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LINEARIZATION OF GROUP STACK ACTIONS AND THE PICARD GROUP OF THE MODULI OF $\mathbf{SL}_r/\mu_s$-BUNDLES ON A CURVE

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ABSTRACT. — In this paper, we consider morphisms of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ which are torsors under a *group stack* $\mathcal{G}$. We show that line bundles on $\mathcal{Y}$ correspond exactly with $\mathcal{G}$-*linearized* line bundles on $\mathcal{X}$ (with a suitable definition of a $\mathcal{G}$-linearization). We use this fact to determine the precise structure of the Picard group of the moduli stack of G-bundles on an algebraic curve when G is any group of type A_n.

RÉSUMÉ. — Dans cet article, on considère les morphismes de champs algébriques $\mathcal{X} \to \mathcal{Y}$ qui sont des torseurs sous un champ en groupes $\mathcal{G}$. Nous prouvons que les fibrés en droites sur $\mathcal{Y}$ correspondent exactement aux fibrés en droites sur $\mathcal{X}$ munis d'une $\mathcal{G}$-linéarisation (avec une définition convenable d'une $\mathcal{G}$-linéarisation). Nous utilisons ceci pour déterminer la structure exacte du groupe de Picard du champ des G-fibrés sur une courbe algébrique lorsque G est un groupe algébrique (non nécessairement simplement connexe) de type A_n.

1. Introduction

Let G be a complex simple group and $\widetilde{G} \twoheadrightarrow G$ the universal covering. For simplicity, let us consider the moduli stack $\mathcal{M}_G$ (resp. $\mathcal{M}_{\widetilde{G}}$) of degree $1 \in \pi_1(G)$ principal G-bundles (resp. $\widetilde{G}$-bundles) over a curve X. In [B-L-S], we have studied the natural morphism

$$\iota : \mathrm{Pic}(\mathcal{M}_G) \longrightarrow \mathrm{Pic}(\mathcal{M}_{\widetilde{G}}),$$

the group $\mathrm{Pic}(\mathcal{M}_{\widetilde{G}})$ being infinite cyclic by [L-S]. It is proved in [B-L-S] that the kernel of ι is naturally identified with the finite group $H^1_{\text{ét}}(X, \pi_1(G)^\vee)$ reducing the study of $\mathrm{Pic}(\mathcal{M}_G)$ to the computation of the cardinality of $\mathrm{Coker}(\iota)$. Among other things, it has been possible to

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perform this computation in the case where $G = \mathbf{PSL}_r$ but not in the case where $G = \mathbf{SL}_r/\mu_s$, where $s \mid r$, although we were able to give partial results. The reason was that the technical background to study the descent of modules through the morphism $p : \mathcal{M}_{\widetilde{G}} \to \mathcal{M}_G$ wasn't at our disposal.

The aim of this paper is to compute $\operatorname{card}\operatorname{Coker}(\iota)$ when $G = \mathbf{SL}_r/\mu_s$.

It turns out to be that p is a torsor under some group stack, not far from a Galois étale cover in the usual schematic picture. Let $f : \mathcal{X} \to \mathcal{Y}$ be a torsor under a group scheme $\mathcal{G}$. We know that a line bundle on $\mathcal{X}$ descends if and only if it has a $\mathcal{G}$-linearization (easy consequence of descent theory). Now, the descent theory of Grothendieck has been adapted to the set-up of fpqc morphisms of stacks in [L-M]. If $\mathcal{G}$ is now only assumed to be a group stack, there is a notion of $\mathcal{G}$-linearization of line bundles on $\mathcal{X}$ (see section 2). One obtains (theorem 4.1) that a line bundle on $\mathcal{X}$ descends if and only if it admits a linearization.

We then use this technical result to compute $\operatorname{card}\operatorname{Coker}(\iota)$ when $G = \mathbf{SL}_r/\mu_s$ (theorem 5.7 and its corollary).

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Notations.

Throughout this paper, all the stacks will be implicitely assumed to be algebraic over an algebraically closed field $\boldsymbol{k}$ and the morphisms locally of finite type. We fix once and for all a projective, smooth, connected genus g curve X and a closed point x of X. For simplicity, we assume $g > 0$ (see remarks 5.6 and 5.10 for the case of $\mathbf{P}^1$). The Picard stack parametrizing families of line bundles of degree 0 on X will be denoted by $\mathcal{J}(X)$ and the jacobian variety of X by JX. If G is an algebraic group over $\boldsymbol{k}$, the quotient stack $\operatorname{Spec}(\boldsymbol{k})/G$ (where G acts trivially on $\operatorname{Spec}(\boldsymbol{k})$) whose category over a $\boldsymbol{k}$-scheme S is the category of G-torsors (or G-bundles) over S will be denoted by BG. If n is an integer and $A = \mathcal{J}(X), JX$ or BG_m we denote by n_A the n^{th}-power morphism $a \mapsto a^n$. We denote by $\mathcal{J}_n$ (resp. J_n) the 0-fiber $A \times_A \operatorname{Spec}(\boldsymbol{k})$ of n_A when $A = \mathcal{J}(X)$ (resp. $A = JX$).

1. Generalities. — Following [Br], for any diagram

$$\begin{tikzcd} A \arrow[r, "h"] & B \arrow[r, bend left, "g"{name=U}] \arrow[r, bend right, "f"'{name=D}] \arrow[from=D, to=U, Rightarrow, "\lambda"'] & C \arrow[r, "\ell"] & D \end{tikzcd}$$

of 2-categories, we will denote by

$$\ell * \lambda : \ell \circ f \Rightarrow \ell \circ g \quad (\text{resp. } \lambda * h : f \circ h \Rightarrow g \circ h)$$

the 2-morphism deduced from λ.

1.1. — For the convenience of the reader, let us prove a simple formal lemma which will be usefull in section 4. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be three 2-categories. Let diagram

$$\begin{tikzcd} & \mathcal{C} \\ \mathcal{A} \arrow[ur, "\delta_0"] \arrow[r, "f"] \arrow[dr, "\delta_1"'] & \mathcal{B} \arrow[u, "d_0"'] \arrow[d, "d_1"] \\ & \mathcal{C} \end{tikzcd} \tag{1.1.1}$$

be a 2-commutative diagram and let $\mu : \delta_0 \Rightarrow \delta_1$ be a 2-morphism.

