Astérisque
312, 2007, p. 191–208

16. THE CONNECTION TO EISENSTEIN SERIES

by

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Abstract. — We consider the non-singular Fourier coefficients of the special value of the derivative of a Siegel-Eisenstein series of genus 3 and weight 2. We identify these coefficients with the arithmetic degrees of non-degenerate intersections of arithmetic modular correspondences.

***Résumé* (Relation avec les séries d'Eisenstein).** — Nous identifions les coefficients de Fourier non-dégénerés d'une valeur spéciale de la dérivée d'une série de Siegel-Eisenstein de genre 3 et de poids 2 avec les degrés arithmétiques des intersections de correspondances modulaires arithmétiques.

Introduction

In a previous chapter [**Go2**] an expression was obtained for the arithmetic intersection number of three modular correspondences $(\mathcal{T}_{m_1} \cdot \mathcal{T}_{m_2} \cdot \mathcal{T}_{m_3})$, when their intersection is of dimension 0. This expression is quite complicated, and involves certain local representation densities $\beta_\ell(Q)$ of quadratic forms and a local intersection multiplicity $\alpha_p(Q)$. It is this expression that is the main result of [**GK**]. However, already in the introduction to their paper, Gross and Keating mention that computations of S. Kudla and D. Zagier strongly suggest that the arithmetic intersection number $(\mathcal{T}_{m_1} \cdot \mathcal{T}_{m_2} \cdot \mathcal{T}_{m_3})$ agrees (up to a constant) with a Fourier coefficient of the restriction of the derivative at $s = 0$ of a Siegel-Eisenstein series of genus 3 and weight 2.

In the intervening years since the publication of [**GK**], Kudla has vastly advanced this idea and has in particular proved the analogue of this statement for the intersection of two Hecke correspondences on Shimura curves [**Ku3**]. In fact, Kudla has proposed a whole program which postulates a relation between special values of

2000 ***Mathematics Subject Classification.*** — 11E08, 11F30, 11F32, 11G18.
Key words and phrases. — Eisenstein series, local Whittaker functions, Siegel-Weil formula, local representation densities.

derivatives of Siegel-Eisenstein series and arithmetic intersection numbers of special cycles on Shimura varieties for orthogonal groups, comp. [**Ku4**].

The purpose of the present chapter is to sketch these ideas of Kudla and to derive from Kudla's various papers on the subject the statement alluded to in the introduction of [**GK**]. We stress that what we have done here is simply a task of compilation, since we do not (and cannot) claim to have mastered the automorphic side of the statement in question. We use the results of Katsurada [**Ka**] on local representation densities of quadratic forms, valid even for $p = 2$, to relate the local intersection multiplicities to the derivatives of certain local Whittaker functions, comp. [**Wd2**]. For $p \neq 2$ the corresponding calculations of representation densities are much older and are based on results of Kitaoka [**Kit**].

We thank S. Kudla for his help with this chapter.

1. Decomposition of the intersections of modular correspondences

1.1. To $m \in \mathbb{Z}_{>0}$ we have associated the Deligne-Mumford stack which parametrizes the category of isogenies of degree m between elliptic curves,

$$\mathcal{T}_m(S) = \{ f \colon E \longrightarrow E' \mid \deg(f) = m \}.$$

Here E and E' are elliptic curves over S. Then $\mathcal{T}_m$ maps by a finite unramified morphism to the stack $\mathcal{M}^{(2)} = \mathcal{M} \times_{\operatorname{Spec}\mathbb{Z}} \mathcal{M}$ parametrizing pairs (E, E') of elliptic curves.

Let now $m_1, m_2, m_3 \in \mathbb{Z}_{>0}$ and consider

$$\mathcal{T}(m_1, m_2, m_3) = \{ \mathbf{f} = (f_1, f_2, f_3) \mid f_i \colon E \longrightarrow E',\ \deg f_i = m_i \},$$

the fiber product of $\mathcal{T}_{m_1}, \mathcal{T}_{m_2}, \mathcal{T}_{m_3}$ over $\mathcal{M}^{(2)}$. Denoting by Q the degree quadratic form on $\operatorname{Hom}(E, E')$, we obtain a disjoint sum decomposition,

$$\mathcal{T}(m_1, m_2, m_3) = \coprod_T \mathcal{T}_T. \tag{1.1}$$

Here

$$\mathcal{T}_T(S) = \{ \mathbf{f} \in \operatorname{Hom}_S(E, E')^3 \mid \frac{1}{2}(\mathbf{f}, \mathbf{f}) = T \},$$

where $(\mathbf{f}, \mathbf{f})$ denotes the matrix (a_{ij}) with $a_{ij} = (f_i, f_j) = Q(f_i + f_j) - Q(f_i) - Q(f_j)$. Note that, due to the positive definiteness of Q, the index set in (1.1) is $\operatorname{Sym}_3(\mathbb{Z})^\vee_{\geq 0}$, the set of positive semi-definite half-integral matrices.

Lemma 1.1. — *Let $T \in \operatorname{Sym}_3(\mathbb{Z})^\vee_{>0}$,* i.e., *$T$ is positive definite. Then there exists a unique prime number p such that $\mathcal{T}_T$ is a finite scheme with support lying over the supersingular locus of $\mathcal{M}^{(2)}_p = \mathcal{M}^{(2)} \otimes_{\mathbb{Z}} \mathbb{F}_p$.*

Proof. — Let $(E, E') \in \mathcal{M}^{(2)}$ be in the image of $\mathcal{T}_T$. Since $\operatorname{Hom}(E, E')$ has rank at least 3, it follows that (E, E') has to be a pair of supersingular elliptic curves in some positive characteristic p. To see that p is uniquely determined by T, note that T is represented by the quadratic space over $\mathbb{Q}$ corresponding to the definite quaternion algebra ramified in p. However, by [**Ku3**, Prop. 1.3], there is only one quadratic space with fixed discriminant which represents T. $\square$

1.2. In this chapter we consider, for $T \in \mathrm{Sym}_3(\mathbb{Z})^\vee_{>0}$, the number

$$\widehat{\deg}(\mathcal{T}_T) = \lg(\mathcal{T}_T) \cdot \log p \ ,$$

where p is the unique prime in the statement of Lemma 1.1, and where

$$\lg(\mathcal{T}_T) = \sum_{\xi \in \mathcal{T}_T(\bar{\mathbb{F}}_p)} e_\xi^{-1} \cdot \lg(\mathcal{O}_{\mathcal{T}_T,\xi}),$$

with $e_\xi = |\operatorname{Aut}(\xi)|$. Our aim is to compare $\widehat{\deg}(\mathcal{T}_T)$ with the T^{th} Fourier coefficient of a certain Siegel-Eisenstein series of genus 3 and weight 2.

We first define a class of Eisenstein series, among which will be the one appearing in our main theorem.

