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PROJECTIVITY OF KÄHLER MANIFOLDS – KODAIRA'S PROBLEM
[after C. Voisin]

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There are various geometric structures that can be studied on a topological manifold M. Depending on one's geometric taste, it is important to know whether M can be endowed with a symplectic form, whether (special) Riemannian metrics can be found or whether M carries an algebraic structure. Often, the existence of a certain geometric structure imposes topological conditions on M. In other words, it may happen that a given topological manifold does simply not allow one's favorite geometry. *E.g.* if M is compact and $b_2(M) = 0$ the manifold M cannot be symplectic, or if $b_1(M) = 1$ no Kähler metrics can exist.

In order to fully understand the relation between two sorts of geometries, it is important to know whether they impose the same topological obstructions. In other words, does the existence of one of the two on a given manifold topological M imply the existence of the other one? This is a report on the work of Claire Voisin [13, 14] that sheds light on an old question, usually attributed to Kodaira, that asks for the topological relation between Kähler geometry and projective geometry.

In the following we let M be a compact manifold that can be endowed with the structure of a complex manifold. Once a complex structure is chosen, one studies Riemannian metrics g that are 'compatible' with it. One possible compatibility condition is to require that g be hermitian, *i.e.*, that the complex structure thought of as an almost complex structure I is orthogonal with respect to g. It is not difficult to see that a hermitian structure can always be found. It is, however, a completely different matter to find a hermitian structure g such that its fundamental form $\omega := g(I\ ,\ )$ is closed, *i.e.*, g satisfies the Kähler condition. Indeed, the classical theory of Kähler manifolds shows that the existence of a Kähler metric imposes strong conditions on the topology of M, which are not satisfied by arbitrary complex or symplectic manifolds. For instance, the odd Betti numbers of a compact Kähler manifold are even, Kähler manifolds are formal and their fundamental groups satisfy further conditions. (In contrast, if only one of the two structures, complex or symplectic, is required, then any finitely presentable group can be realized.)

On the other hand, Kähler manifolds are quite common. Indeed, any complex submanifold of the complex projective space $\mathbb{P}^n$ admits a Kähler metric - the restriction of the Fubini–Study metric is an example. Conversely, one might wonder whether a compact complex manifold that admits a Kähler structure can always be realized as a complex submanifold of $\mathbb{P}^n$ or, in other words, whether the complex structure is projective. This is obviously not the case, general complex tori $\mathbb{C}^n/\Gamma$ ($n \geq 2$) and general K3 surfaces provide counter-examples. In fact, a famous theorem of Kodaira proves that a Kähler manifold is projective if and only if the Kähler metric can be chosen such that the cohomology class of its fundamental form ω is integral, *i.e.*, $[\omega] \in H^2(X, \mathbb{Z})$ (see [6, Thm. 4]).

In these examples one observes that although the given complex structure is not projective, it becomes projective after a small deformation. Kodaira proved that in fact any Kähler surface can be deformed to a projective surface (see [7, Thm. 23] and [8]). Thus, as deforming the complex structure does not change the diffeomorphism type of the manifold, there is no topological difference between compact Kähler surfaces and algebraic surfaces. (Let us also mention that in fact any compact surface X with even $b_1(X)$ is Kähler, *i.e.*, for surfaces the condition to be Kähler is a topological condition. This fails in higher dimensions, due to a famous example of Hironaka [5] of a compact Kähler manifold that deforms to complex manifold which is no longer Kähler.) Note in passing that a similar result holds true for symplectic manifolds: clearly, any given symplectic form ω can be deformed to a symplectic form with integral cohomology class.

Kodaira's problem, which apparently has never been stated by himself in this form, asks for the higher-dimensional version of his result: *Can any compact Kähler manifold be deformed to a projective manifold?*

More in the spirit of the general philosophy explained above, one could ask whether the topological manifold underlying a compact Kähler manifold may also be endowed with the structure of a projective manifold. This question had been open for a very long time. As Kodaira's arguments to prove the two-dimensional case use a great deal of classification theory of surfaces, there was little hope to generalize them to higher dimensions.

Recent work of Claire Voisin fills this gap [11, 13, 14]. She succeeded in showing that topology makes a difference between compact Kähler manifolds and those that are projective. In other words, there exist compact topological manifolds that admit the structure of a Kähler manifold without carrying also the structure of a projective manifold. More precisely, Voisin shows the stronger statement:

THEOREM 0.1 ([13]). — *In any dimension ≥ 4 there exists a compact Kähler manifold X whose rational cohomology ring $H^*(X, \mathbb{Q})$ cannot be realized as the rational cohomology ring of a projective manifold.*

Voisin originally worked with the integral cohomology ring $H^*(X,\mathbb{Z})$, but Deligne then pointed out the stronger version above.

One could wonder whether the answer to these questions would be different if the topological manifold satisfies further conditions, *e.g.* if it is in addition simply-connected. Some of these questions have been addressed and answered by Voisin in [13, 14] and we will comment on them on the way.

Although the examples are obtained by particular constructions, the principal ideas of [13, 14] are of a more general nature and might be applicable in other situations.

The i-th cohomology of a compact Kähler manifold is naturally endowed with a Hodge structure of weight i, which can be polarized (on the primitive part) if the manifold is projective. The idea is to show that there exist compact Kähler manifolds whose cohomology does not admit Hodge structures that are compatible with both, the given cup-product and a polarization. Roughly, there are three steps **A-C**, the first two of which are purely Hodge-theoretical and only the last one has a geometric flavor.

(A) Certain algebraic structures on a rational vector space A are not compatible with any *polarizable* Hodge structure (of weight k) on A.

Remark 0.2. — In the examples, the algebraic structure will be a specific endomorphism $\Phi : A \to A$, but others are in principle possible. That the algebraic structure is not compatible with any polarizable Hodge structure means in the case of an endomorphism Φ that one cannot find a Hodge structure on A such that Φ becomes an endomorphism of it and such that the Hodge structure can be polarized.

(B) Suppose $\bigoplus H^\ell$ is a graded $\mathbb{Q}$-algebra whose direct summands H^ℓ are Hodge structures of weight ℓ and such that the multiplications $H^{\ell_1} \otimes H^{\ell_2} \to H^{\ell_1+\ell_2}$ are homomorphisms of Hodge structures. Suppose furthermore that this $\mathbb{Q}$-algebra structure allows us to detect a subspace $A \subset H^k$ such that: i) $A \subset H^k$ is a Hodge substructure. ii) An algebraic structure as in **(A)** is compatible with this Hodge structure.
Then H^k does not admit a polarization.

Remark 0.3. — Subspaces that are defined purely in terms of the $\mathbb{Q}$-algebra structure do define Hodge substructures. We shall also need a refined version of this, which is due to Deligne.

The compatibility in ii) is more difficult to check, but relies on the same principle. For an endomorphism Φ the idea goes as follows: Firstly, find two Hodge substructures $A, A' \subset H^k$ and a Hodge substructure $\Delta \subset A \oplus A' \subset H^k$ which is the graph of an isomorphism $A \cong A'$. Secondly, prove that under the induced isomorphism of Hodge structures $A \oplus A \cong A \oplus A'$ the graph of Φ is a Hodge substructure.

(C) Construct compact Kähler manifolds such that the above principles apply to its cohomology ring $\bigoplus H^\ell(X,\mathbb{Q})$. Then $H^*(X,\mathbb{Q})$ should not be realizable by a smooth projective variety.

