (it is finite, as R is Noetherian and *local*). The total ring of fractions of R will be denoted by $\mathrm{Frac}(R)$.

Denote by $({}_R\mathrm{Mod})$ the category of all R-modules, by $({}_R\mathrm{Mod})_{ft}$ (resp., $({}_R\mathrm{Mod})_{coft}$) the category of R-modules of finite (resp., co-finite) type (*i.e.*, of modules satisfying the ascending (resp., descending) condition for submodules) and $({}_R\mathrm{Mod})_{\mathrm{fl}} = ({}_R\mathrm{Mod})_{ft} \cap ({}_R\mathrm{Mod})_{coft}$ the category of R-modules of finite length.

2.2. Dualizing functors

Let I be an R-module. The functor

$$D(-) = \mathrm{Hom}_R(-, I) : ({}_R\mathrm{Mod})^{\mathrm{op}} \longrightarrow ({}_R\mathrm{Mod})$$

is *dualizing* if the canonical homomorphism

$$\varepsilon : M \longrightarrow D(D(M)) \tag{2.2.1}$$

is an isomorphism for every $M \in ({}_R\mathrm{Mod})_{\mathrm{fl}}$.

2.3. Matlis duality

2.3.1. Matlis Duality ([LC, Prop. 4.10]; [Br-He, Thm. 3.2.13])
(i) D is dualizing iff I is an injective hull of k (defined, *e.g.*, in [**Br-He**, Def. 3.2.3]).
(ii) Fix such I (it is unique up to a non-unique isomorphism); the functor D is then exact and induces equivalences of categories

$$({}_R\mathrm{Mod})^{\mathrm{op}}_{\mathrm{fl}} \rightleftarrows ({}_R\mathrm{Mod})_{\mathrm{fl}}$$

$$({}_R\mathrm{Mod})^{\mathrm{op}}_{ft} \rightleftarrows ({}_R\mathrm{Mod})_{coft}.$$

The map (2.2.1) is an isomorphism for every M in $({}_R\mathrm{Mod})_{ft}$ or $({}_R\mathrm{Mod})_{coft}$.

2.3.2. From now on, I will be as in 2.3.1 (ii). The functor D, being exact, can be derived trivially. For every complex $M^\bullet$ of R-modules and $n \in \mathbf{Z}$ put

$$D_n(M^\bullet) = \mathrm{Hom}^\bullet_R(M^\bullet, I[n]) = D(M^\bullet)[n]$$

(with the sign conventions of 1.2.1). It follows from 2.3.1 (ii) that the canonical map

$$\varepsilon = \varepsilon_{I[n]} : M \longrightarrow D_n(D_n(M))$$

is an isomorphism for every M in $D_{ft}({}_R\mathrm{Mod})$ or $D_{coft}({}_R\mathrm{Mod})$.

2.3.3. The simplest examples of I are the following:
(i) $I = R$ if $R = k$ is a field.
(ii) $I = K/R$ if R is a (complete) discrete valuation ring with fraction field K.
(iii) $I = R[1/x_1 \dots x_d]/\big(\sum_{i=1}^d R[1/x_1 \dots \widehat{x_i} \dots x_d]\big)$ if $R = k[\![x_1, \dots, x_d]\!]$ is a power series ring.

2.3.4. We shall often use the fact that for every projective (resp.. inductive) system $(M_n)_{n\in\mathbf{N}}$ of R-modules of finite (resp., co-finite) type such that $M = \varprojlim M_n$ (resp., $M = \varinjlim M_n$) is also of finite (resp., co-finite) type, the canonical map $\varinjlim D(M_n) \to D(M)$ (resp., $D(M) \to \varprojlim D(M_n)$) is an isomorphism.

2.3.5. Lemma. — *Let $f : M \to N$ be a homomorphism of R-modules. Then*
(i) $M = 0 \iff D(M) = 0$.
(ii) *The homomorphism $\varepsilon : M \to D(D(M))$ is injective.*
(iii) $f = 0 \iff D(f) = 0$.

Proof. — If $M \neq 0$, choose non-zero $x \in M$, $y \in I$. By 2.3.1 applied to Rx there exists an injective morphism $Rx \to I$; it extends to a morphism $f : M \to I$ satisfying $f(x) \neq 0$. This proves (i) and also shows that $\varepsilon(x) \neq 0$, proving (ii). As regards (iii), the morphism f factors as $f = gh$ with $g : f(M) \to N$ injective ($\Longrightarrow$ $D(g)$ surjective) and $h : M \to f(M)$ surjective ($\Longrightarrow D(h)$ injective). This implies that

$$D(f) = 0 \iff D(g) = 0 \iff D(f(M)) = 0 \overset{(i)}{\iff} f = 0. \quad \square$$

2.3.6. Lemma. — *If M is an R-module of finite (resp., co-finite) type, then every surjective (resp., injective) R-linear endomorphism $f : M \to M$ is bijective.*

Proof. — If M is of finite type, see ([**Mat**, Thm. 2.4]). If M is of co-finite type, the previous statement applied to the dual endomorphism $D(f) : D(M) \to D(M)$ implies that $D(f)$ is bijective, hence so is $f = D(D(f))$. □

2.4. Cohomology with support at $\{\mathfrak{m}\}$

2.4.1. Every R-module M defines a quasi-coherent sheaf $\widetilde{M}$ on $X = \operatorname{Spec}(R)$. Its cohomology with support at the closed point $\{\mathfrak{m}\} \subset X$ will be denoted by

$$H^i_{\{\mathfrak{m}\}}(M) = H^i_{\{\mathfrak{m}\}}(\operatorname{Spec}(R), \widetilde{M}).$$

An explicit complex representing

$$\mathbf{R}\Gamma_{\{\mathfrak{m}\}}(X, \widetilde{M}) \in D^b({}_R\mathrm{Mod})$$

can be constructed by using an exact triangle

$$\text{(2.4.1.1)}\quad \mathbf{R}\Gamma_{\{\mathfrak{m}\}}\big(X, \widetilde{M}\big) \longrightarrow \mathbf{R}\Gamma\big(X, \widetilde{M}\big) \longrightarrow \mathbf{R}\Gamma\big(X - \{\mathfrak{m}\}, \widetilde{M}\big) \longrightarrow \mathbf{R}\Gamma_{\{\mathfrak{m}\}}\big(X, \widetilde{M}\big)[1]$$

2.4.2. First of all, $\mathbf{R}\Gamma\big(X, \widetilde{M}\big)$ is represented by M in degree zero. To get a complex representing $\mathbf{R}\Gamma\big(X - \{\mathfrak{m}\}, \widetilde{M}\big)$, fix a system of parameters of R, *i.e.*, a d-tuple of elements $x_1, \ldots, x_d \in \mathfrak{m}$ such that $R/(x_1, \ldots, x_d)$ has finite length. Then $\mathcal{U} = \{U_i = \operatorname{Spec}(R_{x_i}) \mid i = 1, \ldots, d\}$ is an open covering of $X - \{\mathfrak{m}\}$ such that all intersections $U_{i_0} \cap \cdots \cap U_{i_p} = \operatorname{Spec}(R_{x_{i_0} \cdots x_{i_p}})$ are affine. This implies that the Čech complex $\check{C}^\bullet(M) = \check{C}^\bullet(M, (x_i)) = \check{C}^\bullet\big(\mathcal{U}, \widetilde{M}\big)$ with

$$\check{C}^p(M, (x_i)) = \bigoplus_{i_0 < \cdots < i_p} M_{x_{i_0} \cdots x_{i_p}} = \check{C}^p(R, (x_i)) \otimes_R M \quad (0 \leqslant p < d)$$

and the standard differential

$$(\delta^p A)_{i_0 \cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j A_{i_0 \cdots \widehat{i_j} \cdots i_{p+1}} \quad (i_0 < \cdots < i_{p+1})$$

represents $\mathbf{R}\Gamma\big(X - \{\mathfrak{m}\}, \widetilde{M}\big)$.

2.4.3. It follows from (2.4.1.1) that $\mathbf{R}\Gamma_{\{\mathfrak{m}\}}\big(X, \widetilde{M}\big)$ can be represented by

$$\begin{aligned} C^\bullet(M) = C^\bullet(M, (x_i)) &= \operatorname{Cone}\big(M \xrightarrow{\ i\ } \check{C}^\bullet(M, (x_i))\big)[-1] \\ &= M \xrightarrow{-i} \check{C}^0(M) \xrightarrow{-\delta^0} \cdots \xrightarrow{-\delta^{d-2}} \check{C}^{d-1}(M), \end{aligned}$$

where $i : M \to \oplus_{j=1}^d M_{x_j}$ is the canonical map. The complex $C^\bullet(M)$ is concentrated in degrees $[0, d]$ and is equal to $C^\bullet(R) \otimes_R M = M \otimes_R C^\bullet(R)$.

2.4.4. For example, if $R = \mathbf{Z}_p$, then $C^\bullet(M) = \big[M \xrightarrow{-i} M \otimes_{\mathbf{Z}_p} \mathbf{Q}_p\big]$, $H^0_{\{\mathfrak{m}\}}(M) = M_{\mathrm{tors}}$ and $H^1_{\{\mathfrak{m}\}}(M) = M \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p$.

2.4.5. There is an alternative description of $C^\bullet(M)$ in terms of Koszul complexes ([**LC**, §2]). For a commutative ring A, an A-module M and a sequence $\mathbf{y} = (y_1, \ldots, y_r)$ of elements of A, the Koszul complexes are defined inductively as

$$K^\bullet_A(A, (y_1)) = \big[A \xrightarrow{y_1} A\big]$$

(in degrees $0, 1$),

$$K^\bullet_A(A, \mathbf{y}) = K^\bullet_A(A, (y_1, \ldots, y_r)) = K^\bullet_A(A, (y_1)) \otimes_A K^\bullet_A(A, (y_2, \ldots, y_r))$$

and

$$K^\bullet_A(M, \mathbf{y}) = K^\bullet_A(M, (y_1, \ldots, y_r)) = K^\bullet_A(A, \mathbf{y}) \otimes_A M.$$

If M is a Noetherian A-module, then each cohomology group of $K^\bullet_A(M, \mathbf{y})$ is a Noetherian $A/(y_1, \ldots, y_r)A$-module.

The morphisms $K^\bullet_A(A, (y_j^n)) \to K^\bullet_A(A, (y_j^{n+1}))$, given by the multiplication by y_j^i in degree $i = 0, 1$, define morphisms of complexes $K^\bullet_A(M, \mathbf{y}^n) \to K^\bullet_A(M, \mathbf{y}^{n+1})$, where $\mathbf{y}^n = (y_1^n, \ldots, y_r^n)$. In the situation of 2.4.3, there is a canonical isomorphism of complexes

$$\varinjlim_n K^\bullet_R(M, \mathbf{x}^n) \xrightarrow{\sim} C^\bullet(M, \mathbf{x}).$$

2.4.6. For every complex of R-modules $M^\bullet$ we define $C^\bullet(M^\bullet) = M^\bullet \otimes^\bullet_R C^\bullet(R)$. If $M^\bullet$ has cohomology of finite type, then $C^\bullet(M^\bullet)$ represents $\mathbf{R}\Gamma_{\{\mathfrak{m}\}}(X, M^\bullet)$, hence

$$H^i(C^\bullet(M^\bullet)) = H^i_{\{\mathfrak{m}\}}(M^\bullet).$$

2.4.7. Lemma ([**LC**, §3.10, §6.4]). — *For every $M \in ({}_R\mathrm{Mod})_{ft}$ and $i \geqslant 0$,*

(i) $H^i_{\{\mathfrak{m}\}}(M)$ *is an R-module of co-finite type.*
(ii) $H^i_{\{\mathfrak{m}\}}(M) = 0$ *for* $i < \mathrm{depth}(M)$.
(iii) $H^i_{\{\mathfrak{m}\}}(M) = 0$ *for* $i > \dim(M)$.
(iv) $\dim\big(D(H^i_{\{\mathfrak{m}\}}(M))\big) \leqslant i$.
(v) $H^i_{\{\mathfrak{m}\}}(M) \neq 0$ *for* $i = \dim(M)$ *and* $i = \mathrm{depth}(M)$.

2.5. Local Duality ([RD, Ch. V])

(i) There exists a dualizing complex $\omega \in D^b_{ft}({}_R\mathrm{Mod})$ (unique up to isomorphism) with the property

$$H^i_{\{\mathfrak{m}\}}(\omega) \xrightarrow{\sim} \begin{cases} I, & i = d \\ 0, & i \neq d. \end{cases}$$

We fix an isomorphism $\mathrm{Tr} : H^d_{\{\mathfrak{m}\}}(\omega) \xrightarrow{\sim} I$.

(ii) ω can be represented by a bounded complex of injective R-modules $\omega^\bullet$, supported in degrees $[0, d]$.

(iii) For every R-module M of finite type and $i \in \mathbf{Z}$, the Yoneda pairing

$$H^i_{\{\mathfrak{m}\}}(M) \times \mathbb{E}\mathrm{xt}^{d-i}_R(M,\omega) \longrightarrow H^d_{\{\mathfrak{m}\}}(\omega) \xrightarrow{\ \sim\ } I$$

induces isomorphisms

$$H^i_{\{\mathfrak{m}\}}(M) \xrightarrow{\ \sim\ } D\big(\mathbb{E}\mathrm{xt}^{d-i}_R(M,\omega)\big)$$
$$\mathbb{E}\mathrm{xt}^{d-i}_R(M,\omega) \xrightarrow{\ \sim\ } D\big(H^i_{\{\mathfrak{m}\}}(M)\big).$$

(iv) A triangulated version of (iii): for every object M of $D_{ft}({}_R\mathrm{Mod})$ (resp., $D^{\pm}_{ft}({}_R\mathrm{Mod})$) the canonical map

$$\eta : \mathbf{R}\mathrm{Hom}_R(M,\omega) \longrightarrow D(\mathbf{R}\Gamma_{\{\mathfrak{m}\}}(M)[d])$$

(defined in 2.8.1 below) is an isomorphism in $D_{ft}({}_R\mathrm{Mod})$ (resp., $D^{\mp}_{ft}({}_R\mathrm{Mod})$).

2.6. Grothendieck Duality ([**RD**, Ch. V])

For every object M of $D_{ft}({}_R\mathrm{Mod})$ (resp., $D^{\pm}_{ft}({}_R\mathrm{Mod})$), $\mathscr{D}(M) := \mathbf{R}\mathrm{Hom}_R(M,\omega)$ is an object of $D_{ft}({}_R\mathrm{Mod})$ (resp., $D^{\mp}_{ft}({}_R\mathrm{Mod})$) and the canonical map $\varepsilon = \varepsilon_\omega : M \to \mathscr{D}(\mathscr{D}(M))$ is an isomorphism.

2.7. Remarks

(i) $H^i(\omega) = D(H^{d-i}_{\{\mathfrak{m}\}}(R))$ vanishes for $i > d - \mathrm{depth}(R)$ (resp., $i < 0$) by Lemma 2.4.7 (ii) (resp., 2.4.7 (iii)) and is non-zero for $i = 0$ and $i = d - \mathrm{depth}(R)$ by Lemma 2.4.7 (v). Furthermore, $\dim(H^i(\omega)) \leqslant d - i$, by Lemma 2.4.7 (iv).

(ii) In particular, R is Cohen-Macaulay (*i.e.*, $\mathrm{depth}(R) = d$) iff $\omega \xrightarrow{\sim} H^0(\omega)$ is concentrated in degree zero (in which case $\omega \xrightarrow{\sim} D(H^d_{\{\mathfrak{m}\}}(R))$).

(iii) R is Gorenstein (*i.e.*, R is quasi-isomorphic to a bounded complex of injective R-modules) iff $\omega \xrightarrow{\sim} R$.

(iv) In order to stress their dependence on R we sometimes denote I, D, $\mathscr{D}$, ω by I_R, D_R, $\mathscr{D}_R$, ω_R.

(v) The hyper-cohomology spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_R(M, H^j(\omega)) \Longrightarrow \mathbb{E}\mathrm{xt}^{i+j}_R(M,\omega)$$

implies that

$$\mathbb{E}\mathrm{xt}^0_R(M,\omega) = \mathrm{Hom}_R(M, H^0(\omega)) = \mathrm{Hom}_R\big(M, D(H^d_{\{\mathfrak{m}\}}(R))\big),$$

for every R-module M.

(vi) If $\mathrm{depth}(R_\mathfrak{p}) = 1$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{p}) = 1$, then $\omega \xrightarrow{\sim} H^0(\omega)$ in $D(({}_R\mathrm{Mod})/(\text{pseudo-null}))$, using the language of 2.8.6 below (this follows from 2.7 (i)–(ii) applied to the (non-complete – but see 2.10) localizations $R_\mathfrak{p}$).

2.8. Relating D, $\mathscr{D}$ and Φ

2.8.1. The functors

$$D : D_{ft}({}_R\mathrm{Mod})^{\mathrm{op}} \longleftrightarrow D_{coft}({}_R\mathrm{Mod})$$
$$\mathscr{D} : D_{ft}({}_R\mathrm{Mod})^{\mathrm{op}} \longrightarrow D_{ft}({}_R\mathrm{Mod})$$

(which map $D^{\pm}$ to $D^{\mp}$) are related to

$$\Phi(-) := \mathbf{R}\Gamma_{\{\mathfrak{m}\}}(-)[d] : D_{ft}({}_R\mathrm{Mod}) \longrightarrow D_{coft}({}_R\mathrm{Mod})$$

(which maps $D^{\pm}$ to $D^{\pm}$) as follows.

First of all, we have

$$\Phi(-) = \Big((-)\overset{\mathbf{L}}{\otimes}_R \Phi(R)[-d]\Big)[d].$$

The natural transformation of functors $\eta : \mathscr{D} \Longrightarrow D \circ \Phi$, defined by

$$\begin{aligned}\mathscr{D}(-) = \mathbf{R}\mathrm{Hom}_R(-,\omega) &\longrightarrow \mathbf{R}\mathrm{Hom}_R\Big((-)\overset{\mathbf{L}}{\otimes}_R\Phi(R)[-d], \omega\overset{\mathbf{L}}{\otimes}_R\Phi(R)[-d]\Big)\\ &= \mathbf{R}\mathrm{Hom}_R\big(\Phi(-)[-d], \Phi(\omega)[-d]\big)\\ &\qquad \xrightarrow{t_d} \mathbf{R}\mathrm{Hom}_R(\Phi(-),\Phi(\omega)) \xrightarrow{\mathbf{R}\mathrm{Hom}(\mathrm{id},\mathrm{Tr})} \mathbf{R}\mathrm{Hom}_R(\Phi(-), I)\\ &= D\circ\Phi(-),\end{aligned}$$

is an isomorphism, by local duality 2.5.

The natural transformations

$$\varepsilon_I : \mathrm{id} \Longrightarrow D\circ D, \quad \varepsilon_\omega : \mathrm{id} \Longrightarrow \mathscr{D}\circ\mathscr{D}$$

are isomorphisms, by Matlis and Grothendieck duality, respectively. As a consequence, the natural transformations

$$\psi : \Phi \overset{\varepsilon_I\star\Phi}{\Longrightarrow} D\circ D\circ\Phi \overset{D\star\eta}{\Longrightarrow} D\circ\mathscr{D}$$
$$\xi : \Phi\circ\mathscr{D} \overset{\psi\star\mathscr{D}}{\Longrightarrow} D\circ\mathscr{D}\circ\mathscr{D} \overset{D\star\varepsilon_\omega}{\Longrightarrow} D$$

are isomorphisms, too. It follows from the fact that the composition

$$D \overset{\varepsilon_I\star D}{\Longrightarrow} D\circ D\circ D \overset{D\star\varepsilon_I}{\Longrightarrow} D$$

is the identity (and from an analogous statement for $\mathscr{D}$) that the following diagrams of natural transformations are commutative:

$$\begin{array}{ccc} \Phi & \overset{\Phi\star\varepsilon_\omega}{\Longrightarrow} & \Phi\circ\mathscr{D}\circ\mathscr{D} \\ & \overset{\psi}{\searrow} & \Downarrow{\scriptstyle \xi\star\mathscr{D}} \\ & & D\circ\mathscr{D} \end{array} \qquad \begin{array}{ccc} \mathscr{D} & \overset{\varepsilon_I\star\mathscr{D}}{\Longrightarrow} & D\circ D\circ\mathscr{D} \\ & \overset{\eta}{\searrow} & \Downarrow{\scriptstyle D\star\psi} \\ & & D\circ\Phi \end{array} \qquad \begin{array}{ccc} \mathrm{id} & \overset{\varepsilon_I}{\Longrightarrow} & D\circ D \\ \Downarrow{\scriptstyle \varepsilon_\omega} & & \Downarrow{\scriptstyle D\star\xi} \\ \mathscr{D}\circ\mathscr{D} & \overset{\eta\star\mathscr{D}}{\Longrightarrow} & D\circ\Phi\circ\mathscr{D} \end{array}$$

2.8.2. To sum up: the following *duality diagram* of functors

$$\begin{array}{ccc} D^{\pm}_{ft}({}_R\mathrm{Mod})^{\mathrm{op}} & \overset{\mathscr{D}}{\longleftrightarrow} & D^{\mp}_{ft}({}_R\mathrm{Mod}) \\ \Big\downarrow{\scriptstyle\Phi} & \overset{D}{\times} & \Big\downarrow{\scriptstyle\Phi} \\ D^{\pm}_{coft}({}_R\mathrm{Mod})^{\mathrm{op}} & & D^{\mp}_{coft}({}_R\mathrm{Mod}) \end{array}$$

(and its analogue without $\pm, \mp$) is commutative up to various natural isomorphisms of functors:

- $\varepsilon_I : \mathrm{id} \overset{\sim}{\Longrightarrow} D \circ D$ (Matlis duality).
- $\varepsilon_\omega : \mathrm{id} \overset{\sim}{\Longrightarrow} \mathscr{D} \circ \mathscr{D}$ (Grothendieck duality).
- $\eta : \mathscr{D} \overset{\sim}{\Longrightarrow} D \circ \Phi$ (local duality).

2.8.3. In particular, $\psi(R)$ induces an isomorphism $\Phi(R) \overset{\sim}{\to} D(\mathscr{D}(R)) = D(\omega)$, hence

$$\Phi(-) \overset{\sim}{\longrightarrow} \Big((-)\overset{\mathbf{L}}{\otimes}_R D(\omega)[-d]\Big)[d] \overset{\sim}{\longrightarrow} (-)\overset{\mathbf{L}}{\otimes}_R D(\omega)$$

(the last arrow is given by $s'_{-d}[d]$, in the notation of 1.2.15). This can also be deduced from the adjunction isomorphism

$$\mathrm{adj} : \mathbf{R}\mathrm{Hom}_R\big(A, \mathbf{R}\mathrm{Hom}_R(B, C)\big) \overset{\sim}{\longrightarrow} \mathbf{R}\mathrm{Hom}_R\Big(A\overset{\mathbf{L}}{\otimes}_R B, C\Big)$$

(which is a derived version of 1.2.6; it holds in $D^+({}_R\mathrm{Mod})$ for all $A, B \in D^-({}_R\mathrm{Mod})$, $C \in D^+({}_R\mathrm{Mod})$) applied to $B = D(\omega)$ and $C = I$:

$$\mathbf{R}\mathrm{Hom}_R(-, \omega) \overset{\sim}{\longrightarrow} \mathbf{R}\mathrm{Hom}_R\big(-, D(D(\omega))\big) \overset{\sim}{\longrightarrow} \mathbf{R}\mathrm{Hom}_R\Big((-)\overset{\mathbf{L}}{\otimes}_R D(\omega), I\Big).$$

For example, for $R = \mathbf{Z}_p$ we have $I = \mathbf{Q}_p/\mathbf{Z}_p$, $\omega \overset{\sim}{\to} \mathbf{Z}_p$ and $C^\bullet(\mathbf{Z}_p) = [\mathbf{Z}_p \to \mathbf{Q}_p]$ (in degrees 0 and 1); this is quasi-isomorphic to $\mathbf{Q}_p/\mathbf{Z}_p[-1]$. If M is a free $\mathbf{Z}_p$-module of finite type, then $\mathscr{D}(M) = \mathrm{Hom}_{\mathbf{Z}_p}(M, \mathbf{Z}_p)$, $\Phi(M) = M \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p$ and

$$D(\Phi(M)) \overset{\sim}{\longrightarrow} \mathrm{Hom}_{\mathbf{Z}_p}(M \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p, \mathbf{Q}_p/\mathbf{Z}_p) = \mathscr{D}(M).$$

2.8.4. All of the above makes sense on the level of complexes: fixing a system of parameters x_i of R and a bounded complex of injective R-modules $\omega^\bullet$ representing ω, we have

$$\begin{aligned} \Phi(X^\bullet) &= (X^\bullet \otimes_R C^\bullet((x_i), R))[d] \\ D(X^\bullet) &= \mathrm{Hom}^\bullet_R(X^\bullet, I) \\ \mathscr{D}(X^\bullet) &= \mathrm{Hom}^\bullet_R(X^\bullet, \omega^\bullet). \end{aligned}$$

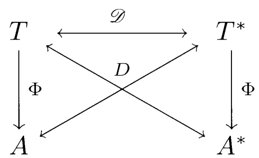
As I is injective, the isomorphism Tr can be represented by a quasi-isomorphism

$$\mathrm{Tr} : \Phi(\omega^\bullet) \longrightarrow I,$$

unique up to homotopy. The morphisms $\eta(X^\bullet), \psi(X^\bullet), \xi(X^\bullet)$ are genuine morphisms of complexes (of course, they are all quasi-isomorphisms). For example, $\eta(X^\bullet)$ is given by

$$\begin{aligned}\mathscr{D}(X^\bullet) = \mathrm{Hom}^\bullet_R(X^\bullet, \omega^\bullet) &\longrightarrow \mathrm{Hom}^\bullet_R(X^\bullet \otimes_R C^\bullet(R), \omega^\bullet \otimes_R C^\bullet(R))\\ \xrightarrow{t_d} \mathrm{Hom}_R((X^\bullet \otimes_R C^\bullet(R))[d], (\omega^\bullet \otimes_R C^\bullet(R))[d]) &\xrightarrow{\mathrm{Hom}^\bullet(\mathrm{id},\mathrm{Tr})} \mathrm{Hom}^\bullet_R(\Phi(X^\bullet), I)\\ &= D \circ \Phi(X^\bullet).\end{aligned}$$

2.8.5. For $T \in D_{ft}({}_R\mathrm{Mod})$ put $T^* = \mathscr{D}(T)$, $A = \Phi(T) = \mathbf{R}\Gamma_{\{\mathfrak{m}\}}(T)[d]$, $A^* = D(T)$. Loosely speaking, we can think of these four objects as being related by the diagram



2.8.6. In the notation of 2.8.5, the hyper-cohomology spectral sequence of $\mathscr{D}$ applied to T^* is given by

$$E_2^{i,j} = \mathrm{Ext}^i_R(H^{-j}(T^*), \omega) \Longrightarrow H^{i+j}(\mathscr{D}(T^*)),$$

i.e.,

$$(2.8.6.1) \qquad E_2^{i,j} = \mathrm{Ext}^i_R(D(H^j(A)), \omega) \Longrightarrow H^{i+j}(T).$$

It follows from local duality 2.5 and Lemma 2.4.7 (iii)–(iv) that $E_2^{i,j} = 0$ for $i < 0$ and that $\mathrm{supp}(E_2^{i,j})$ has codimension $\geqslant i$ in $\mathrm{Spec}(R)$. By 2.7 (v) we have

$$E_2^{0,j} = \mathrm{Hom}_R(D(H^j(A)), H^0(\omega)).$$

Recall that an R-module M of finite (resp., co-finite) type is *pseudo-null* (resp., *co-pseudo-null*) if $\mathrm{supp}(M)$ has codimension $\geqslant 2$ in $\mathrm{Spec}(R)$ (resp., if $D(M)$ is pseudo-null).

It follows that, in the quotient category $({}_R\mathrm{Mod})/(\text{pseudo-null})$, the spectral sequence (2.8.6.1) degenerates to a collection of short exact sequences

$$(2.8.6.2) \quad 0 \longrightarrow \mathrm{Ext}^1_R(D(H^{j-1}(A)), \omega) \longrightarrow H^j(T) \longrightarrow \mathrm{Ext}^0_R(D(H^j(A)), \omega) \longrightarrow 0.$$

Recall also that, for each prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{p}) \leqslant 1$, the localization $M \mapsto M_\mathfrak{p}$ defines an exact functor $({}_R\mathrm{Mod})/(\text{pseudo-null}) \to ({}_{R_\mathfrak{p}}\mathrm{Mod})$.

2.8.7. Proposition. — *In the situation of 2.8.5 there are spectral sequences*

$$\begin{aligned}E_2^{i,j} &= H^{i+d}_{\{\mathfrak{m}\}}(H^j(T)) \Longrightarrow H^{i+j}(A)\\ {}'E_2^{i,j} &= H^{i+d}_{\{\mathfrak{m}\}}\big(D(H^{-j}(A))\big) \Longrightarrow D\big(H^{-i-j}(T)\big).\end{aligned}$$

Proof. — The first spectral sequence E_r is just the hyper-cohomology spectral sequence for Φ. It can be constructed explicitly as follows. Represent T by a complex $M^\bullet$ of R-modules; then A is represented by $N^\bullet = (M^\bullet \otimes_R C^\bullet(R))[d]$. Filter $N^\bullet$ by the subcomplexes

$$F^i N^\bullet = \big(M^\bullet \otimes_R \sigma_{\geqslant i+d}(C^\bullet(R))\big)[d].$$

We have

$$\mathrm{gr}^i_F(N^\bullet) = (M^\bullet \otimes_R C^{i+d}(R))[-i],$$

as $C^\bullet(R)$ is a complex of flat R-modules. The corresponding spectral sequence satisfies

$$E_1^{i,j} = H^j(M^\bullet) \otimes_R C^{i+d}(R) = H^j(T) \otimes_R C^{i+d}(R),$$

hence

$$E_2^{i,j} = H^{i+d}_{\{\mathfrak{m}\}}(H^j(T))$$

as claimed. The second spectral sequence ${}'E_r$ is obtained from (2.8.6.1) by applying D and using local duality 2.5 (iii). □

2.8.8. Lemma. — *For every R-module M of finite type, the R-module $\mathbb{E}\mathrm{xt}^0_R(M,\omega)$ (resp., $H^d_{\{\mathfrak{m}\}}(M)$) is torsion-free (resp., divisible). In particular, $H^0(\omega)$ is torsion-free.*

Proof. — We know from 2.7 (v) that $\mathbb{E}\mathrm{xt}^0_R(M,\omega) = \mathrm{Hom}_R(M, D(H^d_{\{\mathfrak{m}\}}(R)))$. If $r \in R$ does not divide zero, then the exact sequence of local cohomology

$$H^{d-1}_{\{\mathfrak{m}\}}(R/rR) \longrightarrow H^d_{\{\mathfrak{m}\}}(R) \xrightarrow{r} H^d_{\{\mathfrak{m}\}}(R) \longrightarrow H^d_{\{\mathfrak{m}\}}(R/rR) = 0$$

(in which the last term vanishes by Lemma 2.4.7 (iii)) shows that multiplication by r on $D(H^d_{\{\mathfrak{m}\}}(R))$, and hence also on $\mathbb{E}\mathrm{xt}^0_R(M,\omega)$, is injective. It follows that multiplication by r on $H^d_{\{\mathfrak{m}\}}(M) = D(\mathbb{E}\mathrm{xt}^0_R(M,\omega))$ is surjective. The last statement is a consequence of $H^0(\omega) = \mathbb{E}\mathrm{xt}^0_R(R,\omega)$. □

2.8.9. As in 2.8.6, it follows from Lemma 2.4.7 (iii)–(iv) that $E_2^{i,j}, {}'E_2^{i,j}$ in Proposition 2.8.7 are co-pseudo-null for $i \neq 0, -1$. This implies that in the quotient category $({}_R\mathrm{Mod})/(\text{co-pseudo-null})$ the two spectral sequences degenerate into short exact sequences

$$0 \longrightarrow H^d_{\{\mathfrak{m}\}}(H^j(T)) \longrightarrow H^j(A) \longrightarrow H^{d-1}_{\{\mathfrak{m}\}}(H^{j+1}(T)) \longrightarrow 0$$
$$0 \longrightarrow H^d_{\{\mathfrak{m}\}}\big(D(H^j(A))\big) \longrightarrow D(H^j(T)) \longrightarrow H^{d-1}_{\{\mathfrak{m}\}}\big(D(H^{j-1}(A))\big) \longrightarrow 0,$$

(the second one being just D(2.8.6.2)). It follows from Lemma 2.8.8 and Lemma 2.4.7 (iv) that $H^d_{\{\mathfrak{m}\}}(H^j(T))$ (resp., $\mathbb{E}\mathrm{xt}^1_R(D(H^{j-1}(A)),\omega)$) is the maximal R-divisible (resp., R-torsion) subobject of $H^j(A)$ (resp., $H^j(T)$) in $({}_R\mathrm{Mod})/(\text{co-pseudo-null})$ (resp., $({}_R\mathrm{Mod})/(\text{pseudo-null})$).

2.8.10. Much of the previous discussion can be reformulated in terms of an analogue of the duality diagram 2.8.2 for appropriate quotient categories:

$$\begin{array}{ccc}
D_{ft}({}_R\mathrm{Mod}/(\text{pseudo-null}))^{\mathrm{op}} & \xleftrightarrow{\mathscr{D}} & D_{ft}({}_R\mathrm{Mod}/(\text{pseudo-null})) \\
\downarrow{\scriptstyle\Phi} & \times\ D & \downarrow{\scriptstyle\Phi} \\
D_{coft}({}_R\mathrm{Mod}/(\text{co-pseudo-null}))^{\mathrm{op}} & & D_{coft}({}_R\mathrm{Mod}/(\text{co-pseudo-null}))
\end{array}$$

2.8.11. It is sometimes convenient to use another normalization of $\mathscr{D}$ and Φ, namely

$$\mathscr{D}_d(-) = \mathscr{D}(-)[d] = \mathbf{R}\mathrm{Hom}_R(-,\omega)[d] = \mathbf{R}\mathrm{Hom}_R(-,\omega\,[d])$$
$$\Phi_{-d}(-) = \Phi(-)[-d] = \mathbf{R}\Gamma_{\{\mathfrak{m}\}}(-) = (-)\overset{\mathbf{L}}{\otimes}_R\Phi_{-d}(R).$$

Then the map Tr defines a quasi-isomorphism $\mathrm{Tr} : \Phi_{-d}(\omega[d]) \to I$ and the diagram (2.8.2) is replaced by

$$\begin{array}{ccc}
D^{\pm}_{ft}({}_R\mathrm{Mod})^{\mathrm{op}} & \xleftrightarrow{\mathscr{D}_d} & D^{\mp}_{ft}({}_R\mathrm{Mod}) \\
\downarrow{\scriptstyle\Phi_{-d}} & \times\ D & \downarrow{\scriptstyle\Phi_{-d}} \\
D^{\pm}_{coft}({}_R\mathrm{Mod})^{\mathrm{op}} & & D^{\mp}_{coft}({}_R\mathrm{Mod}),
\end{array}$$

which is commutative up to natural isomorphisms of functors

$$\eta_d : \mathscr{D}_d \overset{\sim}{\Longrightarrow} D\circ\Phi_{-d}, \quad \psi_d : \Phi_{-d} \overset{\sim}{\Longrightarrow} D\circ\mathscr{D}_d, \quad \xi_d : \Phi_{-d}\circ\mathscr{D}_d \overset{\sim}{\Longrightarrow} D$$
$$\varepsilon_{\omega[d]} : \mathrm{id} \overset{\sim}{\Longrightarrow} \mathscr{D}_d\circ\mathscr{D}_d, \quad \varepsilon_I : \mathrm{id} \overset{\sim}{\Longrightarrow} D\circ D.$$

Here η_d is given by

$$\begin{aligned}
\mathscr{D}_d(-) = \mathbf{R}\mathrm{Hom}_R(-,\omega[d]) &\longrightarrow \mathbf{R}\mathrm{Hom}_R\Big((-)\overset{\mathbf{L}}{\otimes}_R\Phi_{-d}(R), \omega[d]\overset{\mathbf{L}}{\otimes}_R\Phi_{-d}(R)\Big)\\
&= \mathbf{R}\mathrm{Hom}_R(\Phi_{-d}(-), \Phi_{-d}(\omega[d]))\xrightarrow{\mathbf{R}\mathrm{Hom}(\mathrm{id},\mathrm{Tr})}\mathbf{R}\mathrm{Hom}_R(\Phi_{-d}(-), I)\\
&= D\circ\Phi_{-d}(-),
\end{aligned}$$

and ψ_d, ξ_d are defined as in 2.8.1, with Φ (resp., $\mathscr{D}$) replaced by Φ_{-d} (resp., $\mathscr{D}_d$). Fixing the same data as in 2.8.4, we can define the functors Φ_{-d} and $\mathscr{D}_d$ on the level of complexes; this will be used in the following Lemma.

2.8.12. Lemma

(i) *The morphism of complexes* $\xi_d(R) : \Phi_{-d}(\mathscr{D}_d(R)) = \Phi_{-d}(\omega[d]) \to I$ *is equal to* Tr.

(ii) *For every complex $X^\bullet$ of R-modules with cohomology of finite type, the following diagram of morphisms of complexes is commutative:*

$$\begin{array}{ccc}
(X \otimes_R \mathscr{D}_d(X)) \otimes_R \Phi_{-d}(R) & \xrightarrow{\mathrm{ev}_2 \otimes \mathrm{id}} & \mathscr{D}_d(R) \otimes_R \Phi_{-d}(R) = \Phi_{-d}(\mathscr{D}_d(R)) \\
\downarrow \wr & & \downarrow \mathrm{Tr} \\
X \otimes_R (\mathscr{D}_d(X) \otimes_R \Phi_{-d}(R)) & & I \\
\| & & \uparrow \mathrm{ev}_2 \\
X \otimes_R \Phi_{-d}(\mathscr{D}_d(X)) & \xrightarrow{\mathrm{id} \otimes \xi_d(X)} & X \otimes_R D(X)
\end{array}$$

Proof. — This follows from the definitions. □

2.9. Relation to Pontrjagin duality

In this section we assume that the residue field $k = \mathbf{F}_{p^r}$ is a finite field of characteristic p.

2.9.1. Under this assumption, every R-module M of finite type is compact, Hausdorff and totally disconnected in the $\mathfrak{m}$-adic topology. The Pontrjagin dual of M is equal to

$$\begin{aligned}
M^D &= \mathrm{Hom}_{\mathrm{cont}}(M, \mathbf{R}/\mathbf{Z}) = \mathrm{Hom}_{\mathrm{cont}}(M, \mathbf{Q}_p/\mathbf{Z}_p) \\
&= \varinjlim_n \mathrm{Hom}_{\mathbf{Z}_p}(M/\mathfrak{m}^n M, \mathbf{Q}_p/\mathbf{Z}_p) \\
&= \varinjlim_n \mathrm{Hom}_R(M/\mathfrak{m}^n M, \mathrm{Hom}_{\mathbf{Z}_p}(R/\mathfrak{m}^n, \mathbf{Q}_p/\mathbf{Z}_p)) \\
&= \varinjlim_n \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbf{Z}_p}(R/\mathfrak{m}^n, \mathbf{Q}_p/\mathbf{Z}_p)) \\
&= \mathrm{Hom}_R\Big(M, \varinjlim_n \mathrm{Hom}_{\mathbf{Z}_p}(R/\mathfrak{m}^n, \mathbf{Q}_p/\mathbf{Z}_p)\Big) \\
&= \mathrm{Hom}_R(M, R^D)
\end{aligned}$$

It follows from 2.3.1 (i) and Pontrjagin duality that $R^D = I$ is an injective hull of k, hence $D(M) = M^D$ for every R-module M of finite type.

2.9.2. Similarly, if N is an R-module of co-finite type equipped with discrete topology, then $N = \varinjlim_n N[\mathfrak{m}^n]$ and the adjunction isomorphism

$$N^D = \mathrm{Hom}_{\mathbf{Z}_p}(N, \mathbf{Q}_p/\mathbf{Z}_p) \xrightarrow{\sim} \mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbf{Z}_p}(R, \mathbf{Q}_p/\mathbf{Z}_p))$$

factors through the submodule of the R.H.S. equal to

$$\mathrm{Hom}_R\Big(N, \varinjlim_n \mathrm{Hom}_{\mathbf{Z}_p}(R/\mathfrak{m}^n, \mathbf{Q}_p/\mathbf{Z}_p)\Big) = \mathrm{Hom}_R(N, R^D) = D(N)$$

Thus the functor D coincides with the Pontrjagin dual on both $({}_R\mathrm{Mod})_{ft}$ and $({}_R\mathrm{Mod})_{coft}$.

2.10. Non-complete R

In this section we assume that R is local and Noetherian, but not necessarily complete. As above, $d = \dim(R)$.

2.10.1. Denote by $\widehat{R} = \varprojlim_n R/\mathfrak{m}^n$ the $\mathfrak{m}$-adic completion of R, with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$ (similarly, put $\widehat{M} = \varprojlim_n M/\mathfrak{m}^n M$ for every R-module M). Recall that $\widehat{R}$ is faithfully flat over R ([**Mat**, Thm. 8.14(3)]). All statements in Sect. 2.2–2.4 and 2.4.7(ii)–(iii), (v) are true for R.

2.10.2. Proposition

(i) *An injective hull I_R of $k = R/\mathfrak{m}$ $(= \widehat{R}/\widehat{\mathfrak{m}})$ has a canonical structure of an $\widehat{R}$-module. In fact, $I_R = I_{\widehat{R}}$.*

(ii) *For every R-module of finite type M we have*

$$D_R(M) = \mathrm{Hom}_R(M, I) = \mathrm{Hom}_{\widehat{R}}\big(\widehat{M}, I\big) = D_{\widehat{R}}\big(\widehat{M}\big)$$

(where $I = I_R = I_{\widehat{R}}$).

(iii) *For every M as in (ii), the canonical maps*

$$\mathbf{R}\Gamma_{\{\mathfrak{m}\}}(M) \xrightarrow{\sim} \mathbf{R}\Gamma_{\{\mathfrak{m}\}}(M) \otimes_R \widehat{R} \Big(= \mathbf{R}\Gamma_{\{\mathfrak{m}\}}(M) \overset{\mathbf{L}}{\otimes}_R \widehat{R}\Big) \xrightarrow{\sim} \mathbf{R}\Gamma_{\{\widehat{\mathfrak{m}}\}}\big(\widehat{M}\big)$$

are isomorphisms. In particular, each $H^i_{\{\mathfrak{m}\}}(M)$ has a canonical structure of an $\widehat{R}$-module.

(iv) *For every M as in (ii), each $H^i_{\{\mathfrak{m}\}}(M)$ is an Artinian R-module.*

(v) *For every R-module of finite length N the canonical map $\varepsilon : N \to D_R(D_R(N))$ is an isomorphism.*

Proof. — For (i)–(iii), see [**Br-He**, Ex. 3.2.14, Lemma 3.5.4(d)]. For (iv), see [**Br-Sh**, Thm. 7.1.3]. Finally, (v) follows from $N = \widehat{N}$. □

2.10.3. Dualizing complex. — An object ω_R of $D^b_{ft}({}_R\mathrm{Mod})$ is a *dualizing complex* for R if it can be represented by a bounded complex $\omega^\bullet_R$ of injective R-modules and if it satisfies Grothendieck duality 2.6. If it exists, then ω_R is determined (up to isomorphism) up to a shift $\omega_R \mapsto \omega_R[n]$ ([**RD**, §V.3.1]).

If R is a quotient of a Gorenstein local ring then ω_R exists ([**RD**, §V.10]); the converse also holds [**Kaw**].

2.10.4. Local Duality ([RD, Ch. V]). — Assume that ω_R exists. Then

(i) The undetermined shift in the normalization of ω_R is uniquely determined by the condition

$$H^i_{\{\mathfrak{m}\}}(\omega_R) \xrightarrow{\sim} \begin{cases} I, & i = d \\ 0, & i \neq d. \end{cases}$$

Fix an isomorphism $\mathrm{Tr} : H^d_{\{\mathfrak{m}\}}(\omega_R) \xrightarrow{\sim} I$.

(ii) For every R-module M of finite type and $i \in \mathbf{Z}$, the Yoneda pairing

$$H^i_{\{\mathfrak{m}\}}(M) \times \mathbb{E}\mathrm{xt}^{d-i}_R(M, \omega_R) \longrightarrow H^d_{\{\mathfrak{m}\}}(\omega_R) \xrightarrow{\sim} I$$

induces isomorphisms

$$H^i_{\{\mathfrak{m}\}}(M) \xrightarrow{\sim} D_R\big(\mathbb{E}\mathrm{xt}^{d-i}_R(M, \omega_R)\big)$$

$$\mathbb{E}\mathrm{xt}^{\widehat{d-i}}_R\widehat{(M, \omega_R)} \xrightarrow{\sim} D_R\big(H^i_{\{\mathfrak{m}\}}(M)\big) = D_{\widehat{R}}\big(H^i_{\{\mathfrak{m}\}}(M)\big).$$

(iii) $\omega_{\widehat{R}} = \omega_R \otimes_R \widehat{R} = \omega_R \overset{\mathbf{L}}{\otimes}_R \widehat{R}$ is a dualizing complex for $\widehat{R}$.

(iv) R is universally catenary.

(v) For every prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, the localization $(\omega_R)_{\mathfrak{p}}$ is a dualizing complex for $R_{\mathfrak{p}}$.

(vi) If $R' \supset R$ is a local ring, free of finite rank as an R-module, then $\omega_{R'}$ exists and is isomorphic to $\mathrm{Hom}_R(R', \omega_R) = \mathbf{R}\mathrm{Hom}_R(R', \omega_R)$.

(vii) The statements of 2.7(i)–(iii), (vi) hold.

2.10.5. Lemma. — *Let M be an R-module of finite type. Then*

(i) *If* $\mathfrak{m} \notin \mathrm{Ass}(R)$, *then* $H^0_{\{\mathfrak{m}\}}(M) \subseteq M_{\mathrm{tors}}$.

(ii) *If* $\dim(R) = 1$, *then* $M_{\mathrm{tors}} \subseteq H^0_{\{\mathfrak{m}\}}(M)$.

(iii) *If* $\dim(R) = \mathrm{depth}(R) = 1$, *then* $M_{\mathrm{tors}} = H^0_{\{\mathfrak{m}\}}(M)$.

(iv) $H^d_{\{\mathfrak{m}\}}(M)$ *is R-divisible.*

(v) *If ω_R exists, then $H^0(\omega_R)$ is torsion-free over R.*

Proof. — The statements (i)–(ii) follow from the fact that

$$M_{\mathrm{tors}} = \mathrm{Ker}\left(M \longrightarrow \bigoplus_{\mathfrak{q} \in \mathrm{Ass}(R)} M_{\mathfrak{q}}\right)$$

$$H^0_{\{\mathfrak{m}\}}(M) = \mathrm{Ker}\left(M \longrightarrow \prod_{\mathfrak{q} \in \mathrm{Spec}(R) - \{\mathfrak{m}\}} M_{\mathfrak{q}}\right).$$

Indeed, in (i) we have $\mathrm{Ass}(R) \subsetneq \mathrm{Spec}(R) - \{\mathfrak{m}\}$, while in (ii) $\mathrm{Ass}(R)$ contains all minimal prime ideals $\mathfrak{q} \subset R$, *i.e.*, all elements of $\mathrm{Spec}(R) - \{\mathfrak{m}\}$. The statement (iii) is a combination of (i) and (ii). As regards (iv) and (v), Lemma 2.8.8 and Proposition 2.10.2(iii) imply that $H^d_{\{\mathfrak{m}\}}(M)$ is $\widehat{R}$-divisible and $H^0(\omega_R) \otimes_R \widehat{R}$ is torsion-free over $\widehat{R}$; we conclude by the faithful flatness of $\widehat{R}$ over R (if $r \in R$ does not divide zero in R, it is not a zero divisor in $\widehat{R}$). □

2.10.6. Example. — If $\dim(R) = 1$ and R is *not* Cohen-Macaulay, then

$$R_{\mathrm{tors}} = 0, \quad H^0_{\{\mathfrak{m}\}}(R) \neq 0.$$

2.10.7. Lemma. — *Assume that* $\dim(R) = 1$ *and fix* $x \in \mathfrak{m}$ *such that* $\dim(R/xR) = 0$.

(i) *Let* $C^\bullet = [R \xrightarrow{-i} R_x]$ *be the complex* $C^\bullet(R, x)$ *(in degrees* $0, 1$*) from 2.3.3. Then*

$$C^\bullet \otimes_R C^\bullet = \left[R \xrightarrow{(-i,-i)} R_x \oplus R_x \xrightarrow{(-\mathrm{id},\mathrm{id})} R_x\right]$$

and the morphism of complexes $u : C^\bullet \to C^\bullet \otimes_R C^\bullet$ *given by* id_R *(resp.,* $(\mathrm{id}_{R_x}, \mathrm{id}_{R_x})$*) in degree* 0 *(resp.,* 1*) is a quasi-isomorphism satisfying* $s_{12} \circ u = u$. *The morphism of complexes* $v : C^\bullet \otimes_R C^\bullet \to C^\bullet$ *given by* id_R *(resp., by the projection on the first factor) in degree* 0 *(resp.,* 1*) satisfies* $vu = \mathrm{id}$ *and* uv *is homotopic to the identity.*

(ii) *If* x *is not a zero divisor in* R, *then the canonical map* $R_x \to \mathrm{Frac}(R)$ *is an isomorphism.*

Proof. — Easy exercise. □

2.10.8. Corollary. — *Assume that* $\dim(R) = 1$. *Then the morphisms of complexes*

$$\begin{array}{ccc} (X^\bullet \otimes_R C^\bullet) \otimes_R (Y^\bullet \otimes_R C^\bullet) & \xrightarrow{s_{23}} & (X^\bullet \otimes_R Y^\bullet) \otimes_R (C^\bullet \otimes_R C^\bullet) \\ & & \uparrow{\scriptstyle \mathrm{id}\otimes u} \\ & & (X^\bullet \otimes_R Y^\bullet) \otimes_R C^\bullet \end{array}$$

define a functorial cup product

$$\mathbf{R}\Gamma_{\{\mathfrak{m}\}}(X) \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_{\{\mathfrak{m}\}}(Y) \xrightarrow{\cup} \mathbf{R}\Gamma_{\{\mathfrak{m}\}}\left(X \overset{\mathbf{L}}{\otimes}_R Y\right) \qquad (X, Y \in D^-_{ft}({}_R\mathrm{Mod}))$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{R}\Gamma_{\{\mathfrak{m}\}}(X) \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_{\{\mathfrak{m}\}}(Y) & \xrightarrow{\cup} & \mathbf{R}\Gamma_{\{\mathfrak{m}\}}\left(X \overset{\mathbf{L}}{\otimes}_R Y\right) \\ \downarrow{\scriptstyle s_{12}} & & \downarrow{\scriptstyle (s_{12})_*} \\ \mathbf{R}\Gamma_{\{\mathfrak{m}\}}(Y) \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_{\{\mathfrak{m}\}}(X) & \xrightarrow{\cup} & \mathbf{R}\Gamma_{\{\mathfrak{m}\}}\left(Y \overset{\mathbf{L}}{\otimes}_R X\right). \end{array}$$

Proof. — Combine Lemma 1.2.4 and Lemma 2.10.7. □

2.10.9. Assume that $\dim(R) = \mathrm{depth}(R) = 1$. Then the filtration $\sigma_{\geqslant i} C^\bullet$ induces, as in 2.8.7, a filtration on the complex $X^\bullet \otimes_R C^\bullet$ representing $\mathbf{R}\Gamma_{\{\mathfrak{m}\}}(X^\bullet)$ (for every complex $X^\bullet$ of R-modules with cohomology of finite type). The corresponding spectral sequence E_r from Proposition 2.8.7 for $T = X = X^\bullet \in D_{ft}({}_R\mathrm{Mod})$ degenerates ($E_2 = E_\infty$) into short exact sequences

$$0 \longrightarrow H^1_{\{\mathfrak{m}\}}(H^{i-1}(X)) \longrightarrow H^i_{\{\mathfrak{m}\}}(X) \longrightarrow H^0_{\{\mathfrak{m}\}}(H^i(X)) \longrightarrow 0 \qquad (2.10.9.1)$$

All terms in the above exact sequence are R-torsion (by Lemma 2.10.5 (iii)) and the first term is R-divisible (by Lemma 2.10.5 (iv)). This implies that the cup product on cohomology

$$H^i_{\{\mathfrak{m}\}}(X) \otimes_R H^j_{\{\mathfrak{m}\}}(Y) \longrightarrow H^{i+j}_{\{\mathfrak{m}\}}\left(X \overset{\mathbf{L}}{\otimes}_R Y\right) \qquad (X, Y \in D^-_{ft}({}_R\mathrm{Mod}))$$

induced by the cup product $\cup$ from 2.10.8 factors through

$$\cup_{ij} : H^0_{\{\mathfrak{m}\}}(H^i(X)) \otimes_R H^0_{\{\mathfrak{m}\}}(H^j(Y)) \longrightarrow H^{i+j}_{\{\mathfrak{m}\}}\Big(X \overset{\mathbf{L}}{\otimes}_R Y\Big),$$

i.e.,

$$\cup_{ij} : H^i(X)_{\mathrm{tors}} \otimes_R H^j(Y)_{\mathrm{tors}} \longrightarrow H^{i+j}_{\{\mathfrak{m}\}}\Big(X \overset{\mathbf{L}}{\otimes}_R Y\Big), \tag{2.10.9.2}$$

and satisfies

$$(s_{12})_*(x \cup_{ij} y) = (-1)^{ij} y \cup_{ji} x. \tag{2.10.9.3}$$

We can drop the upper-boundedness condition if we deal not with objects of the derived category, but with complexes of R-modules $X^\bullet, Y^\bullet, Z^\bullet$ with cohomology of finite type. We obtain products

$$H^i_{\{\mathfrak{m}\}}(X^\bullet) \otimes_R H^j_{\{\mathfrak{m}\}}(Y^\bullet) \longrightarrow H^{i+j}_{\{\mathfrak{m}\}}(X^\bullet \otimes_R Y^\bullet)$$

factoring through

$$\cup_{ij} : H^i(X^\bullet)_{\mathrm{tors}} \otimes_R H^j(Y^\bullet)_{\mathrm{tors}} \longrightarrow H^{i+j}_{\{\mathfrak{m}\}}(X^\bullet \otimes_R Y^\bullet)$$

and satisfying (2.10.9.3). If

$$u : X^\bullet \otimes_R Y^\bullet \longrightarrow Z^\bullet$$

is a morphism of complexes of R-modules, then the induced products

$$u_* \circ \cup_{ij} : H^i(X^\bullet)_{\mathrm{tors}} \otimes_R H^j(Y^\bullet)_{\mathrm{tors}} \longrightarrow H^{i+j}_{\{\mathfrak{m}\}}(Z^\bullet)$$

depend only on the homotopy class of u.

2.10.10. Assume that $\dim(R) = \mathrm{depth}(R) = 1$ and that ω_R exists. In the hypercohomology spectral sequence

$$E_2^{i,j} = \mathbb{E}\mathrm{xt}^i_R(H^{-j}(X), \omega_R) \Longrightarrow H^{i+j}(\mathscr{D}(X)) \qquad (X \in D_{ft}({}_R\mathrm{Mod}))$$

we have

$$E_2^{i,j} = \mathrm{Ext}^i_R(H^{-j}(X), H^0(\omega_R))$$

and $E_2^{i,j} = 0$ for $i \neq 0, 1$, which gives short exact sequences

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}^1_R(H^{-j+1}(X), H^0(\omega_R)) &\longrightarrow H^j(\mathscr{D}(X)) \\ &\longrightarrow \mathrm{Hom}_R(H^{-j}(X), H^0(\omega_R)) \longrightarrow 0. \end{aligned} \tag{2.10.10.1}$$

Applying D_R to (2.10.10.1) gives, by local duality 2.10.4(ii), the exact sequences (2.10.9.1) for $i = -j + 1$.

2.10.11. Lemma. — *Assume that* $\dim(R) = \mathrm{depth}(R) = 1$ *and that* ω_R *exists. Then*

(i) $J_R := H^0(\omega_R) \otimes_R \mathrm{Frac}(R)/R$ *is isomorphic to* I_R.

(ii) *The exact sequence* (2.10.10.1) *is isomorphic to*

$$0 \longrightarrow \mathrm{Hom}_R(H^{-j+1}(X)_{\mathrm{tors}}, I_R) \longrightarrow H^j(\mathscr{D}(X)) \longrightarrow \mathrm{Hom}_R(H^{-j}(X), H^0(\omega_R)) \longrightarrow 0.$$

(iii) $H^j(\mathscr{D}(X))_{\rm tors}$ *is isomorphic to*

$$\mathrm{Hom}_R(H^{-j+1}(X)_{\rm tors}, I_R) \xrightarrow{\sim} \mathrm{Hom}_R(H^{-j+1}(X)_{\rm tors}, H^0(\omega_R)\otimes_R \mathrm{Frac}(R)/R).$$

Proof. — As R is Cohen-Macaulay, we have $\omega_R \xrightarrow{\sim} H^0(\omega_R)$ in $D({}_R\mathrm{Mod})$. Fix $x\in\mathfrak{m}$ which is not a zero divisor in R. The R-linear map $\alpha : I_R \to J_R$, induced by $R_x \to \mathrm{Frac}(R)$ and

$$I_R \xrightarrow{(\mathrm{Tr})^{-1}} H^1_{\{\mathfrak{m}\}}(\omega_R) = H^1_{\{\mathfrak{m}\}}(H^0(\omega_R)) = \mathrm{Coker}(H^0(\omega_R) \longrightarrow H^0(\omega_R)\otimes_R R_x),$$

is an isomorphism, since $R_x \to \mathrm{Frac}(R)$ is (Lemma 2.10.7 (ii)). This proves the statement (i).

If M is an R-module of finite type, so is $N = \mathrm{Ext}^1_R(M, H^0(\omega_R))$. As $\mathrm{codim}_R(\mathrm{supp}(N)) \geqslant 1$, we have

$$\mathrm{Ext}^1_R(M, H^0(\omega_R)\otimes_R \mathrm{Frac}(R)) = N\otimes_R \mathrm{Frac}(R) = 0,$$

by Lemma 2.10.5 (iii). It follows that

$$[H^0(\omega_R)\otimes_R \mathrm{Frac}(R) \longrightarrow I_R]$$

is an injective resolution of ω_R, which gives an isomorphism

$$\mathrm{Hom}_R(P, I_R) \xrightarrow{\sim} \mathrm{Ext}^1_R(P, H^0(\omega_R))$$

for every torsion R-module P. Taking $P = M_{\rm tors}$, the long exact sequence of Ext's associated to

$$0 \longrightarrow M/P \xrightarrow{x} M/P \longrightarrow M/(P+xM) \longrightarrow 0$$

shows that $\mathrm{Ext}^1_R(M/P, H^0(\omega_R))/x\mathrm{Ext}^1_R(M/P, H^0(\omega_R)) = 0$, hence $\mathrm{Ext}^1_R(M/P, H^0(\omega_R)) = 0$ by Nakayama's Lemma. It follows that

$$\mathrm{Ext}^1_R(M, H^0(\omega_R)) = \mathrm{Ext}^1_R(M_{\rm tors}, H^0(\omega_R)) \xrightarrow{\sim} \mathrm{Hom}_R(M_{\rm tors}, I_R).$$

Taking $M = H^{-j+1}(X)$ concludes the proof of (ii). Finally, (iii) follows from (ii) and the fact that $H^0(\omega_R)$ is torsion-free (Lemma 2.10.5 (v)). □

2.10.12. Proposition. — *Assume that* $\dim(R) = \mathrm{depth}(R) = 1$ *and that* ω_R *exists. Given complexes of* R*-modules* $X^\bullet, Y^\bullet$ *with cohomology of finite type and a morphism of complexes*

$$u : X^\bullet \otimes_R Y^\bullet \longrightarrow \omega_R^\bullet[n]$$

(where $\omega_R^\bullet$ *is a bounded complex of injective* R*-modules representing* ω_R*), denote by*

$$\cup_{i,1-n-i} : H^i(X^\bullet)_{\rm tors}\otimes_R H^{1-n-i}(Y^\bullet)_{\rm tors} \longrightarrow H^{1-n}_{\{\mathfrak{m}\}}(\omega_R^\bullet[n]) = H^1_{\{\mathfrak{m}\}}(\omega_R^\bullet) \xrightarrow{\sim} I_R \xrightarrow{\sim} H^0(\omega_R)\otimes_R \mathrm{Frac}(R)/R$$

the cup products from 2.10.8 and (2.10.9.2) *induced by* u *and* Tr. *Complete the map*

$$\mathrm{adj}(u) : X^\bullet \longrightarrow \mathrm{Hom}^\bullet_R(Y^\bullet, \omega_R^\bullet[n]) = \mathscr{D}_n(Y^\bullet)$$

to an exact triangle in $D_{ft}({}_R\mathrm{Mod})$

$$X^\bullet \xrightarrow{\mathrm{adj}(u)} \mathscr{D}_n(Y^\bullet) \longrightarrow \mathrm{Err} \longrightarrow X^\bullet[1].$$

Then the morphism

$$\mathrm{adj}(\cup_{i,1-n-i}) : H^i(X^\bullet)_{\mathrm{tors}} \longrightarrow \mathrm{Hom}_R(H^{1-n-i}(Y^\bullet)_{\mathrm{tors}}, I_R) = D(H^{1-n-i}(Y^\bullet)_{\mathrm{tors}})$$

in $({}_R\mathrm{Mod})_{\mathrm{fl}}$ *has the following properties:*

(i) $\mathrm{Ker}(\mathrm{adj}(\cup_{i,1-n-i}))$ *is isomorphic to a subquotient of* $H^{i-1}(\mathrm{Err})$.

(ii) *If* $H^{i-1}(\mathrm{Err})$ *is* R*-torsion, then* $\mathrm{Coker}(\mathrm{adj}(\cup_{i,1-n-i}))$ *(resp.,* $\mathrm{Ker}(\mathrm{adj}(\cup_{i,1-n-i}))$*) is isomorphic to a submodule (resp., a quotient) of* $H^i(\mathrm{Err})$ *(resp., of* $H^{i-1}(\mathrm{Err})$*).*

In particular, if $H^{i-1}(\mathrm{Err}) = H^i(\mathrm{Err}) = 0$*, then* $\mathrm{adj}(\cup_{i,1-n-i})$ *is an isomorphism.*

Proof. — This follows from the Snake Lemma applied to the following diagram, in which the first square is commutative up to a sign and the map f is the isomorphism from Lemma 2.10.11 (ii):

$$\begin{array}{ccccccccc}
 & & & & H^{i-1}(\mathrm{Err}) & & & & \\
 & & & & \big\downarrow & & & & \\
0 & \longrightarrow & H^i(X^\bullet)_{\mathrm{tors}} & \longrightarrow & H^i(X^\bullet) & \longrightarrow & H^i(X^\bullet)/H^i(X^\bullet)_{\mathrm{tors}} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \mathrm{adj}(\cup_{i,1-n-i})} & & \Big\downarrow{\scriptstyle \mathrm{adj}(u)_*} & & \Big\downarrow & & \\
 & & D(H^{1-n-i}(Y^\bullet)_{\mathrm{tors}}) & & & & & & \\
 & & \big\downarrow{\scriptstyle f} & & & & & & \\
0 & \longrightarrow & \mathbb{E}\mathrm{xt}^1_R(H^{1-n-i}(Y^\bullet), \omega_R) & \longrightarrow & H^i(\mathscr{D}_n(Y^\bullet)) & \longrightarrow & \mathbb{E}\mathrm{xt}^0_R(H^{-n-i}(Y^\bullet), \omega_R) & \longrightarrow & 0 \\
 & & & & \big\downarrow & & & & \\
 & & & & H^i(\mathrm{Err}) & & & &
\end{array}$$ □

2.10.13. Torsion submodules

2.10.13.1. The R-torsion submodule of an R-module M is defined as

$$M_{R\text{-tors}} = M_{\mathrm{tors}} = \mathrm{Ker}\big(M \longrightarrow M \otimes_R \mathrm{Frac}(R)\big) = \{m \in M \mid (\exists r \nmid 0 \text{ in } R)\ rm = 0\}.$$

Note that

(i) If $\mathrm{depth}(R) = 0$, then $\mathrm{Frac}(R) = R$ and $M_{R\text{-tors}} = 0$ for all M.

(ii) The set of $r \in R$ dividing zero is equal to the union of the associated primes of R; thus

$$M_{\mathrm{tors}} = \mathrm{Ker}\left(M \longrightarrow \bigoplus_{\mathfrak{p} \in \mathrm{Ass}(R)} M_{\mathfrak{p}}\right).$$

(iii) In particular,

$$(R/\mathfrak{m})_{\mathrm{tors}} = 0 \iff \mathfrak{m} \in \mathrm{Ass}(R) \iff \mathrm{depth}(R) = 0.$$

(iv) Let $\mathscr{S} \subset R$ be a multiplicative subset of R. If $r \in R$ does not divide zero in R, the same is true for its image under the canonical morphism $R \to R_{\mathscr{S}}$, which then induces a homomorphism of $R_{\mathscr{S}}$-algebras

$$\mathrm{Frac}(R)_{\mathscr{S}} = \mathrm{Frac}(R) \otimes_R R_{\mathscr{S}} \longrightarrow \mathrm{Frac}(R_{\mathscr{S}}).$$

Similarly, for every R-module M there is a canonical (injective) homomorphism

$$(M_{R\text{-tors}})_{\mathscr{S}} \longrightarrow (M_{\mathscr{S}})_{R_{\mathscr{S}}\text{-tors}}.$$

(v) The maps in (iv) need not be isomorphisms, even if $\mathscr{S} = R - \mathfrak{p}$ for $\mathfrak{p} \in \mathrm{Spec}(R)$. For example, if $\dim(R) = 2$, $\mathrm{depth}(R) = 0$, $\dim(R_{\mathfrak{p}}) = \mathrm{depth}(R_{\mathfrak{p}}) = 1$, then

$$\mathrm{Frac}(R)_{\mathfrak{p}} = R_{\mathfrak{p}} \neq \mathrm{Frac}(R_{\mathfrak{p}}).$$

2.10.13.2. Recall Serre's conditions

(R_n) $R_{\mathfrak{p}}$ *is regular for all* $\mathfrak{p} \in \mathrm{Spec}(R)$ *with* $\mathrm{ht}(\mathfrak{p}) \leqslant n$.

(S_n) $\mathrm{depth}(R_{\mathfrak{p}}) \geqslant \min(\mathrm{ht}(\mathfrak{p}), n)$ *for all* $\mathfrak{p} \in \mathrm{Spec}(R)$.

The following implications hold ([**EGAIV.2**, §5.7–5.8]):

R is Cohen-Macaulay $\Longleftrightarrow$ R satisfies (S_n) for all $n \geqslant 0 \Longrightarrow R$ satisfies (S_1)
$\Longleftrightarrow$ R has no embedded primes $\Longleftarrow$ R satisfies (R_0) and (S_1)
$\Longleftrightarrow$ R is reduced $\Longleftarrow$ R is a domain.

2.10.13.3. Lemma. — *If R has no embedded primes, then*

(i) *There is a canonical isomorphism*

$$\mathrm{Frac}(R) \xrightarrow{\sim} \prod_{\mathrm{ht}(\mathfrak{p})=0} R_{\mathfrak{p}}.$$

(ii) *The canonical map* $\mathrm{Frac}(R)_{\mathfrak{q}} \to \mathrm{Frac}(R_{\mathfrak{q}})$ *is an isomorphism, for each* $\mathfrak{q} \in \mathrm{Spec}(R)$.

(iii) *If Δ is a finite abelian group, then the canonical map* $\mathrm{Frac}(R)[\Delta] \to \mathrm{Frac}(R[\Delta])$ *is an isomorphism.*

Proof

(i) Combine [**EGAI**, §7.1.8–7.1.9] (cf. [**Bou**, §IV.2.5, Prop. 10(iii)] in the case when R is reduced).

(ii) Fix $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{p}) = 0$. If $\mathfrak{p} \subset \mathfrak{q}$, then $(R_{\mathfrak{p}})_{\mathfrak{q}}$ $(= R_{\mathfrak{p}} \otimes_R R_{\mathfrak{q}}) = R_{\mathfrak{p}} = (R_{\mathfrak{q}})_{\mathfrak{p}}$. If $\mathfrak{p} \not\subset \mathfrak{q}$, then there is $x \in \mathfrak{p}$, $x \notin \mathfrak{q}$. As $x \otimes 1 = 1 \otimes x$ is simultaneously nilpotent and invertible in $R_{\mathfrak{p}} \otimes_R R_{\mathfrak{q}}$, we have $(R_{\mathfrak{p}})_{\mathfrak{q}} = 0$. Applying (i) to both R and $R_{\mathfrak{q}}$ we get isomorphisms

$$\mathrm{Frac}(R)_{\mathfrak{q}} \xrightarrow{\sim} \prod_{\mathrm{ht}(\mathfrak{p})=0} (R_{\mathfrak{p}})_{\mathfrak{q}} = \prod_{\substack{\mathrm{ht}(\mathfrak{p})=0 \\ \mathfrak{p} \subset \mathfrak{q}}} R_{\mathfrak{q}} \xrightarrow{\sim} \mathrm{Frac}(R_{\mathfrak{q}}).$$

(iii) The map in question is injective for arbitrary R. In order to prove surjectivity, we must show that, for every $\alpha \in R[\Delta]$ which is not a zero divisor, the cokernel of the injective multiplication map

$$\mathrm{mult}_\alpha : R[\Delta] \longrightarrow R[\Delta], \quad \mathrm{mult}_\alpha(x) = \alpha x$$

satisfies

$$\mathrm{Coker}(\mathrm{mult}_\alpha) \otimes_R \mathrm{Frac}(R) \stackrel{?}{=} 0.$$

This follows from (i) and the fact that

$$\ell_{R_\mathfrak{p}}(\mathrm{Coker}(\mathrm{mult}_\alpha)_\mathfrak{p}) = \ell_{R_\mathfrak{p}}(\mathrm{Ker}(\mathrm{mult}_\alpha)_\mathfrak{p}) = 0$$

for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{p}) = 0$. □

2.10.13.4. Corollary. — *If R has no embedded primes, then the canonical map*

$$(M_{R\text{-tors}})_\mathfrak{q} \longrightarrow (M_\mathfrak{q})_{R_\mathfrak{q}\text{-tors}}$$

is an isomorphism, for each R-module M and $\mathfrak{q} \in \mathrm{Spec}(R)$.

Proof. — Localize the exact sequence

$$0 \longrightarrow M_{R\text{-tors}} \longrightarrow M \longrightarrow M \otimes_R \mathrm{Frac}(R)$$

at $\mathfrak{q}$ and apply Lemma 2.10.13.3 (ii). □

2.10.14. Assume that ω_R exists, but impose no other conditions on R. In this case the statement of Lemma 2.10.7 (i) holds if we replace $C^\bullet$ by the following complex in degrees $0, 1$:

$$\overline{C}^\bullet = \left[R \xrightarrow{-i} \mathrm{Frac}(R)\right]$$

(note that $C^\bullet = \overline{C}^\bullet$ if $\dim(R) = \mathrm{depth}(R) = 1$, by Lemma 2.10.7 (ii)).

Let $X^\bullet$ and $Y^\bullet$ be complexes of R-modules. The exact sequence

$$0 \longrightarrow H^{i-1}(X^\bullet) \otimes_R (\mathrm{Frac}(R)/R) \longrightarrow H^i(X^\bullet \otimes_R \overline{C}^\bullet) \longrightarrow H^i(X^\bullet)_{\mathrm{tors}} \longrightarrow 0,$$

together with the corresponding sequence for $Y^\bullet$ and the construction of Corollary 2.10.8, yield cup products

$$\cup_{ij} : H^i(X^\bullet)_{\mathrm{tors}} \otimes_R H^j(Y^\bullet)_{\mathrm{tors}} \longrightarrow H^{i+j}((X^\bullet \otimes_R Y^\bullet) \otimes_R \overline{C}^\bullet)$$

satisfying

$$(s_{12})_*(x \cup_{ij} y) = (-1)^{ij} y \cup_{ji} x. \tag{2.10.14.1}$$

If $\omega_R^\bullet$ is a bounded complex of injective R-modules representing ω_R and

$$u : X^\bullet \otimes_R Y^\bullet \longrightarrow \omega_R^\bullet[n]$$

a morphism of complexes, then u induces cup products

$$\cup_{i,1-n-i} : H^i(X^\bullet)_{\mathrm{tors}} \otimes_R H^{1-n-i}(Y^\bullet)_{\mathrm{tors}} \longrightarrow H^1(\omega_R^\bullet \otimes_R \overline{C}^\bullet),$$

with the target sitting in an exact sequence

$$0 \longrightarrow H^0(\omega_R) \otimes_R (\mathrm{Frac}(R)/R) \longrightarrow H^1(\omega_R^\bullet \otimes_R \overline{C}^\bullet) \longrightarrow H^1(\omega_R)_{\mathrm{tors}} \longrightarrow 0.$$

If R is Cohen-Macaulay (resp., if $\mathrm{depth}(R_\mathfrak{p}) = 1$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{p}) = 1$), then $H^1(\omega_R)$ is zero (resp., pseudo-null) and we obtain cup products

$$\cup_{i,1-n-i} : H^i(X^\bullet)_{\mathrm{tors}} \otimes_R H^{1-n-i}(Y^\bullet)_{\mathrm{tors}} \longrightarrow H^0(\omega_R) \otimes_R (\mathrm{Frac}(R)/R)$$

in $({}_R\mathrm{Mod})$ (resp., in $({}_R\mathrm{Mod})/(\text{pseudo-null})$).

If R has no embedded primes, then it follows from Lemma 2.10.13.3 (ii) and Corollary 2.10.13.4 that the localization of $\cup_{i,1-n-i}$ at each $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{p}) = 1$ coincides with the cup product from Proposition 2.10.12, applied to $R_\mathfrak{p}$.

2.10.15. Assume that ω_R exists and $\mathrm{depth}(R_\mathfrak{p}) = 1$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{p}) = 1$. In the category $({}_R\mathrm{Mod})/(\text{pseudo-null})$, the hyper-cohomology spectral sequence

$$E_2^{i,j} = \mathbb{E}\mathrm{xt}^i_R(H^{-j}(X), \omega_R) \Longrightarrow H^{i+j}(\mathscr{D}(X)) \qquad (X \in D_{ft}({}_R\mathrm{Mod}))$$

satisfies

$$E_2^{i,j} = \mathrm{Ext}^i_R(H^{-j}(X), H^0(\omega_R))$$

(by 2.7 (vi)) and $E_2^{i,j} = 0$ for $i \neq 0, 1$. This yields an analogue of (2.10.10.1)

$$0 \longrightarrow \mathrm{Ext}^1_R(H^{-j+1}(X), H^0(\omega_R)) \longrightarrow H^j(\mathscr{D}(X)) \longrightarrow \mathrm{Hom}_R(H^{-j}(X), H^0(\omega_R)) \longrightarrow 0$$

and an isomorphism

$$\mathrm{Ext}^1_R(H^{-j+1}(X), H^0(\omega_R)) \xrightarrow{\sim} H^j(\mathscr{D}(X))_{R\text{-tors}}$$

in $({}_R\mathrm{Mod})/(\text{pseudo-null})$ (using Lemma 2.10.5 (v)).

2.10.16. Proposition. — *Assume that ω_R exists and R has no embedded primes. Then, for each $X \in D_{ft}({}_R\mathrm{Mod})$ and $j \in \mathbf{Z}$, there is an isomorphism in $({}_R\mathrm{Mod})/(\textit{pseudo-null})$*

$$\mathrm{Hom}_R(H^{-j+1}(X)_{\mathrm{tors}}, H^0(\omega_R) \otimes_R (\mathrm{Frac}(R)/R)) \xrightarrow{\sim} \mathrm{Ext}^1_R(H^{-j+1}(X), H^0(\omega_R)) \xrightarrow{\sim} H^j(\mathscr{D}(X))_{\mathrm{tors}},$$

the localization of which at each $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{p}) = 1$ coincides with the isomorphism from Lemma 2.10.11 (iii), applied to $R_\mathfrak{p}$.

Proof. — The construction in 2.10.15 yields an isomorphism in $({}_R\mathrm{Mod})/(\text{pseudo-null})$

$$H^j(\mathscr{D}(X))_{\mathrm{tors}} \xrightarrow{\sim} \mathrm{Ext}^1_R(M, H^0(\omega_R)),$$

where $M = H^{-j+1}(X)$. For each $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{p}) = 1$, the canonical map

$$\mathrm{Ext}^1_R(M, H^0(\omega_R))_\mathfrak{p} = \mathrm{Ext}^1_{R_\mathfrak{p}}(M_\mathfrak{p}, H^0(\omega_R)_\mathfrak{p}) \longrightarrow \mathrm{Ext}^1_{R_\mathfrak{p}}((M_\mathfrak{p})_{R_\mathfrak{p}\text{-tors}}, H^0(\omega_R)_\mathfrak{p}) = \mathrm{Ext}^1_R(M_{R\text{-tors}}, H^0(\omega_R))_\mathfrak{p}$$

is an isomorphism (using Corollary 2.10.13.4 and the proof of Lemma 2.10.11 (ii)). This yields a canonical isomorphism in $({}_R\mathrm{Mod})/(\text{pseudo-null})$

$$\mathrm{Ext}^1_R(M, H^0(\omega_R)) \xrightarrow{\sim} \mathrm{Ext}^1_R(M_{R\text{-tors}}, H^0(\omega_R)).$$

The boundary map associated to the exact sequence

$$0 \longrightarrow H^0(\omega_R) \longrightarrow H^0(\omega_R) \otimes_R \mathrm{Frac}(R) \longrightarrow H^0(\omega_R) \otimes_R \mathrm{Frac}(R)/R \longrightarrow 0$$

gives an isomorphism in $({}_R\mathrm{Mod})$

$$\delta : \mathrm{Hom}_R(M_{R\text{-tors}}, H^0(\omega_R) \otimes_R \mathrm{Frac}(R)/R) \xrightarrow{\sim} \mathrm{Ext}^1_R(M_{R\text{-tors}}, H^0(\omega_R)).$$

Then the composite isomorphism in $({}_R\mathrm{Mod})/(\text{pseudo-null})$

$$\begin{aligned} H^j(\mathscr{D}(X))_{\mathrm{tors}} \xrightarrow{\sim} \mathrm{Ext}^1_R(M, H^0(\omega_R)) \xrightarrow{\sim} \mathrm{Ext}^1_R(M_{R\text{-tors}}, H^0(\omega_R)) \\ \xrightarrow{-\delta^{-1}} \mathrm{Hom}_R(M_{R\text{-tors}}, H^0(\omega_R) \otimes_R \mathrm{Frac}(R)/R) \end{aligned}$$

has the required properties under localization at each $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{p}) = 1$ (the minus sign comes from the fact that the map in 2.10.11 was defined using an injective resolution, hence differs from that defined in terms of δ by a sign). □

2.10.17. Proposition. — *Assume that R is Cohen-Macaulay (resp., R has no embedded primes) and that ω_R exists. Given complexes of R-modules $X^\bullet, Y^\bullet$ with cohomology of finite type and a morphism of complexes*

$$u : X^\bullet \otimes_R Y^\bullet \longrightarrow \omega_R^\bullet[n]$$

as in 2.10.14, let

$$\cup_{i,1-n-i} : H^i(X^\bullet)_{\mathrm{tors}} \otimes_R H^{1-n-i}(Y^\bullet)_{\mathrm{tors}} \longrightarrow H^0(\omega_R) \otimes_R (\mathrm{Frac}(R)/R)$$

be the cup products in $({}_R\mathrm{Mod})$ (resp., in $({}_R\mathrm{Mod})/(\text{pseudo-null})$) defined in 2.10.14. Complete the map

$$\mathrm{adj}(u) : X^\bullet \longrightarrow \mathrm{Hom}^\bullet_R(Y^\bullet, \omega^\bullet_R[n]) = \mathscr{D}_n(Y^\bullet)$$

to an exact triangle in $D_{ft}(({}_R\mathrm{Mod})/(\text{pseudo-null}))$

$$X^\bullet \xrightarrow{\mathrm{adj}(u)} \mathscr{D}_n(Y^\bullet) \longrightarrow \mathrm{Err} \longrightarrow X^\bullet[1].$$

Then the morphism

$$\mathrm{adj}(\cup_{i,1-n-i}) : H^i(X^\bullet)_{\mathrm{tors}} \longrightarrow \mathrm{Hom}_R(H^{1-n-i}(Y^\bullet)_{\mathrm{tors}}, H^0(\omega_R) \otimes_R (\mathrm{Frac}(R)/R))$$

in $({}_R\mathrm{Mod})/(\text{pseudo-null})$ has the following properties:

(i) $\mathrm{Ker}(\mathrm{adj}(\cup_{i,1-n-i}))$ *is isomorphic to a subquotient of $H^{i-1}(\mathrm{Err})$.*

(ii) *If $H^{i-1}(\mathrm{Err})$ is R-torsion, then* $\mathrm{Coker}(\mathrm{adj}(\cup_{i,1-n-i}))$ *(resp.,* $\mathrm{Ker}(\mathrm{adj}(\cup_{i,1-n-i}))$*) is isomorphic to a subobject (resp., a quotient) of $H^i(\mathrm{Err})$ (resp., of $H^{i-1}(\mathrm{Err})$).*

In particular, if $H^{i-1}(\mathrm{Err}) = H^i(\mathrm{Err}) = 0$ in $({}_R\mathrm{Mod})/(\text{pseudo-null})$, then $\mathrm{adj}(\cup_{i,1-n-i})$ *is an isomorphism in $({}_R\mathrm{Mod})/(\text{pseudo-null})$.*

Proof. — The proof of Proposition 2.10.12 applies, with Proposition 2.10.16 replacing Lemma 2.10.11. □

2.10.18. Proposition. — *Let X and Y be R-modules of finite type with support of codimension $\geqslant 1$ in* $\mathrm{Spec}(R)$. *Assume that, for each prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ with* $\mathrm{ht}(\mathfrak{p}) = 1$, *there exists a monomorphism (resp., epimorphism, resp., isomorphism) of $R_{\mathfrak{p}}$-modules $g_{\mathfrak{p}} : X_{\mathfrak{p}} \to Y_{\mathfrak{p}}$. Then there exists a monomorphism (resp., epimorphism, resp., isomorphism) $g : X \to Y$ in $({}_R\mathrm{Mod})/(\text{pseudo-null})$.*

Proof. — For each prime ideal $\mathfrak{p}$ in the finite set

$$A(X) = \{\mathfrak{p} \in \mathrm{supp}(X) \mid \mathrm{ht}(\mathfrak{p}) = 1\},$$

denote by $f_{\mathfrak{p}} : X \to X_{\mathfrak{p}}$ the canonical map. The kernel of

$$f = (f_{\mathfrak{p}}) : X \longrightarrow \bigoplus_{\mathfrak{p} \in A(X)} X_{\mathfrak{p}}$$

is pseudo-null, and so is the cokernel of the canonical map

$$\mathrm{Im}(f) \longrightarrow \bigoplus_{\mathfrak{p} \in A(X)} \mathrm{Im}(f_{\mathfrak{p}}).$$

It is enough, therefore, to consider only the case when $A(X) = A(Y) = \{\mathfrak{p}\}$ and

$$X \subset X_{\mathfrak{p}} \xrightarrow{g_{\mathfrak{p}}} Y_{\mathfrak{p}} \supset Y.$$

There exists $r \in R - \mathfrak{p}$ such that $r \cdot g_{\mathfrak{p}}(X) \subset Y$; restricting the map $r \cdot g_{\mathfrak{p}}$ to X defines the desired morphism of R-modules $g : X \to Y$. □

2.10.19. Corollary. — *If R satisfies (R_1) and X is an R-module of finite type with* $\mathrm{codim}_R(\mathrm{supp}(X)) \geqslant 1$, *then X is isomorphic in $({}_R\mathrm{Mod})/(\text{pseudo-null})$ to*

$$X \xrightarrow{\sim} \bigoplus_{\mathrm{ht}(\mathfrak{p})=1} \bigoplus_{i \geqslant 1} (R/\mathfrak{p}^i)^{n(\mathfrak{p},i)},$$

where $n(\mathfrak{p}, i) \geqslant 0$ and only finitely many $n(\mathfrak{p}, i)$ are non-zero.

Proof. — For each prime ideal $\mathfrak{p} \in A(X)$ (using the same notation as in the proof of Proposition 2.10.18), the localization $R_{\mathfrak{p}}$ is a discrete valuation ring, hence

$$X_{\mathfrak{p}} \xrightarrow{\sim} \bigoplus_{i \geqslant 1} (R_{\mathfrak{p}}/\mathfrak{p}^i R_{\mathfrak{p}})^{n(\mathfrak{p},i)}$$

(where the sum is finite). The claim follows from Proposition 2.10.18 applied to X and

$$Y = \bigoplus_{\mathfrak{p} \in A(X)} \bigoplus_{i \geqslant 1} (R/\mathfrak{p}^i)^{n(\mathfrak{p},i)}. \quad \square$$

2.10.20. Lemma. — *Assume that* $\dim(R) = \operatorname{depth}(R) = 1$ *and that* ω_R *exists. Let* M, N *be* R*-modules of finite type and*

$$h : M \otimes_R N \longrightarrow H^0(\omega_R)$$

a bilinear form. Denote the corresponding adjoint maps by

$$\alpha := \operatorname{adj}(h) : M \longrightarrow \operatorname{Hom}_R(N, H^0(\omega_R))$$
$$\beta := \operatorname{adj}(h \circ s_{12}) : N \longrightarrow \operatorname{Hom}_R(M, H^0(\omega_R)).$$

If, for each minimal prime ideal $\mathfrak{q} \subset R$*, the localization* $\alpha_{\mathfrak{q}}$ *is an isomorphism, then:*

(i) *For each minimal prime ideal* $\mathfrak{q} \subset R$*,* $\beta_{\mathfrak{q}}$ *is an isomorphism.*
(ii) $\operatorname{Ker}(\alpha) = M_{\text{tors}}$, $\operatorname{Ker}(\beta) = N_{\text{tors}}$.
(iii) $D_R(\operatorname{Coker}(\alpha)) \xrightarrow{\sim} \operatorname{Coker}(\beta)$.
(iv) $\ell_R(\operatorname{Coker}(\alpha)) = \ell_R(\operatorname{Coker}(\beta))$.

Proof

(i) We have $H^0(\omega_R)_{\mathfrak{q}} = I_{R_{\mathfrak{q}}}$, hence $h_{\mathfrak{q}} : M_{\mathfrak{q}} \otimes_{R_{\mathfrak{q}}} N_{\mathfrak{q}} \to I_{R_{\mathfrak{q}}}$, $\alpha_{\mathfrak{q}} : M_{\mathfrak{q}} \to D_{R_{\mathfrak{q}}}(N_{\mathfrak{q}})$; thus

$$\beta_{\mathfrak{q}} : N_{\mathfrak{q}} \xrightarrow{\varepsilon} D_{R_{\mathfrak{q}}}(D_{R_{\mathfrak{q}}}(N_{\mathfrak{q}})) \xrightarrow{D_{R_{\mathfrak{q}}}(\alpha_{\mathfrak{q}})} D_{R_{\mathfrak{q}}}(M_{\mathfrak{q}})$$

is an isomorphism by Proposition 2.10.2 (v). As regards (ii), we have $M_{\text{tors}} \subseteq \operatorname{Ker}(\alpha)$, since $H^0(\omega_R)$ is torsion-free. On the other hand,

$$\operatorname{Ker}(\alpha) \subseteq \operatorname{Ker}\left(M \longrightarrow \bigoplus_{\mathfrak{q}} M_{\mathfrak{q}}\right) = M_{\text{tors}},$$

since all $\alpha_{\mathfrak{q}}$ are isomorphisms.

(iii) According to (ii), the map α factors through

$$M/M_{\text{tors}} \longrightarrow \operatorname{Hom}_R(N, H^0(\omega_R))$$

(and similarly for β). Fixing an injective resolution

$$i : H^0(\omega_R) \longrightarrow \omega_R^{\bullet},$$

for each $Z = M, N$ the canonical map

$$\operatorname{Hom}_R(Z/Z_{\text{tors}}, H^0(\omega_R)) \longrightarrow \operatorname{Hom}_R(Z, H^0(\omega_R))$$

resp.,

$$\operatorname{Hom}_R(Z/Z_{\text{tors}}, H^0(\omega_R)) \longrightarrow \operatorname{Hom}^{\bullet}_R(Z/Z_{\text{tors}}, \omega_R^{\bullet}) = \mathscr{D}(Z/Z_{\text{tors}})$$

is an isomorphism (resp., a quasi-isomorphism, by Lemma 2.10.11). This implies that we have an exact triangle in $D_{ft}({}_R\mathrm{Mod})$

$$M/M_{\text{tors}} \xrightarrow{\alpha'} \mathscr{D}(N/N_{\text{tors}}) \longrightarrow \operatorname{Coker}(\alpha)$$

Applying $\mathscr{D}$, we obtain another exact triangle

$$\mathscr{D}(\operatorname{Coker}(\alpha)) \longrightarrow \mathscr{D}(\mathscr{D}(N/N_{\text{tors}})) \xrightarrow{\mathscr{D}(\alpha')} \mathscr{D}(M/M_{\text{tors}}).$$

As $\mathscr{D}(\alpha') \circ \varepsilon = \beta'$ and $\varepsilon : N/N_{\mathrm{tors}} \to \mathscr{D}(\mathscr{D}(N/N_{\mathrm{tors}}))$ is an isomorphism, we obtain (again by Lemma 2.10.11) isomorphisms in $D_{ft}({}_R\mathrm{Mod})$

$$\mathrm{Coker}(\beta) \xrightarrow{\sim} \mathscr{D}(\mathrm{Coker}(\alpha))[1] \xrightarrow{\sim} D(\mathrm{Coker}(\alpha)),$$

hence an isomorphism $\mathrm{Coker}(\beta) \xrightarrow{\sim} D(\mathrm{Coker}(\alpha))$ in $({}_R\mathrm{Mod})$. Finally, (iv) follows from (iii). □

2.10.21. In the situation of Lemma 2.10.20, we use the following notation:

$$\ell_R(\det(h)) := \ell_R(\mathrm{Coker}(\alpha)) = \ell_R(\mathrm{Coker}(\beta)).$$

2.11. Semi-local R

2.11.1. Everything in Sect. 2.2–2.9 has a straightforward generalization to the case when R is an equi-dimensional *semi-local* Noetherian ring, complete with respect to its radical $\mathfrak{m}$. In this case R has finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ and is isomorphic to

$$R \xrightarrow{\sim} R_{\mathfrak{m}_1} \times \cdots \times R_{\mathfrak{m}_r}.$$

Similarly, every R-module M decomposes canonically as

$$M \xrightarrow{\sim} M_{\mathfrak{m}_1} \oplus \cdots \oplus M_{\mathfrak{m}_r},$$

and the theory in 2.2–2.9 applies separately to each $R_{\mathfrak{m}_i}$-module $M_{\mathfrak{m}_i}$.

2.11.2. If R is of the form considered in 2.11.1, so is every finite R-algebra R' (*e.g.* $R' = R[\Delta]$ for a finite abelian group Δ).

CHAPTER 3

CONTINUOUS COHOMOLOGY

In this chapter we develop a formalism of continuous cohomology for a certain class of $R[G]$-modules. Our approach is purely algebraic; the fundamental objects are the "admissible" $R[G]$-modules, even though the cohomology can be defined even for "ind-admissible" modules (filtered inductive limits of admissible modules). Section 3.6 can be ignored; it is unrelated to the rest of the article.

3.1. Properties of R-modules of finite type

We shall repeatedly use the following standard facts about R-modules of finite type ([**Bou**, Ch. III]). Let $f : M \to N$ be an R-linear map between R-modules of finite type; equip both M and N with the $\mathfrak{m}$-adic topology. Then

3.1.1. M and N are Hausdorff and linearly compact.

3.1.2. f is continuous.

3.1.3. $\mathrm{Im}(f)$ is closed in N.

3.1.4. f is strict, *i.e.*, the quotient topology on $\mathrm{Im}(f) = M/\mathrm{Ker}(f)$ coincides with the topology induced from N.

3.1.5. If f is surjective, then it admits a continuous (not necessarily R-linear) section.

3.2. Admissible $R[G]$-modules

Let G be a group acting R-linearly on an R-module M. The action can be described either as a map

$$\lambda_M : G \times M \longrightarrow M, \quad \lambda_M(g, m) = g(m),$$

or by the induced map

$$\rho_M : R[G] \longrightarrow \mathrm{End}_R(M), \quad \rho_M\Big(\sum r_i g_i\Big)(m) = \sum r_i g_i(m).$$

Throughout Chapter 3, G will be a (Hausdorff) topological group.

3.2.1. Definition. — An $R[G]$-module M is **admissible** iff

(i) The image of ρ_M is an R-module of finite type; and

(ii) The map $G \xrightarrow{i} R[G] \xrightarrow{\rho_M} \mathrm{Im}(\rho_M)$ is continuous (if $\mathrm{Im}(\rho_M)$ is equipped with $\mathfrak{m}$-adic topology).

A morphism between admissible $R[G]$-modules M, N is an $R[G]$-linear map $f : M \to N$. Admissible $R[G]$-modules form a full subcategory $\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$ of $\big({}_{R[G]}\mathrm{Mod}\big)$.

3.2.2. Lemma. — *Let M be an admissible $R[G]$-module. Then*

(i) *If $N \subset M$ is an R-submodule of finite type, then $R[G] \cdot N$ is of finite type over R.*

(ii) *M is the union of its $R[G]$-submodules that are of finite type over R.*

(iii) *If $N \subset M$ is an $R[G]$-submodule, then both N and M/N are admissible.*

(iv) *If $f : H \to G$ is a (continuous) homomorphism of topological groups, then f^*M (= M viewed as an H-module) is an admissible $R[H]$-module.*

(v) *If $H \lhd G$ is a closed normal subgroup of G, then M^H is an admissible G/H-module.*

Proof

(i) $R[G] \cdot N$ is contained in the image of

$$\mathrm{Im}(\rho_M) \otimes_R N \longrightarrow \mathrm{End}_R(M) \otimes_R M \xrightarrow{\mathrm{ev}} M.$$

(ii) This follows from (i).

(iii) The commutative diagrams

$$\begin{array}{ccc} R[G] & \xrightarrow{\rho_M} & \mathrm{End}_R(M) \\ \downarrow{\scriptstyle \rho_N} & & \downarrow{\scriptstyle \alpha'} \\ \mathrm{End}_R(N) & \xhookrightarrow{\alpha} & \mathrm{Hom}_R(N, M) \end{array} \qquad \begin{array}{ccc} R[G] & \xrightarrow{\rho_M} & \mathrm{End}_R(M) \\ \downarrow{\scriptstyle \rho_{M/N}} & & \downarrow{\scriptstyle \beta'} \\ \mathrm{End}_R(M/N) & \xhookrightarrow{\beta} & \mathrm{Hom}_R(M, M/N) \end{array}$$

show that both $\mathrm{Im}(\rho_N)$ and $\mathrm{Im}(\rho_{M/N})$ are of finite type over R. The map

$$\mathrm{Im}(\rho_M) \hookrightarrow \alpha'(\mathrm{Im}(\rho_M)) = \alpha(\mathrm{Im}(\rho_N))$$

induced by α' is $\mathfrak{m}$-adically continuous by 3.1.2, hence $G \to \mathrm{Im}(\rho_N)$ is continuous by 3.1.4. The same argument works for $G \to \mathrm{Im}(\rho_{M/N})$.

(iv) This follows from definitions.

(v) The commutative diagram

$$\begin{array}{ccccc} R[G] & \xrightarrow{\text{can}} & R[G/H] & \xrightarrow{\rho_{M^H}} & \mathrm{End}_R(M^H) \\ \downarrow{\scriptstyle \rho_M} & & & & \downarrow{\scriptstyle \beta} \\ \mathrm{End}_R(M) & & \overset{\alpha}{\hookrightarrow} & & \mathrm{Hom}_R(M^H, M) \end{array}$$

shows that $\mathrm{Im}(\rho_{M^H}) = \beta^{-1}\big(\alpha(\mathrm{Im}(\rho_M))\big)$ is of finite type over R. The composite map

$$G \xrightarrow{\text{can}} G/H \xrightarrow{\rho_{M^H} i} \mathrm{Im}(\rho_{M^H}) \xrightarrow{\beta'} \alpha(\mathrm{Im}(\rho_M))$$

is continuous and the maps can, β' (the latter is induced by β) are strict, hence $G/H \to \mathrm{Im}(\rho_{M^H})$ is continuous. □

3.2.3. Corollary. — *$\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$ is an abelian category (satisfying (AB1), (AB2)). Its embedding into $\big({}_{R[G]}\mathrm{Mod}\big)$ preserves finite limits and finite colimits.*

3.2.4. Lemma. — *Let T (resp., A) be an $R[G]$-module of finite (resp., co-finite) type over R. Equip T (resp., A) with $\mathfrak{m}$-adic (resp., discrete) topology. Then T (resp., A) is an admissible $R[G]$-module iff the map $\lambda_T : G \times T \to T$ (resp., $\lambda_A : G \times A \to A$) is continuous.*

Proof. — Let $M = T$ or A. If $\rho_M i : G \to \mathrm{Im}(\rho_M)$ is continuous, so is

$$\lambda_M : G \times M \xrightarrow{\rho_M i \times \mathrm{id}_M} \mathrm{Im}(\rho_M) \times M \xrightarrow{\text{ev}} M.$$

Conversely, assume that λ_T is continuous. By 3.1.4 it is enough to check that $\rho_T i : G \to \mathrm{End}_R(T)$ is continuous. By a version of the Artin-Rees Lemma ([**Bou**, §III.3 Prop. 2]) there is n_0 such that

$$\mathrm{Ker}\big(\mathrm{End}_R(T) \longrightarrow \mathrm{Hom}_R(T, T/\mathfrak{m}^{n+n_0}T)\big) \subset \mathfrak{m}^n \mathrm{End}_R(T) \quad (\forall n \geqslant 0).$$

By continuity of λ_T, for each $n \geqslant 0$ there is a neighbourhood of unity $U_n \subset G$ such that $\rho_M(U_n)$ acts trivially on $T/\mathfrak{m}^{n+n_0}T$; thus $\rho_T(U_n) \subset 1 + \mathfrak{m}^n\mathrm{End}_R(T)$ and $\rho_T i$ is continuous.

If λ_A is continuous, then there is, for each $n \geqslant 0$, a neighbourhood of unity $V_n \subset G$ stabilizing pointwise $A[\mathfrak{m}^n]$. Put $T = D(A)$ with G-action given by $(g(t))(a) = t(g^{-1}(a))$. Then, for $g \in V_n$, $(g-1)T \subseteq A[\mathfrak{m}^n]^\perp = \mathfrak{m}^n T$. This means that λ_T is continuous, hence T is admissible. Admissibility of A follows from the next Proposition. □

3.2.5. Proposition. — *If M, N are admissible $R[G]$-modules, then both $P = M \otimes_R N$ and $Q = \mathrm{Hom}_R(M, N)$ are admissible.*

Proof. — The modules

$$\mathrm{Im}(\rho_P) \subset \mathrm{Im}\big(\mathrm{Im}(\rho_M) \otimes_R \mathrm{Im}(\rho_N) \longrightarrow \mathrm{End}_R(M) \otimes_R \mathrm{End}_R(N) \xrightarrow{\alpha} \mathrm{End}_R(P)\big)$$
$$\mathrm{Im}(\rho_Q) \subset \mathrm{Im}\big(\mathrm{Im}(\rho_M) \otimes_R \mathrm{Im}(\rho_N) \longrightarrow \mathrm{End}_R(M) \otimes_R \mathrm{End}_R(N) \xrightarrow{\beta} \mathrm{End}_R(Q)\big)$$

are both of finite type over R (the maps α, β are given by $(\alpha(f \otimes g))(x \otimes y) = f(x) \otimes g(y)$; $(\beta(f \otimes g))(q) = g \circ q \circ f$). The diagonal action

$$G \xrightarrow{(\rho_M i, \rho_N i)} \mathrm{Im}(\rho_M) \times \mathrm{Im}(\rho_N) \longrightarrow \mathrm{Im}(\rho_M) \otimes_R \mathrm{Im}(\rho_N)$$

is continuous, hence both $\rho_P i$ and $\rho_Q i$ are continuous by 3.1.4. □

3.2.6. Proposition. — *Let $M^\bullet$ be a bounded above complex of admissible $R[G]$-modules with all cohomology groups $H^i(M^\bullet)$ of finite type over R. Then there is a subcomplex $N^\bullet \hookrightarrow M^\bullet$ (of admissible $R[G]$-modules) such that*

(i) *Each N^i is of finite type over R.*
(ii) *The inclusion $N^\bullet \hookrightarrow M^\bullet$ is a quasi-isomorphism.*

Proof. — If $M^i = 0$ for all i then take $N^\bullet = M^\bullet$. If $M^j \neq 0$ but $M^i = 0$ for $i > j$, choose an R-submodule of finite type $X^j \subset M^j$ that surjects onto $H^j(M^\bullet)$. Then $N^j = R[G] \cdot X^j \subset M^j$ is of finite type over R by Lemma 3.2.2(i). Put $Y^{j-1} = d^{-1}(N^j) \supset Z^{j-1} = \mathrm{Ker}(d : M^{j-1} \to M^j)$ and choose an R-submodule of finite type $X^{j-1} \subset Y^{j-1}$ such that $dX^{j-1} = dY^{j-1}$ and that $X^{j-1} \cap Z^{j-1}$ surjects onto $H^{j-1}(M^\bullet)$. Again $N^{j-1} = R[G] \cdot X^{j-1} \subset Z^{j-1}$ is of finite type over R. We put $Y^{j-2} = d^{-1}(N^{j-1})$ and continue this process. □

3.2.7. Corollary. — *Let $A^\bullet$ be a bounded below (resp., bounded) complex of admissible $R[G]$-modules with all cohomology groups $H^i(A^\bullet)$ of co-finite type over R. Then there is a bounded below (resp., bounded) complex $B^\bullet$ of admissible $R[G]$-modules of co-finite type over R and a map of complexes $A^\bullet \to B^\bullet$ which is a quasi-isomorphism.*

Proof. — Applying Proposition 3.2.6 to $M^\bullet = D(A^\bullet)$ we get a subcomplex incl : $N^\bullet \hookrightarrow D(A^\bullet)$. For $B^\bullet := D(N^\bullet)$ the canonical map

$$A^\bullet \xrightarrow{\mathrm{can}} D(D(A^\bullet)) \xrightarrow{D(\mathrm{incl})} D(N^\bullet) = B^\bullet$$

is a composition of two quasi-isomorphisms (using 2.3.2 and 3.2.6). If $A^\bullet$ is bounded, so is $B^\bullet$. □

3.2.8. Proposition. — *Denote by $\left({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\right)_{R\text{-}ft}$ (resp., $\left({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\right)_{R\text{-}coft}$) the category of admissible $R[G]$-modules of finite (resp., co-finite type) over R. Then the embeddings*

$$\left({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\right)_{R\text{-}ft} \longrightarrow \left({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\right) \longleftarrow \left({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\right)_{R\text{-}coft}$$

induce equivalences of categories

$$D^*\left(\left({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\right)_{R\text{-}ft}\right) \xrightarrow{\approx} D^*_{R\text{-}ft}\left({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\right) \qquad (* = -, b)$$

$$D^*\left(\left({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\right)_{R\text{-}coft}\right) \xrightarrow{\approx} D^*_{R\text{-}coft}\left({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\right) \qquad (* = +, b).$$

Proof. — Essential surjectivity follows from Proposition 3.2.6 and Corollary 3.2.7. Full-faithfulness is a general nonsense ([**Ve2**, §III.2.4.1]). □

3.3. Ind-admissible $R[G]$-modules

3.3.1. Definition. — Let M be an $R[G]$-module. Denote by $\mathcal{S}(M)$ the set of $R[G]$-submodules $M_\alpha \subset M$ satisfying

(a) M_α is of finite type over R;

(b) The action $\lambda_{M_\alpha} : G \times M_\alpha \to M_\alpha$ is continuous (with respect to the $\mathfrak{m}$-adic topology on M_α).

3.3.2. Lemma

(i) *If $M_\alpha \in \mathcal{S}(M)$, then $N \in \mathcal{S}(M)$ for every $R[G]$-submodule $N \subset M_\alpha$.*

(ii) *If $f : M \to N$ is a homomorphism of $R[G]$-modules and $M_\alpha \in \mathcal{S}(M)$, then $f(M_\alpha) \in \mathcal{S}(N)$.*

(iii) *If $M_\alpha, M_\beta \in \mathcal{S}(M)$, then $M_\alpha + M_\beta \in \mathcal{S}(M)$.*

Proof. — All one needs to do is to check the condition (b) of the definition. In (i) (resp. (ii)) the continuity of λ_N (resp., $\lambda_{f(M_\alpha)}$) follows from the continuity of λ_{M_α} and the fact that the inclusion $N \hookrightarrow M_\alpha$ (resp., the surjection $f : M_\alpha \to f(M_\alpha)$) is a strict map. The statement (iii) follows from (ii), as $M_\alpha + M_\beta$ is the image of $M_\alpha \oplus M_\beta \in \mathcal{S}(M \oplus M)$ under the sum map $\Sigma : M \oplus M \to M$. $\square$

3.3.3. Corollary. — *Let $f : M \to N$ be a homomorphism of $R[G]$-modules. Then*

$$j(M) := \bigcup_{M_\alpha \in \mathcal{S}(M)} M_\alpha$$

is an $R[G]$-submodule of M, $j(j(M)) = j(M)$ and $f(j(M)) \subseteq j(N)$.

3.3.4. Definition. — An $R[G]$-module M is **ind-admissible** if $M = j(M)$.

3.3.5. Proposition

(i) *Ind-admissible $R[G]$-modules form a full (abelian) subcategory $\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$ of $\left({}_{R[G]}\mathrm{Mod}\right)$, which is stable under subobjects, quotients, colimits and tensor products.*

(ii) *The embedding functor $i : \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right) \to \left({}_{R[G]}\mathrm{Mod}\right)$ is exact and is left adjoint to $j : \left({}_{R[G]}\mathrm{Mod}\right) \to \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$.*

(iii) *The functor j is left exact and preserves injectives; the category $\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$ has enough injectives.*

(iv) *Every admissible $R[G]$-module is ind-admissible.*

(v) *An ind-admissible $R[G]$-module M is admissible iff $\mathrm{Im}(\rho_M)$ is an R-module of finite type.*

(vi) *An $R[G]$-module M of finite (resp., co-finite) type over R is ind-admissible iff it is admissible.*

(vii) *For $M, N \in \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$, the canonical maps*

$$\mathrm{Hom}_{R[G]}(M,N) \xrightarrow{\sim} \varprojlim_{M_\alpha \in \mathcal{S}(M)} \mathrm{Hom}_{R[G]}(M_\alpha, N) \xleftarrow{\sim} \varprojlim_{M_\alpha \in \mathcal{S}(M)} \varinjlim_{N_\beta \in \mathcal{S}(N)} \mathrm{Hom}_{R[G]}(M_\alpha, N_\beta)$$

are both isomorphisms.

(viii) *The categories of ind-objects* $\mathrm{Ind}\big(\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)_{R\text{-}ft}\big)$ *and* $\mathrm{Ind}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$ *are canonically equivalent to* $\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$.

Proof

(i) If $M = j(M)$ is ind-admissible and N is an $R[G]$-submodule of M, then both $N = \bigcup(N \cap M_\alpha)$ and $M/N = \bigcup(M_\alpha/(N \cap M_\alpha))$ are ind-admissible $(M_\alpha \in \mathcal{S}(M))$. This proves stability by subquotients. Every colimit $\varinjlim M(\beta)$ is a quotient of the direct sum $M = \bigoplus M(\beta)$. If each $M(\beta)$ is ind-admissible, so is $M = \bigcup(M(\beta_1)_{\alpha_1} \oplus \cdots \oplus M(\beta_n)_{\alpha_n})$ $(M(\beta_i)_{\alpha_i} \in \mathcal{S}(M(\beta_i)))$. Finally, if $M = j(M)$ and $N = j(N)$, then $M \otimes_R N = \bigcup \mathrm{Im}(M_\alpha \otimes_R N_\beta \to M \otimes_R N) = j(M \otimes_R N)$.

(ii) The functors i, j form an adjoint pair almost by definition; i commutes with finite limits by (i).

(iii) As i is exact, its right adjoint j preserves injectives (and is left exact by adjointness). For every ind-admissible $R[G]$-module M there is a monomorphism $i(M) \to J$ with J injective in $\big({}_{R[G]}\mathrm{Mod}\big)$; then $M \to j(J)$ is a monomorphism with $j(J)$ injective in $\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$.

(iv) Use (i), Lemma 3.2.2(ii) and Lemma 3.2.4.

(v) If M is an ind-admissible $R[G]$-module, then the canonical map

$$u : \mathrm{Im}(\rho_M) \longrightarrow \varprojlim_{M_\alpha \in \mathcal{S}(M)} \mathrm{Im}(\rho_{M_\alpha})$$

is an isomorphism of R-modules. If, in addition, $\mathrm{Im}(\rho_M)$ is an R-module of finite type, then u is a homeomorphism with respect to $\mathfrak{m}$-adic topologies on $\mathrm{Im}(\rho_M)$ and $\mathrm{Im}(\rho_{M_\alpha})$; thus $G \to \mathrm{Im}(\rho_M)$ is continuous and M is admissible.

(vi) This follows from (v).

(vii) The first arrow is an isomorphism by definition of colimits. As regards the second arrow, note that

$$\varinjlim_{N_\beta \in \mathcal{S}(N)} \mathrm{Hom}_{R[G]}(M_\alpha, N_\beta) \longrightarrow \mathrm{Hom}_{R[G]}(M_\alpha, N)$$

is an isomorphism, since the image of any $R[G]$-linear map $M_\alpha \to N$ is of finite type over R, hence is contained in some N_β.

(viii) For a category $\mathcal{C}$, an object of $\mathrm{Ind}(\mathcal{C})$ is a functor $F : J \to \mathcal{C}$, where J is a small filtered category. Morphisms in $\mathrm{Ind}(\mathcal{C})$ are given by

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(F, F') = \varprojlim_{J} \varinjlim_{J'} \mathrm{Hom}_{\mathcal{C}}(F(j), F'(j')).$$

In the special case of $\mathcal{C} = \big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$, associating to F the colimit $\varinjlim_{J} F(j)$ in $\big({}_{R[G]}\mathrm{Mod}\big)$ defines functors

$$\mathrm{Ind}\big(\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)_{R\text{-}ft}\big) \xrightarrow{S} \mathrm{Ind}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big) \xrightarrow{T} \big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big).$$

It follows from (iv) (resp., (vii)) that S (resp., $T \circ S$) is an equivalence of categories. □

3.3.6. Lemma

(i) *If* $M \in \left({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\right)_{R\text{-}ft}$ *and* $N \in \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$, *then* $\mathrm{Hom}_R(M,N) \in \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$.

(ii) *If* $M \in \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$ *and* $H \lhd G$ *is a closed normal subgroup of* G, *then* $M^H \in \left({}^{\text{ind-ad}}_{R[G/H]}\mathrm{Mod}\right)$.

Proof

(i) Write $N = \bigcup N_\beta$ with $N_\beta \in \mathcal{S}(N)$. The $R[G]$-modules $\mathrm{Hom}_R(M, N_\beta)$ are all admissible (hence ind-admissible) by Proposition 3.2.5; it follows from Proposition 3.3.5 (i) that $\mathrm{Hom}_R(M,N) = \varinjlim_\beta \mathrm{Hom}_R(M, N_\beta)$ is ind-admissible as well.

(ii) By Proposition 3.3.5 (i), M^H is ind-admissible as an $R[G]$-module. The claim follows from the fact that $G \to G/H$ is a quotient map (*i.e.*, G/H has the quotient topology). □

3.3.7. Proposition. — *Let* $\widehat{G} = \varprojlim G/U$ *be the pro-finite completion of* G *with the pro-finite topology (*U *runs through all normal subgroups of* G *of finite index). If* k *is finite, then the action* $\rho_M : G \to \mathrm{Aut}_R(M)$ *of* G *on every admissible (resp., ind-admissible)* $R[G]$*-module* M *factors canonically through the natural map* $G \to \widehat{G}$*; this makes* M *into an admissible (resp., ind-admissible)* $R[\widehat{G}]$*-module.*

Proof. — If $M = \bigcup M_\alpha$ ($M_\alpha \in \mathcal{S}(M)$) is ind-admissible, then each group $\mathrm{Aut}_R(M_\alpha)$ is finite and the map

$$\rho_M : G \longrightarrow \varprojlim_\alpha \mathrm{Aut}_R(M_\alpha) \qquad (\subset \mathrm{Aut}_R(M))$$

is continuous with respect to the pro-finite topology on the target, hence factors canonically through a continuous homomorphism $\widehat{G} \to \varprojlim \mathrm{Aut}_R(M_\alpha)$. □

3.3.8. Corollary. — *If the natural map* $G \to \widehat{G}$ *is continuous, then it induces equivalences of categories*

$$\left({}^{\mathrm{ad}}_{R[\widehat{G}]}\mathrm{Mod}\right) \xrightarrow{\approx} \left({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\right), \quad \left({}^{\text{ind-ad}}_{R[\widehat{G}]}\mathrm{Mod}\right) \xrightarrow{\approx} \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right).$$

3.3.9. Proposition. — *Let* $M^\bullet$ *be a bounded above complex of ind-admissible* $R[G]$*-modules with all cohomology groups* $H^i(M^\bullet)$ *of finite type over* R*. Then there is a subcomplex* $N^\bullet \hookrightarrow M^\bullet$ *(of admissible* $R[G]$*-modules) such that*

(i) *Each* N^i *is of finite type over* R*.*

(ii) *The inclusion* $N^\bullet \hookrightarrow M^\bullet$ *is a quasi-isomorphism.*

Proof. — The proof of Proposition 3.2.6 applies word by word. □

3.3.10. Proposition. — *The embeddings*

$$\left({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\right)_{R\text{-}ft} = \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)_{R\text{-}ft} \longrightarrow \left({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\right) \longrightarrow \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$$

induce equivalences of categories

$$D^-\left(\left({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\right)_{R\text{-}ft}\right) = D^-\left(\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)_{R\text{-}ft}\right) \xrightarrow{\approx} D^-_{R\text{-}ft}\left({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\right) \xrightarrow{\approx} D^-_{R\text{-}ft}\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$$

Proof. — As in 3.2.8. □

3.4. Continuous cochains

3.4.1. Let G be a topological group and M an ind-admissible $R[G]$-module.

3.4.1.1. Definition. — **(Non-homogeneous) continuous cochains** of degree $i \geqslant 0$ on G with values in M are defined as

$$C^i_{\mathrm{cont}}(G, M) = \varinjlim_{M_\alpha \in \mathcal{S}(M)} C^i_{\mathrm{cont}}(G, M_\alpha),$$

where $C^i_{\mathrm{cont}}(G, M_\alpha)$ is the R-module of continuous maps $G^i \to M_\alpha$ (M_α is equipped with $\mathfrak{m}$-adic topology). In other words,

$$C^i_{\mathrm{cont}}(G, M_\alpha) = \varprojlim_n C^i_{\mathrm{cont}}(G, M_\alpha/\mathfrak{m}^n M_\alpha).$$

3.4.1.2. The standard differential

$$(\delta c)(g_1, \dots, g_{i+1}) = \\ g_1 c(g_2, \dots, g_{i+1}) + \sum_{j=1}^{i} (-1)^j c(g_1, \dots, g_j g_{j+1}, \dots, g_{i+1}) + (-1)^{i-1} c(g_1, \dots, g_i)$$

maps $C^i_{\mathrm{cont}}(G, M_\alpha)$ to $C^{i+1}_{\mathrm{cont}}(G, M_\alpha)$ (by Lemma 3.2.4), hence

$$\cdots \longrightarrow C^i_{\mathrm{cont}}(G, M) \xrightarrow{\delta^i_M} C^{i+1}_{\mathrm{cont}}(G, M) \longrightarrow \cdots$$

becomes a complex $C^\bullet_{\mathrm{cont}}(G, M)$ of R-modules.

3.4.1.3. Let $M^\bullet$ be a complex of ind-admissible $R[G]$-modules. We define $C^\bullet_{\mathrm{cont}}(G, M^\bullet)$ to be the simple complex associated to $C^j_{\mathrm{cont}}(G, M^i)$: its component of degree n is equal to

$$C^n_{\mathrm{cont}}(G, M^\bullet) = \bigoplus_{i+j=n} C^j_{\mathrm{cont}}(G, M^i)$$

(if $M^\bullet$ is bounded below, then the sum is finite and vanishes for $n << 0$) and the restriction of the differential $\delta^{i+j}_{M^\bullet}$ to $C^j_{\mathrm{cont}}(G, M^i)$ is equal to the sum of

$$(d^i_{M^\bullet})_* : C^j_{\mathrm{cont}}(G, M^i) \longrightarrow C^j_{\mathrm{cont}}(G, M^{i+1})$$

and

$$(-1)^i \delta^j_{M^i} : C^j_{\mathrm{cont}}(G, M^i) \longrightarrow C^{j+1}_{\mathrm{cont}}(G, M^i).$$

This sign rule implies that

$$C^\bullet_{\mathrm{cont}}(G, M^\bullet[1]) = C^\bullet_{\mathrm{cont}}(G, M^\bullet)[1].$$

3.4.1.4. Every morphism $f : M^\bullet \to N^\bullet$ of complexes of ind-admissible $R[G]$-modules induces a morphism of complexes of R-modules

$$f_* : C^\bullet_{\mathrm{cont}}(G, M^\bullet) \longrightarrow C^\bullet_{\mathrm{cont}}(G, N^\bullet)$$

satisfying

$$C^\bullet_{\mathrm{cont}}(G, \mathrm{Cone}(M^\bullet \xrightarrow{f} N^\bullet)) = \mathrm{Cone}(C^\bullet_{\mathrm{cont}}(G, M^\bullet) \xrightarrow{f_*} C^\bullet_{\mathrm{cont}}(G, N^\bullet)).$$

3.4.1.5. It follows from Proposition 3.3.5 (i) that $C^\bullet_{\mathrm{cont}}(G, -)$ commutes with filtered direct limits.

3.4.1.6. Given a continuous homomorphism of topological groups $u : G' \to G$, an ind-admissible $R[G]$ (resp., $R[G']$)-module M (resp., M') and an R-linear map $v : M \to M'$ such that $v(u(g')m) = g'v(m)$ for all $m \in M$, $g' \in G'$, the pair $(u, v) : (G, M) \to (G', M')$ induces a homomorphism

$$f : C^\bullet_{\mathrm{cont}}(G, M) \longrightarrow C^\bullet_{\mathrm{cont}}(G', M')$$

given by $(f(c))(g'_1, \ldots, g'_i) = v\big(c(u(g'_1), \ldots, u(g'_i))\big)$.

3.4.2. Proposition. — *Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be an exact sequence of $R[G]$-modules, with M ind-admissible. Then M', M'' are also ind-admissible and*

$$0 \longrightarrow C^\bullet_{\mathrm{cont}}(G, M') \xrightarrow{\alpha_*} C^\bullet_{\mathrm{cont}}(G, M) \xrightarrow{\beta_*} C^\bullet_{\mathrm{cont}}(G, M'') \longrightarrow 0$$

is an exact sequence of complexes of R-modules. More generally, the statement still holds if we allow M, M', M'' to be complexes of ind-admissible $R[G]$-modules.

Proof. — Ind-admissibility of M', M'' follows from Proposition 3.3.5 (i). Clearly $\mathrm{Ker}(\alpha_*) = 0$ and $\beta_*\alpha_* = 0$. Writing $M = \varinjlim_\alpha M_\alpha$ with $M_\alpha \in \mathcal{S}(M)$, it is enough to consider the case of M of finite type over R. The surjectivity of β_* then follows from 3.1.5 and the equality $\mathrm{Ker}(\beta_*) = \mathrm{Im}(\alpha_*)$ from the fact that α is strict. The statement for complexes is a formal consequence of the statement for modules. □

3.4.3. Corollary. — *The canonical map of complexes $\gamma : [M' \xrightarrow{\alpha} M] \to M''$ (with M, M'' in degree zero) induces a quasi-isomorphism*

$$\gamma_* : C^\bullet_{\mathrm{cont}}(G, [M' \xrightarrow{\alpha} M]) \longrightarrow C^\bullet_{\mathrm{cont}}(G, M'')$$

Proof. — The first complex in the exact sequence

$$0 \longrightarrow C^\bullet_{\mathrm{cont}}(G, [M' \xrightarrow{\mathrm{id}} M']) \longrightarrow C^\bullet_{\mathrm{cont}}(G, [M' \xrightarrow{\alpha} M]) \longrightarrow C^\bullet_{\mathrm{cont}}(G, M'') \longrightarrow 0$$

is acyclic, being equal (up to a shift) to the cone of the identity map on $C^\bullet_{\mathrm{cont}}(G, M')$. □

3.4.4. Proposition. — *Let $M^\bullet$ (resp., $N^\bullet$) be a complex of ind-admissible $R[G]$-modules (resp., flat R-modules). Then the canonical morphisms*

$$C^\bullet_{\mathrm{cont}}(G, M^\bullet) \otimes_R N^\bullet \longrightarrow C^\bullet_{\mathrm{cont}}(G, M^\bullet \otimes_R N^\bullet)$$
$$N^\bullet \otimes_R C^\bullet_{\mathrm{cont}}(G, M^\bullet) \longrightarrow C^\bullet_{\mathrm{cont}}(G, N^\bullet \otimes_R M^\bullet)$$

are isomorphisms of complexes.

Proof. — The morphisms in question are given by the following collections of maps:

$$\begin{array}{ccc} C^i_{\mathrm{cont}}(G, M^a) \otimes_R N^b & \longrightarrow & C^i_{\mathrm{cont}}(G, M^a \otimes_R N^b) \\ \alpha \otimes n & \longmapsto & ((g_1, \ldots, g_i) \longmapsto (-1)^{ib} \alpha(g_1, \ldots, g_i) \otimes n) \end{array}$$

respectively,

$$\begin{array}{ccc} N^a \otimes_R C^j_{\mathrm{cont}}(G, M^b) & \longrightarrow & C^j_{\mathrm{cont}}(G, N^a \otimes_R M^b) \\ n \otimes \alpha & \longmapsto & ((g_1, \ldots, g_j) \longmapsto n \otimes \alpha(g_1, \ldots, g_j)) \end{array}$$

(cf. 3.4.5.2 for the sign conventions). It is sufficient to treat the case when both $M^\bullet = M$ and $N^\bullet = N$ consist of a single module in degree zero. N is admissible, as G acts trivially on it; thus $M \otimes_R N$ is ind-admissible by Proposition 3.3.5 (i). By Lazard's Theorem [**La**], $N = \varinjlim_\beta N_\beta$ is a filtered direct limit of free R-modules of finite type. Writing $M = \varinjlim_\alpha M_\alpha$ ($M_\alpha \in \mathcal{S}(M)$), we have

$$C^\bullet_{\mathrm{cont}}(G, M) \otimes_R N = \varinjlim_{\alpha,\beta} (C^\bullet_{\mathrm{cont}}(G, M_\alpha) \otimes_R N_\beta)$$
$$C^\bullet_{\mathrm{cont}}(G, M \otimes_R N) = \varinjlim_{\alpha,\beta} (C^\bullet_{\mathrm{cont}}(G, M_\alpha \otimes_R N_\beta))$$

However, as $N_\beta \xrightarrow{\sim} R^{n(\beta)}$ for some integer $n(\beta)$, the canonical map

$$C^\bullet_{\mathrm{cont}}(G, M_\alpha) \otimes_R R^{n(\beta)} \longrightarrow C^\bullet_{\mathrm{cont}}\big(G, M_\alpha \otimes_R R^{n(\beta)}\big)$$

is an isomorphism for trivial reasons. The same argument works also for the second morphism. □

3.4.5. Cup products

3.4.5.1. Let A, B be ind-admissible $R[G]$-modules. The cup product

$$\cup : C^i_{\mathrm{cont}}(G, A) \otimes_R C^j_{\mathrm{cont}}(G, B) \longrightarrow C^{i+j}_{\mathrm{cont}}(G, A \otimes_R B)$$

is defined by the usual formula

$$(\alpha \cup \beta)(g_1, \ldots, g_{i+j}) = \alpha(g_1, \ldots, g_i) \otimes (g_1 \cdots g_i)(\beta(g_{i+1}, \ldots, g_{i+j})).$$

As

$$\delta(\alpha \cup \beta) = (\delta\alpha) \cup \beta + (-1)^i \alpha \cup (\delta\beta),$$

the maps $\cup$ define a morphism of complexes

$$\cup : C^\bullet_{\mathrm{cont}}(G, A) \otimes_R C^\bullet_{\mathrm{cont}}(G, B) \longrightarrow C^\bullet_{\mathrm{cont}}(G, A \otimes_R B).$$

This product is associative (with respect to the associativity of $X^\bullet \otimes Y^\bullet$ (1.2.3)):

$$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma).$$

3.4.5.2. Let $A^\bullet$, $B^\bullet$ be complexes of ind-admissible $R[G]$-modules. Recall that

$$C^m_{\mathrm{cont}}(G, A^\bullet) = \bigoplus_{i+a=m} C^i_{\mathrm{cont}}(G, A^a),$$

with differential $d = d_A + (-1)^a \delta$ on $C^i_{\mathrm{cont}}(G, A^a)$, where $d_A : C^i_{\mathrm{cont}}(G, A^a) \to C^i_{\mathrm{cont}}(G, A^{a+1})$ is induced by $d_A : A^a \to A^{a+1}$ and $\delta : C^i_{\mathrm{cont}}(G, A^a) \to C^{i+1}_{\mathrm{cont}}(G, A^a)$ is the cochain differential. Similarly, $d = d_B + (-1)^b \delta$ on $C^j_{\mathrm{cont}}\big(G, B^b\big) \subset C^{j+b}_{\mathrm{cont}}(G, B^\bullet)$. The differential on $C^\bullet = A^\bullet \otimes_R B^\bullet$ is equal to $d_C = d_A \otimes 1 + (-1)^a 1 \otimes d_B$ on $A^a \otimes_R B^b \subset C^{a+b}$. The individual cup products

$$\cup^{ab}_{ij} : C^i_{\mathrm{cont}}(G, A^a) \otimes_R C^j_{\mathrm{cont}}\big(G, B^b\big) \longrightarrow C^{i+j}_{\mathrm{cont}}\big(G, A^a \otimes_R B^b\big)$$

defined in 3.4.5.1 can be combined – with appropriate signs – to the total cup product

$$\cup = \big((-1)^{ib} \cup^{ab}_{ij}\big).$$

The signs are chosen in such a way that

$$\delta(\alpha \cup \beta) = (\delta\alpha) \cup \beta + (-1)^{\deg(\alpha)} \alpha \cup (\delta\beta)$$

($\deg(\alpha) = i + a$ for $\alpha \in C^i_{\mathrm{cont}}(G, A^a)$). As before, this means that $\cup$ defines a morphism of complexes

$$\cup : C^\bullet_{\mathrm{cont}}(G, A^\bullet) \otimes_R C^\bullet_{\mathrm{cont}}(G, B^\bullet) \longrightarrow C^\bullet_{\mathrm{cont}}(G, A^\bullet \otimes_R B^\bullet).$$

Again, this product is associative:

$$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$$

for $\alpha \in C^i_{\mathrm{cont}}(G, A^a)$, $\beta \in C^j_{\mathrm{cont}}(G, B^b)$, $\gamma \in C^k_{\mathrm{cont}}(G, C^c)$, as $(-1)^{ib}(-1)^{(i+j)c} = (-1)^{i(b+c)}(-1)^{jc}$.

3.4.5.3. Let A, B be as in 3.4.5.1. The formulas

$$\mathcal{T} : C^i_{\mathrm{cont}}(G, A) \longrightarrow C^i_{\mathrm{cont}}(G, A)$$
$$(\mathcal{T}(\alpha))(g_1, \ldots, g_i) = (-1)^{i(i+1)/2} g_1 \cdots g_i(\alpha(g_i^{-1}, \ldots, g_1^{-1}))$$

define a morphism of complexes

$$\mathcal{T} : C^\bullet_{\mathrm{cont}}(G, A) \longrightarrow C^\bullet_{\mathrm{cont}}(G, A)$$

which is an involution ($\mathcal{T}^2 = \mathrm{id}$) and is functorially homotopic to the identity (see 3.4.5.5 below for more details).

The transposition $\mathcal{T}$ satisfies the relation

$$\mathcal{T}(\alpha \cup \beta) = (-1)^{ij} (\mathcal{T}\beta) \cup (\mathcal{T}\alpha) \tag{3.4.5.1}$$

(for $\alpha \in C^i_{\mathrm{cont}}(G, A), \beta \in C^j_{\mathrm{cont}}(G, B)$).

3.4.5.4. For $A^\bullet$, $B^\bullet$ as in 3.4.5.2, the involutions $\mathcal{T}$ on $C^i_{\mathrm{cont}}(G, A^a)$ commute with both δ and d_A, hence define a morphism of complexes $\mathcal{T} : C^\bullet_{\mathrm{cont}}(G, A^\bullet) \to C^\bullet_{\mathrm{cont}}(G, A^\bullet)$ (and similarly for $B^\bullet$). Again, $\mathcal{T}$ is an involution homotopic to the identity. The formula (3.4.5.1) implies that $\alpha \in C^i_{\mathrm{cont}}(G, A^a)$ and $\beta \in C^j_{\mathrm{cont}}(G, B^b)$ satisfy

$$(s_{12})_*(\mathcal{T}(\alpha \cup^{ab}_{ij} \beta)) = (-1)^{ab}(-1)^{ij}(\mathcal{T}\beta) \cup^{ba}_{ji} (\mathcal{T}\alpha)$$

(where $(s_{12})_*$ is induced by $s_{12} : A^\bullet \otimes_R B^\bullet \xrightarrow{\sim} B^\bullet \otimes_R A^\bullet$); it follows that

$$(s_{12})_*(\mathcal{T}(\alpha \cup \beta)) = (-1)^{ib}(-1)^{ab}(-1)^{ij}(-1)^{ja}(\mathcal{T}\beta) \cup (\mathcal{T}\alpha) = (-1)^{\deg(\alpha)\deg(\beta)}(\mathcal{T}\beta) \cup (\mathcal{T}\alpha).$$

In other words, the diagram

$$\begin{array}{ccc} C^\bullet_{\mathrm{cont}}(G, A^\bullet) \otimes_R C^\bullet_{\mathrm{cont}}(G, B^\bullet) & \xrightarrow{\cup} & C^\bullet_{\mathrm{cont}}(G, A^\bullet \otimes_R B^\bullet) \\ \big\downarrow {\scriptstyle s_{12} \circ (\mathcal{T} \otimes \mathcal{T})} & & \big\downarrow {\scriptstyle \mathcal{T} \circ (s_{12})_*} \\ C^\bullet_{\mathrm{cont}}(G, B^\bullet) \otimes_R C^\bullet_{\mathrm{cont}}(G, A^\bullet) & \xrightarrow{\cup} & C^\bullet_{\mathrm{cont}}(G, B^\bullet \otimes_R A^\bullet) \end{array}$$

is commutative (all four maps are morphisms of complexes).

3.4.5.5. A homotopy $\mathrm{id} \rightsquigarrow \mathcal{T}$, functorial in both G and M, can be defined as follows.

Let G be any discrete group and $\mathbf{Z}[G]^{\otimes}_{\bullet}$ the standard bar resolution of $\mathbf{Z}$ by free $\mathbf{Z}[G]$-modules:

$$\mathbf{Z}[G]^{\otimes}_i = \bigoplus_{g_1, \ldots, g_i} \mathbf{Z}[G] \cdot [g_1 | \cdots | g_i]$$

with differentials

$$\partial[g_1 | \cdots | g_i] = g_1[g_2 | \cdots | g_i] + \sum_{j=1}^{i-1} (-1)^j [g_1 | \cdots | g_i g_{i+1} | \cdots | g_i] + (-1)^i [g_1 | \cdots | g_{i-1}].$$

The cochain complex $C^\bullet(G, M^\bullet)$ of any complex of G-modules $M^\bullet$ is equal to

$$\mathrm{Hom}^{\bullet,\mathrm{naive}}_{\mathbf{Z}[G]}(\mathbf{Z}[G]^{\otimes}_{\bullet}, M^\bullet).$$

The formula

$$\mathcal{T}[g_1 | \cdots | g_i] = (-1)^{i(i+1)/2} g_1 \cdots g_i [g_i^{-1} | \cdots | g_1^{-1}],$$

extended by $\mathbf{Z}[G]$-linearity, defines an involutive morphism of resolutions $\mathcal{T} : \mathbf{Z}[G]^{\otimes}_{\bullet} \to \mathbf{Z}[G]^{\otimes}_{\bullet}$ lifting the identity on $\mathbf{Z}$. It follows from general properties of projective resolutions that there is a homotopy $a : \mathrm{id} \rightsquigarrow \mathcal{T}$ on $\mathbf{Z}[G]^{\otimes}_{\bullet}$. Moreover, any pair of homotopies $a, a' : \mathrm{id} \rightsquigarrow \mathcal{T}$ on $\mathbf{Z}[G]^{\otimes}_{\bullet}$ is related by a second order homotopy $b : a \rightsquigarrow a'$.

Fixing a homotopy $a : \mathrm{id} \rightsquigarrow \mathcal{T}$ on $\mathbf{Z}[G]^{\otimes}_{\bullet}$ defines a homotopy

$$\mathrm{Hom}^{\bullet,\mathrm{naive}}(a, \mathrm{id}_M) : \mathrm{id} \rightsquigarrow \mathcal{T}$$

on $C^\bullet(G, M)$, functorial in M. Let $G = F$ be a free group on countably many generators g_j ($j \in \mathbf{N}$). The values of $a([g_1 | \cdots | g_i])$, expressed in terms of the generators g_j, define universal formulas for $a : \mathrm{id} \rightsquigarrow \mathcal{T}$ on $\mathbf{Z}[F]^{\otimes}_{\bullet}$, valid for every G. Then $\mathrm{Hom}^{\bullet,\mathrm{naive}}(a, \mathrm{id}_M) : \mathrm{id} \rightsquigarrow \mathcal{T}$ on $C^\bullet(G, M)$ will be functorial in both G and M (as in 3.4.1.6).

If we fix another homotopy $a' : \mathrm{id} \rightsquigarrow \mathcal{T}$ on $\mathbf{Z}[F]^{\otimes}_{\bullet}$, then

$$\mathrm{Hom}^{\bullet,\mathrm{naive}}(b, \mathrm{id}_M) : \mathrm{Hom}^{\bullet,\mathrm{naive}}(a, \mathrm{id}_M) \rightsquigarrow \mathrm{Hom}^{\bullet,\mathrm{naive}}(a', \mathrm{id}_M)$$

is a second order homotopy, functorial in both G and M.

3.5. Continuous (hyper-)cohomology

3.5.1. Let G be a topological group and M (resp., $M^\bullet$) an ind-admissible $R[G]$-module (resp., a complex of ind-admissible $R[G]$-modules).

3.5.1.1. Definition. — The **continuous cohomology** (resp., **hyper-cohomology**) of G with values in M (resp., $M^\bullet$) is defined as

$$H^i_{\mathrm{cont}}(G, M) = H^i(C^\bullet_{\mathrm{cont}}(G, M)), \quad \text{resp.,} \quad H^i_{\mathrm{cont}}(G, M^\bullet) = H^i(C^\bullet_{\mathrm{cont}}(G, M^\bullet)).$$

3.5.1.2. It follows from the exactness of $\varinjlim$ that

$$H^i_{\mathrm{cont}}(G, M) = \varinjlim_{M_\alpha \in \mathcal{S}(M)} H^i_{\mathrm{cont}}(G, M_\alpha).$$

More generally, 3.4.1.5 implies that $H^i_{\mathrm{cont}}(G, -)$ commutes with filtered direct limits.

3.5.1.3. Proposition. — *The functors $H^i_{\mathrm{cont}}(G, -)$ $(i \geqslant 0)$ form a δ-functor on $\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$ with values in $({}_R\mathrm{Mod})$, satisfying $H^0_{\mathrm{cont}}(G, M) = M^G$.*

Proof. — The fact that we have a δ-functor follows from Proposition 3.4.2. The cohomology in degree zero is equal to

$$H^0_{\mathrm{cont}}(G, M) = \varinjlim_{M_\alpha \in \mathcal{S}(M)} H^0_{\mathrm{cont}}(G, M_\alpha)$$

with

$$H^0_{\mathrm{cont}}(G, M_\alpha) = \varprojlim_n (M_\alpha / \mathfrak{m}^n M_\alpha)^G = M_\alpha^G,$$

hence $H^0_{\mathrm{cont}}(G, M) = M^G$ as claimed. □

3.5.2. Any decreasing filtration on $M^\bullet$ by subcomplexes $F^p M^\bullet$ of (necessarily ind-admissible) $R[G]$-modules induces by Proposition 3.4.2 a filtration

$$F^p C^\bullet_{\mathrm{cont}}(G, M^\bullet) = C^\bullet_{\mathrm{cont}}(G, F^p M^\bullet)$$

satisfying

$$\mathrm{gr}^p_F C^\bullet_{\mathrm{cont}}(G, M^\bullet) = C^\bullet_{\mathrm{cont}}(G, \mathrm{gr}^p_F(M^\bullet)).$$

This filtration defines a spectral sequence with

$$E_1^{p,q} = H^{p+q}_{\mathrm{cont}}(G, \mathrm{gr}^p_F(M^\bullet)), \tag{3.5.2.1}$$

which under suitable conditions on the filtration $F^p M^\bullet$ converges to $H^{p+q}_{\mathrm{cont}}(G, M^\bullet)$. We shall need the following two special cases of (3.5.2.1).

3.5.3. The "stupid" filtration $F^pM^\bullet = \sigma_{\geqslant p}M^\bullet$ given by

$$(\sigma_{\geqslant p}M^\bullet)^i = \begin{cases} M^i, & i \geqslant p \\ 0, & i < p \end{cases}$$

satisfies $\mathrm{gr}^p_F(M^\bullet) = M^p[-p]$ and gives rise to the first hyper-cohomology spectral sequence

$$ {}^IE_1^{p,q} = H^q_{\mathrm{cont}}(G, M^p) \Longrightarrow H^{p+q}_{\mathrm{cont}}(G, M^\bullet), \tag{3.5.3.1}$$

which is convergent if $M^\bullet$ is bounded below.

3.5.4. The truncation filtration $F^{-p}M^\bullet = \tau_{\leqslant p}M^\bullet$ on $M^\bullet$ is defined by

$$(\tau_{\leqslant p}M^\bullet)^i = \begin{cases} 0, & i > p \\ Z^p = \mathrm{Ker}\big(M^p \xrightarrow{d} M^{p+1}\big), & i = p \\ M^i, & i < p \end{cases}$$

Its graded quotients are

$$\mathrm{gr}^p_F(M^\bullet) = \Big[M^{-p-1}/Z^{-p-1} \overset{d}{\hookrightarrow} Z^{-p}\Big] \tag{3.5.4.1}$$

(with Z^{-p} in degree $-p$). The cokernel of the map d in (3.5.4.1) is equal to $H^{-p}(M^\bullet)$. Applying Corollary 3.4.3, the spectral sequence (3.5.2.1) becomes – after renumbering – the second hyper-cohomology spectral sequence

$$ {}^{\mathrm{II}}E_2^{p,q} = H^p_{\mathrm{cont}}(G, H^q(M^\bullet)) \Longrightarrow H^{p+q}_{\mathrm{cont}}(G, M^\bullet), \tag{3.5.4.2}$$

which is convergent if $M^\bullet$ is cohomologically bounded below.

3.5.5. Proposition. — *Let $u : M^\bullet \to N^\bullet$ be a quasi-isomorphism of cohomologically bounded below complexes of ind-admissible $R[G]$-modules. Then the induced map*

$$u_* : C^\bullet_{\mathrm{cont}}(G, M^\bullet) \longrightarrow C^\bullet_{\mathrm{cont}}(G, N^\bullet)$$

is again a quasi-isomorphism.

Proof. — The map u induces a morphism of convergent spectral sequences (3.5.4.2)

$$ {}^{\mathrm{II}}E_r(M^\bullet) \longrightarrow {}^{\mathrm{II}}E_r(N^\bullet),$$

which is an isomorphism on E_2. Hence the induced map on the abutments

$$H^i_{\mathrm{cont}}(G, M^\bullet) \longrightarrow H^i_{\mathrm{cont}}(G, N^\bullet)$$

is an isomorphism as well. □

3.5.6. Corollary. — *The functor*

$$\begin{array}{ccc} C^+\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big) & \longrightarrow & C^+({}_R\mathrm{Mod}) \\ M^\bullet & \longmapsto & C^\bullet_{\mathrm{cont}}(G, M^\bullet) \end{array}$$

preserves homotopy, exact sequences and quasi-isomorphisms, hence defines an exact functor

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G,-) : D^+\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big) \longrightarrow D^+({}_R\mathrm{Mod}).$$

3.5.7. As in 2.4.2, fix a system of parameters (x_i) of R. The shifted tensor product

$$M^\bullet \longmapsto (M^\bullet \otimes_R C^\bullet((x_i),R))[d]$$

with the bounded complex of flat R-modules $C^\bullet((x_i),R)$ defines functors (independent of the choice of (x_i))

$$\begin{aligned}\Phi : D^*\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big) &\longrightarrow D^*\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)\\ D^*\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big) &\longrightarrow D^*\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)\end{aligned}$$

(for $* = \varnothing, +, -, b$). If $T \in D^*_{R\text{-}ft}(-)$, then $\Phi(T) \in D^*_{R\text{-}coft}(-)$.

3.5.8. Proposition. — *For every $M \in D^+_{R\text{-}ft}\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$, the canonical map*

$$\Phi(\mathbf{R}\Gamma_{\mathrm{cont}}(G,M)) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G,\Phi(M))$$

is an isomorphism in $D^+({}_R\mathrm{Mod})$.

Proof. — Represent M by a bounded below complex $M^\bullet$ of ind-admissible $R[G]$-modules. The L.H.S. (resp., R.H.S) is represented by the complex $(C^\bullet_{\mathrm{cont}}(G,M^\bullet) \otimes_R C^\bullet(R))[d]$ (resp., $C^\bullet_{\mathrm{cont}}(G,(M^\bullet \otimes_R C^\bullet(R))[d])$). The statement follows from Proposition 3.4.4.

3.5.9. Let $J^\bullet$ be a bounded complex of injective R-modules. The functor

$$M^\bullet \longmapsto \mathrm{Hom}^\bullet_R(M^\bullet, J^\bullet)$$

defined on complexes $M^\bullet$ in $\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$ preserves homotopies, exact sequences and quasi-isomorphisms; it defines an exact functor

$$\mathbf{R}\mathrm{Hom}_R(-,J^\bullet) : D\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)^{\mathrm{op}} \longrightarrow D\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$$

which maps $D^\pm({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod})$ to $D^\mp({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod})$.

If $J'^\bullet$ is another bounded complex of injective R-modules and $J^\bullet \to J'^\bullet$ a quasi-isomorphism, then the induced map

$$\mathrm{Hom}^\bullet_R(M^\bullet, J^\bullet) \longrightarrow \mathrm{Hom}^\bullet_R(M^\bullet, J'^\bullet)$$

is also a quasi-isomorphism. As a result, we obtain a bifunctor

$$\mathbf{R}\mathrm{Hom}_R(-,-) : D\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)^{\mathrm{op}} \times D^b\big(\mathrm{inj} - {}_R\mathrm{Mod}\big) \longrightarrow D\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big).$$

The same argument shows that $\mathrm{Hom}^\bullet_R(M^\bullet, J^\bullet)$ defines a bifunctor

$$\mathbf{R}\mathrm{Hom}_R(-,-) : D^-\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)^{\mathrm{op}} \times D^+({}_R\mathrm{Mod}) \longrightarrow D^+\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big). \quad \square$$

3.5.10. Proposition. — *For every ideal $J \subsetneq R$, the functor*

$$f^* : \big({}_{(R/J)[G]}\mathrm{Mod}\big) \longrightarrow \big({}_{R[G]}\mathrm{Mod}\big)$$

associated to the canonical projection $f : R \to R/J$ has the following property

$$M \in \big({}_{(R/J)[G]}\mathrm{Mod}\big) \text{ is (ind-)admissible } \iff f^*M \in \big({}_{R[G]}\mathrm{Mod}\big) \text{ is (ind-)admissible.}$$

If true, then $C^\bullet_{\mathrm{cont}}(G, M) = C^\bullet_{\mathrm{cont}}(G, f^*M)$.

Proof. — This follows from the definitions and the fact that an R/J-module N is of finite type iff f^*N is of finite type over R. □

3.6. Derived functor cohomology

3.6.1. Derived functors of $(-)^G$

3.6.1.1. Denote by $\Gamma_{\mathrm{der}}(G, -) : \big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big) \to ({}_R\mathrm{Mod})$ the (left exact) functor $M \mapsto M^G$. As the category $\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$ has enough injectives, the right derived functor of $\Gamma_{\mathrm{der}}(G, -)$,

$$\mathbf{R}^+\Gamma_{\mathrm{der}}(G, -) \ : \ D^+\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big) \longrightarrow D^+({}_R\mathrm{Mod}),$$

exists and can be computed using injective resolutions. The cohomological derived functors

$$H^i_{\mathrm{der}}(G, -) = H^i(\mathbf{R}^+\Gamma_{\mathrm{der}}(G, -)) \ : \ \big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big) \longrightarrow ({}_R\mathrm{Mod}) \qquad (i \geqslant 0)$$

form a universal δ-functor.

3.6.1.2. The derived functor $\mathbf{R}^+\Gamma_{\mathrm{der}}(G, M)$ has the usual functoriality properties with respect to pairs of morphisms $u : G' \to G$, $v : M \to M'$ as in 3.4.1.6.

3.6.1.3. In particular, for a subgroup $H \subseteq G$ equipped with induced topology and $g \in G$, the morphisms $u : gHg^{-1} \to H$, $u(ghg^{-1}) = h$, $v : M \to M$, $v(m) = gm$, induce the conjugation map

$$\mathrm{Ad}(g) : \mathbf{R}^+\Gamma_{\mathrm{der}}(H, M) \longrightarrow \mathbf{R}^+\Gamma_{\mathrm{der}}(gHg^{-1}, M).$$

If $H \lhd G$ is a normal subgroup of G, then the maps $\mathrm{Ad}(g)$ define an action of G on $\mathbf{R}^+\Gamma_{\mathrm{der}}(H, M)$. The induced action on cohomology $H^i_{\mathrm{der}}(H, M)$ factors through G/H (each $h \in H$ acts trivially on $H^0_{\mathrm{der}}(H, M) = M^H$, hence on all $H^i_{\mathrm{der}}(H, M)$, by universality of this δ-functor). More precisely, if $J^\bullet$ is an injective resolution of M in $\big({}^{\text{ind-ad}}_{R[H]}\mathrm{Mod}\big)$ and $h \in H$, then $\mathrm{Ad}(h)$ is represented by the identity morphism of $(J^\bullet)^H$; thus the action on $\mathbf{R}^+\Gamma_{\mathrm{der}}(H, M)$ factors through G/H.

3.6.1.4. Still assuming that H is a normal subgroup of G, the conjugation maps $\mathrm{Ad}(g)$ also act on the complex of continuous cochains $C^\bullet_{\mathrm{cont}}(H, M)$. For every $h \in H$ there is a homotopy s between $\mathrm{Ad}(h)$ and the identity map acting on $C^\bullet_{\mathrm{cont}}(H, M)$; it is given by

$$(s_n(c))(h_1, \ldots, h_{n-1}) = \sum_{i=1}^{n} (-1)^{i-1} c(h_1, \ldots, h_{i-1}, h, h^{-1}h_i h, \ldots, h^{-1}h_{n-1}h).$$

This implies that the action of G on $\mathbf{R}\Gamma_{\mathrm{cont}}(H, M)$ factors through G/H.

3.6.2. Proposition

(i) *There is a canonical morphism of functors* $\theta_G : \mathbf{R}^+\Gamma_{\mathrm{der}}(G, -) \to \mathbf{R}\Gamma_{\mathrm{cont}}(G, -)$; *denote by* $\theta^i_G(-) \ : \ H^i_{\mathrm{der}}(G, -) \to H^i_{\mathrm{cont}}(G, -)$ *the corresponding morphism of* δ*-functors.*

(ii) *If* $H \lhd G$ *is a normal subgroup of* G, *then* θ_H *commutes with the action of* G/H *on both sides.*

(iii) *Let* $n \geqslant 1$. *If* $\theta^{n-1}_G(M)$ *is an isomorphism for all* $M \in \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$ *and* $H^n_{\mathrm{cont}}(G, -)$ *is effaceable on* $\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$, *then* $\theta^n_G(M)$ *is also an isomorphism for all* M.

(iv) $H^1_{\mathrm{cont}}(G, -)$ *is effaceable on* $\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$.

(v) $\theta^0_G(M), \theta^1_G(M)$ *are isomorphisms for all* M.

Proof

(i) Let $M^\bullet$ be a bounded below complex in $\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$. Fix a morphism of complexes $M^\bullet \to J^\bullet$ which is a quasi-isomorphism and such that all J^i are injective. Then the morphism $\theta_G(M^\bullet)$ in $D^+({}_R\mathrm{Mod})$ is represented by

$$(J^\bullet)^G \longrightarrow C^\bullet_{\mathrm{cont}}(G, J^\bullet) \xleftarrow{\mathrm{Qis}} C^\bullet_{\mathrm{cont}}(G, M^\bullet).$$

(ii) follows from the definitions and (iii) is a standard general nonsense.

(iv) Given $M \in \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$ we must show that there is a monomorphism $u : M \to E$ in $\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$ such that the induced map on cohomology $u_* : H^1_{\mathrm{cont}}(G, M) \to H^1_{\mathrm{cont}}(G, E)$ is zero. Fix cocycles $c_j \in C^1_{\mathrm{cont}}(G, M_{\alpha(j)})$ (for a suitable index set J and $M_{\alpha(j)} \in \mathcal{S}(M)$) such that their classes in $H^1_{\mathrm{cont}}(G, M)$ generate $H^1_{\mathrm{cont}}(G, M)$ as an R-module. Let E be the R-module $E = M \oplus \bigoplus_{j \in J} R$ with an R-linear action of G given by

$$g(m, \{r_j\}_{j \in J}) = \left(g(m) + \sum_{j \in J} r_j c_j(g), \{r_j\}_{j \in J} \right) \qquad (g \in G).$$

For every finite subset $J_0 \subseteq J$ and $M_\alpha \in \mathcal{S}(M)$, the $R[G]$-module

$$E(J_0, M_\alpha) := \left(M_\alpha + \sum_{j \in J_0} M_{\alpha(j)} \right) \oplus \bigoplus_{j \in J_0} R \subseteq E$$

lies in $\mathcal{S}(E)$; as $E = \bigcup E(J_0, M_\alpha)$, E lies in $\left({}^{\text{ind-ad}}_{R[G]}\text{Mod}\right)$. There is an exact sequence in $\left({}^{\text{ind-ad}}_{R[G]}\text{Mod}\right)$

$$0 \longrightarrow M \xrightarrow{u} E \xrightarrow{v} \bigoplus_{j\in J} R \longrightarrow 0,$$

with u (resp., v) being the canonical inclusion (resp., the projection). For every $i \in J$, the cocycle

$$g \longmapsto u(c_i(g)) = (g-1)\Big(\Big(0, \Big\{r_j = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}\Big\}_{j\in J}\Big)\Big)$$

with values in E is a coboundary, hence u_* is the zero map as required.

(v) $\theta^0_G(M) = \text{id}$; the statement about $\theta^1_G(M)$ follows from (iii) for $n = 1$ and (iv). □

3.6.3. Let $H \lhd G$ be a closed normal subgroup of G. The functor $\Gamma_{\text{der}}(G, -)$ is equal to the composition of $\Gamma_{\text{der}}(G/H, -)$ with the functor

$$\Gamma_{\text{der}}(G, G/H, -) \ : \ \left({}^{\text{ind-ad}}_{R[G]}\text{Mod}\right) \longrightarrow \left({}^{\text{ind-ad}}_{R[G/H]}\text{Mod}\right)$$

sending M to M^H. This functor preserves injectives, since it has an exact left adjoint, namely f^* for $f : R[G] \to R[G/H]$. We have, therefore, a canonical isomorphism of functors

$$\mathbf{R}^+\Gamma_{\text{der}}(G, -) \xrightarrow{\sim} \mathbf{R}^+\Gamma_{\text{der}}(G/H, -) \circ \mathbf{R}^+\Gamma_{\text{der}}(G, G/H, -)$$

and the corresponding spectral sequence

$$E_2^{i,j} = H^i_{\text{der}}(G/H, R^j\Gamma_{\text{der}}(G, G/H, M)) \Longrightarrow H^{i+j}_{\text{der}}(G, M) \tag{3.6.3.1}$$

(for $M \in \left({}^{\text{ind-ad}}_{R[G]}\text{Mod}\right)$).

3.6.4. Proposition. — *Let $H \lhd G$ be a closed normal subgroup of G. Denote by $\text{Res}_{G,H}$ the forgetful functor*

$$\text{Res}_{G,H} : \left({}^{\text{ind-ad}}_{R[G]}\text{Mod}\right) \longrightarrow \left({}^{\text{ind-ad}}_{R[H]}\text{Mod}\right).$$

Then

(i) *There is a canonical morphism of functors*

$$\text{Res}_{G/H,\{1\}} \circ \mathbf{R}^+\Gamma_{\text{der}}(G, G/H, -) \longrightarrow \mathbf{R}^+\Gamma_{\text{der}}(H, -) \circ \text{Res}_{G,H}$$

commuting with the action of G/H on both sides.

(ii) *The induced maps on cohomology*

$$R^q\Gamma_{\text{der}}(G, G/H, M) \longrightarrow H^q_{\text{der}}(H, M)$$

are not isomorphisms in general, even for $q = 1$.

Proof

(i) Denote the adjoint pair of functors from Proposition 3.3.5 (ii) by i_G, j_G. For $M \in \left({}^{\text{ind-ad}}_{R[G]}\text{Mod}\right)$ let $J^\bullet$ be an injective resolution of $i_G(M)$ in $\left({}_{R[G]}\text{Mod}\right)$; then $j_G(J^\bullet)$ (resp., $j_H(J^\bullet)$) is an injective resolution of M (resp., of $\text{Res}_{G,H}(M)$). The inclusions $j_G(J^i) \subseteq j_H(J^i)$ give rise to a morphism of complexes $(j_G(J^\bullet))^H \to (j_H(J^\bullet))^H$, which represents a morphism

$$\text{Res}_{G/H,\{1\}}(\mathbf{R}^+\Gamma_{\text{der}}(G, G/H, M)) \longrightarrow \mathbf{R}^+\Gamma_{\text{der}}(H, \text{Res}_{G,H}(M))$$

in $D^+({}_R\text{Mod})$. This morphism does not depend on the choice of $J^\bullet$ and has the required functoriality properties.

(ii) It is sufficient to find G, H and M such that $H^1_{\text{der}}(H, M) \xrightarrow{\sim} H^1_{\text{cont}}(H, M)$ is *not* an ind-admissible $R[G/H]$-module (since $R^1\Gamma_{\text{der}}(G, G/H, M)$ is ind-admissible). For example, let $p > 2$ be a prime number, $R = \mathbf{Z}_p$, $K = \mathbf{Q}_p(\mu_p)$, $K_\infty = \mathbf{Q}_p(\mu_{p^\infty})$, $G = \text{Gal}(\overline{K}/K)$, $H = \text{Gal}(\overline{K}/K_\infty)$, $\Gamma = G/H = \text{Gal}(K_\infty/K) \xrightarrow{\sim} \mathbf{Z}_p$, $M = \mathbf{Z}_p(1) = \varprojlim_n \mu_{p^n}(\overline{K})$. In this case

$$H^1_{\text{cont}}(H, M) = \varprojlim_n H^1\big(\text{Gal}\big(\overline{K}/K_\infty\big), \mu_{p^n}\big) = \varprojlim_n (K_\infty^* \otimes \mathbf{Z}/p^n\mathbf{Z})$$

is a non-torsion module over the Iwasawa algebra $\mathbf{Z}_p[\![\Gamma]\!]$, hence $H^1_{\text{cont}}(H, M)$ is not an ind-admissible $\mathbf{Z}_p[\Gamma]$-module. □

3.6.5. The whole point of Proposition 3.6.4 (ii) is that the forgetful functor $\text{Res}_{G,H}$ need *not* preserve injectives. Such a pathological behaviour never occurs for discrete groups and modules, when $\text{Res}_{G,H}$ has an exact left adjoint $\mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} (-)$. This point is usually glossed over in standard treatments of the Hochschild-Serre spectral sequence, such as [**We**, § 6.8.2].

3.7. Localization

3.7.1. Let $R_{\mathscr{S}} = \mathscr{S}^{-1}R$ be the localization of R at a multiplicative subset $\mathscr{S} \subset R$. Put

$$\omega_{R_{\mathscr{S}}} = \mathscr{S}^{-1}\omega_R = \omega_R \otimes_R R_{\mathscr{S}} = \omega_R \overset{\mathbf{L}}{\otimes}_R R_{\mathscr{S}};$$

this is an object of $D^b\big({}_{R_{\mathscr{S}}}\text{Mod}\big)$ which can be represented by a bounded complex of injective $R_{\mathscr{S}}$-modules $\omega_R^\bullet \otimes_R R_{\mathscr{S}}$ (the localization $M \mapsto \mathscr{S}^{-1}M$ preserves injectives). Define

$$\mathscr{D}_{R_{\mathscr{S}}}(-) = \mathbf{R}\text{Hom}_{R_{\mathscr{S}}}(-, \omega_{R_{\mathscr{S}}});$$

then the canonical map

$$\varepsilon : M \longrightarrow \mathscr{D}_{R_{\mathscr{S}}}(\mathscr{D}_{R_{\mathscr{S}}}(M))$$

is an isomorphism in $D_{ft}({}_{R_{\mathscr{S}}}\mathrm{Mod})$ for every $M \in D_{ft}({}_{R_{\mathscr{S}}}\mathrm{Mod})$. We have

$$\mathscr{S}^{-1}\mathbf{R}\mathrm{Hom}_R(X, Y) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{R_{\mathscr{S}}}(\mathscr{S}^{-1}X, \mathscr{S}^{-1}Y)$$

for $X \in D^-_{ft}({}_R\mathrm{Mod})$, $Y \in D^+({}_R\mathrm{Mod})$ (resp., $X \in D_{ft}({}_R\mathrm{Mod})$, $Y \in D^b(\mathrm{inj} - {}_R\mathrm{Mod})$).

3.7.2. Definition. — An $R_{\mathscr{S}}[G]$-module M is admissible (resp., ind-admissible) if it is admissible (resp., ind-admissible) as an $R[G]$-module.

Admissible (resp., ind-admissible) $R_{\mathscr{S}}[G]$-modules form a full subcategory $\big({}^{\mathrm{ad}}_{R_{\mathscr{S}}[G]}\mathrm{Mod}\big)$ of $\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$ (resp., $\big({}^{\text{ind-ad}}_{R_{\mathscr{S}}[G]}\mathrm{Mod}\big)$ of $\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$). This notation is slightly ambiguous; *a priori*, these categories depend not only on $R_{\mathscr{S}}$, but also on R.

3.7.3. Lemma. — *If $M \in \big({}^{\text{ind-ad}}_{R_{\mathscr{S}}[G]}\mathrm{Mod}\big)$ is of finite type over $R_{\mathscr{S}}$, then $M \xrightarrow{\sim} \mathscr{S}^{-1}N$ for some $N \in \big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)_{R\text{-}ft} = \big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)_{R\text{-}ft}$.*

Proof. — The $R_{\mathscr{S}}$-module

$$M = \varinjlim_{M_\alpha \in \mathcal{S}(M)} M_\alpha = \varinjlim_{M_\alpha \in \mathcal{S}(M)} \mathscr{S}^{-1}M_\alpha$$

is Noetherian; thus $M = \mathscr{S}^{-1}N$ for some $N = M_\alpha \in \mathcal{S}(M)$. □

3.7.4. Proposition

(i) *All statements of Lemma 3.2.2, Corollary 3.2.3, Proposition 3.2.5–3.2.6, 3.4.4 hold if R is replaced by $R_{\mathscr{S}}$.*

(ii) *If $M \in D^+\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$, then the canonical map*

$$\mathscr{S}^{-1}\mathbf{R}\Gamma_{\mathrm{cont}}(G, M) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G, \mathscr{S}^{-1}M)$$

is an isomorphism in $D^+\big({}_{R_{\mathscr{S}}}\mathrm{Mod}\big)$.

Proof. — (i) is easy; (ii) follows from Proposition 3.4.4 applied to $N = R_{\mathscr{S}}$. □

CHAPTER 4

CONTINUOUS COHOMOLOGY OF PRO-FINITE GROUPS

This chapter treats basic finiteness properties of continuous cohomology of admissible $R[G]$-modules in the case when G is a pro-finite group. Section 4.4 can be ignored; it is unrelated to the rest of the article.

4.1. Basic properties

Throughout Chapter 4, G is a *pro-finite group*, *i.e.*,

$$G = \varprojlim_{U} G/U,$$

where U runs through all open normal subgroups of G (they are all of finite index in G).

4.1.1. Recall from Lemma 3.2.4 that an $R[G]$-module T (resp., A) of finite (resp., co-finite) type over R is admissible if and only if G acts continuously on T equipped with $\mathfrak{m}$-adic topology (resp., on A equipped with discrete topology). Assuming this is the case, each of the G-modules $M := T/\mathfrak{m}^n T, A[\mathfrak{m}^n], A$ is discrete, which means that

$$C^\bullet_{\mathrm{cont}}(G, M) = \varinjlim_{U} C^\bullet(G/U, M^U) = C^\bullet(G, M)$$

is the usual complex of locally constant cochains, hence

$$H^i_{\mathrm{cont}}(G, M) = H^i(G, M) \qquad (i \geqslant 0).$$

4.1.2. Lemma. — *The canonical maps*

$$\varinjlim_{n} C^\bullet_{\mathrm{cont}}(G, A[\mathfrak{m}^n]) \longrightarrow C^\bullet_{\mathrm{cont}}(G, A)$$

$$C^\bullet_{\mathrm{cont}}(G, T) \longrightarrow \varprojlim_{n} C^\bullet_{\mathrm{cont}}(G, T/\mathfrak{m}^n T)$$

are isomorphisms of complexes.

Proof. — This follows from the definitions. □

4.1.3. Corollary. — *For each $i \geqslant 0$ there are canonical isomorphisms*

$$\varinjlim_n H^i(G, A[\mathfrak{m}^n]) \xrightarrow{\sim} H^i(G, A) \xrightarrow{\sim} H^i_{\mathrm{cont}}(G, A)$$

and an exact sequence

$$0 \longrightarrow \varprojlim_n{}^{(1)} H^{i-1}(G, T/\mathfrak{m}^n T) \longrightarrow H^i_{\mathrm{cont}}(G, T) \longrightarrow \varprojlim_n H^i(G, T/\mathfrak{m}^n T) \longrightarrow 0.$$

Proof. — $\varinjlim$ is an exact functor. As regards $\varprojlim$, the projective system of complexes $n \mapsto C^\bullet_{\mathrm{cont}}(G, T/\mathfrak{m}^n T)$ is Mittag-Leffler (in fact surjective, by Proposition 3.4.2), so the usual "universal coefficient theorem" ([**We**, §3.5.8]) applies. □

4.1.4. Lemma. — *Assume that* $\mathrm{char}(k) = p > 0$ *and that $H \lhd G$ is a closed normal subgroup of G, with pro-finite order prime to p. Then the inflation map*

$$\mathrm{inf} : C^\bullet_{\mathrm{cont}}(G/H, M^H) \longrightarrow C^\bullet_{\mathrm{cont}}(G, M)$$

is a quasi-isomorphism.

Proof. — The inflation map is induced by the pair of morphisms $G \to G/H$, $M^H \hookrightarrow M$ (using Lemma 3.3.6 (ii)). According to 3.5.1.2 we can assume that $M = T$ is of finite type over R. Corollary 4.1.3 further reduces to the case of $M = T/\mathfrak{m}^n T$ of finite length over R. In this case M is a p-primary torsion discrete G-module and the statement follows from the degeneration of the Hochschild-Serre spectral sequence

$$E_2^{i,j} = H^i(G/H, H^j(H, M)) \Longrightarrow H^{i+j}(G, M) \tag{4.1.4.1}$$

($E_2^{i,j} = 0$ for $j \neq 0$). □

4.2. Finiteness conditions

4.2.1. Consider the following finiteness conditions on G:

(F) $\ell_R(H^i(G, M)) < \infty$ for every discrete $R[G]$-module M of finite length over R and every $i \geqslant 0$.

(F') $\dim_k H^i(U, k) < \infty$ for every open normal subgroup $U \lhd G$ and every $i \geqslant 0$.

By Shapiro's Lemma, (F) for G implies (F) for every open subgroup of G; in particular (F) implies (F').

4.2.2. Lemma. — *We have implications*

$$(F') \iff (F) \Longrightarrow H^i_{\mathrm{cont}}(G, T) \xrightarrow{\sim} \varprojlim_n H^i(G, T/\mathfrak{m}^n T) \qquad (\forall i \geqslant 0).$$

Proof. — Assume (F') holds. Given M as in (F), there is an open normal subgroup $U \lhd G$ acting trivially on M. Then (F) follows from the Hochschild-Serre spectral sequence (4.1.4.1), as G/U is finite and $\ell_R(H^j(U, M)) < \infty$ by (F') and dévissage. The converse is true by Shapiro's Lemma, as observed in 4.2.1.

Assuming that (F) holds, the $\varprojlim^{(1)}$-term in Corollary 4.1.3 vanishes, as $n \mapsto H^{i-1}(G, T/\mathfrak{m}^n T)$ is a Mittag-Leffler system. $\square$

4.2.3. Proposition. — *If G satisfies (F), then $H^i_{\mathrm{cont}}(G,T)$ (resp., $H^i_{\mathrm{cont}}(G,A)$) is of finite (resp., co-finite) type over R for every $i \geqslant 0$.*

Proof. — Induction on $d = \dim(R)$. There is nothing to prove for $d = 0$. If $d \geqslant 1$, choose $x \in \mathfrak{m}$ such that $\dim(R/xR) = d - 1$. The R-module $M = H^i_{\mathrm{cont}}(G,T) \xrightarrow{\sim} \varprojlim_n M_n$ is the projective limit of a surjective projective system of R-modules of finite length $M_n = \mathrm{Im}(M \to H^i(G, T/\mathfrak{m}^n T))$ satisfying $\mathfrak{m}^n M_n = 0$. Denote by $j_n : M \twoheadrightarrow M_n$ the canonical projection. The exact cohomology sequences of

$$0 \longrightarrow T[x] \longrightarrow T \xrightarrow{x} xT \longrightarrow 0$$

$$0 \longrightarrow xT \longrightarrow T \longrightarrow T/xT \longrightarrow 0$$

(valid by Proposition 3.4.2) together with the induction hypothesis show that M/xM is of finite type over R/xR. Fix an epimorphism $(R/xR)^a \twoheadrightarrow M/xM$ and lift it to a homomorphism of R-modules $f : R^a \to M$. Put $N = \mathrm{Coker}(f)$, $N_n = \mathrm{Coker}(j_n \circ f)$, $K_n = \mathrm{Ker}(j_n \circ f)$. The projective system N_n/xN_n consists of R-modules of finite length, has surjective transition maps and its projective limit

$$\varprojlim_n (N_n/xN_n) = \Big(\varprojlim_n N_n\Big)/x\Big(\varprojlim_n N_n\Big) = 0$$

vanishes, being a quotient of $N/xN = 0$. It follows that, for all n, $N_n/xN_n = 0$, hence $N_n = 0$ by Nakayama's Lemma. The projective systems of exact sequences

$$0 \longrightarrow \mathfrak{m}^n R^a \longrightarrow K_n \longrightarrow K_n/\mathfrak{m}^n R^a \longrightarrow 0, \quad 0 \longrightarrow \mathfrak{m}^n R^a \longrightarrow R^a \longrightarrow (R/\mathfrak{m}^n)^a \longrightarrow 0$$

imply that

$$\varprojlim_n{}^{(1)} K_n = \varprojlim_n{}^{(1)} \mathfrak{m}^n R^a = \varprojlim_n{}^{(1)} R^a = 0,$$

and the exact sequence

$$0 \longrightarrow K_n \longrightarrow R^a \longrightarrow M_n \longrightarrow 0$$

yields

$$R^a \longrightarrow \varprojlim_n M_n (= M) \longrightarrow \varprojlim_n{}^{(1)} K_n = 0,$$

proving that M is of finite type over R. Dually, $P = H^i(G,A) \xrightarrow{\sim} \varinjlim_n P_n$, where $P_n = \mathrm{Im}(H^i(G, A[\mathfrak{m}^n]) \to P)$ is an injective inductive system, and $P[x]$ is of co-finite type over R/xR. Fixing a monomorphism $P[x] \hookrightarrow (I[x])^b$ and extending it to a homomorphism of R-modules $g : P \to I^b$, the vanishing of

$$\varinjlim_n (P_n \cap \mathrm{Ker}(g))[\mathfrak{m}] \subset \varinjlim_n (P_n \cap \mathrm{Ker}(g))[x] \subset \mathrm{Ker}(g)[x] = 0$$

implies that all maps $P_n \hookrightarrow P \to I^b$ are injective; thus $g : P \to I^b$ is also injective, as required. $\square$

4.2.4. Lemma. — *If G satisfies (F), then the canonical maps*

$$\varinjlim_n D\big(C^\bullet_{\mathrm{cont}}(G, T/\mathfrak{m}^n T)\big) \xrightarrow{u} D\Big(\varprojlim_n C^\bullet_{\mathrm{cont}}(G, T/\mathfrak{m}^n T)\Big)$$

$$D\Big(\varinjlim_n C^\bullet_{\mathrm{cont}}\big(G, A[\mathfrak{m}^n]\big)\Big) \xrightarrow{v} \varprojlim_n D\Big(C^\bullet_{\mathrm{cont}}(G, A[\mathfrak{m}^n])\Big)$$

are quasi-isomorphisms.

Proof. — The induced map on cohomology $H^{-i}(u)$ is equal to the composition

$$\varinjlim_n D\big(H^i(G, T/\mathfrak{m}^n T)\big) \xrightarrow{u_1} D\Big(\varprojlim_n H^i(G, T/\mathfrak{m}^n T)\Big) \xrightarrow{u_2} D\big(H^i_{\mathrm{cont}}(G, T)\big)$$

The map u_1 (resp., u_2) is an isomorphism by a combination of 2.3.4 and Proposition 4.2.3 (resp., by Lemma 4.2.2). Similarly, the composition of

$$D(H^i(G, A)) \xrightarrow{H^{-i}(v)} H^{-i}\Big(\varprojlim_n D\big(C^\bullet_{\mathrm{cont}}(G, A[\mathfrak{m}^n])\big)\Big) \xrightarrow{v_2} \varprojlim_n D\big(H^i\big(G, A[\mathfrak{m}^n]\big)\big)$$

is an isomorphism by 2.3.4 and v_2 is an isomorphism by the argument used in the proof of the second implication in Lemma 4.2.2. □

4.2.5. Proposition. — *If G satisfies (F), then the functor $M \mapsto \mathbf{R}\Gamma_{\mathrm{cont}}(G, M)$ maps $D^+_{R\text{-}ft}\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$ (resp., $D^+_{R\text{-}coft}\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$) to $D^+_{ft}({}_R\mathrm{Mod})$ (resp., $D^+_{coft}({}_R\mathrm{Mod})$).*

Proof. — For $M = T$ or A this is the statement of Proposition 4.2.3. The general case follows from the hyper-cohomology spectral sequence (3.5.4.2). □

4.2.6. Lemma. — *If* $\mathrm{char}(k) = p > 0$ *and* $\mathrm{cd}_p(G) = e < \infty$, *then*

(i) $H^i_{\mathrm{cont}}(G, M) = 0$ *for every* $i > e$ *and every ind-admissible* $R[G]$*-module* M.

(ii) *If* $M^\bullet$ *is a bounded below complex of ind-admissible* $R[G]$*-modules with* $H^i(M^\bullet) = 0$ *for* $i > c$, *then* $H^j_{\mathrm{cont}}(G, M^\bullet) = 0$ *for every* $j > c + e$.

Proof

(i) By 3.5.1.2 we can assume that $M = T$ is of finite type over R. It follows from Corollary 4.1.3 that $H^i_{\mathrm{cont}}(G, M) = 0$ for $i > e + 1$. For $i = e + 1$ we have

$$H^{e+1}_{\mathrm{cont}}(G, M) = \varprojlim_n{}^{(1)} H^e(G, T/\mathfrak{m}^n T) = 0,$$

since $n \mapsto H^e(G, T/\mathfrak{m}^n T)$ is a surjective projective system.

(ii) This follows from (i) and the spectral sequence (3.5.4.2). □

4.2.7. Corollary. — *Under the assumptions of 4.2.6,*

(i) *The functor*

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G, -) : D^+\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big) \longrightarrow D^+({}_R\mathrm{Mod})$$

maps $D^b\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$ *to* $D^b({}_R\mathrm{Mod})$.

(ii) *If, in addition, G satisfies (F), then $\mathbf{R}\Gamma_{\mathrm{cont}}(G, -)$ maps $D^*_{R\text{-}ft}\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$ (resp., $D^*_{R\text{-}coft}\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$) to $D^*_{ft}({}_R\mathrm{Mod})$ (resp., $D^*_{coft}({}_R\mathrm{Mod})$) for $* = +, b$.*

Proof. — Combine Proposition 4.2.5 and Lemma 4.2.6. □

4.2.8. Perfect complexes. — Let A be a Noetherian ring. Recall ([**SGA6**, Exp. I, Cor. 5.8.1]) that a complex $M^\bullet$ of A-modules is *perfect* (*i.e.*, there exists a quasi-isomorphism $P^\bullet \to M^\bullet$, where $P^\bullet$ is a bounded complex of projective A-modules of finite type) iff the following conditions are satisfied:

(a) $(\forall i \in \mathbf{Z})\quad H^i(M^\bullet)$ is of finite type over A.
(b) $H^i(M^\bullet) = 0$ for all but finitely many $i \in \mathbf{Z}$.
(c) The complex $M^\bullet$ has finite Tor-dimension, *i.e.*,

$$(\exists c \in \mathbf{Z})\,(\forall N \in ({}_A\mathrm{Mod}))\,(\forall i > c)\quad \mathbb{T}\mathrm{or}_i^A(M^\bullet, N) = 0.$$

Perfect complexes over A form a full subcategory $D_{\mathrm{parf}}({}_A\mathrm{Mod})$ of $D^b({}_A\mathrm{Mod})$. A theorem of Serre and Auslander-Buchsbaum ([**Br-He**, Thm. 2.2.7]) implies that, for our ring R,

$$R \text{ is regular } \iff D_{\mathrm{parf}}({}_R\mathrm{Mod}) = D^b_{ft}({}_R\mathrm{Mod}).$$

One says that $M^\bullet \in D_{\mathrm{parf}}({}_A\mathrm{Mod})$ has *perfect amplitude contained in* $[a,b]$ (notation: $M^\bullet \in D^{[a,b]}_{\mathrm{parf}}({}_A\mathrm{Mod})$) if the complex $P^\bullet$ above can be chosen in such a way that $P^i = 0$ for every $i < a$ and $i > b$. If this is the case, then

$$M^\bullet \in D^{[a,b-1]}_{\mathrm{parf}}({}_A\mathrm{Mod}) \iff H^b(M^\bullet) = 0;$$

more generally,

$$D^{[a,b]}_{\mathrm{parf}}({}_A\mathrm{Mod}) \cap D^{[c,d]}_{\mathrm{parf}}({}_A\mathrm{Mod}) = D^{[\max(a,c),\min(b,d)]}_{\mathrm{parf}}({}_A\mathrm{Mod})$$

([**SGA6**, Exp. I, Lemma 4.13]). The functor $\mathbf{R}\mathrm{Hom}_A(-, A)$ maps $D^{[a,b]}_{\mathrm{parf}}({}_A\mathrm{Mod})$ into $D^{[-b,-a]}_{\mathrm{parf}}({}_A\mathrm{Mod})$.

4.2.9. Proposition. — *Assume that G satisfies (F), $\mathrm{char}(k) = p > 0$ and $\mathrm{cd}_p(G) = e < \infty$. Let $\mathscr{S}$ be a multiplicative subset of R and $R_{\mathscr{S}}$ the corresponding localization. If $M^\bullet$ is a bounded complex of ind-admissible $R[G]$-modules such that $M^\bullet \otimes_R R_{\mathscr{S}} \in D^{[a,b]}_{\mathrm{parf}}({}_{R_{\mathscr{S}}}\mathrm{Mod})$ (if we disregard the G-action), then $\mathbf{R}\Gamma_{\mathrm{cont}}(G, M^\bullet) \otimes_R R_{\mathscr{S}} \in D^{[a,b+e]}_{\mathrm{parf}}({}_{R_{\mathscr{S}}}\mathrm{Mod})$.*

Proof. — It follows from Corollary 4.2.7 that $\mathbf{R}\Gamma_{\mathrm{cont}}(G, M^\bullet)$ is an object of $D^b_{ft}({}_R\mathrm{Mod})$; it remains to verify that $\mathbf{R}\Gamma_{\mathrm{cont}}(G, M^\bullet) \otimes_R R_{\mathscr{S}}$ has finite Tor-dimension over $R_{\mathscr{S}}$. As explained to us by O. Gabber, this follows by a standard "way-out" argument from the fact that $\mathbf{R}\Gamma_{\mathrm{cont}}(G, -)$ commutes with filtered direct limits: let $N^\bullet$ be a bounded complex of $R_{\mathscr{S}}$-modules; consider the canonical map

$$\lambda_{N^\bullet} : \big(C^\bullet_{\mathrm{cont}}(G, M^\bullet) \otimes_R R_{\mathscr{S}}\big) \otimes_{R_{\mathscr{S}}} N^\bullet \longrightarrow C^\bullet_{\mathrm{cont}}\big(G, \big(M^\bullet \otimes_R R_{\mathscr{S}}\big) \otimes_{R_{\mathscr{S}}} N^\bullet\big).$$

As $R_{\mathscr{S}}$ is flat over R, it follows from Proposition 3.4.4 that $\lambda_{N^\bullet}$ is an isomorphism of complexes whenever $N^\bullet$ is a complex of flat $R_{\mathscr{S}}$-modules. Given an arbitrary $R_{\mathscr{S}}$-module N, choose its resolution $F^\bullet$ by free $R_{\mathscr{S}}$-modules:

$$\cdots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow N \longrightarrow 0.$$

Fix $k >> 0$ and consider the truncated complex

$$F_k^\bullet := (\sigma_{\geqslant -k} F^\bullet) : F^{-k} \longrightarrow \cdots \longrightarrow F^0;$$

it satisfies

$$(\forall j < k-b) \quad H^{-j}((M^\bullet \otimes_R R_{\mathscr{S}}) \otimes_{R_{\mathscr{S}}} F_k^\bullet) = \mathbb{T}\mathrm{or}_j^{R_{\mathscr{S}}}(M^\bullet \otimes_R R_{\mathscr{S}}, N)$$

$$(\forall \ell < k-b) \quad H^{-\ell}((C^\bullet_{\mathrm{cont}}(G, M^\bullet) \otimes_R R_{\mathscr{S}}) \otimes_{R_{\mathscr{S}}} F_k^\bullet) = \mathbb{T}\mathrm{or}_\ell^{R_{\mathscr{S}}}(C^\bullet_{\mathrm{cont}}(G, M^\bullet) \otimes_R R_{\mathscr{S}}, N).$$

The cohomology of the bounded complex (of ind-admissible $R[G]$-modules)

$$B^\bullet = (M^\bullet \otimes_R R_{\mathscr{S}}) \otimes_{R_{\mathscr{S}}} F_k^\bullet$$

satisfies

$$(\forall j > b-k) \quad \big[H^j(B^\bullet) \neq 0 \Longrightarrow a \leqslant j \leqslant b\big].$$

It follows from Lemma 4.2.6 and the hyper-cohomology spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{cont}}(G, H^j(B^\bullet)) \Longrightarrow H^{i+j}_{\mathrm{cont}}(G, B^\bullet)$$

that

$$E_2^{i,j} \neq 0 \Longrightarrow [0 \leqslant i \leqslant e \text{ and } (a \leqslant j \leqslant b \text{ or } j \leqslant b-k)].$$

Using the fact that $\lambda_{F_k^\bullet}$ is an isomorphism, we get

$$(\forall \ell < k-b-e)$$
$$\Big[\mathbb{T}\mathrm{or}_\ell^{R_{\mathscr{S}}}(C^\bullet_{\mathrm{cont}}(G, M^\bullet) \otimes_R R_{\mathscr{S}}, N) = H^{-\ell}_{\mathrm{cont}}(G, B^\bullet) \neq 0 \Longrightarrow -b-e \leqslant \ell \leqslant -a\Big].$$

This finishes the proof that $C^\bullet_{\mathrm{cont}}(G, M^\bullet) \otimes_R R_{\mathscr{S}}$ has perfect amplitude contained in $[a, b+e]$, since k can be chosen arbitrarily large. □

4.2.10. Proposition. — *Let $\mathscr{S} \subset R$ be a multiplicative subset. If G satisfies (F), then $\mathbf{R}\Gamma_{\mathrm{cont}}(G, -)$ maps $D^+_{R_{\mathscr{S}}\text{-ft}}\big({}^{\text{ind-ad}}_{R_{\mathscr{S}}[G]}\mathrm{Mod}\big)$ to $D^+_{ft}\big({}_{R_{\mathscr{S}}}\mathrm{Mod}\big)$. If, in addition, $\mathrm{char}(k) = p > 0$ and $\mathrm{cd}_p(G) < \infty$, then $\mathbf{R}\Gamma_{\mathrm{cont}}(G, -)$ maps $D^b_{R_{\mathscr{S}}\text{-ft}}\big({}^{\text{ind-ad}}_{R_{\mathscr{S}}[G]}\mathrm{Mod}\big)$ to $D^b_{ft}\big({}_{R_{\mathscr{S}}}\mathrm{Mod}\big)$.*

Proof. — Combine Lemma 3.7.3, Proposition 3.7.4(ii), Proposition 4.2.5 and Corollary 4.2.7. □

4.3. The duality diagram: T, A, T^*, A^*

4.3.1. Let $T \in D^+_{R\text{-}ft}\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$; put $A = \Phi(T) \in D^+_{R\text{-}coft}\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$. Proposition 3.5.8 then implies that the canonical map

$$\Phi(\mathbf{R}\Gamma_{\mathrm{cont}}(G,T)) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G,A) \tag{4.3.1.1}$$

is an isomorphism in $D^+({}_R\mathrm{Mod})$.

If we assume, in addition, that G satisfies (F), then $\mathbf{R}\Gamma_{\mathrm{cont}}(G,T)$ (resp., $\mathbf{R}\Gamma_{\mathrm{cont}}(G,A)$) lies in $D^+_{ft}({}_R\mathrm{Mod})$ (resp., $D^+_{coft}({}_R\mathrm{Mod})$). Combining the spectral sequence (2.8.6.1) with the isomorphism (4.3.1.1) we get a spectral sequence

$$E_2^{i,j} = \mathbb{E}\mathrm{xt}^i_R(D(H^j_{\mathrm{cont}}(G,A)),\omega) \Longrightarrow H^{i+j}_{\mathrm{cont}}(G,T). \tag{4.3.1.2}$$

4.3.2. The construction from 3.5.9 defines functors

$$\mathscr{D}(-) = \mathbf{R}\mathrm{Hom}_R(-,\omega) \;:\; D_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)^{\mathrm{op}} \longrightarrow D_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$$

$$D(-) = \mathbf{R}\mathrm{Hom}_R(-,I) \;:\; D_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)^{\mathrm{op}} \longleftrightarrow D_{R\text{-}coft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$$

which map $D^{\pm}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$ to $D^{\mp}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$ (hence $D^b\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$ to $D^b\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$). Together with Φ these functors define a duality diagram

$$\begin{array}{ccc}
D_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)^{\mathrm{op}} & \xleftrightarrow{\;\mathscr{D}\;} & D_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big) \\
\Big\downarrow \Phi & \overset{D}{\times} & \Big\downarrow \Phi \\
D_{R\text{-}coft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)^{\mathrm{op}} & & D_{R\text{-}coft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big),
\end{array}$$

commutative up to functorial isomorphisms defined by the same formulas as in 2.8.1. This diagram makes sense for an arbitrary topological group G, not necessarily profinite.

4.3.3. Proposition. — *Let $T \in D^+_{R\text{-}ft}\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$; put $A = \Phi(T)$. Assume that G satisfies (F). Then there are spectral sequences*

$$E_2^{i,j} = H^{i+d}_{\{\mathfrak{m}\}}(H^j_{\mathrm{cont}}(G,T)) \Longrightarrow H^{i+j}_{\mathrm{cont}}(G,A)$$

$$'E_2^{i,j} = H^{i+d}_{\{\mathfrak{m}\}}\big(D(H^{-j}_{\mathrm{cont}}(G,A))\big) \Longrightarrow D(H^{-i-j}_{\mathrm{cont}}(G,T)).$$

Proof. — Apply Proposition 2.8.7 to $\mathbf{R}\Gamma_{\mathrm{cont}}(G,T)$, $\mathbf{R}\Gamma_{\mathrm{cont}}(G,A)$ instead of T, A (which is legitimate by 4.3.1). Of course, the spectral sequence $'E_r$ is just D(4.3.1.2). □

4.3.4. In the special case when $R = \mathbf{Z}_p$ and T is free over $\mathbf{Z}_p$, the spectral sequence $E_2^{i,j}$ degenerates (assuming (F)) into a short exact sequence

$$0 \longrightarrow H^j_{\mathrm{cont}}(G,T) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow H^j_{\mathrm{cont}}(G,A) \longrightarrow H^{j+1}_{\mathrm{cont}}(G,T)_{\mathrm{tors}} \longrightarrow 0, \tag{4.3.4.1}$$

which coincides – up to a sign – with a piece of the cohomology sequence of

$$0 \longrightarrow T \longrightarrow V \longrightarrow A \longrightarrow 0,$$

where $V = T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

4.3.5. Applying 2.8.6 and 2.8.9 to $\mathbf{R}\Gamma_{\mathrm{cont}}(G,T)$, $\mathbf{R}\Gamma_{\mathrm{cont}}(G,A)$ instead of T, A (again assuming (F)) we obtain exact sequences in $({}_R\mathrm{Mod})/(\text{pseudo-null})$

$$0 \longrightarrow \mathbb{E}\mathrm{xt}^1_R(D(H^{j-1}_{\mathrm{cont}}(G,A)),\omega) \longrightarrow H^j_{\mathrm{cont}}(G,T) \longrightarrow \mathbb{E}\mathrm{xt}^0_R(D(H^j_{\mathrm{cont}}(G,A)),\omega) \longrightarrow 0$$

resp., in $({}_R\mathrm{Mod})/(\text{co-pseudo-null})$

$$0 \longrightarrow H^d_{\{\mathfrak{m}\}}(H^j_{\mathrm{cont}}(G,T)) \longrightarrow H^j_{\mathrm{cont}}(G,A) \longrightarrow H^{d-1}_{\{\mathfrak{m}\}}(H^{j+1}_{\mathrm{cont}}(G,T)) \longrightarrow 0$$

$$0 \longrightarrow H^d_{\{\mathfrak{m}\}}\big(D(H^j_{\mathrm{cont}}(G,A))\big) \longrightarrow D(H^j_{\mathrm{cont}}(G,T)) \longrightarrow H^{d-1}_{\{\mathfrak{m}\}}\big(D(H^{j-1}_{\mathrm{cont}}(G,A))\big) \longrightarrow 0,$$

generalizing (4.3.4.1). Again, $H^d_{\{\mathfrak{m}\}}(H^j_{\mathrm{cont}}(G,T))$ (resp., $\mathbb{E}\mathrm{xt}^1_R(D(H^{j-1}_{\mathrm{cont}}(G,A)),\omega)$) is the maximal R-divisible (resp., R-torsion) subobject of $H^j_{\mathrm{cont}}(G,A)$ (resp., $H^j_{\mathrm{cont}}(G,T)$) in $({}_R\mathrm{Mod})/(\text{co-pseudo-null})$ (resp., $({}_R\mathrm{Mod})/(\text{pseudo-null})$).

4.4. Comparing $\mathbf{R}^+\Gamma_{\mathrm{der}}$ and $\mathbf{R}\Gamma_{\mathrm{cont}}$

4.4.1. Proposition. — *If* $\mathrm{cd}_p(G) \leqslant 1$, *then* $\theta_G(M) : \mathbf{R}^+\Gamma_{\mathrm{der}}(G,M) \to \mathbf{R}\Gamma_{\mathrm{cont}}(G,M)$ *is an isomorphism for every* $M \in \big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$.

Proof. — The functors $H^i_{\mathrm{cont}}(G,-)$ are effaceable for $i = 1$ (resp., $i > 1$) by Proposition 3.6.2 (iv) (resp., because they are zero, by Lemma 4.2.6 (i)). The claim follows from Proposition 3.6.2 (iii) by induction. □

4.4.2. Definition. — Denote by $\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$ (resp., $\big({}_{R[G]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$) the full subcategory of $\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$ (resp., of $\big({}_{R[G]}\mathrm{Mod}\big)$) consisting of objects M satisfying $M = \bigcup_{n \geqslant 1} M[\mathfrak{m}^n]$.

4.4.3. For a given $M \in \big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$, every $M_\alpha \in \mathcal{S}(M)$ is an R-module of finite length, hence it is discrete in the $\mathfrak{m}$-adic topology. This implies that the normal subgroup $U = \mathrm{Ker}(G \to \mathrm{Aut}(M_\alpha)) \lhd G$, which acts trivially on M_α, is open in G, hence M is a discrete G-module and $C^\bullet_{\mathrm{cont}}(G,M)$ is the usual complex of locally constant cochains. Using the language of [**Bru**], $\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$ is the category of discrete modules over the pseudo-compact R-algebra

$$R[\![G]\!] = \varprojlim_U R[G/U].$$

As in Proposition 3.3.5 (ii) there is an adjoint pair of functors i', j', where

$$i' : \big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}} \longrightarrow \big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$$

is the (exact) embedding functor and $j'(M) = \bigcup_{n\geqslant 1} M[\mathfrak{m}^n]$. As in Proposition 3.3.5, j' preserves injectives and $\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)_{\{\mathfrak{m}\}}$ has enough injectives of the form $j'(J)$, where J is injective in $\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$. The following statement is *not* an abstract nonsense.

4.4.4. Proposition. — *The embedding functor $i' : \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)_{\{\mathfrak{m}\}} \to \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$ preserves injectives.*

Proof. — Let J be injective in $\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)_{\{\mathfrak{m}\}}$. We must show that for every diagram in $\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$ with exact row

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \xrightarrow{u} & Y \\ & & \Big\downarrow{\scriptstyle f} & & \\ & & J & & \end{array}$$

there is a morphism $g : Y \to J$ such that $f = gu$. A standard argument using Zorn's Lemma reduces the problem to the case when X and Y are of finite type over R. In this case f factors through $X/\mathfrak{m}^n X$ for suitable n. By Artin-Rees Lemma, there is $k \geqslant 0$ such that

$$\mathrm{Ker}(u_{n+k} : X/\mathfrak{m}^{n+k}X \longrightarrow Y/\mathfrak{m}^{n+k}Y)$$

maps to zero in $X/\mathfrak{m}^n X$. This implies that the projection $X/\mathfrak{m}^{n+k} \to X/\mathfrak{m}^n X$ factors through $\mathrm{Im}(u_{n+k})$. In the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \xrightarrow{u} & Y \\ & & \Big\downarrow & & \Big\downarrow \\ 0 & \longrightarrow & \mathrm{Im}(u_{n+k}) & \longrightarrow & Y/\mathfrak{m}^{n+k}Y \\ & & \Big\downarrow & & \\ & & X/\mathfrak{m}^n X & & \\ & & \Big\downarrow & & \\ & & J & & \end{array}$$

both $\mathrm{Im}(u_{n+k})$ and $Y/\mathfrak{m}^{n+k}Y$ are objects of $\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)_{\{\mathfrak{m}\}}$; it follows that there is $h : Y/\mathfrak{m}^{n+k}Y \to J$ extending $\mathrm{Im}(u_{n+k}) \to J$. The composition $g : Y \to Y/\mathfrak{m}^{n+k}Y \xrightarrow{h} J$ satisfies $gu = f$ as required. □

4.4.5. Lemma. — *If G is a finite group, then*

(i) $\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right) = \left({}_{R[G]}\mathrm{Mod}\right)$.

(ii) $C^\bullet_{\mathrm{cont}}(G, M) = C^\bullet(G, M)$, $H^i_{\mathrm{cont}}(G, M) = H^i(G, M)$ $\left(M \in \left({}_{R[G]}\mathrm{Mod}\right), i \geqslant 0\right)$.

(iii) $(\forall i \geqslant 1)$ $H^i_{\mathrm{cont}}(G, -)$ *is effaceable in* $\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right) = \left({}_{R[G]}\mathrm{Mod}\right)$.

(iv) $(\forall i \geqslant 1)$ $H^i_{\mathrm{cont}}(G, -)$ *is effaceable in* $\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)_{\{\mathfrak{m}\}} = \left({}_{R[G]}\mathrm{Mod}\right)_{\{\mathfrak{m}\}}$.

(v) $(\forall i \geqslant 1)$ $H^i_{\mathrm{cont}}(G, -)$ *vanishes on injective objects of* $\left({}_{R[G]}\mathrm{Mod}\right)$ *resp.*, $\left({}_{R[G]}\mathrm{Mod}\right)_{\{\mathfrak{m}\}}$.

Proof. — The statements (i), (ii) follow from the definitions. As regards (iii) and (iv), every $M \in \big({}_{R[G]}\mathrm{Mod}\big)$ embeds to the induced module $\mathrm{Hom}_R(R[G], M)$ which has trivial cohomology. Finally, (v) follows from (iii) and (iv). $\square$

4.4.6. Proposition. — *Let G be a pro-finite group. Then*

(i) *If J is injective in $\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$, then $(\forall i \geqslant 1)$ $H^i_{\mathrm{cont}}(G, J) = H^i_{\mathrm{der}}(G, J) = 0$.*

(ii) *The map*

$$\theta_G(M) : \mathbf{R}^+\Gamma_{\mathrm{der}}(G, M) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G, M)$$

is an isomorphism for every $M \in \big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$.

Proof

(i) Let $U \lhd G$ be an open normal subgroup. The functor $\Gamma_{\mathrm{der}}(G, G/U, -)$: $\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}} \to \big({}^{\text{ind-ad}}_{R[G/U]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$ preserves injectives; thus

$$H^i_{\mathrm{cont}}(G, J) = \varinjlim_{U} H^i_{\mathrm{cont}}(G/U, J^U) = 0 \qquad (\forall i \geqslant 1)$$

by Lemma 4.4.5(v). The equality $H^i_{\mathrm{der}}(G, J) = 0$ (for $i \geqslant 1$) follows from Proposition 4.4.4.

(ii) Let $J^\bullet$ be an injective resolution of M in $\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$. Then $\mathbf{R}^+\Gamma_{\mathrm{der}}(G, M)$ is represented by the complex $(J^\bullet)^G$ (by Proposition 4.4.4) and the canonical morphism $(J^\bullet)^G \to C^\bullet_{\mathrm{cont}}(G, J^\bullet)$ is a quasi-isomorphism by (i) and the spectral sequence (3.5.3.1). $\square$

4.4.7. Lemma. — *Let $H \lhd G$ be a closed normal subgroup of G. Then*

(i) *The functor $\mathrm{Res}_{G,H} : \big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}} \to \big({}^{\text{ind-ad}}_{R[H]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$ preserves injectives.*

(ii) *For every $M \in \big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$ and $j \geqslant 0$ the canonical map $R^j\Gamma_{\mathrm{der}}(G, G/H, M) \to H^j_{\mathrm{der}}(H, M)$ is an isomorphism; it induces an isomorphism between the spectral sequence* (3.6.3.1) *and the Hochschild-Serre spectral sequence* (4.1.4.1).

Proof

(i) Given an injective object J of $\big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$, a monomorphism $u : X \to Y$ in $\big({}^{\text{ind-ad}}_{R[H]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$ and a morphism $f : X \to \mathrm{Res}_{G,H}(J)$, we must show that there is $g : Y \to \mathrm{Res}_{G,H}(J)$ such that $gu = f$. As in the proof of Proposition 4.4.4 one can assume that both X and Y are of finite type over R. In this case there is a normal open subgroup $U \lhd G$ such that $H \cap U$ acts trivially on X and Y. We know that J^U is injective in $\big({}^{\text{ind-ad}}_{R[G/U]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$, hence $\mathrm{Res}_{G,H}(J^U)$ is injective in $\big({}^{\text{ind-ad}}_{R[H/(H\cap U)]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$. This implies that the composite map

$$f' : X = X^{H\cap U} \xrightarrow{f} \mathrm{Res}_{G,H}(J^{H\cap U}) \hookrightarrow \mathrm{Res}_{G,H}(J^U)$$

extends to a map $g' : Y = Y^{H\cap U} \to \mathrm{Res}_{G,H}(J^U)$ such that $g'u = f'$; this defines the required map

$$g : Y \xrightarrow{g'} \mathrm{Res}_{G,H}(J^U) \hookrightarrow \mathrm{Res}_{G,H}(J).$$

(ii) If $J^\bullet$ is an injective resolution of M in $\left({}^{\text{ind-ad}}_{R[G]}\text{Mod}\right)_{\{\mathfrak{m}\}}$, then it follows from (i) that the morphism in Proposition 3.6.4 (i) is represented by the identity map id : $(J^\bullet)^H \to (J^\bullet)^H$. □

4.4.8. Question. — Let $H \lhd G$ be a closed normal subgroup of G such that both $K = H$ and $K = G/H$ satisfy the condition

(∗) K satisfies (F) and $\theta_K(M)$ is an isomorphism for every $M \in \left({}^{\text{ind-ad}}_{R[K]}\text{Mod}\right)$.

Does it follow that G also satisfies (∗)?

4.4.9. A positive answer to 4.4.8 in the simplest non-trivial case, when both H and G/H are topologically cyclic, would considerably simplify our treatment of unramified local conditions in Chapter 7.

4.5. Bar resolution

4.5.1. Proposition. — *Every $M \in \left({}^{\text{ind-ad}}_{R[G]}\text{Mod}\right)$ has a canonical structure of an $R[\![G]\!]$-module, where*

$$R[\![G]\!] = \varprojlim_U R[G/U].$$

Proof. — Writing $M = \bigcup M_\alpha$ $(M_\alpha \in \mathcal{S}(M))$ and $M_\alpha = \varprojlim_n M_\alpha/\mathfrak{m}^n M_\alpha$, the statement follows from the fact that each $M_\alpha/\mathfrak{m}^n M_\alpha$ is a module over $R/\mathfrak{m}^n R[G/U_{\alpha,n}]$, for a suitable open normal subgroup $U_{\alpha,n} \lhd G$. □

4.5.2. The completed tensor products

$$\begin{aligned} R[\![G]\!]^{\widehat{\otimes} i} = R[\![G]\!]\widehat{\otimes}_R \cdots \widehat{\otimes}_R R[\![G]\!] &= \varprojlim_U (R[G/U] \otimes_R \cdots \otimes_R R[G/U]) \\ &= \varprojlim_U R[G/U \times \cdots \times G/U] = R[\![G \times \cdots \times G]\!] = R[\![G^i]\!] \end{aligned}$$

form a pro-finite bar resolution

$$R[\![G]\!]^{\widehat{\otimes}}_\bullet : \cdots \longrightarrow R[\![G]\!]^{\widehat{\otimes} i} \longrightarrow \cdots \longrightarrow R[\![G]\!]$$

of R by projective pseudo-compact $R[\![G]\!]$-modules. For each $M \in \left({}^{\text{ind-ad}}_{R[G]}\text{Mod}\right)_{\{\mathfrak{m}\}}$, the complex

$$\text{Hom}^{\bullet,\text{naive}}_{R[\![G]\!],\text{cont}}(R[\![G]\!]^{\widehat{\otimes}}_\bullet, M)$$

(where the subscript "cont" refers to homomorphisms continuous with respect to the pseudo-compact topology on $R[\![G^i]\!]$ and the discrete topology on M) is canonically isomorphic to $C^\bullet_{\text{cont}}(G, M)$.

4.5.3. Conjugation. — Let G be a discrete group. For each $\sigma \in G$, the formula

$$\lambda_\sigma^n : g_0[g_1|\cdots|g_n] \mapsto g_0\sigma[\sigma^{-1}g_1\sigma|\cdots|\sigma^{-1}g_n\sigma]$$

defines an isomorphism

$$\lambda_\sigma : \mathbf{Z}[G]^{\otimes}_{\bullet} \longrightarrow \mathbf{Z}[G]^{\otimes}_{\bullet}$$

between the bar resolution of $\mathbf{Z}$ and itself. These isomorphisms lift the identity $\mathrm{id} : \mathbf{Z} \to \mathbf{Z}$ and satisfy $\lambda_{\sigma\tau} = \lambda_\tau\lambda_\sigma$ $(\sigma, \tau \in G)$. For every complex of G-modules $M = M^\bullet$, the induced map $\mathrm{Hom}^\bullet(\lambda_\sigma, \mathrm{id})$ on $C^\bullet(G, M) = \mathrm{Hom}^{\bullet,\mathrm{naive}}_{\mathbf{Z}[G]}(\mathbf{Z}[G]^{\otimes}_{\bullet}, M)$ is equal to the conjugation action $\mathrm{Ad}(\sigma)$.

Both λ_σ and id lift the identity on $\mathbf{Z}$. As the bar resolution is projective over $\mathbf{Z}[G]$, there is a homotopy $h_\sigma : \mathrm{id} \rightsquigarrow \lambda_\sigma$, which induces homotopies $h_\sigma(M) : \mathrm{id} \rightsquigarrow \mathrm{Ad}(\sigma)$ on $C^\bullet(G, M)$, functorial in M. If we choose another homotopy $h'_\sigma : \mathrm{id} \rightsquigarrow \lambda_\sigma$, projectivity of the resolution implies that there is a 2-homotopy $H_\sigma : h_\sigma \rightsquigarrow h'_\sigma$, which in turn induces 2-homotopies $H_\sigma(M) : h_\sigma(M) \rightsquigarrow h'_\sigma(M)$, functorial in M.

If $\sigma, \tau \in G$, then the same argument shows that there is a 2-homotopy $H_{\sigma,\tau} : h_{\sigma\tau} \rightsquigarrow \lambda_\tau \star h_\sigma + h_\tau$, inducing 2-homotopies

$$H_{\sigma,\tau}(M) : h_{\sigma\tau}(M) \longrightarrow h_\sigma(M) \star \mathrm{Ad}(\tau) + h_\tau(M),$$

functorial in M.

4.5.4. As in 3.4.5.5, one can apply the above construction in the "universal" case, when G is a free group on countably many generators $\sigma, \tau, g_0, g_1, \ldots$ One obtains homotopies $h_\sigma(M)$ and 2-homotopies $H_{\sigma,\tau}(M)$ functorial in both M and G.

4.5.5. In fact, the formula 3.6.1.4 gives a choice of h_σ

$$h_\sigma : [g_1|\cdots|g_n] \longmapsto \sum_{j=1}^{n+1}(-1)^{j-1}[g_1|\cdots|g_{j-1}|\sigma|\sigma^{-1}g_j\sigma|\cdots|\sigma^{-1}g_n\sigma],$$

which defines such a bi-functorial homotopy $h_\sigma(M) : \mathrm{id} \rightsquigarrow \mathrm{Ad}(\sigma)$. Similarly, the formula

$$H_{\sigma,\tau} : [g_1|\cdots|g_n] \longmapsto \sum_{1\leqslant k<l\leqslant n+2} (-1)^{k+l-1}[g_1|\cdots|g_{k-1}|\tau|\tau^{-1}g_k\tau|\cdots|\tau^{-1}g_{l-2}\tau| \\ \tau^{-1}\sigma\tau|\tau^{-1}\sigma^{-1}g_{l-1}\sigma\tau|\cdots|\tau^{-1}\sigma^{-1}g_n\sigma\tau]$$

defines a bi-functorial 2-homotopy $H_{\sigma,\tau}(M)$.

More generally, if G is a topological group, then the above formulas define $h_\sigma(M)$ and $H_{\sigma,\tau}(M)$ for arbitrary $M \in \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$.

4.6. Euler-Poincaré characteristic

Assume that G is a pro-finite group satisfying (F) and $T^\bullet$ a bounded below complex in $({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod})_{R\text{-}ft}$.

4.6.1. The $\mathfrak{m}$-adic filtration $F^iT^\bullet = \mathfrak{m}^iT^\bullet$ $(i \geqslant 0)$ of $T^\bullet$ gives rise to a spectral sequence (3.5.2.1)

$$E_1^{i,j} = H^{i+j}(G, \mathfrak{m}^iT^\bullet/\mathfrak{m}^{i+1}T^\bullet) \qquad (i \geqslant 0)$$

with the following properties:

4.6.1.1. $(\exists\, c_0)\, (\forall r \geqslant 1)$ $E_r^{i,j} = 0$ whenever $i + j < c_0$ (as $T^\bullet$ is bounded below).

4.6.1.2. $(\forall i, j)$ $\ell_R(E_1^{i,j}) < \infty$ (as G satisfies (F)).

4.6.1.3. $(\forall i, j)\, (\exists\, r_0 = r_0(i, j) \geqslant 1)\, (\forall r \geqslant r_0)$ $E_r^{i,j} = E_{r_0}^{i,j}$ (by 4.6.1.2).

4.6.1.4. $(\forall q)\, (\forall r \geqslant 1)$ $H_r^q := \bigoplus_{i\geqslant 0} E_r^{i,q-i}$ is a graded module (with $E_r^{i,q-i}$ of degree i) over $\mathrm{gr}^\bullet_{\mathfrak{m}}(R) = \bigoplus_{i\geqslant 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. The differentials $(d_r^{i,q-i})$ define a graded homomorphism $d_r^q : H_r^q \to H_r^{q+1}$ of degree r, and $H_{r+1}^q = \mathrm{Ker}(d_r^q)/\mathrm{Im}(d_r^{q-1})$.

4.6.2. Lemma

(i) *Each* H_r^q *is a* $\mathrm{gr}^\bullet_{\mathfrak{m}}(R)$*-module of finite type.*
(ii) $(\forall q)\, (\exists\, r_1 = r_1(q) \geqslant 1)\, (\forall r \geqslant r_1)$ $H_r^q = H_{r_1}^q$.

Proof

(i) As $\mathrm{gr}^\bullet_{\mathfrak{m}}(R)$ is Noetherian, it is enough to consider the case $r = 1$. The hypercohomology spectral sequence

$$'E_1^{p,q} = \bigoplus_{i\geqslant 0} H^q(G, \mathfrak{m}^iT^p/\mathfrak{m}^{i+1}T^p) \Longrightarrow \bigoplus_{i\geqslant 0} H^{p+q}(G, \mathfrak{m}^iT^\bullet/\mathfrak{m}^{i+1}T^\bullet)$$

shows that we can assume that $T^\bullet = T$ is a single module in degree zero.

The exact sequence of graded $\mathrm{gr}^\bullet_{\mathfrak{m}}(R)[G]$-modules (discrete as G-modules)

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow \mathrm{gr}^\bullet_{\mathfrak{m}}(R) \otimes_{R/\mathfrak{m}} T/\mathfrak{m}T \xrightarrow{f} \bigoplus_{i\geqslant 0} \mathfrak{m}^iT/\mathfrak{m}^{i+1}T (= \mathrm{gr}^\bullet_{\mathfrak{m}}(T)) \longrightarrow 0$$

gives an exact cohomology sequence

$$\mathrm{gr}^\bullet_{\mathfrak{m}}(R) \otimes_{R/\mathfrak{m}} H^q(G, T/\mathfrak{m}T) \longrightarrow H_1^q \longrightarrow H^{q+1}(G, \mathrm{Ker}(f)),$$

so it is enough to show that $H^{q+1}(G, X)$ is a $\mathrm{gr}^\bullet_{\mathfrak{m}}(R)$-module of finite type, for every q and every graded $\mathrm{gr}^\bullet_{\mathfrak{m}}(R)[G]$-submodule X of $\mathrm{gr}^\bullet_{\mathfrak{m}}(R) \otimes_{R/\mathfrak{m}} T/\mathfrak{m}T$. By dévissage, we can assume that $T/\mathfrak{m}T$ is a simple $R/\mathfrak{m}[G]$-module. In this case $X = J \otimes_{R/\mathfrak{m}} T/\mathfrak{m}T$, where J is a graded ideal in $\mathrm{gr}^\bullet_{\mathfrak{m}}(R)$, hence $H^{q+1}(G, X) = J \otimes_{R/\mathfrak{m}} H^{q+1}(G, T/\mathfrak{m}T)$ is, indeed, of finite type over $\mathrm{gr}^\bullet_{\mathfrak{m}}(R)$.

(ii) By (i), H_1^q is generated as a $\mathrm{gr}^\bullet_{\mathfrak{m}}(R)$-module by $\bigoplus_{i=0}^{i_0} E_1^{i,q-i}$, for some i_0. We can then take

$$r_1 = \max\{r_0(i, q - i) \mid 0 \leqslant i \leqslant i_0\},$$

by 4.6.1.3. □

4.6.3. For each q and $i \geqslant 0$, put $H^q := H^q_{\mathrm{cont}}(G, T^\bullet)$ and

$$F^i H^q := \mathrm{Im}(H^q_{\mathrm{cont}}(G, \mathfrak{m}^i T^\bullet) \longrightarrow H^q) = \mathrm{Ker}(H^q \longrightarrow H^q(G, T^\bullet/\mathfrak{m}^i T^\bullet)).$$

Then each H^q is an R-module of finite type (by Proposition 4.2.5) and the filtration $F^i H^q$ satisfies

$$F^0 H^q = H^q \tag{4.6.3.1}$$

$$\mathfrak{m}\, F^i H^q \subseteq F^{i+1} H^q \qquad (i \geqslant 0) \tag{4.6.3.2}$$

$$\bigcap_{i \geqslant 0} F^i H^q = 0 \tag{4.6.3.3}$$

The last property holds by the vanishing of the $\varprojlim_n{}^{(1)}$-term in Corollary 4.1.3.

4.6.4. Lemma

(i) *The spectral sequence* E_r *converges to the filtered* R*-module* H^{i+j}*, hence* $\mathrm{gr}^i_F(H^q) \xrightarrow{\sim} E^{i,q-i}_\infty = E^{i,q-i}_{r_1(q)}$.

(ii) *For each* q *the filtration* $F^\bullet H^q$ *is* **good** *in the sense of (*[**Bou**, Def. III.3.1]*),* i.e., *it satisfies* (4.6.3.2) *and* $(\exists\, i_0)\,(\forall i > i_0)\quad F^i H^q = \mathfrak{m}^{i-i_0} F^{i_0} H^q$.

Proof

(i) The spectral sequence E_r comes from the complex $C^\bullet = C^\bullet_{\mathrm{cont}}(G, T^\bullet)$ equipped with the filtration $F^i C^\bullet = C^\bullet_{\mathrm{cont}}(G, \mathfrak{m}^i T^\bullet)$ $(i \geqslant 0)$. As $C^\bullet = F^0 C^\bullet$ and $\bigcap_{i \geqslant 0} F^i C^\bullet = 0$, convergence of E_r follows from [**McCl**, Thm. 3.2].

(ii) The graded $\mathrm{gr}^\bullet_{\mathfrak{m}}(R)$-module $\mathrm{gr}^\bullet_F(H^q)$ is isomorphic to

$$\bigoplus_{i \geqslant 0} E^{i,q-i}_\infty = \bigoplus_{i \geqslant 0} E^{i,q-i}_{r_1(q)} = H^q_{r_1(q)},$$

hence of finite type over $\mathrm{gr}^\bullet_{\mathfrak{m}}(R)$. We conclude by ([**Bou**, Prop. III.3.3]), which applies thanks to (4.6.3.3). □

4.6.5. Hilbert-Samuel functions and multiplicities. — We recall some standard facts from [**Mat**, §13, §14], [**Br-He**, Ch. 4].

If $N = \bigoplus_{i \geqslant 0} N_i$ is a graded $\mathrm{gr}^\bullet_{\mathfrak{m}}(R)$-module of finite type, put

$$P(N, t) := \sum_{i \geqslant 0} \ell_R(N_i) t^i \in \mathbf{Z}[\![t]\!].$$

If M is an R-module of finite type equipped with a *good* filtration $F^\bullet M$ (*i.e.*, such that $(\forall i)\quad \mathfrak{m}\, F^i M \subseteq F^{i+1} M$ and $(\exists\, i_0)\,(\forall i > i_0)\quad F^i M = \mathfrak{m}^{i-i_0} F^{i_0} M$) satisfying $M = F^0 M$, put

$$f(M, F^\bullet, t) := \sum_{i \geqslant 0} \ell_R(M/F^{i+1} M) t^i \in \mathbf{Z}[\![t]\!].$$

In particular, if $F^iM = \mathfrak{m}^iM$ $(i \geqslant 0)$ is the $\mathfrak{m}$-adic filtration, put

$$f(M,t) := f(M,\mathfrak{m}^\bullet,t) = \sum_{i\geqslant 0} \ell_R(M/\mathfrak{m}^{i+1}M)t^i.$$

These generating functions have the following properties:

4.6.5.1. $(1-t)f(M,F^\bullet,t) = P(\mathrm{gr}^\bullet_F(M),t)$.

4.6.5.2. Hilbert's Theorem. — If $N \neq 0$, then $P(N,t)(1-t)^{\dim(N)} \in \mathbf{Z}[t]$ and $P(N,t)(1-t)^{\dim(N)}|_{t=1} > 0$.

The **multiplicity** of M, defined as

$$e_R(M) := (1-t)^{d+1}f(M,t)|_{t=1} = (1-t)^dP(\mathrm{gr}^\bullet_{\mathfrak{m}}(M),t)|_{t=1} \in \mathbf{Z},$$

(where $d = \dim(R)$) satisfies

4.6.5.3.
$$e_R(M) = 0, \quad \text{if } \dim(M) < d$$
$$e_R(M) > 0, \quad \text{if } \dim(M) = d$$

(by 4.6.5.2, as $\dim(M) = \dim(\mathrm{gr}^\bullet_{\mathfrak{m}}(M))$).

4.6.5.4. If $d \geqslant 1$ and $F^\bullet$ is any good filtration on M, then

$$e_R(M) = (1-t)^{d+1}f(M,F^\bullet,t)|_{t=1}.$$

4.6.5.5. ([**Mat**, Thm. 14.7]*)*

$$e_R(M) = \sum_{\mathrm{ht}(\mathfrak{p})=0} e_{R/\mathfrak{p}}(R/\mathfrak{p})\,\ell_{R/\mathfrak{p}}(M_{\mathfrak{p}}).$$

4.6.5.6. In particular, if R is a domain, then

$$e_R(M) = e_R(R)\,\mathrm{rk}_R(M).$$

4.6.6. From now on, assume that $T^\bullet$ is bounded, $\mathrm{char}(k) = p > 0$ and $\mathrm{cd}_p(G) < \infty$. This implies, by Lemma 4.2.6, that $(\exists\, c_1)$ $E_1^{i,j} = 0$ whenever $i + j > c_1$, hence

$$H^q_r = 0 \quad \text{for } q \notin [c_0, c_1].$$

It follows from Lemma 4.6.2 (ii) that

$$(\exists\, r_2)\,(\forall i,j) \quad E^{i,j}_{r_2} = E^{i,j}_\infty.$$

Each H^q_r and $A^q_r := \mathrm{Ker}(d_r : H^q_r \to H^{q+1}_r)$ is a graded $\mathrm{gr}^\bullet_{\mathfrak{m}}(R)$-module of finite type; for each $r \geqslant 1$ put

$$F_r(t) := \sum_q (-1)^q\,P(H^q_r,t) = \sum_{i,j}(-1)^{i+j}\,\ell_R(E^{i,j}_r)\,t^i$$

$$G_r(t) := \sum_q (-1)^q\,P(A^q_r,t) = \sum_{i,j}(-1)^{i+j}\,\ell_R(A^{i,j}_r)\,t^i.$$

According to 4.6.5.2, we have

$$(1-t)^dF_r(t), (1-t)^dG_r(t) \in \mathbf{Z}[t].$$

4.6.7. Proposition

(i) $(\forall r \geqslant 1) \quad F_{r+1}(t) = (1-t^r)G_r(t) + t^r F_r(t)$.

(ii) *If* $d \geqslant 1$, *then*

$$(1-t)^d F_r(t)|_{t=1} = \sum_q (-1)^q \, e_R(H^q_{\mathrm{cont}}(G, T^\bullet)),$$

for every $r \geqslant 1$.

Proof

(i) This follows from the exact sequence

$$0 \longrightarrow A_r^{i-r,j+r-1} \longrightarrow E_r^{i-r,j+r-1} \xrightarrow{d_r} A_r^{i,j} \longrightarrow E_{r+1}^{i,j} \longrightarrow 0.$$

(ii) By (i), the integer $(1-t)^d F_r(t)|_{t=1}$ does not depend on $r \geqslant 1$. For $r \geqslant r_2$ we have $E_r = E_\infty$, hence

$$F_r(t) = \sum_{i,j} (-1)^{i+j} \, \ell_R(E_\infty^{i,j}) \, t^i = \sum_q (-1)^q \, (1-t) f(H^q, F^\bullet, t),$$

where $H^q = H^q_{\mathrm{cont}}(G, T^\bullet)$. We conclude by 4.6.5.4 (which applies, by Lemma 4.6.4 (ii)). □

4.6.8. Lemma. — *Assume that, as before, G satisfies (F), $\mathrm{char}(k) = p > 0$ and $\mathrm{cd}_p(G) < \infty$. Assume, in addition, that there is $c \in \mathbf{Q}$ such that*

$$(\star_c) \qquad \sum_q (-1)^q \, \dim_k H^q(G, M) = c \cdot \dim_k(M)$$

holds, for every discrete $k[G]$-module M with $\dim_k(M) < \infty$. Then, for every bounded complex $M^\bullet$ of discrete $R[G]$-modules of finite length over R, we have

$$\sum_q (-1)^q \, \ell_R(H^q(G, M^\bullet)) = c \sum_q (-1)^q \, \ell_R(M^q).$$

Proof. — Easy dévissage. □

4.6.9. Theorem. — *Assume that G satisfies (F), $\mathrm{char}(k) = p > 0$, $\mathrm{cd}_p(G) < \infty$ and $(\star_c)$. If $T^\bullet$ is a bounded complex in $\left({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\right)_{R\text{-}ft}$, then*

$$\sum_q (-1)^q \, e_R(H^q_{\mathrm{cont}}(G, T^\bullet)) = c \sum_q (-1)^q \, e_R(T^q).$$

Proof. — If $d = 0$, then $e_R(-) = \ell_R(-)$ and the statement reduces to that of Lemma 4.6.8. If $d \geqslant 1$, then Proposition 4.6.7 (ii) gives

$$\sum_q (-1)^q \, e_R(H^q_{\mathrm{cont}}(G, T^\bullet)) = (1-t)^d F_1(t)|_{t=1}.$$

However, Lemma 4.6.8 implies that

$$F_1(t) = \sum_{i,q} (-1)^q \, \ell_R(H^q(G, \mathfrak{m}^i T^\bullet/\mathfrak{m}^{i+1}T^\bullet))t^i = c\sum_{i,q}(-1)^q \, \ell_R(\mathfrak{m}^i T^q/\mathfrak{m}^{i+1}T^q)\, t^i$$
$$= c\sum_q (-1)^q \, P(\mathrm{gr}^\bullet_{\mathfrak{m}}(T^q), t),$$

hence

$$(1-t)^d F_1(t)|_{t=1} = c\sum_q (-1)^q \, e_R(T^q). \quad \square$$

4.6.10. Corollary. — *If R is a domain, then*

$$\sum_q (-1)^q \, \mathrm{rk}_R(H^q_{\mathrm{cont}}(G, T^\bullet)) = c\sum_q (-1)^q \, \mathrm{rk}_R(T^q).$$

CHAPTER 5

DUALITY THEOREMS FOR GALOIS COHOMOLOGY REVISITED

In this chapter we reformulate – and slightly generalize – Tate's (and Poitou's) local and global duality theorems for Galois cohomology. As observed in 0.3, duality with respect to the functor D follows automatically from the classical results for finite modules (cf. 5.2.10); the full duality follows by applying the general result 3.5.8. Throughout Chapter 5 we assume that $k = \mathbf{F}_{p^r}$ is a finite field of characteristic p.

5.1. Classical duality results for Galois cohomology

Let K be a global field of characteristic $\mathrm{char}(K) \neq p$ and S a finite set of primes of K containing all primes above p and all archimedean primes of K (if K is a number field). Denote by S_f the set of non-archimedean primes in S. In Sections 5.1–5.6, we assume that the following condition is satisfied (the general case is treated in 5.7):

(P) If $p = 2$, then K has no real prime.

Fix a separable closure K^{sep} of K. Let K_S be the maximal subextension of K^{sep}/K unramified outside S; denote $G_{K,S} := \mathrm{Gal}(K_S/K)$. For each prime $v \in S$ fix a separable closure K_v^{sep} of K_v and an embedding $K^{\mathrm{sep}} \hookrightarrow K_v^{\mathrm{sep}}$ extending the embedding $K \hookrightarrow K_v$. This defines a continuous homomorphism $\rho_v : G_v = \mathrm{Gal}(K_v^{\mathrm{sep}}/K_v) \xrightarrow{\alpha_v} G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K) \xrightarrow{\pi} G_{K,S}$, hence, for each $M \in ({}^{\text{ind-ad}}_{R[G_{K,S}]}\mathrm{Mod})$, a 'restriction' map

$$\mathrm{res}_v : C^\bullet_{\mathrm{cont}}(G_{K,S}, M) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v, M)$$

Denote by $M_v := \rho_v^*(M) \in ({}^{\text{ind-ad}}_{R[G_v]}\mathrm{Mod})$ the R-module M, equipped with the G_v-action induced by ρ_v.

For $v \in S_f$, our assumptions imply the following ([**Se2, N-S-W**]):

5.1.1. G_v and $G_{K,S}$ satisfy the finiteness condition (F').

5.1.2. $\mathrm{cd}_p(G_v) = \mathrm{cd}_p(G_{K,S}) = 2$.

5.1.3. For every $n \geqslant 0$, local class field theory defines an isomorphism

$$\mathrm{inv}_v : H^2(G_v, \mathbf{Z}/p^n\mathbf{Z}(1)) = \mathrm{Br}(K_v)[p^n] \xrightarrow{\sim} \mathbf{Z}/p^n\mathbf{Z}$$

(where $\mathbf{Z}/p^n\mathbf{Z}(1) = \mu_{p^n}$).

5.1.4. Local duality (Tate). — For every finite discrete $\mathbf{Z}/p^n\mathbf{Z}[G_v]$-module M, the cup product

$$H^i(G_v, M) \times H^{2-i}\big(G_v, \mathrm{Hom}(M, \mathbf{Z}/p^n\mathbf{Z}(1))\big) \xrightarrow{\cup} H^2(G_v, \mathbf{Z}/p^n\mathbf{Z}(1)) \xrightarrow{\sim} \mathbf{Z}/p^n\mathbf{Z}$$

is a perfect pairing of finite $\mathbf{Z}/p^n\mathbf{Z}$-modules ($i = 0, 1, 2$).

5.1.5. Reciprocity law. — The sum of the local invariants $\mathrm{inv}_{S_f} = \sum_{v \in S_f} \mathrm{inv}_v$ defines a short exact sequence

$$0 \longrightarrow H^2(G_{K,S}, \mathbf{Z}/p^n\mathbf{Z}(1)) \longrightarrow \bigoplus_{v \in S_f} H^2(G_v, \mathbf{Z}/p^n\mathbf{Z}(1)) \xrightarrow{\mathrm{inv}_{S_f}} \mathbf{Z}/p^n\mathbf{Z} \longrightarrow 0.$$

5.1.6. Global duality (Poitou-Tate). — For every finite discrete $\mathbf{Z}/p^n\mathbf{Z}[G_{K,S}]$-module M there is an exact sequence of finite $\mathbf{Z}/p^n\mathbf{Z}$-modules

$$\begin{aligned}
0 \longrightarrow H^0(G_{K,S}, M) \longrightarrow \bigoplus_{v \in S_f} H^0(G_v, M) \longrightarrow H^2(G_{K,S}, M^*(1))^* \longrightarrow \\
\longrightarrow H^1(G_{K,S}, M) \longrightarrow \bigoplus_{v \in S_f} H^1(G_v, M) \longrightarrow H^1(G_{K,S}, M^*(1))^* \longrightarrow \\
\longrightarrow H^2(G_{K,S}, M) \longrightarrow \bigoplus_{v \in S_f} H^2(G_v, M) \longrightarrow H^0(G_{K,S}, M^*(1))^* \longrightarrow 0,
\end{aligned}$$

in which $(-)^* = \mathrm{Hom}(-, \mathbf{Z}/p^n\mathbf{Z})$. The maps $H^i(G_{K,S}, M) \to H^i(G_v, M)$ are induced by res_v; the maps $H^i(G_v, M) \to H^{2-i}(G_{K,S}, M^*(1))^*$ by res_v and the pairing 5.1.4; the remaining two maps will be defined in 5.4.3 below.

5.2. Duality for G_v

5.2.1. It follows from 5.1.2–5.1.3 that for every R-module A with trivial action of G_v we have

$$\begin{aligned}
\mathrm{inv}_v : H^2_{\mathrm{cont}}(G_v, A(1)) &\xrightarrow{\sim} A \\
H^i_{\mathrm{cont}}(G_v, A(1)) &= 0 \quad (i > 2).
\end{aligned}$$

Here $A(1) = A \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1) = A \otimes_R R(1)$, which is admissible by Proposition 3.2.5. This implies that the canonical map of complexes

(5.2.1.1) $$A[-2] \xrightarrow{i_v} \tau_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_v, A(1))$$

(in which i_v is induced by the inverse of inv_v) is a quasi-isomorphism. For $A = I$ there is a morphism of complexes

$$I[-2] \xleftarrow{r_v} \tau_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_v, I(1)) \tag{5.2.1.2}$$

which is a homotopy inverse of (5.2.1.1).

More generally, if $A^\bullet$ is a complex of R-modules with trivial action of G_v, then there are canonical morphisms of complexes

$$\begin{aligned} C^\bullet_{\mathrm{cont}}(G_v, A^\bullet(1)) &= \mathrm{Tot}\big(i \longmapsto C^\bullet_{\mathrm{cont}}(G_v, A^i(1))\big) \longrightarrow \tau^{\mathrm{II}}_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_v, A^\bullet(1)) \\ &= \mathrm{Tot}\big(i \longmapsto \tau_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_v, A^i(1))\big) \xleftarrow{i_v} A^\bullet[-2] \end{aligned}$$

defining a canonical map in $D({}_R\mathrm{Mod})$

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, A^\bullet(1)) \longrightarrow A^\bullet[-2]$$

(since i_v, induced by the inverse of inv_v, is a quasi-isomorphism).

If $A^\bullet$ is a bounded below complex of injective R-modules, then i_v has a homotopy inverse

$$r_v = r_{v,A^\bullet} : \tau^{\mathrm{II}}_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_v, A^\bullet(1)) \longrightarrow A^\bullet[-2], \tag{5.2.1.3}$$

unique up to homotopy.

5.2.2. Fix $v \in S_f$, a bounded complex $J = J^\bullet$ of injective R-modules and $r_v = r_{v,J}$ as in (5.2.1.3) for $A^\bullet = J$. If $X^\bullet$ is a bounded complex of admissible $R[G_v]$-modules, so is

$$D_J(X^\bullet) = \mathrm{Hom}^\bullet_R(X^\bullet, J).$$

The evaluation map

$$\mathrm{ev}_2 : X^\bullet \otimes_R D_J(X^\bullet)(1) \longrightarrow J(1)$$

from 1.2.7 and the cup product defined in 3.4.5.2 induce a morphism of complexes

$$\begin{aligned} &C^\bullet_{\mathrm{cont}}(G_v, X^\bullet) \otimes_R C^\bullet_{\mathrm{cont}}(G_v, D_J(X^\bullet)(1)) \\ &\qquad \xrightarrow{\cup} C^\bullet_{\mathrm{cont}}(G_v, J(1)) \longrightarrow \tau^{\mathrm{II}}_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_v, J(1)) \xrightarrow{r_{v,J}} J[-2], \end{aligned}$$

hence by adjunction (see 1.2.6) a morphism of complexes

$$\begin{aligned} \alpha_{J,X^\bullet} : C^\bullet_{\mathrm{cont}}(G_v, X^\bullet) &\longrightarrow \mathrm{Hom}^\bullet_R(C^\bullet_{\mathrm{cont}}(G_v, D_J(X^\bullet)(1)), J[-2]) \\ &= D_{J[-2]}\big(C^\bullet_{\mathrm{cont}}(G_v, D_J(X^\bullet)(1))\big). \end{aligned}$$

This construction gives a well-defined map in $D^b({}_R\mathrm{Mod})$ (independent of the choice of $r_{v,J}$)

$$\alpha_{J,X} : \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, X) \longrightarrow D_{J[-2]}\big(\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, D_J(X)(1))\big)$$

for every $X \in D^b\big({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\big)$ (where $D_J(X) = \mathbf{R}\mathrm{Hom}_R(X, J)$ in the sense of 3.5.9).

In the same way, the evaluation map

$$\mathrm{ev}_1 : D_J(X^\bullet)(1) \otimes_R X^\bullet \longrightarrow J(1)$$

gives rise to a morphism of complexes

$$\alpha'_{J,X^\bullet} : C^\bullet_{\mathrm{cont}}(G_v, D_J(X^\bullet)(1)) \longrightarrow D_{J[-2]}(C^\bullet_{\mathrm{cont}}(G_v, X^\bullet))$$

resp., to a morphism

$$\alpha'_{J,X} : \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, D_J(X)(1)) \longrightarrow D_{J[-2]}(\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, X))$$

in $D^b({}_R\mathrm{Mod})$.

The above cup products define, for each $X \in D^b({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod})$, morphisms in $D^b({}_R\mathrm{Mod})$

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, X)\overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, D_J(X)(1)) \longrightarrow J[-2]$$

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, D_J(X)(1))\overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, X) \longrightarrow J[-2],$$

which induce pairings

(5.2.2.1)
$$H^i_{\mathrm{cont}}(G_v, X) \otimes_R H^j_{\mathrm{cont}}(G_v, D_J(X)(1)) \longrightarrow H^{i+j-2}(J^\bullet)$$

$$H^i_{\mathrm{cont}}(G_v, D_J(X)(1)) \otimes_R H^j_{\mathrm{cont}}(G_v, X) \longrightarrow H^{i+j-2}(J^\bullet)$$

on cohomology.

5.2.3. We shall be interested only in the following two choices of J:

(A) $J = I[n]$ for some $n \in \mathbf{Z}$ (hence $D_J = D_n$ in the notation of 2.3.2) and all cohomology groups of X are of finite (resp., co-finite) type over R.

(B) $J = \omega^\bullet[n]$ for some $n \in \mathbf{Z}$ (hence $D_J = \mathscr{D}_n$ in the notation of 2.8.11) and all cohomology groups of X are of finite type over R.

In both cases the canonical map

$$\varepsilon = \varepsilon_J : X \longrightarrow D_J(D_J(X))$$

is a quasi-isomorphism, by Matlis duality 2.3.2 and Grothendieck duality 2.6, respectively.

5.2.4. Proposition. — *Assume that either*

(i) $J = I[n]$ *and* $X \in D^b_{R\text{-}ft}({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod})$ *or* $X \in D^b_{R\text{-}coft}({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod})$,

or

(ii) $J = \omega^\bullet[n]$ *and* $X \in D^b_{R\text{-}ft}({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod})$.

Then both maps $\alpha_{J,X}, \alpha'_{J,X}$ *are isomorphisms in* $D^b({}_R\mathrm{Mod})$.

Proof. — We consider only $\alpha_{J,X}$; the statement for $\alpha'_{J,X}$ can be proved along the same lines (alternatively, one can use the compatibility result 5.2.7 relating the two maps).

(i) If $A \to B \to C \to A[1]$ is an exact triangle in $D^b({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod})$, then $(\alpha_{J,A}, \alpha_{J,B}, \alpha_{J,C})$ define a map between exact triangles

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, A) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, B) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, C)$$

and

$$D_{J[-2]}\big(\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, D(A)(1))\big) \longrightarrow D_{J[-2]}\big(\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, D(B)(1))\big) \\ \longrightarrow D_{J[-2]}\big(\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, D(C)(1))\big).$$

This means that $\alpha_{J,B}$ is an isomorphism, provided $\alpha_{J,A}$ and $\alpha_{J,C}$ are. Applying this observation to truncations

$$\tau_{\leqslant i-1}X \longrightarrow \tau_{\leqslant i}X \longrightarrow H^i(X)[-i] \longrightarrow (\tau_{\leqslant i-1}X)[1]$$

we reduce to the case when X is a single module in degree zero. Lemma 5.2.5 below further reduces to the case $J = I$. For $X = T \in \big({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\big)_{R\text{-}ft}$ (resp., $X = A \in \big({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\big)_{R\text{-}coft}$) there is a commutative diagram

$$\begin{array}{ccc}
C^\bullet_{\mathrm{cont}}(G_v, T) & \xrightarrow{\ u_1\ } & \varprojlim_n C^\bullet_{\mathrm{cont}}(G_v, T/\mathfrak{m}^n T) \\
\Big\downarrow \alpha_T & & \\
D_{-2}\big(C^\bullet_{\mathrm{cont}}(G_v, D(T)(1))\big) & & \Big\downarrow u_4 \\
\Big\downarrow u_2 & & \\
D_{-2}\big(\varinjlim_n C^\bullet_{\mathrm{cont}}(G_v, D(T/\mathfrak{m}^n T)(1))\big) & \xrightarrow{\ u_3\ } & \varprojlim_n D_{-2}\big(C^\bullet_{\mathrm{cont}}(G_v, D(T/\mathfrak{m}^n T)(1))\big)
\end{array}$$

in $D^b_{ft}({}_R\mathrm{Mod})$ (resp.,

$$\begin{array}{ccc}
\varinjlim_n C^\bullet_{\mathrm{cont}}(G_v, A[\mathfrak{m}^n]) & \xrightarrow{\ u_5\ } & C^\bullet_{\mathrm{cont}}(G_v, A) \\
\Big\downarrow u_6 & & \\
\varinjlim_n D_{-2}\big(C^\bullet_{\mathrm{cont}}(G_v, (D(A)/\mathfrak{m}^n D(A))(1))\big) & & \Big\downarrow \alpha_A \\
\Big\downarrow u_7 & & \\
D_{-2}\Big(\varprojlim_n C^\bullet_{\mathrm{cont}}(G_v, (D(A)/\mathfrak{m}^n D(A))(1))\Big) & \xrightarrow{\ u_8\ } & D_{-2}\big(C^\bullet_{\mathrm{cont}}(G_v, D(A)(1))\big)
\end{array}$$

in $D^b_{coft}({}_R\mathrm{Mod})$). The maps

$$u_4 = \varprojlim_n \alpha_{I,T/\mathfrak{m}^n T}, \quad u_6 = \varinjlim_n \alpha_{I,A[\mathfrak{m}^n]}$$

are quasi-isomorphisms by 5.1.4 (and by the "universal coefficient theorem" for projective limits of Mittag-Leffler systems of complexes, used in the proof of Corollary 4.1.3), u_1, u_2, u_5, u_8 by Lemma 4.1.2 and u_3, u_7 by Lemma 4.2.4. This implies that both $\alpha_{I,T}$ and $\alpha_{I,A}$ are quasi-isomorphisms, as claimed.

(ii) Lemma 5.2.5 below implies that we can assume that $J = \omega^\bullet[d]$ (hence $D_J = \mathscr{D}_d$). Represent X by a bounded complex $X^\bullet$ of admissible $R[G_v]$-modules. The functoriality of the morphisms

$$i_v : A^\bullet[-2] \longrightarrow \tau^{\mathrm{II}}_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_v, A^\bullet(1))$$

from 5.2.1 implies that, for any choices of $r_{v,J}$ for $J = I$ and $J = \omega^\bullet[d]$, the following diagram is commutative up to homotopy:

$$\begin{array}{ccccc} C^\bullet_{\mathrm{cont}}(G_v, \omega^\bullet[d](1)) & \xrightarrow{(\Phi_{-d})_*} & C^\bullet_{\mathrm{cont}}(G_v, \Phi_{-d}(\omega^\bullet[d])(1)) & \xrightarrow{\mathrm{Tr}_*} & C^\bullet_{\mathrm{cont}}(G_v, I(1)) \\ \downarrow{\scriptstyle r_{v,\omega^\bullet[d]}} & & & & \downarrow{\scriptstyle r_{v,I}} \\ (\omega^\bullet[d])[-2] & & \xrightarrow{\mathrm{Tr}\circ\Phi_{-d}[-2]} & & I[-2] \end{array}$$

This implies, by Lemma 2.8.12, that the complexes

$$A = C^\bullet_{\mathrm{cont}}(G_v, X^\bullet), \quad B = C^\bullet_{\mathrm{cont}}(G_v, D(X^\bullet)(1)), \quad B' = C^\bullet_{\mathrm{cont}}(G_v, \mathscr{D}_d(X^\bullet)(1)),$$
$$U = I[-2], \quad U' = (\omega^\bullet[d])[-2], \quad C = C^\bullet((x_i), R) = \Phi_{-d}(R)$$

and the maps f, f' (resp., b, u) induced by $\cup$ and $r_{v,J}$ (resp., by $\xi_d : \Phi_{-d} \circ \mathscr{D}_d \to D$) satisfy the assumptions of Lemma 1.2.13, which in turn implies that the following diagram in $D^b_{ft}({}_R\mathrm{Mod})$ is commutative:

$$\begin{array}{ccc} \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, X) & \xrightarrow{\alpha_{I,X}} & D_{-2}\big(\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, D(X)(1))\big) \\ \downarrow{\scriptstyle \alpha_{\omega[d],X}} & & \downarrow{\scriptstyle D_{-2}((\xi_d(X))_*)} \\ D_{(\omega[d])[-2]}\big(\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, \mathscr{D}_d(X)(1))\big) & & D_{-2}\big(\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, \Phi_{-d}\circ\mathscr{D}_d(X)(1))\big) \\ \downarrow{\scriptstyle \eta_d[-2]} & & \downarrow{\scriptstyle D_{-2}(f)} \\ D_{-2}\circ\Phi_{-d}\big(\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, \mathscr{D}_d(X)(1))\big) & = & D_{-2}\circ\Phi_{-d}\big(\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, \mathscr{D}_d(X)(1))\big) \end{array}$$

The maps $\eta_d[-2]$, $\xi_d(X)$ are isomorphisms by 2.8.11,

$$f : \Phi_{-d}\big(\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, \mathscr{D}_d(X)(1))\big) \longrightarrow \Phi_{-d}\big(\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, \mathscr{D}_d(X)(1))\big)$$

is an isomorphism by Proposition 3.5.8 and $\alpha_{I,X}$ is an isomorphism by (i); hence $\alpha_{\omega[d],X}$ is also an isomorphism.

It remains to prove the following Lemma. ☐

5.2.5. Lemma. — *For every J, $X^\bullet$ and $n \in \mathbf{Z}$ (and a fixed choice of $r_{v,J}$) the map $\alpha_{J[n],X^\bullet}$ is equal to the composite map*

$$\begin{aligned} C^\bullet_{\mathrm{cont}}(G_v, X^\bullet) &\xrightarrow{\alpha_{J,X^\bullet}} D_{J[-2]}\big(C^\bullet_{\mathrm{cont}}(G_v, D_J(X^\bullet)(1))\big) \\ &\xrightarrow{t_n} D_{J[n-2]}\big(C^\bullet_{\mathrm{cont}}(G_v, D_J(X^\bullet)(1))[n]\big) \\ &= D_{J[n-2]}\big(C^\bullet_{\mathrm{cont}}(G_v, D_{J[n]}(X^\bullet)(1))\big). \end{aligned}$$

Proof. — Apply Lemma 1.2.16 to $A^\bullet = C^\bullet_{\mathrm{cont}}(G_v, X^\bullet)$, $B^\bullet = C^\bullet_{\mathrm{cont}}(G_v, D_J(X^\bullet)(1))$, $C^\bullet = J[-2]$ and u the map from 5.2.2 (the composition of $r_{v,J}$ with a truncated cup product). ☐

5.2.6. Theorem. — *If T, $T^* \in D^b_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\big)$, $A, A^* \in D^b_{R\text{-}coft}\big({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\big)$ are related by the duality diagram*

$$\begin{array}{ccc} T & \xleftrightarrow{\ \mathscr{D}\ } & T^* \\ \big\downarrow \Phi & \overset{D}{\times} & \big\downarrow \Phi \\ A & & A^* \end{array}$$

so are

$$\begin{array}{ccc} \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, T) & \xleftrightarrow{\ \mathscr{D}\ } & \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, T^*(1))[2] \\ \big\downarrow \Phi & \overset{D}{\times} & \big\downarrow \Phi \\ \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, A) & & \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, A^*(1))[2] \end{array}$$

(in $D^b_{(co)ft}({}_R\mathrm{Mod})$) and there is a spectral sequence

$$E_2^{i,j} = \mathbb{E}\mathrm{xt}^i_R(H^{2-j}_{\mathrm{cont}}(G_v, T^*(1)), \omega) = \mathbb{E}\mathrm{xt}^i_R(D(H^j_{\mathrm{cont}}(G_v, A)), \omega) \Longrightarrow H^{i+j}_{\mathrm{cont}}(G_v, T).$$

Proof. — We first explain the assumptions: start with $T \in D^b_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\big)$ and put $A = \Phi(T)$, $T^* = D(A)$, $A^* = D(T)$. Then the canonical maps $\Phi(T^*) \to A^*$, $\Phi(\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, T)) \to \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, A)$ and $\Phi\big(\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, T^*(1))\big) \to \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, A^*(1))$ are isomorphisms by 4.3.1; this takes care of the vertical arrows. The diagonal and horizontal arrows are given by the isomorphisms of Proposition 5.2.4. The diagram is commutative up to the canonical isomorphisms from 2.8.1; the spectral sequence follows by applying 2.8.6 to the diagram. □

5.2.7. Proposition. — *Let $f : X^\bullet \to Y^\bullet$ be a morphism of bounded complexes of admissible $R[G_v]$-modules. Fix J and $r_{v,J}$ as in 5.2.1–5.2.2. Then*

(i) *The composite map*

$$C^\bullet_{\mathrm{cont}}(G_v, X^\bullet) \xrightarrow{(\varepsilon_J)_*} C^\bullet_{\mathrm{cont}}\big(G_v, D_J(D_J(X^\bullet))\big) \xrightarrow{\alpha'_{J, D_J(X)(1)}} D_{J[-2]}\big(C^\bullet_{\mathrm{cont}}(G_v, D_J(X^\bullet)(1))\big)$$

resp.,

$$C^\bullet_{\mathrm{cont}}(G_v, D_J(X^\bullet)(1)) \xrightarrow{\alpha_{J, D_J(X)(1)}} D_{J[-2]}\big(C^\bullet_{\mathrm{cont}}\big(G_v, D_J(D_J(X^\bullet))\big)\big) \xrightarrow{D_{J[-2]}((\varepsilon_J)_*)} D_{J[-2]}(C^\bullet_{\mathrm{cont}}(G_v, X^\bullet))$$

is equal to $\alpha_{J,X^\bullet}$ (resp., to $\alpha'_{J,X^\bullet}$).

(ii) *The following diagrams are commutative:*

$$\begin{array}{ccc} C^\bullet_{\mathrm{cont}}(G_v, X^\bullet) & \xrightarrow{\alpha_{J,X^\bullet}} & D_{J[-2]}\big(C^\bullet_{\mathrm{cont}}(G_v, D_J(X^\bullet)(1))\big) \\ \big\downarrow f_* & & \big\downarrow D_{J[-2]}((D_J(f)(1))_*) \\ C^\bullet_{\mathrm{cont}}(G_v, Y^\bullet) & \xrightarrow{\alpha_{J,Y^\bullet}} & D_{J[-2]}\big(C^\bullet_{\mathrm{cont}}(G_v, D_J(Y^\bullet)(1))\big) \end{array}$$

$$\begin{array}{ccc} C^\bullet_{\mathrm{cont}}(G_v, D_J(X^\bullet)(1)) & \xrightarrow{\alpha'_{J,X^\bullet}} & D_{J[-2]}(C^\bullet_{\mathrm{cont}}(G_v, X^\bullet)) \\ \big\uparrow (D_J(f)(1))_* & & \big\uparrow D_{J[-2]}(f_*) \\ C^\bullet_{\mathrm{cont}}(G_v, D_J(Y^\bullet)(1)) & \xrightarrow{\alpha'_{J,Y^\bullet}} & D_{J[-2]}(C^\bullet_{\mathrm{cont}}(G_v, Y^\bullet)) \end{array}$$

(iii) *The composite map*

$$C^\bullet_{\mathrm{cont}}(G_v, X^\bullet) \xrightarrow{\varepsilon_{J[-2]}} D_{J[-2]}(D_{J[-2]}\big(C^\bullet_{\mathrm{cont}}(G_v, X^\bullet))\big) \xrightarrow{D_{J[-2]}(\alpha'_{J,X^\bullet})} D_{J[-2]}\big(C^\bullet_{\mathrm{cont}}(G_v, D_J(X^\bullet)(1))\big)$$

resp.,

$$C^\bullet_{\mathrm{cont}}(G_v, D_J(X^\bullet)(1)) \xrightarrow{\varepsilon_{J[-2]}} D_{J[-2]}\big(D_{J[-2]}\big(C^\bullet_{\mathrm{cont}}(G_v, D_J(X^\bullet)(1))\big)\big) \xrightarrow{D_{J[-2]}(\alpha_{J,X^\bullet})} D_{J[-2]}(C^\bullet_{\mathrm{cont}}(G_v, X^\bullet))$$

is homotopic to $\alpha_{J,X^\bullet}$ *(resp., to* $\alpha'_{J,X^\bullet}$*).*

Proof

(i) The first statement follows from Lemma 1.2.12 and the following commutative diagram (in which $C^\bullet(X^\bullet) = C^\bullet_{\mathrm{cont}}(G_v, X^\bullet)$):

$$\begin{array}{ccccc} C^\bullet(X^\bullet) \otimes_R C^\bullet(D_J(X^\bullet)(1)) & \xrightarrow{\cup} & C^\bullet(X^\bullet \otimes_R D_J(X^\bullet)(1)) & \xrightarrow{(\mathrm{ev}_2)_*} & C^\bullet(J(1)) \\ \big\downarrow (\varepsilon_J)_* \otimes \mathrm{id} & & \big\downarrow (\varepsilon_J \otimes \mathrm{id})_* & & \| \\ C^\bullet\big(D_J(D_J(X^\bullet))\big) \otimes_R C^\bullet(D_J(X^\bullet)(1)) & \xrightarrow{\cup} & C^\bullet(D_J(D_J(X^\bullet)) \otimes_R D_J(X^\bullet)(1)) & \xrightarrow{(\mathrm{ev}_1)_*} & C^\bullet(J(1)) \end{array}$$

(the second square is commutative by Lemma 1.2.9). Exchanging the factors in each tensor product we obtain the second statement.

