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15. CALCULATION OF REPRESENTATION DENSITIES

by

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Abstract. — We calculate for all primes $p \geq 2$ the local representation density of a ternary quadratic form Q over $\mathbb{Z}_p$ in a quadratic space of the form $N \perp H^r$, where N is a quadratic space of rank 4, H is the hyperbolic plane, and $r \geq 0$ is any non-negative integer. Our principal tool is a formula of Katsurada. This defines a rational function $f_{Q,N}$ in p^{-r}. We also determine the derivative of $f_{Q,N}$ and relate it to the arithmetic intersection number of three modular correspondences.

***Résumé* (Calcul de densités de représentation).** — On calcule, pour tous les nombres premiers $p \geq 2$, la densité de représentation locale d'une forme quadratique ternaire Q sur $\mathbb{Z}_p$ dans un espace quadratique de la forme $N \perp H^r$, où N est un espace quadratique de rang 4, H est le plan hyperbolique, et r est un entier ≥ 0. Notre outil principal est une formule de Katsurada. Elle est donnée par une fonction rationnelle $f_{Q,N}$ en p^{-r}. Nous déterminons également la dérivée de $f_{Q,N}$ et nous la relions au nombre d'intersection arithmétique de trois correspondances modulaires.

Introduction

In this note we consider local representation densities of ternary quadratic spaces and derivatives of associated rational functions. These results are used in [**RW**] to relate the arithmetic intersection number of three modular correspondences $(\mathcal{T}_{m_1} \cdot \mathcal{T}_{m_2} \cdot \mathcal{T}_{m_3})$ to a Fourier coefficient of the restriction of the derivative at $s = 0$ of a Siegel–Eisenstein series of genus 3 and weight 2. We also obtain an explicit formula for the integers $\beta_l(Q)$ which occur in [**Go2**].

Let Q and N be quadratic spaces over $\mathbb{Z}_p$ of rank 3 and 4 respectively, and let H be the hyperbolic plane over $\mathbb{Z}_p$. Denote by $\alpha_p(Q, N \perp H^r)$ the local representation density, compare [**Wd1**, 4.3]. This is a rational function $f_{Q,N}(X)$ in $X = p^{-r}$.

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In the first section we consider the case that N is anisotropic and that $r = 0$:

(1) Let D be "the" quaternion division algebra over $\mathbb{Q}_p$ and $N = O_D$ be its maximal order endowed with the reduced norm. Then we compute $\alpha_p(Q, N)$ for any ternary form Q by a direct calculation (Theorem 1.1), following closely [**GK**, section 6].

The value obtained is of course 0 if Q is isotropic, and for anisotropic Q we will see that it does not depend on Q.

In general it is very difficult to compute local representation densities $\alpha_p(Q, N)$, and their computation has a long history. We give only a few references: For $p \neq 2$ a general explicit formula has been given by Hironaka and Sato [**HS**] for arbitrary quadratic spaces Q and N over $\mathbb{Z}_p$. If the rank of Q is 2, Yang has given a formula for $\alpha_p(Q, N)$ in the case of $p = 2$ [**Y1**]. We will use a result of Katsurada [**Ka**] who calculated $\alpha_p(Q, N)$ for arbitrary p and Q in the case that N is an orthogonal sum of copies of the hyperbolic plane H.

In the second section we are interested in the following values:

(2) Let $N = H^2$. Then we specialize Katsurada's formula for $\alpha_p(Q, N \perp H^r) = \alpha_p(Q, H^{r+2})$ to the case where Q is a ternary form and express it in terms of a refinement of the Gross-Keating invariants (see [**B**]) of the ternary form Q. This is done in 2.11.

(3) For Q (ternary and) isotropic we specialize this formula to $r = 0$ and therefore obtain $\alpha_p(Q, H^2)$ (Proposition 2.1) (for Q anisotropic, $\alpha_p(Q, H^2) = 0$).

(4) Finally we calculate for $N = H^2$ and for Q a ternary anisotropic quadratic form the derivative $\frac{\partial}{\partial X} f_{Q,H^2}(X)$ at $X = 1$ (see 2.16).

We remark that the values obtained in (3) and (4) depend only on the Gross-Keating invariants of the ternary form Q although the value in (2) depends on a refinement of these invariants.