LEMMA 1.2. — *Assume that f is an equivalence. There exists a unique 2-morphism*

$$\mu * f^{-1} : d_0 \Rightarrow d_1$$

such that $(\mu * f^{-1}) * f = \mu$.

Proof. — Let ϵ_k, for $k = 0, 1$ be the 2-morphism $d_k \circ f \Rightarrow \delta_k$. Let b be an object of $\mathcal{B}$. Pick an object a of $\mathcal{A}$ and an isomorphism $\alpha : f(a) \xrightarrow{\sim} b$. Let $\varphi_\alpha : d_0(b) \xrightarrow{\sim} d_1(b)$ be the unique isomorphism making the diagram

$$\begin{tikzcd} \delta_0(a) \arrow[r, "\epsilon_0(a)"] \arrow[d, "\mu_a"'] & d_0 \circ f(a) \arrow[r, "d_0(\alpha)"] & d_0(b) \arrow[d, "\varphi_\alpha"] \\ \delta_1(a) \arrow[r, "\epsilon_1(a)"] & d_1 \circ f(a) \arrow[r, "d_1(\alpha)"] & d_1(b) \end{tikzcd}$$

commutative. We have to show that φ_α does not depend on α but only on b. Let $\alpha' : f(a') \xrightarrow{\sim} b$ be another isomorphism. There exists a unique isomorphism $\iota : a' \xrightarrow{\sim} a$ such that $\alpha \circ f(\iota) = \alpha'$. Then one has the equality $\varphi_{\alpha'} = d_1(\alpha) \circ \Phi \circ d_0(\alpha)^{-1}$ where

$$\Phi = \big[d_1 \circ f(\iota)\big] \circ \epsilon_1(a') \circ \mu_{a'} \circ \epsilon_0(a')^{-1} \circ \big[d_0 \circ f(\iota)\big]^{-1}.$$

The functoriality of ϵ_i and μ ensures that one has the equalities

$$d_k \circ f(\iota) \circ \epsilon_k(a') = \epsilon_k(a) \circ \delta_k(\iota)$$

and

$$\mu_a = \delta_1(\iota) \circ \mu_{a'} \circ \delta_0(\iota)^{-1}.$$

This shows the equality

$$\Phi = \epsilon_1(a) \circ \mu_a \circ \epsilon_0(a)^{-1}$$

which proves the equality $\varphi_\alpha = \varphi_{\alpha'}$. We can therefore define μ_b to be the isomorphism φ_α for one isomorphism $\alpha : f(a) \xrightarrow{\sim} b$. We check that the construction is functorial in b and the lemma follows. ∎

2. Linearizations of line bundles on stacks.

Let us first recall the notion of torsor in the stack context according to [Br].

2.1. — Let $f : \mathcal{X} \to \mathcal{Y}$ be a faithfuly flat morphism of stacks. Let us assume that an algebraic *gr*-stack $\mathcal{G}$ acts on f (the product of $\mathcal{G}$ is denoted by $m_\mathcal{G}$ and the unit object by 1). Following [Br], this means that there exists a 1-morphism of $\mathcal{Y}$-stacks $m : \mathcal{G} \times \mathcal{X} \to \mathcal{X}$ and a 2-morphism $\mu : m \circ (m_\mathcal{G} \times \mathrm{Id}_\mathcal{X}) \Rightarrow m \circ (\mathrm{Id}_\mathcal{G} \times m)$ such that the obvious associativity condition (see diagram (6.1.3) in [Br]) is satisfied and such that there exists a 2-morphism $\epsilon : m \circ (1 \times \mathrm{Id}_\mathcal{X}) \Rightarrow \mathrm{Id}_\mathcal{X}$ which is compatible to μ in the obvious sense (see (6.1.4) of [Br]).

Remark 2.2. — To say that m is a morphism of $\mathcal{Y}$-stacks means that the diagram

$$\begin{tikzcd} \mathcal{G} \times \mathcal{X} \arrow[rr, "m"] \arrow[dr] & & \mathcal{X} \arrow[dl] \\ & \mathcal{Y} & \end{tikzcd}$$

is 2-commutative. In other words, if we denote for simplicity the image of a pair of objects $m(g,x)$ by $g{\cdot}x$. This means that there exists a functorial isomorphism $\iota_{g,x} : f(g{\cdot}x) \to f(x)$.

2.3. — Suppose that $\mathcal{G}$ acts on such another morphism $f' : \mathcal{X}' \to \mathcal{Y}$. A morphism $p : \mathcal{X}' \to \mathcal{X}$ will be said *equivariant* if there exists a 2-morphism

$$q : m \circ (\mathrm{Id} \times p) \Rightarrow p \circ m'$$

which is compatible to μ (as in [Br, (6.1.6)]) and ϵ (which is implicit in [Br]) in the obvious sense.

Definition 2.4. — With the above notations, we say that f (or $\mathcal{X}$) is a *$\mathcal{G}$-torsor* over $\mathcal{Y}$ if the morphism $\mathrm{pr}_2 \times m : \mathcal{G} \times \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ is an isomorphism (of stacks) and the geometrical fibers of f are not empty.

REMARK 2.5. — In down to earth terms, this means that if

$$\iota : f(x) \longrightarrow f(x')$$

is an isomorphism in $\mathcal{Y}$ (x, x' being objects of $\mathcal{X}$), there exists an object g of $\mathcal{G}$ and a unique isomorphism $(x, g\cdot x) \xrightarrow{\sim} (x, x')$ which induces ι by way of $\iota_{g,x}$ (*cf.* 2.2).

EXAMPLE 2.6. — Let $\mathcal{M}_X(G_m)$ be the Picard stack of X. Then, the morphism

$$\mathcal{M}_X(G_m) \longrightarrow \mathcal{M}_X(G_m)$$

of multiplication by $n \in \mathbb{Z}$ is a torsor under $B\mu_n \times J_n(X)$ (*cf.* (3.1)).