2. Eisenstein series and the main theorem

2.1. Let B be a quaternion algebra over $\mathbb{Q}$. We denote by $V = V_B$ the quadratic space defined by B, *i.e.*, B with its norm form Q. We note that the idèle class character usually associated to a quadratic space, $x \mapsto (x, (-1)^{n(n-1)/2}\det(V))_{\mathbb{Q}}$ is in this case the trivial character χ_0 ($4 \mid n$, and $\det(V)$ is a square). Let $H = \mathrm{O}(V)$ be the associated orthogonal group. Let $W = \mathbb{Q}^6$, with standard symplectic form $\langle\ ,\ \rangle$ whose matrix with respect to the standard basis is given by $\left(\begin{smallmatrix} 0 & I_3 \\ -I_3 & 0\end{smallmatrix}\right)$. Let $G = \mathrm{Sp}(W) = \mathrm{Sp}_6$, and denote by $P = M.N$ the Siegel parabolic subgroup, with

$$M = \{\, m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1}\end{pmatrix} \mid a \in \mathrm{GL}_3 \,\},$$

$$N = \{\, n(b) = \begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} \mid b \in \mathrm{Sym}_3 \,\}.$$

Let $K = K_\infty.K_f = \prod_v K_v$ be the maximal compact subgroup of $G(\mathbb{A})$ with

$$K_v = \begin{cases} \mathrm{Sp}_6(\mathbb{Z}_p), & \text{if } v = p < \infty; \\ \{\, \left(\begin{smallmatrix} a & b \\ -b & a\end{smallmatrix}\right) \mid a + ib \in \mathrm{U}_3(\mathbb{R}) \,\}, & \text{if } v = \infty. \end{cases} \tag{2.1}$$

We have the Weil representation ω of $G(\mathbb{A}) \times H(\mathbb{A})$ (for the standard additive character ψ of $\mathbb{A}$ with archimedean component $\psi_\infty(x) = \exp(2\pi i x)$ and of conductor zero at all non-archimedean places) on the Schwartz space $\mathcal{S}(V(\mathbb{A})^3)$ (the action of the elements $P(\mathbb{A}) \times H(\mathbb{A})$ are given by simple formulae [**We**], comp. also (4.1) and (4.2) below). In the local version at a place v, we have a representation ω_v of $G(\mathbb{Q}_v) \times H(\mathbb{Q}_v)$ on $\mathcal{S}(V(\mathbb{Q}_v)^3)$.

We have the Iwasawa decomposition

$$G(\mathbb{A}) = P(\mathbb{A})K = N(\mathbb{A})M(\mathbb{A})K.$$

If $g = nm(a)k \in G(\mathbb{A})$, then

$$|a(g)| = |\det(a)|_{\mathbb{A}}$$

is well-defined. For a character χ of $\mathbb{A}^\times/\mathbb{Q}^\times$, we have the induced representation of $G(\mathbb{A})$, corresponding to $s \in \mathbb{C}$,

$$I(s,\chi) = \{\, \Phi\colon G(\mathbb{A}) \to \mathbb{C} \ K\text{-finite function} \mid \Phi(nm(a)g) = \chi(\det(a)) \cdot |a(g)|^{s+2} \cdot \Phi(g)\,\}.$$

For $\varphi \in \mathcal{S}(V(\mathbb{A})^3)$, we set

$$\Phi(g,s) = (\omega(g)\varphi)(0) \cdot |a(g)|^s.$$

In this way, we obtain an intertwining map

$$\mathcal{S}(V(\mathbb{A})^3) \longrightarrow I(0,\chi_0), \quad \varphi \longmapsto \Phi(g,0)\ . \tag{2.2}$$

Note that $|a(g)|$ is a right K-invariant function on $G(\mathbb{A})$, so $\Phi(g,s)$ is a standard section of the induced representation, *i.e.*, its restriction to K is independent of s. We will also need the local version $I(s,\chi_v)$ of the induced representation at a place v and the $G(\mathbb{Q}_v)$-equivariant map

$$\mathcal{S}(V_v^3) \longrightarrow I(0,\chi_{0,v}). \tag{2.3}$$

2.2. Returning to the global situation, we consider the Eisenstein series associated to $\varphi \in \mathcal{S}(V(\mathbb{A})^3)$,

$$E(g,s,\Phi) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \Phi(\gamma g, s).$$

This series is absolutely convergent for $\mathrm{Re}(s) > 2$, and defines an automorphic form. It has a meromorphic continuation and a functional equation with $s = 0$ as its center of symmetry.

We will now make a specific choice of Φ which will define an *incoherent* Eisenstein series. Let $B = \mathrm{M}_2(\mathbb{Q})$ and let $V(\mathbb{Z}_p) = \mathrm{M}_2(\mathbb{Z}_p)$ for any p. We let $\varphi_f = \otimes\varphi_p = \otimes\,\mathrm{char}\ V(\mathbb{Z}_p)$, and let $\Phi_f = \otimes\Phi_p$ be the corresponding factorizable standard section. For Φ_∞ we take the standard section uniquely determined by

$$\Phi_\infty(k,0) = \det(\underline{k})^2,$$

where $k \in K_\infty$ is the image of $\underline{k} \in \mathrm{U}_3(\mathbb{R})$ under the natural identification in (2.1). Then by [**Ku3**, (7.13)], Φ_∞ is the image of the Gaussian φ_∞ under the local map (2.3) for $v = \infty$, where the local quadratic space is V_∞^+, the positive-definite quadratic space corresponding to the Hamilton quaternion algebra over $\mathbb{R}$, and where

$$\varphi_\infty(x) = \exp(-\pi\ \mathrm{tr}(x,x)), \quad x \in (V_\infty^+)^3. \tag{2.4}$$

Since $V_\infty^+ \otimes V(\mathbb{A}_f)$ does not correspond to a quaternion algebra over $\mathbb{Q}$, the standard section $\Phi = \Phi_\infty \otimes \Phi_f$ is *incoherent* in the sense of *loc. cit.*, and hence (*loc. cit.*, Theorem 2.2),

$$E(g,0,\Phi) \equiv 0.$$

Consider the Fourier expansion of $E(g,s,\Phi)$,

$$E(g,s,\Phi) = \sum_{T \in \mathrm{Sym}_3(\mathbb{Q})} E_T(g,s,\Phi),$$

where

$$E_T(g,s,\Phi) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(ng,s,\Phi)\cdot \psi_T(n)^{-1}dn,$$

with

$$\psi_T(n(b)) = \psi(\mathrm{tr}(Tb)), \quad b \in \mathrm{Sym}_3(\mathbb{A}). \tag{2.5}$$

For $T \in \mathrm{Sym}_3(\mathbb{Q})$ with $\det(T) \neq 0$, the Fourier coefficient has an explicit expression as a product

$$E_T(g,s,\Phi) = \prod_v W_{T,v}(g_v,s,\Phi_v), \tag{2.6}$$

see [**Ku3**, (4.4)]. Here $W_{T,v}(g_v,s,\Phi_v)$ is the local Whittaker function, cf. section 5. The local Whittaker functions are entire (cf. [**Ku3**, (4.2) and (4.3)]), and the product (2.6) is absolutely convergent and holomorphic in $s=0$. More precisely, for $\mathrm{Re}(s) > 2$, the identity (2.6) holds and for almost all places p, the local factor at p on the right hand side equals $\zeta_p(s+2)^{-1}\cdot\zeta_p(2s+2)^{-1} = (1-p^{-s-2})\cdot(1-p^{-2s-2})$, and for all places the local factor is an entire function.

2.3. For $T \in \mathrm{Sym}_3(\mathbb{Q})_{>0}$, let

$$\mathrm{Diff}(T,V) = \{\, p \mid T \text{ not represented by } V(\mathbb{Q}_p)\,\}.$$

Then the cardinality $|\mathrm{Diff}(T,V)|$ is odd, cf. [**Ku3**, Cor. 5.2]. Moreover we have $W_{T,p}(g_p,0,\Phi_p) \equiv 0$ for $p \in \mathrm{Diff}(T,V)$, cf. [**Ku3**, Prop. 1.4]. On the other hand, $W_{T,\infty}(g_\infty,0,\Phi_\infty) \neq 0$, cf. [**Ku3**, Prop. 9.5]. Hence

$$\operatorname*{ord}_{s=0} E_T(g,s,\Phi) \geq |\mathrm{Diff}(T,V)|.$$

In particular, if $E'_T(g,0,\Phi) \neq 0$, then $\mathrm{Diff}(T,V) = \{p\}$ for a unique prime number p.

