Séminaire BOURBAKI Juin 2006
58e année, 2005-2006, n° 965, p. 365 à 384

THE WAVE MAP PROBLEM.
SMALL DATA CRITICAL REGULARITY
[after T. Tao]

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1. INTRODUCTION

The purpose of this paper is to describe the wave map problem

$$\Box\phi = -\phi\,(\partial_\alpha\phi\cdot\partial^\alpha\phi)\,,\qquad \phi|_{t=0}=\phi_0,\quad \partial_t\phi|_{t=0}=\phi_1 \tag{1}$$

where ϕ is a map $\phi:\mathbb{R}^{n+1}\to\mathbb{S}^{m-1}\subset\mathbb{R}^m$, and its analogs for other target manifolds, with a specific focus on the small data critical regularity results of T. Tao, contained in the following

Theorem 1 ([30], [31]). — *Let $n\geq 2$ and $s>\frac{n}{2}$. The solution of the Cauchy problem (1) with initial data $(\phi_0,\phi_1)\in(\mathbb{S}^{m-1},T\mathbb{S}^{m-1})$ in $(\dot H^s,\dot H^{s-1})$ can be extended uniquely to a global solution $(\phi(t),\partial_t\phi(t))\in(\dot H^s,\dot H^{s-1})$ on $\mathbb{R}^{n+1}$ provided that the initial data (ϕ_0,ϕ_1) has a sufficiently small $(\dot H^{\frac n2},\dot H^{\frac n2-1})$ norm:*

$$\|\phi_0\|_{\dot H^{\frac n2}(\mathbb{R}^n)}+\|\phi_1\|_{\dot H^{\frac n2-1}(\mathbb{R}^n)}<\epsilon.$$

These results imply that in dimensions $n\geq 3$, despite the fact that the wave map problem is *supercritical* relative to a conserved energy and there exist solutions blowing up in finite time, its classical solutions with $\mathbb{S}^{m-1}$ target can be extended globally in time as long as the initial data has a small scale-invariant $\dot H^{\frac n2}$ norm(1). In the critical dimension $n=2$ the result is particularly exciting as it implies that a solution exists globally as long as it has a small energy.

(1)Here and in what follows we will denote the initial data by $\phi[0]=(\phi_0,\phi_1)$ and will say that $\phi[0]\in H^s$ meaning $(\phi_0,\phi_1)\in H^s\times H^{s-1}$.

The problem (1) arises as an Euler-Lagrange equation corresponding (formally) to the critical points of the Lagrangian density:

$$\mathscr{L}[\phi] \;=\frac{1}{2}(\partial_\alpha\phi\cdot\partial_\beta\phi)\; m^{\alpha\beta}, \tag{2}$$

where $m_{\alpha\beta}$ is the Minkowski metric on $\mathbb{R}^{n+1}$. The density $\mathscr{L}[\phi]$ gives rise to the Minkowski analog of the harmonic map problem on $\mathbb{R}^n$, in which the energy density is given by $\frac{1}{2}\nabla\phi\cdot\nabla\phi$ and the critical points, harmonic maps $\phi:\mathbb{R}^n\to\mathbb{S}^{m-1}$, satisfy the equation

$$\Delta\phi=-\phi(\nabla\phi\cdot\nabla\phi).$$

The equation (1) belongs to the more general class of wave map problems, in which ϕ is a map from an $(n+1)$-dimensional Lorentzian manifold $(\mathscr{M},g)$ to a Riemannian manifold $(\mathscr{N},h)$. The map ϕ is a solution of the Euler-Lagrange equations:

$$D^\alpha\partial_\alpha\phi=0, \tag{3}$$

corresponding to the Lagrangian density:

$$\mathscr{L}[\phi] \;=\; \frac{1}{2}\; h_{ij}\;(\partial_\alpha\phi^i\partial_\beta\phi^j)\; g^{\alpha\beta}. \tag{4}$$

Here $\{\phi^i\}$ denote local coordinates on $\mathscr{N}$. D is the pull-back of the Levi-Civita connection on $T\mathscr{N}$ to the bundle $\phi^*(T\mathscr{N})$. In terms of the local coordinates $\{\phi^i\}$ this pull-back connection acting on sections of $\phi^*(T\mathscr{N})$ reads:

$$D_\alpha \;=\; \nabla_\alpha+\overline{\Gamma}^k_{\alpha j}\;, \qquad\qquad \overline{\Gamma}^k_{\alpha j}\;=\;\Gamma^k_{ij}(\phi)\partial_\alpha\phi^i\;, \tag{5}$$

where Γ^k_{ij} is the Christoffel symbol in the coordinates $\{\phi^i\}$ and ∇ is a covariant derivative on $T\mathscr{M}$. The wave-map equation (3) has the form:

$$\Box_g\phi^k \;=\; -\Gamma^k_{ij}(\phi)\,g^{\alpha\beta}(\partial_\alpha\phi^i\partial_\beta\phi^j). \tag{6}$$

In particular in the case of a wave map problem from Minkowski space $(\mathbb{R}^{n+1},m)$ the map ϕ verifies the equation

$$\Box\phi=-\Gamma(\phi)(\partial_\alpha\phi,\partial^\alpha\phi). \tag{7}$$

The wave map problem appears naturally in solid-state physics, theory of topological solitons, Quantum Field Theory and General Relativity:

Topological solitons. — One of the simplest non-trivial models with *topological soliton* solutions is the $(2+1)$ dimensional Lorentz invariant $O(3)$ classical σ-model which is nothing else but a $(2+1)$-dimensional wave map problem with $\mathbb{S}^2$ target. It arises in the study of a continuum limit of an isotropic anti-ferromagnet, [8]. Topological solitons in this model are the *static* solutions (harmonic maps from $\mathbb{R}^2\to\mathbb{S}^2$ of the equation

$$\Box\phi=-\phi(\partial_\alpha\phi\cdot\partial^\alpha\phi)$$

which minimize the energy (conserved under evolution)

$$E[\phi] = \int_{\mathbb{R}^2} \left(|\partial_t \phi|^2 + |\nabla_x \phi|^2\right) dx$$

in a given homotopy class. Such maps satisfy the Bogomol'nyi equation

$$\partial_i \phi = \pm \epsilon_{ij} \phi \times \partial_j \phi$$

and are thought to represent meta-stable particles, [1]. Here ϵ_{ij} is an anti-symmetric tensor in two dimensions. The important feature of this model, common to all $(2+1)$-dimensional wave map problems, is its *criticality.* Both the equation and the conserved energy $E[\phi]$ are invariant under scaling transformations $\phi(t,x) \to \phi(\lambda t, \lambda x)$. The problem displays a fascinating interplay between the infinite dimensional wave map dynamics defined by (1) and a finite dimensional dynamics generated by restricting the full dynamics to the moduli space of static solutions (e.g. self-shrinking $(\lambda \to 0)$ of harmonic maps), see e.g. [20], ultimately leading to the existence of large data solutions of (1) blowing up in finite time, [23].

