

BULLETIN DE LA S. M. F.

A. IWANIK

The problem of L^p -simple spectrum for ergodic group automorphisms

Bulletin de la S. M. F., tome 119, n° 1 (1991), p. 91-96

http://www.numdam.org/item?id=BSMF_1991__119_1_91_0

© Bulletin de la S. M. F., 1991, tous droits réservés.

L'accès aux archives de la revue « Bulletin de la S. M. F. » (<http://smf.emath.fr/Publications/Bulletin/Presentation.html>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/legal.php>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

THE PROBLEM OF L^p -SIMPLE SPECTRUM FOR ERGODIC GROUP AUTOMORPHISMS

BY

A. IWANIK (*)

RÉSUMÉ. — Soit T un automorphisme d'un groupe abélien compact métrisable. L'isométrie inversible $U_T f = f \circ T$ n'admet pas de fonction cyclique dans l'espace L^p pour $p > 1$. D'autre part, il existe une fonction cyclique pour la norme spectrale dans L^1 .

ABSTRACT. — Let T be an ergodic automorphism of a compact metric abelian group. Then the invertible isometry operator $U_T f = f \circ T$ admits no L^p -cyclic vector in any L^p space, $p > 1$. There exists a cyclic vector for the spectral norm in L^1 .

1. Introduction

Let T be an invertible measure preserving transformation of a probability space (X, B, m) . The associated unitary operator $U_T f(x) = f(Tx)$ acts on $L^2(m)$. The same formula defines an invertible isometry

$$U_T : L^p(m) \longrightarrow L^p(m)$$

for any $1 \leq p \leq \infty$. A function $f \in L^p(m)$ is said to be L^p -cyclic if the linear span of the functions $U_T^n f$ ($n \in \mathbb{Z}$) is dense in $L^p(m)$. If there exists an L^2 -cyclic function then T is said to have *simple spectrum*. Analogously, we say that T has L^p -*simple spectrum* if there exists an L^p -cyclic vector for U_T in $L^p(m)$.

J.-P. THOUVENOT raised the question whether the Bernoulli automorphism has L^1 -simple spectrum. Without solving the problem we present some related results. We shall show that, like for $p = 2$, the ergodic group automorphisms have no L^p -cyclic vectors for $p > 1$ (THEOREM 1). Next we prove that there does exist a cyclic vector for a certain norm weaker than the L^1 -norm (THEOREM 2).

(*) Texte reçu le 6 avril 1990.

A. IWANIK, Institute of Mathematics, Technical University, 50-370 Wrocław, Pologne.

2. $L^p(G)$ is not finitely generated

Throughout the paper we consider an ergodic continuous group automorphism T of a compact metric abelian group G endowed with its probability Haar measure dx . Let $\widehat{G}$ be the dual group. The dual automorphism $\widehat{T}$ is defined by the formula

$$(\widehat{T}\gamma)(x) = \gamma(Tx), \quad (\gamma \in \widehat{G}).$$

By the ergodicity assumption each $\widehat{T}$ -orbit

$$O(\gamma) = \{\widehat{T}^n \gamma : n \in \mathbb{Z}\}, \quad (\gamma \in \widehat{G} \setminus \{1\}),$$

is infinite.

It is known that each $O(\gamma)$ is a Sidon set in $\widehat{G}$, hence a $\Lambda(p)$ -set for any $1 \leq p < \infty$ (see [K], Lemma 3 and [L-R]). Consequently, the set

$$E = O(\gamma_1) \cup \cdots \cup O(\gamma_k),$$

where $\gamma_1, \dots, \gamma_k \in \widehat{G}$, is a $\Lambda(p)$ -set so, for any $2 \leq q < \infty$, there exists a constant C_q such that

$$\|g\|_q \leq C_q \|g\|_2$$

whenever $g \in L^q(G)$ with $\text{supp } \widehat{g} \subset E$.

Now let $1 < p \leq 2$ and $q \geq 2$ with $p^{-1} + q^{-1} = 1$. We define

$$L_E^q(G) = \{g \in L^q(G) : \text{supp } \widehat{g} \subset E\}.$$

If $f \in L^p(G)$ and $g \in L_E^2(G)$ then by Parseval's identity and Hölder inequality we get

$$\left| \sum_{\gamma \in E} \widehat{f}(\gamma) \widehat{g}(\gamma) \right| \leq C_q \|f\|_p \|\widehat{g}\|_2.$$

It follows that $\|\widehat{f}\| \|E\|_2 \leq C_q \|f\|_p < \infty$. Consequently, if $P_E f$ denotes the function determined by the formula

$$(P_E f)^\wedge(\gamma) = \begin{cases} \widehat{f}(\gamma) & \gamma \in E, \\ 0 & \text{otherwise,} \end{cases}$$

then P_E becomes a continuous projection from $L^p(G)$ onto $L_E^2(G)$. Clearly, P_E is well defined on $L^p(G)$ for any $p > 1$.

