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THE NUMBER OF CONJUGACY CLASSES OF ELEMENTS OF THE CREMONA GROUP OF SOME GIVEN FINITE ORDER

by Jérémy Blanc

Abstract. — This note presents the study of the conjugacy classes of elements of some given finite order n in the Cremona group of the plane. In particular, it is shown that the number of conjugacy classes is infinite if n is even, $n = 3$ or $n = 5$, and that it is equal to 3 (respectively 9) if $n = 9$ (respectively if $n = 15$) and to 1 for all remaining odd orders. Some precise representative elements of the classes are given.

Résumé (*Le nombre de classes de conjugaison du groupe de Cremona d'un ordre fini donné*)

Cet article présente l'étude des classes de conjugaisons des éléments d'ordre fini n dans le groupe de Cremona du plan. En particulier, il est montré que le nombre de classes de conjugaisons est infini si n est pair, $n = 3$ ou $n = 5$, et que ce nombre est égal à 3 (respectivement 9) si $n = 9$ (respectivement si $n = 15$) et à 1 pour les nombres entiers impairs restant. Des représentants explicites des classes de conjugaisons sont donnés.

1. Introduction

Let us recall that a *rational transformation* of $\mathbb{P}^2(\mathbb{C})$ is a map of the form

$$(x : y : z) \dashrightarrow (\varphi_1(x, y, z) : \varphi_2(x, y, z) : \varphi_3(x, y, z)),$$

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where $\varphi_1, \varphi_2, \varphi_3 \in \mathbb{C}[x, y, z]$ are homogeneous polynomials of the same degree. If such a map has an inverse of the same type, we say that it is *birational.*

The *Cremona group* is the group of birational transformations of $\mathbb{P}^2(\mathbb{C})$. This group has been studied since the 19^{th} century by many mathematicians. One of the first natural questions that we may ask when we study some group is the following:

QUESTION 1.1. — *Given some positive integer n, how many conjugacy classes of elements of order n exist in the Cremona group?*

First of all, it is important to note that the number of conjugacy classes is at least one, for any integer n, as the linear automorphism

$$(x : y : z) \longmapsto (x : y : \mathrm{e}^{2\mathbf{i}\pi/n} z)$$

is a representative element of one class. It was proved in [2] that all the linear automorphisms of the plane of the same finite order are birationally conjugate (the same is true in any dimension, see [4, Prop. 5]); to find more conjugacy classes we have therefore to show the existence of non-linearizable birational transformations.

The first answer to Question 1.1 was given in [3] for $n = 2$. Infinitely many involutions which are not conjugate are found. Since the proof of [3] is considered as incomplete, a precise and complete one may be found in [1].

In [9], the answer for n prime is given. It is shown that the number of conjugacy classes is infinite for $n = 3, 5$ and is equal to 1 if n is a prime integer ≥ 7.

For other orders, a lot of examples have been given in the ancient articles (for example in [12], [16]) and in many more recent articles, the most recent one being [8]. However, the precise answer to Question 1.1 was not given for n not prime.

In this paper, we answer Question 1.1 for any integer n, proving the following theorems:

THEOREM 1.2. — *For any even integer n, the number of conjugacy classes of elements of order n in the Cremona group is infinite. This is also true for $n = 3, 5$.*

THEOREM 1.3. — *For any odd integer $n \neq 3, 5$, the number of conjugacy classes of elements of order n in the Cremona group is finite. Furthermore this number is equal to 3 (respectively 9) if $n = 9$ (respectively if $n = 15$) and is 1 otherwise.*

This paper is a part of the author's PHD Thesis (the full text is available in [5], the results without proofs have been published in [6]). The results have been a little improved and the arguments have been slightly ameliorated to reduce the length of the proofs. We thank a lot our PHD advisor Thierry Vust, and also Arnaud Beauville, Igor Dolgachev, Ivan Pan and Felice Ronga for helpful discussions.

REMARK 1.4. — Theorem 1.2 and a part of Theorem 1.3 may be proved using the classification of finite Abelian subgroups of the Cremona group made in [5], but the proof is really long and intricate, whereas the results of this paper only require a much shorter and direct proof. Moreover, the precise counting of Theorem 1.3 does not follow from the classification (in general, it is not easy to decide whether or not two elements of the same subgroup are conjugate).

2. Automorphisms of rational surfaces

Let us remark the obvious but important observation: take some birational transformation φ of a rational surface S. Any birational map $\lambda : S \dashrightarrow \mathbb{P}^2$ conjugates φ to the birational transformation $\varphi_\lambda = \lambda \circ \varphi \circ \lambda^{-1}$ of $\mathbb{P}^2$. Although φ_λ is not unique, all the possible φ_λ's form an unique conjugacy class of birational transformations of $\mathbb{P}^2$.

Conversely, taking some birational transformation of $\mathbb{P}^2$, we may conjugate it to a birational transformation of any rational surface. If the order of the transformation is finite, we may furthermore conjugate it to a (*biregular*) *automorphism* of a rational surface. (See for example [10], Theorem 1.4.)

An important family of rational surfaces are the rational surfaces with an ample anticanonical divisor, i.e. the *Del Pezzo surfaces*. These surfaces are $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$, and the blow-up of $1 \leq r \leq 8$ points of $\mathbb{P}^2$ in a general position (i.e. such that no irreducible curve of self-intersection -2 belongs to the surface). There is an extensive literature about this; some descriptions may be found for example in [13]. The *degree* of such a surface is the square of its canonical divisor, and is an integer between 1 and 9; it is 9 for $\mathbb{P}^2$, 8 for $\mathbb{P}^1 \times \mathbb{P}^1$ and $9 - r$ for the blow-up of r points in $\mathbb{P}^2$. Almost all of our examples of rational surfaces will be Del Pezzo surfaces.