Remark 0.4. — This works best for Hodge structures of weight one ($k=1$). In this case $H^1(X,\mathbb{Q})$ of a smooth projective variety X admits a polarized Hodge structure. For the Hodge structure of weight two on $H^2(X,\mathbb{Q})$ one needs an extra argument, for only the primitive part of it admits a polarization.

This report roughly follows these three steps. Some of the algebraic structures in Section 2 might seem rather ad hoc, as their geometric origin is only explained in Section 3. However, I found it helpful for my own understanding to completely separate the arguments that explain why certain $\mathbb{Q}$-algebras cannot be realized as the cohomology of a projective manifold from the part that contains the construction of compact Kähler manifolds that do realize these $\mathbb{Q}$-algebras.

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1. HODGE STRUCTURES (OF WEIGHT ONE AND TWO)

1.1. Recollections

A *Hodge structure of weight* k on a $\mathbb{Q}$-vector space A is given by a direct sum decomposition

$$A_{\mathbb{C}} := A\otimes_{\mathbb{Q}}\mathbb{C} = \bigoplus_{p+q=k} A^{p,q} \quad \text{such that} \quad \overline{A^{p,q}} = A^{q,p}. \tag{1}$$

A direct sum decomposition (1) can also be described in terms of a representation $\rho:\mathbb{C}^*\to \mathrm{Gl}(A_{\mathbb{R}})$ such that the $\mathbb{C}$-linear extension of $\rho(z)$ satisfies $\rho(z)|_{A^{p,q}} = z^p\bar z^q\cdot\mathrm{id}$. The *Hodge classes* of a Hodge structure of weight $2k$ on A are the elements in $A^{k,k}\cap A$.

We shall be particularly interested in Hodge structures of weight one and two.

Remark 1.1. — Recall that Hodge structures of weight one with $A^{p,q}=0$ for $pq\neq 0$ which are integral, *i.e.*, $A=\Gamma_{\mathbb{Q}}$ for some lattice Γ, are in bijection with complex tori. Indeed, to a Hodge structure of weight one on $\Gamma_{\mathbb{Q}}$ given by $\Gamma_{\mathbb{C}} = A^{1,0}\oplus A^{0,1}$ one associates the complex torus $A^{1,0}/\Gamma$, where Γ is identified with its image under the projection $A_{\mathbb{C}}\to A^{1,0}$.

A $\mathbb{Q}$-linear map $\varphi : A \to A'$ is a *morphism* (*of weight* m) *of Hodge structures*

$$A_{\mathbb{C}} = \bigoplus_{p+q=k} A^{p,q} \quad \text{and} \quad A'_{\mathbb{C}} = \bigoplus_{r+s=\ell} A'^{r,s}$$

of weight k and $\ell = k + 2m$, respectively, if $\varphi(A^{p,q}) \subset A'^{p+m,q+m}$. If the two Hodge structures correspond to $\rho : \mathbb{C}^* \to \mathrm{Gl}(A_{\mathbb{R}})$ and $\rho' : \mathbb{C}^* \to \mathrm{Gl}(A'_{\mathbb{R}})$, respectively, then this condition is equivalently expressed by $\varphi(\rho(z)v) = |z|^{2m}\rho'(z)\varphi(v)$ for all $v \in A$ and $z \in \mathbb{C}^*$.

A *Hodge substructure* of a Hodge structure of weight k on A is given by a subspace $A' \subset A$ such that $A'_{\mathbb{C}} = \bigoplus (A^{p,q} \cap A'_{\mathbb{C}})$ or, equivalently, such that $A'_{\mathbb{C}} \subset A_{\mathbb{C}}$ is invariant under the representation $\rho : \mathbb{C}^* \to \mathrm{Gl}(A_{\mathbb{R}})$ that corresponds to the given Hodge structure on A.

The *tensor product* $A \otimes_{\mathbb{Q}} A'$ of two $\mathbb{Q}$-vector spaces A and A' endowed with Hodge structures of weight k and ℓ, respectively, comes with a natural Hodge structure of weight $(k + \ell)$:

$$(A \otimes_{\mathbb{Q}} A')^{r,s} := \bigoplus_{p+p'=r, q+q'=s} A^{p,q} \otimes_{\mathbb{C}} A'^{p',q'}.$$

In other words, the Hodge structure is given by $\rho \otimes \rho'$.

Note that $A_2 := \bigwedge^2 A_1$ of a Hodge structure of weight one A_1 is naturally a Hodge structure of weight two with $A_2^{2,0} := \bigwedge^2 A_1^{1,0}$, $A_2^{1,1} := A_1^{1,0} \otimes A_1^{0,1}$, and $A_2^{0,2} := \bigwedge^2 A_1^{0,1}$.

A *polarization* of a Hodge structure of weight one $A_{\mathbb{C}} = A^{1,0} \oplus A^{0,1}$ is a skew-symmetric form $q \in \bigwedge^2 A^*$ such that

$$A_{\mathbb{C}} \times A_{\mathbb{C}} \longrightarrow \mathbb{C}, \quad (v, w) \longmapsto iq(v, \overline{w}) \tag{2}$$

(where q is extended $\mathbb{C}$-linearly) satisfies the Hodge–Riemann relations:

i) $A^{1,0}$ and $A^{0,1}$ are orthogonal with respect to (2).

ii) The restriction of (2) to $A^{1,0}$ and to $A^{0,1}$ is positive, respectively negative, definite.

Remark 1.2. — With this definition a polarization is always rational. Furthermore, the form q considered as an element of the induced weight-two Hodge structure on $\bigwedge^2 A^*$ is of type $(1, 1)$. Since it is rational, q is a Hodge class (of weight two). Note that any Hodge substructure of a weight-one polarized Hodge structure is naturally polarized.

Example 1.3. — Let X be a compact Kähler manifold of dimension n. The Hodge decomposition

$$H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$$

defines a Hodge structure of weight one on $H^1(X, \mathbb{Q})$.

Suppose X is projective and $\omega \in H^2(X, \mathbb{Z})$ is the class of a hyperplane section, then $q(\alpha) = \int_X \alpha^2 \omega^{n-1}$ is a polarization of the natural Hodge structure of weight one on $H^1(X, \mathbb{Q})$.

If we drop the condition that q be rational, then any Kähler class on a compact Kähler manifold X would yield a form on the Hodge structure of weight one on $H^1(X,\mathbb{Q})$ that satisfies the Hodge–Riemann relations i) and ii).

The notion of a polarization exists for Hodge stuctures of arbitrary weight, but we shall only need it for weight one, explained above, and for weight two. For a Hodge structure of weight two $A_{\mathbb{C}} = A^{2,0} \oplus A^{1,1} \oplus A^{0,2}$ a *polarization* is a symmetric bilinear form $q \in S^2A^*$ such that:

i) The $A^{p,q}$ are pairwise orthogonal with respect to $(v,w) \mapsto q(v,\overline{w})$.

ii) For $0 \neq v \in A^{p,q}$ one has $-i^{p-q}q(v,\overline{v}) > 0$.