2.4. We may now formulate our main theorem.

Theorem 2.1. — *Let* $V = \mathrm{M}_2(\mathbb{Q})$ *and let* $\Phi = \Phi_\infty \otimes \Phi_f$ *be the incoherent standard section as above. Let* $T \in \mathrm{Sym}_3(\mathbb{Q})_{>0}$ *with* $\mathrm{Diff}(T,V) = \{p\}$.

(i) *If* $T \notin \mathrm{Sym}_3(\mathbb{Z})^\vee$, *then* $\mathcal{T}_T = \varnothing$ *and* $\widehat{\deg}(\mathcal{T}_T) = 0$ *and* $E'_T(g,0,\Phi) \equiv 0$.

(ii) *Let* $T \in \mathrm{Sym}_3(\mathbb{Z})^\vee$. *Then* $\mathcal{T}_T$ *has support in characteristic* p. *For* $g = (g_\infty, e, e \ldots) \in G(\mathbb{A})$ *with*

$$g_\infty = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}, \qquad x, y \in \mathrm{Sym}_3(\mathbb{R}), y > 0,$$

let $\tau = g_\infty \cdot i1_3 = x + iy \in \mathfrak{H}_3$. *Then*

$$\det(y)\, \widehat{\deg}(\mathcal{T}_T) \cdot q^T = \kappa \cdot E'_T(g, 0, \Phi),$$

where $q^T = \exp(2\pi i\ \mathrm{tr}(T\tau))$ *and where the negative constant* κ *is independent of* T.

Here $\mathfrak{H}_3 = \{\, \tau \in \mathrm{Sym}_3(\mathbb{C}) \mid \mathrm{Im}(\tau) > 0 \,\}$ is the Siegel upper half space.

The proof of the theorem consists in calculating explicitly both sides of the identity. The first assertion of (i) is obvious and the second is a consequence of section 5 below, where the local Whittaker functions are related to local representation densities (see Proposition 5.2 below). The proof of (ii) will be reduced in section 3 to a statement about local Whittaker functions which will be taken up in sections 4 and 5.

2.5. In the rest of this section we relate the adelic Eisenstein series to the classical Siegel-Eisenstein series, following [**Ku1**, section IV.2]. By strong approximation,

$$G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})K.$$

By our choice of Φ, which is right K_f-invariant, the Eisenstein series $E(g, s, \Phi)$ is determined by its restriction to $G(\mathbb{R})$ (embedded *via* $g_\infty \mapsto (g_\infty, e, e \ldots)$ in $G(\mathbb{A})$).

We have

$$G(\mathbb{Z}) = G(\mathbb{Q}) \cap (G(\mathbb{R}).K_f).$$

Also, $P(\mathbb{Q})\backslash G(\mathbb{Q}) = P(\mathbb{Z})\backslash G(\mathbb{Z})$, hence for $g = g_\infty$,

$$\begin{aligned} E(g, s, \Phi) &= \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \Phi_\infty(\gamma g_\infty, s) \cdot \Phi_f(\gamma, s) \\ &= \sum_{\gamma \in P(\mathbb{Z})\backslash G(\mathbb{Z})} \Phi_\infty(\gamma g_\infty, s). \end{aligned} \tag{2.7}$$

For our choice of Φ_∞ and of $g_\infty = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}$, we have

$$\Phi_\infty(\gamma g_\infty, s) = \det(y)^{\frac{s}{2}+1} \cdot \det(c\tau + d)^{-2} \cdot |\det(c\tau + d)|^{-s},$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_6(\mathbb{Z}).$$

Inserting this expression into the sum (2.7), one obtains from [**Ku1**, IV.2.23], (for $\ell = \rho_n = 2$),

$$E(g, s, \Phi) = \det(y) \cdot E_{\mathrm{class}}(\tau, s), \tag{2.8}$$

where

$$E_{\text{class}}(\tau, s) = \det(y)^{s/2} \sum_{(c,d)} \det(c\tau + d)^{-2} \cdot |\det(c\tau + d)|^{-s}$$

is the classical Siegel Eisenstein series (the sum here ranges over a complete set of representatives of the equivalence classes of pairs of co-prime symmetric integer matrices).

2.6. Using the comparison (2.8) between the adelic and the classical Eisenstein series, we obtain from Theorem 2.1 the following statement. We consider the Fourier expansion of the classical Eisenstein series,

$$E_{\text{class}}(\tau, s) = \sum_{T \in \mathrm{Sym}_3(\mathbb{Z})^\vee} c(T, y, s)\, q^T.$$

Here $\tau = x + iy \in \mathfrak{H}_3$ and $q^T = \exp(2\pi i \ \mathrm{tr}(T\tau))$.

Theorem 2.2. — *Let* $T \in \mathrm{Sym}_3(\mathbb{Z})^\vee_{>0}$.
(1) *Then* $c'(T) = (\frac{\partial}{\partial s} c(T, y, s))_{|s=0}$ *is independent of* y.
(2) *If* $\mathrm{Diff}(T, V) = \{p\}$, *then* $\mathcal{T}_T$ *has support in characteristic* p *and*

$$\widehat{\deg}(\mathcal{T}_T) = \kappa \cdot c'(T)$$

for a negative constant κ *independent of* T.

Corollary 2.3. — *Assume that there is no positive definite binary quadratic form over* $\mathbb{Z}$ *which represents* m_1, m_2 *and* m_3, *so that the divisors* $\mathcal{T}_{m_1}$, $\mathcal{T}_{m_2}$, *and* $\mathcal{T}_{m_3}$ *intersect in dimension* 0, *cf.* [**Go2**, Prop. 3.2]. *Then there exists a constant* κ *independent of* (m_1, m_2, m_3) *such that*

$$(\mathcal{T}_{m_1} \cdot \mathcal{T}_{m_2} \cdot \mathcal{T}_{m_3}) = \kappa \cdot \sum_{\substack{T \in \mathrm{Sym}_3(\mathbb{Z})^\vee_{>0} \\ \mathrm{diag}(T) = (m_1, m_2, m_3)}} c'(T)$$

Proof. — The hypothesis implies that in the disjoint sum (1.1) only positive definite $T \in \mathrm{Sym}_3(\mathbb{Z})^\vee$ occur as indices, comp. [**Go2**, Prop. 3.5]. Therefore the assertion follows from Theorem 2.2. □

3. Use of the Siegel-Weil formula

3.1. Let $\tilde{V}$ be the quadratic space associated to a quaternion algebra $\tilde{B}$ over $\mathbb{Q}$. For $\tilde{\varphi} \in \mathcal{S}(\tilde{V}(\mathbb{A})^3)$, there is the theta series

$$\theta(g, h; \tilde{\varphi}) = \sum_{x \in \tilde{V}(\mathbb{Q})^3} (\omega(g)\tilde{\varphi})(h^{-1}x),$$

and the corresponding theta integral over the orthogonal group $\tilde{H} = \mathrm{O}(\tilde{V})$ associated to $\tilde{V}$,

$$I(g;\tilde{\varphi}) = \int\limits_{\tilde{H}(\mathbb{Q})\backslash\tilde{H}(\mathbb{A})} \theta(g,h;\tilde{\varphi})\,dh.$$

