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11. ENDOMORPHISMS OF QUASI-CANONICAL LIFTS

by

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Abstract. — We present Keating's result on the locus of deformation of an endomorphism of a quasi-canonical lifting. At the same time, this determines the endomorphism ring of the reduction of quasi-canonical liftings to Artin rings.

***Résumé* (Endomorphismes de relèvements quasi-canoniques).** — On donne le résultat de Keating concernant le lieu de déformation d'un endomorphisme d'un relèvement quasi-canonique. En même temps, ceci détermine l'anneau des endomorphismes de la réduction d'un relèvement quasi-canonique à des anneaux artiniens.

In this paper we prove a lifting theorem for endomorphisms of a formal $\mathcal{O}_K$-module to a quasi-canonical lift. For the canonical lift, a similar lifting theorem is proved in [**Ww1**]. This work is due to K. Keating ([**K1**]).

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1. Notation

Let K be a complete discretely valued field, let $\mathcal{O}_K$ be its ring of integers and let π be a uniformizing element of $\mathcal{O}_K$. We will assume that the residue field of $\mathcal{O}_K$ is equal to the field $\mathbb{F}_q$ of characteristic p. Denote by k an algebraic closure of $\mathbb{F}_q$. Let L be a quadratic extension of K and let $A = \widehat{\mathcal{O}}_L^{ur}$ be the completion of the maximal unramified extension of $\mathcal{O}_L$. Denote by M the quotient field of A.

Let F_0 be a formal $\mathcal{O}_K$-module of height 2 over k. By [**Ww1**] Theorem 1.1, the ring of $\mathcal{O}_K$-linear endomorphisms $\operatorname{End}_k F_0$ is isomorphic to the maximal order $\mathcal{O}_D$ in a division algebra D of dimension 4 over K and invariant 1/2. We identify $\operatorname{End}_k F_0$ with $\mathcal{O}_D$. Let F be the canonical lift of F_0 over A with respect to an embedding

$$\mathcal{O}_L \hookrightarrow \mathcal{O}_D.$$

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We consider a quasi-canonical lift F' of F_0 of level s ([**Ww1**, Def. 3.1]). By definition, $\mathrm{End}_{A'} F'$ is an order $\mathcal{O}_s := \mathcal{O}_K + \pi^s \mathcal{O}_L$ in $\mathcal{O}_L$. Note that a quasi-canonical lift of level 0 is a canonical lift and therefore can be defined over A. A quasi-canonical lift of level $s \geq 1$ can be defined over a totally ramified Galois extension M'/M of degree

$$[M' : M] = \begin{cases} q^s + q^{s-1} & \text{if } L/K \text{ is unramified} \\ q^s & \text{if } L/K \text{ is ramified} \end{cases}$$

([**Ww1**, Thm. 3.2]). Denote by A' the ring of integers of M' and denote by π' a uniformizing element of A'. If s is equal to 0, the ring A' is equal to A. Let $e_s = e(A'/\mathcal{O}_K)$ be the ramification index of A' over $\mathcal{O}_K$, *i.e.*,

$$e_s = \begin{cases} 2q^s & \text{if } L/K \text{ is ramified.} \\ q^s + q^{s-1} & \text{if } L/K \text{ is unramified and } s \neq 0. \\ 1 & \text{if } L/K \text{ is unramified and } s = 0. \end{cases}$$

By [**Ww1**, Proposition 4.4 and Proposition 4.6], the endomorphism $[\pi]_{F'}$ is given by a power series

$$[\pi]_{F'} = \pi X + \cdots + uX^q + \cdots + vX^{q^2} + \cdots \in A'[[X]] \tag{1.1}$$

with $v_{\pi'}(u) = 1$ and $v_{\pi'}(v) = 0$.