3. Elements of order 3,5 and of any even order. The proof of Theorem 1.2

Let us give families of conjugacy classes of elements of order 2, 3 and 5 of the Cremona group.

EXAMPLE 3.1 (Birational transformations of order 2). — Let $a_1,\dots,a_n$, $b_1,\dots,b_n$ in $\mathbb{C}$ be all distinct. The birational map

$$((x_1:x_2),(y_1:y_2)) \dashrightarrow \Big((x_1:x_2),\Big(y_2\prod_{i=1}^{n}(x_1-b_ix_2):y_1\prod_{i=1}^{n}(x_1-a_ix_2)\Big)\Big)$$

of $\mathbb{P}^1\times\mathbb{P}^1$ is an involution, which is classically called *de Jonquières involution*. Its fixed points form a smooth curve $\Gamma\subset\mathbb{P}^1\times\mathbb{P}^1$ of equation

$$(y_1)^2\cdot\prod_{i=1}^{n}(x_1-a_ix_2)=(y_2)^2\cdot\prod_{i=1}^{n}(x_1-b_ix_2).$$

The restriction to Γ of the projection $\mathbb{P}^1\times\mathbb{P}^1\to\mathbb{P}^1$ on the first factor is a surjective morphism $\Gamma\to\mathbb{P}^1$ of degree 2, ramified over the points

$$(a_1:1),\dots,(a_n:1),(b_1:1),\dots,(b_n:1).$$

The curve Γ is therefore an *hyperelliptic curve.*

These involutions are birationally equivalent to those of [1], *Example* 2.4 (c).

EXAMPLE 3.2 (Birational transformations of order 3). — Let F be a non-singular form of degree 3 in three variables and let

$$\Gamma=\big\{(x:y:z)\in\mathbb{P}^2 \mid F(x,y,z)=0\big\}$$

be the smooth cubic plane curve associated to it. The surface

$$S=\big\{(w:x:y:z)\in\mathbb{P}^3 \mid w^3=F(x,y,z)\big\}\subset\mathbb{P}^3$$

is thus a smooth cubic surface in $\mathbb{P}^3$, which is rational (it is a Del Pezzo surface of degree 3, see for example [13], Theorem III.3.5). The map $w\mapsto \mathrm{e}^{2\mathbf{i}\pi/3}w$ gives rise to an automorphism of S whose set of fixed points is isomorphic to the elliptic curve Γ.

Such elements generate cyclic groups of order 3, *already given in* [9], *Theorem A, case A1.*

EXAMPLE 3.3 (Birational transformations of order 5). — Let us choose λ,μ in $\mathbb{C}$ such that the surface

$$S=\big\{(w:x:y:z)\in\mathbb{P}(3,1,1,2) \mid w^2=z^3+\lambda x^4z+x(\mu x^5+y^5)\big\}$$

is smooth. The surface S is thus rational (it is a Del Pezzo surface of degree 1, see [13, Thm. III.3.5]) and the map $y\mapsto \mathrm{e}^{2\mathbf{i}\pi/5}y$ gives rise to an automorphism of S whose set of fixed points is the union of the point $(0:0:1:0)$ and the elliptic curve which is the trace on S of the equation $y=0$.

The corresponding cyclic groups of order 5 *were given in* [9], *Theorem A, case A3.*

To prove Theorem 1.2, it remains to give the existence of infinitely many conjugacy classes of elements of order n, for any even integer $n \geq 4$. The elements that we will give are roots of de Jonquières involutions (Example 3.1) and belong to the classical *de Jonquières group*, which is a subgroup of the Cremona group. We now introduce this group.

EXAMPLE 3.4 (The de Jonquières group). — The de Jonquières group is isomorphic to

$$\mathrm{PGL}\left(2, \mathbb{C}(x)\right) \rtimes \mathrm{PGL}(2, \mathbb{C}),$$

where $\mathrm{PGL}(2,\mathbb{C})$ acts naturally on $\mathbb{C}(x)$, as $\mathrm{PGL}(2,\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1)$ is the automorphism group of $\mathbb{P}^1$ and $\mathbb{C}(x) = \mathbb{C}(\mathbb{P}^1)$ is its function field. To the element

$$\left(\begin{pmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & \delta(x) \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \in \mathrm{PGL}\left(2, \mathbb{C}(x)\right) \rtimes \mathrm{PGL}(2, \mathbb{C})$$

we associate the following birational map of $\mathbb{C}^2$:

$$(x,y) \dashrightarrow \left(\frac{ax+b}{cx+d}, \frac{\alpha(x)y+\beta(x)}{\gamma(x)y+\delta(y)}\right).$$

The natural inclusion $\mathbb{C}^2 \subset \mathbb{P}^2(\mathbb{C})$ (resp. $\mathbb{C}^2 \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$) sends the de Jonquières group on the group of birational transformations of $\mathbb{P}^2$ (resp. $\mathbb{P}^1 \times \mathbb{P}^1$) that leave invariant the pencil of lines of $\mathbb{P}^2$ passing through one point (resp. that leave invariant one of the two standard pencils of lines of $\mathbb{P}^1 \times \mathbb{P}^1$).