General Relativity. — The wave map problem on a curved $(2+1)$-dimensional background with an $\mathbb{H}^2$ target arises in the $\mathbb{U}(1)$ symmetry reduction of the Einstein vacuum equations. In this case one starts with a $(\mathbf{M}, \mathbf{g})$ Lorentzian (3+1)-dimensional manifold with Ricci curvature

$$\mathbf{R}_{\alpha\beta} = 0.$$

Under the assumption that $(\mathbf{M}, \mathbf{g})$ is invariant under the group action of $U(1)$ which orbits are space-like the metric $\mathbf{g}$ can be decomposed

$$\mathbf{g} = e^{-2\gamma} g + e^{2\gamma} (\theta)^2$$

where g is a Lorentzian metric on a $(2+1)$-dimensional manifold $\mathscr{N} = (\Sigma \times \mathbb{R})$ and $\theta = dx^3 + A_a dx^a$ with $a = 0, 1, 2$ local coordinates on $\mathscr{N}$ and x^3 a coordinate along the orbit. The equations $\mathbf{R}_{a3} = 0$ (and the assumption of triviality of the first cohomology class of Σ) imply that

$$dA = \frac{1}{2} e^{-4\gamma} \star d\omega,$$

where $*$ is the Hodge dual relative to the metric g and a scalar function ω is called a twist potential. The equation $\mathbf{R}_{33} = 0$ implies that

$$\Box_g \gamma + \frac{1}{2} e^{-4\gamma} g^{ab} \partial_a \omega \, \partial_b \omega = 0,$$
$$\Box_g \omega - 4 g^{ab} \partial_a \omega \, \partial_b \gamma = 0$$

which can be recognized as a wave map equation from $(\Sigma \times \mathbb{R}, g)$ into the hyperbolic space $\mathbb{H}^2$ with the metric

$$2(d\gamma)^2 + \frac{1}{2} e^{-4\gamma} (d\omega)^2.$$

Note that the wave map evolves on a dynamic background with the metric g, which itself depends on the wave map. This coupling is determined by satisfying the remaining equations $\mathbf{R}_{ab} = 0$. The only result available in this fully nonlinear context is a small data global stability in the expanding direction statement for solutions with Σ a compact surface with genus greater than one, metric $g = -dt^2 + t^2\sigma$ with σ a metric of scalar curvature -1 on Σ and the wave map $\phi = 0$, see [6].

Acknowledgments. — The author would like to thank Terry Tao for valuable comments on this paper.

2. SUMMARY OF QUESTIONS AND RESULTS FOR THE WAVE MAP PROBLEM FROM MINKOWSKI SPACE

Traditionally(2), as the wave map equation is a hyperbolic evolution problem, one is interested in the questions of local and global in time existence and uniqueness of solutions, existence of solutions blowing up in finite time and stability of static or other "preferred"(3) solutions. The wave map equation from Minkowski space is invariant under the scaling transformation $\phi(t,x) \to \phi(\lambda t, \lambda x)$, which also preserves the $\dot{H}^{\frac{n}{2}}$ Sobolev norm. On this basis and in view of a geometric nature of the problem, our experience suggests that we could expect(4) that:

Local in time solutions exist and unique for any initial data $\phi[0] \in H^s$ with $s > n/2$.

Solutions with data with a small $\dot{H}^{\frac{n}{2}}$-norm can be extended globally in time.

Large data classical solutions can be extended globally in time for the $(2+1)$-dimensional (*critical*) wave map problem, where the scale invariant space $\dot{H}^1$ coincides with a conserved energy space, at least in the case of a target manifold of negative curvature, in analogy with the harmonic map heat flow.

Large data classical solutions can be extended globally in time for the $(1+1)$-dimensional wave map problem, where the scale invariant space $\dot{H}^{\frac{1}{2}}$ is larger (subcritical) than the energy space.

(2) The connection of the wave problem to QFT and GR may present an additional set of questions.

(3) An example of such a solution is $\phi = \gamma(u)$ where γ is a geodesic on $(\mathscr{N}, h)$ and u verifies the wave equation $\Box u = 0$.

(4) Just on the basis of presented here "evidence" perhaps a more appropriate term here would be "hope" as in some other problems these expectations have not been yet fulfilled or simply turned out to be wrong. For the wave map problem these expectations are more grounded due to the referred to above geometric origin of the problem, which makes available various cancellation properties (e.g. the expression $\partial^\alpha\phi \cdot \partial_\alpha\phi$ is an example of a *null form* eliminating parallel interactions of free waves).

Below we briefly (and incompletely, sometimes referring to just the final result) summarize known results (a good survey of the wave map problem is given in [34]):

Existence and uniqueness of local in time solutions in H^s with $s > n/2$ is in [13], [15] and [11] in dimension $n = 1$.

Small data global existence in $\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2}}$ in dimensions $n \geq 2$ is shown in [30], [31]. Extensions to other targets are in [14], [25], [21], [16], [17], [35].

Large data global existence for the $(1+1)$-dimensional wave map is established in [9], [19].

Existence of large data solutions blowing up in finite time in dimensions $n \geq 3$ is shown in [24], [4].

Stability of a trivial constant wave map and geodesic wave maps is in [27] and stability of certain $(2+1)$-dimensional spherically symmetric solutions is in [18].

For the critical $(2+1)$-dimensional wave map problem existence of solutions blowing up in finite time was proved in [23] for the $\mathbb{S}^2$ target. The large data global existence result is conjectured for the $\mathbb{H}^2$ target.

We should also mention that good results have been obtained for the large data critical $(2+1)$-dimensional wave map problem for solutions with additional *spherical* or *equivariant* symmetry assumptions. It was shown in [7] (for geodesically convex targets), [29] that large data global spherically symmetric solutions can be extended globally and uniquely in time.

