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7. FORMAL MODULI OF FORMAL $\mathcal{O}_K$-MODULES

by

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Abstract. — We define formal $\mathcal{O}_K$-modules and their heights, following Drinfeld. To describe their universal deformations we introduce a formal cohomology group.

***Résumé* (Espaces de modules formels de $\mathcal{O}_K$-modules formels).** — On définit les $\mathcal{O}_K$-modules formels et leurs hauteurs, suivant Drinfeld. Pour décrire leurs déformations universelles, on introduit un groupe de cohomologie formelle.

Notation. — *Except in the proof of Lemma 2.1, all constant coefficients of power series are assumed to be 0.*

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1. Formal modules

Let A, R be commutative rings with 1 and $i : A \to R$ a homomorphism. We also write a instead of $i(a)$ for the image of a under i.

Definition 1.1

1. A *formal A-module over R* is a commutative formal group law $F(X, Y) = X + Y + \cdots \in R[[X, Y]]$ together with a ring homomorphism $\gamma : A \to \mathrm{End}_R(F)$ such that the induced map $A \to \mathrm{End}_R(\mathrm{Lie}F) \cong R$ is equal to the structure map i.
2. For $a \in A$ we write $\gamma(a)(X) = [a]_F(X) = aX + \cdots \in R[[X]]$ for the corresponding endomorphism of F. We will also use the notation $X +_F Y$ instead of $F(X, Y)$.

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3. A *homomorphism of formal A-modules over R* is a homomorphism $\varphi(X): F(X,Y) \to G(X,Y)$ of formal group laws $F(X,Y), G(X,Y)$ over R such that $\varphi \circ \gamma_F(a) = \gamma_G(a) \circ \varphi$ for all $a \in A$. Denote by $\mathrm{Hom}_R(F,G)$ the set of homomorphisms from F to G.

Definition 1.2. — For $r \geq 2$ let $\nu_r = p$, if r is a power of a prime p, and $\nu_r = 1$ else. Denote by

$$C_r(X,Y) = \frac{1}{\nu_r}((X+Y)^r - X^r - Y^r)$$

the *modified binomial form* of degree r.

Consider the functor which assigns to every A-Algebra R the set of formal A-modules over R. It is represented by an algebra Λ_A which is generated by the indeterminate coefficients of the series F and $\gamma(a)$ and whose relations are those which are required by the condition that (F,γ) is a formal module. It has a natural grading: the degree of a coefficient is one less than the degree of the corresponding monomial in X, Y. It is induced by the action of $\mathbb{G}_m$ on $\mathrm{Spf}(A[[t]])$. From this description (or by an elementary calculation) one sees that the grading is compatible with concatenation of power series. The elements of the form ab with $\deg a, \deg b \geq 1$ generate a homogeneous ideal. Let $\tilde{\Lambda}_A$ be the quotient with induced grading $\tilde{\Lambda}_A = \bigoplus \tilde{\Lambda}_A^n$.

Denote by $\mathbb{G}_{a,R}$ the additive formal group law over R. With the canonical R-action $\gamma(a) = aX$, it becomes an R-module over R.

Lemma 1.3. — *If A is an infinite field, then for each formal A-module over A there exists a unique isomorphism with $\mathbb{G}_{a,A}$ whose derivative at zero equals* 1. *In this case there is a canonical isomorphism $\Lambda_A \cong A[c_1, c_2, \ldots]$ as graded algebras where $\deg c_i = i$.*

To prove this lemma, one explicitly computes the desired isomorphism, compare [**D**, Prop. 1.2]. The c_i correspond to the coefficients of a homomorphism to the additive formal group law together with the standard A-module structure.

From now on let K be a complete discretely valued field with finite residue field $\mathbb{F}_q$, where $q = p^l$ for some prime p. Denote by $\mathcal{O}_K$ the ring of integers of K. Let π be a uniformizer.