Example 1.4. — If X is compact Kähler of dimension n, then $H^2(X,\mathbb{Q})$ comes with a natural Hodge structure of weight two $H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$ given by the Hodge decomposition. If X is projective and $\omega \in H^2(X,\mathbb{Z})$ is the class of a hyperplane section, then

$$q(\alpha) = \int_X \alpha^2\omega^{n-2}$$

defines a polarization on the *primitive cohomology*

$$H^2(X,\mathbb{Q})_{\mathrm{p}} := \{\alpha \in H^2(X,\mathbb{Q}) \mid \alpha \wedge \omega^{n-1} = 0\}.$$

Note that due to the Hodge–Riemann bilinear relation $H^{1,1}(X,\mathbb{R}) \cong H^{1,1}(X,\mathbb{R})_{\mathrm{p}} \oplus \mathbb{R}\omega$ does not contain any q-isotropic subspace of dimension ≥ 2. Also, $H^2(X,\mathbb{R})$ does not contain Hodge substructures of dimension ≥ 2 which are q-isotropic.

1.2. Detecting Hodge structures algebraically

The following observation is the key to a general principle, due to Deligne, which allows one to identify Hodge substructures algebraically.

LEMMA 1.5. — *Let $H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$ be a Hodge structure of weight k on a $\mathbb{Q}$-vector space H given by a representation $\rho : \mathbb{C}^* \to \mathrm{Gl}(H_{\mathbb{R}})$ and let $Z \subset H_{\mathbb{C}}$ be an algebraic subset which is invariant under $\rho(\mathbb{C}^*)$. Suppose the span $\langle Z' \rangle$ of an irreducible component $Z' \subset Z$ is of the form $H' \otimes_{\mathbb{Q}} \mathbb{C}$ with $H' \subset H$ a $\mathbb{Q}$-subspace. Then H' is a Hodge substructure of H.*

Proof. — Since $\mathbb{C}^*$ is connected, the $\mathbb{C}^*$-action leaves invariant the irreducible components of Z. Hence, also $\langle Z' \rangle$ is $\mathbb{C}^*$-invariant. For $\langle Z' \rangle = H' \otimes_{\mathbb{Q}} \mathbb{C}$ this is equivalent to saying that $H' \subset H$ is a Hodge substructure. □

In [13, 14] the lemma is applied in various situations. The algebraic set Z is always defined by algebraic conditions on homomorphisms of Hodge structures and thus automatically invariant under $\mathbb{C}^*$. Usually, one starts with several Hodge structures of weight ℓ on $\mathbb{Q}$-vector spaces H^ℓ and homomorphisms of Hodge structures

$H^{\ell_1} \otimes H^{\ell_2} \to H^{\ell_1+\ell_2}$, $a \otimes b \mapsto a \cdot b$. (Think of the cohomology of a smooth projective variety or of a compact Kähler manifold.)

We shall in particular encounter algebraic subsets of the form

$$Z_1 = \{\alpha \in H^k_{\mathbb{C}} \mid \alpha^2 = 0\} \quad \text{or} \quad Z_2 = \left\{a \in H^k_{\mathbb{C}} \mid \operatorname{rk}\left(H^\ell_{\mathbb{C}} \xrightarrow{a\cdot} H^{k+\ell}_{\mathbb{C}}\right) \leq m\right\}.$$

Let us sketch the argument that shows that these sets are $\mathbb{C}^*$-invariant in the example $Z = Z_2$. By definition of the Hodge structure on $H^{\ell_1} \otimes H^{\ell_2}$ and the hypothesis that the multiplication $a \otimes b \mapsto a \cdot b$ is a morphism of Hodge structures, one has $\rho(z)(a) \cdot b = \rho(z)(a \cdot (\rho(z^{-1})(b)))$. Thus, the endomorphism given by multiplication with $\rho(z)(a)$ and a, respectively, differs by automorphisms $\rho(z) \in \mathrm{Gl}(H^{k+\ell}_{\mathbb{R}})$ and $\rho(z^{-1}) \in \mathrm{Gl}(H^\ell_{\mathbb{R}})$. In particular, $\operatorname{rk}(\rho(z)(a)\cdot) = \operatorname{rk}(a\cdot)$ and hence $a \in Z$ if and only if $\rho(z)(a) \in Z$.

Note that it might well happen that $\langle Z \rangle$ is defined over $\mathbb{Q}$, but not $\langle Z' \rangle$.

Let us illustrate the use of Deligne's principle in a concrete situation that will be at the heart of the subsequent discussion. Suppose we are given a graded $\mathbb{Q}$-algebra $\bigoplus H^k$, an integer $\ell \in \mathbb{Z}$ and a subspace $0 \neq H' \subset H^\ell$. Then define for $i \geq 1$ the $\mathbb{Q}$-subspace

$$P_i := \left\{a \in H^2 \mid \left(\bigotimes^i H' \xrightarrow{\cdot a} H^{\ell i+2}\right) = 0\right\}. \tag{3}$$

We shall later fix in addition an integer $m > 1$ and consider the two subspaces

$$P_1 \subset P_m \subset H^2$$

and the algebraic subset of $P_{m\mathbb{C}}$:

$$Z := \{a \in P_{m\mathbb{C}} \mid \operatorname{Ker}\left(H'_{\mathbb{C}} \xrightarrow{\cdot a} H^{\ell+2}_{\mathbb{C}}\right) \neq 0\}. \tag{4}$$

Then Z contains $P_{1\mathbb{C}}$ and we denote its image in $(P_m/P_1)_{\mathbb{C}}$ by $\bar{Z}$ (which is again algebraic). Furthermore, let $e \in Z \cap P_m$ be such that $\mathbb{C}\bar{e} \subset \bar{Z}$ is an irreducible component of $\bar{Z}$.

Corollary 1.6. — *Suppose each H^k is endowed with a Hodge structure of weight k such that the multiplications are morphisms of Hodge structures and such that $H' \subset H^\ell$ is a Hodge substructure. Then*

i) *the $P_i \subset H^2$ are Hodge substructures,*

ii) *the element $\bar{e} \in P_m/P_1$ is of type* $(1,1)$, i.e.*, a Hodge class, and*

iii) $\operatorname{Ker}(H' \xrightarrow{\cdot e} H^{\ell+2})$ *is a Hodge substructure of H^ℓ.*

Proof. — The P_i can be viewed as the kernels of the morphisms of Hodge structures $H^2 \to \left(\bigotimes^i H'\right)^* \otimes H^{\ell i+2}$ and are, therefore, Hodge substructures of H^2.

Deligne's principle shows that $\mathbb{Q}\bar{e} \subset P_m/P_1$ is a Hodge substructure. Since any weight two Hodge structure of rank one is of pure type, one finds $\bar{e} \in (P_m/P_1)^{1,1}$.

In order to prove iii), use the morphism of Hodge structures $P_m/P_1 \otimes H' \to H^{\ell+2}$. □

Remark 1.7

i) The actual description of P_m is of no importance here. We only used $P_1 \subset P_m$ and the condition on e. Note that $e \in P_m$ itself might be of mixed type, *e.g.* it could be arbitrarily modified by rational classes in $P_1^{2,0} \oplus P_1^{0,2}$.

ii) In the applications only the cases $\ell = 1$ and $\ell = 2$ will be considered and, moreover, for $\ell = 1$ we will have $H' = H^1$.

2. THE IMPOSSIBLE ONES

The aim is to exhibit two specific Hodge structures of weight one respectively two which resist polarization. Section 2.1 explains Step **A** of the program, whereas Section 2.2 corresponds to Step **B**.

2.1. Special endomorphisms excluding polarization

Let us start out with an endomorphism $\Phi \in \mathrm{End}(A)$ of a $\mathbb{Q}$-vector space A of dimension $2n$. For any field $\mathbb{Q} \subset K$ we shall denote by Φ_K its K-linear extension. We also use the naturally induced endomorphisms Φ^* and $\bigwedge^2 \Phi^*$ of A^* and $\bigwedge^2 A^*$ respectively.