In this context, we may look at the subgroup of the de Jonquières group that fixes some hyperelliptic curve:

EXAMPLE 3.5 (The group of birational transformations that fix some curve)

Let $g(x) \in \mathbb{C}(x)^*$ be some element which is not a square in $\mathbb{C}(x)$. We denote by $\mathcal{J}_g$ the torus of $\mathrm{PGL}(2,\mathbb{C}(x))$ which is the image in $\mathrm{PGL}(2,\mathbb{C}(x))$ of the subgroup

$$\mathcal{T}_g = \left\{ \begin{pmatrix} \alpha(x) & \beta(x)g(x) \\ \beta(x) & \alpha(x) \end{pmatrix} \;\middle|\; \alpha(x), \beta(x) \in \mathbb{C}(x),\ \alpha \neq 0 \text{ or } \beta \neq 0 \right\}$$

of $\mathrm{GL}(2,\mathbb{C}(x))$. The group $\mathcal{J}_g$ corresponds to the group of birational transformations of the form

$$(x,y) \dashrightarrow \left(x, \frac{\alpha(x)y+\beta(x)g(x)}{\beta(x)y+\alpha(x)}\right).$$

Note that $\mathcal{T}_g$ is isomorphic to the multiplicative group of the field

$$\mathbb{C}(x)\left[\sqrt{g(x)}\right] = \left\{\alpha(x) + \beta(x)\sqrt{g(x)} \;\middle|\; \alpha(x), \beta(x) \in \mathbb{C}(x)\right\}.$$

In the case where $g(x)$ is a polynomial without multiple roots, the field $\mathbb{C}(x)\left[\sqrt{g(x)}\right]$ is the function field $\mathbb{C}(\Gamma)$ of the smooth curve Γ of equation

$y^2 = g(x)$, and the group $\mathcal{J}_g = \mathbb{C}(\Gamma)^*/\mathbb{C}(x)^*$ is the group of elements of the de Jonquières group that fix the curve Γ. (If the degree of $g(x)$ is at least 5, it is in fact the group of birational maps that fix the curve, see [7].)

PROPOSITION 3.6. — *Let $n \geq 1$ be some integer, and let $g(x) \in \mathbb{C}(x)^*$ be such that $g(\mathrm{e}^{2\mathbf{i}\pi/n} \cdot x) = g(x)$. There exists $\nu(x) \in \mathbb{C}(x)$ such that the n-th power of the birational map*

$$\varphi : (x,y) \dashrightarrow \left(\mathrm{e}^{2\mathbf{i}\pi/n} \cdot x, \frac{\nu(x)y + g(x)}{y + \nu(x)}\right)$$

is the de Jonquières involution

$$\varphi^n : (x,y) \dashrightarrow \left(x, \frac{g(x)}{y}\right).$$

Proof. — Note that choosing any $\nu(x) \in \mathbb{C}(x)$, the associated map φ belongs to the de Jonquières group (Example 3.4) and is the composition of $(x,y) \mapsto (\mathrm{e}^{2\mathbf{i}\pi/n} \cdot x, y)$ with an element of $\mathcal{J}_g$ defined in Example 3.5. The n-th power of φ in the de Jonquières group $\mathrm{PGL}(2, \mathbb{C}(x)) \rtimes \mathrm{PGL}(2, \mathbb{C})$ is equal to

$$\left(\begin{pmatrix} \nu(x) & g(x) \\ 1 & \nu(x) \end{pmatrix} \begin{pmatrix} \nu(\xi \cdot x) & g(x) \\ 1 & \nu(\xi \cdot x) \end{pmatrix} \cdots \begin{pmatrix} \nu(\xi^{n-1} \cdot x) & g(x) \\ 1 & \nu(\xi^{n-1} \cdot x) \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right),$$

where $\xi = \mathrm{e}^{2\mathbf{i}\pi/n}$. Since the element $\left(\begin{smallmatrix} \nu(x) & g(x) \\ 1 & \nu(x) \end{smallmatrix}\right) \in \mathcal{J}_g \subset \mathrm{PGL}(2, \mathbb{C}(x))$ is the image of some element of $\mathcal{T}_g \subset \mathrm{GL}(2, \mathbb{C}(x))$ corresponding to

$$\zeta = \left(\nu(x) + \sqrt{g(x)}\right) \in \mathbb{C}(x)\big[\sqrt{g(x)}\,\big]^*,$$

the element $\varphi^n \in \mathcal{J}_g \subset \mathrm{PGL}(2, \mathbb{C}(x))$ is therefore the image of the element

$$\zeta \cdot \sigma(\zeta) \cdot \sigma^2(\zeta) \cdots \sigma^{n-1}(\zeta) \in \mathbb{C}(x)\big[\sqrt{g(x)}\,\big]^*,$$

where σ is the automorphism of $\mathbb{C}(x)\big[\sqrt{g(x)}\,\big]^*$ that sends x on ξx and acts trivially on $\mathbb{C}\big[\sqrt{g(x)}\,\big]$. Let us look at the morphism

$$N : \mathbb{C}(x)[\sqrt{g(x)}]^* \longrightarrow \mathbb{C}(x)[\sqrt{g(x)}]^*, \quad \tau \longmapsto \tau \cdot \sigma(\tau) \cdot \sigma^2(\tau) \cdots \sigma^{n-1}(\tau).$$

All elements of its image are invariant by σ and thus belong to the multiplicative group of the field $\mathbb{C}(x^n)\big[\sqrt{g(x)}\,\big]$. Furthermore, the map N is the norm of the field extension $\mathbb{C}(x)\big[\sqrt{g(x)}\,\big]/\mathbb{C}(x^n)\big[\sqrt{g(x)}\,\big]$.

