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COEFFICIENT SYSTEMS AND SUPERSINGULAR REPRESENTATIONS OF $\mathrm{GL}_2(F)$

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COEFFICIENT SYSTEMS AND SUPERSINGULAR REPRESENTATIONS OF $\mathrm{GL}_2(F)$

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Abstract. — Let F be a non-Archimedean local field with the residual characteristic p. We construct a "good" number of smooth irreducible $\overline{\mathbf{F}}_p$-representations of $\mathrm{GL}_2(F)$, which are supersingular in the sense of Barthel and Livné. If $F = \mathbf{Q}_p$ then results of Breuil imply that our construction gives all the supersingular representations up to the twist by an unramified quasi-character. We conjecture that this is true for an arbitrary F.

***Résumé* (Systèmes de coefficients et représentations supersingulières de $\mathrm{GL}_2(F)$)**

Soit F un corps local non archimédien de caractéristique résiduelle p. Nous construisons le « bon » nombre de $\overline{\mathbf{F}}_p$-représentations lisses et irréductibles de $\mathrm{GL}_2(F)$ qui sont supersingulières au sens de Barthel et Livné. Si $F = \mathbf{Q}_p$, les résultats de Breuil impliquent alors que notre construction donne toutes les représentations supersingulières à la torsion près par un quasi-caractère non ramifié. Nous conjecturons que ceci reste vrai pour F quelconque.

CONTENTS

CHAPTER 1

INTRODUCTION

Recently Breuil in [**4**] has determined the isomorphism classes of the irreducible smooth $\overline{\mathbf{F}}_p$-representations of $\mathrm{GL}_2(\mathbf{Q}_p)$. This allowed him to define a "correspondance semi-simple modulo p pour $\mathrm{GL}_2(\mathbf{Q}_p)$". Under this correspondence the isomorphism classes of irreducible smooth 2-dimensional $\overline{\mathbf{F}}_p$-representations of the Weil group of $\mathbf{Q}_p$ are in bijection with the isomorphism classes of "supersingular" irreducible smooth $\overline{\mathbf{F}}_p$-representations of $\mathrm{GL}_2(\mathbf{Q}_p)$. Moreover, it is conjecturally related to a p-adic correspondence in [**5**]. The term "supersingular" was coined by Barthel and Livné. Roughly speaking a supersingular representation is the $\overline{\mathbf{F}}_p$-analogue of a supercuspidal representation over $\mathbb{C}$, see Definition 1.1.1. Let F be a non-Archimedean local field, with a residue class field $\mathbf{F}_q$ of the characteristic p. All the irreducible smooth $\overline{\mathbf{F}}_p$-representations of $G = \mathrm{GL}_2(F)$, which are <u>not</u> supersingular, have been determined by Barthel and Livné in [**2**] and [**1**], and also by Vignéras in [**18**], with no restrictions on F. However, if $F \neq \mathbf{Q}_p$ then the method of Breuil fails and relatively little is known about the supersingular representations of G.

This paper is an attempt to shed some light on this question. We fix a uniformiser ϖ_F of F and we construct $q(q-1)/2$ pairwise non-isomorphic, irreducible, supersingular, admissible (in the usual smooth sense) representations of G, which admit a central character, such that ϖ_F acts trivially. If $F = \mathbf{Q}_p$ then using the results of Breuil we may show that our construction gives all the supersingular representations up to a twist by an unramified quasi-character. We conjecture that this is true for arbitrary F. If ρ is an irreducible smooth $\overline{\mathbf{F}}_p$-representation of the Weil group W_F of F, then the wild inertia subgroup of W_F acts trivially on ρ, since it is pro-p and normal in W_F. This implies that there are only $q(q-1)/2$ isomorphism classes of irreducible smooth 2-dimensional $\overline{\mathbf{F}}_p$-representations ρ of the Weil group of F such that $(\det\rho)(\mathrm{Fr}) = 1$. Here, Fr is the Frobenius automorphism corresponding to ϖ_F via the local class field theory. So the conjecture would be true if there was a Langlands type of correspondence.

The starting point in this theory is that every pro-p group acting smoothly on an $\overline{\mathbf{F}}_p$-vector space has a non-zero invariant vector. Let I_1 be the unique maximal pro-p subgroup of the standard Iwahori subgroup I of G. Given a smooth representation π of G the Hecke algebra $\mathcal{H} = \mathrm{End}_G(\text{c-}\mathrm{Ind}_{I_1}^G \mathbf{1})$ acts on the I_1-invariants π^{I_1}. It is expected that this functor induces a bijection between the irreducible smooth representations of G and the irreducible modules of $\mathcal{H}$. This happens if $F = \mathbf{Q}_p$. Moreover, if F is arbitrary and π is an irreducible smooth representation of G, which is not supersingular, then π^{I_1} is an irreducible $\mathcal{H}$-module. All the irreducible modules of $\mathcal{H}$ that do not arise this way are called supersingular. They have been determined by Vignéras and we give a list of them in the Definition 2.1.2. There are $q(q-1)/2$ isomorphism classes of irreducible supersingular modules of $\mathcal{H}$ up to a twist by an unramified quasi-character.

Given a supersingular module M of $\mathcal{H}$ we construct two G-equivariant coefficient systems $\mathcal{V}$ and $\mathcal{I}$ on the Bruhat-Tits tree X of $\mathrm{PGL}_2(F)$ and a morphism of G-equivariant coefficient systems between them. Once we pass to the 0-th homology, this induces a homomorphism of G-representations. We show that the image of this homomorphism

$$\pi = \mathrm{Im}(H_0(X, \mathcal{V}) \longrightarrow H_0(X, \mathcal{I}))$$

is a smooth irreducible representation of G, which is supersingular, since π^{I_1} contains a supersingular module M. Moreover, we show that two non-isomorphic irreducible supersingular modules give rise to non-isomorphic representations. However, the question of determining all smooth irreducible representations π of G, such that π^{I_1} contains M, remains open.

We will describe the contents of this paper in more detail. In Section 2 we recall the algebra structure of $\mathcal{H}$ and the definition of supersingular modules.

Sections 3 and 4 deal with some aspects of the $\overline{\mathbf{F}}_p$-representation theory of $\Gamma = \mathrm{GL}_2(\mathbf{F}_q)$. In Section 3 we give two different descriptions of the irreducible $\overline{\mathbf{F}}_p$-representations of Γ, one of them due to Carter and Lusztig [**7**] and the other one due to Brauer and Nesbitt [**3**], and a dictionary between them. Let U be the subgroup of unipotent upper-triangular matrices in Γ, then U is a p-Sylow subgroup of Γ. If ρ is a representation of Γ, then the Hecke algebra $\mathcal{H}_\Gamma = \mathrm{End}_\Gamma(\mathrm{Ind}_U^\Gamma \mathbf{1})$ acts on the U-invariants ρ^U. This functor induces a bijection between the irreducible representations of Γ and the irreducible right modules of $\mathcal{H}_\Gamma$.

Every representation ρ of Γ has an injective envelope $\iota : \rho \hookrightarrow \mathrm{inj}\,\rho$. By this we mean, a representation $\mathrm{inj}\,\rho$ of Γ and an injection ι, such that $\mathrm{inj}\,\rho$ is an injective object in the category of $\overline{\mathbf{F}}_p$-representations of Γ and every non-zero Γ-invariant subspace of $\mathrm{inj}\,\rho$ intersects $\iota(\rho)$ non-trivially. Injective envelopes are unique up to isomorphism. In Section 4 we determine the $\mathcal{H}_\Gamma$-module structure of $(\mathrm{inj}\,\rho)^U$, for an irreducible representation ρ of Γ. This is important to us, so we give two ways of doing it. If

$p = q$ then the dimension of $(\mathrm{inj}\,\rho)^U$ is small and this enables us to give an elementary argument. In general we use the results of Jeyakumar [**10**], where he describes explicitly injective envelopes of irreducible representations of $\mathrm{SL}_2(\mathbf{F}_q)$.

Let $\mathfrak{o}_F$ be the ring of integers of F, let $K = \mathrm{GL}_2(\mathfrak{o}_F)$. The reduction modulo the prime ideal of $\mathfrak{o}_F$ induces a surjection $K \to \Gamma$, let K_1 be the kernel of this map. The Hecke algebra $\mathcal{H}_K = \mathrm{End}_K(\mathrm{Ind}_{I_1}^K \mathbf{1})$ is naturally a subalgebra of $\mathcal{H}$. Let M be a supersingular module of $\mathcal{H}$, then the restriction of M to $\mathcal{H}_K$ is isomorphic to a direct sum of two irreducible modules of $\mathcal{H}_K$. Since $K/K_1 \cong \Gamma$ we may identify representations of K on which K_1 acts trivially with the representations of Γ. This induces an identification $\mathcal{H}_K = \mathcal{H}_\Gamma$. Since the irreducible modules of $\mathcal{H}_\Gamma$ are in bijection with the irreducible representations of Γ, there exists a unique representation $\rho = \rho_M$ of Γ, such that ρ is isomorphic to a direct sum of two irreducible representations of Γ, and $\rho^U \cong M|_{\mathcal{H}_\Gamma}$. Let $\rho \hookrightarrow \mathrm{inj}\,\rho$ be an injective envelope of ρ in the category of $\overline{\mathbf{F}}_p$-representations of Γ. We consider now both ρ and $\mathrm{inj}\,\rho$ as representations of K. We have an exact sequence

$$0 \longrightarrow \rho^{I_1} \longrightarrow (\mathrm{inj}\,\rho)^{I_1}$$

of $\mathcal{H}_K$-modules. The main result of Section 4 are Propositions 4.1.9 ($p = q$), Propositions 4.2.37 and 4.2.38 (general case), which say that there exists an action of $\mathcal{H}$, extending the action of $\mathcal{H}_K$, on $(\mathrm{inj}\,\rho)^{I_1}$, such that the above exact sequence yields an exact sequence

$$0 \longrightarrow M \longrightarrow (\mathrm{inj}\,\rho)^{I_1} \tag{E}$$

of $\mathcal{H}$-modules. The fact that we can extend the action and obtain (E) implies the existence of a certain G-equivariant coefficient system $\mathcal{I}$ on the tree X.

The inspiration to use coefficient systems comes from the works of Schneider and Stuhler [**13**] and [**14**], where the authors work over the complex numbers, and Ronan and Smith [**12**], where the $\overline{\mathbf{F}}_p$ coefficient systems are studied for finite Chevalley groups. We introduce coefficient systems in Section 5. Let σ_1 be an edge on X containing a vertex σ_0. Since, G acts transitively on the vertices of the tree X, the category of G-equivariant coefficient systems is equivalent to a category of diagrams $\mathcal{DIAG}$. The objects of $\mathcal{DIAG}$ are triples (ρ_0, ρ_1, ϕ), where ρ_0 is a smooth representation of $\mathfrak{K}(\sigma_0)$, ρ_1 is a smooth representation of $\mathfrak{K}(\sigma_1)$ and ϕ is a $\mathfrak{K}(\sigma_1)\cap\mathfrak{K}(\sigma_0)$-equivariant homomorphism, $\phi : \rho_1 \to \rho_0$, where $\mathfrak{K}(\sigma_0)$ and $\mathfrak{K}(\sigma_1)$ are the G-stabilisers of σ_0 and σ_1. The proof of equivalence between the two categories is the main result of Section 5. As a corollary we obtain a nice way of passing from "local" to "global" information, see Corollary 5.5.5, and we use this in the construction of $\mathcal{I}$.

More precisely, we start with a supersingular $\mathcal{H}$-module M and find the unique smooth representation $\rho = \rho_M$ of K, such that ρ is isomorphic to a direct sum of two irreducible representations of K, and $\rho^{I_1} \cong M|_{\mathcal{H}_K}$, as above. We then consider an injective envelope $\rho \hookrightarrow \mathrm{Inj}\,\rho$ of ρ in the category of smooth $\overline{\mathbf{F}}_p$-representations of K.

Let σ_1 be an edge on X fixed by I and let σ_0 be a vertex fixed by K. We extend the action of K on $\operatorname{Inj}\rho$ to the action of $F^\times K = \mathfrak{K}(\sigma_0)$, so that a fixed uniformiser acts trivially. We denote this representation by Y_0. Let us assume that we may extend the action of $F^\times I = \mathfrak{K}(\sigma_1) \cap \mathfrak{K}(\sigma_0)$ on $Y_0|_{F^\times I}$ to the action of $\mathfrak{K}(\sigma_1)$. We denote the corresponding representation of $\mathfrak{K}(\sigma_1)$ by Y_1. The triple $Y = (Y_0, Y_1, \mathrm{id})$ is an object in a category $\mathcal{DIAG}$, which is equivalent to the category of G-equivariant coefficient systems on the tree X, by the main result of Section 5. So Y gives us a G-equivariant coefficient system $\mathcal{I}$. Moreover, the restriction maps of $\mathcal{I}$ are all isomorphisms. This implies that

$$H_0(X, \mathcal{I})|_K \cong \operatorname{Inj}\rho.$$

In particular, we have an injection

$$\rho \hookrightarrow \operatorname{Inj}\rho \cong H_0(X, \mathcal{I}_\gamma)|_K,$$

which gives us an exact sequence of vector spaces

$$0 \longrightarrow \rho^{I_1} \longrightarrow H_0(X, \mathcal{I})^{I_1}.$$

We show in Subsection 6.4 that using (E) we may extend the action of $F^\times I$ on $Y_0|_{F^\times I}$ to the action of $\mathfrak{K}(\sigma_1)$, so that the image of ρ^{I_1} in $H_0(X, \mathcal{I})^{I_1}$ is an $\mathcal{H}$-invariant subspace, isomorphic to M as an $\mathcal{H}$-module. We let π be the G-invariant subspace of $H_0(X, \mathcal{I})$ generated by the image of ρ. In Theorem 6.5.2 we prove that π is irreducible and supersingular. We also show that π is the socle of $H_0(X, \mathcal{I})$. The space $H_0(X, \mathcal{I})^{I_1}$ is always finite dimensional, we determine the $\mathcal{H}$-module structure in Proposition 6.4.5. The proofs rely on some general properties of injective envelopes, which we recall in Subsection 6.2. Using injective envelopes we also give a new proof of the criterion for admissibility of a smooth representation of G, which works in a very general context, see Subsection 6.3.

We would like to explain the thinking behind the construction of the coefficient system $\mathcal{V}$ in Subsection 6.1. Let π be a smooth representation of G, generated by its I_1-invariant vectors. We may associate to π a G-equivariant coefficient system $\mathcal{F}_\pi$ as follows. Given a simplex σ on X, we let U^1_σ be the maximal normal pro-p subgroup of the G-stabiliser of σ. With this notation $U^1_{\sigma_1} = I_1$ and $U^1_{\sigma_0} = K_1$. We may consider the coefficient system of invariants $\mathcal{F}_\pi = (\pi^{U^1_\sigma})_\sigma$, where the restriction maps are inclusions. Since π is generated by its I_1-invariants the natural map

$$H_0(X, \mathcal{F}_\pi) \longrightarrow \pi$$

is surjective. If we are working over the complex numbers then a theorem of Schneider and Stuhler in [**13**], says that the above homomorphism is in fact an isomorphism. If we are working over $\overline{\mathbf{F}}_p$, then $H_0(X, \mathcal{F}_\pi)$ can be much bigger than π.

The construction of $\mathcal{V}$ is motivated by the following question. Let M be a supersingular module of $\mathcal{H}$ and suppose that there exists a smooth irreducible $\overline{\mathbf{F}}_p$-representation π of G such that $\pi^{I_1} \cong M$. What can be said about the corresponding coefficient system $\mathcal{F}_\pi$? It is enough to understand the action of K on π^{K_1}. This reduces the question to the representation theory of $\mathrm{GL}_2(\mathbf{F}_q)$. In Corollary 6.1.10 we show that there exists an injection $\mathcal{V} \hookrightarrow \mathcal{F}_\pi$ and hence every π as above is a quotient of $H_0(X, \mathcal{V})$. We would like to point out that although in most cases we do not know whether such π exists, the coefficient system $\mathcal{V}$ is always well defined. Moreover, if π is any non-zero irreducible quotient of $H_0(X, \mathcal{V})$, then we show that π is supersingular, since π^{I_1} contains a supersingular $\mathcal{H}$-module M. This implies that $H_0(X, \mathcal{V})$ is a quotient of one of the spaces considered by Barthel and Livné in [**1**]. Corollary 6.1.8 implies that at least in some cases the quotient map is an isomorphism. Now the Remarque 4.2.6 in [**4**] shows that in general $\dim H_0(X, \mathcal{V})^{I_1}$ is infinite. The irreducible representation π, which we construct in this paper, is a quotient of $H_0(X, \mathcal{V})$, moreover the space π^{I_1} is finite dimensional. Hence, in contrast to the situation over $\mathbb{C}$, in general $H_0(X, \mathcal{V})$ is very far away from being irreducible.

We believe that our construction of irreducible representations will work for other groups. Our strategy could be applied most directly to the group $G = \mathrm{GL}_N(F)$, where N is a prime number. If N is prime then the maximal open, compact-mod-centre subgroups of G are the G-stabilisers of chambers (simplices of maximal dimension) and vertices in the Bruhat-Tits building of G and if we had the equivalent of (E) then the construction of the coefficient system $\mathcal{I}$ and our proofs would carry through. However, in order to do this one needs to understand the $\mathcal{H}_\Gamma$-module structure of $(\mathrm{inj}\,\rho)^U$, (or at least the action of B on $(\mathrm{inj}\,\rho)^U$, at the cost of not knowing $\mathcal{H}$-module structure of $H_0(X, \mathcal{I})^{I_1}$), where ρ is an irreducible $\overline{\mathbf{F}}_p$-representation of $\Gamma = \mathrm{GL}_N(\mathbf{F}_q)$, B is the subgroup of upper-triangular matrices, and U is the subgroup of unipotent upper-triangular matrices of Γ. This might be quite a difficult problem, since already for $N = 2$ the dimension of $(\mathrm{inj}\,\rho)^U$ can be as big as $2^n - 1$, if $q = p^n$.

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1.1. Notation

Let F be a non-Archimedean local field, $\mathfrak{o}_F$ its ring of integers, $\mathfrak{p}_F$ the maximal ideal of $\mathfrak{o}_F$. Let p be the characteristic and let q be the number of elements of the residue class field of F. We fix a uniformiser ϖ_F of F.

Let $G = \mathrm{GL}_2(F)$ and $K = \mathrm{GL}_2(\mathfrak{o}_F)$. Reduction modulo $\mathfrak{p}_F$ induces a surjective homomorphism

$$\mathrm{red} : K \longrightarrow \Gamma = \mathrm{GL}_2(\mathbf{F}_q).$$

Let K_1 be the kernel of red. Let B be the subgroup of Γ of upper triangular matrices. Then

$$B = HU$$

where H is the subgroup of diagonal matrices and U is the subgroup of unipotent matrices in B. It is of importance, that the order of H is prime to p and U is a p-Sylow subgroup of Γ. Let I and I_1 be the subgroups of K, given by

$$I = \mathrm{red}^{-1}(B), \quad I_1 = \mathrm{red}^{-1}(U).$$

Then I is the Iwahori subgroup of G and I_1 is the unique maximal pro-p subgroup of I. Let T be the subgroup of diagonal matrices in K, and let $T_1 = T \cap K_1 = T \cap I_1$. Let N be the normaliser of T in G. We introduce some special elements of N. Let

$$\Pi = \begin{pmatrix} 0 & 1 \\ \varpi_F & 0 \end{pmatrix}, \quad n_s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The images of Π and n_s in N/T, generate it as a group. The normaliser N acts on T by conjugation, and hence it acts on the group of characters of T. This action factors through T, so if $w \in N/T$ and χ is a character of T, we will write χ^w for the character, given by

$$\chi^w(t) = \chi(w^{-1}tw), \quad \forall\, t \in T.$$

Let $\widetilde{B}$ be the group of upper-triangular matrices in G, then $\widetilde{B} = \widetilde{T}\widetilde{U}$ where $\widetilde{T}$ is the group of diagonal matrices in G and $\widetilde{U}$ is the group of unipotent matrices in $\widetilde{B}$.

Definition 1.1.1. — Let π be a smooth irreducible $\overline{\mathbf{F}}_p$-representation of G, such that π admits a central character, then π is called supersingular if π is not a subquotient of $\mathrm{Ind}_{\widetilde{B}}^{G} \chi$, for any smooth quasi-character $\chi : \widetilde{B} \to \widetilde{B}/\widetilde{U} \cong \widetilde{T} \to \overline{\mathbf{F}}_p^{\times}$.

All the representations considered in this paper are over $\overline{\mathbf{F}}_p$, unless it is stated otherwise.

CHAPTER 2

HECKE ALGEBRA

Lemma 2.0.2. — *Let $\mathcal{P}$ be a pro-p group and let π be a smooth non-zero representation of $\mathcal{P}$, then the space $\pi^{\mathcal{P}}$ of $\mathcal{P}$-invariants is non-zero.*

Proof. — We choose a non-zero vector v in π. Let $\rho = \langle \mathcal{P} v\rangle_{\overline{\mathbf{F}}_p}$ be a subspace of π generated by $\mathcal{P}$ and v. Since the action of $\mathcal{P}$ on π is smooth, the stabiliser $\mathrm{Stab}_{\mathcal{P}}(v)$ has finite index in $\mathcal{P}$, hence ρ is finite dimensional. Let $v_1, \dots, v_d$ be an $\overline{\mathbf{F}}_p$ basis of ρ. The group $\mathcal{P}$ acts on ρ and the kernel of this action is given by

$$\operatorname{Ker} \rho = \bigcap_{i=1}^{d} \mathrm{Stab}_{\mathcal{P}}(v_i).$$

In particular, $\operatorname{Ker}\rho$ is an open subgroup of $\mathcal{P}$. Hence, $\mathcal{P}/\operatorname{Ker}\rho$ is a finite group, whose order is a power of p. Now,

$$\rho^{\mathcal{P}} = \rho^{\mathcal{P}/\operatorname{Ker}\rho} \neq 0$$

since $\mathcal{P}/\operatorname{Ker}\rho$ is a finite p-group, see [**15**], § 8, Proposition 26. □

Let π be a smooth representation of G, then

$$\pi^{I_1} \cong \mathrm{Hom}_{I_1}(\mathbf{1}, \pi) \cong \mathrm{Hom}_G(\text{c-Ind}_{I_1}^{G} \mathbf{1}, \pi)$$

by Frobenius reciprocity. Let $\mathcal{H}$ be the Hecke algebra

$$\mathcal{H} = \mathrm{End}_G(\text{c-Ind}_{I_1}^{G} \mathbf{1})$$

then via the above isomorphism π^{I_1} becomes naturally a right $\mathcal{H}$-module. We obtain a functor

$$\mathrm{Rep}_G \longrightarrow \text{Mod-}\mathcal{H}, \quad \pi \longmapsto \pi^{I_1},$$

where Rep_G is a category of smooth $\overline{\mathbf{F}}_p$-representations of G and Mod-$\mathcal{H}$ is the category of right $\mathcal{H}$-modules. Since I_1 is an open pro-p subgroup of G, Lemma 2.0.2 implies that $\pi^{I_1} = 0$ if and only if $\pi = 0$. This functor is our basic tool. The algebra structure of $\mathcal{H}$ is well understood, in a general context of split reductive groups over F, see [**17**]. We recall some of the results below. Since we deal only with GL_2 we can be very explicit. Our notation follows [**7**], where finite groups with split BN-pair are treated.

Definition 2.0.3. — Let $g \in G$ and $f \in \text{c-Ind}_{I_1}^G \mathbf{1}$ we define $T_g \in \mathcal{H}$ by

$$(T_g f)(I_1 g_1) = \sum_{I_1 g_2 \subseteq I_1 g^{-1} I_1 g_1} f(I_1 g_2).$$

Lemma 2.0.4. — *We may write G as a disjoint union*

$$G = \dot{\bigcup_{n \in N/T_1}} I_1 n I_1$$

of double cosets.

Proof. — This follows from the Iwahori decomposition. □

It is immediate that the definition of T_g depends only on the double coset $I_1 g I_1$. The Lemma above implies that it is enough to consider T_n, where $n \in N$ is a representative of a coset in N/T_1.

Definition 2.0.5. — Let $\varphi \in \text{c-Ind}_{I_1}^G \mathbf{1}$ be the unique function such that

$$\operatorname{Supp} \varphi = I_1 \quad \text{and} \quad \varphi(u) = 1, \quad \forall\, u \in I_1.$$

Lemma 2.0.6

(i) *The function φ generates* $\text{c-Ind}_{I_1}^G \mathbf{1}$ *as a G-representation.*

(ii) $\operatorname{Supp} T_n\varphi = I_1 n I_1$ *and $(T_n\varphi)(g) = 1$, for every $g \in I_1 n I_1$. In particular,*

$$T_n \varphi = \sum_{u \in I_1/(I_1 \cap n^{-1} I_1 n)} u n^{-1} \varphi.$$

(iii) *The set $\{T_n \varphi : n \in N/T_1\}$ is an $\overline{\mathbf{F}}_p$-basis of* $(\text{c-Ind}_{I_1}^G \mathbf{1})^{I_1}$.

(iv) *The set $\{T_n : n \in N/T_1\}$ is an $\overline{\mathbf{F}}_p$-basis of $\mathcal{H}$.*

Proof. — Let $g \in G$, then $\operatorname{Supp}(g^{-1}\varphi) = I_1 g$ and $(g^{-1}\varphi)(I_1 g) = 1$. Part (i) follows immediately.

Let $f \in \text{c-Ind}_{I_1}^G \mathbf{1}$, then by examining the definition of T_n, one obtains that $\operatorname{Supp}(T_n f) \subseteq I_1 n \operatorname{Supp} f$. Hence, $\operatorname{Supp}(T_n\varphi) \subseteq I_1 n I_1$. Since T_n is a G-equivariant homomorphism and I_1 acts trivially on φ, it is enough to prove that $(T_n\varphi)(n) = 1$. Since $\operatorname{Supp}\varphi = I_1$, it is immediate from Definition 2.0.3 that $(T_n\varphi)(I_1 n) = \varphi(I_1) = 1$. The last part follows from decomposing $I_1 n I_1$ into right cosets and applying the argument used in Part (i).

Let $n, n' \in N$, and suppose that $nT_1 \neq n'T_1$, then Lemma 2.0.4 implies that $I_1 n I_1 \neq I_1 n' I_1$. By Part (ii) the functions $T_n\varphi$ and $T_{n'}\varphi$ have disjoint support. This implies that the set $\{T_n\varphi : n \in N/T_1\}$ is linearly independent. Any $f \in (\text{c-Ind}_{I_1}^G \mathbf{1})^{I_1}$, is constant on the double cosets $I_1 n I_1$, for $n \in N$, and since $\operatorname{Supp} f$ is compact, f is supported only on finitely many such, hence Lemma 2.0.4 and Part (ii) imply that $\{T_n\varphi : n \in N/T_1\}$ is also a spanning set. Hence we get Part (iii).

Let $\psi \in \mathcal{H}$, Part (i) implies that $\psi = 0$ if and only if $\psi(\varphi) = 0$. This observation coupled with Part (iii) implies Part (iv). □

Corollary 2.0.7. — *Let π be a smooth representation of G and let $v \in \pi^{I_1}$, then the action of T_n on π^{I_1} is given by*

$$vT_n = \sum_{u \in I_1/(I_1 \cap n^{-1} I_1 n)} un^{-1}v.$$

Proof. — The isomorphism $\mathrm{Hom}_G(\text{c-Ind}_{I_1}^G \mathbf{1}, \pi) \cong \pi^{I_1}$ is given explicitly by $\psi \mapsto \psi(\varphi)$. Let ψ be the unique G-invariant homomorphism, such that $\psi(\varphi) = v$, then

$$vT_{n_s} = (\psi \circ T_{n_s})(\varphi) = \psi(T_{n_s}\varphi) = \psi\Bigg(\sum_{u \in I_1/(I_1 \cap n^{-1} I_1 n)} un^{-1}\varphi\Bigg).$$

The last equality follows from Lemma 2.0.6 (ii). Since, ψ is G-invariant, we obtain the Lemma. □

Lemma 2.0.8. — *Let $n', n \in N$ and suppose that n normalises I_1, then*

$$T_{n'}T_n = T_{n'n}, \quad T_n T_{n'} = T_{nn'}.$$

Proof. — Lemma 2.0.6 (i) implies that it is enough to show that the homomorphisms map φ to the same function. Let $f \in \text{c-Ind}_{I_1}^G \mathbf{1}$ then since n normalises I_1 we have $(T_n(f))(g) = f(ng)$ and $T_n\varphi = n^{-1}\varphi$. Now the Lemma follows from Lemma 2.0.6 (ii). □

Let $t \in T$ and let h be the image of t in H, via $T/T_1 \cong H$, we will write T_h for the homomorphism T_t.

Definition 2.0.9. — Let $\chi : H \to \overline{\mathbf{F}}_p^\times$ be a character, we define

$$e_\chi = \frac{1}{|H|} \sum_{h \in H} \chi(h) T_h.$$

Let

$$\varphi_\chi = e_\chi \varphi,$$

then φ_χ is the unique function in $\text{c-Ind}_{I_1}^G \mathbf{1}$ such that

$$\mathrm{Supp}\, \varphi_\chi = I, \quad \varphi_\chi(g) = \chi(gI_1), \quad \forall\, g \in I,$$

via the isomorphism $I/I_1 \cong H$.

Lemma 2.0.10

(i) *$e_\chi^2 = e_\chi$ and $e_\chi e_{\chi'} = 0$, if $\chi \neq \chi'$.*
(ii) *$\mathrm{id} = \sum_\chi e_\chi$, where the sum is taken over all characters $\chi : H \to \overline{\mathbf{F}}_p^\times$.*
(iii) *$e_\chi(\text{c-Ind}_{I_1}^G \mathbf{1}) \cong \text{c-Ind}_I^G \chi$.*

Proof. — We note that H is abelian and the order of H is prime to p. Parts (i) and (ii) follow from the orthogonality relations of characters. Lemma 2.0.6 (i) implies that $e_\chi(\text{c-Ind}_{I_1}^G \mathbf{1})$ is generated by φ_χ and this implies Part (iii). □

Corollary 2.0.11. — *Let π be a smooth representation of G, then I acts on $(\pi^{I_1})e_\chi$ by a character χ. Moreover,*

$$\pi^{I_1} \cong \oplus_\chi (\pi^{I_1})e_\chi.$$

Proof. — The group I acts on π^{I_1}. Since I_1 acts trivially and $I/I_1 \cong H$, which is abelian and of order prime to p, the space π^{I_1} decomposes into one dimensional I invariant subspaces. Corollary 2.0.7 implies that e_χ cuts out the χ-isotypical subspace. The last part follows from Lemma 2.0.10 (ii). □

Lemma 2.0.12

(i) $T_{n_s} e_\chi = e_{\chi^s} T_{n_s}$, $\quad T_\Pi e_\chi = e_{\chi^s} T_\Pi$.
(ii) *If* $\chi = \chi^s$ *then* $T_{n_s}^2 e_\chi = -T_{n_s} e_\chi$.
(iii) *If* $\chi \neq \chi^s$ *then* $T_{n_s}^2 e_\chi = 0$.

Proof. — Part (i) follows from Lemma 2.0.8. Lemma 2.0.6 (i) implies that it is enough to calculate $T_{n_s}^2 e_\chi \varphi = T_{n_s}^2 \varphi_\chi$. Applying Lemma 2.0.6 (ii) twice we obtain $\operatorname{Supp} T_{n_s}^2 \varphi_\chi \subseteq K$. Hence it is enough to do the calculation in the space $\operatorname{Ind}_{I_1}^K \mathbf{1}$. Since K_1 acts trivially on this space, it is enough to do the calculation in the space $\operatorname{Ind}_U^\Gamma \mathbf{1}$. Then the Lemma is a special case of [**7**] Theorem 4.4. □

Lemma 2.0.13. — *Let $m \geqslant 0$ and let $w = \Pi n_s$ then the following hold:*

(i) $I_1 w I_1 w^m I_1 = I_1 w^{m+1} I_1$,
(ii) $I_1 w^{-1} I_1 w^{m+1} \cap I_1 w^m I_1 = I_1 w^m$,
(iii) $T_{w^m} = (T_w)^m = (T_\Pi T_{n_s})^m$.

Proof. — The first two parts can be checked by a direct calculation. For Part (iii) we observe that

$$\operatorname{Supp} T_w T_{w^m} \varphi \subseteq I_1 w \operatorname{Supp} T_{w^m} \varphi = I_1 w I_1 w^m I_1 = I_1 w^{m+1} I_1,$$

where the last equality is Part (i). Part (ii) and Lemma 2.0.6 (ii) imply that

$$(T_w T_{w^m} \varphi)(w^{m+1}) = 1.$$

Since I_1 acts trivially on φ and all the homomorphisms are G-equivariant, we may apply Lemma 2.0.6 (ii) again to obtain

$$T_w T_{w^m} \varphi = T_{w^{m+1}} \varphi.$$

Lemma 2.0.6 (i) implies that $T_w T_{w^m} = T_{w^{m+1}}$. Induction and Lemma 2.0.8 gives us Part (iii). □

Lemma 2.0.14

(i) *Let $n \in N$, then there exists $h \in H$ and integers $a \in \{0,1\}$, $m \geqslant 0$ and $b \in \mathbb{Z}$ such that*

$$T_n = T_\Pi^a (T_\Pi T_{n_s})^m T_\Pi^b T_h$$

where $T_\Pi^{-1} = T_{\Pi^{-1}}$.

(ii) *The elements T_{n_s}, T_Π, $T_{\Pi^{-1}}$ and e_χ, for every character $\chi : H \to \overline{\mathbf{F}}_p^\times$, generate $\mathcal{H}$ as an algebra.*

Proof. — We note that Lemma 2.0.8 implies that T_Π is invertible with $T_\Pi^{-1} = T_{\Pi^{-1}}$ and T_Π^2 is central in $\mathcal{H}$. Every $n \in N$ maybe written as $n = \Pi^a(\Pi n_s)^m \Pi^b t$, where $t \in T$. Lemma 2.0.8 and Lemma 2.0.13(iii) imply Part (i). Hence T_{n_s}, T_Π, $T_{\Pi^{-1}}$ and T_h, for $h \in H$ generate $\mathcal{H}$ as an algebra. Lemma 2.0.8 implies that $T_h e_\chi = \chi(h^{-1})e_\chi$ and hence Lemma 2.0.10 (ii) implies that T_h can be expressed as a linear combination of idempotents e_χ. This gives us Part (ii). ☐

Lemma 2.0.15

(i) *The set $\{e_\chi T_n \varphi : n \in N/T, \chi : H \to \overline{\mathbf{F}}_p^\times\}$ is an $\overline{\mathbf{F}}_p$-basis of* $(\text{c-Ind}_{I_1}^G \mathbf{1})^{I_1}$.

