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RIGIDITY RESULTS FOR BERNOULLI ACTIONS AND THEIR VON NEUMANN ALGEBRAS [after Sorin Popa]

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1. INTRODUCTION

Suppose that a countable group G acts freely and ergodically on the standard probability space (X, μ) preserving the probability measure μ. We are interested in several types of 'isomorphisms' between such actions. Two actions are said to be

(1) *conjugate* if there exist a group isomorphism and a measure space isomorphism satisfying the obvious conjugacy formula;

(2) *orbit equivalent* if there exists a measure space isomorphism sending orbits to orbits, i.e., the *equivalence relations* given by the orbits are isomorphic;

(3) *von Neumann equivalent* if the crossed product von Neumann algebras are isomorphic.

Note that the *crossed product* construction(1) has been introduced by Murray and von Neumann [41], who called it the *group measure space construction*.

It is clear that conjugacy of two actions implies orbit equivalence. Since the crossed product von Neumann algebra can be defined directly from the equivalence relation given by the orbits, orbit equivalence implies von Neumann equivalence. *Rigidity results* provide the converse implications for certain actions of certain groups. This is a highly non-trivial matter. Dye [16, 17] proved that all free ergodic measure preserving actions of groups with polynomial growth on the standard probability space are orbit equivalent. This result was extended to all *amenable groups* by Ornstein and Weiss [45]. Finally, Connes, Feldman and Weiss [10] showed that every ergodic amenable probability measure preserving countable equivalence relation is generated by a free $\mathbb{Z}$-action and is hence unique. Summarizing, for amenable group actions all information on the group, except its amenability, gets lost in the passage to the equivalence relation.

Concerning the relation between orbit equivalence and von Neumann equivalence, it was noted by Feldman and Moore [19] that the *pair* $L^\infty(X,\mu) \subset L^\infty(X,\mu) \rtimes G$ remembers the equivalence relation. The abelian subalgebra $L^\infty(X,\mu)$ is a so-called *Cartan subalgebra*. So, in order to deduce orbit equivalence from von Neumann equivalence, we need certain uniqueness results for Cartan subalgebras, which is an extremely hard problem. Connes and Jones [12] gave the first examples of non orbit equivalent, yet von Neumann equivalent actions.

In this talk, we discuss Popa's recent breakthrough rigidity results for Bernoulli actions(2) of Kazhdan groups. These results open a new era in von Neumann algebra theory, with striking applications in ergodic theory. The heart of Popa's work is his *deformation/rigidity strategy*: he discovered families of von Neumann algebras with a rigid subalgebra but yet with just enough deformation properties in order for the rigid part to be uniquely determined inside the ambient algebra (up to unitary conjugacy). This leads to far reaching classification results for these families of von Neumann algebras. Popa considered the deformation/rigidity strategy for the first time in [54]. In [52], he used it to deduce orbit equivalence from mere von Neumann equivalence between certain group actions and to give the first examples of II_1 factors with trivial fundamental group, through an application of Gaboriau's ℓ^2 Betti numbers of equivalence relations [22]. Deformation/rigidity arguments are again the crucial ingredient in the papers [48, 55, 56, 53] that we discuss in this talk and they are applied in [29], in the study of amalgamated free products. These ideas may lead to

(1) The crossed product von Neumann algebra $L^\infty(X,\mu) \rtimes G$ contains a copy of $L^\infty(X,\mu)$ and a copy of the group G by unitary elements in the algebra, and the commutation relations between both are given by the action of G on (X,μ).

(2) Every discrete group G acts on $(X,\mu) = \prod_{g\in G}(X_0,\mu_0)$, by shifting the Cartesian product. Here (X_0,μ_0) is the standard non-atomic probability space and the action is called the Bernoulli action of G.

many more applications in von Neumann algebra and ergodic theory (see *e.g.* the new papers [28, 58] written since this talk was given).

In the papers discussed in this talk, the *rigidity* comes from the group side and is given by Kazhdan's property (T) [15, 36] and more generally, by the relative property (T) of Kazhdan-Margulis (see [26] and Valette's Bourbaki seminar [63] for details): the groups dealt with contain an infinite normal subgroup with the relative property (T) and are called *w-rigid groups*. Popa discovered a strong *deformation property* shared by the Bernoulli actions, and called it *malleability*. In a sense, a Bernoulli action can be continuously deformed until it becomes orthogonal to its initial position. In order to exploit the tension between the deformation of the action and the rigidity of the group, yet another technique comes in. Using *bimodules* (Connes' correspondences), Popa developed a very strong method to prove that two subalgebras of a von Neumann algebra are unitarily conjugate. Note that he used this bimodule technique in many different settings, see [29, 46, 55, 56, 52, 51].

The following are the two main results of [48, 55, 56] and are discussed below. The *orbit equivalence superrigidity theorem* states that the equivalence relation given by the orbits of a Bernoulli action of a w-rigid group, entirely remembers the group and the action. The *von Neumann strong rigidity theorem* roughly says that whenever a Bernoulli action is von Neumann equivalent with a free ergodic action of a w-rigid group, the actions are actually conjugate. It is the first theorem in the literature deducing conjugacy of actions out of von Neumann equivalence. The methods and ideas behind these far reaching results are fundamentally *operator algebraic* and yield striking theorems in *ergodic theory*.

Some important conventions

All probability spaces in this talk are standard. All actions of countable groups G on (X, μ) are supposed to preserve the probability measure μ. All statements about elements of (X, μ) only hold almost everywhere. A *w-rigid group* is a countable group that admits an infinite normal subgroup with the relative property (T).

Orbit equivalence superrigidity

In [48], the deformation/rigidity technique leads to the following orbit equivalence superrigidity theorem.

Theorem (Theorem 4.4). — *Let $G \curvearrowright (X, \mu)$ be the Bernoulli action of a w-rigid group G as above. Suppose that G does not have finite normal subgroups. If the restriction to $Y \subset X$ of the equivalence relation given by $G \curvearrowright X$ is given by the orbits of $\Gamma \curvearrowright Y$ for some group Γ acting freely and ergodically on Y, then, up to measure zero, $Y = X$ and the actions of G and Γ are conjugate through a group isomorphism.*

The theorem implies as well that the restriction to a Borel set of measure $0 < \mu(Y) < 1$, of the Bernoulli action of a w-rigid group G without finite normal subgroups, yields an ergodic probability measure preserving countable equivalence relation that cannot be generated by a free action of a group. The first examples of this phenomenon – answering a question of Feldman and Moore – were given by Furman in [21]. Dropping the ergodicity, examples were given before by Adams in [1], who also provides examples in the Borel setting.

Popa proves the orbit equivalence superrigidity for the Bernoulli action of G on X using his even stronger *cocycle superrigidity theorem*: any 1-cocycle for the action $G \curvearrowright X$ with values in a discrete group Γ is cohomologous to a homomorphism of G to Γ. The origin of orbit equivalence rigidity and cocycle rigidity theory lies in Zimmer's pioneering work. Zimmer proved in [66] his celebrated cocycle rigidity theorem and used it to obtain the first orbit equivalence rigidity results (see Section 5.2 in [67]). Since Zimmer's theorem deals with cocycles taking values in linear groups, he obtains orbit equivalence rigidity results where both groups are assumed to be linear (see [68]). Furman developed in [20, 21] a new technique and obtains an orbit equivalence superrigidity theorem with quite general ergodic actions of higher rank lattices on one side and an arbitrary free ergodic action on the other side. Note however that Furman's theorem nevertheless depends on Zimmer's cocycle rigidity theorem. We also mention the orbit equivalence superrigidity theorems obtained by Monod and Shalom [39] for certain actions of direct products of hyperbolic groups. An excellent overview of orbit equivalence rigidity theory can be found in Shalom's survey [61].

Zimmer's cocycle rigidity theorem was a deep generalization of Margulis' seminal superrigidity theory [38]. In particular, the mathematics behind involve the theory of algebraic groups and their lattices. On the other hand, Popa's technique to deal with 1-cocycles for Bernoulli actions is intrinsically operator algebraic.

As stated above, Popa uses his powerful *deformation/rigidity strategy* to prove the cocycle superrigidity theorem. Leaving aside several delicate passages, the argument goes as follows. A 1-cocycle γ for the Bernoulli action $G \curvearrowright X$ of a w-rigid group G, can be interpreted in two ways as a 1-cocycle for the diagonal action $G \curvearrowright X \times X$, either as γ_1, only depending on the first variable, either as γ_2, only depending on the second variable. The malleability of the Bernoulli action (this is the deformation property) yields a continuous path joining γ_1 to γ_2. The relative property (T) implies that, in cohomology, the 1-cocycle remains essentially constant along the continuous path. This yields $\gamma_1 = \gamma_2$ in cohomology and the weak mixing property allows to conclude that γ is cohomologous to a homomorphism.

Let (σ_g) be the Bernoulli action of a w-rigid group G on (X, μ). Popa's cocycle superrigidity theorem covers his previous result [54, 57] identifying the 1-cohomology group $H^1(\sigma)$ with the character group $\operatorname{Char} G$. This result allows to compute as well the 1-cohomology for quotients of Bernoulli actions, yielding the following result of [53].

Theorem (Theorem 5.3). — *Let G be a w-rigid group. Then, G admits a continuous family of non-stably*[3] *orbit equivalent actions.*

Note that Popa does not only prove an existence result, but explicitly exhibits a continuous family of mutually non orbit equivalent actions. The existence of a continuum of non orbit equivalent actions of an infinite property (T) group had been established before in a non-constructive way by Hjorth [27], who exhibits a continuous family of actions such that every action in the family is orbit equivalent to at most countably many other actions of the family.

Finally note that the first concrete computations of 1-cohomology for ergodic group actions are due to Moore [40] and Gefter [23].

Von Neumann strong rigidity

The culmination of Popa's work on Bernoulli actions is the following *von Neumann strong rigidity* theorem of [56]; it is the first theorem in the literature that deduces conjugacy of the actions from isomorphism of the crossed product von Neumann algebras.

Theorem (Theorem 9.1). — *Let G be a group with infinite conjugacy classes and $G \curvearrowright (X, \mu)$ its Bernoulli action as above. Let Γ be a w-rigid group that acts freely and ergodically on (Y, η). If*

$$\theta : L^\infty(Y) \rtimes \Gamma \to p(L^\infty(X) \rtimes G)p$$

is a *-isomorphism for some projection $p \in L^\infty(X) \rtimes G$, then $p = 1$, the groups Γ and G are isomorphic and the actions of Γ and G are conjugate through this isomorphism.

Note that in the conditions of the theorem, there is an assumption on the action on one side and an assumption on the group on the other side. As such, it is not a superrigidity theorem: one would like to obtain the same conclusion for any free ergodic action of any group Γ and for the Bernoulli action of a w-rigid ICC group G.

Another type of von Neumann rigidity has been obtained by Popa in [52, 51], deducing orbit equivalence from von Neumann equivalence. We just state the following particular case. Consider the usual action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{T}^2$. Whenever a free and ergodic action of a group Γ with the Haagerup property is von Neumann equivalent with the $\mathrm{SL}(2, \mathbb{Z})$ action on $\mathbb{T}^2$, it actually is orbit equivalent with the latter. One should not hope to deduce a strong rigidity result yielding conjugacy of the actions: Monod and Shalom ([39], Theorem 2.27) proved that any free ergodic action of the

(3) See Definition 4.2.

free group $\mathbb{F}_n$ is orbit equivalent with free ergodic actions of a continuum of non-isomorphic groups. Note that this also follows from Dye's result [16, 17] if we assume that every generator of $\mathbb{F}_n$ acts ergodically.

II_1 factors and their fundamental group

Let G act freely and ergodically on (X,μ). Freeness and ergodicity imply that the crossed product von Neumann algebra $M := L^\infty(X,\mu) \rtimes G$ is a *factor* (the center of the algebra M is reduced to the scalars) and the invariant probability measure yields a finite trace on M. Altogether, we get that M is a so-called *type II_1 factor*.

Another class of II_1 factors arises as follows: for any countable group G, one considers the von Neumann algebra $\mathcal{L}(G)$ generated by the left translation operators on the Hilbert space $\ell^2(G)$. The algebra $\mathcal{L}(G)$ always admits a finite trace and it is a factor if and only if G has infinite conjugacy classes (ICC).

Let M be a II_1 factor with normalized trace τ. The *fundamental group* of M, introduced by Murray and von Neumann [42], is the subgroup of $\mathbb{R}^*_+$ generated by the numbers $\tau(p)$, where p runs through the projections of M satisfying $M \cong pMp$. Murray and von Neumann showed in [42] that the fundamental group of the hyperfinite[(4)] II_1 factor is $\mathbb{R}^*_+$. They also write that there is no reason to believe that the fundamental group of every II_1 factor is $\mathbb{R}^*_+$. However, only forty years later, this intuition was proved to be correct, in a breakthrough paper of Connes [6]. Connes shows that the fundamental group of $\mathcal{L}(G)$ is at most countable when G is an ICC group with Kazhdan's property (T). This can be considered as the first rigidity type result in the theory of von Neumann algebras. It was later refined by Golodets and Nessonov [24] to obtain II_1 factors with countable fundamental group containing a prescribed countable subgroup of $\mathbb{R}^*_+$. However, until Popa's breakthroughs in [55, 52, 51], no precise computation of a fundamental group different from $\mathbb{R}^*_+$ had been obtained.

Note in passing that Voiculescu proved in [64] that the fundamental group of $\mathcal{L}(\mathbb{F}_\infty)$ contains the positive rationals and that it was shown to be the whole of $\mathbb{R}^*_+$ by Rădulescu in [59]. On the other hand, computation of the fundamental group of $\mathcal{L}(\mathbb{F}_n)$ is equivalent with deciding on the (non)-isomorphism of the free group factors (see [18, 60]), which is a famous open problem in the subject.

Specializing the problem of Murray and von Neumann, Kadison [34] posed the following question: does there exist a II_1 factor M not isomorphic to $\mathrm{M}_2(\mathbb{C}) \otimes M$? This question was answered affirmatively by Popa in [52], who showed that, among other examples, $\mathcal{L}(G)$ has trivial fundamental group when $G = \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$. For a more elementary treatment of this example, see [51]. Note that Popa shows in [52] that the fundamental group of $\mathcal{L}(G) = \mathrm{SL}(2,\mathbb{Z}) \ltimes L^\infty(\mathbb{T}^2)$ equals the fundamental group of the equivalence relation given by the orbits of $\mathrm{SL}(2,\mathbb{Z}) \curvearrowright \mathbb{T}^2$. The latter

(4) The hyperfinite II_1 factor is, up to isomorphism, the unique II_1 factor that contains an increasing sequence of matrix algebras with weakly dense union.

reduces to 1 using Gaboriau's ℓ^2 Betti number invariants for equivalence relations, see [22]. We also refer to the Bourbaki seminar by Connes [9] on this part of Popa's œuvre.

In [55], Popa goes much further and constructs II_1 factors with an arbitrary countable fundamental group!

THEOREM (Theorem 7.1). — *Given a countable subgroup $S \subset \mathbb{R}_+^*$ and a w-rigid ICC group G with $\mathcal{L}(G)$ having trivial fundamental group, there exists an action of G on the hyperfinite II_1-factor $\mathcal{R}$ such that the crossed product $\mathcal{R} \rtimes G$ is a II_1 factor with fundamental group S.*

The example par excellence of a group G satisfying the conditions of the theorem, is $G = \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$. Again, Popa does not establish a mere existence result: the actions considered are the so-called Connes-Størmer Bernoulli actions (see [13] and Section 3 below).

Some comments on proving von Neumann strong rigidity

We explain how an isomorphism of crossed products forces, in certain cases, actions to be conjugate.

In a first step, using the *deformation/rigidity strategy*, Popa [55] shows the following result. Suppose that $G \curvearrowright (X,\mu)$ is the Bernoulli action of an infinite group G and consider the crossed product $L^\infty(X,\mu) \rtimes G$. It is shown (see Theorem 6.3 below) that any subalgebra of $L^\infty(X,\mu) \rtimes G$ with the relative property (T) can essentially be unitarily conjugated into $\mathcal{L}(G)$. Again leaving aside several delicate passages, the argument goes as follows. A subalgebra $Q \subset L^\infty(X,\mu) \rtimes G$ with the relative property (T) is viewed in two ways as a subalgebra of $L^\infty(X \times X, \mu \times \mu) \rtimes G$, where G acts diagonally: Q_1 only living on the first variable of $X \times X$ and Q_2 only living on the second one. The malleability of the Bernoulli action implies that the subalgebras Q_1 and Q_2 are joined by a continuous path of subalgebras Q_t. The relative property (T) then ensures that Q_1 and Q_2 are essentially unitarily conjugate. The mixing of the action is used to deduce that Q can essentially be conjugated into $\mathcal{L}(G)$.

Note in passing that the above result remains true when the 'commutative' Bernoulli action is replaced by a 'non-commutative' Connes-Størmer Bernoulli action, which is the crucial ingredient to produce II_1 factors with prescribed countable fundamental groups.

Given an isomorphism $\theta : L^\infty(Y) \rtimes \Gamma \to L^\infty(X) \rtimes G$, where $G \curvearrowright X$ is the Bernoulli action and the group Γ is w-rigid, the previous paragraph implies that θ sends $\mathcal{L}(\Gamma)$ into $\mathcal{L}(G)$, after conjugating by a unitary in the crossed product. Using very precise analytic arguments, Popa [56] succeeds in proving next that also the Cartan subalgebras $L^\infty(Y)$ and $L^\infty(X)$ can be conjugated into each other with a unitary in the crossed product (see Theorem 8.2 below). Having at hand this orbit equivalence and knowing that the group von Neumann algebras can be conjugated into each other, Popa manages to prove conjugacy of the actions.

An important remark should be made here. The results on Bernoulli actions discussed up to now, use the deformation property called *strong malleability* combined with the mixing property of the action. So, they are valid for all strongly malleable mixing actions. The result on the conjugation of the Cartan subalgebras however, uses a much stronger mixing property of Bernoulli actions, called the *clustering property*, which roughly means that the Bernoulli action allows for a natural *tail*. Note in this respect the following conjecture of Neshveyev and Størmer [43]: suppose that the abelian countable groups G and Γ act freely and weakly mixingly on the standard probability space and that they give rise to isomorphic crossed products where the isomorphism sends $\mathcal{L}(G)$ onto $\mathcal{L}(\Gamma)$; then, the Cartan subalgebras are conjugate with a unitary in the crossed product(5).

Outer conjugacy of actions on the hyperfinite II_1 factor

The deformation/rigidity technique first appeared(6) in Popa's paper [54] on the computation of several invariants for (cocycle) actions of w-rigid groups on the hyperfinite II_1 factor. In fact, many ideas exploited in the papers [48, 55, 56, 53, 57] are already present to some extent in the breakthrough paper [54].

Recall that two actions (σ_g) and (ρ_g) of a group G on a factor are said to be *outer conjugate* if there exists an isomorphism Δ such that the conjugate automorphism $\Delta\sigma_g\Delta^{-1}$ equals ρ_g up to an inner automorphism.

The classification up to outer conjugacy of actions of a group G on, say, the hyperfinite II_1 factor is an important subject. This classification has been completed, first for cyclic groups by Connes [5, 3], for finite groups by Jones [31] and finally, for amenable groups by Ocneanu [44]: any two outer(7) actions of an amenable group G on the hyperfinite II_1 factor are outer conjugate (even cocycle conjugate).

Away from amenable groups, Jones proved in [32] that any non-amenable group admits at least two non outer conjugate actions on the hyperfinite II_1 factor. Apart from actions, one also studies *cocycle actions* of a group G on a factor N: families of automorphisms $(\sigma_g)_{g\in G}$ such that $\sigma_g\sigma_h = \sigma_{gh}$ modulo an inner automorphism $\mathrm{Ad}\, u_{g,h}$, where the unitaries $u_{g,h}$ satisfy a 2-cocycle relation.

(5)It is crucial to have conjugation of the Cartan subalgebras through a unitary in the crossed product, which is the hyperfinite II_1 factor. Indeed, thanks to the work of Connes, Feldman and Weiss [10], two Cartan subalgebras are always conjugate with an automorphism of the hyperfinite II_1 factor. But, there exist continuously many non inner conjugate Cartan subalgebras.

(6)The paper [54] circulated since 2001 as a preprint of the MSRI and is the precursor of the papers [48, 55, 56, 53, 57] discussed above.

(7)An outer action is an action (σ_g) such that for $g \neq e$, σ_g is an outer automorphism, i.e., not of the form $\mathrm{Ad}\, u$ for a unitary u in the von Neumann algebra.

In the previously cited works on amenable group actions, it is shown as well that any cocycle action of an amenable group on the hyperfinite II_1 factor is outer conjugate to a genuine action. Popa generalized this result to arbitrary II_1 factors in [50]. In [11], Connes and Jones constructed, for any infinite property (T) group G, examples of cocycle actions of G on the free group factor $\mathcal{L}(\mathbb{F}_\infty)$ that are non outer conjugate to a genuine action.

This brings us to the topic of [54]. Popa introduces two outer conjugacy invariants for a (cocycle) action on a II_1 factor: the fundamental group and the spectrum. These invariants are computed in [54] for the Connes-Størmer Bernoulli actions, yielding the following theorem.

THEOREM (Theorems 10.3 and 10.6). — *Let G be a w-rigid group. Then G admits a continuous family of non outer conjugate actions on the hyperfinite II_1 factor. Also, G admits a continuous family of cocycle actions on the hyperfinite II_1 factor that are non outer conjugate to a genuine action.*

Further remarks

We discussed in detail how Popa recovers information on a group action from the crossed product algebra $L^\infty(X,\mu) \rtimes G$. On the other hand, to what extent a group von Neumann algebra $\mathcal{L}(G)$ remembers the group G? Very little is known on this problem. Connes' celebrated theorem [4] states that all the II_1 factors $\mathcal{L}(G)$ defined by amenable ICC groups G are isomorphic to the hyperfinite II_1 factor. Indeed, they are all *injective*(8) and Connes shows in [4] the uniqueness of the injective II_1 factor. Cowling and Haagerup [14] have shown that the group von Neumann algebras $\mathcal{L}(\Gamma)$ are non-isomorphic if one takes lattices Γ in $\mathrm{Sp}(1,n)$ for different values of n.

Some group von Neumann algebras $\mathcal{L}(G)$ can we written as well as the crossed product by a free ergodic action (but not all, since Voiculescu [65] showed that the free group factors cannot be written in this way). We have for instance $\mathcal{L}(\mathrm{SL}(n,\mathbb{Z}) \ltimes \mathbb{Z}^n) = L^\infty(\mathbb{T}^n) \rtimes \mathrm{SL}(n,\mathbb{Z})$. Another example consists in writing the Bernoulli action crossed product $L^\infty(X,\mu) \rtimes G$ as $\mathcal{L}(\mathbb{Z} \wr G)$, where the *wreath product* group $\mathbb{Z} \wr G$ is defined as the semidirect product $\mathbb{Z} \wr G := (\bigoplus_{g \in G} \mathbb{Z}) \rtimes G$. Popa's von Neumann strong rigidity theorem then implies the following result. It can be considered as a relative version of Connes' conjecture [7], which states that within the class of ICC property (T) groups, $\mathcal{L}(G_1) \cong \mathcal{L}(G_2)$ if and only if $G_1 \cong G_2$. Popa's result 'embeds injectively' the category of w-rigid ICC groups into the category of II_1 factors.

(8) A factor $M \subset \mathrm{B}(H)$ is called injective if there exists a conditional expectation of $\mathrm{B}(H)$ onto M (which of course need not be weakly continuous). A conditional expectation of a von Neumann M onto a von Neumann subalgebra N is a unital, positive, N-N-bimodule map $E : M \to N$.

Corollary. — *When G and Γ are w-rigid ICC groups, $\mathcal{L}(\mathbb{Z} \wr G) \cong \mathcal{L}(\mathbb{Z} \wr \Gamma)$ if and only if $G \cong \Gamma$. Moreover, $\mathcal{L}(\mathbb{Z} \wr G)$ has trivial fundamental group for any w-rigid ICC group G.*

Popa's von Neumann strong rigidity theorem is in fact more precise than the version stated above. As we shall see in Theorem 9.1 below, the strong rigidity theorem allows as well to compute the group $\operatorname{Out} M$ of outer automorphisms of $M = L^\infty(X,\mu) \rtimes G$, where G is a w-rigid ICC group and $G \curvearrowright (X,\mu)$ its Bernoulli action. Then,

$$\operatorname{Out} M \cong \operatorname{Char} G \rtimes \frac{\operatorname{Aut}^*(X,G)}{G} ,$$

where $\operatorname{Aut}^*(X,G)$ is the group of measure space isomorphisms $\Delta : X \to X$ for which there exists a $\delta \in \operatorname{Aut} G$ such that $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$ almost everywhere. Writing $\Delta_g(x) = g \cdot x$, one embeds $G \hookrightarrow \operatorname{Aut}^*(X,G)$. Note moreover that $\operatorname{Aut}^*(X,G)$ obviously contains another copy of G acting by Bernoulli shifts 'on the other side'.

In [29], Ioana, Peterson and Popa apply the strategy of deformation/rigidity in the completely different context of amalgamated free products, yielding the first examples of II_1 factors with trivial outer automorphism group. Much more is done in [29], where actually a von Neumann version of the Bass-Serre theory is developed.