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1. Calculation of the representation density in the anisotropic case

1.1. We fix a prime number p, let D be "the" quaternion division algebra over $\mathbb{Q}_p$, and denote by $N = O_D$ the maximal order of D which we consider as a quadratic space of rank 4 over $\mathbb{Z}_p$ with respect to the reduced norm. Let Q be any ternary quadratic form over $\mathbb{Z}_p$. In this section we are going to calculate the representation density $\alpha_p(Q, N)$.

As N is an anisotropic quadratic space, Q is represented by N if and only if Q is anisotropic. In this case the result is:

Theorem 1.1. — *Let Q be anisotropic. Then*

$$\alpha_p(Q,N) = 2(p+1)(1+\frac{1}{p}).$$

1.2. For the proof we quote the following lemma from [**Ki**] Theorem 5.6.4(e):

Lemma 1.2. — *For any integer $r \in \mathbb{Z}$ we have*

$$\alpha_p(p^r Q, N) = \alpha_p(Q,N).$$

1.3. Proof of Theorem 1.1. — By Lemma 1.2 we can assume that the underlying $\mathbb{Z}_p$-module of the quadratic space Q is a sublattice Λ in O_D such that $\Lambda \not\subset pO_D$.

Clearly any element of $\mathrm{O}(D, \mathrm{Nrd})$ preserves N and hence $\mathrm{O}(D, \mathrm{Nrd})$ acts on

$$\tilde{A}_{p^r}(Q,N) := \{\, \sigma\colon Q/p^rQ \longrightarrow N/p^rN \mid \mathrm{Nrd}(\sigma(x)) \equiv Q(x) \bmod p^r \,\}$$

for all $r \geq 1$. By definition (see [**Wd1**, 4.3]) we have

$$\alpha_p(Q,N) = (p^r)^{-6} \# \tilde{A}_{p^r}(Q,N)$$

for r sufficiently large.

The dual lattice of $N = O_D$ with respect to the pairing associated to the quadratic form is $N^\vee = \mathfrak{m}^{-1} \subset D$ where $\mathfrak{m}$ is the maximal ideal of O_D. We claim that the induced action of $\mathrm{SO}(D, \mathrm{Nrd})$ on

$$\tilde{B}_{p^r}(Q,N) := \{\, \sigma\colon \mathbb{Z}_p^3 \longrightarrow N/p^r\mathfrak{m}^{-1}N \mid \mathrm{Nrd}(\sigma(x)) \equiv Q(x) \bmod p^r \,\}$$

is transitive for $r \geq 1$. For this it suffices to show that $\mathrm{SO}(D, \mathrm{Nrd})$ acts transitively on the set M of all isometries $\tilde{\sigma}\colon Q \to N$. But by Witt's lemma, $\mathrm{O}(D, \mathrm{Nrd})$ acts transitively on M. For every such $\tilde{\sigma}$ the stabilizer in $\mathrm{O}(D, \mathrm{Nrd})$ is nothing but the orthogonal group of the orthogonal complement of the quadratic $\mathbb{Q}_p$-space generated by $\tilde{\sigma}(Q)$. As this complement is a one-dimensional space, we see that $\mathrm{SO}(D, \mathrm{Nrd})$ acts in fact simply transitively on M.

Using [**Wd1**, Lemma 1.6] we identify $\mathrm{SO}(D, \mathrm{Nrd})$ with

$$\{\, (d,d') \in D^\times \times D^\times \mid \mathrm{Nrd}(d) = \mathrm{Nrd}(d') \,\}/\mathbb{Q}_p^\times.$$

This group contains the subgroup of index 2

$$G = \{\, (d,d') \in O_D^\times \times O_D^\times \mid \mathrm{Nrd}(d) = \mathrm{Nrd}(d') \,\}/\mathbb{Z}_p^\times.$$

Therefore G acts with two orbits on $\tilde{B}_{p^r}(Q,N)$. Let $\bar{G}$ be the quotient of G by the subgroup generated by

$$\{\, (d,d') \in G \mid d \equiv d' \equiv 1 \pmod{p^r N^\vee} \,\}$$

and by $1 + p^{r-1}O_{D_p}$ diagonally embedded in G. Then $\bar{G}$ acts faithfully with 2 orbits on $\tilde{B}_{p^r}(Q,N)$. As

$$\#\tilde{A}_{p^r}(Q,N) = (\#\tilde{B}_{p^r}(Q,N)) \cdot (\#(\mathfrak{m}^{-1}/O_D))^3,$$

we see that

$$\#\tilde{A}_{p^r}(Q,N) = 2(\#\bar{G})(\#(\mathfrak{m}^{-1}/O_D))^3 = 2(p+1)^2 p^{6r-7} p^6.$$