(ii) *The set $\{e_\chi T_n : n \in N/T, \chi : H \to \overline{\mathbf{F}}_p^\times\}$ is an $\overline{\mathbf{F}}_p$-basis of $\mathcal{H}$.*

Proof. — Since $e_\chi T_h = \chi(h^{-1})e_\chi$ Lemma 2.0.6 (iii) implies that the set

$$\{e_\chi T_n \varphi : n \in N/T, \chi : H \longrightarrow \overline{\mathbf{F}}_p^\times\}$$

is a spanning set. Since the elements e_χ are orthogonal idempotents it is enough to show that the set $\{e_\chi T_n \varphi : n \in N/T\}$ is linearly independent for a fixed character χ. Lemma 2.0.6 (ii) implies that $\operatorname{Supp} e_\chi T_n \varphi = InI$. Lemma 2.0.4 implies that if $nT \neq n'T$, then $e_\chi T_n \varphi$ and $e_\chi T_{n'} \varphi$ have disjoint support and hence the set is linearly independent. Part (ii) follows from Part (i) and Lemma 2.0.6 (i). ☐

2.1. Supersingular modules

All the irreducible modules of $\mathcal{H}$ have been determined by Vignéras in [**18**]. They naturally split up into two classes.

Proposition 2.1.1. — *Let π be a smooth irreducible representation of G, which admits a central character. Suppose that π is not supersingular, then π^{I_1} is an irreducible $\mathcal{H}$-module.*

Proof. — See [**18**] E.5.1. ☐

The modules as above could be called non-supersingular, we are interested in all the rest.

Definition 2.1.2. — Let $\chi : H \to \overline{\mathbf{F}}_p^\times$ be a character, let $\gamma = \{\chi, \chi^s\}$ and let $\lambda \in \overline{\mathbf{F}}_p^\times$. We define a standard supersingular module M_γ^λ to be a right $\mathcal{H}$-module such that its underlying vector space is 2 dimensional

$$M_\gamma^\lambda = \langle v_1, v_2 \rangle_{\overline{\mathbf{F}}_p}$$

and the action of $\mathcal{H}$ is determined by the following:

(i) If $\chi = \chi^s$ then

$$v_1 e_\chi = v_1, \quad v_1 T_{n_s} = -v_1, \quad v_1 T_\Pi = v_2$$

and

$$v_2 e_\chi = v_2, \quad v_2 T_{n_s} = 0, \quad v_2 T_\Pi = \lambda v_1.$$

(ii) If $\chi \neq \chi^s$ then

$$v_1 e_\chi = v_1, \quad v_1 T_{n_s} = 0, \quad v_1 T_\Pi = v_2$$

and

$$v_2 e_{\chi^s} = v_2, \quad v_2 T_{n_s} = 0, \quad v_2 T_\Pi = \lambda v_1.$$

To show that these relations define an action of $\mathcal{H}$ requires some work, this is done in [**18**].

Lemma 2.1.3. — *The modules M_γ^λ are irreducible and*

$$M_{\gamma'}^{\lambda'} \cong M_\gamma^\lambda$$

if and only if $\gamma' = \gamma$ and $\lambda' = \lambda$.

Proof. — The definition immediately gives that M_γ^λ does not have a 1 dimensional submodule, hence it is irreducible. If $\chi' : H \to \overline{\mathbf{F}}_p^\times$ is a character, such that $\chi' \notin \gamma$ then

$$M_\gamma^\lambda e_{\chi'} = 0.$$

Hence, $\gamma = \gamma'$. The element T_Π^2 acts on M_γ^λ by a scalar λ. Hence, $\lambda = \lambda'$. □

The following Proposition explains why M_γ^λ are called supersingular.

Proposition 2.1.4. — *Let M be an irreducible $\mathcal{H}$ module, such that $M \not\cong \pi^{I_1}$ for any non-supersingular irreducible representation π, then*

$$M \cong M_\gamma^\lambda$$

for some γ and λ.

Proof. — See [**18**] C.2 and E.5.1. □

Corollary 2.1.5. — *Let π be a smooth irreducible representation of G, admitting a central character. Suppose that π^{I_1} contains a submodule isomorphic to M_γ^λ for some γ and λ, then π is supersingular.*

We will also need to consider the following extension of supersingular modules.

DEFINITION 2.1.6. — Let $\chi : H \to \overline{\mathbf{F}}_p^{\times}$ be a character, such that $\chi \neq \chi^s$, let $\gamma = \{\chi, \chi^s\}$ and let $\lambda \in \overline{\mathbf{F}}_p^{\times}$. Let

$$\mathcal{H}^{\lambda} = \mathcal{H}/(T_{\Pi}^2 - \lambda)\mathcal{H}$$

then we define a right $\mathcal{H}$-module L_{γ}^{λ} to be

$$L_{\gamma}^{\lambda} = e_{\chi}\mathcal{H}^{\lambda}/e_{\chi}(T_{\Pi}T_{n_s} - T_{n_s}T_{\Pi})\mathcal{H}^{\lambda}.$$

The definition seems to be asymmetric in χ and χ^s, however the multiplication from the left by T_{Π} induces an isomorphism

$$e_{\chi}\mathcal{H}^{\lambda}/e_{\chi}(T_{\Pi}T_{n_s} - T_{n_s}T_{\Pi})\mathcal{H}^{\lambda} \cong e_{\chi^s}\mathcal{H}^{\lambda}/e_{\chi^s}(T_{\Pi}T_{n_s} - T_{n_s}T_{\Pi})\mathcal{H}^{\lambda},$$

since T_{Π} is a unit in $\mathcal{H}^{\lambda}$.

LEMMA 2.1.7. — *The images of e_{χ}, $e_{\chi}T_{\Pi}$, $e_{\chi}T_{n_s}$ and $e_{\chi}T_{n_s}T_{\Pi}$ in L_{γ}^{λ} form an $\overline{\mathbf{F}}_p$-basis of L_{γ}^{λ}.*

Proof. — This follows from Lemma 2.0.15 (ii) and Lemma 2.0.12 (ii). □

LEMMA 2.1.8. — *There exists a short exact sequence*

$$0 \longrightarrow M_{\gamma}^{\lambda} \longrightarrow L_{\gamma}^{\lambda} \longrightarrow M_{\gamma}^{\lambda} \longrightarrow 0$$

of $\mathcal{H}$-modules.

Proof. — Let v_1 be the image of $e_{\chi}T_{n_s}$ in L_{γ}^{λ} and let v_2 be be image of $e_{\chi}T_{n_s}T_{\Pi}$ in L_{γ}^{λ}. The subspace $\langle v_1, v_2\rangle_{\overline{\mathbf{F}}_p}$ is stable under the action of T_{n_s}, T_{Π} and $e_{\chi'}$, for every $\chi' : H \to \overline{\mathbf{F}}_p^{\times}$. Hence, by Lemma 2.0.14 (ii) the subspace is stable under the action of $\mathcal{H}$. From Lemma 2.0.12 (ii) and Definition 2.1.2 (ii) it follows that $\langle v_1, v_2\rangle_{\overline{\mathbf{F}}_p} \cong M_{\gamma}^{\lambda}$. An easy check shows that $L_{\gamma}^{\lambda}/M_{\gamma}^{\lambda} \cong M_{\gamma}^{\lambda}$. □

LEMMA 2.1.9. — *Let $(\pi, \mathcal{V})$ be a smooth representation of G and let $\xi \in \overline{\mathbf{F}}_p^{\times}$. Let μ_{ξ} be an unramified quasi-character:*

$$\mu_{\xi} : F^{\times} \longrightarrow \overline{\mathbf{F}}_p^{\times}, \quad x \longmapsto \xi^{\mathrm{val}_F(x)}$$

where val_F is the valuation of F. Suppose that π^{I_1} contains M_{γ}^{λ}, where $\gamma = \{\chi, \chi^s\}$ and let V be the underlying vector space of M_{γ}^{λ} in $\mathcal{V}$. If we consider the representation $(\pi \otimes \mu_{\xi} \circ \det, \mathcal{V})$ of G, then the action of $\mathcal{H}$ on V is isomorphic to $M_{\gamma}^{\lambda\xi^{-2}}$.

Proof. — Let

$$V = \langle v_1, v_2\rangle_{\overline{\mathbf{F}}_p}$$

as in Definition 2.1.2. Since μ_{ξ} is unramified, Corollary 2.0.7 implies that the action of T_{n_s} and the idempotents e_{χ} on V does not change. Lemma 2.0.14 (ii) implies that it is enough to check how T_{Π} acts. Since $\det \Pi = -\varpi_F$, twisting by $\mu_{\xi} \circ \det$ gives us

$$v_1 T_{\Pi} = \Pi^{-1}v_1 = \xi^{-1}v_2 \quad \text{and} \quad v_2 T_{\Pi} = \Pi^{-1}v_2 = \xi^{-1}\lambda v_1.$$

Once we replace v_1 by ξv_1 the isomorphism follows from Definition 2.1.2. □

Since, by twisting by an unramified character we may vary λ as we wish, we might as well work with $\lambda = 1$.

Definition 2.1.10. — Let $\gamma = \{\chi, \chi^s\}$ then we define $\mathcal{H}$-modules

$$M_\gamma = M_\gamma^1 \quad \text{and} \quad L_\gamma = L_\gamma^1.$$

2.2. Restriction to $\mathcal{H}_K$

Let $\mathcal{H}_K = \mathrm{End}_K(\mathrm{Ind}_{I_1}^K \mathbf{1})$. The natural isomorphism of K representations

$$\mathrm{Ind}_{I_1}^K \mathbf{1} \cong \{f \in \text{c-}\mathrm{Ind}_{I_1}^G \mathbf{1} : \mathrm{Supp}\, f \subseteq K\}$$

gives an embedding of algebras

$$\mathcal{H}_K \hookrightarrow \mathrm{Hom}_K(\mathrm{Ind}_{I_1}^K \mathbf{1}, \text{c-}\mathrm{Ind}_{I_1}^G \mathbf{1}) \cong \mathrm{Hom}_G(\text{c-}\mathrm{Ind}_{I_1}^G \mathbf{1}, \text{c-}\mathrm{Ind}_{I_1}^G \mathbf{1}) = \mathcal{H}.$$

As an algebra $\mathcal{H}_K$ is generated by T_{n_s} and e_χ, for all characters χ.

Definition 2.2.1. — Let $\chi : H \to \overline{\mathbf{F}}_p^\times$ be a character. Let $J_0(\chi)$ be a set, such that $J_0(\chi) = \varnothing$ if $\chi \neq \chi^s$, and $J_0(\chi) = \{s\}$, if $\chi = \chi^s$. Let J be a subset of $J_0(\chi)$, we define $M_{\chi,J}$ to be a right $\mathcal{H}_K$-module, whose underlying vector space is one dimensional, $M_{\chi,J} = \langle v \rangle_{\overline{\mathbf{F}}_p}$ and the action of $\mathcal{H}_K$ is determined by the following:

$$ve_\chi = v,$$

$$vT_{n_s} = 0 \quad \text{if } s \in J \text{ or } s \notin J_0(\chi), \quad vT_{n_s} = -v, \quad \text{if } s \notin J \text{ and } s \in J_0(\chi).$$

Given χ and J as above, we will denote

$$\overline{J} = J_0(\chi) \backslash J.$$

Lemma 2.2.2. — *Let $\chi : H \to \overline{\mathbf{F}}_p^\times$ be a character and let $\gamma = \{\chi, \chi^s\}$, then*

$$M_\gamma|_{\mathcal{H}_K} \cong M_{\chi,J} \oplus M_{\chi^s,\overline{J}}$$

as $\mathcal{H}_K$-modules, where J is a subset of $J_0(\chi)$. Moreover, if $\chi \neq \chi^s$, then

$$L_\gamma|_{\mathcal{H}_K} \cong (\mathrm{Ind}_I^K \chi \oplus \mathrm{Ind}_I^K \chi^s)^{I_1}$$

as $\mathcal{H}_K$-modules.

Proof. — The first isomorphism follows directly from Definition 2.1.2. Since $J_0(\chi)$ has at most two subsets, it doesn't matter which subset we take. For the second isomorphism we observe that the space $(\mathrm{Ind}_I^K \chi)^{I_1}$ is two dimensional, with the basis $\{\varphi_\chi, T_{n_s}\varphi_{\chi^s}\}$. Moreover, I acts on the basis vectors by characters χ and χ^s respectively. Now

$$\varphi_\chi T_{n_s} = \sum_{u \in I_1/K_1} u n_s^{-1} \varphi_\chi = e_\chi \Bigg(\sum_{u \in I_1/K_1} u n_s^{-1} \varphi \Bigg) = e_\chi T_{n_s} \varphi = T_{n_s} e_{\chi^s} \varphi = T_{n_s} \varphi_{\chi^s}$$

and

$$(T_{n_s}\varphi_{\chi^s})T_{n_s} = \sum_{u\in I_1/K_1} un_s^{-1}T_{n_s}e_{\chi^s}\varphi = T_{n_s}e_{\chi^s}\left(\sum_{u\in I_1/K_1} un_s^{-1}\varphi\right) = e_\chi T_{n_s}^2\varphi = 0$$

and Lemma 2.1.7 allows us to define the obvious isomorphism on the basis. $\square$

CHAPTER 3

IRREDUCIBLE REPRESENTATIONS OF $\mathrm{GL}_2(\mathbf{F}_q)$

3.1. Carter and Lusztig theory

In [**7**] Carter and Lusztig have constructed all irreducible $\overline{\mathbf{F}}_p$-representations of a finite group Γ, which has a 'split BN-pair of characteristic p'. Since $\mathrm{GL}_2(\mathbf{F}_q)$ is a special case of this, we will recall their results. Let Γ be a finite group with a BN-pair (Γ, B, N, S). Let $H = B \cap N$, then H is normal in N, and S is the set of Coxeter generators of $W = N/H$. We additionally require that $B = HU$, where U is a normal subgroup of B, which is a p-group, and H is abelian of order prime to p. Moreover, we assume that $H = \cap_{n\in N} B^n$.

Theorem 3.1.1 ([**7**]). — *Let ρ be an irreducible representation of Γ then*

(i) *the space of U invariants ρ^U is one dimensional;*

(ii) *suppose that the action of B on ρ^U is given by a character $\chi : H \to \overline{\mathbf{F}}_p^{\times}$, via $B \to B/U \cong H$ and let $J = \{s \in S : s \centerdot \rho^U = \rho^U\}$ then the pair (χ, J) determines ρ up to an isomorphism;*

(iii) *conversely, given a character $\chi : H \to \overline{\mathbf{F}}_p^{\times}$, let $J_0(\chi) = \{s \in S : \chi^s = \chi\}$ and let J be a subset of $J_0(\chi)$ then there exists an irreducible representation $\rho_{\chi,J}$ of Γ with the pair (χ, J) as above.*

Proof. — This is [**7**] Corollary 7.5, written out in detail, see also [**11**] Theorem 3.9 and [**8**] Theorem 4.3. and [**6**] § 3.4. □

Let $\mathcal{H}_\Gamma = \mathrm{End}_\Gamma(\mathrm{Ind}_U^\Gamma \mathbf{1})$. We would like to rephrase Theorem 3.1.1 in terms of $\mathcal{H}_\Gamma$-modules. For each $s \in S$ we may choose a representative $n_s \in N$. Moreover, according to [**7**] Lemma 2.2, we can choose n_s in a nice way. The obvious equivalent of Definition 2.0.3 gives an endomorphism $T_n \in \mathcal{H}_\Gamma$ for each $n \in N$. Definition 2.0.9 for each character $\chi : H \to \overline{\mathbf{F}}_p^{\times}$ gives an idempotent $e_\chi \in \mathcal{H}_\Gamma$.

Definition 3.1.2. — Let $\chi : H \to \overline{\mathbf{F}}_p^{\times}$ be a character, and let J be a subset of $J_0(\chi)$ we define $M_{\chi,J}$ to be a right $\mathcal{H}_\Gamma$-module, whose underlying vector space is one dimensional, $M_{\chi,J} = \langle v\rangle_{\overline{\mathbf{F}}_p}$ and the action of $\mathcal{H}_\Gamma$ is determined by the following:

$$ve_\chi = v$$

and for every $s \in S$ we have

$$vT_{n_s} = \begin{cases} 0 & \text{if } s \in J, \\ -v & \text{if } s \in J_0(\chi),\ s \notin J, \\ 0 & \text{if } s \notin J_0(\chi). \end{cases}$$

Corollary 3.1.3. — *The functor of U invariants*

$$\mathrm{Rep}_\Gamma \longrightarrow \mathrm{Mod}\text{-}\mathcal{H}_\Gamma, \quad \rho \longmapsto \rho^U$$

induces a bijection between the irreducible representations of Γ and the irreducible right $\mathcal{H}_\Gamma$-modules. Moreover, if an irreducible representation $\rho_{\chi,J}$ corresponds to the pair (χ, J), in the sense of Theorem 3.1.1 (iii), then

$$\rho^U_{\chi,J} \cong M_{\chi,J}$$

as an $\mathcal{H}_\Gamma$-module.

Proof. — See, [**6**] Theorem 3.32. □

Remark 3.1.4. — Ideally, we would like to have an analogue of the Corollary above for G or more generally for any group of F-points of a reductive group, split over F.

Carter and Lusztig, in [**7**] construct all the irreducible representations $\rho_{\chi,J}$ in a very elegant way. For each pair (χ, J) they define a Γ-equivariant homomorphism

$$\Theta^J_{w_0} : \mathrm{Ind}_B^\Gamma \chi \longrightarrow \mathrm{Ind}_B^\Gamma \chi^{w_0}$$

which depends on the geometry of the Coxeter group W, so that

$$\rho_{\chi,J} \cong \mathrm{Im}\,\Theta^J_{w_0}$$

where w_0 is the unique element of maximal length in W.

From now onwards we specialise to our situation, so that $\Gamma = \mathrm{GL}_2(\mathbf{F}_q)$, B is the subgroup of upper-triangular matrices, U is the subgroup of unipotent upper-triangular matrices, H is the diagonal matrices, N is the normaliser of H in Γ, that is the monomial matrices and $W = N/H$ is isomorphic to the symmetric group on two letters, $W = \{1, s\}$. Let

$$n_s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

be a fixed representative of s in N. In particular, s is the element of the maximal length in W and also the single Coxeter generator, so that $S = \{s\}$. Hence, if $\chi : H \to \overline{\mathbf{F}}_p^\times$, then either $J_0(\chi) = \varnothing$ or $J_0(\chi) = S$. Since

$$K/K_1 \cong \Gamma, \quad I/K_1 \cong B, \quad I_1/K_1 \cong U$$

to ease the notation, we will often identify the spaces

$$\{f : \Gamma \longrightarrow \overline{\mathbf{F}}_p : f(ug) = f(g),\ \forall\, g \in \Gamma,\ \forall\, u \in U\}$$

and

$$\{f \in \text{c-Ind}_{I_1}^{G} \mathbf{1} : \operatorname{Supp} f \subseteq K\}$$

in the natural way. In particular, we will use the same notation for the elements of $\mathcal{H}_K$ and $\mathcal{H}_\Gamma$ and we note that the Definitions 2.2.1 and 3.1.2 coincide.

PROPOSITION 3.1.5. — *For each character $\chi : H \to \overline{\mathbf{F}}_p^{\times}$, such that $\chi = \chi^s$, let*

$$\rho_{\chi,S} = \operatorname{Im}\left((1 + T_{n_s}) : \operatorname{Ind}_B^{\Gamma} \chi \to \operatorname{Ind}_B^{\Gamma} \chi\right)$$

and let

$$\rho_{\chi,\varnothing} = \operatorname{Im}\left(T_{n_s} : \operatorname{Ind}_B^{\Gamma} \chi \to \operatorname{Ind}_B^{\Gamma} \chi\right)$$

then the representations $\rho_{\chi,S}$ and $\rho_{\chi,\varnothing}$ are irreducible. Moreover,

$$\rho_{\chi,S}^{U} = \langle (1 + T_{n_s})\varphi_\chi \rangle_{\overline{\mathbf{F}}_p} \cong M_{\chi,S} \quad \textit{and} \quad \rho_{\chi,\varnothing}^{U} = \langle T_{n_s}\varphi_\chi \rangle_{\overline{\mathbf{F}}_p} \cong M_{\chi,\varnothing}$$

as $\mathcal{H}_\Gamma$-modules. For each character $\chi : H \to \overline{\mathbf{F}}_p^{\times}$, such that $\chi \neq \chi^s$, let

$$\rho_{\chi,\varnothing} = \operatorname{Im}\left(T_{n_s} : \operatorname{Ind}_B^{\Gamma} \chi \to \operatorname{Ind}_B^{\Gamma} \chi^s\right)$$

then the representation $\rho_{\chi,\varnothing}$ is irreducible. Moreover,

$$\rho_{\chi,\varnothing}^{U} = \langle T_{n_s}\varphi_\chi \rangle_{\overline{\mathbf{F}}_p} \cong M_{\chi,\varnothing}$$

as an $\mathcal{H}_\Gamma$-module. Further, these representations are pairwise non-isomorphic, and every irreducible representation of Γ is isomorphic to $\rho_{\chi,J}$, for some character χ and a subset J of $J_0(\chi)$.

Proof. — This is a special case of [**7**] Theorem 7.1 and Corollary 7.5. The isomorphisms of $\mathcal{H}_\Gamma$-modules are given by the Corollary 3.1.3. ☐

REMARK 3.1.6. — Although we do not use this, we note that Frobenius reciprocity gives us

$$\text{c-Ind}_K^G \rho_{\chi,\varnothing} \cong T_{n_s}(\text{c-Ind}_I^G \chi) \leqslant \text{c-Ind}_I^G \chi^s$$

and if $\chi = \chi^s$ then

$$\text{c-Ind}_K^G \rho_{\chi,S} \cong (1 + T_{n_s})(\text{c-Ind}_I^G \chi) \leqslant \text{c-Ind}_I^G \chi.$$

Using this, one can relate the central elements of Vignéras in [**18**] to the 'standard ' endomorphisms T_σ of Barthel and Livné in [**1**].

LEMMA 3.1.7. — *Let $\chi : H \to \overline{\mathbf{F}}_p^{\times}$ be a character, such that $\chi = \chi^s$. Then the homomorphisms $e_\chi(1 + T_{n_s})e_\chi$ and $-e_\chi T_{n_s} e_\chi$ are orthogonal idempotents. In particular,*

$$\operatorname{Ind}_B^{\Gamma} \chi \cong \rho_{\chi,\varnothing} \oplus \rho_{\chi,S}.$$

Moreover, let $\chi' : \mathbf{F}_q^{\times} \to \overline{\mathbf{F}}_p^{\times}$ be a character such that $\chi = \chi' \circ \det$, then

$$\rho_{\chi,S} \cong \chi' \circ \det \quad \textit{and} \quad \rho_{\chi,\varnothing} \cong \operatorname{St} \otimes \chi' \circ \det$$

where St *is the Steinberg representation.*

Proof. — Since $\chi = \chi^s$ we have

$$e_\chi T_{n_s} = T_{n_s} e_\chi \quad \text{and} \quad e_\chi T_{n_s}^2 = -e_\chi T_{n_s}.$$

So the elements above are orthogonal idempotents as claimed. By Proposition 3.1.5, the summands they split off are $\rho_{\chi,S}$ and $\rho_{\chi,\varnothing}$.

Since $\chi = \chi^s$, the character χ must factor through the determinant. So χ extends to a character of Γ and hence

$$\operatorname{Ind}_B^\Gamma \chi \cong \operatorname{Ind}_B^\Gamma \mathbf{1} \otimes \chi' \circ \det .$$

So we may assume that χ is the trivial character. The Bruhat decomposition says that $\Gamma = BsB \cup B$ and hence by Theorem 3.1.1 (ii) $\rho_{\mathbf{1},S} = \mathbf{1}$, the trivial representations of G. This implies that $\rho_{\mathbf{1},\varnothing}$ is the Steinberg representation. $\square$

Corollary 3.1.8. — *Let $\chi : H \to \overline{\mathbf{F}}_p^\times$ be a character, such that $\chi = \chi^s$. Let ρ be any representation of Γ, such that for some $v \in \rho^U$ we have*

$$\langle v \rangle_{\overline{\mathbf{F}}_p} \cong M_{\chi,J}$$

as an $\mathcal{H}_\Gamma$-module. Then

$$\langle \Gamma v \rangle_{\overline{\mathbf{F}}_p} \cong \rho_{\chi,J}$$

as a Γ-representation.

Proof. — Since v is fixed by U there exists a homomorphism $\psi \in \operatorname{Hom}_\Gamma(\operatorname{Ind}_U^\Gamma \mathbf{1}, \rho)$ such that $\psi(\varphi) = v$. The isomorphism of $\mathcal{H}_\Gamma$-modules implies that

$$v = ve_\chi = \psi(e_\chi \varphi) = \psi(\varphi_\chi).$$

Hence, H acts on v by a character χ and

$$\psi(\operatorname{Ind}_U^\Gamma \mathbf{1}) = \psi(e_\chi(\operatorname{Ind}_U^\Gamma \mathbf{1})) = \psi(\operatorname{Ind}_B^\Gamma \chi).$$

If $J = \varnothing$ then

$$\psi((1 + T_{n_s})\varphi_\chi) = v(1 + T_{n_s})e_\chi = 0.$$

Hence, $\rho_{\chi,S}$ is contained in the kernel of ψ. By Lemma 3.1.7

$$\operatorname{Im} \psi \cong \rho_{\chi,\varnothing}.$$

Since, the image is irreducible and contains v we get the result. The proof for $J = S$ is analogous. $\square$

The Corollary has a nice application, which complements [**18**] E.7.1.

Corollary 3.1.9. — *Let π be a smooth representation of G and suppose that there exists a non-zero vector $v \in \pi^{I_1}$ such that*

$$ve_{\mathbf{1}} = v, \quad vT_{n_s} = 0, \quad vT_\Pi = v$$

then G acts trivially on v.

Proof. — As an $\mathcal{H}_K$ module
$$\langle v\rangle_{\overline{\mathbf{F}}_p} \cong M_{\mathbf{1},S}.$$
By Corollary 3.1.8 K acts trivially on v. On the other hand
$$v = vT_\Pi = \Pi^{-1}v.$$
Iwahori decomposition implies that Π and K generate G as a group. Hence G acts trivially on v. □

REMARK 3.1.10. — There is a version of this twisted by a character. This example will lead us to better things. See Remark 5.5.6.

LEMMA 3.1.11. — *Let $\chi : H \to \overline{\mathbf{F}}_p^\times$ be a character, let J be a subset of $J_0(\chi)$, and let $\overline{J} = J_0(\chi)\backslash J$. The sequence of $\mathcal{H}_\Gamma$-modules*
$$0 \longrightarrow M_{\chi,J} \longrightarrow (\mathrm{Ind}_B^\Gamma \chi^s)^U \longrightarrow M_{\chi^s,\overline{J}} \longrightarrow 0$$
is exact. Moreover, it splits if and only if $\chi = \chi^s$.

Proof. — The space $(\mathrm{Ind}_B^\Gamma \chi^s)^U$ is two dimensional, with the basis $\{T_{n_s}\varphi_\chi, \varphi_{\chi^s}\}$.

If $\chi = \chi^s$ then $e_\chi(1+T_{n_s})e_\chi$ and $-e_\chi T_{n_s} e_\chi$ are orthogonal idempotents, which split the sequence.

If $\chi \neq \chi^s$ then $J_0(\chi) = J = \varnothing$ and for every λ, $\mu \in \overline{\mathbf{F}}_p$ we have
$$(\lambda T_{n_s}\varphi_\chi + \mu\varphi_{\chi^s})e_\chi = \lambda T_{n_s}\varphi_\chi, \quad (\lambda T_{n_s}\varphi_\chi + \mu\varphi_{\chi^s})e_{\chi^s} = \mu\varphi_{\chi^s}$$
and
$$(\lambda T_{n_s}\varphi_\chi + \mu\varphi_{\chi^s})T_{n_s} = \mu T_{n_s}\varphi_\chi.$$
Hence $M_{\chi,\varnothing}$ is the only proper submodule, so the sequence cannot split. □

3.2. Alternative description of irreducible representations

Let $V_{d,F}$ be an F vector space of homogeneous polynomials in two variables X and Y of the degree d. The group K acts on $V_{d,F}$ via
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(X^{d-i}Y^i) = (aX+cY)^{d-i}(bX+dY)^i.$$
For $0 \leqslant i \leqslant d$, let
$$m_i = \binom{d}{i} X^{d-i}Y^i$$
where $\binom{d}{i}$ denotes the binomial coefficient. Vectors m_i, for $0 \leqslant i \leqslant d$, form a basis of $V_{d,F}$. Let $V_{d,\mathfrak{o}_F}$ be the $\mathfrak{o}_F$-lattice in $V_{d,F}$ spanned by the m_i, for $0 \leqslant i \leqslant d$. An easy check shows that $V_{d,\mathfrak{o}_F}$ is K invariant. Let
$$V_{d,\mathbf{F}_q} = V_{d,\mathfrak{o}_F} \otimes_{\mathfrak{o}_F} \mathfrak{o}_F/\mathfrak{p}_F.$$

The vectors $m_i \otimes 1$, for $0 \leqslant i \leqslant d$, form an $\mathbf{F}_q$-basis of $V_{d,\mathbf{F}_q}$. The subgroup K_1 acts trivially on $V_{d,\mathbf{F}_q}$, so we consider $V_{d,\mathbf{F}_q}$ as a representation of Γ. Let Fr be the automorphism of Γ, given by

$$\mathrm{Fr} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}.$$

Let ρ be a representation of Γ. We will denote by ρ^{Fr} the representation of Γ given by

$$\rho^{\mathrm{Fr}}(g) = \rho(\mathrm{Fr}(g)).$$

THEOREM 3.2.1. — *Let $\Gamma = \mathrm{GL}_2(\mathbf{F}_q)$ and suppose that $q = p^n$. The isomorphism classes of irreducible $\overline{\mathbf{F}}_p$-representations of Γ are parameterised by pairs $(a, \boldsymbol{r})$, where*

- *a is an integer $1 \leqslant a \leqslant q-1$ and*
- *$\boldsymbol{r}$ is an ordered n-tuple $\boldsymbol{r} = (r_0, r_1, \dots, r_{n-1})$, where $0 \leqslant r_i \leqslant p-1$, for every i.*

Moreover, the irreducible representations of Γ can be realized over $\mathbf{F}_q$ and the irreducible representation corresponding to $(a, \boldsymbol{r})$ is given by

$$V_{\boldsymbol{r},\mathbf{F}_q} \otimes (\det)^a \cong V_{r_0,\mathbf{F}_q} \otimes V_{r_1,\mathbf{F}_q}^{\mathrm{Fr}} \otimes \cdots \otimes V_{r_i,\mathbf{F}_q}^{\mathrm{Fr}^i} \otimes \cdots \otimes V_{r_{n-1},\mathbf{F}_q}^{\mathrm{Fr}^{n-1}} \otimes (\det)^a.$$

Proof. — This is shown in [**3**], see also [**1**] Proposition 1 and [**18**] Ap. 6. We remark that since $\binom{r}{i}$ is a unit in $\mathbf{F}_q$ if $r \leqslant p-1$, our spaces really coincide with the ones considered in [**1**]. □

We fix some embedding $\iota : \mathbf{F}_q \hookrightarrow \overline{\mathbf{F}}_p$ and we will assume that every character $\chi : H \to \overline{\mathbf{F}}_p^\times$ factors through ι. Once we have done that, we will omit ι from our notation. We will denote

$$V_{\boldsymbol{r},\overline{\mathbf{F}}_p} = V_{\boldsymbol{r},\mathbf{F}_q} \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_p.$$

We need a dictionary between the two descriptions.

PROPOSITION 3.2.2. — *Let $\chi : H \to \overline{\mathbf{F}}_p^\times$ and let a be the unique integer, such that $1 \leqslant a \leqslant q-1$ and*

$$\chi\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\right)\right) = \lambda^a \quad \forall\, \lambda \in \mathbf{F}_q^\times$$

and let r be the unique integer, such that $1 \leqslant r \leqslant q-1$ and

$$\chi\left(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)\right) = \lambda^r \quad \forall\, \lambda \in \mathbf{F}_q^\times.$$

Suppose that $r \neq q-1$, and let $\boldsymbol{r} = (r_0, \dots, r_{n-1})$ be the digits of a p-adic expansion of r

$$r = r_0 + r_1 p + \cdots + r_{n-1} p^{n-1}$$

then $\chi \neq \chi^s$ and $\rho_{\chi,\varnothing}$ corresponds to the pair $(a, \boldsymbol{r})$. More precisely

$$\rho_{\chi,\varnothing} \cong V_{r_0,\overline{\mathbf{F}}_p} \otimes \cdots \otimes V_{r_{n-1},\overline{\mathbf{F}}_p}^{\mathrm{Fr}^{n-1}} \otimes (\det)^a.$$

Suppose that $r = q-1$, then $\chi = \chi^s$,

$$\rho_{\chi,\varnothing} \cong V_{p-1,\overline{\mathbf{F}}_p} \otimes \cdots \otimes V_{p-1,\overline{\mathbf{F}}_p}^{\mathrm{Fr}^{n-1}} \otimes (\det)^a \cong \mathrm{St} \otimes (\det)^a$$

and

$$\rho_{\chi,S} \cong V_{0,\overline{\mathbf{F}}_p} \otimes \cdots \otimes V_{0,\overline{\mathbf{F}}_p}^{\mathrm{Fr}^{n-1}} \otimes (\det)^a \cong (\det)^a$$

where St *denotes the Steinberg representation.*

Proof. — Every character $\chi : H \to \overline{\mathbf{F}}_p^{\times}$ is of the form

$$\chi : \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \longmapsto \lambda^c \mu^d$$

for some integers c and d. Moreover, $\chi = \chi^s$ if and only if

$$c - d \equiv 0 \pmod{q-1}.$$

The integers a and r are uniquely determined by the congruences

$$d \equiv a \pmod{q-1} \quad \text{and} \quad c - d \equiv r \pmod{q-1}.$$

By Theorem 3.1.1 if ρ is an irreducible representation of Γ, then $\dim \rho^U = 1$, and by Corollary 3.1.3 the irreducible representations of Γ correspond to the irreducible modules of the Hecke algebra $\mathcal{H}_\Gamma$. Since we have two complete lists of irreducible representations, it is enough to match up the corresponding irreducible modules. We recall that

$$\rho_{\chi,J}^U \cong M_{\chi,J}$$

as $\mathcal{H}_\Gamma$-modules.

We observe that the action of U on $V_{d,\overline{\mathbf{F}}_p}$ fixes the vector $m_0 \otimes 1$. Moreover,

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} m_0 \otimes 1 = \lambda^d m_0 \otimes 1.$$

Let $(a, \boldsymbol{r})$ be any pair parameterising an irreducible representation of Γ and let

$$r = r_0 + r_1 p + \cdots + r_{n-1} p^{n-1}.$$

By picking such $(m_0 \otimes 1)_{r_i}$ in every component of the tensor product we obtain a non-zero vector

$$(m_0 \otimes 1)_{\boldsymbol{r}} = (m_0 \otimes 1)_{r_0} \otimes \cdots \otimes (m_0 \otimes 1)_{r_{n-1}}$$

fixed by U. The vector $(m_0 \otimes 1)_{\boldsymbol{r}}$ spans the space of U invariants, since it is one dimensional. Moreover, since the action on the components of the tensor product is twisted by Fr we obtain

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} (m_0 \otimes 1)_{\boldsymbol{r}} = (\lambda\mu)^a \lambda^r (m_0 \otimes 1)_{\boldsymbol{r}}.$$

Suppose that we start with an arbitrary character $\chi : H \to \overline{\mathbf{F}}_p^{\times}$ and obtain the integers a and r as in the statement of the proposition.