Acknowledgment

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2. PRELIMINARIES AND CONVENTIONS

Von Neumann algebras, traces, almost periodic states and group actions

Throughout $M, \mathcal{M}, N, \mathcal{N}, A, \mathcal{A}$ denote *von Neumann algebras.* Recall that a von Neumann algebra is a non-commutative generalization of a measure space, the algebras $L^\infty(X,\mu)$ being the abelian examples. By definition, a von Neumann algebra is a weakly closed unital *-subalgebra of $\mathrm{B}(H)$ for some Hilbert space H. Whenever $\mathcal{M} \subset \mathrm{B}(H)$ is a von Neumann algebra, the *commutant* of $\mathcal{M}$ is denoted by $\mathcal{M}'$ and consists of the operators in $\mathrm{B}(H)$ commuting with all the operators in $\mathcal{M}$. Von Neumann's *bicommutant* theorem states that $\mathcal{M}'' = \mathcal{M}$ and this equality characterizes von Neumann algebras among the unital *-subalgebras of $\mathrm{B}(H)$. A *factor* is a von Neumann algebra with trivial center, i.e., $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}1$.

A *state* on a von Neumann algebra is a positive linear map $\mathcal{M} \to \mathbb{C}$ satisfying $\omega(1) = 1$. All states are assumed to be *normal*, i.e., continuous with respect to the ultraweak topology on $\mathcal{M}$ (which is equivalent with requiring weak continuity on the unit ball of $\mathcal{M}$). Hence, normal states are the counterparts of probability measures on (X, μ) absolutely continuous with respect to μ. A state ω is said to be *tracial* if $\omega(xy) = \omega(yx)$ for all x, y. A state is said to be *faithful* if the equality $\omega(x) = 0$ for x positive implies that $x = 0$. States are always assumed to be faithful.

The algebras denoted M, N, A are supposed to admit a *faithful normal trace* and if we specify a state on M, N or A, it is always supposed to be a trace. The terminology *finite von Neumann algebra* (N, τ) means a von Neumann algebra N with a faithful normal trace τ.

An action of a countable group on $(\mathcal{M}, \varphi)$ is understood to be an action by automorphisms *leaving the state* φ *invariant*. We denote by (X, μ) the standard probability space without atoms and an action of a countable group on (X, μ) is supposed to preserve the probability measure μ.

If G acts on $(\mathcal{M}, \varphi)$ by automorphisms (σ_g), $\mathcal{M}^\sigma$ denotes the von Neumann subalgebra of elements $x \in \mathcal{M}$ satisfying $\sigma_g(x) = x$ for all $g \in G$. The action (σ_g) is said to be *ergodic* if $\mathcal{M}^\sigma = \mathbb{C}1$.

If φ is a faithful normal state on $\mathcal{M}$, we consider the *centralizer algebra* $\mathcal{M}^\varphi$ of φ consisting of those $x \in \mathcal{M}$ satisfying $\varphi(xy) = \varphi(yx)$ for all y. More generally, for a real number $\lambda > 0$, a λ-*eigenvector* for φ is an element $x \in \mathcal{M}$ satisfying $\varphi(xy) = \lambda\varphi(yx)$ for all $y \in \mathcal{M}$. We say that φ is *almost periodic* (or that $(\mathcal{M}, \varphi)$ is almost periodic), if the λ-eigenvectors span a weakly dense subalgebra of $\mathcal{M}$ when λ runs through $\mathbb{R}^*_+$. If this is the case, $\mathrm{Sp}(\mathcal{M}, \varphi)$ denotes the point spectrum of φ, i.e., the set of $\lambda > 0$ for which there exists a non-zero λ-eigenvector.

A finite von Neumann algebra (P, τ) is said to be *diffuse* if P does not contain a minimal projection. A finite (P, τ) is diffuse if and only if P contains a sequence of unitaries tending weakly to zero. Equivalently, P does not have a direct summand that is a matrix algebra. For instance, the group von Neumann algebra $\mathcal{L}(G)$ (see page 242 for its definition) is diffuse for any infinite group G.

Crossed products

Whenever a countable group G acts by φ-preserving automorphisms (σ_g) on $(\mathcal{M}, \varphi)$, we denote by $\mathcal{M} \rtimes G$ the crossed product, generated by the elements $a \in \mathcal{M}$ and the unitaries $(u_g)_{g \in G}$ such that $u_g a u_g^* = \sigma_g(a)$ for all $a \in \mathcal{M}$ and $g \in G$. We have a natural conditional expectation (see footnote on page 245) given by $E : \mathcal{M} \rtimes G \to \mathcal{M} : E(au_g) = \delta_{g,e}\, a$ and we extend φ to a faithful normal state on $\mathcal{M} \rtimes G$ by the formula $\varphi \circ E$. If φ is tracial, its extension is tracial.

The crossed product M is a factor (hence, a type II_1 factor) in the following (non-exhaustive) list of examples. If $A \subset M$ is an inclusion of von Neumann algebras, we denote by $M \cap A'$ the *relative commutant* consisting of elements in M commuting with all elements of A.

• Suppose that G acts (essentially) freely on (X, μ) and put $M = L^\infty(X) \rtimes G$. Then, $M \cap L^\infty(X)' = L^\infty(X)$ and M is a factor if and only if the G-action is ergodic.

• Suppose that the ICC group G acts on the finite (N, τ) and put $M = N \rtimes G$. Then, $M \cap \mathcal{L}(G)' = N^G$ and M is a factor if and only if the G-action on the center of N is ergodic.

• Suppose that the group G acts on the II_1 factor (N, τ) such that for all $g \neq e$, σ_g is an outer automorphism of N, i.e., an automorphism that cannot be written as $\mathrm{Ad}\, u$ for some unitary $u \in N$. Putting $M = N \rtimes G$, we have $M \cap N' = \mathbb{C}1$ and in particular, M is a factor.

1-cocycles and 1-cohomology

Let the countable group G act on (X, μ). We denote by $g \cdot x$ the action of an element $g \in G$ on $x \in X$ and we denote by (σ_g) the corresponding action of G on $A = L^\infty(X)$ given by $(\sigma_g(F))(x) = F(g^{-1} \cdot x)$. A 1-*cocycle* for (σ_g) with coefficients in a Polish group K is a measurable map

$$\gamma : G \times X \to K \quad \text{satisfying} \quad \gamma(gh, x) = \gamma(g, h \cdot x)\, \gamma(h, x)$$

almost everywhere. Two 1-cocycles γ_1 and γ_2 are said to be *cohomologous* if there exists a measurable map $w : X \to K$ such that

$$\gamma_1(g, x) = w(g \cdot x)\gamma_2(g, x)w(x)^{-1} \quad \text{almost everywhere.}$$

Whenever K is abelian, the 1-cocycles form a group $Z^1(\sigma, K)$ and quotienting by the 1-cocycles cohomologous to the trivial 1-cocycle, we obtain $H^1(\sigma, K)$. Whenever $K = S^1$, we just write $Z^1(\sigma)$ and $H^1(\sigma)$. Several important remarks should be made. Suppose that the action of G on (X, μ) is free and ergodic.

• Write $M = L^\infty(X) \rtimes G$. The group $Z^1(\sigma)$ embeds in $\mathrm{Aut}(M)$, associating with $\gamma \in Z^1(\sigma)$, the automorphism θ_γ of M defined by $\theta_\gamma(a) = a$ for all $a \in L^\infty(X)$ and $\theta_\gamma(u_g) = u_g \gamma(g, \cdot)$. Passing to quotients, $H^1(\sigma)$ embeds into $\mathrm{Out}(M)$.

• $H^1(\sigma)$ is an invariant for (σ_g) up to stable orbit equivalence (see Definition 4.2).

• If (σ_g) is weakly mixing, the group of characters $\mathrm{Char}\, G$ embeds into $H^1(\sigma)$ as 1-cocycles not depending on the space variable x.

The fundamental group of a II_1 factor

Let M be a II_1 factor. If $t > 0$, we define, up to isomorphism, the *amplification* M^t as follows: choose $n \geq 1$ and a projection $p \in \mathrm{M}_n(\mathbb{C}) \otimes M$ with $(\mathrm{Tr} \otimes \tau)(p) = t$. Define $M^t := p(\mathrm{M}_n(\mathbb{C}) \otimes M)p$. The *fundamental group* of M is defined as

$$\mathcal{F}(M) = \{t > 0 \mid M^t \cong M\} .$$

It can be checked that $\mathcal{F}(M)$ is a subgroup of $\mathbb{R}_+^*$.

In Theorem 9.1, a large class of non-isomorphic II_1 factors with trivial fundamental group is obtained. In Theorem 7.1, II_1 factors with a prescribed countable subgroup of $\mathbb{R}^*_+$ as a fundamental group, are constructed.

Quasi-normalizers and almost normal subgroups

Let $Q \subset M$ be a von Neumann subalgebra of M. An element $x \in M$ is said to *quasi-normalize* Q if there exist $x_1, \dots, x_k$ and $y_1, \dots, y_r$ in M such that

$$xQ \subset \sum_{i=1}^{k} Qx_i \quad \text{and} \quad Qx \subset \sum_{i=1}^{r} y_i Q \; .$$

The elements quasi-normalizing Q form a *-subalgebra of M and their weak closure is called the *quasi-normalizer* of Q in M. The inclusion $Q \subset M$ is said to be *quasi-regular* if M is the quasi-normalizer of Q in M.

A typical example arises as follows: let G be a countable group and H an *almost normal subgroup*, which means that $gHg^{-1} \cap H$ is a finite index subgroup of H for every $g \in G$. Equivalently, this means that for any g in G, HgH is the union of finitely many left cosets, as well as the union of finitely many right cosets. So, it is clear that for every almost normal subgroup $H \subset G$, the inclusion $\mathcal{L}(H) \subset \mathcal{L}(G)$ is quasi-regular.

There are some advantages to work with the quasi-normalizer rather than the normalizer. In Lemma 6.5, the following is shown: let $Q \subset M$ be an inclusion of finite von Neumann algebras and let p be a projection in Q. If P denotes the quasi-normalizer of Q in M, the quasi-normalizer of pQp in pMp is pPp. This is no longer true for the actual normalizer.

More background material is available in the appendices. We discuss in Appendix A the *basic construction* $\langle \mathcal{N}, e_B \rangle$ starting from an inclusion $B \subset \mathcal{N}$ of a von Neumann algebra B in the centralizer algebra of $(\mathcal{N}, \varphi)$ (in particular, for an inclusion of finite von Neumann algebras). Appendix B deals with the relative property (T) and its analogue for inclusions of finite von Neumann algebras. In Appendix C is studied the relation between conjugating von Neumann subalgebras with a unitary and the existence of finite-trace bimodules. Finally, Appendix D is devoted to (weakly) mixing actions.

3. THE MALLEABILITY PROPERTY OF BERNOULLI ACTIONS

Popa discovered several remarkable properties of Bernoulli actions. The first one is a deformation property, that he called strong malleability and that is discussed in this section. This notion of malleability, together with its stunning applications, should be considered as one of the major innovations of Popa.

As is well known, the Bernoulli actions are mixing (see Appendix D for definition and results) and this fact is used throughout. But, Popa exploits as well a very strong mixing property of Bernoulli actions that he called the *clustering property.* This will be used in Section 8.

Definition 3.1 (Popa, [55, 57]). — *The action* (σ_g) *of* G *on* $(\mathcal{N}, \varphi)$ *is said to be*

- malleable *if there exists a continuous action* (α_t) *of* $\mathbb{R}$ *on* $(\mathcal{N} \otimes \mathcal{N}, \varphi \otimes \varphi)$ *that commutes with the diagonal action* $(\sigma_g \otimes \sigma_g)$ *and satisfies* $\alpha_1(a \otimes 1) = 1 \otimes a$ *for all* $a \in \mathcal{N}$*;*
- strongly malleable *if there moreover exists an automorphism* β *of* $(\mathcal{N} \otimes \mathcal{N}, \varphi \otimes \varphi)$ *commuting with* $(\sigma_g \otimes \sigma_g)$ *such that* $\beta\alpha_t = \alpha_{-t}\beta$ *for all* $t \in \mathbb{R}$ *and* $\beta(a \otimes 1) = a \otimes 1$ *for all* $a \in \mathcal{N}$ *and such that* β *has period* 2*:* $\beta^2 = \mathrm{id}$.

Remark 3.2. — In [55, 56], Popa uses the term 'malleability' for a larger class of actions: indeed, instead of extending the action from $\mathcal{N}$ to $\mathcal{N} \otimes \mathcal{N}$, he allows for a more general extension to $\widetilde{\mathcal{N}}$, which can typically be a graded tensor square $\mathcal{N} \widehat{\otimes} \mathcal{N}$. This last example occurs when considering Bogolyubov actions. See remark 10.7 for details.

Generalized Bernoulli actions

The main example of a *strongly malleable action* arises as a (generalized) Bernoulli action. Let G be a countable group that acts on the countable set I. Let (X_0, μ_0) be a probability space. The action of G on $(X, \mu) := \prod_{i \in I}(X_0, \mu_0)$ by shifting the infinite product, is called the (generalized) Bernoulli action. The usual Bernoulli action arises by taking $I = G$ with the action of G by translation.

Convention 3.3. — For simplicity, we only deal with Bernoulli actions on the infinite product of *non-atomic* probability spaces and we refer to them as *Bernoulli actions with non-atomic base.* Most of Popa's results also hold for Bernoulli actions on products of atomic spaces. They are no longer malleable but *sub-malleable*, see Definition 4.2 in [55] and Remark 4.6.

Write $A_0 = L^\infty(\mathbb{R}/\mathbb{Z})$. To check that the generalized Bernoulli action is strongly malleable, it suffices to produce an action (α_t) of $\mathbb{R}$ on $A_0 \otimes A_0$ and a period 2 automorphism β of $A_0 \otimes A_0$ such that $\alpha_1(a \otimes 1) = 1 \otimes a$, $\beta(a \otimes 1) = a \otimes 1$ for all $a \in A_0$ and $\beta\alpha_t = \alpha_{-t}\beta$ for all $t \in \mathbb{R}$. One can then take the infinite product of these (α_t) and β. Take the uniquely determined map $f : \mathbb{R}/\mathbb{Z} \to \left]-\frac{1}{2}, \frac{1}{2}\right]$ satisfying $x = f(x) \mod \mathbb{Z}$ for all x. Define the measure preserving flow α_t and the measure preserving transformation β on $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ by the formulae

$$\alpha_t(x, y) = (x + tf(y - x), y + tf(y - x)) \quad \text{and} \quad \beta(x, y) = (x, 2x - y) \, .$$

For $F \in L^\infty(\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z})$, write $\alpha_t(F) = F \circ \alpha_t$ and $\beta(F) = F \circ \beta$.

Popa gives a more functional analytic argument for the strong malleability of the generalized Bernoulli action. Consider $A_0 \otimes A_0$ as being generated by two independent Haar unitaries u and v. We have to construct a one-parameter group (α_t) and a period 2 automorphism β such that $\alpha_1(u) = v$, $\beta(u) = u$ and $\beta\alpha_t = \alpha_{-t}\beta$. Conjugating α_t and β with the automorphism σ determined by $\sigma(u) = u, \sigma(v) = vu$ (note that u and vu are independent generating Haar unitaries), the first requirement changes to $\alpha_1(u) = vu$ and the other requirements remain. Taking $\log : \mathbb{T} \to]-\pi, \pi]$, we can now set $\alpha_t(u) = \exp(t \log v)u$, $\alpha_t(v) = v$ and $\beta(u) = u$, $\beta(v) = v^*$.

Connes-Størmer Bernoulli actions

Apart from 'classical' Bernoulli actions, also the 'non-commutative' Bernoulli actions of Connes and Størmer [13] satisfy Popa's malleability condition. These Connes-Størmer Bernoulli actions provide the main non-commutative examples of *malleable* actions.

Let G be a countable group acting on a countable set I. Let φ_0 be a faithful normal state on $\mathrm{B}(H)$ for some Hilbert space H (finite or infinite-dimensional). Define

$$(\mathcal{N}, \varphi) := \bigotimes_{i \in I} (\mathrm{B}(H), \varphi_0) \ .$$

On $(\mathcal{N}, \varphi)$, G acts by shifting the tensor factors. To prove the malleability, one has to produce an action (α_t) of $\mathbb{R}$ on $(\mathrm{B}(H \otimes H), \varphi_0 \otimes \varphi_0)$ satisfying $\alpha_1(a \otimes 1) = 1 \otimes a$ for all $a \in \mathrm{B}(H)$. Denoting by $P \in \mathrm{B}(H \otimes H)$ the orthogonal projection on the symmetric subspace densely spanned by the vectors $\xi \otimes \mu + \mu \otimes \xi$ for $\xi, \mu \in H$, we define $U_t = P + e^{i\pi t}(1 - P)$ and $\alpha_t = \operatorname{Ad} U_t$. Note that Connes-Størmer Bernoulli actions are not in an obvious way *strongly* malleable. In some cases however, a generalization of strong malleability holds, see 10.7.

The state φ_0 is of the form Tr_Δ for some positive trace-class operator Δ. So, φ is almost periodic and $\mathrm{Sp}(\mathcal{N}, \varphi)$ is the subgroup of $\mathbb{R}^*_+$ generated by the ratios t/s, where t, s belong to the point spectrum of Δ.

4. SUPERRIGIDITY FOR BERNOULLI ACTIONS

In this section, Popa's very strong rigidity results for Bernoulli actions of w-rigid groups are proved: according to the philosophy in the beginning of the introduction, an orbit equivalence rigidity result deduces conjugacy of actions out of their mere orbit equivalence. All these rigidity results follow from the following cocycle superrigidity theorem.

Theorem 4.1 (Popa, [48]). — *Let G be a countable group with infinite normal subgroup H such that (G,H) has the relative property (T). Let G act strongly malleably on (X,μ) and suppose that its restriction to H is weakly mixing. Then, any* 1*-cocycle*

$$\gamma : G \times X \to K$$

with values in a closed subgroup K of the unitary group $\mathcal{U}(B)$ of a finite von Neumann algebra (B,τ), is cohomologous to a homomorphism $\theta : G \to K$.

By regarding $\Gamma \subset \mathcal{U}(\mathcal{L}(\Gamma))$, the theorem covers all 1-cocycles with values in countable groups, which is the crucial ingredient to prove orbit equivalence rigidity results.

The superrigidity theorem for Bernoulli actions proved below, does not only deal with orbit equivalence, but also with *stable orbit equivalence*. There are several ways to introduce this concept, one of them being the following (see *e.g.* [21], where the terminology of weak orbit equivalence is used).

Definition 4.2. — *Let $G \curvearrowright (X,\mu)$ and $\Gamma \curvearrowright (Y,\eta)$ be free and ergodic actions. A* stable orbit equivalence *between these actions is given by a measure space isomorphism $\pi : A \to B$ between non-negligible subsets $A \subset X$ and $B \subset Y$ preserving the restricted equivalence relations: $\pi(A \cap (G \cdot x)) = B \cap (\Gamma \cdot \pi(x))$ for almost all $x \in A$.*

The compression constant *of π is defined as $c(\pi) := \eta(B)/\mu(A)$.*

The maps $\pi_i : A_i \to B_i$ $(i = 1,2)$ define the same stable orbit equivalence *if*

$$\pi_2(A_2 \cap (G \cdot x)) \subset \Gamma \cdot \pi_1(x) \quad \text{for almost all } x \in A_1 \; .$$

Note that this implies that $c(\pi_1) = c(\pi_2)$.

Suppose that $\pi_i : A_i \to B_i$ $(i = 1,2)$ define the same stable orbit equivalence. If, say, $\mu(A_1) \le \mu(A_2)$, there exist ϕ in the full group(9) of the equivalence relation given by the G-orbits and ψ in the full group of the equivalence relation given by the Γ-orbits such that $\phi(A_1) \subset A_2$ and π_1 is the restriction of $\psi \circ \pi_2 \circ \phi$ to A_1.

If $\pi : A \to B$ defines a stable orbit equivalence between the free and ergodic actions $G \curvearrowright (X,\mu)$ and $\Gamma \curvearrowright (Y,\eta)$, one defines as follows a 1-cocycle $\alpha : G \times X \to \Gamma$ for $G \curvearrowright X$ with values in Γ. By ergodicity, we can choose a measurable map $\mathrm{pr}_A : X \to A$ satisfying $\mathrm{pr}_A(x) \in G \cdot x$ almost everywhere and denote $p = \pi \circ \mathrm{pr}_A$. Freeness of the action $\Gamma \curvearrowright Y$, allows to define

$$\alpha : G \times X \to \Gamma : p(g \cdot x) = \alpha(g,x) \cdot p(x)$$

almost everywhere. Taking another π defining the same stable orbit equivalence or choosing another pr_A, yields a cohomologous 1-cocycle.

Given a free and ergodic action $G \curvearrowright (X,\mu)$, there are certain actions that are trivially stably orbit equivalent to $G \curvearrowright X$ and we introduce them in Notation 4.3.

(9)The full group of the equivalence relation defined by G-orbits, consists of the measure space automorphisms $\Delta : X \to X$ satisfying $\Delta(x) \in G \cdot x$ for almost all x.

The superrigidity theorem 4.4 states that for Bernoulli actions of w-rigid groups these are *the only actions* that are stably orbit equivalent to the given Bernoulli action.

Notation 4.3. — Let G act freely and ergodically on (X,μ). Suppose that $\theta : G \to \Gamma$ is a homomorphism with $\operatorname{Ker}\theta$ finite and $\operatorname{Im}\theta$ of finite index in Γ. Define

$$\operatorname{Ind}_G^\Gamma(X,\theta) := G\backslash(X \times \Gamma) \quad \text{where } G \text{ acts on } X \times \Gamma \text{ by} \quad g \cdot (x,s) = (g \cdot x, \theta(g)s)\ .$$

The action of Γ on $\operatorname{Ind}_G^\Gamma(X,\theta)$ given by $t \cdot (x,s) = (x, st^{-1})$ is free, ergodic and finite measure preserving. We also have a canonical stable orbit equivalence between $G \curvearrowright X$ and $\Gamma \curvearrowright \operatorname{Ind}_G^\Gamma(X,\theta)$, with compression constant $[\Gamma : \theta(G)]/|\operatorname{Ker}\theta|$.

THEOREM 4.4 (Popa, [48]). — *Let G be a countable group with infinite normal subgroup H such that (G,H) has the relative property (T). Let G act strongly malleably on (X,μ) and suppose that its restriction to H is weakly mixing.*

Whenever Γ is a countable group acting freely and ergodically on (Y,η) and whenever π defines a stable orbit equivalence between $G \curvearrowright X$ and $\Gamma \curvearrowright Y$, there exist

- *a homomorphism $\theta : G \to \Gamma$ with $\operatorname{Ker}\theta$ finite in G and $\operatorname{Im}\theta$ of finite index in Γ ;*
- *a measure space isomorphism $\Delta : Y \to \operatorname{Ind}_G^\Gamma(X,\theta)$ conjugating the actions $\Gamma \curvearrowright Y$ and $\Gamma \curvearrowright \operatorname{Ind}_G^\Gamma(X,\theta)$,*

such that $\Delta \circ \pi$ defines the canonical stable orbit equivalence between $G \curvearrowright X$ and $\Gamma \curvearrowright \operatorname{Ind}_G^\Gamma(X,\theta)$. In particular, the compression constant $c(\pi)$ equals $[\Gamma : \theta(G)]/|\operatorname{Ker}\theta|$.

Remark 4.5. — Several special instances of Theorem 4.4 should be mentioned. Suppose that the action $G \curvearrowright X$ satisfies the conditions of Theorem 4.4 and denote by $\mathcal{R}$ the equivalence relation given by the G-orbits.

- If we suppose moreover that G *does not have finite normal subgroups*, we get the following result stated in the introduction. If the restriction to $Y \subset X$ of the equivalence relation given by $G \curvearrowright X$ is given by the orbits of $\Gamma \curvearrowright Y$ for some group Γ acting freely and ergodically on Y, then, up to measure zero, $Y = X$ and the actions of G and Γ are conjugate through a group isomorphism.
- The amplified equivalence relation[10] $\mathcal{R}^t$ can be generated by a free action of a group if and only if $t = n/|G_0|$, where $n \in \mathbb{N} \setminus \{0\}$ and G_0 is a finite normal subgroup of G. So, we get many examples of type II_1 equivalence relations that *cannot be generated by a free action of a group.* The first such examples were given by Furman [21], answering a long standing question of Feldman and Moore.

[10] The amplified equivalence relation $\mathcal{R}^t$ is defined as follows. If $t \leq 1$, we restrict $\mathcal{R}$ to a subset of measure t. If $t > 1$, we take a restriction of the obvious type II_1 equivalence relation on $X \times \{1,\dots,n\}$.

• The *fundamental group* of $\mathcal{R}$ is trivial. Note that this fundamental group is defined as the group of $t > 0$ such that t is the compression constant for some stable orbit equivalence between $G \curvearrowright X$ and itself. If $\pi : A \to B$ is a stable orbit equivalence with compression constant $t \geq 1$, Theorem 4.4 implies that $t = n/|\operatorname{Ker}\theta|$, where $\theta : G \to G$ has finite kernel, satisfies $n = [G : \theta(G)]$ and where $G \curvearrowright X$ is conjugate to $G \curvearrowright \operatorname{Ind}_G^G(X, \theta)$. Since the action $G \curvearrowright X$ is weakly mixing, the induction is trivial, i.e., $n = 1$. This implies that $t \leq 1$ and hence, $t = 1$.

