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GEOMETRIC REALIZATIONS OF SUBSTITUTIONS

BY CHARLES HOLTON and LUCA Q. ZAMBONI (*)

ABSTRACT. — Substitution dynamical systems are abstract objects, and it is therefore natural to look for ways of representing them geometrically. In this paper we give geometric realizations of a large class of substitutions. We only require that the substitution be primitive and that the incidence matrix have an eigenvalue α with $0 < |\alpha| < 1$.

RÉSUMÉ. — SYSTÈMES DYNAMIQUES ENGENDRÉS PAR DES SUBSTITIONS. — Les systèmes dynamiques engendrés par des substitutions sont des objets abstraits. Il est pourtant naturel de chercher à les représenter géométriquement. Dans ce travail nous donnons des représentations géométriques d'une grande classe de substitutions. Nous supposerons seulement que la substitution soit primitive et que la matrice de la substitution a une valuer propre α avec $0 < |\alpha| < 1$.

1. Introduction

In [10] the authors associate a free geometric exotic $\mathbb{F}_3$-action on an $\mathbb{R}$-tree to a primitive substitution on three letters. The dynamics of the substitution is encoded in an interval translation mapping $f : [0,1] \to [0,1]$ on three intervals (this was first observed by M. Boshernitzan and I. Kornfeld in [10]). Via a result of D. Gaboriau and G. Levitt in [9], copies of the unit interval are glued together according to the orbit of f to obtain an $\mathbb{R}$-tree with an $\mathbb{F}_3$-action by isometries. The "self similar" nature of the substitution is used to conclude that the resulting $\mathbb{F}_3$-action is both free and exotic.

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The key feature in this construction is the geometric encoding of the dynamics of τ, in this case as a system of partial isometries. G. Rauzy [19] discovered an isomorphism between a certain substitution dynamical system and a Weyl automorphism on the torus $\mathbb{R}^2/\mathbb{Z}^2$. From a different point of view, E. Bombieri and J. Taylor exhibit in [2] a connection between algebraic number theory and the theory of quasicrystals by geometrically encoding the dynamics of certain substitutions on a three letter alphabet. In [4], M. Dekking develops a method of constructing fractal tilings of the plane (including the famous Penrose tiling) using substitutions. Many others, including P. Arnoux [1], M. Boshernitzan-I. Kornfeld [3], S. Ferenczi [7], S. Ito-M. Kimura [13], and B. Solomyak [21] have constructed various types of geometric realizations of substitutions. M. Queffélec asks for which substitutions is it possible to find such a geometric description. (See § VI.5 in [18].)

Let τ be a primitive substitution on the alphabet $\mathcal{A} = \{1, 2, \ldots, r\}$ fixing a point $w_* \in \mathcal{A}^{\mathbb{N}}$. Associated to τ is the dynamical system (X, T), where T is the one-sided shift on $\mathcal{A}^{\mathbb{N}}$ and X the closure in $\mathcal{A}^{\mathbb{N}}$ of the T-orbit of w_*. The primitivity condition assures that (X, T) is minimal and does not depend on the choice of w_*. Following [18] (see pages 140–141), a *complex geometric realization* of τ is a triple $\mathcal{G} = \mathcal{G}(f, \lambda, \boldsymbol{v})$ consisting of a continuous function $f : X \to \mathbb{C}$, a nonzero complex number λ of modulus less than one, and a nonzero vector $\boldsymbol{v} = (v_1, v_2, \ldots, v_r) \in \mathbb{C}^r$, such that for each letter $i \in \mathcal{A}$ and for each point $w \in X$ beginning in i one has

$$f\big(\tau(w)\big) = \lambda f(w) \quad \text{and} \quad f(Tw) = f(w) + v_i.$$

We call $\mathcal{G}$ a *real geometric realization* if $f(X) \subset \mathbb{R}$. We show that associated to each left eigenvector $\boldsymbol{v}_\alpha$ of the incidence matrix M_τ corresponding to a nonzero eigenvalue α of modulus less than one is a complex geometric realization $\mathcal{G} = \mathcal{G}(f, \alpha, \boldsymbol{v}_\alpha)$, and every complex geometric realization of τ arises in this way. (See Theorem 3.4.)

To define the map f, we first define a complex valued function S on the set $\mathcal{A}^*$ of all (finite) words in the alphabet $\mathcal{A}$ by setting

$$S(u) = \sum_{i=1}^{r} |u|_i v_i,$$

where $|u|_i$ is the number of occurrences of the letter i in u. Write $w(n)$ for the initial subword of w of length n. We show (Theorem 2.3) that if $0 < |\alpha| < 1$ then the sequence $S_n := S(w_*(n))$ is bounded and its closure, denoted Ω, is a compact perfect subset of $\mathbb{C}$. Otherwise, $\{S_n\}$ is either unbounded or has finite range. The map $f : X \to \mathbb{C}$ is defined in terms of S as follows : For $w \in X$, let $w_1, w_2, w_3, \dots$ be a sequence of tails of the fixed point w_* converging to w. For each $n \geq 1$ there exist words $u_n \in \mathcal{A}^*$ with $w_* = u_n w_n$, and we set

$$f(w) = \lim_{n\to\infty} S(u_n).$$

If $0 < |\alpha| < 1$ then this limit exists and does not dependent on the choice of the tails w_n. Thus, f is a uniformly continuous function on X whose image is Ω. (See Theorem 3.2.) This gives a natural decomposition $\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_r$ where Ω_i is the image under f of all points w in X beginning in the letter i.

Consider, for example, the substitution τ on three letters given by

$$1 \longmapsto 12323, \quad 2 \longmapsto 121, \quad 3 \longmapsto 23.$$

Corresponding to the eigenvalue $\alpha \approx -.191487884 + .508851779i$ and eigenvector $\boldsymbol{v}_\alpha = ((\alpha-1)^2, 2(\alpha-1), 2)$ of M_τ^t is a complex geometric realization of τ. The image set Ω is shown in Figure 1.1. The Ω_i in this example have nonempty interiors and do not overlap.

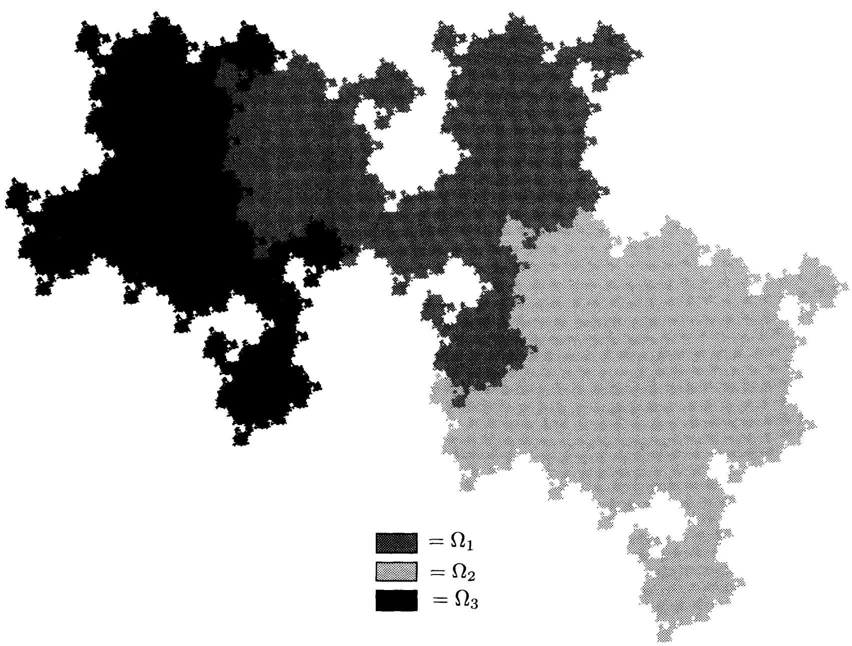


Figure 1.1

We also use S to define a function $\rho : X \times X \longrightarrow \mathbb{R}^{\geq 0}$ as follows : For $w, w' \in X$, choose a sequence $\{a_n\}$ of nonnegative integers such that

$$w(a_n + n) = w(a_n)w'(n),$$

and put

$$\rho(w, w') = \lim_{n \to \infty} \left| S(w(a_n)) \right|.$$

We show that if $0 < |\alpha| < 1$ then this limit exists independent of the choice of $\{a_n\}$ and defines a pseudometric on X whose associated metric space X/ρ is isometric to Ω. (See Theorem 3.1.)

In Appendix A we compare the sequence $\{S_n\}$ to the sequence $\{\delta_n\}$ given by

$$\delta_n = n \cdot (p_1, p_2, \ldots, p_{r-1}) - \left(|w_*(n)|_1, |w_*(n)|_2, \ldots, |w_*(n)|_{r-1} \right) \in \mathbb{R}^{r-1},$$

where $\boldsymbol{p} = (p_1, p_2, \ldots, p_r)$ is the strictly positive eigenvector of M_τ corresponding to the Perron-Frobenius eigenvalue θ, normalized so that $\sum_{i=1}^{r} p_i = 1$. Rauzy (see [19] and [20]) showed (under certain technical conditions) that the sequence $\{\delta_n\}$ is bounded and its closure provides an encoding of τ. As a consequence of Theorem 2.3 we are able to show that if M_τ has $r-1$ distinct eigenvalues of modulus less than one then $\{\delta_n\}$ is bounded. If, however, M_τ has an eigenvalue $\alpha \neq \theta$ with $|\alpha| \geq 1$ and α is not a root of unity, then $\{\delta_n\}$ is unbounded. The substitution in Example 4.7 has two eigenvalues of modulus greater than one and therefore $\{\delta_n\}$ is unbounded, yet the sequence $\{S_n\}$ corresponding to the eigenvalue $\alpha \approx .445041868$ is bounded and its closure $\Omega \subset \mathbb{R}$ gives a geometric encoding of the substitution. (See Figure 4.4.)

In Section 4 we give an alternative characterization of Ω as the limit set of a Mauldin-Williams type graph directed construction. (See [15].) A primitive substitution τ on $\mathcal{A}$ naturally determines a strongly connected directed graph G with vertex set $\mathcal{A}$. To each directed edge e from vertex i to vertex j corresponds a similarity g_e from Ω_j to Ω_i with coefficient of contraction equal to α. We define a continuous surjection h from the set of (rooted) infinite paths in G onto Ω (Theorem 4.3). Each point $x \in \Omega_i$ is thus coded (possibly nonuniquely) by h as an infinite path in G beginning at vertex i. We use this to show that the Hausdorff dimension of Ω is bounded above by

$$\beta = -\frac{\log \theta}{\log |\alpha|} \quad \text{and} \quad \mathcal{H}^\beta(\Omega) < \infty.$$

We give examples for which β is equal to the Hausdorff dimension of Ω.