If $r \neq q-1$, then by above $\chi \neq \chi^s$. Let $\boldsymbol{r}$ be the n-tuple corresponding to r. Since, $\chi \neq \chi^s$, the module $M_{\chi,\varnothing}$ is the only irreducible module of $\mathcal{H}_\Gamma$, which is not killed by the idempotent e_χ. Let $(m_0 \otimes 1)_{\boldsymbol{r}}$ be the vector constructed above. Since, H acts on $(m_0 \otimes 1)_{\boldsymbol{r}}$ via the character χ, we obtain

$$M_{\chi,\varnothing} \cong (V_{r_0,\overline{\mathbf{F}}_p} \otimes \cdots \otimes V^{\mathrm{Fr}^{n-1}}_{r_{n-1},\overline{\mathbf{F}}_p} \otimes (\det)^a)^U$$

as $\mathcal{H}_\Gamma$-modules and that implies the isomorphism between representations.

If $r = q-1$, then $\chi = \chi^s$, and the only $\mathcal{H}_\Gamma$-modules, which are not killed by e_χ, are $M_{\chi,S}$ and $M_{\chi,\varnothing}$. We observe that $V_{0,\overline{\mathbf{F}}_p}$ is just the trivial representation. Let $\mathbf{0} = (0,\dots,0)$, then the representation corresponding to the pair $(a,\mathbf{0})$ is just $\mathbf{1} \otimes (\det)^a$, which is isomorphic to $\rho_{\chi,S}$, by Proposition 3.1.7. The only case left is $\boldsymbol{r} = \boldsymbol{p} - \mathbf{1} = (p-1,\dots,p-1)$, hence

$$M_{\chi,\varnothing} \cong (V_{p-1,\overline{\mathbf{F}}_p} \otimes \cdots \otimes V^{\mathrm{Fr}^{n-1}}_{p-1,\overline{\mathbf{F}}_p} \otimes (\det)^a)^U$$

as $\mathcal{H}_\Gamma$-modules, since the module $M_{\chi,S}$ is already taken. This implies that

$$\rho_{\chi,\varnothing} \cong V_{\boldsymbol{p}-\mathbf{1},\overline{\mathbf{F}}_p} \otimes (\det)^a \cong \mathrm{St} \otimes (\det)^a$$

where the last isomorphism follows from Proposition 3.1.7. □

Corollary 3.2.3. — *Suppose that $q = p^n$ and the representation $\rho_{\chi,J}$ corresponds to the pair $(a,\boldsymbol{r})$. Let $r = r_0 + r_1 p + \cdots + r_{n-1}p^{n-1}$ and let $\overline{J} = J_0(\chi)\backslash J$, where $J_0(\chi) = \{s \in S : \chi^s = \chi\}$. Then*

$$\rho_{\chi^s,\overline{J}} \cong V_{p-1-r_0,\overline{\mathbf{F}}_p} \otimes \cdots \otimes V^{\mathrm{Fr}^{n-1}}_{p-1-r_{n-1},\overline{\mathbf{F}}_p} \otimes (\det)^{a+r}.$$

Proof. — If $r = 0$ or $r = q-1$, then $\boldsymbol{r}$ is of a special form and the isomorphism follows from Proposition 3.2.2.

If $r \neq 0$ and $r \neq q-1$, we observe that

$$\chi^s\big(\big(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\big)\big) = \chi\big(\big(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\big)\big)\chi\big(\big(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\big)\big) = \lambda^{a+r} \quad \forall\, \lambda \in \mathbf{F}_q^\times$$

and

$$\chi^s\big(\big(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\big)\big) = \chi\big(\big(\begin{smallmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{smallmatrix}\big)\big) = \lambda^{-r} \quad \forall\, \lambda \in \mathbf{F}_q^\times.$$

The claim follows from Proposition 3.2.2. □

CHAPTER 4

PRINCIPAL INDECOMPOSABLE REPRESENTATIONS

We will recall some facts from the modular representation theory of finite groups. Let Γ be any finite group. We denote by Rep_Γ the category of $\overline{\mathbf{F}}_p$-representations of Γ and by Irr_Γ the set of isomorphism classes of irreducible representations in Rep_Γ. We note that Rep_Γ is equivalent to the module category of the ring $\overline{\mathbf{F}}_p[\Gamma]$.

PROPOSITION 4.0.4. — *A representation* inj *is an injective object in* Rep_Γ *if and only if it is a projective object in* Rep_Γ.

The isomorphism classes of indecomposable injective (and hence projective) objects in Rep_Γ *are parameterised by* Irr_Γ.

More precisely, if inj *is indecomposable and injective, then the maximal semi-simple submodule* $\mathrm{soc}(\mathrm{inj})$ *and the maximal semi-simple quotient* $\mathrm{inj}/\mathrm{rad}(\mathrm{inj})$ *are both irreducible. Moreover,*

$$\mathrm{soc}(\mathrm{inj}) \cong \mathrm{inj}/\mathrm{rad}(\mathrm{inj}).$$

Conversely, given $\rho \in \mathrm{Irr}_\Gamma$, *there exists a unique up to isomorphism indecomposable, injective object* $\mathrm{inj}\,\rho$ *in* Rep_Γ, *such that*

$$\rho \cong \mathrm{soc}(\mathrm{inj}\,\rho).$$

Proof. — See [**15**], Exercises 14.1 and 14.6. □

We will call indecomposable representations of Γ, which are injective objects in Rep_Γ, principal indecomposable representations.

REMARK 4.0.5. — We note that a monomorphism $\rho \hookrightarrow \mathrm{inj}\,\rho$ is an injective envelope of ρ in Rep_Γ.

COROLLARY 4.0.6. — *We have the following decomposition:*

$$\overline{\mathbf{F}}_p[\Gamma] \cong \bigoplus_{\rho \in \mathrm{Irr}_\Gamma} (\dim \rho)\, \mathrm{inj}\,\rho.$$

Proof. — Since $\overline{\mathbf{F}}_p[\Gamma]$ is an injective and projective object it must decompose into a direct sum of indecomposable injective objects. Since

$$\dim \operatorname{Hom}_\Gamma(\rho, \overline{\mathbf{F}}_p[\Gamma]) = \dim \operatorname{Hom}_{\{1\}}(\rho, \mathbf{1}) = \dim \rho$$

the representation $\operatorname{inj} \rho$ occurs in the decomposition with the multiplicity $\dim \rho$. □

Proposition 4.0.7. — *Let U be a p-Sylow subgroup of Γ. Then a representation ρ is an injective object in $\operatorname{Rep}_\Gamma$ if and only if $\rho|_U$ is an injective object in Rep_U.*

Proof. — This follows easily from [**15**], § 14.4, Lemma 20. □

Proposition 4.0.8. — *Suppose that U is a p-group, then the only irreducible representation is $\mathbf{1}$ and hence the only principal indecomposable representation is $\overline{\mathbf{F}}_p[U]$.*

Proof. — The first part is [**15**], § 8 , Proposition 26, the last part follows from Corollary 4.0.6. □

Corollary 4.0.9. — *Let* inj *be an injective object in $\operatorname{Rep}_\Gamma$ and let U be a p-Sylow subgroup of Γ, then*

$$\dim \operatorname{inj} = \dim \operatorname{inj}^U |U|.$$

Proof. — The restriction $\operatorname{inj}|_U$ is an injective object in Rep_U. By the above Proposition

$$\operatorname{inj}|_U \cong m\overline{\mathbf{F}}_p[U].$$

The multiplicity m is given by: $m = \dim \operatorname{Hom}_U(\mathbf{1}, \operatorname{inj}) = \dim \operatorname{inj}^U$. □

In the rest of the section $\Gamma = \mathrm{GL}_2(\mathbf{F}_q)$ and U is the subgroup of unipotent upper triangular matrices. Given $\rho \in \operatorname{Irr}_\Gamma$ we are going to compute $(\operatorname{inj}\rho)^U$ as an $\mathcal{H}_\Gamma$-module. Once we know the modules we are going to show that if we consider $\operatorname{inj}\rho_{\chi,J}$ and $\operatorname{inj}\rho_{\chi^s,\overline{J}}$ as representations of K, then the action of $\mathcal{H}_K$ on

$$(\operatorname{inj}\rho_{\chi,J} \oplus \operatorname{inj}\rho_{\chi^s,\overline{J}})^{I_1}$$

extends to the action of $\mathcal{H}$, so that if $\chi = \chi^s$ then it is isomorphic to a direct sum of supersingular modules and if $\chi \neq \chi^s$ then it is isomorphic to a direct sum of L_γ and supersingular modules. See Propositions 4.2.37 and 4.2.38 for the precise statement. This calculation, becomes of importance in Section 6.4. Although, the general case includes the case $q = p$, if $q = p$ we give a different, easier way of doing this. When $q = p$, the main result is Proposition 4.1.9.

4.1. The case $q = p$

We start off with no assumption on q.

LEMMA 4.1.1. — *Suppose that $\chi \neq \chi^s$, then there exists an exact sequence*

$$0 \longrightarrow \operatorname{Ind}_B^\Gamma \chi^s \xrightarrow{\psi} \operatorname{inj} \rho_{\chi,\varnothing}$$

of Γ-representations.

Proof. — Since $\operatorname{inj} \rho_{\chi,\varnothing}$ is an injective module, there exists ψ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \rho_{\chi,\varnothing} \arrow[r] \arrow[d] & \operatorname{Ind}_B^\Gamma \chi^s \arrow[dl, dotted, "\psi"] \\ & \operatorname{inj} \rho_{\chi,\varnothing} & \end{tikzcd}$$

commutes. If $\operatorname{Ker}\psi \neq 0$, then $(\operatorname{Ker}\psi)^U$ is a non-zero proper submodule of $(\operatorname{Ind}_B^\Gamma \chi^s)^U$ not containing $M_{\chi,\varnothing}$. By Lemma 3.1.11 this cannot happen. □

COROLLARY 4.1.2. — *Suppose that $\chi \neq \chi^s$ then*

$$\dim \operatorname{inj} \rho_{\chi,\varnothing} \geqslant 2q.$$

Proof. — Corollary 4.0.9 implies that

$$\dim \operatorname{inj} \rho_{\chi,\varnothing} = \dim(\operatorname{inj} \rho_{\chi,\varnothing})^U |U|.$$

The order of U is q and since by Lemma 4.1.1 $\operatorname{Ind}_B^\Gamma \chi^s$ is a subspace of $\operatorname{inj} \rho_{\chi,\varnothing}$, we obtain

$$\dim(\operatorname{inj} \rho_{\chi,\varnothing})^U \geqslant 2.$$ □

LEMMA 4.1.3. — *Suppose that $q = p$ and $\chi \neq \chi^s$ then the sequence of Γ representations*

$$0 \longrightarrow \rho_{\chi,\varnothing} \longrightarrow \operatorname{Ind}_B^\Gamma \chi^s \longrightarrow \rho_{\chi^s,\varnothing} \longrightarrow 0$$

is exact.

REMARK 4.1.4. — This fails if $q \neq p$.

Proof. — The argument below is taken from [**18**] Ap. 6. We know that

$$\rho_{\chi^s,\varnothing} \cong T_{n_s}(\operatorname{Ind}_B^\Gamma \chi^s)$$

and $\rho_{\chi,\varnothing}$ is isomorphic to the subspace of $\operatorname{Ind}_B^\Gamma \chi^s$ generated by $T_{n_s}\varphi_\chi$. Since, $T_{n_s}^2 \varphi_\chi = 0$ we always have

$$\rho_{\chi,\varnothing} \leqslant \operatorname{Ker} T_{n_s}.$$

If $q = p$, then by Proposition 3.2.2 and Corollary 3.2.3 there exists an integer r such that

$$\dim \rho_{\chi,\varnothing} + \dim \rho_{\chi^s,\varnothing} = (r+1) + (p-1-r+1) = p+1 = \dim \operatorname{Ind}_B^\Gamma \chi^s.$$

Hence the sequence is exact. □

Corollary 4.1.5. — *Suppose that $q = p$ and let $\chi : H \to \overline{\mathbf{F}}_p^\times$ be a character, such that $\chi \neq \chi^s$. Let ρ be any representation of Γ, such that for some $v \in \rho^U$*

$$\langle v\rangle_{\overline{\mathbf{F}}_p} \cong M_{\chi,\varnothing}$$

as an $\mathcal{H}_\Gamma$-module. Then

$$\langle \Gamma v\rangle_{\overline{\mathbf{F}}_p} \cong \rho_{\chi,\varnothing}$$

as a Γ-representation.

Remark 4.1.6. — This fails if $p \neq q$, by Remark 4.1.4, it is enough to look at $\operatorname{Ind}_B^\Gamma \chi/\rho_{\chi^s,\varnothing}$.

Proof. — Since v is fixed by U, there exists a homomorphism $\psi \in \operatorname{Hom}_\Gamma(\operatorname{Ind}_U^\Gamma \mathbf{1}, \rho)$ such that $\psi(\varphi) = v$. The isomorphism of $\mathcal{H}_\Gamma$-modules implies that

$$v = ve_\chi = \psi(e_\chi \varphi) = \psi(\varphi_\chi).$$

Hence H acts on v by a character χ and

$$\psi(\operatorname{Ind}_U^\Gamma \mathbf{1}) = \psi(e_\chi(\operatorname{Ind}_U^\Gamma \mathbf{1})) = \psi(\operatorname{Ind}_B^\Gamma \chi).$$

Now

$$\psi(T_{n_s}\varphi_{\chi^s}) = vT_{n_s}e_{\chi^s} = 0.$$

Hence, $\rho_{\chi^s,\varnothing}$ is contained in the kernel of ψ. By Lemma 4.1.3

$$\operatorname{Im}\psi \cong \rho_{\chi,\varnothing}.$$

Since, the image is irreducible and contains v we get the result. □

Lemma 4.1.7. — *Suppose that $q = p$. If $\chi = \chi^s$ then*

$$\dim \operatorname{inj} \rho_{\chi,J} = p.$$

If $\chi \neq \chi^s$ then

$$\dim \operatorname{inj} \rho_{\chi,\varnothing} = 2p.$$

Proof. — Corollary 4.0.6 implies that

$$\begin{aligned}
\dim \overline{\mathbf{F}}_p[\Gamma] &= \sum_{\rho\in \mathrm{Irr}_\Gamma} (\dim\rho)(\dim \operatorname{inj}\rho)\\
&= \sum_{\chi,\chi=\chi^s} (\dim \rho_{\chi,\varnothing})(\dim \operatorname{inj} \rho_{\chi,\varnothing}) + (\dim \rho_{\chi,S})(\dim \operatorname{inj} \rho_{\chi,S})\\
&\quad + \frac{1}{2}\sum_{\chi,\chi\neq\chi^s} (\dim \rho_{\chi,\varnothing})(\dim \operatorname{inj} \rho_{\chi,\varnothing}) + (\dim \rho_{\chi^s,\varnothing})(\dim \operatorname{inj} \rho_{\chi^s,\varnothing}).
\end{aligned}$$

If $\chi = \chi^s$ then Corollary 4.0.9 implies that

$$\dim \operatorname{inj} \rho_{\chi,J} \geqslant p.$$

If $\chi \neq \chi^s$ then Corollary 4.1.2 implies that

$$\dim \operatorname{inj} \rho_{\chi,\varnothing} \geqslant 2p.$$

Lemma 4.1.3 and Lemma 3.1.7 imply that

$$\dim \rho_{\chi,J} + \dim \rho_{\chi^s,\overline{J}} = p+1.$$

We put these inequalities together and we obtain

$$\dim \overline{\mathbf{F}}_p[\Gamma] \geqslant \sum_{\chi} (p+1)p = \dim \overline{\mathbf{F}}_p[\Gamma]$$

So all the inequalities must be equalities and we obtain the lemma. □

Corollary 4.1.8. — *Suppose that $q = p$. If $\chi = \chi^s$ then*

$$\langle \Gamma(\operatorname{inj} \rho_{\chi,J})^U \rangle_{\overline{\mathbf{F}}_p} \cong \rho_{\chi,J}.$$

In particular,

$$(\operatorname{inj} \rho_{\chi,J})^U \cong M_{\chi,J}$$

as an $\mathcal{H}_\Gamma$-module.

If $\chi \neq \chi^s$ then

$$\langle \Gamma(\operatorname{inj} \rho_{\chi,\varnothing})^U \rangle_{\overline{\mathbf{F}}_p} \cong \operatorname{Ind}_B^\Gamma \chi^s.$$

In particular,

$$(\operatorname{inj} \rho_{\chi,J})^U \cong (\operatorname{Ind}_B^\Gamma \chi^s)^U$$

as an $\mathcal{H}_\Gamma$-module.

Proof. — If $\chi = \chi^s$ then we have an exact sequence

$$0 \longrightarrow \rho_{\chi,J} \longrightarrow \operatorname{inj} \rho_{\chi,J}$$

of Γ-representations. Since, by Lemma 4.1.7

$$\dim \rho_{\chi,J}^U = \dim(\operatorname{inj} \rho_{\chi,J})^U$$

we obtain the Corollary. Similarly, if $\chi \neq \chi^s$ then by Lemma 4.1.1 there exists an exact sequence

$$0 \longrightarrow \operatorname{Ind}_B^\Gamma \chi^s \longrightarrow \operatorname{inj} \rho_{\chi,\varnothing}$$

of Γ-representations. Since, by Lemma 4.1.7

$$\dim(\operatorname{Ind}_B^\Gamma \chi^s)^U = \dim(\operatorname{inj} \rho_{\chi,\varnothing})^U$$

we obtain the Corollary. □

Proposition 4.1.9. — *Suppose that $q = p$, let $\chi : H \to \overline{\mathbf{F}}_p^\times$ be a character and let $\gamma = \{\chi, \chi^s\}$. We consider representations* $\operatorname{inj} \rho_{\chi,J}$ *and* $\operatorname{inj} \rho_{\chi,\overline{J}}$ *as representations of K, via*

$$K \longrightarrow K/K_1 \cong \Gamma.$$

If $\chi = \chi^s$ then the action of $\mathcal{H}_K$ on $(\operatorname{inj} \rho_{\chi,\varnothing} \oplus \operatorname{inj} \rho_{\chi,S})^{I_1}$ *extends to the action of $\mathcal{H}$ so that*

$$(\operatorname{inj} \rho_{\chi,\varnothing} \oplus \operatorname{inj} \rho_{\chi,S})^{I_1} \cong M_\gamma.$$

If $\chi \neq \chi^s$ then the action of $\mathcal{H}_K$ on $(\operatorname{inj}\rho_{\chi,\varnothing} \oplus \operatorname{inj}\rho_{\chi^s,\varnothing})^{I_1}$ extends to the action of $\mathcal{H}$ so that

$$(\operatorname{inj}\rho_{\chi,\varnothing} \oplus \operatorname{inj}\rho_{\chi^s,\varnothing})^{I_1} \cong L_\gamma.$$

Proof. — Suppose that $\chi = \chi^s$ then Corollary 4.1.8 says that

$$(\operatorname{inj}\rho_{\chi,\varnothing} \oplus \operatorname{inj}\rho_{\chi,S})^{I_1} \cong \langle T_{n_s}\varphi_\chi\rangle_{\overline{\mathbf{F}}_p} \oplus \langle (1+T_{n_s})\varphi_\chi\rangle_{\overline{\mathbf{F}}_p} \cong M_{\chi,\varnothing} \oplus M_{\chi,S} \cong M_\gamma|_{\mathcal{H}_K}$$

as $\mathcal{H}_K$-modules, where the last isomorphism follows from Lemma 2.2.2. It is enough to define the action of T_Π. If we let

$$(T_{n_s}\varphi_\chi)T_\Pi = (1+T_{n_s})\varphi_\chi \quad \text{and} \quad ((1+T_{n_s})\varphi_\chi)T_\Pi = T_{n_s}\varphi_\chi$$

then this gives us the required action. Suppose that $\chi \neq \chi^s$, then Corollary 4.1.8 and Lemma 2.2.2 imply that

$$(\operatorname{inj}\rho_{\chi,\varnothing} \oplus \operatorname{inj}\rho_{\chi^s,\varnothing})^{I_1} \cong (\operatorname{Ind}_I^K \chi^s \oplus \operatorname{Ind}_I^K \chi)^{I_1} \cong L_\gamma|_{\mathcal{H}_K}$$

as $\mathcal{H}_K$-modules. The space $(\operatorname{Ind}_I^K \chi^s)^{I_1}$ has basis $\{T_{n_s}\varphi_\chi, \varphi_{\chi^s}\}$ and the space $(\operatorname{Ind}_I^K \chi)^{I_1}$ has basis $\{T_{n_s}\varphi_{\chi^s}, \varphi_\chi\}$. It is enough to define the action of T_Π on the basis. If we set

$$\varphi_\chi T_\Pi = \varphi_{\chi^s}, \quad \varphi_{\chi^s} T_\Pi = \varphi_\chi$$

and

$$(T_{n_s}\varphi_\chi)T_\Pi = T_{n_s}\varphi_{\chi^s}, \quad (T_{n_s}\varphi_{\chi^s})T_\Pi = T_{n_s}\varphi_\chi$$

then this gives us the required action. □

4.2. The general case

Our counting argument breaks down if $p \neq q$. The strategy is to restrict to $\mathrm{SL}_2(\mathbf{F}_q)$, where the principal indecomposable representations have been worked out by Jeyakumar in [**10**]. Let

$$\Gamma' = \mathrm{SL}_2(\mathbf{F}_q), \quad B' = B \cap \Gamma', \quad H' = H \cap \Gamma'.$$

We note that U is a subgroup of Γ' and $n_s \in \Gamma'$.

4.2.1. Modular representations of $\mathrm{SL}_2(\mathbf{F}_q)$. — The irreducible $\overline{\mathbf{F}}_p$-representations of $\mathrm{SL}_2(\mathbf{F}_q)$ were determined by Brauer and Nesbitt.

Theorem 4.2.1 ([**3**]). — *Suppose that $q = p^n$. The isomorphism classes of irreducible $\overline{\mathbf{F}}_p$-representations of Γ' are parameterised by n-tuples $\boldsymbol{r} = (r_0, \dots, r_{n-1})$, where $0 \leqslant r_i \leqslant p-1$, for every i. Moreover, every irreducible representation can be realized over $\mathbf{F}_q$ and the representation corresponding to an n-tuple $\boldsymbol{r}$ is given by*

$$V_{\boldsymbol{r},\mathbf{F}_q} \cong V_{r_0,\mathbf{F}_q} \otimes V_{r_1,\mathbf{F}_q}^{\mathrm{Fr}} \otimes \cdots \otimes V_{r_i,\mathbf{F}_q}^{\mathrm{Fr}^i} \otimes \cdots \otimes V_{r_{n-1},\mathbf{F}_q}^{\mathrm{Fr}^{n-1}}$$

where $V_{r_i,\mathbf{F}_q}$ are the spaces of Section 3.2.

COROLLARY 4.2.2. — *Let ρ be an irreducible representation of Γ, then $\rho\,|_{\Gamma'}$ is irreducible. Moreover, given an irreducible representation ρ' of Γ' there exist, precisely $q-1$ isomorphism classes of irreducible representations of Γ, given by $\rho\otimes(\det)^a$, where $0\leqslant a<q-1$, such that*

$$(\rho\otimes(\det)^a)\,|_{\Gamma'}\cong\rho'.$$

Proof. — This is immediate from Theorem 4.2.1 and Theorem 3.2.1. □

REMARK 4.2.3. — By counting dimensions, we may show that

$$(\operatorname{inj}(V_{\boldsymbol{r},\overline{\mathbf{F}}_p}\otimes(\det)^a))\,|_{\Gamma'}\cong\operatorname{inj}V_{\boldsymbol{r},\overline{\mathbf{F}}_p}$$

as Γ'-representations. However, we will obtain this directly later on.

We recall the construction of the indecomposable principal representations for $\mathrm{SL}_2(\mathbf{F}_q)$ as it is done in [**10**]. The idea is to go from the Lie algebra to the universal enveloping algebra and then to the group.

Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{SL}_2(\mathbb{C})$. It has a $\mathbb{C}$-basis

$$e=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad h=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad f=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

Let $\mathcal{U}$ be the universal enveloping algebra of $\mathfrak{g}$. Let $\mathcal{U}_{\mathbb{Z}}$ be a subring of $\mathcal{U}$ generated by the elements

$$\frac{e^k}{k!},\quad\frac{f^k}{k!},\quad\forall\,k\in\mathbb{Z}^+$$

over $\mathbb{Z}$. The ring $\mathcal{U}_{\mathbb{Z}}$ has a $\mathbb{Z}$-basis, which is also a $\mathbb{C}$-basis for $\mathcal{U}$. Let d be a nonnegative integer and let V_d be the irreducible module of $\mathfrak{g}$ of highest weight d. The space V_d has a $\mathbb{C}$-basis of weight vectors m_i, for $0\leqslant i\leqslant d$, and the action of $\mathfrak{g}$ is given by

$$\begin{gathered}em_0=0,\quad em_i=(d-i+1)m_{i-1},\quad 1\leqslant i\leqslant d,\\ fm_d=0,\quad fm_i=(i+1)m_{i+1},\quad 0\leqslant i\leqslant d-1,\\ hm_i=(d-2i)m_i,\quad 0\leqslant i\leqslant d.\end{gathered}$$

Let $V_{d,\mathbb{Z}}$ be a $\mathbb{Z}$-lattice in V_d spanned by m_i, for $0\leqslant i\leqslant d$. We adopt the convention that $m_i=0$ if $i<0$ or $i>d$. Since,

$$\frac{e^k}{k!}m_i=\binom{d-i+k}{d-i}m_{i-k}$$

and

$$\frac{f^k}{k!}m_i=\binom{i+k}{i}m_{i+k}$$

for all $k\in\mathbb{Z}^+$, the lattice $V_{d,\mathbb{Z}}$ is a $\mathcal{U}_{\mathbb{Z}}$-module. Let

$$\widetilde{V}_{d,\mathbf{F}_q}=V_{d,\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbf{F}_q.$$

For every $\lambda \in \mathbf{F}_q$ we define $x(\lambda)$, $y(\lambda) \in \mathrm{End}(\widetilde{V}_{d,\mathbf{F}_q})$, by

$$x(\lambda)(v \otimes 1) = \sum_{k \geqslant 0} \lambda^k \Big(\frac{e^k}{k!} v \otimes 1\Big)$$

and

$$y(\lambda)(v \otimes 1) = \sum_{k \geqslant 0} \lambda^k \Big(\frac{f^k}{k!} v \otimes 1\Big).$$

Since e and f act nilpotently on V_d this sum is well defined. There exists a unique homomorphism

$$\mathrm{SL}_2(\mathbf{F}_q) \longrightarrow \mathrm{End}(\widetilde{V}_{d,\mathbf{F}_q})$$

such that

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \longmapsto x(\lambda) \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \longmapsto y(\lambda).$$

This gives us a representation of Γ'. To ease the notation, we denote

$$m_{i,\mathbf{F}_q} = m_i \otimes 1.$$

We will refer to $\{m_{i,\mathbf{F}_q} : 0 \leqslant i \leqslant d\}$ as the standard basis of $\widetilde{V}_{d,\mathbf{F}_q}$. The action of Γ' is determined by

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} m_{i,\mathbf{F}_q} = \sum_{k=0}^{i} \binom{d-k}{d-i} \lambda^{i-k} m_{k,\mathbf{F}_q},$$

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} m_{i,\mathbf{F}_q} = \sum_{k=i}^{d} \binom{k}{i} \lambda^{k-i} m_{k,\mathbf{F}_q}.$$

This gives

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} m_{i,\mathbf{F}_q} = \lambda^{d-2i} m_{i,\mathbf{F}_q}.$$

At first we resolve the ambiguities in our notation.

LEMMA 4.2.4. — *Let $V_{d,\mathbf{F}_q}$ be a representation of Γ constructed in Section 3.2. Then*

$$V_{d,\mathbf{F}_q}|_{\Gamma'} \cong \widetilde{V}_{d,\mathbf{F}_q}.$$

Proof. — The isomorphism is given by

$$m_i \otimes 1 \longmapsto m_{i,\mathbf{F}_q}.$$

An easy check shows that the isomorphism respects the action of matrices $\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right)$, for all $\lambda \in \mathbf{F}_q$. Since, these matrices generate Γ' we are done. $\square$

The Lemma above is the reason, why we wanted to work over $\mathbf{F}_q$. We drop the tilde from our notation and go to $\overline{\mathbf{F}}_p$.

For each r, such that $0 \leqslant r < p-1$, Jeyakumar finds a Γ'-invariant subspace R_r of the representation $V_{p-1-r,\overline{\mathbf{F}}_p} \otimes V_{p-1,\overline{\mathbf{F}}_p}$, such that $\dim R_r = 2p$. Let $R_{p-1} = V_{p-1,\overline{\mathbf{F}}_p}$, then $\dim R_{p-1} = p$. The main result of [**10**] can be stated as follows.

Theorem 4.2.5 ([**10**]). — *Suppose that $q = p^n$. Let $\boldsymbol{r} = (r_0, \dots, r_{n-1})$ be an n-tuple, such that $0 \leqslant r_i \leqslant p-1$, for every i. Let*

$$R_{\boldsymbol{r}} = R_{r_0} \otimes R_{r_1}^{\mathrm{Fr}} \otimes \cdots \otimes R_{r_{n-1}}^{\mathrm{Fr}^{n-1}}.$$

If $\boldsymbol{r} \neq \mathbf{0}$, then

$$R_{\boldsymbol{r}} \cong \operatorname{inj} V_{\boldsymbol{r}, \overline{\mathbf{F}}_p}.$$

And

$$R_{\mathbf{0}} \cong \operatorname{inj} V_{\mathbf{0}, \overline{\mathbf{F}}_p} \oplus \operatorname{inj} V_{\boldsymbol{p-1}, \overline{\mathbf{F}}_p}$$

where $\boldsymbol{p} - \mathbf{1} = (p-1, \dots, p-1)$ and $\mathbf{0} = (0, \dots, 0)$.

Remark 4.2.6. — Our indices differ slightly from [**10**].

4.2.2. Going from $\mathrm{SL}_2(\mathbf{F}_q)$ to $\mathrm{GL}_2(\mathbf{F}_q)$. — We will recall how the subspaces R_r are constructed and show that they are in fact Γ-invariant. That this should be the case is indicated by Remark 4.2.3. The twisted tensor product will give us principal indecomposable representations of Γ. Since the spaces R_r have a rather concrete description, this will enable us to work out the corresponding $\mathcal{H}_\Gamma$-modules.

Lemma 4.2.7. — *Let V be a representation of Γ and let W be a Γ'-invariant subspace of V. If W is invariant under the action of H, then W is Γ-invariant.*

Proof. — Let $v \in W$ and $g \in \Gamma$. We may write $g = g' g_1$, for some $g' \in \Gamma'$ and $g_1 \in H$. Then

$$gv = g'(g_1 v) \in W.$$

Hence W is Γ-invariant. $\square$

Let r be an integer such that $0 \leqslant r \leqslant p-1$. Let $\{v_i\}$, for $0 \leqslant i \leqslant p-1-r$ be the standard basis of $V_{p-1-r, \overline{\mathbf{F}}_p}$ and let $\{w_j\}$, for $0 \leqslant j \leqslant p-1$ be the standard basis of $V_{p-1, \overline{\mathbf{F}}_p}$.

Definition 4.2.8. — For $0 \leqslant i \leqslant 2p-r-2$, we define vectors E_i in $V_{p-1-r, \overline{\mathbf{F}}_p} \otimes V_{p-1, \overline{\mathbf{F}}_p}$, by

$$E_i = \sum_{k+l=i} v_k \otimes w_l.$$

It is convenient to extend the indexing set to $\mathbb{Z}$ by setting $E_i = 0$, if $i < 0$ or $i > 2p - 2p - r$.

Lemma 4.2.9. — *The sequence of Γ-representations*

$$\begin{aligned} 0 \longrightarrow V_{2p-r-2, \overline{\mathbf{F}}_p} &\longrightarrow V_{p-1-r, \overline{\mathbf{F}}_p} \otimes V_{p-1, \overline{\mathbf{F}}_p} \\ m_{i, \overline{\mathbf{F}}_p} &\longmapsto E_i \end{aligned}$$

is exact.