• The *outer automorphism group* $\operatorname{Out}\mathcal{R} = \operatorname{Aut}\mathcal{R}/\operatorname{Inn}\mathcal{R}$ of $\mathcal{R}$ can be described as follows. Recall first that $\operatorname{Aut}\mathcal{R}$ is defined as the group of orbit equivalences $\Delta : X \to X$ of $G \curvearrowright X$ with itself. The full group (see note on page 252) of $\mathcal{R}$ is a normal subgroup of $\operatorname{Aut}\mathcal{R}$ and denoted by $\operatorname{Inn}\mathcal{R}$. The subgroup $\operatorname{Aut}^*(X, G) \subset \operatorname{Aut}\mathcal{R}$ consists of those Δ satisfying

$$\Delta(g \cdot x) = \delta(g) \cdot \Delta(x) \text{ almost everywhere,}$$

for some group automorphism $\delta \in \operatorname{Aut} G$. For our given $\mathcal{R}$, $\operatorname{Out}\mathcal{R}$ is the image of $\operatorname{Aut}^*(X, G)$ through the quotient map $\operatorname{Aut}\mathcal{R} \to \operatorname{Out}\mathcal{R}$. Weak mixing then implies that $\operatorname{Out}\mathcal{R} \cong \operatorname{Aut}^*(X, G)/G$.

Remark 4.6. — Let G be a group with infinite normal subgroup H with the relative property (T). Let $G \curvearrowright (X, \mu)$ be a strongly malleable action whose restriction to H is weakly mixing. Then, the conclusions of Theorems 4.1 and 4.4 hold as well for all *quotient actions* $G \curvearrowright (Y, \eta)$ provided that the quotient map $X \to Y$ satisfies a *relative weak mixing* property, introduced by Popa in [48] (Definition 2.9). Indeed, if for a measurable map $w : X \to K$ and a homomorphism $\theta : G \to K$, the 1-cocycle $G \times X \to K : (g, x) \mapsto w(g \cdot x)\theta(g)w(x)^{-1}$ actually is a map $G \times Y \to K$, then relative weak mixing imposes that w is already a map $Y \to K$.

Hence, the conclusions of Theorems 4.1 and 4.4 hold for all generalized Bernoulli actions that are free and weakly mixing restricted to H, even starting from an atomic base space.

In fact, Theorem 4.4 follows from the cocycle superrigidity theorem 4.1 and the following classical lemma.

LEMMA 4.7. — *Let $G \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \eta)$ be free ergodic actions that are stably orbit equivalent. If the associated* 1*-cocycle is cohomologous to a homomorphism $\theta : G \to \Gamma$, then the conclusion of Theorem 4.4 holds.*

Proof. — The proof of the lemma consists of two easy translation statements. In the first paragraph, stable orbit equivalence is translated as *measure equivalence* (see *e.g.* [21]): we get a natural space with an infinite measure preserving action of $G \times \Gamma$. In a second paragraph, the conclusion follows using the triviality of the cocycle.

Let $p : X \to Y$ be the equivalence relation preserving map as in the construction of the 1-cocycle α above. Take symmetrically $q : Y \to X$ and the 1-cocycle $\beta : \Gamma \times Y \to G$. We denote by $g \cdot x$ the action of G on X and by $s * y$ the action of Γ on Y. Define commuting actions of G and Γ on $X \times \Gamma$ and $Y \times G$ respectively, by the formulae

$$g \cdot (x, s) \cdot t = (g \cdot x, \alpha(g, x)st) \quad , \qquad s * (y, g) * h = (s * y, \beta(s, y)gh) \; .$$

Following Theorem 3.3 in [21], we prove that there is a natural $G \times \Gamma$-equivariant measure space isomorphism $\Theta : X \times \Gamma \to Y \times G$ satisfying $\Theta(x, s) \in (\Gamma * p(x)) \times G$ for almost all (x, s). Indeed, take measurable maps $X \to G : x \mapsto g_x$ and $Y \to \Gamma : y \mapsto s_y$ such that $q(p(x)) = g_x \cdot x$ and $p(q(y)) = s_y * y$ almost everywhere. Define

$$\Theta : X \times \Gamma \to Y \times G : \Theta(x, s) = (s^{-1} * p(x), \beta(s^{-1}, p(x))g_x)$$
$$\Theta^{-1} : Y \times G \to X \times \Gamma : \Theta^{-1}(y, g) = (g^{-1} * q(y), \alpha(g^{-1}, q(y))s_y) \; .$$

The assumption of the lemma yields a homomorphism $\theta : G \to \Gamma$ and a measurable map $w : X \to \Gamma$ such that $\alpha(g, x) = w(g \cdot x)\theta(g)w(x)^{-1}$ almost everywhere. So, the map $\Psi : X \times \Gamma \to X \times \Gamma : \Psi(x, s) = (x, w(x)s)$ is a measure space isomorphism that is equivariant in the following sense

$$\Psi(g \cdot x, \theta(g)st) = g \cdot \Psi(x, s) \cdot t \; .$$

So, $\Theta \circ \Psi$ conjugates the new commuting actions $g(x, s)t = (g \cdot x, \theta(g)st)$ on $X \times \Gamma$ with the commuting actions on $Y \times G$ given above. In particular, the new action of G on $X \times \Gamma$ has a fundamental domain of finite measure. Having a fundamental domain forces $\operatorname{Ker} \theta$ to be finite, while its being of finite measure imposes $\theta(G)$ to be of finite index in G. Finally, the new action of Γ on the quotient $G \backslash (X \times \Gamma)$ is exactly $\Gamma \curvearrowright \operatorname{Ind}_G^\Gamma(X, \theta)$ and $\Theta \circ \Psi$ induces a conjugacy of the actions $\Gamma \curvearrowright \operatorname{Ind}_G^\Gamma(X, \theta)$ and $\Gamma \curvearrowright Y$. ☐

There is a slightly more general way of writing 'obviously' stably orbit equivalent actions, by first restricting $G \curvearrowright X$ to $G_0 \curvearrowright X_0$, where G_0 is a finite index subgroup of G and $G \curvearrowright X$ is induced from $G_0 \curvearrowright X_0$. Since the superrigid actions in this talk are all weakly mixing, they are not induced in this way.

It remains to prove the cocycle superrigidity theorem 4.1. This proof occupies the rest of the section and consists of several steps.

(0) Using the *weak mixing* property and the fact that $\mathcal{U}(B)$ is a *Polish group with a bi-invariant metric*, restrict to the case $K = \mathcal{U}(B)$.

The 1-cocycle $\gamma : G \times X \to \mathcal{U}(B)$ is then interpreted as a family of unitaries $\gamma_g \in \mathcal{U}(A \otimes B)$, where $A = L^\infty(X, \mu)$. Moreover, strong malleability yields (α_t) and β on $A \otimes A$.

(1) Using the *relative property (T)*, find $t_0 > 0$ and a non-zero partial isometry $a \in A \otimes A \otimes B$ satisfying

$$(*) \qquad (\gamma_g)_{13}(\sigma_g \otimes \sigma_g \otimes \mathrm{id})(a) = a(\alpha_{t_0} \otimes \mathrm{id})((\gamma_g)_{13})$$

for all $g \in H$. We use the notation $(a \otimes b)_{13} := a \otimes 1 \otimes b$ and extend to u_{13} for all $u \in A \otimes B$ by linearity and continuity.

(2) Using the *period* 2 *automorphism given by the strong malleability* and the *weak mixing property of the action restricted to* H, glue together partial isometries, in order to get $(*)$ with $t_0 = 1$, i.e., a non-zero partial isometry $a \in A \otimes A \otimes B$ satisfying

$$(\gamma_g)_{13}(\sigma_g \otimes \sigma_g \otimes \mathrm{id})(a) = a(\gamma_g)_{23}$$

for all $g \in H$.

(3) Deduce from the previous equality, using the *intertwining-by-bimodules technique*, a non-zero partial isometry $v \in A \otimes B$ and partial isometries $\theta(g) \in B$ such that

$$\gamma_g(\sigma_g \otimes \mathrm{id})(v) = v(1 \otimes \theta(g))$$

for all $g \in H$.

(4) Using a *maximality argument*, glue together such partial isometries v in order to get a unitary v satisfying the same formula.

(5) Use the *normality* of H in G and the weak mixing property of the action restricted to H, to extend the formula to $g \in G$.

Lemma 4.8 covers step (0), Lemma 4.9 covers steps (1), (2) and (3), Lemma 4.10 covers step (4) and the final step (5) is done in the proof of the theorem.

To prove step (0) of the program, the essential property of the Polish group $\mathcal{U}(B)$ that we retain is the existence of a *bi-invariant metric* $d(u, v) = \|u - v\|_2$.

Lemma 4.8. — *Let G act weakly mixingly on (X, μ). Let $\mathcal{G}$ be a Polish group with a bi-invariant complete metric d and let $K \subset \mathcal{G}$ be a closed subgroup. Suppose that $\gamma : G \times X \to K$ is a 1-cocycle. Let $v : X \to \mathcal{G}$ be a measurable map and $\theta : G \to \mathcal{G}$ a homomorphism such that*

$$\gamma(g, x) = v(g \cdot x)\theta(g)v(x)^{-1}$$

almost everywhere. Whenever $v_0 \in \mathcal{G}$ is an essential value of the function v, we have $v(x)v_0^{-1} \in K$ almost everywhere and $v_0\theta(g)v_0^{-1} \in K$ for all $g \in G$.

Proof. — Let v_0 be an essential value of the function v. Changing $v(x)$ into $v(x)v_0^{-1}$ and θ into $(\mathrm{Ad}\, v_0) \circ \theta$, we assume that e is an essential value of v and prove that $\theta(g) \in K$ for all $g \in G$ and $v(x) \in K$ almost everywhere.

Denote by d the bi-invariant metric on the $\mathcal{G}$. Choose $\varepsilon > 0$ and $g \in G$. Take $W \subset X$ with $\mu(W) > 0$ such that $d(v(x), 1) < \varepsilon/4$ for all $x \in W$. Take $k \in G$ such that $\mu(k \cdot W \cap W) > 0$ and $\mu((gk)^{-1} \cdot W \cap W) > 0$. If $x \in k \cdot W \cap W$, we have $d(v(x), 1), d(v(k^{-1} \cdot x), 1) < \varepsilon/4$. It follows that $d(\theta(k^{-1}), K) < \varepsilon/2$. In the same way,

$d(\theta(gk), K) < \varepsilon/2$. Together, $d(\theta(g), K) < \varepsilon$. This holds for all $\varepsilon > 0$ and all $g \in G$ and hence, $\theta(G) \subset K$.

Let $\varepsilon > 0$. The formula $v(g \cdot x) = \gamma(g, x)v(x)\theta(g)^*$ almost everywhere, yields that $\{x \in X \mid d(v(x), K) < \varepsilon\}$ is non-negligible and G-invariant, hence, the whole of X. It follows that $v(x) \in K$ almost everywhere. □

We fix the following data and notations.

- Let G be a countable group with infinite normal subgroup H such that (G, H) has the relative property (T). Let G act strongly malleably on (X, μ) and suppose that its restriction to H is weakly mixing. Write $A = L^\infty(X)$ and write the action of G on A as $(\sigma_g(F))(x) = F(g^{-1} \cdot x)$.
- Let $\gamma : G \times X \to \mathcal{U}(B)$ be a 1-cocycle with values in the unitary group of the *II_1 factor* (B, τ). Remark that we can indeed suppose that B is a II_1 factor[(11)]. We write $\gamma_g \in \mathcal{U}(A \otimes B)$, given by $\gamma_g(x) = \gamma(g, g^{-1} \cdot x)$. The 1-cocycle relation becomes
$$\gamma_g(\sigma_g \otimes \mathrm{id})(\gamma_h) = \gamma_{gh} \quad \text{for all } \ g, h \in G \ .$$
- We denote by (ρ_g) the following action of G by automorphisms of $A \otimes B$:
$$\rho_g(a) = \gamma_g(\sigma_g \otimes \mathrm{id})(a)\gamma_g^* \quad \text{for all } \ a \in A \otimes B \ .$$
- We denote by (η_g) the unitary representation of G on $L^2(X) \otimes L^2(B)$ given by
$$\eta_g(a) = \gamma_g(\sigma_g \otimes \mathrm{id})(a) \quad \text{for all } \ a \in A \otimes B \subset L^2(X) \otimes L^2(B) \ .$$
- We denote, for every $t \in \mathbb{R}$, by (π_g^t) the unitary representation on $L^2(X \times X) \otimes L^2(B)$ of G given by
$$\pi_g^t(a) = (\gamma_g)_{13}(\sigma_g \otimes \sigma_g \otimes \mathrm{id})(a)(\alpha_t \otimes \mathrm{id})((\gamma_g)_{13}^*)$$
for all $a \in A \otimes A \otimes B \subset L^2(X \times X) \otimes L^2(B)$. Recall the notation u_{13} determined by $(a \otimes b)_{13} = a \otimes 1 \otimes b$.

We cover steps (1), (2) and (3) of the program in the next lemma.

Lemma 4.9. — *Let $q \in A \otimes B$ be a non-zero projection which is $\rho|_H$-invariant. There exist a non-zero partial isometry $v \in A \otimes B$, a projection $p \in B$ and a homomorphism $\theta : H \to \mathcal{U}(pBp)$ such that $vv^* \leq q$, $v^*v = 1 \otimes p$ and*
$$\gamma_h(\sigma_h \otimes \mathrm{id})(v) = v(1 \otimes \theta(h))$$
for all $h \in H$.

(11) Any finite (B, τ) can be embedded, in a trace-preserving way, into a II_1 factor, *e.g.* into $\left(\bigotimes_{n \in \mathbb{Z}}(B, \tau)\right) \rtimes \mathbb{Z}$ and $\mathcal{U}(B)$ is then a closed subgroup of the unitary group of this II_1 factor.

Proof

Step (1). Note that 1 is a π_G^0-invariant vector. The relative property (T) yields a $t_0 = 2^{-n}$ and a non-zero element $a \in A \otimes A \otimes B$ such that a is $\pi_H^{t_0}$-invariant and such that $\|a - 1\|_2 \leq \|q\|_2/2$. It follows that $a(\alpha_{t_0} \otimes \mathrm{id})(q_{13}) \neq 0$, which remains $\pi_H^{t_0}$-invariant. Taking the polar decomposition of $a(\alpha_{t_0} \otimes \mathrm{id})(q_{13})$, we get a non-zero partial isometry $a \in A \otimes A \otimes B$ which is $\pi_H^{t_0}$-invariant and satisfies $a^*a \leq (\alpha_{t_0} \otimes \mathrm{id})(q_{13})$. Moreover, Proposition D.2 yields

$$aa^* \ , \ (\alpha_{-t_0} \otimes \mathrm{id})(a^*a) \in (A \otimes B)_{13}^{\rho|_H} \ .$$

So, we have a projection $\widetilde{q} \in (A \otimes B)^{\rho|_H}$ such that $\widetilde{q} \leq q$ and

$$a^*a = (\alpha_{t_0} \otimes \mathrm{id})(\widetilde{q}_{13}) \ .$$

Step (2). Whenever a and b are $\pi_H^{t_0}$-invariant, we have that $a(\alpha_{t_0} \otimes \mathrm{id})(b)$ is $\pi_H^{2t_0}$-invariant and that $(\beta \otimes \mathrm{id})(a)$ and $(\alpha_{-t_0} \otimes \mathrm{id})(a^*)$ are $\pi_H^{-t_0}$-invariant. So, if we define

$$a_1 = (\alpha_{t_0} \otimes \mathrm{id})\big((\beta \otimes \mathrm{id})(a^*)a\big)$$

we get that a_1 is $\pi_H^{2t_0}$-invariant and satisfies

$$a_1 a_1^* = \widetilde{q}_{13} \quad \text{and} \quad a_1^* a_1 = (\alpha_{2t_0} \otimes \mathrm{id})(\widetilde{q}_{13}) \ .$$

Iterating the procedure yields at stage n a partial isometry $b \in A \otimes A \otimes B$ which is π_H^1-invariant and satisfies $bb^* = \widetilde{q}_{13}$ and $b^*b = \widetilde{q}_{23}$.

Step (3) Define the (non-zero) operator $T \in \mathrm{B}(L^2(X)) \otimes B$ by

$$(T\xi)(x) = \int_X b(x,y)\xi(y) \, d\mu(y) \quad \text{for all } \ \xi \in L^2(X) \otimes B \ .$$

We get

$$[T, \eta_h] = 0 \ \text{ for } h \in H \ , \quad \widetilde{q}T = T = T\widetilde{q} \ , \quad \|(\mathrm{Tr} \otimes \mathrm{id})(T^*T)\| < \infty \ .$$

Taking a spectral projection P of T, we get a non-zero orthogonal projection P with the same properties as T. It follows that the range of P is a finitely generated right B-submodule of $(L^2(X) \otimes L^2(B))_B$ which is stable under $(\eta_h)_{h \in H}$.

As in Proposition C.1, we get $n \geq 1$, a non-zero projection $p \in \mathrm{M}_n(\mathbb{C}) \otimes B$, a non-zero partial isometry $v \in A \otimes \mathrm{M}_{1,n}(\mathbb{C}) \otimes B$ and a homomorphism $\theta : H \to \mathcal{U}(p(\mathrm{M}_n(\mathbb{C}) \otimes B)p)$ such that

$$\gamma_h(\sigma_h \otimes \mathrm{id})(v) = v(1 \otimes \theta(h)) \ \text{ for } h \in H \ , \quad \widetilde{q}v = v \ , \quad v(1 \otimes p) = v \ .$$

Since v^*v is $(\sigma_h \otimes \mathrm{Ad}\,\theta(h))$-invariant for all $h \in H$, it follows from Proposition D.2 that $v^*v = 1 \otimes p_0$ for some non-zero projection $p_0 \in p(\mathrm{M}_n(\mathbb{C}) \otimes B)p \cap \theta(H)'$. Since p_0 commutes with $\theta(H)$, we can cut down by p_0. Since moreover $\tau(p_0) \leq 1$, we can move p_0 into the upper corner of $\mathrm{M}_n(\mathbb{C}) \otimes B$ and we have found a non-zero partial isometry $v \in A \otimes B$, a non-zero projection $p \in B$ and a homomorphism $\theta : H \to \mathcal{U}(pBp)$ such that $vv^* \leq q$, $v^*v = 1 \otimes p$ and

$$\gamma_h(\sigma_h \otimes \mathrm{id})(v) = v(1 \otimes \theta(h))$$

for all $h \in H$. □

We cover step (4) of the program in the following lemma.

Lemma 4.10. — *There exists a unitary element* $v \in A \otimes B$ *and a homomorphism* $\theta : H \to \mathcal{U}(B)$ *such that*

$$\gamma_h(\sigma_h \otimes \mathrm{id})(v) = v(1 \otimes \theta(h))$$

for all $h \in H$.

Proof. — The proof is a straightforward maximality argument. Consider the set $\mathcal{I}$ of partial isometries $v \in A \otimes B$ for which there exist $p \in B$ and $\theta : H \to \mathcal{U}(pBp)$ satisfying

$$v^*v = 1 \otimes p \quad \text{and} \quad \gamma_h(\sigma_h \otimes \mathrm{id})(v) = v(1 \otimes \theta(h))$$

for all $h \in H$. Partially order $\mathcal{I}$ by extension of partial isometries and let v be a maximal element of $\mathcal{I}$. Write $v^*v = 1 \otimes p$. If $vv^* \neq 1$, put $q = 1 - vv^*$. Then, $q \in (A \otimes B)^{\rho|_H}$ and Lemma 4.9 yields a non-zero partial isometry $w \in A \otimes B$, a projection $e \in B$ and a homomorphism $\theta : H \to \mathcal{U}(eBe)$ such that $ww^* \leq q$, $w^*w = 1 \otimes e$ and

$$\gamma_h(\sigma_h \otimes \mathrm{id})(w) = w(1 \otimes \theta(h))$$

for all $h \in H$. Since $e \preceq 1 - p$ in the II_1 factor B, we contradict the maximality v. ☐

Proof of Theorem 4.1. — Using Lemma 4.8, it is sufficient to prove the existence of a unitary $v \in A \otimes B$ and a homomorphism $\theta : G \to \mathcal{U}(B)$ such that

$$\gamma_g(\sigma_g \otimes \mathrm{id})(v) = v(1 \otimes \theta(g)) \tag{1}$$

for all $g \in G$. Take v and θ as given by Lemma 4.10. Fix $g \in G$ and write

$$\widetilde{v} = \gamma_g(\sigma_g \otimes \mathrm{id})(v) \quad \text{and} \quad \widetilde{\theta}(h) = \theta(g^{-1}hg) \text{ for } h \in H .$$

Obviously, $\gamma_h(\sigma_h \otimes \mathrm{id})(\widetilde{v}) = \widetilde{v}(1 \otimes \widetilde{\theta}(h))$ for all $h \in H$. It follows that

$$(\sigma_h \otimes \mathrm{id})(\widetilde{v}^* v) = (1 \otimes \widetilde{\theta}(h)^*)\widetilde{v}^* v(1 \otimes \theta(h))$$

for all $h \in H$. Since $\widetilde{v}^* v$ is a unitary, the same proof as the one for Proposition D.2, yields a unitary $u \in B$ such that $\widetilde{\theta} = (\mathrm{Ad}\, u)\theta$ and $\widetilde{v} = v(1 \otimes u^*)$. So, for any $g \in G$, we find a unique unitary element $\theta(g) \in \mathcal{U}(B)$ such that (1) holds. By uniqueness, θ is a homomorphism and we are done. ☐

5. NON-ORBIT EQUIVALENT ACTIONS AND 1-COHOMOLOGY

The following theorem is an immediate consequence of Theorem 4.1.

Theorem 5.1 (Popa, Sasyk, [57]). — *Let* G *be a countable group with infinite normal subgroup* H *such that* (G, H) *has the relative property (T). Let* (σ_g) *be the Bernoulli action* (*with non-atomic base*) *of* G *on* (X, μ). *Then,* $H^1(\sigma) = \mathrm{Char}\, G$.

Through the following lemma, one can easily produce non-stable orbit equivalent actions

Lemma 5.2. — *Let G be a countable group and K a compact abelian group. Let $G \times K$ act on (X, μ) and denote by $(\sigma_g \rho_k)$ the corresponding action on $A = L^\infty(X)$. Define $B = A^K$, the algebra of K-fixed points. Denote by (σ_g^K) the restriction of (σ_g) to B. Assume that*

- *(σ_g) is free and weakly mixing,*
- *(σ_g^K) is still free,*
- *$H^1(\sigma) = \operatorname{Char} G$.*

Then, $H^1(\sigma^K) = \operatorname{Char} G \times \operatorname{Sp}(K, \rho)$, where

$$\operatorname{Sp}(K, \rho) = \{\alpha \in \operatorname{Char}(K) \mid \exists u \in \mathcal{U}(A), \rho_k(u) = \alpha(k)u \ \textit{for all}\ k \in K\} .$$

Proof. — Whenever $u \in \mathcal{U}(A)$ and $\rho_k(u) = \alpha(k)u$ for all $k \in K$, we define $\omega_g \in B$ by the formula $\omega_g = u\sigma_g(u^*)$. Using the weak mixing of (σ_g), it is easy to check that we obtain an embedding $\operatorname{Char} G \times \operatorname{Sp}(K, \rho) \hookrightarrow H^1(\sigma^K)$. Suppose on the contrary that the 1-cocycle ω defines an element of $H^1(\sigma^K)$. We regard ω as a 1-cocycle for σ and since $H^1(\sigma) = \operatorname{Char} G$, we find that ω is cohomologous to a character of G. Subtracting this character from ω, we may assume that $\omega_g = u\sigma_g(u^*)$ for some unitary $u \in \mathcal{U}(A)$. Since for any $k \in K$, ω_g is K-invariant and since (σ_g) is weakly mixing, we conclude that there exists $\alpha : K \to S^1$ such that $\rho_k(u) = \alpha(k)u$ for all $k \in K$. But this means that ω is given by an element of $\operatorname{Sp}(K, \rho)$. □

The following proposition immediately follows.

Proposition 5.3 (Popa, [53]). — *Let G be a countable group with infinite normal subgroup H such that (G, H) has the relative property (T). Let Γ be any countably infinite abelian group and $K = \widehat{\Gamma}$. Denote by (σ_g) the Bernoulli action of G on $L^\infty(X, \mu) = \otimes_{g \in G} L^\infty(K, \mathrm{Haar})$ and define $(\rho_k)_{k \in K}$ as the diagonal action on $L^\infty(X, \mu)$ of the translation action of K on $L^\infty(K)$. Define (σ_g^K) as the restriction of (σ_g) to the K-fixed points $L^\infty(X)^K$.*

Then, (σ_g^K) is a free and ergodic action of G satisfying $H^1(\sigma^K) = \operatorname{Char} G \times \Gamma$.

Remark 5.4. — It follows that any countable group G that admits an infinite normal subgroup H such that (G, H) has the relative property (T), admits a continuous family of non-stably orbit equivalent actions. Indeed, $\operatorname{Char} G$ being compact, an isomorphism $\operatorname{Char} G \times \Gamma_1 \cong \operatorname{Char} G \times \Gamma_2$ entails a virtual isomorphism between Γ_1 and Γ_2. It is not hard to exhibit a continuous family of non virtually isomorphic countable abelian groups.