Since this is a finite Galois extension, and the field $\mathbb{C}(x)$ has the C_1-property (by Tsen theorem), the norm N is surjective (see [15], X.7, Prop. 10 and 11). We may thus choose an element $\zeta_0 = \alpha(x) + \beta(x)\sqrt{g(x)}$ whose norm is equal to $\sqrt{g(x)}$. As $\beta(x)$ is certainly not equal to zero, we may choose $\nu(x) = \alpha(x)/\beta(x)$, so that $\zeta = \zeta_0/\beta(x)$ is sent by N on $N(\beta^{-1}) \cdot \sqrt{(g(x)}$, whose image in $\mathrm{PGL}(2, \mathbb{C}(x))$ is $\left(\begin{smallmatrix} 0 & g(x) \\ 1 & 0 \end{smallmatrix}\right)$, as we wanted. $\square$

We give now explicitly a family of examples produced in Proposition 3.6.

Example 3.7. — Let $n = 2m$, where m is an odd integer and let $h \in \mathbb{C}(x)$ be a rational function. We choose α to be the birational transformation

$$\alpha : (x, y) \dashrightarrow \left(\mathrm{e}^{2\mathbf{i}\pi/n} \cdot x, \frac{h(x^m)y - h(x^m)h(-x^m)}{y + h(x^m)} \right).$$

Compute $\alpha^2 : (x, y) \dashrightarrow (\mathrm{e}^{2\mathbf{i}\pi/m} \cdot x, (-h(x^m) \cdot h(-x^m))/y)$ and see that this is the composition of the commuting birational transformations

$$(x, y) \longmapsto (\mathrm{e}^{2\mathbf{i}\pi/m} \cdot x, y) \quad \text{and} \quad (x, y) \dashrightarrow \left(x, \frac{-h(x^m) \cdot h(-x^m)}{y} \right)$$

of order respectively m and 2. Thus, the order of α^2 is $2m = n$ and $\alpha^n = \alpha^{2m}$ is the birational involution

$$\alpha^n : (x, y) \dashrightarrow \left(x, \frac{-h(x^m) \cdot h(-x^m)}{y} \right).$$

We are now able to prove Theorem 1.2, i.e. to show the existence of infinitely many conjugacy classes of elements of order n in the Cremona group, for any even integer n and for $n = 3, 5$.

Proof of Theorem 1.2. — First of all, taking some non-rational curve Γ, any birational transformation sends Γ on a curve birational to it (the same result for a rational curve is false, as the curves may be collapsed on one point). If two birational transformations α, β are conjugate by φ, the element φ sends the non-rational curves fixed by α on the non-rational curves fixed by β. (In fact there is at most one such curve, but we will not need it here.)

Choosing different de Jonquières involutions (Example 3.1), the possible curves fixed are all the hyperelliptic curves. As the number of isomorphism classes of such curves is infinite, we obtain infinitely many conjugacy classes of de Jonquières involutions in the Cremona group. (In fact, there exist some other families, called Geiser and Bertini involution, see [2].)

The same arguments works for elements of order 3 and 5 (Examples 3.2 and 3.3), that may fix all the elliptic curves, whose number of isomorphism classes is also infinite.

Taking $n \geq 2$, and any polynomial $g \in \mathbb{C}[x^n]$ without multiple roots, there exists an element α in the Cremona group which has order $2n$ and such that α^n is the birational involution $(x, y) \dashrightarrow (x, g(x)/y)$ (Proposition 3.6). As this involution fixes the hyperelliptic curve $y^2 = g(x)$, the number of conjugacy classes of such elements (when changing the element g) is infinite. □

4. Elements of odd order ≥ 7. The proof of Theorem 1.3

As it was said in Section 2, any birational transformation of finite order of the plane is conjugate to an automorphism g of some rational surface S. We may then assume that the pair (g, S) is minimal and use the following result, proved in [14].

PROPOSITION 4.1. — *Let g be some automorphism of a rational surface S, such that the pair (g, S) is minimal (i.e. every g-equivariant birational morphism $S \to S'$ is an isomorphism). Then, one of the two following cases occurs:*

- rk $\mathrm{Pic}(S)^g = 1$ *and S is a Del Pezzo surface;*
- rk $\mathrm{Pic}(S)^g = 2$ *and g preserves a conic bundle structure on S (i.e. there exists some morphism $\pi : S \to \mathbb{P}^1$ with fibres isomorphic to $\mathbb{P}^1$, except for a finite number of singular fibres, that consist on the union of two intersecting curves isomorphic to $\mathbb{P}^1$; and g sends any fibre of π on another fibre).*

To prove Theorem 1.3, we enumerate the possibilities of pairs (g, S) where g is an automorphism of odd order ≥ 7, using Proposition 4.1. The following lemma will help us to prove the theorem for Del Pezzo surfaces:

LEMMA 4.2 (Size of the orbits). — *Let S be a Del Pezzo surface, which is the blow-up of $1 \leq r \leq 8$ points of $\mathbb{P}^2$ in general position, and let $G \subset \mathrm{Aut}(S)$ be a finite subgroup of automorphisms with* rk $\mathrm{Pic}(S)^G = 1$. *Then:*

- *$G \neq \{1\}$;*
- *the size of any orbit of the action of G on the set of exceptional divisors is divisible by the degree of S, which is $(K_S)^2$;*
- *in particular, the order of G is divisible by the degree of S.*

Proof. — It is clear that $G \neq \{1\}$, since rk $\mathrm{Pic}(S) > 1$. Let $D_1, D_2, \ldots, D_k$ be k exceptional divisors of S, forming an orbit of G. The divisor $\sum_{i=1}^{k} D_i$ is fixed by G and thus is a multiple of K_S. We can write $\sum_{i=1}^{k} D_i = aK_S$, for some rational number $a \in \mathbb{Q}$. In fact, since aK_S is effective, we have $a < 0$; furthermore $a \in \mathbb{Z}$, since the canonical divisor is not a multiple in $\mathrm{Pic}(S)$. The D_i's being irreducible and rational, we deduce from the adjunction formula $D_i(K_S + D_i) = -2$ that $D_i \cdot K_S = -1$. Hence

$$K_S \cdot \sum_{i=1}^{k} D_i = \sum_{i=1}^{k} K_S \cdot D_i = -k = K_S \cdot aK_S = a(K_S)^2.$$

Consequently, the degree of S divides the size k of the orbit. $\square$

We decompose now our investigations on different surfaces.