Proof. — If $r = p-1$ then this is true trivially. If $r \neq p-1$ then the map is Γ'-equivariant by [**10**] Lemma 4.2. So by Lemma 4.2.7 it is enough to show that it is H-equivariant. Since

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} m_{i,\overline{\mathbf{F}}_p} = \lambda^{2p-r-2-i}\mu^i m_{i,\overline{\mathbf{F}}_p}$$

and

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} E_i = \lambda^{2p-r-2-i}\mu^i E_i$$

we are done. □

Definition 4.2.10 ([**10**]). — Let r be an integer, such that $0 \leqslant r < p-1$. For $0 \leqslant i \leqslant p-r-1$, let a_i be integers defined by the following relation:

$$a_0 = 0 \quad \text{and} \quad a_1 = (p-r-2)!$$

and

$$a_{i+1} = a_i + \frac{(-1)^i(r+1)\dots(r+i)}{(p-r-2)\dots(p-r-i-1)}(a_1 - a_0).$$

Let Z be a vector in $V_{p-1-r,\overline{\mathbf{F}}_p} \otimes V_{p-1,\overline{\mathbf{F}}_p}$ given by

$$Z = a_0(v_0 \otimes w_{p-r-1}) + a_1(v_1 \otimes w_{p-r-2}) + \cdots + a_{p-r-1}(v_{p-r-1} \otimes w_0),$$

and let R_r be a subspace of $V_{p-1-r,\overline{\mathbf{F}}_p} \otimes V_{p-1,\overline{\mathbf{F}}_p}$ given by

$$R_r = \left\langle E_0, \dots, E_{2p-r-2}, Z, \frac{f}{1!}Z, \dots, \frac{f^r}{r!}Z \right\rangle_{\overline{\mathbf{F}}_p}.$$

Moreover, for $r = p-1$ we define

$$R_{p-1} = V_{p-1,\overline{\mathbf{F}}_p}.$$

Proposition 4.2.11. — *Let r be an integer, such that $0 \leqslant r \leqslant p-1$, then R_r is a Γ-invariant subspace of $V_{p-r-1,\overline{\mathbf{F}}_p} \otimes V_{p-1,\overline{\mathbf{F}}_p}$. Moreover, if $r \neq p-1$, then*

$$\dim R_r = 2p$$

and if $r = p-1$, then

$$\dim R_{p-1} = p.$$

Proof. — If $r = p-1$ then there is nothing to prove, since $R_{p-1} = V_{p-1,\overline{\mathbf{F}}_p}$. If $r \neq p-1$ then by [**10**] Theorem 4.7 R_r is Γ'-invariant and $\dim R_r = 2p$. So by Lemma 4.2.7 it is enough to show that R_r is H-invariant. For $v \in V_{p-r-1,\overline{\mathbf{F}}_p}$ and $w \in V_{p-1,\overline{\mathbf{F}}_p}$ we have

$$f(v \otimes w) = fv \otimes w + v \otimes fw.$$

Hence, for $0 \leqslant k \leqslant r$ we have

$$\frac{f^k}{k!}Z \in \langle v_{l+i} \otimes w_{p-r-1-l+j} \mid i+j = k,\ 0 \leqslant l \leqslant p-r-1 \rangle_{\overline{\mathbf{F}}_p}$$

with the usual 'vanishing when not defined' convention. Since

$$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix} v_{l+i}\otimes w_{p-r-1-l+j} = \lambda^{p-r-1-l-i}\mu^{l+i}\lambda^{r+l-j}\mu^{p-r-1-l+j}v_{l+i}\otimes w_{p-r-1-l+j}$$
$$= \lambda^{p-k-1}\mu^{p-r-1+k}v_{l+i}\otimes w_{p-r-1-l+j}$$

the group H acts on each $\frac{f^k}{k!}Z$, for $0\leqslant k\leqslant r$ by a character. We combine this with Lemma 4.2.9 and obtain that R_r is H invariant. □

Lemma 4.2.12. — *We have*

$$\frac{f^k}{k!}E_{p-r-1} = 0$$

if and only if $k\geqslant r+1$*. For* $k\geqslant 1$ *we have*

$$\frac{e^k}{k!}E_{p-r-1} = 0.$$

In particular, U *fixes* E_{p-r-1} *and the action of* H *is given by*

$$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix} E_{p-r-1} = \lambda^r(\lambda\mu)^{p-r-1}E_{p-r-1}.$$

Proof. — If $r=p-1$, then this is trivial. If $r\neq p-1$ then for $k\geqslant 0$ we have

$$\frac{f^k}{k!}E_{p-r-1} = \binom{p-r-1+k}{p-r-1}E_{p-r-1+k}.$$

We observe that $E_{p-r-1+k}$ vanishes trivially, if $k\geqslant p$. If $r+1\leqslant k\leqslant p-1$, then we write $k=r+1+j$, where $0\leqslant j\leqslant p-r-2$. The binomial coefficient becomes

$$\binom{j+p}{p-r-1}.$$

Since $0\leqslant r<p-1$, we have $1\leqslant p-r-1\leqslant p-1$, and since $0\leqslant j<p-r-1$, p divides the binomial coefficient. Hence

$$\frac{f^k}{k!}E_{p-r-1} = 0$$

for $k\geqslant r+1$. If $0\leqslant k\leqslant r$, then $p-r-1\leqslant p-r-1+k\leqslant p-1$ and the binomial coefficient does not vanish. Hence

$$\frac{f^k}{k!}E_{p-r-1} \neq 0$$

for $0\leqslant k\leqslant r$. Let $k\geqslant 0$, then

$$\frac{e^k}{k!}E_{p-r-1} = \binom{p-1+k}{p-1}E_{p-r-1-k}.$$

We observe that $E_{p-r-1-k}$ vanishes trivially, if $k>p-r-1$. Suppose that $1\leqslant k\leqslant p-r-1$, then we may write $k=j-1$, where $0\leqslant j\leqslant p-r-2<p-1$. The binomial coefficient becomes

$$\binom{j+p}{p-1}.$$

Since $j < p-1$, p divides the binomial coefficient, and hence

$$\frac{e^k}{k!}E_{p-r-1} = 0$$

for all $k \geqslant 1$. Since the action of U is given in terms of $e^k/k!$ this implies that U fixes E_{p-r-1}. An easy verification gives us the action of H. ☐

Proposition 4.2.13. — *Let W_r be a subspace of R_r given by*

$$W_r = \langle E_{p-r-1}, \dots, E_{p-1}\rangle_{\overline{\mathbf{F}}_p}.$$

Then W_r is Γ-invariant. Moreover,

$$W_r^U = \langle E_{p-r-1}\rangle_{\overline{\mathbf{F}}_p}$$

and

$$W_r = \langle \Gamma E_{p-r-1}\rangle_{\overline{\mathbf{F}}_p} \cong V_{r,\overline{\mathbf{F}}_p} \otimes (\det)^{p-r-1}.$$

Proof. — If $r = p-1$ then $W_{p-1} \cong V_{p-1,\overline{\mathbf{F}}_p}$ and we are done. Otherwise, since W_r has a basis of eigenvectors for the action of H, it is enough to show that W_r is Γ'-invariant. Since the action of Γ' is given in terms of the action of $\mathcal{U}_{\mathbb{Z}}$ it is enough to show that W_r is invariant under the action of $\mathcal{U}_{\mathbb{Z}}$. Lemma 4.2.12 implies that W_r has a basis $\frac{f^k}{k!}E_{p-r-1}$, for $0 \leqslant k \leqslant r$. We observe that Lemma 4.2.12 also implies that

$$\frac{f^l}{l!}(\frac{f^k}{k!}E_{p-r-1}) = \binom{k+l}{k}\frac{f^{k+l}}{(k+l)!}E_{p-r-1} \in W_r$$

for $0 \leqslant k \leqslant r$ and $l \geqslant 0$. Suppose that $0 \leqslant k \leqslant r$ and $l \geqslant k+1$ then

$$\frac{e^l}{l!}(\frac{f^k}{k!}E_{p-r-1}) = 0.$$

This follows from the multiplication in $\mathcal{U}_{\mathbb{Z}}$, see [**9**] §26.2, and Lemma 4.2.12. If $0 \leqslant l \leqslant k \leqslant r$, then

$$\begin{aligned}\frac{e^l}{l!}(\frac{f^k}{k!}E_{p-r-1}) &= \binom{p-r-1+k}{p-r-1}\frac{e^l}{l!}E_{p-r-1+k}\\ &= \binom{p-r-1+k}{p-r-1}\binom{p-1-k+l}{p-1-k}E_{p-r-1+k-l} \in W_r.\end{aligned}$$

Hence W_r is invariant under the action of $\mathcal{U}_{\mathbb{Z}}$ and hence under the action of Γ.

We know from Lemma 4.2.12 that E_{p-r-1} is fixed by U. The action of H splits W_r^U into a direct sum of one dimensional subspaces. Suppose that $\dim W_r^U \geqslant 2$. Since H acts on each vector $E_{p-r-1+k}$ by a distinct character for $0 \leqslant k \leqslant r$, we must have $E_{p-r-1+j} \in W_r^U$, for some $1 \leqslant j \leqslant r$. This implies that

$$eE_{p-r-1+j} = (p-j)E_{p-r-2+j} = 0.$$

Hence p must divide j and this is impossible. Hence, $\dim W_r^U = 1$.

Since W_r is Γ-invariant, we have

$$\langle \Gamma E_{p-r-1}\rangle_{\overline{\mathbf{F}}_p} \leqslant W_r.$$

We may choose $r+1$ distinct elements λ_i in $\mathbf{F}_q$. Then

$$\begin{pmatrix} 1 & 0 \\ \lambda_i & 1 \end{pmatrix} E_{p-r-1} = \sum_{k=0}^{r} \lambda_i^k \frac{f^k}{k!} E_{p-r-1}.$$

Let A be an $(r+1)\times(r+1)$ matrix, given by $A_{ki} = \lambda_i^k$, for $0 \leqslant i, k \leqslant r$, with the convention that $0^0 = 1$. Then $\det A$ is the Vandermonde determinant, which is non-zero, since all the λ_i are distinct. Hence, A is invertible and

$$\frac{f^k}{k!} E_{p-r-1} \in \langle \Gamma E_{p-r-1}\rangle_{\overline{\mathbf{F}}_p}$$

for all $0 \leqslant k \leqslant r$. Hence, $W_r = \langle \Gamma E_{p-r-1}\rangle_{\overline{\mathbf{F}}_p}$.

Since $\dim W_r^U = 1$ and $W_r = \langle \Gamma W_r^U\rangle_{\overline{\mathbf{F}}_p}$, the representation W_r is irreducible. To decide, which one it is, we may proceed as in the proof of Proposition 3.2.2. Since $r < p-1$, the action of B on W_r^U implies that $W_r \cong V_{r,\overline{\mathbf{F}}_p} \otimes (\det)^{p-r-1}$. ☐

Lemma 4.2.14. — *The vector E_0 is fixed by the action of U. Moreover, H acts on E_0 by*

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} E_0 = \lambda^r (\lambda\mu)^{p-r-1} (\lambda\mu^{-1})^{p-r-1} E_0.$$

Proof. — Since $E_0 = v_0 \otimes w_0$ this is immediate. ☐

Definition 4.2.15. — Suppose that $q = p^n$ and let $\boldsymbol{r} = (r_0, \dots, r_{n-1})$ be the n-tuple such that $0 \leqslant r_i \leqslant p-1$, then we define a representation $R_{\boldsymbol{r}}$ of Γ, given by

$$R_{\boldsymbol{r}} = R_{r_0} \otimes R_{r_1}^{\mathrm{Fr}} \otimes \cdots \otimes R_{r_{n-1}}^{\mathrm{Fr}^{n-1}}$$

where R_{r_i} are Γ-representations of Definition 4.2.10.

Definition 4.2.16. — Suppose that $q = p^n$ and let $\boldsymbol{r} = (r_0, \dots, r_{n-1})$ be an n-tuple, such that $0 \leqslant r_i \leqslant p-1$, for every i. Let $\boldsymbol{\varepsilon} = (\varepsilon_0, \dots, \varepsilon_{n-1})$ be an n-tuple, such that $\varepsilon_i \in \{0,1\}$ for every i. We define a vector

$$b_{\boldsymbol{\varepsilon}} = E_{(1-\varepsilon_0)(p-1-r_0)} \otimes \cdots \otimes E_{(1-\varepsilon_{n-1})(p-1-r_{n-1})}$$

in $R_{\boldsymbol{r}}$, where $E_{(1-\varepsilon_i)(p-1-r_i)}$ is a vector in R_{r_i}, for each $0 \leqslant i \leqslant n-1$.

Definition 4.2.17. — Suppose that $q = p^n$ and let $\boldsymbol{r} = (r_0, \dots, r_{n-1})$ be an n-tuple, such that $0 \leqslant r_i \leqslant p-1$, for $0 \leqslant i \leqslant n-1$. We define $\Sigma_{\boldsymbol{r}}$ to be the set of n-tuples $(\varepsilon_0, \dots, \varepsilon_{n-1})$, such that

$$\varepsilon_i = 0, \quad \text{if } r_i = p-1 \text{ and } \quad \varepsilon_i \in \{0,1\}, \quad \text{otherwise.}$$

We will write $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$.

Remark 4.2.18. — We hope to prevent some notational confusion. Since we want Lemma 4.2.19 to hold and since $\dim R^U_{p-1,\overline{\mathbf{F}}_p} = 1$, if $r_i = p-1$, we have to make a choice for ε_i, between 0 and 1. We choose 0, since then we can state Lemma 4.2.21 in a nice way. However, if $r_i = p-1$, then

$$(1-0)(p-r_i-1) = (1-1)(p-r_i-1) = 0$$

so it does not matter, whether $\varepsilon_i = 0$ or $\varepsilon_i = 1$, and we will exploit this in our notation. We note that the definition of $b_{\boldsymbol{\varepsilon}}$ is independent of the set $\Sigma_{\boldsymbol{r}}$ and we might have $\boldsymbol{\varepsilon} \in \Sigma_{\boldsymbol{r}}$, $\boldsymbol{\varepsilon}' \notin \Sigma_{\boldsymbol{r}}$, but $b_{\boldsymbol{\varepsilon}} = b_{\boldsymbol{\varepsilon}'}$.

Lemma 4.2.19. — *The set $\{b_{\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} \in \Sigma_{\boldsymbol{r}}\}$ is a basis of $R^U_{\boldsymbol{r}}$.*

Proof. — Let r be an integer, such that $0 \leqslant r \leqslant p-1$. If $r = p-1$, then $\dim R_r = p$ and E_0 is in R^U_r. If $0 \leqslant r < p-1$, then $\dim R_r = 2p$ and E_0 and E_{p-1-r} are two linearly independent vectors in R^U_r.

Let $\boldsymbol{r}$ be an n-tuple. Then by above vectors $b_{\boldsymbol{\varepsilon}}$, for $\boldsymbol{\varepsilon} \in \Sigma_{\boldsymbol{r}}$, span a linear subspace of $R^U_{\boldsymbol{r}}$ of dimension $|\Sigma_{\boldsymbol{r}}|$. Also by above, $\dim R_{\boldsymbol{r}} = |\Sigma_{\boldsymbol{r}}|q$. Since, U is a p-Sylow subgroup of Γ' of order q and by Theorem 4.2.5 $R_{\boldsymbol{r}}$ is an injective object in $\mathrm{Rep}_{\Gamma'}$, Corollary 4.0.9 implies that

$$\dim R^U_{\boldsymbol{r}} = |\Sigma_{\boldsymbol{r}}|.$$

Hence, the set $\{b_{\boldsymbol{\varepsilon}} : \varepsilon \in \Sigma_{\boldsymbol{r}}\}$ is a basis of $R^U_{\boldsymbol{r}}$. □

Lemma 4.2.20. — *Let $\boldsymbol{r} = (r_0, \dots, r_{n-1})$ be an n-tuple, with $0 \leqslant r_i \leqslant p-1$, let $\boldsymbol{\varepsilon} = (\varepsilon_0, \dots, \varepsilon_{n-1})$ be an n-tuple such that $\varepsilon_i \in \{0,1\}$, for every i, and let $b_{\boldsymbol{\varepsilon}}$ be a vector in $R^U_{\boldsymbol{r}}$, then the action of H is given by*

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} b_{\boldsymbol{\varepsilon}} = \lambda^r (\lambda\mu)^{q-1-r} (\lambda\mu^{-1})^{\boldsymbol{\varepsilon}\boldsymbol{.}(\boldsymbol{p}-\boldsymbol{r}-\boldsymbol{1})} b_{\boldsymbol{\varepsilon}}$$

where $r = r_0 + r_1 p + \dots + r_{n-1}p^{n-1}$ and

$$\boldsymbol{\varepsilon}\boldsymbol{.}(\boldsymbol{p}-\boldsymbol{r}-\boldsymbol{1}) = \varepsilon_0(p-r_0-1) + \varepsilon_1(p-r_1-1)p + \dots + \varepsilon_{n-1}(p-r_{n-1}-1)p^{n-1}.$$

Proof. — This follows from Proposition 4.2.12 and Lemma 4.2.14. We note that the action on each tensor component is twisted by Fr. □

Lemma 4.2.21. — *Suppose that $q = p^n$ and let $\boldsymbol{r} = (r_0, \dots, r_{n-1})$ be an n-tuple, such that $0 \leqslant r_i \leqslant p-1$, for each i. Let $b_{\boldsymbol{0}}$ be a vector in $R_{\boldsymbol{r}}$. Let*

$$r = r_0 + r_1 p + \dots + r_{n-1}p^{n-1}.$$

Then

$$\langle \Gamma b_{\boldsymbol{0}} \rangle_{\overline{\mathbf{F}}_p} \cong V_{\boldsymbol{r},\overline{\mathbf{F}}_p} \otimes (\det)^{q-1-r}$$

as a Γ-representation.

Proof. — Let $W_{\boldsymbol{r}}$ be the subspace of $R_{\boldsymbol{r}}$ given by

$$W_{\boldsymbol{r}} = W_{r_0} \otimes \cdots \otimes W_{r_{n-1}}$$

with the notation of the Proposition 4.2.13. We have

$$0 \neq \langle \Gamma b_{\boldsymbol{0}} \rangle_{\overline{\mathbf{F}}_p} \leqslant W_{\boldsymbol{r}}.$$

Proposition 4.2.13 applied to every tensor component implies that

$$W_{\boldsymbol{r}} \cong V_{\boldsymbol{r},\overline{\mathbf{F}}_p} \otimes (\det)^{q-1-r}$$

which is irreducible. Hence, we must get the whole of $W_{\boldsymbol{r}}$. □

COROLLARY 4.2.22. — *Let $\chi : H \to \overline{\mathbf{F}}_p^{\times}$ and let a and r be unique integers, such that $1 \leqslant a, r \leqslant q-1$ and*

$$\chi\big(\big(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\big)\big) = \lambda^a \quad \forall \lambda \in \mathbf{F}_q^{\times}, \quad \chi\big(\big(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\big)\big) = \lambda^r \quad \forall \lambda \in \mathbf{F}_q^{\times}.$$

Let $r = r_0 + r_1 p + \cdots + r_{n-1}p^{n-1}$, where $0 \leqslant r_i \leqslant p-1$ for each i, and let $\boldsymbol{r} = (r_0, \ldots, r_{n-1})$. If $\chi \neq \chi^s$ then

$$\operatorname{inj} \rho_{\chi,\varnothing} \cong R_{\boldsymbol{r}} \otimes (\det)^{a+r}.$$

If $\chi = \chi^s$ then

$$\operatorname{inj} \rho_{\chi,\varnothing} \cong R_{\boldsymbol{p-1}} \otimes (\det)^a \cong V_{\boldsymbol{p-1},\overline{\mathbf{F}}_p} \otimes (\det)^a.$$

Proof. — Lemma 4.2.21 implies the existence of an exact sequence

$$0 \longrightarrow V_{\boldsymbol{r},\overline{\mathbf{F}}_p} \longrightarrow R_{\boldsymbol{r}} \otimes (\det)^r$$

of Γ-representations. It is enough to show that $R_{\boldsymbol{r}}$ is an indecomposable injective object in Rep_{Γ}. The rest follows from Propositions 3.2.2 and 4.0.4.

Theorem 4.2.5 says that the restriction of $R_{\boldsymbol{r}}$ to Γ' is indecomposable. In particular, $R_{\boldsymbol{r}}$ must be indecomposable as a Γ-representation. Moreover, Theorem 4.2.5 says that the restrictions of $R_{\boldsymbol{r}}$ to Γ' is an injective object in $\mathrm{Rep}_{\Gamma'}$. Since U is a p-Sylow subgroup of both Γ and Γ', Proposition 4.0.7 implies that $R_{\boldsymbol{r}}$ is an injective object in Rep_{Γ}. Finally, the last isomorphism follows directly from the definition of $R_{\boldsymbol{p-1}}$. □

4.2.3. Computation of $\mathcal{H}_{\Gamma}$-modules. — We will compute the action of T_{n_s} on $R_{\boldsymbol{r}}^U$.

PROPOSITION 4.2.23. — *Let $q = p^n$ and let $\boldsymbol{r} = (r_0, \ldots, r_{n-1})$ be the n-tuple, such that $0 \leqslant r_i \leqslant p-1$, for every i. Let $\boldsymbol{\varepsilon} \in \Sigma_{\boldsymbol{r}}$ and let $b_{\boldsymbol{\varepsilon}}$ be a vector in $R_{\boldsymbol{r}}$.*

(i) *Suppose that for some index j, $\varepsilon_j = 0$ and $r_j \neq p-1$ then*

$$\sum_{u \in U} u n_s^{-1} b_{\boldsymbol{\varepsilon}} = 0.$$

(ii) *Suppose that* $\boldsymbol{r} \neq \mathbf{0}$. *Moreover, suppose that for every* i, *if* $\varepsilon_i = 0$ *then* $r_i = p-1$ *then* $b_{\boldsymbol{\varepsilon}} = b_{\mathbf{1}}$ *and*

$$\sum_{u \in U} u n_s^{-1} b_{\mathbf{1}} = (-1)^{1+|\boldsymbol{r}|} b_{\mathbf{0}}$$

where $|\boldsymbol{r}| = r_0 + r_1 p + \cdots + r_{n-1} p^{n-1}$.

(iii) *Suppose that* $\boldsymbol{r} = \mathbf{0}$ *and* $\boldsymbol{\varepsilon} = \mathbf{1}$, *then*

$$\sum_{u \in U} u n_s^{-1} b_{\mathbf{1}} = -(b_{\mathbf{0}} + b_{\mathbf{1}}).$$

This covers all the possible pairs $(\boldsymbol{r}, \varepsilon)$, *such that* $\varepsilon \in \Sigma_{\boldsymbol{r}}$.

Remark 4.2.24. — We note that $b_{\mathbf{1}}$ is well defined even if $\mathbf{1} \notin \Sigma_{\boldsymbol{r}}$. See Definitions 4.2.16 and 4.2.17.

Proof. — Let r be an integer such that $0 \leqslant r \leqslant p-1$ and let $\varepsilon \in \{0,1\}$ such that $\varepsilon = 0$, if $r = p-1$. Let $E_{(1-\varepsilon)(p-r-1)}$ be a vector in R_r. We observe that

$$n_s^{-1} E_{(1-\varepsilon)(p-r-1)} = (-1)^{p-1+\varepsilon(p-r-1)} E_{p-1+\varepsilon(p-r-1)}.$$

If $r \neq p-1$ this follows from Lemma 4.2.9, and if $r = p-1$, this follows from the isomorphism $R_{p-1} \cong V_{p-1,\overline{\mathbf{F}}_p}$. Moreover, if $\varepsilon = 0$ then Proposition 4.2.13 implies that

$$\frac{e^k}{k!} E_{p-1} = 0 \quad \forall\, k > r$$

and if $\varepsilon = 1$ then Lemma 4.2.9 implies that

$$\frac{e^k}{k!} E_{2p-2-r} = 0 \quad \forall\, k > 2p-2-r.$$

Let $\boldsymbol{r}$ be an n-tuple, $\boldsymbol{r} = (r_0, \ldots, r_{n-1})$, with $0 \leqslant r_i \leqslant p-1$, let $\boldsymbol{\varepsilon} \in \Sigma_{\boldsymbol{r}}$. We recall that

$$b_{\boldsymbol{\varepsilon}} = E_{(1-\varepsilon_0)(p-1-r_0)} \otimes \cdots \otimes E_{(1-\varepsilon_{n-1})(p-1-r_{n-1})}.$$

Hence we may write

$$\sum_{u \in U} u n_s^{-1} b_{\boldsymbol{\varepsilon}} = \operatorname{sgn}_{\boldsymbol{r},\varepsilon} \sum_{\boldsymbol{k}} \Lambda_{\boldsymbol{k}} A_{\boldsymbol{k}}$$

where the sum is taken over all the n-tuples $\boldsymbol{k} = (k_0, \ldots, k_{n-1})$ of non-negative integers, moreover

$$\Lambda_{\boldsymbol{k}} = \sum_{\lambda \in \mathbf{F}_q} \lambda^{k_0 + k_1 p + \cdots + k_{n-1} p^{n-1}}$$

and

$$A_{\boldsymbol{k}} = \frac{e^{k_0}}{k_0!} E_{p-1+\varepsilon_0(p-r_0-1)} \otimes \cdots \otimes \frac{e^{k_{n-1}}}{k_{n-1}!} E_{p-1+\varepsilon_{n-1}(p-r_{n-1}-1)}$$

and

$$\operatorname{sgn}_{\boldsymbol{r},\varepsilon} = (-1)^{q-1-\boldsymbol{\varepsilon} \centerdot (\boldsymbol{p}-\mathbf{1}-\boldsymbol{r})}$$

where $\boldsymbol{\varepsilon} \centerdot (\boldsymbol{p} - \mathbf{1} - \boldsymbol{r}) = \sum_{i=0}^{n-1} \varepsilon_i (p-1-r_i) p^i$. We have acted by n_s^{-1} on each tensor component and then expanded the action of $u \in U$ on each tensor component and rearranged the summation. We will show that the sum on the right hand side vanishes, unless $\boldsymbol{r}$ and $\boldsymbol{\varepsilon}$ are of a special form. We will break up the argument into several lemmas.

LEMMA 4.2.25. — *Let $S(\boldsymbol{r},\varepsilon)$ be the subset of the set of all the n-tuples of non-negative integers $\boldsymbol{k}=(k_0,\dots,k_{n-1})$, such that $\boldsymbol{k}\in S(\boldsymbol{r},\varepsilon)$ if and only if for each i, the following holds:*

(i) *if $\varepsilon_i=0$ then $k_i=r_i$,*
(ii) *if $\varepsilon_i=1$ then $k_i=p-1$ or $k_i=2p-r_i-2$.*

Then

$$\sum_{u\in U} un_s^{-1}b_{\boldsymbol{\varepsilon}}=\operatorname{sgn}_{\boldsymbol{r},\varepsilon}\sum_{\boldsymbol{k}\in S(\boldsymbol{r},\varepsilon)}\Lambda_{\boldsymbol{k}}A_{\boldsymbol{k}}.$$

Proof of Lemma. — If $\varepsilon_i=0$ and $k_i>r_i$ or $\varepsilon_i=1$ and $k_i>2p-2-r_i$ then $A_{\boldsymbol{k}}=0$, since the i-th tensor component of $A_{\boldsymbol{k}}$ vanishes by the argument above. Moreover, Lemma 4.2.9 implies that

$$\frac{e^{k_i}}{k_i!}E_{p-1+\varepsilon_i(p-r_i-1)}\in\langle E_{p-1+\varepsilon_i(p-r_i-1)-k_i}\rangle_{\overline{\mathbf{F}}_p}.$$

The vector $\sum_{u\in U}un_s^{-1}b_{\boldsymbol{\varepsilon}}$ is fixed by U. Since, by Lemma 4.2.19, vectors $b_{\boldsymbol{\varepsilon}'}$, for $\boldsymbol{\varepsilon}'\in\Sigma_{\boldsymbol{r}}$, form a basis of $R_{\boldsymbol{r}}^U$ there exist scalars $\mu_{\varepsilon'}\in\overline{\mathbf{F}}_p$ such that

$$\sum_{u\in U}un_s^{-1}b_\varepsilon=\sum_{\varepsilon'\in\Sigma_{\boldsymbol{r}}}\mu_{\varepsilon'}b_{\varepsilon'}.$$

Hence, it is enough to sum over the n-tuples $\boldsymbol{k}$ of non-negative integers such that, for each i, we have

$$p-1+\varepsilon_i(p-r_i-1)-k_i=(1-\varepsilon_i')(p-1-r_i)$$

for some $\varepsilon_i'\in\{0,1\}$. Hence, k_i is of the form

$$k_i=p-1+(\varepsilon_i-1)(p-r_i-1)\quad\text{or}\quad k_i=p-1+\varepsilon_i(p-r_i-1).$$

If $\varepsilon_i=0$ and k_i is of the form as above then the inequality $k_i\leqslant r_i$ can be fulfilled if and only if $k_i=r_i$. If $\varepsilon_i=1$, then $k_i\leqslant 2p-r_i-2$ implies that $k_i=p-1$ or $k_i=2p-r_i-2$. □

LEMMA 4.2.26. — *Let $\boldsymbol{k}\in S(\boldsymbol{r},\varepsilon)$ and let $k=k_0+k_1p+\cdots+k_{n-1}p^{n-1}$. Suppose that $\Lambda_{\boldsymbol{k}}\neq 0$ then one of the following holds:*

(i) *$\boldsymbol{r}=\boldsymbol{0}$, $\varepsilon=\boldsymbol{1}$ and $\boldsymbol{k}=(2(p-1),\dots,2(p-1))=2(\boldsymbol{p}-\boldsymbol{1})$,*
(ii) *$k=q-1$.*

Proof of Lemma. — Since $\boldsymbol{k}\in S(\boldsymbol{r},\varepsilon)$, for each i we have the inequalities:

$$0\leqslant k_i\leqslant 2p-2-r_i\leqslant 2(p-1).$$

Hence, $0 \leqslant k \leqslant 2(q-1)$, moreover $k = 2(q-1)$ if and only if $\boldsymbol{r} = \mathbf{0}$, $\varepsilon = \mathbf{1}$ and $\boldsymbol{k} = (2(p-1), \dots, 2(p-1))$. If $k = 0$ or $k > 0$ and $q-1$ does not divide k then

$$\Lambda_{\boldsymbol{k}} = \sum_{\lambda \in \mathbf{F}_q} \lambda^k = 0.$$

We note that $0^0 = 1$ comes from the action by the identity matrix. If $k > 0$ and $q-1$ divides k, then

$$\Lambda_{\boldsymbol{k}} = \sum_{\lambda \in \mathbf{F}_q} \lambda^k = -1. \qquad \square$$

LEMMA 4.2.27. — *Let $\boldsymbol{k} \in S(\boldsymbol{r}, \varepsilon)$ and let $k = k_0 + k_1 p + \cdots + k_{n-1} p^{n-1}$. Suppose that $k = q-1$ then $\boldsymbol{k} = (p-1, \dots, p-1) = \boldsymbol{p} - \mathbf{1}$.*

Proof of Lemma. — Since $\boldsymbol{k} \in S(\boldsymbol{r}, \varepsilon)$ we may define integers a_i and a'_i, such that for each i,

$$a_i + a'_i = k_i$$

and $0 \leqslant a_i, a'_i \leqslant p-1$, as follows. If $\varepsilon_i = 0$, then $a_i = r_i$ and $a'_i = 0$. If $\varepsilon_i = 1$ and $k_i = p-1$, then $a_i = p-1$ and $a'_i = 0$. If $\varepsilon_i = 1$ and $k_i = 2p - r_i - 2$, then $a_i = p-1$ and $a'_i = p - 1 - r_i$. Then $q - 1 = k$ implies that

$$a_0 + a_1 p + \cdots + a_{n-1} p^{n-1} = (p - 1 - a'_0) + (p - 1 - a'_1) p + \cdots + (p - 1 - a'_{n-1}) p^{n-1}.$$

Since $0 \leqslant a_i, a'_i \leqslant p-1$, for every i, this implies that

$$a_i = p - 1 - a'_i, \quad \forall\, i.$$

If $\varepsilon_i = 1$ and $k_i = p-1$, then we are done. If $\varepsilon_i = 0$ then by definition $a'_i = 0$ and by above $a_i = p-1$, hence $k_i = a_i + a'_i = p-1$. If $\varepsilon_i = 1$ and $k_i = 2p - 2 - r_i$ then by definition $a_i = p-1$ and by above $a'_i = 0$, hence $k_i = a_i + a'_i = p - 1$. $\square$

We return to the main body of the proof of Proposition 4.2.23.

Suppose that for some index j, we have $\varepsilon_j = 0$ and $r_j \neq p-1$. If $\Lambda_{\boldsymbol{k}} \neq 0$ for some $\boldsymbol{k} \in S(\boldsymbol{r}, \varepsilon)$, then Lemmas 4.2.26 and 4.2.27 imply that either $\boldsymbol{k} = \boldsymbol{p} - \mathbf{1}$ or $\boldsymbol{k} = 2(\boldsymbol{p} - \mathbf{1})$. However, the definition of $S(\boldsymbol{r}, \varepsilon)$ implies that $k_j = r_j < p - 1$. Hence $\Lambda_{\boldsymbol{k}} = 0$ for all $\boldsymbol{k} \in S(\boldsymbol{r}, \varepsilon)$ and Lemma 4.2.25 implies that

$$\sum_{u \in U} u n_s^{-1} b_\varepsilon = 0.$$

So we obtain part (i) of the Proposition. We note that this case includes $\boldsymbol{r} = \mathbf{0}$ and $\boldsymbol{\varepsilon} \neq \mathbf{1}$.

Suppose that $\boldsymbol{r} \neq \mathbf{0}$. Moreover, suppose that $\boldsymbol{r}$ and ε are such that for every i, if $\varepsilon_i = 0$ then $r_i = p-1$. Lemmas 4.2.26 and 4.2.27 imply that if $\boldsymbol{k} \in S(\boldsymbol{r}, \varepsilon)$ and $\Lambda_{\boldsymbol{k}} \neq 0$ then $\boldsymbol{k} = \boldsymbol{p} - \mathbf{1}$. Lemma 4.2.25 implies that

$$\sum_{u \in U} u n_s^{-1} b_\varepsilon = \operatorname{sgn}_{\boldsymbol{r}, \varepsilon}(-1) A_{\boldsymbol{p} - \mathbf{1}}.$$

We will compute what happens on each tensor component of $A_{\boldsymbol{p-1}}$. If $\varepsilon_i = 0$, then by our assumption on $\boldsymbol{r}$ and ε, we have $r_i = p-1$ and

$$\frac{e^{p-1}}{(p-1)!}E_{p-1} = \binom{p-1}{0} E_0 = E_{p-r_i-1}.$$

If $\varepsilon_i = 1$ then

$$\frac{e^{p-1}}{(p-1)!}E_{2p-2-r_i} = \binom{p-1}{0} E_{p-r_i-1} = E_{p-r_i-1}.$$

The above calculation gives us

$$A_{\boldsymbol{p-1}} = E_{p-r_0-1} \otimes \cdots \otimes E_{p-r_{n-1}-1} = b_{\boldsymbol{0}}.$$

Moreover, if $p=2$, then $1=-1$ and if $p \neq 2$ then $(-1)^{p-1+\varepsilon_i(p-1-r_i)} = (-1)^{r_i}$, trivially, if $\varepsilon_i = 1$ and since $r_i = p-1$ if $\varepsilon_i = 0$. Hence, $\operatorname{sgn}_{\boldsymbol{r},\varepsilon} = (-1)^{|\boldsymbol{r}|}$ and

$$\sum_{u \in U} u n_s^{-1} b_{\boldsymbol{\varepsilon}} = (-1)^{|\boldsymbol{r}|+1} b_{\boldsymbol{0}}.$$

We claim that in this case $b_{\boldsymbol{\varepsilon}} = b_{\boldsymbol{1}}$. Indeed, if $r_i \neq p-1$ then our assumption on $\boldsymbol{r}$ and ε implies that $\varepsilon_i = 1$ and if $r_i = p-1$, then

$$(1-\varepsilon_i)(p-1-r_i) = (1-1)(p-1-r_i) = 0.$$

Hence, $b_\varepsilon = b_{\boldsymbol{1}}$, see 4.2.16. This establishes part (ii) of the Proposition.

The only case left is $\boldsymbol{r} = \boldsymbol{0}$ and $\boldsymbol{\varepsilon} = \boldsymbol{1}$. Arguing as before we get that

$$\sum_{u \in U} u n_s^{-1} b_{\boldsymbol{1}} = \operatorname{sgn}_{\boldsymbol{0},\boldsymbol{1}}(-1)(A_{\boldsymbol{p-1}} + A_{2(\boldsymbol{p-1})}).$$

We compute what happens on each tensor component of $A_{2(\boldsymbol{p-1})}$:

$$\frac{e^{2p-2}}{(2p-2)!}E_{2p-2} = \binom{2p-2}{0} E_0 = E_0.$$

And by Definition 4.2.16, $b_{\boldsymbol{1}} = E_0 \otimes \cdots \otimes E_0$. Since $\operatorname{sgn}_{\boldsymbol{0},\boldsymbol{1}} = 1$ we get

$$\sum_{u \in U} u n_s^{-1} b_{\boldsymbol{1}} = -(b_{\boldsymbol{1}} + b_{\boldsymbol{0}}).$$

This establishes part (iii) of the Proposition. ☐

Remark 4.2.28. — We think of $\otimes(\det)^a$ as a twist, that is, it changes the action, but does not change the underlying vector space. Moreover, since $U \leqslant \Gamma'$ and $n_s \in \Gamma'$, Proposition 4.2.23 does not change if we twist the action by $(\det)^a$.