Suppose $\mathcal{G}$ is a real geometric realization of τ, *i.e.*, $\Omega \subset \mathbb{R}$. Associated to $\mathcal{G}$ is a system of partial isometries $\mathcal{I}_{\mathcal{G}} = (I, \phi_i : D_i \to R_i)_{i=1}^{r}$ defined as follows : $I = [a, b]$ is the smallest closed interval containing Ω, and for each $i \in \mathcal{A}$, $D_i = [a_i, b_i]$ is the smallest closed interval containing Ω_i. For $x \in D_i$ set

$$(x)\phi_i = x + S(i) = x + v_i.$$

In Appendix B we derive an algorithm for computing the endpoints of each D_i. We deduce as a consequence of this algorithm that the eigenvector $\boldsymbol{v}_\alpha$ can be chosen so that each endpoint of D_i and each one sided limit point of Ω is a polynomial in the eigenvalue α (with integer coefficients). It is shown in [11] that if the intervals D_i are nonoverlapping and cover I, then $\mathcal{I}_{\mathcal{G}}$ is either an interval exchange mapping or exotic in the sense of [8]. In each case Ω is identified with the limit set I_∞ of $\mathcal{I}_{\mathcal{G}}$ defined by D. Gaboriau in [8].

This work was largely motivated by earlier work of M. Boshernitzan and I. Kornfeld on interval translation mappings. (See [3].) We thank D. Mauldin and M. Urbański for many useful conversations concerning the material in Section 4. We also wish to thank the referee for valuable comments and suggestions.

2. Dynamics of Primitive Substitutions

Let $\mathcal{A} = \{1, 2, \ldots, r\}$ and $\mathcal{A}^*$ be the set of all words of finite length in the alphabet $\mathcal{A}$. We regard the empty word $u_\emptyset$ as the unique element of $\mathcal{A}^*$ of length zero. Set $\mathcal{A}^+ = \mathcal{A}^* \setminus \{u_\emptyset\}$. We write $|w|$ for the length of $w \in \mathcal{A}^*$. If $1 \le j \le r$, let $|w|_j$ be the number of occurrences of the letter j in w, so that

$$\sum_{j=1}^{r} |w|_j = |w|.$$

Let $\mathcal{A}^{\mathbb{N}}$ denote the set of all sequences in $\mathcal{A}$. For $n \ge 0$ and $w \in \mathcal{A}^{\mathbb{N}}$, let $w(n) \in \mathcal{A}^*$ be the initial subword of w of length n. The natural topology on $\mathcal{A}^{\mathbb{N}}$ (the countable product of the discrete topology on $\mathcal{A}$) is metrizable. Specifically, we define the *standard metric* d on $\mathcal{A}^{\mathbb{N}}$ by setting

$$d(w, w') = \mathrm{e}^{-n} \text{ if } w(n) = w'(n) \text{ and } w(n+1) \neq w'(n+1).$$

DEFINITION 2.1. — *A* substitution τ *on the alphabet* $\mathcal{A}$ *is a mapping* $\tau : \mathcal{A} \to \mathcal{A}^+$. *The mapping* τ *extends by concatenation to maps (also denoted* τ*)* $\mathcal{A}^* \to \mathcal{A}^*$ *and* $\mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$. *A substitution* τ *is called* primitive *if there is a positive integer N such that for each pair $(i,j) \in \mathcal{A} \times \mathcal{A}$, the letter j occurs in $\tau^N(i)$. In this work we consider only primitive substitutions.*

Fix a primitive substitution τ on $\mathcal{A}$. A sequence $w_* \in \mathcal{A}^{\mathbb{N}}$ is called a *fixed point* (of τ) if $\tau(w_*) = w_*$, and a *periodic point* if $\tau^m(w_*) = w_*$ for some $m > 0$. Although τ may fail to have a fixed point, every primitive substitution has at least one periodic point. Associated to τ is the topological dynamical system (X,T), where T denotes the one sided shift on $\mathcal{A}^{\mathbb{N}}$ and X the T-orbit closure of a periodic point w_* in $\mathcal{A}^{\mathbb{N}}$. The primitivity of τ assures that (X,T) is independent of the choice of w_* and is *minimal*, *i.e.*, each point $x \in X$ has a dense orbit in X. (See [18].) Let $W(\tau)$ be the set of all words in $\mathcal{A}^*$ which occur in some periodic point w_*. Each nonempty word $w \in W(\tau)$ occurs in X with bounded gap : given w there is a positive integer m such that for each $x \in X$, each subword u of x of length m contains w. (See [18].) For each nonempty word $w \in W(\tau)$, let X_w denote the set of all sequences in X beginning in w.

Let $M_\tau = (m_{i,j})_{r\times r}$ be the *incidence matrix* of τ, *i.e.*,

$$m_{i,j} = \big|\tau(j)\big|_i,$$

and write M_τ^t for its transpose. Because τ is primitive, there exists $N \geq 1$ such that M_τ^N is strictly positive. Given an eigenvector $\boldsymbol{v}_\alpha = (v_1, v_2, \ldots, v_r)$ in $\mathbb{C}^r$ of the matrix M_τ^t corresponding to some eigenvalue $\alpha \in \mathbb{C}$, we define the function $S = S(\tau, \boldsymbol{v}_\alpha) : \mathcal{A}^* \to \mathbb{C}$ by

$$S(w) = \sum_{i=1}^{r} |w|_i v_i. \tag{2.1}$$

Thus $S(u_\emptyset) = 0$ and for each $w, w' \in \mathcal{A}^*$ we have

$$S(ww') = S(w) + S(w'). \tag{2.2}$$

LEMMA 2.2. — *If $w \in \mathcal{A}^*$ then $S(\tau(w)) = \alpha S(w)$.*

Proof. — In view of equation (2.2), it suffices to consider the case where $w = j \in \mathcal{A}$. We have

$$S\big(\tau(j)\big) = \sum_{i=1}^{r} |\tau(j)|_i v_i = \sum_{i=1}^{r} m_{i,j} v_i = \sum_{i=1}^{r} m_{j,i}^t v_i = \alpha v_j = \alpha S(j). \quad \square$$

THEOREM 2.3. — *Let τ be a primitive substitution on the alphabet $\mathcal{A} = \{1, 2, \ldots, r\}$ fixing a point $w_* \in \mathcal{A}^{\mathbb{N}}$. Let $\alpha \in \mathbb{C}$ be an eigenvalue* *of the matrix* M_τ^t, *and let* $\boldsymbol{v}_\alpha = (v_1, v_2, \dots, v_r)$ *be a corresponding eigenvector. Define* $S = S(\tau, \boldsymbol{v}_\alpha) : \mathcal{A}^* \to \mathbb{C}$ *as in* (2.1). *For each* $n \geq 1$, *let* $S_n = S(w_*(n))$. *Then*

1) $\lim_{n\to\infty} \frac{S_n}{n}$ *is equal to zero if and only if* α *is not equal to the Perron-Frobenius eigenvalue* θ *of* M_τ.

2) *If* $|\alpha| > 1$ *then the sequence* $\{S_n\}$ *is unbounded.*

3) *If* $\alpha = 0$ *then the sequence* $\{S_n\}$ *has finite range.*

4) *If* α *is a root of unity then* $\{S_n\}$ *is unbounded or has finite range. If* $|\alpha| = 1$ *and* α *is not a root of unity then* $\{S_n\}$ *is unbounded* (*cf. Examples* 2.8 *and* 2.9)

5) *If* $0 < |\alpha| < 1$ *then the sequence* $\{S_n\}$ *is bounded. Let* $\Omega = \Omega(\tau, \boldsymbol{v}_\alpha)$ *be the closure of* $\{S_n : n \geq 1\}$. *Then* Ω *is a compact perfect subset of* $\mathbb{C}$. (*cf. Example* 2.4 *below.*)

EXAMPLE 2.4. — Let τ be the substitution on $\mathcal{A} = \{1, 2, 3\}$ given by

$$\tau(1) = 132, \quad \tau(2) = 112, \quad \tau(3) = 32.$$

The eigenvalues of

$$M_\tau = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

are the roots of the polynomial

$$p(x) = x^3 - 3x^2 + x - 1.$$

Two are complex and the other is real. Let $\alpha \approx .115353823 + .589742805i$ be one of the complex roots. An eigenvector of M_τ^t corresponding to α is

$$\boldsymbol{v}_\alpha = \begin{pmatrix} (\alpha - 1)^2 \\ 2(\alpha - 1) \\ 2 \end{pmatrix}.$$

Since $0 < |\alpha| < 1$, Ω is a compact perfect subset of $\mathbb{C}$. As in [19], one can show that Ω corresponds to a Weyl automorphism on the torus $\mathbb{R}^2/\mathbb{Z}^2$ and hence gives a tiling of the plane $\mathbb{R}^2$. (See Figure 2.2.)

Proof of Theorem 2.3. — There is no loss of generality in assuming that $\tau(1)$ begins in 1 and that the fixed point w_* of τ is given by $\lim_{n\to\infty} \tau^n(1)$.

