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# ÉQUATIONS DIFFÉRENTIELLES ET SINGULARITÉS. EN L'HONNEUR DE J. M. AROCA

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# DYNAMIQUE TRANSVERSE DE LA LAMINATION DE GHYS-KENYON

*par*

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***Résumé.*** — À partir d'un arbre apériodique et répétitif du graphe de Cayley du groupe abélien libre à deux générateurs décrit par Kenyon, Ghys a construit un exemple de lamination minimale par surfaces de Riemann avec des feuilles euclidiennes et hyperboliques. On démontre que la dynamique transverse de cette lamination est représentée (du point de vue de la mesure) par une machine à sommer binaire. En fait, on peut décrire sa dynamique topologique transverse et montrer ainsi que la lamination de Ghys-Kenyon est affable.

***Abstract* (Transverse dynamics of the Ghys-Kenyon lamination).** — Using an aperiodic and repetitive subtree of the Cayley graph of the free Abelian group with two generators, described by Kenyon, Ghys has constructed an example of minimal Riemann surface lamination having both Euclidean and hyperbolic leaves. We prove that the transverse dynamics of this lamination is represented (in a measurable way) by a 2-adic odometer. In fact, we can describe its topological transverse dynamics, and show that the Ghys-Kenyon lamination is affable.

## 1. Introduction

Il y a des laminations minimales par surfaces de Riemann où les types conformes des feuilles se mélangent. Le premier exemple a été construit par É. Ghys [**4**] à partir d'un arbre apériodique et répétitif décrit par R. Kenyon [**7**]. La construction comporte deux étapes distinctes, valables pour tout sous-graphe répétitif du graphe de Cayley $\mathscr{G}$ d'un groupe infini de type fini $G$. Il s'agit d'abord de construire un espace compact, muni d'un feuilletage par graphes, puis d'obtenir une lamination par surfaces de Riemann. Soit $\mathscr{T} = \mathscr{T}(G)$ l'ensemble des sous-graphes infinis de $\mathscr{G}$ contenant l'élément neutre $e$ de $G$. On munit $\mathscr{T}$ de la *topologie de Gromov-Hausdorff* pour laquelle deux sous-graphes de $\mathscr{G}$ sont proches s'ils coïncident sur une grande boule centrée

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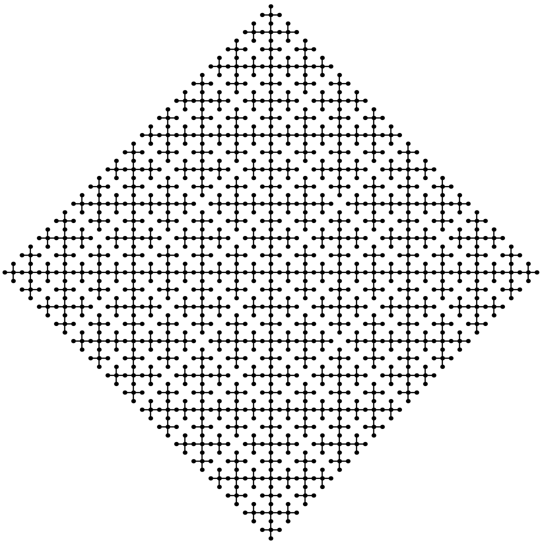

FIGURE 1. Arbre de Kenyon

en $e$. Puisqu'une boule ne contient qu'un nombre fini de sous-graphes, un procédé diagonal classique montre que $\mathscr{T}$ est compact. Grâce à l'action de $G$ sur $\mathscr{G}$, on définit une relation d'équivalence $\mathscr{R}$ qui identifie un arbre $T$ et son translaté $T' = g^{-1}.T$ si $g \in T$. On peut d'ailleurs réaliser $\mathscr{T}$ comme un sous-espace d'un espace métrique compact $\mathcal{T} = \mathcal{T}(\boldsymbol{G})$, muni d'un feuilletage par graphes $\mathcal{F}$ dont toutes les feuilles sont rencontrées par $\mathscr{T}$. Alors $\mathscr{R}$ est induite par $\mathcal{F}$ et les classes d'équivalence sont les ensembles de sommets des feuilles de $\mathcal{F}$.

Pour tout graphe $T \in \mathscr{T}$, l'ensemble $X = \overline{\mathscr{R}[T]}$ est un fermé saturé pour $\mathscr{R}$, appelé l'*enveloppe de* $T$. Il est réalisable comme transversale complète d'un espace feuilleté compact $\boldsymbol{X}$, à savoir la fermeture de la feuille $L_T \in \mathcal{F}$ passant par $T$. Les ensembles $X$ et $\boldsymbol{X}$ sont minimaux si et seulement si le graphe $T$ est *répétitif*, i.e. pour tout nombre réel $r > 0$, il existe un nombre réel $R > 0$ tel que toute boule de rayon $R > 0$ contient une boule qui est l'image par translation de la boule de centre $e$ et rayon $r > 0$. Par ailleurs, l'holonomie de $L_T$ est triviale si et seulement si $T$ est *apériodique*, i.e. $T \neq g.T$ pour tout élément $g \neq e$ de $G$. Ces définitions s'inspirent de définitions analogues pour les pavages [**9, 1**].

Nous appellerons *espace feuilleté de Ghys-Kenyon* la fermeture $\boldsymbol{X}$ de la feuille $L_{T_\infty}$ passant par l'arbre de Kenyon $T_\infty$ (voir la figure 1) dans l'espace $\mathcal{T} = \mathcal{T}(\mathbb{Z}^{\mathbf{2}})$, munie du feuilletage induit par $\mathcal{F}$. Les feuilles sont des sous-arbres répétitifs et apériodiques du graphe de Cayley $\mathcal{Z}^2$ de $\mathbb{Z}^2$. En remplaçant ces arbres par des surfaces, on obtient la lamination $(\boldsymbol{M}, \mathcal{L})$ décrite par É. Ghys. Nous l'appellerons *lamination de Ghys-Kenyon*. Dans ce travail, nous allons récuperer l'espace $\boldsymbol{X}$ par un procédé de construction de sous-arbres répétitifs et apériodiques de $\mathcal{Z}^2$ à partir de suites de 4

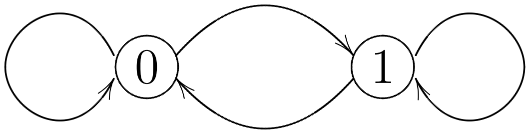

FIGURE 2. Machine à sommer binaire

éléments. Grâce à ce codage, nous montrerons que la dynamique transverse de ce feuilletage est représentée par l'automate de la fig. 2.

Cela signifie que $\mathscr{R}$ est *stablement orbitalement équivalente* à la relation engendrée par la somme $S(x) = x + 1$ définie sur l'anneau des entiers 2-adiques, ou de manière équivalente par la transformation $T: \ \{0,1\}^{\mathbb{N}} \to \ \{0,1\}^{\mathbb{N}}$ définie par :
i) si $\alpha_0 = 0$, alors $T(\alpha)_0 = 1$ et $T(\alpha)_n = \alpha_n$ pour tout $n \geqslant 1$,
ii) si $\alpha_0 = 1$, alors $T(\alpha)_0 = 0$ et $T(\alpha)_1 = T(\sigma(\alpha))_0$ avec $\sigma(\alpha)_n = \alpha_{n+1}$.
Nous compléterons l'étude de l'exemple en décrivant sa dynamique topologique. Nous montrerons ainsi que $\mathscr{L}$ est *affable*, en ce sens que $\mathscr{R}$ est la limite inductive d'une suite de relations d'équivalence étales compactes [**5**]. La dynamique transverse de $\mathscr{L}$ sera ainsi représentée par un système dynamique classique.

## 2. L'espace feuilleté de Gromov-Hausdorff

Soit $S$ un système fini de générateurs de $G$. Le *graphe de Cayley* $\mathscr{G} = \mathscr{G}(G,S)$ est un graphe localement fini non orienté, sans boucle, ni arête multiple, dont les sommets sont les éléments de $G$. Deux sommets $g_1$ et $g_2$ sont reliés par une arête si $g_1^{-1}g_2 \in S$. On appelle *longueur* de $g$ le plus petit nombre d'éléments de $S$ nécessaires pour écrire $g$, i.e. $\mathrm{long}_S(g) = \min\{n \geqslant 1 / g = s_1...s_n \text{ avec } s_i s_{i+1} \neq e\}$. La *distance des $S$-mots* est alors donnée par $d_S(g_1,g_2) = \mathrm{long}_S(g_1^{-1}g_2)$ pour tout couple $g_1, g_2 \in G$. Cette distance se prolonge en une distance sur $\mathscr{G}$ telle que toute arête est isométrique à l'intervalle $[0,1]$. Le graphe de Cayley $\mathscr{G}$ devient ainsi un espace métrique connexe par chemins sur lequel le groupe $G$ agit par isométries.

**2.1. Topologie de Gromov-Hausdorff.** — Soit $\mathscr{T} = \mathscr{T}(G)$ l'ensemble des sous-graphes $T$ de $\mathscr{G}$ contenant l'élément neutre $e$ de $G$. Notons $B_T(e,N)$ (resp. $\overline{B}_T(e,N)$) la boule ouverte (resp. fermée) de centre $e$ et de rayon $N$ et $\mathrm{val}_T(e)$ la valence de $e$, i.e. le nombre d'arêtes issues de $e$. Considérons l'ensemble $A = \{\ N \geqslant 1 \ / \ B_T(e,N) = B_{T'}(e,N)\ \}$ et la quantité

$$R(T,T') = \begin{cases} \sup\ A & si\ A \neq \varnothing, \\ 0 & si\ A = \varnothing, \end{cases}$$

qui appartient à $\mathbb{N} \cup \{+\infty\}$ pour tout couple $T, T' \in \mathscr{T}$. On définit alors la *distance de Gromov-Hausdorff* par $d(T,T') = e^{-R(T,T')}$. C'est une ultramétrique et donc $\mathscr{T}$ est totalement disconnexe. Puisque la boule fermée $\overline{B}_{\mathscr{G}}(e,N)$ ne contient qu'un nombre fini de sous-graphes, un procédé diagonal classique montre que $\mathscr{T}$ est compact. Les sous-graphes finis de $\mathcal{G}$ correspondent aux points isolés de $\mathcal{T}$. Nous noterons désormais $\mathcal{T}$ l'ensemble des sous-graphes *infinis* de $\mathcal{G}$ contenant l'élément neutre $e$. L'avantage de la nouvelle définition est mise en évidence par le fait que $\mathcal{T} = \mathcal{T}(G)$ est alors homéomorphe à l'ensemble de Cantor, sauf si $G = \mathbb{Z}$.

**2.2. Structure feuilletée.** — L'espace $\mathcal{T}$ est muni d'une relation d'équivalence $\mathcal{R}$ qui identifie deux graphes $T$ et $T'$ si $T' = g^{-1}.T$ avec $g \in T$. Toute classe d'équivalence $\mathcal{R}[T]$ peut être alors réalisée comme l'ensemble de sommets d'un graphe $\overline{\mathcal{R}}[T]$. Il suffit de joindre $T' = g^{-1}.T$ et $T'' = h^{-1}.T$ par une arête si $d_S(g,h) = 1$. Le graphe $\overline{\mathcal{R}}[T]$ est donc isomorphe au quotient de $T$ par le groupe de translations $\mathrm{Iso}(T) = \{g \in \mathcal{G} / T = g.T\}$. C'est une feuille de l'espace feuilleté compact fourni par le résultat suivant :

***Théorème de réalisation géométrique 2.2.1.*** — *Il y a un espace compact, métrisable et séparable $\mathcal{T}$, muni d'un feuilletage par graphes $\mathcal{F}$, pour lequel $\mathcal{T}$ est une transversale complète et $\mathcal{R}$ est la relation d'équivalence induite sur $\mathcal{T}$.*

*Démonstration.* — Considérons le sous-espace $\widetilde{\mathcal{T}} = \{(T,g) \in \mathcal{T} \times \mathcal{G} \mid g$ est un sommet de $T\}$ de $\mathcal{T} \times \mathcal{G}$, muni de la pseudodistance $d((T_1,g_1),(T_2,g_2)) = d(g_1^{-1}.T_1, g_2^{-1}.T_2)$. Alors $\mathcal{T}$ est le quotient de $\widetilde{\mathcal{T}}$ par l'action diagonale de $G$ sur $\mathcal{T} \times \mathcal{G}$. Chaque classe d'équivalence $\mathcal{R}[T]$ est obtenue par passage au quotient à partir de l'orbite de $(T,e)$. L'ensemble $\widetilde{U}_{(T_1,g_1)} = \overline{B}_{\widetilde{\mathcal{T}}}((T_1,g_1),e^{-1}) = \{(T_2,g_2) \in \widetilde{\mathcal{T}} / \overline{B}_{g_1^{-1}.T_1}(e,1) = \overline{B}_{g_2^{-1}.T_2}(e,1)\}$ est un ouvert-fermé qui se projette sur l'ouvert-fermé $U_{g_1^{-1}.T_1} = \overline{B}_{\mathcal{T}}(g_1^{-1}.T_1, e^{-1})$. Puisque $\overline{B}_{\mathcal{G}}(e,1)$ ne contient qu'un nombre fini de sous-graphes, les ensembles $\widetilde{U}_{(T_1,g_1)}$ et $U_{g_1^{-1}.T_1}$ définissent des partitions finies de $\widetilde{\mathcal{T}}$ et $\mathcal{T}$ respectivement. Nous allons remplacer $\widetilde{\mathcal{T}}$ par l'ensemble $\widetilde{\mathcal{T}}$ des couples $(T,x)$ où $x$ est un point quelconque de $T$ qui peut appartenir à l'intérieur e̊ d'une arête e de $T$. L'application $\widetilde{\psi}_{(T_1,g_1)} : \big((T_2,g_2),x\big) \in \widetilde{U}_{(T_1,g_1)} \times \overline{B}_{g_1^{-1}.T_1}(e,1) \mapsto (T_2, g_2.x) \in \widetilde{\mathcal{T}}$ est injective en restriction aux ensembles $\widetilde{U}_{(T_1,g_1)} \times B_{g_1^{-1}.T_1}(e,\frac{1}{2})$ et $\widetilde{U}_{(T_1,g_1)} \times \mathring{e}$. Leurs images $\widetilde{V}_{(T_1,g_1)}$ et $\widetilde{V}^{\mathrm{e}}_{(T_1,g_1)}$ sont munies de topologies telles que les restrictions et leurs inverses $\widetilde{\varphi}_{(T_1,g_1)} : \widetilde{V}_{(T_1,g_1)} \to \widetilde{U}_{(T_1,g_1)} \times B_{g_1^{-1}.T_1}(e,\frac{1}{2})$ et $\widetilde{\varphi}^{\mathrm{e}}_{(T_1,g_1)} : \widetilde{V}^{\mathrm{e}}_{(T_1,g_1)} \to \widetilde{U}_{(T_1,g_1)} \times \mathring{e}$ sont des homéomorphismes. On munit $\widetilde{\mathcal{T}}$ de la topologie faible pour laquelle $\widetilde{V}_{(T_1,g_1)}$ et $\widetilde{V}^{\mathrm{e}}_{(T_1,g_1)}$ forment un recouvrement ouvert fini. On vérifie aisément que :

i) l'espace $\widetilde{\mathcal{T}}$ est réalisé comme un sous-espace compact de $\widetilde{\mathcal{T}}$,
ii) l'action de $G$ sur $\widetilde{\mathcal{T}}$ s'étend en une action de $G$ sur $\widetilde{\mathcal{T}}$,
iii) les cartes locales $\widetilde{\varphi}_{(T_1,g_1)}$ et $\widetilde{\varphi}^{\mathrm{e}}_{(T_1,g_1)}$ forment un atlas feuilleté sur $\widetilde{\mathcal{T}}$ qui définit un feuilletage par graphes $\widetilde{\mathcal{F}}$ invariant par l'action de $G$.

Soit $\mathcal{T}$ le quotient de $\widetilde{\mathcal{T}}$ par l'action de $G$. Alors les applications

$$\psi_{g_1^{-1}.T_1} : (g_2^{-1}.T_2, x) \in U_{g_1^{-1}.T_1} \times \overline{B}_{g_1^{-1}.T_1}(e,1) \mapsto x^{-1}.(g_2^{-1}.T_2) \in \mathcal{T}$$

définissent des cartes locales

$$\varphi_{g_1^{-1}.T_1} : V_{g_1^{-1}.T_1} \to U_{g_1^{-1}.T_1} \times B_{g_1^{-1}.T_1}(e, \frac{1}{2}) \quad \text{et} \quad \varphi^{\mathrm{e}}_{g_1^{-1}.T_1} : V^{\mathrm{e}}_{g_1^{-1}.T_1} \to U_{g_1^{-1}.T_1} \times \mathring{\mathrm{e}}$$

et donc l'espace $\mathcal{T}$ possède un atlas feuilleté fini. Il est compact car les plaques sont relativement compactes et les transversales sont compactes. □

**2.3. Structure transverse.** — Nous allons préciser ici la notion de *dynamique transverse* (*mesurable* ou *topologique*) utilisée dans la introduction. D'abord, la relation d'équivalence $\mathcal{R}$ est définie par l'action d'un pseudogroupe de transformations $\Gamma$ engendré par les translations $\tau_g : T \mapsto g^{-1}.T$ associées aux éléments de $G$. Chacune de ces applications est définie sur l'ouvert-fermé $D_g = \{T \in \mathcal{T}/g \in T\}$ de $\mathcal{T}$. Le théorème 2.2.1 montre que $\Gamma$ est le *pseudogroupe d'holonomie de* $\mathcal{F}$ *réduit à* $\mathcal{T}$. Nous utiliserons donc la notion de *dynamique transverse* introduite par A. Haefliger **[6]**.

Néanmoins, si l'holonomie est triviale, la dynamique transverse est représentée par la relation d'équivalence induite sur toute transversale complète. Rappelons qu'une relation d'équivalence $\mathcal{R}$ sur un espace borélien standard $X$ est *mesurable discrète* si les classes d'équivalence sont dénombrables et si le graphe est un borélien de $X \times X$. On appelle *transformation partielle de* $\mathcal{R}$ tout isomorphisme borélien $\varphi : A \to B$ entre parties boréliennes de $X$ dont le graphe $G(\varphi) = \{(x, y) \in X \times X/y = \varphi(x)\} \subset \mathcal{R}$. Une mesure borélienne $\mu$ sur $X$ est dite *invariante pour* $\mathcal{R}$ si elle est invariante pour toute transformation partielle $\varphi$, i.e. $\mu(\varphi^{-1}(B')) = \mu(B')$ pour tout borélien $B' \subset B$. La relation d'équivalence $\mathcal{R}$ sur $\mathcal{T} = \mathcal{T}(G)$ est mesurable discrète car les classes d'équivalence sont dénombrables et le graphe de $\mathcal{R}$ est un borélien de $\mathcal{T} \times \mathcal{T}$ en tant que réunion des graphes des transformations partielles $\tau_g$ définies sur les ouverts-fermés $\overline{B}_{\mathcal{T}}(T, e^{-1})$ (avec $g \in \overline{B}_T(e, 1)$) et de leurs compositions.

***Définition 2.3.1***. — Deux relations d'équivalence mesurées $(\mathcal{R}, X, \mu)$ et $(\mathcal{R}', X', \mu')$ sont dites :

i) *orbitalement équivalentes* si $X$ et $X'$ contiennent des boréliens $Y$ et $Y'$ saturés pour $\mathcal{R}$ et $\mathcal{R}'$ et de mesure totale pour lesquels il existe un isomorphisme borélien $\varphi : Y \to Y'$ tel que $\varphi(\mathcal{R}[x]) = \mathcal{R}'[\varphi(x)]$ pour $\mu$-presque tout $x \in Y$ et $f_*\mu \sim \mu'$ ;

ii) *stablement orbitalement équivalentes* si $X$ et $X'$ contiennent des boréliens $Y$ et $Y'$ dont les saturés pour $\mathcal{R}$ et $\mathcal{R}'$ sont de mesure totale tels que les relations d'équivalence induites $\mathcal{R}|_Y$ et $\mathcal{R}'|_{Y'}$ sont orbitalement équivalentes. Nous dirons alors que $\mathcal{R}$ et $\mathcal{R}'$ représentent une même *dynamique mesurable*.

Toute relation d'equivalence $\mathcal{R}$ sur un espace borélien ou topologique $X$ est munie d'une structure naturelle de groupoïde caractérisée par les données suivantes : l'inclusion $\varepsilon : x \in X \mapsto (x, x) \in \mathcal{R}$ de l'espace des unités $X$ dans $\mathcal{R}$, les projections $\beta : (x, y) \in \mathcal{R} \mapsto x \in X$ et $\alpha : (x, y) \in \mathcal{R} \mapsto y \in X$, l'ensemble des couples composables $\mathcal{R} * \mathcal{R} = \{((x, y), (x', y')) \in \mathcal{R} \times \mathcal{R}/\alpha(x, y) = y = x' = \beta(x', y')\}$, la multiplication partielle $\mu : ((x, y), (x', y')) \in \mathcal{R} * \mathcal{R} \mapsto (x, y') \in \mathcal{R}$ et l'inversion

$\iota : (x,y) \in \mathscr{R} \to (y,x) \in \mathscr{R}$. La relation d'equivalence $\mathscr{R}$ est dite *topologique* si elle l'est comme groupoïde, c'est-à-dire si le graphe de $\mathscr{R}$ est muni d'une topologie (qui en fait un espace localement compact séparé) telle que $\alpha, \beta : \mathscr{R} \to X$ et $\mu : \mathscr{R} * \mathscr{R} \to \mathscr{R}$ sont continues et $\iota : \mathscr{R} \to \mathscr{R}$ est un homéomorphisme. Une telle relation d'équivalence est dite *$\beta$-discrète* si $X$ est ouvert dans $\mathscr{R}$. Pour tout ouvert $U$ de $\mathscr{T}$ et tout élément $g$ de $G$, notons $O(U,g) = \{(T, g^{-1}.T) \in \mathscr{R} / T \in U \cap D_g\}$ le graphe de la translation $\tau_g$ restreinte à $U$. Les ensembles $O(U,g)$ engendrent une topologie sur $\mathscr{R}$, plus fine que celle induite par la topologie produit sur $\mathscr{T} \times \mathscr{T}$, qui en fait une relation d'équivalence topologique $\beta$-discrète.

***Définition 2.3.2***. — Deux relations d'équivalence $\boldsymbol{\beta}$-discrètes $\mathscr{R}$ et $\mathscr{R}'$ sur $X$ et $X'$ sont dites *stablement orbitalement équivalentes* (resp. *isomorphes*) si $X$ et $X'$ contiennent des ouverts $Y$ et $Y'$ qui rencontrent toutes les classes d'équivalence de $\mathscr{R}$ et $\mathscr{R}'$ tels que les relations d'équivalence induites $\mathscr{R}|_Y$ et $\mathscr{R}'|_{Y'}$ sont orbitalement équivalentes (resp. isomorphes).

**2.4. Réalisation géométrique.** — La donnée d'un système fini de générateurs $S$ de $G$ fournit un système fini de générateurs $\Sigma = \{\tau_g / g \in S\}$ de $\Gamma$. Par analogie avec l'action d'un groupe, l'orbite $\Gamma(T) = \mathscr{R}[T]$ est l'ensemble des sommets d'un graphe $\overline{\Gamma}(T) = \overline{\mathscr{R}}[T]$, muni de la distance $d_\Sigma$ définie par la longueur des $\Sigma$-mots. Nous dirons alors que $(\mathscr{R}, \mathscr{T}, \Sigma)$ est une *relation d'équivalence graphée* et que $(\Gamma, \mathscr{T}, \Sigma)$ un *pseudogroupe graphé*. Dans [**8**], le deuxième auteur a prouvé l'extension suivante du théorème 2.2.1 :

***Théorème de réalisation géométrique 2.4.1***. — *Soit $\Gamma$ un pseudogroupe de transformations d'un espace localement compact, métrisable et séparable $X$, muni d'un système fini de générateurs $\Sigma$. Si la fonction de valence* val : $X \to \mathbb{N}$ *est continue, il existe un espace feuilleté localement compact, métrisable et séparable $(\boldsymbol{X}, \mathscr{F})$ tel que $X$ est un fermé qui rencontre toute les feuilles de $\mathscr{F}$ et $\Gamma$ est le pseudogroupe d'holonomie de $\mathscr{F}$ réduit à $X$. En outre, si $X$ est compact, alors $\boldsymbol{X}$ l'est aussi.*

***Théorème de épaissisement 2.4.2***. — *Soit $(\boldsymbol{X}, \mathscr{F})$ un espace feuilleté par graphes localement finis tel que $\Gamma$ admet un système de générateurs dont les domaines et les rangs sont ouverts et fermés dans $X$. Il existe une lamination par surfaces de Riemann $(\boldsymbol{M}, \mathscr{L})$ dont le pseudogroupe d'holonomie réduit à $X$ est égal à $\Gamma$.*

*Démonstration*. — Fixons un entier $k \geqslant 1$ et un réel $\varepsilon > 0$. Considérons la sphère $S^2_\varepsilon$ de rayon $\varepsilon$, munie de la métrique usuelle, puis ôtons une famille de boules géodésiques $\overline{B}(s_1, 2r), \dots, \overline{B}(s_k, 2r)$ de même rayon $2r > 0$ et deux à deux disjointes. Considérons le cylindre $C = S^1_r \times [0, \frac{3}{4}[$. Pour tout $1 \leqslant i \leqslant k$, l'application exponentielle permet de définir un difféomorphisme entre la couronne $C_i = \overline{B}(s_i, 2r) - B(s_i, r)$ et le cylindre $S^1_r \times [0, r]$ qui identifie chaque sphère géodésique de centre $s_i$ et rayon $r \leqslant t \leqslant 2r$ avec le cercle $S^1_r \times \{2r - t\}$. En recollant $k$ copies du cylindre $C$ à la sphère trouée, on obtient une surface $P(k)$ appelée une *pieuvre à $k$ bras*. Les métriques riemanniennes sur les deux parties de la pieuvre se recollent en une métrique riemannienne qui coïncide avec les métriques de départ hors d'un voisinage tubulaire de $C_1 \cup \cdots \cup C_k$. Un choix convenable du rayon $r > 0$ permet de supposer que le cylindre $S^1_r \times ]\frac{1}{4}, \frac{3}{4}[$ ne coupe pas ce voisinage. Nous noterons $P'(k)$ la pieuvre obtenue en remplaçant l'intervalle $[0, \frac{3}{4}[$ par le sous-intervalle $[0, \frac{1}{2}[$.

Notons $X^{\mathrm{val}=k}$ l'ensemble des sommets $x \in X$ tels que $\mathrm{val}(x) = k$ et $\Sigma_x$ l'ensemble d'arêtes $\sigma$ issues de $x$. Soit $\sigma^{-1}$ l'arête opposée, issue de l'extremité $\sigma(x)$ de $\sigma$. Pour tout $x \in X^{\mathrm{val}=k}$, l'ensemble $\Sigma_x$ permet d'étiqueter les bras de $P(k)$. Soit $\boldsymbol{M}$ le quotient de la réunion disjointe $\bigsqcup_{k\in\mathbb{N}} X^{\mathrm{val}=k} \times P(k)$ par la relation d'équivalence qui identifie le point $(x, z, t)$ du cylindre $\{x\} \times S^1_r \times ]\frac{1}{4}, \frac{3}{4}[$ contenu dans le bras de la pieuvre $\{x\} \times P(k)$ d'étiquette $\sigma \in \Sigma_x$ avec le point $(\sigma(x), \overline{z}, 1-t)$ du cylindre $\sigma(x) \times S^1_r \times ]\frac{1}{4}, \frac{3}{4}[$ contenu dans le bras de la pieuvre $\sigma(x) \times P(k')$ d'étiquette $\sigma^{-1} \in \Sigma_{\sigma(x)}$. Ici $\overline{z}$ est le conjugué de $z$ et $k' = \mathrm{val}(\sigma(x))$. Soit $\pi$ la projection. La donnée d'un atlas feuilleté sur $X$ (construit par le procédé décrit dans [**8**]) entraîne l'existence d'un atlas feuilleté sur $\boldsymbol{M}$ définissant une lamination $\mathscr{L}$. La feuille $L_x$ passant par $x$ est la réunion des surfaces $\pi(\{y\} \times P(\mathrm{val}(y)))$ associés aux sommets $y$ de la feuille de $\mathscr{T}$ passant par $x$. En géneral, ces surfaces ne sont pas difféomorphes aux pieuvres $P(\mathrm{val}(y))$ car certaines arêtes issues de $y$ peuvent être des boucles. Néanmoins, en remplaçant $P(\mathrm{val}(y))$ par $P'(\mathrm{val}(y))$, nous obtiendrons des vraies plaques $\pi(\{y\} \times P'(\mathrm{val}(y)))$ difféomorphes aux pieuvres $P'(\mathrm{val}(y))$. En ajoutant les images des bras $\{y\} \times S^1_r \times ]\frac{1}{4}, \frac{3}{4}[$ d'étiquette $\sigma \in \Sigma_y$, on décrit la feuille $L_x$ comme réunion de plaques de $\mathscr{L}$. Puisque les bras des pieuvres sont recollés à l'aide des difféomorphismes qui préservent l'orientation, $L_x$ est une surface de Riemann orientable. L'inclusion naturelle de $\bigsqcup_{k\in\mathbb{N}} X^{\mathrm{val}=k}$ dans $\bigsqcup_{k\in\mathbb{N}} X^{\mathrm{val}=k} \times P(k)$ passe au quotient en un plongement fermé de l'espace de sommets $X$ dans $\boldsymbol{M}$. En identifiant $X$ avec son image dans $\boldsymbol{M}$, on voit que la trace de la feuille $L_x$ coïncide avec l'ensemble de sommets de la feuille de $\mathscr{T}$ passant par $x$. □

**2.5. Graphes répétitifs et ensembles minimaux.** — Le but de ce paragraphe est de caractériser les ensembles minimaux de $(\mathscr{T}, \mathscr{T})$ et $(\boldsymbol{M}, \mathscr{L})$ en adaptant la *propriété d'isomorphisme local* des pavages [**1, 9**].

***Définition 2.5.3.*** — i) Fixons un couple $T, T' \in \mathscr{T}$. Nous dirons qu $T'$ *contient une copie fidèle de la boule* $B_T(x,r)$ et nous écrirons $B_T(x,r) \hookrightarrow T'$ s'il existe $g \in G$ tel que $g.B_T(x,r) = B_{T'}(g.x, r) \subset T'$.

ii) Nous dirons qu'un graphe $T \in \mathscr{T}$ est *répétitif* si pour tout entier $r > 0$, il existe un entier $R > 0$ tel que $B_T(x,r) \hookrightarrow B_T(y,R)$ pour tout couple $x, y \in T$.

Nous adaptons ici une version uniforme de la propriété d'isomorphisme local usuelle. En fait, pour les pavages de type fini, les deux propriétés sont équivalentes. L'analogue pour les graphes fait partie du critère de minimalité suivant (dont l'équivalence $(ii) \Leftrightarrow (iii)$ a été prouvée dans [**2, 4**]) :

***Théorème 2.5.4.*** — *Pour tout $T \in \mathscr{T}$, considérons l'ensemble fermé $X = \overline{\mathscr{R}[T]}$ saturé pour $\mathscr{R}$. Les conditions suivantes sont équivalentes :*

i) *le graphe $T$ est répétitif ;*
ii) *pour tout $r > 0$, il existe $R > 0$ tel que $B_T(e,r) \hookrightarrow B_T(y,R)$ pour tout $y \in T$ ;*
iii) *l'ensemble $X$ est minimal.*

*Démonstration.* — Il suffit de prouver (iii) $\Rightarrow$ (i), mais il convient avant de rappeler brièvement (iii) $\Rightarrow$ (ii). Pour cela, à tout réel $r > 0$, on lui associe une suite croissante d'ouverts $U_R = \{ T' \in X \ / \ B_T(e,r) \hookrightarrow B_{T'}(e,R) \}$ (avec $R \geqslant 1$) qui recouvrent $X$. Puisque $X$ est compact, il existe $R > 0$ tel que $X = U_R$. Pour tout $x \in T$, le graphe $x^{-1}.T \in U_R$ et donc $B_T(e,r) \hookrightarrow B_{x^{-1}.T}(e,R)$, c'est-à-dire qu'il existe $g \in G$ tel que : $g.B_T(e,r) = B_{x^{-1}.T}(g,r) \subset B_{x^{-1}.T}(e,R)$. Alors on a :

$$h.B_T(e,r) = x.B_{x^{-1}.T}(g,r) = B_T(h,r) \subset x.B_{x^{-1}.T}(e,R) = B_T(x,R)$$

avec $h = xg$ et $B_T(e,r) \hookrightarrow B_T(x,R)$. Pour démontrer (iii) $\Rightarrow$ (i), fixons un réel $r > 0$ et un point $x \in T$. Comme auparavant, l'ensemble $X$ est recouvert par une suite croissante d'ouverts $U_R^x = \{T' \in X / B_T(x,r) \hookrightarrow B_{T'}(e,R)\}$ et il existe $R > 0$, qui dépend de $r$ et $x$, tel que $B_T(x,r) \hookrightarrow B_T(y,R)$ pour tout $y \in T$. Pour conclure, il faut pouvoir choisir $R > 0$ indépendant du point $x$. Remarquons tout d'abord que pour tout sommet $g$ de $\mathscr{G}$, la boule $B_{\mathscr{G}}(g,r) = g.B_{\mathscr{G}}(e,r)$. Rappelons aussi que la compacité $\mathscr{T}$ provient du fait que $B_{\mathscr{G}}(e,r)$ ne contient qu'un nombre fini de sous-graphes. Il en est de même pour $B_{\mathscr{G}}(g,r)$. En fait, à translation près, il n'y a qu'un nombre fini de boules de rayon $r > 0$ distinctes. Considérons une famille finie de points $x_1, \dots, x_n \in T$ de manière que les boules $B_T(x_i,r)$ représentent toutes les classes de translations possib ! les. Pour tout $1 \leqslant i \leqslant n$ et tout $y \in T$, on a $B_T(x_i,r) \hookrightarrow B_T(y, R(r,x_i))$. Si on pose $R = \max\{R(r,x_1), \dots, R(r,x_n)\}$, alors $B_T(x,r) \hookrightarrow B_T(y,R)$ pour tout couple $x, y \in T$. $\square$

## 3. L'espace feuilleté de Ghys–Kenyon

Nous donnons ici une nouvelle construction de l'*espace feuilleté de Ghys-Kenyon* [**4**], que n'utilise pas l'*arbre de Kenyon* [**4, 7**], mais ses règles de construction.

**3.1. L'arbre de Kenyon et l'espace feuilleté de Ghys–Kenyon.** — Nous allons commencer par rappeler la construction de l'arbre de Kenyon. Soient $\mathscr{Z}^2$ le graphe de Cayley de $\mathbb{Z}^2$, muni du système de générateurs $\{(\pm 1, 0), (0, \pm 1)\}$, et $T_1$ le sous-arbre de $\mathscr{Z}^2$ décrit dans la figure 3.

Cet arbre est translaté ensuite par le vecteur $(0,2)$, puis l'image est tournée à l'aide des rotations d'angle $\frac{\pi}{2}$, $\pi$ et $\frac{3\pi}{2}$. L'élagage des arêtes terminales contenues dans l'axe horizontal fournit un arbre $T_2$. Si on répète ce procédé, on obtient de même un arbre $T_3$. Par récurrence, on obtient une suite d'arbres $T_n$ qui rencontrent les axes horizontal et vertical suivant les intervalles $[-2^n + 1, 2^n - 1] \times \{0\}$ et $\{0\} \times [-2^n, 2^n]$ respectivement. Nous appellerons *arbre de Kenyon* la réunion $T_\infty = \bigcup_{n \geqslant 1} T_n \subset \mathscr{Z}^2$. C'est un arbre apériodique et répétitif ayant 4 bouts.

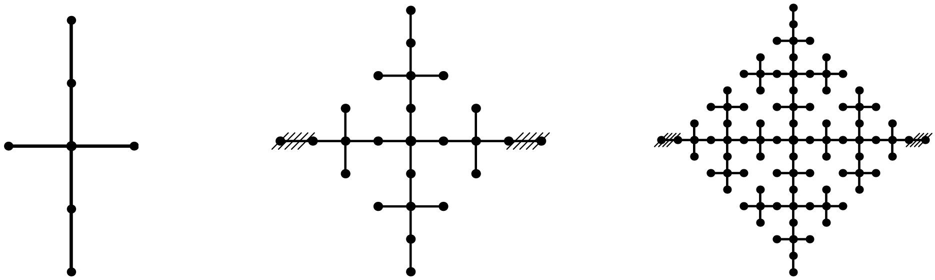

FIGURE 3. Les arbres $T_1$, $T_2$ et $T_3$

Nous appellerons *minimal de Ghys-Kenyon* l'ensemble $X = \overline{\mathscr{R}[T_\infty]}$. D'après les théorèmes 2.2.1 et 2.4.1, il existe un feuilletage par graphes $\mathscr{G}$ d'un espace compact $\boldsymbol{X}$ pour lequel $X$ est une transversale complète et $\mathscr{R}$ est la relation d'équivalence induite par $\mathscr{G}$. Nous appellerons *espace feuilleté de Ghys-Kenyon* ce minimal de l'espace feuilleté de Gromov-Hausdorff $(\mathscr{T}, \mathscr{G})$. En fait, grâce au théorème 2.4.2, on peut remplacer $(\boldsymbol{X}, \mathscr{G})$ par une vraie lamination par surfaces de Riemann $(\boldsymbol{M}, \mathscr{L})$, appelée *lamination de Ghys-Kenyon.*

**3.2. Codage des feuilles.** — Nous allons reconstruire le minimal de Ghys–Kenyon à l'aide d'une application $\Phi : \mathscr{S}_4 \to X$ qui, à toute suite $\alpha = \alpha_0\alpha_1 \cdots \in \mathscr{S}_4 = \{0,1,2,3\}^{\mathbb{N}} = \mathbb{Z}_4^{\mathbb{N}}$, associe un arbre apériodique et répétitif $\Phi(\alpha)$ dans l'enveloppe de $T_\infty$. Nous construirons $\Phi(\alpha)$ de proche en proche en partant du sommet $x_0 = 0$ et de l'arbre trivial $P_0 = \{0\}$. Pour cela, nous commençons par identifier les éléments de $\mathbb{Z}_4$ avec les racines quatrièmes de l'unité grâce à l'application $\mathbf{r} : \mathbb{Z}_4 \to \mathbb{C}$ définie par $\mathbf{r}(k) = e^{\frac{\pi}{2}ik}$. Nous joignons les sommets $x_0$ et $x_1 = \mathbf{r}(\alpha_0)$ par une arête de $\mathcal{Z}^2$, puis nous prenons la réunion des images de cette arête par les rotations de centre $x_1$ et d'angle $\frac{\pi}{2}$, $\pi$ et $\frac{3\pi}{2}$. Nous obtenons ainsi un arbre $P_1 = \Phi(\alpha_0)$. Considérons ensuite l'unique arête de $\mathcal{Z}^2$ qui joint le sommet $x_2 = x_1 + 2\mathbf{r}(\alpha_1)$ avec un sommet de $P_1$. Nous appelons $P_2 = \Phi(\alpha_0\alpha_1)$ la réunion de l'arbre $P_1$ et leurs images par les rotations de centre $x_1$ et d'angle $\frac{\pi}{2}$, $\pi$ et $\frac{3\pi}{2}$. Par récurrence, nous avons une suite de sommets $x_n = x_{n-1} + 2^{n-1}\mathbf{r}(\alpha_{n-1}) = \sum_{i=0}^{n-1} 2^i\mathbf{r}(\alpha_i)$ et une suite croissante de sous-arbres finis $P_n$ de $\mathcal{Z}^2$. Alors $\Phi(\alpha) = \bigcup_{n\geqslant 0} P_n = \bigcup_{n\geqslant 0} \Phi(\alpha_0 \dots \alpha_{n-1})$ est un arbre apériodique et répétitif ayant au plus 2 bouts. Nous appellerons *squelette de* $\Phi(\alpha)$ la suite de sommets $x_0x_1 \dots x_n \dots$ identifiée au chemin d'arêtes obtenu en joignant les sommets $x_n$ et $x_{n+1}$ par $2^n$ arêtes dans la direction $\mathbf{r}(\alpha_i)$. Nous venons de définir une application $\Phi : \mathscr{S}_4 \to \mathscr{T}$.