Remark 4.2.29. — We know that something like

$$\sum_{u \in U} u n_s^{-1} b_{\boldsymbol{1}} = (-1)^{1+|\boldsymbol{r}|} b_{\boldsymbol{0}}$$

has to happen by Lemma 4.1.1.

Lemma 4.2.30. — *Let $b_{\mathbf{1}}$ and $b_{\mathbf{0}}$ be vectors in $R_{\mathbf{0}}$. Then*

$$\langle \Gamma(b_{\mathbf{1}}+b_{\mathbf{0}})\rangle_{\overline{\mathbf{F}}_p} \cong V_{\boldsymbol{p-1},\overline{\mathbf{F}}_p}.$$

Proof. — The vector $b_{\mathbf{1}}+b_{\mathbf{0}}$ is fixed by U. Moreover, by Lemma 4.2.20 H acts trivially on it. By Proposition 4.2.23

$$(b_{\mathbf{1}}+b_{\mathbf{0}})T_{n_s} = \sum_{u\in U} un_s^{-1}(b_{\mathbf{1}}+b_{\mathbf{0}}) = -(b_{\mathbf{1}}+b_{\mathbf{0}}).$$

Hence

$$\langle b_{\mathbf{1}}+b_{\mathbf{0}}\rangle_{\overline{\mathbf{F}}_p} \cong M_{\mathbf{1},\varnothing}$$

as $\mathcal{H}_\Gamma$-module and Lemma 3.1.8 gives us the result. $\square$

Corollary 4.2.31. — *Let $\chi : H \to \overline{\mathbf{F}}_p^\times$ be a character, such that $\chi = \chi^s$ and let a be the unique integer, such that $1 \leqslant a \leqslant q-1$ and*

$$\chi\big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda\end{smallmatrix}\right)\big) = \lambda^a \quad \forall\, \lambda \in \mathbf{F}_q^\times$$

then

$$\operatorname{inj}\rho_{\chi,S} \oplus \operatorname{inj}\rho_{\chi,\varnothing} \cong R_{\mathbf{0}} \otimes (\det)^a.$$

Proof. — This is a rerun of the proof of Corollary 4.2.22. Lemma 4.2.21 and Lemma 4.2.30 imply the existence of an exact sequence

$$0 \longrightarrow V_{\mathbf{0},\overline{\mathbf{F}}_p} \oplus V_{\boldsymbol{p-1},\overline{\mathbf{F}}_p} \longrightarrow R_{\mathbf{0}}$$

of Γ-representations. So it is enough to show that $R_{\mathbf{0}}$ is an injective object in Rep_Γ and that it has at most 2 direct summands. The rest follows from Proposition 4.0.4 and Proposition 3.2.2. Theorem 4.2.5 says that the restriction of $R_{\mathbf{0}}$ to Γ' has exactly 2 direct summands, hence $R_{\mathbf{0}}$ may have at most 2 direct summands. Moreover, Theorem 4.2.5 says that the restriction of $R_{\mathbf{0}}$ to Γ' is an injective object in $\mathrm{Rep}_{\Gamma'}$. Since U is a p-Sylow subgroup of Γ and Γ' contains U, Proposition 4.0.7 implies that $R_{\mathbf{0}}$ is an injective object in Rep_Γ. $\square$

Definition 4.2.32. — Let $\alpha : H \to \overline{\mathbf{F}}_p^\times$ be a character, given by

$$\alpha : \begin{pmatrix} \lambda & 0 \\ 0 & \mu\end{pmatrix} \longmapsto \lambda\mu^{-1}.$$

Lemma 4.2.33. — *Suppose that $q = p^n$ and let $\chi : H \to \overline{\mathbf{F}}_p^\times$ be a character. Let r be the unique integer, such that $0 \leqslant r < q-1$ and*

$$\chi\big(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1}\end{smallmatrix}\right)\big) = \lambda^r \quad \forall\, \lambda \in \mathbf{F}_q^\times.$$

Let $r = r_0 + r_1 p + \cdots + r_{n-1}p^{n-1}$, where $0 \leqslant r_i \leqslant p-1$ for each i, and let

$$\boldsymbol{r} = (r_0, \ldots, r_{n-1}).$$

Let $\boldsymbol{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_{n-1})$ be an n-tuple, such that $\varepsilon_i \in \{0,1\}$ for every i, then

$$(\chi\alpha^{\boldsymbol{\varepsilon}\boldsymbol{\cdot}(\boldsymbol{p-1-r})})^s = \chi\alpha^{(\mathbf{1}-\boldsymbol{\varepsilon})\boldsymbol{\cdot}(\boldsymbol{p-1-r})}.$$

Moreover, if $\boldsymbol{r} = \mathbf{0}$, then we suppose that $\boldsymbol{\varepsilon} \neq \mathbf{0}$ and $\boldsymbol{\varepsilon} \neq \mathbf{1}$, then

$$(\chi\alpha^{\boldsymbol{\varepsilon}\boldsymbol{.}(\boldsymbol{p}-\mathbf{1}-\boldsymbol{r})})^s \neq \chi\alpha^{\boldsymbol{\varepsilon}\boldsymbol{.}(\boldsymbol{p}-\mathbf{1}-\boldsymbol{r})}$$

where $\boldsymbol{\varepsilon}\boldsymbol{.}(\boldsymbol{p}-\mathbf{1}-\boldsymbol{r}) = \sum_{i=0}^{n-1} \varepsilon_i(p-r_i-1)p^i$.

Proof. — Since twisting by s does not affect det we may assume that

$$\chi\big(\big(\begin{smallmatrix}\lambda & 0\\ 0 & \mu\end{smallmatrix}\big)\big) = \lambda^r \quad \forall\, \lambda, \mu \in \mathbf{F}_q^\times.$$

Then the first part of the lemma amounts to

$$\mu^r(\mu\lambda^{-1})^{\boldsymbol{\varepsilon}\boldsymbol{.}(\boldsymbol{p}-\mathbf{1}-\boldsymbol{r})} = \lambda^r(\lambda\mu^{-1})^{q-1-r-\boldsymbol{\varepsilon}\boldsymbol{.}(\boldsymbol{p}-\mathbf{1}-\boldsymbol{r})} = \lambda^r(\lambda\mu^{-1})^{(\mathbf{1}-\boldsymbol{\varepsilon})\boldsymbol{.}(\boldsymbol{p}-\mathbf{1}-\boldsymbol{r})}.$$

For the second part, we observe that the equality holds if and only if

$$\mu^{r+2\boldsymbol{\varepsilon}\boldsymbol{.}(\boldsymbol{p}-\mathbf{1}-\boldsymbol{r})} = \lambda^{r+2\boldsymbol{\varepsilon}\boldsymbol{.}(\boldsymbol{p}-\mathbf{1}-\boldsymbol{r})}$$

for every $\lambda, \mu \in \mathbf{F}_q^\times$. Hence, equality holds if and only if

$$\sum_{i=0}^{n-1}(r_i + 2(p-1-r_i)\varepsilon_i)p^i \equiv 0 \pmod{q-1}.$$

Since, $\varepsilon_i \in \{0,1\}$ we have

$$0 \leqslant r_i + 2(p-1-r_i)\varepsilon_i \leqslant 2(p-1).$$

The congruence implies that $r + 2\boldsymbol{\varepsilon}\boldsymbol{.}(\boldsymbol{p}-\mathbf{1}-\boldsymbol{r})$ must take values $0, q-1$ or $2(q-1)$. The extreme values are obtained if and only if $\boldsymbol{r} = \mathbf{0}$ and $\varepsilon = \mathbf{0}$ or $\boldsymbol{r} = \mathbf{0}$ and $\varepsilon = \mathbf{1}$. By our assumptions, both cases are excluded. If

$$r + 2\boldsymbol{\varepsilon}\boldsymbol{.}(\boldsymbol{p}-\mathbf{1}-\boldsymbol{r}) = q-1$$

then we rewrite this as

$$\sum_{i=0}^{n-1}(p-1-r_i)\varepsilon_i p^i = \sum_{i=0}^{n-1}(p-1-r_i)(1-\varepsilon_i)p^i.$$

Hence, for every i we must have

$$(p-1-r_i)\varepsilon_i = (p-1-r_i)(1-\varepsilon_i).$$

Since $2\varepsilon_i \neq 1$, for every i, this forces $r_i = p-1$, for every i, but $r < q-1$, hence this case is also excluded. □

Definition 4.2.34. — Suppose that $q = p^n$ and let $\boldsymbol{r} = (r_0, \ldots, r_{n-1})$ be an n-tuple, such that $0 \leqslant r_i \leqslant p-1$ for every i. We define

$$\delta \in \Sigma_{\boldsymbol{r}}$$

given by $\delta_i = 1$ if $r_i \neq p-1$ and $\delta_i = 0$ if $r_i = p-1$.

We further define $\Sigma'_{\boldsymbol{r}}$ to be a subset of $\Sigma_{\boldsymbol{r}}$ given by

$$\Sigma'_{\boldsymbol{r}} = \Sigma_{\boldsymbol{r}} \backslash \{\mathbf{0}, \delta\}.$$

Remark 4.2.35. — We note that if $p = q$ or $\boldsymbol{r} = (p-1, \ldots, p-1)$, then $\Sigma'_{\boldsymbol{r}} = \varnothing$ and we always have $b_\delta = b_{\mathbf{1}}$.

Lemma 4.2.36. — *Suppose that $q = p^n$ and let $\chi : H \to \overline{\mathbf{F}}_p^\times$ be a character. Let r be the unique integer, such that $0 \leqslant r < q-1$ and*

$$\chi\left(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)\right) = \lambda^r \quad \forall\, \lambda \in \mathbf{F}_q^\times.$$

Let $r = r_0 + r_1 p + \cdots + r_{n-1}p^{n-1}$, where $0 \leqslant r_i \leqslant p-1$ for each i, and let

$$\boldsymbol{r} = (r_0, \ldots, r_{n-1}).$$

If $r = 0$ then we consider $\operatorname{inj} \rho_{\chi,S}$ and if $r \neq 0$ we consider $\operatorname{inj} \rho_{\chi,\varnothing}$ as representations of K on which K_1 acts trivially.

Suppose that $\boldsymbol{\varepsilon} \in \Sigma'_{\boldsymbol{r}}$. If $r = 0$ then we regard $b_{\boldsymbol{\varepsilon}}$ and $b_{\mathbf{1}-\boldsymbol{\varepsilon}}$ as vectors in $(\operatorname{inj} \rho_{\chi,S})^{I_1}$ via the isomorphism of Corollary 4.2.31. If $r \neq 0$ then we regard $b_{\boldsymbol{\varepsilon}}$ and $b_{\mathbf{1}-\boldsymbol{\varepsilon}}$ as vectors in $(\operatorname{inj} \rho_{\chi,\varnothing})^{I_1}$ via the isomorphism of Corollary 4.2.22.

Then the action of $\mathcal{H}_K$ on $\langle b_{\boldsymbol{\varepsilon}}, b_{\mathbf{1}-\boldsymbol{\varepsilon}}\rangle_{\overline{\mathbf{F}}_p}$ extends to the action of $\mathcal{H}$, so that

$$\langle b_{\boldsymbol{\varepsilon}}, b_{\mathbf{1}-\boldsymbol{\varepsilon}}\rangle_{\overline{\mathbf{F}}_p} \cong M_{\gamma_{\boldsymbol{\varepsilon}}}$$

as an $\mathcal{H}$-module, where

$$\gamma_{\boldsymbol{\varepsilon}} = \gamma_{\mathbf{1}-\boldsymbol{\varepsilon}} = \{\chi\alpha^{\boldsymbol{\varepsilon}\boldsymbol{.}(\boldsymbol{p}-\mathbf{1}-\boldsymbol{r})}, (\chi\alpha^{\boldsymbol{\varepsilon}\boldsymbol{.}(\boldsymbol{p}-\mathbf{1}-\boldsymbol{r})})^s\}.$$

Proof. — To ease the notation, let

$$\psi = \chi\alpha^{\boldsymbol{\varepsilon}\boldsymbol{.}(\boldsymbol{p}-\mathbf{1}-\boldsymbol{r})}.$$

We observe that if $b_{\mathbf{1}-\boldsymbol{\varepsilon}} = b_{\mathbf{0}}$, then $\boldsymbol{\varepsilon} = \delta$ and if $b_{\mathbf{1}-\boldsymbol{\varepsilon}} = b_\delta$ then $\boldsymbol{\varepsilon} = \mathbf{0}$. Since $\boldsymbol{\varepsilon} \in \Sigma'_{\boldsymbol{r}}$ neither of the above can occur.

By Lemma 4.2.20 and taking into account the twist by a power of det, I acts on $b_{\boldsymbol{\varepsilon}}$ via the character ψ. By the same argument and Lemma 4.2.33 I acts on $b_{\mathbf{1}-\boldsymbol{\varepsilon}}$ via the character ψ^s. Hence,

$$b_{\boldsymbol{\varepsilon}} e_\psi = b_{\boldsymbol{\varepsilon}} \quad \text{and} \quad b_{\mathbf{1}-\boldsymbol{\varepsilon}} e_{\psi^s} = b_{\mathbf{1}-\boldsymbol{\varepsilon}}.$$

Moreover, Lemma 4.2.33 says that $\psi \neq \psi^s$. The case $\boldsymbol{r} = \mathbf{0}$ is not a problem, since $\boldsymbol{\varepsilon} \in \Sigma'_{\mathbf{0}}$ implies that $\mathbf{1} - \boldsymbol{\varepsilon} \in \Sigma'_{\mathbf{0}}$. Since H acts on $b_{\boldsymbol{\varepsilon}}$ and $b_{\mathbf{1}-\boldsymbol{\varepsilon}}$ by different characters, they are linearly independent. Proposition 4.2.23 implies that

$$b_{\boldsymbol{\varepsilon}} T_{n_s} = \sum_{u \in I_1/K_1} u n_s^{-1} b_{\boldsymbol{\varepsilon}} = 0 \quad \text{and} \quad b_{\mathbf{1}-\boldsymbol{\varepsilon}} T_{n_s} = \sum_{u \in I_1/K_1} u n_s^{-1} b_{\mathbf{1}-\boldsymbol{\varepsilon}} = 0.$$

Hence, by Lemma 2.2.2

$$\langle b_{\boldsymbol{\varepsilon}}, b_{\mathbf{1}-\boldsymbol{\varepsilon}}\rangle_{\overline{\mathbf{F}}_p} \cong \langle b_{\boldsymbol{\varepsilon}}\rangle_{\overline{\mathbf{F}}_p} \oplus \langle b_{\mathbf{1}-\boldsymbol{\varepsilon}}\rangle_{\overline{\mathbf{F}}_p} \cong M_{\psi,\varnothing} \oplus M_{\psi^s,\varnothing} \cong M_{\gamma_{\boldsymbol{\varepsilon}}}|_{\mathcal{H}_K}$$

as $\mathcal{H}_K$-modules. So we define

$$b_{\boldsymbol{\varepsilon}} T_\Pi = b_{\mathbf{1}-\boldsymbol{\varepsilon}} \quad \text{and} \quad b_{\mathbf{1}-\boldsymbol{\varepsilon}} T_\Pi = b_{\boldsymbol{\varepsilon}}$$

which gives us the required isomorphism of $\mathcal{H}$-modules. $\square$

Proposition 4.2.37. — *Suppose that $q = p^n$ and let $\chi : H \to \overline{\mathbf{F}}_p^{\times}$ be a character, such that $\chi = \chi^s$. We consider the representation*

$$\operatorname{inj} \rho_{\chi,\varnothing} \oplus \operatorname{inj} \rho_{\chi,S}$$

as a representation of K, such that K_1 acts trivially. We may extend the action of $\mathcal{H}_K$ on

$$(\operatorname{inj} \rho_{\chi,\varnothing} \oplus \operatorname{inj} \rho_{\chi,S})^{I_1}$$

to the action of $\mathcal{H}$, such that $(\operatorname{inj} \rho_{\chi,\varnothing} \oplus \operatorname{inj} \rho_{\chi,S})^{I_1}$ as an $\mathcal{H}$-module is isomorphic to a direct sum of 2^{n-1} supersingular modules of $\mathcal{H}$.

More precisely, for every $\boldsymbol{\varepsilon} \in \Sigma_{\mathbf{0}}$ we consider $b_{\boldsymbol{\varepsilon}}$ as vectors in

$$(\operatorname{inj} \rho_{\chi,\varnothing} \oplus \operatorname{inj} \rho_{\chi,S})^{I_1}$$

via the isomorphism of Corollary 4.2.31. Then the action of $\mathcal{H}_K$ can be extended to the action of $\mathcal{H}$ so that

$$\langle b_{\mathbf{0}}, b_{\mathbf{0}} + b_{\mathbf{1}} \rangle_{\overline{\mathbf{F}}_p} \cong M_{\gamma}$$

where $\gamma = \{\chi\}$. If $\boldsymbol{\varepsilon} \in \Sigma'_{\mathbf{0}}$, then

$$\langle b_{\boldsymbol{\varepsilon}}, b_{\mathbf{1}-\boldsymbol{\varepsilon}} \rangle_{\overline{\mathbf{F}}_p} \cong M_{\gamma_{\boldsymbol{\varepsilon}}}$$

where $\gamma_{\boldsymbol{\varepsilon}} = \gamma_{\mathbf{1}-\boldsymbol{\varepsilon}} = \{\chi \alpha^{\boldsymbol{\varepsilon}\boldsymbol{\cdot}(\boldsymbol{p}-\mathbf{1})}, \chi(\alpha^{\boldsymbol{\varepsilon}\boldsymbol{\cdot}(\boldsymbol{p}-\mathbf{1})})^s\}$.

Proof. — Since, by Lemma 4.2.19 $b_{\boldsymbol{\varepsilon}}$ for $\boldsymbol{\varepsilon} \in \Sigma_{\mathbf{0}}$ form a basis of $R^U_{\mathbf{0}}$, the second part implies the first. Since $\Sigma'_{\mathbf{0}} = \Sigma_{\mathbf{0}} \backslash \{\mathbf{0}, \mathbf{1}\}$, the last part of the Proposition is given by Lemma 4.2.36.

Lemmas 4.2.21 and 4.2.30 imply that

$$\langle b_{\mathbf{0}} \rangle_{\overline{\mathbf{F}}_p} \cong M_{\chi,S}, \quad \langle b_{\mathbf{1}} + b_{\mathbf{0}} \rangle_{\overline{\mathbf{F}}_p} \cong M_{\chi,\varnothing}$$

as an $\mathcal{H}_K$-module. Hence, by Lemma 2.2.2

$$\langle b_{\mathbf{0}}, b_{\mathbf{1}} + b_{\mathbf{0}} \rangle_{\overline{\mathbf{F}}_p} \cong M_{\chi,S} \oplus M_{\chi,\varnothing} \cong M_{\gamma}|_{\mathcal{H}_K}$$

as $\mathcal{H}_K$-modules. Hence, if we define

$$b_{\mathbf{0}} T_{\Pi} = b_{\mathbf{0}} + b_{\mathbf{1}} \quad \text{and} \quad (b_{\mathbf{0}} + b_{\mathbf{1}}) T_{\Pi} = b_{\mathbf{0}}$$

we get the required isomorphism. □

Proposition 4.2.38. — *Suppose that $q = p^n$, let $\chi : H \to \overline{\mathbf{F}}_p^{\times}$ be a character, such that $\chi \neq \chi^s$, and let a and r be unique integers, such that $1 \leqslant a, r \leqslant q-1$ and*

$$\chi\big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\right)\big) = \lambda^a \quad \forall\, \lambda \in \mathbf{F}_q^{\times}, \quad \chi\big(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)\big) = \lambda^r \quad \forall\, \lambda \in \mathbf{F}_q^{\times}.$$

Let $r = r_0 + r_1 p + \cdots + r_{n-1} p^{n-1}$, where $0 \leqslant r_i \leqslant p-1$ for each i, and let

$$\boldsymbol{r} = (r_0, \ldots, r_{n-1}).$$

Then

$$\operatorname{inj} \rho_{\chi,\varnothing} \oplus \operatorname{inj} \rho_{\chi^s,\varnothing} \cong R_{\boldsymbol{r}} \otimes (\det)^{a+r} \oplus R_{\boldsymbol{p}-\mathbf{1}-\boldsymbol{r}} \otimes (\det)^a$$

where $\boldsymbol{p} - \mathbf{1} - \boldsymbol{r} = (p-1-r_0, \ldots, p-1-r_{n-1})$.

We regard the representation $\operatorname{inj}\rho_{\chi,\varnothing}\oplus\operatorname{inj}\rho_{\chi^s,\varnothing}$ *as a representation of* K*, on which* K_1 *acts trivially. Let* c *and* d *be the cardinality of the sets:*

$$c=|\{r_i : r_i\neq p-1\}| \quad \textit{and} \quad d=|\{r_i : r_i\neq 0\}|$$

then we may extend the action of $\mathcal{H}_K$ *on*

$$(\operatorname{inj}\rho_{\chi,\varnothing}\oplus\operatorname{inj}\rho_{\chi^s,\varnothing})^{I_1}$$

to the action of $\mathcal{H}$*, such that* $(\operatorname{inj}\rho_{\chi,\varnothing}\oplus\operatorname{inj}\rho_{\chi^s,\varnothing})^{I_1}$ *as an* $\mathcal{H}$*-module is isomorphic to a direct sum of* L_γ *and* $2^{c-1}+2^{d-1}-2$ *supersingular modules of* $\mathcal{H}$*.*

More precisely, let $b_{\boldsymbol{\varepsilon}}$*, for* $\boldsymbol{\varepsilon}\in\Sigma_{\boldsymbol{r}}$*, be a basis of* $(\operatorname{inj}\rho_{\chi,\varnothing})^{I_1}$ *and let* $\overline{b}_{\boldsymbol{\varepsilon}}$*, for* $\boldsymbol{\varepsilon}\in\Sigma_{\boldsymbol{p-1-r}}$*, be a basis of* $(\operatorname{inj}\rho_{\chi^s,\varnothing})^{I_1}$ *via the isomorphism above. Then the action of* $\mathcal{H}_K$ *can be extended to the action of* $\mathcal{H}$ *so that*

$$\langle b_{\mathbf{0}}, b_{\mathbf{1}}, \overline{b}_{\mathbf{0}}, \overline{b}_{\mathbf{1}}\rangle_{\overline{\mathbf{F}}_p}\cong L_\gamma$$

and

$$\langle b_{\mathbf{0}}, \overline{b}_{\mathbf{0}}\rangle_{\overline{\mathbf{F}}_p}\cong M_\gamma$$

where $\gamma=\{\chi,\chi^s\}$*. If* $\boldsymbol{\varepsilon}\in\Sigma'_{\boldsymbol{r}}$*, then*

$$\langle b_{\boldsymbol{\varepsilon}}, b_{\mathbf{1}-\boldsymbol{\varepsilon}}\rangle_{\overline{\mathbf{F}}_p}\cong M_{\gamma_{\boldsymbol{\varepsilon}}}$$

where $\gamma_{\boldsymbol{\varepsilon}}=\gamma_{\mathbf{1}-\boldsymbol{\varepsilon}}=\{\chi\alpha^{\boldsymbol{\varepsilon}\cdot(\boldsymbol{p}-\mathbf{1}-\boldsymbol{r})},(\chi\alpha^{\boldsymbol{\varepsilon}\cdot(\boldsymbol{p}-\mathbf{1}-\boldsymbol{r})})^s\}$*. If* $\boldsymbol{\varepsilon}\in\Sigma'_{\boldsymbol{p-1-r}}$ *then*

$$\langle \overline{b}_{\boldsymbol{\varepsilon}}, \overline{b}_{\mathbf{1}-\boldsymbol{\varepsilon}}\rangle_{\overline{\mathbf{F}}_p}\cong M_{\overline{\gamma}_{\boldsymbol{\varepsilon}}}$$

where $\overline{\gamma}_{\boldsymbol{\varepsilon}}=\overline{\gamma}_{\mathbf{1}-\boldsymbol{\varepsilon}}=\{\chi^s\alpha^{\boldsymbol{\varepsilon}\cdot\boldsymbol{r}},(\chi^s\alpha^{\boldsymbol{\varepsilon}\cdot\boldsymbol{r}})^s\}$*.*

Proof. — The first part of the Proposition follows from Corollary 4.2.22 and Corollary 3.2.3. For the second part we observe that since $\chi\neq\chi^s$, we have $r\neq q-1$ and hence vectors $b_{\mathbf{0}}$, $b_{\mathbf{1}}$, $\overline{b}_{\mathbf{0}}$ and $\overline{b}_{\mathbf{1}}$ are linearly independent. Lemma 4.2.21 implies that

$$\langle b_{\mathbf{0}}\rangle_{\overline{\mathbf{F}}_p}\cong M_{\chi,\varnothing} \quad \text{and} \quad \langle \overline{b}_{\mathbf{0}}\rangle_{\overline{\mathbf{F}}_p}\cong M_{\chi^s,\varnothing}$$

as $\mathcal{H}_K$-modules. Lemma 4.2.20 with the appropriate twist by a power of det says that H acts on $b_{\mathbf{1}}$ by a character $\chi\alpha^{\mathbf{1}\cdot(\boldsymbol{p}-\mathbf{1}-\boldsymbol{r})}$ and H acts on $\overline{b}_{\mathbf{1}}$ by a character $\chi^s\alpha^{\mathbf{1}\cdot\boldsymbol{r}}$. Lemma 4.2.33 implies that

$$\chi\alpha^{\mathbf{1}\cdot(\boldsymbol{p}-\mathbf{1}-\boldsymbol{r})}=\chi^s \quad \text{and} \quad \chi^s\alpha^{\mathbf{1}\cdot\boldsymbol{r}}=\chi.$$

Hence,

$$b_{\mathbf{1}}e_{\chi^s}=b_{\mathbf{1}} \quad \text{and} \quad \overline{b}_{\mathbf{1}}e_{\chi}=\overline{b}_{\mathbf{1}}.$$

Proposition 4.2.23 implies that

$$(-1)^{r+1}b_{\mathbf{1}}T_{n_s}=(-1)^{r+1}\sum_{u\in I_1/K_1}un_s^{-1}b_{\mathbf{1}}=b_{\mathbf{0}}$$

and

$$(-1)^{q-r}\overline{b}_{\mathbf{1}}T_{n_s}=(-1)^{q-r}\sum_{u\in I_1/K_1}un_s^{-1}\overline{b}_{\mathbf{1}}=\overline{b}_{\mathbf{0}}.$$

Hence, by Lemma 2.2.2
$$\langle b_{\mathbf{0}}, b_{\mathbf{1}}, \overline{b}_{\mathbf{0}}, \overline{b}_{\mathbf{1}}\rangle_{\overline{\mathbf{F}}_p} \cong L_\gamma \mid_{\mathcal{H}_K}$$
as $\mathcal{H}_K$-modules. We note that if $p=2$ then $1=-1$ and if $p\neq 2$ then $(-1)^{q-r} = (-1)^{r+1}$. So if we define
$$b_{\mathbf{1}}T_\Pi = \overline{b}_{\mathbf{1}}, \quad \overline{b}_{\mathbf{1}}T_\Pi = b_{\mathbf{1}}, \quad b_{\mathbf{0}}T_\Pi = \overline{b}_{\mathbf{0}}, \quad \overline{b}_{\mathbf{0}}T_\Pi = b_{\mathbf{0}}$$
we get the required isomorphism of $\mathcal{H}$-modules. Moreover,
$$\langle b_{\mathbf{0}}, \overline{b}_{\mathbf{0}}\rangle_{\overline{\mathbf{F}}_p} \cong M_\gamma$$
as $\mathcal{H}$-module. The last part of the Proposition follows from Lemma 4.2.36. Since $\dim(\operatorname{inj}\rho_{\chi,\varnothing})^{I_1} = 2^c$ and $\dim(\operatorname{inj}\rho_{\chi^s,\varnothing})^{I_1} = 2^d$ an easy calculation gives us the number of indecomposable summands. □

Remark 4.2.39. — If $p=q$, then $\Sigma'_{\boldsymbol{r}} = \varnothing$ and Propositions 4.2.37 and 4.2.38 specialise to Proposition 4.1.9.

The following Proposition can be seen as a consolation for the Remark 4.1.6.

Proposition 4.2.40. — *Suppose that $q=p^n$, $\chi\neq\chi^s$ and let ρ be a representation of Γ, such that $\rho^U \cong M_{\chi,\varnothing}\oplus M_{\chi^s,\varnothing}$ as an $\mathcal{H}_\Gamma$-module, and $\rho = \langle \Gamma\rho^U\rangle_{\overline{\mathbf{F}}_p}$, then*
$$\rho \cong \rho_{\chi,\varnothing}\oplus\rho_{\chi^s,\varnothing}.$$

Proof. — If ρ is a semi-simple representation of Γ, then Corollary 3.1.3 implies the Lemma. Suppose that ρ is not semi-simple. Let $\operatorname{soc}(\rho)$ be the maximal semi-simple subrepresentation of ρ. Since ρ is generated by ρ^U as a Γ-representation, the space $(\operatorname{soc}(\rho))^U$ is one dimensional, and hence $\operatorname{soc}(\rho)$ is an irreducible representation of Γ. By Corollary 3.1.3 and symmetry we may assume that
$$\operatorname{soc}(\rho)\cong\rho_{\chi,\varnothing}.$$
Since, $\operatorname{soc}(\rho)$ is irreducible, ρ is an essential extension of $\rho_{\chi,\varnothing}$. By this we mean that every non-zero Γ-invariant subspace of ρ intersects $\rho_{\chi,\varnothing}$ non-trivially. This implies that there exists an exact sequence
$$0\longrightarrow\rho\longrightarrow\operatorname{inj}\rho_{\chi,\varnothing}$$
of Γ-representations. After twisting by a power of determinant we may assume that χ is given by $\chi\big(\left(\begin{smallmatrix}\lambda & 0\\ 0 & \mu\end{smallmatrix}\right)\big) = \lambda^r$, where $0<r<q-1$. The inequalities are strict, since $\chi\neq\chi^s$. Let $\boldsymbol{r}$ be the corresponding n-tuple. Let $\varepsilon\in\Sigma_{\boldsymbol{r}}$ and $b_\varepsilon\in(\operatorname{inj}\rho_{\chi,\varnothing})^U$, then H acts on b_ε by the character $\chi\alpha^{\boldsymbol{\varepsilon\cdot(p-1-r)}}$. In particular, if $\varepsilon'\in\Sigma_{\boldsymbol{r}}$, such that $\varepsilon'\neq\varepsilon$, then H acts on b_ε and $b_{\varepsilon'}$ by different characters. As a consequence of this, the submodule $M_{\chi^s,\varnothing}$ of ρ^U must be mapped to some subspace $\langle b_{\boldsymbol{\varepsilon}}\rangle_{\overline{\mathbf{F}}_p}$ of $(\operatorname{inj}\rho_{\chi,\varnothing})^U$, where $\varepsilon\in\Sigma_{\boldsymbol{r}}$. By examining the action of H, we get that $\chi^s = \chi\alpha^{\boldsymbol{\varepsilon\cdot(p-1-r)}}$. This implies that
$$\varepsilon\boldsymbol{\cdot}(\boldsymbol{p}-\boldsymbol{1}-\boldsymbol{r})+r\equiv 0 \pmod{q-1}.$$

Since $0 < r < q-1$ and $\varepsilon \in \Sigma_{\boldsymbol{r}}$, we have

$$0 < \varepsilon \centerdot (\boldsymbol{p} - \boldsymbol{1} - \boldsymbol{r}) + r \leqslant q - 1.$$

Hence, we get an equality on the right hand side, which implies that, for each i, $(1-\varepsilon_i)(p-1-r_i) = 0$. So $\varepsilon = \delta$, and $b_\varepsilon = b_{\boldsymbol{1}}$, see 4.2.34 and 4.2.35. However, by Proposition 4.2.23 (ii)

$$b_{\boldsymbol{1}} T_{n_s} = (-1)^{r+1} b_{\boldsymbol{0}} \neq 0.$$

We obtain a contradiction, since T_{n_s} kills $M_{\chi^s,\varnothing}$. □

CHAPTER 5

COEFFICIENT SYSTEMS

We closely follow [**13**] and [**14**, § V], where the G-equivariant coefficient systems of $\mathbb{C}$-vector spaces are treated. In fact, the results of this Section do not depend on the underlying field. Our motivation to use coefficient systems stems from [**12**], where the equivariant coefficient systems of $\overline{\mathbf{F}}_p$-vector spaces of finite Chevalley groups are considered.

5.1. Definitions

The Bruhat-Tits tree X of G is the simplicial complex, whose vertices are the similarity classes $[L]$ of $\mathfrak{o}_F$-lattices in a 2-dimensional F-vector space V and whose edges are 1-simplices, given by families $\{[L_0],[L_1]\}$ of similarity classes such that

$$\varpi_F L_0 \subset L_1 \subset L_0.$$

We denote by X_0 the set of all vertices and by X_1 the set of all edges.

Definition 5.1.1. — Let σ be a simplex in X, then we define

$$\mathfrak{K}(\sigma) = \{g \in \mathrm{Aut}_F(V) : g\sigma = \sigma\}.$$

By fixing a basis $\{v_1, v_2\}$ of V we identify G with $\mathrm{Aut}_F(V)$. Let

$$\sigma_0 = [\mathfrak{o}_F v_1 + \mathfrak{o}_F v_2] \quad \text{and} \quad \sigma_1 = \{[\mathfrak{o}_F v_1 + \mathfrak{o}_F v_2], [\mathfrak{o}_F v_1 + \mathfrak{p}_F v_2]\}.$$

Then σ_0 is a vertex and $\mathfrak{K}(\sigma_0) = F^\times K$, and σ_1 is an edge containing a vertex σ_0. Moreover, $\mathfrak{K}(\sigma_1)$ is the group generated by I and Π.

Definition 5.1.2. — A coefficient system $\mathcal{V}$ (of $\overline{\mathbf{F}}_p$-vector spaces) on X consists of

– $\overline{\mathbf{F}}_p$ vector spaces V_σ for each simplex σ of X, and

– linear maps $r_\sigma^{\sigma'} : V_{\sigma'} \to V_\sigma$ for each pair $\sigma \subseteq \sigma'$ of simplices of X such that for every simplex σ, $r_\sigma^\sigma = \mathrm{id}_{V_\sigma}$.

Definition 5.1.3. — We say the group G acts on the coefficient system $\mathcal{V}$, if for every $g \in G$ and for every simplex σ there is given a linear map

$$g_\sigma : V_\sigma \longrightarrow V_{g\sigma},$$

such that

– $g_{h\sigma} \circ h_\sigma = (gh)_\sigma$, for every g, $h \in G$ and for every simplex σ,

– $1_\sigma = \mathrm{id}_{V_\sigma}$ for every simplex σ,

– the following diagram commutes for every $g \in G$ and every pair of simplices $\sigma \subseteq \sigma'$:

$$\begin{tikzcd} V_\sigma \arrow[r, "g_\sigma"] & V_{g\sigma} \\ V_{\sigma'} \arrow[u, "r_\sigma^{\sigma'}"] \arrow[r, "g_{\sigma'}"] & V_{g\sigma'} \arrow[u, "r_{g\sigma}^{g\sigma'}"'] \end{tikzcd} .$$

In particular, the stabiliser $\mathfrak{K}(\sigma)$ acts linearly on V_σ for any simplex σ.