Here the Haar measure dh is normalized so that

$$\mathrm{vol}(\tilde{H}(\mathbb{Q})\backslash\tilde{H}(\mathbb{A})) = 1.$$

We will only consider the case in which the quadratic space $\tilde{V}$ is anisotropic. If $\tilde{\varphi}$ is K-finite, then $I(g;\tilde{\varphi})$ is an automorphic form on $G(\mathbb{A})$. The Siegel-Weil formula [**KR**] states that, if $\tilde{\varphi}$ gives rise to $\tilde{\Phi}$ *via* the map (2.2), then

$$E(g,0,\tilde{\Phi}) = 2\cdot I(g;\tilde{\varphi}). \tag{3.1}$$

Let $T\in \mathrm{Sym}_3(\mathbb{Q})$ with $\det(T)\neq 0$. Then the T^{th}-Fourier coefficient of $I(g;\tilde{\varphi})$ is equal to ([**KR**, (6.21)])

$$I_T(g;\tilde{\varphi}) = \int\limits_{\tilde{H}(\mathbb{Q})\backslash\tilde{H}(\mathbb{A})} \sum_{x\in\tilde{V}(\mathbb{Q})^3_T} (\omega(g)\tilde{\varphi})(h^{-1}x)\,dh$$

where

$$\tilde{V}(\mathbb{Q})^3_T = \{\, x\in\tilde{V}(\mathbb{Q})^3 \mid \frac{1}{2}(x,x) = T\,\}.$$

3.2. We now return to the situation considered in Theorem 2.1. Let $V = \mathrm{M}_2(\mathbb{Q})$ and let Φ be the standard section defined in the previous section. We also fix $T\in \mathrm{Sym}_3(\mathbb{Q})_{>0}$ with $\mathrm{Diff}(T,V) = \{p\}$. Let $\tilde{V}$ be the quadratic space associated to the definite quaternion algebra $\tilde{B} = B^{(p)}$ ramified at p, and unramified at all other finite primes. Note that $\tilde{V}(\mathbb{R}) = V_\infty^+$. We consider the standard section $\tilde{\Phi}$ which is the image of $\tilde{\varphi} = \tilde{\varphi}_\infty\otimes\tilde{\varphi}_f^p\otimes\tilde{\varphi}_p$ under

$$\mathcal{S}(\tilde{V}(\mathbb{A})^3) \longrightarrow I(0,\chi_0),$$

where $\tilde{\varphi}_f^p = \varphi_f^p$, where $\tilde{\varphi}_\infty = \varphi_\infty$ is the Gaussian (2.4) and where $\tilde{\varphi}_p = \mathrm{char}\,\tilde{V}(\mathbb{Z}_p)^3$, with $\tilde{V}(\mathbb{Z}_p)$ the maximal order of the division algebra $B^{(p)}\otimes\mathbb{Q}_p$. Hence $\tilde{\Phi}_\infty = \Phi_\infty$, $\tilde{\Phi}_f^p = \Phi_f^p$ and $\tilde{\Phi}$ is a *coherent* standard section. Comparing the expressions (2.6) for the Fourier coefficients of $E(g,s,\Phi)$ and $E(g,s,\tilde{\Phi})$, we can write, for $g = g_\infty\in G(\mathbb{R})$,

$$E'_T(g,0,\Phi) = \frac{W'_{T,p}(e,0,\Phi_p)}{W_{T,p}(e,0,\tilde{\Phi}_p)}\cdot E_T(g,0,\tilde{\Phi}).$$

We refer to Corollary 5.3 below for a proof of the fact that the denominator here is nonzero. Using the Siegel-Weil formula (3.1) for the anisotropic quadratic space $\tilde{V}$, we can rewrite this as

$$E'_T(g,0,\Phi) = 2\cdot\frac{W'_{T,p}(e,0,\Phi_p)}{W_{T,p}(e,0,\tilde{\Phi}_p)}\cdot I_T(g;\tilde{\varphi}). \tag{3.2}$$

Now the function $\tilde{\varphi}_\infty$ is invariant under $\tilde{H}(\mathbb{R})$. For

$$g_\infty = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}, \qquad x, y \in \mathrm{Sym}_3(\mathbb{R}), y > 0,$$

the value of $\omega(g_\infty)\tilde{\varphi}_\infty$ at $t \in \tilde{V}(\mathbb{R})^3$ with $\frac{1}{2}(t,t) = T$ is equal to

$$(\omega(g_\infty)\tilde{\varphi}_\infty)(t) = \exp(2\pi i \ \mathrm{tr}(T\tau)) \ \det(y).$$

Since $\tilde{H}(\mathbb{R}) = \mathrm{O}(\tilde{V}(\mathbb{R}))$ is compact, we may write using the product measure $dh = d_\infty h \times d_f h$,

$$2 \cdot I_T(g; \tilde{\varphi}) = 2 \det(y) \cdot q^T \cdot \mathrm{vol}(\tilde{H}(\mathbb{R}), d_\infty h) \cdot I_T(\tilde{\varphi}_f), \tag{3.3}$$

where

$$I_T(\tilde{\varphi}_f) = \int\limits_{\tilde{H}(\mathbb{Q})\backslash \tilde{H}(\mathbb{A}_f)} \sum_{x \in \tilde{V}(\mathbb{Q})^3_T} \tilde{\varphi}_f(h^{-1}x) \, d_f h.$$

3.3. Let

$$\tilde{H}' = \{ \tilde{g} = (g, g') \in \tilde{B}^\times \times \tilde{B}^\times \mid \mathrm{Nm}(g) = \mathrm{Nm}(g') \}.$$

Then $\tilde{H}'$ acts on $\tilde{V}$ *via*

$$\tilde{g} \cdot x = (g, g') \cdot x = g' x g^{-1}.$$

This induces an exact sequence, where $\mathbb{G}_m$ lies in the center of $\tilde{H}'$, cf. [**Wd1**, Lemma 1.6],

$$1 \to \mathbb{G}_m \to \tilde{H}' \xrightarrow{\mathrm{pr}} \mathrm{SO}(\tilde{V}) \to 1. \tag{3.4}$$

We fix the Haar measure on $\tilde{H}'(\mathbb{A})$ such that the measure induced by the exact sequence (3.4) on $\mathrm{SO}(\tilde{V})(\mathbb{A})$ is the Tamagawa measure, and with the standard Haar measure on the central idele group $\mathbb{A}^\times$ which is the product of the local measures $\lambda_\ell \, \frac{dx_\ell}{|x_\ell|}$ with convergence factors $\lambda_\ell = 1 - \ell^{-1}$, so that $\mathrm{vol}(\hat{\mathbb{Z}}^\times) = 1$. Let

$$\tilde{K}' = \tilde{H}'(\mathbb{A}_f) \cap ((O_{\tilde{B}} \otimes \hat{\mathbb{Z}})^\times \times (O_{\tilde{B}} \otimes \hat{\mathbb{Z}})^\times).$$

Proposition 3.1. — *Let*

$$|\mathcal{T}_T| = \sum_{\xi \in \mathcal{T}_T(\mathbb{F}_p)} e_\xi^{-1}$$

with $e_\xi = |\mathrm{Aut}(\xi)|$. *Then*

$$|\mathcal{T}_T| = \kappa_1 \cdot I_T(\tilde{\varphi}_f),$$

where $\kappa_1 = 2 \ \mathrm{vol}(\tilde{K}')^{-1}$.