Denote by $A'_n = A'/(\pi')^{n+1}$ the reduction of A' modulo $(\pi')^{n+1}$ and by $F'_n = F' \otimes_{A'} A'_n$ the reduction of F' to A'_n. We obtain

$$\mathcal{O}_s = \mathrm{End}_{A'} F' \subset \cdots \subset \mathrm{End}_{A'_n} F'_n \subset \cdots \subset \mathrm{End}_k F_0 = \mathcal{O}_D$$

([**VZ**, Lem. 2.6]), hence we will consider $\mathrm{End}_{A'_n} F'_n$ as a subring of $\mathrm{End}_k F_0 = \mathcal{O}_D$. We write $\mathrm{End}\, F'_n$ instead of $\mathrm{End}_{A'_n} F'_n$.

For $n \leq e_s$ the ring $A'/(\pi')^n$ is of characteristic p and one can define the height of the module F'_n ([**VZ**, Def. 2.2]). By construction, F'_n is of height 1 if $0 < n \leq e_s$ and F_0 is of height 2. Denote by a_i the coefficients of $[\pi]_{F'}$. Then $v_{\pi'}(a_i) \geq e_s$ if $q \nmid i$, and $v_{\pi'}(a_i) \geq e_s$ if $q \mid i$ and $q^2 \nmid i$.

2. Results

The goal of this paper is to compute the endomorphism rings $\mathrm{End}\, F'_n$ as subrings of $\mathcal{O}_D$. In the case of the canonical lift, these rings are calculated in [**Ww1**]. Denote by $a(k)$ the rational number

$$a(k) = \frac{(q^k - 1)(q + 1)}{q - 1}$$

for every integer k. We have $a(0) = 0$ and $a(k) = (q + 1)(\sum_{i=0}^{k-1} q^i)$ for $k \geq 1$.

Theorem 2.1. — *Let F' be a quasi-canonical lift of F_0 of level s. Let $l \geq 0$ be an integer and let*

$$f_0 \in (\mathcal{O}_s + \pi_D^l \mathcal{O}_D) \setminus (\mathcal{O}_s + \pi_D^{l+1} \mathcal{O}_D).$$

Then f_0 lifts to $\operatorname{End} F'_{n_l - 1}$ and not to $\operatorname{End} F'_{n_l}$ with

$$n_l = n_l(s) = n_l(L/K, s) = \begin{cases} a(\frac{l}{2}) + 1 & \text{if } l \leq 2s \text{ and } l \text{ even.} \\ a(\frac{l-1}{2}) + q^{\frac{l-1}{2}} + 1 & \text{if } l \leq 2s \text{ and } l \text{ odd.} \\ a(s-1) + q^{s-1} + (\frac{l+1}{2} - s)e_s + 1 & \text{if } l \geq 2s - 1. \end{cases}$$

Remark 2.2. — The rational number n_l of the theorem is an integer. Indeed, if L/K is ramified, the ramification index e_s is even. If L/K is unramified and $l \geq 2s$ is even, then

$$\mathcal{O}_s + \pi_D^l \mathcal{O}_D = \mathcal{O}_s + \pi_D^{l+1} \mathcal{O}_D.$$

Theorem 2.3. — *Consider the same situation as in Theorem 2.1. Then* $\operatorname{End} F'_n = \mathcal{O}_s + \pi_D^{j(n)} \mathcal{O}_D$ *where*

$$j(n) = \begin{cases} 2k & \text{if } n \in]a(k-1) + q^{k-1}; a(k)] \text{ for } k < s. \\ 2k+1 & \text{if } n \in]a(k); a(k) + q^k] \text{ for } k < s. \\ k & \text{if } n \in]a(s-1) + q^{s-1} + (\frac{k}{2} - s)e_s; a(s-1) + q^{s-1} + (\frac{k+1}{2} - s)e_s] \\ & \hfill \text{for } k \geq 2s. \end{cases}$$

Note that the above intervals form a disjoint cover of the set of positive integers. The integer $j(n)$ is uniquely determined unless L/K is unramified and $j(n) \geq 2s$. In this case we have $\mathcal{O}_s + \pi_D^{j(n)} \mathcal{O}_D = \mathcal{O}_s + \pi_D^{j(n)+1} \mathcal{O}_D$ for every even $j(n)$.