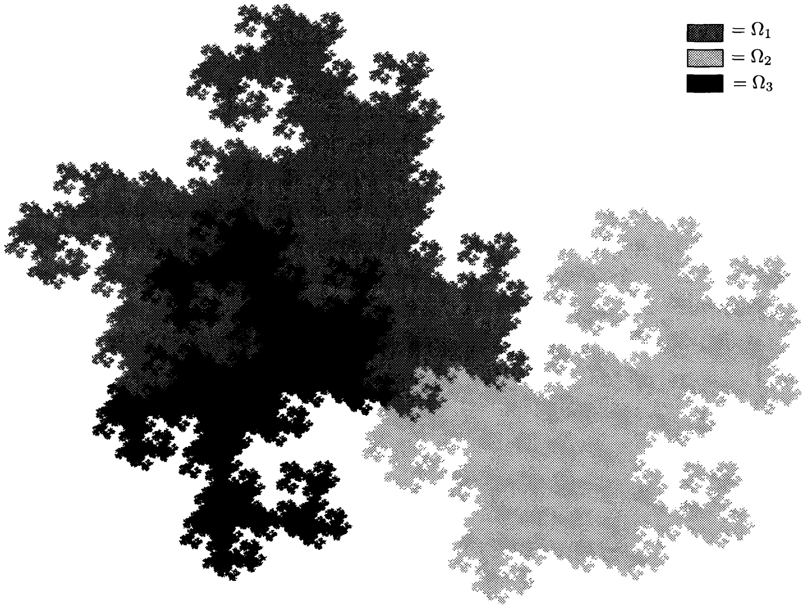


Figure 2.2

Since τ is primitive it follows that the dynamical system (X,T) is uniquely ergodic. (See [16] or [18].) Define $F : X \to \mathbb{C}$ by setting $F(w) = v_i$ if $w \in X_i$, *i.e.*, if w begins in the letter $i \in \mathcal{A}$. Then, for each n we have

$$S_n = \sum_{i=0}^{n-1} F\big(T^i(w_*)\big).$$

By Birkhoff's ergodic theorem

$$\lim_{n\to\infty} \frac{S_n}{n} = \int_X F \,\mathrm{d}\mu$$

where μ is the unique T-invariant probability measure on X. For $u \in \mathcal{A}^+$, $\mu(X_u)$ is equal to the frequency of u in w_*. Moreover, if $u = j$ is an element of $\mathcal{A}$ then $\mu(X_j)$ is equal to the j-th component of the strictly positive eigenvector $\boldsymbol{p} = (p_1, p_2, \dots p_r)$ of the matrix M_τ, corresponding to the Perron-Frobenius eigenvalue θ, normalized so that $\sum_{i=1}^{r} p_i = 1$ (*cf.* [18]).

Thus, decomposing $X = \bigcup_{i=1}^{r} X_i$ we have

$$\lim_{n\to\infty} \frac{S_n}{n} = \int_X F\,\mathrm{d}\mu = \sum_{i=1}^{r} F(X_i)\mu(X_i) = \sum_{i=1}^{r} v_i p_i.$$

This last sum is equal to 0 if and only if $\alpha \neq \theta$. An alternate proof of (1) is given in Appendix A.

To prove (2) we suppose that $|\alpha| > 1$. Let u be an initial subword of w_* with $S(u) \neq 0$. For each $n \geq 1$, $\tau^n(u)$ is an initial subword of w_* and $S(\tau^n(u)) = \alpha^n S(u)$, and hence the sequence $\{S_n\}$ is unbounded.

(3) follows immediately from Lemma 2.2.

To establish (4) and (5) we need :

LEMMA 2.5. — *The range of $\{S_n\}$ is either finite or for each letter $j \in \mathcal{A}$ there is a nonempty word w such that $w_*(|w|+1) = wj$ and $S(w) \neq 0$.*

Proof. — This is clear since each letter $j \in \mathcal{A}$ occurs in w_* with bounded gap. □

Now suppose that α is an m-th root of unity and the range of $\{S_n\}$ is infinite. By Lemma 2.5 there is a nonempty word w such that w_* begins in $w1$ and $S(w) \neq 0$. We can assume that $\alpha = 1$ and that w is a subword of $\tau(1)$, since replacing τ with τ^{mk} for suitable $k \geq 1$ changes neither the fixed point w_* nor the sequence $\{S_n\}$. For each $n \geq 1$ the word $\tau^n(w)\tau^{n-1}(w)\cdots\tau(w)$ is an initial subword of w_* and $S(\tau^n(w)\tau^{n-1}(w)\cdots\tau(w)) = nS(w)$. Letting $n \to \infty$, we see that $\{S_n\}$ is unbounded.

Next, if $|\alpha| = 1$ and α is not a root of unity then the range of $\{S_n\}$ is infinite. Thus we can find w as above and we may assume as before that w is an initial subword of $\tau(1)$. For each $k \geq 1$ we can find numbers $1 < n_1 < n_2 < \cdots < n_k$ with $1 - \mathrm{Re}(\alpha^{n_j}) < 2^{-j}$. (This is because $\{\alpha^q\}$ is dense on the unit circle.) Then $\tau^{n_k}(w)\tau^{n_{k-1}}(w)\cdots\tau^{n_1}(w)$ is an initial subword of w_* and

$$\begin{aligned}
&\left|S(\tau^{n_k}(w)\tau^{n_{k-1}}(w)\cdots\tau^{n_1}(w))\right| \\
&\quad = \left|S(w)\right| \cdot \left|\sum_{j=1}^{k} \alpha^{n_j}\right| \geq \left|S(w)\right| \cdot \left|\sum_{j=1}^{k} \mathrm{Re}(\alpha^{n_j})\right| \\
&\quad > \left|S(w)\right| \sum_{j=1}^{k} \Bigl(1 - \frac{1}{2^j}\Bigr) > (k-1)\left|S(w)\right|.
\end{aligned}$$

Thus $\{S_n\}$ is again unbounded, and (4) is proved.

To prove (5) we assume that $0 < |\alpha| < 1$. Let K be the set of all proper initial subwords (including the empty word) of the words $\tau(1), \tau(2), \ldots, \tau(r)$ and set

$$C = \max\{\, |S(u)| \ : u \in K\}.$$

We will show that the sequence $\{S_n\}$ is bounded in absolute value by the number $C/(1 - |\alpha|)$. The following lemma, due to J.-M. Dumont, is also used in the proof of Theorem 4.3 :

LEMMA 2.6 (See [5].). — *Let $w \in W(\tau)$ be an initial subword of w_*. Then, there exist $m \geq 1$ and words $u_1, u_2, \ldots, u_{m+1} \in K$ and $w_1, w_2, \ldots, w_{m+1} \in W(\tau)$ such that*

1) $|u_1| > 0$ *and* $|w_k| > 0$, $k = 1, 2, \ldots, m+1$;
2) $\tau(1) = u_1 w_1$;
3) $\tau(w_k(1)) = u_{k+1} w_{k+1}$, $k = 1, 2, \ldots, m$, *and*
4) $w = \tau^m(u_1)\tau^{m-1}(u_2) \cdots \tau(u_m) u_{m+1}$.

Let w be any initial subword of w_*. By Lemma 2.6, we can write

$$w = \tau^m(u_1)\tau^{m-1}(u_2) \cdots \tau(u_m) u_{m+1},$$

with each $u_i \in K$. We have

$$\begin{aligned} |S(w)| &= \Big| \sum_{k=0}^{m} S(\tau^{m-k}(u_{k+1})) \Big| \leq \sum_{k=0}^{m} |S(\tau^{m-k}(u_{k+1}))| \\ &= \sum_{k=0}^{m} |\alpha|^{m-k} \, |S(u_{k+1})| \leq C \sum_{k=0}^{m} |\alpha|^{m-k} \leq \frac{C}{1 - |\alpha|} \cdot \end{aligned}$$

It remains to show that if $0 < |\alpha| < 1$, then the set Ω defined in (5) of Theorem 2.3 is a perfect subset of $\mathbb{C}$. Let w be any initial subword of w_* with $S(w) \neq 0$. Then the subsequence $\{S(\tau^n(w))\}_{n=1}^{\infty}$ converges to 0. Next, let $m \geq 1$, and this time choose w to be an initial subword of w_* which is followed by 1 in w_* and with $S(w) \neq 0$ (Lemma 2.5). Then, for each $\epsilon > 0$, there exists $k \geq 1$ such that $0 < |S(\tau^k(w))| < \epsilon$ and $\tau^k(w) w_*(m)$ in an initial subword of w_*. Thus

$$0 < |S(\tau^k(w))| = |S(\tau^k(w) w_*(m)) - S(w_*(m))| < \epsilon,$$

which shows that S_m is also an accumulation point of the sequence $\{S_n\}$. $\square$

COROLLARY 2.7. — *Let τ be a primitive substitution. Suppose that the matrix M_τ has a nonzero eigenvalue α of modulus less than one. Then no fixed point $w_* \in \mathcal{A}^{\mathbb{N}}$ of τ is an eventually periodic sequence, and hence no $w \in X$ is an eventually periodic sequence.*

Proof. — Suppose to the contrary that τ has a fixed point of the form $w_* = u_0 uuu \cdots$. By (5) of Theorem 2.3 we must have $S(u) = 0$ (otherwise the sequence $\{S_n\}$ would be unbounded), and hence the range of $\{S_n\}$ is finite. This contradicts (5) of Theorem 2.3. Minimality of (X, T) allows us to conclude that X contains no eventually periodic sequence. $\square$

We end this section with a few examples illustrating that each of the three possibilities described in (4) of Theorem 2.3 occurs.

EXAMPLE 2.8. — Let $\mathcal{A} = \{1, 2, 3, 4\}$ and set

$$\tau(1) = 1234, \quad \tau(2) = 2312, \quad \tau(3) = 3123, \quad \tau(4) = 1231.$$

If we take the eigenvalue $\alpha = 1$ and eigenvector $(1, -\frac{1}{2}, -\frac{1}{2}, 1)$ of M_τ^t, then the range of the sequence $\{S_n\}$ is $\{0, 1, \frac{1}{2}\}$. Writing

$$w_* = \lim_{n \to \infty} \tau^n(1) = w_1 w_2 w_3 \cdots,$$

we have $w_{3n+1} \in \{1, 4\}$, $w_{3n+2} = 2$, and $w_{3n} = 3$, and thus $S_{3n} = 0$.

On the other hand, corresponding to the eigenvalue $\alpha = -1$, is the eigenvector $(-\frac{2}{3}, \frac{5}{30}, \frac{5}{30}, 1)$. In this case the sequence $\{S_n\}$ is unbounded. In fact, $w_*(6) = 123423$ is followed by 1 in w_*, and $S(w_*(6)) = 1 \neq 0$. It follows from the proof of Theorem 2.3 (4) that $\{S_n\}$ is unbounded.

EXAMPLE 2.9. — Consider the substitution

$$\tau(1) = 12, \quad \tau(2) = 14, \quad \tau(3) = 2, \quad \tau(4) = 3.$$

The characteristic polynomial of M_τ is

$$x^4 - x^3 - x^2 - x + 1 = \Big(x^2 + \tfrac{1}{2}(-1 + \sqrt{13})x + 1\Big)\Big(x^2 + \tfrac{1}{2}(-1 - \sqrt{13})x + 1\Big).$$

The roots of the first quadratic factor are

$$\alpha = \frac{1 - \sqrt{13} \pm \sqrt{2 + 2\sqrt{13}}\,i}{4}$$

which have modulus one. One readily verifies that α is not a root of unity. In this example, the Perron-Frobenius eigenvalue $\theta \approx 1.7220838$ is a Salem number.