Denote the set of all eigenvalues of Φ by $EV(\Phi) := \{\mu_1, \ldots, \mu_{2n}\}$ and by K_Φ the splitting field of the characteristic polynomial of Φ, *i.e.*, $K_\Phi = \mathbb{Q}(\mu_1, \ldots, \mu_{2n})$.

Henceforth, we shall assume that:

HYPOTHESIS 2.1. — i) $\mu_i \notin \mathbb{R}$ *for all* i, *and* ii) $G := \mathrm{Gal}(K_\Phi/\mathbb{Q})$ *acts as the symmetric group* S_{2n} *on* $EV(\Phi)$.

Example 2.2. — It is not difficult to find explicit examples of endomorphisms Φ satisfying these conditions:

– Let $A = \mathbb{Q}^2$, hence $n = 1$, and $\Phi = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then $\{\mu_1, \mu_2\} = \{\pm i\}$.

– Let $A = \mathbb{Q}^4$ and $\Phi = \left(\begin{smallmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{smallmatrix}\right)$. The characteristic polynomial of Φ is $x^4 - x + 1$ whose Galois group is the symmetric group (see [1, Ch.14.6]) and which clearly has no real eigenvalues.

Remark 2.3. — Clearly, ii) implies that $\Phi_{\mathbb{C}} \in \mathrm{End}(A_{\mathbb{C}})$ can be diagonalized. It also yields $\mu_{i_1} \cdot \cdots \cdot \mu_{i_k} \neq \mu_{j_1} \cdot \cdots \cdot \mu_{j_k}$ for any two distinct multi-indices $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$.

LEMMA 2.4. — *Under the assumptions of* 2.1 *the induced endomorphism* $\bigwedge^k \Phi \in \mathrm{End}(\bigwedge^k A)$ *does not admit any non-trivial invariant subspace.*

Proof. — Clearly, the eigenvalues of $\bigwedge^k \Phi$ are $\mu_{i_1} \cdot \cdots \cdot \mu_{i_k}$, $i_1 < \cdots < i_k$. Thus, if $W \subset \bigwedge^k A$ is invariant under $\bigwedge^k \Phi$, then the eigenvalues of $\psi := \bigwedge^k \Phi|_W$ are also of this form. In particular, also ψ can be diagonalized over K_Φ. Suppose $W \neq 0$. Then there exists an eigenvector $v \in W_{K_\Phi}$ with eigenvalue say $\mu_1 \cdot \cdots \cdot \mu_k$.

Being defined over $\mathbb{Q}$, the extension of ψ (and of $\bigwedge^k \Phi$) to an endomorphism of W_{K_Φ} (respectively $\bigwedge^k A_{K_\Phi}$) commutes with the action of the Galois group G on the scalars K_Φ. Hence, with $\mu_1 \cdot \cdots \cdot \mu_k$ also $\mu_{\sigma(1)} \cdot \cdots \cdot \mu_{\sigma(k)}$ is an eigenvalue of ψ for any $\sigma \in G$.

By Remark 2.3, this shows that all $\mu_{i_1} \cdot \cdots \cdot \mu_{i_k}$, $i_1 < \cdots < i_k$, which are pairwise distinct, occur as eigenvalues of ψ. Hence, $\dim(W) = \dim(\bigwedge^k A)$ or, equivalently, $W = \bigwedge^k A$. □

PROPOSITION 2.5. — *Suppose* $\bigwedge^2 \Phi$ *respects a Hodge structure of weight two on* $A_2 := \bigwedge^2 A$ *given by* $\bigwedge^2 A_{\mathbb{C}} = A_2^{2,0} \oplus A_2^{1,1} \oplus A_2^{0,2}$ *with* $A_2^{2,0} \neq 0$. *If* Φ *satisfies* 2.1, *then*

$$A_2^{1,1} \cap \bigwedge^2 A = \{0\},$$

which is equivalent to saying that all Hodge classes of A_2 *are trivial.*

Proof. — As $\bigwedge^2 \Phi_{\mathbb{C}}$ preserves the bidegree (p, q) of elements in $\bigwedge^2 A_{\mathbb{C}}$, the rational subspace $W := A_2^{1,1} \cap \bigwedge^2 A$ is $\bigwedge^2 \Phi$-invariant. Due to the lemma one either has $W = \bigwedge^2 A$, which is excluded by $A_2^{2,0} \neq 0$, or $W = 0$, which proves the assertion. □

COROLLARY 2.6. — *Suppose* $n \geq 2$. *A Hodge structure of weight one* $A_{\mathbb{C}} = A^{1,0} \oplus A^{0,1}$ *that is preserved by* $\Phi_{\mathbb{C}}$ *does not admit a polarization.*

Proof. — A polarization of the Hodge structure $A_{\mathbb{C}} = A^{1,0} \oplus A^{0,1}$ would be given by a special Hodge class q in the induced Hodge structure of weight two on $\bigwedge^2 A^*$. However, there are no non-trivial ones due to the proposition. (Use that Φ^* as well satisfies 2.1.) The assumption $n \geq 2$ is needed in order to ensure that $A_2^{2,0} \neq 0$. □

Remark 2.7. — Observe that Φ preserves the Hodge structure if and only if its graph $\Gamma_\Phi \subset A \oplus A$ is a Hodge substructure.

Example 2.8. — If Φ satisfies i) and ii) of 2.1, one easily constructs Hodge structures of weight one that are preserved by Φ. This will be needed when it actually comes to constructing examples.

Pick n distinct eigenvalues $\lambda_1, \ldots, \lambda_n \in EV(\Phi)$ such that $\lambda_i \neq \bar{\lambda}_j$ for all i, j (note that due to i) no eigenvalue is real) and let $A^{1,0} = \bigoplus_{i=1}^n \mathbb{C}v_i$, where the $v_i \in A_{\mathbb{C}}$ are eigenvectors with eigenvalue λ_i.

With Φ being defined over $\mathbb{Q}$, the complex conjugate $\bar{\lambda}$ of an eigenvalue $\lambda \in EV(\Phi)$ is again an eigenvalue. Thus, with $A^{0,1} := \overline{A^{1,0}}$ one has $A_{\mathbb{C}} = A^{1,0} \oplus A^{0,1}$.

2.2. Identifying the special endomorphisms algebraically

We continue the discussion of Section 1.2 and combine it with endomorphisms Φ of the type studied in Section 2.1.

So, let us consider a $\mathbb{Q}$-vector space A of dimension $2n \geq 4$ together with an endomorphism Φ and let $H^* = \bigoplus_{k=0}^{4n} H^k$ be a graded $\mathbb{Q}$-algebra.

To bring both structures together, we assume that there is a graded inclusion

$$\bigwedge\nolimits^* (A \oplus A) \subset H^*$$

satisfying the following conditions. (We shall apply Corollary 1.6 with $\ell = 1$, $m = 4n-2$, and $H' = H^1$.)