***Proposition 3.1***. — *Le minimal de Ghys-Kenyon $X$ est l'enveloppe $\overline{\mathscr{R}[\Phi(\alpha)]}$ de tout arbre codé $\Phi(\alpha)$. Il se décompose en la réunion disjointe de la classe $\mathscr{R}[T_\infty]$ et de l'ensemble saturé $\bigcup_{\alpha\in\mathscr{S}_4} \mathscr{R}[\Phi(\alpha)]$.*

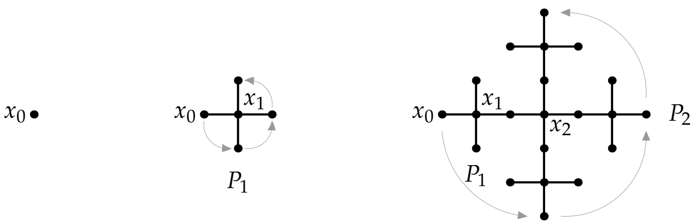


FIGURE 4. Construction de l'arbre $\Phi(\alpha)$

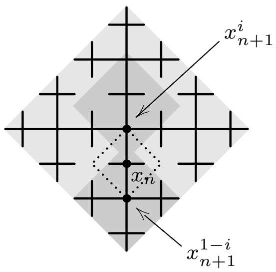


FIGURE 5. Le $(n+1)$-ième code

*Démonstration.* — Vérifions d'abord que $X = \overline{\mathscr{R}[\Phi(\alpha)]}$ pour toute suite $\alpha \in \mathcal{S}_4$. En effet, $T_\infty \in \overline{\mathscr{R}[\Phi(\alpha)]}$ car $B_{T_\infty}(0, 2^n - 1) = B_{\Phi(\alpha)}(x_n, 2^n - 1) - x_n = B_{\Phi(\alpha)-x_n}(0, 2^n - 1)$. Donc $X = \overline{\mathscr{R}[T_\infty]} \subset \overline{\mathscr{R}[\Phi(\alpha)]}$. Mais puisque $\Phi(\alpha)$ est répétitif, on a l'égalité. Pour montrer la deuxième affirmation, on constate que les arbres $T_\infty$ et $\Phi(\alpha)$ sont distincts car ils n'ont pas le même nombre de bouts. Leurs classes d'équivalence $\mathscr{R}[T_\infty]$ et $\mathscr{R}[\Phi(\alpha)]$ le sont aussi. Il faut donc vérifier que tout arbre $T \in X - \mathscr{R}[T_\infty]$ est équivalent à un arbre $\Phi(\alpha)$. En remplaçant $T$ par un translaté $T - v$, nous pourrons supposer que $\mathrm{val}(T) = \mathrm{val}_T(0) = 1$. Nous construirons alors de proche en proche une suite $\alpha \in \mathcal{S}_4$ telle que $T = \Phi(\alpha)$. Psar hypothèse, la sphère $S_T(0,1) = \partial \overline{B}_T(0,1)$ est réduite à un point $x_1$ et $\alpha_0 = \mathbf{r}^{-1}(x_1)$. Supposons connus les codes $\alpha_0 \dots \alpha_n$ et les points $x_0 \dots x_n$ du squelette. Alors la sphère $S_T(x_n, 2^n) = \partial \overline{B}_T(x_n, 2^n)$ vérifie l'une des deux conditions suivantes :

i) $S_T(x_n, 2^n)$ est réduite à un seul point $x_{n+1} = x_n + 2^n v$ où $v \in \mathbb{Z}^4$. Dans ce cas, nous définirons $\alpha_n = \mathbf{r}^{-1}(v)$.
ii) $S_T(x_n, 2^n)$ contient deux points $x^0_{n+1}$ et $x^1_{n+1}$. Dans ce cas, il existe $i \in \{0,1\}$ tel que $B_T(x^i_{n+1}, 2^{n+2} - 1) = B_{T_\infty}(0, 2^{n+2} - 1)$ (voir la figure 5) et nous définirons $x_{n+1} = x^{1-i}_{n+1} = x_n + 2^n v$ et $\alpha_n = \mathbf{r}^{-1}(v)$ avec $v \in \mathbb{Z}^4$.

Par récurrence, nous aurons une suite $\alpha \in \mathcal{S}_4$ telle que $T = \Phi(\alpha)$. $\square$

Considérons l'ouvert-fermé $X^{\leqslant 2} = \{T \in X/\ \mathrm{val}(T) \leqslant 2\}$, le $G_\delta$ dense $Y = X - \mathscr{R}[T_\infty] = \bigcup_{\alpha \in \mathcal{S}_4} \mathscr{R}[\Phi(\alpha)]$ et le borélien $Y^{\leqslant 2} = Y \cap X^{\leqslant 2}$. D'après la preuve de la proposition ci-dessus, l'application de codage $\Phi : \mathcal{S}_4 \to Y^{\leqslant 2}$ est surjective.

**3.3. Relation cofinale.** — Si on munit $\mathcal{S}_4 = \mathbb{Z}_4^{\mathbb{N}}$ de la topologie produit, engendrée par les cylindres $C^{i_0 \dots i_n}_{\beta_0 \dots \beta_n} = \{\alpha \in \mathcal{S}_4/\alpha_{i_0} = \beta_0, \dots, \alpha_{i_n} = \beta_n\}$, $\mathcal{S}_4$ est homéomorphe à l'ensemble de Cantor. Soit $\sigma : \mathcal{S}_4 \to \mathcal{S}_4$ le déplacement de Bernoulli donné par $\sigma(\alpha)_n = \alpha_{n+1}$ pour toute suite $\alpha \in \mathcal{S}_4$ et tout entier $n \geqslant 0$. Deux suites $\alpha$ et $\beta$ dans $\mathcal{S}_4$ sont *cofinales* s'il existe $n \geqslant 0$ tel que $\sigma^n(\alpha) = \sigma^n(\beta)$, c'est-à-dire $\alpha_m = \beta_m$ pour tout $m \geqslant n$. Pour tout couple de suites finies $\alpha_0 \dots \alpha_n$ et $\beta_0 \dots \beta_n$, les arbres finis $\Phi(\alpha_0 \dots \alpha_n)$ et $\Phi(\beta_0 \dots \beta_n)$ sont reliés par $\Phi(\beta_0 \dots \beta_n) = \Phi(\alpha_0 \dots \alpha_n) - v$ où le vecteur $v = \sum_{i=0}^n 2^i \mathbf{r}(\alpha_i) - \sum_{i=0}^n 2^i \mathbf{r}(\beta_i) = \sum_{i=0}^n 2^i \big(\mathbf{r}(\alpha_i) - \mathbf{r}(\beta_i)\big)$. Un argument simple montre alors que :

***Proposition 3.2***. — *Deux arbres codés $\Phi(\alpha)$ et $\Phi(\beta)$ sont $\mathscr{R}$-équivalents si et seulement si les suites $\alpha$ et $\beta$ sont cofinales.*

La remarque précédente montre aussi que $\Phi$ est injective, ce qui nous donne :

***Proposition 3.3***. — *L'application $\Phi : \mathcal{S}_4 \to Y^{\leqslant 2}$ est une bijection*

Signalons que l'expansion binaire des éléments de $\mathbb{Z}_4$ fournit un homéomorphisme entre $\mathcal{S}_4 = \mathbb{Z}_4^{\mathbb{N}}$ et $\mathcal{S}_2 = \mathbb{Z}_2^{\mathbb{N}}$, induit par les substitutions $0 \to 00$, $1 \to 10$, $2 \to 01$ et $3 \to 11$ obtenues en remplaçant $k \in \mathbb{Z}_4$ par un couple d'éléments $a(k)$ et $b(k)$ de $\mathbb{Z}_2$ tels que $k = a(k) + 2b(k)$. Évidemment les relations cofinales sur $\mathcal{S}_4 = \{0,1,2,3\}^{\mathbb{N}}$ et $\mathcal{S}_2 = \{0,1\}^{\mathbb{N}}$ deviennent isomorphes. Il y a d'ailleurs une équivalence orbitale entre la relation cofinale $\mathscr{R}_{\mathrm{cof}}$ sur $\mathcal{S}_2$ et la relation d'équivalence engendrée par la transformation $T : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ décrite dans l'introduction. Sauf les suites $000\dots$ et $111\dots$ qui appartiennent à une même orbite, les classes de cofinalité coïncident avec les orbites de $T$.

**3.4. Dynamique borélienne.** — Empruntée à la théorie des pavages, la notion de *motif* est le bon outil pour décrire la $\sigma$-algèbre des boréliens de $X$. Tout sous-arbre fini $P$ de $\mathcal{Z}^2$ contenant l'origine sera appelé un *motif* de $\mathcal{Z}^2$. Nous dirons que $T \in \mathcal{T}$ *contient le motif $P$ autour d'un sommet $p$* si $P + p \subset T$ et nous définirons $X_{P,p} = \{T \in X/P + p \subset T\}$. Si $p = 0$, nous écrirons simplement $X_P$. Comme pour les pavages [**1**], les ensembles $X_P$ sont des ouverts-fermés de $X$. Néanmoins, les motifs ne suffisent pas pour engendrer la topologie de $X$. En effet, la boule $B = \overline{B}_X(\Phi(00\dots), e^{-1})$ est l'ensemble des arbres $T$ tels que $\overline{B}_T(0,1) = \overline{B}_{\Phi(00\dots)}(0,1) = {}^0\bullet\!\!-\!\!\bullet$, mais il n'y a aucun motif $P$ tel que $X_P \subset B$. En fait,

$$B = X_{{}_0\bullet\!-\!\bullet} - \Big(X_{\begin{smallmatrix}\bullet \\ | \\ \bullet_0\end{smallmatrix}} \cup X_{\bullet\!-\!\bullet_0} \cup X_{\begin{smallmatrix}\bullet^0 \\ | \\ \bullet\end{smallmatrix}}\Big).$$

En général, pour tout arbre $T \in X$ et tout entier $r > 0$, la boule $\overline{B}_X(T, e^{-r})$ est l'ouvert-fermé $X_{(P,A)}$ associé au *motif fin* $(P, A)$ où $P = \overline{B}_T(e, r)$ et $A$ est l'ensemble des arêtes du graphe $\overline{B}_{\mathcal{Z}^2}(e,r) - B_T(e,r-1)$ qui rencontrent $P$. Par conséquent, les ouverts-fermés $X_P$ engendrent la $\sigma$-algèbre des boréliens.

L'application de codage $\Phi$ n'est pas continue, car $S_4$ est compact, mais $Y^{\leqslant 2}$ ne l'est pas. Néanmoins, $\Phi$ a deux propriétés importantes :

***Proposition 3.4.*** — *L'application $\Phi : \mathcal{S}_4 \to Y^{\leqslant 2}$ est borélienne ouverte.*

*Démonstration.* — Pour tout motif $P$, l'ensemble $\Phi^{-1}(X_P) = \bigcup_{\alpha_0\dots\alpha_n \in \mathcal{P}} C^{0\dots n}_{\alpha_0\dots\alpha_n}$ où $\mathcal{P} = \{\alpha_0 \dots \alpha_n / P \subset \Phi(\alpha_0 \dots \alpha_n)\}$. Par ailleurs, on a $\Phi(C^{0\dots n}_{\alpha_0\dots\alpha_n}) = X_{\Phi(\alpha_0\dots\alpha_n)}$. □

Nous pouvons maintenant affirmer que *la dynamique transverse borélienne de la lamination de Ghys-Kenyon est représentée par une machine à sommer binaire.*

**3.5. Propiétés ergodiques.** — Soit $\mathcal{R}$ une relation d'équivalence mesurable discrète sur $X$, munie d'une mesure quasi-invariante ergodique $\mu$. Par analogie avec la classification des facteurs de F. J. Murray et J. von Neumann, on peut distinguer trois types de relations :

1) *Type* $\mathrm{I}_n$ *(avec $n = 1, 2, \dots, \infty$)* : si $\mathcal{R}$ est transitive (avec cardinal $\#X = n$).

2) *Type* $\mathrm{II}_n$ *(avec $n = 1$ ou $\infty$)* : si $\mathcal{R}$ n'est pas transitive et si $\mu$ est équivalente à une mesure (finie ou infinie) invariante pour $\mathcal{R}$.

3) *Type* III : s'il n'existe pas de mesure invariante équivalente à $\mu$.

Si $\mu_4$ est la mesure de probabilité équidistribuée sur $\mathcal{S}_4$, alors $\mathcal{R}_{\mathrm{cof}}$ est de type $\mathrm{II}_1$.

***Proposition 3.5.*** — *La relation d'équivalence $\mathcal{R}$ sur $X$ est de type* $\mathrm{II}_1$.

*Démonstration.* — Pour tout $n \geqslant 1$, notons $B_n$ la boule de centre $T_\infty$ et de rayon $n$ contenue dans $\mathcal{R}[T_\infty]$. L'isomorphisme entre $\overline{\mathcal{R}}[T_\infty]$ et $T_\infty$ identifie $B_n$ avec $B_{T_\infty}(0,n)$. Soit $\mu_n$ la mesure de comptage sur $B_n$. Pour tout motif $P$, on a :

$$\mu_n(X_P) = \frac{\#B_n \cap X_P}{\#B_n} = \frac{\#\{\ p \in B_{T_\infty}(0,n)\ /\ P + p \subset T_\infty\ \}}{\#B_{T_\infty}(0,n))} = \frac{A(P,n)}{V(n)}$$

Quitte à extraire une sous-suite, on peut supposer que $\mu_n$ converge faiblement vers une mesure de probabilité $\mu$. Puisque $X_P$ est un ouvert-fermé, on a :

$$\mu(X_P) = \lim_{n\to\infty} \mu_n(X_P) = \lim_{n\to\infty} \frac{A(P,n)}{V(n)} = \textit{fréquence du motif } P.$$

D'autre part, pour tout sommet $v \in P$, l'ensemble $X_P - v = X_{P-v}$ est l'image de $X_P$ par la translation $\tau_v(T) = T - v$. Si $T_\infty$ contient le motif $P$ autour d'un point $p \in B_{T_\infty}(0, n-r)$, il contient aussi le motif $P - v$ autour du point $p + v \in B_{T_\infty}(0,n)$ avec $r > \| v \|$. Donc

$$| \mu_n(X_P - v) - \mu_n(X_P) | \leqslant \frac{V(n) - V(n-r)}{V(n)} \leqslant \frac{V(n+r) - V(n-r)}{V(n)}$$

pour tout $n \in \mathbb{N}$. Mais puisque la fonction $V(n)$ est à croissance sous-exponentielle, il vient $\lim_{n\to\infty} | \mu_n(X_P - v) - \mu_n(X_P) | = 0$ et donc $\mu$ est invariante pour $\mathcal{R}$. □

***Proposition 3.6.*** — *L'application $\Phi$ définit une équivalence orbitale stable entre les relations d'équivalence mesurées $\mathscr{R}_{\mathrm{cof}}$ sur $\mathscr{S}_4$ et $\mathscr{R}$ sur $X$.*

*Démonstration.* — Puisque le saturé de $Y^{\leqslant 2}$ est de mesure totale, il nous suffit de démontrer que $\Phi : \mathscr{S}_4 \to Y^{\leqslant 2}$ envoie $\mu_4$ sur une mesure équivalente à $\mu|_{Y^{\leqslant 2}}$. Par l'invariance de $\mu$, on a $\mu(X^{\leqslant 2}) = \frac{3}{4}$ et donc $\mu_{X^{\leqslant 2}} = \frac{4}{3}\mu|_{X^{\leqslant 2}}$ est une mesure de probabilité sur $X^{\leqslant 2}$ invariante pour $\mathscr{R}|_{X^{\leqslant 2}}$. L'inverse de $\Phi$ envoie la mesure induite par $\mu_{X^{\leqslant 2}}$ sur une mesure de probabilité sur $\mathscr{S}_4$ invariante pour $\mathscr{R}_{\mathrm{cof}}$. L'unicité ergodique de $\mu_4$ entraîne que $\Phi_*\mu_4 = \mu_{X^{\leqslant 2}}|_{Y^{\leqslant 2}}$. □

***Théorème 3.5.1.*** — *La dynamique transverse mesurable de la lamination de Ghys-Kenyon $(\boldsymbol{M}, \mathscr{L})$ est répresentée par une machine à sommer binaire. En outre, elle est uniquement ergodique.*

Un très joli résultat d'É. Ghys [**3**] permet de parler du type topologique des feuilles génériques de $\mathscr{L}$. De notre cas, on a que :

*i) il y a un ensemble saturé résiduel et de mesure totale dont toutes les feuilles ont exactement un bout ;*

*ii) il y a un ensemble saturé maigre et de mesure nulle constitué par une infinité non dénombrable des feuilles ayant deux bouts ;*

*iii) il y a une seule feuille avec quatre bouts.*

Le point essentiel est de vérifier qu'il y a correspondance biunivoque entre l'ensemble des feuilles ayant deux bouts et l'ensemble des suites de $\mathscr{S}_4$ contenant un nombre fini de détours et une infinité d'aller et retours. Pour toute suite $\alpha \in \mathscr{S}_4$, nous appelons *aller et retour* (resp. *détour*) tout couple $\alpha_n\alpha_{n+1}$ avec $\alpha_n \neq \alpha_{n+1}$ ayant la même (resp. distincte) parité. Cela permet de montrer que l'ensemble des feuilles à deux bouts est non dénombrable de mesure nulle. D'après le lemme 2.6 de [**2**], l'ensemble des feuilles ayant un bout est résiduel.

## 4. Dynamique topologique

Toutes les $\mathscr{R}$-classes du minimal de Ghys-Kenyon sont obtenues à partir des mêmes motifs par un même procédé d'inflation. Nous utiliserons l'inclusion de ces motifs dans les motifs qui résultent de l'inflation pour décrire sa dynamique topologique.

**4.1. Relations d'équivalence affables.** — Une relation d'équivalence $\beta$-discrète $\mathscr{R}$ sur un espace localement compact séparé $X$ est dite *compacte* [**5**] si $\mathscr{R} - \Delta$ est compact où $\Delta$ est la diagonale de $X \times X$.

***Définition 4.1.1*** **([5]).** — Une relation d'équivalence $\mathscr{R}$ définie sur un espace totalement disconnexe $X$ est dite *affable* s'il existe une suite croissante de relations d'équivalence compactes $\mathscr{R}_n$ telle que $\mathscr{R} = \bigcup_{n\in\mathbb{N}} \mathscr{R}_n$. Si on munit $\mathscr{R}$ de la topologie limite inductive, alors $\mathscr{R} = \varinjlim \mathscr{R}_n$ est une relation d'équivalence $\beta$-discrète *approximativement finie* (AF en abrégé).

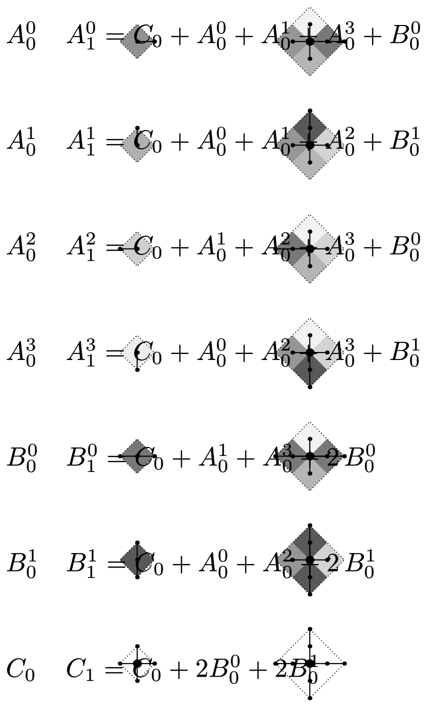


FIGURE 6. Les familles $\mathscr{P}_0$ et $\mathscr{P}_1$ et les règles d'inflation

Un *diagramme de Bratteli* est un graphe orienté $\mathscr{B} = (V, E)$ dont les ensembles de sommets et d'arêtes admettent des décompositions $V = \bigsqcup_{n \geqslant 0} V_n$ et $E = \bigsqcup_{n \geqslant 0} E_n$ où $V_n$ et $E_n$ sont des ensembles finis non vides tels que pour toute arête $e \in E_n$, l'origine $\alpha(e) \in V_n$ et l'extremité $\beta(e) \in V_{n+1}$ [**5**]. On appelle *source* tout sommet $v$ tel que $\beta^{-1}(v) = \varnothing$. Soit $X_{\mathscr{B}}$ l'espace des chemins infinis $e_n e_{n+1} e_{n+2} \dots$ (avec $\alpha(e_{i+1}) = \beta(e_i)$) issus d'une source $\alpha(e_n)$ de $\mathscr{B}$. La relation d'équivalence *cofinal* $\mathscr{R}_{\mathscr{B}}$ sur $X_{\mathscr{B}}$ (qui identifie $e_n e_{n+1} \dots$ et $e'_m, e'_{m+1} \dots$ s'il existe $N \geqslant m, n$ tel que $e'_i = e_i$ pour tout $i \geqslant N$) est affable. En fait, d'après [**5**], toute relation d'équivalence AF sur $X$ est isomorphe à la relation cofinale $\mathscr{R}_{\mathscr{B}}$ sur $X_{\mathscr{B}}$ définie par un diagramme de Bratteli $\mathscr{B}$.

**4.2. Affabilité du minimal de Ghys-Kenyon.** — Soit $\mathscr{P}_n = \{A_n^0, A_n^1, A_n^2, A_n^3, B_n^0, B_n^1, C_n\}$ la famille de *motifs basiques de taille* $n$ définis par $A_n^k = \overline{B}_{T_\infty}(0, 2^n - 1) \cup e_n^k$, $B_n^k = \overline{B}_{T_\infty}(0, 2^n - 1) \cup e_n^k \cup e_n^{k+2}$ et $C_n = \overline{B}_{T_\infty}(0, 2^n)$ où $e_n^k$ est l'arête qui rélie $(2^n - 1)\mathbf{r}(k)$ et $2^n \mathbf{r}(k)$ pour tout $k \in \mathbb{Z}_4$ (voir la figure 6). Deux éléments $T$ et $T'$ de $X - X_{C_n}$ sont $\mathscr{R}_n$*-équivalents* s'il existe un motif basique $P \in \mathscr{P}_n - \{C_n\}$ et deux sommets $v, v' \in P$ avec $\| v \|, \| v \| < 2^n$ tels que $P \subset T - v = T' - v'$. D'autre part, la relation $\mathscr{R}_n$ est triviale sur $X_{C_n}$.

***Proposition 4.1***. — *Les relations d'équivalence $\mathscr{R}_n$ sont compactes et ouvertes dans $\mathscr{R}$ et donc $\mathscr{R}_\infty = \bigcup_{n\in\mathbb{N}} \mathscr{R}_n$ est affable et ouverte dans $\mathscr{R}$.*

*Démonstration*. — Montrons que $\mathscr{R}_n$ est ouverte dans $\mathscr{R}$. Pour tout couple $(T,T') \in \mathscr{R}_n$, il existe un motif $P \in \mathscr{P}_n$ et deux sommets $v, v' \in P$ avec $\| v \|, \| v' \| < 2^n$ tels que $P \subset T - v = T' - v'$. Choisissons $N > 0$ tel que $P + v \subseteq B_T(0,N)$, puis considérons l'ouvert $U = \{T'' \in X / B_{T''}(0,N) = B_T(0,N)\}$ de $X$ et l'ouvert $O(U,w)$ de $\mathscr{R}$ où $w = v - v'$. Pour tout $T'' \in U$, le couple $(T'', T'' - w) \in \mathscr{R}_n$ car $T''$ contient le motif $P$ autour de $v$. Donc $(T,T') \in O(U,v) \subset \mathscr{R}_n$. Alors $\mathscr{R}_n$ est la réunion des ouverts $O(U,w)$ associés motifs $P \in \mathscr{P}_n$ et aux sommets $v, v' \in P$ tels que $\| v \|, \| v' \| < 2^n$. En remplaçant $U$ par l'ouvert-fermé $X_{P,v}$ et $O(U,w)$ par le graphe de la translation $T'' \mapsto T'' - w$ définie sur $X_{P,v}$, nous aurons que $R_n$ est compacte. □

Toutes les classes d'équivalence de $\mathscr{R}$ et $\mathscr{R}_\infty$ sont égales, sauf celle de $T_\infty$ qui se décompose en la réunion de la classe triviale $\{T_\infty\}$ et de quatre classes isomorphes aux composantes connexes de $T_\infty - \{0\}$. La dynamique topologique de $\mathscr{R}_\infty$ est représentée par le diagramme de Bratteli $\mathscr{B} = (V,E)$ où $V_0 = \{0\}$, $V_{n+1} = \mathscr{P}_n = \{A_n^0, A_n^1, A_n^2, A_n^3, B_n^0, B_n^1, C_n\}$ et $P \in \mathscr{P}_n$ est relié par une arête de $E_{n+1}$ à $Q \in \mathscr{P}_{n+1}$ si et seulement si $Q$ contient une copie fidèle de $P$. L'isomorphisme $\Psi : X \to X_{\mathscr{B}}$ entre $\mathscr{R}_\infty$ et $\mathscr{R}_{\mathscr{B}}$ est donné par $\Psi(T) = (e_0, e_1, \dots)$ où $\beta(e_n)$ est l'unique motif $P \in \mathscr{P}_{n+1}$ pour lequel $T - v$ appartient à l'ouvert-fermé $X_{(P,A)}$ avec $v \in P$ et $A$ formé des arêtes de $\overline{B}_{T_\infty}(0, 2^{n+1})$ qui n'appartiennent pas à $P$. Pour tout $T \in X$ avec $\mathrm{val}(T) = 4$, l'origine 0 est l'intersection des translatés de quatre motifs basiques de taille $n$. Nous modifierons alors $\mathscr{R}_n$ pour que 0 devienne équivalent aux autres points du translaté de $A_n^0$ ou de $B_n^0$. Nous obtiendrons ainsi une suite de relations d'équivalence compactes $\mathscr{R}'_n \supset \mathscr{R}_n$. Alors $\mathscr{R}'_\infty = \bigcup_{n\in\mathbb{N}} \mathscr{R}'_n$ est affable. Puisque les bouts de la feuille de $\mathscr{F}$ passant par $T_\infty$ sont partout denses, $\mathscr{R}[T_\infty]$ se décompose en la réunion de quatre orbites denses et donc $\mathscr{R}'_\infty$ est minimale. Nous pouvons maintenant appliquer le corollaire 4.17 de [**5**] :

***Théorème 4.2.1***. — *La relation d'équivalence $\mathscr{R}$ est affable et la dynamique transverse de la lamination de Ghys-Kenyon est répresentée par un système dynamique minimal sur l'ensemble de Cantor.*

## Références


[1] J. Bellissard, R. Benedetti & J.-M. Gambaudo – Spaces of tilings, finite telescopic approximations and gap-labeling, *Comm. Math. Phys.* **261** (2006), p. 1–41.

[2] E. Blanc – Propriétés génériques des laminations, Thèse, UCB Lyon 1, 2001.

[3] É. Ghys – Topologie des feuilles génériques, *Ann. of Math.* **141** (1995), p. 387–422.

[4] ———, Laminations par surfaces de Riemann, in *Dynamique et géométrie complexes (Lyon, 1997)*, Panor. Synthèses, vol. 8, Soc. Math. France, 1999, p. 49–95.

[5] T. Giordano, I. Putnam & C. Skau – Affable equivalence relations and orbit structure of Cantor dynamical systems, *Ergodic Theory Dynam. Systems* **24** (2004), p. 441–475.

[6] A. Haefliger – Pseudogroups of local isometries, *Res. Notes in Math.* **131** (1985), p. 174–197.

[7] R. Kenyon – A group of paths in $\mathbb{R}^2$, *Trans. Amer. Math. Soc.* **348** (1996), p. 3155–3172.

[8] Á. Lozano Rojo – The Cayley foliated space of a graphed pseudogroup, in *XIV Fall Workshop on Geometry and Physics*, *Publ. R. Soc. Mat. Esp.* **10** (2006), p. 267–272.

[9] C. Radin & M. Wolff – Space tilings and local isomorphism, *Geom. Dedicata* **42** (1992), p. 355–360.

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# COMPARAISON DES VALUATIONS DIVISORIELLES

*par*

Charef Beddani

***Résumé.*** — En utilisant la notion de la connexité en codimension un, nous allons donner dans cet article une nouvelle démonstration géométrique du théorème d'Izumi dans deux cas particuliers. Ensuite, nous allons proposer la conjecture suivante : soient $(R, \mathfrak{m})$ un anneau local intègre normal complet et $\nu_1, \nu_2$ deux valuations divisorielles centrées en $\mathfrak{m}$, alors il existe un idéal $\mathfrak{m}$-primaire $I$ de $R$, tel que les centres de $\nu_1$ et $\nu_2$ dans l'éclatement normalisé de Spec$R$ le long de $I$ sont liés en codimension 1. A la fin de ce travail, nous présentons quelques commentaires concernant cette conjecture.

***Abstract* (Comparison of divisorial valuations).** — Using the notion of connexity in codimension one, we give in this paper a new geometric proof of Izumi's theorem in two special cases. We also propose the following conjecture: let $(R, \mathfrak{m})$ be a complete, normal local domain and $\nu_1, \nu_2$ two divisorial valuations centered in $\mathfrak{m}$. Then there exists an $\mathfrak{m}$-primary ideal $I$ of $R$ such that the centers of $\nu_1$ and $\nu_2$ in the normalised blowing up of Spec$R$ along $I$ are linked in codimension 1. At the end of the paper, we make some comments about this conjecture.

## Introduction

Les valuations divisorielles sont des objets fondamentaux pour l'étude de la résolution des singularités. Elles ont été étudiées par Zariski, Abhyankar, Rees, Swanson et beaucoup d'autres. Plus récemment l'étude des valuations pour aborder de manière nouvelle le problème de résolution des singularités a été proposé, notamment par M. Spivakovsky (cf. [**11**]) et B. Teissier (cf. [**12**]). Nous nous intéressons dans cet article à l'étude des valuations divisorielles. Plus précisément, nous présentons dans la première section quelques résultats élémentaires concernant les valuations divisorielles avec leurs démonstrations. La deuxième section est consacrée à la comparaison des valuations divisorielles. Nous donnons une approche géométrique du théorème d'Izumi (cf. [**10**]), en utilisant la notion de connexité en codimension 1 (cf. [**3**]). De manière générale, nous allons montrer le résultat suivant : Soient $(R, \mathfrak{m})$ un anneau

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local intègre nœthérien analytiquement irréductible et $\nu_1$, $\nu_2$ deux valuations divisorielles associées à un idéal $\mathfrak{m}$-primaire $I$ telles que les centres $E_1$ et $E_2$ de $\nu_1$ et $\nu_2$ respectivement dans l'éclatement normalisé $\overline{X}_I$ de Spec $R$ le long de $I$ sont liés en codimension 1. C'est-à-dire, qu'il existe une suite finie

$$Y_1 = E_1, Y_2, ..., Y_{s-1}, Y_s = E_2$$

de composantes irréductibles de $E_I = \mathrm{Proj} \bigoplus_{n\geq 0} \overline{I^n}/I \cdot \overline{I^n}$ telle que pour tout $1 \leq i \leq s-1$, la codimension de $Y_i \cap Y_{i+1}$ dans $Y_{i+1}$ est égale à 1. Alors les topologies $\nu_1$-adique et $\nu_2$-adique sont linéairements équivalentes. Autrement dit, il existe un entier naturel $r$ tel que pour tout élément $x$ non nul appartenant à $R$, nous avons

$$\nu_1(x) \leq r\nu_2(x) \quad \text{et} \quad \nu_2(x) \leq r\nu_1(x).$$

Ensuite, nous en déduisons une nouvelle démonstration du théorème d'Izumi (cf. Théorème 2.1) sur les anneaux analytiquements irréductibles de dimension inférieure ou égale à deux. Les démonstrations connues auparavant de ce théorème en dimension deux (cf. [**7, 5, 10**]) sont basées sur le fait que la matrice $M = (E_i.E_j)_{1\leq i,j\leq s}$ est définie négative. En dimension supérieure où égale à trois, la matrice d'intersection n'a pas de sense. Pour cette raison, D. Rees utilise une démonstration par récurrence sur la dimension de $R$ (cf. [**10**]) quand la dimension de $R$ est supérieure ou égale à trois. La démonstration qu'on donne dans cet article sous quelques hypothèses est directe en dimension quelconque (sans récurrence sur la dimension de $R$). Nous trouvons un remplacement géométrique en dimension supérieure pour la négativité de la matrice $M$ qui est un phénomène spécifique en dimension deux.

Le théorème d'Izumi est toujours vrai sans ajouter la condition « $\nu_1$ et $\nu_2$ sont liées en codimension 1 », c'est la raison pour laquelle nous allons proposer la conjecture suivante : soient $(R, \mathfrak{m})$ un anneau local intègre normal complet et $\nu_1, \nu_2$ deux valuations divisorielles centrées en $\mathfrak{m}$. Alors il existe un idéal $\mathfrak{m}$-primaire $I$ de $R$ tel que les centres de $\nu_1$ et $\nu_2$ dans $\overline{X_I}$ sont liés en codimension 1.

Pour les surfaces (i.e. $\dim R = 2$), nous allons montrer que la conjecture précédente est une conséquence immédiate du théorème principal de Zariski. Ensuite, nous allons suivre les travaux de R. Hartshorne concernant la connexité en codimension (cf. [**3**]), et ceux de M. Spivakovsky sur les valuations divisorielles (cf. [**11**]), pour démontrer que cette conjecture est vraie si $R$ admet une résolution des singularités plongée. En particulier, elle est vraie si $R$ est quasi-excellent de caractéristique zéro ou $\dim R \leq 3$ et $R$ est de type fini sur un corps parfait.

*Remerciement* : Je remercie Mark Spivakovsky, pour les remarques et les conseils qui m'ont permis d'apporter certaines précisions et de rendre plus claires plusieurs parties de ce texte.

## 1. Préliminaires

Tous les anneaux considérés dans cet article sont commutatifs et unitaires. Si $\mathfrak{p}$ est un idéal premier d'un anneau $R$, on note ht $\mathfrak{p}$ la hauteur de $\mathfrak{p}$, et $k(\mathfrak{p})$ le corps résiduel de l'anneau $R_{\mathfrak{p}}$. Si $(Q, \mathfrak{n})$ est un anneau local contenant $R$ tel que $\mathfrak{n} \cap R = \mathfrak{m}$, on note $\text{deg.tr.}_{k(\mathfrak{m})}\, k(\mathfrak{n})$ le degré de transcendance de $k(\mathfrak{n})$ sur $k(\mathfrak{m})$.

***Notation 1.1***. — *Soit $\Gamma$ un groupe commutatif totalement ordonné. Nous adjoignons au groupe $\Gamma$ un élément $\infty$ et nous appelons $\Gamma_\infty$ l'ensemble ainsi obtenu : $\Gamma_\infty = \Gamma \cup \{\infty\}$. Nous munissons cet ensemble d'une relation d'ordre total en posant pour tout $\gamma \in \Gamma$ :*

1. $\gamma \leq \infty$.
2. $\infty + \gamma = \gamma + \infty = \infty + \infty = \infty$.

***Définition 1.2***. — *Soient $(R, \mathfrak{m})$ un anneau local intègre et $K$ son corps de fractions. On appelle valuation de $K$ à valeurs dans un groupe totalement ordonné $\Gamma$, toute application $\nu$ de $K$ dans $\Gamma_\infty$ qui vérifie les conditions suivantes :*

1. *Pour tous $x, y \in K$, $\nu(xy) = \nu(x) + \nu(y)$,*
2. *Pour tous $x, y \in K$, $\nu(x + y) \geq \inf(\nu(x), \nu(y))$,*
3. *Pour tout $x \in K$, $\nu(x) = \infty \Leftrightarrow x = 0$.*

***Définition 1.3***. — *Soient $\Gamma$ un groupe commutatif totalement ordonné et $\nu$ une valuation à valeurs dans $\Gamma$. Le rang rationnel de $\nu$ est un élément de $\mathbb{N} \cup \{+\infty\}$ défini par*

$$\text{rang rat.}\, \nu := \dim_{\mathbb{Q}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Si $\Gamma \simeq \mathbb{Z}$, on dit que la valuation $\nu$ est discrète. Dans ce cas, on note, pour tout $n \in \mathbb{Z}$ :

$$I_n(\nu) = \{x \in R \text{ tel que } \nu(x) \geq n\}.$$

Si $I$ est un idéal de $R$ finiment engendré, on note : $\nu(I) = \min\{\nu(x) \text{ tel que } x \in I\}$. Nous rappelons que si $\nu$ est une valuation discrète de $K$, alors l'ensemble des éléments $x \in K$ tels que $\nu(x) \geq 0$ forme un anneau local. Cet anneau s'appelle l'anneau de valuation associé à $\nu$, on le note $R_\nu$. Son idéal maximal $\mathfrak{m}_\nu$ est défini par $\mathfrak{m}_\nu = \{x \in K \text{ tel que } \nu(x) \geq 1\}$.

***Définition 1.4***. — *Soient $R$ un anneau intègre, $K$ son corps de fractions et $\nu$ une valuation discrète de $K$ centrée dans $R$ en $\mathfrak{p}$ (ie. $R \subset R_\nu$ et $\mathfrak{p} = R \cap \mathfrak{m}_\nu$). La valuation $\nu$ est dite divisorielle si elle vérifie l'égalité suivante*

$$\text{deg.tr.}_{k(\mathfrak{p})}\, k_\nu = \text{ht}\, \mathfrak{p} - 1,$$

*où $k_\nu = R_\nu / \mathfrak{m}_\nu$ est le corps résiduel de $R_\nu$.*

***Définition 1.5***. — *Soient $R$ un anneau intègre et $K$ son corps de fractions.*
*On dit que $R$ est N-2 si pour toute extension finie $L$ de $K$, la clôture intégrale de $R$ dans $L$ est un $R$-module de type fini.*

***Définition 1.6***. — *Un anneau $R$ est dit de Nagata (ou universellement japonais) s'il est nœthérien et pour tout idéal premier $\mathfrak{p}$ de $R$, l'anneau $R/\mathfrak{p}$ est N-2.*

***Lemme 1.7.*** — *Soient $(R, \mathfrak{m})$ un anneau nœthérien local intègre de Nagata et $K$ son corps de fractions. Alors pour toute valuation divisorielle $\nu$ de $K$ centrée dans $R$ en $\mathfrak{m}$, il existe un idéal $\mathfrak{m}$-primaire $I$ de $R$ tel que le centre de $\nu$ dans $\overline{X}_I$ est de codimension 1 dans $\overline{X}_I$.*

*Démonstration.* — On peut choisir $(a_1, a_2, ..., a_d)$ un système de paramètres de $R$ tel que les images de $a_2/a_1, a_3/a_1, ..., a_d/a_1$ dans $R_\nu/\mathfrak{m}_\nu$ sont algébriquements indépendantes sur $k(\mathfrak{m})$. Pour tout $i = 1, 2, ..., d$, soit $s_i = \prod_{j \neq i} \nu(a_j)$. Prenons $I$ l'idéal de $R$ engendré par $a_1^{s_1}, a_2^{s_2}, ..., a_d^{s_d}$. Il est clair d'après le théorème de la dimension (cf. [**8**], Théorème 2.5, Page 333) que le centre de $\nu$ dans $\overline{X}_I$ est de codimension 1 dans $\overline{X}_I$. □

***Définition 1.8.*** — *Soit $(R, \mathfrak{m})$ un anneau local intègre. On dit que $R$ est analytiquement irréductible si le complété $\mathfrak{m}$-adique de $R$ est intègre.*

***Lemme 1.9.*** — *Soient $(R, \mathfrak{m})$ un anneau local nœthérien intègre analytiquement irréductible et $(\widehat{R}, \widehat{\mathfrak{m}})$ le complété $\mathfrak{m}$-adique de $R$. Alors pour toute valuation divisorielle $\nu$ de $R$ centrée en $\mathfrak{m}$, il existe une seule valuation divisorielle $\widehat{\nu}$ de $\widehat{R}$ centrée en $\widehat{\mathfrak{m}}$ telle que pour tout $x \in R$ on a $\widehat{\nu}(x) = \nu(x)$.*

*Démonstration.* — Montrons d'abord l'existence de $\widehat{\nu}$. Nous avons le diagramme commutatif suivant

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[d] & R_\nu \arrow[d] \\ \widehat{R} \arrow[r, "\widehat{i}"] & \widehat{R}_\nu \end{tikzcd}$$

où $\widehat{R}_\nu$ est le completé $\mathfrak{m}_\nu$-adique de $R_\nu$. L'anneau $\widehat{R}_\nu$ est un anneau de valuation discrète. Posons $\widehat{\nu}$ la valuation associée à $\widehat{R}_\nu$ . Nous allons montrer que le morphisme $\widehat{i}$ est injectif. Supposons le contraire (ie. $\mathfrak{p} = \ker \widehat{i} \neq (0)$). Puisque l'anneau $\widehat{R}_\nu$ est intègre, l'idéal $\mathfrak{p}$ est premier. Notons $\mu$ la restriction de $\widehat{\nu}$ à $k(\mathfrak{p})$. Soient $\nu_0$ une valuation de $\widehat{R}_\mathfrak{p}$ centrée en $\mathfrak{p}\widehat{R}_\mathfrak{p}$ et $\nu_1$ une extension de $\mu$ au corps résiduel de $\nu_0$. Prenons $\nu_2 = \nu_0 \circ \nu_1$ la valuation composée avec les valuation $\nu_0$ et $\nu_1$ (cf. [**13**]). Nous avons l'égalité

$$\text{rang rat.}\, \nu_2 = \text{rang rat.}\, \nu_0 + \text{rang rat.}\, \nu_1. \tag{1}$$

Comme $\widehat{R}$ est intègre, la hauteur de $\mathfrak{p}$ est supérieure ou égale à 1. Donc le rang rationnel de $\nu_0$ est supérieur ou égal à 1. Nous en déduisons d'après l'égalité (1) que

$$\text{rang rat.}\, \nu_2 \geq 2. \tag{2}$$

De plus, on a

$$\text{deg.tr.}_{k(\widehat{\mathfrak{m}})}\, k_{\nu_2} \geq \text{deg.tr.}_{k(\mathfrak{m})}\, k_\nu. \tag{3}$$

Les inégalités (2) et (3) donnent

$$\operatorname{deg.tr.}_{k(\widehat{\mathfrak{m}})} k_{\nu_2} + \text{rang rat.}\, \nu_2 \geq 2 + \operatorname{deg.tr.}_{k(\mathfrak{m})} k_\nu = \dim R + 1.$$

Cela est une contradiction avec l'inégalité d'Abhyankar [**1**]. Donc le morphisme $\widehat{i}$ est injectif.

Soit $x$ un élément de $\widehat{R}$. Posons $\widehat{\nu}(x) = s$ et $\widehat{\nu}(\widehat{\mathfrak{m}}) = r_1$. Pour tout entier naturel $n > s/r_1$, il existe un élément $x'$ appartenant à $R$ tel que $x - x' \in \widehat{\mathfrak{m}^n}$. Nous avons

$$\begin{aligned}\widehat{\nu}(x) &= \widehat{\nu}\big((x - x') + x'\big)\\ &= \widehat{\nu}(x')\\ &= \nu(x')\end{aligned}$$

Donc les valuations $\nu$ et $\widehat{\nu}$ ont le même groupe de valeurs (i.e. $\Gamma_\nu = \Gamma_{\widehat{\nu}}$.) Maintenant, soient $x$ et $y \neq 0$ deux éléments appartenant à $\widehat{R}$ tels que $x/y \in R_{\widehat{\nu}}$. Nous pouvons choisir deux éléments $x'$ et $y'$ appartenant à $R$ tels que $\widehat{\nu}(x - x') > \widehat{\nu}(x)$ et $\widehat{\nu}(y - y') > \widehat{\nu}(y)$. Donc $\widehat{\nu}(x) = \nu(x')$ et $\widehat{\nu}(y) = \nu(y')$, par suite $x'/y' \in R_\nu$. En plus nous avons

$$\frac{x}{y} - \frac{x'}{y'} = \frac{x - x'}{x} \cdot \frac{x}{y} + \frac{y' - y}{y} \cdot \frac{x'}{y'} \in \mathfrak{m}_{\widehat{\nu}}.$$

ce qui donne que $\nu$ et $\widehat{\nu}$ ont le même corps résiduel (i.e. $k_{\widehat{\nu}} \backsimeq k_\nu$). Le fait que $\dim \widehat{R} = \dim R$ et $k(\widehat{\mathfrak{m}}) \backsimeq k(\mathfrak{m})$ entraîne que la restriction de la valuation $\widehat{\nu}$ au corps de fractions de $\widehat{R}$, qu'on note aussi $\widehat{\nu}$ est une valuation divisorielle centrée dans $\widehat{R}$ en $\widehat{\mathfrak{m}}$, et elle vérifie bien évidemment $\nu(x) = \widehat{\nu}(x)$ pour tout $x \in R$.