It follows that

$$\alpha_p(Q,N) = p^{-6r} 2(p+1)^2 p^{6r-1} = 2(p+1)(1+\frac{1}{p}).$$

2. Calculation of the representation density in the hyperbolic case

2.1. Again we fix a prime number p. For any element $a \in \mathbb{Q}_p^\times$ we write $\operatorname{ord}(a) \in \mathbb{Z}$ for the p-adic valuation of a.

We denote by H the quadratic space over $\mathbb{Z}_p$ whose underlying module is $\mathbb{Z}_p^2$ and whose matrix with respect to the standard basis is $\left(\begin{smallmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{smallmatrix}\right)$. This means that the quadratic form is given by $\mathbb{Z}_p^2 \ni (x,y) \mapsto xy$.

Note that $H^2 \cong (M_2(\mathbb{Z}_p), \det)$.

Let (M,Q) be any quadratic space over $\mathbb{Z}_p$ of rank 3. In this section we will compute the representation density $\alpha_p(M, H^{r+2})$. In fact, there is a polynomial $f_M(X) \in \mathbb{Q}[X]$ such that $f_M(p^{-r}) = \alpha_p(M, H^{2+r})$ ([**Ka**]). We are interested in

$$f_M(1) = \alpha_p(M, H^2) \tag{2.1}$$

and, for (M,Q) anisotropic, in

$$\frac{\partial}{\partial X} f_M(X)_{|X=1}. \tag{2.2}$$

The first value is given in 2.12 and the second in 2.16.

2.2. We use the formulas by Katsurada [**Ka**] but we express them in terms of the Gross-Keating invariants (cf. [**B**]) of the ternary space (M,Q), an invariant $\tilde{\xi} = \tilde{\xi}(M) \in \{-1,0,1\}$, and an invariant $\eta = \eta(M) \in \{\pm 1\}$.

The invariant η is equal to $+1$ if (M,Q) is isotropic and equal to -1 if (M,Q) is anisotropic.

The Gross-Keating invariants consist of a tuple of integers $GK(M) = (a_1, a_2, a_3)$ such that $0 \leq a_1 \leq a_2 \leq a_3$. In addition, if $a_1 \equiv a_2 \bmod 2$ and $a_2 < a_3$ there is a further invariant $\epsilon_{GK}(M) \in \{\pm 1\}$.

In fact, we will not need the invariant $\epsilon_{GK}(M)$ directly in the sequel, as $\tilde{\xi}(T)$ is a refinement. But we remark that the final expressions for (2.1) and (2.2) depend only on $\eta(M)$ (that is, whether (M,Q) is isotropic or not) and on the Gross-Keating invariants $GK(M)$ and $\epsilon_{GK}(M)$.

If T is the matrix associated to (M,Q) and a $\mathbb{Z}_p$-base of M, we also write $\eta(T)$, $GK(T)$, $\epsilon_{GK}(T)$, and $\tilde{\xi}(T)$.

2.3. Recall the Hilbert symbol $(a,b)_p \in \{\pm 1\}$ for $a,b \in \mathbb{Q}_p^\times$. It is uniquely determined by the following properties (where $a,b,b' \in \mathbb{Q}_p^\times$, $u,v \in \mathbb{Z}_p^\times$):

$$\begin{aligned}(a,b)_p &= (b,a)_p,\\ (a,bb')_p &= (a,b)_p(a,b')_p,\\ (p,p)_p &= (-1,p)_p\end{aligned}$$

and, for p odd, by

$$\begin{aligned}(u,p)_p &= \left(\frac{u}{p}\right),\\ (u,v)_p &= 1,\end{aligned}$$

and, for $p=2$, by

$$\begin{aligned}(u,2)_2 &= \begin{cases} +1, & \text{if } u \equiv \pm 1 \bmod 8,\\ -1, & \text{otherwise,}\end{cases}\\ (u,v)_2 &= \begin{cases} +1, & \text{if } u \text{ or } v \equiv 1 \bmod 4,\\ -1, & \text{otherwise.}\end{cases}\end{aligned}$$