The k-equivariant (co-rotational) solutions of the wave map problem are considered in the case when a target manifold is a surface of revolution. The results in [26] and [28] imply that a solution blows up in finite time only if the energy concentrates (in particular small energy implies regularity), blow-up can not occur at a self-similar rate and at the blow-up a harmonic map can be "bubbled off". We note that in the case of the $\mathbb{S}^2$ target the equation for a k-equivariant wave map takes the form

$$\partial_t^2 u - (\partial_r^2 + \frac{1}{r}\partial_r)u + k^2 \frac{\sin(2u)}{2r^2} = 0$$

for a single scalar function u satisfying the boundary conditions $u(0) = 0$ and $u(\infty) = \pi$.

3. LOCAL SUBCRITICAL THEORY

We begin by discussing the framework for proving (subcritical) local existence and uniqueness results for the wave map problem (7) as it already contains major elements required for the (critical) small data global existence problem.

A general scheme for proving local existence and uniqueness in a Sobolev space H^s can be loosely described as follows:

Identify a space X with the property that X contains solutions of the homogeneous wave equation $\Box w = 0$ with initial data in $w[0] \in H^s$.

Assume a priori that a solution ϕ belongs to X and

$$\|\phi\|_X \leq 2C_0 \tag{8}$$

for some constant C_0.

Express ϕ via a representation

$$\phi = W(t)\phi[0] - \Box^{-1}\left(\Gamma(\phi)\partial^\alpha\phi \cdot \partial_\alpha\phi\right), \tag{9}$$

where $w = W(t)\phi[0]$ is a solution of the homogeneous wave equation $\Box w = 0$ with initial $\phi[0]$ and $v = \Box^{-1}F$ denotes a solution of the inhomogeneous problem $\Box v = F$ with zero initial data at $t = 0$.

Show that

$$\|\Box^{-1}\left(\Gamma(\phi_1)(\partial^\alpha\phi_2 \cdot \partial_\alpha\phi_3)\right)\|_X \leq C \tag{10}$$

for arbitrary functions $\|\phi_i\|_X \leq 2C_0$.

In *the energy method*, in which space X is chosen to be $L_t^\infty H^s$, the representation (9) reads

$$\phi(t) = W(t)\phi[0] - \int_0^t W(t-s)\left(\Gamma(\phi)\partial^\alpha\phi \cdot \partial_\alpha\phi\right)(s)\, ds, \tag{11}$$

and the estimate (10) follows from the standard energy estimates for the wave equation and a choice of a small time interval $[0, T]$ gives local well-posedness in Sobolev spaces H^s with $s > \frac{n}{2} + 1$.

The Strichartz method is based on (11) and combines the energy estimates for the solution of the wave equation with the Strichartz estimates:

$$\|\phi\|_{L_t^q L_x^p} \lesssim \|\phi[0]\|_{\dot{H}^s} + \|\Box\phi\|_{L_t^1 \dot{H}^{s-1}},$$

where

$$\begin{aligned} &\frac{2}{q} \leq (n-1)(\frac{1}{2} - \frac{1}{p}), \quad q \geq 2, \quad (n, q, p) \neq (3, 2, \infty), \\ &\frac{1}{q} + \frac{n}{p} = \frac{n}{2} - s. \end{aligned} \tag{12}$$

The space $X = L_t^\infty H^s \cap L_t^2 W_\infty^1$ for $n \geq 3$ and $X = L_t^\infty H^s \cap L_t^4 W_\infty^1$ for $n = 2$. Local existence and uniqueness of solutions with initial data in H^s with $s > \frac{n}{2} + \frac{1}{2}$ for $n \geq 3$ and $s > \frac{n}{2} + \frac{3}{4}$ for $n = 2$ follow by iterating the wave map equation in the spaces where the energy norm is complemented by the Strichartz norm $\|\partial\phi\|_{L_t^2 L_x^\infty}$ for $n \geq 3$ and $\|\partial\phi\|_{L_t^4 L_x^\infty}$ for $n = 2$, see [22].

The $H^{s,\delta}$ method[(5)] is based on the iteration in the space $X = H^{s,\delta}$ with the norm

$$\|F\|_{H^{s,\delta}} = \|(1 + |\tau| + |\xi|)^s (1 + ||\tau| - |\xi||)^\delta \tilde{F}(\tau, \xi)\|_{L^2_{\tau,\xi}}$$

adapted to the symbol of $\Box$ (i.e., $\Box H^{s,\delta} = H^{s-1,\delta-1}$). For $\delta > 1/2$ the space $H^{s,\delta}$ is smaller than the intersection of energy space $L_t^\infty H^s$ and the $L_t^q L_x^p$ Strichartz spaces consistent with H^s regularity. Iteration in $H^{s,\delta}$ leads to the local existence and uniqueness results in H^s with $s > n/2$, see [13], [15]. The key to this result is the algebra property

$$H^{s,\frac{1}{2}+} \cdot H^{s-1,-\frac{1}{2}+} \subset H^{s-1,-\frac{1}{2}+},$$

and the null form estimate

$$Q_0(H^{s,\frac{1}{2}+}, H^{s,\frac{1}{2}+}) \subset H^{s-1,-\frac{1}{2}+},$$

which both hold with $s > \frac{n}{2}$. Here $H^{s,\frac{1}{2}+}$ stands for the space $H^{s,\delta}$ with $1/2 < \delta < 1/2 + s - n/2$. The null form $Q_0(\phi, \phi) = \partial_\alpha \phi\, \partial^\alpha \phi$ is precisely the expression arising in the nonlinear term of the wave map problem. It has the property that for two solutions of the homogeneous wave equation $\Box\phi = \Box\psi = 0$ we have

$$2Q_0(\phi, \psi) = \Box(\phi\psi).$$

In particular Q_0 eliminates parallel interactions, between ϕ and ψ whose space-time Fourier transform lies on the cone $|\tau|^2 = |\xi|^2$. This special structure of the nonlinearity in the wave map problem is crucial for both the local existence and the small data global existence results in low dimensions.