Hypothesis 2.9

i) $A \oplus A = H^1$,

ii) $H^2 = \bigwedge^2(A \oplus A) \oplus P \oplus R$, *where* $P := P_{4n-2}$ *is defined as in* (3) *and* R *is some subspace,*

iii) $P = P_1 \oplus \bigoplus_{i=1}^4 e_i\mathbb{Q}$, *and*

iv) *the kernel of the multiplication* $H^1 \xrightarrow{\cdot e_i} H^3$, *for* $i = 1, \ldots, 4$, *equals the subspaces* $A \oplus \{0\}$, $\{0\} \oplus A$, $\Delta := \{(a,a) \mid a \in A\}$, *and the graph* Γ_Φ *of* Φ, *respectively. The sum* $\sum \mathrm{Im}(\cdot e_i) \subset H^3$ *is direct.*

Remark 2.10. — Roughly, e_1 and e_2 will be used to detect certain Hodge substructures, e_3 to identify them, and e_4 to view Φ as a homomorphism between them. The auxiliary space R is later only needed in order to construct odd-dimensional examples. Due to Remark 3.4 one could even restrict to the case $P_1 = 0$.

Proposition 2.11. — *Suppose* H^* *and* Φ *meet the conditions of* 2.9 *and* 2.1, *respectively. Then* H^* *cannot be realized as the rational cohomology ring* $H^*(X, \mathbb{Q})$ *of a projective manifold* X.

Proof. — Suppose X is a projective manifold that does realize H^*. In the following we will simply identify $H^*(X, \mathbb{Q})$ with H^*. Thus, each H^k inherits the natural Hodge structure of weight k from $H^k(X, \mathbb{Q})$ and the multiplications $H^{\ell_1} \otimes H^{\ell_2} \to H^{\ell_1+\ell_2}$ are morphisms of Hodge structures.

Corollary 1.6 applies and shows that $A \oplus \{0\}$, $\{0\} \oplus A$, Δ, and the graph Γ_Φ are Hodge substructures of $H^1(X, \mathbb{Q})$. Indeed, the only thing that needs to be checked is that the $\mathbb{C}\bar{e}_i$ define irreducible components of $\bar{Z} \subset (P/P_1)_\mathbb{C}$ (the image of Z as in (4)). This follows from iv): Suppose $\sum a_i e_i \in Z$. Then there exists $0 \neq a \in H^1$ that is annihilated by it. Thus, $a_i(a \cdot e_i) = 0$ for $i = 1, \ldots, 4$. If *e.g.* $a_i \neq 0 \neq a_j$, then $a \in \mathrm{Ker}(\cdot e_i) \cap \mathrm{Ker}(\cdot e_j)$. The description of the kernels shows that this is impossible.

With the identification of the two Hodge structures on $A \oplus \{0\}$ and $\{0\} \oplus A$ via Δ, the graph Γ_Φ allows to view Φ as an endomorphism of the Hodge structure on $A \oplus \{0\}$.

By Corollary 2.6 this Hodge structure does not admit a polarization. Hence, also the Hodge structure $H^1(X,\mathbb{Q})$, of which $A\oplus\{0\}$ is a Hodge substructure, cannot be polarized. This yields a contradiction to the projectivity of X. □

We shall next present a similar result based on an analysis of Hodge structures of weight two.

Let as before A be a $\mathbb{Q}$-vector space of dimension $2n\geq 4$ together with an endomorphism Φ and let $H^*=\bigoplus_{k=0}^{4n}H^k$ be a graded $\mathbb{Q}$-algebra. We assume that there is a graded inclusion $\bigwedge^{2*}(A\oplus A)\subset H^{2*}$ and consider $B_1:=\bigwedge^2 A\oplus\{0\}$, $B_2:=\{0\}\oplus\bigwedge^2 A$, and $H':=B_1\oplus B_2$ as subspaces of H^2. We shall use the notation of Corollary 1.6 with $\ell=2$, $m=2n-1$.

Hypothesis 2.12

i) $H^2=B_1\oplus B_2\oplus P$ *with* $P:=P_{2n-1}$ *as in* (3),

ii) $P=Q_1\oplus Q_2\oplus\mathbb{Q}e_1\oplus\mathbb{Q}e_2$ *for some vector spaces* Q_i *and* $P_1=0$,

iii) $\{\alpha\in H^2_{\mathbb{C}}\mid\alpha^2=0\}=\{\alpha\in B_{1\mathbb{C}}\mid\alpha^2=0\}\cup\{\alpha\in B_{2\mathbb{C}}\mid\alpha^2=0\}$,

iv) $\alpha^2a^{2n-2}=0$ *for all* $\alpha\in B_1$ *and* $a\in P$, *and*

v) *the kernel of the multiplication* $B_1\oplus B_2\xrightarrow{\cdot e_i}H^4$, $i=1,2$, *is the diagonal* $\Delta:=\{(a,a)\}\subset\bigwedge^2 A\oplus\bigwedge^2 A$ *for* $i=1$ *and the graph* $\Gamma_{\bigwedge^2\Phi}$ *for* $i=2$. *Similarly,* $\operatorname{Ker}(\cdot a_i)=B_i$ *for any* $0\neq a_i\in Q_i$. *The sum* $Q_1\cdot B_2+Q_2\cdot B_1+\operatorname{Im}(\cdot e_1)+\operatorname{Im}(\cdot e_2)$ *is direct.*

Proposition 2.13. — *Suppose* H^* *and* Φ *meet the requirements of* 2.12 *and* 2.1, *respectively. Then* H^* *cannot be realized as the rational cohomology ring* $H^*(X,\mathbb{Q})$ *of a projective manifold* X.

Proof. — Suppose X is a projective manifold whose rational cohomology ring $H^*(X,\mathbb{Q})$ can be identified with H^*.

Due to iii) and Lemma 1.5, B_1, B_2, and hence H' are Hodge substructures of H^2. Thus, Corollary 1.6 applies and shows that P is a Hodge substructure. (Note that $\bigwedge^2 A$ is spanned by vectors α with $\alpha^2=0$.)

Due to v), the algebraic set $Z\subset P$ (see notation in Corollary 1.6) contains $\mathbb{C}e_1$ and $\mathbb{C}e_2$ as two irreducible components. Indeed, if $\sum a_i+\sum\eta_ie_i\in Z$ with $a_i\in Q_i$, then some $0\neq b=b_1+b_2\in B_1\oplus B_2$ is annihilated by it. Since the sum of the multiplications is direct, this yields $a_2\cdot b_1=a_1\cdot b_2=\eta_i(b\cdot e_i)=0$. In particular, $a_1\neq 0$ implies $b_2=0$ and $a_2\neq 0$ implies $b_1=0$. Thus, if $\eta_1=\eta_2=0$, then either $a_1\neq 0$ or $a_2\neq 0$. Similarly, if $\eta_1\neq 0=\eta_2$, then $b_1=b_2\neq 0$ and, therefore, $a_1=a_2=0$. Finally, the case $\eta_1\neq 0\neq\eta_2$ is excluded by $\Delta\cap\Gamma_{\bigwedge^2\Phi}=\{(0,0)\}$, which follows from $\mu_i\cdot\mu_j\neq 1$ for all $i\neq j$ and $n\geq 2$. (The argument shows that the other irreducible components are $Q_{i\mathbb{C}}$.)

Thus, by iii) of Corollary 1.6, the diagonal and the graph of $\bigwedge^2 \Phi$ are Hodge substructures of $B_1 \oplus B_2$. In other words, $\bigwedge^2 \Phi$ is an endomorphism of the Hodge structure of $\bigwedge^2 A$ induced by B_1 (or, equivalently, by B_2).

Clearly, $\bigwedge^2 A$ contains a subspace V of dimension at least two such that $0 = \alpha^2 \in H^4$ for all $\alpha \in V$. (For instance, take $V = \langle v_1 \wedge v_2, v_1 \wedge v_3 \rangle$ if $A = \bigoplus \mathbb{Q} v_i$.)