2.7. — Let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. By definition, the datum $\mathcal{L}$ is equivalent to the datum of a morphism $\ell : \mathcal{X} \to BG_m$ (see [L-M, prop. 6.15]). If $\mathcal{L}, \mathcal{L}'$ are two line bundles on $\mathcal{X}$ defined by ℓ, ℓ', we will view an isomorphism $\mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ as a 2-morphism $\ell \Rightarrow \ell'$.

DEFINITION 2.8. — A *$\mathcal{G}$-linearization* of $\mathcal{L}$ is a 2-morphism

$$\lambda : \ell \circ m \Rightarrow \ell \circ \mathrm{pr}_2$$

such that the two diagrams of 2-morphisms

$$\begin{array}{ccc} \ell \circ m \circ (m_{\mathcal{G}} \times \mathrm{Id}_{\mathcal{X}}) & \xRightarrow{\ell * \mu} & \ell \circ m \circ (\mathrm{Id}_{\mathcal{G}} \times m) \\ {\scriptstyle \lambda*(m_{\mathcal{G}}\times \mathrm{Id}_{\mathcal{X}})}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \lambda*(\mathrm{Id}_{\mathcal{G}}\times m)} \\ \ell \circ \mathrm{pr}_2 \circ (m_{\mathcal{G}} \times \mathrm{Id}_{\mathcal{X}}) & \xLeftarrow{\lambda * \mathrm{pr}_{23}} & \ell \circ \mathrm{pr}_2 \circ (\mathrm{Id}_{\mathcal{G}} \times m) \\ \| & & \| \\ \ell \circ \mathrm{pr}_2 \circ \mathrm{pr}_{23} & & \ell \circ m \circ \mathrm{pr}_{23} \end{array} \tag{2.8.1}$$

and

$$\begin{array}{ccc} \ell \circ m \circ (1 \times \mathrm{Id}_{\mathcal{X}}) & \xRightarrow{\ell * \epsilon} & \ell \\ {\scriptstyle \lambda*(1\times \mathrm{Id}_{\mathcal{X}})}\Big\Downarrow & & \| \\ \ell & = & \ell \end{array} \tag{2.8.2}$$

(strictly) commute.

REMARK 2.9. — If g_1, g_2 are objects of $\mathcal{G}$ and d is an object of $\mathcal{X}$, the commutativity of diagram (2.8.1) means that the diagram

$$\begin{tikzcd} \mathcal{L}_{(g_1\cdot g_2)x} \arrow[r, "\sim"'] \arrow[d, "\wr"'] & \mathcal{L}_{g_1(g_2\cdot x)} \arrow[d, "\wr"] \\ \mathcal{L}_x & \mathcal{L}_{g_2\cdot x} \arrow[l, "\sim"'] \end{tikzcd}$$

is commutative and the commutativity of (2.8.2) that the two isomorphisms $\mathcal{L}_{1\cdot x} \simeq \mathcal{L}_x$ defined by the linearization λ and ϵ respectively are the same.

3. An example.

Let me recall that a closed point x of X has been fixed. Let S be a $\boldsymbol{k}$-scheme. The S-points of the jacobian variety of X are by definition isomorphism classes of line bundles on X_S together whith a trivialization along $\{x\} \times S$ (such a pair will be called a *rigidified line bundle*). For the convenience of the reader, let me state this well known lemma which can be found in SGA4, exp. XVIII, (1.5.4).

LEMMA 3.1. — *The Picard stack $\mathcal{J}(X)$ is canonically isomorphic* (*as a $\boldsymbol{k}$-group stack*) *to $JX \times BG_m$.*

Proof. — Let $f : \mathcal{J}(X) \to JX \times BG_m$ be the morphim which associates

• to the line bundle L on X_S the pair $(L \boxtimes L^{-1}_{|\{x\}\times S}, L_{|\{x\}\times S})$ where $\boxtimes$ denotes the external tensor product (this pair is thought of as an object of $JX \times BG_m$ over S);

• to an isomorphism $L \xrightarrow{\sim} L'$ on X_S its restriction to $\{x\} \times S$.

Let $f' : JX \times BG_m \to \mathcal{J}(X)$ be the morphism which associates

• to the pair (L, V) where L is a rigidified bundle on X_S and V a line bundle on S (thought of as an object of $JX \times BG_m$ over S), the line bundle $L \boxtimes_{X_S} V$;

• to an isomorphism $(\ell, v) : (L, V) \xrightarrow{\sim} (L', V')$ the tensor product $\ell \boxtimes_{X_S} v$.

The morphisms f and f' are (quasi)-inverse to each other and are morphisms of $\boldsymbol{k}$-stacks. $\square$

We will identify from now $\mathcal{J}(X)$ and $JX \times BG_m$. Let $\mathcal{L}$ (resp. $\mathcal{P}$ and $\mathcal{T}$) be the universal bundle on $X \times \mathcal{J}(X)$ (resp. on $X \times JX$ and BG_m) and let $\Theta = (\det R\Gamma\mathcal{P})^{-1}$ be the theta line bundle on JX. The isomorphism $\mathcal{L} \xrightarrow{\sim} \mathcal{P} \boxtimes \mathcal{T}$ yields an isomorphism

$$\det R\Gamma\mathcal{L}^n(m\cdot x) \xrightarrow{\sim} \Theta^{-n^2} \boxtimes \mathcal{T}^{(m+1-g)}. \tag{3.1.1}$$

4. Descent of $\mathcal{G}$-line bundles.

The object of this section is to prove the following statement.

THEOREM 4.1. — *Let $f : \mathcal{X} \to \mathcal{Y}$ be a $\mathcal{G}$-torsor as above. Let $\mathrm{Pic}^{\mathcal{G}}(\mathcal{X})$ be the group of isomorphism classes of $\mathcal{G}$-linearized line bundles on $\mathcal{X}$. Then, the pull-back morphism $f^* : \mathrm{Pic}(\mathcal{Y}) \xrightarrow{\sim} \mathrm{Pic}^{\mathcal{G}}(\mathcal{X})$ is an isomorphism.*

The descent theory of Grothendieck has been adapted in the case of algebraic 1-stacks in [L-M], essentially in proposition (6.23).