Montrons maintenant l'unicité de la valuation $\widehat{\nu}$. Soit $\tilde{\nu}$ une autre valuation divisorielle de $K(\widehat{R})$ centrée dans $\widehat{R}$ en $\widehat{\mathfrak{m}}$ qui vérifie $\nu(x) = \tilde{\nu}(x)$ pour tout $x \in R$. Prenons $z$ un élément de $\widehat{R}$. Soient $\widehat{\nu}(z) = s_1$, $\tilde{\nu}(z) = s_2$ et $\tilde{\nu}(\mathfrak{m}) = r_2$. Alors pour tout entier naturel $n \geq \sup(s_1/r_1, s_2/r_2)$, il existe $z' \in R$ tel que $z - z' \in \mathfrak{m}^n\widehat{R}$. Par consequent $\widehat{\nu}(z) = \widehat{\nu}(z')$ et $\tilde{\nu}(z) = \tilde{\nu}(z')$, et comme $z' \in R$ et les valuations $\widehat{\nu}$ et $\tilde{\nu}$ sont égaux sur $R$. Il en résulte que $\widehat{\nu}(z) = \tilde{\nu}(z)$. $\square$

***Lemme 1.10***. — *Soient $(R, \mathfrak{m})$ un anneau nœthérien local intègre analytiquement irréductible et $\overline{R}$ la normalisation de $R$. Alors pour toute valuation divisorielle $\nu$ de $R$ centrée en $\mathfrak{m}$, il existe une valuation divisorielle $\overline{\nu}$ de $\overline{R}$ centrée en $\overline{\mathfrak{m}}$ (l'idéal maximal de $\overline{R}$[(1)]), telle que pour tout $x \in R$, on a $\overline{\nu}(x) = \nu(x)$.*

*Démonstration.* — On a bien l'inclusion :

$$R \subseteq \overline{R} \subseteq R_\nu.$$

Notons $\overline{\nu}$ l'extension de $\nu$ dans $\overline{R}$. Puisque $R$ est analytiquement irréductible, l'anneau $\overline{R}$ est local et $\overline{\mathfrak{m}}$ est son idéal maximal (cf. [**8**]). La valuation $\overline{\nu}$ est donc centrée dans $\overline{R}$ en $\overline{\mathfrak{m}}$. De plus, comme $\nu$ est divisorielle,

$$\operatorname{ht} \mathfrak{m} = \operatorname{ht} \overline{\mathfrak{m}}$$

(1) Sous les hypothèses de ce lemme, la normalisation de $R$ est anneau local (cf. [**8**]), Proposition 2.14, (c), page 344).

et

$$\text{deg.tr.}_{k(\mathfrak{m})}\, k(\overline{\mathfrak{m}}) = 0,$$

il en résulte que la valuation $\overline{\nu}$ est aussi divisorielle. □

## 2. Comparaison des valuations divisorielles

Soient $X = \operatorname{Spec} R$ où $R$ est un anneau nœthérien local analytiquement irréductible, $\mathfrak{m}$ son idéal maximal et $K$ son corps de fractions. Pour tout idéal $I$ de $R$, nous notons $\pi_I : \overline{X}_I \longrightarrow X$ l'éclatement normalisé de $X$ le long de $I$ et $E_I = V(I\mathcal{O}_{\overline{X}_I})_{red}$ le sous-schéma réduit de $\overline{X}_I$ associé au faisceau $I\mathcal{O}_{\overline{X}_I}$. Soient $E_1$ et $E_2$ deux composantes irréductibles de $E_I$. On dit que $E_1$ et $E_2$ sont liées en codimension 1, s'il existe une suite finie $Y_1 = E_1, Y_2, ..., Y_{s-1}, Y_s = E_2$ de composantes irréductibles de $E_I$ telle que pour tout $1 \leq i \leq s-1$, la codimension de $Y_i \cap Y_{i+1}$ dans $Y_{i+1}$ est égale à 1. De même, si $\nu_1$ et $\nu_2$ sont deux valuations divisorielles de $K$ centrées dans $R$ en $\mathfrak{m}$, on dit que $\nu_1$ et $\nu_2$ sont liées en codimension 1 s'il existe un idéal $\mathfrak{m}$-primaire $I$ de $R$ tel que le centre de $\nu_1$ et le centre de $\nu_2$ dans $\overline{X}_I$ sont liés en codimension 1. Rappelons tout d'abord le théorème d'Izumi :

***Théorème 2.1* (Théorème d'Izumi [5, 10]).** — *Soit $(R, \mathfrak{m})$ un anneau nœthérien local analytiquement irréductible. Alors pour toutes valuations divisorielles $\nu_1, \nu_2$ centrées en $\mathfrak{m}$ , il existe $r \in \mathbb{N}^*$ tel que pour tout $x$ non nul appartenant à $R$ on a $\nu_1(x) \leq r\nu_2(x)$.*

Supposons que l'anneau $R$ est analytiquement irréductible et $\nu_1$, $\nu_2$ sont deux valuations divisorielles de $R$ centrées en $\mathfrak{m}$. Le but de cette section est de donner une nouvelle démonstration du théorème d'Izumi dans les cas suivants :

**Cas (I) :** $R$ est de dimension quelconque et les extensions de $\nu_1, \nu_2$ dans le corps de fractions de $\widehat{R}$ sont liées en codimension 1 (cf. Théorème 2.5).

**Cas (II) :** $R$ est de dimension inférieure ou égale à 2 et $\nu_1, \nu_2$ sont deux valuations divisorielles quelconques (cf. Théorème 2.10).

### 2.1. Le cas (I)

***Notation 2.2.*** — *Si $D = \sum n_i D_i$ est un diviseur de Weil d'un schéma $X$ et $Y \subset X$ un sous-schéma de $X$, nous notons :*

$$D \cap |Y| = \sum_{D_i \subset Y} n_i D_i$$

***Proposition 2.3.*** — *Soient $(R, \mathfrak{m})$ un anneau de Nagata analytiquement irréductible et $\nu_1$, $\nu_2$ deux valuations divisorielles de $K(R)$ centrées en $\mathfrak{m}$. Si $\nu_1$ et $\nu_2$ sont liées en codimension 1, alors leurs extensions $\widehat{\nu}_1$ et $\widehat{\nu}_2$ dans $\widehat{R}$ sont également liées en codimension 1.*

*Démonstration.* — Par hypothèse, il existe un idéal $\mathfrak{m}$-primaire $I$ de $R$ tel que les centres de $\nu_1$ et $\nu_2$ sont liés en codimension 1. Soit $E_1$ (resp. $E_2$) le centre de $\nu_1$ et $\nu_2$ dans $\overline{X}_I$, il existe donc une suite finie

$$Y_1 = E_1, Y_2, ..., Y_{s-1}, Y_s = E_2$$

de composantes irréductibles de $E_I$ telles que pour tout $1 \leq i \leq s-1$, la codimension de $Y_i \cap Y_{i+1}$ dans $Y_{i+1}$ est égale à 1. Par conséquent, nous pouvons supposer que la codimension de $E_1 \cap E_2$ dans $E_2$ est égale à 1. Soient

$$S(I) = \bigoplus_{n=0}^{+\infty} \overline{I^n} T^n$$

et

$$S(I\widehat{R}) = \bigoplus_{n=0}^{+\infty} \overline{I^n \widehat{R}} T^n$$

les algèbres graduées définies par les filtrations $\{\overline{I^n}\}_{n=0}^{+\infty}$ et $\{\overline{I^n}\widehat{R}\}_{n=0}^{+\infty}$ respectivement. Les deux diviseurs $E_1$ et $E_2$ sont définis respectivement par deux idéaux premiers homogènes $\mathfrak{q}_1$ et $\mathfrak{q}_2$ de $S(I)$ tels que :

$$\mathrm{ht}\, \mathfrak{q}_1 = \mathrm{ht}\, \mathfrak{q}_2 = 1$$

et

$$IS(I) \subseteq \mathfrak{q}_1 \cap \mathfrak{q}_2.$$

Nous avons donc $R_{\nu_1} = S(I)_{(\mathfrak{q}_1)}$ et $R_{\nu_2} = S(I)_{(\mathfrak{q}_2)}$. Comme le morphisme naturel $R \hookrightarrow \widehat{R}$ est fidèlement plat et $I$ est $\mathfrak{m}$-primaire, $\overline{I^n \widehat{R}} = \overline{I^n}\widehat{R}$.

Par suite,

$$S(I)/IS(I) \cong S(I\widehat{R})/IS(I\widehat{R}).$$

Pour finir la démonstration, on a besoin de l'affirmation suivante :

***Affirmation 2.4.*** — *Pour tout $i = 1, 2$, il existe un unique idéal premier homogène $Q_i$ minimal de $IS(I\widehat{R})$ tel que :*

$$Q_i = \mathfrak{q}_i S(I\widehat{R}).$$

D'abord, admettons que cette affirmation et finissons la preuve de la proposition. On a :

$$(Q_1 + Q_2) = (\mathfrak{q}_1 + \mathfrak{q}_2) S(I\widehat{R}).$$

Par fidèle platitude de l'extension $S(I) \longrightarrow S(I\widehat{R})$, nous obtenons

$$\mathrm{ht}(Q_1 + Q_2) = \mathrm{ht}(\mathfrak{q}_1 + \mathfrak{q}_2).$$

De plus, comme les anneaux $R$ et $\widehat{R}$ sont analytiquement irréductibles, les anneaux $S(I)$ et $S(I\widehat{R})$ sont universellement caténaires (cf. [4], Théorème (18.13), Théorème (18.23)). Donc

$$\begin{aligned}\operatorname{ht}((Q_1+Q_2)/Q_2) &= \operatorname{ht}(Q_1+Q_2) - \operatorname{ht} Q_2 \\ &= \operatorname{ht}(\mathfrak{q}_1+\mathfrak{q}_2) - \operatorname{ht} \mathfrak{q}_2 \\ &= \operatorname{ht}((\mathfrak{q}_1+\mathfrak{q}_2)/\mathfrak{q}_2).\end{aligned}$$

Par construction de $\widehat{\nu}_1$ (resp. $\widehat{\nu}_2$), et l'unicité de $Q_1$ (resp. $Q_2$), on obtient que le centre $\widehat{E}_1$ (resp. $\widehat{E}_2$) de $\widehat{\nu}_1$ (resp. $\widehat{\nu}_2$) dans l'éclatement normalisé de Spec $\widehat{R}$ le long de $I\widehat{R}$ est définie par l'idéal homogène $Q_1$ (resp. $Q_2$). Nous savons que la codimension de $E_1 \cap E_2$ dans $E_2$ est égale à la hauteur de $(\mathfrak{q}_1+\mathfrak{q}_2)/\mathfrak{q}_2$ et ainsi la codimension de $\widehat{E}_1 \cap \widehat{E}_2$ dans $\widehat{E}_2$ est égale à la hauteur de $(Q_1+Q_2)/Q_2$. Par conséquent $\widehat{E}_1$ et $\widehat{E}_2$ sont liés en codimension 1. □

**Démonstration de l'affirmation 2.4 :** Le fait que le morphisme $S(I) \longrightarrow S(I\widehat{R})$ est fidèlement plat entraîne que l'idéal $\mathfrak{q}_1 S(I\widehat{R})$ n'a pas d'idéaux premiers associés plongés. Pour tout idéal premier minimal $\tilde{Q}_1$ de $\mathfrak{q}_1 S(I\widehat{R})$, l'anneau $S(I\widehat{R})_{(\tilde{Q}_1)}$ est un anneau de valuation discrète. La valuation $\widehat{\nu}_1$ associée à cet anneau est une extension de $\nu_1$ à $\widehat{R}$. Comme cette valuation $\widehat{\nu}_1$ est unique (cf. Lemme 1.9), $\mathfrak{q}_1 S(I\widehat{R})$ admet un unique idéal premier $Q_1$ associé, par conséquent $\sqrt{\mathfrak{q}_1 S(I\widehat{R})} = Q_1$. De plus, d'après le lemme 1.9, les valuations $\nu_1$ et $\widehat{\nu}_1$ ont le même groupe de valeurs, ce qui implique que :

$$\nu_1\big(\mathfrak{q}_1 S(I\widehat{R}) \cap S(I\widehat{R})_{(Q_1)}\big) = \widehat{\nu}_1\big(\mathfrak{q}_1 S(I\widehat{R}) \cap S(I\widehat{R})_{(Q_1)}\big) = 1.$$

Donc

$$\mathfrak{q}_1 S(I\widehat{R})_{(Q_1)} = Q_1 S(I\widehat{R})_{(Q_1)}.$$

Ceci donne :

$$\mathfrak{q}_1 S(I\widehat{R})_{Q_1} = Q_1 S(I\widehat{R})_{Q_1}.$$

D'après cette dernière égalité et le fait que $Q_1$ est le seul idéal premier associé à $\mathfrak{q}_1 S(I\widehat{R})$ , il résulte que $\mathfrak{q}_1 S(I\widehat{R}) = Q_1$. De façon analogue, nous montrons que $\mathfrak{q}_2 S(I\widehat{R}) = Q_2$. □

***Théorème 2.5.*** — *Soient $(R,\mathfrak{m})$ un anneau local nœthérien analytiquement irréductible et $\nu_1, \nu_2$ deux valuations divisorielles de $R$ centrées en $\mathfrak{m}$. Si $\widehat{\nu}_1$ et $\widehat{\nu}_2$ sont liées en codimension 1, alors il existe un entier naturel $r \in \mathbb{N}^*$ tel que pour tout $x$ non nul appartenant à $R$, on a :*

$$\nu_1(x) \leq r\nu_2(x).$$

*Démonstration.* — Nous pouvons supposer que $R$ est complet et que $\nu_1, \nu_2$ sont liées en codimension 1. Alors par définition, il existe un idéal $\mathfrak{m}$-primaire $I$ de $R$ tel que le centre de $\nu_1$ et le centre de $\nu_2$ dans $\overline{X}_I$ sont liés en codimension 1. Soit $E_1$ (resp. $E_2$) le centre de $\nu_1$ (resp. $\nu_2$) dans $\overline{X}_I$. Il existe donc une suite finie

$$Y_1 = E_1, Y_2, ..., Y_{s-1}, Y_s = E_2$$

de composantes irréductibles de $E_I$ telles que pour tout $1 \leq i \leq s-1$, la codimension de $Y_i \cap Y_{i+1}$ dans $Y_{i+1}$ est égale à 1. Comme $\overline{X}_I$ est normal et l'anneau $R$ est de Nagata, l'anneau $\mathcal{O}_{\overline{X}_I, Y_i}$ est un anneau de valuation divisorielle de $R$ centrée en $\mathfrak{m}$. Donc pour montrer le théorème, nous pouvons supposer que $E_1 \cap E_2$ est de codimension 1 dans $E_2$. Notons $D$ le diviseur de $E_2$ défini par

$$D := E_1 \cap E_2.$$

Soient $x$ un élément arbitraire non nul appartenant à $R$ et $t \in K(R)$ un paramètre régulier de $R_{\nu_2}$ (i.e. $\nu_2(t) = 1$). Soient $l = \nu_1(x)$ et $m = \nu_2(x)$. En regardant l'ensemble $E_2 \subseteq \mathbb{P}^n_k$ comme une variété projective sur le corps résiduel $k$ de $R$, la restriction de $\frac{x}{t^m}$ sur $E_2$ est une fonction rationnelle sur $E_2$ qui n'est pas identiquement nulle. Soient $V(t)$ le sous-schéma de $\overline{X_I}$ défini par $t$ et $B$ le sous-schéma de $E_2$ défini par

$$B := E_2 \cap (\overline{V(t) - E_2}).$$

Notons $\varphi : \mathbb{P}^n_{\overline{k}} \longrightarrow \mathbb{P}^n_k$ le morphisme naturel induit par l'inclusion $k \subset \overline{k}$, où $\overline{k}$ est une clôture algébrique de $k$, et $\overline{E}_2$ (resp. $\overline{D}$, $\overline{B}$) l'image réciproque de $E_2$ (resp. $D$, $B$) dans $\mathbb{P}^n_{\overline{k}}$.

Soit $\xi$ un point régulier de $\overline{E}_2 - (\overline{D} \cup \overline{B})$. Prenons $L$ un sous-espace linéaire de $\mathbb{P}^n_{\overline{k}}$ de dimension $n - \dim\ E_2 + 1$ qui contient $\xi$ et qui intersecte $\overline{E}_2$ transversalement en $\xi$. L'intersection $C := L \cap \bar{E}_2$ est une courbe, réduite et irréductible dans un voisinage de $\xi$. Soit $Y$ l'unique composante irréductible de $C$ qui passe par $\xi$ ; par construction, $Y$ est réduite. Soient $i : Y \hookrightarrow \mathbb{P}^n_{\overline{k}}$ l'inclusion naturelle, $\psi : \tilde{Y} \longrightarrow Y$ une résolution des singularités de $Y$, et $\tilde{E}_2$ (resp. $\tilde{B}$ et $\tilde{D}$) la pré-image naturelle dans $\tilde{Y}$ de $E_2$ (resp. $B$ et $D$).

Nous avons le diagramme suivant :

$$\begin{tikzcd}
 & B \arrow[d, hook] & & \overline{B} \arrow[d, hook] & & & \tilde{B} \arrow[d, hook] \\
\overline{X}_I & E_2 \arrow[l, hook'] \arrow[r, hook] & \mathbb{P}^n_k & \mathbb{P}^n_{\overline{k}} \arrow[l, "\varphi"'] & \overline{E}_2 \arrow[l, hook', "i_1"'] & Y \arrow[l, hook', "i_2"'] & \tilde{Y} \arrow[l, "\psi"'] \\
 & D \arrow[u, hook] & & \overline{D} \arrow[u, hook] & & & \tilde{D} \arrow[u, hook]
\end{tikzcd}$$

Notons $\phi = \varphi \circ i_1 \circ i_2 \circ \psi$, où $i_1, i_2$ sont les injections naturelles. Soient maintenant $B = B_1 \cup ... \cup B_s$ la décomposition de $B$ en composantes irréductibles. Pour tout

$i \in \{1,2,...,s\}$, soient $V_{i1},...,V_{ik_i}$ toutes les composantes irréductibles de $V \setminus E_2$ qui contiennent $B_i$ pour tous $i \in \{1,...,s\}$ et $j \in \{1,...,k_i\}$, on a :

$$B_i \subset V_{ij} \cap E_2 \text{ et } V_{ij} \neq E_2.$$

En tout point $\xi \in E_2 - B$ la fonction $\frac{x}{t^m}$ est régulière en codimension 1, elle est donc régulière car $\bar{X}_I$ est normal. Soit $\nu_{ij}$ la valuation associée à $V_{ij}$. Pour tous $1 \leq i \leq s$ et $1 \leq j \leq k_i$, soit

$$t_{ij} \in \bigcap_{i=1}^{s} \mathcal{O}_{\overline{X}_I, B_i}$$

tel que :

$$\nu_{ij}(t_{ij}) \geq \nu_{ij}(t). \tag{4}$$

Posons pour tout $1 \leq i \leq s$ :

$$t_i = \prod_{j=1}^{k_i} t_{ij}.$$

Il est clair que la fonction $t_i$ est régulière sur Spec $\mathcal{O}_{\overline{X}_I, B_i}$, et comme $Y \not\subset \overline{B}$, nous pouvons supposer que cette fonction ne s'annule pas identiquement sur $Y$. En particulier le pullback de $t_i$ définit une fonction rationnelle sur $\tilde{Y}$ non identiquement nulle, qui est régulière sur $\phi^{-1}(B_i)$. L'inégalité (4) entraîne :

$$\frac{x}{t^m} \prod_{i=1}^{s} t_i^m \in \bigcap_{i=1}^{s} \mathcal{O}_{\overline{X}_I, B_i}. \tag{5}$$

Soit $H_i$ le diviseur de $\tilde{Y}$ défini par $H_i = (\phi^*(t_i))_0 \cap |\phi^{-1}(B_i)|$ (cf. Notation 2.2). La fonction $\phi^*(\frac{x}{t^m})$ n'a pas de pôles dans $\tilde{Y} - \tilde{B}$ et la fonction $\phi^*(\frac{x}{t^m} \prod_{i=1}^{s} t_i^m)$ n'a pas de pôles dans $\tilde{B}$ (cf. (5)). Alors

$$\deg\,(\phi^*(\frac{x}{t^m}))_\infty \leq \deg\,\big((\phi^*(\prod_{i=1}^{s} t_i^m) \cap |\tilde{B}|)_0\big).$$

Par suite,

$$\deg\,(\phi^*(\frac{x}{t^m}))_\infty \leq m \sum_{i=1}^{s} \deg H_i. \tag{6}$$

Soient $u \in \mathcal{O}_{\overline{X}_I}(-E_1) \setminus \mathcal{O}_{\overline{X}_I}(-E_2)$ et $p = \nu_1(u)$. Nous distinguons deux cas :
<u>Cas 1</u> : $l - m\nu_1(t) \leq 0$.
<u>Cas 2</u> : $l - m\nu_1(t) > 0$.
Dans le cas 1, on a bien $\nu_1(x) \leq \nu_1(t)\nu_2(x)$, ce qui implique le résultat recherché. Il reste maintenant à démontrer le théorème dans le cas 2. Supposons $l - m\nu_1(t) > 0$. Alors par la définition de $u$ et $p$, la fonction $x^{2p}/(t^{2mp}u^{l-m\nu_1(t)})$ s'annule sur $D$. Ainsi sa restriction à $L \cap \overline{E_2}$ s'annule sur $L \cap \overline{D}$. Donc

$$\deg\,(\phi^*(\frac{x}{t^m}))_0 \geq \frac{l - m\nu_1(t)}{2p} \deg \phi^*((u)_0 \cap \tilde{D}) \geq \frac{l - m\nu_1(t)}{2p}. \tag{7}$$

Sachant que tout diviseur d'une fonction rationnelle sur $\tilde{Y}$ est de degré zéro, alors d'après les deux inégalités (6) et (7), nous obtenons

$$\frac{l - m\nu_1(t)}{2p} \leq m \sum_{i=1}^{s} \deg H_i.$$

Par suite,

$$\nu_1(x) \leq \big(\nu_1(t) + 2p \sum_{i=1}^{s} \deg H_i\big)\nu_2(x),$$

(on rappelle que $l = \nu_1(x)$ et $m = \nu_2(x)$).

Ce qui achève la démonstration. □

**2.2. Le cas (II).** — Pour démontrer le théorème d'Izumi dans ce cas (cf. Théorème 2.10), nous allons appliquer le théorème principal de Zariski qui s'énonce comme suit :

***Théorème 2.6*** **(Théorème principal de Zariski, cf.[8])**. — *Soit $f : Y \longrightarrow X$ un schéma projectif sur un schéma localement nœthérien, tel que $f^\sharp : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ est un isomorphisme. Alors pour tout point $x \in X$, le fibre $Y_x$ est connexe.*

***Lemme 2.7*** **(cf. [8], Corollaire 4.4.3)**. — *Soient $X$ un schéma normal et localement nœthérien, et $f : Y \longrightarrow X$ un morphisme birationnel propre. Alors $f^\sharp : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ est un isomorphisme.*

Il est important de noter que le schéma $Y$ dans le lemme 2.7 n'est pas supposé normal. Ci-dessous, on va appliquer ce lemme à l'éclatement normalisé du spectre d'un anneau normal le long de son idéal maximal.

***Corollaire 2.8***. — *Soient $(R, \mathfrak{m})$ un anneau de Nagata intègre et normal de dimension 2, $I$ un idéal $\mathfrak{m}$-primaire de $R$, et $E_1$, $E_2$ deux composantes irréductibles de $E_I$. Alors $E_1$ et $E_2$ sont liées en codimension 1.*

*Démonstration.* — Comme l'application $\pi_I : \overline{X}_I \longrightarrow X = \operatorname{Spec} R$ est un morphisme birationnel propre et que l'anneau $R$ est nœthérien et normal, le morphisme naturel ${\pi_I}^\sharp : \mathcal{O}_X \longrightarrow (\pi_I)_*\mathcal{O}_{\overline{X}_I}$ est un isomorphisme (cf. Lemme 2.7). Donc d'après le théorème principal de Zariski (cf. Théorème 2.6), le diviseur exceptionnel $E_I$ est connexe. Ceci implique que pour toutes composantes irréductibles $E_1$ et $E_2$ de $E_I$, il existe une suite finie

$$Y_1 = E_1, Y_2, \cdots, Y_{s-1}, Y_s = E_2$$

de composantes irréductibles de $E_I$, telle que pour tout $1 \leq i \leq s-1$, nous avons :

$$Y_i \cap Y_{i+1} \neq \varnothing.$$

Comme la dimension de $R$ est égale à 2, cela revient à dire que la codimension de $Y_i \cap Y_{i+1}$ dans $Y_{i+1}$ est égale à 1. Donc les deux composantes irréductibles $E_1$ et $E_2$ sont liées en codimension 1. □

***Corollaire 2.9***. — *Soient $(R, \mathfrak{m})$ un anneau de Nagata normal et de dimension inférieure ou égale à 2, et $\nu_1$ et $\nu_2$ deux valuations divisorielles de $R$ centrées en $\mathfrak{m}$. Alors $\nu_1$ et $\nu_2$ sont liées en codimension* 1.

*Démonstration.* — Si la dimension de $R$ est égale à 1, les deux valuations $\nu_1$ et $\nu_2$ sont égales, donc par définition, elles sont liées en codimension 1. Supposons que la dimension de $R$ est égale à 2. Puisque les valuations $\nu_1$ et $\nu_2$ sont divisorielles, il existe deux idéaux $\mathfrak{m}$-primaires $I_1, I_2$ de $R$ tels que le centre de $\nu_1$ (resp. $\nu_2$) dans $\overline{X}_{I_1}$ (resp. $\overline{X}_{I_2}$) est de codimension 1. Prenons $I = I_1 I_2$, il est clair que cet idéal est aussi $\mathfrak{m}$-primaire et que les centres de $\nu_1$ et de $\nu_2$ dans $\overline{X}_I$ sont liés en codimension 1 (cf. Corollaire 2.8). □

***Théorème 2.10***. — *Soit $(R, \mathfrak{m})$ un anneau local nœthérien analytiquement irréductible de dimension inférieure ou égale à deux. Alors pour tout couple $\nu_1, \nu_2$ de valuations divisorielles centrées en $\mathfrak{m}$, il existe un entier naturel $r \in \mathbb{N}^*$ tel que pour tout $x$ non nul appartenant à $R$, on a :*

$$\nu_1(x) \leq r\nu_2(x).$$

*Démonstration.* — Si la dimension de $R$ est égale à 1, alors :

$$R_{\nu_1} = R_{\nu_2} = \overline{R}.$$

Cela signifie que $\nu_1(x) = \nu_2(x)$ pour tout $x$ non nul appartenant à $R$. Supposons maintenant que $R$ est de dimension égale à 2. D'après les lemmes précédents (cf. Lemme 1.9, lemme 1.10), il nous suffit de montrer le théorème en supposant que $R$ est intègre, normal et complet. Puisque tout anneau nœthérien complet est un anneau de Nagata, il en résulte que $\nu_1$ et $\nu_2$ sont liées en codimension 1 (cf. Corollaire 2.9). Par conséquent, le théorème devient donc une conséquence immédiate du cas (I) (cf. Théorème 2.5). □

## 3. Commentaires

Dans cette section, nous proposons une conjecture concernant le lien entre les valuations divisorielles et la notion de connexité en codimension 1. (cf. **Cas (I)**), et nous donnons certains cas dans lesquelles la conjecture est vraie.

**Conjecture 1**. Soient $X = \operatorname{Spec}(R, \mathfrak{m})$ un schéma affine, intègre, normal et complet, et soient $\nu_1, \nu_2$ deux valuations divisorielles de $R$ centrées en $\mathfrak{m}$. Alors il existe un idéal $\mathfrak{m}$-primaire $I$ de $R$, tel que les centres de $\nu_1$ et $\nu_2$ dans $\overline{X}_I$ sont liés en codimension 1.

*Remarque 3.1.* — *Le corollaire 2.9, nous assure que cette conjecture est vraie si $R$ est de dimension inférieure ou égale à deux.*

***Définition 3.2***. — *Soit $k$ un entier naturel. Un espace topologique nœthérien $X$ est dit connexe en codimension $k$ si pour tout sous-ensemble fermé $Y$ de $X$ de codimension strictement supérieure à $k$, l'ensemble $X - Y$ est connexe.*

Cette définition a été introduite par R. Hartshorne [**3**], en 1962. Il a montré que $X$ est connexe en codimension $k$ si, et seulement si, pour tout couple $(Y, Z)$ de composantes irréductibles de $X$, il existe une suite finie $Y = E_1, Y_2, ..., Y_{s-1}, Y_s = Z$ de composantes irréductibles de $X$ telle que pour tout $1 \leq i \leq s-1$, la codimension de $Y_i \cap Y_{i+1}$ dans $X$ est inférieure ou égale à $k$.

***Définition 3.3***. — *Soit $X$ un schéma intègre. Nous disons que $X$ admet une résolution des singularités plongée, si pour tout sous-schéma fermé $E$ de $X$, il existe une résolution des singularités $\pi : Y \longrightarrow X$ telle que le diviseur $\pi^{-1}(E)$ est à croisements normaux. De plus pour tout point régulier $x$ de $X$ tel que le diviseur $E$ est à croisements normaux, le morphisme $\pi$ est un isomorphisme au-dessus de $x$.*

***Lemme 3.4***. — *Soient $(R, \mathfrak{m})$ un anneau local, $I$ un idéal $\mathfrak{m}$-primaire de $R$, $\pi : X_I \longrightarrow$* Spec $R$ *l'éclatement de* Spec $R$ *le long de $I$, et soit $\mathcal{H}$ un faisceau d'idéaux de $\mathcal{O}_{X_I}$ tel que $V(\mathcal{H}) \subset V(I\mathcal{O}_{X_I})$. Alors le morphisme composé de $\pi$ et de l'éclatement de $X_I$ le long de $\mathcal{H}$ est un éclatement de* Spec $R$ *le long d'un idéal $\mathfrak{m}$-primaire.*

*Démonstration*. — Le faisceau $\mathcal{H}(n) = \mathcal{H} \otimes_{\mathcal{O}_{X_I}} \mathcal{O}_{X_I}(n) = I^n\mathcal{H}$ est engendré par ses sections globales quand $n$ est suffisamment grand (cf. [**8**], Théorème 1.27, Page 167). En considérant $X_I$ comme un sous-schéma fermé de $\mathbb{P}^d_R$, soient $f_1, f_2, \cdots, f_r$ les sections globales de $\mathcal{O}_{\mathbb{P}^d_R}$ qui engendrent le faisceau $\mathcal{H}(n)$, donc elles sont des éléments de $R$, car $\mathcal{H}(n) \subset \mathcal{O}_{\mathbb{P}^d_R}$ et $\mathcal{O}_{\mathbb{P}^d_R}(\mathbb{P}^d_R) = R$. En prenant $J = (f_1, f_2, \cdots, f_r)I$, nous pouvons montrer que la décomposition de $\pi$ et de l'éclatement de $X_I$ le long de $\mathcal{H}$ est exactement l'éclatement de Spec $R$ le long de $J$. Reste maintenant à montrer que $J$ est un idéal $\mathfrak{m}$-primaire, et pour cela, il suffit de montrer que :

$$V(J\mathcal{O}_{X_I}) \subset V(\mathfrak{m}\mathcal{O}_{X_I}). \tag{8}$$

On a :

$$\begin{aligned} V(J\mathcal{O}_{X_I}) &= V(I^n\mathcal{H}) \\ &\subset V(I^n\mathcal{O}_{X_I}) \\ &= V(\mathfrak{m}\mathcal{O}_{X_I}). \end{aligned}$$

Donc on a bien l'inclusion (8). □

***Proposition 3.5***. — *Soient $(R, \mathfrak{m})$ un anneau nœthérien local à singularité isolée et $I$ un idéal $\mathfrak{m}$-primaire de $R$ tel que le schéma $\overline{X}_I$ admet une résolution des singularités plongée. Alors tout couple $(\nu_1, \nu_2)$ de valuations divisorielles de Rees associées à $I$, $\nu_1$ et $\nu_2$ sont liées en codimension 1.*

*Démonstration*. — Soient $E_1$ (resp. $E_2$) les centres de $\nu_1$ (resp. $\nu_2$) dans $\overline{X}_I$. Comme $\overline{X}_I$ admet une résolution des singularités plongée, il existe une résolution des singularités $\pi : Y \longrightarrow \overline{X}_I$ telle que le diviseur exceptionnel $\pi^{-1}(E_I)$ est à croisements normaux. Donc pour tout couple $(D_1, D_2)$ de composantes irréductibles de $\pi^{-1}(E_I)$ telles que $D_1 \cap D_2 \neq \varnothing$, on a :

$$\operatorname{codim}(D_1 \cap D_2, D_1) = 1$$

et

$$\operatorname{codim}(D_1 \cap D_2, D_2) = 1.$$

Le fait que $\pi^{-1}(E_I)$ est de plus connexe (cf. Théorème 2.6) implique qu'il est connexe en codimension 1. Par suite, les transformés stricts de $E_1$ et $E_2$ sont liées en codimension 1. Il nous reste maintenant de démontrer que $Y$ est obtenu d'un éclatement de Spec $R$ le long d'un idéal $\mathfrak{m}$-primaire. Puisque $\mathfrak{m}$ est le seul point singulier de Spec $R$, $\overline{X}_I \backslash V(I\mathcal{O}_{\overline{X}_I}) \simeq \operatorname{Spec} R - \{\mathfrak{m}\}$ est régulier. Nous pouvons donc choisir $\pi$ un éclatement de $\overline{X}_I$ le long d'un faisceau d'idéaux $\mathcal{H}$ de $\mathcal{O}_{\overline{X}_I}$ tel que $V(\mathcal{H}) \subseteq V(I\overline{X}_I)$. En utilisant le lemme 3.4, il est clair que $Y$ est un éclatement de Spec $R$ le long d'un idéal $\mathfrak{m}$-primaire. □

*Remarque 3.6.* — *En admettant que : « tout schéma $X$ excellent fini sur* Spec $\mathbb{Q}$ *admet une résolution de singularités plongée »* [(2)]. *Alors suivant les hypothèses de la première question, si l'anneau $R$ est un anneau excellent sur $\mathbb{Q}$, nous avons toujours une réponse positive a cette question.*

***Corollaire 3.7.*** — *Soit $(R, \mathfrak{m})$ un anneau de Nagata à singularité isolée tel que tout schéma $Y$ de type fini sur $R$ admet une résolution des singularités plongée. Alors tout couple $(\nu_1, \nu_2)$ de valuations divisorielles centrées dans $R$ en $\mathfrak{m}$, $\nu_1$ et $\nu_2$ sont liées en codimension 1.*

## Références


[1] S. Abhyankar – On the valuations centered in a local domain, *Amer. J. Math.* **78** (1956), p. 321–348.

[2] A. Grothendieck – Éléments de géométrie algébrique. IV : Étude locale des schemas et des morphismes de schémas. (Seconde partie).

[3] R. Hartshorne – Complete intersections and connectedness, *Amer. J. Math.* **84** (1962), p. 497–508.

[4] M. Herrmann, S. Ikeda & U. Orbanz – *Equimultiplicity and blowing up*, Springer, 1988.

[5] R. Hübl & I. Swanson – Discrete valuations centered on local domains, *J. Pure Appl. Algebra* **161** (2001), p. 145–166.

[6] C. Huneke & I. Swanson – *Integral closure of ideals, rings, and modules*, London Mathematical Society Lecture Note Series, vol. 336, Cambridge University Press, 2006.

[7] S. Izumi – Linear complementary inequalities for orders in analytic geometry (Łojasiewicz inequalities and strong approximation theorems), *Sūrikaisekikenkyūsho Kōkyūroku* **926** (1995), p. 30–40.


[(2)] Il est possible que cette affirmation soit toujours vraie ; pour plus de détails, nous renvoyons le lecteur à EGA IV 7.9 (cf. [**2**]).

[8] Q. Liu – *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Math., vol. 6, Oxford University Press, 2002.
[9] M. Nagata – *Local rings*, Interscience Tracts in Pure and Applied Mathematics, No. 13, John Wiley & Sons New York-London, 1962.
[10] D. Rees – Izumi's theorem, in *Commutative algebra (Berkeley, CA, 1987)*, Math. Sci. Res. Inst. Publ., vol. 15, Springer, 1989, p. 407–416.
[11] M. Spivakovsky – Valuations in function fields of surfaces, *Amer. J. Math.* **112** (1990), p. 107–156.
[12] B. Teissier – Valuations, deformations and toric geometry, prépublication.
[13] M. Vaquié – Valuations, in *Resolution of singularities (Obergurgl, 1997)*, Progr. Math., vol. 181, Birkhäuser, 2000, p. 539–590.

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# GEVREY CLASS OF THE INFINITESIMAL GENERATOR OF A DIFFEOMORPHISM

*by*

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***Abstract.*** — Let $F$ be an analytic diffeomorphism in $(\mathbb{C}^m, 0)$ tangent to the identity of order $n$. The infinitesimal generator of $F$ is the formal vector field $X$ such that $\operatorname{Exp} X = F$. In this paper we provide an elementary proof of the fact that $X$ belongs to the Gevrey class of order $1/n$.

***Résumé*** **(La classe de Gevrey du générateur infinitésimal d'un difféomorphisme)**
Soit $F$ un difféomorphisme analytique de $\mathbb{C}^m$ tangent à l'identité à l'ordre $n$. Le générateur infinitésimal de $F$ est le champ de vecteurs formel $X$ tel que $\operatorname{Exp} X = F$. Dans cet article nous donnons une preuve élémentaire du fait que $X$ appartient à la classe Gevrey d'ordre $1/n$.

## 1. Introduction

For each couple of integers $m \geq 1$ and $n \geq 2$, let us denote $\hat{\mathfrak{X}}_n(\mathbb{C}^m, 0)$ the module of formal vector fields of order $\geq n$ in $(\mathbb{C}^m, 0)$ and $\widehat{\operatorname{Diff}}_n(\mathbb{C}^m, 0)$ the group of formal diffeomorphisms in $(\mathbb{C}^m, 0)$ tangent to the identity of order $\geq n$, i.e, $F \in \widehat{\operatorname{Diff}}_n(\mathbb{C}^m, 0)$ if and only if $\nu(F) := \min\{\nu_0(x_i \circ F - x_i) | i = 1, \ldots, m\} - 1 \geq n$. For any $X \in \widehat{\mathfrak{X}}_n(\mathbb{C}^m, 0)$, the exponential operator of $X$ is the application $\exp X : \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ defined by the formula

$$\exp X(g) = \sum_{j=0}^{\infty} \frac{1}{j!} X^j(g)$$

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where $X^0(g) = g$ and $X^{j+1}(g) = X(X^j(g))$. It is a classical result (for instance, see [5]) that the application

$$\begin{array}{rcl}\mathrm{Exp} : \widehat{\mathfrak{X}}_n(\mathbb{C}^m, 0) & \to & \widehat{\mathrm{Diff}}_{n-1}(\mathbb{C}^m, 0) \\ X & \mapsto & (\exp X(x_1), \dots, \exp X(x_m))\end{array}$$

is a bijection. The formal vector field $X$ such that $F = \mathrm{Exp}(X)$ is called the *infinitesimal generator* of $F$.

Let $x = (x_1, \dots, x_m)$ and for any $s \in \mathbb{R}$ let $\mathbb{C}[[x]]_s$ denote the subset of elements of $\mathbb{C}[[x]]$ that satisfy the $s$-Gevrey condition, i.e.

$$f(x) = \sum_{k=0}^{\infty} f_k(x) \in \mathbb{C}[[x]]_s \quad \text{if and only if} \quad \sum_{k=0}^{\infty} \frac{f_k(x)}{k!^s} \in \mathbb{C}\{x\},$$

where $f_k(x)$ is homogeneous of degree $k$. Let us observe that 0-Gevrey condition means analyticity, and $\mathbb{C}\{x\} \subset \mathbb{C}[[x]]_s \subset \mathbb{C}[[x]]_t$ if $0 < s < t$. Let $\mathfrak{X}_n(\mathbb{C}^m, 0)_s \subseteq \hat{\mathfrak{X}}_n(\mathbb{C}^m, 0)$ be the set of $s$-Gevrey vector fields $X = \sum_{k=1}^{m} X(x_k) \frac{\partial}{\partial x_k}$ with $X(x_k) \in \mathbb{C}[[x]]_s$ and $\mathrm{Diff}_n(\mathbb{C}^m, 0)_s = \widehat{\mathrm{Diff}}_n(\mathbb{C}^m, 0) \cap (\mathbb{C}[[x]]_s)^m$ the set of $s$-Gevrey diffeomorphisms tangent to the identity of order $\geq n$.