Proposition 4.3. — *Any automorphism of* $\mathbb{P}^1 \times \mathbb{P}^1$ *is birationally conjugate to a linear automorphism of* $\mathbb{P}^2$.

Proof. — Recall first that any automorphism of $\mathbb{P}^1$ fixes a point. We prove that the same is true for the automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$. Indeed, any such automorphism is of the form $(u, v) \mapsto (\alpha(u), \beta(v))$ or $(u, v) \mapsto (\alpha(v), \beta(u))$, for some $\alpha, \beta \in \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2, \mathbb{C})$. The first automorphism fixes the point (a, b), where $a, b \in \mathbb{P}^1$ are points fixed by respectively α and β. The second one fixes the point $(c, \beta(c))$, where $c \in \mathbb{P}^1$ is a point fixed by $\alpha\beta$.

Blowing-up the fixed point, and blowing-down the strict pull-backs of the two lines of $\mathbb{P}^1 \times \mathbb{P}^1$ passing through the fixed point, we conjugate the automorphism to an automorphism of $\mathbb{P}^2$. □

Proposition 4.4. — *Any automorphism of finite odd order of some conic bundle* (*that preserves the conic bundle structure*) *is birationally conjugate to a linear automorphism of* $\mathbb{P}^2$.

Proof. — Let us denote by g the automorphism of odd order of the conic bundle induced by $\pi : S \to \mathbb{P}^1$. Recall that the action of g on the fibres of π induces an automorphism $\overline{g}$ of $\mathbb{P}^1$ of odd order m, whose orbits have all the same size m, except for two fixed points.

Suppose that one fibre F of π is singular. The orbit of F by g is thus a set of singular curves $\{F_1, \ldots, F_n\}$ (where $n = 1$ or $n = m$). Furthermore, g acts on the set T of irreducible components of the F_i's, whose size is even, equal to $2n$. Since the order of g is odd, the action of g on T has two orbits of size n, and two curves of the same orbit do not intersect. This allows us to blow-down one of the two orbits, to obtain a birational g-equivariant morphism from the conic bundle to another one, with fewer singular fibres.

Continuing by this way, we conjugate g to an automorphism of a conic bundle which has no singular fibre. Since the fibration is smooth, the surface is an Hirzebruch surface $\mathbb{F}_k$, for some integer $k \geq 0$. If $k \geq 1$, choose one fibre F invariant by g (there exist at least two such fibres). Since $F \cong \mathbb{P}^1$, g fixes at least two points of F. Blow-up one point of F fixed by g and not lying on the exceptional section of $\mathbb{F}_k$ (the one of self-intersection $-k$); blow-down then the strict pull-back of F, to obtain a g-equivariant birational map $\mathbb{F}_k \dashrightarrow \mathbb{F}_{k-1}$. By this way, we may assume that g acts biregularly on $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and use Proposition 4.3 to achieve the proof. □

Proposition 4.5. — *Any automorphism of finite odd order of a Del Pezzo surface of degree* ≥ 4 *is birationally conjugate to a linear automorphism of* $\mathbb{P}^2$.

Proof. — Recall that a Del Pezzo surface is either $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow-up of some points in $\mathbb{P}^2$ in general position (i.e. such that no irreducible curve of self-intersection ≤ -2 belongs to the surface).

Suppose that g acts on a Del Pezzo surface S of degree ≥ 4. By blowing-down some curves (which gives once again a Del Pezzo surface, with a larger degree), we may assume that g acts minimally on S. If S is $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$, we are done (Proposition 4.3).

Otherwise, either g preserves a conic bundle structure, or $\operatorname{rk}\operatorname{Pic}(S)^g = 1$ (Proposition 4.1). In the first case, g is birationally conjugate to a linear automorphism of $\mathbb{P}^2$ (Proposition 4.4). In the second case, the degree of S divides the order of g (Lemma 4.2), which is odd by hypothesis. The only possibilities for the degree of S are thus 5 or 7. We study now both cases.

If the degree of S is 7, i.e. if S is the blow-up of two distinct points of $\mathbb{P}^2$, there are three exceptional divisors on S. These are the pull-back E_1, E_2 of the two points, and the strict pull-back of the line of $\mathbb{P}^2$ passing through the two points. This configuration implies that the set $\{E_1, E_2\}$ is invariant by any automorphism of the surface, which is thus birationally conjugate to a linear automorphism of $\mathbb{P}^2$.

If the degree of S is 5, i.e. if S is the blow-up of four points of $\mathbb{P}^2$, no three being collinear, we may assume that the points blowed-up are $(1:0:0)$, $(0:1:0)$, $(0:0:1)$ and $(1:1:1)$. The action of the group $\operatorname{Aut}(S)$ of automorphisms of S on the five sets of four skew exceptional divisors of S gives rise to an isomorphism of $\operatorname{Aut}(S)$ to the group Sym_5. The group $\operatorname{Aut}(S)$ is generated by the lift of the group Sym_4 of automorphisms of $\mathbb{P}^2$ that leaves invariant the four points blowed-up, and by the quadratic transformation $(x:y:z) \dashrightarrow (yz:xz:xy)$. The nature of $\operatorname{Aut}(S)$ may be found by direct calculation, and is also well-known for many years (see for example [12], [16], [5], [8]). Using Lemma 4.2, the automorphism g with $\operatorname{rk}\operatorname{Pic}(S)^g = 1$ must have order 5, and is thus birationally conjugate to $(x:y:z) \dashrightarrow (x(z-y):z(x-y):xz)$, which is birationally conjugate to a linear automorphism of $\mathbb{P}^2$ (see [2]). □