Theorem 1.4. — *$\Lambda_{\mathcal{O}_K}$ and $\mathcal{O}_K[g_1, g_2, \ldots]$ are non-canonically isomorphic as graded algebras where $\deg g_i = i$.*

Proof. — First we show that $\tilde{\Lambda}_{\mathcal{O}_K}^{n-1} \cong \mathcal{O}_K$ as $\mathcal{O}_K$-modules for all $n \geq 2$. For each i let F_i and $[a]_i$ denote the polynomials of degree i obtained from the universal formal module by leaving out all summands of higher degree. We write

$$F_n(X,Y) = F_{n-1}(X,Y) + \sum_{i=1}^{n-1} c_i X^i Y^{n-i}$$

and

$$[a]_n = [a]_{n-1} + h(a) X^n.$$

Then the c_i and $h(a)$ generate $\tilde{\Lambda}^{n-1}_{\mathcal{O}_K}$. As F is a formal group law, we obtain $\sum_{i=1}^{n-1} c_i X^i Y^{n-i} = \alpha C_n(X,Y)$ (compare [**H**, Lemma 1.6.6]). Note that we need here that we consider elements in $\tilde{\Lambda}_{\mathcal{O}_K}$ and not in $\Lambda_{\mathcal{O}_K}$ itself. In particular, $\tilde{\Lambda}^{n-1}_{\mathcal{O}_K}$ is generated by α and $h(a)$. The condition that $\gamma : \mathcal{O}_K \to \mathrm{End}(F)$ is a homomorphism implies that modulo $(X,Y)^{n+1}$ we have

$$[ab]_{n-1}(X) + h(ab)X^n = [a]_{n-1}([b]_{n-1}(X) + h(b)X^n) + h(a)(bX)^n,$$

$$\begin{aligned} F_{n-1}([a]_{n-1}(X) + h(a)X^n, [b]_{n-1}(X) + h(b)X^n) + \alpha C_n(aX, bX) \\ = [a+b]_{n-1}(X) + h(a+b)X^n, \end{aligned}$$

and

$$\begin{aligned} &[a]_{n-1}(F_{n-1}(X,Y) + \alpha C_n(X,Y)) + h(a)(X+Y)^n \\ &\qquad = F_{n-1}([a]_{n-1}(X) + h(a)X^n, [a]_{n-1}(Y) + h(a)Y^n) + \alpha C_n(aX, aY). \end{aligned}$$

In $\tilde{\Lambda}^{n-1}_{\mathcal{O}_K}$ this leads to the relations

$$ah(b) + b^n h(a) = h(ab) \tag{1.1}$$

$$h(a+b) - h(a) - h(b) = \alpha C_n(a,b) \tag{1.2}$$

$$(a^n - a)\alpha = \begin{cases} h(a) & \text{if } n \text{ is not a power of a prime} \\ h(a)p' & \text{if } n = p'^l, \end{cases} \tag{1.3}$$

and these are all relations between the generators α, $h(a)$ of $\tilde{\Lambda}^{n-1}_{\mathcal{O}_K}$. If n is invertible in $\mathcal{O}_K$, then (1.3) shows that each $h(a)$ is a multiple of α. If n is a power of p (where $q = p^l$) but not of q itself, then there exists an $a \in \mathcal{O}_K$ with $a^n - a \notin (\pi)$. From (1.1) we obtain $(a^n - a)h(b) = (b^n - b)h(a)$, thus $h(b)$ is a multiple of $h(a)$. Finally (1.2) shows that α is also a multiple of $h(a)$. Now let n be a power of q. By choosing $h(a) \mapsto (a^n - a)/\pi$ and $\alpha \mapsto p/\pi$ we define an epimorphism of $\mathcal{O}_K$-modules $\tilde{\Lambda}^{n-1}_{\mathcal{O}_K} \to \mathcal{O}_K$. It is well defined as (1.1)-(1.3) are the only relations of $\tilde{\Lambda}^{n-1}_{\mathcal{O}_K}$. It remains to prove that $\tilde{\Lambda}^{n-1}_{\mathcal{O}_K}$ is generated by $h(\pi)$. Let $M = \tilde{\Lambda}^{n-1}_{\mathcal{O}_K}/(h(\pi))$, and denote by $\overline{x} \in M$ the image of $x \in \tilde{\Lambda}^{n-1}_{\mathcal{O}_K}$. Then (1.1) shows that $\pi\overline{h(b)} = \overline{h(\pi b)} = \pi^n \overline{h(b)}$, thus $\overline{h(\pi b)} = 0$ for all $b \in \mathcal{O}_K$. Besides, (1.3) shows $(\pi^n - \pi)\overline{\alpha} = \overline{h(\pi)p} = 0$, hence $\pi\overline{\alpha} = 0$, and M is an $\mathbb{F}_q$-vector space. As n is a power of q, (1.1) reduces to $a\overline{h(b)} + b\overline{h(a)} = \overline{h(ab)}$. This shows

$$\overline{h(a)} = \overline{h(a^n)} = n\overline{a^{n-1}h(a)} = 0$$

for all a. Then (1.2) implies that $C_n(a,b)\overline{\alpha} = 0$ for all $a, b \in \mathbb{F}_q$. By [**H**, Lemma 21.3.2], there is an $x \in \mathbb{F}_p$ with $C_n(x,1) \neq 0$ in $\mathbb{F}_p$. Thus $\overline{\alpha} = 0$ and $M = 0$.