2.4. For any symmetric matrix $T \in \mathrm{Sym}_m(\mathbb{Q}_p)$ we denote by $h(T) = h_p(T)$ the Hasse invariant of the associated quadratic space (M,Q). We use the normalization in [**Ki**]. For $m=3$ we have

$$h(T) = \begin{cases} (-1)^{\delta_{2p}}, & \text{if } (M,Q) \text{ is isotropic;}\\ -(-1)^{\delta_{2p}}, & \text{if } (M,Q) \text{ is anisotropic}\end{cases}$$

by [**Ki**, 3.5.1]. Here δ_{2p} is the Kronecker delta.

2.5. In the next sections we recall some results from [**B**] (cf. also [**Y1**]). We start with the case $p>2$. In that case there exists a basis (e_i) of M such that the matrix $T=(t_{ij})$ associated to Q with respect to this basis (*i.e.*, $t_{ij} = \frac{1}{2}(Q(e_i+e_j) - Q(e_i) - Q(e_j))$) is a diagonal matrix. If we write $t_{ii} = u_i p^{a_i}$ for $a_i \in \mathbb{Z}$ and $u_i \in \mathbb{Z}_p^\times$, we can assume that $a_1 \le a_2 \le a_3$. Moreover, if $a_i = a_{i+1}$ we can assume that $u_{i+1} = 1$. Then the Gross-Keating invariants are given as follows. We have

$$GK(T) = (a_1,a_2,a_3).$$

If $a_1 \equiv a_2 \bmod 2$ and $a_2 < a_3$, we have

$$\epsilon_{GK}(T) = \left(\frac{-u_1u_2}{p}\right).$$

We set

$$\tilde{\xi}(T) = \begin{cases} \left(\frac{-u_1u_2}{p}\right), & \text{if } a_1 \equiv a_2 \bmod 2;\\ 0, & \text{if } a_1 \not\equiv a_2 \bmod 2.\end{cases}$$

Finally, let $i, j \in \{1,2,3\}$ with $i \neq j$ and $a_i \equiv a_j \bmod 2$ and define $k \in \{1,2,3\}$ by $\{1,2,3\} \setminus \{i,j\} = \{k\}$. Then T is isotropic if and only if $(-u_i u_j, p)_p = 1$ or $a_k \equiv a_j \bmod 2$.

2.6. Now assume that $p = 2$. In the sequel K will denote one of the matrices

$$H = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad \text{or} \qquad Y := \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

There exists a basis $\mathcal{B}$ of M such that the matrix T associated to Q with respect to $\mathcal{B}$ is of one of the following forms.

Either Q is not diagonalizable (case A). Then we distinguish two subcases:

(A1) $T = \operatorname{diag}(u2^\alpha, 2^\beta K)$ where $\alpha \leq \beta$ are integers and $u \in \mathbb{Z}_2^\times$. Then

$$GK(T) = (\alpha, \beta, \beta).$$

We set

$$\tilde{\xi}(T) = \begin{cases} 1, & \text{if } a_1 \equiv a_2 \bmod 2; \\ 0, & \text{if } a_1 \not\equiv a_2 \bmod 2. \end{cases}$$

(A2) $T = \operatorname{diag}(2^\alpha K, u2^\beta)$ where $\alpha < \beta$ are integers and $u \in \mathbb{Z}_2^\times$. Then

$$GK(T) = (\alpha, \alpha, \beta).$$

In this case $\epsilon_{GK}(T)$ is defined and we have

$$\epsilon_{GK}(T) = \begin{cases} +1 & \text{if } K = H; \\ -1 & \text{if } K = Y. \end{cases}$$

We set $\tilde{\xi}(T) := \epsilon_{GK}(T)$.

In the nondiagonalizable case A, T is isotropic if and only if $K = H$ or $\alpha \equiv \beta \bmod 2$.