(ii) This follows from Lemma 1.2.11.

(iii) The diagram

$$\begin{array}{ccccc} C^\bullet(X^\bullet) \otimes_R C^\bullet(D_J(X^\bullet)(1)) & \xrightarrow{\cup} & C^\bullet(X^\bullet \otimes_R D_J(X^\bullet)(1)) & \xrightarrow{(\mathrm{ev}_2)_*} & C^\bullet(J(1)) \\ \big\downarrow s_{12}\circ(\mathcal{T}\otimes\mathcal{T}) & & \big\downarrow \mathcal{T}\circ(s_{12})_* & & \big\downarrow \mathcal{T} \\ C^\bullet(D_J(X^\bullet)(1)) \otimes_R C^\bullet(X^\bullet) & \xrightarrow{\cup} & C^\bullet(D_J(X^\bullet)(1) \otimes_R X^\bullet) & \xrightarrow{(\mathrm{ev}_1)_*} & C^\bullet(J(1)) \end{array}$$

is commutative by (1.2.7.1) and 3.4.5.4. We apply Lemma 1.2.14 and the fact that $\mathcal{T}$ is homotopic to the identity. □

5.2.8. Corollary. — *If $f : X \to Y$ is a morphism in $D^b({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod})$, then*

$$\alpha_{J,X} = \alpha'_{J,D_J(X)(1)} \circ (\varepsilon_J)_* = D_{J[-2]}(\alpha'_{J,X}) \circ \varepsilon_{J[-2]},$$
$$D_{J[-2]}\big((D_J(f)(1))_*\big) \circ \alpha_{J,X} = \alpha_{J,Y} \circ f_*$$
$$\alpha'_{J,X} = D_{J[-2]}((\varepsilon_J)_*) \circ \alpha_{J,D_J(X)(1)} = D_{J[-2]}(\alpha_{J,X}) \circ \varepsilon_{J[-2]},$$
$$\alpha'_{J,X} \circ (D_J(f)(1))_* = D_{J[-2]}(f_*) \circ \alpha'_{J,Y}$$

(equalities of morphisms in $D^b({}_R\mathrm{Mod})$).

5.2.9. Self-dual case. — If $f : X \to D_J(X)(1)$ is a morphism in $D^b({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod})$, denote the morphism

$$X \xrightarrow{\varepsilon_J} D_J(D_J(X)) = D_J(D_J(X)(1))(1) \xrightarrow{D_J(f)(1)} D_J(X)(1)$$

by g. It follows from (1.2.7.1) and a derived version of Lemma 1.2.10 that the pairings

$$\cup_f : \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, X) \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, X)$$
$$\xrightarrow{\mathrm{id}\otimes f_*} \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, X) \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, D_J(X)(1)) \xrightarrow{(\mathrm{ev}_2)_*} J[-2]$$
$$\cup_g : \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, X) \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, X)$$
$$\xrightarrow{\mathrm{id}\otimes g_*} \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, X) \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, D_J(X)(1)) \xrightarrow{(\mathrm{ev}_2)_*} J[-2]$$

are related by

(5.2.9.1) $$\cup_g = \cup_f \circ s_{12}.$$

In particular,

(5.2.9.2) $$g = \pm f \Longrightarrow \cup_f \circ s_{12} = \pm \cup_f .$$

5.2.10. On the level of cohomology, the diagonal arrows in Theorem 5.2.6 imply that the cup product

$$H^i_{\mathrm{cont}}(G_v, T) \times H^{2-i}_{\mathrm{cont}}(G_v, A^*(1)) \xrightarrow{\cup} H^2_{\mathrm{cont}}(G_v, I(1)) \xrightarrow{\sim} I$$

induces isomorphisms

$$H^i_{\mathrm{cont}}(G_v, T) \xrightarrow{\sim} D\big(H^{2-i}_{\mathrm{cont}}(G_v, A^*(1))\big)$$
$$H^j_{\mathrm{cont}}(G_v, A^*(1)) \xrightarrow{\sim} D(H^{2-j}_{\mathrm{cont}}(G_v, T))$$

(and similarly for the pair $T^*(1), A$). Note that, as remarked in 0.3, these isomorphisms follow immediately from 5.1.4 by a straightforward limit argument (a rather pompous version of which was given in the proof of Proposition 5.2.4 (i)). The only reason for developing the cohomological machinery of Chapters 3–4 was a need to relate cohomology of T and A, as in 4.3.2.

5.2.11. Euler-Poincaré characteristic. — For every non-archimedean prime v of K, the group G_v satisfies the condition $(\star_c)$ from Lemma 4.6.8 with

$$c = \begin{cases} -[K_v : \mathbf{Q}_p], & \text{if } K_v \text{ is a finite extension of } \mathbf{Q}_p \\ 0, & \text{otherwise,} \end{cases}$$

by Tate's local Euler-Poincaré characteristic formula ([**Se2**, §II.5.7, Thm. 5]; [**N-S-W**, Thm. 7.3.1]). As a result, Theorem 4.6.9 applies to G_v and c.

5.3. Cohomology with compact support for $G_{K,S}$

In this section we develop the theory of cohomology groups with compact support. In fact, there are two possible definitions, which differ in their treatment of infinite primes. We use the one appropriate for duality theory (see also the footnote in Sect. 0.7.1).

5.3.1. Cochains with compact support

5.3.1.1. Definition. — Let $M^\bullet$ be a complex of ind-admissible $R[G_{K,S}]$-modules. The complex of **continuous cochains with compact support** with values in $M^\bullet$ is defined as

$$C^\bullet_{c,\mathrm{cont}}(G_{K,S}, M^\bullet) = \mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(G_{K,S}, M^\bullet) \xrightarrow{\mathrm{res}_{S_f}} \bigoplus_{v \in S_f} C^\bullet_{\mathrm{cont}}(G_v, M^\bullet) \right)[-1],$$

where $\mathrm{res}_{S_f} = (\mathrm{res}_v)_{v \in S_f}$.

5.3.1.2. More precisely, the 'restriction' map res_v $(v \in S_f)$ is equal to

$$\mathrm{res}_v = \rho_v^* : C^\bullet_{\mathrm{cont}}(G_{K,S}, M^\bullet) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v, \rho_v^*(M^\bullet)).$$

If we choose another embedding $K^{\mathrm{sep}} \hookrightarrow K_v^{\mathrm{sep}}$, then $\alpha_v : G_v \hookrightarrow G_K$ is replaced by $\alpha'_v = \mathrm{Ad}(\sigma_v) \circ \alpha_v$ (for some $\sigma_v \in G_K$) and the map $\pi(\sigma_v) : \rho_v^*(M^\bullet) \to \rho_v'^*(M^\bullet)$ is an isomorphism of complexes of G_v-modules. In the diagram

$$\begin{array}{ccccccc}
\mathrm{res}_v : & C^\bullet_{\mathrm{cont}}(G_{K,S}, M^\bullet) & \xrightarrow{\mathrm{inf}} & C^\bullet_{\mathrm{cont}}(G_K, \pi^*(M^\bullet)) & \xrightarrow{\alpha_v^*} & C^\bullet_{\mathrm{cont}}(G_v, \rho_v^*(M^\bullet)) \\
& \| & & \downarrow{\scriptstyle \mathrm{Ad}(\sigma_v)} & & \downarrow{\scriptstyle \pi(\sigma_v)_*} \\
\mathrm{res}'_v : & C^\bullet_{\mathrm{cont}}(G_{K,S}, M^\bullet) & \xrightarrow{\mathrm{inf}} & C^\bullet_{\mathrm{cont}}(G_K, \pi^*(M^\bullet)) & \xrightarrow{\alpha_v'^*} & C^\bullet_{\mathrm{cont}}(G_v, \rho_v'^*(M^\bullet))
\end{array}$$

the first (resp., the second) square is commutative up to homotopy (resp., commutative). Choosing bi-functorial homotopies $h_\sigma(M^\bullet) : \mathrm{id} \rightsquigarrow \mathrm{Ad}(\sigma)$ (*e.g.* those from 4.5.5), we obtain homotopies

$$h_v = \alpha_v'^* \star h_{\sigma_v}(M^\bullet) \star \mathrm{inf} : \mathrm{res}'_v \rightsquigarrow \pi(\sigma_v)_* \circ \mathrm{res}_v,$$
$$h = (h_v)_{v \in S_f} : \mathrm{res}'_{S_f} \rightsquigarrow (\pi(\sigma_v)_*) \circ \mathrm{res}_{S_f}.$$

The corresponding morphism of cones

$$\mathrm{Cone}(\mathrm{id}, ((\sigma_v)_*), h) : \mathrm{Cone}(\mathrm{res}_{S_f}) \longrightarrow \mathrm{Cone}(\mathrm{res}'_{S_f})$$

is a homotopy equivalence, with homotopy inverse equal to

$$\mathrm{Cone}(\mathrm{id}, ((\sigma_v^{-1})_*), h') : \mathrm{Cone}(\mathrm{res}'_{S_f}) \longrightarrow \mathrm{Cone}(\mathrm{res}_{S_f}),$$

where $h' = (\alpha_v^* \star h_{\sigma_v^{-1}}(M^\bullet) \star \mathrm{inf})_{v\in S_f}$. Indeed, this follows from 1.1.7 and the fact that there is a 2-homotopy

$$h' \star \mathrm{id} + ((\sigma_v^{-1})_*) \star h = \big(\mathrm{res}_v \star (h_{\sigma_v^{-1}}(M^\bullet) + \mathrm{Ad}(\sigma_v^{-1}) \star h_{\sigma_v}(M^\bullet))\big)_v \star \mathrm{inf} \rightsquigarrow 0.$$

5.3.1.3. The cohomology of $C^\bullet_{c,\mathrm{cont}}(G_{K,S}, M^\bullet)$ will be denoted by $H^i_{c,\mathrm{cont}}(G_{K,S}, M^\bullet)$. As in 3.5.6, the functor $M^\bullet \mapsto C^\bullet_{c,\mathrm{cont}}(G_{K,S}, M^\bullet)$ preserves homotopy, exact sequences and quasi-isomorphisms for cohomologically bounded below complexes, hence defines an exact functor

$$\mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, -) : D^+\big({}^{\text{ind-ad}}_{R[G_{K,S}]}\mathrm{Mod}\big) \longrightarrow D^+({}_R\mathrm{Mod})$$

such that

$$\mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, M) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, M) \xrightarrow{\mathrm{res}_{S_f}} \bigoplus_{v\in S_f} \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, M) \tag{5.3.1.1}$$

is an exact triangle in $D^+({}_R\mathrm{Mod})$ for every $M \in D^+\big({}^{\text{ind-ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)$. In particular, there is an exact sequence

$$\begin{aligned}(5.3.1.2)\qquad \cdots \longrightarrow H^i_{c,\mathrm{cont}}(G_{K,S}, M) \longrightarrow H^i_{\mathrm{cont}}(G_{K,S}, M)\\ \xrightarrow{\mathrm{res}_{S_f}} \bigoplus_{v\in S_f} H^i_{\mathrm{cont}}(G_v, M) \longrightarrow H^{i+1}_{c,\mathrm{cont}}(G_{K,S}, M) \longrightarrow \cdots\end{aligned}$$

5.3.2. Lemma. — *Each of the three functors* $\mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, -)$, $\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, -)$, $\mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, -)$ *maps* $D^*_{R\text{-}ft}\big({}^{\text{ind-ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)$ *(resp.,* $D^*_{R\text{-}coft}\big({}^{\text{ind-ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)$*) to* $D^*_{ft}({}_R\mathrm{Mod})$ *(resp.,* $D^*_{coft}({}_R\mathrm{Mod})$*), for* $* = +, b$.

Proof. — This is true for $\mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, -)$ and $\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, -)$ by Corollary 4.2.7 and 5.1.1–5.1.2. The statement for $\mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, -)$ follows from the exact sequence (5.3.1.2). □

5.3.3. Cup products

5.3.3.1. Let $A^\bullet, B^\bullet$ be complexes of ind-admissible $R[G_{K,S}]$-modules. We shall write elements of

$$C^i_{c,\mathrm{cont}}(G_{K,S}, A^\bullet) = C^i_{\mathrm{cont}}(G_{K,S}, A^\bullet) \oplus \bigoplus_{v\in S_f} C^{i-1}_{\mathrm{cont}}(G_v, A^\bullet)$$

in the form (a, a_S), where $a \in C^i_{\mathrm{cont}}(G_{K,S}, A^\bullet)$, $a_S = (a_v)_{v\in S_f}$, $a_v \in C^{i-1}_{\mathrm{cont}}(G_v, A^\bullet)$; $\deg(a, a_S) = i = \deg(a) = \deg(a_S) + 1$. The differential is given by

$$d(a, a_S) = (da, -\mathrm{res}_{S_f}(a) - da_S).$$

5.3.3.2. The first cup product

$$ {}_c\cup : C^\bullet_{c,\mathrm{cont}}(G_{K,S}, A^\bullet) \otimes_R C^\bullet_{\mathrm{cont}}(G_{K,S}, B^\bullet) \longrightarrow C^\bullet_{c,\mathrm{cont}}(G_{K,S}, A^\bullet \otimes_R B^\bullet) $$

is defined as

$$ (a, a_S)_c\cup b = (a \cup b, a_S \cup \mathrm{res}_{S_f}(b)) $$

and satisfies

$$ d((a, a_S)_c\cup b) = (d(a, a_S))_c\cup b + (-1)^{\deg(a)}(a, a_S)_c\cup db. $$

5.3.3.3. The second cup product

$$ \cup_c : C^\bullet_{\mathrm{cont}}(G_{K,S}, A^\bullet) \otimes_R C^\bullet_{c,\mathrm{cont}}(G_{K,S}, B^\bullet) \longrightarrow C^\bullet_{c,\mathrm{cont}}(G_{K,S}, A^\bullet \otimes_R B^\bullet) $$

is defined as

$$ a\cup_c(b, b_S) = \big(a \cup b, (-1)^{\deg(a)}\mathrm{res}_{S_f}(a) \cup b_S\big) $$

and satisfies

$$ d(a\cup_c(b, b_S)) = (da)\cup_c(b, b_S) + (-1)^{\deg(a)} a\cup_c(d(b, b_S)). $$

5.3.3.4. The involutions $\mathcal{T}$ from 3.4.5.4 define morphisms of complexes

$$ \begin{array}{rcl} \mathcal{T} : C^\bullet_{c,\mathrm{cont}}(G_{K,S}, A^\bullet) & \longrightarrow & C^\bullet_{c,\mathrm{cont}}(G_{K,S}, A^\bullet) \\ (a, a_S) & \longmapsto & (\mathcal{T}(a), \mathcal{T}(a_S)) \end{array} $$

which are again involutions homotopic to the identity. As in 3.4.5.4, the following diagram of morphisms of complexes is commutative (and the same is true if the roles of ${}_c\cup$ and $\cup_c$ are interchanged):

$$ \begin{array}{ccc} C^\bullet_{c,\mathrm{cont}}(G_{K,S}, A^\bullet) \otimes_R C^\bullet_{\mathrm{cont}}(G_{K,S}, B^\bullet) & \xrightarrow{{}_c\cup} & C^\bullet_{c,\mathrm{cont}}(G_{K,S}, A^\bullet \otimes_R B^\bullet) \\ \Big\downarrow{\scriptstyle s_{12}\circ(\mathcal{T}\otimes\mathcal{T})} & & \Big\downarrow{\scriptstyle \mathcal{T}\circ(s_{12})_*} \\ C^\bullet_{\mathrm{cont}}(G_{K,S}, B^\bullet) \otimes_R C^\bullet_{c,\mathrm{cont}}(G_{K,S}, A^\bullet) & \xrightarrow{\cup_c} & C^\bullet_{c,\mathrm{cont}}(G_{K,S}, B^\bullet \otimes_R A^\bullet) \end{array} $$

5.3.3.5. The products ${}_c\cup, \cup_c$ (resp., the involutions $\mathcal{T}$) are compatible with the product $\cup$ (resp., the involution $\mathcal{T}$) defined in3.5.4 (*via* the canonical maps $C^\bullet_{c,\mathrm{cont}}(G_{K,S}, -) \to C^\bullet_{\mathrm{cont}}(G_{K,S}, -)$).

5.3.4. The statements of Propositions 4.2.9–4.2.10 and 4.3.3 also hold for the functor $\mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, -)$.

5.3.5. Euler-Poincaré characteristic. — If M is a finite discrete $\mathbf{F}_p[G_{K,S}]$-module, then Tate's global Euler-Poincaré characteristic formula ([**N-S-W**, Thm. 8.6.14]) implies that

$$ \sum_{q=0}^{3}(-1)^q \dim_{\mathbf{F}_p} H^q_c(G_{K,S}, M) = \sum_{v\mid\infty} \dim_{\mathbf{F}_p}\big(M^{G_v}\big). $$

A straightforward modification of the arguments in Sect. 4.6 gives the following result.

5.3.6. Theorem. — *If $T^\bullet$ is a bounded complex in $\left({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\right)_{R\text{-}ft}$, then*

$$\sum_q (-1)^q\, e_R(H^q_{c,\mathrm{cont}}(G_{K,S},T^\bullet)) = \sum_{v|\infty}\sum_q (-1)^q\, e_R\big((T^q)^{G_v}\big).$$

5.3.7. For R finite and flat over $\mathbf{Z}_p$ (and not necessarily commutative), Flach [**Fl2**] proved a more refined Euler-Poincaré characteristic formula.

5.4. Duality for $G_{K,S}$

5.4.1. The sequence (5.3.1.2) together with 5.1.5 define an isomorphism

$$H^3_{c,\mathrm{cont}}(G_{K,S},\mathbf{Z}/p^n\mathbf{Z}(1)) \xrightarrow{\sim} \mathbf{Z}/p^n\mathbf{Z}.$$

As in 5.2.1 we get

$$H^i_{c,\mathrm{cont}}(G_{K,S},A(1)) \xrightarrow{\sim} \begin{cases} A & i=3 \\ 0 & i>3 \end{cases}$$

for every R-module A with trivial action of $G_{K,S}$, hence a quasi-isomorphism (induced by $\mathrm{inv}_{S_f}^{-1}$)

$$A[-3] \xrightarrow{i} \tau_{\geqslant 3}\, C^\bullet_{c,\mathrm{cont}}(G_{K,S},A(1)). \tag{5.4.1.1}$$

Let $A^\bullet$ be a bounded below complex of R-modules with trivial action of $G_{K,S}$. For each $i\in\mathbf{Z}$ the map

$$\mathrm{res}_{S_f} : H^2_{\mathrm{cont}}(G_{K,S},A^i(1)) \longrightarrow \bigoplus_{v\in S_f} H^2_{\mathrm{cont}}(G_v,A^i(1))$$

is injective, hence the canonical morphism of complexes

$$\tau_{\geqslant 3}\, C^\bullet_{c,\mathrm{cont}}(G_{K,S},A^i(1)) \longrightarrow \mathrm{Cone}\left(\tau_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_{K,S},A^i(1)) \xrightarrow{\mathrm{res}_{S_f}} \bigoplus_{v\in S_f}\tau_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_v,A^i(1))\right)[-1]$$

is a quasi-isomorphism. This implies that the canonical map

$$\begin{aligned}\tau^{\mathrm{II}}_{\geqslant 3}\, C^\bullet_{c,\mathrm{cont}}(G_{K,S},A^\bullet(1)) &= \mathrm{Tot}\big(i\longmapsto \tau_{\geqslant 3}\, C^\bullet_{c,\mathrm{cont}}(G_{K,S},A^i(1))\big) \\ &\longrightarrow \mathrm{Cone}\left(\tau^{\mathrm{II}}_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_{K,S},A^\bullet(1)) \xrightarrow{\mathrm{res}_{S_f}} \bigoplus_{v\in S_f}\tau^{\mathrm{II}}_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_v,A^\bullet(1))\right)[-1]\end{aligned} \tag{5.4.1.2}$$

is also a quasi-isomorphism.

As in 5.2.2, fix a bounded complex $J = J^\bullet$ of injective R-modules. We claim that there is a homotopy equivalence i_S making the following diagram commutative up to homotopy:

$$\begin{array}{ccccc}
\tau^{\mathrm{II}}_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_{K,S}, J(1)) & \xrightarrow{\mathrm{res}_{S_f}} & \bigoplus_{v\in S_f} \tau^{\mathrm{II}}_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_v, J(1)) & \xrightarrow{j} & \mathrm{Cone}(\mathrm{res}_{S_f}) \\
 & & \uparrow{\scriptstyle (i_v)} & & \uparrow{\scriptstyle i_S} \\
 & & \bigoplus_{v\in S_f} J[-2] & \xrightarrow{\Sigma} & J[-2].
\end{array}$$

Indeed, fix $v_0 \in S_f$ and put $i_S = j \circ i_{v_0}$. Then i_S is a quasi-isomorphism, hence a homotopy equivalence (as $J[-2]$ is a bounded below complex of injectives).

Fix homotopy inverses $r_{v,J}$ of i_v ($v \in S_f$) and also a homotopy inverse $r_S = r_{S,J}$ of i_S. Then $r_S \circ j$ is homotopic to $\Sigma \circ (r_{v,J})$. Composing $r_S[-1]$ with the map (5.4.1.2) we obtain a quasi-isomorphism

$$r_J : \tau^{\mathrm{II}}_{\geqslant 3}\, C^\bullet_{c,\mathrm{cont}}(G_{K,S}, J(1)) \longrightarrow J[-3], \tag{5.4.1.3}$$

unique up to homotopy.

5.4.2. Let $X^\bullet$ be a bounded complex of admissible $R[G_{K,S}]$-modules. As in 5.2.2, the cup product ${}_c\cup$, together with the evaluation map ev_2 (resp., ev_1) and the map (5.4.1.3) define morphisms of complexes

$$\begin{aligned}
{}_c\beta_{J,X^\bullet} : C^\bullet_{c,\mathrm{cont}}(G_{K,S}, X^\bullet) &\longrightarrow \mathrm{Hom}^\bullet_R\big(C^\bullet_{\mathrm{cont}}(G_{K,S}, D_J(X^\bullet)(1)), C^\bullet_{c,\mathrm{cont}}(G_{K,S}, J(1))\big) \\
&\longrightarrow \mathrm{Hom}^\bullet_R\big(C^\bullet_{\mathrm{cont}}(G_{K,S}, D_J(X^\bullet)(1)), \tau^{\mathrm{II}}_{\geqslant 3}\, C^\bullet_{c,\mathrm{cont}}(G_{K,S}, J(1))\big) \\
&\xrightarrow{\mathrm{Hom}^\bullet(\mathrm{id}, r_J)} \mathrm{Hom}^\bullet_R(C^\bullet_{\mathrm{cont}}(G_{K,S}, D_J(X^\bullet)(1)), J[-3]) \\
&= D_{J[-3]}\big(C^\bullet_{\mathrm{cont}}(G_{K,S}, D(X^\bullet)(1))\big)
\end{aligned}$$

resp.,

$$\begin{aligned}
{}_c\beta'_{J,X^\bullet} : C^\bullet_{c,\mathrm{cont}}(G_{K,S}, D_J(X^\bullet)(1)) &\longrightarrow \mathrm{Hom}^\bullet_R\big(C^\bullet_{\mathrm{cont}}(G_{K,S}, X^\bullet), C^\bullet_{c,\mathrm{cont}}(G_{K,S}, J(1))\big) \\
&\longrightarrow \mathrm{Hom}^\bullet_R\big(C^\bullet_{\mathrm{cont}}(G_{K,S}, X^\bullet), \tau^{\mathrm{II}}_{\geqslant 3}\, C^\bullet_{c,\mathrm{cont}}(G_{K,S}, J(1))\big) \\
&\xrightarrow{\mathrm{Hom}^\bullet(\mathrm{id}, r_J)} \mathrm{Hom}^\bullet_R(C^\bullet_{\mathrm{cont}}(G_{K,S}, X^\bullet), J[-3]) \\
&= D_{J[-3]}\big(C^\bullet_{\mathrm{cont}}(G_{K,S}, X^\bullet)\big),
\end{aligned}$$

hence canonical maps (independent of the choice of r)

$${}_c\beta_{J,X} : \mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, X) \longrightarrow D_{J[-3]}\big(\mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, D_J(X)(1))\big)$$

resp.,

$${}_c\beta'_{J,X} : \mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, D_J(X)(1)) \longrightarrow D_{J[-3]}(\mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, X))$$

in $D^b({}_R\mathrm{Mod})$ (for every $X \in D^b\big({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)$).

Similarly, using $\cup_c$ instead of ${}_c\cup$, one obtains morphisms of complexes

$$\beta_{c,J,X^\bullet} : C^\bullet_{\mathrm{cont}}(G_{K,S}, X^\bullet) \longrightarrow D_{J[-3]}\big(C^\bullet_{c,\mathrm{cont}}(G_{K,S}, D_J(X^\bullet)(1))\big)$$
$$\beta'_{c,J,X^\bullet} : C^\bullet_{\mathrm{cont}}(G_{K,S}, D_J(X^\bullet)(1)) \longrightarrow D_{J[-3]}(C^\bullet_{c,\mathrm{cont}}(G_{K,S}, X^\bullet))$$

resp., morphisms

$$\beta_{c,J,X} : \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, X) \longrightarrow D_{J[-3]}\big(\mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, D_J(X)(1))\big)$$
$$\beta'_{c,J,X} : \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, D_J(X)(1)) \longrightarrow D_{J[-3]}(\mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, X))$$

in $D^b({}_R\mathrm{Mod})$.

As in 5.2.2, $\cup_c$ and ${}_c\cup$ induce, for each $X \in D^b\big({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)$, cup products

$$\mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, X) \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, D_J(X)(1)) \longrightarrow J[-3]$$
$$\mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, X) \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, D_J(X)(1)) \longrightarrow J[-3]$$

and pairings on cohomology

$$(5.4.2.1) \qquad \begin{aligned} H^i_{c,\mathrm{cont}}(G_{K,S}, X) \otimes_R H^j_{\mathrm{cont}}(G_{K,S}, D_J(X)(1)) &\longrightarrow H^{i+j-3}(J^\bullet) \\ H^i_{\mathrm{cont}}(G_{K,S}, X) \otimes_R H^j_{c,\mathrm{cont}}(G_{K,S}, D_J(X)(1)) &\longrightarrow H^{i+j-3}(J^\bullet), \end{aligned}$$

as well as analogous products in which the roles of X and $D_J(X)(1)$ are interchanged.

5.4.3. Proposition. — *Assume that either*

(i) $J = I[n]$ *and* $X \in D^b_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)$ *or* $X \in D^b_{R\text{-}coft}\big({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)$,
or
(ii) $J = \omega^\bullet[n]$ *and* $X \in D^b_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)$.

Then the maps ${}_c\beta_{J,X}, {}_c\beta'_{J,X}, \beta_{c,J,X}, \beta'_{c,J,X}$ *are isomorphisms in* $D^b({}_R\mathrm{Mod})$.

Proof. — As in the proof of Proposition 5.2.4 one reduces to the case when M is a finite discrete $\mathbf{Z}/p^n\mathbf{Z}[G_{K,S}]$-module. In this case the statement is a variant of the Poitou-Tate global duality theorem (5.1.6) ([**Ni**, Lemma 6.1]; [**N-S-W**, §8.6.13]). This explains the origin of the non-obvious maps in 5.1.6: they are induced by the cup products

$$\cup_c : H^i(G_{K,S}, M) \times H^{3-i}_c(G_{K,S}, M^*(1)) \longrightarrow H^3_c(G_{K,S}, \mathbf{Z}/p^n\mathbf{Z}(1)) \xrightarrow{\sim} \mathbf{Z}/p^n\mathbf{Z}$$
$${}_c\cup : H^j_c(G_{K,S}, M) \times H^{3-j}(G_{K,S}, M^*(1)) \longrightarrow H^3_c(G_{K,S}, \mathbf{Z}/p^n\mathbf{Z}(1)) \xrightarrow{\sim} \mathbf{Z}/p^n\mathbf{Z}. \quad \square$$

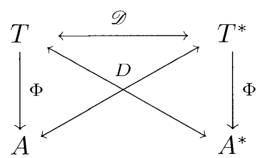
5.4.4. Proposition. — *If $f : X \to Y$ is a morphism in $D^b({}_{R[G_{K,S}]}^{\mathrm{ad}}\mathrm{Mod})$, then*

$$\beta_{c,J,X} = \beta'_{c,J,D_J(X)(1)} \circ (\varepsilon_J)_* = D_{J[-3]}({}_c\beta'_{J,X}) \circ \varepsilon_{J[-3]},$$
$$D_{J[-3]}((D_J(f)(1))_*) \circ \beta_{c,J,X} = \beta_{c,J,Y} \circ f_*$$
$${}_c\beta_{J,X} = {}_c\beta'_{J,D_J(X)(1)} \circ (\varepsilon_J)_* = D_{J[-3]}(\beta'_{c,J,X}) \circ \varepsilon_{J[-3]},$$
$$D_{J[-3]}((D_J(f)(1))_*) \circ {}_c\beta_{J,X} = {}_c\beta_{J,Y} \circ f_*$$
$$\beta'_{c,J,X} = D_{J[-3]}((\varepsilon_J)_*) \circ \beta_{c,J,D_J(X)(1)} = D_{J[-3]}({}_c\beta_{J,X}) \circ \varepsilon_{J[-3]},$$
$$\beta'_{c,J,X} \circ (D_J(f)(1))_* = D_{J[-3]}(f_*) \circ \beta'_{c,J,Y}$$
$${}_c\beta'_{J,X} = D_{J[-3]}((\varepsilon_J)_*) \circ {}_c\beta_{J,D_J(X)(1)} = D_{J[-3]}(\beta_{c,J,X}) \circ \varepsilon_{J[-3]},$$
$${}_c\beta'_{J,X} \circ (D_J(f)(1))_* = D_{J[-3]}(f_*) \circ {}_c\beta'_{J,Y}$$

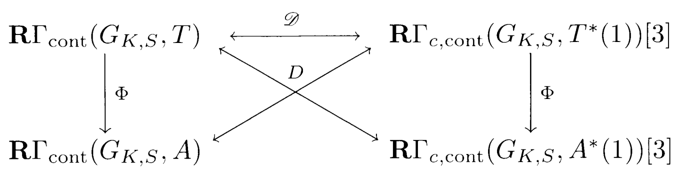
(equalities of morphisms in $D^b({}_R\mathrm{Mod})$).

Proof. — This follows from a variant of Proposition 5.2.7, which is proved in the same way as 5.2.7; the only difference is that the commutative diagram from 3.4.5.4 (with $A^\bullet = X^\bullet$, $B^\bullet = D_J(X^\bullet)(1)$) has to be replaced by an analogous diagram from 5.3.3.4. □

5.4.5. Theorem. — *If $T, T^* \in D^b_{R\text{-}ft}({}_{R[G_{K,S}]}^{\mathrm{ad}}\mathrm{Mod})$, $A, A^* \in D^b_{R\text{-}coft}({}_{R[G_{K,S}]}^{\mathrm{ad}}\mathrm{Mod})$ are related by the duality diagram*



so are



(in $D^b_{(co)ft}({}_R\mathrm{Mod})$) and there are spectral sequences

$$E_2^{i,j} = \mathrm{Ext}^i_R(H^{3-j}_{c,\mathrm{cont}}(G_{K,S}, T^*(1)), \omega)$$
$$= \mathrm{Ext}^i_R(D(H^j_{\mathrm{cont}}(G_{K,S}, A)), \omega) \Longrightarrow H^{i+j}_{\mathrm{cont}}(G_{K,S}, T)$$
$${}'E_2^{i,j} = \mathrm{Ext}^i_R(H^{3-j}_{\mathrm{cont}}(G_{K,S}, T^*(1)), \omega)$$
$$= \mathrm{Ext}^i_R(D(H^j_{c,\mathrm{cont}}(G_{K,S}, A)), \omega) \Longrightarrow H^{i+j}_{c,\mathrm{cont}}(G_{K,S}, T).$$

Proof. — As in 5.2.6, everything follows from Proposition 5.4.3. □

5.5. Duality for Poincaré groups

5.5.1. Recall ([**Ve1**, §4]) that a profinite group G is a *Poincaré group* with respect to a prime number p if the following two conditions are satisfied:

5.5.1.1. $\mathrm{cd}_p(G) = n < \infty$.

5.5.1.2. The abelian groups D_i defined by

$$D_i = \varinjlim_m \varinjlim_U \mathrm{Hom}_{\mathbf{Z}}(H^i(U, \mathbf{Z}/p^m\mathbf{Z}), \mathbf{Q}_p/\mathbf{Z}_p)$$

(where U runs through all open subgroups of G and the transition maps are dual to corestrictions) satisfy

$$D_i \xrightarrow{\sim} \begin{cases} 0, & i \neq n \\ \mathbf{Q}_p/\mathbf{Z}_p, & i = n. \end{cases}$$

There is a natural action of G on each D_i; under the above conditions G acts on D_n by a character

$$\chi : G \longrightarrow \mathbf{Z}_p^*.$$

5.5.2. Examples

(1) G is a compact p-adic Lie group with finite $\mathrm{cd}_p(G)$.
(2) $G = G_v$ in 5.1. In this case $n = 2$ and χ is equal to the cyclotomic character.

5.5.3. The main duality result for Poincaré groups ([**Ve1**, Prop. 4.4]) states that there is a canonical isomorphism

$$\rho : H^n(G, D_n) \xrightarrow{\sim} \mathbf{Q}_p/\mathbf{Z}_p$$

such that for every finite p-primary discrete G-module M and $i \in \mathbf{Z}$ the cup product

$$H^i(G, M) \times H^{n-i}(G, \mathrm{Hom}_{\mathbf{Z}}(M, D_n)) \xrightarrow{\cup} H^n(G, D_n) \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p$$

induces an isomorphism

$$H^{n-i}(G, \mathrm{Hom}_{\mathbf{Z}}(M, D_n)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}}(H^i(G, M), \mathbf{Q}_p/\mathbf{Z}_p).$$

5.5.4. The results and proofs in Sect. 5.2 work for a general Poincaré group G satisfying (F). For example, a generalization of Theorem 5.2.6 yields a digram

$$\begin{array}{ccc} \mathbf{R}\Gamma_{\mathrm{cont}}(G, T) & \xleftrightarrow{\mathscr{D}} & \mathbf{R}\Gamma_{\mathrm{cont}}(G, T^*(\chi))[n] \\ \downarrow{\scriptstyle\Phi} & \overset{D}{\times} & \downarrow{\scriptstyle\Phi} \\ \mathbf{R}\Gamma_{\mathrm{cont}}(G, A) & & \mathbf{R}\Gamma_{\mathrm{cont}}(G, A^*(\chi))[n], \end{array}$$

where $M(\chi)$ denotes M with the G-action twisted by the character χ (for every $\mathbf{Z}_p[G]$-module M).

A generalization of Proposition 5.2.4 gives canonical isomorphisms

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G,X) \xrightarrow{\sim} D_J\big(\mathbf{R}\Gamma_{\mathrm{cont}}(G,D_{J,G}(X))\big),$$

where

$$D_{J,G}(X) = D_J(X)(\chi)[n].$$

5.6. Localization

5.6.1. Let $\mathscr{S} \subset R$ be a multiplicative subset. Everything in Sections 5.2.1–5.2.2, 5.2.7–5.2.9, 5.4.1–5.4.2 and 5.4.4 remains valid if we replace R by $R_{\mathscr{S}}$.

Instead of 5.2.3, we shall be interested in the following case:

5.6.2. $J = \omega^{\bullet}_{R_{\mathscr{S}}}[n]$ for some $n \in \mathbf{Z}$ (hence $D_J = \mathscr{D}_{R_{\mathscr{S}},n}$) and all cohomology groups of X are of finite type over $R_{\mathscr{S}}$.

5.6.3. Proposition. — *Assume that $J = \omega^{\bullet}_{R_{\mathscr{S}}}[n]$ for some $n \in \mathbf{Z}$. Then:*

(i) *For every $X \in D^b_{R_{\mathscr{S}}\text{-}ft}\big({}^{\mathrm{ad}}_{R_{\mathscr{S}}[G_v]}\mathrm{Mod}\big)$, the canonical map*

$$\alpha_{J,X} : \mathbf{R}\Gamma_{\mathrm{cont}}(G_v,X) \longrightarrow \mathscr{D}_{R_{\mathscr{S}},n-2}\big(\mathbf{R}\Gamma_{\mathrm{cont}}\big(G_v,\mathscr{D}_{R_{\mathscr{S}},n}(X)(1)\big)\big)$$

is an isomorphism in $D^b_{ft}\big({}_{R_{\mathscr{S}}}\mathrm{Mod}\big)$ (and the same is true for $\alpha'_{J,X}$).

(ii) *For every $X \in D^b_{R_{\mathscr{S}}-ft}\big({}^{\mathrm{ad}}_{R_{\mathscr{S}}[G_{K,S}]}\mathrm{Mod}\big)$, the canonical map*

$${}_c\beta_{J,X} : \mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S},X) \longrightarrow \mathscr{D}_{R_{\mathscr{S}},n-2}\big(\mathbf{R}\Gamma_{\mathrm{cont}}\big(G_{K,S},\mathscr{D}_{R_{\mathscr{S}},n}(X)(1)\big)\big)$$

is an isomorphism in $D^b_{ft}\big({}_{R_{\mathscr{S}}}\mathrm{Mod}\big)$ (and the same is true for ${}_c\beta'_{J,X}$, $\beta_{c,J,X}$ and $\beta'_{c,J,X}$).

Proof. — By dévissage we reduce to the case when X is an admissible $R_{\mathscr{S}}[G_v]$-module, of finite type over $R_{\mathscr{S}}$. Then $X = \mathscr{S}^{-1}Y$ for some $Y \in \big({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\big)_{R\text{-}ft}$, by Lemma 3.7.3, and $\alpha_{J,X}$ is the localization

$$\mathscr{S}^{-1}\mathbf{R}\Gamma_{\mathrm{cont}}(G_v,Y) \longrightarrow \mathscr{S}^{-1}\big(\mathscr{D}_{R,n-2}\big(\mathbf{R}\Gamma_{\mathrm{cont}}(G_v,\mathscr{D}_{R,n}(Y)(1))\big)\big)$$

of the isomorphism $\alpha_{\omega^{\bullet}_R[n],Y}$ from Proposition 5.2.4. The same argument applies in the case (ii), with 5.4.3 replacing 5.2.4. □

5.7. In the absence of (P)

5.7.1. Assume that the condition (P) from 5.1 fails, *i.e.*, $p = 2$ and K is a number field with at least one real prime. In this case the global duality must also take into account the Tate cohomology groups $\widehat{H}^i(G_v,-)$ for real primes v (where $G_v = \mathrm{Gal}(\mathbf{C}/\mathbf{R})$ has order two). The statements in 5.1 then have to be modified as follows: for every real prime v,

5.7.1.1. G_v and $G_{K,S}$ satisfy the finiteness condition (F').

5.7.1.2. $\mathrm{cd}_2(G_v) = \mathrm{cd}_2(G_{K,S}) = \infty$.

5.7.1.3. For every $n \geqslant 1$, local class field theory defines an isomorphism

$$\mathrm{inv}_v : H^2(G_v, \mathbf{Z}/2^n\mathbf{Z}(1)) \xrightarrow{\sim} \mathbf{Z}/2\mathbf{Z}.$$

5.7.1.4. Local duality. — For every finite discrete $\mathbf{Z}/2^n\mathbf{Z}[G_v]$-module M, the cup product

$$\widehat{H}^i(G_v, M) \times \widehat{H}^{2-i}\big(G_v, \mathrm{Hom}(M, \mathbf{Z}/2^n\mathbf{Z}(1))\big) \xrightarrow{\cup} H^2(G_v, \mathbf{Z}/2^n\mathbf{Z}(1)) \xrightarrow{\sim} \mathbf{Z}/2\mathbf{Z}$$

is a perfect pairing of finite $\mathbf{Z}/2\mathbf{Z}$-modules ($i \in \mathbf{Z}$).

5.7.1.5. Reciprocity law. — The sum of the local invariants $\mathrm{inv}_S = \sum_{v \in S} \mathrm{inv}_v$ defines an exact sequence

$$0 \longrightarrow H^2(G_{K,S}, \mathbf{Z}/2^n\mathbf{Z}(1)) \longrightarrow \bigoplus_{v \in S} H^2(G_v, \mathbf{Z}/2^n\mathbf{Z}(1)) \xrightarrow{\mathrm{inv}_S} \mathbf{Z}/2^n\mathbf{Z} \longrightarrow 0$$

(where $\mathrm{inv}_v = 0$ for each complex prime v).

5.7.1.6. Global duality (Poitou-Tate). — For every finite discrete $\mathbf{Z}/2^n\mathbf{Z}[G_{K,S}]$-module M there is an exact sequence of finite $\mathbf{Z}/2^n\mathbf{Z}$-modules

$$\begin{aligned} 0 \longrightarrow & H^0(G_{K,S}, M) \longrightarrow \bigoplus_{v \in S_f} H^0(G_v, M) \oplus \bigoplus_{K_v = \mathbf{R}} \widehat{H}^0(G_v, M) \\ \longrightarrow H^2(G_{K,S}, M^*(1))^* \longrightarrow & H^1(G_{K,S}, M) \longrightarrow \bigoplus_{v \in S_f} H^1(G_v, M) \oplus \bigoplus_{K_v = \mathbf{R}} \widehat{H}^1(G_v, M) \\ \longrightarrow H^1(G_{K,S}, M^*(1))^* \longrightarrow & H^2(G_{K,S}, M) \longrightarrow \bigoplus_{v \in S_f} H^2(G_v, M) \oplus \bigoplus_{K_v = \mathbf{R}} \widehat{H}^2(G_v, M) \\ \longrightarrow H^0(G_{K,S}, M^*(1))^* \longrightarrow & \ 0, \end{aligned}$$

in which $(-)^* = \mathrm{Hom}(-, \mathbf{Z}/2^n\mathbf{Z})$.

5.7.1.7. For each M as in 5.7.1.6 and $i > 2$, the map

$$\mathrm{res}_S : H^i(G_{K,S}, M) \longrightarrow \bigoplus_{v \in S_f} H^i(G_v, M) \oplus \bigoplus_{K_v = \mathbf{R}} \widehat{H}^i(G_v, M) = \bigoplus_{K_v = \mathbf{R}} \widehat{H}^i(G_v, M)$$

is an isomorphism.

5.7.2. In the present situation, the constructions in 5.3 should be modified as follows. For each real prime v of K, the usual definition of the complete (Tate) cochain complex $\widehat{C}^\bullet_{\mathrm{cont}}(G_v, M)$ of a G_v-module M extends naturally to complexes of G_v-modules by

$$\widehat{C}^n(G_v, M^\bullet) = \bigoplus_{i+j=n} \widehat{C}^j(G_v, M^i)$$

and using the sign rules from 3.4.1.3. The standard cup products ([**C-E**, Ch. XII])

$$\cup : \widehat{C}^\bullet_{\mathrm{cont}}(G_v, M) \otimes \widehat{C}^\bullet_{\mathrm{cont}}(G_v, N) \longrightarrow \widehat{C}^\bullet_{\mathrm{cont}}(G_v, M \otimes N)$$

extend, using the sign rules from 3.4.5.2, to the case of complexes of G_v-modules

$$\text{(5.7.2.1)} \qquad \cup : \widehat{C}^\bullet_{\mathrm{cont}}(G_v, M^\bullet) \otimes \widehat{C}^\bullet_{\mathrm{cont}}(G_v, N^\bullet) \longrightarrow \widehat{C}^\bullet_{\mathrm{cont}}(G_v, M^\bullet \otimes N^\bullet).$$

For every complex $M^\bullet$ of ind-admissible $R[G_{K,S}]$-modules we define

$$\widehat{C}^\bullet_{c,\mathrm{cont}}(G_{K,S}, M^\bullet) = \mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(G_{K,S}, M^\bullet) \xrightarrow{\mathrm{res}_S} \bigoplus_{v \in S_f} C^\bullet_{\mathrm{cont}}(G_v, M^\bullet) \oplus \bigoplus_{K_v = \mathbf{R}} \widehat{C}^\bullet_{\mathrm{cont}}(G_v, M^\bullet) \right)[-1],$$

where the map res_v for a real prime v is given by

$$C^\bullet_{\mathrm{cont}}(G_{K,S}, M^\bullet) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v, M^\bullet) \hookrightarrow \widehat{C}^\bullet_{\mathrm{cont}}(G_v, M^\bullet).$$

The cohomology groups $\widehat{H}^i_{c,\mathrm{cont}}(G_{K,S}, -)$ of $\widehat{C}^\bullet_{c,\mathrm{cont}}(G_{K,S}, -)$ lie in the exact sequence

$$\text{(5.7.2.2)} \quad \cdots \longrightarrow \widehat{H}^i_{c,\mathrm{cont}}(G_{K,S}, M^\bullet) \longrightarrow H^i_{\mathrm{cont}}(G_{K,S}, M^\bullet)$$
$$\xrightarrow{\mathrm{res}_S} \bigoplus_{v \in S_f} H^i_{\mathrm{cont}}(G_v, M^\bullet) \oplus \bigoplus_{K_v = \mathbf{R}} \widehat{H}^i(G_v, M^\bullet) \longrightarrow \widehat{H}^{i+1}_{c,\mathrm{cont}}(G_{K,S}, M^\bullet) \longrightarrow \cdots$$

The functor

$$M^\bullet \longrightarrow \widehat{C}^\bullet_{c,\mathrm{cont}}(G_{K,S}, M^\bullet)$$

gives rise to an exact functor

$$\widehat{\mathbf{R}\Gamma}_{c,\mathrm{cont}}(G_{K,S}, -) : D^b\big({}^{\text{ind-ad}}_{R[G_{K,S}]}\mathrm{Mod}\big) \longrightarrow D({}_R\mathrm{Mod}).$$

The formulas from 5.3.3.2–5.3.3.3 together with (5.7.2.1) define products

$$\widehat{\cup}_c : C^\bullet_{\mathrm{cont}}(G_{K,S}, M^\bullet) \otimes_R \widehat{C}^\bullet_{c,\mathrm{cont}}(G_{K,S}, N^\bullet) \longrightarrow \widehat{C}^\bullet_{c,\mathrm{cont}}(G_{K,S}, M^\bullet \otimes_R N^\bullet)$$
$${}_c\widehat{\cup} : \widehat{C}^\bullet_{c,\mathrm{cont}}(G_{K,S}, M^\bullet) \otimes_R C^\bullet_{\mathrm{cont}}(G_{K,S}, N^\bullet) \longrightarrow \widehat{C}^\bullet_{c,\mathrm{cont}}(G_{K,S}, M^\bullet \otimes_R N^\bullet),$$

for any pair of complexes of ind-admissible $R[G_{K,S}]$-modules $M^\bullet, N^\bullet$.

5.7.3. Lemma

(i) *If $M^\bullet = M$ is concentrated in degree zero, then*

$$(\forall i > 3) \quad \widehat{H}^i_{c,\mathrm{cont}}(G_{K,S}, M) = 0.$$

(ii) *If A is any R-module with trivial action of $G_{K,S}$, then* inv_S *induces an isomorphism*

$$\widehat{H}^3_{c,\mathrm{cont}}(G_{K,S}, A(1)) \xrightarrow{\sim} A.$$

Proof. — By a standard limit procedure one reduces to the case when M is a finite discrete $\mathbf{Z}/2^n\mathbf{Z}[G_{K,S}]$-module, in which case the statement follows from 5.7.1.7 and (5.7.2.2). Similarly, it is sufficient to consider only the case $A = \mathbf{Z}/2^n\mathbf{Z}$ in (ii), when we conclude by 5.7.1.5, 5.7.1.7 and (5.7.2.2). ☐

5.7.4. Proposition. — *If M is a finite discrete $\mathbf{Z}/2^n\mathbf{Z}[G_{K,S}]$-module, then the cup products $\widehat{\cup}_c$, ${}_c\widehat{\cup}$ induce perfect pairings of finite $\mathbf{Z}/2^n\mathbf{Z}$-modules*

$$H^i(G_{K,S}, M) \times \widehat{H}^{3-i}_{c,\mathrm{cont}}\big(G_{K,S}, \mathrm{Hom}(M, \mathbf{Z}/2^n\mathbf{Z}(1))\big) \longrightarrow \widehat{H}^3_{c,\mathrm{cont}}(G_{K,S}, \mathbf{Z}/2^n\mathbf{Z}(1)) \xrightarrow{\sim} \mathbf{Z}/2^n\mathbf{Z}$$

$$\widehat{H}^i_{c,\mathrm{cont}}(G_{K,S}, M) \times H^{3-i}\big(G_{K,S}, \mathrm{Hom}(M, \mathbf{Z}/2^n\mathbf{Z}(1))\big) \longrightarrow \widehat{H}^3_{c,\mathrm{cont}}(G_{K,S}, \mathbf{Z}/2^n\mathbf{Z}(1)) \xrightarrow{\sim} \mathbf{Z}/2^n\mathbf{Z}$$

(for all $i \in \mathbf{Z}$).

Proof. — This follows from 5.7.1.4, 5.7.1.6, 5.7.1.7 and the compatibility of the various products involved. $\square$

5.7.5. All constructions from 5.4 work in the present context, provided that $C^\bullet_{c,\mathrm{cont}}(G_{K,S}, -)$ (resp., $\mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, -)$) is replaced everywhere by $\widehat{C}^\bullet_{c,\mathrm{cont}}(G_{K,S}, -)$ (resp., $\widehat{\mathbf{R}\Gamma}_{c,\mathrm{cont}}(G_{K,S}, -)$). The final result can be summed up as follows: if

$$\begin{array}{ccc} T & \overset{\mathscr{D}}{\longleftrightarrow} & T^* \\ \Big\downarrow \Phi & \overset{D}{\times} & \Big\downarrow \Phi \\ A & & A^* \end{array}$$

is a duality diagram with $T, T^* \in D^b_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)$ and $A, A^* \in D^b_{R\text{-}coft}\big({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)$, then

$$\begin{array}{ccc} \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, T) & \overset{\mathscr{D}}{\longleftrightarrow} & \widehat{\mathbf{R}\Gamma}_{c,\mathrm{cont}}(G_{K,S}, T^*(1))[3] \\ \Big\downarrow \Phi & \overset{D}{\times} & \Big\downarrow \Phi \\ \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, A) & & \widehat{\mathbf{R}\Gamma}_{c,\mathrm{cont}}(G_{K,S}, A^*(1))[3] \end{array}$$

is a duality diagram in $D_{(co)ft}({}_R\mathrm{Mod})$.

5.7.6. As $2 \cdot \widehat{H}^i(G_v, M) = 0$ for each real prime v and each G_v-module M, the canonical map in $D({}_R\mathrm{Mod})$

$$\mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, M^\bullet) \longrightarrow \widehat{\mathbf{R}\Gamma}_{c,\mathrm{cont}}(G_{K,S}, M^\bullet) \qquad \big(M^\bullet \in D^b\big({}^{\text{ind-ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)\big)$$

becomes an isomorphism in $D^b({}_{R[1/2]}\mathrm{Mod})$, where $R[1/2] = R \otimes_{\mathbf{Z}_2} \mathbf{Q}_2$.

CHAPTER 6

SELMER COMPLEXES

Classical Selmer groups consist of elements of $H^1(G_{K,S}, M)$ satisfying suitable local conditions in $H^1(G_v, M)$ at $v \in S$. We work in the derived category, which means that we have to modify this definition and impose local conditions on the level of complexes rather than cohomology. This is done in Sect. 6.1–6.2. The main abstract duality result, Theorem 6.3.4, is deduced from our version of the Poitou-Tate duality (Theorem 5.4.5) using the cup products from Sect. 1.3. The symmetry properties of these duality pairings are investigated in Sect. 6.5–6.6; they require additional data. In Sect. 6.7 we introduce the main example of "elementary" local conditions, following Greenberg [**Gre1, Gre2, Gre3**].

The assumptions of 5.1, including (P), are in force.

6.1. Definition of Selmer complexes

6.1.1. Let $X = X^\bullet$ be a complex of admissible $R[G_{K,S}]$-modules. **Local conditions** for X are given by a collection $\Delta(X) = (\Delta_v(X))_{v \in S_f}$, where each $\Delta_v(X)$ is a local condition at $v \in S_f$, consisting of a morphism of complexes of R-modules

$$i_v^+(X) : U_v^+(X) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v, X).$$

6.1.2. The **Selmer complex** associated to the local conditions $\Delta(X)$ is defined as

$$\widetilde{C}^\bullet_f(G_{K,S}, X; \Delta(X)) =$$

$$\mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(G_{K,S}, X) \oplus \bigoplus_{v \in S_f} U_v^+(X) \xrightarrow{\mathrm{res}_{S_f} - i_S^+(X)} \bigoplus_{v \in S_f} C^\bullet_{\mathrm{cont}}(G_v, X) \right)[-1]$$

(sometimes abbreviated as $\widetilde{C}^\bullet_f(X)$), where $i_S^+(X) = (i_v^+(X))_{v \in S_f}$. Denote by $\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X; \Delta(X))$ (sometimes abbreviated as $\widetilde{\mathbf{R}\Gamma}_f(X)$) the corresponding object of $D({}_R\mathrm{Mod})$ and by $\widetilde{H}^i_f(G_{K,S}, X; \Delta(X))$ (sometimes abbreviated as $\widetilde{H}^i_f(X)$) its cohomology.

If X and all $U_v^+(X)$ have cohomology of finite type over R, resp., of co-finite type over R, resp., are cohomologically bounded above, resp., are cohomologically bounded below, the same is true for $\widetilde{C}_f^\bullet(G_{K,S}, X; \Delta(X))$.

6.1.3. For $v \in S_f$ put

$$U_v^-(X) = \mathrm{Cone}\Big(U_v^+(X) \xrightarrow{-i_v^+(X)} C_{\mathrm{cont}}^\bullet(G_v, X)\Big)$$
$$U_S^\pm(X) = \bigoplus_{v \in S_f} U_v^\pm(X).$$

There are exact sequences of complexes (with maps induced by obvious inclusions resp., projections)

(6.1.3.1)
$$0 \longrightarrow C_{\mathrm{cont}}^\bullet(G_v, X)[-1] \longrightarrow U_v^-(X)[-1] \longrightarrow U_v^+(X) \longrightarrow 0$$
$$0 \longrightarrow C_{c,\mathrm{cont}}^\bullet(G_{K,S}, X) \longrightarrow \widetilde{C}_f^\bullet(G_{K,S}, X; \Delta(X)) \longrightarrow U_S^+(X) \longrightarrow 0$$
$$0 \longrightarrow U_S^-(X)[-1] \longrightarrow \widetilde{C}_f^\bullet(G_{K,S}, X; \Delta(X)) \longrightarrow C_{\mathrm{cont}}^\bullet(G_{K,S}, X) \longrightarrow 0,$$

which give rise to the following exact triangles in $D({}_R\mathrm{Mod})$:

(6.1.3.2)
$$U_v^+(X) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, X) \longrightarrow U_v^-(X) \longrightarrow U_v^+(X)[1]$$
$$U_S^+(X)[-1] \longrightarrow \mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, X) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(X) \longrightarrow U_S^+(X)$$
$$U_S^-(X)[-1] \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(X) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, X) \longrightarrow U_S^-(X).$$

We know from Proposition 5.4.3 that $\mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, X)$ is dual to $\mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, D_J(X)(1))[3]$ (under suitable assumptions). In order to deduce from (6.1.3.2) a duality between $\widetilde{\mathbf{R}\Gamma}_f(X)$ and $\widetilde{\mathbf{R}\Gamma}_f(D_J(X)(1))[3]$, we must ensure that $U_S^+(X)$ is (close to being) isomorphic to the dual of $U_S^-(D_J(X)(1))[2]$. Taking into account the duality between $\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, X)$ and $\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, D_J(X)(1))[2]$, this boils down to a suitable orthogonality relation between $U_v^+(X)$ and $U_v^+(D_J(X)(1))$, for all $v \in S_f$. This notion of orthogonality is introduced and studied in Sect. 6.2.

It is essential to have a *canonical* duality map between $\widetilde{\mathbf{R}\Gamma}_f(X)$ and the dual of $\widetilde{\mathbf{R}\Gamma}_f(D_J(X)(1))[3]$; this is why the additional data 6.2.1.3 enter the picture. The duality map itself is constructed in Sect. 6.3, using the abstract cup products from 1.3.

6.1.4. In the special case when

$$U_S^-(X) \longrightarrow \tau_{\geqslant 1}\, U_S^-(X)$$

is a quasi-isomorphism (which is equivalent to

$$\bigoplus_{v \in S_f} H^i(U_v^+(X)) \xrightarrow{(i_S^+(X))_*} \bigoplus_{v \in S_f} H^i_{\mathrm{cont}}(G_v, X)$$

being bijective for $i \leqslant 0$ and injective for $i = 1$), the canonical map

$$\tau_{\leqslant 0}\, \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X; \Delta(X)) \longrightarrow \tau_{\leqslant 0}\, \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, X)$$

is also a quasi-isomorphism, *i.e.*,

$$\widetilde{H}^i_f(G_{K,S}, X; \Delta(X)) \xrightarrow{\sim} H^i_{\mathrm{cont}}(G_{K,S}, X) \qquad (\forall i \leqslant 0)$$

and

$$\widetilde{H}^1_f(G_{K,S}, X; \Delta(X)) \xrightarrow{\sim} \mathrm{Ker}\left(H^1_{\mathrm{cont}}(G_{K,S}, X) \longrightarrow \bigoplus_{v \in S_f} H^1_{\mathrm{cont}}(G_v, X)/(i^+_v(X))_*\big(H^1(U^+_v(X))\big)\right)$$

coincides with a classical Selmer group given by the local conditions

$$(i^+_v(X))_*\big(H^1(U^+_v(X))\big) \subseteq H^1_{\mathrm{cont}}(G_v, X) \qquad (v \in S_f).$$

6.2. Orthogonal local conditions

Let $J = J^\bullet$, D_J and $r_{v,J}$ (for each $v \in S_f$) be as in 5.2.2.

6.2.1. Assume that X_1, X_2 are complexes of admissible $R[G_{K,S}]$-modules,

$$\pi : X_1 \otimes_R X_2 \longrightarrow J(1)$$

a morphism of complexes of $R[G_{K,S}]$-modules and

$$i^+_v(X_i) : U^+_v(X_i) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v, X_i) \qquad (i = 1, 2;\ v \in S_f)$$

local conditions for X_1, X_2.

6.2.1.1. Typical examples of π are

$$\begin{aligned} \mathrm{ev}_1 &: D_J(X_2)(1) \otimes_R X_2 \longrightarrow J(1) \\ \mathrm{ev}_2 &: X_1 \otimes_R D_J(X_1)(1) \longrightarrow J(1). \end{aligned}$$

In general, π factors as

$$\pi : X_1 \otimes_R X_2 \xrightarrow{\mathrm{adj}(\pi)\otimes\mathrm{id}} D_J(X_2)(1) \otimes_R X_2 \xrightarrow{\mathrm{ev}_1} J(1)$$

and induces another morphism of complexes

$$\pi \circ s_{12} : X_2 \otimes_R X_1 \xrightarrow{s_{12}} X_1 \otimes_R X_2 \xrightarrow{\pi} J(1),$$

which factors into

$$\pi \circ s_{12} : X_2 \otimes_R X_1 \xrightarrow{\mathrm{adj}(\pi\circ s_{12})\otimes\mathrm{id}} D_J(X_1)(1) \otimes_R X_1 \xrightarrow{\mathrm{ev}_1} J(1);$$

thus π also factors as

$$\pi : X_1 \otimes_R X_2 \xrightarrow{\mathrm{id}\otimes\mathrm{adj}(\pi\circ s_{12})} X_1 \otimes_R D_J(X_1)(1) \xrightarrow{\mathrm{ev}_2} J(1).$$

6.2.1.2. For each $v \in S_f$, denote by $\mathrm{prod}_v(X_1, X_2, \pi)$ the morphism of complexes

$$\mathrm{prod}_v(X_1, X_2, \pi) : U_v^+(X_1) \otimes_R U_v^+(X_2) \xrightarrow{i_v^+ \otimes i_v^+} C^\bullet_{\mathrm{cont}}(G_v, X_1) \otimes_R C^\bullet_{\mathrm{cont}}(G_v, X_2)$$
$$\xrightarrow{\cup} C^\bullet_{\mathrm{cont}}(G_v, X_1 \otimes_R X_2) \xrightarrow{\pi_*} C^\bullet_{\mathrm{cont}}(G_v, J(1)) \xrightarrow{\tau^{\mathrm{II}}_{\geqslant 2}} \tau^{\mathrm{II}}_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_v, J(1)).$$

6.2.1.3. Definition. — If there is a null-homotopy

$$h_v = h_v(X_1, X_2, \pi) : \mathrm{prod}_v(X_1, X_2, \pi) \rightsquigarrow 0,$$

we say that $\Delta_v(X_1)$ *is orthogonal to* $\Delta_v(X_2)$ *with respect to* π *and* h_v. Notation:

$$\Delta_v(X_1) \perp_{\pi, h_v} \Delta_v(X_2).$$

6.2.1.4. Definition. — We say that $\Delta(X_1)$ *is orthogonal to* $\Delta(X_2)$ *with respect to* π *and* $h_S = (h_v)_{v \in S_f}$ (notation: $\Delta(X_1) \perp_{\pi, h_S} \Delta(X_2)$) if $\Delta_v(X_1) \perp_{\pi, h_v} \Delta_v(X_2)$ for all $v \in S_f$.

6.2.1.5. If the morphism $\mathrm{prod}_v(X_1, X_2, \pi)$ is equal to zero, which happens very often in practice, then $\Delta_v(X_1) \perp_{\pi, 0} \Delta_v(X_2)$.

6.2.2. Local cup products. — Fix $v \in S_f$ and assume that $\Delta_v(X_1) \perp_{\pi, h_v} \Delta_v(X_2)$.

Recall that, for $i = 1, 2$, the complex

$$U_v^-(X_i)[-1] = \mathrm{Cone}\Big(U_v^+(X_i) \xrightarrow{-i_v^+(X_i)} C^\bullet_{\mathrm{cont}}(G_v, X_i)\Big)[-1]$$

has differential

$$d(b_i, c_i) = (db_i, i_v^+(b_i) - dc_i)$$

(for $b_i \in U_v^+(X_i)^j$, $c_i \in C^{j-1}_{\mathrm{cont}}(G_v, X_i)$, $j = \overline{b_i} = \overline{c_i} + 1 = \overline{(b_i, c_i)}$).

Denote by $\overset{\bullet}{\cup} = \overset{\bullet}{\cup}_\pi = \tau^{\mathrm{II}}_{\geqslant 2} \circ \pi_* \circ \cup$ the truncated cup product with values in $\tau^{\mathrm{II}}_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_v, J(1))$ from 6.2.1.2.

The formulas

$$(b_1, c_1) \cup_{-, h_v} b_2 = c_1 \overset{\bullet}{\cup} i_v^+(b_2) + h_v(b_1 \otimes b_2)$$
$$b_1 \cup_{+, h_v} (b_2, c_2) = (-1)^{\overline{b_1}} i_v^+(b_1) \overset{\bullet}{\cup} c_2 + h_v(b_1 \otimes b_2)$$

(cf. Proposition 1.3.2(i)) define morphisms of complexes

$$\cup_{-, \pi, h_v} : U_v^-(X_1)[-1] \otimes_R U_v^+(X_2) \longrightarrow \big(\tau^{\mathrm{II}}_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_v, J(1))\big)[-1] \xrightarrow{r_{v,J}[-1]} J[-3]$$
$$\cup_{+, \pi, h_v} : U_v^+(X_1) \otimes_R (U_v^-(X_2)[-1]) \longrightarrow \big(\tau^{\mathrm{II}}_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_v, J(1))\big)[-1] \xrightarrow{r_{v,J}[-1]} J[-3],$$

hence, by adjunction, morphisms of complexes

$$u_{-, \pi, h_v} = \mathrm{adj}(\cup_{-, \pi, h_v}) : U_v^-(X_1)[-1] \longrightarrow D_{J[-3]}(U_v^+(X_2))$$
$$u_{+, \pi, h_v} = \mathrm{adj}(\cup_{+, \pi, h_v}) : U_v^+(X_1) \longrightarrow D_{J[-3]}(U_v^-(X_2)[-1]).$$

6.2.3. Error terms. — Assuming that $\Delta_v(X_1) \perp_{\pi,h_v} \Delta_v(X_2)$, put

$$\mathrm{Err}_v(\Delta_v(X_1), \Delta_v(X_2), \pi) = \mathrm{Cone}(u_{+,\pi,h_v}).$$

If $\Delta(X_1) \perp_{\pi,h_S} \Delta(X_2)$, put

$$\mathrm{Err}(\Delta(X_1), \Delta(X_2), \pi) = \bigoplus_{v \in S_f} \mathrm{Err}_v(\Delta_v(X_1), \Delta_v(X_2), \pi).$$

6.2.4. Lemma. — *Fix $v \in S_f$ and assume that $\Delta_v(X_1) \perp_{\pi,h_v} \Delta_v(X_2)$. The following diagram of morphisms of complexes is commutative and the vertical maps define a morphism of exact triangles in $D({}_R\mathrm{Mod})$:*

$$\begin{array}{ccccc}
C^\bullet_{\mathrm{cont}}(G_v, X_1)[-1] & \longrightarrow & U_v^-(X_1)[-1] & \longrightarrow & U_v^+(X_1) \\
\downarrow{\scriptstyle \alpha_{J,X_1}[-1]} & & & & \\
D_{J[-3]}\big(C^\bullet_{\mathrm{cont}}(G_v, D_J(X_1)(1))\big) & & \downarrow{\scriptstyle u_{-,\pi,h_v}} & & \downarrow{\scriptstyle u_{+,\pi,h_v}} \\
\downarrow{\scriptstyle D_{J[-3]}((\mathrm{adj}(\pi\circ s_{12}))_*)} & & & & \\
D_{J[-3]}(C^\bullet_{\mathrm{cont}}(G_v, X_2)) & \xrightarrow{D_{J[-3]}(i_v^+(X_2))} & D_{J[-3]}(U_v^+(X_2)) & \longrightarrow & D_{J[-3]}(U_v^-(X_2)[-1])
\end{array}$$

Proof. — This follows from Lemma 1.2.11 and the following equalities:

$$(0, c_1) \cup_{-,\pi,h_v} b_2 = c_1 \overset{\bullet}{\cup}_\pi i_v^+(b_2)$$
$$(b_1, 0) \cup_{-,\pi,h_v} b_2 = b_1 \cup_{+,\pi,h_v} (b_2, 0). \qquad \square$$

6.2.5. We shall be interested only in the following two cases:

(A) The complexes X_1, X_2 are bounded, $J = I[n]$ for some $n \in \mathbf{Z}$ (hence $D_J = D_n$) and – for $i = 1$ or 2 – all cohomology groups of X_i (resp., of X_{3-i}) are of finite (resp., co-finite) type over R.

(B) The complexes X_1, X_2 are bounded, $J = \omega^\bullet[n]$ for some $n \in \mathbf{Z}$ (hence $D_J = \mathscr{D}_n$) and all cohomology groups of X_1, X_2 are of finite type over R.

6.2.6. Lemma - Definition. — *Assume that one of the conditions (A) or (B) in 6.2.5 is satisfied. Then the following two conditions are equivalent:*

$$\mathrm{adj}(\pi) : X_1 \longrightarrow D_J(X_2)(1) \textit{ is a quasi-isomorphism}$$
$$\iff \mathrm{adj}(\pi \circ s_{12}) : X_2 \longrightarrow D_J(X_1)(1) \textit{ is a Qis.}$$

If they are satisfied, we say that π **is a perfect duality.**

Proof. — Applying Lemma 1.2.14 to $f, g, h = \mathrm{id}$, $\lambda = \pi$, $\mu = \pi \circ s_{12}$, we see that $\mathrm{adj}(\pi \circ s_{12})$ is equal to

$$X_2 \xrightarrow{\varepsilon_J} D_J(D_J(X_2)(1))(1) \xrightarrow{D_J(\mathrm{adj}(\pi))(1)} D_J(X_1)(1)$$

and $\mathrm{adj}(\pi)$ is equal to

$$X_1 \xrightarrow{\varepsilon_J} D_J(D_J(X_1)(1))(1) \xrightarrow{D_J(\mathrm{adj}(\pi\circ s_{12}))(1)} D_J(X_2)(1).$$

The statement follows from the fact that both maps ε_J are quasi-isomorphisms, by Matlis duality 2.3.2 (resp., Grothendieck duality 2.6) in the case 6.2.5(A) (resp., 6.2.5(B)). □

6.2.7. Lemma - Definition. — *Assume that one of the conditions (A) or (B) in 6.2.5 is satisfied and π is a perfect duality. If, for a fixed $v \in S_f$, we have $\Delta_v(X_1) \perp_{\pi,h_v} \Delta_v(X_2)$, then the following two conditions are equivalent:*

$$u_{+,\pi,h_v} \text{ is a quasi-isomorphism} \iff u_{-,\pi,h_v} \text{ is a quasi-isomorphism.}$$

If they are satisfied, we say that $\Delta_v(X_1)$ and $\Delta_v(X_2)$ are othogonal complements of each other with respect to π and h_v; notation: $\Delta_v(X_1) \perp\!\!\perp_{\pi,h_v} \Delta_v(X_2)$. If they are satisfied for all $v \in S_f$, we write $\Delta(X_1) \perp\!\!\perp_{\pi,h_S} \Delta(X_2)$.

Proof. — Under the assumptions (A) or (B) of 6.2.5 the vertical arrow $\alpha_{J,X}[-1]$ (resp., $D_{J[-3]}((\mathrm{adj}(\pi \circ s_{12}))_*)$) in Lemma 6.2.4 is a quasi-isomorphism, by Proposition 5.2.4 (resp., by Definition 6.2.6). □

6.2.8. Corollary. — *Under the assumptions of 6.2.7,*

$$\Delta_v(X_1) \perp\!\!\perp_{\pi,h_v} \Delta_v(X_2) \iff \mathrm{Err}_v(\Delta_v(X_1), \Delta_v(X_2), \pi) \xrightarrow{\sim} 0 \text{ in } D({}_R\mathrm{Mod}).$$

6.3. Global cup products

We are going to apply results of Sect. 1.3 to Selmer complexes.

6.3.1. Assume that we are given X_1, X_2, π and $\Delta(X_1) \perp_{\pi,h_S} \Delta(X_2)$ as in 6.2.1. Our goal is to define a morphism between the exact triangle

$$\mathbf{R}\Gamma_{c,\mathrm{cont}}(G_{K,S}, X_1) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(X_1) \longrightarrow U_S^+(X_1)$$

and the $D_{J[-3]}$-dual of the exact triangle

$$U_S^-(X_2)[-1] \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(X_2) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, X_2).$$

Consider the data of the type 1.3.1.1–1.3.1.4 given by the following objects:

(1) $A_1 = C^\bullet_{\mathrm{cont}}(G_{K,S}, X_1)$ $\quad B_1 = U_S^+(X_1)$ $\quad C_1 = \bigoplus_{v\in S_f} C^\bullet_{\mathrm{cont}}(G_v, X_1)$

$A_2 = C^\bullet_{\mathrm{cont}}(G_{K,S}, X_2)$ $\quad B_2 = U_S^+(X_2)$ $\quad C_2 = \bigoplus_{v\in S_f} C^\bullet_{\mathrm{cont}}(G_v, X_2)$

$A_3 = \tau^{\mathrm{II}}_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_{K,S}, J(1))$ $\quad B_3 = 0$ $\quad C_3 = \bigoplus_{v\in S_f} \tau^{\mathrm{II}}_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_v, J(1))$

(2) $A_j \xrightarrow{f_j} C_j \xrightarrow{g_j} B_j$ given by $f_j = \mathrm{res}_{S_f}$, $g_j = i_S^+(X_j)$ $(j = 1, 2)$, $g_3 = 0$.

(3) Products $\cup_A, \cup_C$ induced by the truncated cup products $\overset{\bullet}{\cup}_\pi$ associated to π (cf. 6.2.2) and $\cup_B = 0$.

(4) Homotopies $h = (h_f, h_g) = (0, h_S)$.

For these data,

$$E_j = \widetilde{C}_f^\bullet(G_{K,S}, X_j; \Delta(X_j)) = \widetilde{C}_f^\bullet(X_j) \qquad (j = 1, 2).$$

As in 5.4.1 we have a quasi-isomorphism $r_J[1] : E_3[1] \to J[-2]$, unique up to homotopy, which makes the diagram

$$\begin{array}{ccc} C_3 & \longrightarrow & E_3[1] \\ \Big\downarrow {\scriptstyle (r_{v,J})} & & \Big\downarrow {\scriptstyle r_J[1]} \\ \bigoplus_{v \in S_f} J[-2] & \xrightarrow{\Sigma} & J[-2] \end{array}$$

commutative up to homotopy.

For every $r \in R$, Proposition 1.3.2 gives cup products

$$\cup_{\pi,r,h} = \cup_{r,h} : E_1 \otimes_R E_2 \longrightarrow E_3 \xrightarrow{r_J} J[-3],$$

hence, by adjunction, morphisms of complexes

$$\gamma_{\pi,r,h_S} = \mathrm{adj}(\cup_{\pi,r,h}) : \widetilde{C}_f^\bullet(X_1) \longrightarrow D_{J[-3]}\big(\widetilde{C}_f^\bullet(X_2)\big).$$

The homotopy class of γ_{π,r,h_S} is independent of the choices of $r \in R$, $(r_{v,J})$ and $r_{S,J}$ from 5.4.1. It may depend on the homotopies (h_v), but it does not change if we replace (h_v) by homotopic homotopies $(\widetilde{h}_v) \rightsquigarrow (h_v)$ (*via* a second order homotopy).

Denote by

$$\gamma_{\pi,h_S} : \widetilde{\mathbf{R}\Gamma}_f(X_1) \longrightarrow D_{J[-3]}\big(\widetilde{\mathbf{R}\Gamma}_f(X_2)\big)$$

the corresponding map in $D({}_R\mathrm{Mod})$.

6.3.2. If X_1, X_2 are bounded and $U_S^+(X_j)$ $(j = 1, 2)$ are both cohomologically bounded above, then $\widetilde{\mathbf{R}\Gamma}_f(X_j)$ $(j = 1, 2)$ both lie in $D^-({}_R\mathrm{Mod})$ and the cup product $\cup_{\pi,r,h}$ induces (assuming that $\Delta(X_1) \perp_{\pi,h_S} \Delta(X_2)$) a cup product

$$\cup_{\pi,h_S} : \widetilde{\mathbf{R}\Gamma}_f(X_1) \overset{\mathbf{L}}{\otimes}_R \widetilde{\mathbf{R}\Gamma}_f(X_2) \longrightarrow J[-3] \tag{6.3.2.1}$$

and pairings on cohomology

$$\widetilde{H}_f^i(G_{K,S}, X_1; \Delta(X_1)) \otimes_R \widetilde{H}_f^j(G_{K,S}, X_2; \Delta(X_2)) \longrightarrow H^{i+j-3}(J^\bullet). \tag{6.3.2.2}$$

6.3.3. Proposition. — *Fix $r_{v,J}$ and $r_{S,J}$. If $\Delta(X_1) \perp_{\pi,h_S} \Delta(X_2)$, then the following diagrams of morphisms of complexes have exact columns and are commutative up to homotopy (the vertical maps are those of 6.1.3 and their $D_{J[-3]}$-duals):*

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
U_S^-(X_1)[-1] & \xrightarrow{u_{-,\pi,h_S}} & D_{J[-3]}(U_S^+(X_2)) \\
\downarrow & & \downarrow \\
\widetilde{C}_f^\bullet(G_{K,S}, X_1; \Delta(X_1)) & \xrightarrow{\gamma_{\pi,0,h_S}} & D_{J[-3]}\big(\widetilde{C}_f^\bullet(G_{K,S}, X_2; \Delta(X_2))\big) \\
\downarrow & & \downarrow \\
C_{\mathrm{cont}}^\bullet(G_{K,S}, X_1) & \xrightarrow{\beta_c} & D_{J[-3]}(C_{c,\mathrm{cont}}^\bullet(G_{K,S}, X_2)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
C_{c,\mathrm{cont}}^\bullet(G_{K,S}, X_1) & \xrightarrow{{}_c\beta} & D_{J[-3]}(C_{\mathrm{cont}}^\bullet(G_{K,S}, X_2)) \\
\downarrow & & \downarrow \\
\widetilde{C}_f^\bullet(G_{K,S}, X_1; \Delta(X_1)) & \xrightarrow{\gamma_{\pi,1,h_S}} & D_{J[-3]}\big(\widetilde{C}_f^\bullet(G_{K,S}, X_2; \Delta(X_2))\big) \\
\downarrow & & \downarrow \\
U_S^+(X_1) & \xrightarrow{u_{+,\pi,h_S}} & D_{J[-3]}(U_S^-(X_2)[-1]) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Above, the maps β_c resp., ${}_c\beta$ are equal to

$$\beta_c : C_{\mathrm{cont}}^\bullet(G_{K,S}, X_1) \xrightarrow{\beta_{c,J,X_1}} D_{J[-3]}\big(C_{c,\mathrm{cont}}^\bullet(G_{K,S}, D_J(X_1)(1))\big) \xrightarrow{D_{J[-3]}(\mathrm{adj}(\pi \circ s_{12})_*)} D_{J[-3]}(C_{c,\mathrm{cont}}^\bullet(G_{K,S}, X_2))$$

$${}_c\beta : C_{c,\mathrm{cont}}^\bullet(G_{K,S}, X_1) \xrightarrow{{}_c\beta_{J,X_1}} D_{J[-3]}\big(C_{\mathrm{cont}}^\bullet(G_{K,S}, D_J(X_1)(1))\big) \xrightarrow{D_{J[-3]}(\mathrm{adj}(\pi \circ s_{12})_*)} D_{J[-3]}(C_{\mathrm{cont}}^\bullet(G_{K,S}, X_2)).$$

Proof. — This follows from Lemma 1.2.11 and the following formulas

$$(a_1, 0, c_1) \cup_{1,h} (a_2, 0, 0) = (a_1 \overset{\bullet}{\cup}_\pi a_2, 0, c_1 \overset{\bullet}{\cup}_\pi \mathrm{res}_{S_f}(a_2))$$

$$(a_1, 0, 0) \cup_{0,h} (a_2, 0, c_2) = (a_1 \overset{\bullet}{\cup}_\pi a_2, 0, (-1)^{\overline{a_1}} \mathrm{res}_{S_f}(a_1) \overset{\bullet}{\cup}_\pi c_2)$$

$$(0, b_1, c_1) \cup_{0,h} (0, b_2, 0) = (0, 0, c_1 \overset{\bullet}{\cup}_\pi (i_S^+(b_2)) + h_S(b_1 \otimes b_2))$$

$$(0, b_1, 0) \cup_{1,h} (0, b_2, c_2) = (0, 0, (-1)^{\overline{b_1}} (i_S^+(b_1)) \overset{\bullet}{\cup}_\pi c_2 + h_S(b_1 \otimes b_2))$$

(valid in our case, since $\cup_B = 0$ and $h_f = 0$). □

6.3.4. Theorem. — *Assume that one of the conditions (A) or (B) in 6.2.5 is satisfied and π is a perfect duality. If $\Delta(X_1) \perp_{\pi,h_S} \Delta(X_2)$, then there is an exact triangle in $D({}_R\mathrm{Mod})$*

$$\widetilde{\mathbf{R}\Gamma}_f(X_1) \xrightarrow{\gamma_{\pi,h_S}} D_{J[-3]}\big(\widetilde{\mathbf{R}\Gamma}_f(X_2)\big) \longrightarrow \mathrm{Err}(\Delta(X_1), \Delta(X_2), \pi).$$

In particular, the map

$$\gamma_{\pi,h_S} : \widetilde{\mathbf{R}\Gamma}_f(X_1) \longrightarrow D_{J[-3]}\big(\widetilde{\mathbf{R}\Gamma}_f(X_2)\big)$$

is an isomorphism in $D({}_R\mathrm{Mod})$ if and only if $\Delta(X_1) \perp\!\!\perp_{\pi,h_S} \Delta(X_2)$.

Proof. — In the second diagram in Proposition 6.3.3, the map ${}_c\beta$ is a quasi-isomorphism by Proposition 5.4.3. This implies that $\mathrm{Cone}(\gamma_{\pi,1,h_S})$ is isomorphic in $D({}_R\mathrm{Mod})$ to $\mathrm{Cone}(u_{+,\pi,h_S}) = \mathrm{Err}(\Delta(X_1), \Delta(X_2), \pi)$, which proves the Theorem (using Corollary 6.2.8). □

6.3.5. On the level of cohomology, Theorem 6.3.4 gives exact sequences

$$\cdots \longrightarrow H^{q-1}(\mathrm{Err}) \longrightarrow \widetilde{H}^q_f(X_1) \longrightarrow H^{q-3}\big(D_J\big(\widetilde{\mathbf{R}\Gamma}_f(X_2)\big)\big) \longrightarrow H^q(\mathrm{Err}) \longrightarrow \cdots$$

(where $\mathrm{Err} = \mathrm{Err}(\Delta(X_1), \Delta(X_2), \pi)$). Under the assumptions of 6.2.5 (A), we have $J = I[n]$ and

$$H^{q-3}\big(D_J\big(\widetilde{\mathbf{R}\Gamma}_f(X_2)\big)\big) = D\big(\widetilde{H}^{3-n-q}_f(X_2)\big).$$

Under the assumptions of 6.2.5 (B), we have $J = \omega^\bullet[n]$. If, in addition, all cohomology groups of $U^+_S(X_2)$ are of finite type over R, then $\widetilde{\mathbf{R}\Gamma}_f(X_2) \in D_{ft}({}_R\mathrm{Mod})$ and there is a spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_R\big(\widetilde{H}^{3-n-j}_f(X_2), \omega\big) \Longrightarrow H^{i+j-3}\big(D_J\big(\widetilde{\mathbf{R}\Gamma}_f(X_2)\big)\big),$$

which degenerates in the category $({}_R\mathrm{Mod})/(\text{pseudo-null})$ into exact sequences

$$0 \longrightarrow E_2^{1,q-1} \longrightarrow H^{q-3}\big(D_J\big(\widetilde{\mathbf{R}\Gamma}_f(X_2)\big)\big) \longrightarrow E_2^{0,q} \longrightarrow 0.$$

The term $E_2^{0,q}$ is torsion-free over R (by Lemma 2.8.8), while $\mathrm{codim}_R(\mathrm{supp}(E_2^{1,q-1})) \geqslant 1$. In particular, there is a monomorphism in $({}_R\mathrm{Mod})/(\text{pseudo-null})$

$$\Big(H^{q-3}\big(D_J\big(\widetilde{\mathbf{R}\Gamma}_f(X_2)\big)\big)\Big)_{R\text{-tors}} \longrightarrow \mathrm{Ext}^1_R\big(\widetilde{H}^{4-n-q}_f(X_2), \omega\big),$$

which is an isomorphism if R has no embedded primes.

6.3.6. In the situation of 6.2.5 (B), there is a straightforward generalization of 6.2.6, 6.3.4 and 6.3.5, if we insist on $\mathrm{adj}(\pi)$ to be a quasi-isomorphism only in $D(({}_R\mathrm{Mod})/(\text{pseudo-null}))$.

6.4. Functoriality of Selmer complexes

Let J be as in Sect. 6.2.

6.4.1. Let X_1, X_2, π be as in 6.2.1 and assume that $Y_1, Y_2, \rho : Y_1 \otimes_R Y_2 \to J(1)$ is another triple of the same kind. Consider the following data:

6.4.1.1. Orthogonal local conditions

$$\Delta(X_1) \perp_{\pi, h_S(X)} \Delta(X_2), \quad \Delta(Y_1) \perp_{\pi, h_S(Y)} \Delta(Y_2).$$

6.4.1.2. Morphisms of complexes of $R[G_{K,S}]$-modules $\lambda_j : X_j \to Y_j$ $(j = 1, 2)$; set

$$\lambda = \lambda_1 \otimes \lambda_2 : X_1 \otimes_R X_2 \longrightarrow Y_1 \otimes_R Y_2.$$

6.4.1.3. Morphisms of complexes of R-modules

$$\beta_j : U_S^+(X_j) \longrightarrow U_S^+(Y_j) \qquad (j = 1, 2).$$

6.4.1.4. Homotopies

$$v_j : i_S^+(Y_j) \circ \beta_j \rightsquigarrow (\lambda_j)_* \circ i_S^+(X_j) \qquad (j = 1, 2).$$

6.4.1.5. A homotopy $k : \rho \circ \lambda \rightsquigarrow \pi$.

6.4.1.6. A second order homotopy

$$K : h_S(Y) \star (\beta_1 \otimes \beta_2) \rightsquigarrow h_S(X) + k_* \star (i_S^+(X_1) \otimes i_S^+(X_2)) + \overset{\bullet}{\cup}_\rho \star (v_1 \otimes v_2)_1$$

$$K^i : \left(U_S^+(X_1) \otimes_R U_S^+(X_2)\right)^i \longrightarrow \bigoplus_{v \in S_f} \left(\tau_{\geqslant 2}^{\mathrm{II}} C_{\mathrm{cont}}^\bullet(G_v, J(1))\right)^{i-2}.$$

Given 6.4.1.1–6.4.1.6, we obtain the following data of the type considered in 1.3.3: $(A_j, B_j, C_j, f_j, g_j, \cup_*, h)$ as in 6.3.1, $(\widetilde{A}_j, \widetilde{B}_j, \widetilde{C}_j, \widetilde{f}_j, \widetilde{g}_j, \widetilde{\cup}_*, \widetilde{h})$ the corresponding objects for (Y_1, Y_2, ρ),

$$\alpha_j, \gamma_j = (\lambda_j)_*, \quad \beta_j = \beta_j \quad (j = 1, 2), \quad \alpha_3, \gamma_3 = \mathrm{id}, \quad \beta_3 = 0,$$

$$u_j = 0, \quad v_j = v_j \quad (j = 1, 2), \quad u_3, v_3 = 0,$$

$$k_\alpha, k_\gamma = k_*, \quad k_\beta = 0, \quad K_f = 0, \quad K_g = K.$$

Applying Proposition 1.3.4, we obtain the following result.

6.4.2. Proposition

(i) *The data 6.4.1.1–6.4.1.4 (it is not necessary to assume orthogonality of local conditions at this point) define morphisms of complexes*

$$\varphi(\lambda_j) = \varphi(\lambda_j, \beta_j, v_j) : \widetilde{C}_f^\bullet(X_j) \longrightarrow \widetilde{C}_f^\bullet(Y_j),$$

given by the formula in Proposition 1.3.4(i). The homotopy class of $\varphi(\lambda_j, \beta_j, v_j)$ is unchanged if v_j is replaced by v_j' related to v_j by a second order homotopy $v_j' \rightsquigarrow v_j$.

(ii) *Given the data 6.4.1.1–6.4.1.6, the following diagram of morphisms of complexes is commutative up to homotopy (for every $r \in R$):*

$$\begin{array}{ccc} \widetilde{C}_f^\bullet(X_1) \otimes_R \widetilde{C}_f^\bullet(X_2) & \xrightarrow{\cup_{\pi, r.h_S(X)}} & J[-3] \\ \big\downarrow{\scriptstyle \varphi(\lambda_1) \otimes \varphi(\lambda_2)} & & \| \\ \widetilde{C}_f^\bullet(Y_1) \otimes_R \widetilde{C}_f^\bullet(Y_2) & \xrightarrow{\cup_{\rho, r.h_S(Y)}} & J[-3]. \end{array}$$

6.4.3. Corollary. — *Given the data 6.4.1.1–6.4.1.6, the following diagram is commutative up to homotopy (for every $r \in R$):*

$$\begin{array}{ccc} \widetilde{C}^{\bullet}_f(X_1) & \xrightarrow{\gamma_{\pi,r,h_S(X)}} & D_{J[-3]}\big(\widetilde{C}^{\bullet}_f(X_2)\big) \\ \downarrow{\scriptstyle \varphi(\lambda_1)} & & \uparrow{\scriptstyle D_{J[-3]}(\varphi(\lambda_2))} \\ \widetilde{C}^{\bullet}_f(Y_1) & \xrightarrow{\gamma_{\rho,r,h_S(Y)}} & D_{J[-3]}\big(\widetilde{C}^{\bullet}_f(Y_2)\big). \end{array}$$

Proof. — Apply Lemma 1.2.11 to the diagram in Proposition 6.4.2 (ii). □

6.4.4. If we are given the data 6.4.1.1–6.4.1.3 such that $h_S(X) = h_S(Y) = 0$, $\pi = \rho\circ\lambda$ and $i^+_S(Y_j)\circ\beta_j = (\lambda_j)_*\circ i^+_S(X_j)$ $(j = 1, 2)$ – which happens quite often in practice – then we can take $v_j = 0$ $(j = 1, 2)$, $k = 0$, $K = 0$. The formula in Proposition 1.3.4 (ii) then gives $H = 0$, which implies that the diagrams in Proposition 6.4.2 (ii) and Corollary 6.4.3 are commutative, not just commutative up to homotopy.

6.4.5. A special case of the functoriality data 6.4.1 occurs if we replace π by a homotopic morphism of complexes $\pi' : X_1 \otimes_R X_2 \to J(1)$. Taking

$$Y_j = X_j, \quad \Delta(Y_j) = \Delta(X_j), \quad \lambda_j = \mathrm{id}, \quad \beta_j = \mathrm{id}, \quad v_j = 0 \quad (j = 1, 2),$$
$$\rho = \pi', \quad k : \pi' \rightsquigarrow \pi,$$

all we need in order to apply Proposition 6.4.2 are new homotopies

$$h'_v = h_v(X_1, X_2, \pi') : \mathrm{prod}'_v = \mathrm{prod}_v(X_1, X_2, \pi') \rightsquigarrow 0$$

and a second order homotopy

$$K : h'_S \rightsquigarrow h_S + k_* \star (i^+_S(X_1) \otimes i^+_S(X_2)).$$

For example, if $\mu : J \to J$ is homotopic to the identity *via* a homotopy $\ell : \mu \rightsquigarrow \mathrm{id}$ and $\pi' = \mu\circ\pi$, then we can take

$$k = \ell \star \pi, \quad h'_v = \mu \star h_v, \quad K = (K_v = -\ell h_v)_{v\in S_f},$$

since

$$K_v : -(d\ell + \ell d) \star h_v = h'_v - h_v \rightsquigarrow -\ell \star (dh_v + h_v d) = \ell \star \mathrm{prod}_v.$$

6.5. Transpositions

In order to exchange the roles of X_1 and X_2 in 6.2.1 we need additional data.

6.5.1. Definition. — Given X as in 6.1.1 and a local condition $\Delta_v(X)$ at $v \in S_f$, a **transposition operator** for $\Delta_v(X)$ is a morphism of complexes

$$\mathcal{T}^+_v(X) : U^+_v(X) \longrightarrow U^+_v(X)$$

such that the diagram

$$\begin{array}{ccc} U_v^+(X) & \xrightarrow{i_v^+(X)} & C^\bullet_{\rm cont}(G_v, X) \\ \big\downarrow {\scriptstyle \mathcal{T}_v^+(X)} & & \big\downarrow {\scriptstyle \mathcal{T}} \\ U_v^+(X) & \xrightarrow{i_v^+(X)} & C^\bullet_{\rm cont}(G_v, X) \end{array}$$

commutes up to homotopy (recall that $\mathcal{T}$ denotes the transposition operator defined in 3.4.5.3–3.4.5.4).

6.5.2. Lemma. — *Assume that we are given $\pi : X_1 \otimes_R X_2 \to J(1)$ as in 6.2.1 and local conditions $\Delta_v(X_j)$, $j = 1, 2$ (for some $v \in S_f$) that both admit transposition operators $\mathcal{T}_v^+(X_j)$. Then the following two conditions are equivalent:*

$$\Delta_v(X_1) \perp_{\pi, h_v} \Delta_v(X_2) \textit{ for some } h_v \iff \Delta_v(X_2) \perp_{\pi\circ s_{12}, h'_v} \Delta_v(X_1) \textit{ for some } h'_v.$$

Proof. — This follows from the fact that the following diagram and its analogue in which the roles of X_1 and X_2 are interchanged are commutative up to homotopy (by 3.4.5.4):

$$\begin{array}{ccc} U_v^+(X_1) \otimes_R U_v^+(X_2) & \xrightarrow{s_{12}\circ(\mathcal{T}_v^+ \otimes \mathcal{T}_v^+)} & U_v^+(X_2) \otimes_R U_v^+(X_1) \\ \big\downarrow {\scriptstyle i_v^+ \otimes i_v^+} & & \big\downarrow {\scriptstyle i_v^+ \otimes i_v^+} \\ C^\bullet_{\rm cont}(G_v, X_1) \otimes_R C^\bullet_{\rm cont}(G_v, X_2) & \xrightarrow{s_{12}\circ(\mathcal{T} \otimes \mathcal{T})} & C^\bullet_{\rm cont}(G_v, X_2) \otimes_R C^\bullet_{\rm cont}(G_v, X_1) \\ \big\downarrow {\scriptstyle \overset{\bullet}{\cup}_\pi} & & \big\downarrow {\scriptstyle \overset{\bullet}{\cup}_{\pi\circ s_{12}}} \\ \tau^{\rm II}_{\geqslant 2} C^\bullet_{\rm cont}(G_v, J(1)) & = & \tau^{\rm II}_{\geqslant 2} C^\bullet_{\rm cont}(G_v, J(1)) \end{array} \quad \square$$

6.5.3. Assume that we are given X_1, X_2, π as in 6.2.1. Consider the following additional data:

6.5.3.1. Orthogonal local conditions

$$\Delta(X_1) \perp_{\pi, h_S} \Delta(X_2), \quad \Delta(X_2) \perp_{\pi\circ s_{12}, h'_S} \Delta(X_1).$$

6.5.3.2. For each $Z = X_1, X_2$ and $v \in S_f$ a transposition operator $\mathcal{T}_v^+(Z) : U_v^+(Z) \to U_v^+(Z)$; put $\mathcal{T}^+(Z) = (\mathcal{T}_v^+(Z))_{v\in S_f}$.

6.5.3.3. For each $Z = X_1, X_2$ and $v \in S_f$ a homotopy

$$V_{Z,v} : i_v^+(Z) \circ \mathcal{T}_v^+(Z) \rightsquigarrow \mathcal{T} \circ i_v^+(Z);$$

put $V_Z = (V_{Z,v})_{v\in S_f}$ (the existence of V_Z follows from 6.5.3.2, by definition).

6.5.3.4. For each $Z = X_1, X_2$ and $v \in S_f$, homotopies

$$k_Z : \mathrm{id} \rightsquigarrow \mathcal{T} \quad \text{on } C^\bullet_{\rm cont}(G_{K,S}, Z)$$
$$k^+_{Z,v} : \mathrm{id} \rightsquigarrow \mathcal{T}_v^+(Z) \quad \text{on } U_v^+(Z)$$
$$k_{Z,v} : \mathrm{id} \rightsquigarrow \mathcal{T} \quad \text{on } C^\bullet_{\rm cont}(G_v, Z)$$

satisfying

$$\mathrm{res}_v \star k_Z = k_{Z,v} \star \mathrm{res}_v$$

and such that there exists a second order homotopy

$$i_v^+(Z) \star k_{Z,v}^+ + V_{Z,v} \rightsquigarrow k_{Z,v} \star i_v^+(Z).$$

6.5.3.5. For each $v \in S_f$, a second order homotopy

$$\begin{aligned} H_v : \mathcal{T} \star h_v(X_1, X_2, \pi) + \overset{\bullet}{\cup}_\pi \star (V_{X_1,v} \otimes V_{X_2,v})_1 \\ \rightsquigarrow h'_v(X_2, X_1, \pi \circ s_{12}) \star \big(s_{12} \circ (\mathcal{T}_v^+(X_1) \otimes \mathcal{T}_v^+(X_2))\big) \end{aligned}$$

$$H_v^i : \big(U_v^+(X_1) \otimes_R U_v^+(X_2)\big)^i \longrightarrow \big(\tau_{\geqslant 2}^{\mathrm{II}}\, C_{\mathrm{cont}}^\bullet(G_v, J(1))\big)^{i-2}.$$

6.5.4. Proposition

(i) *Given the data 6.5.3.1–6.5.3.3, the formula*

$$\mathcal{T}_Z(a, b, c) = (\mathcal{T}(a), \mathcal{T}^+(Z)(b), \mathcal{T}(c) - V_Z(b)) \qquad (Z = X_1, X_2)$$

defines a morphism of complexes

$$\mathcal{T}_Z : \widetilde{C}_f^\bullet(G_{K,S}, Z; \Delta(Z)) \longrightarrow \widetilde{C}_f^\bullet(G_{K,S}, Z; \Delta(Z)).$$

If, in addition, we are given 6.5.3.4, then $\mathcal{T}_Z$ is homotopic to the identity.

(ii) *Given the data 6.5.3.1–6.5.3.3 and 6.5.3.5, then the following diagrams commute up to homotopy (for every $r \in R$).*

$$\begin{array}{ccc} \widetilde{C}_f^\bullet(X_1) \otimes_R \widetilde{C}_f^\bullet(X_2) & \xrightarrow{\cup_{\pi,r,h}} & J[-3] \\ \Big\downarrow {\scriptstyle s_{12} \circ (\mathcal{T}_{X_1} \otimes \mathcal{T}_{X_2})} & & \Big\| \\ \widetilde{C}_f^\bullet(X_2) \otimes_R \widetilde{C}_f^\bullet(X_1) & \xrightarrow{\cup_{\pi \circ s_{12}, 1-r, h'}} & J[-3] \end{array}$$

$$\begin{array}{ccc} \widetilde{C}_f^\bullet(X_1) & \xrightarrow{\gamma_{\pi,r,h_S}} & D_{J[-3]}\big(\widetilde{C}_f^\bullet(X_2)\big) \\ \Big\downarrow {\scriptstyle \varepsilon_{J[-3]} \circ \mathcal{T}_{X_1}} & & \Big\| \\ D_{J[-3]}\big(D_{J[-3]}\big(\widetilde{C}_f^\bullet(X_1)\big)\big) & \xrightarrow{D_{J[-3]}(\gamma_{\pi \circ s_{12}, 1-r, h'_S} \circ \mathcal{T}_{X_2})} & D_{J[-3]}\big(\widetilde{C}_f^\bullet(X_2)\big) \end{array}$$

$$\begin{array}{ccc} \widetilde{C}_f^\bullet(X_2) & \xrightarrow{\gamma_{\pi \circ s_{12}, 1-r, h'_S} \circ \mathcal{T}_{X_2}} & D_{J[-3]}\big(\widetilde{C}_f^\bullet(X_1)\big) \\ \Big\downarrow {\scriptstyle \varepsilon_{J[-3]}} & & \Big\downarrow {\scriptstyle D_{J[-3]}(\mathcal{T}_{X_1})} \\ D_{J[-3]}\big(D_{J[-3]}\big(\widetilde{C}_f^\bullet(X_2)\big)\big) & \xrightarrow{D_{J[-3]}(\gamma_{\pi,r,h_S})} & D_{J[-3]}\big(\widetilde{C}_f^\bullet(X_1)\big) \end{array}$$

Proof

(i) This follows from 1.1.7.

(ii) We are going to apply Proposition 1.3.6 to the data 6.5.3.1–6.5.3.4 together with

$$\mathcal{T}_A, \mathcal{T}_C = \mathcal{T}, \quad \mathcal{T}_B = \mathcal{T}^+(X_1) \text{ resp., } \mathcal{T}^+(X_2),$$
$$\cup'_A, \cup'_C = \overset{\bullet}{\cup}_\pi, \quad \cup'_B = 0$$
$$h'_f = 0, \quad h'_g = h'_S,$$
$$U_j = 0 \quad (j = 1, 2, 3),$$
$$V_3 = 0, \quad V_j = V_{X_j} \quad (j = 1, 2),$$
$$t_\alpha = t_\beta = t_\gamma = 0,$$
$$H_f = 0, \quad H_g = (H_v)_{v \in S_f}.$$

Proposition 1.3.6 then implies that the first diagram is commutative up to homotopy (as $\mathcal{T}_3 = \mathcal{T} \rightsquigarrow \mathrm{id}$ and $\cup'_{1-r,h'} = \cup_{\pi \circ s_{12}, 1-r, h'}$). Commutativity up to homotopy of the second and third diagrams then follows from Corollary 1.3.7. □

6.5.5. Corollary. — *Given the data 6.5.3.1–6.5.3.5 and $r \in R$, the diagram*

$$\begin{array}{ccc} \widetilde{C}^\bullet_f(X_1) \otimes_R \widetilde{C}^\bullet_f(X_2) & \xrightarrow{\cup_{\pi,r,h}} & J[-3] \\ \downarrow{\scriptstyle s_{12}} & & \| \\ \widetilde{C}^\bullet_f(X_2) \otimes_R \widetilde{C}^\bullet_f(X_1) & \xrightarrow{\cup_{\pi \circ s_{12}, 1-r, h'}} & J[-3] \end{array}$$

is commutative up to homotopy and the composite maps

$$\widetilde{C}^\bullet_f(X_1) \xrightarrow{\varepsilon_{J[-3]}} D_{J[-3]}\big(D_{J[-3]}\big(\widetilde{C}^\bullet_f(X_1)\big)\big) \xrightarrow{D_{J[-3]}(\gamma_{\pi \circ s_{12}, 1-r, h'_S})} D_{J[-3]}\big(\widetilde{C}^\bullet_f(X_2)\big)$$

$$\widetilde{C}^\bullet_f(X_2) \xrightarrow{\varepsilon_{J[-3]}} D_{J[-3]}\big(D_{J[-3]}\big(\widetilde{C}^\bullet_f(X_2)\big)\big) \xrightarrow{D_{J[-3]}(\gamma_{\pi, r, h_S})} D_{J[-3]}\big(\widetilde{C}^\bullet_f(X_1)\big)$$

are homotopic to γ_{π,r,h_S} and $\gamma_{\pi \circ s_{12}, 1-r, h'_S}$, respectively. Under the assumptions as in (6.3.2.1), *the corresponding cup products make the diagram*

$$\begin{array}{cccc} \cup_{\pi,h_S} : & \widetilde{\mathbf{R}\Gamma}_f(X_1) \overset{\mathbf{L}}{\otimes}_R \widetilde{\mathbf{R}\Gamma}_f(X_2) & \longrightarrow & J[-3] \\ & \downarrow{\scriptstyle s_{12}} & & \| \\ \cup_{\pi \circ s_{12}, h'_S} : & \widetilde{\mathbf{R}\Gamma}_f(X_2) \overset{\mathbf{L}}{\otimes}_R \widetilde{\mathbf{R}\Gamma}_f(X_1) & \longrightarrow & J[-3] \end{array}$$

commutative in $D^-({}_R\mathrm{Mod})$.

6.6. Self-dual case

6.6.1. Let X and $\Delta(X)$ be as in 6.1.1. Assume that we are given a morphism of complexes of $R[G_{K,S}]$-modules $\pi : X \otimes_R X \to J(1)$ such that $\Delta(X) \perp_{\pi,h_S} \Delta(X)$ for suitable homotopies $h_S = (h_v)_{v \in S_f}$, and that $\pi' := \pi \circ s_{12}$ is equal to

$$\pi' = \pi \circ s_{12} = c \cdot \pi, \qquad c = \pm 1.$$

This implies that $\overset{\bullet}{\cup}_{\pi'} = c \cdot \overset{\bullet}{\cup}_{\pi}$ and

$$\Delta(X) \perp_{\pi', h'_S} \Delta(X), \quad h'_S = c \cdot h_S.$$

The formula in Proposition 1.3.2 (i) implies that the cup products

$$\cup_{\pi,r,h}, \cup_{\pi',r,h'} : \widetilde{C}^{\bullet}_f(X) \otimes_R \widetilde{C}^{\bullet}_f(X) \longrightarrow J[-3]$$

are related by

$$\cup_{\pi',r,h'} = c \cdot \cup_{\pi,r,h}. \tag{6.6.1.1}$$

6.6.2. Proposition. — *Under the assumptions of 6.6.1, assume, in addition, that $Z = X$ admits transposition operators $\mathcal{T}^+_v(X)$ ($v \in S_f$) and the data 6.5.3.3–6.5.3.5, where 6.5.3.5 consists of second order homotopies*

$$H_v : \mathcal{T} \star h_v + \overset{\bullet}{\cup}_{\pi} \star (V_{X,v} \otimes V_{X,v})_1 \rightsquigarrow c \cdot h_v \star \big(s_{12} \circ (\mathcal{T}^+_v(X) \otimes \mathcal{T}^+_v(X))\big) \qquad (v \in S_f).$$

Then the following diagram is commutative up to homotopy:

$$\begin{array}{ccc} \widetilde{C}^{\bullet}_f(X) \otimes_R \widetilde{C}^{\bullet}_f(X) & \xrightarrow{\cup_{\pi,r,h}} & J[-3] \\ \downarrow{\scriptstyle s_{12}} & & \| \\ \widetilde{C}^{\bullet}_f(X) \otimes_R \widetilde{C}^{\bullet}_f(X) & \xrightarrow{c \cdot \cup_{\pi,1-r,h}} & J[-3] \end{array}$$

and the morphism

$$\widetilde{C}^{\bullet}_f(X) \xrightarrow{\varepsilon_{J[-3]}} D_{J[-3]}\big(D_{J[-3]}(\widetilde{C}^{\bullet}_f(X))\big) \xrightarrow{D_{J[-3]}(\gamma_{\pi,r,h_S})} D_{J[-3]}\big(\widetilde{C}^{\bullet}_f(X)\big)$$

is homotopic to $c \cdot \gamma_{\pi,1-r,h_S}$.

Proof. — This is a special case of Corollary 6.5.5 for $X_1 = X_2 = X$, if we take into account (6.6.1.1). □

6.6.3. Corollary. — *If, in addition, $\widetilde{C}^{\bullet}_f(X)$ is cohomologically bounded above, then the cup product* (6.3.2.1)

$$\cup_{\pi,h_S} : \widetilde{\mathbf{R}\Gamma}_f(X) \overset{\mathbf{L}}{\otimes}_R \widetilde{\mathbf{R}\Gamma}_f(X) \longrightarrow J[-3]$$

satisfies

$$\cup_{\pi,h_S} = c \cdot (\cup_{\pi,h_S} \circ s_{12}).$$

6.6.4. Hermitian case. — In 6.6.4–6.6.7 we assume that R is equipped with an involution ι, *i.e.*, with a ring homomorphism $\iota : R \to R$ satisfying $\iota \circ \iota = \mathrm{id}$.

For an R-module X, set $X^{\iota} = X \otimes_{R,\iota} R$; this is an R-module in which

$$r(x \otimes r') = x \otimes rr', \quad (rx) \otimes r' = x \otimes \iota(r)r' \qquad (r, r' \in R,\ x \in X).$$

For any R-modules X, Y there are canonical isomorphisms of R-modules

$$\begin{aligned}
&(6.6.4.1)\quad (X^\iota)^\iota \xrightarrow{\sim} X, && (x\otimes r)\otimes r' \longmapsto \iota(r)r'x,\\
&X^\iota \otimes_R Y^\iota \xrightarrow{\sim} (X\otimes_R Y)^\iota, && (x\otimes r)\otimes(y\otimes r') \longmapsto (x\otimes y)\otimes rr',\\
&\iota(=\iota\otimes \mathrm{id}) : R^\iota \;=\; R\otimes_{R,\iota} R \xrightarrow{\sim} R, && r\times r' \longmapsto \iota(r)r',\\
&\mathrm{Hom}_R(X,Y)^\iota \xrightarrow{\sim} \mathrm{Hom}_R(X^\iota, Y^\iota), && f\otimes r \longmapsto (x\otimes r' \longmapsto f(x)\otimes rr').
\end{aligned}$$

For $f : X \to Y$ we denote by $f^\iota : X^\iota \to Y^\iota$ the image of $f\otimes 1$ under the last isomorphism in (6.6.4.1). With this notation, the isomorphism

$$\iota^\iota : R \xrightarrow{\sim} (R^\iota)^\iota \longrightarrow R^\iota$$

is given by $r \to 1\otimes r$. The functor $X \mapsto X^\iota$ extends in an obvious way to complexes of R-modules, and the isomorphisms (6.6.4.1) also hold for $X^\bullet \otimes_R Y^\bullet$ and $\mathrm{Hom}^\bullet_R(X^\bullet, Y^\bullet)$.

Assume that we are given a morphism of complexes

$$\nu : J^\iota \longrightarrow J$$

(where J is as in 6.2) such that

$$\nu^\iota : J \xrightarrow{\sim} (J^\iota)^\iota \longrightarrow J^\iota$$

is a homotopy inverse of ν.

6.6.5. Let X, $\Delta(X)$ be as in 6.1.1. The local conditions $\Delta(X)$ define local conditions $\Delta(X^\iota) = \Delta(X)^\iota$ for X^ι:

$$i_v^+(X^\iota) = i_v^+(X)^\iota : U_v^+(X)^\iota \longrightarrow C^\bullet_{\mathrm{cont}}(G_v, X)^\iota \xrightarrow{\sim} C^\bullet_{\mathrm{cont}}(G_v, X^\iota)$$

and a canonical isomorphism of complexes

$$\widetilde{C}^\bullet_f(X)^\iota \xrightarrow{\sim} \widetilde{C}^\bullet_f(X^\iota).$$

Assume that $\pi : X\otimes_R X^\iota \to J(1)$ is a morphism of complexes of $R[G_{K,S}]$-modules such that $\Delta(X) \perp_{\pi,h_S} \Delta(X^\iota)$ for suitable homotopies $h_S = (h_v)_{v\in S_f}$, and that $\pi' := \pi\circ s_{12}$ is equal to

$$\pi' = \pi\circ s_{12} = c\cdot(\nu\circ\pi^\iota), \qquad c = \pm 1.$$

It follows that $\overset{\bullet}{\cup}_{\pi'} = c\cdot(\nu\circ(\overset{\bullet}{\cup}_\pi)^\iota)$ and

$$\Delta(X^\iota) \perp_{\pi',h'_S} \Delta(X), \qquad h'_S = c\cdot(\nu\circ h^\iota_S).$$

As in (6.6.1.1), the cup products

$$\cup_{\pi,r,h} : \widetilde{C}^\bullet_f(X)\otimes_R \widetilde{C}^\bullet_f(X)^\iota \longrightarrow J[-3]$$

$$\cup_{\pi',r,h'} : \widetilde{C}^\bullet_f(X)^\iota\otimes_R \widetilde{C}^\bullet_f(X) \longrightarrow J[-3]$$

are related by

$$\cup_{\pi',r,h'} = c\cdot(\nu\circ(\cup_{\pi,r,h})^\iota).$$

6.6.6. Proposition. — *Under the assumptions of 6.6.5, assume, in addition, that $Z = X$ admits transposition operators $\mathcal{T}_v^+(X)$ ($v \in S_f$) and the data 6.5.3.3–6.5.3.4. Applying the functor $M \mapsto M^\iota$, we obtain the same data for $Z = X^\iota$. Assume, furthermore, that the triple (X, X^ι, π) admits the data 6.5.3.3,* i.e., *second order homotopies*

$$H_v : \mathcal{T}\star h_v + \overset{\bullet}{\cup}_\pi \star (V_{X,v}\otimes V_{X^\iota,v})_1 \rightsquigarrow c\cdot(\nu\circ h_v^\iota)\star\big(s_{12}\circ(\mathcal{T}_v^+(X)\otimes\mathcal{T}_v^+(X^\iota))\big) \qquad (v \in S_f).$$

Then the following diagram is commutative up to homotopy:

$$\begin{array}{ccccc} \widetilde{C}_f^\bullet(X)\otimes_R \widetilde{C}_f^\bullet(X)^\iota & \xrightarrow{\cup_{\pi,r,h}} & J[-3] & = & J[-3] \\ \Big\downarrow s_{12} & & \Big\uparrow \nu & & \Big\downarrow \nu^\iota \\ \widetilde{C}_f^\bullet(X)^\iota \otimes_R \widetilde{C}_f^\bullet(X) & \xrightarrow{c\cdot(\cup_{\pi,1-r,h})^\iota} & J^\iota[-3] & = & J^\iota[-3]. \end{array}$$

Proof. — This is a special case of Corollary 6.5.5 for $X_1 = X$, $X_2 = X^\iota$. □

6.6.7. Corollary. — *If, in addition, $\widetilde{C}_f^\bullet(X)$ is cohomologically bounded above, then the cup product* (6.3.2.1)

$$\cup_{\pi,h_S} : \widetilde{\mathbf{R}\Gamma}_f(X)\overset{\mathbf{L}}{\otimes}_R\widetilde{\mathbf{R}\Gamma}_f(X)^\iota \longrightarrow J[-3]$$

satisfies

$$\cup_{\pi,h_S} = c\cdot(\nu\circ(\cup_{\pi,h_S})^\iota\circ s_{12}).$$

6.7. An example of local conditions

In this section we consider local conditions analogous to those studied by Greenberg [**Gre1, Gre2, Gre3**]. Let J be as in 6.2, X, Y complexes of admissible $R[G_{K,S}]$-modules, and $\pi : X\otimes_R Y \to J(1)$ a morphism of complexes of $R[G_{K,S}]$-modules.

6.7.1. Fix $v \in S_f$. Assume that we are given for $Z = X, Y$ a complex of admissible $R[G_v]$-modules Z_v^+ and a morphism of complexes of $R[G_v]$-modules

$$j_v^+(Z) : Z_v^+ \longrightarrow Z.$$

These data define local conditions

$$\Delta_v(Z) : U_v^+(Z) = C^\bullet_{\mathrm{cont}}(G_v, Z_v^+)\xrightarrow{j_v^+(Z)} C^\bullet_{\mathrm{cont}}(G_v, Z).$$

Put

$$Z_v^- = \mathrm{Cone}(Z_v^+\xrightarrow{-j_v^+(Z)} Z);$$

then $U_v^-(Z) = C^\bullet_{\mathrm{cont}}(G_v, Z_v^-)$.

6.7.2. Definition. — We say that $X_v^+ \perp_\pi Y_v^+$ if the morphism

$$X_v^+\otimes_R Y_v^+ \xrightarrow{j^+(X)\otimes j^+(Y)} X\otimes_R Y \xrightarrow{\pi} J(1)$$

is zero.

6.7.3. Lemma

(i) $X_v^+ \perp_\pi Y_v^+ \Longrightarrow \Delta_v(X) \perp_{\pi,0} \Delta_v(Y)$.
(ii) $X_v^+ \perp_\pi Y_v^+ \iff Y_v^+ \perp_{\pi\circ s_{12}} X_v^+$.

Proof. — This follows from the definitions. □

6.7.4. The morphism $\pi \circ (j^+(X) \otimes j^+(Y))$ in 6.7.2 factors through $X \otimes_R Y_v^+$. By adjunction we obtain a morphism of complexes

$$X \longrightarrow \mathrm{Hom}^\bullet_R(Y_v^+, J(1)) = D_J(Y_v^+)(1). \tag{6.7.4.1}$$

If $X_v^+ \perp_\pi Y_v^+$, then (6.7.4.1) induces a morphism of complexes

$$X_v^- \longrightarrow \mathrm{Hom}^\bullet_R(Y_v^+, J(1)) = D_J(Y_v^+)(1). \tag{6.7.4.2}$$

We say that $X_v^+ \perp\!\!\perp_\pi Y_v^+$ if (6.7.4.2) is a quasi-isomorphism.

6.7.5. We shall be interested only in the following two cases:

(A) The complexes X, Y, X_v^+, Y_v^+ are bounded, $J = I[n]$ for some $n \in \mathbf{Z}$ and either all cohomology groups of X, X_v^+ (resp., of Y, Y_v^+) are of finite (resp., co-finite) type over R, or all cohomology groups of X, X_v^+ (resp., of Y, Y_v^+) are of co-finite (resp., finite) type over R.

(B) The complexes X, Y, X_v^+, Y_v^+ are bounded, $J = \omega^\bullet[n]$ for some $n \in \mathbf{Z}$ and all cohomology groups of X, Y, X_v^+, Y_v^+ are of finite type over R.

6.7.6. Proposition. — *Assume that one of the conditions (A) or (B) of 6.7.5 is satisfied, $X_v^+ \perp_\pi Y_v^+$ and π is a perfect duality. Then*

(i) $X_v^+ \perp\!\!\perp_\pi Y_v^+ \iff Y_v^+ \perp\!\!\perp_{\pi\circ s_{12}} X_v^+$.
(ii) $X_v^+ \perp\!\!\perp_\pi Y_v^+ \Longrightarrow \Delta_v(X) \perp\!\!\perp_{\pi,0} \Delta_v(Y)$.
(iii) $Y_v^+ \perp\!\!\perp_{\pi\circ s_{12}} X_v^+ \Longrightarrow \Delta_v(Y) \perp\!\!\perp_{\pi\circ s_{12},0} \Delta_v(X)$.
(iv) *Completing the morphism* (6.7.4.2) *to an exact triangle* $W_v \to X_v^- \to D_J(Y_v^+)(1) \to W_v[1]$ *in* $D^b({}_{R[G_v]}^{\mathrm{ad}}\mathrm{Mod})$, *then there is an isomorphism in* $D^b({}_R\mathrm{Mod})$

$$\mathrm{Err}_v(\Delta_v(X), \Delta_v(Y), \pi) \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, W_v).$$

Proof. — The map (6.7.4.2) and the dual of its analogue for $\pi\circ s_{12}$ fit into a morphism of exact triangles in $D^b({}_{R[G_v]}^{\mathrm{ad}}\mathrm{Mod})$

$$\begin{array}{ccccc} X_v^+ & \longrightarrow & X & \longrightarrow & X_v^- \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}^\bullet_R(Y_v^-, J(1)) & \longrightarrow & \mathrm{Hom}^\bullet_R(Y, J(1)) & \longrightarrow & \mathrm{Hom}^\bullet_R(Y_v^+, J(1)), \end{array} \tag{6.7.6.1}$$

in which the middle vertical arrow is an isomorphism, since π is a perfect duality; this proves (i). As regards (iv), applying the functor $\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, -)$ to (6.7.6.1) we obtain a morphism of exact triangles in $D^b({}_R\mathrm{Mod})$

$$\begin{array}{ccccc} \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, X_v^+) & \longrightarrow & \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, X) & \longrightarrow & \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, X_v^-) \\ \big\downarrow{\scriptstyle u_{+,\pi,0}} & & \big\downarrow & & \big\downarrow{\scriptstyle\lambda} \\ \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, D_J(Y_v^-)(1)) & \longrightarrow & \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, D_J(Y)(1)) & \longrightarrow & \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, D_J(Y_v^+)(1)), \end{array}$$

in which the middle vertical arrow is again an isomorphism. This gives isomorphisms

$$\mathrm{Err}_v(\Delta_v(X), \Delta_v(Y), \pi) = \mathrm{Cone}(u_{+,\pi,0}) \xrightarrow{\sim} \mathrm{Cone}(\lambda)[-1] \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, W_v)$$

in $D^b({}_R\mathrm{Mod})$, proving (iv) and (ii) (hence also (iii), if we replace π by $\pi \circ s_{12}$). □

6.7.7. Proposition. — *Assume that we are given $Z_v^+ \to Z$ $(Z = X, Y)$ satisfying $X_v^+ \perp_\pi Y_v^+$ for all $v \in S_f$. Assume that one of the conditions 6.7.5(A) or (B) is satisfied. Then*

$$\widetilde{\mathbf{R}\Gamma}_f(X) \xrightarrow{\gamma_{\pi,0}} D_{J[-3]}(\widetilde{\mathbf{R}\Gamma}_f(Y)) \longrightarrow \bigoplus_{v \in S_f} \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, W_v),$$

where W_v was defined in Proposition 6.7.6(iv), is an exact triangle in $D^b_{ft}({}_R\mathrm{Mod})$ (resp., $D^b_{coft}({}_R\mathrm{Mod})$). In particular, if $X_v^+ \perp\!\!\perp_\pi Y_v^+$ for all $v \in S_f$, then the map

$$\gamma_{\pi,0} : \widetilde{\mathbf{R}\Gamma}_f(X) \longrightarrow D_{J[-3]}\big(\widetilde{\mathbf{R}\Gamma}_f(Y)\big)$$

is an isomorphism in $D^b_{ft}({}_R\mathrm{Mod})$ (resp., $D^b_{coft}({}_R\mathrm{Mod})$).

Proof. — Apply Theorem 6.3.4. □

6.7.8. Transpositions. — Assume that we are given $Z_v^+ \to Z$ $(Z = X, Y)$ satisfying $X_v^+ \perp_\pi Y_v^+$ for all $v \in S_f$. Then the following objects are the data of the type 6.5.3.1–6.5.3.5:

$$\Delta_v(X) \perp_{\pi,0} \Delta_v(Y), \quad \Delta_v(Y) \perp_{\pi \circ s_{12},0} \Delta_v(X), \quad \mathcal{T}_v^+(Z) = \mathcal{T}, \quad V_{Z,v} = H_v = 0,$$
$$k_Z, k_{Z,v}^+, k_{Z,v}^- \text{ given by a functorial homotopy } \mathrm{id} \rightsquigarrow \mathcal{T}.$$

6.7.9. Assume that X, Y, X_v^+, Y_v^+ $(v \in S_f)$ satisfy the condition 6.7.5(B) with $J = \omega^\bullet$. Write $T = X, T_v^+ = X_v^+$, $T^*(1) = Y$, $T^*(1)_v^+ = Y_v^+$ and put

$$A = D(Y)(1), \quad A_v^+ = D(Y_v^-)(1)$$
$$A^*(1) = D(X)(1), \quad A^*(1)_v^+ = D(X_v^-)(1).$$

If $X_v^+ \perp\!\!\perp_\pi Y_v^+$ for all $v \in S_f$, then the previous discussion and Theorem 6.3.4 imply that the Selmer complexes of T, A, $T^*(1)$, $A^*(1)$ are related by the duality diagram

$$\begin{array}{ccc} \widetilde{\mathbf{R}\Gamma}_f(T) & \overset{\mathscr{D}}{\longleftrightarrow} & \widetilde{\mathbf{R}\Gamma}_f(T^*(1))[3] \\ \downarrow \Phi & \overset{D}{\times} & \downarrow \Phi \\ \widetilde{\mathbf{R}\Gamma}_f(A) & & \widetilde{\mathbf{R}\Gamma}_f(A^*(1))[3] \end{array}$$

(in $D^b_{(\mathrm{co})\mathrm{ft}}({}_R\mathrm{Mod})$). This diagram gives a spectral sequence

$$E_2^{i,j} = \mathbb{E}\mathrm{xt}^i_R\big(\widetilde{H}^{3-j}_f(T^*(1)), \omega\big) = \mathbb{E}\mathrm{xt}^i_R\big(D(\widetilde{H}^j_f(A)), \omega\big) \Longrightarrow \widetilde{H}^{i+j}_f(T).$$

6.8. Localization

6.8.1. Let $\mathscr{S} \subset R$ be a multiplicative subset. Everything in Sections 6.1–6.7 is still valid for $R_{\mathscr{S}}$ instead of R; the only difference is that references to 5.2.3 should be replaced by those to 5.6.2. For example, the same proof as in 5.6.3 gives a localized version of the duality Theorem 6.3.4.

6.9. In the absence of (P)

6.9.1. In the situation of 5.7, we must also consider the complexes $\widehat{C}^\bullet_{\mathrm{cont}}(G_v, X)$ and local conditions

$$U_v^+(X) \longrightarrow \widehat{C}^\bullet_{\mathrm{cont}}(G_v, X)$$

at all real primes v of K. Everything in 6.1–6.7 works with obvious modifications, provided we consider only bounded complexes X, Y. The easiest method is to put $U_v^+(X) = U_v^+(Y) = 0$ for all real primes v; then the complex

$$\bigoplus_{K_v = \mathbf{R}} \mathrm{Err}_v(\Delta_v(X), \Delta_v(Y), \pi)$$

becomes acyclic in $D({}_{R[1/2]}\mathrm{Mod})$, where $R[1/2] = R \otimes_{\mathbf{Z}_2} \mathbf{Q}_2$.

CHAPTER 7

UNRAMIFIED COHOMOLOGY

Let $v \in S_f, v \nmid p$. The aim of this chapter is to define a suitable generalization of unramified cohomology $H^1_{\mathrm{ur}}(G_v, M)$. In the absence of a good Hochschild-Serre spectral sequence on the level of complexes, we use explicit "small" complexes computing continuous cohomology in this case. Unramified local conditions turn out to be orthogonal with respect to the Pontrjagin duality (Proposition 7.5.5, 7.6.6), but not with respect to the Grothendieck duality; generalized local Tamagawa factors appear at this point (7.6.7–7.6.12). Combining unramified local conditions with those from Sect. 6.7, we obtain Selmer complexes associated to the *Greenberg local conditions*; these are studied in Sect. 7.8.

7.1. Notation

7.1.1. We use the standard notation: K_v^{ur} (resp., K_v^t) denotes the maximal unramified (resp., tamely ramified) extension of K_v contained in K_v^{sep} and $I_v = \mathrm{Gal}(K_v^{\mathrm{sep}}/K_v^{\mathrm{ur}})$ (resp., $I_v^w = \mathrm{Gal}(K_v^{\mathrm{sep}}/K_v^t)$) the inertia (resp., wild inertia) group. Put

$$\overline{G}_v = G_v/I_v^w = \mathrm{Gal}(K_v^t/K_v)$$
$$\overline{I}_v = I_v/I_v^w = \mathrm{Gal}(K_v^t/K_v^{\mathrm{ur}}) \quad (= \text{the tame inertia group})$$

7.1.2. For $M \in \left({}_{R[G_v]}^{\text{ind-ad}}\mathrm{Mod}\right)$ we define the unramified local conditions $\Delta_v^{\mathrm{ur}}(M)$ to be

$$U_v^+(M) = C^\bullet_{\mathrm{cont}}(G_v/I_v, M^{I_v}) \xrightarrow{\mathrm{inf}} C^\bullet_{\mathrm{cont}}(G_v, M).$$

The inflation maps induce isomorphisms

$$H^i_{\mathrm{cont}}(G_v/I_v, M^{I_v}) \xrightarrow{\sim} \begin{cases} M^{G_v} & i=0 \\ H^1_{\mathrm{ur}}(G_v, M) = \mathrm{Ker}(H^1_{\mathrm{cont}}(G_v, M) \xrightarrow{\mathrm{res}} H^1_{\mathrm{cont}}(I_v, M)) & i=1 \\ 0 & i>1 \end{cases}$$

(this is well-known for discrete modules; the general case follows by taking limits).

We would like to define $\Delta_v^{\mathrm{ur}}(M^\bullet)$ for a (bounded below) *complex* $M^\bullet$ of ind-admissible $R[G_v]$-modules. The naive definition

$$C^\bullet_{\mathrm{cont}}(G_v/I_v, (M^\bullet)^{I_v}) \xrightarrow{\mathrm{inf}} C^\bullet_{\mathrm{cont}}(G_v, M^\bullet)$$

is not very useful, as it does not factor through the derived category. Note that M^{I_v} is quasi-isomorphic to $\tau_{\leqslant 0}\, C^\bullet_{\mathrm{cont}}(I_v, M)$; it would be natural (especially from a perverse point of view) to define $\Delta_v^{\mathrm{ur}}(M^\bullet)$ as

$$\text{(7.1.2.1)}\quad \text{“}C^\bullet_{\mathrm{cont}}(G_v/I_v, \tau_{\leqslant 0}\, C^\bullet_{\mathrm{cont}}(I_v, M^\bullet)) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v/I_v, C^\bullet_{\mathrm{cont}}(I_v, M^\bullet)) \xrightarrow{\sim} C^\bullet_{\mathrm{cont}}(G_v, M^\bullet)\text{”}.$$

Unfortunately, we have not been able to make sense of the Hochschild-Serre spectral sequence for continuous cohomology even in this very simple case. The problem is, as explained in 3.6.1.4, that in general $\tau_{\leqslant 0}\, C^\bullet_{\mathrm{cont}}(I_v, M^\bullet)$ is not a complex of G_v/I_v-modules (let alone of ind-admissible $R[G_v/I_v]$-modules). Instead of interpreting (7.1.2.1) literally, we use explicit “small” complexes quasi-isomorphic to $C^\bullet_{\mathrm{cont}}(G, M^\bullet)$ for $G = G_v, I_v, G_v/I_v$.

7.2. Complexes $C(M)$

7.2.1. For every $M \in \left({}^{\text{ind-ad}}_{R[G_v]}\mathrm{Mod}\right)$, the inflation map

$$\mathrm{inf} : C^\bullet_{\mathrm{cont}}(\overline{G}_v, M^t) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v, M) \qquad (M^t = M^{I_v^w})$$

is a quasi-isomorphism, by Lemma 4.1.4. This means that it will be sufficient to consider only $C^\bullet_{\mathrm{cont}}(\overline{G}_v, M)$ for “tame” modules $M = M^t \in \left({}^{\text{ind-ad}}_{R[\overline{G}_v]}\mathrm{Mod}\right)$.

Fix a topological generator $t = t_v$ of $\overline{I}_v \xrightarrow{\sim} \widehat{\mathbf{Z}}/\mathbf{Z}_l$ (where $l \neq p$ is the characteristic of the residue field $k(v)$ of v) and a lift $f = f_v \in \overline{G}_v$ of the geometric Frobenius element $\mathrm{Fr}(v) \in G_v/I_v = \overline{G}_v/\overline{I}_v$. Then

$$\overline{G}_v = \langle t\rangle \rtimes \langle f\rangle$$

has topological generators t and f, with a unique relation

$$tf = ft^L, \quad L = |k(v)| = N(v).$$

The element

$$\theta = f(1 + t + \cdots + t^{L-1}) \in \mathbf{Z}[\overline{G}_v]$$

satisfies

$$\theta(t-1) = f(t^L - 1) = (t-1)f$$
$$(\theta - 1)(t-1) = (t-1)(f-1).$$

For every $\overline{G}_v$-module M denote by $C(M)$ the complex

$$C(M) = \left[M \xrightarrow{(f-1,t-1)} M \oplus M \xrightarrow{(1-t,\theta-1)} M\right]$$

in degrees $0,1,2$ and by $C^+(M)$ (resp., $C^-(M)$) the subcomplex of $C(M)$ equal to

$$C^+(M) = \left[M^{t=1} \xrightarrow{f-1} M^{t=1}\right]$$

in degrees $0,1$ (resp., the quotient complex equal to

$$C^-(M) = \left[M/(t-1)M \xrightarrow{\theta-1} M/(t-1)M\right] = \left[M/(t-1)M \xrightarrow{Lf-1} M/(t-1)M\right]$$

in degrees $1,2$). The canonical projections define a quasi-isomorphism

$$C(M)/C^+(M) \xrightarrow{\text{Qis}} C^-(M),$$

hence an exact triangle

$$C^+(M) \longrightarrow C(M) \longrightarrow C^-(M) \longrightarrow C^+(M)[1].$$

On the level of cohomology this gives

$$\begin{array}{ccccccccc} & & H^0(C^+(M)) & \xrightarrow{\sim} & H^0(C(M)) & & & & \\ 0 & \longrightarrow & H^1(C^+(M)) & \longrightarrow & H^1(C(M)) & \longrightarrow & H^1(C^-(M)) & \longrightarrow & 0 \\ & & & & H^2(C(M)) & \xrightarrow{\sim} & H^2(C^-(M)). & & \end{array}$$

We are now going to define functorial quasi-isomorphisms

$$C(M) \xrightarrow{\mu} C^\bullet_{\text{cont}}(\overline{G}_v, M) \xrightarrow{\lambda} C(M) \qquad \left(M \in \left({}^{\text{ind-ad}}_{R[\overline{G}_v]}\text{Mod}\right)\right)$$

satisfying $\lambda \circ \mu = \text{id}$.

7.2.2. Let $G = \langle\sigma\rangle$ be a topologically cyclic pro-finite group with a fixed topological generator σ. Assume that the order of G is divisible by p^∞; then $\text{cd}_p(G) = 1$. In this case the complex

$$\left[M \xrightarrow{\sigma-1} M\right] \quad \left(M \in \left({}^{\text{ind-ad}}_{R[G]}\text{Mod}\right)\right)$$

in degrees $0,1$ is canonically quasi-isomorphic to $C^\bullet_{\text{cont}}(G, M)$. Indeed, writing $M = \varinjlim M_\alpha$ ($M_\alpha \in \mathcal{S}(M)$) and $M_\alpha = \varprojlim M_\alpha/p^n M_\alpha$, it is sufficient to construct functorial quasi-isomorphisms

$$\lambda : C^\bullet_{\text{cont}}(G, M) \longrightarrow \left[M \xrightarrow{\sigma-1} M\right]$$

for discrete p-primary torsion G-modules M (such as $M_\alpha/p^n M_\alpha$ above). The formulas

$$\lambda_0 = \text{id}, \quad \lambda_1(c) = c(\sigma), \quad \lambda_i = 0\,(i > 1)$$

define such a λ. There is another functorial quasi-isomorphism in the opposite direction

$$\mu : \left[M \xrightarrow{\sigma-1} M\right] \longrightarrow C^\bullet_{\text{cont}}(G, M),$$

given by

$$\mu_0 = \text{id}$$
$$(\mu_1(m))(\sigma^a) = (1 + \sigma + \cdots + \sigma^{a-1})m \qquad (a \in \mathbf{N}_0)$$

(with the convention that $1+\sigma+\cdots+\sigma^{a-1}=0$ for $a=0$). This formula defines the values of $\mu_1(m)$ only at $1,\sigma,\sigma^2,\ldots$, but $\mu_1(m)$ extends uniquely by continuity to a continuous 1-cochain (in fact a 1-cocycle).

As $\lambda\circ\mu=\mathrm{id}$, the maps λ,μ induce mutually inverse isomorphisms in the derived category.

All of the above applies, in particular, to $G_v/I_v=\langle f\rangle$ and $\overline{I}_v=\langle t\rangle$.

7.2.3. Proposition. — *The formulas*

$$\begin{aligned}\lambda_0 &= \mathrm{id}\\ \lambda_1(c) &= (c(f),c(t))\\ \lambda_2(z) &= -z(t,f)+z(f,t^L)+f\sum_{i=0}^{L-2} t^i z(t,t^{L-1-i})\\ \lambda_i &= 0 \qquad (i>2)\end{aligned}$$

define functorial quasi-isomorphisms

$$\lambda: C^\bullet_{\mathrm{cont}}(\overline{G}_v,M)\longrightarrow C(M)\quad \big(M\in \big({}^{\text{ind-ad}}_{R[\overline{G}_v]}\mathrm{Mod}\big)\big).$$

Proof. — As in 7.2.2 we can assume that M is a discrete p-primary torsion $\overline{G}_v$-module. Let us first explain the origin of the map λ. We begin with a morphism of complexes

$$\lambda': C^\bullet_{\mathrm{cont}}(\overline{G}_v,M)\longrightarrow \Big[M\xrightarrow{\delta_0}M^{\oplus(2L+1)}\xrightarrow{\delta_1}M^{\oplus 2L}\Big]$$

given by

$$\begin{aligned}\lambda'_0 &= \mathrm{id}\\ \lambda'_1(c) &= (c(f),c(ft),\ldots,c(ft^L),c(t),\ldots,c(t^L))\\ \lambda'_2(z) &= (z(f,t),z(f,t^2),\ldots,z(f,t^L),z(t,t),\ldots,z(t,t^{L-1}),z(t,f)).\end{aligned}$$

The differentials are uniquely determined by $\delta\circ\lambda'=\lambda'\circ\delta$:

$$\begin{aligned}\delta_0(m) &= ((f-1)m,(ft-1)m,\ldots,(ft^L-1)m,(t-1)m,\ldots,(t^L-1)m)\\ \delta_1(x_0,\ldots,x_L,y_1,\ldots,y_L) &= (f(y_1)-x_1+x_0,\ldots,f(y_L)-x_L+x_0,\\ &\qquad t(y_1)-y_2+y_1,\ldots,t(y_{L-1})-y_L+y_1,t(x_0)-x_L+y_1).\end{aligned}$$

We want to construct λ as a composition $\lambda=\lambda''\circ\lambda'$ for a suitable morphism of complexes

$$\lambda'': \Big[M\xrightarrow{\delta_0}M^{\oplus(2L+1)}\xrightarrow{\delta_1}M^{\oplus 2L}\Big]\longrightarrow C(M).$$

It is natural to require $\lambda_0=\mathrm{id}$, $\lambda_1(c)=(c(f),c(t))$, which implies that

$$\lambda''_0=\mathrm{id},\quad \lambda''_1(x_0,\ldots,x_L,y_1,\ldots,y_L)=(x_0,y_1).$$

The condition $\lambda''\circ\delta=\delta\circ\lambda''$ forces us to define

$$\lambda''_2(z_1,\ldots,z_L,u_1,\ldots,u_{L-1},w)=-w+z_L+f\sum_{i=0}^{L-2}t^i u_{L-1-i}.$$

This leads to the formulas for λ and at the same time shows that λ is a morphism of complexes.

Why is λ a quasi-isomorphism? First of all, $(\lambda_0)_* : M^{\overline{G}_v} \xrightarrow{\sim} M^{t=1,f=1}$ is an isomorphism for trivial reasons. The Hochschild-Serre spectral sequence

$$E_2^{i,j} = H^i(\overline{G}_v/\overline{I}_v, H^j(\overline{I}_v, M)) \Longrightarrow H^{i+j}(\overline{G}_v, M)$$

simplifies to

$$0 \longrightarrow H^1(\overline{G}_v/\overline{I}_v, M^{\overline{I}_v}) \longrightarrow H^1(\overline{G}_v, M) \longrightarrow H^1(\overline{I}_v, M)^{\overline{G}_v/\overline{I}_v} \longrightarrow 0$$

and

$$H^2(\overline{G}_v, M) \xrightarrow{\sim} H^1(\overline{G}_v/\overline{I}_v, H^1(\overline{I}_v, M)) \tag{7.2.3.1}$$

(we have dropped the subscript "cont", as we consider usual cohomology groups of discrete modules).

The quasi-isomorphism

$$\lambda^t : C^\bullet_{\mathrm{cont}}(\overline{I}_v, M) \longrightarrow \Big[M \xrightarrow{t-1} M\Big]$$

gives isomorphisms on cohomology $(\lambda_0^t)_* : M^{\overline{I}_v} = M^{t=1}$ and

$$\begin{array}{rccc} (\lambda_1^t)_* : & H^1(\overline{I}_v, M) & \xrightarrow{\sim} & M/(t-1)M \\ & [c] & \longmapsto & c(t) \ (\mathrm{mod}\, (t-1)M). \end{array}$$

Under $(\lambda_1^t)_*$, the action of f on $H^1(\overline{I}_v, M)$ corresponds to the action of θ (= the action of Lf) on $M/(t-1)M$, since

$$(f * c)(t) = f(c(f^{-1}tf)) = f\big(c(t^L)\big) = f(1 + t + \cdots + t^{L-1})c(t) = \theta(c(t)). \tag{7.2.3.2}$$

Similarly, we have a quasi-isomorphism

$$\lambda^f : C^\bullet_{\mathrm{cont}}\big(\overline{G}_v/\overline{I}_v, M^{\overline{I}_v}\big) \longrightarrow \Big[M^{\overline{I}_v} \xrightarrow{f-1} M^{\overline{I}_v}\Big].$$

Taken together, $(\lambda_1^f, \lambda_1, \lambda_1^t)$ induce a map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1\big(\overline{G}_v/\overline{I}_v, M^{\overline{I}_v}\big) & \longrightarrow & H^1(\overline{G}_v, M) & \longrightarrow & H^1(\overline{I}_v, M)^{\overline{G}_v/\overline{I}_v} & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle (\lambda_1^f)_*} & & \big\downarrow{\scriptstyle (\lambda_1)_*} & & \big\downarrow{\scriptstyle (\lambda_1^t)_*} & & \\ 0 & \longrightarrow & H^1(C^+(M)) & \longrightarrow & H^1(C(M)) & \longrightarrow & H^1(C^-(M)) & \longrightarrow & 0. \end{array}$$

This shows that $(\lambda_1)_*$ is an isomorphism.

In degree 2, let us recall an explicit description of the isomorphism (7.2.3.1): for every continuous 2-cocycle $z' \in C^2_{\mathrm{cont}}(\overline{G}_v, M)$ there is a cohomologous 2-cocycle $z = z' + \delta c$ vanishing on $\overline{G}_v \times \overline{I}_v$; such a 2-cocycle is called *normalized*. For fixed $g \in \overline{G}_v$ the function

$$(z_g : h \longmapsto z(h, g)) \in C^1_{\mathrm{cont}}(\overline{I}_v, M)$$

depends only on the coset $\overline{g} = g\overline{I}_v \in \overline{G}_v/\overline{I}_v$ and is a 1-cocycle. Furthermore, the function

$$(\overline{g} \longmapsto [z_g]) \in C^1_{\mathrm{cont}}(\overline{G}_v/\overline{I}_v, H^1(\overline{I}_v, M))$$

is again a 1-cocycle and its class in $H^1(\overline{G}_v/\overline{I}_v, H^1(\overline{I}_v, M))$ corresponds to $[z] = [z'] \in H^2(\overline{G}_v, M)$ under (7.2.3.1).

This recipe, the formulas for λ^t, λ^f and (7.2.3.2) give an isomorphism

$$\begin{array}{rcl} H^2(\overline{G}_v, M) & \xrightarrow{\sim} & (M/(t-1)M)/(\theta-1)(M/(t-1)M) = M/(t-1, 1-\theta)M \\ [z] & \longmapsto & z(t,f) \pmod{(t-1, 1-\theta)M}, \end{array}$$

which coincides with $-(\lambda_2)_*$, since z is normalized. This finishes the proof that λ is a quasi-isomorphism. □

7.2.4. Proposition

(i) *For every discrete p-primary torsion $\overline{G}_v$-module M, the formulas*

$$\begin{aligned} \mu_0 &= \mathrm{id} \\ (\mu_1(m, m'))(f^a t^b) &= (1 + f + \cdots + f^{a-1})m + f^a(1 + t + \cdots + t^{b-1})m' \\ (\mu_2(m))(f^a t^b, f^c t^d) &= -f^a(1 + t + \cdots + t^{b-1})(1 + \theta + \cdots + \theta^{c-1})m \end{aligned}$$

($a, b, c, d \in \mathbf{N}_0$) extend uniquely by continuity to a morphism of complexes

$$\mu : C(M) \longrightarrow C^\bullet_{\mathrm{cont}}(\overline{G}_v, M).$$

(ii) *The morphism μ is a quasi-isomorphism and is functorial in M, hence defines a functorial quasi-isomorphism*

$$\mu : C(M) \longrightarrow C^\bullet_{\mathrm{cont}}(\overline{G}_v, M) \quad \left(M \in \left({}^{\text{ind-ad}}_{R[\overline{G}_v]}\mathrm{Mod}\right)\right)$$

satisfying $\lambda \circ \mu = \mathrm{id}$.

Proof. — We leave it to the reader to check that the function $\mu_1(m, m')$ (resp., $\mu_2(m)$) defined in (i) on a dense subset of $\overline{G}_v$ (resp., $\overline{G}_v \times \overline{G}_v$) extends by continuity to all of $\overline{G}_v$ (resp., $\overline{G}_v \times \overline{G}_v$). In order to verify that μ commutes with differentials it is enough to check this on the above dense subsets, which in turn follows from an explicit calculation based on the following formulas:

$$\begin{aligned} t^a f^b &= f^b t^{aL^b} \\ \theta^b &= f^b(1 + t + \cdots + t^{(L^b - 1)}) \\ \theta^b(t-1) &= (t-1) f^b \\ (1 + t + \cdots + t^{a-1})\theta^b &= f^b(1 + t + \cdots + t^{(aL^b - 1)}) \qquad (a, b \in \mathbf{N}_0). \end{aligned}$$

It follows from the definitions that $\lambda \circ \mu = \mathrm{id}$; thus μ is a quasi-isomorphism. (See 7.4.8 for a more conceptual proof.) □

7.2.5. Corollary. — *The map μ induces isomorphisms*

$$\begin{aligned} (\mu_0)_* : H^0(C^+(M)) &\xrightarrow{\sim} H^0_{\mathrm{cont}}(G_v, M) \\ (\mu_1)_* : H^1(C^+(M)) &\xrightarrow{\sim} H^1_{\mathrm{ur}}(G_v, M) \end{aligned}$$

for every $M \in \left({}^{\text{ind-ad}}_{R[\overline{G}_v]}\mathrm{Mod}\right)$*. In other words,*

$$C^+(M) \longrightarrow C(M) \xrightarrow{\mu} C^\bullet_{\mathrm{cont}}(G_v, M)$$

is an alternative way of defining unramified local conditions $\Delta_v^{\mathrm{ur}}(M)$*.*

7.2.6. Assume that $G = G_1 \times \cdots \times G_r$, where each $G_i = \langle \sigma_i \rangle$ is as in 7.2.2. Consider the Koszul complex $K^\bullet = K^\bullet_\Lambda(\Lambda, \mathbf{x})$ for the sequence $\mathbf{x} = (\sigma_1 - 1, \ldots, \sigma_r - 1)$ over $\Lambda = \mathbf{Z}_p[\![G]\!]$. Then $K^\bullet[r]$ is a Λ-free resolution of $\mathbf{Z}_p$, which implies ([**Bru**, Lemma 4.2 (i)]) that the complex

$$\mathrm{Hom}^{\bullet,\mathrm{naive}}_{\mathbf{Z}_p[\![G]\!],\mathrm{cont}}(K^\bullet[r], M) = \mathrm{Hom}^{\bullet,\mathrm{naive}}_{\mathbf{Z}_p[\![G]\!]}(K^\bullet[r], M) \tag{7.2.6.1}$$

is quasi-isomorphic to $C^\bullet_{\mathrm{cont}}(G, M)$, for every discrete p-primary torsion G-module M. By the usual limit argument (and functoriality of (7.2.6.1)), the same property holds for all $M \in \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$. For example, for $r = 2$, the complex (7.2.6.1) is equal to

$$\Big[M \xrightarrow{(\sigma_1 - 1, \sigma_2 - 1)} M \oplus M \xrightarrow{(1-\sigma_2, \sigma_1 - 1)} M\Big].$$

7.2.7. Continuous homology of $G \xrightarrow{\sim} \mathbf{Z}_p^r$. — Self-duality of the Koszul complex ([**Br-He**, §1.6.10]) implies that the complex (7.2.6.1) is isomorphic to $K^\bullet \otimes_\Lambda M = K^\bullet_\Lambda(M, \mathbf{x})$, hence

$$H^i(G, M) \xrightarrow{\sim} H^i(K^\bullet_\Lambda(M, \mathbf{x})) = H_{r-i}(G, M) \tag{7.2.7.1}$$

(cf. [**Bru**, §4.2 (ii)]).

It is natural to generalize (7.2.7.1) and use it as a *definition* of continuous homology

$$H_{j,\mathrm{cont}}(G, M) := H^{r-j}(K^\bullet_{R[\![G]\!]}(M, \mathbf{x})) \tag{7.2.7.2}$$

of any $R[\![G]\!]$-module M (up to isomorphism, this is independent of the choice of the γ_i's). For example,

$$H_{0,\mathrm{cont}}(G, M) = M_G, \quad H_{r,\mathrm{cont}}(G, M) \xrightarrow{\sim} M^G.$$

If A is a discrete $R[\![G]\!]$-module, then

$$H_{j,\mathrm{cont}}(G, \overline{D}(A)) \xrightarrow{\sim} \overline{D}(H^j(G, A)), \tag{7.2.7.3}$$

where $\overline{D}(-) = D_{R[\![G]\!]}(-)$.

7.3. Explicit resolutions (discrete case)

One can reinterpret the map μ in terms of (a pro-finite version of) Fox's free differential calculus, which we now briefly summarize (see [**Bro**, Ex. II.5.3, Ex. IV.2.3, Ex. IV.2.4]).

7.3.1. Let $F = F(S)$ be the free group on a set S (= generators), and T (= relations) a subset of F. Let $N \lhd F$ be the smallest normal subgroup of F containing T; put $G = F/N$.

The augmentation ideal

$$J_F = \mathrm{Ker}(\mathbf{Z}[F] \longrightarrow \mathbf{Z})$$

is a free left $\mathbf{Z}[F]$-module with basis $s - 1$ ($s \in S$). One defines partial derivatives

$$\frac{\partial}{\partial s} : F \longrightarrow \mathbf{Z}[F] \qquad (s \in S)$$

by the formulas

$$f - 1 = \sum_{s\in S} \left(\frac{\partial f}{\partial s}\right)(s-1) \qquad (f \in F).$$

They satisfy the 1-cocycle identity

$$\frac{\partial(f_1 f_2)}{\partial s} = f_1 \frac{\partial f_2}{\partial s} + \frac{\partial f_1}{\partial s}.$$

Denote by $a \mapsto \overline{a}$ the projections $F \to G$ and $\mathbf{Z}[F] \to \mathbf{Z}[G]$. The latter projection has kernel $J_N \mathbf{Z}[F] = \mathbf{Z}[F] J_N$ (where J_N is the augmentation ideal of $\mathbf{Z}[N]$). Tensoring the exact sequence of $\mathbf{Z}[F]$-modules

$$0 \longrightarrow \bigoplus_{s\in S} \mathbf{Z}[F]e_s \xrightarrow{\partial_1} \mathbf{Z}[F] \longrightarrow \mathbf{Z} \longrightarrow 0$$

(in which $\partial_1(e_s) = s - 1$) with $\mathbf{Z}[G]$ (or, equivalently, taking its homology $H_\bullet(N, -)$), we obtain an exact sequence of $\mathbf{Z}[G]$-modules

$$0 \longrightarrow N^{ab} \xrightarrow{i} \bigoplus_{s\in S} \mathbf{Z}[G]e_s \xrightarrow{\partial_1} \mathbf{Z}[G] \longrightarrow \mathbf{Z} \longrightarrow 0, \tag{7.3.1.1}$$

in which G acts on N^{ab} by conjugation and

$$i(n \pmod{[N,N]}) = \bigoplus_{s\in S} \overline{\left(\frac{\partial n}{\partial s}\right)} e_s, \quad \partial_1(e_s) = \overline{s} - 1 \tag{7.3.1.2}$$

(we use the standard isomorphism $J_N/J_N^2 \xrightarrow{\sim} N^{ab}$, $n - 1 \pmod{J_N^2} \leftrightarrow n \pmod{[N,N]}$).

7.3.2. Define a surjective homomorphism of $\mathbf{Z}[G]$-modules

$$\eta : \bigoplus_{t\in T} \mathbf{Z}[G]e'_t \longrightarrow N^{ab}$$

by

$$\eta(e'_t) = t \pmod{[N,N]}. \tag{7.3.2.1}$$

Then we have

$$\mathrm{cd}(G) \leqslant 2 \iff N^{ab} \text{ is a projective } \mathbf{Z}[G]\text{-module}.$$

When is η an isomorphism? A necessary condition is that $\mathrm{cd}(G) \leqslant 2$ and T being a minimal set of relations of G. It is unclear when is this condition also sufficient. A classical result of Lyndon [**Ly**] states that, for $T = \{t\}$ consisting of one relation, we have

$$\eta \text{ is an isomorphism} \iff \mathrm{cd}(G) \leqslant 2 \iff t \neq u^n \text{ for any } u \in F, n \geqslant 2.$$

For general T, if we *assume* that η is an isomorphism, then

$$\bigoplus_{t\in T} \mathbf{Z}[G]e'_t \xrightarrow{\partial_2} \bigoplus_{s\in S} \mathbf{Z}[G]e_s \xrightarrow{\partial_1} \mathbf{Z}[G] \qquad (\partial_2 = i \circ \eta) \tag{7.3.2.2}$$

is a $\mathbf{Z}[G]$-free resolution of $\mathbf{Z}$ (the "Lyndon-Fox resolution").

Fix a section $\sigma : G \to F$ of the projection $F \to G$. Then the formulas

$$\begin{aligned} &\alpha_0 : [\,] \longmapsto 1 \\ &\alpha_1 : [g] \longmapsto \sum_{s\in S} \overline{\left(\frac{\partial(\sigma(g))}{\partial s}\right)} e_s \\ &\alpha_2 : [g_1, g_2] \longmapsto \sigma(g_1)\sigma(g_2)\sigma(g_1g_2)^{-1} \pmod{[N,N]} \\ &\alpha_i = 0 \qquad (i > 2) \end{aligned} \tag{7.3.2.3}$$

(extended by $\mathbf{Z}[G]$-linearity) define a morphism of complexes

$$\begin{array}{ccccccc} \bigoplus_{g_1,g_2,g_3\in G} \mathbf{Z}[G]\cdot[g_1|g_2|g_3] & \longrightarrow & \bigoplus_{g_1,g_2\in G} \mathbf{Z}[G]\cdot[g_1|g_2] & \longrightarrow & \bigoplus_{g_1} \mathbf{Z}[G]\cdot[g_1] & \longrightarrow & \mathbf{Z}[G]\cdot[\,] \\ \downarrow & & \downarrow{\scriptstyle\alpha_2} & & \downarrow{\scriptstyle\alpha_1} & & \downarrow{\scriptstyle\alpha_0} \\ 0 & \longrightarrow & N^{ab} & \longrightarrow & \bigoplus_{s\in S} \mathbf{Z}[G]e_s & \xrightarrow{\partial_1} & \mathbf{Z}[G] \end{array}$$

from the bar resolution $\mathbf{Z}[G]^{\otimes}_{\bullet}$ to the resolution (7.3.1.1).

If η is an isomorphism, then $(\eta^{-1} \circ \alpha_2, \alpha_1, \alpha_0)$ give an explicit quasi-isomorphism from $\mathbf{Z}[G]^{\otimes}_{\bullet}$ to the Lyndon-Fox resolution (7.3.2.2). Applying $\mathrm{Hom}^{\bullet,\mathrm{naive}}_{\mathbf{Z}[G]}(-, M)$ we obtain functorial quasi-isomorphisms

$$\mu : \left[M \xrightarrow{\delta_1} \bigoplus_{s\in S} M \xrightarrow{\delta_2} \bigoplus_{t\in T} M\right] \longrightarrow C^{\bullet}(G, M)$$

for all G-modules M. Here

$$\begin{aligned} &\delta_1(m) = ((\overline{s} - 1)m)_{s\in S} \\ &\delta_2((m_s)_{s\in S}) = (m'_t)_{t\in T}, \quad m'_t = \overline{\left(\frac{\partial t}{\partial s}\right)} m_s. \end{aligned} \tag{7.3.2.4}$$

7.4. Explicit resolutions (pro-finite case)

Consider now a pro-finite version of the constructions in 7.3.

7.4.1. Let $F = F(S)$ be the free group on a *finite* set S. Denote by $\widehat{F} = \varprojlim F/U$, where U runs through all subgroups of finite index in F, the pro-finite completion of F. The pro-finite group algebra

$$\mathbf{Z}_p[\![\widehat{F}]\!] = \varprojlim_{U} \mathbf{Z}_p[F/U] = \varprojlim_{n,U} \mathbf{Z}/p^n\mathbf{Z}[F/U]$$

has augmentation ideal

$$J_{\widehat{F}} = \varprojlim_{U} J_{F/U} = \varprojlim_{n,U} J_{F/U} \otimes \mathbf{Z}/p^n\mathbf{Z},$$

where

$$J_{F/U} = \operatorname{Ker}(\mathbf{Z}_p[F/U] \longrightarrow \mathbf{Z}_p)$$

is the augmentation ideal of $\mathbf{Z}_p[F/U]$.

7.4.2. Lemma. — *$J_{\widehat{F}}$ is a free left $\mathbf{Z}_p[\![\widehat{F}]\!]$-module with basis $s-1$ ($s \in S$).*

Proof. — Applying $H_\bullet(U,-)$ to the exact sequence

$$0 \longrightarrow J_F \otimes \mathbf{Z}/p^n\mathbf{Z} \longrightarrow \mathbf{Z}/p^n\mathbf{Z}[F] \longrightarrow \mathbf{Z}/p^n\mathbf{Z} \longrightarrow 0,$$

we obtain an exact sequence

$$0 \longrightarrow U^{ab} \otimes \mathbf{Z}/p^n\mathbf{Z} \longrightarrow (J_F/J_U J_F) \otimes \mathbf{Z}/p^n\mathbf{Z} \longrightarrow J_{F/U} \otimes \mathbf{Z}/p^n\mathbf{Z} \longrightarrow 0.$$

The projective system $[U^{ab} \otimes \mathbf{Z}/p^n\mathbf{Z}]_{U,n}$ is ML-zero, hence the map

$$\varprojlim_{n,U} (J_F/J_U J_F) \otimes \mathbf{Z}/p^n\mathbf{Z} \longrightarrow \varprojlim_{n,U} (J_{F/U} \otimes \mathbf{Z}/p^n\mathbf{Z}) = J_{\widehat{F}}$$

is an isomorphism. As J_F is a free $\mathbf{Z}_p[F]$-module with basis $s-1$ ($s \in S$), the same is true for $(J_F/J_U J_F) \otimes \mathbf{Z}/p^n\mathbf{Z}$ as a module over $\mathbf{Z}/p^n\mathbf{Z}[F]/J_U\mathbf{Z}/p^n\mathbf{Z}[F] = \mathbf{Z}/p^n\mathbf{Z}[F/U]$. The claim follows by taking the projective limit. □

7.4.3. Corollary. — *The formula*

$$f - 1 = \sum_{s \in S} \left(\frac{\partial f}{\partial s}\right)(s-1) \qquad (f \in \widehat{F})$$

defines maps (in fact, 1-cocycles)

$$\frac{\partial}{\partial s} : \widehat{F} \longrightarrow \mathbf{Z}_p[\![\widehat{F}]\!] \qquad (s \in S).$$

7.4.4. For a given finite subset $T \subset \widehat{F}$, let $N \lhd \widehat{F}$ be the smallest closed normal subgroup containing T; put $G = \widehat{F}/N$.

The kernel of the projection $\mathbf{Z}_p[\![\widehat{F}]\!] \to \mathbf{Z}_p[\![G]\!]$ is equal to $J_{\widehat{F}}\mathbf{Z}_p[\![N]\!] = \mathbf{Z}_p[\![N]\!]J_{\widehat{F}}$. Taking the completed tensor product of the exact sequence of $\mathbf{Z}_p[\![\widehat{F}]\!]$-modules

$$0 \longrightarrow \bigoplus_{s \in S} \mathbf{Z}_p[\![\widehat{F}]\!]e_s \xrightarrow{\partial_1} \mathbf{Z}_p[\![\widehat{F}]\!] \longrightarrow \mathbf{Z}_p \longrightarrow 0$$

($\partial_1(e_s) = s-1$) with $\mathbf{Z}_p[\![G]\!]$ gives an exact sequence of $\mathbf{Z}_p[\![G]\!]$-modules ([**Bru**, §5.2.2])

$$0 \longrightarrow N^{ab}\widehat{\otimes}\mathbf{Z}_p \xrightarrow{i} \bigoplus_{s\in S}\mathbf{Z}_p[\![G]\!]e_s \xrightarrow{\partial_1} \mathbf{Z}_p[\![G]\!] \longrightarrow \mathbf{Z}_p \longrightarrow 0, \tag{7.4.4.1}$$

in which $N^{ab} = N/[N,N]^{cl}$ ($[N,N]^{cl}$ denotes the closure of $[N,N]$), $N^{ab}\widehat{\otimes}\mathbf{Z}_p = \varprojlim N^{ab}/p^n N^{ab}$ and the maps i, ∂_1 are given by the same formulas as in (7.3.1.2).

7.4.5. According to [**Bru**, Lemma 4.2(i)], cohomology of discrete $\mathbf{Z}_p[\![G]\!]$-modules (= discrete p-primary torsion G-modules) can be computed using arbitrary projective pseudo-compact $\mathbf{Z}_p[\![G]\!]$-resolutions of $\mathbf{Z}_p$. One such resolution is given by the profinite bar resolution

$$\mathbf{Z}_p[\![G]\!]^{\widehat{\otimes}}_{\bullet} : \cdots \longrightarrow \mathbf{Z}_p[\![G]\!]\widehat{\otimes}\cdots\widehat{\otimes}\mathbf{Z}_p[\![G]\!] \longrightarrow \cdots \longrightarrow \mathbf{Z}_p[\![G]\!].$$

Fix a continuous section $\sigma : G \to \widehat{F}$ of the projection $\widehat{F} \to G$. The formulas (7.3.2.3) define a morphism of resolutions of $\mathbf{Z}_p$

$$\alpha : \mathbf{Z}_p[\![G]\!]^{\widehat{\otimes}}_{\bullet} \longrightarrow \left[N^{ab}\widehat{\otimes}\mathbf{Z}_p \xrightarrow{i} \bigoplus_{s\in S}\mathbf{Z}_p[\![G]\!]e_s \xrightarrow{\partial_1} \mathbf{Z}_p[\![G]\!]\right].$$

The formula (7.3.2.1) defines a surjective homomorphism

$$\eta : \bigoplus_{t\in T}\mathbf{Z}_p[\![G]\!]e'_t \longrightarrow N^{ab}\widehat{\otimes}\mathbf{Z}_p,$$

hence a morphism from the complex

$$\bigoplus_{t\in T}\mathbf{Z}_p[\![G]\!]e'_t \xrightarrow{\partial_2 = i\circ\eta} \bigoplus_{s\in S}\mathbf{Z}_p[\![G]\!]e_s \xrightarrow{\partial_1} \mathbf{Z}_p[\![G]\!] \tag{7.4.5.1}$$

to the resolution (7.4.4.1).

As in the discrete case we have ([**Bru**, Thm. 5.2])

$$\mathrm{cd}_p(G) \leqslant 2 \iff N^{ab}\widehat{\otimes}\mathbf{Z}_p \text{ is a projective pseudo-compact } \mathbf{Z}_p[\![G]\!]\text{-module}$$

Under what conditions is η an isomorphism? If G is a pro-p-group, then it is shown in [**Bru**, Cor. 5.3] that

$$\eta \text{ is an isomorphism } \iff \mathrm{cd}_p(G) \leqslant 2 \text{ and } T \text{ is a minimal set of relations of } G.$$

Whenever η is an isomorphism we obtain (as in 7.3.2) functorial quasi-isomorphisms

$$\mu : \left[M \xrightarrow{\delta_1} \bigoplus_{s\in S} M \xrightarrow{\delta_2} \bigoplus_{t\in T} M\right] \longrightarrow C^{\bullet}_{\mathrm{cont}}(G, M)$$

for all discrete p-primary torsion G-modules M (here δ_i are given by (7.3.2.4)). As in 7.2.2, the maps μ define corresponding quasi-isomorphisms for all $M \in \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$.

7.4.6. Let us apply the previous discussion to $G = \overline{G}_v$ with generators f, t, *i.e.*, take $F = F(S)$ for $S = \{\alpha, \beta\}$, $T = \{r = \alpha\beta^L\alpha^{-1}\beta^{-1}\}$ $(L \in \mathbf{N},\ p \nmid L)$. Then $\widehat{F} \to G$ sends α to $\overline{\alpha} = f$ and β to $\overline{\beta} = t$. Every element $g \in G$ can be expressed uniquely as $g = f^a t^b$ with $a \in \widehat{\mathbf{Z}}$, $b \in \prod_{q \neq l} \mathbf{Z}_q \subset \widehat{\mathbf{Z}}$. We define a continuous section $\sigma : G \to \widehat{F}$ by $\sigma(f^a t^b) = \alpha^a \beta^b$. A short calculation shows that

$$\frac{\partial r}{\partial \alpha} = 1 - \alpha\beta^L\alpha^{-1}, \quad \frac{\partial r}{\partial \beta} = \alpha(1 + \beta + \cdots + \beta^{L-1}) - \alpha\beta^L\alpha^{-1}\beta^{-1};$$

thus

$$\overline{\left(\frac{\partial r}{\partial \alpha}\right)} = 1 - t, \quad \overline{\left(\frac{\partial r}{\partial \beta}\right)} = \theta - 1.$$

This implies that the complex (7.4.5.1) is given by

$$\partial_2(e'_r) = (1-t)e_\alpha + (\theta - 1)e_\beta$$
$$\partial_1(e_\alpha) = f - 1, \quad \partial_1(e_\beta) = t - 1,$$

hence the complex

$$\Big[M \xrightarrow{\delta_1} \bigoplus_{s \in S} M \xrightarrow{\delta_2} \bigoplus_{t \in T} M\Big]$$

coincides with

$$C(M) = \Big[M \xrightarrow{(f-1,t-1)} M \oplus M \xrightarrow{(1-t,\theta-1)} M\Big].$$

7.4.7. Lemma. — *For the presentation $G = \overline{G}_v = \widehat{F}/N$ considered in 7.4.6 the map $\eta : \mathbf{Z}_p[\![G]\!] \to N^{ab}\widehat{\otimes}\mathbf{Z}_p$ is an isomorphism.*

Proof. — As $\mathrm{cd}_p(G) = 2$, $N^{ab}\widehat{\otimes}\mathbf{Z}_p$ is a projective pseudo-compact $\mathbf{Z}_p[\![G]\!]$-module; thus η has a section and $\mathbf{Z}_p[\![G]\!] = (N^{ab}\widehat{\otimes}\mathbf{Z}_p) \oplus X$. We want to show that $X = 0$. Let M be an arbitrary discrete p-power torsion G-module. The morphisms of complexes of projective pseudo-compact $\mathbf{Z}_p[\![G]\!]$-modules

$$\begin{array}{ccccccc} \mathbf{Z}_p[\![G]\!]^{\widehat{\otimes}}_{\bullet} & \xrightarrow{\alpha} & L_1 = & [N^{ab}\widehat{\otimes}\mathbf{Z}_p & \xrightarrow{i} & \bigoplus_{s \in S} \mathbf{Z}_p[\![G]\!]e_s & \xrightarrow{\partial_1} & \mathbf{Z}_p[\![G]\!]] \\ & & & \uparrow \eta & & & & \\ & & L_2 = & [\mathbf{Z}_p[\![G]\!]e'_r & \xrightarrow{\partial_2} & \bigoplus_{s \in S} \mathbf{Z}_p[\![G]\!]e_s & \xrightarrow{\partial_1} & \mathbf{Z}_p[\![G]\!]] \end{array}$$

induce morphisms of complexes

$$C^\bullet_{\mathrm{cont}}(G, M) = \mathrm{Hom}^{\bullet,\mathrm{naive}}_{\mathbf{Z}_p[\![G]\!],\mathrm{cont}}(\mathbf{Z}_p[\![G]\!]^{\widehat{\otimes}}_{\bullet}, M)$$
$$\xleftarrow{\alpha^*} \mathrm{Hom}^{\bullet,\mathrm{naive}}_{\mathbf{Z}_p[\![G]\!],\mathrm{cont}}(L_1, M) \xrightarrow{\eta^*} \mathrm{Hom}^{\bullet,\mathrm{naive}}_{\mathbf{Z}_p[\![G]\!],\mathrm{cont}}(L_2, M)$$

such that $\eta^* = \lambda \circ \alpha^*$. As both λ and α^* are quasi-isomorphisms (by Proposition 7.2.3 and $\mathrm{cd}_p(G) = 2$, respectively), so is η^*. This means that $\mathrm{Hom}_{\mathbf{Z}_p[\![G]\!],\mathrm{cont}}(X, M) = 0$ for all M. Writing $X = \varprojlim X_u$ with X_u discrete p-power torsion and taking $M = X_u$, we get $\mathrm{id}_X \in \varprojlim_u \mathrm{Hom}_{\mathbf{Z}_p[\![G]\!],\mathrm{cont}}(X, X_u) = 0$, hence $X = 0$. □

7.4.8. As a corollary of Lemma 7.4.7 we obtain a morphism $\eta^{-1}\alpha = (\eta^{-1}\circ\alpha_2, \alpha_1, \alpha_0)$ from the pro-finite bar resolution of $\overline{G}_v$ to the Lyndon-Fox resolution (7.4.5.1) and functorial quasi-isomorphisms

$$\mu = \mathrm{Hom}^{\bullet,\mathrm{naive}}_{\mathbf{Z}_p[\![\overline{G}_v]\!]}(\eta^{-1}\alpha, \mathrm{id}) : C(M) \longrightarrow C^{\bullet}_{\mathrm{cont}}(\overline{G}_v, M).$$

Let us verify that μ is indeed given by the formulas from Proposition 7.2.4. If $g_1 = f^a t^b$, $g_2 = f^c t^d$ with $a, b, c, d \in \mathbf{N}_0$, then

$$g_1 g_2 = f^{a+c} t^{(bL^c+d)}, \quad n := \sigma(g_1)\sigma(g_2)\sigma(g_1 g_2)^{-1} = \alpha^a \beta^b \alpha^c \beta^{-bL^c} \alpha^{-a-c}.$$

It follows immediately from the definitions that

$$\overline{\left(\frac{\partial(\sigma(g_1))}{\partial\alpha}\right)} e_\alpha + \overline{\left(\frac{\partial(\sigma(g_2))}{\partial\beta}\right)} e_\beta = (1 + f + \cdots f^{a-1}) e_\alpha + f^a(1 + t + \cdots t^{b-1}) e_\beta,$$

which gives the formula for μ_1. A slightly tedious calculation shows that

$$\overline{\left(\frac{\partial n}{\partial\alpha}\right)} e_\alpha + \overline{\left(\frac{\partial n}{\partial\beta}\right)} e_\beta = f^a(1 + t + \cdots t^{b-1})(1 + \theta + \cdots \theta^{c-1})((t-1)e_\alpha + (1-\theta)e_\beta),$$

verifying the formula for μ_2. In fact, this gives an alternative proof of Proposition 7.2.4.

7.4.9. Lemma. — *For every ind-admissible $R[\overline{G}_v]$-module M there exists a homotopy $b_v : \mu\lambda \rightsquigarrow \mathrm{id}$ on $C^{\bullet}_{\mathrm{cont}}(\overline{G}_v, M)$, which is functorial in M and for which there is a 2-homotopy $b_v \star \mu \rightsquigarrow 0$, again functorial in M.*

Proof. — It is enough to consider the case when M is a discrete p-primary torsion $\overline{G}_v$-module. We have

$$\mu = \mathrm{Hom}^{\bullet,\mathrm{naive}}_{\mathbf{Z}_p[\![\overline{G}_v]\!]}(\eta^{-1}\alpha, \mathrm{id}), \quad \lambda = \mathrm{Hom}^{\bullet,\mathrm{naive}}_{\mathbf{Z}_p[\![\overline{G}_v]\!]}(\gamma, \mathrm{id}),$$

where

$$\gamma : \left[\mathbf{Z}_p[\![\overline{G}_v]\!] e'_r \xrightarrow{\partial_2} \bigoplus_{s\in S} \mathbf{Z}_p[\![\overline{G}_v]\!] e_s \xrightarrow{\partial_1} \mathbf{Z}_p[\![\overline{G}_v]\!]\right] \longrightarrow \mathbf{Z}_p[\![\overline{G}_v]\!]^{\widehat{\otimes}}_{\bullet}$$

is given by

$$\gamma_0(1) = [\,], \quad \gamma_1(e_f) = [f], \quad \gamma_1(e_t) = [t]$$

$$\gamma_2(e'_r) = -[t, f] + [f, t^L] + \sum_{i=0}^{L-2} f t^i [t, t^{L-1-i}], \quad \gamma_i = 0 \quad (i > 2).$$

The map $\gamma\circ\eta^{-1}\alpha : \mathbf{Z}_p[\![\overline{G}_v]\!]^{\widehat{\otimes}}_{\bullet} \to \mathbf{Z}_p[\![\overline{G}_v]\!]^{\widehat{\otimes}}_{\bullet}$ is a morphism of projective pseudo-compact $\mathbf{Z}_p[\![\overline{G}_v]\!]$-resolutions lifting the identity on $\mathbf{Z}_p$, which means that there is a homotopy $c_v : \gamma \circ \eta^{-1}\alpha \rightsquigarrow \mathrm{id}$. For any such c_v, the homotopy

$$b_v = \mathrm{Hom}^{\bullet,\mathrm{naive}}(c_v, \mathrm{id}) : \mu\lambda \rightsquigarrow \mathrm{id}$$

has the desired properties. As $\eta^{-1}\alpha \circ \gamma = \mathrm{id}$, both 0 and $\eta^{-1}\alpha \star c_v$ are homotopies $\eta^{-1}\alpha \rightsquigarrow \eta^{-1}\alpha$. Again, projectivity of the pro-finite bar resolution implies that there exists a 2-homotopy $H : \eta^{-1}\alpha \star c_v \rightsquigarrow 0$, which in turn induces the desired 2-homotopy $b_v \star \mu \rightsquigarrow 0$. □

7.5. Duality

7.5.1. Cup products. — For $M, N \in \left({}^{\text{ind-ad}}_{R[\overline{G}_v]}\text{Mod}\right)$ we define

$$\cup : C(M) \otimes_R C(N) \longrightarrow C(M \otimes_R N)$$

to be the composite morphism of complexes

$$C(M) \otimes_R C(N) \xrightarrow{\mu\otimes\mu} C^\bullet_{\text{cont}}(\overline{G}_v, M) \otimes_R C^\bullet_{\text{cont}}(\overline{G}_v, N) \xrightarrow{\cup} C^\bullet_{\text{cont}}(\overline{G}_v, M \otimes_R N) \xrightarrow{\lambda} C(M \otimes_R N).$$

Explicitly, the components of $\cup$

$$\cup_{ab} : C^a(M) \otimes_R C^b(N) \longrightarrow C^{a+b}(M \otimes_R N)$$

are equal to

$$\begin{aligned}
m \cup_{00} n &= m \otimes n \\
m \cup_{01} (n, n') &= (m \otimes n, m \otimes n') \\
(m, m') \cup_{10} n &= (m \otimes f(n), m \otimes t(n)) \\
m \cup_{02} n &= m \otimes n \\
m \cup_{20} n &= m \otimes ft(n) \\
(m, m') \cup_{11} (n, n') &= -m' \otimes t(n) + m \otimes \theta(n') + \sum_{0\leqslant i<j<L} ft^i(m') \otimes ft^j(n').
\end{aligned}$$

7.5.2. *Lemma*. — *For $M, N \in \left({}^{\text{ind-ad}}_{R[\overline{G}_v]}\text{Mod}\right)$, the morphism of complexes*

$$C^+(M) \otimes_R C^+(N) \longrightarrow C(M) \otimes_R C(N) \xrightarrow{\mu\otimes\mu} C^\bullet_{\text{cont}}(\overline{G}_v, M) \otimes_R C^\bullet_{\text{cont}}(\overline{G}_v, N) \xrightarrow{\cup} C^\bullet_{\text{cont}}(\overline{G}_v, M \otimes_R N) \longrightarrow \tau_{\geqslant 2}\, C^\bullet_{\text{cont}}(\overline{G}_v, M \otimes_R N)$$

is equal to zero.

Proof. — We must check that for every $m \in M^{t=1}$ and $n \in N^{t=1}$ the 2-cochain $z = \mu_1(m, 0) \cup \mu_1(n, 0) \in C^2_{\text{cont}}(\overline{G}_v, M \otimes_R N)$ is a coboundary. First of all, z is a cocycle, since both $(m, 0)$ and $(n, 0)$ are. For $a, b, c, d \in \mathbf{N}_0$ we have

$$\begin{aligned}
z(f^a t^b, f^c t^d) &= (1 + f + \cdots + f^{a-1})(m) \otimes f^a t^b (1 + f + \cdots + f^{c-1})(n) \\
&= (1 + f + \cdots + f^{a-1})(m) \otimes f^a (t^b + f t^{bL} + \cdots + f^{c-1} t^{bL^{c-1}})(n) \\
&= (1 + f + \cdots + f^{a-1})(m) \otimes f^a (1 + f + \cdots + f^{c-1})(n),
\end{aligned}$$

which implies that z is a normalized 2-cocycle ($z(f^a t^b, t^d) = 0$). As $\lambda_2(z) = -z(t, f) = 0$, the cohomology class of z vanishes. □

7.5.3. The action of G_v on μ_{p^n} factors through $G_v/I_v = \overline{G}_v/\overline{I}_v$: $t = 1$, $f = L^{-1}$ on μ_{p^n}. It follows that, for every p-primary torsion abelian group A with trivial action of $\overline{G}_v$, we have

$$C(A(1)) = \Big[A \xrightarrow{(L^{-1}-1,0)} A \oplus A \xrightarrow{0} A\Big].$$

The isomorphism inv_v from 5.1.3 is defined (up to a choice of a sign) as the composition

$$H^2(G_v, \mathbf{Z}/p^n\mathbf{Z}(1)) \xleftarrow{\sim} H^2(\overline{G}_v, \mathbf{Z}/p^n\mathbf{Z}(1)) \xrightarrow{\lambda_2} \mathbf{Z}/p^n\mathbf{Z}$$

(with $\lambda_2(z) = -z(t, f)$ for a normalized 2-cocycle z).

7.5.4. Let $M \in \big({}_{R[\overline{G}_v]}^{\mathrm{ad}}\mathrm{Mod}\big)$; then $D(M) \in \big({}_{R[\overline{G}_v]}^{\mathrm{ad}}\mathrm{Mod}\big)$, too. The complex $C(D(M)(1))$ is equal to

$$\Big[D(M) \xrightarrow{(L^{-1}D(f^{-1})-1, D(t^{-1})-1)} D(M) \oplus D(M) \xrightarrow{(D(t^{-1})-1, 1-L^{-1}\theta_{D(M)})} D(M)\Big].$$

Fix r_v as in (5.2.1.2). The truncated cup product

$$C(M) \otimes_R C(D(M)(1)) \longrightarrow \tau_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(\overline{G}_v, I(1)) \xrightarrow{r_v} I[-2]$$

induces, by adjunction, a morphism of complexes $C(M) \to D_{-2}\big(C(D(M)(1))\big)$, which gives rise, by Lemma 7.5.2, to a map of short exact sequences of complexes

(7.5.4.1)

$$\begin{array}{ccccccccc}
0 \longrightarrow & C^+(M) & \longrightarrow & C(M) & \longrightarrow & C(M)/C^+(M) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & D_{-2}\big(C(D(M)(1))/C^+(D(M)(1))\big) & \longrightarrow & D_{-2}\big(C(D(M)(1))\big) & \longrightarrow & D_{-2}\big(C^+(D(M)(1))\big) & \longrightarrow 0
\end{array}$$

7.5.5. Proposition. — *If $M \in \big({}_{R[\overline{G}_v]}^{\mathrm{ad}}\mathrm{Mod}\big)$ is of finite or co-finite type over R, then all three vertical maps in* (7.5.4.1) *are quasi-isomorphisms.*

Proof. — The formulas for $\cup_{ab}$ given in 7.5.1 together with 7.5.3 imply that the induced maps on cohomology for the first (resp., third) vertical arrow in (7.5.4.1) are given by

$$M^{t=1,f=1} \xrightarrow{\mathrm{can}} D\big(D(M)/(D(t^{-1})-1, 1-D(f^{-1}))D(M)\big)$$

$$M^{t=1}/(f-1)M^{t=1} \xrightarrow{-\mathrm{can}} D\big((D(M)/(t-1)D(M))^{D(f^{-1})=1}\big)$$

resp.,

$$(M/(t-1)M)^{Lf=1} \xrightarrow{\mathrm{can}} D\big(D(M)^{D(t^{-1})=1}/(L^{-1}D(f^{-1})-1)\big)$$

$$M/(t-1, 1-Lf)M \xrightarrow{\mathrm{can}} D\big(D(M)^{L^{-1}D(f^{-1})=1, D(t^{-1})=1}\big),$$

hence all vertical arrows are quasi-isomorphisms. □

7.5.6. Of course, Proposition 7.5.5 is nothing else than an "explicit" form of Tate's local duality in the tame case $v \nmid p$. It can be reformulated as follows: for every $M \in \left({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\right)$ of finite or co-finite type over R, the local conditions

$$\Delta_v^{\mathrm{ur}}(M) : C^+\big(M^{I_v^w}\big) \longrightarrow C\big(M^{I_v^w}\big) \xrightarrow{\mu} C^\bullet_{\mathrm{cont}}\big(\overline{G}_v, M^{I_v^w}\big) \xrightarrow{\inf} C^\bullet_{\mathrm{cont}}(G_v, M)$$

and the corresponding local conditions for $D(M)(1)$ are exact orthogonal complements:

$$\Delta_v^{\mathrm{ur}}(M) \perp\!\!\perp_{\mathrm{ev}_2,0} \Delta_v^{\mathrm{ur}}(D(M)(1)).$$

7.5.7. Implicitly, we have used the equality

$$D\big(M^{I_v^w}\big) = D(M)^{I_v^w},$$

which holds for every admissible $R[G_v]$-module M. This follows from the fact that I_v^w acts on M through a finite group of order prime to p, hence semisimply.

7.5.8. Duality for I_v. — For every ind-admissible $R[G_v]$-module M, set $N = M^{I_v^w}$. There are functorial f-equivariant isomorphisms

$$H^0(I_v, M) \xrightarrow{\sim} N^{t=1}, \quad H^1_{\mathrm{cont}}(I_v, M) \xrightarrow{\sim} (N/(t-1)N)(-1),$$

coming from the quasi-isomorphism 7.2.2

$$\Big[N \xrightarrow{t-1} N\Big] \longrightarrow C^\bullet_{\mathrm{cont}}(\overline{I}_v, N) \xrightarrow{\inf} C^\bullet_{\mathrm{cont}}(I_v, M)$$

and the fact that $ftf^{-1} = t^{1/L}$. In particular, evaluation at t gives an f-equivariant isomorphism

$$H^1_{\mathrm{cont}}(I_v, A(1)) \xrightarrow{\sim} A$$

for any R-module A with trivial action of G_v, and the cup products

$$\cup : H^i_{\mathrm{cont}}(I_v, M) \times H^{1-i}_{\mathrm{cont}}(I_v, D(M)(1)) \longrightarrow H^1_{\mathrm{cont}}(I_v, I_R(1)) \xrightarrow{\sim} I_R \quad (i = 0, 1)$$

induce isomorphisms

$$H^i_{\mathrm{cont}}(I_v, M) \xrightarrow{\sim} D\big(H^{1-i}_{\mathrm{cont}}(I_v, D(M)(1))\big) \quad (i = 0, 1) \tag{7.5.8.1}$$

if M is of finite or co-finite type over R. If

$$\begin{array}{ccc} T & \overset{\mathscr{D}}{\longleftrightarrow} & T^* \\ \Big\downarrow{\scriptstyle\Phi} & \overset{D}{\times} & \Big\downarrow{\scriptstyle\Phi} \\ A & & A^* \end{array}$$

is a duality diagram in $D_{R\text{-(co)ft}}\left({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\right)$, then (7.5.8.1) yields, by the same arguments as in the proof of Proposition 5.2.4, a duality diagram

$$\begin{array}{ccc} \mathbf{R}\Gamma_{\mathrm{cont}}(I_v, T) & \overset{\mathscr{D}}{\longleftrightarrow} & \mathbf{R}\Gamma_{\mathrm{cont}}(I_v, T^*(1))[1] \\ \Big\downarrow{\scriptstyle\Phi} & \overset{D}{\times} & \Big\downarrow{\scriptstyle\Phi} \\ \mathbf{R}\Gamma_{\mathrm{cont}}(I_v, A) & & \mathbf{R}\Gamma_{\mathrm{cont}}(I_v, A^*(1))[1] \end{array} \tag{7.5.8.2}$$

in $D_{(co)ft}({}_R\mathrm{Mod})$, with the additional property that all morphisms between cohomology groups induced by (7.5.8.2) are f-equivariant.

7.6. From modules to complexes

7.6.1. We extend the definition of $C(M)$ to complexes using the sign rules of 3.4.1.3. More precisely, let $M^\bullet$ be a complex of ind-admissible $R[\overline{G}_v]$-modules; we put

$$C(M^\bullet) = \bigoplus_{i+j=n} C(M^i)^j,$$

with differentials equal to the sum of

$$(d_M)_* : C(M^i)^j \longrightarrow C(M^{i+1})^j$$

and

$$(-1)^i \delta_{M^i} : C(M^i)^j \longrightarrow C(M^i)^{j+1}.$$

In concrete terms,

$$C(M^\bullet) = M^n \oplus M^{n-1} \oplus M^{n-1} \oplus M^{n-2}$$

with differentials

$$\begin{aligned} d(m_n, m_{n-1}, m'_{n-1}, m_{n-2}) \\ = \big(dm_n, dm_{n-1} + (-1)^n(f-1)m_n, dm'_{n-1} + (-1)^n(t-1)m_n, \\ dm_{n-2} + (-1)^n(t-1)m_{n-1} + (-1)^n(1-\theta)m'_{n-1}\big). \end{aligned}$$

It follows that $C(M^\bullet[1]) = C(M^\bullet)[1]$.

As both $C(M^\bullet)$ and $C^\bullet_{\mathrm{cont}}(\overline{G}_v, M^\bullet)$ are defined using the same sign conventions, the maps λ, μ from 7.2.3–7.2.4 define quasi-isomorphisms

$$C(M^\bullet) \xrightarrow{\mu} C^\bullet_{\mathrm{cont}}(\overline{G}_v, M^\bullet) \xrightarrow{\lambda} C(M^\bullet)$$

satisfying $\lambda \circ \mu = \mathrm{id}$. It also follows that Lemma 7.4.9 holds for bounded below complexes of ind-admissible $R[\overline{G}_v]$-modules.

Similarly, applying the sign conventions of 3.4.5.2 to the cup product 7.5.1 we obtain products

$$\cup : C(M^\bullet) \otimes_R C(N^\bullet) \longrightarrow C(M^\bullet \otimes_R N^\bullet).$$

7.6.2. In order to define an analogue of $C^+(M)$ for $M^\bullet$ we need a slightly different description of $C(M^\bullet)$. Put

$$L(M^\bullet) = \mathrm{Cone}\Big(M^\bullet \xrightarrow{t-1} M^\bullet\Big)[-1],$$

i.e.,

$$L(M^\bullet)^n = M^n \oplus M^{n-1}$$

with differentials

$$d(m_n, m_{n-1}) = (dm_n, (1-t)m_n - dm_{n-1}).$$

The map

$$(f-1,\theta-1):(m_n,m_{n-1})\mapsto((f-1)m_n,(\theta-1)m_{n-1})$$

commutes with the differential of $L(M^\bullet)$; put

$$U(M^\bullet)=\mathrm{Cone}\Big(L(M^\bullet)\xrightarrow{(f-1,\theta-1)}L(M^\bullet)\Big)[-1].$$

Explicitly,

$$U(M^\bullet)^n=L(M^\bullet)^n\oplus L(M^\bullet)^{n-1}=M^n\oplus M^{n-1}\oplus M^{n-1}\oplus M^{n-2}$$

with differentials

$$d(m_n,m_{n-1},m'_{n-1},m_{n-2})=\big(dm_n,-dm_{n-1}+(1-t)m_n,-dm'_{n-1}+(1-f)m_n,\\ dm_{n-2}+(t-1)m'_{n-1}+(1-\theta)m_{n-1}\big).$$

This implies that the map

$$(m_n,m_{n-1},m'_{n-1},m_{n-2})\longmapsto(m_n,(-1)^n m'_{n-1},(-1)^n m_{n-1},m_{n-2})$$

defines an isomorphism of complexes

$$U(M^\bullet)\xrightarrow{\sim}C(M^\bullet). \tag{7.6.2.1}$$

Put

$$U^+(M^\bullet)=\mathrm{Cone}\Big(\tau_{\leqslant 0}\,L(M^\bullet)\xrightarrow{(f-1,\theta-1)}\tau_{\leqslant 0}\,L(M^\bullet)\Big)[-1]$$
$$U^-(M^\bullet)=\mathrm{Cone}\big(\tau_{\geqslant 1}\,L(M^\bullet)\xrightarrow{(f-1,\theta-1)}\tau_{\geqslant 1}\,L(M^\bullet)\big)[-1].$$

Then $U^+(M^\bullet)$ is a subcomplex of $U(M^\bullet)$ and the canonical map $L(M^\bullet)\to\tau_{\geqslant 1}\,L(M^\bullet)$ induces a quasi-isomorphism

$$U(M^\bullet)/U^+(M^\bullet)\xrightarrow{\mathrm{Qis}}U^-(M^\bullet).$$

If $M^\bullet=M$ consists of a single module in degree 0, then $\tau_{\leqslant 0}\,L(M^\bullet)=M^{t=1}$ and

$$U^+(M^\bullet)=\Big[M^{t=1}\xrightarrow{1-f}M^{t=1}\Big]$$

(in degrees 0,1); thus (7.6.2.1) induces an isomorphism

$$U^+(M^\bullet)\xrightarrow{\sim}C^+(M),$$

hence also

$$U^-(M^\bullet)\xrightarrow{\sim}C^-(M).$$

This means that $U^\pm(M^\bullet)$ are generalizations of $C^\pm(M)$ to complexes.

We have canonical morphisms of complexes

$$\mu^+:U^+(M^\bullet)\xrightarrow{\nu}U(M^\bullet)\xrightarrow{\sim}C(M^\bullet)\xrightarrow{\mu}C^\bullet_{\mathrm{cont}}(\overline{G}_v,M^\bullet) \tag{7.6.2.2}$$

7.6.3. Note that

$$\tau_{\leqslant i} L(\tau_{\leqslant i} M^\bullet) = \tau_{\leqslant i} L(M^\bullet)$$
$$\tau_{\geqslant i} L(M^\bullet) = \tau_{\geqslant i} L(\tau_{\geqslant i-1} M^\bullet),$$

hence

$$U^+(\tau_{\leqslant 0} M^\bullet) = U^+(M^\bullet), \quad U^-(M^\bullet) = U^-(\tau_{\geqslant 0} M^\bullet).$$

Similarly,

$$U^+(\sigma_{\geqslant 1} M^\bullet) = 0, \qquad U^-(\sigma_{\geqslant 1} M^\bullet) = U(\sigma_{\geqslant 1} M^\bullet)$$
$$U^-(\sigma_{\leqslant -1} M^\bullet) = 0, \quad U^+(\sigma_{\leqslant -1} M^\bullet) = U(\sigma_{\leqslant -1} M^\bullet).$$

The quotient

$$\tau_{\leqslant i} L(M^\bullet)/L(\tau_{\leqslant i-1} M^\bullet)$$

is canonically quasi-isomorphic to $H^i(M^\bullet)^{t=1}$; this implies that there are exact triangles

$$L(\tau_{\leqslant -1} M^\bullet) \longrightarrow \tau_{\leqslant 0} L(M^\bullet) \longrightarrow H^0(M^\bullet)^{t=1} \longrightarrow L(\tau_{\leqslant -1} M^\bullet)[1]$$

and

$$U(\tau_{\leqslant -1} M^\bullet) \longrightarrow U^+(M^\bullet) \longrightarrow U^+(H^0(M^\bullet)) \longrightarrow U(\tau_{\leqslant -1} M^\bullet)[1]. \tag{7.6.3.1}$$

7.6.4. Lemma. — *If $M^\bullet, N^\bullet$ are complexes of ind-admissible $R[\overline{G}_v]$-modules, then the composite morphism of complexes*

$$U^+(M^\bullet) \otimes_R U^+(N^\bullet) \xrightarrow{\mu^+\otimes\mu^+} C^\bullet_{\mathrm{cont}}(\overline{G}_v, M^\bullet) \otimes_R C^\bullet_{\mathrm{cont}}(\overline{G}_v, N^\bullet)$$
$$\xrightarrow{\cup} C^\bullet_{\mathrm{cont}}(\overline{G}_v, M^\bullet \otimes_R N^\bullet) \longrightarrow \tau_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(\overline{G}_v, M^\bullet \otimes_R N^\bullet)$$

is equal to zero.

Proof. — As $U^+(M^\bullet) = \tau_{\leqslant 1} U^+(\tau_{\leqslant 0} M^\bullet)$, all we have to do is to check that the cup product

$$H^1(U^+(M^\bullet)) \otimes_R H^1(U^+(N^\bullet)) \xrightarrow{\cup} H^2_{\mathrm{cont}}(\overline{G}_v, M^\bullet \otimes_R N^\bullet) \tag{7.6.4.1}$$

vanishes, under the assumptions $M^\bullet = \tau_{\leqslant 0} M^\bullet$, $N^\bullet = \tau_{\leqslant 0} N^\bullet$. The triangle (7.6.3.1) gives exact sequences

$$H^1(U(\tau_{\leqslant -1} X^\bullet)) \longrightarrow H^1(U^+(X^\bullet)) \longrightarrow H^1\big(U^+(H^0(X^\bullet))\big) \longrightarrow 0$$

for $X = M, N$. As $\mathrm{cd}_p(\overline{G}_v) = 2$, the map (7.6.4.1) factors through

$$H^1\big(U^+(H^0(M^\bullet))\big) \otimes_R H^1\big(U^+(H^0(N^\bullet))\big) \xrightarrow{\cup} H^2_{\mathrm{cont}}(\overline{G}_v, H^0(M^\bullet) \otimes_R H^0(N^\bullet)),$$

which is a zero map, by Lemma 7.5.2. □

7.6.5. Fix J as in 5.2.2 satisfying $J = \sigma_{\geqslant 0} J$; then

$$\tau^{\mathrm{II}}_{\geqslant 2} C^{\bullet}_{\mathrm{cont}}(G_v, J(1)) = \tau_{\geqslant 2} C^{\bullet}_{\mathrm{cont}}(G_v, J(1)).$$

Let $M^\bullet$ be a complex of admissible $R[G_v]$-modules. We define unramified local conditions for $M^\bullet$ to be

$$\Delta^{\mathrm{ur}}_v(M^\bullet) : U^+((M^\bullet)^{I^w_v}) \xrightarrow{\nu} U((M^\bullet)^{I^w_v}) \xrightarrow{\sim} C((M^\bullet)^{I^w_v}) \xrightarrow{\mu} C^{\bullet}_{\mathrm{cont}}(\overline{G}_v, (M^\bullet)^{I^w_v}) \xrightarrow{\mathrm{inf}} C^{\bullet}_{\mathrm{cont}}(G_v, M^\bullet).$$

We use the notation

$$C^{\bullet}_{\mathrm{ur}}(G_v, M^\bullet) = U^+((M^\bullet)^{I^w_v}).$$

It follows from Lemma 7.6.4 and 7.5.7 that

$$\Delta^{\mathrm{ur}}_v(M^\bullet) \perp_{\mathrm{ev}_2, 0} \Delta^{\mathrm{ur}}_v(D_J(M^\bullet)(1))$$
$$\Delta^{\mathrm{ur}}_v(D_J(M^\bullet)(1)) \perp_{\mathrm{ev}_1, 0} \Delta^{\mathrm{ur}}_v(M^\bullet).$$

We shall simplify the notation and write

$$\mathrm{Err}^{\mathrm{ur}}_v(D_J, M^\bullet) := \mathrm{Err}_v(\Delta^{\mathrm{ur}}_v(M^\bullet), \Delta^{\mathrm{ur}}_v(D_J(M^\bullet)(1)), \mathrm{ev}_2). \tag{7.6.5.1}$$

7.6.6. Proposition. — *Let $M^\bullet$ be a bounded complex of admissible $R[G_v]$-modules with cohomology of finite (resp., co-finite) type over R. Then*

$$\Delta^{\mathrm{ur}}_v(M^\bullet) \perp\!\!\perp_{\mathrm{ev}_2, 0} \Delta^{\mathrm{ur}}_v(D(M^\bullet)(1))$$
$$\Delta^{\mathrm{ur}}_v(D(M^\bullet)(1)) \perp\!\!\perp_{\mathrm{ev}_1, 0} \Delta^{\mathrm{ur}}_v(M^\bullet).$$

Proof. — By 7.5.7 we can assume that $M^\bullet = (M^\bullet)^{I^w_v}$. Put $N^\bullet = D(M^\bullet)(1)$. By Lemma 7.6.4, the composite morphism of complexes

$$U^+(M^\bullet) \longrightarrow U(M^\bullet) \longrightarrow D_{-2}(U(N^\bullet)) \longrightarrow D_{-2}(U^+(N^\bullet))$$

(the second arrow is defined by adjunction from the truncated cup product) is equal to zero. We must show that the induced map

$$f_{M^\bullet} : U(M^\bullet)/U^+(M^\bullet) \longrightarrow D_{-2}(U^+(N^\bullet))$$

is a quasi-isomorphism. By dévissage it is enough to treat the case when $M^\bullet = M^i[-i]$ is concentrated in degree i. If $i < 0$ then both sides vanish. If $i > 0$ then $U^+(M^\bullet) = 0$, $U^+(N^\bullet) = U(N^\bullet)$ and $f_{M^\bullet}$ is nothing but the duality isomorphism from Proposition 5.2.4 (i), composed with μ. If $i = 0$ then

$$f_{M^\bullet} : C(M^0)/C^+(M^0) \longrightarrow D_{-2}\big(C^+(D(M^0)(1))\big)$$

is an isomorphism by Proposition 7.5.5. □

7.6.7. Proposition. — *Let $T = T^\bullet$ be a bounded complex of admissible $R[G_v]$-modules with cohomology of finite type over R. Then:*

(i) *There is an exact triangle in $D^b_{coft}((_R\mathrm{Mod})/(co\text{-}pseudo\text{-}null))$*

$$\Phi(C^\bullet_{\mathrm{ur}}(G_v, T)) \longrightarrow C^\bullet_{\mathrm{ur}}(G_v, \Phi(T)) \longrightarrow \mathrm{Cone}\Big(H^{d-1}_{\{\mathfrak{m}\}}(H^1_{\mathrm{cont}}(I_v, T)) \xrightarrow{f_v - 1} H^{d-1}_{\{\mathfrak{m}\}}(H^1_{\mathrm{cont}}(I_v, T))\Big)[-1],$$

where $f_v = \mathrm{Fr}(v)$.

(ii) *There is an isomorphism in $D^b_{ft}((_R\mathrm{Mod})/(pseudo\text{-}null))$*

$$\mathscr{D}(\mathrm{Err}^{\mathrm{ur}}_v(\mathscr{D}, T)) \xrightarrow{\sim} \mathrm{Cone}\Big(\mathrm{Ext}^1_R(H^1_{\mathrm{cont}}(I_v, T), \omega) \xrightarrow{f_v^{-1} - 1} \mathrm{Ext}^1_R(H^1_{\mathrm{cont}}(I_v, T), \omega)\Big).$$

(iii) *For each $\mathfrak{q} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{q}) = 0$,*

$$\mathrm{Err}^{\mathrm{ur}}_v(\mathscr{D}, T)_{\mathfrak{q}} \xrightarrow{\sim} 0 \textit{ in } D^b_{ft}(_{R_{\mathfrak{q}}}\mathrm{Mod}).$$

(iv) *Assume that $\mathfrak{p} \in \mathrm{Spec}(R)$, $R_{\mathfrak{p}}$ is a discrete valuation ring, each term $T^i_{\mathfrak{p}}$ is free of finite type over $R_{\mathfrak{p}}$, and $H^1_{\mathrm{cont}}(I_v, T)_{\mathfrak{p}}[\mathfrak{p}]^{f_v=1} = 0$ (equivalently, $\mathrm{Tam}_v(T, \mathfrak{p}) = 0$, in the language of 7.6.10.1 below). Then, for each uniformizer $\varpi \in R_{\mathfrak{p}}$, the canonical (injective) morphism of complexes*

$$C^\bullet_{\mathrm{ur}}(G_v, T)_{\mathfrak{p}}/\varpi C^\bullet_{\mathrm{ur}}(G_v, T)_{\mathfrak{p}} \longrightarrow C^\bullet_{\mathrm{ur}}(G_v, T_{\mathfrak{p}}/\varpi T_{\mathfrak{p}})$$

is a quasi-isomorphism.

Proof. — By 7.5.7 we can assume that $T = T^{I_v^w}$.

(i) Put $L^\bullet = L(T)$, $C^\bullet = C^\bullet(R, (x_i))[d]$ for a fixed system of parameters of R. The spectral sequence

$$E_2^{i,j} = H^{i+d}_{\{\mathfrak{m}\}}(H^j(L^\bullet)) \Longrightarrow H^{i+j}(L^\bullet \otimes_R C^\bullet)$$

degenerates in $\mathcal{C} = (_R\mathrm{Mod})/(\text{co-pseudo-null})$ into exact sequences

$$0 \longrightarrow H^d_{\{\mathfrak{m}\}}(H^j(L^\bullet)) \longrightarrow H^j(L^\bullet \otimes_R C^\bullet) \longrightarrow H^{d-1}_{\{\mathfrak{m}\}}(H^{j+1}(L^\bullet)) \longrightarrow 0.$$

It follows that we have isomorphisms in $\mathcal{C}$

$$H^j((\tau_{\leqslant 0} L^\bullet) \otimes_R C^\bullet) \xrightarrow{\sim} \begin{cases} H^j(L^\bullet \otimes_R C^\bullet) & j < 0 \\ H^d_{\{\mathfrak{m}\}}(H^0(L^\bullet)) & j = 0 \\ 0 & j > 0, \end{cases}$$

hence an exact triangle in $D^b(\mathcal{C})$

$$(\tau_{\leqslant 0} L^\bullet) \otimes_R C^\bullet \longrightarrow \tau_{\leqslant 0}(L^\bullet \otimes_R C^\bullet) \longrightarrow H^{d-1}_{\{\mathfrak{m}\}}(H^1(L^\bullet)) \longrightarrow ((\tau_{\leqslant 0} L^\bullet) \otimes_R C^\bullet)[1],$$

which is equal to

$$\Phi(\tau_{\leqslant 0} L^\bullet) \longrightarrow \tau_{\leqslant 0} \Phi(L^\bullet) \longrightarrow H^{d-1}_{\{\mathfrak{m}\}}(H^1(L^\bullet)) \longrightarrow \Phi(\tau_{\leqslant 0} L^\bullet)[1].$$

The statement follows from the commutative diagram

$$\begin{array}{ccccccc} \Phi(\tau_{\leqslant 0} L^\bullet) & \longrightarrow & \tau_{\leqslant 0}\Phi(L^\bullet) & \longrightarrow & H^{d-1}_{\{\mathfrak m\}}(H^1(L^\bullet)) & \longrightarrow & \Phi(\tau_{\leqslant 0} L^\bullet)[1] \\ \big\downarrow {\scriptstyle \binom{f-1}{\theta-1}} & & \big\downarrow {\scriptstyle \binom{f-1}{\theta-1}} & & \big\downarrow {\scriptstyle f-1} & & \big\downarrow {\scriptstyle \binom{f-1}{\theta-1}} \\ \Phi(\tau_{\leqslant 0} L^\bullet) & \longrightarrow & \tau_{\leqslant 0}\Phi(L^\bullet) & \longrightarrow & H^{d-1}_{\{\mathfrak m\}}(H^1(L^\bullet)) & \longrightarrow & \Phi(\tau_{\leqslant 0} L^\bullet)[1] \end{array}$$

(ii) Let $X = \mathscr{D}(T)(1)$. Applying D_{-2} to the exact triangle

$$\Phi\big(U_v^+(\mathscr{D}(X)(1))\big) \xrightarrow{u} U_v^+(\Phi\circ\mathscr{D}(X)(1)) \longrightarrow \mathrm{Err}_v^{\mathrm{ur}}(\Phi,\mathscr{D}(X)(1))$$

(in which $U_v^+ = C_v^{\mathrm{ur}}$ and the third term is defined to be $\mathrm{Cone}(u)$) and using isomorphisms $\Phi\circ\mathscr{D} \xrightarrow{\sim} D$, $\mathscr{D} \xrightarrow{\sim} D\circ\Phi$, we obtain another triangle

$$D_{-2}\big(\mathrm{Err}_v^{\mathrm{ur}}(\Phi,\mathscr{D}(X)(1))\big) \longrightarrow D_{-2}\big(U_v^+(D(X)(1))\big) \longrightarrow \mathscr{D}_{-2}\big(U_v^+(\mathscr{D}(X)(1))\big),$$

the second term of which is isomorphic to $U_v^-(X)$ (by Proposition 7.6.6). Applying, in turn, $\mathscr{D}_{-2}$, we obtain an exact triangle

$$U_v^+(\mathscr{D}(X)(1)) \longrightarrow \mathscr{D}_{-2}(U_v^-(X)) \longrightarrow \mathscr{D}\circ D\big(\mathrm{Err}_v^{\mathrm{ur}}(\Phi,\mathscr{D}(X)(1))\big),$$

which is nothing but

$$U_v^+(T) \longrightarrow \mathscr{D}_{-2}\big(U_v^-(\mathscr{D}(T)(1))\big) \longrightarrow \mathscr{D}\circ D(\mathrm{Err}_v^{\mathrm{ur}}(\Phi,T)).$$

This implies that

$$\mathrm{Err}_v^{\mathrm{ur}}(\mathscr{D},T) \xrightarrow{\sim} \mathscr{D}\circ D(\mathrm{Err}_v^{\mathrm{ur}}(\Phi,T)), \quad \mathscr{D}(\mathrm{Err}_v^{\mathrm{ur}}(\mathscr{D},T)) \xrightarrow{\sim} D(\mathrm{Err}_v^{\mathrm{ur}}(\Phi,T)).$$

Applying the formula for $\mathrm{Err}_v^{\mathrm{ur}}(\Phi,T)$ given in (i) and local duality

$$D\big(H^{d-1}_{\{\mathfrak m\}}(M)\big) \xrightarrow{\sim} \mathbb{E}\mathrm{xt}^1_R(M,\omega)$$

finishes the proof.

(iii) This follows from (ii), as $\mathrm{codim}_R\big(\mathrm{supp}(\mathbb{E}\mathrm{xt}^1_R(M,\omega))\big) \geqslant 1$ for any R-module of finite type M.

(iv) Replacing in the proof of (i) $L^\bullet$ by $L(T)_{\mathfrak p}$ and $C^\bullet$ by $\big[R_{\mathfrak p} \xrightarrow{\varpi} R_{\mathfrak p}\big]$ (in degrees -1, 0), we obtain a commutative diagram

$$\begin{array}{ccccccc} (\tau_{\leqslant 0} L^\bullet)\otimes_{R_{\mathfrak p}} C^\bullet & \longrightarrow & \tau_{\leqslant 0}(L^\bullet\otimes_{R_{\mathfrak p}} C^\bullet) & \longrightarrow & H^1(L^\bullet)[\mathfrak p] & \longrightarrow & (\tau_{\leqslant 0} L^\bullet)\otimes_{R_{\mathfrak p}} C^\bullet[1] \\ \big\downarrow {\scriptstyle \binom{f-1}{\theta-1}} & & \big\downarrow {\scriptstyle \binom{f-1}{\theta-1}} & & \big\downarrow {\scriptstyle f-1} & & \big\downarrow {\scriptstyle \binom{f-1}{\theta-1}} \\ (\tau_{\leqslant 0} L^\bullet)\otimes_{R_{\mathfrak p}} C^\bullet & \longrightarrow & \tau_{\leqslant 0}(L^\bullet\otimes_{R_{\mathfrak p}} C^\bullet) & \longrightarrow & H^1(L^\bullet)[\mathfrak p] & \longrightarrow & (\tau_{\leqslant 0} L^\bullet)\otimes_{R_{\mathfrak p}} C^\bullet[1] \end{array}$$

in which $f = f_v$ and each row is an exact triangle. As $f-1$ acts bijectively on $H^1(L^\bullet)[\mathfrak p] = H^1_{\mathrm{cont}}(I_v,T)_{\mathfrak p}[\mathfrak p]$ by assumption, the diagram implies that the canonical morphism of complexes

$$C^\bullet_{\mathrm{ur}}(G_v,T)_{\mathfrak p}\otimes_{R_{\mathfrak p}} C^\bullet \longrightarrow C^\bullet_{\mathrm{ur}}(G_v,T_{\mathfrak p}/\varpi T_{\mathfrak p})$$

is a quasi-isomorphism. □

7.6.8. Corollary

(i) *If* $\mathfrak{p} \in \mathrm{Spec}(R)$ *is a prime ideal with* $\dim(R_{\mathfrak{p}}) = \mathrm{depth}(R_{\mathfrak{p}}) = 1$, *then there is an isomorphism in* $D_{ft}({}_{R_{\mathfrak{p}}}\mathrm{Mod})$

$$\mathrm{Err}_v^{\mathrm{ur}}(\mathscr{D}, T)_{\mathfrak{p}} \xrightarrow{\sim} \mathrm{Cone}\Big(H^0_{\{\mathfrak{p}\}}(H^1_{\mathrm{cont}}(I_v, T)_{\mathfrak{p}}) \xrightarrow{f_v - 1} H^0_{\{\mathfrak{p}\}}(H^1_{\mathrm{cont}}(I_v, T)_{\mathfrak{p}})\Big)[-2].$$

(ii) *If every* $\mathfrak{p} \in \mathrm{Spec}(R)$ *with* $\mathrm{ht}(\mathfrak{p}) = 1$ *satisfies* $\mathrm{depth}(R_{\mathfrak{p}}) = 1$ *(*e.g. *if* R *has no embedded primes), then*

$$H^i(\mathrm{Err}_v^{\mathrm{ur}}(\mathscr{D}, T)) \xrightarrow{\sim} 0 \qquad (\forall i \neq 1, 2)$$

in $({}_R\mathrm{Mod})/(\textit{pseudo-null})$.

Proof. — According to Lemma 2.10.11 (ii) there is an isomorphism

$$\mathscr{D}_{R_{\mathfrak{p}}}(W) \xrightarrow{\sim} D_{R_{\mathfrak{p}}}(W)[-1]$$

for every $R_{\mathfrak{p}}$-module W of finite length. Taking $W = \mathrm{Ext}^1_R(H^1_{\mathrm{cont}}(I_v, T), \omega_R)_{\mathfrak{p}}$, we obtain

$$\mathscr{D}_{R_{\mathfrak{p}}}(W) \xrightarrow{\sim} H^0_{\{\mathfrak{p}\}}(H^1_{\mathrm{cont}}(I_v, T)_{\mathfrak{p}})[-1]$$

by local duality. The result then follows from Proposition 7.6.7 (ii). Finally, (ii) is an immediate consequence of (i). □

7.6.9. For example, in the special case when $R = \mathbf{Z}_p$ and T is a single module concentrated in degree zero, free over $\mathbf{Z}_p$, then we have $\Phi(T) = V/T$, where $V = T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, and canonical quasi-isomorphisms

$$\Phi(C^\bullet_{\mathrm{ur}}(G_v, T)) \xrightarrow{\mathrm{Qis}} \mathrm{Cone}\Big(V^{I_v}/T^{I_v} \xrightarrow{f_v - 1} V^{I_v}/T^{I_v}\Big)[-1]$$
$$C^\bullet_{\mathrm{ur}}(G_v, \Phi(T)) \xrightarrow{\mathrm{Qis}} \mathrm{Cone}\Big((V/T)^{I_v} \xrightarrow{f_v - 1} (V/T)^{I_v}\Big)[-1],$$

which yield an exact triangle in $D^b_{ft}({}_{\mathbf{Z}_p}\mathrm{Mod})$

$$\Phi(C^\bullet_{\mathrm{ur}}(G_v, T)) \longrightarrow C^\bullet_{\mathrm{ur}}(G_v, \Phi(T)) \longrightarrow \mathrm{Err}_v^{\mathrm{ur}}(\Phi, T),$$

with $\mathrm{Err}_v^{\mathrm{ur}}(\Phi, T)$ quasi-isomorphic to

$$\mathrm{Err}_v^{\mathrm{ur}}(\Phi, T) \xrightarrow{\mathrm{Qis}} \mathrm{Cone}\Big(Z \xrightarrow{f_v - 1} Z\Big)[-1],$$

where

$$Z = \mathrm{Coker}\big(V^{I_v}/T^{I_v} \longrightarrow (V/T)^{I_v}\big) = H^0(I_v, V/T)/\mathrm{div} \xrightarrow{\sim} H^1(I_v, T)_{\mathrm{tors}}.$$

The $\mathbf{Z}_p$-module Z is finite and

$$\begin{aligned} &H^1(\mathrm{Err}_v^{\mathrm{ur}}(\Phi, T)) \\ &\quad = \mathrm{Coker}\Big(Z \xrightarrow{f_v - 1} Z\Big) = \mathrm{Coker}\big(H^1_{\mathrm{ur}}(G_v, V) \longrightarrow H^1_{\mathrm{ur}}(G_v, V/T)\big) = H^1_{\mathrm{ur}}(G_v, V/T)/\mathrm{div}; \end{aligned}$$

this implies that the cohomology groups $H^i(\mathrm{Err}_v^{\mathrm{ur}}(\Phi, T))$ $(i = 0, 1)$ have the same order, equal to the common order of the groups

$$H^1(I_v, T)_{\mathrm{tors}}^{f_v=1}, \quad H^1_{\mathrm{ur}}(G_v, V/T)/\mathrm{div},$$

which is nothing but the *local Tamagawa factor* appearing in one of the formulations of the Bloch-Kato conjecture ([**Fo-PR**, §I.4.2.2]).

7.6.10. Local Tamagawa factors. — Let R be arbitrary and $T = T^\bullet$ as in Proposition 7.6.7.

7.6.10.1. Definition. — For $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{p}) = 1$, put

$$\mathrm{Tam}_v(T, \mathfrak{p}) = \ell_{R_\mathfrak{p}}(H^0_{\{\mathfrak{p}\}}(M_\mathfrak{p})),$$

where

$$M = H^1_{\mathrm{cont}}(I_v, T)^{f_v=1}.$$

7.6.10.2. If $\mathrm{depth}(R_\mathfrak{p}) = 1$, then

$$H^0_{\{\mathfrak{p}\}}(M_\mathfrak{p}) = (M_\mathfrak{p})_{R_\mathfrak{p}\text{-tors}},$$

by Lemma 2.10.5 (iii).

7.6.10.3. If R has no embedded primes, then

$$(M_{R\text{-tors}})_\mathfrak{p} \xrightarrow{\sim} (M_\mathfrak{p})_{R_\mathfrak{p}\text{-tors}},$$

by Corollary 2.10.13.4.

7.6.10.4. If I_v acts trivially on T^0 and T^1, then

$$H^1_{\mathrm{cont}}(I_v, T) \xrightarrow{\sim} H^0(T)(-1) \oplus H^1(T)(-1),$$

hence

$$\mathrm{Tam}_v(T, \mathfrak{p}) = \ell_{R_\mathfrak{p}}\Big(H^0_{\{\mathfrak{p}\}}\Big(H^0(T)_\mathfrak{p}^{f_v=1}\Big)\Big) + \ell_{R_\mathfrak{p}}\Big(H^0_{\{\mathfrak{p}\}}\Big(H^1(T)_\mathfrak{p}^{f_v=1}\Big)\Big).$$

In particular, if $H^0_{\{\mathfrak{p}\}}(H^i(T)_\mathfrak{p}) = 0$ for $i = 0, 1$, then $\mathrm{Tam}_v(T, \mathfrak{p}) = 0$.

7.6.10.5. If $R = \mathbf{Z}_p$, $\mathfrak{p} = (p)$ and $T = T^0$ is torsion-free over $\mathbf{Z}_p$, then (cf. 7.6.9) $\mathrm{Tam}_v(T, (p))$ is equal to the p-adic valuation of the order of

$$H^1_{\mathrm{cont}}(I_v, T)^{f_v=1}_{\mathbf{Z}_p\text{-tors}},$$

i.e., of the local Tamagawa factor from [**Fo-PR**, §I.4.2.2].

7.6.10.6. Proposition 7.6.7 (more precisely, its proof) implies that

$$\ell_{R_\mathfrak{p}}\big(H^i\big(\mathscr{D}(\mathrm{Err}_v^{\mathrm{ur}}(\mathscr{D}, T))\big)_\mathfrak{p}\big) = \begin{cases} \mathrm{Tam}_v(T, \mathfrak{p}), & i = -1, 0 \\ 0, & i \neq -1, 0. \end{cases}$$

7.6.10.7. If $\operatorname{depth}(R_{\mathfrak p}) = 1$, then Corollary 7.6.8 implies that

$$\ell_{R_{\mathfrak p}}\big(H^i(\mathrm{Err}_v^{\mathrm{ur}}(\mathscr{D}, T))_{\mathfrak p}\big) = \begin{cases} \mathrm{Tam}_v(T, \mathfrak p), & i = 1, 2 \\ 0, & i \neq 1, 2. \end{cases}$$

7.6.10.8. It follows from 7.6.10.6 that

$$\mathrm{Err}_v^{\mathrm{ur}}(\mathscr{D}, T)_{\mathfrak p} \xrightarrow{\sim} 0 \text{ in } D({}_{R_{\mathfrak p}}\mathrm{Mod}) \iff \mathrm{Tam}_v(T, \mathfrak p) = 0.$$

7.6.10.9. If R has no embedded primes, then a combination of 7.6.10.2, 7.6.10.3, 7.6.10.8 shows that

$$\mathrm{Err}_v^{\mathrm{ur}}(\mathscr{D}, T) \xrightarrow{\sim} 0 \text{ in } D(({}_R\mathrm{Mod})/(\text{pseudo-null})) \iff H^1_{\mathrm{cont}}(I_v, T)^{f_v=1}_{R\text{-tors}} \text{ is pseudo-null.}$$

7.6.10.10. The local Tamagawa factor $\mathrm{Tam}_v(T, \mathfrak p)$ is non-zero if and only if $\mathfrak p \in \operatorname{supp}(\mathbb{E}\mathrm{xt}^1_R(M, \omega_R))$, by local duality for $R_{\mathfrak p}$. This implies that, for fixed v and T, there are only finitely many prime ideals $\mathfrak p \in \operatorname{Spec}(R)$, $\operatorname{ht}(\mathfrak p) = 1$, for which $\mathrm{Tam}_v(T, \mathfrak p) \neq 0$.

7.6.10.11. It is proved in 10.2.8 below that

$$\mathrm{Tam}_v(T, \mathfrak p) = \mathrm{Tam}_v(T^*(1), \mathfrak p).$$

7.6.10.12. If $R = \mathcal{O}$ is the ring of integers in a finite extension F of $\mathbf{Q}_p$ and $T = T^0$ is torsion-free over $\mathcal{O}$, let $\varpi \in \mathcal{O}$ be a uniformizer and set $V = T \otimes_{\mathcal{O}} F$ and $A = V/T$. If $V^{I_v} = 0$, then the group $H^1_{\mathrm{cont}}(I_v, T) \xrightarrow{\sim} A^{I_v}$ is finite, hence $\mathrm{Tam}_v(T, (\varpi)) = \ell_{\mathcal{O}}\left(A^{G_v}\right)$. In particular, if $A^{I_v} = 0$, then $\mathrm{Tam}_v(T, (\varpi)) = 0$.

7.6.11. Proposition. — *In the situation of Proposition 7.6.7, assume that $T = \sigma_{\leqslant 0} T$. Then:*

(i) $H^1_{\mathrm{cont}}(I_v, T) = H^1_{\mathrm{cont}}(I_v, H^0(T)) \xrightarrow{\sim} H^0(T)_{I_v}(-1)$.

(ii) *The error term* $\mathrm{Err}_v^{\mathrm{ur}}(\Phi, T)$ *entering into the exact triangle in* $D^b_{coft}({}_R\mathrm{Mod})$

$$\Phi(C^\bullet_{\mathrm{ur}}(G_v, T)) \longrightarrow C^\bullet_{\mathrm{ur}}(G_v, \Phi(T)) \longrightarrow \mathrm{Err}_v^{\mathrm{ur}}(\Phi, T)$$

is isomorphic (in $D^b_{coft}({}_R\mathrm{Mod})$*) to*

$$\mathrm{Err}_v^{\mathrm{ur}}(\Phi, T) \xrightarrow{\sim} \mathrm{Cone}\Big(\tau_{\leqslant -1}\,\Phi(H^1_{\mathrm{cont}}(I_v, T)) \xrightarrow{f_v - 1} \tau_{\leqslant -1}\,\Phi(H^1_{\mathrm{cont}}(I_v, T))\Big)[-2].$$

(iii) *If* $H^1_{\mathrm{cont}}(I_v, T)$ *is zero or a Cohen-Macaulay R-module of dimension $d = \dim(R)$, then* $\mathrm{Err}_v^{\mathrm{ur}}(\Phi, T) \xrightarrow{\sim} 0$ *in* $D^b_{coft}({}_R\mathrm{Mod})$.

(iv) *The cohomology groups of* $\mathrm{Err}_v^{\mathrm{ur}}(\Phi, T)$ *sit in the following exact sequences:*

$$0 \longrightarrow \begin{cases} H^{j+d-2}_{\{\mathfrak m\}}(H)/(f_v - 1), & \text{if } j \leqslant 1 \\ 0, & \text{if } j > 1 \end{cases}$$

$$\longrightarrow H^j(\mathrm{Err}_v^{\mathrm{ur}}(\Phi, T)) \longrightarrow \begin{cases} H^{j+d-1}_{\{\mathfrak m\}}(H)^{f_v=1}, & \text{if } j \leqslant 0 \\ 0, & \text{if } j > 0 \end{cases} \longrightarrow 0,$$

where we have abbreviated $H := H^1_{\mathrm{cont}}(I_v, T)$.

(v) *The following statements are equivalent:*

$$\mathrm{Err}_v^{\mathrm{ur}}(\Phi, T) \xrightarrow{\sim} 0 \text{ in } D^b_{coft}({}_R\mathrm{Mod}) \iff$$
$$(\forall q = 0, \dots, d-1) \quad \alpha_q : H^q_{\{\mathfrak{m}\}}(H^1_{\mathrm{cont}}(I_v, T)) \xrightarrow{f_v - 1} H^q_{\{\mathfrak{m}\}}(H^1_{\mathrm{cont}}(I_v, T)) \text{ is an isomorphism}$$
$$\iff (\forall q = 0, \dots, d-1) \quad \mathrm{Ker}(\alpha_q) = 0.$$

Proof. — As before, we can assume that $T = T^{I_v^{w}}$. The statement (i) follows from (3.5.4.2) and Lemma 4.2.6, as $\mathrm{cd}_p(I_v) = 1$. As regards (ii), let $C^\bullet$ be as in the proof of Proposition 7.6.7. The assumption $T = \sigma_{\leqslant 0}\, T$ implies that $L(T) = \sigma_{\leqslant 1}\, L(T)$ and $\Phi(T) = T \otimes_R C^\bullet = \sigma_{\leqslant 0}\, \Phi(T)$. Tensoring the exact triangle in $D^b_{ft}({}_R\mathrm{Mod})$

$$\tau_{\leqslant 0}\, L(T) \longrightarrow L(T) \longrightarrow H[-1] \longrightarrow (\tau_{\leqslant 0}\, L(T))\,[1]$$

(in which $H = H^1(L(T)) = H^1_{\mathrm{cont}}(I_v, T))$ with the complex $C^\bullet$ (of flat R-modules) yields an exact triangle in $D^b_{coft}({}_R\mathrm{Mod})$

$$(\tau_{\leqslant 0}\, L(T)) \otimes_R C^\bullet \longrightarrow L(T) \otimes_R C^\bullet \longrightarrow H \otimes_R C^\bullet[-1] \longrightarrow (\tau_{\leqslant 0}\, L(T)) \otimes_R C^\bullet[1].$$

As $C^\bullet = \sigma_{\leqslant 0}\, C^\bullet$, applying the truncation $\tau_{\leqslant 0}$ we obtain another exact triangle in $D^b_{coft}({}_R\mathrm{Mod})$

$$(\tau_{\leqslant 0}\, L(T)) \otimes_R C^\bullet \longrightarrow \tau_{\leqslant 0}\, (L(T) \otimes_R C^\bullet)$$
$$\longrightarrow (\tau_{\leqslant -1}\, (H \otimes_R C^\bullet))\,[-1] \longrightarrow (\tau_{\leqslant 0}\, L(T)) \otimes_R C^\bullet[1],$$

which is equal to

$$\Phi\,(\tau_{\leqslant 0}\, L(T)) \longrightarrow \tau_{\leqslant 0}\, (L(\Phi(T))) \longrightarrow (\tau_{\leqslant -1}\, \Phi(H))\,[-1] \longrightarrow \Phi\,(\tau_{\leqslant 0}\, L(T))\,[1];$$

we conclude as in the proof of Proposition 7.6.7 (i).

(iii) If $H = H^1_{\mathrm{cont}}(I_v, T)$ is zero or a Cohen-Macaulay R-module of dimension $d = \dim(R)$, then the complex

$$\tau_{\leqslant -1}\, \Phi(H) \xrightarrow{\sim} \big(\tau_{\leqslant d-1}\, \mathbf{R}\Gamma_{\{\mathfrak{m}\}}(H)\big)\,[d]$$

is acyclic, by Lemma 2.4.7 (ii).

The statement (iv) is a consequence of (ii), and the first equivalence in (v) follows from (iv). The non-trivial implication '$\Longleftarrow$' in (v) follows from Lemma 2.3.6. □

7.6.12. Corollary. — *Under the assumptions of Proposition 7.6.11,*

(i) *There is an isomorphism in* $D^b_{ft}({}_R\mathrm{Mod})$ *(using the notation of* (7.6.5.1)*)*

$$\mathscr{D}(\mathrm{Err}_v^{\mathrm{ur}}(\mathscr{D}, T)) \xrightarrow{\sim} \mathrm{Cone}\Big(\tau_{\geqslant 1}\mathscr{D}(H^1_{\mathrm{cont}}(I_v, T)) \xrightarrow{f_v^{-1} - 1} \tau_{\geqslant 1}\mathscr{D}(H^1_{\mathrm{cont}}(I_v, T))\Big)[1].$$

(ii) *The cohomology groups of* $W := \mathscr{D}(\mathrm{Err}_v^{\mathrm{ur}}(\mathscr{D}, T))$ *sit in the following exact sequences:*

$$0 \longrightarrow \begin{cases} \mathbb{E}\mathrm{xt}_R^{j+1}(H, \omega_R)/(f_v - 1), & \text{if } j \geqslant 0 \\ 0, & \text{if } j < 0 \end{cases}$$

$$\longrightarrow H^j(W) \longrightarrow \begin{cases} \mathbb{E}\mathrm{xt}_R^{j+2}(H, \omega_R)^{f_v = 1}, & \text{if } j \geqslant -1 \\ 0, & \text{if } j < -1 \end{cases} \longrightarrow 0.$$

(iii) *The following statements are equivalent:*

$$\mathrm{Err}_v^{\mathrm{ur}}(\mathscr{D}, T) \xrightarrow{\sim} 0 \text{ in } D_{ft}^b({}_R\mathrm{Mod}) \iff$$
$$(\forall q = 1, \dots, d) \quad \alpha'_q : \mathbb{E}\mathrm{xt}_R^q(H^1_{\mathrm{cont}}(I_v, T), \omega_R) \xrightarrow{f_v^{-1} - 1} \mathbb{E}\mathrm{xt}_R^q(H^1_{\mathrm{cont}}(I_v, T), \omega_R)$$
$$\text{is an isomorphism}$$
$$\iff (\forall q = 1, \dots, d) \quad \mathrm{Coker}(\alpha'_q) = 0.$$

Proof. — As $W \xrightarrow{\sim} D(\mathrm{Err}_v^{\mathrm{ur}}(\Phi, T))$ by the proof of Proposition 7.6.7 (ii), it is enough to apply D to the statements of Proposition 7.6.11 and use local duality 2.5. □

7.6.13. It follows immediately from the definitions that the functor

$$M^\bullet \mapsto U^+\big((M^\bullet)^{I_v^w}\big)$$

respects quasi-isomorphisms and homotopies. As a result, it defines a functor

$$\mathbf{R}\Gamma_{\mathrm{ur}}(G_v, -) : D^*\big({}^{\text{ind-ad}}_{R[G_v]}\mathrm{Mod}\big) \longrightarrow D^b({}_R\mathrm{Mod}), \qquad * = +, b.$$

7.7. Transpositions

Fix $v \in S_f$, $v \nmid p$. We are going to construct transposition operators satisfying 6.5.3.1–6.5.3.5 for the local conditions Δ_v^{ur}.

7.7.1. Lemma

(i) *For every complex of ind-admissible* $R[\overline{G}_v]$*-modules* $M^\bullet$*, the map*

$$C(M^\bullet) \xrightarrow{\mu} C^\bullet_{\mathrm{cont}}(\overline{G}_v, M^\bullet) \xrightarrow{\mathcal{T}} C^\bullet_{\mathrm{cont}}(\overline{G}_v, M^\bullet) \xrightarrow{\lambda} C(M^\bullet)$$

is equal to $\lambda \circ \mu = \mathrm{id}$.

(ii) *For every* $M \in \big({}^{\text{ind-ad}}_{R[\overline{G}_v]}\mathrm{Mod}\big)$ *the map*

$$C^+(M) \xrightarrow{\mu^+} C^\bullet_{\mathrm{cont}}(\overline{G}_v, M) \xrightarrow{\mathcal{T}} C^\bullet_{\mathrm{cont}}(\overline{G}_v, M)$$

is equal to $\mu^+ : C^+(M) \xrightarrow{\nu} C(M) \xrightarrow{\mu} C^\bullet_{\mathrm{cont}}(\overline{G}_v, M)$.

Proof

(i) It is sufficient to consider the case when $M^\bullet = M \in \left({}^{\text{ind-ad}}_{R[\overline{G}_v]}\text{Mod}\right)$. Then the statement is trivial in degree 0. In degree 1, we have to check that $-g(\mu_1(m,m')(g^{-1})) = m$ (resp., $= m'$) for $g = f$ (resp., $g = t$), which follows from the fact that

$$\overline{D(\sigma(g^{-1}))} = \overline{D(\sigma(g)^{-1})} = -\overline{\sigma(g)^{-1}} = -g^{-1}$$

for $(g, D) = (f, \partial/\partial\alpha)$ (resp., $(g, D) = (t, \partial/\partial\beta)$).

In degree 2 we use the fact that $\mu_2(m)$ vanishes on $\overline{G}_v \times \overline{I}_v$; thus

$$\lambda_2 \circ \mathcal{T} \circ \mu_2(m) = \lambda_2\left((g,g') \longmapsto -gg'(\mu_2(m)(g_2^{-1}, g_1^{-1}))\right) = -ft^L(\mu_2(m)(t^{-L}, f^{-1})).$$

As $n := \sigma(t^{-L})\sigma(f^{-1})\sigma(t^{-L}f^{-1})^{-1} = \beta^{-L}\alpha^{-1}\beta\alpha$ satisfies

$$\overline{\left(\frac{\partial n}{\partial \alpha}\right)} e_\alpha + \overline{\left(\frac{\partial n}{\partial \beta}\right)} e_\beta = -f^{-1}t^{-1}[(1-t)e_\alpha + (\theta - 1)e_\beta],$$

we have

$$\mu_2(m)(t^{-L}, f^{-1}) = -f^{-1}t^{-1}(m), \quad \lambda_2 \circ \mathcal{T} \circ \mu_2(m) = -ft^L(-f^{-1}t^{-1}(m)) = m.$$

(ii) We only have to check what happens in degree 1. For $m \in M^{t=1}$ the continuous 1-cochain $\mu_1(m, 0)$ satisfies

$$\mu_1(m,0)(f^a t^b) = (1 + f + \cdots + f^{a-1})m$$

for $a, b \in \mathbf{N}_0$. This implies that $\mu_1(m, 0)$ is a 1-cocycle, hence

$$-g(\mu_1(m,0)(g^{-1})) = \mu_1(m,0)(g). \quad \square$$

7.7.2. Assume that $J = J^\bullet$ is as in 7.6.5, *i.e.*, satisfies $J^\bullet = \sigma_{\geqslant 0} J^\bullet$. Let X, Y be bounded complexes of admissible $R[G_v]$-modules and $\pi : X \otimes_R Y \to J(1)$ a morphism of complexes of $R[G_v]$-modules. The following data 7.7.2.1–7.7.2.5 define transposition operators for the local conditions $\Delta_v^{\text{ur}}(X), \Delta_v^{\text{ur}}(Y)$, satisfying 6.5.3.1–6.5.3.5.

7.7.2.1. Put $h_v = h'_v = 0$; then 6.5.3.1 holds by 7.6.5.

7.7.2.2. Define $\mathcal{T}_v^+(Z) = \text{id} \quad (Z = X, Y)$.

7.7.2.3. By Lemma 7.7.1 (i), $\lambda\mathcal{T}\mu = \lambda\mu = \text{id}$; thus the formula

$$V_{Z,v} = \inf \star b_v \star (\mathcal{T} \circ i_v^+(Z)) \qquad (Z = X, Y)$$

(in which $b_v : \mu\lambda \rightsquigarrow \text{id}$ is as in 7.4.9) defines a homotopy

$$V_{Z,v} : i_v^+(Z) = \inf \circ \mu\nu = \inf \circ \mu\lambda \circ \mathcal{T}\mu\nu \rightsquigarrow \inf \circ \mathcal{T}\mu\nu = \mathcal{T} \circ \inf \circ \mu\nu = \mathcal{T} \circ i_v^+(Z).$$

7.7.2.4. For each $Z = X, Y$, let k_Z and $k_{Z,v}$ be the functorial homotopies $\mathrm{id} \rightsquigarrow \mathcal{T}$ induced by a homotopy a from 3.4.5.5; then $\mathrm{res}_v \star k_Z = k_{Z,v} \star \mathrm{res}_v$ by functoriality. We define $k^+_{Z,v} = 0$. We must show that there is a second order homotopy

$$V_{Z,v} \overset{?}{\rightsquigarrow} k_{Z,v} \star i^+_v(Z).$$

As

$$V_{Z,v} = \inf \star b_v \star \mathcal{T}\mu\nu, \quad k_{Z,v} \star i^+_v(Z) = \inf \star k_{Z^t,v} \star \mu\nu$$

(where $k_{Z^t,v} : \mathrm{id} \rightsquigarrow \mathcal{T}$ on $C^\bullet_{\mathrm{cont}}(\overline{G}_v, Z^t)$ is also induced by a), it is sufficient to show that there is a second order homotopy

$$b_v \star \mathcal{T}\mu \overset{?}{\rightsquigarrow} k_{Z,v} \star \mu$$

for tame $Z = Z^t$. By construction, the L.H.S. (resp., the R.H.S.) is equal to

$$\mathrm{Hom}^{\bullet,\mathrm{naive}}(\eta^{-1}\alpha \circ \mathcal{T} \star c_v, \mathrm{id}_Z) : \mu = \mu\lambda\mathcal{T}\mu \rightsquigarrow \mathcal{T}\mu$$

resp.,

$$\mathrm{Hom}^{\bullet,\mathrm{naive}}(\eta^{-1}\alpha \star a, \mathrm{id}_Z) : \mu \rightsquigarrow \mathcal{T}\mu,$$

in the notation of 7.4.9 (resp., 3.4.5.5). It is enough to observe that the homotopies

$$\eta^{-1}\alpha \circ \mathcal{T} \star c_v,\ \eta^{-1}\alpha \star a : \eta^{-1}\alpha \rightsquigarrow \eta^{-1}\alpha \circ \mathcal{T}$$

between the morphisms of pseudo-compact $\mathbf{Z}_p[\![\overline{G}_v]\!]$-resolutions of $\mathbf{Z}_p$

$$\eta^{-1}\alpha,\ \eta^{-1}\alpha \circ \mathcal{T} : \mathbf{Z}_p[\![\overline{G}_v]\!]^{\widehat{\otimes}}_{\bullet} \longrightarrow \Big[\mathbf{Z}_p[\![\overline{G}_v]\!]e'_r \xrightarrow{\partial_2} \bigoplus_{s\in S} \mathbf{Z}_p[\![\overline{G}_v]\!]e_s \xrightarrow{\partial_1} \mathbf{Z}_p[\![\overline{G}_v]\!]\Big]$$

are 2-homotopic, as both morphisms $\eta^{-1}\alpha, \eta^{-1}\alpha \circ \mathcal{T}$ lift the identity on $\mathbf{Z}_p$ and the resolutions are projective.

7.7.2.5. We have $h_v = h'_v = 0$ by definition. The homotopy

$$h := \overset{\bullet}{\cup}_\pi \star (V_{X,v} \otimes V_{Y,v})_1 : 0 \rightsquigarrow 0$$

is between the zero maps

$$U^+_v(X) \otimes_R U^+_v(Y) \xrightarrow{0} \tau^{\mathrm{II}}_{\geqslant 2}\, C^\bullet_{\mathrm{cont}}(G_v, J(1)).$$

However, the domain (resp., the target) of 0 is concentrated in degrees $\leqslant 2$ (resp., $\geqslant 2$, since $J = \sigma_{\geqslant 0} J$), hence $h = 0$. This means that we can take

$$H_v = 0.$$

We can summarize the previous discussion in the following statement.

7.7.3. Proposition. — *Let $J = \sigma_{\geqslant 0} J$ be a bounded complex of injective R-modules, X and Y bounded complexes of admissible $R[G_v]$-modules and $\pi : X \otimes_R Y \to J(1)$ a morphism of complexes. Then the unramified local conditions $\Delta^{\mathrm{ur}}_v(X), \Delta^{\mathrm{ur}}_v(Y)$ admit transposition operators satisfying 6.5.3.3–6.5.3.5 (with $h_v = h'_v = 0$).*

7.8. Greenberg's local conditions

In this section we develop the theory of Greenberg's local conditions. These seem to be the only local conditions that can be handled by 'elementary' methods.

7.8.1. Fix a subset $\Sigma \subset S_f$ containing all primes above p and put $\Sigma' = S_f - \Sigma$. We are going to combine the local conditions of the type considered in 6.7 (for $v \in \Sigma$) with those from 7.6.5 (for $v \in \Sigma'$). The corresponding Selmer complexes are then analogues of Greenberg's Selmer groups [**Gre2, Gre3, Gre4**].

We consider $J = J^\bullet$ of the form $J = I$ or $J = \omega^\bullet = \sigma_{\geqslant 0} J$ (cf. 2.5(ii)). Let $r_{v,J}$ ($v \in S_f$) be as in 5.2.2. For each $v \in \Sigma'$ fix f_v and t_v as in 7.2.1.

7.8.2. Let $\pi : X \otimes_R Y \to J(1)$ be as in 6.2.1; we assume that π is a perfect duality in the sense of 6.2.6 (in particular, the complexes X, Y are bounded). Assume that we are given, for each $v \in \Sigma$,

$$j_v^+(Z) : Z_v^+ \longrightarrow Z \qquad (Z = X, Y)$$

as in 6.7.1, which satisfy 6.7.5(A) or (B) (in particular, the complexes X_v^+, Y_v^+ are bounded) and such that $X_v^+ \perp\!\!\!\perp_\pi Y_v^+$ for all $v \in \Sigma$. As we have fixed f_v, t_v for all $v \in \Sigma'$, we can define the following local conditions for $Z = X, Y$:

$$\Delta_v(Z) = \begin{cases} C^\bullet_{\text{cont}}(G_v, Z_v^+) \to C^\bullet_{\text{cont}}(G_v, Z), & (v \in \Sigma) \\ C^\bullet_{\text{ur}}(G_v, Z) \to C^\bullet_{\text{cont}}(G_v, Z), & (v \in \Sigma') \end{cases}$$

Our assumptions imply that $\widetilde{\mathbf{R}\Gamma}_f(X), \widetilde{\mathbf{R}\Gamma}_f(Y) \in D^b({}_R\text{Mod})$.

7.8.3. By Proposition 6.7.6,

$$(\forall v \in \Sigma) \quad \Delta_v(X) \perp\!\!\!\perp_{\pi,0} \Delta_v(Y), \quad \Delta_v(Y) \perp\!\!\!\perp_{\pi\circ s_{12},0} \Delta_v(X).$$

By 7.6.5 and Proposition 7.6.6,

$$(\forall v \in \Sigma') \quad \begin{cases} \Delta_v(X) \perp\!\!\!\perp_{\pi,0} \Delta_v(Y), \quad \Delta_v(Y) \perp\!\!\!\perp_{\pi\circ s_{12},0} \Delta_v(X), & \text{if } J = I \\ \Delta_v(X) \perp_{\pi,0} \Delta_v(Y), \quad \Delta_v(Y) \perp_{\pi\circ s_{12},0} \Delta_v(X), & \text{if } J = \omega^\bullet. \end{cases}$$

According to 6.7.8 and Proposition 7.7.3, the local conditions Δ_v admit transpositions satisfying 6.5.3.1–6.5.3.5.

7.8.4. The general machinery of 6.3 and 6.5 then defines, for each $r \in R$, cup products

$$\cup_{\pi,r,0} : \widetilde{C}^\bullet_f(X) \otimes_R \widetilde{C}^\bullet_f(Y) \longrightarrow J[-3]$$
$$\cup_{\pi\circ s_{12},r,0} : \widetilde{C}^\bullet_f(Y) \otimes_R \widetilde{C}^\bullet_f(X) \longrightarrow J[-3]$$

such that

7.8.4.1. The homotopy class of $\cup_{\pi,r,0}$ (resp., $\cup_{\pi\circ s_{12},r,0}$) does not depend on $r \in R$.

7.8.4.2. $\cup_{\pi\circ s_{12},r,0} \circ s_{12}$ is homotopic to $\cup_{\pi,1-r,0}$, hence to $\cup_{\pi,r',0}$, for all $r, r' \in R$.

7.8.4.3. If $J = I$, then the adjoint morphism in $D^b({}_R\mathrm{Mod})$

$$\gamma_{\pi,0} = \mathrm{adj}(\cup_{\pi,r,0}) : \widetilde{\mathbf{R}\Gamma}_f(X) \longrightarrow D_{J[-3]}\big(\widetilde{\mathbf{R}\Gamma}_f(Y)\big)$$

is an isomorphism.

7.8.4.4. If $J = \omega^\bullet$, then there is an exact triangle in $D^b({}_R\mathrm{Mod})$

$$\widetilde{\mathbf{R}\Gamma}_f(X) \xrightarrow{\gamma_X} D_{J[-3]}\big(\widetilde{\mathbf{R}\Gamma}_f(Y)\big) \longrightarrow \bigoplus_{v\in\Sigma'} \mathrm{Err}_v(\Delta_v^{\mathrm{ur}}(X), \Delta_v^{\mathrm{ur}}(Y), \pi).$$

More generally, if one assumes only that $X_v^+ \perp_\pi Y_v^+$ ($v \in \Sigma$), then one has to add to the third term of the triangle the sum

$$\bigoplus_{v\in\Sigma} \mathrm{Err}_v(\Delta_v(X), \Delta_v(Y), \pi),$$

given by the formulas from Proposition 6.7.6 (iv).

7.8.4.5. In $D^b(({}_R\mathrm{Mod})/(\text{pseudo-null}))$, the error terms $\mathrm{Err}_v(\Delta_v^{\mathrm{ur}}(X), \Delta_v^{\mathrm{ur}}(Y), \pi)$ ($v \in \Sigma'$) are given by the formulas in Proposition 7.6.7 (ii) and Corollary 7.6.8 (cf. 7.6.10.6–7.6.10.9). In particular, they vanish after localizing at each prime ideal $\mathfrak{q} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{q}) = 0$.

7.8.4.6. If $X = \sigma_{\leqslant 0}\, X$, then the error terms $\mathrm{Err}_v(\Delta_v^{\mathrm{ur}}(X), \Delta_v^{\mathrm{ur}}(Y), \pi)$ ($v \in \Sigma'$) in $D^b_{ft}({}_R\mathrm{Mod})$ are given by the formulas in Corollary 7.6.12.

7.8.5. In practice, it is often the case that the canonical maps

$$\begin{aligned} \tau_{\leqslant 0}\, X &\longrightarrow X \\ \tau_{\leqslant 0}\, X_v^+ &\longrightarrow X_v^+ \qquad (v \in \Sigma) \end{aligned}$$

are all quasi-isomorphisms. If true, then it follows from Lemma 4.2.6 that the maps

$$\begin{aligned} \tau_{\leqslant 2}\, C^\bullet_{\mathrm{cont}}(G_{K,S}, X) &\longrightarrow C^\bullet_{\mathrm{cont}}(G_{K,S}, X) \\ \tau_{\leqslant 2}\, U_v^+(X) &\longrightarrow U_v^+(X) \qquad (v \in S_f), \end{aligned}$$

and hence

$$\tau_{\leqslant 3}\, \widetilde{C}^\bullet_f(G_{K,S}, X; \Delta(X)) \longrightarrow \widetilde{C}^\bullet_f(G_{K,S}, X; \Delta(X)),$$

are also quasi-isomorphisms.

***7.8.6. Theorem* (Euler-Poincaré characteristic).** — *The Euler-Poincaré characteristic of* $\widetilde{\mathbf{R}\Gamma}_f(X)$ *is equal to*

$$\sum_q (-1)^q\, e_R\big(\widetilde{H}^q_f(X)\big) = \sum_{v|\infty} \sum_q (-1)^q\, e_R\left((X^q)^{G_v}\right) - \sum_{v|p} [K_v : \mathbf{Q}_p] \sum_q (-1)^q\, e_R((X_v^+)^q).$$

Proof. — The middle exact triangle in 6.1.3 implies that the L.H.S. is equal to the sum of

$$(1) = \sum_q (-1)^q \, e_R\left(H^q_{c,\mathrm{cont}}(G_{K,S}, X)\right)$$

and

$$\sum_{v \in S_f} (2)_v = \sum_{v \in S_f} \sum_q (-1)^q \, e_R\big(H^q(U_v^+(X))\big).$$

However,

$$(1) = \sum_{v \mid \infty} \sum_q (-1)^q \, e_R\left((X^q)^{G_v}\right)$$

by Theorem 5.3.6, while $(2)_v = 0$ for $v \in \Sigma'$. Finally, for $v \in \Sigma$, we have

$$(2)_v = \sum_q (-1)^q \, e_R\left(H^q_{\mathrm{cont}}(G_v, X_v^+)\right) = c_v \cdot \sum_q (-1)^q \, e_R((X_v^+)^q)$$

with

$$c_v = \begin{cases} -[K_v : \mathbf{Q}_p], & v \mid p \\ 0, & v \nmid p, \end{cases}$$

by 5.2.11. □

7.8.7. Corollary. — *Assume that:*

(i) $d_\infty^+ = [K_v : \mathbf{R}]^{-1} \sum_q (-1)^q \, e_R\left((X^q)^{G_v}\right)$ *does not depend on* $v \mid \infty$.

(ii) $d_p^+ = \sum_q (-1)^q \, e_R\left((X_v^+)^q\right)$ *does not depend on* $v \mid p$.