3. Geometric Realizations of Primitive Substitutions

Let τ be a primitive substitution on the alphabet $\mathcal{A} = \{1, 2, \ldots, r\}$ fixing a sequence $w_* \in \mathcal{A}^{\mathbb{N}}$. We may assume that w_* begins in 1, so that $w_* = \lim_{n\to\infty} \tau^n(1)$. Let (X, T) be the associated dynamical system defined in Section 2.

Throughout this section we assume that M_τ has a nonzero eigenvalue α of modulus less than one. By Corollary 2.7 no element of X is a periodic sequence. Let $\boldsymbol{v}_\alpha$ be an eigenvector of M_τ^t corresponding to α, $S = S(\tau, \boldsymbol{v}_\alpha) : \mathcal{A}^* \to \mathbb{C}$ as in (2.1), and Ω the closure of $\{S_n\}$ as in (5) of Theorem 2.3.

THEOREM 3.1. — *For $u, w \in X$, let $a_1, a_2, \ldots$ be any sequence of nonnegative integers such that $u(a_n + n) = u(a_n)w(n)$ for each n. Set*

$$\rho(u, w) = \lim_{n\to\infty} \left|S(u(a_n))\right|$$

Then this limit exists independent of the choice of sequence $\{a_n\}$. Moreover, the function $\rho : X \times X \to \mathbb{R}$ defines a pseudometric on X with the following property :

$$\forall u, w \in X, \quad \rho\big(\tau(u), \tau(w)\big) = |\alpha|\, \rho(u, w),$$

and if $u(1) = w(1)$, then $\rho(Tu, Tw) = \rho(u, w)$.

We show that this pseudometric ρ is never equivalent to the standard metric on X. Indeed there exist $w, w' \in X$ with $\rho(w, w') = 0$ and $w(1) \neq w'(1)$. (See Proposition 3.17.)

The associated metric space X/ρ is isometric to Ω :

THEOREM 3.2. — *For $w \in X$, let $\{w_n\}$ be a sequence of tails of w_* converging to w in the standard metric. There exist (unique) nonnegative integers $a_1, a_2, \ldots$ such that, for each $n \geq 1$, $w_* = w_*(a_n)w_n$. Set*

$$f(w) = \lim_{n\to\infty} S\big(w_*(a_n)\big).$$

Then this limit exists independent of the choice of sequence $\{w_n\}$. Hence, $f : X \to \mathbb{C}$ is a uniformly continuous function on X (with respect to the standard metric) and the image of f is Ω. Moreover, for $w, w' \in X$, $f(w) = f(w')$ if and only if $\rho(w, w') = 0$.

DEFINITION 3.3. — *Let τ be a primitive substitution. A* complex geometric realization *of τ is a triple $\mathcal{G} = \mathcal{G}(f, \lambda, \boldsymbol{v})$ consisting of a* *continuous map* $f : X \to \mathbb{C}$, *a nonzero complex number* λ *of modulus less than one, and a nonzero vector* $\boldsymbol{v} = (v_1, v_2, \dots, v_r) \in \mathbb{C}^r$, *such that for all* $i \in \mathcal{A}$ *and for all* $w \in X_i$,

$$f\big(\tau(w)\big) = \lambda f(w) \quad \text{and} \quad f(Tw) = f(w) + v_i.$$

We call $\mathcal{G}$ *a* real geometric realization *if* $f(X) \subset \mathbb{R}$.

THEOREM 3.4. — *Let* τ *be a primitive substitution. Then to each eigenvalue* α *of the matrix* M_τ *with* $0 < |\alpha| < 1$ *and to each eigenvector* $\boldsymbol{v}_\alpha$ *of* M_τ^t *corresponds a geometric realization* $\mathcal{G} = \mathcal{G}(f, \alpha, \boldsymbol{v}_\alpha)$, *where the map* $f : X \to \mathbb{C}$ *is given by Theorem* 3.2. *Conversely, if* $\mathcal{G} = \mathcal{G}(f, \lambda, \boldsymbol{v})$ *is a complex geometric realization of* τ, *then* λ *is an eigenvalue of* M_τ, $\boldsymbol{v}$ *is an eigenvector of* M_τ^t *corresponding to* λ, *and* f *is given by Theorem* 3.2.

REMARK 3.5. — The key in proving each of the above theorems involves establishing the "recognizability" of long initial subswords of w_*. A precise formulation of this recognizability is given in Proposition 3.10. Our proof of Proposition 3.10 is largely due to the work of J.C. Martin in [14]. A similar idea is also found in B. Mossé's proof of Theorem 4.1 in [17] and a slightly different formulation of this proposition appears in [12].

The following lemma is a special case of a result of B. Mossé :

LEMMA 3.6 (*cf.* Theorem 2.4 in [17]). — *There is a positive integer* L *such that* w_* *does not begin with any nonempty subword repeated* L *times.*

Fix L as in the preceeding lemma.

COROLLARY 3.7. — *If* $w_*(n)$ *is an initial subword of* $ww_*(n)$ *for some* $w \in W(\tau)$ *with* $n \geq L|w|$, *then* w *is the empty word.*

Proof. — If $|w| > 0$, then w_* begins with the nonempty word w repeated at least L times, contradicting the choice of L. □

Let $w_0 \in \mathcal{A}^+$. We say that $u = u_1 u_2 \cdots u_k$, each $u_i \in \mathcal{A}$, *minimally* n-*covers* w_0 if $\tau^n(u) = A\tau^n(w_0)B$ for some $A, B \in \mathcal{A}^*$ with $|A| < |\tau^n(u_1)|$ and $|B| < |\tau^n(u_k)|$.

Fix $M \in \mathbb{N}$ such that $\tau^M(1)$ contains at least L copies of each letter of $\mathcal{A}$, *i.e.*, $|\tau^M(1)|_j \geq L$ for each $j \in \mathcal{A}$.

LEMMA 3.8. — *If* $u = u_1 u_2 \cdots u_k$, *each* $u_i \in \mathcal{A}$, *minimally* n-*covers* $\tau^M(1)$, *then the subwords* A *of* $\tau^n(u_1)$ *and* B *of* $\tau^n(u_k)$ *such that* $\tau^n(u) = A\tau^{M+n}(1)B$ *are unique.*

Proof. — Suppose that $\tau^n(u) = A\tau^{M+n}(1)B = C\tau^{M+n}(1)D$ with A and C initial subwords of $\tau^n(u_1)$. It suffices to show that $A = C$. We may assume that $A = Cw$ for some subword w of $\tau^n(u_1)$. Then $C\tau^{M+n}(1)$ is an initial subword of $Cw\tau^{M+n}(1)$, and thus $\tau^{M+n}(1)$ is an initial subword of $w\tau^{M+n}(1)$. Since

$$\left|\tau^{M+n}(1)\right| > L\left|\tau^n(u_1)\right| \geq L\left|w\right|,$$

it follows from Corollary 3.7 that w must be the empty word, and hence, $A = C$. ▯

LEMMA 3.9. — *There is an integer P such that if w is any subword of w_* with $|w| \geq P$ then*

$$\left|\tau^n(w)\right| \geq \left|\tau^{n+M}(1)\right| + 2\max_{a\in\mathcal{A}}\left|\tau^n(a)\right|.$$

Proof. — Using the primitivity of τ, choose P such that if w is a subword of w_* with $|w| \geq P$, then $|w|_a \geq |\tau^M(1)|_a + 2$ for each $a \in \mathcal{A}$. ▯

PROPOSITION 3.10. — *There exists $N \in \mathbb{N}$ such that if*

$$w_*(k+N) = w_*(k)w_*(N)$$

for some $k \in \mathbb{N}$, then there exists $j \in \mathbb{N}$ such that $w_*(k) = \tau(w_*(j))$. In other words, the subword preceeding an occurrence in w_* of a sufficiently long initial subword of w_* is in the image of τ.

Proof. — Fix P as in Lemma 3.9. For each positive integer n, let U_n be the set of all subwords of w_* which minimally n-cover $\tau^M(1)$. Note that $|U_n| < r^P$ for each n and $U_1 \subset U_2 \subset U_3 \subset \cdots$. It follows that the sequence $U_1, U_2, U_3, \ldots$ is eventually constant. Fix $n_0 \in \mathbb{N}$ such that $\bigcup_{n=1}^{\infty} U_n = U_{n_0}$. Set $N = |\tau^{n_0+M+1}(1)|$, and suppose that

$$w_*(k+N) = w_*(k)w_*(N)$$

for some $k > 0$. Let s be the least positive integer for which

$$\left|\tau^{n_0+1}(w_*(s))\right| > k,$$

and let t be the least positive integer for which

$$\left|\tau^{n_0+1}(w_*(t))\right| \geq k+N.$$

For $i = s, s+1, \dots, t$, let u_i be the i-th letter of w_*, and note that

$$w_*(k) = \tau^{n_0+1}\big(w_*(s-1)\big)E$$

for some initial subword E of $\tau^{n_0+1}(u_s)$. Observe that $u_s u_{s+1} \dots u_t$ is an element of $U_{n_0+1} = U_{n_0}$, so that

$$\tau^{n_0}(u_s u_{s+1} \dots u_t) = A\tau^{n_0+M}(1)B$$

and

$$\tau^{n_0+1}(u_s u_{s+1} \dots u_t) = C\tau^{n_0+M+1}(1)D,$$

where A and C are initial subwords of $\tau^{n_0}(u_s)$ and $\tau^{n_0+1}(u_s)$, respectively, and B and D are tails of $\tau^{n_0}(u_t)$ and $\tau^{n_0+1}(u_t)$, respectively. It follows from Lemma 3.8 that $C = \tau(A)$. Now,

$$w_*(k)w_*(N) = \tau^{n_0+1}\big(w_*(s-1)\big)Ew_*(N)$$

is an initial subword of

$$\begin{aligned}\tau^{n_0+1}\big(w_*(t)\big) &= \tau^{n_0+1}\big(w_*(s-1)u_s u_{s+1} \dots u_t\big)\\ &= \tau^{n_0+1}\big(w_*(s-1)\big)Cw_*(N)D.\end{aligned}$$

Therefore, $Ew_*(N)$ is an initial subword of $Cw_*(N)D$ with both C and E initial subwords of $\tau^{n_0+1}(u_s)$. Thus, there is a subword w of $\tau^{n_0+1}(u_s)$ such that either $C = Ew$ or $E = Cw$. In either case, we find that $w_*(N)$ is an initial subword of $ww_*(N)$. Since

$$N \geq L\left|\tau^{n_0+1}(u_s)\right| \geq L|w|,$$

w must be the empty word, by Corollary 3.7. Hence, $C = E$.