Definition 5.1.4. — A G-equivariant coefficient system $(V_\tau)_\tau$ on X is a coefficient system on X together with a G-action , such that the action of the stabiliser $\mathfrak{K}(\sigma)$ of a simplex σ on V_σ is smooth.

Remark 5.1.5. — The definition given in [**14**] § V, requires the action to factor through a discrete quotient.

Let $\mathcal{COEF}_G$ denote the category of all equivariant coefficient systems on X, equipped with the obvious morphisms.

The following observation will turn out to be very useful. Suppose that G acts on a coefficient system $\mathcal{V} = (V_\sigma)_\sigma$. Let τ' be an edge containing a vertex τ. There exists $g \in G$, such that $\tau' = g\sigma_1$ and $\tau = g\sigma_0$. Then

$$V_\tau = g_{\sigma_0} V_{\sigma_0}, \quad V_{\tau'} = g_{\sigma_1} V_{\sigma_1}$$

and

$$r_\tau^{\tau'} = g_{\sigma_0} \circ r_{\sigma_0}^{\sigma_1} \circ (g^{-1})_{\tau'}.$$

5.2. Homology

Let $X_{(0)}$ be the set of vertices on the tree and let $X_{(1)}$ be the set of oriented edges on the tree. We will say that two vertices σ and σ' are neighbours if $\{\sigma, \sigma'\}$ is an edge. And we will write

$$(\sigma, \sigma')$$

to mean a directed edge going from σ to σ'. Let $\mathcal{V} = (V_\tau)_\tau$ be an equivariant coefficient system. We define a space of oriented 0-chains to be

$$C_c^{\mathrm{or}}(X_{(0)}, \mathcal{V}) = \overline{\mathbf{F}}_p\text{-vector space of all maps } \omega : X_{(0)} \longrightarrow \dot{\bigcup_{\sigma \in X_0}} V_\sigma$$

such that

– ω has finite support and
– $\omega(\sigma) \in V_\sigma$ for every vertex σ.

Similarly, the space of oriented 1-chains is

$$C_c^{\mathrm{or}}(X_{(1)},\mathcal{V}) = \overline{\mathbf{F}}_p\text{-vector space of all maps } \omega : X_{(1)} \longrightarrow \dot{\bigcup_{\{\sigma,\sigma'\}\in X_1}} V_{\{\sigma,\sigma'\}}$$

such that

– ω has finite support,
– $\omega((\sigma,\sigma')) \in V_{\{\sigma,\sigma'\}}$,
– $\omega((\sigma',\sigma)) = -\omega((\sigma,\sigma'))$ for every oriented edge (σ,σ').

The group G acts on $C_c^{\mathrm{or}}(X_{(0)},\mathcal{V})$ via

$$(g\omega)(\sigma) = g_{g^{-1}\sigma}(\omega(g^{-1}\sigma))$$

and on $C_c^{\mathrm{or}}(X_{(1)},\mathcal{V})$ via

$$(g\omega)((\sigma,\sigma')) = g_{\{g^{-1}\sigma,g^{-1}\sigma'\}}(\omega((g^{-1}\sigma,g^{-1}\sigma'))).$$

The action on both spaces is smooth.

The boundary map is given by

$$\begin{aligned}\partial : C_c^{\mathrm{or}}(X_{(1)},\mathcal{V}) &\longrightarrow C_c^{\mathrm{or}}(X_{(0)},\mathcal{V})\\ \omega &\longmapsto \Big(\sigma \mapsto \sum_{\sigma'} r_\sigma^{\{\sigma,\sigma'\}}(\omega((\sigma,\sigma')))\Big)\end{aligned}$$

where the sum is taken over all the neighbours of σ. The map ∂ is G-equivariant.

We define $H_0(X,\mathcal{V})$ to be the cokernel of ∂. It is naturally a smooth representation of G.

5.3. Some computations of $H_0(X,\mathcal{V})$

Throughout this section we fix an equivariant coefficient system $\mathcal{V} = (V_\tau)_\tau$, with the restriction maps given by $r_\tau^{\tau'}$. Our first lemma follows immediately from the definition of ∂.

Lemma 5.3.1. — *Let ω be an oriented 1-chain supported on a single edge $\tau = \{\sigma,\sigma'\}$. Let*

$$v = \omega((\sigma,\sigma')).$$

Then

$$\partial(\omega) = \omega_\sigma - \omega_{\sigma'},$$

where ω_σ and $\omega_{\sigma'}$ are 0-chains supported only on σ and σ' respectively. Moreover,

$$\omega_\sigma(\sigma) = r_\sigma^\tau(v) \quad \textit{and} \quad \omega_{\sigma'}(\sigma') = r_{\sigma'}^\tau(v).$$

LEMMA 5.3.2. — *Let ω be a* 0*-chain supported on a single vertex σ. Suppose that the restriction map $r^{\sigma_1}_{\sigma_0}$ is an injection, then the image of ω in $H_0(X,\mathcal{V})$ is non-zero.*

Proof. — Since every restriction map is conjugate to $r^{\sigma_1}_{\sigma_0}$ by some element of G, it follows that every restriction map is injective.

Let ω' be a non-zero oriented 1-chain. We may think of the support of ω' as the union of edges of a finite subgraph $\mathcal{T}$ of X. Since all the restriction maps are injective, Lemma 5.3.1 implies that $\partial(\omega')$ will not vanish on the vertices of $\mathcal{T}$, which have only one neighbour in $\mathcal{T}$. In particular, $\partial(\omega')$ will be supported on at least 2 vertices. Hence, $\omega\notin\partial C^{\mathrm{or}}_c(X_{(1)},\mathcal{V})$. ☐

LEMMA 5.3.3. — *Let ω be* 0*-chain. Suppose that the restriction map $r^{\sigma_1}_{\sigma_0}$ is surjective, then there exists a* 0*-chain ω_0, supported on a single vertex σ_0, such that*

$$\omega+\partial C^{\mathrm{or}}_c(X_{(1)},\mathcal{V})=\omega_0+\partial C^{\mathrm{or}}_c(X_{(1)},\mathcal{V}).$$

Proof. — Since every restriction map is conjugate to $r^{\sigma_1}_{\sigma_0}$ by some element of G, it follows that every restriction map is surjective.

It is enough to prove the statement when ω is supported on a single vertex τ, since an arbitrary 0-chain is a finite sum of such. If $\tau=\sigma_0$ then we are done. Otherwise, there exists a directed path going from σ_0 to τ, consisting of finitely many directed edges $(\sigma_0,\tau_1),\dots,(\tau_m,\tau)$.

We argue by induction on m. Let $v=\omega(\tau)$. Since $r^{\{\tau_m,\tau\}}_{\tau}$ is surjective there exists $v'\in V_{\{\tau_m,\tau\}}$, such that

$$r^{\{\tau_m,\tau\}}_{\tau}(v')=v.$$

Let ω' be an oriented 1-chain supported on the single edge $\{\tau_m,\tau\}$ with $\omega'((\tau_m,\tau))=v'$. By Lemma 5.3.1 $\omega+\partial(\omega')$ is supported on a single vertex τ_m. Since, the number of edges in the directed path has decreased by one, the claim follows from induction. ☐

The following special case will be used in the calculations of modules of the Hecke algebra.

LEMMA 5.3.4. — *Let ω_0 be a* 0*-chain supported on a single vertex σ_0. Let*

$$v_0=\omega_0(\sigma_0)$$

and suppose that there exists $v_1\in V_{\sigma_1}$, such that

$$r^{\sigma_1}_{\sigma_0}(v_1)=v_0.$$

Let ω' be a 0*-chain supported on a single vertex σ_0 with*

$$\omega'(\sigma_0)=r^{\sigma_1}_{\sigma_0}((\Pi^{-1})_{\sigma_1}(v_1)),$$

then

$$\Pi^{-1}\omega_0+\partial C^{\mathrm{or}}_c(X_{(1)},\mathcal{V})=\omega'+\partial C^{\mathrm{or}}_c(X_{(1)},\mathcal{V}).$$

Proof. — We observe that $\Pi\sigma_0 = \Pi^{-1}\sigma_0$ and $\sigma_1 = \{\sigma_0, \Pi\sigma_0\}$. The 0-chain $\Pi^{-1}\omega_0$ is supported on a single vertex $\Pi\sigma_0$ with

$$(\Pi^{-1}\omega_0)(\Pi\sigma_0) = (\Pi^{-1})_{\sigma_0}(v_0).$$

Let ω_1 be an oriented 1-chain supported on a single edge σ_1 with

$$\omega_1((\sigma_0, \Pi\sigma_0)) = (\Pi^{-1})_{\sigma_1}(v_1).$$

From Lemma 5.3.1 we know that $\partial(\omega_1)$ is supported only on σ_0 and $\Pi\sigma_0$. Moreover,

$$\begin{aligned}\partial(\omega_1)(\Pi\sigma_0) &= r^{\sigma_1}_{\Pi\sigma_0}(\omega_1((\Pi\sigma_0, \sigma_0)) = r^{\sigma_1}_{\Pi\sigma_0}(-(\Pi^{-1})_{\sigma_1}(v_1))\\ &= -(r^{\sigma_1}_{\Pi\sigma_0} \circ (\Pi^{-1})_{\sigma_1})(v_1) = -((\Pi^{-1})_{\sigma_0} \circ r^{\sigma_1}_{\sigma_0})(v_1) = -(\Pi^{-1})_{\sigma_0}(v_0)\end{aligned}$$

and

$$\partial(\omega_1)(\sigma_0) = r^{\sigma_1}_{\sigma_0}(\omega_1((\sigma_0, \Pi\sigma_0))) = r^{\sigma_1}_{\sigma_0}((\Pi^{-1})_{\sigma_1}(v_1)).$$

Hence

$$\partial(\omega_1) = \omega' - \Pi^{-1}\omega_0$$

and that establishes the claim. □

PROPOSITION 5.3.5. — *Suppose that the restriction map $r^{\sigma_1}_{\sigma_0}$ is an isomorphism of vector spaces. Then*

$$H_0(X,\mathcal{V})|_{\mathfrak{K}(\sigma_0)} \cong V_{\sigma_0}$$

and

$$H_0(X,\mathcal{V})|_{\mathfrak{K}(\sigma_1)} \cong V_{\sigma_1}.$$

Moreover, the diagram

$$\begin{array}{ccc} V_{\sigma_0} & \xrightarrow{\cong} & H_0(X,\mathcal{V}) \\ {\scriptstyle r^{\sigma_1}_{\sigma_0}}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{id}} \\ V_{\sigma_1} & \xrightarrow{\cong} & H_0(X,\mathcal{V}) \end{array}$$

of $F^\times I$-representations commutes.

Proof. — Let $C^{\mathrm{or}}_c(\sigma_0,\mathcal{V})$ be a subspace of $C^{\mathrm{or}}_c(X_{(0)},\mathcal{V})$ consisting of the 0-chains whose support lies in the simplex σ_0, with the understanding that the 0-chain which vanishes on every simplex is supported on the empty simplex. Let $\jmath$ be the composition

$$\jmath : C^{\mathrm{or}}_c(\sigma_0,\mathcal{V}) \hookrightarrow C^{\mathrm{or}}_c(X_{(0)},\mathcal{V}) \longrightarrow H_0(X,\mathcal{V}).$$

Then $\jmath$ is $\mathfrak{K}(\sigma_0)$ equivariant. Moreover, Lemma 5.3.2 says that $\jmath$ is an injection and Lemma 5.3.3 says that it is a surjection. Hence

$$\jmath : C^{\mathrm{or}}_c(\sigma_0,\mathcal{V}) \cong H_0(X,\mathcal{V})|_{\mathfrak{K}(\sigma_0)}.$$

Let ev_0 be the map

$$\begin{aligned}\mathrm{ev}_0 : C^{\mathrm{or}}_c(\sigma_0,\mathcal{V}) &\longrightarrow V_{\sigma_0}\\ \omega &\longmapsto \omega(\sigma_0)\end{aligned}$$

then ev_0 is an isomorphism of $\mathfrak{K}(\sigma_0)$-representations. Hence

$$\jmath \circ (\mathrm{ev}_0)^{-1} : V_{\sigma_0} \cong H_0(X, \mathcal{V})|_{\mathfrak{K}(\sigma_0)}.$$

Since $\mathcal{V}$ is G-equivariant, the map $r_{\sigma_0}^{\sigma_1}$ is $F^\times I = \mathfrak{K}(\sigma_1) \cap \mathfrak{K}(\sigma_0)$-equivariant and since it is isomorphism of vector spaces, we obtain that

$$\jmath \circ (\mathrm{ev}_0)^{-1} \circ r_{\sigma_0}^{\sigma_1} : V_{\sigma_1}|_{F^\times I} \cong H_0(X, \mathcal{V})|_{F^\times I}.$$

We claim that this isomorphism is in fact $\mathfrak{K}(\sigma_1)$-equivariant. Let $v_1 \in V_{\sigma_1}$, let $v_0 = r_{\sigma_0}^{\sigma_1}(v_1)$ and let $\omega_0 \in C_c^{\mathrm{or}}(\sigma_0, \mathcal{V})$, such that $\omega_0(\sigma_0) = v_0$. Then

$$(\jmath \circ (\mathrm{ev}_0)^{-1} \circ r_{\sigma_0}^{\sigma_1})(v_1) = \omega_0 + \partial C_c^{\mathrm{or}}(X_{(1)}, \mathcal{V}).$$

By Lemma 5.3.4

$$\Pi^{-1}\omega_0 + \partial C_c^{\mathrm{or}}(X_{(1)}, \mathcal{V}) = \omega' + \partial C_c^{\mathrm{or}}(X_{(1)}, \mathcal{V}),$$

where $\omega' \in C_c^{\mathrm{or}}(\sigma_0, \mathcal{V})$ with $\omega'(\sigma_0) = r_{\sigma_0}^{\sigma_1}((\Pi^{-1})_{\sigma_1}(v_1))$. This implies that

$$\Pi^{-1}(\jmath \circ (\mathrm{ev}_0)^{-1} \circ r_{\sigma_0}^{\sigma_1})(v_1) = (\jmath \circ (\mathrm{ev}_0)^{-1} \circ r_{\sigma_0}^{\sigma_1})((\Pi^{-1})_{\sigma_1}(v_1)).$$

Since Π^{-1} and $F^\times I$ generate $\mathfrak{K}(\sigma_1)$ this proves the claim.

The commutativity of the diagram follows from the way the isomorphisms are constructed. □

5.4. Constant functor

The content of this Section is essentially [**12**] Lemma 1.1 and Theorem 1.2. Let Rep_G be the category of smooth $\overline{\mathbf{F}}_p$-representations of G. Let π be a smooth representation of G with the underlying vector space $\mathcal{W}$. Let σ be a simplex on the tree X, we set

$$(\mathcal{K}_\pi)_\sigma = \mathcal{W}.$$

If σ and σ' are two simplices, such that $\sigma \subseteq \sigma'$ then we define the restriction map

$$r_\sigma^{\sigma'} = \mathrm{id}_{\mathcal{W}}.$$

For every $g \in G$ and every simplex σ in X we define a linear map g_σ by

$$g_\sigma : (\mathcal{K}_\pi)_\sigma \longrightarrow (\mathcal{K}_\pi)_{g\sigma}, \quad v \longmapsto \pi(g)v.$$

This gives a G-equivariant coefficient system on X, which we denote by $\mathcal{K}_\pi$.

Definition 5.4.1. — We define the constant functor

$$\mathcal{K} : \mathrm{Rep}_G \longrightarrow \mathcal{COEF}_G, \quad \pi \longmapsto \mathcal{K}_\pi.$$

Lemma 5.4.2. — *Let π be a smooth representation of G, then*

$$H_0(X, \mathcal{K}_\pi) \cong \pi$$

as a G-representation.

Proof. — We have an evaluation map

$$\mathrm{ev} : C_c^{\mathrm{or}}(X_{(0)}, \mathcal{K}_\pi) \longrightarrow \pi, \quad \omega \longmapsto \sum_{\sigma \in X_{(0)}} \omega(\sigma).$$

Since the restriction maps are just $\mathrm{id}_{\mathcal{W}}$, Lemma 5.3.1 implies that the image of the boundary map $\partial C_c^{\mathrm{or}}(X_{(1)}, \mathcal{K}_\pi)$ is contained in the kernel of ev. Hence, we get a G-equivariant map

$$H_0(X, \mathcal{K}_\pi) \longrightarrow \pi.$$

It is enough to show that this is an isomorphism of vector spaces, and this is implied by Proposition 5.3.5. □

PROPOSITION 5.4.3. — *Let $\mathcal{V} = (V_\sigma)_\sigma$ be a G-equivariant coefficient system with the restriction maps $r_\sigma^{\sigma'}$ and let $(\pi, \mathcal{W})$ be a smooth representation of G, then*

$$\mathrm{Hom}_{\mathcal{COEF}_G}(\mathcal{V}, \mathcal{K}_\pi) \cong \mathrm{Hom}_G(H_0(X, \mathcal{V}), \pi).$$

Proof. — Any morphism of G-equivariant coefficient systems will induce a G-equivariant homomorphism in the 0-th homology. Hence by Lemma 5.4.2 we have a map

$$\mathrm{Hom}_{\mathcal{COEF}_G}(\mathcal{V}, \mathcal{K}_\pi) \longrightarrow \mathrm{Hom}_G(H_0(X, \mathcal{V}), \pi).$$

We will construct an inverse of this. Let $\phi \in \mathrm{Hom}_G(H_0(X, \mathcal{V}), \pi)$, let σ be a vertex on the tree X, let v be a vector in V_σ, and let $\omega_{\sigma,v}$ be a 0-chain, such that

$$\mathrm{Supp}\, \omega_{\sigma,v} \subseteq \sigma, \quad \omega_{\sigma,v}(\sigma) = v,$$

then we define

$$\phi_\sigma : V_\sigma \longrightarrow \mathcal{W}, \quad v \longmapsto \phi(\omega_{\sigma,v} + \partial C_c^{\mathrm{or}}(X_{(1)}, \mathcal{V})).$$

Let τ be an edge in X with vertices σ and σ', we define

$$\phi_\tau : V_\tau \longrightarrow \mathcal{W}, \quad v \longmapsto \phi_\sigma(r_\sigma^\tau(v)).$$

Lemma 5.3.1 implies that the definition of ϕ_τ does not depend on the choice of vertex. Hence, the collection of linear maps $(\phi_\sigma)_\sigma$ is a morphism of coefficient systems, which induces ϕ on the 0-th homology. An easy check shows that $(\phi_\sigma)_\sigma$ respect the G-action on $\mathcal{V}$ and $\mathcal{K}_\pi$. □

5.5. Diagrams

DEFINITION 5.5.1. — Let $\mathcal{DIAG}$ be the category, whose objects are diagrams

$$\begin{array}{c} D_0 \\ r \big\uparrow \\ D_1 \end{array}$$

where (ρ_0, D_0) is a a smooth $\overline{\mathbf{F}}_p$-representation of $\mathfrak{K}(\sigma_0)$, (ρ_1, D_1) is a smooth $\overline{\mathbf{F}}_p$-representation of $\mathfrak{K}(\sigma_1)$, and $r \in \mathrm{Hom}_{F^\times I}(D_1, D_0)$.

The morphisms between two objects (D_0, D_1, r) and (D_0', D_1', r') are pairs (ψ_0, ψ_1), such that $\psi_0 \in \mathrm{Hom}_{\mathfrak{K}(\sigma_0)}(D_0, D_0')$, $\psi_1 \in \mathrm{Hom}_{\mathfrak{K}(\sigma_1)}(D_1, D_1')$ and the diagram:

$$\begin{tikzcd} D_0 \arrow[r, "\psi_0"] & D_0' \\ D_1 \arrow[u, "r"] \arrow[r, "\psi_1"] & D_1' \arrow[u, "r'"'] \end{tikzcd}$$

of $F^\times I$ representations commutes.

The main result of this section is Theorem 5.5.4, which says that the categories $\mathcal{DIAG}$ and $\mathcal{COEF}_G$ are equivalent. It is easier to work with objects of $\mathcal{DIAG}$ than the coefficient systems.

Definition 5.5.2. — Let $\mathcal{V} = (V_\sigma)_\sigma$ be an object in $\mathcal{COEF}_G$. Let $\mathcal{D} : \mathcal{COEF}_G \to \mathcal{DIAG}$ be a functor, given by

$$\mathcal{V} \longmapsto \begin{tikzcd} V_{\sigma_0} \\ V_{\sigma_1} \arrow[u, "r_{\sigma_0}^{\sigma_1}"] \end{tikzcd} .$$

We will construct a functor $\mathcal{C} : \mathcal{DIAG} \to \mathcal{COEF}_G$ and show that the functors $\mathcal{C}$ and $\mathcal{D}$ induce an equivalence of categories. The key point here is that G acts transitively on the vertices of X.

5.5.1. Underlying vector spaces. — Let $D = (D_0, D_1, r)$ be an object in $\mathcal{DIAG}$. Let $i \in \{0, 1\}$, we define $\text{c-Ind}_{\mathfrak{K}(\sigma_i)}^G \rho_i$, to be a representation of G whose underlying vector space consists of functions

$$f : G \longrightarrow D_i$$

such that

$$f(kg) = \rho_i(k) f(g) \quad \forall\, g \in G, \quad \forall\, k \in \mathfrak{K}(\sigma_i)$$

and $\operatorname{Supp} f$ is compact modulo the centre. The group G acts by the right translations, that is

$$(gf)(g_1) = f(g_1 g).$$

Let τ be a vertex on the tree X, then there exists $g \in G$, such that $\tau = g\sigma_0$. Let

$$\mathcal{F}_\tau = \{f \in \text{c-Ind}_{\mathfrak{K}(\sigma_0)}^G \rho_0 : \operatorname{Supp} f \subseteq \mathfrak{K}(\sigma_0) g^{-1}\}.$$

The space $\mathcal{F}_\tau$ is independent of the choice of g. Let τ' be an edge on the tree X, then there exists $g \in G$ such that $\tau' = g\sigma_1$. We define

$$\mathcal{F}_{\tau'} = \{f \in \text{c-Ind}_{\mathfrak{K}(\sigma_1)}^G \rho_1 : \operatorname{Supp} f \subseteq \mathfrak{K}(\sigma_1) g^{-1}\}.$$

We observe that $\mathcal{F}_{\tau'}$ is also independent of the choice of g.

5.5.2. Restriction maps. — Let $i \in \{0,1\}$, then $\mathcal{F}_{\sigma_i}$ is naturally isomorphic to D_i as a $\mathfrak{K}(\sigma_i)$ representation. The isomorphism is given by

$$\mathrm{ev}_i : \mathcal{F}_{\sigma_i} \longrightarrow D_i, \quad f \longmapsto f(1).$$

The inverse is given by

$$\mathrm{ev}_i^{-1} : D_i \longrightarrow \mathcal{F}_{\sigma_i}, \quad v \longmapsto f_v$$

where $f_v(k) = \rho_i(k)v$, if $k \in \mathfrak{K}(\sigma_i)$, and 0 otherwise. Let

$$r_{\sigma_0}^{\sigma_1} = \mathrm{ev}_0^{-1} \circ r \circ \mathrm{ev}_1 .$$

Then $r_{\sigma_0}^{\sigma_1}$ is an $F^\times I$-equivariant map from $\mathcal{F}_{\sigma_1}$ to $\mathcal{F}_{\sigma_0}$. If $v \in D_1$ then it sends

$$r_{\sigma_0}^{\sigma_1} : f_v \longmapsto f_{r(v)}.$$

We observe, for the purposes of Theorem 5.5.4, that

$$\widetilde{D} = (\mathcal{F}_{\sigma_0}, \mathcal{F}_{\sigma_1}, r_{\sigma_0}^{\sigma_1})$$

is an object of $\mathcal{DIAG}$. Moreover, $\mathrm{ev} = (\mathrm{ev}_0, \mathrm{ev}_1)$ is an isomorphism of diagrams between D and $\widetilde{D}$. We will show later on that ev induces a natural transformation between certain functors.

Let τ' be an edge containing a vertex τ, then there exists $g \in G$, such that $\tau = g\sigma_0$ and $\tau' = g\sigma_1$. Moreover, g can only be replaced by gk, where $k \in \mathfrak{K}(\sigma_0) \cap \mathfrak{K}(\sigma_1) = F^\times I$. We define

$$r_\tau^{\tau'} : \mathcal{F}_{\tau'} \longrightarrow \mathcal{F}_\tau, \quad f \longmapsto g r_{\sigma_0}^{\sigma_1}(g^{-1} f)$$

where g acts on the space $\text{c-Ind}_{\mathfrak{K}(\sigma_0)}^G D_0$ and g^{-1} on the space $\text{c-Ind}_{\mathfrak{K}(\sigma_1)}^G D_1$. Since, r is $F^\times I$-equivariant we have

$$\rho_0(k) \circ r_{\sigma_0}^{\sigma_1} \circ \rho_1(k^{-1}) = r_{\sigma_0}^{\sigma_1}$$

for all $k \in F^\times I$. Hence, the map $r_\tau^{\tau'}$ is independent of the choice of g. Explicitly, let $v = f(g^{-1})$, then

$$r_\tau^{\tau'} : f \longmapsto g f_{r(v)}.$$

Let τ be any simplex then we define the map $r_\tau^\tau = \mathrm{id}_{\mathcal{F}_\tau}$.

5.5.3. G-action. — So far from a diagram we have constructed a coefficient system. We need to show that G acts on it. Let $i \in \{0,1\}$ and let $f \in \text{c-Ind}_{\mathfrak{K}(\sigma_i)}^G D_i$. For any $g \in G$ we have

$$\mathrm{Supp}(gf) = (\mathrm{Supp}\, f) g^{-1}.$$

Hence for any simplex τ we obtain a linear map

$$g_\tau : \mathcal{F}_\tau \longrightarrow \mathcal{F}_{g\tau}, \quad f \longmapsto gf.$$

Moreover, $1_\tau = \mathrm{id}_{\mathcal{F}_\tau}$ and $g_{h\tau} \circ h_\tau = (gh)_\tau$, for any $g, h \in G$. Let τ' be an edge containing a vertex τ. We need to show that the diagram:

$$\begin{tikzcd} \mathcal{F}_\tau \arrow[r, "g_\tau"] & \mathcal{F}_{g\tau} \\ \mathcal{F}_{\tau'} \arrow[u, "r_\tau^{\tau'}"] \arrow[r, "g_{\tau'}"] & \mathcal{F}_{g\tau'} \arrow[u, "r_{g\tau}^{g\tau'}"'] \end{tikzcd}$$

commutes. There exists $g_1 \in G$ such that $\tau = g_1\sigma_0$ and $\tau' = g_1\sigma_1$. Moreover, such g_1 is determined up to a multiple g_1k, where $k \in F^\times I$. Let $f \in \mathcal{F}_{\tau'}$ and let $v = f(g_1^{-1})$, then

$$r_\tau^{\tau'}(f) = g_1 f_{r(v)}.$$

Hence

$$(g_\tau \circ r_\tau^{\tau'})(f) = gg_1 f_{r(v)}.$$

Since $g\tau' = gg_1\sigma_1$, $g\tau = gg_1\sigma_0$ and $(gf)((gg_1)^{-1}) = f(g_1^{-1}) = v$ we obtain

$$(r_{g\tau}^{g\tau'} \circ g_{\tau'})(f) = r_{g\tau}^{g\tau'}(gg_1 f_v) = gg_1 f_{r(v)}.$$

Hence the diagram commutes.

5.5.4. Morphisms. — Let $D' = (D_0', D_1', r')$ be another diagram, let $\psi = (\psi_0, \psi_1)$ be a morphism of diagrams

$$\psi : D \longrightarrow D'$$

and let $\mathcal{F}' = (\mathcal{F}_\tau')_\tau$ be a coefficient system associated to D' via the construction above. Let τ be any simplex on the tree. If τ is a vertex let $i = 0$ and if τ is an edge, let $i = 1$. There exists some $g \in G$ such that $\tau = g\sigma_i$. Let $f \in V_\tau$ and let $v = f(g^{-1})$ we define a map

$$\psi_\tau : \mathcal{F}_\tau \longrightarrow \mathcal{F}_\tau', \quad f \longmapsto g f_{\psi_i(v)}$$

where $f_{\psi_i(v)}$ is the unique function in $\mathcal{F}_{\sigma_i}'$, such that $f_{\psi_i(v)}(1) = \psi_i(v)$. Since the map ψ_i is $\mathfrak{K}(\sigma_i)$-equivariant, ψ_τ is independent of the choice of g.

We will show that the maps $(\psi_\tau)_\tau$ are compatible with the restriction maps. Let τ' be an edge containing a vertex τ. We claim that the diagram

$$\begin{tikzcd} \mathcal{F}_\tau \arrow[r, "\psi_\tau"] & \mathcal{F}_\tau' \\ \mathcal{F}_{\tau'} \arrow[u, "r_\tau^{\tau'}"] \arrow[r, "\psi_{\tau'}"] & \mathcal{F}_{\tau'}' \arrow[u, "(r')_\tau^{\tau'}"'] \end{tikzcd}$$

commutes. There exists $g \in G$ such that $\tau = g\sigma_0$ and $\tau' = g\sigma_1$. Let $f \in \mathcal{F}_{\tau'}$ and let $v = f(g^{-1})$. Then

$$(\psi_\tau \circ r_\tau^{\tau'})(f) = \psi_\tau(g f_{r(v)}) = g f_{\psi_0(r(v))}$$

and

$$((r')_\tau^{\tau'} \circ \psi_{\tau'})(f) = (r')_\tau^{\tau'}(g f_{\psi_1(v)}) = g f_{r'(\psi_1(v))}.$$

Since (ψ_0, ψ_1) is a morphism of diagrams

$$\psi_0(r(v)) = r'(\psi_1(v)).$$

Hence the diagram commutes as claimed and $(\psi_\tau)_\tau$ are compatible with the restriction maps.

Finally, we will show that the maps $(\psi_\tau)_\tau$ are compatible with the G-action. Let τ be any simplex on the tree. To ease the notation, for every $h \in G$ we denote by h_τ the action of h on both $(\mathcal{F}_\tau)_\tau$ and $(\mathcal{F}'_\tau)_\tau$. Let τ be a simplex on the tree X and let $h \in G$. We claim that the diagram

$$\begin{array}{ccc} \mathcal{F}_{h\tau} & \xrightarrow{\psi_{h\tau}} & \mathcal{F}'_{h\tau} \\ {\scriptstyle h_\tau}\uparrow & & \uparrow{\scriptstyle h_\tau} \\ \mathcal{F}_\tau & \xrightarrow{\psi_\tau} & \mathcal{F}'_\tau \end{array}$$

commutes. If τ is an edge let $i = 1$, if τ is a vertex let $i = 0$. There exists $g \in G$, such that $\tau = g\sigma_i$. Let $f \in \mathcal{F}_\tau$ and let $v = f(g^{-1})$, then

$$\psi_{h\tau}(h_\tau(f)) = \psi_{h\tau}(hgf_v) = hgf_{\psi_i(v)}$$

and

$$h_\tau(\psi_\tau(f)) = h_\tau(gf_{\psi_i(v)}) = hgf_{\psi_i(v)}.$$

Hence, the diagram commutes as claimed and the collection $(\psi_\tau)_\tau$ defines a morphism of equivariant coefficient systems.

5.5.5. Equivalence

Definition 5.5.3. — Let $\mathcal{C}$ be a functor

$$\mathcal{C} : \mathcal{DIAG} \longrightarrow \mathcal{COEF}_G$$

which sends a diagram D to the coefficient system $(\mathcal{F}_\tau)_\tau$ as above.

One needs to check that given three diagrams and two morphisms between them

$$D \xrightarrow{\psi} D' \xrightarrow{\psi'} D''$$

we have

$$\mathcal{C}(\psi' \circ \psi) = \mathcal{C}(\psi') \circ \mathcal{C}(\psi).$$

However, that is immediate from the construction of $\mathcal{C}(\psi)$ in Section 5.5.4.

Theorem 5.5.4. — *The functors $\mathcal{C}$ and $\mathcal{D}$ induce an equivalence of categories between $\mathcal{DIAG}$ and $\mathcal{COEF}_G$.*

Proof. — Let $D = (D_0, D_1, r)$ be an object in $\mathcal{DIAG}$. Then

$$(\mathcal{D} \circ \mathcal{C})(D) = \widetilde{D} = (\mathcal{F}_{\sigma_0}, \mathcal{F}_{\sigma_1}, r_{\sigma_0}^{\sigma_1})$$

with the notation of Section 5.5.2. The isomorphism

$$\mathrm{ev} : \widetilde{D} \cong D$$

of Section 5.5.2 is given by the evaluation at 1. We claim that it induces an isomorphism of functors between $\mathcal{D} \circ \mathcal{C}$ and $\mathrm{id}_{\mathcal{DIAG}}$. We only need to check what happens to morphisms. Let $D' = (D_0', D_1', r')$ be another object in the category of diagrams and let $\psi = (\psi_0, \psi_1)$ be a morphism

$$\psi : D \longrightarrow D'.$$

Let $(\mathcal{D} \circ \mathcal{C})(D') = \widetilde{D}' = (\mathcal{F}_{\sigma_0}', \mathcal{F}_{\sigma_1}', (r')_{\sigma_0}^{\sigma_1})$ and let

$$(\mathcal{D} \circ \mathcal{C})(\psi) = \widetilde{\psi} = (\widetilde{\psi}_0, \widetilde{\psi}_1)$$

be a morphism induced by a functor $\mathcal{D} \circ \mathcal{C}$. We need to show that the diagram:

$$\begin{tikzcd} \widetilde{D}' \arrow[r, "\mathrm{ev}"] & D' \\ \widetilde{D} \arrow[u, "\widetilde{\psi}"] \arrow[r, "\mathrm{ev}"] & D \arrow[u, "\psi"'] \end{tikzcd}$$

commutes. Let $i \in \{0, 1\}$, let $f \in \mathcal{F}_{\sigma_i}$ and let $v = f(1)$ then

$$(\psi_i \circ \mathrm{ev}_i)(f) = \psi_i(v).$$

From Section 5.5.4 $\widetilde{\psi}_i(f)$ is the unique function in $\mathcal{F}_{\sigma_i}'$, taking value $\psi_i(v)$ at 1. Hence

$$(\mathrm{ev}_i \circ \widetilde{\psi}_i)(f) = \psi_i(v).$$

This implies that the diagram commutes.