Proof. — We choose a finite set of double coset representatives $h_j \in \tilde{H}'(\mathbb{A}_f)$ such that

$$\tilde{H}'(\mathbb{A}_f) = \coprod_j \tilde{H}'(\mathbb{Q}) h_j \tilde{K}'.$$

Since each double coset $\tilde{H}'(\mathbb{Q})h_j\tilde{K}'$ is stable under $\widehat{\mathbb{Z}}^\times\mathbb{Q}^\times = \mathbb{A}_f^\times$, we obtain a disjoint decomposition,

$$\mathrm{SO}(\tilde{V})(\mathbb{A}_f) = \coprod_j \mathrm{SO}(\tilde{V})(\mathbb{Q})\mathrm{pr}(h_j)\mathrm{pr}(\tilde{K}').$$

Let

$$\tilde{\Gamma}'_j = \tilde{H}'(\mathbb{Q}) \cap h_j\tilde{K}'h_j^{-1}.$$

Note that $\mathrm{vol}(\mathrm{SO}(\tilde{V})(\mathbb{Q})\backslash\mathrm{SO}(\tilde{V})(\mathbb{A})) = 2$. We have

$$\tilde{H}(\mathbb{A}_f) = \mathrm{SO}(\tilde{V})(\mathbb{A}_f) \rtimes \mu_2(\mathbb{A}_f).$$

Hence

$$\begin{aligned}
1 &= \mathrm{vol}(\mathrm{O}(\tilde{V})(\mathbb{Q})\backslash\mathrm{O}(\tilde{V})(\mathbb{A}))\\
&= \frac{1}{2}\mathrm{vol}(\mathrm{SO}(\tilde{V})(\mathbb{Q})\backslash\mathrm{O}(\tilde{V})(\mathbb{A}))\\
&= \frac{1}{2}\mathrm{vol}(\mathrm{SO}(\tilde{V})(\mathbb{Q})\backslash\mathrm{SO}(\tilde{V})(\mathbb{A}))\,\mathrm{vol}(\mu_2(\mathbb{A}))\\
&= \mathrm{vol}(\mu_2(\mathbb{A}))
\end{aligned}$$

and therefore

$$\mathrm{vol}(\mu_2(\mathbb{Q})\backslash\mu_2(\mathbb{A})) = \frac{1}{2}.$$

Let us normalize the Haar measure on $\mu_2(\mathbb{R})$ by $\mathrm{vol}(\mu_2(\mathbb{R})) = 1$. Then we get $\mathrm{vol}(\mu_2(\mathbb{Q})\backslash\mu_2(\mathbb{A}_f)) = \frac{1}{2}$. Then we obtain as in [**Ku3**, (7.28)],

$$\begin{aligned}
I_T(\tilde{\varphi}_f) &= \int\limits_{\mathrm{SO}(\tilde{V})(\mathbb{Q})\backslash\mathrm{SO}(\tilde{V})(\mathbb{A}_f)} \int\limits_{\mu_2(\mathbb{Q})\backslash\mu_2(\mathbb{A}_f)} \sum_{x\in\tilde{V}(\mathbb{Q})^3_T} \tilde{\varphi}_f(h^{-1}cx)\,d_fh\,dc\\
&= \frac{1}{2}\int\limits_{\mathrm{SO}(\tilde{V})(\mathbb{Q})\backslash\ \mathrm{SO}(\tilde{V})(\mathbb{A}_f)} \sum_{x\in\tilde{V}(\mathbb{Q})^3_T} \tilde{\varphi}_f(h^{-1}x)\,d_fh\\
&= \frac{1}{2}\sum_j \int\limits_{\mathrm{SO}(\tilde{V})(\mathbb{Q})\backslash\mathrm{SO}(\tilde{V})(\mathbb{Q})\mathrm{pr}(h_j)\mathrm{pr}(\tilde{K}')} \sum_{x\in\tilde{V}(\mathbb{Q})^3_T} \tilde{\varphi}_f(h^{-1}x)\,d_fh\\
&= \frac{1}{2}\cdot\mathrm{vol}(\mathrm{pr}(\tilde{K}'))\cdot\sum_j\sum_{x\in\tilde{V}(\mathbb{Q})^3_T}\frac{1}{|\tilde{\Gamma}_{j,x}|}\cdot\tilde{\varphi}_f(h_j^{-1}x).
\end{aligned}$$

Here $\tilde{\Gamma}_{j,x}$ is the image of $\tilde{\Gamma}'_{j,x}$ in $\mathbb{Q}^\times\backslash\tilde{H}'(\mathbb{Q}) = \mathrm{SO}(\tilde{V})(\mathbb{Q})$. Therefore we have $|\tilde{\Gamma}_{j,x}| = \frac{1}{2}\cdot|\tilde{\Gamma}'_{j,x}|$. Note that $\tilde{\Gamma}_{j,x}$ is trivial since x spans a three-dimensional subspace of the 4-dimensional space $\tilde{V}$.

To make the connection with $\mathcal{T}_T$, note that the supersingular locus of $\mathcal{M}_p^{(2)}$ can be written as a double coset space (cf. [**Mi**, 6]),

$$(\mathcal{M}^{(2)})^{ss} = \tilde{H}'(\mathbb{Q})\backslash\tilde{H}'(\mathbb{A}_f)/\tilde{K}'.$$

Here we chose (E_0, E_0) as a base point, such that $\tilde{K}'$ is the stabilizer of the Tate module $\hat{T}(E_0) \times \hat{T}(E_0)$ (completed by the Dieudonné module at p). To $\tilde{g} = (g, g') \in \tilde{H}'(\mathbb{A}_f)$ corresponds $E_g \times E_{g'}$ with the diagonal isogeny,

$$(g, g') \colon E_0 \times E_0 \longrightarrow E_g \times E_{g'}.$$

The lattice $\mathrm{Hom}(E_g, E_{g'})$ in $\tilde{V}(\mathbb{Q}) = \mathrm{Hom}(E_0, E_0) \otimes \mathbb{Q}$ is given by

$$\begin{aligned}\mathrm{Hom}(E_g, E_{g'}) &= \{\, y \in \tilde{B} \mid yg(\hat{T}(E_0)) \subset g'\hat{T}(E_0) \,\} \\ &= \{\, y \in \tilde{B} \mid g'^{-1}yg \in \tilde{V}(\hat{\mathbb{Z}}) \,\} \\ &= \{\, y \in \tilde{B} \mid \tilde{g}^{-1}y \in \tilde{V}(\hat{\mathbb{Z}}) \,\}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}|\mathcal{T}_T| &= \sum_{[y,\tilde{g}] \in \tilde{H}'(\mathbb{Q}) \backslash (\tilde{V}^3(\mathbb{Q})_T \times \tilde{H}'(\mathbb{A}_f)/\tilde{K})} \tilde{\varphi}_f(\tilde{g}^{-1} \cdot y) \\ &= \sum_j \sum_{x \in \tilde{V}^3(\mathbb{Q})_T} \tilde{\varphi}_f(h_j^{-1} \cdot x) \\ &= 2 \cdot \mathrm{vol}(\mathrm{pr}(\tilde{K}'))^{-1} \cdot I_T(\tilde{\varphi}_f).\end{aligned}$$

Since $\mathrm{vol}(\tilde{K}') = \mathrm{vol}(\mathrm{pr}(\tilde{K}'))$, the result follows. □

3.4. The next result will be proved in section 5.6.

Theorem 3.2. — *The lengths of the local rings $\mathcal{O}_{\mathcal{T}_{T,\xi}}$ at all points $\xi \in \mathcal{T}_T(\bar{\mathbb{F}}_p)$ are all equal to*

$$\lg(\mathcal{O}_{\mathcal{T}_{T,\xi}}) = -\frac{2}{(p-1)^2} \cdot \frac{W'_{T,p}(e, 0, \Phi_p)}{W_{T,p}(e, 0, \tilde{\Phi}_p)} \cdot (\log p)^{-1}.$$

3.5. We will now prove Theorem 2.1 using Theorem 3.2. Let

$$\begin{aligned}H' &= \{\tilde{g} = (g, g') \in \mathrm{GL}_2 \times \mathrm{GL}_2 \mid \det(g) = \det(g')\}, \\ K' &= H'(\mathbb{A}_f) \cap (\mathrm{GL}_2(\hat{\mathbb{Z}}) \otimes \mathrm{GL}_2(\hat{\mathbb{Z}})).\end{aligned}$$

Then $\tilde{H}'$ is an inner form of H'.