The challenge in strengthening these results to obtain a global scale-invariant critical statement lied in the fact that it would require a scale-invariant homogeneous version of the space $H^{\frac{n}{2},\frac{1}{2}}$

$$\|F\|_{\dot{H}^{s,\delta}} = \|(|\tau| + |\xi|)^s (||\tau| - |\xi||)^\delta \tilde{F}(\tau, \xi)\|_{L^2_{\tau,\xi}}.$$

The space $\dot{H}^{\frac{n}{2},\frac{1}{2}}$ is not suitable (in fact it is not even well defined) for the solution of the critical wave map problem in particular in view of the *division* and *summation*

(5) In the context of the well-posedness theory for hyperbolic equations the $H^{s,\delta}$ spaces were first used in the work of Klainerman-Machedon [12]. They are closely connected with the $X^{s,b}$ spaces introduced and used by Bourgain for the Schrödinger and KdV equations in [2], [3].

problems one faces in the process of constructing (i.e., proving a priori estimates) the wave map via a representation (9)

$$\phi = W(t)\phi[0] - \Box^{-1}\left(\Gamma(\phi)(\partial^\alpha\phi\cdot\partial_\alpha\phi)\right).$$

First we can modify the space $\dot H^{\frac{n}{2},\frac12}$ by improving its summability properties relative to the distance to the cone $||\tau|-|\xi||$ while keeping the space scale-invariant. We assume that the space-time Fourier transform $\tilde F(\tau,\xi)$ has support in the region $|\xi|\approx 2^k$ and define

$$\|F\|_{\dot H^{s,\frac12,1}_k} = 2^{ks}\sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\|m(2^{-j}\,||\tau|-|\xi||)\tilde F(\tau,\xi)\|_{L^2_{\tau,\xi}}, \tag{13}$$

where m is a smooth bump function supported on the interval $[1/4,4]$ and equal to one on $[1/2,2]$. The division problem arises already at the level of attempting to implement the general existence scheme described above assuming that ϕ is a linear combination of *finitely* many $H^{\frac{n}{2},\frac12,1}_k$ *atoms.* The summation problem requires handling square summable combinations of $H^{\frac{n}{2},\frac12,1}_k$ pieces.

4. TATARU'S RESULT

In [32], [33] Tataru solved the division problem under the assumption that the initial has a small critical Besov norm $\dot B^{\frac{n}{2}}_{2,1}$ instead of a larger Sobolev space $\dot H^{\frac{n}{2}}$. The summation problem was avoided by putting together the $H^{\frac{n}{2},\frac12,1}_k$ pieces in ℓ^1. Such a space is only consistent with free waves with initial data in the smaller Besov space $\dot B^{\frac{n}{2}}_{2,1}$. The solution of the division problem requires enlarging the atomic space built on $H^{\frac{n}{2},\frac12,1}_k$ atoms by adding a component which lies in a *null frame* space. Without going into much details, the null frame spaces are motivated by the following observation. Let w be a unit frequency solution of the homogeneous wave equation $\Box w = 0$ whose spatial Fourier support lies in an angular sector Ω. For simplicity we may assume that w is a $+$ wave and thus it can be represented as a superposition of the traveling waves

$$w = \int_{\mathbb{R}^n} e^{it|\xi|+ix\cdot\xi}w_0(\xi)\,d\xi = \int_\Omega\int_{\lambda\sim1} e^{i\lambda(t+ix\cdot\omega)}w_0(\lambda\omega)\lambda^{n-1}d\lambda\,d\omega = \int_\Omega w_\omega(t+x\cdot\omega)d\omega,$$

where

$$w_\omega(s) = \int_{\lambda\sim1} e^{is\lambda}w_0(\lambda\omega)\lambda^{n-1}d\lambda.$$

For a fixed ω let $u_\omega = t + x\cdot\omega$ denote a variable parametrizing the corresponding null (i.e., the length of the tangent vector $(1,\omega)$ with respect to the Minkowski metric is zero) direction and let x_ω denote the variables $(t-x\cdot\omega, x-x\cdot\omega)$. Then the traveling wave decomposition has the property that

$$\int_\Omega \|u_\omega\|_{L^2_{u_\omega} L^\infty_{x_\omega}} \lesssim |\Omega|^{\frac12}\|w_0\|_{L^2_x} \tag{14}$$

In addition for any $\theta \notin \Omega$

$$\|w\|_{L^\infty_{u_\theta} L^2_{x_\theta}} \lesssim \operatorname{dist}(\theta,\Omega)^{-1}\|w_0\|_{L^2_x} \tag{15}$$

The iteration space(6) X is composed from the X_k atoms, where each space X_k contains functions F with spatial Fourier support in the region $|\xi| \sim 2^k$ and

$$\|F\|_{X_k} = \inf\big(\|F_0\|_{H_k^{\frac n2,\frac12,1}} + \sum_{\ell\in\mathbb{Z}_+}\|F_\ell\|_{Y_{k,\ell}}\big),$$

where the infimum is taken with respect to all possible decompositions $F = F_0 + \sum_\ell F_\ell$, the functions F_ℓ are supported at distance $\leq 2^{k-2\ell}$ from the cone $\tau^2 = |\xi|^2$ in Fourier space. Each of the spaces $Y_{k,\ell}$ is also an atomic space, where

$$\|G\|_{Y_{k,\ell}} = 2^{\frac n2 k}\inf\sum_{m=1}^\infty |a_m|,$$

where the infimum is taken with respect to the decompositions $G = \sum_{m=1}^\infty a_m G_m$ and each atom G_m, in addition to the above requirements on its Fourier support, is assumed to verify the following conditions. Let K_ℓ denote a collection of $(2^{2\ell(n-1)})$ spherical caps of size $2^{-\ell}$ covering the unit sphere of directions in ξ space and P_κ denote the associated projection in Fourier space on the cap $\kappa \in K_\ell$. Then we require that for each κ there exists an angle $\omega_\kappa \notin 2\kappa$ such that

$$\sum_{\kappa\in K_\ell}\Big(2^{-2k}(\operatorname{dist}(\omega_\kappa,\kappa))^{-2}\|\Box P_\kappa G_m\|^2_{L^1_{u_{\omega_\kappa}} L^2_{x_{\omega_\kappa}}} + (\operatorname{dist}(\omega_\kappa,\kappa))^2\|P_\kappa G_m\|^2_{L^\infty_{u_{\omega_\kappa}} L^2_{x_{\omega_\kappa}}}\Big) \leq 1.$$

Thus constructed space X is shown to satisfy the following properties:

(1) $X \subset C^0_t \dot B^{\frac n2}_{2,1} \cap C^1_t \dot B^{\frac n2 -1}_{2,1}$ and X contains solutions of the homogeneous wave equation with $(\dot B^{\frac n2}_{2,1} \times \dot B^{\frac n2 -1}_{2,1})$ initial data.