Proof. — This theorem follows from Theorem 2.1. □

Remark 2.4. — If F' is the canonical lift of F_0, *i.e.*, if $s = 0$, Theorem 2.1 and Theorem 2.3 have already been proved in [**Ww1**] Theorem 1.4. We obtain in this case

$$\operatorname{End} F_n = \mathcal{O}_L + \pi_L^n \mathcal{O}_D$$

and

$$n_l(0) = \begin{cases} l + 1 & \text{if } L/K \text{ is ramified.} \\ \frac{l+1}{2} & \text{if } L/K \text{ is unramified.} \end{cases}$$

3. Proofs

We will assume in the following that s is greater or equal than 1. We will split the proof of Theorem 2.1 into two propositions similar to the proof of Theorem 1.1 in [**Vi**].

Proposition 3.1. — *Let $l \leq 2s+1$ and let $s \geq 1$. Let*

$$f_0 \in (\mathcal{O}_s + \pi_D^l \mathcal{O}_D) \setminus (\mathcal{O}_s + \pi_D^{l+1}\mathcal{O}_D).$$

Then f_0 lifts to $\operatorname{End} F'_{n_l-1} \setminus \operatorname{End} F'_{n_l}$ *with*

$$n_l = \begin{cases} a(\frac{l}{2}) + 1 & \text{if } l \leq 2s \text{ and } l \text{ even.} \\ a(\frac{l-1}{2}) + q^{\frac{l-1}{2}} + 1 & \text{if } l \leq 2s \text{ and } l \text{ odd.} \\ a(s-1) + q^{s-1} + e_s + 1 & \text{if } l = 2s+1. \end{cases}$$

Proposition 3.2. — *Let $s \geq 1$ and let $f_0 \in \operatorname{End} F'_{n-1} \setminus \operatorname{End} F'_n$ with $n \geq \frac{e_s - 1}{q-1}$. Then $[\pi] \circ f_0$ lifts to $\operatorname{End} F'_{n'-1} \setminus \operatorname{End} F'_{n'}$ with $n' = n + e_s$.*

Proof of Theorem 2.1. — Theorem 2.1 follows by induction from Proposition 3.1 and Proposition 3.2. Let $l > 2s+1$ and let $f_0 \in (\mathcal{O}_s + \pi_D^l \mathcal{O}_D) \setminus (\mathcal{O}_s + \pi_D^{l+1}\mathcal{O}_D)$. Write $f_0 = c + [\pi]_{F_0} \circ g_0$ with $c \in \mathcal{O}_s$ and $g_0 \in \pi_D^{l-2}\mathcal{O}_D \setminus (\mathcal{O}_s + \pi_D^{l-1}\mathcal{O}_D)$. By induction g_0 lifts to $\operatorname{End} F'_{n_{l-2}-1} \setminus \operatorname{End} F'_{n_{l-2}}$ with

$$\begin{aligned} n_{l-2} &= a(s-1) + q^{s-1} + (\frac{l-1}{2} - s)e_s + 1 \\ &= \frac{2q^s - 2}{q-1} + (\frac{l-1}{2} - s)e_s \geq \frac{e_s - 1}{q-1}. \end{aligned}$$

By Proposition 3.2 the endomorphism $[\pi]_{F_0} \circ g_0$, hence f_0, lifts to $\operatorname{End} F'_{n'-1} \setminus \operatorname{End} F'_{n'}$ with $n' = n_{l-2} + e_s = n_l$. □

Remark 3.3. — We now split the proof of Proposition 3.1 into two cases. As we will see below, we can use the results of [**Vi**] in the case $n_l + 1 \leq e_s$. Note that n_l is a strictly increasing sequence.

An easy computation shows that there exists an integer l_0 such that $n_{l_0} + 1 \leq e_s < n_{l_0+1}$. We obtain

- $l_0 = 2s$ if L/K is ramified and $q \geq 3$.
- $l_0 = 2s - 1$ if $\begin{cases} L/K \text{ is unramified and } q \geq 3. \\ L/K \text{ is unramified, } q = 2 \text{ and } s = 1. \\ L/K \text{ is ramified and } q = 2. \end{cases}$
- $l_0 = 2s - 2$ if L/K is unramified, $q = 2$ and $s \neq 1$.