We now fix Haar measures on $\tilde{H}'(\mathbb{A})$ and on $H'(\mathbb{A})$ following [**Ku3**, p. 573]. More precisely, in *loc. cit.* Kudla defines for any quaternion algebra B over $\mathbb{Q}$ a Haar measure on $(B \otimes \mathbb{A})^\times$ which is decomposed, *i.e.*, the explicit product of local Haar measures on $(B \otimes \mathbb{Q}_v)^\times$ for all places v. By our fixed choice of Haar measure on $\mathbb{A}^\times$, we therefore also obtain a decomposed Haar measure on $H(B)'(\mathbb{A})$, where

$$H(B)' = \{\tilde{g} = (g, g') \in B^\times \times B^\times \mid \mathrm{Nm}(g) = \mathrm{Nm}(g')\}.$$

By *loc. cit.*, the induced Haar measure on $SO(V(B))(\mathbb{A})$ is the Tamagawa measure, as required above.

We apply this construction to $B = \mathrm{M}_2(\mathbb{Q})$ and to $B = \tilde{B} = B^{(p)}$, the definite quaternion algebra, ramified at p and unramified at all other finite places. Then we have for these Haar measures (comp. [**Ku3**, Lemma 14.10]),

$$\frac{\mathrm{vol}(K'_p)}{\mathrm{vol}(\tilde{K}'_p)} = (p-1)^2$$

and

$$\frac{\mathrm{vol}(K')}{\mathrm{vol}(\tilde{K}')} = (p-1)^2.$$

Hence

$$\begin{aligned} q^T \cdot \widehat{\deg}(\mathcal{T}_T) &= q^T \ \lg(\mathcal{O}_{\mathcal{T}_T}) \cdot \log p \\ &= q^T \ \lg(\mathcal{O}_{\mathcal{T}_{T,\xi}}) \cdot |\mathcal{T}_T| \cdot \log p \\ &= -\frac{2}{(p-1)^2 \cdot \mathrm{vol}(\tilde{K}')} \cdot 2 \cdot \frac{W'_{T,p}(e,0,\Phi_p)}{W_{T,p}(e,0,\tilde{\Phi}_p)} \cdot q^T \cdot I_T(\tilde{\varphi}_f) \\ &= -\frac{2}{\mathrm{vol}(K')} \cdot E'_T(g,0,\Phi) \det(y)^{-1} \ v^{-1}, \end{aligned}$$

where we used (3.2) and (3.3) in the last step, and where $v = \mathrm{vol}(\tilde{H}(\mathbb{R}), d_\infty h)$. This proves the main theorem with the negative constant $\kappa = -\frac{2}{\mathrm{vol}(K')} \cdot v^{-1}$.

4. The Weil representation

4.1. The remainder of this chapter is devoted to the proof of Theorem 3.2. This is a purely local statement.

We fix a prime number p and change our notation: We replace V by $V \otimes \mathbb{Q}_p$, G by $G \otimes \mathbb{Q}_p$, ψ by its localization ψ_p (of conductor zero), etc. At the same time we consider a more general situation.

4.2. Instead of the quadratic space associated to a quaternion algebra, we now let V be any $\mathbb{Q}_p$-vector space and $(\ ,\ )$ a symmetric nondegenerate bilinear form on V. Then $Q(x) = \frac{1}{2}(x,x)$ is a quadratic form on V.

We assume that $n := \dim(V)$ is even. In fact, we will only need the case $V = B \perp H^r$ where B is a quaternion algebra over $\mathbb{Q}_p$ endowed with the reduced norm, and where H^r is the orthogonal sum of r copies of the hyperbolic plane H.

We denote by $\det(V)$ the image in $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ of the determinant of the matrix $((v_i,v_j))_{ij}$ where $(v_1,\dots,v_n)$ is some basis of V. As in 2.1 we have the quadratic character χ_V of $\mathbb{Q}_p^\times$ associated to V given by

$$\chi_V(x) = (x, (-1)^{n(n-1)/2}\det(V))_p = (x, (-1)^{n/2}\det((v_i,v_j))_{ij})_p,$$

where $(\ ,\ )_p$ denotes the Hilbert symbol.

4.3. Let $(W, \langle\ ,\ \rangle)$ be the space $\mathbb{Q}_p^{2m}$ endowed with the standard symplectic form whose matrix with respect to the standard basis is given by $\left(\begin{smallmatrix} 0 & I_m \\ -I_m & 0 \end{smallmatrix}\right)$. We consider W as vector space of row vectors, in particular the canonical GL_{2m}-action is from the right. To prove Theorem 3.2 we will need only the case $m = 3$.

As in 2.1 we denote by $P = MN$ the Siegel parabolic subgroup of $G = \mathrm{Sp}_{2m}(\mathbb{Q}_p)$ over $\mathbb{Q}_p$ where

$$M = \{\, m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \mid a \in \mathrm{GL}_m(\mathbb{Q}_p) \,\},$$

$$N = \{\, n(b) = \begin{pmatrix} I_m & b \\ 0 & I_m \end{pmatrix} \mid b \in \mathrm{Sym}_m(\mathbb{Q}_p) \,\}.$$

Let $K = \mathrm{Sp}_{2m}(\mathbb{Z}_p) \subset G$ the standard maximal compact subgroup and set

$$w := \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \in G.$$

4.4. In the sequel we let $a \in \mathrm{GL}_m$ act on $V^m = V \otimes \mathbb{Q}_p^m$ via right multiplication, which we denote by $x \mapsto xa$.

Moreover for $x, y \in V^m$ we set

$$(x, y) := ((x_i, y_j))_{ij} \in \mathrm{Sym}_m(\mathbb{Q}_p).$$

4.5. Associated to the quadratic space V and the fixed additive character ψ there is a Weil representation ω_V of G on the vector space $\mathcal{S}(V^m)$ of Schwartz functions on V^m. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, $\varphi \in \mathcal{S}(V^m)$, and $x \in V^m$ we have by [**Ku2**, Prop. 4.3] (cf. also [**Rao**, Lemma 3.2], and [**We**]),

$$\begin{aligned}(\omega_V(g)(\varphi))(x) &= \gamma(V, \psi, g) \\ &\quad \cdot \int_{V^m/\mathrm{Ker}(c)} \psi(\mathrm{tr}(\tfrac{1}{2}(xa, xb) + (xb, yc) + \tfrac{1}{2}(yc, yd)))\, \varphi(xa + yc)\, d_g y\end{aligned}$$

where $\gamma(V, \psi, g)$ is a certain 8^{th} root of unity depending on V, ψ, and g such that $\gamma(V, \psi, e) = 1$ and where $d_g y$ is a suitable Haar measure. We make this more explicit in three special cases:

$$(\omega_V(m(a))\varphi)(x) = \chi_V(\det a)|\det a|^{n/2}\varphi(xa), \tag{4.1}$$

$$(\omega_V(n(b))\varphi)(x) = \psi(\tfrac{1}{2}\,\mathrm{tr}((x, x)b))\varphi(x), \tag{4.2}$$

$$(\omega_V(w^{-1}))\varphi)(x) = \gamma(V) \int_{V^m} \psi(-\,\mathrm{tr}((x, y)))\, \varphi(y)\, dy \tag{4.3}$$

where in (4.3) dy is the Haar measure on V^m which is self dual for Fourier transform and where $\gamma(V) = \gamma(V, \psi, w^{-1})$ is the 8^{th} root of unity explicitly given in [**Ku3**, A.4].