Let $\mathcal{X}_\bullet \to \mathcal{Y}$ be the (augmented) simplicial complex of stacks coskeleton of f (as defined in [De, (5.1.4)] for instance). By proposition (6.23) of [L-M], one just has to construct a cartesian $\mathcal{O}_{D_\bullet}$-module $\mathcal{L}_\bullet$ such that $\mathcal{L}_0$ is the $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{L}$ to prove the theorem. The n-th piece $\mathcal{X}_n$ is inductively defined by

$$\mathcal{X}_0 = \mathcal{X}, \quad \mathcal{X}_n = \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}_{n-1} \quad \text{for } n > 0.$$

Let $p_n : \mathcal{X}_n \to \mathcal{X}$ be the projection on the first factor. It is the simplicial morphism associated to the map

$$\tilde{p}_n : \begin{cases} \Delta_0 \to \Delta_n, \\ 0 \longmapsto 0. \end{cases}$$

Let $\mathcal{L}_n$ be the line bundle defined by the morphism (see (2.7))

$$\ell_n : \mathcal{X}_n \xrightarrow{p_n} \mathcal{X} \xrightarrow{\ell} BG_m. \tag{4.1.1}$$

4.2. — Let δ_i (resp. s_j) be the face (resp. degeneracy) operators (see for instance [De, 5.1.1]). By abuse of notation, we use the same notation for δ_j, s_j and their image by $\mathcal{X}$). The category $(\Delta_\bullet)$ is generated by the face and degeneracy operators with the following relations (see for instance the proposition VII.5.2, p. 174 of [McL])

$$\delta_i \circ \delta_j = \delta_{j+1} \circ \delta_i \qquad \text{if} \quad i \le j, \tag{4.2.1}$$

$$s_j \circ s_i = s_i \circ s_{j+1} \qquad \text{if} \quad i \le j, \tag{4.2.2}$$

$$s_j \circ \delta_i = \begin{cases} \delta_i \circ s_{j-1} & \text{if} \quad i < j, \\ 1 & \text{if} \quad i = j,\ i = j+1, \\ \delta_{i-1} \circ s_j & \text{if} \quad i > j+1. \end{cases} \tag{4.2.3}$$

Therefore, the datum of a cartesian $\mathcal{O}_{\mathcal{X}_\bullet}$-module $\mathcal{L}_\bullet$ is equivalent to the data of isomorphisms

$$\alpha_j : \delta_j^* \mathcal{L}_n \xrightarrow{\sim} \mathcal{L}_{n+1}, \quad j = 0, \dots, n+1,$$

$$\beta_j : s_j^* \mathcal{L}_{n+1} \xrightarrow{\sim} \mathcal{L}_n, \quad j = 0, \dots, n,$$

(where n is a non negative integer) which are compatible with relations (4.2.1), (4.2.2) and (4.2.32).

Let n be a non negative integer.

4.3. — We have first to define for $j = 0, \dots, n+1$ an isomorphism

$$\alpha_j : \delta_j^* \mathcal{L}_n \xrightarrow{\sim} \mathcal{L}_{n+1}.$$

The line bundle $\delta_j^* \mathcal{L}_n$ is defined by the morphism $\ell \circ p_n \circ \delta_j : \mathcal{X}_{n+1} \to BG_m$ and $\tilde{p}_n \circ \delta_j$ is associated to the map

$$\begin{cases} \Delta_0 \longrightarrow \Delta_{n+1}, \\ \;0 \longmapsto \delta_j(0). \end{cases}$$

• If $j \neq 0$, one has therefore $\tilde{p}_n \circ \delta_j = \tilde{p}_{n+1}$ and $\delta_j^* \mathcal{L}_n = L_{n+1}$. We define α_j by the identity in this case.

• Suppose now that $j = 0$. Let $\pi_n : \mathcal{X}_n \to \mathcal{X}_1$ be the projection on the two first factors (associated to the canonical inclusion $\Delta_1 \hookrightarrow \Delta_n$). The commutativity of the two diagrams

$$\begin{array}{ccc} \mathcal{X}_{n+1} & \xrightarrow{\delta_0} & \mathcal{X}_n \\ {\scriptstyle \pi_{n+1}}\downarrow & & \downarrow{\scriptstyle p_n} \\ \mathcal{X}_1 & \xrightarrow{\delta_0} & \mathcal{X} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{X}_{n+1} & \xrightarrow{p_{n+1}} & \mathcal{X} \\ \downarrow{\scriptstyle \pi_{n+1}} & & \uparrow{\scriptstyle \delta_1} \\ \mathcal{X}_1 & = & \mathcal{X}_1 \end{array}$$

allows to reduce the problem to the construction of an isomorphism

$$\delta_0^* \mathcal{L} \xrightarrow{\sim} \delta_1^* \mathcal{L}$$

where $\delta_i : \mathcal{X}_1 \to \mathcal{X}$ for $i = 0, 1$ are the face morphisms or, what amounts to the same, to the construction of a 2-morphism $\nu : \ell \circ \delta_0 \Rightarrow \ell \circ \delta_1$ (the morphism α_j will be $\alpha_j = \nu * \pi_{n+1}$). Now the diagram

$$\begin{tikzcd} & BG_m \\ \mathcal{G} \times \mathcal{X} \arrow[ur, "\ell \circ m"] \arrow[r, "\mathrm{pr}_2 \times m"] \arrow[dr, "\ell \circ \mathrm{pr}_2"'] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \arrow[u, "\ell \circ \delta_0"'] \arrow[d, "\ell \circ \delta_1"] \\ & BG_m \end{tikzcd} \tag{4.3.1}$$

is strictly commutative and $\mathrm{pr}_2 \times m$ is an equivalence by definition of a torsor. According to lemma 1.2, the 2-morphism λ induces a canonical 2-morphism

$$\lambda * (\mathrm{pr}_2 \times m)^{-1} : \ell \circ \delta_0 \Rightarrow \ell \circ \delta_1$$

which is the required 2-morphism ν.

4.4. — We then have to define for $j = 0, \dots, n$ an isomorphism

$$\beta_j : s_j^* \mathcal{L}_{n+1} \xrightarrow{\sim} \mathcal{L}_n.$$

The line bundle $s_j^* \mathcal{L}$ is defined by the morphism $\ell \circ p_{n+1} \circ s_j$ and $p_{n+1} \circ s_j$ is associated to the canonical inclusion $\Delta_0 \hookrightarrow \Delta_n$ which means $p_{n+1} \circ s_j = p_n$. Therefore, one has a canonical isomorphism $s_j^* \mathcal{L}_{n+1} = \mathcal{L}_n$ and we define β_j to be the identity.