Hence in all cases $\tilde{\Lambda}^{n-1}_{\mathcal{O}_K} \cong \mathcal{O}_K$, and we have an epimorphism of graded algebras $\mathcal{O}_K[g_1, g_2, \dots] \to \Lambda_{\mathcal{O}_K}$. Here g_i is a lift of a generator of $\tilde{\Lambda}^i_{\mathcal{O}_K}$. The construction of the isomorphism $\Lambda_K \cong K[c_1, c_2, \dots]$ in Lemma 1.3 implies that the canonical morphism $\Lambda_{\mathcal{O}_K} \otimes K \to K[c_1, c_2, \dots]$ which is compatible with the grading is also surjective. Comparing dimensions one sees that the epimorphism $\mathcal{O}_K[g_1, g_2, \dots] \to \Lambda_{\mathcal{O}_K}$ is an isomorphism. $\square$

2. Heights

Let $\mathcal{O}_K$ be as above and let R be a local $\mathcal{O}_K$-algebra of characteristic p with residue field k.

Lemma 2.1. — *Let F, G be formal $\mathcal{O}_K$-modules over R and let $\alpha \in \mathrm{Hom}_R(F,G) \setminus \{0\}$. Then there is a unique integer $h = \mathrm{ht}(\alpha) \geq 0$ and $\beta \in R[[X]]$ with $\alpha(X) = \beta(X^{q^h})$ and $\beta'(0) \neq 0$. The integer h is called the* height $\mathrm{ht}(\alpha)$ of α.

This lemma is analogous to the corresponding result over a field, compare [**H**, 18.3.1]. For $\alpha = 0$ we set $\mathrm{ht}(\alpha) = \infty$.

Proof. — We first show that $\alpha(X) = \beta(X^{p^n})$ for some β with $\beta'(0) \neq 0$. To do this we assume $\alpha(X) \neq 0$ with $(\partial\alpha/\partial X)(0) = 0$ and show that $\alpha(X) = \beta(X^p)$ for some homomorphism β of (not necessarily the same) formal group laws. The claim then follows by induction.

Partial differentiation of $\alpha(F(X,Y)) = G(\alpha(X),\alpha(Y))$ with respect to Y gives

$$\frac{\partial\alpha}{\partial X}(F(X,Y))\frac{\partial F}{\partial Y}(X,Y) = \frac{\partial G}{\partial Y}(\alpha(X),\alpha(Y))\frac{\partial\alpha}{\partial X}(Y).$$

Substituting $Y = 0$ and using $(\partial\alpha/\partial X)(0) = 0$ we obtain

$$\frac{\partial\alpha}{\partial X}(X)\frac{\partial F}{\partial Y}(X,0) = 0.$$

As $(\partial F/\partial Y)(X,0) = 1 + a_1X + \cdots \in R[[X]]^\times$, we obtain $\frac{\partial\alpha}{\partial X}(X) = 0$. Hence $\alpha(X) = \beta(X^p)$ for some $\beta \in R[[X]]$. Let σ_*F be the formal group law obtained from F by raising each coefficient to the pth power. Then an easy calculation shows that β is a homomorphism from σ_*F to G.

We now have to show that p^n is a power of q. Let $a \in \mathcal{O}_K$. Then

$$[a]_G(\alpha(X)) = \alpha([a]_F(X)) = \beta'(0)i(a)^{p^n}X^{p^n} + \cdots$$

and on the other hand

$$[a]_G(\alpha(X)) = \beta'(0)i(a)X^{p^n} + \cdots .$$

This implies $\beta'(0)(i(a) - i(a^{p^n})) = 0$ with $\beta'(0) \neq 0$, hence $i(a) - i(a^{p^n}) = i(a - a^{p^n})$ maps to 0 in k. Thus $a^{p^n} = a$ for all $a \in \mathbb{F}_q$ and p^n is a power of q. □

Definition 2.2. — The *height* of a formal $\mathcal{O}_K$-module F over R is

$$\mathrm{ht}(F) = \begin{cases} h & \text{if } [\pi]_F \text{ has height } h \\ \infty & \text{if } [\pi]_F = 0. \end{cases}$$

Remark 2.3. — This definition is different from the definition of height of a formal module given in [**H**], where it is defined as the height of the reduction of the module over the residue field.