5. Local Whittaker functions and representation densities

5.1. We keep the notation of section 4 and assume from now on that $m = 3$ and hence $G = \mathrm{Sp}_6(\mathbb{Q}_p)$, and $n = 4$.

For $s \in \mathbb{C}$ let $I(s, \chi_V)$ be the degenerate principal series representation of G induced from P, *i.e.*, $I(s, \chi_V)$ consists of K-finite functions $\Phi(\cdot, s)\colon G \to \mathbb{C}$ such that

$$\Phi(nm(a)g, s) = \chi_V(\det a)|\det a|^{s+2}\Phi(g, s)$$

for all $n \in N$, $a \in \mathrm{GL}_3(\mathbb{Q}_p)$, and $g \in G$.

We also set for $T \in \mathrm{Sym}_3(\mathbb{Q}_p)$, as in (2.5),

$$\psi_T\colon N \longrightarrow \mathbb{C}^\times, \qquad \psi_T(n(b)) = \psi(\mathrm{tr}(Tb)).$$

5.2. For $s \in \mathbb{C}$, $\Phi \in I(s, \chi_V)$, $T \in \mathrm{Sym}_3(\mathbb{Q}_p)$ with $\det(T) \neq 0$, and $g \in G$ we define the local Whittaker function by

$$W_T(g, s, \Phi) = \int_N \Phi(w^{-1}n(b)g, s)\, \psi_T(n(b))^{-1}\, db$$

where db is the Haar measure on $\mathrm{Sym}_3(\mathbb{Q}_p)$ which is selfdual with respect to the pairing

$$\psi_N\colon\ \mathrm{Sym}_3(\mathbb{Q}_p) \times \mathrm{Sym}_3(\mathbb{Q}_p) \longrightarrow \mathbb{C}, \qquad (b, b') \longmapsto \psi(\mathrm{tr}(bb')).$$

As the conductor of ψ is zero, we have

$$\{\, b \in \mathrm{Sym}_3(\mathbb{Q}_p) \mid \psi_N(b, b') = 1 \text{ for all } b' \in \mathrm{Sym}_3(\mathbb{Z}_p) \,\} = \mathrm{Sym}_3(\mathbb{Z}_p)^\vee. \tag{5.1}$$

Therefore

$$\mathrm{vol}_{db}(\mathrm{Sym}_3(\mathbb{Z}_p))\mathrm{vol}_{db}(\mathrm{Sym}_3(\mathbb{Z}_p)^\vee) = 1.$$

As the index of $\mathrm{Sym}_3(\mathbb{Z}_p)$ in $\mathrm{Sym}_3(\mathbb{Z}_p)^\vee$ is $2^{3\delta_{2p}}$, we obtain

$$\mathrm{vol}_{db}(\mathrm{Sym}_3(\mathbb{Z}_p)) = 2^{-(3/2)\delta_{2p}}. \tag{5.2}$$

It is known that $W_T(g, s, \Phi)$ converges for $\mathrm{Re}(s) > 2$ and admits a holomorphic continuation to the entire complex plane, if Φ is standard, *i.e.*, if its restriction to K is independent of s [**Ku3**, Prop. 1.4]. Moreover, we will see in Proposition 5.2 below that $W_T(e, s, \Phi)$ is a polynomial in p^{-s}.

5.3. For $\varphi \in \mathcal{S}(V^3)$ we set

$$\Phi(g,s) = (\omega(g)\varphi)(0) \cdot |a(g)|^s.$$

It follows from (4.1) and (4.2) that $\Phi(g,s) \in I(s,\chi_V)$. In this way, we obtain a G-equivariant map similar to (2.2),

$$\mathcal{S}(V^3) \longrightarrow I(0,\chi_V), \qquad \varphi \longmapsto \Phi(g,0).$$

5.4. For $r \geq 0$ we denote by the H_r the quadratic space $\mathbb{Q}_p^{2r}$ whose associated bilinear form has the matrix $\frac{1}{2}\left(\begin{smallmatrix} 0 & I_r \\ I_r & 0 \end{smallmatrix}\right)$ with respect to the standard basis, and set

$$V_r = V \perp H_r.$$

It is known [**Ku3**, Lemma A.2] that $\omega_{V_r} = \omega_V \otimes \omega_{H_r}$ as representations of G on $\mathcal{S}(V_r^3) = \mathcal{S}(V^3) \otimes \mathcal{S}(H_r^3)$.

We also recall Lemma A.3 from [**Ku3**] (see also [**Ral**, Remark II.3.2]):

Lemma 5.1. — *Let $\varphi_r^0 \in \mathcal{S}(H_r^3)$ be the characteristic function of $\mathrm{M}_{2r,3}(\mathbb{Z}_p)$ and $\varphi \in \mathcal{S}(V^3)$ with associated $\Phi(g,s) \in I(s,\chi_V)$. Set $\varphi^{[r]} = \varphi \otimes \varphi_r^0 \in \mathcal{S}(V_r^3) = \mathcal{S}(V^3) \otimes \mathcal{S}(H_r^3)$. Then we have for all $g \in G$ and $r \geq 0$*

$$\Phi(g,r) = (\omega_{V_r}(g)\varphi^{[r]})(0).$$

5.5. We fix a $\mathbb{Z}_p$-lattice L of V such that $(\ ,\ )$ is integral on L. Choose a $\mathbb{Z}_p$-basis of L and let S_r be the matrix associated to the quadratic form on $V_r = V \oplus H_r$ with respect to the chosen basis of L and the standard basis of H_r. In particular, the matrix of the bilinear form $(\ ,\ )$ with respect to the chosen base of L equals $2S_0$.