6. INTERTWINING RIGID SUBALGEBRAS OF CROSSED PRODUCTS

The major aim of the rest of the talk is to prove Popa's *von Neumann strong rigidity theorem* for Bernoulli actions of w-rigid groups, deducing conjugacy of actions out of their mere von Neumann equivalence. This is more difficult, but nevertheless related to the orbit equivalence superrigidity Theorem 4.4. In particular, the crucial Lemma 6.1 below, is the von Neumann counterpart to Lemma 4.9, covering steps (1), (2) and (3) of the program on page 255. It states that in a crossed product $M := N \rtimes G$ by a malleable mixing action, a subalgebra $Q \subset M$ with the relative property (T), can be essentially conjugated into $\mathcal{L}(G)$.

But, the aim of this section is not only preparation to the von Neumann strong rigidity theorem. The results are applied as well in the next section in order to construct II_1 factors with prescribed countable fundamental groups. For this reason, we need to deal with actions on non-tracial (but almost-periodic) algebras.

We refer to page 243 for a rough explanation of the idea of the proof of Lemma 6.1. It is another application of Popa's *deformation/rigidity strategy.* The deformation property of *malleability* is played against the relative property (T). For this, we need the notion of relative property (T) for an inclusion $Q \subset M$ of finite von Neumann algebras (see Definition B.2). The mixing property of the action has several von Neumann algebraic consequences that are used throughout and proved in Appendix D. Finally, in order to actually conjugate (essentially) Q into $\mathcal{L}(G)$, Popa's *intertwining-by-bimodules* technique is used (see Appendix C).

Lemma 6.1. — *Given a* strongly malleable *mixing action of a countable group G on an almost periodic $(\mathcal{N}, \varphi)$, write $N = \mathcal{N}^\varphi$. Let $Q \subset N \rtimes G$ be a* diffuse *subalgebra with the* relative property (T)*. Denote by P the quasi-normalizer of Q in $N \rtimes G$ and suppose that there is no non-zero homomorphism from P to an amplification of N.*

Then, there exist $\gamma > 0$, $n \geq 1$ and a non-zero partial isometry $v \in \mathrm{M}_{n,1}(\mathbb{C}) \otimes (\mathcal{N} \rtimes G)$ which is a γ-eigenvector for φ and satisfies

$$v^*v \in P \cap Q' \, , \quad vPv^* \subset \mathrm{M}_n(\mathbb{C}) \otimes \mathcal{L}(G) \, .$$

Proof. — In the course of this proof, we use the following terminology: given subalgebras Q_1, Q_2 of a von Neumann algebra, an element a is said to be Q_1-Q_2-finite, if there exists finite families (a_i) and (b_i) such that

$$aQ_2 \subset \sum_{i=1}^{n} Q_1 a_i \quad \text{and} \quad Q_1 a \subset \sum_{i=1}^{m} b_i Q_2 \, .$$

Hence, the Q-Q-finite elements are nothing else but the elements quasi-normalizing Q.

Step (1), using relative property (T). Take (α_t) and β as in Definition 3.1. Write $\widetilde{N} = (\mathcal{N} \otimes \mathcal{N})^{\varphi \otimes \varphi}$ and $\widetilde{M} = \widetilde{N} \rtimes G$. Write $M = N \rtimes G$ and consider M as a subalgebra of $\widetilde{M}$ by considering $\mathcal{N} \otimes 1 \subset \mathcal{N} \otimes \mathcal{N}$. Extend (α_t) and β to $\widetilde{M}$. The relative property (T) yields $t_0 = 2^{-n}$ and a non-zero element $w \in \widetilde{M}$ such that $xw = w\alpha_{t_0}(x)$ for all $x \in Q$.

Step (2), finding a non-zero element $a \in \widetilde{M}$ that is Q-$\alpha_1(Q)$-finite, using the period 2-automorphism β. Denote by $\mathcal{P}$ the *-algebra of Q-Q-finite elements in M. By definition, P is the weak closure of $\mathcal{P}$. Whenever $y \in \mathcal{P}$, the element $\alpha_{t_0}(\beta(w^*)yw)$ is Q-$\alpha_{2t_0}(Q)$-finite. It suffices to find y such that $\beta(w^*)yw$ is non-zero, since we can then continue to find a non-zero Q-$\alpha_1(Q)$-finite element a in $\widetilde{M}$. Denote by p the supremum of all range projections of elements yw, where $y \in \mathcal{P}$. We have to prove that $p\beta(w) \neq 0$. By construction, $p \in \widetilde{M} \cap P'$ and $pw = w$. From Proposition D.5 (and here we use that there is no non-zero homomorphism from P to an amplification of N), $\widetilde{M} \cap P' \subset M$ and so, $p \in M$. But, β acts trivially on M and we obtain $p\beta(w) = \beta(pw) = \beta(w) \neq 0$.

Step (3), using the intertwining-by-bimodules technique to conclude. Denote by $f \in \langle \widetilde{M}, e_{\alpha_1(M)} \rangle \cap Q'$ the orthogonal projection onto the closure of $Q\, a\, \alpha_1(M)$ in $L^2(\widetilde{M})$ and remark that $\widehat{\varphi}(f) < +\infty$. Denoting by $\mathcal{F} : \langle (\mathcal{N} \otimes \mathcal{N}) \rtimes G, e_{(1\otimes\mathcal{N})\rtimes G} \rangle \to \langle \mathcal{N} \rtimes G, e_{\mathcal{L}(G)} \rangle$ the $\widehat{\varphi}$-preserving conditional expectation, it follows that

$$\mathcal{F}(f) \in \langle \mathcal{N} \rtimes G, e_{\mathcal{L}(G)} \rangle \cap Q' \quad \text{with} \quad \widehat{\varphi}(\mathcal{F}(f)) < \infty \,.$$

Moreover, $\mathcal{F}(f) \neq 0$ since $\mathcal{F}$ is faithful.

>From Proposition C.1, we get $\gamma > 0$, $n \geq 1$, $p \in \mathrm{M}_n(\mathbb{C}) \otimes \mathcal{L}(G)$, a homomorphism $\theta : Q \to p(M_n(\mathbb{C}) \otimes \mathcal{L}(G))p$ and a non-zero partial isometry $w \in M_{1,n}(\mathbb{C}) \otimes (\mathcal{N} \rtimes G)$ such that w is a γ-eigenvector for φ and $xw = w\theta(x)$ for all $x \in Q$. It follows that $w^*w \in p(M_n(\mathbb{C}) \otimes (N \rtimes G))p \cap \theta(Q)'$, which is included in $p(M_n(\mathbb{C}) \otimes \mathcal{L}(G))p$ by Theorem D.4. Also $ww^* \in M \cap Q'$ and hence, w^*Qw is a diffuse subalgebra of $p(M_n(\mathbb{C}) \otimes \mathcal{L}(G))p$. Applying once more Theorem D.4, we get $w^*Pw \subset p(M_n(\mathbb{C}) \otimes \mathcal{L}(G))p$. Since obviously $M \cap Q' \subset P$, we can take $v = w^*$ to conclude. □

Remark 6.2. — If P is a factor, it is sufficient to assume *malleability* instead of *strong malleability*. Indeed, looking back at the proof, let $a \in \widetilde{M}$ be a Q-$\alpha_{t_0}(Q)$-finite element. Then, $a\alpha_{t_0}(ya)$ is Q-$\alpha_{2t_0}(Q)$-finite for every $y \in \widetilde{M}$ that quasi-normalizes Q. Denote by $\widetilde{P}$ the quasi-normalizer of Q in $\widetilde{M}$. It is then sufficient to show that $\widetilde{P}$ is factorial, to obtain at least one y such that $a\alpha_{t_0}(ya) \neq 0$. As in the proof above, $\widetilde{M} \cap P' \subset M$. Since $\widetilde{P}$ contains P, it follows that $\widetilde{M} \cap \widetilde{P}' \subset M \cap P' = \mathcal{Z}(P) = \mathbb{C}1$. So, we are done.

In two cases, a unitary intertwiner v can be found. The first case is easy and follows immediately: assume G to be ICC and the quasi-normalizer P to be a factor. It is crucial to allow as well for an amplification in order to apply the result when dealing with the fundamental group of the crossed product $N \rtimes G$.

THEOREM 6.3 (Popa, [55]). — *Given a malleable mixing action of an ICC group G on an almost periodic $(\mathcal{N}, \varphi)$, write $N = \mathcal{N}^{\varphi}$ and $M = N \rtimes G$. Let $t > 0$ and let $Q \subset M^t$ be a* diffuse *subalgebra with the* relative property (T). *Denote by P the quasi-normalizer of Q in M^t. Suppose that P is a* factor *and that there is no non-zero homomorphism from P to an amplification of N. Realize $M^t = p(\mathrm{M}_n(\mathbb{C}) \otimes M)p$.*

Then, there exist $\gamma > 0$, $k \geq 1$ and $v \in \mathrm{M}_{n,k}(\mathbb{C}) \otimes (\mathcal{N} \rtimes G)$ a γ-eigenvector for φ, such that

$$v^*v = p\;, \quad q := vv^* \in \mathrm{M}_k(\mathbb{C}) \otimes \mathcal{L}(G)\;, \quad vPv^* \subset \mathcal{L}(G)^{t\gamma}\;,$$

where we have realized $\mathcal{L}(G)^{t\gamma} := q(\mathrm{M}_k(\mathbb{C}) \otimes \mathcal{L}(G))q$.

Proof. — Choose a projection $q \in \mathrm{M}_k(\mathbb{C}) \otimes Q$ with trace s where $s = 1/t$. Write $Q^s := q(\mathrm{M}_k(\mathbb{C}) \otimes Q)q$ and $P^s := q(\mathrm{M}_k(\mathbb{C}) \otimes P)q$. We consider $Q^s \subset P^s \subset M$. Clearly, Q^s is diffuse, $Q^s \subset M$ has the relative property (T) by Proposition B.6 and P^s is the quasi-normalizer of Q^s by Lemma 6.5. So, Lemma 6.1 (with Remark 6.2) yields a partial isometry v which is a γ-eigenvector for φ and satisfies $v^*v \in P^s$, $vP^sv^* \subset \mathcal{L}(G)^{\gamma}$. Since both P^s and $\mathcal{L}(G)$ are factors, we can move around v using partial isometries in matrix algebras over P and $\mathcal{L}(G)$ to conclude. $\square$

In the tracial case, assuming G to be ICC is sufficient.

THEOREM 6.4 (Popa, [55]). — *Given a strongly malleable mixing action of an ICC group G on a finite (N, τ), let $t > 0$ and let $Q \subset (N \rtimes G)^t$ be a* diffuse *subalgebra with the* relative property (T). *Denote by P the quasi-normalizer of Q in $(N \rtimes G)^t$ and suppose that there is no non-zero homomorphism from P to an amplification of N.*

Then, there exists a unitary element $v \in (N \rtimes G)^t$ such that $vPv^* \subset \mathcal{L}(G)^t$.

Proof. — Write $M = N \rtimes G$. Below we prove the existence of a partial isometry $v \in M^t$ satisfying $v^*v \in P \cap Q'$ and $vPv^* \subset \mathcal{L}(G)^t$. Since any projection $p \in P \cap Q'$ of trace s yields an inclusion $pQ \subset pPp \subset M^{st}$ satisfying the assumptions of the theorem, a maximality argument combined with the factoriality of $\mathcal{L}(G)$ then allows to conclude.

Choose a projection $q \in \mathrm{M}_k(\mathbb{C}) \otimes Q$ with trace s where $s = 1/t$. Write $Q^s := q(\mathrm{M}_k(\mathbb{C}) \otimes Q)q$ and $P^s := q(\mathrm{M}_k(\mathbb{C}) \otimes P)q$ as in the proof of the previous theorem. From Lemma 6.1, we get a partial isometry $w \in M$ satisfying $w^*w \in P^s \cap (Q^s)'$ and $wP^sw^* \subset \mathcal{L}(G)$. Let e be the smallest projection in $P \cap Q'$ satisfying $w^*w \leq 1 \otimes e$. Moving around w using partial isometries in matrix algebras over Q and $\mathcal{L}(G)$, we find a partial isometry $v \in M^t$ satisfying $v^*v = e$ and $vPv^* \subset \mathcal{L}(G)^t$. $\square$

LEMMA 6.5. — *Let $Q \subset M$ be an inclusion of finite von Neumann algebras and p a non-zero projection in Q. If P denotes the quasi-normalizer of q in M, then pPp is the quasi-normalizer of pQp in pMp.*

Proof. — Denote by $\widetilde{P}$ the quasi-normalizer of pQp in pMp. We only prove the inclusion $pPp \subset \widetilde{P}$, the converse inclusion being analogous. Let z be a central projection in Q such that $z = \sum_{i=1}^{n} v_i v_i^*$ with v_i partial isometries in Q and $v_i^* v_i \leq p$.

If now $x \in M$ quasi-normalizes Q, we write $p_0 = pz$ and claim that $p_0 x p_0$ quasi-normalizes pQp. Indeed, if $xQ \subset \sum_{k=1}^{r} Q x_k$, it is readily checked that

$$p_0 x p_0 \, pQp \subset \sum_{k,i} pQp \, v_i^* x_k p \; .$$

Since the central support of p in Q can be approximated arbitrary well by such special central projections z, p_0 approximates arbitrary well p and we have proved that $pPp \subset \widetilde{P}$. □

7. FUNDAMENTAL GROUPS OF TYPE II_1 FACTORS

Recall that we denote the fundamental group of a II_1 factor M by $\mathcal{F}(M) \subset \mathbb{R}^*_+$ and that $\mathrm{Sp}(\mathcal{N}, \varphi) \subset \mathbb{R}^*_+$ denotes the point spectrum of the modular automorphism group of an almost periodic state φ on $\mathcal{N}$.

Theorem 7.1 (Popa, [55]). — *Let G be an ICC group that admits an infinite almost normal subgroup H with the relative property (T). Let (σ_g) be a malleable mixing action of G on the almost periodic injective $(\mathcal{N}, \varphi)$. Denote $M := \mathcal{N}^\varphi \rtimes G$. One has*

$$\mathrm{Sp}(\mathcal{N}, \varphi) \subset \mathcal{F}(M) \subset \mathrm{Sp}(\mathcal{N}, \varphi)\mathcal{F}(\mathcal{L}(G)) \; .$$

In particular, if $\mathcal{L}(G)$ has trivial fundamental group, $\mathcal{F}(M) = \mathrm{Sp}(\mathcal{N}, \varphi)$.

Proof. — As shown by Golodets and Nessonov [24], the inclusion $\mathrm{Sp}(\mathcal{N}, \varphi) \subset \mathcal{F}(M)$ holds. Indeed, let $s \in \mathrm{Sp}(\mathcal{N}, \varphi)$ and take an s-eigenvector $v \in \mathcal{N}$, that we may suppose to be a partial isometry. Write $p = v^*v$ and $q = vv^*$. Then, $p, q \in \mathcal{N}^\varphi \subset M$, $\varphi(q) = s\varphi(p)$ and $\mathrm{Ad}\, v$ yields an isomorphism of pMp with qMq. Hence, $s \in \mathcal{F}(M)$.

Suppose $t \in \mathcal{F}(M)$ and let $\theta : M \to M^t$ be a *-isomorphism. Since H is almost normal in G, $\mathcal{L}(G)$ is contained in the quasi-normalizer of $\mathcal{L}(H)$ in M. Moreover, $\mathcal{L}(H)$ is diffuse since H is infinite. So, it follows from Theorem D.4 that the quasi-normalizer of $\mathcal{L}(H)$ in M is exactly $\mathcal{L}(G)$ and, in particular, a factor. Since $\mathcal{N}^\varphi$ is an injective von Neumann algebra with finite trace φ, it follows from Remark B.4 that there is no non-zero homomorphism from $\mathcal{L}(G)$ to an amplification of $\mathcal{N}^\varphi$.

Write $\mathcal{M} = \mathcal{N} \rtimes G$, $Q = \theta(\mathcal{L}(H))$ and $P = \theta(\mathcal{L}(G))$. Realize $M^t := p(\mathrm{M}_n(\mathbb{C}) \otimes M)p$, where p is chosen in $\mathrm{M}_n(\mathbb{C}) \otimes \mathcal{L}(H)$. By Proposition B.5, the inclusion $Q \subset P$ has the relative property (T). Increasing n if necessary, the previous paragraph and Theorem 6.3 yield $s \in \mathrm{Sp}(\mathcal{M}, \varphi)$ and $v \in \mathrm{M}_n(\mathbb{C}) \otimes \mathcal{M}$ such that v is an s-eigenvector for φ, $v^*v = p$, $q := vv^* \in \mathrm{M}_n(\mathbb{C}) \otimes \mathcal{L}(G)$ and $vPv^* \subset q(\mathrm{M}_n(\mathbb{C}) \otimes \mathcal{L}(G))q$. We claim that this inclusion is an equality. Then, we have shown that $\mathcal{L}(G)$ and $\mathcal{L}(G)^{ts}$ are isomorphic, which yields $ts \in \mathcal{F}(\mathcal{L}(G))$ and hence, $t \in \mathrm{Sp}(\mathcal{N},\varphi)\mathcal{F}(\mathcal{L}(G))$. So, this ends the proof.

Changing q to an equivalent projection in $\mathrm{M}_n(\mathbb{C}) \otimes \mathcal{L}(G)$, we may assume that $q \in \mathrm{M}_n(\mathbb{C}) \otimes \mathcal{L}(H)$. Write $Q_1 \subset P_1 \subset M$ as

$$Q_1 := \theta^{-1}\big(v^*(\mathrm{M}_n(\mathbb{C}) \otimes \mathcal{L}(H))v\big) \quad \text{and} \quad P_1 := \theta^{-1}\big(v^*(\mathrm{M}_n(\mathbb{C}) \otimes \mathcal{L}(G))v\big) .$$

The inclusion $Q_1 \subset M = N \rtimes G$ has the relative property (T), P_1 is the quasi-normalizer of Q_1 and $\mathcal{L}(G) \subset P_1$. We have to prove that $\mathcal{L}(G) = P_1$.

By Theorem 6.3, there exist a $w \in \mathrm{M}_{k,1}(\mathbb{C}) \otimes \mathcal{M}$, an r-eigenvector for φ satisfying $w^*w = 1$ and $wP_1w^* \subset \mathcal{L}(G)^r$. Since $\mathcal{L}(G) \subset P_1$, Theorem D.4 yields $w \in \mathrm{M}_{k,1}(\mathbb{C}) \otimes \mathcal{L}(G)$. But then, $\mathcal{L}(G) = P_1$ and we are done. □

Corollary 7.2. — *Let G be an ICC group that admits an infinite almost normal subgroup with the relative property (T). Suppose that $\mathcal{L}(G)$ has trivial fundamental group. Let Tr_Δ be the faithful normal state on $\mathrm{B}(H)$ given by $\mathrm{Tr}_\Delta(a) = \mathrm{Tr}(\Delta a)$ and define $(\mathcal{N},\varphi) = \bigotimes_{g\in G}(\mathrm{B}(H),\mathrm{Tr}_\Delta)$, with Connes-Størmer Bernoulli action $G \curvearrowright (\mathcal{N},\varphi)$. Write $M := \mathcal{N}^\varphi \rtimes G$.*

Then, $\mathcal{F}(M)$ is the subgroup of $\mathbb{R}^*_+$ generated by the ratios λ/μ for λ,μ belonging to the point spectrum of Δ. In particular, for every countable subgroup $S \subset \mathbb{R}^*_+$, there exists a type II_1 factor with separable predual whose fundamental group is S.

Popa showed in [52] that, among other examples, $\mathcal{L}(G)$ has trivial fundamental group when $G = \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$. Note that Popa shows in [52] that the fundamental group of $\mathcal{L}(G) = \mathrm{SL}(2,\mathbb{Z}) \ltimes L^\infty(\mathbb{T}^2)$ equals the fundamental group of the equivalence relation given by the orbits of $\mathrm{SL}(2,\mathbb{Z}) \curvearrowright \mathbb{T}^2$. The latter reduces to 1 using Gaboriau's ℓ^2 Betti number invariants for equivalence relations, see [22].

It is an open problem whether there exist II_1 factors with separable predual and uncountable fundamental group different from $\mathbb{R}^*_+$.

8. FROM VON NEUMANN EQUIVALENCE TO ORBIT EQUIVALENCE

The following is an immediate consequence of Theorem 6.4.

Proposition 8.1. — *Let G be an ICC group with a strongly malleable mixing action on the probability space (X,μ). Write $M = L^\infty(X) \rtimes G$. Let Γ be a countable group that admits an almost normal infinite subgroup Γ_0 such that (Γ,Γ_0) has the relative property (T). Suppose that Γ acts on the probability space (Y,η).*

Let p be a projection in $\mathcal{L}(G)$ and

$$\theta : L^\infty(Y) \rtimes \Gamma \to p(L^\infty(X) \rtimes G)p$$

a *-isomorphism. Then, there exists a unitary $v \in pMp$ such that $v\theta(\mathcal{L}(\Gamma))v^* \subset p\mathcal{L}(G)p$.

Proof. — We apply Theorem 6.4, observing that $\mathcal{L}(\Gamma)$ is included in the quasi-normalizer P of $\mathcal{L}(\Gamma_0)$ in $L^\infty(Y) \rtimes \Gamma$. Using Remark B.4, it follows that there is no non-zero homomorphism from P to an amplification of $L^\infty(X)$. □

From now on, specify $G \curvearrowright (X, \mu)$ to be the *Bernoulli action.* The following preliminary result is proved: an isomorphism between crossed products sending one group algebra into the other, makes the Cartan subalgebras conjugate. The final aim is Theorem 9.1 below, which states that the actions are necessarily conjugate.

THEOREM 8.2 (Popa, [56]). — *Let G be an infinite group and, for μ_0 non-atomic, $G \curvearrowright (X, \mu) = \prod_{g \in G}(X_0, \mu_0)$, its Bernoulli action. Let Γ be an infinite group that acts freely and weakly mixingly on the probability space (Y, η). Write $A = L^\infty(X)$ and $B = L^\infty(Y)$. Let p be a projection in $\mathcal{L}(G)$ and*

$$\theta : B \rtimes \Gamma \to p(A \rtimes G)p$$

a *-isomorphism. Suppose that $\theta(\mathcal{L}(\Gamma)) \subset p\mathcal{L}(G)p$. Then,

- *there exists a partial isometry $u \in A \rtimes G$ satisfying $u^*u = p$, $e := uu^* \in A$ and $u\theta(B)u^* = eA$;*
- *the equality $\theta(\mathcal{L}(\Gamma)) = p\mathcal{L}(G)p$ holds.*

Later on, Proposition 8.1 and Theorem 8.2 are combined to prove that the actions of Γ and G are conjugate through a group isomorphism of Γ and G. The proof of Theorem 8.2 certainly is the most technical and analytically subtle part of this talk.

Notations 8.3. — We fix several notations used throughout the lemmas needed to prove Theorem 8.2.

- We fix an infinite group G and write $A_0 = L^\infty(X_0)$, $(A, \tau) = \bigotimes_{g \in G}(A_0, \tau_0)$. For every finite subset $K \subset G$, we write $A_{K^c} := \bigotimes_{g \notin K}(A_0, \tau_0)$. Write $M = A \rtimes G$ and denote by τ the tracial state on M.
- We use $\eta : M \to L^2(M)$ to identify an element of the algebra M with its corresponding vector in the Hilbert space $L^2(M)$.
- For a finite subset $K \subset G$, we denote by $e_{\check{K}}$ the orthogonal projection onto the closure of $\operatorname{span}\{\eta(A_{K^c}u_g) \mid g \in G\}$ in $L^2(M)$ and we denote by $p_{\check{K}}$ the orthogonal projection onto the closure of $\operatorname{span}\{\eta(Au_k) \mid k \in G \setminus K\}$ in $L^2(M)$.
- We do not write the isomorphism θ. We simply suppose that $B \rtimes \Gamma = p(A \rtimes G)p$ in such a way that $\mathcal{L}(\Gamma) \subset p\mathcal{L}(G)p$. Of course, τ is as well the trace on $B \rtimes \Gamma$, but non-normalized.
- The elements of Γ are denoted by s, t and the action of Γ on B by $(\rho_s)_{s \in \Gamma}$. The elements of G are denoted by g, h and the action of G on A by $(\sigma_g)_{g \in G}$.

• Denote by $(\nu_s)_{s\in\Gamma}$ the canonical unitaries generating $\mathcal{L}(\Gamma)$ and by $(u_g)_{g\in G}$ the canonical unitaries generating $\mathcal{L}(G)$.

We first explain the idea of the proof of Theorem 8.2. Elements in the image of $e_{\check{K}}$ for K large are thought of as *living far away space-wise*, while elements in the image of $p_{\check{K}}$ for K large are thought of as *living far away group-wise*. In order to show that B can be conjugated into A, one shows first that sufficiently many elements of B are not living far away group-wise. This suffices to construct a B-A-subbimodule of $L^2(M)$ which is finitely generated as an A-module. To obtain elements of B that are not living far away group-wise, two lemmas are used:

• if an element of B lives far away space-wise, it does not live far away group wise (Lemma 8.4);

• if $b \in B$ and $s_n \to \infty$ in Γ, the elements $\rho_{s_n}(b)$ are more and more living far away space-wise (Lemma 8.5).