Then

$$\sum_q (-1)^q \, e_R\big(\widetilde{H}^q_f(X)\big) = [K : \mathbf{Q}](d_\infty^+ - d_p^+).$$

***7.8.8. Proposition* (Change of S)**. — *Let X be an ind-admissible $R[G_{K,S}]$-module and $S' \supset S$ a finite set of primes of K. Then:*

(i) *The canonical morphisms of complexes*

$$\mathrm{inf} : C^\bullet_{\mathrm{cont}}(G_{K,S}, X) \longrightarrow C^\bullet_{\mathrm{cont}}(G_{K,S'}, X),$$

$$\mathrm{res} : C^\bullet_{\mathrm{cont}}(G_{K,S}, X) \longrightarrow \bigoplus_{v \in S'-S} C^\bullet_{\mathrm{cont}}(G_v/I_v, X)$$

(the second depending on the choice of embeddings $\overline{K} \hookrightarrow \overline{K}_v$ for all $v \in S' - S$) give rise to an exact triangle

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, X) \xrightarrow{(\mathrm{inf},\mathrm{res})} \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S'}, X) \oplus \bigoplus_{v \in S'-S} \mathbf{R}\Gamma_{\mathrm{ur}}(G_v, X) \xrightarrow{(0,\mathrm{inf})} \bigoplus_{v \in S'-S} \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, X).$$

(ii) *Assume that we are given local conditions* $\Delta_v(X)$ *for all* $v \in S_f$. *Defining* $U_v^+(X) = C^\bullet_{\mathrm{ur}}(G_v, X)$ *for all* $v \in S' - S$ *and keeping the given local conditions for* $v \in S_f$, *there is a canonical isomorphism*

$$\mathbf{R}\Gamma\big(G_{K,S}, X; (\Delta_v(X))_{v\in S_f}\big) \xrightarrow{\sim} \mathbf{R}\Gamma\big(G_{K,S'}, X; (\Delta_v(X))_{v\in S'_f}\big).$$

Proof. — By a standard limit argument we can assume that $X = M$ is a p-primary torsion discrete $G_{K,S}$-module. As recalled in 9.2.1 below, M defines an étale sheaf M_{et} on $\mathrm{Spec}(\mathcal{O}_{K,S})$ and $\mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, M)$ is canonically isomorphic to $\mathbf{R}\Gamma(\mathrm{Spec}(\mathcal{O}_{K,S}), M_{\mathrm{et}})$. The statement of (i) then follows from the excision triangles for étale cohomology (cf. [**Mi**, p. 214]):

$$\bigoplus_{v\in S'-S} \mathbf{R}\Gamma_{\{v\}}(\mathcal{O}_v^h, M_{\mathrm{et}}) \longrightarrow \mathbf{R}\Gamma(\mathrm{Spec}(\mathcal{O}_{K,S}), M_{\mathrm{et}}) \longrightarrow \mathbf{R}\Gamma(\mathrm{Spec}(\mathcal{O}_{K,S'}), M_{\mathrm{et}}),$$

$$\mathbf{R}\Gamma_{\{v\}}(\mathcal{O}_v^h, M_{\mathrm{et}}) \longrightarrow \mathbf{R}\Gamma(\mathcal{O}_v^h, M_{\mathrm{et}}) \longrightarrow \mathbf{R}\Gamma(\mathrm{Spec}(K_v), M_{\mathrm{et}}) \quad (v \in S' - S)$$

(where $\mathcal{O}_v^h$ denotes the henselianization of $\mathcal{O}_{K,S}$ at v), if we take into account canonical isomorphisms

$$\mathbf{R}\Gamma(\mathcal{O}_v^h, M_{\mathrm{et}}) \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cont}}(G_v/I_v, M), \quad \mathbf{R}\Gamma(\mathrm{Spec}(K_v), M_{\mathrm{et}}) \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, M).$$

The statement (ii) is a straightforward consequence of (i). □

7.8.9. Corollary - Definition. — *Under the assumptions of Proposition 7.8.8(ii), the cohomology groups* $\widetilde{H}^i_f\big(G_{K,S'}, X; (\Delta_v(X))_{v\in S'_f}\big)$ *do not depend – up to a canonical isomorphism – on the choice of* S'. *We shall denote them by* $\widetilde{H}^i_f(K, X) = \widetilde{H}^i_f(K, X; \Delta(X))$.

7.8.10. Localization. — The above discussion works whenever R is replaced by $R_{\mathscr{S}}$, under the assumption 6.7.5 (B), localized at $\mathscr{S}$. For example, if $\mathscr{S} = R - \mathfrak{p}$ for $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{p}) = 1$, then everything in 7.6.10 holds if T is a bounded complex of admissible $R_{\mathfrak{p}}[G_v]$-modules with cohomology of finite type over $R_{\mathfrak{p}}$. Another example is provided by the following Proposition.

7.8.11. Proposition **(Euler-Poincaré characteristic: self-dual case)**

Assume that, in the situation of 7.8.6, R *is an integral domain with fraction field* $\mathcal{K} = \mathrm{Frac}(R)$ *of characteristic* $\mathrm{char}(\mathcal{K}) = 0$, *all complexes* $X = H^0(X)$ *and* $X_v^+ = H^0(X_v^+)$ $(v \in \Sigma)$ *are concentrated in degree zero and all morphisms* $X_v^+ \to X$ $(v \in \Sigma)$ *are injective. Put* $V = X \otimes_R \mathcal{K}$, $V_v^\pm = X_v^\pm \otimes_R \mathcal{K}$ *(where* $X_v^- = X/X_v^+$, $v \in \Sigma$*). Assume, in addition, that* V *is a simple* $\mathcal{K}[G_{K,S}]$*-module and that there exists a non-degenerate skew-symmetric* $G_{K,S}$*-equivariant bilinear form* $V \otimes_{\mathcal{K}} V \to \mathcal{K}(1)$, *which induces isomorphisms of* $\mathcal{K}[G_v]$*-modules*

$$V_v^\pm \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}}(V_v^\mp, \mathcal{K})(1)$$

for all $v \in \Sigma$. *Then, for each homomorphism* $\chi : G_{K,S} \to R^*$,

$$\mathrm{rk}_R\, \widetilde{H}^1_f(X \otimes \chi) = \mathrm{rk}_R\, \widetilde{H}^2_f(X \otimes \chi) = \mathrm{rk}_R\, \widetilde{H}^1_f(X \otimes \chi^{-1}) = \mathrm{rk}_R\, \widetilde{H}^2_f(X \otimes \chi^{-1})$$

(with respect to Greenberg's local conditions given by $X_v^+ \otimes \chi$ *resp.,* $X_v^+ \otimes \chi^{-1}$*).*

Proof. — For $\alpha \in \{\chi, \chi^{-1}\}$, put

$$h^q_\alpha = \dim_{\mathcal{K}}\, \widetilde{H}^q_f(V \otimes \alpha) = \mathrm{rk}_R\, \widetilde{H}^q_f(X \otimes \alpha).$$

Self-duality of V and of the local conditions V_v^+ $(v \in \Sigma)$ imply that, by the localized duality theorem (cf. 7.8.4.4–7.8.4.5),

$$\widetilde{H}^q_f(V \otimes \chi) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}}\big(\widetilde{H}^{3-q}_f(V \otimes \chi^{-1}), \mathcal{K}\big) \implies h^q_\chi = h^{3-q}_{\chi^{-1}}. \tag{7.8.11.1}$$

For each $v \in \Sigma$, V_v^+ is a Lagrangian (= maximal isotropic) subspace of V; it follows that

$$(\forall v \mid p) \quad \dim_{\mathcal{K}}(V_v^+ \otimes \alpha) = \dim_{\mathcal{K}}(V)/2 = \mathrm{rk}_R(X)/2.$$

Self-duality $V \xrightarrow{\sim} V^*(1)$ implies that, for each real embedding $K \hookrightarrow \mathbf{R}$, the corresponding complex conjugation acts on V by a matrix with eigenvalues $+1, -1$, each with the same multiplicity $\dim_{\mathcal{K}}(V)/2 = \mathrm{rk}_R(X)/2$; thus

$$(\forall v \mid \infty) \quad [K_v : \mathbf{R}]^{-1} \dim_{\mathcal{K}} H^0(G_v, V \otimes \alpha) = \dim_{\mathcal{K}}(V)/2 = \mathrm{rk}_R(X)/2.$$

Applying Corollary 7.8.7, we obtain (using 7.8.5)

$$\sum_{q=0}^{3} (-1)^q\, h^q_\alpha = 0 \qquad (\alpha = \chi, \chi^{-1}) \tag{7.8.11.2}$$

However,

$$h^0_\chi = h^0_{\chi^{-1}} (= h^3_\chi) = 0, \tag{7.8.11.3}$$

because V is an irreducible representation. Combining (7.8.11.1)–(7.8.11.3), we obtain

$$h^1_\chi = h^2_\chi = h^1_{\chi^{-1}} = h^2_{\chi^{-1}},$$

as required. □

CHAPTER 8

IWASAWA THEORY

In this chapter we study continuous cohomology in towers K_∞/K, where $\mathrm{Gal}(K_\infty/K) = \Gamma \xrightarrow{\sim} \mathbf{Z}_p^r \times \Delta$, for a finite abelian group Δ. Our main tool is Shapiro's Lemma, which allows us to reduce to statements about cohomology over K, but replaces R by the bigger coefficient ring $R[\![\Gamma]\!]$. Once we establish a correspondence between the duality diagrams over K and over K_∞ (Sect. 8.4) and the compatibility of the Greenberg local conditions with Shapiro's Lemma (Sect. 8.5–8.8), we can apply the duality formalism over K to the induced modules and obtain – after an analysis of the local Tamagawa factors – Iwasawa-theoretical duality results (Sect. 8.9). In Sect. 8.10 we study an abstract version of "Mazur's Control Theorem"; in our context it is a consequence of a fundamental base-change property of Selmer complexes (Proposition 8.10.1).

Throughout Chapter 8 we assume that the residue field of R is a finite field of characteristic p.

8.1. Shapiro's Lemma

Let us recall basic facts about Shapiro's Lemma ([**Ve1**, §1.1-3]; [**Bro**, §III.6.2, Ex. III.8.2]).

8.1.1. Let $U \subset G$ be an open subgroup of a pro-finite group G. For every discrete U-module X, the induced module

$$\mathrm{Ind}_U^G(X) = \{f : G \longrightarrow X \mid f \text{ locally constant}, f(ug) = uf(g)\ \forall u \in U, \forall g \in G\}$$

is a discrete G-module with respect to the (left) action

$$(g * f)(g') = f(g'g).$$

The inclusion $U \hookrightarrow G$ and the map

$$\begin{aligned} \delta_U : \mathrm{Ind}_U^G(X) &\longrightarrow X \\ f &\longmapsto f(1) \end{aligned}$$

define a morphism of pairs $(G, \mathrm{Ind}_U^G(X)) \to (U, X)$ in the sense of 3.4.1.6, hence a morphism of complexes (functorial in X)

$$\mathrm{sh} : C^{\bullet}_{\mathrm{cont}}(G, \mathrm{Ind}_U^G(X)) \longrightarrow C^{\bullet}_{\mathrm{cont}}(U, X).$$

Shapiro's Lemma asserts that sh is a quasi-isomorphism.

If X, Y are discrete U-modules, then there is a commutative diagram

$$\begin{array}{ccc} C^{\bullet}_{\mathrm{cont}}(G, \mathrm{Ind}_U^G(X)) \otimes_{\mathbf{Z}} C^{\bullet}_{\mathrm{cont}}(G, \mathrm{Ind}_U^G(Y)) & \xrightarrow{\cup} & C^{\bullet}_{\mathrm{cont}}(G, \mathrm{Ind}_U^G(X \otimes_{\mathbf{Z}} Y)) \\ \big\downarrow{\scriptstyle \mathrm{sh}\otimes\mathrm{sh}} & & \big\downarrow{\scriptstyle \mathrm{sh}} \\ C^{\bullet}_{\mathrm{cont}}(U, X) \otimes_{\mathbf{Z}} C^{\bullet}_{\mathrm{cont}}(U, Y) & \xrightarrow{\cup} & C^{\bullet}_{\mathrm{cont}}(U, X \otimes_{\mathbf{Z}} Y), \end{array}$$

in which the upper horizontal arrow is induced by the map

$$\begin{array}{ccc} \mathrm{Ind}_U^G(X) \otimes_{\mathbf{Z}} \mathrm{Ind}_U^G(Y) & \longrightarrow & \mathrm{Ind}_U^G(X \otimes_{\mathbf{Z}} Y) \\ f_1 \otimes f_2 & \longmapsto & (g \longmapsto f_1(g) \otimes f_2(g)). \end{array}$$

8.1.2. Restriction. — If $V \subset U$ is another open subgroup of G, then there is an inclusion $\mathrm{Ind}_U^G(X) \subset \mathrm{Ind}_V^G(X)$ making the following diagram of morphisms of complexes commutative:

$$\begin{array}{ccc} C^{\bullet}_{\mathrm{cont}}(G, \mathrm{Ind}_U^G(X)) & \xrightarrow{\mathrm{sh}} & C^{\bullet}_{\mathrm{cont}}(U, X) \\ \big\downarrow{\scriptstyle \mathrm{incl}_*} & & \big\downarrow{\scriptstyle \mathrm{res}} \\ C^{\bullet}_{\mathrm{cont}}(G, \mathrm{Ind}_V^G(X)) & \xrightarrow{\mathrm{sh}} & C^{\bullet}_{\mathrm{cont}}(V, X). \end{array}$$

8.1.3. If X is a discrete G-module, then there are two other natural discrete G-modules isomorphic to $\mathrm{Ind}_U^G(X)$ (more precisely, to $\mathrm{Ind}_U^G(X_0)$, where X_0 is equal to X as a set, but viewed as an U-module), namely

$$X_U = \mathbf{Z}[G/U] \otimes_{\mathbf{Z}} X = \left\{ \sum \beta \otimes x_\beta \mid \beta \in G/U, x_\beta \in X \right\}$$

$${}_U X = \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[G/U], X) \xrightarrow{\sim} \{a : G/U \longrightarrow X\}$$

with G-action

$$g(\beta \otimes x) = g\beta \otimes gx, \quad (ga)(\beta) = g(a(g^{-1}\beta)).$$

Denote by $\delta_\beta : G/U \to \mathbf{Z}$ Kronecker's delta-function

$$\delta_\beta(\beta') = \begin{cases} 1 & \beta = \beta' \\ 0 & \beta \neq \beta'. \end{cases}$$

Then the following formulas define G-equivariant isomorphisms, functorial in X:

$$\begin{aligned}
\mathrm{Ind}_U^G(X) &\xrightarrow{\sim} {}_UX, & f &\longmapsto \big(gU \longmapsto g(f(g^{-1}))\big)\\
X_U &\xrightarrow{\sim} {}_UX, & \sum \beta\otimes x_\beta &\longmapsto \sum x_\beta\delta_\beta\\
\mathrm{Ind}_U^G(X) &\xrightarrow{\sim} X_U, & f &\longmapsto \sum_{gU\in G/U} gU\otimes g\big(f(g^{-1})\big).
\end{aligned}$$

We use the above isomorphisms to pass freely between $\mathrm{Ind}_U^G(X)$, X_U and ${}_UX$. The map δ_U from 8.1.1 will then be, indeed, identified with Kronecker's delta-function $\delta_U : X_U \to X$, $\delta_U(\sum \beta\otimes x_\beta) = x_U$.

8.1.4. Corestriction. — Fix a section $U\backslash G$, $\alpha\mapsto\overline{\alpha}$ of the canonical projection $G\to U\backslash G$ (*i.e.*, a set of coset representatives $G=\bigcup U\overline{\alpha}_i$). For every discrete G-module X, the formula

$$(\mathrm{cor}(c))(g_1,\dots,g_n) = \sum_{\alpha\in U\backslash G} \overline{\alpha}^{-1}c\big(\overline{\alpha}g_1(\overline{\alpha g_1})^{-1},\dots,\overline{\alpha g_1\cdots g_{n-1}}g_n(\overline{\alpha g_1\cdots g_n})^{-1}\big)$$

defines a morphism of complexes

$$\mathrm{cor} : C^\bullet_{\mathrm{cont}}(U,X)\longrightarrow C^\bullet_{\mathrm{cont}}(G,X)$$

inducing the corestriction maps on cohomology (see [**Bro**, §III.9 (D)], for a conceptual explanation of this formula). In fact, the homotopy class of cor is independent of any choices: if cor' corresponds to another section $\alpha\mapsto\overline{\alpha}'$, then the formula

$$\begin{aligned}
&(h^{n+1}(c))(g_1,\dots,g_n) =\\
&\sum_{\alpha\in U\backslash G}\overline{\alpha}^{-1}\sum_{i=0}^{n-1}(-1)^i c\big(\overline{\alpha}g_1(\overline{\alpha g_1})^{-1},\dots,\overline{\alpha g_1\cdots g_{i-1}}g_i(\overline{\alpha g_1\cdots g_i})^{-1},\\
&\qquad\overline{\alpha g_1\cdots g_{i-1}}g_i(\overline{\alpha g_1\cdots g_i}')^{-1},\overline{\alpha g_1\cdots g_i}'g_{i+1}(\overline{\alpha g_1\cdots g_{i+1}}')^{-1},\\
&\qquad\qquad\dots,\overline{\alpha g_1\cdots g_{n-1}}'g_n(\overline{\alpha g_1\cdots g_n}')^{-1}\big)
\end{aligned}$$

defines a homotopy $h : \mathrm{cor}\rightsquigarrow\mathrm{cor}'$.

If $V\subset U$ is another open subgroup of G, fix coset representatives

$$G=\bigcup_i U\overline{\alpha}_i,\quad U=\bigcup_j V\overline{\beta}_j.$$

Then the composition of the corresponding corestriction morphisms

$$C^\bullet_{\mathrm{cont}}(V,X)\xrightarrow{\mathrm{cor}}C^\bullet_{\mathrm{cont}}(U,X)\xrightarrow{\mathrm{cor}}C^\bullet_{\mathrm{cont}}(G,X)$$

is equal to the corestriction map associated to the coset decomposition

$$G=\bigcup_{i,j}V\overline{\beta}_j\overline{\alpha}_i.$$

8.1.5. Lemma. — *For every discrete G-module X, the composite morphism*

$$C^{\bullet}_{\mathrm{cont}}(G, {}_U X) \xrightarrow{\sim} C^{\bullet}_{\mathrm{cont}}(G, \mathrm{Ind}^G_U(X)) \xrightarrow{\mathrm{sh}} C^{\bullet}_{\mathrm{cont}}(U, X) \xrightarrow{\mathrm{cor}} C^{\bullet}_{\mathrm{cont}}(G, X)$$

(where cor *depends on a fixed section of $G \to U\backslash G$) is homotopic to the map induced by*

$$\mathrm{Tr} : {}_U X \longrightarrow X, \quad a \longmapsto \sum_{\beta \in G/U} a(\beta).$$

Proof. — The formula

$$(h^n(c))(g_1, \ldots, g_{n-1}) = \sum_{\alpha \in U\backslash G} \sum_{i=0}^{n-1} (-1)^i c\big(g_1, \ldots, g_i, (\overline{\alpha g_1 \cdots g_i})^{-1}, \overline{\alpha g_1 \cdots g_i} g_{i+1} (\overline{\alpha g_1 \cdots g_{i+1}})^{-1}, \ldots, \overline{\alpha g_1 \cdots g_{n-2}} g_{n-1} (\overline{\alpha g_1 \cdots g_{n-1}})^{-1}\big)(\overline{\alpha}^{-1}),$$

e.g.

$$h^1(c) = \sum_{\alpha \in U\backslash G} c(\overline{\alpha}^{-1})(\overline{\alpha}^{-1}),$$

defines a homotopy $h : \mathrm{Tr}_* \rightsquigarrow \mathrm{cor} \circ \mathrm{sh}$. □

8.1.6. Functoriality. — Let $V \subset U$ be open subgroups of G. The previous discussion gives the following functoriality properties of $\mathrm{Ind}^G_U(X)$, ${}_U X$ and X_U (for variable U).

8.1.6.1. Restriction. — The map res is induced by the inclusion $\mathrm{Ind}^G_U(X) \hookrightarrow \mathrm{Ind}^G_V(X)$, which corresponds to the map

$${}_U X \longrightarrow {}_V X$$

induced by the canonical projection

$$\mathrm{pr} : \mathbf{Z}[G/V] \longrightarrow \mathbf{Z}[G/U]$$

resp., to the trace map

$$\mathrm{Tr} : X_U \longrightarrow X_V, \quad gU \otimes x \longmapsto \sum_{u \in U/V} guV \otimes x.$$

8.1.6.2. Corestriction. — The homotopy class of cor corresponds to the homotopy class of the map

$$C^{\bullet}_{\mathrm{cont}}(G, {}_V X) \longrightarrow C^{\bullet}_{\mathrm{cont}}(G, {}_U X)$$

induced by

$$\mathrm{Tr} : \mathbf{Z}[G/U] \longrightarrow \mathbf{Z}[G/V], \quad gU \longmapsto \sum_{u \in U/V} guV,$$

resp., to the map

$$C^{\bullet}_{\mathrm{cont}}(G, X_V) \longrightarrow C^{\bullet}_{\mathrm{cont}}(G, X_U)$$

induced by

$$\mathrm{pr} : \mathbf{Z}[G/V] \longrightarrow \mathbf{Z}[G/U].$$

8.1.6.3. Conjugation. — If $U \lhd G$ is an open normal subgroup of G, then the formula

$$(\mathrm{Ad}(gU)f)(g') = \big({}^{gU}f\big)(g') = g\big(f(g^{-1}g')\big)$$

defines a (left) G/U-action on $\mathrm{Ind}_U^G(X)$ commuting with the action of G. This Ad-action of G/U corresponds to the action

$$({}^{gU}a)(g'U) = a(g'gU)$$

on ${}_UX$ (resp., to the action

$${}^{gU}\left(\sum_{hU\in G/U} hU \otimes x_{hU}\right) = \sum_{hU\in G/U} hg^{-1}U \otimes x_{hU}$$

on X_U) and corresponds *via* sh to the homotopy action of G/U on $C^\bullet_{\mathrm{cont}}(U,X)$, defined in 3.6.1.4.

More precisely, for each $g \in G$, the commutative diagram of morphisms of pairs

$$\begin{array}{ccc}
(G, \mathrm{Ind}_U^G(X)) & \xrightarrow{(\mathrm{incl},\mathrm{can})} & (U,X) \\
\big\downarrow{\scriptstyle (\mathrm{id},\mathrm{Ad}(gU))} & & \\
(G, \mathrm{Ind}_U^G(X)) & & \big\downarrow{\scriptstyle (\mathrm{Ad}(g^{-1}),g)} \\
\big\downarrow{\scriptstyle (\mathrm{Ad}(g^{-1}),g)} & & \\
(G, \mathrm{Ind}_U^G(X)) & \xrightarrow{(\mathrm{incl},\mathrm{can})} & (U,X)
\end{array}$$

induces a commutative diagram of morphisms of complexes

$$\begin{array}{ccc}
C^\bullet_{\mathrm{cont}}(G, \mathrm{Ind}_U^G(X)) & \xrightarrow{\mathrm{sh}} & C^\bullet_{\mathrm{cont}}(U,X) \\
\big\downarrow{\scriptstyle \mathrm{Ad}(gU)_*} & & \\
C^\bullet_{\mathrm{cont}}(G, \mathrm{Ind}_U^G(X)) & & \big\downarrow{\scriptstyle \mathrm{Ad}(g)} \\
\big\downarrow{\scriptstyle \mathrm{Ad}(g)} & & \\
C^\bullet_{\mathrm{cont}}(G, \mathrm{Ind}_U^G(X)) & \xrightarrow{\mathrm{sh}} & C^\bullet_{\mathrm{cont}}(U,X).
\end{array}$$

In the left column, the two vertical maps commute with each other. A homotopy $h_g : \mathrm{id} \rightsquigarrow \mathrm{Ad}(g)$ from 4.5.3 induces a homotopy

$$\begin{aligned}(\mathrm{sh}\circ \mathrm{Ad}(gU)_*) \star h_g : \mathrm{sh}\circ \mathrm{Ad}(gU)_* &\rightsquigarrow \mathrm{sh}\circ \mathrm{Ad}(gU)_* \circ \mathrm{Ad}(g) \\ &= \mathrm{sh}\circ \mathrm{Ad}(g)\circ \mathrm{Ad}(gU)_* = \mathrm{Ad}(g)\circ \mathrm{sh}.\end{aligned}$$

8.1.6.4. Products. — If, in the situation of 8.1.6.3, G/U is abelian, then X_U becomes a $\mathbf{Z}[G/U][G]$-module, with G (resp., G/U) acting as in 8.1.3 (resp., in 8.1.6.3). Let $(X_U)^\iota$ be the following $\mathbf{Z}[G/U][G]$-module: as a G-module, $(X_U)^\iota = X_U$, but gU acts on $(X_U)^\iota$ as $g^{-1}U$ acted on X_U. The map

$$\begin{aligned} m_U : X_U \otimes_{\mathbf{Z}[G/U]} (Y_U)^\iota &\longrightarrow \mathbf{Z}[G/U] \otimes_{\mathbf{Z}} (X \otimes_{\mathbf{Z}} Y) \\ [g_1U] \otimes x \otimes [g_2U] \otimes y &\longmapsto [g_2g_1^{-1}U] \otimes x \otimes y \end{aligned}$$

is a morphism of $\mathbf{Z}[G/U][G]$-modules, provided we let G (resp., $\mathbf{Z}[G/U]$) act trivially (resp., by multiplication) on the factor $\mathbf{Z}[G/U]$ on the R.H.S. If G/V is also abelian, then the diagram

$$\begin{array}{ccc} X_V \otimes_{\mathbf{Z}[G/V]} (Y_V)^\iota & \xrightarrow{m_V} & \mathbf{Z}[G/V] \otimes_{\mathbf{Z}} (X \otimes_{\mathbf{Z}} Y) \\ \big\downarrow{\scriptstyle \mathrm{pr}\otimes\mathrm{pr}} & & \big\downarrow{\scriptstyle \mathrm{pr}\otimes\mathrm{id}} \\ X_U \otimes_{\mathbf{Z}[G/U]} (Y_U)^\iota & \xrightarrow{m_U} & \mathbf{Z}[G/U] \otimes_{\mathbf{Z}} (X \otimes_{\mathbf{Z}} Y) \end{array}$$

is commutative.

8.1.6.5. Lemma. — *If G/U is abelian, then the diagram*

$$\begin{array}{ccccc} C^\bullet_{\mathrm{cont}}(G, X_U) \otimes_{\mathbf{Z}} C^\bullet_{\mathrm{cont}}(G, (Y_U)^\iota) & \xrightarrow{\mathrm{sh}\otimes\mathrm{sh}} & C^\bullet_{\mathrm{cont}}(U, X) \otimes_{\mathbf{Z}} C^\bullet_{\mathrm{cont}}(U, Y) & \xrightarrow{\cup} & C^\bullet_{\mathrm{cont}}(U, X \otimes_{\mathbf{Z}} Y) \\ \big\downarrow & & & & \big\downarrow{\scriptstyle \mathrm{cor}} \\ C^\bullet_{\mathrm{cont}}(G, X_U) \otimes_{\mathbf{Z}[G/U]} C^\bullet_{\mathrm{cont}}(G, (Y_U)^\iota) & \xrightarrow{\cup\circ(m_U)_*} & \mathbf{Z}[G/U] \otimes_{\mathbf{Z}} C^\bullet_{\mathrm{cont}}(G, X \otimes_{\mathbf{Z}} Y) & \xrightarrow{\delta_U} & C^\bullet_{\mathrm{cont}}(G, X \otimes_{\mathbf{Z}} Y) \end{array}$$

is commutative up to homotopy.

Proof. — This follows from Lemma 8.1.5 and the commutative diagram

$$\begin{array}{ccccc} C^\bullet_{\mathrm{cont}}(U, X) \otimes_{\mathbf{Z}} C^\bullet_{\mathrm{cont}}(U, Y) & & \xrightarrow{\qquad\qquad\cup\qquad\qquad} & & C^\bullet_{\mathrm{cont}}(U, X \otimes_{\mathbf{Z}} Y) \\ \big\uparrow{\scriptstyle \mathrm{sh}\otimes\mathrm{sh}} & & & & \big\uparrow{\scriptstyle \mathrm{sh}} \\ C^\bullet_{\mathrm{cont}}(G, X_U) \otimes_{\mathbf{Z}} C^\bullet_{\mathrm{cont}}(G, (Y_U)^\iota) & \xrightarrow{\cup} & C^\bullet_{\mathrm{cont}}(G, X_U \otimes_{\mathbf{Z}} Y_U) & \xrightarrow{q_*} & C^\bullet_{\mathrm{cont}}(G, (X \otimes_{\mathbf{Z}} Y)_U) \\ \big\downarrow & & & & \big\downarrow{\scriptstyle \mathrm{Tr}_*} \\ C^\bullet_{\mathrm{cont}}(G, X_U) \otimes_{\mathbf{Z}[G/U]} C^\bullet_{\mathrm{cont}}(G, (Y_U)^\iota) & \xrightarrow{\cup\circ(m_U)_*} & \mathbf{Z}[G/U] \otimes_{\mathbf{Z}} C^\bullet_{\mathrm{cont}}(G, X \otimes_{\mathbf{Z}} Y) & \xrightarrow{\delta_U} & C^\bullet_{\mathrm{cont}}(G, X \otimes_{\mathbf{Z}} Y), \end{array}$$

where

$$q : X_U \otimes_{\mathbf{Z}} Y_U \longrightarrow (X \otimes_{\mathbf{Z}} Y)_U$$

is the morphism of G-modules given by the formula

$$q : \sum_{\beta,\beta'} \beta \otimes x_\beta \otimes \beta' \otimes y_{\beta'} \mapsto \sum_\beta \beta \otimes x_\beta \otimes y_\beta. \quad \square$$

8.1.7. Semilocal case

8.1.7.1. Assume that $\alpha : \overline{G} \to G$ is a continuous homomorphism of pro-finite groups and $U \lhd G$ is an open normal subgroup of G. Then $\overline{U} = \alpha^{-1}(U)$ is an open normal subgroup of $\overline{G}$ and $\alpha : \overline{G}/\overline{U} \to G/U$ is injective.

Fix coset representatives $\sigma_i \in G$ of

$$G/U = \bigcup_i \sigma_i\, \alpha(\overline{G}/\overline{U}) = \bigcup_i \alpha(\overline{G}/\overline{U})\sigma_i^{-1}$$

and set, for each i,

$$\alpha_i : \overline{G} \xrightarrow{\alpha} G \xrightarrow{\mathrm{Ad}(\sigma_i)} G,$$
$$\alpha_i : \overline{G}/\overline{U} \xrightarrow{\alpha} G/U \xrightarrow{\mathrm{Ad}(\sigma_i)} G/U.$$

For every discrete G-module X, the above coset decomposition yields a decomposition of the $\overline{G}$-module $\alpha^*(X_U)$ into a direct sum

$$\alpha^*(\mathbf{Z}[G/U] \otimes_{\mathbf{Z}} X) = \bigoplus_i \alpha^*(\mathbf{Z}[\alpha(\overline{G}/\overline{U})]\sigma_i^{-1} \otimes_{\mathbf{Z}} X) = \bigoplus_i \alpha^*(X_U)_i;$$

denote by $\mathrm{pr}_i : \alpha^*(X_U) \to \alpha^*(X_U)_i$ the projection on the i-th factor. The isomorphism of $\overline{G}$-modules

$$\sigma_i \otimes \sigma_i : \alpha^*(X_U)_i \xrightarrow{\sim} \alpha_i^*(\mathbf{Z}[\alpha_i(\overline{G}/\overline{U})] \otimes_{\mathbf{Z}} X),$$

together with the canonical identification

$$\begin{aligned} \alpha_i^*(\mathbf{Z}[\alpha_i(\overline{G}/\overline{U})] \otimes_{\mathbf{Z}} X) &\xrightarrow{\sim} (\alpha_i^* X)_{\overline{U}} \\ \alpha_i^*(\overline{g}\overline{U}) \otimes x &\longmapsto [\overline{g}\overline{U}] \otimes x, \end{aligned}$$

give a $\overline{G}$-isomorphism

$$\omega_i' : \alpha^*(X_U)_i \xrightarrow{\sim} (\alpha_i^* X)_{\overline{U}}.$$

Putting all $\omega_i = \omega_i' \circ \mathrm{pr}_i$ together, we obtain a $\overline{G}$-isomorphism

$$\omega = (\omega_i) : \alpha^*(X_U) \xrightarrow{\sim} \bigoplus_i (\alpha_i^* X)_{\overline{U}}.$$

8.1.7.2. The commutative diagram of morphisms of pairs

$$\begin{array}{ccccc} (G, X_U) & \xrightarrow{(\mathrm{Ad}(\sigma_i^{-1}),\sigma_i)} & (G, X_U) & \xrightarrow{\mathrm{sh}} & (U, X) \\ \big\downarrow{\scriptstyle (\alpha,\mathrm{id})} & & & & \big\downarrow{\scriptstyle (\alpha_i,\mathrm{id})} \\ (\overline{G}, \alpha^*(X_U)) & \xrightarrow{(\mathrm{id},\omega_i)} & (\overline{G}, (\alpha_i^* X)_{\overline{U}}) & \xrightarrow{\mathrm{sh}} & (\overline{U}, \alpha_i^* X) \end{array}$$

induces a commutative diagram of morphisms of complexes

$$\begin{array}{ccccc} C^\bullet(G, X_U) & \xrightarrow{\mathrm{Ad}(\sigma_i)} & C^\bullet(G, X_U) & \xrightarrow{\mathrm{sh}} & C^\bullet(U, X) \\ \big\downarrow{\scriptstyle \alpha^*} & & & & \big\downarrow{\scriptstyle \alpha_i^*} \\ C^\bullet\big(\overline{G}, (\alpha^*(X_U))\big) & \xrightarrow{\omega_i} & C^\bullet(\overline{G}, (\alpha_i^* X)_{\overline{U}}) & \xrightarrow{\mathrm{sh}} & C^\bullet(\overline{U}, \alpha_i^* X). \end{array}$$

This implies that the following diagram

$$\begin{array}{ccc} C^\bullet(G, X_U) & \xrightarrow{\omega\circ\alpha^*} & \bigoplus_i C^\bullet(\overline{G}, (\alpha_i^* X)_{\overline{U}}) \\ \big\downarrow \text{sh} & & \big\downarrow \text{sh} \\ C^\bullet(U, X) & \xrightarrow{(\alpha_i^*)} & \bigoplus_i C^\bullet(\overline{U}, \alpha_i^* X) \end{array}$$

is commutative up to the homotopy $h = (\alpha_i^* \circ \text{sh} \circ h_{\sigma_i})_i : (\alpha_i^*) \circ \text{sh} \rightsquigarrow \text{sh} \circ \omega \circ \alpha^*$, which induces a quasi-isomorphism (functorial in X)

$$\text{Cone}(\text{sh}, \text{sh} \circ \omega, h) : \text{Cone}(\alpha^*) \longrightarrow \text{Cone}((\alpha_i^*)). \tag{8.1.7.1}$$

Above, h_σ denotes fixed bi-functorial homotopies $h_\sigma(M) : \text{id} \rightsquigarrow \text{Ad}(\sigma)$, *e.g.* those from 4.5.5. As any two choices of $h_\sigma(M)$ are 2-homotopic, it follows from 1.1.7 that the homotopy class of the quasi-isomorphism (8.1.7.1) does not depend on the choice of h_σ.

8.1.7.3. Conjugation

8.1.7.3.1. Fix $g \in G$. Then we have, for each i,

$$g^{-1}\sigma_i = u_i^{-1}\sigma_{g(i)}\alpha(\overline{g}_i), \qquad u_i \in U,\ \overline{g}_i \in \overline{G}.$$

The $\overline{G}$-isomorphisms $\sigma_i : \alpha^* X \xrightarrow{\sim} \alpha_i^* X$ give rise to morphisms of complexes

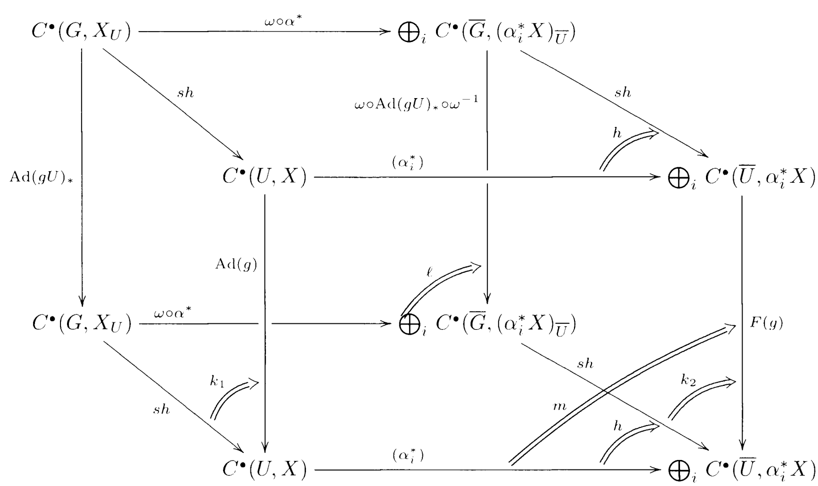
$$F(g)_i : C^\bullet(\overline{U}, \alpha_{g(i)}^* X) \xrightarrow{(\sigma_{g(i)}^{-1})_*} C^\bullet(\overline{U}, \alpha^* X) \xrightarrow{\text{Ad}(\overline{g}_i^{-1})} C^\bullet(\overline{U}, \alpha^* X) \xrightarrow{(\sigma_i)_*} C^\bullet(\overline{U}, \alpha_i^* X)$$

and

$$F(g) = (F(g)_i)_i \ : \ \bigoplus_i C^\bullet(\overline{U}, \alpha_i^* X) \longrightarrow \bigoplus_i C^\bullet(\overline{U}, \alpha_i^* X),$$

functorial in X.

8.1.7.3.2. The faces of the cubic diagram



commute up to the following homotopies:

$$\begin{aligned}
\ell &= 0 \\
h &= ((\alpha_i^* \circ \mathrm{sh}) \star h_{\sigma_i})_i && \text{(as in 8.1.7.2)} \\
m &= -((F(g)_i \circ \alpha_{g(i)}^*) \star h_{u_i})_i && (\text{as } \alpha_i^* \circ \mathrm{Ad}(g) = F(g)_i \circ \alpha_{g(i)}^* \circ \mathrm{Ad}(u_i)) \\
k_1 &= (\mathrm{sh} \circ \mathrm{Ad}(gU)_*) \star h_g && \text{(as in 8.1.6.3)} \\
k_2 &= -((F(g)_i \circ \mathrm{sh}) \star h_{\overline{g}_i})_i && (\text{as } \mathrm{sh} \circ \omega_i \circ \mathrm{Ad}(gU)_* = F(g)_i \circ \mathrm{sh} \circ \omega_{g(i)} \circ \mathrm{Ad}(\overline{g}_i)).
\end{aligned}$$

8.1.7.3.3. Lemma. — *The boundary of the cube in 8.1.7.3.2 is trivialized by a 2-homotopy*

$$(\alpha_i^*) \star k_1 + m \star \mathrm{sh} + F(g) \star h - k_2 \star \alpha^* - h \star (\mathrm{Ad}(gU)_*) \rightsquigarrow 0.$$

Proof. — This can be proved, *e.g.*, by a brute force calculation based on the existence of bifunctorial 2-homotopies $H_{\sigma,\tau}(M)$ from 4.5.5. The details of the unilluminating calculation are omitted. □

8.1.7.3.4. Corollary. — *The following diagram*

$$\begin{array}{ccc}
\mathrm{Cone}(\alpha^*) & \xrightarrow{\mathrm{Cone}(\mathrm{sh},\mathrm{sh}\circ\omega,h)} & \mathrm{Cone}((\alpha_i^*)) \\
\big\downarrow {\scriptstyle \mathrm{Cone}(\mathrm{Ad}(gU)_*,\mathrm{Ad}(gU)_*,0)} & & \big\downarrow {\scriptstyle \mathrm{Cone}(\mathrm{Ad}(g),F(g),m)} \\
\mathrm{Cone}(\alpha^*) & \xrightarrow{\mathrm{Cone}(\mathrm{sh},\mathrm{sh}\circ\omega,h)} & \mathrm{Cone}((\alpha_i^*))
\end{array}$$

is commutative up to homotopy.

Proof. — This follows from 1.1.8 and Lemma 8.1.7.3.3. □

8.1.7.4. Restriction

8.1.7.4.1. Assume that $V \lhd G$, $V \subset U$ is another open normal subgroup of G; put $\overline{V} = \alpha^{-1}(V)$. Fix coset representatives $\tau_j \in G$ of

$$G/V = \bigcup_j \tau_j\, \alpha(\overline{G}/\overline{V}) = \bigcup_j \alpha(\overline{G}/\overline{V})\tau_j^{-1}.$$

Then

$$G = \bigcup_j V\tau_j\, \alpha(\overline{G}) = \bigcup_i U\sigma_i\, \alpha(\overline{G})$$

and for each j we have $U\tau_j\,\alpha(\overline{G}) = U\sigma_i\,\alpha(\overline{G})$ for unique $i = \pi(j)$. In particular,

$$\tau_j = u_{ij}\sigma_i\alpha(\overline{g}_{ij}), \quad u_{ij} \in U,\ \overline{g}_{ij} \in \overline{G} \quad (i = \pi(j)).$$

Set

$$\beta_j : \overline{G} \xrightarrow{\alpha} G \xrightarrow{\mathrm{Ad}(\tau_j)} G$$

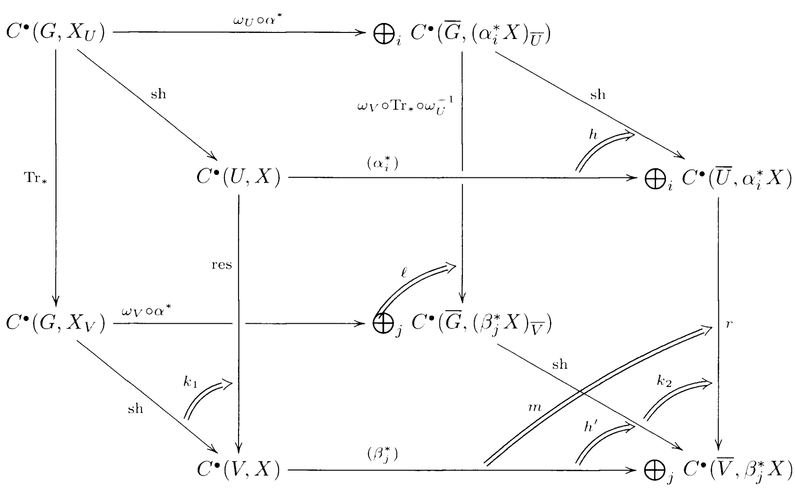
and define a morphism of complexes (functorial in X)

$$r = (r_{ij}) \ : \ \bigoplus_i C^\bullet(\overline{U}, \alpha_i^* X) \longrightarrow \bigoplus_j C^\bullet(\overline{V}, \beta_j^* X)$$

by

$$r_{ij} : C^\bullet(\overline{U}, \alpha_i^* X) \xrightarrow{(\sigma_i^{-1})_*} C^\bullet(\overline{U}, \alpha^* X) \xrightarrow{\mathrm{Ad}(\overline{g}_{ij}^{-1})} C^\bullet(\overline{U}, \alpha^* X)$$
$$\xrightarrow{\mathrm{res}} C^\bullet(\overline{V}, \alpha^* X) \xrightarrow{(\tau_j)_*} C^\bullet(\overline{V}, \beta_j^* X).$$

8.1.7.4.2. The faces of the cubic diagram



commute up to the following homotopies:

$$\begin{aligned}
\ell &= 0 \\
k_1 &= 0 \\
h &= ((\alpha_i^* \circ \mathrm{sh}) \star h_{\sigma_i})_i && \text{(as in 8.1.7.2)} \\
h' &= ((\beta_j^* \circ \mathrm{sh}) \star h_{\tau_j})_j && \text{(as in 8.1.7.2)} \\
m &= ((\beta_j \circ \mathrm{res}) \star h_{u_{ij}})_j && \text{(as } r \circ (\alpha_i^*) = (\beta_j^*) \circ \mathrm{res} \circ \mathrm{Ad}(u_{ij})) \\
k_2 &= -((r \circ \mathrm{sh}) \star h_{\overline{g}_{ij}})_j && \text{(as } \mathrm{sh} \circ \omega_{V,j} \circ \mathrm{Tr}_* = r \circ \mathrm{sh} \circ \omega_{U,i} \circ \mathrm{Ad}(\overline{g}_{ij}))
\end{aligned}$$

(above, $i = \pi(j)$).

8.1.7.4.3. Lemma. — *The boundary of the cube in 8.1.7.4.2 is trivialized by a 2-homotopy*

$$m \star \mathrm{sh} + r \star h - k_2 \star \alpha^* - h' \star \mathrm{Tr}_* \rightsquigarrow 0.$$

Proof. — Again, this can be proved by an explicit calculation, the details of which are omitted. □

8.1.7.4.4. Corollary. — *The following diagram*

$$\begin{array}{ccc}
\mathrm{Cone}(\omega_U \circ \alpha^*) & \xrightarrow{\mathrm{Cone}(\mathrm{sh}, sh, h)} & \mathrm{Cone}((\alpha_i^*)) \\
\Big\downarrow {\scriptstyle \mathrm{Cone}(\mathrm{Tr}_*, \omega_V \circ \mathrm{Tr}_* \circ \omega_U^{-1}, 0)} & & \Big\downarrow {\scriptstyle \mathrm{Cone}(\mathrm{res}, r, m)} \\
\mathrm{Cone}(\omega_V \circ \alpha^*) & \xrightarrow{\mathrm{Cone}(\mathrm{sh}, \mathrm{sh}, h')} & \mathrm{Cone}((\beta_j^*))
\end{array}$$

is commutative up to homotopy.

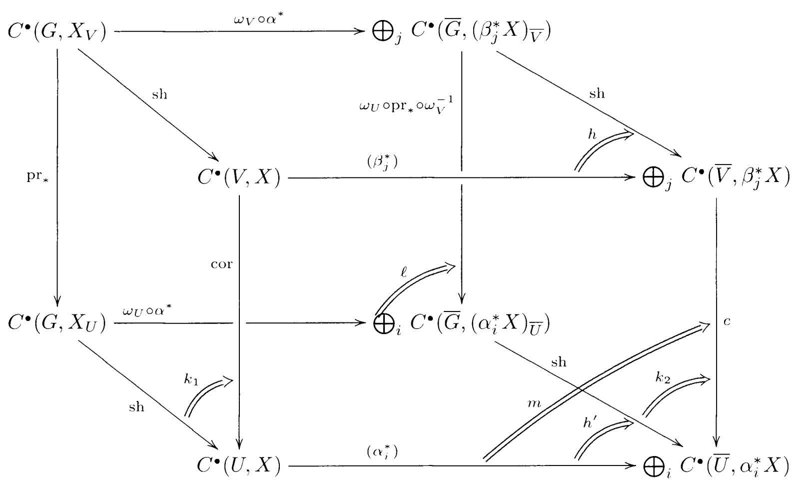
Proof. — This follows from 1.1.8 and Lemma 8.1.7.4.3. □

8.1.7.5. Corestriction

In the notation of 8.1.7.4.1, there is a similar construction of 'corestriction' morphisms

$$c : \bigoplus_j C^\bullet(\overline{V}, \beta_j^* X) \longrightarrow \bigoplus_i C^\bullet(\overline{U}, \alpha_i^* X),$$

yielding the cubic diagram



and the square

$$\begin{array}{ccc}
\mathrm{Cone}(\omega_V \circ \alpha^*) & \xrightarrow{\mathrm{Cone}(\mathrm{sh},\mathrm{sh},h)} & \mathrm{Cone}((\beta_j^*)) \\
\big\downarrow {\scriptstyle \mathrm{Cone}(\mathrm{pr}_*, \omega_U \circ \mathrm{pr}_* \circ \omega_V^{-1}, 0)} & & \big\downarrow {\scriptstyle \mathrm{Cone}(\mathrm{cor}, c, m)} \\
\mathrm{Cone}(\omega_U \circ \alpha^*) & \xrightarrow{\mathrm{Cone}(\mathrm{sh},\mathrm{sh},h)} & \mathrm{Cone}((\alpha_i^*)),
\end{array}$$

commutative up to homotopy. The details are omitted.

8.1.7.6. The constructions in 8.1.7.1–8.1.7.5 have the following useful variant. As the action of σ_i^{-1} defines an isomorphism of $\overline{G}$-modules

$$\sigma_i^{-1} : \alpha_i^* X \xrightarrow{\sim} \alpha^* X,$$

the modified map $\widetilde{\omega} = (\sigma_i^{-1} \circ \omega_i)$ yields a $\overline{G}$-isomorphism

$$\widetilde{\omega} : \alpha^*(X_U) \xrightarrow{\sim} \bigoplus_i (\alpha^* X)_{\overline{U}}.$$

If we use consistently $\widetilde{\omega}$ instead of ω, then the map $F(g)$ has to be replaced by $\widetilde{F}(g) = \big(\widetilde{F}(g)_i\big)_i$, where

$$\widetilde{F}(g)_i : C^\bullet(\overline{U}, \alpha^* X)_{g(i)} \xrightarrow{\mathrm{Ad}(\overline{g}_i^{-1})} C^\bullet(\overline{U}, \alpha^* X)_i,$$

where each subscript refers to the corresponding summand in the direct sum

$$\bigoplus_i \alpha_i^* X \xrightarrow{\sim} \bigoplus_i \alpha^* X.$$

Similarly, the map r_{ij} has to be replaced by

$$\widetilde{r}_{ij} : C^\bullet(\overline{U}, \alpha^* X)_i \xrightarrow{\mathrm{Ad}(\overline{g}_{ij}^{-1})} C^\bullet(\overline{U}, \alpha^* X)_i \xrightarrow{\mathrm{res}} C^\bullet(\overline{V}, \alpha^* X)_j.$$

If α is injective, then the morphisms of pairs

$$(\alpha_i, \sigma_i^{-1}) : (\sigma_i \alpha(\overline{U}) \sigma_i^{-1}, X) \longrightarrow (\overline{U}, \alpha^* X)$$

induces an isomorphism of complexes

$$C^\bullet(\sigma_i \alpha(\overline{U})\sigma_i^{-1}, X) \xrightarrow{\sim} C^\bullet(\overline{U}, \alpha^* X).$$

The latter complex is isomorphic to $C^\bullet(\overline{U}, \alpha_i^* X)$, *via* $\sigma_i : \alpha^* X \xrightarrow{\sim} \alpha_i^* X$.

8.2. Shapiro's Lemma for ind-admissible modules

Recall that R is assumed to have finite residue field of characteristic p. Let $U \subset G$ be as in 8.1.

8.2.1. For $M \in \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$ we define $R[G]$-modules

$$M_U = M \otimes_R R[G/U], \quad {}_U M = \mathrm{Hom}_R(R[G/U], M)$$

with the G-action given by the formulas in 8.1.3. These modules have the following properties:

(i) If the action of G on M is discrete, then M_U (resp., ${}_U M$) coincides with the corresponding object defined in 8.1.3.

(ii) The formulas in 8.1.3 define an isomorphism $M_U \xrightarrow{\sim} {}_U M$, functorial in M.

(iii) The functors $M \mapsto M_U$ and $M \mapsto {}_U M$ commute with arbitrary direct and inverse limits.

In particular, if M is of finite type over R, then the canonical maps

$$M_U/\mathfrak{m}^n M_U \xrightarrow{\sim} (M/\mathfrak{m}^n M)_U, \quad M_U \xrightarrow{\sim} \varprojlim_n (M/\mathfrak{m}^n M)_U$$

are isomorphisms (and similarly for ${}_U M$).

These observations imply that $M_U, {}_U M \in \left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right)$.

8.2.2. Writing $M = \varinjlim_{M_\alpha \in \mathcal{S}(M)} M_\alpha$ and each M_α as $M_\alpha = \varprojlim_n M_\alpha/\mathfrak{m}^n M_\alpha$, Shapiro's Lemma applied to each $M_\alpha/\mathfrak{m}^n M_\alpha$ (and the fact that $M_\alpha/\mathfrak{m}^n M_\alpha$ is a surjective projective system) gives quasi-isomorphisms

$$\begin{aligned} C^\bullet_{\mathrm{cont}}(G, (M_\alpha)_U) &\xrightarrow{\sim} \varprojlim C^\bullet_{\mathrm{cont}}(G, (M_\alpha/\mathfrak{m}^n M_\alpha)_U) \\ &\xrightarrow{\sim} \varprojlim C^\bullet_{\mathrm{cont}}(G, \mathrm{Ind}_U^G(M_\alpha/\mathfrak{m}^n M_\alpha)) \xrightarrow{\mathrm{sh}} \varprojlim C^\bullet_{\mathrm{cont}}(U, M_\alpha/\mathfrak{m}^n M_\alpha) = C^\bullet_{\mathrm{cont}}(U, M_\alpha) \end{aligned}$$

and

$$C^\bullet_{\mathrm{cont}}(G, M_U) \xrightarrow{\sim} \varinjlim_{\alpha} C^\bullet_{\mathrm{cont}}(G, (M_\alpha)_U) \xrightarrow{\mathrm{sh}} \varinjlim_{\alpha} C^\bullet_{\mathrm{cont}}(U, M_\alpha) = C^\bullet_{\mathrm{cont}}(U, M).$$

The same argument applies to ${}_UM$.

8.2.3. The definitions above extend in an obvious way to complexes $M^\bullet$ of ind-admissible $R[G]$-modules. If at least one of the following conditions is satisfied:

(A) $M^\bullet$ is bounded below;

(B) $\mathrm{cd}_p(G) < \infty$,

then both morphisms

$$C^\bullet_{\mathrm{cont}}(G, M^\bullet_U) \longrightarrow C^\bullet_{\mathrm{cont}}(U, M^\bullet) \longleftarrow C^\bullet_{\mathrm{cont}}(G, {}_UM^\bullet)$$

are quasi-isomorphisms (using the spectral sequence (3.5.3.1)).

8.2.4. Put $R_U = R[G/U]$ and denote by

$$\iota : R_U \longrightarrow R_U$$

the R-linear involution induced by the map $g \mapsto g^{-1}$ on G/U. Both M_U and ${}_UM$ are (left) $R_U[G]$-modules, with the action of R_U given by the Ad-action of G/U described in 8.1.6.3.

If G/U is *abelian*, then the action of $x \in R_U$ on $M_U = M \otimes_R R_U$ (resp., on ${}_UM = \mathrm{Hom}_R(R_U, M)$) is given by $\mathrm{id} \otimes \iota(x)$ (resp., $\mathrm{Hom}(x, \mathrm{id})$).

8.2.5. Lemma. — *Assume that G/U is abelian. Then*

(i) *R_U is an equidimensional (of dimension d) semilocal complete Noetherian ring.*

(ii) *If M is an (ind-)admissible $R[G]$-module, then ${}_UM$, M_U are (ind-)admissible $R_U[G]$-modules.*

(iii) *If M is of finite (resp., co-finite) type over R, then ${}_UM$, M_U are of finite (resp., co-finite) type over R_U.*

Proof. — Everything follows from the fact that R_U is a free R-module of finite type. □

8.2.6. Corollary. — *If G/U is abelian, then both functors $M \mapsto M_U$, $M \mapsto {}_UM$ map $\left({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\right)_{R-*}$ to $\left({}^{\mathrm{ad}}_{R_U[G]}\mathrm{Mod}\right)_{R_U-*}$ (and similarly for* ind-ad*), with $* = ft, coft$. Under the canonical quasi-isomorphisms*

$$C^\bullet_{\mathrm{cont}}(G, M_U) \xrightarrow{\mathrm{sh}} C^\bullet_{\mathrm{cont}}(U, M) \xleftarrow{\mathrm{sh}} C^\bullet_{\mathrm{cont}}(G, {}_UM)$$

the action of R_U on $M_U, {}_UM$ corresponds to the homotopy action of G/U on $C^\bullet_{\mathrm{cont}}(U, M)$.

8.3. Infinite extensions

8.3.1. Let $H \lhd G$ be a closed normal subgroup of G. Put $\Gamma = G/H$,

$$\mathscr{U} = \{U \subset G \mid U \text{ open subgroup, } U \supset H\}$$
$$\overline{R} = R[\![\Gamma]\!] = \varprojlim_{U \in \mathscr{U}} R[G/U].$$

For every $M \in \left({}_{R[G]}^{\text{ind-ad}}\text{Mod}\right)$, the $R_U[G]$-modules M_U (resp., ${}_UM$) for variable $U \in \mathscr{U}$ form a projective (resp., an inductive) system with transition maps induced by the projections $\text{pr} : R_V \to R_U$ (for $V \subset U$, $U, V \in \mathscr{U}$). Denote by

$$\mathscr{F}_\Gamma(M) = \varprojlim_{U \in \mathscr{U}} M_U, \quad F_\Gamma(M) = \varinjlim_{U \in \mathscr{U}} {}_UM$$

the corresponding limits; they are both (left) $\overline{R}[G]$-modules. We define $\mathscr{F}_\Gamma(M^\bullet)$, $F_\Gamma(M^\bullet)$ for a complex $M^\bullet$ of ind-admissible $R[G]$-modules by the same formulas (termwise).

8.3.2. Lemma

(i) *If $M^\bullet$ is a complex of ind-admissible $R[G]$-modules, so is $F_\Gamma(M^\bullet)$ and the canonical map*

$$\varinjlim_U C^\bullet_{\text{cont}}(G, {}_UM^\bullet) \longrightarrow C^\bullet_{\text{cont}}(G, F_\Gamma(M^\bullet))$$

is an isomorphism of complexes.

(ii) *If M is contained in $\left({}_{R[G]}^{\text{ind-ad}}\text{Mod}\right)_{\{\mathfrak{m}\}}$, so is $F_\Gamma(M)$ and the composite morphism*

$$\varinjlim_U C^\bullet_{\text{cont}}(G, {}_UM) \xrightarrow{\text{sh}} \varinjlim_U C^\bullet_{\text{cont}}(U, M) \xrightarrow{\text{res}} C^\bullet_{\text{cont}}(H, M)$$

is an quasi-isomorphism. The same is true for complexes $M^\bullet$ of such modules, provided they satisfy 8.2.3(A) or (B).

Proof. — It is enough to consider only the case $M^\bullet = M$.

(i) $F_\Gamma(M)$ is ind-admissible, since each ${}_UM$ is; then apply 3.4.1.5.

(ii) $F_\Gamma(M)$ is supported at $\{\mathfrak{m}\}$, since each ${}_UM$ is; in particular, both M and $F_\Gamma(M)$ are discrete G-modules. The map res is an isomorphism by [**Se2**, §I.2.2, Prop. 8] and sh is a quasi-isomorphism. □

8.3.3. Corollary. — *If $M^\bullet$ is a complex in $\left({}_{R[G]}^{\text{ind-ad}}\text{Mod}\right)_{\{\mathfrak{m}\}}$ satisfying 8.2.3(A) or (B), then there is a canonical quasi-isomorphism*

$$C^\bullet_{\text{cont}}(G, F_\Gamma(M^\bullet)) \longrightarrow C^\bullet_{\text{cont}}(H, M^\bullet)$$

such that the $\overline{R}$-action on $F_\Gamma(M^\bullet)$ corresponds to the homotopy action of $G/H = \Gamma$ on the R.H.S.

8.3.4. For any $M \in \left({}^{\text{ind-ad}}_{R[G]}\text{Mod}\right)$ the complexes $C^\bullet_{\text{cont}}(G, M_U)$ form a projective system indexed by $U \in \mathscr{U}$, with surjective transition maps. The projective limit

$$\varprojlim_U C^\bullet_{\text{cont}}(G, M_U)$$

is a complex of $\overline{R}$-modules; denote by

$$\mathbf{R}\Gamma_{\text{Iw}}(G, H; M) \in D({}_{\overline{R}}\text{Mod})$$

the corresponding object of the derived category and by

$$H^i_{\text{Iw}}(G, H; M) = H^i(\mathbf{R}\Gamma_{\text{Iw}}(G, H; M))$$

its cohomology. The same notation will be used for complexes $M^\bullet$ of ind-admissible $R[G]$-modules (the subscript "Iw" stands for "Iwasawa" – this notation is due to Fontaine).

8.3.5. Proposition. — *Let $M \in \left({}^{\text{ind-ad}}_{R[G]}\text{Mod}\right)$. Then*

(i) *There is a spectral sequence*

$$E_2^{i,j} = \varprojlim_{U,\text{cor}}{}^{(i)} \left(H^j_{\text{cont}}(U, M)\right) \Longrightarrow H^{i+j}_{\text{Iw}}(G, H; M).$$

(ii) *If M is of finite type over R and if G satisfies (F), then $E_2^{i,j} = 0$ for $i \neq 0$ and*

$$H^j_{\text{Iw}}(G, H; M) = \varprojlim_{U,\text{cor}} H^j_{\text{cont}}(U, M).$$

(iii) *Assume that M is of finite type over R, $\mathscr{U}$ contains a cofinal chain $U_1 \supset U_2 \supset U_3 \supset \cdots$ and the pro-finite order of Γ is divisible by p^∞. Then $H^0_{\text{Iw}}(G, H; M) = 0$.*

Proof

(i) Consider the two hyper-cohomology spectral sequences for the functor $\varprojlim$ and the projective system $C^\bullet_{\text{cont}}(G, M_U)$:

$${}^{I}E_1^{i,j} = \varprojlim_U{}^{(j)} C^i_{\text{cont}}(G, M_U) \Longrightarrow H^{i+j} \Longleftarrow {}^{\text{II}}E_2^{i,j} = \varprojlim_U{}^{(i)} \left(H^j_{\text{cont}}(G, M_U)\right).$$

For each i, the projective system $C^i_{\text{cont}}(G, M_U)$ is "weakly flabby" in the sense of [**Je**, Lemma 1.3 (ii)], as

$$\varprojlim_U C^i_{\text{cont}}(G, M_U) \longrightarrow C^i_{\text{cont}}(G, M_V)$$

is surjective for every $V \in \mathscr{U}$. This implies that ${}^{I}E_1^{i,j} = 0$ for $j \neq 0$, hence $H^n = H^n(i \mapsto {}^{I}E_1^{i,0}) = H^n_{\text{Iw}}(G, H; M)$. By Lemma 8.1.5, the transition maps in

$${}^{\text{II}}E_2^{i,j} \xrightarrow{\sim} \varprojlim_{U \in \mathscr{U}}{}^{(i)} \left(H^j_{\text{cont}}(U, M)\right)$$

are given by the corestriction, as claimed.

(ii) The same argument as in (i) yields a spectral sequence

$$E_2^{i,j} = \varprojlim_{U,n}{}^{(i)} \left(H^j_{\mathrm{cont}}(U, M/\mathfrak{m}^n M)\right) \Longrightarrow H^{i+j}_{\mathrm{Iw}}(G, H; M).$$

As G satisfies (F), the projective system $H^j_{\mathrm{cont}}(U, M/\mathfrak{m}^n M)$ (indexed by $\mathscr{U} \times \mathbf{N}$) consists of R-modules of finite length, hence $E_2^{i,j} = 0$ for $i \neq 0$ ([**Je**, Cor. 7.2]). It follows that

$$H^j_{\mathrm{Iw}}(G, H; M) \xrightarrow{\sim} E_2^{0,j} = \varprojlim_U \varprojlim_n \left(H^j_{\mathrm{cont}}(U, M/\mathfrak{m}^n M)\right) \xrightarrow{\sim} \varprojlim_U H^j_{\mathrm{cont}}(U, M)$$

(the last isomorphism by Lemma 4.2.2).

(iii) As M is of finite type over R, the sequence of invariants

$$M^{U_1} \subseteq M^{U_2} \subseteq \cdots$$

stabilizes: $M^{U_n} = N$ for all $n \geqslant n_0$. This implies that

$$H^0_{\mathrm{Iw}}(G, H; M) = \varprojlim_{n,\mathrm{cor}} H^0_{\mathrm{cont}}(U_n, M) = \varprojlim_{n \geqslant n_0} N,$$

with the transition maps given by the multiplication by $[U_n : U_{n'}]$ $(n' \geqslant n \geqslant n_0)$. For fixed $n \geqslant n_0$, the power of p dividing $[U_n : U_{n'}]$ tends to infinity as $n' \to \infty$, hence

$$\mathrm{Im}\left(H^0_{\mathrm{Iw}}(G, H; M) \longrightarrow H^0_{\mathrm{cont}}(U_n, M)\right) \subseteq \bigcap_{n' \geqslant n} [U_n : U_{n'}] \cdot N = 0,$$

proving the claim. □

8.3.6. In 8.3.5 (i) we can replace M by a complex $M^\bullet$ of ind-admissible $R[G]$-modules, provided 8.2.3 (A) or (B) holds. In 8.3.5 (ii) we have to assume, in addition, that $M^\bullet$ has cohomology groups of finite type over R (cf. proof of Proposition 4.2.5).

This implies that $\mathbf{R}\Gamma_{\mathrm{Iw}}(G, H; -)$ induces an exact functor

$$\mathbf{R}\Gamma_{\mathrm{Iw}}(G, H; -) : D^*\left({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\right) \longrightarrow D^*\left({}_{\overline{R}}\mathrm{Mod}\right)$$

for $* = +$ (resp., for $* = +, \varnothing$, provided $\mathrm{cd}_p(G) < \infty$).

8.4. Infinite Abelian extensions

We retain the notation and assumptions of 8.3.

8.4.1. In practice one is usually interested in the case when $\Gamma = G/H$ is a p-adic Lie group. In this paper we consider only the case of *abelian* Γ. According to Lemma 4.1.4, cohomological invariants do not change if Γ (resp., G) is replaced by its pro-p-Sylow subgroup $\Gamma(p)$ (resp., by the inverse image of $\Gamma(p) \subset G/H$ in G). Without loss of generality we shall, therefore, assume that

$$\Gamma = \Gamma_0 \times \Delta,$$

where Δ is a finite abelian group and Γ_0 is isomorphic to $\mathbf{Z}_p^r$ for some $r \geqslant 1$. In this case

$$\overline{R} = R[\![\Gamma]\!] = \overline{R}_0 \otimes_R R[\Delta], \quad \overline{R}_0 = R[\![\Gamma_0]\!].$$

A choice of an isomorphism $\Gamma_0 \xrightarrow{\sim} \mathbf{Z}_p^r$ (*i.e.*, a choice of a $\mathbf{Z}_p$-basis $\gamma_1, \ldots, \gamma_r$ of Γ_0) gives an isomorphism of rings

$$\overline{R}_0 \xrightarrow{\sim} R[\![X_1, \ldots, X_r]\!], \quad \gamma_i \mapsto 1 + X_i.$$

In particular, $\overline{R}$ is an equidimensional semilocal complete Noetherian ring, of dimension $d + r$. Let $\overline{\mathfrak{m}} \subset \overline{R}$ be the radical of $\overline{R}$.

8.4.2. Denote by

$$\chi_\Gamma : G \twoheadrightarrow \Gamma \hookrightarrow R[\Gamma]^* \quad (\subset \overline{R}^*)$$

the tautological one-dimensional representation of G over $R[\Gamma]$.

For every $(R[\Gamma])[G]$-module M and $n \in \mathbf{Z}$ we construct new $(R[\Gamma])[G]$-modules $M < n >$ resp., M^ι as follows: as an $R[\Gamma]$-module, $M < n >$ coincides with M, but the action of G is given by

$$g_{M<n>} = \chi_\Gamma(g)^n g_M \qquad (g \in G).$$

M^ι coincides with M as an $R[G]$-module, but the action of $R[\Gamma]$ is given by

$$x_{M^\iota} = \iota(x)_M \qquad (x \in R[\Gamma]).$$

A more functorial definition of M^ι is $M \otimes_{R[\Gamma],\iota} R[\Gamma]$, as in 6.6.4. With this notation (which applies, in particular, to $\overline{R}[G]$-modules), we have

$$M < n >^\iota \xrightarrow{\sim} M^\iota < -n > .$$

8.4.3. Injective hulls. — We first relate $I_{\overline{R}}$ to I_R.

8.4.3.1. Lemma. — *As an $\overline{R}[G]$-module, $F_\Gamma(I_R) < 1 >$ is isomorphic to the Pontrjagin dual of $\overline{R}$ (the action of G on I_R and $\overline{R}$ being trivial).*

Proof. — As in 2.9, fix an isomorphism of R-modules

$$I_R \xrightarrow{\sim} \widehat{R} = \varinjlim_n \mathrm{Hom}_{\mathbf{Z}_p}(R/\mathfrak{m}^n R, \mathbf{Q}_p/\mathbf{Z}_p).$$

This induces an isomorphism of $R[G/U]$-modules

$$\begin{aligned}&\mathrm{Hom}_R(R[G/U], I_R)\\ &\quad \xrightarrow{\sim} \varinjlim_n \mathrm{Hom}_{R/\mathfrak{m}^n R}(R/\mathfrak{m}^n R[G/U], \mathrm{Hom}_{\mathbf{Z}_p}(R/\mathfrak{m}^n R, \mathbf{Q}_p/\mathbf{Z}_p))\\ &\qquad \xrightarrow{\sim} \varinjlim_n \mathrm{Hom}_{\mathbf{Z}_p}(R/\mathfrak{m}^n R[G/U], \mathbf{Q}_p/\mathbf{Z}_p),\end{aligned}$$

hence an isomorphism of $\overline{R}$-modules between

$$F_\Gamma(I_R) = \varinjlim_{U \in \mathscr{U}} \mathrm{Hom}_R(R[G/U], I_R)$$

(cf. 8.2.4) and

$$\varinjlim_{U,n} \mathrm{Hom}_{\mathbf{Z}_p}(R/\mathfrak{m}^n R[G/U], \mathbf{Q}_p/\mathbf{Z}_p),$$

which is nothing but the Pontrjagin dual of

$$\varprojlim_{U,n} R/\mathfrak{m}^n R[G/U] = \overline{R}.$$

The action of $g \in G$ on $F_\Gamma(I_R)$ is given by

$$(g * f)(x) = g\big(f(g^{-1}(x))\big) = f(g^{-1}(x)) = f(\chi_\Gamma(g)^{-1}x) = \chi_\Gamma(g)^{-1} f(x),$$

hence G acts trivially on $F_\Gamma(I_R) < 1 >$. □

8.4.3.2. Corollary. — *The $\overline{R}$-module $\overline{I} = I_{\overline{R}} := F_\Gamma(I_R) < 1 >$ is an injective hull of $\overline{R}/\overline{\mathfrak{m}}$ over $\overline{R}$.*

8.4.4. The functor $\mathscr{F}_\Gamma$

8.4.4.1. Proposition. — *Let $M \in \big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)$. Then*

(i) *If M is of finite type over R, then there are canonical isomorphisms of $\overline{R}[G]$-modules*

$$\mathscr{F}_\Gamma(M) \xrightarrow{\sim} (M \otimes_R \overline{R}) < -1 >,$$
$$\mathscr{F}_\Gamma(M)^\iota \xrightarrow{\sim} (M \otimes_R \overline{R}^\iota) < -1 > \xrightarrow{\sim} (M \otimes_R \overline{R}) < 1 >,$$

functorial in M. In particular, $\mathscr{F}_\Gamma(M)$ is of finite type over $\overline{R}$.

(ii) *If $M \in \big({}^{\text{ind-ad}}_{R[G]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$, then $F_\Gamma(M) \in \big({}^{\text{ind-ad}}_{\overline{R}[G]}\mathrm{Mod}\big)_{\{\overline{\mathfrak{m}}\}}$.*

(iii) *If M is of co-finite type over R, then $F_\Gamma(M)$ is of co-finite type over $\overline{R}$.*

Proof

(i) The canonical map of $\overline{R}$-modules

$$N \otimes_R \overline{R} = N \otimes_R \varprojlim_U R_U \longrightarrow \varprojlim_U (N \otimes_R R_U) \qquad (8.4.4.1.1)$$

is an isomorphism for every R-module N of finite type. For $N = M$, the natural action of $\overline{R}[G]$ on $\mathscr{F}_\Gamma(M) = \varprojlim_U (M \otimes_R R_U)$ is the following (cf. 8.1.3 and 8.2.4):

$$x \in \overline{R} \text{ acts by } \mathrm{id} \otimes \iota(x)$$
$$g \in G \text{ acts by } g \otimes \chi_\Gamma(g).$$

this implies that (8.4.4.1.1) and ι induce the following isomorphisms of $\overline{R}[G]$-modules:

$$\mathscr{F}_\Gamma(M) = (M \otimes_R \overline{R}^\iota) < 1 > \xrightarrow{\mathrm{id}\otimes\iota} (M \otimes_R \overline{R}) < -1 > .$$

The statement about $\mathscr{F}_\Gamma(M)^\iota$ is proved in the same way.

(ii) This follows from Lemma 8.3.2.

(iii) It is sufficient to treat the case $\Gamma = \Gamma_0$, when $\overline{R} = \overline{R}_0$ is local (cf. proof of Lemma 8.2.5). We must show that $\dim_k(F_\Gamma(M)[\overline{\mathfrak{m}}]) < \infty$ (here $k = R/\mathfrak{m} = \overline{R}/\overline{\mathfrak{m}}$). As $F_\Gamma(M[\mathfrak{m}])[\overline{\mathfrak{m}}] = F_\Gamma(M)[\overline{\mathfrak{m}}]$, we can assume that $M = M[\mathfrak{m}]$ and hence, by dévissage, that $M = k$. In this case $\dim_k(F_\Gamma(k)[\overline{\mathfrak{m}}]) = 1$ by Lemma 8.4.3.1 applied to $R = k$. $\square$

8.4.4.2. Proposition. — *For every $M \in \big({}^{\text{ind-ad}}_{R[G]}\text{Mod}\big)_{R\text{-}ft}$, the canonical morphism of complexes*

$$C^\bullet_{\text{cont}}(G, \mathscr{F}_\Gamma(M)) \xrightarrow{\sim} \varprojlim_U C^\bullet_{\text{cont}}(G, M_U)$$

is an isomorphism, hence inducing an isomorphism

$$\mathbf{R}\Gamma_{\text{cont}}(G, \mathscr{F}_\Gamma(M)) \xrightarrow{\sim} \mathbf{R}\Gamma_{\text{Iw}}(G, H; M)$$

in $D^+({}_{\overline{R}}\text{Mod})$.

Proof. — We must show that the two projective limit topologies on

$$\mathscr{F}_\Gamma(M) \xrightarrow{\sim} \varprojlim_n \mathscr{F}_\Gamma(M)/\overline{\mathfrak{m}}^n \mathscr{F}_\Gamma(M) \xrightarrow{\sim} \varprojlim_{U,n} M_U/\mathfrak{m}^n M_U$$

coincide (the first (resp., second) isomorphism follows from the fact that $\mathscr{F}_\Gamma(M)$ is $\overline{\mathfrak{m}}$-adically (resp., $\mathfrak{m}$-adically) complete). We can assume that $\Gamma = \Gamma_0$, in which case $\overline{R} = \overline{R}_0 \xrightarrow{\sim} R[\gamma_1 - 1, \ldots, \gamma_r - 1]$ is local, with $\overline{\mathfrak{m}} = \mathfrak{m}R + (\gamma_1 - 1, \ldots, \gamma_r - 1)$. Put $J_U = \operatorname{Ker}(\overline{R} \to R[G/U])$; then

$$M_U/\mathfrak{m}^n M_U = \mathscr{F}_\Gamma(M)/(J_U + \mathfrak{m}^n \overline{R})\mathscr{F}_\Gamma(M)$$

and the claim follows from the fact (used in the proof of the isomorphism $\overline{R} \xrightarrow{\sim} R[\gamma_1 - 1, \ldots, \gamma_r - 1]$) that the systems of ideals $\{\overline{\mathfrak{m}}^n\}_{n\in\mathbf{N}}$ and $\{J_U + \mathfrak{m}^n\overline{R}\}_{(U,n)\in\mathscr{U}\times\mathbf{N}}$ of $\overline{R}$ are cofinal in each other. $\square$

8.4.4.3. It follows from Lemma 8.3.2 and Proposition 8.4.4.1 that the functors F_Γ and $\mathscr{F}_\Gamma$ derive (for each $* = \varnothing, +, -, b$) trivially to exact functors

$$F_\Gamma : D^*\big(\big({}^{\text{ad}}_{R[G]}\text{Mod}\big)_{R\text{-}coft}\big) \longrightarrow D^*\big(\big({}^{\text{ad}}_{\overline{R}[G]}\text{Mod}\big)_{\overline{R}\text{-}coft}\big)$$

$$\mathscr{F}_\Gamma : D^*\big(\big({}^{\text{ad}}_{R[G]}\text{Mod}\big)_{R\text{-}ft}\big) \longrightarrow D^*\big(\big({}^{\text{ad}}_{\overline{R}[G]}\text{Mod}\big)_{\overline{R}\text{-}ft}\big),$$

hence, using Proposition 3.2.8, to functors

$$F_\Gamma : D^*_{R\text{-}coft}\big({}^{\text{ad}}_{R[G]}\text{Mod}\big) \approx D^*\big(\big({}^{\text{ad}}_{R[G]}\text{Mod}\big)_{R\text{-}coft}\big) \longrightarrow D^*_{\overline{R}\text{-}coft}\big({}^{\text{ad}}_{\overline{R}[G]}\text{Mod}\big) \qquad (* = +, b)$$

$$\mathscr{F}_\Gamma : D^*_{R\text{-}ft}\big({}^{\text{ad}}_{R[G]}\text{Mod}\big) \approx D^*\big(\big({}^{\text{ad}}_{R[G]}\text{Mod}\big)_{R\text{-}ft}\big) \longrightarrow D^*_{\overline{R}\text{-}ft}\big({}^{\text{ad}}_{\overline{R}[G]}\text{Mod}\big) \qquad (* = -, b).$$

Proposition 8.4.4.2 implies that there is a canonical isomorphism in $D^+({}_{\overline{R}}\text{Mod})$

$$\mathbf{R}\Gamma_{\text{cont}}(G, \mathscr{F}_\Gamma(M)) \xrightarrow{\sim} \mathbf{R}\Gamma_{\text{Iw}}(G, H; M), \quad \big(M \in D^b_{R\text{-}ft}\big({}^{\text{ad}}_{R[G]}\text{Mod}\big)\big).$$

8.4.5. Relating the functors F_Γ and $\mathscr{F}_\Gamma$. — Our next goal is to investigate the relationship between the functors $F_\Gamma, \mathscr{F}_\Gamma$ and the duality diagrams 2.8.2 over the rings R and $\overline{R}$. We denote the functors $(D_R, \mathscr{D}_R, \Phi_R)$ (resp., $(D_{\overline{R}}, \mathscr{D}_{\overline{R}}, \Phi_{\overline{R}})$) appearing in these diagrams by $(D, \mathscr{D}, \Phi)$ (resp., $(\overline{D}, \overline{\mathscr{D}}, \overline{\Phi})$).

8.4.5.1. Lemma. — *For every $M \in ({}_{R[G]}^{\mathrm{ad}}\mathrm{Mod})_{R\text{-}coft}$ there are canonical isomorphisms of $\overline{R}[G]$-modules*

$$F_\Gamma(M) \xrightarrow{\sim} \mathrm{Hom}_{\overline{R}}(D(M) \otimes_R \overline{R}, \overline{I}) < -1 > \xrightarrow{\sim} \overline{D}((D(M) \otimes_R \overline{R}) < 1 >)$$
$$\xrightarrow{\sim} \overline{D}(\mathscr{F}_\Gamma(D(M))^\iota)$$
$$\overline{D} \circ F_\Gamma(M) \xrightarrow{\sim} \iota \circ \mathscr{F}_\Gamma \circ D(M).$$

Proof. — The first isomorphism is given by

$$F_\Gamma(M) \xrightarrow{\sim} \varinjlim_U \mathrm{Hom}_R(R[G/U], \mathrm{Hom}_R(D(M), I))$$
$$\xrightarrow{\sim} \varinjlim_U \mathrm{Hom}_R(R[G/U] \otimes_R D(M), I) \xrightarrow{\sim} \varinjlim_U \mathrm{Hom}_R(D(M), \mathrm{Hom}_R(R[G/U], I))$$
$$\xrightarrow{\sim} \mathrm{Hom}_R(D(M), \varinjlim_U \mathrm{Hom}_R(R[G/U], I)) = \mathrm{Hom}_R(D(M), \overline{I} < -1 >)$$
$$\xrightarrow{\sim} \mathrm{Hom}_{\overline{R}}(D(M) \otimes_R \overline{R}, \overline{I}) < -1 >,$$

using Lemma 8.4.3.1 (and the fact that the direct limit can be interchanged with Hom_R, since $D(M)$ is of finite type over R). The second isomorphism follows by applying $\overline{D}$ and the isomorphism $\varepsilon : \mathrm{id} \xrightarrow{\sim} \overline{D} \circ \overline{D}$.

8.4.5.2. Fix γ_i $(i = 1, \ldots, r)$ as in 8.4.1 and put $X_i = \gamma_i - 1 \in \overline{R}_0$. Fix a system of parameters $x_1, \ldots, x_d$ of R; then $x_1, \ldots, x_d, X_1, \ldots, X_r$ form a system of parameters of $\overline{R}_0$. Let $C^\bullet_R(\Gamma_0)$ be the following complex in degrees $[0, r]$ (and differentials as in 2.4.3):

$$\Big[\overline{R}_0 \longrightarrow \bigoplus_i (\overline{R}_0)_{X_i} \longrightarrow \bigoplus_{i<j} (\overline{R}_0)_{X_i X_j} \longrightarrow \cdots \longrightarrow (\overline{R}_0)_{X_1 \cdots X_r}\Big];$$

put $C^\bullet_R(\Gamma) = C^\bullet_R(\Gamma_0) \otimes_R R[\Delta]$. □

8.4.5.3. Lemma

(i) *$H^i(C^\bullet_R(\Gamma_0)) = 0$ for $i \neq r$.*

(ii) *$H^r(C^\bullet_R(\Gamma_0))$ is a free R-module.*

(iii) *$C^\bullet(\overline{R}_0, (x, X))$ is isomorphic to $C^\bullet(R, x) \otimes_R C^\bullet_R(\Gamma_0)$.*

(iv) *The complex $C^\bullet(\overline{R}, (x, X)) := C^\bullet(\overline{R}_0, (x, X)) \otimes_R R[\Delta]$ is isomorphic to $C^\bullet(R, x) \otimes_R C^\bullet_R(\Gamma)$.*

Proof

(i)-(ii) $C^\bullet_R(\Gamma_0) \xrightarrow{\sim} \varinjlim_n K^\bullet_{\overline{R}_0}(\overline{R}_0, (X^n))$ and each Koszul complex $K^\bullet_{\overline{R}_0}(\overline{R}_0, (X^n))$ is quasi-isomorphic to $\overline{R}_0/(X_1^n, \ldots, X_r^n)\overline{R}_0[-r]$.

(iii) Both sides are isomorphic to $\varinjlim_n K^\bullet_{\overline{R}_0}(\overline{R}_0, (x^n, X^n))$; (iv) follows from (iii). □

8.4.5.4. Lemma. — *If M, N are R-modules, then there is a canonical isomorphism of $R[\Delta]$-modules*

$$\mathrm{Hom}_R(M[\Delta], N) \xrightarrow{\sim} \mathrm{Hom}_R(M, N)[\Delta]^\iota$$

given by the formula

$$f \longmapsto \left(\sum_{\delta\in\Delta} (m \longmapsto f(m \otimes \delta)) \otimes \delta \right).$$

Above, $M[\Delta] = M \otimes_R R[\Delta]$ and Δ acts on the L.H.S. by $(\delta * f)(m\otimes\delta') = f(m\otimes\delta\delta')$.

Proof. — Direct calculation. □

8.4.5.5. Lemma. — *There are isomorphisms of $\overline{R}$-modules*

$$I_R \otimes_R H^r(C^\bullet_R(\Gamma)) \xrightarrow{\sim} I_{\overline{R}_0} \otimes_R R[\Delta] \xrightarrow{\sim} I_{\overline{R}}.$$

Proof. — As remarked in the proof of Lemma 8.4.5.3, $H^r(C^\bullet_R(\Gamma_0))$ is isomorphic to the inductive limit

$$\varinjlim_n \overline{R}_0/(X_1^n, \ldots, X_r^n)\overline{R}_0,$$

where the transition map is given by the multiplication by $X_1 \cdots X_r$. The pairings

$$(\, , \,)_n : \overline{R}_0/(X_1^n, \ldots, X_r^n)\overline{R}_0 \otimes_R \overline{R}_0/(X_1^n, \ldots, X_r^n)\overline{R}_0 \longrightarrow R$$

$$f \otimes g \longmapsto \mathrm{Res}_{,X_1=\cdots=X_r=0} \frac{fg}{(X_1\cdots X_r)^n}$$

satisfy

$$(f \pmod{(X_1^n, \ldots, X_r^n)}, g)_n = (f, X_1 \cdots X_r g)_{n+1}.$$

Tensoring them with I_R, the corresponding adjoint maps define an isomorphism of projective systems

$$\left(\overline{R}_0/(X_1^n, \ldots, X_r^n)\overline{R}_0\right)_{n\geqslant 1} \xrightarrow{\sim} \left(\mathrm{Hom}_R\left(I_R \otimes_R (\overline{R}_0/(X_1^n, \ldots, X_r^n)\overline{R}_0), I_R\right)\right)_{n\geqslant 1}.$$

Applying the same argument to the Artinian rings $R/\mathfrak{m}^l$ instead of R and passing to the projective limit with respect to (n, l), we obtain an isomorphism of $\overline{R}_0$-modules between $\overline{R}_0$ and the Pontrjagin dual of the discrete $\overline{R}_0$-module $I_R \otimes_R H^r(C^\bullet_R(\Gamma_0))$. Applying Pontrjagin duality again, we obtain an isomorphism

$$I_R \otimes_R H^r(C^\bullet_R(\Gamma_0)) \xrightarrow{\sim} I_{\overline{R}_0},$$

which implies that

$$I_R \otimes_R H^r(C^\bullet_R(\Gamma)) = I_R \otimes_R H^r(C^\bullet_R(\Gamma_0)) \otimes_R R[\Delta] \xrightarrow{\sim} I_{\overline{R}_0} \otimes_R R[\Delta].$$

Finally, it follows from Lemma 8.4.3.1 and 8.4.5.4 that there are isomorphisms of $\overline{R}$-modules

$$I_{\overline{R}} \xrightarrow{\sim} \mathrm{Hom}_{\overline{R}_0}(\overline{R}_0[\Delta], I_R) \xrightarrow{\sim} I_{\overline{R}_0} \otimes_R R[\Delta]^\iota \xrightarrow{\mathrm{id}\otimes\iota} I_{\overline{R}_0} \otimes_R R[\Delta]. \quad \square$$

8.4.5.6. Lemma. — *There are isomorphisms in* $D^b({}_{\overline{R}_0}\mathrm{Mod})$ *(resp., in* $D^b({}_{\overline{R}}\mathrm{Mod})$*)*

$$\omega_{\overline{R}_0} \xrightarrow{\sim} \omega_R \overset{\mathbf{L}}{\otimes}_R \overline{R}_0 = \omega_R \otimes_R \overline{R}_0, \quad \omega_{\overline{R}} \xrightarrow{\sim} \omega_R \overset{\mathbf{L}}{\otimes}_R \overline{R} = \omega_R \otimes_R \overline{R}.$$

Proof. — The isomorphisms

$$C^\bullet(R,(x)) \overset{\mathbf{L}}{\otimes}_R \omega_R \xrightarrow{\sim} I_R[-d], \quad I_R \otimes_R H^r(C^\bullet_R(\Gamma_0)) \xrightarrow{\sim} I_{\overline{R}_0}$$

in $D^b({}_R\mathrm{Mod})$ (resp., in $({}_{\overline{R}_0}\mathrm{Mod})$) together with Lemma 8.4.5.3 yield an isomorphism in $D^b_{coft}({}_{\overline{R}_0}\mathrm{Mod})$

$$\begin{aligned}\Phi_{\overline{R}_0}\Big(\omega_R \overset{\mathbf{L}}{\otimes}_R \overline{R}_0\Big)[-d-r] &= C^\bullet(\overline{R}_0,(x,X)) \overset{\mathbf{L}}{\otimes}_{\overline{R}_0} (\omega_R \overset{\mathbf{L}}{\otimes}_R \overline{R}_0) \\ &= \Big(C^\bullet(R,(x)) \overset{\mathbf{L}}{\otimes}_R \omega_R\Big) \overset{\mathbf{L}}{\otimes}_R C^\bullet_R(\Gamma_0) \xrightarrow{\sim} I_R[-d] \otimes_R H^r(C^\bullet_R(\Gamma_0))[-r] \xrightarrow{\sim} I_{\overline{R}_0}[-d-r],\end{aligned}$$

hence an isomorphism in $D^b_{ft}({}_{\overline{R}_0}\mathrm{Mod})$

$$\omega_R \overset{\mathbf{L}}{\otimes}_R \overline{R}_0 \xrightarrow{\sim} \mathscr{D}_{\overline{R}_0} \circ D_{\overline{R}_0} \circ \Phi_{\overline{R}_0}\Big(\omega_R \overset{\mathbf{L}}{\otimes}_R \overline{R}_0\Big) \xrightarrow{\sim} \mathscr{D}_{\overline{R}_0}(\overline{R}_0) = \omega_{\overline{R}_0}.$$

As $\overline{R} = \overline{R}_0[\Delta]$ is free of finite rank over $\overline{R}_0$, we have

$$\omega_{\overline{R}} \xrightarrow{\sim} \mathrm{Hom}_{\overline{R}_0}(\overline{R}, \omega_{\overline{R}_0}) = \mathrm{Hom}_R(R[\Delta], R) \otimes_R \omega_{\overline{R}_0}.$$

By Lemma 8.4.5.4 we have isomorphisms of $R[\Delta]$-modules

$$\mathrm{Hom}_R(R[\Delta], R) \xrightarrow{\sim} R[\Delta]^\iota \xrightarrow{\iota} R[\Delta]$$

("relative Gorenstein property of group rings"), which gives the second isomorphism. □

8.4.6. Comparison of the duality diagrams over R and $\overline{R}$

8.4.6.1. Let X, Y be bounded complexes in $({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod})_{R\text{-}ft}$. Applying the discussion from 8.1.6.4 to all $U \in \mathscr{U}$ and passing to the projective limit, we obtain a morphism of (bounded) complexes in $({}^{\mathrm{ad}}_{\overline{R}[G]}\mathrm{Mod})_{\overline{R}\text{-}ft}$

$$m : \mathscr{F}_\Gamma(X) \otimes_{\overline{R}} \mathscr{F}_\Gamma(Y)^\iota \longrightarrow \overline{R} \otimes_R (X \otimes_R Y)$$

(with G acting trivially on the factor $\overline{R}$ on the R.H.S.). This pairing can also be written explicitly (with a slight permutation of the factors) as

$$\begin{aligned}&(X \otimes_R \overline{R}^\iota) < 1 > \otimes_{\overline{R}} \big((Y \otimes_R \overline{R}^\iota) < 1 > \big)^\iota \\ &\xrightarrow{\mathrm{id}\otimes\iota\otimes\mathrm{id}\otimes\mathrm{id}} (X \otimes_R \overline{R}) < -1 > \otimes_{\overline{R}} (Y \otimes_R \overline{R}) < 1 >= (X \otimes_R Y) \otimes_R \overline{R}.\end{aligned}$$

8.4.6.2. Dualizing complexes. — For each $S = R, \overline{R}$, fix a complex $\omega^\bullet_S = \sigma_{\geqslant 0}\, \omega^\bullet_S$ of injective S-modules representing ω_S. There exists a quasi-isomorphism (unique up to homotopy)

$$\varphi : \omega^\bullet_R \otimes_R \overline{R} \longrightarrow \omega^\bullet_{\overline{R}},$$

which we fix, once for all.

8.4.6.3. Given X and Y as in 8.4.6.1 and a morphism of complexes

$$\pi : X \otimes_R Y \longrightarrow \omega_R^\bullet(1),$$

we obtain a morphism of complexes

$$\overline{\pi} : \mathscr{F}_\Gamma(X) \otimes_{\overline{R}} \mathscr{F}_\Gamma(Y)^\iota \xrightarrow{m} \overline{R} \otimes_R (X \otimes_R Y) \xrightarrow{\mathrm{id}\otimes\pi} \overline{R} \otimes_R \omega_R^\bullet(1) \xrightarrow{\varphi} \omega_{\overline{R}}^\bullet(1).$$

The corresponding adjoint map is equal to

$$\begin{aligned}\mathrm{adj}(\overline{\pi}) : \mathscr{F}_\Gamma(X) \xrightarrow{\sim} \overline{R} \otimes_R X < -1 > &\xrightarrow{\mathrm{id}\otimes\mathrm{adj}(\pi)<-1>} \overline{R} \otimes_R \mathrm{Hom}_R^\bullet(Y, \omega_R^\bullet(1)) < -1 > \\ &= \mathrm{Hom}_{\overline{R}}^\bullet(\overline{R} \otimes_R Y < 1 >, \overline{R} \otimes_R \omega_R^\bullet(1)) \\ &= \mathrm{Hom}_{\overline{R}}^\bullet(\mathscr{F}_\Gamma(Y)^\iota, \overline{R} \otimes_R \omega_R^\bullet(1)) \xrightarrow{\varphi_*} \mathrm{Hom}_{\overline{R}}^\bullet(\mathscr{F}_\Gamma(Y)^\iota, \omega_{\overline{R}}^\bullet(1)).\end{aligned}$$

8.4.6.4. Lemma. — *There exist functorial isomorphisms*

$$\iota \circ \mathscr{F}_\Gamma \circ \mathscr{D}(X) \xrightarrow{\sim} \overline{\mathscr{D}} \circ \mathscr{F}_\Gamma(X), \quad \overline{\Phi} \circ \mathscr{F}_\Gamma(X) \xrightarrow{\sim} F_\Gamma \circ \Phi(X) \quad \big(X \in D^b_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)\big).$$

Proof. — By Proposition 3.2.6 there exists a bounded complex Y in $\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)_{R\text{-}ft}$ and a quasi-isomorphism $f : Y \to \mathscr{D}(X) = \mathrm{Hom}_R^\bullet(X, \omega_R^\bullet)$. Applying 8.4.6.3 to the pairing

$$\pi : X \otimes_R Y \xrightarrow{\mathrm{id}\otimes f} X \otimes_R \mathscr{D}(X) \xrightarrow{\mathrm{ev}_2} \omega_R^\bullet,$$

we obtain a morphism of complexes

$$\mathrm{adj}(\overline{\pi}) : \mathscr{F}_\Gamma(Y)^\iota \longrightarrow \mathrm{Hom}_{\overline{R}}^\bullet(\mathscr{F}_\Gamma(X), \omega_{\overline{R}}^\bullet) = \overline{\mathscr{D}} \circ \mathscr{F}_\Gamma(X),$$

hence a morphism in $D^b_{\overline{R}\text{-}ft}\big({}^{\mathrm{ad}}_{\overline{R}[G]}\mathrm{Mod}\big)$

$$\alpha_X : \iota \circ \mathscr{F}_\Gamma \circ \mathscr{D}(X) = \mathscr{F}_\Gamma(\mathscr{D}(X))^\iota \xrightarrow{(\mathscr{F}_\Gamma(f)^\iota)^{-1}} \mathscr{F}_\Gamma(Y)^\iota \xrightarrow{\mathrm{adj}(\overline{\pi})} \overline{\mathscr{D}} \circ \mathscr{F}_\Gamma(X),$$

which is functorial in X. In order to prove that α_X is an isomorphism, we can disregard the action of G, *i.e.*, consider only $X \in D^b_{ft}({}_R\mathrm{Mod})$, $\mathscr{F}_\Gamma(X) = X \otimes_R \overline{R}$ and replace ι by the identity map. As we are dealing with "way-out functors" ([**RD**, §I.7]), standard dévissage reduces the claim to the fact that $\alpha_R = \varphi$ is an isomorphism in $D^b_{ft}({}_{\overline{R}}\mathrm{Mod})$.

As regards the second isomorphism, we combine α_X with Lemma 8.4.5.1, obtaining functorial isomorphisms

$$F_\Gamma\circ\Phi(X) \xrightarrow{\sim} \overline{D}\circ\iota\circ\mathscr{F}_\Gamma\circ D\circ\Phi(X) \xrightarrow{\sim} \overline{D}\circ\iota\circ\mathscr{F}_\Gamma\circ\mathscr{D}(X) \xrightarrow{\sim} \overline{D}\circ\overline{\mathscr{D}}\circ\mathscr{F}_\Gamma(X) \xrightarrow{\sim} \overline{\Phi}\circ\mathscr{F}_\Gamma(X).$$

□

8.4.6.5. Corollary. — *If, in the situation of 8.4.6.3,*

$$X \xrightarrow{\mathrm{adj}(\pi)} \mathrm{Hom}_R^\bullet(Y, \omega_R^\bullet(1)) \longrightarrow W \longrightarrow X[1]$$

is an exact triangle in $D^b_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$, *then*

$$\mathscr{F}_\Gamma(X) \xrightarrow{\mathrm{adj}(\overline{\pi})} \mathrm{Hom}_{\overline{R}}^\bullet(\mathscr{F}_\Gamma(Y)^\iota, \omega_{\overline{R}}^\bullet(1)) \longrightarrow \mathscr{F}_\Gamma(W) \longrightarrow \mathscr{F}_\Gamma(X)[1]$$

is an exact triangle in $D^b_{\overline{R}\text{-}ft}\big({}^{\mathrm{ad}}_{\overline{R}[G]}\mathrm{Mod}\big)$. *In particular,*

$$\pi \text{ is a perfect duality over } R \iff \overline{\pi} \text{ is a perfect duality over } \overline{R}.$$

8.4.6.6. The isomorphisms from Lemma 8.4.5.1 and 8.4.6.4 can be summarized as follows.

If $T, T^* \in D^b_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$ and $A, A^* \in D^b_{R\text{-}coft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$ are related by the duality diagram

$$\begin{array}{ccc} T & \overset{\mathscr{D}}{\longleftrightarrow} & T^* \\ \Big\downarrow{\scriptstyle \Phi} & \overset{D}{\times} & \Big\downarrow{\scriptstyle \Phi} \\ A & & A^* \end{array}$$

over R, then $\mathscr{F}_\Gamma(T), \mathscr{F}_\Gamma(T^*) \in D^b_{\overline{R}\text{-}ft}\big({}^{\mathrm{ad}}_{\overline{R}[G]}\mathrm{Mod}\big)$ and $F_\Gamma(A), F_\Gamma(A^*) \in D^b_{\overline{R}\text{-}coft}\big({}^{\mathrm{ad}}_{\overline{R}[G]}\mathrm{Mod}\big)$ are related by the duality diagram

$$\begin{array}{ccc} \mathscr{F}_\Gamma(T) & \overset{\overline{\mathscr{D}}}{\longleftrightarrow} & \mathscr{F}_\Gamma(T^*)^\iota \\ \Big\downarrow{\scriptstyle \overline{\Phi}} & \overset{\overline{D}}{\times} & \Big\downarrow{\scriptstyle \overline{\Phi}} \\ F_\Gamma(A) & & F_\Gamma(A^*)^\iota \end{array}$$

over $\overline{R}$.

8.4.6.7. Proposition

(i) *Let* $T \in D^+\big(\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)_{R\text{-}ft}\big)$; *put* $A = \Phi(T) \in D^+_{R\text{-}coft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$. *Then there are canonical isomorphisms*

$$\overline{\Phi}(\mathbf{R}\Gamma_{\mathrm{Iw}}(G,H;T)) \xrightarrow{\sim} \overline{\Phi}\big(\mathbf{R}\Gamma_{\mathrm{cont}}(G,\mathscr{F}_\Gamma(T))\big) \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cont}}(G,F_\Gamma(A)) \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cont}}(H,A)$$

in $D^+\big({}_{\overline{R}}\mathrm{Mod}\big)$.

(ii) *Let* $T \in D^b_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$; *put* $A = \Phi(T) \in D^b_{R\text{-}coft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$. *If G satisfies* (F) *(resp., if G satisfies* (F) *and* $\mathrm{cd}_p(G) < \infty$*), then the above isomorphisms take place in* $D^+_{coft}\big({}_{\overline{R}}\mathrm{Mod}\big)$ *(resp., in* $D^b_{coft}\big({}_{\overline{R}}\mathrm{Mod}\big)$*) and there is a spectral sequence*

$$E_2^{i,j} = \mathbb{E}\mathrm{xt}^i_{\overline{R}}(\overline{D}(H^j_{\mathrm{cont}}(H,A)),\omega_{\overline{R}}) \Longrightarrow H^{i+j}_{\mathrm{Iw}}(G,H;T) = \varprojlim_{U\in\mathscr{U}} H^{i+j}_{\mathrm{cont}}(U,T)$$

(all terms of which are $\overline{R}$*-modules of finite type).*

Proof. — Represent T by a bounded below complex $T^\bullet$ in $\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)_{R\text{-}ft}$. Lemma 8.4.6.4 applied to each component of $T^\bullet$ then yields an isomorphism

$$\overline{\Phi}\circ\mathscr{F}_\Gamma(T) \xrightarrow{\sim} F_\Gamma\circ\Phi(A)$$

in $D^+_{R\text{-}coft}\big({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod}\big)$. The statement of (i) follows from 4.3.1, applied to the pair $\mathscr{F}_\Gamma(T)$ and $F_\Gamma(A)$ over $\overline{R}$, while (ii) follows from (i) and 4.3.1. □

8.4.6.8. A variant of the above spectral sequence for $R = \mathbf{Z}_p$ and T consisting of a single module was constructed by Jannsen [**Ja3**], who also considered the case of non-abelian $\Gamma = G/H$.

8.4.7. Dihedral case

8.4.7.1. Assume that we are given the following *dihedral data.*

8.4.7.1.1. An exact sequence of pro-finite groups

$$1 \longrightarrow G \longrightarrow G^+ \longrightarrow \{\pm 1\} \longrightarrow 1$$

such that

8.4.7.1.2. H is a normal subgroup of G^+,

8.4.7.1.3. The quotient exact sequence of groups

$$1 \longrightarrow \Gamma \longrightarrow \Gamma^+ \longrightarrow \{\pm 1\} \longrightarrow 1$$

(in which $\Gamma^+ = G^+/H$) is split and $-1 \in \{\pm 1\}$ acts on Γ as $\iota : g \mapsto g^{-1}$.

The last condition can be reformulated by saying that there exists an element $\tau \in \Gamma^+ - \Gamma$ such that

$$\tau^2 = 1, \quad \tau\gamma\tau^{-1} = \gamma^{-1} \qquad (\gamma \in \Gamma). \tag{8.4.7.4}$$

If M is an $R[\Gamma^+]$-module, then the map

$$m \longmapsto \tau(m) \qquad (m \in M)$$

defines an isomorphism of $R[\Gamma]$-modules

$$M \xrightarrow{\sim} M^\iota.$$

The same is true for complexes of $R[\Gamma^+]$-modules.

In particular, if $T \in \big({}^{\mathrm{ad}}_{R[G^+]}\mathrm{Mod}\big)_{R\text{-}ft}$, then $\mathbf{R}\Gamma_{\mathrm{Iw}}(G,H;T)$ can be represented not just by $C^\bullet_{\mathrm{cont}}(G,\mathscr{F}_\Gamma(T))$, but by $C^\bullet_{\mathrm{cont}}(G^+,\mathscr{F}_{\Gamma^+}(T))$, where

$$\mathscr{F}_{\Gamma^+}(T) = \varprojlim_{U^+} T \otimes_R R[G^+/U^+],$$

where U^+ runs through all open subgroups of G^+ containing H. Similarly, for every $M \in \big({}^{\text{ind-ad}}_{R[G^+]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$, $\mathbf{R}\Gamma_{\mathrm{cont}}(H,M)$ can be represented by $C^\bullet_{\mathrm{cont}}(G^+, F_{\Gamma^+}(M))$, where

$$F_{\Gamma^+}(M) = \varinjlim_{U^+} \mathrm{Hom}_R(R[G^+/U^+], M).$$

Applying the previous discussion to the complexes $C^\bullet_{\mathrm{cont}}(G^+,\mathscr{F}_{\Gamma^+}(T))$ and $C^\bullet_{\mathrm{cont}}(G^+, F_{\Gamma^+}(M))$ we obtain the following statement.

8.4.7.2. Proposition. — *Assume that we are given the dihedral data 8.4.7.1.1–8.4.7.1.3. If $T \in D^b_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G^+]}\mathrm{Mod}\big)$ and $M \in D^+\big(\big({}^{\text{ind-ad}}_{R[G^+]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}\big)$, then the action of τ from* (8.4.7.4) *gives isomorphisms*

$$\mathbf{R}\Gamma_{\mathrm{Iw}}(G,H;T)^\iota \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{Iw}}(G,H;T), \quad \mathbf{R}\Gamma_{\mathrm{cont}}(H,M)^\iota \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cont}}(H,M)$$

in $D^+\big({}_{\overline{R}}\mathrm{Mod}\big)$.

8.4.8. Descent

8.4.8.1. Proposition

(i) *For every* $M \in D^+\big(\big({}^{\text{ind-ad}}_{R[G]}\text{Mod}\big)_{\{\mathfrak{m}\}}\big)$ *there is a spectral sequence*

$$E_2^{i,j} = H^i_{\text{cont}}(\Gamma, H^j_{\text{cont}}(H, M)) \Longrightarrow H^{i+j}_{\text{cont}}(G, M).$$

(ii) *If* $\Gamma = \Gamma_0 \xrightarrow{\sim} \mathbf{Z}_p^r$, *then for every* $T \in D^+\big(\big({}^{\text{ad}}_{R[G]}\text{Mod}\big)_{R\text{-}ft}\big)$ *there is a canonical isomorphism in* $D^+({}_R\text{Mod})$

$$\mathbf{R}\Gamma_{\text{Iw}}(G, H; T) \overset{\mathbf{L}}{\otimes}_{\overline{R}} R \xrightarrow{\sim} \mathbf{R}\Gamma_{\text{cont}}(G, T)$$

(where the product is taken with respect to the augmentation map $\overline{R} \to R$*), which induces a (homological) spectral sequence*

$$'E^2_{i,j} = H_{i,\text{cont}}(\Gamma, H^{-j}_{\text{Iw}}(G, H; T)) \Longrightarrow H^{-i-j}_{\text{cont}}(G, T)$$

(where $H_{i,\text{cont}}(\Gamma, -)$ *was defined in* (7.2.7.2)*). If* G *satisfies* (F)*, then each term* $'E^2_{i,j}$ *is an* R*-module of finite type.*

Proof

(i) This is just the Hochschild-Serre spectral sequence for discrete G-modules.

(ii) Represent T by a bounded below complex in $\big({}^{\text{ad}}_{R[G]}\text{Mod}\big)_{R\text{-}ft}$, which will also be denoted by T. The elements $\gamma_1 - 1, \dots, \gamma_r - 1$ form a regular sequence $\mathbf{x}$ in $\overline{R}$ and the augmentation map defines an isomorphism $\overline{R}/\mathbf{x}\overline{R} \xrightarrow{\sim} R$. For each $i = 0, \dots, r$, let $H_i \subset \Gamma$ be the subgroup topologically generated by $\gamma_1, \dots, \gamma_i$ and put $\overline{R}_i = R[\![\Gamma/H_i]\!]$ (*e.g.*, $\overline{R}_0 = \overline{R}$, $\overline{R}_r = R$). For each $i = 1, \dots, r$, the tautological exact sequence

$$0 \longrightarrow \overline{R}_{i-1} \xrightarrow{\gamma_i - 1} \overline{R}_{i-1} \longrightarrow \overline{R}_i \longrightarrow 0$$

yields, upon tensoring with T, an exact sequence of complexes in $\big({}^{\text{ad}}_{R[G]}\text{Mod}\big)$

$$0 \longrightarrow \mathscr{F}_{\Gamma/H_{i-1}}(T) \xrightarrow{\gamma_i - 1} \mathscr{F}_{\Gamma/H_{i-1}}(T) \longrightarrow \mathscr{F}_{\Gamma/H_i}(T) \longrightarrow 0,$$

hence an exact sequence of complexes of $\overline{R}_{i-1}$-modules

$$\text{(8.4.8.1.1)} \quad 0 \longrightarrow C^\bullet_{\text{cont}}(G, \mathscr{F}_{\Gamma/H_{i-1}}(T)) \xrightarrow{\gamma_i - 1} C^\bullet_{\text{cont}}(G, \mathscr{F}_{\Gamma/H_{i-1}}(T)) \longrightarrow C^\bullet_{\text{cont}}(G, \mathscr{F}_{\Gamma/H_i}(T)) \longrightarrow 0,$$

which can be rewritten as an isomorphism of complexes

$$C^\bullet_{\text{cont}}(G, \mathscr{F}_{\Gamma/H_{i-1}}(T)) \otimes_{\overline{R}_{i-1}} \Big[\overline{R}_{i-1} \xrightarrow{\gamma_i - 1} \overline{R}_{i-1}\Big] \xrightarrow{\sim} C^\bullet_{\text{cont}}(G, \mathscr{F}_{\Gamma/H_i}(T))$$

(with the complex $[\gamma_i - 1 : \overline{R}_{i-1} \to \overline{R}_{i-1}]$ supported in degrees $-1, 0$). This yields a canonical isomorphism in $D({}_{\overline{R}_i}\text{Mod})$

$$\mathbf{R}\Gamma_{\text{cont}}(G, \mathscr{F}_{\Gamma/H_{i-1}}(T)) \overset{\mathbf{L}}{\otimes}_{\overline{R}_{i-1}} \overline{R}_i \xrightarrow{\sim} \mathbf{R}\Gamma_{\text{cont}}(G, \mathscr{F}_{\Gamma/H_i}(T)).$$

Iterating this construction r-times, we obtain the desired isomorphism

$$\mathbf{R}\Gamma_{\mathrm{Iw}}(G,H;T)\overset{\mathbf{L}}{\otimes}_{\overline{R}}R=\mathbf{R}\Gamma_{\mathrm{cont}}(G,\mathscr{F}_\Gamma(T))\overset{\mathbf{L}}{\otimes}_{\overline{R}}R\xrightarrow{\sim}\mathbf{R}\Gamma_{\mathrm{cont}}(G,T)$$

in $D({}_R\mathrm{Mod})$.

The augmentation map $f:\overline{R}\to R$ defines an exact functor $f^*:({}_R\mathrm{Mod})\to\big({}_{\overline{R}}\mathrm{Mod}\big)$ ($f^*M=M$ as an abelian group, with $\overline{r}\in\overline{R}$ acting on f^*M by $f(\overline{r})$). As the Koszul complex $K^\bullet_{\overline{R}}(\overline{R},\mathbf{x})$ is an $\overline{R}$-free resolution of $f^*R[-r]$, we obtain from the previous result an isomorphism in $D({}_{\overline{R}}\mathrm{Mod})$

$$f^*\mathbf{R}\Gamma_{\mathrm{cont}}(G,T)\xrightarrow{\sim}\mathbf{R}\Gamma_{\mathrm{cont}}(G,\mathscr{F}_\Gamma(T))\otimes_{\overline{R}}K^\bullet_{\overline{R}}(\overline{R},\mathbf{x})[r],$$

where the R.H.S. is represented by the complex

$$\begin{aligned}K^\bullet&=C^\bullet_{\mathrm{cont}}(G,\mathscr{F}_\Gamma(T))\otimes_{\overline{R}}K^\bullet_{\overline{R}}(\overline{R},\mathbf{x})[r]\\&=C^\bullet_{\mathrm{cont}}(G,\mathscr{F}_\Gamma(T)\otimes_{\overline{R}}K^\bullet_{\overline{R}}(\overline{R},\mathbf{x}))[r]=K^\bullet_{\overline{R}}(C^\bullet_{\mathrm{cont}}(G,\mathscr{F}_\Gamma(T)),\mathbf{x})[r].\end{aligned}$$

The stupid filtration $\sigma_{\geqslant i}$ on $K^\bullet_{\overline{R}}(\overline{R},\mathbf{x})$ induces a finite decreasing filtration $F^iK^\bullet$ on $K^\bullet$. The corresponding spectral sequence (3.5.2.1) is given by

$${}'E_1^{i,j}=K^{i+r}_{\overline{R}}(H^j_{\mathrm{cont}}(G,\mathscr{F}_\Gamma(T)),\mathbf{x})\Longrightarrow H^{i+j}_{\mathrm{cont}}(G,K^\bullet)=H^{i+j}_{\mathrm{cont}}(G,T).$$

It follows from (7.2.7.2) that

$${}'E_2^{i,j}=H_{-i,\mathrm{cont}}\big(\Gamma,H^j_{\mathrm{cont}}(G,\mathscr{F}_\Gamma(T))\big)=H_{-i,\mathrm{cont}}(\Gamma,H^j_{\mathrm{Iw}}(G,H;T)),$$

which proves the result if we pass to the homological notation ${}'E^m_{i,j}={}'E_m^{-i,-j}$.

Finally, if G satisfies (F), then each cohomology group $M^j=H^j_{\mathrm{cont}}(G,\mathscr{F}_\Gamma(T))$ is an $\overline{R}$-module of finite type, hence the cohomology groups of the Koszul complex $K^\bullet_{\overline{R}}(M^j,\mathbf{x})$ are $\overline{R}/\mathbf{x}\overline{R}$-modules of finite type. ☐

8.4.8.2. Corollary. — *Assume that $\Gamma\xrightarrow{\sim}\mathbf{Z}_p^r$ and $T\in D^+\big(({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod})_{R\text{-}ft}\big)$. Then*

(i) *If $\Gamma\xrightarrow{\sim}\mathbf{Z}_p$, then ${}'E^2={}'E^\infty$ degenerates into short exact sequences*

$$0\longrightarrow H^j_{\mathrm{Iw}}(G,H;T)_\Gamma\longrightarrow H^j_{\mathrm{cont}}(G,T)\longrightarrow H^{j+1}_{\mathrm{Iw}}(G,H;T)^\Gamma\longrightarrow 0.$$

(ii) *If $\mathrm{cd}_p(G)=e<\infty$ and $\tau_{\leqslant n}T\xrightarrow{\sim}T$, then*

$${}'E^2_{0,-e-n}=H^{e+n}_{\mathrm{Iw}}(G,H;T)_\Gamma\xrightarrow{\sim}H^{e+n}_{\mathrm{cont}}(G,T).$$

8.4.8.3. Proposition. — *Assume that $T\in D^+\big(({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod})_{R\text{-}ft}\big)$. Let $\Gamma'\subset\Gamma=G/H$ be a closed subgroup of Γ isomorphic to $\mathbf{Z}_p^{r'}$ ($r'\leqslant r$) and $H'\subset G$ its inverse image in G. Then there is a canonical isomorphism in $D^+({}_{R[\![\Gamma/\Gamma']\!]}\mathrm{Mod})$*

$$\mathbf{R}\Gamma_{\mathrm{Iw}}(G,H;T)\overset{\mathbf{L}}{\otimes}_{\overline{R}}R[\![\Gamma/\Gamma']\!]\xrightarrow{\sim}\mathbf{R}\Gamma_{\mathrm{Iw}}(G,H';T)$$

(where the product is taken with respect to the canonical map induced by the projection $\Gamma \to \Gamma/\Gamma'$*), which induces a (homological) spectral sequence*

$$'E^2_{i,j} = H_{i,\mathrm{cont}}(\Gamma', H^{-j}_{\mathrm{Iw}}(G,H;T)) \Longrightarrow H^{-i-j}_{\mathrm{Iw}}(G,H';T).$$

Proof. — This follows from the same argument as in the proof of Proposition 8.4.8.1 (ii) if we replace $\mathbf{x} = (\gamma_1 - 1, \dots, \gamma_r - 1)$ by $\mathbf{x}' = (\gamma'_1 - 1, \dots, \gamma'_{r'} - 1)$, where $\gamma'_1, \dots, \gamma'_{r'}$ are topological generators of Γ', as $\overline{R}/\mathbf{x}'\overline{R} = R[\![\Gamma/\Gamma']\!]$. ☐

8.4.8.4. Corollary. — *Under the assumptions of 8.4.8.3,*

(i) *If* $\Gamma' \xrightarrow{\sim} \mathbf{Z}_p$*, then* $'E^2 = {}'E^\infty$ *degenerates into short exact sequences*

$$0 \longrightarrow H^j_{\mathrm{Iw}}(G,H;T)_{\Gamma'} \longrightarrow \mathbf{R}\Gamma_{\mathrm{Iw}}(G,H';T) \longrightarrow H^{j+1}_{\mathrm{Iw}}(G,H;T)^{\Gamma'} \longrightarrow 0.$$

(ii) *If* $\mathrm{cd}_p(G) = e < \infty$ *and* $\tau_{\leqslant n} T \xrightarrow{\sim} T$*, then*

$$'E^2_{0,-e-n} = H^{e+n}_{\mathrm{Iw}}(G,H;T)_{\Gamma'} \xrightarrow{\sim} H^{e+n}_{\mathrm{Iw}}(G,H';T).$$

8.4.8.5. Proposition. — *Assume that* $\Gamma \xrightarrow{\sim} \mathbf{Z}_p^r$*,* G *satisfies* (F) *and* $\mathrm{cd}_p(G) < \infty$*. Let* $\mathfrak{p} \in \mathrm{Spec}(R)$*; denote by* $\overline{\mathfrak{p}} \in \mathrm{Spec}(\overline{R})$ *the inverse image of* $\mathfrak{p}$ *under the augmentation map* $\overline{R} \to R$*. If* $T \in D^b\big(({}^{\mathrm{ad}}_{R[G]}\mathrm{Mod})_{R\text{-}ft}\big)$ *satisfies* $\mathbf{R}\Gamma_{\mathrm{cont}}(G,T)_{\mathfrak{p}} \xrightarrow{\sim} 0$ *in* $D^b_{ft}({}_{R_{\mathfrak{p}}}\mathrm{Mod})$*, then* $\mathbf{R}\Gamma_{\mathrm{cont}}(G,\mathscr{F}_\Gamma(T))_{\overline{\mathfrak{p}}} \xrightarrow{\sim} 0$ *in* $D^b_{ft}\big({}_{\overline{R}_{\overline{\mathfrak{p}}}}\mathrm{Mod}\big)$*.*

Proof. — According to Proposition 8.4.8.1 (ii), there is a spectral sequence in $\big({}_{R_{\mathfrak{p}}}\mathrm{Mod}\big)_{ft}$

$$E^2_{i,j} = H_{i,\mathrm{cont}}(\Gamma, H^{-j})_{\mathfrak{p}} \Longrightarrow H^{-i-j}_{\mathrm{cont}}(G,T)_{\mathfrak{p}} = 0,$$

where we have denoted

$$H^{-j} := H^{-j}_{\mathrm{cont}}(G,\mathscr{F}_\Gamma(T)) \in \big({}_{\overline{R}}\mathrm{Mod}\big)_{ft}.$$

If, for some $j \in \mathbf{Z}$, we have $E^2_{0,j} = 0$, then we deduce from 7.2.7 and Lemma 8.10.5 below (applied to $B = R_{\mathfrak{p}}$, $M = (H^{-j})_{\mathfrak{p}}$) that $E^2_{i,j} = 0$ vanishes for every $i \geqslant 0$. As the spectral sequence has only finitely many non-zero terms and $E^2_{i,j} = 0$ for $i < 0$, induction on j shows that

$$(\forall j \in \mathbf{Z}) \quad 0 = E^2_{0,j} = (H^{-j}_\Gamma)_{\mathfrak{p}} = (H^{-j}_{\overline{\mathfrak{p}}})_\Gamma = (H^{-j}_{\overline{\mathfrak{p}}})/J,$$

where $J = \mathrm{Ker}(R[\![\Gamma]\!] \to R)$ denotes the augmentation ideal. As $J \subseteq \overline{\mathfrak{p}}$, Nakayama's Lemma implies that $H^{-j}_{\overline{\mathfrak{p}}} = 0$ for all $j \in \mathbf{Z}$, as claimed. ☐

8.5. Duality theorems in Iwasawa theory

Assume that we are in the situation of 5.1, with K a number field (totally imaginary if $p = 2$, because of the condition (P)).

8.5.1. Notation. — Assume that K_∞/K is a Galois extension contained in K_S, with Galois group $\mathrm{Gal}(K_\infty/K) = \Gamma = \Gamma_0 \times \Delta$, where $\Gamma_0 \xrightarrow{\sim} \mathbf{Z}_p^r$ $(r \geqslant 1)$ and Δ is a finite abelian group. The results of 8.4 then apply to $G = G_{K,S} = \mathrm{Gal}(K_S/K)$ and $H = \mathrm{Gal}(K_S/K_\infty)$. We shall use a simplified notation $\mathbf{R}\Gamma_{\mathrm{Iw}}(K_\infty/K, -)$ for $\mathbf{R}\Gamma_{\mathrm{Iw}}(G_{K,S}, \mathrm{Gal}(K_S/K_\infty); -)$ and $H^i_{\mathrm{Iw}}(K_\infty/K, -)$ for its cohomology, even though these objects depend, in general, also on S.

Let $\{K_\alpha\}$ be the set of all finite extensions of K contained in K_∞; put $\Gamma_\alpha = \mathrm{Gal}(K_\infty/K_\alpha)$. If v is a prime of K, then v_α (resp., v_∞) will usually denote a prime of K_α (resp., of K_∞) above v.

8.5.2. Let $v \in S_f$. As Γ is abelian, the decomposition (resp., inertia) group of any prime $v_\infty \mid v$ in K_∞/K is independent of v_∞; denote it by Γ_v (resp., $I(\Gamma_v)$). Both groups $I(\Gamma_v) \subset \Gamma_v$ are closed subgroups of Γ and the quotient $\Gamma_v/I(\Gamma_v)$ is isomorphic to $\mathbf{Z}_p$ or $\mathbf{Z}/p^n\mathbf{Z}$, for some $n \geqslant 0$. The subgroups $I(\Gamma_v)$ $(v \in S_f)$ generate a subgroup of finite index in Γ, since the maximal abelian extension of K unramified at all finite primes is finite over K. If $v \in S_f$ and $v \nmid p$, then $I(\Gamma_v) \subseteq \Gamma_{\mathbf{Z}_p\text{-tors}} = \Delta$, by class field theory. In particular, there is $v \in S_f$ such that $v \mid p$ and $|I(\Gamma_v)| = \infty$. If there is only one prime v above p in K, then $(\Gamma : I(\Gamma_v)) < \infty$.

8.5.3. Semilocal theory

8.5.3.1. Let K'/K be a finite Galois subextension of K_S/K. Denote by S' the set of primes of K' above S; then $G_{K',S'} = \mathrm{Gal}(K_S/K')$. For fixed $v \in S_f$ we can apply the general discussion in 8.1.7 to the groups $U = G_{K'} = \mathrm{Gal}(\overline{K}/K') \lhd G = G_K = \mathrm{Gal}(\overline{K}/K)$ and $\overline{G} = G_v = \mathrm{Gal}(\overline{K}_v/K_v)$. As in 5.1, the fixed embedding of $\overline{K} \hookrightarrow \overline{K}_v$ defines an injective homomorphism $\alpha = \alpha_v : \overline{G} \hookrightarrow G$ and a prime $v'_0 \in S'_f$ above v; then $\overline{U} = G_{v'_0} = \mathrm{Gal}(\overline{K}_v/K'_{v'_0})$.

For each $\sigma \in G$, the subgroup $\overline{U}_\sigma = \sigma\alpha(\overline{U})\sigma^{-1} \subset U$ depends only on the double coset $U\sigma\alpha(\overline{G}) \in U\backslash G/\alpha(\overline{G})$, and is equal to the decomposition group $G_{\sigma(v'_0)} \subset U$ of the prime $\sigma(v'_0)$. Fixing coset representatives $\sigma_i \in G$ (with $\sigma_0 = 1$)

$$G = \bigcup_i U\sigma_i\alpha(\overline{G})$$

as in 8.1.7, the set of primes $\{v' \mid v\}$ in K coincides with $\{v'_i = \sigma_i(v'_0)\}$. For every complex X of discrete G-modules we have, as in 8.1.7.6, isomorphisms of complexes

$$C^\bullet(G_{\sigma_i(v'_0)}, X) \xrightarrow{\sim} C^\bullet(\overline{U}, \alpha^* X) \xrightarrow{\sim} C^\bullet(\overline{U}, \alpha_i^* X), \tag{8.5.3.1}$$

induced by the morphisms of pairs $(\alpha_i, \sigma_i^{-1})$ resp., (id, σ_i). The results of 8.1.7 then give, for every complex X of discrete $G_{K,S}$-modules, the following:

8.5.3.2. A quasi-isomorphism (8.1.7.1)[−1]

$$\mathrm{sh}_c : C_c^\bullet(G_{K,S}, X_U) \longrightarrow C_c^\bullet(G_{K',S'}, X),$$

which is functorial in X, the homotopy class of which is independent of any choices, and which makes the bottom half of the following diagram with exact rows commutative up to homotopy (the top half is commutative in the usual sense):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{v\in S_f} C^\bullet(G_v, X_U)[-1] & \longrightarrow & C_c^\bullet(G_{K,S}, X_U) & \longrightarrow & C^\bullet(G_{K,S}, X_U) & \longrightarrow & 0 \\
 & & \downarrow \wr & & \| & & \| & & \\
0 & \longrightarrow & \bigoplus_{v\in S_f}\bigoplus_{v'|v} C^\bullet(G_v, \mathbf{Z}[G_v/G_{v'}]\otimes_{\mathbf{Z}} X)[-1] & \longrightarrow & C_c^\bullet(G_{K,S}, X_U) & \longrightarrow & C^\bullet(G_{K,S}, X_U) & \longrightarrow & 0 \\
 & & \downarrow \mathrm{sh}[-1] & & \downarrow \mathrm{sh}_c & & \downarrow \mathrm{sh} & & \\
0 & \longrightarrow & \bigoplus_{v'\in S'_f} C^\bullet(G_{v'}, X)[-1] & \longrightarrow & C_c^\bullet(G_{K',S'}, X) & \longrightarrow & C^\bullet(G_{K',S'}, X) & \longrightarrow & 0
\end{array}$$

8.5.3.3. For each $g \in G_{K,S}$, 8.1.7.3 gives a morphism

$$\mathrm{Ad}(g)_c = (\mathrm{Ad}(g), F(g), m)[-1] : C_c^\bullet(G_{K',S'}, X) \longrightarrow C_c^\bullet(G_{K',S'}, X),$$

which is functorial in X and makes the following diagrams commutative up to homotopy:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{v'\in S'_f} C^\bullet(G_{v'}, X)[-1] & \longrightarrow & C_c^\bullet(G_{K',S'}, X) & \longrightarrow & C^\bullet(G_{K',S'}, X) & \longrightarrow & 0 \\
 & & \downarrow F(g)[-1] & & \downarrow \mathrm{Ad}(g)_c & & \downarrow \mathrm{Ad}(g) & & \\
0 & \longrightarrow & \bigoplus_{v'\in S'_f} C^\bullet(G_{v'}, X)[-1] & \longrightarrow & C_c^\bullet(G_{K',S'}, X) & \longrightarrow & C^\bullet(G_{K',S'}, X) & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccc}
C_c^\bullet(G_{K,S}, X_U) & \xrightarrow{\mathrm{sh}_c} & C_c^\bullet(G_{K',S'}, X) \\
\downarrow \mathrm{Ad}(gU)_* & & \downarrow \mathrm{Ad}(g)_c \\
C_c^\bullet(G_{K,S}, X_U) & \xrightarrow{\mathrm{sh}_c} & C_c^\bullet(G_{K',S'}, X)
\end{array}$$

8.5.3.4. If $K \subset K' \subset K''$ are two finite Galois subextensions of K_S/K, 8.1.7.4–8.1.7.5 give restriction and corestriction morphisms of complexes

$$\begin{aligned}
\mathrm{res}_c &: C_c^\bullet(G_{K',S'}, X) \longrightarrow C_c^\bullet(G_{K'',S''}, X) \\
\mathrm{cor}_c &: C_c^\bullet(G_{K'',S''}, X) \longrightarrow C_c^\bullet(G_{K',S'}, X),
\end{aligned}$$

which are functorial in X and make the following diagrams commutative up to homotopy:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{v'\in S'_f} C^\bullet(G_{v'}, X)[-1] & \longrightarrow & C_c^\bullet(G_{K',S'}, X) & \longrightarrow & C^\bullet(G_{K',S'}, X) & \longrightarrow & 0 \\
 & & \downarrow \mathrm{res}[-1] & & \downarrow \mathrm{res}_c & & \downarrow \mathrm{res} & & \\
0 & \longrightarrow & \bigoplus_{v''\in S''_f} C^\bullet(G_{v''}, X)[-1] & \longrightarrow & C_c^\bullet(G_{K'',S''}, X) & \longrightarrow & C^\bullet(G_{K'',S''}, X) & \longrightarrow & 0,
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{v''\in S''_f} C^\bullet(G_{v''}, X)[-1] & \longrightarrow & C_c^\bullet(G_{K'',S''}, X) & \longrightarrow & C^\bullet(G_{K'',S''}, X) & \longrightarrow & 0 \\
 & & \downarrow \mathrm{cor}[-1] & & \downarrow \mathrm{cor}_c & & \downarrow \mathrm{cor} & & \\
0 & \longrightarrow & \bigoplus_{v'\in S'_f} C^\bullet(G_{v'}, X)[-1] & \longrightarrow & C_c^\bullet(G_{K',S'}, X) & \longrightarrow & C^\bullet(G_{K',S'}, X) & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccc} C_c^\bullet(G_{K,S},X_U) & \xrightarrow{\mathrm{sh}_c} & C_c^\bullet(G_{K',S'},X) \\ \downarrow{\scriptstyle \mathrm{Tr}_*} & & \downarrow{\scriptstyle \mathrm{res}_c} \\ C_c^\bullet(G_{K,S},X_V) & \xrightarrow{\mathrm{sh}_c} & C_c^\bullet(G_{K'',S''},X), \end{array} \qquad \begin{array}{ccc} C_c^\bullet(G_{K,S},X_V) & \xrightarrow{\mathrm{sh}_c} & C_c^\bullet(G_{K'',S''},X) \\ \downarrow{\scriptstyle \mathrm{pr}_*} & & \downarrow{\scriptstyle \mathrm{cor}_c} \\ C_c^\bullet(G_{K,S},X_U) & \xrightarrow{\mathrm{sh}_c} & C_c^\bullet(G_{K',S'},X) \end{array}$$

8.5.3.5. As all morphisms in 8.5.3.2–8.5.3.5 are functorial in X, they induce morphisms valid for arbitrary ind-admissible $R[G_{K,S}]$-modules X (more generally, for complexes of such modules).

8.5.4. For every bounded below complex T (resp., M) in $\big({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)_{R\text{-}ft}$ (resp., in $\big({}^{\text{ind-ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$), we denote

$$C^\bullet_{c,\mathrm{Iw}}(K_\infty/K,T) = C^\bullet_{c,\mathrm{cont}}(G_{K,S},\mathscr{F}_\Gamma(T))$$
$$C^\bullet_c(K_S/K_\infty,M) = C^\bullet_{c,\mathrm{cont}}(G_{K,S},F_\Gamma(M)).$$

The corresponding objects of $D^+({}_{\overline{R}}\mathrm{Mod})$ will be denoted by $\mathbf{R}\Gamma_{c,\mathrm{Iw}}(K_\infty/K,T)$ (resp., $\mathbf{R}\Gamma_c(K_S/K_\infty,M)$) and their cohomology by $H^i_{c,\mathrm{Iw}}(K_\infty/K,T)$ (resp., $H^i_c(K_S/K_\infty,M)$). This is in line with the notation $\mathbf{R}\Gamma_{\mathrm{Iw}}(K_\infty/K,T)$ (from 8.5.1) for the object of $D\big({}_{\overline{R}}\mathrm{Mod}\big)$ represented by $C^\bullet_{\mathrm{cont}}(G_{K,S},\mathscr{F}_\Gamma(T))$. Similarly, let $\mathbf{R}\Gamma(K_S/K_\infty,M)$ be the object of $D^+\big({}_{\overline{R}}\mathrm{Mod}\big)$ represented by $C^\bullet_{\mathrm{cont}}(G_{K,S},F_\Gamma(M))$ and $H^i(K_S/K_\infty,M)$ its cohomology. As in 8.5.1, the notation for $\mathbf{R}\Gamma_{c,\mathrm{Iw}}$ is slightly ambiguous, since S does not appear explicitly.

8.5.5. Proposition

(i) *For every bounded below complex M in $\big({}^{\text{ind-ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$, the morphisms* sh_c *induce isomorphisms of $\overline{R}$-modules*

$$\varinjlim_{\mathrm{res}} H^i_{c,\mathrm{cont}}(G_{K',S'},M) \xleftarrow{\ \sim\ } \varinjlim_{U,\mathrm{Tr}_*} H^i_{c,\mathrm{cont}}(G_{K,S},M_U)$$
$$\xrightarrow{\ \sim\ } \varinjlim_{U} H^i_{c,\mathrm{cont}}(G_{K,S},{}_UM) \xrightarrow{\ \sim\ } H^i_{c,\mathrm{cont}}(K_S/K_\infty,M).$$

(ii) *For every bounded below complex T in $\big({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)_{R\text{-}ft}$, the canonical morphism of complexes*

$$C^\bullet_{c,\mathrm{cont}}(G,\mathscr{F}_\Gamma(M)) \xrightarrow{\ \sim\ } \varprojlim_{U} C^\bullet_{c,\mathrm{cont}}(G,M_U)$$

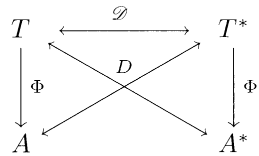
is an isomorphism, and the morphisms sh_c *induce isomorphisms of $\overline{R}$-modules*

$$\varprojlim_{\mathrm{cor}} H^i_{c,\mathrm{cont}}(G_{K',S'},T) \xleftarrow{\ \sim\ } \varprojlim_{U,\mathrm{pr}_*} H^i_{c,\mathrm{cont}}(G_{K,S},T_U) \xrightarrow{\ \sim\ } H^i_{c,\mathrm{Iw}}(K_\infty/K,T).$$

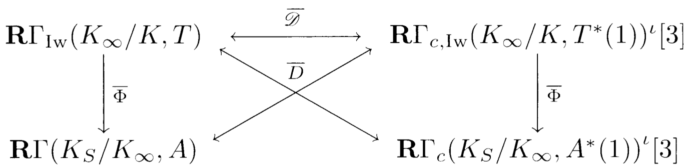
Above, K'/K runs through finite subextensions of K_∞/K and $U = \mathrm{Gal}(K_\infty/K')$.

Proof. — This follows from 8.5.3.2–8.5.3.4 and the corresponding statements for $C^\bullet_{\mathrm{cont}}(G,-)$ and $H^i_{\mathrm{cont}}(G,-)$ (with $G = G_{K,S}, G_v$), proved in Lemma 8.3.2, Proposition 8.3.5 and Proposition 8.4.4.2. □

8.5.6. Theorem. — *If $T, T^* \in D^b_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)$, $A, A^* \in D^b_{R\text{-}coft}\big({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)$ are related by the duality diagram*



over R, then



are related in the same way in $D^b_{(co)ft}\big({}_{\overline{R}}\mathrm{Mod}\big)$ and there are spectral sequences

$$E_2^{i,j} = \mathbb{E}\mathrm{xt}^i_{\overline{R}}\big(\overline{D}(H^j(K_S/K_\infty, A)), \omega_{\overline{R}}\big) = \mathbb{E}\mathrm{xt}^i_{\overline{R}}\big(H^{3-j}_{c,\mathrm{Iw}}(K_\infty/K, T^*(1)), \omega_{\overline{R}}\big)^\iota \Longrightarrow H^{i+j}_{\mathrm{Iw}}(K_\infty/K, T)$$

$${}'E_2^{i,j} = \mathbb{E}\mathrm{xt}^i_{\overline{R}}\big(\overline{D}(H^j_c(K_S/K_\infty, A)), \omega_{\overline{R}}\big) = \mathbb{E}\mathrm{xt}^i_{\overline{R}}\big(H^{3-j}_{\mathrm{Iw}}(K_\infty/K, T^*(1)), \omega_{\overline{R}}\big)^\iota \Longrightarrow H^{i+j}_{c,\mathrm{Iw}}(K_\infty/K, T).$$

Proof. — By Proposition 3.2.8 we can assume that T, T^* (resp., A, A^*) are represented by bounded complexes of admissible $R[G_{K,S}]$-modules, of finite (resp., co-finite) type over R. The statement then follows from Theorem 5.4.5 applied to the duality diagram 8.4.6.6, if we take into account 8.3.2, 8.3.5, 8.4.4.2 and 8.5.5. □

8.5.7. The following fact was implicitly used in 8.5.6: for every $X \in D\big({}_{\overline{R}}\mathrm{Mod}\big)$ and $i \in \mathbf{Z}$ there is a canonical isomorphism

$$\mathbb{E}\mathrm{xt}^i_{\overline{R}}\big(X, \omega_{\overline{R}}\big)^\iota \xrightarrow{\sim} \mathbb{E}\mathrm{xt}^i_{\overline{R}}\big(X^\iota, \omega_{\overline{R}}\big).$$

This follows from the fact that the isomorphism $\iota : \overline{R} \xrightarrow{\sim} \overline{R}^\iota$ of $\overline{R}$-modules induces isomorphisms in $D\big({}_{\overline{R}}\mathrm{Mod}\big)$

$$\omega_{\overline{R}} = \omega \otimes_R \overline{R} \xrightarrow{\sim} \omega \otimes_R \overline{R}^\iota = (\omega_{\overline{R}})^\iota$$

and

$$\overline{\mathscr{D}}(X)^\iota \xrightarrow{\sim} \overline{\mathscr{D}}(X^\iota).$$

8.5.8. Over each finite subextension K_α/K of K_∞/K, the pairing

$$\mathrm{ev}_2 : T \otimes_R T^*(1) \longrightarrow \omega_R(1)$$

induces cup products (5.4.2.1)

$$\langle\ ,\ \rangle_\alpha : H^i_{c,\mathrm{cont}}(G_{K_\alpha,S_\alpha}, T) \otimes_R H^j_{\mathrm{cont}}(G_{K_\alpha,S_\alpha}, T^*(1)) \longrightarrow H^{i+j-3}(\omega_R)$$

(where S_α denotes the set of primes of K_α above S). The pairing

$$\overline{\mathrm{ev}}_2 : \mathscr{F}_\Gamma(T) \otimes_{\overline{R}} \mathscr{F}_\Gamma(T^*(1))^\iota \longrightarrow \omega_{\overline{R}}(1)$$

(cf. 8.4.6.3) induces, in turn, products on the projective limits

$$\begin{aligned}\langle\ ,\ \rangle : \big(\varprojlim_\alpha H^i_{c,\mathrm{cont}}(G_{K_\alpha,S_\alpha}, T)\big) \otimes_{\overline{R}} \big(\varprojlim_\alpha H^j_{\mathrm{cont}}(G_{K_\alpha,S_\alpha}, T^*(1))\big)^\iota \longrightarrow H^{i+j-3}(\omega_{\overline{R}}) \\ = H^{i+j-3}(\omega_R) \otimes_R \overline{R} = \varprojlim_\alpha \big(H^{i+j-3}(\omega_R) \otimes_R R[\mathrm{Gal}(K_\alpha/K)]\big)\end{aligned}$$

(and similarly for the products in which the roles of cohomology and cohomology with compact support are interchanged).

8.5.9. Proposition. — *In the situation of 8.5.8,*

$$\langle (x_\alpha), (y_\alpha) \rangle = \Bigg(\sum_{\sigma \in \mathrm{Gal}(K_\alpha/K)} \langle x_\alpha, \sigma y_\alpha \rangle_\alpha \otimes [\sigma] \Bigg)_\alpha$$

(where σ acts on $H^j_{\mathrm{cont}}(G_{K_\alpha,S_\alpha}, T^*(1))$ by conjugation).

Proof. — This follows from the proof of the corresponding local statement 8.11.10 and the commutative diagrams 8.5.3.3–8.5.3.4. ☐

8.6. Local conditions and Shapiro's Lemma

Let K and S be as in 8.5.

8.6.1. Assume that $K \subset K' \subset K''$ are finite Galois subextensions of K_S/K. For each $v \in S_f$, the fixed embedding $\overline{K} \hookrightarrow \overline{K}_v$ induces primes v'_0 and v''_0 of K' and K'', respectively, with completions $K_v \subset K'_{v'_0} \subset K''_{v''_0} \subset \overline{K}_v$ and absolute Galois groups

$$\overline{G} = \mathrm{Gal}(\overline{K}_v/K_v) \supset \overline{U} = \mathrm{Gal}\big(\overline{K}_v/K'_{v'_0}\big) \supset \overline{V} = \mathrm{Gal}\big(\overline{K}_v/K''_{v''_0}\big).$$

Both $\overline{U}$ and $\overline{V}$ are normal in $\overline{G}$. As in 8.5.3, we have an injective homomorphism $\alpha : \overline{G} \hookrightarrow G = \mathrm{Gal}(\overline{K}/K)$ such that $\overline{U} = \alpha^{-1}(U)$ and $\overline{V} = \alpha^{-1}(V)$, where $U = \mathrm{Gal}(\overline{K}/K') \supset V = \mathrm{Gal}(\overline{K}/K'')$.

8.6.2. Let $X = X^\bullet$ be a complex of ind-admissible $R[G]$-modules. We shall abuse the notation and write X for $\alpha^* X$, *i.e.*, for the same object, but viewed as a complex of $R[\overline{G}]$-modules *via* the map $\alpha : \overline{G} \hookrightarrow G$. As in 8.2, we let $X_{\overline{U}} = X \otimes_R R[\overline{G}/\overline{U}]$ (and similarly for $X_{\overline{V}}$).

Assume that we are given the following local conditions (*i.e.*, morphisms of complexes of R-modules)

$$\begin{aligned}
i_v^+(X_{\overline{U}}) : U_v^+(X_{\overline{U}}) &\longrightarrow C^\bullet_{\mathrm{cont}}(\overline{G}, X_{\overline{U}})\\
i_v^+(X_{\overline{V}}) : U_v^+(X_{\overline{V}}) &\longrightarrow C^\bullet_{\mathrm{cont}}(\overline{G}, X_{\overline{V}})\\
i_{v_0'}^+(X) : U_{v_0'}^+(X) &\longrightarrow C^\bullet_{\mathrm{cont}}(\overline{U}, X)\\
i_{v_0''}^+(X) : U_{v_0''}^+(X) &\longrightarrow C^\bullet_{\mathrm{cont}}(\overline{V}, X)
\end{aligned}$$

together with the following data:

8.6.2.1. Morphisms of complexes

$$\begin{aligned}
\mathrm{sh}^+ : U_v^+(X_{\overline{U}}) &\longrightarrow U_{v_0'}^+(X)\\
\mathrm{sh}^+ : U_v^+(X_{\overline{V}}) &\longrightarrow U_{v_0''}^+(X),
\end{aligned}$$

which are quasi-isomorphisms and make the diagram

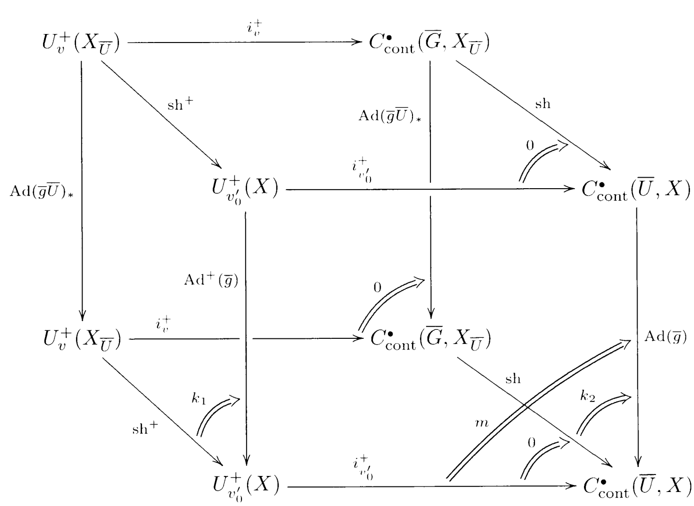
$$\begin{array}{ccc}
U_v^+(X_{\overline{U}}) & \xrightarrow{\;i_v^+\;} & C^\bullet_{\mathrm{cont}}(\overline{G}, X_{\overline{U}})\\
\big\downarrow{\scriptstyle \mathrm{sh}^+} & & \big\downarrow{\scriptstyle \mathrm{sh}}\\
U_{v_0'}^+(X) & \xrightarrow{\;i_{v_0'}^+\;} & C^\bullet_{\mathrm{cont}}(\overline{U}, X)
\end{array}$$

(and its variant with $\overline{U}$ replaced by $\overline{V}$ and v_0' by v_0'') commutative.

8.6.2.2. For each $\overline{g} \in \overline{G}$ a morphism of complexes

$$\mathrm{Ad}^+(\overline{g}) : U_{v_0'}^+(X) \longrightarrow U_{v_0'}^+(X)$$

such that the faces of the following cubic diagram



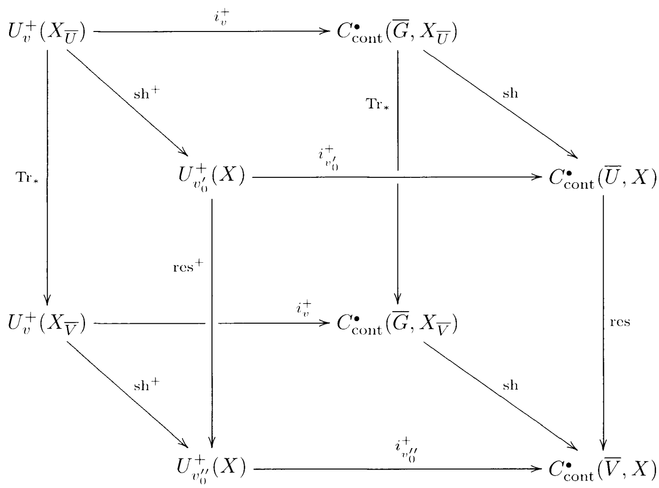
commute up to the indicated homotopies and the boundary of the cube is trivialized by a 2-homotopy

$$i^+_{v'_0} \star k_1 + m \star \mathrm{sh}^+ - k_2 \star i^+_v \rightsquigarrow 0.$$

8.6.2.3. A morphism of complexes

$$\mathrm{res}^+ : U^+_{v'_0}(X) \longrightarrow U^+_{v''_0}(X)$$

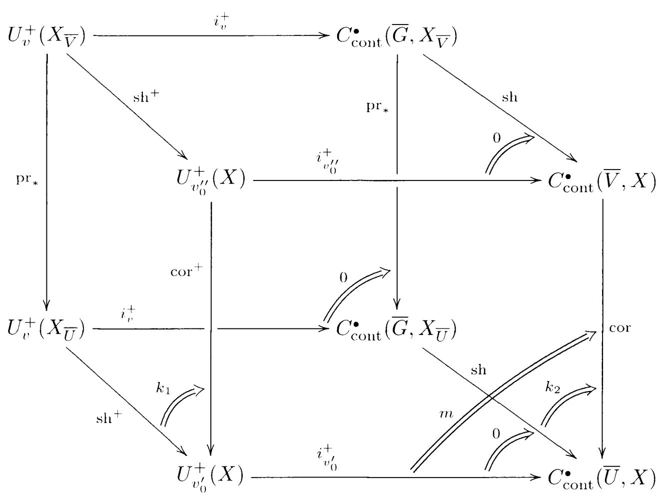
such that each face of the following cubic diagram is commutative:



8.6.2.4. A morphism of complexes

$$\mathrm{cor}^+ : U^+_{v''_0}(X) \longrightarrow U^+_{v'_0}(X)$$

such that the faces of the cubic diagram



(in which the vertical arrow cor is defined using fixed coset representatives $\overline{U} = \bigcup \overline{V}\overline{\alpha}_i$) are commutative up to the indicated homotopies and the boundary of the cube is trivialized by a 2-homotopy

$$i^+_{v'_0} \star k_1 + m \star \mathrm{sh}^+ - k_2 \star i^+_v \rightsquigarrow 0.$$

8.6.3. Fixing coset representatives $\sigma_i, \tau_j \in G$ of

$$G = \bigcup_i U\sigma_i\alpha(\overline{G}) = \bigcup_j V\tau_j\alpha(\overline{G})$$

as in 8.1.7 (with $\sigma_0 = \tau_0 = 1$), we obtain from 8.5.3 isomorphisms of complexes

$$\bigoplus_{v'|v} C^\bullet_{\mathrm{cont}}(G_{v'}, X) \xrightarrow{\sim} \bigoplus_i C^\bullet_{\mathrm{cont}}(\overline{U}, \alpha^* X) \xrightarrow{\sim} \bigoplus_i C^\bullet_{\mathrm{cont}}(\overline{U}, \alpha_i^* X)$$

$$\bigoplus_{v''|v} C^\bullet_{\mathrm{cont}}(G_{v''}, X) \xrightarrow{\sim} \bigoplus_j C^\bullet_{\mathrm{cont}}(\overline{V}, \alpha^* X) \xrightarrow{\sim} \bigoplus_j C^\bullet_{\mathrm{cont}}(\overline{V}, \beta_j^* X)$$

(where $v' \in S'$, $v'' \in S''$).

We define the local condition for X at each $v' = \sigma_i(v'_0)$ to be the composition of

$$i^+_{v'_0} : U^+_{v'_0}(X) \longrightarrow C^\bullet_{\mathrm{cont}}(\overline{U}, X) = C^\bullet_{\mathrm{cont}}(\overline{U}, \alpha^* X)$$

with the inverse of the isomorphism (8.5.3.1)

$$C^\bullet_{\mathrm{cont}}(G_{v'}, X) \xrightarrow{\sim} C^\bullet_{\mathrm{cont}}(\overline{U}, \alpha^* X)$$

(and similarly for the local condition at each $v'' = \tau_j(v''_0)$).

The isomorphism

$$\widetilde{\omega}_U : \alpha^*(X_U) \xrightarrow{\sim} \bigoplus_i (\alpha^* X)_{\overline{U}}$$

(resp., its analogue $\widetilde{\omega}_V$ for X_V) together with $i^+_v(X_{\overline{U}})$ (resp., $i^+_v(X_{\overline{V}})$) define local conditions at v for X_U (resp., X_V).

8.6.4. Putting together the diagrams from 8.1.7.2–8.1.7.5 (modified in accordance with 8.1.7.6) and 8.6.2.1–8.6.2.4, we obtain the following generalizations of 8.5.3:

8.6.4.1. A quasi-isomorphism

$$\mathrm{sh}_f : \widetilde{C}^\bullet_f(G_{K,S}, X_U; \Delta(X_U)) \longrightarrow \widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X)),$$

which is functorial in X, the homotopy class of which is independent of any choices, and such that the following diagram with exact columns is commutative up to homotopy:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\bigoplus_{v\in S_f} C^\bullet_{\mathrm{cont}}(G_v, X_U)[-1] & \xrightarrow{\mathrm{sh}[-1]} & \bigoplus_{v'\in S'_f} C^\bullet_{\mathrm{cont}}(G_{v'}, X)[-1] \\
\downarrow & & \downarrow \\
\widetilde{C}^\bullet_f(G_{K,S}, X_U; \Delta(X_U)) & \xrightarrow{\mathrm{sh}_f} & \widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X)) \\
\downarrow & & \downarrow \\
C^\bullet_{\mathrm{cont}}(G_{K,S}, X_U) \oplus \bigoplus_{v\in S_f} U_v^+(X_U) & \xrightarrow{(\mathrm{sh},\mathrm{sh}^+)} & C^\bullet_{\mathrm{cont}}(G_{K',S'}, X) \oplus \bigoplus_{v'\in S'_f} U_{v'}^+(X) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

8.6.4.2. For each $g \in G_{K,S}$ a morphism of complexes

$$\mathrm{Ad}(g)_f : \widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X)) \longrightarrow \widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X)),$$

which is functorial in X and makes the following diagrams commutative up to homotopy:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\bigoplus_{v'\in S'_f} C^\bullet_{\mathrm{cont}}(G_{v'}, X)[-1] & \xrightarrow{F(g)[-1]} & \bigoplus_{v'\in S'_f} C^\bullet_{\mathrm{cont}}(G_{v'}, X)[-1] \\
\downarrow & & \downarrow \\
\widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X)) & \xrightarrow{\mathrm{Ad}(g)_f} & \widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X)) \\
\downarrow & & \downarrow \\
C^\bullet_{\mathrm{cont}}(G_{K',S'}, X) \oplus \bigoplus_{v'\in S'_f} U_{v'}^+(X) & \xrightarrow{(\mathrm{Ad}(g),\mathrm{Ad}^+(g))} & C^\bullet_{\mathrm{cont}}(G_{K',S'}, X) \oplus \bigoplus_{v'\in S'_f} U_{v'}^+(X) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

$$\begin{array}{ccc}
\widetilde{C}^\bullet_f(G_{K,S}, X_U; \Delta(X_U)) & \xrightarrow{\mathrm{sh}_f} & \widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X)) \\
\big\downarrow{\scriptstyle \mathrm{Ad}(gU)_*} & & \big\downarrow{\scriptstyle \mathrm{Ad}(g)_f} \\
\widetilde{C}^\bullet_f(G_{K,S}, X_U; \Delta(X_U)) & \xrightarrow{\mathrm{sh}_f} & \widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X))
\end{array}$$

8.6.4.3. Morphisms of complexes

$$\begin{aligned}
\mathrm{res}_f &: \widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X)) \longrightarrow \widetilde{C}^\bullet_f(G_{K'',S''}, X; \Delta(X)) \\
\mathrm{cor}_f &: \widetilde{C}^\bullet_f(G_{K'',S''}, X; \Delta(X)) \longrightarrow \widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X)),
\end{aligned}$$

which are functorial in X and make the following diagrams commutative up to homotopy:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\bigoplus_{v'\in S'_f} C^\bullet_{\mathrm{cont}}(G_{v'}, X)[-1] & \xrightarrow{\mathrm{res}[-1]} & \bigoplus_{v''\in S''_f} C^\bullet_{\mathrm{cont}}(G_{v''}, X)[-1] \\
\downarrow & & \downarrow \\
\widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X)) & \xrightarrow{\mathrm{res}_f} & \widetilde{C}^\bullet_f(G_{K'',S''}, X; \Delta(X)) \\
\downarrow & & \downarrow \\
C^\bullet_{\mathrm{cont}}(G_{K',S'}, X) \oplus \bigoplus_{v'\in S'_f} U^+_{v'}(X) & \xrightarrow{(\mathrm{res},\mathrm{res}^+)} & C^\bullet_{\mathrm{cont}}(G_{K'',S''}, X) \oplus \bigoplus_{v''\in S''_f} U^+_{v''}(X) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

$$\begin{array}{ccc}
\widetilde{C}^\bullet_f(G_{K,S}, X_U; \Delta(X_U)) & \xrightarrow{\mathrm{sh}_f} & \widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X)) \\
\Big\downarrow{\scriptstyle \mathrm{Tr}_*} & & \Big\downarrow{\scriptstyle \mathrm{res}_f} \\
\widetilde{C}^\bullet_f(G_{K,S}, X_V; \Delta(X_V)) & \xrightarrow{\mathrm{sh}_f} & \widetilde{C}^\bullet_f(G_{K'',S''}, X; \Delta(X))
\end{array}$$

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\bigoplus_{v''\in S''_f} C^\bullet_{\mathrm{cont}}(G_{v''}, X)[-1] & \xrightarrow{\mathrm{cor}[-1]} & \bigoplus_{v'\in S'_f} C^\bullet_{\mathrm{cont}}(G_{v'}, X)[-1] \\
\downarrow & & \downarrow \\
\widetilde{C}^\bullet_f(G_{K'',S''}, X; \Delta(X)) & \xrightarrow{\mathrm{cor}_f} & \widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X)) \\
\downarrow & & \downarrow \\
C^\bullet_{\mathrm{cont}}(G_{K'',S''}, X) \oplus \bigoplus_{v''\in S''_f} U^+_{v''}(X) & \xrightarrow{(\mathrm{cor},\mathrm{cor}^+)} & C^\bullet_{\mathrm{cont}}(G_{K',S'}, X) \oplus \bigoplus_{v'\in S'_f} U^+_{v'}(X) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

$$\begin{array}{ccc}
\widetilde{C}^\bullet_f(G_{K,S}, X_V; \Delta(X_V)) & \xrightarrow{\mathrm{sh}_f} & \widetilde{C}^\bullet_f(G_{K'',S''}, X; \Delta(X)) \\
\Big\downarrow{\scriptstyle \mathrm{pr}_*} & & \Big\downarrow{\scriptstyle \mathrm{cor}_f} \\
\widetilde{C}^\bullet_f(G_{K,S}, X_U; \Delta(X_U)) & \xrightarrow{\mathrm{sh}_f} & \widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X)).
\end{array}$$

8.6.4.4. Lemma

(i) *For $g, g' \in G_{K,S}$, the action of* $\mathrm{Ad}(gg')_f$ *on* $\widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X))$ *is homotopic to* $\mathrm{Ad}(g)_f \circ \mathrm{Ad}(g')_f$. *If $g \in G_{K',S'} \subset G_{K,S}$, then* $\mathrm{Ad}(g)_f$ *is homotopic to the identity.*

(ii) $\mathrm{cor}_f \circ \mathrm{res}_f$ *is homotopic to* $[K'' : K'] \cdot \mathrm{id}$ *on* $\widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X))$.

(iii) *If K''/K' is a Galois extension, then the action of* $\mathrm{res}_f \circ \mathrm{cor}_f$ *on* $\widetilde{C}^\bullet_f(G_{K'',S''}, X; \Delta(X))$ *is homotopic to* $\sum_{\sigma\in \mathrm{Gal}(K''/K')} \mathrm{Ad}(\widetilde{\sigma})_f$, *where $\widetilde{\sigma} \in G_{K',S'}$ is any lift of σ.*

(iv) *If K''/K' is a Galois extension of degree $d = [K'' : K']$, then* res_f *induces isomorphisms*

$$\widetilde{H}^i_f(G_{K',S'}, X; \Delta(X)) \otimes_R R[1/d] \xrightarrow{\sim} \big(\widetilde{H}^i_f(G_{K'',S''}, X; \Delta(X)) \otimes_R R[1/d]\big)^{\mathrm{Gal}(K''/K')}.$$

Proof

(i) This follows from the last diagram in 8.6.4.2 and equalities $\mathrm{Ad}(gg'U) = \mathrm{Ad}(gU)\mathrm{Ad}(g'U)$, $\mathrm{Ad}(U) = \mathrm{id}$.

(ii)-(iii) Combine the third and the last diagram in 8.6.4.3 with the formulas $\mathrm{pr} \circ \mathrm{Tr} = [U : V] \cdot \mathrm{id}$, $\mathrm{Tr} \circ \mathrm{pr} = \sum_{\sigma V \in U/V} \mathrm{Ad}(\sigma V)$.

(iv) This follows from (ii) and (iii). □

8.7. Functoriality of the unramified local conditions

Before proceeding further we must clarify various functorial properties of the objects studied in Chapter 7. Fix $v \in S_f$ not dividing p and consider the group $G = \overline{G}_v = \langle t \rangle \rtimes \langle f \rangle$ introduced in 7.2.1 (*i.e.*, $tf = ft^L$, where $L = \ell^r$ is a power of the residue characteristic $\ell \neq p$ of v); in order to emphasize dependence on f and t, we include them in the notation. Throughout Sect. 8.7, M will be a p-primary torsion discrete G-module and $M^\bullet$ a bounded below complex of such modules.

8.7.1. Dependence on f, t. — Any change of topological generators of G is given by

$$f' = ft^A, \quad t' = t^B \qquad (A \in \widehat{\mathbf{Z}}/\mathbf{Z}_\ell,\ B \in (\widehat{\mathbf{Z}}/\mathbf{Z}_\ell)^*).$$

If $A \in \mathbf{N}_0$ and $B \in \mathbf{N}$, then the formulas

$$\alpha_0(m) = m, \quad \alpha_2(m) = (1 + t + \cdots + t^{B-1})m$$
$$\alpha_1(m, m') = (m + f(1 + t + \cdots + t^{A-1})m', (1 + t + \cdots + t^{B-1})m')$$

define a morphism of complexes

$$\alpha = \alpha(f, t; f', t') : C(M, f, t) \longrightarrow C(M, f', t')$$

(where $C(M, f, t) = C(M)$ was defined in 7.2.1) such that the composition

$$C(M, f, t) \xrightarrow{\alpha} C(M, f', t') \xrightarrow{\mu_{f',t'}} C^\bullet(G, M)$$

is equal to $\mu_{f,t}$ (defined in 7.2.4). For general A, B one can define α by continuity, or by using the pro-finite differential calculus from 7.4. The sign conventions of 7.6 yield morphisms of complexes

$$\alpha : U(M^\bullet, f, t) \xrightarrow{\sim} C(M^\bullet, f, t) \xrightarrow{\alpha} C(M^\bullet, f', t') \xrightarrow{\sim} U(M^\bullet, f', t').$$

The morphisms α are transitive in the following sense. If

$$f'' = f'(t')^{A'}, \quad t'' = (t')^{B'}$$

(hence $f'' = ft^{A''}$, $t'' = t^{B''}$ with $A'' = A+BA'$, $B'' = BB'$), then the matrix equality

$$\begin{pmatrix} 1 & f'\big(1+t'+\cdots+(t')^{A'-1}\big) \\ 0 & 1+t'+\cdots+(t')^{B'-1} \end{pmatrix} \begin{pmatrix} 1 & f\big(1+t+\cdots+t^{A-1}\big) \\ 0 & 1+t+\cdots+t^{B-1} \end{pmatrix} = \begin{pmatrix} 1 & f\big(1+t+\cdots+t^{A''-1}\big) \\ 0 & 1+t+\cdots+t^{B''-1} \end{pmatrix}$$

implies that

$$\alpha(f', t'; f'', t'') \circ \alpha(f, t; f', t') = \alpha(f, t; f'', t'').$$

In particular, each $\alpha(f, t; f', t')$ is an isomorphism of complexes.

8.7.2. Restriction. — Let $U = \langle t^B \rangle \rtimes \langle f^A \rangle$, where $A, B \in \mathbf{N}$ and $(L, B) = 1$. This is an open subgroup of G with a similar structure: $f' = f^A$ and $t' = t^B$ satisfy $t'f' = f'(t')^{L'}$ with $L' = L^A$. The formulas

$$\mathrm{res}_0(m) = m, \quad \mathrm{res}_1(m, m') = ((1+f+\cdots+f^{A-1})m, (1+t+\cdots+t^{B-1})m'),$$
$$\mathrm{res}_2(m) = (1+t+\cdots+t^{B-1})(1+\theta+\cdots+\theta^{A-1})m$$

define a morphism of complexes

$$\mathrm{res} : C(M, f, t) \longrightarrow C(M, f^A, t^B)$$

making the following diagram commutative:

$$\begin{array}{ccc} C(M, f, t) & \xrightarrow{\mu_{f,t}} & C^\bullet(G, M) \\ \big\downarrow \mathrm{res} & & \big\downarrow \mathrm{res} \\ C(M, f^A, t^B) & \xrightarrow{\mu_{f^A, t^B}} & C^\bullet(U, M). \end{array}$$

8.7.3. Restriction - unramified case. — Assume that $B = 1$, *i.e.*, $U = \langle t \rangle \rtimes \langle f^A \rangle$ ($A \in \mathbf{N}$). In this case $U = \langle t' \rangle \rtimes \langle f' \rangle$ is an open normal subgroup of G with

$$f' = f^A, \quad t' = t, \quad t'f' = f'(t')^{L'} \quad (L' = L^A),$$
$$\theta' = f'(1+t'+\cdots+(t')^{L'-1}) = \theta^A.$$

The formula

$$\mathrm{res} = (\mathrm{id}, \mathrm{id}, 1+f+\cdots+f^{A-1}, 1+\theta+\cdots+\theta^{A-1})$$

defines a morphism of complexes

$$\mathrm{res} : U(M^\bullet, f, t) \longrightarrow U(M^\bullet, f^A, t),$$

which maps the subcomplex $U^+(M^\bullet, f, t)$ to $U^+(M^\bullet, f^A, t)$, reduces to the map res from 8.7.2 if $M^\bullet = M$, and makes the following diagram commutative:

$$\begin{array}{ccc} U(M^\bullet, f, t) & \xrightarrow{\mu_{f,t}} & C^\bullet(G, M^\bullet) \\ \big\downarrow \mathrm{res} & & \big\downarrow \mathrm{res} \\ U(M^\bullet, f^A, t) & \xrightarrow{\mu_{f^A,t}} & C^\bullet(U, M^\bullet). \end{array}$$

8.7.4. Corestriction - unramified case. — Under the assumptions of 8.7.3, the formula

$$\mathrm{cor} = (1 + f + \cdots + f^{A-1}, 1 + \theta + \cdots + \theta^{A-1}, \mathrm{id}, \mathrm{id})$$

defines a morphism of complexes

$$\mathrm{cor} : U(M^\bullet, f^A, t) \longrightarrow U(M^\bullet, f, t),$$

which maps the subcomplex $U^+(M^\bullet, f^A, t)$ to $U^+(M^\bullet, f, t)$. A short calculation (based on the formulas in Proposition 7.2.4 (i) and its proof) shows that the diagram

$$\begin{array}{ccc} U(M^\bullet, f^A, t) & \xrightarrow{\mu_{f^A,t}} & C^\bullet(U, M^\bullet) \\ \big\downarrow \mathrm{cor} & & \big\downarrow \mathrm{cor} \\ U(M^\bullet, f, t) & \xrightarrow{\mu_{f,t}} & C^\bullet(G, M^\bullet) \end{array}$$

commutes, where the right vertical arrow cor is defined using the coset representatives $\overline{\alpha}_i = f^{-i}$ $(0 \leqslant i < A)$.

8.7.5. Shapiro's Lemma - unramified case. — Still keeping the notation of 8.7.3, we define a morphism of complexes

$$\mathrm{sh} : U(M^\bullet_U, f, t) \longrightarrow U(M^\bullet, f^A, t)$$

to be the composite map

$$U(M^\bullet_U, f, t) \xrightarrow{\mathrm{res}} U(M^\bullet_U, f^A, t) \xrightarrow{(\delta_U)_*} U(M^\bullet, f^A, t),$$

where δ_U was defined in 8.1.3. As μ commutes with res and is functorial in $M^\bullet$, it follows from 8.1.3 that the following diagram is commutative:

$$\begin{array}{ccc} U(M^\bullet_U, f, t) & \xrightarrow{\mu_{f,t}} & C^\bullet(G, M^\bullet_U) \\ \big\downarrow \mathrm{sh} & & \big\downarrow \mathrm{sh} \\ U(M^\bullet, f^A, t) & \xrightarrow{\mu_{f^A,t}} & C^\bullet(U, M^\bullet). \end{array}$$

We know that both horizontal arrows and the right vertical arrow are quasi-isomorphisms; thus the left vertical arrow is also a Qis.

8.7.6. Shapiro's Lemma for U^+ - unramified case. — The restriction of the morphism sh from 8.7.5 to the subcomplex $U^+(M_U^\bullet, f, t)$ defines a morphism of complexes

$$\mathrm{sh}^+ : U^+(M_U^\bullet, f, t) \xrightarrow{\mathrm{res}} U^+(M_U^\bullet, f^A, t) \xrightarrow{(\delta_U)_*} U^+(M^\bullet, f^A, t).$$

We claim that sh^+ is also a quasi-isomorphism. By dévissage and (7.6.3.1), it is enough to consider the following two cases:

(i) $M^\bullet = \tau_{\geqslant -1} M^\bullet$, in which case $U^+(M^\bullet) = U(M^\bullet)$ and we can apply 8.7.5.

(ii) $M^\bullet = M^0 = M$: putting $N = M^{t=1}$, we have $(M_U)^{t=1} = N_U$, and it remains to check that the composite morphism

$$g : \left[N_U \xrightarrow{f-1} N_U\right] \xrightarrow{(\mathrm{id}, 1+f+\cdots+f^{A-1})} \left[N_U \xrightarrow{f^A-1} N_U\right] \xrightarrow{(\delta_U, \delta_U)} \left[N_U \xrightarrow{f^A-1} N_U\right]$$

is a quasi-isomorphism. This follows from the fact that the quasi-isomorphism μ from 7.2.2 (with respect to $\sigma = f$) maps g to

$$\mathrm{sh} : C^\bullet(\langle f \rangle, N_U) \xrightarrow{\mathrm{res}} C^\bullet(\langle f^A \rangle, N_U) \xrightarrow{(\delta_U)_*} C^\bullet(\langle f^A \rangle, N_U).$$

8.7.7. Conjugation - unramified case. — For each $g \in G$, the morphism of complexes

$$\mathrm{Ad}(g) : U(M^\bullet, f^A, t) \xrightarrow{\mu_{f^A,t}} C^\bullet(U, M^\bullet) \xrightarrow{\mathrm{Ad}(g)} C^\bullet(U, M^\bullet) \xrightarrow{\lambda_{f^A,t}} U(M^\bullet, f^A, t)$$

maps the subcomplex $U^+(M^\bullet, f^A, t)$ to itself.

8.7.8. As all morphisms in 8.7.1–8.7.7 are functorial in $M^\bullet$, they extend, by the usual limit procedure, to the case when $M^\bullet$ is a complex of ind-admissible $R[\overline{G}_v]$-modules.

8.8. Greenberg's local conditions in Iwasawa theory

8.8.1. Let $\Sigma \subset S_f$ and $\Sigma' = S_f - \Sigma$ be as in 7.8.1, and K_∞/K as in 8.5.1. Throughout sections 8.8–8.9 we assume that the following condition holds:

(U) Each prime $v \in \Sigma'$ is unramified in K_∞/K.

This condition is automatically satisfied if $\Delta = 0$. As in 7.8, fix f_v and t_v for each $v \in \Sigma'$.

8.8.2. Let T (resp., M) be a bounded below complex in $\left({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\right)_{R\text{-}ft}$ (resp., in $\left({}^{\text{ind-ad}}_{R[G_{K,S}]}\mathrm{Mod}\right)_{\{\mathfrak{m}\}}$). Assume that we are given, for each $v \in \Sigma$, a bounded below complex T_v^+ (resp., M_v^+) in $\left({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\right)_{R\text{-}ft}$ (resp., in $\left({}^{\text{ind-ad}}_{R[G_v]}\mathrm{Mod}\right)_{\{\mathfrak{m}\}}$) and a morphism of complexes of $R[G_v]$-modules $T_v^+ \to T$ (resp., $M_v^+ \to M$).

8.8.3. The data from 8.8.2 define Greenberg's local conditions

$$\Delta_v(Z) = \begin{cases} C^\bullet_{\mathrm{cont}}(G_v, Z_v^+) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v, Z), & (v \in \Sigma) \\ C^\bullet_{\mathrm{ur}}(G_v, Z) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v, Z), & (v \in \Sigma') \end{cases}$$

for $Z = T, M, T_U, M_U, \mathscr{F}_\Gamma(T), F_\Gamma(M)$ (where U is an open subgroup of $G = G_K$).

Recall that we have fixed an embedding $\overline{K} \hookrightarrow \overline{K}_v$, for each $v \in S_f$. If K'/K is a finite subextension of K_∞/K and $v_0' \mid v$ the prime of K' induced by $\overline{K} \hookrightarrow \overline{K}_v$, then $G_{v_0'} \subset G_v$ and we can define Greenberg's local conditions at v_0' for $Z = T, M$ by

$$\Delta_{v_0'}(Z) = \begin{cases} C^\bullet_{\mathrm{cont}}(G_{v_0'}, Z_v^+) \longrightarrow C^\bullet_{\mathrm{cont}}(G_{v_0'}, Z), & (v \in \Sigma) \\ C^\bullet_{\mathrm{ur}}(G_{v_0'}, Z) \longrightarrow C^\bullet_{\mathrm{cont}}(G_{v_0'}, Z), & (v \in \Sigma'). \end{cases}$$

8.8.4. We must check that the local conditions for $Z = T, M$ defined in 8.8.3 satisfy the properties 8.6.2.1–8.6.2.4 (for $K \subset K' \subset K'' \subset K_\infty$).

8.8.4.1. This is straightforward for $v \in \Sigma$: in this case $\mathrm{Ad}^+(\overline{g}) = \mathrm{Ad}(\overline{g})$, $\mathrm{sh}^+ = \mathrm{sh}$, $\mathrm{res}^+ = \mathrm{Tr}_*$, $\mathrm{cor}^+ = \mathrm{pr}_*$, all homotopies in 8.6.2.4 are zero, while in 8.6.2.2 we have $m = 0$ and k_1, k_2 are as in 8.1.6.3 (bi-functorial). Finally, the boundaries of the cubes in 8.6.2.2 and 8.6.2.4 are both trivialized by the zero 2-homotopy, as

$$i^+_{v_0'} \star k_1 + m \star \mathrm{sh}^+ - k_2 \star i^+_v = 0$$

in both cases.

8.8.4.2. Assume that $v \in \Sigma'$ and write $f = f_v$, $t = t_v$. Let $X = X^\bullet$ be a complex in $\left({}^{\text{ind-ad}}_{R[G_v]}\mathrm{Mod}\right)$. It will be enough to verify 8.6.2.1–8.6.2.4 in the case when X is a complex of discrete p-primary torsion $\overline{G}_v$-modules, provided that all morphisms, homotopies and 2-homotopies are functorial in X.

As in 7.5.7, we can assume that the wild inertia I_v^w acts trivially on X; it is then sufficient to consider a variant of 8.6.2.1–8.6.2.4, in which $\overline{G} \supset \overline{U} \supset \overline{V}$ are replaced by their quotients by I_v^w. We shall abuse the notation and denote these quotients again by $\overline{G}, \overline{U}, \overline{V}$. Note that the assumption (U) implies that

$$\overline{G} = \langle t \rangle \rtimes \langle f \rangle, \quad \overline{U} = \langle t \rangle \rtimes \langle f^A \rangle, \quad \overline{V} = \langle t \rangle \rtimes \langle f^{AA'} \rangle$$

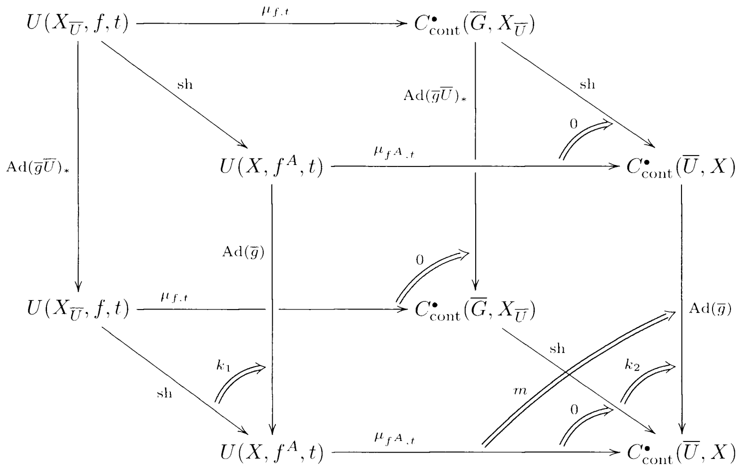
for some $A, A' \in \mathbf{N}$.

The morphisms sh^+ (resp., res^+) satisfying 8.6.2.1 (resp., 8.6.2.3) were defined in 8.7.6 (resp., 8.7.3).

8.8.4.3. Conjugation. — For $\overline{g} \in \overline{G}$ the morphism

$$\mathrm{Ad}^+(\overline{g}) = \mathrm{Ad}(\overline{g}) : U^+(X, f^A, t) \longrightarrow U^+(X, f^A, t)$$

was defined in 8.7.7. As the inclusion $U^+(X, f^A, t) \hookrightarrow U(X, f^A, t)$ commutes with $\mathrm{Ad}(\overline{g})$ and sh (and everything in sight is functorial in X), it is enough to consider the following cubic diagram, which involves $U(X)$ instead of $U^+(X)$



(and also check that k_1 preserves $U^+(-)$). The faces of the cube are commutative up to the following homotopies:

$$m = b \star (\mathrm{Ad}(\overline{g}) \circ \mu_{f^A,t}) \qquad \text{(where } b : \mu_{f^A,t} \circ \lambda_{f^A,t} \rightsquigarrow \mathrm{id} \text{ is as in 7.4.9)}$$
$$k_1 = (\lambda_{f^A,t} \circ \mathrm{sh}) \star h_{\overline{g}} \star (\mu_{f,t} \circ \mathrm{Ad}(\overline{g}\overline{U})_*) \quad \text{(as } \mathrm{Ad}(\overline{g}) \circ \mathrm{sh} = \lambda_{f^A,t} \circ \mathrm{Ad}(\overline{g}) \circ \mu_{f,t} \circ \mathrm{sh})$$
$$k_2 = (\mathrm{sh} \circ \mathrm{Ad}(\overline{g}\overline{U})_*) \star h_{\overline{g}} \qquad \text{(as in 8.1.6.3).}$$

We now check that the homotopy k_1 maps

$$U^+(X_{\overline{U}}, f, t)^i \subset U(X_{\overline{U}}, f, t)^i$$

to

$$U^+(X_{\overline{U}}, f^A, t)^{i-1} \subset U(X_{\overline{U}}, f^A, t)^{i-1}.$$

As $\sigma_{\geqslant 2}\, U^+(-) = 0$ and $\sigma_{\leqslant -1}\, U^+(-) = \sigma_{\leqslant -1}\, U(-)$, we only have to consider the case $i = 1$. However, the inclusion $k_1(U^+(-)^1) \subseteq U^+(-)^0$ follows from

$$dk_1\,(U^+(-)^1) = (dk_1 + k_1 d)U^+(-)^1 \subseteq U^+(-)^0.$$

Finally, it remains to verify that the homotopy

$$h := \mu_{f^A,t} \star k_1 - k_2 \star \mu_{f,t} + m \star \mathrm{sh}$$

is 2-homotopic to zero. We have 2-homotopies

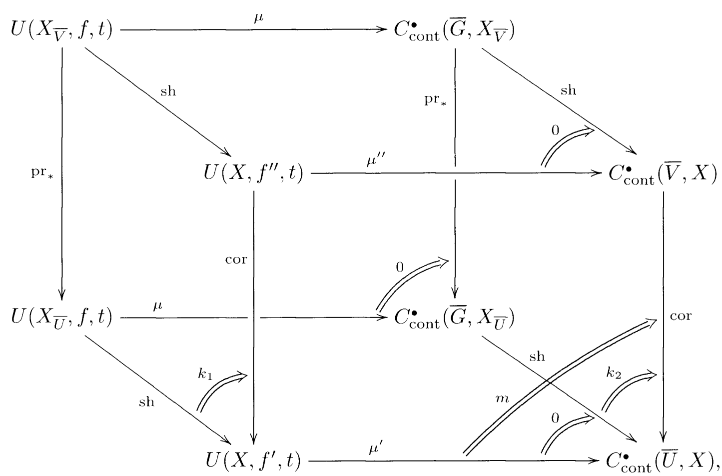
$$\begin{aligned}
&\mu_{f^A,t} \star k_1 - k_2 \star \mu_{f,t} = ((\mu_{f^A,t} \circ \lambda_{f^A,t} - \mathrm{id}) \circ \mathrm{sh}) \star h_{\overline{g}} \star (\mathrm{Ad}(\overline{g}\overline{U})_* \circ \mu_{f,t}) \\
&= -((db+bd)\circ \mathrm{sh}) \star h_{\overline{g}} \star (\mathrm{Ad}(\overline{g}\overline{U})_* \circ \mu_{f,t}) \rightsquigarrow -b \star (d(\mathrm{sh} \star h_{\overline{g}}) + (\mathrm{sh} \star h_{\overline{g}})d) \circ \mathrm{Ad}(\overline{g}\overline{U})_* \circ \mu_{f,t} \\
&\qquad = b \star (\mathrm{sh} \circ (\mathrm{id} - \mathrm{Ad}(\overline{g})) \circ \mathrm{Ad}(\overline{g}\overline{U})_* \circ \mu_{f,t}),
\end{aligned}$$

hence

$$h \rightsquigarrow b \star (\mathrm{sh} \circ \mathrm{Ad}(\overline{g}\overline{U})_* \circ \mu_{f,t}) = b \star (\mu_{f^A,t} \circ \mathrm{sh} \circ \mathrm{Ad}(\overline{g}\overline{U})_*),$$

which is 2-homotopic to zero, by Lemma 7.4.9. This completes the verification of 8.6.2.2.

8.8.4.4. Corestriction. — In order to verify 8.6.2.4, it is enough, as in 8.8.4.3, to consider the following cubic diagram (in which $f' = f^A$, $f'' = f^{AA'}$, $\mu = \mu_{f,t}$, $\mu' = \mu_{f',t}$, $\mu'' = \mu_{f'',t}$)



in which the morphism

$$\mathrm{cor} : C^\bullet_{\mathrm{cont}}(\overline{V}, X) \longrightarrow C^\bullet_{\mathrm{cont}}(\overline{U}, X)$$

is defined in terms of the coset representatives $(f'')^{-i}$, $0 \leqslant i < A'$.

We shall give the details only in the case when $\overline{G} = \overline{U}$ (*i.e.*, $A = 1$, $f = f'$), and leave the general case to the reader.

The faces of the cube are commutative up to the following homotopies:

$$\begin{aligned} m &= 0 && \text{(by 8.7.4)} \\ k_1 &= h \star \mathrm{res} && \text{(with } h \text{ defined below)} \\ k_2 &= \text{defined as in 8.1.5} \end{aligned}$$

In order to define k_1, it is enough to consider the case when X is concentrated in degree zero. The first step is to construct a homotopy

$$h : (\delta_{\overline{U}})_* \circ \mathrm{pr}_* \rightsquigarrow \mathrm{cor} \circ (\delta_{\overline{V}})_* \circ \mathrm{res}$$

between

$$C(X_{\overline{V}}, f', t) \xrightarrow{\mathrm{pr}_*} C(X_{\overline{U}}, f', t) \xrightarrow{(\delta_{\overline{U}})_*} C(X, f', t)$$

and

$$C(X_{\overline{V}}, f', t) \xrightarrow{\mathrm{res}} C(X_{\overline{V}}, f'', t) \xrightarrow{(\delta_{\overline{V}})_*} C(X, f'', t) \xrightarrow{\mathrm{cor}} C(X, f', t).$$

Writing an arbitrary element of $X_{\overline{V}}$ as

$$x = \sum_{j=0}^{A'-1} [(f')^j \overline{V}] \otimes x_j, \quad x_j = \delta_{(f')^j \overline{V}}(x) \in X,$$

the homotopy h is given by the formulas

$$h^0(x, x') = \sum_{j=0}^{A'-1} \delta_{(f')^j \overline{V}} \big((1 + f' + \cdots + (f')^{j-1})x\big) = \sum_{j=0}^{A'-1} (x_j + f' x_{j-1} + \cdots + (f')^{j-1} x_1)$$

$$h^1(x) = (0, \sum_{j=0}^{A'-1} (1 + \theta' + \cdots + (\theta')^{j-1}) x_{A'-j})$$

(note that we are considering only the special case when $f' = f$, hence $\theta' = \theta$). Having constructed h, we define

$$k_1 = h \star \mathrm{res} : \mathrm{sh} \circ \mathrm{pr}_* \rightsquigarrow \mathrm{cor} \circ \mathrm{sh},$$

where

$$\mathrm{res} : C(X_{\overline{V}}, f, t) \longrightarrow C(X_{\overline{V}}, f', t)$$

was defined in 8.7.3–8.7.4. The same argument as in 8.8.4.3 shows that k_1 maps $U^+(-)^i$ to $U^+(-)^{i-1}$.

We claim that we can take zero for the 2-homotopy in 8.6.2.4, *i.e.*,

$$\mu' \star k_1 - k_2 \star \mu = 0.$$

Indeed, in degree 1,

$$k_2^0(c) = \sum_{j=0}^{A'-1} \delta_{f^j \overline{V}}(c(f^j)),$$

hence we have (for $x, x' \in X_{\overline{V}}$)

$$k_2^0 \circ \mu_1(x, x') = \sum_{j=0}^{A'-1} \delta_{f^j \overline{V}}\big((1 + f + \cdots + f^{j-1})x\big) = h^0(x, x').$$

In degree 2, let $g = f^a t^b$ with $a, b \in \mathbf{N}_0$. Then

$$\big(k_2^1(z)\big)(g) = \sum_{i=0}^{A'-1} \delta_{f^i \overline{V}}\big(z(f^i, f^{-i} g (\overline{f^{-i} g})^{-1}) - z(g, (\overline{f^{-i} g})^{-1})\big),$$

where

$$(\overline{f^{-i} g})^{-1} = f^{j(i)}, \quad 0 \leqslant j(i) < A', \quad j(i) \equiv i - a \ (\mathrm{mod}\, A').$$

If $z = \mu_2(x)$ (for $x \in X_{\overline{V}}$), then the formulas in 7.2.4(i) give

$$\begin{aligned}\left(k_2^1 \circ \mu_2(x)\right)(f^a t^b) &= \sum_{i=0}^{A'-1} \delta_{f^i\overline{V}}\big(f^a(1+t+\cdots+t^{b-1})(1+\theta+\cdots+\theta^{j(i)-1})x\big)\\ &= f^a(1+t+\cdots+t^{b-1})\sum_{j=1}^{A'-1}(1+\theta+\cdots+\theta^{j-1})x_{A'-j}\\ &= ((\mu_1' \circ k_1^1)(x))(f^a t^b),\end{aligned}$$

as claimed. This concludes the verification of 8.6.2.4.

8.8.5. In analogy to 8.5.4, we define

$$\begin{aligned}\widetilde{C}^\bullet_{f,\mathrm{Iw}}(K_\infty/K,T) &= \widetilde{C}^\bullet_f\big(G_{K,S},\mathscr{F}_\Gamma(T);\Delta\big(\mathscr{F}_\Gamma(T)\big)\big)\\ \widetilde{C}^\bullet_f(K_S/K_\infty,M) &= \widetilde{C}^\bullet_f\big(G_{K,S},F_\Gamma(M);\Delta\big(F_\Gamma(M)\big)\big).\end{aligned}$$

The corresponding objects of $D({}_{\overline{R}}\mathrm{Mod})$ will be denoted by $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K,T)$ (resp., $\widetilde{\mathbf{R}\Gamma}_f(K_S/K_\infty,M)$) and their cohomology by $\widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K,T)$ (resp., $\widetilde{H}^i_f(K_S/K_\infty,M)$).

8.8.6. Proposition

(i) *For every bounded below complex M in $\big({}^{\text{ind-ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)_{\{\mathfrak{m}\}}$, the morphisms* sh_f *induce isomorphisms of $\overline{R}$-modules*

$$\begin{aligned}\varinjlim_{\mathrm{res}} \widetilde{H}^i_f(G_{K',S'},M;\Delta(M)) &\xleftarrow{\sim} \varinjlim_{U,\mathrm{Tr}_*} \widetilde{H}^i_f(G_{K,S},M_U;\Delta(M_U))\\ &\xrightarrow{\sim} \varinjlim_{U} \widetilde{H}^i_f(G_{K,S},{}_UM;\Delta({}_UM)) \xrightarrow{\sim} \widetilde{H}^i_f(K_S/K_\infty,M).\end{aligned}$$

(ii) *For every bounded below complex T in $({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod})_{R\text{-}ft}$, the canonical morphism of complexes*

$$\widetilde{C}^\bullet_f\big(G_{K,S},\mathscr{F}_\Gamma(T);\Delta\big(\mathscr{F}_\Gamma(T)\big)\big) \xrightarrow{\sim} \varprojlim_U \widetilde{C}^\bullet_f(G_{K,S},T_U;\Delta(T_U))$$

is an isomorphism, and the morphisms sh_f *induce isomorphisms of $\overline{R}$-modules*

$$\varprojlim_{\mathrm{cor}} \widetilde{H}^i_f(G_{K',S'},T;\Delta(T)) \xleftarrow{\sim} \varprojlim_{U,\mathrm{pr}_*} \widetilde{H}^i_f(G_{K,S},T_U;\Delta(T_U)) \xrightarrow{\sim} \widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K,T).$$

Above, K'/K runs through finite subextensions of K_∞/K and $U = \mathrm{Gal}(K_\infty/K')$.

Proof. — This will follow from 8.6.4 (which applies in our case, thanks to 8.8.4) in the same way as Proposition 8.5.5 does from 8.5.3.2–8.5.3.4, once we check that the canonical maps

$$U_v^+(\mathscr{F}_\Gamma(T)) \longrightarrow \varprojlim_U U_v^+(T_U) \longrightarrow \varprojlim_{U,n} U_v^+((T/\mathfrak{m}^nT)_U) \qquad (v \in S_f)$$

are all isomorphisms (the local conditions for $Z = (T/\mathfrak{m}^n T)_U$ are defined as in 8.8.3). For $v \in \Sigma$ this follows from Proposition 8.3.5 and 8.4.4.2. For $v \in \Sigma'$, it is sufficient to observe that the condition (U) from 8.8.1 implies that

$$L(T_U) = L(T)_U, \quad L(\mathscr{F}_\Gamma(T)) = \mathscr{F}_\Gamma(L(T)). \quad \square$$

8.8.7. Proposition. — *Let X be a bounded below complex in $\left({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\right)$ equipped with Greenberg's local conditions defined by $X_v^+ \to X$ (where X_v^+ is a bounded below complex in $\left({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\right)$, for each $v \in \Sigma$). Let K'/K be a finite Galois subextension of K_S/K, of degree $d = [K' : K]$ which is not a zero divisor on R, such that all primes $v \in \Sigma'$ are unramified in K'/K. Then, for each homomorphism $\chi : \mathrm{Gal}(K'/K) \to R[1/d]^*$, the maps $i_v^+ \otimes \mathrm{id} : X_v^+ \otimes \chi \to X \otimes \chi$ define Greenberg's local conditions for $X \otimes \chi$ and there are canonical isomorphisms*

$$\widetilde{H}^i_f(G_{K,S}, X \otimes \chi; \Delta(X \otimes \chi)) \otimes_R R[1/d] \xrightarrow{\sim} \left(\widetilde{H}^i_f(G_{K',S'}, X; \Delta(X)) \otimes_R R[1/d]\right)^{\chi^{-1}},$$

where we denote, for any $R[1/d][\mathrm{Gal}(K'/K)]$-module M,

$$M^{\chi^{-1}} = M^{(\chi^{-1})} = \left\{x \in M \mid (\forall g \in \mathrm{Gal}(K'/K)) \quad g(x) = \chi^{-1}(g)x\right\}.$$

Proof. — This follows from Lemma 8.6.4.4 (iv) applied to $X \otimes_R \otimes R[1/d] \otimes \chi$, as there is a canonical isomorphism of $\mathrm{Gal}(K'/K)$-modules

$$\widetilde{H}^i_f(G_{K',S'}, X \otimes_R \otimes R[1/d] \otimes \chi; \Delta(X \otimes_R \otimes R[1/d] \otimes \chi)) \xrightarrow{\sim} \widetilde{H}^i_f(G_{K',S'}, X; \Delta(X)) \otimes_R \otimes R[1/d] \otimes \chi$$

(with respect to the Ad_f-action on the cohomology groups $\widetilde{H}^i_f$). $\square$

***8.8.8. Proposition* (self-dual case)**. — *Assume that we are in the situation of 7.8.11 (in particular, R is an integral domain with fraction field $\mathcal{K}$ of characteristic zero, all complexes X and X^+ are concentrated in degree zero, and there exists a skew-symmetric isomorphism between $V := X \otimes_R \mathcal{K}$ and $V^*(1) := \mathrm{Hom}_{\mathcal{K}}(V, \mathcal{K})(1)$, for which each subspace $V_v^+ := X_v^+ \otimes_R \mathcal{K} \subset V$ ($v \in \Sigma$) is maximal isotropic). Let K'/K be as in 8.8.7. Then:*

(i) *For each homomorphism $\chi : \mathrm{Gal}(K'/K) \to \mathcal{K}^*$, we have (using the notation from 7.8.9 and 8.8.7)*

$$\dim_{\mathcal{K}} \widetilde{H}^1_f(K', V)^{\chi} = \dim_{\mathcal{K}} \widetilde{H}^1_f(K', V)^{\chi^{-1}} = \dim_{\mathcal{K}} \widetilde{H}^2_f(K', V)^{\chi} = \dim_{\mathcal{K}} \widetilde{H}^2_f(K', V)^{\chi^{-1}}.$$

(ii) *Assume that there exists an intermediate field $K \subset K_1 \subset K'$ such that $2 \nmid [K' : K_1]$ and $\mathrm{Gal}(K_1/K)$ is an abelian group of order 2^t ($t \geqslant 0$). If $K_1 = K$, set $U = \{K\}$. If $K_1 \neq K$, set $U = \{L \mid L \subset K_1,\ [L : K] = 2\}$. Then*

$$\dim_{\mathcal{K}} \widetilde{H}^1_f(K', V) \equiv \sum_{\chi:\mathrm{Gal}(K_1/K)\to\{\pm 1\}} \dim_{\mathcal{K}} \widetilde{H}^1_f(K, V \otimes \chi) \equiv \sum_{L \in U} \dim_{\mathcal{K}} \widetilde{H}^1_f(L, V) \pmod 2.$$

Proof

(i) Combine Proposition 7.8.11 with the isomorphisms $\widetilde{H}^i_f(K, V \otimes \chi^{\pm 1}) \xrightarrow{\sim} \widetilde{H}^i_f(K', V)^{\chi^{\mp 1}}$ from Proposition 8.8.7.

(ii) As every finite group of odd order is solvable, there exist fields $K \subset K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_n = K'$ such that each Galois group $G_i = \mathrm{Gal}(K_{i+1}/K_i)$ $(i = 0, \ldots, n-1)$ is abelian (and $2 \nmid |G_i|$ for $i > 0$). After adjoining to R suitable roots of unity, we can assume that all characters of $G_0, \ldots, G_{n-1}$ have values in $\mathcal{K}^*$. In the sum

$$\dim_{\mathcal{K}} \widetilde{H}^1_f(K_{i+1}, V) = \sum_{\chi : G_i \to \mathcal{K}^*} \dim_{\mathcal{K}} \widetilde{H}^1_f(K_{i+1}, V)^{\chi},$$

the contributions of χ and χ^{-1} are the same, by (i); thus

$$\dim_{\mathcal{K}} \widetilde{H}^1_f(K_{i+1}, V) \equiv \sum_{\substack{\chi : G_i \to \mathcal{K}^* \\ \chi = \chi^{-1}}} \dim_{\mathcal{K}} \widetilde{H}^1_f(K_{i+1}, V)^{\chi} \pmod 2.$$

If $i > 0$, then the only character of G_i satisfying $\chi = \chi^{-1}$ is the trivial character, for which $\widetilde{H}^1_f(K_{i+1}, V)^1 = \widetilde{H}^1_f(K_i, V)$, by Lemma 8.6.4.4. It follows that

$$\dim_{\mathcal{K}} \widetilde{H}^1_f(K_n, V) \equiv \cdots \equiv \dim_{\mathcal{K}} \widetilde{H}^1_f(K_1, V) \pmod 2,$$

which proves the claim if $K_1 = K$. If $K_1 \neq K$, then the set of characters of G_0 satisfying $\chi = \chi^{-1}$ consists of the trivial character and the characters $\eta_{L/K} : G_0 \twoheadrightarrow \mathrm{Gal}(L/K) \xrightarrow{\sim} \{\pm 1\}$ $(L \in U)$, hence

$$\begin{aligned}\dim_{\mathcal{K}} \widetilde{H}^1_f(K_1, V) &\equiv \sum_{\chi : G_0 \to \{\pm 1\}} \dim_{\mathcal{K}} \widetilde{H}^1_f(K, V \otimes \chi) \\ &= \dim_{\mathcal{K}} \widetilde{H}^1_f(K, V) + \sum_{L \in U} \dim_{\mathcal{K}} \widetilde{H}^1_f(K_1, V)^{\eta_{L/K}} \\ &= (1 - |U|) \dim_{\mathcal{K}} \widetilde{H}^1_f(K, V) + \sum_{L \in U} \dim_{\mathcal{K}} \widetilde{H}^1_f(L, V) \pmod 2,\end{aligned}$$

which proves the claim, since $r \geqslant 1$ and $1 - |U| = 2 - 2^r \equiv 0 \pmod 2$. □

8.8.9. Proposition. — *Let X, Y be bounded below complexes in $\left({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\right)$ equipped with Greenberg's local conditions defined by $Z_v^+ \to Z$ $(Z = X, Y)$, where Z_v^+ is a bounded below complex in $\left({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\right)$, for each $v \in \Sigma$. Let K'/K be a finite Galois subextension of K_S/K such that all primes $v \in \Sigma'$ are unramified in K'/K. Let $\pi : X \otimes_R Y \to \omega^\bullet(1)$ be a morphism of complexes such that $(\forall v \in \Sigma)$ $X_v \perp_\pi Y_v$. Then, for each $g \in \mathrm{Gal}(K'/K)$, the diagram*

$$\begin{array}{ccc} \widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X)) \otimes_R \widetilde{C}^\bullet_f(G_{K',S'}, Y; \Delta(Y)) & \xrightarrow{\cup_{\pi,r,0}} & \omega^\bullet[-3] \\ \Big\downarrow {\scriptstyle \mathrm{Ad}_f(g) \otimes \mathrm{Ad}_f(g)} & & \| \\ \widetilde{C}^\bullet_f(G_{K',S'}, X; \Delta(X)) \otimes_R \widetilde{C}^\bullet_f(G_{K',S'}, Y; \Delta(Y)) & \xrightarrow{\cup_{\pi,r,0}} & \omega^\bullet[-3] \end{array}$$

is commutative up to homotopy.

Proof. — Replacing K' by the fixed field of the cyclic group generated by g, we can assume that $\mathrm{Gal}(K'/K)$ is abelian. In this case the statement follows from Lemma 8.1.6.5 combined with 8.8.4 and the fact that the maps inv_v commute with corestrictions (cf. 9.2.2 below). □

8.9. Duality theorems in Iwasawa theory revisited

Assume that we are in the situation of 8.8.1. In particular, each prime $v \in \Sigma'$ is unramified in K_∞/K.

8.9.1. As in 8.4.6.2, fix a bounded complex $\omega_R^\bullet = \sigma_{\geqslant 0}\,\omega_R^\bullet$ (resp., $\omega_{\overline{R}}^\bullet = \sigma_{\geqslant 0}\,\omega_{\overline{R}}^\bullet$) of injective R-modules (resp., injective $\overline{R}$-modules) representing ω_R (resp., $\omega_{\overline{R}}$) and a quasi-isomorphism

$$\varphi : \omega_R^\bullet \otimes_R \overline{R} \longrightarrow \omega_{\overline{R}}^\bullet.$$

Assume that we are given the following data:

8.9.1.1. Bounded complexes $T, T^*(1)$ in $\big({}^{\mathrm{ad}}_{R[G_{K.S}]}\mathrm{Mod}\big)_{R\text{-}ft}$ and $T_v^+, T^*(1)_v^+$ $(v \in \Sigma)$ in $\big({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\big)_{R\text{-}ft}$.

8.9.1.2. Morphisms of complexes of $R[G_v]$-modules

$$j_v^+(Z) : Z_v^+ \longrightarrow Z \qquad (v \in \Sigma;\ Z = T, T^*(1)).$$

8.9.1.3. A morphism of complexes of $R[G_{K,S}]$-modules

$$\pi : T \otimes_R T^*(1) \longrightarrow \omega_R^\bullet(1),$$

which is a perfect duality in the sense of 6.2.6, *i.e.*, such that its adjoint

$$\mathrm{adj}(\pi) : T \longrightarrow \mathrm{Hom}_R^\bullet(T^*(1), \omega_R^\bullet(1))$$

is a quasi-isomorphism.

8.9.2. For $Z = T, T^*(1)$ consider

$$\mathscr{F}_\Gamma(Z) = (Z \otimes_R \overline{R}) < -1 >, \quad \mathscr{F}_\Gamma(Z)_v^+ = \mathscr{F}_\Gamma(Z_v^+) = (Z_v^+ \otimes_R \overline{R}) < -1 > \ \ (v \in \Sigma).$$

Then $\mathscr{F}_\Gamma(Z)$ (resp., $\mathscr{F}_\Gamma(Z)_v^+$) is a complex in $\big({}^{\mathrm{ad}}_{\overline{R}[G_{K.S}]}\mathrm{Mod}\big)_{\overline{R}\text{-}ft}$ (resp., in $\big({}^{\mathrm{ad}}_{\overline{R}[G_v]}\mathrm{Mod}\big)_{\overline{R}\text{-}ft}$). The morphisms $j_v^+(Z)$ induce morphisms of complexes of $\overline{R}[G_v]$-modules

$$j_v^+(\mathscr{F}_\Gamma(Z)) : \mathscr{F}_\Gamma(Z)_v^+ \longrightarrow \mathscr{F}_\Gamma(Z) \qquad (v \in \Sigma;\ Z = T, T^*(1)).$$

As in 8.4.6.1, 8.4.6.3, π induces a morphism of complexes of $\overline{R}[G_{K,S}]$-modules

$$\begin{aligned}\overline{\pi} : \mathscr{F}_\Gamma(T) \otimes_{\overline{R}} \mathscr{F}_\Gamma(T^*(1))^\iota = (T \otimes_R \overline{R}) < -1 > \otimes_{\overline{R}} ((T^*(1) \otimes_R \overline{R}) < -1 >)^\iota \\ \xrightarrow{\mathrm{id}\otimes(\mathrm{id}\otimes\iota)} (T \otimes_R \overline{R}) < -1 > \otimes_{\overline{R}} (T^*(1) \otimes_R \overline{R}) < 1 > \\ = (T \otimes_R \overline{R}) \otimes_{\overline{R}} (T^*(1) \otimes_R \overline{R}) \xrightarrow{\sim} (T \otimes_R T^*(1)) \otimes_R \overline{R} \\ \xrightarrow{\pi\otimes\mathrm{id}} \omega_R^\bullet(1) \otimes_R \overline{R} \xrightarrow{\varphi} \omega_{\overline{R}}^\bullet(1),\end{aligned}$$

the adjoint of which is equal to

$$\begin{aligned}\mathrm{adj}(\overline{\pi}) : \mathscr{F}_\Gamma(T) = (T \otimes_R \overline{R}) < -1 > \\ \xrightarrow{(\mathrm{adj}(\pi)\otimes\mathrm{id})<-1>} \mathrm{Hom}_R^\bullet(T^*(1) \otimes_R \overline{R}, \omega_R^\bullet \otimes_R \overline{R}(1)) < -1 > \\ \longrightarrow \mathrm{Hom}_{\overline{R}}^\bullet(T^*(1) \otimes_R \overline{R}, \omega_{\overline{R}}^\bullet(1)) < -1 > .\end{aligned}$$

By Corollary 8.4.6.5, $\overline{\pi}$ is again a perfect duality. In other words, the data 8.9.1.1–8.9.1.3 induce the same kind of data for $\mathscr{F}_\Gamma(T)$ and $\mathscr{F}_\Gamma(T^*(1))^\iota$, this time over $\overline{R}$.

8.9.3. Lemma. — *Fix $v \in \Sigma$. Then:*

(i) $T_v^+ \perp_\pi T^*(1)_v^+ \Longrightarrow \mathscr{F}_\Gamma(T)_v^+ \perp_{\overline{\pi}} (\mathscr{F}_\Gamma(T^*(1))^\iota)_v^+$.
(ii) $T_v^+ \perp\!\!\perp_\pi T^*(1)_v^+ \Longrightarrow \mathscr{F}_\Gamma(T)_v^+ \perp\!\!\perp_{\overline{\pi}} (\mathscr{F}_\Gamma(T^*(1))^\iota)_v^+$.

Proof. — The statement (i) is trivial and (ii) follows from Corollary 8.4.6.5. □

8.9.4. Let $T_v^-, T^*(1)_v^-$ $(v \in \Sigma)$ be as in 6.7.1. We apply the construction from 6.7.9 and define bounded complexes $A, A^*(1)$ (resp., $A_v^+, A^*(1)_v^+$ $(v \in \Sigma)$) in $\left({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\right)_{R\text{-}coft}$ (resp., in $\left({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\right)_{R\text{-}coft}$) by

$$A = D(T^*(1))(1), \quad A^*(1) = D(T)(1), \quad A_v^+ = D(T^*(1)_v^-)(1), \quad A^*(1)_v^+ = D(T_v^-)(1).$$

Applying D to the canonical morphisms $T^*(1) \to T^*(1)_v^-$ and $T \to T_v^-$ we obtain morphisms of complexes of $R[G_v]$-modules

$$A_v^+ \longrightarrow A, \quad A^*(1)_v^+ \longrightarrow A^*(1).$$

In the notation of 6.7.4, we have

$$T_v^+ \perp\!\!\perp_{\mathrm{ev}_2} A^*(1)_v^+, \quad A_v^+ \perp\!\!\perp_{\mathrm{ev}_1} T^*(1)_v^+, \tag{8.9.4.1}$$

with respect to the evaluation morphisms

$$\begin{aligned}\mathrm{ev}_2 : T \otimes_R A^*(1) \longrightarrow I_R(1) \\ \mathrm{ev}_1 : A \otimes_R T^*(1) \longrightarrow I_R(1).\end{aligned}$$

8.9.5. For $Z = A, A^*(1)$ and $v \in \Sigma$, put

$$F_\Gamma(Z)^+_v = F_\Gamma(Z^+_v).$$

It follows from Lemma 8.4.5.1 that the adjoint of the evaluation morphism

$$\begin{aligned} \mathrm{ev}_2 : \mathscr{F}_\Gamma(T) \otimes_{\overline{R}} F_\Gamma(A^*(1))^\iota &= (T \otimes_R \overline{R}) < -1 > \otimes_{\overline{R}} \varinjlim_U \mathrm{Hom}^\bullet_R(R[G/U] < 1 >, A^*(1))^\iota \\ &\longrightarrow \varinjlim_U \mathrm{Hom}^\bullet_R(R[G/U], I_R)(1) = I_{\overline{R}}(1) \end{aligned}$$

is a quasi-isomorphism

$$\mathrm{adj}(\mathrm{ev}_2) : \mathscr{F}_\Gamma(T) \longrightarrow \mathrm{Hom}^\bullet_{\overline{R}}(F_\Gamma(A^*(1))^\iota, I_{\overline{R}}(1));$$

a similar statement holds for

$$\mathrm{ev}_1 : F_\Gamma(A) \otimes_{\overline{R}} \mathscr{F}_\Gamma(T^*(1))^\iota \longrightarrow I_{\overline{R}}(1).$$

Combining Lemma 8.4.5.1 with (8.9.4.1) shows that, for each $v \in \Sigma$,

$$\mathscr{F}_\Gamma(T)^+_v \perp\!\!\perp_{\mathrm{ev}_2} (F_\Gamma(A^*(1))^\iota)^+_v, \qquad F_\Gamma(A)^+_v \perp\!\!\perp_{\mathrm{ev}_1} (\mathscr{F}_\Gamma(T^*(1))^\iota)^+_v.$$

8.9.6. Assume, from now on until the end of Section 8.9, that

$$T^+_v \perp_\pi T^*(1)^+_v \qquad (\forall v \in \Sigma).$$

For each $v \in \Sigma$, define

$$W_v \in D^b_{R\text{-}ft}\big({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\big)$$

as in Proposition 6.7.6 (iv); it sits in an exact triangle

$$W_v \longrightarrow T^-_v \xrightarrow{\mathrm{adj}(\pi)} \mathscr{D}\left((T^*(1))^+_v\right)(1) \longrightarrow W_v[1].$$

According to Corollary 8.4.6.5,

$$\mathscr{F}_\Gamma(W_v) \longrightarrow \mathscr{F}_\Gamma(T^-_v) \xrightarrow{\mathrm{adj}(\pi)} \overline{\mathscr{D}}\big(\mathscr{F}_\Gamma((T^*(1))^+_v)^\iota\big)(1) \longrightarrow \mathscr{F}_\Gamma(W_v)[1]$$

is an exact triangle in $D^b_{\overline{R}\text{-}ft}\big({}^{\mathrm{ad}}_{\overline{R}[G_v]}\mathrm{Mod}\big)$.

Applying the discussion in 7.8 and 8.8 to $\mathscr{F}_\Gamma(T)$, $\mathscr{F}_\Gamma(T^*(1))^\iota$, $F_\Gamma(A)$, $F_\Gamma(A^*(1))^\iota$ (for which we consider the unramified local conditions Δ^{ur}_v at all $v \in \Sigma'$), we obtain the following (below, $\overline{D} = D_{\overline{R}}$ and $\overline{\mathscr{D}} = \mathscr{D}_{\overline{R}}$):

8.9.6.1. Isomorphisms in $D^b_{ft}\big({}_{\overline{R}}\mathrm{Mod}\big)$ (resp., $D^b_{coft}\big({}_{\overline{R}}\mathrm{Mod}\big)$)

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, T) \xrightarrow{\sim} \overline{D}\Big(\widetilde{\mathbf{R}\Gamma}_f(K_S/K_\infty, A^*(1))^\iota\Big)[-3]$$

$$\widetilde{\mathbf{R}\Gamma}_f(K_S/K_\infty, A) \xrightarrow{\sim} \overline{D}\Big(\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1))^\iota\Big)[-3].$$

8.9.6.2. Induced isomorphisms on cohomology

$$\widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K, T) \xrightarrow{\sim} \overline{D}\Big(\widetilde{H}^{3-i}_f(K_S/K_\infty, A^*(1))^\iota\Big)$$

$$\widetilde{H}^i_f(K_S/K_\infty, A) \xrightarrow{\sim} \overline{D}\Big(\widetilde{H}^{3-i}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1))^\iota\Big).$$

8.9.6.3. A pairing in $D^b_{ft}({}_{\overline{R}}\mathrm{Mod})$

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K,T)\otimes^{\mathbf{L}}_{\overline{R}}\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K,T^*(1))^{\iota}\longrightarrow\omega_{\overline{R}}[-3], \tag{8.9.6.3.1}$$

whose adjoint map sits in an exact triangle in $D^b_{ft}({}_{\overline{R}}\mathrm{Mod})$

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K,T)\longrightarrow\overline{\mathscr{D}}\Big(\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K,T^*(1))^{\iota}\Big)[-3]\longrightarrow\bigoplus_{v\in S_f}\mathrm{Err}_v, \tag{8.9.6.3.2}$$

where the error terms Err_v are as follows:

$$(\forall v\in\Sigma')\quad \mathrm{Err}_v=\mathrm{Err}^{\mathrm{ur}}_v(\overline{\mathscr{D}},\mathscr{F}_\Gamma(T))=\mathrm{Err}_v(\Delta^{\mathrm{ur}}_v(\mathscr{F}_\Gamma(T)),\Delta^{\mathrm{ur}}_v(\mathscr{F}_\Gamma(T^*(1))^{\iota}),\overline{\pi})$$

are as in Proposition 7.6.7 (ii) and Corollary 7.6.8;

$$(\forall v\in\Sigma)\quad \mathrm{Err}_v=\mathbf{R}\Gamma_{\mathrm{cont}}(G_v,\mathscr{F}_\Gamma(W_v)),$$

by Proposition 6.7.6 (iv).

8.9.6.4. If $T^+_v\perp\!\!\!\perp_\pi T^*(1)^+_v$ for all $v\in\Sigma$, and $\overline{\mathfrak{q}}\in\mathrm{Spec}(\overline{R})$ with $\mathrm{ht}(\overline{\mathfrak{q}})=0$, then the localization of (8.9.6.3.2) at $\overline{\mathfrak{q}}$ induces isomorphisms

$$\begin{aligned}\widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K,T)_{\overline{\mathfrak{q}}}&\overset{\sim}{\longrightarrow}\mathscr{D}_{\overline{R}_{\overline{\mathfrak{q}}}}\Big(\big(\widetilde{H}^{3-j}_{f,\mathrm{Iw}}(K_\infty/K,T^*(1))^{\iota}\big)_{\overline{\mathfrak{q}}}\Big)\\&=D_{\overline{R}_{\overline{\mathfrak{q}}}}\Big(\big(\widetilde{H}^{3-j}_{f,\mathrm{Iw}}(K_\infty/K,T^*(1))^{\iota}\big)_{\overline{\mathfrak{q}}}\Big).\end{aligned} \tag{8.9.6.4.1}$$

It follows that

$$\ell_{\overline{R}_{\overline{\mathfrak{q}}}}\big(\widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K,T)_{\overline{\mathfrak{q}}}\big)=\ell_{\overline{R}_{\overline{\mathfrak{q}}}}\Big(\big(\widetilde{H}^{3-j}_{f,\mathrm{Iw}}(K_\infty/K,T^*(1))^{\iota}\big)_{\overline{\mathfrak{q}}}\Big). \tag{8.9.6.4.2}$$

In particular, if $\Gamma=\Gamma_0\overset{\sim}{\to}\mathbf{Z}_p^r$ and R is a domain, then

$$\begin{aligned}\mathrm{cork}_{\overline{R}}\,\widetilde{H}^j_f(K_S/K_\infty,A)&=\mathrm{rk}_{\overline{R}}\,\widetilde{H}^{3-j}_{f,\mathrm{Iw}}(K_\infty/K,T^*(1))\\&=\mathrm{rk}_{\overline{R}}\,\widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K,T)\\&=\mathrm{cork}_{\overline{R}}\,\widetilde{H}^{3-j}_f(K_S/K_\infty,A^*(1)).\end{aligned} \tag{8.9.6.4.3}$$

8.9.7. Error terms and local Tamagawa factors for $\mathscr{F}_\Gamma(T)$. — We are going to investigate the error terms in 8.9.6.3 under the assumption (U) from 8.8.1 (which is automatically satisfied if $\Delta=0$). As $\Gamma=\Gamma_0\times\Delta$, $\Gamma_0\overset{\sim}{\to}\mathbf{Z}_p^r$ $(r\geqslant 1)$, we have $\overline{R}\overset{\sim}{\to}R[\Delta][\![X_1,\ldots,X_r]\!]$ $(X_i=\gamma_i-1)$. Our first goal is to show that the cohomology groups of $\mathrm{Err}^{\mathrm{ur}}_v$ (for $v\nmid p$) are very often pseudo-null over $\overline{R}$. We use repeatedly the fact that $\mathscr{F}_\Gamma(T)=\mathscr{F}_{\Gamma_0}(\mathscr{F}_\Delta(T))$.

8.9.7.1. Lemma

(i) $\{\overline{\mathfrak{p}}\in\mathrm{Spec}(\overline{R})\mid\mathrm{ht}(\overline{\mathfrak{p}})=0\}=\{\mathfrak{p}\overline{R}\mid\mathfrak{p}\in\mathrm{Spec}(R[\Delta]),\ \mathrm{ht}(\mathfrak{p})=0\}$.

(ii) *Let* $M\in({}_{R[\Delta]}\mathrm{Mod})_{ft}$ *and* $\mathfrak{p}\in\mathrm{Spec}(R[\Delta])$, $\mathrm{ht}(\mathfrak{p})=0$.

Then:

$$\mathfrak{p} \in \operatorname{supp}(M) \iff \overline{\mathfrak{p}} := \mathfrak{p}\overline{R} \in \operatorname{supp}(M \otimes_{R[\Delta]} \overline{R})$$

and

$$\ell_{\overline{R}_{\overline{\mathfrak{p}}}}((M \otimes_{R[\Delta]} \overline{R})_{\overline{\mathfrak{p}}}) = \ell_{R[\Delta]_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Proof

(i) Note that, for every ideal $I \subset R[\Delta]$, $\overline{R}/I\overline{R} \xrightarrow{\sim} (R[\Delta]/I)[\![X_1, \ldots, X_r]\!]$. If $\overline{\mathfrak{p}} \in \operatorname{Spec}(\overline{R})$ has $\operatorname{ht}(\overline{\mathfrak{p}}) = 0$, put $\mathfrak{p} = R[\Delta] \cap \overline{\mathfrak{p}} \in \operatorname{Spec}(R[\Delta])$. Then $\mathfrak{p}\overline{R} \subset \overline{\mathfrak{p}}$, hence $\mathfrak{p}\overline{R} = \overline{\mathfrak{p}}$. If there is $\mathfrak{q} \in \operatorname{Spec}(R[\Delta])$ with $\mathfrak{q} \subsetneq \mathfrak{p}$, then $\mathfrak{q}\overline{R} \subsetneq \mathfrak{p}\overline{R} = \overline{\mathfrak{p}}$, which contradicts $\operatorname{ht}(\overline{\mathfrak{p}}) = 0$; thus $\operatorname{ht}(\mathfrak{p}) = 0$. Conversely, if $\mathfrak{p} \in \operatorname{Spec}(R[\Delta])$ has $\operatorname{ht}(\mathfrak{p}) = 0$ and $\overline{\mathfrak{p}} = \mathfrak{p}\overline{R}$, then $\overline{R}/\overline{\mathfrak{p}} = (R[\Delta]/\mathfrak{p})[\![X_1, \ldots, X_r]\!]$ is a domain, hence $\overline{\mathfrak{p}} \in \operatorname{Spec}(\overline{R})$. If $\operatorname{ht}(\overline{\mathfrak{p}}) \neq 0$, then there is $\overline{\mathfrak{q}} \in \operatorname{Spec}(\overline{R})$, $\overline{\mathfrak{q}} \subsetneq \overline{\mathfrak{p}}$, $\operatorname{ht}(\overline{\mathfrak{q}}) = 0$. By the above argument, $\overline{\mathfrak{q}} = \mathfrak{q}\overline{R}$ for $\mathfrak{q} = \overline{\mathfrak{q}} \cap R[\Delta]$, $\operatorname{ht}(\mathfrak{q}) = 0$. As $\mathfrak{q} \subset \mathfrak{p}$, we have $\mathfrak{q} = \mathfrak{p}$, hence $\overline{\mathfrak{q}} = \mathfrak{q}\overline{R} = \mathfrak{p}\overline{R} = \overline{\mathfrak{p}}$; this contradiction proves that $\operatorname{ht}(\overline{\mathfrak{p}}) = 0$.

(ii) We have

$$M_{\mathfrak{p}} \neq 0 \iff M/\mathfrak{p}M \neq 0 \iff (M \otimes_{R[\Delta]} \overline{R})/(\mathfrak{p}\overline{R})(M \otimes_{R[\Delta]} \overline{R}) \neq 0 \iff (M \otimes_{R[\Delta]} \overline{R})_{\mathfrak{p}\overline{R}} \neq 0,$$

which proves the first statement. By dévissage, it is enough to check the equality of lengths for $M = R[\Delta]/\mathfrak{p}$, in which case both sides are equal to one. □

8.9.7.2. Proposition. — *Let $j \geqslant 0$. Given $M \in ({}_R\mathrm{Mod})_{ft}$ with $\operatorname{codim}_R(\operatorname{supp}(M)) \geqslant j$, $f \in \operatorname{Aut}_R(M)$ and $u \in \overline{R}^*$, let N_0, N_1 be the $\overline{R}$-modules defined by the exact sequence*

$$0 \longrightarrow N_0 \longrightarrow M \otimes_R \overline{R} \xrightarrow{f \otimes u - 1} M \otimes_R \overline{R} \longrightarrow N_1 \longrightarrow 0.$$

For $i = 0, 1$, let

$$B_i = \operatorname{supp}(N_i) \cap \{\overline{\mathfrak{p}} \in \operatorname{Spec}(\overline{R}) \mid \operatorname{ht}(\overline{\mathfrak{p}}) \leqslant j\}.$$

(i) *If $u \in R[\Delta]^*$, then $N_i = M_i \otimes_{R[\Delta]} \overline{R}$ and*

$$B_i = \{\mathfrak{p}\overline{R} \mid \mathfrak{p} \in \operatorname{Spec}(R[\Delta]), \operatorname{ht}(\mathfrak{p}) = j, \mathfrak{p} \in \operatorname{supp}(M_i)\},$$

where M_0, M_1 are the $R[\Delta]$-modules defined by the exact sequence

$$0 \longrightarrow M_0 \longrightarrow M \otimes_R R[\Delta] \xrightarrow{fu-1} M \otimes_R R[\Delta] \longrightarrow M_1 \longrightarrow 0.$$

For each $\overline{\mathfrak{p}} = \mathfrak{p}\overline{R} \in B_i$,

$$\ell_{\overline{R}_{\overline{\mathfrak{p}}}}((N_i)_{\overline{\mathfrak{p}}}) = \ell_{R[\Delta]_{\mathfrak{p}}}((M_i)_{\mathfrak{p}}).$$

(ii) *If, for each maximal ideal $\mathfrak{m}_\Delta$ of $R[\Delta]$, $u \pmod{\mathfrak{m}_\Delta\overline{R}} \in (\overline{R}/\mathfrak{m}_\Delta\overline{R})^* = (R[\Delta]/\mathfrak{m}_\Delta)[\![X_1, \ldots, X_r]\!]^*$ is not contained in $(R[\Delta]/\mathfrak{m}_\Delta)^*$, then both sets B_0, B_1 are empty.*

Proof. — If $j > d$, then $M = 0$ and there is nothing to prove. If $j \leqslant d$, then there is an ideal $I \subset \operatorname{Ann}_R(M)$ with $\dim(R/I) = d - j$. Replacing R by R/I, we can assume that $j = 0$.

(i) As $\overline{R}$ is flat over R, we have $N_i = M_i \otimes_{R[\Delta]} \overline{R}$; the rest of the statement follows from Lemma 8.9.7.1.

(ii) Let $\mathfrak{p} \in \mathrm{Spec}(R[\Delta])$, $\mathrm{ht}(\mathfrak{p}) = 0$; put $\overline{\mathfrak{p}} = \mathfrak{p}\overline{R}$. We must show that $(N_i)_{\overline{\mathfrak{p}}} = 0$ for $i = 0, 1$ (again using Lemma 8.9.7.1 (i). The $R[\Delta]$-module $M[\Delta] := M \otimes_R R[\Delta]$ admits a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M[\Delta]$ with graded quotients $M_{t+1}/M_t \xrightarrow{\sim} R[\Delta]/\mathfrak{q}_t$, for some $\mathfrak{q}_t \in \mathrm{Spec}(R[\Delta])$. By dévissage, we can assume that $M[\Delta] \xrightarrow{\sim} R[\Delta]/\mathfrak{q}_t$. If $\mathfrak{q}_t \neq \mathfrak{p}$, then $(M \otimes_R \overline{R})_{\mathfrak{p}} = (M[\Delta] \otimes_{R[\Delta]} \overline{R})_{\mathfrak{p}} = 0$ by Lemma 8.9.7.1 (ii). If $\mathfrak{q}_t = \mathfrak{p}$, then we can replace $R[\Delta]$ by $R[\Delta]/\mathfrak{p}$, hence assume that $R[\Delta]$ is a domain and $\mathfrak{p} = (0)$. In this case $f \in R[\Delta]^*$, $N_0 = 0$ and

$$N_1 = \overline{R}/(u - f^{-1})\overline{R},$$

which is $\overline{R}$-torsion; thus $(N_1)_{(0)} = 0$. □

8.9.7.3. Proposition

(i) *For each non-archimedean prime $v \nmid p$ of K which is unramified in K_∞/K, there is an isomorphism in $D^b\big(({}_{\overline{R}}\mathrm{Mod})/(\text{pseudo-null})\big)$ (using the notation of* (7.6.5.1)*)*

$$\overline{\mathscr{D}}\big(\mathrm{Err}_v^{\mathrm{ur}}(\overline{\mathscr{D}}, \mathscr{F}_\Gamma(T))\big) \xrightarrow{\sim} \begin{cases} \mathscr{F}_{\Gamma_0}\big(\mathscr{D}_{R[\Delta]}\big(\mathrm{Err}_v^{\mathrm{ur}}(\mathscr{D}_{R[\Delta]}, \mathscr{F}_\Delta(T))\big)\big), & |\Gamma_v| < \infty \quad (\Longleftrightarrow \Gamma_v \subset \Delta) \\ 0, & |\Gamma_v| = \infty. \end{cases}$$

(ii) *If $\overline{\mathfrak{p}} \in \mathrm{Spec}(\overline{R})$, $\mathrm{ht}(\overline{\mathfrak{p}}) = 1$ and $\mathrm{Err}_v^{\mathrm{ur}}(\overline{\mathscr{D}}, \mathscr{F}_\Gamma(T))_{\overline{\mathfrak{p}}}$ is not acyclic (*i.e., *not isomorphic to* 0 *in $D({}_{\overline{R}_{\overline{\mathfrak{p}}}}\mathrm{Mod})$), then*

$$\Gamma_v \subset \Delta, \quad \overline{\mathfrak{p}} = \mathfrak{p}\overline{R}, \quad \mathfrak{p} \in \mathrm{Spec}(R[\Delta]), \quad \mathrm{ht}(\mathfrak{p}) = 1$$

and

$$\mathrm{Tam}_v(\mathscr{F}_\Gamma(T), \overline{\mathfrak{p}}) = \mathrm{Tam}_v(\mathscr{F}_\Delta(T), \mathfrak{p}) \neq 0.$$

Proof. — As v is unramified in K_∞/K, we have

$$H^1_{\mathrm{cont}}(I_v, \mathscr{F}_\Gamma(T)) \xrightarrow{\sim} \mathscr{F}_\Gamma(H^1_{\mathrm{cont}}(I_v, T)). \tag{8.9.7.3.1}$$

Let N be an R-module of finite type and $P_\bullet \to N$ a resolution of N by free R-modules of finite type. Then $P_\bullet \otimes_R \overline{R}$ is a free resolution of $N \otimes_R \overline{R}$, hence

$$\mathrm{Hom}^\bullet_R(P_\bullet, \omega^\bullet_R) \otimes_R \overline{R} \xrightarrow{\sim} \mathrm{Hom}^\bullet_{\overline{R}}(P_\bullet \otimes_R \overline{R}, \omega^\bullet_R \otimes_R \overline{R})$$

gives a canonical isomorphism in $D({}_{\overline{R}}\mathrm{Mod})$

$$\mathscr{D}(N) \otimes_R \overline{R} \xrightarrow{\sim} \overline{\mathscr{D}}(N \otimes_R \overline{R}). \tag{8.9.7.3.2}$$

Applying this observation to $N = H^1_{\mathrm{cont}}(I_v, T)$, we obtain isomorphisms of $\overline{R}[G_v/I_v]$-modules

$$\begin{aligned} \mathbb{E}\mathrm{xt}^1_{\overline{R}}(\mathscr{F}_\Gamma(N), \omega_{\overline{R}}) &\xrightarrow{\sim} \mathbb{E}\mathrm{xt}^1_{\overline{R}}(N \otimes_R \overline{R}, \omega_{\overline{R}}) < 1 > \\ &\xrightarrow{\sim} (\mathbb{E}\mathrm{xt}^1_R(N, \omega_R) \otimes_R \overline{R}) < 1 > \xrightarrow{\sim} \mathscr{F}_\Gamma(\mathbb{E}\mathrm{xt}^1_R(N, \omega_R))^\iota. \end{aligned}$$

Let $M = \mathbb{E}\mathrm{xt}^1_R(N,\omega_R)$, $f = f_v^{-1} \in \mathrm{Aut}_R(M)$ and $u =$ the image of $f_v^{-1} \in \Gamma_v$ under $\Gamma_v \hookrightarrow \Gamma \hookrightarrow \overline{R}^*$. It follows from the above observation and Proposition 7.6.7 (ii) that

$$\overline{\mathscr{D}}\big(\mathrm{Err}_v^{\mathrm{ur}}(\overline{\mathscr{D}},\mathscr{F}_\Gamma(T))\big) = \mathrm{Err}_v(\Delta_v^{\mathrm{ur}}(\mathscr{F}_\Gamma(T)), \Delta_v^{\mathrm{ur}}(\mathscr{F}_\Gamma(T^*(1))^\iota), \overline{\pi})$$

is isomorphic in $D\big(\big({}_{\overline{R}}\mathrm{Mod}\big)/(\text{pseudo-null})\big)$ to the following complex in degrees $-1, 0$:

$$K^\bullet = \Big[M \otimes_R \overline{R} \xrightarrow{f\otimes u-1} M \otimes_R \overline{R}\Big].$$

If $|\Gamma_v| < \infty$, then $\Gamma_v \subset \Delta$, $u \in R[\Delta]^*$ and $K^\bullet$ is isomorphic to

$$\Big[M \otimes_R R[\Delta] \xrightarrow{f_v^{-1}u-1} M \otimes_R R[\Delta]\Big] \otimes_{R[\Delta]} \overline{R}.$$

Both statements (i), (ii) in this case follow from Proposition 8.9.7.2 (i) (for $j = 1$) and Proposition 7.6.7 (ii), this time applied to $R[\Delta]$ and $\mathscr{F}_\Delta(T)$.

If $|\Gamma_v| = \infty$, then there is a $\mathbf{Z}_p$-basis $\gamma_1, \ldots, \gamma_r \in \Gamma$ and $n \geqslant 0$ such that $u = \delta u' \gamma_1^{p^n} = \delta u'(1 + X_1)^{p^n}$, with $\delta \in \Delta$, $u' \in \mathbf{Z}_p^*$. Applying Proposition 8.9.7.2 (ii) with $j = 1$, we deduce that the cohomology groups of $K^\bullet$ are pseudo-null over $\overline{R}$. ☐

8.9.7.4. Corollary. — *For each non-archimedean prime $v \nmid p$ of K which is unramified in K_∞/K, there is an isomorphism in the category $D^b\big(\big({}_{\overline{R}}\mathrm{Mod}\big)/(\text{pseudo-null})\big)$*

$$\mathrm{Err}_v^{\mathrm{ur}}(\overline{\mathscr{D}},\mathscr{F}_\Gamma(T)) \xrightarrow{\sim} \begin{cases} \mathscr{F}_{\Gamma_0}\big(\mathrm{Err}_v^{\mathrm{ur}}(\mathscr{D}_{R[\Delta]},\mathscr{F}_\Delta(T))\big), & |\Gamma_v| < \infty \\ 0, & |\Gamma_v| = \infty. \end{cases}$$

Proof. — Apply $\overline{\mathscr{D}}$ to the statement (i) in Proposition 8.9.7.3 (and use (8.9.7.3.2)). ☐

8.9.7.5. For each $\mathfrak{q} \in \mathrm{Spec}(R[\Delta])$ and $M \in \big({}_{\overline{R}}\mathrm{Mod}\big)$, we put $M_\mathfrak{q} = M \otimes_{R[\Delta]} R[\Delta]_\mathfrak{q}$; this is an $\overline{R}_\mathfrak{q}$ ($=\overline{R} \otimes_{R[\Delta]} R[\Delta]_\mathfrak{q}$)-module.

8.9.7.6. Proposition. — *Assume that $T = \sigma_{\leqslant 0} T$ and $v \nmid p$ is a non-archimedean prime of K, unramified in K_∞/K.*

(i) *If $|\Gamma_v| < \infty$, then there is an isomorphism in $D^b_{ft}\big({}_{\overline{R}}\mathrm{Mod}\big)$*

$$\overline{\mathscr{D}}\big(\mathrm{Err}_v^{\mathrm{ur}}(\overline{\mathscr{D}},\mathscr{F}_\Gamma(T))\big) \xrightarrow{\sim} \mathscr{F}_{\Gamma_0}\Big(\mathscr{D}_{R[\Delta]}\big(\mathrm{Err}_v^{\mathrm{ur}}(\mathscr{D}_{R[\Delta]},\mathscr{F}_\Delta(T))\big)\Big).$$

(ii) *If $H^1_{\mathrm{cont}}(I_v,\mathscr{F}_\Delta(T)) = H^1_{\mathrm{cont}}(I_v, H^0(\mathscr{F}_\Delta(T))) \xrightarrow{\sim} H^0(\mathscr{F}_\Delta(T))_{I_v}(-1)$ is zero or a Cohen-Macaulay $R[\Delta]$-module of dimension $d = \dim(R) = \dim(R[\Delta])$, then*

$$\mathrm{Err}_v^{\mathrm{ur}}(\overline{\mathscr{D}},\mathscr{F}_\Gamma(T)) \xrightarrow{\sim} 0 \text{ in } D^b_{ft}\big({}_{\overline{R}}\mathrm{Mod}\big).$$

(iii) *If $\mathfrak{q} \in \mathrm{Spec}(R[\Delta])$, $\mathrm{ht}(\mathfrak{q}) = 0$, then*

$$\mathrm{Err}_v^{\mathrm{ur}}(\overline{\mathscr{D}},\mathscr{F}_\Gamma(T))_\mathfrak{q} \xrightarrow{\sim} 0 \text{ in } D^b_{ft}\big({}_{\overline{R}_\mathfrak{q}}\mathrm{Mod}\big).$$

(iv) *The following statements are equivalent (cf. Corollary 7.6.12 (iii)):*

$$\mathrm{Err}_v^{\mathrm{ur}}(\overline{\mathscr{D}},\mathscr{F}_\Gamma(T)) \xrightarrow{\sim} 0 \text{ in } D^b_{ft}\big({}_{\overline{R}}\mathrm{Mod}\big)$$
$$\iff \mathrm{Err}_v^{\mathrm{ur}}(\mathscr{D}_{R[\Delta]},\mathscr{F}_\Delta(T)) \xrightarrow{\sim} 0 \text{ in } D^b_{ft}\big({}_{R[\Delta]}\mathrm{Mod}\big).$$

(v) *If* $\dim(R) = 1$, *then*

$$\mathrm{Err}_v^{\mathrm{ur}}(\overline{\mathscr{D}}, \mathscr{F}_\Gamma(T)) \xrightarrow{\sim} 0 \text{ in } D^b_{ft}({}_{\overline{R}}\mathrm{Mod}) \iff (\forall \mathfrak{m}_\Delta \in \mathrm{Max}(R))\ \mathrm{Tam}_v(\mathscr{F}_\Delta(T), \mathfrak{m}_\Delta) = 0.$$

Proof. — We apply Proposition 7.6.11 and Corollary 7.6.12 to $\mathscr{F}_\Gamma(T) = \mathscr{F}_{\Gamma_0}(\mathscr{F}_\Delta(T)) = \sigma_{\leqslant 0}\mathscr{F}_\Gamma(T)$ over $\overline{R}$. Combining (8.9.7.3.1)–(8.9.7.3.2), we obtain

$$\tau_{\geqslant 1}\overline{\mathscr{D}}\left(H^1_{\mathrm{cont}}(I_v, \mathscr{F}_\Gamma(T))\right) \xrightarrow{\sim} \mathscr{F}_\Gamma\left(\tau_{\geqslant 1}\mathscr{D}(H^1_{\mathrm{cont}}(I_v, T))\right)^\iota;$$

Corollary 7.6.12 (i) then yields an isomorphism in $D^b_{ft}({}_{\overline{R}}\mathrm{Mod})$

(8.9.7.6.1) $\overline{\mathscr{D}}\left(\mathrm{Err}_v^{\mathrm{ur}}(\overline{\mathscr{D}}, \mathscr{F}_\Gamma(T))\right)$

$$\xrightarrow{\sim} \mathrm{Cone}\big(f_v^{-1} \otimes \chi_{\Gamma_0}(f_v)^{-1} - 1 : \tau_{\geqslant 1}\mathscr{D}_{R[\Delta]}(H) \otimes_{R[\Delta]} \overline{R} \longrightarrow \tau_{\geqslant 1}\mathscr{D}_{R[\Delta]}(H) \otimes_{R[\Delta]} \overline{R}\big)[1]$$

(where $H = H^1_{\mathrm{cont}}(I_v, \mathscr{F}_\Delta(T))$).

(i) If $|\Gamma_v| < \infty$, then $\chi_{\Gamma_0}(f_v) = 1$, hence (8.9.7.6.1) and the flatness of $\overline{R}$ over R imply (using Corollary 7.6.12 (i) again, this time for $\mathscr{F}_\Delta(T)$ over $R[\Delta]$) that

$$\overline{\mathscr{D}}\left(\mathrm{Err}_v^{\mathrm{ur}}(\overline{\mathscr{D}}, \mathscr{F}_\Gamma(T))\right) \xrightarrow{\sim} \mathrm{Err}_v^{\mathrm{ur}}(\mathscr{D}_{R[\Delta]}, \mathscr{F}_\Delta(T)) \otimes_{R[\Delta]} \overline{R},$$

as required.

(ii) In this case the complex $\tau_{\geqslant 1}\mathscr{D}_{R[\Delta]}\big(H^1_{\mathrm{cont}}(I_v, \mathscr{F}_\Delta(T))\big)$ is acyclic.

(iii) The complex $\tau_{\geqslant 1}\mathscr{D}_{R[\Delta]}\big(H^1_{\mathrm{cont}}(I_v, \mathscr{F}_\Delta(T))\big)_\mathfrak{q}$ is acyclic, by Lemma 2.4.7 (iv) and local duality 2.5.

(iv) Put, for each $q \geqslant 0$,

$$M_q := \mathbb{E}\mathrm{xt}^q_{R[\Delta]}\big(H^1_{\mathrm{cont}}(I_v, \mathscr{F}_\Delta(T))), \omega_{R[\Delta]}\big) \in \big({}_{R[\Delta]}\mathrm{Mod}\big)_{ft}.$$

According to Corollary 7.6.12 (iii), we have

$$\mathrm{Err}_v^{\mathrm{ur}}(\mathscr{D}_{R[\Delta]}, \mathscr{F}_\Delta(T)) \xrightarrow{\sim} 0 \iff (\forall q = 1, \dots, d) \quad \mathrm{Coker}(f_v^{-1} - 1 : M_q \longrightarrow M_q) = 0,$$

$$\mathrm{Err}_v^{\mathrm{ur}}(\overline{\mathscr{D}}, \mathscr{F}_\Gamma(T)) \xrightarrow{\sim} 0 \iff (\forall q = 1, \dots, d) \quad \mathrm{Coker}(f_v^{-1} \otimes \chi_{\Gamma_0}(f_v)^{-1} - 1 : M_q \otimes_{R[\Delta]} \overline{R} \longrightarrow M_q \otimes_{R[\Delta]} \overline{R}) = 0.$$

However, as $\chi_{\Gamma_0}(f_v) - 1$ is contained in the radical $\overline{\mathfrak{m}}$ of $\overline{R}$, Nakayama's Lemma implies that

$$\begin{aligned}
&\mathrm{Coker}(f_v^{-1} \otimes \chi_{\Gamma_0}(f_v)^{-1} - 1 : M_q \otimes_{R[\Delta]} \overline{R} \longrightarrow M_q \otimes_{R[\Delta]} \overline{R}) = 0 \\
&\iff 0 = \mathrm{Coker}(f_v^{-1} \otimes 1 - 1 : M_q \otimes_{R[\Delta]} \overline{R}/\overline{\mathfrak{m}} \longrightarrow M_q \otimes_{R[\Delta]} \overline{R}/\overline{\mathfrak{m}}) \\
&\qquad = \mathrm{Coker}(f_v^{-1} - 1 : M_q \otimes_{R[\Delta]} R[\Delta]/\mathfrak{m} \longrightarrow M_q \otimes_{R[\Delta]} R[\Delta]/\mathfrak{m}) \\
&\qquad \iff \mathrm{Coker}(f_v^{-1} - 1 : M_q \longrightarrow M_q) = 0,
\end{aligned}$$

where we have denoted by $\mathfrak{m}$ the radical of $R[\Delta]$.

(v) Combine (iv) with 7.6.10.8 (for $\mathfrak{p} = \mathfrak{m}_\Delta$). □

8.9.7.7. Proposition. — *Assume that* $v \in \Sigma$ *and* $\mathfrak{q}_0 \in \mathrm{Spec}(R[\Delta])$; *denote by* $\overline{\mathfrak{p}} \in \mathrm{Spec}(\overline{R})$ *the inverse image of* $\mathfrak{q}_0$ *under the projection map* $\overline{R} = R[\Delta][\![\Gamma_0]\!] \to R[\Delta]$.

(i) *If* $\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, \mathscr{F}_\Delta(W_v))_{\mathfrak{q}_0} \xrightarrow{\sim} 0$ *in* $D^b_{ft}({}_{R[\Delta]_{\mathfrak{q}_0}}\mathrm{Mod})$, *then* $\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, \mathscr{F}_\Gamma(W_v))_{\overline{\mathfrak{p}}} \xrightarrow{\sim} 0$ *in* $D^b_{ft}({}_{\overline{R}_{\overline{\mathfrak{p}}}}\mathrm{Mod})$.

(ii) *If* $W_v = H^0(W_v)$ *is concentrated in degree 0 and* $\mathbf{R}\Gamma_{\mathrm{ur}}(G_v, \mathscr{F}_\Delta(W_v))_{\mathfrak{q}_0} \xrightarrow{\sim} 0$ *in* $D^b_{ft}({}_{R[\Delta]_{\mathfrak{q}_0}}\mathrm{Mod})$, *then* $\mathbf{R}\Gamma_{\mathrm{ur}}(G_v, \mathscr{F}_\Gamma(W_v))_{\overline{\mathfrak{p}}} \xrightarrow{\sim} 0$ *in* $D^b_{ft}({}_{\overline{R}_{\overline{\mathfrak{p}}}}\mathrm{Mod})$.

Proof. — We use the fact that $\mathscr{F}_\Gamma(W_v) = \mathscr{F}_{\Gamma_0}(\mathscr{F}_\Delta(W_v))$. The statement (i) follows from Proposition 8.4.8.5 applied to $R[\Delta]$ instead of R, $\mathfrak{p} = \mathfrak{q}_0$, $G = G_v$ and $T = \mathscr{F}_\Delta(W_v)$.

As W_v is concentrated in degree 0, we have

$$\mathbf{R}\Gamma_{\mathrm{ur}}(G_v, \mathscr{F}_\Delta(W_v)) = \mathbf{R}\Gamma_{\mathrm{cont}}(G_v/I_v, \mathscr{F}_\Delta(W_v)^{I_v}),$$

$$\mathbf{R}\Gamma_{\mathrm{ur}}(G_v, \mathscr{F}_\Gamma(W_v)) = \mathbf{R}\Gamma_{\mathrm{cont}}(G_v/I_v, \mathscr{F}_\Gamma(W_v)^{I_v}).$$

On the other hand, the assumption (U) from 8.8.1 implies that

$$\mathscr{F}_\Gamma(W_v)^{I_v} = \mathscr{F}_\Gamma(W_v^{I_v}) = \mathscr{F}_{\Gamma_0}(\mathscr{F}_\Delta(W_v)^{I_v}) = \mathscr{F}_{\Gamma_0}(\mathscr{F}_\Delta(W_v^{I_v}));$$

we apply again Proposition 8.4.8.5 with $G = G_v/I_v$ and $T = \mathscr{F}_\Delta(W_v^{I_v})$. □

8.9.8. Theorem. — *Denote*

$$S_{\mathrm{bad}} = \{\mathfrak{p}\overline{R} \mid \mathfrak{p} \in \mathrm{Spec}(R[\Delta]), \mathrm{ht}(\mathfrak{p}) = 1, (\exists v \in \Sigma')\ |\Gamma_v| < \infty, \mathrm{Tam}_v(\mathscr{F}_\Delta(T), \mathfrak{p}) \neq 0\}.$$

Let $\overline{\mathfrak{p}} \in \mathrm{Spec}(\overline{R})$, $\mathrm{ht}(\overline{\mathfrak{p}}) = 1$, $\overline{\mathfrak{p}} \notin S_{\mathrm{bad}}$. *Assume that, either:*

(i) $(\forall v \in \Sigma)$ $T_v^+ \perp\!\!\!\perp_\pi T^*(1)_v^+$; *or:*

(ii) $\Gamma = \Gamma_0 \times \Delta = \Gamma_0' \times \Gamma_0'' \times \Delta$, $\Gamma_0' \xrightarrow{\sim} \mathbf{Z}_p$, $\mathfrak{q}_0 \in \mathrm{Spec}(R[\Delta])$, $\mathrm{ht}(\mathfrak{q}_0) = 0$, $\overline{\mathfrak{p}} = \mathfrak{q}_0\overline{R} + (\gamma - 1)\overline{R} \in \mathrm{Spec}(\overline{R})$, $(\forall v \in \Sigma)$ $\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, \mathscr{F}_\Delta(W_v))_{\mathfrak{q}_0} \xrightarrow{\sim} 0$ *in* $D^b_{ft}({}_{R[\Delta]_{\mathfrak{q}_0}}\mathrm{Mod})$.

Then the localized duality map

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, T)_{\overline{\mathfrak{p}}} \longrightarrow \mathscr{D}_{\overline{R}_{\overline{\mathfrak{p}}}}\big((\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1))^\iota)_{\overline{\mathfrak{p}}}\big)[-3]$$

is an isomorphism in $D^b_{ft}({}_{\overline{R}_{\overline{\mathfrak{p}}}}\mathrm{Mod})$, *inducing exact sequences of* $\overline{R}_{\overline{\mathfrak{p}}}$*-modules of finite type*

$$0 \longrightarrow \big(\mathbb{E}\mathrm{xt}^1_{\overline{R}}(\widetilde{H}^{4-j}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1)), \omega_{\overline{R}})^\iota\big)_{\overline{\mathfrak{p}}}$$
$$\longrightarrow \widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, T)_{\overline{\mathfrak{p}}} \longrightarrow \big(\mathbb{E}\mathrm{xt}^0_{\overline{R}}(\widetilde{H}^{3-j}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1)), \omega_{\overline{R}})^\iota\big)_{\overline{\mathfrak{p}}} \longrightarrow 0.$$

Proof. — We apply 7.8.4.4 with $X = \mathscr{F}_\Gamma(T)$ over $\overline{R}$; for each $v \in \Sigma$ (resp., $v \in \Sigma'$), the localization at $\overline{\mathfrak{p}}$ of the error term Err_v vanishes, by Proposition 8.9.7.7 (resp., Proposition 8.9.7.3 (ii)). □

8.9.9. Theorem. — *Assume that $T_v^+ \perp\!\!\!\perp_\pi T^*(1)_v^+$ ($\forall v \in \Sigma$). If $S_{\mathrm{bad}} = \varnothing$ (e.g., if ($\forall v \in \Sigma'$) $|\Gamma_v| = \infty$), then*

$$\begin{array}{ccc} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, T) & \overset{\overline{\mathscr{D}}}{\longleftrightarrow} & \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1))^\iota[3] \\ \Big\downarrow \overline{\Phi} & \overline{D} & \Big\downarrow \overline{\Phi} \\ \widetilde{\mathbf{R}\Gamma}_f(K_S/K_\infty, A) & & \widetilde{\mathbf{R}\Gamma}_f(K_S/K_\infty, A^*(1))^\iota[3] \end{array}$$

is a duality diagram in $D^b_{ft}\big(({}_{\overline{R}}\mathrm{Mod})/(\text{pseudo-null})\big)$ (top row) resp., $D^b_{coft}\big(({}_{\overline{R}}\mathrm{Mod})/(\text{co-pseudo-null})\big)$ (bottom row), inducing exact sequences in $({}_{\overline{R}}\mathrm{Mod})/(\text{pseudo-null})$

$$0 \longrightarrow \mathbb{E}\mathrm{xt}^1_{\overline{R}}\big(\widetilde{H}^{4-j}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1)), \omega_{\overline{R}}\big)^\iota$$
$$\longrightarrow \widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, T) \longrightarrow \mathbb{E}\mathrm{xt}^0_{\overline{R}}\big(\widetilde{H}^{3-j}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1)), \omega_{\overline{R}}\big)^\iota \longrightarrow 0.$$

These sequences yield monomorphisms in $({}_{\overline{R}}\mathrm{Mod})/(\text{pseudo-null})$

$$\widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, T)_{\overline{R}\text{-tors}} \longrightarrow \mathbb{E}\mathrm{xt}^1_{\overline{R}}\big(\widetilde{H}^{4-j}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1)), \omega_{\overline{R}}\big)^\iota,$$

which are isomorphisms if R has no embedded primes.