Proposition 4.5 is false for Del Pezzo surfaces of degree at most 3. We give now some examples:

EXAMPLE 4.6. — Let

$$S_F = \{(w:x:y:z) \in \mathbb{P}^3 \mid w^3 + x^3 + y^3 + z^3 = 0\}$$

be the *Fermat cubic surface*, which is a Del Pezzo surface of degree 3 (see [13], Theorem III.3.5). The elements

$$\rho_1 : (w:x:y:z) \longmapsto (w : \mathrm{e}^{2\mathrm{i}\pi/3} y : z : x),$$

$$\rho_2 : (w:x:y:z) \longmapsto (w : \mathrm{e}^{4\mathrm{i}\pi/3} y : z : x)$$

are automorphisms of S_F. For $i=1,2$, the element

$$(\rho_i)^3 : (w:x:y:z) \longmapsto (w : \mathrm{e}^{i2\mathbf{i}\pi/3}x : \mathrm{e}^{i2\mathbf{i}\pi/3}y : \mathrm{e}^{i2\mathbf{i}\pi/3}z)$$

fixes the elliptic curve which is the intersection of S with the plane $w=0$, and corresponds to an element of order 3 described in Example 3.2. Since $(\rho_i)^3$ is not birationally conjugate to a linear automorphism of $\mathbb{P}^2$, the same occurs for ρ_i.

EXAMPLE 4.7. — Let

$$S_{15} = \left\{(w:x:y:z) \in \mathbb{P}(3,1,1,2) \mid w^2 = z^3 + x(x^5+y^5)\right\},$$

be a special Del Pezzo surface of degree 1 (see [13], Theorem III.3.5). The element

$$\theta : (w:x:y:z) \longmapsto (w : x : \mathrm{e}^{2\mathbf{i}\pi/5}y : \mathrm{e}^{2\mathbf{i}\pi/3}z)$$

is an automorphism of the surface S_{15} which has order 15. Since θ^3 (which is an element described in Example 3.3) fixes an elliptic curve, it is not birationally conjugate to a linear automorphism of $\mathbb{P}^2$, and thus the same occurs for θ.

PROPOSITION 4.8. — *Let g be some birational map of $\mathbb{P}^2$ of finite odd order ≥ 7. Then, g is birationally conjugate either to a linear automorphism of $\mathbb{P}^2$, or to one of the elements ρ_1,ρ_2 described in Example 4.6, or to one of the elements $\theta,\theta^2,\theta^4,\theta^7,\theta^8,\theta^{11},\theta^{13},\theta^{14}$, where θ is described in Example 4.7.*

Proof. — As we already mentioned, every birational map of $\mathbb{P}^2$ is birationally conjugate to an automorphism of a rational surface S (see for example [10], Theorem 1.4). Supposing that the action is minimal (i.e. that every g-equivariant birational morphism $S \to S'$ is an isomorphism), either g preserves a conic bundle structure on S or S is a Del Pezzo surface (Proposition 4.1).

In the first case, the automorphism is birationally conjugate to a linear automorphism of $\mathbb{P}^2$, since it has odd order (Proposition 4.4).

In the second case, if the surface has degree ≥ 4, the automorphism is birationally conjugate to a linear automorphism of $\mathbb{P}^2$ (Proposition 4.5). Otherwise, applying Lemma 4.2, the degree of the surface is 1 or 3 and divides the order of the automorphism. We enumerate the possibilities:

— *Assume that the degree of S is* 3*, and the order of g is a multiple of* 3.

The linear system $|K_S|$ gives rise to the canonical embedding of S in $\mathbb{P}^3$, whose image is a smooth cubic surface (see for example [13, Thm. III.3.5]). Since g leaves invariant the linear system $|K_S|$, it is the restriction of a linear automorphism of $\mathbb{P}^3$.

Suppose first that S is isomorphic to the Fermat cubic surface S_F, whose equation is $w^3+x^3+y^3+z^3=0$, and whose group of automorphisms is $(\mathbb{Z}/3\mathbb{Z})^3 \rtimes \mathrm{Sym}_4$, where $(\mathbb{Z}/3\mathbb{Z})^3$ is the 3-torsion of $\mathrm{PGL}(4,\mathbb{C})$ and Sym_4 is the group of permutations of the variables. Since the order of g is odd and at least 7, its image in Sym_4 is an element of order 3. The elements of order 3 of Sym_4 being all conjugate, g is conjugate to an element of the form

$$(w:x:y:z)\longmapsto(w:ay:bz:cx),$$

for some a,b,c in the 3-torsion of $\mathbb{C}^*$. We conjugate g by the automorphism $(w:x:y:z)\mapsto(w:bcx:y:cz)$ of S_F and obtain the automorphism $(w:x:y:z)\mapsto(w:abcy:z:x)$. Since the order of g is not 3, abc is not equal to 1 and is thus a primitive 3-th root of unity. The two possible cases give the elements ρ_1 and ρ_2 described in Example 4.6.

We proceed now to the study of general cubic surfaces. We denote by G the group generated by g, and by h one of the two elements of order 3 of G. Up to isomorphism (and to the choice of h), three possibilities occur (we use the notation $\omega=\mathrm{e}^{2\mathbf{i}\pi/3}$):

1) *The automorphism h is $(w:x:y:z)\mapsto(\omega w:\omega x:y:z)$.* — The equation of S is thus $L_3(w,x)+L_3'(y,z)=0$, where L_3,L_3' are homogeneous forms of degree 3; this implies that the surface is isomorphicto the Fermat cubic surface, a case already studied.