To pass from the approximate inequalities in Lemmas 8.4, 8.5 to exact inequalities, the powerful technique of ultraproducts is applied. This allows to conjugate B into A at least on the level of the ultrapower algebra. But this is sufficient to return to earth and conjugate B into A.

LEMMA 8.4. — *For every $\varepsilon > 0$ there exist finite subsets $K, L \subset G$ such that*

$$\|p_{\check{K}}\eta(x)\|^2 \leq 3\|(1 - e_{\check{L}})\eta(x)\| + \varepsilon$$

for all $x \in B$ with $\|x\| \leq 1$.

Proof. — We make the following claim.
Claim. For every $a \in M$ with $\|a\| \leq 1$ and every $\varepsilon > 0$, there exist $K, L \subset G$ finite such that

$$|\langle a \cdot \eta(x) \cdot a^*, p_{\check{K}}\eta(x)\rangle| \leq 3\|(1 - e_{\check{L}})\eta(x)\| + \|E_{\mathcal{L}(G)}(a)\|_2 + \varepsilon$$

for all $x \in M$ with $\|x\| \leq 1$. To deduce the lemma from this claim it is then sufficient to prove that B contains unitaries a with $\|E_{\mathcal{L}(G)}(a)\|_2$ arbitrary small and to use the commutativity of B in order to get $a \cdot \eta(x) \cdot a^* = \eta(x)$ for $x \in B$.

To prove the claim, choose $a \in M$ with $\|a\| \leq 1$ and $\varepsilon > 0$. By the Kaplansky density theorem, we may assume that $a \in \operatorname{span}\{A_{F_0}u_g \mid g \in F_1\}$ for some finite subsets $F_0, F_1 \subset G$. We may assume as well that $e \in F_1$. Put $L = F_1^{-1}F_0$ and $K = LF_0^{-1}$. It is an excellent Bernoulli exercise to check that

$$e_{\check{L}}(a \cdot \xi) = e_{\check{L}}(E_{\mathcal{L}(G)}(a) \cdot \xi) \text{ for } \xi \in \operatorname{Im} e_{\check{L}}, \qquad e_{\check{L}}(\xi \cdot a) = (e_{\check{L}}\xi) \cdot a \text{ for } \xi \in \operatorname{Im} p_{\check{K}}.$$

Take $x \in M$ with $\|x\| \leq 1$. We obtain that

$$(*) \qquad |\langle a \cdot \eta(x) \cdot a^*, p_{\check{K}}\eta(x)\rangle| \leq \|e_{\check{L}}(a \cdot \eta(x))\| + \|(1 - e_{\check{L}})\big((p_{\check{K}}\eta(x)) \cdot a\big)\| .$$

In $(*)$, the second term equals

$$\|\big((1 - e_{\check{L}})p_{\check{K}}\eta(x)\big) \cdot a\| \leq \|(1 - e_{\check{L}})\eta(x)\| .$$

The first term of $(*)$, is bounded by

$$(**)\qquad \|e_{\check{L}}(a \cdot (e_{\check{L}}\eta(x)))\| + \|(1-e_{\check{L}})\eta(x)\| \; .$$

In $(**)$, the first term equals

$$\begin{aligned}\|e_{\check{L}}(E_{\mathcal{L}(G)}(a) \cdot (e_{\check{L}}\eta(x)))\| &\leq \|E_{\mathcal{L}(G)}(a) \cdot (e_{\check{L}}\eta(x))\| \\ &\leq \|E_{\mathcal{L}(G)}(a) \cdot \eta(x)\| + \|(1-e_{\check{L}})\eta(x)\| \\ &\leq \|E_{\mathcal{L}(G)}(a)\|_2 + \|(1-e_{\check{L}})\eta(x)\| \; .\end{aligned}$$

We have shown that

$$|\langle a \cdot \eta(x) \cdot a^*, p_{\check{K}}\eta(x)\rangle| \leq 3\|(1-e_{\check{L}})\eta(x)\| + \|E_{\mathcal{L}(G)}(a)\|_2$$

for all $x \in M$ with $\|x\| \leq 1$, which proves the claim.

It remains to show that, for every $\varepsilon > 0$, there exists a unitary $u \in B$ such that $\|E_{\mathcal{L}(G)}(u)\|_2 < \varepsilon$. If not, it follows from Proposition C.1 that there exist $n \geq 1$, a projection $q \in \mathrm{M}_n(\mathbb{C}) \otimes \mathcal{L}(G)$, a homomorphism $\theta : B \to q(\mathrm{M}_n(\mathbb{C}) \otimes \mathcal{L}(G))q$ and a non-zero partial isometry $v \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes pM$ satisfying $v^*v \leq q$ and $bv = v\theta(b)$ for all $b \in B$. Using Theorem D.4, $v^*v \in \mathrm{M}_n(\mathbb{C}) \otimes \mathcal{L}(G)$ and we may assume that $v^*v = q$. Then, v^*Bv is a diffuse subalgebra of $q(\mathrm{M}_n(\mathbb{C}) \otimes \mathcal{L}(G))q$. Since the normalizer of B in pMp is the whole of pMp, it follows from Theorem D.4 that $v^*Mv \subset q(\mathrm{M}_n(\mathbb{C}) \otimes \mathcal{L}(G))q$. Since $v^*Mv = q(\mathrm{M}_n(\mathbb{C}) \otimes M)q$, this is a contradiction. □

Lemma 8.5. — *For every $b \in B$, $\varepsilon > 0$ and $L \subset G$ finite, there exists $K \subset \Gamma$ finite such that*

$$\|(1-e_{\check{L}})\eta(\rho_s(b))\| < \varepsilon$$

for all $s \in \Gamma \setminus K$.

Proof. — We again make a claim.
Claim. For every $a \in M$ with $\|a\| \leq 1$, $L \subset G$ finite and $\varepsilon > 0$, there exists $K_1 \subset G$ finite such that

$$\|(1-e_{\check{L}})\eta(vaw)\| \leq \varepsilon + \|(1-p_{\check{K}_1})\eta(v)\|$$

for all $v, w \in \mathcal{L}(G)$ with $\|w\| \leq 1$.

The lemma follows easily from the claim: given $K_1 \subset G$ finite and $\varepsilon > 0$, we can take $K \subset \Gamma$ finite such that $\|(1-p_{\check{K}_1})\eta(\nu_s)\| < \varepsilon$ for all $s \in \Gamma \setminus K$. It remains to observe that $\rho_s(b) = \nu_s b \nu_s^*$ and $\nu_s \in \mathcal{L}(\Gamma) \subset \mathcal{L}(G)$.

To prove the claim, choose $a \in M$ with $\|a\| \leq 1$ and $\varepsilon > 0$. By the Kaplansky density theorem, we may assume that $a \in \operatorname{span}\{A_F u_g \mid g \in G\}$ for some finite subset $F \subset G$. Given $L \subset G$ finite, we put $K_1 = LF^{-1}$ and leave as an exercise to check that

$$(p_{\check{K}_1}\eta(v)) \cdot (aw) \in \operatorname{Im} e_{\check{L}} \;\text{ for all } v, w \in \mathcal{L}(G).$$

The claim follows immediately. □

Lemma 8.6. — *For every* $b \in B$, $E_{\mathcal{L}(G)}(b) = \dfrac{\tau(b)}{\tau(p)} p$. *Hence,* $\mathcal{L}(\Gamma) = p\mathcal{L}(G)p$.

Proof. — We have to prove the following: if $b \in B$ and $\tau(b) = 0$, then $E_{\mathcal{L}(G)}(b) = 0$. Take such a $b \in B$ with $\tau(b) = 0$. Since Γ acts weakly mixingly on B, we take a sequence $s_n \to \infty$ in Γ such that $\rho_{s_n}(b) \to 0$ in the weak topology.

Combining Lemmas 8.4 and 8.5, we find a finite subset $K \subset G$ and n_0 such that $\|p_{\check{K}} \eta(\rho_{s_n}(b))\|^2 \leq \varepsilon$ for all $n \geq n_0$. Denote by f the orthogonal projection of $L^2(M)$ onto the closure of $\eta(\mathcal{L}(G))$. Since f and $p_{\check{K}}$ commute, we find that $\|p_{\check{K}} \eta\big(E_{\mathcal{L}(G)}(\rho_{s_n}(b))\big)\|^2 \leq \varepsilon$ for all $n \geq n_0$. On the other hand, $E_{\mathcal{L}(G)}(\rho_{s_n}(b))$ tends weakly to 0 and belongs to $\mathcal{L}(G)$. Hence,

$$\|(1 - p_{\check{K}}) \eta\big(E_{\mathcal{L}(G)}(\rho_{s_n}(b))\big)\|^2 \to 0$$

when $n \to \infty$. We conclude that for n sufficiently large, $\|E_{\mathcal{L}(G)}(\rho_{s_n}(b))\|_2^2 \leq 2\varepsilon$. But, for every n,

$$\|E_{\mathcal{L}(G)}(\rho_{s_n}(b))\|_2 = \|\nu_{s_n} E_{\mathcal{L}(G)}(b) \nu_{s_n}\|_2 = \|E_{\mathcal{L}(G)}(b)\|_2 \, .$$

It follows that $\|E_{\mathcal{L}(G)}(b)\|_2^2 \leq 2\varepsilon$ for all $\varepsilon > 0$, which proves that $E_{\mathcal{L}(G)}(b) = 0$.

Since $pMp = B \rtimes \Gamma$ and $\mathcal{L}(\Gamma) \subset p\mathcal{L}(G)p$, it suffices to apply $E_{\mathcal{L}(G)}$ to obtain that $p\mathcal{L}(G)p = \mathcal{L}(\Gamma)$. ☐

Let us warm up the ultraproduct machinery to finish the proof of Theorem 8.2.

Notations 8.7. — We introduce the following notations.

- Let ω be a free ultrafilter on $\mathbb{N}$ and define the ultrapower algebra M^ω, containing A^ω as a maximal abelian subalgebra. Denote by $A^\omega_\infty \subset A^\omega$ the *tail algebra* for the Bernoulli action, defined as

$$A^\omega_\infty := \bigcap_{\substack{F \subset G \\ F \text{ finite}}} (A_{F^c})^\omega \, .$$

Observe that A^ω_∞, as a subalgebra of M^ω, is normalized by the unitaries $(u_g)_{g \in G}$.

- Denote by $A^\omega_\infty \rtimes G$ the von Neumann subalgebra of M^ω generated by A^ω_∞ and $\mathcal{L}(G)$.
- We define $\chi := B^\omega \cap p(A^\omega_\infty \rtimes G)p$.

Lemmas 8.4 and 8.5 can be reinterpreted to yield elements of χ.

Lemma 8.8. — *The following results hold.*

(1) *A bounded sequence* (b_n) *in* B *represents an element of* χ *if and only if*

$$\lim_{n \to \omega} \|(1 - e_{\check{L}}) \eta(b_n)\| = 0 \quad \textit{for every finite subset } L \subset G.$$

(2) *When* $s_n \to \infty$ *in* Γ *and* $b \in B$, *the sequence* $(\rho_{s_n}(b))$ *represents an element in* χ.

(3) *If a bounded sequence* (b_n) *in* B *represents an element of* χ, *then* $b_n - \tau_1(b_n)p$ *tends to* 0 *weakly. Here* $\tau_1 := \tau(\cdot)/\tau(p)$ *denotes the normalized trace on* pMp.

Proof

(1) If $(a_n) \in A_\infty^\omega$ and $g \in G$, clearly $\lim_{n\to\omega} \|(1 - e_{\check{L}})\eta(a_n u_g)\| = 0$. Hence, the same holds if we replace $(a_n u_g)$ by any element of $A_\infty^\omega \rtimes G$. Conversely, let $b \in B^\omega$ be represented by the bounded sequence (b_n) in B such that (1) holds. For any finite $K \subset G$, define $z_K \in M^\omega$ by the sequence $\big(\sum_{g\in K} E_A(b_n u_g^*)u_g\big)$. Our assumption yields that $z_K \in A_\infty^\omega \rtimes G$ for all K. From Lemma 8.4 it follows that $\|z_K - b\|_2 \to 0$, if $K \to G$. Hence, $b \in A_\infty^\omega \rtimes G$.

(2) This follows using Lemma 8.5 and statement (1).

(3) Using Lemma 8.6, it suffices to check that $b_n - E_{\mathcal{L}(G)}(b_n)$ tends to 0 weakly. This is true for any (b_n) in $A_\infty^\omega \rtimes G$. □

In the next lemma, χ is shown to be sufficiently big.

Lemma 8.9. — *One has* $pM^\omega p \cap \chi' = B^\omega$.

Proof. — We first claim that the action $(\rho_s)_{s\in\Gamma}$ is 2-mixing (see Definition D.6). We have to prove that for all $a, b, c \in B$,

$$|\tau(a\rho_s(b)\rho_t(c)) - \tau(a)\tau(\rho_s(b)\rho_t(c))| \to 0$$

when $s, t \to \infty$.

Suppose that the bounded sequence (d_n) represents an element $d \in \chi$. By (3) in Lemma 8.8, $d_n - \tau_1(d_n)p \to 0$ weakly and hence,

$$|\tau_1(ad_n) - \tau_1(a)\tau_1(d_n)| \to 0$$

for all $a \in B$. Fix $a, b, c \in B$ and take sequences $s_n, t_n \to \infty$ in Γ. From (2) in Lemma 8.8, we get that the sequences $(\rho_{s_n}(b))$ and $(\rho_{t_n}(c))$ represent elements of χ. Since χ is a von Neumann algebra, the sequence $(\rho_{s_n}(b)\rho_{t_n}(c))$ represents an element of χ as well. Applying the previous paragraph to this sequence, we have proved the claim. Combining the 2-mixing of the action $(\rho_s)_{s\in\Gamma}$ with Lemma D.7, we are done. □

Proof of Theorem 8.2. — We first claim that there exists a non-zero $a \in p\langle M^\omega, e_{A^\omega}\rangle^+ p \cap \chi'$ with $\widehat{\tau}(a) < \infty$. As usual, $\widehat{\tau}$ denotes the semi-finite trace on the basic construction $\langle M^\omega, e_{A^\omega}\rangle$, see Appendix A.

There exists a finite subset $K \subset G$ such that

$$\lim_{n\to\omega} \|p_{\check{K}}\eta(b_n)\| \leq \frac{1}{2}$$

for all (b_n) in the unit ball of χ. Indeed, if not, write G as an increasing union of finite subsets K_n and choose $b_n \in B$ with $\|b_n\| \leq 1$, $\|(1 - e_{\check{K}_n})\eta(b_n)\| \leq 1/n$ and $\|p_{\check{K}_n}\eta(b_n)\| > 1/2$. This yields a contradiction with Lemma 8.4.

Define the projection $f_K \in \langle M^\omega, e_{A^\omega}\rangle$ as $f_K = \sum_{g\in K} u_g^* e_{A^\omega} u_g$. Clearly $\widehat{\tau}(f_K) < \infty$. Denote by a the (unique) element in the ultraweakly closed convex hull of $\{b f_K b^* \mid b \in \mathcal{U}(\chi)\}$. By construction $\widehat{\tau}(a) < \infty$ and $a \in \chi'$. To obtain the claim, we have to show that $a \neq 0$. Whenever (b_n) represents $b \in \mathcal{U}(\chi)$, we have

$$\widehat{\tau}(e_{A^\omega} b f_K b^* e_{A^\omega}) = \lim_{n\to\omega} \|(1 - p_{\check{K}})\eta(b_n)\|^2 \geq 3/4 \,.$$

Hence, $\widehat{\tau}(e_{A^\omega} a e_{A^\omega}) \neq 0$ and $a \neq 0$. This proves the claim stated in the beginning of the proof.

It follows from Lemma 8.9 and Theorem C.3 that there exists a non-zero partial isometry $v \in M^\omega$ satisfying $v^*v \in B^\omega$, $vv^* \in A^\omega$ and $vB^\omega v^* \subset A^\omega$. Take partial isometries $v_n \in M$ such that $e_n := v_n^* v_n \in B$, $v_n v_n^* \in A$ and (v_n) represents v. It follows that there exists n such that

$$\|v_n b v_n^* - E_A(v_n b v_n^*)\|_2 < \frac{1}{2}\|e_n\|_2$$

for all $b \in B$ with $\|b\| \leq 1$. Indeed, if not, we find a sequence of elements $b_n \in B$ with $\|b_n\| \leq 1$ and $\|v_n b_n v_n^* - E_A(v_n b v_n^*)\|_2 \geq \frac{1}{2}\|e_n\|_2$. Since (b_n) defines an element in B^ω, taking the limit $n \to \omega$ yields a contradiction.

If we write $f = v_n v_n^* \in A$, $A_1 := fA$ and $B_1 := v_n B v_n^*$ as subalgebras of fMf, we have, after normalization of the trace, $\|b - E_{A_1}(b)\|_2 \leq \frac{1}{2}$ for all $b \in B_1$ with $\|b\| \leq 1$. Hence, (4) in Proposition C.1 is satisfied and an application of Theorem C.3 concludes the proof of Theorem 8.2. $\square$

9. STRONG RIGIDITY FOR VON NEUMANN ALGEBRAS

Suppose that G acts on (A,τ) by $(\sigma_g)_{g\in G}$ and Γ on (B,τ) by $(\rho_s)_{s\in\Gamma}$. A *conjugation* of both actions is a pair (Δ,δ) of isomorphisms $\Delta : B \to A$, $\delta : \Gamma \to G$ satisfying $\Delta(\rho_s(b)) = \sigma_{\delta(s)}(\Delta(b))$, for all $b \in B$ and $s \in \Gamma$. Associated with the conjugation (Δ,δ) is of course the obvious isomorphism of crossed products $\theta_{\Delta,\delta} : B \rtimes \Gamma \to A \rtimes G$.

Whenever G acts on (A,τ) and $\alpha : G \to S^1$ is a character, we have an obvious automorphism θ_α of the crossed product $A \rtimes G$ defined as fixing pointwise A and $\theta_\alpha(u_g) = \alpha(g)u_g$.

THEOREM 9.1 (Popa, [56]). — *Let G be an ICC group acting and $G \curvearrowright (X,\mu)$ its Bernoulli action (with non-atomic base). Let Γ be a countable group that admits an almost normal infinite subgroup Γ_0 such that (Γ,Γ_0) has the relative property (T). Suppose that Γ acts freely on the probability space (Y,η). Let p be a projection in $L^\infty(X) \rtimes G$ and*

$$\theta : L^\infty(Y) \rtimes \Gamma \to p(L^\infty(X) \rtimes G)p$$

*a *-isomorphism. Then, $p = 1$ and there exist a unitary $u \in L^\infty(X) \rtimes G$, a conjugation (Δ, δ) of the actions through a group isomorphism $\delta : \Gamma \to G$ and a character α on G such that*

$$\theta = \operatorname{Ad} u \circ \theta_\alpha \circ \theta_{\Delta,\delta} \; .$$

Theorem 9.1 admits the following corollary stated in the introduction.

COROLLARY 9.2. — *Let G be a w-rigid group and denote by $M_G := L^\infty(X) \rtimes G$ the crossed product of the Bernoulli action $G \curvearrowright (X, \mu)$ with non-atomic base. Then, for w-rigid ICC groups G and Γ, we have $M_G \cong M_\Gamma$ if and only if $G \cong \Gamma$. Moreover, all M_G for G w-rigid ICC, have trivial fundamental group.*

The corollary is an immediate consequence of Theorem 8.2 and the orbit equivalence superrigidity Theorem 4.4. Indeed, let G and Γ be w-rigid ICC groups with Bernoulli actions on (X, μ) and (Y, η), respectively. If p is a projection in $L^\infty(X) \rtimes G$ and $\theta : L^\infty(Y) \rtimes \Gamma \to p(L^\infty(X) \rtimes G)p$ is a *-isomorphism, we have to prove that $p = 1$ and that Γ and G are isomorphic. Combining Proposition 8.1 and Theorem 8.2, we may assume that $p \in L^\infty(X)$ and $\theta(L^\infty(Y)) = L^\infty(X)p$. Hence, θ defines a stable orbit equivalence between $\Gamma \curvearrowright Y$ and $G \curvearrowright X$. So, Theorem 4.4 allows to conclude.

Refining the reasoning above, Theorem 9.1 is proved. First, taking a further reduction, it is shown that we may assume that the action $\Gamma \curvearrowright Y$ is weakly mixing. So, Proposition 8.1 and Theorem 8.2 can be applied and yield a stable orbit equivalence of $\Gamma \curvearrowright Y$ and $G \curvearrowright X$. Associated with this stable orbit equivalence is a cocycle. The unitary that conjugates $\mathcal{L}(\Gamma)$ into $\mathcal{L}(G)$ (its existence is the contents of Proposition 8.1) is reinterpreted as making cohomologous this cocycle to a homomorphism into $\mathcal{U}(\mathcal{L}(G))$. Using the weak mixing property through an application of Lemma 4.8, the homomorphism can be assumed to take values in G itself. This yields the conjugacy of the actions.

Proof of Theorem 9.1. — Write $A = L^\infty(X)$ and $B = L^\infty(Y)$. Write $M = A \rtimes G$ and identify through θ, $B \rtimes \Gamma = p(A \rtimes G)p$. First applying Proposition 8.1, we may assume that $p \in \mathcal{L}(G)$ and $\mathcal{L}(\Gamma) \subset p\mathcal{L}(G)p$. We claim that there exist a finite index subgroup $\Gamma_1 \subset \Gamma$ and a Γ_1-invariant projection $p_1 \in B \cap \mathcal{L}(G)$ such that the Γ-action on B is induced from the Γ_1-action on $p_1 B$ obtained by restriction, and such that the Γ_1-action on $p_1 B$ is weakly mixing.

Whenever $V \subset B$ is a finite-dimensional Γ-invariant subspace, it follows from Theorem D.4 that $V \subset p\mathcal{L}(G)p$. Also, $B \cap \mathcal{L}(G)$ is a Γ-invariant von Neumann subalgebra of B. By the ergodicity of the Γ-action on B, this invariant subalgebra is either diffuse or atomic. If it is diffuse and since it commutes with B, it would follow from Theorem D.4 that $B \subset p\mathcal{L}(G)p$ and hence, $pMp \subset p\mathcal{L}(G)p$, a contradiction. So, $B \cap \mathcal{L}(G)$ is atomic, hence finite-dimensional, and it suffices to take a minimal projection $p_1 \in B \cap \mathcal{L}(G)$. This proves the claim.

It now suffices to prove the theorem under the additional assumption that the action of Γ on B is weakly mixing. We apply Theorem 8.2. Conjugating again, we obtain the following situation: a projection $q \in A$ and a partial isometry $v \in M$ such that $vv^* = p \in \mathcal{L}(G)$, $v^*v = q$ and $B \rtimes \Gamma = q(A \rtimes G)q$ in such a way that $B = qA$ and $v\mathcal{L}(\Gamma)v^* = p\mathcal{L}(G)p$. The theorem follows from Proposition 9.3 below. □

In the proof of Theorem 9.1, we used the following proposition. It is a weaker version of Theorem 5.2 in [56], but sufficient for our purposes. It also provides a generalization and simpler proof for the main result in [43] by Neshveyev and Størmer.

Proposition 9.3 (Popa, [56]). — *Let G be an infinite group that acts freely and weakly mixingly on (X, μ). Let Γ be an infinite group that acts freely on (Y, η). Write $A = L^\infty(X)$ and $B = L^\infty(Y)$. Suppose that q is a projection in A such that*

$$B \rtimes \Gamma = q(A \rtimes G)q \quad \textit{with} \quad B = qA\,.$$

Suppose that there exists a partial isometry $v \in A \rtimes G$ satisfying $v^*v = q$, $vv^* = p \in \mathcal{L}(G)$ and $v\mathcal{L}(\Gamma)v^* = p\mathcal{L}(G)p$.

- *If G has no finite normal subgroups, $q = 1$.*
- *If $q = 1$, there exists $w \in \mathcal{U}(\mathcal{L}(G))$ such that, writing $\widetilde{v} = wv$, $\widetilde{v}$ normalizes $B = A$ and $\widetilde{v}\nu_s\widetilde{v}^* = \alpha(s)u_{\delta(s)}$ for some $\alpha \in$ Char(Γ) and some group isomorphism $\delta : \Gamma \to G$.*

We write this rather pedantic formulation of the proposition, to cover at the same time its application in the proof of Theorem 9.1 (where G is ICC and hence, without finite normal subgroups) and the result of [43] (where G is an any abelian group, but $q = 1$ from the beginning).

Proof. — We make use of the canonical embedding $\eta : A \rtimes G \to A \overline{\otimes} \ell^2(G)$ of the crossed product into the Hilbert-W*-module $A \overline{\otimes} \ell^2(G)$ given by $\eta(u_g a) = a \otimes \delta_{g^{-1}}$ for all $g \in G$ and $a \in A$. Here $(\delta_g)_{g \in G}$ is the canonical orthonormal basis of $\ell^2(G)$. We identify $A \overline{\otimes} \ell^2(G) = L^\infty(X, \ell^2(G))$ and we make act $\mathcal{L}(G)$ on $\ell^2(G)$ on the left and the right: $u_g \delta_h = \delta_{gh}$ and $\delta_h u_g = \delta_{hg}$. At the same time, we regard $\mathcal{L}(G) \subset \ell^2(G)$.