(2) $X \cdot X \subset X$ and $X \cdot \Box X \subset \Box X$ – bilinear estimates.

5. TAO'S RESULT IN HIGHER DIMENSIONS

We now discuss the result contained in [30], in which the ($\dot H^{\frac n2}$ critical) small data global existence problem was completely solved for the wave map equation (1) on Minkowski space $\mathbb{R}^{n+1}$ for the $\mathbb{S}^{m-1}$ target manifold in dimensions $n \geq 5$. The

(6)The description given below is somewhat imprecise as some of the spatial Fourier localizations should be in fact space-time Fourier localizations.

solution of the problem is particularly elegant in this case as the iteration spaces are essentially as in the Strichartz method.

5.1. Tools

A consistent theme in [30] is micro-localization. In this context it simply means that all involved functions(7) are decomposed into parts each oscillating with frequencies in a given dyadic interval and all interactions are viewed in terms of interactions between such individual pieces. The framework for this is provided by the Littlewood-Paley theory and paradifferential calculus. According to it an arbitrary function $f(x)$ can be decomposed with the help of the Littlewood-Paley projections P_k:

$$f = \sum_{k\in\mathbb{Z}} f_k, \qquad f_k = P_k f = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\chi(2^{-k}|\xi|)\hat{f}(\xi)\,d\xi$$

where

$$\sum_{k\in\mathbb{Z}} \chi(2^{-k}r) = 1, \quad \forall r \neq 0$$

and χ is a smooth non-negative cut-off function supported on $[1/2, 2]$. We set

$$f_{<k} = \sum_{m<k} f_m, \qquad f_{>k} = \sum_{m>k} f_m.$$

The following capture usefulness of such decompositions:

$$\| |\nabla|^s f_k\|_{L^p} \sim 2^{ks}\|f_k\|_{L^p}, \qquad \forall\, 1 \le p \le \infty,$$
$$\| |f_k\|_{L^p} \lesssim 2^{k(\frac{n}{q}-\frac{n}{p})}\|f_k\|_{L^q}, \qquad \forall\, 1 \le q \le p \le \infty.$$

The first relation reflects the fact that f_k oscillates at the frequency 2^k, while the second is called the Bernstein inequality and is simply a prototype of the Sobolev inequality.

The product of two different Littlewood-Paley pieces $f^1_{k_1}$ and $f^2_{k_2}$ can be characterized as follows(8):

If $k_1 > k_2$ then the Fourier support $f^1_{k_1} f^2_{k_2}$ is essentially still contained in the dyadic region $|\xi| \sim 2^k$.

If $k_1 = k_2$ then the Fourier support $f^1_{k_1} f^2_{k_2}$ is essentially still contained in the region $|\xi| \le 2^k$.

(7) A wave map $\phi : \mathbb{R}^{n+1} \to \mathbb{S}^{m-1}$ is viewed as an $\mathbb{R}^m$-valued function with $|\phi| = 1$.
(8) For simplicity of exposition we will make no distinction between the relations $k_1 > k_2$ and $k_1 \gg k_2$.

The analysis requires a micro-local version of the Strichartz spaces S_k adapted to frequency localized functions and consistent with the critical $\dot H^{\frac{n}{2}}$ regularity. Define the norm in S_k by

$$\|\psi\|_{S_k} = \sup_{q,p} 2^{(\frac{1}{q}+\frac{n}{p})k}\left(\|\psi\|_{L^q_t L^p_x} + 2^{-k}\|\partial_t\psi\|_{L^q_t L^p_x}\right), \tag{16}$$

where the sup is taken over all Strichartz *admissible* exponents described in (12). The Strichartz estimates applied to a function ψ_k localized at the frequency 2^k imply:

$$\|\psi_k\|_{S_k} \lesssim \|\psi_k[0]\|_{\dot H^{\frac{n}{2}}} + 2^{\frac{n-2}{2}k}\|\Box\psi_k\|_{L^1_t L^2_x} \tag{17}$$

In particular, one easily obtains the following strengthened version of the Strichartz estimates:

$$\left(\sum_{k\in\mathbb{Z}}\|\psi_k\|^2_{S_k}\right)^{\frac12} \lesssim \|\psi[0]\|_{\dot H^{\frac{n}{2}}} + \|\Box\psi\|_{L^1_t \dot H^{\frac{n}{2}-1}}.$$

A very useful notion introduced in [30] is that of an σ-*envelope*: $c = \{c_k\}_{k\in\mathbb{Z}} \in \ell_2$, which has the property that

$$c_k \le 2^{\sigma|k-k'|}c_{k'}, \qquad \forall\, k,k' \in \mathbb{Z}.$$

Any ℓ^2 sequence a lies under a σ-envelope c (for instance)

$$c_k = \sum_{k'\in\mathbb{Z}} 2^{-\sigma|k-k'|}a_{k'}.$$

In particular the sequence $\|\phi[0]_k\|_{\dot H^{\frac{n}{2}}}$ associated with initial data for the wave map ϕ lies under a σ-envelope c with the property that

$$\sum_{k\in\mathbb{Z}} c_k^2 < \epsilon^2.$$

5.2. The setup

Existence, uniqueness and propagation of regularity[(9)] follow from proving estimates in the space $X = S(c)$ defined by the norm

$$\|\phi\|_{S(c)} = \sup_k c_k^{-1}\|\phi_k\|_{S_k}.$$

This means that assuming that

$$\|\phi_k\|_{S_k} \le 2C_0 c_k$$

for some sufficiently large constant C_0 the result will follow if one can prove the stronger estimate

$$\|\phi_k\|_{S_k} \le C_0 c_k.$$

(9) Restriction on the parameter σ limits propagation of regularity to the values of Sobolev exponent $n/2 < s < n/2+\sigma$. It is not however standard that a propagation of regularity in a Sobolev space H^s with $s > n/2$ immediately implies the same statement in all higher Sobolev spaces.

As opposed to the local existence scheme one is not able to prove the estimate

$$\|\Box^{-1}\left(\Gamma(\phi^1)(\partial^\alpha\phi^2\cdot\partial_\alpha\phi^3)\right)\|_{S(c)}\leq C_0$$

for functions $\|\phi^i\|_{S(c)}\leq 2C_0$. The problem requires renormalization!