Let $\varphi \in \mathcal{S}(V^3)$ be the characteristic function of L^3 with associated $\Phi = \Phi(g,s) \in I(s,\chi_V)$. Then the local Whittaker function $W_T(e,s,\Phi)$ interpolates the local representation densities:

Proposition 5.2. — *For all integers $r \geq 0$ we have*

$$W_T(e,r,\Phi) = 2^{-(15/2)\delta_{2p}} |\det S_0|^{3/2} \gamma(V) \alpha_p(T,S_r),$$

where we denote by $\alpha_p(\ ,\ )$ the local representation density as normalized in [**Wd1**, (4.4)]. *In particular, $W_T(e,s,\Phi)$ is a polynomial in $X = p^{-s}$.*

Proof. — The right hand side is a polynomial in $X = p^{-r}$ [**Kit**] and the left hand side is an entire function in r. Hence it suffices to show the identity for $r > 2$. Now we have

$$\begin{aligned}
W_T(e,r,\Phi) &= \int\limits_{\mathrm{Sym}_3(\mathbb{Q}_p)} \Phi\big(w^{-1}n(b), r\big)\psi\big(-\operatorname{tr}(Tb)\big)\,db \\
&\overset{(5.1)}{=} \int\limits_{\mathrm{Sym}_3(\mathbb{Q}_p)} \big(\omega_{V_r}(w^{-1}n(b))\varphi^{[r]}\big)(0)\psi\big(-\operatorname{tr}(Tb)\big)\,db \\
&\overset{(4.3)}{=} \int\limits_{\mathrm{Sym}_3(\mathbb{Q}_p)} \gamma(V)\int\limits_{V_r^3} \psi\big(\frac{1}{2}\operatorname{tr}(b(y,y))\big)\cdot\varphi^{[r]}(y)\,dy\,\psi\big(-\operatorname{tr}(Tb)\big)\,db \\
&= \gamma(V)\lim_{t\to\infty} \int\limits_{p^{-t}\,\mathrm{Sym}_3(\mathbb{Z}_p)}\int\limits_{V_r^3} \psi\big(\operatorname{tr}(b(\frac{1}{2}(y,y)-T))\big)\cdot\varphi^{[r]}(y)\,dy\,db \\
&\overset{(5.1)}{=} \gamma(V)\lim_{t\to\infty}\operatorname{vol}_{db}(p^{-t}\,\mathrm{Sym}_3(\mathbb{Z}_p))\cdot \int\limits_{\substack{y\in V_r^3 \\ \frac{1}{2}(y,y)-T\in p^t\,\mathrm{Sym}_3(\mathbb{Z}_p)^\vee}} \varphi^{[r]}(y)\,dy \\
&\overset{(5.2)}{=} \gamma(V)\lim_{t\to\infty} 2^{-(3/2)\delta_{2p}}p^{6t} \int\limits_{\substack{y\in \mathrm{M}_{2r+4,3}(\mathbb{Z}_p) \\ {}^t yS_ry-T\in p^t\,\mathrm{Sym}_3(\mathbb{Z}_p)^\vee}} dy.
\end{aligned}$$

Now $\{\, y \in \mathrm{M}_{2r+4,3}(\mathbb{Z}_p) \mid {}^t yS_ry - T \in p^t\,\mathrm{Sym}_3(\mathbb{Z}_p)^\vee \,\}$ is a union of

$$\tilde{A}_{p^t}(T,S_r) := \#\{\, y \in \mathrm{M}_{2r+4,3}(\mathbb{Z}_p/2p^t\mathbb{Z}_p) \mid {}^t yS_ry - T \in p^t\,\mathrm{Sym}_3(\mathbb{Z}_p)^\vee \,\}$$

cosets for $2p^t\mathrm{M}_{2r+4,3}(\mathbb{Z}_p)$. Moreover, by the definition of dy (4.3) we have

$$\begin{aligned}
\operatorname{vol}_{dy}(\mathrm{M}_{2r+4,3}(\mathbb{Z}_p)) &= |\det 2S_r|^{3/2} \\
&= |\det 2S_0|^{3/2} \\
&= 2^{-6\delta_{2p}}|\det S_0|^{3/2},
\end{aligned}$$

and hence

$$\operatorname{vol}_{dy}(2p^t\mathrm{M}_{2r+4,3}(\mathbb{Z}_p)) = 2^{-6\delta_{2p}}|\det S_0|^{3/2}2^{-3(4+2r)\delta_{2p}}p^{-t3(4+2r)}.$$

Therefore $W_T(e,r,\Phi)$ is equal to

$$\gamma(V)2^{-6\delta_{2p}}|\det S_0|^{3/2}2^{(-(3/2)-3(4+2r))\delta_{2p}}\lim_{t\to\infty} p^{6t-t3(4+2r)}\tilde{A}_{p^t}(T,S_r).$$

Now we have

$$\tilde{A}_{p^t}(T,S_r) = 2^{3(4+2r)\delta_{2p}}A_{p^t}(T,S_r)$$

with

$$A_{p^t}(T,S_r) = \#\{\, y \in \mathrm{M}_{2r+4,3}(\mathbb{Z}_p/p^t\mathbb{Z}_p) \mid {}^t yS_ry - T \in p^t\,\mathrm{Sym}_3(\mathbb{Z}_p)^\vee \,\}.$$

By definition we have

$$\alpha_p(T, S_r) = \lim_{t\to\infty} p^{6t-t3(4+2r)} A_{p^t}(T, S_r)$$

and this proves the proposition. □

Corollary 5.3. — *For Φ as in Proposition 5.2, $W_T(e,0,\Phi) \neq 0$ if and only if T is represented by S_0.*

5.6. We will now prove Theorem 3.2.

As $\alpha_p(T, S_r)$ is a rational function in $X = p^{-r}$, it follows from Proposition 5.2 that

$$W'_T(e,0,\Phi) = -\log(p) 2^{-(15/2)\delta_{2p}} |\det S_0|^{3/2} \gamma(V) \frac{\partial}{\partial X} \alpha_p(T, S_r)_{|X=1}. \tag{5.3}$$

Let D be the division quaternion algebra over $\mathbb{Q}_p$ and denote by $\mathcal{O}_D$ its maximal order. We denote by $S = S_0$ (resp. $\tilde{S} = \tilde{S}_0$) the matrix associated to the quadratic space $V = \mathrm{M}_2(\mathbb{Z}_p)$ (resp. $\tilde{V} = \mathcal{O}_D$) endowed with the reduced Norm. Then we have (see *e.g.*, [**Wd1**, (4.5) and (4.6)])

$$\begin{aligned} |\det(S_0)| &= 2^{4\delta_{2p}}, \\ |\det(\tilde{S}_0)| &= 2^{4\delta_{2p}} p^{-2}. \end{aligned}$$

Moreover we have by the explicit formulas in the Appendix of [**Ku3**]

$$\gamma(V) = -\gamma(\tilde{V}).$$

Using the notation of Theorem 3.2, we therefore have by Proposition 5.2 and (5.3)

$$\frac{W'_{T,p}(e,0,\Phi_p)}{W_{T,p}(e,0,\tilde{\Phi}_p)} (\log p)^{-1} = p^3 \frac{\frac{\partial}{\partial X}\alpha_p(T, S_r)_{|X=1}}{\alpha_p(T, \tilde{S}_0)}. \tag{5.4}$$

But now by [**Wd2**, Theorem 1.1 and 2.16] we have

$$\alpha_p(T, \tilde{S}_0) = 2(p+1)^2 p^{-1} \tag{5.5}$$

and

$$\frac{\partial}{\partial X}\alpha_p(T, S_r)_{|X=1} = -p^{-4}(p^2-1)^2 \mathrm{lg}(\mathcal{O}_{\mathcal{T}_T,\xi}). \tag{5.6}$$

Therefore we have

$$\begin{aligned} \mathrm{lg}(\mathcal{O}_{\mathcal{T}_T,\xi}) \log(p) &\underset{(5.6)}{\overset{(5.5)}{=}} -\frac{p^4}{(p^2-1)^2} \frac{2(p+1)^2}{p} \frac{\frac{\partial}{\partial X}\alpha_p(T, S_r)_{|X=1}}{\alpha_p(T, \tilde{S}_0)} \\ &\overset{(5.4)}{=} -\frac{2}{(p-1)^2} \frac{W'_T(e,0,\Phi_p)}{W_T(e,0,\tilde{\Phi}_p)} \end{aligned}$$

which proves the theorem.

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