Denote $S^1G := S^1 \times G$ that we identify in the obvious way with a closed subgroup of $\mathcal{U}(\mathcal{L}(G))$. We identified $Y \subset X$ such that Γ acts on Y, $B = L^\infty(Y)$, $A = L^\infty(X)$ and $q = \chi_Y$. We have the orbit equivalence $q(A \rtimes G)q = B \rtimes \Gamma$ with $B = qA$. This yields a one-cocycle $\gamma : \Gamma \times Y \to S^1G$ given by

$$\eta(z\nu_s)(x) = \eta(z)(s * x)\, \gamma(s, x)$$

for all $z \in A \rtimes G$ and where we use $s * x$ to denote the action of an element $s \in \Gamma$ on $x \in Y$. We claim that the partial isometry v makes γ cohomologous to a homomorphism.

Observe that $E_{\mathcal{L}(G)}(vav^*) = \tau(p)^{-1}\tau(a)p$ for all $a \in B$. Indeed,

$$E_{\mathcal{L}(G)}(vav^*) = \tau(p)^{-1} E_{v\mathcal{L}(\Gamma)v^*}(vav^*) = \tau(p)^{-1} v E_{\mathcal{L}(\Gamma)}(a) v^* = \tau(p)^{-1}\tau(a)p\,.$$

We first study the function $w := \tau(p)^{1/2}\eta(v) \in L^\infty(Y, \ell^2(G))$. Suppose that $w_0 \in \mathcal{L}(G)$ is an essential value of this function. We find a decreasing sequence of non-zero projections q_n in B such that $\|\tau(p)^{1/2}\eta(v)q_n - q_n \otimes w_0\|_\infty \to 0$, where we use the uniform norm for functions in $L^\infty(X, \ell^2(G))$. So, we have a sequence $\varepsilon_n \to 0$ such that $\|(\tau(p)^{1/2}v - w_0)q_n\|_2 \le \varepsilon_n \|q_n\|_2$, where we use the norm of $L^2(M)$. In $L^1(M)$, we obtain that $\tau(q_n)^{-1}\|\tau(p)vq_nv^* - w_0q_nw_0^*\|_1 \to 0$. Applying $E_{\mathcal{L}(G)}$ it follows that $\|p - w_0w_0^*\|_1 \to 0$ and hence $w_0w_0^* = p$. We have shown that for almost all $y \in Y$,

$$w(y) \in \mathcal{L}(G) \quad \text{and} \quad w(y)w(y)^* = p\;.$$

Since we can replace v by w_0^*v, we may assume that p is an essential value of the function w.

Define the homomorphism $\pi : \Gamma \to \mathcal{U}(p\mathcal{L}(G)p) : \pi(s) = v\nu_s v^*$. For every $s \in \Gamma$, $v\nu_s = \pi(s)v$. Applying η, this yields,

$$w(s * x)\; \gamma(s, x) = \pi(s)\; w(x) \quad \text{for almost all } x \in Y. \tag{2}$$

If $q = 1$, Lemma 4.8 yields that $\pi(s) \in S^1G$ for all $s \in \Gamma$ and $w(x) \in S^1G$ for almost all $x \in X$. The latter implies that v normalizes the Cartan subalgebra $A = B$. The former allows to define the group isomorphism $\delta : \Gamma \to G$ and the character $\alpha : \Gamma \to S^1$ such that $\pi(s) = \alpha(s)\delta(s)$ for all $s \in \Gamma$. So, we are done in the case $q = 1$.

It remains to show that $p = 1$ when G has no finite normal subgroups. The orbit equivalence allows as well for an inverse 1-cocycle: define $W = \{(g, x) \in G \times Y \mid x \in Y, g \cdot x \in Y\}$. We use the notation $g \cdot x$ to denote the action of an element $g \in G$ on $x \in X$. Then, the 1-cocycle $\mu : W \to S^1\Gamma$ is well defined and related to γ by the formula

$$g = \gamma(\mu_{\text{group}}(g, x), x)\; \mu_{\text{scal}}(g, x)$$

for almost all $(g, x) \in W$. Here we split up explicitly $\mu = \mu_{\text{scal}}\mu_{\text{group}}$. Plugging the previous equality into (2) yields

$$w(g \cdot x)\; u_g = \pi(\mu(g, x))\; w(x) \quad \text{for almost all } (g, x) \in W. \tag{3}$$

Since p is an essential value of the function w and since π takes values in the unitaries of $p\mathcal{L}(G)p$, arguing exactly as in the proof of Lemma 4.8, yields that for any $g \in G$, pu_g is arbitrary close to a unitary and hence, u_g and p commute for all $g \in G$. So, p is a central projection in $\mathcal{L}(G)$ and it follows that $w(x) \in \mathcal{U}(p\mathcal{L}(G)p)$ for almost all $x \in Y$. Conjugating equation (3) with v^*, implies that the cocycle $\mu : W \to S^1\Gamma$ is cohomologous, as a cocycle with values in $\mathcal{U}(\mathcal{L}(\Gamma))$, to the homomorphism $g \mapsto v^*u_gv$. It follows from Lemma 4.8 that $v^*u_gv \in S^1\Gamma$ for all $g \in G$. On $S^1\Gamma$, the trace τ takes the values 0 and $\tau(p)S^1$. Hence, for all $g \in G$, we have

$$\tau(u_gp) = \tau(u_gvv^*) = \tau(v^*u_gv) \in \{0\} \cup S^1\tau(p)\;.$$

We also know that p is a central projection in $\mathcal{L}(G)$. It is an excellent exercise to deduce from all this that p is of the form $\sum_{g \in K} \beta(k)u_k$ for some finite normal subgroup

$K \subset G$ and an $\mathrm{Ad}\, G$-invariant character $\beta \in \mathrm{Char}\, K$. Hence, $K = \{e\}$, $p = 1$ and we are done. □

10. OUTER CONJUGACY OF w-RIGID GROUP ACTIONS ON THE HYPERFINITE II_1 FACTOR

We discuss some of the results of Popa [54] on (cocycle) actions of w-rigid groups on the hyperfinite II_1 factor. As explained in the introduction, the paper [54] is the precursor to all of Popa's papers on rigidity of Bernoulli actions.

Definition 10.1. — *A* cocycle action *of a countable group G on a von Neumann algebra N consists of automorphisms $(\sigma_g)_{g \in G}$ of N and unitaries $(u_{g,h})_{g,h \in G}$ in N satisfying*

$$\sigma_g \sigma_h = (\mathrm{Ad}\, u_{g,h}) \sigma_{gh} \ , \quad u_{g,h}\, u_{gh,k} = \sigma_g(u_{h,k})\, u_{g,hk} \ , \quad \sigma_e = \mathrm{id} \quad \textit{and} \quad u_{e,e} = 1 \ ,$$

for all $g, h, k \in G$.

A cocycle action (σ_g) of G on N is said to be outer conjugate *to a cocycle action (ρ_g) of G on M if there exists an isomorphism $\Delta : N \to M$ such that $\Delta \sigma_g \Delta^{-1} = \rho_g$* mod $\mathrm{Inn}\, M$ *for all $g \in G$.*

Note that a stronger notion of conjugacy exists, called *cocycle conjugacy*, where it is imposed that $\Delta \sigma_g \Delta^{-1} = (\mathrm{Ad}\, w_g) \rho_g$, with unitaries (w_g) making the 2-cocycles for σ and ρ cohomologous. In the case of an outer conjugacy between cocycle actions on a factor, the associated 2-cocycles are only made cohomologous up to a scalar-valued 2-cocycle.

Cocycle actions on a II_1 factor can be obtained by reducing an action by a projection. Let (σ_g) be an action of G on the II_1 factor N. Whenever p is a non-zero projection in N, choose partial isometries $w_g \in N$ such that $w_g w_g^* = p$ and $w_g^* w_g = \sigma_g(p)$. This is possible because (σ_g) preserves the trace and hence, p and $\sigma_g(p)$ are equivalent projections since they have the same trace. Define

$$\sigma_g^p \in \mathrm{Aut}(pNp) : \sigma_g^p(x) = w_g \sigma_g(x) w_g^* \quad \text{and} \quad u_{g,h} \in \mathcal{U}(pNp) : u_{g,h} = w_g \sigma_g(w_h) w_{gh}^* \ . \tag{4}$$

It is easily checked that (σ_g^p) is a cocycle action of G on the II_1 factor pNp and that its outer conjugacy class does not depend on the choice of w_g.

Definition 10.2. — *Let (σ_g) be an action of the countable group G on the II_1 factor N. Whenever $t > 0$, the cocycle action (σ_g^t) of G on N^t is defined by reducing the action $(\mathrm{id} \otimes \sigma_g)$ of G on $\mathrm{M}_n(\mathbb{C}) \otimes N$ by a projection p with $(\mathrm{Tr} \otimes \tau)(p) = t$, as in* (4).

The fundamental group $\mathcal{F}(\sigma)$ *of the action σ is defined as the group of $t > 0$ such that (σ_g^t) and (σ_g) are outer conjugate.*

It is clear that $\mathcal{F}(\sigma)$ is an outer conjugacy invariant for (σ_g). The following theorem computes the fundamental group for Connes-Størmer Bernoulli actions of w-rigid groups.

Theorem 10.3 (Popa, [54]). — *Let $(\mathcal{N},\varphi)$ be an almost periodic von Neumann algebra and suppose that $N := \mathcal{N}^\varphi$ is a II_1 factor. Suppose that the countable group G admits an infinite normal subgroup H with the relative property (T) and that (σ_g) is a malleable action of G on $(\mathcal{N},\varphi)$ whose restriction to H is weakly mixing.*

If we still denote by (σ_g) the restricted action of G on the II_1 factor N, then $\mathcal{F}(\sigma) = \mathrm{Sp}(\mathcal{N},\varphi)$.

Proof. — If $s \in \mathrm{Sp}(\mathcal{N},\varphi)$, we take a non-zero partial isometry $v \in \mathcal{N}$ which is an s-eigenvector for φ. Denote $p = v^*v$ and $q = vv^*$. Then, $\mathrm{Ad}\, v$ outer conjugates (σ_g^p) and (σ_g^q). Since $s = \frac{\varphi(q)}{\varphi(p)}$, it follows that $s \in \mathcal{F}(\sigma)$.

Conversely, suppose that $s \in \mathcal{F}(\sigma)$. We have to prove that $s \in \mathrm{Sp}(\mathcal{N},\varphi)$. We may clearly assume that $0 < s < 1$ and take a projection $p \in N$ and elements $w_g \in N$ such that $\varphi(p) = s$, $w_g w_g^* = p$ and $w_g^* w_g = \sigma_g(p)$ for all $g \in G$ and such that $\rho_g(x) = w_g \sigma_g(x) w_g^*$ defines a genuine action of G on pNp that is conjugate to (σ_g). We only retain that (ρ_g) is a genuine action and that its restriction $\rho|_H$ is weakly mixing.

Let (α_t) be the one-parameter group on $\mathcal{N} \otimes \mathcal{N}$ given by the malleability of (σ_g). As in the proof of Lemma 4.9, the relative property (T) yields $t_0 = 1/n$ and a non-zero partial isometry $a \in (\mathcal{N} \otimes \mathcal{N})^{\varphi\otimes\varphi}$ such that $aa^* \leq p \otimes 1$, $a^*a \leq \alpha_{t_0}(p \otimes 1)$ and

$$(w_g \otimes 1)(\sigma_g \otimes \sigma_g)(a) = a\alpha_{t_0}(w_g \otimes 1) \quad \text{for all} \quad g \in H\ .$$

Weak mixing of $\sigma|_H$ on $\mathcal{N}$ and of $\rho|_H$ on pNp implies that $aa^* = p \otimes 1$ and $a^*a = \alpha_{t_0}(p \otimes 1)$. Taking $b := a\alpha_{t_0}(a) \cdots \alpha_{(n-1)t_0}(a)$, we get a partial isometry $b \in (\mathcal{N} \otimes \mathcal{N})^{\varphi\otimes\varphi}$ satisfying $bb^* = p \otimes 1$, $b^*b = 1 \otimes p$ and

$$(w_g \otimes 1)(\sigma_g \otimes \sigma_g)(b) = b(1 \otimes w_g) \quad \text{for all} \quad g \in H\ .$$

Continuing as in the proof of Lemma 4.9, Step (3), we obtain the following data: a non-zero partial isometry $v \in p\mathcal{N} \otimes \mathrm{M}_{1,n}(\mathbb{C})$ which is a γ-eigenvector for φ and satisfies $v^*v = 1$ as well as $w_g(\sigma_g \otimes \mathrm{id})(v) = v(1 \otimes \theta(g))$ for all $g \in H$, where $\theta : G \to \mathcal{U}(n)$ is a projective representation. The ergodicity of $\rho|_H$ yields $vv^* = p$ and hence, $\mathrm{Ad}\, v$ conjugates the actions $\rho|_H$ and $(\rho_g \otimes \mathrm{Ad}\,\theta(g))_{g\in H}$. Since $1 \otimes \mathrm{M}_n(\mathbb{C})$ is an invariant subspace of the latter, weak mixing of $\rho|_H$ imposes $n = 1$. Since $vv^* = p$, $v^*v = 1$ and v is a γ-eigenvector, we conclude that $s = 1/\gamma \in \mathrm{Sp}(\mathcal{N},\varphi)$. □

In Section 3, Connes-Størmer Bernoulli actions were shown to be malleable and mixing. The following corollary is then clear.

COROLLARY 10.4. — *Let G be a countable group that admits an infinite normal subgroup with the relative property (T). Let Tr_Δ be the faithful normal state on $\mathrm{B}(H)$ given by $\mathrm{Tr}_\Delta(a) = \mathrm{Tr}(\Delta a)$ and define $(\mathcal{N}, \varphi) = \bigotimes_{g \in G}(\mathrm{B}(H), \mathrm{Tr}_\Delta)$, with Connes-Størmer Bernoulli action $G \curvearrowright (\mathcal{N}, \varphi)$. Write $\mathcal{R} := \mathcal{N}^\varphi$ and denote by (σ_g) the restricted action of G. Then, $\mathcal{F}(\sigma)$ is the subgroup of $\mathbb{R}^*_+$ generated by the ratios λ/μ between λ, μ in the point spectrum of Δ.*

In particular, G admits a continuous family of non outer conjugate actions on the hyperfinite II_1 factor $\mathcal{R}$.

In Theorem 10.3 the following question was studied: when is the cocycle action (σ^t_g) outer conjugate to (σ_g)? Another natural question is: when is the cocycle action (σ^t_g) outer conjugate to a genuine action? The following remark shows that (σ^t_g) is always outer conjugate to a genuine action when (σ_g) is a Connes-Størmer Bernoulli action on the centralizer of $\otimes_{g \in G}(\mathrm{B}(H), \varphi_0)$ for φ_0 *non-tracial.* On the other hand, for φ_0 the *trace* on $\mathrm{M}_2(\mathbb{C})$ and t not an integer, (σ^t_g) is not outer conjugate to a genuine action, see Theorem 10.6 below.

Remark 10.5. — Let $(\mathcal{N}, \varphi)$ be an almost periodic factor with $N := \mathcal{N}^\varphi$ a type II_1 factor and φ non-tracial (note that this implies that $\mathcal{N}$ is a factor of type III_λ with $0 < \lambda \leq 1$). Suppose that the group G acts on $(\mathcal{N}, \varphi)$ and denote by (σ_g) the restriction of this action to N. Then, for any $t > 0$, (σ^t_g) is outer conjugate to a genuine action.

For simplicity of notation, suppose $t \leq 1$ and let $p \in N$ be a projection with $\varphi(p) = t$. We can write a series $t = \sum_n \gamma_n$ with $\gamma_n \in \mathrm{Sp}(\mathcal{N}, \varphi)$. Write $p = \sum_n p_n$ for some mutually orthogonal projections p_n in N with $\varphi(p_n) = \gamma_n$. Take partial isometries $v_n \in \mathcal{N}$ such that v_n is a γ_n-eigenvector for φ and $v_n^* v_n = 1$, $v_n v_n^* = p_n$. Define for $g \in G$, the element $w_g \in N$ as

$$w_g := \sum_n v_n \sigma_g(v_n^*) \; .$$

It is easy to check that $w_g w_g^* = p$, $w_g^* w_g = \sigma_g(p)$ for all $g \in G$ and $w_g \sigma_g(w_h) = w_{gh}$ for all $g, h \in G$. Writing $\sigma^p_g(x) = w_g \sigma_g(x) w_g^*$ for $x \in pNp$, it follows that (σ^p_g) is a genuine action of G on pNp and a way to write (σ^t_g).

THEOREM 10.6 (Popa, [54]). — *Suppose that the countable group G admits an infinite normal subgroup H with the relative property (T). Denote by (σ_g) the Bernoulli action of G on $\mathcal{R} = \otimes_{g \in G}(M_2(\mathbb{C}), \tau)$. For $t > 0$, the cocycle action (σ^t_g) is outer conjugate to a genuine action if and only if $t \in \mathbb{N}_0$.*

Observe moreover that it follows from Theorem 10.3 that, for different values of $t > 0$, the cocycle actions (σ^t_g) are mutually non outer conjugate.

Proof. — Given (σ_g^t) outer conjugate to a genuine action (ρ_g), we can start off in the same way as in the proof of 10.3, but we do not know anymore that $\rho|_H$ is weakly mixing (or even, that ρ is ergodic). So, in order to make the passage from 'an intertwiner for α_{t_0}' to 'an intertwiner for α_1', we need the extra data of *strong malleability*, as in the proof of Lemma 4.9. But, the Connes-Størmer Bernoulli action (σ_g) is not strongly malleable in the sense of Definition 3.1 in an obvious way. So, we need a more flexible notion, essentially replacing tensor products by graded tensor products, see Remark 10.7 below.

Let $t > 0$ and suppose that (σ_g^t) is outer conjugate to a genuine action. So, we can take $k \in \mathbb{N}$, a projection $p \in \mathcal{R} \otimes \mathrm{M}_k(\mathbb{C})$ with $(\tau \otimes \mathrm{Tr})(p) = t$ and partial isometries $w_g \in \mathcal{R} \otimes \mathrm{M}_k(\mathbb{C})$ such that $w_g w_g^* = p$, $w_g^* w_g = (\sigma_g \otimes \mathrm{id})(p)$ and such that $\rho_g(x) = w_g(\sigma_g \otimes \mathrm{id})(x) w_g^*$ defines an action of G on $\mathcal{R}^t := p(\mathrm{M}_k(\mathbb{C}) \otimes \mathcal{R})p$. Let $q \leq p$ be any non-zero projection in $\mathcal{R}^t$ invariant under $\rho|_H$. We shall prove that q dominates a non-zero projection q_0, invariant under $\rho|_H$ and with $(\tau \otimes \mathrm{Tr})(q) \in \mathbb{N}$. This of course proves that $(\tau \otimes \mathrm{Tr})(p) \in \mathbb{N}$.

Combining Remark 10.7 and the proof of Lemma 4.9, we find a non-zero partial isometry $v \in \mathcal{R} \otimes \mathrm{M}_{k,n}(\mathbb{C})$ and a projective representation $\theta : G \to U(n)$ such that $v^* v = 1$, $vv^* \leq q$ and such that $w_g(\sigma_g \otimes \mathrm{id})(v) = v(1 \otimes \theta(g))$ for all $g \in H$. Putting $q_0 = vv^*$, we are done. □

Remark 10.7. — The Connes-Størmer Bernoulli action (σ_g) of the group G on $N := \otimes_{g \in G} \mathrm{M}_2(\mathbb{C})$ satisfies the following form of strong malleability: the II_1 factor N is $\mathbb{Z}/2\mathbb{Z}$-graded, the action (σ_g) commutes with the grading and the graded tensor square $N \mathbin{\widehat{\otimes}} N$ is equipped with a one-parameter group of automorphisms (α_t) and a period 2 automorphism β, all commuting with the grading and satisfying

$$\alpha_1(x \mathbin{\widehat{\otimes}} 1) = 1 \mathbin{\widehat{\otimes}} x\,, \quad \beta(x \mathbin{\widehat{\otimes}} 1) = x \mathbin{\widehat{\otimes}} 1 \quad \text{and} \quad \beta\alpha_t\beta = \alpha_{-t} \quad \text{for all } x \in N, t \in \mathbb{R}.$$

To check that the Bernoulli action indeed admits such a graded strong malleability, it suffices to construct the grading and (α_t), β on the level of $\mathrm{M}_2(\mathbb{C})$ and take the infinite product.

More generally however, for any real Hilbert space $H_\mathbb{R}$, one considers the complexified Clifford *-algebra $\mathrm{Cliff}(H_\mathbb{R})$, generated by self-adjoint elements $s(\xi)$, $\xi \in H_\mathbb{R}$ with relations

$$s(\xi)^2 = \|\xi\|^2 \quad \text{for all} \quad \xi \in H_\mathbb{R} \quad \text{and} \quad \xi \mapsto s(\xi) \quad \mathbb{R}\text{-linear}.$$

The *-algebra $\mathrm{Cliff}(H_\mathbb{R})$ admits an obvious $\mathbb{Z}/2\mathbb{Z}$-grading such that the elements $s(\xi)$ have odd degree. Also, $\mathrm{Cliff}(H_\mathbb{R})$ has a natural tracial state yielding the hyperfinite II_1 factor after completion if $H_\mathbb{R}$ is of infinite dimension. Clearly, any orthogonal representation on $H_\mathbb{R}$ extends to an action on $\mathrm{Cliff}(H_\mathbb{R})$ preserving the grading. Finally, we have a canonical isomorphism $\mathrm{Cliff}(H_\mathbb{R} \oplus K_\mathbb{R}) \cong \mathrm{Cliff}(H_\mathbb{R}) \mathbin{\widehat{\otimes}} \mathrm{Cliff}(K_\mathbb{R})$.

If one notes that $\mathrm{Cliff}(\mathbb{R}^2) \cong \mathrm{M}_2(\mathbb{C})$, one defines α_t and β on $\mathrm{Cliff}(\mathbb{R}^2 \oplus \mathbb{R}^2)$ by the formulas

$$\alpha_t\ (s\begin{pmatrix}\xi\\ \eta\end{pmatrix}) = s(\begin{pmatrix}\cos\frac{\pi t}{2} & -\sin\frac{\pi t}{2}\\ \sin\frac{\pi t}{2} & \cos\frac{\pi t}{2}\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}) \quad \text{and} \quad \beta\ (s\begin{pmatrix}\xi\\ \eta\end{pmatrix}) = s\begin{pmatrix}\xi\\ -\eta\end{pmatrix}\ .$$

The above procedure shows that also the so-called *Bogolyubov actions* are strongly malleable in a graded way.

APPENDIX A
THE BASIC CONSTRUCTION AND HILBERT MODULES

Let $(\mathcal{N}, \varphi)$ be a von Neumann algebra with almost periodic faithful normal state φ and let $B \subset \mathcal{N}^\varphi$ be a von Neumann subalgebra of the centralizer algebra. A particularly interesting case, is the one where φ is a trace and where we consider an inclusion $B \subset (N, \tau)$. We briefly explain the so-called *basic construction* von Neumann algebra $\langle \mathcal{N}, e_B \rangle$, introduced in [62, 2] and used extensively by Jones [33] in his seminal work on subfactors. We refer to [8, 25, 33] for further reading and briefly explain what is needed in this talk.

The basic construction $\langle \mathcal{N}, e_B \rangle$ is defined as the von Neumann subalgebra of $\mathrm{B}(L^2(\mathcal{N}))$ generated by $\mathcal{N}$ and the orthogonal projection e_B of $L^2(\mathcal{N})$ onto $L^2(B) \subset L^2(\mathcal{N})$. It can be checked that $\langle \mathcal{N}, e_B \rangle$ consists of those operators $T \in \mathrm{B}(L^2(\mathcal{N}))$ that commute with the right module action of B: $T(\xi b) = T(\xi)b$ for all $\xi \in L^2(\mathcal{N})$ and $b \in B$.

The basic construction $\langle \mathcal{N}, e_B \rangle$ comes equipped with a canonical normal semi-finite faithful weight $\widehat{\varphi}$ satisfying

$$\widehat{\varphi}(x e_B y) = \varphi(xy) \quad \text{for all } x, y \in \mathcal{N}.$$

If φ is a tracial state, $\widehat{\varphi}$ is a semi-finite trace.

Let (B, τ) be a finite von Neumann algebra with faithful tracial state τ. Whenever K is a right B-module, the commutant B' of B on K is a semi-finite von Neumann algebra that admits a canonical semi-finite trace τ', characterized by the formula

$$\tau'(TT^*) = \tau(T^*T) \text{ whenever } \quad T : L^2(B) \to K \quad \text{is bounded and right } B\text{-linear.}$$

Observe that for every bounded right B-linear map $T : L^2(B) \to K$, the element TT^* belongs to B' and T^*T belongs to B, acting on the left on $L^2(B)$.

When B is a factor, one defines $\dim_B(K) := \tau'(1)$ and calls $\dim_B(K)$ the *coupling constant*. It is a complete invariant for countably generated B-modules, which means the following: if $\dim_B(K) = +\infty$, K is isomorphic to $\ell^2(\mathbb{N}) \otimes L^2(B)$ as a right B-module and if $\dim_B(K) = t$ and $p \in \mathrm{M}_n(\mathbb{C}) \otimes B$ is a projection with $(\mathrm{Tr} \otimes \tau)(p) = t$, then K is isomorphic with $pL^2(B)^{\oplus n}$ as a right B-module.