Apart from U_T we shall consider the operator $\widehat{U}_T$ acting on $c_0(\widehat{G})$ by

$$\widehat{U}_T \xi(\gamma) = \xi(\widehat{T}^{-1} \gamma).$$

By a direct computation we have $(U_T f)^\sim = \widehat{U}_T \hat{f}$ for any $f \in L^1(G)$. Since E is T -invariant, we obtain

$$U_T P_E f = P_E U_T f, \quad (f \in L^p(G)).$$

In other words, the following diagram commutes

$$\begin{array}{ccc} L^p(G) & \xrightarrow{U_T} & L^p(G) \\ P_E \downarrow & & \downarrow P_E \\ L_E^2(G) & \xrightarrow{U_T} & L_E^2(G) \end{array}$$

THEOREM 1. — *Let $p > 1$ and $f_1, \dots, f_r$ be any finite collection in $L^p(G)$. Then the linear span of the functions $U_T^n f_j$ ($n \in \mathbb{Z}$, $j = 1, \dots, r$) is not dense in $L^p(G)$.*

Proof. — Fix any $k > r$ and let E be the union of k disjoint orbits,

$$E = O(\gamma_1) \cup \dots \cup O(\gamma_k), \quad (\gamma_1, \dots, \gamma_k \in \widehat{G} \setminus \{1\}).$$

The unitary operator U_T restricted to $L_{O(\gamma_j)}^2(G)$ has simple Lebesgue spectrum since $U_T \gamma = \widehat{T} \gamma$. Consequently,

$$U_T|_{L_E^2(G)}$$

has Lebesgue spectrum of multiplicity k so the invariant subspace generated by the $r < k$ vectors $P_E f_1, \dots, P_E f_r$ is not dense in $L_E^2(G)$. By looking at the diagram we infer that the functions $U_T^n f_j$, ($n \in \mathbb{Z}$, $j = 1, \dots, r$) cannot be linearly dense in $L^p(G)$.

3. Cyclic function for a weaker norm

For the rest of this paper we consider the spectral norm

$$\|f\|_F = \|\hat{f}\|_\infty$$

on $L^1(G)$. The convergence in $\|\cdot\|_F$ is simply the uniform convergence of Fourier coefficients, and clearly $\|f\|_F \leq \|f\|_1$ for any $f \in L^1(G)$. Evidently, U_T is a $\|\cdot\|_F$ isometry.

Our aim is to prove the existence of a $\|\cdot\|_F$ -cyclic function for U_T acting on $L^1(G)$.

First we shall identify $\widehat{G} \setminus \{1\}$ with the product space $\mathbb{N} \times \mathbb{Z}$ where (i, j) represents the character $\widehat{T}^{-j}\gamma_i$ for a fixed cross section $\gamma_1, \gamma_2, \dots$ of the infinite $\widehat{T}$ -orbits in $\widehat{G}$. Now $\widehat{U}_T$ restricted to $c_0(\mathbb{N} \times \mathbb{Z})$ becomes the translation operator S on $c_0(\mathbb{N} \times \mathbb{Z})$,

$$(S\xi)(i, j) = \xi(i, j+1).$$

We shall often write $\xi_i(j) = \xi(i, j)$.

LEMMA. — *A vector $\xi \in c_0(\mathbb{N} \times \mathbb{Z})$ is c_0 -cyclic with respect to S iff for every $\mu \in \ell^1(\mathbb{N} \times \mathbb{Z})$*

$$\sum \mu_i * \xi_i = 0 \implies \mu = 0.$$

Proof. — First note that ξ is cyclic iff the operator

$$K : \ell^1(\mathbb{Z}) \longrightarrow c_0(\mathbb{N} \times \mathbb{Z})$$

defined by $(K\lambda)(i, j) = (\lambda * \xi_i)(j)$ has a dense range. Equivalently, ξ is cyclic iff the adjoint operator

$$K^* : \ell^1(\mathbb{N} \times \mathbb{Z}) \longrightarrow \ell^\infty(\mathbb{Z})$$

is one-to-one. But for any $\lambda \in \ell^1(\mathbb{Z})$ and $\mu \in \ell^1(\mathbb{N} \times \mathbb{Z})$ we have

$$\begin{aligned} \langle K\lambda, \mu \rangle &= \sum_{i,j} (\lambda * \xi_i)(j) \mu(i, j) \\ &= \sum_{i,j} \sum_n \lambda(n) \xi_i(j-n) \mu_i(j) \\ &= \sum_n \sum_i \lambda(n) (\tilde{\xi}_i * \mu_i)(n) \\ &= \left\langle \lambda, \sum_i \tilde{\xi}_i * \mu_i \right\rangle, \end{aligned}$$

where $\tilde{\xi}_i(j) = \xi_i(-j)$. This means

$$K^* \mu = \sum_i \tilde{\xi}_i * \mu_i.$$

Since ξ is cyclic iff $\tilde{\xi}$ is cyclic, we obtain the desired condition.