Lemma 2.4. — *Let R be as above and let (F, γ_F) be the formal $\mathcal{O}_K$-module corresponding to a homomorphism $\varphi : \Lambda_{\mathcal{O}_K} \to R$. Then $\mathrm{ht}(F) = \min\{i | \varphi(g_{q^i-1}) \neq 0\}$.*

Proof. — In the proof of Theorem 1.4 we identified the generator g_{q^i-1} of $\tilde{\Lambda}^{q^i-1}_{\mathcal{O}_K}$ with the coefficient of X^{q^i} of $[\pi](X)$. □

The following lemma reduces the examination of formal modules over fields and of their deformations to formal modules of an especially simple form. For a proof see [**D**, Prop. 1.7].

Lemma 2.5. — *Let (F, γ) be a formal $\mathcal{O}_K$-module of height $h < \infty$ over a separably closed field k of characteristic $p > 0$. Then F is isomorphic to a formal module (F', γ') over k with*

$$\begin{array}{rcll} F'(X,Y) & \equiv & X+Y & (\operatorname{mod} \deg q^h), \\ [a]_{F'}(X) & \equiv & aX & (\operatorname{mod} \deg q^h), \\ [\pi]_{F'}(X) & = & X^{q^h}. & \end{array}$$

Such modules are called normal modules.

Fix an integer $h > 1$ and let F_0 be a formal $\mathcal{O}_K$-module of height h over k. Assume that R is a local artinian $\mathcal{O}_K$-algebra with maximal ideal $\mathfrak{m}$ and residue field k. Let $I \lhd R$ be an ideal. We set $\overline{R} = R/I$. If F is a lift of F_0 over R, we set $\overline{F} := F \otimes_R \overline{R}$.

Lemma 2.6. — *Let F, G be lifts of F_0 over R. Then the reduction map*

$$\operatorname{Hom}_R(F,G) \to \operatorname{Hom}_{\overline{R}}(\overline{F}, \overline{G}) \tag{2.1}$$

is injective.

Proof. — The reduction map in (2.1) is the composition of finitely many maps

$$\operatorname{Hom}_{R_{n+1}}(F \otimes R_{n+1}, G \otimes R_{n+1}) \to \operatorname{Hom}_{R_n}(F \otimes R_n, G \otimes R_n),$$

where $R_n = R/I_n$ with $I_n = I \cap \mathfrak{m}^n$. We may therefore assume that $\mathfrak{m} \cdot I = 0$. Then I is a finite dimensional k-vector space, and we have $I^2 = 0$. Let $\alpha(X) = a_1 X + a_2 X^2 + \dots$ be a homomorphism from F to G such that $\alpha(X) \equiv 0 \pmod{I}$. We get

$$\alpha([\pi]_F(X)) = [\pi]_G(\alpha(X)) = 0.$$

Since $\operatorname{ht}(F_0) < \infty$, we have $[\pi]_F(X) \neq 0 \pmod{\mathfrak{m}}$, thus $\alpha = 0$ which proves the lemma. □

From now on we may consider $\operatorname{Hom}_R(F,G)$ as a subset of $\operatorname{Hom}_{\overline{R}}(\overline{F}, \overline{G})$.

3. Deformations of modules, formal cohomology

Let F be a formal $\mathcal{O}_K$-module of height $h < \infty$ over k, and let M be a finite dimensional k-vector space. A *symmetric 2-cocycle* of F with coefficients in M is a collection of power series $\Delta(X,Y) \in M[[X,Y]]$ and $\{\delta_a(X) \in M[[X]]\}_{a\in\mathcal{O}_K}$ satisfying

$$\Delta(X,Y) = \Delta(Y,X) \tag{3.1}$$

$$\Delta(X,Y) + \Delta(F(X,Y),Z) = \Delta(Y,Z) + \Delta(X,F(Y,Z)) \tag{3.2}$$

$$\delta_a(X) + \delta_a(Y) + \Delta([a]_F(X),[a]_F(Y)) = i(a)\Delta(X,Y) + \delta_a(F(X,Y)) \tag{3.3}$$

$$\delta_a(X) + \delta_b(X) + \Delta([a]_F(X),[b]_F(X)) = \delta_{a+b}(X) \tag{3.4}$$

$$i(a)\delta_b(X) + \delta_a([b]_F(X)) = \delta_{ab}(X). \tag{3.5}$$

For any $\Psi \in M[[X]]$, the *coboundary of* Ψ is the symmetric 2-cocycle $(\Delta^\Psi, \{\delta_a^\Psi\})$ with

$$\Delta^\Psi(X,Y) \quad = \quad \Psi(F(X,Y)) - \Psi(X) - \Psi(Y) \tag{3.6}$$

$$\delta_a^\Psi(X) \quad = \quad \Psi([a]_F(X)) - i(a)\Psi(X). \tag{3.7}$$

The coboundaries form a subspace of the vector space $Z^2(F,M)$ of symmetric 2-cocycles. The quotient of the symmetric 2-cocycles by the coboundaries is a k-vector space denoted $H^2(F,M)$.