We will prove the following result

***Theorem 1.1.*** — *For any $s \geq \frac{1}{n-1}$ the application* Exp *gives a bijection*

$$\mathrm{Exp} : \mathfrak{X}_n(\mathbb{C}^m, 0)_s \to \mathrm{Diff}_{n-1}(\mathbb{C}^m, 0)_s.$$

*In particular, the infinitesimal generator of any tangent to the identity analytic diffeomorphism $F$ is $\frac{1}{\nu(F)}$-Gevrey.*

In general, $X$ may be divergent for a convergent $F$, for instance, Szekeres [7] and Baker [2] proved that every entire holomorphic function tangent to the identity of order $k$ in dimension 1 has a non-convergent infinitesimal generator, Ahern and Rosay [1] proved that this kind of diffeomorphisms cannot be the time-1 map of a $C^{3k+3}$-vector field, and finally J. Rey [6] showed that they cannot be the time-1 map of a $C^{k+1}$-vector field, which is the best possible bound. Thus, the map $\mathrm{Exp} : \mathfrak{X}_n(\mathbb{C}^m, 0)_0 \to \mathrm{Diff}_{n-1}(\mathbb{C}^m, 0)_0$ is not surjective for any couple of positive integers $m, n$. In addition, in dimension 1, using resummation arguments, it is proved that if an analytic diffeomorphism $f(x) = x + a_{k+1}x^{k+1} + \cdots$ with $a_{k+1} \neq 0$ has a divergent infinitesimal generator $X$, then $X$ is $k$-summable, so $X$ is Gevrey of order $\frac{1}{k}$, but not smaller (see [4], [3] and [5]). Therefore, the condition $s \geq \frac{1}{n-1}$ is necessary.

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## 2. Technical estimations

In this paper, we take the following notations:

- $h_k(x)$ will denote the homogeneous polynomial $\sum\limits_{\substack{\alpha\in\mathbb{N}^m\\|\alpha|=k}} x^\alpha$.
- $H_{s,n}(x)$ the series $\sum\limits_{q=n}^{\infty}(q+m-n)!^s h_q(x)$.
- $\frac{\partial}{\partial x}$ the differential operator $\sum\limits_{k=1}^{m}\frac{\partial}{\partial x_k}$.

For formal series $f(x)=\sum_\alpha f_\alpha x^\alpha$ and $g(x)=\sum_\alpha g_\alpha x^\alpha$, we say that $f\preceq g$ if $|f_\alpha|\le|g_\alpha|$ for any $\alpha\in\mathbb{N}^m$. We get in this way a partial order in $\mathbb{C}[[x]]$, and also in $\hat{\mathfrak{X}}_n(\mathbb{C}^m,0)$ and $\widehat{\mathrm{Diff}}_n(\mathbb{C}^m,0)$, working on the component function. From the definition of Gevrey condition, it can be seen that $X\in\mathfrak{X}_n(\mathbb{C}^m,0)_s$ if and only if there exists $a\in\mathbb{R}^+$ such that, for all $q\ge n$,

$$\mathrm{Coef}_q(X)\preceq(q+m-n)!^s a^q h_q(x)\frac{\partial}{\partial x},$$

where $\mathrm{Coef}_q(X)$ denotes the homogeneous term of $X$ of degree $q$. Thus $X\in\mathfrak{X}_n(\mathbb{C}^m,0)_s$ if and only if there exists $a\in\mathbb{R}^+$ such that $X\preceq H_{s,n}(ax)\frac{\partial}{\partial x}$.
We need the following technical lemmas:

***Lemma 2.1***. — *For every $k,l\in\mathbb{N}^*$*

$$h_k\frac{\partial}{\partial x}h_l\preceq(l+m-1)\min\left\{\binom{k+m-1}{m-1},\binom{l+m-2}{m-1}\right\}h_{k+l-1}.$$

*Proof*. — Observe that

$$\begin{aligned}\frac{\partial}{\partial x}h_l &= \sum_{k=1}^{m}\frac{\partial}{\partial x_k}\sum_{\substack{\alpha\in\mathbb{N}^m\\|\alpha|=l}}x^\alpha=\sum_{k=1}^{m}\sum_{\substack{\alpha\in\mathbb{N}^m\\|\alpha|=l}}\alpha_k\frac{x^\alpha}{x_k}\\ &= \sum_{\substack{\beta\in\mathbb{N}^m\\|\beta|=l-1}}\sum_{k=1}^{m}(\beta_k+1)x^\beta=(l+m-1)h_{l-1}\end{aligned}$$

Now, the coefficient of $x^\alpha$ in the product $h_k(x)h_{l-1}(x)$ is less than or equal to the minimum between the number of monomials of $h_k$ and the number of monomials of $h_{l-1}$, and the number of monomials of $h_j$ is $\binom{j+m-1}{m-1}$, that corresponds to the number of ordered partitions of $j$ in $m$ parts; therefore,

$$h_k\frac{\partial}{\partial x}h_l=(l+m-1)h_kh_{l-1}\preceq(l+m-1)\binom{\min\{k,l-1\}+m-1}{m-1}h_{k+l-1}.\qquad\square$$

***Lemma 2.2***. — *Let $\Theta(y)=\sum\limits_{j=n}^{\infty}\binom{m-1+j}{m-1}y^{j-n}$. Then $\Theta(y)$ converges for any $|y|<1$.*

*Proof.* — Since $\sum_{j=n}^{\infty} y^{m-1+j} = \frac{y^{m+n-1}}{1-y}$ converges for any $|y|<1$ then

$$\Theta(y) = \frac{1}{(m-1)!}\frac{1}{y^n}\frac{d^{m-1}}{dy^{m-1}}\left(\frac{y^{m+n-1}}{1-y}\right)$$

converges for any $|y|<1$. □

***Lemma 2.3***. — *For any $s>0$ and integers $m\geq 1$ and $n\geq 2$, the sequence $\{b_q\}_{q\geq 2n-1}$ given by*

$$b_q = \sum_{j=n}^{\lfloor\frac{q+1}{2}\rfloor}\left(\frac{(j+m-n)!(q-j+1+m-n)!}{m!(q+m-n)!}(q-j+m)^{n-1}\right)^s\binom{j+m-1}{m-1},$$

*is bounded.*

*Proof.* — Observe that

$$\begin{aligned}\frac{(q-j+m)^{n-1}}{(q-j+2+m-n)\cdots(q-j+m)} &< \left(\frac{q-j+m}{q-j+2+m-n}\right)^{n-1}\\ &\leq \left(\frac{\frac{q-1}{2}+m}{\frac{q-1}{2}+2+m-n}\right)^{n-1} \leq \left(\frac{m+n-1}{m+1}\right)^{n-1}\end{aligned}$$

then

$$b_n \leq \left(\frac{m+n-1}{m+1}\right)^{s(n-1)}\sum_{j=n}^{\lfloor\frac{q+1}{2}\rfloor}\left(\frac{(j+m-n)!(q-j+m)!}{m!(q+m-n)!}\right)^s\binom{j+m-1}{m-1}.$$

In addition

$$\frac{m+1}{q+m-j+1} < \frac{m+2}{q+m-j+2} < \cdots < \frac{j+m-n}{q+m-n}$$

and

$$\frac{j+m-n}{q+m-n} \leq \frac{\frac{q+1}{2}+m-n}{q+m-n} \leq \max\left\{\frac{1}{2}, \frac{m}{m+n-1}\right\} = C_{m,n} < 1;$$

from lemma 2.2,

$$b_q < \left(\frac{m+n-1}{m+1}\right)^{s(n-1)}\Theta(C_{m,n}^s).$$

□

***Proposition 2.4***. — *Let $s\geq\frac{1}{n-1}$, $X\in\widehat{\mathfrak{X}}_n(\mathbb{C}^m,0)$ and $a\in\mathbb{R}^+$ such that*

$$\mathrm{Coef}_q(X) \preceq (q+m-n)!^s a^q h_q(x)\frac{\partial}{\partial x}$$

*for all $n\leq q\leq N$, and let us denote $A = 2m!^s\left(\frac{m+n-1}{m+1}\right)^{s(n-1)}\Theta(C_{m,n}^s)$. For every $q,k$ with $n\leq q\leq N+k-1$,*

$$\mathrm{Coef}_q(X^k) \preceq (aA)^{k-1}(q+m-n)!^s a^q h_q(x)\frac{\partial}{\partial x},$$

*Proof.* — Since $X^k = \sum_{i=1}^{m} X^k(x_i)\frac{\partial}{\partial x_i}$, it is enough to prove the affirmation for $X^k(x_i)$, where $i \in \{1, 2, \dots, m\}$. Let us write $X = \sum_{j=n}^{\infty} X_j$, where $X_j$ is homogeneous of degree $j$. We will proceed by induction on $k$; if $k = 1$, by hypothesis

$$X_q(x_i) \preceq (q+m-n)!^s a^q h_q(x) \quad \text{for every } n \leq q \leq N.$$

Suppose that the lemma is true for every $k \leq p$, then, since the order of $X^j$ is greater than or equal to $(n-1)j+1$, $\mathrm{Coef}_q(X^{p+1}) = 0$ for $n \leq q \leq (n-1)p+n-1$ and for $(n-1)p+n \leq q \leq N+p$ we have

$$\begin{aligned}
&\mathrm{Coef}_q(X^{p+1}(x_i)) = \mathrm{Coef}_q(X(X^p(x_i))) = \mathrm{Coef}_q\Big(\sum_{j=n}^{\infty} X_j(X^p(x_i))\Big)\\
&= \sum_{j=n}^{q-(n-1)p} X_j \,\mathrm{Coef}_{q+1-j}(X^p(x_i))\\
&\preceq \sum_{j=n}^{q-(n-1)p} (j+m-n)!^s a^j h_j(x)\tfrac{\partial}{\partial x}\Big((aA)^{p-1}(q-j+1+m-n)!^s a^{q+1-j} h_{q+1-j}(x)\Big)\\
&\preceq \sum_{j=n}^{q-n+1} (j+m-n)!^s (q-j+1+m-n)!^s (q-j+m)\tbinom{\min\{j,q-j\}+m-1}{m-1} A^{p-1}a^{q+p}h_q,\\
&\preceq 2\sum_{j=n}^{\lfloor\frac{q+1}{2}\rfloor} ((j+m-n)!(q-j+1+m-n)!(q+m-j)^{n-1})^s \tbinom{j+m-1}{m-1} A^{p-1}a^{q+p}h_q.
\end{aligned}$$

Now, observe that

$$b_q m!^s (q+m-n)!^s = \sum_{j=n}^{\lfloor\frac{q+1}{2}\rfloor} ((j+m-n)!(q-j+1+m-n)!(q-j+m)^{n-1})^s \binom{j+m-1}{m-1},$$

where $\{b_q\}$ is the sequence defined in lemma 2.3; it follows that

$$\begin{aligned}
\mathrm{Coef}_q(X^{p+1}(x_i)) &\preceq 2b_q m!^s (q+m-n)!^s A^{p-1}a^{q+p}h_q\\
&\preceq (q+m-n)!^s (aA)^p a^q h_q
\end{aligned}$$

□

## 3. Proof of theorem 1.1.

To prove that the application $\mathrm{Exp} : \mathfrak{X}_n(\mathbb{C}^m,0)_s \to \mathrm{Diff}_{n-1}(\mathbb{C}^m,0)_s$ is well defined for $s \geq \frac{1}{n-1}$, let $X \in \mathfrak{X}_n(\mathbb{C}^m,0)_s$, $a > 0$ be such that $X \preceq H_{s,n}(ax)$, and $A$ as in proposition 2.4.

Then by proposition 2.4 we have

$$\begin{aligned}
\mathrm{Coef}_q(\exp X(x_j)) &= \sum_{k=1}^{\infty} \frac{1}{k!}\mathrm{Coef}_q(X^k(x_j))\\
&\preceq \sum_{k=1}^{\infty} \frac{1}{k!}(aA)^{k-1}(q+m-n)!^s a^q h_q(x)
\end{aligned}$$

therefore $\operatorname{Exp}(X) \preceq \sum_{k=1}^{\infty} \frac{(aA)^{k-1}}{k!} H_{s,n}(ax)$. Now, to prove that Exp is surjective, let us consider a diffeomorphism $F(x) = (x_1 + f_1(x), \dots, x_m + f_m(x)) \in \mathrm{Diff}_{n-1}(\mathbb{C}^m, 0)_s$ where $f_j(x) = \sum_{q=n}^{\infty} f_{j,q}(x) \in \mathbb{C}[[x]]_s$ and $f_{j,q}(x)$ is an homogeneous polynomial of degree $q$. Then there exists $a > 0$ such that $f_{j,q}(x) \preceq (q+m-n)!^s a^q h_q(x)$. Observe that, making a linear change of coordinates, we can suppose that $a$ is small enough such that $\sum_{k=2}^{\infty} \frac{1}{k!}(2aA)^{k-1} \leq \frac{1}{2}$. If $X = \sum_{q=n}^{\infty} X_q$ is the infinitesimal generator of $F(x)$, we will show by induction on $q$ that

$$X_q \preceq (q+m-n)!^s (2a)^q h_q(x) \frac{\partial}{\partial x}.$$

For $q = n$

$$X_n(x_j) = f_{j,n}(x) \preceq m!^s a^n h_n(x) \preceq m!^s (2a)^n h_n(x).$$

Suppose that the claim is true for any integer between $n$ and $q$, it follows that

$$f_{j,q+1}(x) = \mathrm{Coef}_{q+1}\Big(\sum_{k=1}^{\infty} \frac{1}{k!} X^k(x_j)\Big) = X_{q+1}(x_j) + \sum_{k=2}^{q} \frac{1}{k!} \mathrm{Coef}_{q+1}\big(X^k(x_j)\big),$$

using proposition 2.4

$$\begin{aligned} X_{q+1}(x_j) &\preceq (q+1+m-n)!^s a^{q+1} h_{q+1}(x) \\ &\qquad + \sum_{k=2}^{\infty} \frac{1}{k!}(2aA)^{k-1}(q+1+m-n)!^s (2a)^{q+1} h_{q+1}(x) \\ &\preceq \left(\frac{1}{2^{q+1}} + \sum_{k=2}^{\infty} \frac{1}{k!}(2aA)^{k-1}\right)(q+1+m-n)!^s (2a)^{q+1} h_{q+1}(x) \\ &\preceq (q+1+m-n)!^s (2a)^{q+1} h_{q+1}(x), \end{aligned}$$

in other words $X \preceq H_{s,n}(2a)\frac{\partial}{\partial x}$. □

## 4. Case $0 \leq s < \frac{1}{n-1}$

As we indicated in the introduction, in this case, there exists $F \in \mathrm{Diff}_{n-1}(\mathbb{C}^m, 0)_s$ such that its infinitesimal generator is not $s$-Gevrey , but the reciprocal is true, i.e.

***Proposition 4.1.*** — *Let $0 \leq s \leq \frac{1}{n-1}$, and $X \in \mathfrak{X}_n(\mathbb{C}^m, 0)_s$. Then* $\operatorname{Exp}(X) \in \mathrm{Diff}_{n-1}(\mathbb{C}^m, 0)_s$.

Observe that the case $s = 0$ is a classical result about the existence of solution of an analytic differential equation. To prove this proposition in the case $s > 0$ we need the following lemma

***Lemma 4.2.*** — *Let $t, r \in \mathbb{R}$ such that $0 < t < 1$ and $1 - t < r < 1$. Let $\{a_k\}$ be the sequence defined by $a_1 = a > 0$ and for $k \geq 1$, $a_{k+1} = \sup_{q \in \mathbb{N}^*} \sqrt[q+k]{\frac{(q+m)^{1-t}}{(k+1)^r}} a_k$. Then $\{a_k\}$ is increasing and convergent.*

*Proof.* — Taking $q \gg k$ it is clear that $\sqrt[q+k]{\frac{(q+m)^{1-t}}{(k+1)^r}} > 1$, and then $a_{k+1} > a_k$. Now, we know by Bernoulli inequality that

$$\sqrt[q+k]{\frac{q+m}{(k+1)^{\frac{r}{1-t}}}} < 1 + \frac{1}{q+k}\left(\frac{q+m}{(k+1)^{\frac{r}{1-t}}} - 1\right) < 1 + \frac{1}{(k+1)^{\frac{r}{1-t}}}$$

for $k > m$, so

$$a_{k+1} < \left(1 + \frac{1}{(k+1)^{\frac{r}{1-t}}}\right)^{1-t} a_k < \left(\prod_{j=m+1}^{k+1}\left(1 + \frac{1}{j^{\frac{r}{1-t}}}\right)\right)^{1-t} a_m,$$

and since $\frac{r}{1-t} > 1$ it follows that $\{a_k\}$ is bounded, thereby it is convergent. □

*Proof of proposition 4.1.* — If $s \in (0, \frac{1}{n-1})$, $X \in \mathfrak{X}_n(\mathbb{C}^m, 0)_s$ and $a \in \mathbb{R}^+$ such that $X \preceq H_{s,n}(ax)\frac{\partial}{\partial x}$ then for $t = s(n-1)$, $r \in (1-t, 1)$ and $\{a_k\}$ as in lemma 4.2, using the arguments of proposition 2.4 and the fact that $k^r a_k^{k+q-1} \geq (q+m)^{1-t} a_{k-1}^{k+q-1}$ for every $q \geq 2$, we can prove that

$$X^k \preceq (a_k A)^{k-1} k!^r H_{s,n}(a_k x)\frac{\partial}{\partial x},$$

where $A = 2m!^s\left(\frac{m+n-1}{m+1}\right)^{s(n-1)}\Theta(C^s_{m,n})$. Let $c = \lim_{k\to\infty} a_k$. Therefore we have

$$\mathrm{Coef}_q(\exp(X)(x_j)) = \sum_{k=1}^{\infty}\frac{1}{k!}\mathrm{Coef}_q(X^k(x_j)) \preceq \sum_{k=1}^{\infty}\frac{(cA)^{k-1}}{k!^{1-r}}(m+q-n)!^s c^q h_q(x)$$

Thus $\mathrm{Exp}(X) \preceq \sum_{k=1}^{\infty}\frac{(cA)^{k-1}}{k!^{1-r}} H_{s,n}(cx)\frac{\partial}{\partial x}$. □

## References


[1] P. Ahern & J.-P. Rosay – "Entire functions, in the classification of differentiable germs tangent to the identity, in one or two variables", *Trans. Amer. Math. Soc.* **347** (1995), p. 543–572.

[2] I. N. Baker – "Non-embeddable functions with a fixpoint of multiplier 1", *Math. Z.* **99** (1967), p. 377–384.

[3] J. Écalle – "Théorie itérative: introduction à la théorie des invariants holomorphes", *J. Math. Pures Appl. (9)* **54** (1975), p. 183–258.

[4] B. Malgrange – "Sommation des séries divergentes", *Exposition. Math.* **13** (1995), p. 163–222.

[5] J. Martinet & J.-P. Ramis – "Classification analytique des équations différentielles non linéaires résonnantes du premier ordre", *Ann. Sci. École Norm. Sup. (4)* **16** (1983), p. 571–621 (1984).

[6] J. Rey – "Difféomorphismes résonnants de $(\mathbb{C}, 0)$", Ph.D. Thesis, Univ. Paul Sabatier, Toulouse, 1996.

[7] G. Szekeres – "Regular iteration of real and complex functions", *Acta Math.* **100** (1958), p. 203–258.

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# ON THE LIMIT OF FAMILIES OF ALGEBRAIC SUBVARIETIES WITH UNBOUNDED VOLUME

*by*

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*Dedicated to José Manuel Aroca on the occasion of his $60^{th}$ birthday*

***Abstract.*** — We prove that the limit of a sequence of generic semi-algebraic sets given by a finite number of formulas always exists and is a semi-algebraic set that can be explicitly given as a Boolean expression involving the primitives of the additive forms of the formulas.

***Résumé*** **(Sur la limite des familles de subvarietés algébriques sans volume borné)**
On prouve que la limite d'une suite d'ensembles semi-algébriques génériques donnés par un nombre fini de formules existe toujours et est un ensemble semi-algébrique, ensemble qui peut être donné explicitement comme une expression booléenne impliquant les primitives des formes additives de formules.

## 1. Introduction

Bishop [**2**] proved that the limit set of a sequence of complex purely $k$-dimensional algebraic subvarieties whose real volumes are uniformly bounded is again a purely $k$-dimensional algebraic subvariety. On the other hand, there are many reasons why one should be interested in analyzing the limit sets of algebraic subvarieties with unbounded volume. One reason is the existence of families of algebraic curves of increasing degree that are integrals of families of polynomials differential equations on the plane with bounded degree, a badly understood phenomenon related to the sixteenth Hilbert Problem (see [**4**], for instance). Another reason is that, despite the existence of topologically complicated limit sets of curves with unbounded volume (see [**6**], for instance), much can be said about the limit sets of algebraic subvarieties which lie in a family of subvarieties with finite complexity (see [**5**] for a definition of this concept).

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In this paper we consider the limit sets of one-parameter families of algebraic subvarieties, indexed by a natural number $n$, defined by a finite number of equations, each equation defined by a *formula*. Informally, a formula is a polynomial expression in which $n$ appears in exponents only. Associated to each formula there is a *height*, which is the maximum number of nested $n$-th powers that appear in it. Here is the formal definition:

***Definition 1***. — *Formulas* and their *heights* are defined recursively as follows:

1. Every polynomial $F \in \mathbb{C}[X_1, \ldots, X_m]$ is a formula of height zero.
2. If $F_1$ and $F_2$ are formulas, then $F_1 + F_2$ and $F_1 F_2$ are formulas of height $\max(h_1, h_2)$, where $h_i$ is the height of $F_i$.
3. If $F$ is a formula of height $h$, then $F^n$ is a formula of height $h + 1$.

A formula of height zero is also called a *primitive* formula; it is simply a complex polynomial.

At times we shall need to evaluate a formula $F$ at a point $z \in \mathbb{C}^m$ and for a particular $n$. In this case, we shall write $F(z; n)$.

The height is a measure of the complexity of the formula: it measures how the degree increases with $n$. A formula of height $h$ has degree proportional to $n^h$. More precisely, the degree of a formula of height $h$ is $\Theta(n^h)$, using Landau's asymptotic notation as modified by Knuth [**3**].

An example of a formula of height 3 is

$$xy^2(((x^2 - y + 1)^n - 1)^n + x)^n + (xy)^n + (y^n - 1)^2 + 1.$$

Note that the degree is $2n^3 + 3 = \Theta(n^3)$.

The same polynomial family may be given by different formulas. For instance,

$$(x^n + y)^2 = (x^n)^2 + 2x^n y + y^2.$$

For our purposes, a convenient way to handle this issue is to express formulas in additive form. A formula is in *additive form* when it can be expressed as

$$Q_1 A_1^n + Q_2 A_2^n + \cdots + Q_l A_l^n - P,$$

where $Q_1, \ldots, Q_l$, and $P$ are primitive formulas and $A_1, \ldots, A_l$ are arbitrary subformulas (necessarily of smaller height than the original formula). As we shall see later, additive forms help us to use induction on the height when working with formulas.

***Lemma 1***. — *Every formula can be written in additive form.*

*Proof*. — The proof is by induction on the number of operations required to obtain the formula according to Definition 1. If $F$ is a primitive formula, then we can take $l = 0$ and $P = -F$. If $F = A^n$, then $F$ is already in additive form because we can take $l = 1$, $Q_1 = 1$, $A_1 = A$, and $P = 0$. If $F = A + B$, then by induction $A$ and $B$ can be expressed in additive form, whose combination gives an additive form for $F$. If $F = AB$, then again by induction $A$ and $B$ can be expressed in additive form. By performing the multiplication $AB$ on their additive forms, we get an additive form for $F$. $\square$

As an example of the procedure described in the proof above, $(x^n+y)^2$ can be written in additive form as $(x^2)^n+(2y)x^n+y^2$. Note that the expression $(x^n)^2+2x^ny+y^2$ given earlier for $(x^n+y)^2$ is *not* in additive form.

***Definition 2***. — The *limit* (as $n\to\infty$) of a sequence $(\Omega_n)$ of subsets of $\mathbb{C}^m$ is the set $\lim\Omega_n$ of points that are limits of sequences of points lying in a subsequence of $(\Omega_n)$. More precisely,

$$\lim\Omega_n=\{\,z\in\mathbb{C}^m:\exists(z_n),z_n\to z,\exists(k_n),k_n\to\infty,\ z_n\in\Omega_{k_n}\text{ for sufficiently large }n\,\}.$$

Thus, according to this definition, the family of real curves $x^{2n}+y^{2n}=1$ converges to the border of the unit square given by $x^2\le 1$, $y^2\le 1$. Actually, the definition of limit applies to the curves $x^n+y^n=1$ (note that we now allow both even and odd exponents). These curves converge to the union of the border of the unit square with the two rays given by $x=-y$, $x^2\ge 1$ (the curves actually alternate between these two limit sets, but our definition of limit covers this). Considered as a family of complex curves, $x^n+y^n=1$ has as limit set the subset of $\mathbb{C}^2$ given by $\partial([|x|<1]\cap[|y|<1])\cup[|x|=|y|>1]$, as it is easy to verify.

We shall consider two situations: limit sets in $\mathbb{R}^m$ of families of algebraic subvarieties given by a finite number of formulas and limit sets in $\mathbb{C}^m$ of families of complex algebraic subvarieties.

In the real case it turns out that it is easier to describe the limits of semi-algebraic subsets, instead of algebraic subsets. Semi-algebraic subsets will also play a role in the complex case. An *algebraic subvariety* is the set of points that satisfy a polynomial equation $f(z)=0$. For simplicity, we shall write this set as $[f=0]$. A *semi-algebraic set* in $\mathbb{R}^m$ is one given by a Boolean expression on subsets of the form $[f>0]$ or $[f\ge 0]$. We shall also deal with *basic closed semi-algebraic subsets*, which are the solutions of a system of polynomial inequalities: $[f_1\ge 0,\dots,f_k\ge 0]$, and with *basic open semi-algebraic subsets*, which are given by strict inequalities: $[f_1>0,\dots,f_k>0]$.

One main difficulty in the theory of semi-algebraic sets is that the closure of a basic open semi-algebraic set is not necessarily the corresponding basic closed semi-algebraic set obtained by relaxing the strict inequalities. That is, the closure of $[f_1>0,\dots,f_k>0]$ is not always $[f_1\ge 0,\dots,f_k\ge 0]$. Nor is the interior of a closed semi-algebraic set equal to the corresponding basic open semi-algebraic set obtained by restricting the inequalities. That is, the interior of $[f_1\ge 0,\dots,f_k\ge 0]$ is not always $[f_1>0,\dots,f_k>0]$. However, these statements are true *generically*, in two senses: (i) they are true if we perturb the polynomials slightly, and (ii) relaxing or restricting the inequalities only adds or removes lower dimensional components. We say that a basic closed semi-algebraic set is *generic* when it coincides with the closure of the corresponding basic open semi-algebraic set obtained by restricting the inequalities. In other words, a basic closed semi-algebraic set given by $[f_1\ge 0,\dots,f_k\ge 0]$ is generic when $[f_1\ge 0,\dots,f_k\ge 0]=\text{closure}[f_1>0,\dots,f_k>0]$. A *generic algebraic set* is, by definition, the boundary of a generic semi-algebraic subset. For a full discussion of real algebraic and semi-algebraic sets, see the book by Benedetti and Risler [**1**].

Our main result is the following:

***Theorem 1***. — *The limit of a sequence of generic semi-algebraic sets given by a finite number of formulas always exists and is a semi-algebraic set that can be explicitly given as a Boolean expression involving the primitives of the additive forms of the formulas.*

The corresponding algebraic version is also valid:

***Theorem 2***. — *The limit of a sequence of generic algebraic sets given by a finite number of formulas always exists and is an algebraic set that can be explicitly given as a Boolean expression involving the primitives of the additive forms of the formulas.*

In the complex case, the limit set of a family of algebraic sets given by a finite number of formulas has also an underlying semi-algebraic structure in the sense that it projects, by means of a rational map, onto a proper real semi-algebraic subset defined by expressions involving the absolute values of the primitives of the formulas. More precisely, we have the following result:

***Theorem 3***. — *The limit of a sequence of generic algebraic subsets given by a finite number of formulas with complex coefficients always exists; it is a subset with a complex structure obtained by means of a rational pull-back on semi-algebraic subsets defined explicitly in terms of Boolean expressions involving the absolute values of the primitives of the formulas.*

As an example of the situation in the complex case, we consider the following generalization of the $x^n+y^n=1$ example given earlier. Let $A_1$, $A_2$, and $P$ be complex polynomials. Then

$$\lim[A_1^n + A_2^n = P] = \partial([|A_1| < 1] \cap [|A_2| < 1] \cap [P \neq 0]) \cup [|A_1| = |A_2| > 1]).$$

This limit can be also understood as the pull-back by the polynomial map

$$(A_1, A_2)\colon \mathbb{C}^2 \to \mathbb{C}^2$$

of the Reinhardt preimage of the semi-algebraic subset of $\mathbb{R}^2$ given by the second member of the equation above, where the axes of $\mathbb{R}^2$ are taken as $|A_1|$ and $|A_2|$.

## 2. The real case

We start with the simplest cases and continue to more complicated cases until we reach general formulas in additive form. To simplify the exposition, we assume that *all semi-algebraic sets are generic* and we consider only formulas in which *all $n$-th powers are even.*

The simplest non-trivial formula of height 1 is $A^{2n} - P$, where $A$ and $P$ are real polynomials. We want to describe the limit of the algebraic subsets $[A^{2n} = P]$. As mentioned before, it is simpler to describe the limit of the semi-algebraic sets $\Omega_n = [A^{2n} \leq P]$. The strategy in the following lemma and in all subsequent lemmas in this section will be to give a candidate $\Omega$ for $\Omega_\infty = \lim \Omega_n$ and to show that $\Omega_\infty \subseteq \Omega$ and $\Omega \subseteq \Omega_\infty$, thus establishing that $\Omega_\infty = \Omega$.

All lemmas in this section say that the limit of a formula can be expressed as a Boolean combination of formulas of smaller height. Thus, they will provide a basis for proving Theorem 1 by induction on the height of the formula.

***Lemma 2***. — *Let $A$ and $P$ be polynomials. Then* $\lim[A^{2n} \leq P] = [A^2 \leq 1, P \geq 0]$.

*Proof.* — Let $\Omega_n = [A^{2n} \leq P]$, $\Omega_\infty = \lim \Omega_n$, and $\Omega = [A^2 \leq 1, P \geq 0]$. We shall show that $\Omega_\infty = \Omega$.

Take $z \in \Omega_\infty$. Then, by definition, there are sequences $z_n \to z$ and $k_n \to \infty$ with $z_n \in \Omega_{k_n}$, that is, $A(z_n)^{2k_n} \leq P(z_n)$. Since $A(z_n)^{2k_n} \geq 0$, we get $P(z_n) \geq 0$ and hence $P(z) = \lim P(z_n) \geq 0$. Moreover, the sequence $(P(z_n))$ is bounded and so $P(z_n) \leq L$ for some $L > 0$. This implies that $A(z_n)^2 \leq P(z_n)^{1/k_n} \leq L^{1/k_n}$. Therefore, $A(z)^2 = \lim A(z_n)^2 \leq \lim L^{1/k_n} = 1$. Hence, $z \in \Omega$.

Reciprocally, take $z \in \Omega$. Since $\Omega$ is generic, we have that $z = \lim z_n$, with $z_n \in [A^2 < 1, P > 0]$. From $A(z_n)^2 < 1$ we get that $A(z_n)^{2k} \to 0$ as $k \to \infty$. Since $P(z_n) > 0$, there is a $k_n$ such that $A(z_n)^{2k_n} < P(z_n)$, that is, $z_n \in \Omega_{k_n}$. By increasing $k_n$ beyond $n$ if necessary to get $k_n \to \infty$, we conclude that $z \in \Omega_\infty$. □

The genericity hypothesis is essential to the lemma as stated. Although the proof shows that $\Omega_\infty \subseteq \Omega$ even when $\Omega$ is not generic, the reverse inclusion is not always true when $\Omega$ is not generic. The following example gives a taste of how things are more complicated in the general case. Let $A = y(y-1)^2 + 1$ and $P = x^2(x-1)$. Note that $[P \geq 0]$ is not the closure of $[P > 0]$ because $[P \geq 0]$ contains the line $[x = 0]$, which is not in the closure of $[P > 0]$ since $P$ is negative around $x = 0$. Similarly, $[A^2 \leq 1]$ is not the closure of $[A^2 < 1]$ because of the line $[y = 1]$. As a consequence, $[A = 1, P \geq 0]$ is only partially contained in $\lim[A^n \leq P]$; only $[A = 1, P \geq 1]$ is part of the limit set. This example is typical of what happens in general: $\lim[A^{2n} \leq P]$ is equal to $[A^2 \leq 1, P \geq 0]$, except that $P \geq 1$ when $A = 1^+$, and $A = 1$ when $P = 0^-$.

The next lemma generalizes Lemma 2 and the $x^n + y^n = 1$ example given in §1:

***Lemma 3***. — *Let $A_1, \ldots, A_k$ and $P$ be polynomials. Then*

$$\lim[A_1^{2n} + \cdots + A_k^{2n} \leq P] = \bigcap_{i=1}^{k} \lim[A_i^{2n} \leq P] = [A_1^2 \leq 1, \ldots, A_k^2 \leq 1, P \geq 0].$$

*Proof.* — Take $z \in \lim[A_1^{2n} + \cdots + A_k^{2n} \leq P]$. Then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A_i(z_n)^{2k_n} \leq A_1(z_n)^{2k_n} + \cdots + A_k(z_n)^{2k_n} \leq P(z_n)$. So

$$\lim[A_1^{2n} + \cdots + A_k^{2n} \leq P] \subseteq \bigcap_{i=1}^{k} \lim[A_i^{2n} \leq P] = \bigcap_{i=1}^{k} [A_i^2 \leq 1, P \geq 0],$$

by Lemma 2. Hence $z \in [A_1^2 \leq 1, \ldots, A_k^2 \leq 1, P \geq 0]$.

Reciprocally, take $z \in [A_1^2 \leq 1, \dots, A_k^2 \leq 1, P \geq 0]$. Then $z = \lim z_n$ with $A_i(z_n) < 1$ and $P(z_n) > 0$. Since $A_i(z_n)^{2r} \to 0$ as $r \to \infty$, we have $A_i(z_n)^{2k_n} < P(z_n)/k$ for sufficiently large $k_n$. As in Lemma 2, we can ensure that $k_n \to \infty$ and conclude that $z \in \lim[A_1^{2n} + \cdots + A_k^{2n} \leq P]$. □

The next lemma generalizes Lemma 2 for formulas of larger height.

***Lemma 4***. — *Let $A$ be a formula and $P$ be a primitive formula. Then*

$$\lim[A^{2n} \leq P] = \lim[A^2 \leq 1] \cap [P \geq 0].$$

*Proof.* — Take $z \in \lim[A^{2n} \leq P]$. Then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A(z_n; k_n)^{2k_n} \leq P(z_n)$. Clearly, $P(z) = \lim P(z_n) \geq 0$. As in Lemma 2, the sequence $(P(z_n))$ is bounded and we have $A(z_n; k_n)^{2k_n} \leq L$ for some $L > 0$. This implies $A(z_n; k_n)^2 \leq L^{1/k_n}$. Clearly, $\lim[A^2 \leq L^{1/n}] = \lim[A^2 \leq 1]$, because $L^{1/n} \to 1$. Hence, $z \in \lim[A^2 \leq 1] \cap [P \geq 0]$. Reciprocally, take $z \in \lim[A^2 \leq 1] \cap [P \geq 0]$. Assume for the moment that $P(z) > 0$. Since $[P \geq 0]$ is generic, there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A(z_n; k_n)^2 \leq 1$ and $P(z_n) > 0$. Since $(A(z_n; k_n)^{2k_n}/P(z_n))$ is bounded we have $A(z_n; k_n)^{2k_n} \leq LP(z_n)$, for some $L > 0$. Since $\lim[A^{2n} \leq LP] = \lim[A^{2n} \leq P]$, we conclude that $z \in \lim[A^{2n} \leq P]$. Finally, if $P(z) = 0$, then $z = \lim z_n$ with $P(z_n) > 0$, again because $[P \geq 0]$ is generic. Since $\lim[A^{2n} \leq P]$ is closed, we conclude that $z \in \lim[A^{2n} \leq P]$. □

The next lemma handles the reverse inequality.

***Lemma 5***. — *Let $A$ be a formula and $P$ be a primitive formula. Then*

$$\lim[A^{2n} \geq P] = [P \leq 0] \cup (\lim[A^2 \geq 1] \cap [P \geq 0]).$$

*Proof.* — Take $z \in \lim[A^{2n} \geq P]$. Then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A(z_n; k_n)^{2k_n} \geq P(z_n)$. So, either $P(z) \leq 0$, or $P(z) > 0$ and $A(z_n; k_n)^2 \geq P(z_n)^{1/k_n}$. Since $\lim[P^{-1/n} A^2 \geq 1] = \lim[A^2 \geq 1]$, we obtain $z \in [P \leq 0] \cup (\lim[A^2 \geq 1] \cap [P \geq 0])$. Reciprocally, take $z \in [P \leq 0] \cup (\lim[A^2 \geq 1] \cap [P \geq 0])$. Since $[P \geq 0]$ is generic, there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A(z_n; k_n)^2 \geq 1$ and $P(z_n) > 0$. As in Lemma 4, we may assume that $P(z) > 0$, and then the sequence $(A(z_n; k_n)^{2k_n}/P(z_n))$ is bounded below by $L > 0$, i.e., $A(z_n; k_n)^{2k_n} \geq LP(z_n)$. Since $\lim[A^{2n} \geq LP] = \lim[A^{2n} \geq P]$, we conclude that $z \in \lim[A^{2n} \geq P]$. □

***Lemma 6***. — *Let $A$ be a formula and $P$ and $Q$ be primitive formulas. Then*

$$\lim[QA^{2n} \leq P] = ([Q > 0] \cap \lim[A^{2n} \leq P]) \cup ([Q < 0] \cap \lim[A^{2n} \geq -P]) \cup [Q = 0, P \geq 0].$$

*Proof.* — If $Q(z) > 0$ and $z \in \lim[QA^{2n} \leq P]$, then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $Q(z_n)A(z_n; k_n)^{2k_n} \leq P(z_n) \leq L$, with $L > 0$. Since $\lim[Q^{1/n} A^2 \leq L^{1/n}] = \lim[A^2 \leq 1]$ and $P(z) \geq 0$, we obtain that $z \in [Q > 0] \cap \lim[A^{2n} \leq P]$ by Lemma 4. If $Q(z) < 0$, then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A(z_n; k_n)^{2k_n} \geq -P(z_n)/Q(z_n)$. By Lemma 5, either $-P(z)/Q(z) \leq 0$ or $z \in$

$\lim[A^2 \geq 1, -P/Q \geq 0]$, or equivalently $P(z) \leq 0$ or $\lim[A^2 \geq 1, P(z) \geq 0]$, i.e., $z \in \lim[A^{2n} \geq -P]$. □

By setting $Q = A$ in the limit above when $A$ is a primitive formula, we get an expression for $\lim[A^{2n+1} \leq P]$, and from this an expression for $\lim[A^n \leq P]$, which should convince the reader that restricting to even powers simplifies the exposition.

***Lemma 7.*** — *Let $A$ and $B$ be formulas and $P$ be a primitive formula. Then*

$$\lim[A^{2n} \leq P + B^{2n}] = (\lim[B^2 < 1] \cap \lim[A^{2n} \leq P]) \cup (\lim[B^2 \geq 1] \cap \lim[A^2 \leq B^2]).$$

*Proof.* — Take $z \in \lim[A^{2n} \leq P + B^{2n}]$. Then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A(z_n;k_n)^{2k_n} \leq P(z_n) + B(z_n;k_n)^{2k_n}$. If $B(z_n;k_n) < 1$, then $B(z_n;k_n)^{2k_n} \to 0$ and we have $P(z) \geq 0$ and $A(z_n;k_n)^{2k_n} \leq L$, where $L$ is a constant. Thus $A(z_n;k_n)^2 \leq L^{1/k_n}$ and so $z \in [P \geq 0] \cap \lim[A^2 \leq 1] = \lim[A^{2n} \leq P]$ by Lemma 4. So we get $z \in \lim[B^2 < 1] \cap \lim[A^{2n} \leq P]$. If $\lim B(z_n;k_n)^2 \geq 1$, then for $n$ large $P(z_n) \leq KB(z_n;k_n)^{2k_n}$ for some constant $K > 0$. Thus $A(z_n;k_n)^{2k_n} \leq (K + 1)B(z_n;k_n)^{2k_n}$, so $z \in \lim[A^{2n} \leq B^{2n}]$. Reciprocally, if $z \in \lim[B^2 < 1] \cap \lim[A^{2n} \leq P]$, then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A(z_n;k_n)^{2k_n} \leq P(z_n) \leq P(z_n) + B(z_n;k_n)^{2k_n}$. If $z \in \lim[B^2 \geq 1] \cap \lim[A^{2n} \leq B^{2n}]$, then we have two possibilities: either $P(z) > 0$ and then $A(z_n;k_n)^{2k_n} \leq B(z_n;k_n)^{2k_n} \leq B(z_n;k_n)^{2k_n} + P(z_n)$, or $P(z) \leq 0$ and then for $n$ large $B(z_n;k_n)^{2k_n} > -2P(z_n)$. Since $\lim[A^{2n} \leq B^{2n}] = \lim[2A^{2n} \leq B^{2n}]$ we can write $2A(z_n;k_n)^{2k_n} \leq B(z_n;k_n)^{2k_n} = 2B(z_n;k_n)^{2k_n} - B(z_n;k_n)^{2k_n} < 2B(z_n;k_n)^{2k_n} + 2P(z_n)$, i.e., $A(z_n;k_n)^{2k_n} \leq B(z_n;k_n)^{2k_n} + P(z_n)$. □

The next lemma shows that Lemma 7 is an important tool for the general case:

***Lemma 8.*** — *Let $A_1, \ldots, A_k, B_1, \ldots, B_l$ be formulas and $P$ be a primitive formula. Then*

$$\begin{aligned}&\lim[A_1^{2n} + \cdots + A_k^{2n} \leq P + B_1^{2n} + \cdots + B_l^{2n}] = \\ &\quad = \bigcap_{i=1}^{k} \lim[A_i^{2n} \leq P + B_1^{2n} + \cdots + B_l^{2n}] = \bigcap_{i=1}^{k} \bigcup_{j=1}^{l} \lim[A_i^{2n} \leq P + B_j^{2n}].\end{aligned}$$

*Proof.* — Define $P_1 := P + B_1^{2n} + \cdots + B_l^{2n}$. We first show that

$$\lim[A_1^{2n} + \cdots + A_k^{2n} \leq P_1] = \bigcap_{i=1}^{k} \lim[A_i^{2n} \leq P_1].$$

Indeed, $\lim[A_1^{2n} + \cdots + A_k^{2n} \leq P_1] \subseteq \bigcap_{i=1}^{k} \lim[A_i^{2n} \leq P_1]$, because $A_i^{2n} \leq A_1^{2n} + \cdots + A_k^{2n} \leq P_1$. Reciprocally, $\bigcap_{i=1}^{k} \lim[A_i^{2n} \leq P_1] \subseteq \lim[A_1^{2n} + \cdots + A_k^{2n} \leq P_1]$. because $\lim[A_i^{2n} \leq P_1] = \lim[A_i^{2n} \leq (1/k)P_1]$, as in Lemma 3.