2) *The automorphism h is $(w:x:y:z)\mapsto(\omega w:x:y:z)$.* — In this case, h fixes an elliptic curve Γ, which is the intersection of S with the plane of equation $w=0$ (h corresponds to an element described in Example 3.2). Note that G commutes with h so it leaves invariant Γ and also the plane $w=0$. The action of the group G on the curve Γ must thus be cyclic of odd order at least 3 and corresponds to the action of a cyclic subgroup of $\mathrm{PGL}(3,\mathbb{C})$ on a smooth cubic curve. If the action is a translation, it does not have fixed points and corresponds to the action of $(x:y:z)\mapsto(x:\omega y:\omega^2 z)$ on a plane cubic curve of equation $x^3+y^3+w^3+\lambda xyz=0$. But this is not possible, since the group obtained by lifting this action is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$ and thus is not cyclic. It remains the case of an automorphism of an elliptic curve, which has fixed points. The only possibility is an element of order 3, that acts on the curve of equation $x^3+y^3+z^3=0$. But this case yields once again the Fermat cubic surface.

3) *The automorphism h is $(w:x:y:z)\mapsto(\omega w:\omega^2 x:y:z)$.* — We see now that this case is incompatible with the hypothesis on g. Note that the action of h on $\mathbb{P}^3$ fixes the line L_{yz} of equation $y=z=0$ and thus the group G itself leaves invariant this line. If $L_{yz}\subset S$, the rank of $\mathrm{rk}\ \mathrm{Pic}(S)^G$ is at least 2. Otherwise, the equation of S is of the form $L_3(w,x)+L_1(w,x)yz+y^3+z^3=0$, where L_3 and L_1 are homogeneous form of degree respectively 3 and 1, and L_3 has three distinct roots. The action of G on the three points of $L_{yz}\cap S$ gives an exact sequence $1\to\langle h\rangle\to G\to\mathrm{Sym}_3$. Since the order of g is odd and at

least 7, the image at the right is a cyclic group of order 3 and g has order 9. A quick calculation shows that this is not possible.

— *Assume now that the degree of S is* 1.

The linear system $|-2K_S|$ induces a degree 2 morphism onto a quadric cone in $Q \subset \mathbb{P}^3$, ramified over the vertex v of Q and a smooth curve C of genus 4. Moreover C is the intersection of Q with a cubic surface (see [1], [9], [8]). Note that a quadric cone is isomorphic to the weighted projective plane $\mathbb{P}(1,1,2)$ and the ramification curve C has equation of degree 6 there. Up to a change of coordinates, we may thus assume that the surface S has the equation

$$w^2 = z^3 + F_4(x,y)z + F_6(x,y)$$

in the weighted projective space $\mathbb{P}(3,1,1,2)$, where F_4 and F_6 are forms of respective degree 4 and 6 (see [13], Theorem III.3.5). Remark that multiple roots of F_6 are not roots of F_4, since S is non-singular, and the point $v = (1:0:0:1) = (-1:0:0:1)$ is the vertex of the quadric.

The double covering of the quadric $Q \cong \mathbb{P}(1,1,2)$ gives an exact sequence

$$1 \to \langle\sigma\rangle \to \mathrm{Aut}(S) \to \mathrm{Aut}(Q)_C,$$

where $\mathrm{Aut}(Q)_C$ denote the automorphisms of Q that leaves invariant the ramification curve $C = \{(x:y:z) \mid z^3 + zF_4(x,y) + F_6(x,y) = 0\}$. (In fact we can prove that the right homomorphism is surjective, but we will not need it here.) A quick calculation shows that any element of $\mathrm{Aut}(Q)_C$ belongs to $\mathrm{P}(\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C}))$. This implies that

$$\mathrm{Aut}(S) \subset \mathrm{P}(\mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})).$$

Note that $|K_S|$ is a pencil of elliptic curves, parametrised by the (x,y)-coordinates, which has one base point, $v = (1:0:0:1)$. Any automorphism of S acts thus on the elliptic bundle and fixes the vertex v of Q. This induces another exact sequence

$$1 \to G_S \to \mathrm{Aut}(S) \xrightarrow{\pi} \mathrm{Aut}(\mathbb{P}^1),$$

where

$$G_S = \begin{cases} \langle\, (w:x:y:z) \mapsto (-w:x:y:\mathrm{e}^{2\mathbf{i}\pi/3}z)\,\rangle \cong \mathbb{Z}/6\mathbb{Z} & \text{if } F_4 = 0,\\ \langle\, (w:x:y:z) \mapsto (-w:x:y:z)\,\rangle \cong \mathbb{Z}/2\mathbb{Z} & \text{otherwise.}\end{cases}$$

The involution that belongs to G_S is classically called "Bertini involution". Denoting by G the group generated by our automorphism g, the group $G \cap G_S$ is either trivial or cyclic of order 3. We study the two different cases.

1) *The group $G \cap G_S$ is trivial.* — In this case, the action of G on the elliptic pencil (which is cyclic and diagonal) has the same order as g, which is by hypothesis at least 7. As both F_4 and F_6 are preserved by this action,

both are monomials. Then, either y^2 or x^2 divides F_6, which implies that F_4 is a multiple of x^4 or y^4. (Recall that the double roots of F_6 are not roots of F_4.) Up to an exchange of coordinates, we may thus suppose that $F_4 = x^4$ and $F_6 = y^6$ or $F_6 = xy^5$.