Now assume that T is diagonalizable over $\mathbb{Z}_2$ (case B), *i.e.*, there exists a basis such that $T = \operatorname{diag}(u_1 2^{\beta_1}, u_2 2^{\beta_2}, u_3 2^{\beta_3})$ where $0 \leq \beta_1 \leq \beta_2 \leq \beta_3$ are integers and $u_i \in \mathbb{Z}_2^\times$. Then there are four subcases (here our subdivision of cases is different from [**R**]):

(B1) $\beta_1 \not\equiv \beta_2 \bmod 2$. Then

$$GK(T) = (\beta_1, \beta_2, \beta_3 + 2).$$

We set $\tilde{\xi}(T) := 0$.

(B2) $\beta_1 \equiv \beta_2 \bmod 2$ and ($u_1 u_2 \equiv 1 \bmod 4$ or $\beta_3 = \beta_2$)). Then

$$GK(T) = (\beta_1, \beta_2 + 1, \beta_3 + 1).$$

We set $\tilde{\xi}(T) := 0$.

(B3) $\beta_1 \equiv \beta_2 \bmod 2$, $\beta_3 = \beta_2 + 1$, and $u_1 u_2 \equiv -1 \bmod 4$. Then

$$GK(T) = (\beta_1, \beta_2 + 1, \beta_3 + 1).$$

We set $\tilde{\xi}(T) := (-u_1 u_2, 2)_2$ where $(\ ,\ )_2$ denotes the Hilbert symbol.

(B4) $\beta_1 \equiv \beta_2 \bmod 2$, $\beta_3 > \beta_2 + 1$, and $u_1u_2 \equiv -1 \bmod 4$. Then

$$GK(T) = (\beta_1, \beta_2 + 2, \beta_3).$$

In this case $\epsilon_{GK}(T)$ is defined and we have

$$\epsilon_{GK}(T) = (-u_1u_2, 2)_2.$$

We set $\tilde{\xi}(T) := \epsilon_{GK}(T)$.

Finally, let $i, j \in \{1,2,3\}$ with $i \neq j$ and $\beta_i \equiv \beta_j \bmod 2$ and define $k \in \{1,2,3\}$ by $\{1,2,3\} \setminus \{i,j\} = \{k\}$. Then T is isotropic if and only if

$$(-u_ku_j, -u_iu_j)_2 = (-u_iu_j, 2)_2^{\beta_k+\beta_j}.$$

2.7. Going through the cases in 2.5 and 2.6 we see that there are the following possibilities for the value of $\tilde{\xi}$ if T is anisotropic:

- If $a_1 \not\equiv a_2 \bmod 2$, we either have $\tilde{\xi} = 0$ or we have $\tilde{\xi} = -1$ and $a_3 = a_2 + 1$.
- If $a_1 \equiv a_2 \bmod 2$, we always have $a_2 \not\equiv a_3 \bmod 2$ and $\tilde{\xi} = -1$.

If T is isotropic, the possibilities for the value of $\tilde{\xi}$ are the following:

- If $a_1 \not\equiv a_2 \bmod 2$, we either have $\tilde{\xi} = 0$ or we have $\tilde{\xi} = 1$ and $a_3 = a_2 + 1$.
- If $a_1 \equiv a_2 \bmod 2$, we either have $\tilde{\xi} = 1$ or we have $\tilde{\xi} = -1$ and $a_2 \equiv a_3 \bmod 2$.

2.8. By [**Ka**] there exists a polynomial $f_M(X) = f_T(X) \in \mathbb{Q}[X]$ such that $f_T(p^{-r}) = \alpha_p(M, H^{2+r})$. We use the formulas from [**Ka**] to compute f_T. Indeed, by *loc. cit.* p. 417 and p. 428 we have

$$f_T(X) = \tilde{\gamma}_p(T;X)\tilde{F}_p(T;X)$$

with $\tilde{\gamma}_p(T;X) = \gamma_p(T;p^{-2}X)$ and $\tilde{F}_p(T;X) = F_p(T;p^{-2}X)$ where $\gamma_p(T;X)$ and $F_p(T;X)$ are the rational functions defined in *loc. cit.* p. 417 and p. 451 respectively. Thus

$$\tilde{\gamma}_p(T;X) = (1 - p^{-2}X)(1 - p^{-2}X^2).$$

The function $\tilde{F}_p(T;X)$ is more complicated. We will express it in the next sections using the Gross-Keating invariants $GK(T)$ and the invariant $\tilde{\xi}(T)$.