Proof. — This follows from Theorem 8.9.8. □

8.9.10. Recall that we have established in 8.9.6.2 isomorphisms of $\overline{R}$-modules

$$\widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, T) \xrightarrow{\sim} \overline{D}\big(\widetilde{H}^{3-j}_f(K_S/K_\infty, A^*(1))^\iota\big)$$
$$\widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, T^*(1)) \xrightarrow{\sim} \overline{D}\big(\widetilde{H}^{3-j}_f(K_S/K_\infty, A)^\iota\big).$$

8.9.11. Theorem. — *Assume that $T_v^+ \perp\!\!\!\perp_\pi T^*(1)_v^+$ ($\forall v \in \Sigma$) and $T = \sigma_{\leqslant 0} T$. Then, for each $\mathfrak{q} \in \mathrm{Spec}(R[\Delta])$ with $\mathrm{ht}(\mathfrak{q}) = 0$, the localized duality map*

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, T)_{\mathfrak{q}} \longrightarrow \mathscr{D}_{\overline{R}}\big(\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1))^\iota\big)_{\mathfrak{q}}[-3]$$

is an isomorphism in $D^b_{ft}({}_{R_{\mathfrak{q}}}\mathrm{Mod})$, inducing a spectral sequence

$$E_2^{i,j} = \big(\mathbb{E}\mathrm{xt}^i_{\overline{R}}(\widetilde{H}^{3-j}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1)), \omega_{\overline{R}})^\iota\big)_{\mathfrak{q}} \Longrightarrow \widetilde{H}^{i+j}_{f,\mathrm{Iw}}(K_\infty/K, T)_{\mathfrak{q}}.$$

Proof. — We apply 7.8.4.4 with $X = \mathscr{F}_\Gamma(T)$ over $\overline{R}$; by Proposition 8.9.7.6 (iii), the error terms vanish after tensoring with $R[\Delta]_{\mathfrak{q}}$. The spectral sequence is obtained from 2.8.6, by applying the same tensor product. □

8.9.12. Theorem. — *Assume that $T = \sigma_{\leqslant 0} T$ and ($\forall v \in \Sigma$) $T_v^+ \perp\!\!\!\perp_\pi T^*(1)_v^+$. For $v \in \Sigma$, set $H_v = H^1_{\mathrm{cont}}(I_v, \mathscr{F}_\Delta(T))$. If*

($\forall v \in \Sigma'$) ($\forall q = 1, \dots, d$)

$$\mathrm{Coker}\big(f_v^{-1} - 1 : \mathrm{Ext}^q_{R[\Delta]}(H_v, \omega_{R[\Delta]}) \longrightarrow \mathrm{Ext}^q_{R[\Delta]}(H_v, \omega_{R[\Delta]})\big) = 0$$

(if $d = \dim(R) = 1$*, then this condition is equivalent to* $(\forall v \in \Sigma')\,(\forall \mathfrak{m}_\Delta \in \operatorname{Max}(R[\Delta]))\ \operatorname{Tam}_v(\mathscr{F}_\Delta(T), \mathfrak{m}_\Delta) = 0$*), then*

$$\begin{array}{ccc}
\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, T) & \overset{\overline{\mathscr{D}}}{\longleftrightarrow} & \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1))^\iota[3] \\
\downarrow{\scriptstyle \overline{\Phi}} & \overline{D}\ (\text{crossed arrows}) & \downarrow{\scriptstyle \overline{\Phi}} \\
\widetilde{\mathbf{R}\Gamma}_f(K_S/K_\infty, A) & & \widetilde{\mathbf{R}\Gamma}_f(K_S/K_\infty, A^*(1))^\iota[3]
\end{array}$$

is a duality diagram in $D^b_{(co)ft}({}_{\overline{R}}\mathrm{Mod})$*, inducing a spectral sequence*

$$E_2^{i,j} = \mathbb{E}\mathrm{xt}^i_{\overline{R}}\big(\widetilde{H}^{3-j}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1)), \omega_{\overline{R}}\big)^\iota \Longrightarrow \widetilde{H}^{i+j}_{f,\mathrm{Iw}}(K_\infty/K, T).$$

Proof. — This time the error terms vanish in $D^b_{ft}({}_{\overline{R}}\mathrm{Mod})$, thanks to Proposition 8.9.7.6 (iv) and Corollary 7.6.12 (iii). □

8.9.13. As in 8.5.8, the pairings

$$\pi : T \otimes_R T^*(1) \longrightarrow \omega^\bullet_R(1), \quad \overline{\pi} : \mathscr{F}_\Gamma(T) \otimes_{\overline{R}} \mathscr{F}_\Gamma(T^*(1))^\iota \longrightarrow \omega^\bullet_{\overline{R}}(1)$$

define – for each finite subextension K_α/K of K_∞/K – cup products (6.3.2.2)

$$\langle\ ,\ \rangle_\alpha : \widetilde{H}^i_f(G_{K_\alpha,S_\alpha}, T) \otimes_R \widetilde{H}^j_f(G_{K_\alpha,S_\alpha}, T^*(1)) \longrightarrow H^{i+j-3}(\omega_R)$$

and

$$\begin{aligned}\langle\ ,\ \rangle : \Big(\varprojlim_\alpha \widetilde{H}^i_f(G_{K_\alpha,S_\alpha}, T)\Big) \otimes_{\overline{R}} \Big(\varprojlim_\alpha \widetilde{H}^j_f(G_{K_\alpha,S_\alpha}, T^*(1))\Big)^\iota &\longrightarrow H^{i+j-3}(\omega_{\overline{R}}) \\ = H^{i+j-3}(\omega_R) \otimes_R \overline{R} &= \varprojlim_\alpha \Big(H^{i+j-3}(\omega_R) \otimes_R R[\mathrm{Gal}(K_\alpha/K)]\Big).\end{aligned}$$

8.9.14. Proposition. — *In the situation of 8.9.13,*

$$\langle (x_\alpha), (y_\alpha) \rangle = \left(\sum_{\sigma \in \mathrm{Gal}(K_\alpha/K)} \langle x_\alpha, \sigma y_\alpha \rangle_\alpha \otimes [\sigma] \right)_\alpha$$

(where σ *acts on* $\widetilde{H}^j_f(G_{K_\alpha,S_\alpha}, T^*(1))$ *by the conjugation action from 8.6.4.2).*

Proof. — As in the proof of 8.5.9, one reduces to the proof of the corresponding local statement 8.11.10, this time using 8.6.4.2–8.6.4.3. □

8.9.15. Theorem. — *The Euler-Poincaré characteristic of* $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, T)$ *is equal to*

$$\begin{aligned}\sum_q (-1)^q\, e_{\overline{R}}\big(\widetilde{H}^q_{f,\mathrm{Iw}}(K_\infty/K, T)\big) = \\ \sum_{v \mid \infty} \sum_q (-1)^q\, e_{\overline{R}}\left((T^q)^{G_v}\right) - \sum_{v \mid p} [K_v : \mathbf{Q}_p] \sum_q (-1)^q\, e_{\overline{R}}\left((T_v^+)^q\right).\end{aligned}$$

Proof. — This follows from Theorem 7.8.6 and the fact that

$$e_{\overline{R}}(M \otimes_R \overline{R}) = e_R(M),$$

for every R-module M of finite type. □

8.9.16. Corollary. — *Under the assumptions (i)–(ii) of Corollary 7.8.7,*

$$\sum_q (-1)^q\, e_{\overline{R}}\left(\widetilde{H}^q_{f,\mathrm{Iw}}(K_\infty/K,T)\right) = [K:\mathbf{Q}](d^+_\infty - d^+_p).$$

8.10. Control Theorems

The goal of this section is to prove analogues of Mazur's control theorem for classical Selmer groups in our context, generalizing the results of 8.4.8. We assume that we are in the situation of 8.8.1 with $\Gamma = \Gamma_0 \xrightarrow{\sim} \mathbf{Z}_p^r$ (hence the condition (U) from 8.8.1 is automatically satisfied). Let T and T_v^+ be as in 8.8.2; throughout Section 8.10 we assume that the complexes T and T_v^+ are bounded. The key point is the following exact control theorem for Selmer complexes, from which we deduce in Proposition 8.10.4 (ii) and 8.10.8 (ii) somewhat less precise control results for cohomology groups.

8.10.1. Proposition. — *There is a canonical isomorphism in $D^b({}_R\mathrm{Mod})$*

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K,T)\overset{\mathbf{L}}{\otimes}_{\overline{R}} R \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_f(T)$$

(where the product is taken with respect to the augmentation map $\overline{R} \to R$) and a (homological) spectral sequence

$$E^2_{i,j} = H_{i,\mathrm{cont}}\big(\Gamma, \widetilde{H}^{-j}_{f,\mathrm{Iw}}(K_\infty/K,T)\big) \Longrightarrow \widetilde{H}^{-i-j}_f(T),$$

in which each term $E^2_{i,j}$ is an R-module of finite type.

Proof. — The same argument as in the proof of Proposition 8.4.8.1 (ii) applies: the exact sequences (8.4.8.1.1) for $G = G_{K,S}, G_v$ ($v \in S_f$) and analogous sequences for the local conditions $U_v^+(-)$ ($v \in S_f$) yield an exact sequence

$$0 \longrightarrow \widetilde{C}^\bullet_f(\mathscr{F}_{\Gamma/H_{i-1}}(T)) \xrightarrow{\gamma_i - 1} \widetilde{C}^\bullet_f(\mathscr{F}_{\Gamma/H_{i-1}}(T)) \longrightarrow \widetilde{C}^\bullet_f(\mathscr{F}_{\Gamma/H_i}(T)) \longrightarrow 0,$$

and we conclude as in the proof of 8.4.8.1 □

8.10.2. Corollary. — *If $\Gamma \xrightarrow{\sim} \mathbf{Z}_p$, then there are natural exact sequences*

$$0 \longrightarrow \widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K,T)_\Gamma \longrightarrow \widetilde{H}^j_f(T) \longrightarrow \widetilde{H}^{j+1}_{f,\mathrm{Iw}}(K_\infty/K,T)^\Gamma \longrightarrow 0.$$

8.10.3. Consider the following condition:

8.10.3.1. The canonical maps

$$\tau_{\leqslant 0}\,T \longrightarrow T$$
$$\tau_{\leqslant 0}\,T_v^+ \longrightarrow T_v^+ \qquad (v\in\Sigma)$$

are all quasi-isomorphisms.

If satisfied, then the same holds for $\mathscr{F}_\Gamma(T)$ and $\mathscr{F}_\Gamma(T_v^+)$. This implies, by 7.8.5, that the maps

$$\tau_{\leqslant 3}\,\widetilde{\mathbf{R}\Gamma}_f(T) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(T) \tag{8.10.3.2}$$
$$\tau_{\leqslant 3}\,\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K,T) \longrightarrow \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K,T)$$

are also quasi-isomorphisms, hence

$$\widetilde{H}^3_f(T) = \widetilde{H}^3_f(H^0(T)) \xrightarrow{\ \sim\ } D\big(\widetilde{H}^0_f(D(H^0(T))(1))\big)$$
$$\widetilde{H}^3_{f,\mathrm{Iw}}(K_\infty/K,T) = \widetilde{H}^3_{f,\mathrm{Iw}}(K_\infty/K,H^0(T)) \xrightarrow{\ \sim\ } D\big(\widetilde{H}^0_f(K_S/K_\infty, D(H^0(T))(1))\big)^\iota$$

(for the induced Greenberg's local conditions $H^0(T_v)\to H^0(T)$ $(v\in\Sigma)$). In particular, $\widetilde{H}^3_{f,\mathrm{Iw}}(K_\infty/K,T)$ is an R-module of finite type. The spectral sequence from 8.10.1 yields an isomorphism

$$\widetilde{H}^3_{f,\mathrm{Iw}}(K_\infty/K,T)_\Gamma \xrightarrow{\ \sim\ } \widetilde{H}^3_f(T). \tag{8.10.3.3}$$

8.10.4. Proposition. — *Assume that 8.10.3.1 is satisfied.*

(1) *If $\mathscr{S}\subset R$ is a multiplicative set such that $\widetilde{H}^3_f(T)_{\mathscr{S}}=0$, then:*

  (i) $(\forall i\geqslant 0)$ $(E^2_{i,-3})_{\mathscr{S}} = H_{i,\mathrm{cont}}(\Gamma,\widetilde{H}^3_{f,\mathrm{Iw}}(K_\infty/K,T))_{\mathscr{S}}=0.$

  (ii) *After localizing at $\mathscr{S}$, the canonical map*

$$\mathrm{edge}:\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K,T)_\Gamma \longrightarrow \widetilde{H}^2_f(T)$$

  (which is an edge map in the spectral sequence E^m) becomes an isomorphism, i.e.,

$$\mathrm{Ker}(\mathrm{edge})_{\mathscr{S}} = \mathrm{Coker}(\mathrm{edge})_{\mathscr{S}} = 0.$$

(2) *In particular, taking $\mathscr{S}=R-\mathfrak{p}$ with $\mathfrak{p}\in\mathrm{Spec}(R)$, then*

$$\mathrm{supp}_R(\mathrm{Ker}(\mathrm{edge}))\cup\mathrm{supp}_R(\mathrm{Coker}(\mathrm{edge}))\cup\mathrm{supp}_R(E^2_{i,-3})\subseteq\mathrm{supp}_R\big(\widetilde{H}^3_f(T)\big).$$

Proof. — It is sufficient to prove (1). By (8.10.3.2), the assumption 8.10.3.1 implies that the spectral sequence E^r from Proposition 8.10.1 satisfies $E^2_{i,j}=0$ for $j<-3$ or $i<0$. As

$$E^2_{0,-2} \xrightarrow{\ \sim\ } \widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K,T)_\Gamma$$

and the map in (ii) is an edge homomorphism for E_r, it follows that the statement (ii) is a consequence of (i) (for $i=1,2$). In order to prove (i), consider

$$M = \widetilde{H}^3_{f,\mathrm{Iw}}(K_\infty/K,T)_{\mathscr{S}}.$$

This is an $R_{\mathscr{S}}$-module of finite type satisfying

$$M_\Gamma \xrightarrow{\sim} \big(E^2_{0,-3}\big)_{\mathscr{S}} \xrightarrow{\sim} \widetilde{H}^3_f(T)_{\mathscr{S}} = 0.$$

The claim (i) then follows from 7.2.7 and the following Lemma, applied to M, $B = R_{\mathscr{S}}$ and $t_i = \gamma_i - 1$ $(1 \leqslant i \leqslant r)$. □

8.10.5. Lemma. — *Let B be a commutative ring, M a B-module of finite type and $t_1, \dots, t_r \in \mathrm{End}_B(M)$ mutually commuting endomorphisms (*i.e., *$t_i t_j = t_j t_i$ for all i, j). View M as a module over $B' = B[T_1, \dots, T_r]$, with T_i acting as t_i. Then: the Koszul complex $K^\bullet = K^\bullet_{B'}(M, (T_1, \dots, T_r))$ is acyclic $\iff$ $H^r(K^\bullet) = 0$.*

Proof. — The implication '$\Longrightarrow$' is trivial. In order to prove '$\Longleftarrow$', it is enough to show that the localization $K^\bullet_{\mathfrak{m}}$ is acyclic, for each maximal ideal $\mathfrak{m} \subset B'$. If $T_1, \dots, T_r \in \mathfrak{m}$, then the assumption $H^r(K^\bullet) = M/(T_1, \dots, T_r)M = 0$ implies that $M/\mathfrak{m}M = 0$, hence $M = 0$ ($\Longrightarrow K^\bullet = 0$) by Nakayama's Lemma. If, for some $i = 1, \dots, r$, $T_i \notin \mathfrak{m}$, then T_i is invertible in $B'_{\mathfrak{m}}$, hence $K^\bullet_{\mathfrak{m}}$ is acyclic ([**Br-He**, Prop. 1.6.5 (c)]). □

8.10.6. The above results admit a dual formulation. Assume that A (resp., A_v^+, $v \in \Sigma$) is a bounded complex in $\big({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)$ (resp., in $\big({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\big)$), with cohomology of co-finite type over R, and $A_v^+ \to A$ $(v \in \Sigma)$ is a morphism of complexes of $R[G_v]$-modules. Define A_v^- as in 6.7.1.

According to Proposition 3.2.6 there exists a bounded complex T^* in $\big({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\big)_{R\text{-}ft}$ and a quasi-isomorphism

$$T^* \longrightarrow D(A).$$

For each $v \in \Sigma$, the complex

$$F_v = \mathrm{Cone}(T^* \oplus D(A_v^-) \longrightarrow D(A))[-1]$$

in $\big({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\big)$ is bounded, has cohomology of finite type over R and sits in a diagram

$$\begin{array}{ccc} F_v & \longrightarrow & D(A_v^-) \\ \big\downarrow & & \big\downarrow \\ T^* & \longrightarrow & D(A), \end{array}$$

in which both horizontal arrows are quasi-isomorphisms and the cones of the two vertical arrows are canonically isomorphic. Applying Proposition 3.2.6 once again, there exists a bounded complex $(T^*(1))_v^+$ in $\big({}^{\mathrm{ad}}_{R[G_v]}\mathrm{Mod}\big)_{R\text{-}ft}$ and a quasi-isomorphism $(T^*(1))_v^+ \to F_v(1)$.

The duality results 7.8.4.3 and 8.9.6.1 give isomorphisms

$$\widetilde{\mathbf{R}\Gamma}_f(A) \xrightarrow{\sim} D_R\big(\widetilde{\mathbf{R}\Gamma}_f(T^*(1))\big)[-3] \tag{8.10.6.1}$$

$$\widetilde{\mathbf{R}\Gamma}_f(K_S/K_\infty, A) \xrightarrow{\sim} D_{\overline{R}}\big(\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1))\big)^\iota[-3]$$

in $D^b\big({}_R\mathrm{Mod}\big)$ and $D^b\big({}_{\overline{R}}\mathrm{Mod}\big)$, respectively.

Applying $D_{\overline{R}}$ to the Koszul complex $K^{\bullet}_{\overline{R}}\big(\widetilde{C}^{\bullet}_f\big(\mathscr{F}_\Gamma(T^*(1))\big), \mathbf{x}\big)$ appearing implicitly in the proof of Proposition 8.10.1 and using (7.2.7.3) together with (8.10.6.1), we obtain isomorphisms in $D^b\big({}_{\overline{R}}\mathrm{Mod}\big)$

$$\widetilde{\mathbf{R}\Gamma}_f(A) \xrightarrow{\sim} K^{\bullet}_{\overline{R}}(\widetilde{C}^{\bullet}_f(K_S/K_\infty, A)) \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma, \widetilde{C}^{\bullet}_f(K_S/K_\infty, A)),$$

hence a spectral sequence

$$\overline{E}_2^{i,j} = H^i_{\mathrm{cont}}\big(\Gamma, \widetilde{H}^j_f(K_S/K_\infty, A)\big) \Longrightarrow \widetilde{H}^{i+j}_f(A) \qquad (8.10.6.2)$$

dual to $E^m_{i,j}$ (in particular, each $\overline{E}_2^{i,j}$ is an R-module of co-finite type). By construction, the edge homomorphisms of E^r and $\overline{E}_r$ are dual to each other, *i.e.*, the diagram

$$\begin{array}{ccc} \widetilde{H}^i_f(A) & \xrightarrow{\sim} & D_R\big(\widetilde{H}^{3-i}_f(T^*(1))\big) \\ \big\downarrow & & \big\downarrow \\ \widetilde{H}^i_f(K_S/K_\infty, A)^{\Gamma} & \xrightarrow{\sim} & D_{\overline{R}}\big((\widetilde{H}^{3-i}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1))^{\iota})_\Gamma\big) \end{array} \qquad (8.10.6.3)$$

is commutative.

8.10.7. Consider the following condition:

8.10.7.1. The canonical maps

$$A \longrightarrow \tau_{\geqslant 0}\, A$$
$$A_v^+ \longrightarrow \tau_{\geqslant 0}\, A_v^+ \qquad (v \in \Sigma)$$

are all quasi-isomorphisms.

If 8.10.7.1 holds, then the maps

$$\widetilde{\mathbf{R}\Gamma}_f(A) \longrightarrow \tau_{\geqslant 0}\, \widetilde{\mathbf{R}\Gamma}_f(A) \qquad (8.10.7.2)$$

$$\widetilde{\mathbf{R}\Gamma}_f(K_S/K_\infty, A) \longrightarrow \tau_{\geqslant 0}\, \widetilde{\mathbf{R}\Gamma}_f(K_S/K_\infty, A)$$

are quasi-isomorphisms, $T^*(1)$ and $(T^*(1))_v^+$ satisfy 8.10.3.1 and

$$\widetilde{H}^0_f(A) = \widetilde{H}^0_f(H^0(A)), \quad \widetilde{H}^0_f(K_S/K_\infty, A) = \widetilde{H}^0_f(K_S/K_\infty, H^0(A)).$$

In particular, $\widetilde{H}^0_f(K_S/K_\infty, A)$ is an R-module of co-finite type. Combining (8.10.3.3) with (8.10.6.3), we obtain an isomorphism

$$\widetilde{H}^0_f(A) \xrightarrow{\sim} \widetilde{H}^0_f(K_S/K_\infty, A)^{\Gamma}. \qquad (8.10.7.3)$$

8.10.8. Proposition. — *Assume that 8.10.7.1 is satisfied.*

(1) *If $\mathscr{S} \subset R$ is a multiplicative set such that $D(\widetilde{H}^0_f(A))_{\mathscr{S}} = 0$, then:*

(i) $(\forall i \geqslant 0)$ $\big(D\big(\overline{E}_2^{i,0}\big)\big)_{\mathscr{S}} = D\big(H^i_{\mathrm{cont}}\big(\Gamma, \widetilde{H}^0_f(K_S/K_\infty, A)\big)\big)_{\mathscr{S}} = 0.$

(ii) *The kernel and cokernel of the canonical restriction map*

$$\mathrm{res}_{K_\infty/K} : \widetilde{H}^1_f(A) \longrightarrow \widetilde{H}^1_f(K_S/K_\infty, A)^{\Gamma}$$

(which is an edge map in the spectral sequence $\overline{E}_m$) satisfy

$$D(\mathrm{Ker}(\mathrm{res}_{K_\infty/K}))_{\mathscr{S}} = D(\mathrm{Coker}(\mathrm{res}_{K_\infty/K}))_{\mathscr{S}} = 0.$$

(2) *In particular, taking* $\mathscr{S} = R - \mathfrak{p}$ *with* $\mathfrak{p} \in \operatorname{Spec}(R)$*, then*

$$\operatorname{supp}_R\big(D(\operatorname{Ker}(\operatorname{res}_{K_\infty/K}))\big) \cup \operatorname{supp}_R\big(D(\operatorname{Coker}(\operatorname{res}_{K_\infty/K}))\big) \cup \operatorname{supp}_R(D(\overline{E}_2^{i,0})) \subseteq \operatorname{supp}_R\big(D(\widetilde{H}^0_f(A))\big).$$

Proof. — This follows from Proposition 8.10.4 applied to $T^*(1)$ and the commutative diagram (8.10.6.3). □

8.10.9. The results in 8.10.1, 8.10.4, 8.10.8 can be further generalized as follows: in 8.10.9–8.10.12, we do not require that $\Delta = 0$, but instead assume that the condition (U) from 8.4.8 is satisfied, and that $\Gamma' \subset \Gamma$ is a closed subgroup isomorphic to $\Gamma' \xrightarrow{\sim} \mathbf{Z}_p^{r'}$ for some $r' \leqslant r$. Put $K'_\infty = (K_\infty)^{\Gamma'}$ and $\overline{R}' = R[\![\Gamma/\Gamma']\!]$.

8.10.10. Proposition. — *There is a canonical isomorphism in* $D^b\big({}_{\overline{R}'}\mathrm{Mod}\big)$

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K,T)\overset{\mathbf{L}}{\otimes}_{\overline{R}}\overline{R}' \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K'_\infty/K,T) \qquad (8.10.10.1)$$

(where the product is taken with respect to the canonical projection $\overline{R} = R[\![\Gamma]\!] \to \overline{R}' = R[\![\Gamma/\Gamma']\!]$*) and a (homological) spectral sequence*

$${}'E^2_{i,j} = H_{i,\mathrm{cont}}\big(\Gamma', \widetilde{H}^{-j}_{f,\mathrm{Iw}}(K_\infty/K,T)\big) \Longrightarrow \widetilde{H}^{-i-j}_{f,\mathrm{Iw}}(K'_\infty/K,T),$$

in which each term ${}'E^2_{i,j}$ *is an* $\overline{R}'$*-module of finite type. The pairings* (8.9.6.3.1) *are compatible with respect to the isomorphisms* (8.10.10.1).

Proof. — The proof of Proposition 8.10.1 applies, with the same modification as in the proof of Proposition 8.4.8.3. The compatibility with the pairings (8.9.6.3.1) follows from the definitions. □

8.10.11. Proposition. — *Assume that 8.10.3.1 is satisfied. If* $\mathscr{S} \subset R$ *is a multiplicative set such that* $\widetilde{H}^3_{f,\mathrm{Iw}}(K'_\infty/K,T)_{\mathscr{S}} = 0$ *(recall from 8.10.3 that* $\widetilde{H}^3_{f,\mathrm{Iw}}(K'_\infty/K,T)$ *is an* R*-module of finite type), then:*

(i) $(\forall i \geqslant 0)$ $({}'E^2_{i,-3}) \otimes_R R_{\mathscr{S}} = H_{i,\mathrm{cont}}(\Gamma', \widetilde{H}^3_{f,\mathrm{Iw}}(K_\infty/K,T)) \otimes_R R_{\mathscr{S}} = 0.$

(ii) *The canonical map*

$$\mathrm{edge} : \widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K,T)_{\Gamma'} \longrightarrow \widetilde{H}^2_{f,\mathrm{Iw}}(K'_\infty/K,T)$$

(which is an edge map in the spectral sequence ${}'E^m$*) satisfies*

$$\operatorname{Ker}(\mathrm{edge}) \otimes_R R_{\mathscr{S}} = \operatorname{Coker}(\mathrm{edge}) \otimes_R R_{\mathscr{S}} = 0.$$

Proof. — As in the proof of Proposition 8.10.4, apply Lemma 8.10.5 to $B = R_{\mathscr{S}}$, $M = \widetilde{H}^3_{f,\mathrm{Iw}}(K_\infty/K,T) \otimes_R R_{\mathscr{S}}$ and $t_i = \gamma'_i - 1$ $(1 \leqslant i \leqslant r')$. □

8.10.12. Proposition. — *Assume that 8.10.7.1 is satisfied.*

If $\mathscr{S} \subset R$ is a multiplicative set such that

$$\big(D_R\big(\widetilde{H}^0_f(K_S/K'_\infty, A)\big)\big) \otimes_R R_{\mathscr{S}} = 0,$$

then:

(i) $(\forall i \geqslant 0)$ $(D_{\overline{R}'}({'\overline{E}}_2^{i,0})) \otimes_R R_{\mathscr{S}} = \big(D_{\overline{R}'}\big(H^i_{\mathrm{cont}}(\Gamma', \widetilde{H}^0_f(K_S/K_\infty, A))\big)\big) \otimes_R R_{\mathscr{S}} = 0.$

(ii) *The kernel and cokernel of the canonical map*

$$\mathrm{res}_{K_\infty/K'_\infty} : \widetilde{H}^1_f(K_S/K'_\infty, A) \longrightarrow \widetilde{H}^1_f(K_S/K_\infty, A)^{\Gamma'}$$

satisfy

$$\big(D_{\overline{R}'}(\mathrm{Ker}(\mathrm{res}_{K_\infty/K'_\infty}))\big) \otimes_R R_{\mathscr{S}} = \big(D_{\overline{R}'}(\mathrm{Coker}(\mathrm{res}_{K_\infty/K'_\infty}))\big) \otimes_R R_{\mathscr{S}} = 0.$$

Proof. — As in 8.10.6 there is a spectral sequence

$${'\overline{E}}_2^{i,j} = H^i_{\mathrm{cont}}\big(\Gamma', \widetilde{H}^j_f(K_S/K_\infty, A)\big) \Longrightarrow \widetilde{H}^{i+j}_f(K_S/K'_\infty, A)$$

such that the edge homomorphisms of ${'E}^m$ and ${'\overline{E}}_m$ are related by the following commutative diagram:

$$\begin{array}{ccc} \widetilde{H}^i_f(K_S/K'_\infty, A) & \xrightarrow{\sim} & D_{\overline{R}'}\big(\widetilde{H}^{3-i}_{f,\mathrm{Iw}}(K'_\infty/K, T^*(1))^{\iota}\big) \\ \downarrow & & \downarrow \\ \widetilde{H}^i_f(K_S/K_\infty, A)^{\Gamma'} & \xrightarrow{\sim} & D_{\overline{R}}\big((\widetilde{H}^{3-i}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1))^{\iota})_{\Gamma'}\big). \end{array}$$

The statement follows from Proposition 8.10.11 applied to $T^*(1)$. □

8.10.13. Proposition. — *Assume that 8.10.3.1 is satisfied and $\widetilde{H}^3_f(T) = 0$. Then:*

(i) $\widetilde{H}^3_{f,\mathrm{Iw}}(K_\infty/K, T) = 0.$

(ii) *The canonical map*

$$\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, T)_{\Gamma'} \longrightarrow \widetilde{H}^2_{f,\mathrm{Iw}}(K'_\infty/K, T)$$

is an isomorphism.

Proof

(i) $M = \widetilde{H}^3_{f,\mathrm{Iw}}(K_\infty/K, T)$ is an $\overline{R}$-module of finite type satisfying $M_\Gamma = \widetilde{H}^3_f(T) = 0$ (using (8.10.3.3)), hence $M = 0$ by Nakayama's Lemma.

(ii) Apply Proposition 8.10.11 with $\mathscr{S} = \{1\}$. □

8.10.14. Proposition. — *Assume that 8.10.7.1 is satisfied and $\widetilde{H}^0_f(A) = 0$. Then:*

(i) $\widetilde{H}^0_f(K_S/K_\infty, A) = 0.$

(ii) *The canonical map*

$$\mathrm{res}_{K_\infty/K'_\infty} : \widetilde{H}^1_f(K_S/K'_\infty, A) \longrightarrow \widetilde{H}^1_f(K_S/K_\infty, A)^{\Gamma'}$$

is an isomorphism.

Proof

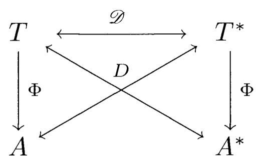
(i) $N = \widetilde{H}^0_f(K_S/K_\infty, A)$ is an $\overline{R}$-module of co-finite type satisfying $\overline{D}(N)_\Gamma = \overline{D}(N^\Gamma) = \overline{D}(\widetilde{H}^0_f(A)) = 0$ (by (8.10.7.3)); Nakayama's Lemma implies that $\overline{D}(N) = 0$, hence $N = 0$.

(ii) Apply Proposition 8.10.12 with $\mathscr{S} = \{1\}$. $\square$

8.11. Iwasawa theory over local fields

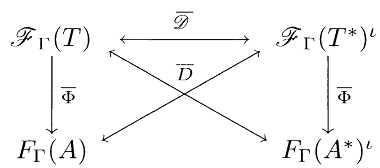
8.11.1. Assume that F is a local field of characteristic $\mathrm{char}(F) \neq p$, with finite residue field k_F. Let F_∞/F be a Galois extension with $\Gamma = \mathrm{Gal}(F_\infty/F) = \Gamma_0 \times \Delta$, where $\Gamma_0 \overset{\sim}{\to} \mathbf{Z}_p^r$ $(r \geqslant 1)$ and Δ is a finite abelian group. If $\mathrm{char}(k_F) \neq p$, then $r = 1$ and F_∞ is a finite abelian extension of the unique unramified $\mathbf{Z}_p$-extension of F; thus the only 'interesting' case is when F is a finite extension of $\mathbf{Q}_p$. As usual, put $G_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ and $G_{F_\infty} = \mathrm{Gal}(F^{\mathrm{sep}}/F_\infty)$.

8.11.2. If



is a duality diagram with $T, T^* \in D^b\big(({}^{\mathrm{ad}}_{R[G_F]}\mathrm{Mod})_{R\text{-}ft}\big)$ and $A, A^* \in D^b\big(({}^{\mathrm{ad}}_{R[G_F]}\mathrm{Mod})_{R\text{-}coft}\big)$, then the functors $\mathscr{F}_\Gamma$ and F_Γ from 8.3–8.4 define a duality diagram over $\overline{R}$

(8.11.2.1)



with $\mathscr{F}_\Gamma(T), \mathscr{F}_\Gamma(T^*) \in D^b\big(({}^{\mathrm{ad}}_{\overline{R}[G_F]}\mathrm{Mod})_{\overline{R}\text{-}ft}\big)$ and $F_\Gamma(A)$, $F_\Gamma(A^*) \in D^b\big(({}^{\mathrm{ad}}_{\overline{R}[G_F]}\mathrm{Mod})_{\overline{R}\text{-}coft}\big)$. Define

$$\mathbf{R}\Gamma_{\mathrm{Iw}}(F_\infty/F, T) = \mathbf{R}\Gamma_{\mathrm{cont}}(G_F, \mathscr{F}_\Gamma(T)).$$

As in the global case, we have

$$C^\bullet_{\mathrm{cont}}(G_F, F_\Gamma(A)) \overset{\sim}{\longrightarrow} \varinjlim_\alpha C^\bullet_{\mathrm{cont}}(G_{F_\alpha}, A)$$

$$H^i_{\mathrm{cont}}(G_F, F_\Gamma(A)) \overset{\sim}{\longrightarrow} \varinjlim_\alpha H^i_{\mathrm{cont}}(G_{F_\alpha}, A) \overset{\sim}{\longrightarrow} H^i_{\mathrm{cont}}(G_{F_\infty}, A)$$

$$H^i_{\mathrm{Iw}}(F_\infty/F, T) := H^i(\mathbf{R}\Gamma_{\mathrm{Iw}}(F_\infty/F, T)) \overset{\sim}{\longrightarrow} \varprojlim_\alpha H^i_{\mathrm{cont}}(G_{F_\alpha}, T),$$

where F_α/F runs through all finite subextensions of F_∞/F. Applying Theorem 5.2.6 to (8.11.2.1) we obtain a duality diagram

(8.11.2.2)
$$\begin{array}{ccc} \mathbf{R}\Gamma_{\mathrm{Iw}}(F_\infty/F,T) & \overset{\overline{\mathscr{D}}}{\longleftrightarrow} & \mathbf{R}\Gamma_{\mathrm{Iw}}(F_\infty/F,T^*(1))^\iota[2] \\ \downarrow{\scriptstyle \overline{\Phi}} & \overset{\overline{D}}{\times} & \downarrow{\scriptstyle \overline{\Phi}} \\ \mathbf{R}\Gamma_{\mathrm{cont}}(G_{F_\infty},A) & & \mathbf{R}\Gamma_{\mathrm{cont}}(G_{F_\infty},A^*(1))^\iota[2] \end{array}$$

in $D^b_{(co)ft}({}_{\overline{R}}\mathrm{Mod})$. On the level of cohomology, (8.11.2.2) yields isomorphisms

(8.11.2.3)
$$H^i_{\mathrm{Iw}}(F_\infty/F,T)^\iota \xrightarrow{\sim} \overline{D}(H^{2-i}_{\mathrm{cont}}(G_{F_\infty},A^*(1)))$$

and a spectral sequence

(8.11.2.4)
$$\begin{aligned} E_2^{i,j} = \mathbb{E}\mathrm{xt}^i_{\overline{R}}(H^{2-j}_{\mathrm{Iw}}(F_\infty/F,T^*(1))^\iota,\omega_{\overline{R}}) = \mathbb{E}\mathrm{xt}^i_{\overline{R}}(\overline{D}(H^j_{\mathrm{cont}}(G_{F_\infty},A)),\omega_{\overline{R}}) \\ \Longrightarrow H^{i+j}_{\mathrm{Iw}}(F_\infty/F,T). \end{aligned}$$

For simplicity, we shall use in the rest of Sect. 8.11 the following notation:

$$E^i(-) = \mathbb{E}\mathrm{xt}^i_{\overline{R}}(-,\omega_{\overline{R}}), \quad H^i_{\mathrm{Iw}}(-) = H^i_{\mathrm{Iw}}(F_\infty/F,-).$$

Note that the isomorphisms (8.11.2.3) are obtained from Tate's local duality isomorphisms over F_α

$$H^i_{\mathrm{cont}}(G_{F_\alpha},T) \xrightarrow{\sim} D(H^{2-i}_{\mathrm{cont}}(G_{F_\alpha},A^*(1)))$$

by taking the projective limit.

8.11.3. Lemma. — *If T is concentrated in degree zero, then:*

(i) $(\forall i \neq 1,2)$ $H^i_{\mathrm{Iw}}(T) = 0$.

(ii) $H^2_{\mathrm{Iw}}(T) \xrightarrow{\sim} \overline{D}(H^0(G_{F_\infty},A^*(1))^\iota)$ *is an $\overline{R}$-module of finite type of dimension* $\leqslant d = \dim(R)$.

(iii) $(\forall q < r)$ $E^q(H^2_{\mathrm{Iw}}(T)) = 0$.

Proof

(i) The group $H^i_{\mathrm{Iw}}(T)$ vanishes for $i < 0$ (resp., $i > 2$) as $\tau_{<0}T = 0$ (resp., as $\tau_{>0}T = 0$ and $\mathrm{cd}_p(G_F) = 2$). Finally, $H^0_{\mathrm{Iw}}(T) = 0$ by Proposition 8.3.5 (iii).

(ii) The statement (ii) is just (8.11.2.3) for $i = 2$, combined with the fact that $\overline{D}\big(H^0(G_{F_\infty},A^*(1))\big)$ is a quotient of $D(A^*(1)) = T(-1)$. As $\dim(H^2_{\mathrm{Iw}}(T)) \leqslant d$, local duality over $\overline{R}$ and Lemma 2.4.7 (iii) show that

$$\overline{D}\big(E^q(H^2_{\mathrm{Iw}}(T))\big) \xrightarrow{\sim} H^{d+r-q}_{\{\overline{\mathfrak{m}}\}}(H^2_{\mathrm{Iw}}(T)) = 0$$

for $q < r$. □

8.11.4. Proposition. — *If both T and $T^*(1)$ are concentrated in degree zero, then:*

(i) *There is an exact sequence*

$$\begin{aligned} 0 &\longrightarrow E^1\big(H^2_{\mathrm{Iw}}(T^*(1))\big)^{\iota} \longrightarrow H^1_{\mathrm{Iw}}(T) \longrightarrow E^0\big(H^1_{\mathrm{Iw}}(T^*(1))\big)^{\iota} \\ &\longrightarrow E^2\big(H^2_{\mathrm{Iw}}(T^*(1))\big)^{\iota} \longrightarrow H^2_{\mathrm{Iw}}(T) \longrightarrow E^1\big(H^1_{\mathrm{Iw}}(T^*(1))\big)^{\iota} \\ &\longrightarrow E^3\big(H^2_{\mathrm{Iw}}(T^*(1))\big)^{\iota} \longrightarrow 0 \end{aligned}$$

and isomorphisms

$$E^q\big(H^1_{\mathrm{Iw}}(T^*(1))\big) \xrightarrow{\sim} E^{q+2}\big(H^2_{\mathrm{Iw}}(T^*(1))\big) \qquad (q \geqslant 2).$$

(ii) *If $r > 1$, then $H^2_{\mathrm{Iw}}(T)$ is pseudo-null over $\overline{R}$.*

(iii) *If $r > 1$ and R is regular, then $H^1_{\mathrm{Iw}}(T)$ injects into $E^0\big(H^1_{\mathrm{Iw}}(T^*(1))\big)^{\iota} = \mathrm{Hom}_{\overline{R}}(H^1_{\mathrm{Iw}}(T^*(1)), H^0(\omega_{\overline{R}}))$ and is torsion-free over $\overline{R}$.*

Proof

(i) This follows from the spectral sequence (8.11.2.4) and the vanishing of $H^q_{\mathrm{Iw}}(Z)$ ($Z = T, T^*(1)$) for $q \neq 1, 2$.

(ii) The statement (ii) follows from Lemma 8.11.3 (ii). As regards (iii), regularity of R (hence of $\overline{R}$) implies that $E^1(M) = 0$ for any pseudo-null $\overline{R}$-module M; we conclude by (ii) and the fact that $H^0(\omega_{\overline{R}})$ is torsion-free over $\overline{R}$. □

8.11.5. Proposition. — *If $R = \mathcal{O}$ is a discrete valuation ring (finite over $\mathbf{Z}_p$), $\Gamma \xrightarrow{\sim} \mathbf{Z}_p^r$ ($r \geqslant 1$), T is supported in degree zero and is torsion-free over $\mathcal{O}$, then:*

(i) *There are isomorphisms of $\overline{R}$-modules*

$$E^q\big(H^2_{\mathrm{Iw}}(T^*(1))\big)^{\iota} \xrightarrow{\sim} \begin{cases} 0, & q \neq r, r+1 \\ H^0(G_{F_\infty}, T), & q = r \\ H^0(G_{F_\infty}, A)/B, & q = r+1, \end{cases}$$

where B is the maximal $\mathcal{O}$-divisible submodule of $H^0(G_{F_\infty}, A)$.

(ii) *The torsion submodule of $H^1_{\mathrm{Iw}}(T)$ is isomorphic to*

$$H^1_{\mathrm{Iw}}(T)_{\overline{R}\text{-tors}} \xrightarrow{\sim} \begin{cases} H^0(G_{F_\infty}, T), & r = 1 \\ 0, & r > 1. \end{cases}$$

(iii) *$H^1_{\mathrm{Iw}}(T)$ contains no non-zero pseudo-null $\overline{R}$-submodules.*

Proof. — The proof of Lemma 9.1.6 below applies in the present situation and yields (i). As $T_{\mathcal{O}\text{-tors}} = 0$, $T^*(1)$ is also concentrated in degree zero, hence the exact sequence from Proposition 8.11.4 (i) gives

$$H^1_{\mathrm{Iw}}(T)_{\overline{R}\text{-tors}} \xrightarrow{\sim} E^1\big(H^2_{\mathrm{Iw}}(T^*(1))\big)^{\iota} \xrightarrow{\sim} \begin{cases} H^0(G_{F_\infty}, T), & r = 1 \\ 0, & r > 1, \end{cases}$$

using (i). Finally, as $H^2_{\mathrm{Iw}}(T^*(1))$ is $\overline{R}$-torsion, (iii) follows from 9.1.3 (vi). □

8.11.6. Universal norms. — We define

$$N_\infty H^i_{\mathrm{cont}}(G_F, T) = \mathrm{Im}\left(H^i_{\mathrm{Iw}}(T)_\Gamma \longrightarrow H^i_{\mathrm{cont}}(G_F, T)\right) = \bigcap_\alpha \mathrm{Im}\left(H^i_{\mathrm{cont}}(G_{F_\alpha}, T) \xrightarrow{\mathrm{cor}} H^i_{\mathrm{cont}}(G_F, T)\right),$$

where F_α are as in 8.11.1. By (8.11.2.3), there is an isomorphism

$$D(H^i_{\mathrm{cont}}(G_F, T)/N_\infty) \xrightarrow{\sim} \mathrm{Ker}\left(H^{2-i}_{\mathrm{cont}}(G_F, A^*(1)) \xrightarrow{\mathrm{res}} H^{2-i}_{\mathrm{cont}}(G_{F_\infty}, A^*(1))\right). \quad (8.11.6.1)$$

If T is concentrated in degree zero, so is $A^*(1)$, hence

$$D(H^1_{\mathrm{cont}}(G_F, T)/N_\infty) \xrightarrow{\sim} H^1_{\mathrm{cont}}\left(\Gamma, H^0(G_{F_\infty}, A^*(1))\right), \quad (8.11.6.2)$$

by the Hochschild-Serre spectral sequence (note that $A^*(1)$ is a discrete G_F-module).

8.11.7. Proposition. — *Assume that $T \in D^b_{R\text{-}ft}({}^{\mathrm{ad}}_{R[G_F]}\mathrm{Mod})$ and $\Gamma \xrightarrow{\sim} \mathbf{Z}_p^r$. Then:*

(i) *There is a (homological) spectral sequence*

$$E^2_{i,j} = H_{i,\mathrm{cont}}(\Gamma, H^{-j}_{\mathrm{Iw}}(T)) \Longrightarrow H^{-i-j}_{\mathrm{cont}}(G_F, T),$$

in which all terms $E^2_{i,j}$ are R-modules of finite type.

(ii) *If $\tau_{\leqslant 0} T \xrightarrow{\sim} T$ and $\mathscr{S} \subset R$ is a multiplicative set such that $D\big(H^0(G_F, D(H^0(T))(1))\big)_{\mathscr{S}} = 0$, then, for all i,*

$$(\forall j < -2) \quad (E^2_{i,j})_{\mathscr{S}} = 0$$

and the localization at $\mathscr{S}$ of the canonical map $H^1_{\mathrm{Iw}}(T)_\Gamma \to H^1_{\mathrm{cont}}(G_F, T)$ is an isomorphism of $R_{\mathscr{S}}$-modules. In particular, if $\mathrm{codim}_R\big(\mathrm{supp}_R\big(D\big(H^0(G_F, D(H^0(T))(1))\big)\big)\big) \geqslant 1$, then

$$\mathrm{codim}_R(\mathrm{supp}_R(H^1_{\mathrm{cont}}(G_F, T)/N_\infty)) \geqslant 1.$$

Proof. — Proposition 8.4.8.1 (ii) gives (i). As $\tau_{\leqslant 0} T \xrightarrow{\sim} T$ and $\mathrm{cd}_p(G_F) = 2$, we have $H^j_{\mathrm{Iw}}(T) = 0$ for $j > 2$. The remaining statements follow by the same argument as in the proof of Proposition 8.10.4 (ii). □

8.11.8. Corollary. — *Let $R = \mathcal{O}$, $\Gamma \xrightarrow{\sim} \mathbf{Z}_p^r$ and T be as in Proposition 8.11.5. If $H^0(G_F, T^*(1)) = 0$, then the group $H^1_{\mathrm{cont}}(G_F, T)/N_\infty$ is finite.*

Proof. — This is a special case of Proposition 8.11.7, but it can be proved more directly as follows: the assumption $H^0(G_F, T^*(1)) = 0$ is equivalent to the finiteness of

$$H^0(G_F, A^*(1)) = H^0\left(\Gamma, H^0(G_{F_\infty}, A^*(1))\right)$$

(where $A^* = D(T)$). Lemma 8.10.5 then implies that

$$H^1(\Gamma, H^0(G_{F_\infty}, A^*(1)))$$

is finite, too. We conclude by (8.11.6.2). □

8.11.9. Over each finite subextension F_α/F of F_∞/F, the pairing

$$\mathrm{ev}_2 : T \otimes_R T^*(1) \longrightarrow \omega_R(1)$$

induces cup products (5.2.2.1)

$$\langle\ ,\ \rangle_\alpha : H^i_{\mathrm{cont}}(G_{F_\alpha}, T) \otimes_R H^j_{\mathrm{cont}}(G_{F_\alpha}, T^*(1)) \longrightarrow H^{i+j-2}(\omega_R).$$

Similarly, the pairing

$$\overline{\mathrm{ev}}_2 : \mathscr{F}_\Gamma(T) \otimes_{\overline{R}} \mathscr{F}_\Gamma(T^*(1))^\iota \longrightarrow \omega_{\overline{R}}(1)$$

(cf. 8.4.6.3) induces products on the "Iwasawa cohomology"

$$\begin{aligned}\langle\ ,\ \rangle : \Big(\varprojlim_\alpha H^i_{\mathrm{cont}}(G_{F_\alpha}, T)\Big) \otimes_{\overline{R}} \Big(\varprojlim_\alpha H^j_{\mathrm{cont}}(G_{F_\alpha}, T^*(1))\Big)^\iota &\longrightarrow H^{i+j-2}(\omega_{\overline{R}}) \\ = H^{i+j-2}(\omega_R) \otimes_R \overline{R} &= \varprojlim_\alpha \left(H^{i+j-2}(\omega_R) \otimes_R R[\mathrm{Gal}(F_\alpha/F)]\right).\end{aligned}$$

8.11.10. Proposition. — *In the situation of 8.11.9,*

$$\langle (x_\alpha), (y_\alpha) \rangle = \left(\sum_{\sigma \in \mathrm{Gal}(F_\alpha/F)} \langle x_\alpha, \sigma y_\alpha \rangle_\alpha \otimes [\sigma] \right)_\alpha$$

(where σ acts on $H^j_{\mathrm{cont}}(G_{F_\alpha}, T^*(1))$ by conjugation).

Proof. — As the pairing $\langle\ ,\ \rangle$ is $\overline{R}$-bilinear, it suffices to check that the coefficient at [1] in the projection of the L.H.S. to $H^{i+j-2}(\omega_R) \otimes_R R[\mathrm{Gal}(F_\alpha/F)]$ is equal to $\langle x_\alpha, y_\alpha \rangle_\alpha$. This, in turn, is a consequence of Lemma 8.1.6.5 and the fact that the local invariant maps inv_v commute with corestriction (cf. 9.2.2 below). □

8.12. In the absence of (P)

8.12.1. The discussion in 6.9 applies to $\overline{R}$ and $\mathscr{F}_\Gamma(X)$, $\mathscr{F}_\Gamma(Y)^\iota$.

CHAPTER 9

CLASSICAL IWASAWA THEORY

In this chapter we apply the duality results from Chapter 8 to classical Iwasawa theory, obtaining new results on ideal class groups in $\mathbf{Z}_p^r$-extensions of number fields (Sect. 9.4–9.5). In Sect. 9.6 we compare groups arising in our theory to classical Selmer groups and in Sect. 9.7 we show that well-behaved perfect complexes naturally appear in this context. These results were used in the work of Mazur and Rubin [**M-R2**] on "organizing modules" in Iwasawa theory of elliptic curves.

Let K_∞/K be as in 8.5.1 (hence we also assume that (P) from 5.1 holds).

9.1. Generalities

9.1.1. We consider the following special case of 8.5: $R = \mathcal{O}$ is the ring of integers in a finite extension F of $\mathbf{Q}_p$ and $\Gamma \xrightarrow{\sim} \mathbf{Z}_p^r$ ($r \geqslant 1$); we fix a prime element $\varpi \in \mathcal{O}$. In this case $I \xrightarrow{\sim} F/\mathcal{O}$ and the ring $\overline{R} = R[\![\Gamma]\!]$ is equal to the usual Iwasawa algebra

$$\Lambda = \mathcal{O}[\![\Gamma]\!] \xrightarrow{\sim} \mathcal{O}[\![X_1, \ldots, X_r]\!],$$

which is a regular ring of dimension $\dim(\Lambda) = r + 1$. This implies that $\omega_\Lambda \xrightarrow{\sim} \Lambda$ (by 2.7 (iii), since regular local rings are Gorenstein).

9.1.2. In the category $({}_\Lambda\mathrm{Mod})/(\text{pseudo-null})$, every Λ-module of finite type M is isomorphic to

$$\Lambda^a \oplus \bigoplus_{i=1}^{b} \Lambda/f_i\Lambda \qquad (f_i \in \Lambda - \{0\}).$$

The element

$$\mathrm{char}_\Lambda(M) = \mathrm{char}_A(M) = f_1 \cdots f_b \ (\mathrm{mod}\ \Lambda^*) \in \mathrm{Frac}(\Lambda)^*/\Lambda^*$$

("the characteristic power series of M") depends only on the isomorphism class of M in $({}_\Lambda\mathrm{Mod})/(\text{pseudo-null})$.

9.1.3. For every Λ-module of finite type M and $i \geqslant 0$, put

$$E^i(M) = \mathrm{Ext}^i_\Lambda(M, \Lambda)$$

($E^1(M)$ is the "Iwasawa adjoint" of M). These Λ-modules have the following properties.

(i) $E^i(M)$ is a Λ-module of finite type.

(ii) $\mathrm{codim}_\Lambda\big(\mathrm{supp}(E^i(M))\big) \geqslant i$.

(iii) $E^i(M) = 0$ for $i < \dim(\Lambda) - \dim(M)$.

(iv) $E^0(M)$ is a reflexive Λ-module.

(v) The canonical map $E^1(M) \to E^1(M_{\mathrm{tors}})$ is an isomorphism in $({}_\Lambda\mathrm{Mod})/(\text{pseudo-null})$.

(vi) $E^1(M_{\mathrm{tors}})$ has no non-zero pseudo-null submodules.

(vii) $E^1(M_{\mathrm{tors}})$ is isomorphic to M_{tors} in $({}_\Lambda\mathrm{Mod})/(\text{pseudo-null})$.

(viii) If M is a Cohen-Macaulay Λ-module, then $E^i(M) = 0$ for $i \neq \dim(\Lambda) - \dim(M)$.

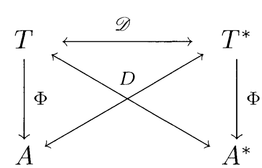
(ix) If $M = \Lambda/(x_1, \ldots, x_a)\Lambda$ is the quotient of Λ by a regular sequence of length a, then $E^a(M) \xrightarrow{\sim} M$.

For (i), take a finitely generated free resolution of M; (ii), (iii) and (viii) follow from Lemma 2.4.7 (ii)–(iv) and Local Duality 2.5; (vi) and (vii) are proved in [**PR1**, Ch. I, Prop. 8], while (ix) follows from the Koszul resolution of M. As the kernel (resp., cokernel) of the canonical map $\varepsilon : M \to E^0(E^0(M))$ is torsion (resp., pseudo-null), the exact sequence of Ext's together with (vi) and (vii) prove (iv). Finally, the exact sequence of Ext's together with (ii) show that, for each $x \in \Lambda - \{0\}$, $E^1(M/M_{\mathrm{tors}})/x$ is pseudo-null; it follows that $E^1(M/M_{\mathrm{tors}})$ itself is pseudo-null, proving (v).

9.1.4. Let T be a free $\mathcal{O}$-module of finite type equipped with a continuous $\mathcal{O}$-linear action of $G_{K,S}$. Then the $\mathcal{O}[G_{K,S}]$-modules

$$T^* = \mathrm{Hom}_{\mathcal{O}}(T, \mathcal{O})$$
$$A^* = \mathrm{Hom}_{\mathcal{O}}(T, F/\mathcal{O}) = T^* \otimes_{\mathcal{O}} F/\mathcal{O}$$
$$A = \mathrm{Hom}_{\mathcal{O}}(T^*, F/\mathcal{O}) = T \otimes_{\mathcal{O}} F/\mathcal{O}$$

are related by the duality diagram



over $\mathcal{O}$.

According to Theorem 8.5.6, there is a spectral sequence

$$(9.1.4.1)\quad E_2^{i,j} = E^i\big(H^{3-j}_{c,\mathrm{Iw}}(K_\infty/K, T^*(1))\big)^\iota = E^i\big(D_\Lambda(H^j(K_S/K_\infty, A))\big) \Longrightarrow H^{i+j}_{\mathrm{Iw}}(K_\infty/K, T).$$

Above, $D_\Lambda(M)$ coincides with the Pontrjagin dual of M, for every Λ-module of finite or co-finite type. The Λ-action on $D_\Lambda(M) = \mathrm{Hom}_\Lambda(M, D_\Lambda(\Lambda)) = \mathrm{Hom}_{\mathcal{O},\mathrm{cont}}(M, F/\mathcal{O})$ is given by $(\lambda \cdot f)(m) = f(\lambda m)$.

Similarly, there is a spectral sequence

$$(9.1.4.2)\quad {}'E_2^{i,j} = E^i\big(H^{3-j}_{\mathrm{Iw}}(K_\infty/K, T^*(1))\big)^\iota = E^i(D_\Lambda(H^j_c(K_S/K_\infty, A))) \Longrightarrow H^{i+j}_{c,\mathrm{Iw}}(K_\infty/K, T).$$

Together with 9.1.3 (ii), these spectral sequences imply that

$$\mathrm{cork}_\Lambda\, H^j(K_S/K_\infty, A) = \mathrm{rk}_\Lambda\, H^j_{\mathrm{Iw}}(K_\infty/K, T)$$
$$\mathrm{cork}_\Lambda\, H^j_c(K_S/K_\infty, A) = \mathrm{rk}_\Lambda\, H^j_{c,\mathrm{Iw}}(K_\infty/K, T)$$

9.1.5. Lemma

(i) ${}'E_2^{i,j} = 0$ *for* $j \neq 1, 2$.

(ii) *The spectral sequence* ${}'E_r$ *induces isomorphisms of* Λ*-modules*

$$E^0\big(H^2_{\mathrm{Iw}}(K_\infty/K, T^*(1))\big) \xrightarrow{\sim} H^1_{c,\mathrm{Iw}}(K_\infty/K, T)^\iota$$
$$E^i\big(H^2_{\mathrm{Iw}}(K_\infty/K, T^*(1))\big) \xrightarrow{\sim} E^{i-2}\big(H^1_{\mathrm{Iw}}(K_\infty/K, T^*(1))\big) \qquad (i > 3)$$

and an exact sequence

$$\begin{aligned}0 \longrightarrow E^1\big(H^2_{\mathrm{Iw}}(K_\infty/K, T^*(1))\big) &\longrightarrow H^2_{c,\mathrm{Iw}}(K_\infty/K, T)^\iota \\ \longrightarrow E^0\big(H^1_{\mathrm{Iw}}(K_\infty/K, T^*(1))\big) &\longrightarrow E^2\big(H^2_{\mathrm{Iw}}(K_\infty/K, T^*(1))\big) \\ \longrightarrow H^3_{c,\mathrm{Iw}}(K_\infty/K, T)^\iota &\longrightarrow E^1\big(H^1_{\mathrm{Iw}}(K_\infty/K, T^*(1))\big) \\ &\longrightarrow E^3\big(H^2_{\mathrm{Iw}}(K_\infty/K, T^*(1))\big) \longrightarrow 0.\end{aligned}$$

(iii) *(cf.* [**Gre2**, Prop. 4]*)* $D_\Lambda\big(H^2(K_S/K_\infty, A^*(1))\big)$ *is a reflexive* Λ*-module.*

Proof

(i) As $\mathrm{cd}_p(G_{K,S}) = 2$, we have ${}'E_2^{i,j} = 0$ for $j \neq 1, 2, 3$. Put $T_0 = \bigcup_\alpha H^0(\mathrm{Gal}(K_S/K_\alpha), T) \subseteq T$; then $H^0_{\mathrm{Iw}}(K_\infty/K, T) = H^0_{\mathrm{Iw}}(K_\infty/K, T_0)$. The transition maps in the projective system $H^0(\mathrm{Gal}(K_S/K_\alpha), T_0)$ are given, ultimately, by the multiplication by $[K_\beta : K_\alpha]$. Hence $H^0_{\mathrm{Iw}}(K_\infty/K, T_0) = 0$ and ${}'E_2^{i,3} = 0$.

(ii) This follows from (i).

(iii) By (ii), $D_\Lambda\big(H^2(K_S/K_\infty, A^*(1))\big) \xrightarrow{\sim} H^1_{c,\mathrm{Iw}}(K_\infty/K, T)^\iota$ is isomorphic to $E^0\big(H^2_{\mathrm{Iw}}(K_\infty/K, T^*(1))\big)$, which is reflexive by 9.1.3 (iv). □

9.1.6. Lemma. — *There is a canonical isomorphism of* Λ*-modules*

$$E_2^{i,0} \xrightarrow{\sim} \begin{cases} 0, & i \neq r, r+1 \\ T^{G_{K_\infty}}, & i = r \\ A^{G_{K_\infty}}/B, & i = r+1, \end{cases}$$

where $B = T^{G_{K_\infty}} \otimes_{\mathcal{O}} F/\mathcal{O}$ *is the maximal* $\mathcal{O}$*-divisible submodule of* $A^{G_{K_\infty}}$.

Proof. — Let $M = D_\Lambda\big(A^{G_{K_\infty}}\big)$. We distinguish three cases:

(i) $A^{G_{K_\infty}} = H^0(\mathrm{Gal}(K_S/K_\infty), A)$ is finite.
(ii) $A^{G_{K_\infty}} = A$.
(iii) The general case.

(i) As M is a Λ-module of finite length, we have $E^i(M) = 0$ for $i \neq \dim(\Lambda) = r+1$, by 9.1.3 (iii). Local duality implies that

$$E^{r+1}(M) \xrightarrow{\sim} D_\Lambda\big(H^0_{\{\mathfrak{m}\}}(M)\big) = D_\Lambda(M) = A^{G_{K_\infty}}.$$

(ii) As an $\mathcal{O}$-module, $M = D_\Lambda(A) = T^*$ is free of finite rank. Thus $E^i(M) = 0$ for $i \neq \dim(\Lambda) - 1 = r$, by 9.1.3 (viii). The exact sequence of Ext's associated to

$$0 \longrightarrow M \xrightarrow{p^n} M \longrightarrow M/p^n M \longrightarrow 0$$

together with case (i) give

$$E^r(M)/p^n E^r(M) \xrightarrow{\sim} E^{r+1}(M/p^n M) \xrightarrow{\sim} A[p^n],$$

hence

$$E^r(M) \xrightarrow{\sim} \varprojlim_n E^r(M)/p^n E^r(M) = T_p(A) = T.$$

(iii) Note that B is a Λ-submodule of $A^{G_{K_\infty}}$. The exact sequence of Ext's associated to

$$0 \longrightarrow D_\Lambda\big(A^{G_{K_\infty}}/B\big) \longrightarrow M \longrightarrow D_\Lambda(B) \longrightarrow 0$$

together with cases (i) and (ii) give isomorphisms

$$\begin{aligned} E^{r+1}(M) &\xrightarrow{\sim} E^{r+1}\big(D_\Lambda\big(A^{G_{K_\infty}}/B\big)\big) \xrightarrow{\sim} A^{G_{K_\infty}}/B \\ E^r(M) &\xleftarrow{\sim} E^r(D_\Lambda(B)) \xrightarrow{\sim} T_p(B) = T^{G_{K_\infty}} \\ E^i(M) &\xrightarrow{\sim} 0 \qquad (i \neq r, r+1). \end{aligned}$$

□

9.1.7. Corollary. — *There is a canonical isomorphism of Λ-modules*

$$H^1_{\mathrm{Iw}}(K_\infty/K, T)_{\mathrm{tors}} \xrightarrow{\sim} E_2^{1,0} \xrightarrow{\sim} \begin{cases} T^{G_{K_\infty}}, & r = 1 \\ 0, & r > 1. \end{cases}$$

9.2. Cohomology of $\mathbf{Z}_p(1)$

In this section, $\mathcal{O} = \mathbf{Z}_p$.

9.2.1. Every finite discrete $G_{K,S}$-module M determines an étale sheaf M_{et} on $\mathrm{Spec}(\mathcal{O}_{K,S})$. If the order of M is a unit in $\mathcal{O}_{K,S}$, then the spectral sequence

$$E_2^{i,j} = \varinjlim_L H^i(\mathrm{Gal}(L/K), H^j(\mathrm{Spec}(\mathcal{O}_{L,S})_{\text{et}}, M_{\text{et}})) \Longrightarrow H^{i+j}(\mathrm{Spec}(\mathcal{O}_{K,S})_{\text{et}}, M_{\text{et}}),$$

in which L runs through finite subextensions of K_S/K, degenerates into isomorphisms

$$E_2^{i,0} = H^i(G_{K,S}, M) \xrightarrow{\sim} H^i(\mathrm{Spec}(\mathcal{O}_{K,S})_{\text{et}}, M_{\text{et}}).$$

More precisely, there is a canonical isomorphism

$$\mathbf{R}\Gamma(\mathrm{Spec}(\mathcal{O}_{K,S})_{\text{et}}, M_{\text{et}}) \xrightarrow{\sim} \mathbf{R}\Gamma_{\text{cont}}(G_{K,S}, M)$$

in $D^+(Ab)$.

The exact sequence of sheaves

$$0 \longrightarrow \mu_{p^n} \longrightarrow \mathbf{G}_m \xrightarrow{p^n} \mathbf{G}_m \longrightarrow 0$$

on $\mathrm{Spec}(\mathcal{O}_{K,S})_{\text{et}}$ and the standard description of the Brauer group $\mathrm{Br}(K)$ yield the following exact sequences (cf. [**Sch1**]):

$$\mu_{p^n}(K) \xrightarrow{\sim} H^0(G_{K,S}, \mu_{p^n})$$

$$0 \longrightarrow \mathcal{O}_{K,S}^* \otimes \mathbf{Z}/p^n\mathbf{Z} \longrightarrow H^1(G_{K,S}, \mu_{p^n}) \longrightarrow \mathrm{Pic}(\mathcal{O}_{K,S})[p^n] \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Pic}(\mathcal{O}_{K,S}) \otimes \mathbf{Z}/p^n\mathbf{Z} \longrightarrow H^2(G_{K,S}, \mu_{p^n}) \longrightarrow \bigoplus_{v \in S_f} \mathbf{Z}/p^n\mathbf{Z} \xrightarrow{\Sigma} \mathbf{Z}/p^n\mathbf{Z} \longrightarrow 0.$$

Passing to the inductive (resp., projective) limit with respect to n, we obtain

(9.2.1.1)
$$H^0(G_{K,S}, \mathbf{Q}_p/\mathbf{Z}_p(1)) \xrightarrow{\sim} \mu_{p^\infty}(K)$$

$$0 \longrightarrow \mathcal{O}_{K,S}^* \otimes \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow H^1(G_{K,S}, \mathbf{Q}_p/\mathbf{Z}_p(1)) \longrightarrow \mathrm{Pic}(\mathcal{O}_{K,S})[p^\infty] \longrightarrow 0$$

$$0 \longrightarrow H^2(G_{K,S}, \mathbf{Q}_p/\mathbf{Z}_p(1)) \longrightarrow \bigoplus_{v \in S_f} \mathbf{Q}_p/\mathbf{Z}_p \xrightarrow{\Sigma} \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow 0$$

$$H^i(G_{K,S}, \mathbf{Q}_p/\mathbf{Z}_p(1)) = 0 \qquad (i > 2).$$

resp.,

(9.2.1.2)
$$\mathcal{O}_{K,S}^* \otimes \mathbf{Z}_p \xrightarrow{\sim} H^1_{\text{cont}}(G_{K,S}, \mathbf{Z}_p(1))$$

$$0 \longrightarrow \mathrm{Pic}(\mathcal{O}_{K,S})[p^\infty] \longrightarrow H^2_{\text{cont}}(G_{K,S}, \mathbf{Z}_p(1)) \longrightarrow \bigoplus_{v \in S_f} \mathbf{Z}_p \xrightarrow{\Sigma} \mathbf{Z}_p \longrightarrow 0$$

$$H^i_{\text{cont}}(G_{K,S}, \mathbf{Z}_p(1)) = 0 \qquad (i \neq 1, 2).$$

In particular,

$$\mathrm{rk}_{\mathbf{Z}_p} H^1_{c,\text{cont}}(G_{K,S}, \mathbf{Z}_p) = \mathrm{rk}_{\mathbf{Z}_p} H^2_{\text{cont}}(G_{K,S}, \mathbf{Z}_p(1)) = |S_f| - 1$$

$$\mathrm{rk}_{\mathbf{Z}_p} H^2_{c,\text{cont}}(G_{K,S}, \mathbf{Z}_p) = \mathrm{rk}_{\mathbf{Z}_p} H^1_{\text{cont}}(G_{K,S}, \mathbf{Z}_p(1)) = r_1 + r_2 + |S_f| - 1$$

$$\mathrm{rk}_{\mathbf{Z}_p} H^q_{c,\text{cont}}(G_{K,S}, \mathbf{Z}_p) = \mathrm{rk}_{\mathbf{Z}_p} H^{3-q}_{\text{cont}}(G_{K,S}, \mathbf{Z}_p(1)) = 0 \qquad (q \neq 1, 2),$$

hence

$$\sum_q (-1)^q \operatorname{rk}_{\mathbf{Z}_p} H^q_{c,\mathrm{cont}}(G_{K,S}, \mathbf{Z}_p) = r_1 + r_2 = \sum_{v|\infty} \operatorname{rk}_{\mathbf{Z}_p} H^0_{\mathrm{cont}}(G_v, \mathbf{Z}_p),$$

in line with Theorem 5.3.6.

9.2.2. Write $K_\infty = \bigcup K_\alpha$ as a union of finite extensions of K. Let S_α be the set of all primes of K_α above S. If $K_\alpha \subset K_\beta$ and $w_\beta \in S_\beta$ lies above $v_\alpha \in S_\alpha$, then the corresponding local Brauer groups are related by

$$\begin{array}{ccccccc} \mathrm{Br}((K_\alpha)_{v_\alpha}) & \xrightarrow{\mathrm{res}} & \mathrm{Br}((K_\beta)_{w_\beta}) & \quad & \mathrm{Br}((K_\alpha)_{v_\alpha}) & \xleftarrow{\mathrm{cor}} & \mathrm{Br}((K_\beta)_{w_\beta}) \\ \downarrow \mathrm{inv}_{v_\alpha} & & \downarrow \mathrm{inv}_{w_\beta} & & \downarrow \mathrm{inv}_{v_\alpha} & & \downarrow \mathrm{inv}_{w_\beta} \\ \mathbf{Q}/\mathbf{Z} & \xrightarrow{n_{\alpha\beta}} & \mathbf{Q}/\mathbf{Z} & & \mathbf{Q}/\mathbf{Z} & \xleftarrow{\mathrm{id}} & \mathbf{Q}/\mathbf{Z}, \end{array}$$

where $n_{\alpha\beta} = [(K_\beta)_{w_\beta} : (K_\alpha)_{v_\alpha}]$. Put

$$E'_\alpha = \mathcal{O}^*_{K_\alpha, S_\alpha} \otimes \mathbf{Z}_p, \quad A'_\alpha = \mathrm{Pic}(\mathcal{O}_{K_\alpha, S_\alpha})[p^\infty]$$
$$E'_\infty = \varinjlim_\alpha E'_\alpha, \quad A'_\infty = \varinjlim_\alpha A'_\alpha, \quad X'_\infty = \varprojlim_\alpha A'_\alpha.$$

It is well-known that X'_∞ and $D_\Lambda(A'_\infty)$ are torsion Λ-modules of finite type ([**Gre1**, proof of Thm. 1]).

Applying (9.2.1.1)–(9.2.1.2) to each K_α and using the above description of the transition maps for the local Brauer groups, we obtain the following exact sequences (and isomorphisms):

(9.2.2.1)
$$\begin{aligned} H^i(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p(1)) &= 0 \qquad (i > 2) \\ H^0(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p(1)) &\xrightarrow{\sim} \mu_{p^\infty}(K_\infty) \\ 0 \longrightarrow E'_\infty \otimes \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow H^1(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p(1)) &\longrightarrow A'_\infty \longrightarrow 0 \\ H^2(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p(1)) &\xrightarrow{\sim} \bigoplus_{\substack{v \in S_f \\ \Gamma_v = 0}} D_\Lambda(\Lambda). \end{aligned}$$

(9.2.2.2)
$$\begin{aligned} H^i_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1)) &= 0 \qquad (i \neq 1, 2) \\ H^1_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1)) &\xrightarrow{\sim} \varprojlim_\alpha E'_\alpha \\ 0 \longrightarrow X'_\infty \longrightarrow H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1)) &\longrightarrow \bigoplus_{v \in S_f} \Lambda_v \longrightarrow \mathbf{Z}_p \longrightarrow 0, \end{aligned}$$

where $\Gamma_v \subset \Gamma$ is the decomposition group of any prime v_∞ of K_∞ above v, and $\Lambda_v = \mathbf{Z}_p[\![\Gamma/\Gamma_v]\!]$. In particular, the Λ-module

(9.2.2.3)
$$D_\Lambda\big(H^2(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p(1))\big) \xrightarrow{\sim} \bigoplus_{\substack{v \in S_f \\ \Gamma_v = 0}} \Lambda \xrightarrow{\sim} H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))/H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))_{\mathrm{tors}}$$

is free and

$$(9.2.2.4) \qquad 0 \longrightarrow X'_\infty \longrightarrow H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))_{\mathrm{tors}} \longrightarrow \bigoplus_{\substack{v\in S_f\\ \Gamma_v\neq 0}} \Lambda_v \longrightarrow \mathbf{Z}_p \longrightarrow 0$$

is an exact sequence of torsion Λ-modules.

9.3. Pseudo-null submodules

9.3.1. Proposition **(cf. [Gre2, Prop. 5]).** — *If, in the notation of 9.1, $H^2(K_S/K_\infty, A) = 0$, then $D_\Lambda(H^1(K_S/K_\infty, A))$ has no non-zero pseudo-null submodules.*

Proof. — Lemma 9.1.5 (ii) applied to $T^*(1)$ instead of T (combined with the fact that $E^0(-)$ is torsion-free) yields isomorphisms

$$D_\Lambda(H^1(K_S/K_\infty, A))_{\mathrm{tors}} \xrightarrow{\sim} H^2_{c,\mathrm{Iw}}(K_\infty/K, T^*(1))^\iota_{\mathrm{tors}} \xrightarrow{\sim} E^1(H^2_{\mathrm{Iw}}(K_\infty/K, T)) = E^1(H^2_{\mathrm{Iw}}(K_\infty/K, T)_{\mathrm{tors}}),$$

where the last equality follows from

$$\mathrm{rk}_\Lambda H^2_{\mathrm{Iw}}(K_\infty/K, T) = \mathrm{rk}_\Lambda D_\Lambda(H^2(K_S/K_\infty, A)) = 0.$$

We conclude by 9.1.3 (vi). □

9.3.2. Corollary **([Ng, Thm. 3.1]).** — *Let M_∞ be the maximal pro-p-abelian extension of K_∞, unramified outside the primes above S. If $H^2(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p) = 0$ (*i.e.,* if K_∞ satisfies the "weak Leopoldt conjecture"), then* $\mathrm{Gal}(M_\infty/K_\infty)$ *has no non-zero pseudo-null submodules.*

Proof. — We have

$$D_\Lambda(\mathrm{Gal}(M_\infty/K_\infty)) \xrightarrow{\sim} H^1(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p)^\iota$$

(the involution ι appears, because of different sign conventions for the Γ- and Λ- action on $D_\Lambda(-)$). Apply Proposition 9.3.1 with $T = \mathbf{Z}_p$. □

9.3.3. Proposition. — *$D_\Lambda(A'_\infty)$ has no non-zero pseudo-null submodules.*

Proof. — By (9.2.2.1), $D_\Lambda(A'_\infty)$ is contained in

$$D_\Lambda\big(H^1(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p(1))\big)_{\mathrm{tors}} \xrightarrow{\sim} H^2_{c,\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p)^\iota_{\mathrm{tors}} \xrightarrow{\sim} E^1\big(H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))\big),$$

where the last isomorphism follows from the spectral sequence (9.1.4.2) for $T = \mathbf{Z}_p$. As the quotient $H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))/H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))_{\mathrm{tors}}$ is free over Λ (by (9.2.2.3)), it follows that

$$E^1\big(H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))\big) = E^1\big(H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))_{\mathrm{tors}}\big);$$

we again conclude by 9.1.3 (vi). □

9.3.4. Greenberg ([**Gre6**, Thm. 2]) recently proved the following generalization of Proposition 9.3.1 (and of Lemma 9.1.5 (iii)): denote, using the notation of 9.2.2,

$$Ш^j(K_\infty, A) = \operatorname{Ker}\left(H^j(K_S/K_\infty, A) \longrightarrow \bigoplus_{v\in S_f} H^j_{\mathrm{loc},v}(K_\infty, A)\right)$$

$$H^j_{\mathrm{loc},v}(K_\infty, A) = \varinjlim_\alpha \bigoplus_{v_\alpha | v} H^j(G_{v_\alpha}, A) = H^j(G_v, F_\Gamma(A))$$

$$H^j_{\mathrm{Iw}}((K_\infty/K)_v, T) = \varprojlim_\alpha \bigoplus_{v_\alpha | v} H^j_{\mathrm{cont}}(G_{v_\alpha}, T) = H^j_{\mathrm{cont}}(G_v, \mathscr{F}_\Gamma(T)).$$

Then the Λ-module $D_\Lambda(Ш^2(K_\infty, A))$ is reflexive; furthermore, if $Ш^2(K_\infty, A) = 0$, then $D_\Lambda(H^1(K_S/K_\infty, A))$ has no non-zero pseudo-null submodules. We present in 9.3.5–9.3.7 below an alternative proof of Greenberg's result.

9.3.5. Lemma

(i) *The sequence*

$$0 \longrightarrow Ш^2(K_\infty, A) \longrightarrow H^2(K_S/K_\infty, A) \longrightarrow \bigoplus_{v\in S_f} H^2_{\mathrm{loc},v}(K_\infty, A) \longrightarrow 0$$

is exact.

(ii) *For each $v \in S_f$, the Pontrjagin dual of $H^2_{\mathrm{loc},v}(K_\infty, A)$ is a free Λ-module of rank*

$$u_v = \begin{cases} 0, & \Gamma_v \neq 0 \\ \mathrm{rk}_{\mathcal{O}}\, H^0(G_v, T^*(1)), & \Gamma_v = 0. \end{cases}$$

(iii) *The composite map (induced by the isomorphism from the proof of Lemma 9.1.5 (iii))*

$$H^2_{\mathrm{Iw}}(K_\infty/K, T) \longrightarrow E^0\big(E^0(H^2_{\mathrm{Iw}}(K_\infty/K, T))\big)$$
$$\xrightarrow{\sim} E^0\big(D_\Lambda(H^2(K_S/K_\infty, A))\big) \longrightarrow \bigoplus_{\substack{v\in S_f\\ \Gamma_v=0}} E^0\big(D_\Lambda(H^2_{\mathrm{loc},v}(K_\infty, A))\big) \xrightarrow{\sim} \Lambda^u$$

(where $u = \sum_{v\in S_f} u_v$) is surjective.

(iv) *The sequence obtained by applying $E^0 \circ D_\Lambda$ to the exact sequence from (i) is exact.*

Proof

(i), (ii) The Poitou-Tate exact sequence 5.1.6 yields, in the limit over all finite subextensions K_α/K of K_∞/K, an exact sequence

$$0 \longrightarrow Ш^2(K_\infty, A) \longrightarrow H^2(K_S/K_\infty, A)$$
$$\longrightarrow \bigoplus_{v\in S_f} H^2_{\mathrm{loc},v}(K_\infty, A) \longrightarrow D_\Lambda\left(H^0_{\mathrm{Iw}}(K_\infty/K, T^*(1))\right)^\iota \longrightarrow 0;$$

the Λ-module $H^0_{\mathrm{Iw}}(K_\infty/K, T^*(1))$ vanishes, by (the proof of) Lemma 9.1.5(i). The same argument also shows that the Λ-module $D_\Lambda(H^2_{\mathrm{loc},v}(K_\infty, A)) = H^0_{\mathrm{Iw}}((K_\infty/K)_v, T^*(1))^\iota$ vanishes if $\Gamma_v \neq 0$. If $\Gamma_v = 0$, then $H^0_{\mathrm{Iw}}((K_\infty/K)_v, T^*(1)) = \mathscr{F}_\Gamma\big(H^0(G_v, T^*(1))\big)$ is a free Λ-module of rank u_v.

(iii) After applying $\iota \circ D_\Lambda$, it is enough to show that the map

$$\begin{aligned}\bigoplus_{\substack{v\in S_f\\ \Gamma_v=0}} \iota\circ D_\Lambda \circ E^0\big(\mathscr{F}_\Gamma\big(H^0(G_v, T^*(1))\big)\big)^\iota \\ = \bigoplus_{\substack{v\in S_f\\ \Gamma_v=0}} F_\Gamma\big(\Phi_{\mathcal{O}}\big(H^0(G_v, T^*(1))\big)\big) = \bigoplus_{\substack{v\in S_f\\ \Gamma_v=0}} F_\Gamma(H^0(G_v, A^*(1))_{\mathrm{div}}) \\ \hookrightarrow \bigoplus_{\substack{v\in S_f\\ \Gamma_v=0}} H^0_{\mathrm{loc},v}(K_\infty, A^*(1)) \longrightarrow H^1_c(K_S/K_\infty, A^*(1))\end{aligned}$$

(where we have used Lemma 8.4.6.4 for the first equality) is injective; this follows from the exact sequence

$$0 \longrightarrow H^0(K_S/K_\infty, A^*(1)) \longrightarrow \bigoplus_{v\in S_f} H^0_{\mathrm{loc},v}(K_\infty, A^*(1)) \longrightarrow H^1_c(K_S/K_\infty, A^*(1))$$

and the fact that there exists $v \in S_f$ for which $\Gamma_v \neq 0$.

(iv) We must show that the map

$$\begin{aligned}E^0\big(D_\Lambda(H^2(K_S/K_\infty, A))\big) \\ \longrightarrow \bigoplus_{v\in S_f} E^0\big(D_\Lambda(H^2_{\mathrm{loc},v}(K_\infty, A))\big) = \bigoplus_{\substack{v\in S_f\\ \Gamma_v=0}} E^0\big(D_\Lambda(H^2_{\mathrm{loc},v}(K_\infty, A))\big)\end{aligned}$$

is surjective; this follows from (iii). □

9.3.6. Proposition (**[Gre6**, Thm. 2]). — *The Λ-module $D_\Lambda(\text{Ш}^2(K_\infty, A))$ is reflexive (in particular, $\text{Ш}^2(K_\infty, A) = 0 \iff \mathrm{cork}_\Lambda\, \text{Ш}^2(K_\infty, A) = 0$).*

Proof. — By Lemma 9.3.5, there is an exact sequence

$$0 \longrightarrow \Lambda^u \longrightarrow D_\Lambda(H^2(K_S/K_\infty, A)) \longrightarrow D_\Lambda(\text{Ш}^2(K_\infty, A)) \longrightarrow 0; \tag{9.3.6.1}$$

moreover, the sequence obtained by applying E^0 to (9.3.6.1)

$$\begin{aligned}0 \longrightarrow E^0\big(D_\Lambda(\text{Ш}^2(K_\infty, A))\big) \\ \longrightarrow E^0\big(D_\Lambda(H^2(K_S/K_\infty, A))\big) \longrightarrow E^0(\Lambda^u) \longrightarrow 0\end{aligned} \tag{9.3.6.2}$$

is also exact. As $E^0(\Lambda^u) \xrightarrow{\sim} \Lambda^u$ is a free Λ-module, it follows that the double dual of (9.3.6.1)

$$\begin{aligned}0 \longrightarrow E^0(E^0(\Lambda^u)) \longrightarrow E^0\big(E^0\big(D_\Lambda(H^2(K_S/K_\infty, A))\big)\big) \\ \longrightarrow E^0\big(E^0\big(D_\Lambda(\text{Ш}^2(K_\infty, A))\big)\big) \longrightarrow 0\end{aligned}$$

is also exact. As both Λ^u and $D_\Lambda(H^2(K_S/K_\infty, A))$ are reflexive (by Lemma 9.1.5 (iii)), we deduce from the Snake Lemma that the third module $D_\Lambda(\text{Ш}^2(K_\infty, A))$ is also reflexive. □

9.3.7. Proposition. — *If* $\text{Ш}^2(K_\infty, A) = 0$, *then:*

(i) $H^2_{\mathrm{Iw}}(K_\infty/K, T)/H^2_{\mathrm{Iw}}(K_\infty/K, T)_{\mathrm{tors}} \xrightarrow{\sim} \Lambda^u$ *is a free* Λ*-module of rank* u.

(ii) *There is an isomorphism of* Λ*-modules* $\left(D_\Lambda(H^1(K_S/K_\infty, A))\right)_{\mathrm{tors}} \xrightarrow{\sim} E^1\left(H^2_{\mathrm{Iw}}(K_\infty/K, T)_{\mathrm{tors}}\right)$.

(iii) *(*[**Gre6**, Thm. 2]*) The* Λ*-module* $D_\Lambda(H^1(K_S/K_\infty, A))$ *has no non-zero pseudo-null submodules.*

Proof

(i) The assumption $\text{Ш}^2(K_\infty, A) = 0$ implies, thanks to Lemma 9.3.5 (i)–(ii), that

$$\mathrm{rk}_\Lambda\, H^2_{\mathrm{Iw}}(K_\infty/K, T) = \mathrm{cork}_\Lambda\, H^2(K_S/K_\infty, A) = \sum_{v\in S_f} \mathrm{cork}_\Lambda\, H^2_{\mathrm{loc},v}(K_\infty, A) = u.$$

According to Lemma 9.3.5 (iii), there exists a surjection $H^2_{\mathrm{Iw}}(K_\infty/K, T) \twoheadrightarrow \Lambda^u$; its kernel must be equal to $H^2_{\mathrm{Iw}}(K_\infty/K, T)_{\mathrm{tors}}$.

(ii) It follows from Lemma 9.1.5 (ii) that

$$\left(D_\Lambda(H^1(K_S/K_\infty, A))\right)_{\mathrm{tors}} = H^2_{c,\mathrm{Iw}}(K_\infty/K, T)^\iota_{\mathrm{tors}} = E^1(H^2_{\mathrm{Iw}}(K_\infty/K, T)).$$

On the other hand, (i) implies that $E^1(H^2_{\mathrm{Iw}}(K_\infty/K, T)) = E^1(H^2_{\mathrm{Iw}}(K_\infty/K, T)_{\mathrm{tors}})$.

(iii) This follows from (ii), by 9.1.3 (vi). □

9.4. Relating A'_∞ and X'_∞

9.4.1. It is generally expected that X'_∞ and $D_\Lambda(A'_\infty)$ are isomorphic in $({}_\Lambda\mathrm{Mod})/$(pseudo-null); in particular, their characteristic power series should coincide.

It is known that $D_\Lambda(A'_\infty)$ is isomorphic to $E^1(X'_\infty)$ in $({}_\Lambda\mathrm{Mod})/$(pseudo-null) (resp., in $({}_\Lambda\mathrm{Mod})$), provided $r = 1$ ([**Iw**, Thm. 11]) (resp., $|S_f| = 1$ ([**McCa2**, Thm. 8])).

In this section we define a canonical homomorphism of Λ-modules

$$\alpha' : X'_\infty \longrightarrow E^1(D_\Lambda(A'_\infty)),$$

show that $\mathrm{Coker}(\alpha')$ is very close to being pseudo-null and that

$$\mathrm{char}_\Lambda(D_\Lambda(A'_\infty)) \;\big|\; \mathrm{char}_\Lambda(X'_\infty).$$

9.4.2. Denote by $F^* H^n_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))$ the filtration induced by the spectral sequence (9.1.4.1) for $T = \mathbf{Z}_p(1)$:

$$E_2^{i,j} = E^i\big(D_\Lambda\big(H^j(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p(1))\big)\big) \Longrightarrow H^{i+j}_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1)).$$

As $\operatorname{codim}_\Lambda(\operatorname{supp}(E_2^{i,j})) \geqslant i$ and

$$E_2^{i,2} = \begin{cases} 0, & i \neq 0 \\ \Lambda\text{-free}, & i = 0 \end{cases}$$

by (9.2.2.1), we have $F^1 H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1)) = H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))_{\mathrm{tors}}$.

The terms $E_2^{i,0}$ can be determined from Lemma 9.1.6: writing $\mu_{p^\infty}(K_\infty) = \mu_{p^m}$ with $m \in \mathbf{N} \cup \{\infty\}$ and

$$\varepsilon = \begin{cases} 0, & m = \infty \\ 1, & m < \infty, \end{cases}$$

then

$$E_2^{i,0} = \begin{cases} 0, & i \neq r + \varepsilon \\ \mathbf{Z}_p/p^m\mathbf{Z}_p(1), & i = r + \varepsilon \end{cases}$$

(with the convention $\mathbf{Z}_p/p^\infty\mathbf{Z}_p = \mathbf{Z}_p$). The spectral sequence E_r induces a map

$$\delta : H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))_{\mathrm{tors}} \longrightarrow E_2^{1,1}$$

with pseudo-null kernel and cokernel. More precisely, the previous discussion and the exact sequence

$$E_2^{2,0} \longrightarrow H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))_{\mathrm{tors}} \xrightarrow{\delta} E_2^{1,1} \xrightarrow{d_2^{1,1}} E_2^{3,0}$$

imply the following result (cf. [**McCa2**, Thm. 13]).

9.4.3. Lemma

(i) *If* $r + \varepsilon \neq 2, 3$, *then* δ *is an isomorphism.*

(ii) *If* $r + \varepsilon = 2$ *(resp.,* $r + \varepsilon = 3$*), then there is an exact sequence*

$$\mathbf{Z}_p/p^m\mathbf{Z}_p(1) \longrightarrow H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))_{\mathrm{tors}} \xrightarrow{\delta} E_2^{1,1} \longrightarrow 0$$

resp.,

$$0 \longrightarrow H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))_{\mathrm{tors}} \xrightarrow{\delta} E_2^{1,1} \longrightarrow \mathbf{Z}_p/p^m\mathbf{Z}_p(1).$$

9.4.4. We shall use the following notation:

$$Y = D_\Lambda(E'_\infty \otimes \mathbf{Q}_p/\mathbf{Z}_p)$$

$$S_{\mathrm{ur}} = \{v \in S_f \mid v \text{ is unramified in } K_\infty/K\}$$

$$Z_{\mathrm{ur}} = \bigoplus_{\substack{v \in S_{\mathrm{ur}} \\ \Gamma_v \neq 0}} \Lambda_v$$

$$Z = \operatorname{Ker}\Bigg(\bigoplus_{v \in S_f \Gamma_v \neq 0} \Lambda_v \longrightarrow \mathbf{Z}_p\Bigg).$$

The canonical projection $Z \to Z_{\mathrm{ur}}$ is surjective; denote its kernel by Z_{ram}.

9.4.5. The map δ is a bridge between two exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X'_\infty & \xrightarrow{i} & H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))_{\mathrm{tors}} & \xrightarrow{i'} & Z & \longrightarrow 0 \\ & & \vdots \downarrow \alpha' & & \downarrow \delta & & & \\ E^2(Y) \longleftarrow & & E^1(D_\Lambda(A'_\infty)) & \xleftarrow{j} & E_2^{1,1} & & \longleftarrow E^1(Y) & \longleftarrow 0 \end{array} \tag{9.4.5.1}$$

The top (resp., bottom) row is equal to (9.2.2.4) (resp., to $E^\bullet \circ D_\Lambda$ applied to (9.2.2.1)). We define

$$\alpha' : X'_\infty \longrightarrow E^1(D_\Lambda(A'_\infty))$$

to be the composite map $\alpha' = j \circ \delta \circ i$.

Conjecturally, both $\mathrm{Ker}(\alpha')$ and $\mathrm{Coker}(\alpha')$ are pseudo-null. The morphism δ is invertible in the category $({}_\Lambda\mathrm{Mod})/(\text{pseudo-null})$, hence defines a morphism

$$\gamma : E^1(Y) \longrightarrow E_2^{1,1} \xrightarrow{\delta^{-1}} H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))_{\mathrm{tors}} \longrightarrow Z$$

in the latter category. The statement that $\mathrm{Ker}(\alpha')$ is pseudo-null (resp., $\mathrm{Coker}(\alpha')$ is pseudo-null) is equivalent to $\mathrm{Ker}(\gamma)=0$ (resp., $\mathrm{Coker}(\gamma)=0$) in $({}_\Lambda\mathrm{Mod})/(\text{pseudo-null})$.

9.4.6. Definition. — A prime ideal $\mathfrak{p} \in \mathrm{Spec}(\Lambda)$ with $\mathrm{ht}(\mathfrak{p}) = 1$ is **exceptional** if there is $v \in S_f - S_{\mathrm{ur}}$ such that $\Gamma_v = \langle \gamma_v \rangle \xrightarrow{\sim} \mathbf{Z}_p$ and $\gamma_v - 1 \in \mathfrak{p}$.

9.4.7. Proposition. — *If* $\mathfrak{p} \in \mathrm{Spec}(\Lambda)$ *with* $\mathrm{ht}(\mathfrak{p}) = 1$ *is not exceptional, then* $\mathrm{Coker}(\alpha')_{\mathfrak{p}} = 0$.

Proof. — We must show that $\mathrm{Coker}(\gamma)_{\mathfrak{p}} = 0$. The assumption on $\mathfrak{p}$ implies that $(Z_{\mathrm{ram}})_{\mathfrak{p}} = 0$, which means that it is enough to show that the composite map

$$\gamma_{\mathrm{ur}} : E^1(Y) \xrightarrow{\gamma} Z \longrightarrow Z_{\mathrm{ur}}$$

has $\mathrm{Coker}(\gamma_{\mathrm{ur}}) = 0$ in $({}_\Lambda\mathrm{Mod})/(\text{pseudo-null})$.

For each $v \in S_f$ there is a semi-local version of the spectral sequence E_r, namely

$$ {}^vE_2^{i,j} = E^i\left(D_\Lambda\left(\bigoplus_{v_\infty | v} H^j\big((K_\infty)_{v_\infty}, \mathbf{Q}_p/\mathbf{Z}_p(1)\big)\right)\right) \Longrightarrow H^{i+j}_{\mathrm{Iw}}((K_\infty/K)_v, \mathbf{Z}_p(1)), \tag{9.4.7.1}$$

where we denote

$$\bigoplus_{v_\infty | v} H^j((K_\infty)_{v_\infty}, -) = \varinjlim_{\alpha} \bigoplus_{v_\alpha | v} H^j((K_\alpha)_{v_\alpha}, -).$$

As $\Gamma_v \xrightarrow{\sim} \mathbf{Z}_p^{r(v)}$ with $0 \leqslant r(v) \leqslant r$, the ring Λ_v is a quotient of Λ by a regular sequence of length $r(v)$. As in (9.2.2.1) and 9.4.2, we have

$${}^vE_2^{i,2} \xrightarrow{\sim} \begin{cases} 0, & i \neq 0 \quad \text{or} \quad \Gamma_v \neq 0 \\ \Lambda, & i = 0,\, \Gamma_v = 0. \end{cases}$$

The invariant maps inv_{v_α} for $v_\alpha \mid v$ together with a choice of $v_\infty \mid v$ induce an isomorphism

$$H^2_{\mathrm{Iw}}((K_\infty/K)_v, \mathbf{Z}_p(1)) \xrightarrow{\sim} \Lambda_v,$$

which implies that

$$F^1 H^2_{\mathrm{Iw}}((K_\infty/K)_v, \mathbf{Z}_p(1)) = \begin{cases} H^2_{\mathrm{Iw}}((K_\infty/K)_v, \mathbf{Z}_p(1)), & \Gamma_v \neq 0 \\ 0, & \Gamma_v = 0. \end{cases}$$

As in the global case, we have $\mu_{p^\infty}((K_\infty)_{v_\infty}) = \mu_{p^{m(v)}}$ for some $m(v) \in \mathbf{N} \cup \{\infty\}$. Put

$$\varepsilon(v) = \begin{cases} 0, & m(v) = \infty \\ 1, & m(v) < \infty; \end{cases}$$

then

$${}^vE_2^{i,0} = \begin{cases} 0, & i \neq r(v) + \varepsilon(v) \\ \mathbf{Z}_p/p^{m(v)}\mathbf{Z}_p[\![\Gamma/\Gamma_v]\!], & i = r(v) + \varepsilon(v). \end{cases}$$

The exact sequence

$${}^vE_2^{2,0} \longrightarrow F^1 H^2_{\mathrm{Iw}}((K_\infty/K)_v, \mathbf{Z}_p(1)) \xrightarrow{\delta_v} {}^vE_2^{1,1} \xrightarrow{{}^v d_2^{1,1}} {}^vE_2^{3,0}$$

implies that, in the case $r(v) = 1$, δ_v is an isomorphism in $({}_\Lambda\mathrm{Mod})/(\text{pseudo-null})$.

Fix $v \in S_{\mathrm{ur}}$. For each K_α, the valuations $v_\alpha \mid v$ define surjective maps

$$\bigoplus_{v_\alpha \mid v} H^1((K_\alpha)_{v_\alpha}, \mathbf{Q}_p/\mathbf{Z}_p(1)) \xrightarrow{(v_\alpha)} \bigoplus_{v_\alpha \mid v} \mathbf{Q}_p/\mathbf{Z}_p,$$

which induce in the limit, after dualization, injective maps

$$\beta_v : \Lambda_v \hookrightarrow D_\Lambda\left(\bigoplus_{v_\alpha \mid v} H^1((K_\alpha)_{v_\alpha}, \mathbf{Q}_p/\mathbf{Z}_p(1))\right).$$

In the global situation, the composite map

$$E'_\infty \otimes \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow \bigoplus_{v \in S_{\mathrm{ur}}} \bigoplus_{v_\alpha \mid v} H^1((K_\alpha)_{v_\alpha}, \mathbf{Q}_p/\mathbf{Z}_p(1)) \xrightarrow{(v_\alpha)} \bigoplus_{v \in S_{\mathrm{ur}}} \bigoplus_{v_\alpha \mid v} \mathbf{Q}_p/\mathbf{Z}_p$$

is also surjective, giving rise to an injective map

$$\beta : \bigoplus_{v \in S_{\mathrm{ur}}} \Lambda_v \xrightarrow{(\beta_v)} \bigoplus_{v \in S_{\mathrm{ur}}} D_\Lambda\left(\bigoplus_{v_\alpha \mid v} H^1((K_\alpha)_{v_\alpha}, \mathbf{Q}_p/\mathbf{Z}_p(1))\right) \longrightarrow Y.$$

The induced maps on E^1 sit in the following commutative diagram:

$$\begin{array}{ccccc}
H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))_{\mathrm{tors}} & \longrightarrow & \bigoplus_{\substack{v \in S_{\mathrm{ur}} \\ \Gamma_v \neq 0}} F^1 H^2_{\mathrm{Iw}}((K_\infty/K)_v, \mathbf{Z}_p(1)) & \xrightarrow{\sim} & \bigoplus_{\substack{v \in S_{\mathrm{ur}} \\ \Gamma_v \neq 0}} \Lambda_v = Z_{\mathrm{ur}} \\
\big\downarrow \delta & & \big\downarrow (\delta_v) & & \\
E_2^{1,1} & \longrightarrow & \bigoplus_{\substack{v \in S_{\mathrm{ur}} \\ \Gamma_v \neq 0}} {}^vE_2^{1,1} & & \\
\big\uparrow & & \big\downarrow E^1(\beta_v) & & \\
E^1(Y) & \xrightarrow{E^1(\beta)} & \bigoplus_{\substack{v \in S_{\mathrm{ur}} \\ \Gamma_v \neq 0}} E^1(\Lambda_v) & &
\end{array}$$

For each $v \in S_{\mathrm{ur}}$, the condition $\Gamma_v \neq 0$ implies that $r(v) = 1$; thus δ_v is an isomorphism. We know that $\mathrm{Ker}(\delta)$ and $\mathrm{Coker}(\delta)$ are pseudo-null. This is also true for $\mathrm{Coker}(E^1(\beta))$, by the injectivity of β, hence also for $\mathrm{Coker}(E^1(\beta_v))$. As $E^1(\Lambda_v) \xrightarrow{\sim} \Lambda_v$ (by 9.1.3(ix)), it follows that the composite map $E^1(\beta_v) \circ (\delta_v)$ has pseudo-null kernel and cokernel. Putting all this together, we see that the composite morphism in $({}_{\Lambda}\mathrm{Mod})/(\text{pseudo-null})$

$$E^1(Y) \longrightarrow E_2^{1,1} \xrightarrow{\delta^{-1}} H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))_{\mathrm{tors}} \longrightarrow \bigoplus_{\substack{v\in S_{\mathrm{ur}}\\ \Gamma_v\neq 0}} F^1 H^2_{\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1)) \xrightarrow{\sim} Z_{\mathrm{ur}},$$

which is equal to γ_{ur}, has $\mathrm{Coker}(\gamma_{\mathrm{ur}}) = 0$. This proves the claim. □

9.4.8. Corollary. — *If, for every $v \in S_f$ ramified in K_∞/K, we have $\Gamma_v \xrightarrow{\sim} \mathbf{Z}_p^{r(v)}$ with $r(v) \geqslant 2$, then the map*

$$\alpha' : X'_\infty \longrightarrow E^1(D_\Lambda(A'_\infty))$$

has pseudo-null cokernel.

Proof. — Under these assumptions, Z_{ram} is pseudo-null and there are no exceptional $\mathfrak{p} \in \mathrm{Spec}(\Lambda)$. □

9.4.9. Ranks

9.4.9.1. Definition. — Let A be a Noetherian domain with fraction field F, and M an A-module of finite type. The rank of M is

$$\mathrm{rk}_A(M) = \dim_F(M \otimes_A F).$$

If $\mathrm{rk}_A(M) = 0$ and $\mathfrak{p} \in \mathrm{Spec}(A)$ has height $\mathrm{ht}(\mathfrak{p}) = 1$, put

$$e_{\mathfrak{p}}(M) = \ell_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty.$$

9.4.9.2. Let $\Gamma = \Gamma_0 \times \Gamma'$, where $\Gamma_0 \xrightarrow{\sim} \mathbf{Z}_p$ and $\Gamma' \xrightarrow{\sim} \mathbf{Z}_p^{r-1}$ $(r \geqslant 1)$. Fix a topological generator $\gamma_0 \in \Gamma_0$; then $\Lambda \xrightarrow{\sim} \Lambda'[\![\gamma_0 - 1]\!]$, where $\Lambda' = \mathbf{Z}_p[\![\Gamma']\!]$. For each $n \geqslant 0$, put $\omega_n = \gamma_0^{p^n}$, $\nu_n = \omega_n/\omega_{n-1}$ (where $\omega_{-1} = 1$). Then each (ν_n) is a prime ideal in Λ with

$$\mathrm{rk}_{\Lambda'}(\Lambda/\nu_n\Lambda) = \varphi(p^n).$$

If $\mathfrak{p} \in \mathrm{Spec}(\Lambda)$ and $\mathrm{ht}(\mathfrak{p}) = 1$, then

$$e_{\mathfrak{p}}(\mathbf{Z}_p[\![\Gamma/p^n\Gamma_0]\!]) = e_{\mathfrak{p}}(\Lambda/\omega_n\Lambda) = \begin{cases} 1, & \mathfrak{p} = (\nu_i), \quad 0 \leqslant i \leqslant n \\ 0, & \text{otherwise.} \end{cases}$$

9.4.9.3. Lemma. — *Let M be a Λ-module of finite type. For each $n \geqslant 0$ we have*

$$(M_{\mathrm{tors}})_{(\nu_n)} \xrightarrow{\sim} \bigoplus_{i=1}^{k_n} \Lambda_{(\nu_n)}/(\nu_n^{m(n,i)}) \qquad (m(n,i) \geqslant 1).$$

Then, for each $n \geqslant 0$,

(i) $0 \leqslant k_n \leqslant e_{(\nu_n)}(M_{\mathrm{tors}})$.

(ii) $\prod_{j\geqslant 0}(\nu_j)^{k_j}$ *divides* $\mathrm{char}_\Lambda(M_{\mathrm{tors}})$.
(iii) $\mathrm{rk}_{\Lambda'}(M/\omega_n M) = p^n\,\mathrm{rk}_\Lambda(M) + \sum_{j=0}^n k_j\,\varphi(p^j)$.
(iv) $\mathrm{rk}_{\Lambda'}(M/\omega_n M) = p^n\,\mathrm{rk}_\Lambda(M) + O(1)$ *for* $n\to\infty$.

Proof

(i), (ii) $\mathrm{char}_\Lambda(M_{\mathrm{tors}})$ is divisible by $\prod \nu_j^{e_j}$, where $e_j = e_{\nu_j}(M_{\mathrm{tors}}) = \sum_{i=1}^{k_j} m(j,i) \geqslant k_j \geqslant 0$.

(iii) Fix $n\geqslant 0$. In $({}_\Lambda\mathrm{Mod})/(\text{pseudo-null})$, M is isomorphic to $\Lambda^a\oplus M_1\oplus M_2$, where $a=\mathrm{rk}_\Lambda(M)$, M_1, M_2 are Λ-torsion,

$$M_1 = \bigoplus_{j=0}^{n}\bigoplus_{i=1}^{k_j} \Lambda/\nu_j^{m(j,i)}\Lambda$$

and $\mathrm{supp}(M_2)\cap\{\nu_0,\dots,\nu_n\}=\varnothing$. Then

$$\mathrm{rk}_{\Lambda'}(M_1/\omega_n M_1) = \sum_{j=0}^{n} k_j\,\mathrm{rk}_{\Lambda'}(\Lambda/\nu_j\Lambda) = \sum_{j=0}^{n} k_j\,\varphi(p^j),$$
$$\mathrm{rk}_{\Lambda'}(M_2/\omega_n M_2) = 0,\quad \mathrm{rk}_{\Lambda'}(\Lambda^a/\omega_n\Lambda^a) = p^n a.$$

The equality (iii) follows. Finally, (iv) is an immediate consequence of (iii). □

9.4.9.4. Proposition. — *Let M be a Λ-module of finite type, where $\Lambda = \mathbf{Z}_p[\![\Gamma]\!]$, $\Gamma \xrightarrow{\sim} \mathbf{Z}_p^r$, $r\geqslant 1$. Assume that we are given a finite collection $\{\Gamma_v\subseteq\Gamma \mid v\in S'\}$ of non-zero closed subgroups of Γ, an integer $a\geqslant 0$ and a real number $C\geqslant 0$ such that*

$$\left|\mathrm{rk}_{\mathbf{Z}_p}(M_{\Gamma_\alpha}) - a[\Gamma:\Gamma_\alpha]\right| \leqslant C\sum_{v\in S'} |\Gamma_\alpha\backslash\Gamma/\Gamma_v|$$

holds for every open subgroup Γ_α of Γ. Then $\mathrm{rk}_\Lambda(M)=a$.

Proof. — Induction on r. If $r=1$, then $\Gamma_\alpha = p^n\Gamma$, $[\Gamma:\Gamma_\alpha]=p^n$ and $|\Gamma_\alpha\backslash\Gamma/\Gamma_v| = O(1)$ (for each $v\in S'$), hence $\mathrm{rk}_{\mathbf{Z}_p}(M_{p^n\Gamma}) = p^n a + O(1)$; the result then follows from Lemma 9.4.9.3 (iv).

Assume that $r>1$ and the statement has been proved for $r-1$. Choose a decomposition $\Gamma=\Gamma_0\times\Gamma'$ as in 9.4.9.2 such that we have, for all $v\in S'$, $\Gamma_v\not\subset\Gamma_0$. Fix $n\geqslant 0$ and put $M' = M_{p^n\Gamma_0}$, viewed as a Λ'-module. For every open subgroup Γ'_β of Γ' we have

$$\begin{aligned}\left|\mathrm{rk}_{\mathbf{Z}_p}(M'_{\Gamma'_\beta}) - ap^n[\Gamma':\Gamma'_\beta]\right| &= \left|\mathrm{rk}_{\mathbf{Z}_p}(M_{p^n\Gamma_0\times\Gamma'_\beta}) - a[\Gamma:(p^n\Gamma_0\times\Gamma'_\beta)]\right|\\ &\leqslant C\sum_{v\in S'}|((p^n\Gamma_0\backslash\Gamma_0)\times(\Gamma'_\beta\backslash\Gamma'))/\Gamma_v| \leqslant p^n C\sum_{v\in S'}|\Gamma'_\beta\backslash\Gamma'/\Gamma'_v|,\end{aligned}$$

where $\Gamma'_v = \mathrm{Im}(\Gamma_v\hookrightarrow\Gamma\to\Gamma/\Gamma_0\xrightarrow{\sim}\Gamma')\neq 0$. The induction hypothesis implies that $\mathrm{rk}_{\Lambda'}(M/\omega_n M) = \mathrm{rk}_{\Lambda'}(M') = ap^n$, hence $\mathrm{rk}_\Lambda(M)=a$, by Lemma 9.4.9.3 (iv). □

9.4.9.5. Proposition. — *Let* $\Lambda = \mathbf{Z}_p[\![\Gamma]\!] = \Lambda'[\![\gamma_0 - 1]\!]$ *be as in 9.4.9.2. Assume that* M *is a* Λ*-module of finite type,* $b \geqslant 0$ *an integer,* $\{\Gamma_v \subseteq \Gamma \mid v \in S'\}$ *a finite set of closed subgroups of* Γ *and* $\{X_v \mid v \in S'\}$ *a collection of (right)* Γ_v*-sets such that*

$$(\star)\qquad \mathrm{rk}_{\mathbf{Z}_p}(M_{\Gamma_\alpha}) = b\,[\Gamma : \Gamma_\alpha] + \sum_{v\in S'} |(X_v \times (\Gamma_\alpha\backslash\Gamma))/\Gamma_v| - 1$$

for each open subgroup Γ_α *of* Γ*. Put*

$$a = b + \sum_{\substack{v\in S'\\ \Gamma_v = 0}} |X_v|.$$

Then

(i) *For each* $n \geqslant 0$,

$$\mathrm{rk}_{\Lambda'}(M/\omega_n M) = p^n a + \sum_{\substack{v\in S'\\ 0\neq\Gamma_v\subseteq\Gamma_0}} |(X_v \times (p^n\Gamma_0\backslash\Gamma_0))/\Gamma_v| + \begin{cases} -1, & r = 1\\ 0, & r > 1.\end{cases}$$

(ii) $\mathrm{rk}_\Lambda(M) = a$.

(iii) *If* $|X_v| = 1$ *for each* $v \in S'$*, then*

$$\mathrm{rk}_{\Lambda'}(M/\omega_n M) = p^n a + \sum_{j=0}^{n} k_j\,\varphi(p^j) \qquad (\forall n \geqslant 0),$$

where

$$k_n - k_{n+1} = |\{v \in S' \mid \Gamma_v \subseteq \Gamma_0, [\Gamma_0 : \Gamma_v] = p^n\}| + \begin{cases} -1, & r = 1,\ n = 0\\ 0, & \text{otherwise},\end{cases}$$

and $\mathrm{char}_\Lambda(M_{\mathrm{tors}})$ *is divisible by* $\prod_{n\geqslant 0}(\nu_n)^{k_n}$.

Proof. — Induction on r. If $r = 1$, then $\Gamma = \Gamma_0$, $\Lambda' = \mathbf{Z}_p$, $p^n\Gamma_0 = \Gamma_\alpha$ is open in Γ, $[\Gamma : \Gamma_\alpha] = p^n$ and $M/\omega_n M = M_{\Gamma_\alpha}$, hence (i) is just $(\star)$. For $v \in S'$ we have

$$\lim_{n\to\infty} \frac{|(X_v \times (p^n\Gamma\backslash\Gamma))/\Gamma_v|}{p^n} = \begin{cases} |X_v|, & \Gamma_v = 0\\ 0, & \Gamma_v \neq 0;\end{cases}$$

applying Lemma 9.4.9.3(iv) gives (ii). The statement (iii) follows from (i) and Lemma 9.4.9.3(iii).

Assume that $r > 1$ and the statement holds for $r - 1$. Fix $n \geqslant 0$ and put $M' = M/\omega_n M = M_{p^n\Gamma_0}$, viewed as a Λ'-module. The assumption $(\star)$ implies that we have, for every open subgroup Γ'_β of Γ',

$$\mathrm{rk}_{\mathbf{Z}_p}(M'_{\Gamma'_\beta}) = \mathrm{rk}_{\mathbf{Z}_p}(M_{p^n\Gamma_0\times\Gamma'_\beta}) = ap^n\,[\Gamma' : \Gamma'_\beta] + \sum_{\substack{v\in S'\\ \Gamma_v\neq 0}} a_\beta(v) - 1$$

with

$$a_\beta(v) = |(X_v \times (p^n\Gamma_0\backslash\Gamma_0) \times (\Gamma'_\beta\backslash\Gamma'))/\Gamma_v|.$$

Fix $v \in S'$ such that $\Gamma_v \neq 0$. If $\Gamma_v \subseteq \Gamma_0$, then $[\Gamma_0 : \Gamma_v] < \infty$ and

$$a_\beta(v) = [\Gamma' : \Gamma'_\beta] \cdot |(X_v \times (p^n\Gamma_0\backslash\Gamma_0))/\Gamma_v|.$$

If $\Gamma_v \not\subseteq \Gamma_0$, then $\Gamma'_v := \mathrm{Im}(\Gamma_v \hookrightarrow \Gamma \to \Gamma/\Gamma_0 \xrightarrow{\sim} \Gamma') \neq 0$ and

$$a_\beta(v) \leqslant p^n|X_v| \cdot |\Gamma'_\beta\backslash\Gamma'/\Gamma_v|.$$

Applying Proposition 9.4.9.4 to M' and Λ', we obtain (i). The statements (ii) and (iii) follow from (i) by the same argument as in the case $r = 1$. □

9.4.9.6. In the remainder of Sect. 9.4.9, let $\mathcal{O}$ be as in 9.1.1, $\Gamma \xrightarrow{\sim} \mathbf{Z}_p^r$ $(r \geqslant 1)$ and $\Lambda = \mathcal{O}[\![\Gamma]\!] \xrightarrow{\sim} \mathcal{O}[\![X_1, \dots, X_r]\!]$ $(X_i = \gamma_i - 1)$. For each open subgroup $\Gamma_\alpha \subset \Gamma$, set

$$J_\alpha = \mathrm{Ker}\,(\Lambda \longrightarrow \mathcal{O}[\Gamma/\Gamma_\alpha])\,;$$

then

$$\Lambda/J_\alpha = \mathcal{O}[\Gamma/\Gamma_\alpha], \quad \mathrm{rk}_{\mathcal{O}}(\Lambda/J_\alpha) = (\Gamma : \Gamma_\alpha).$$

9.4.9.7. Proposition. — *Let $\Gamma_\alpha \subset \Gamma$ be an open subgroup. For each Λ-module of finite type M, define*

$$\chi_\alpha(M) := \sum_{i=0}^{r} (-1)^i \,\mathrm{rk}_{\mathcal{O}}\, \mathrm{Tor}_i^\Lambda(M, \Lambda/J_\alpha).$$

(i) *The integer $\chi_\alpha(M)$ is well-defined.*

(ii) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of Λ-modules of finite type, then $\chi_\alpha(M) = \chi_\alpha(M') + \chi_\alpha(M'')$.*

(iii) *If $f \in \Lambda - \{0\}$, then $\chi_\alpha(\Lambda/f\Lambda) = 0$.*

(iv) $\chi_\alpha(\Lambda) = (\Gamma : \Gamma_\alpha)$.

(v) $\chi_\alpha(M) = \mathrm{rk}_\Lambda(M)\,(\Gamma : \Gamma_\alpha)$, *for each Λ-module of finite type M.*

Proof. — There exists a set of topological generators $\gamma_1, \dots, \gamma_r$ of Γ and integers $n_1, \dots, n_r \geqslant 0$ such that Γ_α is topologically generated by $\gamma_1^{p^{n_1}}, \dots, \gamma_r^{p^{n_r}}$. This implies that the ideal J_α is generated by the regular sequence $\mathbf{x} = (x_1, \dots, x_r)$, where $x_i = \gamma_i^{p^{n_i}} - 1$; thus $M \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda/J_\alpha$ is represented by the Koszul complex $K_\Lambda^\bullet(M, \mathbf{x})[r]$. The statements (i) and (ii) follow from this description of $\mathrm{Tor}_i^\Lambda(M, \Lambda/J_\alpha) = H^{r-i}(K_\Lambda^\bullet(M, \mathbf{x}))$, while (iii) is a consequence of (ii), applied to the exact sequence

$$0 \longrightarrow \Lambda \xrightarrow{f} \Lambda \longrightarrow \Lambda/f\Lambda \longrightarrow 0.$$

The formula $\chi_\alpha(\Lambda) = \mathrm{rk}_{\mathcal{O}}(\Lambda/J_\alpha) = (\Gamma : \Gamma_\alpha)$ is immediate. In order to prove (v), note that there exists a filtration

$$M = M_0 \supset M_1 \supset \cdots \supset M_t = 0$$

by Λ-submodules such that each graded quotient M_j/M_{j+1} is isomorphic to Λ or $\Lambda/f_j\Lambda$ $(f_j \in \Lambda - \{0\})$. Applying (ii)–(iv), we deduce that

$$\chi_\alpha(M) = \sum_{j=0}^{t-1} \chi_\alpha(M_j/M_{j+1}) = (\Gamma : \Gamma_\alpha) \sum_{j=0}^{t-1} \mathrm{rk}_\Lambda(M_j/M_{j+1}) = (\Gamma : \Gamma_\alpha)\,\mathrm{rk}_\Lambda(M). \quad \square$$

9.4.9.8. Proposition. — *Let M be a Λ-module of finite type.*

(i) *If* $\mathrm{rk}_\Lambda(M) = 0$, *then*

$$\lim_{\Gamma_\alpha \to 0} \frac{\mathrm{rk}_{\mathcal{O}}(M_{\Gamma_\alpha})}{(\Gamma : \Gamma_\alpha)} = 0.$$

(ii) *For each* $i \geqslant 1$,

$$\lim_{\Gamma_\alpha \to 0} \frac{\mathrm{rk}_{\mathcal{O}}\, \mathrm{Tor}_i^\Lambda(M, \Lambda/J_\alpha)}{(\Gamma : \Gamma_\alpha)} = 0.$$

(iii)
$$\lim_{\Gamma_\alpha \to 0} \frac{\mathrm{rk}_{\mathcal{O}}(M_{\Gamma_\alpha})}{(\Gamma : \Gamma_\alpha)} = \mathrm{rk}_\Lambda(M).$$

Above, Γ_α *runs through all open subgroups of* Γ.

Proof. — We apply dévissage in the following form: if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence such that (i) (resp., (ii)) holds for M' and M'', then it also holds for M.

(i) If M is $\mathcal{O}$-torsion, then (i) holds for trivial reasons. By dévissage, we can replace M by $M/M_{\mathcal{O}\text{-tors}}$, hence assume that M has no $\mathcal{O}$-torsion. Fix a uniformizing element $\pi \in \mathcal{O}$ and set $k = \mathcal{O}/\pi\mathcal{O}$, $\overline{\Lambda} = \Lambda/\pi\Lambda = k[\![\Gamma]\!]$, $\overline{M} = M/\pi M$. The assumption $M_{\mathcal{O}\text{-tors}} = 0$ implies that $\mathrm{rk}_{\overline{\Lambda}}(\overline{M}) = 0$; as $\mathrm{rk}_{\mathcal{O}}(M_{\Gamma_\alpha}) \leqslant \dim_k(\overline{M}_{\Gamma_\alpha})$, it is enough to show that

$$\lim_{\Gamma_\alpha \to 0} \frac{\dim_k(\overline{M}_{\Gamma_\alpha})}{(\Gamma : \Gamma_\alpha)} \overset{?}{=} 0.$$

There exists a filtration

$$\overline{M} = N_0 \supset N_1 \supset \cdots \supset N_t = 0$$

by $\overline{\Lambda}$-submodules such that each graded quotient N_j/N_{j+1} is isomorphic to $\overline{\Lambda}/g_j\overline{\Lambda}$ ($g_j \in \overline{\Lambda} - \{0\}$). By dévissage, it is sufficient to consider the case $\overline{M} = \overline{\Lambda}/g\overline{\Lambda}$ ($g \in \overline{\Lambda} - \{0\}$). We have $\overline{\Lambda} = k[\![X_1, \ldots, X_r]\!]$ ($X_i = \gamma_i - 1$). After renumbering the topological generators γ_i of Γ, we can assume that

$$C := \dim_k\big(\overline{\Lambda}/(g, X_2, \ldots, X_r)\big) = \dim_k\big(\overline{M}/(X_2, \ldots, X_r)\overline{M}\big) < \infty.$$

An induction argument then shows that

$$\begin{aligned}(\forall n_i \geqslant 1) \quad & \dim_k\big(\overline{M}/(X_1^{n_1}, \ldots, X_r^{n_r})\overline{M}\big) \\ & \leqslant \dim_k\big(\overline{M}/(X_2^{n_2}, \ldots, X_r^{n_r})\overline{M}\big) \leqslant C(n_2 - 1)\cdots(n_r - 1) < C n_2 \cdots n_r.\end{aligned}$$

If Γ_α is topologically generated by $\gamma_i^{p^{\alpha_i}}$ $(i = 1, \ldots, r)$, then $J_\alpha\overline{\Lambda} = (X_1^{p^{\alpha_1}}, \ldots, X_r^{p^{\alpha_r}})$ and

$$\lim_{\Gamma_\alpha \to 0} \frac{\dim_k(\overline{M}_{\Gamma_\alpha})}{(\Gamma : \Gamma_\alpha)} = \frac{\dim_k(\overline{M}/J_\alpha\overline{M})}{p^{\alpha_1 + \cdots + \alpha_r}} < \frac{C}{p^{\alpha_1}},$$

which proves (i).

(ii) Fix $i \geqslant 1$. Filter M as in the proof of Proposition 9.4.9.7 (v); by dévissage, we reduce the proof of (ii) to the case $M = \Lambda$ or $M = \Lambda/f\Lambda$ ($f \in \Lambda - \{0\}$). If $M = \Lambda$ or ($M = \Lambda/f\Lambda$ and $i \geqslant 2$), then $\mathrm{Tor}_i^\Lambda(M, \Lambda/J_\alpha) = 0$. In the remaining case $M = \Lambda/f\Lambda$ and $i = 1$, the exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^\Lambda(M, \Lambda/J_\alpha) \longrightarrow \Lambda/J_\alpha \xrightarrow{f} \Lambda/J_\alpha \longrightarrow M_{\Gamma_\alpha} \longrightarrow 0$$

implies that $\mathrm{rk}_\mathcal{O}\, \mathrm{Tor}_1^\Lambda(M, \Lambda/J_\alpha) = \mathrm{rk}_\mathcal{O}(M_{\Gamma_\alpha})$, hence (ii) follows from (i).

(iii) As $M_{\Gamma_\alpha} = M \otimes_\Lambda \Lambda/J_\alpha = \mathrm{Tor}_0^\Lambda(M, \Lambda/J_\alpha)$, the statement follows from (ii) and Proposition 9.4.9.7 (v). □

9.4.9.9. Note that the statements of Proposition 9.4.9.4 and Proposition 9.4.9.5 (ii) follow directly from Proposition 9.4.9.8 (iii).

9.4.10. We now return to the extension K_∞/K and the Λ-modules

$$Y = D_\Lambda(E'_\infty \otimes \mathbf{Q}_p/\mathbf{Z}_p)$$

$$Z = \mathrm{Ker}\left(\bigoplus_{\substack{v \in S_f \\ \Gamma_v \neq 0}} \Lambda_v \longrightarrow \mathbf{Z}_p\right).$$

For a number field L, we use the standard notation $r_1(L)$ (resp., $r_2(L)$) for the number of real (resp., complex) primes of L.

9.4.11. Proposition

(i) *For every open subgroup $\Gamma_\alpha \subseteq \Gamma$, the canonical map*

$$i_\alpha : E'_\alpha \otimes \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow (E'_\infty \otimes \mathbf{Q}_p/\mathbf{Z}_p)^{\Gamma_\alpha}$$

has finite kernel and cokernel.

(ii) $\mathrm{rk}_\Lambda(Y) = r_1(K) + r_2(K) + |\{v \in S_f \mid \Gamma_v = 0\}|$.

(iii) *We have the divisibility of characteristic power series*

$$\mathrm{char}_\Lambda(Z) \,\big|\, \mathrm{char}_\Lambda(Y_{\mathrm{tors}}).$$

Proof

(i) The following diagram is commutative and has exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & H^1(\Gamma_\alpha, \mu_{p^\infty}(K_\infty)) & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & E'_\alpha \otimes \mathbf{Q}_p/\mathbf{Z}_p & \longrightarrow & H^1(K_S/K_\alpha, \mathbf{Q}_p/\mathbf{Z}_p(1)) & \longrightarrow & A'_\alpha & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_\alpha} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (E'_\infty \otimes \mathbf{Q}_p/\mathbf{Z}_p)^{\Gamma_\alpha} & \longrightarrow & H^1(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p(1))^{\Gamma_\alpha} & \longrightarrow & (A'_\infty)^{\Gamma_\alpha} & & \\
 & & & & \downarrow & & & & \\
 & & & & H^2(\Gamma_\alpha, \mu_{p^\infty}(K_\infty)) & & & &
\end{array}$$

By the Snake Lemma (and the finiteness of A'_α), it is enough to show that $H^j(\Gamma_\alpha, \mu_{p^\infty}(K_\infty))$ is finite for $j = 1, 2$. This is clear if $\mu_{p^\infty} \not\subset K_\infty$. If $\mu_{p^\infty} \subset K_\infty$, then the cyclotomic character

$$\chi_\alpha : \Gamma_\alpha \hookrightarrow \Gamma \longrightarrow \mathrm{Aut}_{\mathbf{Z}_p}(\mu_{p^\infty}) = \mathbf{Z}_p^*$$

has infinite image $\mathrm{Im}(\chi_\alpha) \xrightarrow{\sim} \mathbf{Z}_p$. The Hochschild-Serre spectral sequence

$$E_2^{i,j} = H^i(\Gamma_\alpha/\Gamma'_\alpha, H^j(\Gamma'_\alpha, \mu_{p^\infty})) \Longrightarrow H^{i+j}(\Gamma_\alpha, \mu_{p^\infty})$$

for $\Gamma'_\alpha = \mathrm{Ker}(\chi_\alpha) \subset \Gamma_\alpha$ degenerates into exact sequences

$$0 \longrightarrow H^{j-1}(\Gamma'_\alpha, \mu_{p^\infty})_{\Gamma_\alpha/\Gamma'_\alpha} \longrightarrow H^j(\Gamma_\alpha, \mu_{p^\infty}) \longrightarrow H^j(\Gamma'_\alpha, \mu_{p^\infty})^{\Gamma_\alpha/\Gamma'_\alpha} \longrightarrow 0,$$

in which

$$H^j(\Gamma'_\alpha, \mu_{p^\infty}) = (\mu_{p^\infty})^{\oplus\binom{r-1}{j}},$$

with $\Gamma_\alpha/\Gamma'_\alpha$ acting *via* χ_α. This implies that the groups

$$H^j(\Gamma_\alpha, \mu_{p^\infty}) \xrightarrow{\sim} (\mu_{p^\infty}(K_\alpha))^{\oplus\binom{r-1}{j}}$$

are finite for all $j \geqslant 0$.

(ii) The statement (i) together with Dirichlet's theorem on units imply that

$$(9.4.11.1)\quad \mathrm{rk}_{\mathbf{Z}_p}(Y_{\Gamma_\alpha}) = \mathrm{rk}_{\mathbf{Z}}(\mathcal{O}^*_{K_\alpha,S_\alpha}) = (r_1(K)+r_2(K))[\Gamma:\Gamma_\alpha] + \sum_{v\in S_f} |\Gamma_\alpha\backslash\Gamma/\Gamma_v| - 1$$

holds for every open subgroup Γ_α of Γ (note that $r_i(K_\alpha) = [K_\alpha : K]\, r_i(K)$, thanks to the assumption (P)); apply Proposition 9.4.9.5 (ii).

(iii) Put $S' = \{v \in S_f \mid \Gamma_v \xrightarrow{\sim} \mathbf{Z}_p\}$; for $v \in S'$ let

$$\Gamma_v^{\mathrm{sat}} := \Gamma \cap (\Gamma_v \otimes_{\mathbf{Z}_p} \mathbf{Q}_p) \subset \Gamma \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

be the saturation of Γ_v in Γ. Put $S'' = \{\Gamma_v^{\mathrm{sat}} \mid v \in S'\}$ and let $\pi : S' \to S''$ be the map $\pi(v) = \Gamma_v^{\mathrm{sat}}$.

Fix $\Gamma_0 \in S''$; for every $v \in \pi^{-1}(\Gamma_0)$ we have $\Gamma_v = p^{n(v)}\Gamma_0$ for suitable $n(v) \geqslant 0$. Then

$$\mathrm{char}_\Lambda(Z) = \prod_{\Gamma_0\in S''}\ \prod_{v\in\pi^{-1}(\Gamma_0)} \omega_{n(v)}(\Gamma_0) \cdot \begin{cases} 1, & \text{if } r > 1 \\ \frac{1}{\gamma-1}, & \text{if } \Gamma = \langle\gamma\rangle \xrightarrow{\sim} \mathbf{Z}_p, \end{cases}$$

where $\omega_n(\Gamma_0) = \gamma_0^{p^n} - 1$, for a fixed topological generator γ_0 of Γ_0. The desired divisibility

$$\mathrm{char}_\Lambda(Z) \,\big|\, \mathrm{char}_\Lambda(Y_{\mathrm{tors}})$$

then follows from Proposition 9.4.9.5 (iii) and the formula (9.4.11.1). □

9.4.12. Proposition. — *The characteristic power series* $\mathrm{char}_\Lambda(X'_\infty)$ *is divisible by* $\mathrm{char}_\Lambda\big(E^1(D_\Lambda(A'_\infty))\big) = \mathrm{char}_\Lambda(D_\Lambda(A'_\infty))$.

Proof. — The map δ in the diagram (9.4.5.1) is an isomorphism in $({}_\Lambda\mathrm{Mod})/$(pseudo-null), hence

$$\mathrm{char}_\Lambda(X'_\infty)\,\mathrm{char}_\Lambda(Z) = \mathrm{char}_\Lambda(E^1(Y))\,\mathrm{char}_\Lambda\big(E^1(D_\Lambda(A'_\infty))\big) = \mathrm{char}_\Lambda(Y_{\mathrm{tors}})\,\mathrm{char}_\Lambda(D_\Lambda(A'_\infty)).$$

However, $\mathrm{char}_\Lambda(Z)$ divides $\mathrm{char}_\Lambda(Y_{\mathrm{tors}})$, by Proposition 9.4.11 (iii). □

9.5. Relating A_∞ and X_∞

9.5.1. For every finite subextension K_α/K of K_∞/K, put

$$E_\alpha = \mathcal{O}^*_{K_\alpha}\otimes\mathbf{Z}_p,\qquad A_\alpha = \mathrm{Pic}(\mathcal{O}_{K_\alpha})[p^\infty]$$

and let

$$E_\infty = \varinjlim_\alpha E_\alpha,\quad A_\infty = \varinjlim_\alpha A_\alpha,\quad X_\infty = \varprojlim_\alpha A_\alpha.$$

In the case when $r=1$, Iwasawa ([**Iw**, Thm. 11]) showed that there is a surjective morphism of Λ-modules

$$A_\infty \longrightarrow D_\Lambda(E^1(X_\infty)),$$

with pseudo-null kernel.

In this section we construct a canonical morphism in $({}_\Lambda\mathrm{Mod})/$(pseudo-null)

$$\alpha : X_\infty \longrightarrow E^1(D_\Lambda(A_\infty))$$

and show that $\mathrm{Coker}(\alpha)$ is close to zero in $({}_\Lambda\mathrm{Mod})/$(pseudo-null).

9.5.2. From now on, let $S_f=\{v\mid p\}$. We first relate $\mathrm{Pic}(\mathcal{O}_K)[p^\infty]$ to $\widetilde{H}^1_f(G_{K,S},\mathbf{Q}_p/\mathbf{Z}_p(1);\Delta(\mathbf{Q}_p/\mathbf{Z}_p(1)))$ for appropriate local conditions $\Delta(\mathbf{Q}_p/\mathbf{Z}_p(1))$.

For each $v\mid p$ the valuation v defines a morphism of complexes

$$\tau_{\leqslant 1}C^\bullet_{\mathrm{cont}}(G_v,\mathbf{Z}/p^n\mathbf{Z}(1)) \longrightarrow H^1(G_v,\mathbf{Z}/p^n\mathbf{Z}(1))[-1] = K_v^*\otimes\mathbf{Z}/p^n\mathbf{Z}[-1]\xrightarrow{v\otimes\mathrm{id}}\mathbf{Z}/p^n\mathbf{Z}[-1];$$

put

$$U_v^+(\mathbf{Z}/p^n\mathbf{Z}(1)) = \mathrm{Cone}\left(\tau_{\leqslant 1}C^\bullet_{\mathrm{cont}}(G_v,\mathbf{Z}/p^n\mathbf{Z}(1))\xrightarrow{v\otimes\mathrm{id}}\mathbf{Z}/p^n\mathbf{Z}[-1]\right)[-1].$$

This complex is equipped with a canonical morphism

$$i_v^+ : U_v^+(\mathbf{Z}/p^n\mathbf{Z}(1)) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v,\mathbf{Z}/p^n\mathbf{Z}(1));$$

put

$$U_v^-(\mathbf{Z}/p^n\mathbf{Z}(1)) = \mathrm{Cone}(-i_v^+).$$

The Pontrjagin dual of $U_v^-(\mathbf{Z}/p^n\mathbf{Z}(1))$ is quasi-isomorphic to $C^\bullet_{\mathrm{cont}}(G_v/I_v,\mathbf{Z}/p^n\mathbf{Z})[2]$, hence to the complex

$$\left[\mathbf{Z}/p^n\mathbf{Z}\xrightarrow{f_v-1}\mathbf{Z}/p^n\mathbf{Z}\right] = \left[\mathbf{Z}/p^n\mathbf{Z}\xrightarrow{0}\mathbf{Z}/p^n\mathbf{Z}\right] = \mathbf{Z}/p^n\mathbf{Z}[2]\oplus\mathbf{Z}/p^n\mathbf{Z}[1]$$

(in degrees $-2,-1$). It follows that $U_v^-(\mathbf{Z}/p^n\mathbf{Z}(1))$ is quasi-isomorphic to

$$\mathbf{Z}/p^n\mathbf{Z}[-2]\oplus\mathbf{Z}/p^n\mathbf{Z}[-1].$$

Passing to the inductive (resp., projective) limit with respect to n, we obtain local conditions $\Delta(\mathbf{Q}_p/\mathbf{Z}_p(1))$ (resp., $\Delta(\mathbf{Z}_p(1))$). The exact triangles

$$\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},X(1))\longrightarrow\mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S},X(1))\longrightarrow\bigoplus_{v\mid p}(X[-2]\oplus X[-1])\qquad(X=\mathbf{Z}_p,\mathbf{Q}_p/\mathbf{Z}_p)$$

together with (9.2.1.1)–(9.2.1.2) give

(9.5.2.1)
$$\begin{aligned}&\widetilde{H}^0_f(G_{K,S},\mathbf{Q}_p/\mathbf{Z}_p(1))\xrightarrow{\sim}\mu_{p^\infty}(K)\\0\longrightarrow\mathcal{O}_K^*\otimes\mathbf{Q}_p/\mathbf{Z}_p\longrightarrow&\widetilde{H}^1_f(G_{K,S},\mathbf{Q}_p/\mathbf{Z}_p(1))\longrightarrow\mathrm{Pic}(\mathcal{O}_K)[p^\infty]\longrightarrow0\\&\widetilde{H}^3_f(G_{K,S},\mathbf{Q}_p/\mathbf{Z}_p(1))\xrightarrow{\sim}\mathbf{Q}_p/\mathbf{Z}_p\\&\widetilde{H}^i_f(G_{K,S},\mathbf{Q}_p/\mathbf{Z}_p(1))=0\qquad(i\neq0,1,3),\end{aligned}$$

resp.,

(9.5.2.2)
$$\begin{aligned}&\widetilde{H}^1_f(G_{K,S},\mathbf{Z}_p(1))\xrightarrow{\sim}\mathcal{O}_K^*\otimes\mathbf{Z}_p\\&\widetilde{H}^2_f(G_{K,S},\mathbf{Z}_p(1))\xrightarrow{\sim}\mathrm{Pic}(\mathcal{O}_K)[p^\infty]\\&\widetilde{H}^3_f(G_{K,S},\mathbf{Z}_p(1))\xrightarrow{\sim}\mathbf{Z}_p\\&\widetilde{H}^i_f(G_{K,S},\mathbf{Z}_p(1))\ =\ 0\qquad(i\neq1,2,3).\end{aligned}$$

9.5.3. The corresponding local conditions in the limit over K_∞/K are equal to

$$U_v^+\big(F_\Gamma(\mathbf{Q}_p/\mathbf{Z}_p(1))\big)=\mathrm{Cone}\left(\tau_{\leqslant1}C^\bullet_{\mathrm{cont}}\big(G_v,F_\Gamma(\mathbf{Q}_p/\mathbf{Z}_p(1))\big)\xrightarrow{v_\alpha\otimes\mathrm{id}}\varinjlim_\alpha\bigoplus_{v_\alpha\mid v}\mathbf{Q}_p/\mathbf{Z}_p[-1]\right)[-1]$$

resp.,

$$U_v^+\big(\mathscr{F}_\Gamma(\mathbf{Z}_p(1))\big)=\mathrm{Cone}\left(\tau_{\leqslant1}C^\bullet_{\mathrm{cont}}(G_v,\mathscr{F}_\Gamma(\mathbf{Z}_p(1)))\xrightarrow{v_\alpha\otimes\mathrm{id}}\varprojlim_\alpha\bigoplus_{v_\alpha\mid v}\mathbf{Z}_p[-1]\right)[-1].$$

For $K_\alpha\subset K_\beta$, the transition maps

$$\bigoplus_{v_\alpha\mid v}\left(\mathbf{Q}_p/\mathbf{Z}_p\longrightarrow\bigoplus_{w_\beta\mid v_\alpha}\mathbf{Q}_p/\mathbf{Z}_p\right)\quad\text{resp.,}\quad\bigoplus_{v_\alpha\mid v}\left(\bigoplus_{w_\beta\mid v_\alpha}\mathbf{Z}_p\longrightarrow\mathbf{Z}_p\right)$$

are given by the ramification indices $e(w_\beta \mid v_\alpha)$ (resp., the inertia degrees $f(w_\beta \mid v_\alpha)$). It follows that

$$H^1\big(U_v^-\big(F_\Gamma(\mathbf{Q}_p/\mathbf{Z}_p(1))\big)\big) = \varinjlim_\alpha \bigoplus_{v_\alpha \mid v} \mathbf{Q}_p/\mathbf{Z}_p \xrightarrow{\sim} \begin{cases} 0, & I(\Gamma_v) \neq 0 \\ D_\Lambda(\Lambda_v), & I(\Gamma_v) = 0 \end{cases}$$

$$H^1\big(U_v^-\big(\mathscr{F}_\Gamma(\mathbf{Z}_p(1))\big)\big) = \varprojlim_\alpha \bigoplus_{v_\alpha \mid v} \mathbf{Z}_p \xrightarrow{\sim} \begin{cases} 0, & [\Gamma_v : I(\Gamma_v)] = \infty \\ \Lambda_v, & [\Gamma_v : I(\Gamma_v)] < \infty. \end{cases}$$

Combined with the functoriality of the local Brauer groups (cf. 9.2.2), this gives isomorphisms in $D({}_\Lambda\mathrm{Mod})$

$$U_v^-\big(F_\Gamma(\mathbf{Q}_p/\mathbf{Z}_p(1))\big) \xrightarrow{\sim} \begin{cases} D_\Lambda(\Lambda)[-1] \oplus D_\Lambda(\Lambda)[-2], & \Gamma_v = 0 \\ D_\Lambda(\Lambda_v)[-1], & \Gamma_v \neq 0 = I(\Gamma_v) \\ 0, & I(\Gamma_v) \neq 0 \end{cases} \tag{9.5.3.1}$$

resp.,

$$U_v^-\big(\mathscr{F}_\Gamma(\mathbf{Z}_p(1))\big) \xrightarrow{\sim} \begin{cases} \Lambda_v[-2] \oplus \Lambda_v[-1], & [\Gamma_v : I(\Gamma_v)] < \infty \\ \Lambda_v[-2], & [\Gamma_v : I(\Gamma_v)] = \infty. \end{cases} \tag{9.5.3.2}$$

9.5.4. For each $v \mid p$ there are canonical isomorphisms in $D({}_\Lambda\mathrm{Mod})$

$$\begin{aligned} \mathscr{D}_\Lambda(\Lambda_v) &\xrightarrow{\sim} \Lambda_v[-r(v)] \\ \Phi_\Lambda(\Lambda_v) &\xrightarrow{\sim} D_\Lambda(\Lambda_v)[r(v)], \end{aligned} \tag{9.5.4.1}$$

where $\Gamma_v \xrightarrow{\sim} \mathbf{Z}_p^{r(v)}$. Put

$$S_{\mathrm{ex}} = \{v \mid p \,:\, r(v) = 1,\ v \text{ is ramified in } K_\infty/K\}.$$

9.5.5. Lemma

(i) *If v is unramified in K_∞/K, then there is a canonical isomorphism in $D({}_\Lambda\mathrm{Mod})$*

$$\mathscr{D}_\Lambda\big(U_v^-\big(\mathscr{F}_\Gamma(\mathbf{Z}_p(1))\big)\big) \xrightarrow{\sim} D_\Lambda\big(U_v^-\big(F_\Gamma(\mathbf{Q}_p/\mathbf{Z}_p(1))\big)\big).$$

(ii) *If v is ramified in K_∞/K and $r(v) > 1$, then there are isomorphisms in $D(({}_\Lambda\mathrm{Mod})/(pseudo\text{-}null))$*

$$\mathscr{D}_\Lambda\big(U_v^-\big(\mathscr{F}_\Gamma(\mathbf{Z}_p(1))\big)\big) \xrightarrow{\sim} D_\Lambda\big(U_v^-\big(F_\Gamma(\mathbf{Q}_p/\mathbf{Z}_p(1))\big)\big) \xrightarrow{\sim} 0.$$

(iii) *If $v \in S_{\mathrm{ex}}$, then there are isomorphisms in $D({}_\Lambda\mathrm{Mod})$*

$$\mathscr{D}_\Lambda\big(U_v^-\big(\mathscr{F}_\Gamma(\mathbf{Z}_p(1))\big)\big) \xrightarrow{\sim} \Lambda_v \oplus \Lambda_v[1], \quad D_\Lambda\big(U_v^-\big(F_\Gamma(\mathbf{Q}_p/\mathbf{Z}_p(1))\big)\big) \xrightarrow{\sim} 0.$$

Proof. — This follows from (9.5.3.1)–(9.5.3.2), (9.5.4.1) and the fact that Λ_v is pseudo-null if $r(v) > 1$. □

9.5.6. The Selmer complexes $\widetilde{\mathbf{R}\Gamma}_f(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p(1))$ (resp., $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1)))$ represented by $C_f^\bullet(G_{K,S}, F_\Gamma(\mathbf{Q}_p/\mathbf{Z}_p(1)); \Delta)$ (resp., by $C_f^\bullet(G_{K,S}, \mathscr{F}_\Gamma(\mathbf{Z}_p(1)); \Delta))$ – for the local conditions defined in 9.5.3 – have cohomology equal to

$$\widetilde{H}^i_f(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p(1)) = \varinjlim_\alpha \widetilde{H}^i_f(G_{K_\alpha,S_\alpha}, \mathbf{Q}_p/\mathbf{Z}_p(1))$$
$$\widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1)) = \varprojlim_\alpha \widetilde{H}^i_f(G_{K_\alpha,S_\alpha}, \mathbf{Z}_p(1)).$$

It follows from (9.5.2.1)–(9.5.2.2) that we have

(9.5.6.1)
$$\begin{aligned} \widetilde{H}^0_f(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p(1)) &\xrightarrow{\sim} \mu_{p^\infty}(K_\infty) \\ 0 \longrightarrow E_\infty \longrightarrow \widetilde{H}^1_f(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p(1)) &\longrightarrow A_\infty \longrightarrow 0 \\ \widetilde{H}^i_f(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p(1)) &= 0 \qquad (i > 1), \end{aligned}$$

resp.,

(9.5.6.2)
$$\begin{aligned} \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1)) &\xrightarrow{\sim} \varprojlim_\alpha E_\alpha \\ \widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1)) &\xrightarrow{\sim} X_\infty \\ \widetilde{H}^3_{f,\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1)) &\xrightarrow{\sim} \mathbf{Z}_p \\ \widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1)) &= 0 \qquad (i \neq 1, 2, 3). \end{aligned}$$

9.5.7. The exact triangles in $D({}_\Lambda\mathrm{Mod})$

$$\widetilde{\mathbf{R}\Gamma}_f(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p(1)) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}\big(G_{K,S}, F_\Gamma(\mathbf{Q}_p/\mathbf{Z}_p(1))\big) \longrightarrow \bigoplus_{v|p} U_v^-\big(F_\Gamma(\mathbf{Q}_p/\mathbf{Z}_p(1))\big)$$

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1)) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}\big(G_{K,S}, \mathscr{F}_\Gamma(\mathbf{Z}_p(1))\big) \longrightarrow \bigoplus_{v|p} U_v^-\big(\mathscr{F}_\Gamma(\mathbf{Z}_p(1))\big)$$

together with the canonical isomorphisms

$$D_\Lambda\Big(\mathbf{R}\Gamma_{\mathrm{cont}}\big(G_{K,S}, F_\Gamma(\mathbf{Q}_p/\mathbf{Z}_p(1))\big)\Big) \xrightarrow{\sim} D_\Lambda\Big(\Phi_\Lambda\Big(\mathbf{R}\Gamma_{\mathrm{cont}}\big(G_{K,S}, \mathscr{F}_\Gamma(\mathbf{Z}_p(1))\big)\Big)\Big) \xrightarrow{\sim} \mathscr{D}_\Lambda\Big(\mathbf{R}\Gamma_{\mathrm{cont}}\big(G_{K,S}, \mathscr{F}_\Gamma(\mathbf{Z}_p(1))\big)\Big)$$

and Lemma 9.5.5 yield an exact triangle in $D(({}_\Lambda\mathrm{Mod})/(\text{pseudo-null}))$

$$D_\Lambda\big(\widetilde{\mathbf{R}\Gamma}_f(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p(1))\big) \longrightarrow \mathscr{D}_\Lambda\big(\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1))\big) \longrightarrow \bigoplus_{v\in S_{\mathrm{ex}}} \big(\Lambda_v[2] \oplus \Lambda_v[1]\big).$$

Applying $\mathscr{D}_\Lambda$ we obtain another exact triangle in $D(({}_\Lambda\mathrm{Mod})/(\text{pseudo-null}))$

(9.5.7.1)
$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, \mathbf{Z}_p(1)) \longrightarrow W \longrightarrow \bigoplus_{v\in S_{\mathrm{ex}}} (\Lambda_v[-1] \oplus \Lambda_v[-2]),$$

in which

$$W = \mathscr{D}_\Lambda\big(D_\Lambda\big(\widetilde{\mathbf{R}\Gamma}_f(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p(1))\big)\big).$$

9.5.8. The hyper-cohomology spectral sequence

$$E_2^{i,j} = E^i\big(D_\Lambda\big(\widetilde{H}^j_f(K_S/K_\infty, \mathbf{Q}_p/\mathbf{Z}_p(1))\big)\big) \Longrightarrow H^{i+j}(W)$$

satisfies

$$E_2^{i,j} = 0 \quad (j \neq 0,1), \qquad \operatorname{codim}_\Lambda\big(\operatorname{supp}\big(E_2^{i,j}\big)\big) \geqslant i.$$

In particular, we have isomorphisms in $({}_\Lambda\mathrm{Mod})/(\text{pseudo-null})$

$$\begin{aligned} H^j(W) &= 0 \qquad (j \neq 0,1,2) \\ H^2(W) &\xrightarrow{\sim} E_2^{1,1}. \end{aligned}$$

Combining E^i(9.5.6.1) with the cohomology sequence of the triangle (9.5.7.1), we obtain a diagram in $({}_\Lambda\mathrm{Mod})/(\text{pseudo-null})$

(9.5.8.1)

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & E^1(D_\Lambda(E_\infty \otimes \mathbf{Q}_p/\mathbf{Z}_p)) & & & & \\ & & & & \downarrow & & & & \\ \bigoplus_{v\in S_{\mathrm{ex}}} \Lambda_v & \longrightarrow & X_\infty & \longrightarrow & H^2(W) & \longrightarrow & \bigoplus_{v\in S_{\mathrm{ex}}} \Lambda_v & \longrightarrow & \mathbf{Z}_p \longrightarrow 0 \\ & & & & \downarrow & & & & \\ & & & & E^1(D_\Lambda(A_\infty)) & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

with exact row and column. This diagram defines a morphism in $({}_\Lambda\mathrm{Mod})/(\text{pseudo-null})$

$$\alpha : X_\infty \longrightarrow H^2(W) \longrightarrow E^1(D_\Lambda(A_\infty)).$$

9.5.9. Proposition. — *If* $\mathfrak{p} \in \operatorname{Spec}(\Lambda)$ *with* $\operatorname{ht}(\mathfrak{p}) = 1$ *is not exceptional (in the sense of 9.4.6), then* $\operatorname{Coker}(\alpha)_\mathfrak{p} = 0$.

Proof. — The assumption on $\mathfrak{p}$ implies that

$$\Big(\bigoplus_{v\in S_{\mathrm{ex}}} \Lambda_v\Big)_\mathfrak{p} = 0.$$

The statement follows by localizing the diagram (9.5.8.1) at $\mathfrak{p}$. □

9.5.10. Corollary. — *If, for every* $v \in S_f$ *ramified in* K_∞/K, *we have* $\Gamma_v \xrightarrow{\sim} \mathbf{Z}_p^{r(v)}$ *with* $r(v) \geqslant 2$, *then the map*

$$\alpha : X_\infty \longrightarrow E^1(D_\Lambda(A_\infty))$$

is an epimorphism in $({}_\Lambda\mathrm{Mod})/(\textit{pseudo-null})$.

Proof. — Under these assumptions there are no exceptional $\mathfrak{p} \in \mathrm{Spec}(\Lambda)$. □

9.6. Comparison with classical Selmer groups

9.6.1. Let $\mathcal{O}, T, T^*, A, A^*$ be as in 9.1.4; put $V = T \otimes_{\mathcal{O}} F$, $V^* = T^* \otimes_{\mathcal{O}} F$. Assume that we are given, for each $v \in \Sigma = \{v \mid p\}$, an $F[G_v]$-submodule $V_v^+ \subset V$. Put

$$T_v^+ = T \cap V_v^+, \quad A_v^+ = V_v^+/T_v^+ \subset V/T = A, \quad X_v^- = X/X_v^+ \qquad (X = T, V, A;\ v \in \Sigma)$$

and, for each $v \mid p$,

$$V^*(1)_v^{\pm} = \mathrm{Hom}_F(V_v^{\mp}, F)(1),\ T^*(1)_v^{\pm} = \mathrm{Hom}_{\mathcal{O}}(T_v^{\mp}, \mathcal{O})(1),\ A^*(1)_v^{\pm} = V^*(1)_v^{\pm}/T^*(1)_v^{\pm}.$$

These data induce Greenberg's local conditions for $X = T, V, A$ (and also for $X = T^*(1), V^*(1), A^*(1)$) with

$$U_v^+(X) = \begin{cases} C^\bullet_{\mathrm{cont}}(G_v, X_v^+), & (v \in \Sigma) \\ C^\bullet_{\mathrm{cont}}(G_v/I_v, X^{I_v}), & (v \in \Sigma'), \end{cases}$$

where $\Sigma' = \{v \in S_f\ ;\ v \nmid p\}$, hence the corresponding Selmer complexes $\widetilde{\mathbf{R}\Gamma}_f(X)$ and their cohomology groups $\widetilde{H}^i_f(X)$. Greenberg [**Gre2, Gre3**] defined his Selmer groups (resp., strict Selmer groups) as

$$S_X(K) = \mathrm{Ker}\Big(H^1_{\mathrm{cont}}(G_{K,S}, X) \longrightarrow \bigoplus_{v \in \Sigma} H^1_{\mathrm{cont}}(I_v, X_v^-) \ \oplus \bigoplus_{v \in \Sigma'} H^1_{\mathrm{cont}}(I_v, X)\Big)$$

$$S_X^{\mathrm{str}}(K) = \mathrm{Ker}\Big(H^1_{\mathrm{cont}}(G_{K,S}, X) \longrightarrow \bigoplus_{v \in \Sigma} H^1_{\mathrm{cont}}(G_v, X_v^-) \oplus \bigoplus_{v \in \Sigma'} H^1_{\mathrm{cont}}(I_v, X)\Big).$$

9.6.2. These groups satisfy the following properties: there is an exact sequence

$$0 \longrightarrow S_X^{\mathrm{str}}(K) \longrightarrow S_X(K) \longrightarrow \bigoplus_{v \mid p} H^1_{\mathrm{cont}}(G_v/I_v, (X_v^-)^{I_v}),$$

isomorphisms

$$S_T(K) \otimes_{\mathcal{O}} F \xrightarrow{\sim} S_V(K), \quad S_T^{\mathrm{str}}(K) \otimes_{\mathcal{O}} F \xrightarrow{\sim} S_V^{\mathrm{str}}(K)$$

and canonical injective maps

$$S_T(K) \otimes_{\mathcal{O}} F/\mathcal{O} \hookrightarrow S_A(K), \quad S_T^{\mathrm{str}}(K) \otimes_{\mathcal{O}} F/\mathcal{O} \hookrightarrow S_A^{\mathrm{str}}(K)$$

with finite cokernels.

9.6.3. Lemma. — *For each $X = T, V, A$ there is an exact sequence*

$$0 \longrightarrow \widetilde{H}^0_f(X) \longrightarrow X^{G_K} \longrightarrow \bigoplus_{v \mid p} (X_v^-)^{G_v} \longrightarrow \widetilde{H}^1_f(X) \longrightarrow S_X^{\mathrm{str}}(K) \longrightarrow 0.$$

Proof. — This follows from the exact triangle

$$\widetilde{\mathbf{R}\Gamma}_f(X) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, X) \longrightarrow \bigoplus_{v \in S_f} U_v^-(X)$$

and the fact that $U_v^-(X)$ (defined in 6.1.3) is quasi-isomorphic to $C^\bullet_{\mathrm{cont}}(G_v/I_v, H^1_{\mathrm{cont}}(I_v, X))[-1]$ (resp., $C^\bullet_{\mathrm{cont}}(G_v, X_v^-)$) if $v \nmid p$ (resp., $v \mid p$). □

9.6.4. Corollary. — *If* $V^{G_K} = 0$, *then*

$$\mathrm{cork}_{\mathcal{O}}\big(\widetilde{H}^1_f(A)\big)\ \big(= \dim_F\big(\widetilde{H}^1_f(V)\big)\big) = \mathrm{cork}_{\mathcal{O}}(S_A^{\mathrm{str}}(K)) + \bigoplus_{v \mid p} \dim_F\big((V_v^-)^{G_v}\big).$$

$$\mathrm{cork}_{\mathcal{O}}\big(\widetilde{H}^1_f(A)\big) \geqslant \mathrm{cork}_{\mathcal{O}}(S_A(K)).$$

Proof. — Combine the exact sequence of Lemma 9.6.3 with the statements from 9.6.2 and the equality

$$\dim_F\big(H^1_{\mathrm{cont}}\big(G_v/I_v, (V_v^-)^{I_v}\big)\big) = \dim_F\big((V_v^-)^{G_v}\big). \quad \square$$

9.6.5. Let $K \subset K_\infty \subset K_S$, with $\Gamma = \mathrm{Gal}(K_\infty/K) \xrightarrow{\sim} \mathbf{Z}_p^r$ $(r \geqslant 1)$. We define

$$S_A(K_\infty) = \varinjlim_{\alpha} S_A(K_\alpha), \quad S_A^{\mathrm{str}}(K_\infty) = \varinjlim_{\alpha} S_A^{\mathrm{str}}(K_\alpha),$$

where K_α, as usual, runs through all finite subextensions of K_∞/K. The exact sequence of Lemma 9.6.3 yields, in the limit, an exact sequence

$$A^{G_{K_\infty}} \longrightarrow \bigoplus_{v \mid p} \bigoplus_{v_\infty \mid v} (A_v^-)^{G_{v_\infty}} \longrightarrow \widetilde{H}^1_f(K_S/K_\infty, A) \longrightarrow S_A^{\mathrm{str}}(K_\infty) \longrightarrow 0. \tag{9.6.5.1}$$

For each $v \mid p$ there is an isomorphism of Λ_v-modules

$$D_{\Lambda_v}\left(\bigoplus_{v_\infty \mid v} (A_v^-)^{G_{v_\infty}}\right) \xrightarrow{\ \sim\ } D\big((A_v^-)^{G_{v_\infty}}\big) \otimes_{\mathcal{O}} \Lambda_v;$$

thus

$$\mathrm{cork}_\Lambda\, \widetilde{H}^1_f(K_S/K_\infty, A) = \mathrm{cork}_\Lambda\, S_A^{\mathrm{str}}(K_\infty) + \bigoplus_{\substack{v \mid p \\ \Gamma_v = 0}} \dim_F\big((V_v^-)^{G_v}\big). \tag{9.6.5.2}$$

If we write, as usual, $\Gamma_v \xrightarrow{\sim} \mathbf{Z}_p^{r(v)}$, then there is an isomorphism in $({}_\Lambda\mathrm{Mod})/$(pseudo-null)

$$D((A_v^-)^{G_{v_\infty}}) \otimes_{\mathcal{O}} \Lambda_v \xrightarrow{\ \sim\ } \begin{cases} 0, & r(v) \geqslant 2 \\ D\big((A_v^-)^{G_{v_\infty}}_{\mathcal{O}\text{-div}}\big) \otimes_{\mathcal{O}} \Lambda_v, & r(v) = 1. \end{cases}$$

9.6.6. Proposition. — *The canonical surjective map*

$$\beta : \widetilde{H}^1_f(K_S/K_\infty, A) \longrightarrow S_A^{\mathrm{str}}(K_\infty)$$

has the following properties.

(i) *If* $(V_v^-)^{G_{v\infty}} = 0$ *for all primes* $v \mid p$, *then*

$$D_\Lambda(\mathrm{Ker}(\beta)) \otimes \mathbf{Q} = 0.$$

(ii) *Assume that no* $v \mid p$ *splits completely in* K_∞/K, *and that* $(V_v^-)^{G_{v\infty}} = 0$ *for all primes* $v \mid p$ *satisfying* $r(v) = 1$. *Then* $D_\Lambda(\mathrm{Ker}(\beta))$ *is* Λ*-pseudo-null.*

(iii) *If* $(A_v^-)^{G_v} = 0$ *for all primes* $v \mid p$, *then* $(A_v^-)^{G_{v\infty}} = 0$ *for all* $v \mid p$ *and* β *is an isomorphism.*

Proof. — The statements (i) and (ii) follow from the discussion in 9.6.5. As regards (iii), for each $v \mid p$, $N := D\big((A_v^-)^{G_{v\infty}}\big)$ is a Λ_v-module of finite type satisfying $N_{\Gamma_v} = D\big((A_v^-)^{G_v}\big) = 0$; thus $N = 0$ by Nakayama's Lemma, hence $(A_v^-)^{G_{v\infty}} = D(N) = 0$. The exact sequence (9.6.5.1) then implies that $\mathrm{Ker}(\beta) = 0$. □

9.6.7. Abelian varieties. — In the rest of 9.6 we let $\mathcal{O} = \mathbf{Z}_p$, $F = \mathbf{Q}_p$. Let B be an abelian variety over K with good reduction outside S_f. Then $T = T_p(B)$ is a representation of $G_{K,S}$ and the Weil pairing identifies $T^*(1)$ with $T_p(\widehat{B})$, where $\widehat{B}$ denotes the dual abelian variety; hence $A = B[p^\infty]$ and $A^*(1) = \widehat{B}[p^\infty]$. Let S be any finite set of primes of K containing all primes of bad reduction of B and all primes dividing $p\infty$. Put, as before, $\Sigma = \{v \mid p\}$, $\Sigma' = S_f - \Sigma$.

9.6.7.1. The classical Selmer groups for the p^n-descent

$$\mathrm{Sel}(B/K, p^n) = \mathrm{Ker}\left(H^1(G_{K,S}, B[p^n]) \longrightarrow \bigoplus_{v \in S_f} H^1(G_v, B[p^n])/\mathrm{Im}(\delta_{v,n})\right)$$

are defined by the local conditions

$$\mathrm{Im}\left(\delta_{v,n} : B(K_v) \otimes \mathbf{Z}/p^n\mathbf{Z} \hookrightarrow H^1(G_v, B[p^n])\right).$$

The corresponding discrete and compact Selmer groups

$$\mathrm{Sel}(B/K, p^\infty) = \varinjlim_n \mathrm{Sel}(B/K, p^n) = \mathrm{Ker}\left(H^1(G_{K,S}, A) \longrightarrow \bigoplus_{v \in S_f} H^1(G_v, A)/L_v(A)\right)$$

$$S_p(B/K) = \varprojlim_n \mathrm{Sel}(B/K, p^n) = \mathrm{Ker}\left(H^1(G_{K,S}, T) \longrightarrow \bigoplus_{v \in S_f} H^1(G_v, T)/L_v(T)\right),$$

where we have put

$$L_v(A) = \varinjlim_n \mathrm{Im}(\delta_{v,n}) = \mathrm{Im}\left(B(K_v) \otimes \mathbf{Q}_p/\mathbf{Z}_p \hookrightarrow H^1(G_v, A)\right)$$

$$L_v(T) = \varprojlim_n \mathrm{Im}(\delta_{v,n}) = \mathrm{Im}\left(B(K_v)\widehat{\otimes}\mathbf{Z}_p \longrightarrow H^1_{\mathrm{cont}}(G_v, T)\right),$$

depend only on the $G_{K,S}$-modules $A = B[p^\infty]$ and $T = T_p(B)$, respectively. More precisely, they coincide with the Bloch-Kato Selmer groups

$$\mathrm{Sel}(B/K, p^\infty) = H^1_f(K, A), \quad S_p(B/K) = H^1_f(K, T)$$

([**B-K**, §3.11]). If $v \nmid p$, then

$$L_v(A) = 0, \quad L_v(T) = H^1_{\mathrm{cont}}(G_v, T). \tag{9.6.7.1}$$

9.6.7.2. We shall consider only the following "elementary" case:

$$\Sigma = \{v \mid p\} = \Sigma_o \cup \Sigma_t,$$

where

(Ord) $(\forall v \in \Sigma_o)$ B has good ordinary reduction at v.

(Tor) $(\forall v \in \Sigma_t)$ B has completely toric reduction at v.

Under this assumption, for each $v \mid p$ there is a canonical sub-$\mathbf{Q}_p[G_v]$-module $V_v^+ \hookrightarrow V = T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ as in 9.6.1, with V_v^+ arising from the kernel of the reduction map at v (resp., from a p-adic uniformization by a torus) in the case (Ord) (resp., (Tor)).

In either case, $V_v^- = V/V_v^+$ is an unramified G_v-module. For $v \in \Sigma_t$, the geometric Frobenius element f_v acts on V_v^- by an element of finite order, while for $v \in \Sigma_o$ all eigenvalues of f_v acting on V_v^- are v-Weil numbers of weight -1 (see 12.4.8.1 below).

The dual abelian variety $\widehat{B}$ is isogeneous to B, which implies that, for each $v \in \Sigma_o$ (resp., $v \in \Sigma_t$), $\widehat{B}$ also has good ordinary (resp., completely toric) reduction at v. This means that the same construction defines sub-$\mathbf{Q}_p[G_v]$-modules $V^*(1)_v^+ \hookrightarrow V^*(1) = T^*(1) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p = T_p(\widehat{B}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ $(v \in \Sigma)$, which coincide with the abstract modules defined in 9.6.1.

Any polarization $\lambda : B \to \widehat{B}$ defines an injective morphism of $\mathbf{Z}_p[G_{K,S}]$-modules $T = T_p(B) \hookrightarrow T^*(1) = T_p(\widehat{B})$ with finite cokernel. For each $v \mid p$, the sub-$\mathbf{Z}_p[G_v]$-module $T_v^+ \subset T$ is mapped into $T^*(1)_v^+$, again with finite cokernel. In other words, the Weil pairing associated to λ defines a skew-symmetric bilinear form

$$V \otimes_{\mathbf{Q}_p} V \longrightarrow \mathbf{Q}_p(1),$$

which induces isomorphisms of $\mathbf{Q}_p[G_{K,S}]$-modules

$$V \xrightarrow{\sim} V^*(1) = \mathrm{Hom}_{\mathbf{Q}_p}(V, \mathbf{Q}_p)(1)$$

resp., of $\mathbf{Q}_p[G_v]$-modules

$$V_v^\pm \xrightarrow{\sim} (V_v^\mp)^*(1) = \mathrm{Hom}_{\mathbf{Q}_p}(V_v^\mp, \mathbf{Q}_p)(1).$$

The following results are well-known (cf. [**Co-Gr, Gre5**]); we record them for the sake of completeness.

9.6.7.3. Lemma

(i) *There are exact sequences*

$$0 \longrightarrow S_T^{\mathrm{str}}(K) \longrightarrow S_p(B/K) \longrightarrow \bigoplus_{v|p} H^1_{\mathrm{cont}}(G_v, T_v^-)_{\mathrm{tors}} \oplus \bigoplus_{v\in\Sigma'} H^1_{\mathrm{cont}}(G_v, T)/H^1_{\mathrm{ur}}(G_v, T)$$

$$0 \longrightarrow \mathrm{Sel}(B/K, p^\infty) \longrightarrow S_A^{\mathrm{str}}(K) \longrightarrow \bigoplus_{v|p} \mathrm{Im}\big(H^1(G_v, A_v^+) \longrightarrow H^1(G_v, A)\big)/\mathrm{div} \oplus \bigoplus_{v\in\Sigma'} H^1_{\mathrm{ur}}(G_v, A).$$

(ii) *For each $v \in \Sigma'$, the groups $H^1_{\mathrm{cont}}(G_v, T)/H^1_{\mathrm{ur}}(G_v, T)$ and $H^1_{\mathrm{ur}}(G_v, A)$ are finite, of common order equal to $p^{\mathrm{Tam}_v(T,(p))}$.*

(iii) *For each $v \mid p$,*

$$H^1_{\mathrm{cont}}(G_v, T_v^-)_{\mathrm{tors}} \xrightarrow{\sim} H^0(G_v, A_v^-)/\mathrm{div}$$
$$D\left(\mathrm{Im}\left(H^1(G_v, A_v^+) \longrightarrow H^1(G_v, A)\right)/\mathrm{div}\right) \subseteq H^0(G_v, A^*(1)_v^-)/\mathrm{div}.$$

(iv) *The groups*

$$S_p(B/K)/S_T^{\mathrm{str}}(K), \quad S_A^{\mathrm{str}}(K)/\mathrm{Sel}(B/K, p^\infty)$$

are finite.

Proof

(i) For $v \mid p$,

$$L_v(A) = \mathrm{Im}\left(H^1(G_v, A_v^+) \longrightarrow H^1(G_v, A)\right)_{\mathrm{div}}$$

(cf. [**Gre5**, Prop. 2.2 and pp. 69–70] in the case $\dim(B) = 1$). Replacing B by $\widehat{B}$ and applying Tate's local duality, we obtain

$$L_v(T) = \mathrm{Ker}\left(H^1_{\mathrm{cont}}(G_v, T) \longrightarrow H^1_{\mathrm{cont}}(G_v, T_v^-)/\mathrm{tors}\right).$$

Combining these expressions with (9.6.7.1), we obtain (i).

(ii) The self-dual $\mathbf{Q}_p[G_v]$-module $V \xrightarrow{\sim} V^*(1)$ is known to satisfy the weight-monodromy conjecture (see, *e.g.*, [**Ja2**, §5, §7]), which implies that $V^{G_v} = 0$, hence $H^1_{\mathrm{ur}}(G_v, V) = 0$ ($\Longrightarrow H^1_{\mathrm{ur}}(G_v, A)$ is finite). Applying the duality isomorphism

$$D(H^1_{\mathrm{cont}}(G_v, T)/H^1_{\mathrm{ur}}(G_v, T)) \xrightarrow{\sim} H^1_{\mathrm{ur}}(G_v, A^*(1)),$$

the statement then follows from 7.6.9 and 7.6.10.11.

(iii) The isomorphism in the first row is standard. By Tate's local duality, the L.H.S. of the second row is isomorphic to

$$\mathrm{Im}\left(H^1_{\mathrm{cont}}(G_v, T^*(1)) \longrightarrow H^1_{\mathrm{cont}}(G_v, T^*(1)_v^-)\right)_{\mathrm{tors}} \subseteq H^1_{\mathrm{cont}}(G_v, T^*(1)_v^-)_{\mathrm{tors}} = H^0(G_v, A^*(1)_v^-)/\mathrm{div}.$$

(iv) This is a consequence of (i)–(iii). □

9.6.7.4. For K_∞/K as in 9.6.5, put

$$\mathrm{Sel}(B/K_\infty) = \varinjlim_\alpha \mathrm{Sel}(B/K_\alpha, p^\infty), \quad S_p(B/K_\infty) = \varprojlim_\alpha S_p(B/K_\alpha);$$

then

$$\mathrm{Sel}(B/K_\infty, p^\infty) \subseteq S_A^{\mathrm{str}}(K_\infty), \quad \varprojlim_\alpha S_T^{\mathrm{str}}(K_\alpha) \subseteq S_p(B/K_\infty). \tag{9.6.7.1}$$

9.6.7.5. Proposition. — *Assume that each $v \mid p$ is ramified in K_∞/K. Then:*

(i) *There exist exact sequences of Λ-modules of finite type*

$$0 \longrightarrow \varprojlim_\alpha S_T^{\mathrm{str}}(K_\alpha) \longrightarrow S_p(B/K_\infty) \longrightarrow \bigoplus_{\substack{v\in\Sigma' \\ \Gamma_v=0}} D\big(H^1_{\mathrm{ur}}(G_v, A^*(1))\big) \otimes_{\mathbf{Z}_p} \Lambda$$

$$\bigoplus_{\substack{v\in\Sigma' \\ \Gamma_v=0}} D(H^1_{\mathrm{ur}}(G_v, A)) \otimes_{\mathbf{Z}_p} \Lambda \longrightarrow D_\Lambda(S_A^{\mathrm{str}}(K_\infty)) \longrightarrow D_\Lambda(\mathrm{Sel}(B/K_\infty, p^\infty)) \longrightarrow 0.$$

(ii) *Put*

$$c = \max\{\mathrm{Tam}_v(T,(p)) \mid v \in \Sigma',\ v \textit{ splits completely in } K_\infty/K\}.$$

Then

$$p^c \cdot (S_A^{\mathrm{str}}(K_\infty)/\mathrm{Sel}(B/K_\infty, p^\infty)) = p^c \cdot \Big(S_p(B/K_\infty)/\varprojlim_\alpha S_T^{\mathrm{str}}(K_\alpha)\Big) = 0.$$

In particular, if $\mathrm{Tam}_v(T,(p)) = 0$ *for all $v \in \Sigma'$ that split completely in K_∞/K, then*

$$\mathrm{Sel}(B/K_\infty, p^\infty) = S_A^{\mathrm{str}}(K_\infty), \quad S_p(B/K_\infty) = \varprojlim_\alpha S_T^{\mathrm{str}}(K_\alpha).$$

Proof

(i) Fix $v \mid p$, a finite sub-extension K_α/K of K_∞/K and a prime $v_\alpha \mid v$ of K_α. Our assumption implies that, possibly after replacing K_α by a finite extension contained in K_∞, there exists a $\mathbf{Z}_p$-extension $K_\alpha = F \subset F_1 \subset \cdots \subset F_\infty = \bigcup_n F_n \subset K_\infty$ ($\mathrm{Gal}(F_n/F) \xrightarrow{\sim} \mathbf{Z}/p^n\mathbf{Z}$), which is totally ramified at v_α. As the G_v-modules $A_v^-, A^*(1)_v^-$ are unramified, the corestriction maps in the projective systems

$$\big(H^0(G_{v_n}, A_v^-)\big)_{n\geqslant 1}, \quad \big(H^0(G_{v_n}, A^*(1)_v^-)\big)_{n\geqslant 1}$$

(where v_n denotes the unique prime of F_n above v_α) are given by multiplication by p, hence the corresponding projective limits vanish. It follows that, if we apply Lemma 9.6.7.3 (i), (iii) over each finite sub-extension K_α/K of K_∞/K and pass to the limit, the terms corresponding to $v \mid p$ will disappear and we shall be left (using Tate's local duality) with the exact sequences

$$0 \longrightarrow \varprojlim_\alpha S_T^{\mathrm{str}}(K_\alpha) \longrightarrow S_p(B/K_\infty) \longrightarrow \bigoplus_{v\in\Sigma'} M_v(A^*(1))$$

$$\bigoplus_{v\in\Sigma'} M_v(A) \longrightarrow D_\Lambda(S_A^{\mathrm{str}}(K_\infty)) \longrightarrow D_\Lambda(\mathrm{Sel}(B/K_\infty, p^\infty)) \longrightarrow 0.$$

Here we have used the notation

$$M_v(X) := D_\Lambda\left(\varinjlim_{\alpha} \bigoplus_{v_\alpha | v} H^1_{\mathrm{ur}}(G_{v_\alpha}, X)\right),$$

for each $v \in \Sigma'$ and any p-primary torsion discrete G_v-module X. Fix $v \in \Sigma'$; as v is unramified in K_∞/K, we have either $\Gamma_v \xrightarrow{\sim} \mathbf{Z}_p$, or $\Gamma_v = 0$.

If $\Gamma_v \xrightarrow{\sim} \mathbf{Z}_p$, then the tower of local fields $(K_\alpha)_{v_\alpha}$ exhausts the maximal pro-p-unramified extension of any fixed $(K_{\alpha_0})_{v_{\alpha_0}}$; this implies that $M_v(X) = 0$ (for any X as above).

If $\Gamma_v = 0$, then v splits completely in K_∞/K, hence

$$M_v(X) = D(H^1_{\mathrm{ur}}(G_{v_\alpha}, X)) \otimes_{\mathbf{Z}_p} \Lambda.$$

The statement (i) is proved.

(ii) This follows from (i), as the groups $H^1_{\mathrm{ur}}(G_v, X)$ $(X = A, A^*(1))$ have common order, equal to $p^{\mathrm{Tam}_v(T,(p))}$ (for each $v \in \Sigma'$). □

9.6.7.6. Lemma

(i) *If* $v \in \Sigma_o$, *then* $H^0(G_v, V_v^-) = 0$ *and the group* $H^0(G_v, A_v^-)$ *is finite.*

(ii) *If* $v \in \Sigma_t$, *let* $\mathcal{T}_v$ *be the torus over* K_v *associated to* B *and denote by* $t(v)$ *(*$0 \leqslant t(v) \leqslant \dim(B)$*) the dimension of the maximal* K_v*-split subtorus of* $\mathcal{T}_v$. *Then*

$$H^0(G_v, V_v^-) \xrightarrow{\sim} \mathbf{Q}_p^{\oplus t(v)}, \quad H^0(G_v, A_v^-) \xrightarrow{\sim} (\mathbf{Q}_p/\mathbf{Z}_p)^{\oplus t(v)}.$$

(iii) *We have*

$$\mathrm{rk}_{\mathbf{Z}_p} \widetilde{H}^1_f(T) = \mathrm{cork}_{\mathbf{Z}_p} \mathrm{Sel}(B/K, p^\infty) + \sum_{v \in \Sigma_t} t(v).$$

Proof

(i) All eigenvalues of f_v acting on V_v^- have absolute values $(Nv)^{-1/2}$, hence there are no f_v-invariants.

(ii) This follows from the fact that T_v^- is isomorphic, as an G_v/I_v-module, to $X_*(\mathcal{T}_v) \otimes_{\mathbf{Z}} \mathbf{Z}_p$, where $X_*(\mathcal{T}_v)$ is the cocharacter group of the torus $\mathcal{T}_v$. Finally, (iii) follows from (i)–(ii) and the exact sequence from Lemma 9.6.3 (as $V^{G_K} = 0$). □

9.6.7.7. Corollary. — *If* $\dim(B) = 1$, i.e., *if* $B = E$ *is an elliptic curve, then*

$$\mathrm{rk}_{\mathbf{Z}_p} \widetilde{H}^1_f(T) = \mathrm{cork}_{\mathbf{Z}_p} \mathrm{Sel}(E/K, p^\infty) + |\{v \in \Sigma_t \mid E \text{ has split multiplicative reduction at } v\}|.$$

9.6.7.8. Proposition. — *For* K_∞/K *as in 9.6.5,*

$$\mathrm{cork}_\Lambda\, S_A^{\mathrm{str}}(K_\infty) = \mathrm{cork}_\Lambda\, \mathrm{Sel}(B/K_\infty, p^\infty).$$

Proof. — In view of the proof of Proposition 9.6.7.5 (i), it is enough to show that, for each prime $v \mid p$ which is unramified in K_∞/K, the Λ-module

$$N_v = \varprojlim_{\alpha,\mathrm{cor}} \bigoplus_{v_\alpha \mid v} H^0(G_{v_\alpha}, A^*(1)_v^-)/\mathrm{div} \qquad \Big(K_\infty = \bigcup K_\alpha,\ [K_\alpha : K] < \infty\Big)$$

satisfies $\mathrm{rk}_\Lambda(N_v) = 0$. If $\Gamma_v = 0$, then $N_v = \Lambda \otimes_{\mathbf{Z}_p} (H^0(G_v, A^*(1)_v^-)/\mathrm{div})$ is killed by some power of p. If $\Gamma_v \neq 0$, then $\Gamma_v \xrightarrow{\sim} \mathbf{Z}_p$ and N_v is a $\mathbf{Z}_p[\Gamma/\Gamma_v]$-module of finite type, hence $\mathrm{rk}_\Lambda(N_v) = 0$. □

9.7. Duality and perfectness

The notation from 9.6.1 is in force.

9.7.1. For each intermediate field $K \subset L \subset K_\infty$, put $\Gamma^L = \mathrm{Gal}(K_\infty/L)$, $\Gamma_L = \mathrm{Gal}(L/K) = \Gamma/\Gamma^L$, $\Lambda_L = \mathscr{O}[\![\Gamma_L]\!]$. Greenberg's local conditions associated to the data from 9.6.1 define "Selmer complexes"

$$\widetilde{\mathbf{R}\Gamma}_f(K_S/L, Y) \in D^b_{coft}(_{\Lambda_L}\mathrm{Mod}), \quad \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, Z) \in D^b_{ft}(_{\Lambda_L}\mathrm{Mod})$$
$$(Y = A, A^*(1);\ Z = T, T^*(1)),$$

which do not, in fact, depend on S (by Proposition 7.8.8), and whose cohomology groups $\widetilde{H}^i_f(K_S/L, Y)$ (resp., $\widetilde{H}^i_{f,\mathrm{Iw}}(L/K, Z)$) are equal, respectively, to the inductive (resp., projective) limit of $\widetilde{H}^i_f(L', Y)$ (resp., of $\widetilde{H}^i_f(L', Z)$), where L'/K runs through all finite subextensions of L/K (by Proposition 8.8.6).

We also put

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, V) = \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, T) \otimes_{\Lambda_L} (\Lambda_L \otimes \mathbf{Q}) \in D^b_{ft}(_{\Lambda_L \otimes \mathbf{Q}}\mathrm{Mod})$$

(and similarly for $V^*(1)$). These Selmer complexes have the following properties.

9.7.2. Proposition

(i) *There is an isomorphism*

$$D_{\Lambda_L}\big(\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, T)\big) \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_f(K_S/L, A^*(1))^\iota[3].$$

(ii) $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, Z) \in D^{[0,3]}_{\mathrm{parf}}(_{\Lambda_L}\mathrm{Mod})$ *($Z = T, T^*(1)$).*

(iii) *If $[L : K] = \infty$, then $\widetilde{H}^0_{f,\mathrm{Iw}}(L/K, Z) = 0$ ($Z = T, T^*(1)$).*

(iv) *If $\widetilde{H}^0_f(K, A^*(1)) = 0$, then $\widetilde{H}^0_f(K_S/L, A^*(1)) = 0$ and $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, T) \in D^{[0,2]}_{\mathrm{parf}}(_{\Lambda_L}\mathrm{Mod})$.*

(v) *If $\widetilde{H}^0_f(K_S/L, A^*(1))$ is finite, then $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, V) \in D^{[0,2]}_{\mathrm{parf}}(_{\Lambda_L \otimes \mathbf{Q}}\mathrm{Mod})$.*

Proof

(i) See 8.9.6.1.

(ii) By definition, $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,Z)$ is represented by the complex

$$\mathrm{Cone}\,(X_1\oplus X_2\oplus X_2'\longrightarrow X_3)\,[-1],$$

where

$$X_1=C^{\bullet}_{\mathrm{cont}}(G_{K,S},\mathscr{F}_{\Gamma_L}(Z)),\quad X_2=\bigoplus_{v\mid p}C^{\bullet}_{\mathrm{cont}}(G_v,\mathscr{F}_{\Gamma_L}(Z_v^+)),$$
$$X_2'=\bigoplus_{v\in\Sigma'}C^{\bullet}_{\mathrm{cont}}\big(G_v/I_v,\mathscr{F}_{\Gamma_L}\big(Z^{I_v}\big)\big),\quad X_3=\bigoplus_{v\in S_f}C^{\bullet}_{\mathrm{cont}}(G_v,\mathscr{F}_{\Gamma_L}(Z)).$$

As each $\mathscr{O}$-module Z,Z_v^+,Z^{I_v} is free, the corresponding Λ_L-modules $\mathscr{F}_{\Gamma_L}(Z)$, $\mathscr{F}_{\Gamma_L}(Z_v^+)$, $\mathscr{F}_{\Gamma_L}(Z^{I_v})$ are also free. Applying Proposition 4.2.9, we obtain that

$$X_1,X_2,X_3\in D^{[0,2]}_{\mathrm{parf}}\big({}_{\Lambda_L}\mathrm{Mod}\big),\quad X_2'\in D^{[0,1]}_{\mathrm{parf}}\big({}_{\Lambda_L}\mathrm{Mod}\big)$$

(as $\mathrm{cd}_p(G_{K,S})=\mathrm{cd}_p(G_v)=2$, $\mathrm{cd}_p(G_v/I_v)=1$), which proves the claim.

(iii) This follows from the fact that the projective limit of the groups $\widetilde{H}^0_f(L',Z)\subseteq H^0(L',Z)$ vanishes, by Proposition 8.3.5 (iii) (cf. the proof of Lemma 9.1.5 (i)).

(iv) It follows from Proposition 8.10.14 that $\widetilde{H}^0_f(K_S/L,A^*(1))=0$, which in turns implies the vanishing of

$$\widetilde{H}^3_{f,\mathrm{Iw}}(L/K,T)\xrightarrow{\sim}D\big(\widetilde{H}^0_f(K_S/L,A^*(1))\big)^{\iota}=0.$$

As noted in 4.2.8, this is sufficient to prove the claim. The same argument proves (v), as

$$\widetilde{H}^3_{f,\mathrm{Iw}}(L/K,V)\xrightarrow{\sim}D\big(\widetilde{H}^0_f(K_S/L,A^*(1))\big)^{\iota}\otimes\mathbf{Q}=0$$

in this case. □

9.7.3. Proposition. — *Let $K\subset L'\subset L\subset K_\infty$ be arbitrary intermediate fields. Then:*

(i) *There is a canonical isomorphism in $D^b_{ft}({}_{\Lambda_{L'}}\mathrm{Mod})$*

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,Z)\overset{\mathbf{L}}{\otimes}_{\Lambda_L}\Lambda_{L'}\xrightarrow{\sim}\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L'/K,Z)\qquad(Z=T,T^*(1)).$$

(ii) *There are natural pairings in $D^b_{ft}({}_{\Lambda_L}\mathrm{Mod})$*

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T)\overset{\mathbf{L}}{\otimes}_{\Lambda_L}\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T^*(1))^{\iota}\longrightarrow\Lambda_L[-3],$$

compatible with the isomorphisms from (i). Denote by

$$\alpha_T:\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T)\longrightarrow\mathscr{D}_{\Lambda_L}\big(\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T^*(1))\big)^{\iota}[-3]$$

the corresponding adjoint map.

(iii) *The map*

$$\alpha_V=\alpha_T\otimes\mathbf{Q}:\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,V)\longrightarrow\mathscr{D}_{\Lambda_L\otimes\mathbf{Q}}\big(\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,V^*(1))\big)^{\iota}[-3]$$

is an isomorphism in $D^b_{ft}({}_{\Lambda_L\otimes\mathbf{Q}}\mathrm{Mod})$.

(iv) *The following conditions are equivalent:*

$$\alpha_T \textit{ is an isomorphism in } D^b_{ft}({}_{\Lambda_L}\mathrm{Mod}) \iff (\forall v \in \Sigma') \quad \mathrm{Tam}_v(T,(\varpi)) \ (= \mathrm{Tam}_v(T^*(1),(\varpi))) = 0.$$

Proof

(i) Applying Proposition 8.10.10 to $K'_\infty = L, L'$, we obtain canonical isomorphisms

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,Z) \overset{\mathbf{L}}{\otimes}_{\Lambda_L} \Lambda_{L'} \xrightarrow{\sim} \Big(\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K,Z) \overset{\mathbf{L}}{\otimes}_{\Lambda} \Lambda_L\Big) \overset{\mathbf{L}}{\otimes}_{\Lambda_L} \Lambda_{L'} \xrightarrow{\sim} \mathbf{R}\Gamma_{f,\mathrm{Iw}}(K_\infty/K,Z) \overset{\mathbf{L}}{\otimes}_{\Lambda} \Lambda_{L'} \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L'/K,Z).$$

(ii) These are the duality pairings (8.9.6.3.1) (cf. Proposition 8.10.10).

(iii) This is a special case of Theorem 8.9.11.

(iv) Proposition 8.9.7.6 (iv), (v) and Theorem 8.9.12 apply to the extension L/K. In particular,

$$\alpha_T \text{ is an isomorphism in } D^b_{ft}({}_{\Lambda_L}\mathrm{Mod}) \iff (\forall v \in \Sigma') \quad \mathrm{Err}_v(\mathscr{D}_{\Lambda_L}, \mathscr{F}_{\Gamma_L}(T)) = 0 \iff (\forall v \in \Sigma') \quad \mathrm{Err}_v(\mathscr{D}, T) = 0 \iff (\forall v \in \Sigma') \quad \mathrm{Tam}_v(T,(\varpi)) = 0. \qquad \square$$

9.7.4. Proposition

(i) *If* $\widetilde{H}^0_f(K_S/L,A)$ *is finite, then* $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,V) \in D^{[1,3]}_{\mathrm{parf}}({}_{\Lambda_L\otimes\mathbf{Q}}\mathrm{Mod})$.

(ii) *If* $\widetilde{H}^0_f(K_S/L,Y)$ *(*$Y = A, A^*(1)$*) are finite, then* $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,Z) \in D^{[1,2]}_{\mathrm{parf}}({}_{\Lambda_L\otimes\mathbf{Q}}\mathrm{Mod})$ *(*$Z = V, V^*(1)$*).*

Proof

(i) According to Proposition 9.7.2 (v) (applied to A and $V^*(1)$), we have $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,V^*(1)) \in D^{[0,2]}_{\mathrm{parf}}({}_{\Lambda_L\otimes\mathbf{Q}}\mathrm{Mod})$. Applying $\mathscr{D}_{\Lambda_L}[-3]$ and using the duality isomorphism of Proposition 9.7.3 (iii), we obtain the claim.

The statement (ii) follows from (i) and Proposition 9.7.2 (v). $\square$

9.7.5. Proposition. — *Assume that* $(\forall v \in \Sigma')$ $\mathrm{Tam}_v(T,(\varpi)) = 0$*. Then:*

(i) *If* $\widetilde{H}^0_f(K,A) = 0$*, then* $\widetilde{H}^0_f(K_S/L,A) = 0$ *and* $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T) \in D^{[1,3]}_{\mathrm{parf}}({}_{\Lambda_L}\mathrm{Mod})$.

(ii) *If* $\widetilde{H}^0_f(K,Y) = 00$ *(*$Y = A, A^*(1)$*), then* $\widetilde{H}^0_f(K_S/L,Y) = 0$ *and* $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,Z) \in D^{[1,2]}_{\mathrm{parf}}({}_{\Lambda_L}\mathrm{Mod})$ *(*$Z = T, T^*(1)$*).*

Proof. — The proof of 9.7.4 applies, using Proposition 9.7.2 (iv) and 9.7.3 (iv) instead of 9.7.2 (v) and 9.7.3 (iii). $\square$

***9.7.6. Proposition* (Self-dual case)**. — *Assume that there exists an isomorphism of* $F[G_{K,S}]$*-modules* $j : V \xrightarrow{\sim} V^*(1)$ *which is skew-symmetric (*i.e., $j^*(1) = -j$*) and satisfies* $j(V_v^+) = V^*(1)_v^+$ *(*$v \in \Sigma$*). Then:*

(i) *j induces an isomorphism in* $D^b_{ft}({}_{\Lambda_L\otimes\mathbf{Q}}\mathrm{Mod})$

$$j_* : \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,V) \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,V^*(1)).$$

(ii) *The induced pairing*

$$\cup : \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,V) \overset{\mathbf{L}}{\otimes}_{\Lambda_L} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,V)^{\iota}$$
$$\xrightarrow{\mathrm{id}\otimes j_*} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,V) \overset{\mathbf{L}}{\otimes}_{\Lambda_L} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,V^*(1))^{\iota} \longrightarrow \Lambda_L[-3]$$

is skew-Hermitian, i.e., *satisfies* $\cup^{\iota}\circ s_{12} = -\nu^{\iota}\circ\cup$, *in the notation of Corollary 6.6.7.*

(iii) *If* $\widetilde{H}^0_f(K_S/L,A)$ *is finite, then* $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,V) \in D^{[1,2]}_{\mathrm{parf}}({}_{\Lambda_L\otimes\mathbf{Q}}\mathrm{Mod})$.

(iv) *If* $\widetilde{H}^0_f(K_S/L,A)$ *is finite and* $\mathrm{cork}_{\Lambda_L}(\widetilde{H}^1_f(K_S/L,A)) = 0$, *then*

$$(\forall i \neq 2) \quad \widetilde{H}^i_{f,\mathrm{Iw}}(L/K,V) = 0,$$

the $\Lambda_L\otimes\mathbf{Q}$*-module*

$$\widetilde{H}^2_{f,\mathrm{Iw}}(L/K,V) \xrightarrow{\sim} D_{\Lambda_L}\big(\widetilde{H}^1_f(K_S/L,A)\big)^{\iota}\otimes\mathbf{Q}$$

is torsion and $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,V)$ *can be represented by a complex*

$$\mathrm{Cone}\big(M \xrightarrow{u} M\big)[-2],$$

where M *is a free* $\Lambda_L\otimes\mathbf{Q}$*-module of finite type and* u *an injective endomorphism of* M*.*

Proof. — The statement (i) is trivial and (ii) follows from Corollary 6.6.7 (which applies thanks to 7.7.2).

(iii) As $A^*(1)$ (resp., $A^*(1)^+_v$, $v\in\Sigma$) differs from A (resp., from A^+_v) by a finite group, $\widetilde{H}^0_f(K_S/L,A^*(1))$ is also finite, hence (iii) follows from Proposition 9.7.4 (ii).

(iv) By (iii), the $\Lambda_L\otimes\mathbf{Q}$-modules of finite type $H^i := \widetilde{H}^i_{f,\mathrm{Iw}}(L/K,V)$ vanish for $i\neq 1,2$, and H^1 is torsion-free. On the other hand, the duality isomorphisms 9.7.2 (i) and 9.7.3 (iii) imply that

$$\mathrm{rk}_{\Lambda_L\otimes\mathbf{Q}}(H^1) = \mathrm{rk}_{\Lambda_L\otimes\mathbf{Q}}(H^2) = \mathrm{cork}_{\Lambda_L}(\widetilde{H}^1_f(K_S/L,A)) = 0,$$

hence $H^1 = 0$ vanishes and H^2 is a torsion $\Lambda_L\otimes\mathbf{Q}$-module. It follows that $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,V)$ can be represented by a complex $\big[P^1 \xrightarrow{u} P^2\big]$ (in degrees $1,2$), where P^1, P^2 are projective (hence free) $\Lambda_L\otimes\mathbf{Q}$-modules of the same rank (hence isomorphic to each other) and $\mathrm{Ker}(u) = 0$. □

***9.7.7. Proposition* (Integral self-dual case).** — *In the situation of 9.7.6, assume that* $j(T) = T^*(1)$ *(hence* $j(T^+_v) = T^*(1)^+_v$ *for all* $v\in\Sigma$*). Then:*

(i) *j induces an isomorphism in* $D^b_{ft}({}_{\Lambda_L}\mathrm{Mod})$

$$j_* : \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T) \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T^*(1)).$$

(ii) *The induced pairing*

$$\cup : \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T) \overset{\mathbf{L}}{\otimes}_{\Lambda_L} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T)^{\iota}$$
$$\xrightarrow{\mathrm{id}\otimes j_*} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T) \overset{\mathbf{L}}{\otimes}_{\Lambda_L} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T^*(1))^{\iota} \longrightarrow \Lambda_L[-3]$$

is skew-Hermitian.

(iii) *If* $\widetilde{H}^0_f(K,A) = 0$ *and* $(\forall v \in \Sigma')$ $\mathrm{Tam}_v(T,(\varpi)) = 0$, *then* $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T) \in D^{[1,2]}_{\mathrm{parf}}({}_{\Lambda_L}\mathrm{Mod})$.

(iv) *If, under the assumptions of (iii),* $\mathrm{cork}_{\Lambda_L}\big(\widetilde{H}^1_f(K_S/L,A)\big) = 0$, *then*

$$(\forall i \neq 2) \quad \widetilde{H}^i_{f,\mathrm{Iw}}(L/K,T) = 0,$$

the Λ_L*-module*

$$\widetilde{H}^2_{f,\mathrm{Iw}}(L/K,T) \xrightarrow{\sim} D_{\Lambda_L}\big(\widetilde{H}^1_f(K_S/L,A)\big)^{\iota}$$

is torsion and $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T)$ *can be represented by a complex*

$$\mathrm{Cone}\big(M \xrightarrow{u} M\big)[-2],$$

where M *is a free* Λ_L*-module of finite type and* u *an injective endomorphism of* M.

Proof. — The proof of Proposition 9.7.6 applies, using 9.7.5 instead of 9.7.4. □

9.7.8. More general local conditions. — In Sect. 9.6 and 9.7.1–9.7.7, we considered only Greenberg's local conditions for $\Sigma = \{v \mid p\}$. It is often useful to consider Greenberg's local conditions associated to an arbitrary intermediate set $\{v \mid p\} \subset \Sigma \subset S_f$: define, for each $v \in \Sigma^{(p)} := \Sigma - \{v \mid p\}$,

$$(\forall X = T, V, A) \quad X_v^+ = 0, \quad X_v^- = X_v$$
$$(\forall X = T^*(1), V^*(1), A^*(1)) \quad X_v^+ = X_v, \quad X_v^- = 0.$$

We incorporate Σ into the notation by writing

$$\widetilde{\mathbf{R}\Gamma}_{f,\Sigma}(L,X), \quad \widetilde{\mathbf{R}\Gamma}_{f,\Sigma}(K_S/L,Y), \quad \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw},\Sigma}(L/K,Z)$$

for the Selmer complexes associated to such local conditions (and we drop Σ from the notation if $\Sigma = \{v \mid p\}$).

9.7.9. Proposition. — *Assume that, for each finite extension* K' *of* K *contained in* K_∞, *each* $v \in \Sigma^{(p)}$ *and each prime* $v' \mid v$ *of* K', *we have*

$$(*) \qquad H^0(G_{v'},V) = 0 \quad (G_{v'} = \mathrm{Gal}(\overline{K}_v/K'_{v'})).$$

Then, for each intermediate field $K \subset L \subset K_\infty$:

(i) *If* $[L:K] < \infty$, *then the canonical maps*

$$\widetilde{\mathbf{R}\Gamma}_{f,\Sigma}(L,V) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(L,V),$$
$$\widetilde{\mathbf{R}\Gamma}_f(L,V^*(1)) \longrightarrow \widetilde{\mathbf{R}\Gamma}_{f,\Sigma}(L,V^*(1))$$

are isomorphisms in $D^b_{ft}({}_F\mathrm{Mod})$.

(ii) *If* $\overline{\mathfrak{p}} \in \operatorname{Spec}(\Lambda_L \otimes \mathbf{Q})$ *is contained in the augmentation ideal of* $\Lambda_L \otimes \mathbf{Q}$*, then the canonical maps*

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw},\Sigma}(L/K,V)_{\overline{\mathfrak{p}}} \longrightarrow \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,V)_{\overline{\mathfrak{p}}},$$
$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,V^*(1))_{\overline{\mathfrak{p}}} \longrightarrow \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw},\Sigma}(L/K,V^*(1))_{\overline{\mathfrak{p}}}$$

are isomorphisms in $D^b_{ft}({}_{\Lambda_L\otimes\mathbf{Q}}\mathrm{Mod})$.

(iii) *For each* $j \in \mathbf{Z}$ *and each minimal prime ideal* $\overline{\mathfrak{q}} \in \operatorname{Spec}(\Lambda_L)$*, the ranks*

$$\begin{aligned}\mathrm{rk}_{(\Lambda_L)_{\overline{\mathfrak{q}}}} \widetilde{H}^j_{f,\mathrm{Iw},\Sigma}(L/K,T)_{\overline{\mathfrak{q}}} &= \mathrm{rk}_{(\Lambda_L)_{\overline{\mathfrak{q}}}}\big(\widetilde{H}^{3-j}_{f,\mathrm{Iw},\Sigma}(L/K,T^*(1))^{\iota}\big)_{\overline{\mathfrak{q}}} \\ &= \mathrm{rk}_{(\Lambda_L)_{\overline{\mathfrak{q}}}}\big(D_{\Lambda_L}\big(\widetilde{H}^j_{f,\Sigma}(K_S/L,A)\big)\big)_{\overline{\mathfrak{q}}} = \mathrm{rk}_{(\Lambda_L)_{\overline{\mathfrak{q}}}}\big(D_{\Lambda_L}\big(\widetilde{H}^{3-j}_{f,\Sigma}(K_S/L,A^*(1))\big)^{\iota}\big)_{\overline{\mathfrak{q}}}\end{aligned}$$

do not depend on Σ *(and vanish for* $j \neq 1,2$*).*

Proof

(i) There are exact triangles in $D^b_{ft}({}_F\mathrm{Mod})$

$$\widetilde{\mathbf{R}\Gamma}_{f,\Sigma}(L,V) \longrightarrow \widetilde{\mathbf{R}\Gamma}_{f}(L,V) \longrightarrow \bigoplus_{v\in\Sigma^{(p)}}\bigoplus_{w\mid v}\mathbf{R}\Gamma_{\mathrm{ur}}(G_w,V)$$

$$\widetilde{\mathbf{R}\Gamma}_{f}(L,V^*(1)) \longrightarrow \widetilde{\mathbf{R}\Gamma}_{f,\Sigma}(L,V^*(1)) \longrightarrow \bigoplus_{v\in\Sigma^{(p)}}\mathscr{D}_F\left(\bigoplus_{w\mid v}\mathbf{R}\Gamma_{\mathrm{ur}}(G_w,V)\right)^{\iota}[-2],$$

where $G_w = \mathrm{Gal}(\overline{K}_v/L_w)$ and $\mathbf{R}\Gamma_{\mathrm{ur}}(G_w,V) = \mathbf{R}\Gamma_{\mathrm{cont}}(G_w/I_w,V^{I_w})$. As $\dim H^1_{\mathrm{ur}}(G_w,V) = \dim H^0(G_w,V)$, the assumption (*) for $K' = L$ implies that $\mathbf{R}\Gamma_{\mathrm{ur}}(G_w,V) \xrightarrow{\sim} 0$.

(ii) There are exact triangles in $D^b_{ft}({}_{\Lambda_L\otimes\mathbf{Q}}\mathrm{Mod})$

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw},\Sigma}(L/K,V) \longrightarrow \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,V) \longrightarrow \bigoplus_{v\in\Sigma^{(p)}}\mathbf{R}\Gamma_{\mathrm{ur}}(G_v,\mathscr{F}_\Gamma(V))$$

$$\begin{aligned}\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,V^*(1)) \longrightarrow \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw},\Sigma}(L/K,V^*(1))& \\ \longrightarrow \bigoplus_{v\in\Sigma^{(p)}}\mathscr{D}_{\Lambda_L\otimes\mathbf{Q}}\big(\mathbf{R}\Gamma_{\mathrm{ur}}(G_v,\mathscr{F}_\Gamma(V))\big)^{\iota}[-2].&\end{aligned}$$

Write $\Gamma_L = \mathrm{Gal}(L/K)$ as a product $\Gamma_L = \Gamma_{L,0}\times\Delta_L$, where $|\Delta_L| < \infty$ and $\Gamma_{L,0} \xrightarrow{\sim} \mathbf{Z}_p^r$. Applying (the proof of) Proposition 8.9.7.7(ii) with $R = \mathscr{O}$, $\Gamma_0 = \Gamma_{L,0}$, $\Delta = \Delta_L$, $W_v = T$ and all minimal prime ideals $\mathfrak{q}_0 \in \operatorname{Spec}(\Delta_L)$, and using the assumption (*) with $K' = L^{\Gamma_{L,0}}$, we obtain

$$\big(\forall v \in \Sigma^{(p)}\big) \quad \mathbf{R}\Gamma_{\mathrm{ur}}(G_v,\mathscr{F}_\Gamma(V))_{\overline{\mathfrak{p}}} = 0.$$

(iii) This follows from (ii), 8.9.6.1 and (8.9.6.4.2). □

9.7.10. Let B be an abelian variety defined over K, which has good reduction outside S_f, and such that $\{v \mid p\} = \Sigma_o \cup \Sigma_t$ in the notation of 9.6.7.2. The assumption of Proposition 9.7.9 are then satisfied for $T = T_p(B)$ ($V = V_p(B) \xrightarrow{\sim} V^*(1)$), since V_v satisfies the monodromy-weight conjecture for each $v \nmid p$ (as remarked in the proof of Lemma 9.6.7.3 (ii)).

9.7.11. In the situation 9.7.7 (iv), it was proved in [**M-R2**, Prop. 6.5] that the duality isomorphism α_T can be represented by a skew-Hermitian pairing on the module M, provided that $p \neq 2$ and M is "minimal" in a suitable sense.

CHAPTER 10

GENERALIZED CASSELS-TATE PAIRINGS

Let K and S be as in 5.1. In this chapter we reformulate the duality results from Chapters 6 and 8 in terms of generalized Cassels-Tate pairings. The general mechanism is very simple: in the situation of 6.2.5 (B) with $n = 0$, the discussion in 6.3.5 gives homomorphisms (under suitable finiteness assumptions on the local conditions)

$$\widetilde{H}^i_f(X_1)_{R\text{-tors}} \longrightarrow \mathrm{Ext}^1_R\big(\widetilde{H}^j_f(X_2), \omega_R\big) \qquad (i+j=4)$$

with controlled[(1)] kernels and cokernels. Observing that, for every R-module M of finite type, the group $\mathrm{Ext}^1_R(M, \omega_R)$ is very close[(1)] to $\mathrm{Hom}_R(M_{R\text{-tors}}, H^0(\omega_R) \otimes_R (\mathrm{Frac}(R)/R))$, we obtain bilinear forms

$$\widetilde{H}^i_f(X_1)_{R\text{-tors}} \otimes_R \widetilde{H}^j_f(X_2)_{R\text{-tors}} \longrightarrow H^0(\omega_R) \otimes_R (\mathrm{Frac}(R)/R) \qquad (i+j=4)$$

in $({}_R\mathrm{Mod})/(\text{pseudo-null})$ (sometimes even in $({}_R\mathrm{Mod})$) with controlled kernels. These pairings – at least for $i = j = 2$ – are natural generalizations of the Cassels-Tate (and Flach [**Fl1**]) pairings in our context. In the self-dual case, we obtain skew-symmetric (or skew-Hermitian) pairings, from which we deduce various parity results (Sect. 10.6–10.7).

10.1. A topological analogue

10.1.1. Assume that X is a connected compact oriented topological manifold of dimension D. The cohomology groups $H^i(X, \mathbf{Z})$ are finitely generated abelian groups; the exact sequences

$$0 \longrightarrow H^i(X, \mathbf{Z}) \otimes \mathbf{Z}/n\mathbf{Z} \longrightarrow H^i(X, \mathbf{Z}/n\mathbf{Z}) \longrightarrow H^{i+1}(X, \mathbf{Z})[n] \longrightarrow 0$$

give, in the limit,

$$(10.1.1.1) \qquad 0 \longrightarrow H^i(X, \mathbf{Z}) \otimes \mathbf{Q}/\mathbf{Z} \longrightarrow H^i(X, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^{i+1}(X, \mathbf{Z})_{\mathrm{tors}} \longrightarrow 0$$

(1) At least if R has no embedded primes

and

$$H^i(X, \widehat{\mathbf{Z}}) := \varprojlim_n H^i(X, \mathbf{Z}/n\mathbf{Z}) \xrightarrow{\sim} H^i(X, \mathbf{Z}) \otimes \widehat{\mathbf{Z}}.$$

For $m|n$, Poincaré duality and the orientation $H^D(X, A) \xrightarrow{\sim} A$ give perfect pairings

$$\begin{array}{ccccc} H^i(X, \mathbf{Z}/m\mathbf{Z}) & \times & H^{D-i}(X, \mathbf{Z}/m\mathbf{Z}) & \longrightarrow & \mathbf{Z}/m\mathbf{Z} \\ \downarrow & & \uparrow & & \downarrow \\ H^i(X, \mathbf{Z}/n\mathbf{Z}) & \times & H^{D-i}(X, \mathbf{Z}/n\mathbf{Z}) & \longrightarrow & \mathbf{Z}/n\mathbf{Z}, \end{array}$$

which induce in the limit Pontrjagin duality

$$H^i(X, \mathbf{Q}/\mathbf{Z}) \times H^{D-i}(X, \widehat{\mathbf{Z}}) \longrightarrow \mathbf{Q}/\mathbf{Z}.$$

The orthogonal complement of the maximal divisible subgroup

$$H^i(X, \mathbf{Q}/\mathbf{Z})_{\text{div}} = H^i(X, \mathbf{Z}) \otimes \mathbf{Q}/\mathbf{Z} \subset H^i(X, \mathbf{Q}/\mathbf{Z})$$

is the torsion subgroup

$$H^{D-i}(X, \widehat{\mathbf{Z}})_{\mathbf{Z}\text{-tors}} = H^{D-i}(X, \mathbf{Z})_{\mathbf{Z}\text{-tors}}.$$

As a consequence, we obtain perfect pairings of finite abelian groups

$$\langle\ ,\ \rangle_{i+1,D-i} : H^{i+1}(X, \mathbf{Z})_{\text{tors}} \times H^{D-i}(X, \mathbf{Z})_{\text{tors}} \longrightarrow \mathbf{Q}/\mathbf{Z}.$$

More precisely, Poincaré duality identifies (10.1.1.1) with the Pontrjagin dual of

$$0 \longleftarrow \big(H^{D-i}(X, \mathbf{Z})/\text{tors}\big) \otimes \widehat{\mathbf{Z}} \longleftarrow H^{D-i}(X, \widehat{\mathbf{Z}}) \longleftarrow H^{D-i}(X, \mathbf{Z})_{\text{tors}} \longleftarrow 0.$$

10.1.2. The pairing $\langle a, b\rangle_{i+1,D-i}$ can be described in terms of cocycles as follows. Represent a (resp., b) by a (singular) cocycle $\alpha \in Z^{i+1}(X, \mathbf{Z})$ (resp., $\beta \in Z^{D-i}(X, \mathbf{Z})$). There exist $n \geqslant 1$ and $\alpha' \in C^i(X, \mathbf{Z})$ (resp., $\beta' \in C^{D-i-1}(X, \mathbf{Z})$) such that $n\alpha = d\alpha'$, $n\beta = d\beta'$. Then

$$\alpha' \cup \beta, \quad \beta' \cup \alpha \in C^D(X, \mathbf{Z})$$
$$d(\alpha' \cup \beta) = n(\alpha \cup \beta), \quad d(\beta' \cup \alpha) = n(\beta \cup \alpha) \in C^D(X, n\mathbf{Z}).$$

This means that

$$\frac{1}{n}(\alpha' \cup \beta) \ (\operatorname{mod} C^D(X, \mathbf{Z})), \quad \frac{1}{n}(\beta' \cup \alpha) \ (\operatorname{mod} C^D(X, \mathbf{Z}))$$

are elements of $Z^D(X, \frac{1}{n}\mathbf{Z}/\mathbf{Z})$; their cohomology classes in

$$H^D\left(X, \frac{1}{n}\mathbf{Z}/\mathbf{Z}\right) \xrightarrow{\sim} \frac{1}{n}\mathbf{Z}/\mathbf{Z} \subset \mathbf{Q}/\mathbf{Z}$$

are equal to $\langle a, b\rangle_{i+1,D-i}$ and $\langle b, a\rangle_{D-i,i+1}$, respectively.

10.1.3. As

$$d(\beta'\cup\alpha') = n(\beta\cup\alpha')+(-1)^{D-i-1}n(\beta'\cup\alpha) = (-1)^{i(D-i)}n(\alpha'\cup\beta)+(-1)^{D-i-1}n(\beta'\cup\alpha)$$

is a coboundary in $C^D(X,\mathbf{Z})$, we have

$$(-1)^{i(D-i)}\langle a,b\rangle_{i+1,D-i} + (-1)^{D-i-1}\langle b,a\rangle_{D-i,i+1} = 0,$$

hence

$$\langle b,a\rangle_{D-i,i+1} = (-1)^{(D-i)(i+1)}\langle a,b\rangle_{i+1,D-i} = (-1)^{D(i+1)}\langle a,b\rangle_{i+1,D-i}. \tag{10.1.3.1}$$

In particular, if $D = 2n-1$, then the pairing

$$\langle\ ,\ \rangle_{n,n} : H^n(X,\mathbf{Z})_{\mathrm{tors}} \times H^n(X,\mathbf{Z})_{\mathrm{tors}} \longrightarrow \mathbf{Q}/\mathbf{Z}$$

satisfies

$$\langle b,a\rangle_{n,n} = (-1)^n\langle a,b\rangle_{n,n}.$$

10.1.4. The pairings $\langle\ ,\ \rangle_{i,j}$ can be described in a more abstract way in terms of a composition of two cup products in the derived category.

For every complex $A^\bullet$ of abelian groups put

$$\mathbf{R}\Gamma_!(A^\bullet) = A^\bullet \otimes_{\mathbf{Z}} \Big[\mathbf{Z}\xrightarrow{-\mathrm{incl}}\mathbf{Q}\Big], \quad H^i_!(A^\bullet) = H^i(\mathbf{R}\Gamma_!(A^\bullet)),$$

where the complex $[\mathbf{Z}\to\mathbf{Q}]$ is in degrees $0,1$. This defines an exact functor

$$\mathbf{R}\Gamma_! : D^*({}_{\mathbf{Z}}\mathrm{Mod}) \longrightarrow D^*({}_{\mathbf{Z}}\mathrm{Mod}) \qquad (* = \varnothing,+,-,b).$$

The cohomology sequence associated to the exact sequence of complexes

$$0 \longrightarrow (A^\bullet\otimes\mathbf{Q})[-1] \longrightarrow \mathbf{R}\Gamma_!(A^\bullet) \longrightarrow A^\bullet \longrightarrow 0$$

gives

$$0 \longrightarrow H^{i-1}(A^\bullet)\otimes\mathbf{Q}/\mathbf{Z} \longrightarrow H^i_!(A^\bullet) \longrightarrow H^i(A^\bullet)_{\mathrm{tors}} \longrightarrow 0.$$

For $A,B\in D^-({}_{\mathbf{Z}}\mathrm{Mod})$, the same construction as in 2.10.7–2.10.8 defines a canonical product

$$\mathbf{R}\Gamma_!(A)\overset{\mathbf{L}}{\otimes}_{\mathbf{Z}}\mathbf{R}\Gamma_!(B) \longrightarrow \mathbf{R}\Gamma_!\Big(A\overset{\mathbf{L}}{\otimes}_{\mathbf{Z}}B\Big).$$

The induced cup products

$$H^i_!(A)\otimes_{\mathbf{Z}} H^j_!(B) \longrightarrow H^{i+j}_!\Big(A\overset{\mathbf{L}}{\otimes}_{\mathbf{Z}}B\Big)$$

factor through

$$H^i(A)_{\mathrm{tors}}\otimes_{\mathbf{Z}} H^j(B)_{\mathrm{tors}} \longrightarrow H^{i+j}_!\Big(A\overset{\mathbf{L}}{\otimes}_{\mathbf{Z}}B\Big).$$

10.1.5. Applying this construction to $A = B = \mathbf{R}\Gamma(X,\mathbf{Z})$ and the truncated cup product

$$\mathbf{R}\Gamma(X,\mathbf{Z})\overset{\mathbf{L}}{\otimes}_{\mathbf{Z}}\mathbf{R}\Gamma(X,\mathbf{Z}) \longrightarrow \tau_{\geqslant D}\,\mathbf{R}\Gamma(X,\mathbf{Z}) \xrightarrow{\ \sim\ } \mathbf{Z}[-D],$$

we obtain products

$$\mathbf{R}\Gamma_!(\mathbf{R}\Gamma(X,\mathbf{Z}))\overset{\mathbf{L}}{\otimes}_{\mathbf{Z}}\mathbf{R}\Gamma_!(\mathbf{R}\Gamma(X,\mathbf{Z})) \longrightarrow \mathbf{R}\Gamma_!(\mathbf{Z}[-D]) \xrightarrow{\ \sim\ } \mathbf{Q}/\mathbf{Z}[-D-1] \tag{10.1.5.1}$$

and

$$\cup_{i+1,D-i} : H^{i+1}(X,\mathbf{Z})_{\mathrm{tors}} \otimes_{\mathbf{Z}} H^{D-i}(X,\mathbf{Z})_{\mathrm{tors}} \longrightarrow H_!^{D+1}(\mathbf{Z}[-D]) = \mathbf{Q}/\mathbf{Z}.$$

Alternatively, there are canonical isomorphisms in $D({}_{\mathbf{Z}}\mathrm{Mod})$

$$\mathbf{R}\Gamma_!(\mathbf{R}\Gamma(X,\mathbf{Z})) \xrightarrow{\ \sim\ } \mathbf{R}\Gamma(X,[\mathbf{Z}\longrightarrow\mathbf{Q}]) \xrightarrow{\ \sim\ } \mathbf{R}\Gamma(X,\mathbf{Q}/\mathbf{Z}[-1]),$$

and the product (10.1.5.1) is induced by

$$\mathbf{Q}/\mathbf{Z}[-1]\overset{\mathbf{L}}{\otimes}_{\mathbf{Z}}\mathbf{Q}/\mathbf{Z}[-1] \xrightarrow{\ \sim\ } \mathbf{Q}/\mathbf{Z}[-1]$$

and

$$\mathbf{R}\Gamma(X,\mathbf{Q}/\mathbf{Z}[-1])\overset{\mathbf{L}}{\otimes}_{\mathbf{Z}}\mathbf{R}\Gamma(X,\mathbf{Q}/\mathbf{Z}[-1]) \longrightarrow \tau_{\geqslant D+1}\,\mathbf{R}\Gamma(X,\mathbf{Q}/\mathbf{Z}[-1]) \xrightarrow{\ \sim\ } \mathbf{Q}/\mathbf{Z}[-D-1].$$

10.1.6. Lemma. — *The pairings*

$$\langle\ ,\ \rangle_{i+1,D-i},\ \cup_{i+1,D-i} : H^{i+1}(X,\mathbf{Z})_{\mathrm{tors}} \otimes_{\mathbf{Z}} H^{D-i}(X,\mathbf{Z})_{\mathrm{tors}} \longrightarrow \mathbf{Q}/\mathbf{Z}$$

from 10.1.2 (resp., 10.1.5) are related by $\cup = (-1)^{D-1}\langle\ ,\ \rangle$.

Proof. — In the notation of 10.1.2,

$$\widetilde{\alpha} = \alpha\otimes 1 + (-1)^{i+1}\alpha'\otimes\frac{1}{n}, \quad \widetilde{\beta} = \beta\otimes 1 + (-1)^{D-i}\beta'\otimes\frac{1}{n}$$

are cocycles of degrees $i+1$ (resp., $D-i$) in $\mathbf{R}\Gamma_!(C^\bullet(X,\mathbf{Z}))$ lifting α (resp., β). Using the notation from Lemma 2.10.7 (i), the cup product $a\cup_{i+1,D-i} b$ is represented by

$$\begin{aligned} v\circ s_{23}(\widetilde{\alpha}\otimes\widetilde{\beta}) = v\Big(&\alpha\otimes\beta\otimes 1\otimes 1 + (-1)^{D-i}\alpha\otimes\beta'\otimes 1\otimes\frac{1}{n} + \\ &(-1)^{D+1}\alpha'\otimes\beta\otimes\frac{1}{n}\otimes 1 + (-1)^{i}\alpha'\otimes\beta'\otimes\frac{1}{n}\otimes\frac{1}{n}\Big) \\ &= (\alpha\cup\beta)\otimes 1 + (-1)^{D-i}(\alpha\cup\beta')\otimes\frac{1}{n}. \end{aligned}$$

As $\alpha\cup\beta\in Z^{D+1}(X,\mathbf{Z}) = dC^D(X,\mathbf{Z})$, it follows that $a\cup_{i+1,D-i} b$ is represented by

$$(-1)^{D-i}\frac{1}{n}(\alpha\cup\beta') \pmod{C^D(X,\mathbf{Z})},$$

hence

$$\begin{aligned} a\cup_{i+1,D-i} b &= (-1)^{D-i}(-1)^{(i+1)(D-i-1)}\langle b,a\rangle_{D-i,i+1} \\ &= (-1)^{D-i}(-1)^{(i+1)(D-i-1)}(-1)^{(i+1)(D-i)}\langle a,b\rangle_{i+1,D-i} \\ &= (-1)^{D-1}\langle a,b\rangle_{i+1,D-i}. \end{aligned}$$

□

10.2. Abstract Cassels-Tate pairings

In this section we construct generalized Cassels-Tate pairings, mimicking the discussion in 10.1.4. Throughout 10.2, we assume that R has no embedded primes ($\Longleftrightarrow$ R satisfies (S_1)).

10.2.1. Assume that $J = \omega_R^\bullet[n]$ for some $n \in \mathbf{Z}$ (hence $D_J = \mathscr{D}_n$). Let X_1, X_2 be bounded complexes of admissible $R[G_{K,S}]$-modules with cohomology of finite type over R, and

$$\pi : X_1 \otimes_R X_2 \longrightarrow J(1)$$

a morphism of complexes of $R[G_{K,S}]$-modules. Finally, assume that we are given orthogonal local conditions

$$\Delta(X_1) \perp_{\pi,h_S} \Delta(X_2)$$

such that the complexes $U_S^+(X_i)$ $(i = 1, 2)$ have cohomology of finite type over R.

10.2.2. Under the assumptions of 10.2.1, the Selmer complexes $\widetilde{C}_f^\bullet(X_i) = \widetilde{C}_f^\bullet(G_{K,S}, X_i; \Delta(X_i))$ $(i = 1, 2)$ also have cohomology of finite type over R. Recall from 6.3.1 the cup products

$$\cup_{\pi,r,h} : \widetilde{C}_f^\bullet(X_1) \otimes_R \widetilde{C}_f^\bullet(X_2) \longrightarrow J[-3] = \omega_R^\bullet[n-3] \qquad (r \in R)$$

and their adjoints

$$\gamma_{\pi,r,h_S} = \mathrm{adj}(\cup_{\pi,r,h}) : \widetilde{C}_f^\bullet(X_1) \longrightarrow \mathrm{Hom}_R^\bullet\big(\widetilde{C}_f^\bullet(X_2), \omega_R^\bullet[n-3]\big).$$

The construction from 2.10.14 applied to $\cup_{\pi,r,h}$ defines cup products (independent of $r \in R$)

$$\cup_{\pi,h_S,i,j} : \widetilde{H}_f^i(X_1)_{R\text{-tors}} \otimes_R \widetilde{H}_f^j(X_2)_{R\text{-tors}} \longrightarrow H^0(\omega_R) \otimes_R (\mathrm{Frac}(R)/R) \quad (i+j = 4-n)$$

in $({}_R\mathrm{Mod})/(\text{pseudo-null})$ (or even in $({}_R\mathrm{Mod})$, if R is Cohen-Macaulay). Furthermore, we obtain from 2.10.15–2.10.17 isomorphisms in $({}_R\mathrm{Mod})/(\text{pseudo-null})$

$$\begin{aligned}\mathrm{Hom}_R\big(\widetilde{H}_f^j(X_2)_{R\text{-tors}}, H^0(\omega_R) \otimes_R (\mathrm{Frac}(R)/R)\big) &\xrightarrow{\sim} \mathbb{E}\mathrm{xt}_R^1\big(\widetilde{H}_f^j(X_2), \omega_R\big) \\ &\xrightarrow{\sim} H^i\big(D_{J[-3]}\big(\widetilde{\mathbf{R}\Gamma}_f(X_2)\big)\big)_{R\text{-tors}}, \qquad (i+j = 4-n)\end{aligned}$$

the composition of which with

$$\begin{aligned}\mathrm{adj}(\cup_{\pi,h_S,i,j}) &: \widetilde{H}_f^i(X_1)_{R\text{-tors}} \\ &\longrightarrow \mathrm{Hom}_R\big(\widetilde{H}_f^j(X_2)_{R\text{-tors}}, H^0(\omega_R) \otimes_R (\mathrm{Frac}(R)/R)\big) \qquad (i+j = 4-n)\end{aligned}$$

coincides, up to a sign, with the restriction of the map

$$(\gamma_{\pi,r,h_S})_* : \widetilde{H}_f^i(X_1) \longrightarrow H^i\big(D_{J[-3]}\big(\widetilde{\mathbf{R}\Gamma}_f(X_2)\big)\big).$$

10.2.3. Theorem. — *Under the assumptions of 10.2.1, let* $i + j = 4 - n$ *and* $\mathrm{Err} = \mathrm{Err}(\Delta(X_1), \Delta(X_2), \pi)$. *Assume that* π *is a perfect duality in the sense of 6.2.6. Then, in the category* $({}_R\mathrm{Mod})/(\textit{pseudo-null})$,

(i) $\mathrm{Ker}(\mathrm{adj}(\cup_{\pi,h_S,i,j}))$ *is isomorphic to a subquotient of* $H^{i-1}(\mathrm{Err})$.

(ii) *If* $H^{i-1}(\mathrm{Err})$ *is* R*-torsion, then* $\mathrm{Coker}(\mathrm{adj}(\cup_{\pi,h_S,i,j}))$ *(resp.,* $\mathrm{Ker}(\mathrm{adj}(\cup_{\pi,h_S,i,j}))$*) is isomorphic to a subobject (resp., a quotient) of* $H^i(\mathrm{Err})$ *(resp., of* $H^{i-1}(\mathrm{Err})$*).*

In particular, if $H^{i-1}(\mathrm{Err}) = H^i(\mathrm{Err}) \xrightarrow{\sim} 0$ *in* $({}_R\mathrm{Mod})/(\textit{pseudo-null})$*, then* $\mathrm{adj}(\cup_{\pi,h_S,i,j})$ *is an isomorphism (again in* $({}_R\mathrm{Mod})/(\textit{pseudo-null})$*).*

Proof. — This follows from Proposition 2.10.17 applied to $\cup_{\pi,r,h}$ and Theorem 6.3.4. □

10.2.4. Proposition. — *Assume that, in addition to 10.2.1, the local conditions* $\Delta(X_i)$ $(i = 1, 2)$ *admit transposition data 6.5.3.1–6.5.3.5. Then the cup products associated to* π *and* $\pi \circ s_{12} : X_2 \otimes_R X_1 \to X_1 \otimes_R X_2 \to J(1)$ *are related by*

$$x \cup_{\pi,h_S,i,j} y = (-1)^{ij}\, y \cup_{\pi\circ s_{12},h'_S,j,i} x \qquad (i + j = 4 - n).$$

Proof. — This follows from Corollary 6.5.5 and (2.10.14.1). □

***10.2.5. Proposition* (Self-dual case)**. — *Assume that, in 10.2.1,* $X_i = X$, $\Delta(X_i) = \Delta(X)$ $(i = 1, 2)$, $\pi' := \pi \circ s_{12} = c \cdot \pi$ *with* $c = \pm 1$ *and* $\Delta(X)$ *admit transposition operators as in Proposition 6.6.2. Then*

$$x \cup_{\pi,h_S,i,j} y = c(-1)^{ij}\, y \cup_{\pi,h_S,j,i} x \qquad (i + j = 4 - n).$$

In particular, if $n = 0$, *then the bilinear form*

$$\cup_{\pi,h_S,2,2} : \widetilde{H}^2_f(X)_{R\text{-tors}} \otimes_R \widetilde{H}^2_f(X)_{R\text{-tors}} \longrightarrow H^0(\omega_R) \otimes_R (\mathrm{Frac}(R)/R)$$

is symmetric (resp., skew-symmetric) if $c = +1$ *(resp.,* $c = -1$*).*

Proof. — Combine Proposition 6.6.2 and Proposition 10.2.4. □

10.2.6. In the case $n = 0$, the formula

$$y \cup_{\pi,h_S,j,i} x = c(-1)^i\, x \cup_{\pi,h_S,i,j} y \qquad (i + j = 4)$$

in Proposition 10.2.5 should be compared to (10.1.3.1) for $D = 3$:

$$\langle b, a\rangle_{j,i} = (-1)^i \langle a, b\rangle_{i,j} \quad (i + j = 4).$$

The extra factor c comes from the fact that X is identified with its dual $D_J(X)(1) = \mathscr{D}(X)(1)$ by using the bilinear form π of parity c.

10.2.7. Hermitian case

10.2.7.1. Assume that R is equipped with an involution $\iota : R \to R$, and that $\nu : J^\iota \to J$ is as in 6.6.4; denote by

$$\nu_* : H^0(\omega_R)^\iota \longrightarrow H^0(\omega_R)$$

the map induced by $\nu[-n] : J^\iota[-n] = (\omega_R^\bullet)^\iota \to J[-n]$.

Assume that we are in the situation of 6.6.5, *i.e.*,

$$(X_1, \Delta(X_1)) = (X, \Delta(X)), \quad (X_2, \Delta(X_2)) = (X^\iota, \Delta(X)^\iota), \quad \Delta(X) \perp_{\pi,h_S} \Delta(X)^\iota$$

and

$$\pi \circ s_{12} = c \cdot (\nu \circ \pi^\iota), \quad c = \pm 1.$$

It is sometimes more convenient to view the cup products from 10.2.2

$$\cup_{\pi,h_S,i,j} : \widetilde{H}^i_f(X)_{R\text{-tors}} \otimes_R \widetilde{H}^j_f(X)^\iota_{R\text{-tors}} \longrightarrow H^0(\omega_R) \otimes_R (\mathrm{Frac}(R)/R) \qquad (i+j = 4-n)$$

as Hermitian pairings

$$\langle\ ,\ \rangle_{\pi,h_S,i,j} : \widetilde{H}^i_f(X)_{R\text{-tors}} \times \widetilde{H}^j_f(X)_{R\text{-tors}} \longrightarrow H^0(\omega_R) \otimes_R (\mathrm{Frac}(R)/R) \qquad (i+j = 4-n)$$

satisfying

$$\langle \alpha x, \beta y\rangle_{\pi,h_S,i,j} = \alpha\, \iota(\beta)\, \langle x, y\rangle_{\pi,h_S,i,j} \qquad (\alpha, \beta \in R).$$

10.2.7.2. Proposition. — *Assume that X is as in 10.2.7.1 and admits transposition operators as in Proposition 6.6.6. Then*

$$\langle x, y\rangle_{\pi,h_S,i,j} = c(-1)^{ij}\, (\nu_* \otimes \iota)\big(\langle y, x\rangle_{\pi,h_S,j,i}\big) \qquad (i+j = 4-n).$$

In particular, if $n = 0$, then the Hermitian form

$$\langle\ ,\ \rangle = \langle\ ,\ \rangle_{\pi,h_S,2,2} : \widetilde{H}^2_f(X)_{R\text{-tors}} \times \widetilde{H}^2_f(X)_{R\text{-tors}} \longrightarrow H^0(\omega_R) \otimes_R (\mathrm{Frac}(R)/R)$$

satisfies

$$\langle \alpha x, \beta y\rangle = \alpha\, \iota(\beta)\, \langle x, y\rangle, \quad \langle x, y\rangle = c \cdot (\nu_* \otimes \iota)(\langle y, x\rangle) \qquad (\alpha, \beta \in R).$$

Proof. — Combine Proposition 6.6.6 and (2.10.14.1). □

10.2.8. Duality of error terms. — For $v \nmid p$, let X, Y be bounded complexes of admissible $R[G_v]$-modules with cohomology of finite type over R and $\pi : X \otimes_R Y \to \omega_R^\bullet(1) = \sigma_{\geqslant 0}\, \omega_R^\bullet(1)$ a morphism of complexes of $R[G_v]$-modules. Assume that $\mathfrak{p} \in \mathrm{Spec}(R)$ satisfies $\dim(R_\mathfrak{p}) = \mathrm{depth}(R_\mathfrak{p}) = 1$ and $\pi_\mathfrak{p} : X_\mathfrak{p} \otimes_{R_\mathfrak{p}} Y_\mathfrak{p} \to (\omega_R^\bullet)_\mathfrak{p}(1)$ is a perfect duality over $R_\mathfrak{p}$. By Corollary 7.6.8 (i), there are isomorphisms in $D_{ft}({}_{R_\mathfrak{p}}\mathrm{Mod})$

$$\mathrm{Err}_v(\Delta_v^{\mathrm{ur}}(X), \Delta_v^{\mathrm{ur}}(Y), \pi)_\mathfrak{p} \xrightarrow{\ \sim\ } \Big[H^0_{\{\mathfrak{p}\}}(H^1_{\mathrm{cont}}(I_v, X)_\mathfrak{p}) \xrightarrow{f_v - 1} H^0_{\{\mathfrak{p}\}}(H^1_{\mathrm{cont}}(I_v, X)_\mathfrak{p})\Big]$$

$$\mathrm{Err}_v(\Delta_v^{\mathrm{ur}}(Y), \Delta_v^{\mathrm{ur}}(X), \pi \circ s_{12})_\mathfrak{p} \xrightarrow{\ \sim\ } \Big[H^0_{\{\mathfrak{p}\}}(H^1_{\mathrm{cont}}(I_v, Y)_\mathfrak{p}) \xrightarrow{f_v - 1} H^0_{\{\mathfrak{p}\}}(H^1_{\mathrm{cont}}(I_v, Y)_\mathfrak{p})\Big],$$

where the complexes on the R.H.S. are in degrees 1, 2. The duality theory for I_v described in 7.5.8 comes from the cup product

$$(10.2.8.1)\quad C^\bullet_{\mathrm{cont}}(I_v, X)\otimes_R C^\bullet_{\mathrm{cont}}(I_v, Y) \longrightarrow C^\bullet_{\mathrm{cont}}(I_v, \omega^\bullet_R(1)) \longrightarrow \tau^{\mathrm{II}}_{\geqslant 1}\, C^\bullet_{\mathrm{cont}}(I_v, \omega^\bullet_R(1)) \xrightarrow{\mathrm{Qis}} \omega^\bullet_R[-1].$$

Localizing (10.2.8.1) at $\mathfrak{p}$, the construction in 2.10.7–2.10.9 gives rise to products

$$\cup_{ij} : H^0_{\{\mathfrak{p}\}}(H^i_{\mathrm{cont}}(I_v, X)_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} H^0_{\{\mathfrak{p}\}}(H^j_{\mathrm{cont}}(I_v, Y)_{\mathfrak{p}}) \longrightarrow H^1_{\{\mathfrak{p}\}}((\omega^\bullet_R)_{\mathfrak{p}}) \xrightarrow{\sim} I_{R_{\mathfrak{p}}} \qquad (i+j=2),$$

which induce f_v-equivariant isomorphisms of $R_{\mathfrak{p}}$-modules of finite length

$$(10.2.8.2)\qquad H^0_{\{\mathfrak{p}\}}(H^i_{\mathrm{cont}}(I_v, X)_{\mathfrak{p}}) \xrightarrow{\sim} D_{R_{\mathfrak{p}}}(H^0_{\{\mathfrak{p}\}}(H^j_{\mathrm{cont}}(I_v, Y)_{\mathfrak{p}})) \qquad (i+j=2),$$

by a localized version of the duality (7.5.8.2) and Proposition 2.10.12. Using the f_v-equivariance of (10.2.8.2) for $i=j=1$, we obtain isomorphisms

$$(10.2.8.3)\quad H^q(\mathrm{Err}_v(\Delta^{\mathrm{ur}}_v(X), \Delta^{\mathrm{ur}}_v(Y), \pi)_{\mathfrak{p}}) \xrightarrow{\sim} D_{R_{\mathfrak{p}}}(H^{3-q}(\mathrm{Err}_v(\Delta^{\mathrm{ur}}_v(Y), \Delta^{\mathrm{ur}}_v(X), \pi\circ s_{12})_{\mathfrak{p}})) \qquad (q=1,2).$$

Comparing the lengths (over $R_{\mathfrak{p}}$) of the both sides in (10.2.8.3), we obtain

$$\mathrm{Tam}_v(X, \mathfrak{p}) = \mathrm{Tam}_v(Y, \mathfrak{p}).$$

10.3. Greenberg's local conditions

In this section we investigate the abstract pairings from Sect. 10.2 in the context of Greenberg's local conditions (including Iwasawa theory). Throughout 10.3, we assume that R has no embedded primes.

10.3.1. Everything in 10.2 works under the assumptions of 7.8.2: let $J = \omega^\bullet_R = \sigma_{\geqslant 0} J$, $X_1 = X$, $X_2 = Y$ and $\pi : X\otimes_R Y \to J(1)$ be as in 6.7.5(B) (in particular, all complexes X, Y, X^+_v, Y^+_v are bounded) and $X^+_v \perp_\pi Y^+_v$ for all $v\in\Sigma$. Under these assumptions, the local conditions $\Delta(Z)$ $(Z = X, Y)$ defined in 7.8.2 satisfy

$$\Delta(X) \perp_{\pi,0} \Delta(Y)$$

and admit transposition operators satisfying 6.5.3.1– 6.5.3.5, by 7.8.3.

If π is a perfect duality, then the cohomology of the error terms $\mathrm{Err}_v(\Delta^{\mathrm{ur}}_v(X), \Delta^{\mathrm{ur}}_v(Y), \pi)$ for $v\in\Sigma'$ are given in $({}_R\mathrm{Mod})/(\text{pseudo-null})$ by 7.8.4.5. In particular,

$$\bigoplus_{v\in\Sigma'} H^i(\mathrm{Err}_v(\Delta_v(X), \Delta_v(Y), \pi)) \xrightarrow{\sim} 0 \qquad (\forall i\neq 1,2)$$

in $({}_R\mathrm{Mod})/(\text{pseudo-null})$.

If π is a perfect duality, then the error terms $\mathrm{Err}_v(\Delta_v(X), \Delta_v(Y), \pi)$ for $v\in\Sigma$ are given by Proposition 6.7.6(iv). In particular, if $X^+_v \perp\!\!\perp_\pi Y^+_v$, then $\mathrm{Err}_v(\Delta_v(X), \Delta_v(Y), \pi) \xrightarrow{\sim} 0$ in $D^b({}_R\mathrm{Mod})$.

10.3.2. Specializing the results of 10.2.2–10.2.4, the cup products

$$\cup_{\pi,r,0} : \widetilde{C}^\bullet_f(X) \otimes_R \widetilde{C}^\bullet_f(Y) \longrightarrow J[-3] = \omega^\bullet_R[-3] \qquad (r \in R)$$

give rise to pairings

$$\cup_{\pi,0,i,j} : \widetilde{H}^i_f(X)_{R\text{-tors}} \otimes_R \widetilde{H}^j_f(Y)_{R\text{-tors}} \longrightarrow H^0(\omega_R) \otimes_R (\mathrm{Frac}(R)/R) \qquad (i+j=4)$$

in $({}_R\mathrm{Mod})/(\text{pseudo-null})$ (or even in $({}_R\mathrm{Mod})$, if R is Cohen-Macaulay), satisfying

(10.3.2.1) $$x \cup_{\pi,0,i,j} y = (-1)^{ij}\, y \cup_{\pi\circ s_{12},0,j,i} x \qquad (i+j=4).$$

The kernels and cokernels of the adjoint maps

$$\mathrm{adj}(\cup_{\pi,0,i,j}) : \widetilde{H}^i_f(X)_{R\text{-tors}} \longrightarrow \mathrm{Hom}_R\big(\widetilde{H}^j_f(Y)_{R\text{-tors}}, H^0(\omega_R) \otimes_R (\mathrm{Frac}(R)/R)\big) \quad (i+j=4)$$

are as in Theorem 10.2.3.

If K'/K is a finite Galois subextension of K_S/K in which all primes $v \in \Sigma'$ are unramified, then

(10.3.2.2) $$(\forall g \in \mathrm{Gal}(K'/K)) \quad \mathrm{Ad}_f(g)(x) \cup_{\pi,0,i,j} \mathrm{Ad}_f(g)(y) = x \cup_{\pi,0,i,j} y,$$

by Proposition 8.8.9.

In the self-dual case, *i.e.*, if $Y = X$, $Y_v^+ = X_v^+$ $(v \in \Sigma)$ and $\pi \circ s_{12} = c \cdot \pi$ with $c = \pm 1$, then Proposition 10.2.5 implies that $\cup_{\pi,0,2,2}$ is symmetric (resp., skew-symmetric) if $c = +1$ (resp., $c = -1$).

10.3.3. Iwasawa theory

10.3.3.1. Let K_∞/K be as in 8.8.1, with $\Gamma = \mathrm{Gal}(K_\infty/K) = \Gamma_0 \times \Delta$, where $\Gamma_0 \xrightarrow{\sim} \mathbf{Z}_p^r$ $(r \geqslant 1)$ and Δ is a finite abelian group. We assume that the condition (U) from 8.8.1 is satisfied, *i.e.*, each prime $v \in \Sigma'$ is unramified in K_∞/K (this is automatic if $\Gamma = \Gamma_0$). The ring $\overline{R} = R[\![\Gamma]\!]$ is equipped with the canonical R-linear involution $\iota : \overline{R} \to \overline{R}$ induced by $\gamma \to \gamma^{-1}$ $(\gamma \in \Gamma)$. As $\overline{R} = \overline{R}_0[\Delta]$ with $\overline{R}_0 = R[\![\Gamma_0]\!] \xrightarrow{\sim} R[\![X_1, \dots, X_r]\!]$, it follows from Lemma 2.10.13.3 (iii) that $\mathrm{Frac}(\overline{R}) = \mathrm{Frac}(\overline{R}_0)[\Delta]$.

As in 8.4.6.2, we fix, for each $S = R, \overline{R}$, a complex $\omega^\bullet_S = \sigma_{\geqslant 0}\, \omega^\bullet_S$ of injective S-modules representing ω_S. There exists a quasi-isomorphism (unique up to homotopy)

$$\varphi : \omega^\bullet_R \otimes_R \overline{R} \longrightarrow \omega^\bullet_{\overline{R}}\,.$$

Fix φ; then there exists a morphism of complexes of $\overline{R}$-modules

$$\nu : \big(\omega^\bullet_{\overline{R}}\big)^\iota \longrightarrow \omega^\bullet_{\overline{R}}$$

(again unique up to homotopy) making the following diagram commutative:

$$\begin{array}{ccc} \omega^\bullet_R \otimes_R \overline{R}^\iota & \xrightarrow{\varphi^\iota} & \big(\omega^\bullet_{\overline{R}}\big)^\iota \\ \big\downarrow{\scriptstyle \mathrm{id}\otimes\iota} & & \big\downarrow{\scriptstyle \nu} \\ \omega^\bullet_R \otimes_R \overline{R} & \xrightarrow{\varphi} & \omega^\bullet_{\overline{R}}\,. \end{array}$$

Fix such a morphism ν; then the map $\nu^\iota : \omega^\bullet_{\overline{R}} \to \big(\omega^\bullet_{\overline{R}}\big)^\iota$ is a homotopy inverse of ν.

The induced maps on cohomology

$$\nu_* : H^*\big(\omega^\bullet_{\overline{R}}\big)^\iota = H^*(\omega_R) \otimes_R \overline{R}^\iota \longrightarrow H^*(\omega_R) \otimes_R \overline{R} = H^*\big(\omega^\bullet_{\overline{R}}\big)$$

are equal to $\mathrm{id} \otimes \iota$. By abuse of language, we shall denote ν_*, as well as

$$\nu_* \otimes \iota : \big(H^0(\omega_{\overline{R}}) \otimes_{\overline{R}} (\mathrm{Frac}(\overline{R})/\overline{R})\big)^\iota \longrightarrow H^0(\omega_{\overline{R}}) \otimes_{\overline{R}} (\mathrm{Frac}(\overline{R})/\overline{R}),$$

by ι.

10.3.3.2. We need a slightly stronger version of the assumptions 6.7.5 (B): we require that, in each degree $i \in \mathbf{Z}$, the components $X^i, Y^i, (X_v^+)^i, (Y_v^+)^i$ $(v \in \Sigma)$ are of finite type over R. The recipe 8.9.2 then defines Greenberg's local conditions for

$$\mathscr{F}_\Gamma(Z) = (Z \otimes_R \overline{R}) < -1 >, \quad \mathscr{F}_\Gamma(Z)^\iota = (Z \otimes_R \overline{R}^\iota) < -1 > \qquad (Z = X, Y)$$

over $\overline{R}$, together with a pairing

$$\overline{\pi} : \mathscr{F}_\Gamma(X) \otimes_{\overline{R}} \mathscr{F}_\Gamma(Y)^\iota \xrightarrow{\mathscr{F}(\pi)} \omega^\bullet_R(1) \otimes_R \overline{R} \xrightarrow{\varphi(1)} \omega^\bullet_{\overline{R}}(1),$$

under which

$$\mathscr{F}_\Gamma(X)_v^+ \perp_{\overline{\pi}} (\mathscr{F}_\Gamma(Y)^\iota)_v^+ \qquad (v \in \Sigma).$$

As in 6.6.5, we have

$$\Delta(\mathscr{F}_\Gamma(Y)^\iota) = \Delta(\mathscr{F}_\Gamma(Y))^\iota, \quad \widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(Y)^\iota) = \widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(Y))^\iota.$$

If π is a perfect duality, so is $\overline{\pi}$, by Corollary 8.4.6.5; if, in addition, $X_v^+ \perp\perp_\pi X_v^+$ $(v \in \Sigma)$, then

$$\mathscr{F}_\Gamma(X)_v^+ \perp\perp_{\overline{\pi}} (\mathscr{F}_\Gamma(Y)^\iota)_v^+.$$

10.3.3.3. As R has no embedded primes, the ring $\overline{R}$ has the same property. This implies that the theory from 10.2 (as specialized in 10.3.1–10.3.2) applies to $\mathscr{F}_\Gamma(X)$, $\mathscr{F}_\Gamma(Y)^\iota$ and $\overline{\pi}$: the cup products

$$\cup_{\overline{\pi},r,0} : \widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(X)) \otimes_{\overline{R}} \widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(Y))^\iota \longrightarrow \omega^\bullet_{\overline{R}}[-3] \qquad (r \in \overline{R})$$

induce pairings

$$\cup_{\overline{\pi},0,i,j} : \widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K, X)_{\overline{R}\text{-tors}} \otimes_{\overline{R}} \widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, Y)^\iota_{\overline{R}\text{-tors}} \longrightarrow H^0(\omega_{\overline{R}}) \otimes_{\overline{R}} (\mathrm{Frac}(\overline{R})/\overline{R}) \qquad (i + j = 4)$$

in $\big({}_{\overline{R}}\mathrm{Mod}\big)/(\text{pseudo-null})$ (or even in $\big({}_{\overline{R}}\mathrm{Mod}\big)$, if R – hence $\overline{R}$ – is Cohen-Macaulay). Similarly,

$$\pi \circ s_{12} : Y \otimes_R X \xrightarrow{s_{12}} X \otimes_R Y \xrightarrow{\pi} \omega^\bullet_R(1)$$

gives rise to

$$\overline{\pi \circ s_{12}} : \mathscr{F}_\Gamma(Y) \otimes_{\overline{R}} \mathscr{F}_\Gamma(X)^\iota \xrightarrow{\mathscr{F}(\pi \circ s_{12})} \omega^\bullet_R(1) \otimes_R \overline{R} \xrightarrow{\varphi(1)} \omega^\bullet_{\overline{R}}(1)$$

and – using the notation from 6.6.4 – cup products

$$(\cup_{\overline{\pi \circ s_{12}},r,0})^\iota : \widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(Y))^\iota \otimes_{\overline{R}} \widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(X)) \longrightarrow \omega^\bullet_{\overline{R}}[-3] \quad (r \in \overline{R})$$

and

$$(\cup_{\overline{\pi\circ s_{12}},0,j,i})^{\iota} : \widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K,Y)^{\iota}_{\overline{R}\text{-tors}} \otimes_{\overline{R}} \widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K,X)_{\overline{R}\text{-tors}} \longrightarrow \big(H^0(\omega_{\overline{R}}) \otimes_{\overline{R}} (\mathrm{Frac}(\overline{R})/\overline{R})\big)^{\iota} \qquad (i+j=4).$$

As in 10.2.7.1, it is sometimes convenient to view $\cup_{\overline{\pi},0,i,j}$ and $(\cup_{\overline{\pi\circ s_{12}},0,j,i})^{\iota}$ as Hermitian pairings

$$(10.3.3.1)\quad \langle\ ,\ \rangle_{\overline{\pi},0,i,j} : \widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K,X)_{\overline{R}\text{-tors}} \times \widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K,Y)_{\overline{R}\text{-tors}} \longrightarrow H^0(\omega_{\overline{R}}) \otimes_{\overline{R}} (\mathrm{Frac}(\overline{R})/\overline{R})$$

$$\langle\ ,\ \rangle^{\iota}_{\overline{\pi\circ s_{12}},0,j,i} : \widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K,Y)^{\iota}_{\overline{R}\text{-tors}} \times \widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K,X)_{\overline{R}\text{-tors}} \longrightarrow \big(H^0(\omega_{\overline{R}}) \otimes_{\overline{R}} (\mathrm{Frac}(\overline{R})/\overline{R})\big)^{\iota}$$

satisfying

$$\begin{aligned} \langle \alpha x, \beta y\rangle_{\overline{\pi},0,i,j} &= \alpha\, \iota(\beta)\, \langle x,y\rangle_{\overline{\pi},0,i,j} \\ \langle \beta y, \alpha x\rangle^{\iota}_{\overline{\pi\circ s_{12}},0,j,i} &= \iota(\beta)\, \alpha\ \langle y,x\rangle^{\iota}_{\overline{\pi\circ s_{12}},0,j,i} \qquad (\alpha,\beta\in\overline{R};\ i+j=4) \end{aligned}$$

In order to avoid any confusion, we stress that the action of $\iota(\beta)\,\alpha$ in the second formula is with respect to the $\overline{R}$-module structure $(-)^{\iota}$ on the R.H.S.