We now proceed to show that

$$\lim[A^{2n} \leq P + B_1^{2n} + \cdots + B_l^{2n}] = \bigcup_{j=1}^{l} \lim[A^{2n} \leq P + B_j^{2n}].$$

On the one hand, it is clear that $\lim[A^{2n} \le P+B_j^{2n}] \subseteq \lim[A^{2n} \le P+B_1^{2n}+\cdots+B_l^{2n}]$. On the other hand, if $z \in \lim[A^{2n} \le P + B_1^{2n} + \cdots + B_l^{2n}]$, then we have a relation $B_{i_1}(z)^2 \le \cdots \le B_{i_l}(z)^2$, where $i_1, \ldots, i_l = 1, \ldots, l$. Then $A(z_n; k_n)^{2k_n} \le P(z_n) + B_1(z_n; k_n)^{2k_n} + \cdots + B_l(z_n; k_n)^{2k_n} \le P(z_n) + lB_{i_l}(z_n; k_n)^{2k_n}$, i.e., $z \in \lim[A^{2n} \le P + lB_{i_l}^{2n}] = \lim[A^{2n} \le P + B_{i_l}^{2n}]$. □

The next lemma is the stepping stone to the proof of Theorem 1. Its proof is similar to that of Lemma 6, and we leave it to the reader.

***Lemma 9***. — *Let $A_1, \ldots, A_k, B_1, \ldots, B_l$ be formulas and $P, Q_1, \ldots, Q_k, R_1, \ldots, R_l$ be primitive formulas. Then*

$$\begin{aligned}&\lim[Q_1 A_1^{2n} + \cdots + Q_k A_k^{2n} \le P + R_1 B_1^{2n} + \cdots + R_l B_l^{2n}]\\ &\quad = \lim[A_1^{2n} + \cdots + A_k^{2n} \le P + B_1^{2n} + \cdots + B_l^{2n}]\end{aligned}$$

*provided that $Q_1, \ldots, Q_k, R_1, \ldots, R_l$ are positive.* □

We are now ready to prove Theorem 1.

*Proof of Theorem 1.* — By Lemma 1, every formula can be expressed in additive form and the question is reduced to determining

$$\lim[Q_1 A_1^{2n} + \cdots + Q_k A_k^{2n} \le P + R_1 B_1^{2n} + \cdots + R_l B_l^{2n}],$$

where $Q_1, \ldots, Q_k, R_1, \ldots, R_l$ are positive, since the complete limit can be written as a finite union of expressions as above. By Lemma 9 it is enough to find

$$\lim[A_1^{2n} + \cdots + A_k^{2n} \le P + B_1^{2n} + \cdots + B_l^{2n}],$$

which by Lemma 8 is

$$\bigcap_{i=1}^{k} \bigcup_{j=1}^{l} \lim[A_i^{2n} \le P + B_j^{2n}].$$

Thus, it is enough to find the limit of formulas of the type $[A^{2n} \le P+B^{2n}]$. Proceeding by induction on the height $h$ of $A^n - B^n - P$, we have by Lemma 7 that for $h = 0$ $\lim[A^{2n} \le P + B^{2n}] = [B^2 < 1] \cap [A^2 \le 1] \cap [P \ge 0] \cup [B^2 \ge 1] \cap [A^2 \le B^2]$ and so this limit can be given by a Boolean expression involving the primitives of the formula. Again, by Lemma 7 if $h > 0$, then $\lim[A^{2n} \le P + B^{2n}] = \lim[B^2 \le 1] \cap \lim[[A^2 \le 1] \cap [P \ge 0] \cup \lim[B^2 \ge 1] \cap \lim[A^2 \le B^2]$ is expressed in terms of limits of formulas of height smaller than $h$. Thus by induction hypothesis we conclude that $\lim[A^{2n} \le P + B^{2n}]$ exists, has a semi-algebraic structure, and can be given in terms of a Boolean expression involving the primitives of the formula. □

## 3. The complex case

Consider now a formula of height $h \geq 1$ written in additive form: $Q_1 A_1^n + \cdots + Q_l A_l^n - P$, where $Q_1, \ldots, Q_l$, and $P$ are complex polynomials in $m$ variables and $A_1, \ldots, A_l$ are formulas of height $\leq h-1$. We wish to describe $\lim[Q_1 A_1^n + \cdots + Q_l A_l^n = P]$. As in the real case, we start with the simplest situation, $\lim[A^n = P]$.

***Lemma 10***. — *Let $A$ and $P$ be complex polynomials. Suppose that $P \not\equiv 0$ and that $A$ and $P$ are independent in the sense that $P \nmid dP \wedge dA$ in the region where $|A| < 1$. Then $\lim[A^n = P] = \partial([|A| < 1] \cup [P \neq 0])$.*

*Proof.* — Let $z \in \lim[A^n = P]$. Then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A(z_n)^{k_n} = P(z_n)$. There are two possibilities: $|A(z)| < 1$, then $|A(z_n)| < 1$ for large $n$ and $P(z) = \lim P(z_n) = 0$, i.e., $z \in [|A| < 1, P = 0]$; and $|A(z)| = 1$, then $z \in [|A| = 1] = \overline{[|A| = 1] \cup [P \neq 0]}$. Since $\partial([|A| < 1] \cap [P \neq 0]) = \overline{[|A| < 1]} \cap [P = 0] \cup [|A| = 1] \cap \overline{[P \neq 0]}$, we obtain that $z \in \partial([|A| < 1] \cap [P \neq 0])$.

Conversely, we wish to prove that $\overline{|A| < 1]} \cap [P = 0] \cup [|A| = 1] \cap \overline{[P \neq 0]} \subseteq \lim[A^n = P]$. Since $\lim[A^n = P]$ is closed, it is enough to show that

$$[|A| < 1] \cap [P = 0] \cup [|A| = 1] \cap [P \neq 0] \subseteq \lim[A^n = P].$$

First take $z \in [|A| < 1] \cap [P = 0]$. Then $|A(z)| < 1$ and $P(z) = 0$. In the plane $(A, P)$ the graph $G_{k_n}$ of the map $P = A^{k_n}$ approaches any point $(A, 0)$ with $|A| < 1$ as $k_n \to \infty$. Thus, given $\varepsilon > 0$ there is $N$ such that for each $n \geq N$ the point $(A(z), \xi_n) \in G_{k_n}$ satisfies $|\xi_n| < \varepsilon$. Since $S := A^{-1}(A(z)) \cap P^{-1}(P(z))$ is an algebraic subvariety of codimension $\geq 2$, there is a 1-disc $z \in U_\varepsilon \subseteq A^{-1}(A(z))$, in general position with $S$, such that $P|_{U_\varepsilon}$ is a covering map of $U_\varepsilon$ over a neighborhood of $0 \in \mathbb{C}$. Thus, for $k_n$ large enough, there is $w_n \in U_\varepsilon$ such that $P(w_n) = \xi_n$. Since $A(w_n) = A(z)$ and $(A(z), \xi_n) \in G_{k_n}$ we obtain that $P(w_n) = A(w_n)^{k_n}$. Clearly, $w_n \to z$ and so $z \in \lim[A^n = P]$.

Suppose now that $z \in [|A| = 1] \cap [P \neq 0]$. Then $|A(z)| = 1$ and $P(z) \neq 0$. In the plane $(A, P)$ the horizontal line through the point $(0, P(z))$ intersects the graph $G_{k_n}$ of the map $P = A^{k_n}$ in $k_n$ points over the points $\mathfrak{A}_n = \{P(z)^{1/k_n}\}$ in the $A$-axis. For each of the points $w \in A^{-1}(\mathfrak{A}_n)$ we have $P(z) = P(w) = A(w)^{k_n}$. Since $|P(z)|^{1/k_n} \to 1$, the graph $G_{k_n}$ approaches the set$|A| = 1$, thus the set $\mathfrak{A}_n$ tends to fill the unitary circle. Therefore for each $n$ we can find $w_n \in A^{-1}(\mathfrak{A}_n)$, $w_n \to z$, such that $P(w_n) = A(w_n)^{k_n}$. □

***Lemma 11***. — $\lim(|A|^n = |P|) = \partial([|A| < 1] \cap [|P| \neq 0]) = \lim[A^n = P]$.

*Proof.* — Same as above. □

***Lemma 12***. — *Suppose $P$ and $Q$ are polynomials, not identically zero, and let $A$ be a formula of positive height $h$. Assuming that, for $n$ large, $P \nmid dP \wedge dA$ and $Q \nmid dQ \wedge dA$, we have*

$$\lim[QA = P] = \partial(\lim[|A| < 1] \cap [P \neq 0]) \cup \partial([\lim[|A| > 1] \cap [Q \neq 0]).$$

*Proof.* — Take $z \in \lim[QA^n = P]$. Then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $Q(z_n)A(z_n;k_n)^{k_n} = P(z_n)$. We have the following possibilities:

- $\lim|A(z_n;k_n)| < 1$. Then for, $n$ large, $|A(z_n;k_n)| < 1$ and $P(z) = \lim P(z_n) = 0$. Thus $z \in \lim[|A| < 1] \cap [P = 0]$.
- $\lim|A(z_n;k_n)| = 1$. Then $z \in \lim[|A| = 1] = \lim[|A| = 1] \cap \overline{[P \neq 0]} = \lim[|A| = 1] \cap \overline{[Q \neq 0]}$
- $\lim|A(z_n;k_n)| > 1$. Then for $n$ large $|A(z_n;k_n)| > 1$ and $Q(z) = \lim Q(z_n) = \lim P(z_n)A(z_n;k_n)^{-k_n} = 0$.

Reciprocally, if $z \in \lim[|A| < 1] \cap [P = 0]$, then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $P(z) = 0$ and $\lim|A(z_n;k_n)| < 1$. Assume that $Q(z) \neq 0$. Let $\mathfrak{D} = \{w : |A(w;k_n)| \leq 1, n \geq 1\}$. Then $\mathfrak{D} \neq \varnothing$ and since $(A(\cdot\,,k_n)$ is bounded in $\mathfrak{D}$, it is a normal family. Then there is a subsequence, say $(A(\cdot\,;k_n))$, which converges to a holomorphic function $A$, i.e., $\lim A(w;k_n) = A(w)$. Since $|A(z)| < 1$, we have $\xi_n = A(z)^{l_n} \to 0$ as $l_n \to \infty$. As by hypothesis $S_n := A^{-1}(A(z);k_n) \cap (P/Q)^{-1}((P/Q)(z))$ is a codimension 2 algebraic subvariety for $n$ large, there is a neighborhood $z \in U$ such that $(P/Q)|_{U\cap A^{-1}(A(z);k_n)}$ projects onto a neighborhood of $0 \in \mathbb{C}$. Thus, there is $w_n \in U \cap A^{-1}(A(z);k_n)$, such that $(P/Q)(w_n) = \xi_n$. Therefore $P(w_n) = Q(w_n)\xi_n = Q(w_n)A(z)^{l_n} = Q(w_n)A(w_n;k_n)^{l_n}$. Clearly, $w_n \to z$ and so $z \in \lim[QA^n = P]$. Similarly, if $z \in \lim[|A| = 1]$ we have that $z \in \lim[QA^n]$. On the other hand, if $z \in \lim[|A| > 1] \cap [Q = 0]$ then $Q(z) = 0$ and $\lim|A(z_n;k_n)| > 1$. If $P = 0$ then $z \in \lim[QA^n = P]$. We assume $P(z) \neq 0$. Define the domain $\widetilde{\mathfrak{D}} = \{w : |A(w;k_n)^{-1}| < 1, n > 1\}$. On $\widetilde{\mathfrak{D}}$ the sequence $(A(\cdot\,;k_n)^{-1})$ is normal and converges to a holomorphic function $B$, i.e., $\lim A(w;k_n)^{-1} = B(w)^{-1}$. Thus $|B(z)| > 1$ and $\eta_n = B(z)^{-l_n} \to 0$ as $l_n \to \infty$. By hypothesis $A^{-1}(B(z);k_n) \cap Q^{-1}(0)$ is a codimension 2 algebraic subvariety for $n$ large. Then, since $P(z) \neq 0$, there is a neighborhood $z \in U$ such that $Q/P|^{-1}_{U\cap A(\cdot\,;k_n)}(B(z))$ projects over a neighborhood of $0 \in \mathbb{C}$. Thus there is $w_n \in A^{-1}(B(z);k_n) \cap U$ such that $(Q/P)(w_n) = \eta_n = B(z)^{-l_n} = A(w_n;k_n)^{-l_n}$ or $Q(w_n)A(w_n;k_n)^{l_n} = P(w_n)$. Clearly, $w_n \to z$ and so $z \in \lim[QA^n = P]$. $\square$

**3.1. Example.** — Let us compute $\lim[(A^n + P)^n = Q]$.

$$
\begin{aligned}
\lim[(A^n + P)^n = Q] &= \partial(\lim[|A^n + P| < 1] \cap [Q \neq 0]) \\
&= \overline{\lim[|A^n + P| < 1]} \cap [Q = 0]) \cup (\lim[|A^n| = 1] \cap \overline{[Q \neq 0]}) \\
\lim[|A^n + P| < 1] &= \lim[|A|^n < 1 + |P|] \\
&= [|A|^2 < 1] \\
\lim[|A|^n = 1] &= [|A| = 1] \\
\text{Thus,}\qquad & \\
\lim[(A^n + P)^n = Q] &= [|A|^2 \leq 1] \cap [Q = 0] \cup \overline{[|A| = 1] \cap [Q \neq 0]} \\
&= [|A|^2 \leq 1] \cap [Q = 0] \cup [|A| = 1].
\end{aligned}
$$

Thus this limit is the pull back by a rational map of a Reinhardt variety over a semi-algebraic subset of $\mathbb{R}^2$. This example reflects pretty well the general picture described in Theorem 3.

Suppose that $Q_1, \dots, Q_l, P \in \mathbb{C}[x_1, \dots, x_m]$. In what follows we will write

$$\begin{aligned} \mathcal{Q}\mathcal{A}^n &= [Q_1A_1^n + \cdots + Q_lA_l^n = P] \\ \mathcal{Q}\mathcal{A}^n(\hat{i}) &= [Q_1A_1^n + \cdots + \widehat{Q_iA_i^n} + \cdots + Q_lA_l^n = P] \end{aligned}$$

***Lemma 13***. — *Suppose that $Q_1, \dots, Q_l, P \in \mathbb{C}[x_1, \dots, x_m]$ and assume that $Z_{Q_i}$, $Z_{Q_j}$, $Z_{Q_k}$ intersect in general position if $i \neq j \neq k \neq i$. Let $A_1, \dots, A_l$ be formulas of positive height $h$. Then*

1. $\lim[Q_1A_1^n + \cdots + Q_lA_l^n = P] = \bigcup_{i,j=1}^l \lim[|A_i| = |A_j| > 1] \cup (Z_{Q_i} \cap Z_{Q_j}) \bigcup_{i=1}^l \lim[|A_i| < 1] \cap \lim[Q_1A_1^n + \cdots + \widehat{Q_iA_i^n} + \cdots + Q_lA_l^n = P]$
2. $\lim[Q_1A_1^n + \cdots + Q_lA_l^n = P] = \bigcup_{i,j=1}^l \lim[|A_i| = |A_j| > 1] \cup (Z_{Q_i} \cap Z_{Q_j}) \cup \partial(\lim[|A_1| < 1] \cap \cdots \cap \lim[|A_l| < 1] \cap [P \neq 0])$

*Proof.* — Write $\mathcal{R}_1 := \bigcup_{i=1}^l \lim[|A_i| < 1$, $\mathcal{R}_2 := \bigcap_{i=1}^l \lim[|A_i| \geq 1]$. Then Lemma 13 follows from the next two lemmas. □

***Lemma 14***. — *We have:*

$$\begin{aligned} &\lim[Q_1A_1^n + \cdots + Q_lA_l^n = P] \cap \mathcal{R}_1 \\ &= \bigcup_{i=1}^l \lim[|A_i| < 1] \cap \lim[Q_1A_1^n + \cdots + \widehat{Q_iA_i^n} + \cdots + Q_lA_l^n = P] \\ &= \partial(\lim[|A_1| < 1] \cap \cdots \cap \lim[|A_l| < 1] \cap [P \neq 0]) \end{aligned}$$

*Proof.* — Let $z \in \lim[Q_1A_1^n + \cdots + Q_lA_l^n = P] \cap \mathcal{R}_1$. Then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $Q_1(z_n)A_1(z_n;k_n)^{k_n} + \cdots + Q_l(z_n)A_l(z_n;k_n)^{k_n} = P(z_n)$. Suppose first that $\lim |A_1(z_n;k_n)| < 1$. Then $\varepsilon_n(z_n) = Q_1(z_n)A_1(z_n;k_n)^{k_n} \to 0$ as $n \to \infty$ and if we define

$$f_n(z_n) := P(z_n) - Q_2(z_n)A_2(z_n;k_n)^{k_n} - \cdots - Q_l(z_n)A_l(z_n;k_n)^{k_n},$$

then we have $f_n(z_n) = \varepsilon_n(z_n)$. Let $Z_n = f_n^{-1}(0)$ and $Z = \lim Z_n$. We claim that $z \in Z$. Indeed, if $z \notin Z$ then there are neighborhoods $z \in V$ and $Z \subseteq W$ with $V \cap W = \varnothing$. For $n$ large $z_n \in V$ and $f_n^{-1}(\varepsilon_n) \subseteq W$, a contradiction since $f_n(z_n) = \varepsilon_n$ and $f_n(z_n) = \varepsilon_n$ and $z_n \to z$. Then there is $w_n \in Z_n = f_n^{-1}(0)$, $w_n \to z$, i.e., $f_n(w_n) = 0, w_n \to z$. This means that

$$Q_2(w_n)A_{2k_n}(w_n)^{k_n} + \cdots + Q_l(w_n)A_{lk_n}(w_n)^{k_n} = P(w_n)$$

and so $z \in \lim[|A_1| < 1] \cap \lim[Q_2A_2^n + \cdots + Q_lA_l^n = P]$. Similarly, if $z \in \mathcal{R}_1$, then

$$z \in \bigcup_{i=1}^l \lim[|A_i| < 1] \cap \lim[Q_1A_1^n + \cdots + \widehat{Q_iA_i^n} + \cdots + Q_lA_l^n = P].$$

Reciprocally, suppose there are sequences $z_n \to z$ and $w_n \to z$ such that $f_n(w_n) = 0$ and $\varepsilon_n(z_n) \to 0$. Then $Z_n = f_n^{-1}(0) \to Z$. Since $w_n \in Z_n$ and $w_n \to z$, then $z \in Z$. Therefore for any $\delta$ small positive there is $y_n \in \varepsilon_n^{-1}(\delta) \cap f_n^{-1}(\delta) \neq \varnothing$, i.e., $\varepsilon_n(y_n) = f_n(y_n)$. We will show now that for the points in the region $\mathscr{R}_1$ we have

$$\begin{aligned}
&\partial([|A_1| < 1] \cap \cdots \cap [|A_l| < 1] \cap [P \neq 0]) = \\
&= \bigcup_{i=1}^{l} [|A_i| < 1] \cap \lim[Q_1 A_1^n + \cdots + \widehat{Q_i A_i^n} + \cdots + Q_l A_l^n = P] \qquad (**) \\
&= \lim[Q_1 A_1^n + \cdots + Q_l A_l^n = P].
\end{aligned}$$

We proceed by induction on $l$. For $l = 2$ we have, by Lemma 12,

$$\begin{aligned}
&\partial[[|A_1| < 1] \cap [|A_2| < 1 \cap [P \neq 0]] = \\
&= [|A_1| < 1] \cap \partial([|A_2| < 1] \cap [P \neq 0]) \cup [|A_2| < 1] \cap \partial([|A_1| < 1] \cap [P \neq 0]) \\
&= [|A_1| < 1] \cap \lim[Q_2 A_2^n = P] \cup [|A_2| < 1] \cap \lim[Q_1 A_1^n = P] \\
&= \lim[Q_1 A_1^n + Q_2 A_2^n = P].
\end{aligned}$$

For $l > 2$, we have

$$\begin{aligned}
&\partial([|A_1| < 1)] \cap \cdots \cap (|A_l| < 1) \cap [P \neq 0]) = \\
&= \bigcup_{i=1}^{l} (|A_i| < 1) \cap \partial([|A_1| < 1] \cap \cdots \cap \widehat{(|A_i|)} \cap \cdots \cap (|A_l| < 1) \cap [P \neq 0]) \\
&= \bigcup_{i=1}^{l} (|A_i| < 1) \cap \lim[Q_1 A_1^n + \cdots + \widehat{Q_i A_i^n} + \cdots + Q_l A_l^n = P] \\
&= \lim[Q_1 A_1^n + \cdots + Q_l A_l^n = P] = \lim[Q_1 A_1^n + \cdots + Q_l A_l^n = P],
\end{aligned}$$

where the last two equalities are derived by induction hypothesis on $(**)$. □

***Lemma 15***. — *We have:*

$$\lim[Q_1 A_1^n + \cdots + Q_l A_l^n = P] \cap \mathscr{R}_2 = \bigcup_{i,j=1}^{l} \lim[|A_i| = |A_j| > 1] \cup (Z_{Q_i} \cap Z_{Q_j})$$

*Proof*. — Suppose now that

$$z \in \lim[|A_1| > 1] \cap \cdots \cap \lim[|A_q| > 1] \cup \lim[|A_{q+1}| = 1] \cap \cdots \cap \lim[|A_l| = 1].$$

Then $q \neq 1$ and

$$Q_1(z_n)(A_1)(z_n; k_n)^{k_n} + \cdots + Q_q(z_n)(A_q)(z_n; k_n)^{k_n} = R(z_n),$$

where

$$R(z_n) := P(z_n) - Q_{q+1}(z_n)(A_{q+1})(z_n; k_n)^{k_n} - \cdots - Q_l(z_n)(A_l)(z_n; k_n)^{k_n}$$

is locally bounded at $z$. For any $i, j = 1, \dots, q$, $i \neq j$, we can write the next inequality where, for simplicity, we wrote $i = 1$ and $j = 2$:

$$|Q_1(z_n)||A_1(z_n; k_n)|^{k_n} - |Q_2(z_n)||A_2(z_n; k_n)|^{k_n} \leq \sum_{t=3}^{q} |Q_t(z_n)||A_t(z_n; k_n)|^{k_n} + |R(z_n)|.$$

Thus, dividing both members of this expression by $\prod_{t=3}^{q} |A_t(z_n)|^{k_n}$, we obtain a left member locally bounded at $z$. Then there is a bounded sequence $\{\lambda_n\}$ such that

$$|Q_1(z_n)||A_1(z_n)|^{k_n} / \prod_{t=3}^{q} |A_t(z_n)|^{k_n} = \lambda_n |Q_2(z_n)||A_2(z_n)|^{k_n} / \prod_{t=3}^{q} |A_t(z_n)|^{k_n},$$

i.e.,

$$|Q_1(z_n)||A_1(z_n)|^{k_n} = \lambda_n |Q_2(z_n)||A_2(z_n)|^{k_n}.$$

Thus, either $z \in Z_{Q_1} \cap Z_{Q_2}$, or $|A_1(z_n)| = (\lambda_n |Q_2(z_n)|/|Q_1(z_n)|)^{1/k_n} A_2(z_n)$. Therefore, $\lim[|A_1(z_n; k_n)|] = \lim[|A_2(z_n; k_n)|]$. Thus, $z \in \bigcup_{i,j=1}^{q} [\lim[|A_i|] = \lim[|A_j|]] \cup (Z_{Q_i} \cap Z_{Q_j})$. This shows that for $l > 1$,

$$\lim[Q_1 A_1^n + \cdots + Q_l A_l^n = P] \cap \mathscr{R}_2 \subseteq [\lim[|A_i|] = \lim[|A_j|]] \cup (Z_{Q_i} \cap Z_{Q_j}) \qquad (*)$$

We now proceed to show the converse to $(*)$. Suppose $z \in [|A_i| = |A_j| > 1] \cup (Z_{Q_i} \cap Z_{Q_j})$. For simplicity take $i = 1$, $j = 2$ and $z \in [|A_1| = |A_2| > 1]$, $|A_i(z)| > 1$, $i = 1, \dots, k$, $|A_j(z)| \leq 1$, $j = k+1, \dots, l$. Consider the expression

$$a_n := Q_1(A_1/A_3 \dots A_k)^n + Q_2(A_2/A_3 \dots A_k)^n$$

We claim that the curve $a_n = 0$ approaches $z$ as $n \to \infty$. Indeed, from $a_n(w) = 0$ we obtain

$$(A_1/A_2)^n(w) = -(Q_2/Q_1)(w).$$

For any $w$ close to $z$ such that $\arg(-Q_2/Q_1)(w)$ is irrational we have that $(-Q_2/Q_1)(w)^{1/n}$ approaches the circle of center $0 \in \mathbb{C}$ and radius 1 as $n \to \infty$. Therefore $(A_1/A_2)(z)$ is in the closure of the sequence $((-Q_2/Q_1(w))^{1/n})_n$. On the other hand, if

$$b_n := P/(A_3 \cdots A_k)^n - 1/(A_3 \cdots A_k)^n \sum_{j=3}^{l} Q_j A_j^n,$$

then $b_n(z) \to 0$ as $n \to \infty$. Therefore the curve $b_n = 0$ approaches $z$ as $n \to \infty$. Thus there is $z_n \in [a_n = b_n], z_n \to z$, i.e.,

$$(Q_1 A_1^n/(A_3 \cdots A_k)^n + Q_2 A_2^n/(A_3 \cdots A_k)^n)(z_n) = 1/(A_3 \cdots A_k)^n (P - \sum_{j=3}^{l} Q_j A_j^n)(z_n)$$

or $Q_1(z_n) A_1(z_n)^n + \cdots + Q_l(z_n) A_l(z_n)^n = P(z_n)$. $\square$

***Lemma 16.*** — *Suppose that $Q_1A_1^n+\cdots+Q_lA_l^n$ is a formula of positive height $h$. Then*

$$\begin{aligned}&\lim[|Q_1A_1^n+\cdots+Q_lA_l^n|<1]\\ &\quad=\bigcup_{i=1}^{l}\lim[|A_i<1|]\cap\lim[|Q_1A_1^n+\cdots+\widehat{Q_iA_i^n}+\cdots+Q_lA_l^n|<1]\\ &\qquad\bigcup_{i,j=1,i\neq j}^{l}[\lim|A_i|=\lim|A_j|>1]\cup(Z_{Q_i}\cap Z_{Q_j}).\end{aligned}$$

*Proof.* — Let $z\in\lim[|Q_1A_1^n+\cdots+Q_lA_l^n|\leq 1]$. There is $z_n\to z$ and $k_n\to\infty$ such that $|Q_1(z_n)A_1(z_n)^{k_n}+\cdots+Q_l(z_n)A_l(z_n)^{k_n}|<1$. Suppose that $\lim|A_1(z_n;k_n)|<1$, then $\varepsilon_n=|Q_1A_1(z_n)^{k_n}|\to 0$ and there is $w_n\to z$ such that $|Q_2(w_n)A_2(w_n)^{k_n}+\cdots+Q_l(w_n)A_l(w_n)^{k_n}|<1$. Therefore, $z\in\bigcup_{i=1}^{l}\lim[|A_i|<1]$, then $z\in\bigcup_{i=1}^{l}\lim[|A_i<1|]\cap\lim[|Q_1A_1^n+\cdots+\widehat{Q_iA_i^n}+\cdots+Q_lA_l^n|<1$. On the other hand, if $z\in\bigcap_{i=1}^{q}\lim[|A_i|>1]\bigcap_{j=q+1}^{l}\lim[|A_j<1|]$, then

$$Q_1(z_n)A_1(z_n;k_n)^{k_n}+\cdots+Q_q(z_n)A_q(z_n;k_n)^{k_n}\leq 1+S(z_n),$$

where $S(z_n)=|\sum_{j=q+1}^{l}Q_j(z_n)A_j(z_n)^{k_n}|$, is locally bounded at $z$. Proceeding as in Lemma 13, we obtain that for any $i,j=1,\dots,q$ either $z\in(\lim|A_i|=\lim|A_j|)$ or $z\in Z_{Q_i}\cap Z_{Q_j}$. The proof of the converse follows the same line of arguments of Lemma 13. □

*Proof of Theorem 2.* — In order to describe $\lim[Q_1A_1^n+\cdots+Q_lA_l^n=P]$ we first use induction on $l$ by means of Lemma 13, which reduces the problem to describing $\lim[QA^n=P]$ and $\lim[|A|<1]$ where $QA^n-P$ has height $h\geq 1$. Then we proceed by induction on $h$. For $h=1$ Lemma 12 gives $\lim[QA^n=P]=\partial(\lim[|A|<1)\cap(P\neq 0)]\cup\partial(\lim[|A|>1]\cap[Q\neq 0])$, which reduces the problem to height $h-1$. It only remains to find $\lim[|A|<1]$ and this follows from Lemma 14.

Thus we have shown that this limit can be expressed by algebraic relations between $|A_1|,\dots,|A_l|$ and $|P|$. □

## 4. Algebraic curves as integrals of differential equations

***Lemma 17.*** — *Given polynomials $A$ and $P$, there is a family $(\mathcal{X}_n)$ of polynomial vector fields of fixed degree such that $[A^{2n}=P]$ is an integral curve of $\mathcal{X}_n$.*

*Proof.* — Let $\mathcal{X}_n$ be the field corresponding to the following differential equation:

$$\dot{x}=-2nPA_y+P_yA,\qquad \dot{y}=2nPA_x-P_xA.$$

Let $f=A^{2n}-P$. Then

$$\dot{x}f_x+\dot{y}f_y=2n(P_yA_x-P_xA_y)f,$$

as can be easily verified. This shows that $[f=0]$ is an integral curve of $\mathcal{X}_n$. □

Thus, we have curves of increasing degree that are integral curves of polynomial fields of fixed degree. The next lemma says that in this case the field is essentially unique. The following proof is essentially due to B. Scárdua.

***Lemma 18.*** — *Suppose that $[f_n = 0]$ is a family of polynomial curves indexed by their degree. Assume that each curve is an integral curve of two differential equations of bounded degree: $\omega_n = 0$ and $\Omega_n = 0$. Then, for $n$ large enough, $\omega_n = 0$ and $\Omega_n = 0$ define the same foliation.*

*Proof.* — Forget the indices, for simplicity.

The hypotheses imply that

$$\begin{aligned} df \wedge \omega &= f\ell dx \wedge dy \\ df \wedge \Omega &= fL dx \wedge dy, \end{aligned}$$

where $\ell$ and $L$ are polynomials.

Assume that $\omega \wedge \Omega \neq 0$.

If $df \wedge \Omega \neq 0$, then we can write

$$\omega = \alpha df + \beta\Omega.$$

The coefficients $\alpha$ and $\beta$ are determined as follows:

$$\omega \wedge \Omega = \alpha df \wedge \Omega \quad \Rightarrow \quad \alpha = \frac{\omega \wedge \Omega}{df \wedge \Omega}$$

$$df \wedge \omega = \beta df \wedge \Omega \quad \Rightarrow \quad \beta = \frac{df \wedge \omega}{df \wedge \Omega}.$$

Therefore

$$\beta = \frac{\ell}{L}, \quad \alpha = \frac{\omega \wedge \Omega}{fL dx \wedge dy}$$

and so

$$\omega = \frac{\omega \wedge \Omega}{dx \wedge dy} \cdot \frac{df}{fL} + \frac{\ell}{L}\Omega,$$

or

$$L\omega = \frac{\omega \wedge \Omega}{dx \wedge dy} \cdot \frac{df}{f} + \ell\Omega.$$

Assume that $f$ is irreducible. Since $L\omega - \ell\Omega$ has bounded degree, we must have that $f dx \wedge dy$ divides $\omega \wedge \Omega$, that is,

$$\omega \wedge \Omega = f\mu dx \wedge dy,$$

for some polynomial $\mu$. Hence, $L\omega = \mu df + \ell\Omega$.

Now $\partial\ell = \partial\omega - 1$ and $\partial L = \partial\Omega - 1$, and so $\mu df$ has bounded degree. Since $df_n \to \infty$ we conclude that $\mu_n = 0$ for large $n$.

If $df \wedge \Omega = 0$, then we take $df \wedge \omega \neq 0$. If both expressions vanish identically, then $\omega$, $\Omega$, and $df$ define the same foliation. □

Moreover, as the next lemma indicates, formulas that are more complicated than $A^{2n} = P$ are not likely to be integral curves of fields of fixed degree.

***Lemma 19.*** — *Let $A$, $B$, and $P$ be bivariate polynomials such that $A(0,0) = 0 = B(0,0)$ and $(A,B) = 1$. Then the curves in the family $A^n + B^n = P$ are not integral curves of a family of polynomial fields of degree 2.*

*Proof.* — Suppose that $A$ and $B$ have degree $k$ and $P$ has degree $j$. Let $f = A^n + B^n - P$. Suppose that $f$ is an integral curve of the 1-form

$$\omega = adx + bdy,$$

with $a$ and $b$ polynomials of degree 2. Then

$$df \wedge \omega = fLdx \wedge dy,$$

with $L$ a polynomial of degree 1. This equation is equivalent to

$$(nA^{n-1}A_x + nB^{n-1}B_x - P_x)b - (nA^{n-1}A_y + nB^{n-1}B_y - P_y)a = (A^n + B^n - P)L.$$

For $n$ large, because $A(0,0) = 0 = B(0,0)$, we obtain

$$P_x b - P_y a = PL \tag{1}$$

$$nA^{n-1}(A_x b - A_y a) + nB^{n-1}(B_x b - B_y a) = (A^n + B^n)L. \tag{2}$$

Because $(A,B) = 1$, we get

$$\begin{aligned} n(A_x b - A_y a) &= AL \\ n(B_x b - B_y a) &= BL \end{aligned} \tag{3}$$

(The proof is at the end.)

Suppose that $P$ is homogeneous of degree $j$, $A$ and $B$ are homogeneous of degree $k$, and $a$ and $b$ are homogeneous of degree 2, in equations (1) and (3). This is not a restriction because it suffices to compare the homogeneous parts of highest degree in these equations.

Equation (3) can be written as

$$n\begin{pmatrix} -A_y & A_x \\ -B_y & B_x \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = L\begin{pmatrix} A \\ B \end{pmatrix}$$

If $\Delta = A_x B_y - A_y B_x$, then

$$n\begin{pmatrix} a \\ b \end{pmatrix} = \frac{L}{\Delta}\begin{pmatrix} B_x & -A_x \\ B_y & -A_y \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

Since, by Euler's formula, $kA = A_x x + A_y y$ and $kB = B_x x + B_y y$, we get

$$\begin{aligned} n\begin{pmatrix} a \\ b \end{pmatrix} &= \frac{L}{k\Delta}\begin{pmatrix} B_x & -A_x \\ B_y & -A_y \end{pmatrix}\begin{pmatrix} A_x & A_y \\ B_x & B_y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \frac{L}{k\Delta}\begin{pmatrix} 0 & -\Delta \\ \Delta & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \frac{L}{k}\begin{pmatrix} -y \\ x \end{pmatrix}, \end{aligned}$$

which implies that

$$a = -\frac{L}{kn}y, \qquad b = \frac{L}{kn}x.$$

From (1), we get

$$P_x \frac{L}{kn}x + P_y \frac{L}{kn}y = PL,$$

that is,

$$P_x x + P_y y = nkP,$$

which implies that $P$ is homogeneous of degree $nk$. Since $n$ is arbitrarily large and $P$ has a fixed degree, this is cannot happen. Therefore, $f$ is not an integral curve of $\omega$.

We still have to prove that (2) implies (3). In fact, let $\alpha = A_x b - A_y a$ and $\beta = B_x b - B_y a$. Then

$$nA^{n-1}\alpha + nB^{n-1}\beta = L(A^n + B^n),$$

that is,

$$A^{n-1}(n\alpha - LA) = B^{n-1}(-n\beta + LB).$$

Since $(A, B) = 1$, this implies that $A^{n-1}|(-n\beta + LB)$ and $B^{n-1}|(n\alpha - LA)$. Hence, there is a polynomial $\lambda$ such that

$$\begin{aligned} \lambda A^{n-1} &= -n\beta + LB \\ \lambda B^{n-1} &= (n\alpha - LA) \end{aligned}$$

Comparing degrees, we get $\lambda = 0$ for large $n$. Therefore,

$$(-n\beta + LB) = 0 = (n\alpha - LA),$$

as claimed. □

Define the *length* of a formula as the minimum number of its primitives of degree $\geq 1$. So, for instance, the formula

$$(x+1)^{2n} + ((x-y-1)^n + y)^n + y^2 - 1$$

has length 4.

Suppose that $\mathscr{C}$ is a family of curves given by the zeros of a formula of positive height. Let $l$ be the length of the formula and assume that the curves defined by the zeros of its primitives intersect transversely in the complex domain. If $\mathscr{V}$ is a family of vector fields of degree $k$ such that the elements of $\mathscr{C}$ are integral curves of the corresponding elements of $\mathscr{V}$, then $l \leq k^2 + k + 1$, as this last expression is the number of singular points of the elements of $\mathscr{V}$. In particular if $l > k^2 + k + 1$ the elements in $\mathscr{C}$ can not be integral curves of a family of polynomial vector fields of degree $\leq k$.

***Theorem 4***. — *Every generic basic closed one-dimensional semi-algebraic set in the plane is the limit of an family of algebraic curves that are integral curves of a family of polynomial vector fields of fixed degree.*

*Proof.* — Let $\Omega$ be a generic basic closed semi-algebraic set. It is known (but hard to prove) that every basic open semi-algebraic set in the plane can actually be given by *two* inequalities [**1**]. Since $\Omega$ is generic, this also applies to $\Omega$ and we can write $\Omega = [P \geq 0, Q \geq 0]$. We shall show that $\Omega = \lim[A^{2n} \leq P]$ for

$$A = \frac{Q}{n} - 1.$$

Indeed,

$$[A^2 \leq 1] = [(\frac{Q}{n} - 1)^2 \leq 1] = [0 \leq Q \leq 2n]$$

Hence,

$$\{\, z : A^2(z) \leq 1, \text{ for sufficiently large } n \,\} = [Q \geq 0].$$

and so

$$\{\, z : P(z) \geq 0,\ A^2(z) \leq 1, \text{ for sufficiently large } n \,\} = [P \geq 0, Q \geq 0].$$

Lemma 2 then says that

$$\lim[A^{2n} \leq P] = [P \geq 0, Q \geq 0] = \Omega,$$

if $[A^2 \leq 1, P \geq 0]$ is generic for sufficiently large $n$.

As mentioned in §4, the curves $[A^{2n} = P]$ are integral curves of a family of polynomial vector fields of fixed degree. (Note that, although $A$ has coefficients that depend on $n$, the vector fields are still of fixed degree. The general case is described in Lemma 20 below.) □

***Lemma 20.*** — *Let*

$$A(z;n) = \sum_{j=-k}^{\ell} a_j(z) n^j$$

*be a real polynomial in $z$ and $n$. Then*

$$\lim_{n\to\infty} [A(z;n)^{2n} \leq P(z)] = \begin{cases} \varnothing, & \text{if } a_j \not\equiv 0 \text{ for some } j \geq 1; \\ [a_0^2 < 1, 0 \leq P] \cup [a_0^2 = 1, e^{2a_{-1}(z)} \leq P(z)], & \text{otherwise.} \end{cases}$$

*Proof.* — First, notice that if $a_j \not\equiv 0$ for some $j \geq 1$, then the limit is empty.

Next, suppose that $z \in \lim[A^{2n} \leq P]$. Then, there is a sequence $z_n \to z$ such that

$$\left(\frac{a_{-k}(z_n)}{n^k} + \frac{a_{-k+1}(z_n)}{n^{k-1}} + \cdots + \frac{a_{-1}(z_n)}{n} + a_0(z_n)\right)^{2n} \leq P(z_n).$$

If $a_0(z) = 0$, then $0 \leq P(z)$.

Finally, assume that $a_0(z) \neq 0$. Then, $a_0(z_n) \neq 0$ for $n$ large enough. Letting

$$B(w;n) = \frac{a_{-1}(w)}{a_0(w)} \cdot \frac{1}{n} + \cdots + \frac{a_{-k}(w)}{a_0(w)} \cdot \frac{1}{n^k},$$

we have

$$a_0(z_n)^{2n}(1 + B(z_n;n))^{2n} \leq P(z_n).$$

For $n$ large enough, we have $|B(z_n;n)| < 1$ and then

$$n \log a_0(z_n)^2 + 2n \log(1 + B(z_n;n)) \leq \log P(z_n).$$

If $a_0(z)^2 = 1$, we have $2a_{-1}(z) \leq \log P(z)$, i.e., $e^{2a_{-1}(z)} \leq P(z)$. If $a_0(z)^2 < 1$, we have $0 \leq P(z)$. $\square$

## References

[1] R. Benedetti & J.-J. Risler – *Real algebraic and semi-algebraic sets*, Actualités Mathématiques, Hermann, 1990.

[2] E. Bishop – "Conditions for the analyticity of certain sets", *Michigan Math. J.* **11** (1964), p. 289–304.

[3] D. E. Knuth – "Big omicron and big omega and big theta", *SIGACT News* **8** (1976), p. 18–24.