2.9. By [**Ka**] we have

$$\begin{aligned}\tilde{F}_p(T;X) = &\sum_{i=0}^{\hat{\delta}} \sum_{j=0}^{\tilde{\delta}'/2-i-1} p^{i+j}X^{i+2j} \\ &+ \eta p^{(\tilde{\delta}'-2)/2} X^{\delta-\tilde{\delta}'+2} \sum_{i=0}^{\hat{\delta}} \sum_{j=0}^{\tilde{\delta}'/2-i-1} p^{-j}X^{i+2j} \\ &+ \tilde{\xi}^2 p^{\tilde{\delta}'/2} X^{\tilde{\delta}'-\hat{\delta}} \sum_{i=0}^{\hat{\delta}} \sum_{j=0}^{\delta-2\tilde{\delta}'+\hat{\delta}} \tilde{\xi}^j X^{i+j}\end{aligned} \tag{2.3}$$

where η, δ, $\hat{\delta}$, and $\tilde{\delta}'$ are the invariants defined on p. 450 of *loc. cit.* (note that in *loc. cit.* the definitions of $\tilde{\delta}$ and $\hat{\delta}$ have to be interchanged).

2.10. Going through all the cases in 2.5 and 2.6 one sees that η, δ, $\hat{\delta}$, and $\tilde{\delta}'$ can be expressed as follows (where $GK(T) = (a_1, a_2, a_3)$ are the Gross-Keating invariants):

$$\eta = \begin{cases} +1 & \text{if } T \text{ is isotropic,} \\ -1 & \text{if } T \text{ is anisotropic,} \end{cases} \tag{2.4}$$

$$\delta = a_1 + a_2 + a_3, \tag{2.5}$$

$$\hat{\delta} = a_1, \tag{2.6}$$

$$\tilde{\delta}' = \begin{cases} a_1 + a_2, & \text{if } a_1 \equiv a_2 \bmod 2, \\ a_1 + a_2 + 1, & \text{if } a_1 \not\equiv a_2 \bmod 2, \end{cases} \tag{2.7}$$

2.11. If we set

$$\sigma := \begin{cases} 2, & \text{if } a_1 \equiv a_2 \bmod 2, \\ 1, & \text{if } a_1 \not\equiv a_2 \bmod 2, \end{cases}$$

we can rewrite (2.3) using the invariants η, (a_1, a_2, a_3), and $\tilde{\xi}$:

$$\begin{aligned} \tilde{F}_p(T, X) = & \sum_{i=0}^{a_1} \sum_{j=0}^{(a_1+a_2-\sigma)/2-i} p^{i+j} X^{i+2j} \\ & + \eta \sum_{i=0}^{a_1} \sum_{j=0}^{(a_1+a_2-\sigma)/2-i} p^{(a_1+a_2-\sigma)/2-j} X^{a_3+\sigma+i+2j} \\ & + \tilde{\xi}^2 p^{(a_1+a_2-\sigma+2)/2} \sum_{i=0}^{a_1} \sum_{j=0}^{a_3-a_2+2\sigma-4} \tilde{\xi}^j X^{a_2-\sigma+2+i+j}. \end{aligned} \tag{2.8}$$

2.12. We now specialize to $r = 0$, *i.e.*, $X = 1$. In that case we have

$$\alpha_p(T, H^2) = f_T(1) = (1 - p^{-2})^2 \tilde{F}_p(T, 1).$$

If we set $\beta_p(T) := \tilde{F}_p(T, 1)$, it follows from (2.8) that

$$\begin{aligned} \beta_p(T) = (1+\eta)\Big(& \sum_{i=0}^{a_1-1} (i+1)p^i + \sum_{i=a_1}^{(a_1+a_2-\sigma)/2} (a_1+1)p^i \Big) \\ & + p^{(a_1+a_2-\sigma+2)/2}(a_1+1) R_{\tilde{\xi}} \end{aligned} \tag{2.9}$$

where

$$R_{\tilde{\xi}} = \begin{cases} 0, & \text{if } \tilde{\xi} = 0 \\ 0, & \text{if } \tilde{\xi} = -1 \text{ and } a_3 \not\equiv a_2 \bmod 2; \\ a_3 - a_2 + 2\sigma - 3, & \text{if } \tilde{\xi} = 1; \\ 1, & \text{if } \tilde{\xi} = -1 \text{ and } a_3 \equiv a_2 \bmod 2. \end{cases}$$