10.3.3.4. Lemma. — *The following diagram is commutative:*

$$\begin{array}{cccccc} \overline{\pi}: & \mathscr{F}_\Gamma(X)\otimes_{\overline{R}}\mathscr{F}_\Gamma(Y)^{\iota} & \xrightarrow{\mathscr{F}(\pi)} & \omega^{\bullet}_R(1)\otimes_R \overline{R} & \xrightarrow{\varphi(1)} & \omega^{\bullet}_{\overline{R}}(1) \\ & \big\downarrow s_{12} & & \big\uparrow \mathrm{id}\otimes\iota & & \big\uparrow \nu(1) \\ (\overline{\pi\circ s_{12}})^{\iota}: & \mathscr{F}_\Gamma(Y)^{\iota}\otimes_{\overline{R}}\mathscr{F}_\Gamma(X) & \xrightarrow{\mathscr{F}(\pi\circ s_{12})^{\iota}} & \omega^{\bullet}_R(1)\otimes_R \overline{R}^{\iota} & \xrightarrow{\varphi^{\iota}(1)} & (\omega^{\bullet}_{\overline{R}})^{\iota}(1). \end{array}$$

Proof. — Commutativity of the left (resp., right) square follows from the commutative diagram

$$\begin{array}{ccc} \overline{R}\otimes_{\overline{R}}\overline{R}^{\iota} & \xrightarrow{\mathrm{id}\otimes\iota} & \overline{R} \\ \big\downarrow s_{12} & & \big\uparrow \iota \\ \overline{R}^{\iota}\otimes_{\overline{R}}\overline{R} & \xrightarrow{(\mathrm{id}\otimes\iota)^{\iota}} & \overline{R}^{\iota} \end{array}$$

(resp., from 10.3.3.1). □

10.3.3.5. Proposition. — *Under the assumptions of 10.3.3.2, the Hermitian pairings* (10.3.3.1) *are related by*

$$\langle \alpha x,\beta y\rangle_{\overline{\pi},0,i,j} = (-1)^{ij}\,\iota\big(\langle \beta y,\alpha x\rangle^{\iota}_{\overline{\pi\circ s_{12}},0,j,i}\big) \qquad (i+j=4).$$

Proof. — As in the proof of Corollary 6.5.5, the existence of transposition operators for $\mathscr{F}_\Gamma(X)$ and $\mathscr{F}_\Gamma(Y)^{\iota}$, together with Lemma 10.3.3.4, imply that the following diagram is commutative up to homotopy (for any $r \in \overline{R}$):

$$\begin{array}{lccc} \cup_{\overline{\pi},r,0} : & \widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(X)) \otimes_{\overline{R}} \widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(Y))^\iota & \longrightarrow & \omega^\bullet_{\overline{R}}[-3] \\ & \downarrow s_{12} & & \uparrow \nu[-3] \\ (\cup_{\overline{\pi\circ s_{12}},1-r,0})^\iota : & \widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(Y))^\iota \otimes_{\overline{R}} \widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(X)) & \longrightarrow & (\omega^\bullet_{\overline{R}})^\iota[-3]. \end{array}$$

The result follows by applying (2.10.14.1). □

10.3.3.6. Alternatively, one can reformulate the formula in Proposition 10.3.3.5 as

$$x \cup_{\overline{\pi},0,i,j} y = (-1)^{ij}\,\iota\,(y\,(\cup_{\overline{\pi\circ s_{12}},0,j,i})^\iota\,x) \qquad (i+j=4),$$

where

$$x \in \widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K,X)_{\overline{R}\text{-tors}}, \quad y \in \widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K,Y)^\iota_{\overline{R}\text{-tors}} \qquad (i+j=4).$$

10.3.4. Self-dual case

10.3.4.1. Back to the situation of 10.3.1, assume that $Y = X$, $\Delta(Y) = \Delta(X)$ and $\pi : X \otimes_R X \to J(1)$ satisfies

$$\pi \circ s_{12} = c\cdot\pi, \qquad c = \pm 1.$$

10.3.4.2. Proposition

(i) *The pairings*

$$\cup_{\pi,0,i,j} : \widetilde{H}^i_f(X)_{R\text{-tors}} \otimes_R \widetilde{H}^j_f(X)_{R\text{-tors}} \longrightarrow H^0(\omega_R)\otimes_R (\mathrm{Frac}(R)/R) \qquad (i+j=4)$$

satisfy

$$x \cup_{\pi,0,i,j} y = c(-1)^{ij}\, y \cup_{\pi,0,j,i} x \qquad (i+j=4).$$

In particular, $\cup_{\pi,0,2,2}$ *is symmetric (resp., skew-symmetric) if* $c=+1$ *(resp.,* $c=-1$*).*

(ii) *If* X, X_v^+ $(v\in\Sigma)$ *satisfy the assumptions of 10.3.3.2, then the pairings*

$$\cup_{\overline{\pi},0,i,j} : \widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K,X)_{\overline{R}\text{-tors}} \otimes_{\overline{R}} \widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K,X)^\iota_{\overline{R}\text{-tors}} \longrightarrow H^0(\omega_{\overline{R}})\otimes_{\overline{R}} (\mathrm{Frac}(\overline{R})/\overline{R}) \qquad (i+j=4)$$

satisfy

$$x \cup_{\overline{\pi},0,i,j} y = c(-1)^{ij}\,\iota\,(y\,(\cup_{\overline{\pi\circ s_{12}},0,j,i})^\iota\,x) \qquad (i+j=4).$$

In particular, $\cup_{\overline{\pi},0,2,2}$ *is symmetric Hermitian (resp., skew-Hermitian) if* $c=+1$ *(resp.,* $c=-1$*).*

Proof

(i) This is a special case of Proposition 10.2.5.

(ii) The statement follows from 10.3.3.6, since

$$\overline{\pi\circ s_{12}} = c\cdot\overline{\pi}, \quad \cup_{\overline{\pi\circ s_{12}},0,j,i} = c\cdot\cup_{\overline{\pi},0,j,i}.$$

Alternatively, one can apply Proposition 10.2.7.2, as

$$\overline{\pi}\circ s_{12} = \nu\circ(\overline{\pi\circ s_{12}})^\iota = c\cdot(\nu\circ\overline{\pi}^\iota). \quad \square$$

10.3.5. Iwasawa theory - dihedral case. — Let X, Y and π be as in 10.3.3.2.

10.3.5.1. Assume that K_∞/K extends to a dihedral Galois extension K_∞/K^+, *i.e.*, $[K : K^+] = 2$ and $\Gamma^+ = \mathrm{Gal}(K_\infty/K^+)$ is a semi-direct product $\Gamma^+ = \Gamma \rtimes \{1, \tau\}$ for some $\tau \in \Gamma^+ - \Gamma$ such that

$$\tau^2 = 1, \quad \tau\gamma\tau^{-1} = \gamma^{-1} \qquad (\gamma \in \Gamma)$$

(which is then true for *every* $\tau \in \Gamma^+ - \Gamma$).

We also assume that Greenberg's local conditions for X, Y are defined over K^+, in the following sense:

10.3.5.1.1. There is a finite set of primes S^+ of K^+ such that

$$S = \{v : (\exists v^+ \in S^+)\, v | v^+\}$$

(hence K_S/K^+ is a Galois extension).

10.3.5.1.2. Both X and Y are (bounded) complexes of (admissible) $R[\mathrm{Gal}(K_S/K^+)]$-modules and the morphism π is $\mathrm{Gal}(K_S/K^+)$-equivariant.

10.3.5.1.3. For a suitable subset $\Sigma^+ \subset S_f^+$,

$$\Sigma = \{v \in S_f : (\exists v^+ \in \Sigma^+)\, v|v^+\}, \quad \Sigma' = \{v \in S_f : (\exists v^+ \in S_f^+ - \Sigma^+)\, v|v^+\}.$$

10.3.5.1.4. $(\forall Z = X, Y)\, (\forall v^+ \in \Sigma^+)$ there is a (bounded) complex of (admissible) $R[G_{v^+}]$-modules $Z^+_{v^+}$ and a morphism of complexes of $R[G_{v^+}]$-modules

$$j^+_{v^+}(Z) : Z^+_{v^+} \longrightarrow Z$$

such that, for each $v \in \Sigma$, $v|v^+$, the restriction of $j^+_{v^+}(Z)$ to $G_v \subset G_{v^+}$ coincides with $j^+_v(Z)$.

10.3.5.1.5. Each prime $v \in \Sigma'$ is unramified in K/K^+.

10.3.5.2. In order to simplify the notation, put $G^+ = \mathrm{Gal}(K_S/K^+)$, $G = G_{K,S} = \mathrm{Gal}(K_S/K)$; denote by $\rho : G^+ \to \Gamma^+$ the canonical projection and fix $\overline{\tau} \in \rho^{-1}(\tau)$. The tautological character

$$\chi_\Gamma : G \xrightarrow{\rho} \Gamma \hookrightarrow \overline{R}^*$$

satisfies

$$\chi_\Gamma(\overline{\tau}^2) = 1, \quad \chi_\Gamma(\overline{\tau} g \overline{\tau}^{-1}) = \chi_\Gamma(g^{-1}) \qquad (g \in G).$$

The R-algebra

$$R[\![\Gamma^+]\!] = \overline{R} \oplus \overline{R}\tau = \overline{R} \oplus \tau\overline{R} \tag{10.3.5.1}$$

is equipped with an involution ι, induced by $\gamma^+ \mapsto (\gamma^+)^{-1}$ $(\gamma^+ \in \Gamma^+)$, which extends ι on $\overline{R}$.

10.3.5.3. For $Z = X, Y$, Shapiro's Lemma gives a quasi-isomorphism

$$\mathrm{sh} : C^{\bullet}_{\mathrm{cont}}\big(G^+, \mathrm{Ind}_G^{G^+}(\mathscr{F}_\Gamma(Z))\big) \longrightarrow C^{\bullet}_{\mathrm{cont}}(G, \mathscr{F}_\Gamma(Z))$$

(and similarly for $\mathscr{F}_\Gamma(Z)^\iota$). Recall that

$$\mathscr{F}_\Gamma(Z) = (Z \otimes_R \overline{R}) < -1 >, \quad \mathscr{F}_\Gamma(Z)^\iota = (Z \otimes_R \overline{R}^\iota) < -1 > .$$

Denote by $\mathscr{F}_{\Gamma^+}(Z)$ the following complex in $\big({}^{\mathrm{ad}}_{\overline{R}[G^+]}\mathrm{Mod}\big)_{\overline{R}\text{-}ft}$:

$$\mathscr{F}_{\Gamma^+}(Z) = Z \otimes_R R[\![\Gamma^+]\!],$$

with $r \in \overline{R}$ and $g^+ \in G^+$ acting by

$$r(z \otimes a) = z \otimes ra, \quad g^+(z \otimes a) = g^+z \otimes a\rho(g^+)^{-1}, \qquad (z \in Z,\ a \in R[\![\Gamma^+]\!])$$

respectively. Note that $\mathscr{F}_{\Gamma^+}(Z)^\iota$ is canonically isomorphic to

$$Z \otimes_R R[\![\Gamma^+]\!],$$

where $r \in \overline{R}$ and $g^+ \in G^+$ act by

$$r(z \otimes a) = z \otimes \iota(r)a, \quad g^+(z \otimes a) = g^+z \otimes a\rho(g^+)^{-1} \qquad (z \in Z,\ a \in R[\![\Gamma^+]\!]).$$

The decomposition (10.3.5.1) defines canonical isomorphisms of $\overline{R}[G]$-modules

$$\begin{aligned}\mathscr{F}_{\Gamma^+}(Z) &\xrightarrow{\sim} (Z \otimes_R \overline{R}) < -1 > \oplus (Z \otimes_R \overline{R})\tau < -1 > \\ &\xrightarrow{\sim} (Z \otimes_R \overline{R}) < -1 > \oplus (Z \otimes_R \overline{R}) < 1 > \tau \\ \mathscr{F}_{\Gamma^+}(Z)^\iota &\xrightarrow{\sim} \big(Z \otimes_R \overline{R}^\iota\big) < -1 > \oplus \big(Z \otimes_R \overline{R}^\iota\big)\tau < -1 > \\ &\xrightarrow{\sim} \big(Z \otimes_R \overline{R}^\iota\big) < -1 > \oplus \big(Z \otimes_R \overline{R}^\iota\big) < 1 > \tau.\end{aligned}$$

10.3.5.4. Lemma

(i) *For $Z = X, Y$, the formula $f \mapsto f(1) + (\overline{\tau}^{-1} \otimes 1) f(\overline{\tau})(1 \otimes \tau)$ defines isomorphisms of complexes in $\big({}^{\mathrm{ad}}_{\overline{R}[G^+]}\mathrm{Mod}\big)_{\overline{R}\text{-}ft}$*

$$\begin{aligned}\mathrm{Ind}_G^{G^+}\big((Z \otimes_R \overline{R}) < -1 >\big) \ (= \mathrm{Ind}_G^{G^+}(\mathscr{F}_\Gamma(Z))) &\xrightarrow{\sim} \mathscr{F}_{\Gamma^+}(Z) \\ \mathrm{Ind}_G^{G^+}\Big(\big(Z \otimes_R \overline{R}^\iota\big) < -1 > \Big) \ (= \mathrm{Ind}_G^{G^+}(\mathscr{F}_\Gamma(Z)^\iota)) &\xrightarrow{\sim} \mathscr{F}_{\Gamma^+}(Z)^\iota \\ \mathrm{Ind}_G^{G^+}\big((X \otimes_R Y) \otimes_R \overline{R}\big) &\xrightarrow{\sim} (X \otimes_R Y) \otimes_R \overline{R}[\Gamma^+/\Gamma],\end{aligned}$$

which make the following diagram commutative:

$$\begin{array}{ccc}\mathrm{Ind}_G^{G^+}\big((X \otimes_R \overline{R}) < -1 > \big) \otimes_{\overline{R}} \mathrm{Ind}_G^{G^+}\big((Y \otimes_R \overline{R}^\iota) < -1 > \big) & \xrightarrow{\mathrm{id}\otimes(\mathrm{id}\otimes\iota)} & \mathrm{Ind}_G^{G^+}\big((X \otimes_R Y) \otimes_R \overline{R}\big) \\ \downarrow \wr & & \downarrow \wr \\ \mathscr{F}_{\Gamma^+}(X) \otimes_{\overline{R}} \mathscr{F}_{\Gamma^+}(Y)^\iota & \xrightarrow{\mathrm{prod}} & (X \otimes_R Y) \otimes_R \overline{R}[\Gamma^+/\Gamma].\end{array}$$

Above, the map prod *is given by*

$$\mathrm{prod} : (x \otimes (a_1 + a_2\tau)) \otimes (y \otimes (b_1 + b_2\tau)) \longmapsto (x \otimes y) \otimes (a_1\iota(b_1) + a_2\iota(b_2)\tau) \qquad (a_j, b_j \in \overline{R}).$$

(ii) *The formulas*

$$u_X : x \otimes (a_1 + a_2\tau) \longmapsto x \otimes (\iota(a_2) + \iota(a_1)\tau)$$
$$v_Y : y \otimes (b_1 + b_2\tau) \longmapsto y \otimes (\iota(b_2) + \iota(b_1)\tau)$$

define isomorphisms of complexes of $\overline{R}[G^+]$*-modules*

$$u_X : \mathscr{F}_{\Gamma^+}(X) \xrightarrow{\sim} \mathscr{F}_{\Gamma^+}(X)^\iota, \quad v_Y : \mathscr{F}_{\Gamma^+}(Y)^\iota \xrightarrow{\sim} \mathscr{F}_{\Gamma^+}(Y)$$

satisfying $v_Z = u_Z^{-1}$ $(Z = X, Y)$ *and making the following diagram commutative:*

$$\begin{array}{ccc} \mathscr{F}_{\Gamma^+}(X) \otimes_{\overline{R}} \mathscr{F}_{\Gamma^+}(Y)^\iota & \xrightarrow{\mathrm{prod}} & (X \otimes_R Y) \otimes_R \overline{R}[\Gamma^+/\Gamma] \\ \Big\downarrow u_X \otimes v_Y & & \Big\uparrow \mathrm{id}_{X\otimes Y} \otimes (\iota \otimes \tau) \\ \mathscr{F}_{\Gamma^+}(X)^\iota \otimes_{\overline{R}} \mathscr{F}_{\Gamma^+}(Y) & \xrightarrow{\mathrm{prod}^\iota} & (X \otimes_R Y) \otimes_R \overline{R}^\iota[\Gamma^+/\Gamma]. \end{array}$$

Proof

(i) Let $f \in \mathrm{Ind}_G^{G^+}((Z \otimes_R \overline{R}) < -1 >)$; then $f : G^+ \to Z \otimes_R \overline{R}$ is a function satisfying

$$f(gg^+) = (g \otimes \chi_\Gamma(g)^{-1})f(g^+) \qquad (g \in G,\ g^+ \in G^+),$$

on which $g_1^+ \in G^+$ acts by $(g_1^+ * f)(g^+) = f(g^+g_1^+)$. In particular,

$$(\overline{\tau} * f)(1) = f(\overline{\tau}), \qquad (g * f)(1) = (g \otimes \chi_\Gamma(g)^{-1})f(1),$$
$$(\overline{\tau} * f)(\overline{\tau}) = (\overline{\tau}^2 \otimes 1)f(1), \quad (g * f)(\overline{\tau}) = f((\overline{\tau}g\overline{\tau}^{-1})\overline{\tau}) = (\overline{\tau}g\overline{\tau}^{-1} \otimes \chi_\Gamma(g))f(\overline{\tau}).$$

$(g \in G)$. Putting

$$j(f) = f(1) + (\overline{\tau}^{-1} \otimes 1)f(\overline{\tau})(1 \otimes \tau) = f_1 + f_2\tau,$$

where $f_i \in Z \otimes_R \overline{R}$ are equal to

$$f_1 = f(1), \quad f_2 = (\overline{\tau}^{-1} \otimes 1)f(\overline{\tau}),$$

then

$$j(\overline{\tau} * f) = f(\overline{\tau}) + (\overline{\tau}^{-1} \otimes 1)(\overline{\tau}^2 \otimes 1)f(1)(1 \otimes \tau) = (\overline{\tau} \otimes 1)(f_2 + f_1\tau) = \overline{\tau} * j(f)$$
$$j(g * f) = (g \otimes \chi_\Gamma(g)^{-1})f_1 + (g \otimes \chi_\Gamma(g))f_2\tau = g * j(f) \qquad (g \in G).$$

This shows that the (bijective) map

$$j : \mathrm{Ind}_G^{G^+}((Z \otimes_R \overline{R}) < -1 >) \longrightarrow \mathscr{F}_{\Gamma^+}(Z)$$

is a homomorphism of $\overline{R}[G^+]$-modules, hence an isomorphism. The same argument also works for

$$j^\iota : \mathrm{Ind}_G^{G^+}\big((Z \otimes_R \overline{R}^\iota) < -1 >\big) \xrightarrow{\sim} \mathscr{F}_{\Gamma^+}(Z)^\iota.$$

The corresponding statement for $(X \otimes_R Y) \otimes_R \overline{R}$ is just one of the isomorphisms from 8.1.3.

As regards the commutativity of the diagram involving prod, let

$$f_X \in \mathrm{Ind}_G^{G^+}((X \otimes_R \overline{R}) < -1 >), \quad f_Y \in \mathrm{Ind}_G^{G^+}\big((Y \otimes_R \overline{R}^\iota) < -1 > \big).$$

Writing

$$\begin{aligned} j(f_X) &= u_1 + u_2\tau, \quad u_1 = f_X(1), \quad u_2 = (\overline{\tau}^{-1} \otimes 1) f_X(\overline{\tau}) \in X \otimes_R \overline{R} \\ j^\iota(f_Y) &= v_1 + v_2\tau, \quad v_1 = f_Y(1), \quad v_2 = (\overline{\tau}^{-1} \otimes 1) f_Y(\overline{\tau}) \in Y \otimes_R \overline{R}^\iota, \end{aligned}$$

we must express the values

$$w_1 = f(1), \quad w_2 = (\overline{\tau}^{-1} \otimes \overline{\tau}^{-1} \otimes 1) f(\overline{\tau}) \in (X \otimes_R Y) \otimes_R \overline{R}$$

of the function

$$f(g^+) = f_X(g^+) \otimes (\mathrm{id} \otimes \iota) f_Y(g^+) \qquad (g^+ \in G^+)$$

in terms of u_i, v_i. As

$$w_1 = u_1 \otimes (\mathrm{id} \otimes \iota) v_1, \quad w_2 = u_2 \otimes (\mathrm{id} \otimes \iota) v_2,$$

the formula for prod which makes the diagram commutative follows.

(ii) Clearly $u_Z \circ v_Z = \mathrm{id}$, $v_Z \circ u_Z = \mathrm{id}$. The remaining statement follows from the commutativity of the following diagram:

$$\begin{array}{llcl} \mathrm{prod}: & (x \otimes (a_1 + a_2\tau)) \otimes (y \otimes (b_1 + b_2\tau)) & \longmapsto & (x \otimes y) \otimes (a_1\iota(b_1) + a_2\iota(b_2)\tau) \\ & \big\downarrow u_X \otimes v_Y & & \big\uparrow \mathrm{id} \otimes (\iota \otimes \tau) \\ \mathrm{prod}^\iota: & (x \otimes (\iota(a_2) + \iota(a_1)\tau)) \otimes (y \otimes (\iota(b_2) + \iota(b_1)\tau)) & \longmapsto & (x \otimes y) \otimes (b_2\iota(a_2) + b_1\iota(a_1)\tau). \end{array}$$

□

10.3.5.5. In this section we show that the assumptions 10.3.5.1.1–10.3.5.1.5 allow us to replace Selmer complexes $\widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(Z))$ $(Z = X, Y)$ by certain generalized Selmer complexes $\widetilde{C}^\bullet_f(G^+, \mathscr{F}_{\Gamma^+}(Z))$.

As a first step we define, for each $v^+ \in S^+$, local conditions $\Delta_{v^+}(\mathscr{F}_{\Gamma^+}(Z))$ $(Z = X, Y)$ by

$$i^+_{v^+}(\mathscr{F}_{\Gamma^+}(Z)) : \begin{cases} U^+_{v^+}(\mathscr{F}_{\Gamma^+}(Z)) = C^\bullet_{\mathrm{cont}}(G_{v^+}, \mathscr{F}_{\Gamma^+}(Z^+_{v^+})) \longrightarrow C^\bullet_{\mathrm{cont}}(G_{v^+}, \mathscr{F}_{\Gamma^+}(Z)), \\ \hfill (v^+ \in \Sigma^+) \\ U^+_{v^+}(\mathscr{F}_{\Gamma^+}(Z)) = C^\bullet_{\mathrm{ur}}(G_{v^+}, \mathscr{F}_{\Gamma^+}(Z)) \longrightarrow C^\bullet_{\mathrm{cont}}(G_{v^+}, \mathscr{F}_{\Gamma^+}(Z)), \\ \hfill (v^+ \in S^+_f - \Sigma^+) \end{cases}$$

and the corresponding generalized Selmer complex for G^+ by

$$\widetilde{C}^\bullet_f(G^+,\mathscr{F}_{\Gamma^+}(Z)) = \mathrm{Cone}\Bigg(C^\bullet_{\mathrm{cont}}(G^+,\mathscr{F}_{\Gamma^+}(Z)) \oplus \bigoplus_{v^+\in S_f^+} U^+_{v^+}(\mathscr{F}_{\Gamma^+}(Z)) \xrightarrow{\mathrm{res}_{S_f^+}-(i^+_{v^+})} \bigoplus_{v^+\in S_f^+} C^\bullet_{\mathrm{cont}}(G_{v^+},\mathscr{F}_{\Gamma^+}(Z))\Bigg)[-1]$$

(and similarly for $\mathscr{F}_{\Gamma^+}(Z)^\iota$).

Lemma 10.3.5.4 together with a variant of the theory from 8.6–8.8 (which applies thanks to the assumptions 10.3.5.1.1–10.3.5.1.5) give functorial quasi-isomorphisms of complexes of $\overline{R}$-modules

$$\mathrm{sh}_f^+ : \widetilde{C}^\bullet_f(G^+,\mathscr{F}_{\Gamma^+}(Z)) \xleftarrow{\ \sim\ } \widetilde{C}^\bullet_f(G^+,\mathrm{Ind}_G^{G^+}(\mathscr{F}_{\Gamma}(Z))) \xrightarrow{\mathrm{sh}_f} \widetilde{C}^\bullet_f(\mathscr{F}_{\Gamma^+}(Z)) \qquad (Z = X, Y).$$

It follows from Lemma 10.3.5.4 (i) that the morphisms of complexes $\pi : X \otimes_R Y \to \omega^\bullet_R(1)$ and

$$\overline{\pi} : \mathscr{F}_{\Gamma}(X) \otimes_{\overline{R}} \mathscr{F}_{\Gamma}(Y)^\iota \xrightarrow{\mathscr{F}(\pi)} \omega^\bullet_R(1) \otimes_R \overline{R} \xrightarrow{\varphi(1)} \omega^\bullet_{\overline{R}}(1)$$

induce products

$$\overline{\pi}^+ : \mathscr{F}_{\Gamma^+}(X) \otimes_{\overline{R}} \mathscr{F}_{\Gamma^+}(Y)^\iota \longrightarrow \omega^\bullet_R(1) \otimes_R \overline{R}[\Gamma^+/\Gamma] \xrightarrow{\varphi(1)} \omega^\bullet_{\overline{R}}(1) \otimes_{\overline{R}} \overline{R}[\Gamma^+/\Gamma]$$

and

$$\overset{\bullet}{\cup}_{\overline{\pi}^+} : C^\bullet_{\mathrm{cont}}(G_{v^+},\mathscr{F}_{\Gamma^+}(X)) \otimes_{\overline{R}} C^\bullet_{\mathrm{cont}}(G_{v^+},\mathscr{F}_{\Gamma^+}(Y))^\iota \longrightarrow \big(\omega^\bullet_{\overline{R}} \otimes_{\overline{R}} \overline{R}[\Gamma^+/\Gamma]\big)[-2] \qquad (v^+ \in S_f^+),$$

with respect to which

$$U^+_{v^+}(\mathscr{F}_{\Gamma^+}(X)) \perp_{\overline{\pi}^+,0} U^+_{v^+}(\mathscr{F}_{\Gamma^+}(Y)^\iota) \qquad (v^+ \in S_f^+).$$

Applying the construction from 6.3.1 and using 8.5.3, we obtain cup products ($r \in \overline{R}$)

$$\begin{aligned}\cup_{\overline{\pi}^+,r,0} : \widetilde{C}^\bullet_f(G^+,\mathscr{F}_{\Gamma^+}(X)) \otimes_{\overline{R}} \widetilde{C}^\bullet_f(G^+,\mathscr{F}_{\Gamma^+}(Y))^\iota &\longrightarrow \tau_{\geqslant 3}\mathrm{Cone}\Bigg(C^\bullet_{\mathrm{cont}}(G^+,\omega^\bullet_{\overline{R}}(1) \otimes_{\overline{R}} \overline{R}[\Gamma^+/\Gamma]) \\ &\xrightarrow{\mathrm{res}_{S_f^+}} \bigoplus_{v^+\in S_f^+} C^\bullet_{\mathrm{cont}}(G_{v^+},\omega^\bullet_{\overline{R}}(1) \otimes_{\overline{R}} \overline{R}[\Gamma^+/\Gamma])\Bigg)[-1] \\ &\xrightarrow{\mathrm{sh}_c} \tau_{\geqslant 3}\, C^\bullet_{c,\mathrm{cont}}(G_{K,S},\omega^\bullet_{\overline{R}}(1)) \longrightarrow \omega^\bullet_{\overline{R}}[-3]\end{aligned}$$

such that the diagram

$$\begin{array}{ccc} \widetilde{C}^\bullet_f(G^+,\mathscr{F}_{\Gamma^+}(X)) \otimes_{\overline{R}} \widetilde{C}^\bullet_f(G^+,\mathscr{F}_{\Gamma^+}(Y))^\iota & \xrightarrow{\cup_{\overline{\pi}^+,r,0}} & \omega^\bullet_{\overline{R}}[-3] \\ \Big\downarrow \mathrm{sh}^+_f\otimes sh^+_f & & \Big\| \\ \widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(X)) \otimes_{\overline{R}} \widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(Y))^\iota & \xrightarrow{\cup_{\overline{\pi},r,0}} & \omega^\bullet_{\overline{R}}[-3] \end{array}$$

is commutative up to homotopy. This implies that the pairings $\cup_{\overline{\pi},0,i,j}$ from 10.3.3.3 can also be defined using $\cup_{\overline{\pi}^+,r,0}$ instead of $\cup_{\overline{\pi},r,0}$.

10.3.5.6. Lemma. — *The following diagram is commutative up to homotopy:*

$$\begin{array}{ccc} \widetilde{C}^\bullet_f(G^+,\mathscr{F}_{\Gamma^+}(X)) \otimes_{\overline{R}} \widetilde{C}^\bullet_f(G^+,\mathscr{F}_{\Gamma^+}(Y))^\iota & \xrightarrow{\cup_{\overline{\pi}^+,r,0}} & \omega^\bullet_{\overline{R}}[-3] \\ \Big\downarrow (u_X)_*\otimes(v_Y)_* & & \Big\uparrow \nu[-3] \\ \widetilde{C}^\bullet_f(G^+,\mathscr{F}_{\Gamma^+}(X))^\iota \otimes_{\overline{R}} \widetilde{C}^\bullet_f(G^+,\mathscr{F}_{\Gamma^+}(Y)) & \xrightarrow{(\cup_{\overline{\pi}^+,r,0})^\iota} & (\omega^\bullet_{\overline{R}})^\iota[-3]. \end{array}$$

Proof. — Using the previous discussion, the statement follows from Lemma 10.3.5.4 (ii) and Proposition 6.4.2. Note that the action of τ on $\overline{R}[\Gamma^+/\Gamma]$ in the right vertical arrow in 10.3.5.4 (ii) corresponds to the homotopy action of $\mathrm{Ad}(\tau)$ on $\tau_{\geqslant 3}\, C^\bullet_{c,\mathrm{cont}}(G_{K,S}, J(1))$, which makes the following diagram commutative up to homotopy:

$$\begin{array}{ccc} \tau_{\geqslant 3}\, C^\bullet_{c,\mathrm{cont}}(G_{K,S}, J(1)) & \xrightarrow{\mathrm{Qis}} & J[-3] \\ \Big\downarrow \mathrm{Ad}(\tau) & & \Big\| \\ \tau_{\geqslant 3}\, C^\bullet_{c,\mathrm{cont}}(G_{K,S}, J(1)) & \xrightarrow{\mathrm{Qis}} & J[-3]. \end{array}$$ □

10.3.5.7. Corollary. — *The pairings*

$$\cup_{\overline{\pi},0,i,j} : \widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K, X)_{\overline{R}\text{-tors}} \otimes_{\overline{R}} \widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, X)^\iota_{\overline{R}\text{-tors}} \longrightarrow H^0(\omega_{\overline{R}}) \otimes_{\overline{R}} (\mathrm{Frac}(\overline{R})/\overline{R}) \quad (i+j=4)$$

satisfy

$$x \cup_{\overline{\pi},0,i,j} y = \iota\left(u_X(x)\, (\cup_{\overline{\pi},0,i,j})^\iota\, v_Y(y)\right) \qquad (i+j=4).$$

Proof. — This follows from Lemma 10.3.5.6 and (2.10.14.1). □

***10.3.5.8. Proposition* (Self-dual dihedral case)**. — *If, under the assumptions 10.3.5.1.1–10.3.5.1.5, $Y = X$, $\Delta(Y) = \Delta(X)$ and $\pi : X \otimes_R X \to \omega^\bullet_R(1)$ satisfies*

$$\pi \circ s_{12} = c\cdot\pi, \quad c = \pm 1,$$

then the formula

$$(x,y)_{i,j} := x \cup_{\overline{\pi},0,i,j} u_X(y) \qquad (i+j=4)$$

defines $\overline{R}$-bilinear pairings

$$(\ ,\ )_{i,j} : \widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K, X)_{\overline{R}\text{-tors}} \otimes_{\overline{R}} \widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, X)_{\overline{R}\text{-tors}} \longrightarrow H^0(\omega_{\overline{R}}) \otimes_{\overline{R}} (\mathrm{Frac}(\overline{R})/\overline{R}) \qquad (i+j=4)$$

in $({}_{\overline{R}}\mathrm{Mod})/(\textit{pseudo-null})$ *(or even in* $({}_{\overline{R}}\mathrm{Mod})$*, if* R *is Cohen-Macaulay) satisfying*

$$(x,y)_{i,j} = c(-1)^{ij}\,(y,x)_{j,i} \qquad (i+j=4).$$

In particular, $(\ ,\ )_{2,2}$ *is a symmetric (resp., skew-symmetric) bilinear form if* $c=+1$ *(resp.,* $c=-1$*).*

Proof. — Combining Corollary 10.3.5.7 with Proposition 10.3.4.2, we obtain

$$\begin{aligned}(x,y)_{i,j} &= x \cup_{\overline{\pi},0,i,j} u_X(y) = \iota\big(u_X(x)(\cup_{\overline{\pi},0,i,j})^{\iota} y\big)\\ &= c(-1)^{ij}\, y \cup_{\overline{\pi},0,j,i} u_X(x) = c(-1)^{ij}\,(y,x)_{j,i}.\end{aligned}$$ □

10.3.5.9. It is very likely that the conclusions of Corollary 10.3.5.7 and Proposition 10.3.5.8 still hold if the assumption 10.3.5.1.5 is weakened (it was included above in order to apply 8.7.6).

10.3.5.10. There is an alternative way of constructing $\overline{R}$-bilinear generalized Cassels-Tate pairings on $\widetilde{H}^*_{f,\mathrm{Iw}}(K_\infty/K,-)_{\overline{R}\text{-tors}}$ in the dihedral situation of 10.3.5.1. Mimicking the definition of an invariant bilinear form on irreducible two-dimensional representations of a dihedral group, we define a ($\overline{R}$-bilinear) pairing

$$\lambda : \mathscr{F}_{\Gamma^+}(X) \otimes_{\overline{R}} \mathscr{F}_{\Gamma^+}(Y) \longrightarrow (X \otimes_R Y) \otimes_R \overline{R}$$

by the formula

$$(10.3.5.1) \qquad \lambda(x \otimes (a_1 + \tau a_2) \otimes y \otimes (b_1 + \tau b_2)) = x \otimes y \otimes (a_1 \iota(b_2) + \iota(a_2) b_1).$$

It is easy to check that this pairing is, indeed, $\overline{R}$-bilinear, symmetric

$$(10.3.5.2) \qquad \lambda \circ s_{12} = (s_{12} \otimes \mathrm{id}) \circ \lambda$$

and G^+-invariant

$$(\forall g^+ \in G^+) \quad (g^+ \otimes \mathrm{id}) \circ \lambda = \lambda \circ (g^+ \otimes g^+).$$

If we denote by $\overline{\pi}'$ the composite pairing

$$\overline{\pi}' : \mathscr{F}_{\Gamma^+}(X) \otimes_{\overline{R}} \mathscr{F}_{\Gamma^+}(Y) \xrightarrow{\lambda} (X \otimes_R Y) \otimes_R \overline{R} \xrightarrow{\pi\otimes\mathrm{id}} \omega^\bullet_R(1) \otimes_R \overline{R} \xrightarrow{\varphi(1)} \omega^\bullet_{\overline{R}}(1),$$

then we have, for each $v^+ \in \Sigma^+$,

$$\mathscr{F}_{\Gamma^+}(X^+_{v^+}) \perp_{\overline{\pi}'} \mathscr{F}_{\Gamma^+}(Y^+_{v^+}).$$

If π is a perfect duality, so is $\overline{\pi}'$, by Corollary 8.4.6.5. If, in addition,

$$X^+_{v^+} \perp\!\!\perp_\pi Y^+_{v^+},$$

then

$$\mathscr{F}_{\Gamma^+}(X^+_{v^+}) \perp\!\!\perp_{\overline{\pi}'} \mathscr{F}_{\Gamma^+}(Y^+_{v^+})$$

(again by Corollary 8.4.6.5).

The cup products ($r \in \overline{R}$)

$$\cup_{\overline{\pi}',r,0} : \widetilde{C}^\bullet_f(G^+, \mathscr{F}_{\Gamma^+}(X)) \otimes_{\overline{R}} \widetilde{C}^\bullet_f(G^+, \mathscr{F}_{\Gamma^+}(Y)) \longrightarrow \omega^\bullet_{\overline{R}}[-3]$$

together with the quasi-isomorphisms sh_f^+ from 10.3.5.5 give rise to $\overline{R}$-bilinear pairings

$$\cup_{\overline{\pi}',0,i,j} : \widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K, X)_{\overline{R}\text{-tors}} \times \widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, Y)_{\overline{R}\text{-tors}} \longrightarrow H^0(\omega_{\overline{R}}) \otimes_{\overline{R}} (\mathrm{Frac}(\overline{R})/\overline{R}) \quad (i+j=4)$$

in $({}_{\overline{R}}\mathrm{Mod})/(\text{pseudo-null})$ (or even in $({}_{\overline{R}}\mathrm{Mod})$, if R is Cohen-Macaulay).

In the self-dual case, when $Y = X$, $\Delta(Y) = \Delta(X)$ and $\pi \circ s_{12} = c \cdot \pi$ ($c = \pm 1$), the symmetry property (10.3.5.2) implies that $\overline{\pi}' \circ s_{12} = c \cdot \overline{\pi}'$. Applying Proposition 10.2.5, we deduce that

$$x \cup_{\overline{\pi}',0,i,j} y = c(-1)^{ij} y \cup_{\overline{\pi}',0,j,i} x \qquad (i+j=4).$$

In particular, the pairing $\cup_{\overline{\pi}',0,2,2}$ is $\overline{R}$-bilinear symmetric (resp., skew-symmetric) if $c = +1$ (resp., if $c = -1$).

10.4. Localized Cassels-Tate pairings

In this section we drop the earlier assumptions on R (*i.e.*, R can have embedded primes), but we fix a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ satisfying $\dim(R_\mathfrak{p}) = \mathrm{depth}(R_\mathfrak{p}) = 1$ and consider only Selmer complexes localized at $\mathfrak{p}$.

10.4.1. We fix $\omega_R^\bullet = \sigma_{\geqslant 0}\, \omega_R^\bullet$ as usual and put $\omega_{R_\mathfrak{p}}^\bullet = (\omega_R^\bullet)_\mathfrak{p}$ – this is a bounded complex of injective $R_\mathfrak{p}$-modules representing $\omega_{R_\mathfrak{p}}$. Let Y_1, Y_2 be bounded complexes of admissible $R_\mathfrak{p}[G_{K,S}]$-modules (in the sense of 3.7.2) with cohomology of finite type over $R_\mathfrak{p}$ and

$$\pi(\mathfrak{p}) : Y_1 \otimes_{R_\mathfrak{p}} Y_2 \longrightarrow \omega_{R_\mathfrak{p}}^\bullet[n](1) \qquad (n \in \mathbf{Z})$$

a morphism of complexes of $R_\mathfrak{p}[G_{K,S}]$-modules. Assume that we are also given orthogonal local conditions

$$\Delta(Y_1) \perp_{\pi(\mathfrak{p}),h_S(\mathfrak{p})} \Delta(Y_2)$$

such that $U_S^+(Y_i)$ ($i = 1,2$) have cohomology of finite type over $R_\mathfrak{p}$. Under these assumptions, the Selmer complexes $\widetilde{C}_f^\bullet(Y_i) = \widetilde{C}_f^\bullet(G_{K,S}, Y_i; \Delta(Y_i))$ ($i = 1,2$) also have cohomology of finite type over $R_\mathfrak{p}$.

10.4.2. Example. — If $\pi : X_1 \otimes_R X_2 \to \omega_R^\bullet[n](1)$ and $\Delta(X_1) \perp_{\pi,h_S} \Delta(X_2)$ are as in 10.2.1, then the localizations

$$(Y_i, \Delta(Y_i), \pi(\mathfrak{p}), h_S(\mathfrak{p})) = ((X_i)_\mathfrak{p}, \Delta(X_i)_\mathfrak{p}, \pi_\mathfrak{p}, (h_S)_\mathfrak{p})$$

satisfy the assumptions of 10.4.1 and

$$\widetilde{C}_f^\bullet(Y_i) = \widetilde{C}_f^\bullet(X_i)_\mathfrak{p} \qquad (i = 1,2).$$

10.4.3. Under the assumptions of 10.4.1, a localized version of the cup products from 6.3.1

$$\cup_{\pi(\mathfrak{p}),r,h} : \widetilde{C}^{\bullet}_f(Y_1) \otimes_{R_\mathfrak{p}} \widetilde{C}^{\bullet}_f(Y_2) \longrightarrow \omega^{\bullet}_{R_\mathfrak{p}}[n-3] \qquad (r \in R_\mathfrak{p})$$

together with the construction in 2.10.7–2.10.9 define cup products

$$\cup_{\pi(\mathfrak{p}),h_S(\mathfrak{p}),i,j} : \widetilde{H}^i_f(Y_1)_{R_\mathfrak{p}\text{-tors}} \otimes_{R_\mathfrak{p}} \widetilde{H}^j_f(Y_2)_{R_\mathfrak{p}\text{-tors}} \longrightarrow H^0(\omega_{R_\mathfrak{p}}) \otimes_{R_\mathfrak{p}} (\mathrm{Frac}(R_\mathfrak{p})/R_\mathfrak{p}) \xrightarrow{\sim} I_{R_\mathfrak{p}} \qquad (i+j=4-n)$$

in $({}_{R_\mathfrak{p}}\mathrm{Mod})$ (independent of $r \in R_\mathfrak{p}$). If, in Example 10.4.2, R has no embedded primes, then $\cup_{\pi(\mathfrak{p}),r,h}$ (for $r \in R$) and $\cup_{\pi(\mathfrak{p}),h_S(\mathfrak{p}),i,j}$ are obtained from the corresponding products $\cup_{\pi,r,h}$ and $\cup_{\pi,h_S,i,j}$ from 10.2.2 by localizing at $\mathfrak{p}$.

If $\pi(\mathfrak{p})$ is a perfect duality in the sense that the map

$$\mathrm{adj}(\pi(\mathfrak{p})) : Y_1 \longrightarrow \mathrm{Hom}^{\bullet}_{R_\mathfrak{p}}(Y_2, \omega^{\bullet}_{R_\mathfrak{p}}[n](1)),$$

or, equivalently,

$$\mathrm{adj}(\pi(\mathfrak{p}) \circ s_{12}) : Y_2 \longrightarrow \mathrm{Hom}^{\bullet}_{R_\mathfrak{p}}(Y_1, \omega^{\bullet}_{R_\mathfrak{p}}[n](1)),$$

is a quasi-isomorphism (cf. 6.2.6), then the adjoint map

$$\gamma_{\pi(\mathfrak{p}),r,h_S(\mathfrak{p})} = \mathrm{adj}(\cup_{\pi(\mathfrak{p}),r,h}) : \widetilde{C}^{\bullet}_f(Y_1) \longrightarrow \mathrm{Hom}^{\bullet}_{R_\mathfrak{p}}\big(\widetilde{C}^{\bullet}_f(Y_2), \omega^{\bullet}_{R_\mathfrak{p}}[n-3]\big)$$

appears in the following exact triangle in $D_{ft}({}_{R_\mathfrak{p}}\mathrm{Mod})$:

$$\widetilde{\mathbf{R}\Gamma}_f(Y_1) \xrightarrow{\gamma_{\pi(\mathfrak{p}),r,h_S(\mathfrak{p})}} \mathscr{D}_{R_\mathfrak{p}}\big(\widetilde{\mathbf{R}\Gamma}_f(Y_2)\big)[n-3] \longrightarrow \mathrm{Err}(\Delta(Y_1), \Delta(Y_2), \pi(\mathfrak{p})) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(Y_1)[1].$$

Again, in the situation of 10.4.2, we have $\gamma_{\pi(\mathfrak{p}),r,h_S(\mathfrak{p})} = (\gamma_{\pi,r,h_S})_\mathfrak{p}$ (if $r \in R$) and

$$\mathrm{Err}(\Delta(Y_1), \Delta(Y_2), \pi(\mathfrak{p})) = \mathrm{Err}(\Delta(X_1), \Delta(X_2), \pi)_\mathfrak{p}.$$

Lemma 2.10.11 gives isomorphisms of $R_\mathfrak{p}$-modules

$$D_{R_\mathfrak{p}}\big(\widetilde{H}^j_f(Y_2)_{R_\mathfrak{p}\text{-tors}}\big) \xrightarrow{\sim} \mathrm{Hom}_{R_\mathfrak{p}}\big(\widetilde{H}^j_f(Y_2)_{R_\mathfrak{p}\text{-tors}}, H^0(\omega_{R_\mathfrak{p}}) \otimes_{R_\mathfrak{p}} (\mathrm{Frac}(R_\mathfrak{p})/R_\mathfrak{p})\big) \xrightarrow{\sim} \mathrm{Ext}^1_{R_\mathfrak{p}}\big(\widetilde{H}^j_f(Y_2), \omega_{R_\mathfrak{p}}\big) \xrightarrow{\sim} H^i\big(\mathscr{D}_{R_\mathfrak{p}}\big(\widetilde{\mathbf{R}\Gamma}_f(Y_2)\big)[n-3]\big)_{R_\mathfrak{p}\text{-tors}}, \qquad (i+j=4-n)$$

the composition of which with

$$\mathrm{adj}(\cup_{\pi(\mathfrak{p}),h_S(\mathfrak{p}),i,j}) : \widetilde{H}^i_f(Y_1)_{R_\mathfrak{p}\text{-tors}} \longrightarrow D_{R_\mathfrak{p}}\big(\widetilde{H}^j_f(Y_2)_{R_\mathfrak{p}\text{-tors}}\big) \qquad (i+j=4-n)$$

coincides, up to a sign, with the restriction of the map

$$(\gamma_{\pi(\mathfrak{p}),r,h_S(\mathfrak{p})})_* : \widetilde{H}^i_f(Y_1) \longrightarrow H^i\big(\mathscr{D}_{R_\mathfrak{p}}(\widetilde{\mathbf{R}\Gamma}_f(Y_2))[n-3]\big).$$

10.4.4. Theorem. — *Under the assumptions of 10.4.1, let* $i+j=4-n$ *and* $\mathrm{Err} = \mathrm{Err}(\Delta(Y_1), \Delta(Y_2), \pi(\mathfrak{p}))$. *Assume that* $\pi(\mathfrak{p})$ *is a perfect duality. Then, in the category* $({}_{R_\mathfrak{p}}\mathrm{Mod})$,

(i) $\mathrm{Ker}(\mathrm{adj}(\cup_{\pi(\mathfrak{p}),h_S(\mathfrak{p}),i,j}))$ *is isomorphic to a subquotient of* $H^{i-1}(\mathrm{Err})$.

(ii) *If* $H^{i-1}(\mathrm{Err})$ *is* $R_{\mathfrak{p}}$*-torsion, then* $\mathrm{Coker}(\mathrm{adj}(\cup_{\pi(\mathfrak{p}),h_S(\mathfrak{p}),i,j}))$ *(respectively,* $\mathrm{Ker}(\mathrm{adj}(\cup_{\pi(\mathfrak{p}),h_S(\mathfrak{p}),i,j}))$*) is isomorphic to a submodule (resp., a quotient) of* $H^i(\mathrm{Err})$ *(resp., of* $H^{i-1}(\mathrm{Err})$*).*

In particular, if $H^{i-1}(\mathrm{Err}) = H^i(\mathrm{Err}) \xrightarrow{\sim} 0$*, then* $\mathrm{adj}(\cup_{\pi,h_S,i,j})$ *is an isomorphism of* $R_{\mathfrak{p}}$*-modules.*

Proof. — This follows from Proposition 2.10.12 applied to $\cup_{\pi(\mathfrak{p}),r,h}$. ☐

10.4.5. Under the assumptions of 10.4.1, the pairings $\cup_{\pi(\mathfrak{p}),h_S(\mathfrak{p}),i,j}$ satisfy the obvious analogues of 10.2.4–10.2.7, with the same proofs.

10.5. Greenberg's local conditions - localized version

Let $R, \mathfrak{p}, \omega_R^\bullet$ and $\omega_{R_{\mathfrak{p}}}^\bullet$ be as in 10.4.

10.5.1. A special case of the data from 10.4.1 is the following: $Y_1 = X(\mathfrak{p})$, $Y_2 = Y(\mathfrak{p})$ and $\pi(\mathfrak{p}) : X(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} Y(\mathfrak{p}) \to \omega_{R_{\mathfrak{p}}}^\bullet(1)$ are as in 10.4.1, $S_f = \Sigma \cup \Sigma'$ are as in 7.8, and for each $Z = X(\mathfrak{p}), Y(\mathfrak{p})$ and $v \in \Sigma$ we are given a bounded complex of admissible $R_{\mathfrak{p}}[G_v]$-modules Z_v^+ with cohomology of finite type over $R_{\mathfrak{p}}$ and a morphism of complexes of $R_{\mathfrak{p}}[G_v]$-modules

$$j_v^+(Z) : Z_v^+ \longrightarrow Z \qquad (v \in \Sigma);$$

we assume that

$$X(\mathfrak{p})_v^+ \perp_{\pi(\mathfrak{p})} Y(\mathfrak{p})_v^+ \qquad (v \in \Sigma).$$

These data define the usual local conditions ($Z = X(\mathfrak{p}), Y(\mathfrak{p})$)

$$\Delta_v(Z) = \begin{cases} C^\bullet_{\mathrm{cont}}(G_v, Z_v^+) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v, Z), & (v \in \Sigma) \\ C^\bullet_{\mathrm{ur}}(G_v, Z) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v, Z), & (v \in \Sigma') \end{cases}$$

satisfying

$$\Delta(X(\mathfrak{p})) \perp_{\pi(\mathfrak{p}),0} \Delta(Y(\mathfrak{p})),$$

which admit transposition operators satisfying a localized version of 6.5.3.1–6.5.3.5. If $\pi(\mathfrak{p})$ is a perfect duality, then the error terms

$$\mathrm{Err}_v = \mathrm{Err}_v(\Delta_v(X(\mathfrak{p})), \Delta_v(Y(\mathfrak{p})), \pi(\mathfrak{p}))$$

in $D_{ft}({}_{R_{\mathfrak{p}}}\mathrm{Mod})$ are as follows:

For $v \in \Sigma$,

$$\mathrm{Err}_v \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, W_v),$$

where W_v sits in an exact triangle

$$W_v \longrightarrow X(\mathfrak{p})_v^- \longrightarrow \mathscr{D}_{R_{\mathfrak{p}}}(Y(\mathfrak{p})_v^+) \longrightarrow W_v[1]$$

in $D\big({}^{\mathrm{ad}}_{R_{\mathfrak{p}}[G_v]}\mathrm{Mod}\big)$, by a localized version of Proposition 6.7.6 (iv).

For $v \in \Sigma'$,

$$\mathrm{Err}_v \xrightarrow{\sim} \left[H^0_{\{\mathfrak{p}\}}\left(H^1_{\mathrm{cont}}(I_v, X(\mathfrak{p}))\right) \xrightarrow{f_v - 1} H^0_{\{\mathfrak{p}\}}\left(H^1_{\mathrm{cont}}(I_v, X(\mathfrak{p}))\right)\right],$$

where the complex on the R.H.S. is in degrees $1, 2$, and

$$\ell_{R_\mathfrak{p}}\left(H^i(\mathrm{Err}_v)\right) = \begin{cases} \mathrm{Tam}_v(X(\mathfrak{p}), \mathfrak{p}), & i = 1, 2 \\ 0, & i \neq 1, 2, \end{cases}$$

by a localized version of 7.6.10.7.

10.5.2. Example. — Let $\pi : X \otimes_R Y \to \omega^\bullet_R(1)$ be as in 10.3.1; then the localized data

$$(X(\mathfrak{p}), Y(\mathfrak{p}), \pi(\mathfrak{p}), X(\mathfrak{p})^+_v, Y(\mathfrak{p})^+_v) = (X_\mathfrak{p}, Y_\mathfrak{p}, \pi_\mathfrak{p}, (X^+_v)_\mathfrak{p}, (Y^+_v)_\mathfrak{p})$$

are as in 10.5.1 and

$$\mathrm{Err}_v = \mathrm{Err}_v(\Delta(X), \Delta(Y), \pi)_\mathfrak{p} \qquad (v \in S_f).$$

10.5.3. Under the assumptions of 10.5.1, we obtain from 10.4 bilinear forms in $({}_{R_\mathfrak{p}}\mathrm{Mod})$

$$\cup_{\pi(\mathfrak{p}),0,i,j} : \widetilde{H}^i_f(X(\mathfrak{p}))_{R_\mathfrak{p}\text{-tors}} \otimes_{R_\mathfrak{p}} \widetilde{H}^j_f(Y(\mathfrak{p}))_{R_\mathfrak{p}\text{-tors}} \longrightarrow I_{R_\mathfrak{p}} \qquad (i + j = 4)$$

satisfying the formula (10.3.2.1). The kernels and cokernels of the adjoint maps

$$\mathrm{adj}(\cup_{\pi(\mathfrak{p}),0,i,j}) : \widetilde{H}^i_f(X(\mathfrak{p}))_{R_\mathfrak{p}\text{-tors}} \longrightarrow D_{R_\mathfrak{p}}\left(\widetilde{H}^j_f(Y(\mathfrak{p}))_{R_\mathfrak{p}\text{-tors}}\right) \qquad (i + j = 4)$$

are as in Theorem 10.4.4.

10.5.4. Iwasawa theory. — Let $\Gamma = \mathrm{Gal}(K_\infty/K)$ and $\overline{R} = R[\![\Gamma]\!]$ be as in 10.3.3. Fix φ and ν as in 10.3.3.1. Let $\overline{R}_{\overline{\mathfrak{p}}} \in \mathrm{Spec}(\overline{R})$ be a prime ideal satisfying $\dim(\overline{R}_{\overline{\mathfrak{p}}}) = \mathrm{depth}(\overline{R}_{\overline{\mathfrak{p}}}) = 1$. One can obtain data of the type 10.5.1 over $\overline{R}$ as follows.

10.5.4.1. Assume that $\mathfrak{p} \in \mathrm{Spec}(R)$, $\mathfrak{p} \supset R \cap \overline{\mathfrak{p}}$ and we are given the data from 10.5.1 (except that we *do not* assume that $\dim(R_\mathfrak{p}) = \mathrm{depth}(R_\mathfrak{p}) = 1$) such that all $Z(\mathfrak{p})^i, (Z(\mathfrak{p})^+_v)^i$ ($Z = X, Y$, $v \in \Sigma$, $i \in \mathbf{Z}$) are of finite type over $R_\mathfrak{p}$. Then a localized version of the recipe in 8.9.2 defines Greenberg's local conditions for

$$\mathscr{F}_\Gamma\big(Z(\mathfrak{p})\big)_{\overline{\mathfrak{p}}} = \big(Z(\mathfrak{p}) \otimes_R \overline{R}\big)_{\overline{\mathfrak{p}}} < -1 >= \big(Z(\mathfrak{p}) \otimes_{R_\mathfrak{p}} \overline{R}_{\overline{\mathfrak{p}}}\big) < -1 >$$

$$\mathscr{F}_\Gamma\big(Z(\mathfrak{p})\big)^\iota_{\overline{\mathfrak{p}}} = \big(Z(\mathfrak{p}) \otimes_R \overline{R}^\iota\big)_{\overline{\mathfrak{p}}} < -1 >= \big(Z(\mathfrak{p}) \otimes_{R_\mathfrak{p}} \overline{R}^\iota_{\overline{\mathfrak{p}}}\big) < -1 >,$$

together with a pairing

$$\pi(\overline{\mathfrak{p}}) : \mathscr{F}_\Gamma\big(X(\mathfrak{p})\big)_{\overline{\mathfrak{p}}} \otimes_{\overline{R}_{\overline{\mathfrak{p}}}} \mathscr{F}_\Gamma\big(Y(\mathfrak{p})\big)^\iota_{\overline{\mathfrak{p}}} \longrightarrow \omega^\bullet_{\overline{R}_{\overline{\mathfrak{p}}}}(1),$$

under which

$$\left(\mathscr{F}_\Gamma\big(X(\mathfrak{p})\big)_{\overline{\mathfrak{p}}}\right)^+_v \perp_{\pi(\overline{\mathfrak{p}})} \left(\mathscr{F}_\Gamma\big(Y(\mathfrak{p})\big)^\iota_{\overline{\mathfrak{p}}}\right)^+_v \qquad (v \in \Sigma).$$

Above, $(-)^\iota_{\overline{\mathfrak{p}}}$ is a shorthand for $((-)^\iota)_{\overline{\mathfrak{p}}}$.

10.5.4.2. Lemma. — *In the situation of 10.5.4.1, assume that the localization of* $\pi(\mathfrak{p})$ *at the prime ideal* $\mathfrak{q} = R \cap \overline{\mathfrak{p}} \in \mathrm{Spec}(R)$ *is a perfect pairing* $\pi(\mathfrak{p})_{\mathfrak{q}} : X(\mathfrak{p})_{\mathfrak{q}} \otimes_{R_{\mathfrak{q}}} Y(\mathfrak{p})_{\mathfrak{q}} \to \omega^{\bullet}_{R_{\mathfrak{q}}}(1)$ *over* $R_{\mathfrak{q}}$ *and* $(X(\mathfrak{p})^+_v)_{\mathfrak{q}} \perp\!\!\perp_{\pi(\mathfrak{p})_{\mathfrak{q}}} (Y(\mathfrak{p})^+_v)_{\mathfrak{q}}$ *for all* $v \in \Sigma$. *Then* $\pi(\overline{\mathfrak{p}})$ *is a perfect pairing and*

$$\left(\mathscr{F}_{\Gamma}\left(X(\mathfrak{p})\right)_{\overline{\mathfrak{p}}}\right)^+_v \perp\!\!\perp_{\pi(\overline{\mathfrak{p}})} \left(\mathscr{F}_{\Gamma}\left(Y(\mathfrak{p})\right)^{\iota}_{\overline{\mathfrak{p}}}\right)^+_v \qquad (\forall v \in \Sigma).$$

Proof. — This follows from the flatness of $\overline{R}_{\overline{\mathfrak{p}}}$ over $R_{\mathfrak{q}}$ and the fact that

$$\mathscr{F}_{\Gamma}\left(Z(\mathfrak{p})\right)_{\overline{\mathfrak{p}}} < 1 > = \left(Z(\mathfrak{p})_{\mathfrak{q}}\right) \otimes_{R_{\mathfrak{q}}} \overline{R}_{\overline{\mathfrak{p}}},$$

$$\mathscr{F}_{\Gamma}\left(Z(\mathfrak{p})^+_v\right)_{\overline{\mathfrak{p}}} < 1 > = \left((Z(\mathfrak{p})^+_v)_{\mathfrak{q}}\right) \otimes_{R_{\mathfrak{q}}} \overline{R}_{\overline{\mathfrak{p}}} \qquad (Z = X, Y). \qquad \square$$

10.5.4.3. Alternatively, given $\pi : X \otimes_R Y \to \omega^{\bullet}_R(1)$ and Z^+_v $(Z = X, Y,\ v \in \Sigma)$ as in 10.3.1, we can localize $\mathscr{F}_{\Gamma}(X)$, $\mathscr{F}_{\Gamma}(Y)^{\iota}$ and $\overline{\pi}$ from 10.3.3.2 at $\overline{\mathfrak{p}}$.

10.5.5. Under the assumptions of 10.5.4.1 (resp., 10.5.4.3), we can apply 10.5.3 and obtain bilinear forms in $({}_{\overline{R}_{\overline{\mathfrak{p}}}}\mathrm{Mod})$ (for $i + j = 4$)

$$(10.5.5.1)\quad \cup_{\pi(\overline{\mathfrak{p}}),0,i,j} : \left(\widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K, X(\mathfrak{p}))_{\overline{\mathfrak{p}}}\right)_{(\overline{R}_{\overline{\mathfrak{p}}})\text{-tors}} \otimes_{\overline{R}_{\overline{\mathfrak{p}}}} \left(\widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, Y(\mathfrak{p}))^{\iota}_{\overline{\mathfrak{p}}}\right)_{(\overline{R}_{\overline{\mathfrak{p}}})\text{-tors}} \longrightarrow I_{\overline{R}_{\overline{\mathfrak{p}}}}$$

resp.,

$$(10.5.5.2)\quad \cup_{\overline{\pi}_{\overline{\mathfrak{p}}},0,i,j} : \left(\widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K, X)_{\overline{\mathfrak{p}}}\right)_{(\overline{R}_{\overline{\mathfrak{p}}})\text{-tors}} \otimes_{\overline{R}_{\overline{\mathfrak{p}}}} \left(\widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, Y)^{\iota}_{\overline{\mathfrak{p}}}\right)_{(\overline{R}_{\overline{\mathfrak{p}}})\text{-tors}} \longrightarrow I_{\overline{R}_{\overline{\mathfrak{p}}}}.$$

All results in 10.3.3–10.3.5 hold (with the same proofs) for the pairings (10.5.5.1)–(10.5.5.2).

If R has no embedded primes (and if the data 10.5.4.1 are of the kind considered in 10.5.2), then the pairings (10.5.5.1)-(10.5.5.2) are obtained from those in 10.3.3.3 by localizing at $\overline{\mathfrak{p}}$.

10.6. Discrete valuation rings with involution

10.6.1. An *involution* on a (commutative) ring $\mathcal{O}$ is a ring isomorphism $\iota : \mathcal{O} \to \mathcal{O}$ satisfying $\iota \circ \iota = \mathrm{id}$. A typical example is the standard involution on $\overline{R} = R[\![\Gamma]\!]$.

10.6.1.1. If $\mathcal{O}$ is a discrete valuation ring with prime element $\varpi \in \mathcal{O}$, then each involution $\iota : \mathcal{O} \to \mathcal{O}$ maps the maximal ideal $\varpi\mathcal{O}$ to itself, hence induces an involution ι_k on the residue field $k = \mathcal{O}/\varpi\mathcal{O}$ and $\iota(\varpi)/\varpi \in \mathcal{O}^*$ is a unit; denote by $\varepsilon \in k^*$ its image in k^*, which satisfies

$$\varepsilon \cdot \iota_k(\varepsilon) = 1.$$

If we replace ϖ by $\varpi' = u\varpi$ $(u \in \mathcal{O}^*)$, then ε is replaced by $\varepsilon' = (\iota_k(\overline{u})/\overline{u})\varepsilon$, where $\overline{u} \in k^*$ denotes the image of u in k^*.

10.6.1.2. If $\iota_k = \mathrm{id}$, then $\varepsilon = \pm 1$, hence

$$\iota(\varpi) \equiv \pm\varpi \pmod{\varpi^2}.$$

10.6.1.3. Lemma. — *If* $\iota_k = \mathrm{id}$, $\varepsilon = 1$ *and* $2 \in \mathcal{O}^*$, *then* $\iota = \mathrm{id}$.

Proof. — Assume that $\iota(u) \neq u$ for some $u \in \mathcal{O}^*$; then $\iota(u) = u + v\varpi^n$, where $v \in \mathcal{O}^*$ and $n \geqslant 1$. This implies that $0 = v\varpi^n + \iota(v)\iota(\varpi)^n \equiv 2v\varpi^n \pmod{\varpi^{n+1}}$, which is impossible. Thus ι acts trivially on $\mathcal{O}^*$ and also on ϖ, as $\iota(\varpi) = \iota(1+\varpi) - \iota(1) = \varpi$. $\square$

10.6.1.4. If $\iota_k \neq \mathrm{id}$, then k is a quadratic Galois extension of $k_+ = k^{\iota_k=1}$, with $\mathrm{Gal}(k/k_+) = \{\mathrm{id}, \iota_k\}$. Hilbert's Theorem 90 implies that there is $\overline{u} \in k^*$ satisfying $\varepsilon = \overline{u}/\iota_k(\overline{u})$; replacing ϖ by $\varpi' = u\varpi$ (where $u \in \mathcal{O}^*$ is any lift of $\overline{u} \in k^*$), we have

$$\iota(\varpi') \equiv \varpi' \pmod{\varpi'^2}.$$

10.6.2. A typical example of a discrete valuation ring with involution is the localization $\mathcal{O} = \overline{R}_{\overline{\mathfrak{p}}}$ of $\overline{R} = R[\![\Gamma]\!]$ at a regular prime ideal $\overline{\mathfrak{p}} \in \mathrm{Spec}(\overline{R})$ satisfying $\mathrm{ht}(\overline{\mathfrak{p}}) = 1$ and $\iota(\overline{\mathfrak{p}}) = \overline{\mathfrak{p}}$.

10.6.3. In particular, assume that R is an integral domain, flat over $\mathbf{Z}_p$, and that $\Gamma \xrightarrow{\sim} \mathbf{Z}_p$ $(\gamma \mapsto 1)$. If

$$f = (\gamma-1)^n + a_1(\gamma-1)^{n-1} + \cdots + a_n \in R[\gamma-1] \qquad (a_1, \ldots, a_n \in \mathfrak{m})$$

is an irreducible distinguished polynomial of degree $n \geqslant 1$, then $\overline{\mathfrak{p}} = (f)$ is a prime ideal $\overline{\mathfrak{p}} \subset \overline{R} = R[\![\Gamma]\!] \xrightarrow{\sim} R[\![\gamma-1]\!]$ satisfying $\mathrm{ht}(\overline{\mathfrak{p}}) = 1$ and $\overline{\mathfrak{p}} \cap R = (0)$. The localization $\mathcal{O} = \overline{R}_{\overline{\mathfrak{p}}} = R[\![\gamma-1]\!]_{(f)}$ is a discrete valuation ring with prime element f. Factorizing f into linear factors

$$f = (\gamma-\alpha_1)\cdots(\gamma-\alpha_n)$$

over a suitable finite extension of $\mathrm{Frac}(R)$, we have

$$\overline{\mathfrak{p}} = \iota(\overline{\mathfrak{p}}) \Longrightarrow Z(f) := \{\alpha_1, \ldots, \alpha_n\} = \{\alpha_1^{-1}, \ldots, \alpha_n^{-1}\}.$$

There are three mutually exclusive cases:

10.6.3.1. $1 \in Z(f) \Longrightarrow \overline{\mathfrak{p}} = (\gamma-1)$, $k = \mathrm{Frac}(R)$, $\iota_k = \mathrm{id}$, $\varepsilon = -1$.

10.6.3.2. $-1 \in Z(f) \Longrightarrow \overline{\mathfrak{p}} = (\gamma+1)$, $p = 2$, $k = \mathrm{Frac}(R)$, $\iota_k = \mathrm{id}$, $\varepsilon = -1 = 1$.

10.6.3.3. $\{\pm 1\} \cap Z(f) = \varnothing \Longrightarrow \gamma \not\equiv \gamma^{-1} \pmod{f\mathcal{O}} \Longrightarrow \iota_k \neq \mathrm{id}$.

10.6.4. Lemma. — *Let* $\mathcal{O}$ *be a discrete valuation ring with involution* ι. *Assume that either* $2 \in \mathcal{O}^*$, *or that* $\mathcal{O}$ *is complete. Then there exist a prime element* $\varpi \in \mathcal{O}$ *and* $\varepsilon = \pm 1$ *satisfying* $\iota(\varpi) = \varepsilon\varpi$.

Proof. — According to 10.6.1.2 and 10.6.1.4 there exist $\varepsilon = \pm 1$ and a prime element $\varpi_0 \in \mathcal{O}$ satisfying $\iota(\varpi_0) \equiv \varepsilon\varpi_0 \pmod{\varpi_0^2}$. If $2 \in \mathcal{O}^*$, then the elements $x_{\pm\varepsilon} := \varpi_0 \pm \varepsilon\iota(\varpi_0)$ satisfy $\iota(x_{\pm\varepsilon}) = \pm\varepsilon x_{\pm\varepsilon}$, and at least one of them is a prime element of $\mathcal{O}$. If $\mathcal{O}$ is complete, it is enough to construct a sequence of prime elements $\varpi_1, \varpi_2, \ldots$ satisfying

$$\varpi_n \equiv \varpi_{n-1} \pmod{\varpi_0^{n+1}}, \quad \iota(\varpi_n) \equiv \varepsilon\varpi_n \pmod{\varpi_0^{n+2}} \qquad (n \geqslant 1),$$

as $\varpi = \lim \varpi_n$ will have the desired property. Given ϖ_n ($n \geqslant 0$), put $u = (\iota(\varpi_n) - \varepsilon\varpi_n)/\varpi_n^{n+2} \in \mathcal{O}$. If $u \notin \mathcal{O}^*$, let $\varpi_{n+1} = \varpi_n$. If $u \in \mathcal{O}^*$, denote by $\overline{u} \in k^*$ its image in k^*. The exactness of the sequence

$$\cdots \xrightarrow{\mathrm{id}-\iota_k} k \xrightarrow{\mathrm{id}+\iota_k} k \xrightarrow{\mathrm{id}-\iota_k} k \xrightarrow{\mathrm{id}+\iota_k} \cdots$$

implies that there exists $\overline{a} \in k$ satisfying $(\varepsilon^{n-1}\iota_k - \mathrm{id})\overline{a} = -\varepsilon\overline{u}$. The element $\varpi_{n+1} = \varpi_n + a\varpi_n^{n+2}$ (where $a \in \mathcal{O}$ is any lift of $\overline{a}$) has the desired properties, as $\varpi_{n+1} \equiv \varpi_n \pmod{\varpi_0^{n+2}}$ and

$$\begin{aligned}\iota(\varpi_{n+1}) - \varepsilon\varpi_{n+1} &= (u - \varepsilon a)\varpi_n^{n+2} + \iota(a)(\varepsilon\varpi_n + u\varpi_n^{n+2})^{n+2} \\ &\equiv (u + \varepsilon^n\iota(a) - \varepsilon a)\varpi_n^{n+2} \equiv 0 \pmod{\varpi_0^{n+3}}.\end{aligned}$$

□

10.6.5. Lemma. — *Let $\mathcal{O}$ be a discrete valuation ring with fraction field F, $\varpi \in \mathcal{O}$ a prime element and $k = \mathcal{O}/\varpi\mathcal{O}$ the residue field. Assume that $\varepsilon \in \{\pm 1\}$ and $\iota : \mathcal{O} \to \mathcal{O}$ is an involution satisfying $\iota(\varpi) \equiv \varepsilon\varpi \pmod{\varpi^2}$. Let N be an $\mathcal{O}$-module of co-finite type and*

$$\langle\ ,\ \rangle : N \times N \longrightarrow F/\mathcal{O}$$

a skew-Hermitian form, i.e., *a bi-additive map satisfying*

$$\langle \lambda x, \mu y\rangle = \lambda\iota(\mu)\langle x, y\rangle, \quad \langle y, x\rangle = -\iota(\langle x, y\rangle) \qquad (\forall \lambda, \mu \in \mathcal{O}, \forall x, y \in N).$$

Assume that the kernel of $\langle\ ,\ \rangle$ is equal to N_{div} (= the maximal $\mathcal{O}$-divisible submodule of N). Then:

(i) *The k-vector space $F^1 := N[\varpi]$ has a canonical decreasing filtration*

$$F^1 \supseteq F^2 \supseteq \cdots$$

by the k-subspaces $F^j = N[\varpi] \cap \varpi^{j-1}N$.

(ii) $F^\infty := \bigcap_{j\geqslant 1} F^j = N[\varpi] \cap N_{\mathrm{div}}$.

(iii) *The formula*

$$\left\langle \varpi^{j-1}x, \varpi^{j-1}y\right\rangle_{j,\varpi} := \varpi^j\left(\langle x, y\rangle \pmod{\varpi^{-j+1}}\right) \in \mathcal{O}/\varpi\mathcal{O} = k \qquad (x, y \in N[\varpi^j])$$

defines an $(-\varepsilon^j)$-Hermitian form

$$\langle\ ,\ \rangle_{j,\varpi} : F^j \otimes_k F^j \longrightarrow k,$$

i.e., *a bi-additive map satisfying*

$$\langle \lambda x, \mu y\rangle_{j,\varpi} = \lambda\iota_k(\mu)\langle x, y\rangle_{j,\varpi},\ \langle y, x\rangle_{j,\varpi} = -\varepsilon^j\iota_k(\langle x, y\rangle_{j,\varpi}) \quad (\forall \lambda, \mu \in k, \forall x, y \in F^j),$$

with kernel equal to F^{j+1}, *hence a non-degenerate* $(-\varepsilon^j)$*-Hermitian form*

$$\langle\ ,\ \rangle_{j,\varpi} : \mathrm{gr}_F^j \times \mathrm{gr}_F^j \longrightarrow k$$

on $\mathrm{gr}_F^j = F^j/F^{j+1}$.

(iv) *If we replace* $\varpi \in \mathcal{O}$ *by another prime element* ϖ' *satisfying* $\iota(\varpi') \equiv \varepsilon\varpi' \pmod{\varpi'^2}$, *then*

$$\langle\ ,\ \rangle_{j,\varpi'} = u^{2-j}\langle\ ,\ \rangle_{j,\varpi},$$

where $u \in k^*$ *is the image of* $\varpi'/\varpi \in \mathcal{O}^*$ *in* k^* *(it satisfies* $\iota_k(u) = u$*).*

(v) *Assume that* $\iota_k = \mathrm{id}$ *and* $\varepsilon^j = 1$. *If* $2 \in \mathcal{O}^*$ *(more generally, if the pairing* $\langle\ ,\ \rangle$ *is alternating,* i.e., *if* $\langle x, x\rangle = 0$ *for all* $x \in N$*), then* $\langle\ ,\ \rangle_{j,\varpi}$ *is a symplectic (= alternating and non-degenerate) form on* gr_F^j, *hence*

$$\dim_k(F^j) \equiv \dim_k(F^{j+1}) \pmod 2.$$

(vi) *Assume that* $\iota = \mathrm{id}$. *If* $2 \in \mathcal{O}^*$ *(more generally, if the pairing* $\langle\ ,\ \rangle$ *is alternating), then*

$$(\forall j \geqslant 1) \quad \dim_k(F^j) \equiv \dim_k(F^\infty) \pmod 2.$$

In particular,

$$\dim_k(N[\varpi])(= \dim_k(F^1)) \equiv \mathrm{cork}_{\mathcal{O}}(N)(= \dim_k(F^\infty)) \pmod 2.$$

Proof. — Elementary linear algebra. □

***10.6.6. Lemma* (Dihedral case).** — *In the situation of 10.6.5, assume that we are given a bijective additive map* $\tau : N \to N$ *satisfying* $\tau \circ \tau = \mathrm{id}$ *and*

$$\tau(\lambda x) = \iota(\lambda)\tau(x), \quad \langle \tau x, \tau y\rangle = \iota(\langle x, y\rangle) \qquad (\forall \lambda \in \mathcal{O}, \forall x, y \in N).$$

Then:

(i) *The formula* $(x, y) = \langle x, \tau y\rangle$ *defines an* $\mathcal{O}$*-bilinear skew-symmetric pairing* $(\ ,\ ) : N \times N \to F/\mathcal{O}$ *with kernel* N_{div}.

(ii) *The filtration* F^j *on* $F^1 = N[\varpi]$ *is* τ*-stable and the pairings* $\langle\ ,\ \rangle_{j,\varpi}$ *satisfy*

$$\langle \tau x, \tau y\rangle_{j,\varpi} = \varepsilon^j \iota_k(\langle x, y\rangle_{j,\varpi}) \qquad (x, y \in \mathrm{gr}_F^j).$$

(iii) *If* $2 \in \mathcal{O}^*$ *(more generally, if the pairing* $(\ ,\ )$ *is alternating), then*

$$(\forall j \geqslant 1) \quad \dim_k(F^j) \equiv \dim_k(F^\infty) \pmod 2.$$

In particular,

$$\dim_k(N[\varpi]) (= \dim_k(F^1)) \equiv \mathrm{cork}_{\mathcal{O}}(N) (= \dim_k(F^\infty)) \pmod 2.$$

Proof. — (i) and (ii) follow from the definition of $(\ ,\ )$, while (iii) is a consequence of Lemma 10.6.5 applied to the pairing $(\ ,\ )$ and the trivial involution id. □

10.6.7. The conclusion of Lemma 10.6.6 (iii) can be proved without introducing the pairing $(\ ,\ )$, at least in the following special case.

10.6.8. Lemma. — *In the situation of 10.6.6, assume that* $\varepsilon = -1$, $\iota_k = \mathrm{id}$ *and* $2 \in \mathcal{O}^*$. *For* $\eta, \eta' = \pm$ *denote by* $\langle\ ,\ \rangle_{j,\varpi}^{\eta\eta'}$ *the restriction of* $\langle\ ,\ \rangle_{j,\varpi}$ *to* $(\mathrm{gr}_F^j)^\eta \times (\mathrm{gr}_F^j)^{\eta'}$, *where* $(\mathrm{gr}_F^j)^\pm = (\mathrm{gr}_F^j)^{\tau=\pm 1}$. *Let* $j \geqslant 1$*; then:*

(i) *If* $2 \nmid j$, *then* $\langle\ ,\ \rangle_{j,\varpi}^{++} = \langle\ ,\ \rangle_{j,\varpi}^{--} = 0$ *and*

$$\langle\ ,\ \rangle_{j,\varpi}^{+-} : (\mathrm{gr}_F^j)^+ \times (\mathrm{gr}_F^j)^- \longrightarrow k$$

is a non-degenerate k*-bilinear form. In particular,*

$$\dim_k(\mathrm{gr}_F^j)^+ = \dim_k(\mathrm{gr}_F^j)^-, \quad \dim_k(\mathrm{gr}_F^j) = 2\dim_k(\mathrm{gr}_F^j)^+ \equiv 0 \pmod 2.$$

(ii) *If* $2|j$, *then* $\langle\ ,\ \rangle_{j,\varpi}^{+-} = \langle\ ,\ \rangle_{j,\varpi}^{-+} = 0$ *and* $\langle\ ,\ \rangle_{j,\varpi}^{++}$ *(resp.,* $\langle\ ,\ \rangle_{j,\varpi}^{--}$*) is a symplectic (= alternating and non-degenerate) form on* $(\mathrm{gr}_F^j)^+$ *(resp., on* $(\mathrm{gr}_F^j)^-$*). In particular,*

$$\dim_k(\mathrm{gr}_F^j)^\pm \equiv 0 \pmod 2, \quad \dim_k(\mathrm{gr}_F^j) \equiv 0 \pmod 2.$$

(iii) *We have*

$$(\forall j \geqslant 1) \quad \dim_k(F^j) \equiv \dim_k(F^\infty) \pmod 2.$$

In particular,

$$\dim_k(N[\varpi]) \,(= \dim_k(F^1)) \equiv \mathrm{cork}_{\mathcal{O}}(N) \,(= \dim_k(F^\infty)) \pmod 2.$$

Proof

(i), (ii) As

$$\langle \tau x, \tau y\rangle_{j,\varpi} = \varepsilon^j \langle x, y\rangle_{j,\varpi} = (-1)^j \langle x, y\rangle_{j,\varpi} \qquad (x, y \in \mathrm{gr}_F^j)$$

and $-1 \neq 1$ in k, it follows that $\langle\ ,\ \rangle_{j,\varpi}^{\eta\eta'} = 0$ if $\eta' \neq (-1)^j\eta$. This implies, by non-degeneracy of $\langle\ ,\ \rangle_{j,\varpi}$, that the pairings $\langle\ ,\ \rangle_{j,\varpi}^{\eta\eta'}$ are non-degenerate if $\eta' = (-1)^j\eta$. Finally, if $2|j$, then the pairing $\langle\ ,\ \rangle_{j,\varpi}$ (hence each $\langle\ ,\ \rangle_{j,\varpi}^{\eta\eta}$) is skew-symmetric by Lemma 10.6.5 (iii), hence alternating (as $2 \neq 0$ in k).

The statement (iii) is an immediate consequence of (i) and (ii). □

10.7. Skew-Hermitian and symplectic pairings on generalized Selmer groups

10.7.1. In this section we investigate skew-Hermitian and skew-symmetric pairings arising from the constructions in 10.1–10.5. In self-dual situations, the localizations of generalized Cassels-Tate pairings at regular prime ideals of height one give rise to pairings of the type considered in Lemma 10.6.5, from which one can deduce various parity results. As we shall see in 11.8, in the special case 10.6.3.1 the corresponding pairings on the graded pieces gr_F^j can be identified with (higher) height pairings.

10.7.2. Symplectic pairings revisited. — In the situation of 10.6.5, the $\mathcal{O}$-module N is isomorphic to

$$N \xrightarrow{\sim} (F/\mathcal{O})^s \oplus \bigoplus_{i\geqslant 1} (\mathcal{O}/\varpi^i)^{n_i},$$

with $n_i \geqslant 0$ and only finitely many n_i non-zero,

$$N_{\mathrm{div}} \xrightarrow{\sim} (F/\mathcal{O})^s, \quad s = \mathrm{cork}_{\mathcal{O}}(N)$$

and

$$\dim_k(N[\varpi]) = s + \sum_{i\geqslant 1} n_i$$
$$\dim_k(F^j) = s + \sum_{i\geqslant j} n_i \qquad (j \geqslant 1).$$

Assume that $\iota = \mathrm{id}$ and that the pairing $\langle\ ,\ \rangle$ is alternating (which is automatic if $2 \in \mathcal{O}^*$). The existence of the symplectic forms on the graded quotients gr^j_F implies that

$$n_i \equiv 0 \pmod 2 \qquad (\forall i \geqslant 1).$$

In particular, if $\langle\ ,\ \rangle$ is non-degenerate, then $s = 0$ and

$$N \xrightarrow{\sim} M \oplus M, \quad M = \bigoplus_{i\geqslant 1} (\mathcal{O}/\varpi^i)^{n_i/2}.$$

Note that M can be chosen to be a Lagrangian (= maximal isotropic) submodule of N.

If the form $\langle\ ,\ \rangle$ is merely skew-symmetric (*i.e.*, $\langle x, y\rangle = -\langle y, x\rangle$ for all $x, y \in N$) and non-degenerate, then $2\langle\ ,\ \rangle$ is alternating, but possibly degenerate, with kernel $N' \subset N$ annihilated by 2. By the previous remark, we have

$$N/N' \xrightarrow{\sim} M' \oplus M'.$$

Denoting the image of the map

$$\begin{aligned} M' \hookrightarrow N/N' &\longrightarrow N \\ n + N' &\longmapsto 2n \end{aligned}$$

by $M \subset N$, then

$$M \oplus M \subset N, \quad 2 \cdot (N/(M \oplus M)) = 0.$$

10.7.3. The abstract linear algebra construction from Lemma 10.6.5 appears in several contexts, *e.g.* as the “Jantzen filtration” in representation theory [**Jan**] (as pointed out to us by L. Clozel). For us, the most important example comes from the classical descent theory on elliptic curves.

Let E be an elliptic curve over a number field K and S a finite set of primes of K, which contains all primes above p, all archimedean primes and all primes of bad reduction of E. The classical Selmer group for the p^n-descent on E (over K) is defined by the following cartesian diagram:

$$\begin{array}{ccc} \mathrm{Sel}(E/K, p^n) & \hookrightarrow & H^1(G_{K,S}, E[p^n]) \\ \downarrow & & \downarrow \\ \bigoplus_{v\in S} E(K_v)\otimes \mathbf{Z}/p^n\mathbf{Z} & \hookrightarrow & \bigoplus_{v\in S} H^1(G_v, E[p^n]). \end{array}$$

Passing to the limit,

$$\mathrm{Sel}(E/K, p^\infty) = \varinjlim_n \mathrm{Sel}(E/K, p^n)$$

is a $\mathbf{Z}_p$-module of co-finite type sitting in an exact sequence

$$0 \longrightarrow E(K)\otimes \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow \mathrm{Sel}(E/K, p^\infty) \longrightarrow Ш(E/K)[p^\infty] \longrightarrow 0,$$

where $Ш(E/K)$ is the Tate-Šafarevič group of E over K. The image of the canonical map

$$\mathrm{Im}\big[\mathrm{Sel}(E/K, p^n) \longrightarrow \mathrm{Sel}(E/K, p^\infty)\big] =: \widetilde{\mathrm{Sel}}(E/K, p^n)$$

coincides with $\mathrm{Sel}(E/K, p^\infty)[p^n]$ and sits in an exact sequence

$$0 \longrightarrow (E(K)/\mathrm{tors})\otimes \mathbf{Z}/p^n\mathbf{Z} \longrightarrow \widetilde{\mathrm{Sel}}(E/K, p^n) \longrightarrow Ш(E/K)[p^n] \longrightarrow 0.$$

Putting

$$r = \mathrm{rk}_{\mathbf{Z}}(E(K)), \quad s_n = \max\{j \geqslant 0 \mid (\mathbf{Z}/p^n\mathbf{Z})^j \subseteq \widetilde{\mathrm{Sel}}(E/K, p^n)\}$$

(for each $n \geqslant 1$), then

$$s_1 \geqslant s_2 \geqslant \cdots \geqslant s_{n_0} = s_{n_0+1} = \cdots = s_\infty \geqslant r,$$

$$\mathrm{Sel}(E/K, p^\infty) \xrightarrow{\sim} (\mathbf{Q}_p/\mathbf{Z}_p)^{s_\infty} \oplus \bigoplus_{i\geqslant 1} (\mathbf{Z}/p^i\mathbf{Z})^{s_i - s_{i+1}},$$

$$s_\infty - r = \mathrm{cork}_{\mathbf{Z}_p}(Ш(E/K)[p^\infty])$$

(as $Ш(E/K)$ is conjecturally finite, it is expected that $s_\infty = r$).

The integer s_n (for fixed p) is classically called "the number of n-th descents". It was first observed by Selmer that, in all available examples, the integers $s_i - s_{i+1}$ (as well as $s_i - r$) always turned out to be *even*. Selmer's observation concerning $s_i - s_{i+1}$ was conceptually explained by Cassels [**Ca2**], who constructed a non-degenerate alternating pairing (subsequently generalized by Tate)

$$\langle\ ,\ \rangle : \mathrm{Sel}(E/K, p^\infty) \times \mathrm{Sel}(E/K, p^\infty) \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p$$

with kernel $\mathrm{Sel}(E/K, p^\infty)_{\mathrm{div}}$. The filtration F^j on $F^1 = \mathrm{Sel}(E/K, p^\infty)[p] = \widetilde{\mathrm{Sel}}(E/K, p)$ (constructed as in Lemma 10.6.5, with $\mathcal{O} = \mathbf{Z}_p$, $\iota = \mathrm{id}$ and $\varpi = p$) then coincides with

$$F^j = \mathrm{Im}\Big[\widetilde{\mathrm{Sel}}(E/K, p^j) \longrightarrow \widetilde{\mathrm{Sel}}(E/K, p)\Big], \tag{10.7.3.1}$$

i.e.,

$$s_j = \dim_{\mathbf{Z}/p\mathbf{Z}}(F^j)$$

is equal to the number of first descents that can be "extended" to j-th descents. The fact that F^2 (defined by (10.7.3.1)) is the kernel of a suitable alternating form on F^1 was first established in a special case in [**Ca1**]; this was the starting point of the general construction of $\langle\ ,\ \rangle$, given in [**Ca2**].

10.7.4. Lemma. — *Let N be an R-module of finite length and*

$$\langle\ ,\ \rangle : N \times N \longrightarrow I_R$$

*a symplectic pairing (*i.e., *$\langle x, x\rangle = 0$ for all $x \in N$ and the adjoint map*

$$j = \mathrm{adj}(\langle\ ,\ \rangle) : N \longrightarrow D(N)$$

is an isomorphism). Then:

(i) *For each Lagrangian (maximal isotropic) submodule $M \subset N$, the map j induces an isomorphism*

$$N/M \xrightarrow{\sim} D(M).$$

(ii) $\ell_R(N) \equiv 0 \pmod 2$.

Proof

(i) Lagrangian submodules exist (since N is Noetherian); let $i : M \hookrightarrow N$ be one of them. As M is isotropic, the (surjective) map

$$N \xrightarrow{j} D(N) \xrightarrow{D(i)} D(M)$$

factors through

$$j' : N/M \longrightarrow D(M).$$

If $x + M \in \mathrm{Ker}(j')$ and $x \notin M$, then $M + Rx \supsetneq M$ is also isotropic - contradiction. Thus j' is injective.

(ii) $\ell_R(N) = \ell_R(M) + \ell_R(D(M)) = 2\,\ell_R(M)$. □

10.7.5. Abstract self-dual case. — Pairings of the kind considered in 10.6.5 and 10.7.4 naturally arise in the following context.

Consider the following assumptions:

10.7.5.1. $\mathfrak{p} \in \mathrm{Spec}(R)$ satisfies $\dim(R_{\mathfrak{p}}) = \mathrm{depth}(R_{\mathfrak{p}}) = 1$.

10.7.5.2. In 10.4.1, we have $Y_i = Y$, $\Delta(Y_i) = \Delta(Y)$ $(i = 1, 2)$, $\pi(\mathfrak{p}) : Y \otimes_{R_{\mathfrak{p}}} Y \to \omega^{\bullet}_{R_{\mathfrak{p}}}(1)$ satisfies $\pi(\mathfrak{p}) \circ s_{12} = -\pi(\mathfrak{p})$ and the local conditions $\Delta(Y) \perp_{\pi(\mathfrak{p}), h_S(\mathfrak{p})} \Delta(Y)$ admit (a localized version of) the transposition data as in 6.6.2.

10.7.5.3. $H^i\big(\mathrm{Err}(\Delta(Y), \Delta(Y), \pi(\mathfrak{p}))\big) = 0$ for $i = 1, 2$.

10.7.5.4. 2 is invertible in $R_{\mathfrak{p}}$.

10.7.5.5. $R_{\mathfrak{p}}$ is a discrete valuation ring.

10.7.6. Proposition

(i) *Under the assumptions 10.7.5.1–10.7.5.3, the pairing* $\langle\ ,\ \rangle = \cup_{\pi(\mathfrak{p}),h_S(\mathfrak{p}),2,2}$ *on* $N = \widetilde{H}^2_f(Y)_{R_\mathfrak{p}\text{-tors}}$ *is a non-degenerate skew-symmetric* $R_\mathfrak{p}$*-bilinear form*

$$\langle\ ,\ \rangle : N \times N \longrightarrow I_{R_\mathfrak{p}},$$

i.e., $\langle x, y\rangle = -\langle y, x\rangle$ *for all* $x, y \in N$ *and the adjoint map* $\mathrm{adj}(\langle\ ,\ \rangle) : N \to D_{R_\mathfrak{p}}(N)$ *is an isomorphism.*

(ii) *Under the assumptions 10.7.5.1–10.7.5.4,* $\langle\ ,\ \rangle$ *is a symplectic pairing as in Lemma 10.7.4 (over* $R_\mathfrak{p}$*) and*

$$\ell_{R_\mathfrak{p}}(N) \equiv 0 \pmod 2.$$

(iii) *Under the assumptions 10.7.5.2–10.7.5.5,* $\langle\ ,\ \rangle$ *is a symplectic pairing as in Lemma 10.6.5 (over* $\mathcal{O} = R_\mathfrak{p}$*) with* $N_{\mathrm{div}} = 0$ *and*

$$N \xrightarrow{\sim} M \oplus M$$

for a suitable Lagrangian submodule $M \subset N$*.*

(iv) *Under the assumptions 10.7.5.2–10.7.5.3, 10.7.5.5, there is an* $R_\mathfrak{p}$*-submodule* $M \subset N$ *such that*

$$M \oplus M \subset N, \quad 2 \cdot (N/(M \oplus M)) = 0.$$

Proof

(i), (ii) A localized version of Proposition 10.2.5 implies that the pairing $\langle\ ,\ \rangle$ is skew-symmetric (hence alternating, under the assumption 10.7.5.4). It is also non-degenerate, by Theorem 10.4.4 and the assumption 10.7.5.3.

The remaining statements follow by applying Lemma 10.6.5, 10.7.4 and the discussion in 10.7.2. ☐

10.7.7. Self-dual Greenberg's local conditions. — Consider the following assumptions:

10.7.7.1. $\mathfrak{p} \in \mathrm{Spec}(R)$ satisfies $\dim(R_\mathfrak{p}) = \mathrm{depth}(R_\mathfrak{p}) = 1$.

10.7.7.2. In 10.5.1, we have $X(\mathfrak{p}) = Y(\mathfrak{p})$, $X(\mathfrak{p})^+_v = Y(\mathfrak{p})^+_v$ $(v \in \Sigma)$, $\pi(\mathfrak{p}) : X(\mathfrak{p}) \otimes_{R_\mathfrak{p}} X(\mathfrak{p}) \to \omega^\bullet_{R_\mathfrak{p}}(1)$ satisfies $\pi(\mathfrak{p}) \circ s_{12} = -\pi(\mathfrak{p})$ and $X(\mathfrak{p})^+_v \perp_{\pi(\mathfrak{p})} X(\mathfrak{p})^+_v$ $(v \in \Sigma)$.

10.7.7.3. $H^i_{\mathrm{cont}}(G_v, W_v) = 0$ $(i = 1, 2;\ v \in \Sigma)$, $\mathrm{Tam}_v(X(\mathfrak{p}), \mathfrak{p}) = 0$ $(v \in \Sigma')$.

10.7.7.4. 2 is invertible in $R_\mathfrak{p}$.

10.7.7.5. $R_\mathfrak{p}$ is a discrete valuation ring.

10.7.7.6. In each degree $i \in \mathbf{Z}$, the $R_\mathfrak{p}$-modules $X(\mathfrak{p})^i$, $(X(\mathfrak{p})^+_v)^i$ $(v \in \Sigma)$ are torsion-free.

10.7.8. Under the assumptions 10.7.7.2–10.7.7.6, fix a prime element $\varpi \in R_{\mathfrak{p}}$. Then

$$X_{[\mathfrak{p}]} := X(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} R_{\mathfrak{p}}/\varpi$$

is a complex in $\left({}^{\mathrm{ad}}_{R[G_{K,S}]}\mathrm{Mod}\right)$, which can also be viewed as a complex in $\left({}^{\mathrm{ad}}_{(R/\mathfrak{p})[G_{K,S}]}\mathrm{Mod}\right)$. The complex of continuous cochains $C^\bullet_{\mathrm{cont}}(G_{K,S}, X_{[\mathfrak{p}]})$ is the same in both cases, by Proposition 3.5.10. The same is true for

$$(X_{[\mathfrak{p}]})_v^+ := X(\mathfrak{p})_v^+ \otimes_{R_{\mathfrak{p}}} R_{\mathfrak{p}}/\varpi$$

as complexes of G_v-modules ($v \in \Sigma$). The maps

$$j_v^+(X_{[\mathfrak{p}]}) : (X_{[\mathfrak{p}]})_v^+ \longrightarrow X_{[\mathfrak{p}]} \qquad (v \in \Sigma),$$

induced by $j_v^+(X(\mathfrak{p}))$, define Greenberg's local conditions for $X_{[\mathfrak{p}]}$, the Selmer complex $\widetilde{C}^\bullet_f(X_{[\mathfrak{p}]})$ and the corresponding object $\widetilde{\mathbf{R}\Gamma}_f(X_{[\mathfrak{p}]}) \in D({}_{\kappa(\mathfrak{p})}\mathrm{Mod})$, where $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\varpi = \mathrm{Frac}(R/\mathfrak{p})$ is the residue field of $\mathfrak{p}$.

By Proposition 7.6.7(iv), the assumptions 10.7.7.3 and 10.7.7.6 give an exact triangle

$$\widetilde{\mathbf{R}\Gamma}_f(X(\mathfrak{p})) \xrightarrow{\varpi} \widetilde{\mathbf{R}\Gamma}_f(X(\mathfrak{p})) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(X_{[\mathfrak{p}]}) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(X(\mathfrak{p}))[1]$$

in $D({}_{R_{\mathfrak{p}}}\mathrm{Mod})$. The corresponding cohomology sequence yields short exact sequences

$$(10.7.8.1) \qquad 0 \longrightarrow \widetilde{H}^i_f(X(\mathfrak{p})) \otimes_{R_{\mathfrak{p}}} R_{\mathfrak{p}}/\varpi \longrightarrow \widetilde{H}^i_f(X_{[\mathfrak{p}]}) \longrightarrow \widetilde{H}^{i+1}_f(X(\mathfrak{p}))[\varpi] \longrightarrow 0,$$

which imply that $\widetilde{\mathbf{R}\Gamma}_f(X_{[\mathfrak{p}]}) \in D_{ft}({}_{\kappa(\mathfrak{p})}\mathrm{Mod})$.

10.7.9. Proposition. — *Under the assumptions 10.7.7.2–10.7.7.6,*

(i) *The $\kappa(\mathfrak{p})$-vector space $F^1 := \widetilde{H}^1_f(X_{[\mathfrak{p}]})$ has a canonical decreasing filtration by subspaces $F^1 \supseteq F^2 \supseteq \cdots$ satisfying*

$$F^\infty := \bigcap_{j \geqslant 1} F^j = \widetilde{H}^1_f(X(\mathfrak{p})) \otimes_{R_{\mathfrak{p}}} R_{\mathfrak{p}}/\varpi.$$

(ii) *There are natural symplectic pairings*

$$\langle\ ,\ \rangle_{j,\varpi} : \mathrm{gr}^j_F \times \mathrm{gr}^j_F \longrightarrow \kappa(\mathfrak{p})$$

depending on ϖ, with

$$\langle\ ,\ \rangle_{j,\varpi'} = (\varpi'/\varpi \pmod{\varpi}))^{2-j} \langle\ ,\ \rangle_{j,\varpi}.$$

(iii) $\dim_{\kappa(\mathfrak{p})}\left(\widetilde{H}^1_f(X_{[\mathfrak{p}]})\right) \equiv \dim_{\kappa(\mathfrak{p})}(F^\infty) \pmod 2$.

(iv) *If $\widetilde{H}^0_f(X_{[\mathfrak{p}]}) = 0$, then*

$$\dim_{\kappa(\mathfrak{p})}\left(\widetilde{H}^1_f(X_{[\mathfrak{p}]})\right) \equiv \mathrm{rk}_{R_{\mathfrak{p}}}\left(\widetilde{H}^1_f(X(\mathfrak{p}))\right) \pmod 2.$$

Proof. — By Proposition 10.7.6 (iii), $\langle\ ,\ \rangle = \cup_{\pi(\mathfrak{p}),0,2,2}$ is a non-degenerate alternating pairing

$$\langle\ ,\ \rangle : N \times N \longrightarrow \mathrm{Frac}(R_{\mathfrak{p}})/R_{\mathfrak{p}}$$

on

$$N = \big(\widetilde{H}^2_f(X(\mathfrak{p}))\big)_{R_{\mathfrak{p}}\text{-tors}}.$$

Applying Lemma 10.6.5 to this pairing, we obtain a filtration F^j on $N[\varpi]$ and symplectic pairings on its graded quotients. Taking the pull-back of these objects by the canonical map

$$\widetilde{H}^1_f(X_{[\mathfrak{p}]}) \longrightarrow \widetilde{H}^2_f(X(\mathfrak{p}))[\varpi] = N[\varpi]$$

from (10.7.8.1), we obtain the statements (i)–(iii).

As regards (iv), the assumption $\widetilde{H}^0_f(X_{[\mathfrak{p}]}) = 0$ implies (again by (10.7.8.1)) that $\widetilde{H}^1_f(X(\mathfrak{p}))$ is torsion-free, hence free, over $R_{\mathfrak{p}}$; thus

$$\dim_{\kappa(\mathfrak{p})}(F^\infty) = \mathrm{rk}_{R_{\mathfrak{p}}}\big(\widetilde{H}^1_f(X(\mathfrak{p}))\big). \quad \square$$

10.7.10. Note that $\mathfrak{p}$ contains a unique minimal prime ideal $\mathfrak{q} \subsetneq \mathfrak{p}$, and

$$\mathrm{rk}_{R_{\mathfrak{p}}}\big(\widetilde{H}^1_f(X(\mathfrak{p}))\big) = \dim_{\mathrm{Frac}(R_{\mathfrak{p}})}\big(\widetilde{H}^1_f(X(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} \mathrm{Frac}(R_{\mathfrak{p}}))\big)$$

depends only on

$$X(\mathfrak{q}) := X(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} \mathrm{Frac}(R_{\mathfrak{p}}) = X(\mathfrak{p})_{\mathfrak{q}}$$

(a complex of admissible $R_{\mathfrak{q}}[G_{K,S}]$-modules) and

$$X(\mathfrak{q})^+_v := \big(X(\mathfrak{p})^+_v\big)_{\mathfrak{q}} \qquad (v \in \Sigma).$$

10.7.11. Self-dual dihedral case in Iwasawa theory. — Consider the following assumptions:

10.7.11.1. We are given $K^+ \subset K \subset K_\infty$ as in 10.3.5.1.

10.7.11.2. In 10.3.3.2, we have $X = Y$, $X^+_v = Y^+_v$ $(v \in \Sigma)$, $\pi : X \otimes_R X \to \omega^\bullet_R(1)$ satisfies $\pi \circ s_{12} = -\pi$, $X^+_v \perp_\pi X^+_v$ $(\forall v \in \Sigma)$ and the conditions 10.3.5.1.1–10.3.5.1.5 are satisfied for X, X^+_v $(v \in \Sigma)$.

10.7.11.3. $\overline{\mathfrak{p}} \in \mathrm{Spec}(\overline{R})$ is a prime ideal satisfying $\dim(\overline{R}_{\overline{\mathfrak{p}}}) = \mathrm{depth}(\overline{R}_{\overline{\mathfrak{p}}}) = 1$; put $\mathfrak{q}_0 = R[\Delta] \cap \overline{\mathfrak{p}} \in \mathrm{Spec}(R[\Delta])$ and $\mathfrak{q} = \mathfrak{q}_0 \cap R \in \mathrm{Spec}(R)$.

10.7.11.4. If $\mathfrak{q}$ is not a minimal prime ideal of R ($\Longrightarrow \mathrm{ht}(\mathfrak{q}) = 1$), then $\mathrm{Tam}_v(\mathscr{F}_\Delta(X), \mathfrak{q}_0) = 0$ for all $v \in \Sigma'$ satisfying $\Gamma_v \subset \Delta$.

10.7.11.5. 2 is invertible in $\overline{R}_{\overline{\mathfrak{p}}}$.

10.7.11.6. $\overline{R}_{\overline{\mathfrak{p}}}$ is a discrete valuation ring.

10.7.11.7. The localization of π at $\mathfrak{q}$ is a perfect pairing $\pi_{\mathfrak{q}} : X_{\mathfrak{q}} \otimes_{R_{\mathfrak{q}}} X_{\mathfrak{q}} \to \omega^\bullet_{R_{\mathfrak{q}}}(1)$ over $R_{\mathfrak{q}}$ and $(X^+_v)_{\mathfrak{q}} \perp\!\!\perp_{\pi_{\mathfrak{q}}} (X^+_v)_{\mathfrak{q}}$ for all $v \in \Sigma$.

10.7.11.8. The localization of π at $\mathfrak{q}$ is a perfect pairing $\pi_{\mathfrak{q}} : X_{\mathfrak{q}} \otimes_{R_{\mathfrak{q}}} X_{\mathfrak{q}} \longrightarrow \omega^{\bullet}_{R_{\mathfrak{q}}}(1)$ over $R_{\mathfrak{q}}$.

10.7.11.9. $(\forall v \in \Sigma)$ $\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, \mathscr{F}_\Delta(W_v))_{\mathfrak{q}_0} \xrightarrow{\sim} 0$ in $D^b_{ft}\big({}_{R[\Delta]_{\mathfrak{q}_0}}\mathrm{Mod}\big)$, where W_v sits in an exact triangle in $D^b\big({}^{\mathrm{ad}}_{R_{\mathfrak{q}}[G_v]}\mathrm{Mod}\big)$

$$W_v \longrightarrow (X_v^-)_{\mathfrak{q}} \longrightarrow \mathscr{D}_{R_{\mathfrak{q}}}\big((X_v^+)_{\mathfrak{q}}\big) \longrightarrow W_v[1].$$

10.7.11.10. $\Gamma = \Gamma_0' \times \Gamma_0'' \times \Delta$, $\Gamma_0' = \langle\gamma\rangle \xrightarrow{\sim} \mathbf{Z}_p$, $\overline{\mathfrak{p}} = \mathfrak{q}_0\overline{R} + (\gamma - 1)\overline{R}$, $\mathfrak{q}_0 \in \mathrm{Spec}(R[\Delta])$, $\mathrm{ht}(\mathfrak{q}_0) = 0$, $\mathfrak{q} = \mathfrak{q}_0 \cap R \in \mathrm{Spec}(R)$.

10.7.12. Proposition

(i) *Under the assumptions 10.7.11.1–10.7.11.5, 10.7.11.7 (resp., 10.7.11.1–10.7.11.7), there is a symplectic pairing over* $\overline{R}_{\overline{\mathfrak{p}}}$

$$\langle\ ,\ \rangle : N \times N \longrightarrow I_{\overline{R}_{\overline{\mathfrak{p}}}} \quad (\textit{resp.},\ N \times N \longrightarrow \mathrm{Frac}(\overline{R}_{\overline{\mathfrak{p}}})/\overline{R}_{\overline{\mathfrak{p}}})$$

on

$$N = \big(\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, X)\big)_{(\overline{R}_{\overline{\mathfrak{p}}})\text{-tors}} = \Big(\big(\overline{D}\big(\widetilde{H}^1_f(K_S/K_\infty, D(X)(1))^{\iota}\big)\big)_{\overline{\mathfrak{p}}}\Big)_{(\overline{R}_{\overline{\mathfrak{p}}})\text{-tors}}$$

as in Lemma 10.7.4 (resp., in Lemma 10.6.5, with $\mathcal{O} = \overline{R}_{\overline{\mathfrak{p}}}$ *and* $N_{\mathrm{div}} = 0$*) and* $\ell_{\overline{R}_{\overline{\mathfrak{p}}}}(N) \equiv 0 \pmod 2$ *(resp., there is a submodule* $M \subset N$ *satisfying* $M \oplus M = N$*).*

(ii) *Under the assumptions 10.7.11.1, 10.7.11.2, 10.7.11.4, 10.7.11.6–10.7.11.7 (resp., 10.7.11.1, 10.7.11.2 10.7.11.8–10.7.11.10 (*$\Longrightarrow$ *10.7.11.6)), there is a non-degenerate skew-symmetric pairing*

$$\langle\ ,\ \rangle : N \times N \longrightarrow \mathrm{Frac}(\overline{R}_{\overline{\mathfrak{p}}})/\overline{R}_{\overline{\mathfrak{p}}}$$

and a submodule $M \subset N$ *satisfying*

$$M \oplus M \subset N, \quad 2 \cdot (N/(M \oplus M)) = 0.$$

Proof. — A localized version of Proposition 10.3.5.8 (and a choice of $\tau, \overline{\tau}$ as in 10.3.5.2) defines a skew-symmetric pairing

$$\langle\ ,\ \rangle = (\ ,\ )_{2,2} : N \times N \longrightarrow I_{\overline{R}_{\overline{\mathfrak{p}}}},$$

which is non-degenerate by Theorem 10.4.4, Theorem 8.9.8 and the relevant assumptions of 10.7.11 (using Lemma 10.5.4.2). The rest follows from Lemma 10.6.5, 10.7.4 and the discussion in 10.7.2. □

10.7.13. Consider the following global version of the assumptions from 10.7.11:

10.7.13.1. = 10.7.11.1

10.7.13.2. = 10.7.11.2

10.7.13.3. R (hence also $\overline{R}$) has no embedded primes.

10.7.13.4. Γ_v is infinite for each $v \in \Sigma'$.

10.7.13.5. 2 is invertible in $\overline{R}$.

10.7.13.6. R (hence also $\overline{R}$) satisfies (R_1).

10.7.13.7. The condition 10.7.11.7 is satisfied for all prime ideals $\mathfrak{q} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{q}) = 0$.

10.7.13.8. The condition 10.7.11.7 is satisfied for all prime ideals $\mathfrak{q} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{q}) = 1$.

10.7.14. Proposition. — *Under the assumptions 10.7.13.1–10.7.13.8 (resp., 10.7.13.1–10.7.13.3, 10.7.13.6–10.7.13.7), the $\overline{R}$-module of finite type*

$$N = \big(\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, X)\big)_{\overline{R}\text{-tors}} = \Big(\big(\overline{D}\big(\widetilde{H}^1_f(K_S/K_\infty, D(X)(1))^{\iota}\big)\big)\Big)_{\overline{R}\text{-tors}}$$

has a subobject $M \hookrightarrow N$ in $\big({}_{\overline{R}}\mathrm{Mod}\big)/(\text{pseudo-null})$ such that

$$M \oplus M \xrightarrow{\sim} N$$

in $\big({}_{\overline{R}}\mathrm{Mod}\big)/(\text{pseudo-null})$ (resp.,

$$2 \cdot (N/(M \oplus M \oplus N')) = 0, \quad N' \xrightarrow{\sim} \bigoplus_{\overline{\mathfrak{p}} \in A} \bigoplus_{i \geqslant 1} \big(\overline{R}/\overline{\mathfrak{p}}^i\big)^{a(\mathfrak{p},i)}$$

in $\big({}_{\overline{R}}\mathrm{Mod}\big)/(\text{pseudo-null})$), where

$$A = \big\{\overline{\mathfrak{p}} \in \mathrm{Spec}(\overline{R}) \mid \mathrm{ht}(\overline{\mathfrak{p}}) = 1,\ (\exists \mathfrak{p} \in \mathrm{Spec}(R))\, \mathrm{ht}(\mathfrak{p}) = 1,\ \mathfrak{p}\overline{R} \subseteq \overline{\mathfrak{p}}\big\}.$$

In particular, if $\Gamma = \Gamma_0 \xrightarrow{\sim} \mathbf{Z}_p^r$, then

$$A = \big\{\mathfrak{p}\overline{R} \mid \mathfrak{p} \in \mathrm{Spec}(R),\ \mathrm{ht}(\mathfrak{p}) = 1\big\}.$$

Proof. — Assume that 10.7.13.1–10.7.13.8 hold. By Corollary 2.10.19, N is isomorphic in $\big({}_{\overline{R}}\mathrm{Mod}\big)/(\text{pseudo-null})$ to

$$N \xrightarrow{\sim} \bigoplus_{\substack{\overline{\mathfrak{p}} \in \mathrm{Spec}(\overline{R}) \\ \mathrm{ht}(\overline{\mathfrak{p}})=1}} \bigoplus_{i \geqslant 1} (\overline{R}/\overline{\mathfrak{p}}^i)^{n(\overline{\mathfrak{p}},i)}.$$

The conditions 10.7.11.1–10.7.11.6 (resp., 10.7.11.7), follow from 10.7.13.1–10.7.13.6 (resp., from 10.7.13.7–10.7.13.8 applied to $\mathfrak{q} = \overline{\mathfrak{p}} \cap R$). As

$$N_{\overline{\mathfrak{p}}} \xrightarrow{\sim} \bigoplus_{i \geqslant 1} \big(\overline{R}_{\overline{\mathfrak{p}}}/\overline{\mathfrak{p}}^i \overline{R}_{\overline{\mathfrak{p}}}\big)^{n(\overline{\mathfrak{p}},i)},$$

Proposition 10.7.12 (i) implies that each exponent $n(\overline{\mathfrak{p}}, i)$ is even, hence

$$N \xrightarrow{\sim} M \oplus M$$

in $\big({}_{\overline{R}}\mathrm{Mod}\big)/(\text{pseudo-null})$, where

$$M = \bigoplus_{\substack{\overline{\mathfrak{p}} \in \mathrm{Spec}(\overline{R}) \\ \mathrm{ht}(\overline{\mathfrak{p}})=1}} \bigoplus_{i \geqslant 1} \big(\overline{R}/\overline{\mathfrak{p}}^i\big)^{n(\overline{\mathfrak{p}},i)/2}.$$

If we only assume 10.7.13.1–10.7.13.3 and 10.7.13.6–10.7.13.7, then Proposition 10.7.12 (ii) shows that, for each $\overline{\mathfrak{p}} \notin A$, there is an $(\overline{R}_{\overline{\mathfrak{p}}})$-submodule

$$M(\overline{\mathfrak{p}}) \oplus M(\overline{\mathfrak{p}}) \subset N_{\overline{\mathfrak{p}}}$$

satisfying

$$2 \cdot \big(N_{\overline{\mathfrak{p}}}/(M(\overline{\mathfrak{p}}) \oplus M(\overline{\mathfrak{p}}))\big) = 0.$$

Writing

$$M(\overline{\mathfrak{p}}) = \bigoplus_{i \geqslant 1} (\overline{R}_{\overline{\mathfrak{p}}}/\overline{\mathfrak{p}}^i \overline{R}_{\overline{\mathfrak{p}}})^{m(\overline{\mathfrak{p}},i)},$$

Proposition 2.10.18 and Corollary 2.10.19 then show the existence of the required $M \hookrightarrow N$, isomorphic to

$$M \xrightarrow{\sim} \bigoplus_{\substack{\mathrm{ht}(\overline{\mathfrak{p}})=1 \\ \overline{\mathfrak{p}} \notin A}} (\overline{R}/\overline{\mathfrak{p}}^i)^{m(\overline{\mathfrak{p}},i)}. \quad \square$$

10.7.15. Theorem. — *Assume that 10.7.11.1–10.7.11.2 hold and that $X = H^0(X)$ and $X_v^+ = H^0(X_v^+)$ $(\forall v \in \Sigma)$ are concentrated in degree zero. Let $\mathfrak{q}_0 \in \mathrm{Spec}(R[\Delta])$ be a minimal prime ideal such that 10.7.11.8–10.7.11.9 hold for $\mathfrak{q} = \mathfrak{q}_0 \cap R \in \mathrm{Spec}(R)$. Assume, in addition, that $\kappa(\mathfrak{q}_0) = R[\Delta]_{\mathfrak{q}_0}$ is a field of characteristic $\mathrm{char}(\kappa(\mathfrak{q}_0)) \neq 2$ and $\widetilde{H}^0_f(K_0, X)_{\mathfrak{q}_0} = 0$, where $K_0 := K_\infty^{\Gamma_0}$ and $\widetilde{H}^j_f(K_0, X) := \widetilde{H}^j_f(K, \mathscr{F}_\Delta(X))$. Denote by $\overline{\mathfrak{q}} = \mathfrak{q}_0 \overline{R}$ the unique minimal prime ideal of $\overline{R}$ containing $\mathfrak{q}_0$ ($\Longrightarrow \kappa(\overline{\mathfrak{q}}) = \overline{R}_{\overline{\mathfrak{q}}}$ is a field containing $\kappa(\mathfrak{q}_0)$). Then:*

(i) *Denoting by $\iota : R[\Delta] \to R[\Delta]$ (resp., $\iota : \overline{R} \to \overline{R}$) the standard involution, then*

$$\begin{aligned}
\dim_{\kappa(\mathfrak{q}_0)} \widetilde{H}^1_f(K_0, X)_{\mathfrak{q}_0} &= \dim_{\kappa(\mathfrak{q}_0)} \widetilde{H}^2_f(K_0, X)_{\mathfrak{q}_0} \\
&= \dim_{\kappa(\mathfrak{q}_0)} \big(\widetilde{H}^1_f(K_0, X)^\iota\big)_{\mathfrak{q}_0} = \dim_{\kappa(\mathfrak{q}_0)} \big(\widetilde{H}^2_f(K_0, X)^\iota\big)_{\mathfrak{q}_0} \\
\dim_{\kappa(\overline{\mathfrak{q}})} \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K, X)_{\overline{\mathfrak{q}}} &= \dim_{\kappa(\overline{\mathfrak{q}})} \widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, X)_{\overline{\mathfrak{q}}} \\
&= \dim_{\kappa(\overline{\mathfrak{q}})} \big(\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K, X)^\iota\big)_{\overline{\mathfrak{q}}} = \dim_{\kappa(\overline{\mathfrak{q}})} \big(\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, X)^\iota\big)_{\overline{\mathfrak{q}}}.
\end{aligned}$$

(ii) *The dimensions from (i) satisfy*

$$\begin{aligned}
\dim_{\kappa(\mathfrak{q}_0)} \widetilde{H}^2_f(K_0, X)_{\mathfrak{q}_0} &\geqslant \dim_{\kappa(\overline{\mathfrak{q}})} \widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, X)_{\overline{\mathfrak{q}}} \\
\dim_{\kappa(\mathfrak{q}_0)} \widetilde{H}^2_f(K_0, X)_{\mathfrak{q}_0} &\equiv \dim_{\kappa(\overline{\mathfrak{q}})} \widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, X)_{\overline{\mathfrak{q}}} \pmod 2.
\end{aligned}$$

In particular, if $\Delta = 0$ ($\Longrightarrow K_0 = K$, $\mathfrak{q}_0 = \mathfrak{q}$) and R is a domain ($\Longrightarrow \mathfrak{q} = (0)$), then

$$\mathrm{rk}_R\big(\widetilde{H}^1_f(X)\big) = \mathrm{rk}_R\big(\widetilde{H}^2_f(X)\big), \ \mathrm{rk}_{\overline{R}}\big(\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K, X)\big) = \mathrm{rk}_{\overline{R}}\big(\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, X)\big)$$

$$\mathrm{rk}_R\big(\widetilde{H}^2_f(X)\big) \geqslant \mathrm{rk}_{\overline{R}}\big(\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, X)\big),$$

$$\mathrm{rk}_R\big(\widetilde{H}^2_f(X)\big) \equiv \mathrm{rk}_{\overline{R}}\big(\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, X)\big) \pmod 2.$$

(iii) *Let* $\Gamma_0' \subset \Gamma_0$ *be an open subgroup,* $K_0' = K_\infty^{\Gamma_0'}$, $\Delta' = \Gamma_0/\Gamma_0'$ *and* $\mathfrak{q}_0' \in \mathrm{Spec}(R[\Delta\times\Delta'])$ *a minimal prime ideal above* $\mathfrak{q}_0 \in \mathrm{Spec}(R[\Delta])$ *such that* $\widetilde{H}^0_f(K_0', X)_{\mathfrak{q}_0'} = 0$*, where* $\widetilde{H}^j_f(K_0', X) := \widetilde{H}^j_f(K, \mathscr{F}_{\Delta\times\Delta'}(X))$*. Assume that* $(\forall v \in \Sigma)$ $\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, \mathscr{F}_{\Delta\times\Delta'}(W_v))_{\mathfrak{q}_0'} \xrightarrow{\sim} 0$ *in* $D^b_{ft}({}_{R[\Delta\times\Delta']_{\mathfrak{q}_0'}}\mathrm{Mod})$*. Assume, in addition, that the field* $\kappa(\mathfrak{q}_0)$ *has characteristic zero. Then* $\kappa(\mathfrak{q}_0') := R[\Delta\times\Delta']_{\mathfrak{q}_0'}$ *is a field of finite degree over* $\kappa(\mathfrak{q}_0)$ *and*

$$\dim_{\kappa(\mathfrak{q}_0')} \widetilde{H}^1_f(K_0', X)_{\mathfrak{q}_0'} = \dim_{\kappa(\mathfrak{q}_0')} \widetilde{H}^2_f(K_0', X)_{\mathfrak{q}_0'}$$

$$\dim_{\kappa(\mathfrak{q}_0')} \widetilde{H}^2_f(K_0', X)_{\mathfrak{q}_0'} \equiv \dim_{\kappa(\overline{\mathfrak{q}})} \widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, X)_{\overline{\mathfrak{q}}} \equiv \dim_{\kappa(\mathfrak{q}_0)} \widetilde{H}^2_f(K_0, X)_{\mathfrak{q}_0}, \pmod 2$$

$$\dim_{\kappa(\mathfrak{q}_0')} \widetilde{H}^2_f(K_0', X)_{\mathfrak{q}_0'} \geqslant \dim_{\kappa(\overline{\mathfrak{q}})} \widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, X)_{\overline{\mathfrak{q}}}.$$

Proof

(i) The assumption 10.7.11.1 yields, by 10.3.5.4–10.3.5.5, isomorphisms of $R[\Delta]$-modules (resp., of $\overline{R}$-modules)

$$\widetilde{H}^j_f(K_0, X)^\iota \xrightarrow{\sim} \widetilde{H}^j_f(K_0, X), \qquad (10.7.15.1)$$

$$\widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, X)^\iota \xrightarrow{\sim} \widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, X).$$

According to Theorem 8.9.8, localizing the morphism

$$\gamma_{\pi_\Delta} : \widetilde{\mathbf{R}\Gamma}_f(\mathscr{F}_\Delta(X)) \longrightarrow \mathbf{R}\mathrm{Hom}_{R[\Delta]}\big(\widetilde{\mathbf{R}\Gamma}_f(\mathscr{F}_\Delta(X)), \omega_{R[\Delta]}\big)^\iota[-3]$$

at $\mathfrak{q}_0$, we obtain isomorphisms

$$\widetilde{H}^j_f(K_0, X)_{\mathfrak{q}_0} \xrightarrow{\sim} \mathrm{Hom}_{\kappa(\mathfrak{q}_0)}\big((\widetilde{H}^{3-j}_f(K_0, X)^\iota)_{\mathfrak{q}_0}, \kappa(\mathfrak{q}_0)\big), \qquad (10.7.15.2)$$

which proves the first half of (i). In order to prove the second half, fix a chain of subgroups $\Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_r = 0$ satisfying $\Gamma_i/\Gamma_{i+1} \xrightarrow{\sim} \mathbf{Z}_p$ $(i = 0, \ldots, r-1)$ and put $K_{\infty,i} = K_\infty^{\Gamma_i}$ $(K_0 = K_{\infty,0} \subset \cdots \subset K_{\infty,r} = K_\infty)$, $R_i = R[\![\Gamma/\Gamma_i]\!] = R[\![\Gamma_0/\Gamma_i]\!][\Delta]$, $\mathfrak{q}_i = \mathfrak{q}_0 R_i$ $(R_0 = R[\Delta]$, $R_r = \overline{R})$.

For each $i = 0, \ldots, r-1$, we denote by $\mathfrak{p}_{i+1} \in \mathrm{Spec}(R_{i+1})$ the inverse image of $\mathfrak{q}_i$ under the augmentation map $R_{i+1} = R_i[\![\Gamma_i/\Gamma_{i+1}]\!] \twoheadrightarrow R_i$. Then $S_{i+1} := (R_{i+1})_{\mathfrak{p}_{i+1}}$ is a discrete valuation ring with residue field $\kappa(\mathfrak{q}_i) := (R_i)_{\mathfrak{q}_i}$ (and uniformizing element $\gamma_i - 1$, for any topological generator γ_i of Γ_i/Γ_{i+1}).

According to Theorem 8.9.8, the localization of the morphism

$$\gamma_{\overline{\pi}} : \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, X) \longrightarrow \mathbf{R}\mathrm{Hom}_{\overline{R}}\big(\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, X), \omega_{\overline{R}}\big)^\iota[-3]$$

at $\mathfrak{p}_r \in \mathrm{Spec}(\overline{R})$ is an isomorphism. Localizing further at the minimal prime ideal $\overline{\mathfrak{q}} \subset \mathfrak{p}_r$, we obtain isomorphisms

$$\widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, X)_{\overline{\mathfrak{q}}} \xrightarrow{\sim} \mathrm{Hom}_{\kappa(\overline{\mathfrak{q}})}\big((\widetilde{H}^{3-j}_{f,\mathrm{Iw}}(K_\infty/K, X)^\iota)_{\overline{\mathfrak{q}}}, \kappa(\overline{\mathfrak{q}})\big);$$

the second half of (i) follows.

(ii) Denote, for each $i=0,\dots,r$,

$$N_i = \widetilde{H}^2_{f,\mathrm{Iw}}(K_{\infty,i}/K, X) \in ({}_{R_i}\mathrm{Mod})_{ft}.$$

A localized version of Proposition 10.3.5.8 yields, for each $i=0,\dots,r-1$, a skew-symmetric (hence alternating, as $\mathrm{char}(\kappa(\mathfrak{q}_i)) \neq 2$) S_{i+1}-bilinear form

$$\big((N_{i+1})_{\mathfrak{p}_{i+1}}\big)_{S_{i+1}\text{-tors}} \times \big((N_{i+1})_{\mathfrak{p}_{i+1}}\big)_{S_{i+1}\text{-tors}} \longrightarrow \mathrm{Frac}(S_{i+1})/S_{i+1}. \tag{10.7.15.3}$$

The assumptions on $\mathfrak{q}_0$ imply, by Lemma 10.5.4.2, that

$$(\forall i=0,\dots,r-1)\ (\forall v\in\Sigma)\quad \mathscr{F}_{\Gamma/\Gamma_{i+1}}(X_v^+)_{\mathfrak{p}_{i+1}} \perp_{\pi_{i+1}} \mathscr{F}_{\Gamma/\Gamma_{i+1}}(X_v^+)_{\mathfrak{p}_{i+1}}.$$

The cohomology sequences of

$$0 \longrightarrow \mathscr{F}_{\Gamma/\Gamma_{i+1}}(W_v) \xrightarrow{\gamma_i - 1} \mathscr{F}_{\Gamma/\Gamma_{i+1}}(W_v) \longrightarrow \mathscr{F}_{\Gamma/\Gamma_i}(W_v) \longrightarrow 0$$

give rise to exact sequences

$$\begin{aligned} 0 \longrightarrow H^j_{\mathrm{cont}}(G_v, \mathscr{F}_{\Gamma/\Gamma_{i+1}}(W_v))_{\mathfrak{p}_{i+1}}/\mathfrak{p}_{i+1} \longrightarrow H^j_{\mathrm{cont}}(G_v, \mathscr{F}_{\Gamma/\Gamma_i}(W_v))_{\mathfrak{q}_i} \\ \longrightarrow H^{j+1}_{\mathrm{cont}}(G_v, \mathscr{F}_{\Gamma/\Gamma_{i+1}}(W_v))_{\mathfrak{p}_{i+1}}[\mathfrak{p}_{i+1}] \longrightarrow 0. \end{aligned}$$

According to the assumptions, the middle term vanishes for $i=0$ (and all j). Induction on i, together with the Nakayama Lemma, imply that

$$(\forall i=0,\dots,r-1)\quad \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, \mathscr{F}_{\Gamma/\Gamma_{i+1}}(W_v))_{\mathfrak{p}_{i+1}} \xrightarrow{\sim} 0,$$

hence

$$\bigoplus_{v\in\Sigma} \mathrm{Err}_v(\Delta(\mathscr{F}_{\Gamma/\Gamma_{i+1}}(X)), \Delta(\mathscr{F}_{\Gamma/\Gamma_{i+1}}(X)), \pi_{i+1})_{\mathfrak{p}_{i+1}} = 0.$$

As $\mathfrak{p}_{i+1} \neq \mathfrak{p}R_{i+1}$ for any $\mathfrak{p} \in \mathrm{Spec}(R_i)$, we also have

$$\bigoplus_{v\in\Sigma'} \mathrm{Err}^{\mathrm{ur}}_v(\Delta(\mathscr{F}_{\Gamma/\Gamma_{i+1}}(X)), \Delta(\mathscr{F}_{\Gamma/\Gamma_{i+1}}(X)), \pi_{i+1})_{\mathfrak{p}_{i+1}} = 0.$$

It follows from Theorem 10.4.4 that the pairing (10.7.15.3) is non-degenerate, hence

$$\big((N_{i+1})_{\mathfrak{p}_{i+1}}\big)_{\mathrm{tors}} \xrightarrow{\sim} M_{i+1} \oplus M_{i+1}$$

for some S_{i+1}-module M_{i+1} of finite length. As $\widetilde{H}^3_f(K_0, X)_{\mathfrak{q}_0} = 0$ (by (10.7.15.2) for $j=0$ and (10.7.15.1)), Proposition 8.10.11 implies that the canonical map

$$(N_{i+1})_{\Gamma_i/\Gamma_{i+1}} \longrightarrow N_i$$

becomes an isomorphism after localizing at $\mathfrak{q}_i$, hence

$$(N_{i+1})_{\mathfrak{p}_{i+1}}/\mathfrak{p}_{i+1}(N_{i+1})_{\mathfrak{p}_{i+1}} \xrightarrow{\sim} (N_i)_{\mathfrak{q}_i}.$$

As

$$(N_{i+1})_{\mathfrak{p}_{i+1}} \xrightarrow{\sim} S_{i+1}^{\oplus a_{i+1}} \oplus M_{i+1} \oplus M_{i+1},$$

where

$$a_{i+1} := \mathrm{rk}_{S_{i+1}}(N_{i+1})_{\mathfrak{p}_{i+1}} = \dim_{\kappa(\mathfrak{q}_{i+1})}(N_{i+1})_{\mathfrak{q}_{i+1}},$$

it follows that

(10.7.15.4) $a_i = \dim_{\kappa(\mathfrak{q}_i)}(N_i)_{\mathfrak{q}_i} = a_{i+1} + 2\dim_{\kappa(\mathfrak{q}_i)}(M_{i+1}/\mathfrak{p}_{i+1}M_{i+1}) \equiv a_{i+1} \pmod 2$

and $a_i \geqslant a_{i+1}$.

By induction, we deduce that

$$a_0 \geqslant a_1 \geqslant \cdots \geqslant a_r$$

$$a_0 = \dim_{\kappa(\mathfrak{q}_0)} \widetilde{H}^2_f(K_0, X)_{\mathfrak{q}_0} \equiv a_r = \dim_{\kappa(\overline{\mathfrak{q}})} \widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, X)_{\overline{\mathfrak{q}}} \pmod 2,$$

proving (ii).

(iii) The prime ideal $\mathfrak{q}''_0 = \mathfrak{q}'_0 \cap R[\Delta'] \in \mathrm{Spec}(R[\Delta'])$ defines a tautological character $\chi : \Delta' \to \big(R[\Delta']_{\mathfrak{q}''_0}\big)^*$. Replacing K'_0 by the fixed field of $\mathrm{Ker}(\chi)$ and using Lemma 8.6.4.4 (iv) (which applies, since $[K'_0 : K_0]$ is invertible in $\kappa(\mathfrak{q}_0) = R[\Delta]_{\mathfrak{q}_0}$), we reduce to the case when Δ' is cyclic. In this case we have

$$K \subset K_0 \subset K'_0 \subset K_{\infty,1} \subset K_\infty$$

for a suitable choice of $\Gamma_1, \dots, \Gamma_{r-1}$ as in the proof of (ii). Let $\mathfrak{p}' \in \mathrm{Spec}(R_1)$ be the inverse image of $\mathfrak{q}'_0$ under the canonical surjection

$$R_1 = R[\![\Gamma/\Gamma_1]\!] = R[\![\mathrm{Gal}(K_{\infty,1}/K)]\!] \twoheadrightarrow R[\![\mathrm{Gal}(K'_0/K)]\!] = R[\![\Gamma/\Gamma'_0]\!] = R[\Delta \times \Delta'];$$

then $S'_1 := (R_1)_{\mathfrak{p}'}$ is a discrete valuation ring with residue field $R[\Delta \times \Delta']_{\mathfrak{q}'_0} = \kappa(\mathfrak{q}'_0)$. The same argument as in the proof of (ii) then shows that

$$((N_1)_{\mathfrak{p}'})_{S'_1\text{-tors}} \xrightarrow{\sim} M'_1 \oplus M'_1$$

and

$$(N_1)_{\mathfrak{p}'}/\mathfrak{p}'(N_1)_{\mathfrak{p}'} \xrightarrow{\sim} \widetilde{H}^2_f(K'_0, X)_{\mathfrak{q}'_0},$$

hence

$$\begin{aligned} \dim_{\kappa(\mathfrak{q}'_0)} \widetilde{H}^2_f(K'_0, X)_{\mathfrak{q}'_0} &= \dim_{\kappa(\mathfrak{q}_1)}(N_1)_{\mathfrak{q}_1} + 2\dim_{\kappa(\mathfrak{q}_1)}(M'_1/\mathfrak{p}'M'_1) \\ &\equiv \dim_{\kappa(\mathfrak{q}_1)}(N_1)_{\mathfrak{q}_1} = a_1 \equiv a_r = \dim_{\kappa(\overline{\mathfrak{q}})} \widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, X)_{\overline{\mathfrak{q}}} \pmod 2 \end{aligned}$$

and

$$\dim_{\kappa(\mathfrak{q}'_0)} \widetilde{H}^2_f(K'_0, X)_{\mathfrak{q}'_0} \geqslant a_1 \geqslant a_r = \dim_{\kappa(\overline{\mathfrak{q}})} \widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K, X)_{\overline{\mathfrak{q}}},$$

by (10.7.15.4). □

10.7.16. An important special case in which the results of 10.7.14–10.7.15 apply is the following: $R = \mathcal{O}$ is a discrete valuation ring with fraction field F as in 9.1.1, $\overline{R} = \mathcal{O}[\![\Gamma]\!] = \Lambda \xrightarrow{\sim} \mathcal{O}[\Delta][\![X_1, \dots, X_r]\!]$, $X = T$ is as in 9.1.4, Greenberg's local conditions are given by exact sequences of $\mathcal{O}[G_v]$-modules

$$0 \longrightarrow T_v^+ \longrightarrow T \longrightarrow T_v^- \longrightarrow 0 \quad (v \in \Sigma)$$

(with T_v^- free over $\mathcal{O}$) and π comes from a skew-symmetric bilinear form

$$j : T \otimes_{\mathcal{O}} T \longrightarrow \mathcal{O}(1)$$

satisfying $j(T_v^+ \otimes_{\mathcal{O}} T_v^+) = 0$ (for all $v \in \Sigma$) and inducing an isomorphism

$$\mathrm{adj}(j) \otimes \mathrm{id} : V \xrightarrow{\sim} \mathrm{Hom}_F(V, F)(1) = V^*(1),$$

where $V = T \otimes_{\mathcal{O}} F$. For each $v \in \Sigma$, this isomorphism induces an exact sequence of $F[G_v]$-modules

$$(10.7.16.1) \qquad 0 \longrightarrow W_v \longrightarrow V_v^- \longrightarrow \mathrm{Hom}_F(V_v^+, F)(1) \longrightarrow 0,$$

where $V_v^{\pm} = T_v^{\pm} \otimes_{\mathcal{O}} F$. As in 8.9.4, define

$$A^*(1) = \mathrm{Hom}_{\mathcal{O}}(T, F/\mathcal{O})(1) = T^*(1) \otimes_{\mathcal{O}} F/\mathcal{O},$$
$$A = \mathrm{Hom}_{\mathcal{O}}(T^*(1), F/\mathcal{O})(1) = T \otimes_{\mathcal{O}} F/\mathcal{O}$$

and, for each $v \in \Sigma$, Greenberg's local conditions

$$T^*(1)_v^{\pm} = \mathrm{Hom}_{\mathcal{O}}(T_v^{\mp}, \mathcal{O})(1), \quad A_v^+ = \mathrm{Hom}_{\mathcal{O}}(T^*(1)_v^-, F/\mathcal{O})(1) = T_v^+ \otimes_{\mathcal{O}} F/\mathcal{O} \subset A$$
$$A^*(1)_v^+ = \mathrm{Hom}_{\mathcal{O}}(T_v^-, F/\mathcal{O})(1) = T^*(1)_v^+ \otimes_{\mathcal{O}} F/\mathcal{O} \subset A^*(1),$$
$$A_v^- = A/A_v^+, \quad A^*(1)_v^- = A^*(1)/A^*(1)_v^+$$

for $A, T^*(1), A^*(1)$. The map $\mathrm{adj}(j) : T \to T^*(1)$ is injective and its cokernel is $\mathcal{O}$-torsion. It induces maps $A \to A^*(1)$ and, for each $v \in \Sigma$, $T_v^{\pm} \to T^*(1)_v^{\pm}$, $A_v^{\pm} \to A^*(1)_v^{\pm}$, hence also

$$\widetilde{\mathbf{R}\Gamma}_f(X) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(X^*(1)), \qquad (X = T, A;\ Y = T)$$
$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, Y) \longrightarrow \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, Y^*(1)).$$

The exact sequences (10.7.16.1) give rise to exact triangles

$$(10.7.16.2) \qquad V_v^+ \longrightarrow V^*(1)_v^+ \longrightarrow W_v^*(1)$$

in $D^b({}_{F[G_v]}\mathrm{Mod})$ ($v \in \Sigma$). It follows that, for each finite subextension L/K of K_∞/K, there is an exact triangle

$$(10.7.16.3) \qquad \widetilde{\mathbf{R}\Gamma}_f(L, V) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(L, V^*(1)) \longrightarrow \bigoplus_{v\in\Sigma}\bigoplus_{w\mid v} \mathbf{R}\Gamma_{\mathrm{cont}}(G_w, W_v^*(1))$$

in $D^b({}_F\mathrm{Mod})$ (where $G_w = \mathrm{Gal}(\overline{K}_v/L_w)$), and an exact triangle

$$(10.7.16.4) \quad \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, V) \longrightarrow \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, V^*(1)) \longrightarrow \bigoplus_{v\in\Sigma} \mathbf{R}\Gamma_{\mathrm{cont}}(G_v, \mathscr{F}_\Gamma(W_v^*(1)))$$

in $D^b_{ft}({}_{\Lambda\otimes\mathbf{Q}}\mathrm{Mod})$, where $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, V) = \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, T) \otimes_O F$ (and similarly for $V^*(1)$).

We are going to apply Theorem 10.7.15 to $X = T$, $X_v^+ = T_v^+$ and $\pi = j$, assuming that the conditions 10.7.11.1–10.7.11.2 are satisfied.

More precisely, let $\Gamma_0' \subset \Gamma_0$ be an open subgroup, $K_0' = K_\infty^{\Gamma_0'}$ and

$$(10.7.16.5) \qquad \chi : \Delta \longrightarrow (\mathcal{O}')^*, \quad \chi' : \Delta' = \Gamma_0/\Gamma_0' \longrightarrow (\mathcal{O}')^*$$

a pair of characters with values in the ring of integers $\mathcal{O}'$ of a finite extension F' of F. For every $\mathcal{O}[\Delta]$-module M (resp., every $\mathcal{O}[\Delta\times\Delta']$-module N) put

$$M^{(\chi)} = M \otimes_{\mathcal{O}[\Delta],\chi} \mathcal{O}', \quad N^{(\chi\times\chi')} = N \otimes_{\mathcal{O}[\Delta\times\Delta'],\chi\times\chi'} \mathcal{O}'.$$

Then

$$\mathfrak{q}_0 := \mathrm{Ker}(\chi : \mathcal{O}[\Delta] \longrightarrow \mathcal{O}'), \quad \mathfrak{q}_0' := \mathrm{Ker}(\chi\times\chi' : \mathcal{O}[\Delta\times\Delta'] \longrightarrow \mathcal{O}')$$

are minimal prime ideals as in Theorem 10.7.15, satisfying (by Proposition 8.8.7)

$$\begin{aligned}(10.7.16.6)\qquad \widetilde{H}^j_f(K_0,V)_{\mathfrak{q}_0} \otimes_{\kappa(\mathfrak{q}_0)} F' &= \widetilde{H}^j_f(K_0,V)^{(\chi)} = \widetilde{H}^j_f(K, V\otimes\chi^{-1}),\\ \widetilde{H}^j_f(K_0',V)_{\mathfrak{q}_0'} \otimes_{\kappa(\mathfrak{q}_0')} F' &= \widetilde{H}^j_f(K_0',V)^{(\chi\times\chi')} = \widetilde{H}^j_f(K, V\otimes(\chi\times\chi')^{-1}).\end{aligned}$$

10.7.17. Theorem. — *In the situation of 10.7.16, assume that 10.7.11.1–10.7.11.2 hold and*

$$(\forall v\in\Sigma)\,(\forall v_0 \mid v \text{ in } K_0)\quad H^0_{\mathrm{cont}}(G_{v_0}, W_v) = H^0_{\mathrm{cont}}(G_{v_0}, W_v^*(1)) = 0.$$

Then, for any pair of characters χ,χ' as in (10.7.16.5), *we have:*

(i) *For each $j\in\mathbf{Z}$, the canonical maps*

$$\widetilde{H}^j_f(K_0,V)^{(\chi)} \longrightarrow \widetilde{H}^j_f(K_0,V^*(1))^{(\chi)}$$

$$\widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K,V)^{(\chi)} \otimes_{\Lambda^{(\chi)}} \mathrm{Frac}(\Lambda^{(\chi)}) \longrightarrow \widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K,V^*(1))^{(\chi)} \otimes_{\Lambda^{(\chi)}} \mathrm{Frac}(\Lambda^{(\chi)})$$

are isomorphisms.

(ii) *We have*

$$\begin{aligned}\dim_{F'} \widetilde{H}^1_f(K_0,V)^{(\chi)} &= \dim_{F'} \widetilde{H}^2_f(K_0,V)^{(\chi)} = \dim_{F'} \widetilde{H}^1_f(K_0,V)^{(\chi^{-1})}\\ &= \dim_{F'} \widetilde{H}^2_f(K_0,V)^{(\chi^{-1})}\end{aligned}$$

$$\begin{aligned}\mathrm{rk}_{\Lambda^{(\chi)}} \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K,T)^{(\chi)} &= \mathrm{rk}_{\Lambda^{(\chi)}} \widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K,T)^{(\chi)} = \mathrm{rk}_{\Lambda^{(\chi^{-1})}} \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K,T)^{(\chi^{-1})}\\ &= \mathrm{rk}_{\Lambda^{(\chi^{-1})}} \widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K,T)^{(\chi^{-1})}.\end{aligned}$$

(iii) *If $\widetilde{H}^0_f(K_0,V)^{(\chi)}=0$, then*

$$\begin{aligned}\dim_{F'} \widetilde{H}^1_f(K_0,V)^{(\chi)} &\equiv \mathrm{rk}_{\Lambda^{(\chi)}} \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K,T)^{(\chi)} \pmod 2,\\ \dim_{F'} \widetilde{H}^1_f(K_0,V)^{(\chi)} &\geqslant \mathrm{rk}_{\Lambda^{(\chi)}} \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K,T)^{(\chi)}.\end{aligned}$$

In particular, if $\Delta = 0$, then

$$\mathrm{rk}_{\mathcal{O}} \widetilde{H}^1_f(T) = \mathrm{rk}_{\mathcal{O}} \widetilde{H}^2_f(T), \quad \mathrm{rk}_\Lambda \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K,T) = \mathrm{rk}_\Lambda \widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K,T)$$

$$\mathrm{rk}_{\mathcal{O}} \widetilde{H}^1_f(T) \equiv \mathrm{rk}_\Lambda \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K,T) \pmod 2, \quad \mathrm{rk}_{\mathcal{O}} \widetilde{H}^1_f(T) \geqslant \mathrm{rk}_\Lambda \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K,T).$$

(iv) *Assume that*

$$(\forall v \in \Sigma)\,(\forall v_0' \mid v \text{ in } K_0') \quad H^0_{\mathrm{cont}}(G_{v_0'}, W_v) = H^0_{\mathrm{cont}}(G_{v_0'}, W_v^*(1)) = 0.$$

Then, for each $j \in \mathbf{Z}$, *the canonical map*

$$\widetilde{H}^j_f(K_0', V)^{(\chi\times\chi')} \longrightarrow \widetilde{H}^j_f(K_0', V^*(1))^{(\chi\times\chi')}$$

is an isomorphism. If, in addition, $\widetilde{H}^0_f(K_0', V)^{(\chi\times\chi')} = 0$, *then*

$$\begin{aligned}\dim_{F'} \widetilde{H}^1_f(K_0', V)^{(\chi\times\chi')} &= \dim_{F'} \widetilde{H}^2_f(K_0', V)^{(\chi\times\chi')} = \dim_{F'} \widetilde{H}^1_f(K_0', V)^{(\chi\times\chi')^{-1}} \\ &= \dim_{F'} \widetilde{H}^2_f(K_0', V)^{(\chi\times\chi')^{-1}}\end{aligned}$$

and

$$\dim_{F'} \widetilde{H}^1_f(K_0', V)^{(\chi\times\chi')} \equiv \mathrm{rk}_{\Lambda(\chi)} \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K, T)^{(\chi)} \equiv \dim_{F'} \widetilde{H}^1_f(K_0, V)^{(\chi)} \pmod 2$$

$$\dim_{F'} \widetilde{H}^1_f(K_0', V)^{(\chi\times\chi')} \geqslant \mathrm{rk}_{\Lambda(\chi)} \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K, T)^{(\chi)}.$$

(v) *Assume that* $\widetilde{H}^0_f(K_0', V)^{(\chi)} = 0$, $\dim_{F'} \widetilde{H}^1_f(K_0, V)^{(\chi)} \equiv 1 \pmod 2$ *and*

$$(\forall v \in \Sigma)\,(\forall v_0' \mid v \text{ in } K_0') \quad H^0_{\mathrm{cont}}(G_{v_0'}, W_v) = H^0_{\mathrm{cont}}(G_{v_0'}, W_v^*(1)) = 0;$$

then $\dim_{F'} \widetilde{H}^1_f(K_0', V)^{(\chi)} \geqslant [K_0' : K_0]$.

Proof. — This is a special case of Theorem 10.7.15. However, in view of the importance of the congruence

$$\dim_{F'} \widetilde{H}^1_f(K_0', V)^{(\chi\times\chi')} \equiv \dim_{F'} \widetilde{H}^1_f(K_0, V)^{(\chi)} \pmod 2$$

for the arithmetic applications proved in Chapter 12 below, we repeat the key arguments in this simplified setting.

(i) Fix a prime v_0 of K_0 above $v \in \Sigma$. By assumption, the cohomology groups $H^0_{\mathrm{cont}}(G_{v_0}, W_v^*(1))$ and $H^2_{\mathrm{cont}}(G_{v_0}, W_v^*(1)) \xrightarrow{\sim} H^0_{\mathrm{cont}}(G_{v_0}, W_v)^*$ vanish; the Euler characteristic formula (Theorem 4.6.9 and 5.2.11)

$$\sum_{q=0}^{2} (-1)^q \dim_F H^q_{\mathrm{cont}}(G_{v_0}, W_v^*(1)) = 0$$

implies that $H^1_{\mathrm{cont}}(G_{v_0}, W_v^*(1))$ also vanishes. The first half of (i) then follows from the exact triangle (10.7.16.3) for $L = K_0$. Proposition 8.4.8.5 applied to the extension K_∞/K_0, $R = \mathcal{O}$, $\mathfrak{p} = (0)$ and $T = \mathscr{F}_\Delta(T(W_v)^*(1))$ (where $T(W_v)^*(1) \subset W_v^*(1)$ is an arbitrary G_v-stable $\mathcal{O}$-lattice) shows that $\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, \mathscr{F}_\Gamma(W_v^*(1))) \otimes_\Lambda \mathrm{Frac}(\Lambda) \xrightarrow{\sim} 0$; the second half of (i) then follows from (10.7.16.4).

(ii) This follows from (i), the duality isomorphisms

$$\widetilde{H}^j_f(K_0, V)^{(\chi)} \xrightarrow{\sim} \mathrm{Hom}_{F'}\big(\widetilde{H}^{3-j}_f(K_0, V^*(1))^{(\chi^{-1})}, F'\big)$$

$$\widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, T) \otimes_\Lambda \mathrm{Frac}(\Lambda) \xrightarrow{\sim} \mathrm{Hom}_\Lambda\big(\widetilde{H}^{3-j}_{f,\mathrm{Iw}}(K_\infty/K, T^*(1)), \Lambda\big)^\iota \otimes_\Lambda \mathrm{Frac}(\Lambda)$$

and the fact that the action of τ (= a lift of the non-trivial element of $\mathrm{Gal}(K/K^+)$) interchanges the eigenspaces for χ and χ^{-1}.

(iv) As $\chi'(\Gamma_0)$ is a finite cyclic group, χ' factors through a quotient $\overline{\Gamma}_0$ of Γ_0, which is isomorphic to $\mathbf{Z}_p$. We can then replace K_∞ by the fixed field of $\mathrm{Ker}(\Gamma_0 \to \overline{\Gamma}_0)$ and assume that $\Gamma_0 \xrightarrow{\sim} \mathbf{Z}_p$, hence $\mathrm{Ker}(\chi') = p^n\Gamma_0$ for some $n \geqslant 0$. Let $\mathfrak{p}' \in \mathrm{Spec}(\Lambda')$ be the augmentation ideal of $\Lambda' := \mathcal{O}'[\![\Gamma_0]\!]$. Proposition 10.7.12 (ii) implies that there is an exact sequence of $\Lambda'_{\mathfrak{p}'}$-modules

$$0 \longrightarrow X \oplus X \longrightarrow \left(\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K,T)^{(\chi)}\right)_{\mathfrak{p}'} \longrightarrow \left(\Lambda'_{\mathfrak{p}'}\right)^{\oplus r} \longrightarrow 0, \tag{10.7.17.1}$$

where

$$r = \mathrm{rk}_{\Lambda'_{\mathfrak{p}'}}\left(\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K,T)^{(\chi)}\right)_{\mathfrak{p}'} \geqslant 0.$$

Thanks to the assumptions

$$\widetilde{H}^0_f(K_0,V)^{(\chi)} = \widetilde{H}^0_f(K'_0,V)^{(\chi\times\chi')} = 0,$$

Proposition 8.10.4 applies with $R = \mathcal{O}$ and $\mathscr{S} = R - \{0\}$ to the extensions K_∞/K_0 and K_∞/K'_0; one obtains isomorphisms

$$\left(\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K,T)^{(\chi)}_{\Gamma_0}\right) \otimes_{\mathcal{O}'} F' \xrightarrow{\sim} \widetilde{H}^2_f(K_0,V)^{(\chi)}$$
$$\left(\widetilde{H}^2_{f,\mathrm{Iw}}(K_\infty/K,T)^{(\chi\times\chi')}_{p^n\Gamma_0}\right) \otimes_{\mathcal{O}'} F' \xrightarrow{\sim} \widetilde{H}^2_f(K'_0,V)^{(\chi\times\chi')}.$$

The exact sequence (10.7.17.1) then implies that

$$\dim_{F'} \widetilde{H}^2_f(K_0,V)^{(\chi)} = r + 2\dim_{F'}(X_{\Gamma_0})$$
$$\dim_{F'} \widetilde{H}^2_f(K'_0,V)^{(\chi\times\chi')} = r + 2\dim_{F'}(X_{p^n\Gamma_0})^{(\chi')},$$

hence

$$\dim_{F'} \widetilde{H}^2_f(K_0,V)^{(\chi)} \equiv r \equiv \dim_{F'} \widetilde{H}^2_f(K'_0,V)^{(\chi\times\chi')} \pmod 2.$$

As $\widetilde{H}^2_f(K_0,V)^{(\chi)}$ is dual to

$$\widetilde{H}^1_f(K_0,V^*(1))^{(\chi^{-1})} = \widetilde{H}^1_f(K_0,V)^{(\chi^{-1})} \xrightarrow{\sim} \widetilde{H}^1_f(K_0,V)^{(\chi)}$$

(the last isomorphism being given by the action of τ) and $\widetilde{H}^2_f(K'_0,V)^{(\chi\times\chi')}$ is dual to

$$\widetilde{H}^1_f(K'_0,V^*(1))^{(\chi\times\chi')^{-1}} = \widetilde{H}^1_f(K'_0,V)^{(\chi\times\chi')^{-1}} \xrightarrow{\sim} \widetilde{H}^1_f(K'_0,V)^{(\chi\times\chi')},$$

we obtain the desired congruence

$$\dim_{F'} \widetilde{H}^1_f(K_0,V)^{(\chi)} \equiv \dim_{F'} \widetilde{H}^1_f(K'_0,V)^{(\chi\times\chi')} \pmod 2.$$

(v) We can assume that F' contains the values of all characters of $\Delta' = \mathrm{Gal}(K_0'/K_0)$; then

$$\widetilde{H}^1_f(K_0', V)^{(\chi)} = \bigoplus_{\chi'} \widetilde{H}^1_f(K_0', V)^{(\chi\times\chi')},$$

where χ' runs through all characters of Δ'. As

$$\dim_{F'} \widetilde{H}^1_f(K_0', V)^{(\chi\times\chi')} \equiv \dim_{F'} \widetilde{H}^1_f(K_0, V)^{(\chi)} \equiv 1 \pmod 2$$

by (iv), it follows that

$$\widetilde{H}^1_f(K_0', V)^{(\chi)} \geqslant \sum_{\chi'} 1 = |\Delta'| = [K_0' : K_0],$$

as claimed. □

10.7.18. For example, if K is an imaginary quadratic field and K_∞/K its anticyclotomic $\mathbf{Z}_p$-extension, then $K^+ = \mathbf{Q}$ and we are in the situation considered in [**Ne3**]. If $\Sigma^+ = \{p\}$ and S_f^+ consists of all rational primes dividing pN, where $N \geqslant 1$ is an integer not divisible by p such that all primes dividing N are unramified (resp., split) in $K/\mathbf{Q}$, then the condition 10.3.5.1.5 is satisfied (resp., the condition 10.3.5.1.5 is satisfied and no prime in Σ' splits completely in K_∞/K).

If E is an elliptic curve over $\mathbf{Q}$ with good ordinary reduction at p and N is the conductor of E, then $T = T_p(E)$ and $A = E[p^\infty]$ are as in 10.7.16 (for $\mathcal{O} = \mathbf{Z}_p, F = \mathbf{Q}_p$), with j given by the Weil pairing and $T_p^- = T_p(\mathcal{E} \otimes_{\mathbf{Z}_p} \mathbf{F}_p)$, where $\mathcal{E}$ is a proper smooth model of E over $\mathbf{Z}_p$.

The Selmer group $\widetilde{H}^1_f(A)$ (resp., the Λ-module $\widetilde{H}^1_f(K_S/K_\infty, A)$) differs from $\mathrm{Sel}(E/K, p^\infty)$ (resp., from

$$\mathrm{Sel}(E/K_\infty, p^\infty) = \varinjlim_{\alpha} \mathrm{Sel}(E/K_\alpha, p^\infty),$$

where K_α/K are the finite subextensions of K_∞/K) by a finite group (resp., by a Λ-module killed by a power of p), by Proposition 9.6.6 combined with Proposition 9.6.7.5. As the group $H^0(G_{K,S}, A) = E(K)[p^\infty]$ is finite, the assumptions of Theorem 10.7.17 (i)–(iii) are satisfied and the congruence in 10.7.17 (iii) becomes

(10.7.18.1) $$\mathrm{cork}_{\mathbf{Z}_p}(\mathrm{Sel}(E/K, p^\infty)) \equiv \mathrm{cork}_\Lambda(\mathrm{Sel}(E/K_\infty, p^\infty)) \pmod 2,$$

as in [**Ne3**, Thm. B] (assuming that each prime dividing N is unramified in $K/\mathbf{Q}$). More precisely, if all primes dividing N are split in $K/\mathbf{Q}$, then Proposition 10.7.12 implies that

$$\big(D_\Lambda(\mathrm{Sel}(E/K_\infty, p^\infty))\big)_{\Lambda\text{-tors}} \xrightarrow{\sim} M \oplus M \oplus N'$$

in $({}_\Lambda\mathrm{Mod})/(\text{pseudo-null})$, where $2^n \cdot N' = 0$ for some $n \geqslant 0$ (in particular, $N' \xrightarrow{\sim} 0$ in $({}_\Lambda\mathrm{Mod})/(\text{pseudo-null})$ if $p \neq 2$); this proves ([**Ne3**, Lemma 2.5]).

It may be useful to spell out explicitly the following special case of Theorem 10.7.17 (which generalizes the congruence (10.7.18.1)) in a situation considered by Mazur and Rubin ([**M-R3**]).

10.7.19. Proposition. — *Let $K^+ \subset K \subset K_\infty$ be as in 10.3.5.1, with $\Gamma = \mathrm{Gal}(K_\infty/K) \xrightarrow{\sim} \mathbf{Z}_p^r$; set $\Lambda = \mathbf{Z}_p[\![\Gamma]\!]$. If E is an elliptic curve over K^+ with good ordinary reduction at all primes above p, then*

$$\mathrm{cork}_{\mathbf{Z}_p} \mathrm{Sel}(E/K, p^\infty) \equiv \mathrm{cork}_\Lambda \mathrm{Sel}(E/K_\infty, p^\infty) \pmod 2$$
$$\mathrm{cork}_{\mathbf{Z}_p} \mathrm{Sel}(E/K, p^\infty) \geqslant \mathrm{cork}_\Lambda \mathrm{Sel}(E/K_\infty, p^\infty).$$

If, in addition, $\mathrm{cork}_{\mathbf{Z}_p} \mathrm{Sel}(E/K, p^\infty) \equiv 1 \pmod 2$*, then*

$$\mathrm{cork}_{\mathbf{Z}_p} \mathrm{Sel}(E/K', p^\infty) \geqslant [K' : K],$$

for each finite subextension K'/K of K_∞/K (cf. [**M-R3**, Thm. 3.1]*).*

Proof. — Fix a finite set S^+ of primes of K^+ containing all archimedean primes, all primes above p and all primes at which E has bad reduction. Consider Greenberg's local conditions associated to the data 10.3.5.1.2–10.3.5.1.4 with $R = \mathbf{Z}_p$, $X = Y = T_p(E) = T$, $\pi : T_p(E) \otimes_{\mathbf{Z}_p} T_p(E) \to \mathbf{Z}_p(1)$ given by the Weil pairing, $\Sigma^+ = S_f^+$ and, for each $v^+ \in S_f^+$,

$$T_p(E)^+_{v^+} = \begin{cases} T_p(E)^+_{v^+} \text{ defined in 9.6.7.2}, & v^+ \mid p \\ 0, & v^+ \nmid p. \end{cases}$$

As $\Sigma' = \varnothing$, we do not have to worry about the condition 10.3.5.1.5. As $V = T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p = V_p(E) \xrightarrow{\sim} V^*(1)$ satisfies the monodromy-weight conjecture at each non-archimedean prime not dividing p ([**Ja2**, §5, §7]), we have

$$H^0(G_w, V) = H^0(G_w, V^*(1)) = 0, \qquad (G_w = \mathrm{Gal}(\overline{K}_v/L_w))$$

for each finite extension L/K and each non-archimedean prime $w \mid v \nmid p$ of L. In particular, the assumption of Theorem 10.7.17 is satisfied for $K_0 = K$ and each $v \in \Sigma$ (note that $W_v = 0$ if $v \mid p$, while $W_v = V_v$ if $v \nmid p$).

Applying Theorem 10.7.17 (iii) (observing that $\widetilde{H}^0_f(K, V) \subset H^0(G_K, V) = 0$), we obtain, using the notation from 9.7.8,

$$\mathrm{rk}_{\mathbf{Z}_p} \widetilde{H}^1_{f,\Sigma}(K, T_p(E)) \equiv \mathrm{rk}_\Lambda \widetilde{H}^1_{f,\mathrm{Iw},\Sigma}(K_\infty/K, T_p(E)) \pmod 2,$$
$$\mathrm{rk}_{\mathbf{Z}_p} \widetilde{H}^1_{f,\Sigma}(K, T_p(E)) \geqslant \mathrm{rk}_\Lambda \widetilde{H}^1_{f,\mathrm{Iw},\Sigma}(K_\infty/K, T_p(E)).$$

On the other hand, Proposition 9.7.9 (i) together with Lemma 9.6.7.6 (iii) show that

$$\mathrm{rk}_{\mathbf{Z}_p} \widetilde{H}^1_{f,\Sigma}(K, T_p(E)) = \mathrm{rk}_{\mathbf{Z}_p} \widetilde{H}^1_f(K, T_p(E)) = \mathrm{cork}_{\mathbf{Z}_p} \mathrm{Sel}(E/K, p^\infty).$$

Similarly, Proposition 9.7.9 (iii) together with (8.9.6.4.3), (9.6.5.2) and Proposition 9.6.7.8 imply that

$$\begin{aligned}\mathrm{rk}_\Lambda\, \widetilde{H}^1_{f,\mathrm{Iw},\Sigma}(K_\infty/K, T_p(E)) &= \mathrm{rk}_\Lambda\, \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K, T_p(E)) \\ &= \mathrm{cork}_\Lambda\, \widetilde{H}^1_f(K_S/K_\infty, E[p^\infty]) = \mathrm{cork}_\Lambda\, S^{\mathrm{str}}_A(K_\infty) = \mathrm{cork}_\Lambda\, \mathrm{Sel}(E/K_\infty, p^\infty).\end{aligned}$$

If $\mathrm{cork}_{\mathbf{Z}_p}\,\mathrm{Sel}(E/K, p^\infty) \equiv 1 \pmod 2$, then Theorem 10.7.17 (v) combined with the previous discussion yields the inequality $\mathrm{cork}_{\mathbf{Z}_p}\,\mathrm{Sel}(E/K', p^\infty) \geqslant [K' : K]$. $\square$

10.7.20. In Chapter 12, we generalize Proposition 10.7.19, as well as the main parity result of [**Ne3**], to the case of Hilbert modular forms.

10.8. Comparison with the Flach pairing

For simplicity, assume that the condition (P) from 5.1 holds.

10.8.1. Let $R = \mathcal{O}$, $F = \mathrm{Frac}(\mathcal{O})$ and $T, A, T^*(1), A^*(1)$ be as in 9.1.1–9.1.4. Set $V = T \otimes_{\mathcal{O}} F$, $V^*(1) = T^*(1) \otimes_{\mathcal{O}} F$ and assume that we are given, for each $v \in S_f$, $F[G_v]$-submodules $V_v^+ \subset V$, $V^*(1)_v^+ \subset V^*(1)$ satisfying $V_v^+ \perp_{\mathrm{ev}_2} V^*(1)_v^+$; set $T_v^+ = T \cap V_v^+$, $T^*(1)_v^+ = T^*(1) \cap V^*(1)_v^+$.

We denote the inclusions $T \hookrightarrow V$, $T^*(1) \hookrightarrow V^*(1)$ (resp., $X_v^+ \hookrightarrow X$, for $X = T, V, T^*(1), V^*(1)$) by i (resp., by i_v^+). For each function f with values in V, we denote by f° the corresponding function with values in $V/T = A$ (and similarly for $V^*(1)$ and $V^*(1)/T^*(1) = A^*(1)$, resp., F and $F/\mathcal{O}$).

10.8.2. These data define Greenberg style Selmer groups

$$S(X) = \mathrm{Ker}\left[H^1(G_{K,S}, X) \longrightarrow \bigoplus_{v \in S_f} H^1(G_v, X)/L_v^+(X)\right], \qquad (X = A, A^*(1))$$

where

$$\begin{aligned}L_v^+(A) &= \mathrm{Im}\left(H^1_{\mathrm{cont}}(G_v, V_v^+) \longrightarrow H^1(G_v, A)\right),\\ L_v^+(A^*(1)) &= \mathrm{Im}\left(H^1_{\mathrm{cont}}(G_v, V^*(1)_v^+) \longrightarrow H^1(G_v, A^*(1))\right).\end{aligned}$$

10.8.3. For example, assume that $\mathcal{O} = \mathbf{Z}_p$, B is an abelian variety over K with good reduction outside S_f satisfying $\{v \mid p\} = \Sigma_o \cup \Sigma_t$ (in the notation of 9.6.7.2), $T = T_p(B)$, $T^*(1) = T_p(\widehat{B})$, $A = B[p^\infty]$ and $A^*(1) = \widehat{B}[p^\infty]$. If we define $V_v^+, V^*(1)_v^+$ as in 9.6.7.2 if $v \mid p$ (resp., set $V_v^+ = V^*(1)_v^+ = 0$ if $v \nmid p$), then we have, for each $v \in S_f$,

$$L_v^+(A) = \mathrm{Im}\left(B(K_v) \otimes \mathbf{Q}_p/\mathbf{Z}_p \hookrightarrow H^1(G_v, A)\right)$$

(and similarly for $A^*(1)$ and $\widehat{B}$), hence

$$S(A) = \mathrm{Sel}(B/K, p^\infty), \quad S(A^*(1)) = \mathrm{Sel}(\widehat{B}/K, p^\infty).$$

10.8.4. The data from 10.8.1 also give rise, for each $X = T, V, T^*(1), V^*(1)$, to Selmer complexes

$$\widetilde{C}^\bullet_f(X) := \widetilde{C}^\bullet_f(G_{K,S}, X; \Delta(X))$$

associated to the local conditions

$$\Delta_v(X) : C^\bullet_{\text{cont}}(G_v, X_v^+) \longrightarrow C^\bullet_{\text{cont}}(G_v, X).$$

We denote their cohomology groups by $\widetilde{H}^j_f(X)$.

10.8.5. Flach's pairing. — Flach's construction [**Fl1**] yields an $\mathcal{O}$-bilinear pairing

$$\langle\ ,\ \rangle_{\text{Flach}} : S(A) \times S(A^*(1)) \longrightarrow F/\mathcal{O}.$$

In order to define this pairing, fix cohomology classes

$$[\alpha] \in S(A), \quad [\alpha'] \in S(A^*(1))$$

represented by 1-cocycles

$$\alpha \in C^1(G_{K,S}, A), \quad d\alpha = 0, \quad \alpha' \in C^1(G_{K,S}, A^*(1)), \quad d\alpha' = 0.$$

Lift these cocycles to 1-cochains

$$a_1 \in C^\bullet_{\text{cont}}(G_{K,S}, V), \quad b_1 \in C^\bullet_{\text{cont}}(G_{K,S}, V^*(1));$$

then

$$da_1 = -i(a_2), \quad db_1 = -i(b_2), \quad a_2 \in C^2_{\text{cont}}(G_{K,S}, T), \quad b_2 \in C^2_{\text{cont}}(G_{K,S}, T^*(1)).$$

The definition of $S(X)$ implies that there exist, for each $v \in S_f$, elements

$$\begin{aligned} (10.8.5.1) \quad & a^+_{1,v} \in C^1_{\text{cont}}(G_v, V_v^+), \quad A_{0,v} \in C^0_{\text{cont}}(G_v, V) = V, \quad A_{1,v} \in C^1_{\text{cont}}(G_v, T) \\ & b^+_{1,v} \in C^1_{\text{cont}}(G_v, V^*(1)_v^+), \quad B_{0,v} \in C^0_{\text{cont}}(G_v, V^*(1)) = V^*(1), \\ & B_{1,v} \in C^1_{\text{cont}}(G_v, T^*(1)) \end{aligned}$$

satisfying

$$\begin{aligned} \text{res}_v(a_1) &= -i_v^+(a^+_{1,v}) + dA_{0,v} + i(A_{1,v}), \quad da^+_{1,v} = 0 \\ \text{res}_v(b_1) &= -i_v^+(b^+_{1,v}) + dB_{0,v} + i(B_{1,v}), \quad db^+_{1,v} = 0. \end{aligned}$$

The coboundary of the 3-cochain

$$da_1 \cup b_1 \in C^3_{\text{cont}}(G_{K,S}, F(1))$$

satisfies

$$d(da_1 \cup b_1) = i(a_2 \cup b_2) \in i\left(C^4_{\text{cont}}(G_{K,S}, \mathcal{O}(1))\right),$$

which means that $(da_1 \cup b_1)^\circ$ (= the image of $da_1 \cup b_1$ modulo cochains with values in $\mathcal{O}(1)$) is a 3-cocycle:

$$(da_1 \cup b_1)^\circ \in C^3_{\text{cont}}(G_{K,S}, F/\mathcal{O}(1)), \quad d(da_1 \cup b_1)^\circ = 0.$$

As $H^3(G_{K,S}, F/\mathcal{O}(1)) = 0$, there exists $\varepsilon \in C^2_{\text{cont}}(G_{K,S}, F/\mathcal{O}(1))$ such that

$$-a_2 \cup b_1^\circ = (da_1 \cup b_1)^\circ = d\varepsilon.$$

For each $v \in S_f$, the 1-cocycle

$$\widetilde{\beta}_v := -i_v^+(b_{1,v}^+) + dB_{0,v} \in C^1_{\rm cont}(G_v, V^*(1)), \quad d\widetilde{\beta}_v = 0$$

is a lift of $\mathrm{res}_v(b_1)^\circ \in C^1_{\rm cont}(G_v, V^*(1)/T^*(1))$. As

$$d(\mathrm{res}_v(a_1) \cup \widetilde{\beta}_v)^\circ = -\mathrm{res}_v(a_2) \cup \widetilde{\beta}_v^\circ = -\mathrm{res}_v(a_2 \cup b_1^\circ) = \mathrm{res}_v(d\varepsilon) = d(\mathrm{res}_v(\varepsilon)),$$

the 2-cochain

$$c_v := (\mathrm{res}_v(a_1) \cup \widetilde{\beta}_v)^\circ - \mathrm{res}_v(\varepsilon) \in C^2_{\rm cont}(G_v, F(1)/\mathcal{O}(1)), \quad dc_v = 0$$

is a 2-cocycle. Flach defined

$$\langle [\alpha], [\alpha'] \rangle_{\rm Flach} = \sum_{v \in S_f} \mathrm{inv}_v(c_v) \in F/\mathcal{O} \tag{10.8.5.2}$$

(and showed that this element depends only on $[\alpha]$ and $[\alpha']$).

10.8.6. The pairing from 10.2.2. — The choice of the cocycles in (10.8.5.1) determines 2-cochains

$$x = (a_2, 0, (A_{1,v})) \in \widetilde{C}^2_f(G_{K,S}, T), \quad dx = 0,$$
$$y = (b_2, 0, (B_{1,v})) \in \widetilde{C}^2_f(G_{K,S}, T^*(1)), \quad dy = 0,$$

where we have used the notation from 1.3.1 and 6.3.1:

$$\widetilde{C}^n_f(G_{K,S}, T) = C^n_{\rm cont}(G_{K,S}, T) \oplus \bigoplus_{v \in S_f} C^n_{\rm cont}(G_v, T_v^+) \oplus \bigoplus_{v \in S_f} C^{n-1}_{\rm cont}(G_v, T),$$

with the differentials given by

$$d(t_n, (t_{n,v}^+), (t_{n-1,v})) = (dt_n, (dt_{n,v}^+), (-\mathrm{res}_v(t_n) + i_v^+(t_{n,v}^+) - dt_{n-1,v})).$$

The formula

$$i(x) = \big(i(a_2), 0, (i(A_{1,v}))\big) = (-d(a_1), 0, (\mathrm{res}_v(a_1) + i_v^+(a_{1,v}^+) - dA_{0,v})) = dX,$$
$$X = (-a_1, (a_{1,v}^+), (A_{0,v}))$$

shows that $i(x) \in d\,\widetilde{C}^1_f(G_{K,S}, V)$ is a coboundary, hence the class of x is contained in

$$[x] \in \mathrm{Ker}\big(\widetilde{H}^2_f(T) \longrightarrow \widetilde{H}^2_f(V)\big) = \widetilde{H}^2_f(T)_{\rm tors}.$$

Similarly, the class of y satisfies $[y] \in \widetilde{H}^2_f(T^*(1))_{\rm tors}$. We wish to relate the product of $[x]$ and $[y]$ under the pairing

$$\cup_{2,2} = \cup_{\mathrm{ev}_2,0,2,2} : \widetilde{H}^2_f(T)_{\rm tors} \times \widetilde{H}^2_f(T^*(1))_{\rm tors} \longrightarrow F/\mathcal{O}$$

(which was defined in 10.2.2) to the product (10.8.5.2).

10.8.7. Proposition. — *In the situation of 10.8.5–10.8.6,*

$$[x] \cup_{2,2} [y] = -\langle [\alpha], [\alpha'] \rangle_{\rm Flach}.$$

Proof. — Let $C^\bullet = \big[\mathcal{O} \xrightarrow{-i} F\big]$ be the complex from Lemma 2.10.7 (i) (in degrees 0,1). The tensor product complex $\widetilde{C}^\bullet_f(T) \otimes_{\mathcal{O}} C^\bullet$ has differentials

$$\begin{aligned} \widetilde{C}^n_f(T) \oplus \widetilde{C}^{n-1}_f(V) &\longrightarrow \widetilde{C}^{n+1}_f(T) \oplus \widetilde{C}^n_f(V) \\ (x_n, X_{n-1}) &\longmapsto (dx_n, (-1)^{n-1} i(x_n) + dX_{n-1}), \end{aligned}$$

which means that the pair

$$a = (x, X) \in (\widetilde{C}^\bullet_f(T) \otimes_{\mathcal{O}} C^\bullet)^2$$

is a 2-cocyle lifting $x \in \widetilde{C}^2_f(T)$. Similarly,

$$b = (y, Y) \in (\widetilde{C}^\bullet_f(T^*(1)) \otimes_{\mathcal{O}} C^\bullet)^2, \quad Y = (-b_1, (b^+_{1,v}), (B_{0,v})),$$

is a 2-cocyle lifting $y \in \widetilde{C}^2_f(T^*(1))$. In order to compute $[x] \cup_{2,2} [y]$, we must first determine $v(s_{23}(a \cup b))$, where v is the map defined in Lemma 2.10.7 (i) and the cup product $a \cup b$ is computed using the product

$$\cup_{r,0} : \widetilde{C}^\bullet_f(T) \otimes_{\mathcal{O}} \widetilde{C}^\bullet_f(T^*(1)) \longrightarrow C^\bullet_c(\mathcal{O}(1)),$$

from Proposition 1.3.2 (i), for any fixed value $r \in \mathcal{O}$:

$$v(s_{23}(a \cup b)) = (x \cup_{r,0} y,\, x \cup_{r,0} Y) \in (C^\bullet_c(\mathcal{O}(1)) \otimes_{\mathcal{O}} C^\bullet)^4.$$

Taking $r = 1$, we compute

$$\begin{aligned} x \cup_{1,0} y &= (a_2, 0, (A_{1,v})) \cup_{1,0} (b_2, 0, (B_{1,v})) \\ &= \big(a_2 \cup b_2, (A_{1,v} \cup \mathrm{res}_v(b_v))\big) \in C^4_{c,\mathrm{cont}}(G_{K,S}, \mathcal{O}(1)) \\ x \cup_{1,0} Y &= (a_2, 0, (A_{1,v})) \cup_{1,0} (-b_1, (b^+_{1,v}), (B_{0,v})) \\ &= \big(-a_2 \cup b_1, (-A_{1,v} \cup \mathrm{res}_v(b_1))\big) \in C^3_{c,\mathrm{cont}}(G_{K,S}, F(1)). \end{aligned}$$

The next step is to reduce $v(s_{23}(a \cup b))$ modulo cochains with values in $\mathcal{O}(1)$, obtaining a 3-cocycle

$$\begin{gathered} v(s_{23}(a \cup b))^\circ = \big(-a_2 \cup b_1, (-A_{1,v} \cup \mathrm{res}_v(b_1))\big)^\circ \in C^3_{c,\mathrm{cont}}(G_{K,S}, F(1)/\mathcal{O}(1)), \\ d(v(s_{23}(a \cup b))^\circ) = 0, \end{gathered}$$

whose image under the isomorphism

$$H^3_{c,\mathrm{cont}}(G_{K,S}, F(1)/\mathcal{O}(1)) \xrightarrow{\sim} F/\mathcal{O}$$

gives the desired product $[x] \cup_{2,2} [y]$. As

$$\begin{aligned} &\big(-a_2 \cup b_1, (-A_{1,v} \cup \mathrm{res}_v(b_1))\big)^\circ \\ &\qquad = (-a_2 \cup b_1^\circ, (-A_{1,v} \cup \widetilde{\beta}_v)^\circ) = d(\varepsilon, 0) + (0, (\mathrm{res}_v(\varepsilon) - (A_{1,v} \cup \widetilde{\beta}_v)^\circ)), \end{aligned}$$

we have

$$[x] \cup_{2,2} [y] = \sum_{v \in S_f} \mathrm{inv}_v(e_v),$$

where
$$e_v = \mathrm{res}_v(\varepsilon) - (A_{1,v} \cup \widetilde{\beta}_v)^\circ \in C^2_{\mathrm{cont}}(G_v, F(1)/\mathcal{O}(1)), \quad de_v = 0.$$
However,
$$\begin{aligned}(\forall v \in S_f) \quad c_v + e_v &= ((dA_{0,v} - i_v^+(a_{1,v}^+)) \cup \widetilde{\beta}_v)^\circ \\ &= \big[(dA_{0,v} - i_v^+(a_{1,v}^+)) \cup (dB_{0,v} - i_v^+(b_{1,v}^+))\big]^\circ = \big[d\big(A_{0,v} \cup (dB_{0,v} - i_v^+(b_{1,v}^+))\big)\big]^\circ\end{aligned}$$
is a coboundary, hence $\mathrm{inv}_v(c_v + e_v) = 0$, which implies that
$$[x] \cup_{2,2} [y] = -\langle [\alpha], [\alpha'] \rangle_{\mathrm{Flach}},$$
as claimed. □

CHAPTER 11

R-VALUED HEIGHT PAIRINGS

This chapter is devoted to various generalizations of p-adic height pairings in the context of the Greenberg local conditions. The pairings are introduced and studied in the full generality in Sect. 11.1–11.2. They are compared to the classical p-adic height pairings for Galois representations in Sect. 11.3–11.4. Higher (or "derived") height pairings are studied in Sect. 11.5. An abstract descent formalism for $\mathbf{Z}_p$-extensions is developed in Sect. 11.6; it is applied to formulas of the Birch and Swinnerton-Dyer type in Sect. 11.7. For $\mathbf{Z}_p$-extensions, the higher height pairings can be related to the generalized Cassels-Tate pairing from Chapter 10 (Sect. 11.8). In the self-dual dihedral case one deduces various parity results in Sect. 11.9.

11.1. Definition of the height pairing

11.1.1. Let K be a number field, S a finite set of primes of K containing all primes above p and all archimedean primes, and K_∞/K a subextension of K_S/K with $\mathrm{Gal}(K_\infty/K) = \Gamma = \Gamma_0 \times \Delta$, where $\Gamma_0 \xrightarrow{\sim} \mathbf{Z}_p^r$ ($r \geqslant 0$) and Δ is a finite abelian p-group. We assume that the condition (P) from 5.1 holds (*i.e.*, K has no real primes if $p = 2$) and that

(Fl(Γ)) $\Gamma_R := \Gamma \otimes_{\mathbf{Z}_p} R$ is a flat (hence free) R-module.

The condition (Fl(Γ)) is automatically satisfied if $\Delta = 0$, while in the case $\Delta \neq 0$ it simply says that R is annihilated by the exponent of Δ.

11.1.2. Denote by $J \subset \overline{R}$ the augmentation ideal of $\overline{R} = R[\![\Gamma]\!]$. The first two graded quotients of the J-adic filtration of $\overline{R}$ are isomorphic to $\overline{R}/J \xrightarrow{\sim} R$ and

$$J/J^2 \xrightarrow{\sim} \Gamma_R, \quad \gamma - 1 \pmod{J^2} \longmapsto \gamma \otimes 1 \qquad (\gamma \in \Gamma).$$

It follows from (Fl(Γ)) that $\overline{R}/J, J/J^2$ and $\overline{R}/J^2$ are flat R-modules.

11.1.3. Let X, Y be complexes of admissible $R[G_{K,S}]$-modules of finite type over R, and $\pi : X \otimes_R Y \to \omega_R^\bullet(1)$ a morphism of complexes (where $\omega_R^\bullet = \sigma_{\geqslant 0}\, \omega_R^\bullet$ is a complex of injective R-modules representing ω_R). We assume that X, Y are equipped with Greenberg's local conditions 7.8.2 associated to

$$X_v^+ \longrightarrow X, \quad Y_v^+ \longrightarrow Y \qquad (v \in \Sigma \subset S_f),$$

where X_v^+, Y_v^+ are complexes of admissible $R[G_v]$-modules of finite type over R, satisfying $X_v^+ \perp_\pi Y_v^+$ $(v \in \Sigma)$. We also assume that the condition (U) from 8.8.1 holds, *i.e.*, each prime $v \in \Sigma' = S_f - \Sigma$ is unramified in K_∞/K; this is automatic if $\Delta = 0$.

Fixing a quasi-isomorphism

$$\omega_R^\bullet \otimes_R \overline{R} \longrightarrow \omega_{\overline{R}}^\bullet$$

as in 10.3.3.1, we obtain a morphism of complexes of admissible $\overline{R}[G_{K,S}]$-modules

$$\overline{\pi} : \mathscr{F}_\Gamma(X) \otimes_{\overline{R}} \mathscr{F}_\Gamma(Y)^\iota \xrightarrow{\mathscr{F}_\Gamma(\pi)} \omega_R^\bullet(1) \otimes_R \overline{R} \longrightarrow \omega_{\overline{R}}^\bullet(1).$$

As $\mathscr{F}_\Gamma(Z) = (Z \otimes_R \overline{R}) < -1 >$ for $Z = X, Y$, the sequence of complexes

$$0 \longrightarrow \mathscr{F}_\Gamma(Z) \otimes_{\overline{R}} J/J^2 \longrightarrow \mathscr{F}_\Gamma(Z) \otimes_{\overline{R}} \overline{R}/J^2 \longrightarrow \mathscr{F}_\Gamma(Z) \otimes_{\overline{R}} \overline{R}/J \longrightarrow 0 \qquad (Z = X, Y)$$

in $\left({}^{\mathrm{ad}}_{\overline{R}[G_{K,S}]}\mathrm{Mod}\right)_{R\text{-}ft}$ is exact and isomorphic to

$$0 \longrightarrow Z \otimes_R \Gamma_R \longrightarrow (Z \otimes_R \overline{R}/J^2) < -1 > \longrightarrow Z \longrightarrow 0, \tag{11.1.3.1}$$

using $(\mathrm{Fl}(\Gamma))$ and observing that $\mathrm{Im}(\chi_\Gamma) \in 1 + J$, hence

$$Z < n > \otimes_R J^r/J^{r+1} \xrightarrow{\sim} Z \otimes_R J^r/J^{r+1} \qquad (r \geqslant 0,\ n \in \mathbf{Z}).$$

The first two terms in (11.1.3.1) inherit Greenberg's local conditions from those for Z, as in 8.9.2. The assumptions (U) and $(\mathrm{Fl}(\Gamma))$ imply that

$$0 \longrightarrow U_S^+(Z \otimes_R \Gamma_R) \longrightarrow U_S^+((Z \otimes_R \overline{R}/J^2) < -1 >) \longrightarrow U_S^+(Z) \longrightarrow 0 \qquad (Z = X, Y)$$

is an exact sequence of complexes and

$$U_S^+(Z \otimes_R \Gamma_R) = U_S^+(Z) \otimes_R \Gamma_R \quad (Z = X, Y).$$

It follows that

$$0 \longrightarrow \widetilde{C}_f^\bullet(Z \otimes_R \Gamma_R) \longrightarrow \widetilde{C}_f^\bullet((Z \otimes_R \overline{R}/J^2) < -1 >) \longrightarrow \widetilde{C}_f^\bullet(Z) \longrightarrow 0 \qquad (Z = X, Y)$$

is also an exact sequence of complexes and

$$\widetilde{C}_f^\bullet(Z \otimes_R \Gamma_R) = \widetilde{C}_f^\bullet(Z) \otimes_R \Gamma_R \quad (Z = X, Y).$$

As a result, we obtain (for $Z = X, Y$) exact triangles in $D_{ft}({}_R\mathrm{Mod})$

$$\begin{aligned} \widetilde{\mathbf{R}\Gamma}_f(Z) \otimes_R \Gamma_R \longrightarrow &\widetilde{\mathbf{R}\Gamma}_f((Z \otimes_R \overline{R}/J^2) < -1 >) \\ &\longrightarrow \widetilde{\mathbf{R}\Gamma}_f(Z) \xrightarrow{\beta} \widetilde{\mathbf{R}\Gamma}_f(Z)[1] \otimes_R \Gamma_R. \end{aligned} \tag{11.1.3.2}$$

These triangles can be constructed more directly, by applying the derived tensor product

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, Z)\overset{\mathbf{L}}{\otimes}_{\overline{R}}(-)$$

to the exact triangle

$$J/J^2 \longrightarrow \overline{R}/J^2 \longrightarrow R \longrightarrow J/J^2[1]$$

(using Proposition 8.10.1 and the assumption (Fl(Γ))).

11.1.4. The "height pairing" associated to the data $(X, Y, \pi, X_v^+, Y_v^+)$ is defined as the following morphism in $D_{ft}({}_R\mathrm{Mod})$:

$$\begin{aligned} h = h_\pi : \widetilde{\mathbf{R}\Gamma}_f(X) &\xrightarrow{\beta} \widetilde{\mathbf{R}\Gamma}_f(X)[1] \otimes_R \Gamma_R \\ &\xrightarrow{\gamma_{\pi,0}[1]} \mathbf{R}\mathrm{Hom}_R\big(\widetilde{\mathbf{R}\Gamma}_f(Y), \omega_R[-2]\big) \otimes_R \Gamma_R = \mathscr{D}_{-2}\big(\widetilde{\mathbf{R}\Gamma}_f(Y)\big) \otimes_R \Gamma_R. \end{aligned}$$

This induces maps on cohomology

$$\widetilde{H}^i_f(X) \longrightarrow H^i\big(\mathscr{D}_{-2}(\widetilde{\mathbf{R}\Gamma}_f(Y))\big) \otimes_R \Gamma_R \longrightarrow \mathrm{Hom}_R\big(\widetilde{H}^j_f(Y), H^{i+j-2}(\omega_R) \otimes_R \Gamma_R\big),$$

i.e., bilinear maps

$$\widetilde{h}_{\pi,i,j} : \widetilde{H}^i_f(X) \otimes_R \widetilde{H}^j_f(Y) \longrightarrow H^{i+j-2}(\omega_R) \otimes_R \Gamma_R.$$

The only (?) interesting case is that of $i + j = 2$, when we obtain pairings

$$\widetilde{h}_{\pi,i,j} : \widetilde{H}^i_f(X) \otimes_R \widetilde{H}^j_f(Y) \longrightarrow H^0(\omega_R) \otimes_R \Gamma_R \qquad (i + j = 2),$$

of which $\widetilde{h}_{\pi,1,1}$ is the most important one.

If all complexes X, Y, X_v^+, Y_v^+ (hence also $\widetilde{C}^\bullet_f(X)$, $\widetilde{C}^\bullet_f(Y)$) are cohomologically bounded above, then $h = \mathrm{adj}(\widetilde{h})$, where

$$\begin{aligned} \widetilde{h} = \widetilde{h}_\pi : \widetilde{\mathbf{R}\Gamma}_f(X)\overset{\mathbf{L}}{\otimes}_R\widetilde{\mathbf{R}\Gamma}_f(Y) &\xrightarrow{\beta\otimes\mathrm{id}} \widetilde{\mathbf{R}\Gamma}_f(X)[1] \overset{\mathbf{L}}{\otimes}_R \widetilde{\mathbf{R}\Gamma}_f(Y) \otimes_R \Gamma_R \\ &\xrightarrow{\cup_{\pi,0}[1]} \omega_R[-2] \otimes_R \Gamma_R \end{aligned}$$

is a pairing in $D^-_{ft}({}_R\mathrm{Mod})$.

11.1.5. If $K \subset K'_\infty \subset K_\infty$ is a subextension such that $\Gamma'_R := \mathrm{Gal}(K'_\infty/K) \otimes_{\mathbf{Z}_p} R$ is flat over R, then the various height "pairings" h' associated to K'_∞/K are obtained from h by applying the canonical projection $\Gamma_R \twoheadrightarrow \Gamma'_R$.

11.1.6. The above constructions apply, in particular, in the case when R, X, Y are of the form $R = R_0[\![\Gamma']\!]$, $X = \mathscr{F}_{\Gamma'}(X_0)$, $Y = \mathscr{F}_{\Gamma'}(Y_0)$ for suitable $\Gamma' = \mathrm{Gal}(K'_\infty/K) \xrightarrow{\sim} \mathbf{Z}_p^{r'}$. This is the situation considered in [**PR3**], where K is an imaginary quadratic field and K_∞ (resp., K'_∞) is the cyclotomic (resp., anti-cyclotomic) $\mathbf{Z}_p$-extension of K.

11.1.7. More generally, let L/K be any subextension of K_∞/K. As in 9.7.1, put $\Gamma^L = \mathrm{Gal}(K_\infty/L)$, $\Gamma_L = \mathrm{Gal}(L/K) = \Gamma/\Gamma^L$ and

$$J^L = \mathrm{Ker}(R[\![\Gamma^L]\!] \longrightarrow R), \quad J_L = \mathrm{Ker}(R[\![\Gamma]\!] \longrightarrow R[\![\Gamma_L]\!]).$$

For each $r \geqslant 0$, there is a canonical isomorphism of $R[\![\Gamma_L]\!]$-modules

$$J_L^r/J_L^{r+1} \xrightarrow{\sim} (J^L)^r/(J^L)^{r+1} \otimes_R R[\![\Gamma_L]\!]. \tag{11.1.7.1}$$

In particular, for $r = 1$,

$$J^L/(J^L)^2 \xrightarrow{\sim} \Gamma^L \otimes_{\mathbf{Z}_p} R = \Gamma_R^L \Longrightarrow J_L/J_L^2 \xrightarrow{\sim} \Gamma_R^L \otimes_R R[\![\Gamma_L]\!].$$

If we replace the flatness condition (Fl(Γ)) by

(Fl(Γ^L)) $\Gamma_R^L = \Gamma^L \otimes_{\mathbf{Z}_p} R$ is a flat (hence free) R-module

(which is automatic if $\Gamma \xrightarrow{\sim} \mathbf{Z}_p^r$), then the construction from 11.1.3 works if one replaces everywhere J by J_L (with the proviso that J_L^r/J_L^{r+1} is no longer isomorphic to $J_L^r/J_L^{r+1} < n >$ as a Galois module): using (8.10.10.1), one obtains an exact sequence

$$0 \longrightarrow \mathscr{F}_{\Gamma_L}(Z) \otimes_R \Gamma_R^L \longrightarrow \mathscr{F}_\Gamma(Z)/J_L^2 \longrightarrow \mathscr{F}_{\Gamma_L}(Z) \longrightarrow 0 \tag{11.1.7.2}$$

generalizing (11.1.3.1), an exact triangle

$$\begin{aligned}\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, Z) \otimes_R \Gamma_R^L \longrightarrow \widetilde{\mathbf{R}\Gamma}_f((Z \otimes_R \overline{R})/J_L^2 < -1 >) \longrightarrow \\ \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, Z) \xrightarrow{\beta} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, Z)[1] \otimes_R \Gamma_R^L\end{aligned} \tag{11.1.7.3}$$

generalizing (11.1.3.2) and pairings

$$\begin{aligned}&\widetilde{h}_{\pi,L/K} : \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X) \overset{\mathbf{L}}{\otimes}_{R[\![\Gamma_L]\!]} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, Y)^\iota \\ &\xrightarrow{\beta\otimes\mathrm{id}} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X)[1] \overset{\mathbf{L}}{\otimes}_{R[\![\Gamma_L]\!]} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, Y)^\iota \otimes_R \Gamma_R^L \xrightarrow{\cup_{\pi,0}[1]} \omega_{R[\![\Gamma_L]\!]}[-2] \otimes_R \Gamma_R^L\end{aligned} \tag{11.1.7.4}$$

$$\begin{aligned}\widetilde{h}_{\pi,L/K,i,j} : \widetilde{H}^i_{f,\mathrm{Iw}}(L/K, X) \otimes_{R[\![\Gamma_L]\!]} \widetilde{H}^j_{f,\mathrm{Iw}}(L/K, Y)^\iota \\ \longrightarrow H^{i+j-2}(\omega_R) \otimes_R R[\![\Gamma_L]\!] \otimes_R \Gamma_R^L.\end{aligned} \tag{11.1.7.5}$$

As in 11.1.3, the exact triangle (11.1.7.3) can also be obtained by applying

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K, Z) \overset{\mathbf{L}}{\otimes}_{\overline{R}} (-)$$

to the exact triangle

$$J_L/J_L^2 \longrightarrow \overline{R}/J_L^2 \longrightarrow R[\![\Gamma_L]\!] \longrightarrow J_L/J_L^2[1].$$

11.1.8. What is the relation between (11.1.7.4)–(11.1.7.5) and the pairings constructed in 11.1.4? In the special case when the exact sequence

$$0 \longrightarrow \Gamma^L \longrightarrow \Gamma \longrightarrow \Gamma_L \longrightarrow 0 \tag{11.1.8.1}$$

splits (*i.e.*, when $\Gamma = \Gamma^L \times \Gamma_L$), then the construction from 11.1.7 reduces to that in 11.1.3, applied to

$$\overline{\pi} : \mathscr{F}_{\Gamma_L}(X) \otimes_{R[\![\Gamma_L]\!]} \mathscr{F}_{\Gamma_L}(Y)^\iota \longrightarrow \omega_R^\bullet(1) \otimes_R R[\![\Gamma_L]\!]$$

and $K'_\infty = K_\infty^{\Gamma_L}$ instead of $\pi : X \otimes_R Y \to \omega_R^\bullet(1)$ and K_∞, as $\mathrm{Gal}(K'_\infty/K) = \Gamma^L$; cf. 11.1.6.

If the sequence (11.1.8.1) does not split, then the following Proposition shows that at least the height pairings on cohomology (11.1.7.5) reduce to those from 11.1.4.

11.1.9. Proposition. — *Let $\{L_\alpha/K\}$ be the set of all finite subextensions of L/K. Then the height pairing*

$$\widetilde{h}_{\pi,L/K,i,j} : \Big(\varprojlim_\alpha \widetilde{H}^i_f(L_\alpha, X)\Big) \otimes_{R[\![\Gamma_L]\!]} \Big(\varprojlim_\alpha \widetilde{H}^j_f(L_\alpha, Y)\Big)^\iota \longrightarrow \varprojlim_\alpha \big(H^{i+j-2}(\omega_R) \otimes_R \Gamma^L_R \otimes_R R[\mathrm{Gal}(L_\alpha/K)]\big) \tag{11.1.9.1}$$

is given by the formula

$$\widetilde{h}_{\pi,L/K,i,j}\left((x_\alpha),(y_\alpha)\right) = \left(\sum_{\sigma\in\mathrm{Gal}(L_\alpha/K)} \widetilde{h}_{\pi_\alpha,i,j}(x_\alpha, \sigma y_\alpha) \otimes [\sigma]\right)_\alpha,$$

where

$$\widetilde{h}_{\pi_\alpha,i,j} : \widetilde{H}^i_f(L_\alpha, X) \otimes_R \widetilde{H}^j_f(L_\alpha, Y) \longrightarrow H^{i+j-2}(\omega_R) \otimes_R (\mathrm{Gal}(K_\infty/L_\alpha) \otimes_{\mathbf{Z}_p} R)$$

is the height pairing 11.1.4 associated to the extension K_∞/L_α (if L_α is sufficiently large, then the flatness condition $(\mathrm{Fl}(\mathrm{Gal}(K_\infty/L_\alpha)))$ *holds, thanks to the assumption* $(\mathrm{Fl}(\Gamma^L))$*).*

Proof. — This follows from the definitions and the formula proved in Proposition 8.9.14. □

11.2. Symmetry of the height pairing

11.2.1. Theorem

(i) *If all complexes X, Y, X_v^+, Y_v^+ are cohomologically bounded above, then the diagram*

$$\begin{array}{cccc} \widetilde{h}_\pi : & \widetilde{\mathbf{R}\Gamma}_f(X) \overset{\mathbf{L}}{\otimes}_R \widetilde{\mathbf{R}\Gamma}_f(Y) & \longrightarrow & \omega_R[-2] \otimes_R \Gamma_R \\ & \big\downarrow s_{12} & & \| \\ \widetilde{h}_{\pi\circ s_{12}} : & \widetilde{\mathbf{R}\Gamma}_f(Y) \overset{\mathbf{L}}{\otimes}_R \widetilde{\mathbf{R}\Gamma}_f(X) & \longrightarrow & \omega_R[-2] \otimes_R \Gamma_R \end{array}$$

is commutative in $D^-_{ft}({}_R\mathrm{Mod})$.

(ii) *The pairings* $\widetilde{h}_{\pi,i,j}$ *satisfy*

$$\widetilde{h}_{\pi,i,j}(x,y) = (-1)^{ij}\,\widetilde{h}_{\pi\circ s_{12},j,i}(y,x).$$

11.2.2. Corollary. — *If* $X = Y$, $X_v^+ = Y_v^+$ $(v \in \Sigma)$ *and* $\pi \circ s_{12} = c\cdot\pi$ *with* $c = \pm 1$, *then the bilinear form*

$$\widetilde{h}_{\pi,1,1} : \widetilde{H}^1_f(X)\otimes_R \widetilde{H}^1_f(X) \longrightarrow H^0(\omega_R)\otimes_R \Gamma_R$$

is symmetric (resp., skew-symmetric) if $c = -1$ *(resp.,* $c = +1$*).*

Proof of Theorem 11.2.1. — It is enough to prove (i), since (ii) follows from (i) applied to $\tau_{\leqslant N} Z$,$\tau_{\leqslant N} Z_v^+$ $(Z = X, Y)$ for big enough $N >> 0$. The first step is to observe that the "Bockstein map"

$$\beta : \widetilde{\mathbf{R}\Gamma}_f(Z) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(Z)[1]\otimes_R \Gamma_R \qquad (Z = X, Y)$$

in $D_{ft}({}_R\mathrm{Mod})$ can be represented by a canonical morphism of complexes

$$\beta_E : \widetilde{C}^\bullet_f(Z) \longrightarrow \widetilde{C}^\bullet_f(Z)[1]\otimes_R \Gamma_R.$$

This follows from the construction 1.3.11 applied to the complexes

$$\begin{aligned}
A &= C^\bullet_{\mathrm{cont}}(G_{K,S}, Z) & B &= U^+_S(Z) & C &= \bigoplus_{v\in S_f} C^\bullet_{\mathrm{cont}}(G_v, Z)\\
A' &= C^\bullet_{\mathrm{cont}}(G_{K,S}, Z') & B' &= U^+_S(Z') & C' &= \bigoplus_{v\in S_f} C^\bullet_{\mathrm{cont}}(G_v, Z')\\
A'' &= C^\bullet_{\mathrm{cont}}(G_{K,S}, Z'') & B'' &= U^+_S(Z'') & C'' &= \bigoplus_{v\in S_f} C^\bullet_{\mathrm{cont}}(G_v, Z'')
\end{aligned}$$

for

$$Z' = \mathscr{F}_\Gamma(Z)\otimes_{\overline{R}} \overline{R}/J^2, \quad Z'' = Z\otimes_R \Gamma_R$$

and the morphisms of complexes

$$f, f', f'' = \mathrm{res}_{S_f}, \quad g, g', g'' = i^+_S(-),$$

as

$$E = \widetilde{C}^\bullet_f(Z), \quad E' = \widetilde{C}^\bullet_f(Z'), \quad E'' = \widetilde{C}^\bullet_f(Z'') = \widetilde{C}^\bullet_f(Z)\otimes_R \Gamma_R$$

in the notation of 1.3.11. As a result, the discussion in 1.3.11 yields the data 1.3.8.1–1.3.8.2 for the complexes

$$A_j = C^\bullet_{\mathrm{cont}}(G_{K,S}, Z_j), \quad B_j = U^+_S(Z_j), \quad C_j = \bigoplus_{v\in S_f} C^\bullet_{\mathrm{cont}}(G_v, Z_j)$$

from 6.3.1 (where $j = 1, 2$ and $Z_1 = X$, $Z_2 = Y$). In fact, in the present situation we can make the formulas (1.3.11.1)–(1.3.11.2) very explicit, as follows. The canonical projection

$$\mathrm{pr}_\Gamma : G_{K,S} = \mathrm{Gal}(K_S/K) \twoheadrightarrow \mathrm{Gal}(K_\infty/K) = \Gamma$$

and the map

$$\Gamma \longrightarrow \Gamma_R \quad (\gamma \longmapsto \gamma\otimes 1)$$

define a tautological 1-cocycle

$$z : G_{K,S} \longrightarrow\!\!\!\!\!\!\rightarrow \Gamma \hookrightarrow \Gamma_R, \quad z \in C^1_{\mathrm{cont}}(G_{K,S}, \Gamma_R), \quad dz = 0.$$

For each $v \in S_f$, a fixed embedding $\overline{K} \hookrightarrow \overline{K}_v$ induces primes $v_\infty | v$ of K_∞ and $v_\alpha | v$ of K_α (for all finite subextensions K_α/K of K_∞/K). The localization of z at v, which is equal to

$$\mathrm{res}_v(z) : G_v \hookrightarrow G_{K,S} \xrightarrow{z} \Gamma_R,$$

factors through

$$z_v : G_v \longrightarrow\!\!\!\!\!\!\rightarrow \Gamma_v \longrightarrow (\Gamma_v)_R := \Gamma_v \otimes_{\mathbf{Z}_p} R,$$

where

$$\Gamma_v = \mathrm{Gal}((K_\infty)_{v_\infty}/K_v) = \varprojlim_\alpha \mathrm{Gal}((K_\alpha)_{v_\alpha}/K_v) \subset \Gamma$$

is the decomposition group of v in K_∞/K. □

11.2.3. Lemma. — *The Bockstein map β in the exact triangle*

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G, Z) \otimes_R \Gamma_R \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G, (Z \otimes_R \overline{R}/J^2) < -1 >)$$
$$\longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G, Z) \xrightarrow{\beta} \mathbf{R}\Gamma_{\mathrm{cont}}(G, Z)[1] \otimes_R \Gamma_R \quad (Z = X, Y;\ G = G_{K,S}, G_v)$$

is induced by the morphism of complexes

$$\beta : C^\bullet_{\mathrm{cont}}(G_{K,S}, Z) \longrightarrow C^\bullet_{\mathrm{cont}}(G_{K,S}, Z \otimes_R \Gamma_R)[1]$$
$$c \longmapsto -z \cup c$$

(and similarly for G_v, when z has to be replaced by $\mathrm{res}_v(z)$*).*

Proof. — It is enough to consider the case $G = G_{K,S}$ and $Z = Z^n$ being concentrated in degree $n \in \mathbf{Z}$. For each $i \geqslant 0$, the epimorphism

$$C^{i+n}_{\mathrm{cont}}\big(G, \mathscr{F}_\Gamma(Z) \otimes_{\overline{R}} \overline{R}/J^2\big) = C^i_{\mathrm{cont}}\big(G, (Z^n \otimes_R \overline{R}/J^2) < -1 >\big)$$
$$\longrightarrow C^i_{\mathrm{cont}}\big(G, Z^n\big) = C^{i+n}_{\mathrm{cont}}(G, Z)$$

has a canonical section s given by

$$(s(c))(g_1, \dots, g_i) = c(g_1, \dots, g_i) \otimes 1.$$

A short calculation shows that

$$(-1)^n\big(d(s(c)) - s(d(c))\big)(g_1, \dots, g_{i+1})$$
$$= g_1 c(g_2, \dots, g_{i+1}) \otimes (\mathrm{pr}_\Gamma(g_1)^{-1} - 1) \pmod{J^2}$$
$$= -g_1 c(g_2, \dots, g_{i+1}) \otimes \mathrm{pr}_\Gamma(g_1) = -(-1)^n (z \cup c)(g_1, \dots, g_{i+1}),$$

where the sign $(-1)^n$ on the L.H.S. (resp., the R.H.S.) comes from the sign rule 3.4.1.3 (resp., 3.4.5.2). It follows that

$$(d \circ s - s \circ d)(c) = -z \cup c,$$

which proves the result, by 1.1.4. □

11.2.4. Lemma. — *For each* $v \in \Sigma'$,

(i) *the Bockstein map* β *in the exact triangle*

$$\mathbf{R}\Gamma_{\mathrm{ur}}(G_v, Z) \otimes_R \Gamma_R \longrightarrow \mathbf{R}\Gamma_{\mathrm{ur}}(G_v, (Z \otimes_R \overline{R}/J^2) < -1 >) \longrightarrow \mathbf{R}\Gamma_{\mathrm{ur}}(G_v, Z) \xrightarrow{\beta} \mathbf{R}\Gamma_{\mathrm{ur}}(G_v, Z)[1] \otimes_R \Gamma_R \quad (Z = X, Y)$$

is induced by the morphism of complexes

$$\beta : U_v^+(Z) \longrightarrow U_v^+(Z \otimes_R \Gamma_R)$$
$$\beta(m_n, m_{n-1}, m'_{n-1}, m_{n-2}) = (0, 0, f_v(m_n) \otimes z(f_v), \theta_v(m_{n-1}) \otimes z(f_v)),$$

using the notation of 7.6.2.

(ii) *The morphism* β *is also equal to the restriction of the composite morphism*

$$U_v(Z) \xrightarrow{\mu} C^\bullet_{\mathrm{cont}}(G_v, Z) \xrightarrow{\beta} C^\bullet_{\mathrm{cont}}(G_v, Z \otimes_R \Gamma_R)[1] \xrightarrow{\lambda[1]} U_v(Z)[1] \otimes_R \Gamma_R$$

to the subcomplex $U_v^+(Z) \subset U_v(Z)$, *where* $\beta(c) = -z \cup c$ *is as in Lemma 11.2.3.*

Proof. — The condition (U) implies that, in the notation of 7.6.2, we have

$$U_v^+((Z \otimes_R \overline{R}/J^2) < -1 >) = (U_v^+(Z) \otimes_R \overline{R}/J^2) < -1 >$$

(and similarly for $U_v(-) = U(-)$). The epimorphism

$$(U_v^+(Z) \otimes_R \overline{R}/J^2) < -1 > \longrightarrow U_v^+(Z)$$

has a canonical section s given by

$$s(c) = c \otimes 1$$

(and similarly for $U_v(-)$). The explicit formula for the differential on $U_v(-)$ from 7.6.2 gives, as in the proof of Lemma 11.2.3,

$$\begin{aligned}(d \circ s - s \circ d)(m_n, m_{n-1}, m'_{n-1}, m_{n-2}) = \\ (0, 0, -f_v(m_n) \otimes \mathrm{pr}_\Gamma(f_v)^{-1}, -\theta_v(m_{n-1}) \otimes \mathrm{pr}_\Gamma(f_v)^{-1}) \\ = (0, 0, f_v(m_n) \otimes z(f_v), \theta_v(m_{n-1}) \otimes z(f_v)),\end{aligned}$$

as claimed in (i). A short calculation based on the explicit formulas for λ and μ from Propositions 7.2.3–7.2.4 shows that the composite map $\lambda[1] \circ \beta \circ \mu$ is given by the same formula, proving (ii). ☐

11.2.5. To sum up, the construction from 1.3.11 yields morphisms of complexes

$$\beta_W : W \longrightarrow W[1] \otimes_R \Gamma_R, \quad (W = A, B, C, E),$$

where A, B, C and $E = \widetilde{C}_f^\bullet(Z)$ $(Z = X, Y)$ are as above and

$$\beta_E(a, b, c) = (\beta_A(a), \beta_B(b), -\beta_C(c) - u(a) + v(b)),$$

with

$$\begin{aligned}
\beta_A(a) &= -z \cup a, && a \in C^i_{\mathrm{cont}}(G_{K,S}, Z)\\
\beta_C(c)_v &= -\mathrm{res}_v(z) \cup c_v, && c_v \in C^{i-1}_{\mathrm{cont}}(G_v, Z),\ v \in S_f\\
\beta_B(b)_v &= \begin{cases} -\mathrm{res}_v(z) \cup b_v, & v \in \Sigma,\ b_v \in C^i_{\mathrm{cont}}(G_v, Z_v^+)\\ -\lambda(\mathrm{res}_v(z) \cup \mu(b_v)), & v \in \Sigma',\ b_v \in C^i_{\mathrm{ur}}(G_v, Z).\end{cases}
\end{aligned}$$

As the section s in the proof of Lemma 11.2.3 is functorial in G, we have

$$u = 0.$$

For the same reason, the Σ-components of $v(b)$ vanish, while the Σ'-components can be described explicitly using the recipe from 1.3.11. We obtain, therefore, morphisms $\beta_{j,W}$ for $W = A, B, C$ and homotopies $u_j = 0, v_j$ (where $j = 1$ resp., $j = 2$ for $Z = X$ resp., $Z = Y$) as in 1.3.8.1–1.3.8.2.

11.2.6. The next step is to verify the conditions 1.3.8.3–1.3.8.5, with the products $\cup_A, \cup_B = 0$ and $\cup_C$ as in 6.3.1.

In 1.3.8.3 we can take $h_B = 0$, while h_A and h_C are constructed using the following diagram, in which $C^\bullet(-)$ stands for $C^\bullet_{\mathrm{cont}}(G, -)$ with $G = G_{K,S}$ or G_v:

(11.2.6.1)

$$\begin{array}{ccc}
C^\bullet(X) \otimes_R C^\bullet(Y) & \xrightarrow{\mathrm{id}} & C^\bullet(X) \otimes_R C^\bullet(Y)\\
\| & & \|\\
C^\bullet(X) \otimes_R R \otimes_R C^\bullet(Y) & \xrightarrow{\mathrm{id}} & R \otimes_R C^\bullet(X) \otimes_R C^\bullet(Y)\\
\big\downarrow{\scriptstyle \mathrm{id}\otimes(-z)\otimes\mathrm{id}} & & \big\downarrow{\scriptstyle (-z)\otimes\mathrm{id}\otimes\mathrm{id}}\\
C^\bullet(X) \otimes_R (C^\bullet(\Gamma_R)[1]) \otimes_R C^\bullet(Y) & \xrightarrow{s_{12}\otimes\mathrm{id}} & C^\bullet(\Gamma_R)[1] \otimes_R C^\bullet(X) \otimes_R C^\bullet(Y)\\
\big\downarrow{\scriptstyle \cup\otimes\mathrm{id}} & & \big\downarrow{\scriptstyle \cup\otimes\mathrm{id}}\\
C^\bullet(X \otimes_R \Gamma_R)[1] \otimes_R C^\bullet(Y) & \xrightarrow{\mathrm{id}} & C^\bullet(X \otimes_R \Gamma_R)[1] \otimes_R C^\bullet(Y)\\
\big\downarrow{\scriptstyle \cup} & & \big\downarrow{\scriptstyle \cup}\\
C^\bullet(X \otimes_R Y \otimes_R \Gamma_R)[1] & \xrightarrow{\mathrm{id}} & C^\bullet(X \otimes_R Y \otimes_R \Gamma_R)[1]
\end{array}$$

The composite map

$$C^\bullet(X) \otimes_R C^\bullet(Y) \longrightarrow C^\bullet(X \otimes_R Y \otimes_R \Gamma_R)[1]$$

in the left (resp., right) column is equal to $\mathrm{id}\otimes\beta_2$ (resp., $\beta_1\otimes\mathrm{id}$). The first, second and the fourth squares from the top are all commutative, while the commutative diagram

$$\begin{array}{ccc}
C^\bullet(X) \otimes_R (C^\bullet(\Gamma_R)[1]) & \xrightarrow{s_{12}\circ(\mathcal{T}\otimes\mathcal{T})} & C^\bullet(\Gamma_R)[1] \otimes_R C^\bullet(X)\\
\big\downarrow{\scriptstyle \cup} & & \big\downarrow{\scriptstyle \cup}\\
C^\bullet(X \otimes_R \Gamma_R)[1] & \xleftarrow{\mathcal{T}} & C^\bullet(X \otimes_R \Gamma_R)[1]
\end{array}$$

together with the bifunctorial homotopy $a : \mathrm{id} \rightsquigarrow \mathcal{T}$ from 3.4.5.5 show that the third square from the top in (11.2.6.1) is commutative up to the homotopy

$$k \otimes \mathrm{id} : \cup \otimes \mathrm{id} \rightsquigarrow (\cup \otimes \mathrm{id}) \circ (s_{12} \otimes \mathrm{id}),$$

where

$$k = -a \star (\cup \circ s_{12} \circ (\mathcal{T} \otimes \mathcal{T})) - (\mathcal{T} \circ \cup \circ s_{12}) \star (a \otimes a)_1.$$

We define the homotopy

$$h_W : \mathrm{id} \otimes \beta_{2,W} \rightsquigarrow \beta_{1,W} \otimes \mathrm{id} \qquad (W = A, C)$$

by the formula

$$h_W = \cup \star (k \otimes \mathrm{id}) \star (\mathrm{id} \otimes (-z) \otimes \mathrm{id}),$$

with $G = G_{K,S}$ (resp., $G = G_v$, $v \in S_f$) if $W = A$ (resp., $W = C$).

As

$$h_f = u_1 = u_2 = 0$$

and

$$f_3[1] \star h_A = h_C \star (f_1 \otimes f_2) \tag{11.2.6.2}$$

(the last equality following from the bifunctoriality of the homotopy $a : \mathrm{id} \rightsquigarrow \mathcal{T}$ used in the construction of h_A and h_C), the condition 1.3.8.4 is satisfied with $H_f = 0$.

It remains to verify the condition 1.3.8.5, which boils down to the statement that, for each $v \in S_f$, the v-component m_v of the homotopy

$$m = \cup_C[1] \star (v_1 \otimes g_2) - \cup_C[1] \star (g_1 \otimes v_2) - h_C \star (g_1 \otimes g_2)$$
$$m_v^i : \left(U_v^+(X) \otimes_R U_v^+(Y)\right)^i \longrightarrow \left(\tau_{\geqslant 2}^{\mathrm{II}} C_{\mathrm{cont}}^\bullet(G_v, \omega_R^\bullet(1)) \otimes_R \Gamma_R[1]\right)^{i-1}$$

is 2-homotopic to zero. For $v \in \Sigma$ we have

$$m_v = -h_C \star (g_1 \otimes g_2) = -h_C \star (g_1 \otimes g_2) + g_3[1] \star h_B = 0,$$

as in (11.2.6.2). For $v \in \Sigma'$, the only possibly non-zero component of m_v is

$$m_v^2 : H^1_{\mathrm{ur}}(X) \otimes_R H^1_{\mathrm{ur}}(Y) \longrightarrow H^0(\omega_R) \otimes_R \Gamma_R.$$

It is enough to show that $m_v^2 = 0$ (hence 1.3.8.5 holds with $H_g = 0$). Firstly, m_v^2 factors through

$$H^1_{\mathrm{ur}}(X) \otimes_R H^1_{\mathrm{ur}}(\mathscr{D}(X)(1)), \tag{11.2.6.3}$$

so we can assume that $Y = \mathscr{D}(X)(1)$ and $\pi = \mathrm{ev}_2$. The R-module (11.2.6.3) is annihilated by $Nv - 1$, which proves that $m_v^2 = 0$ in the case when R is flat over $\mathbf{Z}_p$. The general case follows by observing that the morphism

$$\mathrm{adj}(m_v^2) : H^1_{\mathrm{ur}}(X) \longrightarrow \mathrm{Hom}_R(H^1_{\mathrm{ur}}(\mathscr{D}(X)(1)), H^0(\omega_R) \otimes_R \Gamma_R)$$

is given by a "universal" formula which does not depend on R, as all homotopies h_C, v_1, v_2 are given by such universal formulas.

This completes the verification of the conditions 1.3.8.3–1.3.8.5.

11.2.7. We are now ready to conclude the proof of Theorem 11.2.1 (i). We apply Proposition 1.3.10 to $E_1 = \widetilde{C}^\bullet_f(X)$, $E_2 = \widetilde{C}^\bullet_f(Y)$ and $E_3 = \omega^\bullet_R[-3]$ (as in 6.3.1). This is legitimate, since Greenberg's local conditions admit the transposition data 1.3.5.1–1.3.5.7, by 6.7.8 and Proposition 7.7.3. The morphisms $\mathcal{T}_j$ for $j = 1, 2$ (resp., for $j = 3$) are homotopic to the identity by Proposition 6.5.4 (i) (resp., because $\omega^\bullet_R$ is a complex of injective R-modules). Finally, we use our earlier observation that the Bockstein map

$$\beta : \widetilde{\mathbf{R}\Gamma}_f(Z) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(Z)[1] \otimes_R \Gamma_R \qquad (Z = X, Y)$$

is represented by the morphism of complexes

$$\beta_E : \widetilde{C}^\bullet_f(Z) \longrightarrow \widetilde{C}^\bullet_f(Z)[1] \otimes_R \Gamma_R$$

given by the formula from Proposition 1.3.9 (i). The statement of Theorem 11.2.1 (i) then follows from Proposition 1.3.10.

11.2.8. In the definition of the height pairings in 11.1.4 we let the Bockstein map β act on $\widetilde{\mathbf{R}\Gamma}_f(X)$ and $\widetilde{H}^i_f(X)$. If we chose instead the opposite convention and define

$$\widetilde{h}'_\pi : \widetilde{\mathbf{R}\Gamma}_f(X) \overset{\mathbf{L}}{\otimes}_R \widetilde{\mathbf{R}\Gamma}_f(Y) \xrightarrow{\mathrm{id}\otimes\beta} \widetilde{\mathbf{R}\Gamma}_f(X) \overset{\mathbf{L}}{\otimes}_R \big(\widetilde{\mathbf{R}\Gamma}_f(Y)[1]\big) \otimes_R \Gamma_R \xrightarrow{\cup_{\pi,0}[1]} \omega_R[-2] \otimes_R \Gamma_R,$$

then the proof of Theorem 11.2.1 (namely the appeal to Proposition 1.3.9) shows that $\widetilde{h}'_\pi = \widetilde{h}_\pi$ (and $\widetilde{h}'_{\pi,i,j} = \widetilde{h}_{\pi,i,j}$).

11.2.9. On the other hand, if we replace $\mathscr{F}_\Gamma(X)/J^2 = (X \otimes_R \overline{R}/J^2) < -1 >$ in 11.1.3 by $\mathscr{F}_\Gamma(X)^\iota/J^2 = (X \otimes_R \overline{R}/J^2) < 1 >$, then the map β is replaced by $-\beta$ and the height pairings $\widetilde{h}$, $\widetilde{h}_{\pi,i,j}$ change sign.

11.3. Comparison with classical p-adic height pairings

In contrast to various classical constructions of $\mathbf{Q}_p$-valued height pairings on Selmer groups ([**Ne2, PR4, PR5**]), the definition in 11.1 requires no additional assumptions apart from the existence of orthogonal Greenberg's local conditions $X^+_v \perp_\pi Y^+_v$ ($v \in \Sigma$). In this (and the following) section we relate the pairing $\widetilde{h}_{\pi,1,1}$ for $R = \mathbf{Z}_p$ to the height pairing constructed in terms of universal norms.

11.3.1. Assume that $\Gamma = \mathrm{Gal}(K_\infty/K) \xrightarrow{\sim} \mathbf{Z}_p^r$ ($r \geqslant 1$) and that we are given the following data:

11.3.1.1. Two $\mathbf{Q}_p$-representations X, Y of $G_{K,S}$ (*i.e.*, finite-dimensional vector spaces over $\mathbf{Q}_p$ with a continuous $\mathbf{Q}_p$-linear action of $G_{K,S}$).

11.3.1.2. A $G_{K,S}$-map

$$\pi : X \otimes_{\mathbf{Q}_p} Y \longrightarrow \mathbf{Q}_p(1)$$

such that

$$\mathrm{adj}(\pi) : X \longrightarrow \mathrm{Hom}_{\mathbf{Q}_p}(Y, \mathbf{Q}_p(1)) = Y^*(1)$$

is an isomorphism.

11.3.1.3. For each $v \in S_f$, $\mathbf{Q}_p$-subrepresentations of G_v

$$X_v^+ \subset X, \quad Y_v^+ \subset Y$$

such that

$$(\forall v|p) \quad X_v^+ \perp\!\!\perp_\pi Y_v^+, \quad (\forall v \nmid p) \quad X_v^+ = Y_v^+ = 0.$$

Putting

$$X_v^- = X/X_v^+, \quad Y_v^- = Y/Y_v^+ \qquad (v \in S_f),$$

it follows that $\mathrm{adj}(\pi)$ induces isomorphisms

$$X_v^\pm \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Q}_p}(Y_v^\mp, \mathbf{Q}_p(1)) = (Y_v^\mp)^*(1) \qquad (v|p).$$

We also assume that the following conditions hold:

11.3.1.4. $(\forall v \in S_f) \quad H^0(G_v, X) = H^0(G_v, Y) = 0.$

11.3.1.5. $(\forall v|p) \quad H^0(G_v, X_v^-) = H^0(G_v, Y_v^-) = 0.$

Observe that 11.3.1.1–11.3.1.5 hold in the context of [**Ne2**, §6.9] (where X, Y were denoted by $V, V^*(1)$).

11.3.2. There are two natural Greenberg's local conditions for $Z = X, Y$: $\Delta(Z)$ with $\Sigma = S_f$ (resp., $\Delta'(Z)$ with $\Sigma = \{v|p\}$) and the subrepresentations Z_v^+ $(v \in \Sigma)$ as in 11.3.1.3.

The assumptions 11.3.1.2–11.3.1.4 imply that

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G_v, Z) = \mathbf{R}\Gamma_{\mathrm{ur}}(G_v, Z) \xrightarrow{\sim} 0 \qquad (v \in S_f, v \nmid p;\ Z = X, Y)$$

in $D({}_{\mathbf{Q}_p}\mathrm{Mod})$; it follows that the canonical maps

$$\widetilde{\mathbf{R}\Gamma}_f(Z) := \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Z; \Delta(Z)) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Z; \Delta'(Z)) \qquad (Z = X, Y)$$

are isomorphisms in $D^b_{ft}({}_{\mathbf{Q}_p}\mathrm{Mod})$. The condition 11.3.1.5 implies that, under the canonical map

$$\widetilde{\mathbf{R}\Gamma}_f(Z) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, Z),$$

the cohomology group

$$\widetilde{H}^1_f(Z) := \widetilde{H}^1_f(G_{K,S}, Z; \Delta(Z))$$

is identified with the Selmer group defined by Greenberg [**Gre2, Gre3**]

$$\begin{aligned}\mathrm{Sel}(Z) &= \mathrm{Ker}\Big(H^1_{\mathrm{cont}}(G_{K,S}, Z) \longrightarrow \bigoplus_{v\in S_f} H^1_{\mathrm{cont}}(G_v, Z_v^-)\Big)\\ &= \mathrm{Ker}\Big(H^1_{\mathrm{cont}}(G_{K,S}, Z) \longrightarrow \bigoplus_{v\mid p} H^1_{\mathrm{cont}}(G_v, Z_v^-)\Big).\end{aligned} \qquad (Z = X, Y)$$

11.3.3. Under the assumptions 11.3.1.1–11.3.1.5 there is a canonical (up to a sign) height pairing ([**Ne2, PR4, PR5**])

$$h^{\mathrm{norm}} = h^{\mathrm{norm}}_{\pi} : \mathrm{Sel}(X) \otimes_{\mathbf{Q}_p} \mathrm{Sel}(Y) \longrightarrow \Gamma \otimes_{\mathbf{Z}_p} \mathbf{Q}_p,$$

the definition of which is recalled in 11.3.7 below.

Fix a $G_{K,S}$-invariant $\mathbf{Z}_p$-lattice $T(X) \subset X$. This determines a $G_{K,S}$-invariant $\mathbf{Z}_p$-lattice $T(Y) \subset Y$ such that $\mathrm{adj}(\pi)$ defines an isomorphism

$$\mathrm{adj}(\pi) : T(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}_p}(T(Y), \mathbf{Z}_p(1)) = T(Y)^*(1),$$

i.e.,

$$\pi : T(X) \otimes_{\mathbf{Z}_p} T(Y) \longrightarrow \mathbf{Z}_p(1)$$

is a perfect duality over $R = \mathbf{Z}_p$. For each $v \in S_f$, put

$$T(Z)_v^+ = T(Z) \cap Z_v^+, \quad T(Z)_v^- = T(Z)/T(Z)_v^+ \qquad (Z = X, Y);$$

then $\mathrm{adj}(\pi)$ induces isomorphisms

$$T(X)_v^{\pm} \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}_p}(T(Y)_v^{\mp}, \mathbf{Z}_p(1)) = (T(Y)_v^{\mp})^*(1) \quad (v \mid p).$$

Let $\Delta(T(Z))$ $(Z = X, Y)$ be Greenberg's local conditions with $\Sigma = S_f$ and the above $T(Z)_v^+$ (*i.e.*, $T(Z)_v^+ = 0$ if $v \nmid p$). Then the Selmer complexes

$$\widetilde{\mathbf{R}\Gamma}_f(T(Z)) := \widetilde{\mathbf{R}\Gamma}_f\big(G_{K,S}, T(Z); \Delta(T(Z))\big) \in D^b_{ft}({}_{\mathbf{Z}_p}\mathrm{Mod}) \qquad (Z = X, Y)$$

satisfy

$$\widetilde{\mathbf{R}\Gamma}_f(T(Z)) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_f(Z) \quad (Z = X, Y),$$

hence the pairing from 11.1.4

$$\widetilde{h}_{\pi,1,1} : \widetilde{H}^1_f(T(X)) \otimes_{\mathbf{Z}_p} \widetilde{H}^1_f(T(Y)) \longrightarrow \Gamma$$

induces a pairing

$$\widetilde{h}_{\pi,1,1} : \mathrm{Sel}(X) \otimes_{\mathbf{Q}_p} \mathrm{Sel}(Y) = \widetilde{H}^1_f(X) \otimes_{\mathbf{Q}_p} \widetilde{H}^1_f(Y) \longrightarrow \Gamma \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

The main goal of Sect. 11.3 is to show that

$$\widetilde{h}_{\pi,1,1} = -h^{\mathrm{norm}}_{\pi}.$$

In order to simplify the arguments we work exclusively with the $\mathbf{Q}_p$-representations Z, Z_v^+, not with the lattices $T(Z), T(Z)_v^+$ $(Z = X, Y)$. This means that we apply the constructions in 11.1.3–11.1.4 to the exact sequences

$$0 \longrightarrow \widetilde{C}^{\bullet}_f(Z \otimes_{\mathbf{Z}_p} \Gamma) \longrightarrow \widetilde{C}^{\bullet}_f(\mathscr{F}_{\Gamma}(Z) \otimes_{\Lambda} \Lambda/J^2) \longrightarrow \widetilde{C}^{\bullet}_f(Z) \longrightarrow 0,$$

obtained from

$$0 \longrightarrow \widetilde{C}^{\bullet}_{f}(T(Z) \otimes_{\mathbf{Z}_p} \Gamma) \longrightarrow \widetilde{C}^{\bullet}_{f}(\mathscr{F}_{\Gamma}(T(Z)) \otimes_{\Lambda} \Lambda/J^2) \longrightarrow \widetilde{C}^{\bullet}_{f}(T(Z)) \longrightarrow 0$$

by taking the tensor product $- \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Above, $\Lambda = \mathbf{Z}_p[\![\Gamma]\!]$ and

$$\mathscr{F}_{\Gamma}(Z) := \mathscr{F}_{\Gamma}(T(Z)) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

11.3.4. Universal norms. — Let v be a non-archimedean prime of K, V a $\mathbf{Q}_p$-representation of G_v and $T \subset V$ a G_v-stable $\mathbf{Z}_p$-lattice. Fixing an embedding $\overline{K} \hookrightarrow \overline{K}_v$, we obtain a prime $v_\alpha | v$ in each finite subextension K_α/K of K_∞/K. We define the *universal norms* with respect to K_∞/K at v as

$$\begin{aligned} N_\infty H^1_{\mathrm{cont}}(G_v, T) &= \bigcap_{\alpha} \mathrm{Im}\left(H^1_{\mathrm{cont}}(G_{v_\alpha}, T) \xrightarrow{\mathrm{cor}} H^1_{\mathrm{cont}}(G_v, T)\right) \\ &= \mathrm{Im}\left(H^1_{\mathrm{cont}}(G_v, \mathscr{F}_{\Gamma_v}(T)) \longrightarrow H^1_{\mathrm{cont}}(G_v, T)\right) \\ &= \mathrm{Im}\left(H^1_{\mathrm{cont}}(G_v, \mathscr{F}_{\Gamma}(T)) \longrightarrow H^1_{\mathrm{cont}}(G_v, T)\right). \end{aligned}$$

We also define

$$N_\infty H^1_{\mathrm{cont}}(G_v, V) = (N_\infty H^1_{\mathrm{cont}}(G_v, T)) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \subset H^1_{\mathrm{cont}}(G_v, V)$$

(this vector space over $\mathbf{Q}_p$ depends only on V, not on T).

11.3.5. Local reciprocity maps. — Let v be a non-archimedean prime of K. The local reciprocity map

$$\mathrm{rec}_v : K_v^* \longrightarrow G_v^{ab}$$

is related to the invariant map

$$\mathrm{inv}_v : H^2(G_v, \mathbf{Q}/\mathbf{Z}(1)) = \mathrm{Br}(K_v) \xrightarrow{\sim} \mathbf{Q}/\mathbf{Z}$$

by the formula

$$\chi(\mathrm{rec}_v(a)) = \mathrm{inv}_v(a \cup \delta\chi), \tag{11.3.5.1}$$

where

$$a \in K_v^*, \quad \chi \in H^1(G_v, \mathbf{Q}/\mathbf{Z}) = \mathrm{Hom}_{\mathrm{cont}}(G_v^{ab}, \mathbf{Q}/\mathbf{Z})$$

and $\delta\chi \in H^2(G_v, \mathbf{Z})$ is the coboundary associated to the exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Q} \longrightarrow \mathbf{Q}/\mathbf{Z} \longrightarrow 0$$

([**Se1**, Prop. XI.2]). If we work with cohomology with finite coefficients, *i.e.*, with the following exact sequences of (discrete) G_v-modules

$$0 \longrightarrow \mu_n \longrightarrow \overline{K}_v^* \xrightarrow{n} \overline{K}_v^* \longrightarrow 0$$

$$0 \longrightarrow \mathbf{Z} \xrightarrow{n} \mathbf{Z} \longrightarrow \mathbf{Z}/n\mathbf{Z} \longrightarrow 0$$

and elements

$$a \in K_v^* = H^0(G_v, \overline{K}_v^*), \qquad \delta a \in H^1(G_v, \mu_n)$$

$$\chi \in H^1(G_v, \mathbf{Z}/n\mathbf{Z}) = \mathrm{Hom}_{\mathrm{cont}}(G_v^{ab}, \mathbf{Z}/n\mathbf{Z}), \qquad \delta\chi \in H^2(G_v, \mathbf{Z}),$$

we obtain from (11.3.5.1) and the vanishing of

$$0 = \mathrm{inv}_v(\delta(a \cup \chi)) = \mathrm{inv}_v(\delta a \cup \chi + a \cup \delta\chi)$$

another useful formula

$$\chi(\mathrm{rec}_v(a)) = -\mathrm{inv}_v(\delta a \cup \chi) = \mathrm{inv}_v(\chi \cup \delta a). \tag{11.3.5.2}$$

Taking $n = p^r$ and passing to the projective limit, we obtain isomorphisms

$$\mathrm{rec}_v : K_v^* \widehat{\otimes} \mathbf{Z}_p \ (= \varprojlim_r K_v^* \otimes \mathbf{Z}/p^r\mathbf{Z}) \xrightarrow{\ \sim\ } G_v^{ab} \widehat{\otimes} \mathbf{Z}_p$$

$$\delta : K_v^* \widehat{\otimes} \mathbf{Z}_p \xrightarrow{\ \sim\ } H^1_{\mathrm{cont}}(G_v, \mathbf{Z}_p(1)).$$

Recall from 11.2 the tautological projection

$$z_v : G_v^{ab} \widehat{\otimes} \mathbf{Z}_p \twoheadrightarrow \Gamma_v;$$

denote by ℓ_v the composite map

$$\ell_v = z_v \circ \mathrm{rec}_v \circ \delta^{-1} : H^1_{\mathrm{cont}}(G_v, \mathbf{Z}_p(1)) \twoheadrightarrow \Gamma_v.$$

The formula (11.3.5.2) implies that

$$\ell_v(c) = \mathrm{inv}_v(z_v \cup c), \quad \forall c \in H^1_{\mathrm{cont}}(G_v, \mathbf{Z}_p(1)). \tag{11.3.5.3}$$

As

$$\mathrm{Ker}(\ell_v) = N_\infty H^1_{\mathrm{cont}}(G_v, \mathbf{Z}_p(1)),$$

the map ℓ_v induces an isomorphism

$$\ell_v : H^1_{\mathrm{cont}}(G_v, \mathbf{Z}_p(1))/N_\infty \xrightarrow{\ \sim\ } \Gamma_v.$$

11.3.6. Lemma. — *For each $v \mid p$,*

$$H^1_{\mathrm{cont}}(G_v, Z_v^+) = N_\infty H^1_{\mathrm{cont}}(G_v, Z_v^+) \qquad (Z = X, Y).$$

Proof. — The statement is trivial if $\Gamma_v = 0$, so we can assume that $\Gamma_v \neq 0$. In this case $H^0(G_v, (Z_v^+)^*(1)) = 0$ by 11.3.1.5, and we conclude by Corollary 8.11.8. □

11.3.7. Definition of h^{norm}. — Let

$$x \in C^1_{\mathrm{cont}}(G_{K,S}, X), \quad dx = 0, \quad y \in C^1_{\mathrm{cont}}(G_{K,S}, Y), \quad dy = 0$$

be 1-cocycles with cohomology classes contained in

$$[x] \in \mathrm{Sel}(X) \subset H^1_{\mathrm{cont}}(G_{K,S}, X), \quad [y] \in \mathrm{Sel}(Y) \subset H^1_{\mathrm{cont}}(G_{K,S}, Y).$$

Then y defines an extension of $G_{K,S}$-modules

$$0 \longrightarrow Y \xrightarrow{\sigma^*(1)} Y_y \longrightarrow \mathbf{Q}_p \longrightarrow 0$$

with extension class $[Y_y] = [y]$. Dualizing, we obtain an extension

$$0 \longrightarrow \mathbf{Q}_p(1) \longrightarrow Y_y^*(1) \xrightarrow{\sigma} X \longrightarrow 0.$$

For $v \in S_f$, put

$$Y_y^*(1)_v^+ = \sigma^{-1}(X_v^+), \quad Y_y^*(1)_v^- = Y_y^*(1)/Y_y^*(1)_v^+.$$

The coboundary map δ in the exact cohomology sequence

$$0 \longrightarrow H^1_{\mathrm{cont}}(G_{K,S}, \mathbf{Q}_p(1)) \longrightarrow H^1_{\mathrm{cont}}(G_{K,S}, Y_y^*(1)) \xrightarrow{\sigma_*} H^1_{\mathrm{cont}}(G_{K,S}, X) \xrightarrow{\delta} H^2_{\mathrm{cont}}(G_{K,S}, \mathbf{Q}_p(1))$$

is equal, up to a sign, to the cup product with $[y]$ under $\pi : X \otimes_{\mathbf{Q}_p} Y \to \mathbf{Q}_p(1)$. As the map

$$\mathrm{res}_{S_f} : H^2_{\mathrm{cont}}(G_{K,S}, \mathbf{Q}_p(1)) \longrightarrow \bigoplus_{v \in S_f} H^2_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1))$$

is injective and $X_v^+ \perp_\pi Y_v^+$ for all $v \in S_f$, we have

$$\delta([x]) = 0.$$

Fix a lift of $[x]$

$$[\widehat{x}] \in \sigma_*^{-1}([x]) \subset H^1_{\mathrm{cont}}(G_{K,S}, Y_y^*(1));$$

it is determined up to an element of $H^1_{\mathrm{cont}}(G_{K,S}, \mathbf{Q}_p(1)) = \mathcal{O}_{K,S}^* \otimes_{\mathbf{Z}} \mathbf{Q}_p$.

For each $v \in S_f$, denote by

$$[\widehat{x}_v] = \mathrm{res}_v([\widehat{x}]) \in H^1_{\mathrm{cont}}(G_v, Y_y^*(1))$$

the localization of $[\widehat{x}]$ at v. The assumptions 11.3.1.3–11.3.1.5 imply that the following diagram has exact rows and columns:

(11.3.7.1)

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \longrightarrow H^1_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1)) & \longrightarrow & H^1_{\mathrm{cont}}(G_v, Y_y^*(1)_v^+) & \longrightarrow & H^1_{\mathrm{cont}}(G_v, X_v^+) & \longrightarrow & H^2_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1)) \\
\| & & \downarrow{\scriptstyle i_v^+} & & \downarrow{\scriptstyle i_v^+} & & \| \\
0 \longrightarrow H^1_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1)) & \longrightarrow & H^1_{\mathrm{cont}}(G_v, Y_y^*(1)) & \longrightarrow & H^1_{\mathrm{cont}}(G_v, X) & \xrightarrow{\delta_v} & H^2_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1)).
\end{array}$$

As δ_v is (again up to a sign) given by the cup product with $[y_v] = \mathrm{res}_v([y])$ (hence $\delta_v([x_v]) = 0$) and $[x_v] = i_v^+([x_v^+])$ for some $[x_v^+] \in H^1_{\mathrm{cont}}(G_v, X_v^+)$ by assumption, an easy diagram chase shows that there is a (unique) cohomology class

$$[\widehat{x}_v^+] \in H^1_{\mathrm{cont}}(G_v, Y_y^*(1)_v^+)$$

satisfying

$$i_v^+([\widehat{x}_v^+]) = [\widehat{x}_v].$$

We claim that the following diagram also has exact rows and columns:

$$(11.3.7.2)\qquad \begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
H^1_{\mathrm{cont}}(G_v, \mathscr{F}_{\Gamma_v}(\mathbf{Q}_p(1))) & \xrightarrow{\mathrm{pr}_*} & H^1_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1)) & \xrightarrow{\ell_v} & \Gamma_v \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \\
\downarrow & & \downarrow & & \\
H^1_{\mathrm{cont}}(G_v, \mathscr{F}_{\Gamma_v}(Y_y^*(1)_v^+)) & \xrightarrow{\mathrm{pr}_*} & H^1_{\mathrm{cont}}(G_v, Y_y^*(1)_v^+) & & \\
\downarrow{\scriptstyle \mathscr{F}_{\Gamma_v}(\sigma)_*} & & \downarrow{\scriptstyle \sigma_*} & & \\
H^1_{\mathrm{cont}}(G_v, \mathscr{F}_{\Gamma_v}(X_v^+)) & \xrightarrow{\mathrm{pr}_*} & H^1_{\mathrm{cont}}(G_v, X_v^+) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \delta_v} & & \\
H^2_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1)) & \xrightarrow{\mathrm{id}} & H^2_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1)) & & \\
\| & & \| & & \\
\mathbf{Q}_p & \xrightarrow{\mathrm{id}} & \mathbf{Q}_p & &
\end{array}$$

This follows from the fact that $\mathrm{Im}(\mathrm{pr}_*) = N_\infty$ and the description of the universal norms N_∞ for $\mathbf{Q}_p(1)$ (resp., for X_v^+) given in 11.3.5 (resp., in Lemma 11.3.6). This implies that there is a cohomology class

$$[\mathscr{F}_{\Gamma_v}(x_v^+)] \in H^1_{\mathrm{cont}}(G_v, \mathscr{F}_{\Gamma_v}(Y_y^*(1)_v^+))$$

which maps to

$$[x_v^+] = \sigma_*([\widehat{x}_v^+]) \in H^1_{\mathrm{cont}}(G_v, X_v^+)$$

under the map $\mathrm{pr}_* \circ \mathscr{F}_{\Gamma_v}(\sigma)_*$. The difference

$$u_v = [\widehat{x}_v^+] - \mathrm{pr}_*([\mathscr{F}_{\Gamma_v}(x_v^+)])$$

is an element of

$$u_v \in H^1_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1))$$

and the value

$$\ell_v(u_v) \in \Gamma_v \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

does not depend on the choice of $[\mathscr{F}_{\Gamma_v}(x_v^+)]$.

If $v \nmid p$, then $X_v^+ = 0$, $Y_y^*(1)_v^+ = \mathbf{Q}_p(1)$ and we have

$$u_v = [\widehat{x}_v^+] \in H^1_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1)) = K_v^* \widehat{\otimes} \mathbf{Q}_p;$$

the latter group is isomorphic to $\mathbf{Q}_p$ under the valuation map

$$v \otimes 1 : K_v^* \widehat{\otimes} \mathbf{Q}_p \xrightarrow{\sim} \mathbf{Q}_p.$$

The height pairing h_π^{norm} is defined by the formula

$$h_\pi^{\mathrm{norm}}([x],[y]) = \sum_{v\in S_f} \ell_v(u_v) \in \Gamma \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

This is well-defined, since another choice of $[\widehat{x}]$ replaces each u_v by $u_v + u$ for some $u \in H^1_{\mathrm{cont}}(G_{K,S}, \mathbf{Q}_p(1))$ and

$$\sum_{v\in S_f} \ell_v(u) = 0$$

for such a global cohomology class $u \in \mathcal{O}^*_{K,S} \otimes_{\mathbf{Z}} \mathbf{Q}_p$.

11.3.8. If $v \in S_f$ is unramified in K_∞/K, then the map

$$z_v \circ \mathrm{rec}_v : K_v^* \longrightarrow \Gamma_v$$

factors through the valuation $v : K_v^* \to \mathbf{Z}$. As the direct sum of the valuations

$$\mathcal{O}^*_{K,S} \otimes_{\mathbf{Z}} \mathbf{Q}_p \xrightarrow{(v\otimes \mathrm{id})} \bigoplus_{v\in S_f} \mathbf{Q}_p$$

is surjective, it follows that there is always a lift $[\widehat{x}]$ of $[x]$ with vanishing $\ell_v(u_v) = 0$ for all $v \in S_f$ unramified in K_∞/K (in particular, such that $[\widehat{x}_v^+] = 0$ for all $v \in S_f$, $v \nmid p$).

11.3.9. Theorem. — *Under the assumptions 11.3.1.1–11.3.1.5, the height pairings*

$$h_\pi^{\mathrm{norm}}, \widetilde{h}_{\pi,1,1} : \mathrm{Sel}(X) \otimes_{\mathbf{Q}_p} \mathrm{Sel}(Y) = \widetilde{H}^1_f(X) \otimes_{\mathbf{Q}_p} \widetilde{H}^1_f(Y) \longrightarrow \mathbf{Q}_p$$

are related by

$$h_\pi^{\mathrm{norm}} = -\widetilde{h}_{\pi,1,1}.$$

Proof. — Let $[x_f] \in \widetilde{H}^1_f(X)$ and $[y_f] \in \widetilde{H}^1_f(Y)$ be the cohomology classes of 1-cocycles

$$x_f = (x, (x_v^+), (\lambda_v)) \in \widetilde{C}^1_f(X), \quad dx_f = 0, \quad y_f = (y, (y_v^+), (\mu_v)) \in \widetilde{C}^1_f(Y), \quad dy_f = 0,$$

where

$$\begin{aligned} &x \in C^1_{\mathrm{cont}}(G_{K,S}, X), && dx = 0\\ &x_v^+ \in C^1_{\mathrm{cont}}(G_v, X_v^+), && dx_v^+ = 0 \quad (v \in S_f)\\ &\lambda_v \in C^0_{\mathrm{cont}}(G_v, X) = X, && i_v^+(x_v^+) = x_v + d\lambda_v \quad (v \in S_f)\end{aligned}$$

and similarly for y, y_v^+, μ_v. Recall that $X_v^+ = Y_v^+ = 0$ if $v \nmid p$, hence $x_v^+ = y_v^+ = 0$ for such v.