When (B,τ) is an arbitrary finite von Neumann algebra with faithful tracial state τ, the situation is slightly more complicated. If $E_{\mathcal{Z}}$ denotes the center valued trace, i.e., the unique τ-preserving conditional expectation $E_{\mathcal{Z}} : B \to \mathcal{Z}(B)$ of B onto the center of B, we know that $E_{\mathcal{Z}}(xy) = E_{\mathcal{Z}}(yx)$ for all $x, y \in B$ and that $p \preceq q$ if and only if $E_{\mathcal{Z}}(p) \leq E_{\mathcal{Z}}(q)$ whenever p and q are projections in B. Moreover, whenever the Hilbert space K is a right B-module and τ a faithful tracial state on B, we denote by B' the commutant of B on K as above and construct a normal, semi-finite positive linear map

$$E'_{\mathcal{Z}} : (B')^+ \to \{ \text{ positive elements affiliated with } \mathcal{Z}(B) \ \}$$

satisfying $E'_{\mathcal{Z}}(x^*x) = E'_{\mathcal{Z}}(xx^*)$ for all x and such that

$$E'_{\mathcal{Z}}(TT^*) = E_{\mathcal{Z}}(T^*T) \ \text{ whenever } \ T : L^2(B) \to K \ \text{ is bounded and right } B\text{-linear.}$$

The positive affiliated element $E'_{\mathcal{Z}}(1)$ of $\mathcal{Z}(B)$ provides a complete invariant for countably generated right B-modules. First note that the B-module K is finitely generated, i.e., of the form $pL^2(B)^{\oplus n}$ for some projection $p \in \mathrm{M}_n(\mathbb{C}) \otimes B$, if and only if $E'_{\mathcal{Z}}(1)$ is bounded. In that case $E'_{\mathcal{Z}}(1) = (\mathrm{Tr} \otimes E_{\mathcal{Z}})(p)$.

Note that $\tau' = \tau \circ E'_{\mathcal{Z}}$. So, if $\tau'(1) < \infty$, it follows that $E'_{\mathcal{Z}}(1)$ is not necessarily bounded, but τ-integrable. This implies that $E'_{\mathcal{Z}}(1)z$ is bounded for projections $z \in \mathcal{Z}(B)$ with trace arbitrary close to 1. So, we have shown the following lemma.

Lemma A.1. — *Let K be a right B-module and τ a normal faithful tracial state on B. Denote by τ' the canonical semi-finite trace on the commutant B' of B on K. If $\tau'(1) < \infty$, there exists for any $\varepsilon > 0$ a central projection $z \in \mathcal{Z}(B)$ with $\tau(z) \geq 1 - \varepsilon$ and such that the B-module Kz is finitely generated, i.e., of the form $pL^2(B)^{\oplus n}$ for some projection $p \in \mathrm{M}_n(\mathbb{C}) \otimes B$.*

Returning to the basic construction for the inclusion $B \subset \mathcal{N}$, with $B \subset \mathcal{N}^\varphi$, we observe that the restriction of φ defines a tracial state on B and that $\langle \mathcal{N}, e_B \rangle$ is the commutant of B on $L^2(\mathcal{N})$. Using the previous paragraph, $\langle \mathcal{N}, e_B \rangle$ comes equipped with a canonical semi-finite trace φ'. If φ is tracial on $\mathcal{N}$, it is easily checked that $\widehat{\varphi} = \varphi'$. If φ is no longer a trace, but an almost periodic state, we denote by p_γ the orthogonal projection of $L^2(\mathcal{N})$ on the γ-eigenvectors for φ. Note that p_γ belongs to $\langle \mathcal{N}, e_B \rangle$ because $B \subset \mathcal{N}^\varphi$. It is easy to check that

$$\widehat{\varphi}(x) = \sum_{\gamma \in \mathrm{Sp}(\mathcal{N},\varphi)} \widehat{\varphi}(p_\gamma x p_\gamma) \quad \text{and} \quad \varphi'(x) = \sum_{\gamma \in \mathrm{Sp}(\mathcal{N},\varphi)} \gamma^{-1} \widehat{\varphi}(p_\gamma x p_\gamma)$$

for all $x \in \langle \mathcal{N}, e_B \rangle^+$. In particular, $\widehat{\varphi}$ is tracial and a multiple of φ' on $p_\gamma \langle \mathcal{N}, e_B \rangle p_\gamma$, for all $\gamma \in \mathrm{Sp}(\mathcal{N}, \varphi)$.

APPENDIX B

RELATIVE PROPERTY (T) AND II_1 FACTORS

A countable group G has Kazhdan's *property (T)* if every unitary representation of G that admits a sequence of almost invariant unit vectors, admits a non-zero G-invariant vector. More generally, a pair (G, H) consisting of a countable group G with subgroup H is said to have the *relative property (T)* of Kazhdan-Margulis [26, 15, 36, 37], if every unitary representation of G that admits a sequence of almost invariant unit vectors, admits a non-zero H-invariant vector. The main example is the pair $(\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$.

A countable group G is said to be *amenable* if the regular representation on $\ell^2(G)$ admits a sequence of almost invariant unit vectors. Hence, an amenable property (T) group is finite and an amenable group does not have an infinite subgroup with the relative property (T).

Below, we need the following alternative characterization of relative property (T) due to Jolissaint (see Theorem 1.2(a3) in [30]). The pair (G, H) has the relative property (T) if and only if every unitary representation of G admitting a sequence of almost invariant unit vectors, admits a non-zero H-invariant finite dimensional subspace.

The notion of property (T) has been defined for II_1 factors by Connes and Jones [11]. Unitary representations of groups are replaced by *bimodules* (*Connes' correspondences*, see [7, 49]). Popa [52] defined the relative property (T) for an inclusion of finite von Neumann algebras $Q \subset P$ and we explain it in this appendix.

A P-P bimodule is a Hilbert space H with a left and a right (normal, unital) action of P. We write $x\xi$, resp. ξx for the left, resp. right action of P on H.

Terminology B.1. — Let (P, τ) be a von Neumann algebra with a faithful normal tracial state τ. If K is a P-P-bimodule and (ξ_n) a sequence of unit vectors in K, we say that

– (ξ_n) is *almost central* if $\|x\xi_n - \xi_n x\| \to 0$ for all $x \in P$;
– (ξ_n) is *almost tracial* if $\|\langle \xi_n, \cdot\xi_n\rangle - \tau\| \to 0$ and $\|\langle \xi_n, \xi_n\cdot\rangle - \tau\| \to 0$.

A vector ξ is said to be *Q-central* for some von Neumann subalgebra $Q \subset P$ if $x\xi = \xi x$ for all $x \in Q$.

Definition B.2 (Popa, [52]). — *Let (P, τ) be a von Neumann algebra with a faithful normal tracial state τ. The inclusion $Q \subset P$ is said to have the* relative property (T) *if any P-P bimodule that admits a sequence of almost central almost tracial unit vectors, admits a sequence of almost tracial Q-central unit vectors.*

Remark B.3. — One might wonder why almost traciality is assumed in the definition of relative property (T). In applications (as the ones Popa's work), it is crucial that an inclusion $Q \subset P$ with the relative property (T) remains relative (T) when cutting down with a projection of Q (see Proposition B.6). Now look at the following example: we take a II_1 factor P, two von Neumann subalgebras $Q_1, Q_2 \subset P$ and we consider the inclusion of $Q_1 \oplus Q_2 \subset \mathrm{M}_2(\mathbb{C}) \otimes P$. If one would define naively the relative property (T) by imposing that any P-P bimodule admitting almost central vectors, admits a non-zero Q-central vector, then the inclusion $Q_1 \oplus Q_2 \subset \mathrm{M}_2(\mathbb{C}) \otimes P$ would have the relative property (T) if *one of the inclusions* $Q_1 \subset P$, $Q_2 \subset P$ has the relative property (T). And hence, Proposition B.6 would not hold.

Remark B.4. — A finite von Neumann algebra (P, τ) with faithful normal tracial state τ is said to be *injective* (or *amenable*) if the coarse Hilbert P-P-bimodule $L^2(P) \otimes L^2(P)$ defined by $a \cdot \xi \cdot b = (a \otimes 1)\xi(1 \otimes b)$ contains a sequence of almost central almost tracial vectors. It is then clear that an injective (P, τ) does not contain a diffuse subalgebra $Q \subset P$ with the relative property (T). More generally, if $Q \subset P$ is diffuse with the relative property (T), there is no non-zero normal homomorphism from P to an injective finite von Neumann algebra.

A lot can be said about relative property (T) in the setting of von Neumann algebras, see the papers of Peterson and Popa [47, 52]. In this talk, only two easy results are shown, which suffices for the applications in the rest of the talk.

Proposition B.5. — *Let G be a countable group with subgroup H. Then, (G, H) has the relative property (T) if and only if the inclusion $\mathcal{L}(H) \subset \mathcal{L}(G)$ has the relative property (T) in the sense of Definition B.2.*

Proof. — First suppose that (G, H) has the relative property (T). Let K be an $\mathcal{L}(G)$-$\mathcal{L}(G)$-bimodule with an almost central almost τ-tracial sequence of unit vectors (ξ_n), for some faithful normal tracial state τ on $\mathcal{L}(G)$. Define the representation $\pi(g)\xi = u_g \xi u_g^*$ of G on K. Choose $\varepsilon > 0$. Using the stronger version of relative property (T), we can take a $\pi(H)$-invariant unit vector ξ and $n \in \mathbb{N}$ such that

$$\|\xi - \xi_n\| < \frac{\varepsilon}{3}\ , \quad \|\langle \xi_n, \cdot \xi_n \rangle - \tau\| < \frac{\varepsilon}{3}\ , \quad \|\langle \xi_n, \xi_n \cdot \rangle - \tau\| < \frac{\varepsilon}{3}\ .$$

Since a $\pi(H)$-invariant vector is $\mathcal{L}(H)$-central, we have found an $\mathcal{L}(H)$-central unit vector ξ satisfying

$$\|\langle \xi, \cdot \xi \rangle - \tau\| < \varepsilon\ , \quad \|\langle \xi, \xi \cdot \rangle - \tau\| < \varepsilon\ .$$

It follows that K admits a sequence of almost tracial $\mathcal{L}(H)$-central vectors.

Conversely, suppose that the inclusion $\mathcal{L}(H) \subset \mathcal{L}(G)$ has the relative property (T) in the sense of Definition B.2. Let $\pi : G \to \mathcal{U}(K_0)$ be a unitary representation of G that admits a sequence (ξ_n) of almost invariant unit vectors. As stated above, it is sufficient to prove that K_0 admits a non-zero finite-dimensional $\pi(H)$-invariant subspace. Define $K = \ell^2(G) \otimes K_0$, which we turn into an $\mathcal{L}(G)$-$\mathcal{L}(G)$-bimodule by the formulas

$$u_g \cdot (\delta_h \otimes \xi) = \delta_{gh} \otimes \pi(g)\xi \quad \text{and} \quad (\delta_h \otimes \xi) \cdot u_g = \delta_{hg} \otimes \xi$$

for all $g, h \in G$, $\xi \in K_0$. It is clear that $(\delta_e \otimes \xi_n)$ is a sequence of almost central almost tracial unit vectors. So, K admits a non-zero $\mathcal{L}(H)$-central vector μ. Considering μ as an element in $\ell^2(G, K_0)$, we get that $\mu(hgh^{-1}) = \pi(h)\mu(g)$ for all $h \in H, g \in G$. Take $g \in G$ such that $\mu(g) \neq 0$. Since $\mu \in \ell^2(G, K_0)$, we conclude that $\{hgh^{-1} \mid h \in H\}$ is finite. But then, the linear span of $\{\mu(hgh^{-1}) \mid h \in H\}$ is a finite-dimensional $\pi(H)$-invariant subspace of K_0. □

Proposition B.6. — *Let P be a II_1 factor and $Q \subset P$ an inclusion having the relative property (T). If $p \in Q$ is a non-zero projection, $pQp \subset pPp$ has the relative property (T).*

Proof. — Write $Q_1 = pQp$ and $P_1 = pPp$. Since P is a II_1 factor, we can take partial isometries $v_1, \dots, v_k \in P$ satisfying $v_1 = p$, $v_i^* v_i \leq p$ and $\sum_{i=1}^k v_i v_i^* = 1$. Let K_1 be a P_1-P_1-bimodule admitting the almost central almost tracial sequence of unit vectors (ξ_n). Define K as the induced P-P-bimodule: put a scalar product on $Pp\, K_1\, pP$ by the formula

$$\langle x\xi y^*, a\mu b^* \rangle = \langle \xi, (x^*a)\mu(b^*y) \rangle \quad \text{for all } x, y, a, b \in Pp,\ \xi, \mu \in K_1 .$$

Up to normalization, the sequence $\sum_{i=1}^k v_i \xi_n v_i^*$ is almost central almost tracial in the P-P-bimodule K. Hence, K admits an almost tracial sequence (μ_n) of Q-central vectors. Up to normalization, $(p\mu_n) = (\mu_n p)$ defines an almost tracial sequence of pQp-central vectors in K_1. □

The above proposition remains valid when (P, τ) is just von Neumann algebra with faithul tracial state τ, but the proof becomes slightly more involved.

APPENDIX C
INTERTWINING SUBALGEBRAS USING BIMODULES

The fundamental problem in the whole of this talk is to decide when two von Neumann subalgebras $P, B \subset M$ can be conjugated one into the other: $uPu^* \subset B$ for some $u \in \mathcal{U}(M)$. The usage of the basic construction in this respect goes back to Christensen [2], who used it to study conjugacy of uniformly close subalgebras. A major innovation came with the work of Popa [55, 52], who managed to prove conjugacy results for arbitrary subalgebras, still using the basic construction.

Roughly, Proposition C.1 below says the following. Let $P, B \subset M$ be von Neumann subalgebras of a finite von Neumann algebra (M, τ). Then, the following are equivalent.

• A corner of P can be conjugated into a corner of B.

• $L^2(M)$ contains a non-zero P-B-subbimodule which is finitely generated as a B-module.

• The basic construction $\langle M, e_B\rangle$ contains a positive element a, commuting with P and satisfying $0 < \widehat{\tau}(a) < +\infty$, where $\widehat{\tau}$ is the canonical semi-finite trace on $\langle M, e_B\rangle$.

The relation between the second and the third condition is clear: the orthogonal projection p_K onto a P-B-subbimodule K of $L^2(M)$ belongs to $\langle M, e_B\rangle \cap P'$ and $\widehat{\tau}(p_K) < \infty$ is essentially equivalent to K being a finitely generated B-module.

We reproduce from [55, 52] two results needed in this talk.

Proposition C.1 (Popa, [55, 52]). — *Let $(\mathcal{M}, \varphi)$ be a von Neumann algebra with an almost periodic faithful normal state φ. Let $P, B \subset \mathcal{M}^\varphi$ be von Neumann subalgebras. Then, the following statements are equivalent.*

(1) *There exist $n \geq 1$, $\gamma > 0$, $v \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes \mathcal{M}$, a projection $p \in \mathrm{M}_n(\mathbb{C}) \otimes B$ and a homomorphism $\theta : P \to p(\mathrm{M}_n(\mathbb{C}) \otimes B)p$ such that v is a non-zero partial isometry which is a γ-eigenvector for φ, $v^*v \leq p$ and*

$$xv = v\theta(x) \quad \text{for all} \quad x \in P\ .$$

(2) *There exists a non-zero element $w \in \mathcal{M}$ such that $Pw \subset \sum_{k=1}^n w_k B$ for some finite family w_k in $\mathcal{M}$.*

(3) *There exists a non-zero element $a \in \langle \mathcal{M}, e_B\rangle^+ \cap P'$ with $\widehat{\varphi}(a) < \infty$. Here $\langle \mathcal{M}, e_B\rangle$ denotes the basic construction for the inclusion $B \subset \mathcal{M}$, with its canonical almost periodic semi-finite weight $\widehat{\varphi}$.*

(4) *There is no sequence of unitaries (u_n) in P such that $\|E_B(au_nb)\|_2 \to 0$ for all $a, b \in \mathcal{M}$.*

Of course, if one wants to deal as well with the non-separable case, one should take a net instead of a sequence in statement (4).

Proof

(1) $\Rightarrow$ (2). Taking a non-zero component of v, this is trivial.

(2) $\Rightarrow$ (3). Since P and B are in the centralizer algebra $\mathcal{M}^\varphi$ and φ is almost periodic, we can assume that $w, w_1, \dots, w_n$ are all γ-eigenvectors for φ. Note that, whenever $w \in \mathcal{M}$ is a γ-eigenvector, the projection of $L^2(\mathcal{M})$ onto the closure of wB yields a projection $f \in \langle \mathcal{M}, e_B\rangle$ and f is the range projection of we_Bw^*. It follows that $\widehat{\varphi}(f) \leq \gamma$. In the same way, the projection onto the closure of $\sum_{k=1}^n w_k B$ has finite $\widehat{\varphi}$-weight. Hence, the projection f onto the closure of PwB in $L^2(\mathcal{M})$ satisfies the requirements in (3).

(3) $\Rightarrow$ (1). If p_γ denotes the orthogonal projection of $L^2(\mathcal{M})$ onto the γ-spectral subspace of φ, we know that $\widehat{\varphi}(a) = \sum_\gamma \widehat{\varphi}(p_\gamma a p_\gamma)$ and we can replace a by $p_\gamma a p_\gamma \neq 0$. Taking a spectral projection of the form $\chi_{[\delta,+\infty[}(a)$, we obtain an orthogonal projection $f \in \langle \mathcal{M}, e_B\rangle^+ \cap P'$ with $\widehat{\varphi}(f) < \infty$ and the range of f contained in the γ-spectral

subspace of φ. Hence, the range of f is a non-zero P-B-sub-bimodule of $L^2(\mathcal{M})_\gamma$ with finite trace over B. Cutting down by a central projection of B (see Lemma A.1), we get a P-B-sub-bimodule $H \subset L^2(\mathcal{M})_\gamma$ which is finitely generated over B. Hence, we can take $n \geq 1$, a projection $p \in \mathrm{M}_n(\mathbb{C}) \otimes B$ and a B-module isomorphism

$$\psi : pL^2(B)^{\oplus n} \to H .$$

Since H is a P-module, we get a homomorphism $\theta : P \to p(\mathrm{M}_n(\mathbb{C}) \otimes B)p$ satisfying $x\psi(\xi) = \psi(\theta(x)\xi)$ for all $x \in P$ and $\xi \in H$. Define $e_i \in L^2(B)^{\oplus n}$ as $e_i = (0, \dots, 1, \dots, 0)$ and $\xi \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes H$ as $\xi_i = \psi(pe_i)$. The polar decomposition of the vector ξ yields an isometry $v \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes \mathcal{M}$ belonging to the γ-spectral subspace for φ. A direct computation shows that $xv = v\theta(x)$ for all $x \in P$, as well as $v^*v \leq p$.

(1) $\Rightarrow$ (4). Suppose that we have all the data of (1). If (u_n) is a sequence of unitaries in P such that $\|E_B(au_nb)\|_2 \to 0$ for all $a, b \in \mathcal{M}$, it follows that $\|(\mathrm{id} \otimes E_B)(v^*u_nv)\|_2 \to 0$ when $n \to \infty$. But, $\|(\mathrm{id} \otimes E_B)(v^*u_nv)\|_2 = \|(\mathrm{id} \otimes E_B)(v^*v)\theta(u_n)\|_2 = \|(\mathrm{id} \otimes E_B)(v^*v)\|_2$. We conclude that $v = 0$, a contradiction.

(4) $\Rightarrow$ (3). By (4), we can take $\varepsilon > 0$ and $K \subset \mathcal{M}$ finite such that for all unitaries $u \in P$, $\max_{a,b \in K} \|E_B(aub)\|_2 \geq \varepsilon$. Define the element $c = \sum_{b \in K} be_Bb^*$ in $\langle \mathcal{M}, e_B\rangle^+$. Note that $\widehat{\varphi}(c) < \infty$. Let $d \in \langle \mathcal{M}, e_B\rangle^+$ be the element of minimal L^2-norm (with respect to $\widehat{\varphi}$) in the L^2-closed convex hull of $\{ucu^* \mid u \in \mathcal{U}(P)\}$. By uniqueness of the element of minimal L^2-norm, it follows that $d \in \langle \mathcal{M}, e_B\rangle^+ \cap P'$ and by construction $\widehat{\varphi}(d) < \infty$. It remains to show that $d \neq 0$. But, for all $u \in \mathcal{U}(P)$, we have

$$\sum_{a \in K} \widehat{\varphi}(e_Ba\, ucu^*\, a^*e_B) = \sum_{a,b \in K} \|E_B(aub)\|_2^2 \geq \varepsilon^2 .$$

It follows that $\sum_{a \in K} \widehat{\varphi}(e_Bada^*e_B) \geq \varepsilon^2$ and $d \neq 0$. □

Lemma C.2. — *Let M be a finite von Neumann algebra and $B \subset M$ a maximal abelian subalgebra.*

- *If $q \in M$ is an abelian projection, there exists $v \in M$ satisfying $v^*v = q$ and $vMv^* \subset B$.*
- *If M is of finite type I and $P_0 \subset M$ an abelian von Neumann subalgebra, there exists a unitary $u \in M$ such that $uP_0u^* \subset B$.*

Proof. — We do not provide a full proof of this classical lemma: see paragraph 6.4 in [35] for the necessary background. The following indications shall allow the reader to fill in the proof.

For the first statement, it suffices to find a projection in B which is equivalent with q, i.e., $v \in M$ with $v^*v = q$ and $vv^* \in B$. Since B is maximal abelian, we have $vMv^* \subset B$.

For the second statement: since M is of finite type I and $L^\infty(X) = B \subset M$ is maximal abelian, the partial isometries in M normalizing B induce an equivalence relation with finite orbits on X. Taking a fundamental domain for this equivalence relation, we can easily conclude. Of course, a proper proof can be given in operator algebraic terms: if M is of type I_n and $B \subset M$ maximal abelian, we can write 1 as the sum of n equivalent abelian projections contained in B. Embedding $P_0 \subset P \subset M$ with P maximal abelian, we can do the same with P and then, P and B are unitary conjugate. $\square$

Theorem C.3 (Popa, [52]). — *Let (M, τ) be a finite von Neumann algebra and $P_0, B \subset M$ abelian subalgebras. Suppose that B is maximal abelian and $P := M \cap P_0'$ abelian (hence, maximal abelian). The following statements are equivalent.*

(1) *There exists a non-zero $v \in M$ such that $P_0 v \subset \sum_{k=1}^n v_k B$ for some finite set of elements (v_k) in B.*

(2) *There exists a non-zero $a \in \langle M, e_B \rangle^+ \cap P_0'$ satisfying $\widehat{\tau}(a) < \infty$. Here $\langle M, e_B \rangle$ denotes the basic construction for the inclusion $B \subset M$ and $\widehat{\tau}$ is the canonical semi-finite trace on it.*

(3) *There exists a non-zero partial isometry $v \in M$ such that $v^* v \in P$, $p := vv^* \in B$ and $vPv^* = Bp$.*

If moreover M is a factor and P and B are Cartan subalgebras, a fourth statement is equivalent:

(4) *There exists a unitary $u \in M$ such that $uPu^* = B$.*

Proof. — Given Proposition C.1, it suffices to prove that (2) implies (3) as well as (4) under the additional assumption that M is factorial and P and D are Cartan.

Using Proposition C.1, we take $n \geq 1$, a projection $p \in \mathrm{M}_n(\mathbb{C}) \otimes B$, a non-zero partial isometry $w \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes M$ and a homomorphism $\theta : P_0 \to p(\mathrm{M}_n(\mathbb{C}) \otimes B)p$ such that $xw = w\theta(x)$ for all $x \in P_0$. We can replace p by an equivalent projection in $\mathrm{M}_n(\mathbb{C}) \otimes B$ and take $p = \mathrm{diag}(p_1, \ldots, p_n)$. Then, $\mathrm{diag}(p_1 B, \ldots, p_n B)$ is a maximal abelian subalgebra of the finite type I algebra $p(\mathrm{M}_n(\mathbb{C}) \otimes B)p$. Since P_0 is abelian, Lemma C.2 allows to suppose that $\theta(P_0) \subset \mathrm{diag}(p_1 B, \ldots, p_n B)$. Hence, we can cut down θ and w by one of the projections $(0, \ldots, p_i, \ldots, 0)$ and suppose from the beginning that $n = 1$.

Write $q := w^* w$, $e := ww^* \in P$ and $A := pMp \cap \theta(P_0)'$. Then, $q \in A$ and $qAq = w^*(eMe \cap (Pe)')w = w^* P w$, which is abelian. Since A is finite and $pB \subset A$ maximal abelian, Lemma C.2 gives $u \in A$ satisfying $uu^* = q$ and $u^* A u \subset pB$. Writing $v = u^* w^*$, we have $vPv^* \subset B$ and $v^* v = e$. Write $f := vv^* \in B$. Hence, $eP \subset v^* B v \subset eMe$. Since $v^* B v$ is abelian, it follows that $eP = v^* B v$ and so, $vPv^* = fB$.