The following lemma is due to Keating, see [**K2**, Lemma 2.1].

Lemma 3.1. — *A cocycle* $(\Delta;\{\delta_a\}) \in Z^2(F,M)$ *is zero if and only if* $\delta_\pi(X) = 0$.

Proof. — If the cocyle is zero, then clearly $\delta_\pi(X) = 0$. Assume conversely that $\delta_\pi(X) = 0$. Substituting $a = \pi$ in (3.3) gives

$$\Delta([\pi]_F(X),[\pi]_F(Y)) = 0,$$

since $\delta_\pi(X) = 0$ and $i(\pi) = 0$. As $[\pi]_F(X) \neq 0$, this implies $\Delta(X,Y) = 0$. Condition (3.5) with $a = \pi$ together with $\delta_\pi(X) = 0$ shows $\delta_{\pi b}(X) = 0$. The same formula with $b = \pi$ and a arbitrary gives $\delta_a([\pi]_F(X)) = 0$. This implies that $\delta_a(X) = 0$, so all components of the cocycle are zero. □

In the following let R denote a local artinian $\mathcal{O}_K$-algebra with maximal ideal $\mathfrak{m}$ and residue field k. Let $I \subseteq \mathfrak{m}$ be an ideal with $\mathfrak{m}I = 0$. Then I is a k-vector space. We set $\overline{R} = R/I$. If F_0 is a formal module over k and F is a lift of F_0 over R, denote by $\overline{F} = F \otimes_R \overline{R}$ the reduction modulo I. The reduction modulo $\mathfrak{m}$ of power series over R is denoted by $\cdot^*$.

Proposition 3.2. — *In the setting above let* F_0 *be a formal* $\mathcal{O}_K$*-module over* k *and let* $F, G \in R[[X,Y]]$ *be formal* $\mathcal{O}_K$*-modules with* $F^* = G^* = F_0$. *For* $\varphi(X) \in R[[X]]$ *let* $\overline{\varphi} \in \overline{R}[[X]]$ *be the image. Assume that* $\overline{\varphi}$ *is a homomorphism from* $\overline{F}$ *to* $\overline{G}$. *Then*

1. *There is an element of* $Z^2(F_0,I)$ *defined by*

$$\Delta = \varphi(F(X,Y)) -_G \varphi(X) -_G \varphi(Y)$$
$$\delta_a = \varphi([a]_F(X)) -_G [a]_G(\varphi(X)).$$

2. $(\Delta;\{\delta_a\}_a) = 0$ *if and only if* $\varphi(X) \in \mathrm{Hom}_R(F,G)$.
3. *The class of* $(\Delta;\{\delta_a\}_a)$ *in* $H^2(F_0,I)$ *is independent of the choice of the lift* φ *of* $\overline{\varphi}$. *It vanishes if and only if* $\overline{\varphi} \in \mathrm{Hom}_R(F,G) \subseteq \mathrm{Hom}_{\overline{R}}(\overline{F},\overline{G})$. *If* $(\Delta;\{\delta_a\})$ *is the coboundary of* ψ, *the lift of* $\overline{\varphi}$ *to a homomorphism over* R *is given by* $\varphi -_G \psi$.

Proof. — Applying φ to the left hand side of the associativity law for F

$$(X +_F Y) +_F Z = X +_F (Y +_F Z) \tag{3.8}$$

and using the definition of Δ, we get

$$\varphi(X) +_G \varphi(Y) +_G \varphi(Z) +_G \Delta(X, Y) +_G \Delta(X +_F Y, Z). \tag{3.9}$$

Applying φ to the right hand side of (3.8), we get

$$\varphi(X) +_G \varphi(Y) +_G \varphi(Z) +_G \Delta(X, Y +_F Z) +_G \Delta(Y, Z). \tag{3.10}$$

From (3.10) and (3.9) we obtain

$$\Delta(X, Y) +_G \Delta(X +_F Y, Z) = \Delta(X, Y +_F Z) +_G \Delta(Y, Z). \tag{3.11}$$

Using the assumption $\mathfrak{m} \cdot I = 0$, we see that (3.11) implies the second cocycle rule

$$\Delta(X, Y) + \Delta(X +_{F_0} Y, Z) = \Delta(X, Y +_{F_0} Z) + \Delta(Y, Z). \tag{3.12}$$

The other cocycle rules are proved in a similar manner, replacing (3.8) by the commutativity resp. the distributivity law of F. This proves 1.