Proof of Proposition 3.1 in the case of $n_l + 1 \leq e_s$. — Since $A'/\widehat{\mathcal{O}}_K$ is a totally ramified extension of ramification index e_s, we obtain for $n \leq e_s$ an isomorphism of $\mathcal{O}_K$-algebras

$$\begin{aligned} A'/(\pi')^n &\cong (\widehat{\mathcal{O}}_K/(\pi))[\pi']/(\pi')^n \\ &\cong k[t]/(t)^n. \end{aligned}$$

Let $f_0 \in (\mathcal{O}_s + \pi_D^l \mathcal{O}_D) \setminus (\mathcal{O}_s + \pi_D^{l+1}\mathcal{O}_D)$ with $n_l + 1 \leq e_s$, *i.e.*, with $l \leq l_0$ (Rem. 3.3). Then F'_{n_l} is a lift of F_0 of height 1 over $k[t]/(t)^{n_l+1}$ and we will prove the proposition by using the results of [**Vi**].

We have $\mathcal{O}_s + \pi_D^l \mathcal{O}_D = \mathcal{O}_K + \pi_D^l \mathcal{O}_D$ for $l \leq 2s$ and by an easy computation $\mathcal{O}_s + \pi_D^{2s+1}\mathcal{O}_D = \mathcal{O}_K + \pi_D^{2s+1}\mathcal{O}_D$ if L/K is ramified. Hence [**Vi**] Theorem 1.1, shows that f_0 lifts to $\operatorname{End} F'_{n_l-1} \setminus \operatorname{End} F'_{n_l}$. This proves the proposition in this case. □

3.1. Let f_0 be an element of $\operatorname{End} F'_{n-1}$. By $f_{n-1} \in A'_{n-1}[[X]]$ we always denote the unique lift of f_0 as an endomorphism of F'_{n-1}. Let $f \in A'[[X]]$ be a lift of f_{n-1} as a power series without constant coefficient. As we are interested in endomorphisms of formal groups, we make the general assumption that all power series in this article have no constant coefficient. We write f_k for the residue class of the power series f in $A'_k[[X]]$. Denote by ϵ the commutator

$$\epsilon = f \circ [\pi]_{F'} -_{F'} [\pi]_{F'} \circ f \in A'[[X]]$$

using the additive operation on $A'[[X]]$ induced by F'. Then ϵ has coefficients in $(\pi')^n$ because f_{n-1} is an endomorphism of F'_{n-1}.

The main technique to prove the lifting theorem is the cohomology theory as in [**VZ**]. Denote by I_n the 1-dimensional k-vector space $(\pi')^n/(\pi')^{n+1}$. Consider the cohomology group $H^2(F_0, I_n)$ as in [**VZ**] Chapter 3. For $f_{n-1} \in \operatorname{End} F'_{n-1}$ one can define a cocycle $(\Delta, \{\delta_a\}) \in H^2(F_0, I_n)$. Then f_{n-1} lifts to $\operatorname{End} F'_n$ if and only if $(\Delta, \{\delta_a\}) \equiv 0$, *i.e.*, if and only if δ_π is a power series in X^{q^2} ([**VZ**] Prop. 3.2, Cor. 3.4). We have

$$\delta_\pi = \epsilon \bmod (\pi')^{n+1}.$$

Lemma 3.4. — *The cohomology group $H^2(F_0, I_n)$ is a k-vector space of dimension 1. For a cocycle $(\Delta, \{\delta_a\}) \in H^2(F_0, I_n)$, the element $\delta_\pi = \beta(X^q)$ is a power series in X^q and $(\Delta, \{\delta_a\}) \not\equiv 0$ if and only if $\beta'(0) \neq 0$.*