In the first case, the equation of the surface is $w^2 = z^3 + x^4z + y^6$, and its group of automorphisms is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$, generated by the Bertini involution $(w : x : y : z) \mapsto (-w : x : y : z)$ and $(w : x : y : z) \mapsto (iw : x : \mathrm{e}^{2i\pi/12} : -z)$. This case is thus not possible, since the order of g is odd and at least 7.

In the second case, the equation of the surface is $w^2 = z^3 + x^4z + xy^5$ and its group of automorphisms is isomorphic to $\mathbb{Z}/20\mathbb{Z}$, generated by $(w : x : y : z) \mapsto (iw : x : \mathrm{e}^{2i\pi/10}y : -z)$. We obtain once again a contradiction.

2) *The group $G \cap G_S$ is cyclic of order* 3*, generated by* $(w : x : y : z) \mapsto (w : x : y : \mathrm{e}^{2i\pi/3}z)$. — In this case, $L_4 = 0$, so L_6 has exactly six distinct roots. Since the action of G on these roots must be of odd order ≥ 3, it must be of order 3 or 5. The element g is of the form $(w : x : y : z) \mapsto (\lambda_w w : x : \alpha y : \lambda_z z)$, for some $\lambda_w, \lambda_z \in \mathbb{C}^*$, where α is a n-th root of the unity, and $n = 3$ or $n = 5$. This implies that L_6 is, up to a linear change on x and y, respectively $x^6 + ax^3y^3 + y^6$, for some $a \in \mathbb{C}$, or $x(x^5 + y^5)$. Thus, we have $\lambda_w^2 = 1$ and $\lambda_z^3 = 1$. Since the order of g is odd and at least 7, this shows that $\lambda_w = 1$, $n = 5$ and λ_z is a 3-th root of the unity. The surface S is thus the surface S_{15} of Example 4.7, and the automorphism is one power of θ, which has order 15. $\square$

We are now able to prove Theorem 1.3:

Proof of Theorem 1.3. — Using Proposition 4.8 above, and the fact that all the linear automorphisms of some given finite order are birationally conjugate (see [2]), there exists exactly one single conjugacy class of elements of the Cremona group of some given odd order $n \neq 3, 5, 9, 15$, which is represented by the linear automorphism $\alpha_n : (x : y : z) \mapsto (x : y : \mathrm{e}^{2i\pi/n}z)$.

Using the same results, the elements of order 9 of the Cremona group are birationally conjugate to one of the three elements α_9, ρ_1, ρ_2, where α_9 is the automorphism $\alpha_9 : (x : y : z) \mapsto (x : y : \mathrm{e}^{2i\pi/9}z)$ of $\mathbb{P}^2$ and ρ_1, ρ_2 are the automorphisms of the Fermat cubic surface S_F described in Example 4.6. It remains to show that these three elements are not birationally conjugate. Firstly, since $(\rho_1)^2$ and $(\rho_2)^2$ both fix an elliptic curve, neither of them is birationally conjugate to a linear automorphism of $\mathbb{P}^2$. Thus α_9 is neither conjugate to ρ_1, nor to ρ_2. Secondly, the elements $(\rho_1)^2$ and $(\rho_2)^2$ are diagonal in $\mathrm{PGL}(4, \mathbb{C})$ and have distinct eigenvalues (up to multiplication), so are not conjugate by an element of $\mathrm{PGL}(4, \mathbb{C})$. This implies that ρ_1, ρ_2, which are elements of the group of automorphisms of the Fermat cubic surface S_F, are not conjugate in this group. Suppose now that these two elements are conjugate by some birational transformation φ of S_F. Then, since φ is G-equivariant, where $G \cong \mathbb{Z}/9\mathbb{Z}$, we may factorise it into a composition of elementary G-equivariant links (see for example [11, Thm. 2.5]). Since our surface is of Del Pezzo type ($S \in \{\mathbb{D}\}$ in the notation of [11]), the first link is of type I or II. The classification of elementary links (see [11, Thm. 2.6]) shows that the only possiblity for the link is to be the Geiser or Bertini involution of a surface obtained by the blow-up of one or two points invariant by G. A Geiser (respectively Bertini) involution of a Del Pezzo surface of degree 2 (respectively 1) commutes with any automorphism of the surface, thus the elementary link conjugates ρ_1 to itself. Since ρ_1 and ρ_2 are not conjugate in $\mathrm{Aut}(S_F)$, these elements are neither birationally conjugate. Summing up, there are three conjugacy classes of elements of order 9 in the Cremona group, represented by α_9, ρ_1 and ρ_2.

The case of elements of order 15 is similar. Using once again Proposition 4.8 and [2], any element of order 15 of the Cremona group is birationally conjugate either to $\alpha_{15} : (x : y : z) \mapsto (x : y : \mathrm{e}^{2i\pi/9} z)$, or to one of the generators of the group $\langle\theta\rangle$, generated by the automorphism $\theta \in \mathrm{Aut}(S_{15})$ described in Example 4.7. Since the 5-torsion of $\langle\theta\rangle$ fixes an elliptic curve, no generator of $\langle\theta\rangle$ is birationally conjugate to α_{15}. Note that the group of automorphisms of S_{15} is isomorphic to $\mathbb{Z}/30\mathbb{Z}$, generated by θ and the Bertini involution. Two distinct elements of the group $\langle\theta\rangle \subset \mathrm{Aut}(S_{15})$ are thus not conjugate by an automorphism of S_{15}. Since the classification of elementary links gives no satisfactory elementary link, the same argument as before shows that the elements are not birationally conjugate. There are thus exactly nine conjugacy classes of elements of order 15 in the Cremona group, represented by α_{15}, θ, θ^2, θ^4, θ^7, θ^8, θ^{11}, θ^{13} and θ^{14}. □

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