Part 2 of the proposition is a straightforward consequence of the definition of $(\Delta; \{\delta_a\})$. To prove 3., we continue with the notation used in the proof of 1. Let $\varphi'(X)$ be another lift of $\overline{\varphi}$, and let $(\Delta'; \{\delta'_a\})$ be the cocycle it defines. We can write $\varphi' = \varphi +_G \psi$, with $\psi \in I[[X]]$. Then

$$\begin{aligned}\varphi'([\pi]_F(X)) &= [\pi]_G(\varphi(X)) +_G \delta_\pi(X) +_G \psi([\pi]_F(X)) \\ &= [\pi]_G(\varphi'(X)) +_G (\delta_\pi(X) +_G \psi([\pi]_F(X))).\end{aligned}$$

For the second equality we have used that $I\mathfrak{m} = 0$. We conclude that $\delta'_\pi(X) - \delta_\pi(X) = \psi([\pi]_F(X))$ is the π-component of the coboundary of ψ. Then Lemma 3.1 implies that the two cocycles differ by the coboundary of ψ. Hence $(\Delta; \{\delta_a\})$ and $(\Delta'; \{\delta'_a\})$ lie in the same class in $H^2(F_0, I)$. It follows from 2. that this class vanishes if and only if $\overline{\varphi} \in \mathrm{Hom}_R(F, G)$. This completes the proof of 3. and the proposition. ☐

Lemma 3.3. — *In the setting of Proposition 3.2 let (F, γ) be a lift of F_0 to R and let $\overline{F}$ be the reduction to $\overline{R}$.*

1. *Proposition 3.2 defines a bijection between deformations of $\overline{F}$ to R and cocycles in $Z^2(F_0, I)$. Its inverse is given by assigning to $(\Delta; \{\delta_a\})$ the deformation $F_\Delta(X, Y) = X +_F Y +_F \Delta(X, Y)$ and $\gamma_\delta(a) = \gamma(a) +_F \delta_a$.*
2. *Two cocycles are in the same cohomology class if and only if the corresponding deformations are isomorphic via an isomorphism which lifts the identity of $\overline{F}$.*

Proof. — For the first assertion we have to check that $(F_\Delta, \gamma_\delta)$ is a formal module. From $I^2 = 0$ we obtain that the equations (3.1) to (3.5) also hold with F replaced by F_Δ. These equations immediately imply that $(F_\Delta, \gamma_\delta)$ is a formal module. For F_Δ, F and $\varphi = X$ we obtain the cocycle $(\Delta, \{\delta_a\})$. Then the second assertion follows from Proposition 3.2, 3. ☐

Corollary 3.4. — *Let F_0, R, and I be as above with $\mathrm{char}(R) = p$, $\mathrm{ht}(F_0) = h$, and $(\Delta; \{\delta_a\}) \in Z^2(F_0, I)$.*

1. *Let $g \leq h$. Then $\delta_\pi(X) \equiv 0 \pmod{X^{q^{g-1}+1}}$ if and only if $\delta_\pi \in I[[X^{q^g}]]$.*

2. *The following are equivalent:*
   (a) *The cocycle* $(\Delta; \{\delta_a\})$ *is the coboundary of some* $\psi(X) \in I[[X]]$.
   (b) $\delta_\pi \in I[[X^{q^h}]]$.
   (c) *Let* (F, γ) *be a lift of* F_0 *to a formal* $\mathcal{O}_K$*-module over* R. *Then the identity of* $\overline{F}$ *lifts to an isomorphism between* (F, γ) *and* $(F_\Delta, \gamma_\delta)$.

   If these conditions are satisfied, $(\Delta; \{\delta_a\})$ *is the coboundary of* $\psi = d \circ \beta^{-1}$ *where* $d(X^{q^h}) = \delta_\pi(X)$ *and* $\beta(X^{q^h}) = [\pi]_{F_0}(X)$.