Assume now that M is a factor and that $P, B \subset M$ are Cartan subalgebras. Whenever u_1 is a unitary in M normalizing P and u_2 is a unitary in M normalizing B, $u_2 v u_1$ moves as well a corner of P into a corner of B. A maximality argument yields (4). $\square$

APPENDIX D
SOME RESULTS ON (WEAKLY) MIXING ACTIONS

An action of a countable group G on $(\mathcal{A},\varphi)$ is said to be *ergodic* if the scalars are the only G-invariant elements of $\mathcal{A}$. Equivalently, the multiples of 1 are the only G-invariant vectors in $L^2(\mathcal{A},\varphi)$. Stronger notions of ergodicity are the mixing and weak mixing properties.

Definition D.1. — *An action of a countable group G on $(\mathcal{A},\varphi)$ is said to be*

• mixing *if for every $a,b\in\mathcal{A}$, $\varphi(a\sigma_g(b))\to\varphi(a)\varphi(b)$ when $g\to\infty$;*

• weakly mixing *if for every $a_1,\dots,a_n\in\mathcal{A}$ and $\varepsilon>0$, there exists $g\in G$ such that $|\varphi(a_i\sigma_g(a_j))-\varphi(a_i)\varphi(a_j)|<\varepsilon$ for all $i,j=1,\dots,n$.*

For the convenience of the reader, we prove the following classical equivalent characterizations for weakly mixing actions.

Proposition D.2. — *Let a countable group G act on the finite von Neumann algebra (A,τ) by automorphisms (σ_g). Then, the following statements are equivalent.*

(1) *The action (σ_g) is weakly mixing.*

(2) *For every $a_1,\dots,a_k\in\mathcal{A}$ with $\tau(a_i)=0$, there exists a sequence $g_n\to\infty$ in G such that $\sigma_{g_n}(a_i)\to 0$ weakly for all $i=1,\dots,k$.*

(3) $\mathbb{C}1$ *is the only finite-dimensional invariant subspace of $L^2(A)$.*

(4) $\mathbb{C}1$ *is the only finite-dimensional invariant subspace of A.*

(5) *For every action (α_g) of G on a finite von Neumann algebra (M,τ), $(A\otimes M)^{\sigma\otimes\alpha}=1\otimes M^{\alpha}$.*

(6) *The diagonal action of G on $A\otimes A$ is ergodic.*

Proof. — The implications (1) ⇒ (2) ⇒ (3) ⇒ (4), as well as (5) ⇒ (6), being obvious, we prove two implications below.

(4) ⇒ (5). Suppose that $X\in(A\otimes M)^{\sigma\otimes\alpha}$. Denote by η the canonical embeddings $M\to L^2(M)$ and $A\to L^2(A)$. Define the Hilbert-Schmidt operator $T:\overline{L^2(M)}\to L^2(A):T\overline{\xi}=\eta\big((\mathrm{id}\otimes\omega_{\xi,\eta(1)})(X)\big)$. Note that the image of T is contained in $\eta(A)$ and that TT^* commutes with the unitary representation (π_g) on $L^2(A)$ given by $\pi_g\eta(a)=\eta(\sigma_g(a))$. Moreover, TT^* is trace-class. Taking a spectral projection, we find a G-invariant finite-dimensional subspace of A. By (4), the image of T is included in $\mathbb{C}\eta(1)$, which means that $X\in 1\otimes M^{\alpha}$.

(6) ⇒ (1). Suppose that (σ_g) is not weakly mixing. We find $\varepsilon>0$ and $a_1,\dots,a_n$ with $\tau(a_i)=0$ and $\sum_{i,j=1}^n|\tau(a_j^*\sigma_g(a_i))|^2\geq\varepsilon$ for every $g\in G$. Define the vector $\xi=\sum_{i=1}^n a_i\otimes a_i^*$ in $L^2(A\otimes A)$. Let ξ_1 be the element of minimal norm in the closed convex hull of $\{(\pi_g\otimes\pi_g)\xi\mid g\in G\}$. Since for any $g\in G$,

$$\langle\xi,(\pi_g\otimes\pi_g)(\xi)\rangle=\sum_{i,j=1}^n|\tau(a_j^*\sigma_g(a_i))|^2\geq\varepsilon$$

we conclude that $\xi_1 \neq 0$. Moreover, by the uniqueness of ξ_1, we get that ξ_1 is $(\pi_g \otimes \pi_g)$-invariant. By construction ξ_1 is orthogonal to 1 and we have obtained a contradiction with (6). □

LEMMA D.3. — *Let $(\mathcal{M}, \varphi)$ be an almost periodic von Neumann algebra and $P \subset B \subset \mathcal{M}^\varphi$ von Neumann subalgebras of the centralizer algebra $\mathcal{M}^\varphi$. Suppose that there exists a sequence of unitaries (u_n) in P such that*

$$\|E_B(au_nb)\|_2 \to 0 \quad \textit{whenever} \quad a, b \in \operatorname{Ker} E_B \ ,$$

where $E_B : \mathcal{M} \to B$ is the φ-preserving conditional expectation. If $x \in \mathcal{M}$ is such that $Px \subset \sum_{k=1}^n x_k B$ for a finite family of elements $x_k \in \mathcal{M}$, then $x \in B$.

More generally, any P-B-sub-bimodule of $L^2(\mathcal{M})$ that is finitely generated as a B-module, is contained in $L^2(B)$.

Proof. — Let $H_0 \subset L^2(\mathcal{M})$ be a P-B-subbimodule that is finitely generated as a B-module. We have to prove that $H_0 \subset L^2(B)$. Cutting down with a central projection in $\mathcal{Z}(B)$ and using almost periodicity, we may assume that H_0 is generated by the entries of a γ-eigenvector $\xi \in (\mathrm{M}_{1,n}(\mathbb{C}) \otimes \mathcal{M})p$, with $p \in \mathrm{M}_n(\mathbb{C}) \otimes B$ and $\theta : P \to p(\mathrm{M}_n(\mathbb{C}) \otimes B)p$ a homomorphism satisfying $a\xi = \xi\theta(a)$ for all $a \in P$. We have to prove that all entries of ξ belong to $L^2(B)$.

In the polar decomposition of ξ, the positive part $|\xi|$ commutes with $\theta(P)$ and is affiliated with $\mathrm{M}_n(\mathbb{C}) \otimes \mathcal{M}^\varphi$. So, cutting down ξ by spectral projections of $|\xi|$, we may moreover assume that $\xi \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes \mathcal{M}$. Our assumptions imply that

$$\|(\mathrm{id} \otimes E_B)(\xi^* u_n \xi) - (\mathrm{id} \otimes E_B)(\xi)^* \ u_n \ (\mathrm{id} \otimes E_B)(\xi)\|_2 \to 0 \ .$$

Since $u_n\xi = \xi\theta(u_n)$ and $\theta(u_n) \in \mathrm{M}_n(\mathbb{C}) \otimes B$, it follows that

$$(\mathrm{id} \otimes E_B)(\xi^*\xi) = (\mathrm{id} \otimes E_B)(\xi)^*(\mathrm{id} \otimes E_B)(\xi) \ .$$

This implies that the entries of ξ belong to B and we are done. □

THEOREM D.4 (Popa, [55]). — *Suppose that G acts mixingly on an almost periodic $(\mathcal{N}, \varphi)$ and write $\mathcal{M} = \mathcal{N} \rtimes G$. Let $p \in \mathrm{M}_n(\mathbb{C}) \otimes \mathcal{L}(G)$ a projection with (non-normalized) trace t and write $\mathcal{L}(G)^t = p(\mathrm{M}_n(\mathbb{C}) \otimes \mathcal{L}(G))p$, $\mathcal{M}^t = p(\mathrm{M}_n(\mathbb{C}) \otimes \mathcal{M})p$. If $P \subset \mathcal{L}(G)^t$ is a diffuse von Neumann subalgebra, any P-$\mathcal{L}(G)^t$-sub-bimodule of $L^2(\mathcal{M}^t)$ that is finitely generated as an $\mathcal{L}(G)^t$-module, is contained in $L^2(\mathcal{L}(G)^t)$.*

So, under the conditions of Theorem D.4, if $x \in \mathcal{M}^t$ such that

$$Px \subset \sum_{k=1}^n x_k \mathcal{L}(G)^t$$

for a finite family $x_k \in \mathcal{M}^t$, then $x \in \mathcal{L}(G)^t$.

Proof. — We claim that whenever (x_n) is a bounded sequence in $\mathcal{L}(G)$ that weakly tends to 0,

$$\|E_{\mathcal{L}(G)}(ax_nb)\|_2 \to 0$$

when $n \to \infty$, for all $a, b \in \mathrm{Ker}(E_{\mathcal{L}(G)})$. Here $E_{\mathcal{L}(G)} : \mathcal{M} \to \mathcal{L}(G)$ is the φ-preserving conditional expectation. It suffices to prove the claim when $a, b \in \mathcal{N}$ with $\varphi(a) = \varphi(b) = 0$. Writing $x_n = \sum_{g \in G} x_n(g) u_g$, we have

$$\|E_{\mathcal{L}(G)}(ax_nb)\|_2^2 = \sum_{g \in G} |x_n(g)\varphi(a\sigma_g(b))|^2 \ .$$

Take $C > 0$ such that $\|x_n\| \leq C$ for all n. Choose $\varepsilon > 0$. Since (σ_g) is a mixing action, take $K \subset G$ finite such that $|\varphi(a\sigma_g(b))|^2 \leq \varepsilon/(2C^2)$ for all $g \in G \setminus K$. Since x_n tends weakly to 0, $x_n(g) \to 0$ for every g. Hence, take n_0 such that for $n \geq n_0$, $\sum_{g \in K} |x_n(g)\varphi(a\sigma_g(b))|^2 < \varepsilon/2$. Since $\sum_g |x_n(g)|^2 \leq C^2$ for all n, we obtain that $\|E_{\mathcal{L}(G)}(ax_nb)\|_2^2 \leq \varepsilon$ for all $n \geq n_0$, which proves the claim.

It is then clear that any sequence of unitaries (u_n) in P tending weakly to 0 satisfies the conditions of Lemma D.3 with $B = \mathcal{L}(G)^t$ and $M = M^t$. □

Proposition D.5 (Popa, [55]). — *Suppose that G acts mixingly on the almost periodic $(\mathcal{N}, \varphi)$ and arbitrarily on the almost periodic $(\mathcal{A}, \psi)$. Consider the diagonal action on $\mathcal{A} \otimes \mathcal{N}$. Write $M = \mathcal{A}^\psi \rtimes G$ and $\widetilde{M} = (\mathcal{A} \otimes \mathcal{N})^{\psi \otimes \varphi} \rtimes G$. Let $P \subset M$ be a diffuse subalgebra such that there is no non-zero homomorphism from P to an amplification of $\mathcal{A}^\psi$. If $x \in \widetilde{M}$ and $Px \subset \sum_{k=1}^n x_k M$, we have $x \in M$.*

Proof. — Write $A = \mathcal{A}^\psi$. It follows from Proposition C.1 that there exists a sequence of unitaries (u_n) in P such that $\|E_A(u_n u_g)\|_2 \to 0$ for all $g \in G$. Let $E : (\mathcal{A} \otimes \mathcal{N}) \rtimes G \to \mathcal{A} \rtimes G$ be the unique state-preserving conditional expectation. By Lemma D.3, it suffices to check that $\|E(au_nb)\|_2 \to 0$ for all $a, b \in \mathrm{Ker}\, E$. It moreover suffices to check this last statement for $a, b \in \mathcal{N}$ with $\varphi(a) = \varphi(b) = 0$. Writing $u_n = \sum_g u_n(g) u_g$ with $u_n(g) \in A$, we have

$$\|E(au_nb)\|_2^2 = \sum_{g \in G} |\varphi(a\sigma_g(b))|^2 \, \|u_n(g)\|_2^2 \ .$$

We conclude the proof in exactly the same way as the proof of Theorem D.4. □

Finally, the notion of a 2-mixing action is introduced. Definition D.1 of a mixing action comes down to the notion of a 1-mixing action.

Definition D.6. — *An action of a countable group G on $(\mathcal{A}, \varphi)$ is said to be* 2-mixing *if*

$$\varphi(a\sigma_g(b)\sigma_h(c)) \to \varphi(a)\varphi(b)\varphi(c) \quad \textit{when } g, h, g^{-1}h \to \infty.$$

Note that any 2-mixing action is mixing and satisfies

$$|\varphi(a\sigma_g(b)\sigma_h(c)) - \varphi(a)\varphi(\sigma_g(b)\sigma_h(c))| \to 0 \quad \text{when } g, h \to \infty.$$

Conversely, this last statement characterizes 2-mixing actions.

Lemma D.7. — *Let $(\sigma_g)_{g\in G}$ be a free 2-mixing action of a countable group G on (X, μ). Write $A = L^\infty(X, \mu)$. For every $\varepsilon > 0$, there exists a finite partition of 1 in A given by $1 = q_1 + \cdots + q_n$ with q_i projections in A and satisfying*

$$\limsup_{g\to\infty} \Big\| \sum_{k=1}^{n} \sigma_g(q_k) x \sigma_g(q_k) \Big\|_2^2 \leq \varepsilon \|x\|_2^2 \tag{5}$$

for all $x \in A \rtimes G$ with $E_A(x) = 0$.

Proof. — Choose $\varepsilon > 0$. Combining freeness and the mixing property, we take a finite partition of 1 in A given by $1 = q_1 + \cdots + q_n$ with q_i projections in A and satisfying

$$\sum_{k=1}^{n} \tau(q_k \sigma_g(q_k)) \leq \varepsilon$$

for all $g \neq e$. We claim that (5) holds for all $x \in A \rtimes G$ with $E_A(x) = 0$. It is sufficient to check this for $x = \sum_{h\in F} a_h u_h$ for some finite subset $F \subset G$ not containing e. Then,

$$\Big\| \sum_{k=1}^{n} \sigma_g(q_k) x \sigma_g(q_k) \Big\|_2^2 = \sum_{h\in F, k=1}^{n} \tau(a_h^* a_h \sigma_g(q_k) \sigma_{hg}(q_k)) \ .$$

When $g \to \infty$, the right hand side is arbitrary close to

$$\sum_{h\in F, k=1}^{n} \tau(a_h^* a_h) \tau(\sigma_g(q_k)\sigma_{hg}(q_k)) = \sum_{h\in F, k=1}^{n} \tau(a_h^* a_h) \tau(q_k \sigma_{g^{-1}hg}(q_k)) \leq \varepsilon \|x\|^2 \ .$$

So, we are done. □

REFERENCES

[1] S. Adams – « An equivalence relation that is not freely generated », *Proc. Amer. Math. Soc.* **102** (1988), no. 3, p. 565–566.

[2] E. Christensen – « Subalgebras of a finite algebra », *Math. Ann.* **243** (1979), no. 1, p. 17–29.

[3] A. Connes – « Outer conjugacy classes of automorphisms of factors », *Ann. Sci. École Norm. Sup. (4)* **8** (1975), no. 3, p. 383–419.

[4] ———, « Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$ », *Ann. of Math. (2)* **104** (1976), no. 1, p. 73–115.

[5] ———, « Periodic automorphisms of the hyperfinite factor of type II_1 », *Acta Sci. Math. (Szeged)* **39** (1977), no. 1-2, p. 39–66.

[6] ———, « A factor of type II_1 with countable fundamental group », *J. Operator Theory* **4** (1980), no. 1, p. 151–153.

[7] ———, « Classification des facteurs », in *Operator algebras and applications, Part 2 (Kingston, Ont., 1980)*, Proc. Sympos. Pure Math., vol. 38, Amer. Math. Soc., Providence, 1982, p. 43–109.

[8] ———, « Indice des sous facteurs, algèbres de Hecke et théorie des nœuds [d'après Vaughan Jones] », in *Séminaire Bourbaki (1984/85)*, Astérisque, vol. 133-134, Soc. Math. France, Paris, 1986, p. 289–308.

[9] ———, « Nombres de Betti L^2 et facteurs de type II_1 [d'après D. Gaboriau et S. Popa] », in *Séminaire Bourbaki (2002/2003)*, Astérisque, vol. 294, Soc. Math. France, Paris, 2004, p. 321–333.

[10] A. Connes, J. Feldman & B. Weiss – « An amenable equivalence relation is generated by a single transformation », *Ergodic Theory Dynam. Systems* **1** (1981), p. 431–450.

[11] A. Connes & V. F. R. Jones – « Property (T) for von Neumann algebras », *Bull. London Math. Soc.* **17** (1985), p. 57–62.

[12] ———, « A II_1 factor with two nonconjugate Cartan subalgebras », *Bull. Amer. Math. Soc. (N.S.)* **6** (1982), p. 211–212.

[13] A. Connes & E. Størmer – « Entropy for automorphisms of II_1 von Neumann algebras », *Acta Math.* **134** (1975), p. 289–306.

[14] M. Cowling & U. Haagerup – « Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one », *Invent. Math.* **96** (1989), p. 507–549.

[15] C. Delaroche & A. Kirillov – « Sur les relations entre l'espace dual d'un groupe et la structure de ses sous-groupes fermés [d'après D. A. Kazhdan] », in *Séminaire Bourbaki*, Astérisque, vol. 10, Soc. Math. France, Paris, 1995, exp. no. 343 (février 1968), p. 507–528.

[16] H. A. Dye – « On groups of measure preserving transformation. I », *Amer. J. Math.* **81** (1959), p. 119–159.

[17] ———, « On groups of measure preserving transformations. II », *Amer. J. Math.* **85** (1963), p. 551–576.

[18] K. Dykema – « Interpolated free group factors », *Pacific J. Math.* **163** (1994), no. 1, p. 123–135.

[19] J. Feldman & C. C. Moore – « Ergodic equivalence relations, cohomology, and von Neumann algebras, II », *Trans. Amer. Math. Soc.* **234** (1977), p. 325–359.

[20] A. Furman – « Gromov's measure equivalence and rigidity of higher rank lattices », *Ann. of Math. (2)* **150** (1999), no. 3, p. 1059–1081.

[21] ———, « Orbit equivalence rigidity », *Ann. of Math. (2)* **150** (1999), no. 3, p. 1083–1108.

[22] D. Gaboriau – « Invariants l^2 de relations d'équivalence et de groupes », *Publ. Math. Inst. Hautes Études Sci.* (2002), no. 95, p. 93–150.

[23] S. L. Gefter – « Cohomology of the ergodic action of a T-group on the homogeneous space of a compact Lie group », in *Operators in function spaces and problems in function theory*, Naukova Dumka, Kiev, 1987, Russian, p. 77–83.

[24] V. Y. Golodets & N. I. Nessonov – « T-property and nonisomorphic full factors of types II and III », *J. Funct. Anal.* **70** (1987), p. 80–89.

[25] F. M. Goodman, P. de la Harpe & V. F. R. Jones – *Coxeter graphs and towers of algebras*, Springer-Verlag, New York, 1989.

[26] P. de la Harpe & A. Valette – *La propriété (T) de Kazhdan pour les groupes localement compacts*, Astérisque, vol. 175, Soc. Math. France, Paris, 1989.

[27] G. Hjorth – « A converse to Dye's theorem », *Trans. Amer. Math. Soc.* **357** (2005), no. 8, p. 3083–3103 (electronic).

[28] A. Ioana – « Rigidity results for wreath product II_1 factors », preprint, `math.OA/0606574`.

[29] A. Ioana, J. Peterson & S. Popa – « Amalgamated Free Products of w-Rigid Factors and Calculation of their Symmetry Groups », *Acta Math.*, to appear; `math.OA/0505589`.

[30] P. Jolissaint – « On property (T) for pairs of topological groups », *Enseign. Math. (2)* **51** (2005), no. 1-2, p. 31–45.

[31] V. F. R. Jones – « Actions of finite groups on the hyperfinite type II_1 factor », *Mem. Amer. Math. Soc.* **28** (1980), no. 237.

[32] ______, « A converse to Ocneanu's theorem », *J. Operator Theory* **10** (1983), no. 1, p. 61–63.

[33] ______, « Index for subfactors », *Invent. Math.* **72** (1983), no. 1, p. 1–25.

[34] R. V. Kadison – « Problems on von Neumann algebras », Baton Rouge Conference, 1967.

[35] R. V. Kadison & J. R. Ringrose – *Fundamentals of the theory of operator algebras II*, Academic Press, Orlando, 1986.

[36] D. A. Kazhdan – « Connection of the dual space of a group with the structure of its closed subgroups », *Funct. Anal. Appl.* **1** (1967), p. 63–65.

[37] G. A. Margulis – « Finitely-additive invariant measures on Euclidean spaces », *Ergodic Theory Dynam. Systems* **2** (1982), no. 3-4, p. 383–396 (1983).

[38] ______, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 17, Springer-Verlag, Berlin, 1991.

[39] N. Monod & Y. Shalom – « Orbit equivalence rigidity and bounded cohomology », *Ann. of Math. (2)* **164** (2006), p. 825–878.

[40] C. C. Moore – « Ergodic theory and von Neumann algebras », in *Operator algebras and applications, Part 2 (Kingston, Ont., 1980)*, Proc. Sympos. Pure Math., vol. 38, Amer. Math. Soc., Providence, 1982, p. 179–226.

[41] F. J. Murray & J. von Neumann – « On rings of operators », *Ann. of Math. (2)* **37** (1936), p. 116–229.

[42] ———, « On rings of operators, IV », *Ann. of Math. (2)* **44** (1943), p. 716–808.

[43] S. Neshveyev & E. Størmer – « Ergodic Theory and Maximal Abelian Subalgebras of the Hyperfinite Factor », *J. Funct. Anal.* **195** (2002), p. 239–261.

[44] A. Ocneanu – *Actions of discrete amenable groups on von Neumann algebras*, Lecture Notes in Mathematics, vol. 1138, Springer-Verlag, Berlin, 1985.

[45] D. S. Ornstein & B. Weiss – « Ergodic theory of amenable group actions », *Bull. Amer. Math. Soc. (N.S.)* **2** (1980), p. 161–164.

[46] N. Ozawa & S. Popa – « Some prime factorization results for type II_1 factors », *Invent. Math.* **156** (2004), p. 223–234.

[47] J. Peterson & S. Popa – « On the Notion of Relative Property (T) for Inclusions of Von Neumann Algebras », *J. Funct. Anal.* **219** (2005), p. 469–483.

[48] S. Popa – « Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups », *Invent. Math.*, to appear; `math.GR/0512646`.

[49] ———, « Correspondences », INCREST preprint, 1986.

[50] ———, « Sous-facteurs, actions des groupes et cohomologie », *C. R. Acad. Sci. Paris Sér. I Math.* **309** (1989), no. 12, p. 771–776.

[51] ———, « On the fundamental group of type II_1 factors », *Proc. Natl. Acad. Sci. USA* **101** (2004), no. 3, p. 723–726 (electronic).

[52] ———, « On a class of type II_1 factors with Betti numbers invariants », *Ann. of Math. (2)* **163** (2006), no. 3, p. 809–899.

[53] ———, « Some computations of 1-cohomology groups and construction of non-orbit-equivalent actions », *J. Inst. Math. Jussieu* **5** (2006), no. 2, p. 309–332.

[54] ———, « Some rigidity results for non-commutative Bernoulli shifts », *J. Funct. Anal.* **230** (2006), no. 2, p. 273–328.

[55] ———, « Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups, Part I », *Invent. Math.* **165** (2006), p. 369–408.

[56] ———, « Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups, Part II », *Invent. Math.* **165** (2006), p. 409–451.

[57] S. Popa & R. Sasyk – « On the cohomology of actions of groups by Bernoulli shifts », *Ergodic Dynam. Systems* **27** (2007), p. 241–251.

[58] S. Popa & S. Vaes – « Strong rigidity of generalized Bernoulli actions and computations of their symmetry groups », preprint, `math.OA/0605456`.

[59] F. Rădulescu – « The fundamental group of the von Neumann algebra of a free group with infinitely many generators is $\mathbb{R}_+^*$ », *J. Amer. Math. Soc.* **5** (1992), no. 3, p. 517–532.

[60] ———, « Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group, of noninteger index », *Invent. Math.* **115** (1994), no. 2, p. 347–389.

[61] Y. Shalom – « Measurable group theory », in *European Congress of Mathematics*, Eur. Math. Soc., Zürich, 2005, p. 391–423.

[62] C. F. Skau – « Finite subalgebras of a von Neumann algebra », *J. Funct. Anal.* **25** (1977), no. 3, p. 211–235.

[63] A. Valette – « Nouvelles approches de la propriété (T) de Kazhdan », Astérisque, vol. 294, Soc. Math. France, Paris, 2004, p. 97–124.

[64] D. Voiculescu – « Circular and semicircular systems and free product factors », in *Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris 1989)*, Progr. Math., vol. 92, Birkhäuser Boston, Boston, MA, 1990, p. 45–60.

[65] ———, « The analogues of entropy and of Fisher's information measure in free probability theory. III. The absence of Cartan subalgebras », *Geom. Funct. Anal.* **6** (1996), no. 1, p. 172–199.

[66] R. J. Zimmer – « Strong rigidity for ergodic actions of semisimple Lie groups », *Ann. of Math. (2)* **112** (1980), no. 3, p. 511–529.

[67] ———, *Ergodic theory and semisimple groups*, Monographs in Mathematics, vol. 81, Birkhäuser Verlag, Basel, 1984.

[68] ———, « Groups generating transversals to semisimple Lie group actions », *Israel J. Math.* **73** (1991), no. 2, p. 151–159.

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