Hence, by the Hodge–Riemann bilinear relations this excludes $V \subset H^{1,1}(X)$ (see Example 1.4). Therefore, $\bigwedge^{2,0} A \neq 0$ and, hence, the Hodge structure $\bigwedge^2 A$ does not contain any Hodge class (see Proposition 2.5).

This shows that all Hodge classes of H^2 are contained in P. In particular, any hyperplane class $[\omega]$ is contained in P. On the other hand, due to iv) one has $\alpha^2.[\omega]^{2n-2} = 0$ for all $\alpha \in B_1$, but $H^2(X, \mathbb{Q})$ can clearly not contain a Hodge substructure of dimension ≥ 2 which is isotropic with respect to the polarization (see Example 1.4). This yields the contradiction. □

3. CONSTRUCTION OF EXAMPLES

So far we have explained how Voisin is able to exclude certain Hodge structures on $\mathbb{Q}$-algebras from being realized by the cohomology of a projective manifold. It remains to find compact Kähler manifolds which do realize these structures and which, therefore, are topologically different from any projective manifold.

The first two examples are obtained as blow-ups of well-known Kähler manifolds and the following general facts will be used tacitly throughout (see [3, 4, 12]). Let $\pi : \widetilde{X} \to X$ be the blow-up of a compact complex manifold X along a submanifold $i : Y \hookrightarrow X$ of codimension $c \geq 2$. The exceptional divisor $j : E = \pi^{-1}(Y) \hookrightarrow \widetilde{X}$ is isomorphic to $\mathbb{P}(\mathcal{N}_{Y/X})$ and $\pi|_E$ equals the projection $\pi_Y : \mathbb{P}(\mathcal{N}_{Y/X}) \to Y$. In the following, cohomology will be considered with coefficients in $\mathbb{Q}$.

• If X is Kähler, then $\widetilde{X}$ is Kähler.

• If a submanifold $Z \subset X$ intersects Y transversally, then the proper transform, which is by definition the closure of $\pi^{-1}(Z \setminus Y)$, is the blow-up $\widetilde{Z} \to Z$ along $Y \cap Z$.

• The natural morphisms $\pi^* : H^k(X) \to H^k(\widetilde{X})$ and

$$H^{k-2(\ell+1)}(Y) \xrightarrow{\pi_Y^*} H^{k-2(\ell+1)}(E) \xrightarrow{\cdot h^\ell} H^{k-2}(E) \xrightarrow{j_*} H^k(\widetilde{X}),$$

where $h := \mathrm{c}_1(\mathcal{O}_{\pi_Y}(1))$, induce isomorphisms

$$H^k(\widetilde{X}) \cong H^k(X) \oplus \bigoplus_{i=k-2(c-1)}^{k-2} H^i(Y).$$

In particular, $H^2(\widetilde{X}) \cong H^2(X) \oplus \mathbb{Q}e$ if $e := [E] \in H^2(\widetilde{X})$ and Y is connected.

• Moreover,

$$\varphi_e : H^k(X) \xrightarrow{\pi^*} H^k(\widetilde{X}) \xrightarrow{\cdot e} H^{k+2}(\widetilde{X})$$

equals

$$H^k(X) \xrightarrow{i^*} H^k(Y) \xrightarrow{\pi_Y^*} H^k(E) \xrightarrow{j_*} H^{k+2}(\widetilde{X}).$$

In particular, $\mathrm{Ker}(H^k(X) \xrightarrow{\varphi_e} H^{k+2}(\widetilde{X})) = \mathrm{Ker}(H^k(X) \xrightarrow{i^*} H^k(Y))$.

• If $Y = Y_1 \sqcup Y_2$ and accordingly $E = E_1 \sqcup E_2$, then for $k = 1$ the sum $\sum \mathrm{Im}(\varphi_{e_i}) \subset H^3(\widetilde{X}) = H^3(X) \oplus H^1(Y_1) \oplus H^1(Y_2)$ is direct and similar for $k = 2$ the sum $\sum \mathrm{Im}(\varphi_{e_i}) \subset H^4(\widetilde{X}) \cong H^4(X) \oplus \bigoplus H^2(Y_i) \oplus \bigoplus H^0(Y_i)$ is direct. (Note that the degree zero terms only occur if $c \geq 3$.) This principle can be generalized to the case that Y_1, Y_2 intersect transversally and that $\pi : \widetilde{X} \to X$ is obtained from first blowing up along Y_1 and then along the proper transform of Y_2.

3.1. Voisin's first example

Let Φ be an endomorphism of a $\mathbb{Q}$-vector space A of dimension $2n \geq 4$ satisfying Hypothesis 2.1. By passing to $k\Phi$ for some $0 \neq k \in \mathbb{Z}$ if necessary, we may assume that Φ^* preserves a maximal lattice $\Gamma \subset A^*$. Consider the complex torus $T := A^{1,0^*}/\Gamma$, where $A_{\mathbb{C}} = A^{1,0} \oplus A^{0,1}$ is a Hodge structure as in Example 2.8. Then there exist natural isomorphisms $H^1(T, \mathbb{Q}) \cong A$ and $H^{1,0}(T) \cong A^{1,0}$. The endomorphism Φ^* induces an endomorphism of T which shall also be denoted Φ^*.

Remark 3.1. — The complex tori T and $T \times T$ are not projective due to Corollary 2.6, but they are, as all other complex tori, deformation equivalent and hence homeomorphic to abelian varieties.

Voisin's first example constructed in [13] is a compact Kähler manifold X obtained as a blow-up of $T \times T$.

Consider the following submanifolds of $T \times T$:

$$\Delta_1 := \{(x, -x)\}, \quad \Delta_2 := \{(x, -\Phi^*(x))\}, \quad T_1 := \{0\} \times T, \quad T_2 := T \times \{0\},$$

which meet pairwise transversally. (*E.g.*, via the first projection the tangent space of $\Delta_1 \cap \Delta_2$ in an intersection point $z = (x, y)$ is identified with $\mathrm{Ker}(\mathrm{id} - \Phi^*)$, but 1 is not an eigenvalue of Φ.)

Let $z_1, \ldots, z_M \in T \times T$ be the finitely many intersection points of all the pairwise intersections. Then consider the blow-up $\pi_1 : \widetilde{T \times T} \to T \times T$ in these points. The proper transforms of the four submanifolds $\widetilde{\Delta}_1, \widetilde{\Delta}_2, \widetilde{T}_1, \widetilde{T}_2$ are pairwise disjoint submanifolds of $\widetilde{T \times T}$. Thus, the blow-up $\pi_2 : X \to \widetilde{T \times T}$ along the union $\widetilde{\Delta}_1 \cup \widetilde{\Delta}_2 \cup \widetilde{T}_1 \cup \widetilde{T}_2$ is a compact Kähler manifold.

We shall denote by $F_1, \ldots, F_M \subset X$ the proper transform of the exceptional divisors of π_1 and by $E_1 \to \widetilde{T}_1$, $E_2 \to \widetilde{T}_2$, $E_3 \to \widetilde{\Delta}_1$, $E_4 \to \widetilde{\Delta}_4$ the exceptional divisors of π_2. Their cohomology classes shall be called $f_1, \ldots, f_M, e_1, \ldots, e_4 \in H^2(X, \mathbb{Q})$. It is the

second blow-up π_2 and its exceptional classes $e_1, \ldots, e_4$ that are important; the first blow-up π_1 is only needed in order to ensure the smoothness of X.