Clearly, $\tau^{n_0}(w_*(s))$ is an initial subword of w_*, and A, by definition, is an initial subword of $\tau^{n_0}(u_s)$. Then $\tau^{n_0}(w_*(s-1))A$ is an initial subword of w_*. Setting $j = |\tau^{n_0}(w_*(s-1))A|$, we obtain

$$\begin{aligned}w_*(k) &= \tau^{n_0+1}\big(w_*(s-1)\big)E = \tau^{n_0+1}\big(w_*(s-1)\big)C\\ &= \tau\big(\tau^{n_0}\big(w_*(s-1)\big)A\big) = \tau\big(w_*(j)\big),\end{aligned}$$

as required. $\square$

Remark 3.11. — Proposition 3.10 remains valid when we replace τ with τ^k for $k \geq 1$, as each fixed point of τ is a fixed point of τ^k.

Proof of Theorems 3.1, 3.2 and 3.4. — Fix $w \in X$ and let $w_1, w_2, \ldots,$ $a_1, a_2, \ldots$ and f be as in the statement of Theorem 3.2.

We show first that the limit (3.2) exists. By (5) of Theorem 2.3, there is a bound B for $|S_n|$. Let $\epsilon > 0$, and fix $k \in \mathbb{N}$ such that $|\alpha^k|B < \epsilon$. By Proposition 3.10 and Remark 3.11, there is a natural number Q such that if $w_*(n+Q) = w_*(n)w_*(Q)$, then $w_*(n) = \tau^k(w_*(b))$ for some integer $b \geq 0$. By primitivity of τ there exists $n_1 > Q$ such that $w(n_1) = w(n_1 - Q)w_*(Q)$. Since the w_i converge to w, there is an integer n_2 such that $w_i(n_1) = w(n_1)$ for each $i \geq n_2$. Observe that, for $i \geq n_2$,

$$w_*(a_i)w(n_1 - Q)w_*(Q) = w_*(a_i)w_i(n_1) = w_*(a_i + n_1).$$

Thus, by Proposition 3.10 and Remark 3.11, for each $i \geq n_2$, there exists an integer $b_i \geq 0$ such that

$$\tau^k\big(w_*(b_i)\big) = w_*(a_i)w(n_1 - Q).$$

We have, for $i \geq n_2$,

$$\begin{aligned}\big|S(w_*(a_i)) + S(w(n_1 - Q))\big| &= \big|S(w_*(a_i)w(n_1 - Q))\big| \\ &= \big|\alpha^k S(w_*(b_i))\big| \leq |\alpha^k|B < \epsilon,\end{aligned}$$

so that $S(w_*(a_i))$ is a Cauchy sequence. Thus, the limit exists and f is well defined.

Having established that the limit in (3.2) exists and is independent of the choice of tails $\{w_n\}_{n=1}^{\infty}$ converging to w, it follows that the limit in (3.2) defines a continuous, hence uniformly continuous, function f on X. If w is a tail of w_*, then $f(w) = S(w')$, where $w_* = w'w$. Thus, $f(X) = \Omega$.

LEMMA 3.12. — *If $u \in X$, then $f(\tau(u)) = \alpha f(u)$.*

Proof. — Let $u_1, u_2, \ldots$ be sequence of tails of w_* converging to u, and choose a sequence $a_1, a_2, \ldots$ of nonnegative integers such that, for each n, $w_* = w_*(a_n)u_n$. Note that $\tau(u) = \lim_{n\to\infty} \tau(u_n)$. Since

$$w_* = \tau(w_*) = \tau\big(w_*(a_n)\big)\tau(u_n),$$

we have

$$f\big(\tau(u)\big) = \lim_{n\to\infty} S\big(\tau(w_*(a_n))\big) = \lim_{n\to\infty} \alpha S\big(w_*(a_n)\big) = \alpha f(u). \quad \square$$

LEMMA 3.13. — *If $u \in X$ begins in the letter $i \in \mathcal{A}$, then*

$$f \circ T(u) = f(u) + S(i).$$

Proof. — Let $u_1, u_2, \ldots$ be sequence of tails of w_* converging to u, and choose a sequence $a_1, a_2, a_3, \ldots$ of nonnegative integers such that $w_* = w_*(a_n)u_n$ for each n. Note that $T(u_n)$ converges to $T(u)$ and $w_* = w_*(a_n + 1)T(u_n)$, so that

$$\begin{aligned} f \circ T(u) &= \lim_{n\to\infty} S\big(w_*(a_n + 1)\big) \\ &= \lim_{n\to\infty} \big(S(w_*(a_n)) + S(i)\big) = f(u) + S(i), \end{aligned}$$

as required. □

Lemmas 3.12 and 3.13 conclude the proof of the first part of Theorem 3.4. To prove the converse, suppose that $\mathcal{G} = \mathcal{G}(f, \lambda, \boldsymbol{v})$ is a geometric realization of τ. Set $S_{\boldsymbol{v}} : \mathcal{A}^* \to \mathbb{C}$ by

$$S_{\boldsymbol{v}}(u) = \sum_{i=1}^{r} |u|_i v_i$$

where v_i denotes the ith entry of $\boldsymbol{v}$. Then, for $u \in W(\tau)$, $S_{\boldsymbol{v}}(u)$ can be computed in terms of f by $S_{\boldsymbol{v}}(u) = f(w') - f(w)$ where w and w' are tails of w_* with $w = uw'$. Then, for each $i \in \mathcal{A}$ it follows that $S_{\boldsymbol{v}}(\tau(i)) = \lambda S_{\boldsymbol{v}}(i)$ so that λ must be an eigenvalue of M_τ^t and $\boldsymbol{v}$ a corresponding eigenvector. This concludes the proof of Theorem 3.4.

The next proposition completes the proof of Theorem 3.2.

PROPOSITION 3.14. — *For all $u, w \in X$, the limit $\rho(u, w)$ defined in* (3.1) *exists and is equal to $|f(w) - f(u)|$.*

Proof. — Let $\{a_n\}$ be the sequence of nonnegative integers given in Theorem 3.1. Since τ is primitive, there exist b_n such that

$$w_*(b_n + a_n + n) = w_*(b_n)u(a_n + n)$$

for each $n \in \mathbb{N}$. Then

$$\begin{aligned} w_*(b_n + a_n + n) &= w_*(b_n)u(a_n + n) \\ &= w_*(b_n)u(a_n)w(n) \\ &= w_*(b_n + a_n)w(n) \end{aligned}$$

so that

$$\lim_{n\to\infty} S(w_*(b_n + a_n)) = f(w) \quad \text{and} \quad \lim_{n\to\infty} S(w_*(b_n)) = f(u).$$

Thus,

$$\begin{aligned}
|f(w) - f(u)| &= \big|\lim_{n\to\infty} S(w_*(b_n + a_n)) - \lim_{n\to\infty} S(w_*(b_n))\big| \\
&= \big|\lim_{n\to\infty} S(w_*(b_n)u(a_n)) - \lim_{n\to\infty} (S(w_*(b_n)))\big| \\
&= \big|\lim_{n\to\infty} (S(w_*(b_n)) + S(u(a_n))) - \lim_{n\to\infty} (S(w_*(b_n)))\big| \\
&= \big|\lim_{n\to\infty} S(u(a_n))\big| = \lim_{n\to\infty} \big|S(u(a_n))\big| = \rho(u, w). \quad \square
\end{aligned}$$

It follows from Proposition 3.14 that $\rho : X \times X \to \mathbb{R}^{\geq 0}$ of Theorem 3.1 defines a pseudometric on X.

COROLLARY 3.15. — *For all $u, w \in X$, $\rho(\tau(u), \tau(w)) = |\alpha|\, \rho(u, w)$.*

Proof. — One has :

$$\begin{aligned}
\rho(\tau(u), \tau(w)) &= \big|f(\tau(u)) - f(\tau(w))\big| = \big|\alpha\big| \cdot \big|f(u) - f(w)\big| \\
&= |\alpha|\, \rho(u, w). \quad \square
\end{aligned}$$

COROLLARY 3.16. — *If $u, w \in X$ begin in the letter i, then*

$$\rho\big(T(u), T(w)\big) = \rho(u, w).$$

Proof. — One has :

$$\begin{aligned}
\rho(Tu, Tw) &= \big|f(Tu) - f(Tw)\big| = \big|f(u) + v_i - f(w) - v_i\big| \\
&= \big|f(u) - f(w)\big| = \rho(u, w). \quad \square
\end{aligned}$$

This concludes the proof of Theorem 3.1. $\square$

Theorem 3.2 implies that each point $x \in \Omega$ is encoded by a point $w \in X$. This yields a natural decomposition $\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_r$ where $\Omega_i = f(X_i)$. Thus, if $x \in \Omega_i$, then x is encoded by a point $w \in X$ with $w(1) = i$. Two points $w, w' \in X$ sufficiently close in the standard metric are close with respect to ρ; however, the following proposition shows that ρ is never a metric.

PROPOSITION 3.17. — *There exist distinct letters $i, j \in \mathcal{A}$ such that $\Omega_i \cap \Omega_j \neq \emptyset$.*

Proof. — Suppose to the contrary that for all $i \neq j$, we have $\Omega_i \cap \Omega_j = \emptyset$. Choose $\epsilon > 0$ such that for all points $x, y \in \Omega$ if $|x - y| < \epsilon$ then x and y belong to the same Ω_i. There exists a positive integer N such that for all $w, w' \in X$ if $w(N) = w'(N)$ then $\rho(w, w') = |f(w) - f(w')| < \epsilon$.