4.5. — We have to show that the data $\mathcal{L}_\bullet$ and α_j, β_j, for $j \geq 0$ define a line bundle on the simplicial stack $\mathcal{X}_\bullet$ as explained in 4.2. Notice that the fact that the definition of the β_j is compatible with relations (4.2.2) is tautological (β_j is the identity on the relevant $\mathcal{L}_n$).

4.6. — In terms of ℓ, relation (4.2.1) means the following. We have the two strictly 2-commutative diagrams

$$\begin{tikzcd}
\mathcal{X}_{n+2} \arrow[r, "\delta_i"] \arrow[dr, "p_{n+2}"'] & \mathcal{X}_{n+1} \arrow[r, "\delta_j"] \arrow[d, "p_{n+1}"'] & \mathcal{X}_n \arrow[dl, "p_n"] \\
 & \mathcal{X} \arrow[r, "\ell"'] & BG_m
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{X}_{n+2} \arrow[r, "\delta_{j+1}"] \arrow[dr, "p_{n+2}"'] & \mathcal{X}_{n+1} \arrow[r, "\delta_i"] \arrow[d, "p_{n+1}"'] & \mathcal{X}_n \arrow[dl, "p_n"] \\
 & \mathcal{X} \arrow[r, "\ell"'] & BG_m
\end{tikzcd}$$

inducing the two 2-morphisms

$$\alpha_i \circ (\alpha_j * \delta_i) : \ell \circ p_n \circ \delta_j \circ \delta_i \xRightarrow{\alpha_j * \delta_i} \ell \circ p_{n+1} \circ \delta_i \xRightarrow{\alpha_i} \ell \circ p_{n+2}$$

and

$$\alpha_{j+1} \circ (\alpha_i * \delta_{j+1}) : \ell \circ p_n \circ \delta_i \circ \delta_{j+1} \xRightarrow{\alpha_i * \delta_{j+1}} \ell \circ p_{n+1} \circ \delta_{j+1} \xRightarrow{\alpha_{j+1}} \ell \circ p_{n+2}.$$

The relation (4.2.1) means exactly the equality

$$(4.2.1') \qquad \alpha_i \circ (\alpha_j * \delta_i) = \alpha_{j+1} \circ (\alpha_i * \delta_{j+1}), \quad \text{for } i \leq j.$$

- If $j = 0$, the relation (4.2.1′) is just by definition of α_j the condition (2.8.1) (see remark 2.9).
- If $j > 0$, both isomorphisms α_j and α_{j+1} are the relevant identities and the relation (4.2.1′) is tautological.

4.7. — The only non tautological relation in (4.2.3) corresponds to the equality $s_0 \circ \delta_0 = 1$ in $(\Delta_\bullet)$ which means as before that $\alpha_0 * \delta_0$ is the identity functor of $\ell \circ p_n = \ell \circ p_n \circ \delta_0 \circ s_0$. But, this is exactly the meaning of the relation (2.8.2) (see remark 2.9).

5. Application to the Picard groups of some moduli spaces.

Let us choose three integers r, s, d such that

$$r \geq 2 \quad \text{and} \quad s \mid r \mid ds.$$

If G is the group $\mathbf{SL}_r/\mu_s$ we denote as in [B-L-S] by $\mathcal{M}_G(d)$ the (connected) moduli stack of G-bundles on X of degree $\exp(2i\pi d/r) \in H^2_{\text{ét}}(X, \mu_s) = \mu_s$ and by $\mathcal{M}_{\mathbf{SL}_r}(d)$ the moduli stack of rank r vector bundles and determinant $\mathcal{O}(d{\cdot}x)$. If $r = s$ (*i.e.* $G = \mathbf{PSL}_r$), the natural morphism of algebraic stacks

$$\pi : \mathcal{M}_{\mathbf{SL}_r}(d) \longrightarrow \mathcal{M}_G(d)$$

is a $\mathcal{J}_r$-torsor (see the corollary of proposition 2 of [Gr] for instance). Let me explain how to deal with the general case.

5.1. — Let E be a rank r vector bundle on X_S endowed with an isomorphism

$$\tau : D^{r/s} \xrightarrow{\sim} \det(E)$$

where D is some line bundle. Let me define the $\mathbf{SL}_r/\mu_s$-bundle $\pi(E)$ associated to E (more precisely to the pair (E, τ)).

Definition 5.2.

• An *s-trivialization* of E on an étale neighborhood $T \to X_S$ is a triple (M, α, σ) where

$\alpha : D \xrightarrow{\sim} M^s$ is an isomorphism (M is a line bundle on T);

$\sigma : M^{\oplus r} \xrightarrow{\sim} E_T$ is an isomorphism;

$\det(\sigma) \circ \alpha^{r/s} : D^{r/s} \xrightarrow{\sim} \det(E)$ is equal to τ.

• Two s-trivializations (M, α, σ) and (M', α', σ') of E will be said *equivalent* if there exists an isomorphism $\iota : M \xrightarrow{\sim} M'$ such that $\iota^s \circ \alpha = \alpha'$.

The principal homogeneous space

$$T \longmapsto \{\text{equivalence classes of } s\text{-trivializations of } E_T\}$$

defines the $\mathbf{SL}_r/\mu_s$-bundle $\pi(E)^\dagger$. Now, the construction is obviously functorial and therefore defines the morphism $\pi : \mathcal{M}_{\mathbf{SL}_r}(d) \to \mathcal{M}_G(d)$ (observe that an object E of $\mathcal{M}_{\mathbf{SL}_r}(d)$ has determinant $\mathcal{O}(ds/r \cdot x)^{r/s}$). Let L be a line bundle and (M, α, τ) an s-trivialization of E_T. Then, $(M \otimes L, \alpha \otimes \mathrm{Id}_{L^s}, \sigma \otimes \mathrm{Id}_L)$ is an s-trivialization of $E \otimes L$ (which has determinant $(D \otimes L^s)^{r/s}$). This shows that there exists a canonical functorial isomorphism

$$\pi(E) \xrightarrow{\sim} \pi(E \otimes L) \tag{5.2.1}$$

In particular, π is $\mathcal{J}_s$-equivariant.