Conversely, we need to show that the functor $\mathcal{C} \circ \mathcal{D}$ is isomorphic to $\mathrm{id}_{\mathcal{COEF}_G}$. Let $\mathcal{V} = (V_\tau)_\tau$ be a G-equivariant coefficient system with the restriction maps $t_\tau^{\tau'}$. Then $\mathcal{D}(\mathcal{V})$ is a diagram given by:

$$\begin{tikzcd} V_{\sigma_0} \, . \\ V_{\sigma_1} \arrow[u, "t_{\sigma_0}^{\sigma_1}"] \end{tikzcd}$$

Let $k \in \mathfrak{K}(\sigma_0)$ then it acts on V_{σ_0} by a linear map k_{σ_0}. Similarly, if $k \in \mathfrak{K}(\sigma_1)$ then it acts on on V_{σ_1} by a linear map k_{σ_1}. Let

$$(\mathcal{C} \circ \mathcal{D})(\mathcal{V}) = \mathcal{F} = (\mathcal{F}_\tau)_\tau$$

with the restriction maps $r_\tau^{\tau'}$. We will construct a canonical isomorphism $\mathrm{ev} = (\mathrm{ev}_\tau)_\tau$

$$\mathrm{ev} : \mathcal{F} \cong \mathcal{V}$$

of G equivariant coefficient systems. Let τ be a simplex on the tree. If τ is a vertex let $i = 0$ and if τ is an edge let $i = 1$. There exists $g \in G$ such that $\tau = g\sigma_i$. For $f \in \mathcal{F}_\tau$ we let $v = f(g^{-1})$. Then v is a vector in V_{σ_i}. We define a map ev_τ, by

$$\mathrm{ev}_\tau : \mathcal{F}_\tau \longrightarrow V_\tau, \quad f \longmapsto g_{\sigma_i} v$$

where g_{σ_i} is the linear map coming from the G action on $\mathcal{V}$. If we replace g by gk, for some $k \in \mathfrak{K}(\sigma_i)$, then

$$(gk)_{\sigma_i}(f((gk)^{-1})) = (g_{\sigma_i} \circ k_{\sigma_i} \circ k_{\sigma_i}^{-1})(f(g^{-1})) = g_{\sigma_i}(f(g^{-1})).$$

Hence, the map ev_τ is independent of the choice of g. Moreover, ev_τ is an isomorphism of vector spaces with the inverse given as follows. Let $w \in V_\tau$, let $v = (g^{-1})_\tau w$, then v is a vector in W_{σ_i}. Let f_v be the unique function in $\mathcal{F}_\tau$ such that $f_v(1) = v$. Then $(\mathrm{ev}_\tau)^{-1}$ is given by

$$(\mathrm{ev}_\tau)^{-1} : V_\tau \longrightarrow \mathcal{F}_\tau, \quad w \longmapsto g f_v$$

where the action by g is on the space $\operatorname{c-Ind}_{\mathfrak{K}(\sigma_i)}^G V_{\sigma_i}$.

The collection of maps $(\mathrm{ev}_\tau)_\tau$ is G-equivariant. Let $h \in G$, then hf belongs to the space $\mathcal{F}_{h\tau}$ and

$$\mathrm{ev}_{h\tau}(hf) = (hg)_{\sigma_i}((hf)((hg)^{-1})) = (h_\tau \circ g_{\sigma_i})(f(g^{-1})) = h_\tau(\mathrm{ev}_\tau(f)).$$

We need to show that the maps ev_τ are compatible with the restriction maps. Let τ' be an edge containing a vertex τ. We need to show that the diagram

$$\begin{tikzcd} \mathcal{F}_\tau \arrow[r, "\mathrm{ev}_\tau"] & V_\tau \\ \mathcal{F}_{\tau'} \arrow[u, "r_\tau^{\tau'}"] \arrow[r, "\mathrm{ev}_{\tau'}"] & V_{\tau'} \arrow[u, "t_\tau^{\tau'}"'] \end{tikzcd}$$

commutes. There exists $g \in G$ such that $\tau = g\sigma_0$ and $\tau' = g\sigma_1$. Let f be a function in $\mathcal{F}_{\tau'}$. Let $v_1 = f(g^{-1})$, then v_1 is a vector in V_{σ_1}. Let $v_0 = t_{\sigma_0}^{\sigma_1}(v_1)$. Then $r_\tau^{\tau'}(f)$ is the unique function of $\mathcal{F}_\tau$ taking value v_0 at g^{-1}. Hence

$$(\mathrm{ev}_\tau \circ r_\tau^{\tau'})(f) = g_{\sigma_0} v_0.$$

On the other hand

$$(t_\tau^{\tau'} \circ \mathrm{ev}_{\tau'})(f) = t_\tau^{\tau'}(g_{\sigma_1} v_1).$$

The action of G on $\mathcal{V}$ respects the restriction maps, in the sense that the diagram:

$$\begin{tikzcd} V_{\sigma_0} \arrow[r, "g_{\sigma_0}"] & V_\tau \\ V_{\sigma_1} \arrow[u, "t_{\sigma_0}^{\sigma_1}"] \arrow[r, "g_{\sigma_1}"] & V_{\tau'} \arrow[u, "t_\tau^{\tau'}"'] \end{tikzcd} \quad .$$

commutes. Hence,

$$t_\tau^{\tau'}(g_{\sigma_1} v_1) = g_{\sigma_0} v_0.$$

Hence our original diagram commutes and $\mathrm{ev} = (\mathrm{ev}_\tau)_\tau$ defines an isomorphism of G-equivariant coefficient systems.

In order to show that the morphism ev induces an isomorphism of functors between $\mathcal{C} \circ \mathcal{D}$ and $\mathrm{id}_{\mathcal{COEF}_G}$ we need to check what happens to the morphisms. However the proof is almost identical to the one given for $\mathcal{DIAG}$ so we omit it. $\square$

Corollary 5.5.5. — *Let (ρ_0, V_0) be a smooth representation of $\mathfrak{K}(\sigma_0)$ and (ρ_1, V_1) a smooth representation of $\mathfrak{K}(\sigma_1)$. Suppose that there exists an $F^\times I$-equivariant isomorphism*

$$r : V_1 \cong V_0,$$

then there exists a unique (up to isomorphism) smooth representation π of G, such that

$$\pi|_{\mathfrak{K}(\sigma_0)} \cong \rho_0, \quad \pi|_{\mathfrak{K}(\sigma_1)} \cong \rho_1$$

and the diagram

$$\begin{tikzcd} V_0 \arrow[r, "\cong"] & \pi \\ V_1 \arrow[u, "r"] \arrow[r, "\cong"] & \pi \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

of $F^\times I$-representations commutes.

Proof. — Let D be the object in $\mathcal{DIAG}$, given by $D = (V_0, V_1, r)$. Let $\mathcal{C}(D)$ be a coefficient system corresponding to D, with the restriction maps $r_\tau^{\tau'}$. Since $(\mathcal{D} \circ \mathcal{C})(D) \cong D$ and r is an isomorphism, the map $r_{\sigma_0}^{\sigma_1}$ is an isomorphism and Proposition 5.3.4 implies that $H_0(X, \mathcal{C}(D))$ satisfies the conditions of the Corollary.

The statement of the Corollary can be rephrased as follows: there exists a unique up to isomorphism smooth representation π of G, such that

$$D \cong \mathcal{D}(\mathcal{K}_\pi).$$

If π' was another such, then

$$\mathcal{D}(\mathcal{K}_{\pi'}) \cong D \cong \mathcal{D}(\mathcal{K}_\pi).$$

Hence, by Theorem 5.5.4

$$\mathcal{K}_{\pi'} \cong \mathcal{K}_\pi.$$

Lemma 5.4.2 implies that

$$\pi' \cong H_0(X, \mathcal{K}_{\pi'}) \cong H_0(X, \mathcal{K}_\pi) \cong \pi$$

and we obtain uniqueness. $\square$

Remark 5.5.6. — Let $\widetilde{W}$ be a subgroup of G generated by s and Π. The Iwahori decomposition says that $G = I\widetilde{W}I$. Let π be a representation constructed as above, $v \in \pi$ and $g \in G$. Then gv may be determined by decomposing $g = u_1 w u_2$, where $u_1, u_2 \in I$, $w \in \widetilde{W}$, and then chasing around the diagram.

The simplest example illustrating 5.5.5 is the trivial diagram $\widetilde{\mathbf{1}} = (\mathbf{1}, \mathbf{1}, \mathrm{id})$. The proof of Corollary 3.1.9 can be reinterpreted as a construction of a morphism $\widetilde{\mathbf{1}} \hookrightarrow \mathcal{D}(\mathcal{K}_\pi)$. This gives us an injection of G representations

$$\mathbf{1} \cong H_0(X, \mathcal{C}(\widetilde{\mathbf{1}})) \longhookrightarrow H_0(X, \mathcal{K}_\pi) \cong \pi.$$

CHAPTER 6

SUPERSINGULAR REPRESENTATIONS

6.1. Coefficient systems $\mathcal{V}_\gamma$

Let $\chi : H \to \overline{\mathbf{F}}_p^\times$ be a character, and let $\rho_{\chi,J}$ be an irreducible representation of Γ, with the notations of Section 3. We consider χ as a character of I and $\rho_{\chi,J}$ as a representation of K, via

$$K \longrightarrow K/K_1 \cong \Gamma \quad \text{and} \quad I \longrightarrow I/I_1 \cong H.$$

Let $\widetilde{\rho}_{\chi,J}$ be the extension of $\rho_{\chi,J}$ to $F^\times K$ such that our fixed uniformiser ϖ_F acts trivially, and let $\widetilde{\chi}$ be the extension of χ to $F^\times I$, such that ϖ_F acts trivially. The space of I_1-invariants of $\widetilde{\rho}_{\chi,J}$ is one dimensional and $F^\times I$ acts on it via the character $\widetilde{\chi}$. We fix a vector $v_{\chi,J}$ such that

$$\rho_{\chi,J}^{I_1} = \langle v_{\chi,J}\rangle_{\overline{\mathbf{F}}_p}.$$

Lemma 6.1.1. — *There exists a unique action of $\mathfrak{K}(\sigma_1)$ on $(\widetilde{\rho}_{\chi,J} \oplus \widetilde{\rho}_{\chi^s,\overline{J}})^{I_1}$, extending the action of $F^\times I$, such that*

$$\Pi^{-1} v_{\chi,J} = v_{\chi^s,\overline{J}} \quad \textit{and} \quad \Pi^{-1} v_{\chi^s,\overline{J}} = v_{\chi,J}.$$

Moreover, with this action

$$(\widetilde{\rho}_{\chi,J} \oplus \widetilde{\rho}_{\chi^s,\overline{J}})^{I_1} \cong \operatorname{Ind}_{F^\times I}^{\mathfrak{K}(\sigma_1)} \widetilde{\chi}$$

as $\mathfrak{K}(\sigma_1)$-representations.

Proof. — We note that if $t \in T$ is a diagonal matrix then $\Pi t \Pi^{-1} = sts$, hence $(\widetilde{\chi})^\Pi \cong \widetilde{\chi^s}$ as representations of $F^\times I$ and Mackey's decomposition gives us

$$(\operatorname{Ind}_{F^\times I}^{\mathfrak{K}(\sigma_1)} \widetilde{\chi})|_{F^\times I} \cong \widetilde{\chi} \oplus \widetilde{\chi^s}.$$

Since

$$(\widetilde{\rho}_{\chi,J} \oplus \widetilde{\rho}_{\chi^s,\overline{J}})^{I_1} \cong \widetilde{\chi} \oplus \widetilde{\chi^s}$$

as $F^\times I$-representation, we can extend the action. Explicitly, we consider $f \in \operatorname{Ind}_{F^\times I}^{\mathfrak{K}(\sigma_1)} \widetilde{\chi}$, such that $\operatorname{Supp} f = F^\times I$ and $f(g) = \widetilde{\chi}(g)$, for all $g \in F^\times I$. Then the map

$$f \longmapsto v_{\chi,J}, \quad \Pi^{-1} f \longmapsto v_{\chi^s,\overline{J}}$$

induces the required isomorphism. Since, Π and $F^\times I$ generate $\mathfrak{K}(\sigma_1)$ the action is unique. □

Definition 6.1.2. — Let $\chi : H \to \overline{\mathbf{F}}_p^\times$ be a character, and let $\gamma = \{\chi, \chi^s\}$ we define D_γ to be an object in $\mathcal{DIAG}$, given by

$$\begin{array}{c} \widetilde{\rho}_{\chi,J} \oplus \widetilde{\rho}_{\chi^s,\overline{J}} \\ \Big\uparrow \\ (\widetilde{\rho}_{\chi,J} \oplus \widetilde{\rho}_{\chi^s,\overline{J}})^{I_1} \end{array}$$

where the action of $\mathfrak{K}(\sigma_1)$ on $(\widetilde{\rho}_{\chi,J} \oplus \widetilde{\rho}_{\chi^s,\overline{J}})^{I_1}$ is given by Lemma 6.1.1. Moreover, we define $\mathcal{V}_\gamma$ to be a coefficient system, given by

$$\mathcal{V}_\gamma = \mathcal{C}(D_\gamma).$$

Lemma 6.1.3. — *The diagram D_γ is independent up to isomorphism of the choices made for $v_{\chi,J}$ and $v_{\chi^s,\overline{J}}$.*

Proof. — Suppose that instead we choose vectors $v'_{\chi,J}$ and $v'_{\chi^s,\overline{J}}$ and let D'_γ be the corresponding diagram. Since, the spaces $\rho_{\chi,J}^{I_1}$ and $\rho_{\chi^s,\overline{J}}^{I_1}$ are one dimensional there exist $\lambda, \mu \in \overline{\mathbf{F}}_p^\times$, such that

$$\lambda v_{\chi,J} = v'_{\chi,J}, \quad \mu v_{\chi^s,\overline{J}} = v'_{\chi^s,\overline{J}}.$$

The isomorphism

$$\lambda \operatorname{id} \oplus \mu \operatorname{id} : \widetilde{\rho}_{\chi,J} \oplus \widetilde{\rho}_{\chi^s,\overline{J}} \longrightarrow \widetilde{\rho}_{\chi,J} \oplus \widetilde{\rho}_{\chi^s,\overline{J}}$$

induces an isomorphism of diagrams $D_\gamma \cong D'_\gamma$. □

Since D_γ and $\mathcal{D}(\mathcal{V}_\gamma)$ are canonically isomorphic, to ease the notation, we identify them. Let $\omega_{\chi,J}$, $\omega_{\chi^s,\overline{J}} \in C_c^{\mathrm{or}}(X_{(0)}, \mathcal{V}_\gamma)$ supported on a single vertex σ_0, such that

$$\omega_{\chi,J}(\sigma_0) = v_{\chi,J} \quad \text{and} \quad \omega_{\chi^s,\overline{J}}(\sigma_0) = v_{\chi^s,\overline{J}}.$$

Let

$$\overline{\omega}_{\chi,J} = \omega_{\chi,J} + \partial C_c^{\mathrm{or}}(X_{(1)}, \mathcal{V}_\gamma) \quad \text{and} \quad \overline{\omega}_{\chi^s,\overline{J}} = \omega_{\chi^s,\overline{J}} + \partial C_c^{\mathrm{or}}(X_{(1)}, \mathcal{V}_\gamma)$$

be their images in $H_0(X, \mathcal{V}_\gamma)$.

Lemma 6.1.4. — *We have*

$$\langle \overline{\omega}_{\chi,J}, \overline{\omega}_{\chi^s,\overline{J}} \rangle_{\overline{\mathbf{F}}_p} \cong M_\gamma$$

as right $\mathcal{H}$-modules.

Proof. — Since the restriction maps in $\mathcal{V}_\gamma$ are injective, Lemma 5.3.2 says that $\overline{\omega}_{\chi,J}$ and $\overline{\omega}_{\chi^s,\overline{J}}$ are non-zero. We have

$$\langle v_{\chi,J}\rangle_{\overline{\mathbf{F}}_p} = (\widetilde{\rho}_{\chi,J})^{I_1} \cong M_{\chi,J} \quad \text{and} \quad \langle v_{\chi^s,\overline{J}}\rangle_{\overline{\mathbf{F}}_p} = (\widetilde{\rho}_{\chi^s,\overline{J}})^{I_1} \cong M_{\chi^s,\overline{J}}$$

as $\mathcal{H}_K$-modules. Hence $\overline{\omega}_{\chi,J}$ and $\overline{\omega}_{\chi^s,\overline{J}}$ are fixed by I_1 and

$$\langle \overline{\omega}_{\chi,J}\rangle_{\overline{\mathbf{F}}_p} \oplus \langle \overline{\omega}_{\chi^s,\overline{J}}\rangle_{\overline{\mathbf{F}}_p} \cong M_{\chi,J} \oplus M_{\chi^s,\overline{J}}$$

as $\mathcal{H}_K$-modules. Corollary 2.0.7 and Lemma 5.3.4 imply that

$$\overline{\omega}_{\chi,J} T_\Pi = \Pi^{-1}\overline{\omega}_{\chi,J} = \overline{\omega}_{\chi^s,\overline{J}} \quad \text{and} \quad \overline{\omega}_{\chi^s,\overline{J}} T_\Pi = \Pi^{-1}\overline{\omega}_{\chi^s,\overline{J}} = \overline{\omega}_{\chi,J}.$$

Hence

$$\langle \overline{\omega}_{\chi,J}, \overline{\omega}_{\chi^s,\overline{J}}\rangle_{\overline{\mathbf{F}}_p} \cong M_\gamma$$

as $\mathcal{H}$-modules. □

LEMMA 6.1.5. — *The vector $\overline{\omega}_{\chi,J}$ (resp. $\overline{\omega}_{\chi^s,\overline{J}}$) generates $H_0(X,\mathcal{V}_\gamma)$ as a G-representation.*

Proof. — Lemma 5.3.4 implies that $\Pi^{-1}\overline{\omega}_{\chi,J} = \overline{\omega}_{\chi^s,\overline{J}}$. Hence, it is enough to show that $\omega_{\chi,J}$ and $\omega_{\chi^s,\overline{J}}$ generate $C_c^{\mathrm{or}}(X_{(0)},\mathcal{V}_\gamma)$ as a G-representation. Since, $\rho_{\chi,J}$ and $\rho_{\chi^s,\overline{J}}$ are irreducible K-representations, $\omega_{\chi,J}$ and $\omega_{\chi^s,\overline{J}}$ will generate the space

$$C_c^{\mathrm{or}}(\sigma_0,\mathcal{V}_\gamma) = \{\omega \in C_c^{\mathrm{or}}(X_{(0)},\mathcal{V}_\gamma) : \operatorname{Supp}\omega \subseteq \sigma_0\}$$

as a K-representation. Since the action of G on the vertices of X is transitive, the space $C_c^{\mathrm{or}}(\sigma_0,\mathcal{V}_\gamma)$ will generate $C_c^{\mathrm{or}}(X_{(0)},\mathcal{V}_\gamma)$ as a G-representation. □

COROLLARY 6.1.6. — *Let π be a non-zero irreducible quotient of $H_0(X,\mathcal{V}_\gamma)$, then π is a supersingular representation.*

Proof. — Lemma 6.1.5 implies that the images of $\overline{\omega}_{\chi,J}$ and $\overline{\omega}_{\chi^s,\overline{J}}$ in π are non-zero. Hence, by Lemma 6.1.4, π^{I_1} will contain a supersingular module M_γ, then Corollary 2.1.5 implies that π is supersingular. □

PROPOSITION 6.1.7. — *Let π be a smooth representation of G and suppose that there exists $v_1, v_2 \in \pi^{I_1}$ such that*

$$\langle Kv_1\rangle_{\overline{\mathbf{F}}_p} \cong \rho_{\chi,J}, \quad \langle Kv_2\rangle_{\overline{\mathbf{F}}_p} \cong \rho_{\chi^s,\overline{J}}, \quad \Pi^{-1}v_1 = v_2, \quad \Pi^{-1}v_2 = v_1,$$

then there exists a G-equivariant map $\phi : H_0(X,\mathcal{V}_\gamma) \to \pi$ such that

$$\phi(\overline{\omega}_{\chi,J}) = v_1 \quad \textit{and} \quad \phi(\overline{\omega}_{\chi^s,\overline{J}}) = v_2$$

where $\gamma = \{\chi,\chi^s\}$.

Proof. — By Lemma 5.4.2 and Theorem 5.5.4, it is enough to construct a morphism of diagrams $D_\gamma \to \mathcal{D}(\mathcal{K}_\pi)$. However, such morphism is immediate. □

Corollary 6.1.8. — *Let π be a smooth representation of G and suppose that one of the following holds: $\chi = \chi^s$, or $p = q$, then*

$$\mathrm{Hom}_G(H_0(X, \mathcal{V}_\gamma), \pi) \cong \mathrm{Hom}_{\mathcal{H}}(M_\gamma, \pi^{I_1}).$$

Remark 6.1.9. — This fails if $q \neq p$ and $\chi \neq \chi^s$. Proposition 6.4.5 gives an example.

Proof. — Lemmas 6.1.4 and 6.1.5 imply that we always have an injection

$$\mathrm{Hom}_G(H_0(X, \mathcal{V}_\gamma), \pi) \hookrightarrow \mathrm{Hom}_{\mathcal{H}}(M_\gamma, \pi^{I_1}).$$

By Lemma 2.2.2 $M_\gamma|_{\mathcal{H}_K} \cong M_{\chi,J} \oplus M_{\chi^s,\overline{J}}$. Under the assumptions made, Corollaries 2.0.7, 3.1.8 and respectively 4.1.5 give us vectors $v_1, v_2 \in \pi^{I_1}$ as in Proposition 6.1.7, hence the injection is an isomorphism. □

Corollary 6.1.10. — *Let π be a smooth representation, and suppose that $\pi^{I_1} \cong M_\gamma$, then*

$$\dim \mathrm{Hom}_G(H_0(X, \mathcal{V}_\gamma), \pi) = 1.$$

Proof. — It is enough to consider the case $p \neq q$ and $\chi \neq \chi^s$. Since Corollary 6.1.8 implies the statement in the other cases. Let $\rho = \langle K\pi^{I_1}\rangle_{\overline{\mathbf{F}}_p}$, then $\rho^{I_1} = \pi^{I_1}$. Hence

$$\rho^{I_1} \cong M_\gamma|_{\mathcal{H}_K} \cong M_{\chi,\varnothing} \oplus M_{\chi^s,\varnothing}$$

as an $\mathcal{H}_K$-module. Proposition 4.2.40 implies that $\rho \cong \rho_{\chi,\varnothing} \oplus \rho_{\chi^s,\varnothing}$. The action of Π on π^{I_1} is given by Corollary 2.0.7. Now we may apply Proposition 6.1.7 to get a non-zero homomorphism. So the dimension is at least one. The module M_γ is irreducible, and Lemmas 6.1.4 and 6.1.5 imply that the dimension is at most one. □

6.2. Injective envelopes

For the convenience of the reader we recall some general facts about injective envelopes. Let $\mathcal{K}$ be a pro-finite group and let $\mathrm{Rep}_{\mathcal{K}}$ be the category of smooth $\overline{\mathbf{F}}_p$-representations of $\mathcal{K}$. We assume that $\mathcal{K}$ has an open normal pro-p subgroup $\mathcal{P}$.

Definition 6.2.1. — Let $\pi \in \mathrm{Rep}_{\mathcal{K}}$ and let ρ be a $\mathcal{K}$-invariant subspace of π. We say that π is an essential extension of ρ if for every non-zero $\mathcal{K}$-invariant subspace π' of π, we have $\pi' \cap \rho \neq 0$.

Let $\rho \in \mathrm{Rep}_{\mathcal{K}}$ and let Inj be an injective object in $\mathrm{Rep}_{\mathcal{K}}$. A monomorphism $\iota : \rho \hookrightarrow \mathrm{Inj}$ is called an injective envelope of ρ, if Inj is an essential extension of $\iota(\rho)$.

Proposition 6.2.2. — *Every representation $\rho \in \mathrm{Rep}_{\mathcal{K}}$ has an injective envelope $\iota : \rho \hookrightarrow \mathrm{Inj}\,\rho$. Moreover, injective envelopes are unique up to isomorphism.*

Proof. — [**16**], § 3.1. □

LEMMA 6.2.3. — *Let* Inj *be an injective object in* $\mathrm{Rep}_{\mathcal{K}}$ *and let* $\iota : \rho \to \mathrm{Inj}\,\rho$ *be an injective envelope of* ρ *in* $\mathrm{Rep}_{\mathcal{K}}$. *Let* ϕ *be a monomorphism* $\phi : \rho \hookrightarrow \mathrm{Inj}$, *then there exists a monomorphism* $\psi : \mathrm{Inj}\,\rho \hookrightarrow \mathrm{Inj}$ *such that* $\phi = \psi \circ \iota$.

Proof. — Since Inj is an injective object there exists ψ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \rho \arrow[r, "\iota"] \arrow[d, "\phi"'] & \mathrm{Inj}\,\rho \arrow[dl, dotted, "\psi"] \\ & \mathrm{Inj} & \end{tikzcd}$$

of $\mathcal{K}$-representations commutes. Since ϕ is an injection $\mathrm{Ker}\,\psi \cap \iota(\rho) = 0$. This implies that $\mathrm{Ker}\,\psi = 0$, as $\mathrm{Inj}\,\rho$ is an essential extension of $\iota(\rho)$. □

LEMMA 6.2.4. — *Let* $\rho \in \mathrm{Rep}_{\mathcal{K}}$ *be an irreducible representation and let* $\iota : \rho \hookrightarrow \mathrm{Inj}\,\rho$ *be an injective envelope of* ρ *in* $\mathrm{Rep}_{\mathcal{K}}$, *then* $\rho \hookrightarrow (\mathrm{Inj}\,\rho)^{\mathcal{P}}$ *is an injective envelope of* ρ *in* $\mathrm{Rep}_{\mathcal{K}/\mathcal{P}}$.

Proof. — We note that since $\mathcal{P}$ is an open normal pro-p subgroup of $\mathcal{K}$ and ρ is irreducible, Lemma 2.0.2 implies that $\mathcal{P}$ acts trivially on ρ. Hence, $\iota(\rho)$ is a subspace of $(\mathrm{Inj}\,\rho)^{\mathcal{P}}$. Moreover, $(\mathrm{Inj}\,\rho)^{\mathcal{P}}$ is an essential extension of $\iota(\rho)$, since $\mathrm{Inj}\,\rho$ is an essential extension of $\iota(\rho)$.

Let $\mathcal{L} : \mathrm{Rep}_{\mathcal{K}/\mathcal{P}} \to \mathrm{Rep}_{\mathcal{K}}$ be a functor sending a representation ξ to its inflation $\mathcal{L}(\xi)$ to a representation of $\mathcal{K}$, via $\mathcal{K} \to \mathcal{K}/\mathcal{P}$. Then

$$\mathrm{Hom}_{\mathcal{K}/\mathcal{P}}(\xi, (\mathrm{Inj}\,\rho)^{\mathcal{P}}) \cong \mathrm{Hom}_{\mathcal{K}}(\mathcal{L}(\xi), \mathrm{Inj}\,\rho)$$

where the isomorphism is canonical. Since, the functor $\mathcal{L}$ is exact and $\mathrm{Inj}\,\rho$ is an injective object in Rep_{K}, the functor $\mathrm{Hom}_{\mathcal{K}/\mathcal{P}}(*, (\mathrm{Inj}\,\rho)^{\mathcal{P}})$ is exact. Hence, $(\mathrm{Inj}\,\rho)^{\mathcal{P}}$ is an injective object in $\mathrm{Rep}_{\mathcal{K}/\mathcal{P}}$, which establishes the Lemma. □

DEFINITION 6.2.5. — Let $\pi \in \mathrm{Rep}_{\mathcal{K}}$, we denote by $\mathrm{soc}\,\pi$ the subspace of π, generated by all irreducible subrepresentations of π.

LEMMA 6.2.6. — *Let* $\rho \in \mathrm{Rep}_{\mathcal{K}}$ *be irreducible, and let* $\iota : \rho \hookrightarrow \mathrm{Inj}\,\rho$ *be an injective envelope of* ρ, *then* $\mathrm{soc}(\mathrm{Inj}\,\rho) \cong \rho$.

Proof. — Let τ be any non-zero $\mathcal{K}$ invariant subspace of $\mathrm{Inj}\,\rho$, which is irreducible as a representation of $\mathcal{K}$. Since $\mathrm{Inj}\,\rho$ is an essential extension of $\iota(\rho)$ and ρ is irreducible, we have $\tau = \iota(\rho)$. Hence, $\mathrm{soc}(\mathrm{Inj}\,\rho) = \iota(\rho)$. □

6.3. Admissibility

Let $\mathcal{G}$ be a locally pro-finite group and let $\mathrm{Rep}_{\mathcal{G}}$ be the category of smooth $\overline{\mathbf{F}}_p$-representations of $\mathcal{G}$.

DEFINITION 6.3.1. — A representation $\pi \in \mathrm{Rep}_{\mathcal{G}}$ is called admissible, if for every open subgroup $\mathcal{K}$ of $\mathcal{G}$, the space $\pi^{\mathcal{K}}$ of $\mathcal{K}$-invariants is finite dimensional.

THEOREM 6.3.2. — *Suppose that $\mathcal{G}$ has an open pro-p subgroup $\mathcal{P}$. A representation $\pi \in \mathrm{Rep}_{\mathcal{G}}$ is admissible if and only if $\pi^{\mathcal{P}}$ is finite dimensional.*

Proof. — If π is admissible, then $\pi^{\mathcal{P}}$ is finite dimensional. Suppose that $\pi^{\mathcal{P}}$ is finite dimensional and let $\mathbf{1} \hookrightarrow \mathrm{Inj}\,\mathbf{1}$ be an injective envelope of the trivial representation in $\mathrm{Rep}_{\mathcal{P}}$, then there exists ψ, such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi^{\mathcal{P}} \arrow[r] \arrow[d] & \pi|_{\mathcal{P}} \arrow[dl, dotted, "\psi"] \\ & (\dim \pi^{\mathcal{P}})\,\mathrm{Inj}\,\mathbf{1} & \end{tikzcd}$$

of $\mathcal{P}$-representations commutes. This implies that $(\mathrm{Ker}\,\psi)^{\mathcal{P}} = 0$, and hence by Lemma 2.0.2, ψ is injective.

Let $\mathcal{K}$ be any open subgroup of $\mathcal{G}$. Since $\mathcal{P}$ is an open compact subgroup of $\mathcal{G}$, we may choose an open subgroup $\mathcal{P}'$ of $\mathcal{G}$ such that $\mathcal{P}'$ is a subgroup of $\mathcal{P} \cap \mathcal{K}$ and $\mathcal{P}'$ is normal in $\mathcal{P}$. It is enough to show that $\pi^{\mathcal{P}'}$ is finite dimensional. Since ψ is an injection, it is enough to show that $(\mathrm{Inj}\,\mathbf{1})^{\mathcal{P}'}$ is finite dimensional. Since $\mathcal{P}$ is pro-p and $\mathcal{P}'$ is a normal open subgroup of $\mathcal{P}$, Lemma 6.2.4 and Proposition 4.0.8 imply that

$$(\mathrm{Inj}\,\mathbf{1})^{\mathcal{P}'} \cong \overline{\mathbf{F}}_p[\mathcal{P}/\mathcal{P}']$$

which is finite dimensional. $\square$

6.4. Coefficient systems $\mathcal{I}_\gamma$

Let $\chi : H \to \overline{\mathbf{F}}_p^{\times}$ be a character, and let

$$\rho_{\chi,J} \hookrightarrow \mathrm{Inj}\,\rho_{\chi,J}, \quad \rho_{\chi^s,\overline{J}} \hookrightarrow \mathrm{Inj}\,\rho_{\chi^s,\overline{J}}$$

be injective envelopes of $\rho_{\chi,J}$ and $\rho_{\chi^s,\overline{J}}$ in Rep_K, respectively. We may extend the action of K to the action of $F^\times K$, so that our fixed uniformiser ϖ_F acts trivially. We get an exact sequence

$$0 \longrightarrow \widetilde{\rho}_{\chi,J} \oplus \widetilde{\rho}_{\chi^s,\overline{J}} \longrightarrow \widetilde{\mathrm{Inj}}\rho_{\chi,J} \oplus \widetilde{\mathrm{Inj}}\rho_{\chi^s,\overline{J}}$$

of $F^\times K$-representations. This gives a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{\rho}_{\chi,J} \oplus \widetilde{\rho}_{\chi^s,\overline{J}} \arrow[r] & \widetilde{\mathrm{Inj}}\rho_{\chi,J} \oplus \widetilde{\mathrm{Inj}}\rho_{\chi^s,\overline{J}} \\ 0 \arrow[r] & (\widetilde{\rho}_{\chi,J} \oplus \widetilde{\rho}_{\chi^s,\overline{J}})^{I_1} \arrow[r] \arrow[u] & \widetilde{\mathrm{Inj}}\rho_{\chi,J} \oplus \widetilde{\mathrm{Inj}}\rho_{\chi^s,\overline{J}} \arrow[u] \end{tikzcd}$$

of $F^\times I$-representations. We will show that we may extend the action of $F^\times I$ on $(\widetilde{\mathrm{Inj}}\rho_{\chi,J} \oplus \widetilde{\mathrm{Inj}}\rho_{\chi^s,\overline{J}})|_{F^\times I}$ to the action of $\mathfrak{K}(\sigma_1)$, so that we get an object Y_γ in $\mathcal{DIAG}$, together with an embedding $D_\gamma \hookrightarrow Y_\gamma$. Since the categories $\mathcal{DIAG}$ and $\mathcal{COEF}_G$ are equivalent, this will give us an embedding of coefficient systems $\mathcal{V}_\gamma \hookrightarrow \mathcal{I}_\gamma$. We will show that the image

$$\pi_\gamma = \mathrm{Im}(H_0(X, \mathcal{V}_\gamma) \longrightarrow H_0(X, \mathcal{I}_\gamma))$$

is an irreducible supersingular representation of G. All the hard work was done in Propositions 4.2.37 and 4.2.38, the construction of Y_γ and the proof of irreducibility follow from the 'general non-sense' of Section 6.2. This gives hope that similar construction might work for other groups.

LEMMA 6.4.1. — *Let ρ be an irreducible representation of K and let*

$$\rho \hookrightarrow \mathrm{Inj}\,\rho$$

be an injective envelope of ρ in Rep_K, then

$$(\mathrm{Inj}\,\rho)|_I \cong \bigoplus_\chi \dim \mathrm{Hom}_H(\chi, (\mathrm{inj}\,\rho)^U)\,\mathrm{Inj}\,\chi$$

where the sum is taken over all irreducible representations of H, which we identify with the irreducible representations of I and

$$\rho \hookrightarrow \mathrm{inj}\,\rho, \quad \chi \hookrightarrow \mathrm{Inj}\,\chi$$

are the injective envelopes of ρ in Rep_Γ and of χ in Rep_I, respectively.