The composition $\pi := \pi_1 \circ \pi_2 : X \to T \times T$ induces a graded inclusion $\bigwedge^*(A \oplus A) = H^*(T \times T, \mathbb{Q}) \subset H^*(X, \mathbb{Q})$.

PROPOSITION 3.2. — *The conditions* i)-iv) *of* 2.9 *are satisfied.*

Proof. — The condition i) is obvious, as X and $T \times T$ are homeomorphic away from subsets of real codimension ≥ 2. Since $H^2(T \times T, \mathbb{Q}) \cong \bigwedge^2 H^1(T \times T, \mathbb{Q})$, one has $H^2(X, \mathbb{Q}) \cong \bigwedge^2(A \oplus A) \oplus \bigoplus_{i=1}^M \mathbb{Q} f_i \oplus \bigoplus_{i=1}^4 \mathbb{Q} e_i$.

A class in $\bigwedge^{4n-2} H^1(X, \mathbb{Q}) = \bigwedge^{4n-2} H^1(T \times T, \mathbb{Q})$ can be thought of as a linear combination of fundamental classes of subsets of real codimension $4n-2$ in $T \times T$ in general position, whose pull-back clearly avoids the exceptional divisors $F_1, \ldots, F_M, E_1, \ldots, E_4$ which all live over subsets of real codimension > 2. This yields ii) with $P = \langle f_1, \ldots, f_M, e_1, \ldots, e_4 \rangle$ and $R = 0$.

A similar argument yields iii), where $P_1 = \langle f_1, \ldots, f_M \rangle$. Finally, condition iv) is proved by applying the above general remarks on the cohomology of a blow-up and by using the explicit description of Δ_1, Δ_2, T_1, and T_2. □

Together with Proposition 2.11 this yields

COROLLARY 3.3. — *The rational homotopy type of the compact Kähler manifold X of dimension $2n \geq 4$ is not realized by any projective manifold.* □

Note that this time the result has been phrased in terms of the rational homotopy type rather than in terms of the rational cohomology. Both statements are equivalent due to [2] and the fact that the fundamental group is abelian in our situation.

Remark 3.4. — One could also avoid the initial point blow-ups and instead successively blow up T_1, T_2, Δ_1, Δ_2, respectively their proper transforms. The above arguments remain valid, only that in this case $P_1 = 0$.

In order to fully prove Theorem 0.1 it remains to construct examples of odd dimension. These are obtained as products $X' := X \times \mathbb{P}^1$, where X is one of the compact Kähler manifolds above. Once more the conditions i)-iv) of 2.9 are satisfied, but this time $R = H^2(\mathbb{P}^1, \mathbb{Q})$. The rest of the argument is unaffected by this modification.

Remark 3.5. — In [13] it is first shown that the integral cohomology $H^*(X, \mathbb{Z})$ of the above constructed Kähler manifold cannot be realized by a projective manifold. The proof of this weaker statement does not rely on Deligne's principle, but uses the Albanese morphism instead.

One finds in [13] also an example, due to Deligne, of a compact Kähler manifold whose complex cohomology $H^*(X, \mathbb{C})$ cannot be realized by a projective manifold. The manifold X is again obtained as a blow-up of $T \times T$.

3.2. Simply-connected examples

One might wonder whether the fundamental group is responsible for the fact that the above constructed compact Kähler manifold is topologically different from any projective manifold. This question leads Voisin to her second example, which is simply-connected. Roughly, the simply-connected Kähler manifold is obtained from the first one by dividing by the $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-action, which is induced by the standard involution on the two factors.

On the one hand, the construction is simpler in the sense that blowing up T_1 and T_2 can be avoided, which was needed before to detect certain Hodge substructures. As it turns out, the analogous Hodge structures in the simply-connected case can be described directly. (As the examples will be simply-connected, one cannot work with Hodge structures of weight one. Therefore, Voisin analyses the weight-two Hodge structure on $H^2(X,\mathbb{Q})$ instead.) On the other hand, due to the (mild) singularities of $T/\pm$, the construction is slightly more involved, as we first have to desingularize.

In [13] Voisin proceeds as follows. Start with a torus $T = A^{1,0^*}/\Gamma$ as in Section 3.1. In particular, T comes with an endomorphism Φ^*. Next, consider the quotient $T/\pm$ of T by the standard involution $z \mapsto \pm z$ and its desingularization $K \to T/\pm$ obtained by a simple blow-up of all the two-torsion points. Equivalently, one may first blow up the two-torsion points $\widetilde{T} \to T$ and then take the quotient $K = \widetilde{T}/\pm$ by the induced involution. The latter description shows that K is smooth and Kähler. (Indeed, a general result of Varouchas [10] proves that for a surjection $\pi : X \to X'$ whose fibres are all of dimension $\dim(X) - \dim(X')$ the manifold X' is Kähler if X is so.) Viewing K as the desingularization of $T/\pm$, shows that it is simply-connected, for $T/\pm$ is.

The endomorphism $-\Phi^*$ of T descends to an endomorphism $-\bar{\Phi}^*$ of $T/\pm$ and we consider its graph $\Gamma_{-\bar{\Phi}^*} \subset (T/\pm)\times(T/\pm)$.

In the last step, one first blows up $K \times K$ along the anti-diagonal $\Delta_1 := \{(a,-a)\}$ and then along the proper transform Γ' of $\Gamma_{-\bar{\Phi}^*}$. (Note that Γ' is smooth. This can be seen by passing via $\widetilde{T}\times\widetilde{T} \to T\times T$.)

Thus, the resulting variety X is indeed a Kähler manifold. We let $\pi : X \to K\times K$ be the composition of the two blow-ups. The two exceptional divisors $E_1 \to \Delta$ and $E_2 \to \Gamma'$ yield distinguished cohomology classes $e_1, e_2 \in H^2(X,\mathbb{Z})$.

PROPOSITION 3.6. — *Let $n \geq 3$. Then the conditions* i)-v) *of* 2.12 *are satisfied.*

Proof. — Since the involution of T acts trivially on $H^2(T,\mathbb{Q})$, one has $H^2(T/\pm,\mathbb{Q}) \cong H^2(T,\mathbb{Q}) = A$ and $H^2(K,\mathbb{Q}) = A \oplus \bigoplus \mathbb{Q}f_j$, where f_i are the classes corresponding to the exceptional divisors F_i over the two-torsion points.

Thus, $H^2(X,\mathbb{Q}) = H^2(K\times K,\mathbb{Q}) \oplus \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 = H^2((T/\pm)\times(T/\pm),\mathbb{Q}) \oplus Q_1 \oplus Q_2 \oplus \mathbb{Q}e_1 \oplus \mathbb{Q}e_2$, where Q_i is the pull-back of $\bigoplus \mathbb{Q}f_j$ under the i-th projection onto K.

It is easy to see that $P := Q_1 \oplus Q_2 \oplus \mathbb{Q}e_1 \oplus \mathbb{Q}e_2$ is indeed the subspace that is annihilated by $S^{2n-1}H^2((T/\pm)\times(T/\pm),\mathbb{Q})$. This proves i).

Since $\bigwedge^2 A$ is spanned by elements a with $a^2 = 0$ and no non-trivial linear combination of $f_{1j} := \pi_1^* f_j, f_{2j} := \pi_2^* f_j, e_1$, and e_2 has this property, condition iii) follows. It is here that one needs the assumption $n \geq 3$. The verification of condition v) is straightforward; use the explicit description of the classes e_1 and e_2.