Proof. — By [**VZ**, Lemma 2.5], every formal module over k is isomorphic to a normal module. Then [**VZ**, Proposition 3.6], shows that $H^2(F_0, k)$ is a k-vector space of dimension 1. A basis is given by a cocycle $(\Delta, \{\delta_a\})$ such that $\delta_\pi = \beta(X^q)$ is a power series in X^q with $\beta'(0) \neq 0$. This proves the lemma. □

Remark 3.5. — Let $f_0 \in \operatorname{End} F'_{n-1}$. By Lemma 3.4 the power series ϵ is a power series in X^q modulo $(\pi')^{n+1}$,

$$\epsilon \equiv aX^q + \dots \bmod (\pi')^{n+1}. \tag{3.1}$$

Furthermore, $v_{\pi'}(a) \geq n$ and $v_{\pi'}(a) = n$ if and only if $f_0 \notin \operatorname{End} F'_n$.

Lemma 3.6. — *Let $f_0 \in \operatorname{End} F'_{n-1}$ and let $k = \min\{n + e_s, 1 + qn\}$. Then $[\pi]_{F_0} \circ f_0$ lifts to $\operatorname{End} F'_{k-1}$.*

(i) *If $1 + qn < n + e_s$, the endomorphism $[\pi]_{F_0} \circ f_0$ lifts to $\operatorname{End} F'_k$.*

(ii) *If* $k = n + e_s$ *and* $f_0 \notin \operatorname{End} F'_n$, *the endomorphism* $[\pi]_{F_0} \circ f_0$ *does not lift to* $\operatorname{End} F'_k$.

Proof. — We use the notations of 3.1. By equation (1.1) we obtain

$$\begin{aligned}[\pi]_{F'} \circ f \circ [\pi]_{F'} -_{F'} [\pi]^2_{F'} \circ f &= [\pi]_{F'} \circ \epsilon \\ &= \pi\epsilon + \cdots + u\epsilon^q + \cdots + v\epsilon^{q^2} + \dots \end{aligned} \tag{3.2}$$

Since ϵ has coefficients in $(\pi')^n$, we have

$$[\pi]_{F'} \circ \epsilon \equiv 0 \bmod (\pi')^k.$$

Thus $[\pi]_{F'_{k-1}} \circ f_{k-1}$ commutes with $[\pi]_{F'_{k-1}}$, hence it is an element of $\operatorname{End} F'_{k-1}$ ([**VZ**, Cor. 3.1]). We obtain by (3.2)

$$\delta_\pi([\pi]_{F'_{k-1}} \circ f_{k-1}) \equiv \pi\epsilon + \cdots + u\epsilon^q + \cdots + v\epsilon^{q^2} + \dots \bmod (\pi')^{k+1}. \tag{3.3}$$

If $1 + qn < n + e_s$, the power series (3.3) is a power series in X^{q^2} as ϵ is a power series in X^q modulo $(\pi')^{n+1}$. Hence $[\pi]_{F_0} \circ f_0$ lifts to $\operatorname{End} F'_k$.

If $k = n + e_s$, we obtain

$$\delta_\pi([\pi]_{F'_{k-1}} \circ f_{k-1}) \equiv \pi a X^q + \dots \bmod (\pi')^{k+1}$$

with $v_{\pi'}(\pi a) = n + e_s$. Hence $[\pi]_{F'_{n'-1}} \circ f_{n'-1}$ does not lift to $\operatorname{End} F'_k$. $\square$

Proof of Proposition 3.2. — Since $n \geq \frac{e_s-1}{q-1}$, we obtain $\min\{n+e_s, 1+qn\} = n+e_s = n'$. The proposition follows from Lemma 3.6. $\square$

Proof of Proposition 3.1 in the case of $n \geq e_s$. — By Remark 3.3 we have to prove the following cases.

1. L/K unramified and $l = 2s + 1$.
2. L/K ramified and $l = 2s + 1$.
3. L/K ramified, $q = 2$ and $l = 2s$.
4. L/K unramified, $q = 2$, $l = 2s - 1$ and $s \neq 1$.