Lemma 5.3. — *The natural morphism of algebraic stacks*

$$\pi : \mathcal{M}_{\mathbf{SL}_r}(d) \longrightarrow \mathcal{M}_G(d)$$

is a $\mathcal{J}_s$-torsor.

Proof. — Let E, E' be two rank r vector bundles on X_S (with determinant equal to $\mathcal{O}(d{\cdot}x)$) and let $\iota : \pi(E) \xrightarrow{\sim} \pi(E)'$ be an isomorphism. As in the proof of the lemma 13.4 of [B-L-S], we have the exact sequence of sets

$$1 \to \mu_s \longrightarrow \mathrm{Isom}(E, E') \longrightarrow \mathrm{Isom}\big(\pi(E), \pi(E)'\big) \xrightarrow{\pi_{E,E'}} H^1_{\text{ét}}(X_S, \mu_s).$$

Let L be a μ_s-torsor such that $\pi_{E,E'}(\iota) = [L]$. Then, $\pi(E \otimes L)$ is equal to $\pi(E)$, $\pi_{E\otimes L, E'} = 0$ and ι is induced by an isomorphism $E \otimes L \xrightarrow{\sim} E'$ well defined up to multiplication by μ_s. The lemma follows. $\square$

5.4. — Let $\mathcal{U}$ be the universal bundle on $X \times \mathcal{M}_{\mathbf{SL}_r}(d)$. We would like to know which power of the determinant bundle $\mathcal{D} = (\det R\Gamma\mathcal{U})^{-1}$ on $\mathcal{M}_{\mathbf{SL}_r}(d)$ descends to $\mathcal{M}_G(d)$. As in I.3 of [B-L-S], the rank r bundle

$$\mathcal{F} = \mathcal{L}^{\oplus(r-1)} \oplus \mathcal{L}^{1-r}(d{\cdot}x)$$

on $X \times \mathcal{J}(X)$ has determinant $\mathcal{O}(d{\cdot}x)$ and therefore defines a morphism

$$f : \mathcal{J}(X) = JX \times BG_m \longrightarrow \mathcal{M}_{\mathbf{SL}_r}(d)$$

which is $\mathcal{J}_s$-equivariant.

† We see here a G-bundle as a formal homogeneous space under G.

The vector bundle

$$\mathcal{F}' = \mathcal{O}^{\oplus(r-1)} \oplus \mathcal{L}^{-r/s}(d{\cdot}x)$$

on $X \times \mathcal{J}(X)$ has determinant $[\mathcal{L}^{-1}(ds/r \cdot x)]^{r/s}$. The G-bundle $\pi(\mathcal{F}')$ on $X \times \mathcal{J}(X)$ defines a morphism $f' : \mathcal{J} \to \mathcal{M}_G(d)$. The relation

$$\mathcal{L} \otimes (\mathrm{Id}_X \times s_{\mathcal{J}})^* * (\mathcal{F}') = \mathcal{F}$$

and (5.2.1) gives an isomorphism

$$\pi(\mathcal{F}) = (\mathrm{Id}_X \times s_{\mathcal{J}})^* \pi(\mathcal{F}')$$

which means that the diagram

$$\begin{array}{ccc} \mathcal{J}(X) & \xrightarrow{f} & \mathcal{M}_{\mathbf{SL}_r}(d) \\ {\scriptstyle s_{\mathcal{J}}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{J}(X) & \xrightarrow{f'} & \mathcal{M}_G(d) \end{array} \tag{5.4.1}$$

is 2-commutative. Exactly as in I.3 of [B-L-S], let me prove the

LEMMA 5.5. — *The line bundle $f^*\mathcal{D}^k$ on $\mathcal{J}(X)$ descends through $s_{\mathcal{J}}$ if and only if k multiples of $s/(s, r/s)$.*

Proof. — Let $\chi = r(g-1) - d$ be the opposite of the Euler characteristic of ($\boldsymbol{k}$-)points of $\mathcal{M}_{\mathbf{SL}_r}(d)$. By (3.1.1), one has an isomorphism

$$f^*\mathcal{D}^k \xrightarrow{\sim} \Theta^{kr(r-1)} \boxtimes \mathcal{T}^{k\chi}.$$

The theory of Mumford groups says that $\Theta^{kr(r-1)}$ descends through s_J if and only if k is a multiple of $s/(s, r/s)$. The line bundle $\mathcal{T}^{k\chi}$ on BG_m descends through s_{BG_m} if and only if $k\chi$ is a multiple of s. The lemma follows from the above isomorphism and from the observation that the condition $s \mid r \mid ds$ forces $s\chi$ to be a multiple of s. $\square$

REMARK 5.6. — If $g = 0$, the jacobian J is a point and the condition on Θ is empty. The only condition in this case is that $k\chi$ is a multiple of s.

Let me recall that $\mathcal{D}$ is the determinant bundle on $\mathcal{M}_{\mathbf{SL}_r}(d)$ and $G = \mathbf{SL}_r/\mu_s$.

THEOREM 5.7. — *Assume that the characteristic of $\boldsymbol{k}$ is 0. The integers k such that $\mathcal{D}^k$ descends to $\mathcal{M}_G(d)$ are the multiple of $s/(s, r/s)$.*

By the proposition 1.5 of [BLS], one gets the

COROLLARY 5.8. — *The natural morphism*

$$\operatorname{Pic}\big(\mathcal{M}_G(d)\big) \longrightarrow \operatorname{Pic}\big(\mathcal{M}_{\mathbf{SL}_r}(d)\big) = \mathbb{Z}\cdot\mathcal{D}$$

makes the Picard group of $\mathcal{M}_G(d)$ an extension of $\mathbb{Z} = \mathbb{Z}\cdot\mathcal{D}^{s/(s,r/s)}$ by $H^1_{\text{ét}}(X,\mathbb{Z}/d\mathbb{Z}) \xrightarrow{\sim} (\mathbb{Z}/d\mathbb{Z})^{2g}$.