To conclude, we have to verify condition iv). One can show that for all $\alpha \in B_1$ expressions of the form $\alpha^2 \cdot P(f_{ij}, e_1, e_2)$ with P a polynomial of degree $2n-2$ are indeed trivial. Here are a few of the necessary arguments. Firstly, $f_{ij}^k = 0$ for all $k > n$. Secondly, the classes $f_{ij} \cdot e_k$ and $e_1 \cdot e_2$ are supported over finitely many points in $(T/\pm) \times (T/\pm)$ and, hence as α is pulled back from there, one has $\alpha \cdot (f_{ij} \cdot e_k) = \alpha \cdot (e_1 \cdot e_2) = 0$. Thirdly, $\alpha \cdot f_{1j} = 0$. Thus, the only combinations that need to be checked are $\alpha^2 \cdot e_i^{2n-2}$. We may assume that $E_i = \mathbb{P}(\Omega_T)$ and that $\pi|_{E_i}$ is the natural projection $p : E \to T$. Then one shows that $e_i|_{E_i} = \mathrm{c}_1(\mathcal{O}_p(-1))$ and thus reduces to $0 = p^*\alpha_T^2.\mathrm{c}_1(\mathcal{O}_p(-1))^{2n-3}$, which follows from $\mathrm{c}_1(\mathcal{O}_p(-1))^k = 0$ for $k \geq n$ and the assumption $n \geq 3$. □

Together with Proposition 2.13 this yields

COROLLARY 3.7. — *The rational homotopy type of the compact simply-connected Kähler manifold X of dimension $2n \geq 6$ is not realized by any projective manifold.* □

Odd-dimensional examples can again be produced by taking products with $\mathbb{P}^1$. In 2.12 only i) and iii) have to be modified. In i) one has $H^2 = B_1 \oplus B_2 \oplus P \oplus R$ with $R = H^2(\mathbb{P}^1, \mathbb{Q})$ and in iii) $R_{\mathbb{C}}$ will provide another irreducible component. The arguments are not affected by this modification. This yields C. Voisin's second counter-example:

THEOREM 3.8 ([13]). — *In any dimension ≥ 6 there exists a simply-connected compact Kähler manifold which does not have the rational homotopy type of a projective manifold.*

Once more, instead of working with the rational homotopy type one could equivalently say that $H^*(X, \mathbb{Q})$ is not realized as the cohomology ring of a projective manifold (see [2]).

Remark 3.9. — Inspired by Voisin's examples, Oguiso studies in [9] simply-connected compact Kähler manifolds of dimension $d \geq 4$ which are not projective, but rigid, *i.e.*, which do not allow any deformations at all and, therefore, cannot be deformed to projective ones in particular. In the case of simply-connected examples one can no longer work with Hodge structures of weight one. Thus, K3 surfaces (or, more generally, compact hyperkähler manifolds) with their very special but rich Hodge structures of weight two provide a reservoir of potentially interesting examples. Roughly, the special endomorphisms of tori used by Voisin are in [9] replaced by special automorphisms of K3 surfaces which are described completely by their action on the second cohomology.

However, the methods in [9] fall short of proving that the examples do not have the rational homotopy type of projective manifolds. It seems likely, nevertheless, that four-dimensional simply-connected examples could eventually be produced in this way.

3.3. The birational Kodaira problem

Right after [13] had appeared, modifications of the original problem have been proposed. For many problems in complex algebraic geometry it is natural not to restrict to projective or Kähler manifolds, but to allow manifolds that are birational or bimeromorphic to those. Passing to a bimeromorphic model often changes the topology drastically, but in a somewhat controlled manner. So, modifying Kodaira's problem in this sense seems natural also from a topological point of view.

More precisely, the compact Kähler manifolds constructed in [13] are both bimeromorphic to compact Kähler manifolds which do have the homotopy type of projective manifolds. *E.g.* in the first example, described in Section 3.1, the Kähler manifold X was constructed as a blow-up of a torus whose underlying manifold carries also the structure of a projective manifold. In other words, after a controlled topological modification the original topological manifold underlying X has been transformed to one that does carry a projective structure. So, one could ask whether this is true for any Kähler manifold. Again, the answer is negative.

THEOREM 3.10 ([14]). — *There exist compact Kähler manifolds X of dimension $2n \geq 10$ such that no complex manifold bimeromorphic to it has the rational homotopy type of a projective manifold.*

The principal ideas in [14] are similar to those in [13]. Roughly, one tries to detect certain Hodge structures in terms of the multiplicative structure of the cohomology ring and to derive a contradiction to the existence of a polarization on the (primitive) second cohomology of a projective manifold. Technically, the arguments are more involved and we only give an idea of the actual construction.

The construction of the birational counter-examples in [14] starts again with the same torus T of dimension $n \geq 4$ and an endomorphism Φ satisfying 2.1. If $\mathcal{P}$ denotes the Poincaré bundle on $T \times \widehat{T}$, then let $E := \mathcal{P} \oplus \mathcal{P}^{-1}$ and $E_\Phi := (\Phi, \mathrm{id})^* E$. In the next step one considers the fibre product $\mathbb{P}(E) \times_{T \times \widehat{T}} \mathbb{P}(E_\Phi)$ and its quotient Q by the action of $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ given by natural lifts of $(-\mathrm{id}, \mathrm{id})$ and $(\mathrm{id}, -\mathrm{id})$. Then any Kähler desingularization X of Q will work. Note that these examples are bimeromorphic to a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over $K \times \widehat{K}$, where $K \to T/\pm$ is the desingularization considered in the simply-connected case.

The reason that one is able to control in this example all bimeromorphic models by cohomological methods is due to the fact that there exist only few subvarieties of positive dimension.

4. FURTHER COMMENTS

This is still not the end. Why not allowing topological changes that are not obtained by bimeromorphic maps? One could ask whether there always exists another complex structure on X (*e.g.* one obtained by a deformation) such that a bimeromorphic model of this new one has the rational homotopy type of a projective manifold. So, more formally, if one introduces the equivalence relation between complex manifolds generated by deformations and bimeromorphic correspondences, one might ask whether any compact Kähler manifold is equivalent to a projective manifold.

Continuing in this direction, one could allow singular varieties or certain ramified covers in order to enlarge the equivalence classes. Would the answer to Kodaira's problem be different then? Most of these questions are open for the time being, but see the comments in [11].

In another direction, it could be interesting to see whether the birational geometry does matter in these questions. The above counter-example for the birational Kodaira problem is, by construction, of Kodaira dimension $-\infty$. For the time being the techniques do not seem to produce examples of non-negative Kodaira dimension.

As has been mentioned, topologically there is no difference between compact Kähler surfaces and projective surfaces. Due to the examples of Voisin, the situation changes drastically in dimension ≥ 4 (or rather ≥ 6 if one prefers simply-connected manifolds). What seems open, however, is the three-dimensional case:

Does there exist a compact Kähler threefold which is not homeomorphic to a projective manifold?

Since we mentioned fundamental groups in the beginning, let us point out that the following problem is also still open:

Does there exist a group that is the fundamental group of a compact Kähler manifold, but not of a projective manifold?

A question of a more general nature is the following:

Are there topological, cohomological, ... conditions that decide whether a compact Kähler manifold can also be endowed with a complex structure which is projective?

Nothing seems to be known in this direction and the examples show that if such conditions can be found at all, they cannot be formulated purely in terms of the fundamental group.

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