It follows from Theorem 3.4 that for all $w, w' \in X$ if $w(N) = w'(N)$ then $w(n) = w'(n)$ for all $n \geq N$. In particular, this would imply that the set X is finite, contradicting that it surjects via f onto the perfect set Ω. $\square$

REMARK 3.18. — It is important to understand the extent of information lost when considering the complex geometric realization of a substitution rather than the original system. Let t_j be the restriction to Ω_j of the translation $x \mapsto x + v_j$. If $x \in \Omega_i$ and $\tau(i) = i_1 i_2 \ldots i_\ell$ then the partial isometry $t_{i_\ell} \cdots t_{i_2} t_{i_1}$ is defined at αx and sends it to $\alpha t_i(x)$. Thus, if there exists only one word in the t_j sending αx to $\alpha t_i(x)$ then the value of $\tau(i)$ can be found from the geometric realization. One case where this occurs is when the generators are independent in the sense of [8]. The authors in [11] give a criterion which assures that the generators be independent.

4. The Graph Directed Construction

Let τ be a primitive substitution on the alphabet $\mathcal{A} = \{1, 2, \ldots, r\}$ fixing the sequence $w_* = \lim_{n\to\infty} \tau^n(1)$ in $\mathcal{A}^{\mathbb{N}}$. Let $\boldsymbol{v}_\alpha$ be an eigenvector of M_τ^t corresponding to an eigenvalue α with $0 < |\alpha| < 1$ and $\mathcal{G} = \mathcal{G}(f, \alpha, \boldsymbol{v}_\alpha)$ the associated complex geometric realization from Theorem 3.4. Denote by Ω the image of f. The substitution τ naturally determines a strongly connected directed graph G with vertex set $\mathcal{A}$, and $\mathcal{G}$ associates similarities $g_e : \Omega_{t(e)} \to \Omega_{o(e)}$ to each directed edge e, where $t(e)$ and $o(e)$ denote the terminal vertex and the initial vertex of the directed edge e, as follows.

Define a directed graph $G = G(\tau)$

- with vertex set $V(G) = \mathcal{A}$ and
- with edge set $E(G) = \{(i, j) \,:\, j \in \mathcal{A} \text{ and } 0 \leq i < |\tau(j)|\}$.

We interpret $(i, j) \in E(G)$ to be a directed edge from $(T^i \circ \tau(j))(1)$ to j.

EXAMPLE 4.1. — Consider τ defined by

$$\tau(1) = 12, \quad \tau(2) = 13, \quad \tau(3) = 2.$$

The graph $G = G(\tau)$ is shown in Figure 4.3.

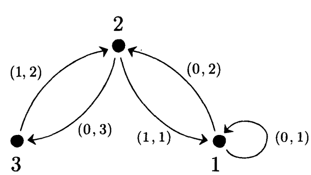


Figure 4.3

LEMMA 4.2. — *The graph G is strongly connected.*

Proof. — This follows immediately from the primitivity of τ. □

To each edge $(i,j) \in E(G)$, we associate a contraction

$$g_{(i,j)} : \Omega_j \longrightarrow \Omega_{(T^i \circ \tau(j))(1)}$$

given by

$$g_{(i,j)}(z) = \alpha z + S\big((\tau(j))(i)\big)$$

where $S = S(\tau, \boldsymbol{v}_\alpha)$ is given by equation (2.1). We check that $g_{(i,j)}$ is well defined : If $z \in \Omega_j$, then there is a point $w \in X_j$ with $f(w) = z$, and $T^i \circ \tau(w) \in X_{(T^i \circ \tau(j))(1)}$. By Lemmas 3.12 and 3.13, we have

$$\begin{aligned} f\big(T^i \circ \tau(w)\big) &= S\big((\tau(w))(i)\big) + f\big(\tau(w)\big) \\ &= S\big((\tau(j))(i)\big) + \alpha f(w) \\ &= S\big((\tau(j))(i)\big) + \alpha z = g_{(i,j)}(z). \end{aligned} \tag{4.1}$$

Hence, $g_{(i,j)}(z) \in \Omega_{(T^i \circ \tau(j))(1)}$ as required.

For $p \in \mathbb{N}$, let $G(p)$ be the set of directed paths of length p in G and G^∞ the set of (rooted) infinite directed paths in G.

THEOREM 4.3. — *Fix for each $j \in \mathcal{A}$ a representative $x_j \in \Omega_j$. Define $h : G^\infty \to \mathbb{C}$ by*

$$h\big((i_n, j_n)_{n=1}^\infty\big) = \lim_{n\to\infty} g_{(i_1,j_1)} \circ g_{(i_2,j_2)} \circ \cdots \circ g_{(i_n,j_n)}(x_{j_n}). \tag{4.2}$$

Then

1) *h is well defined, uniformly continuous (with respect to the standard metric on G^∞) and independent of the choice of the x_j.*

2) *$h(G^\infty) = \Omega$.*

Proof. — Let $(i_n, j_n)_{n=1}^\infty \in G^\infty$. It follows from (4.1) that each of the compositions considered in (4.2) is well defined. We have

$$\begin{aligned} h\big((i_n, j_n)_{n=1}^\infty\big) &= \lim_{m\to\infty} g_{(i_1,j_1)} \circ g_{(i_2,j_2)} \circ \cdots \circ g_{(i_m,j_m)}(x_{j_m}) \\ &= \lim_{m\to\infty} \Big(\sum_{l=1}^{m} \alpha^{\ell-1} S\big((\tau(j_\ell))(i_\ell)\big) + \alpha^m x_{j_m}\Big) \\ &= \sum_{\ell=1}^{\infty} \alpha^{\ell-1} S\big((\tau(j_\ell))(i_\ell)\big). \end{aligned}$$

Thus the limit in (4.2) exists and is independent of the choice of the x_j. To see that h is uniformly continuous we observe that if $(i'_n, j'_n)_{n=1}^{\infty} \in G^{\infty}$ with $(i_\ell, j_\ell) = (i'_\ell, j'_\ell)$ for each $\ell \leq m$ then

$$\begin{aligned}
&\big|h((i_n, j_n)_{n=1}^{\infty}) - h((i'_n, j'_n)_{n=1}^{\infty})\big| \\
&\qquad\qquad \leq 2 \sum_{\ell \geq m+1} |\alpha|^{\ell-1} \max_{(i,j)\in E(G)} \big|S((\tau(j))(i))\big|
\end{aligned}$$

where the right hand side depends only on m and tends to 0 as $m \to \infty$.

To establish (2) we first note that since Ω is closed,

$$\begin{aligned}
&h((i_n, j_n)_{n=1}^{\infty}) \\
&\quad = \lim_{n\to\infty} g_{(i_1,j_1)} \circ g_{(i_2,j_2)} \circ \cdots \circ g_{(i_n,j_n)}(x_{j_n}) \in \Omega_{(T^{i_1}\circ\tau(j_1))(1)} \subset \Omega,
\end{aligned}$$

so that $h(G^{\infty}) \subset \Omega$.

Now $h(G^{\infty})$ is closed (h is continuous and G^{∞} is compact). To prove $\Omega \subset h(G^{\infty})$ it suffices to show that $f(T^N(w_*)) \in h(G^{\infty})$ for each N, since the set of tails of w_* is dense in X. Fix $N \in \mathbb{N}$. Applying Lemma 2.6 to the initial subword $w = w_*(N)$ of w_*, there exist $m \geq 1$ and words $u_1, u_2, \ldots, u_{m+1}, w_1, w_2, \ldots, w_{m+1} \in W(\tau)$ satisfying conditions (1)–(4) of Lemma 2.6. Conditions (2) and (3) assure that

$$\big(|u_{m+1}|, w_m(1)\big), \big(|u_m|, w_{m-1}(1)\big), \ldots, \big(|u_2|, w_1(1)\big), \big(|u_1|, 1\big) \in G(m+1),$$

while condition (4) gives

$$w = T^{|u_{m+1}|} \circ \tau \circ T^{|u_m|} \circ \tau \circ \cdots \circ T^{|u_1|} \circ \tau(w_*).$$

Since $(0,1)$ is an edge from 1 to 1 in G, it follows that

$$\sigma = \big(|u_{m+1}|, w_m(1)\big), \ldots, \big(|u_2|, w_1(1)\big), \big(|u_1|, 1\big), (0,1), (0,1), \ldots \in G^{\infty}.$$

By (4.1) and (1) of Theorem 4.3, we have

$$\begin{aligned}
h(\sigma) &= \lim_{n\to\infty} g_{\sigma(1)} \circ \cdots \circ g_{\sigma(m+1)} \circ g_{(0,1)} \circ g_{(0,1)} \circ \cdots g_{\sigma(n)=(0,1)}(0) \\
&= g_{\sigma(1)} \circ \cdots \circ g_{\sigma(m+1)}(0) \\
&= f\big(T^{|u_{m+1}|} \circ \tau \circ T^{|u_m|} \circ \tau \circ \cdots \circ T^{|u_1|} \circ \tau(w_*)\big) \\
&= f(w),
\end{aligned}$$

where $\sigma(i)$ denotes the ith edge of the infinite path σ. Thus $f(T^N(w_*))$ is an element of $h(G^{\infty})$ as required. $\square$

Theorem 4.3 gives an alternate symbolic representation of points in Ω in terms of infinite paths in G. Each point $x \in \Omega_i$ is coded (possibly nonuniquely) by an infinite path $\sigma \in G^\infty$ whose initial vertex is equal to i. The characterization of the limit set Ω given in Theorem 4.3 is analogous to the Mauldin-Williams graph directed construction [15] for the directed graph $G = G(\tau)$ with respect to the similarity maps $g_{(i,j)}$. In [15] the authors assume the existence of nonoverlapping open sets $\{J_i\}$ (indexed by the vertices of G) having nonempty interiors and satisfying the following conditions (commonly referred to as the "open set condition" or OSC) : For each directed edge e in G, the similarity g_e maps $J_{t(e)}$ into $J_{o(e)}$ and if e and e' are distinct edges with $o(e) = o(e')$ then $g_e(J_{t(e)})$ and $g_{e'}(J_{t(e')})$ are nonoverlapping. Although in [15] the associated limit set is defined in terms of the $\{J_i\}$, it is not difficult to see that this same limit set can be generated in a pointwise manner as in Theorem 4.3. The importance of OSC is that it gives an exact value of the Hausdorff dimension of the limit set Ω. In the absence of OSC we have only an upper bound on the Hausdorff dimension :

THEOREM 4.4. — *The Hausdorff dimension of Ω is bounded above by the number*

$$\beta = -\frac{\log \theta}{\log |\alpha|}$$

where θ is the Perron-Frobenius eigenvalue of the incidence matrix M_τ. Moreover, $\mathcal{H}^\beta(\Omega) < \infty$.