It follows from Lemma 11.2.3 that

$$\beta([x_f]) \in \widetilde{H}^2_f(X \otimes_{\mathbf{Z}_p} \Gamma)$$

is represented by the 2-cocycle

$$\beta(x_f) = (-z \cup x, (-z_v \cup x_v^+), (z_v \cup \lambda_v)) \in \widetilde{C}^2_f(X \otimes_{\mathbf{Z}_p} \Gamma),$$

where $z \in C^1_{\mathrm{cont}}(G_{K,S}, \Gamma)$ is the tautological cocycle given by the projection $G_{K,S} \twoheadrightarrow \Gamma$. By definition,

$$\widetilde{h}_{\pi,1,1}([x_f],[y_f]) = [\beta(x_f) \cup_{\pi,0} y_f] \in H^3_{c,\mathrm{cont}}(G_{K,S}, \mathbf{Q}_p(1) \otimes_{\mathbf{Z}_p} \Gamma) \xrightarrow{\sim} \Gamma \otimes_{\mathbf{Z}_p} \mathbf{Q}_p. \quad \square$$

11.3.10. Lemma. — *Represent the cohomology classes* $[\widehat{x}] \in H^1_{\mathrm{cont}}(G_{K,S}, Y_y^*(1))$, $[\widehat{x}_v^+] \in H^1_{\mathrm{cont}}(G_v, Y_y^*(1)_v^+)$ $(v \in S_f)$ *from the construction of* h_π^{norm} *by 1-cocycles* $\widehat{x} \in C^1_{\mathrm{cont}}(G_{K,S}, Y_y^*(1))$, $\widehat{x}_v^+ \in C^1_{\mathrm{cont}}(G_v, Y_y^*(1)_v^+)$ $(v \in S_f)$. *As* $i_v^+(\widehat{x}_v^+) = \widehat{x}_v + d\widehat{\lambda}_v$ *for suitable* $\widehat{\lambda}_v \in C^0_{\mathrm{cont}}(G_v, Y_y^*(1)) = Y_y^*(1)$ $(v \in S_f)$, *we obtain a 1-cocycle* $(\widehat{x}, (\widehat{x}_v^+), (\widehat{\lambda}_v)) \in \widetilde{C}^1_f(Y_y^*(1))$. *Then*

$$\beta(\widehat{x}, (\widehat{x}_v^+), (\widehat{\lambda}_v)) = \big(\widehat{a}, \big(\widehat{b}_v\big), (\widehat{c}_v)\big) \in C^2_f(Y_y^*(1)),$$

where

$$\widehat{b}_v = -z_v \cup u_v \in C^2_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1) \otimes_{\mathbf{Z}_p} \Gamma) \qquad (v \in S_f)$$

and u_v *is as in 11.3.7.*

Proof. — The diagram (11.3.7.2) has an analogue in which each $\mathscr{F}_{\Gamma_v}$ is replaced by $\mathscr{F}_\Gamma$ and $\mathbf{Q}_p$ at the bottom of the left column is replaced by $\mathbf{Z}_p[\![\Gamma/\Gamma_v]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. This implies that, for each $v \in S_f$, the element

$$\widehat{x}_v^+ - u_v \in C^1_{\mathrm{cont}}(G_v, Y_y^*(1)_v^+)$$

is equal to the image $\mathrm{pr}_*\,(\mathscr{F}_\Gamma(x_v^+))$ of a *1-cocycle*

$$\mathscr{F}_\Gamma(x_v^+) \in C^1_{\mathrm{cont}}(G_v, \mathscr{F}_\Gamma(Y_y^*(1)_v^+)), \quad d\left(\mathscr{F}_\Gamma(x_v^+)\right) = 0.$$

The definition of β and Lemma 11.2.3 then yield

$$\widehat{b}_v = \beta(\widehat{x}_v^+) = \beta(u_v) = -z_v \cup u_v,$$

as claimed. $\square$

11.3.11. Lemma. — *Assume that* $\beta([x_f]) \in \widetilde{H}^2_f(X \otimes_{\mathbf{Z}_p} \Gamma)$ *is represented by a 2-cocycle of the form* $e = (a, (b_v), (c_v)) = \sigma_*(\widehat{e})$, *where* $\widehat{e} = \big(\widehat{a}, \big(\widehat{b}_v\big), (\widehat{c}_v)\big) \in \widetilde{C}^2_f(Y_y^*(1) \otimes_{\mathbf{Z}_p} \Gamma)$ *is also a 2-cocycle (*i.e., $d\widehat{e} = 0$*). Then*

$$[\beta(x_f) \cup y_f] = \sum_{v \in S_f} \mathrm{inv}_v\big(i_v^+\big(\widehat{b}_v\big) \cup \widehat{y}_v + i_v^+(b_v) \cup \mu_v\big),$$

where $\widehat{y} \in C^0_{\mathrm{cont}}(G_{K,S}, Y_y) = Y_y$ *maps to* $1 \in C^0_{\mathrm{cont}}(G_{K,S}, \mathbf{Q}_p) = \mathbf{Q}_p$ *and satisfies* $d\widehat{y} = (\sigma^*(1))_*(y)$. *If, in addition,*

$$\widehat{b}_v \in C^2_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1) \otimes_{\mathbf{Z}_p} \Gamma) \subset C^2_{\mathrm{cont}}(G_v, Y_y^*(1)_v^+ \otimes_{\mathbf{Z}_p} \Gamma) \qquad (\forall v \in S_f),$$

then

$$[\beta(x_f) \cup y_f] = \sum_{v \in S_f} \mathrm{inv}_v\big(\widehat{b}_v\big) \in \Gamma \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

Proof. — We have

$$\begin{aligned} e\cup_1 y_f &= (a,(b_v),(c_v))\cup_1 (y,(y_v^+),(\mu_v))\\ &= (a\cup y,(c_v\cup y_v + i_v^+(b_v)\cup\mu_v))\in C^3_{c,\mathrm{cont}}(G_{K,S},\mathbf{Q}_p(1)\otimes_{\mathbf{Z}_p}\Gamma),\end{aligned}$$

where $\cup_1$ denotes the product $\cup_{r,h}$ from Proposition 1.3.2(i) for $r=1$ and $h=0$. As

$$a\cup y\in C^3_{\mathrm{cont}}(G_{K,S},\mathbf{Q}_p(1)\otimes_{\mathbf{Z}_p}\Gamma),\quad d(a\cup y)=0$$

and $H^3_{\mathrm{cont}}(G_{K,S},\mathbf{Q}_p(1)\otimes_{\mathbf{Z}_p}\Gamma)=0$, we have

$$a\cup y = d\varepsilon,\quad \varepsilon\in C^2_{\mathrm{cont}}(G_{K,S},\mathbf{Q}_p(1)\otimes_{\mathbf{Z}_p}\Gamma).$$

Then

$$\begin{aligned} d(\varepsilon,0) &= (d\varepsilon,(-\varepsilon_v))\in C^3_{c,\mathrm{cont}}(G_{K,S},\mathbf{Q}_p(1)\otimes_{\mathbf{Z}_p}\Gamma)\\ e\cup_1 y_f - d(\varepsilon,0) &= (0,(c_v\cup y_v + i_v^+(b_v)\cup\mu_v+\varepsilon_v)),\end{aligned}$$

which implies that the cohomology class

$$[e\cup_1 y_f]\in H^3_{c,\mathrm{cont}}(G_{K,S},\mathbf{Q}_p(1)\otimes_{\mathbf{Z}_p}\Gamma)\xrightarrow{\sim}\Gamma\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$$

is equal to

(11.3.11.1) $$[e\cup_1 y_f]=\sum_{v\in S_f}\mathrm{inv}_v(c_v\cup y_v + i_v^+(b_v)\cup\mu_v+\varepsilon_v).$$

We are now going to use the fact that $e=\sigma_*(\widehat{e})$, $d\widehat{e}=0$. For each $v\in S_f$, the commutative diagram

$$\begin{array}{ccc} C^2_{\mathrm{cont}}(G_v,Y_y^*(1)\otimes_{\mathbf{Q}_p}Y\otimes_{\mathbf{Z}_p}\Gamma) & \xrightarrow{\sigma_*\otimes\mathrm{id}} & C^2_{\mathrm{cont}}(G_v,X\otimes_{\mathbf{Q}_p}Y\otimes_{\mathbf{Z}_p}\Gamma)\\ \downarrow{\scriptstyle \mathrm{id}\otimes(\sigma^*(1))_*} & & \downarrow{\scriptstyle \pi_*}\\ C^2_{\mathrm{cont}}(G_v,Y_y^*(1)\otimes_{\mathbf{Q}_p}Y_y\otimes_{\mathbf{Z}_p}\Gamma) & \longrightarrow & C^2_{\mathrm{cont}}(G_v,\mathbf{Q}_p(1)\otimes_{\mathbf{Z}_p}\Gamma)\end{array}$$

shows that

(11.3.11.2) $$c_v\cup y_v=\widehat{c}_v\cup(d\widehat{y})_v\in C^2_{\mathrm{cont}}(G_v,\mathbf{Q}_p(1)\otimes_{\mathbf{Z}_p}\Gamma).$$

Combining (11.3.11.1)–(11.3.11.2) and using the formulas

$$0=\mathrm{inv}_v(d(\widehat{c}_v\cup\widehat{y}_v))=\mathrm{inv}_v(d\widehat{c}_v\cup\widehat{y}_v-\widehat{c}_v\cup d\widehat{y}_v),\quad i^+(\widehat{b}_v)=\widehat{a}_v+d\widehat{c}_v$$

(the second one following from $d\widehat{e}=0$), we obtain

$$\begin{aligned} [e\cup_1 y_f] &= \sum_{v\in S_f}\mathrm{inv}_v(i^+(\widehat{b}_v)\cup\widehat{y}_v + i^+(b_v)\cup\mu_v+(\varepsilon-\widehat{a}\cup\widehat{y})_v)\\ &= \sum_{v\in S_f}\mathrm{inv}_v(i^+(\widehat{b}_v)\cup\widehat{y}_v + i^+(b_v)\cup\mu_v),\end{aligned}$$

as the sum of the local invariants of the global 2-cocycle

$$\varepsilon-\widehat{a}\cup\widehat{y}\in C^2_{\mathrm{cont}}(G_{K,S},\mathbf{Q}_p(1)\otimes_{\mathbf{Z}_p}\Gamma),\quad d(\varepsilon-\widehat{a}\cup\widehat{y})=0,$$

vanishes. This proves the first statement of the Lemma. In the special case when

$$\widehat{b}_v\in C^2_{\mathrm{cont}}(G_v,\mathbf{Q}_p(1)\otimes_{\mathbf{Z}_p}\Gamma)\qquad(\forall v\in S_f),$$

we have

$$i^+(b_v)\cup\mu_v=0,\quad i^+(\widehat{b}_v)\cup\widehat{y}_v=i^+(\widehat{b}_v)\cup 1=i^+(\widehat{b}_v).\quad\square$$

11.3.12. We are now ready to conclude the proof of Theorem 11.3.9: combining the statements of Lemma 11.3.10–11.3.11, we obtain (using (11.3.5.3))

$$\begin{aligned}\widetilde{h}_{\pi,1,1}([x_f],[y_f])=[\beta(x_f)\cup y_f]=&\sum_{v\in S_f}\mathrm{inv}_v(-z_v\cup u_v)\\ =&-\sum_{v\in S_f}\ell_v(u_v)=-h_\pi^{\mathrm{norm}}([x],[y]).\end{aligned}$$

11.3.13. Plater [**Pl**] used universal norms to construct $\mathrm{Frac}(R)$-valued height pairings on Selmer groups associated to certain two-dimensional Galois representations

$$G_{K,S}\longrightarrow \mathrm{GL}_2(R),$$

where R is an integral domain, finite and flat over $\mathbf{Z}_p[\![\Gamma]\!]$, $\Gamma\xrightarrow{\sim}\mathbf{Z}_p$. It is very likely that the proof of Theorem 11.3.9 applies to the situation considered in [Pl]; we have not checked the details, though.

11.3.14. Rubin's formula. — Assume that

$$[x_f]=[(x,(x_v^+),(\lambda_v))]\in\widetilde{H}^1_f(X),\quad [y_f]=[(y,(y_v^+),(\mu_v))]\in\widetilde{H}^1_f(Y)$$

are as in the proof of Theorem 11.3.9 and that there exists a cohomology class

$$[x_{\mathrm{Iw}}]\in H^1_{\mathrm{Iw}}(K_\infty/K,X)=H^1_{\mathrm{cont}}(G_{K,S},\mathscr{F}_\Gamma(X))$$

such that

$$\mathrm{pr}_*([x_{\mathrm{Iw}}])=[x]\in H^1_{\mathrm{cont}}(G_{K,S},X).\tag{11.3.14.1}$$

The commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
&&&&&& \bigoplus_v H^0(G_v,X_v^-)=0\\
&&&&&& \big\downarrow{\scriptstyle\beta=0}\\
&&&&&& \bigoplus_v H^1(G_v,X_v^-)\otimes\Gamma_R\\
&&&&&& \big\downarrow{\scriptstyle i}\\
&&&& H^1(G_{K,S},\mathscr{F}_\Gamma(X)/J^2) & \xrightarrow{\mathrm{res}_{S_f}} & \bigoplus_v H^1(G_v,\mathscr{F}_\Gamma(X_v^-)/J^2)\\
&&&& \big\downarrow{\scriptstyle pr_*} && \big\downarrow{\scriptstyle pr_*}\\
0=\bigoplus_v H^0(G_v,X_v^-) & \xrightarrow{\partial=0} & \widetilde{H}^1_f(X) & \longrightarrow & H^1(G_{K,S},X) & \xrightarrow{\mathrm{res}_{S_f}} & \bigoplus_v H^1(G_v,X_v^-)\\
\big\downarrow{\scriptstyle -\beta=0} && \big\downarrow{\scriptstyle\beta} && \big\downarrow{\scriptstyle\beta} &&\\
\bigoplus_v H^1(G_v,X_v^-)\otimes\Gamma_R & \xrightarrow{\partial} & \widetilde{H}^2_f(X)\otimes\Gamma_R & \longrightarrow & H^2(G_{K,S},X)\otimes\Gamma_R &&
\end{array}$$

(in which the direct sums are taken over $v\in S_f$ and the subscript "cont" is suppressed in all cohomology groups) shows that there is a unique collection of cohomology classes

$$[Dx_{\mathrm{Iw}}]=[(Dx_{\mathrm{Iw}})_v],\quad (Dx_{\mathrm{Iw}})_v\in H^1_{\mathrm{cont}}(G_v,X_v^-)\otimes_R\Gamma_R\qquad(v\in S_f)$$

satisfying

$$i([Dx_{\mathrm{Iw}}])=\mathrm{res}_{S_f}([x_{\mathrm{Iw}}]\ (\mathrm{mod}\ J^2)).$$

The pair $([x_{\mathrm{Iw}}],[Dx_{\mathrm{Iw}}])$ is an analogue of $(\mathbf{z},\mathrm{Der}_\rho(\mathbf{z}))$ in the situation considered by Rubin [**Ru**]. The following result is a variant of Rubin's theorem ([**Ru**, Thm. 3.2 (ii)]) in our context.

11.3.15. Proposition. — *In the situation of 11.3.14 (still assuming 11.3.1.1–11.3.1.5), we have*

$$\widetilde{h}_{\pi,1,1}([x_f],[y_f]) = -\sum_{v\in S_f} \mathrm{inv}_v((Dx_{\mathrm{Iw}})_v \cup y_v^+).$$

Proof. — The assumption (11.3.14.1) implies that the 2-cocycle a in

$$\beta(x_f) = (a,(b_v),(c_v))$$

is a coboundary,

$$a = dA, \quad A \in C^1_{\mathrm{cont}}(G_{K,S},X).$$

For each $v\in S_f$, we have

$$i_v^+(b_v) = d(c_v + A_v),$$

hence

$$\beta([x_f]) = \partial([(c_v^- + A_v^-)_v]),$$

where $(-)_v^-$ denotes the image of $(-)_v$ under the map

$$C^1_{\mathrm{cont}}(G_v,X) \longrightarrow C^1_{\mathrm{cont}}(G_v,X_v^-).$$

In the notation used in the proof of Lemma 11.3.11, we can take $\varepsilon = A\cup y$, hence

$$\begin{aligned}
\widetilde{h}_{\pi,1,1}([x_f],[y_f]) &= \sum_{v\in S_f} \mathrm{inv}_v(c_v\cup y_v + i_v^+(b_v)\cup\mu_v + \varepsilon_v)\\
&= \sum_{v\in S_f} \mathrm{inv}_v((c_v+A_v)\cup y_v + d(c_v+A_v)\cup\mu_v)\\
&= \sum_{v\in S_f} \mathrm{inv}_v((c_v+A_v)\cup(y_v+d\mu_v)) = \sum_{v\in S_f} \mathrm{inv}_v((c_v+A_v)\cup i_v^+(y_v^+))\\
&= \sum_{v\in S_f} \mathrm{inv}_v((c_v^- + A_v^-)\cup y_v^+).
\end{aligned}$$

On the other hand, Lemma 1.2.19 implies that

$$\beta([x_f]) = -\partial([Dx_{\mathrm{Iw}}]),$$

hence there is

$$B\in C^1_{\mathrm{cont}}(G_{K,S},X), \quad dB = 0$$

satisfying

$$c_v^- + A_v^- = -(Dx_{\mathrm{Iw}})_v + B_v^- \qquad (\forall v\in S_f)$$

(where Dx_{Iw} is a 1-cocycle representing $[Dx_{\mathrm{Iw}}]$). Thus

$$\widetilde{h}_{\pi,1,1}([x_f],[y_f]) = \sum_{v\in S_f} \mathrm{inv}_v((-(Dx_{\mathrm{Iw}})_v + B_v^-)\cup y_v^+),$$

but the sum

$$\begin{aligned}\sum_{v\in S_f} \mathrm{inv}_v(B_v^- \cup y_v^+) = \sum_{v\in S_f} \mathrm{inv}_v(B_v \cup i_v^+(y_v^+)) &= \sum_{v\in S_f} \mathrm{inv}_v(B_v \cup (y_v + d\mu_v)) \\ &= \sum_{v\in S_f} \mathrm{inv}_v((B\cup y)_v - d(B_v\cup \mu_v)) = 0\end{aligned}$$

vanishes. Proposition follows. □

11.4. Comparison with classical p-adic height pairings (bis)

In this section we slightly relax the conditions on the Galois representations X, Y from 11.3, so that $\widetilde{H}^1_f(Z)$ no longer coincides with $\mathrm{Sel}(Z)$ ($Z = X, Y$). The difference between the two groups is an algebraic counterpart of trivial zeros of p-adic L-functions (cf. [**M-T-T**]). In this case there are two 'natural' height pairings

$$\mathrm{Sel}(X)\otimes_{\mathbf{Q}_p} \mathrm{Sel}(Y) \longrightarrow \Gamma\otimes_{\mathbf{Z}_p} \mathbf{Q}_p,$$

namely the one constructed *via* universal norms, and the "canonical" height pairing. A relation between them is proved in [**Ne2**, Thm. 7.13], in the context of p-adic Hodge theory.

11.4.1. Assume that $\Gamma = \mathrm{Gal}(K_\infty/K) \xrightarrow{\sim} \mathbf{Z}_p^r$ ($r \geqslant 1$) and that we are given the data

11.4.1.1. $=$ 11.3.1.1

11.4.1.2. $=$ 11.3.1.2

11.4.1.3. $=$ 11.3.1.3

satisfying

11.4.1.4. $=$ 11.3.1.4.

For each prime $v \mid p$ in K, put

$$\begin{aligned} A_v &= H^0(G_v, X_v^-) = H^0(G_v, (Y_v^+)^*(1)) \\ B_v &= H^0(G_v, Y_v^-) = H^0(G_v, (X_v^+)^*(1)).\end{aligned}$$

Taking the pull-back (resp., push-forward) of the tautological exact sequence

$$0 \longrightarrow X_v^+ \longrightarrow X \longrightarrow X_v^- \longrightarrow 0$$

with respect to the inclusion $A_v \hookrightarrow X_v^-$ (resp., the projection $\rho_v : X_v^+ \twoheadrightarrow B_v^*(1)$), we obtain a short exact sequence of G_v-modules

$$0 \longrightarrow B_v^*(1) \longrightarrow W_v \longrightarrow A_v \longrightarrow 0.$$

Similarly, applying $B_v \hookrightarrow Y_v^-$ and $\rho'_v : Y_v^+ \twoheadrightarrow A_v^*(1)$ to

$$0 \longrightarrow Y_v^+ \longrightarrow Y \longrightarrow Y_v^- \longrightarrow 0,$$

we obtain an exact sequence

$$0 \longrightarrow A_v^*(1) \longrightarrow W'_v \longrightarrow B_v \longrightarrow 0.$$

We denote the corresponding extension classes by

$$q_v(X) := [W_v] \in \mathrm{Ext}^1_{\mathbf{Q}_p[G_v]}(A_v, B_v^*(1)) = \mathrm{Hom}_{\mathbf{Q}_p}(A_v, B_v^*) \otimes_{\mathbf{Q}_p} H^1_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1))$$
$$q_v(Y) := [W'_v] \in \mathrm{Ext}^1_{\mathbf{Q}_p[G_v]}(B_v, A_v^*(1)) = \mathrm{Hom}_{\mathbf{Q}_p}(B_v, A_v^*) \otimes_{\mathbf{Q}_p} H^1_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1)).$$

Equivalently, they are equal to the maps

$$\begin{aligned} q_v(X) : A_v = H^0(G_v, X_v^-) &\xrightarrow{\partial_v} H^1_{\mathrm{cont}}(G_v, X_v^+) \xrightarrow{(\rho_v)_*} H^1_{\mathrm{cont}}(G_v, B_v^*(1)) \\ &= B_v^* \otimes_{\mathbf{Q}_p} H^1_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1)) \\ q_v(Y) : B_v = H^0(G_v, Y_v^-) &\xrightarrow{\partial'_v} H^1_{\mathrm{cont}}(G_v, Y_v^+) \xrightarrow{(\rho'_v)_*} H^1_{\mathrm{cont}}(G_v, A_v^*(1)) \\ &= A_v^* \otimes_{\mathbf{Q}_p} H^1_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1)). \end{aligned}$$

If $q \in K_v^* \widehat{\otimes} \mathbf{Q}_p = H^1_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1))$ is the extension class of an exact sequence of G_v-modules

$$0 \longrightarrow \mathbf{Q}_p(1) \xrightarrow{\alpha} E \xrightarrow{\beta} \mathbf{Q}_p \longrightarrow 0,$$

then q^{-1} (we write the group law multiplicatively) is the extension class of

$$0 \longrightarrow \mathbf{Q}_p(1) \xrightarrow{\beta^*(1)} E^*(1) \xrightarrow{\alpha^*(1)} \mathbf{Q}_p \longrightarrow 0.$$

Applying this remark to W_v and $W'_v = W_v^*(1)$, we see that

$$q_v(Y) = q_v(X)^{-1}$$

under the canonical isomorphism

$$\mathrm{Hom}_{\mathbf{Q}_p}(A_v, B_v^*) \xrightarrow{\sim} A_v^* \otimes_{\mathbf{Q}_p} B_v^* \xrightarrow{\sim} B_v^* \otimes_{\mathbf{Q}_p} A_v^* \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Q}_p}(B_v, A_v^*).$$

Any homomorphism of vector spaces over $\mathbf{Q}_p$

$$F : H^1_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1)) \longrightarrow U$$

induces $\mathbf{Q}_p$-linear maps

$$F(q_v(X)) : A_v \longrightarrow B_v^* \otimes_{\mathbf{Q}_p} U, \quad F(q_v(Y)) : B_v \longrightarrow A_v^* \otimes_{\mathbf{Q}_p} U$$

satisfying

$$F(q_v(Y)) = -F(q_v(X)) \in A_v^* \otimes_{\mathbf{Q}_p} B_v^* \otimes_{\mathbf{Q}_p} U.$$

This applies, in particular, to the valuation

$$v = v \otimes 1 : H^1_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1)) = K_v^* \widehat{\otimes} \mathbf{Q}_p \longrightarrow \mathbf{Q}_p$$

and the map

$$\ell_v : H^1_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1)) \longrightarrow \Gamma_v \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

We assume that the following conditions 11.4.1.5–11.4.1.7 are satisfied:

11.4.1.5. $(\forall v|p)$ $v(q_v(X)) : A_v \to B_v^*$ is an isomorphism (equivalently, $v(q_v(Y)) : B_v \to A_v^*$ is an isomorphism).

We denote by

$$S_{\rm sp} = \{v|p;\ A_v \neq 0\} = \{v|p;\ B_v \neq 0\}$$

the set of "special" primes of K; new phenomena arise at such primes.

11.4.1.6. $(\forall v \in S_{\rm sp})$ either $\Gamma_v = 0$ or $H^0(G_{v_\infty}, X_v^-/A_v) = H^0(G_{v_\infty}, Y_v^-/B_v) = 0$ (where we put $G_{v_\infty} = \mathrm{Gal}(\overline{K}_v/(K_\infty)_{v_\infty})$ for a fixed prime v_∞ of K_∞ above v).

11.4.1.7. $(\forall v \in S_{\rm sp})$ $\ell_v(q_v(X)) : A_v \to B_v^* \otimes_{\mathbf{Z}_p} \Gamma_v$ is surjective (equivalently, $\ell_v(q_v(Y)) : B_v \to A_v^* \otimes_{\mathbf{Z}_p} \Gamma_v$ is surjective).

11.4.2. We now list the most important consequences of 11.4.1.1–11.4.1.7. For each $v \in S_{\rm sp}$, the boundary maps

$$\partial_v : A_v \hookrightarrow H^1_{\rm cont}(G_v, X_v^+), \quad \partial_v' : B_v \hookrightarrow H^1_{\rm cont}(G_v, Y_v^+) \quad (v \in S_{\rm sp})$$

are injective and admit canonical splittings

$$\mathrm{spl}_v : H^1_{\rm cont}(G_v, X_v^+) \xrightarrow{(\rho_v)_*} H^1_{\rm cont}(G_v, B_v^*(1)) \xrightarrow{v} B_v \xrightarrow{v(q_v(X))^{-1}} A_v$$
$$\mathrm{spl}_v' : H^1_{\rm cont}(G_v, Y_v^+) \xrightarrow{(\rho_v')_*} H^1_{\rm cont}(G_v, A_v^*(1)) \xrightarrow{v} A_v \xrightarrow{v(q_v(Y))^{-1}} B_v.$$

As in 11.3.2, we have

$$\mathbf{R}\Gamma_{\rm cont}(G_v, Z) = \mathbf{R}\Gamma_{\rm ur}(G_v, Z) \xrightarrow{\sim} 0 \qquad (v \in S_f, v \nmid p;\ Z = X, Y)$$

in $D(_{\mathbf{Q}_p}\mathrm{Mod})$; hence the Selmer complexes

$$\widetilde{\mathbf{R}\Gamma}_f(Z) := \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Z; \Delta(Z)) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Z; \Delta'(Z)) \qquad (Z = X, Y)$$

(with the local conditions defined as in 11.3.2) are again isomorphic in $D^b_{ft}(_{\mathbf{Q}_p}\mathrm{Mod})$. The image of the canonical map

$$\widetilde{H}^1_f(Z) \longrightarrow H^1_{\rm cont}(G_{K,S}, Z) \quad (Z = X, Y)$$

is again equal to the Selmer group $\mathrm{Sel}(Z)$ from 11.3.2, but the map can have a non-trivial kernel. Indeed, the third of the exact sequences in 6.1.3 yields exact sequences

$$0 \longrightarrow \bigoplus_{v \in S_{\rm sp}} A_v \longrightarrow \widetilde{H}^1_f(X) \longrightarrow \mathrm{Sel}(X) \longrightarrow 0$$
$$0 \longrightarrow \bigoplus_{v \in S_{\rm sp}} B_v \longrightarrow \widetilde{H}^1_f(Y) \longrightarrow \mathrm{Sel}(Y) \longrightarrow 0.$$

These sequences have canonical splittings

$$\mathrm{spl} : \widetilde{H}^1_f(X) \longrightarrow \bigoplus_{v\in S_{\mathrm{sp}}} A_v, \quad [(x,(x_v^+),(\lambda_v))] \longmapsto (\mathrm{spl}_v([x_v^+]))$$
$$\mathrm{spl}' : \widetilde{H}^1_f(Y) \longrightarrow \bigoplus_{v\in S_{\mathrm{sp}}} B_v, \quad [(y,(y_v^+),(\mu_v))] \longmapsto (\mathrm{spl}'_v([y_v^+])).$$

The induced decompositions

$$\widetilde{H}^1_f(X) \xrightarrow{\sim} \mathrm{Sel}(X) \oplus \bigoplus_{v\in S_{\mathrm{sp}}} A_v, \quad \widetilde{H}^1_f(Y) \xrightarrow{\sim} \mathrm{Sel}(Y) \oplus \bigoplus_{v\in S_{\mathrm{sp}}} B_v$$

can be characterized as follows: the lift

$$\mathrm{Sel}(X) \longrightarrow \widetilde{H}^1_f(X), \quad [x] \longmapsto [(x,(x_v^+),(\lambda_v))]$$

satisfies

$$\rho_v([x_v^+]) \in (\mathcal{O}_v^*\widehat{\otimes}\mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_v^* \subset (K_v^*\widehat{\otimes}\mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_v^* \qquad (\forall v \in S_{\mathrm{sp}})$$

(and similarly for Y).

The conditions 11.4.1.5–11.4.1.7 also imply that

$$S_{\mathrm{sp}} = S^0_{\mathrm{sp}} \cup S^1_{\mathrm{sp}}, \quad S^i_{\mathrm{sp}} = \{v \in S_{\mathrm{sp}} \mid \Gamma_v \xrightarrow{\sim} \mathbf{Z}_p^i\}.$$

11.4.3. Lemma. — *For each $v \mid p$, the canonical maps*

$$H^1_{\mathrm{cont}}(G_v, X_v^+)/N_\infty \xrightarrow{(\rho_v)_*} H^1_{\mathrm{cont}}(G_v, B_v^*(1))/N_\infty \xrightarrow{\ell_v} B_v^* \otimes_{\mathbf{Z}_p} \Gamma_v$$
$$H^1_{\mathrm{cont}}(G_v, Y_v^+)/N_\infty \xrightarrow{(\rho'_v)_*} H^1_{\mathrm{cont}}(G_v, A_v^*(1))/N_\infty \xrightarrow{\ell_v} A_v^* \otimes_{\mathbf{Z}_p} \Gamma_v$$

are isomorphisms.

Proof. — This is trivial for $\Gamma_v = 0$, so we can assume that $\Gamma_v \neq 0$. If $v \notin S_{\mathrm{sp}}$, then $A_v = B_v = 0$ and the statement follows from Lemma 11.3.6. If $v \in S_{\mathrm{sp}}$, then 11.4.1.6 implies that

$$H^1_{\mathrm{cont}}\big(G_v, \mathrm{Ker}\big(X_v^+ \xrightarrow{\rho_v} B_v^*(1)\big)\big)/N_\infty = 0,$$

by the same argument as in the proof of Lemma 11.3.6. The fact that the map

$$(\rho_v)_* : H^1_{\mathrm{cont}}(G_v, X_v^+)/N_\infty \longrightarrow H^1_{\mathrm{cont}}(G_v, B_v^*(1))/N_\infty$$

is an isomorphism then follows from the Snake Lemma applied to the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
H^1_{\mathrm{cont}}\big(G_v, \mathscr{F}_{\Gamma_v}\big(\mathrm{Ker}\big(X_v^+ \xrightarrow{\rho_v} B_v^*(1)\big)\big)\big) & \longrightarrow & H^1_{\mathrm{cont}}(G_v, \mathscr{F}_{\Gamma_v}(X_v^+)) & \longrightarrow & H^1_{\mathrm{cont}}\big(G_v, \mathscr{F}_{\Gamma_v}(B_v^*(1))\big) & \longrightarrow & 0 \\
\downarrow \mathrm{pr}_* & & \downarrow \mathrm{pr}_* & & \downarrow \mathrm{pr}_* & & \\
H^1_{\mathrm{cont}}\big(G_v, \mathrm{Ker}\big(X_v^+ \xrightarrow{\rho_v} B_v^*(1)\big)\big) & \longrightarrow & H^1_{\mathrm{cont}}(G_v, X_v^+) & \longrightarrow & H^1_{\mathrm{cont}}(G_v, B_v^*(1)) & \to & 0
\end{array}$$

(note that the zeros on the right come from the assumption 11.4.1.6 and the local Tate duality). ☐

11.4.4. Definition of h^{norm}. — Under the assumptions 11.4.1.1–11.4.1.7 one can define a height pairing

$$h_{\pi}^{\mathrm{norm}} : \mathrm{Sel}(X) \otimes_{\mathbf{Q}_p} \mathrm{Sel}(Y) \longrightarrow \Gamma \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

by modifying the construction from 11.3.7 at the special primes $v \in S_{\mathrm{sp}}$ as follows.

Let x, y, Y_y and $[\widehat{x}]$ be as in 11.3.7. The top of the diagram (11.3.7.1) has to be replaced by

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ H^0(G_v, Y_y^*(1)_v^-) & \xrightarrow{\sim} & H^0(G_v, X_v^-) \\ \downarrow \partial_v & & \downarrow \partial_v \\ H^1_{\mathrm{cont}}(G_v, Y_y^*(1)_v^+) & \longrightarrow & H^1_{\mathrm{cont}}(G_v, X_v^+). \end{array}$$

This implies that $\delta_v \circ i_v^+ \circ \partial_v = 0$, and as before we obtain a uniquely determined cohomology class

$$[\widehat{x}_v^+] \in H^1_{\mathrm{cont}}(G_v, Y_y^*(1)_v^+), \quad i_v^+([\widehat{x}_v^+]) = [\widehat{x}_v].$$

The diagram (11.3.7.2) also has to be modified by adding

$$\begin{array}{ccc} H^1_{\mathrm{cont}}(G_v, Y_y^*(1)_v^+) & \xleftarrow{\partial_v} & H^0(G_v, Y_y^*(1)_v^-) \\ \downarrow \sigma_* & & \| \\ H^1_{\mathrm{cont}}(G_v, X_v^+) & \xleftarrow{\partial_v} & H^0(G_v, X_v^-). \end{array}$$

There exist

$$[\mathscr{F}_{\Gamma_v}(x_v^+)] \in H^1_{\mathrm{cont}}(G_v, \mathscr{F}_{\Gamma_v}(Y_y^*(1)_v^+)), \quad t_v \in H^0(G_v, X_v^-) = A_v$$

such that

$$[x_v^+] + \partial_v(t_v) = \mathrm{pr}_* \circ \mathscr{F}_{\Gamma_v}(\sigma)_*([\mathscr{F}_{\Gamma_v}(x_v^+)]).$$

Then

$$u_v = [\widehat{x}_v^+] - \mathrm{pr}_*([\mathscr{F}_{\Gamma_v}(x_v^+)]) + \partial_v(t_v)$$

is an element of

$$u_v \in H^1_{\mathrm{cont}}(G_v, \mathbf{Q}_p(1))$$

and the height pairing h_{π}^{norm} is defined by the formula

$$h_{\pi}^{\mathrm{norm}}([x], [y]) = \sum_{v \in S_f} \ell_v(u_v) \in \Gamma \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

11.4.5. Definition of h^{can}. — One defines the "canonical height"

$$h_\pi^{\mathrm{can}} : \mathrm{Sel}(X) \otimes_{\mathbf{Q}_p} \mathrm{Sel}(Y) \longrightarrow \mathbf{Q}_p$$

as follows. For each $v \in S_{\mathrm{sp}}$, the subgroup of units

$$H^1_f(G_v, B_v^*(1)) := (\mathcal{O}_v^* \widehat{\otimes} \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_v^* \subset (K_v^* \widehat{\otimes} \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_v^* = H^1_{\mathrm{cont}}(G_v, B_v^*(1))$$

(resp., $H^1_f(G_v, A_v^*(1)) \subset H^1_{\mathrm{cont}}(G_v, A_v^*(1)))$ is complementary to $\mathrm{Im}(q_v(X))$ (resp., $\mathrm{Im}(q_v(Y))$), hence the localizations at v define morphisms

$$\begin{aligned}\widetilde{\ }_v : \mathrm{Sel}(X) &\xrightarrow{\mathrm{res}_v} \mathrm{Ker}(H^1_{\mathrm{cont}}(G_v, X) \longrightarrow H^1_{\mathrm{cont}}(G_v, X_v^-)) \\ &= \mathrm{Coker}(\partial_v : H^0(G_v, X_v^-) \longrightarrow H^1_{\mathrm{cont}}(G_v, X_v^+)) \\ &\xrightarrow{(\rho_v)_*} \mathrm{Coker}\left(q_v(X) : A_v \longrightarrow H^1_{\mathrm{cont}}(G_v, B_v^*(1))\right) \xleftarrow{\sim} H^1_f(G_v, B_v^*(1)),\end{aligned}$$

$$\widetilde{\ }_v : [x] \longmapsto [\widetilde{x}_v] \in H^1_f(G_v, B_v^*(1))$$

and

$$\mathrm{Sel}(Y) \longrightarrow H^1_f(G_v, A_v^*(1)), \quad [y] \longmapsto [\widetilde{y}_v].$$

We define

$$\begin{aligned}(11.4.5.1) \quad h_\pi^{\mathrm{can}}([x],[y]) &:= h_\pi^{\mathrm{norm}}([x],[y]) \\ &+ \sum_{v \in S^1_{\mathrm{sp}}} \ell_v(q_v(X))^{-1} \ell_v([\widetilde{x}_v]) \ell_v([\widetilde{y}_v]) \in \Gamma \otimes_{\mathbf{Z}_p} \mathbf{Q}_p,\end{aligned}$$

where the term on the R.H.S. should be interpreted as follows: we have

$$\ell_v([\widetilde{x}_v]) \in B_v^* \otimes_{\mathbf{Z}_p} \Gamma_v, \quad \ell_v(q_v(X))^{-1} \ell_v([\widetilde{x}_v]) \in A_v$$

and the latter element of A_v is evaluated against

$$\ell_v([\widetilde{y}_v]) \in A_v^* \otimes_{\mathbf{Z}_p} \Gamma_v.$$

Although we have used (11.4.5.1) as a definition, in the situation of ([**Ne2**, § 7]) one defines h_π^{can} directly and (11.4.5.1) then becomes the statement of ([**Ne2**, Thm. 7.13 (2)]).

11.4.6. Theorem. — *Under the assumptions 11.4.1.1–11.4.1.7, let $[x_f] \in \widetilde{H}^1_f(X)$, $[y_f] \in \widetilde{H}^1_f(Y)$ be the cohomology classes of 1-cocycles $x_f = (x, (x_v^+), (\lambda_v)) \in \widetilde{C}^1_f(X)$, $y_f = (y, (y_v^+), (\mu_v)) \in \widetilde{C}^1_f(Y)$ (using the same notation as in the proof of Theorem 11.3.9). Then*

$$\widetilde{h}_{\pi,1,1}([x_f],[y_f]) = -h_\pi^{\mathrm{norm}}([x],[y]) - \sum_{v \in S^1_{\mathrm{sp}}} \ell_v(q_v(X))^{-1} \ell_v(\rho_v(x_v^+)) \ell_v(\rho'_v(y_v^+)).$$

Proof. — We indicate only the necessary modifications to the proof of Theorem 11.3.9 at $v \in S_{\mathrm{sp}}$. Firstly, the statement of Lemma 11.3.10 has to be replaced by

$$\widehat{b}_v = z_v \cup (\partial_v(t_v) - u_v),$$

where

$$t_v \in H^0(G_v, Y_y^*(1)_v^-) = H^0(G_v, X_v^-) = A_v$$

is as in 11.4.4 (the proof of the Lemma remains the same). Secondly, we have

$$i_v^+(\partial_v(t_v)) = d\widetilde{t}_v, \quad \widetilde{t}_v \in C^0_{\mathrm{cont}}(G_v, Y_y^*(1)), \quad \widetilde{t}_v \longmapsto t_v.$$

It follows that

$$i_v^+(\widehat{b}_v) = -z_v \cup u_v + z_v \cup d\widetilde{t}_v, \quad i_v^+(b_v) \cup \mu_v = z_v \cup \sigma_*(d\widetilde{t}_v) \cup \mu_v,$$

hence

$$\begin{aligned}
&\ell_v(u_v) + \mathrm{inv}_v(i_v^+(\widehat{b}_v) \cup \widehat{y}_v + i_v^+(b_v) \cup \mu_v) \\
&= \mathrm{inv}_v(z_v \cup d\widetilde{t}_v \cup \widehat{y}_v + z_v \cup d\sigma_*(\widetilde{t}_v) \cup \mu_v) = -\mathrm{inv}_v(z_v \cup \widetilde{t}_v \cup d\widehat{y}_v + z_v \cup \sigma_*(\widetilde{t}_v) \cup d\mu_v) \\
&\quad = -\mathrm{inv}_v(z_v \cup (\widetilde{t}_v \cup (\sigma^*(1))_*(y_v) + \sigma_*(\widetilde{t}_v) \cup d\mu_v)) = -\mathrm{inv}_v(z_v \cup \widetilde{t}_v \cup i_v^+(y_v^+)) \\
&\qquad = -\mathrm{inv}_v(z_v \cup t_v \cup y_v^+) = -\ell_v(t_v \cup y_v^+) = -\ell_v(t_v \cup \rho'_v(y_v^+)).
\end{aligned}$$

The first formula in Lemma 11.3.11 (which still holds in our case) yields

$$\begin{aligned}
\widetilde{h}_{\pi,1,1}([x_f],[y_f]) = [\beta(x_f) \cup y_f] &= -\sum_{v \in S_f} \ell_v(u_v) - \sum_{v \in S_{\mathrm{sp}}} \ell_v(t_v \cup \rho'_v(y_v^+)) \\
&= -h_\pi^{\mathrm{norm}}([x],[y]) - \sum_{v \in S^1_{\mathrm{sp}}} \ell_v(t_v \cup \rho'_v(y_v^+))
\end{aligned}$$

(as $\ell_v = 0$ for $v \in S^0_{\mathrm{sp}}$). For each $v \in S^1_{\mathrm{sp}}$, the composite map

$$A_v \xrightarrow{\partial_v} H^1_{\mathrm{cont}}(G_v, X_v^+)/N_\infty \xrightarrow{\rho_v} H^1_{\mathrm{cont}}(G_v, B_v^*(1))/N_\infty \xrightarrow{\ell_v} B_v^* \otimes_{\mathbf{Z}_p} \Gamma_v$$

is equal to $\ell_v(q_v(X))$, which means that we can take

$$t_v = \ell_v(q_v(X))^{-1}\ell_v(\rho_v(x_v^+)),$$

hence the correction term in the above formula is equal to

$$\ell_v(t_v \cup \rho'_v(y_v^+)) = \ell_v(q_v(X))^{-1}\ell_v(\rho_v(x_v^+))\ell_v(\rho'_v(y_v^+)).$$

Theorem is proved. □

11.4.7. Corollary. — *With respect to the canonical decompositions defined in 11.4.2:*

$$\widetilde{H}^1_f(X) \xrightarrow{\sim} \mathrm{Sel}(X) \oplus \bigoplus_{v \in S_{\mathrm{sp}}} A_v, \quad \widetilde{H}^1_f(Y) \xrightarrow{\sim} \mathrm{Sel}(Y) \oplus \bigoplus_{v \in S_{\mathrm{sp}}} B_v,$$

the pairing $\widetilde{h}_{\pi,1,1}$ *vanishes on* $A_v \otimes \widetilde{H}^1_f(Y)$ *(resp.,* $\widetilde{H}^1_f(X) \otimes B_v$*) for all* $v \in S^0_{\mathrm{sp}}$*, and is given by the formulas*

$$\widetilde{h}_{\pi,1,1}([x],[y]) = -h^{\mathrm{can}}_{\pi}([x],[y]), \quad \widetilde{h}_{\pi,1,1}([a_v],[y]) = -\ell_v([\widetilde{y}_v])(a_v),$$

$$\widetilde{h}_{\pi,1,1}([x],[b_w]) = -\ell_w([\widetilde{x}_w])(b_w)$$

$$\widetilde{h}_{\pi,1,1}([a_v],[b_w]) = \begin{cases} -\ell_v(q_v(X))(a_v)(b_w), & v = w, \\ 0, & v \neq w, \end{cases}$$

where $[x] \in \mathrm{Sel}(X)$, $[y] \in \mathrm{Sel}(Y)$, $v, w \in S^1_{\mathrm{sp}}$, $a_v \in A_v$, $b_w \in B_w$.

11.4.8. In matrix terms, Corollary 11.4.7 says that the restriction of $\widetilde{h}_{\pi,1,1}$ to the subspaces

$$\widetilde{H}^1_f(X)^{(1)} = \mathrm{Sel}(X) \oplus \bigoplus_{S^1_{\mathrm{sp}}} A_v \subset \widetilde{H}^1_f(X), \quad \widetilde{H}^1_f(Y)^{(1)} = \mathrm{Sel}(Y) \oplus \bigoplus_{S^1_{\mathrm{sp}}} B_v \subset \widetilde{H}^1_f(Y)$$

has matrix

$$\widetilde{h}_{\pi,1,1} = -\begin{pmatrix} h^{\mathrm{can}}_{\pi} & \ell_v \circ \widetilde{\ }_v \\ \ell_v \circ \widetilde{\ }_v & \ell_v(q_v(X)) \end{pmatrix}$$

11.4.9. Proposition. — *Under the assumptions 11.4.1.1–11.4.1.7, the pairing*

$$\widetilde{h}_{\pi,1,1} : \widetilde{H}^1_f(X)^{(1)} \otimes_{\mathbf{Q}_p} \widetilde{H}^1_f(Y)^{(1)} \longrightarrow \Gamma \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

is non-degenerate if and only if

$$h^{\mathrm{norm}}_{\pi} : \mathrm{Sel}(X) \otimes_{\mathbf{Q}_p} \mathrm{Sel}(Y) \longrightarrow \Gamma \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

is. If true, then

$$\det\big(-\widetilde{h}_{\pi,1,1}\big) = \det(h^{\mathrm{norm}}_{\pi}) \prod_{v \in S^1_{\mathrm{sp}}} \det(\ell_v(q_v(X))$$

in

$$\det(\mathrm{Sel}(X))^* \otimes \det(\mathrm{Sel}(Y))^* \otimes \bigotimes_{S^1_{\mathrm{sp}}} (\det(A_v)^* \otimes \det(B_v)^*).$$

Proof. — As in [**Ne2**, §7.13 (3)], this follows from Corollary 11.4.7 and the matrix identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}. \quad \square$$

11.4.10. If $K = \mathbf{Q}$ and X is the one-dimensional Galois representation associated to a non-trivial Dirichlet character $\chi \neq 1$ whose restriction to G_p is trivial, then the formulas from 11.4.7 (for the local conditions $X^+_v = 0$, $Y^+_v = Y = X^*(1)$) yield the statement of [**Bu-Gr**, Prop. 9.3 (iii)]).

11.5. Higher height pairings

In this section we define analogues of the "derived heights" of Bertolini-Darmon [**B-D1, B-D2**] in our setting.

11.5.1. Let K_∞/K, X, Y be as in 11.1. Fix an integer $k \geqslant 1$ and assume that (P), (U) and the following strengthening of $(\mathrm{Fl}(\Gamma))$ hold:

$(\mathrm{Fl}_k(\Gamma))$ For all $i = 0, \dots, k$, J^i/J^{i+1} is a flat (hence free) R-module.

The condition $(\mathrm{Fl}_k(\Gamma))$ implies that, for all $0 \leqslant i \leqslant j \leqslant k+1$, the exact sequence

$$0 \longrightarrow J^j/J^{k+1} \longrightarrow J^i/J^{k+1} \longrightarrow J^i/J^j \longrightarrow 0 \tag{11.5.1.1}$$

consists of free R-modules, hence splits. If $\Delta = 0$, then $(\mathrm{Fl}_k(\Gamma))$ holds for all $k \geqslant 1$. If $\Delta = \mathbf{Z}/p^m\mathbf{Z}$ and R is an $\mathbf{Z}/p^m\mathbf{Z}$-algebra, then $(Fl_{p-1}(\Gamma))$ holds.

11.5.2. The condition $(\mathrm{Fl}_k(\Gamma))$ implies that the J-adic filtrations on

$$\begin{aligned} \mathscr{F}_\Gamma(X)/J^{k+1} &= (X \otimes_R \overline{R}/J^{k+1}) < -1 >, \\ \mathscr{F}_\Gamma(Y)^\iota/J^{k+1} &= (Y \otimes_R \overline{R}/J^{k+1}) < 1 > \end{aligned} \tag{11.5.2.1}$$

have graded quotients

$$\begin{aligned} \mathrm{gr}^i_J(\mathscr{F}_\Gamma(X)/J^{k+1}) &= (X \otimes_R J^i/J^{i+1}) < -1 >= X \otimes_R J^i/J^{i+1} \\ \mathrm{gr}^i_J(\mathscr{F}_\Gamma(Y)^\iota/J^{k+1}) &= \ (Y \otimes_R J^i/J^{i+1}) < 1 > \ = Y \otimes_R J^i/J^{i+1} \end{aligned} \qquad (0 \leqslant i \leqslant k)$$

($\mathrm{gr}^i_J = 0$ for $i > k$). As in 11.1, the given Greenberg's local conditions for X, Y induce similar local conditions for all

$$(X \otimes_R J^i/J^j) < -1 >, \quad (Y \otimes_R J^i/J^j) < 1 >$$

The assumptions (U), $(\mathrm{Fl}_k(\Gamma))$ imply that the filtrations $F^\bullet_J$ on $\widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(X)/J^{k+1})$ (resp., $\widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(Y)^\iota/J^{k+1})$) induced by the J-adic filtrations on the coefficient modules (11.5.2.1) satisfy

$$\mathrm{gr}^i_{F_J}\left(\widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(X)/J^{k+1})\right) = \begin{cases} \widetilde{C}^\bullet_f(X) \otimes_R J^i/J^{i+1}, & (0 \leqslant i \leqslant k) \\ 0, & \text{otherwise,} \end{cases}$$

and similarly for $\mathscr{F}_\Gamma(Y)^\iota/J^{k+1}$ (one can show that the filtrations $F^\bullet_J$ are the J-adic ones, but we do not need this fact). It follows that the associated spectral sequences

$$\begin{aligned} E_r(X) &= E_r\big(\widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(X)/J^{k+1}), F^\bullet_J\big) \\ {}'E_r(Y) &= E_r\big(\widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(Y)^\iota/J^{k+1}), F^\bullet_J\big) \end{aligned} \tag{11.5.2.2}$$

have initial terms

$$E_1^{i,j}(X) = \widetilde{H}^{i+j}_f(X) \otimes_R J^i/J^{i+1}, \quad {}'E_1^{i,j}(Y) = \widetilde{H}^{i+j}_f(Y) \otimes_R J^i/J^{i+1} \qquad (0 \leqslant i \leqslant k)$$

(and $E_1^{i,j} = {}'E_1^{i,j} = 0$ for $i > k$). The differential

$$d_1^{0,j} : \widetilde{H}^j_f(X) \longrightarrow \widetilde{H}^{j+1}_f(X) \otimes_R J/J^2 = \widetilde{H}^{j+1}_f(X) \otimes_R \Gamma_R$$

coincides with the coboundary map induced by the morphism

$$\beta : \widetilde{\mathbf{R}\Gamma}_f(X) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(X)[1] \otimes_R J/J^2$$

from 11.3. Other differentials $d_1^{i,j}$ are also induced by β:

$$d_1^{i,j} : E_1^{i,j} = E_1^{0,i+j} \otimes_R J^i/J^{i+1} \xrightarrow{d_1^{0,i+j}\otimes \mathrm{id}} E_1^{0,i+j+1} \otimes_R J/J^2 \otimes_R J^i/J^{i+1}$$
$$\xrightarrow{\mathrm{id}\otimes \mathrm{mult}} E_1^{0,i+j+1} \otimes_R J^{i+1}/J^{i+2} = E_1^{i+1,j}.$$

More generally, multiplication induces maps

$$E_r^{i,j} \otimes_R J^l/J^{l+1} \longrightarrow E_r^{i+l,j-l} \qquad (i, l \geqslant 0,\ r \geqslant 1)$$

compatible with the differentials. Almost the same remarks apply to the differentials $'d_r$ in the spectral sequence $'E_r(Y)$, the only difference being that

$$('E_r(Y), 'd_r) = (E_r(Y), (-1)^r d_r).$$

For $r = 1$, the equality

$$'d_1^{0,j} = -d_1^{0,j} : \widetilde{H}_f^j(Y) \longrightarrow \widetilde{H}_f^{j+1}(Y) \otimes_R J/J^2$$

follows from the description of β in 11.2.5, as the presence of the involution ι changes z into $-z$.

If $k \geqslant k' \geqslant 1$, then the canonical morphism of spectral sequences

$$E_*\big(\widetilde{C}_f^\bullet(\mathscr{F}_\Gamma(X)/J^{k+1})\big) \longrightarrow E_*\big(\widetilde{C}_f^\bullet(\mathscr{F}_\Gamma(X)/J^{k'+1})\big)$$

induces isomorphisms in the region

$$E_r^{i,j}(-) \xrightarrow{\sim} E_r^{i,j}(-') \qquad (0 \leqslant i \leqslant k' - r + 1).$$

11.5.3. A slightly modified construction from 5.4.1 can be applied to the J-adic truncation

$$\mathscr{F}(\pi)_{k+1} : (\mathscr{F}_\Gamma(X)/J^{k+1}) \otimes_{\overline{R}} (\mathscr{F}_\Gamma(Y)^\iota/J^{k+1}) \longrightarrow \omega_R^\bullet \otimes_R (\overline{R}/J^{k+1})(1)$$

of the pairing

$$\mathscr{F}(\pi) : \mathscr{F}_\Gamma(X) \otimes_{\overline{R}} \mathscr{F}_\Gamma(Y)^\iota \longrightarrow \omega_R^\bullet \otimes_R \overline{R}(1),$$

yielding cup products (depending on various choices, as in 5.4.1 and 6.3.1)

$$\text{(11.5.3.1)} \quad \cup : \widetilde{C}_f^\bullet(\mathscr{F}_\Gamma(X)/J^{k+1}) \otimes_{\overline{R}} \widetilde{C}_f^\bullet(\mathscr{F}_\Gamma(Y)^\iota/J^{k+1})$$
$$\longrightarrow \tau_{\geqslant 3}^{\mathrm{II}}\, C_{c,\mathrm{cont}}^\bullet(G_{K,S}, \omega_R^\bullet \otimes_R \overline{R}/J^{k+1}(1)) = \tau_{\geqslant 3}^{\mathrm{II}}\, C_{c,\mathrm{cont}}^\bullet(G_{K,S}, \omega_R^\bullet(1)) \otimes_R \overline{R}/J^{k+1}$$
$$\longrightarrow (\omega_R^\bullet \otimes_R \overline{R}/J^{k+1})[-3],$$

the homotopy classes of which are independent of any choices. These cup products are compatible with the filtrations in the sense that

$$F_J^i \cup F_J^{i'} \subseteq (\omega_R^\bullet \otimes_R J^{i+i'}/J^{k+1})[-3],$$

hence they induce products

$$\cup_r : E_r^{i,j}(X) \otimes_R {'E_r^{i',j'}}(Y) \longrightarrow E_r^{i+i',j+j'}\big((\omega_R^\bullet \otimes_R \overline{R}/J^{k+1})[-3], J^\bullet\big)$$

satisfying

$$d_r(x \cup_r y) = (d_r x) \cup_r y + (-1)^{\overline{x}} x \cup_r ({}'d_r y) = (d_r x) \cup_r y + (-1)^{i+j} x \cup_r ({}'d_r y).$$

As the sequences (11.5.1.1) split, the spectral sequence for the J-adic filtration on $\omega_R^\bullet \otimes_R \overline{R}/J^{k+1}$ has vanishing differentials, hence degenerates into

$$E_r^{n,n'}\big((\omega_R^\bullet \otimes_R \overline{R}/J^{k+1})[-3], J^\bullet\big) = E_1^{n,n'} = \begin{cases} H^{n+n'-3}(\omega_R) \otimes_R J^n/J^{n+1}, & (0 \leqslant n \leqslant k) \\ 0, & \text{otherwise.} \end{cases}$$

To sum up, we have constructed products (for each $r \geqslant 1$)

$$\cup_r : E_r^{i,j}(X) \otimes_R {}'E_r^{i',j'}(Y) \longrightarrow \begin{cases} H^{i+i'+j+j'-3}(\omega_R) \otimes_R J^{i+i'}/J^{i+i'+1}, & (0 \leqslant i, i', \, i+i' \leqslant k) \\ 0, & \text{otherwise} \end{cases}$$

satisfying

$$(d_r x) \cup_r y + (-1)^{i+j} x \cup_r ({}'d_r y) = 0. \tag{11.5.3.2}$$

For $r = 1$, the cup product

$$\cup_1 : E_1^{0,j}(X) \otimes_R {}'E_1^{0,j'}(Y) = \widetilde{H}_f^j(X) \otimes_R \widetilde{H}_f^{j'}(Y) \longrightarrow H^{j+j'-3}(\omega_R)$$

is induced by the cup products

$$\widetilde{C}_f^\bullet(X) \otimes_R \widetilde{C}_f^\bullet(Y) \longrightarrow \omega_R^\bullet[-3].$$

11.5.4. The spectral sequences (11.5.2.1) induce decreasing filtrations

$$\mathrm{Fil}_\Gamma^r \widetilde{H}_f^i(X) := E_r^{0,i}(X), \quad {}'\mathrm{Fil}_\Gamma^r \widetilde{H}_f^i(Y) := {}'E_r^{0,i}(Y) \qquad (1 \leqslant r \leqslant k+1)$$

on

$$\widetilde{H}_f^i(X) = \mathrm{Fil}_\Gamma^1 \supset \mathrm{Fil}_\Gamma^2 \supset \cdots \supset \mathrm{Fil}_\Gamma^{k+1}$$
$$\widetilde{H}_f^i(Y) = {}'\mathrm{Fil}_\Gamma^1 \supset {}'\mathrm{Fil}_\Gamma^2 \supset \cdots \supset {}'\mathrm{Fil}_\Gamma^{k+1}.$$

As ${}'d_r = (-1)^r d_r$, we have ${}'\mathrm{Fil}_\Gamma^r = \mathrm{Fil}_\Gamma^r$. If we replace k by $k' \leqslant k$, then the terms Fil_Γ^r for $r \leqslant k'+1$ do not change.

11.5.5. Definition of the higher order height pairings. — The pairings

$$\widetilde{h}_{\pi,i,j}^{(r)} : \mathrm{Fil}_\Gamma^r\big(\widetilde{H}_f^i(X)\big) \otimes_R \mathrm{Fil}_\Gamma^r\big(\widetilde{H}_f^j(Y)\big) = E_r^{0,i}(X) \otimes_R {}'E_r^{0,j}(Y)$$
$$\xrightarrow{d_r^{0,i} \otimes \mathrm{id}} E_r^{r,i-r+1}(X) \otimes_R {}'E_r^{0,j}(Y) \xrightarrow{\cup_r} H^{i+j-2}(\omega_R) \otimes_R J^r/J^{r+1} \qquad (1 \leqslant r \leqslant k)$$

do not depend on $k \geqslant r$ (assuming that $(\mathrm{Fl}_k(\Gamma))$ holds), and for $r = 1$ coincide with the pairings defined in 11.1.4:

$$\widetilde{h}_{\pi,i,j}^{(1)} = \widetilde{h}_{\pi,i,j}.$$

As before, it is the case $i+j=2$ which is interesting, when we obtain pairings with values in $H^0(\omega_R)\otimes_R J^r/J^{r+1}$. We denote by

$$_L\mathrm{Ker}\Big(\widetilde{h}^{(r)}_{\pi,i,j}\Big)\subset \mathrm{Fil}^r_\Gamma\big(\widetilde{H}^i_f(X)\big),\quad \mathrm{Ker}_R\Big(\widetilde{h}^{(r)}_{\pi,i,j}\Big)\subset \mathrm{Fil}^r_\Gamma\big(\widetilde{H}^j_f(Y)\big)$$

the left (resp., right) kernel of $\widetilde{h}^{(r)}_{\pi,i,j}$.

11.5.6. Proposition

(i) *The pairings*

$$\widetilde{h}^{(r)}_{\pi,i,j}:\mathrm{Fil}^r_\Gamma\big(\widetilde{H}^i_f(X)\big)\otimes_R \mathrm{Fil}^r_\Gamma\big(\widetilde{H}^j_f(Y)\big)\longrightarrow H^{i+j-2}(\omega_R)\otimes_R J^r/J^{r+1}\qquad (1\leqslant r\leqslant k)$$

satisfy

$$\widetilde{h}^{(r)}_{\pi\circ s_{12},j,i}(y,x)=(-1)^{ij+r-1}\,\widetilde{h}^{(r)}_{\pi,i,j}(x,y).$$

(ii) *In particular, if $X=Y$, $X_v^+=Y_v^+$ $(v\in\Sigma)$, $\pi\circ s_{12}=c\cdot\pi$, $c=\pm1$, then the pairing*

$$\widetilde{h}^{(r)}_{\pi,1,1}:\mathrm{Fil}^r_\Gamma\big(\widetilde{H}^1_f(X)\big)\otimes_R \mathrm{Fil}^r_\Gamma\big(\widetilde{H}^1_f(X)\big)\longrightarrow H^{0}(\omega_R)\otimes_R J^r/J^{r+1}\qquad (1\leqslant r\leqslant k)$$

is symmetric (resp., skew-symmetric) if $c=(-1)^r$ (resp., $c=(-1)^{r-1}$).

(iii) *For each $r\leqslant k$,*

$$\mathrm{Fil}^{r+1}_\Gamma\big(\widetilde{H}^i_f(X)\big)\subseteq {}_L\mathrm{Ker}\big(\widetilde{h}^{(r)}_{\pi,i,j}\big),\quad \mathrm{Fil}^{r+1}_\Gamma\big(\widetilde{H}^j_f(Y)\big)\subseteq \mathrm{Ker}_R\big(\widetilde{h}^{(r)}_{\pi,i,j}\big).$$

(iv) *For each $r\leqslant k$ and $i+j=2$,*

$$\big(\mathrm{Fil}^r_\Gamma(\widetilde{H}^i_f(X))\big)_{R\text{-tors}}\subseteq {}_L\mathrm{Ker}(\widetilde{h}^{(r)}_{\pi,i,j}),\quad \big(\mathrm{Fil}^r_\Gamma(\widetilde{H}^j_f(Y))\big)_{R\text{-tors}}\subseteq \mathrm{Ker}_R(\widetilde{h}^{(r)}_{\pi,i,j}).$$

Proof. — The formula (11.5.3.2) yields

$$\begin{aligned}\widetilde{h}^{(r)}_{\pi\circ s_{12},j,i}(y,x)&=(d_ry)\cup_r x=-(-1)^j y\cup_r ({}'d_rx)=-(-1)^j y\cup_r(-1)^r d_rx\\&=(-1)^{j+r-1}(-1)^{(i+1)j}(d_rx)\cup_r y=(-1)^{ij+r-1}\widetilde{h}^{(r)}_{\pi,i,j}(x,y),\end{aligned}$$

proving (i) (and its special case (ii)). If $x\in\mathrm{Fil}^{r+1}_\Gamma$, then $d_rx=0$ and $\widetilde{h}^{(r)}(x,y)=(d_rx)\cup_r y=0$ vanishes by the definition of $\widetilde{h}^{(r)}$. If $y\in\mathrm{Fil}^{r+1}_\Gamma$, then $\widetilde{h}^{(r)}(x,y)=\pm\widetilde{h}^{(r)}(y,x)=0$ by (i); this proves (iii). Finally, (iv) follows from the fact that $H^0(\omega_R)\otimes_R J^r/J^{r+1}$ is torsion-free over R, by Lemma 2.8.8 and the assumption $(\mathrm{Fl}_k(\Gamma))$. □

11.5.7. Proposition. — *Assume that $(\mathrm{Fl}_1(\Gamma))$ holds, $\mathfrak{p}\in\mathrm{Spec}(R)$, $R_\mathfrak{p}$ has no embedded primes, and that for each $\mathfrak{q}\in Q=\{\mathfrak{q}\in\mathrm{Spec}(R)\mid \mathfrak{q}\subset\mathfrak{p},\mathrm{ht}(\mathfrak{q})=0\}$, the localized pairing $\pi_\mathfrak{q}:X_\mathfrak{q}\otimes_{R_\mathfrak{q}}Y_\mathfrak{q}\to(\omega_R^\bullet)_\mathfrak{q}(1)$ is a perfect duality (over $R_\mathfrak{q}$) and $(\forall v\in\Sigma)$ $(X_v^+)_\mathfrak{q}\perp\!\!\!\perp_{\pi_\mathfrak{q}}(Y_v^+)_\mathfrak{q}$. Then*

$$_L\mathrm{Ker}(\widetilde{h}^{(1)}_{\pi,i,2-i})_\mathfrak{p}=\mathrm{Ker}\left(\widetilde{H}^i_f(X)_\mathfrak{p}\longrightarrow\bigoplus_{\mathfrak{q}\in Q}\widetilde{H}^i_f(X)_\mathfrak{q}/\big(\mathrm{Fil}^2_\Gamma\big(\widetilde{H}^i_f(X)\big)\big)_\mathfrak{q}\right)\qquad(\forall i).$$

Proof. — For every $R_{\mathfrak{p}}$-module M of finite type,

$$M_{R_{\mathfrak{p}}\text{-tors}} = \mathrm{Ker}\Big(M \longrightarrow \bigoplus_{\mathfrak{q}\in Q} M_{\mathfrak{q}}\Big).$$

As $H^0(\omega_R)_{\mathfrak{p}}$ is torsion-free over $R_{\mathfrak{p}}$ (by Lemma 2.10.5 (v)), the right vertical arrow of the following commutative diagram is injective:

$$\begin{array}{rccc} \big(\widetilde{h}^{(1)}_{\pi,i,2-i}\big)_{\mathfrak{p}}: & \widetilde{H}^i_f(X)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \widetilde{H}^{2-i}_f(Y)_{\mathfrak{p}} & \longrightarrow & (H^0(\omega_R)\otimes_R \Gamma_R)_{\mathfrak{p}} \\ & \downarrow & & \downarrow \\ \big(\big(\widetilde{h}^{(1)}_{\pi,i,2-i}\big)_{\mathfrak{q}}\big): & \bigoplus_{\mathfrak{q}\in Q} \widetilde{H}^i_f(X)_{\mathfrak{q}} \otimes_{R_{\mathfrak{q}}} \widetilde{H}^{2-i}_f(Y)_{\mathfrak{q}} & \longrightarrow & \bigoplus_{\mathfrak{q}\in Q}(H^0(\omega_R)\otimes_R \Gamma_R)_{\mathfrak{q}}. \end{array}$$

This implies that it is sufficient to prove the result for $\mathfrak{p} = \mathfrak{q}$. The exact triangle in $D({}_R\mathrm{Mod})$

$$\widetilde{\mathbf{R}\Gamma}_f(X) \longrightarrow \mathbf{R}\mathrm{Hom}_R\big(\widetilde{\mathbf{R}\Gamma}_f(Y), \omega_R[-3]\big) \longrightarrow \mathrm{Err}(\Delta(X), \Delta(Y), \pi)$$

gives, after localizing at $\mathfrak{q}$, isomorphisms

$$\mathrm{adj}(\cup_1)_{\mathfrak{q}} : \widetilde{H}^{i+1}_f(X)_{\mathfrak{q}} \xrightarrow{\sim} \mathrm{Hom}_{R_{\mathfrak{q}}}\big(\widetilde{H}^{2-i}_f(Y)_{\mathfrak{q}}, H^0(\omega_R)_{\mathfrak{q}}\big).$$

As the height pairing is defined by the formula

$$\widetilde{h}^{(1)}_{\pi,i,2-i}(x,y) = \big(d^{0,i}_1 x\big) \cup_1 y,$$

it follows that

$${}_L\mathrm{Ker}\big(\widetilde{h}^{(1)}_{\pi,i,2-i}\big)_{\mathfrak{q}} = \mathrm{Ker}\big(d^{0,i}_1\big)_{\mathfrak{q}} = \big(\mathrm{Fil}^2_\Gamma\big(\widetilde{H}^i_f(X)\big)\big)_{\mathfrak{q}},$$

as claimed. □

11.5.8. If $\Delta = 0$, *i.e.*, $\Gamma \xrightarrow{\sim} \mathbf{Z}_p^{r(\Gamma)}$ ($r(\Gamma) \geqslant 1$), then ($\mathrm{Fl}_k(\Gamma)$) holds for $k \geqslant 1$. The cup products in (11.5.3.1) are compatible as k varies and induce a cup product

$$\cup : \widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(X)) \otimes_{\overline{R}} \widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(Y)^\iota) \longrightarrow (\omega^\bullet_R \otimes_R \overline{R})[-3],$$

the homotopy class of which does not depend on any choices. The corresponding spectral sequences

$$E_r(X) = E_r\big(\widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(X)), F^\bullet_J\big), \quad 'E_r(Y) = E_r\big(\widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(Y)^\iota), F^\bullet_J\big)$$

then satisfy

$$E^{i,j}_1(X) = \widetilde{H}^{i+j}_f(X) \otimes_R J^i/J^{i+1}, \quad 'E^{i,j}_1(Y) = \widetilde{H}^{i+j}_f(Y) \otimes_R J^i/J^{i+1}$$

for all $i \geqslant 0$. Similarly, the decreasing filtrations Fil^r_Γ on $\widetilde{H}^i_f(X)$ ($Z = X, Y$) and the higher height pairings $\widetilde{h}^{(r)}_{\pi,i,j}$ are defined for all $r \geqslant 1$, and the intersection

$$\mathrm{Fil}^\infty_\Gamma\big(\widetilde{H}^i_f(Z)\big) := \bigcap_{r\geqslant 1} \mathrm{Fil}^r_\Gamma\big(\widetilde{H}^i_f(Z)\big) \qquad (Z = X, Y)$$

is equal to the submodule of universal norms

$$\mathrm{Fil}^\infty_\Gamma\big(\widetilde{H}^i_f(Z)\big) = \mathrm{Im}\big(\widetilde{H}^i_{f,\mathrm{Iw}}(K_\infty/K, Z) \longrightarrow \widetilde{H}^i_f(Z)\big) \qquad (Z = X, Y).$$

***11.5.9. Proposition* (Dihedral case).** — *Assume that* ($\mathrm{Fl}_k(\Gamma)$) *holds,* K_∞/K^+ *is as in 10.3.5.1 and the conditions 10.3.5.1.1–10.3.5.1.5 are satisfied. Then*

(i) *Fix* $\tau \in \Gamma^+ - \Gamma$ *and its lift* $\widetilde{\tau} \in G(K_S/K^+)$*. Then the action of* $\mathrm{Ad}(\widetilde{\tau})_f$ *defines an involution* τ *on* $\widetilde{H}^i_f(X)$ *and* $\widetilde{H}^j_f(Y)$*, which does not depend on any choices.*

(ii) *The filtrations* Fil^r_Γ *on* $\widetilde{H}^i_f(X)$ *and* $\widetilde{H}^j_f(Y)$ *are* τ*-stable.*

(iii) *The pairings* $\widetilde{h}^{(r)}_{\pi,i,j}$ *satisfy*

$$\widetilde{h}^{(r)}_{\pi,i,j}(\tau x, \tau y) = (-1)^r\, \widetilde{h}^{(r)}_{\pi,i,j}(x, y) \qquad (1 \leqslant r \leqslant k).$$

Proof

(i) This follows from Lemma 8.6.4.4 (i).

(ii) The spectral sequence $E^{\bullet,\bullet}_r(X)$ (resp., ${}'E^{\bullet,\bullet}_r(Y)$) can be defined using the complex $\widetilde{C}^\bullet_f(G^+, \mathscr{F}_{\Gamma^+}(X))$ (resp., $\widetilde{C}^\bullet_f(G^+, \mathscr{F}_{\Gamma^+}(Y)^\iota)$) – equipped with the filtration induced by the J-adic filtration on the coefficient modules – and the filtered quasi-isomorphism sh^+_f from 10.3.5.5. This implies that the differentials commute with τ, proving the claim.

(iii) Lemma 10.3.5.6 implies that the products

$$\cup_r : E^{i,j}_r(X) \otimes_R {}'E^{i',j'}_r(Y) \longrightarrow H^{i+i'+j+j'-3}(\omega_R) \otimes_R J^{i+i'}/J^{i+i'+1} \quad (0 \leqslant i, i',\ i+i' \leqslant k)$$

satisfy

$$\tau x \cup_r \tau y = \iota(x \cup_r y) = (-1)^{i+i'} x \cup_r y,$$

hence

$$\begin{aligned}\widetilde{h}^{(r)}_{\pi,i,j}(\tau x, \tau y) &= d^{0,i}_r(\tau x) \cup_r \tau y = \tau(d^{0,i}_r(x)) \cup_r \tau y \\ &= (-1)^r d^{0,i}_r(x) \cup_r y = (-1)^r\, \widetilde{h}^{(r)}_{\pi,i,j}(x, y),\end{aligned}$$

for all $x \in E^{0,i}_r(X)$, $y \in {}'E^{0,j}_r(Y)$. □

11.5.10. Rubin's formula revisited. — Howard [**Ho3**, Thm. 3.4] generalized Rubin's result [**Ru**, Thm. 3.2 (ii)] to higher height pairings for $\mathbf{Z}_p$-extensions (defined as in 11.8.4 below). This result holds in our context, too: assume that we are in the situation of 11.5.1 with $\Sigma = S_f$ and that ($\mathrm{Fl}_k(\Gamma)$) holds. Each element $[x_f] \in \mathrm{Fil}^r_\Gamma \widetilde{H}^1_f(X) = E^{0,1}_r(X)$ ($r \leqslant k$) can be represented by a triple

$$x_{f,\mathrm{Iw}} = \big(x_{\mathrm{Iw}}, (x^+_{\mathrm{Iw},v}), (\alpha_v)\big) \in Z^{0,1}_r\big(\widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(X)), J^\bullet\big),$$

where

$$x_{\mathrm{Iw}} \in C^1_{\mathrm{cont}}(G_{K,S}, \mathscr{F}_\Gamma(X)), \quad x^+_{\mathrm{Iw},v} \in C^1_{\mathrm{cont}}(G_v, \mathscr{F}_\Gamma(X^+_v))$$
$$\alpha_v \in C^0_{\mathrm{cont}}(G_v, \mathscr{F}_\Gamma(X)) \quad (v \in S_f)$$

and

$$\begin{aligned}\widehat{e} := dx_{f,\mathrm{Iw}} &= \big(dx_{\mathrm{Iw}}, (dx^+_{\mathrm{Iw},v}), (-(x_{\mathrm{Iw}})_v + i^+_v(x^+_{\mathrm{Iw},v}) - d\alpha_v)\big) \\ &= \big(\widehat{a}, (\widehat{b}_v), (\widehat{c}_v)\big) \in \widetilde{C}^2_f(J^r \mathscr{F}_\Gamma(X)).\end{aligned}$$

Denote by $e = (a, (b_v), (c_v))$ the image of $\widehat{e}$ in

$$\widetilde{C}^2_f(J^r\mathscr{F}_\Gamma(X)/J^{r+1}\mathscr{F}_\Gamma(X)) = \widetilde{C}^2_f(X)\otimes_R J^r/J^{r+1}.$$

Similarly, an element $[y_f] \in \widetilde{H}^1_f(Y)$ can be represented by a 1-cocycle $y_f = (y, (y_v^+), (\mu_v))$, where

$$y \in C^1_{\mathrm{cont}}(G_{K,S}, Y), \quad y_v^+ \in C^1_{\mathrm{cont}}(G_v, Y_v^+), \quad \mu_v \in C^0_{\mathrm{cont}}(G_v, Y) \quad (v \in S_f)$$

and

$$dy_f = (dy, (dy_v^+), (-y_v + i_v^+(y_v^+) - d\mu_v)) = 0.$$

11.5.11. Proposition. — *Assume that* $dx_{\mathrm{Iw}} = 0$. *For each* $v \in S_f$, $i_v^-((x_{\mathrm{Iw}})_v) \in C^1_{\mathrm{cont}}(G_v, J^r\mathscr{F}_\Gamma(X_v^-))$ *is a 1-cocycle; denote by*

$$[(D^r x_{\mathrm{Iw}})_v] \in H^1_{\mathrm{cont}}(G_v, \mathscr{F}_\Gamma(X_v^-)\otimes_R J^r/J^{r+1}) = H^1_{\mathrm{cont}}(G_v, X_v^-)\otimes_R J^r/J^{r+1}$$

the cohomology class of its reduction modulo $J^{r+1}\mathscr{F}_\Gamma(X_v^-)$. *Then*

(i) *For* $1 \leqslant j < r \leqslant k$, $\widetilde{h}^{(j)}_{\pi,1,1}([x_f],[y_f]) = 0$.

(ii) $\widetilde{h}^{(r)}_{\pi,1,1}([x_f],[y_f]) = -\sum_{v\in S_f} \mathrm{inv}_v\left([(D^r x_{\mathrm{Iw}})_v]\cup[y_v^+]\right) \quad (r \leqslant k)$.

Proof

(i) This follows from Proposition 11.5.6 (iii).

(ii) By definition, $d_r^{0,1}([x_f])$ is represented by $e = (0, (b_v), (c_v))$; the same argument as in the proof of Proposition 11.3.15 (this time with $A = \varepsilon = 0$) shows that

$$\widetilde{h}^{(r)}_{\pi,1,1}([x_f],[y_f]) = \left[d_r^{0,1}(x_f)\cup_r y_f\right] = \sum_{v\in S_f}\mathrm{inv}_v\left([i_v^-(c_v)\cup y_v^+]\right)$$
$$= \sum_{v\in S_f}\mathrm{inv}_v([i_v^-(c_v)]\cup[y_v^+]).$$

However, for each $v \in S_f$,

$$[i_v^-(c_v)] = -[(i_v^-(x_{\mathrm{Iw}})_v) \pmod{J^{r+1}\mathscr{F}_\Gamma(X_v^-)}] = -[(D^r x_{\mathrm{Iw}})_v]. \quad \square$$

11.5.12. Localized height pairings. — If $\mathscr{S} \subset R$ is a multiplicative subset, let $\overline{\mathscr{S}} \subset \overline{R}$ be its inverse image under the augmentation map. The constructions in 11.1–11.2, 11.5 all work if we replace $(R, \overline{R}, J)$ by $\left(R_{\mathscr{S}}, \overline{R}_{\overline{\mathscr{S}}}, J_{\overline{\mathscr{S}}}\right)$.

11.5.13. As in 11.1.7, one can generalize the constructions from 11.5.2–11.5.5, by considering an intermediate field $K \subset L \subset K_\infty$ and replacing the ideal J by J_L and the assumption $(\mathrm{Fl}_k(\Gamma))$ by $(\mathrm{Fl}_k(\Gamma^L))$. The analogues of the spectral sequences (11.5.2.2) have initial terms

$$E_1^{i,j}(X) = \widetilde{H}^{i+j}_{f,\mathrm{Iw}}(L/K, X)\otimes_R (J^L)^i/(J^L)^{i+1},$$
$$'E_1^{i,j}(Y) = \widetilde{H}^{i+j}_{f,\mathrm{Iw}}(L/K, Y)^\iota\otimes_R (J^L)^i/(J^L)^{i+1} \qquad (0 \leqslant i \leqslant k)$$

As in 11.5.4, they define decreasing filtrations

$$\mathrm{Fil}^r_\Gamma \widetilde{H}^i_{f,\mathrm{Iw}}(L/K,X) := E^{0,i}_r(X), \quad {}'\mathrm{Fil}^r_\Gamma \widetilde{H}^i_{f,\mathrm{Iw}}(L/K,Y)^\iota := {}'E^{0,i}_r(Y) \qquad (1 \leqslant r \leqslant k+1)$$

on

$$\widetilde{H}^i_{f,\mathrm{Iw}}(L/K,X) = E^{0,i}_1(X), \quad \widetilde{H}^i_{f,\mathrm{Iw}}(L/K,Y)^\iota = {}'E^{0,i}_1(Y)$$

and higher height pairings

$$\widetilde{h}^{(r)}_{\pi,L/K,i,j} : \mathrm{Fil}^r_\Gamma \widetilde{H}^i_{f,\mathrm{Iw}}(L/K,X) \otimes_{R[\![\Gamma_L]\!]} {}'\mathrm{Fil}^r_\Gamma \widetilde{H}^i_{f,\mathrm{Iw}}(L/K,Y)^\iota \longrightarrow H^{i+j-2}(\omega_R) \otimes_R R[\![\Gamma_L]\!] \otimes_R (J^L)^r/(J^L)^{r+1}, \qquad (1 \leqslant r \leqslant k)$$

which generalize the pairings (11.1.7.5).

11.6. Bockstein spectral sequence and descent

In the case when $\Gamma \xrightarrow{\sim} \mathbf{Z}_p$ $(\gamma \mapsto 1)$ we have $\overline{R} = R[\![\Gamma]\!] \xrightarrow{\sim} R[\![T]\!]$ $(T = \gamma - 1)$ and the height pairings $\widetilde{h}_{\pi,i,j}$ are defined in terms of the Bockstein maps

$$\beta_i : H^i(M^\bullet/TM^\bullet) \longrightarrow H^{i+1}(M^\bullet/TM^\bullet),$$

i.e., the coboundary maps associated to the exact sequence of complexes

$$0 \longrightarrow M^\bullet/TM^\bullet \xrightarrow{T} M^\bullet/T^2M^\bullet \longrightarrow M^\bullet/TM^\bullet \longrightarrow 0,$$

where

$$M^\bullet = \widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(X)) = \widetilde{C}^\bullet_f((X \otimes_R R[\![T]\!]) < -1 >).$$

As observed in 11.5.2, the maps β_i are equal to the differentials d_1 in the spectral sequence $E_r(X)$ associated to the filtration $F^\bullet_J$ induced on $M^\bullet$ by the T-adic filtration on $\mathscr{F}_\Gamma(X)$. In our case, $F^\bullet_J$ coincides with the T-adic filtration on $M^\bullet$, hence $E_r(X)$ becomes, essentially, the "Bockstein spectral sequence". In this section we list some elementary properties of this spectral sequence; they play an important role in a descent formalism relating the "leading term of $\det_{\overline{R}}(M^\bullet) = \det_{\overline{R}}\left(\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(K_\infty/K,X)\right)$" to certain cohomological invariants over K.

11.6.1. Bockstein spectral sequence

11.6.1.1. We shall work with triples (A,Λ,T), where A is a (non-zero) Noetherian ring, Λ is an A-algebra, $T \in \Lambda$ is an element of Λ not dividing zero and such that the composite map $A \to \Lambda \to \Lambda/T\Lambda$ is an isomorphism (we use this map to identify A with $\Lambda/T\Lambda$). A typical example is provided by $(R, R[\![T]\!], T)$.

11.6.1.2. If (A,Λ,T) are as in 11.6.1.1 and $\mathfrak{p} \in \mathrm{Spec}(A)$, denote by $\overline{\mathfrak{p}} \in \mathrm{Spec}(\Lambda)$ its inverse image under the projection $\Lambda \to \Lambda/T\Lambda = A$. Then the triple $(A_\mathfrak{p}, \Lambda_{\overline{\mathfrak{p}}}, T)$ is again of the type considered in 11.6.1.1.

11.6.1.3. We say that a Λ-module M is T-*flat* if the multiplication by T on M is injective. This is equivalent to the vanishing of $\mathrm{Tor}_1^{\Lambda}(M, A) = 0$, by the exact sequence of Tor's associated to

$$0 \longrightarrow \Lambda \xrightarrow{T} \Lambda \longrightarrow A \longrightarrow 0.$$

11.6.1.4. From now on (until the end of 11.6), let $M^\bullet$ be a cohomologically bounded complex of T-flat Λ-modules. It is equipped with the T-adic filtration $F^iM^\bullet = T^iM^\bullet$ (*i.e.*, $F^iM^\bullet = M^\bullet$ for $i \leqslant 0$), which has graded quotients isomorphic to $\mathrm{gr}^i_F(M^\bullet) \xrightarrow{\sim} M^\bullet/TM^\bullet$ ($i \geqslant 0$). By definition, the corresponding spectral sequence $E_r = E_r(M^\bullet, F^\bullet)$ satisfies $E_r^{i,j} = Z_r^{i,j}/B_r^{i,j}$, where

$$\begin{aligned} Z_r^{i,j} &= \{a \in F^iM^{i+j} \mid da \in F^{i+r}M^{i+j+1}\} \\ B_r^{i,j} &= \big(d(F^{i-r+1}M^{i+j-1}) + F^{i+1}M^{i+j}\big) \cap Z_r^{i,j}, \end{aligned} \qquad (r \geqslant 1)$$

hence multiplication by T induces isomorphisms

$$\begin{aligned} &T_* : Z_r^{i,j} \xrightarrow{\sim} Z_r^{i+1,j-1} && (i \geqslant 0), \\ &T_* : B_r^{i,j} \xrightarrow{\sim} B_r^{i+1,j-1} && \\ &T_* : E_r^{i,j} \xrightarrow{\sim} E_r^{i+1,j-1}, && (i \geqslant r-1) \end{aligned}$$

monomorphisms

$$T_* : B_r^{i,j} \hookrightarrow B_r^{i+1,j-1} \qquad (0 \leqslant i < r-1)$$

and epimorphisms

$$T_* : E_r^{i,j} \twoheadrightarrow E_r^{i+1,j-1} \qquad (0 \leqslant i < r-1).$$

In particular, multiplication by T^{-i} induces isomorphisms

$$\begin{aligned} &(T^{-i})_* : Z_r^{i,j} \xrightarrow{\sim} Z_r^{i+j} := M^{i+j} \cap d^{-1}(T^rM^{i+j+1}) && (i \geqslant 0) \\ &(T^{-i})_* : B_r^{i,j} \xrightarrow{\sim} B_r^{i+j} := (T^{-i})_* B_r^{i,j} && (i \geqslant r-1). \end{aligned}$$

This means that, in the "stable range" $i \geqslant r-1$, the terms $W_r^{i,j}$ ($W = Z, B, E$) depend only on $i+j$ and r, as

$$(T^{-i})_* : (Z_r^{i,j}, B_r^{i,j}, E_r^{i,j}) \xrightarrow{\sim} (Z_r^{i+j}, B_r^{i+j}, E_r^{i+j}) \qquad (i \geqslant r-1) \tag{11.6.1.1}$$

(where $E_r^n := Z_r^n/B_r^n$) is an isomorphism. The differentials $d_r^{i,j}$ define, under the isomorphisms (11.6.1.1), "stable" differentials

$$d_r^n : E_r^n \longrightarrow E_r^{n+1},$$

which depend on T. If we replace T by uT ($u \in \Lambda^*$), then the isomorphism (11.6.1.1) is multiplied by u^{-i}; this implies that d_r^n is replaced by $(\overline{u})^{-r}d_r^n$, where $\overline{u} = u \pmod{T\Lambda} \in A^*$.

The simply graded "stable" spectral sequence (E_r^n, d_r^n) is usually referred to as the "Bockstein spectral sequence" It arises from the exact couple

$$H^*(M^\bullet) \xrightarrow{T} H^*(M^\bullet) \longrightarrow H^*(M^\bullet/TM^\bullet) \tag{11.6.1.2}$$

(cf. [**We**, §5.9.9–5.9.12]).

11.6.1.5. In the special case when

$$(\forall j \neq j_0, j_0+1) \qquad E_1^{0,j} = 0,$$

we have

$$(\forall i+j \neq j_0, j_0+1, \, \forall r \geqslant 1) \qquad E_r^{i,j} = 0$$

and the maps

$$T_* : E_r^{i,j} \longrightarrow E_r^{i+1,j-1} \qquad (r \geqslant 1)$$

are isomorphisms for all pairs $(i,j) \neq (0, j_0+1)$.

11.6.2. In concrete terms, the spectral sequence E_r^n can be described as follows:

$$E_1^n = H^n(M^\bullet/TM^\bullet)$$

and

$$d_1^n : H^n(M^\bullet/TM^\bullet) \longrightarrow H^{n+1}(M^\bullet/TM^\bullet)$$

is the coboundary map associated to

$$0 \longrightarrow M^\bullet/TM^\bullet \xrightarrow{T} M^\bullet/T^2M^\bullet \longrightarrow M^\bullet/TM^\bullet \longrightarrow 0.$$

Put

$$H^n = H^n(M^\bullet)$$

and let

$$c_r^n : T^{r-1}H^n \cap H^n[T] \longrightarrow H^n/(TH^n + H^n[T^{r-1}]) \qquad (r \geqslant 1)$$

be the map sending $T^{r-1}a$ to the class of a. For $r=0$, let

$$c_0^n : H^n \xrightarrow{T} H^n$$

be the multiplication by T. The description of E_r^n in terms of the exact couple (11.6.1.2) shows that, for each $r \geqslant 1$, there are exact sequences

(11.6.2.1)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^n/(TH^n + H^n[T^{r-1}]) & \longrightarrow & E_r^n & \xrightarrow{j_r^n} & T^{r-1}H^{n+1} \cap H^{n+1}[T] & \longrightarrow & 0 \\
 & & \downarrow 0 & & \downarrow d_r^n & & \downarrow 0 & & \\
0 & \longrightarrow & H^{n+1}/(TH^{n+1} + H^{n+1}[T^{r-1}]) & \xrightarrow{i_r^{n+1}} & E_r^{n+1} & \longrightarrow & T^{r-1}H^{n+2} \cap H^{n+2}[T] & \longrightarrow & 0,
\end{array}$$

in which $d_r^n = i_r^{n+1} \circ c_r^{n+1} \circ j_r^n$. Indeed, for $r=1$, these are nothing but the exact sequences

$$0 \longrightarrow H^n/TH^n \longrightarrow H^n(M^\bullet/TM^\bullet) \longrightarrow H^{n+1}[T] \longrightarrow 0$$

coming from the cohomology sequence of

$$0 \longrightarrow M^\bullet \xrightarrow{T} M^\bullet \longrightarrow M^\bullet/TM^\bullet \longrightarrow 0.$$

Given (11.6.2.1) for $r \geqslant 1$, it can be viewed as an exact sequence of complexes

$$0 \longrightarrow \bigoplus_n (H^n/(TH^n + H^n[T^{r-1}]))[-n] \longrightarrow (E_r^\bullet, d_r) \longrightarrow \bigoplus_n (T^{r-1}H^n \cap H^n[T])[-n+1] \longrightarrow 0,$$

for which the corresponding boundary map in $D({}_A\mathrm{Mod})$ is equal to

$$(c_r^n[-n+1])_n : \bigoplus_n (T^{r-1}H^n \cap H^n[T])[-n+1] \longrightarrow \bigoplus_n (H^n/(TH^n + H^n[T^{r-1}]))[-n+1].$$

Taking cohomology gives exact sequences

$$0 \longrightarrow \mathrm{Coker}(c_r^n) \longrightarrow H^n(E_r^\bullet, d_r) \longrightarrow \mathrm{Ker}(c_r^{n+1}) \longrightarrow 0, \tag{11.6.2.2}$$

i.e., (11.6.2.1) for $r+1$. Noting that, for each $r \geqslant 1$, multiplication by T^{r-1} induces an isomorphism

$$T^{r-1} : H^n/(TH^n + H^n[T^{r-1}]) \xrightarrow{\sim} T^{r-1}H^n/T^rH^n,$$

we see that (11.6.2.1) can be written as

$$0 \longrightarrow T^{r-1}H^n/T^rH^n \longrightarrow E_r^n \longrightarrow T^{r-1}(H^{n+1}[T^r]) \longrightarrow 0,$$

and

$$c_r^n : T^{r-1}(H^n[T^r]) \hookrightarrow T^{r-1}H^n \twoheadrightarrow T^{r-1}H^n/T^rH^n.$$

11.6.3. Lemma. — *If each $H^n = H^n(M^\bullet)$ is a Λ-module of finite type, then*

$$(\exists r_0)\,(\forall r \geqslant r_0)\,(\forall n) \quad E_r^n = E_{r_0}^n \xrightarrow{\sim} T^{r_0-1}(H^n[T^{r_0}]) \xrightarrow{\sim} T^{r-1}(H^n[T^r]).$$

We denote this stationary value by E_∞^n.

Proof. — As $M^\bullet$ is cohomologically bounded and each H^n is a Noetherian Λ-module, there exists $r_0 \geqslant 1$ such that $H^n[T^\infty] = H^n[T^{r_0-1}]$, for all n. Then

$$E_r^n = H^n/(TH^n + H^n[T^{r-1}]) = H^n/(TH^n + H^n[T^{r_0-1}]) = E_{r_0}^n$$

for all $r \geqslant r_0$. □

11.6.4. Lemma 11.6.3 states that, if H is a Λ-module of finite type, then the inductive system of A-modules of finite type $T^{r-1}H/T^rH$ (with transition maps given by T) becomes stationary. Its limit value is isomorphic to

$$\varinjlim_r T^{r-1}H/T^rH \xrightarrow{\sim} H/(TH + H[T^\infty]).$$

11.6.5. Lemma. — *Assume that A is a local ring,* $\dim(A) = 0$ *and each $H^n = H^n(M^\bullet)$ is a Λ-module of finite type. Then:*

(i) *The following conditions are equivalent:*

$$(\forall n) \quad \ell_\Lambda(H^n) < \infty \iff (\exists r_0)(\forall n) \quad E_{r_0}^n = 0 \iff (\forall n) \quad E_\infty^n = 0.$$

(ii) *If the equivalent conditions of (i) are satisfied, then*

$$(\forall r \geqslant 1) \quad 0 = \sum_n (-1)^n \ell_A(E_r^n)$$

$$\sum_n (-1)^n \ell_\Lambda(H^n) = \sum_{r\geqslant 1} \sum_n (-1)^n n\, \ell_A(E_r^n).$$

In particular, if the complex $(E_1^\bullet, d_1)$ *is acyclic, then*

$$\sum_n (-1)^n \ell_\Lambda(H^n(M^\bullet)) = \sum_n (-1)^n n\, \ell_A(H^n(M^\bullet/TM^\bullet)).$$

Proof

(i) If each H^n has finite length over Λ, then there is $r_0 \geqslant 1$ such that $T^{r_0-1}H^n = 0$ for all n; then $E_r^n = 0$ for all $r \geqslant r_0$. Conversely, if $E_\infty^n = 0$, then $T^r H^n = T^{r+1}H^n$ for some r; Nakayama's Lemma then implies $T^r H^n = 0$, hence $\ell_\Lambda(H^n) < \infty$.

(ii) By (i), $E_r^n = 0$ for $r \geqslant r_0$ (and all n). The following exact sequences of Λ-modules of finite length

$$0 \longrightarrow T^{r-1}H^n/T^rH^n \longrightarrow E_r^n \longrightarrow T^{r-1}(H^{n+1}[T^r]) \longrightarrow 0$$

$$0 \longrightarrow H^{n+1}[T^{r-1}] \longrightarrow H^{n+1}[T^r] \xrightarrow{T^{r-1}} H^{n+1}[T] \longrightarrow H^{n+1}[T]/(T^{r-1}H^{n+1}[T^r]) \longrightarrow 0$$

$$0 \longrightarrow H^{n+1}[T^r] \longrightarrow H^{n+1} \xrightarrow{T^r} T^rH^{n+1} \longrightarrow 0$$

imply that

$$\ell_A(E_r^n) = \ell_\Lambda(E_r^n) = \ell_\Lambda(T^{r-1}H^n/T^rH^n) + \ell_\Lambda(T^{r-1}H^{n+1}/T^rH^{n+1}),$$

hence

$$\sum_n (-1)^n \ell_A(E_r^n) = 0 \qquad (r \geqslant 1)$$

and

$$\sum_n (-1)^n n\, \ell_A(E_r^n) = \sum_n (-1)^n \ell_\Lambda(T^{r-1}H^n/T^rH^n)$$

$$\sum_{r\geqslant 1} \sum_n (-1)^n n\, \ell_A(E_r^n) = \sum_n (-1)^n \ell_\Lambda(H^n). \qquad \square$$

11.6.6. Lemma - Definition. — *Assume that A is a local ring,* $\dim(A) = 1$ *and Λ is catenary. Then the following properties of a Λ-module of finite type H are equivalent:*

$$\ell_A\Big(\varinjlim_r T^{r-1}H/T^rH\Big) < \infty \iff \operatorname{codim}_\Lambda(\operatorname{supp}(H)) \geqslant 1.$$

If they are satisfied, we put

$$a_A(H) := \ell_A\Big(\varinjlim_r T^{r-1}H/T^rH\Big).$$

If $K^\bullet$ is a cohomologically bounded complex of Λ-modules such that each cohomology module $H^n(K^\bullet)$ is of finite type over Λ and satisfies (i), we put

$$a_A(H(K^\bullet)) := \sum_n (-1)^n a_A(H^n(K^\bullet)).$$

Proof. — Set

$$Q = \{\mathfrak{q} \in \operatorname{Spec}(A) \mid \operatorname{ht}(\mathfrak{q}) = 0\}, \quad \overline{Q} = \{\overline{\mathfrak{q}} \in \operatorname{Spec}(\Lambda) \mid \mathfrak{q} \in Q\}$$

(using the notation of 11.6.1.2). For each $\overline{\mathfrak{q}} \in \overline{Q}$ we have $\dim(\Lambda_{\overline{\mathfrak{q}}}) = 1$; Lemma 11.6.5 (i) applied to $(A_{\mathfrak{q}}, \Lambda_{\overline{\mathfrak{q}}}, H_{\overline{\mathfrak{q}}})$ shows that the following conditions are equivalent:

$$\begin{aligned}(\forall \mathfrak{q} \in Q) \quad \ell_{\Lambda_{\overline{\mathfrak{q}}}}(H_{\overline{\mathfrak{q}}}) < \infty \iff (\forall \mathfrak{q} \in Q) \quad & \Big(\varinjlim_r T^{r-1}H/T^rH\Big)_{\mathfrak{q}} = 0 \\ \iff & \ell_A\Big(\varinjlim_r T^{r-1}H/T^rH\Big) < \infty.\end{aligned}$$

In order to conclude the proof of the Lemma, it remains to show that each minimal prime ideal $\mathfrak{q}_\Lambda \subset \Lambda$ is contained in some $\overline{\mathfrak{q}} \in \overline{Q}$. As Λ is catenary, $\dim_\Lambda(\Lambda/\mathfrak{q}_\Lambda) = 2$ and $\dim_\Lambda(\Lambda/\mathfrak{q}_\Lambda + T\Lambda) \geqslant 1$, hence there is $P \in \operatorname{Spec}(\Lambda)$ with $P \neq \mathfrak{m}_\Lambda$ and $P \supseteq \mathfrak{q}_\Lambda + T\Lambda$. This implies that $P = \overline{\mathfrak{q}}$ for some $\mathfrak{q} \in \operatorname{Spec}(A)$, $\mathfrak{q} \neq \mathfrak{m}_A$, hence $\mathfrak{q} \in Q$.

11.6.7. If A is a discrete valuation ring, then Λ is a regular local ring (hence a unique factorization domain) of dimension $\dim(\Lambda) = 2$ and T is an irreducible element of Λ. If H is a finitely generated torsion Λ-module, then its "characteristic power series"

$$\operatorname{char}_\Lambda(H) \in (\Lambda - \{0\})/\Lambda^* \subset \operatorname{Frac}(\Lambda)^*/\Lambda^*$$

(defined as in 9.1.2) is of the form

$$\operatorname{char}_\Lambda(H) = T^{r(H)} \operatorname{char}^*_\Lambda(H),$$

where

$$r(H) = \ell_{\Lambda_{(T)}}\left(H_{(T)}\right), \quad \operatorname{char}^*_\Lambda(H) \notin T\Lambda/\Lambda^*.$$

We define

$$\operatorname{char}^*_\Lambda(H)(0) := \operatorname{char}^*_\Lambda(H) \pmod{T\Lambda} \in A/A^* = (\Lambda/T\Lambda)/\Lambda^* \subset \operatorname{Frac}(A)^*/A^*.$$

The integer $r(H) \geqslant 0$ is "the order of vanishing" of $\operatorname{char}_\Lambda(H)$ at $T = 0$, while $\operatorname{char}^*_\Lambda(H)(0)$ is its "leading term". These invariants (which do not change if we multiply T by a unit of Λ) are usually studied in Iwasawa theory for $A = \mathbf{Z}_p$, $\Lambda = \mathbf{Z}_p[\![T]\!]$. □

11.6.8. Lemma. — *If A is a discrete valuation ring and H a finitely generated torsion Λ-module, then*

$$a_A(H) = \operatorname{ord}_A(\operatorname{char}^*_\Lambda(H)(0)).$$

Proof. — If H is pseudo-null over Λ, then $T^r H = T^{r+1} H$ for some r, hence $a_A(H) = 0$. As $a_A(-)$ is additive in exact sequences, the structure theory for torsion Λ-modules implies that it is enough to consider the case of a cyclic module: $H = \Lambda/T^n f\Lambda$, where $f \in \Lambda$, $f(0) \neq 0$ (where $f(0) := f \pmod{T} \in \Lambda/T\Lambda = A$). As $T^n H \xrightarrow{\sim} \Lambda/f\Lambda$ and $a_A(H) = a_A(T^n H)$, we can assume that $n = 0$. Then the map $T : H \to H$ is injective, hence

$$a_A(H) = \ell_A(H/TH) = \ell_A(\Lambda/(f,T)\Lambda) = \ell_A(A/f(0)A) = \mathrm{ord}_A(f(0)). \quad \square$$

11.6.9. Euler-Poincaré characteristic - notation. — Let B be a ring and $K^\bullet$ a bounded (resp., cohomologically bounded) complex of B-modules. If each K^n (resp., each $H^n(K^\bullet)$) is a B-module of finite length, put

$$\chi_B(K^\bullet) := \sum_n (-1)^n \ell_B(K^n)$$

resp.,

$$\chi_B(H(K^\bullet)) := \sum_n (-1)^n \ell_B(H^n(K^\bullet)).$$

If $\chi_B(K^\bullet)$ is defined, so is $\chi_B(H(K^\bullet))$, and the two integers coincide.

11.6.10. Proposition. — *Assume that A is a local ring,* $\dim(A) = 1$, Λ *is catenary and each* $H^n = H^n(M^\bullet)$ *is a Λ-module of finite type with* $\mathrm{codim}_\Lambda(\mathrm{supp}(H^n)) \geqslant 1$. *Then:*

(i) $(\exists r_0 \geqslant 1)\,(\forall n)\ \ \ell_A(E^n_{r_0}) < \infty$. *Fix such* r_0.

(ii) $(\forall r \geqslant r_0)\,(\forall n)\ \ \ell_A(E^n_r) < \infty$.

(iii) $(\forall r \geqslant r_0)\ \chi_A(E^\bullet_r, d_r) = \chi_A(E^\bullet_{r+1}, d_{r+1})$.

(iv) $(\forall r \geqslant r_0)\quad a_A(H(M^\bullet)) = \chi_A(E^\bullet_r, d_r) = \sum_n (-1)^n \big(\ell_A(\mathrm{Coker}(c^n_{r-1})) - \ell_A(\mathrm{Ker}(c^n_{r-1}))\big)$.

(v) *If* $(\forall n)\ \ \ell_A(E^n_1) < \infty$, *then*

$$a_A(H(M^\bullet)) = \sum_n (-1)^n \left(\ell_A(H^n/TH^n) - \ell_A(H^n[T])\right).$$

(vi) *If* $(E^\bullet_1, d_1)_{\mathfrak{q}}$ *is acyclic for all minimal prime ideals* $\mathfrak{q} \subset A$, *then*

$$a_A(H(M^\bullet)) = \sum_n (-1)^n \big(\ell_A(\mathrm{Coker}(H^n[T] \longrightarrow H^n/TH^n)) - \ell_A(\mathrm{Ker}(H^n[T] \longrightarrow H^n/TH^n))\big).$$

Proof. — The statement (i) follows from Lemma 11.6.5(i) applied to $(A_{\mathfrak{q}}, \Lambda_{\overline{\mathfrak{q}}}, M^\bullet_{\overline{\mathfrak{q}}})$ instead of $(A, \Lambda, M^\bullet)$, for all minimal prime ideals $\mathfrak{q} \subset A$. The statements (ii) and (iii) are immediate consequences of (i), as $H^*(E^\bullet_r) = E^*_{r+1}$. For (iv), note that there exists $s \geqslant r_0$ such that $E^n_s = E^n_\infty$ (for all n). Then

$$a_A(H(M^\bullet)) = \sum_n (-1)^n \ell_A(E^n_\infty) = \chi_A(E^\bullet_s)$$

(by Lemma 11.6.3) and we have

$$\chi_A(E_s^\bullet) = \chi_A(E_r^\bullet) = \sum_n (-1)^n \big(\ell_A(\mathrm{Coker}(c_{r-1}^n)) - \ell_A(\mathrm{Ker}(c_{r-1}^n))\big)$$

for all $r \geqslant r_0$, by (iii) and (11.6.2.2). The statements (v) and (vi) are special cases of (iv), for $r = r_0 = 1$ and $r = r_0 = 2$, respectively.

11.6.11. In the special case when $A = \mathbf{Z}_p$, $\Lambda = \mathbf{Z}_p[\![T]\!]$ and $H^n = 0$ for $n \neq 0$, then the statements (v) and (vi) in Proposition 11.6.10 boil down to the following well-known facts about the finitely generated torsion Λ-module $H = H^0$:

If $\mathrm{char}_\Lambda(H)(0) \neq 0$, then

$$\mathrm{ord}_p\left(\mathrm{char}_\Lambda(H)(0)\right) = \mathrm{ord}_p\left(|H_\Gamma|/|H^\Gamma|\right).$$

If the canonical map $c : H^\Gamma \longrightarrow H_\Gamma$ has finite kernel and cokernel, then

$$\mathrm{ord}_p\left(\mathrm{char}_\Lambda^*(H)(0)\right) = \mathrm{ord}_p\left(|\mathrm{Coker}(c)|/|\mathrm{Ker}(c)|\right). \quad \square$$

11.6.12. If A is a discrete valuation ring, then

$$a_A(H(M^\bullet)) = \mathrm{ord}_A\left(\mathrm{char}_\Lambda^*(H(M^\bullet))(0)\right),$$

where

(11.6.12.1) $$\mathrm{char}_\Lambda^*(H(M^\bullet))(0) := \prod_n \left(\mathrm{char}_\Lambda^*(H^n(M^\bullet))(0)\right)^{(-1)^n}$$

(by Lemma 11.6.8). The R.H.S. of (11.6.12.1) can be reinterpreted as follows: as Λ is regular, $M^\bullet$ is a perfect complex over Λ, with Λ-torsion cohomology. This implies that there is a *canonical* (up to a sign) isomorphism

$$\det_\Lambda(M^\bullet) \otimes_\Lambda \mathrm{Frac}(\Lambda) \xrightarrow{\sim} \mathrm{Frac}(\Lambda);$$

it identifies $\det_\Lambda(M^\bullet)$ with the invertible Λ-module

$$\Lambda \cdot \mathrm{char}_\Lambda(H(M^\bullet))^{-1} \subset \mathrm{Frac}(\Lambda),$$

where $\mathrm{char}_\Lambda(H(M^\bullet))$ is defined analogously to (11.6.12.1). Similarly, in the situation of Proposition 11.6.10, for each $r \geqslant r_0 - 1$ there is a canonical (up to a sign) isomorphism

$$\det_A(E_r^\bullet, d_r) \otimes_A \mathrm{Frac}(A) \xrightarrow{\sim} \mathrm{Frac}(A);$$

it identifies $\det_A(E_r^\bullet, d_r)$ with

$$A \cdot \mathrm{char}_A(H(E_r^\bullet, d_r))^{-1} \subset \mathrm{Frac}(A),$$

where

$$\mathrm{char}_A(H(E_r^\bullet, d_r)) = \prod_n \left(\mathrm{char}_A(E_{r+1}^n)\right)^{(-1)^n}.$$

This means that we can reformulate Proposition 11.6.10 (iv) in more suggestive terms as follows:

(11.6.12.2) $$\det_\Lambda^*(M^\bullet)(0) = \det_A(E_r^\bullet, d_r) \qquad (\forall r \geqslant r_0 - 1).$$

This formulation is used, in a special case, in [**Bu-Gr**] (the idea of using determinants in this context is due to Kato [**Ka1**]). It is not clear whether one can make sense of (11.6.12.2) for more general one-dimensional local rings A; the problem is (even for $r = 1$) to find appropriate conditions guaranteeing that the complex $(E_r^\bullet, d_r)$ is perfect over A.

11.7. Formulas of the Birch and Swinnerton-Dyer type

Perrin-Riou [**PR2, PR3, PR4**] and Schneider [**Sch2, Sch3**] computed the leading term of the characteristic power series of a Selmer group arising in Iwasawa theory of abelian varieties. Their results (further generalized in [**PR5**] and [**Pl**]) involved p-adic variants of various terms appearing in the conjecture of Birch and Swinnerton-Dyer. In this section we deduce similar formulas in our context, as an application of the formalism from 11.6.

11.7.1. Assume that we are in the situation of 11.1.3, with Γ isomorphic to $\mathbf{Z}_p$ (hence the condition $(\mathrm{Fl}_k(\Gamma))$ from 11.5.1 holds for all $k \geqslant 1$). Fixing an isomorphism $\Gamma \xrightarrow{\sim} \mathbf{Z}_p$, we identify $\overline{R} = R[\![\Gamma]\!]$ with $R[\![T]\!]$ in the usual way ($T = \gamma - 1$).

For a prime ideal $\mathfrak{p}_0 \in \mathrm{Spec}(R)$ consider the following conditions:

11.7.1.1. $\pi_{\mathfrak{p}_0} : X_{\mathfrak{p}_0} \otimes_{R_{\mathfrak{p}_0}} Y_{\mathfrak{p}_0} \longrightarrow (\omega_R^\bullet)_{\mathfrak{p}_0}(1)$ is a perfect duality (over $R_{\mathfrak{p}_0}$).

11.7.1.2. $(\forall v \in \Sigma)$ $(X_v^+)_{\mathfrak{p}_0} \perp\!\!\!\perp_{\pi_{\mathfrak{p}_0}} (Y_v^+)_{\mathfrak{p}_0}$.

11.7.1.3. $\tau_{\leqslant 0}\, \widetilde{\mathbf{R}\Gamma}_f(X)_{\mathfrak{p}_0} \xrightarrow{\sim} \tau_{\leqslant 0}\, \widetilde{\mathbf{R}\Gamma}_f(Y)_{\mathfrak{p}_0} \xrightarrow{\sim} 0$ in $D\big({}_{R_{\mathfrak{p}_0}}\mathrm{Mod}\big)$.

11.7.2. The spectral sequence $E_r^{i,j}(X)$ (resp., ${}'E_r^{i,j}(Y)$) from 11.5.8 is equal to the spectral sequence $E_r^{i,j}(M^\bullet, T^\bullet)$ (resp., $E_r^{i,j}((N^\bullet)^\iota, T^\bullet)$) from 11.6.1.4 for $A = R$, $\Lambda = R[\![T]\!]$, where

$$M^\bullet = \widetilde{C}_f^\bullet(\mathscr{F}_\Gamma(X)) = \widetilde{C}_f^\bullet((X \otimes_R R[\![T]\!]) < -1 >)$$

$$(N^\bullet)^\iota = \widetilde{C}_f^\bullet(\mathscr{F}_\Gamma(Y)^\iota) = \widetilde{C}_f^\bullet((Y \otimes_R R[\![T]\!]) < 1 >).$$

As in 11.6.1.4, we have the corresponding stable (simply graded) spectral sequences $E_r^n(X), {}'E_r^n(Y)$ and the surjective maps

$$(T^{-i})_* : E_r^{i,j}(X) \twoheadrightarrow E_r^{i+j}(X), \quad (T^{-i})_* : {}'E_r^{i,j}(Y) \twoheadrightarrow {}'E_r^{i+j}(Y), \qquad (i \geqslant 0).$$

As

$$\iota(T) = (1+T)^{-1} - 1 \equiv -T \pmod{T^2},$$

we have

$$({}'E_r^n(Y), {}'d_r^n) = (E_r^n(Y), (-1)^r d_r^n).$$

The cup products $\cup_r$ from 11.5.3 are compatible with T_*, hence induce "stable" cup products $\cup_r^{\mathrm{st}}$ fitting to the following commutative diagram:

(11.7.2.1)
$$\begin{array}{lccc} \cup_r : & E_r^{i,j}(X) \otimes_R {}'E_r^{i',j'}(Y) & \longrightarrow & H^{i+i'+j+j'-3}(\omega_R) \otimes_R R \cdot T^{i+i'} \\ & \Big\downarrow {\scriptstyle (T^{-i})_* \otimes (T^{-i'})_*} & & \Big\downarrow {\scriptstyle (T^{-i-i'})_*} \\ \cup_r^{\mathrm{st}} : & E_r^n(X) \otimes_R {}'E_r^{n'}(Y) & \longrightarrow & H^{n+n'-3}(\omega_R) \end{array}$$

(in which $n = i+j$, $n' = i'+j'$, $i, i' \geqslant 0$, $r \geqslant 1$). The formula (11.5.3.2) holds also for $\cup_r^{\mathrm{st}}$, hence the adjoint map $\mathrm{adj}(\cup_r^{\mathrm{st}})$ induces, for each $q \in \mathbf{Z}$, a morphism of complexes

$$\mathrm{adj}_q(\cup_r^{\mathrm{st}}) : (E_r^\bullet(X), d_r) \longrightarrow \mathrm{Hom}_R^\bullet(({}'E^\bullet(Y), {}'d_r), H^q(\omega_R)[-3+q]).$$

The most interesting case is that of $q = 0$:

$$\mathrm{adj}_0(\cup_r^{\mathrm{st}}) : E_r^\bullet(X) \longrightarrow \mathrm{Hom}_R^\bullet({}'E^\bullet(Y), H^0(\omega_R)[-3]) \qquad (r \geqslant 1).$$

These maps determine inductively each other, as the components of $\mathrm{adj}_0(\cup_{r+1}^{\mathrm{st}})$ are equal to

(11.7.2.2)
$$\begin{aligned} E_{r+1}^n(X) = H^n(E_r^n(X)) &\xrightarrow{\mathrm{adj}_0(\cup_r^{\mathrm{st}})_*} H^n(\mathrm{Hom}_R^\bullet({}'E^\bullet(Y), H^0(\omega_R)[-3])) \\ &\longrightarrow \mathrm{Hom}_R({}'E^{3-n}(Y), H^0(\omega_R)). \end{aligned}$$

For each $r \geqslant 1$, we define "stable" higher height pairings by the formula

$$\widetilde{h}_{\pi,i,j}^{(r),\mathrm{st}} : E_r^i(X) \otimes_R {}'E_r^j(Y) \xrightarrow{d_r^i \otimes \mathrm{id}} E_r^{i+1}(X) \otimes_R {}'E_r^j(Y) \xrightarrow{\cup_r^{\mathrm{st}}} H^{i+j-2}(\omega_R).$$

These pairings satisfy the formula in Proposition 11.5.6 (ii) and depend on the choice of $\gamma \in \Gamma$; if we replace γ by γ^c ($c \in \mathbf{Z}_p^*$), then T is replaced by

$$(1+T)^c - 1 \equiv cT \pmod{T^2},$$

hence d_r^i and $\widetilde{h}_{\pi,i,j}^{(r),\mathrm{st}}$ are multiplied by c^{-r}. The commutative diagram

$$\begin{array}{lccc} \widetilde{h}_{\pi,i,j}^{(r)} : & E_r^{0,i}(X) \otimes_R {}'E_r^{0,j}(Y) & \longrightarrow & H^{i+j-2}(\omega_R) \otimes_R R \cdot T^r \\ & \Big\downarrow {\scriptstyle 1_* \otimes 1_*} & & \Big\downarrow {\scriptstyle (T^{-r})_*} \\ \widetilde{h}_{\pi,i,j}^{(r),\mathrm{st}} : & E_r^i(X) \otimes_R {}'E_r^j(Y) & \longrightarrow & H^{i+j-2}(\omega_R) \end{array}$$

(induced by (11.7.2.1)) shows that $\widetilde{h}_{\pi,i,j}^{(r)}$ factors through the canonical surjective maps

$$\mathrm{Fil}_\Gamma^r\big(\widetilde{H}_f^i(X)\big) = E_r^{0,i}(X) \twoheadrightarrow E_r^i(X), \quad \mathrm{Fil}_\Gamma^r\big(\widetilde{H}_f^j(Y)\big) = {}'E_r^{0,j}(Y) \twoheadrightarrow {}'E_r^j(Y).$$

11.7.3. Proposition. — *Assume that* $\mathfrak{q} \in \mathrm{Spec}(R)$, $\mathrm{ht}(\mathfrak{q}) = 0$ *and the conditions 11.7.1.1–11.7.1.2 are satisfied for* $\mathfrak{p}_0 = \mathfrak{q}$. *Then:*

(i) *The localized maps*

$$\mathrm{adj}_0(\cup_r^{\mathrm{st}})_{\mathfrak{q}} : E_r^n(X)_{\mathfrak{q}} \xrightarrow{\sim} \mathrm{Hom}_{R_{\mathfrak{q}}}({}'E_r^{3-n}(Y)_{\mathfrak{q}}, H^0(\omega_R)_{\mathfrak{q}}) \qquad (r \geqslant 1)$$

are all isomorphisms.

(ii) $(\forall i)$ ${}_L\mathrm{Ker}\big(\widetilde{h}_{\pi,i,2-i}^{(r),\mathrm{st}}\big)_{\mathfrak{q}} = \mathrm{Ker}\big((d_r^i)_{\mathfrak{q}} : E_r^i(X)_{\mathfrak{q}} \to E_r^{i+1}(X)_{\mathfrak{q}}\big)$.

Proof

(i) For $r=1$ the statement follows from the duality Theorem 6.3.4, as observed in the proof of Proposition 11.5.7. Assume that the claim is proved for $r \geqslant 1$; then the localization at $\mathfrak{q}$ of the first (resp., the second) arrow in (11.7.2.2) is an isomorphism by the induction hypothesis (resp., because $H^0(\omega_R)_{\mathfrak{q}} = I_{R_{\mathfrak{q}}}$ is an injective $R_{\mathfrak{q}}$-module), proving the statement for $r+1$.

The part (ii) follows from (i) and the definition of $\widetilde{h}^{(r),\mathrm{st}}$. □

11.7.4. Proposition. — *Assume that* $\mathfrak{p} \in \mathrm{Spec}(R)$, $R_{\mathfrak{p}}$ *has no embedded primes and the conditions (11.7.1.1-3) are satisfied for all* $\mathfrak{p}_0 \in Q = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{q} \subset \mathfrak{p},\ \mathrm{ht}(\mathfrak{q}) = 0\}$. *Then*

$$\begin{aligned} {}_L\mathrm{Ker}\big(\widetilde{h}^{(r)}_{\pi,1,1}\big)_{\mathfrak{p}} &= \mathrm{Ker}\left(\widetilde{H}^1_f(X)_{\mathfrak{p}} \longrightarrow \bigoplus_{\mathfrak{q}\in Q} \widetilde{H}^1_f(X)_{\mathfrak{q}} / \big(\mathrm{Fil}^{r+1}_{\Gamma}\big(\widetilde{H}^1_f(X)\big)\big)_{\mathfrak{q}}\right) \\ \mathrm{Ker}_R\big(\widetilde{h}^{(r)}_{\pi,1,1}\big)_{\mathfrak{p}} &= \mathrm{Ker}\left(\widetilde{H}^1_f(Y)_{\mathfrak{p}} \longrightarrow \bigoplus_{\mathfrak{q}\in Q} \widetilde{H}^1_f(Y)_{\mathfrak{q}} / \big(\mathrm{Fil}^{r+1}_{\Gamma}\big(\widetilde{H}^1_f(Y)\big)\big)_{\mathfrak{q}}\right). \end{aligned} \qquad (r \geqslant 1)$$

Proof. — The symmetry property from Proposition 11.5.6 (i) (which also holds for the stable height pairings) shows that it is sufficient to consider only the left kernel. The same argument as in the proof of Proposition 11.5.7 reduces to the case $\mathfrak{p} = \mathfrak{q}$, $\mathrm{ht}(\mathfrak{q}) = 0$. The assumption 11.7.1.3 and the duality isomorphisms

$$\widetilde{H}^i_f(X)_{\mathfrak{q}} \xrightarrow{\ \sim\ } \mathrm{Hom}_{R_{\mathfrak{q}}}\big(\widetilde{H}^{3-i}_f(Y)_{\mathfrak{q}}, H^0(\omega_R)_{\mathfrak{q}}\big)$$

imply that

$$(\forall j \neq 1,2) \quad \widetilde{H}^j_f(Z)_{\mathfrak{q}} = 0 \qquad (Z = X, Y).$$

This means that $E^{i,j}_r(X)_{\mathfrak{q}}, {}'E^{i,j}_r(Y)_{\mathfrak{q}}$ satisfy 11.6.1.5 for $j_0 = 1$, hence all the maps

$$(11.7.4.1) \qquad 1_* : E^{0,1}_r(X)_{\mathfrak{q}} \xrightarrow{\ \sim\ } E^1_r(X)_{\mathfrak{q}}, \quad 1_* : {}'E^{0,1}_r(Y)_{\mathfrak{q}} \xrightarrow{\ \sim\ } {}'E^1_r(Y)_{\mathfrak{q}}, \qquad (r \geqslant 1)$$

are isomorphisms. It follows that

$$\big(\widetilde{h}^{(r)}_{\pi,1,1}\big)_{\mathfrak{q}} = \big(\widetilde{h}^{(r),\mathrm{st}}_{\pi,1,1}\big)_{\mathfrak{q}}$$

and

$$\begin{aligned} {}_L\mathrm{Ker}\big(\widetilde{h}^{(r)}_{\pi,1,1}\big)_{\mathfrak{q}} &= \mathrm{Ker}\big((d^1_r)_{\mathfrak{q}} : E^1_r(X)_{\mathfrak{q}} \longrightarrow E^2_r(X)_{\mathfrak{q}}\big) \\ &\xrightarrow{\ \sim\ } \mathrm{Ker}\big((d^{0,1}_r)_{\mathfrak{q}} : E^{0,1}_r(X)_{\mathfrak{q}} \longrightarrow E^{r,2-r}(X)_{\mathfrak{q}}\big) = \mathrm{Fil}^r_{\Gamma}\big(\widetilde{H}^1_f(X)\big)_{\mathfrak{q}}, \end{aligned}$$

by Proposition 11.7.3 (ii) and (11.7.4.1). □

11.7.5. We now fix $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{p}) = 1$ and put $Q = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{q} \subset \mathfrak{p}, \mathfrak{q} \neq \mathfrak{p}\}$ (*i.e.*, $\mathrm{ht}(\mathfrak{q}) = 0$ for all $\mathfrak{q} \in Q$). We are going to apply the discussion from 11.6 to the localized objects $A = R_{\mathfrak{p}}$, $\Lambda = \overline{R}_{\overline{\mathfrak{p}}}$ (using the notation from 11.6.1.2) and $M^\bullet = \widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(X))_{\overline{\mathfrak{p}}}$ (resp., $(N^\bullet)^\iota = \widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(Y)^\iota)_{\overline{\mathfrak{p}}}$). The reader should keep in mind the following simplest case: $R = \mathbf{Z}_p$, $\overline{R} = \mathbf{Z}_p[\![T]\!]$, $\mathfrak{p} = (p)$, $\overline{\mathfrak{p}} = (p, T)$, $A = \mathbf{Z}_p$, $\Lambda = \mathbf{Z}_p[\![T]\!]$, $\mathfrak{q} = (0)$, $\overline{\mathfrak{q}} = (T)$.

11.7.6. Proposition. — *Let $\mathfrak{q} \in Q$ and assume that the conditions 11.7.1.1–11.7.1.3 hold for $\mathfrak{p}_0 = \mathfrak{q}$. Then:*

(i) $(\forall j \neq 1, 2)$ $\widetilde{H}^j_f(Z)_{\mathfrak{q}} = 0 \quad (Z = X, Y)$.

(ii) *The following conditions are equivalent:*

$$\begin{aligned}(\forall n) \quad \ell_{\overline{R}_{\overline{\mathfrak{q}}}}\big(\widetilde{H}^n_{f,\mathrm{Iw}}(X)_{\overline{\mathfrak{q}}}\big) < \infty \iff (\forall n) \quad \ell_{\overline{R}_{\overline{\mathfrak{q}}}}\big(\widetilde{H}^n_{f,\mathrm{Iw}}(Y)_{\overline{\mathfrak{q}}}\big) < \infty \\ \iff (\exists r \geqslant 1) \quad \mathrm{Fil}^{r+1}_\Gamma\big(\widetilde{H}^1_f(X)\big)_{\mathfrak{q}} = \mathrm{Fil}^{r+1}_\Gamma\big(\widetilde{H}^1_f(Y)\big)_{\mathfrak{q}} = 0 \\ \iff (\exists r \geqslant 1) \quad \big(\widetilde{h}^{(r)}_{\pi,1,1}\big)_{\mathfrak{q}} : \mathrm{Fil}^r_\Gamma\big(\widetilde{H}^1_f(X)\big)_{\mathfrak{q}} \times \mathrm{Fil}^r_\Gamma\big(\widetilde{H}^1_f(Y)\big)_{\mathfrak{q}} \longrightarrow H^0(\omega_R)_{\mathfrak{q}}\end{aligned}$$

has trivial left and right kernels.

If these equivalent conditions hold, then:

(iii) $(\forall r \geqslant 1)$ $\ell_{R_{\mathfrak{q}}}\big(\mathrm{Fil}^r_\Gamma\big(\widetilde{H}^1_f(X)\big)_{\mathfrak{q}}\big) = \ell_{R_{\mathfrak{q}}}\big(\mathrm{Fil}^r_\Gamma\big(\widetilde{H}^1_f(Y)\big)_{\mathfrak{q}}\big)$, $\ell_{R_{\mathfrak{q}}}\big(\widetilde{H}^1_f(Z)_{\mathfrak{q}}\big) = \ell_{R_{\mathfrak{q}}}\big(\widetilde{H}^2_f(Z)_{\mathfrak{q}}\big)$ $(Z = X, Y)$.

(iv) $(\forall n \neq 2)$ $\widetilde{H}^n_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}} = 0 \quad (Z = X, Y)$.

(v) *If $r_0 \geqslant 1$ is the smallest positive integer for which $\mathrm{Fil}^{r_0+1}_\Gamma\big(\widetilde{H}^1_f(X)\big)_{\mathfrak{q}} = 0$, then*

$$\ell_{\overline{R}_{\overline{\mathfrak{q}}}}\big(\widetilde{H}^2_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}}\big) = \sum_{r=1}^{r_0} \ell_{R_{\mathfrak{q}}}\big(\mathrm{Fil}^r_\Gamma\big(\widetilde{H}^1_f(X)\big)_{\mathfrak{q}}\big) \qquad (Z = X, Y).$$

(vi) *$r_0 \geqslant 1$ from (v) is the smallest positive integer for which*

$$\big(\widetilde{h}^{(r_0)}_{\pi,1,1}\big)_{\mathfrak{q}} : \mathrm{Fil}^{r_0}_\Gamma\big(\widetilde{H}^1_f(X)\big)_{\mathfrak{q}} \times \mathrm{Fil}^{r_0}_\Gamma\big(\widetilde{H}^1_f(Y)\big)_{\mathfrak{q}} \longrightarrow H^0(\omega_R)_{\mathfrak{q}}$$

has trivial left and right kernels.

(vii) $\ell_{\overline{R}_{\overline{\mathfrak{q}}}}\big(\widetilde{H}^2_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}}\big) \geqslant \ell_{R_{\mathfrak{q}}}\big(\widetilde{H}^1_f(X)_{\mathfrak{q}}\big)$ $(Z = X, Y)$, *with equality occurring if and only if*

$$\big(\widetilde{h}_{\pi,1,1}\big)_{\mathfrak{q}} : \widetilde{H}^1_f(X)_{\mathfrak{q}} \times \widetilde{H}^1_f(Y)_{\mathfrak{q}} \longrightarrow H^0(\omega_R)_{\mathfrak{q}}$$

has trivial left and right kernels.

Proof. — The statement (i) was proved in 11.7.4. As regards (ii), Lemma 11.6.5 (ii) applied to $A_{\mathfrak{q}}, \Lambda_{\overline{\mathfrak{q}}}$ and $M^\bullet_{\overline{\mathfrak{q}}}$ (resp., $(N^\bullet)^\iota_{\overline{\mathfrak{q}}}$) shows (if we take into account (i)) equivalences

$$\begin{aligned}(11.7.6.1) \quad (\forall n)\ \ell_{\overline{R}_{\overline{\mathfrak{q}}}}\big(\widetilde{H}^n_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}}\big) < \infty \\ \iff (\exists r \geqslant 1)\ E^1_r(Z)_{\mathfrak{q}} = E^2_r(Z)_{\mathfrak{q}} = 0 \qquad (Z = X, Y).\end{aligned}$$

The duality result in Proposition 11.7.3 (i) gives

$$\ell_{R_{\mathfrak{q}}}\left(E_r^i(X)_{\mathfrak{q}}\right)=\ell_{R_{\mathfrak{q}}}\left(E_r^{3-i}(Y)_{\mathfrak{q}}\right), \tag{11.7.6.2}$$

and (11.7.4.1) yields

$$E_r^1(Z)_{\mathfrak{q}} \xrightarrow{\sim} \mathrm{Fil}_\Gamma^r\big(\widetilde{H}_f^1(Z)\big)_{\mathfrak{q}} \qquad (Z=X,Y). \tag{11.7.6.3}$$

Combining the statements (11.7.6.1)–(11.7.6.3) we obtain (ii). Assuming that the equivalent conditions in (ii) are satisfied, Lemma 11.6.5 (ii) gives

$$0=\ell_{R_{\mathfrak{q}}}\left(E_r^2(Z)_{\mathfrak{q}}\right)-\ell_{R_{\mathfrak{q}}}\left(E_r^1(Z)_{\mathfrak{q}}\right),$$

which proves (iii), if we take into account (11.7.6.2)–(11.7.6.3). For $Z=X,Y$, the exact sequences

$$0\longrightarrow \widetilde{H}^n_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}}/T\widetilde{H}^n_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}}\longrightarrow \widetilde{H}^n_f(Z)_{\mathfrak{q}}\longrightarrow \widetilde{H}^{n+1}_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}}[T]\longrightarrow 0$$

together with (i) and the Nakayama Lemma show that

$$(\forall n\neq 1,2)\quad \widetilde{H}^n_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}}=0,\quad \widetilde{H}^1_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}}[T]=0,$$

which proves (iv), since

$$\ell_{\overline{R}_{\overline{\mathfrak{q}}}}\big(\widetilde{H}^1_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}}\big)<\infty.$$

The formula (v) follows from (iii), (iv), Lemma 11.6.5 (ii) and (11.7.6.2)–(11.7.6.3):

$$\begin{aligned}\ell_{\overline{R}_{\overline{\mathfrak{q}}}}\big(\widetilde{H}^2_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}}\big)&=\sum_{r\geqslant 1}(2\ell_{R_{\mathfrak{q}}}(E_r^2(Z)_{\mathfrak{q}})-\ell_{R_{\mathfrak{q}}}(E_r^1(Z)_{\mathfrak{q}}))\\&=\sum_{r\geqslant 1}\ell_{R_{\mathfrak{q}}}(E_r^1(X)_{\mathfrak{q}})=\sum_{r\geqslant 1}\ell_{R_{\mathfrak{q}}}\big(\mathrm{Fil}_\Gamma^r\big(\widetilde{H}^1_f(X)\big)_{\mathfrak{q}}\big).\end{aligned}$$

Finally, the statement (vi) follows from Proposition 11.7.4 for $\mathfrak{p}=\mathfrak{q}$, and (vii) is a direct consequence of (v) and (vi). □

11.7.7. Lemma. — *Assume that, for each $\mathfrak{q}\in Q$, the conditions* (11.7.6.1)–(11.7.6.3) *(for $\mathfrak{p}_0=\mathfrak{q}$) and 11.7.6 (ii) hold. Then, for each $Z=X,Y$:*

(i) $(\forall n)$ $\mathrm{codim}_{\overline{R}_{\overline{\mathfrak{p}}}}\big(\mathrm{supp}\big(H^n_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{p}}}\big)\big)\geqslant 1$, *hence the integer $a_{R_{\mathfrak{p}}}\big(H^n_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{p}}}\big)$ is defined.*

(ii) *If $r_0\geqslant 1$ and* $\mathrm{codim}_{R_{\mathfrak{p}}}(\mathrm{supp}(\mathrm{Fil}_\Gamma^{r_0+1}(\widetilde{H}^1_f(X))_{\mathfrak{p}}))\geqslant 1$, *then*

$$\sum_n(-1)^n a_{R_{\mathfrak{p}}}(H^n_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{p}}})=\chi_{R_{\mathfrak{p}}}(H(E^\bullet_{r_0}(Z)_{\mathfrak{p}}))=\chi_{R_{\mathfrak{p}}}(E^\bullet_{r_0+1}(Z)_{\mathfrak{p}}).$$

Proof. — The condition 11.7.6 (ii) for any $\mathfrak{q}\in Q$ implies (i), while (ii) follows from Proposition 11.6.10 (iv) (for $A=R_{\mathfrak{p}}$, $\Lambda=\overline{R}_{\overline{\mathfrak{p}}}$, $M^\bullet=\widetilde{C}^\bullet_f(\mathscr{F}_\Gamma(X))_{\overline{\mathfrak{p}}}$) and Proposition 11.7.6 (iv). □

11.7.8. Recall that, for each $r \geqslant 1$, we have a morphism of complexes

$$\mathrm{adj}_0(\cup_r^{\mathrm{st}}) : E_r^\bullet \longrightarrow \mathrm{Hom}_R^\bullet({}'E_r^\bullet(Y), H^0(\omega_R)[-3])$$

such that the composite map

$$\mathrm{adj}_0(\cup_r^{\mathrm{st}})_{i+1} \circ d_r : E_r^i(X) \longrightarrow \mathrm{Hom}_R({}'E_r^{2-i}(Y), H^0(\omega_R))$$

is equal to $\mathrm{adj}\big(\widetilde{h}_{\pi,i,2-i}^{(r),\mathrm{st}}\big)$:

(11.7.8.1)

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d_r} & E_r^i(X) & \xrightarrow{d_r} & E_r^{i+1}(X) & \xrightarrow{d_r} & \cdots \\
 & & \downarrow & \searrow^{\mathrm{adj}(\widetilde{h})} & \downarrow & & \\
\cdots & \longrightarrow & \mathrm{Hom}_R({}'E_r^{3-i}(Y), H^0(\omega_R)) & \longrightarrow & \mathrm{Hom}_R({}'E_r^{2-i}(Y), H^0(\omega_R)) & \longrightarrow & \cdots
\end{array}$$

We form a new complex $E_r^\bullet(X,Y,\pi)$ by traversing the diagonal arrow with $i = 1$ in (11.7.8.1):

$$\begin{aligned}
E_r^\bullet(X,Y,\pi) :\ \cdots \longrightarrow E_r^0(X) \longrightarrow E_r^1(X) \longrightarrow \mathrm{Hom}_R({}'E_r^1(Y), H^0(\omega_R)) \\
\longrightarrow \mathrm{Hom}_R({}'E_r^0(Y), H^0(\omega_R)) \longrightarrow \cdots
\end{aligned}$$

For $r = 1$, the complex $E_1^\bullet(X,Y,\pi)$ is equal to

$$\begin{aligned}
\cdots \longrightarrow \widetilde{H}_f^0(X) \xrightarrow{d_1^0} \widetilde{H}_f^1(X) \xrightarrow{\mathrm{adj}(\widetilde{h}_{\pi,1,1})} \mathrm{Hom}_R\big(\widetilde{H}_f^1(Y), H^0(\omega_R)\big) \\
\xrightarrow{\mathrm{Hom}(d_1^0,\mathrm{id})} \mathrm{Hom}_R\big(\widetilde{H}_f^0(Y), H^0(\omega_R)\big) \longrightarrow \cdots
\end{aligned}$$

11.7.9. Lemma. — *Assume that the conditions 11.7.1.1–11.7.1.3 hold for* $\mathfrak{p}_0 = \mathfrak{p}$ *and* $\mathrm{depth}(R_\mathfrak{p}) = 1 (= \dim(R_\mathfrak{p}))$. *Then:*

(i) $(\forall r \geqslant 1)\,(\forall i \neq 1,2,3)$ $E_r^i(Z)_\mathfrak{p} = 0$, $E_r^3(Z)_\mathfrak{p}$ *is* $R_\mathfrak{p}$*-torsion* ($Z = X, Y$).

(ii) *The only possibly non-zero components of the localization of* (11.7.8.1) *at* $\mathfrak{p}$ *are*

$$\begin{array}{ccccccc}
0 & \longrightarrow & E_r^1(X)_\mathfrak{p} & \longrightarrow & E_r^2(X)_\mathfrak{p} & \longrightarrow & E_r^3(X)_\mathfrak{p} \\
 & & \downarrow & \searrow & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_{R_\mathfrak{p}}({}'E_r^2(Y)_\mathfrak{p}, H^0(\omega_R)_\mathfrak{p}) & \longrightarrow & \mathrm{Hom}_{R_\mathfrak{p}}({}'E_r^1(Y)_\mathfrak{p}, H^0(\omega_R)_\mathfrak{p}) & \longrightarrow & 0
\end{array}$$

Proof. — It is sufficient to prove the Lemma for $r=1$, when $E_1^i(Z)=\widetilde{H}_f^i(Z)$. The assumptions 11.7.1.1–11.7.1.2 yield the exact triangle

$$\widetilde{\mathbf{R}\Gamma}_f(X)_{\mathfrak{p}} \longrightarrow \mathbf{R}\mathrm{Hom}_{R_{\mathfrak{p}}}\big(\widetilde{\mathbf{R}\Gamma}_f(Y)_{\mathfrak{p}}, \omega_{R_{\mathfrak{p}}}\big) \longrightarrow \mathrm{Err}(\Delta(X),\Delta(Y),\pi)_{\mathfrak{p}}$$

in $D_{ft}({}_{R_{\mathfrak{p}}}\mathrm{Mod})$, which induces the following two exact sequences (by Lemma 2.10.11 (ii)):

(11.7.9.1)

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & D_{R_{\mathfrak{p}}}((\widetilde{H}_f^{4-i}(Y)_{\mathfrak{p}})_{R_{\mathfrak{p}}-\mathrm{tors}}) & & \\
 & & & & \downarrow & & \\
\cdots \longrightarrow H^{i-1}(\mathrm{Err})_{\mathfrak{p}} & \longrightarrow & \widetilde{H}_f^i(X)_{\mathfrak{p}} & \longrightarrow & H^i(\mathbf{R}\mathrm{Hom}_{R_{\mathfrak{p}}}(\widetilde{\mathbf{R}\Gamma}_f(Y)_{\mathfrak{p}},\omega_{R_{\mathfrak{p}}})) & \longrightarrow & H^i(\mathrm{Err})_{\mathfrak{p}} \longrightarrow \cdots \\
 & & & & \downarrow & & \\
 & & & & \mathrm{Hom}_{R_{\mathfrak{p}}}(\widetilde{H}_f^{3-i}(Y)_{\mathfrak{p}}, H^0(\omega_R)_{\mathfrak{p}}) & & \\
 & & & & \downarrow & & \\
 & & & & 0, & &
\end{array}$$

where the error terms $H^i(\mathrm{Err})_{\mathfrak{p}}$ vanish for $i\neq 1,2$ and are torsion over $R_{\mathfrak{p}}$ for $i=1,2$. As $\widetilde{H}_f^i(X)_{\mathfrak{p}}=\widetilde{H}_f^i(Y)_{\mathfrak{p}}=0$ for $i\leqslant 0$ by the assumption 11.7.1.3, it follows from (11.7.9.1) that $\widetilde{H}_f^i(X)_{\mathfrak{p}}=0$ for $i>3$ and that $\widetilde{H}_f^3(X)_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$-torsion. Interchanging X,Y and replacing π by $\pi\circ s_{12}$ gives the same result for Y, proving (i). The statement (ii) follows from (i) and the fact that $H^0(\omega_R)_{\mathfrak{p}}$ is torsion-free over $R_{\mathfrak{p}}$. □

11.7.10. Under the assumptions of Lemma 11.7.9, the diagram (11.7.9.1) yields a complex

(11.7.10.1)
$$\begin{aligned}
0 &\longrightarrow \widetilde{H}_f^1(X_{\mathfrak{p}})_{\mathrm{tors}} \longrightarrow D_{R_{\mathfrak{p}}}(\widetilde{H}_f^3(Y_{\mathfrak{p}})_{\mathrm{tors}}) \longrightarrow H^1(\mathrm{Err}_{\mathfrak{p}}) \longrightarrow \\
&\longrightarrow \widetilde{H}_f^2(X_{\mathfrak{p}})_{\mathrm{tors}} \longrightarrow D_{R_{\mathfrak{p}}}(\widetilde{H}_f^2(Y_{\mathfrak{p}})_{\mathrm{tors}}) \longrightarrow H^2(\mathrm{Err}_{\mathfrak{p}}) \longrightarrow \\
&\longrightarrow \widetilde{H}_f^3(X_{\mathfrak{p}})_{\mathrm{tors}} \longrightarrow D_{R_{\mathfrak{p}}}(\widetilde{H}_f^1(Y_{\mathfrak{p}})_{\mathrm{tors}}) \longrightarrow 0,
\end{aligned}$$

in which we write

$$\widetilde{H}_f^i(Z_{\mathfrak{p}})_{\mathrm{tors}} = (\widetilde{H}_f^i(Z)_{\mathfrak{p}})_{R_{\mathfrak{p}}\text{-tors}}, \quad \mathrm{Err}=\mathrm{Err}(\Delta(X),\Delta(Y),\pi).$$

The complex (11.7.10.1) is acyclic everywhere except possibly at the terms $H^i(\mathrm{Err}_{\mathfrak{p}})$; denote its cohomology at these two terms by

$$H(H^i(\mathrm{Err}_{\mathfrak{p}})) = H(H^i(\mathrm{Err}(\Delta(X),\Delta(Y),\pi)_{\mathfrak{p}})) \qquad (i=1,2).$$

According to 7.6.10.7 we have

$$\ell_{R_{\mathfrak{p}}}\big(H^i(\mathrm{Err}_{\mathfrak{p}})\big) = \sum_{v\in\Sigma'} \mathrm{Tam}_v(X,\mathfrak{p}) \qquad (i=1,2),$$

hence

$$0 \leqslant \ell_{R_{\mathfrak{p}}}\left(H(H^i(\mathrm{Err}_{\mathfrak{p}}))\right) \leqslant \sum_{v\in\Sigma'} \mathrm{Tam}_v(X,\mathfrak{p}). \tag{11.7.10.2}$$

The generalized Cassels-Tate pairings 10.5.3 and the isomorphisms (10.2.8.3) show that the $D_{R_{\mathfrak{p}}}$-dual of (11.7.10.1) is isomorphic to the same sequence in which (X,Y,π) are replaced by $(Y,X,\pi\circ s_{12})$. This implies that

$$\begin{aligned}\ell_{R_{\mathfrak{p}}}&\left(H(H^i(\mathrm{Err}(\Delta(X),\Delta(Y),\pi)_{\mathfrak{p}}))\right)\\ &= \ell_{R_{\mathfrak{p}}}\left(H(H^{3-i}(\mathrm{Err}(\Delta(Y),\Delta(X),\pi\circ s_{12})_{\mathfrak{p}}))\right) \qquad (i=1,2)\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{3}(-1)^i\ell_{R_{\mathfrak{p}}}&\left(\widetilde{H}^i_f(X_{\mathfrak{p}})_{\mathrm{tors}}\right) + \ell_{R_{\mathfrak{p}}}\left(H(H^2(\mathrm{Err}(\Delta(X),\Delta(Y),\pi)_{\mathfrak{p}}))\right)\\ =\sum_{i=1}^{3}(-1)^i\ell_{R_{\mathfrak{p}}}&\left(\widetilde{H}^i_f(Y_{\mathfrak{p}})_{\mathrm{tors}}\right) + \ell_{R_{\mathfrak{p}}}\left(H(H^2(\mathrm{Err}(\Delta(Y),\Delta(X),\pi\circ s_{12})_{\mathfrak{p}}))\right).\end{aligned} \tag{11.7.10.3}$$

11.7.11. Theorem. — *Assume that* $\mathrm{depth}(R_{\mathfrak{p}}) = \dim(R_{\mathfrak{p}}) = 1$ *and that the conditions* (11.7.1.1)–(11.7.1.3) *hold for* $\mathfrak{p}_0 = \mathfrak{p}$. *If, for all* $\mathfrak{q}\in Q$, *the pairings*

$$\left(\widetilde{h}_{\pi,1,1}\right)_{\mathfrak{q}} : \widetilde{H}^1_f(X)_{\mathfrak{q}} \times \widetilde{H}^1_f(Y)_{\mathfrak{q}} \longrightarrow H^0(\omega_R)_{\mathfrak{q}}$$

have trivial left and right kernels, then we have (for $Z = X, Y$*):*

(i) $(\forall\mathfrak{q}\in Q)\ \ell_{\overline{R}_{\overline{\mathfrak{q}}}}\left(\widetilde{H}^n_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}}\right) = \begin{cases}\ell_{R_{\mathfrak{q}}}\left(\widetilde{H}^1_f(X)_{\mathfrak{q}}\right), & n=2\\ 0, & n\neq 2.\end{cases}$

(ii) $(\forall n)\ \widetilde{H}^n_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{p}}}$ *is an* $(\overline{R}_{\overline{\mathfrak{p}}})$*-module of finite type and* $\mathrm{codim}_{\overline{R}_{\overline{\mathfrak{p}}}}\big(\mathrm{supp}\big(\widetilde{H}^n_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{p}}}\big)\big) \geqslant 1$.

(iii) *(The formula of the Birch and Swinnerton-Dyer type)*

$$\begin{aligned}\sum_n(-1)^n a_{R_{\mathfrak{p}}}&\left(\widetilde{H}^n_{f,\mathrm{Iw}}(X)_{\overline{\mathfrak{p}}}\right) = \ell_{R_{\mathfrak{p}}}\left(\det\left(\left(\widetilde{h}_{\pi,1,1}\right)_{\mathfrak{p}}\right)\right)\\ &+\sum_{i=1}^{3}(-1)^i\ell_{R_{\mathfrak{p}}}\left(\left(\widetilde{H}^i_f(X)_{\mathfrak{p}}\right)_{R_{\mathfrak{p}}\text{-tors}}\right) + \ell_{R_{\mathfrak{p}}}(H(H^2(\mathrm{Err}(\Delta(X),\Delta(Y),\pi)_{\mathfrak{p}})),\end{aligned}$$

where the first term on the right is defined as in 2.10.21, i.e., *is equal to*

$$\ell_{R_{\mathfrak{p}}}\Big(\mathrm{Coker}\big(\mathrm{adj}\big((\widetilde{h}_{\pi,1,1})_{\mathfrak{p}}\big) : \widetilde{H}^1_f(X)_{\mathfrak{p}} \longrightarrow \mathrm{Hom}_{R_{\mathfrak{p}}}\left(\widetilde{H}^1_f(Y)_{\mathfrak{p}}, H^0(\omega_R)_{\mathfrak{p}}\right)\big)\Big),$$

and

$$0 \leqslant \ell_{R_{\mathfrak{p}}}\left(H(H^2(\mathrm{Err}(\Delta(X),\Delta(Y),\pi)_{\mathfrak{p}}))\right) \leqslant \sum_{v\in\Sigma'}\mathrm{Tam}_v(X,\mathfrak{p}).$$

Proof. — As the conditions (11.7.1.1)–(11.7.1.3) hold for $\mathfrak{p}_0 = \mathfrak{p}$, they are also satisfied by all $\mathfrak{p}_0 = \mathfrak{q}\in Q$. Thus (i) follows from Proposition 11.7.6 (vii), and (ii) is a consequence of (i). In order to prove (iii), Lemma 11.7.7 (ii) for $r_0 = 1$ together with Lemma 11.7.9 give

$$\begin{aligned}\sum_n (-1)^n a_{R_{\mathfrak p}}\big(\widetilde{H}^n_{f,\mathrm{Iw}}(X)_{\overline{\mathfrak q}}\big) &= \chi_{R_{\mathfrak p}}(H(E_1^\bullet(X)_{\mathfrak p}))\\ &= \chi_{R_{\mathfrak p}}(H(\sigma_{\leqslant 2}\, E_1^\bullet(X)_{\mathfrak p})) - \ell_{R_{\mathfrak p}}\big(\big(\widetilde{H}^3_f(X)_{\mathfrak p}\big)_{R_{\mathfrak p}\text{-tors}}\big).\end{aligned}$$

The exact triangle

$$\sigma_{\leqslant 2}\, E_1^\bullet(X)_{\mathfrak p} \longrightarrow E_1^\bullet(X,Y,\pi)_{\mathfrak p} \longrightarrow \mathrm{Cone}((\lambda_r^2)_{\mathfrak p})[-2], \tag{11.7.11.1}$$

where λ_r^n is a shorthand for

$$\lambda_r^n = \mathrm{adj}_0(\cup_r^{\mathrm{st}})_n : E_r^n(X) \longrightarrow \mathrm{Hom}_R({}'E_r^{3-n}(Y), H^0(\omega_R)),$$

then yields

$$\chi_{R_{\mathfrak p}}(H(\sigma_{\leqslant 2}\, E_1^\bullet(X)_{\mathfrak p})) = \chi_{R_{\mathfrak p}}(H(E_1^\bullet(X,Y,\pi)_{\mathfrak p})) + \ell_{R_{\mathfrak p}}(\mathrm{Ker}((\lambda_1^2)_{\mathfrak p})) - \ell_{R_{\mathfrak p}}(\mathrm{Coker}((\lambda_1^2)_{\mathfrak p})).$$

As the complex $E_1^\bullet(X,Y,\pi)_{\mathfrak p})$ coincides with

$$\Big[\mathrm{adj}\big(\big(\widetilde{h}_{\pi,1,1}\big)_{\mathfrak p}\big) : \widetilde{H}^1_f(X)_{\mathfrak p} \longrightarrow \mathrm{Hom}_{R_{\mathfrak p}}\big(\widetilde{H}^1_f(Y)_{\mathfrak p}, H^0(\omega_R)_{\mathfrak p}\big)\Big]$$

in degrees 1, 2 (by Lemma 11.7.9 (ii)), it follows that

$$\chi_{R_{\mathfrak p}}(H(E_1^\bullet(X,Y,\pi)_{\mathfrak p})) = \ell_{R_{\mathfrak p}}\big(\det\big(\big(\widetilde{h}_{\pi,1,1}\big)_{\mathfrak p}\big)\big) - \ell_{R_{\mathfrak p}}\big(\big(\widetilde{H}^1_f(X)_{\mathfrak p}\big)_{R_{\mathfrak p}\text{-tors}}\big).$$

A diagram chase on (11.7.9.1) for $i = 2$ shows that

$$\mathrm{Ker}\big((\lambda_1^2)_{\mathfrak p}\big) = \big(\widetilde{H}^2_f(X)_{\mathfrak p}\big)_{R_{\mathfrak p}\text{-tors}}, \quad \mathrm{Coker}\big((\lambda_1^2)_{\mathfrak p}\big) \xrightarrow{\sim} H^2(H^2(\mathrm{Err}_{\mathfrak p})).$$

Putting everything together, we obtain (iii). □

11.7.12. By (11.7.10.3), Lemma 2.10.20 (iv) and Theorem 11.7.11 (iii),

$$\sum_n (-1)^n a_{R_{\mathfrak p}}\big(\widetilde{H}^n_{f,\mathrm{Iw}}(X)_{\overline{\mathfrak p}}\big) = \sum_n (-1)^n a_{R_{\mathfrak p}}\big(\widetilde{H}^n_{f,\mathrm{Iw}}(Y)_{\overline{\mathfrak p}}\big).$$

This can also be deduced from the exact triangle

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(X)_{\overline{\mathfrak p}} \longrightarrow \mathscr{D}_{\overline{R}_{\overline{\mathfrak p}}}\big(\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(Y)_{\overline{\mathfrak p}}\big)^{\iota}[-3] \longrightarrow \mathrm{Err}(\mathscr{F}_\Gamma(X), \mathscr{F}_\Gamma(X)^\iota, \mathscr{F}_\Gamma(\pi))_{\overline{\mathfrak p}},$$

as the contributions from the error term Err cancel each other.

11.7.13. Attentive readers will have noticed that the statements and proofs in 11.7.5–11.7.12 depend only on $Z_{\mathfrak p}, (Z_v^+)_{\mathfrak p}$, $\pi_{\mathfrak p}$ ($Z = X, Y$). This implies that everything in these sections works in the context of 10.5.1.

11.7.14. Many results of Sect. 11.7 also hold in the case when $\Gamma \xrightarrow{\sim} \mathbf{Z}/p^m\mathbf{Z}$ ($m \geqslant 1$), $p^m \cdot R = 0$, $\dim(R) = 0$, $r \leqslant p-1$ (as $\overline{R}/J^p \xrightarrow{\sim} R[T]/(T^p)$, where $T = \gamma - 1$ for a fixed generator $\gamma \in \Gamma$).

11.7.15. A variant of Theorem 11.7.11 in non-commutative Iwasawa theory is proved in [**Bu-Ve**, Thm. 6.7].

11.8. Higher height pairings and the Cassels-Tate pairing

In this section we assume that $\Gamma \xrightarrow{\sim} \mathbf{Z}_p$. We show that, in this case, the J-adic graded quotients of the generalized Cassels-Tate pairing over $\overline{R}$ coincide (up to a sign) with higher height pairings.

11.8.1. Consider first the abstract situation: let (A, Λ, T) be as in 11.6.1.1 and denote the complex $\left[\Lambda \xrightarrow{-i} \Lambda_T\right]$ in degrees $0, 1$ (where i denotes the canonical map) by $C^\bullet$. For a complex $X^\bullet$ of Λ-modules denote by $H^i_!(X^\bullet)$ the cohomology groups of the complex $\mathbf{R}\Gamma_!(X^\bullet) = X^\bullet \otimes_\Lambda C^\bullet$. As in 2.10.7–2.10.9, there is an exact sequence of T-primary torsion Λ-modules

$$0 \longrightarrow H^{i-1}(X^\bullet) \otimes_\Lambda (\Lambda_T/\Lambda) \longrightarrow H^i_!(X^\bullet) \longrightarrow H^i(X^\bullet)[T^\infty] \longrightarrow 0$$

(the first term of which is T-divisible), quasi-isomorphisms

$$C^\bullet \xrightarrow{u} C^\bullet \otimes_\Lambda C^\bullet \xrightarrow{v} C^\bullet, \quad vu = \mathrm{id}, \quad uv \rightsquigarrow \mathrm{id}$$

and products

$$(X^\bullet \otimes_\Lambda C^\bullet) \otimes_\Lambda (Y^\bullet \otimes_\Lambda C^\bullet) \xrightarrow{s_{23}} (X^\bullet \otimes_\Lambda Y^\bullet) \otimes_\Lambda (C^\bullet \otimes_\Lambda C^\bullet) \xrightarrow{\mathrm{id} \otimes v} (X^\bullet \otimes_\Lambda Y^\bullet) \otimes_\Lambda C^\bullet$$

(note that both $\mathrm{id} \otimes v$ and $\mathrm{id} \otimes u$ are quasi-isomorphisms), which induce products

$$H^i_!(X^\bullet) \otimes_\Lambda H^j_!(Y^\bullet) \longrightarrow H^{i+j}_!(X^\bullet \otimes_\Lambda Y^\bullet)$$

factoring through

$$\cup_{ij} : H^i(X^\bullet)[T^\infty] \otimes_\Lambda H^j(Y^\bullet)[T^\infty] \longrightarrow H^{i+j}_!(X^\bullet \otimes_\Lambda Y^\bullet).$$

If $X^\bullet$ is a complex of T-flat Λ-modules, then the canonical projection $C^\bullet \to (\Lambda_T/\Lambda)[-1]$ induces isomorphisms

$$H^i_!(X^\bullet) \xrightarrow{\sim} H^{i-1}(X^\bullet \otimes_\Lambda (\Lambda_T/\Lambda)).$$

11.8.2. Assume that we are given the following data:

11.8.2.1. An involution $\iota : \Lambda \to \Lambda$ satisfying $\iota(T) \equiv \varepsilon T \pmod{T^2}$, where $\varepsilon = \pm 1$. The induced involution on $A = \Lambda/T\Lambda$ will also be denoted by ι.

11.8.2.2. Complexes of T-flat Λ-modules $M^\bullet, N^\bullet, P^\bullet$.

11.8.2.3. A morphism of complexes $\pi : M^\bullet \otimes_\Lambda (N^\bullet)^\iota \to P^\bullet[-3]$, where $(N^\bullet)^\iota = N^\bullet \otimes_{\Lambda,\iota} \Lambda$.

The construction recalled in 11.8.1 then yields products

$$\cup_{\pi,i,j} : H^i(M^\bullet)[T^\infty] \otimes_\Lambda H^j(N^\bullet)[T^\infty]^\iota \longrightarrow H^{i+j-4}(P^\bullet \otimes_\Lambda (\Lambda_T/\Lambda)),$$

which induces for each $r \geqslant 1$ pairings

$$H^i(M^\bullet)[T^r] \otimes_\Lambda H^j(N^\bullet)[T^r]^\iota \longrightarrow H^{i+j-4}(P^\bullet \otimes_\Lambda (T^{-r}\Lambda/\Lambda))$$

and

$$\langle\ ,\ \rangle^{(r)}_{\pi,i,j} : \frac{H^i(M^\bullet)[T^r]}{H^i(M^\bullet)[T^{r-1}]} \otimes_A \frac{H^j(N^\bullet)[T^r]^\iota}{H^j(N^\bullet)[T^{r-1}]^\iota} \longrightarrow H^{i+j-4}(P^\bullet \otimes_\Lambda (T^{-r}\Lambda/T^{-r+1}\Lambda)).$$

11.8.3. For each of the complexes $Z^\bullet = M^\bullet, N^\bullet, P^\bullet[-3]$ the T-adic filtration on $Z^\bullet$ (resp., on $(Z^\bullet)^\iota$) induces a spectral sequence $(E_r^{i,j}(Z^\bullet), d_r)$ (resp., $({}'E_r^{i,j}(Z^\bullet), {}'d_r) = (E_r^{i,j}(Z^\bullet), \varepsilon^r d_r)$) and the corresponding stable spectral sequence $(E_r^n(Z^\bullet), d_r)$ (resp., $({}'E_r^n(Z^\bullet), {}'d_r) = (E_r^n(Z^\bullet), \varepsilon^r d_r)$). As in 11.5.3 we obtain from 11.8.2.3 products

$$\cup_r : E_r^{i,j}(M^\bullet) \otimes_A {}'E_r^{i',j'}(N^\bullet)^\iota \longrightarrow E_r^{i+i',j+j'}(P^\bullet[-3])$$

satisfying

$$d_r(x \cup_r y) = (d_r x) \cup_r y + (-1)^{i+j} x \cup_r ({}'d_r y).$$

We assume, from now on, that

$$E_1^{\bullet,\bullet}(P^\bullet[-3]) = E_\infty^{\bullet,\bullet}(P^\bullet[-3]). \tag{11.8.3.1}$$

This allows us to define abstract height pairings by the same recipe as in 11.5.5:

$$\begin{aligned} h^{(r)}_{\pi,i,j} : E_r^{0,i}(M^\bullet) \otimes_A {}'E_r^{0,j}(N^\bullet)^\iota &\xrightarrow{d_r^{0,i}\otimes \mathrm{id}} E_r^{r,i+1-r}(M^\bullet) \otimes_A {}'E_r^{0,j}(N^\bullet)^\iota \\ &\xrightarrow{\cup_r} E_r^{r,i+j+1-r}(P^\bullet[-3]) = E_1^{r,i+j+1-r}(P^\bullet[-3]) = H^{i+j-2}(P^\bullet \otimes_\Lambda (T^r\Lambda/T^{r+1}\Lambda)) \end{aligned}$$

(note that in 11.5.3 the involution ι acts trivially on R, hence ${}'E_r^{i',j'}(Y^\bullet) = {}'E_r^{i',j'}(Y^\bullet)^\iota$ then). Recall from 11.6.2 that the differentials (depending on T) in $E_r^n(Z^\bullet)$ are given by the following formula:

$$d_r^n : E_r^n(Z^\bullet) \xrightarrow{j_r^n} T^{r-1}(H^{n+1}(Z^\bullet)[T^r]) \xrightarrow{T^{1-r}} \frac{H^{n+1}(Z^\bullet)[T^r]}{H^{n+1}(Z^\bullet)[T^{r-1}]} \xrightarrow{i_r^{n+1}} E_r^{n+1}(Z^\bullet).$$

11.8.4. Proposition. — *The pairing*

$$\begin{aligned} h : E_r^{0,i}(M^\bullet) \otimes_A {}'E_r^{0,j}(N^\bullet)^\iota &\xrightarrow{1_*\otimes 1_*} E_r^i(M^\bullet) \otimes_A {}'E_r^j(N^\bullet)^\iota \xrightarrow{(T^{1-r}j_r^i \otimes T^{1-r}j_r^j)} \\ &\longrightarrow \frac{H^{i+1}(M^\bullet)[T^r]}{H^{i+1}(M^\bullet)[T^{r-1}]} \otimes_A \frac{H^{j+1}(N^\bullet)[T^r]^\iota}{H^{j+1}(N^\bullet)[T^{r-1}]^\iota} \xrightarrow{f} H^{i+j-2}(P^\bullet \otimes_\Lambda (T^{-r}\Lambda/T^{-r+1}\Lambda)) \xrightarrow{T^{2r}} \\ &\longrightarrow H^{i+j-2}(P^\bullet \otimes_\Lambda (T^r\Lambda/T^{r+1}\Lambda)) \end{aligned}$$

(where the arrow f is equal to $f = \langle\ ,\ \rangle^{(r)}_{\pi,i+1,j+1}$) is equal to $h = (-1)^{j+1} h^{(r)}_{\pi,i,j}$.

Proof. — An element $a \in E_r^{0,i}(M^\bullet)$ (resp., $b \in {}'E_r^{0,j}(N^\bullet)^\iota$) is represented by $\alpha' \in M^i$ (resp., $\beta' \in (N^j)^\iota$) satisfying $d\alpha' = T^r\alpha$ (resp., $d\beta' = T^r\beta$) for some $\alpha \in M^{i+1}$ (resp., $\beta \in (N^{j+1})^\iota$). Then $h^{(r)}_{\pi,i,j}(a \otimes b) = d_r^{0,i}(a) \cup_r b \in E_r^{r,i+j+1-r}(P^\bullet[-3])$ is represented by $d\alpha' \cup \beta' = T^r\alpha \cup \beta'$. On the other hand, the same calculation as in the proof of Lemma 10.1.6 shows that the value of the pairing $h(a \otimes b)$ is represented by $T^{2r} v \circ s_{23}(\widetilde{\alpha} \cup \widetilde{\beta})$, where

$$\widetilde{\alpha} = \alpha \otimes 1 + (-1)^{i+1}\alpha' \otimes T^{-r}, \quad \widetilde{\beta} = \beta \otimes 1 + (-1)^{j+1}\beta' \otimes T^{-r},$$

i.e., by

$$T^{2r}\left((\alpha\cup\beta)\otimes 1+(-1)^{j+1}(\alpha\cup\beta')\otimes T^{-r}\right)\ (\operatorname{mod} P^{\bullet}[-3]\otimes_{\Lambda}T^{r+1}\Lambda),$$

which proves the claim. ☐

11.8.5. Assume that we are in the situation of 11.1.3 with $\Gamma \xrightarrow{\sim} \mathbf{Z}_p$ ($\gamma \mapsto 1$) and $\mathfrak{q} \in \operatorname{Spec}(R)$ is a prime ideal with $\operatorname{ht}(\mathfrak{q}) = 0$ such that $R_{\mathfrak{q}}$ is an integral domain ($\iff$ $R_{\mathfrak{q}}$ is a field). Denote by $\overline{\mathfrak{q}} \subset \overline{R} = R[\![\Gamma]\!] = R[\![\gamma-1]\!]$ the inverse image of $\mathfrak{q}$ under the augmentation map; the localization $\Lambda = \overline{R}_{\overline{\mathfrak{q}}}$ is a discrete valuation ring with involution ι (induced by the standard involution on $\overline{R}$), $T = \gamma - 1$ is a prime element of Λ and $\iota(T) \equiv -T \pmod{T^2}$.

The discussion in 11.8.1–11.8.4 applies to the complexes

$$M^{\bullet} = \widetilde{C}^{\bullet}_f(\mathscr{F}_{\Gamma}(X))_{\overline{\mathfrak{q}}},\quad N^{\bullet} = \widetilde{C}^{\bullet}_f(\mathscr{F}_{\Gamma}(Y))_{\overline{\mathfrak{q}}},\quad P^{\bullet} = (\omega^{\bullet}_R\otimes_R\overline{R})_{\overline{\mathfrak{q}}}$$

and the cup product $\cup : M^{\bullet}\otimes_{\Lambda}(N^{\bullet})^{\iota} \to P^{\bullet}[-3]$ from (11.5.3.1).

If $i + j = 2$, then the pairing $\langle\ ,\ \rangle^{(r)}_{\pi,i+1,j+1}$ defined in 11.8.2 is equal to the r-th graded quotient (with respect to the T-adic=$\overline{\mathfrak{q}}$-adic filtration) of the localized generalized Cassels-Tate pairing

$$\left(\widetilde{H}^{i+1}_{f,\mathrm{Iw}}(K_{\infty}/K,X)_{\overline{\mathfrak{q}}}\right)_{(\overline{R}_{\overline{\mathfrak{q}}})\text{-tors}}\otimes_{\overline{R}_{\overline{\mathfrak{q}}}}\left(\widetilde{H}^{j+1}_{f,\mathrm{Iw}}(K_{\infty}/K,Y)^{\iota}_{\overline{\mathfrak{q}}}\right)_{(\overline{R}_{\overline{\mathfrak{q}}})\text{-tors}} \longrightarrow \operatorname{Frac}(\overline{R}_{\overline{\mathfrak{q}}})/\overline{R}_{\overline{\mathfrak{q}}}\qquad (i+j=2).$$

According to Proposition 11.8.4, it is equal (up to a sign) to the localization at $\overline{\mathfrak{q}}$ of the higher height pairing $\widetilde{h}^{(r)}_{\pi,i,j}$ from 11.5.5.

In the special case $R = \mathbf{Z}_p$, Howard [**Ho3**] uses this description as a definition of higher height pairings.

11.9. Higher height pairings and parity results

In this section we continue to assume that $\Gamma \xrightarrow{\sim} \mathbf{Z}_p$, $\gamma \mapsto 1$. See [**Ho3**] for similar results in the special case $R = \mathbf{Z}_p$.

11.9.1. Proposition. — *Assume that we are in the situation of 11.7.1 and the conditions 11.7.1.1–11.7.1.3 hold for $\mathfrak{p}_0 = \mathfrak{q} \in \operatorname{Spec}(R)$, $\operatorname{ht}(\mathfrak{q}) = 0$. Let $\overline{\mathfrak{q}} \subset \overline{R} = R[\![T]\!] = R[\![\gamma - 1]\!]$ be the inverse image of $\mathfrak{q}$ under the augmentation map. Then, for each $Z = X, Y$ and $n \in \mathbf{Z}$,*

(i) *There is an exact sequence*

$$0 \longrightarrow \widetilde{H}^{n}_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}}/T\widetilde{H}^{n}_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}} \longrightarrow \widetilde{H}^{n}_{f}(Z)_{\mathfrak{q}} \longrightarrow \widetilde{H}^{n+1}_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}}[T] \longrightarrow 0.$$

(ii) $(\forall n \neq 1, 2)$ $\widetilde{H}^{n}_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}} = 0$; $\widetilde{H}^{1}_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}}[T] = 0$.

(iii) *For each* $r \geqslant 1$, $\mathrm{Fil}^r_\Gamma \widetilde{H}^1_f(Z)_{\mathfrak{q}}$ *is equal to the inverse image of* $T^{r-1}\big(\widetilde{H}^2_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}}[T^r]\big)$*; in particular,* $\mathrm{Fil}^\infty_\Gamma \widetilde{H}^1_f(Z)_{\mathfrak{q}}$ *coincides with the submodule of universal norms* $\mathrm{Im}\big(\widetilde{H}^1_{f,\mathrm{Iw}}(Z)_{\overline{\mathfrak{q}}}/T \hookrightarrow \widetilde{H}^1_f(Z)_{\mathfrak{q}}\big)$.

(iv) *For each* $r \geqslant 1$, $h^{(r)}_{\pi,1,1}$ *induces a non-degenerate paring on the graded quotients*

$$(\ , \ )_r : \mathrm{gr}^r \widetilde{H}^1_f(X)_{\mathfrak{q}} \times \mathrm{gr}^r \widetilde{H}^1_f(Y)_{\mathfrak{q}} \longrightarrow H^0(\omega_R)_{\mathfrak{q}}.$$

(v) *If* $X = Y$, $X_v^+ = Y_v^+$ $(v \in \Sigma)$ *and* $\pi \circ s_{12} = -\pi$, *then the pairing* $(\ , \ )_r$ *is symmetric (resp, skew-symmetric) for* r *odd (resp.,* r *even).*

Proof. — The statements (i) and (ii) were proved in the course of the proof of Proposition 11.7.6; (iii) follows from (11.6.2.1) and 11.6.1.5 (which applies with $j_0 = 1$, by the proof of Proposition 11.7.4). Finally, (iv) (resp., (v)) follows from Proposition 11.7.4 (resp., Proposition 11.5.6 (ii)). □

***11.9.2. Proposition* (Dihedral case).** — . *In the situation of Proposition 11.9.1, assume that we are given* K_∞/K^+ *as in 10.3.5.1 satisfying 10.3.5.1.1–10.3.5.1.5. For* $\varepsilon, \varepsilon' = \pm$, *denote by* $(\ , \ )_r^{\varepsilon\varepsilon'}$ *the restriction of* $(\ , \ )_r$ *to* $\mathrm{gr}^r \widetilde{H}^1_f(X)_{\mathfrak{q}}^{\varepsilon} \times \mathrm{gr}^r \widetilde{H}^1_f(Y)_{\mathfrak{q}}^{\varepsilon'}$, *where* $(-)^\pm = (-)^{\tau=\pm 1}$. *Assume that 2 is invertible in* $R_{\mathfrak{q}}$. *Then*

(i) *For* r *odd,* $(\ , \ )_r^{++} = (\ , \ )_r^{--} = 0$ *and the pairings*

$$(\ , \ )_r^{\pm\mp} : \mathrm{gr}^r \widetilde{H}^1_f(X)_{\mathfrak{q}}^{\pm} \times \mathrm{gr}^r \widetilde{H}^1_f(Y)_{\mathfrak{q}}^{\mp} \longrightarrow H^0(\omega_R)_{\mathfrak{q}}$$

are non-degenerate. In particular,

$$\ell_{R_{\mathfrak{q}}}\big(\mathrm{gr}^r \widetilde{H}^1_f(X)_{\mathfrak{q}}^{\pm}\big) = \ell_{R_{\mathfrak{q}}}\big(\mathrm{gr}^r \widetilde{H}^1_f(Y)_{\mathfrak{q}}^{\mp}\big).$$

(ii) *For* r *even,* $(\ , \ )_r^{+-} = (\ , \ )_r^{-+} = 0$ *and the pairings*

$$(\ , \ )_r^{\pm\pm} : \mathrm{gr}^r \widetilde{H}^1_f(X)_{\mathfrak{q}}^{\pm} \times \mathrm{gr}^r \widetilde{H}^1_f(Y)_{\mathfrak{q}}^{\pm} \longrightarrow H^0(\omega_R)_{\mathfrak{q}}$$

are non-degenerate.

Proof. — By Proposition 11.5.9,

$$(\tau x, \tau y)_r = (-1)^r (x, y)_r,$$

hence

$$(\ , \ )_r^{\varepsilon\varepsilon'} = (-1)^r \varepsilon\varepsilon' (\ , \ )_r^{\varepsilon\varepsilon'}.$$

As 2 is invertible in $R_{\mathfrak{q}}$, it follows that $(\ , \ )_r^{\varepsilon\varepsilon'} = 0$ if $(-1)^r\varepsilon\varepsilon' = -1$. The rest follows from the non-degeneracy of $(\ , \ )_r$. □

***11.9.3. Proposition* (Self-dual dihedral case).** — *In the situation of Proposition 11.9.2, assume that* $X = Y$, $X_v^+ = Y_v^+$ $(v \in \Sigma)$ *and* $\pi \circ s_{12} = -\pi$. *Then*

(i) *For* r *odd,*

$$\ell_{R_{\mathfrak{q}}}\big(\mathrm{gr}^r \widetilde{H}^1_f(X)_{\mathfrak{q}}\big) = 2\ell_{R_{\mathfrak{q}}}\big(\mathrm{gr}^r \widetilde{H}^1_f(X)_{\mathfrak{q}}^{\pm}\big) \equiv 0 \pmod 2.$$

(ii) *For r even,* $(\ ,\ )_r^{\pm\pm}$ *is a symplectic (= non-degenerate alternating) pairing on* $\mathrm{gr}^r \widetilde{H}^1_f(X)_{\mathfrak{q}}^{\pm}$ *and*

$$\ell_{R_{\mathfrak{q}}}\big(\mathrm{gr}^r \widetilde{H}^1_f(X)_{\mathfrak{q}}^{\pm}\big) \equiv 0 \pmod 2.$$

(iii) $\ell_{R_{\mathfrak{q}}}\big(\widetilde{H}^1_f(X)_{\mathfrak{q}}/(\mathrm{Fil}^{\infty}_{\Gamma})_{\mathfrak{q}}\big) \equiv 0 \pmod 2$.

(iv) *Let* $\mathfrak{q}' = \mathfrak{q}\overline{R} \in \mathrm{Spec}(\overline{R})$. *Then*

$$\ell_{R_{\mathfrak{q}}}\big(\widetilde{H}^1_f(X)_{\mathfrak{q}}\big) \equiv \ell_{\overline{R}_{\mathfrak{q}'}}\big(\widetilde{H}^1_{f,\mathrm{Iw}}(X)_{\mathfrak{q}'}\big) \pmod 2.$$

Proof

(i) The lengths of $\mathrm{gr}^r \widetilde{H}^1_f(X)_{\mathfrak{q}}^{\pm}$ are equal, by Proposition 11.9.2 (i).

(ii) This follows from Proposition 11.9.2 (ii) and Lemma 10.7.4 (ii).

(iii) Combining (i) and (ii), we obtain

$$\ell_{R_{\mathfrak{q}}}\big(\widetilde{H}^1_f(X)_{\mathfrak{q}}/(\mathrm{Fil}^{\infty}_{\Gamma})_{\mathfrak{q}}\big) = \sum_{r\geqslant 1} \ell_{R_{\mathfrak{q}}}\big(\mathrm{gr}^r \widetilde{H}^1_f(X)_{\mathfrak{q}}\big) \equiv 0 \pmod 2.$$

(iv) As $\widetilde{H}^1_{f,\mathrm{Iw}}(X)_{\overline{\mathfrak{q}}}[T] = 0$, we have

$$\ell_{R_{\mathfrak{q}}}\big((\mathrm{Fil}^{\infty}_{\Gamma})_{\mathfrak{q}}\big) = \ell_{R_{\mathfrak{q}}}\big(\widetilde{H}^1_{f,\mathrm{Iw}}(X)_{\overline{\mathfrak{q}}}/T\big) = \ell_{\overline{R}_{\mathfrak{q}'}}\big(\widetilde{H}^1_{f,\mathrm{Iw}}(X)_{\mathfrak{q}'}\big). \qquad \square$$

11.9.4. In the special case when $R_{\mathfrak{q}}$ is an integral domain ($\Longleftrightarrow$ $R_{\mathfrak{q}}$ is a field), we have $H^0(\omega_R)_{\mathfrak{q}} = R_{\mathfrak{q}}$ and $\overline{R}_{\overline{\mathfrak{q}}}$ is a discrete valuation ring with prime element $T = \gamma - 1$. The statement (resp., the proof) of Proposition 11.9.3 then boils down to that of Lemma 10.6.8, as the pairing $\langle\ ,\ \rangle_{r,T}$ from *loc. cit.* coincides, up to a sign, with the pairing $(\ ,\ )_r$, by 11.8.5.

CHAPTER 12

PARITY OF RANKS OF SELMER GROUPS

In this chapter we generalize the parity results of [**Ne3**] to elliptic curves over totally real number fields and to Selmer groups associated to Hilbert modular forms of parallel (even) weight.

12.1. The general setup

12.1.1. Let F and L be number fields (contained in a fixed algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$) and M a "motive" over F with coefficients in L. The L-function $L(M,s) = \sum_{n=1}^{\infty} a_n n^{-s}$ (assuming it is well-defined) is a Dirichlet series with coefficients $a_n \in L$. For each embedding $\iota : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, define $L(\iota M, s) = \sum_{n=1}^{\infty} \iota(a_n) n^{-s}$. Conjecturally, $L(\iota M, s)$ admits a meromorphic continuation to $\mathbf{C}$ and a functional equation of the form

$$L_\infty(\iota M, s)\, L(\iota M, s) \stackrel{?}{=} \varepsilon(\iota M, s)\, L_\infty(\iota M^*(1), -s)\, L(\iota M^*(1), -s),$$

where

$$L_\infty(\iota M, s) = \prod_{v \mid \infty} L_v(\iota M, s)$$

is a suitable product of Γ-factors (independent of ι) and

$$\varepsilon(\iota M, s) = \prod_v \varepsilon_v(\iota M, s) = \iota(\varepsilon(M, 0)) \cdot f(M)^{-s},$$

where $\varepsilon(M,0) \in \overline{\mathbf{Q}}^*$ is the global ε-factor of M and $f(M) \in \mathbf{N}$ is the conductor of M.

12.1.2. If M is a pure motive of weight $w \neq -2$, the Bloch-Kato Conjectures ([**B-K, Fo-PR, Fo1**]) predict that

$$(C_{BK}) \qquad \operatorname{ord}_{s=0} L(\iota M, s) \stackrel{?}{=} \dim_L\, H^1_{f,\mathcal{M}}(F, M^*(1)),$$

where $H^1_{f,\mathcal{M}}$ is a suitable motivic cohomology group (in particular, the L.H.S. should not depend on ι).

For each prime $\mathfrak{p} \mid p$ of L, let $M_\mathfrak{p}$ be the $\mathfrak{p}$-adic realization of M; this is a continuous $L_\mathfrak{p}$-representation of $G_{F,S}$, for a suitable finite set of primes S. Bloch and Kato conjectured that the "$\mathfrak{p}$-adic realization map" induces an isomorphism

$$(C_\mathfrak{p}) \qquad H^1_{f,\mathcal{M}}(F, M^*(1)) \otimes_L L_\mathfrak{p} \xrightarrow{\sim} H^1_f(F, M^*_\mathfrak{p}(1)),$$

where the R.H.S. is the Bloch-Kato generalized Selmer group. Combining (C_{BK}) with $(C_\mathfrak{p})$ leads to a much weaker (and more accessible) version of the conjecture (C_{BK}), namely

$$(C_{BK,\mathfrak{p}}) \qquad \mathrm{ord}_{s=0}\, L(\iota M, s) \stackrel{?}{=} \dim_{L_\mathfrak{p}}\ H^1_f(F, M^*_\mathfrak{p}(1)).$$

12.1.3. We shall concentrate on the case of a **self-dual** (pure) motive $M \xrightarrow{\sim} M^*(1)$. In this case $w = -1$ and $s = 0$ is the center of symmetry of the conjectural functional equation

$$(C_{FE}) \qquad L_\infty(\iota M, s)\, L(\iota M, s) \stackrel{?}{=} \varepsilon(\iota M, s)\, L_\infty(\iota M, -s)\, L(\iota M, -s),$$
$$\varepsilon(\iota M, s) = \iota(\varepsilon(M, 0)) \cdot f(M)^{-s}.$$

For each prime $\mathfrak{p} \mid p$ of L, the induced isomorphism of Galois representations $M_\mathfrak{p} \xrightarrow{\sim} M^*_\mathfrak{p}(1)$ should be skew-symmetric. The conjecture $(C_{BK,\mathfrak{p}})$ then reads as

$$(12.1.3.1) \qquad \mathrm{ord}_{s=0}\, L(\iota M, s) \stackrel{?}{=} h^1_f(F, M_\mathfrak{p}) := \dim_{L_\mathfrak{p}}\ H^1_f(F, M_\mathfrak{p}).$$

12.1.4. Examples

(i) An archetypal example of a self-dual pure motive is given by $M = h^1(E)(1)$, where E is an elliptic curve over F. In this case we have $L = \mathbf{Q}$, $L(\iota M, s) = L(E/F, s+1)$, $\mathfrak{p} = p$, $M_\mathfrak{p} = T_p(E) \otimes \mathbf{Q} = V_p(E)$, the isomorphism $M_\mathfrak{p} \xrightarrow{\sim} M^*_\mathfrak{p}(1)$ is induced by the Weil pairing, $H^1_{f,\mathcal{M}}(F, M) = E(F) \otimes \mathbf{Q}$ and there is an exact sequence

$$0 \longrightarrow E(F) \otimes \mathbf{Q}_p \longrightarrow H^1_f(F, V_p(E)) \longrightarrow (T_p Ш(E/F)) \otimes \mathbf{Q} \longrightarrow 0.$$

This means that

$$(12.1.4.1) \qquad \mathrm{ord}_{s=0}\, L(\iota M, s) = \mathrm{ord}_{s=1}\, L(E/F, s),$$
$$h^1_f(F, V_p(E)) = \mathrm{rk}_\mathbf{Z} E(F) + \mathrm{cork}_{\mathbf{Z}_p} Ш(E/F)[p^\infty],$$

hence the conjecture (C_{BK}) (resp., $(C_\mathfrak{p})$) is equivalent to the Conjecture of Birch and Swinnerton-Dyer (resp., to the finiteness of the p-primary part of $Ш(E/F)$).

(ii) More generally, let A be an abelian variety over F and L a totally real number field with $\mathcal{O}_L \hookrightarrow \mathrm{End}_F(A)$, $\dim(A) = [L : \mathbf{Q}]$. Then $M = h^1(A)(1)$ is self-dual as a rank 2 motive with coefficients in L, $M_\mathfrak{p} = V_\mathfrak{p}(A)$, $L(\iota M, s) = L(\iota A/F, s+1)$, $H^1_{f,\mathcal{M}}(F, M) = A(F) \otimes \mathbf{Q}$, there is an exact sequence

$$0 \longrightarrow A(F) \otimes_{\mathcal{O}_L} L_\mathfrak{p} \longrightarrow H^1_f(F, V_\mathfrak{p}(A)) \longrightarrow (T_\mathfrak{p} Ш(A/F)) \otimes \mathbf{Q} \longrightarrow 0$$

and

$$\text{(12.1.4.2)} \qquad \begin{gathered} \operatorname{ord}_{s=0} L(\iota M, s) = \operatorname{ord}_{s=1} L(\iota A/F, s), \\ h^1_f(F, V_{\mathfrak{p}}(A)) = \operatorname{rk}_{\mathcal{O}_L} A(F) + \operatorname{cork}_{\mathcal{O}_{L,\mathfrak{p}}} Ш(A/F)[\mathfrak{p}^\infty]. \end{gathered}$$

12.1.5. Returning to the case of an arbitrary self-dual pure motive $M \xrightarrow{\sim} M^*(1)$, the general recipe [**Se4**, §3] implies that

$$\operatorname{ord}_{s=0} L_\infty(\iota M, s) = 0,$$

i.e., the central point $s = 0$ is "critical" in the sense of Deligne [**De3**]. The functional equation (C_{FE}) (if available) then yields

$$\text{(12.1.5.1)} \qquad (-1)^{\operatorname{ord}_{s=0} L(\iota M, s)} = \varepsilon(M, 0) \in \{\pm 1\}.$$

We shall be interested in the conjecture (12.1.3.1) modulo 2, *i.e.*, in

$$\text{(12.1.5.2)} \qquad \operatorname{ord}_{s=0} L(\iota M, s) \stackrel{?}{\equiv} h^1_f(F, M_{\mathfrak{p}}) \pmod 2.$$

Thanks to (12.1.5.1), this conjectural congruence can be reformulated (assuming the validity of (C_{FE})) as

$$\text{(12.1.5.3)} \qquad (-1)^{h^1_f(F, M_{\mathfrak{p}})} \stackrel{?}{=} \varepsilon(M, 0) = \prod_v \varepsilon_v(M, 0).$$

The local ε-factors $\varepsilon_v(M, 0) \in \{\pm 1\}$ depend only on the Galois representation $M_{\mathfrak{p}}$ (assuming the standard compatibilities between $M_{\mathfrak{p}}$ and other realizations of M):

If $v \mid \infty$, then $\varepsilon_v(M, 0)$ depends only on the Hodge numbers of the de Rham realization M_{dR} of M (and on the action of the complex conjugation at v on $M_{\mathfrak{p}}$ if v is a real prime). These Hodge numbers can be read off from the comparison isomorphism $M_{\mathrm{dR}} \otimes B_{\mathrm{dR}} \xrightarrow{\sim} M_{\mathfrak{p}} \otimes B_{\mathrm{dR}}$ (at a fixed prime of F above p). If $v \nmid \infty$, then the action of G_v on $M_{\mathfrak{p}}$ gives rise to a representation $WD_v(M_{\mathfrak{p}})$ of the Weil-Deligne group of F_v (for $v \nmid p$, this is a classical fact ([**De2**, §8.4]); for $v \mid p$ one uses the potential semistability of $M_{\mathfrak{p}}$ at v and applies the recipe in [**Fo2**] (cf. [**Sa**, remarks before Thm. 1])). One defines, using the notation of [**De2**, (5.5.2), §8.12],

$$\varepsilon_v(M_{\mathfrak{p}}, 0) = \varepsilon_v(WD_v(M_{\mathfrak{p}}), \psi_v, dx_{\psi_v}, 0),$$

where ψ_v is any non-trivial additive character of F_v and dx_{ψ_v} the corresponding self-dual Haar measure on F_v. The value of $\varepsilon_v(M_{\mathfrak{p}}, 0)$ is equal to ± 1 ([**De2**, (5.7.1), §8.12]) and does not depend on ψ_v ([**De2**, (5.3)–(5.4), §8.12]).

It is expected that the isomorphism class of $WD_v(M_{\mathfrak{p}})$ is defined over L and does not depend on $\mathfrak{p}$; if true, then the local ε-factor

$$\varepsilon_v(M, 0) := \varepsilon_v(M_{\mathfrak{p}}, 0) \in \{\pm 1\}$$

is well-defined. Assuming this independence on $\mathfrak{p}$, one can reformulate (12.1.5.3) as

$$(C_{\mathrm{sgn}}(M_{\mathfrak{p}})) \qquad (-1)^{h^1_f(F,M_{\mathfrak{p}})} \stackrel{?}{=} \varepsilon(M_{\mathfrak{p}},0) = \prod_v \varepsilon_v(M_{\mathfrak{p}},0).$$

12.1.6. The conjectural equality $(C_{\mathrm{sgn}}(M_{\mathfrak{p}}))$ does not involve the motive M, only its $\mathfrak{p}$-adic realization $M_{\mathfrak{p}}$. Moreover, it makes sense to consider $(C_{\mathrm{sgn}}(V))$ for an arbitrary self-dual $\mathfrak{p}$-adic Galois representation $V \xrightarrow{\sim} V^*(1)$ of G_F which is geometric in the sense of [**Fo-PR**, §II.3.1.1] and [**Fo-Ma**, §I.1], *i.e.*, unramified outside a finite set S of primes of F and potentially semistable at all primes above p.

We propose to study the **variation** of the conjecture $(C_{\mathrm{sgn}}(V))$ in p-adic families of Galois representations. In vague terms, assume that

(a) $\mathscr{X}$ is an irreducible p-adic analytic space.

(b) $V^{(\lambda)}$ is a family of $L_{\mathfrak{p}}$-representations of $G_{F,S}$ depending analytically on a parameter $\lambda \in \mathscr{X}(L_{\mathfrak{p}})$.

(c) Each representation $V^{(\lambda)}$ is equipped with a skew-symmetric isomorphism $V^{(\lambda)} \xrightarrow{\sim} \left(V^{(\lambda)}\right)^*(1)$, varying analytically in λ.

(d) There is a Zariski dense subset $\mathscr{X}^{\mathrm{mot}} \subset \mathscr{X}(L_{\mathfrak{p}})$ of "motivic" values of the parameter for which $V^{(\lambda)}$ is the $\mathfrak{p}$-adic realization of a (pure) motive $M^{(\lambda)}$.

The first observation is that neither side of the conjecture $(C_{\mathrm{sgn}}(V^{(\lambda)}))$ is, in general, constant on $\mathscr{X}^{\mathrm{mot}}$. The point is that both terms are related to the **complex-valued L-function** $L(\iota M^{(\lambda)}, s)$, while a good p-adic behaviour can only be expected from a suitable **p-adic L-function** $L_p(\lambda, s)$ (which depends on fixed embeddings $\iota : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$).

In favorable circumstances, such a p-adic L-function satisfies, for **critical** values $(\lambda, n) \in \mathscr{X}^{\mathrm{mot}} \times \mathbf{Z}$, an interpolation property of the form

$$L_p(\lambda, n) = \mathrm{Eu}(\lambda, n)\, \frac{L(\iota M^{(\lambda)}, n)}{\Omega(\iota, \lambda, n)},$$

where $\Omega(\iota, \lambda, n)$ is a suitable period and

$$\mathrm{Eu}(\lambda, n) = \prod_{v \mid p} \mathrm{Eu}_v(\lambda, n)$$

a product of appropriate Euler factors at primes of F above p.

Whenever one of the Euler factors $\mathrm{Eu}_v(\lambda, 0)$ vanishes, it forces the function $s \mapsto L_p(\lambda, s)$ to have a "trivial" (or "exceptional") zero at $s = 0$. In a self-dual situation, this phenomenon was first studied by Mazur-Tate-Teitelbaum [**M-T-T**], who observed that the presence of a trivial zero at $\lambda \in \mathscr{X}^{\mathrm{mot}}$ may cause a mismatch between the signs in the functional equations of $L\big(\iota M^{(\lambda)}, s\big)$ and $s \mapsto L_p(\lambda, s)$. One would expect, in general, that an appropriate functional equation for the "two-variable" p-adic L-function $L_p(\lambda, s)$ will induce functional equations for each "one-variable" p-adic L-function $s \mapsto L_p(\lambda, s)$ $(\lambda \in \mathscr{X}^{\mathrm{mot}})$ with a **common sign**

$$\widetilde{\varepsilon}\big(V^{(\lambda)}, 0\big) = \widetilde{\varepsilon} = \pm 1,$$

while

$$\begin{aligned} \varepsilon\big(M^{(\lambda)}, 0\big) &= \varepsilon\big(V^{(\lambda)}, 0\big) \\ &= \widetilde{\varepsilon}\big(V^{(\lambda)}, 0\big) \prod_{v \mid p} (\text{contribution from a possible trivial zero of } \mathrm{Eu}_v(\lambda, 0)) \end{aligned}$$

($i.e.$, the presence of a trivial zero at $\lambda \in \mathscr{X}^{\mathrm{mot}}$ causes a jump in the value of $\varepsilon\big(M^{(\lambda)}\big) = \varepsilon\big(V^{(\lambda)}\big)$). This happens, for example, in the case of the two-variable p-adic L-function associated to a Hida family of p-ordinary modular forms [**G-S**].

12.1.7. As proposed first in a special case in [**M-T-T**], a p-adic variant of the conjecture (12.1.3.1) for $M = M^{(\lambda)}$ $(\lambda \in \mathscr{X}^{\mathrm{mot}})$ should read as

$$\text{(12.1.7.1)} \qquad \mathrm{ord}_{s=0}\, L_p(\lambda, s) \overset{?}{=} \widetilde{h}^1_f\big(F, V^{(\lambda)}\big) := \dim_{L_{\mathfrak{p}}} \widetilde{H}^1_f\big(F, V^{(\lambda)}\big),$$

where $\widetilde{H}^1_f\big(F, V^{(\lambda)}\big)$ is an appropriate "extended Selmer group", which incorporates $H^1_f\big(F, V^{(\lambda)}\big)$, as well as the trivial zeros of $\mathrm{Eu}(\lambda, 0)$.

If $M^{(\lambda)} = h^1(E)(1)$ for an elliptic curve E over F (say, with semistable reduction at all primes above p), then the Euler factor $\mathrm{Eu}_v(\lambda, 0)$ at a prime $v \mid p$ vanishes $\iff$ E has split multiplicative reduction at v. The appropriate extended Selmer group was defined in this case in [**M-T-T**].

12.1.8. The theory developed in the previous chapters leads naturally to extended Selmer groups, at least in the **quasi-ordinary case**. More precisely, assume that V is a (continuous) $L_{\mathfrak{p}}$-representation of $G_{F,S}$ (with S containing all archimedean primes and all primes above p in F), which is equipped with the following data:

12.1.8.1. A skew-symmetric isomorphism of $L_{\mathfrak{p}}[G_{F,S}]$-modules $V \xrightarrow{\sim} V^*(1)$.

12.1.8.2. For each prime $v \mid p$ of F, an exact sequence of $L_{\mathfrak{p}}[G_v]$-modules

$$0 \longrightarrow V_v^+ \longrightarrow V \longrightarrow V_v^- \longrightarrow 0,$$

for which the self-duality isomorphism $V \xrightarrow{\sim} V^*(1)$ from (a) induces isomorphisms of $L_{\mathfrak{p}}[G_v]$-modules $V_v^+ \xrightarrow{\sim} (V_v^-)^*(1)$, $V_v^- \xrightarrow{\sim} (V_v^+)^*(1)$.

These data define Greenberg's local conditions 7.8.2

$$U_v^+(V) = \begin{cases} C^\bullet_{\mathrm{cont}}(G_v, V_v^+), & v \mid p \\ C^\bullet_{\mathrm{cont}}(G_v/I_v, V^{I_v}), & v \in S_f,\ v \nmid p \end{cases}$$

for V (with $\Sigma = \{v \mid p\}$). The cohomology groups of the corresponding Selmer complex $\widetilde{C}^{\bullet}_f(G_{F,S}, V; \Delta(V))$ do not depend on S (by Proposition 7.8.8(ii)); we denote them by $\widetilde{H}^j_f(F, V)$. The third triangle in (6.1.3.2) gives rise to an exact sequence

$$0 \longrightarrow \bigoplus_{v|p} H^0(G_v, V_v^-) \longrightarrow \widetilde{H}^1_f(F, V) \longrightarrow H^1_{\mathrm{cont}}(G_{F,S}, V) \longrightarrow \bigoplus_{v|p} H^1(G_v, V_v^-) \oplus \bigoplus_{v \in S_f, v\nmid p} H^1(I_v, V)^{G_v/I_v},$$

which boils down, under suitable additional assumptions (cf. 12.5.9.2 below), to

$$0 \longrightarrow \bigoplus_{v|p} H^0(G_v, V_v^-) \longrightarrow \widetilde{H}^1_f(F, V) \longrightarrow H^1_f(F, V) \longrightarrow 0. \tag{12.1.8.3}$$

We interpret $\widetilde{H}^1_f(F, V)$ as an "extended Selmer group" and each term $H^0(G_v, V_v^-)$ (if non-trivial) as a contribution from a trivial zero.

For example, if E is an elliptic curve over F with ordinary reduction (= multiplicative or good ordinary reduction) at each prime above p, then $V = V_p(E)$ is naturally equipped with the data 12.1.8.1–12.1.8.1 (see 9.6.7.2) and the sequence (12.1.8.3) is exact, with

$$\dim_{\mathbf{Q}_p} H^0(G_v, V_p(E)) = \begin{cases} 1, & E \text{ has split multiplicative reduction at } v \\ 0, & \text{otherwise,} \end{cases}$$

by Lemma 9.6.3, 9.6.7.3 and 9.6.7.6(ii). See [**M-T-T**] for a more explicit definition of $\widetilde{H}^1_f(F, V)$ in this special case.

12.1.9. Assume that the representation V from 12.1.8.1–12.1.8.2 is potentially semistable at all primes above p and that the sequence (12.1.8.3) is exact. For each prime v of F, we define

$$\widetilde{\varepsilon}_v(V, 0) = \varepsilon_v(V, 0) \cdot \begin{cases} (-1)^{\dim_{L_{\mathfrak{p}}} H^0(G_v, V_v^-)}, & v \mid p \\ 1, & v \nmid p \end{cases}$$

and we set

$$\widetilde{\varepsilon}(V, 0) = \prod_v \widetilde{\varepsilon}_v(V, 0) = \varepsilon(V, 0) \prod_{v|p} (-1)^{\dim_{L_{\mathfrak{p}}} H^0(G_v, V_v^-)} \in \{\pm 1\}.$$

With this notation, the conjecture $(C_{\mathrm{sgn}}(V))$ is equivalent to

$$(-1)^{\widetilde{h}^1_f(F,V)} \stackrel{?}{=} \widetilde{\varepsilon}(V, 0) = \prod_v \widetilde{\varepsilon}_v(V, 0). \tag{$\widetilde{C}_{\mathrm{sgn}}(V)$}$$

If the above discussion applies to the representation $V = V^{(\lambda)}$ ($\lambda \in \mathscr{X}^{\mathrm{mot}}$) from 12.1.6, then $\widetilde{\varepsilon}(V, 0)$ should be equal to the sign in the functional equation of the p-adic L-function $s \mapsto L_p(\lambda, s)$, hence $(\widetilde{C}_{\mathrm{sgn}}(V))$ would be just the conjecture (12.1.7.1) modulo 2.

12.1.10. Assume that each member $V = V^{(\lambda)}$ $(\lambda \in \mathscr{X}(L_\mathfrak{p}))$ of the family of Galois representations considered in 12.1.6 is equipped with the structure 12.1.8.1–12.1.8.2, depending analytically on λ.

It seems very likely that the following principles hold under very general circumstances:

(I) $\widetilde{h}^1_f(F, V) \pmod 2$ does not depend on $\lambda \in \mathscr{X}^{\mathrm{mot}}$.

(IIa) $(\forall v \nmid p\infty)$ $\varepsilon_v\big(V^{(\lambda)}, 0\big)$ does not depend on $\lambda \in \mathscr{X}^{\mathrm{mot}}$.

(IIb) $\prod_{v \mid p\infty} \widetilde{\varepsilon}_v\big(V^{(\lambda)}, 0\big)$ does not depend on $\lambda \in \mathscr{X}^{\mathrm{mot}}$.

(III) The presence of a non-trivial Euler system for $V^{(\lambda)}$ implies the conjectural equality (12.1.3.1) (resp., (12.1.7.1)), provided the L.H.S. is equal to 0 or 1.

Once the principles (I) and (II) are established, then the validity of the conjecture $\big(\widetilde{C}_{\mathrm{sgn}}\big(V^{(\lambda)}\big)\big)$ $(\Longleftrightarrow$ $\big(C_{\mathrm{sgn}}\big(V^{(\lambda)}\big)\big))$ does not depend on $\lambda \in \mathscr{X}^{\mathrm{mot}}$. In order to prove $\big(\widetilde{C}_{\mathrm{sgn}}\big(V^{(\lambda)}\big)\big)$ for all $\lambda \in \mathscr{X}^{\mathrm{mot}}$, it then remains to find one motivic value λ to which the principle (III) would apply.

12.1.11. In [**N-P**] (resp., in [**Ne3**]), this strategy was applied to a Hida family of classical p-ordinary modular forms (resp., to anticyclotomic Iwasawa theory of an elliptic curve with good ordinary reduction at p). In both cases, the principles (I) and (a global version of) (II) were established. The results of [**N-P**] were conditional, as the principle (III) applied only half-way. The relevant Euler system arguments were available ([**Ka2, Ne1**]); what was missing was the proof that the Euler systems in question were non-trivial. In the anticyclotomic situation considered in [**Ne3**], the principle (III) was available, thanks to the proof of Mazur's conjecture on Heegner points ([**Cor, Va**]).

This chapter offers a simultaneous generalization of [**N-P**] and [**Ne3**] to (twists of p-ordinary) Hilbert modular forms of parallel weight. Our main results are stated in 12.2.3, 12.2.6 below. In the proof, one applies the principles (I) and (II) in three different contexts:

(a) level raising congruences;
(b) twisted Hida family;
(c) dihedral Iwasawa theory.

In each case, the principle (I) follows from the existence of a suitable symplectic form, which is provided by the generalized Cassels-Tate pairing from Chapter 10. The principle (IIa) appears to be quite general (cf. Corollary 12.7.14.3 and Proposition 12.8.1.4 below); by contrast, we prove (IIb) in each case by an explicit calculation.

The principle (III) applies in the context of dihedral Iwasawa theory, namely to CM points on Shimura curves. The Euler system argument is provided by a generalization of Kolyvagin's original method ([**Ne4**, Thm. 3.2]), while the non-triviality of the Euler system follows from the proof of the generalized Mazur conjecture ([**Cor-Va**, Thm. 4.1]). Finally, a descent argument due to R. Taylor ([**Tay4**, §6]) allows us to treat two-dimensional self-dual Galois representations over totally real number fields that are only potentially modular (of parallel weight) and potentially ordinary. This applies, in particular, to elliptic curves and abelian varieties of GL(2)-type with a totally real coefficient field (modulo potential modularity results that seem to be well-known to the experts). The final result is stated in 12.2.8.

12.1.12. The ad hoc approach to the cases (IIb) and (IIc) in the present chapter is superseded by the subsequent work [**Ne5**], in which the principle (II) is established in a rather general context, namely for families of Galois representations which are pure and satisfy the Pančiškin condition at all primes above p.

12.2. The parity results

In this section we state our main parity results and describe the main steps of the proofs. The proofs themselves will occupy the rest of Chapter 12.

Let F be a totally real number field. Fix a prime number p and embeddings $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$.

12.2.1. Let $f \in S_k(\mathfrak{n}, \varphi)$ be a (non-zero) cuspidal Hilbert modular form of parallel weight $(k, \dots, k)$, level $\mathfrak{n} \subset \mathcal{O}_F$ and character $\varphi : \mathbf{A}_F^*/F^* \to \mathbf{C}^*$ (see 12.3 below for a more detailed discussion). We assume that f is a newform, which implies that f is an eigenform for all Hecke operators

$$T(\mathfrak{a})f = \lambda_f(\mathfrak{a})f \quad (\mathfrak{a} \subset \mathcal{O}_F).$$

Let L be a sufficiently large number field (considered as a subfield of $\overline{\mathbf{Q}}$ *via* ι_∞) containing all Hecke eigenvalues $\lambda_f(\mathfrak{a})$ and the values of φ. We shall be free to replace L by a finite extension, if necessary. The embedding ι_p induces a prime ideal $\mathfrak{p}|p$ of L. There is a continuous two-dimensional representation $V(f)$ of $G_{F,S}$ over $L_\mathfrak{p}$ associated to f (where $S = \{v \mid p\mathfrak{n}\infty\}$). It is characterized by the fact that, for each prime $v \notin S$ of F, the Euler factor

$$L_v(f, s) = \left(1 - \lambda_f(v)(Nv)^{-s} + \varphi(v)(Nv)^{k-1-2s}\right)^{-1}$$

of the standard L-function of f

$$L(f, s) = \prod_{v \nmid \infty} L_v(f, s) = \sum_{\mathfrak{a}} \frac{\lambda_f(\mathfrak{a})}{(N\mathfrak{a})^s}$$

coincides with

$$L_v(V(f), s) = \det\left(1 - \mathrm{Fr}(v)_{\mathrm{geom}}\,(Nv)^{-s} \mid V(f)^{I_v}\right)^{-1}.$$

12.2.2. Assume that $2|k$ and that there exists a character $\chi : \mathbf{A}_F^*/F^* \to \mathbf{C}^*$ satisfying $\chi^{-2} = \varphi$. Fix such a χ and denote by $g = f \otimes \chi \in S_k(\mathfrak{n}(g), 1)$ the unique newform satisfying $V(g) = V(f) \otimes \chi$ (above, $\mathfrak{n}(g)$ denotes the level of g). The Galois representation

$$V := V(g)(k/2) = V(f)(k/2) \otimes \chi$$

satisfies $\Lambda^2 V \xrightarrow{\sim} L_{\mathfrak{p}}(1)$, hence it is self-dual in the sense of 12.1.8.1. For each prime $v \nmid p\infty$ of F we have

$$L_v(V, s) = L_v(g, s + k/2).$$

The L-function $L(g, s + k/2)$ satisfies a functional equation of the form (C_{FE}). We are interested in the following variant of the conjectural equality (12.1.5.2):

$$r_{\mathrm{an}}(F, g) := \mathrm{ord}_{s=k/2} L(g, s) \stackrel{?}{\equiv} h^1_f(F, V) \pmod 2. \tag{12.2.2.1}$$

If F'/F is a finite solvable Galois extension (not necessarily totally real), then there is an automorphic form $g' = BC_{F'/F}(g)$ on $\mathrm{GL}_2(\mathbf{A}_{F'})$ satisfying, for each prime w of F' above a prime $v \nmid p\mathfrak{n}(g)\infty$ of F, the equality

$$L_w(g', s) = \det\left(1 - \mathrm{Fr}(w)_{\mathrm{geom}}\,(Nw)^{-s} \mid V(g)^{I_w}\right)^{-1}.$$

If F' is totally real or if g does not have CM, then the form g' is cuspidal. One can generalize (12.2.2.1) to the base change form g', namely

$$r_{\mathrm{an}}(F', g) := \mathrm{ord}_{s=k/2} L(g', s) \stackrel{?}{\equiv} h^1_f(F', V) \pmod 2. \tag{12.2.2.2}$$

Our first result in this direction is the following Theorem.

12.2.3. Theorem. — *Let $g = f \otimes \chi \in S_k(\mathfrak{n}(g), 1)$, where f is p-ordinary,* i.e., $(\forall v \mid p)$ $\mathrm{ord}_{\mathfrak{p}}(\lambda_f(v)) = 0$. *Let F'/F be a finite 2-abelian extension. Assume that at least one of the following assumptions (1)–(4) holds.*

(1) $2 \nmid [F : \mathbf{Q}]$.

(2) *There exists a character $\mu : \mathbf{A}_F^*/F^* \to \{\pm 1\}$ such that the level $\mathfrak{n}(g \otimes \mu)$ of the newform $g \otimes \mu$ is not a square.*

(3) *The following conditions (i)-(iii) are satisfied:*

(i) *p is prime to $k!\,\mathfrak{n}(g) d_{F'/\mathbf{Q}} w_2(L(g))$, where $L(g)$ is the (totally real) number field generated by the Hecke eigenvalues of g and $w_2(L(g))$ denotes the order of $H^0(G_{L(g)}, \mathbf{Q}/\mathbf{Z}(2))$.*

(ii) *The residual representation $\overline{\rho}_{\mathfrak{p}} : G_F \to \mathrm{Aut}(T/\mathfrak{p}T)$ (where $T \subset V$ is a G_F-stable $\mathcal{O}_{L,\mathfrak{p}}$-lattice) is irreducible.*

(iii) *If g has no CM, then there exists a basis of $T/\mathfrak{p}T$ in which $\mathrm{Im}(\overline{\rho}_{\mathfrak{p}}) \supseteq \mathrm{SL}_2(\mathbf{F}_p)$.*

(4) *For each character $\alpha : \mathrm{Gal}(F'/F) \to \{\pm 1\}$ there exists a congruence $f \otimes \alpha \equiv f_\alpha \left(\mathrm{mod}\, \mathfrak{p}^{M(f\otimes\alpha, \mathfrak{p})}\right)$ in the sense of (12.10.3.1), where $f_\alpha \in S_k(\mathfrak{n}(f \otimes \alpha) Q_\alpha, \chi^{-2})$ is a p-ordinary newform, $Q_\alpha = \mathfrak{q}_{\alpha,1} \cdots \mathfrak{q}_{\alpha,r_\alpha}$, $\mathfrak{q}_{\alpha,j} \nmid p\mathfrak{n}(f \otimes \alpha)\mathrm{cond}(\chi)$ are distinct prime ideals satisfying $(\forall j = 1, \ldots, r_\alpha)$ $\lambda_{f_\alpha}(\mathfrak{q}_{\alpha,j}) = -\chi^{-1}(\mathfrak{q}_{\alpha,j})(N\mathfrak{q}_{\alpha,j})^{k/2-1}$ and $M(f \otimes \alpha, \mathfrak{p})$*

is a certain constant (more precisely, $M(f\otimes\alpha,\mathfrak{p})$ is any value of M satisfying the assumptions of Proposition 12.8.4.14 for the form $f\otimes\alpha$ instead of f).
Then: for each finite Galois extension F''/F' of odd degree,

$$r_{\mathrm{an}}(F'',g)\equiv h^1_f(F'',V)\pmod 2.$$

Proof. — See 12.10. □

12.2.4. Remarks

(i) It is likely that our methods can be used to prove a similar result for quasi-ordinary Hilbert modular forms of non-parallel weight.

(ii) The condition 12.2.3 (2) is satisfied if there exists a prime v of F for which $\pi(g)_v$ (the local component of the automorphic representation $\pi(g)$ associated to g) is a twisted Steinberg representation: in this case there exists $\mu:\mathbf{A}_F^*/F^*\to\{\pm1\}$ such that $\mathrm{ord}_v(\mathfrak{n}(g\otimes\mu))=1$.

(iii) The condition 12.2.3 (3i) implies that $p>3$ (as $w_2(\mathbf{Q})=24$) and that g is p-ordinary (as p is prime to $\mathfrak{n}(g)$).

(iv) The condition 12.2.3 (3) rules out only finitely many prime ideals $\mathfrak{p}$ of L ([**Dim**, Prop. 3.1 (i)]).

(v) In 12.2.3 (4), instead of requiring the existence of congruences $f\otimes\alpha\equiv f_\alpha\ \left(\mathrm{mod}\,\mathfrak{p}^{M(f\otimes\alpha,\mathfrak{p})}\right)$ separately for each $\alpha:\mathrm{Gal}(F'/F)\to\{\pm1\}$, it would be sufficient to assume that, for each $M\geqslant1$, there is a congruence $f\equiv f_{(M)}\ (\mathrm{mod}\,\mathfrak{p}^M)$ with an appropriate p-ordinary newform $f_{(M)}\in S_k(\mathfrak{n}(f\otimes\alpha)Q_{(M)},\chi^{-2})$. One could then take $f_\alpha=f_{(M)}\otimes\alpha$, for any $M\geqslant\max\{M(f\otimes\alpha,\mathfrak{p})\mid\alpha:\mathrm{Gal}(F'/F)\to\{\pm1\}\}$.

(vi) It is possible that the existence of such congruences $f\equiv f_{(M)}\ (\mathrm{mod}\,\mathfrak{p}^M)$ could be established (under suitable additional assumptions) using the techniques of [**K-R**].

(vii) In fact, it would be sufficient to assume the existence of congruences $f'\equiv f'_{(M)}\ (\mathrm{mod}\,\mathfrak{p}^M)$ for a suitable fixed element f' of the Hida family containing f.

(viii) In the original version of 12.2.3 it was assumed that F''/F is an abelian extension. The current formulation was suggested by a reading of [**M-R1**].

(ix) In the case when $F''=F=\mathbf{Q}$, $\chi=1$ and $g=f\in S_k(N,1)$ is p-ordinary, Skinner and Urban [**S-U**] proved that $2\nmid r_{\mathrm{an}}(\mathbf{Q},g)\implies h^1_f(\mathbf{Q},V)\geqslant1$.

(x) Combining Theorem 12.2.3 (and Theorem 12.2.8 below) with Theorem 10.7.17 (v), we obtain a generalization of [**M-R3**, Cor. 3.6–3.7]. See Sect. 12.12 for more details.

12.2.5. One can combine Theorem 12.2.3 with base change [**Lan**] (cf. also [**A-C, J-PS-S**]), obtaining the following result.

12.2.6. Corollary. — *Let $g=f\otimes\chi\in S_k(\mathfrak{n}(g),1)$, where f is p-ordinary. Let F_1/F be a finite solvable extension of totally real number fields and F_1'/F_1 a finite 2-abelian extension. Assume that at least one of the following conditions (1)–(3) holds:*

(1) $2\nmid[F_1:\mathbf{Q}]$.

(2) *There exists a prime v of F such that $\pi(g)_v$ is a twisted Steinberg representation.*

(3) *The condition 12.2.3(3) is satisfied and p is unramified in $F_1'/\mathbf{Q}$.*

Then: for each finite Galois extension of odd degree F_1''/F_1',

$$r_{\mathrm{an}}(F_1'', g) \equiv h^1_f(F_1'', V) \pmod 2.$$

Proof. — This follows from Theorem 12.2.3 applied to the base change form $g_1 = BC_{F_1/F}(g)$. In view of 12.2.4(ii), the condition 12.2.6(2) implies that 12.2.3(2) holds for g_1. If the condition 12.2.6(3) holds, then 12.2.3(3iii) holds for g_1 over F_1, since $\mathrm{Gal}(F_1/F)$ is solvable and $\mathrm{SL}_2(\mathbf{F}_p)$ has no non-trivial solvable quotients (as $p > 3$, by 12.2.3(3i) for g). □

12.2.7. Furthermore, combining Corollary 12.2.6 with the descent arguments of [**Tay3, Tay4**], we deduce the following theorem.

12.2.8. Theorem. — *Let F and L be totally real number fields, F_0/F an abelian 2-extension and A an abelian variety over F satisfying $\mathcal{O}_L \subset \mathrm{End}_F(A)$ and $\dim(A) = [L : \mathbf{Q}]$; assume that A is potentially modular in the sense of 12.11.3(1) below*[1]*. Fix an embedding $\iota : L \hookrightarrow \mathbf{R}$. Let p be a prime number such that A has potentially ordinary (= potentially good ordinary or potentially totally multiplicative) reduction at each prime of F above p; let $\mathfrak{p}$ a prime of L above p. Assume that at least one of the following conditions holds:*

(1) *A is modular over F in the sense of 12.11.3(1)(ii) (with $F' = F$) and $2 \nmid [F : \mathbf{Q}]$.*

(2) *A does not have potentially good reduction everywhere.*

(3) *A has potentially good reduction everywhere, A has good ordinary reduction at each prime of F above p, the prime number p is unramified in $F_0/\mathbf{Q}$ and does not divide $w_2(L)$ ($\Longrightarrow p > 3$). If A does not have CM, assume, in addition, that there exists a basis of $A[\mathfrak{p}]$ in which $\mathrm{Im}(G_F \to \mathrm{Aut}(A[\mathfrak{p}])) \supseteq \mathrm{SL}_2(\mathbf{F}_p)$.*

Then: for each finite Galois extension of odd degree F_1/F_0,

$$\mathrm{rk}_{\mathcal{O}_L} A(F_1) + \mathrm{cork}_{\mathcal{O}_{L,\mathfrak{p}}} Ш(A/F_1)[\mathfrak{p}^\infty] \equiv \mathrm{ord}_{s=1} L(\iota A/F_1, s) \pmod 2.$$

Proof. — See 12.11. □

12.2.9. If A does not have CM, then $\mathrm{Im}(G_F \to \mathrm{Aut}(A[\mathfrak{p}]))$ contains $\mathrm{SL}_2(\mathbf{F}_p)$ for all but finitely many $\mathfrak{p}$ ([**Dim**, Prop. 3.8] + potential modularity; see also [**Ri**, §5]).

[1] Potential modularity of A seems to be well-known to the experts [**Tay5**]; a proof is expected to be written down in a forthcoming thesis of a student of R. Taylor.

12.2.10. Corollary. — *Let F be a totally real number field, F_0/F an abelian 2-extension, E an elliptic curve over F which is potentially modular in the sense of 12.11.3(1) below*[(2)] *and p a prime number such that E has potentially ordinary (= potentially good ordinary or potentially multiplicative) reduction at each prime of F above p. Assume that at least one of the following conditions holds:*

(1) *E is modular over F and $2 \nmid [F : \mathbf{Q}]$.*

(2) *$j(E) \notin \mathcal{O}_F$.*

(3) *$j(E) \in \mathcal{O}_F$, E has good ordinary reduction at each prime of F above p, the prime number p is unramified in $F_0/\mathbf{Q}$ and $p > 3$. If E does not have CM, assume, in addition, that* $\mathrm{Im}(G_F \to \mathrm{Aut}(E[p])) \supseteq \mathrm{SL}_2(\mathbf{F}_p)$.

Then: for each finite Galois extension of odd degree F_1/F_0,

$$\mathrm{rk}_{\mathbf{Z}} E(F_1) + \mathrm{cork}_{\mathbf{Z}_p} \text{Ш}(E/F_1)[p^\infty] \equiv \mathrm{ord}_{s=1} L(E/F_1, s) \pmod 2.$$

***12.2.11. Example* (cf. [M-R3, § 5.3]).** — Let $a(x) \in \mathbf{Q}[X]$ be an irreducible polynomial of degree $\deg(a) = 4$ with non-real roots $\alpha, \overline{\alpha}, \beta, \overline{\beta}$; denote by $b(x) \in \mathbf{Q}[X]$ its cubic resolvent with (real) roots $\gamma_1 = \alpha\overline{\alpha} + \beta\overline{\beta}$, $\gamma_2 = \alpha\beta + \overline{\alpha}\overline{\beta}$, $\gamma_3 = \alpha\overline{\beta} + \overline{\alpha}\beta$. Assume that the splitting field $K = \mathbf{Q}(\alpha, \overline{\alpha}, \beta, \overline{\beta})$ of $a(X)$ has maximal degree $[K : \mathbf{Q}] = 24$, in other words that $\mathrm{Gal}(K/\mathbf{Q}) = S_4$.

Let $\mathbf{Q} \subset k \subset K$ be an intermediate field such that $3 \mid [k : \mathbf{Q}]$ and k is not equal to the fixed field of a transposition in S_4. This implies that exactly one of the following two cases occurs:

(0) k is an abelian extension of degree 2 or 4 of the splitting field $K_0 = \mathbf{Q}(\gamma_1, \gamma_2, \gamma_3)$ of $b(X)$ (which is a totally real S_3-extension of $\mathbf{Q}$);

(j) $(j = 1, 2, 3)$ k is an extension of degree 1 or 2 of at least one of the (totally real, non-normal) cubic fields $K_j = \mathbf{Q}(\gamma_j)$.

Let E be an elliptic curve over $\mathbf{Q}$ with potentially ordinary reduction at a prime p. Our results imply that the congruence

$$\mathrm{rk}_{\mathbf{Z}} E(k) + \mathrm{cork}_{\mathbf{Z}_p} \text{Ш}(E/k)[p^\infty] \equiv \mathrm{ord}_{s=1} L(E/k, s) \pmod 2$$

holds in the following situations:

(a) $j(E) \notin \mathbf{Z}$ [by 12.2.10(1), for $F = K_j$ $(j = 0, 1, 2, 3)$, $F_1 = F_0 = k$].

(b) $j(E) \in \mathbf{Z}$, $[k : K_j] \leqslant 2$ for some $j = 1, 2, 3$ [by 12.2.3(1) and 12.11.5(iv) (or by 12.2.10(2)) applied to $F = K_j$, $F'' = F' = k$ and the cubic base change $g = BC_{F/\mathbf{Q}}(g_E)$ (see [J-PS-S] for the non-Galois cubic base change) of the classical modular form $g_E \in S_2(N_E, 1)$ associated to E].

(2) See the previous footnote.

(c) $j(E) \in \mathbf{Z}$, $[k : K_0] = 2$ or 4, $p > 3$ is unramified in $k/\mathbf{Q}$, E has good ordinary reduction at p, $E[p]$ is an irreducible $\mathbf{F}_p[G_{\mathbf{Q}}]$-module; if E has no CM, then $\mathrm{Im}\big(G_{\mathbf{Q}} \to \mathrm{Aut}(E[p])\big) = \mathrm{Aut}(E[p])$ [by 12.2.6(3) for $F_1 = K_0$, $F_1'' = F_1' = k$, $g = g_E$].

12.2.12. A general outline of the proof of Theorem 12.2.3 was given in 12.1.11. We now describe the main steps in more detail.

12.2.12.1. Reduction to the case $F'' = F' = F$ for twisted forms $g \otimes \alpha$ ($\alpha : \mathrm{Gal}(F'/F) \to \{\pm 1\}$). — There is a tower of fields $F = F_0 \subset F_1 = F' \subset \cdots \subset F_n = F''$ such that each Galois group $G_i = \mathrm{Gal}(F_{i+1}/F_i)$ is abelian (and of odd order for $i > 0$). For each $i = 0, \ldots, n-1$, we have

$$r_{\mathrm{an}}(F_{i+1}, g) = \sum_{\alpha} r_{\mathrm{an}}(F_i, g \otimes \alpha), \quad h^1_f(F_{i+1}, V) = \sum_{\alpha} h^1_f(F_i, V \otimes \alpha),$$

where α runs through all characters of G_i. If $\alpha^2 \neq 1$, then the contributions of α and α^{-1} to both terms are the same; this implies that, for each $i = 1, \ldots, n-1$,

$$r_{\mathrm{an}}(F_{i+1}, g) \equiv r_{\mathrm{an}}(F_i, g) \pmod 2, \quad h^1_f(F_{i+1}, V) \equiv h^1_f(F_i, V) \pmod 2,$$

which reduces to the case $F'' = F' = F_1$, and

$$r_{\mathrm{an}}(F', g) \equiv \sum_{\alpha : G_0 \to \{\pm 1\}} r_{\mathrm{an}}(F, g \otimes \alpha) \pmod 2,$$
$$h^1_f(F', V) \equiv \sum_{\alpha : G_0 \to \{\pm 1\}} h^1_f(F, V \otimes \alpha) \pmod 2,$$

which further reduces to the case $F'' = F' = F$ for the twisted forms $g \otimes \alpha$ ($\alpha : \mathrm{Gal}(F'/F) \to \{\pm 1\}$).

12.2.12.2. Reduction to the assumptions 12.2.3(1)–(2). — One establishes the principles

(I) $\widetilde{h}^1_f(F, V \otimes \alpha) \equiv \widetilde{h}^1_f(F, V_1) \pmod 2$ $\qquad (V_1 = V(f_\alpha)(k/2) \otimes \chi)$
(II) $\widetilde{\varepsilon}(V \otimes \alpha, 0) = \widetilde{\varepsilon}(V_1, 0)$

for the "level raising congruences" $f \otimes \alpha \equiv f_\alpha \pmod{\mathfrak{p}^M}$, whose existence is guaranteed by the condition 12.2.3(4) (resp., to analogous congruences $\pmod{\mathfrak{p}}$, whose existence is deduced from the condition 12.2.3(3)).

The key point in the proof of (II) is the fact that, for each prime $v \nmid p\infty$, the local $\varepsilon_{0,v}$-constants preserve congruences modulo $\mathfrak{p}^M$ ([**De2**, Thm. 6.5] for $M = 1$; [**Ya**, Thm. 5.1] for $M > 1$).

12.2.12.3. Reduction to the case $2 \nmid [F : \mathbf{Q}]$ or $(\exists P \mid p)\,(\exists \mathfrak{q} \neq P)\; 2 \nmid \mathrm{ord}_{\mathfrak{q}}(\mathfrak{n}(g))$

This follows from an argument analogous to that from 12.2.12.1 for a suitable quadratic (resp., cyclic cubic) extension of F.

12.2.12.4. Further reduction to the case $k = 2$. — One embeds (the p-stabilization of) f to a Hida family; such a family contains a form f' of weight $(2, \ldots, 2)$. We show that the principles (I) and (II) hold in this context:

$$\widetilde{h}^1_f(F, V) \equiv \widetilde{h}^1_f(F, V') \pmod 2, \quad \widetilde{\varepsilon}(V, 0) = \widetilde{\varepsilon}(V', 0),$$

where $g' = f' \otimes \chi' \in S_2(\mathfrak{n}(g'), 1)$ and $V' = V(g')(1)$.

12.2.12.5. Switch to a Shimura curve. — Thanks to the previous reduction steps, one can assume that $g \in S_2(\mathfrak{n}(g), 1)$ is associated by the Jacquet-Langlands correspondence to an automorphic form g_B (with trivial central character) on $B^*_{\mathbf{A}}$, where B is the quaternion algebra over F ramified at all but one archimedean primes (and at the prime $\mathfrak{q}$ from 12.2.12.3 if $2 \mid [F : \mathbf{Q}]$). This implies that there exists a Shimura curve X over F and a surjection $V_p(J(X)) \otimes_{\mathbf{Q}_p} L_{\mathfrak{p}} \twoheadrightarrow V$.

12.2.12.6. Dihedral Iwasawa theory. — By construction, there exists a prime $P \mid p$ of F at which B does not ramify. We fix such a prime and a suitable totally imaginary quadratic extension K/F which embeds into B. For example, we can take any K in which all primes dividing $\mathfrak{n}(g)P$ split (resp., in which $\mathfrak{q}$ is inert and all remaining primes dividing $\mathfrak{n}(g)P$ split) if $2 \nmid [F : \mathbf{Q}]$ (resp., if $2 \mid [F : \mathbf{Q}]$).

One considers the tower of ring class fields

$$K \subset K[1] \subset K[P] \subset \cdots \subset K[P^\infty] = \bigcup_{n=1}^{\infty} K[P^n];$$

the Galois group $G = \mathrm{Gal}(K[P^\infty]/K)$ is abelian, its torsion subgroup is finite and the quotient group G/G_{tors} is isomorphic to $\mathbf{Z}_p^{r_P}$, where $r_P = [F_P : \mathbf{Q}_p]$. The extension $K[P^\infty]/F$ is dihedral: any lift τ of the non-trivial element of $\mathrm{Gal}(K/F)$ satisfies $\tau g \tau^{-1} = g^{-1}$, for all $g \in G$.

Set $K_\infty = K[P^\infty]^{G_{\mathrm{tors}}}$; then $\Gamma = \mathrm{Gal}(K_\infty/K) \xrightarrow{\sim} \mathbf{Z}_p^{r_P}$. If $\beta : \Gamma \to L_{\mathfrak{p}}(\beta)^*$ is a character of finite order (where $L_{\mathfrak{p}}(\beta)$ is a finite extension of $L_{\mathfrak{p}}$), put $K_\beta = K_\infty^{\mathrm{Ker}(\beta)}$ and

$$h^1_f(K, V \otimes \beta) := \dim_{L_{\mathfrak{p}}(\beta)} H^1_f(K, V \otimes \beta) = \dim_{L_{\mathfrak{p}}(\beta)} \left(H^1_f(K_\beta, V) \otimes \beta\right)^{\mathrm{Gal}(K_\beta/K)}.$$

For K chosen as above we have principle (II)

$$r_{\mathrm{an}}(K, g) \equiv r_{\mathrm{an}}(K, g \otimes \beta) \equiv 1 \pmod 2,$$

where

$$r_{\mathrm{an}}(K, g \otimes \beta) = \mathrm{ord}_{s=1} L(g \times \theta(\beta), s)$$

denotes the order of vanishing of the Rankin-Selberg L-function associated to g and to the theta series of β. We also have a version of the principle (I), namely

$$h^1_f(K, V) \equiv h^1_f(K, V \otimes \beta) \pmod 2.$$

12.2.12.7. *CM* **points.** — The Shimura curve X contains a large supply of CM points defined over the ring class fields $K[P^n]$; they give rise to cohomology classes in $H^1_f(K[P^n], V)$. A fundamental result of Cornut-Vatsal [**Cor-Va**, Thm. 4.1] ("generalized Mazur's conjecture") states that, for $n >> 0$, there exists a CM point defined over $K[P^n]$ such that the corestriction from $K[P^n]$ to $K[P^n] \cap K_\infty$ of the corresponding cohomology class yields a non-zero element of $(H^1_f(K_\beta, V) \otimes \beta)^{\mathrm{Gal}(K_\beta/K)}$, for a suitable character of finite order $\beta : \Gamma \to \overline{L}^*_{\mathfrak{p}}$. An Euler system argument ([**Ne4**, Thm. 3.2]) then shows that

$$h^1_f(K, V \otimes \beta) = 1.$$

Combined with 12.2.12.6, this implies that

$$r_{\mathrm{an}}(K, g) \equiv h^1_f(K, V) \pmod 2.$$

12.2.12.8. Descent to F (varying K). — Denoting by η the quadratic character associated to K/F, the result established in 12.2.12.7 can be written as

$$r_{\mathrm{an}}(F, g) + r_{\mathrm{an}}(F, g \otimes \eta) \equiv h^1_f(F, V) + h^1_f(F, V \otimes \eta) \pmod 2.$$

In order to eliminate one of the contributions on each side, we vary K as in [**Ne3**]; applying the non-vanishing results of [**Wa2**] and [**F-H**], the generalized Gross-Zagier formula [**Zh1, Zh2**] and an Euler system argument [**Ne4**], we obtain in the end the desired congruence

$$r_{\mathrm{an}}(F, g) \equiv h^1_f(F, V) \pmod 2.$$

12.2.13. The contents of this chapter is the following: in 12.3 (resp., 12.4) we recall basic properties of Hilbert modular forms of parallel weight and the corresponding automorphic representations (resp. of the corresponding Galois representations). In 12.5 we investigate p-ordinary forms and their twists. In 12.6, 12.7 and 12.8 we establish, respectively, the principles (I) and (II) in the context of dihedral Iwasawa theory, Hida families and level raising congruences. In 12.9 we prove a parity result over ring class fields, from which we deduce in 12.10 Theorem 12.2.3. In 12.11, we prove Theorem 12.2.8, by combining 12.2.3 with potential modularity results and R. Taylor's descent arguments [**Tay4**]. The results on the Euler system of Heegner points used in the steps 12.2.12.7–12.2.12.8 are proved in [**Ne4**].

12.3. Hilbert modular forms

Let F be a totally real number field with ring of integers $\mathcal{O}_F$. We shall consider only Hilbert modular forms over F of parallel weight $(k, k, \dots, k)$ ("of weight k"), where $k \geqslant 1$. There are several competing normalizations and conventions concerning such forms; we shall use the language of representation theory, but for the reader's convenience we briefly recall the dictionary between the classical and representation-theoretical approaches.

The reciprocity isomorphism of class field theory will be normalized by letting uniformizers correspond to *geometric* Frobenius elements. We shall use it to identify, for any pro-finite abelian group A, a continuous homomorphism $\alpha : \mathbf{A}_F^*/F^* \to A$ (resp., $F_v^* \to A$) with a continuous homomorphism $G_F \to A$ (resp., $G_v \to A$). This applies, in particular, to finite abelian groups A, such as $A = \mu_n(\mathbf{C}) \subset \mathbf{C}$.

12.3.1. For a non-zero ideal $\mathfrak{n} \subset \mathcal{O}_F$ let

$$U_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F) \mid c \equiv 0 \pmod{\widehat{\mathfrak{n}}} \right\} = \prod_{v\nmid\infty} U_0(\mathfrak{n})_v$$

$$U_1(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F) \mid c, d-1 \equiv 0 \pmod{\widehat{\mathfrak{n}}} \right\} = \prod_{v\nmid\infty} U_1(\mathfrak{n})_v$$

be the standard open compact subgroups of $\mathrm{GL}_2(\widehat{F})$ (where $\widehat{\mathcal{O}}_F = \mathcal{O}_F \otimes \widehat{\mathbf{Z}} = \prod_{v\nmid\infty} \mathcal{O}_v$, $\widehat{\mathfrak{n}} = \mathfrak{n} \otimes \widehat{\mathbf{Z}}$, $\widehat{F} = \widehat{\mathcal{O}}_F \otimes \mathbf{Q} = \mathbf{A}_F^\infty$). Write an element of $K_\infty = \prod_{v\mid\infty} SO(2)$ as $k_\infty = (r(\theta_v))_{v\mid\infty}$, where

$$r(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and put $\rho(k_\infty) = \prod_{v\mid\infty} \exp(i\theta_v)$. Let $Z \subset \mathrm{GL}_2$ be the subgroup of scalar matrices and define

$$\mathrm{sgn}(z_\infty) = \mathrm{sgn}\left(\begin{pmatrix} y_\infty & 0 \\ 0 & y_\infty \end{pmatrix} \right) = \prod_{v\mid\infty} \mathrm{sgn}(y_v), \quad \Big(z_\infty \in Z(F\otimes\mathbf{R}) = \prod_{v\mid\infty} Z(F_v)\Big).$$

A **Hilbert modular form** of weight $k \geqslant 1$ and level $U_1(\mathfrak{n})$ is a continuous function $f : \mathrm{GL}_2(\mathbf{A}_F) \to \mathbf{C}$ satisfying the functional equation

$$f(\gamma z_\infty g u k_\infty) = \rho(k_\infty)^{-k} \mathrm{sgn}(z_\infty)^k f(g),$$
$$\gamma \in \mathrm{GL}_2(F),\ z_\infty \in Z(F\otimes\mathbf{R}),\ u \in U_1(\mathfrak{n}),\ k_\infty \in K_\infty$$

such that, for fixed $g^\infty \in \mathrm{GL}_2(\widehat{F})$, the function

$$|y_\infty|^{-k/2} f\left(\begin{pmatrix} y_\infty & x_\infty \\ 0 & 1 \end{pmatrix} g^\infty \right) \tag{12.3.1.1}$$

is holomorphic in the variables $x_\infty + iy_\infty = (x_v + iy_v)_{v\mid\infty}$ ($|y_\infty| = \prod_{v\mid\infty} |y_v|$) and is bounded as $|y_v| \to \infty$ (see [**Oh2**, Prop. 5.1.2]). Such functions form a finite-dimensional complex vector space $M_k(\mathfrak{n})$. The subspace of **cusp forms** $S_k(\mathfrak{n}) \subset M_k(\mathfrak{n})$ consists of those functions for which the expression (12.3.1.1) tends to zero as $|y_v| \to +\infty$ for all $v\mid\infty$.

12.3.2. The operators

$$(S(b)f)(g) = f\left(\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} g\right) \qquad (b \in \mathbf{A}_F^*,\ f \in M_k(\mathfrak{n}))$$

define a representation of the ray class group

$$I_{\mathfrak{n}\infty} = F^* \backslash \mathbf{A}_F^* / (F \otimes \mathbf{R})_+^* (1 + \widehat{\mathfrak{n}})$$

modulo $\mathfrak{n}\infty$ (where ∞ is a shorthand for the sum of all archimedean primes of F) on $M_k(\mathfrak{n})$ (resp., $S_k(\mathfrak{n})$). Under this action, the spaces $M_k(\mathfrak{n}) \supset S_k(\mathfrak{n})$ decompose into direct sums

$$M_k(\mathfrak{n}) = \bigoplus_{\varphi} M_k(\mathfrak{n}, \varphi), \quad S_k(\mathfrak{n}) = \bigoplus_{\varphi} S_k(\mathfrak{n}, \varphi),$$

where φ runs through all characters $\varphi : I_{\mathfrak{n}\infty} \to \mathbf{C}^*$ satisfying the parity condition

$$\varphi_v(-1) = (-1)^k \qquad (\forall v | \infty)$$

and

$$X_k(\mathfrak{n}, \varphi) = \{ f \in X_k(\mathfrak{n}) \mid (\forall b \in \mathbf{A}_F^*)\ S(b)f = \varphi(b) f \} \qquad (X = M, S).$$

In concrete terms, elements of $M_k(\mathfrak{n}, \varphi)$ satisfy the functional equation

$$f(\gamma g z u k_\infty) = \varphi(z) \varphi^\infty(u) \rho(k_\infty)^{-k} f(g),$$
$$\gamma \in \mathrm{GL}_2(F),\ z \in Z(\mathbf{A}_F),\ u \in U_0(\mathfrak{n}),\ k_\infty \in K_\infty,$$

where we have denoted

$$\varphi^\infty : U_0(\mathfrak{n})/U_1(\mathfrak{n}) \longrightarrow \mathbf{C}^*, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} U_1(\mathfrak{n}) \longrightarrow \prod_{v | \mathfrak{n}} \varphi_v(d_v)$$

and

$$\varphi : Z(\mathbf{A}_F) \longrightarrow \mathbf{C}^*, \quad \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \longmapsto \varphi(z).$$

Let h be the narrow class number of F. The recipe from [**Sh**, §2], [**Oh2**, Prop. 5.1.2] identifies each $f \in M_k(\mathfrak{n}, \varphi)$ with an h-tuple of holomorphic functions $(f_1, \ldots, f_h) : \mathcal{H}^{[F:\mathbf{Q}]} \to \mathbf{C}^h$ on the self-product of the upper half-plane $\mathcal{H} = \{ z \in \mathbf{C} \mid \mathrm{Im}(z) > 0 \}$, which satisfy the classical weight k transformation formulas with respect to suitable congruence subgroups of $\mathrm{GL}_2(F)$.

12.3.3. For each non-archimedean prime v of F we denote by dg_v the (bi-invariant) Haar measure on $\mathrm{GL}_2(F_v)$ normalized by $\int_{\mathrm{GL}_2(\mathcal{O}_v)} dg_v = 1$ and by $dg = \prod_{v \nmid \infty} dg_v$ the product Haar measure on $\mathrm{GL}_2(\widehat{F})$. The Hecke algebra $\mathcal{H}(U_1(\mathfrak{n}) \backslash \mathrm{GL}_2(\widehat{F}) / U_1(\mathfrak{n}))$ of $U_1(\mathfrak{n})$-biinvariant compactly supported functions $\alpha : \mathrm{GL}_2(\widehat{F}) \to \mathbf{C}$ with the convolution product

$$(\alpha * \beta)(h) = \int_{\mathrm{GL}_2(\widehat{F})} \alpha(g) \beta(g^{-1} h) \, dg$$

acts on $M_k(\mathfrak{n})$ (and on its subspaces $M_k(\mathfrak{n},\varphi)$, $S_k(\mathfrak{n},\varphi)$) by the formula

$$(\alpha\cdot f)(h) = \int_{\mathrm{GL}_2(\widehat{F})} \alpha(g) f(hg)\, dg.$$

For each square-free ideal $\mathfrak{a}\subset\mathcal{O}_F$, the classical Hecke operator $T_{\mathfrak{n}}(\mathfrak{a}) : M_k(\mathfrak{n}) \to M_k(\mathfrak{n})$ corresponds to the action of

$$(N\mathfrak{a})^{k/2-1}\cdot\left(\int_A dg\right)^{-1}\cdot \text{the characteristic function of } A = U_1(\mathfrak{n})\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} U_1(\mathfrak{n}),$$

where $a\in\widehat{F}^*$ is any finite idèle satisfying $\mathrm{div}(a)=\mathfrak{a}$ (*i.e.*, $\mathrm{ord}_v(a_v)=\mathrm{ord}_v(\mathfrak{a})$ for all $v\nmid\infty$).

If $\mathfrak{n}'|\mathfrak{n}$, then $M_k(\mathfrak{n}')\subset M_k(\mathfrak{n})$. The operators $T_{\mathfrak{n}}(\mathfrak{a})$, $T_{\mathfrak{n}'}(\mathfrak{a})$ do not always agree on $M_k(\mathfrak{n}')$; they do if every prime ideal dividing $(\mathfrak{a},\mathfrak{n})$ also divides $\mathfrak{n}'$. If there is no danger of confusion we shall write $T(\mathfrak{a})$ instead of $T_{\mathfrak{n}}(\mathfrak{a})$.

As in the classical case $F=\mathbf{Q}$, the Hecke operators $T(\mathfrak{a})$ for $(\mathfrak{a},\mathfrak{n})=(1)$ acting on $M_k(\mathfrak{n})$ satisfy the formal identity

$$\sum_{(\mathfrak{a},\mathfrak{n})=(1)} T(\mathfrak{a})(N\mathfrak{a})^{-s} = \prod_{v\nmid\mathfrak{n}}\left(1 - T(v)(Nv)^{-s} + S(v)(Nv)^{k-1-2s}\right)^{-1}.$$

12.3.4. Newforms [**Miy**]. — Assume that $f\in S_k(\mathfrak{n},\varphi)$ is a (non-zero) cuspidal eigenform for all operators $T(\mathfrak{a})$ with $(\mathfrak{a},\mathfrak{n})=(1)$:

$$T(\mathfrak{a})f=\lambda_f(\mathfrak{a})f,\qquad ((\mathfrak{a},\mathfrak{n})=(1),\ \lambda_f(\mathfrak{a})\in\mathbf{C}).$$

If $\mathfrak{n}'|\mathfrak{n}$ and $f'\in S_k(\mathfrak{n}',\varphi)-\{0\}$, we shall write $f'\sim f$ iff

$$T(\mathfrak{a})f'=\lambda_f(\mathfrak{a})f'\qquad ((\mathfrak{a},\mathfrak{n})=(1)).$$

If $v|\mathfrak{n}$ is a prime ideal, we say that f is v-**new** if there is no $f'\in S_k(\mathfrak{n}/v,\varphi)-\{0\}$ satisfying $f'\sim f$. If this is the case, then f is also an eigenform for $T(v)=T_{\mathfrak{n}}(v)$:

$$T(v)f=\lambda_f(v)f,\qquad (\lambda_f(v)\in\mathbf{C}).$$

We say that f is a **newform of level** $\mathfrak{n}$ if f is v-new at all prime ideals $v|\mathfrak{n}$. If this is the case, then f is an eigenform for all Hecke operators $T(\mathfrak{a})=T_{\mathfrak{n}}(\mathfrak{a})$

$$T(\mathfrak{a})f=\lambda_f(\mathfrak{a})f\qquad (\mathfrak{a}\subset\mathcal{O}_F)$$

and the L-function of f

$$L(f,s):=\sum_{\mathfrak{a}}\lambda_f(\mathfrak{a})(N\mathfrak{a})^{-s}$$

(absolutely convergent for sufficiently large $\mathrm{Re}(s)>>0$) is equal to the Euler product

$$L(f,s)=\prod_{v|\mathfrak{n}}\left(1-\lambda_f(v)(Nv)^{-s}\right)^{-1}\prod_{v\nmid\mathfrak{n}}\left(1-\lambda_f(v)(Nv)^{-s}+\varphi(v)(Nv)^{k-1-2s}\right)^{-1}.$$

In general, there exists a smallest ideal $\mathfrak{n}'|\mathfrak{n}$ (with respect to divisibility) and $f'\in S_k(\mathfrak{n}',\varphi)-\{0\}$ such that $f'\sim f$. The form f' is unique up to a scalar multiple and is a newform of level $\mathfrak{n}(f) := \mathfrak{n}'$. We say that f' is the **newform associated to** f. We also put

$$L(f,s) := L(f',s).$$

If f is a newform of level $\mathfrak{n}$ and $\chi : \mathbf{A}_F^*/F^* \longrightarrow \mathbf{C}^*$ a character of finite order of conductor $\mathrm{cond}(\chi)$, then there is a unique newform g (up to a scalar multiple) of level dividing $\mathrm{cond}(\chi)^2\mathfrak{n}$ such that

$$\lambda_g(v) = \chi(v)\lambda_f(v) \qquad (\forall v \nmid \mathrm{cond}(\chi)\mathfrak{n}).$$

We shall write $g = f \otimes \chi$.

12.3.5. We shall use the language of representation theory, which we briefly recall. Assume that $f \in S_k(\mathfrak{n},\varphi)$ $(k \geqslant 1)$ is a non-zero cuspidal eigenform for all Hecke operators $T(\mathfrak{a})$ with $(\mathfrak{a},\mathfrak{n}) = (1)$. The right regular action of $\mathrm{Lie}(\mathrm{GL}_2(F \otimes \mathbf{R})) \times \mathrm{GL}_2(\widehat{F})$ on f then generates an irreducible automorphic representation $\pi = \pi(f) = \otimes_v' \pi_v$ of $\mathrm{GL}_2(\mathbf{A}_F)$. This representation has central character φ, *i.e.*,

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix}\right) = \varphi(a)I \qquad (\forall a \in \mathbf{A}_F^*).$$

The isomorphism class of $\pi(f)$ depends only on the newform f' associated to f. The representation associated to $g = f \otimes \chi$ is equal to

$$\pi(f \otimes \chi) = \pi(f) \otimes \chi := \pi(f) \cdot (\chi \circ \det)$$

(and has central character $\varphi\chi^2$). The dual representation to $\pi = \pi(f)$ is isomorphic to

$$\widetilde{\pi} \xrightarrow{\ \sim\ } \pi \otimes \varphi^{-1}$$

(and has central character φ^{-1}).

12.3.6. For each non-archimedean prime v of F, the local representation $\pi_v = \pi(f)_v$ of $\mathrm{GL}_2(F_v)$ is one of the following types:

12.3.6.1. (Irreducible) principal series representation $\pi(\mu,\mu')$, in which $\mathrm{GL}_2(F_v)$ acts by right translations on the space $\mathcal{B}(\mu,\mu')$ of locally constant functions $f : \mathrm{GL}_2(F_v) \to \mathbf{C}$ satisfying

$$f\left(\begin{pmatrix} a & b \\ 0 & a'\end{pmatrix} g\right) = \mu(a)\mu'(a')|a/a'|^{1/2} f(g).$$

Above, $\mu,\mu' : F_v^* \longrightarrow \mathbf{C}^*$ are characters, $|\cdot| = |\cdot|_v : F_v^* \longrightarrow \mathbf{R}_+^*$ is the normalized valuation $(|\varpi_v|_v = (Nv)^{-1}$ for any uniformizer $\varpi_v \in \mathcal{O}_v)$, $\mu/\mu' \neq |\cdot|^{\pm 1}$. The representation $\pi(\mu,\mu')$ is isomorphic to $\pi(\mu',\mu)$; its central character is equal to $\mu\mu'$ and the dual representation is isomorphic to $\pi(\mu^{-1},(\mu')^{-1})$. If $\nu : F_v^* \to \mathbf{C}^*$ is another character, then $\pi(\mu,\mu') \otimes \nu \xrightarrow{\sim} \pi(\mu\nu,\mu'\nu)$.

12.3.6.2. Twisted Steinberg representation (= special representation)

$$\mathrm{St}(\mu) = \mathrm{St} \otimes \mu \subset \mathcal{B}(\mu|\cdot|^{1/2}, \mu|\cdot|^{-1/2})$$

(where $\mu : F_v^* \to \mathbf{C}^*$ is a character). The central character of $\mathrm{St}(\mu)$ is equal to μ^2; the dual representation is isomorphic to $\mathrm{St}(\mu^{-1})$. If $\nu : F_v^* \to \mathbf{C}^*$ is a character, then $\mathrm{St}(\mu) \otimes \nu \xrightarrow{\sim} \mathrm{St}(\mu\nu)$.

12.3.6.3. Supercuspidal representations

12.3.6.4. For each archimedean prime $v|\infty$, π_v is the Harish-Chandra module associated to the discrete series representation of weight k (resp., to the limit discrete series representation) if $k \geqslant 2$ (resp., if $k = 1$).

12.3.7. The local L-factors as defined by Jacquet and Langlands ([**J-L**, Prop. 3.5, Prop. 3.6]) are given by the formulas

$$L_v(\pi_v, s) = \begin{cases} \Gamma_{\mathbf{C}}\big(s + \frac{k-1}{2}\big), & v|\infty \\ L_v(\mu, s) L_v(\mu', s), & \pi_v = \pi(\mu, \mu') \\ L_v\big(\mu, s + \frac{1}{2}\big), & \pi_v = \mathrm{St}(\mu) \\ 1, & \pi_v \quad \text{supercuspidal}, \end{cases}$$

where $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ and, for each character $\mu : F_v^* \to \mathbf{C}^*$ $(v \nmid \infty)$,

$$L_v(\mu, s) = \begin{cases} (1 - \mu(v)(Nv)^{-s})^{-1}, & \mu \text{ unramified} \\ 1, & \mu \text{ ramified}. \end{cases}$$

12.3.8. For each non-archimedean prime $v \nmid \infty$ of F, the local Hecke algebra $\mathcal{H}_v$ of locally constant functions $\alpha : \mathrm{GL}_2(F_v) \to \mathbf{C}$ with compact support (with respect to the convolution product defined by the measure dg_v) acts on the representation space of π_v by the operator

$$\int_{\mathrm{GL}_2(F_v)} \alpha(g_v)\pi_v(g_v)\, dg_v.$$

For each $m \geqslant 1$, the action of the primitive Hecke operator $T(v^m)_{\mathrm{prim}}$ $(T(v)_{\mathrm{prim}} = T(v))$ corresponds to the action of

$$(Nv^m)^{k/2-1} \cdot \left(\int_A dg_v\right)^{-1} \cdot \text{the characteristic function of } A = U_v \begin{pmatrix} \varpi_v^m & 0 \\ 0 & 1 \end{pmatrix} U_v,$$

where ϖ_v is any prime element of $\mathcal{O}_v$ and $U = U_1(\mathfrak{n})$.

If $v \nmid \mathfrak{n}$, then $\pi_v = \pi(\mu, \mu')$ for unramified characters $\mu, \mu' : F_v^* \to \mathbf{C}^*$ ("unramified principal series"). The characteristic functions of the double cosets

$$U_v \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U_v, \quad U_v \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix} U_v$$

(note that $U_v = \mathrm{GL}_2(\mathcal{O}_v)$ in this case) act on the one-dimensional space $\pi_v^{U_v}$ of spherical vectors by the scalars $(Nv)^{1/2}(\mu(v) + \mu'(v))$ and $(\mu\mu')(v)$, respectively. In other words, the "Hecke polynomial"

$$1 - \left[U_v \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U_v\right] X + \left[U_v \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix} U_v\right] (Nv) X^2$$

acts on $\pi_v^{U_v}$ as

$$H_v(\pi_v, X) = (1 - (Nv)^{1/2}\mu(v)X)(1 - (Nv)^{1/2}\mu'(v)X).$$

The classical local L-factor of f

$$L_v(f,s) = \left(1 - \lambda_f(v)(Nv)^{-s} + \varphi(v)(Nv)^{k-1-2s}\right)^{-1}$$

is related to the local Hecke L-factor

$$L_v^H(\pi_v, s) = H_v(\pi_v, (Nv)^{-s})^{-1}$$

and to the local Jacquet-Langlands L-factor

$$L_v(\pi_v, s) = \left[(1 - \mu(v)(Nv)^{-s})(1 - \mu'(v)(Nv)^{-s})\right]^{-1}$$

by

$$L_v(f,s) = L_v^H\left(\pi_v, s - \left(\frac{k}{2} - 1\right)\right) = L_v\left(\pi_v, s - \frac{k-1}{2}\right); \tag{12.3.8.1}$$

in other words,

$$\lambda_f(v) = (Nv)^{(k-1)/2}(\mu(v) + \mu'(v)).$$

If $v|\mathfrak{n}$ and f is v-new, then it is still true that

$$L_v(f,s) := \left(1 - \lambda_f(v)(Nv)^{-s}\right)^{-1} = L_v\left(\pi_v, s - \frac{k-1}{2}\right). \tag{12.3.8.2}$$

In particular, if $\pi_v = \mathrm{St}(\mu)$, then

$$\lambda_f(v) = \begin{cases} \mu(v)(Nv)^{k/2-1}, & \mu \text{ unramified} \\ 0, & \mu \text{ ramified.} \end{cases} \tag{12.3.8.3}$$

12.3.9. For each prime ideal $v|\mathfrak{n}$, the exponent of v in $\mathfrak{n}(f)$ is determined purely by the local representation $\pi_v = \pi(f)_v$, namely $\mathrm{ord}_v(\mathfrak{n}(f)) = o(\pi_v)$, where $o(\pi_v)$ ([**Cas**]) denotes the minimal integer $n \geqslant \mathrm{ord}_v(\mathrm{cond}(\varphi))$ for which there exists a non-zero vector in the representation space of π_v on which $U_0(v^n)_v$ acts by the character

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \varphi_v(d)$$

(equivalently, for which π_v admits a non-zero vector invariant under the action of $U_1(v^n)_v$). If we denote, for any character $\mu : F_v^* \to \mathbf{C}^*$, the exponent of the conductor of μ by

$$o(\mu) := \min\{n \geqslant 0 \mid \mu((1 + \varpi_v^n \mathcal{O}_v)^*) = \{1\}\},$$

then we have ([**Cas**, proof of Thm. 1])

$$o(\pi_v) = \begin{cases} o(\mu) + o(\mu'), & \pi_v = \pi(\mu, \mu') \\ 1, & \pi_v = \mathrm{St}(\mu),\ \mu \text{ unramified} \\ 2\, o(\mu), & \pi_v = \mathrm{St}(\mu),\ \mu \text{ ramified} \\ \geqslant 2, & \pi_v \quad \text{supercuspidal.} \end{cases} \tag{12.3.9.1}$$

In particular, if φ_v is unramified, then

$$o(\pi_v) = 1 \iff \pi_v = \mathrm{St}(\mu),\ \mu \text{ unramified} \tag{12.3.9.2}$$

and

$$2 \nmid o(\pi_v),\ o(\pi_v) > 1 \Longrightarrow \pi_v \text{ supercuspidal.} \tag{12.3.9.3}$$

12.3.10. Lemma. — *Let $f \in S_k(\mathfrak{n}, \varphi)$ $(k \geqslant 1)$ be a non-zero eigenform for all $T(\mathfrak{a})$ with $(\mathfrak{a}, \mathfrak{n}) = (1)$; let $\pi(f)$ be the corresponding automorphic representation.*

(i) *For a non-archimedean prime $v \nmid \infty$ of F, the following conditions are equivalent:*

$$\pi(f)_v = \mathrm{St}(\mu) \text{ with } \mu \text{ unramified} \iff \mathrm{ord}_v(\mathfrak{n}(f)) = 1 \text{ and } \varphi_v \text{ is unramified.}$$

If they are satisfied, then

$$\lambda_f(v)^2 = \varphi(v)(Nv)^{k-2}.$$

(ii) *If $v|\mathfrak{n}$ and f is v-new, then the following conditions are equivalent:*

$$\lambda_f(v) \neq 0 \iff \pi(f)_v = \begin{cases} \mathrm{St}(\mu), & \mu \text{ unramified} \\ \pi(\mu, \mu'), & \mu \text{ unramified.} \end{cases}$$

If $\lambda_f(v) \neq 0$, then the two cases are distinguished as follows:

$$\pi(f)_v = \mathrm{St}(\mu),\ \mu \text{ unramified} \iff \mathrm{ord}_v(\mathfrak{n}) = 1,\ \mathrm{ord}_v(\mathrm{cond}(\varphi)) = 0$$
$$\pi(f)_v = \pi(\mu, \mu'),\ \mu \text{ unramified} \iff \mathrm{ord}_v(\mathfrak{n}) = \mathrm{ord}_v(\mathrm{cond}(\varphi))\ (= o(\mu')).$$

Proof

(i) The two conditions are equivalent, by (12.3.9.2). If they are satisfied, then

$$\lambda_f(v)^2 = \mu(v)^2(Nv)^{k-2} = \varphi(v)(Nv)^{k-2},$$

by (12.3.8.3).