Note that $l \geq 2$. Let f_0 be an element of $(\mathcal{O}_s + \pi_D^l \mathcal{O}_D) \setminus (\mathcal{O}_s + \pi_D^{l+1}\mathcal{O}_D)$. Write $f_0 = c + [\pi]_{F_0} \circ g_0$ with $c \in \mathcal{O}_s$ and $g_0 \in \pi_D^{l-2}\mathcal{O}_D \setminus (\mathcal{O}_s + \pi_D^{l-1}\mathcal{O}_D)$. Since elements of $\mathcal{O}_s$ lift to $\operatorname{End} F'$, it is enough to show that $[\pi]_{F_0} \circ g_0$ satisfies the claim. As g_0 is an element of $\pi_D^{2s-1}\mathcal{O}_D \setminus (\mathcal{O}_s + \pi_D^{2s}\mathcal{O}_D)$, it lifts to $\operatorname{End} F'_{n-1} \setminus \operatorname{End} F'_n$ with $n = n_{l-2}$. We have

$$n = a(s-1) + q^{s-1} + 1 = \frac{2q^s - 2}{q - 1}. \tag{3.4}$$

In the first case, we obtain $n_{l-2} \geq \frac{e_s-1}{q-1}$ and the claim follows from the case $l = 2s - 1$ from Proposition 3.2.

Now consider the other cases. Note that in these cases $n + 1 \leq e_s$ (Rem. 3.3). Let $n' = n_l$. We have to show that $[\pi]_{F_0} \circ g_0$ lifts to $\operatorname{End} F'_{n'-1} \setminus \operatorname{End} F'_{n'}$. An easy calculation shows that in each case $n' = qn + 2$. By equation (3.4) we see that $e_s + n = qn$ in the second case, and $e_s + n > qn + 2$ in the other cases. Now we can use Lemma 3.6 (ii) to see that $\pi \circ g_0$ lifts to $\operatorname{End} F'_{n'-1}$. Let $h_{n'-1} \in \operatorname{End} F'_{n'-1}$ be a lift of $[\pi]_{F'_0} \circ g_0$. It remains to show that $h_{n'-1}$ does not lift to $\operatorname{End} F'_{n'}$, *i.e.*, $\delta_\pi(h_{n'-1}) \in A'_{n'}[[X]]$ is not equal to zero modulo $(X)^{q^2}$.

Let $h_{n'} \in A'_{n'}[[X]]$ be a lift of $h_{n'-1}$ as a power series. Then $h_{n'} = [\pi]_{F'_{n'}} \circ g_{n'} +_{F'_{n'}} \psi$ with a power series $\psi = bX + \cdots \in (\pi')^{n'-1}[[X]]$. Using the notation of 3.1, we obtain

$$\begin{aligned} \delta_\pi(h_{n'-1}) &= \delta_\pi(\pi_{F'_{n'-1}} \circ g_{n'-1}) +_{F'_{n'}} \psi \circ [\pi]_{F'_{n'}} -_{F'_{n'}} [\pi]_{F'_{n'}} \circ \psi \\ &\equiv [\pi]_{F'_{n'}} \circ \epsilon +_{F'_{n'}} \psi \circ [\pi]_{F'_{n'}} \bmod (\pi')^{n'+1}. \end{aligned} \tag{3.5}$$

By (3.1) we obtain from equation (3.5)

$$\delta_\pi(h_{n'-1}) \equiv (\pi a + bu)X^q + \ldots \bmod (\pi')^{n'+1}.$$

It is sufficient to prove the following claim.

Claim. — We have

$$\pi a + bu \not\equiv 0 \bmod (\pi')^{n'+1}.$$

Indeed, we have $\delta_\pi(h_{n'-1}) \equiv 0 \bmod (\pi')^{n'}$ since $h_{n'-1}$ is an endomorphism. We obtain from equation (3.5) that

$$\begin{aligned} \delta_\pi(h_{n'-1}) &\equiv (u\epsilon^q + \cdots + v\epsilon^{q^2} + \ldots) +_{F'_{n'}} (bvX^{q^2} + \ldots) \bmod (\pi')^{n'} \\ &\equiv (ua^q + bv)X^{q^2} + \ldots \bmod (\pi')^{n'}, \end{aligned}$$

hence we have

$$ua^q + bv \equiv 0 \bmod (\pi')^{n'}. \tag{3.6}$$

Since $v_{\pi'}(a) = n$ (Rem. 3.5) and $n' = qn + 2$, we obtain that $v_{\pi'}(b) = n' - 1$.