Proof. — For each directed path σ in G, put

$$J_\sigma = g_{\sigma(1)} \circ g_{\sigma(2)} \circ \cdots \circ g_{\sigma(|\sigma|)}(\Omega_{t(\sigma)}),$$

so that $\operatorname{diam}(J_\sigma) = |\alpha|^{|\sigma|} \operatorname{diam}(\Omega_{t(\sigma)})$ goes to zero as $|\sigma| \to \infty$. Thus,

$$\bigcup_{\sigma \in G(p)} J_\sigma = \Omega$$

for each p. It follows that

$$\begin{aligned}
\mathcal{H}^\beta(\Omega) &\le \liminf_{p\to\infty} \sum_{\sigma\in G(p)} \big(|\alpha|^p \max_{i\in\mathcal{A}} \operatorname{diam}\Omega_i\big)^\beta \\
&= \max_{i\in\mathcal{A}} (\operatorname{diam}\Omega_i)^\beta \liminf_{p\to\infty} \sum_{\sigma\in G(p)} \frac{1}{\theta^p} \\
&= \max_{i\in\mathcal{A}} (\operatorname{diam}\Omega_i)^\beta \liminf_{p\to\infty} \frac{\#(G(p))}{\theta^p}.
\end{aligned}$$

Since the number of paths in G of length p is equal to the sum of the entries of the matrix M_τ^p and $\lim_{p\to\infty} \theta^{-p} M_\tau^p$ exists (Proposition V.7 in [18],) we get

$$\mathcal{H}^\beta(\Omega) < \infty.$$

Thus, the Hausdorff dimension of Ω is bounded above by β. $\square$

REMARK 4.5. — We are interested in a condition which guarantees that the map f given by Theorem 3.2 be one-to-one off a set of μ-measure zero, where μ is the unique T-invariant probability measure on X. We state some partial results here. The proofs can be found in [12], which is an extension of the material of this section.

- If $\mathcal{H}^\beta(\Omega) > 0$ then the restriction of f to each X_i is one-to-one off a set of measure zero.
- If $\mathcal{H}^\beta(\Omega) > 0$ and the substitution τ has only one periodic point then f is one-to-one off a set of measure zero.

We do not know whether the condition $\mathcal{H}^\beta(\Omega) > 0$ guarantees that f is one-to-one off a set of μ-measure zero.

Theorem 4.4 suggests that some properties of Ω depend only on the incidence matrix M_τ rather than on the actual substitution. For example, as an immediate consequence of Theorem 4.4 :

COROLLARY 4.6. — *If $|\alpha|\theta < 1$, then Ω is a Cantor set. If $\alpha \in \mathbb{C} \setminus \mathbb{R}$ and $|\alpha|^2\theta < 1$, then Ω has no interior.*

EXAMPLE 4.7. — Let τ be defined by

$$\tau(1) = 12, \quad \tau(2) = 13, \quad \tau(3) = 2.$$

Let $\mathcal{G}$ be the geometric realization of τ corresponding to the eigenvalue $\alpha \approx .445041868$ and eigenvector

$$v_\alpha = \begin{pmatrix} 1-\alpha^2 \\ -\alpha \\ -1 \end{pmatrix}$$

of M_τ^t corresponding to α. The Perron-Frobenius eigenvalue of M_τ is $\theta \approx 1.80193774$. Since $|\alpha|\theta < 1$, Ω is a Cantor set. An argument similar to that in §5 of [3] shows that the Hausdorff dimension of Ω is equal to $\beta = -\log\theta/\log|\alpha| \approx .727361811$ and that $0 < \mathcal{H}^\beta(\Omega) < \infty$. (See Figure 4.4.)

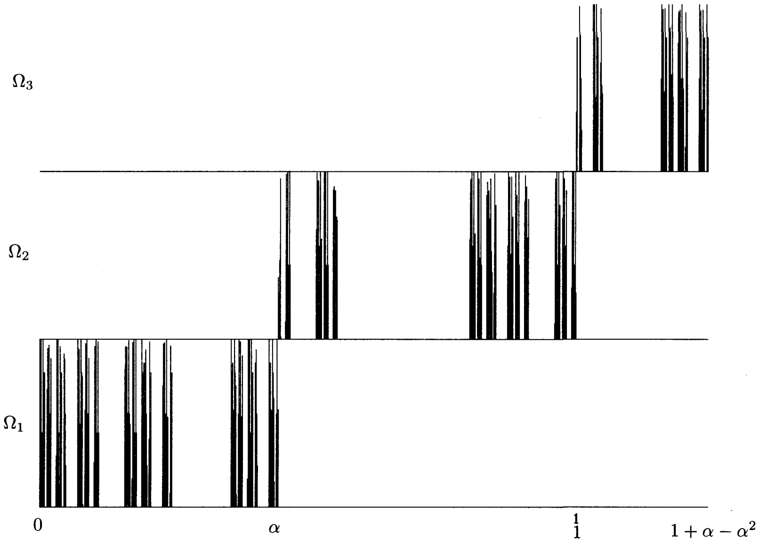


Figure 4.4

EXAMPLE 4.8. — Let τ be defined by

$$\tau(1) = 14, \quad \tau(2) = 3, \quad \tau(3) = 12, \quad \tau(4) = 2.$$

Let $\mathcal{G}$ be the geometric realization of τ corresponding to the eigenvalue $\alpha \approx .332923890 + .670769077\,i$ of M_τ and eigenvector

$$\boldsymbol{v}_\alpha = \begin{pmatrix} \alpha^3 - \alpha \\ \alpha \\ \alpha^2 \\ 1 \end{pmatrix}$$

of M_τ^t. The Perron-Frobenius eigenvalue is $\theta \approx 1.51287640$. Since $|\alpha|^2\theta < 1$, Ω has no interior. (See Figure 4.5.)

EXAMPLE 4.9. — Let τ be the substitution given in Example 2.4. Let $\mathcal{G}$ be the geometric realization of τ corresponding to the complex eigenvalue α and eigenvector $\boldsymbol{v}_\alpha$ given in Example 2.4. As in [19], one can show that Ω has a nonempty interior, and hence the Hausdorff dimension of Ω is equal to $\beta = 2$.

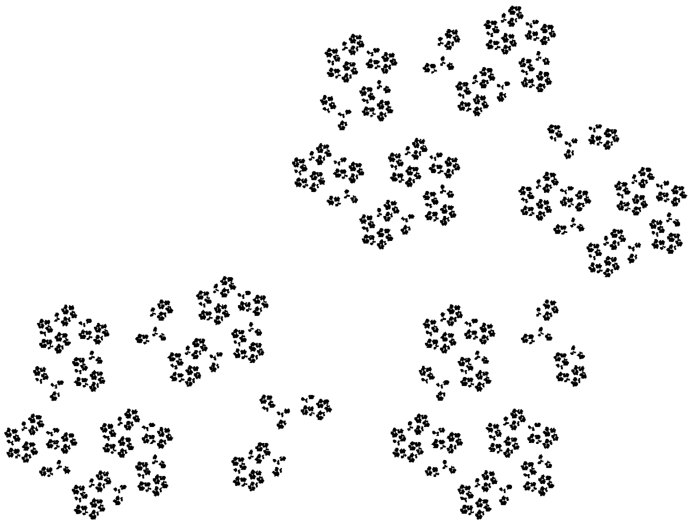


Figure 4.5

Appendix A

Let τ be a primitive substitution on $\mathcal{A} = \{1, 2, \ldots, r\}$ fixing a point $w_* \in \mathcal{A}^{\mathbb{N}}$. Let θ be the Perron-Frobenius eigenvalue of M_τ and $\boldsymbol{p} = (p_1, p_2, \ldots, p_r)$ the strictly positive eigenvector of M_τ (corresponding to θ), normalized so $\sum_{i=1}^{r} p_i = 1$. We consider the sequence $\{\delta_n\}_{n=1}^{\infty}$ defined by

$$\text{(A.1)} \qquad \delta_n = n \cdot (p_1, p_2, \ldots, p_{r-1}) - \big(|w_*(n)|_1, |w_*(n)|_2, \ldots, |w_*(n)|_{r-1}\big) \in \mathbb{R}^{r-1}.$$

Under certain conditions $\{\delta_n\}$ is bounded and its closure provides a geometric encoding of the dynamics of τ. (See [19] and [20].)

Let $\boldsymbol{v}_\alpha = (v_1, v_2, \ldots, v_r) \in \mathbb{C}^r$ be an eigenvector of M_τ^t corresponding to an eigenvalue $\alpha \neq \theta$, and define $S = S(\tau, \boldsymbol{v}_\alpha) : \mathcal{A}^* \to \mathbb{C}$ as in (2.1). We make no assumptions on the modulus of α. To compare the behavior of the sequence $\{S_n\}$ of Theorem 2.3 with that of $\{\delta_n\}$, define

$$\text{(A.2)} \quad \hat{\delta}_n = n(p_1, p_2, \ldots, p_r) - \big(|w_*(n)|_1, |w_*(n)|_2, \ldots, |w_*(n)|_r\big) \in \mathbb{R}^r.$$

We note that $\{\hat{\delta}_n\}_{n=1}^{\infty}$ is bounded if and only $\{\delta_n\}_{n=1}^{\infty}$ is bounded. In fact,

$$\big|w_*(n)\big|_r = n - \sum_{i=1}^{r-1} \big|w_*(n)\big|_i,$$

and $p_r = 1 - \sum_{i=1}^{r-1} p_i$. Taking the dot product of both sides of (A.2) with $\boldsymbol{v}_\alpha$ yields

$$S_n = -\langle \boldsymbol{v}_\alpha, \hat{\delta}_n \rangle \tag{A.3}$$

since $\langle \boldsymbol{v}_\alpha, \boldsymbol{p} \rangle = 0$.

REMARK A.1. — Dividing both sides of (A.3) by n and taking the limit as n goes to infinity gives another proof of (1) of Theorem 2.3 since $\lim_{n\to\infty} \dfrac{\hat{\delta}_n}{n} = 0$. (See [18].)