Proof of the theorem. — By lemma 5.5 and diagram (5.4.1), we just have to prove that $\mathcal{D}^k$ effectively descends when $k = s/(s,r/s)$. By theorem 4.1 and lemma 5.3, this means exactly that $\mathcal{D}^k$ has a $\mathcal{J}_s$-linearization. We know by lemma 5.5 that the pull-back $f^*\mathcal{D}^k$ has such a linearization.

LEMMA. — *The pull-back morphism*

$$\operatorname{Pic}\big(\mathcal{J}_s \times \mathcal{M}_{SL_r}(d)\big) \longrightarrow \operatorname{Pic}\big(\mathcal{J}_s \times \mathcal{J}(X)\big)$$

is injective.

Proof. — By lemma 3.1, one is reduced to prove that the natural morphism

$$\operatorname{Pic}\big(B\mu_s \times \mathcal{M}_{SL_r}(d)\big) \longrightarrow \operatorname{Pic}\big(B\mu_s \times \mathcal{J}(X)\big)$$

is injective. Let $\mathcal{X}$ be any stack. The canonical morphism $\mathcal{X} \to \mathcal{X} \times B\mu_s$ is a μ_s-torsor (with the trivial action of μ_s on $\mathcal{X}$). By theorem 4.1, one has the equality

$$\operatorname{Pic}(\mathcal{X} \times B\mu_s) = \operatorname{Pic}^{\mu_s}(\mathcal{X}).$$

Assume further that $H^0(\mathcal{X},\mathcal{O}) = \boldsymbol{k}$. The later group is then canonically isomorphic to

$$\operatorname{Pic}(\mathcal{X}) \times \operatorname{Hom}(\mu_s, G_m) = \operatorname{Pic}(\mathcal{X}) \times \operatorname{Pic}(B\mu_s).$$

Eventually, we get a functorial isomorphism

$$\iota_{\mathcal{X}} : \operatorname{Pic}(\mathcal{X} \times B\mu_s) \xrightarrow{\sim} \operatorname{Pic}(\mathcal{X}) \times \operatorname{Pic}(B\mu_s). \tag{5.9.1}$$

By [L-S], the Picard group of $\mathcal{M}_{\mathbf{SL}_r}(d)$ is the free abelian group $\mathbb{Z}\cdot\mathcal{D}$ and the formula (3.1.1) proves that

$$f^* : \operatorname{Pic}\big(\mathcal{M}_{\mathbf{SL}_r}(d)\big) \longrightarrow \operatorname{Pic}\big(\mathcal{J}(X)\big)$$

is an injection. The diagram

$$\begin{tikzcd}
\mathrm{Pic}(\mathcal{M}_{\mathbf{SL}_r}(d)) \times \mathrm{Pic}(B\mu_s) \arrow[r, hook] \arrow[d, "\iota_{\mathcal{M}}"', "\wr"] & \mathrm{Pic}(\mathcal{J}(X)) \times \mathrm{Pic}(B\mu_s) \arrow[d, "\iota_{\mathcal{J}}"', "\wr"] \\
\mathrm{Pic}(\mathcal{M}_{\mathbf{SL}_r}(d) \times B\mu_s) \arrow[r] & \mathrm{Pic}(\mathcal{J}(X) \times B\mu_s)
\end{tikzcd}$$

is commutative and the lemma follows from this commutative diagram. $\square$

Let $\mathcal{H}$ (resp. $\mathcal{H}_{\mathcal{J}}$) be the line bundle on $\mathcal{J}_s \times \mathcal{M}_{\mathbf{SL}_r}(d)$ (resp. $\mathcal{J}_s \times \mathcal{J}(X)$)

$$\mathcal{H} = \mathcal{H}om(m_{\mathcal{M}}^* \mathcal{D}^k, \mathrm{pr}_2^* \mathcal{D}^k)$$

$$\big(\text{resp. } \mathcal{H}_{\mathcal{J}} = \mathcal{H}om(m_{\mathcal{M}}^* f^* \mathcal{D}^k, \mathrm{pr}_2^* f^* \mathcal{D}^k)\big).$$

Let us choose a $\mathcal{J}_s$-linearization $\lambda_{\mathcal{J}}$ of $f^*\mathcal{D}^k$. It defines a trivialization of the line bundle $\mathcal{H}_{\mathcal{J}}$. The equivariance of f implies (*cf.* 2.3) that there exists a (compatible) 2-morphism

$$q : m_{\mathcal{M}} \circ (\mathrm{Id} \times f) \Longrightarrow f \circ m_{\mathcal{J}}$$

making the diagram

$$\begin{tikzcd}
\mathcal{J}_s \times \mathcal{J}(X) \arrow[r, "m_{\mathcal{J}}"] \arrow[d, "\mathrm{Id} \times f"'] & \mathcal{J}(X) \arrow[d, "f"] \\
\mathcal{J}_s \times \mathcal{M}_{\mathbf{SL}_r}(d) \arrow[r, "m_{\mathcal{M}}"] & \mathcal{M}_{\mathbf{SL}_r}(d)
\end{tikzcd}$$

2-commutative. The 2-morphism q defines an isomorphism from the pull-back $m_{\mathcal{M}}^* \mathcal{D}^k$ on $\mathcal{J}_s \times \mathcal{J}(X)$ to $m_{\mathcal{J}}^*(f^*\mathcal{D}^k)$. The pull-back of $\mathrm{pr}_2^* \mathcal{D}^k$ on $\mathcal{J}_s \times \mathcal{J}(X)$ is tautologically isomorphic to $\mathrm{pr}_2^*(f^*\mathcal{D}^k)$. The preceding isomorphisms induce an isomorphism

$$(\mathrm{Id} \times f)^* \mathcal{H} \xrightarrow{\sim} \mathcal{H}_J.$$

The later line bundle being trivial, so is $(\mathrm{Id} \times f)^*\mathcal{H}$. The lemma above proves therefore that $\mathcal{H}$ itself is *trivial*. Each ($\boldsymbol{k}$-)point j of $\mathcal{J}_s$ defines a morphism