2.13. If T is anisotropic we have $\alpha_p(T, H^2) = \beta_p(T) = 0$, as a three dimensional anisotropic space cannot be represented by a four-dimensional hyperbolic space. Alternatively this follows also from (2.9): By (2.4) we have $\eta = -1$ and hence it suffices to show that $R_{\tilde{\xi}} = 0$ if T is anisotropic. By 2.7 we are in one of the following two cases:
(a) $\tilde{\xi} = 0$;
(b) $\tilde{\xi} = -1$ and $a_2 \not\equiv a_3 \bmod 2$.
In both cases we have $R_{\tilde{\xi}} = 0$ by definition.

2.14. If T is isotropic, (2.9) gives Proposition 6.25 of [**GK**]:

Proposition 2.1. — *Let T be isotropic. Then:*
(1) *If $a_1 \not\equiv a_2 \bmod 2$, we have*

$$\beta_p(T) = 2\Big(\sum_{i=0}^{a_1-1}(i+1)p^i + \sum_{i=a_1}^{(a_1+a_2-\sigma)/2}(a_1+1)p^i\Big).$$

(2) *If $a_1 \equiv a_2 \bmod 2$ and $\tilde{\xi} = 1$, we have*

$$\beta_p(T) = 2\Big(\sum_{i=0}^{a_1-1}(i+1)p^i + \sum_{i=a_1}^{(a_1+a_2-\sigma)/2}(a_1+1)p^i\Big) + (a_1+1)(a_3-a_2+1)p^{(a_1+a_2)/2}.$$

(3) *If $a_1 \equiv a_2 \bmod 2$ and $\tilde{\xi} = -1$, we have*

$$\beta_p(T) = 2\Big(\sum_{i=0}^{a_1-1}(i+1)p^i + \sum_{i=a_1}^{(a_1+a_2-\sigma)/2}(a_1+1)p^i\Big) + (a_1+1)p^{(a_1+a_2)/2}.$$

Proof. — We have $\eta = 1$, and by 2.7 we are in one of the following cases:
(a) $a_1 \not\equiv a_2 \bmod 2$ and $\tilde{\xi} = 0$;
(b) $a_1 \not\equiv a_2 \bmod 2$, $\tilde{\xi} = 1$, and $a_3 = a_2 + 1$;
(c) $a_1 \equiv a_2 \bmod 2$ and $\tilde{\xi} = 1$;
(d) $a_1 \equiv a_2 \bmod 2$, $\tilde{\xi} = -1$, and $a_2 \equiv a_3 \bmod 2$.

In case (a), we have $R_{\tilde{\xi}} = 0$ by definition, and in case (b) we also have $R_{\tilde{\xi}} = a_3 - a_2 + 2\sigma - 3 = 0$. This proves (1).

In case (c), we have $R_{\tilde{\xi}} = a_3 - a_2 + 1$ and therefore (2).

In case (d), we have $R_{\tilde{\xi}} = 1$ which implies (3) □

Corollary 2.2. — *Set $\Delta(T) = \frac{1}{2}\det(2T) = 4\det(T)$ and assume that T is isotropic. Then $\beta_p(T) = 1$ if $\mathrm{ord}_p(\Delta(T)) = 0$.*

Proof. — For $p>2$ the equality $\operatorname{ord}_p(\Delta(T))=0$ is equivalent to $a_1=a_2=a_3=0$ by definition of the Gross–Keating invariants (see 2.5). For $p=2$ the condition $\operatorname{ord}_p(\Delta)=0$ implies that we are in case (A1) of 2.6 with $\alpha=\beta=0$ and $K=H$. Therefore we have again $a_1=a_2=a_3=0$. Hence the corollary follows for all p from Proposition 2.1. □