THEOREM A.2. — *If M_τ has $r-1$ distinct eigenvalues of modulus less than one then $\{\delta_n\}$ is bounded. On the other hand, if for some eigenvalue $\alpha \neq \theta$ the associated sequence $\{S_n\}$ is unbounded, then so is $\{\delta_n\}$. In particular, if M_τ has an eigenvalue $\alpha \neq \theta$ with $|\alpha| \geq 1$ and α is not a root of unity, then the sequence $\{\delta_n\}_{n=1}^{\infty}$ is unbounded.*

Proof. — If M_τ has $r-1$ distinct eigenvalues of modulus less than one, then it follows from (5) of Theorem 2.3 that $\{\delta_n\}$ is bounded. In fact, we see from (A.3) that the projection of $\{\hat{\delta}_n\}$ onto the the space perpendicular to $\boldsymbol{p}$ is bounded. It also follows from (A.3) that if for some eigenvector $\boldsymbol{v}_\alpha$ the associated sequence $\{S_n\}$ is unbounded, then so is $\{\delta_n\}$. The rest follows from (2) and (4) of Theorem 2.3. □

In Example 4.7, M_τ has two eigenvalues of modulus greater than one and therefore the sequence $\{\delta_n\}$ is unbounded. However, the sequence $\{S_n\}$, associated to the nonzero eigenvalue α of modulus less than one, is bounded and its closure provides an encoding of τ.

REMARK A.3. — Example 2.9 shows that if the characteristic polynomial of M_τ is irreducible and θ is a unitary Salem number, then $\{\delta_n\}$ need not be bounded.

In case $r = 3$, S_n and δ_n differ by a linear mapping of $\mathbb{R}^2$. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ we have

$$S_n = \begin{pmatrix} a_1 - a_3 & a_2 - a_3 \\ b_1 - b_3 & b_2 - b_3 \end{pmatrix} \begin{pmatrix} |w_*(n)|_1 \\ |w_*(n)|_2 \end{pmatrix} + n \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}$$

where a_i and b_i denote the real and imaginary parts, respectively, of the entry v_i of $\boldsymbol{v}_\alpha$. Thus, multiplying both sides of (A.1) by the matrix

$$\begin{pmatrix} a_1 - a_3 & a_2 - a_3 \\ b_1 - b_3 & b_2 - b_3 \end{pmatrix}$$

yields

$$\begin{pmatrix} a_1 - a_3 & a_2 - a_3 \\ b_1 - b_3 & b_2 - b_3 \end{pmatrix} \delta_n = n \begin{pmatrix} a_1 - a_3 & a_2 - a_3 \\ b_1 - b_3 & b_2 - b_3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + n \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} - S_n.$$

Since $\langle \boldsymbol{v}_\alpha, \boldsymbol{p} \rangle = 0$ it follows that

$$\begin{pmatrix} a_1 - a_3 & a_2 - a_3 \\ b_1 - b_3 & b_2 - b_3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = - \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}$$

and hence

$$S_n = \begin{pmatrix} a_3 - a_1 & a_3 - a_2 \\ b_3 - b_1 & b_3 - b_2 \end{pmatrix} \delta_n$$

as required.

Appendix B

Assume that M_τ has a real eigenvalue α with $0 < |\alpha| < 1$. Let $\boldsymbol{v}_\alpha = (v_1, v_2, \ldots, v_r)$ be a real eigenvector of M_τ^t and $\mathcal{G}$ the corresponding real geometric realization of Theorem 3.4. For each $a \in \mathcal{A}$, let $D_a \subset \mathbb{R}$ be the smallest closed interval containing Ω_a. We give an algorithm for computing the endpoints of the intervals D_a.

Fix $a \in \mathcal{A}$. We first compute the right endpoint of D_a. Replacing τ by τ^2 if necessary, we can assume that $\alpha > 0$. Recall from Section 4 that for each point $x \in \Omega_a$ there is an infinite path $\gamma \in G^\infty$ beginning at the vertex a of G with $h(\gamma) = x$. (See Theorem 4.3.) Thus

$$\sup \Omega_a = h(\sigma) = \max\bigl\{h(\gamma) \,:\, \gamma \in G^\infty,\ o(\gamma) = a\bigr\}$$

for some path σ with $o(\sigma) = a$.

For each directed edge $e = (i,j)$, set $\ell(e) = S(\tau(j)(i))$. Recall from Section 4 that (i,j) represents a directed edge from $(T^i \circ \tau(j))(1)$ to j. By Theorem 4.3

$$h(\sigma) = \sum_{n=1}^{\infty} \alpha^{n-1} \ell(\sigma(n))$$

where $\sigma(n)$ denotes the nth edge of σ. It follows from maximality of $h(\sigma)$ that for all $n \in \mathbb{N}$,

$$\ell(\sigma(n)) = \max\bigl\{\ell(e) \,:\, e \in E(G),\ o(e) = o(\sigma(n)),\ t(e) = t(\sigma(n))\bigr\}.$$

For each pair of vertices i and j of G for which there is a directed edge from i to j, choose a directed edge $e_{ij} \in E(G)$ (from i to j) which maximizes ℓ. Let $H_{\max}$ be the directed graph with vertex set $\mathcal{A}$ and with edge set $\{e_{ij}\}$. In other words, $H_{\max}$ is obtained from G by erasing all but one directed edge between any ordered pair of vertices and retaining one that maximizes the weight function ℓ. In view of the above remarks, we can assume that σ is an infinite path in $H_{\max}$. Moreover, since any ordered pair of vertices in $H_{\max}$ is joined by at most one directed edge, we may regard σ as a sequence $\{t_n\}_{n=0}^{\infty}$ of vertices in $H_{\max}$.

Let q be the least positive integer such that $t_p = t_q$ for some $p < q$. Since

$$h(\sigma) = h\big((t_n)_{n=0}^{\infty}\big) = \sum_{n=0}^{\infty} \alpha^n \ell(e_{t_n t_{n+1}})$$

is maximized, it follows that for each k,

$$\sup \Omega_{t_k} = \sum_{n=k}^{\infty} \alpha^{n-k} \ell(e_{t_n t_{n+1}}).$$

This implies

$$\sum_{n=p}^{\infty} \alpha^{n-p} \ell(e_{t_n t_{n+1}}) = \sum_{n=q}^{\infty} \alpha^{n-q} \ell(e_{t_n t_{n+1}}).$$

Hence, we may assume that the sequence (t_n) becomes periodic at t_p, with period $q - p$.

We compute $h(\sigma) = \sup \Omega_a$ as follows : First consider the finite set $P_{\max}^{(a)}$ of all triples of the form (b, σ_1, σ_2) where b is a vertex of $H_{\max}$, σ_2 is a simple loop in $H_{\max}$ based at b, and σ_1 a geodesic in $H_{\max}$ from a to the loop determined by σ_2 whose terminal vertex is b. In case $a = b$, we take σ_1 to be the empty path.

Then

$$\sup \Omega_a = \max \Big\{ w(\sigma_1) + \alpha^{|\sigma_1|} w(\sigma_2) \sum_{n=0}^{\infty} \alpha^{|\sigma_2| n} \ : \ (b, \sigma_1, \sigma_2) \in P_{\max}^{(a)} \Big\}$$

where for each path γ,

$$w(\gamma) = \sum_{n=1}^{|\gamma|} \alpha^{n-1} \ell\big(\gamma(n)\big).$$

Note that if each entry of $\boldsymbol{v}_\alpha$ is a rational function in α, then so is $\sup \Omega_a$ for each letter $a \in \mathcal{A}$.

We define analogously a subgraph $H_{\min}$ of G to obtain a similar characterization of $\inf \Omega_a$. A similar argument also shows that each one sided limit point of Ω can be represented by an eventually periodic path in G of the form $\sigma_1 \sigma_2 \sigma_2 \sigma_2 \cdots$ where σ_2 is a simple loop in G.

As a consequence of this algorithm, we have :

THEOREM B.1. — *There is an eigenvector $\boldsymbol{v}_\alpha$ all of whose entries are polynomials in α such that each endpoint of D_i and each one sided limit point of Ω is a polynomial in α.*

Proof. — We first choose $\boldsymbol{v}_\alpha$ so that each entry is a rational polynomial in α. Then, by the above algorithm we can clear denominators to ensure that each entry of $\boldsymbol{v}_\alpha$ and each endpoint of D_i is a polynomial in α. Finally, as each one sided limit point of Ω is encoded by an infinite path in G of the form $\sigma_1\sigma_2\sigma_2\sigma_2\cdots$, where σ_2 is a simple loop in G, the result follows. $\square$

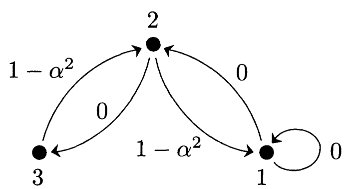
We illustrate this algorithm with an example.

EXAMPLE B.2. — Let τ be the substitution given in Example 4.7. That is, $\tau(1) = 12$, $\tau(2) = 13$, and $\tau(3) = 2$. Fix the eigenvector

$$v_\alpha = \begin{pmatrix} 1-\alpha^2 \\ -\alpha \\ -1 \end{pmatrix}$$

of M_τ^t corresponding to the eigenvalue $\alpha \approx .445041867$.

In this case $G = H_{\min} = H_{\max}$. The graph G together with the ℓ-value of each edge is shown in Figure B.1.



$G = H_{\max} = H_{\min}$

Figure B.1

Then, $\sup \Omega_1$ is the largest of the numbers

$$h(11111111\ldots) = 0, \quad h(123232323\ldots) = \alpha^2, \quad h(12121212\ldots) = \alpha.$$

Similarly, $\sup \Omega_2$ is the largest of

$$h(21212121\ldots) = 1, \quad h(23232323\ldots) = \alpha, \quad h(21111111\ldots) = 1-\alpha^2,$$

and $\sup \Omega_3$ is the largest of

$$\begin{aligned} h(32111111\ldots) &= -2\alpha^2 - \alpha + 2, \\ h(321212121\ldots) &= 1 + \alpha - \alpha^2, \\ h(32323232\ldots) &= 1. \end{aligned}$$

Since $H_{\min} = H_{\max}$, it follows that the infimum of each Ω_i is the smallest member of each of the sets of numbers given above. Thus,

$$D_1 = [0, \alpha], \quad D_2 = [\alpha, 1], \quad D_3 = [1, 1 + \alpha - \alpha^2].$$

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