5.3. Micro-linearization of the wave-map equation

The next step is to project the wave map equation

$$\Box\phi=-\phi(\partial_\alpha\phi\cdot\partial^\alpha\phi)$$

on the frequency 2^k with the help of Littlewood-Paley projection P_k thus deriving an equation for ϕ_k

$$\Box\phi_k=-\sum_{k_1,k_2,k_3}P_k\left(\phi_{k_1}(\partial_\alpha\phi_{k_2}\cdot\partial^\alpha\phi_{k_3})\right).$$

Using scale invariance each ϕ_k can be seen to satisfy the same equation as ϕ_0

$$\Box\phi_0=-\sum_{k_1,k_2,k_3\sim 0}P_0\left(\phi_{k_1}(\partial_\alpha\phi_{k_2}\cdot\partial^\alpha\phi_{k_3})\right) \tag{18}$$

and it suffices to prove that

$$\|\phi_0\|_{S_0}\leq C_0c_0.$$

(This is where the envelope idea becomes very helpful.)

Next one says that F is an acceptable error term if $\|F\|_{L^1_tL^2_x}\leq\epsilon C_0^3c_0$ and thus by Strichartz estimates (17) its contribution to the $\|\phi_0\|_{S_0}$ is less than the allowed C_0c_0 and thus can be discarded.

The main contribution of the nonlinear term in (18) is identified by rewriting (18) in the form

$$\Box\phi_0=-2\phi_{<0}(\partial_\alpha\phi_{<0}\cdot\partial^\alpha\phi_0)+F, \tag{19}$$

and claiming that the term F is an acceptable error. Before tackling F one sets the following

Rules of the game: place the terms with a higher frequency in the norm requiring fewer[(10)] derivatives (e.g. $L^2_tL^4_x$) and terms with a lower frequency in the norm requiring more (e.g. $L^\infty_tL^\infty_x$). Almost all estimates will appear as if they have "extra room" and thus are not scale invariant. This is merely an illusion due to the fact that we measure low and high frequency relative to a fixed benchmark frequency. That is, a low frequency will mean $k<0$ while the high frequency will mean that $k>0$. The actual gain occurs only through relative ratios of frequencies of functions involved in the products and is entirely consistent with an overall scale invariance.

(10) The number of derivatives is reflected in the exponential factors placed in front of the Strichartz norms in the definition of S_k in (16). For example the $L^2_tL^4_x$ norm "costs" only $1/2+n/4$ derivatives while the $L^\infty_tL^\infty_x$ norm requires $n/2$ derivatives.

Using the paradifferential calculus rules for products described above, one sees that the error term essentially contains[11] low-high-high, high-high-high, high-low-high and 0-low-low interactions. For example, the low-high-high interaction can be handled as follows (with the help of the envelope properties):

$$\begin{aligned}\Big\|\sum_{k_2\sim k_3>0}\phi_{<0}(\partial_\alpha\phi_{k_2}\cdot\partial^\alpha\phi_{k_3})\Big\|_{L^1_tL^2_x} &\lesssim \sum_{k_2\sim k_3>0}\|\phi_{<0}\|_{L^\infty_tL^\infty_x}\|\partial_\alpha\phi_{k_2}\|_{L^2_tL^4_x}\|\partial^\alpha\phi_{k_3}\|_{L^2_tL^4_x}\\ &\lesssim \sum_{k_2\sim k_3>0}2^{-(k_2+k_3)\frac{n-2}{4}}\|\phi_{k_2}\|_{S_{k_2}}\|\phi_{k_3}\|_{S_{k_3}}\\ &\lesssim \epsilon C^2c_0\sum_{k_2\sim k_3>0}2^{-(k_2+k_3)\frac{n-2}{4}+\sigma k_2}\lesssim \epsilon C^2 c_0\end{aligned}$$

provided that σ is sufficiently small. Similarly for the high-low-high interactions

$$\begin{aligned}&\Big\|\textstyle\sum_{k_1\sim k_2>0,k_3<0}\phi_{k_1}(\partial_\alpha\phi_{k_2}\cdot\partial^\alpha\phi_{k_3})\Big\|_{L^1_tL^2_x}\\ &\quad\lesssim \sum_{k_1\sim k_2,k_3<0}\|\partial_\alpha\phi_{k_3}\|_{L^\infty_tL^\infty_x}\|\phi_{k_1}\|_{L^2_tL^4_x}\|\partial^\alpha\phi_{k_3}\|_{L^2_tL^4_x}\\ &\quad\lesssim \sum_{k_1\sim k_2>0,k_3<0}2^{k_3-k_1}2^{-(k_1+k_3)\frac{n-2}{4}}\times\|\phi_{k_1}\|_{S_{k_1}}\|\phi_{k_2}\|_{S_{k_2}}\|\phi_{k_3}\|_{S_{k_3}}\lesssim\epsilon^2C^3c_0.\end{aligned}$$

The remaining cases can be treated in a similar fashion.

The principal term $\phi_{<0}(\partial_\alpha\phi_{<0}\cdot\partial^\alpha\phi_0)$ cannot be iterated away by means of Strichartz estimates (or even with the help of more sophisticated spaces and estimates) as can be seen from the following argument. The term $\phi_{<0}$ has to be placed in $L^\infty_tL^\infty_x$, which means that one needs an estimate

$$\|\partial_\alpha\phi_{<0}\cdot\partial^\alpha\phi_0\|_{L^1_tL^2_x}\lesssim\|\phi_{<0}\|_{L^2_tL^p_x}\|\phi_0\|_{L^2_tL^r_x}\lesssim\left(\sum_{k<0}\|\phi_k\|^2_{S_k}\right)^{\frac12}\|\phi_0\|_{S_0},$$

where $1/2=1/p+1/r$. By a scaling argument the term with $\phi_{<0}$ requires $p\geq 2n$. The condition $1/2=1/p+1/r$ then implies that $r\leq 2n/(n-1)$. However the admissible Strichartz exponents (q,r) with $q=2$ lie in the range $[2\frac{n-1}{n-3},\infty]$ inconsistent with $r\leq 2n/(n-1)$.