2.15. From now on we assume that T is anisotropic. We are going to calculate

$$f_T'(1)=\frac{\partial}{\partial X}f_T(X)_{|X=1}.$$

As T is anisotropic we have $\tilde{F}_p(T;1)=0$ and therefore

$$f_T'(1)=\tilde{\gamma}_p(T,1)\frac{\partial}{\partial X}\tilde{F}_p(T;X)_{|X=1} \tag{2.10}$$

$$=(1-p^{-2})^2\frac{\partial}{\partial X}\tilde{F}_p(T;X)_{|X=1}. \tag{2.11}$$

Using (2.8) we see that

$$\frac{\partial}{\partial X}\tilde{F}_p(T;X)_{|X=1}=F_1+F_2+F_3.$$

Here

$$\begin{aligned}F_1&=\sum_{i=0}^{a_1}\sum_{j=0}^{(a_1+a_2-\sigma)/2-i}(i+2j)p^{i+j}\\&=\sum_{l=0}^{a_1-1}\frac{3}{2}(l+1)lp^l+\sum_{l=a_1}^{(a_1+a_2-\sigma)/2}(a_1+1)(2l-\frac{a_1}{2})p^l,\end{aligned}$$

and

$$\begin{aligned}F_2&=-\sum_{i=0}^{a_1}\sum_{j=0}^{(a_1+a_2-\sigma)/2-i}(a_3+\sigma+i+2j)p^{(a_1+a_2-\sigma)/2-j}\\&=-\sum_{i=0}^{a_1}\sum_{j=i}^{(a_1+a_2-\sigma)/2}(a_1+a_2+a_3+i-2j)p^j\\&=-\sum_{l=0}^{a_1-1}(l+1)(a_1+a_2+a_3-\frac{3}{2}l)p^l\\&\quad-\sum_{l=a_1}^{(a_1+a_2-\sigma)/2}(a_1+1)(\frac{3}{2}a_1+a_2+a_3-3l)p^l\end{aligned}$$

and hence

$$F_1 + F_2 = \sum_{l=0}^{a_1-1} (l+1)(3l - a_1 - a_2 - a_3)p^l + \sum_{l=a_1}^{(a_1+a_2-\sigma)/2} (a_1+1)(4l - 2a_1 - a_2 - a_3)p^l,$$

and

$$F_3 = p^{(a_1+a_2-\sigma+2)/2} \frac{a_1+1}{2} A_{\tilde{\xi}}$$

with

$$A_{\tilde{\xi}} = \begin{cases} 0, & \text{if } \tilde{\xi} = 0; \\ (a_3 - a_2 + 2\sigma - 3)(a_1 + a_2 + a_3), & \text{if } \tilde{\xi} = 1; \\ a_2 - a_3 - 2\sigma + 3, & \text{if } \tilde{\xi} = -1,\ a_2 \not\equiv a_3 \bmod 2; \\ 3a_3 - a_2 + a_1 + 4\sigma - 8, & \text{if } \tilde{\xi} = -1,\ a_2 \equiv a_3 \bmod 2. \end{cases}$$

2.16. We distinguish two cases. The first case is $a_1 \not\equiv a_2 \bmod 2$, *i.e.*, $\sigma = 1$. By 2.7 we either have $\tilde{\xi} = 0$ and hence $A_{\tilde{\xi}} = 0$ or we have $\tilde{\xi} = -1$ and $a_3 = a_2 + 1$ and hence again $A_{\tilde{\xi}} = 0$. Therefore we see that for $a_1 \not\equiv a_2 \bmod 2$ we have

$$\frac{\partial}{\partial X} \tilde{F}_p(T;X)_{|X=1} = \sum_{l=0}^{a_1-1} (l+1)(3l - a_1 - a_2 - a_3)p^l + \sum_{l=a_1}^{(a_1+a_2-1)/2} (a_1+1)(4l - 2a_1 - a_2 - a_3)p^l. \tag{2.12}$$

The second case is $a_1 \equiv a_2 \bmod 2$, *i.e.*, $\sigma = 2$. Then we have $a_3 \not\equiv a_2 \bmod 2$ and hence

$$\frac{\partial}{\partial X} \tilde{F}_p(T;X)_{|X=1} = \sum_{l=0}^{a_1-1} (l+1)(3l - a_1 - a_2 - a_3)p^l + \sum_{l=a_1}^{(a_1+a_2-2)/2} (a_1+1)(4l - 2a_1 - a_2 - a_3)p^l + p^{(a_1+a_2)/2} \frac{a_1+1}{2}(a_2 - a_3 - 1). \tag{2.13}$$

Therefore we see by [**R**, Theorem 1.1] that in either case

$$\frac{\partial}{\partial X} \tilde{F}_p(T;X)_{|X=1} = -\lg(\mathcal{O}_{\mathcal{T}_T,\xi}).$$

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