[4] J. V. Pereira – "On the Poincaré problem for foliations of general type", *Math. Ann.* **323** (2002), p. 217–226.

[5] J.-J. Risler – "Additive complexity and zeros of real polynomials", *SIAM J. Comput.* **14** (1985), p. 178–183.

[6] G. Stolzenberg – *Volumes, limits, and extensions of analytic varieties*, Lecture Notes in Mathematics, No. 19, Springer, 1966.

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# THE SPACE OF GENERALIZED FORMAL POWER SERIES SOLUTIONS OF AN ORDINARY DIFFERENTIAL EQUATION

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***Abstract.*** — We prove that the set of truncations of generalized power series solutions of an ordinary differential equations is contained in a semi-algebraic set of dimension bounded by twice the order of the differential equation.

***Résumé* (L'espace des séries formelles généralisées qui sont solution d'une équation différentielle ordinaire)**
Nous montrons que l'ensemble des troncations de séries généralisées qui sont solutions d'une équation différentielle ordinaire est contenu dans un ensemble semi-algébrique dont la dimension est bornée par le double de l'ordre de l'équation différentielle.

## 1. Introduction

Consider a polynomial differential equation $F(\partial_0(y), \ldots, \partial_n(y)) = 0$, where $F(y_0, \ldots, y_n)$ is a polynomial in the variables $y_0, \ldots, y_n$ with coefficients in $\mathbb{C}[x^{\mathbb{R}}]$ (polynomials with real exponents). We are interested in series solutions of $(F = 0)$ of the form $\sum_{i=1}^{\infty} c_i\, x^{\mu_i}$, where $c_i \in \mathbb{C}$ and $\mu_i \in \mathbb{R}$ with $\mu_1 < \mu_2 \cdots$ (so called *generalized power series*). D.Y. Grigor'ev and M. Singer describe in [**5**] a parametric version of the Newton polygon process applied to $F$, which for each integer $k$, gives rise to a semi-algebraic subset $\mathrm{NIC}_k^\star(F) \subseteq \mathbb{R}^{3k}$ so that the space of truncations of length $k$ of generalized power series solution of $(F = 0)$ is included in $\mathrm{NIC}_k^\star(F)$. The main contribution of this paper is to prove that the dimension of this semi-algebraic set is bounded by $2n$. More precisely, its *adapted dimension* (see subsection 3.2) is bounded by $n$. The adapted dimension is a proper measure of the *number of free parameters* (real or complex, coefficient or exponent) which have been introduced

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along the Newton polygon process in a parametric family of power series solution of a differential equation.

Briot and Bouquet [**1**] in 1856 use the Newton polygon for studying first order and first degree ordinary differential equations and Fine [**4**] in 1889 gives a description of the method for ordinary differential equation of arbitrary order. In section 2 we present a brief introduction to its classical version. In section 4 we introduce the notion of parametric Newton polygon: specifically, we define it and give some technical results about *parametric polynomials* which will be used in the proof of the main theorem.

In section 3 we state the main theorem and give a straightforward proof for the case $k = 1$. The general case is dealt with in section 5.

## 2. Newton polygon of an ODE

A *well-ordered* series with complex coefficients and real exponents is a series $\phi(x) = \sum_{\alpha\in S} c_\alpha\, x^\alpha$, where $c_\alpha \in \mathbb{C}$, and $S$ is a well ordered subset of $\mathbb{R}$. If there exist a finitely generated semi-group $\Gamma$ of $\mathbb{R}_{\geq 0}$ and $\gamma \in \mathbb{R}$, such that, $S \subseteq \gamma+\Gamma$, then we say that $\phi(x)$ is a *grid-based* series (this terminology comes from [**6**].) Let $\mathbb{C}((x))^w$ and $\mathbb{C}((x))^g$ be the sets of well-ordered series and of grid-based series, respectively. We denote $\mathbb{C}[x^{\mathbb{R}}]$ the subring of series in $\mathbb{C}((x))^g$ with finite support (*polynomials*, so to speak). It is well-know (see [**7**], for example), that both $\mathbb{C}((x))^w$ and $\mathbb{C}((x))^g$ are actually fields. Both are differential rings with the usual inner operations and the differential operator $\partial = x\frac{d}{dx}$:

$$\partial\left(\sum c_\alpha\, x^\alpha\right) = \sum \alpha\, c_\alpha\, x^\alpha.$$

Denote by $\partial_0$ the identity operator and for positive integer $i$, $\partial_i = \partial \circ \partial_{i-1}$.

Let $F(y_0,\dots,y_n)$ be a polynomial in the variables $y_0,\dots,y_n$ with coefficients in $\mathbb{C}[x^{\mathbb{R}}]$. The differential equation

$$F\big(\partial_0(y), \partial_1(y), \dots, \partial_n(y)\big) = 0$$

will be denoted by $F(y) = 0$. Notice that any polynomial ordinary differential equation can be rewritten in this form.

We are interested in solutions of $F(y) = 0$ in the field $\mathbb{C}((x))^w$. By virtue of [**2, 5, 6**], all of them are actually in $\mathbb{C}((x))^g$.

Write $F$ in a uniquely, using the standard multiindex notation $y^\rho = y_0^{\rho_0}\cdots y_n^{\rho_n}$ ( where $\rho = (\rho_0,\dots,\rho_n)$) as

$$F = \sum_{\alpha,\rho} A_{\alpha,\rho}\, x^\alpha\, y^\rho, \text{ with } A_{\alpha,\rho} \in \mathbb{C},$$

where $\alpha$ and $\rho$ run over finite subsets of $\mathbb{R}$ and $\mathbb{N}^{n+1}$ respectively. The *cloud of points* of $F$ is the set

$$\mathscr{P}(F) = \{(\alpha, |\rho|) : A_{\alpha,\rho} \neq 0\},$$

where $|\rho| = \rho_0 + \cdots + \rho_n$. The Newton polygon $\mathcal{N}(F)$ of $F$ is the convex hull of

$$\bigcup_{P \in \mathcal{P}(F)} (P + \{(a,0) \mid a \geq 0\}) \ .$$

Notice that $\mathcal{N}(F)$ has a finite number of vertices, all of whose ordinates are non-negative integers.

Given a line $L \subseteq \mathbb{R}^2$ with slope $-1/\mu$, we say that $\mu$ is the *inclination* of $L$. Let $\mu \in \mathbb{R}$, we denote $L(F;\mu)$ the supporting line of $\mathcal{N}(F)$ with inclination $\mu$ (i.e. the only line $L$ with inclination $\mu$ such that $\mathcal{N}(F)$ is contained in the right closed half-plane defined by $L$ and $L \cap \mathcal{N}(F) \neq \varnothing$). More precisely, $L(F;\mu)$ is the set of points $(a,b)$ in $\mathbb{R}^2$ such that $a + \mu b = \nu(F;\mu)$, where $\nu(F;\mu) = \min\{\alpha + \mu\,|\rho|\,; A_{\alpha,\rho} \neq 0\}$.

For any $\mu \in \mathbb{R}$, define the polynomial

$$\Phi_{(F;\mu)}(\mathfrak{c}) = \sum_{(\alpha,|\rho|) \in L(F;\mu)} A_{\alpha,\rho}\, \mu^{w(\rho)}\, \mathfrak{c}^{|\rho|} \in \mathbb{C}[\mathfrak{c}], \tag{1}$$

where $w(\rho) = \rho_1 + 2\rho_2 + \cdots + n\rho_n$. The *Newton polygon data* of $F$ will be the set of vertices $v_0, \ldots, v_t$ (ordered with decreasing ordinate), the sides $[v_i, v_{i+1}]$, $0 \leq i < t$, the *indicial polynomials* associated to each vertex $v$:

$$\Psi_{(F;v)}(\mathfrak{m}) = \sum_{(\alpha,|\rho|) = v} A_{\alpha,\rho}\, \mathfrak{m}^{w(\rho)} \in \mathbb{C}[\mathfrak{m}]. \tag{2}$$

and the *characteristic polynomials* associated to each side $[v_i, v_{i+1}]$:

$$\Phi_{(F;[v_i,v_{i+1}])}(\mathfrak{c}) = \Phi_{(F;\mu_{[v_i,v_{i+1}]})}(\mathfrak{c}),$$

where $\mu_{[v_i,v_{i+1}]}$ is the inclination of side $[v_i, v_{i+1}]$.

**2.1. Necessary Initial Conditions.** — Given a well-ordered formal power series $y(x) = \sum_{\alpha \in S} c_\alpha\, x^\alpha$, its *order*, $\mathrm{ord}(y(x))$, is infinity if $y(x) = 0$ and $\min\{\alpha \in S \mid c_\alpha \neq 0\}$ otherwise.

***Lemma 1***. — *Let $y(x) = c\,x^\mu + \sum_{\alpha > \mu} c_\alpha\, x^\alpha \in \mathbb{C}((x))^w$ be a solution of the differential equation $F(y) = 0$. Then*

$$\Phi_{(F;\mu)}(c) = 0.$$

*where $c$ may be zero. In particular, if $y(x) = 0$ is a solution of $F(y) = 0$ then $\Phi_{(F;\mu)}(0) = 0$ for all $\mu$.*

*Proof.* — Developing $F$

$$\begin{aligned}&F(c\,x^{\mu}+\cdots)=\\&\sum_{\alpha,\rho}A_{\alpha,\rho}\,x^{\alpha}\,(c\,x^{\mu}+\cdots)^{\rho_0}(\mu c\,x^{\mu}+\cdots)^{\rho_1}\cdots(\mu^{n}c\,x^{\mu}+\cdots)^{\rho_n}=\\&\sum_{\alpha,\rho}\left\{A_{\alpha,\rho}\;c^{|\rho|}\,\mu^{w(\rho)}x^{\alpha+\mu|\rho|}+\cdots\right\}=\\&\left\{\sum_{\alpha+\mu\,|\rho|=\nu(F;\mu)}A_{\alpha,\rho}\;c^{|\rho|}\,\mu^{w(\rho)}\right\}x^{\nu(F;\mu)}+\cdots,\end{aligned}$$

where dots $\cdots$ stand for monomials of order greater than the exponent of $x$ in the preceding term. The lemma follows from the fact that $\alpha+\mu\,|\rho|=\nu(F;\mu)$ if and only if $(\alpha,|\rho|)\in L(F;\mu)$. $\square$

***Notation 1***. — *Let $\varphi\in\mathbb{C}((x))^g$ and $F(y_0,\dots,y_n)\in\mathbb{C}((x))^g[y_0,\dots,y_n]$, denote*

$$F(\varphi+y)=F(\varphi+y_0,\partial(\varphi)+y_1,\dots,\partial_n(\varphi)+y_n)\in\mathbb{C}((x))^g[y_0,\dots,y_n].$$

***Definition 1***. — *Given $F(y_0,\dots,y_n)$ and a positive integer $k$, define the set of* necessary $k$-initials conditions, $\mathrm{NIC}_k(F)$, *to be the subset of $(\mathbb{R}\times\mathbb{C})^k$ of the points $(\mu_1,c_1,\dots,\mu_k,c_k)\in(\mathbb{R}\times\mathbb{C})^k$ such that*

$$\mu_1<\cdots<\mu_k,\ \ \textit{and}$$
$$\Phi_{(F_1;\mu_1)}(c_1)=0,\dots,\Phi_{(F_k;\mu_k)}(c_k)=0,$$

*where $F_1(y)=F(y)$ and $F_{i+1}(y)=F_i(c_i\,x^{\mu_i}+y)$, for $1\le i<k$.*

*Define the $\mathrm{NIC}^*_k(F)=\mathrm{NIC}_k(F)\cap(\mathbb{R}\times\mathbb{C}^*)^k$, where $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$.*

***Corollary 1***. — *If $y(x)=\sum_{i=1}^{k}c_ix^{\mu_i}+\sum_{\mu_k<\alpha}c_\alpha x^\alpha$ is a solution of $F(y)=0$ with $\mu_1<\cdots<\mu_k$, then*

$$(\mu_1,c_1,\dots,\mu_k,c_k)\in\mathrm{NIC}_k(F).$$

***Corollary 2***. — *Let $v_0,\dots,v_t$ be the vertices of $\mathcal{N}(F)$, ordered by decreasing ordinate. Let $\mu_i$, $1\le i\le t$ be the inclination of the side $[v_{i-1},v_i]$. Set $\mu_0=-\infty$ and $\mu_{t+1}=+\infty$. The subset $\mathrm{NIC}_1(F)\subseteq(\mathbb{R}\times\mathbb{C})$ is semi-algebraic. Moreover, $\mathrm{NIC}^*_1(F)$ is the finite union of the semi-algebraic sets corresponding to the sides of the Newton polygon of $F$:*

$$\{(\mu,c)\in\mathbb{R}\times\mathbb{C}^*\,;\mu=\mu_i,\ \textit{and}\ \Phi_{(F;\mu_i)}(c)=0\},\quad 1\le i\le t,$$

*and the semi-algebraic sets corresponding to the vertices:*

$$\{(\mu,c)\in\mathbb{R}\times\mathbb{C}^*\,;\mu_i<\mu<\mu_{i+1},\ \textit{and}\ \Psi_{(F;v_i)}(\mu)=0\},\quad 0\le i\le t.$$

*Proof.* — Let $\mu\in\mathbb{R}$, $\mu_i<\mu<\mu_{i+1}$, for some $0\le i\le t$. As $L(F;\mu)\cap\mathcal{N}(F)=v_i$ and $\Phi_{(F;\mu)}(\mathfrak{c})=\mathfrak{c}^h\,\Psi_{(F;v_i)}(\mu)$, (where $h$ is the ordinate of $v_i$) then, for $c\ne0$ and $\mu_i<\mu<\mu_{i+1}$, one has $\Phi_{(F;\mu)}(c)=0$ if and only if $\Psi_{(F;v_i)}(\mu)=0$, and we are done. $\square$

Let $\mu \in \mathbb{R}$ be a real number and fix a point $(a,h) \in \mathbb{R} \times \mathbb{N}$.

***Definition 2***. — *We say that $(a,h)$ belongs to the* red part with respect to $\mu$ *of the Newton polygon of $F(y)$ if $h \geq 1$ and either $(a,h)$ is the vertex of $\mathcal{N}(F)$ with minimum ordinate or it belongs to a side of $\mathcal{N}(F)$ with inclination greater than $\mu$.*

Notice that if the red part with respect to $\mu$ of $\mathcal{N}(F)$ is empty, then there are no generalized power series solution of $(F=0)$ of order greater than $\mu$: the vertex $(a,h)$ with minimum ordinate has $h=0$ and all the sides of $\mathcal{N}(F)$ have inclination less than or equal to $\mu$, hence for $\gamma > \mu$, the polynomial $\Phi_{(F;\mu)}(\mathfrak{c})$ is a non-zero constant and by Corollary 2 the set $\mathrm{NIC}^*_1(F)$ is empty. The reciprocal is not true as Example 1 (page 65) shows.

***Lemma 2***. — *Let $(\mu_1, c_1, \dots, \mu_k, c_k) \in \mathrm{NIC}^*_k(F)$, $\varphi = \sum_{j=1}^k c_j\, x^{\mu_j}$ and $F_{k+1}(y) = F(\varphi + y)$. The red part of $\mathcal{N}(F_{k+1}(y))$ with respect to $\mu_k$ nonempty.*

*Proof*. — Let $(\mu, c) \in \mathrm{NIC}^*_1(F)$ and consider $G = F(c\,x^{\mu} + y)$. The red part of the Newton polygon of $G$ with respect to $\mu$ is not empty. To see this, let $v_0, \dots, v_t$ be the vertices of $\mathcal{N}(F)$ ordered by decreasing ordinate and let $v_k$ be the vertex with highest ordinate in $L(F;\mu) \cap \mathcal{N}(F)$. The ordinate of this $v_k$ is greater than zero because otherwise $\Phi_{(F;\mu)}(\mathfrak{c})$ would be a nonzero constant, in contradiction with the fact that $\Phi_{(F;\mu)}(c) = 0$.

Returning to the main argument, given a monomial $M = x^{\alpha} y_0^{\rho_0} \cdots y_n^{\rho_n}$, one may write

$$M(c\,x^{\mu} + y) = x^{\alpha} \prod_{i=0}^{n} (c\,\mu^{i}\, x^{\mu} + y_i)^{\rho_i} = M + R, \tag{3}$$

where the points corresponding to the monomials of $R$ have ordinate less than $|\rho|$ and belong to the line with inclination $\mu$ passing through $(\alpha, |\rho|)$. If $w$ is the intersection of $L(F;\mu)$ with the axis of abscissas, then the cloud of points $\mathcal{P}(G)$ of $G$ is contained in the positive convex hull of $\{v_0, \dots, v_k, w\}$. The coefficient of $G$ corresponding to $w$ is precisely $\Phi_{(F;\mu)}(c) = 0$, hence $w \notin \mathcal{P}(G)$. Moreover, $\{v_0, \dots, v_k\} \subseteq \mathcal{P}(G)$, because of (3). Therefore $v_0, \dots, v_k$ are vertices of $\mathcal{N}(G)$. Hence either $v_k$ is the vertex of $\mathcal{N}(G)$ with minimum ordinate or there exists a side of $\mathcal{N}(G)$ with inclination greater than $\mu$ and we are done. □

***Example 1* (See Figure 1)**. — *Let $F = x^{-1}\, y_0^6\, y_1 + y_0^2\, y_1 + x\, y_0^2 - 3\,x\, y_0\, y_1 - x^2\, y_0 + 2\,x^2\, y_1 + x^5$. The point $(1,1) \in \mathrm{NIC}^*(F)$. Let $G = F(x+y)$. The red part of $\mathcal{N}(G)$ with respect to $\mu = 1$ is vertex $v'_2$ and point $p$. In this example, there are no solutions of $(G=0)$ of order greater than $1$.*

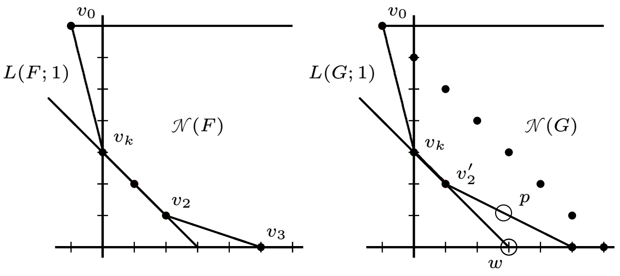


FIGURE 1. Newton polygons of $F$ and $G$ from Lemma 2 and Example 1

## 3. Main result

In this section we introduce the notions of *truncation* of well-ordered power series and of *adapted dimension*, and proceed to state the main result: the truncation of length $k$ of the solutions of the differential equation $(F=0)$ is contained in a semi-algebraic subset of $(\mathbb{R}\times\mathbb{C})^k$ of adapted dimension less than or equal to the order of $F$.

The adapted dimension is a proper measure of the *number of free parameters* (real or complex, coefficient or exponent) which have been introduced along the Newton polygon process in a parametric family of power series solution of a differential equation. Heuristically, when one introduces an exponent as a free parameter in the solution space then one must also introduce a coefficient as a free parameter. The simplest non-trivial case is the equation $F(y)=y_1^2-y_0\,y_2=0$, whose solutions are $c\,x^{\mu}$ for any $\mu\in\mathbb{R}$ and $c\in\mathbb{C}$, (*adapted dimension* 2).

**3.1. Truncations.** — For any positive integer $k$ and real $\beta$, the truncation of length $k$ to the right of $\beta$ is a map $\mathrm{Tr}_{k;\beta}:\mathbb{C}((x))^w\to(\mathbb{R}\times\mathbb{C})^k$ defined as follows. If $y(x)=0$, then

$$\mathrm{Tr}_{k;\beta}(y(x))=((\beta+1,0),(\beta+2,0),\ldots,(\beta+k,0)),$$

otherwise, $y(x)=c\,x^{\mu}+\sum_{\alpha>\mu}c_\alpha\,x^{\alpha}=c\,x^{\mu}+\bar{y}(x)$, with $c\neq0$ and in this case

$$\mathrm{Tr}_{k;\beta}(y(x))=((\mu,c),\mathrm{Tr}_{k-1;\mu}(\bar{y}(x))).$$

Finally, the *truncation of length* $k$ is $\mathrm{Tr}_k=\mathrm{Tr}_{k;0}$. For instance,

$$\mathrm{Tr}_4(x^{-0.5}+x^{\pi})=((-0.5,1),(\pi,1),(\pi+1,0),(\pi+2,0))\in(\mathbb{R}\times\mathbb{C})^4$$

***Remark 1.*** — *Let $\mathcal{M}_s$ be the subset of $\mathbb{C}[x^{\mathbb{R}}]$ of the elements with exactly $s$ monomials. Then $\mathrm{Tr}_k(\mathcal{M}_s)$ is a semi-algebraic subset of $(\mathbb{R}\times\mathbb{C})^k$.*

***Remark 2.*** — *By corollary 1, if $y(x)$ is a solution of $(F=0)$, then $\mathrm{Tr}_k(y)\subseteq\mathrm{NIC}_k(F)$.*

**3.2. Adapted dimension of cells of** $(\mathbb{R}\times\mathbb{C})^k$**.** — In this section we use some known notions and results of real algebraic geometry for which we refer to the reader to [**3**] (or any other standard text on the subject).

Given a finite family $P_1,\dots,P_r\in\mathbb{R}[X_1,\dots,X_t]$, we say that a subset $C$ of $\mathbb{R}^t$ is $(P_1,\dots,P_r)$-*invariant* if every polynomial $P_i$ has constant sign ($>0$, $<0$, or $=0$) on $C$. A $\mathscr{C}^\infty$*-cylindrical algebraic decomposition* of $\mathbb{R}^t$ adapted to $P_1,\dots,P_r$ is a cylindrical algebraic decomposition $\mathscr{C}$ all of whose cells are $(P_1,\dots,P_r)$-invariant $\mathscr{C}^\infty$-manifolds and such that the defining functions of the cells of $\mathscr{C}$ are $\mathscr{C}^\infty$.

Algorithms for constructing $\mathscr{C}^\infty$-cylindrical algebraic decomposition for a given family of polynomials are well-known (see for instance [**3**]).

Let $\mathscr{C}=\{\mathscr{C}_1,\dots,\mathscr{C}_t\}$ be a $\mathscr{C}^\infty$-cylindrical algebraic decomposition of $\mathbb{R}^t$. Let $C$ a cell of $\mathscr{C}$. Let $i_1<\cdots<i_d\le k$ such that the restriction $\tilde\pi_C: C\to\mathbb{R}^d$ to $C$ of the projection $\tilde\pi(r_1,\dots,r_k)=(r_{i_1},\dots,r_{i_d})$ is a local diffeomorphism of $C$ onto an open subset of $\mathbb{R}^d$. We choose $\tilde\pi_C$ such that $(i_1,i_2,\dots,i_d)$ is minimal with respect to the lexicographical order. In particular, $\tilde\pi_C$ is a local system of coordinates of $C$ at any point $\alpha\in C$ and $d$ is the dimension of $C$. We call $\tilde\pi_C$ the *standard system of coordinates* of the cell $C$ with respect to the cylindrical algebraic decomposition $\mathscr{C}$. We denote $I_C$ the $d$-uple $I_C=(i_1,\dots,i_d)$. The derivations $\frac{\partial}{\partial r_{i_1}}\Big|_\alpha,\dots,\frac{\partial}{\partial r_{i_d}}\Big|_\alpha$ span the tangent space of $C$ at $\alpha$. One proves easily that

$$\frac{\partial}{\partial r_{i_j}}\bigg|_\alpha (r_s|_C)=0,\quad s<i_j,\quad 1\le j\le d. \tag{4}$$

***Remark 3***. — *The sequence $I_C=(i_1,\dots,i_d)$ is characterized as follows: let $i_C$ be the inclusion of $C$ in $\mathbb{R}^k$ and $\alpha\in C$ any point. Then $j\notin I_C$ if and only if $i_C^*(d\,r_j)_\alpha$ depends linearly on $\{i_C^*(d\,r_s)_\alpha\mid s<j\}$.*

We identify $(\mathbb{R}\times\mathbb{C})^k$ with $\mathbb{R}^{3k}$ as follows: let $(r_1,\dots,r_{3k})$ be the coordinate functions of $\mathbb{R}^{3k}$ and $(\mu_1,c_1,\dots,\mu_k,c_k)$ the coordinate functions of $(\mathbb{R}\times\mathbb{C})^k$. For $1\le t\le k$, let

$$\begin{aligned}\mu_t &= r_{3(t-1)+1},\quad\text{and}\\ c_t &= r_{3(t-1)+2}+\sqrt{-1}\,r_{3(t-1)+3}.\end{aligned} \tag{5}$$

***Definition 3* (Adapted dimension)**. — *Let $C$ be a cell of a $\mathscr{C}^\infty$-cylindrical algebraic decomposition $\mathscr{C}$ of $(\mathbb{R}\times\mathbb{C})^k=\mathbb{R}^{3k}$ and $\tilde\pi_C(r_1,\dots,r_{3k})=(r_{i_1},\dots,r_{i_d})$ be the standard system of coordinates of $C$ with respect to $\mathscr{C}$. For each $t$, $1\le t\le k$, define $d_t$ as follows:*

1. *$d_t=2$ if $3(t-1)+1\in\{i_1,\dots,i_d\}$.*
2. *$d_t=1$ if either $3(t-1)+2$ or $3(t-1)+3$ belongs to $\{i_1,\dots,i_d\}$ and we are not in case (1).*
3. *$d_t=0$ otherwise.*

*The* adapted dimension *of $C$ is* $\dim_{\mathrm{a}}(C)=d_1+\cdots+d_k$.

***Remark 4***. — *Certainly,* $\dim(C)\le 2\dim_{\mathrm{a}}(C)$.

***Lemma 3***. — *Let $\mathscr{C}^1$ and $\mathscr{C}^2$ be two $\mathscr{C}^\infty$-cylindrical algebraic decompositions of* $(\mathbb{R}\times\mathbb{C})^k$. *Let $C$ be a cell of $\mathscr{C}^1$, and assume that $C = C'_1 \cup \cdots \cup C'_s$,where $C'_i$ is a cell of $\mathscr{C}^2$ for all $i$. Then $\dim_{\mathrm{a}}(C'_i) \leq \dim_{\mathrm{a}}(C)$ for $1 \leq i \leq s$ and there exists $j \in \{1,\ldots,s\}$ such that $\dim_{\mathrm{a}}(C'_j) = \dim_{\mathrm{a}}(C)$, $\dim C'_j = \dim C$ and $I_{C'_j} = I_C$.*

*Proof.* — From the characterization given in Remark 3 and the fact that linear dependency is preserved by the pull-back of the inclusion of $C'_i$ into $C$, one infers that $I_{C'_i} \subseteq I_C$, which implies that $\dim_{\mathrm{a}} C'_i \leq \dim_{\mathrm{a}} C$. Then there must exist an index $j$ with $\dim C'_j = \dim C$, whence $I_{C'_j} = I_C$ and $\dim_{\mathrm{a}} C'_j = \dim_{\mathrm{a}} C$. □

**3.3. Main result.** — Let $F(y_0,\ldots,y_n)$ be a polynomial in the variables $y_0,\ldots,y_n$ with coefficients in $\mathbb{C}[x^{\mathbb{R}}]$.

***Theorem 1***. — *Let $Sol(F)$ the set of solutions of the differential equation $F(y) = 0$ in $\mathbb{C}((x))^g$. For any positive integer $k$, there exists a $\mathscr{C}^\infty$-cylindrical algebraic decomposition $\mathscr{C}$ of $(\mathbb{R}\times\mathbb{C})^k$ and a finite number number of cells $C_1,\ldots,C_s$ of $\mathscr{C}$ such that:*

- $\mathrm{Tr}_k(\mathrm{Sol}(F)) \subseteq C_1 \cup \cdots \cup C_s$, *and*
- $\dim_{\mathrm{a}}(C_i) \leq n$, *for $1 \leq i \leq s$.*

*As a consequence,* $\dim(C_i) \leq 2\,n$, *for $1 \leq i \leq s$.*

We end this section doing a technical reduction for the proof of Theorem 1 and, for the sake of clarity, giving a simple proof of case $k = 1$.

**Claim:** it is enough to prove the theorem substituting $\mathrm{NIC}^*_k(F)$ for $\mathrm{Tr}_k(\mathrm{Sol}(F))$ in the statement.

*Proof of the claim:* let $\mathscr{M}_s$ be the subset of $\mathbb{C}[x^{\mathbb{R}}]$ of "polynomials" with exactly $s$ monomials, and $\mathscr{M}_{\geq s}$ the subset of $\mathbb{C}((x))^w$ of series with at least $s$ monomials. Certainly,

$$\mathrm{Tr}_k(\mathrm{Sol}(F)) = \mathrm{Tr}_k\big(\,\mathrm{Sol}(F)\cap\mathscr{M}_{\geq k}\big) \cup \bigcup_{s=0}^{k} \mathrm{Tr}_k\big(\,\mathrm{Sol}(F)\cap\mathscr{M}_s\big). \tag{6}$$

By definition $\mathrm{Tr}_k\big(\,\mathrm{Sol}(F)\cap\mathscr{M}_{\geq k}\big) \subseteq (\mathbb{R}\times\mathbb{C}^*)^k$, so that by corollary 1,

$$\mathrm{Tr}_k\big(\,\mathrm{Sol}(F)\cap\mathscr{M}_{\geq k}\big) \subseteq \mathrm{NIC}^*_k(F). \tag{7}$$

Let $0 \leq s < k$, and consider the differentiable semi-algebraic function $F_s : (\mathbb{R}\times\mathbb{C}^*)^s \to (\mathbb{R}\times\mathbb{C})^{k-s}$ given by

$$F_s\big((\mu_1,c_1),\ldots,(\mu_s,c_s)\big) = \big((\mu_s+1,0),\ldots,(\mu_s+k-s,0)\big).$$

One sees easily that $\mathrm{Tr}_k\big(\,\mathrm{Sol}(F)\cap\mathscr{M}_s\big)$ is the graph of $F_s$ restricted to $\mathrm{Tr}_s\big(\,\mathrm{Sol}(F)\cap\mathscr{M}_s\big)$. As above, $\mathrm{Tr}_s\big(\,\mathrm{Sol}(F)\cap\mathscr{M}_s\big) \subseteq \mathrm{NIC}^*_s(F)$ and also, the adapted and the usual dimensions of $\mathrm{NIC}^*_s(F)$ are (respectively) equal to the adapted and usual dimensions of the graph of $F_s$ restricted to $\mathrm{NIC}^*_s(F)$, which finishes. □

Follows a straightforward proof of the theorem for $k = 1$.

*Proof of the theorem for $k = 1$.* — In this case, the first result of the theorem is just Corollary 2.

For the second result, notice that the adapted dimension of any cell in $\mathbb{R} \times \mathbb{C}^*$ is less than or equal to 2. Hence, it is enough to prove that $\dim_a(C) \leq n$ for $n = 0$ and $n = 1$, which we do separately.

If $n = 0$, the only monomial of $F(y)$ corresponding to point $(a, b) \in \mathbb{R} \times \mathbb{N}$ is exactly $A_{a,b} x^a y_0^b$, whence the polynomial $\Phi_{F;\mu_i}(C)$ is nonzero and has only a finite number of roots. The polynomial $\Psi_{(F;v)}(\mathfrak{m}) = A_v$ is clearly a nonzero constant. From these two facts, it follows that the dimension of $NIC_1^*(F)$ is zero

For $n = 1$, in order to prove that $\dim_a(C) \leq 1$, it suffices to show that the projection $(\mu, c) \mapsto \mu$ cannot belong to a local coordinate system of $\mathrm{NIC}_1^*(F)$. Let $v = (a, b)$ be a vertex of the Newton polygon. Since $n = 1$, all the monomials of $F(y)$ corresponding to $v$ are of the form $A_{a,(\rho_0,\rho_1)} x^a y_0^{\rho_0} y_1^{\rho_1}$ with $\rho_0 + \rho_1 = b$. Hence,

$$\Psi_{(F;v)}(\mathfrak{m}) = \sum_{j=0}^{b} A_{a,(b-j,j)}\, \mathfrak{m}^j.$$

and $\Psi_{(F;v)}(\mathfrak{m})$ cannot be zero because for some $j$, $A_{a,(b-j,j)} \neq 0$. The image of $\mathrm{NIC}_1^*(F)$ by the projection $(\mu, c) \mapsto \mu$ is thus a finite number of points and we are done for $k = 1$. □

## 4. Parametric polynomials and parametric Newton polygon

In this section we define the parametric Newton polygon data of a parametric differential polynomial. A parametric polynomial is a finite sum of the form $\sum_{i \in I} c_i\, x^{\mu_i}$, where $\mu_i$ and $c_i$ are respectively real and complex semi-algebraic $\mathscr{C}^\infty$-functions on a semi-algebraic $\mathscr{C}^\infty$-submanifold $C$. A parametric differential polynomial $H$ is a polynomial in $y_0, \ldots, y_n$ whose coefficients are parametric polynomials. For any parameter $\phi \in C$, the value $H_\phi$ of $H$ at $\phi$ is an ordinary differential polynomial. The parametric polygon data of $H$ will be defined as a family of functional objects on $C$ whose "values" are classical Newton polygon data (vertices, slopes, characteristics polynomials, etc) in such a way that their values at $\phi$ coincide with the Newton polygon data of $H_\phi$.

In order to define the *parametric Newton polygon data* of $H$, some semi-algebraic properties on the family of exponents of $x$ and on the real and imaginary parts of the coefficients of $H$ are required. They are gathered in the notion of *invariance* on a cell $C$.

Specifically, for a parametric polynomial $H = \sum_{i \in I} c_i\, x^{\mu_i}$ to be invariant on $C$ we require that the family of exponents $\mathscr{E} = \{\mu_i ; i \in I\}$ is totally ordered on $C$ and that none of the coefficients $c_i$ vanishes at any point of $C$. This way, both the minimum of $\mathscr{E}$ and the (function) coefficient of $x^\theta$ in $H$, provided $\theta \in \mathscr{E}$, are well defined. Moreover, the value of the minimum of $\mathscr{E}$ at every point $\phi \in C$ is the minimum of $\{\mu_i(\phi) ; i \in I\}$. For technical reasons one needs also to be able to compare the coefficients with functions belonging to some family of functions $E$ (for instance, some constant functions or the coordinate functions of the cell). This gives rise to the notion of invariance with respect to a family $E$.

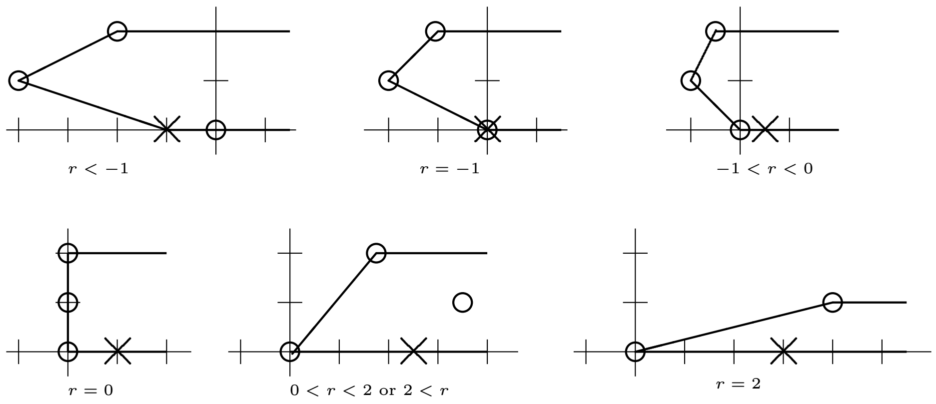


FIGURE 2.

All the above is probably better understood with an example. In Figure 2 are shown the different *shapes* (and points) of the Newton polygon of

$$H = 1 + x^{1+r} + (1+r^2)\, x^{2\,r}\, y_3 + (2-r)\, x^r\, y_0 y_1, \quad r \in \mathbb{R}.$$

One should imagine $r$ "moving" on $\mathbb{R}$ "from left to right" giving rise to the six essentially different shapes of the Newton polygon. One can see how three *exceptional* situations can happen (in the same order as in the figure): (a) two or more points of the polygon collide, (b) a point in the interior of the polygon collides with a side, and (c) a point disappears from the polygon. These are respectively the cases $r=-1$, $r=0$ and $r=2$ in Figure 2. All the equations describing those events are of semi-algebraic nature, so that there exists a cylindrical algebraic decomposition of the parameter space such that $H$ is invariant on each cell. In our example, the cells are the sets defined by the equations in $r$ below each diagram.

In the cell $r<-1$, the parametric Newton polygon data is composed of the sequence of functions $V_0=(r,2), V_1=(2r,1), V_2=(1+r,0)$, the indicial polynomials corresponding to each vertex ($\Psi_{(H;V_0)}(\mathfrak{m}) = (2-r)\mathfrak{m}$, $\Psi_{(H;V_1)}(\mathfrak{m}) = (1+r^2)\mathfrak{m}^3$, and $\Psi_{(H;V_2)}(\mathfrak{m})=1$), and the characteristic polynomials corresponding to the sides: $\Phi_{(H;[V_0,V_1])}(\mathfrak{c}) = \Psi_{(H;V_0)}(r)\mathfrak{c}^2 + \Psi_{(H;V_1)}(r)\mathfrak{c}$, and $\Phi_{(H;[V_1,V_2])}(\mathfrak{c}) = \Psi_{(H;V_1)}(1-r)\mathfrak{c}^2 + \Psi_{(H;V_0)}(1-r)\mathfrak{c}$. Similarly for the other cells.

In the proof of Theorem 1 we shall differentiate the parametric polynomials with respect to the "parameters". As the class of parametric polynomials is not closed under such derivations (due to $d(x^r)/dr$), we need to consider a larger family including polynomials in $\log x$, which can be related to the space of mappings from $C \times \tilde{\mathbb{C}}$ to $\mathbb{C}$, where $\tilde{\mathbb{C}}$ is the Riemann surface of the logarithm and $C$ is a semi-algebraic smooth manifold.

Throughout this section, $C \subseteq \mathbb{R}^n$ denotes a nonempty semi-algebraic $\mathscr{C}^\infty$ submanifold.

### 4.1. Parametric polynomials

***Definition 4***. — *An $\mathscr{N}$-function on $C$ is a semi-algebraic smooth function $f : C \to \mathbb{R}$. An $\mathscr{N}_\mathbb{C}$-function is a function $c : C \to \mathbb{C}$ of the form $c = a + \sqrt{-1}\, b$, where $a, b$ are $\mathscr{N}$-functions on $C$.*

*Let $\tilde{\mathbb{C}}$ be the Riemann surface of the logarithm. A function*

$$H : C \times \tilde{\mathbb{C}} \to \mathbb{C}$$

*is called an $\mathscr{N}_\mathbb{X}$-function over $C$ if there exist a finite number of functions $c_i \in \mathscr{N}_\mathbb{C}(C)$ and $\mu_i \in \mathscr{N}(C)$, $1 \leq i \leq k$, such that*

$$H(\phi, x) = \sum_{j=1}^{k} c_i(\phi)\, x^{\mu_i(\phi)}, \quad \textit{for all } (\phi, x) \in C \times \tilde{\mathbb{C}}. \tag{8}$$

*Denote $\mathscr{N}(C)$, $\mathscr{N}_\mathbb{C}(C)$ and $\mathscr{N}_\mathbb{X}(C)$ the rings of $\mathscr{N}$, $\mathscr{N}_\mathbb{C}$ or $\mathscr{N}_\mathbb{X}$-functions over $C$, respectively.*

*$\mathscr{N}_\mathbb{X}(C)[\log x]$ is the set of finite sums*

$$H(\phi, x) = \sum_{i,j} c_{i,j}(\phi)\, x^{\mu_{i,j}(\phi)}\, (\log x)^j, \quad \textit{for all } (\phi, x) \in C \times \tilde{\mathbb{C}}.$$

*where $c_{i,j} \in \mathscr{N}_\mathbb{C}(C)$ and $\mu_{i,j} \in \mathscr{N}(C)$, $1 \leq i \leq k$, $0 \leq j \leq s$.*

The following result shows that $\mathscr{N}_\mathbb{X}(c)[\log x]$ is actually the set of "polynomials" in $\log x$ with coefficients in $\mathscr{N}_\mathbb{X}(C)$:

***Lemma 4***. — *Any $H \in \mathscr{N}_\mathbb{X}(C)[\log x]$, can be written uniquely as $H = \sum_{j=0}^{s} H_j\, (\log x)^j$, where $H_j \in \mathscr{N}_\mathbb{X}(C)$.*

*Proof*. — Let $\mathscr{O}(\tilde{\mathbb{C}})$ be the differential ring of holomorphic functions on $\tilde{\mathbb{C}}$ with the derivation $\delta = x\frac{\partial}{\partial x}$, which is an integral domain. The map sending $\sum c_\alpha\, x^\alpha \in \mathbb{C}[x^\mathbb{R}]$ to the holomorphic function $\tilde{\mathbb{C}} \ni x \mapsto \sum c_\alpha\, x^\alpha$ is an injective differential ring homomorphism. The result follows from $\log x \in \mathscr{O}(\tilde{\mathbb{C}})$ being algebraically independent over the quotient ring of $\mathbb{C}[x^\mathbb{R}]$. □

4.1.1. *Derivations with respect the parameters.* — Assume that $C$ is a cell of a $\mathscr{C}^\infty$-cylindrical algebraic decomposition and $\pi_C(r_1, \ldots, r_s) = (r_{i_1}, \ldots, r_{i_d})$ its standard system of coordinates. It is known that if $f$ is an $\mathscr{N}$-function then its partial derivatives $\frac{\partial f}{\partial r_{i_j}}$ are also $\mathscr{N}$-functions. The operator $\frac{\partial}{\partial r_{i_j}}$ acts as a derivation on $\mathscr{N}_\mathbb{X}(C)[\log x]$ as follows: if $\mu \in \mathscr{N}(C)$ and $c \in \mathscr{N}_\mathbb{C}(C)$, then

$$\frac{\partial}{\partial r_{i_j}}(c\, x^\mu\, \log^s x) = \left( \frac{\partial c}{\partial r_{i_j}}\, x^\mu + c\, \frac{\partial \mu}{\partial r_{i_j}}\, x^\mu\, \log x \right) \log^s x. \tag{9}$$

4.1.2. *The notion of invariance*

***Notation 2.*** — *Let $H \in \mathcal{N}_{\mathbb{X}}(C)[\log x]$. For $\phi \in C$, denote by $H(\phi)$ the function $\tilde{\mathbb{C}} \ni x \mapsto H(\phi, x)$.*

*If $H \in \mathcal{N}_{\mathbb{X}}(C)$, then $H(\phi) \in \mathbb{C}[x^{\mathbb{R}}]$; for $\tau \in \mathcal{N}(C)$, denote by $[H]_\tau$ the function $[H]_\tau : C \to \mathbb{C}$ such that $[H]_\tau(\phi)$ is the value of the coefficient of $x^{\tau(\phi)}$ in $H(\tau)$. This $[H]_\tau$ is a semi-algebraic function but it is not smooth in general.*

*If $H = \sum_j H_j \log^j x \in \mathcal{N}_{\mathbb{X}}(C)[\log x]$, we write $[H]_\tau = \sum_j [H_j]_\tau \log^j x$.*

*We shall denote $<_C$ the partial order over $\mathcal{N}(C)$ given by $\mu <_C \mu'$ if and only if $\mu(\phi) < \mu'(\phi)$ for all $\phi \in C$.*

***Definition 5.*** — *Let $H \in \mathcal{N}_{\mathbb{X}}(C)$ and $E$ be a finite subset of $\mathcal{N}(C)$. We say that $H$ is* invariant on $C$ with respect to $E$, *if either $H = 0$ or there exist a finite subset $\mathcal{E} \subseteq \mathcal{N}(C)$ and functions $c_\theta \in \mathcal{N}_{\mathbb{C}}(C)$ for each $\theta \in \mathcal{E}$ with the following properties:*

1. *For all $(\phi, x) \in C \times \tilde{\mathbb{C}}$, $H(\phi, x) = \sum_{\theta\in\mathcal{E}} c_\theta(\phi)\, x^{\theta(\phi)}$.*
2. *The set $\mathcal{E} \cup E$ is totally ordered with respect to $<_C$.*
3. *For every $\theta \in \mathcal{E}$ and every $\phi \in C$, $c_\theta(\phi) \neq 0$.*

We remark that a set $\mathcal{E}$ satisfying (1), (2) and (3) is uniquely determined and independent of $E$: by (2), its elements are ordered $\theta_1 <_C \theta_2 <_C \cdots <_C \theta_s$ and for each $\phi \in C$, $\theta_1(\phi) < \theta_2(\phi) < \cdots < \theta_s(\phi)$ are the exponents of $x$ in $H(\phi)$.