$$\mathcal{M}_{\mathbf{SL}_r}(d) \to \mathcal{J}_s \times \mathcal{M}_{\mathbf{SL}_r}(d) \quad \big(\text{resp. } \mathcal{J}(X) \to \mathcal{J}_s \times \mathcal{J}(X)\big);$$

let me denote by $\mathcal{H}_j$ (resp. $f^*\mathcal{H}_j$) the pull-back of $\mathcal{H}$ (resp. $(\mathrm{Id} \times f)^*\mathcal{H}$) by this morphism. The pull-back morphism

$$H^0\big(\mathcal{J}_s \times \mathcal{M}_{\mathbf{SL}_r}(d), \mathcal{H}\big) \longrightarrow H^0\big(\mathcal{J}_s \times \mathcal{J}(X), (\mathrm{Id} \times f)^*\mathcal{H}\big)$$

can be identified to the direct sum

$$\bigoplus_{j\in\mathcal{J}_s(\boldsymbol{k})} H^0\big(\mathcal{M}_{\mathbf{SL}_r}(d),\mathcal{H}_j\big) \longrightarrow H^0\big(\mathcal{J}(X),f^*\mathcal{H}_j\big).$$

As

$$H^0\big(\mathcal{M}_{\mathbf{SL}_r}(d),\mathcal{O}\big) = H^0\big(\mathcal{J}(X),\mathcal{O}\big) = \boldsymbol{k}, \tag{5.9.2}$$

this morphism is a direct sum of non-zero homomorphisms of vector spaces of dimension 1 and therefore an isomorphism. In particular, a linearization $\lambda_{\mathcal{J}}$ of $f^*\mathcal{D}^k$ defines canonicaly an isomophism

$$\lambda_{\mathcal{M}} : m_{\mathcal{M}}^*\mathcal{D}^k \xrightarrow{\sim} \mathrm{pr}_2^*\mathcal{D}^k$$

such that $(\mathrm{Id}\times f)^*\lambda_{\mathcal{M}} = \lambda_{\mathcal{J}}$.

Explicitely, $\lambda_{\mathcal{M}}$ is characterized as follows. Let x be an object of $\mathcal{M}_{\mathbf{SL}_r}(d)$ over a connected scheme S and g an object of $\mathcal{J}_s(S) = \mathcal{J}_s(\boldsymbol{k})$. The preceding dicussion means that the functorial isomorphisms

$$\lambda_{\mathcal{M}}(g,x) : \mathcal{D}^k_{g\cdot x} \xrightarrow{\sim} \mathcal{D}^k_x$$

are determined when x lies in the essential image of f. In this case, let us choose an isomorphism $f(x') \xrightarrow{\sim} x$ (inducing an isomorphism $g{\cdot}f(x') \xrightarrow{\sim} g{\cdot}x$). Then, the diagram of isomorphisms of line bundles on S

$$\begin{tikzcd}
L'_{x'} = L_{f(x')} \arrow[r] \arrow[d, "\lambda_{\mathcal{J}}(g,x')"'] & L_x & \\
L'_{g\cdot x'} = L_{f(g\cdot x')} \arrow[r, "q_{g,x'}"] & L_{g\cdot f(x')} \arrow[r] & L_{g\cdot x} \arrow[ul, "\lambda_{\mathcal{M}}(g,x)"']
\end{tikzcd}$$

is commutative (where $L = \mathcal{D}^k$ and $L' = f^*\mathcal{D}^k$).

Now, the pull-back of $\lambda_{\mathcal{M}}$ on $\mathcal{J}_s \times \mathcal{J}(X)$ satisfies conditions (2.8.1) and (2.8.2). Using (5.9.2) and the equivariance of f as above, this shows that $\lambda_{\mathcal{M}}$ is a linearization. For instance, keeping the notation above, let us verify condition (2.8.2). We have to check that the isomorphism ι of L induced by ϵ is the identity. As above, it is enough to check that on $\mathcal{J}(X)$. With a slight abuse of notations, the two diagrams

$$\begin{tikzcd}
L'_{x'} = L_{f(x')} \arrow[r] & L_x & \\
L'_{1\cdot x'} = L_{f(1\cdot x')} \arrow[u, "\lambda_{\mathcal{J}}(1,x')"'] \arrow[r, "q_{1,x'}"] & L_{1\cdot f(x')} \arrow[r] & L_{1\cdot x} \arrow[ul, "\lambda_{\mathcal{M}}(1,x)"']
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
L_x \arrow[r, "\iota"] & L_x \arrow[d, "\epsilon(x)"] \\
 & L_{1\cdot x} \arrow[ul, "\lambda_{\mathcal{M}}(1,x)"]
\end{tikzcd}$$

are commutative (the commutativity of the first diagram follows from the equivariance of f and the commutativity of the second diagram follows exactly from the definition of ι). Because $\lambda_{\mathcal{J}}$ is a linearization, condition (2.8.2) shows that the diagram

$$\begin{tikzcd} L'_{x'} \arrow[r, equal] & L'_{x'} \\ & L_{1\cdot x'} \arrow[ul, "\lambda_{\mathcal{J}}(1{,}x')"] \arrow[u, "\epsilon'(x')"'] \end{tikzcd}$$

is commutative. It follows that the equality $\iota = \mathrm{Id}$ will follow from the commutativity of the diagram

$$\begin{tikzcd} L_{f(1\cdot x')} \arrow[r, "\epsilon'"] \arrow[d, "q_{1,x'}"'] & L_{f(x')} \arrow[d, equal] \\ L_{1\cdot f(x')} \arrow[r, "\epsilon"] & L_{f(x')}. \end{tikzcd} \tag{5.9.3}$$

But q is compatible with ϵ as required in 2.3. The diagram

$$\begin{tikzcd} f(1.x') \arrow[r, "\epsilon'"] \arrow[d, "q_{1,x'}"'] & f(x') \arrow[d, equal] \\ 1\cdot f(x') \arrow[r, "\epsilon"] & f(x') \end{tikzcd}$$

is therefore commutative from which the commutativity of (5.9.3) follows. One would check condition (2.8.1) in an analogous way. □

REMARK 5.10. — In the case $g = 0$, the condition is an in remark 5.6.

REMARK 5.11. — This linearization can be certainly also deduced from a careful analysis of the first section of [Fa], but the method above seems simpler.

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