5.4. The renormalization procedure

One begins with an anti-symmetrization trick, used previously in the context of regularity theory for harmonic maps in [10], [5]. Recall that ϕ takes values in a unit sphere $\mathbb{S}^{m-1}\subset\mathbb{R}^m$ and write relative to the standard coordinates on $\mathbb{R}^m$

$$\phi^i_{<0}(\partial_\alpha\phi^j_{<0}\partial^\alpha\phi^j_0)=B^{ij}_\alpha\partial^\alpha\phi^j_0.$$

(11) We will consistently ignore all the terms arising from the commutators with the Littlewood-Paley projections.

The $m \times m$ matrices B_α can be anti-symmetrized as follows. The expression

$$\partial_\alpha \phi^i_{<0}(\phi^j_{<0}\partial^\alpha \phi^j_0) = \partial_\alpha \phi^i_{<0}(\partial^\alpha \phi^j_{<0}\phi^j_0) - \partial_\alpha \phi^i_{<0}\partial^\alpha(\phi^j_{<0}\phi^j_0).$$

Both of the terms on the right hand side above are error terms similar to F_0 satisfying a good $L^1_t L^2_x$ estimate. In the first term a derivative has been successfully transferred to a lower frequency making in a 0-low-low term. On the other hand the expression

$$(\phi^j_{<0}\phi^j_0) = P_0(|\phi|^2) + P_0(\text{high-high})$$

can be identified, modulo terms contributing to an acceptable error term, with the projection of $|\phi|^2$ on the unit frequencies. Since $\phi \in \mathbb{S}^{m-1}$ we have $|\phi| = 1$ and $P_0(|\phi|^2) = 0$.

This argument allows one to replace

$$\phi^i_{<0}(\partial_\alpha \phi^j_{<0}\partial^\alpha \phi^j_0) = B^{ij}_\alpha \partial^\alpha \phi^j_0 = A^{ij}_\alpha \partial^\alpha \phi^j_0,$$

where A_α are the anti-symmetric matrices

$$A_\alpha = \phi_{<0}(\partial_\alpha \phi_{<0}, \cdot) - \partial_\alpha \phi_{<0}(\phi_{<0}, \cdot)$$

and $(\cdot, \cdot)$ is the standard scalar product on $\mathbb{R}^m$. This anti-symmetry property is crucial for the following renormalization procedure. The unit frequency part of the wave-map ϕ is replaced by a new dynamic variable

$$w = U\phi_0$$

with U an almost orthogonal matrix nonlinearly dependent on ϕ. The map w verifies the equation

$$\Box w = -2UA_\alpha \partial^\alpha \phi_0 + 2\partial_\alpha U \partial^\alpha \phi_0 + \Box U \phi_0$$

and this change of variables is motivated by the attempt to eliminate the troublesome term $A_\alpha \partial^\alpha \phi_0$ by setting

$$\partial_\alpha U = UA_\alpha. \tag{20}$$

Solubility of the above transport equations depends on the Frobenius condition[12]

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] = 0. \tag{21}$$

Given the explicit form of A_α the condition (21) is not satisfied. However,

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] = [A_\alpha, A_\beta] \approx \partial\phi_{<0} \cdot \partial\phi_{<0}\,.$$

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[12] Equation (20) would suggest that A is a trivial $O(n)$ connection, which of course requires that its curvature vanishes. It turns out indeed that A is a low frequency portion of the pull-back connection $\phi_*(\nabla)$ on the bundle $\phi_*(T\mathbb{S}^{m-1})$ over $\mathbb{R}^{n+1}$.

is quadratic in the derivatives of a low frequency part of ϕ (and thus in particular only of size ϵ^2). The equation (20) is then solved approximately with the defect giving rise to an acceptable error term. The construction is recursive with[13]

$$U_k := (P_k\phi(P_{<k}\phi,\cdot) - P_{<k}\phi(P_k\phi,\cdot))\,U_{<k},$$
$$U_{<k} := I + \sum_{k'<k} U_{k'}$$

This construction can be motivated by the following argument. In the original equation for ϕ_0 the term $A_\alpha \partial_0^\phi$ can be replaced by $\partial_\alpha\Lambda\partial^\alpha\phi_0$ with

$$\Lambda = \sum_{k<0} (P_k\phi(P_{<k}\phi,\cdot) - P_{<k}\phi(P_k\phi,\cdot))$$

at the expense of generating an error term with an acceptable $L_t^1L_x^2$ bound: the difference between A_α and $\tilde A_\alpha = \partial_\alpha\Lambda$ involves terms where derivatives transferred to terms with lower frequencies. The equation (20) then reads

$$\partial_\alpha U = \partial_\alpha\Lambda\, U$$

with the solution given by

$$U = e^\Lambda = e^{\sum_{k<0}(P_k\phi(P_{<k}\phi,\cdot)-P_{<k}\phi(P_k\phi,\cdot))} \approx$$

and thus if we set

$$U_{<k} = e^{\sum_{k'<k}\left(P_{k'}\phi(P_{<k'}\phi,\cdot)-P_{<k'}\phi(P_{k'}\phi,\cdot)\right)}$$

we have

$$\begin{aligned} U_k &= \left(e^{(P_k\phi(P_{<k}\phi,\cdot)-P_{<k}\phi(P_k\phi,\cdot))} - I\right) e^{\sum_{k'<k}\left(P_{k'}\phi(P_{<k'}\phi,\cdot)-P_{<k'}\phi(P_{k'}\phi,\cdot)\right)} \\ &\approx (P_k\phi(P_{<k}\phi,\cdot) - P_{<k}\phi(P_k\phi,\cdot))\,U_{<k}. \end{aligned}$$

It remains to show that the remaining terms on the right hand side of the equation for w are acceptable error terms so that

$$\Box w = F \tag{22}$$

and that the transformation U preserves the space S_0. The latter follows from the estimates

$$\|U\|_{L_t^\infty L_x^\infty} \lesssim 1, \qquad \|U\|_{L_t^\infty L_x^\infty}, \qquad \|\partial_\alpha U\|_{L_t^\infty L_x^\infty} \lesssim \epsilon.$$

The standard[14] Strichartz estimates applied to the equation (22) imply that

$$\|w\|_{S_0} \lesssim \|w[0]\|_{L^2} + C_0^3\epsilon c_0 \lesssim \|\phi[0]_0\|_{L^2} + C_0^3\epsilon c_0 \lesssim 2c_0$$

and the desired estimate for ϕ_0 follows.

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(13) For technical reasons the sum in the second definition should extend only to a finite large negative integer $-M$.