COROLLARY. — *If $f \in L^1(G)$ has absolutely convergent Fourier series then f is not L^1 -cyclic for U_T .*

Proof. — Suppose to the contrary that $\hat{f} \in \ell^1(\hat{G})$ and f is L^1 -cyclic. Then $\hat{f}$ is $c_0(\hat{G})$ -cyclic for $\hat{U}_T$. By identifying $\hat{G} \setminus \{1\}$ with $\mathbb{N} \times \mathbb{Z}$ as above, we would obtain a $c_0(\mathbb{N} \times \mathbb{Z})$ -cyclic vector $\xi = \hat{f}|_{\mathbb{N} \times \mathbb{Z}} \in \ell^1(\mathbb{N} \times \mathbb{Z})$ for S . Since clearly $\xi_i \neq 0$ for every $i \in \mathbb{N}$, we can define a nonzero vector μ in $\ell^1(\mathbb{N} \times \mathbb{Z})$ by letting $\mu_1 = \xi_2$, $\mu_2 = -\xi_1$ and $\mu_i = 0$ for $i \geq 2$. Now

$$\sum \xi_i * \mu_i = 0$$

which contradicts the Lemma.

We prove now the existence of a $\|\cdot\|_F$ -cyclic function.

THEOREM 2. — *There exists $f \in L^2(G)$ such that the linear span of the functions $U_T^n f$ ($n \in \mathbb{Z}$) is dense in $\|\cdot\|_F$.*

Proof. — Since $U_T 1 = 1$ and

$$\frac{1}{n}(f + U_T f + \cdots + U_T^{n-1} f) \longrightarrow \int f(x) dx$$

in $L^1(G)$, it suffices to find a $\|\cdot\|_F$ -cyclic vector for the subspace

$$\{f \in L^1(G) : \int f(x) dx = 0\}.$$

Equivalently, we shall find a $c_0(\mathbb{N} \times \mathbb{Z})$ -cyclic vector $\xi \in \ell^2(\mathbb{N} \times \mathbb{Z})$ for S .

Let $Q_1, Q_2, \dots$ be disjoint countable dense subsets of the unit interval $(0, 1)$. For each Q_n pick an atomic probability measure ν_n whose set of atoms coincides with Q_n . Now fix a convergent series $\sum a_n < \infty$, with $a_n > 0$, and define

$$g_n(t) = a_n \nu_n([0, t])$$

for $0 \leq t < 1$. The functions g_n are right continuous and the set of discontinuity points of g_n coincides with Q_n .

Moreover, the functions

$$h_n(e^{2\pi i t}) = g_n(t), \quad (0 \leq t < 1),$$

satisfy the conditions

$$\sum \|h_n\|_2 \leq \sum \|h_n\|_\infty = \sum a_n < \infty.$$

(We can identify $[0, 1)$ with $\mathbb{T}$ and g_n with h_n .)

Now we let $\xi_n = \hat{h}_n$, where the Fourier transform is taken in the sense of the $\mathbb{T}$ - $\mathbb{Z}$ duality. We shall show that ξ is $c_0(\mathbb{N} \times \mathbb{Z})$ -cyclic. By the LEMMA it suffices to prove that any $\mu \in \ell^1(\mathbb{N} \times \mathbb{Z})$ which satisfies

$$\sum \mu_n * \xi_n = 0$$

must in fact vanish. Let $u_n \in C(\mathbb{T})$ be such that $\hat{u}_n = \mu_n$. Then

$$(h_n u_n)^\wedge = \hat{h}_n * \hat{u}_n = \xi_n * \mu_n.$$

The condition $\sum \xi_n * \mu_n = 0$ now implies

$$\sum h_n u_n = 0 \quad \text{a.e.,}$$

where the series converges in $L^2(\mathbb{T})$. Since $|h_n| \leq a_n$ and $|u_n| \leq \|\mu\|$, the series converges uniformly. By the right continuity of the g_n the sum $\sum h_n u_n$ is also right continuous. This implies

$$\sum h_n(x) u_n(x) = 0$$

everywhere. To end the proof we show that the latter condition forces

$$u_1 = u_2 = \cdots = 0,$$

whence $\mu = 0$. To see this suppose, to the contrary, that e.g. $u_1 \neq 0$. Then there exists an arc $J \subset \mathbb{T}$ with

$$|u_1(x)| \geq \varepsilon > 0$$

for $x \in J$. We have

$$h_1 = - \sum_{n \geq 2} \frac{u_n}{u_1} h_n$$

on J . The latter series is uniformly convergent on J , so its sum is continuous at each continuity point of all the h_n 's, $n \geq 2$, in particular on $Q_1 \cap J$. On the other hand, each of these points is an atom of ν_1 hence a discontinuity for h_1 , a contradiction.

Acknowledgements : I would like to thank S. FERENCZI for bringing the problem to my attention and M. BOZEJKO for helpful discussions.

BIBLIOGRAPHY

- [K] KRZYZEWSKI (K.). — On regularity of measurable solutions of a cohomology equation, *Bull. Polish Acad. Sci. Math.*, t. **37**, 1989, p. 279–287.
- [L-R] LOPEZ (J.M.) and ROSS (K.A.). — *Sidon sets*. — Dekker, New York, 1975.