*The set $\mathcal{E}$ will be denoted $\mathcal{E}(H)$. By definition $\mathcal{E}(0) = \varnothing$.*

***Definition 6.*** — *An element $H = \sum_{j=0}^{s} H_j \log^j x \in \mathcal{N}_{\mathbb{X}}(C)[\log x]$ is* invariant with respect to $E$ *if each $H_j \in \mathcal{N}_{\mathbb{X}}(C)$, $0 \leq j \leq t$ is invariant on $C$ with respect to $E$ and the set $\mathcal{E}(H) = \cup_j \mathcal{E}(H_j)$ is totally ordered on $C$.*

***Lemma 5.*** — *Let $H \in \mathcal{N}_{\mathbb{X}}(C)$ be invariant on $C$ with respect to $E$. If $\tau \in E \cup \mathcal{E}(H)$, then $[H]_\tau \in \mathcal{N}_{\mathbb{C}}(C)$. If $\tau \notin \mathcal{E}(H)$, then $[H]_\tau = 0$. In particular, $H = \sum_{\tau\in\mathcal{E}(H)\cup E} [H]_\tau\, x^\theta$.*

*Proof.* — If $\tau \notin \mathcal{E}(H)$, then for each $\phi \in C$, $\tau(\phi) \notin \{\theta(\phi) \mid \theta \in \mathcal{E}(H)\}$ because $E \cup \mathcal{E}(H)$ is totally ordered. If $\tau \in \mathcal{E}$ then $[H]_\tau = c_\tau \in \mathcal{N}_{\mathbb{C}}(C)$, using the notation of (1) in Definition 5. □

***Corollary 3.*** — *Let $H, G \in \mathcal{N}_{\mathbb{C}}(C)[\log x]$ be such that $H, G$ and $HG$ are invariant on $C$ with respect to $E$. Let $\tau \in E$ be such that $\tau - \theta \in E$ for all $\theta \in \mathcal{E}(G)$. Then*

$$[H\,G]_\tau = \sum_{\theta\in\mathcal{E}(G)} [H]_{\tau-\theta}\, [G]_\theta\, .$$

***Corollary 4.*** — *Let $H^1, \dots, H^t \in \mathcal{N}_{\mathbb{X}}(C)[\log x]$ be invariant on $C$ with respect to $E$. Assume that $\sum_{j=1}^{s} H^j$, is also invariant on $C$ with respect to $E$. Let $\tau \in E$, then $\left[\sum_{j=1}^{s} H^j\right]_\tau = \sum_{j=1}^{s} \left[H^j\right]_\tau$. If $\frac{\partial}{\partial r_{i_j}}$ be a vector field on $C$, then $\frac{\partial}{\partial r_{i_j}} \left[\sum_{j=1}^{s} H^j\right]_\tau = \sum_{j=1}^{s} \left[\frac{\partial}{\partial r_{i_j}} H^j\right]_\tau$.*

The following result is a consequence of the fact that *invariance* is a *semi-algebraic* property.

***Lemma 6***. — *Let $H^1, \dots, H^t \in \mathcal{N}_{\mathbb{X}}(C)[\log x]$, $E$ a finite subset of $\mathcal{N}(C)$ and $\mathcal{P}$ a finite family of polynomials in $\mathbb{R}[r_1, \dots, r_n]$. There exists a $\mathcal{C}^\infty$-cylindrical algebraic decomposition $\mathcal{C}$ of $\mathbb{R}^n$ adapted to $\mathcal{P}$ such that $C$ is a finite union of cells of $\mathcal{C}$ and for any cell $C' \in \mathcal{C}$ and any $s$, $1 \le s \le t$, $H^s$ is invariant on $C'$ with respect to $E|_{C'} = \{f|_{C'} \mid f \in E\}$.*

**4.2. The Parametric Newton polygon.** — A *parametric differential polynomial* is an element $H(y) \in \mathcal{N}_{\mathbb{X}}(C)[y_0 \dots, y_n]$. Let $d$ be the total degree in the indeterminates $y_0, \dots, y_n$ of $H(y)$. Write uniquely

$$H(y) = \sum_{|\rho| \le d} H_\rho \, y_0^{\rho_0} \cdots y_n^{\rho_n}, \quad H_\rho \in \mathcal{N}_{\mathbb{X}}(C).$$

We proceed to the definition of the parametric Newton polygon (and its data) of $H(y)$ on $C$. This notion requires several properties on the coefficients of $H(y)$, expressed technically in Definition 8. The first one (condition (a)) is *invariance* on a cell, which lets us speak of *monomials* of $H(y)$ and their coefficients (no monomial disappears or appears inside a cell). In (b) we require that for each height (each ordinate) one can define the leftmost point at that height of the cloud of $H(y)$. We follow the usual algorithm to compute the positive convex hull: starting from the top-leftmost point, which will be the first vertex, we determine inductively the following ones. This is possible, for example (and this is what we impose in (c)) if the "slopes" appearing in the polygon are totally ordered in C:

***Definition** 7*. — *A parametric differential polynomial $H(y)$ is* invariant on $C$ with respect to a finite subset $E \subseteq \mathcal{N}(C)$ *if the following conditions hold:*

(a) *For all $\rho \in \mathbb{R}$, $H_\rho$ is invariant on $C$ with respect to $E$.*
(b) *Let $h$ be an integer, $0 \le h \le d$, and set*

$$\mathcal{E}_h(H) = \bigcup_{|\rho| = h} \mathcal{E}(H_\rho).$$

*Then $E \cup \mathcal{E}_h(H)$ is totally ordered for all $h$, $0 \le h \le d$.*

(c) *For $\mathcal{E}_h(H) \ne \varnothing$, let $\theta_h = \min \mathcal{E}_h(H)$. Then the union of $E$ with the set of functions on $C$ given by*

$$\frac{\theta_{h_2} - \theta_{h_1}}{h_1 - h_2}, \quad \text{for } 0 \le h_2 < h_1 \le d, \quad \mathcal{E}_{h_1}(H) \ne \varnothing \ne \mathcal{E}_{h_2}(H),$$

*is totally ordered.*

*We say that $H(y) \in \mathcal{N}_{\mathbb{X}}(C)[\log x][y_0 \dots, y_n]$ is* invariant on $C$ with respect to $E$ *if, writing $H(y) = \sum_{j=0}^{s} H_j(y) \log^j x$, then each $H_j(y)$ is invariant on $C$ with respect to $E$. $H(y)$ is just* invariant *if it is invariant with respect to the empty set.*

We proceed to "build up" the Newton polygon. Assume $H(y) \in \mathscr{N}_{\mathbb{X}}(C)[y_0 \dots, y_n]$ is invariant on $C$. The *parametric Newton polygon of $H(y)$ with respect to $C$* is just a sequence $V_0, V_1, \dots, V_t$ of vertices, each being a pair $(\theta, h)$ where $h$ is an integer, $0 \le h \le d$ and $\theta$ belongs to $\mathscr{E}_h(C)$. These vertices are defined inductively:

***Definition 8.*** — *Let $V_0 = (\theta_d, d)$, where $d$ is the total degree of $H(y)$ on $y_0, \dots, y_n$. Assume that vertex $V_i = (\theta_{h_i}, h_i)$ has been defined. If $\bigcup_{h<h_i} \mathscr{E}_h(H)$ is empty, then we have finished, Otherwise, set $V_{i+1} = (\theta_{h_{i+1}}, h_{i+1})$, where $h_{i+1}$ is the minimum of those $h < h_i$ such that*

$$\frac{\theta_h - \theta_{h_i}}{h_i - h} = \min\{\frac{\theta_{h'} - \theta_{h_i}}{h_i - h'} \mid h' < h_i,\ \mathscr{E}_{h'}(C) \neq \varnothing\}.$$

*The* parametric Newton polygon *of $H(y)$ with respect to $C$ as the sequence $V_0, \dots, V_i$. The* sides *are the sets $[V_i, V_{i+1}]$ for $i = 0, \dots, t-1$:*

$$[V_i, V_{i+1}] = \{V_i\} \cup \{(\theta_h, h) \mid h_i > h \ge h_{i+1},\ \frac{\theta_h - \theta_{h_i}}{h_i - h} = \frac{\theta_{h_{i+1}} - \theta_{h_i}}{h_i - h_{i+1}}\}.$$

*The* inclination *of side $[V_i, V_{i+1}]$ is $\mu_{[V_i, V_{i+1}]} = \frac{\theta_{h_{i+1}} - \theta_{h_i}}{h_i - h_{i+1}} \in \mathscr{N}(C)$.*

One can write uniquely

$$H(y) = \sum_{|\rho| \le d} \sum_{\theta \in \mathscr{E}_{|\rho|}(H)} H_{\rho,\theta}\, x^\theta y_0^{\rho_0} \dots y_n^{\rho_n}, \quad H_{\rho,\theta} \in \mathscr{N}_{\mathbb{C}}(C).$$

Given a vertex $V = (\theta_h, h)$ and a side $[V_i, V_{i+1}]$, define the *indicial* and *characteristic* polynomials as follows (respectively):

$$\begin{aligned}\Psi_{(H;V)}(\mathfrak{m}) &= \sum_{|\rho|=h} H_{\rho,\theta_h}\ \mathfrak{m}^{w(\rho)} \in \mathscr{N}_{\mathbb{C}}[\mathfrak{m}],\\ \Phi_{(H;[V_i,V_{i+1}])}(\mathfrak{c}) &= \sum_{(\theta,|\rho|) \in [V_i, V_{i+1}]} H_{\rho,\theta}\ \mu_{[V_i,V_{i+1}]}^{w(\rho)}\ \mathfrak{c}^{|\rho|} \in \mathscr{N}_{\mathbb{C}}[\mathfrak{c}].\end{aligned}$$

***Definition 9.*** — *The* parametric Newton polygon data of $H(y)$ with respect to $C$ *is the family of vertices $V_0, \dots, V_t$, sides $[V_i, V_{i+1}]$, $0 \le i < t$, and polynomials $\Psi_{(H;V_i)}(\mathfrak{m})$, $0 \le i \le t$ and $\Phi_{(H;[V_i,V_{i+1}])}(\mathfrak{c})$, $0 \le i \le t-1$.*

***Lemma 7.*** — *Let $H(y)$ be invariant on $C$ with respect to $E$ and $C' \subseteq$ be a semi-algebraic $\mathscr{C}^\infty$-submanifold. Then $H(y)$ is invariant on $C'$ with respect to $E$ and the parametric Newton polygon data of $H(y)$ with respect to $C'$ is the natural restriction of the parametric Newton polygon data of $H(y)$ with respect to $C$. In particular, if $V = (\theta, h)$ is a vertex with respect to $C$, then $V|_{C'} = (\theta|_{C'}, h)$ is a vertex with respect to $C'$.*

*Proof.* — If $E' \subseteq \mathscr{N}(C)$ is totally ordered with respect to $<_C$, then $E'|_{C'} = \{\tau|_{C'} \mid \tau \in E'\}$ is totally ordered with respect to $<_{C'}$. The minimum of $E'|_{C'}$ is the restriction to $C'$ of the minimum of $E$. This implies that if $G \in \mathscr{N}_{\mathbb{X}}(C)$ is invariant with respect to $E$, then $G|_{C'} \in \mathscr{N}_{\mathbb{X}}(C')$ is invariant with respect to $E|_{C'}$ and $\mathscr{E}(G) = \mathscr{E}(G|_{C'})$.

Condition (a) of Definition 7 holds for $H|_{C'}$. Moreover, $\mathcal{E}_h(H) = \mathcal{E}_h(H|_{C'})$ so that also conditions (b) and (c) are satisfied. As one can write

$$H(y) = \sum_{|\rho|\le d} \sum_{\theta\in\mathcal{E}_{|\rho|}(H)} H_{\rho,\theta}|_{C'}\, x^{\theta|_{C'}} y_0^{\rho_0} \dots y_n^{\rho_n}, \quad H_{\rho,\theta} \in \mathcal{N}_{\mathbb{C}}(C'),$$

then $\Psi_{(H|_{C'};V_i)}(\mathfrak{m})$ and $\Phi_{(H|_{C'};[V_i|_{C'},V_{i+1}|_{C'}])}(\mathfrak{c})$ are (respectively) the restrictions to $C'$ of the polynomials $\Psi_{(H;V_i)}(\mathfrak{m})$ and $\Phi_{(H;[V_i,V_{i+1}])}(\mathfrak{c})$. □

***Remark 5***. — *The above lemma holds for $C'$ a single point. Namely, for any $\phi \in C$, denote*

$$H_\phi(y) = \sum_{\rho,\theta} H_{\rho,\theta}(\phi)\, x^{\theta(\phi)}\, y^\rho \in \mathbb{C}[x^{\mathbb{R}}][y_0,\dots,y_n].$$

*The vertices of the Newton polygon of $H_\phi(y)$ are precisely the points $V_0(\phi),\dots,V_t(\phi)$, where $V_i(\phi) = (\theta_{h_i}(\phi), h_i)$. Moreover, the (differential) monomials of $H_\phi(y)$ whose corresponding points belong to the side $[V_i(\phi), V_{i+1}(\phi)]$ are precisely the monomials $H_{\rho,\theta}(\phi)\, x^{\theta(\phi)}\, y^\rho$, where $(\theta,|\rho|) \in [V_i, V_{i+1}]$. Hence,*

$$\Psi_{(H;V)}(\phi,\mathfrak{m}) = \Psi_{(H_\phi(y);V(\phi))}(\mathfrak{m}), \tag{10}$$

$$\Phi_{(H;[V_i,V_{i+1}])}(\phi,\mathfrak{c}) = \Phi_{(H_\phi(y);\mu_{[V_i,V_{i+1}]}(\phi))}(\mathfrak{c}). \tag{11}$$

From the semialgebraic nature of the properties required in Definition 7, one infers

***Lemma 8***. — *Let $H^1(y),\dots,H^t(y) \in \mathcal{N}_{\mathbb{X}}(C)[\log x][y_0\dots,y_n]$, let $E \subseteq \mathcal{N}(C)$ be a finite subset, and $\mathcal{P}$ a finite set of polynomials. There exists a $\mathcal{C}^\infty$-cylindrical decomposition $\mathcal{C}$ adapted to $\mathcal{P}$ such that $C$ is a finite union of cells of $\mathcal{C}$ and for each cell $C' \in \mathcal{C}$ with $C' \subseteq C$, and for each $j$, $H^j(Y)$ is invariant on $C'$ with respect to $E$.*

## 5. Proof of the Main Theorem

We start with the differential equation $F(y) = 0$, where

$$F(y) = \sum_{a\in S} \sum_{|\rho|\le d} A_{a,\rho}\, x^a\, y_0^{\rho_0} \cdots y_n^{\rho_n} \in \mathbb{C}[x^{\mathbb{R}}][y_0,\dots,y_n],$$

where $S$ is finite subset of $\mathbb{R}$. Rewrite it in the following way

$$F(y) = \sum_{a\in S} f_a(y)\, x^a, \quad \text{where } f_a(y) \in \mathbb{C}[y_0,\dots,y_n].$$

Before proceeding, we need to provide some notation. Then we shall prove in Lemma 9 that $NIC_k^\star(F)$ is semi-algebraic and state and prove Proposition 1, which is the cornerstone of the present paper, from which Theorem 1 will follow.

Let $(\mu_1, c_1,\dots,\mu_k, c_k)$ denote the coordinate functions on $(\mathbb{R}\times\mathbb{C})^k$ and $(r_1,\dots,r_{3k})$ those on $\mathbb{R}^{3k}$ with the identification given in (5). Define

$$\begin{aligned} \varphi = \varphi_0 &= c_1\, x^{\mu_1} + \dots + c_k\, x^{\mu_k} \in \mathcal{N}_{\mathbb{X}}(\mathbb{R}^{3k}), \\ \varphi_s &= \mu_1^s\, c_1\, x^{\mu_1} + \dots + \mu_k^s\, c_k\, x^{\mu_k} \in \mathcal{N}_{\mathbb{X}}(\mathbb{R}^{3k}), \end{aligned}$$

for any non-negative integer $s$, and let

$$\mathcal{F}(y) = F(y_0 + \varphi_0, \dots, y_n + \varphi_n) \in \mathcal{N}_{\mathbb{X}}(\mathbb{R}^{3k})[y_0, \dots, y_n].$$

For any $\phi \in (\mathbb{R} \times \mathbb{C})^k$, $\mathcal{F}_\phi(y) \in \mathbb{C}[x^{\mathbb{R}}][y_0, \dots, y_n]$ denotes the value of $\mathcal{F}(y)$ at $\phi$. One has

$$\mathcal{F}_\phi(y) = F(y_0 + \varphi_0(\phi), \dots, y_n + \varphi_n(\phi)) = F(y + \varphi(\phi))$$

because $\varphi_s(\phi) = \partial_s(\varphi(\phi))$ for $s \in \mathbb{N}$.

***Lemma 9***. — *The set* $\mathrm{NIC}_k^*(F) \subseteq (\mathbb{R} \times \mathbb{C}^*)^k$ *is semi-algebraic for all* $k \geq 1$.

*Proof*. — We proceed by induction on $k$, the case $k = 1$ having already been proved in Corollary 2.

Assume that $\mathrm{NIC}_k^*(F)$ is a semi-algebraic subset of $(\mathbb{R} \times \mathbb{C})^k$. From Lemma 8, there exists a $\mathcal{C}^\infty$-cylindrical algebraic decomposition $\mathcal{C}$ of $(\mathbb{R} \times \mathbb{C})^k$ such that $\mathcal{F}(y)$ in invariant on each cell of $\mathcal{C}$ and $\mathrm{NIC}_k^*(F)$ is the union of some of these cells.

Let $\phi \in (\mathbb{R} \times \mathbb{C})^k$ and $\phi' = (\phi, m_{k+1}, b_{k+1}) \in (\mathbb{R} \times \mathbb{C})^{k+1}$. One sees easily that $\phi' \in \mathrm{NIC}_{k+1}^*(F)$ if and only if both $\phi \in \mathrm{NIC}_k^*(F)$ and $(m_{k+1}, b_{k+1}) \in \mathrm{NIC}_1^*(\mathcal{F}_\phi(y))$. Hence it is enough to prove that for any cell $C \in \mathcal{C}$ the set

$$A_C = \{(\phi, \mu, c) \mid \phi \in C,\ (\mu, c) \in \mathrm{NIC}_1^*(\mathcal{F}_\phi(y))\}$$

is semi-algebraic. If $C$ is contained in the complement of $\mathrm{NIC}_k^*(F)$, then $A_C = \varnothing$. Assume that $C \subseteq \mathrm{NIC}_k^*(F)$ and let $V_0, V_1, \dots, V_t$ be the vertices of the parametric Newton Polygon of $\mathcal{F}(y)$ with respect to $C$. By Remark 5 the vertices of the Newton polygon of $\mathcal{F}_\phi(y)$ are $V_0(\phi), \dots, V_t(\phi)$. From the proof of case $k = 1$ (Corollary 2) and equations (10) and (11), one infers that $A_C$ is the union of the semi-algebraic sets given by the following conditions:

$$\phi \in C,\ \mu = \mu_{[V_{i-1}, V_i]}(\phi),\ \text{ and } \Phi_{(\mathcal{F};[V_{i-1},V_i])}(\phi, c) = 0,$$

for $1 \leq i \leq t$, and

$$\phi \in C,\ \mu_{[V_{i-1}, V_i]}(\phi) < \mu < \mu_{[V_i, V_{i+1}]}(\phi),\ \text{ and } \Psi_{(\mathcal{F};V_i)}(\phi, \mu) = 0,$$

for $0 \leq i \leq t$, (where by definition $\mu_{[V_{-1}, V_0]}(\phi) = -\infty$ and $\mu_{[V_t, V_{t+1}]}(\phi) = \infty$). These conditions are semi-algebraic, so $A_C$ is semi-algebraic and so is $\mathrm{NIC}_{k+1}^*(F)$. □

Given a nonempty $\mathcal{C}^\infty$-differentiable semi-algebraic manifold $C \subseteq \mathrm{NIC}_k^*(F)$, let $E_0 = \{\mu_k\} \subseteq \mathcal{N}(C)$. Assume that $\mathcal{F}(y)$ is invariant on $C$ with respect to $E_0$, let $V_0, \dots, V_t$ be the vertices of the parametric Newton Polygon of $\mathcal{F}(y)$ on $C$ and let $(\theta_h, h) \in [V_i, V_{i+1}]$ for some $0 \leq i < t$.

***Definition 10***. — *With the above notation, we say that* $(\theta_h, h)$ *is in the* red part with respect to $\mu_k$ of the parametric Newton Polygon of $\mathcal{F}(y)$ on $C$ *if* $h \geq 1$ *and either the inclination* $\mu_{[V_i, V_{i+1}]} > \mu_k$ *or* $(\theta_h, h) = V_t$.

Notice that the definition makes sense because since $\mathcal{F}(y)$ is invariant on $C$ with respect to $E_0$, any inclination $\mu_{[V_i, V_{i+1}]}$ can be compared with the function $\mu_k$.

***Lemma 10.*** — *In the conditions of the above definition, the red part with respect to $\mu_k$ of the parametric Newton Polygon of $\mathcal{F}(y)$ on $C$ is nonempty.*

*Proof.* — Let $\phi \in C$. Since $\phi \in \mathrm{NIC}_k^*(\mathcal{F}_\phi(y))$, by Lemma 2, the red part with respect to $\mu_k(\phi)$ of the Newton polygon of $\mathcal{F}_\phi(y)$ is nonempty. The vertices of the Newton polygon of $\mathcal{F}_\phi(y)$ are $V_0(\phi), \dots, V_t(\phi)$. Hence, either there exists a side $[V_i(\phi), V_{i+1}(\phi)]$ with inclination $\mu_{[V_i,V_{i+1}]}(\phi)$ greater than $\mu_k(\phi)$, or $V_t(\phi)$ has ordinate greater than zero. If $\mu_k(\phi) < \mu_{[V_i,V_{i+1}]}(\phi)$, then $\mu_k <_C \mu_{[V_i,V_{i+1}]}$, because $\mu_k \in E_0$ and $V_i$ is in the red part. Otherwise, if $V_t(\phi) = (\theta_t(\phi), h_t)$ with $h_t \geq 1$, then $V_t$ is in the red part. $\square$

Let $\mathrm{NIC}_k^{*,>}(F)$ denote the following semi-algebraic set:

$$\mathrm{NIC}_k^{*,>}(F) = \mathrm{NIC}_k^*(F) \cap \{\mu_1 > 0\}$$

and let $\mathcal{C}$ be a $\mathcal{C}^\infty$-cylindrical algebraic decomposition of $(\mathbb{R} \times \mathbb{C})^k$ such that $\mathrm{NIC}_k^{*,>}(F)$ is the union of some cells of $\mathcal{C}$. $\mathrm{NIC}_k^{*,>}(F)$ is defined as semi-algebraic set by a finite family of polynomials $\mathcal{Q}$ and each cell $C_i \in \mathcal{C}$ by a finite family $\mathcal{P}_i$. Set $\mathcal{P} = \mathcal{Q} \bigcup \cup_i \mathcal{P}_i$. Fix a cell $C \in \mathcal{C}$ and let $I_C = (i_1, \dots, i_d)$ (so that $d = \dim(C)$). Denote $d_a = \dim_{\mathrm{a}}(C)$.

***Proposition 1.*** — *With the notation above, the adapted dimension of $C$ with respect to $\mathcal{C}$ is less than or equal to the order of $F$: $d_a \leq \operatorname{ord}(F)$.*

Before starting the proof, let us introduce some useful notation.

Given $\lambda \in \mathbb{N}^{n+1}$ and $f(y) \in \mathbb{C}[y_0, \dots, y_n]$, let

$$f^{(\lambda)}(\varphi) = \frac{\partial^{|\lambda|} f}{\partial^{\lambda_0} y_0 \dots \partial^{\lambda_n} y_n}(\varphi_0, \dots, \varphi_n) \in \mathcal{N}_{\mathbb{X}}(\mathbb{R}^{3k}).$$

By the Taylor expansion formula,

$$f_a(y_0 + \varphi_0, \dots, y_n + \varphi_n) = \sum_{|\lambda| \leq d} \frac{1}{\lambda!} f_a^{(\lambda)}(\varphi)\, y^\lambda \in \mathcal{N}_{\mathbb{X}}(\mathbb{R}^{3k})[y_0, \dots, y_n],$$

where $\lambda! = \lambda_0! \cdots \lambda_n!$ and $y^\lambda = y_0^{\lambda_0} \cdots y_n^{\lambda_n}$. Hence

$$\mathcal{F}(y) = \sum_{a \in S} \sum_{|\lambda| \leq d} \frac{1}{\lambda!} f_a^{(\lambda)}(\varphi)\, x^a\, y^\lambda.$$

$F_\lambda$ will denote the coefficient of $y^\lambda$ in $\mathcal{F}(y)$:

$$F_\lambda = \sum_{a \in S} \frac{1}{\lambda!} f_a^{(\lambda)}(\varphi)\, x^a \in \mathcal{N}_{\mathbb{X}}(\mathbb{R}^{3k}),$$

so that $\mathcal{F}(y) = \sum_{|\lambda| \leq d} F_\lambda\, y^\lambda$.

*Proof of Proposition 1.* — Let $E_0 = \{\mu_k\}$. By Lemma 8 there exists a $\mathcal{C}^\infty$-cylindrical algebraic decomposition $\mathcal{C}^1$ of $(\mathbb{R} \times \mathbb{C})^k$ adapted to $\mathcal{P}$ such that $\mathcal{F}(y)$ and $\frac{\partial}{\partial r_{i_j}} \mathcal{F}(y)$, for $1 \leq j \leq d$, are all invariant with respect to $E_0$ on any cell of $\mathcal{C}^1$ contained in $C$. In particular, $C$ is a finite union of cells of $\mathcal{C}^1$. By Lemma 3, there exists a cell $C_1$ of $\mathcal{C}^1$

such that $\dim_{\mathrm{a}} C_1 = d_a$ and $I_{C_1} = I_{C_0}$. Let $\mathscr{P}_1$ be a family of polynomials defining $C_1$ as semi-algebraic set.

Let $V_0, \dots, V_l$ be the vertices of the parametric Newton polygon of $\mathscr{F}(y)$ on $C_1$ (in decreasing order or height: $h_0 > h_1 > \cdots > h_l$). Let $\theta_h$ be as in Definition 8, so that $V_s = (\theta_{h_s}, h_s)$, $0 \leq s \leq l$. Given a side $[V_s, V_{s+1}]$ of the Polygon and a height $h \in \mathbb{N}$ with $h_s \geq h \geq h_{s+1}$, we denote by $\tau_h$ the following value (see Figure 2): $\tau_h = \theta_{h_s} + (h_s - h)\mu_{[V_s, V_{s+1}]} \in \mathscr{N}(C_1)$. Notice that $\tau_{h_j} = \theta_{h_j}$ for $0 \leq j \leq l$. Given $h$ with $\mathscr{E}_h(\mathscr{F}) \neq \varnothing$, if $(\theta_h, h) \in [V_s, V_{s+1}]$, then $\tau_h = \theta_h$, otherwise $\tau_h < \theta_h$.

We shall later need to take coefficients with respect to the functions $\mu_s$ and $\tau_h - \mu_s$, and compare $\tau_h + \mu_s$ with $\tau_{h-1}$. For simplicity, let $E_1$ denote the subset of $\mathscr{N}(C_1)$ composed of $\tau_h$, $\mu_s$, $\tau_h - \mu_s$, and $\tau_h + \mu_s$ for $h_0 \geq h \geq h_l$ and $1 \leq s \leq k$.

Let $\mathscr{H} \subset \mathscr{N}_{\mathbb{X}}(C_1)$ be the set composed of the following functions:

$$F_\lambda,\ \frac{\partial F_\lambda}{\partial r_{i_j}},\ f_a^{(\lambda')}(\varphi)x^a,\ \frac{\partial}{\partial r_{i_j}} f_a^{(\lambda)}(\varphi)\, x^a,\ f_a^{(\lambda)}(\varphi)\, x^a\, \frac{\partial \varphi_s}{\partial r_{i_j}},\ \frac{\partial \varphi_j}{\partial r_i},$$

for all $|\lambda| \leq d$, $1 \leq j \leq d$ and $a \in S$. Fix a $\mathscr{C}^\infty$-cylindrical algebraic decomposition $\mathscr{C}^2$ adapted to $\mathscr{P}_1$ such that any element of $\mathscr{H}$ is invariant with respect to $E_1$ on any cell of $\mathscr{C}^2$. As above, $C_1$ is a finite union of cells of $\mathscr{C}^2$ and we may choose a cell $C_2$ of $\mathscr{C}^2$ such that $C_2 \subseteq C_1$, $\dim_{\mathrm{a}} C_2 = d_a$ and $I_{C_2} = I_{C_0}$.

By Lemma 7, $\mathscr{F}(y)$ and $\frac{\partial \mathscr{F}(y)}{\partial r_{i_j}}$ are invariant on $C_2$ with respect to $E_0$ and the parametric Newton Polygon of $\mathscr{F}(y)$ on $C_2$ has vertices $V_0|_{C_2}, \dots, V_t|_{C_2}$. Therefore, we may write uniquely

$$\mathscr{F}(y) = \sum_{|\lambda| \leq d} \ \sum_{\theta \in E_1 \cup \mathscr{E}_{|\lambda|}(\mathscr{F})} F_{\theta,\lambda}\, x^\theta\, y^\lambda, \quad F_{\theta,\lambda} \in \mathscr{N}_{\mathbb{C}}(C).$$

Let $(\theta_h, h) \in [V_s, V_{s+1}]$ be in the red part with respect to $\mu_k$ of the parametric Newton polygon of $\mathscr{F}(y)$ on $C_2$ (recall that $h \geq 1$).

Take $i \in \{i_1, \dots, i_d\}$ and let $t$ the minimum integer greater than or equal to $i/3$, so that the corresponding $r_i$ in (5) is $\mu_i$, $\Re(c_i)$ or $\Im(c_i)$ (real and imaginary parts). Let $\lambda' \in \mathbb{N}^{n+1}$ such that $|\lambda'| = h - 1$. Fix $t \in \{1 \dots, k\}$ and let $\tau = \theta_h + \mu_t \in E_1|_{C_2}$.

We claim that $F_{\tau,\lambda'} = 0$: if $(\theta_h, h) = V_l$ then $\mathscr{E}_{h-1} = \varnothing$ and by Lemma 5 $[F_{\lambda'}]_\tau = 0$; if $(\theta_h, h) \in [V_s, V_{s+1}]$ then $\mu_t < \mu_{[V_s, V_{s+1}]}$, so $\tau < \theta_{h-1}$ and $\tau \notin \mathscr{E}_{h-1}(\mathscr{F})$ and again by Lemma 5 $[F_{\lambda'}]_\tau = 0$. Therefore $0 = \frac{\partial F_{\tau,\lambda'}}{\partial r_i}$.

On the other hand, by direct computation

$$\begin{aligned}
\frac{\partial F_{\tau,\lambda'}}{\partial r_i} &= \frac{\partial}{\partial r_i}\left[\sum_a \frac{1}{\lambda'!} f_a^{(\lambda')}(\varphi)x^a\right]_\tau \overset{(a)}{=} \sum_a \frac{1}{\lambda'!}\left[\frac{\partial}{\partial r_i} f_a^{(\lambda')}(\varphi)x^a\right]_\tau \\
&= \sum_a \left[\frac{1}{\lambda'!}\sum_{j=0}^n f_a^{(\lambda'+e_j)}(\varphi)\frac{\partial \varphi_j}{\partial r_i}x^a\right]_\tau \\
&\overset{(b)}{=} \sum_{j=0}^n (\lambda'_j + 1)\left[\sum_a \frac{1}{(\lambda'+e_j)!} f_a^{(\lambda'+e_j)}(\varphi)\, x^a \frac{\partial \varphi_j}{\partial r_i}\right]_\tau
\end{aligned} \tag{12}$$

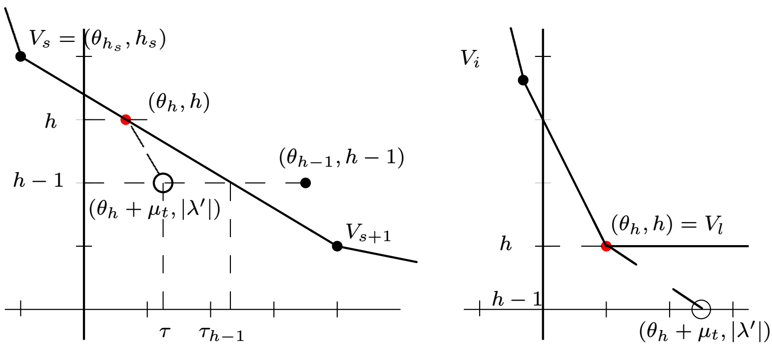


FIGURE 3. Two possibilities for a point in the red part.

$$\stackrel{(c)}{=} \sum_{j=0}^{n} (\lambda'_j + 1) \sum_{\theta \in \mathscr{E}(\frac{\partial \varphi_j}{\partial r_i})} \left[ \sum_a \frac{1}{(\lambda' + e_j)!} f_a^{(\lambda'+e_j)}(\varphi)\, x^a \right]_{\tau-\theta} \left[ \frac{\partial \varphi_j}{\partial r_i} \right]_\theta$$

$$\stackrel{(d)}{=} \sum_{j=0}^{n} (\lambda'_j + 1) \sum_{\theta \in \{\mu_1, \dots, \mu_k\}} F_{\tau-\theta, \lambda'+e_j} \left[ \frac{\partial \varphi_j}{\partial r_i} \right]_\theta$$

Where $e_j$ is the element of $\mathbb{N}^{j+1}$ $(0, \dots, 0, 1, 0, \dots, 0)$ where the 1 appears in the $j-$th place *counting from 0.*

Equality $(a)$ is a consequence of Corollary 4 and the fact that all members of $\mathscr{H}$ are invariant on $C_2$ with respect to $E_1$ (cf. Lemma 7). For $(b)$, we just rewrite $\frac{1}{\lambda!} = \frac{\lambda_j+1}{(\lambda+e_j)!}$. Equality $(d)$ follows from the definition of $F_{\lambda'+e_j}$ and the fact that it is invariant in $C_2$ with respect to $E_1$. Finally, in order to get $(c)$, we use Corollary 3 together with the inclusion $\mathscr{E}(\frac{\partial \varphi_j}{\partial r_j}) \subseteq \{\mu_1, \dots, \mu_k\}$ which is proved as follows:

From relations (5) and equation (9) one gets

$$\frac{\partial}{\partial r_i}(\mu_j^s\, c_j\, x^{\mu_j}) = (s\mu_j^{s-1} \frac{\partial \mu_j}{\partial r_i}\, c_j + \mu_j^s \frac{\partial c_j}{\partial r_i} + \mu_j^s\, c_j\, \frac{\partial \mu_j}{\partial r_i} \log x)\, x^{\mu_j}, \quad 1 \le j \le k.$$

and since $\varphi_s = \sum_{j=1}^k \mu_j^s\, c_j\, x^{\mu_j} \in \mathscr{N}_{\mathbb{X}}(C_2)$, then $\mathscr{E}(\frac{\partial \varphi_s}{\partial r_i}) \subseteq \{\mu_1, \dots, \mu_s\}$.

From equation (4), $\frac{\partial}{\partial r_i}(r_j) = 0$ if $j < i$ and $\frac{\partial}{\partial r_i}(r_i) = 1$, so that

$$\begin{aligned}
&\frac{\partial}{\partial r_i}(\mu_j^s\, c_j\, x^{\mu_j}) = 0, \quad \text{for } j < t, \\
&\frac{\partial}{\partial r_i}(\mu_t^s\, c_t\, x^{\mu_t}) = (s\mu_t^{s-1}\, c_t + \mu_t^s \frac{\partial c_t}{\partial r_i} + \mu_t^s\, c_t\, \log x)\, x^{\mu_t}, (i = 1 \bmod 3), \\
&\frac{\partial}{\partial r_i}(\mu_t^s\, c_t\, x^{\mu_t}) = \mu_t^s \frac{\partial c_t}{\partial r_i}\, x^{\mu_t}, \quad (i = 2, 3 \mod 3).
\end{aligned} \tag{13}$$

Therefore one may write

$$\frac{\partial \varphi_j}{\partial r_i} = \frac{\partial}{\partial r_i}(\mu_t^j\, c_t\, x^{\mu_t}) + \sum_{s=t+1}^{k} \left[\frac{\partial \varphi_j}{\partial r_i}\right]_{\mu_s}.$$

For $s > t$, $\tau - \mu_s < \theta_h$, so that $F_{\tau-\mu_s,\lambda'+e_j} = 0$ and the last member of equation (12) is

$$\begin{aligned} 0 &= \sum_{j=0}^{n}(\lambda'_j + 1) \sum_{\theta\in\{\mu_1,\dots,\mu_k\}} F_{\tau-\theta,\lambda'+e_j} \left[\frac{\partial \varphi_j}{\partial r_i}\right]_{\theta} = \\ &= \sum_{j=0}^{n}(\lambda'_j + 1)\ F_{\theta_h,\lambda'+e_j}\ \frac{\partial}{\partial r_i}(\mu_t^j\, c_t\, x^{\mu_t}). \end{aligned}$$

If $i = 1 \mod 3$, from (13) and the fact that $c_t$ does not vanish in $C_2 \subseteq \mathrm{NIC}^*_k(F)$, one infers the following two linear equations

$$0 = \sum_{j=0}^{n}(\lambda'_j + 1)\, \mu_t^j\, F_{\theta_h,\lambda'+e_j}, \tag{14}$$

$$0 = \sum_{j=0}^{n}(\lambda'_j + 1)\, j\, \mu_t^{j-1}\, F_{\theta_h,\lambda'+e_j}. \tag{15}$$

(If $i = 2$ or $0 \mod 3$ then only (14) appears). Letting $i$ run over $\{i_1, \dots, i_d\}$, one obtains a linear system of equations in the variables $F_{\theta_h,\lambda'+e_0}, \dots, F_{\theta_h,\lambda'+e_n}$. An elementary argument of linear algebra shows that it has rank $d_a = \dim_{\mathrm{a}}(C_2)$. Hence, if $d_a > n$ its only solution is the trivial one, which implies that the red part of the parametric Newton polygon of $\mathcal{F}(y)$ on $C_2$ is empty, contradicting Lemma 10. $\square$

*Proof of Theorem 1.* Proceed by contradiction. Assume that there exists a cell $C$ in $\mathrm{NIC}^*_k(F)$ with adapted dimension greater than $n$ and let $\phi \in C$, $m = \mu_1(\phi) - 1$ and consider the differential polynomial $G(y) = F(x^m y)$. The set $\mathrm{NIC}^*_k(G)$ is the image of $\mathrm{NIC}^*_k(F)$ under the translation $T(x, y) = (x - m, y)$ so that $T(C) \cap \mathrm{NIC}^{*>}_k(G)$ is nonempty. Since translations preserve the adapted dimension of cells, there must exist a cell in $\mathrm{NIC}^{*>}_k(G)$ with adapted dimension greater than the order of $G$, which is equal to the order of $F$, against Proposition 1. $\square$

## References


[1] C. Briot & J. Bouquet – “Propriétés des fonctions définies par des équations différentielles”, *Journal de l'École polytechnique* **36** (1856), 133–198.

[2] J. Cano – “On the series defined by differential equations, with an extension of the Puiseux polygon construction to these equations”, *Analysis* **13** (1993), p. 103–119.

[3] M. Coste – *An introduction to semialgebraic geometry*, Istituti Editoriali e Poligrafici Internazionali, Pisa, 2000.

[4] H. B. Fine – “On the Functions Defined by Differential Equations, with an Extension of the Puiseux Polygon Construction to these Equations”, *Amer. J. Math.* **11** (1889), p. 317–328.