(14) not quite, as the Fourier support of w is not supported on unit frequencies, since ϕ_0 has been distorted by U. This can be corrected however without too much difficulty.

6. TAO'S RESULT IN LOWER DIMENSIONS

The (critical) small data global existence result for the wave map problem with the $\mathbb{S}^{m-1}$ target for the remaining dimensions $2 \leq n \leq 4$ is contained in [31]. Given the space constraints it is very difficult to do the justice to that paper here. The proof is a true tour de force. We will only give a (superficial) description of the spaces used in the proof.

As the problem is shifted to the lower dimensions the dispersion properties of the wave equation become weaker and one starts "losing" various key Strichartz estimates (e.g. $L_t^2 L_x^4$ in dimensions $n \leq 4$ and even the $L_t^2 L_x^\infty$ in dimensions $n \leq 3$). This eventually means that the renormalization procedure has to be combined with a refinement of ideas that led to the solution of the division problem. In particular the null structure of the nonlinearity, which did not have much effect on the higher dimensional problem, becomes crucial. In the iteration procedure the Strichartz based spaces are replaced by a combination of the $H^{\frac{n}{2},\frac{1}{2}}$ and null frame spaces. The iteration space $X = S(c)$, associated with an envelope $c = \{c_k\}$, is built of the following parts:

$$\|\phi\|_{S(c)} = \|\phi\|_{L_t^\infty L_x^\infty} + \sup_k c_k^{-1} \|\phi\|_{S[k]}.$$

The first L^∞ component is very important for the algebra property and reflects the fact that the wave map $\phi \in \mathbb{S}^m$ and thus $|\phi| = 1$. The dyadic spaces $S[k]$ in turn are defined by the norm

$$\|\phi\|_{S[k]} := \|\nabla_{x,t}\phi\|_{L_t^\infty \dot{H}^{\frac{n}{2}-1}} + \|\nabla_{x,t}\phi\|_{\dot{H}_k^{\frac{n}{2}-1,\frac{1}{2},\infty}} + \sup_{\pm} \sup_{\ell > 0} \left(\sum_{\kappa \in K_\ell} \|P_{k,\pm\kappa} Q^{\pm}_{<k-2\ell} \phi\|^2_{S[k,\kappa]} \right)^{\frac{1}{2}}.$$

The first term on the right represents the usual energy space. The second is an $H^{s,\delta}$ type space where the index ∞ reflects an ℓ^∞ norm with respect to a dyadic distance to the cone $|\tau|^2 = |\xi|^2$. The last ingredient is a null frame space built with the help of the Fourier projections $P_{k,\pm\kappa}$ and $Q_j^{\pm}$ restricting the Fourier transform of a function to the region of (τ, ξ) with $|\tau \pm |\xi|| \sim 2^j$ (coming from the Q_j action) $|\xi| \sim 2^k$ (coming from P_k part) and a spherical cap $\xi/|\xi| \in \kappa$ of size $2^{-\ell}$ for $\kappa \in K_\ell$. Finally the space $S[k, \kappa]$ is defined by the norm

$$\|\phi\|_{S[k,\kappa]} = 2^{\frac{nk2}{2}} \|\phi\|_{NFA^*[\kappa]} + |\kappa|^{-\frac{1}{2}} 2^{\frac{k}{2}} \|\phi\|_{PW[\kappa]} + 2^{\frac{nk}{2}} \|\phi\|_{L_t^\infty L_x^2},$$

where

$$\|\phi\|_{NFA^*[\kappa]} = \sup_{\omega \notin 2\kappa} \operatorname{dist}(\omega, \kappa) \|\phi\|_{L^\infty_{u_\omega} L^2_{x_\omega}},$$

is motivated by the property (15) of free waves, and the space $PW[\kappa]$ is an atomic Banach space whose atoms are functions ϕ with

$$\|\phi\|_{L^2_{u_\omega} L^\infty_{x_\omega}} \leq 1$$

for some $\omega \in \kappa$ and its definition is motivated by the property (14).

One of the important new ingredients is an appearance of a true trilinear estimate which deals with the nonlinearity $\phi(\partial_\alpha \phi \cdot \partial^\alpha \phi)$ and provides an exponential gain in the ratio of frequencies in the case where the frequency of the first term is larger than one of the other two frequencies.

7. EXTENSIONS TO OTHER TARGET MANIFOLDS

The work in [30], [31] has already had a serious impact on the field. In particular a lot of effort has been concentrated on the extension of the (critical) small data global existence result to other target manifolds.

In a more intrinsic interpretation of the wave map problem it is cast as a system of equations for the derivatives of the wave map ϕ. For a given orthonormal e_a on $(\mathscr{N}, h)$ denote

$$\phi^a_\alpha = h(\partial_\alpha \phi, e_a).$$

We set $A^a_{b\alpha} = h(\nabla_{\phi^*(\partial_\alpha)} e_a, e_b)$ to be the pull-back of the Levi-Civita connection ∇ on $(\mathscr{N}, h)$ to the bundle $\phi^*(T\mathscr{N})$, represented by anti-symmetric matrices A_α. Then if ϕ is a wave map the components $\Phi = (\phi^a_\alpha)$ satisfy the equation

$$\Box \Phi = -2A_\alpha \cdot \partial^\alpha \Phi + E, \tag{23}$$

where E is a term cubic in Φ. The problem (23) can be micro-linearized similar to (19). The low frequency of the connection A_α is split with the help of a Hodge decomposition and a Coulomb gauge. Its gradient part then renormalized following Tao's approach. This led to an extension of Tao's result to a large class of target manifolds in dimensions $n \geq 5$ in [14]. Intuitively, in this picture, the procedure of a Hodge decomposition and renormalization corresponds to a choice of an orthonormal frame e_a with the property that the connection A_α, $\alpha = 0, ..., n$ satisfies the Coulomb gauge condition $\sum_{i=1}^n \partial^i A_i = 0$. This was made even more explicit in [25], where Tao's work was extended to more general targets in dimensions $n \geq 4$. The geometric choice of a global Coulomb gauge avoided a micro-linearization of the equation. Similar extension for $n \geq 4$ was obtained in [21]. In [16], [17] the results were extended to general targets in dimensions $n \geq 3$ and a hyperbolic space $\mathbb{H}^2$ in dimension $n \geq 2$. In [35] the result was extended to targets isometrically embedded (with bounded geometry) into $\mathbb{R}^m$ in dimensions $n \geq 2$.

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