We first consider the last two cases. In these cases, we have $e_s + n > n'$. Therefore, $\pi a \equiv 0 \bmod (\pi')^{n'+1}$ and the claim is satisfied. Thus the proposition is proved in these cases.

Now consider the second case. Let

$$g = \alpha X + \cdots \in A'[[X]].$$

Since $n + 1 \leq e_s$, we obtain from the definition of ϵ

$$\epsilon \equiv u(\alpha - \alpha^q)X^q + \ldots \bmod (\pi')^{n+1},$$

hence

$$a = u(\alpha - \alpha^q).$$

As $v_{\pi'}(a) = n$, we have $v_{\pi'}(\alpha) = n - 1$.

Using equation (3.6), we obtain

$$\pi a + bu \equiv \pi a - v^{-1}u^2 a \equiv \pi u\alpha - v^{-1}u^{q+2}\alpha^q \bmod (\pi')^{n'+1}.$$

The idea is to analyze the solutions of the equation

$$\pi\alpha - v^{-1}u^{q+1}\alpha^q \equiv 0 \bmod (\pi')^{n'}. \tag{3.7}$$

There are q different solutions of this equation for $\alpha \in (\pi')^{n-1}/(\pi')^n$. We will identify these solutions as first coefficients of endomorphisms corresponding to elements of $\mathcal{O}_s$.

Consider the following general situation. Let f_0 and f_0' be two elements of $\pi_D^{2s+1}\mathcal{O}_D$ which are not equivalent modulo $\pi_D^{2s+2}\mathcal{O}_D$. As before, we write $f_0 = [\pi]_{F_0'} \circ g_0$ and $f_0' = [\pi]_{F_0'} \circ g_0'$. We obtain

$$g_0 - g_0' \in \pi_D^{2s-1}\mathcal{O}_D \setminus \pi_D^{2s}\mathcal{O}_D = \pi_D^{2s-1}\mathcal{O}_D \setminus (\mathcal{O}_s + \pi_D^{2s}\mathcal{O}_D).$$

Hence the endomorphism $g_0 - g_0'$ lifts to $\operatorname{End} F_{n-1}' \setminus \operatorname{End} F_n'$. Write $g = \alpha X + \dots$ and $g' = \alpha' X + \dots$ as before. We obtain $v_{\pi'}(\alpha - \alpha') = n - 1$, hence α and α' are not equivalent modulo $(\pi')^n$. Thus different equivalence classes of endomorphisms belong to different equivalence classes of coefficients. As L/K is a ramified extension in the division algebra D, we have

$$\begin{aligned}(\mathcal{O}_s \cap \pi_D^{2s+1}\mathcal{O}_D)/\pi_D^{2s+2}\mathcal{O}_D &= ((\mathcal{O}_K + \pi^s\mathcal{O}_L) \cap \pi_D^{2s+1}\mathcal{O}_D)/\pi_D^{2s+2}\mathcal{O}_D \\ &= \pi^{s+1}\mathcal{O}_L/(\pi_D^{2s+2}\mathcal{O}_D \cap \pi^{s+1}\mathcal{O}_L) \\ &= \mathcal{O}_L/\pi_L\mathcal{O}_L \cong \mathbb{F}_q.\end{aligned}$$

Thus the q different solutions of (3.7) correspond to the equivalence classes of endomorphisms of $\mathcal{O}_s$ in $\operatorname{End} F_{n'}'$. By our assumption $[\pi]_{F_0'} \circ g_0 \notin \mathcal{O}_s + \pi_D^{2s+2}\mathcal{O}_D$, hence equation (3.7) is not satisfied which proves the claim. □

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