Proof. — If $\delta_\pi(X) \equiv 0 \pmod{X^{q^{g-1}+1}}$ then

$$[\pi]_{F_\Delta}(X) = \delta_\pi(X) +_F [\pi]_F(X) \equiv 0 \pmod{(X^{q^{g-1}+1})},$$

thus $\mathrm{ht}(F_\Delta) > g - 1$. This shows that $\delta_\pi(X) = [\pi]_{F_\Delta}(X) -_F [\pi]_F(X)$ is a power series in X^{q^g}. The other assertion of 1. is trivial. The equivalence of (a) and (c) of 2. follows from Lemma 3.3. From Lemma 3.1 we see that $(\Delta; \{\delta_a\}) = (\Delta^\psi; \{\delta_a^\psi\})$ for some ψ if and only if $\delta_\pi(X) = \delta_\pi^\psi(X) = \psi([\pi]_F(X)) = \psi([\pi]_{F_0}(X))$. Here the last two equations follow from $I\mathfrak{m} = 0$. As $\mathrm{ht}(F_0) = h$, this implies (b). On the other hand assume (b) and let $d(X^{q^h}) = \delta_\pi(X)$ and $\beta(X^{q^h}) = [\pi]_{F_0}(X)$. Then the π-component of the coboundary of $\psi = d \circ \beta^{-1}$ is δ_π. □

Let $\hat{\mathcal{O}}_K^{nr}$ be the completion of the maximal unramified extension of $\mathcal{O}_K$. Denote by $\hat{\mathcal{O}}_K^{nr}[[t]] = \hat{\mathcal{O}}_K^{nr}[[t_1, \ldots, t_{h-1}]]$ the power series ring over $\hat{\mathcal{O}}_K^{nr}$ in $h - 1$ variables. Let $k = \hat{\mathcal{O}}_K^{nr}/(\pi)$.

Lemma 3.5. — *Let* (F, γ_F) *be a normal* $\mathcal{O}_K$*-module over* k *of height* $h < \infty$. *Then there exists a formal* $\mathcal{O}_K$*-module* (Γ, γ) *over* $\hat{\mathcal{O}}_K^{nr}[[t]]$ *which over* k *reduces to* F *with the following property: For* $1 \le i \le h - 1$ *denote by* (Γ_i, γ_i) *the reduction to* $\hat{\mathcal{O}}_K^{nr}[[t]]/(t_1, \ldots, t_{i-1})$. *Then*

$$\gamma_i(\pi)(X) \equiv \pi X + t_i X^{q^i} \pmod{\deg(q^i + 1)}. \tag{3.13}$$

Proof. — The module F corresponds to a map $\overline{\varphi} : \Lambda_{\mathcal{O}_K} \cong \mathcal{O}_K[g_1, g_2, \ldots] \to k$ with $g_i \mapsto 0$ for all $i < q^h - 1$. Let $\varphi : \Lambda_{\mathcal{O}_K} \to \hat{\mathcal{O}}_K^{nr}$ be a lift with the same property. We choose

$$f_i = \begin{cases} t_j & \text{if } i = q^j - 1 \text{ with } 1 \le j < h - 1 \\ \varphi(g_i) & \text{else.} \end{cases}$$

Let Γ be the formal $\mathcal{O}_K$-module corresponding to the map $\Lambda_{\mathcal{O}_K} \to \hat{\mathcal{O}}_K^{nr}[[t]]$ which maps g_i to f_i. Then for (Γ_i, γ_i) we see that g_{q^i-1} is the first generator which is mapped to a nonzero element in $\hat{\mathcal{O}}_K^{nr}[[t]]/(t_1, \ldots, t_{i-1})$. From the description of $\tilde{\Lambda}_{\mathcal{O}_K}^{q^i-1}$ in the proof of Theorem 1.4 we see that $\gamma_i(\pi)(X)$ has the desired form. □

Note that a proof of this result can also be found in [**GH**, Section 12].

Let (F, γ_F) be a normal formal $\mathcal{O}_K$-module of height $h < \infty$ over k. Let (Γ, γ) be the deformation over $\hat{\mathcal{O}}_K^{nr}[[t]]$ defined in Lemma 3.5. Let (Γ^i, γ^i) be the reduction of (Γ, γ) to $k[[t_i]]/(t_i)^2 = R_i$ and let $(F, \gamma_F)_{R_i}$ be the base change of (F, γ_F) to R_i.