Proof. — If χ is an irreducible representation of I, then Lemma 2.0.2 implies that I_1 acts trivially on χ. Since $I/I_1 \cong H$, the irreducible representations of I and H coincide. Moreover, since H is abelian, all the irreducible representations of H are one dimensional. Since, the order of H is prime to p, all the representations of H are semi-simple. Therefore

$$(\mathrm{Inj}\,\rho)^{I_1} \cong \bigoplus_\chi m_\chi \chi$$

as a representation of I, where the multiplicity m_χ of χ is given by

$$m_\chi = \dim \mathrm{Hom}_I(\chi, \mathrm{Inj}\,\rho).$$

Lemma 6.2.4 implies that $(\mathrm{Inj}\,\rho)^{K_1} \cong \mathrm{inj}\,\rho$ as representations of $K/K_1 \cong \Gamma$. Corollary 4.0.6 implies that $\mathrm{inj}\,\rho$ is finite dimensional. In particular, m_χ is finite for every χ. Moreover,

$$\mathrm{Hom}_I(\chi, \mathrm{Inj}\,\rho) \cong \mathrm{Hom}_I(\chi, (\mathrm{Inj}\,\rho)^{K_1}) \cong \mathrm{Hom}_B(\chi, \mathrm{inj}\,\rho) \cong \mathrm{Hom}_H(\chi, (\mathrm{inj}\,\rho)^U).$$

Hence, $m_\chi = \dim \mathrm{Hom}_H(\chi, (\mathrm{inj}\,\rho)^U)$. We consider an exact sequence

$$0 \longrightarrow (\mathrm{Inj}\,\rho)^{I_1} \longrightarrow (\mathrm{Inj}\,\rho)|_I$$

of I-representations. The restriction $(\mathrm{Inj}\,\rho)|_I$ is an injective object in Rep_I. Lemma 6.2.3 implies that

$$(\mathrm{Inj}\,\rho)|_I \cong \mathcal{N} \oplus \bigoplus_{\chi} m_\chi \,\mathrm{Inj}\,\chi$$

for some representation $\mathcal{N}$. Since Rep_H is semi-simple and $\mathrm{Inj}\,\chi$ is an essential extension of χ, Lemma 6.2.4 implies that $(\mathrm{Inj}\,\chi)^{I_1} \cong \chi$. By comparing the dimensions of I_1-invariants of both sides we get that $\dim \mathcal{N}^{I_1} = 0$ and Lemma 2.0.2 implies that $\mathcal{N} = 0$. □

LEMMA 6.4.2. — *Let $\chi : H \to \overline{\mathbf{F}}_p^\times$ be a character. We consider χ and χ^s as one dimensional representations of I, via $I/I_1 \cong H$. Let*

$$\chi \hookrightarrow \mathrm{Inj}\,\chi, \quad \chi^s \hookrightarrow \mathrm{Inj}\,\chi^s$$

be injective envelopes of χ and χ^s in Rep_I, respectively. Let V_1 be the underlying vector space of $\mathrm{Inj}\,\chi$ and let V_2 be the underlying vector space of $\mathrm{Inj}\,\chi^s$. Further, let v_1 and v_2 be vectors in V_1 and V_2 respectively, such that

$$\langle v_1\rangle_{\overline{\mathbf{F}}_p} = (\mathrm{Inj}\,\chi)^{I_1}, \quad \langle v_2\rangle_{\overline{\mathbf{F}}_p} = (\mathrm{Inj}\,\chi^s)^{I_1}.$$

Then there exists an action of $\mathfrak{K}(\sigma_1)$ on $V_1 \oplus V_2$, extending the action of I, so that our fixed uniformiser ϖ_F acts trivially and

$$\Pi^{-1} v_1 = v_2, \quad \Pi^{-1} v_2 = v_1.$$

Proof. — Let $t \in T$ be any diagonal matrix, then $sts = \Pi t \Pi^{-1}$. Hence

$$\chi^s \cong \chi^\Pi$$

as I-representations, where χ^Π denotes the action of I, on the underlying vector space of χ, twisted by Π. So we get an exact sequence

$$0 \longrightarrow \chi^s \longrightarrow (\mathrm{Inj}\,\chi)^\Pi$$

of I-representations. Twisting by Π is an exact functor in Rep_I and

$$\mathrm{Hom}_I(\xi, (\mathrm{Inj}\,\chi)^\Pi) \cong \mathrm{Hom}_I(\xi^\Pi, \mathrm{Inj}\,\chi).$$

Since $\mathrm{Inj}\,\chi$ is an injective object in Rep_I, this implies that $(\mathrm{Inj}\,\chi)^\Pi$ is an injective object in Rep_I. Since $\mathrm{Inj}\,\chi$ is an essential extension of χ, $(\mathrm{Inj}\,\chi)^\Pi$ is an essential extension of χ^s. Since injective envelopes are unique up to isomorphism, there exists an isomorphism ϕ of I-representations

$$\phi : (\mathrm{Inj}\,\chi)^\Pi \cong \mathrm{Inj}\,\chi^s.$$

The proof of Lemma 6.4.1 shows that the space $(\mathrm{Inj}\,\chi)^{I_1}$ is one dimensional. Hence, after replacing ϕ by a scalar multiple we may assume that $\phi(v_1) = v_2$. We may extend the action of I on V_1 and V_2 to the action of $F^\times I$ by making ϖ_F act trivially. We denote the corresponding representations by $\widetilde{\mathrm{Inj}}\chi$ and $\widetilde{\mathrm{Inj}}\chi^s$. For trivial reasons

$$\phi : (\widetilde{\mathrm{Inj}}\chi)^\Pi \cong \widetilde{\mathrm{Inj}}\chi^s.$$

We consider the induced representation $\operatorname{Ind}_{F^\times I}^{\mathfrak{K}(\sigma_1)} \widetilde{\operatorname{Inj}}\chi$. Let ev_1 and ev_Π be the evaluation maps at 1 and Π respectively, then we get an $F^\times I$-equivariant isomorphism:

$$\operatorname{Ind}_{F^\times I}^{\mathfrak{K}(\sigma_1)} \widetilde{\operatorname{Inj}}\chi \cong V_1 \oplus V_2, \quad f \longmapsto \mathrm{ev}_1(f) + \phi(\mathrm{ev}_\Pi(f)).$$

The action of $\mathfrak{K}(\sigma_1)$ on the left hand side gives us the action of $\mathfrak{K}(\sigma_1)$ on $V_1 \oplus V_2$. Let $v \in V_1$ and $w \in V_2$, then the action of Π^{-1} is given by

$$\Pi^{-1}(v+w) = \phi^{-1}(w) + \phi(v)$$

and hence $\Pi^{-1}v_1 = v_2$ and $\Pi^{-1}v_2 = v_1$. □

We will construct a diagram Y_γ. This will involve making some choices. Suppose that $q = p^n$, let $\chi : H \to \overline{\mathbf{F}}_p^\times$ be a character and let $\gamma = \{\chi, \chi^s\}$. We consider an irreducible representation $\rho_{\chi,J}$ of K. Lemma 3.2.2 gives us a pair $(\boldsymbol{r}, a)$, where $\boldsymbol{r}$ is the usual n-tuple and a is an integer modulo $q-1$. Let $\rho_{\chi,J} \hookrightarrow \operatorname{Inj}\rho_{\chi,J}$ be an injective envelope of $\rho_{\chi,J}$ in Rep_K. Let $\mathcal{W}_{\boldsymbol{r}}$ be the underlying vector space of $\operatorname{Inj}\rho_{\chi,J}$. We may assume that $\mathcal{W}_{\boldsymbol{r}}$ depends only on the n-tuple $\boldsymbol{r}$. Since, if $\chi' = \chi \otimes (\det)^c$, then $\rho_{\chi',J} \cong \rho_{\chi,J} \otimes (\det)^c$ and a simple argument shows that $\rho_{\chi',J} \hookrightarrow (\operatorname{Inj}\rho_{\chi,J}) \otimes (\det)^c$ is an injective envelope of $\rho_{\chi',J}$ in Rep_K. Let

$$Y_{\gamma,0} = (\widetilde{\operatorname{Inj}}\rho_{\chi,J} \oplus \widetilde{\operatorname{Inj}}\rho_{\chi^s,\overline{J}}, \mathcal{W}_{\boldsymbol{r}} \oplus \mathcal{W}_{\boldsymbol{p-1-r}})$$

where tilde denotes the extension of the action of K to the action of $F^\times K$, so that ϖ_F acts trivially. We are going to construct an action of $\mathfrak{K}(\sigma_1)$ on $Y_{\gamma,0}|_{F^\times I}$, which extends the action of $F^\times I$, and this will give us Y_γ. However, this can be done in a lot of ways, and not all of them suit our purposes. Lemma 6.2.4 and Remark 4.0.5 imply that

$$(Y_{\gamma,0})^{K_1} \cong \operatorname{inj}\rho_{\chi,J} \oplus \operatorname{inj}\rho_{\chi^s,\overline{J}}$$

as K-representations, where on the right hand side we adopt the notation of Propositions 4.2.37 and 4.2.38. In particular,

$$(Y_{\gamma,0})^{I_1} \cong (\operatorname{inj}\rho_{\chi,J} \oplus \operatorname{inj}\rho_{\chi^s,\overline{J}})^{I_1}$$

as $\mathcal{H}_K$-modules. In Lemma 4.2.19 we have worked out a basis consisting of eigenvectors for the action of I of (a model of) $(\operatorname{inj}\rho_{\chi,J} \oplus \operatorname{inj}\rho_{\chi^s,\overline{J}})^{I_1}$. The above isomorphism gives us a basis $\mathcal{B}_\gamma$ of $(Y_{\gamma,0})^{I_1}$. Lemma 6.4.1 gives an $F^\times I$-equivariant decomposition:

$$\zeta : \mathcal{W}_{\boldsymbol{r}} \oplus \mathcal{W}_{\boldsymbol{p-1-r}} \cong \bigoplus_{b \in \mathcal{B}_\gamma} \mathcal{W}(b)$$

such that $\zeta(b) \in \mathcal{W}(b)$, for every $b \in \mathcal{B}_\gamma$, and the representation, given by the action of I on $\mathcal{W}(b)$, is an injective object in Rep_I, which is also an essential extension of $\langle\zeta(b)\rangle_{\overline{\mathbf{F}}_p}$. To simplify things we view ζ as identification and omit it from our notation.

If $\chi = \chi^s$ then we pair up the basis vectors as in Proposition 4.2.37:

$$\mathcal{B}_\gamma = \{b_{\mathbf{0}}, b_{\mathbf{0}} + b_{\mathbf{1}}\} \bigcup_{\{\varepsilon, \mathbf{1}-\varepsilon\} \subseteq \Sigma'_{\mathbf{0}}} \{b_\varepsilon, b_{\mathbf{1}-\varepsilon}\}.$$

If $\chi \neq \chi^s$ then we pair up the basis vectors as in Proposition 4.2.38:

$$\mathcal{B}_\gamma = \{b_{\mathbf{0}}, \overline{b}_{\mathbf{0}}\} \cup \{b_{\mathbf{1}}, \overline{b}_{\mathbf{1}}\} \bigcup_{\{\varepsilon, \mathbf{1}-\varepsilon\} \subseteq \Sigma'_{\boldsymbol{r}}} \{b_\varepsilon, b_{\mathbf{1}-\varepsilon}\} \bigcup_{\{\varepsilon, \mathbf{1}-\varepsilon\} \subseteq \Sigma'_{\boldsymbol{p-1-r}}} \{\overline{b}_\varepsilon, \overline{b}_{\mathbf{1}-\varepsilon}\}.$$

Let $\{b, b'\}$ be any such pair and suppose that I acts on b via a character ψ, then I will act on b', via a character ψ^s. We denote

$$\mathcal{W}(b, b') = \mathcal{W}(b) \oplus \mathcal{W}(b').$$

Lemma 6.4.2 implies that there exists an action of $\mathfrak{K}(\sigma_1)$ on $\mathcal{W}(b, b')$, extending the action of $F^\times I$, such that

$$\Pi^{-1} b = b', \quad \Pi^{-1} b' = b.$$

This amounts to fixing an isomorphism of vector spaces $\phi : \mathcal{W}(b) \cong \mathcal{W}(b')$, such that $\phi(b) = b'$ and which induces an isomorphism of I representations $\phi : (\operatorname{Inj} \psi)^\Pi \cong \operatorname{Inj} \psi^s$.

If $\chi = \chi^s$ then $Y_{\gamma,0}$ decomposes into $F^\times I$-invariant subspaces:

$$\mathcal{W}(b_{\mathbf{0}}, b_{\mathbf{0}} + b_{\mathbf{1}}) \bigoplus_{\{\varepsilon, \mathbf{1}-\varepsilon\} \subseteq \Sigma'_{\mathbf{0}}} \mathcal{W}(b_\varepsilon, b_{\mathbf{1}-\varepsilon}).$$

If $\chi \neq \chi^s$ then $Y_{\gamma,0}$ decomposes into $F^\times I$-invariant subspaces:

$$\mathcal{W}(b_{\mathbf{0}}, \overline{b}_{\mathbf{0}}) \oplus \mathcal{W}(b_{\mathbf{1}}, \overline{b}_{\mathbf{1}}) \bigoplus_{\{\varepsilon, \mathbf{1}-\varepsilon\} \subseteq \Sigma'_{\boldsymbol{r}}} \mathcal{W}(b_\varepsilon, b_{\mathbf{1}-\varepsilon}) \bigoplus_{\{\varepsilon, \mathbf{1}-\varepsilon\} \subseteq \Sigma'_{\boldsymbol{p-1-r}}} \mathcal{W}(\overline{b}_\varepsilon, \overline{b}_{\mathbf{1}-\varepsilon}).$$

Let $Y_{\gamma,1}$ be a representation of $\mathfrak{K}(\sigma_1)$, whose underlying vector space is $\mathcal{W}_{\boldsymbol{r}} \oplus \mathcal{W}_{\boldsymbol{p-1-r}}$, and the action of $\mathfrak{K}(\sigma_1)$ extends the action of $F^\times I$ on each direct summand, as it was done for $\mathcal{W}(b, b')$.

Definition 6.4.3. — Let Y_γ be an object in $\mathcal{DIAG}$, given by

$$Y_\gamma = (Y_{\gamma,0}, Y_{\gamma,1}, \mathrm{id})$$

and let $\mathcal{I}_\gamma$ be the corresponding coefficient system

$$\mathcal{I}_\gamma = \mathcal{C}(Y_\gamma).$$

Remark 6.4.4. — The definition of Y_γ depends on all the choices we have made.

Proposition 6.4.5. — *Let $\chi : H \to \overline{\mathbf{F}}_p^\times$ be a character and let $\gamma = \{\chi, \chi^s\}$. Suppose that $\chi = \chi^s$, then*

$$H_0(X, \mathcal{I}_\gamma)^{I_1} \cong M_\gamma \bigoplus_{\{\varepsilon, \mathbf{1}-\varepsilon\} \subseteq \Sigma'_{\mathbf{0}}} M_{\gamma_\varepsilon}$$

as $\mathcal{H}$-modules, where $\gamma_{\boldsymbol{\varepsilon}} = \gamma_{\mathbf{1}-\boldsymbol{\varepsilon}} = \{\chi \alpha^{\boldsymbol{\varepsilon} \cdot (\boldsymbol{p-1})}, \chi(\alpha^{\boldsymbol{\varepsilon} \cdot (\boldsymbol{p-1})})^s\}$. Suppose that $\chi \neq \chi^s$, then

$$H_0(X, \mathcal{I}_\gamma)^{I_1} \cong L_\gamma \bigoplus_{\{\varepsilon, \mathbf{1}-\varepsilon\} \subseteq \Sigma'_{\boldsymbol{r}}} M_{\gamma_{\boldsymbol{\varepsilon}}} \bigoplus_{\{\varepsilon, \mathbf{1}-\varepsilon\} \subseteq \Sigma'_{\boldsymbol{p-1-r}}} M_{\overline{\gamma}_\varepsilon}$$

as $\mathcal{H}$-modules, where $\gamma_{\boldsymbol{\varepsilon}} = \gamma_{\mathbf{1}-\boldsymbol{\varepsilon}} = \{\chi \alpha^{\boldsymbol{\varepsilon} \cdot (\boldsymbol{p-1-r})}, (\chi \alpha^{\boldsymbol{\varepsilon} \cdot (\boldsymbol{p-1-r})})^s\}$ and $\overline{\gamma}_{\boldsymbol{\varepsilon}} = \overline{\gamma}_{\mathbf{1}-\boldsymbol{\varepsilon}} = \{\chi^s \alpha^{\boldsymbol{\varepsilon} \cdot \boldsymbol{r}}, (\chi^s \alpha^{\boldsymbol{\varepsilon} \cdot \boldsymbol{r}})^s\}$.

Proof. — In Propositions 4.2.37 and 4.2.38 we have showed that we may extend the action of $\mathcal{H}_K$ on $(\operatorname{inj}\rho_{\chi,J}\oplus\operatorname{inj}\rho_{\chi^s,\overline{J}})^{I_1}$ to the action of $\mathcal{H}$, so that the resulting modules are isomorphic to the ones considered above. We will show that $H_0(X,\mathcal{I}_\gamma)^{I_1}$ realizes this extension. By Proposition 5.3.5 (or alternatively Corollary 5.5.5) we have

$$H_0(X,\mathcal{I}_\gamma)|_{\mathfrak{K}(\sigma_0)}\cong Y_{\gamma,0},\quad H_0(X,\mathcal{I}_\gamma)|_{\mathfrak{K}(\sigma_1)}\cong Y_{\gamma,1}$$

as $\mathfrak{K}(\sigma_0)$ and $\mathfrak{K}(\sigma_1)$-representations, respectively. Moreover, the diagram

$$\begin{tikzcd} Y_{\gamma,0} \arrow[r, "\cong"] & H_0(X,\mathcal{I}_\gamma) \\ Y_{\gamma,1} \arrow[u, "\mathrm{id}"] \arrow[r, "\cong"] & H_0(X,\mathcal{I}_\gamma) \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

of $F^\times I$-representations commutes. So

$$(Y_{\gamma,0})^{I_1}\cong H_0(X,\mathcal{I}_\gamma)^{I_1}$$

as $\mathcal{H}_K$-modules. Lemma 6.2.4 implies that

$$H_0(X,\mathcal{I}_\gamma)^{I_1}\cong(\operatorname{inj}\rho_{\chi,J}\oplus\operatorname{inj}\rho_{\chi^s,\overline{J}})^{I_1}$$

as $\mathcal{H}_K$-modules, and we know the right hand side from Propositions 4.2.37 and 4.2.38. It remains to determine the action of T_Π. Corollary 2.0.7 implies that for every $v\in H_0(X,\mathcal{I}_\gamma)^{I_1}$ we have

$$vT_\Pi=\Pi^{-1}v.$$

Hence the action of T_Π is determined by the isomorphism

$$Y_{\gamma,1}\cong H_0(X,\mathcal{I}_\gamma)|_{\mathfrak{K}(\sigma_1)}.$$

Since $\mathcal{B}_\gamma$ is a basis of $(Y_{\gamma,0})^{I_1}$, it is enough to know how Π^{-1} acts on the basis vectors. Let $\mathcal{W}(b,b')$ be one of the $\mathfrak{K}(\sigma_1)$-invariant subspaces of $Y_{\gamma,1}$, as before. We have extended the action of $F^\times I$ on $Y_{\gamma,0}|_{F^\times I}$ to $\mathfrak{K}(\sigma_1)$ so that

$$\Pi^{-1}b=b',\quad \Pi^{-1}b'=b.$$

Hence, if we consider $\mathcal{B}_\gamma$ also as a basis of $H_0(X,\mathcal{I}_\gamma)^{I_1}$ we have

$$bT_\Pi=b',\quad b'T_\Pi=b.$$

Now the statement of the Proposition is just a realization of Propositions 4.2.37 and 4.2.38. □

6.5. Construction

Now we will construct an embedding $D_\gamma\hookrightarrow Y_\gamma$. Suppose that $\chi=\chi^s$, then we consider vectors $b_{\mathbf{0}}$ and $b_{\mathbf{0}}+b_{\mathbf{1}}$ in $(Y_{\gamma,0})^{I_1}$. Lemmas 4.2.21 and 4.2.30 imply that

$$\langle Kb_{\mathbf{0}}\rangle_{\overline{\mathbf{F}}_p}\cong\widetilde{\rho}_{\chi,S},\quad\langle K(b_{\mathbf{0}}+b_{\mathbf{1}})\rangle_{\overline{\mathbf{F}}_p}\cong\widetilde{\rho}_{\chi,\varnothing}$$

as $F^{\times}K$-representations. We have constructed the action of $\mathfrak{K}(\sigma_1)$ on $Y_{\gamma,1}$ so that

$$\Pi^{-1}b_{\mathbf{0}} = b_{\mathbf{0}} + b_{\mathbf{1}}, \quad \Pi^{-1}(b_{\mathbf{0}} + b_{\mathbf{1}}) = b_{\mathbf{0}}.$$

Suppose that $\chi \neq \chi^s$, then we consider vectors $b_{\mathbf{0}}$ and $\overline{b}_{\mathbf{0}}$ in $(Y_{\gamma,0})^{I_1}$. Lemmas 4.2.21 implies that

$$\langle Kb_{\mathbf{0}}\rangle_{\overline{\mathbf{F}}_p} \cong \widetilde{\rho}_{\chi,\varnothing}, \quad \langle K\overline{b}_{\mathbf{0}}\rangle_{\overline{\mathbf{F}}_p} \cong \widetilde{\rho}_{\chi^s,\varnothing}$$

as $F^{\times}K$-representations. We have constructed the action of $\mathfrak{K}(\sigma_1)$ on $Y_{\gamma,1}$ so that

$$\Pi^{-1}b_{\mathbf{0}} = \overline{b}_{\mathbf{0}}, \quad \Pi^{-1}\overline{b}_{\mathbf{0}} = b_{\mathbf{0}}.$$

Hence, in both cases we get an embedding $D_\gamma \hookrightarrow Y_\gamma$ in the category $\mathcal{DIAG}$. This gives us an embedding of G equivariant coefficient systems $\mathcal{V}_\gamma \hookrightarrow \mathcal{I}_\gamma$.

Definition 6.5.1. — Let π_γ be a representation of G, given by

$$\pi_\gamma = \operatorname{Im}(H_0(X, \mathcal{V}_\gamma) \longrightarrow H_0(X, \mathcal{I}_\gamma)).$$

Theorem 6.5.2. — *For each $\gamma = \{\chi, \chi^s\}$, the representation π_γ is irreducible and supersingular. Moreover, $\pi_\gamma^{I_1}$ contains an $\mathcal{H}$-submodule isomorphic to M_γ. Further, if*

$$\pi_\gamma \cong \pi_{\gamma'}$$

then $\gamma = \gamma'$.

Proof. — Lemma 5.3.2 implies that π_γ is non-zero. So by Corollary 6.1.6 it is enough to prove that π_γ is irreducible. To ease the notation we identify the underlying vector spaces of $Y_{\gamma,0}$ and $H_0(X, \mathcal{I}_\gamma)$. If $\chi = \chi^s$ then Lemma 6.1.5 implies that

$$\pi_\gamma = \langle Gb_{\mathbf{0}}\rangle_{\overline{\mathbf{F}}_p} = \langle G(b_{\mathbf{0}} + b_{\mathbf{1}})\rangle_{\overline{\mathbf{F}}_p}.$$

If $\chi \neq \chi^s$ then Lemma 6.1.5 implies that

$$\pi_\gamma = \langle Gb_{\mathbf{0}}\rangle_{\overline{\mathbf{F}}_p} = \langle G\overline{b}_{\mathbf{0}}\rangle_{\overline{\mathbf{F}}_p}.$$

This can be rephrased in a different way. By Proposition 5.3.5 we have

$$H_0(X, \mathcal{I}_\gamma)|_K \cong \operatorname{Inj}\rho_{\chi,J} \oplus \operatorname{Inj}\rho_{\chi^s,\overline{J}}$$

as K-representations. Lemma 6.2.6 implies that

$$\rho_{\chi,J} \oplus \rho_{\chi^s,\overline{J}} \cong \operatorname{soc}(H_0(X, \mathcal{I}_\gamma)|_K).$$

Hence, if $\chi = \chi^s$ then

$$(\operatorname{soc}(H_0(X, \mathcal{I}_\gamma)|_K))^{I_1} = \langle b_{\mathbf{0}}, b_{\mathbf{0}} + b_{\mathbf{1}}\rangle_{\overline{\mathbf{F}}_p}$$

and if $\chi \neq \chi^s$ then

$$(\operatorname{soc}(H_0(X, \mathcal{I}_\gamma)|_K))^{I_1} = \langle b_{\mathbf{0}}, \overline{b}_{\mathbf{0}}\rangle_{\overline{\mathbf{F}}_p}$$

and hence

$$\pi_\gamma = \langle G(\operatorname{soc}(H_0(X, \mathcal{I}_\gamma)|_K))^{I_1}\rangle_{\overline{\mathbf{F}}_p}.$$

The key point is that $(\mathrm{soc}(H_0(X,\mathcal{I}_\gamma)|_K))^{I_1}$ is an $\mathcal{H}$-invariant subspace of $H_0(X,\mathcal{I}_\gamma)^{I_1}$, moreover

$$(\mathrm{soc}(H_0(X,\mathcal{I}_\gamma)|_K))^{I_1} \cong M_\gamma$$

as an $\mathcal{H}$-module. This can be deduced either from Lemma 6.1.4 or from the module computation in Proposition 6.4.5.

Suppose that π' is non-zero G-invariant subspace of π_γ then by Lemma 2.0.2 $(\pi')^{K_1} \neq 0$, and hence $\mathrm{soc}(\pi'|_K) \neq 0$. We apply Lemma 2.0.2 again to obtain $(\mathrm{soc}(\pi'|_K))^{I_1} \neq 0$. We have trivially $\mathrm{soc}(\pi'|_K) \subseteq \mathrm{soc}(H_0(X,\mathcal{I}_\gamma)|_K)$. Hence

$$0 \neq (\mathrm{soc}(\pi'|_K))^{I_1} \leqslant (\mathrm{soc}(H_0(X,\mathcal{I}_\gamma)|_K))^{I_1} \cap (\pi')^{I_1}.$$

Since the spaces $(\mathrm{soc}(H_0(X,\mathcal{I}_\gamma)|_K))^{I_1}$ and $(\pi')^{I_1}$ are $\mathcal{H}$-invariant subspaces of $H_0(X,\mathcal{I}_\gamma)^{I_1}$, and $(\mathrm{soc}(H_0(X,\mathcal{I}_\gamma)|_K))^{I_1}$ is an irreducible $\mathcal{H}$-module, we get

$$(\mathrm{soc}(H_0(X,\mathcal{I}_\gamma)|_K))^{I_1} \leqslant (\pi')^{I_1}$$

and this implies that $\pi' = \pi_\gamma$. Hence π_γ is irreducible.

Suppose that $\pi_\gamma \cong \pi_{\gamma'}$, then this induces an isomorphism of vector spaces

$$\phi : (\mathrm{soc}(\pi_\gamma|_K))^{I_1} \cong (\mathrm{soc}(\pi_{\gamma'}|_K))^{I_1}.$$

The argument above implies that both spaces are $\mathcal{H}$-invariant and Corollary 2.0.7 implies that ϕ is an isomorphism of $\mathcal{H}$-modules. Hence,

$$M_\gamma \cong (\mathrm{soc}(\pi_\gamma|_K))^{I_1} \cong (\mathrm{soc}(\pi_{\gamma'}|_K))^{I_1} \cong M_{\gamma'}.$$

Lemma 2.1.3 implies that $\gamma = \gamma'$. □

Corollary 6.5.3. — *The representation $H_0(X,\mathcal{I}_\gamma)$ is an essential extension of π_γ in Rep_G. In particular,*

$$\pi_\gamma \cong \mathrm{soc}(H_0(X,\mathcal{I}_\gamma)),$$

where $\mathrm{soc}(H_0(X,\mathcal{I}_\gamma))$ is the subspace of $H_0(X,\mathcal{I}_\gamma)$ generated by all the irreducible subrepresentations.

Proof. — Let π be a non-zero G-invariant subspace of $H_0(X,\mathcal{I}_\gamma)$. The proof of Theorem 6.5.2 shows that $(\mathrm{soc}(H_0(X,\mathcal{I}_\gamma)|_K))^{I_1}$ is a subspace of π^{I_1}. This implies that π_γ is a subspace of π. The last part is immediate. □

6.5.1. Twists by unramified quasi-characters. — Let $\lambda \in \overline{\mathbf{F}}_p^\times$, we define an unramified quasi-character $\mu_\lambda : F^\times \to \overline{\mathbf{F}}_p^\times$, by

$$\mu_\lambda(x) = \lambda^{\mathrm{val}_F(x)}.$$

Lemma 6.5.4. — *Suppose that $\pi_\gamma \otimes \mu_\lambda \circ \det \cong \pi_{\gamma'}$, then $\gamma = \gamma'$ and $\lambda = \pm 1$.*

Proof. — Our fixed uniformiser ϖ_F acts on $\pi_\gamma \otimes \mu_\lambda \circ \det$, by a scalar λ^2, and it acts trivially on $\pi_{\gamma'}$. Hence, $\lambda = \pm 1$. By Lemma 2.1.9 $M_\gamma \otimes \mu_{-1} \circ \det \cong M_\gamma$, and hence by the argument of 6.5.2 $M_{\gamma'} \cong M_\gamma$, which implies that $\gamma = \gamma'$. □

Proposition 6.5.5. — *Suppose that $q = p$, then $\pi_\gamma \otimes (\mu_{-1} \circ \det) \cong \pi_\gamma$.*

Proof. — By Corollary 6.5.3 it is enough to show that $Y_\gamma \otimes (\mu_{-1} \circ \det) \cong Y_\gamma$ in $\mathcal{DIAG}$. We claim that we always have

$$Y_{\gamma,1} \cong Y_{\gamma,1} \otimes (\mu_{-1} \circ \det)$$

as $\mathfrak{K}(\sigma_1)$-representations. Since $F^\times I$ is contained in the kernel of $\mu_{-1} \circ \det$, it is enough to examine the action of Π. We recall that the action of $\mathfrak{K}(\sigma_1)$ was defined, by fixing a certain isomorphism $\phi : \mathcal{W}(b) \cong \mathcal{W}(b')$, and then letting Π^{-1} act on $\mathcal{W}(b, b') = \mathcal{W}(b) \oplus \mathcal{W}(b')$ by

$$\Pi^{-1}(v + w) = \phi^{-1}(w) + \phi(v).$$

Let ι_1 be an $F^\times I$-equivariant isomorphism

$$\iota_1 : \mathcal{W}(b) \oplus \mathcal{W}(b') \cong \mathcal{W}(b) \oplus \mathcal{W}(b'), \quad v + w \longmapsto v - w,$$

then, since $\mu_{-1}(\det(\Pi^{-1})) = -1$, we have

$$\Pi^{-1} \otimes \mu_{-1}(\det(\Pi^{-1}))(\iota_1(v + w)) = \phi^{-1}(w) - \phi(v) = \iota_1(\Pi^{-1}(v + w)).$$

Hence $\mathcal{W}(b, b') \cong \mathcal{W}(b, b') \otimes (\mu_{-1} \circ \det)$ as $\mathfrak{K}(\sigma_1)$-representations and hence $Y_{\gamma,1} \cong Y_{\gamma,1} \otimes (\mu_{-1} \circ \det)$ as $\mathfrak{K}(\sigma_1)$-representations. Since $F^\times K$ is contained in the kernel of $\mu_{-1} \circ \det$ we also have $Y_{\gamma,0} \cong Y_{\gamma,0} \otimes (\mu_{-1} \circ \det)$. However, to define an isomorphism in $\mathcal{DIAG}$ we need to find $\iota_0 : Y_{\gamma,0} \cong Y_{\gamma,0}$, which is compatible with ι_1 via the restriction maps. If $p = q$ this is easy, since if $\chi = \chi^s$, then

$$\mathcal{W}_{\boldsymbol{r}} \oplus \mathcal{W}_{\boldsymbol{p-1-r}} = \mathcal{W}(b_{\mathbf{0}}) \oplus \mathcal{W}(b_{\mathbf{0}} + b_{\mathbf{1}})$$

and if $\chi \neq \chi^s$ then

$$\mathcal{W}_{\boldsymbol{r}} \oplus \mathcal{W}_{\boldsymbol{p-1-r}} = (\mathcal{W}(b_{\mathbf{0}}) \oplus \mathcal{W}(b_{\mathbf{1}})) \oplus (\mathcal{W}(\overline{b}_{\mathbf{0}}) \oplus \mathcal{W}(\overline{b}_{\mathbf{1}}))$$

and the subspaces that Π 'swaps' come from different injective envelopes. Note, that this is not the case if $q \neq p$. Hence, if we define

$$\iota_0 : \mathcal{W}_{\boldsymbol{r}} \oplus \mathcal{W}_{\boldsymbol{p-1-r}} \cong \mathcal{W}_{\boldsymbol{r}} \oplus \mathcal{W}_{\boldsymbol{p-1-r}}, \quad v + w \longmapsto v - w$$

then $\iota = (\iota_0, \iota_1)$ is an isomorphism $\iota : Y_\gamma \cong Y_\gamma \otimes (\mu_{-1} \circ \det)$. □

Lemma 6.5.6. — *The representations $H_0(X, \mathcal{I}_\gamma)$ and π_γ are admissible.*

Proof. — Proposition 6.4.5, Lemma 6.3.2. □

Our main result can be summarised as follows.

Theorem 6.5.7. — *Let ϖ_F be a fixed uniformiser, then there exists at least $q(q-1)/2$ pairwise non-isomorphic, irreducible, supersingular, admissible representations of G, which admit a central character, such that ϖ_F acts trivially.*

Proof. — There are precisely $q(q-1)/2$ orbits $\gamma = \{\chi, \chi^s\}$. Then the statement follows from Theorem 6.5.2 and Corollary 6.5.6. Each π_γ admits a central character, since $H_0(X, \mathcal{V}_\gamma)$ admits a central character. If $\lambda \in \mathfrak{o}_F^\times$, then it acts on $H_0(X, \mathcal{V}_\gamma)$ by a scalar

$$\chi\big(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)\big) = \chi^s\big(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)\big)$$

and ϖ_F acts trivially by construction. □

If $F = \mathbf{Q}_p$ then we may apply the results of Breuil [**4**].

Corollary 6.5.8. — *Suppose that $F = \mathbf{Q}_p$, then π_γ is independent up to isomorphism of the choices made in the construction of Y_γ. Moreover, if π is an irreducible supersingular representation of G, admitting a central character, then there exists $\lambda \in \overline{\mathbf{F}}_p^\times$, unique up to a sign, and a unique γ, such that*

$$\pi \cong \pi_\gamma \otimes (\mu_\lambda \circ \det).$$

Proof. — In [**4**] Breuil has determined all the supersingular representations, in the case of $F = \mathbf{Q}_p$. As a consequence, by [**18**] Theorem E.7.2, the functor of I_1-invariants, $\mathrm{Rep}_G \to \mathrm{Mod}\text{-}\mathcal{H}$, $\pi \mapsto \pi^{I_1}$ induces a bijection between the isomorphism classes of irreducible supersingular representations with a central character and isomorphism classes of supersingular right modules of $\mathcal{H}$. In particular, there are precisely $p(p-1)/2$ isomorphism classes of supersingular representations with a central character, such that ϖ_F acts trivially. By Theorem 6.5.7 our construction yields at least $p(p-1)/2$ such representations. Hence π_γ does not depend up to isomorphism on the choices made for Y_γ.

Let π be any supersingular representation of G with a central character. We may always twist π by an unramified quasi-character, so that ϖ_F acts trivially. Hence by above

$$\pi \cong \pi_\gamma \otimes (\mu_\lambda \circ \det)$$

and by Lemma 6.5.4 and Proposition 6.5.5, γ is determined uniquely and λ up to ± 1. □

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