The statement (ii) follows from (i) combined with (12.3.8.2) and (12.3.9.1). □

12.3.11. The statement of Lemma 12.3.10 (ii) should be compared to the fact that, for $v \nmid \mathfrak{n}(f)$,

$$\mathrm{ord}_v(\mathfrak{n}(f)) = \mathrm{ord}_v(\mathrm{cond}(\varphi)) = 0, \quad \pi(f)_v = \pi(\mu, \mu'), \quad \mu, \mu' \text{ unramified.}$$

12.3.12. Assume that $f \in S_k(\mathfrak{n}, \varphi)$ is a (non-zero) newform of level $\mathfrak{n}$; let $\pi = \pi(f)$ be the corresponding automorphic representation. The complete L-functions

$$L(\pi, s) = \prod_v L_v(\pi_v, s), \quad \widehat{L}(f, s) = \prod_{v|\infty} \Gamma_{\mathbf{C}}(s) \prod_{v\nmid\infty} L_v(f, s)$$

are related by

$$\widehat{L}(f, s) = L\left(\pi, s - \frac{k-1}{2}\right)$$

(see (12.3.8.1)–(12.3.8.2)) and satisfy the functional equation

$$L(\pi, s) = \varepsilon(\pi, s) L(\widetilde{\pi}, 1-s) = \varepsilon(\pi, s) L(\pi \otimes \varphi^{-1}, 1-s),$$

where

$$\varepsilon(\pi, s) = \prod_v \varepsilon_v(\pi_v, s, \psi_v)$$

is the product of the local ε-factors defined in [**J-L**] (above, $\psi = (\psi_v)$ is a non-trivial (unitary) additive character $\psi : \mathbf{A}_F/F \to \mathbf{C}^*$). Each local ε-factor is of the form $a \cdot b^s$, where $a \in \mathbf{C}^*$ and $b > 0$. Their basic properties are summarized below:

12.3.13. Lemma

(i) *If* $\psi'_v(x) = \psi_v(ax)$ $(a \in F_v^*)$, *then*

$$\varepsilon_v(\pi_v, s, \psi'_v) = \varphi_v(a)|a|_v^{2s-1} \varepsilon_v(\pi_v, s, \psi_v).$$

(ii) *For each character* $\mu : F_v^* \to \mathbf{C}^*$, $\varepsilon_v(\pi_v \otimes \mu, s, \psi_v)\, \varepsilon_v(\widetilde{\pi}_v \otimes \mu^{-1}, 1-s, \psi_v) = \varphi_v(-1)$.

(iii) *If* $v \nmid \mathfrak{n}$ *and* $\mathrm{Ker}(\psi_v) = \mathcal{O}_v$, *then* $\varepsilon_v\big(\pi_v, \frac{1}{2}, \psi_v\big) = 1$.

(iv) *If* $\varphi_v = 1$, *then* $\varepsilon_v\big(\pi_v, \frac{1}{2}\big) := \varepsilon_v\big(\pi_v, \frac{1}{2}, \psi_v\big)$ *does not depend on* ψ_v *and is equal to* $\varepsilon_v\big(\pi_v, \frac{1}{2}\big) = \pm 1$.

(v) *If* $\pi_v = \pi(\mu, \mu^{-1})$, *then* $\varepsilon_v\big(\pi_v, \frac{1}{2}\big) = \mu(-1)$.

(vi) *If* $\pi_v = \mathrm{St}(\mu)$, $\mu^2 = 1$, *then*

$$\varepsilon_v\big(\pi_v, \tfrac{1}{2}\big) = \mu(-1) \times \begin{cases} -1, & \mu = 1 \\ 1, & \mu \neq 1. \end{cases}$$

(vii) *If* $v|\infty$ *and* $2|k$, *then* $\varepsilon_v\big(\pi_v, \frac{1}{2}\big) = (-1)^{k/2}$.

Proof

(i), (ii) [**J-L**, remarks after Thm. 2.18] (the formula (ii) follows from the comparison of the local functional equations of $L_v(\pi_v \otimes \mu, s)$ and $L_v(\widetilde{\pi}_v \otimes \mu^{-1}, s)$).

(iii) As $\pi_v = \pi(\mu, \mu')$ with unramified characters μ, μ', we have $\varepsilon_v(\pi_v, s, \psi_v) = \varepsilon_v(\mu, s, \psi_v)\, \varepsilon_v(\mu', s, \psi_v) = 1$.

(iv) Independence on ψ_v follows from (i), the equality $\varepsilon_v\big(\pi_v, \frac{1}{2}\big)^2 = 1$ from (ii) (with $\mu = 1$).

(v) In this case

$$\varepsilon_v\big(\pi(\mu,\mu^{-1}),\tfrac{1}{2}\big) = \varepsilon_v\big(\mu,\tfrac{1}{2},\psi_v\big)\,\varepsilon_v\big(\mu^{-1},\tfrac{1}{2},\psi_v\big) = \mu(-1),$$

where the latter equality follows from the comparison of the local functional equations for $L_v(\mu,s)$ and $L_v(\mu^{-1},1-s)$.

(vi) By definition,

$$\varepsilon_v(\mathrm{St}(\mu),s,\psi_v) = \varepsilon_v\big(\mu,s+\tfrac{1}{2},\psi_v\big)\,\varepsilon_v\big(\mu,s-\tfrac{1}{2},\psi_v\big)\cdot\begin{cases}\dfrac{L_v\big(\mu^{-1},\frac{1}{2}-s\big)}{L_v\big(\mu,s-\frac{1}{2}\big)}, & \mu \text{ unramified}\\ 1, & \mu \text{ ramified.}\end{cases}$$

If we let $s \to \frac{1}{2}$, then the product of the two ε-factors tends to $\mu(-1)$ (cf. the proof of (v)), while the remaining term tends to -1 (resp., 1) if $\mu = 1$ (resp., if $\mu \neq 1$).

(vii) This is well-known; one can use, for example, the corresponding formulas for the representations of the Weil group of $\mathbf{R}$. □

12.3.14. We shall be particularly interested in the case when $f \in S_k(\mathfrak{n},1)$ is a newform of level $\mathfrak{n}$ with trivial character ($\Longrightarrow 2 \mid k$). In this case

$$r_{\mathrm{an}}(F,f) := \mathrm{ord}_{s=k/2}L(f,s) = \mathrm{ord}_{s=1/2}L(\pi(f),s)$$

and

$$(-1)^{r_{\mathrm{an}}(F,f)} = \varepsilon\big(\pi(f),\tfrac{1}{2}\big) = \prod_v \varepsilon_v\big(\pi(f)_v,\tfrac{1}{2}\big),$$

(as $\Gamma_{\mathbf{C}}(k/2) \neq 0,\infty$) with each local ε-factor equal to $\varepsilon_v\big(\pi(f)_v,\frac{1}{2}\big) = \pm 1$.

12.4. Galois representations associated to Hilbert modular forms

From now on, fix a prime number p and embeddings $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$.

12.4.1. Assume that $f \in S_k(\mathfrak{n},\varphi)$ ($k \geqslant 1$) is a non-zero cuspidal eigenform for all Hecke operators $T(\mathfrak{a})$ with $(\mathfrak{a},\mathfrak{n}) = (1)$. According to [**Sh**, Prop. 1.3], there exists a number field $L \subset \overline{\mathbf{Q}}$ such that $\iota_\infty(\mathcal{O}_L)$ contains the values of φ and all Hecke eigenvalues $\lambda_f(\mathfrak{a})$ ($(\mathfrak{a},\mathfrak{n}) = (1)$); fix such a field L. The embedding ι_p induces a prime ideal $\mathfrak{p} \mid p$ of L.

12.4.2. There exists a continuous two-dimensional representation $V(f) = V_{\mathfrak{p}}(f)$ of $G_F = \mathrm{Gal}(\overline{F}/F)$ over $L_{\mathfrak{p}}$ which is unramified outside $\mathfrak{n}p$ and satisfies

$$\det(1 - \mathrm{Fr}(v)\,X \mid V(f)) = 1 - \lambda_f(v)X + \varphi(v)(Nv)^{k-1}X^2$$

for all prime ideals $v \nmid \mathfrak{n}p$ of $\mathcal{O}_F$ (here $\mathrm{Fr}(v) = \mathrm{Fr}(v)_{\mathrm{geom}}$ denotes the *geometric* Frobenius element at v). The representation $V(f)$ is absolutely irreducible, by [**Tay2**, Prop. 3.1] (and the fact that each complex conjugation acts on $V(f)$ by a matrix with distinct eigenvalues $\pm 1 \in L_{\mathfrak{p}}$). This implies, by the Čebotarev density theorem, that $V(f)$ is unique up to isomorphism, depends only on the newform associated to f, and $V(f\otimes\chi)=V(f)\otimes\chi$ for every character of finite order $\chi:\mathbf{A}_F^*/F^*\to\mathbf{C}^*$. The determinant of $V(f)$ is equal to

$$\det(V(f))=\Lambda^2V(f)\xrightarrow{\sim}L_{\mathfrak{p}}(1-k)\otimes\varphi,$$

hence the dual of $V(f)$ is isomorphic to

$$V(f)^*\xrightarrow{\sim}V(f)(k-1)\otimes\varphi^{-1}.$$

12.4.3. For each prime ideal $v\nmid p$, the restriction $V(f)_v$ of $V(f)$ to the decomposition group $G_v=\mathrm{Gal}(\overline{F}_v/F_v)$ is related to $\pi(f)_v$ by the local Langlands correspondence as follows: for each character $\chi:F_v^*\to\mathbf{C}^*$ of finite order and each non-trivial additive character $\psi_v:F_v\to\mathbf{C}^*$,

$$\begin{aligned}L_v(\pi(f)_v\otimes\chi,s)&=L_v\big(V(f)_v\otimes\chi,s+\tfrac{k-1}{2}\big)\\ \varepsilon_v(\pi(f)_v\otimes\chi,s,\psi_v)&=\varepsilon_v\big(V(f)_v\otimes\chi,\psi_v,dx_{\psi_v},s+\tfrac{k-1}{2}\big),\end{aligned}\tag{12.4.3.1}$$

where

$$L_v(V(f)_v\otimes\chi,s)=\det\big(1-(Nv)^{-s}\mathrm{Fr}(v)\mid(V(f)_v\otimes\chi)^{I_v}\big)^{-1}$$

and the local ε-factor on the R.H.S. is the one defined in [**De2**, §8.12] (above, dx_{ψ_v} is the self-dual Haar measure on F_v with respect to ψ_v). Moreover, the conductors also agree:

$$o(\pi(f)_v)=\mathrm{ord}_v(\text{the Artin conductor of }V(f)).\tag{12.4.3.2}$$

These results are due, under varying degrees of generality, to Eichler, Shimura, Deligne ($F=\mathbf{Q}$), Langlands, Carayol, Ohta, Rogawski-Tunnell, Wiles, R. Taylor, Blasius-Rogawski. We shall be particularly interested in the ordinary case, treated in [**Wi**].

12.4.4. In concrete terms, (12.4.3.1) implies the following (again, for a prime ideal $v\nmid p$):

12.4.4.1. If $\pi(f)_v=\pi(\mu,\mu')$, then I_v acts on $V(f)_v$ through a finite quotient and the semi-simplification of $V(f)_v$ is isomorphic to

$$V(f)_v^{ss}\xrightarrow{\sim}L_{\mathfrak{p}}\otimes\mu|\cdot|^{(1-k)/2}\oplus L_{\mathfrak{p}}\otimes\mu'|\cdot|^{(1-k)/2}$$

(thus I_v acts on $V(f)_v$ by $\mu|_{\mathcal{O}_v^*}\oplus\mu'|_{\mathcal{O}_v^*}$).

12.4.4.2. If $\pi(f)_v=\mathrm{St}(\mu)$, then the representation $V(f)_v$ is reducible and I_v acts on $V(f)_v$ through an infinite quotient. There is an exact sequence of $L_{\mathfrak{p}}[G_v]$-modules

$$0\longrightarrow L_{\mathfrak{p}}\otimes\mu|\cdot|^{1-k/2}\longrightarrow V(f)_v\longrightarrow L_{\mathfrak{p}}\otimes\mu|\cdot|^{-k/2}\longrightarrow 0,$$

whose extension class in

$$H^1_{\mathrm{cont}}(G_v,\mathrm{Hom}(L_{\mathfrak{p}}\otimes\mu|\cdot|^{-k/2},L_{\mathfrak{p}}\otimes\mu|\cdot|^{1-k/2}))=H^1_{\mathrm{cont}}(G_v,L_{\mathfrak{p}}(1))=F_v^*\widehat{\otimes}L_{\mathfrak{p}}\xrightarrow{\sim}L_{\mathfrak{p}}$$

is non-zero. In particular, if μ is unramified, then I_v acts on $V(f)_v$ through its tame quotient $I_v^t = I_v/I_v^w$, and any topological generator of I_v^t acts on $V(f)_v$ by an endomorphism A satisfying $(A-1)^2 = 0 \neq A-1$.

12.4.4.3. If $\pi(f)_v$ is supercuspidal, then I_v acts on $V(f)_v$ irreducibly, through a finite quotient. More precisely,

– either $V(f)_v$ is monomial, *i.e.*, $V(f)_v \xrightarrow{\sim} \mathrm{Ind}_{G_w}^{G_v}(\mu)$, where E_w/F_v is a ramified quadratic extension and $\mu : E_w^* \longrightarrow L_{\mathfrak{p}}^*$ does not factor through $N_{E_w/F_v} : E_w^* \longrightarrow F_v^*$ (hence μ is ramified),

– or $V(f)_v$ is exceptional ($\Longrightarrow v \mid 2$), in which case there exists a Galois extension of F_v with Galois group isomorphic to A_3 or S_3, over which $V(f)_v$ becomes monomial ([**W**]).

12.4.4.4. Let k be even. It follows from [**Wa1**, Lemma I.2.3] that, for each prime $v \nmid p\mathfrak{n}\infty$, the local representation $\pi(f)_v$ is defined over L, in the sense of [**Wa1**, §I.1]. According to [**Wa1**, Cor. I.8.3], the global representation $\pi(f)$ is also defined over L, and so are the local representation $\pi(f)_v$ for all $v \nmid \infty$.

As k is even, the correspondence between $V(f)_v$ and $\pi(f)_v$ normalized as in (12.4.3.1) commutes with the action of automorphisms of $\overline{\mathbf{Q}}$, which implies that

$$(\forall v \nmid p\infty)\,(\forall g \in W(\overline{F}_v/F_v)) \quad \det(1 - gX \mid V(f)_v) \in L[X].$$

12.4.5. Lemma. — *Assume that $f \in S_k(\mathfrak{n}, \varphi)$ $(k \geqslant 1)$ is a newform of level $\mathfrak{n}$. Then, for each prime ideal $v \nmid p$ of $\mathcal{O}_F$,*

(i) *I_v acts trivially on $V(f)$ $\Longleftrightarrow$ $\pi(f)_v$ is in the unramified principal series $\Longleftrightarrow$ $v \nmid \mathfrak{n}$.*

(ii) *If φ_v is unramified, then the following conditions are equivalent:*

$$\dim_{L_{\mathfrak{p}}}(V(f)^{I_v}) = 1 \iff \pi(f)_v = \mathrm{St}(\mu),\ \mu \textit{ unramified} \iff \mathrm{ord}_v(\mathfrak{n}) = 1.$$

Proof

(i) This is a consequence of (12.4.3.1).

The second equivalence in (ii) is just (12.3.9.2), while the first implication '$\Longleftarrow$' follows from 12.4.4.2. If $\pi_v(f) = \mathrm{St}(\mu)$ with μ ramified or if $\pi_v(f)$ is supercuspidal, then $V(f)^{I_v} = 0$. If $\pi_v(f) = \pi(\mu, \mu')$, then $\mu\mu' = \varphi_v$ is unramified; thus

$$V(f)^{I_v} = \begin{cases} V(f), & \mu, \mu' \text{ unramified} \\ 0, & \mu, \mu' \text{ ramified.} \end{cases}$$

This proves the first implication '$\Longrightarrow$'. □

12.4.6. Corollary. — *Assume that $f \in S_k(\mathfrak{n}, 1)$ is a newform of level $\mathfrak{n}$ and trivial central character ($\Longrightarrow 2 \mid k$). Then, for each prime ideal $v \mid \mathfrak{n}$, $v \nmid p$,*

$$\det(-\mathrm{Fr}(v) \mid V(f)(k/2)^{I_v}) \\ = \begin{cases} -(Nv)^{-k/2}\lambda_f(v) = -(Nv)^{-1}\mu(v), & \pi(f)_v = \mathrm{St}(\mu)\ (\mu^2 = 1),\ \mu \textit{ unramified} \\ 1, & \textit{otherwise.} \end{cases}$$

12.4.7. The Jacquet-Langlands correspondence. — Fix an archimedean prime $\tau_1 : F \hookrightarrow \mathbf{R}$ of F. If $\mathcal{R}$ is a finite set of non-archimedean primes of F satisfying

$$|\mathcal{R}| \equiv [F : \mathbf{Q}] - 1 \pmod 2, \tag{12.4.7.1}$$

then there exists a unique quaternion algebra B over F ramified at the set

$$\mathrm{Ram}(B) = \{v \mid \infty,\ v \neq \tau_1\} \cup \mathcal{R}.$$

Assume that the form f satisfies

$$(\forall v \in \mathcal{R}) \quad \pi(f)_v \text{ is not in the principal series} \tag{12.4.7.2}$$

(if $2 \nmid [F : \mathbf{Q}]$, then the conditions (12.4.7.1-2) are automatically satisfied for $\mathcal{R} = \varnothing$). According to [**J-L**, Thm. 16.1] (see also [**G-J**, §8]; [**Ro**, §3]), there exists an irreducible automorphic representation π' of $B^*_{\mathbf{A}}$ such that

$$(\forall v \notin \mathrm{Ram}(B)) \quad \pi'_v \xrightarrow{\sim} \pi(f)_v.$$

If $k \geqslant 2$, then the Galois representation $V(f)$ can be constructed as a quotient of the étale cohomology group $H^1_{\mathrm{et}}(X \otimes_F \overline{F}, \mathscr{F})$ of a suitable local system $\mathscr{F}$ on a Shimura curve X associated to B and π' ([**Oh1, Oh2**]; [**Car**, §2]). If $v \nmid \mathfrak{n}(f)$, then there exists X as above with good reduction at v.

12.4.8. Purity (Ramanujan's Conjecture)

12.4.8.1. Definition. — Let v be a non-archimedean prime of a number field K and $n \in \mathbf{Z}$. A **v-Weil number of weight** n is an algebraic number $\alpha \in \overline{\mathbf{Q}}$ such that

$$(\exists i \in \mathbf{Z}) \quad (Nv)^i\alpha \text{ is an algebraic integer;}$$
$$(\forall \sigma : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}) \quad |\sigma(\alpha)| = (Nv)^{n/2}.$$

12.4.8.2. Theorem. — *Let $f \in S_k(\mathfrak{n}, \varphi)$ ($k \geqslant 2$) be a non-zero cuspidal eigenform for all Hecke operators $T(\mathfrak{a})$ with $(\mathfrak{a}, \mathfrak{n}) = (1)$. Let $v \nmid p\infty$ be a prime of F and $g \in G_v$ a lift of the geometric Frobenius $\mathrm{Fr}(v) \in G_v/I_v$. Then the two eigenvalues of g acting on $V(f)$ are v-Weil numbers of weights*

$$\begin{cases} k-1,\ k-1 & \textit{if} \quad \pi(f)_v \neq \mathrm{St}(\mu) \\ k,\ k-2 & \textit{if} \quad \pi(f)_v = \mathrm{St}(\mu). \end{cases}$$

Proof. — The case $\pi(f)_v = \mathrm{St}(\mu)$ is easy, thanks to 12.4.4.2. Assume $\pi(f)_v \neq \mathrm{St}(\mu)$. In the case $F = \mathbf{Q}$ and $v \nmid (f)$ (resp., $2 \nmid [F : \mathbf{Q}]$ and $v \nmid (f)$) the statement is deduced from Weil's conjectures in [**De1**] (resp., in [**Oh2**, Thm. 1.4.1]). If v does not lie in a certain exceptional finite set, the result is proved in [**B-L**]. The general case is treated in [**Bl**]. □

12.4.8.3. Letting p (and $\mathfrak{p}$) vary, one can deduce from Theorem 12.4.8.2 and 12.4.3–12.4.4 an 'automorphic' version of Ramanujan's Conjecture. For example, if v is a non-archimedean prime of F such that $\pi(f)_v = \pi(\mu, \mu')$ is in the principal series, then, for each uniformizer $\varpi_v \in \mathcal{O}_v$, both $\mu(\varpi_v)$ and $\mu'(\varpi_v)$ are v-Weil numbers of weight 0.

12.4.8.4. Proposition. — *Let $f \in S_k(\mathfrak{n}, \varphi)$ be as in 12.4.8.2; assume that the weight $k \in 2\mathbf{Z}$ is even. Let $v \nmid p\infty$ be a prime of F and E_w/F_v a finite extension. Set*

$$J = \begin{cases} \varnothing, & \text{if} \quad \pi(f)_v \neq \mathrm{St}(\mu) \\ \{k/2 - 1, k/2 + 1\}, & \text{if} \quad \pi(f)_v = \mathrm{St}(\mu). \end{cases}$$

Then, for each character of finite order $\beta : G_w = \mathrm{Gal}(\overline{F}_v/E_w) \to L_{\mathfrak{p}}^*$, we have

$$(\forall j \in \mathbf{Z} - J) \quad \mathbf{R}\Gamma_{\mathrm{cont}}(G_w, V(f)(j) \otimes \beta) \xrightarrow{\sim} 0$$

in $D^b_{ft}({}_{L_{\mathfrak{p}}}\mathrm{Mod})$.

Proof. — Let $g \in G_w$ be a lift of the geometric Frobenius $\mathrm{Fr}(w) \in G_w/I_w$. If $\pi(f)_v \neq \mathrm{St}(\mu)$, then all eigenvalues of g acting on $V = V(f)(j) \otimes \beta$ are w-Weil numbers of weight $k - 1 - 2j \neq 0$ (as k is even); thus $H^0_{\mathrm{cont}}(G_w, V) = 0$. As $V(f)(j)^*(1) = V(f)^*(1-j) \xrightarrow{\sim} V(f \otimes \varphi^{-1})(k-j)$, the same argument applied to $f \otimes \varphi^{-1}$ shows that $H^2_{\mathrm{cont}}(G_w, V) = H^0_{\mathrm{cont}}(G_w, V^*(1))^* = 0$. The vanishing of $H^1_{\mathrm{cont}}(G_w, V)$ then follows from the Euler characteristic formula (Theorem 4.6.9 and 5.2.11)

$$\sum_{q=0}^{2} (-1)^q \dim_{L_{\mathfrak{p}}} H^q_{\mathrm{cont}}(G_w, V) = 0.$$

If $\pi(f)_v = \mathrm{St}(\mu)$, then $V^{I_w} \subset L_{\mathfrak{p}}(1 - k/2 + j) \otimes \beta \cdot (\mu \circ N_w)$, where N_w denotes the norm from the extension E_w/F_v. This means that either $V^{I_w} = 0$, or g acts on the one-dimensional $L_{\mathfrak{p}}$-vector space V^{I_w} by a w-Weil number of weight $k - 2 - 2j$; thus $H^0_{\mathrm{cont}}(G_w, V) = 0$ for $j \neq k/2 - 1$. The same argument as above shows that $H^2_{\mathrm{cont}}(G_w, V) = 0$ for $j \neq k/2 + 1$, hence $H^1_{\mathrm{cont}}(G_w, V) = 0$ for $j \neq k/2 \pm 1$. □

12.4.9. p-adic results. — Let $f \in S_k(\mathfrak{n}, \varphi)$ ($k \geqslant 2$) be as in 12.4.8.2.

12.4.9.1. One can formulate a conjectural variant of (12.4.3.1) for a prime $v \mid p$ of F dividing p. If the conditions (12.4.7.1)–(12.4.7.2) hold for suitable $\mathcal{R}$, then the corresponding analogue of the first equality in (12.4.3.1) was proved by T. Saito ([**Sa**, Thm. 1])[(3)]. If the form f is p-ordinary, then this compatibility can be established directly (even in the case when $\mathcal{R}$ does not exist); see Lemma 12.5.4 (iii) below.

12.4.9.2. Proposition. — *Let $v \mid p$ be a prime of F which does not divide $\mathfrak{n}(f)$. Assume that at least one of the following conditions holds:*

(1) (12.4.7.1)–(12.4.7.2) *hold for suitable* $\mathcal{R}$.
(2) $k = 2$ *and the residual representation of* $V(f)$ *is absolutely irreducible.*
(3) v *is unramified in* $F/\mathbf{Q}$.

Then $V(f)_v$ is a crystalline representation of G_v whose Hodge-Tate weights are contained in the interval $[1-k, 0]$ (if (1) holds, then the Hodge-Tate weights are known to be equal to $1-k$ and 0).

Proof. — In the case (1), the result follows from the geometric construction of $V(f)_v$ ([**Oh1, Oh2**]) and the p-adic comparison results of Fontaine-Messing, Faltings and Tsuji. In general, $V(f)_v$ is constructed as a "limit" of finite subquotients of crystalline representation of G_v whose Hodge-Tate weights are contained in the interval $[1-k, 0]$; a conjecture of Fontaine asserts that $V(f)_v$ itself is crystalline (and its Hodge-Tate weights are contained in the same interval). This was proved by R. Taylor ([**Tay2**]) in the case (2). The case (3) is due to Breuil ([**Bre**]) if $k < p$, and to Berger ([**Be**]) without restriction on k (a forthcoming work of Berger-Colmez is expected to treat the case of ramified v). □

12.4.10. Tamagawa factors

12.4.10.1. Definition. — If $L \subset \overline{\mathbf{Q}}$ is a number field and $j \geqslant 1$, denote by $w_j(L)$ the order of the (finite cyclic) group $H^0(G_L, \mathbf{Q}/\mathbf{Z}(j))$.

12.4.10.2. In particular, if $\zeta \in \overline{\mathbf{Q}}$ is a primitive root of unity of order $n \geqslant 1$, then

$$\zeta \in L \iff n \mid w_1(L)$$
$$\zeta + \zeta^{-1} \in L \implies n \mid w_2(L).$$

12.4.10.3. Proposition. — *Let $f \in S_k(\mathfrak{n}, \varphi)$ ($k \geqslant 1$) be a newform and L the number field generated by the Hecke eigenvalues of f and the values of φ. Let $\mathfrak{p} \mid p$ be a prime of L and $v \nmid p\infty$ a prime of F such that φ_v is unramified. Let $T(f) \subset V(f)$ be a G_F-stable $\mathcal{O}_{L,\mathfrak{p}}$-lattice. Assume that $p \nmid w_2(L)$ and that either $\pi(f)_v$ is in the ramified principal series, or $\pi(f)_v$ is supercuspidal. Then*

$$(V(f)/T(f))^{I_v} = 0, \quad \mathrm{Tam}_v(T(f), \mathfrak{p}) = 0.$$

(3) The general case was recently proved by M. Kisin ([**Kis**]); this result implies that the assumptions (1)–(3) in Proposition 12.4.9.2 can be removed.

Proof. — The assumptions imply that there exists an element $g \in I_v \subset G_v$ which acts on $V(f)_v$ non-trivially (necessarily by an automorphism of finite order). As $\det(g \mid V(f)_v) = \varphi_v(g) = 1$, the eigenvalues of g are equal to ζ, ζ^{-1}, where ζ is a primitive root of unity of order $n \geqslant 2$. As

$$\zeta + \zeta^{-1} = \mathrm{Tr}(g \mid V(f)_v) \in L$$

(by 12.4.4.4), we have $n \mid w_2(L)$. The assumption $p \nmid w_2(L)$ implies that $p \nmid n$, hence $(1-\zeta)(1-\zeta^{-1}) \in L^*$ is a $\mathfrak{p}$-adic unit. In particular, $g-1$ acts invertibly on $T(f)$ and $V(f)/T(f)$, hence

$$V(f)^{I_v} = 0, \quad H^1_{\mathrm{cont}}(I_v, T(f)) = (V(f)/T(f))^{I_v} = 0, \quad \mathrm{Tam}_v(T(f), \mathfrak{p}) = 0. \quad \square$$

12.5. Ordinary Hilbert modular forms

Recall that we have fixed a prime number p and embeddings $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. For non-zero ideals $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_F$, denote by $\mathfrak{a}^{(\mathfrak{b})} := \mathfrak{a}/(\mathfrak{a}, \mathfrak{b}^\infty)$ the prime-to-$\mathfrak{b}$ part of $\mathfrak{a}$.

12.5.1. Definition. — A non-zero cusp form $f \in S_k(\mathfrak{n}, \varphi)$ ($k \geqslant 2$) is p-**ordinary** if it is an eigenform for all Hecke operators $T(\mathfrak{a}) = T_\mathfrak{n}(\mathfrak{a})$ with $\big(\mathfrak{a}, \mathfrak{n}^{(p)}\big) = (1)$:

$$T(\mathfrak{a})f = \lambda_f(\mathfrak{a})f, \qquad \big(\mathfrak{a}, \mathfrak{n}^{(p)}\big) = (1)$$

and if

$$(\forall v \mid p) \quad |\iota_p(\lambda_f(v))|_p = 1.$$

12.5.2. If $v \mid p$ but $v \nmid \mathfrak{n}$, then the restriction of the Hecke operator $T_{\mathfrak{n}v}(v)$ to $S_k(\mathfrak{n}, \varphi)$ does not coincide with $T_\mathfrak{n}(v)$; this implies that the inclusion $S_k(\mathfrak{n}, \varphi) \subset S_k(\mathfrak{n}v, \varphi)$ does not preserve p-ordinary forms. However, there is a canonical procedure of "v-stabilization" ([**Wi**, p. 538]) which associates to f from 12.5.1 an ordinary form $f^{0v} \in S_k(\mathfrak{n}v, \varphi)$ satisfying

$$T_{\mathfrak{n}v}(\mathfrak{a})f^{0v} = \lambda_f(\mathfrak{a})f^{0v} \qquad (\mathfrak{a}, \mathfrak{n}^{(v)}) = (1)$$

(hence $f^{0v} \sim f$). In the classical case $F = \mathbf{Q}$ ($v = p$), f (resp., f^{0v}) corresponds to a classical cusp form $\Phi(z)$ (resp., $\Phi^0(z)$) defined on the upper half plane and

$$\Phi^0(z) = \Phi(z) - \lambda_f(p)^{-1}\varphi(p)\, p^{k-1}\Phi(pz).$$

In general, one can apply the v-stabilization with respect to all primes $v \mid p$ which do not divide $\mathfrak{n}$ and obtain the "p-stabilization" of f, which is an ordinary form $f^0 \in S_k(\mathrm{lcm}(\mathfrak{n}, (p)), \varphi)$ satisfying

$$T_{(p)\mathfrak{n}}(\mathfrak{a})f^0 = \lambda_f(\mathfrak{a})f^0 \qquad \big(\mathfrak{a}, \mathfrak{n}^{(p)}\big) = (1)$$

(hence $f^0 \sim f$). There is a theory of 'newforms' for v- and p-stabilized cusp forms (see [**Hi3**, p. 318]; [**Wi**, (1.2.2)]).

12.5.3. For p-ordinary cusp forms f the representation $V(f)$ was constructed in full generality by Wiles ([**Wi**]), who also described its restriction $V(f)_v$ to G_v for all $v \mid p$ (note that our representation is dual to the one considered in [**Wi**], as we prefer geometric Frobenius elements). Namely, for each $v \mid p$ there is an exact sequence of $L_{\mathfrak{p}}[G_v]$-modules

$$0 \longrightarrow V(f)_v^+ \longrightarrow V(f)_v \longrightarrow V(f)_v^- \longrightarrow 0, \quad \dim_{L_{\mathfrak{p}}} V(f)_v^{\pm} = 1,$$

in which

$$V(f)_v^+ = V(f)^{I_v} \xrightarrow{\sim} (V(f)_v^-)^*(1-k) \otimes \varphi_v \tag{12.5.3.1}$$

is an unramified G_v-module on which $\mathrm{Fr}(v)$ ($= \mathrm{Fr}(v)_{\mathrm{geom}}$) acts by the scalar $\alpha_v(f) \in \mathcal{O}_{L,\mathfrak{p}}^*$, where

$$\begin{cases} \alpha_v(f) = \lambda_f(v), & \text{if } v \mid \mathfrak{n} \\ (1-\alpha_v(f)X)(1-\varphi(v)\alpha_v(f)^{-1}(Nv)^{k-1}X) & \\ \qquad = 1 - \lambda_f(v)X + \varphi(v)(Nv)^{k-1}X^2, & \text{if } v \nmid \mathfrak{n}. \end{cases}$$

The subspace $V(f)_v^+ \subset V(f)$ is uniquely determined by these properties.

The ordinarity assumption implies, by 12.3.10–12.3.11, that either

$$\mathrm{ord}_v(\mathfrak{n}(f)) = \mathrm{ord}_v(\mathrm{cond}(\varphi)) \geqslant 0 \text{ or } \mathrm{ord}_v(\mathfrak{n}(f)) = 1,\ \mathrm{ord}_v(\mathrm{cond}(\varphi)) = 0.$$

More precisely, we have the following dichotomy:

12.5.4. Lemma. — *If $f \in S_k(\mathfrak{n}, \varphi)$ $(k \geqslant 2)$ is p-ordinary, then for each $v \mid p$ there are two mutually exclusive possibilities:*

(i) *If* $\mathrm{ord}_v(\mathfrak{n}(f)) = \mathrm{ord}_v(\mathrm{cond}(\varphi)) \geqslant 0$, *then*

$$\pi(f)_v = \pi(\mu_+, \mu_-), \quad \mu_+\mu_- = \varphi_v, \quad \mu_+ \text{ is unramified}, \quad o(\mu_-) = \mathrm{ord}_v(\mathfrak{n}(f)),$$

$$V(f)_v^+ = L_{\mathfrak{p}} \otimes \mu_+ |\cdot|^{(1-k)/2}, \quad V(f)_v^- = L_{\mathfrak{p}}(1-k) \otimes \mu_- |\cdot|^{(k-1)/2},$$

$$\alpha_v(f) = (Nv)^{(k-1)/2}\mu_+(v)$$

and $\alpha_v(f)$ is a v-Weil number of weight $(k-1)/2$. The representation $V(f)_v$ is de Rham; it is crystalline over a finite extension E_w of F_v $\iff$ the character $\mu_- \circ N_{E_w/F_v} = \mu_- \circ N_w : E_w^* \to \mathbf{C}^*$ is unramified.

(ii) *If* $\mathrm{ord}_v(\mathfrak{n}(f)) = 1$, $\mathrm{ord}_v(\mathrm{cond}(\varphi)) = 0$, *then*

$$\pi(f)_v = \mathrm{St}(\mu), \quad \mu^2 = \varphi_v, \quad \mu \text{ is unramified}, \quad k = 2,$$

$$V(f)_v^+ = L_{\mathfrak{p}} \otimes \mu, \quad V(f)_v^- = L_{\mathfrak{p}}(-1) \otimes \mu, \quad \alpha_v(f) \text{ is a root of unity}.$$

The representation $V(f)_v$ is semistable, but not crystalline (over any finite extension E_w of F_v): its extension class in

$$H^1_{\mathrm{cont}}(G_v, \mathrm{Hom}_{L_{\mathfrak{p}}}(V(f)_v^-, V(f)_v^+)) = H^1_{\mathrm{cont}}(G_v, L_{\mathfrak{p}}(1)) \xrightarrow{\sim} F_v^* \widehat{\otimes} L_{\mathfrak{p}}$$

is not contained in $H^1_f(G_v, L_{\mathfrak{p}}(1)) \xrightarrow{\sim} \mathcal{O}_v^* \widehat{\otimes} L_{\mathfrak{p}}$.

(iii) *In the case (i) (resp., (ii)), the representation of the Weil-Deligne group of* F_v *associated to* $D_{\rm pst}(V(f)_v)$ *as in* [**Fo-PR**, §I.2.2.5, §I.1.3.2] *is given by the direct sum of characters* $\mu_+|\cdot|^{(1-k)/2}$ *and* $\mu_-|\cdot|^{(1-k)/2}$ *(resp., is equal to* $\mu|\cdot|^{-1}\otimes\mathrm{sp}(2)$*), hence it corresponds to* $\pi(f)_v=\pi(\mu_+,\mu_-)$ *(resp., to* $\pi(f)_v=\mathrm{St}(\mu)$*) by the local Langlands correspondence (normalized as in 12.4.3).*

Proof. — Fix $v\mid p$. In the case (i), we know from 12.3.10–12.3.11 that $\pi(f)_v=\pi(\mu,\mu')$ with μ unramified and $\mu\mu'=\varphi_v$. Denote by $\mu_\pm: F_v^*\to L_{\mathfrak p}^*$ the characters such that G_v acts on $V(f)_v^+$ by $\mu_+|\cdot|^{(1-k)/2}$, and on $V(f)_v^-$ by $L_{\mathfrak p}(1-k)\otimes\mu_-|\cdot|^{(k-1)/2}$. If $v\mid\mathfrak n(f)$, then

$$L_v(\mu,s)=L_v(\pi(f)_v,s)=(1-\lambda_f(v)(Nv)^{-s-(k-1)/2})^{-1},$$

which implies that

$$(\mu|\cdot|^{(1-k)/2})(v)=\lambda_f(v)=\alpha_v(f)=(\mu_+|\cdot|^{(1-k)/2})(v)\Longrightarrow\ \mu=\mu_+.$$

It follows from (12.5.3.1) that $\mu_-=\varphi_v/\mu_+=\mu'$. Finally, 12.4.8.3 implies that $\mu_+(v)$ is a v-Weil number of weight 0 (hence $\alpha_v(f)$ is a v-Weil number of weight $(k-1)/2$).

If $v\nmid\mathfrak n(f)$, but $v\mid\mathfrak n$, then the theory of 'newforms' for v-stabilized cusp forms implies that there exists a p-ordinary form $g\in S_k\big(\mathfrak n^{(v)},\varphi\big)$ such that, for each ideal $\mathfrak a$ with $\big(\mathfrak a,\mathfrak n^{(v)}\big)=(1)$,

$$T_{\mathfrak n}(\mathfrak a)f=\lambda_f(\mathfrak a)f,\quad T_{\mathfrak n^{(v)}}(\mathfrak a)g=\lambda_f(\mathfrak a)g.$$

This implies that it is sufficient to prove (i) for g, *i.e.*, we can assume that $v\nmid\mathfrak n$. In this case μ' is also unramified and

$$L_v(\mu,s)L_v(\mu',s)=L_v(\pi(f)_v,s)=\big(1-\lambda_f(v)(Nv)^{-s-(k-1)/2}+\varphi(v)(Nv)^{-2s}\big)^{-1},$$

hence

$$\{\mu(v),\mu'(v)\}=\big\{\alpha_v(f)(Nv)^{-(k-1)/2},\varphi(v)\alpha_v(f)^{-1}(Nv)^{(k-1)/2}\big\}=\{\mu_+(v),\mu_-(v)\}.$$

Exchanging μ and μ' if necessary (and noting that $\mu_+(v)\neq\mu_-(v)$, as their p-adic valuations are distinct), we obtain $\mu=\mu_+$, $\mu'=\mu_-$. Again, 12.4.8.3 implies that both $\mu_\pm(v)$ are v-Weil numbers of weight 0.

The extension class of

$$0\longrightarrow V(f)_v^+\longrightarrow V(f)_v\longrightarrow V(f)_v^-\longrightarrow 0 \tag{12.5.4.1}$$

is contained in

$$\begin{aligned}H^1_{\rm cont}(G_v,\mathrm{Hom}_{L_{\mathfrak p}}(V(f)_v^-,V(f)_v^+))&=H^1_{\rm cont}(G_v,(V(f)_v^+)^{\otimes 2}\otimes\varphi_v^{-1}(k-1))\\&=H^1_{\rm cont}(G_v,L_{\mathfrak p}(k-1)\otimes\beta),\end{aligned}$$

where $\beta: G_v/I_v\to L_{\mathfrak p}^*$ is an unramified character whose value at the geometric Frobenius

$$\beta(v)=\beta(\mathrm{Fr}(v))=\alpha_v(f)^2\varphi(v)^{-1}=(Nv)^{k-1}\mu_+(v)/\mu_-(v)$$

is a v-Weil number of weight $k-1\geqslant 1$ (thus $\beta\neq 1$).

It follows from [**B-K**, §3.9] that, for any unramified character $\nu : G_v/I_v \to L_{\mathfrak{p}}^*$, we have

$$(12.5.4.2) \qquad \begin{aligned} (\forall n > 1) \quad & H^1_{\mathrm{cont}}(G_v, L_{\mathfrak{p}}(n) \otimes \nu) = H^1_f(F_v, L_{\mathfrak{p}}(n) \otimes \nu) \\ \nu \neq 1 \Longrightarrow \quad & H^1_{\mathrm{cont}}(G_v, L_{\mathfrak{p}}(1) \otimes \nu) = H^1_f(F_v, L_{\mathfrak{p}}(1) \otimes \nu). \end{aligned}$$

Applying (12.5.4.2) to $\nu = \beta$, we deduce that $V(f)_v$ is a crystalline representation of G_v.

Back to the case $v \mid \mathfrak{n}(f)$; we see that the extension class of (12.5.4.1) is contained in $H^1_{\mathrm{cont}}(G_v, L_{\mathfrak{p}}(k-1) \otimes \beta)$, where

$$\beta = (\mu_+/\mu_-)|\cdot|^{1-k} = \mu_+^2 \varphi_v^{-1}|\cdot|^{1-k} : G_v \longrightarrow L_{\mathfrak{p}}^*$$

is a product of an unramified character by a character of finite order; [**B-K**, §3.9] then implies that

$$(\forall n \geqslant 1) \quad H^1_{\mathrm{cont}}(G_v, L_{\mathfrak{p}}(n) \otimes \beta) = H^1_g(F_v, L_{\mathfrak{p}}(n) \otimes \beta),$$

hence $V(f)_v$ is a de Rham representation of G_v. If $V(f)_v$ is crystalline over E_w, so is $V(f)_v^- = L_{\mathfrak{p}}(1-k) \otimes \mu_-|\cdot|^{(k-1)/2}$, hence $\mu_- \circ N_w$ is unramified. Conversely, if $\mu_- \circ N_w$ is unramified, applying (12.5.4.2) to E_w and to the restriction β' of β to G_w, we deduce that $V(f)_v$ is a crystalline representation of G_w (note that $\beta' \neq 1$, as $\beta'(\mathrm{Fr}(w))$ is a w-Weil number of weight $k-1 \geqslant 1$).

(ii) We have $\pi(f)_v = \mathrm{St}(\mu)$, by Lemma 12.3.10. The formula

$$L_v(\mu, s + \tfrac{1}{2}) = L_v(\pi(f)_v, s) = (1 - \lambda_f(v)(Nv)^{-s-(k-1)/2})^{-1}$$

implies that

$$(\mu|\cdot|^{1-k/2})(v) = \lambda_f(v) = \alpha_v(f),$$

hence G_v acts on $V(f)_v^+$ by $\mu|\cdot|^{1-k/2}$ ($\Longrightarrow \mu$ is unramified) and on $V(f)_v^- \xrightarrow{\sim} (V(f)_v^+)^*(1-k) \otimes \varphi_v$ by $L_{\mathfrak{p}}(1-k) \otimes (\mu|\cdot|^{1-k/2})^{-1}\mu^2 = L_{\mathfrak{p}}(1-k) \otimes \mu|\cdot|^{k/2-1}$. As

$$|\iota_p(\varphi(v)(Nv)^{k-2})|_p = |\iota_p(\lambda_f(v)^2)|_p = 1,$$

it follows that $k = 2$, hence $\alpha_v(f) = \mu(v)$, which is a root of unity (as $\mu^2 = \varphi_v$ has finite order).

The fact that the representation $V(f)_v$ is semistable, but not crystalline, follows from the comparison result of T. Saito ([**Sa**, Thm. 1]). One can also argue geometrically: $V(f)_v$ is a direct factor of the first étale cohomology group of the Jacobian of a Shimura curve associated to a quaternion algebra over F ramified at v, but whose level subgroup is maximal at v. The results of Čerednik [**Če**] (see also [**Dr**] and [**Bo-Car**] in the case $F = \mathbf{Q}$) imply that the Jacobian in question has purely toric reduction at v, which proves the desired result.

(iii) This follows from the following two facts: firstly, if $\chi : G_v \to L_{\mathfrak{p}}^*$ is a potentially unramified one-dimensional representation and if $n \in \mathbf{Z}$, then the representation of the Weil-Deligne group of F_v associated to $L_{\mathfrak{p}}(n) \otimes \chi$ is the one-dimensional representation given by $\chi|\cdot|^n$; secondly, that the monodromy operator N on $D_{\mathrm{pst}}(V(f)_v)$ satisfies $N=0$ (resp., $N\neq 0$) in the case (i) (resp., (ii)), as $V(f)_v$ is (resp., is not) potentially crystalline. □

12.5.5. Assume that $f\in S_k(\mathfrak{n},\varphi)$ is p-ordinary and there exists a character $\chi : \mathbf{A}_F^*/F^*\to\mathbf{C}^*$ (with values contained in $\iota_\infty(L)$) such that $\varphi=\chi^{-2}$ ($\Longrightarrow k$ is even). Fix such a character χ; it is determined up to multiplication by a quadratic character $\mathbf{A}_F^*/F^*\to\{\pm1\}$. The form

$$g:=f\otimes\chi\in S_k(\mathfrak{n}(g),1)$$

is a newform of level $\mathfrak{n}(g)$ dividing $\mathrm{cond}(\chi)^2\mathfrak{n}$ and trivial character. The twisted Galois representation

$$V=V(g)(k/2)=V(f)(k/2)\otimes\chi$$

of G_F is self-dual in the sense that

$$\Lambda^2V\xrightarrow{\sim}L_{\mathfrak{p}}(1),\quad V\xrightarrow{\sim}V^*(1) \tag{12.5.5.1}$$

and, for each prime $v\mid p$ of F, the twisted representations of G_v

$$V_v^{\pm}=V(f)_v^{\pm}(k/2)\otimes\chi_v$$

sit in an exact sequence of $L_{\mathfrak{p}}[G_v]$-modules

$$0\longrightarrow V_v^+\longrightarrow V_v\longrightarrow V_v^-\longrightarrow 0, \tag{12.5.5.2}$$

which is self-dual with respect to the isomorphism (12.5.5.1), *i.e.*, $V_v^{\pm}\xrightarrow{\sim}(V_v^{\mp})^*(1)$.

If E_w/F_v (again for $v\mid p$) is a finite extension, we put

$$\alpha_w(f)=\alpha_v(f)^{[\kappa(w):\kappa(v)]},$$

where $\kappa(w)/\kappa(v)$ is the corresponding extension of residue fields. If, in addition, the character

$$\chi_v\circ N_w : E_w^*\longrightarrow F_v^*\longrightarrow\mathbf{C}^*\quad(N_w=N_{E_w/F_v})$$

is unramified, then we put

$$\alpha_w(g)=\alpha_w(f)(\chi_v\circ N_w)(w).$$

For example, if $\pi(f)_v=\mathrm{St}(\lambda)$ with λ unramified, then $\alpha_v(f)=\lambda(v)$ and $\alpha_w(f)=(\lambda\circ N_w)(w)$; if, in addition, $\chi_v\circ N_w$ is unramified, then

$$\pi(g)_v=\mathrm{St}(\mu),\quad \mu=\lambda\chi_v,\quad \mu^2=1,\quad \mu\circ N_w \text{ is unramified},$$
$$\alpha_w(g)=((\lambda\chi_v)\circ N_w)(w)=(\mu\circ N_w)(w).$$

In general, Lemma 12.5.4 implies that we have, for each prime $v \mid p$ of F, either

$$\pi(g)_v = \pi(\mu_+\chi_v, (\mu_+\chi_v)^{-1}), \quad \mu_+ \text{ unramified}, \quad \mathrm{ord}_v(\mathfrak{n}(g)) = 2\,\mathrm{ord}_v(\mathrm{cond}(\chi))$$

or

$$\pi(g)_v = \mathrm{St}(\mu), \quad \mu^2 = 1, \quad k = 2, \quad \mu\chi_v^{-1} \text{ unramified},$$
$$\mathrm{ord}_v(\mathfrak{n}(g)) = \max(1, 2\,\mathrm{ord}_v(\mathrm{cond}(\chi))).$$

12.5.6. We wish to relate two kinds of Selmer groups associated to $V \otimes \beta$ (where β is a character of finite order) over a finite extension of F: our $\widetilde{H}^1_f(V \otimes \beta)$ (with respect to Greenberg's local conditions defined by (12.5.5.2)) and Bloch-Kato's $H^1_f(V \otimes \beta)$ (cf. [**N-P**, §3.1.7] for $F = \mathbf{Q}$).

Recall first the local situation: let $v \mid p$ be a prime of F and W a continuous $\mathbf{Q}_p$-representation of G_v; put $W^*(1) = \mathrm{Hom}_{\mathbf{Q}_p}(W, \mathbf{Q}_p(1))$. Bloch-Kato [**B-K**, §3.7.2] defined subspaces

$$H^1_e(F_v, W) \subset H^1_f(F_v, W) \subset H^1_g(F_v, W) \subset H^1(F_v, W) = H^1_{\mathrm{cont}}(G_v, W)$$

satisfying the following properties: if W is a de Rham representation of G_v, then

12.5.6.1. The extension class of a short exact sequence of continuous $\mathbf{Q}_p$-representation of G_v

$$0 \longrightarrow W \longrightarrow W' \longrightarrow \mathbf{Q}_p \longrightarrow 0$$

is contained in $H^1_g(F_v, W)$ (resp., in $H^1_f(F_v, W)$) iff W' is a de Rham representation of G_v (resp., a crystalline representation of G_v, provided that W is crystalline).

12.5.6.2. $H^1_e(F_v, W)$ (resp., $H^1_f(F_v, W)$) is the exact annihilator of $H^1_g(F_v, W^*(1))$ (resp., of $H^1_f(F_v, W^*(1))$) under the local Tate duality

$$H^1(F_v, W) \times H^1(F_v, W^*(1)) \xrightarrow{\cup} H^2(F_v, \mathbf{Q}_p(1)) \xrightarrow{\sim} \mathbf{Q}_p.$$

12.5.6.3. $H^1_e(F_v, W)$ is a quotient of $D_{\mathrm{dR}}(W)/F^0$.

12.5.6.4. $H^1_f(F_v, W)/H^1_e(F_v, W)$ is a quotient of $D_{\mathrm{cris}}(W)/(f-1)$.

12.5.6.5. For each exact sequence of $\mathbf{Q}_p$-representations of G_v

$$0 \longrightarrow W^+ \longrightarrow W \longrightarrow W^- \longrightarrow 0$$

($\Longrightarrow$ both W^+ and W^- are de Rham representations), the induced sequence

$$0 \longrightarrow H^0(F_v, W^+) \longrightarrow H^0(F_v, W) \longrightarrow H^0(F_v, W^-)$$
$$\longrightarrow H^1_g(F_v, W^+) \longrightarrow H^1_g(F_v, W) \longrightarrow H^1_g(F_v, W^-)$$

is also exact.

We shall also study the subspaces $H^1_*(E_w, W)$ over finite extensions E_w of F_v; in this case we use the notation

$$D_{\mathrm{cris},w}(W) = H^0(E_w, W \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}}), \quad D_{\mathrm{dR},w}(W) = H^0(E_w, W \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})$$

for Fontaine's functors over E_w.

12.5.7. Lemma. — *Under the assumptions of 12.5.5, let E_w/F_v (where $v \mid p$) be a finite extension and $\beta : G_w = \mathrm{Gal}(\overline{F}_v/E_w) \to L_{\mathfrak{p}}^*$ a character of finite order. Then:*

(1) (i) $H^1_e(E_w, V_v^- \otimes \beta) = 0$, $H^1_g(E_w, V_v^+ \otimes \beta) = H^1(E_w, V_v^+ \otimes \beta)$.

(ii) $D_{\mathrm{cris},w}(V_v^+ \otimes \beta)^{f=1} = H^0(E_w, V_v^+ \otimes \beta) = 0$.

(iii) $H^1_e(E_w, V_v^+ \otimes \beta) = H^1_f(E_w, V_v^+ \otimes \beta)$, $H^1_f(E_w, V_v^- \otimes \beta) = H^1_g(E_w, V_v^- \otimes \beta)$.

(2) *If* $D_{\mathrm{cris},w}(V_v^- \otimes \beta^{\pm 1})^{f=1} = 0$, *then:*

(i) $D_{\mathrm{cris},w}(V \otimes \beta^{\pm 1})^{f=1} = 0$.

(ii) $(\forall W = V, V_v^\pm)$ $H^0(E_w, W \otimes \beta^{\pm 1}) = 0$, $H^1_e(E_w, W \otimes \beta^{\pm 1}) = H^1_f(E_w, W \otimes \beta^{\pm 1}) = H^1_g(E_w, W \otimes \beta^{\pm 1})$.

(iii) $(\forall * = e, f, g)$ $H^1_*(E_w, V_v^- \otimes \beta^{\pm 1}) = 0$, $H^1_*(E_w, V_v^+ \otimes \beta^{\pm 1}) = H^1(E_w, V_v^+ \otimes \beta^{\pm 1})$.

(iv) *The map* $H^1(E_w, V_v^+ \otimes \beta^{\pm 1}) \to H^1(E_w, V \otimes \beta^{\pm 1})$ *is injective and its image is equal to* $H^1_f(E_w, V \otimes \beta^{\pm 1})$.

Proof

(1) As $V_v^-(k/2-1) \otimes \chi_v$ is an unramified representation of G_v and $k/2 - 1 \geqslant 0$, it follows that $D_{\mathrm{dR},w}(V_v^- \otimes \beta)/F^0 = 0$, hence $H^1_e(E_w, V_v^- \otimes \beta) = 0$, by 12.5.6.3. Applying the duality 12.5.6.2 and replacing β by its inverse we obtain the second statement of (i). As $V_v^+(-k/2) \otimes \chi_v^{-1}$ is an unramified representation and $k \neq 0$, we have $H^0(E_w, V_v^+ \otimes \beta) \subseteq D_{\mathrm{cris},w}(V_v^+ \otimes \beta)^{f=1} = 0$, proving (ii). Combining (ii) with 12.5.6.4 yields the first statement of (iii); the second follows from the duality 12.5.6.2 (replacing β by its inverse).

(2) As both $D_{\mathrm{cris},w}(V_v^\pm \otimes \beta^{\pm 1})^{f=1} = 0$ vanish, so do $D_{\mathrm{cris},w}(V \otimes \beta^{\pm 1})^{f=1}$, $H^0(E_w, W \otimes \beta^{\pm 1})$ and $H^1_f(E_w, W \otimes \beta^{\pm 1})/H^1_e(E_w, W \otimes \beta^{\pm 1})$ $(W = V, V_v^\pm)$, thanks to 12.5.6.4. Applying the duality 12.5.6.2, we obtain the equality $H^1_f = H^1_g$ in (ii). The statements in (iii) follow from (ii) and (1)(i). Finally, (iv) is a consequence of (iii) and the exact sequence 12.5.6.5 associated to

$$0 \longrightarrow V_v^+ \otimes \beta^{\pm 1} \longrightarrow V \otimes \beta^{\pm 1} \longrightarrow V_v^- \otimes \beta^{\pm 1} \longrightarrow 0. \quad \square$$

12.5.8. Proposition. — *Under the assumptions of 12.5.5, let E_w/F_v (where $v \mid p$) be a finite extension and $\beta : G_w = \mathrm{Gal}(\overline{F}_v/E_w) \to L_{\mathfrak{p}}^*$ a character of finite order. Then the kernel (resp., the image) of the map $H^1(E_w, V_v^+ \otimes \beta) \to H^1(E_w, V \otimes \beta)$ is equal to $H^0(E_w, V_v^- \otimes \beta)$ (resp., to $H^1_f(E_w, V \otimes \beta)$),* i.e., *there is an exact sequence*

$$0 \longrightarrow H^0(E_w, V_v^- \otimes \beta) \longrightarrow H^1(E_w, V_v^+ \otimes \beta) \longrightarrow H^1_f(E_w, V \otimes \beta) \longrightarrow 0.$$

Moreover,

$$\dim_{L_{\mathfrak{p}}} H^0(E_w, V_v^- \otimes \beta) = \begin{cases} 1, & \pi(g)_v = \mathrm{St}(\mu)\ (\mu^2 = 1),\ \beta \cdot (\mu \circ N_w) = 1 \\ 0, & \textit{otherwise.} \end{cases}$$

In particular, if $\beta = 1$*, then*

$$\begin{aligned}\dim_{L_{\mathfrak{p}}} H^0(E_w, V_v^-) &= \begin{cases} 1, & \pi(f)_v = \mathrm{St}(\lambda)\ (\lambda \text{ unramified}),\ \chi_v \circ N_w \text{ unramified, } \alpha_w(g) = 1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \pi(g)_v = \mathrm{St}(\mu)\ (\mu = \lambda\chi_v,\ \mu^2 = 1),\ \mu \circ N_w \text{ unramified, } \alpha_w(g) = 1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \pi(g)_v = \mathrm{St}(\mu)\ (\mu^2 = 1),\ \mu \circ N_w = 1 \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

Proof. — We analyze separately the two cases distinguished in Lemma 12.5.4. If $\pi(f)_v = \pi(\mu_+, \mu_-)$, then $\pi(g)_v = \pi(\mu_+\chi_v, \mu_-\chi_v)$; we claim that

$$D_{\mathrm{cris},w}(V_v^- \otimes \beta^{\pm 1})^{f=1} = 0 \tag{12.5.8.1}$$

in this case (which implies the statement of the Proposition, thanks to Lemma 12.5.7 (2) (ii), (iv)). Fix $\nu \in \{\beta, \beta^{-1}\}$; if the character $\nu \cdot ((\mu_-\chi_v) \circ N_w)$ is ramified, then $D_{\mathrm{cris},w}(V_v^- \otimes \nu) = 0$. If it is unramified, then $D_{\mathrm{cris},w}(V_v^- \otimes \nu)$ is a free $L_{\mathfrak{p}} \otimes_{\mathbf{Q}_p} (E_w \cap \mathbf{Q}_p^{\mathrm{ur}})$-module of rank one, on which f acts by $a \otimes \sigma$, where σ is the absolute (arithmetic) Frobenius and $a = (\nu\mu_-\chi_v)(\mathrm{Fr}(w))(Nw)^{-1/2}$ is w-Weil number of weight -1, by 12.4.8.3; thus (12.5.8.1) holds in this case, too.

If $\pi(f)_v = \mathrm{St}(\lambda)$ (λ unramified), then $\pi(g)_v = \mathrm{St}(\mu)$, where $\mu = \lambda\chi_v$, $\mu^2 = 1$. As $V_v^+ = L_{\mathfrak{p}}(1) \otimes \mu$ and $V_v^- = L_{\mathfrak{p}} \otimes \mu$, we have, for each $\nu \in \{\beta, \beta^{-1}\}$,

$$\dim_{L_{\mathfrak{p}}} D_{\mathrm{cris},w}(V_v^- \otimes \nu)^{f=1} = \begin{cases} 1, & \nu \cdot (\mu \circ N_w) = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{12.5.8.2}$$

If $\beta \cdot (\mu \circ N_w) \neq 1$, then $\beta^{-1} \cdot (\mu \circ N_w) \neq 1$ either, since $\mu^2 = 1$; thanks to (12.5.8.2), we conclude again by appealing to Lemma 12.5.7 (2) (ii), (iv).

In the remaining case $\beta \cdot (\mu \circ N_w) = 1$, the exact sequence of $L_{\mathfrak{p}}[G_w]$-modules

$$0 \longrightarrow V_v^+ \otimes \beta \longrightarrow V \otimes \beta \longrightarrow V_v^- \otimes \beta \longrightarrow 0$$

is isomorphic to a non-crystalline Kummer extension

$$0 \longrightarrow L_{\mathfrak{p}}(1) \longrightarrow V \otimes \beta \longrightarrow L_{\mathfrak{p}} \longrightarrow 0.$$

The exactness of the sequence

$$0 \longrightarrow L_{\mathfrak{p}} = H^0(E_w, V_v^- \otimes \beta) \longrightarrow H^1(E_w, V_v^+ \otimes \beta) \longrightarrow H^1_f(E_w, V \otimes \beta) \longrightarrow 0$$

then follows from the calculation in [**N-P**, §3.1.6].

In the special case $\beta = 1$, we have

$$\begin{aligned}\mu \circ N_w = 1 \iff & \mu \circ N_w \text{ is unramified and } (\mu \circ N_w)(w) = 1 \\ \iff & \chi_v \circ N_w \text{ is unramified and } \alpha_v(g) = 1,\end{aligned}$$

which proves the remaining statements. □

12.5.9. Comparison of Selmer groups. — Let $f, \chi, g = f \otimes \chi$ and $V = V(g)(k/2)$ be as in 12.5.5.

12.5.9.1. Fix a finite set S of primes of F such that $S \supset \{v \mid p\mathfrak{n}(g)\infty\}$, a finite subextension E/F of F_S/F and a (continuous) character of finite order $\beta : \mathrm{Gal}(F_S/E) = G_{E,S} \to L_{\mathfrak{p}}^*$. The Bloch-Kato Selmer group

$$H^1_f(E, V \otimes \beta) \\ = \mathrm{Ker}\left(H^1_{\mathrm{cont}}(G_{E,S}, V \otimes \beta) \longrightarrow \bigoplus_{v \in S_f} \bigoplus_{w \mid v} H^1_{\mathrm{cont}}(G_w, V \otimes \beta_w)/H^1_f(E_w, V \otimes \beta_w) \right)$$

(where w runs through all primes of E above v, and $H^1_f(E_w, -) = H^1_{\mathrm{ur}}(E_w, -)$ if $v \nmid p$) does not depend on the choice of S, by [**Fo-PR**, §II.1.3.2].

Fix a subset $\Sigma \subset S_f$ containing all primes dividing p, set $\Sigma' = S_f - \Sigma$ and define, for each $v \in S_f$ and $w \mid v$, Greenberg's local conditions $\Delta_{\Sigma,w}(V \otimes \beta)$ for $V \otimes \beta$ as follows:

$$U_w^+(V \otimes \beta) = \begin{cases} C^\bullet_{\mathrm{cont}}(G_w, V_v^+ \otimes \beta_w), & v \mid p \\ 0, & v \in \Sigma,\ v \nmid p \\ C^\bullet_{\mathrm{cont}}(G_w/I_w, (V \otimes \beta_w)^{I_w}), & v \in \Sigma'. \end{cases}$$

12.5.9.2. Proposition - Definition. — *Under the assumptions of 12.5.9.1,*

(i) *Let $w \nmid p\infty$ be a prime of E; then the complexes $C^\bullet_{\mathrm{cont}}(G_w, V \otimes \beta_w)$ and $C^\bullet_{\mathrm{cont}}(G_w/I_w, (V \otimes \beta_w)^{I_w})$ (where $G_w = \mathrm{Gal}(\overline{E}_w/E_w)$) are acyclic.*

(ii) *Up to a canonical isomorphism, the Selmer complex $\widetilde{\mathbf{R}\Gamma}_f(G_{E,S}, V \otimes \beta; \Delta_\Sigma(V \otimes \beta))$ does not depend on the choice of S and Σ; we denote it by $\widetilde{\mathbf{R}\Gamma}_f(E, V \otimes \beta)$ and its cohomology groups by $\widetilde{H}^i_f(E, V \otimes \beta)$.*

(iii) *There is an exact sequence*

$$0 \longrightarrow \bigoplus_{v \mid p} \bigoplus_{w \mid v} H^0(E_w, V_v^- \otimes \beta_w) \longrightarrow \widetilde{H}^1_f(E, V \otimes \beta) \longrightarrow H^1_f(E, V \otimes \beta) \longrightarrow 0,$$

in which $v \mid p$ is a prime of F and

$$\dim_{L_{\mathfrak{p}}} H^0(E_w, V_v^- \otimes \beta_w) = \begin{cases} 1, & \pi(g)_v = \mathrm{St}(\mu)\ (\mu^2 = 1),\ \beta_w \cdot (\mu \circ N_w) = 1 \\ 0, & \textit{otherwise.} \end{cases}$$

(iv) *Let E'/E be a finite subextension of F_S/E such that β factors through $\beta : \mathrm{Gal}(E'/E) \to L_{\mathfrak{p}}^*$. Then, in the notation of Proposition 8.8.7, there are canonical isomorphisms*

$$\widetilde{H}^q_f(E, V \otimes \beta) \xrightarrow{\sim} \widetilde{H}^q_f(E', V)^{(\beta^{-1})}, \quad H^1_f(E, V \otimes \beta) \xrightarrow{\sim} H^1_f(E', V)^{(\beta^{-1})}.$$

Proof

(i) This follows from the fact that the inflation map

$$\inf : C^\bullet_{\mathrm{cont}}\big(G_w/I_w, (V\otimes\beta_w)^{I_w}\big) \longrightarrow C^\bullet_{\mathrm{cont}}(G_w, V\otimes\beta_w)$$

induces monomorphisms on cohomology and that the complex $C^\bullet_{\mathrm{cont}}(G_w, V\otimes\beta_w)$ is acyclic (by Proposition 12.4.8.4).

(ii) Independence on S (resp., on Σ) follows from Proposition 7.8.8 (resp., from (i)).

(iii) It follows from (i) that the bottom row of (6.1.3.2) boils down to the following exact triangle:

$$\widetilde{\mathbf{R}\Gamma}_f(E, V\otimes\beta) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G_{E,S}, V\otimes\beta) \longrightarrow \bigoplus_{v|p}\bigoplus_{w|v}\mathbf{R}\Gamma_{\mathrm{cont}}(G_w, V_v^-\otimes\beta_w).$$

In the corresponding cohomology sequence

$$\begin{aligned} &H^0(G_{E,S}, V\otimes\beta) \longrightarrow \bigoplus_{v|p}\bigoplus_{w|v} H^0(E_w, V_v^-\otimes\beta_w)\\ &\qquad \longrightarrow \widetilde{H}^1_f(E, V\otimes\beta) \longrightarrow H^1_{\mathrm{cont}}(G_{E,S}, V\otimes\beta) \xrightarrow{r} \bigoplus_{v|p}\bigoplus_{w|v} H^1_{\mathrm{cont}}(G_w, V_v^-\otimes\beta)\end{aligned}$$

the group $H^0(G_{E,S}, V\otimes\beta) = 0$ vanishes by Proposition 12.4.8.4, $H^0(E_w, V_v^-\otimes\beta_w)$ was computed in Proposition 12.5.8 and $\mathrm{Ker}(r) = H^1_f(E, V\otimes\beta)$ (by combining (i) with Proposition 12.5.8).

(iv) This follows from Proposition 8.8.7 (the same proof also applies to the Bloch-Kato Selmer groups). □

12.5.9.3. Corollary - Definition. — *Denote*

$$\widetilde{h}^i_f(E, V\otimes\beta) = \dim_{L_\mathfrak{p}} \widetilde{H}^i_f(E, V\otimes\beta), \quad h^1_f(E, V\otimes\beta) = \dim_{L_\mathfrak{p}} H^1_f(E, V\otimes\beta);$$

then

$$\widetilde{h}^1_f(E, V\otimes\beta) - h^1_f(E, V\otimes\beta) = \big|\{w : w|v|p,\ \pi(g)_v = \mathrm{St}(\mu)\ (\mu^2 = 1),\ \beta_w\cdot(\mu\circ N_w) = 1\}\big|.$$

12.5.9.4. Proposition - Definition. — *Under the assumptions of 12.5.9.1, define*

$$\widetilde{\varepsilon}\big(\pi(g), \tfrac12\big) = \prod_v \widetilde{\varepsilon}_v\big(\pi(g)_v, \tfrac12\big),$$

$$\widetilde{\varepsilon}_v\big(\pi(g)_v, \tfrac12\big) = \varepsilon_v\big(\pi(g)_v, \tfrac12\big)\cdot\begin{cases}-1, & \textit{if } v \mid p,\ \pi(g)_v = \mathrm{St}\\ 1, & \textit{otherwise.}\end{cases}$$

Then we have

$$\widetilde{\varepsilon}\big(\pi(g), \tfrac12\big)/\varepsilon\big(\pi(g), \tfrac12\big) = (-1)^{\widetilde{h}^1_f(F,V) - h^1_f(F,V)}$$

and

$$\begin{aligned}(-1)^{\widetilde{h}^1_f(F,V)-h^1_f(F,V)+r_{\rm an}(F,g)} &= \widetilde{\varepsilon}\big(\pi(g),\tfrac{1}{2}\big)\\ &= \prod_{v\nmid p\infty}\chi_v(-1)\,\varepsilon_v\big(\pi(g)_v,\tfrac{1}{2}\big)\prod_{v\mid\infty}\chi_v(-1)(-1)^{k/2}.\end{aligned}$$

Proof. — Applying Proposition 12.5.9.2 (iii) to $E=F$ and $\beta=1$, we obtain

$$(-1)^{\widetilde{h}^1_f(F,V)-h^1_f(F,V)} = \prod_{\substack{v\mid p\\ \pi(g)_v=\mathrm{St}}}(-1) = \widetilde{\varepsilon}\big(\pi(g),\tfrac{1}{2}\big)/\varepsilon\big(\pi(g),\tfrac{1}{2}\big).$$

Fix a prime $v \mid p$ of F. According to Lemma 12.5.4 there are two possible cases: either $\pi(f)_v = \pi(\mu_+,\mu_-)$ with μ_+ unramified, which implies that $\pi(g)_v = \pi(\mu_+\chi_v,(\mu_+\chi_v)^{-1})$ and

$$\varepsilon_v\big(\pi(g)_v,\tfrac{1}{2}\big) = (\mu_+\chi_v)(-1) = \chi_v(-1)$$

(by Lemma 12.3.13 (v)), or $\pi(f)_v = \mathrm{St}(\lambda)$ with λ unramified, which implies that $\pi(g)_v = \mathrm{St}(\mu)$ with $\mu = \lambda\chi_v$ and

$$\varepsilon_v\big(\pi(g)_v,\tfrac{1}{2}\big) = \chi_v(-1)\cdot\begin{cases}-1, & \text{if } \mu=1\\ 1, & \text{if } \mu\neq 1\end{cases}$$

(by Lemma 12.3.13 (vi), as $\mu(-1)=\chi_v(-1)$). Combining the last two formulas, we obtain

$$\prod_{\substack{v\mid p\\ \pi(g)_v=\mathrm{St}}}(-1) = \prod_{v\mid p}\chi_v(-1)\,\varepsilon_v\big(\pi(g)_v,\tfrac{1}{2}\big).$$

The second statement follows from the product formulas

$$\prod_v \chi_v(-1) = 1,\qquad \prod_v \varepsilon_v\big(\pi(g)_v,\tfrac{1}{2}\big) = (-1)^{r_{\rm an}(F,g)}$$

and Lemma 12.3.13 (vii). □

12.5.9.5. Proposition. — *Let K (resp., $L_\mathfrak{p}$) be a finite extension of $\mathbf{Q}$ (resp., of $\mathbf{Q}_p$), $S \supset \{v \mid p\infty\}$ a finite set of primes of K and W a finite-dimensional vector space over $L_\mathfrak{p}$ equipped with a continuous $L_\mathfrak{p}$-linear action of $G_{K,S}$. Assume that, for each prime $v \mid p$ of K, the representation W_v (= W considered as a representation of G_v) is de Rham. Denote by $(-)^* = \mathrm{Hom}_{L_\mathfrak{p}}(-,L_\mathfrak{p})$ the $L_\mathfrak{p}$-dual.*

(i) *For each prime v of K, denote*

$$h^i_v(W) := \dim_{L_\mathfrak{p}} H^i_{\rm cont}(G_v,W),\quad h^1_{v,f}(W) := \dim_{L_\mathfrak{p}} H^1_f(K_v,W)\quad (v\nmid\infty).$$

Then

$$h^0_v(W) - h^1_{v,f}(W) = \begin{cases}0, & v\nmid p\infty\\ -\dim_{L_\mathfrak{p}}(D_{\mathrm{dR},v}(W_v)/F^0), & v\mid p\end{cases}$$

(ii) *There is an exact sequence*

$$0 \longrightarrow H^0(G_{K,S},W) \longrightarrow \bigoplus_{v\in S_f} H^0(G_v,W) \longrightarrow H^2_{\mathrm{cont}}(G_{K,S},W^*(1))^* \longrightarrow H^1_f(K,W)$$
$$\longrightarrow \bigoplus_{v\in S_f} H^1_f(K_v,W) \longrightarrow H^1_{\mathrm{cont}}(G_{K,S},W^*(1))^* \longrightarrow H^1_f(K,W^*(1))^* \longrightarrow 0.$$

(iii) *Denoting* $h^i(W) := \dim_{L_\mathfrak{p}} H^i_{\mathrm{cont}}(G_{K,S},W)$ *and* $h^1_f(W) := \dim_{L_\mathfrak{p}} H^1_f(K,W)$, *then the Euler-Poincaré characteristic* $\chi_f(W) := h^0(W) - h^1_f(W) + h^1_f(W^*(1)) - h^0(W^*(1))$ *is equal to*

$$\chi_f(W) = \sum_{v\mid\infty} h^0_v(W) - \sum_{v\mid p} \dim_{L_\mathfrak{p}}(D_{\mathrm{dR},v}(W_v)/F^0).$$

(iv) *If* $W = V\otimes\alpha$, *where* $\alpha : G_{K,S} \to L_\mathfrak{p}^*$ *is a character of finite order and* $V \overset{\sim}{\to} V^*(1)$ *is self-dual, then* $\chi_f(W) = 0$. *In particular, if* $H^0(G_{K,S},V\otimes\alpha) = H^0(G_{K,S},V\otimes\alpha^{-1}) = 0$, *then* $h^1_f(V\otimes\alpha) = h^1_f(V\otimes\alpha^{-1})$.

Proof

(i) For $v \nmid p\infty$ (resp., for $v \mid p$) the result follows from the exact sequence

$$0 \longrightarrow H^0(G_v,W) \longrightarrow W^{I_v} \xrightarrow{\mathrm{Fr}(v)-1} W^{I_v} \longrightarrow H^1_f(K_v,W) \longrightarrow 0$$

(resp., from the bottom exact sequence in [**B-K**, §3.8.4]).

(ii) See [**Fo-PR**, §II.2.2.1].

(iii) The duality Theorem 5.4.5 implies that $h^i(W^*(1)) = h^{3-i}_c(W) := \dim_{L_\mathfrak{p}} H^{3-i}_{c,\mathrm{cont}}(G_{K,S},W)$. Applying the Euler characteristic formula 5.3.6, we deduce that

$$\chi(W^*(1)) := h^0(W^*(1)) - h^1(W^*(1)) + h^2(W^*(1)) = h^1_c(W) - h^2_c(W) + h^3_c(W)$$
$$= -\sum_{v\mid\infty} h^0_v(W).$$

On the other hand, it follows from the exact sequence (ii) that

$$\chi_f(W) + \chi(W^*(1)) = \sum_{v\in S_f} (h^0_v(W) - h^1_{v,f}(W)) \overset{\text{(i)}}{=} -\sum_{v\mid p} \dim_{L_\mathfrak{p}}(D_{\mathrm{dR},v}(W_v)/F^0).$$

Subtracting the two formulas yields the result.

(iv) In this case $W^*(1) = V\otimes\alpha^{-1}$ and

$$(\forall v \mid \infty) \quad h^0_v(V\otimes\alpha) = [K_v : \mathbf{R}]\dim(V)/2.$$

If $v \mid p$, then

$$2\dim_{L_\mathfrak{p}} F^0 D_{\mathrm{dR},v}(V_v) = \dim_{L_\mathfrak{p}} F^0 D_{\mathrm{dR},v}(V_v) + \dim_{L_\mathfrak{p}} F^0 D_{\mathrm{dR},v}(V^*(1)_v)$$
$$= \dim_{L_\mathfrak{p}} D_{\mathrm{dR},v}(V_v) = [K_v : \mathbf{Q}_p]\dim(V)$$
$$\dim_{L_\mathfrak{p}}(D_{\mathrm{dR},v}(W_v)/F^0) = \dim_{L_\mathfrak{p}}(D_{\mathrm{dR},v}(V_v)/F^0) = [K_v : \mathbf{Q}_p]\dim(V)/2,$$

hence

$$\chi_f(W) \overset{\text{(iii)}}{=} \left(\sum_{v\mid\infty}[K_v:\mathbf{R}] - \sum_{v\mid p}[K_v:\mathbf{Q}_p]\right)\dim(V)/2 = 0. \quad \square$$

12.5.10. Proposition. — *Let $g \in S_k(\mathfrak{n},\varphi)$ ($k \geqslant 2$) be a newform. The following conditions are equivalent:*

(i) *There exists a character of finite order $\chi : \mathbf{A}_F^*/F^* \to \mathbf{C}^*$ such that $g\otimes\chi^{-1}$ is p-ordinary.*

(ii) *There exists a finite totally real solvable extension F'/F such that the form $g' := BC_{F'/F}(g)$ is p-ordinary.*

Proof

(i) $\Longrightarrow$ (ii) According to [**A-T**, ch. 10, Thm. 5], there exists a character of finite order $\chi' : \mathbf{A}_F^*/F^* \to \mathbf{C}^*$ such that

$$(\forall v \mid p) \quad \chi'_v = \chi_v, \qquad (\forall v \mid \infty) \quad \chi'_v = 1.$$

For such χ', the form $g\otimes\chi'^{-1}$ is p-ordinary, $F' := \overline{F}^{\mathrm{Ker}(\chi')}$ is a totally real cyclic extension of F and the base change form $BC_{F'/F}(g) = BC_{F'/F}(g\otimes\chi'^{-1})$ is p-ordinary.

(ii) $\Longrightarrow$ (i) Fix a number field L as in 12.4.1; let $u \nmid p\infty$ be a prime of L and $V(g) = V_u(g)$ the L_u-representation of G_F associated to g. Let $v \mid p$ be a prime of F and $v' \mid v$ a prime of F'. We claim that $\pi(g)_v$ is not supercuspidal: if it were, then there would exist intermediate fields $F_v \subset K_w \subset K'_{w'} \subset F'_{v'}$ with $\mathrm{Gal}(K_w/F_v) \xrightarrow{\sim} \{1\}, A_3$ or S_3, $\mathrm{Gal}(K'_{w'}/K_w) = \{1,\sigma\}$ and $V(g)_w \xrightarrow{\sim} \mathrm{Ind}_{G_{w'}}^{G_w}(\mu)$ monomial, where $\mu : (K'_{w'})^* \to \overline{\mathbf{Q}}^*$,

$$\pi(g')_{v'} = \pi(\mu_1,\mu_2), \quad \mu_1 = \mu\circ N_{F'_{v'}/K'_{w'}}, \quad \mu_2 = {}^{\sigma}\mu\circ N_{F'_{v'}/K'_{w'}} \qquad ({}^{\sigma}\mu(h) = \mu(\sigma^{-1}h\sigma)).$$

There exists an integer $n \geqslant 1$ such that μ^n is unramified. If π is a uniformizer of $K'_{w'}$, then

$$({}^{\sigma}\mu)^n(\pi) = \mu^n(\sigma^{-1}(\pi)) = \mu^n(\sigma)\,\mu^n(\sigma^{-1}(\pi)/\pi) = \mu^n(\pi),$$

which implies that $({}^{\sigma}\mu/\mu)^n = 1$, hence $(\mu_2/\mu_1)^n = 1$. In particular, the numbers $\iota_p(\mu_1^n(v'))$ and $\iota_p(\mu_2^n(v'))$ have the same p-adic valuations, which contradicts the p-ordinarity of g'.

This contradiction shows that $\pi(g)_v$ is not supercuspidal. As g' is p-ordinary, we must have

$$\pi(g)_v = \begin{cases} \pi(\mu_1^{(v)},\mu_2^{(v)}), & \left|\iota_p\big((\mu_1^{(v)}\circ N_{F'_{v'}/F_v})(v')(Nv')^{(k-1)/2}\big)\right|_p = 1 \\ \mathrm{St}(\mu_1^{(v)}), & \mu_1^{(v)} \text{ of finite order.} \end{cases}$$

Applying again [**A-T**, ch. 10, Thm. 5], there exists a character of finite order $\chi : \mathbf{A}_F^*/F^* \to \mathbf{C}^*$ such that $\mu_1^{(v)}\chi_v^{-1}$ is unramified, for all primes $v \mid p$; the form $f\otimes\chi^{-1}$ is then p-ordinary. $\square$

12.6. L-functions over ring class fields

Throughout this section F is a totally real number field, K/F a totally imaginary quadratic extension, $\eta = \eta_{K/F} : \mathbf{A}_F^*/F^* \longrightarrow \{\pm 1\}$ the quadratic character associated to K/F and τ the non-trivial element of $\mathrm{Gal}(K/F)$. We denote $N = N_{K/F}$ and $N_w = N_{K_w/F_v}$, $\mathrm{Tr}_w = \mathrm{Tr}_{K_w/F_v}$ (if w is a prime of K above a prime v of F).

12.6.1. Ring class fields and characters

12.6.1.1. If $\beta : \mathbf{A}_K^*/K^* \longrightarrow \mathbf{C}^*$ is a (continuous) character of finite order, then there is a unique automorphic representation $\theta(\beta)$ of $\mathrm{GL}_2(\mathbf{A}_F)$ satisfying

$$L_v(\theta(\beta) \otimes \mu, s) = \prod_{w|v} L_w(\beta_w \cdot (\mu \circ N_w), s)$$

for all primes v of F and all characters $\mu : F_v^* \longrightarrow \mathbf{C}^*$; it corresponds to the two-dimensional representation $\mathrm{Ind}_{G_F}^{G_K}(\beta)$ of G_K. The representation $\theta(\beta)$ has central character $\eta \cdot (\beta|_{\mathbf{A}_F^*})$; it is cuspidal iff β does not factor as

$$\mathbf{A}_K^*/K^* \xrightarrow{N} \mathbf{A}_F^*/F^* \xrightarrow{\chi} \mathbf{C}^*.$$

If $\mu : \mathbf{A}_F^*/F^* \longrightarrow \mathbf{C}^*$ is a character of finite order, then $\theta(\beta) \otimes \mu = \theta(\beta \cdot (\mu \circ N))$. In the classical case $F = \mathbf{Q}$, $\theta(\beta)$ corresponds to the theta series of weight 1

$$\sum \beta(\mathfrak{a})\, q^{N\mathfrak{a}} \qquad (-L(0,\chi)/2 \text{ if } \beta = \chi \circ N).$$

More generally, one can define $\theta(\beta)$ for algebraic Hecke characters of K; see 12.6.5 below.

12.6.1.2. The local component of $\theta(\beta)$ at a prime v of F has the following properties ([**J-L**, Thm. 4.6]):

12.6.1.2.1. If $v|\infty$, then $\theta(\beta)_v$ is the limit discrete series representation.

12.6.1.2.2. If $v \nmid \infty$ splits in K/F into $v\mathcal{O}_K = ww'$, then

$$\theta(\beta)_v = \pi(\beta_w, \beta_{w'}),$$

where β_w (resp., $\beta_{w'}$) is considered as a character of $F_v^* = K_w^*$ (resp., of $F_v^* = K_{w'}^*$).

12.6.1.2.3. If $v \nmid \infty$ is inert or ramified in K/F ($v\mathcal{O}_K = w$ or w^2), then

$$\theta(\beta)_v = \begin{cases} \pi(\mu, \mu\eta_v), & \text{if } \beta_w = \mu \circ N_w \\ \text{supercuspidal}, & \text{if } \beta_w \neq \mu \circ N_w \end{cases}$$

and

$$\varepsilon_v(\theta(\beta)_v, s, \psi_v) = \varepsilon_v(\eta_v \cdot \beta_w|_{F_v^*}, s, \psi_v)\, \varepsilon_w(\beta_w, s, \psi_v \circ \mathrm{Tr}_w).$$

12.6.1.3. In the situation of 12.6.1.2.3 there is a purely local version of $\theta(\beta)$: one can associate to every continuous character $\beta : K_w^* \to \mathbf{C}^*$ an irreducible admissible representation $\theta_v(\beta)$ of $\mathrm{GL}_2(F_v)$ satisfying

$$L_v(\theta_v(\beta)\otimes\mu, s) = L_w(\beta\cdot(\mu\circ N_w), s) \qquad (\forall \mu : F_v^* \longrightarrow \mathbf{C}^*)$$

and 12.6.1.2.3.

12.6.1.4. For a non-zero ideal $c \subset \mathcal{O}_F$ put $\mathcal{O}_c = \mathcal{O}_F + c\mathcal{O}_K \subset \mathcal{O}_K$ and $\widehat{\mathcal{O}}_c = \mathcal{O}_c \otimes \widehat{\mathbf{Z}} = \widehat{\mathcal{O}}_F + c\widehat{\mathcal{O}}_K \subset \widehat{\mathcal{O}}_K = O_K \otimes \widehat{\mathbf{Z}} \subset \widehat{K} = K \otimes \widehat{\mathbf{Z}}$. The abelian extension $K[c]/K$ corresponding to the subgroup $K^*\cdot \mathbf{A}_F^* \cdot (K\otimes\mathbf{R})^* \cdot \widehat{\mathcal{O}}_c^* \subset \mathbf{A}_K^*$ is called the **ring class field** of K of conductor c. The extension $K[c]/F$ is Galois; more precisely,

$$\mathrm{Gal}(K[c]/F) \xrightarrow{\sim} \mathrm{Gal}(K[c]/K) \rtimes \{1,\tau\}$$

is a generalized dihedral group, since each lift of τ to $\mathrm{Gal}(K[c]/F)$ acts by conjugation on

$$\mathrm{Gal}(K[c]/K) \xrightarrow{\sim} \mathbf{A}_K^*/K^*\mathbf{A}_F^*(K\otimes\mathbf{R})^*\widehat{\mathcal{O}}_c^* \xrightarrow{\sim} \widehat{K}^*/K^*\widehat{F}^*\widehat{\mathcal{O}}_c^*$$

as $g \mapsto g^{-1}$.

12.6.1.5. We say that β from 12.6.1.1 is a **ring class character** if $\beta|_{\mathbf{A}_F^*} = 1$ ($\Longrightarrow \beta\circ\tau = \beta^{-1}$). We shall denote by $c(\beta)$ the smallest non-zero ideal $c \subset \mathcal{O}_F$ (with respect to divisibility) such that $\beta|_{\widehat{\mathcal{O}}_c^*} = 1$. In this case the level of $\theta(\beta)$ divides $d_{K/F}\, c(\beta)^2$ and its central character is equal to η, which implies that

$$\widetilde{\theta(\beta)} = \theta(\beta)\otimes\eta^{-1} = \theta(\beta\cdot(\eta\circ N)^{-1}) = \theta(\beta).$$

12.6.1.6. Proposition. — *Let $\beta : \mathbf{A}_K^*/K^*\mathbf{A}_F^* \to \mathbf{C}^*$ be a ring class character of finite order, $v \nmid \infty$ a prime of F which does not split in K/F, w the unique prime of K above v and $\mu : F_v^* \to \{\pm 1\}$ a character.*

(i) *If v is inert in K/F and $v \nmid c(\beta)$, then $\beta_w = 1$. In particular,*

$$\beta_w = \mu\circ N_w \iff \mu = 1 \text{ or } \mu = \eta_v \iff \mu \text{ is unramified.}$$

(ii) *If $v \mid c(\beta)$, then β_w is ramified. In particular,*

$$\beta_w = \mu\circ N_w \Longrightarrow \mu \text{ is ramified.}$$

(iii) *If v is ramified in K/F and $v \nmid c(\beta)$, then $\beta_w^2 = 1$ and there is a unique unramified character $\lambda : F_v^* \to \{\pm 1\}$ such that $\beta_w = \lambda\circ N_w$.*

Proof. — Easy exercise. □

12.6.2. Rankin-Selberg L-functions

12.6.2.1. Let $f \in S_k(\mathfrak{n},\varphi) - \{0\}$ ($k \geqslant 1$) be a newform of level $\mathfrak{n}$ and $\pi = \pi(f)$ the corresponding automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$. One can associate to π its **base change to** K, which is an automorphic representation π_K of $\mathrm{GL}_2(\mathbf{A}_K)$ ([**Jac**, Thm. 20.6]). If $\beta : \mathbf{A}_K^*/K^* \to \mathbf{C}^*$ is a character of finite order, then the local L-factors and ε-factors of $\pi_K \otimes \beta$ coincide with the corresponding factors of the Rankin-Selberg product $\pi \times \theta(\beta)$:

$$L_v(\pi \times \theta(\beta), s) = \prod_{w|v} L_w(\pi_K \otimes \beta, s)$$
$$\varepsilon_v(\pi \times \theta(\beta), s, \psi_v) = \prod_{w|v} \varepsilon_w(\pi_K \otimes \beta, s, \psi_v \circ \mathrm{Tr}_w).$$

The functional equation

$$L(\pi \times \theta(\beta), s) = \varepsilon(\pi \times \theta(\beta), s)\, L(\widetilde{\pi} \times \widetilde{\theta(\beta)}, 1-s)$$
$$\varepsilon(\pi \times \theta(\beta), s) = \prod_{v} \varepsilon_v(\pi \times \theta(\beta), s, \psi_v)$$

yields a functional equation for the L-function

$$L(f_K, \beta, s) := \prod_{v\nmid\infty} L_v\Big(\pi_K \otimes \beta, s - \frac{k-1}{2}\Big) = \prod_{v\nmid\infty} L_v\Big(\pi \times \theta(\beta), s - \frac{k-1}{2}\Big),$$

in particular for

$$L(f_K, s) = \prod_{v\nmid\infty} L_v\Big(\pi_K, s - \frac{k-1}{2}\Big) = \prod_{v\nmid\infty} L_v\Big(\pi, s - \frac{k-1}{2}\Big) \prod_{v\nmid\infty} L_v\Big(\pi \otimes \eta, s - \frac{k-1}{2}\Big).$$

If $V(f)$ is the $\mathfrak{p}$-adic representation of G_F associated to f, then we have, for each prime $w \nmid p\infty$ of K,

$$L_w(\pi_K \otimes \beta, s) = L_w\Big(\mathrm{Res}_{G_F}^{G_K}(V(f)) \otimes \beta, s + \frac{k-1}{2}\Big).$$

12.6.2.2. We shall use the following explicit formulas ([**Jac**, Thm. 15.1]):

12.6.2.2.1. If $v \mid \infty$, then

$$L_v(\pi \times \theta(\beta), s) = \Gamma_{\mathbf{C}}\big(s + \tfrac{k-1}{2}\big), \quad \varepsilon_v\big(\pi \times \theta(\beta), \tfrac{1}{2}, \psi_v\big) = 1.$$

12.6.2.2.2. If $v \nmid \infty$ and $\pi_v = \pi(\mu, \mu')$, then

$$L_v(\pi \times \pi', s) = L_v(\pi' \otimes \mu, s)\, L_v(\pi' \otimes \mu', s)$$
$$\varepsilon_v(\pi \times \pi', s, \psi_v) = \varepsilon_v(\pi' \otimes \mu, s, \psi_v)\, \varepsilon_v(\pi' \otimes \mu', s, \psi_v).$$

12.6.2.2.3. If $v \nmid \infty$ and $\pi_v = \mathrm{St}(\mu)$, then

$$L_v(\pi \times \pi', s) = L_v\big(\pi' \otimes \mu, s + \tfrac{1}{2}\big)$$
$$\varepsilon_v(\pi \times \pi', s, \psi_v) = \varepsilon_v\big(\pi' \otimes \mu, s + \tfrac{1}{2}, \psi_v\big)\, \varepsilon_v\big(\pi' \otimes \mu, s - \tfrac{1}{2}, \psi_v\big)\, \frac{L_v\big(\frac{1}{2} - s, \widetilde{\pi}' \otimes \mu^{-1}\big)}{L_v\big(s - \frac{1}{2}, \pi' \otimes \mu\big)}$$

12.6.2.3. Let $g \in S_k(\mathfrak{n}, 1)$ be a newform of level $\mathfrak{n}$ with trivial central character ($\Longrightarrow k \in 2\mathbf{Z}$), $\pi = \pi(g)$ the corresponding automorphic representation and $\beta : \mathbf{A}_K^*/K^*\mathbf{A}_F^* \to \mathbf{C}^*$ a ring class character of finite order. We denote

$$r_{\rm an}(K, g, \beta) := \mathrm{ord}_{s=k/2} L(g_K, \beta, s) = \mathrm{ord}_{s=1/2} L(\pi \times \theta(\beta), s)$$

$$r_{\rm an}(K, g) := r_{\rm an}(K, g, 1) = \mathrm{ord}_{s=k/2} L(g, s)L(g \otimes \eta, s) = \mathrm{ord}_{s=1/2} L(\pi, s)L(\pi \otimes \eta, s).$$

The functional equation

$$L(\pi \times \theta(\beta), s) = \varepsilon(\pi \times \theta(\beta), s)\, L(\pi \times \theta(\beta), 1 - s)$$

implies (as in 12.3.4, we use the fact that the archimedean L-factors $\prod_{v|\infty} L_v(\pi \times \theta(\beta), s)$ have neither zero nor pole at $s = 1/2$) that

$$(12.6.2.1) \qquad (-1)^{r_{\rm an}(K,g,\beta)} = \varepsilon(\beta) = \prod_v \varepsilon(\beta)_v, \quad (-1)^{r_{\rm an}(K,g)} = \varepsilon = \prod_v \varepsilon_v,$$

where we have denoted, for each prime v of F,

$$\varepsilon(\beta)_v := \varepsilon_v\big(\pi \times \theta(\beta), \tfrac{1}{2}, \psi_v\big), \quad \varepsilon_v := \varepsilon(1)_v$$

(for a fixed additive character $\psi : \mathbf{A}_F/F \to \mathbf{C}^*$).

12.6.2.4. Proposition. — *In the notation of 12.6.2.3, let $v \nmid \infty$ be a prime of F.*

(i) *If $\pi_v = \pi(\mu, \mu^{-1})$, then $\varepsilon(\beta)_v = \eta_v(-1)$.*

(ii) *If $\pi_v = \mathrm{St}(\mu)$ ($\mu^2 = 1$), then*

$$\varepsilon(\beta)_v = \eta_v(-1) \cdot \begin{cases} -1, & \text{if } v \text{ is not split in } K/F \text{ and } \beta_w = \mu \circ N_w \\ 1, & \text{otherwise.} \end{cases}$$

(iii) *If π_v is supercuspidal, assume that $v \nmid d_{K/F}$ and, if v is inert in K/F, then $v \nmid c(\beta)$. Then*

$$\varepsilon(\beta)_v = \eta_v(v)^{o(\pi_v)} = \eta_v(v)^{\mathrm{ord}_v(\mathfrak{n})}.$$

Proof

(i) It follows from 12.6.2.2.2 that

$$\varepsilon(\beta)_v = \varepsilon_v\big(\theta(\beta)_v \otimes \mu, \tfrac{1}{2}, \psi_v\big)\, \varepsilon_v\big(\theta(\beta)_v \otimes \mu^{-1}, \tfrac{1}{2}, \psi_v\big).$$

If $v\mathcal{O}_K = ww'$ ($w \neq w'$), then $\theta(\beta)_v = \pi(\beta_w, \beta_{w'}) = \pi(\beta_w, \beta_w^{-1})$, hence

$$\begin{aligned}\varepsilon(\beta)_v &= \varepsilon_v\big(\mu\beta_w, \tfrac{1}{2}\big)\, \varepsilon_v\big(\mu^{-1}\beta_w^{-1}, \tfrac{1}{2}\big)\, \varepsilon_v\big(\mu\beta_w^{-1}, \tfrac{1}{2}\big)\, \varepsilon_v\big(\mu^{-1}\beta_w, \tfrac{1}{2}\big) \\ &= (\mu\beta_w)(-1)\, (\mu\beta_w^{-1})(-1) = 1 = \eta_v(-1).\end{aligned}$$

If v does not split in K/F and w is the unique prime of K above v, then we obtain from 12.6.1.2.3

$$\begin{aligned}\varepsilon_v\big(\theta(\beta)_v \otimes \mu, \tfrac{1}{2}, \psi_v\big) &= \varepsilon_v\big(\theta_v(\beta_w \cdot (\mu \circ N_w)), \tfrac{1}{2}, \psi_v\big) \\ &= \varepsilon_v\big(\eta_v\mu^2 \cdot \beta_w|_{F_v^*}, \tfrac{1}{2}, \psi_v\big)\, \varepsilon_w\big(\beta_w \cdot (\mu \circ N_w), \tfrac{1}{2}, \psi_v \circ \mathrm{Tr}_w\big) \\ \varepsilon_v\big(\theta(\beta)_v \otimes \mu^{-1}, \tfrac{1}{2}, \psi_v\big) &= \varepsilon_v\big(\eta_v\mu^{-2} \cdot \beta_w|_{F_v^*}, \tfrac{1}{2}, \psi_v\big)\, \varepsilon_w\big(\beta_w \cdot (\mu \circ N_w)^{-1}, \tfrac{1}{2}, \psi_v \circ \mathrm{Tr}_w\big).\end{aligned}$$

As $\beta_w|_{F_v^*} = 1$ and

$$\begin{aligned}\varepsilon_w\big(\beta_w\cdot(\mu\circ N_w)^{-1},\tfrac12,\psi_v\circ\mathrm{Tr}_w\big) &= \varepsilon_w\big((\beta_w\cdot(\mu\circ N_w)^{-1})\circ\tau,\tfrac12,\psi_v\circ\mathrm{Tr}_w\big)\\ &= \varepsilon_w\big(\beta_w^{-1}\cdot(\mu\circ N_w)^{-1},\tfrac12,\psi_v\circ\mathrm{Tr}_w\big),\end{aligned}$$

it follows that

$$\begin{aligned}\varepsilon(\beta)_v &= \varepsilon_v\big(\eta_v\mu^2,\tfrac12\big)\varepsilon_v\big(\eta_v\mu^{-2},\tfrac12\big)\varepsilon_w\big(\beta_w\cdot(\mu\circ N_w),\tfrac12\big)\varepsilon_w\big(\beta_w^{-1}\cdot(\mu\circ N_w)^{-1},\tfrac12\big)\\ &= (\eta_v\mu^2\beta_w\cdot(\mu\circ N_w))(-1) = \eta_v(-1).\end{aligned}$$

(ii) It follows from 12.6.2.2.3 that $\varepsilon(\beta)_v = A\cdot B$, where

$$\begin{aligned}A &= \varepsilon_v\big(\theta(\beta)_v\otimes\mu|\cdot|^{1/2},\tfrac12,\psi_v\big)\,\varepsilon_v\big(\theta(\beta)_v\otimes\mu|\cdot|^{-1/2},\tfrac12,\psi_v\big)\\ B &= \lim_{s\to1/2}\frac{L_v\big(\theta(\beta)_v\otimes\mu,\tfrac12-s\big)}{L_v\big(\theta(\beta)_v\otimes\mu,s-\tfrac12\big)}\end{aligned}$$

The same calculation as in (i) shows that $A = \eta_v(-1)$.

In order to compute B, we use the fact that, for any character $\chi : F_v^*\to\mathbf{C}^*$,

$$B(\chi) := \lim_{s\to0}\frac{L_v(\chi,s)}{L_v(\chi,-s)} = \begin{cases}-1, & \chi = 1\\ 1, & \chi\neq 1.\end{cases}$$

If v splits in K/F, then 12.6.1.2.2 implies that

$$B = B(\beta_w)\,B(\beta_{w'}) = B(\beta_w)\,B(\beta_w^{-1}) = 1.$$

If v does not split in K/F, let w be the unique prime of K above v. If $\beta_w\neq\nu\circ N_w$, then $\theta(\beta)_v$ is supercuspidal, hence $B = 1$. If $\beta_w = \nu\circ N_w$ ($\nu : F_v^*\to\mathbf{C}^*$), then 12.6.1.2.3 implies that

$$B = B(\mu\nu)\,B(\mu\nu\eta_v) = \begin{cases}-1, & \nu = \mu,\mu\eta_v\\ 1, & \nu\neq\mu,\mu\eta_v.\end{cases}$$

As η_v induces an isomorphism $\eta_v : F_v^*/N_w(K_w^*)\xrightarrow{\sim}\{\pm1\}$, we have

$$\nu = \mu,\mu\eta_v \iff \eta\circ N_w = \mu\circ N_w.$$

Putting everything together, we obtain the formula (ii).

(iii) By assumption, v is unramified in K/F. If $v\mathcal{O}_K = ww'$, then $\theta(\beta)_v = \pi(\beta_w,\beta_w^{-1})$, hence

$$\varepsilon(\beta)_v = \varepsilon_v\big(\pi_v\otimes\beta_w,\tfrac12,\psi_v\big)\,\varepsilon_v\big(\pi_v\otimes\beta_w^{-1},\tfrac12,\psi_v\big) = 1 = \eta_v(v),$$

by Lemma 12.3.13 (ii).

If v is inert in K/F and w is the unique prime of K above v, then $\beta_w : K_w^*\to\mathbf{C}^*$ is unramified and trivial on F_v^*; it follows that $\beta_w = 1$, hence $\theta(\beta)_v = \pi(1,\eta_v)$ and

$$\varepsilon(\beta)_v = \varepsilon_v\big(\pi_v,\tfrac12,\psi_v\big)\,\varepsilon_v\big(\pi_v\otimes\eta_v,\tfrac12,\psi_v\big).$$

Let $n(\psi_v)$ be the minimal integer $n \in \mathbf{Z}$ such that $\psi_v(v^{-n}\mathcal{O}_{F,v}) = 1$. As η_v is unramified, we have

$$\frac{\varepsilon_v\big(\pi_v \otimes \eta_v, \frac{1}{2}, \psi_v\big)}{\varepsilon_v\big(\pi_v, \frac{1}{2}, \psi_v\big)} = \eta_v(v)^{o(\pi_v)+2n(\psi_v)} = \eta_v(v)^{o(\pi_v)} = (-1)^{o(\pi_v)},$$

which implies that

$$\varepsilon(\beta)_v = \varepsilon_v\big(\pi_v, \tfrac{1}{2}, \psi_v\big)^2\, \eta_v(v)^{o(\pi_v)} = \eta_v(v)^{o(\pi_v)}. \quad \square$$

12.6.3. Global ε-factors for ring class characters. — Let $g \in S_k(\mathfrak{n}, 1)$, $\pi = \pi(g)$ and $\beta : \mathbf{A}_K^*/K^*\mathbf{A}_F^* \to \mathbf{C}^*$ be as in 12.6.2.3.

12.6.3.1. Define the "Steinberg part" of the level of g by

$$\mathfrak{n}_{\mathrm{St}} = \mathfrak{n}^{(\mathfrak{a})}, \text{ where } \mathfrak{a} = \prod_{\pi_v \neq \mathrm{St}(\mu)} v \quad (\mathrm{ord}_v(\mathfrak{n}_{\mathrm{St}}) > 0 \iff \pi_v = \mathrm{St}(\mu)).$$

12.6.3.2. Definition. — We define

$$R(\beta)^0 := \{v \mid \mathfrak{n}_{\mathrm{St}} : \pi_v = \mathrm{St}(\mu),\, \beta_w = \mu \circ N_w,\, v \text{ ramified in } K/F\}$$
$$R(\beta)^- := \{v \mid \mathfrak{n}_{\mathrm{St}} : \pi_v = \mathrm{St}(\mu),\, \beta_w = \mu \circ N_w,\, v \text{ inert in } K/F\}$$

(above, w is the unique prime of K above v). Note that

$$R(1)^0 := \{v \mid \mathfrak{n}_{\mathrm{St}} : \pi_v = \mathrm{St}, \text{ or } \mathrm{St}(\eta_v),\, v \text{ ramified in } K/F\}$$
$$R(1)^- := \{v \mid \mathfrak{n}_{\mathrm{St}} : \pi_v = \mathrm{St}, \text{ or } \mathrm{St}(\eta_v),\, v \text{ inert in } K/F\}$$

12.6.3.3. Proposition

(i) $R(\beta)^- \cap \{v \nmid c(\beta)\} = R(1)^- \cap \{v \nmid c(\beta)\}$.
(ii) $R(\beta)^0 \cap \{v \mid c(\beta)\} \cap R(1)^0 = R(\beta)^- \cap \{v \mid c(\beta)\} \cap R(1)^- = \varnothing$.

Proof. — This follows from Proposition 12.6.1.6 (i)–(ii). □

12.6.3.4. Proposition. — *If the order of β is odd, then*

(i) $R(\beta)^0 \cap \{v \nmid c(\beta)\} = R(1)^0 \cap \{v \nmid c(\beta)\}$.
(ii) $(R(\beta)^0 \cup R(\beta)^-) \cap \{v \mid c(\beta)\} = \varnothing$.
(iii) $R(\beta)^0 = R(1)^0 \cap \{v \nmid c(\beta)\}$, $R(\beta)^- = R(1)^- \cap \{v \nmid c(\beta)\}$.

Proof. — The statement (i) (resp., (ii)) follows from Proposition 12.6.1.6 (iii) (resp., (ii)), while (iii) is a consequence of (i) and (ii). □

12.6.3.5. Consider the following conditions:

$H(K)$ **:** $(\forall v \mid d_{K/F})\quad \pi_v$ is not supercuspidal.
$H(K,\beta)$ **:** $H(K) \wedge ((\forall v \mid c(\beta)$ inert in $K/F)\quad \pi_v$ is not supercuspidal).

12.6.3.6. Proposition. — *Assume that $H(K)$ holds. Then*

$$\varepsilon = (-1)^{[F:\mathbf{Q}]}\, \eta\big(\mathfrak{n}^{(d_{K/F})}\big)\, (-1)^{|R(1)^0|}.$$

Proof. — Proposition 12.6.2.4 (for $\beta = 1$) implies that

$$(-1)^{[F:\mathbf{Q}]}\,\varepsilon = \varepsilon \prod_{v\mid\infty} \eta_v(-1) = \prod_{v\nmid d_{K/F}\infty} \eta_v(v)^{o(\pi_v)} \prod_{v\in R(1)^0} (-1). \quad \square$$

12.6.3.7. Corollary. — *If* $(\forall v \mid d_{K/F})$ π_v *is in the principal series, then* $\varepsilon = (-1)^{[F:\mathbf{Q}]}\,\eta\big(\mathfrak{n}^{(d_{K/F})}\big)$.

In particular, if $(d_{K/F}, \mathfrak{n}) = (1)$, *then* $\varepsilon = (-1)^{[F:\mathbf{Q}]}\,\eta(\mathfrak{n})$.

12.6.3.8. Proposition. — *Assume that* $H(K,\beta)$ *holds. Then*

$$\varepsilon(\beta) = (-1)^{[F:\mathbf{Q}]}\,\eta\big(\mathfrak{n}^{(d_{K/F})}\big)\,(-1)^{|R(\beta)^0|}(-1)^{|R(\beta)^-\cap\{v\mid c(\beta)\}|}(-1)^{|R(1)^-\cap\{v\mid c(\beta)\}|}.$$

Proof. — Proposition 12.6.2.4 implies that

$$(12.6.3.1)\qquad (-1)^{|R(\beta)^0|+|R(\beta)^-|}\,\varepsilon(\beta) = (-1)^{|R(1)^0|+|R(1)^-|}\,\varepsilon.$$

Combining this formula with Proposition 12.6.3.3 (i) and 12.6.3.6 yields the result. □

12.6.3.9. Corollary

(i) *If* $H(K,\beta)$ *holds and* β *has odd order, then*

$$\varepsilon(\beta) = (-1)^{[F:\mathbf{Q}]}\,\eta\big(\mathfrak{n}^{(d_{K/F})}\big)\,(-1)^{|R(1)^0\cap\{v\nmid c(\beta)\}|}(-1)^{|R(1)^-\cap\{v\mid c(\beta)\}|}.$$

(ii) *If* $H(K,\beta)$ *holds and* $(d_{K/F}\,c(\beta), \mathfrak{n}_{\mathrm{St}}) = (1)$, *then* $\varepsilon(\beta) = \varepsilon = (-1)^{[F:\mathbf{Q}]}\eta\big(\mathfrak{n}^{(d_{K/F})}\big)$.

12.6.3.10. Define, for each prime v of F,

$$\widetilde{\varepsilon(\beta)}_v := \varepsilon(\beta)_v \cdot \begin{cases} -1, & \text{if } v \mid p, v \text{ is not split in } K/F,\ \pi_v = \mathrm{St}(\mu),\ \beta_w = \mu\circ N_w \\ 1, & \text{otherwise} \end{cases}$$

(where w is the unique prime of K above v) and set

$$\widetilde{\varepsilon}_v = \widetilde{\varepsilon(1)}_v, \quad \widetilde{\varepsilon(\beta)} = \prod_v \widetilde{\varepsilon(\beta)}_v, \quad \widetilde{\varepsilon} = \prod_v \widetilde{\varepsilon}_v.$$

It follows from the definitions that

$$(12.6.3.1)\qquad \widetilde{\varepsilon(\beta)}/\varepsilon(\beta) = (-1)^{|R(\beta)^0\cap\{v\mid p\}|}(-1)^{|R(\beta)^-\cap\{v\mid p\}|}.$$

Combining this formula with (12.6.3.1), we obtain, assuming $H(K,\beta)$, that

$$(12.6.3.2)\quad (-1)^{|R(\beta)^0\cap\{v\nmid p\}|}(-1)^{|R(\beta)^-\cap\{v\nmid p\}|}\,\widetilde{\varepsilon(\beta)} = (-1)^{|R(1)^0\cap\{v\nmid p\}|}(-1)^{|R(1)^-\cap\{v\nmid p\}|}\,\widetilde{\varepsilon}.$$

12.6.3.11. Consider the following conditions:

$H(K,p)$: $H(K) \ \wedge\ \big(\forall v \mid d^{(p)}_{K/F}\big)\ \pi_v$ is in the principal series.

$H(K,\beta,p)$: $H(K,\beta)\ \wedge\ H(K,p)$.

12.6.3.12. Proposition. — *Assume that $H(K)$ holds. Then*

$$\widetilde{\varepsilon} = (-1)^{[F:\mathbf{Q}]}\,\eta\big(\mathfrak{n}^{(d_{K/F})}\big)\,(-1)^{|R(1)^0\cap\{v\nmid p\}|}(-1)^{|R(1)^-\cap\{v|p\}|}.$$

Proof. — Combine (12.6.3.1) (for $\beta = 1$) with Proposition 12.6.3.6. □

12.6.3.13. Corollary. — *Assume that $H(K,p)$ holds. Then*

$$\widetilde{\varepsilon} = (-1)^{[F:\mathbf{Q}]}\,\eta\big(\mathfrak{n}^{(d_{K/F})}\big)\,(-1)^{|R(1)^-\cap\{v|p\}|}.$$

If, in addition, $(\forall v \mid p)$ π_v is not supercuspidal, then $\widetilde{\varepsilon} = (-1)^{[F:\mathbf{Q}]}\,\eta\big(\mathfrak{n}^{(pd_{K/F})}\big)$.

Proof. — The first formula follows from Proposition 12.6.3.12. If $v \mid p$ is unramified in K/F, then

$$\eta_v(v)^{o(\pi_v)} = \begin{cases} -1, & \text{if } \pi_v = \mathrm{St}\otimes\mu,\ \mu \text{ unramified},\ v \text{ inert in } K/F\\ 1, & \text{if } \pi_v = \mathrm{St}\otimes\mu,\ \mu \text{ ramified}\\ 1, & \text{if } v \text{ is split in } K/F\\ 1, & \text{if } \pi_v = \pi(\mu,\mu^{-1}),\end{cases}$$

which proves the second formula. □

12.6.3.14. Proposition. — *Assume that $H(K,\beta)$ holds. Then*

$$\widetilde{\varepsilon(\beta)} = (-1)^{[F:\mathbf{Q}]}\eta\big(\mathfrak{n}^{(d_{K/F})}\big)(-1)^{|R(1)^-\cap\{v|pc(\beta)\}|}(-1)^{|R(\beta)^0\cap\{v\nmid p\}|}(-1)^{|R(\beta)^-\cap\{v|c(\beta),\, v\nmid p\}|}.$$

Proof. — Combining Proposition 12.6.3.8 with (12.6.3.1), we obtain

$$\widetilde{\varepsilon(\beta)}\,(-1)^{[F:\mathbf{Q}]}\,\eta\big(\mathfrak{n}^{(d_{K/F})}\big)\,(-1)^{|R(\beta)^0\cap\{v\nmid p\}|} = (-1)^{|R(\beta)^-\cap\{v|p\}|}(-1)^{|R(\beta)^-\cap\{v|c(\beta)\}|}(-1)^{|R(1)^-\cap\{v|c(\beta)\}|}.$$

The R.H.S. can be simplified as follows: we have

$$|R(\beta)^-\cap\{v\mid p\}| + |R(\beta)^-\cap\{v\mid c(\beta)\}| = |R(\beta)^-\cap\{v\mid p,\ v\nmid c(\beta)\}| + |R(\beta)^-\cap\{v\mid c(\beta),\ v\nmid p\}|$$

and

$$|R(\beta)^-\cap\{v\mid p,\ v\nmid c(\beta)\}| = |R(1)^-\cap\{v\mid p,\ v\nmid c(\beta)\}|$$

(using Proposition 12.6.3.4 (i)), hence

$$|R(\beta)^-\cap\{v\mid p,\ v\nmid c(\beta)\}| + |R(1)^-\cap\{v\mid c(\beta)\}| = |R(1)^-\cap\{v\mid pc(\beta)\}|,$$

which proves the desired formula. □

12.6.3.15. Corollary

(i) *If $H(K,\beta)$ holds and β has odd order, then*

$$\widetilde{\varepsilon(\beta)} = (-1)^{[F:\mathbf{Q}]}\,\eta\big(\mathfrak{n}^{(d_{K/F})}\big)\,(-1)^{|R(1)^0\cap\{v\nmid pc(\beta)\}|}(-1)^{|R(1)^-\cap\{v|pc(\beta)\}|}.$$

(ii) *If $H(K,\beta,p)$ holds and β has odd order, then*

$$\widetilde{\varepsilon(\beta)} = (-1)^{[F:\mathbf{Q}]}\,\eta\big(\mathfrak{n}^{(d_{K/F})}\big)\,(-1)^{|R(1)^-\cap\{v|pc(\beta)\}|}.$$

(iii) *Assume that $H(K,\beta)$ holds, $(c(\beta),\mathfrak{n}_{\mathrm{St}}^{(p)})=(1)$ and either $H(K,p)$ holds or β has odd order. Then $\widetilde{\varepsilon(\beta)}=\widetilde{\varepsilon}$.*

12.6.4. Ring class characters and twists of p-ordinary forms

12.6.4.1. Let $f\in S_k(\mathfrak{n},\varphi)$, $g=f\otimes\chi\in S_k(\mathfrak{n}(g),1)$ and $V=V(g)(k/2)=V(f)(k/2)\otimes\chi$ be as in 12.5.5 (*i.e.*, f is p-ordinary and $\varphi=\chi^{-2}$). Fix a ring class character of finite order $\beta:\mathbf{A}_K^*/K^*\mathbf{A}_F^*\to\mathbf{C}^*$.

12.6.4.2. Using the embeddings ι_∞ and ι_p, we consider β as a character with values in $L_{\mathfrak{p}}^*$ (after enlarging L, if necessary). Denote the fixed field of β by H_β. According to 12.5.9.2 (iv) there are canonical isomorphisms

$$\widetilde{H}^1_f(K,V\otimes\beta)\xrightarrow{\sim}\widetilde{H}^1_f(H_\beta,V)^{(\beta^{-1})},\quad \widetilde{H}^1_f(K,V\otimes\beta^{-1})\xrightarrow{\sim}\widetilde{H}^1_f(H_\beta,V)^{(\beta)}.$$

The action of $\mathrm{Ad}_f(\tau)$ (for any lift τ to $\mathrm{Gal}(H_\beta/F)$) induces an isomorphism

$$\widetilde{H}^1_f(H_\beta,V)^{(\beta^{-1})}\xrightarrow{\sim}\widetilde{H}^1_f(H_\beta,V)^{(\beta)},$$

hence

$$\widetilde{h}^1_f(K,V\otimes\beta)=\widetilde{h}^1_f(K,V\otimes\beta^{-1}).$$

The same results hold for H^1_f (and h^1_f).

12.6.4.3. Proposition. — *Under the assumptions of 12.6.4.1 we have*

$$(-1)^{\widetilde{h}^1_f(K,V\otimes\beta)-h^1_f(K,V\otimes\beta)}=\widetilde{\varepsilon(\beta)}/\varepsilon(\beta),$$

$$(-1)^{\widetilde{h}^1_f(K,V\otimes\beta)-h^1_f(K,V\otimes\beta)+r_{\mathrm{an}}(K,g,\beta)}=\widetilde{\varepsilon(\beta)}.$$

Proof. — It follows from (12.6.2.1) that it is enough to prove the first formula. According to 12.5.9.3,

$$\begin{aligned}\widetilde{h}^1_f(K,V\otimes\beta)-h^1_f(K,V\otimes\beta)&=\sum_{v|p}\sum_{w|v}\dim_{L_{\mathfrak{p}}}H^0(K_w,V_v^-\otimes\beta_w)\\&=\sum_{\substack{v|p\\ \pi_v=\mathrm{St}(\mu)}}\sum_{w|v}\begin{cases}1, & \beta_w=\mu\circ N_w\\ 0, & \beta_w\neq\mu\circ N_w.\end{cases}\end{aligned}$$

The contribution from each prime $v\mid p$ that splits in K/F ($v\mathcal{O}_K=ww'$) is even, as $\beta_{w'}=\beta_w^{-1}$, hence

$$\beta_{w'}=\mu\circ N_{w'}\iff\beta_w=\mu\circ N_w.$$

The parity of the contributions from the remaining primes yields exactly the sign $\widetilde{\varepsilon(\beta)}/\varepsilon(\beta)$. □

12.6.4.4. We shall be interested in the validity of the following congruence:

$$(\star)\quad \widetilde{h}^1_f(K, V\otimes\beta) - h^1_f(K, V\otimes\beta) + r_{\mathrm{an}}(K, g, \beta) \overset{?}{\equiv} \widetilde{h}^1_f(K, V) - h^1_f(K, V) + r_{\mathrm{an}}(K, g) \pmod 2.$$

It follows from 12.6.4.3 that $(\star)$ holds $\iff \widetilde{\varepsilon(\beta)} = \widetilde{\varepsilon}$.

12.6.4.5. Proposition. — *The congruence $(\star)$ holds in any of the following cases:*

(i) *$H(K,\beta)$ holds, $(c(\beta), \mathfrak{n}(g)^{(p)}_{\mathrm{St}}) = (1)$ and either $H(K,p)$ holds or β has odd order.*

(ii) *$H(K)$ holds, $c(\beta)$ divides a power of p and either $H(K,p)$ holds or β has odd order.*

(iii) *$(\mathfrak{n}(g), d_{K/F}) = (1)$ and $c(\beta)$ divides a power of p.*

Proof

(i) Combine Proposition 12.6.4.3 and Corollary 12.6.3.15 (iii).

The statement (ii) is a special case of (i), as π_v is not supercuspidal for any $v \mid p$ (hence $H(K,\beta)$ follows from $H(K)$ in this case). Similarly, (iii) is a special case of (ii). □

12.6.4.6. Fix a non-zero ideal $c \subset \mathcal{O}_F$ prime to p and a non-empty set $P_1, \dots, P_s \mid p$ of prime ideals above p in F. Set

$$K[cP_1^\infty\cdots P_s^\infty] = \bigcup_{n\geqslant 1} K[cP_1^n\cdots P_s^n],\quad G(cP_1^\infty\cdots P_s^\infty) = \mathrm{Gal}(K[cP_1^\infty\cdots P_s^\infty]/K).$$

The torsion subgroup $G(cP_1^\infty\cdots P_s^\infty)_{\mathrm{tors}}$ of $G = G(cP_1^\infty\cdots P_s^\infty)$ is finite and the quotient group G/G_{tors} is isomorphic to $\mathbf{Z}_p^a$, where

$$a = \sum_{i=1}^{s} [F_{P_i} : \mathbf{Q}_p].$$

12.6.4.7. Proposition. — *Assume that $\beta, \beta' : G = G(cP_1^\infty\cdots P_s^\infty) \to \overline{L}_{\mathfrak{p}}^*$ are characters of finite order satisfying $\beta|_{G_{\mathrm{tors}}} = \beta'|_{G_{\mathrm{tors}}}$. Then:*

(i) *If $w \nmid p\infty$ is a prime of K, then $\beta_w|_{\mathcal{O}^*_{K,w}} = \beta'_w|_{\mathcal{O}^*_{K,w}}$.*

(ii) *The ideals $c(\beta)^{(p)} = c(\beta')^{(p)}$ are equal and they divide c.*

(iii) *$R(\beta)^0 \cap \{v \nmid p\} = R(\beta')^0 \cap \{v \nmid p\}$ and $R(\beta)^- \cap \{v \nmid p\} = R(\beta')^- \cap \{v \nmid p\}$.*

(iv) *If $H(K,\beta)$ holds, so does $H(K,\beta')$ and we have*

$$\widetilde{h}^1_f(K, V\otimes\beta) - h^1_f(K, V\otimes\beta) + r_{\mathrm{an}}(K, g, \beta) \equiv \widetilde{h}^1_f(K, V\otimes\beta') - h^1_f(K, V\otimes\beta') + r_{\mathrm{an}}(K, g, \beta') \pmod 2.$$

(v) *If $H(K,\beta)$ holds, then*

$$h^1_f(K, V\otimes\beta) - r_{\mathrm{an}}(K,g,\beta) \equiv h^1_f(K, V\otimes\beta') - r_{\mathrm{an}}(K,g,\beta') \pmod 2.$$

Proof

(i) As the pro-p-part of $\mathcal{O}^*_{K,w}$ is finite, the image of $\mathcal{O}^*_{K,w}/\mathcal{O}^*_{F,v}$ (where v is the prime of F induced by w) in G is contained in G_{tors}, which implies that $\beta_w = \beta'_w$. The statements (ii) and (iii) are straightforward consequences of (i).

(iv) As $c(\beta)^{(p)} = c(\beta')^{(p)}$ and π_v is not supercuspidal at any prime $v \mid p$, $H(K,\beta')$ follows from $H(K,\beta)$. Applying Proposition 12.6.3.14 to β and β', we obtain $\widetilde{\varepsilon(\beta)} = \widetilde{\varepsilon(\beta')}$, which is equivalent to the statement (iv), by Proposition 12.6.4.3.

(v) We use the fundamental parity result proved in Theorem 10.7.17. Fix an isomorphism $G \overset{\sim}{\to} G_{\mathrm{tors}}\times H$, $H = \mathbf{Z}_p^{[F:\mathbf{Q}]}$, and put $K_0 = K[cP_1^\infty\cdots P_s^\infty]^H$. The character β (resp., β') decomposes as a product $\beta = \alpha_0\times\alpha$ (resp., $\beta' = \alpha_0\times\alpha'$), where

$$\alpha_0 = \beta|_{G_{\mathrm{tors}}} = \beta'|_{G_{\mathrm{tors}}} : G_{\mathrm{tors}} \longrightarrow \overline{L}^*_{\mathfrak{p}}, \quad \alpha : H \longrightarrow \overline{L}^*_{\mathfrak{p}}, \quad \alpha' : H \longrightarrow \overline{L}^*_{\mathfrak{p}}.$$

The parity result 10.7.17 (iii) then yields

$$\widetilde{h}^1_f(K, V\otimes\beta) \equiv \widetilde{h}^1_f(K, V\otimes\alpha_0) \pmod 2$$
$$\widetilde{h}^1_f(K, V\otimes\beta') \equiv \widetilde{h}^1_f(K, V\otimes\alpha_0) \pmod 2,$$

hence

$$\widetilde{h}^1_f(K, V\otimes\beta) \equiv \widetilde{h}^1_f(K, V\otimes\beta') \pmod 2.$$

Combined with (iv), this proves (v). □

12.6.4.8. Corollary. — *Assume that $\big(\mathfrak{n}(g), c\,d_{K/F}^{(P_1\cdots P_s)}\big) = (1)$ and that $\beta,\beta' : G = G(cP_1^\infty\cdots P_s^\infty)\to \overline{L}^*_{\mathfrak{p}}$ are characters of finite order satisfying $\beta|_{G_{\mathrm{tors}}} = \beta'|_{G_{\mathrm{tors}}}$. Then*

$$h^1_f(K, V\otimes\beta) - r_{\mathrm{an}}(K,g,\beta) \equiv h^1_f(K, V\otimes\beta') - r_{\mathrm{an}}(K,g,\beta') \pmod 2.$$

Proof. — The assumption $\big(\mathfrak{n}(g), c\,d_{K/F}^{(P_1\cdots P_s)}\big) = (1)$ implies that $H(K,\beta)$ holds. We apply Proposition 12.6.4.7 (v). □

12.6.4.9. Proposition - Definition. — *Assume that $\big(\mathfrak{n}(g), c\,d_{K/F}^{(P_1\cdots P_s)}\big) = (1)$ and define*

$$\mathscr{M}(g,K;P_1,\dots,P_s) := \{v \mid p \text{ in } F,\ v\notin\{P_1,\dots,P_s\},\ v \text{ inert in } K/F,\ \mathrm{ord}_v(\mathfrak{n}(g)) = 1\}$$
$$m(g) = m(g,K;P_1,\dots,P_s) := |\mathscr{M}(g,K;P_1,\dots,P_s)|.$$

(i) *We have $\eta_{K/F}\big(\mathfrak{n}(g)^{(P_1\cdots P_s)}/\mathfrak{n}(g)^{(p)}\big) = (-1)^{m(g)}$.*

(ii) *If $\big(\mathfrak{n}(g)^{(P_1\cdots P_s)}_{\mathrm{St}}, (p)\big) = (1)$ (*e.g.*, if $k\neq 2$), then $m(g) = 0$.*

(iii) *Let $\beta : G(cP_1^\infty\cdots P_s^\infty)\to\overline{L}^*_{\mathfrak{p}}$ be a character of finite order; then*

$$\widetilde{\varepsilon(\beta)} = (-1)^{[F:\mathbf{Q}]+m(g)}\,\eta_{K/F}\big(\mathfrak{n}(g)^{(P_1\cdots P_s)}\big).$$

If $(P_1 \cdots P_s)^n \mid c(\beta)$ *and* $n >> 0$, *then*

$$\widetilde{h}^1_f(K, V \otimes \beta) - h^1_f(K, V \otimes \beta) = m(g), \quad \varepsilon(\beta) = (-1)^{[F:\mathbf{Q}]}\, \eta_{K/F}\big(\mathfrak{n}(g)^{(P_1 \cdots P_s)}\big).$$

Proof

(i), (ii) Let $v \mid ((p), \mathfrak{n}(g))$, $v \neq P_1, \dots, P_s$ be a prime of F; then $\pi(g)_v$ is not supercuspidal and v is unramified in K/F, hence

$$\begin{aligned}\eta_v(o(\pi(g)_v)) = -1 &\iff v \text{ is inert in } K/F,\ 2 \nmid \mathrm{ord}_v(\mathfrak{n}(g))\\ &\iff v \text{ is inert in } K/F,\ \pi(g)_v = \mathrm{St}(\mu),\ \mu \text{ unramified}\\ &\iff v \text{ is inert in } K/F,\ \mathrm{ord}_v(\mathfrak{n}(g)) = 1,\end{aligned}$$

which proves both claims (above, $\eta = \eta_{K/F}$).

(iii) We use Proposition 12.6.3.14 to compute $\widetilde{\varepsilon(\beta)}$ (the condition $H(K, \beta)$ holds, as observed in the proof of Corollary 12.6.4.8). Our assumptions imply that

$$R(\beta)^0 \cap \{v \nmid p\} = R(\beta)^- \cap \{v \mid c(\beta),\ v \nmid p\} = \varnothing$$

$$R(1)^- \cap \{v \mid pc(\beta)\} = \{v \mid p,\ v \text{ inert in } K/F,\ \mathrm{ord}_v(\mathfrak{n}(g)) = 1\},$$

hence

$$\begin{aligned}\widetilde{\varepsilon(\beta)}\,(-1)^{[F:\mathbf{Q}]} &= \eta_{K/F}\big(\mathfrak{n}(g)^{(d_{K/F})}\big)(-1)^{m(g)}(-1)^{|\{j=1,\dots s \mid P_j \text{ inert in } K/F,\, 2 \nmid \mathrm{ord}_{P_j}(\mathfrak{n}(g))\}|}\\ &= \eta_{K/F}\big(\mathfrak{n}(g)^{(P_1 \cdots P_s)}\big)(-1)^{m(g)}.\end{aligned}$$

Assume now that $(P_1 \cdots P_s)^n \mid c(\beta)$ and $n >> 0$. We use Corollary 12.5.9.3 to compute $\widetilde{h}^1_f(K, V \otimes \beta) - h^1_f(K, V \otimes \beta)$. Let $v \mid p$ be a prime of F satisfying $\pi(g)_v = \mathrm{St}(\mu)$ ($\mu^2 = 1$). Fix a prime $w \mid v$ of K and put $N_w = N_{K_w/F_v}$. We must investigate the condition $\beta_w = \mu \circ N_w$ appearing in 12.5.9.3.

If $v = P_i$ ($i = 1, \dots, s$) and $n >> 0$, then $\beta_w^2 \neq 1$, hence $\beta_w \neq \mu \circ N_w$.

Assume that $v \notin \{P_1, \dots, P_s\}$. By assumption, $v \nmid d_{K/F}c(\beta)$, hence $N_w(\mathcal{O}^*_{K,w}) = \mathcal{O}^*_{F,v}$ and $\beta_w(\mathcal{O}^*_{K,w}) = \{1\}$; in particular, $\beta_w \neq \mu \circ N_w$ if μ is ramified.

If v splits in K/F and $n >> 0$, then $\beta_w^2(\mathrm{Fr}(w)) \neq 1$, hence $\beta_w \neq \mu \circ N_w$.

If v is inert in K/F (and $n \geqslant 0$ is arbitrary), then $\beta_w = 1$ and

$$\big[\beta_w = \mu \circ N_w \iff \mu \text{ is unramified} \iff \mathrm{ord}_v(\mathfrak{n}(g)) = 1\big],$$

by Proposition 12.6.1.6 (i). Putting all the cases together, we deduce from Corollary 12.5.9.3 that

$$(\forall n >> 0) \quad \widetilde{h}^1_f(K, V \otimes \beta) - h^1_f(K, V \otimes \beta) = m(g).$$

Finally, Proposition 12.6.4.3 implies that

$$(\forall n >> 0) \quad \varepsilon(\beta) = \widetilde{\varepsilon(\beta)}\,(-1)^{\widetilde{h}^1_f(K,V\otimes\beta) - h^1_f(K,V\otimes\beta)} = (-1)^{[F:\mathbf{Q}]}\eta_{K/F}\big(\mathfrak{n}(g)^{(P_1 \cdots P_s)}\big). \quad \square$$

12.6.4.10. Trivial zeros in $K[cP_1^\infty\cdots P_s^\infty]$. — In the situation of Proposition 12.6.4.9, assume that $K_\infty\subset K[cP_1^\infty\cdots P_s^\infty]$ is a subfield such that

$$(\forall n\geqslant 1)\,\exists\beta:\quad \Gamma=\mathrm{Gal}(K_\infty/K)\longrightarrow \overline{L}_{\mathfrak{p}}^{*}\text{ of finite order with }(P_1\cdots P_s)^n\mid c(\beta).$$

Write $\Gamma=\Gamma_0\times\Delta$, $\Gamma_0\xrightarrow{\sim}\mathbf{Z}_p^r$ $(r\geqslant 1)$, $\Delta=\Gamma_{\mathrm{tors}}$ $(|\Delta|<\infty)$, and fix a character $\chi:\Delta\to(\mathcal{O}')^*$, where $\mathcal{O}'$ is a discrete valuation ring, finite over $\mathcal{O}=\mathcal{O}_{L,\mathfrak{p}}$. Set $\Lambda=\mathcal{O}[\![\Gamma]\!]$ and, for each $\mathcal{O}[\Delta]$-module M, $M^{(\chi)}:=M\otimes_{\mathcal{O}[\Delta],\chi}\mathcal{O}'$. Fix a G_F-stable $\mathcal{O}$-lattice $T\subset V$ and put $A=V/T$,

$$(\forall v\mid p)\quad T_v^+=V_v^+\cap T,\quad T_v^-=T/T_v^+,\quad A_v^\pm=V_v^\pm/T_v^\pm.$$

Consider the contribution of the trivial zeros to $\widetilde{H}^1_f(K_S/K_\infty,A)$ (cf. (9.6.5.1)):

$$Z=\bigoplus_{v\mid p}\bigoplus_{w\mid v}Z_w,\qquad Z_w=\varinjlim_{\alpha}\bigoplus_{w_\alpha\mid w}H^0(G_{w_\alpha},A_v^-),$$

where K_α/K runs through the finite subextensions of K_∞/K, and $w_\alpha\mid w\mid v$ are primes of K_α,K and F, respectively.

12.6.4.11. Proposition. — *In the situation of 12.6.4.10, we have, for each prime $v\mid p$ of F,*

$$\mathrm{cork}_{\Lambda^{(\chi)}}\bigoplus_{w\mid v}Z_w^{(\chi)}=\begin{cases}1, & v\in\mathscr{M}(g,K;P_1,\ldots,P_s)\\ 0, & v\notin\mathscr{M}(g,K;P_1,\ldots,P_s),\end{cases}$$
$$\mathrm{cork}_{\Lambda^{(\chi)}}Z^{(\chi)}=m(g,K;P_1,\ldots,P_s).$$

Proof. — Fix a prime $w\mid p$ of K and denote by v the induced prime of F. As Z_w is a module of co-finite type over $\mathcal{O}[\![\Gamma/\Gamma_w]\!]$ (where $\Gamma_w\subset\Gamma$ is the decomposition group of w in K_∞/K), we have $\mathrm{cork}_{\Lambda^{(\chi)}}Z_w^{(\chi)}=0$ whenever $|\Gamma_w|=\infty$. If $v\in\{P_1,\ldots,P_s\}$, then $|\Gamma_w|\geqslant|I(\Gamma_w)|=\infty$. Assume that $v\notin\{P_1,\ldots,P_s\}$; then v is unramified in K/F. If v splits in K/F, then $|\Gamma_w|=\infty$. If v is inert in K/F, then $\Gamma_w=0$, hence

$$D_\Lambda(Z_w)=D_{\mathcal{O}}(H^0(G_w,A_v^-))\otimes_{\mathcal{O}}\Lambda.$$

Applying Proposition 12.5.8, we obtain, in this inert case,

$$\mathrm{cork}_{\Lambda^{(\chi)}}Z_w^{(\chi)}=\mathrm{cork}_{\mathcal{O}}H^0(G_w,A_v^-)=\begin{cases}1, & \mathrm{ord}_v(\mathfrak{n}(g))=1\\ 0, & \text{otherwise.}\end{cases}$$

Putting all cases together yields the claim. □

12.6.4.12. Proposition. — *In the situation of 12.6.4.10, let $K_\infty=K[cP_1^\infty\cdots P_s^\infty]$, $\Gamma=\mathrm{Gal}(K_\infty/K)=G(cP_1^\infty\cdots P_s^\infty)$. Enlarge S so that $S\supset\{v\mid c\}$; then $K_S\supset K_\infty$. Define $\delta\in\{0,1\}$ by the formula*

$$(-1)^\delta=(-1)^{[F:\mathbf{Q}]}\,\eta_{K/F}\big(\mathfrak{n}(g)^{(P_1\cdots P_s)}\big).$$

Assume that the following condition holds:

$$(C(g,\chi)) \quad (\forall n \geqslant 1)\,\exists \beta : \Gamma \longrightarrow \overline{L}_{\mathfrak{p}}^{*} \text{ of finite order with } (P_1 \cdots P_s)^n \mid c(\beta),$$
$$\beta|_{\Delta} = \chi^{-1}, \quad h^1_f(K, V \otimes \beta) = \delta.$$

Then:

(i) $(\forall j = 1,2)$ $\mathrm{rk}_{\Lambda(\chi^{\pm 1})} \widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, T)^{(\chi^{\pm 1})} = \mathrm{cork}_{\Lambda(\chi^{\pm 1})} \widetilde{H}^j_f(K_S/K_\infty, A)^{(\chi^{\pm 1})} = \delta + m(g, K; P_1, \ldots, P_s)$.

(ii) $\mathrm{cork}_{\Lambda(\chi^{\pm 1})} S_A^{\mathrm{str}}(K_\infty)^{(\chi^{\pm 1})} = \delta$.

(iii) *Let* $K \subset K'_\infty \subset K_\infty$ *be any intermediate field; set* $\Gamma' = \mathrm{Gal}(K'_\infty/K)$, $\Lambda' = \mathcal{O}[\![\Gamma']\!]$. *If* χ *factors through a character* $\chi' : \Gamma'_{\mathrm{tors}} \to (\mathcal{O}')^*$, *then we have, for* $j = 1, 2$,

$$\mathrm{rk}_{\Lambda'(\chi'^{\pm 1})} \widetilde{H}^j_{f,\mathrm{Iw}}(K'_\infty/K, T)^{(\chi'^{\pm 1})} = \mathrm{cork}_{\Lambda'(\chi'^{\pm 1})} \widetilde{H}^j_f(K_S/K'_\infty, A)^{(\chi'^{\pm 1})} \geqslant \delta + m(g, K; P_1, \ldots, P_s)$$
$$\mathrm{cork}_{\Lambda'(\chi'^{\pm 1})} \widetilde{H}^j_f(K_S/K'_\infty, A)^{(\chi'^{\pm 1})} \equiv \delta + m(g, K; P_1, \ldots, P_s) \pmod 2.$$

Proof

(i), (ii) The equality of the various (co)ranks and their independence on the choices of S, Σ and T follow from Proposition 7.8.8, Theorem 10.7.17 (ii) and Proposition 9.7.9 (the assumptions are satisfied, thanks to Proposition 12.4.8.4). Denote their common value by m and put $m(g) := m(g, K; P_1, \ldots, P_s)$. The exact sequence (9.6.5.1) (in which $\Sigma = \{v \mid p\}$) together with Proposition 12.6.4.11 imply that

$$m(g) \leqslant m = m(g) + \mathrm{cork}_{\Lambda(\chi^{\pm 1})} S_A^{\mathrm{str}}(K_\infty)^{(\chi^{\pm 1})}. \tag{12.6.4.1}$$

Fix $n >> 0$ and choose β as in the condition $C(g, \chi)$. Proposition 12.5.9.2 together with Proposition 12.6.4.9 (iii) show that

$$m(g) \leqslant \widetilde{h}^1_f(K, V \otimes \beta) = m(g) + h^1_f(K, V \otimes \beta) = m(g) + \delta \leqslant m(g) + 1. \tag{12.6.4.2}$$

On the other hand, set $K_0 = K_\infty^{\Gamma_0}$, $K'_0 = K_\infty^{\mathrm{Ker}(\beta|_{\Gamma_0})}$, and denote by $\chi' : \mathrm{Gal}(K'_0/K_0) \to \overline{L}_{\mathfrak{p}}^{*}$ the character induced by $\beta^{-1}|_{\Gamma_0}$ (this has nothing to do with χ' that appears in (iii)). After enlarging $\mathcal{O}'$ if necessary, we can assume that $\mathrm{Im}(\chi') \subset \mathcal{O}'$. The character β^{-1} coincides with the composite map

$$\beta^{-1} : \Gamma = \Delta \otimes \Gamma_0 \twoheadrightarrow \Delta \otimes \mathrm{Gal}(K'_0/K_0) \xrightarrow{\chi \times \chi'} \overline{L}_{\mathfrak{p}}^{*},$$

hence

$$H^1_f(K'_0, V)^{(\chi \times \chi')} = H^1_f(K, V \otimes (\chi \times \chi')^{-1}) = H^1_f(K, V \otimes \beta)$$
$$\widetilde{H}^1_f(K'_0, V)^{(\chi \times \chi')} = \widetilde{H}^1_f(K, V \otimes (\chi \times \chi')^{-1}) = \widetilde{H}^1_f(K, V \otimes \beta),$$

by Proposition 8.8.7. Applying Theorem 10.7.17 (iv), we deduce that

$$m \leqslant \widetilde{h}^1_f(K, V \otimes \beta) = m(g) + \delta, \quad m \equiv \widetilde{h}^1_f(K, V \otimes \beta) \equiv m(g) + \delta \pmod 2,$$

which implies, thanks to (12.6.4.1)–(12.6.4.2), that

$$m = m(g) + \delta, \quad \mathrm{cork}_{\Lambda(\chi^{\pm 1})}\, S_A^{\mathrm{str}}(K_\infty)^{(\chi^{\pm 1})} = m - m(g) = \delta.$$

(iii) The equality of the various (co)ranks follows as in the proof of (i); denote their common value by m'. Applying Theorem 10.7.17 (iii) to both K_∞/K and K'_∞/K, we see that

$$m \equiv \widetilde{h}^1_f(K, V \otimes \chi^{-1}) = \widetilde{h}^1_f(K, V \otimes \chi'^{-1}) \equiv m' \pmod 2.$$

The inequality $m \geqslant m'$ is a consequence of Proposition 8.10.11 (ii). □

12.6.5. Hilbert modular forms with *CM*

12.6.5.1. Lemma. — *Let $f \in S_k(\mathfrak{n}, \varphi) - \{0\}$ be a newform of level $\mathfrak{n}$ and weight $k \geqslant 2$. The following conditions are equivalent:*

(i) *There exists a quadratic extension K'/F such that $\pi(f) \xrightarrow{\sim} \pi(f) \otimes \eta_{K'/F}$.*

(ii) *There exists a quadratic extension K'/F such that $V(f) \xrightarrow{\sim} V(f) \otimes \eta_{K'/F}$ (where $\eta_{K'/F} : G_F \to \{\pm 1\}$ is the quadratic character with kernel $\mathrm{Ker}(\eta_{K'/F}) = G_{K'}$).*

Proof. — This follows from the compatibility (12.4.3.1) and the fact that, for each finite set S of primes of F, the representation $\pi(f)$ (resp., $V(f)$) is uniquely determined by the set of local representations $\{\pi(f)_v \mid v \notin S\}$ (resp., $\{V(f)_v \mid v \notin S\}$). □

12.6.5.2. Proposition - Definition. — *If the conditions (i), (ii) of Lemma 12.6.5.1 are satisfied, then:*

(i) *The field $K := K'$ is totally imaginary (we say that f* **has** *CM* **by** *K).*

(ii) *There exists a character $\lambda : \mathbf{A}_K^*/K^* \to \mathbf{C}^*$ satisfying*

$$(\forall w \mid \infty \text{ in } K)\,(\forall x \in K_w^* = \mathbf{C}^*) \quad \lambda_w(x) = x^{1-k}$$

$$(\forall v \text{ prime of } F)\,(\forall \mu : F_v^* \longrightarrow \mathbf{C}^*) \quad L_v(\pi(f)_v \otimes \mu, s) = \prod_{w \mid v} L_w(\lambda_w \cdot (\mu \circ N_w), s + (k-1)/2).$$

In other words, $\pi(f) \otimes |\cdot|^{(1-k)/2}$ is isomorphic to the automorphic induction $I_{K/F}(\lambda)$ and

$$(\forall v \text{ prime of } F) \quad L_v(f, s) = \prod_{w \mid v} L_w(\lambda, s).$$

(iii) $\mathfrak{n} = d_{K/F}\, N_{K/F}(\mathrm{cond}(\lambda))$.

(iv) *If a non-archimedean prime v of F splits in K/F ($v\mathcal{O}_K = ww'$), then $\pi(f)_v = \pi(\lambda_w, \lambda_{w'}) \otimes |\cdot|^{(k-1)/2}$.*

(v) *The central character of $\pi(f)$ is equal to $\eta_{K/F}(\lambda|_{\mathbf{A}_F^*})|\cdot|^{k-1}$.*

(vi) *If the central character of $\pi(f)$ is trivial and v is a non-archimedean prime of F which is unramified in K/F, then $2 \mid \mathrm{ord}_v(\mathfrak{n})$.*

Proof. — The statements (i) and (ii) are well-known: if $V(f) \xrightarrow{\sim} V(f) \otimes \eta_{K'/F}$, then we have (possibly after replacing $L_{\mathfrak{p}}$ by a finite extension) $V(f)|_{G_{K'}} \xrightarrow{\sim} \chi \oplus \chi \circ \rho$, for some character $\chi : G_{K'} \to L_{\mathfrak{p}}^*$ (where $\mathrm{Gal}(K'/F) = \{1, \rho\}$). As $V(f)|_{G_{K'}}$ is a semi-simple L-rational abelian representation of $G_{K'}$, it follows from [**He**, Thm. 2] (see also [**Se3**, §III.2.3, Thm. 2]) that χ is the Galois representation associated to an algebraic Hecke character ψ of K'. As $V(f) \xrightarrow{\sim} \mathrm{Ind}_{G_{K'}}^{G_F}(\chi)$, we have $L(f,s) = L(\lambda, s)$ (Euler factor by Euler factor), where $\lambda : \mathbf{A}_{K'}^*/K'^* \to \mathbf{C}^*$ is the idèle class character associated to ψ and the fixed embedding $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ ([**Sc**, ch. 0, §6]). Comparing the archimedean terms in the functional equations of $L(f,s)$ and $L(\lambda, s)$ (and using the fact that $k - 1 \neq 0$), we deduce that $K' = K$ is totally imaginary and the infinity type of ψ is equal to $(k-1)\sum_{\sigma \in \Phi} \sigma$, for some CM type Φ of K.

The formula (iii) follows from [**A-T**, ch. 11, Thm. 18] and (12.4.3.2), while (iv) is a consequence of (ii). The central character of $I_{K/F}(\lambda)$ is equal to $\eta_{K/F}(\lambda|_{\mathbf{A}_F^*})$, which proves (v). In order to prove (vi), we need to show that $2 \mid \mathrm{ord}_v(\mathrm{cond}(\lambda))$, by (iii). This divisibility is automatic if v is inert in K/F. If v splits in K/F, then $\pi(f)_v = \pi(\mu, \mu^{-1})$ (by (iv)), hence $\mathrm{ord}_v(\mathfrak{n}) = o(\pi(f)_v) = 2\, o(\mu)$ is even. □

12.7. Hida families of Hilbert modular forms

In this section we apply the methods of 10.7.7–10.7.10 to big Galois representations associated to ordinary families of Hilbert modular forms. The parity results proved in 12.7.15 generalize, among others, Theorem A' from [**N-P**].

12.7.1. Fix an ideal $\mathfrak{n} \subset \mathcal{O}_F$ prime to p. We shall be interested in ordinary eigenforms $f \in S_k(\mathfrak{n}p^r, \phi)$ for $k \geqslant 2, r \geqslant 1$. Such forms have central characters defined on

$$Z = \varprojlim_r I_{\mathfrak{n}p^r\infty},$$

where $I_{\mathfrak{a}\infty}$ is the ray class group modulo $\mathfrak{a}\infty$ (recall that ∞ denotes the sum of all archimedean primes of F). The torsion subgroup Z_{tors} is finite and $Z/Z_{\mathrm{tors}} \xrightarrow{\sim} \mathbf{Z}_p^{1+\delta}$, where $\delta \geqslant 0$ is the default of the Leopoldt conjecture for F and p. In particular, $\delta = 0$ if F is an abelian extension of $\mathbf{Q}$.

Let S_0 be the set of all primes of F dividing $\mathfrak{n}p\infty$. Recall that we have normalized the isomorphism of class field theory by

$$\begin{aligned} Z &\xrightarrow{\sim} G_{F,S_0}^{ab} \\ \mathfrak{q} &\longmapsto \mathrm{Fr}(\mathfrak{q})_{\mathrm{geom}}. \end{aligned}$$

With this normalization, the composite map

$$Z \xrightarrow{\sim} G_{F,S_0}^{ab} \twoheadrightarrow \mathrm{Gal}(F(\mu_{p^\infty})/F) \hookrightarrow \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \xrightarrow{\chi_{\mathrm{cycl}}^{-1}} \mathbf{Z}_p^* \qquad (12.7.1.1)$$

sends $\mathfrak{q}$ to its norm $N\mathfrak{q} \in \mathbf{Z}_p^*$. Putting

$$q = \begin{cases} p, & p \neq 2 \\ 4, & p = 2, \end{cases}$$

the torsion subgroup of $\mathbf{Z}_p^*$

$$(\mathbf{Z}_p^*)_{\mathrm{tors}} = \begin{cases} \mu_{p-1}(\mathbf{Z}_p), & p \neq 2 \\ \{\pm 1\}, & p = 2 \end{cases}$$

has a canonical complement $1 + q\mathbf{Z}_p \subset \mathbf{Z}_p^*$. The corresponding projections

$$\omega : \mathbf{Z}_p^* \longrightarrow (\mathbf{Z}_p^*)_{\mathrm{tors}}, \quad \kappa : \mathbf{Z}_p^* \longrightarrow 1 + q\mathbf{Z}_p$$

define a decomposition

$$(\omega, \kappa) : \mathbf{Z}_p^* \xrightarrow{\sim} (\mathbf{Z}_p^*)_{\mathrm{tors}} \times 1 + q\mathbf{Z}_p.$$

Let $\mathbf{Q}_\infty/\mathbf{Q}$ be the unique $\mathbf{Z}_p$-extension of $\mathbf{Q}$, *i.e.*, the fixed field of $\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})_{\mathrm{tors}} \subset \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$. Then the map

$$\kappa \circ \chi_{\mathrm{cycl}}^{-1} : \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q}) \xrightarrow{\sim} 1 + q\mathbf{Z}_p$$

is an isomorphism; denote by $\Gamma \subset 1 + q\mathbf{Z}_p$ the image of

$$\mathrm{Gal}(F\mathbf{Q}_\infty/F) \hookrightarrow \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q}) \xrightarrow{\kappa\circ\chi_{\mathrm{cycl}}^{-1}} 1 + q\mathbf{Z}_p$$

(Γ is isomorphic to $\mathbf{Z}_p$).

12.7.2. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbf{Q}_p$ (*e.g.* of $L_\mathfrak{p}$); put $\Lambda = \mathcal{O}[\![\Gamma]\!]$. The tautological representation $\chi_\Gamma : \Gamma \hookrightarrow \Lambda^*$ defines a big Galois representation

$$\chi_\Gamma : G_{F,S_0} \twoheadrightarrow \mathrm{Gal}(F\mathbf{Q}_\infty/F) \xrightarrow{\kappa\circ\chi_{\mathrm{cycl}}^{-1}} \Gamma \hookrightarrow \Lambda^*,$$

for which

$$\chi_\Gamma(\mathrm{Fr}(\mathfrak{q})_{\mathrm{geom}}) = \chi_\Gamma(\kappa(N\mathfrak{q})) \qquad (\mathfrak{q} \nmid \mathfrak{n}p).$$

Fix a topological generator $\gamma \in \Gamma$ (one can take, for example, $\gamma = (1+q)^{p^e}$, where $p^e = (1 + q\mathbf{Z}_p : \Gamma)$). For each pair (k, ε), where $k \in \mathbf{Z}$, $k \geqslant 2$ and

$$\varepsilon : \Gamma \longrightarrow \mu_{p^\infty}(\mathcal{O}) \subset \mathcal{O}^*$$

is a character of Γ of finite order, put

$$P_{k,\varepsilon} = \chi_\Gamma(\gamma) - \varepsilon(\gamma)\gamma^{k-2} \in \Lambda$$

(where $\gamma^{k-2} \in \Gamma \subset 1 + q\mathbf{Z}_p \subset \mathcal{O}^*$). Then $\Lambda/P_{k,\varepsilon}\Lambda \xrightarrow{\sim} \mathcal{O}$ and the representation

$$\chi_\Gamma \pmod{P_{k,\varepsilon}} : G_{F,S_0} \longrightarrow (\Lambda/P_{k,\varepsilon}\Lambda)^* = \mathcal{O}^*$$

sends $\mathrm{Fr}(\mathfrak{q}) = \mathrm{Fr}(\mathfrak{q})_{\mathrm{geom}}$ (for each $\mathfrak{q} \nmid \mathfrak{n}p$) to $\varepsilon \circ \kappa(N\mathfrak{q}) \cdot \kappa(N\mathfrak{q})^{k-2}$, *i.e.*,

$$\chi_\Gamma \pmod{P_{k,\varepsilon}} = (\varepsilon \circ \kappa \circ N) \cdot (\kappa \circ N)^{k-2},$$

by the Čebotarev density theorem. More generally, if $\mathcal{O}'$ is the ring of integers of a finite extension of $\mathrm{Frac}(\mathcal{O})$, $k' \geqslant 2$ and $\varepsilon' : \Gamma \to (\mathcal{O}')^*$ a character of finite order, then

$$P_{k',\varepsilon'}\Lambda' \in \mathrm{Spec}(\Lambda') \qquad (\Lambda' = \mathcal{O}'[\![\Gamma]\!])$$

induces a prime ideal

$$P_{k',\varepsilon'}\Lambda' \cap \Lambda \in \mathrm{Spec}(\Lambda)$$

with residue field $\mathrm{Frac}(\mathcal{O})(\varepsilon'(\Gamma))$, and the representation

$$\chi_\Gamma \pmod{P_{k',\varepsilon'}\Lambda' \cap \Lambda} : G_{F,S_0} \longrightarrow \mathcal{O}(\varepsilon'(\Gamma))^*$$

is equal to $(\varepsilon' \circ \kappa \circ N) \cdot (\kappa \circ N)^{k'-2}$.

12.7.3. For any level $\mathfrak{n}' \subset \mathcal{O}_F$ one defines $S_k(\mathfrak{n}', \mathbf{Z}) \subset S_k(\mathfrak{n}')$ ($k \geqslant 2$) to be the subgroup consisting of cusp forms f whose all Fourier coefficients $c(\mathfrak{a}, f) \in \mathbf{Z}$ are rational integers. Then

$$S_k(\mathfrak{n}', \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} = S_k(\mathfrak{n}')$$

and one defines

$$S_k(\mathfrak{n}', \mathcal{O}) = S_k(\mathfrak{n}', \mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{O}.$$

The Hecke algebra

$$h_k(\mathfrak{n}', \mathcal{O}) \qquad (k \geqslant 2)$$

is the $\mathcal{O}$-algebra generated by $T_{\mathfrak{n}'}(\mathfrak{a}) = T(\mathfrak{a})$, $S_{\mathfrak{n}'}(\mathfrak{b})$ (for all integral ideals $\mathfrak{a}$ in $\mathcal{O}_F$ and all integral ideals $\mathfrak{b}$ prime to $\mathfrak{n}'$) acting on $S_k(\mathfrak{n}', \mathcal{O})$.

If $\mathfrak{n}p | \mathfrak{n}'$, then $U_p := T_{\mathfrak{n}p}(p) = T_{\mathfrak{n}'}(p)$ and

$$e = \lim_{r \to \infty} U_p^{r!}$$

(Hida's *ordinary projector*, $e^2 = e$) act on $S_k(\mathfrak{n}', \mathcal{O})$. The *ordinary part* of the Hecke algebra $h_k(\mathfrak{n}', \mathcal{O})$ is defined as

$$h_k^{\mathrm{ord}}(\mathfrak{n}', \mathcal{O}) = e\, h_k(\mathfrak{n}', \mathcal{O}) \qquad (k \geqslant 2,\ \mathfrak{n}p | \mathfrak{n}').$$

For fixed $k \geqslant 2$ and variable $r \geqslant 1$, the Hecke algebras $h_k(\mathfrak{n}p^r, \mathcal{O})$ (as well as their ordinary parts) form a projective system under the transition maps $T(\mathfrak{a}) \mapsto T(\mathfrak{a})$. According to [**Hi2**, Thm. 3.2], the projective limits

$$\varprojlim_r h_k(\mathfrak{n}p^r, \mathcal{O}) \qquad (k \geqslant 2)$$

($= h_{k \cdot t, (k-1) \cdot t}(\mathfrak{n}p^\infty; \mathcal{O})$ in the notation of [**Hi2**]) are canonically isomorphic to each other, *i.e.*, do not depend on $k \geqslant 2$. We denote by

$$h^{\mathrm{ord}}(\mathfrak{n}; \mathcal{O}) = e \varprojlim_r h_k(\mathfrak{n}p^r, \mathcal{O}) = \varprojlim_r h_k^{\mathrm{ord}}(\mathfrak{n}p^r, \mathcal{O}) \qquad (k \geqslant 2)$$

the ordinary part of this limit ($= \mathbf{h}_0^{\mathrm{ord}}(\mathfrak{n}; \mathcal{O})$ in the notation of [**Hi2**]).

The $\mathcal{O}$-algebra $h^{\mathrm{ord}}(\mathfrak{n};\mathcal{O})$ contains the images of all Hecke operators $T(\mathfrak{a}) = T_{\mathfrak{n}p}(\mathfrak{a})$ and $S(\mathfrak{b}) = S_{\mathfrak{n}p}(\mathfrak{b})$ (for all $\mathfrak{a}$ and $\mathfrak{b}$ prime to $\mathfrak{n}p$). The map $\mathfrak{b} \mapsto S(\mathfrak{b})$ for $k = 2$ defines a homomorphism of $\mathcal{O}$-algebras

$$\mathcal{O}[\![Z]\!] \longrightarrow h^{\mathrm{ord}}(\mathfrak{n};\mathcal{O}).$$

Fix a splitting of the canonical projection

$$Z \longrightarrow Z/Z_{\mathrm{tors}} =: W,$$

compatible *via* $\chi_{\mathrm{cycl}}^{-1}$ with the canonical splitting of

$$\mathbf{Z}_p^* \longrightarrow \mathbf{Z}_p^*/(\mathbf{Z}_p^*)_{\mathrm{tors}}$$

given by $1 + q\mathbf{Z}_p$. This induces an injection

$$\Lambda(W) := \mathcal{O}[\![W]\!] \hookrightarrow \mathcal{O}[\![Z]\!] \xrightarrow{\sim} \mathcal{O}[Z_{\mathrm{tors}}] \otimes_{\mathcal{O}} \Lambda(W).$$

The ring $\Lambda(W)$ is isomorphic to $\mathcal{O}[\![T_0,\ldots,T_\delta]\!]$ and $h^{\mathrm{ord}}(\mathfrak{n};\mathcal{O})$ is a torsion-free $\Lambda(W)$-module of finite type, by [**Hi2**, Thm. 3.3].

The fixed decomposition $Z \xrightarrow{\sim} Z_{\mathrm{tors}} \times W$ allows us to decompose any character $\phi : Z \to \mathcal{O}^*$ as $\phi = \phi_{\mathrm{tame}}\phi_W$, where ϕ_{tame} (resp., ϕ_W) denotes the restriction of ϕ to Z_{tors} (resp., to W).

Fix also a splitting of the surjective homomorphism $N : W \twoheadrightarrow \Gamma$ (given by the norm map $\mathrm{Fr}(\mathfrak{q}) \mapsto N\mathfrak{q}$), *i.e.*, a decomposition $W \xrightarrow{\sim} W_{nL} \times \Gamma$, where $W_{nL} := \mathrm{Ker}(N)$ denotes the "non-Leopoldt" part of W (of course, $W_{nL} = 0 \iff$ the Leopoldt Conjecture holds for (F,p)).

For a fixed character of finite order $\phi_{nL} : W_{nL} \to \mu_{p^\infty}(\mathcal{O}) \subset \mathcal{O}^*$ we define

$$h^{\mathrm{ord}}_{\phi_{nL}}(\mathfrak{n};\mathcal{O}) = h^{\mathrm{ord}}(\mathfrak{n};\mathcal{O}) \otimes_{\mathcal{O}[\![W_{nL}\times\Gamma]\!],\phi_{nL}\times\mathrm{id}} \mathcal{O}[\![\Gamma]\!] = h^{\mathrm{ord}}(\mathfrak{n};\mathcal{O}) \otimes_{\mathcal{O}[\![W_{nL}]\!],\phi_{nL}} \mathcal{O};$$

this a finite Λ-algebra ($\Lambda = \mathcal{O}[\![\Gamma]\!]$).

12.7.4. An **arithmetic point** of $h^{\mathrm{ord}}_{\phi_{nL}}(\mathfrak{n};\mathcal{O})$ is a prime ideal $\mathcal{P} \in \mathrm{Spec}(h^{\mathrm{ord}}_{\phi_{nL}}(\mathfrak{n};\mathcal{O}))$ whose restriction to $\Lambda = \mathcal{O}[\![\Gamma]\!]$ is equal to $P_{k',\varepsilon'}\Lambda' \cap \Lambda \in \mathrm{Spec}(\Lambda)$ for some $k' \geqslant 2$ and $\varepsilon' : \Gamma \to (\mathcal{O}')^*$ as in 12.7.2.

We define a ϕ_{nL}**-arithmetic point** of $h^{\mathrm{ord}}(\mathfrak{n};\mathcal{O})$ to be the image in $\mathrm{Spec}(h^{\mathrm{ord}}(\mathfrak{n};\mathcal{O}))$ of an arithmetic point $\mathcal{P} \in \mathrm{Spec}(h^{\mathrm{ord}}_{\phi_{nL}}(\mathfrak{n};\mathcal{O}))$ under the canonical map

$$\mathrm{Spec}(h^{\mathrm{ord}}_{\phi_{nL}}(\mathfrak{n};\mathcal{O})) \hookrightarrow \mathrm{Spec}(h^{\mathrm{ord}}(\mathfrak{n};\mathcal{O})).$$

For such $\mathcal{P}$, the images of $T(\mathfrak{a})$ and $S(\mathfrak{b})$ in $h^{\mathrm{ord}}_{\phi_{nL}}(\mathfrak{n};\mathcal{O})/\mathcal{P}$ are equal to (the images under ι_p of) the Hecke eigenvalues of an ordinary normalized Hecke eigenform

$$f_{\mathcal{P}} \in S_k(\mathfrak{n}p^{r(\mathcal{P})}, \phi_{\mathcal{P}}),$$

where $r(\mathcal{P}) \geqslant 1$ and $\phi_{\mathcal{P}} : Z \to \mathcal{O}'^*$ is a character of finite order with "wild part"

$$(\phi_{\mathcal{P}})_W = \phi_{nL} \cdot (\varepsilon \circ \kappa \circ N)$$

(see [**Hi2**, Cor. 3.5], for $v = 0$).

12.7.5. Conversely, assume that

$$f \in S_k(\mathfrak{n}p^r, \phi) \qquad (k \geqslant 2)$$

is an ordinary normalized Hecke eigenform such that $\mathrm{Frac}(\mathcal{O})$ contains (the images under ι_p of) all Hecke eigenvalues of f and all values of ϕ, and that the restriction of ϕ to W_{nL} is equal to ϕ_{nL}. Assume, in addition, that f is a *p-stabilized newform* in the sense of [**Wi**, p. 538]. This means that $r \geqslant 1$ and that the (ordinary) normalized newform f' associated to f has level $\mathfrak{n}'$ divisible by $\mathfrak{n}$. This implies that, for each prime ideal $v \mid p$,

$$1 - \lambda_{f'}(v)X + \phi(v)(Nv)^{k-1}X^2 = (1 - \lambda_f(v))(1 - \phi(v)(Nv)^{k-1}\lambda_f(v)^{-1}X) \qquad v \nmid \mathfrak{n}'$$
$$1 - \lambda_{f'}(v)X = 1 - \lambda_f(v)X \qquad v | \mathfrak{n}'.$$

The "wild part" of ϕ is equal to

$$\phi_W = \phi_{nL} \cdot (\varepsilon \circ \kappa \circ N)$$

for some character of finite order

$$\varepsilon : \Gamma \longrightarrow \mu_{p^\infty}(\mathcal{O}) \subset \mathcal{O}^*.$$

By [**Hi2**, Cor. 3.5], there exists an arithmetic point $\mathcal{P}$ of $h^{\mathrm{ord}}_{\phi_{nL}}(\mathfrak{n};\mathcal{O})$ above $(P_{k,\varepsilon}) \in \mathrm{Spec}(\Lambda)$ such that

$$T(v) \pmod{\mathcal{P}} = \lambda_{f'}(v) = \lambda_f(v)$$

for all prime ideals ideals $v \nmid p$. This collection of Hecke eigenvalues occurs in $h^{\mathrm{ord}}_k(\mathfrak{n}p^{r'}, \mathcal{O})$ only once, for each $r' \geqslant r$. This follows from the fact that the level of f' is divisible by $\mathfrak{n}$ and the standard description of ordinary eigenforms in terms of newforms ([**Wi**, (1.2.2)]). This implies that $\mathcal{P}$ is uniquely determined by f and that

$$f = f_{\mathcal{P}}.$$

One can say more: the multiplicity one statement alluded to above implies that

$$\Lambda_{(P_{k,\varepsilon})} \hookrightarrow h^{\mathrm{ord}}_{\phi_{nL}}(\mathfrak{n};\mathcal{O})_{\mathcal{P}}$$

is an unramified extension of discrete valuation rings (we are grateful to A. Wiles for explaining this fact to us). In the case $F = \mathbf{Q}$, this is proved in [**Hi1**, Cor. 1.4]. The general abstract principle is summarized in Lemma 12.7.6 below, which has to be applied to $A = h^{\mathrm{ord}}(\mathfrak{n};\mathcal{O})$, $B = \mathcal{O}[\![W]\!]$, $J = \mathrm{Ker}(\phi_{nL} \times \mathrm{id} : B = \mathcal{O}[\![W]\!] \twoheadrightarrow \mathcal{O}[\![\Gamma]\!] = \Lambda)$, $B/J = \Lambda$, $A/JA = h^{\mathrm{ord}}_{\phi_{nL}}(\mathfrak{n};\mathcal{O})$, $Q_0 = (P_{k,\varepsilon})$. The assumption (a) (resp., (b)) holds by [**Hi2**, Thm. 3.3] (resp., by Hida's "control theorem" [**Hi2**, Thm. 3.4] and the multiplicity one statement for f).

12.7.6. Lemma. — *Let $B \subset A$ be commutative Noetherian rings, $J \subset B$ an ideal, $P_0 \in \mathrm{Spec}(A/JA)$. Let $P \in \mathrm{Spec}(A)$, $Q = P \cap B \in \mathrm{Spec}(B)$ and $Q_0 \in \mathrm{Spec}(B/J)$ be induced by P_0. Assume that*

(a) *As a B-module, A is of finite type and torsion-free.*

(b) *Let* $M = (P/QA) \otimes_B B_Q/QB_Q$, $N = (A/P) \otimes_B B_Q/QB_Q$. *In the canonical exact sequence of* A/QA*-modules*

$$0 \longrightarrow M \longrightarrow (A/QA) \otimes_{B/Q} B_Q/QB_Q \longrightarrow N \longrightarrow 0,$$

N *occurs in the middle term* $A \otimes_B B_Q/QB_Q$ *with multiplicity one in the sense that*

$$\mathrm{Hom}_{A/QA}(M, N) = 0.$$

Then

(i) $QA_P = PA_P$, $Q_0(A/JA)_{P_0} = P_0(A/JA)_{P_0}$.

(ii) *If* $(B/J)_{Q_0}$ *is a discrete valuation ring with prime element* $x \pmod J$ *(*$x \in B$*), the same is true for* $(A/JA)_{P_0}$.

Proof

(i) The assumption (a) implies that the map $B_Q \to A_P$ is injective. Both $M_P = PA_P/QA_P$ and $N_P = A_P/PA_P$ are finitely generated modules over the Noetherian local ring $S = A_P/QA_P$, with $N_P = S/\mathfrak{m}_S$ equal to the residue field of S. Localizing (b) at P, we obtain

$$0 = \mathrm{Hom}_S(M_P, N_P) = \mathrm{Hom}_{S/\mathfrak{m}_S}(M_P/\mathfrak{m}_S M_P, S/\mathfrak{m}_S),$$

hence $M_P/\mathfrak{m}_S M_P = 0$. Nakayama's Lemma implies that $M_P = 0$, proving $QA_P = PA_P$. As

$$(A/JA)_{P_0} = A_P \otimes_B (B/J)_{Q_0} = A_P \otimes_B (B/J) = A_P/JA_P,$$

we have

$$Q_0(A/JA)_{P_0} = QA_P/JA_P = PA_P/JA_P = P_0(A/JA)_{P_0},$$

proving (i).

The statement (ii) follows from (i). □

12.7.7. Returning to the situation of 12.7.5, $\mathcal{P}$ contains a unique prime ideal $\mathcal{P}_{\min} \subsetneq \mathcal{P}$ of $h^{\mathrm{ord}}_{\phi_{nL}}(\mathfrak{n}; \mathcal{O})$, necessarily minimal. Put $R = h^{\mathrm{ord}}_{\phi_{nL}}(\mathfrak{n}; \mathcal{O})/\mathcal{P}_{\min}$, $\overline{\mathcal{P}} = \mathcal{P}/\mathcal{P}_{\min} \in \mathrm{Spec}(R)$. Then R is a local domain, finite over Λ, $R_{\overline{\mathcal{P}}}$ is a discrete valuation ring with prime element $P_{k,\varepsilon}$ and $\mathrm{Frac}(R) = \mathrm{Frac}(R_{\overline{\mathcal{P}}})$ is a finite extension of $\mathrm{Frac}(\Lambda)$.

The image of the canonical map $\mathrm{Spec}(R) \to \mathrm{Spec}(\Lambda)$ is closed (as R is a finitely generated Λ-module) and contains $\mathrm{Spec}(\Lambda_{(P_{k,\varepsilon})})$, hence is equal to $\mathrm{Spec}(\Lambda)$. As a result, for each pair $k' \geqslant 2$, $\varepsilon' : \Gamma \to (\mathcal{O}')^*$ (with $\mathcal{O}'$ as in 12.7.2), there is an arithmetic point $\mathcal{P}' \in \mathrm{Spec}(h^{\mathrm{ord}}_{\phi_{nL}}(\mathfrak{n}; \mathcal{O}))$ containing $\mathcal{P}_{\min}$ and lying above $P_{k',\varepsilon'}\Lambda' \cap \Lambda \in \mathrm{Spec}(\Lambda)$; put $\overline{\mathcal{P}}' = \mathcal{P}'/\mathcal{P}_{\min} \in \mathrm{Spec}(R)$.

The homomorphism

$$\lambda : h^{\mathrm{ord}}(\mathfrak{n}; \mathcal{O}) \longrightarrow\!\!\!\!\!\rightarrow h^{\mathrm{ord}}_{\phi_{nL}}(\mathfrak{n}; \mathcal{O})/\mathcal{P}_{\min} = R \hookrightarrow \mathrm{Frac}(R)$$

is *minimal* in the sense of [**Hi2**, p. 317], since $f = f_{\mathcal{P}}$ is a p-stabilized newform. This implies, by [**Hi2**, Thm. 3.6, Cor. 3.5] that the form $f_{\mathcal{P}'}$ associated to $\mathcal{P}'$ as in 12.7.4 is again a p-stabilized newform

$$f_{\mathcal{P}'} \in S_{k'}(\mathfrak{n}p^{r(\mathcal{P}')}, \phi_{\mathcal{P}'}),$$

where (the images under ι_p of) all Hecke eigenvalues of $f_{\mathcal{P}'}$ are contained in $\mathcal{O}'$,

$$(\phi_{\mathcal{P}'})_W = \phi_{nL} \circ (\varepsilon' \circ \kappa \circ N)$$

and such that the character

$$\phi_0 := \phi_{\mathcal{P}'}^{\mathrm{tame}}(\omega \circ N)^{k'-2} = \phi_{\mathcal{P}}^{\mathrm{tame}}(\omega \circ N)^{k-2} \tag{12.7.7.1}$$

does not depend on $\mathcal{P}'$. One can again apply Lemma 12.7.6 to show that

$$\Lambda_{P_{k',\varepsilon'}\Lambda' \cap \Lambda} \hookrightarrow (h^{\mathrm{ord}}_{\phi_{nL}}(\mathfrak{n}; \mathcal{O}))_{\mathcal{P}'} = R_{\overline{\mathcal{P}}'}$$

is an unramified extension of discrete valuation rings. Without loss of generality, we can assume that the ring $\mathcal{O}'$ used in the definition of $\overline{\mathcal{P}}'$ coincides with the ring of integers of the residue field $\kappa(\overline{\mathcal{P}}') = R_{\overline{\mathcal{P}}'}/\overline{\mathcal{P}}' R_{\overline{\mathcal{P}}'} = \kappa(\mathcal{P}')$ of $\overline{\mathcal{P}}'$.

12.7.8. According to [**Wi**, Thm. 2.2.1] there is a unique (up to equivalence) continuous Galois representation

$$\rho : G_{F,S_0} \longrightarrow \mathrm{GL}_2(\mathrm{Frac}(R))$$

satisfying

$$\det(1 - \rho(\mathrm{Fr}(\mathfrak{q}))\, X) = 1 - \lambda(T(\mathfrak{q}))X + (\phi_0\phi_{nL})(\mathfrak{q})\chi_\Gamma(\kappa \circ N(\mathfrak{q}))(N\mathfrak{q})X^2$$

for all prime ideals $\mathfrak{q} \nmid \mathfrak{n}p$, where λ and ϕ_0 were defined in 12.7.7 (more precisely, ρ is the dual of the representation constructed in [**Wi**], as we use $\mathrm{Fr}(\mathfrak{q}) = \mathrm{Fr}(\mathfrak{q})_{\mathrm{geom}}$ instead of $\mathrm{Fr}(\mathfrak{q})_{\mathrm{arith}}$). This representation is (absolutely) irreducible (as its residual representation modulo $\overline{\mathcal{P}}$ is (absolutely) irreducible; see 12.7.9 below) and is continuous in the sense that its representation space $V(\lambda)$ is an admissible $R[G_{F,S_0}]$-module. According to [**Wi**, Thm. 2.2.2], for each prime ideal $v \mid p$ in $\mathcal{O}_F$ there is an exact sequence of $\mathrm{Frac}(R)[G_v]$-modules

$$0 \longrightarrow V(\lambda)_v^+ \longrightarrow V(\lambda) \longrightarrow V(\lambda)_v^- \longrightarrow 0,$$

such that each $V(\lambda)_v^{\pm}$ is one-dimensional over $\mathrm{Frac}(R)$, I_v acts trivially on $V(\lambda)_v^+$ and $\mathrm{Fr}(v)$ acts on $V(\lambda)_v^+$ by $\lambda(T(v))$. These properties determine the subspace $V(\lambda)_v^+$ uniquely.

12.7.9. For each $\mathcal{P}'$ as in 12.7.7, $R_{\overline{\mathcal{P}}'}$ is a discrete valuation ring with fraction field $\mathrm{Frac}(R)$, which implies that $V(\lambda)$ contains a G_{F,S_0}-invariant $(R_{\overline{\mathcal{P}}'})$-lattice $\overline{T}(\mathcal{P}') \subset V(\lambda)$. Its reduction modulo $\overline{\mathcal{P}}'$

$$\overline{T}(\mathcal{P}')/\overline{\mathcal{P}}'\overline{T}(\mathcal{P}')$$

is a two-dimensional representation of G_{F,S_0} over the residue field $\kappa(\overline{\mathcal{P}}') = \kappa(\mathcal{P}')$ satisfying

$$\begin{aligned}\det(1 - \rho(\mathrm{Fr}(\mathfrak{q}))\, X \mid \overline{T}(\mathcal{P}')/\overline{\mathcal{P}}'\overline{T}(\mathcal{P}'))\\ = 1 - \lambda_{f'_{\mathcal{P}}}(\mathfrak{q})X + (\phi_0\phi_{nL})(\mathfrak{q})(\varepsilon' \circ \kappa \circ N)(\mathfrak{q})(\kappa \circ N)^{k'-2}(\mathfrak{q})(N\mathfrak{q})X^2\\ = 1 - \lambda_{f'_{\mathcal{P}}}(\mathfrak{q})X + \phi_{\mathcal{P}'}(\mathfrak{q})(N\mathfrak{q})^{k-1}X^2 = \det(1 - \mathrm{Fr}(\mathfrak{q})\, X \mid V(f_{\mathcal{P}'}))\end{aligned}$$

for all prime ideals $\mathfrak{q} \nmid \mathfrak{n}p$. Absolute irreducibility of $V(f_{\mathcal{P}'})$ and the Čebotarev density theorem imply that there is an isomorphism of $\kappa(\mathcal{P}')[G_{F,S_0}]$-modules

$$\overline{T}(\mathcal{P}')/\overline{\mathcal{P}}'\overline{T}(\mathcal{P}') \xrightarrow{\sim} V(f_{\mathcal{P}'}),$$

(unique up to an element of $\kappa(\mathcal{P}')^*$) and that the only G_{F,S_0}-invariant $(R_{\overline{\mathcal{P}}'})$-lattices in $V(\lambda)$ are $(\overline{\mathcal{P}}')^i\overline{T}(\mathcal{P}')$ $(i \in \mathbf{Z})$.

For each prime ideal $v \mid p$ in $\mathcal{O}_F$, put

$$\overline{T}(\mathcal{P}')_v^+ = \overline{T}(\mathcal{P}') \cap \overline{V}(\lambda)_v^+, \quad \overline{T}(\mathcal{P}')_v^- = \overline{T}(\mathcal{P}')/\overline{T}(\mathcal{P}')_v^+.$$

Both $\overline{T}(\mathcal{P}')_v^\pm$ are $R_{\overline{\mathcal{P}}'}[G_v]$-modules, free of rank one over $R_{\overline{\mathcal{P}}'}$, and the exact sequence of $\kappa(\mathcal{P}')[G_v]$-modules

$$0 \longrightarrow \overline{T}(\mathcal{P}')_v^+/\overline{\mathcal{P}}'\overline{T}(\mathcal{P}')_v^+ \longrightarrow \overline{T}(\mathcal{P}')/\overline{\mathcal{P}}'\overline{T}(\mathcal{P}') \longrightarrow \overline{T}(\mathcal{P}')_v^-/\overline{\mathcal{P}}'\overline{T}(\mathcal{P}')_v^- \longrightarrow 0$$

is isomorphic to

$$0 \longrightarrow V(f_{\mathcal{P}'})_v^+ \longrightarrow V(f_{\mathcal{P}'}) \longrightarrow V(f_{\mathcal{P}'})_v^- \longrightarrow 0$$

(by the uniqueness of $V(f_{\mathcal{P}'})_v^+$). Strictly speaking, the notation $V(f_{\mathcal{P}'})$ is ambiguous, as it does not specify the field of coefficients of the representation. Above, we have implicitly used $\kappa(\mathcal{P}')$ as the field of coefficients.

Note that the determinant of the representation ρ is equal to

$$\Lambda^2 V(\lambda) = \mathrm{Frac}(R)(-1) \otimes \chi_\Gamma\phi_0\phi_{nL}.$$

We would like to define a suitable twist

$$\mathscr{V} = \text{“}V(\lambda)(1) \otimes (\chi_\Gamma\phi_0\phi_{nL})^{-1/2}\text{”}$$

which would be self-dual in the sense that

$$\Lambda^2\mathscr{V} \xrightarrow{\sim} \mathrm{Frac}(R)(1),$$

i.e., $\mathscr{V}$ would be isomorphic to

$$\mathscr{V} \xrightarrow{\sim} \mathscr{V}^*(1) = \mathrm{Hom}_{\mathrm{Frac}(R)}(\mathscr{V}, \mathrm{Frac}(R))(1)$$

(cf. [**N-P**, §3.2.3]). This is indeed possible (after a quadratic base change if $p = 2$) under the following assumption:

12.7.9.1. For $f = f_{\mathcal{P}}$ from 12.7.5, $k \in 2\mathbf{Z}$ and there exists a (continuous) character $\chi : \mathbf{A}_F^*/F^* \to \mathbf{C}^*$ such that $\phi := \phi_{\mathcal{P}} = \chi^{-2}$; fix such χ and a finite set $S \supset S_0$ of primes of F such that χ is unramified outside S.

12.7.10. Assume that 12.7.9.1 is satisfied and $p \neq 2$. As the groups $W \xrightarrow{\sim} \mathbf{Z}_p^{1+\delta}$ and $\Gamma \xrightarrow{\sim} \mathbf{Z}_p$ are uniquely 2-divisible, the representations $\phi_{nL} : W_{nL} \to \mathcal{O}^*$, $\phi_W : W \to \mathcal{O}^*$, $\varepsilon : \Gamma \to \mathcal{O}^*$, $\kappa : \mathbf{Z}_p^* \to 1 + q\mathbf{Z}_p$ and $\chi_\Gamma : G_{F,S} \twoheadrightarrow \Gamma \hookrightarrow \Lambda^*$ have canonical square roots

$$\phi_{nL}^{1/2} : W_{nL} \xrightarrow{\frac{1}{2}} W_{nL} \xrightarrow{\phi_{nL}} \mathcal{O}^*, \quad \phi_W^{1/2} : W \xrightarrow{\frac{1}{2}} W \xrightarrow{\phi_W} \mathcal{O}^*, \quad \varepsilon^{1/2} : \Gamma \xrightarrow{\frac{1}{2}} \Gamma \xrightarrow{\varepsilon} \mathcal{O}^*,$$

$$\kappa^{1/2} : \mathbf{Z}_p^* \xrightarrow{\kappa} 1 + q\mathbf{Z}_p \xrightarrow{\frac{1}{2}} 1 + q\mathbf{Z}_p, \quad \chi_\Gamma^{1/2} : G_{F,S} \twoheadrightarrow \Gamma \xrightarrow{\frac{1}{2}} \Gamma \hookrightarrow \Lambda^*$$

satisfying

$$\phi_W^{1/2} = \phi_{nL}^{1/2} \cdot \left(\varepsilon^{1/2} \circ \kappa \circ N\right),$$

$$\chi_\Gamma^{1/2} \pmod{P_{k,\varepsilon}} \xrightarrow{\sim} \left(\varepsilon^{1/2} \circ \kappa \circ N\right) \cdot \left(\kappa^{1/2} \circ N\right)^{k-2} = \left(\varepsilon^{1/2} \circ \kappa \circ N\right) \cdot (\kappa \circ N)^{k/2-1}.$$

Put

$$\chi_W : \mathbf{A}_F^*/F^* \twoheadrightarrow Z \twoheadrightarrow W \xrightarrow{(\phi_W^{1/2})^{-1}} \mathcal{O}^*,$$

$$\chi_{\mathrm{tame}} := \chi\chi_W^{-1} \quad (\Longrightarrow \chi_{\mathrm{tame}}^{-2} = \phi_{\mathrm{tame}} = \phi_0(\omega \circ N)^{2-k})$$

and define

$$\begin{aligned}\mathscr{V} &= V(\lambda)(1) \otimes \left(\chi_\Gamma^{1/2}\phi_{nL}^{1/2}\right)^{-1} \chi_{\mathrm{tame}}(\omega \circ N)^{1-k/2} \\ &= V(\lambda)(1) \otimes (\chi_\Gamma^{1/2})^{-1}(\varepsilon^{1/2} \circ \kappa \circ N)(\omega \circ N)^{1-k/2}\chi.\end{aligned}$$

This is again a two-dimensional representation of $G_{F,S}$ over $\mathrm{Frac}(R)$ (admissible as an $R[G_{F,S}]$-module) satisfying

$$\Lambda^2\mathscr{V} = \Lambda^2 V(\lambda)(2) \otimes \chi_\Gamma^{-1}\phi_{nL}^{-1}\phi_0^{-1} = \mathrm{Frac}(R)(1).$$

For each prime ideal $v \mid p$ in $\mathcal{O}_F$, there is an exact sequence of $\mathrm{Frac}(R)[G_v]$-modules

$$0 \longrightarrow \mathscr{V}_v^+ \longrightarrow \mathscr{V} \longrightarrow \mathscr{V}_v^- \longrightarrow 0,$$

where

$$\mathscr{V}_v^\pm = V(\lambda)_v^\pm(1) \otimes \left(\chi_\Gamma^{1/2}\phi_{nL}^{1/2}\right)^{-1}\chi_{\mathrm{tame}}(\omega \circ N)^{1-k/2}.$$

If we fix an isomorphism of $\mathrm{Frac}(R)[G_{F,S}]$-modules

$$\mathscr{V} \xrightarrow{\sim} \mathscr{V}^*(1) = \mathrm{Hom}_{\mathrm{Frac}(R)}(\mathscr{V}, \mathrm{Frac}(R))(1)$$

(it is unique up to a scalar in $\mathrm{Frac}(R)^*$), it induces isomorphisms of $\mathrm{Frac}(R)[G_v]$-modules

$$\mathscr{V}_v^\pm \xrightarrow{\sim} (\mathscr{V}_v^\mp)^*(1),$$

by the uniqueness of $V(\lambda)_v^+$.

The only $G_{F,S}$-invariant $(R_{\overline{\mathcal{P}}})$-lattices in $\mathscr{V}$ are $\overline{\mathcal{P}}^i T(\mathcal{P})$ $(i \in \mathbf{Z})$, where

$$T(\mathcal{P}) = \overline{T}(\mathcal{P})(1) \otimes (\chi_\Gamma^{1/2}\phi_{nL}^{1/2})^{-1}\chi_{\text{tame}}(\omega \circ N)^{1-k/2}.$$

The residual representation at $\overline{\mathcal{P}}$

$$T(\mathcal{P})/\overline{\mathcal{P}}T(\mathcal{P}) = (\overline{T}(\mathcal{P})/\overline{\mathcal{P}T}(\mathcal{P}))(1) \otimes \left(\chi_\Gamma^{1/2}\phi_{nL}^{1/2}\right)^{-1}\chi_{\text{tame}}(\omega \circ N)^{1-k/2}$$

is isomorphic to

$$V(f_{\mathcal{P}})(1) \otimes N^{1-k/2}\chi = V(f_{\mathcal{P}})(k/2) \otimes \chi = V(g_{\mathcal{P}})(k/2),$$

where the twisted cusp form

$$g_{\mathcal{P}} := f_{\mathcal{P}} \otimes \chi \in S_k(\mathfrak{n}(g_{\mathcal{P}}), 1)$$

has trivial central character. For each $v \mid p$, the $\kappa(\mathcal{P})[G_v]$-module

$$T(\mathcal{P})_v^{\pm}/\overline{\mathcal{P}}T(\mathcal{P})_v^{\pm}$$

is isomorphic to

$$V(g_{\mathcal{P}})(k/2)_v^{\pm} := V(f_{\mathcal{P}})_v^{\pm}(k/2) \otimes \chi_v.$$

Let $\mathcal{P}'$ be as in 12.7.7, with $k' \in 2\mathbf{Z}$. The only $G_{F,S}$-invariant $(R_{\overline{\mathcal{P}}'})$-lattices in $\mathscr{V}$ are $(\overline{\mathcal{P}}')^i T(\mathcal{P}')$ $(i \in \mathbf{Z})$, where

$$T(\mathcal{P}') = \overline{T}(\mathcal{P}')(1) \otimes \left(\chi_\Gamma^{1/2}\phi_{nL}^{1/2}\right)^{-1}\chi_{\text{tame}}(\omega \circ N)^{1-k/2}.$$

Let

$$(\varepsilon')^{1/2} : \Gamma \xrightarrow{\frac{1}{2}} \Gamma \xrightarrow{\varepsilon'} (\mathcal{O}')^*$$

be the canonical square root of ε'. The residual representation of $\mathscr{V}$ at $\overline{\mathcal{P}}'$ is isomorphic to

$$\begin{aligned} T(\mathcal{P}')/\overline{\mathcal{P}}'T(\mathcal{P}') \xrightarrow{\sim} V(f_{\mathcal{P}'})(1) \otimes \left(\frac{\varepsilon^{1/2}}{(\varepsilon')^{1/2}} \circ \kappa \circ N\right)(\omega \circ N)^{1-k/2}\chi(\kappa \circ N)^{1-k'/2} \\ = V(f_{\mathcal{P}'})(k'/2) \otimes \chi' = V(g_{\mathcal{P}'})(k'/2), \end{aligned}$$

where

$$\chi' = \chi \cdot (\omega \circ N)^{(k'-k)/2}\left(\frac{\varepsilon^{1/2}}{(\varepsilon')^{1/2}} \circ \kappa \circ N\right) : \mathbf{A}_F^*/F^* \longrightarrow \mathbf{C}^*$$

is a character satisfying

$$\begin{aligned} (\chi')^{-2} &= \phi(\omega \circ N)^{k-k'}\left(\frac{\varepsilon}{\varepsilon'} \circ \kappa \circ N\right) = \phi_{nL}\phi_{\text{tame}}(\omega \circ N)^{k-k'}(\varepsilon' \circ \kappa \circ N) \\ &= \phi_{nL}\phi'_{\text{tame}}(\varepsilon' \circ \kappa \circ N) = \phi_{\mathcal{P}'} =: \phi' \end{aligned}$$

and the twisted cusp form

$$g_{\mathcal{P}'} := f_{\mathcal{P}'} \otimes \chi' \in S_{k'}(\mathfrak{n}(g_{\mathcal{P}'}), 1)$$

has trivial central character. Similarly, for each $v \mid p$, the $\kappa(\mathcal{P}')[G_v]$-module

$$T(\mathcal{P}')_v^{\pm}/\overline{\mathcal{P}}'T(\mathcal{P}')_v^{\pm}$$

is isomorphic to

$$V(g_{\mathcal{P}'})(k'/2)_v^{\pm} := V(f_{\mathcal{P}'})_v^{\pm}(k'/2) \otimes \chi'_v.$$

For each archimedean prime $v|\infty$ of F, we have

$$(\chi'_v/\chi_v)(-1) = (-1)^{(k'-k)/2}.$$

If the character ε' has a sufficiently large order, then

$$(\forall v \mid p) \quad \mathrm{ord}_v(\mathrm{cond}(\chi')) >> 0,$$

hence

$$(\forall v \mid p) \quad (\chi'_v)^2(I_v) \neq 1,$$

which implies that

$$(\mathfrak{n}(g_{\mathcal{P}'})_{\mathrm{St}}, (p)) = (1).$$

12.7.11. Assume that 12.7.9.1 is satisfied and $p = 2$, hence $\Gamma \xrightarrow{\sim} \mathbf{Z}_2$. Multiplication by 2

$$\Gamma \xrightarrow{2} \Gamma$$

induces an injective homomorphism of $\mathcal{O}$-algebras

$$i : \Lambda = \mathcal{O}[\![\Gamma]\!] \longrightarrow \Lambda,$$

for which

$$\Lambda = i(\Lambda) + \chi_\Gamma(\gamma) i(\Lambda).$$

We replace the assumption 12.7.9.1 by

12.7.11.1. For $f = f_{\mathcal{P}}$ from 12.7.5, $k \in 2\mathbf{Z}$ and there exist (continuous) characters $\chi : \mathbf{A}_F^*/F^* \to \mathbf{C}^*$, $\widetilde{\varepsilon} : \Gamma \to \mathcal{O}^*$ such that $\phi := \phi_{\mathcal{P}} = \chi^{-2}$ and $\varepsilon = \widetilde{\varepsilon}^2$; fix such χ and $\widetilde{\varepsilon}$, as well as a finite set $S \supset S_0$ of primes of F such that χ is unramified outside S.

In order to define a square root "$\chi_\Gamma^{1/2}$", we must extend the scalars $R \subset \widetilde{R}$ so that "$\chi_\Gamma^{1/2}$"$(\gamma) \in \widetilde{R}$. We distinguish two cases.

(A) There exists $r \in \mathrm{Frac}(R)$ satisfying $r^2 = \chi_\Gamma(\gamma)$. We define $\widetilde{R} = R[r] = R + Rr \subset \mathrm{Frac}(R)$.

(B) There is no $r \in \mathrm{Frac}(R)$ satisfying $r^2 = \chi_\Gamma(\gamma)$. We define $\widetilde{R}$ to be the image of $R[X]$ in the field $\mathrm{Frac}(R)[X]/(X^2 - \chi_\Gamma(\gamma))$ and $r \in \widetilde{R}$ to be the image of X; then $\widetilde{R} = R + Rr$.

In both cases $\widetilde{R}$ is a local domain, finite over Λ, with $\mathrm{Frac}(\widetilde{R})/\mathrm{Frac}(R)$ an extension of degree one (resp., two) in the case (A) (resp., (B)). As $P_{k,\varepsilon}$ factorizes in $\widetilde{R}$ as

$$P_{k,\varepsilon} = (r - \widetilde{\varepsilon}(\gamma)\gamma^{k/2-1})(r + \widetilde{\varepsilon}(\gamma)\gamma^{k/2-1}),$$

there exists a unique prime ideal $\widetilde{\mathcal{P}} \in \mathrm{Spec}(\widetilde{R})$ above $\overline{\mathcal{P}} \in \mathrm{Spec}(R)$ such that

$$r - \widetilde{\varepsilon}(\gamma)\gamma^{k/2-1} \in \widetilde{\mathcal{P}}.$$

The base change to $\widetilde{R}$ of the tautological representation $\chi_\Gamma : G_{F,S} \twoheadrightarrow \Gamma \hookrightarrow \Lambda^*$ has a square root (depending on the choice of the square root $r \in \widetilde{R}$ of $\chi_\Gamma(\gamma)$)

$$\chi_\Gamma^{1/2} : G_{F,S} \twoheadrightarrow \Gamma \xrightarrow{\chi_\Gamma^{1/2}} \widetilde{R}^*$$

given by

$$\chi_\Gamma^{1/2}(\gamma^x) = r^x \qquad (x \in \mathbf{Z}_2).$$

Its reduction modulo $\widetilde{\mathcal{P}}$ is isomorphic to

$$\chi_\Gamma^{1/2} \pmod{\widetilde{\mathcal{P}}} \xrightarrow{\sim} (\widetilde{\varepsilon} \circ \kappa \circ N) \cdot (\kappa \circ N)^{k/2-1}.$$

As in the case $p \neq 2$, we define

$$\mathscr{V} = \big(V(\lambda) \otimes_{\mathrm{Frac}(R)} \mathrm{Frac}(\widetilde{R})\big)(1) \otimes \big(\chi_\Gamma^{1/2}\big)^{-1} (\widetilde{\varepsilon} \circ \kappa \circ N)(\omega \circ N)^{1-k/2} \chi.$$

This is a two-dimensional representation of $G_{F,S}$ over $\mathrm{Frac}(\widetilde{R})$ (admissible as an $\widetilde{R}[G_{F,S}]$-module) satisfying

$$\Lambda^2 \mathscr{V} = \mathrm{Frac}(\widetilde{R})(1)$$

and sitting in exact sequences of $\mathrm{Frac}(\widetilde{R})[G_v]$-modules

$$0 \longrightarrow \mathscr{V}_v^+ \longrightarrow \mathscr{V} \longrightarrow \mathscr{V}_v^- \longrightarrow 0 \qquad (v|p),$$

defined analogously as in the case $p \neq 2$. Put

$$T(\widetilde{\mathcal{P}}) := \big(\overline{T}(\mathcal{P}) \otimes_{R_{\overline{\mathcal{P}}}} \widetilde{R}_{\widetilde{\mathcal{P}}}\big)(1) \otimes \big(\chi_\Gamma^{1/2}\big)^{-1} (\widetilde{\varepsilon} \circ \kappa \circ N)(\omega \circ N)^{1-k/2} \chi;$$

this is a $G_{F,S}$-invariant $(\widetilde{R}_{\widetilde{\mathcal{P}}})$-lattice in $\mathscr{V}$, with $T(\widetilde{\mathcal{P}})/\widetilde{\mathcal{P}}T(\widetilde{\mathcal{P}})$ a two-dimensional representation of $G_{F,S}$ over $\kappa(\widetilde{\mathcal{P}}) = \widetilde{R}_{\widetilde{\mathcal{P}}}/\widetilde{\mathcal{P}}\widetilde{R}_{\widetilde{\mathcal{P}}}$ isomorphic to

$$V(f_{\mathcal{P}})(1) \otimes N^{1-k/2}\chi = V(f_{\mathcal{P}})(k/2) \otimes \chi = V(g_{\widetilde{\mathcal{P}}})(k/2),$$

where the twisted cusp form

$$g_{\widetilde{\mathcal{P}}} = f_{\mathcal{P}} \otimes \chi \in S_k(\mathfrak{n}(g_{\widetilde{\mathcal{P}}}), 1)$$

has trivial central character.

More generally, let $\mathcal{P}'$ be as in 12.7.7, and assume that the following condition is satisfied:

12.7.11.2. $k' \in 2\mathbf{Z}$ and there exists a character $\widetilde{\varepsilon}' : \Gamma \to (\mathcal{O}')^*$ such that $\varepsilon' = (\widetilde{\varepsilon}')^2$; fix such $\widetilde{\varepsilon}'$.

For each $a \in \mathbf{Z}/2\mathbf{Z}$ there exists a unique prime ideal $\widetilde{\mathcal{P}}' = \widetilde{\mathcal{P}}'_a \in \mathrm{Spec}(\widetilde{R})$ above $\overline{\mathcal{P}}' \in \mathrm{Spec}(R)$ such that

$$r - (-1)^a \widetilde{\varepsilon}'(\gamma)\gamma^{k'/2-1} \in \widetilde{\mathcal{P}}'.$$

The reduction of $\chi_\Gamma^{1/2}$ modulo $\widetilde{\mathcal{P}}'$ is isomorphic to

$$\chi_\Gamma^{1/2} \pmod{\widetilde{\mathcal{P}}'} \xrightarrow{\sim} (\widetilde{\varepsilon}' \circ \kappa \circ N) \cdot (\kappa \circ N)^{k'/2-1} \chi_{F_1/F}^a,$$

where

$$\chi_{F_1/F} : G_{F,S} \twoheadrightarrow \Gamma \twoheadrightarrow \Gamma/2\Gamma \xrightarrow{\sim} \{\pm 1\}$$

is the quadratic character associated to the first non-trivial layer

$$F \subset F_1 \subset F_2 \subset \cdots \subset F_\infty = F\mathbf{Q}_\infty$$

in the cyclotomic $\mathbf{Z}_2$-extension of F (for example, if $\sqrt{2} \notin F$, then $F_1 = F(\sqrt{2})$).

As before,

$$T(\widetilde{\mathcal{P}}') := \big(\overline{T}(\mathcal{P}') \otimes_{R_{\overline{\mathcal{P}}'}} \widetilde{R}_{\widetilde{\mathcal{P}}'}\big)(1) \otimes \big(\chi_\Gamma^{1/2}\big)^{-1} (\widetilde{\varepsilon} \circ \kappa \circ N) \cdot (\omega \circ N)^{1-k/2} \chi$$

is a $G_{F,S}$-invariant $(\widetilde{R}_{\widetilde{\mathcal{P}}'})$-lattice in $\mathscr{V}$, with $T(\widetilde{\mathcal{P}}')/\widetilde{\mathcal{P}}' T(\widetilde{\mathcal{P}}')$ isomorphic to the two-dimensional representation (over $\kappa(\widetilde{\mathcal{P}}')$)

$$\begin{aligned} V(f'_{\mathcal{P}})(1) \otimes \big(\tfrac{\widetilde{\varepsilon}}{\widetilde{\varepsilon}'} \circ \kappa \circ N\big)(\kappa \circ N)^{1-k'/2}(\omega \circ N)^{1-k/2} \chi^a_{F_1/F} \chi& \\ = V(f'_{\mathcal{P}})(k'/2) \otimes \chi' = V(g'_{\mathcal{P}})(k'/2),& \end{aligned}$$

where

$$\chi' = \chi'_a = \left(\frac{\widetilde{\varepsilon}}{\widetilde{\varepsilon}'} \circ \kappa \circ N\right)(\omega \circ N)^{(k'-k)/2} \chi^a_{F_1/F} \chi : \mathbf{A}_F^*/F^* \longrightarrow \mathbf{C}^*$$

is a character satisfying

$$(\chi')^{-2} = \phi_{\mathcal{P}'} =: \phi'$$

and the twisted cusp form

$$g_{\widetilde{\mathcal{P}}'} := f_{\mathcal{P}'} \otimes \chi' \in S_{k'}(\mathfrak{n}(g_{\widetilde{\mathcal{P}}'}), 1)$$

has trivial central character. The rest of the discussion from 12.7.10 has obvious analogues in the present context; in particular,

$$(\forall v | \infty) \quad (\chi'_v/\chi_v)(-1) = (-1)^{(k'-k)/2}.$$

12.7.12. We are almost ready to apply the results of 10.7.7–10.7.10 to Selmer complexes arising in the present situation under the assumption 12.7.9.1 (resp., 12.7.11.1–12.7.11.2) if $p \neq 2$ (resp., if $p = 2$). In order to simplify the notation, we shall consider in Sect. 12.7.12–12.7.15 only the case $p \neq 2$. For $p = 2$, one has to replace $(R, \mathcal{P}, \mathcal{P}')$ by $\big(\widetilde{R}, \widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}'\big)$. Even though the condition (P) from 5.1 is not satisfied, we shall consider in 12.7.13, 12.7.15 only cohomology with coefficients in $\mathbf{Q}_2$-vector spaces, which means that the contribution from the real places vanishes (see the discussion in 6.9).

Fix a prime element $\varpi_{\mathcal{P}}$ (resp., $\varpi_{\mathcal{P}'}$) of $R_{\overline{\mathcal{P}}}$ (resp., of $R_{\overline{\mathcal{P}}'}$) and denote the fraction field of R by $\mathscr{L} = \mathrm{Frac}(R)$. Fix an isomorphism of $\mathscr{L}[G_{F,S}]$-modules

(12.7.12.1) $$\Lambda^2 \mathscr{V} \overset{\sim}{\longrightarrow} \mathscr{L}(1)$$

(it is unique up to an element of $\mathscr{L}^*$); under the induced isomorphism

$$\mathscr{V} \overset{\sim}{\longrightarrow} \mathscr{V}^*(1) = \mathrm{Hom}_{\mathscr{L}}(\mathscr{V}, \mathscr{L})(1),$$

the $G_{F,S}$-invariant lattices $T(\mathcal{P}) \subset \mathscr{V}$ and

$$T(\mathcal{P})^*(1) = \mathrm{Hom}_{R_{\overline{\mathcal{P}}}}(T(\mathcal{P}), R_{\overline{\mathcal{P}}})(1) \subset \mathscr{V}^*(1)$$

differ by $\varpi_{\mathcal{P}}^i$, for some $i \in \mathbf{Z}$. Multiplying (12.7.12.1) by a suitable scalar in $\mathscr{L}^*$, we obtain a skew-symmetric bilinear form

$$\pi : T(\mathcal{P}) \otimes_{R_{\overline{\mathcal{P}}}} T(\mathcal{P}) \longrightarrow R_{\overline{\mathcal{P}}}(1) \tag{12.7.12.2}$$

such that

$$\mathrm{adj}(\pi) : T(\mathcal{P}) \xrightarrow{\sim} T(\mathcal{P})^*(1)$$

is an isomorphism. The form π being skew-symmetric (hence alternating, since $\mathscr{L}$ has characteristic zero), the map $\mathrm{adj}(\pi)$ induces, for each $v \mid p$, a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(\mathcal{P})_v^+ & \longrightarrow & T(\mathcal{P}) & \longrightarrow & T(\mathcal{P})_v^- & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \wr & & \downarrow \beta & & \\ 0 & \longrightarrow & (T(\mathcal{P})_v^-)^*(1) & \longrightarrow & T(\mathcal{P})^*(1) & \longrightarrow & (T(\mathcal{P})_v^+)^*(1) & \longrightarrow & 0. \end{array}$$

As $\mathrm{Ker}(\alpha) = \mathrm{Coker}(\beta) = 0$ and each $T(\mathcal{P})_v^{\pm}$ is free of rank one over $R_{\overline{\mathcal{P}}}$, both $\mathrm{Ker}(\beta) = \mathrm{Coker}(\alpha) \subset T(\mathcal{P})_v^-$ must vanish, hence $\mathrm{adj}(\pi)$ induces isomorphisms

$$(\forall v|p) \quad T(\mathcal{P})_v^{\pm} \xrightarrow{\sim} (T(\mathcal{P})_v^{\mp})^*(1).$$

In other words, if we replace $R_{\overline{\mathcal{P}}}$ on the R.H.S. of (12.7.12.2) by its canonical injective resolution

$$[\mathscr{L} \longrightarrow \mathscr{L}/R_{\overline{\mathcal{P}}}],$$

then

$$(\forall v|p) \quad T(\mathcal{P})_v^+ \perp\!\!\perp_\pi T(\mathcal{P})_v^+.$$

12.7.13. Selmer complexes in Hida families

12.7.13.1. As in 12.5.9.1, fix a subset $\Sigma \subset S_f$ containing all primes dividing p and set $\Sigma' = S_f - \Sigma$. In order to simplify the notation, we denote

$$V = T(\mathcal{P})/\overline{\mathcal{P}}T(\mathcal{P}) = V(g_{\mathcal{P}})(k/2), \quad V' = T(\mathcal{P}')/\overline{\mathcal{P}}'T(\mathcal{P}') = V(g_{\mathcal{P}'})(k'/2).$$

Each of the $R[G_{F,S}]$-modules $X = T(\mathcal{P}), T(\mathcal{P}'), V, V', \mathscr{V}$ is admissible. More precisely, X is an $R_X[G_{F,S}]$-module, where $R_X = R_{\overline{\mathcal{P}}}, R_{\overline{\mathcal{P}'}}, \kappa(\mathcal{P}), \kappa(\mathcal{P}'), \mathscr{L}$, respectively.

We have morphisms of $R_X[G_{F,S}]$-modules

$$\pi_X : X \otimes_{R_X} X \longrightarrow R_X(1),$$

which are equal, respectively, to: π, π', the reduction of π modulo $\overline{\mathcal{P}}$, the reduction of π' modulo $\overline{\mathcal{P}}'$ and the pairing induced by (12.7.12.1) ($= \pi \otimes 1$).

Define, for each X, Greenberg's local conditions by the following formulas ($\forall v \in \Sigma$):

$$X_v^+ = \begin{cases} X_v^+, & v \mid p, \\ 0, & v \nmid p, \end{cases} \quad X_v^- = \begin{cases} X_v^-, & v \mid p, \\ X, & v \nmid p. \end{cases} \tag{12.7.13.1}$$

They satisfy, for each $v \in \Sigma$,

$$(12.7.13.2) \qquad T(\mathcal{P})_v^{\pm}/\overline{\mathcal{P}}T(\mathcal{P})_v^{\pm} = V_v^{\pm}, \quad T(\mathcal{P}')_v^{\pm}/\overline{\mathcal{P}}'T(\mathcal{P}')_v^{\pm} = (V')_v^{\pm},$$
$$T(\mathcal{P})_v^{\pm} \otimes_{R_{\overline{\mathcal{P}}}} \mathscr{L} = T(\mathcal{P}')_v^{\pm} \otimes_{R_{\overline{\mathcal{P}}'}} \mathscr{L} = \mathscr{V}_v^{\pm}$$

and

$$(12.7.13.3) \qquad \begin{cases} X_v^+ \perp_{\pi_X} X_v^+ & v \nmid p \\ X_v^+ \perp\!\!\perp_{\pi_X} X_v^+ & v \mid p. \end{cases}$$

12.7.13.2. Let E/F be a finite subextension of F_S/F and $\beta : G_{E,S} = \mathrm{Gal}(F_S/E) \to \mathscr{O}^*$ a (continuous) character of finite order. For each of the $R[G_{F,S}]$-modules $X = T(\mathcal{P}), T(\mathcal{P}'), V, V', \mathscr{V}$ we define Greenberg's local conditions $\Delta_\Sigma(X \otimes \beta)$ for the admissible $R[G_{E,S}]$-module $X \otimes \beta$ as in 12.5.9.1: if w is a prime of E above $v \in S_f$, set

$$U_w^+(X \otimes \beta) = \begin{cases} C^\bullet_{\mathrm{cont}}(G_w, X_v^+ \otimes \beta_w), & v \mid p \\ 0, & v \in \Sigma,\ v \nmid p \\ C^\bullet_{\mathrm{cont}}(G_w/I_w, (X \otimes \beta_w)^{I_w}), & v \in \Sigma' \end{cases}$$

(above, $G_w = \mathrm{Gal}(\overline{F}_v/E_w)$). For each X, the map π_X induces a morphism of $R_X[G_{E,S}]$-modules

$$\pi_{X,\beta} : (X \otimes \beta) \otimes_{R_X} (X \otimes \beta^{-1}) \longrightarrow R_X(1)$$

satisfying, for each $v \in \Sigma$,

$$\begin{cases} (X \otimes \beta)_v^+ \perp_{\pi_{X,\beta}} (X \otimes \beta^{-1})_v^+ & v \nmid p \\ (X \otimes \beta)_v^+ \perp\!\!\perp_{\pi_{X,\beta}} (X \otimes \beta^{-1})_v^+ & v \mid p. \end{cases}$$

12.7.13.3. Proposition. — *Under the assumptions of 12.7.13.1–12.7.13.2, let* $X \in \{T(\mathcal{P}), T(\mathcal{P}'), V, V', \mathscr{V}\}$.

(i) *Let $w \nmid p\infty$ be a prime of E; then the complexes $C^\bullet_{\mathrm{cont}}(G_w, X \otimes \beta_w)$ and $C^\bullet_{\mathrm{cont}}(G_w/I_w, (X \otimes \beta_w)^{I_w})$ are acyclic.*

(ii) *Up to a canonical isomorphism, the Selmer complex $\widetilde{\mathbf{R}\Gamma}_f(G_{E,S}, X \otimes \beta; \Delta_\Sigma(X \otimes \beta)) \in D^b_{ft}({}_{R_X}\mathrm{Mod})$ does not depend on the choice of S and Σ; we denote it by $\widetilde{\mathbf{R}\Gamma}_f(E, X \otimes \beta)$ and its cohomology groups by $\widetilde{H}^i_f(E, X \otimes \beta)$.*

(iii) *If $X \in \{V, V'\}$, then there is an exact sequence*

$$0 \longrightarrow \bigoplus_{v \mid p} \bigoplus_{w \mid v} H^0(E_w, X_v^- \otimes \beta_w) \longrightarrow \widetilde{H}^1_f(E, X \otimes \beta) \longrightarrow H^1_f(E, X \otimes \beta) \longrightarrow 0.$$

Proof

(i) For $X = V, V'$, the statement was proved in Proposition 12.5.9.2(i). As in the proof of *loc. cit.*, it is enough to show that $\mathbf{R}\Gamma_{\mathrm{cont}}(G_w, X \otimes \beta_w)$ is acyclic, for

$X = T(\mathcal{P}), T(\mathcal{P}'), \mathscr{V}$. This follows from Lemma 12.7.15.6 below applied to the exact triangle

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G_w, T(\mathcal{P}) \otimes \beta_w) \xrightarrow{\varpi_{\mathcal{P}}} \mathbf{R}\Gamma_{\mathrm{cont}}(G_w, T(\mathcal{P}) \otimes \beta_w) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G_w, V)$$

and its analogue for $T(\mathcal{P}')$, together with the isomorphism

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G_w, T(\mathcal{P}) \otimes \beta_w) \otimes_{R_{\overline{\mathcal{P}}}} \mathscr{L} \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cont}}(G_w, \mathscr{V} \otimes \beta_w).$$

(ii) The proof of 12.5.9.2 (ii) applies.

(iii) This was proved in 12.5.9.2 (iii). □

12.7.13.4. Proposition. — *Under the assumptions of 12.7.13.1–12.7.13.2,*

(i) *There is an exact triangle in* $D^b_{ft}({}_{R_{\overline{\mathcal{P}}}}\mathrm{Mod})$

$$\widetilde{\mathbf{R}\Gamma}_f(E, T(\mathcal{P}) \otimes \beta) \xrightarrow{\varpi_{\mathcal{P}}} \widetilde{\mathbf{R}\Gamma}_f(E, T(\mathcal{P}) \otimes \beta) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(E, V \otimes \beta)$$

inducing short exact sequences

$$\begin{aligned} 0 \longrightarrow \widetilde{H}^j_f(E, T(\mathcal{P}) \otimes \beta)/\varpi_{\mathcal{P}} \widetilde{H}^j_f(E, T(\mathcal{P}) \otimes \beta) &\longrightarrow \widetilde{H}^j_f(E, V \otimes \beta) \\ &\longrightarrow \widetilde{H}^{j+1}_f(E, T(\mathcal{P}) \otimes \beta)[\varpi_{\mathcal{P}}] \longrightarrow 0 \end{aligned}$$

(and similarly for $T(\mathcal{P}')$*).*

(ii) *There are canonical isomorphisms in* $D^b_{ft}({}_{\mathscr{L}}\mathrm{Mod})$

$$\widetilde{\mathbf{R}\Gamma}_f(E, T(\mathcal{P}) \otimes \beta) \otimes_{R_{\overline{\mathcal{P}}}} \mathscr{L} \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_f(E, \mathscr{V} \otimes \beta) \xleftarrow{\sim} \widetilde{\mathbf{R}\Gamma}_f(E, T(\mathcal{P}') \otimes \beta) \otimes_{R_{\overline{\mathcal{P}}'}} \mathscr{L}.$$

(iii) $\widetilde{H}^1_f(E, T(\mathcal{P}) \otimes \beta)$ *(resp.,* $\widetilde{H}^1_f(E, T(\mathcal{P}') \otimes \beta)$*) is a free* $(R_{\overline{\mathcal{P}}})$*-module (resp., a free* $(R_{\overline{\mathcal{P}}'})$*-module) of rank*

$$\widetilde{h}^1_f(E, \mathscr{V} \otimes \beta) := \dim_{\mathscr{L}} \widetilde{H}^1_f(E, \mathscr{V} \otimes \beta).$$

(iv) *The duality morphism in* $D^b_{ft}({}_{R_{\overline{\mathcal{P}}}}\mathrm{Mod})$

$$\mathrm{adj}(\pi_{T(\mathcal{P}),\beta}) : \widetilde{\mathbf{R}\Gamma}_f(E, T(\mathcal{P}) \otimes \beta) \longrightarrow \mathbf{R}\mathrm{Hom}_{R_{\overline{\mathcal{P}}}}\big(\widetilde{\mathbf{R}\Gamma}_f(E, T(\mathcal{P}) \otimes \beta^{-1}), R_{\overline{\mathcal{P}}}\big)[-3]$$

is an isomorphism, which induces a non-degenerate $(R_{\overline{\mathcal{P}}})$*-bilinear form*

$$\langle\ ,\ \rangle : \widetilde{H}^2_f(E, T(\mathcal{P}) \otimes \beta)_{\mathrm{tors}} \times \widetilde{H}^2_f(E, T(\mathcal{P}) \otimes \beta^{-1})_{\mathrm{tors}} \longrightarrow \mathrm{Frac}(R_{\overline{\mathcal{P}}})/R_{\overline{\mathcal{P}}} = \mathscr{L}/R_{\overline{\mathcal{P}}},$$

*where the subscript "*tors*" refers to the* $(R_{\overline{\mathcal{P}}})$*-torsion. If* $\beta = \beta^{-1}$*, then this form is skew-symmetric. Similar statements hold for* $\mathcal{P}'$*.*

(v) *Fix a* G_F*-stable* R*-lattice* $\mathcal{T} \subset \mathscr{V}$ *(*i.e.*, an* R*-submodule of finite type satisfying* $\mathscr{L}\mathcal{T} = \mathscr{V}$*). Then, for each prime* $w \nmid p\infty$ *of* E*,*

$$\mathrm{Tam}_w(\mathcal{T} \otimes \beta^{\pm 1}, \overline{\mathcal{P}}) = \mathrm{Tam}_w(\mathcal{T} \otimes \beta^{\pm 1}, \overline{\mathcal{P}}') = 0.$$

Proof. — We are going to use the independence of $\widetilde{\mathbf{R}\Gamma}_f(E, X \otimes \beta)$ on Σ proved in Proposition 12.7.13.3 (ii). Throughout the proof of (i)–(iv) we let $\Sigma = S_f$.

(i) The statement follows from the exact sequences

$$0 \longrightarrow C^{\bullet}_{\mathrm{cont}}(G, T(\mathcal{P}) \otimes \beta) \xrightarrow{\varpi_{\mathcal{P}}} C^{\bullet}_{\mathrm{cont}}(G, T(\mathcal{P}) \otimes \beta) \longrightarrow C^{\bullet}_{\mathrm{cont}}(G, V \otimes \beta) \longrightarrow 0$$

(for $G = G_{E,S}, G_w$) and

$$0 \longrightarrow C^{\bullet}_{\mathrm{cont}}(G_w, T(\mathcal{P})^+_v \otimes \beta) \xrightarrow{\varpi_{\mathcal{P}}} C^{\bullet}_{\mathrm{cont}}(G_w, T(\mathcal{P})^+_v \otimes \beta) \longrightarrow C^{\bullet}_{\mathrm{cont}}(G_w, V^+_v \otimes \beta) \longrightarrow 0$$

(cf. (12.7.13.2)), which are compatible with respect to the 'restriction' maps res_w.

(ii) This follows from the functorial isomorphisms of complexes (cf. Proposition 3.4.4)

$$C^{\bullet}_{\mathrm{cont}}(G, M) \otimes_{R_{\overline{\mathcal{P}}}} \mathscr{L} \xrightarrow{\sim} C^{\bullet}_{\mathrm{cont}}(G, M \otimes_{R_{\overline{\mathcal{P}}}} \mathscr{L})$$

for $(G, M) = (G_{E,S}, T(\mathcal{P}) \otimes \beta), (G_w, T(\mathcal{P})^+_v \otimes \beta)$.

(iii) Taking into account (i) and (ii), it is sufficient to show that

$$\widetilde{H}^1_f(E, T(\mathcal{P}) \otimes \beta)[\varpi_{\mathcal{P}}] = 0,$$

which follows from (i) and the vanishing of

$$\widetilde{H}^0_f(E, V \otimes \beta) \subseteq H^0(G_{E,S}, V \otimes \beta) = 0,$$

proved in Proposition 12.4.8.4 (and similarly for $\mathcal{P}'$).

(iv) We apply the localized version of the duality Theorem 6.3.4. For $w|v|p$ (resp., for $w|v$, $v \in S_f, v \nmid p$) the error term $\mathrm{Err}_w(T(\mathcal{P}) \otimes \beta, T(\mathcal{P}) \otimes \beta^{-1}, \pi_{T(\mathcal{P}),\beta})$ vanishes in $D^b_{ft}({}_{R_{\overline{\mathcal{P}}}}\mathrm{Mod})$ by Proposition 6.7.6 (ii) (resp., by Proposition 6.7.6 (iv) combined with Proposition 12.7.13.3 (i). This implies that the map $\mathrm{adj}(\pi_{T(\mathcal{P}),\beta})$ is, indeed, an isomorphism. The induced generalized Cassels-Tate pairing on the torsion submodules of $\widetilde{H}^2_f(-)$ is non-degenerate, by Theorem 10.4.4, and skew-symmetric (in the case $\beta = \beta^{-1}$), by Proposition 10.2.5 (as the pairing $\pi_{T(\mathcal{P}),\beta}$ is skew-symmetric).

(v) Let v be the prime of F below w. By 7.6.10.4, it is enough to treat the case $v \in S_f$. Consider the duality morphisms

$$\gamma^{\pm}_{\Sigma} : \widetilde{\mathbf{R}\Gamma}_f(G_{E,S}, T(\mathcal{P}) \otimes \beta^{\pm 1}; \Delta_{\Sigma}) \longrightarrow \mathbf{R}\mathrm{Hom}_{R_{\overline{\mathcal{P}}}}\big(\widetilde{\mathbf{R}\Gamma}_f(G_{E,S}, T(\mathcal{P}) \otimes \beta^{\mp 1}; \Delta_{\Sigma})), R_{\overline{\mathcal{P}}}\big)[-3]$$

for arbitrary $\{v \mid p\} \subset \Sigma \subset S_f$. As observed in the proof of (iv), the map $\gamma^{\pm}_{S_f}$ is an isomorphism. On the other hand, $\mathrm{Cone}\big(\gamma^{\pm}_{\{v|p\}}\big)$ is isomorphic to

$$\bigoplus_{\substack{v \in S_f \\ v \nmid p}} \bigoplus_{w|v} \mathrm{Err}^{\mathrm{ur}}_w(T(\mathcal{P}) \otimes \beta^{\pm 1}, T(\mathcal{P}) \otimes \beta^{\mp 1}, \pi_{T(\mathcal{P}),\beta^{\pm 1}}) = \bigoplus_{\substack{v \in S_f \\ v \nmid p}} \bigoplus_{w|v} \mathrm{Err}_w.$$

The independence of $\widetilde{\mathbf{R}\Gamma}_f(G_{E,S}, T(\mathcal{P}) \otimes \beta^{\pm 1}; \Delta_{\Sigma})$ on Σ, proved in Proposition 12.7.13.3 (ii), implies that each term Err_w is acyclic, hence

$$\mathrm{Tam}_w(\mathcal{T} \otimes \beta^{\pm 1}, \overline{\mathcal{P}}) = \ell_{R_{\mathcal{P}}}(H^1(\mathrm{Err}_w)) = 0$$

(by 7.6.10.7). The same proof applies to $\overline{\mathcal{P}}'$. □

12.7.13.5. Corollary. — *In the situation of 12.7.13.1–12.7.13.2, assume that $\beta = \beta^{-1}$. Then:*

(i) *There exist canonical decreasing filtrations*

$$\widetilde{H}^1_f(E, V \otimes \beta) = F^1 \supseteq F^2 \supseteq \cdots, \quad \widetilde{H}^1_f(E, V' \otimes \beta) = {}'F^1 \supseteq {}'F^2 \supseteq \cdots$$

by $\kappa(\mathcal{P})$-subspaces (resp., $\kappa(\mathcal{P}')$-subspaces) such that

$$F^\infty := \bigcap_{j\geqslant 1} F^j = \mathrm{Im}\big(\widetilde{H}^1_f(E, T(\mathcal{P}) \otimes \beta)/\varpi_{\mathcal{P}} \hookrightarrow \widetilde{H}^1_f(E, V \otimes \beta)\big)$$

$${}'F^\infty := \bigcap_{j\geqslant 1} {}'F^j = \mathrm{Im}\big(\widetilde{H}^1_f(E, T(\mathcal{P}') \otimes \beta)/\varpi_{\mathcal{P}'} \hookrightarrow \widetilde{H}^1_f(E, V' \otimes \beta)\big)$$

and

$$\dim_{\kappa(\mathcal{P})} F^\infty = \dim_{\kappa(\mathcal{P}')} {}'F^\infty = \widetilde{h}^1_f(E, \mathscr{V} \otimes \beta).$$

(ii) *For each $j \geqslant 1$ there exist symplectic (= alternating and non-degenerate) pairings*

$$\langle\ ,\ \rangle_{j,\pi,\varpi_{\mathcal{P}}} : \mathrm{gr}^j_F \times \mathrm{gr}^j_F \longrightarrow \kappa(\mathcal{P}),$$

$$\langle\ ,\ \rangle_{j,\pi',\varpi_{\mathcal{P}'}} : \mathrm{gr}^j_{'F} \times \mathrm{gr}^j_{'F} \longrightarrow \kappa(\mathcal{P}')$$

on $\mathrm{gr}^j_F = F^j/F^{j+1}$ and $\mathrm{gr}^j_{'F} = {}'F^j/{}'F^{j+1}$.

(iii) *For each $j \geqslant 1$ we have*

$$\dim_{\kappa(\mathcal{P})} F^j \equiv \dim_{\kappa(\mathcal{P}')} {}'F^j \equiv \widetilde{h}^1_f(E, \mathscr{V} \otimes \beta) \pmod 2.$$

In particular,

$$\widetilde{h}^1_f(E, V \otimes \beta) = \dim_{\kappa(\mathcal{P})} F^1 \equiv \widetilde{h}^1_f(E, \mathscr{V} \otimes \beta) \equiv \dim_{\kappa(\mathcal{P}')} {}'F^1 = \widetilde{h}^1_f(E, V' \otimes \beta) \pmod 2.$$

Proof. — Apply Lemma 10.6.5 (with $\iota = \mathrm{id}$ and $\varepsilon = 1$) to the symplectic forms from Proposition 12.7.13.4 (iv) (as the fields $\kappa(\mathcal{P}), \kappa(\mathcal{P}')$ are of characteristic zero, there is no difference between "skew-symmetric" and "alternating"). □

12.7.13.6. The subspace $F^\infty \subsetneq F^1 = \widetilde{H}^1_f(E, V \otimes \beta)$ can be interpreted as the "generic subspace" of F^1, corresponding to those cohomology classes that can be lifted to (the twist by β of) the whole Hida family λ. The last statement of Corollary 12.7.13.5 (iii) then says that the parity of the dimensions of $\widetilde{H}^1_f(E, V \otimes \beta)$ is constant in the whole Hida family, and coincides with the parity of the rank of the common "generic subspace", generalizing Theorem A' from [**N-P**].

The dependence of the pairings from 12.7.13.5 (ii) on $\varpi_{\mathcal{P}}$ (resp., on $\varpi_{\mathcal{P}'}$) is given by the formula from Proposition 10.7.9 (ii). For example, if $\kappa(\mathcal{P}') = \mathrm{Frac}(\mathcal{O})$ (*e.g.* if $\mathcal{P}' = \mathcal{P}$), then we can take $\varpi_{\mathcal{P}'} = P_{k',\varepsilon'}$. This choice is not completely canonical, as it depends on a fixed topological generator $\gamma \in \Gamma$. If we replace γ by $\gamma^{\mathrm{new}} = \gamma^c$ ($c \in \mathbf{Z}_p^*$), then

$$\frac{\varpi_{\mathcal{P}'}^{\mathrm{new}}}{\varpi_{\mathcal{P}'}} = \frac{\chi_\Gamma(\gamma)^c - (\varepsilon'(\gamma)\gamma^{k'-2})^c}{\chi_\Gamma(\gamma) - \varepsilon'(\gamma)\gamma^{k'-2}} \equiv c\,(\varepsilon'(\gamma)\gamma^{k'-2})^{c-1} \pmod{\varpi_{\mathcal{P}'}},$$

hence

$$\langle\ ,\ \rangle_{j,\pi',\varpi_{\mathcal{P}'}^{\mathrm{new}}} = u^{2-j}\,\langle\ ,\ \rangle_{j,\pi',\varpi_{\mathcal{P}'}}, \quad u = c\,(\varepsilon'(\gamma)\gamma^{k'-2})^{c-1} \in \mathbf{Z}_p^*.$$

There is another ambiguity in the construction of $\langle\ ,\ \rangle_{j,\pi',\varpi_{\mathcal{P}'}}$, namely the choice of an isomorphism $\mathscr{V} \xrightarrow{\sim} \mathscr{V}^*(1)$ compatible with the chosen $G_{F,S}$-invariant lattice $T(\mathcal{P}') \subset V$.

If the degree $[F:\mathbf{Q}]$ is *odd*, then this ambiguity can be avoided as follows: the representation $V(\lambda)$ can be constructed geometrically, using the Tate modules of Shimura curves associated to a quaternion algebra over F which is ramified at all but one archimedean primes of F (and no finite primes). Such a geometric construction yields both a *canonical* $G_{F,S}$-invariant R-lattice $\overline{T} \subset V(\lambda)$ and a *canonical* isomorphism

$$\Lambda^2 V(\lambda) = \mathrm{Frac}(R)(-1) \otimes \chi_\Gamma \phi_0 \phi_{nL},$$

well-behaved with respect to $\overline{T}$. In [**N-P**], this geometric construction was summarized in the classical case $F = \mathbf{Q}$ (we would like to use this opportunity to remark that some of the results from [**N-P**, § 1.6] had earlier been obtained by Ohta [**Oh3**]). As we now work consistently with $(R_{\overline{\mathcal{P}}})$-modules, the auxiliary assumptions used in [**N-P**], such as $p > 3$ and irreducibility of certain residual representations, are no longer necessary.

12.7.13.7. If E is an elliptic curve over $\mathbf{Q}$ with good ordinary reduction at p, then $L(E/\mathbf{Q}, s) = L(f, s)$ for a p-ordinary eigenform $f = \sum_{n \geqslant 1} a_n q^n \in S_2(\Gamma_0(N), 1)$ with integral coefficients. Applying Corollary 12.7.13.5 to the p-stabilization of f (over $\mathbf{Q}$, and to $\beta = 1$), we obtain the result stated in 0.15.2.

***12.7.13.8. Proposition* (Dihedral case).** — *In the situation of 12.7.13.1–12.7.13.2, assume that* $[E:F] = 2$ *and* $\beta \circ \tau = \beta^{-1}$, *where* τ *is the non-trivial element of* $\mathrm{Gal}(E/F)$. *Set* $E_\beta = \overline{E}^{\mathrm{Ker}(\beta)}$ *and fix a lift of* τ *to* $\mathrm{Gal}(E_\beta/F)$ *(which will also be denoted by* τ*). Then:*

(i) $\mathrm{Ad}_f(\tau)$ *induces an isomorphism*

$$\widetilde{H}^2_f(E, T(\mathcal{P}) \otimes \beta) = \widetilde{H}^2_f(E_\beta, T(\mathcal{P}))^{(\beta^{-1})} \xrightarrow{\sim} \widetilde{H}^2_f(E_\beta, T(\mathcal{P}))^{(\beta)} = \widetilde{H}^2_f(E, T(\mathcal{P}) \otimes \beta^{-1})$$

such that the induced $(R_{\overline{\mathcal{P}}})$*-bilinear form* $[x,y] = \langle x, \mathrm{Ad}_f(\tau) y\rangle$

$$[\ ,\ ] : \widetilde{H}^2_f(E, T(\mathcal{P}) \otimes \beta)_{\mathrm{tors}} \times \widetilde{H}^2_f(E, T(\mathcal{P}) \otimes \beta)_{\mathrm{tors}} \longrightarrow \mathrm{Frac}(R_{\overline{\mathcal{P}}})/R_{\overline{\mathcal{P}}} = \mathscr{L}/R_{\overline{\mathcal{P}}}$$

is non-degenerate and skew-symmetric.

(ii) *All conclusions of Corollary 12.7.13.5 hold.*

Proof

(i) This follows from Proposition 12.5.9.2 (iv) and (10.3.2.2) (taking $\Sigma = S_f$ and $\Sigma' = \varnothing$, in order to ensure that all primes in Σ' are unramified in E/F).

(ii) The proof of Corollary 12.7.13.5 applies to [,]. □

12.7.13.9. The statements of Proposition 12.7.13.4, 12.7.13.8 also hold, with minor modifications, if $\beta : G_{E,S} \to \mathcal{O}(\beta)^*$ is a character of finite order with values in a discrete valuation ring $\mathcal{O}(\beta)$, finite over $\mathcal{O}$. As $R/\overline{\mathcal{P}} = \mathcal{O}$, the field $\mathrm{Frac}(\mathcal{O})$ is algebraically closed in $\mathrm{Frac}(R) = \mathscr{L}$, which implies that the ring $R(\beta) := R \otimes_{\mathcal{O}} \mathcal{O}(\beta)$ is an integral domain. Fix $\overline{\mathcal{P}}'(\beta) \in \mathrm{Spec}(R(\beta))$ above $\overline{\mathcal{P}}' \in \mathrm{Spec}(R)$; it follows from [**Mat**, Thm. 23.7 (ii)] that $R(\beta)_{\overline{\mathcal{P}}'(\beta)}$ is a discrete valuation ring, finite over $R_{\overline{\mathcal{P}}'}$. We define an admissible $R(\beta)_{\overline{\mathcal{P}}'(\beta)}[G_{F,S}]$-module

$$T(\mathcal{P}') \otimes \beta := T(\mathcal{P}') \otimes_{R_{\overline{\mathcal{P}}'}} R(\beta)_{\overline{\mathcal{P}}'(\beta)},$$

with $g \in G_{F,S}$ acting as $g \otimes \beta(g)$ (and similarly for $(T(\mathcal{P}') \otimes \beta)^+_v := T(\mathcal{P}')^+_v \otimes \beta$, for $v \mid p$). The Galois representation

$$(T(\mathcal{P}') \otimes \beta)/\overline{\mathcal{P}}'(\beta)(T(\mathcal{P}') \otimes \beta)$$

resp.,

$$(T(\mathcal{P}')^+_v \otimes \beta)/\overline{\mathcal{P}}'(\beta)(T(\mathcal{P}')^+_v \otimes \beta) \qquad (v \mid p)$$

is isomorphic to $V' \otimes \beta$ (resp., $(V')^+_v \otimes \beta$), with coefficients in $\kappa(\overline{\mathcal{P}}'(\beta))$. The discussion in 12.7.13.2–12.7.13.8 goes through, provided that one replaces R by $R(\beta)$ and $\overline{\mathcal{P}}'$ by $\overline{\mathcal{P}}'(\beta)$.

12.7.14. ε-factors in Hida families. — We continue to use the notation of 12.7.13.

12.7.14.1. Proposition. — *Let $v \nmid p\infty$ be a prime of F. Assume that $\pi(g_{\mathcal{P}'})_v = \mathrm{St}(\mu)$ ($\mu^2 = 1$). Then:*

(i) *There is an exact sequence of $\mathscr{L}[G_v]$-modules*

$$0 \longrightarrow \mathscr{L}(1) \otimes \mu \longrightarrow \mathscr{V}_v \longrightarrow \mathscr{L} \longrightarrow 0.$$

(ii) *For each arithmetic point $\mathcal{P}'' \in \mathrm{Spec}(h^{\mathrm{ord}}_{\phi_n L}(\mathfrak{n}; \mathcal{O}))$ containing $\mathcal{P}_{\min}$ there is an exact sequence of $\kappa(\mathcal{P}'')$-modules*

$$0 \longrightarrow \kappa(\mathcal{P}'')(1) \otimes \mu \longrightarrow V''_v \longrightarrow \kappa(\mathcal{P}'') \longrightarrow 0$$

(in which $V'' = V(g_{\mathcal{P}''})(k''/2)$) and $\pi(g_{\mathcal{P}''})_v = \mathrm{St}(\mu) = \pi(g_{\mathcal{P}'})_v$.

Proof. — After multiplying χ by a suitable character $\mathbf{A}_F^*/F^* \to \{\pm 1\}$, we can assume that μ is unramified at v. In this case the wild inertia group I_v^w acts through a finite quotient on $T(\mathcal{P}')$ and trivially on $V' = V(g_{\mathcal{P}'})(k'/2) = T(\mathcal{P}')/\overline{\mathcal{P}}' T(\mathcal{P}')$ (cf. 12.4.4.2); it follows that I_v^w acts trivially on $T(\mathcal{P}')$ (hence on $\mathscr{V}$).

Fix a topological generator $t \in I_v^t = I_v/I_v^w$ of the tame inertia group and a lift $f \in G_v$ of the geometric Frobenius element. The relation $tf = ft^{Nv}$ implies that the set of eigenvalues of t acting on $\mathscr{V}$ is stable under the map $\lambda \mapsto \lambda^{Nv}$, hence $t' = t^{(Nv)^2-1}$ acts unipotently on $\mathscr{V}$, $(t'-1)^2 = 0$ on $\mathscr{V}$. On the other hand, it follows from 12.4.4.2 that $(t-1)^2 = 0 \neq t-1$ on $T(\mathcal{P}')/\overline{\mathcal{P}}'T(\mathcal{P}')$, which implies that $t'-1 \neq 0$ on $\mathscr{V}$, hence $T(\mathcal{P}')^{I_v} = T(\mathcal{P}')^{t=1}$ is a free $R_{\overline{\mathcal{P}}'}$-module of rank one. Counting the dimensions, we infer that the sequence

$$0 \longrightarrow T(\mathcal{P}')^{t'=1} \xrightarrow{\varpi_{\mathcal{P}'}} T(\mathcal{P}')^{t'=1} \longrightarrow (V')^{t'=1} \longrightarrow 0$$

is exact; taking invariants with respect to the finite cyclic group $\langle t\rangle/\langle t'\rangle$ we obtain an exact sequence

$$0 \longrightarrow T(\mathcal{P}')^{I_v} \xrightarrow{\varpi_{\mathcal{P}'}} T(\mathcal{P}')^{I_v} \longrightarrow (V')^{I_v} \longrightarrow 0. \tag{12.7.14.1.1}$$

In particular,

$$\dim_L \mathscr{V}^{t=1} = \dim_L \mathscr{V}^{I_v} = \mathrm{rk}_{R_{\overline{\mathcal{P}}'}} T(\mathcal{P}')^{I_v} = 1.$$

As $\Lambda^2\mathscr{V} \xrightarrow{\sim} \mathscr{L}$ as a representation of I_v and $\mathrm{Ker}(t-1|\mathscr{V}) \neq 0$, the operator $t-1$ is nilpotent on $\mathscr{V}$; thus $t-1$ induces an isomorphism of $\mathscr{L}[G_v/I_v]$-modules (both of which are one-dimensional $\mathscr{L}$-vector spaces)

$$\mathscr{V}/\mathscr{V}^{I_v} = \mathscr{V}/\mathscr{V}^{t=1} \xrightarrow{\sim} \mathscr{V}(-1)^{t=1} = \mathscr{V}(-1)^{I_v}.$$

Let $\beta : G_v/I_v \to \mathscr{L}^*$ be the unramified character through which G_v acts on $\mathscr{V}/\mathscr{V}^{I_v}$; then $\mathscr{V}^{I_v} \xrightarrow{\sim} \mathscr{L}(1) \otimes \beta$ (as an $\mathscr{L}[G_v]$-module) and $\mathscr{L}(1) = \Lambda^2\mathscr{V} \xrightarrow{\sim} \mathscr{L}(1) \otimes \beta^2$, hence $\beta^2 = 1$. The exact sequence (12.7.14.1.1) implies that $(V')^{I_v} \xrightarrow{\sim} \kappa(\mathcal{P}')(1) \otimes \beta$, hence $\beta = \mu$, thanks to 12.4.4.2; this finishes the proof of (i).

(ii) As

$$V'' = V(g_{\mathcal{P}''})(k''/2) = T(\mathcal{P}'')\overline{\mathcal{P}}''T(\mathcal{P}'')$$

is a subquotient of $\mathscr{V}$, the group I_v^w acts trivially on V''. The inclusion

$$T(\mathcal{P}'')^{I_v}/\varpi_{\mathcal{P}''}T(\mathcal{P}'')^{I_v} = \kappa(\mathcal{P}'')(1) \otimes \mu \subset (T(\mathcal{P}'')/\varpi_{\mathcal{P}''}T(\mathcal{P}''))^{I_v} = (V'')^{I_v}$$

implies that

$$d_v := \dim_{\kappa(\mathcal{P}'')}(V'')^{I_v} \geqslant 1,$$

hence

$$L_v\big(\pi(g_{\mathcal{P}''})_v, s+\tfrac{1}{2}\big) = L_v(V'', s) = \big[(1-\mu(v)(Nv)^{-1-s})(1-a(Nv)^{-s})\big]^{-1},$$

where

$$a = \begin{cases} 0, & d_v = 1 \\ 1, & d_v = 2. \end{cases}$$

The purity result 12.4.8.2 implies that $d_v = 1$ and $\pi(g_{\mathcal{P}''})_v = \mathrm{St}(\mu)$, which concludes the proof. □

12.7.14.2. Proposition. — *In the situation of 12.7.10, set $\pi = \pi(g_{\mathcal{P}})$, $\pi' = \pi(g_{\mathcal{P}'})$. Let $v \nmid p\infty$ be a prime of F.*

(i) *If* $\pi_v \neq \mathrm{St}(\mu)$, *then* I_v *acts on* $\mathscr{V}_v$ *through a finite quotient.*

(ii) *If* $\pi_v = \pi(\mu, \mu^{-1})$, *then* $\pi'_v = \pi(\mu', (\mu')^{-1})$, *where* μ'/μ *is an unramified character.*

(iii) *If* π_v *is supercuspidal, then* $\pi'_v = \pi_v$.

Proof

(i) The same proof as in the classical case ([**Se-Ta**, App.]) shows that there is an open normal subgroup $U \lhd I_v$ which acts unipotently on $\mathscr{V}$, *i.e.*$(\forall u \in U)$ $(u-1)^2 = 0$ on $\mathscr{V}$. If U does not act trivially on $\mathscr{V}$, then there exists an arithmetic point $\mathcal{P}'' \in \mathrm{Spec}(h^{\mathrm{ord}}_{\phi_n L}(\mathfrak{n}; \mathcal{O}))$ containing $\mathcal{P}_{\min}$ and an element $u \in U$ such that $(u-1)^2 = 0 \neq u - 1$ on $V'' = V(g_{\mathcal{P}''})(k''/2)$, thus $\pi(g_{\mathcal{P}''})_v = \mathrm{St}(\mu)$, by 12.4.4. Applying Proposition 12.7.14.1 (with the pair $\mathcal{P}'', \mathcal{P}$ playing the role of $\mathcal{P}', \mathcal{P}''$), we obtain $\pi_v = \mathrm{St}(\mu)$. This contradiction implies that I_v acts on $\mathscr{V}$ through the finite quotient I_v/U.

(ii) For each $g \in I_v/U$, the trace $\mathrm{Tr}(g|\mathscr{V})$ is contained in $\overline{\mathbf{Q}}_p \cap \mathscr{L} = \mathrm{Frac}(\mathcal{O}) = L_{\mathfrak{p}}$, hence

$$\mathrm{Tr}(g|\mathscr{V}) = \mathrm{Tr}(g|T(\mathcal{P}')) = \mathrm{Tr}(g|V'_v) \in L_{\mathfrak{p}}.$$

This implies that there is, for a suitable $L_{\mathfrak{p}}$-embedding of fields $\kappa(\mathcal{P}') \hookrightarrow \overline{\mathscr{L}}$, an isomorphism of $\overline{\mathscr{L}}[I_v]$-modules

$$\alpha : V'_v \otimes_{\kappa(\mathcal{P}')} \overline{\mathscr{L}} \xrightarrow{\sim} \mathscr{V}_v \otimes_{\mathscr{L}} \overline{\mathscr{L}}.$$

Replacing $\mathcal{P}'$ by $\mathcal{P}$, we obtain an equality of traces

$$(12.7.14.2.1) \qquad (\forall g \in I_v/U) \quad \mathrm{Tr}(g|V_v) = \mathrm{Tr}(g|V'_v) \in L_{\mathfrak{p}},$$

hence an isomorphism of $\overline{\mathbf{Q}}_p[I_v]$-modules

$$(12.7.14.2.2) \qquad V_v \otimes_{L_{\mathfrak{p}}} \overline{\mathbf{Q}}_p \xrightarrow{\sim} V'_v \otimes_{\kappa(\mathcal{P}')} \overline{\mathbf{Q}}_p$$

(for some $L_{\mathfrak{p}}$-embedding $\kappa(\mathcal{P}') \hookrightarrow \overline{\mathbf{Q}}_p$).

If $\pi_v = \pi(\mu, \mu^{-1})$, then (12.7.14.2.2) together with 12.4.4 imply that $\pi'_v = \pi(\mu', (\mu')^{-1})$, with μ'/μ unramified.

(iii) If π_v is supercuspidal, then V_v is an absolutely irreducible I_v-module; by (12.7.14.2.1)–(12.7.14.2.2), the same is true for $\mathscr{V}_v$ and V'_v (hence π'_v is also supercuspidal). The action of a lift $f \in G_v$ of the geometric Frobenius on V'_v defines an isomorphism of $\kappa(\mathcal{P}')[I_v]$-modules $\beta : V'_v \xrightarrow{\sim} {}^f V'_v$ (where $g \in I_v$ acts on ${}^f V'_v$ by $fgf^{-1}|V'_v$). It follows that

$$\alpha \circ (\beta \otimes 1) \circ \alpha^{-1} \in \mathrm{Isom}_{\overline{\mathscr{L}}[I_v]}(\mathscr{V}_v \otimes_{\mathscr{L}} \overline{\mathscr{L}}, {}^f\mathscr{V}_v \otimes_{\mathscr{L}} \overline{\mathscr{L}}),$$

hence

$$(12.7.14.2.3) \qquad \alpha \circ (\beta \otimes 1) \circ \alpha^{-1} = c \cdot f|\mathscr{V}$$

for some $c \in \overline{\mathscr{L}}^*$. Comparing the determinants

$$\det(\beta) = \det(f|V'_v) = \det(f|\mathscr{V}) = (Nv)^{-1} \in \mathbf{Q}_p^*,$$

we see that $c^2 = 1$, hence $c = \pm 1 \in \mathbf{Q}_p^* \subset \overline{\mathscr{L}}^*$.

As the map $\kappa(\mathcal{P}')[I_v] \to \mathrm{End}_{\kappa(\mathcal{P}')[I_v]}(V'_v)$ is surjective, we can assume (after replacing f by fg for suitable $g \in I_v$) that

$$u := \mathrm{Tr}(f|V'_v) \in \kappa(\mathcal{P}')$$

is non-zero. It follows from (12.7.14.2.3) that $u = cu$, hence $c = 1$, which implies that α is an isomorphism of $\overline{\mathscr{L}}[G_v]$-modules. Applying the same argument to V_v, we deduce that (12.7.14.2.2) is an isomorphism of $\overline{\mathbf{Q}}_p[G_v]$-modules, hence $\pi_v \xrightarrow{\sim} \pi'_v$. □

12.7.14.3. Corollary. — *Let $v \nmid p\infty$ be a prime of F.*

(i) *If $\pi(g_{\mathcal{P}})_v = \pi(\mu, \mu^{-1})$ (resp., $\pi(g_{\mathcal{P}})_v \neq \pi(\mu, \mu^{-1})$), then there is, for a suitable $L_{\mathfrak{p}}$-embedding $\kappa(\mathcal{P}') \hookrightarrow \overline{\mathbf{Q}}_p$, an isomorphism of $\overline{\mathbf{Q}}_p[I_v]$-modules (resp., of $\overline{\mathbf{Q}}_p[G_v]$-modules)*

$$V_v \otimes_{L_{\mathfrak{p}}} \overline{\mathbf{Q}}_p \xrightarrow{\sim} V'_v \otimes_{\kappa(\mathcal{P}')} \overline{\mathbf{Q}}_p.$$

(ii) $\mathrm{ord}_v(n(g_{\mathcal{P}})) = \mathrm{ord}_v(n(g_{\mathcal{P}'}))$.