Proposition 3.6. — *For F as above we have $\dim_k H^2(F,k) = h-1$. The cocycles $(\Delta^i;\{\delta^i_a\})$ associated to the pairs of deformations $(F,\gamma_F)_{R_i}$ and (Γ^i,γ^i) with values in $t_iR_i \cong k$ satisfy*

$$\delta^i_\pi \equiv t_i X^{q^i} \pmod{\deg q^i + 1}. \tag{3.14}$$

Their classes form a basis for $H^2(F,k)$.

Proof. — Equation (3.14) immediately follows from (3.13). Corollary 3.4, 2. shows that the π-components of coboundaries are power series in X^{q^h}. Thus (3.14) implies that the classes of the cocycles $(\Delta^i;\{\delta^i_a\})$ are linearly independent in $H^2(F,k)$. Let $(\Delta;\{\delta_a\}) \in H^2(F,k)$. Then by Corollary 3.4, 1., δ_π is of the form $\beta(X^{q^g})$ with $\beta'(0) \neq 0$. If $g<h$ we subtract a suitable multiple of $(\Delta^g;\{\delta^g_a\})$ to annihilate the coefficient of X^{q^g}. In this way we can inductively represent the cocycle $(\Delta;\{\delta_a\})$ as a linear combination of the $(\Delta^i;\{\delta^i_a\})$ plus a cocycle whose π-component is congruent to 0 modulo $X^{q^{h-1}+1}$. Hence by Corollary 3.4, the cohomology class is a linear combination of the classes of the $(\Delta^i;\{\delta^i_a\})$. ☐

Definition 3.7. — Let R be a local ring with maximal ideal $\mathfrak{m}$. For a power series f with coefficients in R let f^* be the reduction modulo $\mathfrak{m}$. A $*$-*isomorphism* between $\mathcal{O}_K$-modules F,G over R is an isomorphism $\varphi \in \mathrm{Hom}_R(F,G)$ with $\varphi^*(X)=X$.

Let F be a fixed $\mathcal{O}_K$-module of height $h<\infty$ over $k=\hat{\mathcal{O}}_K^{nr}/(\pi)$. We consider the functor $\mathcal{D}_F$ which assigns to each complete local noetherian $\hat{\mathcal{O}}_K^{nr}$-algebra R with residue field k and maximal ideal $\mathfrak{m}$ the set of $*$-isomorphism classes of formal $\mathcal{O}_K$-modules over R that modulo $\mathfrak{m}$ reduce to F.

***Theorem 3.8* (Universal deformation).** — *Let (F,γ_F) be an $\mathcal{O}_K$-module over k of height $h<\infty$. Then $\mathcal{D}_F$ is represented by $\hat{\mathcal{O}}_K^{nr}[[t]]$.*

Proof. — As k is separably closed, Lemma 2.5 shows that we may assume (F,γ_F) to be normal. Let (Γ,γ) be the deformation over $\hat{\mathcal{O}}_K^{nr}[[t]]$ of Lemma 3.5. Let $(\Phi,\gamma_\Phi) \in \mathcal{D}_F(R)$ for some complete local noetherian $\hat{\mathcal{O}}_K^{nr}$-algebra R with residue field k and maximal ideal $\mathfrak{m}$. As R is complete, it is enough to show that for each $r\in\mathbb{N}$ the following holds: If the projection Φ_r of Φ to $R/\mathfrak{m}^r$ corresponds to a homomorphism $\varphi_r : \hat{\mathcal{O}}_K^{nr}[[t]] \to R/\mathfrak{m}^r$, then there is a unique lift $\varphi_{r+1} : \hat{\mathcal{O}}_K^{nr}[[t]] \to R/\mathfrak{m}^{r+1}$ of φ_r corresponding to Φ_{r+1}.

Let ψ be any lift of φ_r to $R/\mathfrak{m}^{r+1}[[X]]$. Then the pair of deformations $\psi(\Gamma,\gamma)$, $(\Phi_{r+1},\gamma_{\Phi_{r+1}})$ corresponds to an element of $H^2(F,\mathfrak{m}^r/\mathfrak{m}^{r+1})$, hence to a uniquely defined linear combination of the Δ^i with coefficients a_i in $\mathfrak{m}^r/\mathfrak{m}^{r+1}$. Let $\varphi_{r+1}(t_i) = \psi(t_i)+a_i$. Then by Corollary 3.4, the deformations Φ_{r+1} and $\varphi_{r+1}(\Gamma,\gamma)$ of F over $R/\mathfrak{m}^{r+1}$ are isomorphic via an isomorphism which lifts the given isomorphism over $R/\mathfrak{m}^r$. As the classes of the Δ^i are linearly independent, φ_{r+1} is unique. ☐

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