(iii) $\varepsilon_v\big(\pi(g_{\mathcal{P}})_v, \tfrac{1}{2}\big) = \varepsilon_v\big(\pi(g_{\mathcal{P}'})_v, \tfrac{1}{2}\big)$.

Proof

(i) This follows from (the proof of) Proposition 12.7.14.1 and 12.7.14.2.

(ii), (iii) If $\pi(g_{\mathcal{P}})_v \neq \pi(\mu, \mu^{-1})$, then $\pi(g_{\mathcal{P}})_v = \pi(g_{\mathcal{P}'})_v$, hence

$$\mathrm{ord}_v(n(g_{\mathcal{P}})) = o(\pi(g_{\mathcal{P}})_v) = o(\pi(g_{\mathcal{P}'})_v) = \mathrm{ord}_v(n(g_{\mathcal{P}'}))$$
$$\varepsilon_v\big(\pi(g_{\mathcal{P}})_v, \tfrac{1}{2}\big) = \varepsilon_v\big(\pi(g_{\mathcal{P}'})_v, \tfrac{1}{2}\big).$$

If $\pi(g_{\mathcal{P}})_v = \pi(\mu, \mu^{-1})$, then $\pi(g_{\mathcal{P}'})_v = \pi(\mu', (\mu')^{-1})$ with μ'/μ unramified, hence

$$\mathrm{ord}_v(n(g_{\mathcal{P}})) = 2\,o(\mu) = 2\,o(\mu') = \mathrm{ord}_v(n(g_{\mathcal{P}'}))$$
$$\varepsilon_v\big(\pi(g_{\mathcal{P}})_v, \tfrac{1}{2}\big) = \mu(-1) = \mu'(-1) = \varepsilon_v\big(\pi(g_{\mathcal{P}'})_v, \tfrac{1}{2}\big).$$ □

12.7.14.4. Proposition. — *In the situation of 12.7.10, set $g = g_{\mathcal{P}}$, $g' = g_{\mathcal{P}'}$, $V = V(g)(k/2)$, $V' = V(g')(k'/2)$.*

(i) $\widetilde{\varepsilon}\big(\pi(g), \tfrac{1}{2}\big) = \widetilde{\varepsilon}\big(\pi(g'), \tfrac{1}{2}\big)$.

(ii) $\widetilde{h}^1_f(F,V) - h^1_f(F,V) + r_{\mathrm{an}}(F,g) \equiv \widetilde{h}^1_f(F,V') - h^1_f(F,V') + r_{\mathrm{an}}(F,g') \pmod 2$.

(iii) $\widetilde{h}^1_f(F,V) \equiv \widetilde{h}^1_f(F,V') \pmod 2$.

(iv) $h^1_f(F,V) - r_{\mathrm{an}}(F,g) \equiv h^1_f(F,V') - r_{\mathrm{an}}(F,g') \pmod 2$.

Proof

(i) According to Proposition 12.5.9.4, (ii) is a reformulation of (i) and

$$\widetilde{\varepsilon}(\pi(g), \tfrac{1}{2}) = \prod_{v \nmid p\infty} \chi_v(-1)\,\varepsilon_v(\pi(g)_v, \tfrac{1}{2}) \prod_{v|\infty} \chi_v(-1)\,(-1)^{k/2}$$
$$\widetilde{\varepsilon}(\pi(g'), \tfrac{1}{2}) = \prod_{v \nmid p\infty} \chi'_v(-1)\,\varepsilon_v(\pi(g')_v, \tfrac{1}{2}) \prod_{v|\infty} \chi'_v(-1)\,(-1)^{k'/2}.$$

Fix a prime v of F. If $v \nmid p\infty$, then $\chi_v(-1) = \chi'_v(-1)$ (as the character χ'/χ is unramified at v) and $\varepsilon_v(\pi(g)_v, \frac{1}{2}) = \varepsilon_v(\pi(g')_v, \frac{1}{2})$ (by Corollary 12.7.14.3). If $v \mid \infty$, then $\chi_v(-1)\,(-1)^{k/2} = \chi'_v(-1)\,(-1)^{k'/2}$, as observed in 12.7.10. The statements (i), (ii) follow.

(iii) This is a special case of Corollary 12.7.13.5 and (iv) is a combination of (ii) and (iii). □

12.7.14.5. Proposition **(Dihedral case)**. — *In the situation of 12.7.14.4, let K/F be a totally imaginary quadratic extension and $\beta : \mathbf{A}_K^*/K^*\mathbf{A}_F^* \to \mathbf{C}^*$ a ring class character of finite order.*

(i) *If v is a prime of F, then $\widetilde{\varepsilon}_v\big(\pi \times \theta(\beta), \frac{1}{2}\big) = \widetilde{\varepsilon}_v\big(\pi' \times \theta(\beta), \frac{1}{2}\big)$.*

(ii) $\widetilde{h}^1_f(K, g, \beta) - h^1_f(K, g, \beta) + r_{\mathrm{an}}(K, g, \beta) \equiv \widetilde{h}^1_f(K, g', \beta) - h^1_f(K, g', \beta) + r_{\mathrm{an}}(K, g', \beta) \pmod 2$.

(iii) $\widetilde{h}^1_f(K, g, \beta) \equiv \widetilde{h}^1_f(K, g', \beta) \pmod 2$.

(iv) $h^1_f(K, g, \beta) - r_{\mathrm{an}}(K, g, \beta) \equiv h^1_f(K, g', \beta) - r_{\mathrm{an}}(K, g', \beta) \pmod 2$.

(v) *g has CM by $K \iff g'$ has CM by K.*

Proof

(i) If $v \mid \infty$, then both sides are equal to 1. Assume that $v \nmid p\infty$. As $\pi_v = \pi'_v$ if $\pi_v \neq \pi(\mu, \mu^{-1})$ by Proposition 12.7.14.1–12.7.14.2, we have to treat only the case $\pi_v = \pi(\mu, \mu^{-1})$. In this case $\pi'_v = \pi(\mu', (\mu')^{-1})$ by Proposition 12.7.14.2 (ii); applying Proposition 12.6.2.4 (i) we obtain

$$\varepsilon_v\big(\pi \times \theta(\beta), \tfrac{1}{2}\big) = \eta_v(-1) = \varepsilon_v\big(\pi' \times \theta(\beta), \tfrac{1}{2}\big).$$

If $v \mid p$, then neither π_v nor π'_v is supercuspidal. Applying Proposition 12.6.2.4 (i)–(ii) we obtain

$$\widetilde{\varepsilon}_v\big(\pi \times \theta(\beta), \tfrac{1}{2}\big) = \eta_v(-1) = \widetilde{\varepsilon}_v\big(\pi' \times \theta(\beta), \tfrac{1}{2}\big).$$

(ii) Multiply together the equalities (i) over all primes v of F and apply Proposition 12.6.4.3.

(iii) This follows from 12.7.13.8 (ii).

(iv) Combine (ii) and (iii).

(v) If g has CM by K, so does f, which implies that each prime of F above p splits in K/F. There exists a cyclotomic Hida family containing the p-stabilization f^0 of f, in which all arithmetic points correspond to forms with CM by K ([**H-T**, §4]). As f^0 is a p-stabilized newform, the corresponding primitive family is unique, hence corresponds to the morphism λ; thus f' also has CM by K, and so does g'. The converse follows by the same argument. □

12.7.15. Hida families and Iwasawa theory. — We continue to use the notation of 12.7.13.

12.7.15.1. In the situation of 12.7.13.1–12.7.13.2, fix an abelian extension E_∞ of E contained in F_S, for which $\Gamma := \mathrm{Gal}(E_\infty/E) = \Gamma_0 \times \Delta$, $\Gamma_0 \xrightarrow{\sim} \mathbf{Z}_p^r$ $(r \geqslant 1)$, $|\Delta| < \infty$. Assume that all primes of E above Σ' are unramified in E_∞/E, which is automatic if $\Delta = 0$ or if $\Sigma' = \varnothing$.

12.7.15.2. Set $\overline{R} = R[\![\Gamma]\!] = R[\Delta][\![\Gamma_0]\!]$, $\Lambda = \mathcal{O}[\![\Gamma]\!] = \mathcal{O}[\Delta][\![\Gamma_0]\!]$. As $R/\overline{\mathcal{P}} = \mathcal{O}$, the field $\mathrm{Frac}(\mathcal{O})$ is algebraically closed in the field $\mathrm{Frac}(R) = \mathscr{L}$, which implies that the formula $\mathfrak{q} \mapsto \overline{\mathfrak{q}} = \mathfrak{q}\overline{R}$ defines a bijection

$$\{\mathfrak{q} \in \mathrm{Spec}(\Lambda),\ \mathrm{ht}(\mathfrak{q}) = 0\} \xrightarrow{\sim} \{\overline{\mathfrak{q}} \in \mathrm{Spec}(\overline{R}),\ \mathrm{ht}(\overline{\mathfrak{q}}) = 0\};$$

fix such $\mathfrak{q}$ and $\overline{\mathfrak{q}} = \mathfrak{q}\overline{R}$.

Recall that we can (and will) assume that the discrete valuation ring $\mathcal{O}'$ coincides with the normalization of $R/\overline{\mathcal{P}}'$; thus $\mathrm{Frac}(\mathcal{O}') = \kappa(\overline{\mathcal{P}}')$. Set $\Lambda' = \mathcal{O}'[\![\Gamma]\!] = \mathcal{O}'[\Delta][\![\Gamma_0]\!]$ and denote by $u' : R[\Delta] \to R/\overline{\mathcal{P}}'[\Delta] \to \mathcal{O}'[\Delta]$ the canonical map. The formulas

$$\mathfrak{q}' \mapsto \mathfrak{q}' \cap \mathcal{O}'[\Delta], \quad \mathfrak{q}' \mapsto \overline{\mathfrak{p}}' = \overline{\mathcal{P}}'\overline{R} + u'^{-1}(\mathfrak{q}' \cap \mathcal{O}'[\Delta])\overline{R} \tag{12.7.15.1}$$

define bijections between

$$\text{the fibre of } \mathrm{Spec}(\Lambda') \longrightarrow \mathrm{Spec}(\Lambda) \text{ above } \mathfrak{q}$$

and, respectively,

$$\text{the fibre of } \mathrm{Spec}(\mathcal{O}'[\Delta]) \longrightarrow \mathrm{Spec}(\mathcal{O}[\Delta]) \text{ above } \mathfrak{q} \cap \mathcal{O}[\Delta]$$

and

$$\{\overline{\mathfrak{p}}' \in \mathrm{Spec}(\overline{R}) \mid \mathrm{ht}(\overline{\mathfrak{p}}') = 1,\ \overline{\mathfrak{p}}' \supset \overline{\mathfrak{q}},\ \overline{\mathfrak{p}}' \supset \overline{\mathcal{P}}'\overline{R}\},$$

(by Lemma 8.9.7.1). Fix $\mathfrak{q}' \in \mathrm{Spec}(\Lambda')$ above $\mathfrak{q}$ and define $\overline{\mathfrak{p}}' \in \mathrm{Spec}(\overline{R})$ by the formula (12.7.15.1). The localization $\overline{R}_{\overline{\mathfrak{p}}'}$ is a discrete valuation ring with prime element $\varpi_{\mathcal{P}'}$ and residue field

$$\kappa(\overline{\mathfrak{p}}') = \Lambda'_{\mathfrak{q}'} = \kappa(\mathfrak{q}').$$

More precisely, there exists a character $\chi : \Delta \to \overline{L}_{\mathfrak{p}}^*$ such that $\mathfrak{q}' = \mathrm{Ker}(\chi : \mathcal{O}'[\Delta] \to \overline{L}_{\mathfrak{p}})$. Fix such χ and denote by $\mathcal{O}'(\chi)$ the image of $\mathcal{O}'[\Delta]$ under χ; it is a discrete valuation ring, finite over $\mathcal{O}'$. For each $\mathcal{O}'[\Delta]$-module M, denote

$$M^{(\chi)} := M \otimes_{\mathcal{O}'[\Delta],\chi} \mathcal{O}'(\chi)$$

(as in 10.7.16). Using this notation, the normalization of $\overline{R}/\overline{\mathfrak{p}}'$ is equal to $\mathcal{O}'(\chi)[\![\Gamma_0]\!] = \Lambda'^{(\chi)}$, hence

$$\kappa(\overline{\mathfrak{p}}') = \mathrm{Frac}(\Lambda'^{(\chi)}) = \Lambda'_{\mathfrak{q}'}. \tag{12.7.15.2}$$

The fraction field of $\overline{R}_{\overline{\mathfrak{p}}'}$ is equal to

$$\mathrm{Frac}(\overline{R}_{\overline{\mathfrak{p}}'}) = \mathrm{Frac}(\overline{R}_{\overline{\mathfrak{q}}}) = \kappa(\overline{\mathfrak{q}}) = \mathrm{Frac}(\overline{R}/\overline{\mathfrak{q}}) = \mathrm{Frac}((R \otimes_{\mathcal{O}} \mathcal{O}(\chi))[\![\Gamma_0]\!]).$$

12.7.15.3. Fix a $G_{F,S}$-stable R-lattice (resp., $\mathcal{O}'$-lattice) $\mathcal{T} \subset \mathscr{V}$ (resp., $T' \subset V'$), and denote, for each $v \mid p$ in F,

$$\mathcal{T}_v^+ = \mathcal{T} \cap \mathscr{V}_v^+, \quad \mathscr{F}_\Gamma(\mathcal{T})_v^+ = \mathscr{F}_\Gamma(\mathcal{T}_v^+), \quad (T')_v^+ = T' \cap (V')_v^+, \quad \mathscr{F}_\Gamma(T')_v^+ = \mathscr{F}_\Gamma((T')_v^+).$$

For each intermediate set $\{v \mid p\} \subset \Sigma \subset S_f$, we define Greenberg's local conditions $\Delta_\Sigma(X)$ for $X = \mathscr{F}_\Gamma(\mathcal{T})$ and $\mathscr{F}_\Gamma(T')$ by the formulas from 12.7.13.2 (with $\beta = 1$).

12.7.15.4. As $\mathcal{T}_{\overline{\mathcal{P}}'} := R_{\overline{\mathcal{P}}'}\mathcal{T} = \varpi_{\mathcal{P}'}^{b'} T(\mathcal{P}')$ for some $b' \in \mathbf{Z}$, we have

$$(\forall v \mid p) \quad (\mathcal{T}_v^+)_{\overline{\mathcal{P}}'} := R_{\overline{\mathcal{P}}'}\mathcal{T}_v^+ = \varpi_{\mathcal{P}'}^{b'} T(\mathcal{P}')_v^+$$

and

$$\begin{aligned} (12.7.15.1) \quad \mathscr{F}_\Gamma(\mathcal{T}) \otimes_{\overline{R}} \overline{R}_{\overline{\mathfrak{p}}'} &= \mathcal{T} \otimes_R \overline{R}_{\overline{\mathfrak{p}}'} < -1 > \\ &= \mathcal{T}_{\overline{\mathcal{P}}'} \otimes_{R_{\overline{\mathcal{P}}'}} \overline{R}_{\overline{\mathfrak{p}}'} < -1 > = \varpi_{\mathcal{P}'}^{b'} T(\mathcal{P}') \otimes_{R_{\overline{\mathcal{P}}'}} \overline{R}_{\overline{\mathfrak{p}}'} < -1 > \\ (\forall v \mid p) \quad \mathscr{F}_\Gamma(\mathcal{T}_v^+) \otimes_{\overline{R}} \overline{R}_{\overline{\mathfrak{p}}'} &= \varpi_{\mathcal{P}'}^{b'} T(\mathcal{P}')_v^+ \otimes_{R_{\overline{\mathcal{P}}'}} \overline{R}_{\overline{\mathfrak{p}}'} < -1 > . \end{aligned}$$

As $T(\mathcal{P}')/\varpi_{\mathcal{P}'}T(\mathcal{P}') = V'$ and $\overline{R}_{\overline{\mathfrak{p}}'}/\varpi_{\mathcal{P}'}\overline{R}_{\overline{\mathfrak{p}}'} = \mathrm{Frac}(\Lambda'^{(\chi)})$, the formulas (12.7.15.1) imply that multiplication by $\varpi_{\mathcal{P}'}^{-b'}$, followed by reduction modulo $\varpi_{\mathcal{P}'}$, induces isomorphisms

$$\begin{aligned} (12.7.15.2) \quad \mathscr{F}_\Gamma(\mathcal{T})_{\overline{\mathfrak{p}}'}/\varpi_{\mathcal{P}'}\mathscr{F}_\Gamma(\mathcal{T})_{\overline{\mathfrak{p}}'} &\xrightarrow{\sim} \mathscr{F}_\Gamma(T') \otimes_{\Lambda',\chi} \mathrm{Frac}(\Lambda'^{(\chi)}) = \mathscr{F}_\Gamma(T')_{\mathfrak{q}'} \\ (\forall v \mid p) \quad \mathscr{F}_\Gamma(\mathcal{T}_v^+)_{\overline{\mathfrak{p}}'}/\varpi_{\mathcal{P}'}\mathscr{F}_\Gamma(\mathcal{T}_v^+)_{\overline{\mathfrak{p}}'} &\xrightarrow{\sim} \mathscr{F}_\Gamma((T')_v^+) \otimes_{\Lambda',\chi} \mathrm{Frac}(\Lambda'^{(\chi)}) = \mathscr{F}_\Gamma((T')_v^+)_{\mathfrak{q}'} \end{aligned}$$

of $\mathrm{Frac}(\Lambda'^{(\chi)})[G_F]$-modules (resp., $\mathrm{Frac}(\Lambda'^{(\chi)})[G_v]$-modules), which are compatible with respect to the inclusion maps $\mathcal{T}_v^+ \subset \mathcal{T}$ and $(T')_v^+ \subset T'$.

The pairing

$$\begin{aligned} \pi_{\mathcal{T},\overline{\mathfrak{p}}'} : \mathscr{F}_\Gamma(\mathcal{T})_{\overline{\mathfrak{p}}'} \otimes_{\overline{R}_{\overline{\mathfrak{p}}'}} \left(\mathscr{F}_\Gamma(\mathcal{T})^\iota\right)_{\overline{\mathfrak{p}}'} = \\ \left(\varpi_{\mathcal{P}'}^{b'} T(\mathcal{P}') \otimes_{R_{\overline{\mathcal{P}}'}} \overline{R}_{\overline{\mathfrak{p}}'} < -1 > \right) \otimes_{\overline{R}_{\overline{\mathfrak{p}}'}} \left(\varpi_{\mathcal{P}'}^{b'} T(\mathcal{P}') \otimes_{R_{\overline{\mathcal{P}}'}} \overline{R}_{\overline{\mathfrak{p}}'} < 1 > \right) \\ \xrightarrow{\pi_{T(\mathcal{P}')}\otimes\mathrm{id}} \varpi_{\mathcal{P}'}^{2b'} \overline{R}_{\overline{\mathfrak{p}}'}(1) \xrightarrow{\varpi_{\mathcal{P}'}^{-2b'}} \overline{R}_{\overline{\mathfrak{p}}'}(1) \end{aligned}$$

is a perfect duality, satisfying

$$(12.7.15.3) \qquad (\forall v \mid p) \quad \mathscr{F}_\Gamma(\mathcal{T}_v^+)_{\overline{\mathfrak{p}}'} \perp\perp_{\pi_{\mathcal{T},\overline{\mathfrak{p}}'}} \mathscr{F}_\Gamma(\mathcal{T}_v^+)_{\overline{\mathfrak{p}}'}$$

12.7.15.5. Proposition

(i) *There exists an exact triangle in* $D^b_{ft}({}_{\overline{R}_{\overline{\mathfrak{p}}'}}\mathrm{Mod})$

$$\widetilde{\mathbf{R}\Gamma}_f(G_{E,S}, \mathscr{F}_\Gamma(\mathcal{T}); \Delta_\Sigma)_{\overline{\mathfrak{p}}'} \xrightarrow{\varpi_{\mathcal{P}'}} \widetilde{\mathbf{R}\Gamma}_f(G_{E,S}, \mathscr{F}_\Gamma(\mathcal{T}); \Delta_\Sigma)_{\overline{\mathfrak{p}}'} \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(G_{E,S}, \mathscr{F}_\Gamma(T'); \Delta_\Sigma)_{\mathfrak{q}'},$$

which gives rise to exact sequences

$$\begin{aligned} 0 \longrightarrow \widetilde{H}^j_f(G_{E,S}, \mathscr{F}_\Gamma(\mathcal{T}); \Delta_\Sigma)_{\overline{\mathfrak{p}}'}/\overline{\mathfrak{p}}' \longrightarrow \widetilde{H}^j_f(G_{E,S}, \mathscr{F}_\Gamma(T'); \Delta_\Sigma)_{\mathfrak{q}'} \\ \longrightarrow \widetilde{H}^{j+1}_f(G_{E,S}, \mathscr{F}_\Gamma(\mathcal{T}); \Delta_\Sigma)_{\overline{\mathfrak{p}}'}[\overline{\mathfrak{p}}'] \longrightarrow 0, \end{aligned}$$

where $\widetilde{H}^j_f(G_{E,S}, \mathscr{F}_\Gamma(T'); \Delta_\Sigma)_{\mathfrak{q}'} = \widetilde{H}^j_f(G_{E,S}, \mathscr{F}_\Gamma(T'); \Delta_\Sigma) \otimes_{\Lambda',\chi} \mathrm{Frac}(\Lambda'^{(\chi)})$.

(ii) *Up to a canonical isomorphism,* $\widetilde{\mathbf{R}\Gamma}_f(G_{E,S},\mathscr{F}_\Gamma(T');\Delta_\Sigma)_{\mathfrak{q}'} \in D^b_{ft}({}_{\Lambda'_{\mathfrak{q}'}}\mathrm{Mod})$ *does not depend on the choice of* S*,* Σ *and* T'*; we denote it by* $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(E_\infty/E,T')_{\mathfrak{q}'}$ *and its cohomology by* $\widetilde{H}^j_{f,\mathrm{Iw}}(E_\infty/E,T')_{\mathfrak{q}'}$*.*

(iii) *Up to a canonical isomorphism,* $\widetilde{\mathbf{R}\Gamma}_f(G_{E,S},\mathscr{F}_\Gamma(\mathcal{T});\Delta_\Sigma)_{\overline{\mathfrak{p}}'} \in D^b_{ft}({}_{\overline{R}_{\overline{\mathfrak{p}}'}}\mathrm{Mod})$ *does not depend on the choice of* S *and* Σ*; we denote it by* $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})_{\overline{\mathfrak{p}}'}$ *and its cohomology by* $\widetilde{H}^j_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})_{\overline{\mathfrak{p}}'}$*.*

(iv) *Up to a canonical isomorphism,* $\widetilde{\mathbf{R}\Gamma}_f(G_{E,S},\mathscr{F}_\Gamma(\mathcal{T});\Delta_\Sigma)_{\overline{\mathfrak{q}}} \in D^b_{ft}({}_{\kappa(\overline{\mathfrak{q}})}\mathrm{Mod})$ *does not depend on the choice of* S*,* Σ *and* $\mathcal{T}$*; we denote it by* $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})_{\overline{\mathfrak{q}}}$ *and its cohomology by* $\widetilde{H}^j_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})_{\overline{\mathfrak{q}}}$*.*

(v) *For each* $j\in\mathbf{Z}$*, the image of* $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})_{\overline{\mathfrak{p}}'}$ *in* $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})_{\overline{\mathfrak{q}}}$ *depends only on* $\mathcal{T}_{\overline{\mathcal{P}}'} = \varpi^{b'}_{\overline{\mathcal{P}}'}T(\mathcal{P}')$*.*

Proof

(i) The same argument as in the proof of Proposition 12.7.13.4 (i) applies (using (12.7.15.2) instead of (12.7.13.2)).

(ii) This follows from Proposition 9.7.9 (ii).

(iii) Independence on S was proved in Proposition 7.8.8 (ii). In order to prove independence on Σ, denote by $Y\in D^b_{ft}({}_{\overline{R}_{\overline{\mathfrak{p}}'}}\mathrm{Mod})$ the cone of the canonical map $\widetilde{\mathbf{R}\Gamma}_f(G_{E,S},\mathscr{F}_\Gamma(\mathcal{T});\Delta_\Sigma)_{\overline{\mathfrak{p}}'}\to\widetilde{\mathbf{R}\Gamma}_f(G_{E,S},\mathscr{F}_\Gamma(\mathcal{T});\Delta_{S_f})_{\overline{\mathfrak{p}}'}$. It follows from (i) and (ii) that $Y\overset{\mathbf{L}}{\otimes}_{\overline{R}_{\overline{\mathfrak{p}}'}}\kappa(\overline{\mathfrak{p}}')$ is acyclic; thus Y is also acyclic, by Lemma 12.7.15.6 below.

(iv) This follows from (iii) and the fact that $\mathcal{T}\otimes_R\overline{R}_{\overline{\mathfrak{q}}}$ does not depend on $\mathcal{T}$, since

$$\widetilde{\mathbf{R}\Gamma}_f(G_{E,S},\mathscr{F}_\Gamma(\mathcal{T});\Delta_\Sigma)_{\overline{\mathfrak{q}}}=\widetilde{\mathbf{R}\Gamma}_f(G_{E,S},\mathcal{T}\otimes_R\overline{R}_{\overline{\mathfrak{q}}}<-1>;\Delta_\Sigma).$$

(v) Similarly, we use

$$\widetilde{\mathbf{R}\Gamma}_f(G_{E,S},\mathscr{F}_\Gamma(\mathcal{T});\Delta_\Sigma)_{\overline{\mathfrak{p}}'}=\widetilde{\mathbf{R}\Gamma}_f(G_{E,S},\mathcal{T}_{\overline{\mathcal{P}}'}\otimes_{R_{\overline{\mathcal{P}}'}}\overline{R}_{\overline{\mathfrak{p}}'}<-1>).\quad\square$$

12.7.15.6. Lemma. — *Let* R *be a Noetherian ring and* $X\in D_{ft}({}_R\mathrm{Mod})$*. Assume that* $r\in R$ *in not a zero divisor, but is contained in the radical of* R*. If* $Y:=X\overset{\mathbf{L}}{\otimes}_R R/rR\overset{\sim}{\to}0$ *in* $D_{ft}({}_{R/rR}\mathrm{Mod})$*, then* $X\overset{\sim}{\to}0$ *in* $D_{ft}({}_R\mathrm{Mod})$*.*

Proof. — The free resolution $[R\xrightarrow{r}R]$ of R/rR gives rise to injections $H^j(X)/rH^j(X)\hookrightarrow H^j(Y)$. As each $H^j(Y)$ vanishes by assumption, Nakayama's Lemma implies that $H^j(X)=0$, too. $\square$

12.7.15.7. Proposition

(i) *The duality morphism*

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})_{\overline{\mathfrak{p}}'}\longrightarrow\mathbf{R}\mathrm{Hom}_{\overline{R}_{\overline{\mathfrak{p}}'}}\big((\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})^\iota)_{\overline{\mathfrak{p}}'},\overline{R}_{\overline{\mathfrak{p}}'}\big)[-3]$$

is an isomorphism in $D^b_{ft}({}_{\overline{R}_{\overline{\mathfrak{p}}'}}\mathrm{Mod})$*.*

(ii) *The corresponding generalized Cassels-Tate pairing*

$$\langle\ ,\ \rangle : \big(\widetilde{H}^2_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})_{\overline{\mathfrak{p}}'}\big)_{(\overline{R}_{\overline{\mathfrak{p}}'})\text{-tors}} \times \big(\widetilde{H}^2_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})_{\overline{\mathfrak{p}}'}\big)_{(\overline{R}_{\overline{\mathfrak{p}}'})\text{-tors}} \longrightarrow \mathrm{Frac}(\overline{R}_{\overline{\mathfrak{p}}'})/\overline{R}_{\overline{\mathfrak{p}}'}$$

is a non-degenerate skew-Hermitian form.

(iii) $\widetilde{H}^1_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})_{\overline{\mathfrak{p}}'}$ *is a free* $(\overline{R}_{\overline{\mathfrak{p}}'})$*-module of rank equal to*

$$\mathrm{rk}_{\overline{R}_{\overline{\mathfrak{p}}'}}\, \widetilde{H}^1_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})_{\overline{\mathfrak{p}}'} = \dim_{\kappa(\overline{\mathfrak{q}})}\, \widetilde{H}^1_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})_{\overline{\mathfrak{q}}}.$$

Proof

(i) Apply the localized version of the duality Theorem 6.3.4, for the local conditions Δ_{S_f}. Let w be a prime of E above a prime $v \in S_f$. If $v \mid p$, then the error term Err_w vanishes, thanks to (12.7.15.3). If $v \nmid p$, then $\mathrm{Err}_w = 0$, by combining Proposition 8.9.7.7 with Proposition 12.7.13.3 (i) for $X = T(\mathcal{P}')$.

(ii) The pairing $\langle\ ,\ \rangle$ is non-degenerate, by Theorem 10.4.4 and (i). It is skew-Hermitian, by a localized version of Proposition 10.3.4.2.

(iii) The proof of Proposition 12.7.13.4 (iii) applies, using Proposition 12.7.15.5 (i) instead of Proposition 12.7.13.4 (i). □

***12.7.15.8. Proposition* (Dihedral case).** — *Assume that* $E = K$, *where* K *is as in 12.6, and* $E_\infty = K_\infty \subset K[\infty] = \bigcup K[c]$ *(in the notation of 12.6.1.5). Fix a lift* $\tau \in \mathrm{Gal}(K_\infty/F)$ *of the non-trivial element of* $\mathrm{Gal}(K/F)$*. Then:*

(i) *The map* u_X *from Lemma 10.3.5.4 (for* $X = \mathcal{T}$*) induces an isomorphism*

$$u_{\mathcal{T}} : \widetilde{\mathbf{R}\Gamma}_f(G_{K,S},\mathscr{F}_\Gamma(\mathcal{T});\Delta_\Sigma) \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_f(G_{K,S},\mathscr{F}_\Gamma(\mathcal{T});\Delta_\Sigma)^\iota.$$

(ii) *The formula* $[x,y] := \langle x, u_{\mathcal{T}}(y)\rangle$ *defines a non-degenerate, skew-symmetric* $(\overline{R}_{\overline{\mathfrak{p}}'})$*-bilinear pairing*

$$\big(\widetilde{H}^2_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})_{\overline{\mathfrak{p}}'}\big)_{(\overline{R}_{\overline{\mathfrak{p}}'})\text{-tors}} \times \big(\widetilde{H}^2_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})_{\overline{\mathfrak{p}}'}\big)_{(\overline{R}_{\overline{\mathfrak{p}}'})\text{-tors}} \longrightarrow \mathrm{Frac}(\overline{R}_{\overline{\mathfrak{p}}'})/\overline{R}_{\overline{\mathfrak{p}}'}.$$

(iii) *We have*

$$\mathrm{rk}_{\Lambda'(x)}\, \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K,T')^{(x)} \equiv \dim_{\kappa(\overline{\mathfrak{q}})}\, \widetilde{H}^1_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})_{\overline{\mathfrak{q}}} \pmod 2$$
$$\mathrm{rk}_{\Lambda'(x)}\, \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K,T')^{(x)} \geqslant \dim_{\kappa(\overline{\mathfrak{q}})}\, \widetilde{H}^1_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})_{\overline{\mathfrak{q}}}.$$

Proof

(i), (ii) This follows from Lemma 10.3.5.4 and Proposition 10.3.5.8.

(iii) Applying Lemma 10.6.5 to the pairing from (ii), we deduce that

$$\dim_{\kappa(\overline{\mathfrak{p}}')}\, \widetilde{H}^2_{f,\mathrm{Iw}}(E_\infty/E,\mathcal{T})_{\overline{\mathfrak{p}}'}[\overline{\mathfrak{p}}'] \equiv 0 \pmod 2;$$

the statement then follows from the exact sequence in Proposition 12.7.15.5 (i) (with $j = 1$). □

12.8. Level raising

In this section we assume that $f \in S_k(\mathfrak{n}, \varphi)$ is a p-ordinary newform of level $\mathfrak{n}$ and $\chi : \mathbf{A}_F^*/F^* \to \mathbf{C}^*$ is a character (with values contained in $\iota_\infty(L)$) satisfying $\chi^{-2} = \varphi$ ($\Longrightarrow k$ is even). For simplicity, we shall denote $\mathcal{O} = \mathcal{O}_{L_\mathfrak{p}}$.

12.8.1. Congruences between newforms and ε-factors

12.8.1.1. As in 12.5.5, we put $g = f \otimes \chi \in S_k(\mathfrak{n}(g), 1)$ (which is a newform of level dividing $\mathrm{cond}(\chi)^2\mathfrak{n}$) and $V = V(g)(k/2) = V(f)(k/2) \otimes \chi$, $V_v^\pm = V(f)^\pm(k/2) \otimes \chi_v$ $(\forall v \mid p)$.

Fix a G_F-stable $\mathcal{O}$-lattice $T(f) \subset V(f)$ and put $T(g) = T(f) \otimes \chi \subset V(g)$, $T = T(g)(k/2) = T(f)(k/2) \otimes \chi \subset V$. The absolute irreducibility of $V(f)$ implies that there exists an integer $c \geqslant 0$ such that

$$\mathrm{Im}(\mathcal{O}[G_F] \longrightarrow \mathrm{End}_\mathcal{O}(T)) \supseteq \mathfrak{p}^c \, \mathrm{End}_\mathcal{O}(T).$$

12.8.1.2. Fix an integer $M \geqslant 1$ and assume that we are given a newform $g_1 \in S_k(\mathfrak{n}(g)Q, 1)$ of level $\mathfrak{n}(g_1)$ divisible by Q such that

12.8.1.2.1. $Q = \mathfrak{q}_1 \cdots \mathfrak{q}_s$ is a product of $s \geqslant 1$ distinct prime ideals not dividing $(p)\mathrm{cond}(\chi)\mathfrak{n}(g)$.

12.8.1.2.2. $(\forall i = 1, \ldots, s)$ $\lambda_{g_1}(\mathfrak{q}_i) = -(N\mathfrak{q}_i)^{k/2-1}$ ($\Longleftrightarrow \pi(g_1)_{\mathfrak{q}_i} = \mathrm{St}(\mu)$, $\mu^2 = 1$, μ unramified, $\mu(\mathfrak{q}_i) = -1$, by Lemma 12.3.10 (i)).

12.8.1.2.3. For all primes $v \nmid \mathrm{cond}(\chi)\mathfrak{n}pQ$ contained in a set of density 1,

$$\lambda_{g_1}(v) \equiv \lambda_g(v) \pmod{\mathfrak{p}^{M+6c}}$$

(a congruence holding in $\mathcal{O}$).

12.8.1.2.4. The newform $f_1 = g_1 \otimes \chi^{-1}$ is p-ordinary (this is automatically true if M is large enough).

12.8.1.2.5. If $v \nmid Q$ is a non-archimedean prime of F, then $[\mathrm{ord}_v(\mathfrak{n}(g)) = 1 \iff \mathrm{ord}_v(\mathfrak{n}(g_1)) = 1]$.

12.8.1.3. The congruence 12.8.1.2.3 implies, by Proposition 12.8.3.1 below and the Čebotarev density theorem, that there exists a G_F-stable $\mathcal{O}$-lattice $T_1 \subset V_1 = V(g_1)(k/2) = V(f_1)(k/2) \otimes \chi$ and an isomorphism of $\mathcal{O}[G_F]$-modules

$$\bar{j} : T/\mathfrak{p}^M T \xrightarrow{\sim} T_1/\mathfrak{p}^M T_1. \tag{12.8.1.1}$$

As usual, we put $A = V/T, A_1 = V_1/T_1$ and, for each prime $v \mid p$,

$$T_v^+ = V_v^+, \quad T_v^- = T/T_v^+, \quad A_v^\pm = V_v^\pm/T_v^\pm, \quad A[\mathfrak{p}^M]_v^\pm = A_v^\pm[\mathfrak{p}^M].$$

Thanks to the assumption 12.8.1.2.4, we can also define (for each $v \mid p$)

$$(V_1)_v^{\pm} = V(f_1)_v^{\pm}(k/2) \otimes \chi_v, \quad (T_1)_v^{+} = (V_1)_v^{+},$$
$$(T_1)_v^{-} = T_1/(T_1)_v^{+}, \quad (A_1)_v^{\pm} = (V_1)_v^{\pm}/(T_1)_v^{\pm}.$$

Let $\pi(g)$ (resp., $\pi(g_1)$) be the automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ associated to g (resp., to g_1).

12.8.1.4. Proposition. — *Let v be a prime of F. The local ε-factors $\varepsilon_v\big(\pi_v, \frac{1}{2}\big)$ ($\pi = \pi(g), \pi(g_1)$) have the following properties:*

(i) *If $v \nmid \mathfrak{n}(g)Q\infty$, then $\varepsilon_v\big(\pi(g)_v, \frac{1}{2}\big) = \varepsilon_v\big(\pi(g_1)_v, \frac{1}{2}\big) = 1$.*

(ii) *If $v \mid \infty$, then $\varepsilon_v\big(\pi(g)_v, \frac{1}{2}\big) = \varepsilon_v\big(\pi(g_1)_v, \frac{1}{2}\big) = (-1)^{k/2}$.*

(iii) *If $v \mid Q$, then $\varepsilon_v\big(\pi(g)_v, \frac{1}{2}\big) = \varepsilon_v\big(\pi(g_1)_v, \frac{1}{2}\big) = 1$.*

(iv) *If $v \mid \mathfrak{n}(g)$, $v \nmid p$ and $M \geqslant 1 + \mathrm{ord}_{\mathfrak{p}}(2)$ ($+\,\mathrm{ord}_{\mathfrak{p}}(Nv+1)$ if $\mathrm{ord}_v(\mathfrak{n}(g)) = 1$), then $\varepsilon_v\big(\pi(g)_v, \frac{1}{2}\big) = \varepsilon_v\big(\pi(g_1)_v, \frac{1}{2}\big)$.*

Proof. — We only have to prove (iii) and (iv). If $v = \mathfrak{q}_i$ ($i = 1, \ldots, s$), then $v \nmid \mathfrak{n}(g)$, hence $\varepsilon_v\big(\pi(g)_v, \frac{1}{2}\big) = 1$. On the other hand, $\pi(g_1)_v = \mathrm{St}(\mu)$ with μ unramified and $\mu(v) = -1$, hence $\varepsilon_v\big(\pi(g_1)_v, \frac{1}{2}\big) = \mu(-1) = 1$ (by Lemma 12.3.13). It remains to prove (iv): let $v \mid \mathfrak{n}(g)$, $v \nmid p$. According to (12.4.3.1), we have

$$\varepsilon_v\big(\pi(g)_v, \tfrac{1}{2}\big) = \varepsilon_v(V, \psi_v, dx_v) = \varepsilon_{0,v}(V, \psi_v, dx_v) \det\big(-\mathrm{Fr}(v)_{\mathrm{geom}}, V^{I_v}\big)^{-1}, \tag{12.8.1.4.1}$$

where dx_v is the self-dual Haar measure with respect to ψ_v (and similarly for g_1 and V_1). As

$$T/2\mathfrak{p}T = T/\mathfrak{p}^{1+\mathrm{ord}_{\mathfrak{p}}(2)}T \xrightarrow{\sim} T_1/\mathfrak{p}^{1+\mathrm{ord}_{\mathfrak{p}}(2)}T_1 = T_1/2\mathfrak{p}T_1 \tag{12.8.1.4.2}$$

by (12.8.1.1), it follows from [**De2**, Thm. 6.5] (resp., [**Ya**, Thm. 5.1]) in the case $p \neq 2$ (resp., $p = 2$) that

$$\varepsilon_{0,v}(V, \psi_v, dx_v) \equiv \varepsilon_{0,v}(V_1, \psi_v, dx_v) \pmod{\mathfrak{p}^{1+\mathrm{ord}_{\mathfrak{p}}(2)}}. \tag{12.8.1.4.3}$$

As both values $\varepsilon_v\big(\pi_v, \frac{1}{2}\big)$ ($\pi = \pi(g), \pi(g_1)$) are equal to ± 1, it is enough to show that they are congruent modulo $2\mathfrak{p}$. In view of (12.8.1.4.1), (12.8.1.4.3), it suffices to establish the following congruence (holding in $\mathcal{O}^*$):

$$\det\big(-\mathrm{Fr}(v)_{\mathrm{geom}}, V^{I_v}\big)^{-1} \overset{?}{\equiv} \det\big(-\mathrm{Fr}(v)_{\mathrm{geom}}, V_1^{I_v}\big)^{-1} \pmod{\mathfrak{p}^{1+\mathrm{ord}_{\mathfrak{p}}(2)}}.$$

Combining the assumption 12.8.1.2.5 with Lemma 12.4.5(ii), we see that

$$V^{I_v} \neq 0 \iff V_1^{I_v} \neq 0;$$

if this is the case, then $\pi(g)_v = \mathrm{St}(\mu)$ ($\mu^2 = 1$), $\pi(g_1)_v = \mathrm{St}(\mu_1)$ ($\mu_1^2 = 1$), the characters μ and μ_1 are unramified and there are exact sequences of $\mathcal{O}[G_v]$-modules

$$0 \longrightarrow \mathcal{O}(1) \otimes \mu \longrightarrow T \longrightarrow \mathcal{O} \otimes \mu \longrightarrow 0, \quad 0 \longrightarrow \mathcal{O}(1) \otimes \mu_1 \longrightarrow T_1 \longrightarrow \mathcal{O} \otimes \mu_1 \longrightarrow 0.$$

It follows that

$$\begin{cases} \mathcal{O}/\mathfrak{p}^M \subseteq ((T/\mathfrak{p}^M T)(-1))^{G_v} & \text{if } \mu(v) = 1 \\ 2(Nv+1) \cdot ((T/\mathfrak{p}^M T)(-1))^{G_v} = 0 & \text{if } \mu(v) = -1 \end{cases}$$

and

$$\begin{cases} \mathcal{O}/\mathfrak{p}^M \subseteq ((T_1/\mathfrak{p}^M T_1)(-1))^{G_v} & \text{if } \mu_1(v) = 1 \\ 2(Nv+1) \cdot ((T_1/\mathfrak{p}^M T_1)(-1))^{G_v} = 0 & \text{if } \mu_1(v) = -1. \end{cases}$$

The isomorphism (12.8.1.4.2) together with the assumption $M > \mathrm{ord}_{\mathfrak{p}}(2(Nv+1))$ then imply that $\mu(v) = \mu_1(v)$ $(= \pm 1)$, hence

$$\det\big(-\mathrm{Fr}(v)_{\mathrm{geom}}, V^{I_v}\big)^{-1} = -\mu(v)Nv = -\mu_1(v)Nv = \det\big(-\mathrm{Fr}(v)_{\mathrm{geom}}, V_1^{I_v}\big)^{-1},$$

which concludes the proof. □

12.8.1.5. Corollary. — *If* $M \geqslant 1 + \mathrm{ord}_{\mathfrak{p}}(2) + \max\{\mathrm{ord}_{\mathfrak{p}}(Nv+1) \mid v \nmid p,\ \mathrm{ord}_v(\mathfrak{n}(g)) = 1\}$, *then*

$$\widetilde{\varepsilon}\big(\pi(g), \tfrac{1}{2}\big) = \widetilde{\varepsilon}\big(\pi(g_1), \tfrac{1}{2}\big)$$

$$\widetilde{h}^1_f(F,g) - h^1_f(F,g) + r_{\mathrm{an}}(F,g) \equiv \widetilde{h}^1_f(F,g_1) - h^1_f(F,g_1) + r_{\mathrm{an}}(F,g_1) \pmod 2$$

Proof. — Combining Proposition 12.5.9.4 and 12.8.1.4, we obtain

$$\begin{aligned}(-1)^{\widetilde{h}^1_f(F,g) - h^1_f(F,g) + r_{\mathrm{an}}(F,g)} &= \widetilde{\varepsilon}\big(\pi(g), \tfrac{1}{2}\big) \\ &= \prod_{v \nmid p\infty} \chi_v(-1)\, \varepsilon_v\big(\pi(g)_v, \tfrac{1}{2}\big) \prod_{v \mid \infty} \chi_v(-1)(-1)^{k/2} \\ &= \prod_{v \nmid p\infty} \chi_v(-1)\, \varepsilon_v\big(\pi(g_1)_v, \tfrac{1}{2}\big) \prod_{v \mid \infty} \chi_v(-1)(-1)^{k/2} \\ &= \widetilde{\varepsilon}\big(\pi(g_1), \tfrac{1}{2}\big) = (-1)^{\widetilde{h}^1_f(F,g_1) - h^1_f(F,g_1) + r_{\mathrm{an}}(F,g_1)}. \end{aligned}$$ □

12.8.2. Linear algebra. — Fix a prime element ϖ of $\mathcal{O} = \mathcal{O}_{L_{\mathfrak{p}}}$.

12.8.2.1. Definition. — Let $\alpha, \beta \geqslant 0$ be integers and $f : X \to Y$ a homomorphism of $\mathcal{O}$-modules. We say that f is an (α,β)**-morphism** if $\mathfrak{p}^\alpha \cdot \mathrm{Ker}(f) = 0$ and $\mathfrak{p}^\beta \cdot \mathrm{Coker}(f) = 0$.

12.8.2.2. Lemma

(i) *If* $f : X \to Y$ *(resp.,* $f' : Y \to Z$*) is an* (α,β)*-morphism (resp., an* (α',β')*-morphism), then* $f'f : X \to Z$ *is an* $(\alpha+\alpha', \beta+\beta')$*-morphism.*

(ii) *If* $f : X \to Y$ *is an* (α,β)*-morphism, then there exists a homomorphism* $g : Y \to X$ *satisfying* $fg = \varpi^{\alpha+\beta} \cdot \mathrm{id}_Y$, $gf = \varpi^{\alpha+\beta} \cdot \mathrm{id}_X$ *(*$\Longrightarrow$ *g is an* $(\alpha+\beta, \alpha+\beta)$*-morphism).*

(iii) *If* $f : X \to Y$ *is an* (α,β)*-morphism, so is the induced morphism* $f_n : \mathfrak{p}^n X \to \mathfrak{p}^n Y$ *(for each* $n \geqslant 0$*).*

Proof. — Elementary exercise. □

12.8.2.3. Proposition. — *Let $k, M \geqslant 1$ and $r, r_1, \alpha, \beta \geqslant 0$ be integers, W and W_1 two $\mathcal{O}/\mathfrak{p}^M$-modules of finite length and*

$$f : (\mathcal{O}/\mathfrak{p}^M)^{\oplus r} \oplus W^{\oplus k} \longrightarrow (\mathcal{O}/\mathfrak{p}^M)^{\oplus r_1} \oplus W_1^{\oplus k}$$

an (α, β)-morphism. Write $W = \mathcal{O}/\mathfrak{p}^{n_1} \oplus \cdots \oplus \mathcal{O}/\mathfrak{p}^{n_g}$ ($g \geqslant 0$, $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_g$). Put $n_0 = M$, $n_{g+1} = 0$ and assume that there exists an index $i \in \{1, \ldots, g\}$ such that $n_i > n_{i+1} + (\alpha + \beta)$ (this is automatically true if $M > (\alpha + \beta)(g + 1)$ or if $\mathfrak{p}^{M-1-\alpha-\beta} W = 0$). Then

$$r \equiv r_1 \pmod{k}.$$

Proof. — If M is an $\mathcal{O}$-module of finite length and $n \geqslant 1$, put

$$s_n(M) := \max\{t \geqslant 0 \mid (\mathcal{O}/\mathfrak{p}^n)^{\oplus t} \subseteq M\}.$$

Using Lemma 12.8.2.2 (iii), we can replace f by $f_{n_{i+1}}$, hence assume that $n_g > \alpha + \beta$. The surjections

$$\big(\mathcal{O}/\mathfrak{p}^M\big)^{\oplus r} \oplus W^{\oplus k} \twoheadrightarrow \operatorname{Im}(f) \twoheadrightarrow \big(\mathcal{O}/\mathfrak{p}^{M-\alpha}\big)^{\oplus r} \oplus (\mathfrak{p}^\alpha W)^{\oplus k}$$

resp., the inclusions

$$\mathfrak{p}^\beta (\mathcal{O}/\mathfrak{p}^M)^{\oplus r_1} \oplus (\mathfrak{p}^\beta W_1)^{\oplus k} \subseteq \operatorname{Im}(f) \subseteq (\mathcal{O}/\mathfrak{p}^M)^{\oplus r_1} \oplus W_1^{\oplus k}$$

imply that

$$s_1(\operatorname{Im}(f)) = s_{\beta+1}(\operatorname{Im}(f)) = r + k\, s_1(W)$$

resp.,

$$r_1 + k\, s_{\beta+1}(W_1) = r_1 + k\, s_1(\mathfrak{p}^\beta W_1)^{\oplus k} \leqslant s_1(\operatorname{Im}(f)) = s_{\beta+1}(\operatorname{Im}(f)) \leqslant r_1 + k\, s_{\beta+1}(W_1).$$

It follows that

$$r + k\, s_1(W) = s_1(\operatorname{Im}(f)) = r_1 + k\, s_{\beta+1}(W_1) \implies r \equiv r_1 \pmod{k}. \quad \square$$

12.8.2.4. The following example shows that the assumption $n_i - n_{i+1} > \alpha + \beta$ is necessary: the map

$$f : \big(\mathcal{O}/\mathfrak{p}^M\big) \oplus \big(\mathcal{O}/\mathfrak{p}^{M-1}\big)^{\oplus 2} \oplus \cdots \oplus \big(\mathcal{O}/\mathfrak{p}\big)^{\oplus 2} \longrightarrow (\mathcal{O}/\mathfrak{p}^{M-1})^{\oplus 2} \oplus \cdots \oplus (\mathcal{O}/\mathfrak{p})^{\oplus 2}$$

$$f(y_0, x_1, y_1, \ldots, x_{M-1}, y_{M-1}) = (\overline{y}_0, x_1, \overline{y}_1, x_2, \ldots, \overline{y}_{M-2}, x_{M-1})$$

(where $\overline{y}_i = y_i \pmod{\mathfrak{p}^{M-i-1}}$) is a $(1, 0)$-morphism

$$f : \mathcal{O}/\mathfrak{p}^M \oplus W^{\oplus 2} \longrightarrow W_1^{\oplus 2}.$$

12.8.2.5. Lemma. — *If G is a topological group and $f : X \to Y$ an (α, β)-morphism of ind-admissible $\mathcal{O}[G]$-modules, then the induced maps*

$$H^i_{\mathrm{cont}}(G, X) \longrightarrow H^i_{\mathrm{cont}}(G, Y)$$

are all $(\alpha + \beta, \alpha + \beta)$-morphisms.

Proof. — Combine the cohomology sequences associated to

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow X \longrightarrow \mathrm{Im}(f) \longrightarrow 0, \quad 0 \longrightarrow \mathrm{Im}(f) \longrightarrow Y \longrightarrow \mathrm{Coker}(f) \longrightarrow 0. \quad \square$$

12.8.3. Congruences between traces and congruences between representations

12.8.3.1. Proposition. — *Let Φ be a finite extension of $\mathbf{Q}_p$, $\mathcal{O} \subset \Phi$ its ring of integers, $\varpi \in \mathcal{O}$ a uniformizer of $\mathcal{O}$ and G a group. Let T_1 be an $\mathcal{O}[G]$-module, free of rank $n \geqslant 1$ as an $\mathcal{O}$-module.*

(i) *If $T_1 \otimes_{\mathcal{O}} \Phi$ is an absolutely simple $\Phi[G]$-module, then*

$$(\exists c \geqslant 0) \quad \mathrm{Im}(\mathcal{O}[G] \longrightarrow \mathrm{End}_{\mathcal{O}}(T_1)) \supset \varpi^c \mathrm{End}_{\mathcal{O}}(T_1).$$

(ii) *Assume that $N \geqslant 1$ an integer and T_2 is an $\mathcal{O}/\varpi^{N+5c}\mathcal{O}[G]$-module, free of rank n as an $\mathcal{O}/\varpi^{N+5c}\mathcal{O}$-module, satisfying*

$$(\forall g \in G) \quad \mathrm{Tr}_{\mathcal{O}}(g|T_1) \equiv \mathrm{Tr}_{\mathcal{O}}(g|T_2) \pmod{\varpi^{N+5c}},$$

where $c \geqslant 0$ is as in (i). Then there exists a homomorphism of $\mathcal{O}[G]$-modules $\alpha : T_1/\varpi^N T_1 \to T_2/\varpi^N T_2$ satisfying $\varpi^c \,\mathrm{Ker}(\alpha) = \varpi^c \,\mathrm{Coker}(\alpha) = 0$. If $N \geqslant c$, then there is an $\mathcal{O}[G]$-submodule $\varpi^c T_2 \subset T_2' \subset T_2$ and an isomorphism of $\mathcal{O}$-modules $T_1/\varpi^{N-c}T_1 \xrightarrow{\sim} T_2'/\varpi^{N-c}T_2'$.

Proof

(i) As $V_1 = T_1 \otimes_{\mathcal{O}} \Phi$ is an absolutely simple $\Phi[G]$-module, we have ([**Cu-Re**, §3.43, §3.32])

$$\mathrm{End}_{\Phi[G]}(V_1) = \Phi,$$
$$\mathrm{Im}(\mathcal{O}[G] \longrightarrow \mathrm{End}_{\mathcal{O}}(T_1)) \otimes_{\mathcal{O}} \Phi = \mathrm{Im}(\Phi[G] \longrightarrow \mathrm{End}_{\Phi}(V_1)) = \mathrm{End}_{\Phi}(V_1),$$

which proves the claim.

(ii) Our assumptions imply that

$$(12.8.3.1.1) \qquad (\forall a \in \mathcal{O}[G]) \quad \mathrm{Tr}_{\mathcal{O}}(a|T_1) \equiv \mathrm{Tr}_{\mathcal{O}}(a|T_2) \pmod{\varpi^{N+5c}}.$$

Fix $\mathcal{O}$-module isomorphisms $\mathcal{O}^{\oplus n} \xrightarrow{\sim} T_1$, $\left(\mathcal{O}/\varpi^{N+5c}\mathcal{O}\right)^{\oplus n} \xrightarrow{\sim} T_2$; they induce isomorphisms $\mathrm{End}_{\mathcal{O}}(T_1) \xrightarrow{\sim} M_n(\mathcal{O})$, $\mathrm{End}_{\mathcal{O}}(T_2) \xrightarrow{\sim} M_n(\mathcal{O}/\varpi^{N+5c}\mathcal{O})$. Put

$$B = \mathrm{Im}\big(\mathcal{O}[G] \longrightarrow \mathrm{End}_{\mathcal{O}}(T_1/\varpi^{N+5c}T_1) \oplus \mathrm{End}_{\mathcal{O}}(T_2/\varpi^{N+5c}T_2)\big) \subset M_n\big(\mathcal{O}/\varpi^{N+5c}\mathcal{O}\big)^{\oplus 2}$$

and denote by $i_j : B \to M_n\big(\mathcal{O}/\varpi^{N+5c}\mathcal{O}\big)$ $(j = 1, 2)$ the projections of B on the two factors. Their kernels $I_j = \mathrm{Ker}(i_j)$ are bilateral ideals of B satisfying $I_1 \cap I_2 = 0$. We claim that $\varpi^{2c} i_1(I_2) = 0$: indeed, fix $x \in i_1(I_2)$. As $i_1(I_2)$ is a bilateral ideal in $i_1(B) \supset \varpi^c M_n\big(\mathcal{O}/\varpi^{N+5c}\mathcal{O}\big)$, $i_1(I_2)$ contains the bilateral ideal $J = \varpi^c M_n\big(\mathcal{O}/\varpi^{N+5c}\mathcal{O}\big) x \varpi^c M_n\big(\mathcal{O}/\varpi^{N+5c}\mathcal{O}\big)$ of $M_n\big(\mathcal{O}/\varpi^{N+5c}\mathcal{O}\big)$, which is necessarily of the form $J = \varpi^k M_n\big(\mathcal{O}/\varpi^{N+5c}\mathcal{O}\big)$ $(2c \leqslant k \leqslant N + 5c)$. It follows from (12.8.3.1.1) that

$$(\forall A \in i_1(I_2)) \quad \mathrm{Tr}(A) = 0 \in \mathcal{O}/\varpi^{N+5c}\mathcal{O};$$

taking $A = \mathrm{diag}\big(\varpi^k, 0, \dots, 0\big)$, we obtain $\varpi^k \equiv 0 \pmod{\varpi^{N+5c}}$, hence $J = 0$. As $\varpi^{2c}x \in J$, we have $\varpi^{2c} i_1(I_2) = 0$, hence $\varpi^{2c} I_2 \subset I_1 \cap I_2 = 0$.

In the special case $c = 0$ we can conclude as follows: as $I_2 = 0$, the inequalities

$$\begin{aligned}\ell_{\mathcal{O}}\big(M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big)\big) \geqslant \ell_{\mathcal{O}}(i_2(B)) = \ell_{\mathcal{O}}(B) &= \ell_{\mathcal{O}}(i_1(B)) + \ell_{\mathcal{O}}(I_1)\\ &= \ell_{\mathcal{O}}\big(M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big)\big) + \ell_{\mathcal{O}}(I_1)\end{aligned}$$

imply that $I_1 = 0$ and $i_2(B) = M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big)$, hence both maps $i_1, i_2 : B \xrightarrow{\sim} M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big)$ are isomorphisms of $\mathcal{O}$-algebras. The isomorphism $i_2 i_1^{-1} : M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big) \xrightarrow{\sim} M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big)$ is necessarily inner (cf. Lemma 12.8.3.2 below), which implies that

$$B = \big\{\big(A, gAg^{-1}\big) \mid A \in M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big)\big\}$$

for some $g \in \mathrm{GL}_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big)$. Multiplication by g then induces the desired isomorphism of $\mathcal{O}[G]$-modules

$$T_1/\varpi^N T_1 \xrightarrow{\sim} \big(\mathcal{O}/\varpi^N\mathcal{O}\big) \xrightarrow{g} \big(\mathcal{O}/\varpi^N\mathcal{O}\big) \xrightarrow{\sim} T_2/\varpi^N T_2.$$

If $c \geqslant 1$, then the multivalued 'map' $i_1(b) \mapsto i_2(b)$ induces an isomorphism of $\mathcal{O}$-algebras $i_1(B)/i_1(I_2) \xrightarrow{\sim} i_2(B)/i_2(I_1)$, hence a homomorphism of $\mathcal{O}$-algebras

$$\begin{aligned}f : i_1(B) &\longrightarrow i_1(B)/i_1(I_2) \xrightarrow{\sim} i_2(B)/i_2(I_1)\\ &\longrightarrow M_n\big(\mathcal{O}/\varpi^{N+5c}\mathcal{O}\big)/M_n\big(\mathcal{O}/\varpi^{N+5c}\mathcal{O}\big)\big[\varpi^{2c}\big] \xrightarrow{\sim} M_n\big(\mathcal{O}/\varpi^{N+3c}\mathcal{O}\big).\end{aligned}$$

As $\varpi^c M_n\big(\mathcal{O}/\varpi^{N+5c}\mathcal{O}\big) \subset i_1(B)$, Lemma 12.8.3.2 below implies that there exists a matrix $g \in M_n(\mathcal{O}) \cap \mathrm{GL}_n(\Phi)$ such that $\varpi^c g^{-1} \in M_n(\mathcal{O})$ and

$$\big(\forall A \in M_n\big(\mathcal{O}/\varpi^{N+5c}\mathcal{O}\big)\big) \quad f(\varpi^c A) \equiv gA\varpi^c g^{-1} \pmod{\varpi^{N+c}}.$$

In particular, for $A \in i_1(B)$ we obtain a congruence

$$(\forall A \in i_1(B)) \quad f(A)g \equiv gA \ \big(\mathrm{mod}\, \varpi^N M_n\big(\mathcal{O}/\varpi^{N+5c}\mathcal{O}\big)g\big),$$

which implies that multiplication by $g \in M_n(\mathcal{O})$ induces a morphism of $\mathcal{O}[G]$-modules

$$\alpha : T_1/\varpi^N T_1 \xrightarrow{\sim} \big(\mathcal{O}/\varpi^N\mathcal{O}\big) \xrightarrow{g} \big(\mathcal{O}/\varpi^N\mathcal{O}\big) \xrightarrow{\sim} T_2/\varpi^N T_2.$$

Multiplication by $\varpi^c g^{-1} \in M_n(\mathcal{O})$ induces a morphism of $\mathcal{O}[G]$-modules $\beta : T_2/\varpi^N T_2 \to T_1/\varpi^N T_1$ satisfying $\beta\alpha = \alpha\beta = \varpi^c$; thus $\varpi^c \,\mathrm{Ker}(\alpha) = \varpi^c \,\mathrm{Coker}(\alpha) = 0$.

If, in addition, $N \geqslant c$, let $T_2' \subset T_2$ be the inverse image of $\mathrm{Im}(\alpha)$ under the canonical projection $T_2 \to T_2/\varpi^N T_2$; this is an $\mathcal{O}[G]$-submodule of T_2 containing $\varpi^c T_2$. The surjective homomorphism of $\mathcal{O}[G]$-modules (induced by α)

$$T_1/\varpi^N T_1 \twoheadrightarrow T_2'/\varpi^N T_2 \twoheadrightarrow T_2'/\varpi^{N-c} T_2'$$

factors through $\alpha' : T_1/\varpi^{N-c} T_1 \twoheadrightarrow T_2'/\varpi^{N-c} T_2'$; as $\ell_{\mathcal{O}}\big(T_1/\varpi^{N-c}T_1\big) = n(N-c) = \ell_{\mathcal{O}}\big(T_2'/\varpi^{N-c}T_2'\big)$, it follows that α' is an isomorphism. □

12.8.3.2. Lemma. — *Let $\mathcal{O}$ be a discrete valuation ring with a uniformizer $\varpi \in \mathcal{O}$. Assume that $n \geqslant 1$, $c \geqslant 0$, $N \geqslant 2c+1$ are integers and $f : \big(\mathcal{O}/\varpi^{N+c}\mathcal{O}\big)\cdot I + \varpi^c M_n\big(\mathcal{O}/\varpi^{N+c}\mathcal{O}\big) \to M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big)$ an $\mathcal{O}$-algebra homomorphism. Then there exists a non-singular matrix $g \in M_n(\mathcal{O})$ such that $\varpi^c g^{-1} \in M_n(\mathcal{O})$ and*

$$\big(\forall A \in M_n\big(\mathcal{O}/\varpi^{N+c}\mathcal{O}\big)\big) \quad f(\varpi^c A) \equiv gA\,\varpi^c g^{-1} \ \big(\mathrm{mod}\, \varpi^{N-2c} M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big)\big).$$

Proof. — If $c = 0$, then $f : M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big) \to M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big)$ is an $\mathcal{O}$-algebra homomorphism, in fact an isomorphism (as $\ell_{\mathcal{O}}\big(M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big)\big) < \infty$, it is enough to check that $\mathrm{Ker}(f) = 0$; but $\mathrm{Ker}(f)$ is a bilateral ideal in $M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big)$ not containing $\varpi^{N-1}I$, hence $\mathrm{Ker}(f) \subseteq \varpi^N M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big) = 0$). If $N = 1$, we conclude by the Skolem-Noether theorem ([**Cu-Re**, §3.63]). If $N > 1$, we can assume, by induction, that $f(A) \equiv A \ \big(\mathrm{mod}\, \varpi^{N-1}\big)$ for all $A \in M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big)$ (after replacing $f(A)$ by $g^{-1}f(A)g$ for suitable $g \in \mathrm{GL}_n(\mathcal{O})$). Writing $f(A) = A + \varpi^{N-1}h(A)$, the reduction modulo ϖ $A \mapsto h(A) \ (\mathrm{mod}\, \varpi)$ defines an $\mathcal{O}$-linear map

$$\overline{h} : M_n(\mathcal{O}/\varpi\mathcal{O}) \longrightarrow M_n(\mathcal{O}/\varpi\mathcal{O}), \quad \overline{h}(I) = 0, \quad \overline{h}(AB) = \overline{h}(A)B + A\overline{h}(B).$$

Substituting for A, B various elementary matrices, an easy calculation shows that there exists a matrix $\overline{H} \in M_n\big(\mathcal{O}/\varpi\mathcal{O}\big)$ such that $\overline{h}(A) = [\overline{H}, A] = \overline{H}A - A\overline{H}$ for all A. Choosing a lift $H \in M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big)$ of $\overline{H}$, we obtain $f(A) = (I + \varpi^{N-1}H)A(I + \varpi^{N-1}H)^{-1}$, as required.

If $c \geqslant 1$, consider $M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big)$ acting on $T/\varpi^N T = \overline{T} = \big(\mathcal{O}/\varpi^N\mathcal{O}\big)^{\oplus n}$, where $T = \mathcal{O}^{\oplus n}$. Let $\overline{U} \subset \overline{T}$ be the $\mathcal{O}/\varpi^N\mathcal{O}$-submodule of $\overline{T}$ generated by $f(\varpi^c A)\overline{T}$, for all $A \in M_n\big(\mathcal{O}/\varpi^{N+c}\mathcal{O}\big)$. It follows from

$$f(\varpi^c A)f(\varpi^c A') = f(\varpi^{2c}AA') = \varpi^c f(\varpi^c AA') \quad (A,\, A' \in M_n(\mathcal{O}/\varpi^{N+c}\mathcal{O}))$$

that

$$\varpi^c\overline{T} = f(\varpi^c I)\overline{T} \subset \overline{U}, \quad f(\varpi^c A)\overline{U} \subset \varpi^c\overline{U} \quad \big(A \in M_n\big(\mathcal{O}/\varpi^{N+c}\mathcal{O}\big)\big).$$

Let $U \subset T$ be the inverse image of $\overline{U}$ under the projection $T \to T/\varpi^N T$. Fix a matrix $s \in M_n(\mathcal{O})$ such that $s(T) = U$; then $\varpi^c s^{-1} \in M_n(\mathcal{O})$, as $\varpi^c T \subset U$. For each $A \in M_n(\mathcal{O}/\varpi^{N+c}\mathcal{O})$, the $\mathcal{O}$-linear map

$$u(A) : \overline{T} \xrightarrow{\ s\ } \overline{U} \xrightarrow{f(\varpi^c A)} \varpi^c\overline{U} \xrightarrow{\varpi^c s^{-1}} \varpi^{2c}\overline{T} \xleftarrow{\ \varpi^{2c}\ } \overline{T}/\overline{T}\big[\varpi^{2c}\big] = \big(\mathcal{O}/\varpi^{N-2c}\mathcal{O}\big)^{\oplus n}$$

satisfies $u(A)u(A') = u(AA')$, hence the map $u : M_n(\mathcal{O}/\varpi^{N+c}\mathcal{O}) \to M_n(\mathcal{O}/\varpi^{N-2c}\mathcal{O})$, $A \mapsto u(A)$, is a homomorphism of $\mathcal{O}$-algebras, factoring through $\overline{u} : M_n(\mathcal{O}/\varpi^{N-2c}\mathcal{O}) \to M_n(\mathcal{O}/\varpi^{N-2c}\mathcal{O})$. The case $c = 0$ treated above implies that there exists $t \in \mathrm{GL}_n(\mathcal{O}/\varpi^{N-2c}\mathcal{O})$ such that $\overline{u}(A) = tAt^{-1}$ for all $A \in M_n(\mathcal{O}/\varpi^{N-2c}\mathcal{O})$, hence

$$\varpi^c s^{-1} f(\varpi^c A)s \equiv \varpi^{2c}\big(tAt^{-1} \ \big(\mathrm{mod}\, \varpi^{N-2c}\big)\big),$$

$$\varpi^{2c} f(\varpi^c A) \equiv \varpi^{2c}\big(gA\,\pi^c g^{-1} \ \big(\mathrm{mod}\, \varpi^{N-2c}\big)\big)$$

holds for all $A \in M_n\big(\mathcal{O}/\varpi^N\mathcal{O}\big)$, if we put $g = st$. □

12.8.4. Comparison of Selmer groups

12.8.4.1. Let S be the set of primes of F dividing $p\mathfrak{n}(g)Q\infty$, $\Sigma = \{v \mid p\}$ and $\Sigma' = S_f - \Sigma$. We shall consider the groups

$$\widetilde{H}^i_f(Z) := \widetilde{H}^i_f(G_{F,S}, Z; \Delta(Z))$$

for

$$Z = T, V, A, T_1, V_1, A_1, A[\mathfrak{p}^k] = T/\mathfrak{p}^k T, A_1[\mathfrak{p}^k] = T_1/\mathfrak{p}^k T_1 \qquad (k \geqslant 1)$$

and Greenberg's local conditions

$$U_v^+(Z) = \begin{cases} C^\bullet_{\mathrm{cont}}(G_v, Z_v^+), & v \mid p \\ C^\bullet_{\mathrm{cont}}(G_v/I_v, Z^{I_v}), & v \in \Sigma' \end{cases} \tag{12.8.4.1}$$

(if $Z = T, V, A, A[\mathfrak{p}^k]$, then we obtain the same groups if we replace S by $S - \{\mathfrak{q}_1, \ldots, \mathfrak{q}_s\}$, by Corollary 7.8.9). Set

$$r := \widetilde{h}^1_f(F, V) = \dim_{L_\mathfrak{p}} \widetilde{H}^1_f(V), \quad r_1 := \widetilde{h}^1_f(F, V_1) = \dim_{L_\mathfrak{p}} \widetilde{H}^1_f(V_1);$$

the goal of Sect. 12.8.4 is to show that $r \equiv r_1 \pmod 2$, provided that M is large enough. Here is a sketch of the argument: we have

$$\widetilde{H}^1_f(A) \xrightarrow{\sim} (L_\mathfrak{p}/\mathcal{O})^{\oplus r} \oplus \widetilde{Ш}(A), \quad \widetilde{H}^1_f(A_1) \xrightarrow{\sim} (L_\mathfrak{p}/\mathcal{O})^{\oplus r_1} \oplus \widetilde{Ш}(A_1),$$

where

$$\widetilde{Ш}(A) = \widetilde{H}^1_f(A)/\widetilde{H}^1_f(A)_{\mathrm{div}}, \quad \widetilde{Ш}(A_1) = \widetilde{H}^1_f(A_1)/\widetilde{H}^1_f(A_1)_{\mathrm{div}}$$

are $\mathcal{O}$-modules of finite length. We show that the $\mathcal{O}$-modules $\widetilde{H}^1_f(A)[\mathfrak{p}^M]$ and $\widetilde{H}^1_f(A_1)[\mathfrak{p}^M]$ are almost the same and, using the generalized Cassels-Tate pairing, that $\widetilde{Ш}(A)$ (resp., $\widetilde{Ш}(A_1)$) is close to being isomorphic to $W \oplus W$ (resp., $W_1 \oplus W_1$). The result then follows from Proposition 12.8.2.3.

12.8.4.2. Proposition. — *For each prime $v \mid p$ of F, put*

$$a_v = \min\left\{\mathrm{ord}_\mathfrak{p}\left((N^{k-1}_{F_v/\mathbf{Q}_p} \cdot \chi_v^2)(u) - 1\right) \mid u \in \mathcal{O}^*_{F,v}\right\}$$

and define $a = \max_{v|p}(a_v)$,

$$j = \varpi^a \bar{j} : A[\mathfrak{p}^M] \longrightarrow A_1[\mathfrak{p}^M].$$

Then we have, for each $v \mid p$,

$$j\left(A[\mathfrak{p}^M]^+_v\right) \subseteq A_1[\mathfrak{p}^M]^+_v$$

and the map $j_v^+ : A[\mathfrak{p}^M]^+_v \to A_1[\mathfrak{p}^M]^+_v$ (resp., $j_v^- : A[\mathfrak{p}^M]^-_v \to A_1[\mathfrak{p}^M]^-_v$) induced by j is an $(a, 2a)$-morphism (resp., a $(2a, a)$-morphism).

Proof. — In the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A[\mathfrak{p}^M]^+_v & \xrightarrow{i_v^+} & A[\mathfrak{p}^M] & \longrightarrow & A[\mathfrak{p}^M]^-_v & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \bar{j}} & & & & \\ 0 & \longrightarrow & A_1[\mathfrak{p}^M]^+_v & \longrightarrow & A_1[\mathfrak{p}^M] & \xrightarrow{(i_1)_v^-} & A_1[\mathfrak{p}^M]^-_v & \longrightarrow & 0, \end{array}$$

the G_v-modules $Z_v^+(-k/2)\otimes\chi_v^{-1}$ and $Z_v^-(k/2-1)$ are unramified ($Z = A[\mathfrak{p}^M], A_1[\mathfrak{p}^M]$). This implies that $\mathrm{Im}((i_1)_v^-\circ\overline{j}\circ i_v^+)$ is killed by $(\chi_{\mathrm{cycl}}^{k-1}\chi_v^2)(g_v)-1$, for each $g_v \in I_v$ (where $\chi_{\mathrm{cycl}} : G_v \to \mathbf{Z}_p^*$ is the cyclotomic character), hence by $(N_{F_v/\mathbf{Q}_p}^{k-1}\cdot\chi_v^2)(u)-1$, for each $u \in \mathcal{O}_{F,v}^*$ (recall that the isomorphism of class field theory $I(\mathrm{Gal}(\mathbf{Q}_p^{ab}/\mathbf{Q}_p)) \xrightarrow{\sim} \mathbf{Z}_p^*$ is given by χ_{cycl}, not $\chi_{\mathrm{cycl}}^{-1}$, as we are using geometric Frobenius elements). This shows that $j\big(A[\mathfrak{p}^M]_v^+\big) \subseteq A_1[\mathfrak{p}^M]_v^+$, as claimed. The statements about $j_v^\pm$ follow from the exact sequence

$$0 \longrightarrow \mathrm{Ker}(j_v^+) \longrightarrow \mathrm{Ker}(j) \longrightarrow \mathrm{Ker}(j_v^-) \xrightarrow{\partial} \mathrm{Coker}(j_v^+) \longrightarrow \mathrm{Coker}(j) \longrightarrow \mathrm{Coker}(j_v^-) \longrightarrow 0,$$

as j is an (a,a)-morphism and $\mathfrak{p}^a\cdot\mathrm{Im}(\partial) = 0$. ☐

12.8.4.3. Corollary. — *The maps j and j_v^+ $(v \mid p)$ induce a $(3a,3a)$-morphism $\widetilde{H}_f^1(A[\mathfrak{p}^M]) \to \widetilde{H}_f^1(A_1[\mathfrak{p}^M])$.*

Proof. — Combine Proposition 12.8.4.2 with Lemma 12.8.2.5. ☐

12.8.4.4. Proposition

(i) $(\forall v \in \Sigma')\,(\forall Z = A, A_1)$ *$H^0(G_v, Z)$ is finite.*

(ii) *Put* $b = \max_{v\in\Sigma'}(b_v)$ *and* $b_1 = \max_{v\in\Sigma'}(b_{1,v})$, *where* $b_v = \mathrm{Tam}_v(T,\mathfrak{p})$, $b_{1,v} = \mathrm{Tam}_v(T_1,\mathfrak{p})$. *If*

$$M \geqslant \max\big\{b+1,\ \mathrm{ord}_{\mathfrak{p}}(2),\ 1+\max\{\mathrm{ord}_{\mathfrak{p}}(2(Nv+1)) \mid v\nmid p,\ \mathrm{ord}_v(\mathfrak{n}(g)) = 1\}\big\},$$

then

$$(\forall v \in \Sigma' - \{\mathfrak{q}_1,\dots,\mathfrak{q}_s\})\ b_v = b_{v,1}, \quad (\forall i = 1,\dots,s)\ b_{\mathfrak{q}_i} = 0, \quad b_{1,\mathfrak{q}_i} = \mathrm{ord}_{\mathfrak{p}}(2).$$

(iii) $(\forall v \in \Sigma')$ $\ell_{\mathcal{O}}(H^1_{\mathrm{ur}}(G_v, A)) = b_v$, $\ell_{\mathcal{O}}(H^1_{\mathrm{ur}}(G_v, A_1)) = b_{1,v}$.

(iv) $(\forall v \in \Sigma')$ $\mathfrak{p}^{b_v}\cdot H^i(G_v/I_v, A^{I_v}/\mathfrak{p}^M A^{I_v}) = 0$, $\mathfrak{p}^{b_{1,v}}\cdot H^i(G_v/I_v, A_1^{I_v}/\mathfrak{p}^M A_1^{I_v}) = 0$.

Proof

(i) This follows from the vanishing $H^0(G_v, V) = H^0(G_v, V_1) = 0$ (Proposition 12.4.8.4).

(ii) Fix $i \in \{1,\dots,s\}$. As T is unramified at $\mathfrak{q}_i$, we have $b_{\mathfrak{q}_i} = 0$. In order to compute $b_{1,\mathfrak{q}_i}$, we use the exact sequence of $\mathcal{O}[G_{\mathfrak{q}_i}]$-modules

$$0 \longrightarrow \mathcal{O}(1)\otimes\mu_1 \longrightarrow T_1 \longrightarrow \mathcal{O}\otimes\mu_1 \longrightarrow 0,$$

in which μ_1 is an unramified character satisfying $\mu_1(\mathfrak{q}_i) = -1$. As $I_{\mathfrak{q}_i}$ acts trivially on $T_1/\mathfrak{p}^M T_1 \xrightarrow{\sim} T/\mathfrak{p}^M T$, we have

$$H^1_{\mathrm{cont}}(I_{\mathfrak{q}_i}, T_1)_{\mathrm{tors}} \xrightarrow{\sim} \mathcal{O}/\mathfrak{p}^{M'}\otimes\mu_1$$

for some $M' \geqslant M \geqslant \mathrm{ord}_{\mathfrak{p}}(2)$, hence

$$H^1_{\mathrm{cont}}(I_{\mathfrak{q}_i}, T_1)_{\mathrm{tors}}^{\mathrm{Fr}(\mathfrak{q}_i)=1} \xrightarrow{\sim} \mathcal{O}/2\mathcal{O} \implies b_{1,\mathfrak{q}_i} = \mathrm{ord}_{\mathfrak{p}}(2).$$

If $v \in \Sigma' - \{\mathfrak{q}_1, \dots, \mathfrak{q}_s\}$, then

$$b_v = \ell_{\mathcal{O}}\Big(H^1(I_v, T)_{\mathrm{tors}}^{\mathrm{Fr}(v)=1}\Big) = \ell_{\mathcal{O}}\Big(\big(A^{I_v}/(A^{I_v})_{\mathrm{div}}\big)^{\mathrm{Fr}(v)=1}\Big),$$

and similarly for A_1 and $b_{1,v}$.

If $\mathrm{ord}_v(\mathfrak{n}(g)) \neq 1$, then $V^{I_v} = V_1^{I_v} = 0$ (by Lemma 12.4.5 (ii)), hence $\big(A^{I_v}\big)_{\mathrm{div}} = \big(A_1^{I_v}\big)_{\mathrm{div}} = 0$, which implies that

$$b_v = \ell_{\mathcal{O}}(H^0(G_v, A)) = \ell_{\mathcal{O}}(H^0(G_v, A_1)) = b_{1,v}$$

(using the fact that $\mathfrak{p}^{b_v}$ kills $H^0(G_v, A)$ and $A\big[\mathfrak{p}^{b_v+1}\big] = A_1\big[\mathfrak{p}^{b_v+1}\big]$).

If $\mathrm{ord}_v(\mathfrak{n}(g)) = 1$, then the proof of Proposition 12.8.1.4 shows that there are exact sequences of $\mathcal{O}[G_v]$-modules

$$0 \longrightarrow \mathcal{O}(1) \otimes \mu \longrightarrow T \longrightarrow \mathcal{O} \otimes \mu \longrightarrow 0, \quad 0 \longrightarrow \mathcal{O}(1) \otimes \mu \longrightarrow T_1 \longrightarrow \mathcal{O} \otimes \mu \longrightarrow 0$$

(where $\mu : G_v \to \{\pm 1\}$ is an unramified character). Consider the corresponding (non-trivial) extension classes

$$[T], [T_1] \in H^1_{\mathrm{cont}}(G_v, \mathcal{O}(1)) = \mathcal{O}_{F,v}^* \widehat{\otimes} \mathcal{O} \xrightarrow{\sim} \mathcal{O}$$

(the last isomorphism being induced by the valuation ord_v). It follows from the exact sequence

$$0 \longrightarrow \mathcal{O}(1)\otimes\mu \longrightarrow T^{I_v} \longrightarrow \mathcal{O}\otimes\mu \xrightarrow{[T]\otimes 1} \mathcal{O}\otimes\mu \longrightarrow H^1_{\mathrm{cont}}(I_v, T) \longrightarrow \mathcal{O}(-1)\otimes\mu \longrightarrow 0$$

that

$$b_v = \begin{cases} \ell_{\mathcal{O}}(\mathcal{O}/[T]\mathcal{O}), & \text{if } \mu(v) = 1 \\ \ell_{\mathcal{O}}(\mathcal{O}/([T], 2)\mathcal{O}), & \text{if } \mu(v) = -1 \end{cases}$$

(and similarly for T_1 and $b_{1,v}$). If $\mu(v) = 1$, then

$$\ell_{\mathcal{O}}\big((T/\mathfrak{p}^k T)^{I_v}\big) = k + \min(k, b_v), \quad \ell_{\mathcal{O}}\big((T_1/\mathfrak{p}^k T_1)^{I_v}\big) = k + \min(k, b_{1,v}) \qquad (\forall k \geqslant 0).$$

The assumption $M > b_v$ then implies that

$$2M > M + b_v = \ell_{\mathcal{O}}\big((T/\mathfrak{p}^M T)^{I_v}\big) = \ell_{\mathcal{O}}\big((T_1/\mathfrak{p}^M T_1)^{I_v}\big) = M + \min(M, b_{1,v}) \Longrightarrow b_v = b_{1,v}.$$

If $\mu(v) = -1$, then

$$\begin{aligned} &\text{either} \quad \ell_{\mathcal{O}}(\mathcal{O}/[T]\mathcal{O}) \geqslant \mathrm{ord}_{\mathfrak{p}}(2) \Longrightarrow b_v = \mathrm{ord}_{\mathfrak{p}}(2) \\ &\text{or} \quad \ell_{\mathcal{O}}(\mathcal{O}/[T]\mathcal{O}) < \mathrm{ord}_{\mathfrak{p}}(2) \Longrightarrow b_v = \ell_{\mathcal{O}}(\mathcal{O}/[T]\mathcal{O}) = \ell_{\mathcal{O}}\big((T/\mathfrak{p}^M T)^{I_v}\big) - M \end{aligned}$$

(and similarly for T_1 and $b_{1,v}$), hence

$$\begin{aligned} b_v &= \min\big(\mathrm{ord}_{\mathfrak{p}}(2), \ell_{\mathcal{O}}\big((T/\mathfrak{p}^M T)^{I_v}\big) - M\big) \\ &= \min\big(\mathrm{ord}_{\mathfrak{p}}(2), \ell_{\mathcal{O}}\big((T_1/\mathfrak{p}^M T_1)^{I_v}\big) - M\big) = b_{1,v}. \end{aligned}$$

(iii) It is enough to treat A: the first term in the exact sequence

$$\big((A^{I_v})_{\mathrm{div}}\big)/(\mathrm{Fr}(v)-1) \longrightarrow H^1_{\mathrm{ur}}(G_v, A) = (A^{I_v})/(\mathrm{Fr}(v)-1) \longrightarrow \big(A^{I_v}/(A^{I_v})_{\mathrm{div}}\big)/(\mathrm{Fr}(v)-1) \longrightarrow 0$$

is a quotient of $H^1_{\mathrm{ur}}(G_v, V) = (V^{I_v})/(\mathrm{Fr}(v)-1) = 0$ (as $\dim H^1_{\mathrm{ur}}(G_v, V) = \dim H^0(G_v, V) = 0$), while the the third term has the same length as

$$\big(A^{I_v}/\big(A^{I_v}\big)_{\mathrm{div}}\big)^{\mathrm{Fr}(v)=1} = H^1_{\mathrm{cont}}(I_v, T)^{\mathrm{Fr}(v)=1}_{\mathrm{tors}}.$$

(iv) As $X := A^{I_v}/\mathfrak{p}^M A^{I_v}$ is finite, we have

$$\begin{aligned}\ell_{\mathcal{O}}(H^0(G_v/I_v, X)) &= \ell_{\mathcal{O}}(H^1(G_v/I_v, X)) = \ell_{\mathcal{O}}(X/(\mathrm{Fr}(v)-1)X)\\ &\leqslant \ell_{\mathcal{O}}\big((A^{I_v})/(\mathrm{Fr}(v)-1)\big) = \ell_{\mathcal{O}}(H^1_{\mathrm{ur}}(G_v, A)) = b_v.\end{aligned}$$

The argument for A_1 is the same. □

12.8.4.5. Lemma. — *$\mathfrak{p}^c \cdot \widetilde{H}^0_f(A) = 0$. If $M > c$, then $\mathfrak{p}^c \cdot \widetilde{H}^0_f(A_1) = 0$.*

Proof. — By definition of c, we have $\widetilde{H}^0_f(A) \subset H^0(G_F, A) \subset A[\mathfrak{p}^c]$. If $M > c$, then $A_1[\mathfrak{p}^{c+1}] = A[\mathfrak{p}^{c+1}]$, hence $\widetilde{H}^0_f(A_1) \subset H^0(G_F, A_1) \subset A_1[\mathfrak{p}^c]$. □

12.8.4.6. Proposition. — *The canonical morphism $\widetilde{H}^1_f\big(A\big[\mathfrak{p}^M\big]\big) \to \widetilde{H}^1_f\big(A\big)\big[\mathfrak{p}^M\big]$ (resp., $\widetilde{H}^1_f(A_1[\mathfrak{p}^M]) \to \widetilde{H}^1_f(A_1)[\mathfrak{p}^M]$) is a (c, b)-morphism (resp., a (c, b_1)-morphism).*

Proof. — The natural exact sequences of complexes

$$0 \longrightarrow C^\bullet_{\mathrm{ur}}(G_v, A[\mathfrak{p}^M]) \longrightarrow C^\bullet_{\mathrm{ur}}(G_v, A) \xrightarrow{\varpi^M} C^\bullet_{\mathrm{ur}}(G_v, A) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v/I_v, A^{I_v}/\mathfrak{p}^M A^{I_v}) \longrightarrow 0 \qquad (v \in \Sigma')$$

$$0 \longrightarrow C^\bullet_{\mathrm{cont}}(G_v, A[\mathfrak{p}^M]^+_v) \longrightarrow C^\bullet_{\mathrm{cont}}(G_v, A^+_v) \xrightarrow{\varpi^M} C^\bullet_{\mathrm{cont}}(G_v, A^+_v) \longrightarrow 0 \qquad (v \mid p)$$

give rise to exact triangles

$$\widetilde{\mathbf{R}\Gamma}_f\big(A\big[\mathfrak{p}^M\big]\big) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(A) \xrightarrow{u} Y, \quad Y \xrightarrow{u'} \widetilde{\mathbf{R}\Gamma}_f(A) \longrightarrow \bigoplus_{v \in \Sigma'} C^\bullet_{\mathrm{cont}}\big(G_v/I_v, A^{I_v}/\mathfrak{p}^M A^{I_v}\big)$$

such that $u' \circ u = \varpi^M \cdot \mathrm{id}$. It follows from Proposition 12.8.4.4 (iv) and Lemma 12.8.2.5 that the maps $H^i(u') : H^i(Y) \to \widetilde{H}^i_f(A)$ are (b, b)-morphisms, hence the $\mathcal{O}$-modules

$$\mathrm{Coker}\big(\widetilde{H}^i_f\big(A\big[\mathfrak{p}^M\big]\big) \longrightarrow \widetilde{H}^i_f\big(A\big)\big[\mathfrak{p}^M\big]\big) \subseteq \mathrm{Ker}(H^i(u'))$$

are all killed by $\mathfrak{p}^b$. According to Lemma 12.8.4.5, the kernel

$$\mathrm{Ker}\big(\widetilde{H}^1_f\big(A\big[\mathfrak{p}^M\big]\big) \longrightarrow \widetilde{H}^1_f\big(A\big)\big[\mathfrak{p}^M\big]\big) \subseteq H^0(X) \subseteq \widetilde{H}^0_f(A)$$

is killed by $\mathfrak{p}^c$. The argument for A_1 is the same (if $M \leqslant c$, then the statement about the kernel is trivially true). □

12.8.4.7. Proposition. — *The canonical map* $\widetilde{\mathbf{R}\Gamma}_f(T)\overset{\mathbf{L}}{\otimes}_{\mathcal{O}} L_{\mathfrak{p}}/\mathcal{O} \to \widetilde{\mathbf{R}\Gamma}_f(A)$ *(resp.,* $\widetilde{\mathbf{R}\Gamma}_f(T_1)\overset{\mathbf{L}}{\otimes}_{\mathcal{O}} L_{\mathfrak{p}}/\mathcal{O} \to \widetilde{\mathbf{R}\Gamma}_f(A_1)$*) defines, for each* i*, a* (b,b)*-morphism* $\widetilde{H}^{i+1}_f(T)_{\mathrm{tors}} \to \widetilde{H}^i_f(A)/\widetilde{H}^i_f(A)_{\mathrm{div}}$ *(resp., a* (b_1,b_1)*-morphism* $\widetilde{H}^{i+1}_f(T_1)_{\mathrm{tors}} \to \widetilde{H}^i_f(A_1)/\widetilde{H}^i_f(A_1)_{\mathrm{div}}$*).*

Proof. — In the exact triangle

$$Z = \widetilde{\mathbf{R}\Gamma}_f(T)\overset{\mathbf{L}}{\otimes}_{\mathcal{O}} L_{\mathfrak{p}}/\mathcal{O} \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(A) \longrightarrow \bigoplus_{v\in\Sigma'} \mathrm{Err}_v(T),$$

the cohomology groups $H^i(\mathrm{Err}_v(T))$ vanish for $i \neq 0,1$ (resp., are killed by $\mathfrak{p}^{b_v}$ for $i=0,1$), by 7.6.9. Combining the corresponding cohomology sequence with the Snake Lemma applied to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{H}^i_f(T)\otimes_{\mathcal{O}} L_{\mathfrak{p}}/\mathcal{O} & \longrightarrow & H^i(Z) & \longrightarrow & \widetilde{H}^{i+1}_f(T)_{\mathrm{tors}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{H}^i_f(A)_{\mathrm{div}} & \longrightarrow & \widetilde{H}^i_f(A) & \longrightarrow & \widetilde{H}^i_f(A)/\widetilde{H}^i_f(A)_{\mathrm{div}} & \longrightarrow & 0 \end{array}$$

yields the result for T, A (as the left vertical map is surjective). The argument for T_1, A_1 is the same. □

12.8.4.8. Corollary. — *The maps from 12.8.4.7 induce a* $(b,2b)$*-morphism (resp., a* $(b_1,2b_1)$*-morphism)* $\widetilde{H}^2_f(T)[\mathfrak{p}^M] \to \big(\widetilde{H}^1_f(A)/\widetilde{H}^1_f(A)_{\mathrm{div}}\big)[\mathfrak{p}^M]$ *(resp.,* $\widetilde{H}^2_f(T_1)[\mathfrak{p}^M] \to \big(\widetilde{H}^1_f(A_1)/\widetilde{H}^1_f(A_1)_{\mathrm{div}}\big)[\mathfrak{p}^M]$*).*

12.8.4.9. Proposition

(i) *There exists an injective morphism of* $\mathcal{O}[G_F]$*-modules* $\nu : T \to T^*(1)$ *satisfying* $\nu(T) \not\subset \mathfrak{p}T^*(1)$ *and* $\mathfrak{p}^c\cdot\mathrm{Coker}(\nu) = 0$*. This morphism is skew-symmetric (*i.e., $\nu^*(1) = -\nu$*) and unique up to a scalar multiple by an element of* $\mathcal{O}^*$*. For each prime* $v \mid p$ *of* F*,* $\nu(T_v^+)\otimes_{\mathcal{O}} L_{\mathfrak{p}} = (V_v^-)^*(1)$*.*

(ii) *If* $M > c$*, then there exists an injective morphism of* $\mathcal{O}[G_F]$*-modules* $\nu_1 : T_1 \to T_1^*(1)$ *satisfying* $\nu_1(T_1) \not\subset \mathfrak{p}T_1^*(1)$ *and* $\mathfrak{p}^c\cdot\mathrm{Coker}(\nu_1) = 0$*. It is skew-symmetric (*i.e., $\nu_1^*(1) = -\nu_1$*) and unique up to a scalar multiple by an element of* $\mathcal{O}^*$*. For each prime* $v \mid p$ *of* F*,* $\nu_1((T_1)_v^+)\otimes_{\mathcal{O}} L_{\mathfrak{p}} = ((V_1)_v^-)^*(1)$*.*

Proof

(i) We know that there exists a skew-symmetric (*i.e.*, such that $\nu_V^*(1) = -\nu_V$) isomorphism of $L_{\mathfrak{p}}[G_F]$-modules $\nu_V : V \xrightarrow{\sim} V^*(1)$. Let $k \in \mathbf{Z}$ be the smallest integer such that $\varpi^k\nu_V(T) \subseteq T^*(1)$ and put $\nu = \varpi^k\nu_V$. As $\nu(T) \subset T^*(1)$ are G_F-stable $\mathcal{O}$-lattices in $V^*(1) \xrightarrow{\sim} V$ satisfying $\nu(T) \not\subset \mathfrak{p}T^*(1)$, we must have $\mathfrak{p}^cT^*(1) \subseteq \nu(T)$. The absolute irreducibility of $V \xrightarrow{\sim} V^*(1)$ implies that ν_V is unique up to a scalar multiple in $L_{\mathfrak{p}}^*$, hence ν is unique up to an element of $\mathcal{O}^*$. If $v \mid p$, then the vanishing of

$$\mathrm{Hom}_{\mathcal{O}[G_v]}(V_v^+, (V_v^+)^*(1)) = \mathrm{Hom}_{\mathcal{O}[G_v]}(V_v^+, V_v^-) = 0$$

implies that $\nu(V_v^+) = (V_v^-)^*(1)$.

(ii) The isomorphism of $\mathcal{O}[G_F]$-modules $T/\mathfrak{p}^{c+1}T \xrightarrow{\sim} T_1/\mathfrak{p}^{c+1}T_1$ implies that the image of the subring $R_1 := \mathrm{Im}(\mathcal{O}[G_F] \to \mathrm{End}_{\mathcal{O}}(T_1)) \subseteq \mathrm{End}_{\mathcal{O}}(T_1) = E$ in $\mathrm{End}_{\mathcal{O}}(T_1/\mathfrak{p}^{c+1}T_1) = E/\mathfrak{p}^{c+1}E$ contains $\mathfrak{p}^c\mathrm{End}_{\mathcal{O}}(T_1/\mathfrak{p}^{c+1}T_1) = \mathfrak{p}^cE/\mathfrak{p}^{c+1}E$; thus $(R_1 \cap \mathfrak{p}^cE) + \mathfrak{p}^{c+1}E = \mathfrak{p}^cE$. Nakayama's Lemma then implies that $R_1 \cap \mathfrak{p}^cE = \mathfrak{p}^cE$, hence $R_1 \supseteq \mathfrak{p}^c\mathrm{End}_{\mathcal{O}}(T_1)$. We apply the argument from (i) to T_1. □

12.8.4.10. Corollary. — *Consider Greenberg's local conditions 12.8.4.1 for $Z = T^*(1)$ (with $T^*(1)_v^+ = T^*(1) \cap (V_v^-)^*(1)$ for $v \mid p$), and similarly for $T_1^*(1)$. The map ν induces, for each i, a (c,c)-morphism $\widetilde{H}^i_f(T) \to \widetilde{H}^i_f(T^*(1))$. If $M > c$, then ν_1 induces, for each i, a (c,c)-morphism $\widetilde{H}^i_f(T_1) \to \widetilde{H}^i_f(T_1^*(1))$.*

Proof. — It follows from Proposition 12.8.4.9(i) that ν defines an injective morphism of complexes $\nu_* : \widetilde{C}^\bullet_f(T) \to \widetilde{C}^\bullet_f(T^*(1))$. As $\mathfrak{p}^c$ kills $T^*(1)_v^+/\nu(T_v^+)$ (resp., $T^*(1)^{I_v}/\nu(T^{I_v})$) for $v \mid p$ (resp., for $v \in \Sigma'$), it also kills $\mathrm{Coker}(\nu_*)$. We conclude by Lemma 12.8.2.5 (and similarly for T_1 and ν_1). □

12.8.4.11. Proposition

(i) *There exists an $\mathcal{O}$-module W' of finite length and a $(b + c + 2\,\mathrm{ord}_{\mathfrak{p}}(2), 0)$-morphism $\widetilde{H}^2_f(T)_{\mathrm{tors}} \to W' \oplus W'$.*

(ii) *If $M > c$, then there exists an $\mathcal{O}$-module W_1' of finite length and a $(b + c + 2\,\mathrm{ord}_{\mathfrak{p}}(2), 0)$-morphism $\widetilde{H}^2_f(T_1) \to W_1' \oplus W_1'$.*

Proof

(i) The generalized Cassels-Tate pairing

$$\widetilde{H}^2_f(T)_{\mathrm{tors}} \times \widetilde{H}^2_f(T^*(1))_{\mathrm{tors}} \longrightarrow L_{\mathfrak{p}}/\mathcal{O}$$

defined in 10.2.2 has left kernel killed by $2\mathfrak{p}^b$, according to Theorem 10.2.3 (the factor 2 comes from the contribution of the archimedean primes). The induced pairing

$$\langle\ ,\ \rangle : \widetilde{H}^2_f(T)_{\mathrm{tors}} \times \widetilde{H}^2_f(T)_{\mathrm{tors}} \xrightarrow{\mathrm{id}\times 2\nu_*} \widetilde{H}^2_f(T)_{\mathrm{tors}} \times \widetilde{H}^2_f(T^*(1))_{\mathrm{tors}} \longrightarrow L_{\mathfrak{p}}/\mathcal{O}$$

is alternating, by Proposition 10.2.5. As $2\mathfrak{p}^c \cdot \mathrm{Coker}(2\nu_*) = 0$, by Corollary 12.8.4.10, it follows that the kernel of $\langle\ ,\ \rangle$ is killed by $4\mathfrak{p}^{b+c}$. The quotient of $\widetilde{H}^2_f(T)_{\mathrm{tors}}$ by the kernel of $\langle\ ,\ \rangle$ then admits a symplectic pairing with values in $L_{\mathfrak{p}}/\mathcal{O}$, hence is isomorphic to $W' \oplus W'$ for some $\mathcal{O}$-module of finite length W'.

(ii) The same argument applies to T_1. □

12.8.4.12. Corollary

(i) *Put $W = W'[\mathfrak{p}^M]$. Then there is a $(b+c+2\,\mathrm{ord}_{\mathfrak{p}}(2), b+c+2\,\mathrm{ord}_{\mathfrak{p}}(2))$-morphism $\widetilde{H}^2_f(T)[\mathfrak{p}^M] \to W \oplus W$.*

(ii) *If $M > c$, put $W_1 = W_1'[\mathfrak{p}^M]$. Then there is a $(b+c+2\,\mathrm{ord}_{\mathfrak{p}}(2), b+c+2\,\mathrm{ord}_{\mathfrak{p}}(2))$-morphism $\widetilde{H}^2_f(T_1)[\mathfrak{p}^M] \to W_1 \oplus W_1$.*

12.8.4.13. Putting together 12.8.4.6–12.8.4.12, we obtain, for

$$M \geqslant \max\big(b+1, c+1, \mathrm{ord}_{\mathfrak{p}}(2), 1+\max\{\mathrm{ord}_{\mathfrak{p}}(2(Nv+1)) \mid v \nmid p,\ \mathrm{ord}_v(\mathfrak{n}(g)) = 1\}\big), \tag{12.8.4.1}$$

a chain of (α_i,β_i)-morphisms f_i

$$\begin{aligned}(\mathcal{O}/\mathfrak{p}^M)^{\oplus r} \oplus W \oplus W &\xrightarrow{f_1} (\mathcal{O}/\mathfrak{p}^M)^{\oplus r} \oplus \widetilde{H}^2_f(T)[\mathfrak{p}^M]\\ &\xrightarrow{f_2} \widetilde{H}^1_f(A)[\mathfrak{p}^M] \xrightarrow{f_3} \widetilde{H}^1_f(A[\mathfrak{p}^M]) \xrightarrow{f_4} \widetilde{H}^1_f(A_1[\mathfrak{p}^M]) \xrightarrow{f_5} \widetilde{H}^1_f(A_1)[\mathfrak{p}^M]\\ &\xrightarrow{f_6} (\mathcal{O}/\mathfrak{p}^M)^{\oplus r_1} \oplus \widetilde{H}^2_f(T_1)[\mathfrak{p}^M] \xrightarrow{f_7} (\mathcal{O}/\mathfrak{p}^M)^{\oplus r_1} \oplus W_1 \oplus W_1\end{aligned}$$

with

$$\begin{aligned}(\alpha_1,\beta_1) &= (2b+2c+4\,\mathrm{ord}_{\mathfrak{p}}(2), 2b+2c+4\,\mathrm{ord}_{\mathfrak{p}}(2))\\ (\alpha_2,\beta_2) &= (b, 2b)\\ (\alpha_3,\beta_3) &= (b+c, b+c)\\ (\alpha_4,\beta_4) &= (3a, 3a)\\ (\alpha_5,\beta_5) &= (b_1+c, b_1+c)\\ (\alpha_6,\beta_6) &= (3b_1, 3b_1)\\ (\alpha_7,\beta_7) &= (b_1+c+2\,\mathrm{ord}_{\mathfrak{p}}(2), b_1+c+2\,\mathrm{ord}_{\mathfrak{p}}(2))\end{aligned}$$

Their composition is an (α,β)-morphism

$$(\mathcal{O}/\mathfrak{p}^M)^{\oplus r} \oplus W \oplus W \longrightarrow (\mathcal{O}/\mathfrak{p}^M)^{\oplus r_1} \oplus W_1 \oplus W_1, \tag{12.8.4.2}$$

where

$$\begin{aligned}\alpha &= 3a+4b+5b_1+5c+6\,\mathrm{ord}_{\mathfrak{p}}(2) \leqslant 3a+9b+5c+10\,\mathrm{ord}_{\mathfrak{p}}(2)\\ \beta &= 3a+5b+5b_1+5c+6\,\mathrm{ord}_{\mathfrak{p}}(2) \leqslant 3a+10b+5c+11\,\mathrm{ord}_{\mathfrak{p}}(2)\end{aligned}$$

(as $b_1 \leqslant b + \mathrm{ord}_{\mathfrak{p}}(2)$).

12.8.4.14. Proposition. — *Let $t \geqslant 0$ be an integer such that $\mathfrak{p}^t \cdot \widetilde{H}^2_f(T)_{\mathrm{tors}} = 0$. If M satisfies (12.8.4.1) and $M > t + 6a + 19b + 10c + 21\,\mathrm{ord}_{\mathfrak{p}}(2)$, then*

(i) $\widetilde{h}^1_f(F,V) = r \equiv r_1 = \widetilde{h}^1_f(F,V_1) \pmod 2$.

(ii) $h^1_f(F,V) - r_{\mathrm{an}}(F,g) \equiv h^1_f(F,V_1) - r_{\mathrm{an}}(F,g_1) \pmod 2$.

Proof

(i) The assumption $\mathfrak{p}^t \cdot \widetilde{H}^2_f(T)_{\mathrm{tors}} = 0$ implies that $\mathfrak{p}^t W = 0$. Applying Proposition 12.8.2.3 to the morphism (12.8.4.2) then gives $r \equiv r_1 \pmod 2$ (as $M > t+\alpha+\beta$). The statement (ii) is a consequence of (i) and Corollary 12.8.1.5. □

12.8.4.15. Proposition. — *Assume that $p \neq 2$, the residual representation of V is absolutely irreducible, $(\forall v \mid \mathfrak{n}(g), v \nmid p)$ $\mathrm{Tam}_v(T,\mathfrak{p}) = 0$ and $(\forall v \mid p)\,(\exists u \in \mathcal{O}^*_{F,v})$ $\mathfrak{p} \nmid \big((N^{k-1}_{F_v/\mathbf{Q}_p} \cdot \chi^2_v)(u) - 1\big)$ (the last condition holds with $u = -1$ if $[F_v : \mathbf{Q}_p]$ is odd for all*

$v \mid p$). Then the conclusions of 12.8.4.14 hold if $M \geqslant 1$ and $M \geqslant 1+\max\{\mathrm{ord}_{\mathfrak{p}}(Nv+1) \mid v \nmid p,\ \mathrm{ord}_v(\mathfrak{n}(g)) = 1\}$.

Proof. — The assumptions imply that $a = b = c = \mathrm{ord}_{\mathfrak{p}}(2) = 0$, hence (12.8.4.2) is an isomorphism, which proves the congruence $r \equiv r_1 \pmod 2$. We conclude as in 12.8.4.14. □

12.9. Parity results in the dihedral case

In this section we assume that $f \in S_k(\mathfrak{n}, \varphi)$, $g = f \otimes \chi \in S_k(\mathfrak{n}(g), 1)$ and $V = V(g)(k/2) = V(f)(k/2) \otimes \chi$ are as in 12.6.4.1 (*i.e.*, f is p-ordinary and $\varphi = \chi^{-2}$).

12.9.1. Fix a prime $P \mid p$ of F. Throughout 12.9, we assume that the following condition is satisfied:

12.9.1.1. If $2 \mid [F : \mathbf{Q}]$, then $(\exists \mathfrak{q} \neq P)\ 2 \nmid \mathrm{ord}_{\mathfrak{q}}(\mathfrak{n}(g))$.

If $2 \mid [F : \mathbf{Q}]$, then fix such a prime $\mathfrak{q}$. It follows from the remarks at the end of 12.5.5 that

$$\text{if } k \neq 2, \text{ then } \mathfrak{q} \nmid p.$$

12.9.2. Fix a totally imaginary quadratic extension K/F satisfying the following two conditions (in which $\eta = \eta_{K/F} : \mathbf{A}_F^*/F^* \to \{\pm 1\}$ denotes the quadratic character associated to K/F):

12.9.2.1. $\big(d_{K/F}, \mathfrak{n}(g)^{(P)}\big) = (1)$.

12.9.2.2. $\eta\big(\mathfrak{n}(g)^{(P)}\big) = (-1)^{[F:\mathbf{Q}]-1}$.

For example, any K in which all primes of F dividing $\mathfrak{n}(g)^{(P)}$ split (resp., in which $\mathfrak{q}$ is inert and all primes $v \neq \mathfrak{q}$ dividing $\mathfrak{n}(g)^{(P)}$ split) will do if $2 \nmid [F : \mathbf{Q}]$ (resp., if $2 \mid [F : \mathbf{Q}]$).

12.9.3. Similarly as in 12.6.4.6, we set

$$K[P^\infty] = \bigcup_{n \geqslant 1} K[P^n], \quad G(P^\infty) = \mathrm{Gal}(K[P^\infty]/K),$$
$$K[p^\infty] = \bigcup_{n \geqslant 1} K[p^n], \quad G(p^\infty) = \mathrm{Gal}(K[p^\infty]/K).$$

As $K[P^\infty] \subset K[p^\infty]$, there is a canonical epimorphism $G(p^\infty) \to G(P^\infty)$. The torsion subgroup $G(P^\infty)_{\mathrm{tors}}$ of $G(P^\infty)$ is finite and the quotient $G(P^\infty)/G(P^\infty)_{\mathrm{tors}}$ is isomorphic to $\mathbf{Z}_p^{r_P}$, where $r_P = [F_P : \mathbf{Q}_p]$. Fix a character $\beta_0 : G(P^\infty)_{\mathrm{tors}} \to \overline{L}_{\mathfrak{p}}^*$.

12.9.4. Lemma. — *Assume that K/F is as in 12.9.2 and $\beta : G(P^\infty) \to \mathbf{C}^*$ is a ring class character of conductor $c(\beta) = P^n$ ($n \geqslant 0$). If P splits in K/F or if $n >> 0$, then $r_{\mathrm{an}}(K, g, \beta) \equiv 1 \pmod 2$.*

Proof. — We apply Proposition 12.6.3.8. If P splits in K/F, then $\eta(P) = 1$ and $\mathfrak{n}(g)^{(d_{K/F}P)} = (1)$, which implies that

$$R(\beta)^0 = R(\beta)^0 \cap \{P\} = \varnothing = R(\beta)^- \cap \{P\}$$

for any ring class character β, hence

$$\varepsilon(\beta) = (-1)^{[F:\mathbf{Q}]}\,\eta(\mathfrak{n}(g)) = (-1)^{[F:\mathbf{Q}]}\,\eta\big(\mathfrak{n}(g)^{(P)}\big) = -1,$$

by 12.9.2.2. Similarly, if $c(\beta) = P^n$ with $n >> 0$, then

$$R(\beta)^0 = R(\beta)^0 \cap \{P\} = \varnothing = R(\beta)^- \cap \{P\}.$$

If P is ramified in K/F, then $\mathfrak{n}(g)^{(d_{K/F})} = \mathfrak{n}(g)^{(P)}$ and $R(1)^- \cap \{P\} = \varnothing$, hence

$$\varepsilon(\beta) = (-1)^{[F:\mathbf{Q}]}\,\eta\big(\mathfrak{n}(g)^{(P)}\big) = -1.$$

If P is inert in K/F, then

$$-\varepsilon(\beta) = (-1)^{[F:\mathbf{Q}]-1}\,\eta(\mathfrak{n}(g))\,(-1)^{|R(1)^- \cap \{P\}|} = \eta(P)^{\mathrm{ord}_P(\mathfrak{n}(g))}\,(-1)^{|R(1)^- \cap \{P\}|} = 1,$$

where the last equality follows from the formulas used in the proof of Corollary 12.6.3.13 (recall that $\pi(g)_P$ is not supercuspidal, since $P \mid p$ and f is p-ordinary). $\square$

12.9.5. Theorem. — *Let $g = f \otimes \chi \in S_k(\mathfrak{n}(g), 1)$, $P \mid p$, K/F and $\beta_0 : G(P^\infty)_{\mathrm{tors}} \to \overline{L}_\mathfrak{p}^*$ be as in 12.9.1–12.9.3. Assume that the form g does not have CM by any totally imaginary quadratic extension K' of F contained in $K[P^\infty]^{\mathrm{Ker}(\beta_0)}$. If $k \neq 2$, assume that $\big(d_{K/F}^{(P)}, (p)\big) = 1$. Then, for any ring class field character of finite order $\beta : G(p^\infty) \to \overline{L}_\mathfrak{p}^*$ such that $\beta|_{G(p^\infty)_{\mathrm{tors}}}$ is induced by β_0* via *the canonical map $G(p^\infty)_{\mathrm{tors}} \to G(P^\infty)_{\mathrm{tors}}$, we have*

$$r_{\mathrm{an}}(K, g, \beta) \equiv h^1_f(K, V \otimes \beta) \pmod 2.$$

If $c(\beta) = P^n$ and (P splits in K/F or $n >> 0$), then

$$r_{\mathrm{an}}(K, g, \beta) \equiv h^1_f(K, V \otimes \beta) \equiv 1 \pmod 2.$$

Proof. — The proof consists of three steps, the last of which is inspired by the proof of Theorem B in [**Ne3**].

12.9.5.1. Reduction to the case $k = 2$. — If $k = 2$, then we go directly to 12.9.5.2.

If $k \neq 2$, then we can embed (the p-stabilization f^0 of) f into a Hida family, as in 12.7.5: there exists an arithmetic point $\mathcal{P}$ above $(P_{k,\varepsilon})$ such that $f^0 = f_\mathcal{P}$.

If $p \neq 2$ (resp., if $p = 2$), then the assumption 12.7.9.1 (resp., 12.7.11.1) is satisfied – possibly after enlarging $L_\mathfrak{p}$ if $p = 2$ – and we have $g = g_\mathcal{P}$ (resp., $g = g_{\widetilde{\mathcal{P}}}$). As in 12.7.9–12.7.11, choose a character ε' (resp., $\widetilde{\varepsilon}'$) of sufficiently large order and an arithmetic point $\mathcal{P}'$ (resp., $\widetilde{\mathcal{P}}'$) above $P_{2,\varepsilon'}\Lambda' \cap \Lambda$. We obtain a p-ordinary cuspidal eigenform $f' = f_{\mathcal{P}'}$ of weight $(2, \ldots, 2)$ and its twist $g' = f' \otimes \chi' \in S_2(\mathfrak{n}(g'), 1)$.

We must check that the form g' also satisfies 12.9.1.1 and 12.9.2.1–12.9.2.2. We use repeatedly the equality $\mathfrak{n}(g)^{(p)} = \mathfrak{n}(g')^{(p)}$ proved in Corollary 12.7.14.3 (ii). Firstly, if $2 \mid [F : \mathbf{Q}]$ and if $2 \nmid \mathrm{ord}_{\mathfrak{q}}(\mathfrak{n}(g))$, then $\mathfrak{q} \nmid p$ (as $k \neq 2$), hence $2 \nmid \mathrm{ord}_{\mathfrak{q}}(\mathfrak{n}(g'))$; thus 12.9.1.1 holds. Secondly, the assumption $\left(d_{K/F}^{(P)}, (p)\right) = 1$ implies that $\left(d_{K/F}, \mathfrak{n}(g')^{(P)}\right) = (1)$, proving 12.9.2.1. Thirdly, we have assumed that the order of ε' was large. This implies that $(\mathfrak{n}(g')_{\mathrm{St}}, (p)) = (1)$, hence $\pi(g')_v$ is in the principal series ($\Longrightarrow 2 \mid \mathrm{ord}_v(\mathfrak{n}(g'))$), for each prime $v \mid p$. The same holds for g (as $k \neq 2$), hence

$$\eta_{K/F}(\mathfrak{n}(g')^{(P)}) = \eta_{K/F}(\mathfrak{n}(g')^{(p)}) = \eta_{K/F}(\mathfrak{n}(g)^{(p)}) = \eta_{K/F}(\mathfrak{n}(g)^{(P)}),$$

proving 12.9.2.2. Finally, the form g' has CM by a quadratic extension K' of F iff g has (by 12.7.14.5 (v)), which implies that the assumptions of Theorem 12.9.5 are also satisfied by g'.

The statement of Proposition 12.7.14.5 (iv) then shows that Theorem 12.9.5 holds for g' iff it holds for g.

12.9.5.2. Passage to a Shimura curve. — Thanks to 12.9.5.1, we can assume that $k = 2$. Put

$$\mathcal{R} = \{v \mid \mathfrak{n}(g)^{(P)},\ 2 \nmid \mathrm{ord}_v(\mathfrak{n}(g)),\ v \text{ is inert in } K/F\};$$

the condition 12.9.2.2 implies that $|\mathcal{R}| \equiv [F : \mathbf{Q}] - 1 \pmod 2$. Fix an archimedean prime τ_1 of F and denote by B the quaternion algebra B over F ramified at the set $\mathrm{Ram}(B) = \{v|\infty,\ v \neq \tau_1\} \cup \mathcal{R}$. By construction,

$$(\forall v \in \mathcal{R}) \quad \pi(g)_v \text{ is not in the principal series}$$

(using (12.3.9.2)–(12.3.9.3)), which implies (cf. 12.4.7) that $\pi(g)$ is associated by the Jacquet-Langlands correspondence to an irreducible automorphic representation π' of $B^*_{\mathbf{A}}$ with trivial central character. More precisely, we have

$$(\forall v \notin \mathrm{Ram}(B)) \quad \pi'_v \xrightarrow{\sim} \pi(g)_v$$

(as representations of $B^*_v \xrightarrow{\sim} \mathrm{GL}_2(F_v)$).

Fix an F-embedding $K \hookrightarrow B$ (it exists, since $K \otimes_F F_v$ is a field for each prime $v \in \mathrm{Ram}(B)$). One can describe an explicit level subgroup of π' as follows (see [**Zh2**, § 1.2]; [**Cor-Va**, (6)]): write $\mathfrak{n}(g) = \mathfrak{n}(g)^{(P)} P^{\delta}$ ($\delta \geqslant 0$) and fix an Eichler order $R_0 \subset B$ of level P^{δ} such that the conductor of the $\mathcal{O}_F$-order $\mathcal{O} = \mathcal{O}_K \cap R_0 \subset \mathcal{O}_K$ is a power of P. By construction, there exists an ideal $J \subset \mathcal{O}_K$ such that

$$N_{K/F}(J) \prod_{\lambda \in \mathcal{R}} \lambda = \mathfrak{n}(g)^{(P)}.$$

Fix such an ideal J and put $R = \mathcal{O} + (J \cap \mathcal{O}) \cdot R_0$. Then we have, for each finite prime v of F,

$$\dim_{\mathbf{C}}(\pi'_v)^{R^*_v} = 1 \qquad (12.9.5.1)$$

([**Zh1**, Thm. 3.2.2]).

In what follows, we are going to use the notation from [**Ne4**] (see also [**Cor-Va**]). Denote by N_H the Shimura curve over F associated to the subgroup $H\widehat{F}^* \subset \widehat{B}^*$, where $H = \widehat{R}^*$. The complex points of $N_H \otimes_{F,\tau_1} \mathbf{C}$ are equal to $B^*\backslash\big((\mathbf{C}-\mathbf{R}) \times \widehat{B}^*/H\widehat{F}^*\big)$, with B^* acting on $\mathbf{C}-\mathbf{R}$ *via* a fixed isomorphism $B \otimes_{F,\tau_1} \mathbf{R} \xrightarrow{\sim} M_2(\mathbf{R})$. If $F = \mathbf{Q}$ and $\mathcal{R} = \varnothing$, then N_H is a classical modular curve and we put $N_H^* = N_H \cup \{\text{cusps}\}$. In all other cases N_H is a proper curve over $\mathrm{Spec}(F)$ and we put $N_H^* = N_H$. In general, the curve N_H^* is not geometrically irreducible; one defines its "Jacobian" as $J(N_H^*) = \underline{\mathrm{Pic}}^\circ_{N_H^*/F}$.

The multiplicity one result (12.9.5.1) implies that the representation π' is generated by an automorphic form Φ (unique up to a scalar multiple) of level H. In geometric terms, such a form defines a one-dimensional subspace

$$\mathbf{C}\cdot\Phi \subset \Gamma(N_H^*, \Omega^1_{N_H^*/F}) \otimes_{F,\tau_1} \mathbf{C} = \Gamma(J(N_H^*), \Omega^1_{J(N_H^*)/F}) \otimes_{F,\tau_1} \mathbf{C},$$

on which the level H Hecke algebra $\mathbf{T}_H$ of $\widehat{B}^*$ acts with the same eigenvalues as it does on the one-dimensional space

$$(\pi'^{\infty})^H = \otimes'_{v\nmid\infty}(\pi'_v)^{R_v^*}.$$

The quotient abelian variety

$$A_1 = J(N_H^*)/\mathrm{Ann}_{\mathbf{T}_H}(\mathbf{C}\cdot\Phi)\cdot J(N_H^*)$$

is an abelian variety of GL_2-type defined over F. More precisely, the number field L_1 generated by the Hecke eigenvalues of $\mathbf{T}_H$ acting on $\mathbf{C}\cdot\Phi$ is totally real, it is equipped with a natural isomorphism $L_1 \xrightarrow{\sim} \mathrm{End}_F(A_1)\otimes\mathbf{Q}$, and its degree is equal to $[L_1 : \mathbf{Q}] = \dim(A_1)$. Denoting by $\mathfrak{p}_1$ the prime of L_1 induced by ι_p, then the generalized Eichler-Shimura relation implies that there is an isomorphism of $G_{F,S}$-modules

$$V_p(A_1)\otimes_{(L_1)_{\mathfrak{p}_1}} L_{\mathfrak{p}} \xrightarrow{\sim} V(g)(1) = V$$

(cf. the discussion in [**Ne4**, § 1]).

12.9.5.3. *CM* **points.** — The set CM_H of *CM* points by K inside

$$(N_H \otimes_{F,\tau_1}\mathbf{C})(\mathbf{C}) = B^*\backslash\big((\mathbf{C}-\mathbf{R})\times\widehat{B}^*/H\widehat{F}^*\big)$$

is equal to $K^*\backslash(\{z\}\times\widehat{B}^*/H\widehat{F}^*)$, where $z \in \mathbf{C}-\mathbf{R}$ is the unique fixed point of

$$K^* \hookrightarrow B^* \hookrightarrow (B\otimes_{F,\tau_1}\mathbf{R})^* \xrightarrow{\sim} \mathrm{GL}_2(\mathbf{R})$$

with $\mathrm{Im}(z) > 0$. For each $g \in \widehat{B}^*$, the *CM* point $x = [z,g] \in \mathrm{CM}_H$ represented by $(z, gH\widehat{F}^*)$ is defined over the ring class field $K[c]$, whose conductor $c \subset \mathcal{O}_F$ is determined by

$$\widehat{\mathcal{O}}_c^* = \widehat{K}^* \cap gH\widehat{F}^*g^{-1} = \widehat{K}^* \cap g\widehat{R}^*\widehat{F}^*g^{-1}.$$

We say[(4)] that c is the conductor of x and we write $x \in \mathrm{CM}_H(c)$.

(4) Note that the conventions concerning ring class fields used by Cornut and Vatsal in [**Cor-Va**] are **not** the same as ours, as they treat Hilbert modular forms with a possibly non-trivial (unramified) character.

The next step is to map CM points to A_1. In the classical case $F = \mathbf{Q}$, $\mathcal{R} = \varnothing$ this is done using a suitable multiple $m(\infty)$ of the cusp ∞; in the remaining cases, there is a morphism

$$\iota = \iota_{H\widehat{F}^*} : N_H^* \longrightarrow J(N_H^*)$$

defined using a suitable multiple of the so-called "Hodge class" ([**Zh1**, p. 30]; [**Cor-Va**, §3.5], for M_H^* instead of N_H^*, [**Ne4**, §1.19]). Denote by ι_1 the composite map

$$\iota_1 : \mathrm{CM}_H \hookrightarrow N_H \hookrightarrow N_H^* \xrightarrow{\iota} J(N_H^*) \longrightarrow A_1.$$

By construction, the quaternion algebra B is unramified at P and $R_P \subset B_P \xrightarrow{\sim} M_2(F_P)$ is an Eichler order of level P^δ. This implies that for each $n >> 0$ there exists a "good" CM point $x \in \mathrm{CM}_H(P^n)$ in the sense of [**Cor-Va**, Def. 1.6].

The main result of [**Cor-Va**] in the indefinite case (Theorem 4.1) in the case of a trivial central character $\omega = 1$ states that, for each $n >> 0$ and every "good" $x \in \mathrm{CM}_H(P^n)$, there exists a ring class character $\beta : G(P^\infty) \to L_\mathfrak{p}^*$ (after enlarging $L_\mathfrak{p}$ if necessary) satisfying $c(\beta) = P^n$ and $\beta|_{G(P^\infty)_{\mathrm{tors}}} = \beta_0$, for which

$$e_\beta(\iota_1(x)) = \sum_{\sigma \in \mathrm{Gal}(K[P^n]/K)} \beta^{-1}(\sigma)\,\sigma(\iota_1(x)) \neq 0 \in A_1(K[P^n]) \otimes \mathbf{C}. \tag{12.9.5.1}$$

Fix such $n >> 0$, $x \in \mathrm{CM}_H(P^n)$ and β. As

$$H(P^n)^{\mathrm{Ker}(\beta)} \subset H(P^\infty)^{\mathrm{Ker}(\beta_0)},$$

our assumptions imply that the form g does not have CM by any totally imaginary quadratic extension K'/F contained in $H(P^n)^{\mathrm{Ker}(\beta)}$ (hence the abelian variety A_1 does not acquire CM over any such extension K'). This fact together with the non-triviality statement (12.9.5.1) imply, by an Euler system argument ([**Ne4**, Thm. 3.2]), that

$$h_f^1(K, V \otimes \beta) = 1.$$

On the other hand, we can also assume that $n >> 0$ is big enough in order to apply Lemma 12.9.4, which then yields

$$r_{\mathrm{an}}(K, g, \beta) \equiv 1 \equiv h_f^1(K, V \otimes \beta) \pmod 2$$

for the chosen ring class character $\beta : G(P^\infty) \to L_\mathfrak{p}^*$. Applying Corollary 12.6.4.8, we deduce the desired congruence

$$r_{\mathrm{an}}(K, g, \beta') \equiv h_f^1(K, V \otimes \beta') \pmod 2$$

for every ring class character of finite order $\beta' : G(p^\infty) = \mathrm{Gal}(K[p^\infty]/K) \to L_\mathfrak{p}^*$ for which $\beta'|_{G(p^\infty)_{\mathrm{tors}}}$ is induced by β_0 *via* the canonical map $G(p^\infty)_{\mathrm{tors}} \to G(P^\infty)_{\mathrm{tors}}$. If, in addition, $c(\beta') = P^n$ and (P splits in K/F or $n >> 0$), then $r_{\mathrm{an}}(K, g, \beta') \equiv 1 \pmod 2$, by Lemma 12.9.4. Theorem is proved. □

12.9.6. Proposition. — *Let $g = f \otimes \chi \in S_k(\mathfrak{n}(g), 1)$, $P \mid p$ and K/F be as in 12.9.1–12.9.2. We say that a totally imaginary quadratic extension K'/F is* **exceptional**, *if $K' \subset K[P^\infty]$ and the form g has CM by K'.*

(i) *If K'/F is exceptional, then $d_{K'/F} = P^m$ $(m \geqslant 0)$.*

(ii) *If $2 \mid [F:\mathbf{Q}]$, then there is no exceptional extension K'/F.*

(iii) *If K'/F is exceptional, $p \neq 2$ and $K' \subset K[P^\infty]^{G(P^\infty)_{\rm tors}}$, then $K' = K$ and $d_{K/F} = P^m$ $(m \geqslant 0)$.*

(iv) *If $2 \nmid [F:\mathbf{Q}]$ and if P splits in K/F, then there is no exceptional $K' \subset K[P^\infty]^{G(P^\infty)_{\rm tors}}$.*

Proof

(i) We have $d_{K'/F} \mid \mathfrak{n}(g)$ (since g has CM by K'). As $K[P^\infty]/F$ is ramified only at primes dividing $d_{K/F}P\infty$, any finite prime ramified in K'/F must divide

$$(\mathfrak{n}(g), d_{K/F}P) = \begin{cases} (1), & \text{if } P \nmid \mathfrak{n}(g) \\ P, & \text{if } P \mid \mathfrak{n}(g). \end{cases}$$

(ii) Assume that K'/F is exceptional. As $d_{K'/F} = P^m$ by (i), each prime $v \mid \mathfrak{n}(g)^{(P)}$ is unramified in K'/F, hence $2 \mid \operatorname{ord}_v(\mathfrak{n}(g))$, by Proposition 12.6.5.2(vi). This means that the ideal $\mathfrak{n}(g)^{(P)}$ is a square, which contradicts 12.9.2.2.

(iii) As $\operatorname{Gal}(K[P^\infty]^{G(P^\infty)_{\rm tors}}/K) = G(P^\infty)/G(P^\infty)_{\rm tors} \xrightarrow{\sim} \mathbf{Z}_p^{r_P}$ is a pro-p-group and $p \neq 2$, the only quadratic subextension K'/F of $K[P^\infty]^{G(P^\infty)_{\rm tors}}/K$ is $K' = K$; we apply (i).

(iv) Class field theory gives and exact sequence

$$0 \longrightarrow \mathcal{O}_K^*/\mathcal{O}_F^* \longrightarrow G' \longrightarrow \operatorname{Gal}(K[P^\infty]/K[1]) \longrightarrow 0,$$

in which $G' = (\mathcal{O}_{F,P}^* \times \mathcal{O}_{F,P}^*)/\Delta(\mathcal{O}_{F,P}^*)$, where Δ is the diagonal map. Put $\mu = (\mathcal{O}_{F,P}^*)_{\rm tors}$; this is a finite group and we have $G'_{\rm tors} = \operatorname{Im}(\mu \times \{1\} \to G')$. It follows that the reciprocity map

$$\mathbf{A}_K^*/K^*\mathbf{A}_F^* \longrightarrow \operatorname{Gal}\big(K[P^\infty]^{G(P^\infty)_{\rm tors}}/K\big)$$

factors through

$$\mathbf{A}_K^*/K^*\mathbf{A}_F^*\Big(\big(\mu \times \{1\}\big) \times \prod_{w \nmid P\infty} \mathcal{O}_{K,w}^*\Big). \tag{12.9.6.1}$$

If K'/F is any totally imaginary quadratic extension contained in $K[P^\infty]^{G(P^\infty)_{\rm tors}}$ with $d_{K'/F} = P^m$ $(m \geqslant 0)$, applying the norm $N_{K'/F}$ to (12.9.6.1) shows that the quadratic character $\eta' = \eta_{K'/F} : \mathbf{A}_F^*/F^* \to \{\pm 1\}$ associated to K'/F factors through

$$\mathbf{A}_F^*/F^*\mathbf{A}_F^{*2}\Big(\mu \times \prod_{v \nmid P\infty} \mathcal{O}_{F,v}^*\Big).$$

As $-1 \in \mu$, it follows that the idèle $x = (x_v)$ defined by

$$x_v = \begin{cases} 1, & v \mid \infty \\ -1, & v \nmid \infty \end{cases}$$

is contained in $\mathrm{Ker}(\eta')$, which contradicts the fact that

$$\eta'(-x) = \prod_{v\mid\infty} \eta_v(-1) = (-1)^{[F:\mathbf{Q}]} = -1.$$

Combined with (i), this contradiction proves the statement (iv). ☐

12.9.7. Theorem. — *Let $g = f \otimes \chi \in S_k(\mathfrak{n}(g), 1)$, $P \mid p$ and K/F be as in 12.9.1–12.9.2. Denote by $K_\infty = K[p^\infty]^{G(p^\infty)_{\mathrm{tors}}}$ the unique $\mathbf{Z}_p^{[F:\mathbf{Q}]}$-extension of K contained in $K[p^\infty]$. If $2 \nmid [F:\mathbf{Q}]$, assume that, either*

(i) *P splits in K/F, or*

(ii) *$p \neq 2$ and (g does not have CM by K or there exists a finite prime $v \neq P$ ramified in K'/F).*

Then, for each character of finite order $\beta : \mathrm{Gal}(K_\infty/K) \to \overline{L}_{\mathfrak{p}}^*$, we have

$$r_{\mathrm{an}}(K, g, \beta) \equiv h_f^1(K, V \otimes \beta) \pmod 2.$$

If $c(\beta) = P^n$ and (P splits in K/F or $n >> 0$), then

$$r_{\mathrm{an}}(K, g, \beta) \equiv h_f^1(K, V \otimes \beta) \equiv 1 \pmod 2.$$

Proof. — Thanks to the assumptions (i) or (ii), Proposition 12.9.6 implies that g does not have CM by any totally imaginary quadratic extension K'/F contained in $K[P^\infty]^{G(P^\infty)_{\mathrm{tors}}}$; we apply Theorem 12.9.5 with $\beta_0 = 1$. ☐

12.9.8. Theorem. — *Let $g = f\otimes\chi \in S_2(\mathfrak{n}(g), 1)$, $P \mid p$, K/F and $\beta_0 : G(P^\infty)_{\mathrm{tors}} \to \overline{L}_{\mathfrak{p}}^*$ be as in 12.9.1–12.9.3. Set $\mathcal{O} = \mathcal{O}_{L,\mathfrak{p}}$ and $\Lambda = \mathcal{O}[\![G(P^\infty)]\!]$. Fix a G_F-stable $\mathcal{O}$-lattice $T \subset V$ and put $A = V/T$. Assume that g does not have CM by any totally imaginary quadratic extension K'/F contained in $K[P^\infty]^{\mathrm{Ker}(\beta_0)}$. Then we have, using the notation from 10.7.16 and 12.6.4.9–12.6.4.10:*

(i) $(\forall j = 1, 2)$ $\mathrm{rk}_{\Lambda(\beta_0^{\pm 1})} \widetilde{H}^j_{f,\mathrm{Iw}}(K[P^\infty]/K, T)^{(\beta_0^{\pm 1})} = \mathrm{cork}_{\Lambda(\beta_0^{\pm 1})} \widetilde{H}^j_f(K_S/K[P^\infty], A)^{(\beta_0^{\pm 1})}$ $= m(g, K; P) + 1.$

(ii) $\mathrm{cork}_{\Lambda(\beta_0^{\pm 1})} S_A^{\mathrm{str}}(K[P^\infty])^{(\beta_0^{\pm 1})} = 1.$

(iii) *If R, $g = g_{\mathcal{P}}$ and $g' = g_{\overline{\mathcal{P}}}$ are as in 12.7.10 (resp. 12.7.11), denote $\overline{R} = R[\![G(P^\infty)]\!]$ and $\overline{\mathfrak{q}} = \overline{R} \cdot I_{\beta_0} \in \mathrm{Spec}(\overline{R})$ ($\mathrm{ht}(\overline{\mathfrak{q}}) = 0$), where $I_{\beta_0} = \mathrm{Ker}(\beta_0 : \mathcal{O}[G(P^\infty)_{\mathrm{tors}}] \to \overline{L}_{\mathfrak{p}})$. Fix $\mathcal{T} \subset \mathcal{V}$ as in 12.7.15.3. If 12.9.1.1 and 12.9.2.1–12.9.2.2 hold for $\mathfrak{n}(g')$ and if $m(g', K; P) = 0$, then*

$$(\forall j = 1, 2) \quad \mathrm{rk}_{\overline{R}_{\overline{\mathfrak{q}}}} \widetilde{H}^j_{f,\mathrm{Iw}}(K[P^\infty]/K, \mathcal{T})_{\overline{\mathfrak{q}}} = \mathrm{rk}_{\overline{R}_{\overline{\mathfrak{q}}}} \big(\widetilde{H}^j_{f,\mathrm{Iw}}(K[P^\infty]/K, \mathcal{T})^\iota\big)_{\overline{\mathfrak{q}}} = 1.$$

Proof

(i), (ii) The proof of Theorem 12.9.5 shows that the condition $C(g, \beta_0)$ from Proposition 12.6.4.12 is satisfied (with $c = (1)$, $s = 1$, $P_1 = P$, $\delta = 1$); we apply Proposition 12.6.4.12 (i)–(ii).

(iii) The four ranks coincide, by Proposition 12.7.15.7 (i) and 12.7.15.8 (i); denote their common value by h, and set

$$m' = \mathrm{rk}_{\Lambda'^{(\beta_0)}} \widetilde{H}^1_{f,\mathrm{Iw}}(K[P^\infty]/K, T')^{(\beta_0)},$$

using the notation of 12.7.15. According to Proposition 12.7.15.8 (iii), we have

$$h \leqslant m', \quad h \equiv m' \pmod 2;$$

on the other hand, $m' = m(g', K; P) + 1 = 1$, by (i) applied to g' (note that g' does not have CM by any totally imaginary quadratic extension K'/F contained in $K[P^\infty]^{\mathrm{Ker}(\beta_0)}$, by Proposition 12.7.14.5), hence $h = 1$.

□

12.9.9. Corollary. — *Let $g = f \otimes \chi \in S_2(\mathfrak{n}(g), 1)$, $P \mid p$ and K/F be as in 12.9.1–12.9.2; set $\Lambda = \mathcal{O}[\![G(P^\infty)]\!]$. If g does not have CM by any totally imaginary quadratic extension K'/F contained in $K[P^\infty]$ (which is automatic if $2 \mid [F:\mathbf{Q}]$), then:*

(i) *For each $\mathfrak{q} \in \mathrm{Spec}(\Lambda)$ with $\mathrm{ht}(\mathfrak{q}) = 0$,*

$$(\forall j = 1, 2) \quad \dim_{\kappa(\mathfrak{q})} \big(D_\Lambda\big(\widetilde{H}^j_f(K_S/K[P^\infty], A)\big)\big)_{\mathfrak{q}} = m(g, K; P) + 1$$
$$\dim_{\kappa(\mathfrak{q})} \big(D_\Lambda(S^{\mathrm{str}}_A(K[P^\infty]))\big)_{\mathfrak{q}} = 1.$$

(ii) *If $g = g_{\mathcal{P}}$ and $g' = g_{\overline{\mathcal{P}}}$ are as in 12.7.10 (resp. 12.7.11), set $\overline{R} = R[\![G(P^\infty)]\!]$ and choose $\mathcal{T} \subset \mathcal{V}$ as in 12.7.15.3. Assume that 12.9.1.1 and 12.9.2.1–12.9.2.2 hold for $\mathfrak{n}(g')$ and $m(g', K; P) = 0$. Then, for each $\overline{\mathfrak{q}} \in \mathrm{Spec}(\overline{R})$ with $\mathrm{ht}(\overline{\mathfrak{q}}) = 0$,*

$$(\forall j = 1, 2) \quad \dim_{\kappa(\overline{\mathfrak{q}})} \widetilde{H}^j_{f,\mathrm{Iw}}(K[P^\infty]/K, \mathcal{T})_{\overline{\mathfrak{q}}} = 1.$$

12.9.10. Howard [**Ho2**] showed (combine his Thm. B with [**Cor-Va**, Thm. 4.1]) that if $g = f \in S_2(\mathfrak{n}(g), 1)$ is p-ordinary, $P = (p)$, $\eta_{K/F}(\mathfrak{n}(g)) = (-1)^{[F:\mathbf{Q}]-1}$ and a few more assumptions hold, then

$$\mathrm{cork}_{\mathcal{O}[\![\Gamma]\!]} S^{\mathrm{str}}_A(K_\infty) = 1,$$

where $K_\infty = K[P^\infty]^{G(P^\infty)_{\mathrm{tors}}}$ and $\Gamma = \mathrm{Gal}(K_\infty/K)$.

12.9.11. Theorem. — *Let $g = f \otimes \chi \in S_k(\mathfrak{n}(g), 1)$, $P \mid p$ and K/F be as in 12.9.1. Fix a character $\beta_0 : G(P^\infty)_{\mathrm{tors}} \to \overline{L}^*_{\mathfrak{p}}$. Assume that $\big((p)\mathfrak{n}(g), d^{(P)}_{K/F}\big) = (1)$, $\eta_{K/F}\big(\mathfrak{n}(g)^{(p)}\big) = (-1)^{[F:\mathbf{Q}]-1}$ and g does not have CM by any totally imaginary quadratic extension K'/F contained in $K[P^\infty]^{\mathrm{Ker}(\beta_0)}$. Then we have, using the notation from 12.9.8:*

(i) $(\forall j = 1, 2) \quad \mathrm{rk}_{\overline{R}_{\overline{\mathfrak{q}}}} \widetilde{H}^j_{f,\mathrm{Iw}}(K[P^\infty]/K, \mathcal{T})_{\overline{\mathfrak{q}}} = \mathrm{rk}_{\overline{R}_{\overline{\mathfrak{q}}}} \big(\widetilde{H}^j_{f,\mathrm{Iw}}(K[P^\infty]/K, \mathcal{T})^\iota\big)_{\overline{\mathfrak{q}}} = 1.$

(ii) *For all but finitely many arithmetic points $\mathcal{P}'$ of R, all (co)ranks appearing in Proposition 12.9.8 (i) for $T' \subset V' = V(g_{\mathcal{P}'})$ and $A' = V'/T'$, are equal to* 1.

Proof. — The statement (ii) is an immediate consequence of (i) and the exact sequence in Proposition 12.7.15.5 (i). In order to prove (i), choose an arithmetic point

$\mathcal{P}'$ as in 12.9.5.1 and set $g' = g_{\mathcal{P}'}$. As observed in 12.9.5.1, we have $m(g', K; P) = 0$ and $\mathfrak{n}(g')^{(p)} = \mathfrak{n}(g)^{(p)}$, hence $((p)\mathfrak{n}(g'), d_{K/F}^{(P)}) = (1)$ and

$$\eta_{K/F}\big(\mathfrak{n}(g')^{(P)}\big) = (-1)^{m(g',K;P)}\,\eta_{K/F}\big(\mathfrak{n}(g')^{(p)}\big) = \eta_{K/F}\big(\mathfrak{n}(g)^{(p)}\big) = (-1)^{[F:\mathbf{Q}]-1},$$

where the first equality follows from Proposition 12.6.4.9 (i). Furthermore, g' does not have CM by any totally imaginary quadratic extension K'/F contained in $K[P^\infty]^{\mathrm{Ker}(\beta_0)}$, thanks to Proposition 12.7.14.5. It follows that Theorem 12.9.8 (iii) applies to g', which proves (i). □

12.9.12. Corollary. — *Let $g = f \otimes \chi \in S_k(\mathfrak{n}(g), 1)$, $P \mid p$ and K/F be as in 12.9.1. Assume that $((p)\mathfrak{n}(g), d_{K/F}^{(P)}) = (1)$, $\eta_{K/F}(\mathfrak{n}(g)^{(p)}) = (-1)^{[F:\mathbf{Q}]-1}$ and g does not have CM by any totally imaginary quadratic extension K'/F contained in $K[P^\infty]$ (which is automatic if $2 \mid [F:\mathbf{Q}]$). Write $g = g_{\mathcal{P}}$, put $\overline{R} = R[\![G(P^\infty)]\!]$ and choose $\mathcal{T} \subset \mathcal{V}$ as in 12.7.15.3. Then*

$$(\forall \overline{\mathfrak{q}} \in \mathrm{Spec}(\overline{R}),\ \mathrm{ht}(\overline{\mathfrak{q}}) = 0)\ (\forall j = 1,2) \quad \dim_{\kappa(\overline{\mathfrak{q}})} \widetilde{H}^j_{f,\mathrm{Iw}}(K[P^\infty]/K, \mathcal{T})_{\overline{\mathfrak{q}}} = 1.$$

12.9.13. Proposition. — *In the situation of 12.6.4.9, assume that $g = f \otimes \chi \in S_k(\mathfrak{n}(g), 1)$ and $\big((p)\mathfrak{n}(g),\, c\,d_{K/F}^{(P_1\cdots P_s)}\big) = (1)$. Set $K_\infty = K[cP_1^\infty \cdots P_s^\infty]$, $\Gamma = \mathrm{Gal}(K_\infty/K)$ and fix a character $\beta_0 : \Gamma_{\mathrm{tors}} \to \overline{L}_{\mathfrak{p}}^*$. Assume that there exist infinitely many arithmetic points $\mathcal{P}'$ of R for which the condition $C(g_{\mathcal{P}'}, \beta_0)$ from 12.6.4.12 holds. Then*

$$(\forall j = 1,2) \quad \mathrm{rk}_{\overline{R}_{\overline{\mathfrak{p}}}} \widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, \mathcal{T})_{\overline{\mathfrak{p}}} = \mathrm{rk}_{\overline{R}_{\overline{\mathfrak{p}}}} \big(\widetilde{H}^j_{f,\mathrm{Iw}}(K[P^\infty]/K, \mathcal{T})^\iota\big)_{\overline{\mathfrak{p}}} = \delta,$$

where $\delta \in \{0,1\}$ is characterized by the formula $\eta_{K/F}\big(\mathfrak{n}(g)^{(p)}\big) = (-1)^{[F:\mathbf{Q}]+\delta}$.

Proof. — As $m(g_{\mathcal{P}'}) := m(g_{\mathcal{P}'}, K; P_1, \ldots, P_s) = 0$ for all but finitely many arithmetic points $\mathcal{P}'$ of R, we choose $\mathcal{P}'$ such that $g' := g_{\mathcal{P}'}$ satisfies $C(g', \beta_0)$ and $m(g') = 0$. As in the proof of Theorem 12.9.11, we have $\big((p)\mathfrak{n}(g'), c\,d_{K/F}^{(P_1\cdots P_s)}\big) = (1)$ and

$$\eta_{K/F}\big(\mathfrak{n}(g')^{(P_1\cdots P_s)}\big) = (-1)^{m(g')}\,\eta_{K/F}\big(\mathfrak{n}(g')^{(p)}\big) = \eta_{K/F}\big(\mathfrak{n}(g)^{(p)}\big) = (-1)^{[F:\mathbf{Q}]+\delta},$$

hence we deduce from Proposition 12.6.4.12 that the value of

$$m := \mathrm{rk}_{\Lambda'(\beta_0)} \widetilde{H}^j_{f,\mathrm{Iw}}(K_\infty/K, V(g'))$$

is equal to $m = m(g') + \delta = \delta$. On the other hand, the rank

$$h := \mathrm{rk}_{\overline{R}_{\overline{\mathfrak{p}}}} \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty/K, \mathcal{T})_{\overline{\mathfrak{p}}}$$

satisfies $h \leqslant m$, $h \equiv m \pmod 2$, by Proposition 12.7.15.8 (iii). It follows that $h = \delta$, as claimed. □

12.10. Proof of Theorem 12.2.3

In this section we prove Theorem 12.2.3. Recall that $f \in S_k(\mathfrak{n}, \chi^{-2})$ is p-ordinary, $g = f \otimes \chi \in S_k(\mathfrak{n}(g), 1)$ and $V = V(g)(k/2) = V(f)(k/2) \otimes \chi \xrightarrow{\sim} V^*(1)$.

12.10.1. Descent properties. — Assume that F_0/F is a finite solvable extension (not necessarily totally real) and F_1/F_0 a finite abelian extension. Denote by g_0 and g_1 the base change of g to F_0 and F_1, respectively (strictly speaking, we are abusing the language, as we should speak about the base change of the corresponding automorphic representations). Let $\widehat{\Delta}$ be the character group of $\Delta = \mathrm{Gal}(F_1/F_0)$. Enlarging L if necessary, we can assume that all characters $\alpha \in \widehat{\Delta}$ have values in L^*. We are going to use the relations

$$(12.10.1.1) \qquad L(g_1, s) = \prod_{\alpha\in\widehat{\Delta}} L(g_0 \otimes \alpha, s), \quad H^1_f(F_1, V) = \bigoplus_{\alpha\in\widehat{\Delta}} H^1_f(F_0, V \otimes \alpha),$$
$$\widetilde{H}^1_f(F_1, V) = \bigoplus_{\alpha\in\widehat{\Delta}} \widetilde{H}^1_f(F_0, V \otimes \alpha)$$

(see Proposition 12.5.9.2 (iv)). If $\alpha \in \widehat{\Delta}$, $\alpha^2 \neq 1$, then the functional equation relating $L(g_0 \otimes \alpha, s)$ and $L(g_0 \otimes \alpha^{-1}, s)$ yields

$$r_{\mathrm{an}}(F_0, g_0 \otimes \alpha) = r_{\mathrm{an}}(F_0, g_0 \otimes \alpha^{-1})$$

(the archimedean L-factors take non-zero finite values at the central point). Similarly, Proposition 7.8.11 implies that

$$\widetilde{h}^1_f(F_0, V \otimes \alpha) = \widetilde{h}^1_f(F_0, V \otimes \alpha^{-1}).$$

The H^0-terms for α and α^{-1} in 12.5.9.2 (iii) have the same dimension, hence

$$h^1_f(F_0, V \otimes \alpha) = h^1_f(F_0, V \otimes \alpha^{-1}).$$

The preceding discussion yields

$$r_{\mathrm{an}}(F_1, g_1) \equiv \sum_{\alpha:\Delta\to\{\pm1\}} r_{\mathrm{an}}(F_0, g_0 \otimes \alpha) \pmod 2$$
$$h^1_f(F_1, V) \equiv \sum_{\alpha:\Delta\to\{\pm1\}} h^1_f(F_0, V \otimes \alpha) \pmod 2,$$

which implies that

$$(12.10.1.2) \quad r_{\mathrm{an}}(F_1, g_1) - h^1_f(F_1, V) \equiv \sum_{\alpha:\Delta\to\{\pm1\}} \big(r_{\mathrm{an}}(F_0, g_0 \otimes \alpha) - h^1_f(F_0, V \otimes \alpha)\big) \pmod 2.$$

12.10.2. Reduction to the case $F'' = F' = F$ for the twists $g \otimes \alpha$ ($\alpha : \mathrm{Gal}(F'/F) \to \{\pm1\}$). — As each finite group of odd order is solvable, there exists a tower of fields $F' = F_0 \subset F_1 \subset \cdots \subset F_n = F''$ in which each layer F_{i+1}/F_i is an abelian extension of odd degree. Applying (12.10.1.2) to each F_{i+1}/F_i, as well as to F'/F, we deduce that

$$r_{\mathrm{an}}(F'', g) - h^1_f(F'', V) \equiv \sum_{\alpha:\mathrm{Gal}(F'/F)\to\{\pm1\}} \big(r_{\mathrm{an}}(F, g \otimes \alpha) - h^1_f(F, V \otimes \alpha)\big) \pmod 2,$$

which means that it suffices to prove the congruences

(12.10.2.1) $$r_{\rm an}(F, g\otimes\alpha) \stackrel{?}{\equiv} h^1_f(F, V\otimes\alpha) \pmod 2$$

for all $\alpha : \mathrm{Gal}(F'/F) \to \{\pm 1\}$.

12.10.3. Reduction to the assumptions 12.2.3 (1)–(3). — Assume that, for each $\alpha : \mathrm{Gal}(F'/F) \to \{\pm 1\}$, there exist distinct prime ideals $\mathfrak{q}_{\alpha,j} \nmid (p)\mathfrak{n}(f\otimes\alpha)\mathrm{cond}(\chi)$ $(j = 1, \ldots, r_\alpha,\ r_\alpha \geqslant 1)$ of F and a p-ordinary newform $f_\alpha \in S_k(\mathfrak{n}(f\otimes\alpha)Q_\alpha, \chi^{-2})$ $(Q_\alpha = \mathfrak{q}_{\alpha,1}\cdots\mathfrak{q}_{\alpha,r_\alpha})$ such that

$$(\forall j = 1, \ldots, r_\alpha) \quad \lambda_{f_\alpha}(\mathfrak{q}_{\alpha,j}) = -\chi^{-1}(\mathfrak{q}_{\alpha,j})(N\mathfrak{q}_{\alpha,j})^{k/2-1}$$

and

(12.10.3.1) $$\lambda_{f_\alpha}(v) \equiv \lambda_{f\otimes\alpha}(v) \pmod{\mathfrak{p}^M}$$

for a set of primes $v \nmid (p)\mathfrak{n}(f\otimes\alpha)Q_\alpha$ of density 1, where M satisfies (12.8.4.1) for $g\otimes\alpha$ and

$$M > t + 6a + 19b + 10c + 21\,\mathrm{ord}_{\mathfrak{p}}(2)$$

(again for $g\otimes\alpha$), in the notation of 12.8.4.

We put $g_\alpha = f_\alpha\otimes\chi \in S_k(\mathfrak{n}(g_\alpha), 1)$ and $V_\alpha = V(g_\alpha)(k/2) = V(f_\alpha)(k/2)\otimes\chi$. Proposition 12.8.4.14 (ii) applies to $g\otimes\alpha$ and g_α, yielding

$$r_{\rm an}(F, g\otimes\alpha) - h^1_f(F, V\otimes\alpha) \equiv r_{\rm an}(F, g_\alpha) - h^1_f(F, V_\alpha) \pmod 2,$$

which means that in order to prove (12.10.2.1) it is enough to establish the congruences

$$r_{\rm an}(F, g_\alpha) \stackrel{?}{\equiv} h^1_f(F, V_\alpha) \pmod 2$$

for the forms g_α. These forms satisfy the assumption 12.2.3 (2), as

$$\mathrm{ord}_{\mathfrak{q}_1}(\mathfrak{n}(g_\alpha)) = \mathrm{ord}_{\mathfrak{q}_1}(\mathfrak{n}(f_\alpha)) = 1.$$

12.10.4. Reduction to the assumptions 12.2.3 (1)–(2). — Consider the case when g satisfies the condition 12.2.3 (3), but not 12.2.3 (1) nor 12.2.3 (2). This implies, in particular, that $p > 3$, $2 \mid [F : \mathbf{Q}]$ and $\pi(g)_v \neq \mathrm{St}(\mu)$ for all primes v of F. As $\mathfrak{n}(g)$ is prime to p, it follows from the remarks at the end of 12.5.5 that χ is unramified at all $v \mid p$, hence the form g itself is p-ordinary.

12.10.4.1. Lemma. — *Fix a G_F-stable $\mathcal{O} = \mathcal{O}_{L,\mathfrak{p}}$-lattice $T \subset V$ and denote by $\rho_{\mathfrak{p}} : G_F \to \mathrm{Aut}_{\mathcal{O}}(T)$ the Galois action on T. Then $-1 \in \mathrm{Im}(\rho_{\mathfrak{p}})$.*

Proof. — According to 12.2.3 (3iii), if g does not have CM, then there exists a choice of a basis $T \xrightarrow{\sim} \mathcal{O}^{\oplus 2}$ such that the image of the reduction $\overline{\rho}_{\mathfrak{p}} = \rho_{\mathfrak{p}} \pmod{\mathfrak{p}}$ of $\rho_{\mathfrak{p}} : G_F \to \mathrm{Aut}_{\mathcal{O}}(T) \xrightarrow{\sim} \mathrm{GL}_2(\mathcal{O})$ satisfies $\mathrm{Im}(\overline{\rho}_{\mathfrak{p}}) \supseteq \mathrm{SL}_2(\mathbf{F}_p)$. As $p > 3$, it follows from [**Se3**, §IV.3.4, Lemma 3] that $\mathrm{Im}(\rho_{\mathfrak{p}}) \supseteq \mathrm{SL}_2(\mathbf{Z}_p)$, hence $-1 \in \mathrm{Im}(\rho_{\mathfrak{p}})$.

If g has CM by a totally imaginary quadratic extension K/F, then the restriction of $\rho_{\mathfrak{p}}$ to G_K is associated ([**Sc**, Ch. 0, §5]) to an algebraic Hecke character

$\psi : \mathbf{A}_K^* \to (L')^*$ ($[L' : L] = 2$) of infinity type $\sum_{\sigma\in\Phi}\big((\frac{k}{2}-1)\sigma - \frac{k}{2}\,c\circ\sigma\big)$, where Φ is a CM type of K and c the complex conjugation. As $\mathfrak{n}(g)$ is prime to p, so is the conductor of ψ, hence the composite map

$$\varphi : \prod_{v\mid p}\mathcal{O}_{F,v}^* \longrightarrow \mathbf{A}_F^*/F^* \longrightarrow \mathbf{A}_K^*/K^* \xrightarrow{\mathrm{rec}_K} G_K^{ab} \xrightarrow{\rho_{\mathfrak{p}}|G_K} (L'\otimes_L L_{\mathfrak{p}})^* \subset \mathrm{GL}_2(L_{\mathfrak{p}})$$

is given by $\varphi((u_v)_{v\mid p}) = \prod_{v\mid p} u_v$ (provided we normalize the reciprocity map rec_K *via* the *geometric* Frobenius elements). In particular, $-1 = \varphi((-1,1,\ldots,1)) \in \mathrm{Im}(\rho_{\mathfrak{p}} \mid G_K) \subset \mathrm{Im}(\rho_{\mathfrak{p}})$. □

12.10.4.2. Corollary. — *For each character* $\alpha : \mathrm{Gal}(F'/F) \to \{\pm1\}$, $-1 \in \mathrm{Im}(\rho_{\mathfrak{p}}\otimes\alpha)$.

Proof. — The condition 12.2.3(3) for g implies that the same condition is satisfied by $g\otimes\alpha$ (for example, $\mathfrak{n}(g\otimes\alpha) \mid \mathfrak{n}(g)\,\mathrm{cond}(\alpha)^2$ is prime to p, since $\mathrm{cond}(\alpha) \mid d_{F'/F}$ and $p \nmid d_{F'/\mathbf{Q}}$), hence Lemma 12.10.4.1 applies to $g\otimes\alpha$. □

12.10.4.3. Level raising for $g\otimes\alpha$**.** — Fix $\alpha : \mathrm{Gal}(F'/F) \to \{\pm1\}$ and let $E_{g\otimes\alpha} \subset \mathcal{O}_{L(g)}$ be the non-zero ideal associated to $g\otimes\alpha$ in [**Tay1**, Thm. 1]. Set $m = \mathrm{ord}_{\mathfrak{p}}(E_{g\otimes\alpha}\mathcal{O}_L)$ and denote by E the fixed field of the kernel of

$$\rho_{\mathfrak{p}}\otimes\alpha \pmod{\mathfrak{p}^{m+1}} : G_F \longrightarrow \mathrm{Aut}_{\mathcal{O}}(T/\mathfrak{p}^{m+1}T) \xrightarrow{\sim} \mathrm{GL}_2(\mathcal{O}/\mathfrak{p}^{m+1}\mathcal{O}).$$

As $\mathrm{Gal}(E/F) = \mathrm{Im}\big(\rho_{\mathfrak{p}}\otimes\alpha \pmod{\mathfrak{p}^{m+1}}\big) \subset \mathrm{GL}_2(\mathcal{O}/\mathfrak{p}^{m+1}\mathcal{O})$ contains -1 (by Corollary 12.10.4.2), the Čebotarev density theorem implies that there exists a prime ideal $\mathfrak{q} \nmid (p)\,\mathfrak{n}(g\otimes\alpha)\,\mathrm{cond}(\chi)$ of F such that

$$(\rho_{\mathfrak{p}}\otimes\alpha)(\mathrm{Fr}(\mathfrak{q})) \equiv -I \pmod{\mathfrak{p}^{m+1}} \qquad (\mathrm{Fr}(\mathfrak{q}) = \mathrm{Fr}(\mathfrak{q})_{\mathrm{geom}}).$$

Fix such $\mathfrak{q}$; then

$$\begin{aligned} 1 - (N\mathfrak{q})^{-k/2}\lambda_{g\otimes\alpha}(\mathfrak{q})X + (N\mathfrak{q})X^2 &= \det(1-\mathrm{Fr}(\mathfrak{q})X \mid V\otimes\alpha) \\ &\equiv \det(1+X \mid V\otimes\alpha) = (1+X)^2 \pmod{\mathfrak{p}^{m+1}}, \end{aligned}$$

hence

$$N\mathfrak{q} - 1 \equiv \lambda_{g\otimes\alpha}(\mathfrak{q}) + 2 \equiv 0 \pmod{\mathfrak{p}^{m+1}},$$
$$\mathrm{ord}_{\mathfrak{p}}(\lambda_{g\otimes\alpha}(\mathfrak{q})^2 - (N\mathfrak{q}+1)^2) - \mathrm{ord}_{\mathfrak{p}}(E_{g\otimes\alpha}\mathcal{O}_L) \geqslant 1.$$

Applying [**Tay1**, Thm. 1] and [**De-Se**, Lemma 6.11], we deduce that there exists $g_1 \in S_k(\mathfrak{n}(g\otimes\alpha)\mathfrak{q}, 1)$, which is an eigenform for all $T(v)$ ($v \neq \mathfrak{q}$), its Hecke eigenvalues are contained in a number field $L' \supset L$ and satisfy

$$(12.10.4.1) \qquad (\forall v \neq \mathfrak{q}) \quad \lambda_{g_1}(v) \equiv \lambda_{g\otimes\alpha}(v) \pmod{\mathfrak{p}'},$$

where $\mathfrak{p}'$ is a prime of L' above $\mathfrak{p}$. Moreover, $\mathrm{ord}_{\mathfrak{q}}(\mathfrak{n}(g_1)) = 1$, hence g_1 is also an eigenform for $T(\mathfrak{q})$.

Choose a G_F-stable $\mathcal{O}'$-lattice $T_1 \subset V_1 = V(g_1)(k/2)$ ($\mathcal{O}' = \mathcal{O}_{L',\mathfrak{p}'}$) and set $T' = T \otimes_{\mathcal{O}} \mathcal{O}'$. The congruence (12.10.4.1) implies that

$$(\forall \sigma \in G_F) \quad \mathrm{Tr}(\sigma \mid T_1/\mathfrak{p}'T_1) = \mathrm{Tr}(\sigma \mid T'/\mathfrak{p}'T' \otimes \alpha).$$

As $T/\mathfrak{p}T$ is an absolutely irreducible representation of G_F (it is irreducible, by 12.2.3(3ii), and the complex conjugation acts on it with two distinct eigenvalues ± 1 (recall that $p \neq 2$) contained in the residue field $\mathcal{O}/\mathfrak{p}$), it follows that

$$T_1/\mathfrak{p}'T_1 \xrightarrow{\sim} T'/\mathfrak{p}'T' \tag{12.10.4.2}$$

are isomorphic, absolutely irreducible representations of G_F.

We are going to verify that Proposition 12.8.4.15 applies (with $M = 1$) to the pair of congruent forms $g \otimes \alpha$ and g_1 (with $(\mathcal{O}, \mathfrak{p})$ replaced by $(\mathcal{O}', \mathfrak{p}')$).

12.10.4.4. Proposition

(i) $\pi(g_1)_{\mathfrak{q}} = \mathrm{St}(\mu)$, *where* μ *is unramified,* $\mu(\mathfrak{q}) = -1$, *and* $\lambda_{g_1}(\mathfrak{q}) = -(N\mathfrak{q})^{k/2-1}$.

(ii) *The form* g_1 *is* p*-ordinary and its level* $\mathfrak{n}(g_1)$ *is prime to* p.

(iii) $(\forall v \neq \mathfrak{q})$ $\mathrm{ord}_v(\mathfrak{n}(g_1)) \neq 1$.

Proof

(i) As $\mathrm{ord}_{\mathfrak{q}}(\mathfrak{n}(g_1)) = 1$, we have $\pi(g_1)_{\mathfrak{q}} = \mathrm{St}(\mu)$, where μ is unramified and $\mu(\mathfrak{q}) = \pm 1$. As

$$T_1^{I_{\mathfrak{q}}} \twoheadrightarrow T_1^{I_{\mathfrak{q}}}/\mathfrak{p}'T_1^{I_{\mathfrak{q}}} \subset T_1/\mathfrak{p}'T_1 \xrightarrow{\sim} T'/\mathfrak{p}'T'$$

with $\mathrm{Fr}(\mathfrak{q}) = \mathrm{Fr}(\mathfrak{q})_{\mathrm{geom}}$ acting on $T_1^{I_{\mathfrak{q}}}$ (resp., on $T'/\mathfrak{p}'T'$) by the scalar $(N\mathfrak{q})^{-k/2}\lambda_{g_1}(\mathfrak{q}) = \mu(\mathfrak{q})(N\mathfrak{q})^{-1}$ (resp., by -1), it follows that

$$\mu(\mathfrak{q}) \equiv -N\mathfrak{q} \equiv -1 \pmod{\mathfrak{p}'},$$

hence (since $p \neq 2$)

$$\mu(\mathfrak{q}) = -1, \quad \lambda_{g_1}(\mathfrak{q}) = (N\mathfrak{q})^{k/2-1}\mu(\mathfrak{q}) = -(N\mathfrak{q})^{k/2-1}.$$

(ii) Firstly, $\mathfrak{n}(g_1) \mid \mathfrak{n}(g\otimes\alpha)\mathfrak{q} \mid \mathfrak{n}(g)\,\mathrm{cond}(\alpha)^2\mathfrak{q} \mid \mathfrak{n}(g)\, d_{F'/F}^2\mathfrak{q}$, which means that $\mathfrak{n}(g_1)$ is prime to p. Fix a prime $v \mid p$ of F. According to Proposition 12.4.9.2, $V(g_1)_v$ is a crystalline representation of G_v with Hodge-Tate weights equal to $1-k$ and 0. Denote by $T(g_1) \subset V(g_1)$ the lattice $T_1(-k/2)$. The isomorphism (12.10.4.2) combined with (12.5.3.1) for $g \otimes \alpha$ implies that $(T(g_1)/\mathfrak{p}'T(g_1))^{I_v}$ is a one-dimensional subspace of $T(g_1)/\mathfrak{p}'T(g_1)$. An easy exercise in Fontaine-Laffaille theory (which applies to $T(g_1)_v$, as v is unramified in $F/\mathbf{Q}$ and $k-1 < p-1$) shows that $T(g_1)^{I_v} \neq 0$. T. Saito's comparison result ([**Sa**, Thm. 1]) then implies that $\lambda_{g_1}(v) \in \mathcal{O}'^*$, hence the form g_1 is p-ordinary.

(iii) As $\mathfrak{n}(g_1) \mid \mathfrak{n}(g \otimes \alpha)\mathfrak{q}$, it is enough to consider primes v dividing $\mathfrak{n}(g \otimes \alpha)$ (hence prime to p). As $\pi(g)_v \neq \mathrm{St}(\mu)$ by assumption, it follows from Proposition 12.4.10.3 and the assumption $p \nmid w_2(L(g))$ that $((T/\mathfrak{p}T) \otimes \alpha)^{I_v} = 0$. The isomorphism (12.10.4.2) implies that $(T_1/\mathfrak{p}'T_1)^{I_v} = 0$, hence $V_1^{I_v} = 0$, which proves the claim (by Lemma 12.4.5 (ii)). □

12.10.4.5. Proposition

(i) *The integers $a, b, c \geqslant 0$ associated to $g \otimes \alpha$ (over $\mathcal{O}'$) in 12.8.4.2, 12.8.4.4 (ii) and 12.8.1.1, respectively, are equal to $a = b = c = 0$, and the condition* (12.8.4.1) *is satisfied for $M = 1$.*

(ii) $r_{\mathrm{an}}(F, g \otimes \alpha) - h^1_f(F, V \otimes \alpha) \equiv r_{\mathrm{an}}(F, g_1) - h^1_f(F, V_1) \pmod 2$.

Proof

(i) As $T'/\mathfrak{p}'T' \otimes \alpha$ is an absolutely irreducible representation of G_F, we have $c = 0$.

Fix $v \mid p$ in F. By assumption, $F_v/\mathbf{Q}_p$ and χ_v are unramified, and $\alpha^2 = 1$, hence

$$N^{k-1}_{F_v/\mathbf{Q}_p} \cdot (\chi_v \alpha)^2 (\mathcal{O}^*_{F,v}) = (\mathbf{Z}^*_p)^{k-1} \not\subset 1 + p\mathbf{Z}_p$$

(as $k - 1 < p - 1$), which implies that $a_v = 0$ (see 12.8.4.2) and $a = 0$.

Fix a prime $v \mid \mathfrak{n}(g \otimes \alpha)$. By assumptions, $v \nmid p$ and $\pi(g \otimes \alpha)_v$ is in the ramified principal series, or a supercuspidal representation. As shown in Proposition 12.4.10.3, in either case the assumption $p \nmid w_2(L(g))$ implies that $((V/T) \otimes \alpha)^{I_v} = 0$, hence $\mathrm{Tam}_v(T \otimes \alpha, \mathfrak{p}) = 0$ ($\Longrightarrow$ $\mathrm{Tam}_v(T' \otimes \alpha, \mathfrak{p}') = 0$). This implies that $b_v = 0$ (see 12.8.4.4 (ii)) and $b = 0$.

The condition (12.8.4.1) is satisfied with $M = 1$, thanks to Proposition 12.10.4.4 (iii) and the fact that $b = c = \mathrm{ord}_{\mathfrak{p}'}(2) = 0$.

(ii) This follows from Proposition 12.8.4.15, which applies to the forms $g \otimes \alpha \equiv g_1 \pmod{\mathfrak{p}'}$ (over $\mathcal{O}'$) with $M = 1$, thanks to (i). □

12.10.4.6. To sum up, the congruences (12.10.2.1) follow from

$$r_{\mathrm{an}}(F, g_1) \overset{?}{\equiv} h^1_f(F, V_1) \pmod 2,$$

for the forms g_1 associated to all $g \otimes \alpha$ ($\alpha : \mathrm{Gal}(F'/F) \to \{\pm 1\}$) as in 12.10.4.3. As $\mathrm{ord}_{\mathfrak{q}}(g_1) = 1$, these forms satisfy the assumption 12.2.3 (2).

In other words, thanks to 12.10.1–12.10.4, it is enough to prove the congruence

$$r_{\mathrm{an}}(F, g) \overset{?}{\equiv} h^1_f(F, V) \pmod 2, \qquad (12.10.4.1)$$

in the case when at least one of the assumptions 12.2.3 (1)–(2) holds.

12.10.5. Reduction of the assumption 12.2.3 (2) to the case when $\mathfrak{n}(g)$ is not a square. — Assume that $\mathfrak{n}(g)$ is a square and that there exists a quadratic extension F_1/F and a prime $\mathfrak{q}$ of F for which $2 \nmid \mathrm{ord}_{\mathfrak{q}}(\mathfrak{n}(g \otimes \mu))$, where $\mu = \eta_{F_1/F}$ is the quadratic character associated to F_1/F. As $\mathrm{ord}_{\mathfrak{q}}(\mathfrak{n}(g \otimes \mu))$ depends only on the local representation $\pi(g)_{\mathfrak{q}} \otimes \mu_{\mathfrak{q}}$, we can assume that the field F_1 is totally real. Denote by $f_1 = BC_{F_1/F}(f)$ (resp., $g_1 = BC_{F_1/F}(g)$) the base change of f (resp., g)

to F_1. The form f_1 is again p-ordinary and $g_1 = f_1 \otimes \chi_1$, where $\chi_1 = \chi \otimes N_{F_1/F}$. Our assumptions imply that

$$\mathrm{ord}_{\mathfrak{q}}(\mathfrak{n}(g)) \equiv 0 \not\equiv 1 \equiv \mathrm{ord}_{\mathfrak{q}}(\mathfrak{n}(g \otimes \mu)) \pmod 2,$$

hence $\mathfrak{q}$ is ramified in F_1/F; denote by $\mathfrak{q}_1$ the only prime of F_1 above $\mathfrak{q}$. If W is any $L_{\mathfrak{p}}$-representation of G_F, denote by W_1 its restriction to G_{F_1}. The formula

$$d_{F_1/F}^{\dim(W)} N_{F_1/F}(\mathrm{cond}(W_1)) = \mathrm{cond}(W)\,\mathrm{cond}(W \otimes \mu)$$

([**A-T**], ch. 11, Thm. 18) for $W = V(g)$ together with (12.4.3.2) imply that

$$\mathrm{ord}_{\mathfrak{q}_1}(\mathfrak{n}(g_1)) \equiv \mathrm{ord}_{\mathfrak{q}}(\mathfrak{n}(g)) + \mathrm{ord}_{\mathfrak{q}}(\mathfrak{n}(g \otimes \mu)) \equiv 1 \pmod 2.$$

It follows that neither $\mathfrak{n}(g_1)$, nor $\mathfrak{n}(g \otimes \mu)$ is a square. The formulas (12.10.1.1) for F_1/F imply that

$$r_{\mathrm{an}}(F, g) - h^1_f(F, V) = (r_{\mathrm{an}}(F_1, g) - h^1_f(F_1, V)) - (r_{\mathrm{an}}(F, g \otimes \mu) - h^1_f(F, V \otimes \mu)),$$

which means that it is enough to prove (12.10.4.1) for the forms $g \otimes \mu$ (over F) and g_1 (over F_1), whose levels are not squares.

12.10.6. Reduction of the assumption 12.2.3(2) to the case when there exists a prime $P \mid p$ such that $\mathfrak{n}(g)^{(P)}$ is not a square. — Assume that F has only one prime P above p, that $\mathfrak{n}(g)^{(P)}$ is a square and $2 \nmid \mathrm{ord}_P(\mathfrak{n}(g))$. Thanks to [**A-T**, ch. 10, Thm. 5], there exists a cyclic extension F_2/F of degree $[F_2 : F] = 3$ (hence F_2 is totally real) in which P splits: $P\mathcal{O}_{F_2} = Q_1 Q_2 Q_3$. Let $g_2 = BC_{F_2/F}(g)$ be the base change of g to F_2. For each $j = 1, 2, 3$, we have $\pi(g_2)_{Q_j} = \pi(g)_P$, hence $2 \nmid \mathrm{ord}_{Q_j}(\mathfrak{n}(g_2))$. The congruence (12.10.1.2) for F_2/F reads as

$$r_{\mathrm{an}}(F_2, g_2) - h^1_f(F_2, V) \equiv r_{\mathrm{an}}(F, g) - h^1_f(F, V) \pmod 2,$$

hence it is enough to prove (12.10.4.1) for the form g_2 over F_2, for which $\mathfrak{n}(g_2)^{(Q_1)}$ is not a square.

12.10.7. Reduction to the case when 12.9.1.1 holds and $F'' = F$. — To sum up the results of 12.10.1–12.10.6, we have reduced Theorem 12.2.3 to the following statement: if the condition 12.9.1.1 holds, then

$$r_{\mathrm{an}}(F, g) \overset{?}{\equiv} h^1_f(F, V) \pmod 2. \tag{12.10.7.1}$$

12.10.8. Reduction to the case when 12.9.1.1 holds, $k = 2$ and $F'' = F$

We assume that $g = f \otimes \chi \in S_k(\mathfrak{n}(g), 1)$, f is p-ordinary, $k \neq 2$ and that $P \mid p$ is a prime of F for which 12.9.1.1 holds (in the case when $[F : \mathbf{Q}]$ is even).

As in 12.9.5.1, we embed (the p-stabilization of) f into a Hida family as $f_{\mathcal{P}}$ for a suitable arithmetic points $\mathcal{P}$ and apply the discussion from 12.7.9–12.7.11 (possibly after slightly enlarging $L_{\mathfrak{p}}$ if $p = 2$) to obtain a p-ordinary form $f' = f_{\mathcal{P}'}$ of weight $(2, \ldots, 2)$ in the same family, the corresponding twist $g' = f' \otimes \chi' \in S_2(n(g'), 1)$ (with

$\chi' = \chi \cdot (\omega \circ N)^{1-k/2})$ and the Galois representation $V' = V(g')(1)$. As remarked in 12.9.5.1, g' also satisfies 12.9.1.1. According to Proposition 12.7.14.4 (iv), we have

$$r_{\rm an}(F, g) - h^1_f(F, V) \equiv r_{\rm an}(F, g') - h^1_f(F, V') \pmod 2.$$

In particular, the desired congruence (12.10.7.1) for g follows from the analogous result for g'.

12.10.9. Proof in the case $F'' = F$, $k = 2$, 12.9.1.1 holds. — We assume that $g = f \otimes \chi \in S_2(\mathfrak{n}(g), 1)$, f is p-ordinary and that $P \mid p$ is a prime of F for which 12.9.1.1 holds (if $[F : \mathbf{Q}]$ is even). Our goal is to prove (12.10.7.1).

If K/F is any totally imaginary quadratic extension satisfying 12.9.2.1–12.9.2.2 in which P splits, then we have

$$r_{\rm an}(K, g) \equiv h^1_f(K, V) \equiv 1 \pmod 2, \tag{12.10.9.1}$$

by Theorem 12.9.7 applied to $\beta = 1$. Denoting by $\eta_{K/F}$ the quadratic character over F associated to K/F, then we have

$$r_{\rm an}(K, g) = r_{\rm an}(F, g) + r_{\rm an}(F, g \otimes \eta_{K/F})$$

$$h^1_f(K, V) = h^1_f(F, V) + h^1_f(F, V \otimes \eta_{K/F}) \tag{12.10.9.2}$$

(by (12.10.1.1) for K/F). We now apply the argument used in [**Ne3**] to deduce Thm. A from Thm. B.

Case (I): $2 \nmid r_{\rm an}(F, g)$. — By [**Wa2**, Thm. 4], [**F-H**, Thm. B (1)] there exists K/F satisfying 12.9.2.1–12.9.2.2 in which P splits and for which $r_{\rm an}(F, g \otimes \eta_{K/F}) = 0$. Put $g_0 = g \otimes \eta_{K/F}$ and $V_0 = V \otimes \eta_{K/F} = V(g_0)(1)$. As $(d_{K/F}, \mathfrak{n}(g)) = (1)$, we have $\mathrm{ord}_v(\mathfrak{n}(g)) = \mathrm{ord}_v(\mathfrak{n}(g_0))$ for any prime $v \mid \mathfrak{n}(g)$. In particular, the form also g_0 satisfies the condition 12.9.1.1. If K'/F is any totally imaginary quadratic extension satisfying

$$(d_{K'/F}, \mathfrak{n}(g_0)) = (1), \quad \eta_{K'/F}(\mathfrak{n}(g_0)) = (-1)^{[F:\mathbf{Q}]-1}$$

in which P splits, then we have

$$r_{\rm an}(F, g_0 \otimes \eta_{K'/F}) = r_{\rm an}(K', g_0) - r_{\rm an}(F, g_0) = r_{\rm an}(K', g_0) \equiv 1 \pmod 2,$$

by Corollary 12.6.3.7 applied to g_0 and K'/F. As g_0 satisfies 12.9.1.1, such fields K' exist; [**F-H**, Thm. B (2)] implies that there exists K' for which $r_{\rm an}(F, g_0 \otimes \eta_{K'/F}) = 1$ (and such that g_0 does not have CM by K'), hence

$$r_{\rm an}(K', g_0) = r_{\rm an}(F, g_0 \otimes \eta_{K'/F}) + r_{\rm an}(F, g_0) = 1 + 0 = 1. \tag{12.10.9.3}$$

Applying the discussion from 12.9.5.2 to g_0 instead to g, we obtain an abelian variety A_0. According to Zhang's generalization of the Gross-Zagier formula ([**Zh1**, Thm. C]; [**Zh2**, Thm. 1.2.1]), it follows from (12.10.9.3) that a certain Heegner point $y \in A_0(K')$ is non-torsion. An Euler system argument ([**Ne4**, Thm. 3.2]) together with the assumption that g_0 does not have CM by K' imply that

$$H^1_f(K', V_0) = L_{\mathfrak{p}} \cdot \delta y,$$

where $\delta : A_0(K') \to H^1_f(K', V_0)$ is the standard Kummer map. Moreover, the non-trivial element $\tau \in \mathrm{Gal}(K'/F)$ acts on $\delta y \in H^1_f(K', V_0)$ as

$$\tau(\delta y) = -\varepsilon(\pi(g_0), \tfrac{1}{2})\, \delta y = -\delta y$$

(by [**Ti**, Lemma 9.1]). As

$$H^1_f(K', V_0)^{\tau=1} = H^1_f(F, V_0), \quad H^1_f(K', V_0)^{\tau=-1} = H^1_f(F, V_0 \otimes \eta_{K'/F}),$$

it follows that $H^1_f(F, V_0) = 0$, *i.e.*, that

$$h^1_f(F, V_0) = h^1_f(F, V \otimes \eta_{K/F}) = 0.$$

Combining

$$r_{\mathrm{an}}(F, g \otimes \eta_{K/F}) = h^1_f(F, V \otimes \eta_{K/F}) = 0$$

with (12.10.9.1)–(12.10.9.2), we obtain the desired congruence (12.10.7.1).

Case (II): $2 \mid r_{\mathrm{an}}(F, g)$. — Let K/F be any totally imaginary quadratic extension satisfying 12.9.2.1–12.9.2.2 in which P splits. It follows from (12.10.9.1)–(12.10.9.2) that $2 \nmid r_{\mathrm{an}}(F, g \otimes \eta_{K/F})$. The same argument as in Case (I) shows that the form $g \otimes \eta_{K/F}$ satisfies 12.9.1.1, hence Case (I) applies to it:

$$r_{\mathrm{an}}(F, g \otimes \eta_{K/F}) \equiv h^1_f(F, V \otimes \eta_{K/F}) \equiv 1 \pmod 2.$$

The relations (12.10.9.1)–(12.10.9.2) then imply

$$r_{\mathrm{an}}(F, g) = r_{\mathrm{an}}(K, g) - r_{\mathrm{an}}(F, g \otimes \eta_{K/F}) \equiv h^1_f(K, V) - h^1_f(F, V \otimes \eta_{K/F}) = h^1_f(F, V) \pmod 2.$$

This completes the proof of Theorem 12.2.3.

12.11. Proof of Theorem 12.2.8

In this section we prove Theorem 12.2.8. The following base change result, which does not require any ordinarity assumption, will not be used in the proof of 12.2.8. However, we record it here for future reference.

12.11.1. Proposition. — *Let $f \in S_2(\mathfrak{n}, 1)$ be a newform over a totally real number field F such that $\pi(f)_{v_0} = \mathrm{St}(\mu_0)$ ($\mu_0^2 = 1$) at some prime v_0 of F. Let L be the number field generated by the Hecke eigenvalues of f and $\mathfrak{p}$ a non-archimedean prime of L; set $V = V_{\mathfrak{p}}(f)(1)$. Then there exists a finite solvable totally real extension F'/F such that the base change form $f' = BC_{F'/F}(f)$ satisfies the following properties:*

(i) *The level $\mathfrak{n}(f')$ of f' is square-free. Equivalently, each local representation $\pi(f')_{v'}$ (where v' is a non-archimedean prime of F') is either in the unramified principal series, or is an unramified twist of the Steinberg representation.*

(ii) $h^1_f(F', V) - r_{\mathrm{an}}(F', f') \equiv h^1_f(F, V) - r_{\mathrm{an}}(F, f) \pmod 2$.

Proof. — Let λ be a non-archimedean prime of L relatively prime to $N\mathfrak{n}$. For each prime $v \mid \mathfrak{n}$ of F, there exists a finite Galois extension E_w/F_v such that the restriction of $V_\lambda(f)$ to $G_w = \mathrm{Gal}(\overline{F}_v/E_w)$ is semistable in the following sense: either the inertia group I_w acts trivially on $V_\lambda(f)$, or there is an exact sequence of $L_\mathfrak{p}[G_w]$-modules

$$0 \longrightarrow L_\mathfrak{p}(1) \otimes \mu \longrightarrow V_\lambda(f)_w \longrightarrow L_\mathfrak{p} \otimes \mu \longrightarrow 0,$$

where $\mu : G_w \to \{\pm 1\}$ is unramified and I_w acts on $V_\lambda(f)_w$ through an infinite quotient.

More precisely, it follows from 12.4.4.3 that one can choose E_w in such a way that the Galois group $H_v := \mathrm{Gal}(E_w/F_v)$ is of the following form:

– If $\pi(f)_v = \pi(\mu, \mu^{-1})$, then H_v ic cyclic (isomorphic to $\mu(\mathcal{O}^*_{F,v})$).

– If $\pi(f)_v = \mathrm{St}(\mu)$ ($\mu^2 = 1$), then H_v is trivial (resp., cyclic of order 2) if μ is unramified (resp., ramified).

– If $\pi(f)$ is supercuspidal and not exceptional, then $H_v \xrightarrow{\sim} D_n = \mathbf{Z}/n\mathbf{Z} \rtimes \{\pm 1\}$ ($n \geqslant 2$) is dihedral.

– If $\pi(f)$ is supercuspidal exceptional ($\Longrightarrow v \mid 2$), then H_v is an extension of A_3 or S_3 by D_n ($n \geqslant 2$).

This implies that

– If $2 \nmid |H_v|$, then H_v is cyclic.

– If $2 \mid |H_v|$, then either there exists an epimorphism $H_v \twoheadrightarrow \mathbf{Z}/2\mathbf{Z}$, or there exists an epimorphism $H_v \twoheadrightarrow \mathbf{Z}/3\mathbf{Z}$ with kernel D_n ($n \geqslant 2$).

Assume that at least one of the groups H_v is not trivial (otherwise we can take $F' = F$).

Step 1.— If the set $\Sigma(F) = \{v \mid \mathfrak{n},\ \exists\, H_v \twoheadrightarrow \{\pm 1\}\}$ is not empty, choose, for each $v \in \Sigma(F)$, an epimorphism $H_v \twoheadrightarrow \{\pm 1\}$; denote by $\varphi^{(v)} : G_v \twoheadrightarrow H_v \twoheadrightarrow \{\pm 1\}$ the corresponding (ramified) quadratic character. According to [**A-T**, ch. 10, Thm. 5], there exists a totally real global character $\varphi : \mathbf{A}^*_F/F^* \to \{\pm 1\}$ such that $(\forall v \in \Sigma(F))\ \varphi_v = \varphi^{(v)}$. By construction, $\pi(f \otimes \varphi)_{v_0} = \mathrm{St}\big(\mu_0 \varphi^{(v_0)}\big)$ is an unramified twist of the Steinberg representation, hence $\varepsilon_{v_0}\big(f \otimes \varphi, \frac{1}{2}\big) = -(\mu_0 \varphi^{(v_0)})(v_0)$.

We are going to define another character $\varphi' : \mathbf{A}^*_F/F^* \to \{\pm 1\}$. If $\varepsilon\big(\pi(f) \otimes \varphi, \frac{1}{2}\big) = 1$, set $\varphi' = 1$. If $\varepsilon\big(\pi(f) \otimes \varphi, \frac{1}{2}\big) = -1$, then there exists $\varphi' : \mathbf{A}^*_F/F^* \to \{\pm 1\}$ of conductor prime to $\mathfrak{n}\,\mathrm{cond}(\varphi)$, which is totally real and satisfies

$$(\forall v \mid \mathfrak{n}\,\mathrm{cond}(\varphi)) \quad \varphi'(v) = \begin{cases} -1, & v = v_0 \\ 1, & v \neq v_0. \end{cases}$$

The local formulas from Proposition 12.6.2.4 imply that

$$\varepsilon\big(\pi(f) \otimes \varphi, \tfrac{1}{2}\big)\, \varepsilon\big(\pi(f) \otimes \varphi\varphi', \tfrac{1}{2}\big) = -1,$$

hence $\varepsilon\big(\pi(f)\otimes\varphi\varphi', \frac{1}{2}\big) = 1$, in either case. It follows from [**F-H**, Thm. B(1)] that there exists a (non-trivial) character $\chi : \mathbf{A}_F^*/F^* \to \{\pm 1\}$ such that

$$(\forall v \mid \infty\,\mathfrak{n}\,\mathrm{cond}(\varphi))\quad \chi_v = (\varphi\varphi')_v,\quad r_{\mathrm{an}}(F, f\otimes\chi) = 0.$$

The first property implies that $\chi = \eta_{F_1/F}$ is associated to a totally real quadratic extension of F; the second that $h_f^1(F, V\otimes\chi) = 0$ (as in 12.10.9), hence

$$\begin{aligned}&h_f^1(F_1, V) - r_{\mathrm{an}}(F_1, f)\\ &\quad = (h_f^1(F,V) - r_{\mathrm{an}}(F,f)) + (h_f^1(F, V\otimes\chi) - r_{\mathrm{an}}(F, f\otimes\chi)) = h_f^1(F,V) - r_{\mathrm{an}}(F,f).\end{aligned}$$

This means that, in order to prove the Proposition, we can replace F by F_1. Repeating this process, if necessary, we reduce to the case when each group H_v (if non-trivial) is either cyclic of odd order, or an extension of A_3 by D_n ($n \geqslant 2$).

Step 2. — By Step 1, we can assume that, for each v with H_v non-trivial, there exists an epimorphism $H_v \twoheadrightarrow \mathbf{Z}/n(v)\mathbf{Z}$ ($2 \nmid n(v)$), whose kernel is either trivial, or is isomorphic to D_n ($n \geqslant 2$). Denote by $\chi^{(v)} : G_v \twoheadrightarrow H_v \twoheadrightarrow \mathbf{Z}/n(v)\mathbf{Z}$ the corresponding character of G_v. Again, [**A-T**, ch. 10, Thm. 5] implies that there exists a global character $\chi : \mathbf{A}_F^*/F^* \to \mathbf{Z}/n\mathbf{Z}$ ($n = \mathrm{lcm}(n(v))$, $2 \nmid n$)such that $\chi_v = \chi^{(v)}$ for each prime v with $H_v \neq \{1\}$. As n is odd, the field $F_2 := \overline{F}^{\mathrm{Ker}(\chi)}$ is a totally real cyclic extension of F satisfying $H_w = \{1\}$ or $H_w = D_n$ ($n \geqslant 2$), for each prime $w \nmid \infty$ of F_2. As in 12.10.1–12.10.2, we have

$$h_f^1(F_2, V) - r_{\mathrm{an}}(F_2, f) \equiv h_f^1(F,V) - r_{\mathrm{an}}(F,f) \pmod 2,$$

which means that we can replace F by F_2.

Step 3.— By Step 2, we can assume that all non-trivial groups H_v are isomorphic to D_n ($n \geqslant 2$). Repeating again Step 1 (r times, if $2^r \| 2n$) and then Step 2, we obtain the desired field F'. □

12.11.2. Lemma. — *Let A, F, L and $\mathfrak{p}$ be as in Theorem 12.2.8; set $V = V_{\mathfrak{p}}(A)$. Let E/F be a finite extension and E'/E a finite Galois extension with Galois group $\Delta = \mathrm{Gal}(E'/E)$.*

(i) *$H_f^1(E', V) = H_f^1(E, V\otimes_{L_{\mathfrak{p}}} L_{\mathfrak{p}}[\Delta])$, with G_E acting on $V\otimes_{L_{\mathfrak{p}}} L_{\mathfrak{p}}[\Delta]$ as in 8.1.3.*

(ii) *If $\alpha : \Delta \to L_{\mathfrak{p}}(\alpha)^*$ is a character with values in a finite extension $L_{\mathfrak{p}}(\alpha)$ of $L_{\mathfrak{p}}$, then*

$$H_f^1(E, V\otimes\alpha) = (H_f^1(E', V)\otimes\alpha)^{\Delta} = (H_f^1(E', V)\otimes_{L_{\mathfrak{p}}} L_{\mathfrak{p}}[\Delta])^{(\alpha^{-1})}.$$

Denote $h_f^1(E, V\otimes\alpha) = \dim_{L_{\mathfrak{p}}(\alpha)} H_f^1(E, V\otimes\alpha)$.

(iii) *If Δ is abelian and all characters of Δ have values in a finite extension M of $L_{\mathfrak{p}}$, then*

$$H_f^1(E', V)\otimes_{L_{\mathfrak{p}}} M \xrightarrow{\sim} \bigoplus_{\alpha:\Delta\to M^*} H_f^1(E, V\otimes\alpha),\quad h_f^1(E', V) = \sum_{\alpha:\Delta\to M^*} h_f^1(E, V\otimes\alpha).$$

If $\alpha^2 = 1$*, then* $V \otimes \alpha = V_{\mathfrak{p}}(A) \otimes \alpha = V_{\mathfrak{p}}(A_\alpha)$*, where* A_α *is the abelian variety over* F *obtained by twisting* A *with* $\alpha \in \mathrm{Hom}_{\mathrm{cont}}(G_F, \{\pm 1\}) = H^1(G_F, \{\pm 1\})$.

(iv) *For each* $\alpha : \Delta \to L_{\mathfrak{p}}(\alpha)^*$,

$$h^1_f(E, V \otimes \alpha) = h^1_f(E, V \otimes \alpha^{-1}).$$

(v) *If* Δ *is abelian, then*

$$h^1_f(E', V) \equiv \sum_{\alpha:\Delta\to\{\pm 1\}} h^1_f(E, V \otimes \alpha) \equiv \sum_{\alpha:\Delta\to\{\pm 1\}} h^1_f(E, V_{\mathfrak{p}}(A_\alpha)) \pmod 2.$$

(vi) *If* $E \subset E_1 \subset E'$*, where* E_1/E *is a 2-abelian extension and* $2 \nmid [E' : E_1]$*, then*

$$h^1_f(E', V) \equiv \sum_{\alpha:\mathrm{Gal}(E_1/E)\to\{\pm 1\}} h^1_f(E, V \otimes \alpha) \pmod 2.$$

Proof. — The statement (i) is a consequence of Shapiro's Lemma; (ii) and (iii) follow from the Δ-equivariance of the isomorphism in (i), and (iv) was proved in Proposition 12.5.9.5 (iv). Finally, (v) and (vi) are deduced from (iii) and (iv) as in the proof of Proposition 8.8.8 (ii) (cf. 12.10.1). □

12.11.3. Proposition - Definition. — *Let* F *and* L *be totally real number fields and* A *an abelian variety over* F *with* $\mathcal{O}_L \subset \mathrm{End}_F(A)$ *and* $\dim(A) = [L : \mathbf{Q}]$.

(1) *We say that* A *is* **potentially modular** *if, for each finite set* S *of non-archimedean primes of* F*, there exists a totally real finite Galois extension* F'/F *such that:*

(i) F'/F *is unramified at* S.

(ii) $A \otimes_F F'$ *is modular in the sense that there exists a Hilbert modular eigenform* $g' \in S_2(\mathfrak{n}(g'), 1)$ *over* F' *with Hecke eigenvalues in* L *such that we have, for each embedding* $\iota : L \hookrightarrow \mathbf{R}$*, an equality of* L*-functions (Euler factor by Euler factor)*

$$L(\iota A/F', s) = L(\iota g', s).$$

Equivalently, for each non-archimedean prime λ *of* L*, the* λ*-adic Galois representations* $V_\lambda(A)|_{G_{F'}}$ *and* $V_\lambda(g')(1)$ *of* $G_{F'}$ *are isomorphic.*

(2) *(*[**Tay3**, proof of Thm. 2.4]*). If (1)(i)–(ii) hold for* F'*, then, for each intermediate field* $F \subset F'' \subset F'$ *with* $\mathrm{Gal}(F'/F'')$ *solvable, there exists a newform* $g'' \in S_2(\mathfrak{n}(g''), 1)$ *over* F'' *such that* $BC_{F'/F''}(g'') = g'$ *and, for each embedding* $\iota : L \hookrightarrow \mathbf{R}$,

$$L(\iota A/F'', s) = L(\iota g'', s)$$

(Euler factor by Euler factor). Equivalently, $V_\lambda(A)|_{G_{F''}}$ *is isomorphic to* $V_\lambda(g'')(1)$*, for each* λ *as in (ii).*

12.11.4. Potential modularity of any A as in 12.11.3 seems to be well-known to the experts [**Tay5**]; a proof is expected to appear in a forthcoming thesis of a student of R. Taylor.

12.11.5. Proposition. — *In the situation of 12.11.3(2),*

(i) *If $w \mid p$ is a prime of F'' such that $A \otimes_F F''$ has totally multiplicative reduction at w, then $\pi(g'')_w = \mathrm{St}(\mu)$, $\mu^2 = 1$, μ is unramified and $\lambda_{g''}(w) = \mu(w) = \pm 1$.*

(ii) *If $w \mid p$ is a prime of F'' such that $A \otimes_F F''$ has good ordinary reduction at w, then $\mathrm{ord}_{\mathfrak{p}}(\lambda_{g''}(w)) = 0$.*

(iii) *If $A \otimes_F F''$ has ordinary (= good ordinary or totally multiplicative) reduction at all primes of F'' above p, then g'' is p-ordinary (with respect to each embedding $L \hookrightarrow \overline{\mathbf{Q}}_p$).*

(iv) *If A has potentially ordinary (= potentially good ordinary or potentially totally multiplicative) reduction at all primes of F above p, then there exists a character of finite order $\chi : \mathbf{A}^*_{F''}/F''^* \to \overline{\mathbf{Q}}^*$ such that $g'' \otimes \chi^{-1}$ is p-ordinary (with respect to each embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$).*

Proof. — Fix a non-archimedean prime $u \nmid p$ of L.

(i) It follows from the analytic uniformization of $A \otimes_F F''_w$ that there is an exact sequence of $L_u[G_w]$-modules

$$0 \longrightarrow L_u(1) \otimes \chi_1 \longrightarrow V_u(A)_w \longrightarrow L_u \otimes \chi_2 \longrightarrow 0,$$

where $\chi_1, \chi_2 : F''^*_w \to L^*_u$ are unramified characters of finite order, and that the inertia group I_w acts on $V_u(A)_w$ through an infinite quotient. The latter condition implies that $\chi_1 = \chi_2$; on the other hand, $\chi_1\chi_2 = 1$, since $\Lambda^2 V_u(A) = L_u(1)$. We conclude by 12.4.4.3.

(ii) The assumption of good ordinary reduction implies that I_w acts trivially on $V_u(A)$ and

$$\det(1 - \mathrm{Fr}(w)_{\mathrm{arith}} X \mid V_u(A)) = (1 - \alpha_w X)(1 - \beta_w X),$$

where α_w, β_w are contained in a quadratic extension L' of L and, for each prime $\mathfrak{p}' \mid \mathfrak{p}$ in L', exactly one of the valuations $\mathrm{ord}_{\mathfrak{p}'}(\alpha_w), \mathrm{ord}_{\mathfrak{p}'}(\beta_w)$ is zero, while the other is positive. It follows that $\mathrm{ord}_{\mathfrak{p}}(\lambda_{g''}(w)) = \mathrm{ord}_{\mathfrak{p}}(\alpha_w + \beta_w) = 0$.

(iii) This is a consequence of (i) and (ii).

(iv) For each prime $v \mid p$ in F'' there exists a finite Galois extension of F''_v over which $A \otimes_F F''_v$ has ordinary reduction. A repeated application of [**A-T**, ch. 10, Thm. 5] implies that there exists a finite totally real solvable extension K/F'' such that $A \otimes_F K$ has ordinary reduction at all primes above p. According to (iii) applied to $A \otimes_F K$, the form $g_K = BC_{F''/K}(g'')$ over K is p-ordinary. The existence of χ then follows from Proposition 12.5.10. □

12.11.6. Application of Brauer's Theorem ([Tay4, §6]). — Let A, F, L and $\mathfrak{p} \mid p$ be as in Theorem 12.2.8; set $V = V_{\mathfrak{p}}(A)$. Assume that E/F is a finite solvable extension and $\alpha : G_E \to \overline{\mathbf{Q}}^*$ a character of finite order. Fixing an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, we view α as a character with values in a finite extension of $L_{\mathfrak{p}}$.

Let F'/F be as in 12.11.3. According to a version of Brauer's Theorem due to L. Solomon ([**Cu-Re**, Thm. 15.10]), there exist intermediate fields $E \cap F' \subset F_j \subset F'$ and integers $n_j \in \mathbf{Z}$ (for j running through a finite set $J(E)$) such that each Galois group $\mathrm{Gal}(F'/F_j)$ is solvable and the trivial representation of $\mathrm{Gal}(F'/E \cap F')$ is equal, as a virtual representation, to

$$1_{\mathrm{Gal}(F'/E\cap F')} = \sum_{j\in J(E)} n_j \,\mathrm{Ind}_{\mathrm{Gal}(F'/F_j)}^{\mathrm{Gal}(F'/E\cap F')}(1_{\mathrm{Gal}(F'/F_j)}). \tag{12.11.6.1}$$

As

$$\mathrm{Gal}(EF'/E) \xrightarrow{\sim} \mathrm{Gal}(F'/E\cap F'), \quad \mathrm{Gal}(EF'/EF_j) \xrightarrow{\sim} \mathrm{Gal}(F'/F_j),$$

we deduce from (12.11.6.1) and the projection formula an equality of virtual representations of G_E

$$V|_{G_E} \otimes \alpha = \sum_{j\in J(E)} n_j \,\mathrm{Ind}_{G_{EF_j}}^{G_E}((V\otimes\alpha)|_{G_{EF_j}}). \tag{12.11.6.2}$$

On the other hand, Proposition 12.11.3(2) implies that $V|_{G_{F_j}} \xrightarrow{\sim} V_{\mathfrak{p}}(g_j)(1)$ for some newform $g_j \in S_2(\mathfrak{n}(g_j), 1)$ over F_j, hence

$$(V\otimes\alpha)|_{G_{EF_j}} \xrightarrow{\sim} V_{\mathfrak{p}}(BC_{EF_j/F_j}(g_j))(1) \otimes \alpha|_{G_{EF_j}}. \tag{12.11.6.3}$$

Combining (12.11.6.2)–(12.11.6.3), we obtain

$$L(\iota A/E, \alpha, s) = \prod_{j\in J(E)} L(BC_{EF_j/F_j}(g_j)\otimes\alpha, s)^{n_j}. \tag{12.11.6.4}$$

This formula gives rise to a meromorphic continuation of the L.H.S. to $\mathbf{C}$ and a functional equation relating $L(\iota A/E, \alpha, s)$ to $L(\iota A/E, \alpha^{-1}, 2-s)$ (as in [**Tay4**, Thm. 6.6]). Putting $r_{\mathrm{an}}(\iota A/F, \alpha) := \mathrm{ord}_{s=1} L(\iota A/E, \alpha, s)$, we deduce from (12.11.6.2)–(12.11.6.4) the following formulas:

$$r_{\mathrm{an}}(\iota A/E, \alpha) = \sum_{j\in J(E)} n_j \, r_{\mathrm{an}}(EF_j, BC_{EF_j/F_j}(g_j)\otimes\alpha) \tag{12.11.6.5}$$

$$h^1_f(E, V\otimes\alpha) = \sum_{j\in J(E)} n_j \, h^1_f(EF_j, V\otimes\alpha). \tag{12.11.6.6}$$

12.11.7. Descent. — In the situation of 12.11.6, assume that E'/E is a finite abelian extension with Galois group $\Delta = \mathrm{Gal}(E'/E)$. Applying (12.11.6.5) to each term in the product

$$L(\iota A/E', s) = \prod_{\alpha\in\widehat{\Delta}} L(\iota A/E, \alpha, s)$$

(where $\widehat{\Delta}$ denotes the group of characters of Δ) and using the fact that, for each $\alpha \in \widehat{\Delta}$,

$$r_{\mathrm{an}}(EF_j, BC_{EF_j/F_j}(g_j) \otimes \alpha) = r_{\mathrm{an}}(EF_j, BC_{EF_j/F_j}(g_j) \otimes \alpha^{-1})$$

by the functional equation, we obtain

$$r_{\mathrm{an}}(\iota A/E') \equiv \sum_{\alpha:\Delta\to\{\pm 1\}} r_{\mathrm{an}}(\iota A/E, \alpha) \pmod 2. \tag{12.11.7.1}$$

Similarly, Lemma 12.11.2 (v)–(vi) yields

$$h^1_f(E', V) \equiv \sum_{\alpha:\Delta\to\{\pm 1\}} h^1_f(E, V \otimes \alpha) \pmod 2. \tag{12.11.7.2}$$

The following special cases of (12.11.7.1)-(12.11.7.2) are of particular interest:

(A) If $2 \nmid [E' : E]$, then

$$r_{\mathrm{an}}(\iota A/E') \equiv r_{\mathrm{an}}(\iota A/E) \pmod 2, \quad h^1_f(E', V) \equiv h^1_f(E, V) \pmod 2. \tag{12.11.7.3}$$

(B) If $E = F$, then we have, for each character $\alpha : \Delta \to \{\pm 1\}$,

$$L(\iota A/F, \alpha, s) = L(\iota A_\alpha/F, s)$$

(in the notation from 12.11.2 (iii)). In particular, the congruences (12.11.7.1)–(12.11.7.2) read as follows:

$$\begin{aligned} r_{\mathrm{an}}(\iota A/E') &\equiv \sum_{\alpha:\Delta\to\{\pm 1\}} r_{\mathrm{an}}(\iota A_\alpha/F) \pmod 2 \\ h^1_f(E', V_{\mathfrak{p}}(A)) &\equiv \sum_{\alpha:\Delta\to\{\pm 1\}} h^1_f(F, V_{\mathfrak{p}}(A_\alpha)) \pmod 2. \end{aligned} \tag{12.11.7.4}$$

12.11.8. Reducing Theorem 12.2.8 to the case $F_1 = F$ for A_α. — Returning to the proof of Theorem 12.2.8, there exists a tower of fields $F \subset F_0 = E_0 \subset \cdots \subset E_n = F_1$ in which each Galois group $\mathrm{Gal}(E_{i+1}/E_i)$ is abelian, of odd degree. Applying (12.11.7.3) (resp., (12.11.7.4)) to each E_{i+1}/E_i (resp., to F_0/F), we obtain

$$h^1_f(F_1, V_{\mathfrak{p}}(A)) - r_{\mathrm{an}}(\iota A/F_1) \equiv \sum_{\alpha:\Delta\to\{\pm 1\}} (h^1_f(F, V_{\mathfrak{p}}(A_\alpha)) - r_{\mathrm{an}}(\iota A_\alpha/F)) \pmod 2$$

(where $\Delta = \mathrm{Gal}(F_0/F)$). It is enough, therefore, to prove Theorem 12.2.8 for each A_α over $F_1 = F$.

12.11.9. The case 12.2.8 (1). — In this case one can take $F' = F$ in 12.11.3 (1): A is associated to a newform $g \in S_2(\mathfrak{n}, 1)$ in the sense that $V_\lambda(A) = V_\lambda(g)(1)$, for all non-archimedean primes λ of L. Moreover, Proposition 12.11.5 (iv) (for $F'' = F$) implies that g is of the form $g = f \otimes \chi$, where $f \in S_2(\mathfrak{n}(f), \chi^{-2})$ is p-ordinary. The result then follows from Theorem 12.2.3 (1) applied to g.

12.11.10. The case 12.2.8 (2). — Applying the discussion in 12.11.6 to $E = F$, we obtain, for each $j \in J(F)$, a newform $g_j \in S_2(\mathfrak{n}(g_j), 1)$ over a totally real extension F_j/F such that $V|_{G_{F_j}} = V_{\mathfrak{p}}(A)|_{G_{F_j}} = V_{\mathfrak{p}}(g_j)(1)$. As A does not have potentially good reduction everywhere, there exists a prime v_j of F_j such that $\pi(g_j)_{v_j} = \mathrm{St}(\mu_j)$. Thanks to (12.11.6.5)–(12.11.6.6), we have, for each $\alpha : \mathrm{Gal}(F_0/F) \to \{\pm 1\}$,

$$h^1_f(F, V_{\mathfrak{p}}(A_\alpha)) - r_{\mathrm{an}}(\iota A_\alpha/F) = \sum_{j \in J(F)} n_j(h^1_f(F_j, V \otimes \alpha) - r_{\mathrm{an}}(F_j, g_j \otimes \alpha)). \tag{12.11.10.1}$$

According to Proposition 12.11.5 (iv), for each $j \in J(F)$ there exists a character of finite order $\chi_j : \mathbf{A}^*_{F_j}/F^*_j \to \overline{\mathbf{Q}}^*$ such that $f_j := g_j \otimes \chi_j^{-1}$ is p-ordinary. Applying Theorem 12.2.3 (2) to the forms $g_j \otimes \alpha = f_j \otimes \alpha\chi_j$ over F_j, we obtain

$$h^1_f(F_j, V \otimes \alpha) \equiv r_{\mathrm{an}}(F_j, g_j \otimes \alpha) \pmod 2. \tag{12.11.10.2}$$

Combining (12.11.10.1)–(12.11.10.2), we obtain the desired congruences

$$h^1_f(F, V_{\mathfrak{p}}(A_\alpha)) \equiv r_{\mathrm{an}}(\iota A_\alpha/F) \pmod 2, \tag{12.11.10.3}$$

which prove (thanks to 12.11.8) Theorem 12.2.8 in the case when A does not have potentially good reduction everywhere.

12.11.11. The case 12.2.8 (3). — If A is modular over F, *i.e.*, if $V_{\mathfrak{p}}(A) = V_{\mathfrak{p}}(f)(1)$ for some newform $f \in S_2(\mathfrak{n}(f), 1)$ over F, then the result follows from Theorem 12.2.3 (3).

In general, choose F'/F as in Proposition 12.11.3 (1), unramified at all primes above p. As in 12.11.10, we obtain intermediate fields $F \subset F_j \subset F'$ and newforms $g_j \in S_2(\mathfrak{n}(g_j), 1)$ over F_j, for which (12.11.10.1) holds (for each $\alpha : \mathrm{Gal}(F_0/F) \to \{\pm 1\}$). According to Proposition 12.11.5 (iii), the forms g_j are p-ordinary. We are going to check that the assumptions 12.2.3 (3) (i)–(iii) are satisfied for each form $g_j \otimes \alpha$ over F_j (with the field F_0 playing the role of the field F' in 12.2.3). Firstly, 12.2.3 (3) (i) follows automatically from the assumptions 12.2.8 (ii).

As regards 12.2.3 (3) (ii), this will follow from 12.2.3 (3) (iii) if A does not have CM. If A has CM, so does $g_j \otimes \alpha$ (over a totally imaginary quadratic extension K_j of F_j). Our assumptions imply that the level of $g_j \otimes \alpha$ is prime to p, hence the restriction of the Galois representation $V_{\mathfrak{p}}(g_j \otimes \alpha)(1)$ to G_{K_j} is associated to an algebraic Hecke character of infinity type $\sum_{\sigma \in \Phi} -\sigma$ (for some CM type Φ of K_j) of conductor prime to p. This implies that, in a suitable basis of $T/\mathfrak{p}T$ (where we have denoted $T = T_{\mathfrak{p}}(A)$), the image of the residual representation $(\overline{\rho}_{\mathfrak{p}} \otimes \alpha)(G_{F_j}) \subset \mathrm{Aut}(T/\mathfrak{p}T)$ contains the normalizer of a Cartan subgroup of $\mathrm{SL}_2(\mathbf{F}_p)$, hence acts irreducibly on $T/\mathfrak{p}T$.

Finally, if A does not have CM, nor does g_j; we must check that, for a suitable choice of a basis of $T/\mathfrak{p}T$, $(\overline{\rho}_{\mathfrak{p}} \otimes \alpha)(G_{F_j}) \supset \mathrm{SL}_2(\mathbf{F}_p)$. Let K' be the fixed field of $\mathrm{Ker}(\overline{\rho}_{\mathfrak{p}} \otimes \alpha : G_F \to \mathrm{Aut}(T/\mathfrak{p}T))$; set $G = \mathrm{Im}(\overline{\rho}_{\mathfrak{p}} \otimes \alpha) = \mathrm{Gal}(K'/F)$, $K := F_j \cap E'$. Let $c \in G$ be the image of the complex conjugation (for a fixed embedding $\overline{F} \hookrightarrow \mathbf{C}$). As F_j is totally real, $\mathrm{Gal}(K'/K)$ contains all conjugates gcg^{-1}, $g \in G$. By assumption, G contains a group conjugate to $\mathrm{SL}_2(\mathbf{F}_p)$; Lemma 12.11.12 below implies that $\mathrm{Gal}(K'/K)$ also contains a group conjugate to $\mathrm{SL}_2(\mathbf{F}_p)$, which is precisely the condition 12.2.3 (3) (iii) for the form $g_j \otimes \alpha$ over F_j.

To sum up, Theorem 12.2.3 (3) applies to $g_j \otimes \alpha$, yielding

$$h^1_f(F_j, V \otimes \alpha) \equiv r_{\mathrm{an}}(F_j, g_j \otimes \alpha) \pmod 2.$$

Together with (12.11.10.1), this implies (12.11.10.3), finishing the proof of Theorem 12.2.8.

12.11.12. Lemma. — *Let k be a finite extension of $\mathbf{F}_p$ ($p \neq 2$) and G a subgroup of $\mathrm{GL}_2(k)$ containing an element c with eigenvalues ± 1. Denote by H the subgroup of $\mathrm{GL}_2(k)$ generated by $\{gcg^{-1} \mid g \in G\}$. If G contains a subgroup conjugate to $\mathrm{SL}_2(\mathbf{F}_p)$ ("G is large enough"), the same is true for H.*

Proof. — According to the list of subgroups of $\mathrm{PGL}_2(k)$ given in [**Dic**, §260], the statement will follow from the following claims:

(i) H acts irreducibly on k^2.

(ii) H is not contained in the normalizer $N = N(C)$ of a Cartan subgroup $C \subset \mathrm{GL}_2(k)$.

(iii) PH (= the image of H in $\mathrm{PGL}_2(k)$) is not isomorphic to A_4, S_4, A_5.

(i) Note that $c \in H$ and that the only c-stable lines in k^2 are $X_\pm = \mathrm{Ker}(c \mp I)$. If H acts reducibly on k^2, then there exists $\varepsilon = +, -$ such that X_ε is stable by all $g^{-1}cg$ ($g \in G$). This implies that, for each $g \in G$, the line $g(X_\varepsilon)$ is c-stable, hence equal to one of the $X_\pm$. As a result, G is contained in the normalizer of the split Cartan subgroup $C = \{g \in \mathrm{GL}_2(k) \mid g(X_\pm) = X_\pm\}$, which contradicts our assumptions; thus H acts irreducibly on k^2.

(ii) Assume that $H \subset N = N(C)$.

Case (1): $c \in C$. As the eigenvalues of c are contained in k, C is split and there is a basis of k^2 in which

$$c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}, \quad N = C \cup \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\}.$$

As G is large enough, there exists a unipotent element $g \in G - \{I\}$. As $p \nmid |N|$, $g \notin N$, hence $gcg^{-1} \in N - C$. A straightforward matrix calculation shows that the conditions $gcg^{-1} \in N - C$ and $g^2cg^{-2} \in N$ imply that there exists $x, t \in k^*$ such that $g = x\left(\begin{smallmatrix} 1 & t \\ \pm t^{-1} & \mp 1 \end{smallmatrix}\right)$ (resp., $g = x\left(\begin{smallmatrix} 1 & t \\ \pm it^{-1} & \mp i \end{smallmatrix}\right)$, $i \in k$, $i^2 = -1$) if $g^2cg^{-2} \in C$ (resp., if $g^2cg^{-2} \in N - C$). However, none of these matrices is unipotent.

Case (2): $H \subset N$, $c \notin C$. After extending scalars to a quadratic extension k'/k, there is a basis of k'^2 in which C, N are as in (1) and $c = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Again, a straightforward calculation shows that, for $g \in G - N$, the conditions $gcg^{-1}, g^{-1}cg \in N$ impose such severe conditions on the matrix elements of g that g cannot be unipotent, which again contradicts the assumption that G is large enough.

(iii) As A_4 is not generated by its elements of order 2, we cannot have $PH \xrightarrow{\sim} A_4$. In order to rule out the remaining two cases, note that the group $\{g \in \mathrm{GL}_2(k) \mid gcg^{-1} \in k^*c\}$ coincides with the normalizer of the split Cartan subgroup $C = \{g \in \mathrm{GL}_2(k) \mid gcg^{-1} = c\}$ of $\mathrm{GL}_2(k)$. As $G \supset h\mathrm{SL}_2(\mathbf{F}_p)h^{-1}$ for some $h \in \mathrm{GL}_2(k)$, it follows that the number of elements of order 2 in PH is equal at least to

$$\left|\mathrm{Im}(\{gcg^{-1} \mid g \in G\} \longrightarrow \mathrm{PGL}_2(k))\right| \geqslant \frac{|\mathrm{SL}_2(\mathbf{F}_p)|}{2\,|\mathrm{SL}_2(\mathbf{F}_p) \cap h^{-1}Ch|} \geqslant \frac{|\mathrm{SL}_2(\mathbf{F}_p)|}{2(p-1)} = \frac{p(p+1)}{2}$$

(where we have used the fact that $\mathrm{SL}_2(\mathbf{F}_p) \cap h^{-1}Ch$ is contained in a split Cartan subgroup of $\mathrm{SL}_2(\mathbf{F}_p)$, hence its order divides $p-1$). If $PH \xrightarrow{\sim} S_4$ or A_5, then $p \nmid |PH|$ ([**Dic**, §260]). On the other hand, the inequalities

$$(\forall p \nmid |S_4|) \quad \left|\{\text{elements of order 2 in} S_4\}\right| = 9 < \frac{p(p+1)}{2}$$

$$(\forall p \nmid |A_5|) \quad \left|\{\text{elements of order 2 in } A_5\}\right| = 15 < \frac{p(p+1)}{2}$$

imply that $PH \neq S_4, A_5$. □

12.12. Systematic growth of Selmer groups

In this section we combine Theorem 12.2.3 and 12.2.8 with Theorem 10.7.17 (v), obtaining a generalization of [**M-R3**, Cor. 3.6-7]. Incidentally, note that Lemma 12.11.2 (vi) can be incorporated into the proof of [**M-R3**, Thm. 11.4 (i)], eliminating the unnatural assumption about the finiteness of the p-primary components of Ш$(E/-)$ in [**M-R3**, Cor. 3.6].

12.12.1. Proposition. — *Let $g = f \otimes \chi \in S_k(\mathfrak{n}(g), 1)$ and $F \subset F' \subset F''$ be as in Theorem 12.2.3. Assume that at least one of the conditions 12.2.3 (1)-(4) is satisfied. Assume, in addition, that*

(i) *F''_∞/F'' is a $\mathbf{Z}_p^r$-extension ($r \geqslant 1$), dihedral (in the sense of 8.4.7) over a certain subfield F''_+ ($[F'' : F''_+] = 2$, $F \subset F''_+$).*

(ii) *For each prime $v \mid p$ in F we have $\pi(g)_v \neq \mathrm{St}(\mu)$ (which is automatic if $k \neq 2$).*

(iii) $r_{\mathrm{an}}(F', g) \equiv 1 \pmod 2$.

Then, for each finite sub-extension $F'' \subset M \subset F''_\infty$,

$$h^1_f(M, V(g)(k/2)) \geqslant [M : F''].$$

Proof. — Denote $V = V(g)(k/2)$. The assumption (ii) implies, by Proposition 12.5.9.2 (iii), that $h^1_f(E, V) = \widetilde{h}^1_f(E, V)$, for each finite extension E/F. According to 12.10.2, we have

$$r_{\mathrm{an}}(F'', g) \equiv r_{\mathrm{an}}(F', g) \equiv 1 \pmod 2,$$

hence $\widetilde{h}^1_f(F'', V) = h^1_f(F'', V) \equiv 1 \pmod 2$, by Theorem 12.2.3. Theorem 10.7.17 (v) (applied with the Greenberg local conditions associated to $\Sigma' = \varnothing$) then yields, for each field M as in the statement of the Proposition, $h^1_f(M,V) = \widetilde{h}^1_f(M,V) \geqslant [M : F'']$. $\square$

12.12.2. Proposition. — *Let A and $F \subset F_0 \subset F_1$ be as in Theorem 12.2.8. Assume that at least one of the conditions 12.2.8(1)–(3) is satisfied. Assume, in addition, that*

(i) *A has good ordinary reduction at each prime of F above p.*

(ii) *$F_{1,\infty}/F_1$ is a $\mathbf{Z}_p^r$-extension ($r \geqslant 1$), dihedral over a certain subfield F_1^+ ($[F_1 : F_1^+] = 2$, $F \subset F_1^+ \subset F_1$).*

(iii) $\mathrm{ord}_{s=1} L(\iota A/F_0, s) \equiv 1 \pmod 2$.

Then, for each finite sub-extension $F_1 \subset M \subset F_{1,\infty}$,

$$\mathrm{rk}_{\mathcal{O}_L} A(M) + \mathrm{cork}_{\mathcal{O}_{L,\mathfrak{p}}} Ш(A/M)[\mathfrak{p}^\infty] \geqslant [M : F_1].$$

Proof. — Denote $V = V_{\mathfrak{p}}(A)$ and use the Greenberg local conditions for V as in 9.7.8, with $\Sigma' = \varnothing$. Thanks to Lemma 9.6.3, 9.6.7.3, 9.6.7.6 and Proposition 9.7.9, we have

$$\mathrm{rk}_{\mathcal{O}_L} A(M) + \mathrm{cork}_{\mathcal{O}_{L,\mathfrak{p}}} Ш(A/M)[\mathfrak{p}^\infty] = h^1_f(M,V) = \widetilde{h}^1_f(M,V),$$

for each finite extension M/F. The proof of Proposition 12.12.1 then applies with obvious modifications (one uses Theorem 12.2.8 instead of 12.2.3). $\square$

12.13. Concluding remarks

12.13.1. Attentive readers will have noticed that the proofs in this chapter rely only on duality theorems for Selmer complexes associated to Greenberg's local conditions with $\Sigma' = \varnothing$. In other words, the theory from Chapter 7 has not been used.

12.13.2. It would be of interest to try to generalize the techniques of [**Ki**] to the situation of Corollary 12.2.10 in the case when E has good supersingular reduction at some prime above p. It would be even more desirable to have a proof of a general version of Theorem 12.9.5 for $k = 2$ (without the quasi-ordinarity assumption) using a general crystalline machinery in the spirit of [**PR6**].

ERRATA

Proposition - Definition 12.6.4.9. — The sentence "If v splits in K/F..." in the proof of (iii) is incorrect, in general. As a result, one has to add the following assumption to Proposition 12.6.4.9(iii): no prime $v \mid p$ that splits in K/F divides $\mathfrak{n}(g)_{\mathrm{St}}^{(P_1 \cdots P_s)}$. This assumption also has to be added to Proposition 12.6.4.12, but not to Proposition 12.6.4.11.

Proposition 12.9.6. — In fact, if $2 \nmid [F : \mathbf{Q}]$, then there is no exceptional extension K'/F, either. If it existed, put $\eta' = \eta_{K'/F}$. As K'/F is unramified outside ∞P, we have $1 = \prod_v \eta'_v(-1) = \prod_{v \mid \infty P} \eta'_v(-1) = (-1)^{[F:\mathbf{Q}]} \eta'_P(-1)$, hence $\eta'_P(-1) = -1$. In particular, P is ramified in K'/F. As f is p-ordinary, $\pi_P(g)$ is not supercuspidal, which implies (by 12.6.1.2.3) that $\pi(g)_P = \pi(\mu, \mu\eta'_P)$ for some $\mu : F_P^* \to \mathbf{C}^*$. The central character of $\pi(g)_P$ is trivial; thus $\mu^2 \eta'_P = 1$ and $\eta'_P(-1) = \mu(-1)^{-2} = 1$, contradiction.

As a result, we can omit the assumption "Assume that g does not have CM by..." in Theorem 12.9.5, Theorem 12.9.8, Corollary 12.9.9, Theorem 12.9.11 and Corollary 12.9.12. Similarly, in Theorem 12.9.7, we can omit the assumptions (i), (ii) in the case $2 \nmid [F : \mathbf{Q}]$.

Theorem 12.9.8, Corollary 12.9.9. — Add the following assumption: no prime $v \mid p$ that splits in K/F divides $\mathfrak{n}(g)_{\mathrm{St}}^{(P)} \mathfrak{n}(g')_{\mathrm{St}}^{(P)}$.

Proposition 12.9.13. — In the proof, we choose $\mathcal{P}'$ such that g' also satisfies $(\mathfrak{n}(g')_{\mathrm{St}}, (p)) = (1)$.

LIST OF SYMBOLS

BIBLIOGRAPHY

[A-C] J. Arthur & L. Clozel – *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Mathematics Studies, vol. 120, Princeton Univ. Press, Princeton, 1989.

[A-T] E. Artin & J. Tate – *Class field theory*, 2nd ed., Addison-Wesley, Redwood City, 1990.

[Be] L. Berger – Limites de représentations cristallines, *Compositio Math.* **140** (2004), p. 1473–1498.

[B-B-M] P. Berthelot, L. Breen & W. Messing – *Théorie de Dieudonné cristalline, II*, Lect. Notes in Math., vol. 930, Springer, Berlin, 1982.

[B-D1] M. Bertolini & H. Darmon – Derived heights and generalized Mazur-Tate regulators, *Duke Math. J.* **76** (1994), p. 75–111.

[B-D2] ———, Derived p-adic heights, *Amer. J. Math.* **117** (1995), p. 1517–1554.

[B-D3] ———, Heegner points on Mumford-Tate curves, *Invent. Math.* **126** (1996), p. 413–456.

[Bl] D. Blasius – Hilbert modular forms and the Ramanujan conjecture, preprint available at `http://www.arxiv.org/abs/math.NT/0511007`.

[B-K] S. Bloch & K. Kato – L-functions and Tamagawa numbers of motives, in *The Grothendieck Festschrift, I*, Progress in Mathematics, vol. 86, Birkhäuser, Boston, Basel, Berlin, 1990, p. 333–400.

[Bou] N. Bourbaki – *Commutative algebra. Chapters 1–7*, Springer, Berlin, 1989.

[Bo-Car] J.-F. Boutot & H. Carayol – Uniformisation p-adique des courbes de Shimura : les théorèmes de Čerednik et de Drinfeld, Astérisque, vol. 196-197, Soc. Math. France, Paris, 1991, p. 45–158.

[B-C-D-T] C. Breuil, B. Conrad, F. Diamond & R. Taylor – On the modularity of elliptic curves over **Q**: wild 3-adic exercises, *J. Amer. Math. Soc.* **14** (2001), p. 843–939.

[Bre] C. Breuil – Une remarque sur les représentations locales p-adiques et les congruences entre formes modulaires de Hilbert, *Bull. Soc. Math. France* **127** (1999), p. 459–472.

[Br-Sh] M. Brodmann & R. Sharp – *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Stud. in Adv. Math., vol. 60, Cambridge Univ. Press, Cambridge, 1998.

[Bro] K. Brown – *Cohomology of groups*, Grad. Texts in Math., vol. 87, Springer, New York, Berlin, 1982.

[Bru] A. Brumer – Pseudocompact algebras, profinite groups and class formations, *J. Algebra* **4** (1966), p. 442–470.

[Br-He] W. Bruns & J. Herzog – *Cohen-Macaulay rings*, Cambridge Stud. in Adv. Math., vol. 39, Cambridge Univ. Press, Cambridge, 1993.

[B-L] J.-L. Brylinski & J.-P. Labesse – Cohomologie d'intersection et fonctions L de certaines variétés de Shimura, *Ann. Sci. École Norm. Sup.* **17** (1984), p. 361–412.

[Bu-Fl1] D. Burns & M. Flach – Tamagawa numbers for motives with (non-commutative) coefficients, *Doc. Math.* **6** (2001), p. 501–570, (electronic).

[Bu-Fl2] ______, Tamagawa numbers for motives with (non-commutative) coefficients, II, *Amer. J. Math.* **125** (2003), p. 475–512.

[Bu-Gr] D. Burns & C. Greither – On the Equivariant Tamagawa Number Conjecture for Tate motives, *Invent. Math.* **153** (2003), p. 303–359.

[Bu-Ve] D. Burns & O. Venjakob – On the leading terms of Zeta isomorphisms and p-adic L-functions in non-commutative Iwasawa theory, *Doc. Math.*, extra vol.: John Coates' Sixtieth Birthday (2006), p. 165–209.

[Car] H. Carayol – Sur les représentations ℓ-adiques attachées aux formes modulaires de Hilbert, *Ann. Sci. École Norm. Sup.* **19** (1986), p. 409–469.

[C-E] H. Cartan & S. Eilenberg – *Homological algebra*, Princeton Univ. Press, Princeton, 1956.

[Cas] W. Casselman – On some results of Atkin and Lehner, *Math. Ann.* **201** (1973), p. 301–314.

[Ca1] J. Cassels – Arithmetic on curves of genus 1. I. On a conjecture of Selmer, *J. Reine Angew. Math.* **202** (1959), p. 52–99.

[Ca2] ———, Arithmetic on curves of genus 1. IV. Proof of the Hauptvermutung, *J. Reine Angew. Math.* **211** (1962), p. 95–112.

[Če] I. ČEREDNIK – Uniformization of algebraic curves by discrete arithmetic subgroups of $PGL_2(k_w)$ with compact quotient spaces, *Mat. Sbornik* **29** (1976), p. 55–78.

[Co-Gr] J. COATES & R. GREENBERG – Kummer theory for abelian varieties over local fields, *Invent. Math.* **124** (1996), p. 129–174.

[Col] P. COLMEZ – Théorie d'Iwasawa des représentations de de Rham d'un corps local, *Ann. of Math. (2)* **148** (1998), p. 485–571.

[Con] B. CONRAD – *Grothendieck duality and base change*, Lect. Notes in Math., vol. 1750, Springer, Berlin, 2000.

[Cor] C. CORNUT – Mazur's conjecture on higher Heegner points, *Invent. Math.* **148** (2002), p. 495–523.

[Cor-Va] C. CORNUT & V. VATSAL – Nontriviality of Rankin-Selberg L-functions and CM points, to appear in *L-functions and Galois representations (Durham, July 2004)*, Cambridge Univ. Press.

[Cu-Re] C. CURTIS & I. REINER – *Methods of Representation Theory, I*, Wiley, New York, 1981.

[De1] P. DELIGNE – Formes modulaires et représentations ℓ-adiques, in *Séminaire Bourbaki, 1968/69* (Berlin), Lect. Notes in Math., vol. 179, Springer, 1971, Exp. 355.

[De2] ———, Les constantes des équations fonctionnelles des fonctions L, in *Modular functions of one variable, II (Antwerp, 1972)*, Lect. Notes in Math., vol. 349, Springer, Berlin, 1973, p. 501–597.

[De3] ———, Valeurs de fonctions L et périodes d'intégrales, in *Automorphic forms, representations and L-functions (Corvallis, 1977)*, Proc. Symposia Pure Math., vol. 33/II, Amer. Math. Soc., Providence, 1979, p. 313–346.

[De4] ———, La conjecture de Weil, II, *Publ. Math. de l'I.H.É.S.* **52** (1980), p. 137–252.

[De-Se] P. DELIGNE & J.-P. SERRE – Formes modulaires de poids 1, *Ann. Sci. École Norm. Sup.* **7** (1974), p. 507–530.

[Dic] L. DICKSON – *Linear groups: With an exposition of the Galois field theory*, Dover, New York, 1958.

[Dim] M. DIMITROV – Galois representations modulo p and cohomology of Hilbert modular varieties, *Ann. Sci. École Norm. Sup.* **38** (2005), p. 505–551.

[Dr] V. Drinfeld – Coverings of p-adic symmetric domains, *Funkc. Anal. i Priloz.* **10** (1976), p. 29–40.

[Fl1] M. Flach – A generalization of the Cassels-Tate pairing, *J. Reine Angew. Math.* **412** (1990), p. 113–127.

[Fl2] ______, Euler characteristics in relative K-groups, *Bull. London Math. Soc.* **32** (2000), p. 272–284.

[Fo1] J.-M. Fontaine – Valeurs spéciales des fonctions L des motifs, in *Séminaire Bourbaki, 1991/92*, Astérisque, vol. 206, Soc. Math. France, Paris, 1992, Exp. 751, p. 205–249.

[Fo2] ______, Représentations ℓ-adiques potentiellement semi-stables, in *Périodes p-adiques (Bures-sur-Yvette, 1988)*, Astérisque, vol. 223, Soc. Math. France, Paris, 1994, p. 321–347.

[Fo-Ma] J.-M. Fontaine & B. Mazur – Geometric Galois representations, in *Elliptic Curves, Modular Forms and Fermat's Last Theorem* (J. Coates & S. Yau, eds.), International Press, Boston, 2nd ed., 1997, p. 190–227.

[Fo-PR] J.-M. Fontaine & B. Perrin-Riou – Autour des conjectures de Bloch et Kato : cohomologie galoisienne et valeurs de fonctions L, in *Motives (Seattle, 1991)*, Proc. Symposia in Pure Math., vol. 55/I, Amer. Math. Soc., Providence, 1994, p. 599–706.

[F-H] S. Friedberg & J. Hoffstein – Nonvanishing theorems for automorphic L-functions on $GL(2)$, *Ann. of Math. (2)* **142** (1995), p. 385–423.

[Fu-Ka] T. Fukaya & K. Kato – A formulation of conjectures on p-adic zeta functions in non-commutative Iwasawa theory, to appear in *Proc. St. Petersburg Math. Soc.*

[G-J] S. Gelbart & H. Jacquet – Forms of $GL(2)$ from the analytic point of view, in *Automorphic forms, representations and L-functions (Corvallis, 1977)*, Proc. Symposia Pure Math, vol. 33/I, Amer. Math. Soc., Providence, 1979, p. 213–251.

[Gre1] R. Greenberg – The Iwasawa invariants of Γ-extensions of a fixed number field, *Amer. J. Math.* **95** (1973), p. 204–214.

[Gre2] ______, Iwasawa theory for p-adic representations, in *Algebraic number theory*, Adv. Stud. Pure Math., vol. 17, Academic Press, Boston, 1989, p. 97–137.

[Gre3] ______, Iwasawa theory for motives, in *L-functions and arithmetic (Durham, 1989)*, London Math. Soc. Lect. Note Ser., vol. 153, Cambridge Univ. Press, Cambridge, 1991, p. 211–233.

[Gre4] ———, Iwasawa theory and p-adic deformations of motives, in *Motives (Seattle, 1991)*, Proc. Symposia in Pure Math., vol. 55/II, Amer. Math. Soc., Providence, 1994, p. 193–223.

[Gre5] ———, Iwasawa theory for elliptic curves, in *Arithmetic theory of elliptic curves (Cetraro, 1997)*, Lect. Notes in Math., vol. 1716, Springer, Berlin, 1999, p. 51–144.

[Gre6] ———, On the structure of certain Galois cohomology groups, *Doc. Math.*, extra vol.: John Coates' Sixtieth Birthday (2006), p. 335–391.

[G-S] R. Greenberg & G. Stevens – p-adic L-functions and p-adic periods of modular forms, *Invent. Math.* **111** (1993), p. 407–447.

[He] G. Henniart – Représentations l-adiques abéliennes, in *Séminaire de Théorie des Nombres de Paris, 1980/81* (M.-J. Bertin, ed.), Progress in Math., vol. 22, Birkhäuser, Boston, 1982, p. 107–126.

[Hi1] H. Hida – Galois representations into $GL_2(\mathbf{Z}_p[\![X]\!])$ attached to ordinary cusp forms, *Invent. Math.* **85** (1986), p. 545–613.

[Hi2] ———, On p-adic Hecke algebras for GL_2 over totally real fields, *Ann. of Math. (2)* **128** (1988), p. 295–384.

[Hi3] ———, On nearly ordinary Hecke algebras for $GL(2)$ over totally real fields, in *Algebraic number theory*, Adv. Stud. Pure Math., vol. 17, Academic Press, Boston, 1989, p. 139–169.

[H-T] H. Hida & J. Tilouine – On the anticyclotomic main conjecture for CM fields, *Invent. Math.* **117** (1994), p. 89–147.

[Ho1] B. Howard – The Heegner point Kolyvagin system, *Compositio Math.* **140** (2004), p. 1439–1472.

[Ho2] ———, Iwasawa Theory of Heegner points on abelian varieties of GL_2-type, *Duke Math. J.* **124** (2004), p. 1–45.

[Ho3] ———, Derived p-adic heights and p-adic L-functions, *Amer. J. Math.* **126** (2004), p. 1315–1340.

[Iw] K. Iwasawa – On $\mathbf{Z}_\ell$-extensions of algebraic number fields, *Ann. of Math. (2)* **98** (1973), p. 246–326.

[Jac] H. Jacquet – *Automorphic forms on $GL(2)$, II*, Lect. Notes in Math., vol. 278, Springer, Berlin, 1972.

[J-L] H. Jacquet & R. Langlands – *Automorphic forms on $GL(2)$*, Lect. Notes in Math., vol. 114, Springer, Berlin, 1970.

[J-PS-S] H. Jacquet, I. Piatetski-Shapiro & J. Shalika – Relèvement cubique non normal, *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), no. 12, p. 567–571.

[Ja1] U. Jannsen – Iwasawa modules up to isomorphism, in *Algebraic number theory*, Adv. Stud. Pure Math., vol. 17, Academic Press, Boston, 1989, p. 171–207.

[Ja2] ———, On the ℓ-adic cohomology of varieties over number fields and its Galois cohomology, in *Galois Groups over* $\mathbf{Q}$ (Y. Ihara, K. Ribet & J.-P. Serre, eds.), MSRI Publ., vol. 16, Springer, New York, 1989, p. 315–360.

[Ja3] ———, A spectral sequence for Iwasawa adjoints, unpublished manuscript, 1994.

[Jan] J.-C. Jantzen – *Moduln mit einem höchsten Gewicht*, Lect. Notes in Math., vol. 750, Springer, Berlin, 1979.

[Je] C. Jensen – *Les foncteurs dérivés de* $\varprojlim$ *et leurs applications en théorie des modules*, Lect. Notes in Math., vol. 254, Springer, Berlin, New York, 1972.

[Ka1] K. Kato – Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions *via* B_{dR}, I, in *Arithmetic algebraic geometry (Trento, 1991)*, Lect. Notes in Math., vol. 1553, Springer, Berlin, 1993, p. 50–163.

[Ka2] ———, p-adic Hodge theory and values of zeta functions of modular forms, in *Cohomologies p-adiques et applications arithmétiques, III*, Astérisque, vol. 295, Soc. Math. France, Paris, 2004, p. 117–290.

[Kaw] T. Kawasaki – On Macaulayfication of Noetherian schemes, *Trans. Amer. Math. Soc.* **352** (2000), p. 2517–2552.

[K-R] C. Khare & R. Ramakrishna – Finiteness of Selmer groups and deformation rings, *Invent. Math.* **154** (2003), p. 179–198.

[Ki] B. Kim – The Parity Theorem of Elliptic Curves at Primes with Supersingular Reduction, to appear in *Compositio Math.*

[Kis] M. Kisin – Potentially semi-stable deformation rings, preprint, 2005.

[Lan] R. Langlands – *Base change for* $GL(2)$, Annals of Mathematics Studies, vol. 96, Princeton Univ. Press, Princeton, 1980.

[La] D. Lazard – Sur les modules plats, *C. R. Acad. Sci. Paris* **258** (1964), p. 6313–6316.

[Ly] R. Lyndon – Cohomology theory of groups with a single defining relation, *Ann. of Math. (2)* **52** (1950), p. 650–665.

[Mat] H. Matsumura – *Commutative ring theory*, 2nd ed., Cambridge Stud. in Adv. Math., vol. 8, Cambridge Univ. Press, Cambridge, 1989.

[M-T-T] B. Mazur, J. Tate & J. Teitelbaum – On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer, *Invent. Math.* **84** (1986), p. 1–48.

[Maz1] B. MAZUR – Rational points of abelian varieties with values in towers of number fields, *Invent. Math.* **18** (1972), p. 183–266.

[Maz2] ______, Modular curves and arithmetic, in *Proceedings of the ICM 1983 (Warsaw), I* PWN, Warsaw, 1984, p. 185–211.

[M-R1] B. MAZUR & K. RUBIN –Studying the growth of Mordell-Weil, *Doc. Math.* (2003), p. 585–607, extra vol., (electronic).

[M-R2] ______, Organizing the arithmetic of elliptic curves, *Adv. in Math.* **198** (2005), p. 504–546.

[M-R3] ______, Finding large Selmer groups, *J. Diff. Geom.* **70** (2005), p. 1–22.

[M-R4] ______, Finding large Selmer rank *via* an arithmetic theory of local constants, to appear. in *Ann. of Math.*

[McCa1] W. MCCALLUM – Duality theorems in multivariable Iwasawa Theory of Number Fields, preprint, 1995.

[McCa2] ______, Greenberg's conjecture and units in multiple $\mathbf{Z}_p$-extensions, *Amer. J. Math.* **123** (2001), p. 909–930.

[McCl] J. MCCLEARY – *A user's guide to spectral sequences*, 2nd ed., Cambridge Stud. in Adv. Math., vol. 58, Cambridge Univ. Press, Cambridge, 2000.

[Mi] J. MILNE – *Arithmetic duality theorems*, Persp. in Math., vol. 1, Academic Press, Boston, 1986.

[Miy] T. MIYAKE – On automorphic forms on GL_2 and Hecke operators, *Ann. of Math. (2)* **94** (1971), p. 174–189.

[Ne1] J. NEKOVÁŘ – Kolyvagin's method for Chow groups of Kuga-Sato varieties, *Invent. Math.* **107** (1992), p. 99–125.

[Ne2] ______, On p-adic height pairings, in *Séminaire de Théorie des Nombres de Paris, 1990/91* (S. David, ed.), Progress in Math., vol. 108, Birkhäuser, Boston, 1993, p. 127–202.

[Ne3] ______, On the parity of ranks of Selmer groups, II, *C. R. Acad. Sci. Sér. I Math.* **332** (2001), no. 2, p. 99–104.

[Ne4] ______, The Euler system method for CM points on Shimura curves, to appear in *L-functions and Galois representations (Durham, July 2004)*, Cambridge Univ. Press.

[Ne5] ______, On the parity of ranks of Selmer groups, III, preprint, 2006.

[N-P] J. NEKOVÁŘ & A. PLATER – On the parity of ranks of Selmer groups, *Asian J. Math.* **4** (2000), no. 2, p. 437–498.

[Ng] T. Nguyen-Quang-Do – Formations de classes et modules d'Iwasawa, in *Number theory (Noordwijkerhout, 1983)*, Lect. Notes in Math., vol. 1068, Springer, Berlin, 1984, p. 167–185.

[Ni] W. Nizioł – Cohomology of crystalline representations, *Duke Math. J.* **71** (1993), p. 747–791.

[N-S-W] J. Neukirch, A. Schmidt & K. Wingberg – *Cohomology of number fields*, Grund. der Math. Wiss., vol. 323, Springer, Berlin, 2000.

[Oh1] M. Ohta – On l-adic representations attached to automorphic forms, *Japan. J. Math. (N.S.)* **8** (1982), p. 1–47.

[Oh2] ______, On the zeta function of an abelian scheme over the Shimura curve, *Japan. J. Math.* **9** (1983), p. 1–25.

[Oh3] ______, On the p-adic Eichler-Shimura isomorphism for Λ-adic cusp forms, *J. Reine Angew. Math.* **463** (1995), p. 49–98.

[PR1] B. Perrin-Riou – *Arithmétique des courbes elliptiques et théorie d'Iwasawa*, Mém. Soc. Math. France (N.S.), vol. 17, Soc. Math. France, Paris, 1984.

[PR2] ______, Descente infinie et hauteur p-adique sur les courbes elliptiques à multiplication complexe, *Invent. Math.* **70** (1982/83), p. 369–398.

[PR3] ______, Fonctions L p-adiques, théorie d'Iwasawa et points de Heegner, *Bull. Soc. Math. France* **115** (1987), p. 399–456.

[PR4] ______, Théorie d'Iwasawa et hauteurs p-adiques (cas des variétés abéliennes), in *Séminaire de Théorie des Nombres de Paris, 1990/91* (S. David, ed.), Progress in Math., vol. 108, Birkhäuser, Boston, 1993, p. 203–220.

[PR5] ______, Théorie d'Iwasawa et hauteurs p-adiques, *Invent. Math.* **109** (1992), p. 137–185.

[PR6] ______, Théorie d'Iwasawa des représentations p-adiques sur un corps local, *Invent. Math.* **115** (1994), p. 81–161.

[Pl] A. Plater – Height pairings in families of deformations, *J. Reine Angew. Math.* **486** (1997), p. 97–127.

[Ri] K. Ribet – Galois action on division points of Abelian varieties with real multiplications, *Amer. J. Math.* **98** (1976), p. 751–804.

[Ro] J. Rogawski – Modular forms, the Ramanujan conjecture and the Jacquet-Langlands correspondence, in *Discrete groups, expanding graphs and invariant measures* (A. Lubotzky, ed.), Progress in Math., vol. 125, Birkhäuser, Basel, 1994, appendix.

[Ru] K. RUBIN – Abelian varieties, p-adic heights and derivatives, in *Algebra and number theory (Essen, 1992)*, de Gruyter, Berlin, 1994, p. 247–266.

[Sa] T. SAITO – Hilbert modular forms and p-adic Hodge theory, preprint, 2001.

[Sc] N. SCHAPPACHER – *Periods of Hecke Characters*, Lect. Notes in Math., vol. 1301, Springer, Berlin, 1988.

[Sch1] P. SCHNEIDER – Über gewisse Galoiscohomologiegruppen, *Math. Z.* **168** (1979), p. 181–205.

[Sch2] ———, Iwasawa L-functions of varieties over algebraic number fields. A first approach, *Invent. Math.* **71** (1983), p. 251–293.

[Sch3] ———, p-adic height pairings, II, *Invent. Math.* **79** (1985), p. 329–374.

[Sch-Te] P. SCHNEIDER & J. TEITELBAUM – Algebras of p-adic distributions and admissible representations, *Invent. Math.* **153** (2003), p. 145–196.

[Se1] J.-P. SERRE –*Corps locaux*, Hermann, Paris, 1962.

[Se2] ———, *Cohomologie Galoisienne*, third ed., Lect. Notes in Math., vol. 5, Springer, Berlin, 1965.

[Se3] ———, *Abelian l-adic representations and elliptic curves*, W.A. Benjamin, New York, Amsterdam, 1968.

[Se4] ———, Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures), in *Séminaire Delange-Pisot-Poitou, 1969/70*, Collected Papers II, vol. 19, p. 581–592.

[Se-Ta] J.-P. SERRE & J. TATE – Good reduction of abelian varieties, *Ann. of Math. (2)* **88** (1968), p. 492–517.

[Sh] G. SHIMURA – The special values of the zeta functions associated to Hilbert modular forms, *Duke Math. J.* **45** (1978), p. 637–679.

[S-U] C. SKINNER & E. URBAN – Sur les déformations p-adiques de certaines représentations automorphes, *J. Inst. Math. Jussieu* **5** (2006), p. 629–698.

[Tay1] R. TAYLOR –On Galois representations associated to Hilbert modular forms, *Invent. Math.* **98** (1989), p. 265–280.

[Tay2] ———, On Galois representations associated to Hilbert modular forms, II, in *Elliptic Curves, Modular Forms and Fermat's Last Theorem* (J. Coates & S. Yau, eds.), International Press, Boston, 1995, p. 185–191.

[Tay3] ———, On icosahedral Artin representations, II, *Amer. J. Math.* **125** (2003), p. 549–566.

[Tay4] ______, On the meromorphic continuation of degree two L-functions, *Doc. Math.*, extra vol.: John Coates' Sixtieth Birthday (2006), p. 729–779.

[Tay5] ______, private communication.

[Ti] Y. TIAN – Euler systems of CM points on Shimura curves, Ph.D. Thesis, Columbia University, 2003.

[Va] V. VATSAL – Special values of anticyclotomic L-functions, *Duke Math. J.* **116** (2003), p. 219–261.

[Ve1] J.-L. VERDIER – Appendix to [**Se2**].

[Ve2] ______, *Des catégories dérivées des catégories abéliennes*, Astérisque, vol. 239, Soc. Math. France, Paris, 1996.

[Wa1] J.-L. WALDSPURGER – Quelques propriétés arithmétiques de certaines formes automorphes sur GL(2), *Compositio Math.* **54** (1985), p. 121–171.

[Wa2] ______, Correspondances de Shimura et quaternions, *Forum Math.* **3** (1991), p. 219–307.

[We] C. WEIBEL – *An introduction to homological algebra*, Cambridge Stud. in Adv. Math., vol. 38, Cambridge Univ. Press, Cambridge, 1994.

[W] A. WEIL – Exercices dyadiques, *Invent. Math.* **27** (1974), p. 1–22.

[Wi] A. WILES – On ordinary λ-adic representations associated to modular forms, *Invent. Math.* **94** (1988), p. 529–573.

[Win] K. WINGBERG – Duality theorems for abelian varieties over $\mathbf{Z}_p$-extensions, in *Algebraic number theory*, Adv. Stud. Pure Math, vol. 17, Academic Press, Boston, 1989, p. 471–492.

[Ya] S. YASUDA – Local constants in torsion rings, preprint, 2001.

[Zh1] S.-W. ZHANG –Heights of Heegner points on Shimura curves, *Ann. of Math. (2)* **153** (2001), p. 27–147.

[Zh2] ______, Gross-Zagier formula for GL_2, *Asian J. Math.* **5** (2001), p. 183–290.

[LC] A. GROTHENDIECK – *Local cohomology*, Lect. Notes in Math., vol. 41, Springer, Berlin, New York, 1967.

[RD] R. HARTSHORNE – *Residues and duality*, Lect. Notes in Math., vol. 20, Springer, Berlin, New York, 1966.

[EGAI] A. GROTHENDIECK – Éléments de géométrie algébrique, I, *Publ. Math. de l'I.H.É.S.* **4** (1960).

[EGAIV.2] ———, Éléments de géométrie algébrique, IV, Seconde Partie, *Publ. Math. de l'I.H.É.S.* **24** (1965).

[SGA6] ———, *Théorie des Intersections et Théorème de Riemann-Roch*, Lect. Notes in Math., vol. 225, Springer, Berlin, Heidelberg, New York, 1971.