[5] D. Y. Grigor′ev & M. F. Singer – "Solving ordinary differential equations in terms of series with real exponents", *Trans. Amer. Math. Soc.* **327** (1991), p. 329–351.

[6] J. v. d. Hoeven – "Asymptotique automatique", Thèse de doctorat, Université Paris 7, 1997.

[7] ———, *Transseries and real differential algebra*, Lecture Notes in Math., vol. 1888, Springer, 2006.

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# UNE PREUVE GALOISIENNE DE L'IRRÉDUCTIBILITÉ AU SENS DE NISHIOKA-UMEMURA DE LA PREMIÈRE ÉQUATION DE PAINLEVÉ

*par*

Guy Casale

*À José Manuel Aroca pour son soixantième anniversaire*

***Résumé.*** — Cet article fait suite à un précédent. Nous utilisons le groupoïde de Galois calculé dans *loc. cit.* pour prouver que la première équation de Painlevé est irréductible au sens de Painlevé-Nishioka-Umemura. Pour cela nous prouvons que l'algèbre de Lie du groupoïde de Galois d'une équation réductible admet une suite croissante d'idéaux dont le premier est composé des champs tangents au feuilletage (donné par l'équation), le dernier est l'algèbre de Lie du groupoïde de Galois et les quotients de deux idéaux successifs sont de type linéaire. Ce n'est pas le cas pour $P_1$.

***Abstract*** **(Galoisian proof of Nishioka-Umemura irreducibility of first Painlevé equation)**
This article follows a previous one. The Galois groupoid computed in *loc. cit.* is used to prove irreducibility in Painlevé-Nishioka-Umemura sense of the first Painlevé equation. We prove that the Lie algebra of the Galois groupoid of a reducible equation gets an increasing sequence of ideals such that: the first is the algebra of vector fields tangent to the foliation given by the equation, the last is the Lie algebra of the Galois groupoid, the quotient of two successive ideals is a Lie Algebra with linear type. This is not the case for $P_1$.

La question de l'irréductibilité d'une équation différentielle a été étudiée de manière approfondie par P. Painlevé depuis les *Leçons de Stokholm* [**19**]. Une première définition d'équation différentielle ordinaire réductible a été donnée par P. Painlevé et sera ensuite formalisée par K. Nishioka [**16**]. Une équation d'ordre $n$ est dite réductible si on peut exprimer une solution rationnellement après avoir résolu successivement des équations différentielles linéaires, abéliennes (dont les solutions sont des fonctions abéliennes) ou d'ordre strictement plus petit que $n$.

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***Mots clefs.*** — Équations différentielles, irréductibilité, groupoïde de Galois.

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Après avoir étudié l'irréductibilité des équations du premier ordre, Painlevé se pose la question de l'irréductibilité de la première des équations d'ordre deux sans singularité mobile qu'il a découvertes :

$$(P_1) \qquad \frac{d^2y}{dx^2} = 6y^2 + x.$$

Il prouve dans [**17**] qu'au moins une solution de cette équation est irréductible puis affirme dans [**18**] avoir déterminé le « groupe de rationalité de J. Drach » de cette équation et prouver ainsi son irréductibilité « absolue ». Le groupe de rationalité utilisé par P. Painlevé provient d'une tentative de J. Drach de mettre en place une théorie « de la rationalité » (ou « de Galois ») valide pour toute équation différentielle [**4**]. Malheureusement, les travaux de J. Drach sont entachés d'erreurs.

À la fin des années soixante-dix, l'école japonaise reprend et continue les travaux de Painlevé sur les équations sans singularité mobile. Une preuve de l'irréductibilité de $P_1$ est enfin obtenue par K. Nishioka [**16**] puis par H. Umemura [**23, 22**]. Récemment l'étude géométrique des variétés de conditions initiales des équations sans singularité mobile [**21**] a permis à M.-H. Saito et H. Terajima d'obtenir une autre preuve de l'irréductibilité de cette équation. Aucune de ces preuves n'utilise une « théorie de Galois générale ».

Ce type de théorie a été mis en place indépendamment par H. Umemura [**24, 25**] et B. Malgrange [**15**] à la fin du vingtième siècle, achevant les travaux de J. Drach et E. Vessiot ([**26, 27**]). Dans cet article nous présentons une nouvelle preuve de l'irréductibilité de la première équation de Painlevé utilisant le groupoïde de Galois de cette équation [**3, 15**]. Ce dernier a été calculé dans [**3**] en complétant les calculs de J. Drach [**5**]. La détermination du groupoïde de Galois permet de montrer différents types de résultats concernant la réductibilité ou l'irréductibilité d'une équation. L'irréductibilité au sens de Drach-Vessiot ainsi que l'irréductibilité au sens des feuilletages de $P_1$ ont été prouvées dans [**3**]. Nous expliquons ici les relations entre le groupoïde de Galois d'une équation et son irréductibilité au sens de Painlevé-Nishioka-Umemura dans le cas particulier de la première équation de Painlevé.

Cet article est constitué de six parties. Dans la première, nous rappelons les définitions d'irréductibilité et de modèles pour des corps différentiels. Nous étudions ensuite rapidement la géométrie transverse des feuilletages donnés par les types d'extensions utilisés pour « réduire » une équation. Dans la troisième partie nous rappelons la définition du groupoïde de Galois d'un feuilletage suivant B. Malgrange [**15**] et présentons ensuite quelques lemmes élémentaires dans une quatrième partie. Après avoir fait les rappels nécessaires sur les algèbres de Lie de champs de vecteurs formels, nous prouvons l'irréductibilité au sens de Nishioka-Umemura de la première équation de Painlevé.

Je remercie B. Malgrange dont les remarques ont permis d'améliorer grandement ce texte.

## 1. Définitions et modèles géométriques

Pour les notions de géométrie algébrique utilisées dans cet article, nous renvoyons le lecteur à **[9]** et pour celles d'algèbre différentielle à **[20]**. Commençons par rappeler la définition de réductibilité d'une équation différentielle du second ordre suivant Nishioka-Umemura **[16, 19, 22]**. Les corps différentiels $(K,\delta)$ seront toujours des corps de type fini sur $\mathbb{C}$ et auront pour corps de constantes $K^\delta = \mathbb{C}$.

***Définition 1.1* (réductibilité [16, 22]).** — *Soit $(K,\delta)$ un corps différentiel ordinaire et $E : \delta^2 y = F(y,\delta y) \in K(y,\delta y)$ une* équation différentielle du second ordre *sur $K$. L'équation $E$ est dite* réductible *s'il existe une solution dans une extension différentielle $L$ de $K$ construite de la manière suivante :*

$$K = K_0 \subset K_1 \cdots \subset K_m = L$$

*avec pour tout $i$,*

- *soit $K_i \subset K_{i+1}$ est une* extension algébrique*,*
- *soit $K_i \subset K_{i+1}$ est une* extension de Picard-Vessiot*, c'est-à-dire $K_{i+1} = K_i(f_j^p; 1 \leq p, j \leq n)$ avec $\delta f_j^p = \sum_k A_j^k f_k^p$, $A_j^k \in K_i$ et $K_i^\delta = K_{i+1}^\delta$.*
- *soit $K_i \subset K_{i+1}$ est une* extension abélienne*, c'est-à-dire $K_{i+1} = K_i(\varphi_j(a_1,\ldots,a_n); 1 \leq j \leq n)$ les $\varphi_j$ formant une base de transcendance du corps des fonctions d'une variété abélienne définie sur $\mathbb{C}$, les $a_j$ appartenant à $K_i$ et $K_i^\delta = K_{i+1}^\delta$,*
- *soit $K_i \subset K_{i+1}$ est une* extension d'ordre un*, c'est-à-dire $K_{i+1} = K_i(z)$ avec $P(z,\delta z) = 0$, $P \in K_i(z,\delta z)$ et $K_i^\delta = K_{i+1}^\delta$.*

*Une extension différentielle $K \subset L$ du type précédent sera dite* réductrice*. Les extensions intermédiaires $(K_i \subset K_{i+1})$ décrites ci-dessus seront dites* élémentaires.

Dans la suite le corps différentiel de base sera $(\mathbb{C}(x), \frac{d}{dx})$. Nous allons décrire les modèles géométriques des extensions élementaires. Le corps $\mathbb{C}(x)$ sera le corps des fractions de la droite affine $\mathbb{A}^1(\mathbb{C})$ et sa structure différentielle sera donnée par le champ de vecteurs $\frac{\partial}{\partial x}$.

***Définition 1.2***. — *Soient $\mathbb{C} \subset K$ une extension de corps de type fini et $\delta$ une dérivation de $K$. Un modèle pour l'extension différentielle $(K,\delta)$ de $\mathbb{C}$ est une variété algébrique affine $Y$ sur $\mathbb{C}$ de corps de fractions $K$ munie du champ de vecteurs $\delta_Y$ induit par $\delta$.*

Soient $K \subset L$ une extension différentielle de type fini, $(Y,\delta_Y)$ et $(Z,\delta_Z)$ des modèles respectivement de $K$ et de $L$. L'inclusion des corps donne une application rationnelle dominante $\pi : Z \dashrightarrow Y$ et la compatibilité des dérivations dit que le champ $\delta_Z$ est $\pi$-projetable d'image $\delta_Y$.

***Définition 1.3***. — *Une application rationnelle $\varphi : Z \dashrightarrow Y$ entre variétés munies des champs de vecteurs respectifs $\delta_Z$ et $\delta_Y$ sera dite différentielle si $\overline{\varphi(Z)}$ est $\delta_Y$-invariante et $\delta_Z$ est $\varphi$-projetable sur $\delta_Y|_{\overline{\varphi(Z)}}$*

Les applications différentielles entre modèles induites par les morphismes élémentaires seront appelées élémentaires. Voici une description succincte de ces applications.

**Extensions algébriques.** — En restriction à des ouverts convenables, elles correspondent aux applications finies étales $Z \to Y$. Le champ sur $Y$ se relève de manière unique sur $Z$.

**Extensions de Picard-Vessiot.** — Ces applications se construisent à partir de $(Y, \delta_Y)$ en considérant le produit $Y \times GL_n(\mathbb{C})$ muni du champ de vecteurs

$$\delta_{Y \times GL_n(\mathbb{C})} = \delta_Y + \sum_{j,k,p} A_j^k g_k^p \frac{\partial}{\partial g_j^p}$$

où les $g_j^p$ sont les coordonnées standard de $GL_n(\mathbb{C})$ et $(A_j^k) \in GL_n(\mathbb{C}(Y))$. Une extension de Picard-Vessiot est une sous-variété $\delta_{Y \times GL_n(\mathbb{C})}$-invariante minimale $Z$ de $Y \times GL_n(\mathbb{C})$ telle que la projection induite de $Z$ sur $Y$ soit dominante. Par construction, cette projection est différentielle et il n'est pas difficile de montrer que l'extension de corps induite ne dépend pas du choix de $Z$.

**Extensions abéliennes.** — Soient $\Gamma$ un réseau de $\mathbb{C}^n$ tel que $A = \mathbb{C}^n/\Gamma$ soit une variété abélienne, $\mathbb{C}(A)$ le corps des fonctions $\Gamma$-périodiques et $\varphi_1, \ldots, \varphi_n$ une base de transcendance de $\mathbb{C}(A)$ sur $\mathbb{C}$ . Il existe des fonctions algébriques de $n$ variables $F_{i,j}$, $1 \leq i, j \leq n$, telles que

$$\frac{\partial \varphi_i}{\partial x_j} = F_{i,j}(\varphi_1, \ldots, \varphi_n).$$

Soient $a_1, \ldots, a_n$ $n$ fonctions sur une variété $(Y, \delta_Y)$. Considérons $Y \times A$ muni du champ de vecteurs

$$\delta_{Y \times A} = \delta_Y + \sum_{i,j} \delta_Y(a_j) F_{i,j}(\varphi_1, \ldots, \varphi_n) \frac{\partial}{\partial \varphi_i}.$$

Les trajectoires de ce champ de vecteurs sont les graphes des fonctions sur $Y$ à valeurs dans $A$ données par

$$\varphi_1(a_1 + c_1, \ldots, a_n + c_n), \ldots, \varphi_n(a_1 + c_1, \ldots, a_n + c_n),$$

les $c$ étant des constantes de $\delta_Y$. Un extension abélienne est une sous-variété $\delta_{Y \times A}$-invariante minimale $Z$ de $Y \times A$ telle que la projection induite de $Z$ sur $Y$ soit dominante.

**Extensions d'ordre un.** — Une telle extension est donnée par une variété irréductible $Z$ sur $Y$ de dimension relative un et par un relevé $\delta_Z$ de $\delta_Y$ sans intégrales première rationnelles.

***Définition 1.1bis*** *Soient $(X, \delta_X)$ une variété algébrique affine sur $\mathbb{C}$ muni d'un champ de vecteurs et $\pi : (X, \delta_X) \dashrightarrow (\mathbb{A}^1(\mathbb{C}), \frac{\partial}{\partial x})$ une application différentielle dominante. Le champ $\delta_X$ est dit* réductible *s'il existe une famille d'applications différentielles dominantes $\pi_i : (Y_i, \delta_i) \dashrightarrow (Y_{i-1}, \delta_{i-1})$ pour $1 \leq i \leq m$ de type* élémentaires *avec $(Y_0, \delta_0) = (\mathbb{A}^1(\mathbb{C}), \frac{\partial}{\partial x})$ et une application différentielle $\varphi : (Y_m, \delta_m) \dashrightarrow (X, \delta_X)$ dite* réductrice.

## 2. Structures transverses des extensions réductrices

Les structures transverses que nous étudierons sont données par des suites de formes rationnelles commençant par une base de formes nulles sur les trajectoires du champ de vecteurs et satisfaisant à certaines identités différentielles. Ces suites sont aussi appelées suites de Godbillon-Vey ou équations de structures. Soient $(Y, \delta_Y)$ une variété munie d'un champ de vecteurs, $N_Y$ le $\mathbb{C}(Y)$-espace vectoriel des formes sur $Y$ s'annulant sur $\delta_Y$ et $d_Y$ une forme rationnelle telle que $d_Y(\delta_Y) = 1$.

**Extensions algébriques.** — Si $Z$ est un revêtement étale de $Y$, le relevé de $\delta_Y$ est unique et l'annulateur de ce relevé est $N_Z = \mathbb{C}(Z) \otimes_{\mathbb{C}(Y)} N_Y$. Pour cette raison, la géométrie transverse locale de ces extensions est triviale.

**Extensions de Picard-Vessiot.** — Soit $Z$ une extension de Picard-Vessiot de $Y$. Les formes de $N_Y$ s'annulent sur $\delta_Z$. Une famille génératrice de l'espace des formes s'annulant sur $\delta_Z$ complétant $N_Y$ est donnée par la matrice de formes suivante :

$$\Theta = G^{-1}dG - G^{-1}AGd_Y$$

où $G$ est la matrice des coordonnées $(g_j^p)$ de $GL_n(\mathbb{C})$ restreinte à $Z$. On vérifie que

$$d\Theta = -\Theta \wedge \Theta \text{ modulo } N_Y.$$

Cette identité traduit la structure de feuilletage de Lie linéaire de ce type d'équation au-dessus des trajectoires de $\delta_Y$ (*i.e.* modulo $N_Y$) ([**7**]).

**Extensions abéliennes.** — Soient $\Gamma$ un réseau de $\mathbb{C}^n$ tel que $A = \mathbb{C}^n/\Gamma$ soit une variété abélienne, $\mathbb{C}(A)$ le corps des fonctions $\Gamma$-périodiques, $\varphi_1, \ldots, \varphi_n$ une base de transcendance de $\mathbb{C}(A)$ sur $\mathbb{C}$ et $F_{i,j}$, $1 \leq i, j \leq n$, des fonctions algébriques de $n$ variables telles que

$$\frac{\partial \varphi_i}{\partial x_j} = F_{i,j}(\varphi_1, \ldots, \varphi_n).$$

Considérons $a_1, \ldots, a_n$ $n$ fonctions sur une variété $(Y, \delta_Y)$ et $Z \subset Y \times A$ une extension abélienne de $Y$ de champ de vecteurs

$$\delta_Z = \delta_Y + \sum_{i,j} \delta_Y(a_j) F_{i,j}(\varphi_1, \ldots, \varphi_n) \frac{\partial}{\partial \varphi_i}.$$

Nous noterons $(F_{i,j}^{-1})_{i,j}$ la matrice inverse de $(F_{i,j})_{i,j}$. Une famille génératrice de l'espace des formes s'annulant sur $\delta_Z$ complétant $N_Y$ est donnée par

$$\eta_j = \sum_i F_{i,j}^{-1} d\varphi_i - \delta_Y(a_j) d_Y.$$

Ces formes sont fermées modulo $N_Y$.

**Extensions d'ordre un.** — Soit $Z \dashrightarrow Y$ une extension d'ordre un et $\theta$ une forme rationnelle s'annulant sur $\delta_Z$ indépendante des formes de $N_Y$. Il existe une suite de formes rationnelles sur $Z$ satisfaisant aux égalités :

$$\begin{aligned} d\theta &= \theta \wedge \theta_1 \mod N_Y, \\ d\theta_1 &= \theta \wedge \theta_2 \mod N_Y, \end{aligned}$$

et pour tout $n \in \mathbb{N}$ :

$$d\theta_n = \theta \wedge \theta_{n+1} + \sum_{k=1}^{n} \binom{n}{k} \theta_k \wedge \theta_{n-k+1} \mod N_Y.$$

Une telle suite est appelée suite de Godbillon-Vey générale de codimension un modulo $N_Y$.

**Structure transverse d'une extension réductrice.** — Soit

$$(Y_m, \delta_m) \overset{\pi^m_{m-1}}{\dashrightarrow} (Y_{m-1}, \delta_{m-1}) \overset{\pi^{m-1}_{m-2}}{\dashrightarrow} \dots \overset{\pi^2_1}{\dashrightarrow} (Y_1, \delta_1) \overset{\pi^1_0}{\dashrightarrow} (\mathbb{A}^1, \frac{\partial}{\partial x})$$

une extension réductrice de la droite affine. Le feuilletage défini par le champ de vecteurs $\delta_m$ sur la variété $Y_m$ admet une structure géométrique transverse particulière. Le $\mathbb{C}(Y_m)$-espace vectoriel des formes qui s'annulent sur $\delta_m$ est engendré par une famille filtrée de formes $\Omega = \cup_{i=1}^m \Omega(i)$ telle que $\Omega(i)$ engendre $\mathbb{C}(Y_m) \otimes_{\mathbb{C}(Y_i)} N_{Y_i}$ et $\Omega(i) - \Omega(i-1)$ soit

- une famille de formes satisfaisant à des identités différentielles de type linéaire modulo $N_{Y_{i-1}}$,
- une famille de formes fermées modulo $N_{Y_{i-1}}$,
- ou bien une forme intégrable modulo $N_{Y_{i-1}}$.

Ces formes différentielles donnent une famille de feuilletages sur $Y_m$ :

$$\mathcal{F}_m \subset \mathcal{F}_{m-1} \subset \dots \subset \mathcal{F}_1,$$

la géométrie transverse de $\mathcal{F}_i$ relative à $\mathcal{F}_{i-1}$ étant linéaire, de translation ou de codimension un.

## 3. Groupoïde de Galois

Soit $Y$ une variété algébrique affine lisse et irréductible sur $\mathbb{C}$ de dimension $n$. Nous noterons $J^*(Y)$ l'espace des difféomorphismes formels $\overline{y} : \widehat{Y, a} \to \widehat{Y, b}$, pour tout couple de points $(a, b)$. Cet espace est un groupoïde pro-algébrique sur $Y$ pour les lois de composition et d'inversion évidentes **[13, 15]**. Nous noterons $Y$ la variété source et $\overline{Y}$ la variété but. Son anneau de fonctions régulières est

$$\mathcal{O}(J^*(Y)) = \mathcal{O}(Y) \otimes_{\mathbb{C}} \mathcal{O}(\overline{Y}) \left[\det[y_i^{\epsilon(j)}],\ \overline{y}_i^{\alpha} \ |1 \leq i \leq n, \alpha \in \mathbb{N}^n, 0 < |\alpha|\right],$$

où $\epsilon(j)$ est le vecteur de coordonnées nulles sauf la $j$-ième égale à un. Le faisceau structural de $J^*(Y)$ est le faisceau sur $Y \times \overline{Y}$ donné par

$$\mathcal{O}_Y \otimes_{\mathbb{C}} \mathcal{O}_{\overline{Y}} \left[\det[y_i^{\epsilon(j)}],\ \overline{y}_i^{\alpha}\ |1 \leq i \leq n, \alpha \in \mathbb{N}^n, 0 < |\alpha|\right].$$

Il fait de cet espace une pro-variété affine au-dessus de $Y \times \overline{Y}$ au sens de [**14**].

Dans des coordonnées locales $(y_1, \dots, y_n)$ sur $Y$ et $(\overline{y}_1, \dots, \overline{y}_n)$ les mêmes coordonnées sur $\overline{Y}$, cet anneau est muni de dérivations dites totales définies par :

$$D_i = \frac{\partial}{\partial y_i} + \sum_{\substack{\alpha \in \mathbb{N}^n \\ 1 \leq j \leq n}} \overline{y}_j^{\alpha+\epsilon(i)} \frac{\partial}{\partial \overline{y}_j^{\alpha}}$$

où $\overline{y}_j^0$ est égal à $\overline{y}_j$.

Ces champs définissent une connexion $D : \mathcal{O}_{J^*(Y)} \to \mathcal{O}_{J^*(Y)} \otimes_{\mathcal{O}_Y} \Omega_Y^1$ en recollant sur $Y \times \overline{Y}$ les formules locales $Df = \sum_i D_i f \otimes dy_i$

Cet espace est muni de projections naturelles sur les espaces de jets d'ordre $q$ : $J_q^*(Y)$ d'anneau

$$\mathcal{O}(J_q^*(Y)) = \mathcal{O}(Y) \otimes_{\mathbb{C}} \mathcal{O}(\overline{Y}) \left[\det[y_i^{\epsilon(j)}],\ \overline{y}_i^{\alpha}\ |1 \leq i \leq n, \alpha \in \mathbb{N}^n, 0 < |\alpha| \leq q\right].$$

Les flèches précédentes induisent une structure de groupoïde algébrique sur les $J_q^*(Y)$ et une connexion $D : \mathcal{O}(J_q^*(Y)) \to \mathcal{O}(J_{q+1}^*(Y)) \otimes_{\mathcal{O}(Y)} \Omega^1(Y)$

***Définition 3.1* (Malgrange [15])**. — *Un $\mathcal{D}$-groupoïde de Lie sur $Y$ est un sous-espace $G \subset J^*(Y)$ défini par un idéal radiciel différentiel de $\mathcal{O}(J^*(Y))$ tel qu'en dehors d'une sous-variété $S$ de $Y$, la restriction de $G$ soit un sous-groupoïde de $J^*(Y - S)$ (au sens schématique).*

Un groupoïde de Lie $G$ sur $Y$ se projette sur des sous-variétés $G_q$ des espaces $J_q^*(Y)$. Celles-ci sont des sous-groupoïdes algébriques de $J_q^*(U)$ au-dessus d'un ouvert Zariski dense convenable $U$ de $Y$.

***Exemples 3.2***. — 1°) *Soit $\mathcal{F}$ un feuilletage de $Y$ donné par un système de formes $\omega_i$, $1 \leq i \leq q$, intégrable (i.e. $\bigwedge_{i=1}^q \omega_i \wedge d\omega_j = 0$ pour tout $j$). Les difféomorphismes formels préservant le feuilletage*[(1)] *sont les solutions formelles d'un $\mathcal{D}$-groupoïde de Lie dont l'idéal est engendré par les coordonnées des équations $\bigwedge_{i=1}^q \omega_i \wedge \overline{y}^* \omega_j = 0$ pour $1 \leq j \leq q$. Il sera noté* $\mathrm{Aut}(\mathcal{F})$

2°) *Soit $\pi : Y \dashrightarrow X$ une application rationnelle. Elle définit un feuilletage sur $Y$ dont le $\mathcal{D}$-groupoïde de Lie d'invariance sera noté* $\mathrm{Aut}(\pi)$.

3°) *On peut aussi considérer le $\mathcal{D}$-groupoïde de Lie d'invariance de $\pi$ : $Inv(\pi)$ défini par les coordonnées de $\pi(\overline{y}) - \pi(y) = 0$ qui est naturellement inclus dans le précédent.*

(1) Aussi appelés difféomorphismes basiques.

4°) *Soient $\theta$ une $p$-forme et $\omega$ une $q$-forme sur $Y$, le groupoïde d'invariance de $\theta$ modulo $\omega$ : $Inv(\theta \mod \omega)$ est le $\mathcal{D}$-groupoïde de Lie dont l'idéal est différentiellement engendré par les coordonnées de $(\overline{y}^*\theta - \theta) \wedge \omega = 0$*

***Remarque 3.3***. — *Les transformations tangentes aux feuilles d'un feuilletage n'étant pas caractérisées a priori par des équations différentielles, le troisième exemple n'a pas de sens pour un feuilletage sans intégrales premières rationnelles.*

Un $\mathcal{D}$-groupoïde de Lie a une $\mathcal{D}$-algèbre de Lie formée par les champs de vecteurs formels dont les flots appartiennent au $\mathcal{D}$-groupoïde. Soient $(y_1, \dots, y_n)$ des coordonnées locales sur $Y$. Les champs de vecteurs $\sum_i a_i(y)\frac{\partial}{\partial y_i}$ dont les flots sont solutions d'une équation $E \in \mathcal{O}(J^*(Y))$ d'ordre $q$ sont eux mêmes solutions d'une équation d'ordre $q$ appelée équation linéarisée le long de l'identité :

$$\mathcal{L}(E) = \sum_{i,\alpha} \frac{\partial E}{\partial y_i^\alpha}(\mathrm{id})\ a_i^\alpha$$

où id est l'application de $Y$ dans $J^*(Y)$ telle que $\mathrm{id}^* y_i = y_i$, $\mathrm{id}^*\overline{y}_i = y_i$, $\mathrm{id}^*\overline{y}_i^{\epsilon(j)} = \delta_i^j$ et $\mathrm{id}^*\overline{y}_i^\alpha = 0$ pour $|\alpha| > 1$, et $a_i^\alpha$ est la dérivée d'ordre $|\alpha|$ de $a_i$ par rapport à $y$.

***Définition 3.4***. — *Soit $I \subset \mathcal{O}(J^*(Y))$ l'idéal d'un $\mathcal{D}$-groupoïde de Lie $G$. L'idéal $\mathcal{L}(I)$ engendré par $\{\mathcal{L}(E), E \in I\}$ décrit au-dessus d'un ouvert Zariski dense $U$ de $Y$ un sous-fibré vectoriel de l'espace $J(TY)$ des champs de vecteurs formels sur $Y$. C'est la $\mathcal{D}$-algèbre de Lie de $G$ notée $\mathcal{L}G$. Le $\mathbb{C}$-espace vectoriel des champs de vecteurs formels solutions de $\mathcal{L}G$ en $p \in U$ sera noté $\mathfrak{g}_p$ ou $\mathfrak{g}$ si aucune confusion ne porte sur $p$.*

En dehors de $S$, les solutions formelles de la $\mathcal{D}$-algèbre de Lie d'un $\mathcal{D}$-groupoïde de Lie sont stables par le crochet de Lie [**15**]. Dans *loc. cit.* une définition de $\mathcal{D}$-algèbre de Lie dont les objets ci-dessus ne sont qu'un cas particulier est donnée. Cette notion englobe en particulier les feuilletages (algébriques singuliers) de $Y$. Avec cette définition, il arrive qu'une $\mathcal{D}$-algèbre de Lie ne soit pas la $\mathcal{D}$-algèbre d'un $\mathcal{D}$-groupoïde de Lie. Dans le cas d'un feuilletage, cela conduit à la définition suivante :

***Définition 3.5* ([15])**. — *Soit $\mathcal{F}$ un feuilletage (algébrique, singulier) de $Y$. Le groupoïde de Galois de $\mathcal{F}$ est le plus petit $\mathcal{D}$-groupoïde de Lie $G$ tel que $\mathcal{F} \subset \mathcal{L}G$. Il sera noté* $\mathrm{Gal}(\mathcal{F})$.

***Remarque 3.6***. — *Soient $I$ l'idéal de $G$ et $Ann(\mathcal{F})$ l'idéal différentiel des équations différentielles en les $a_i$ s'annulant sur les champs $\sum a_i\partial/\partial x_i$ tangents à $\mathcal{F}$. L'inclusion $\mathcal{F} \subset \mathcal{L}G$ doit se lire $\mathcal{L}(I) \subset Ann(\mathcal{F})$.*

Un point essentiel dans la preuve de l'existence de cet objet est le théorème suivant dont nous nous servirons dans la suite.

***Théorème 3.7*** **([15], thm 4.5.1.)**. — *Soient $G_1$ et $G_2$ deux groupoïdes de Lie sur $Y$ d'idéaux respectifs $I$ et $J$. L'idéal $\sqrt{I+J}$ est l'idéal d'un groupoïde de Lie sur $Y$ noté $G_1 \cap G_2$.*

Cet objet généralise le groupe de Galois différentiel d'une équation différentielle linéaire. Dans le cas d'un modèle $(Y, \delta_Y)$ d'un corps différentiel ordinaire, nous noterons $\mathcal{F}_Y$ le feuilletage de $Y$ défini par le champ de vecteurs $\delta_Y$. Le groupoïde $\mathrm{Gal}(\mathcal{F}_Y)$ sera aussi appelé groupoïde de Galois du corps différentiel $(\mathbb{C}(Y), \delta_Y)$. Dans le cas d'une extension de $(\mathbb{C}(x), \frac{\partial}{\partial x})$, $\pi : (Y, \delta_Y) \to (\mathbb{A}^1, \frac{\partial}{\partial x})$, la partie significative de $\mathrm{Gal}(\mathcal{F}_Y)$ est le $\mathcal{D}$-groupoïde de Lie $\mathrm{Gal}(\mathcal{F}_Y) \cap Inv(\pi)$. Lorsque cette extension est de type Picard-Vessiot, on obtient ainsi un fibré en groupe au-dessus d'un ouvert de $\mathbb{A}^1$ dont la fibre est un représentant du groupe de Galois abstrait donné par la théorie de Picard-Vessiot.

***Exemples 3.8***. — 1°) *Soit $\mathcal{F}$ un feuilletage de Lie linéaire* [**7**] *donné par une matrice de $n^2$ formes rationnelles $\Theta = (\theta_i^j)_{ij}$ vérifiant*

$$d\Theta = \Theta \wedge \Theta.$$

*Les champs $X$ tangents au feuilletage $(\Theta(X) = 0)$ préservent la forme $\Theta$ :*

$$L_X \Theta = \iota_X d\Theta + d\iota_X \Theta = 0.$$

*Le groupoïde de Galois de $\mathcal{F}$ est donc inclus dans $Inv(\Theta)$, le groupoïde d'invariance des formes $\theta_i^j$, $1 \leq i, j \leq n$.*

2°) *Soit $(Z, \delta_Z) \dashrightarrow (Y, \delta_Y)$ une extension de Picard-Vessiot. On note $\mathcal{F}_Z$ le feuilletage défini par $\delta_Z$, $\mathcal{F}_Y$ le feuilletage que l'on obtient en ramenant le feuilletage sur $Z$ défini par $\delta_Y$. Soient $\omega_1, \ldots, \omega_q$ une base de formes s'annulant sur $\mathcal{F}_Y$ et $\theta_i^j$, $1 \leq i, j \leq n$, les formes définissant $\mathcal{F}_Z$ satisfaisant $d\theta_i^j = \sum_k \theta_i^k \wedge \theta_k^j \mod \omega_1, \ldots, \omega_q$. On a évidemment $\mathrm{Gal}(\mathcal{F}_Z) \subset \mathrm{Gal}(\mathcal{F}_Y) \subset \mathrm{Aut}(\mathcal{F}_Y)$. De plus comme les champs $X$ tangents à $\mathcal{F}_Z$ satisfont à $L_X(\theta_i^j) = 0 \mod \omega_1, \ldots, \omega_q$, $1 \leq i, j \leq n$, on a $\mathrm{Gal}(\mathcal{F}_Z) \subset \mathrm{Gal}(\mathcal{F}_Y) \cap Inv(\theta_i^j \mod \omega_1, \ldots, \omega_q \,;\, 1 \leq i, j \leq n)$.*

***Théorème 3.9*** **([3])**. — *Nous appellerons groupoïde de Galois de la première équation de Painlevé le groupoïde de Galois du feuilletage défini par $\delta_P = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + (6y^2+x)\frac{\partial}{\partial z}$ sur $\mathbb{A}^3$. Le groupoide de Galois de la première équation de Painlevé est le groupoïde de Lie d'invariance $Inv(\gamma)$ de la 2-forme $\gamma = dy \wedge dz - z dx \wedge dz + (6y^2 + x) dx \wedge dy$.*

La partie significative du groupoïde de Galois de $(\mathbb{A}^3, \delta_P) \to (\mathbb{A}^1, \frac{\partial}{\partial x})$ est $Inv(\gamma) \cap Inv(x)$. Dans des coordonnées analytiques locales $x, h, k$ avec $dh \wedge dk = \gamma$, ses solutions sont de la forme :

$$\begin{cases} \overline{x} = x \\ \overline{h} = \overline{h}(h,k) \\ \overline{k} = \overline{k}(h,k) \end{cases} , \quad \text{avec} \quad \frac{\partial(\overline{h}, \overline{k})}{\partial(h,k)} = 1.$$

## 4. Quelques lemmes sur les groupoïdes de Lie

Soit $Y$ une variété algébrique affine lisse et irréductible sur $\mathbb{C}$ de dimension $n$. Notons $R_q(Y)$ le fibré des repères d'ordre $q$ sur $Y$. Ce fibré est l'espace des jets d'ordre $q$ en 0 d'applications inversibles $(\mathbb{C}^n,0) \to Y$. Il s'identifie à la fibre en $p \in Y$ de la projection source $J_q^*(Y) \to Y$ via la composition par un repère d'ordre $q$ en $p$. C'est aussi un fibré principal sous l'action du groupe $\Gamma_q^n$ des jets d'ordre $q$ en 0 d'applications inversibles $(\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$. L'application $\lambda : R_q(Y) \times R_q(Y) \to J_q^*(Y)$ défini par $\lambda(r,s) = r \circ s^{-1}$ permet d'associer à un sous-groupoïde algébrique $G_q$ de $J_q^*(Y)$ une relation d'équivalence algébrique $EG_q = \lambda^{-1}G_q$ sur $R_q(Y)$. Réciproquement, toute relation d'équivalence algébrique sur $R_q(Y)$ invariante sous l'action diagonale de $\Gamma_q^n$ sur $R_q(Y) \times R_q(Y)$ se projette sur un sous-groupoïde algébrique de $J_q^*(Y)$.

***Définition 4.1***. — *Soit $\delta$ un champ de vecteurs sur $Y$ s'écrivant en coordonnées locales $\sum_i a_i \frac{\partial}{\partial y_i}$. Ce champ se prolonge en un champ $\Gamma_q^n$-invariant sur $R_q(Y)$ noté $\delta^R$ et donné en coordonnées locales par $\sum_{i,\alpha} D^\alpha a_i \frac{\partial}{\partial y_i^\alpha}$. Ce prolongement est compatible avec le crochet de Lie.*

Soient $G$ un groupoïde de Lie sur $Y$ et $S$ une sous-variété fermée de $Y$ telle que $G$ soit un sous-groupoïde de $J^*(Y-S)$. La définition de son algèbre de Lie se lit sur les repères de la manière suivante ([**15**] paragraphe 3.2.). Les champs de vecteurs formels solutions de $\mathscr{L}G$ sont les champs dont les prolongements sont tangents aux orbites de la relation d'équivalence $EG_q$ au-dessus de $Y-S$ pout tout $q \in \mathbb{N}$. Autrement dit $\mathscr{L}G$ définit sur tous les espaces de repères un feuilletage algébrique $\Gamma_q^n$-invariant dont les feuilles sont les composantes irréductibles des orbites de $EG_q$. D'après [**6**], le corps des invariants rationnels de cette relation d'équivalence a un degré de transcendance sur $\mathbb{C}$ égal à la codimension du feuilletage. Parmi les feuilletages algébriques $\Gamma_q^n$-invariant de $R_q(Y)$, ceux provenant de relations d'équivalence sont ceux vérifiant la condition précédente. Un théorème de X. Gomez-Mont [**8**] assure que cette condition est vérifiée lorsque toutes les feuilles sont algébriques.

***Lemme 4.2***. — *Soit $\varphi : Z \dashrightarrow Y$ une application dominante entre deux variétés algébriques affines lisses et irréductibles sur $\mathbb{C}$. Elle induit un morphisme de groupoïde*

$$\varphi_\star : \mathrm{Aut}(\varphi) \dashrightarrow J^*(Y).$$

*Si $H$ est $\mathscr{D}$-groupoïde de Lie sur $Y$, sa préimage par $\varphi_\star$ est un $\mathscr{D}$-groupoïde de Lie sur $Z$ noté $\varphi_\star^{-1}H$.*

*Démonstration.* — Commençons par définir le morphisme d'anneau

$$\varphi^\# : \mathscr{O}_{J^*(Y)}(U \times \overline{U}) \to \mathscr{O}_{\mathrm{Aut}(\varphi)}(V \times \overline{V})$$

définissant $\varphi_\star$ au-dessus d'un ouvert affine Zariski dense $V \times \overline{V}$ de $Z \times \overline{Z}$. Soient $U$ et $V$ des ouverts affines Zariski dense de $Y$ et $Z$ tels que

- $\varphi(U) = V$,
- $U$ et $V$ sont des revêtements non ramifiés de $\mathbb{A}^n$ et $\mathbb{A}^{n+m}$,

- ces revêtements conjuguent $\varphi$ à la projection de $\mathbb{A}^{n+m}$ sur l'espace $\mathbb{A}^n$ des $n$ premières coordonnées.

Notons $\overline{U}$ et $\overline{V}$ les mêmes ouverts sur les copies $\overline{Y}$ et $\overline{Z}$ de $Y$ et $Z$. Nous appellerons $y_i$ (resp. $\overline{y}_i$) les $n$ coordonnées induites sur $U$ et $V$ (resp. $\overline{U}$ et $\overline{V}$) par $\mathbb{A}^n$ et $t_j$ (resp. $\overline{t}_j$) des $m$ coordonnées sur $V$ (resp. $\overline{V}$) provenant de $\mathbb{A}^{n+m}$. L'anneaux des fonctions régulières de $J^*(Y)$ au-dessus de $U \times \overline{U}$ est

$$\mathcal{O}_{J^*(Y)}(U \times \overline{U}) = \mathcal{O}(U) \otimes_{\mathbb{C}} \mathcal{O}(\overline{U}) \left[\, \overline{y}_i^{\alpha} \;|1 \leq i \leq n; \alpha \in \mathbb{N}^n; 0 < |\alpha|\right]$$

et l'anneau des fonctions régulières de $\mathrm{Aut}(\varphi)$ au-dessus de $V \times \overline{V}$ est

$$\begin{aligned}\mathcal{O}_{\mathrm{Aut}(\varphi)}(V \times \overline{V}) = \mathcal{O}(V) \otimes_{\mathbb{C}} \mathcal{O}(\overline{V}) & \\ & \left[\, \overline{y}_i^{\alpha}, \overline{t_j}^{\beta} \;|1 \leq i \leq n; 1 \leq j \leq m-n; \alpha \in \mathbb{N}^n; \alpha \in \mathbb{N}^m; 0 < |\alpha|, |\beta|\right].\end{aligned}$$

On définit l'application $\varphi^{\#}$ de $\mathcal{O}_{J^*(Y)}(U \times \overline{U})$ dans $\mathcal{O}_{\mathrm{Aut}(\varphi)}(V \times \overline{V})$ en étendant l'inclusion de $\mathcal{O}(U) \otimes_{\mathbb{C}} \mathcal{O}(\overline{U})$ dans $\mathcal{O}(V) \otimes_{\mathbb{C}} \mathcal{O}(\overline{V})$ par l'identification des variables portant le même nom. Cette application est différentielle (*i.e.* commute aux connexions) et induit une application $\varphi_\star$ sur les difféomorphismes formels.

Vérifions que cette application est un morphisme de groupoïde. Dans notre situation, un morphisme de groupoïde est un couple d'applications

$$\phi_0 : V \to U$$
$$\phi : \mathrm{Aut}(\varphi)|_{V \times \overline{V}} \to J^*(Y)|_{U \times \overline{U}}$$

telles que $\mathit{source} \circ \phi = \phi_0 \circ \mathit{source}$, $\mathit{but} \circ \phi = \phi_0 \circ \mathit{but}$ et $\phi \circ c = c \circ (\phi \times \phi)$. Les notations *source* et *but* sont assez explicites, $c$ designe la loi de composition $(J, \mathit{source}) \times_Z (J, \mathit{but}) \to J$ ainsi que les applications induites sur les sous-variétes et leurs ouverts où cela a un sens et enfin $\phi \times \phi$ designe l'application de

$$\mathrm{Aut}(\varphi)|_{V \times \overline{V}} \;\times_{\overline{V}}\; \mathrm{Aut}(\varphi)|_{\overline{V} \times \overline{\overline{V}}} \to J^*(Y)|_{U \times \overline{U}} \;\times_{\overline{U}}\; J^*(Y)|_{\overline{U} \times \overline{\overline{U}}}$$

induite par $\phi$ et $\phi_0$. Ces applications doivent être des morphismes de variétés algébriques. Nous noterons $\phi_0^*$ et $\phi^*$ les morphismes d'anneaux induits. Ils doivent vérifier les diagrammes duaux de ceux donnés ci-dessus. On vérifie aisément que le couple $\varphi^* : \mathcal{O}(U) \to \mathcal{O}(V)$ et $\varphi^{\#}$ satisfont à ces diagrammes.

On vérifie tout aussi facilement que l'image réciproque d'un sous-groupoïde $H$ de $J^*(Y)|_{U \times \overline{U}}$ par $\varphi_\star$ est un groupoïde que nous appellerons $\varphi_\star^{-1} H$. □

***Définition 4.3***. — *Soient $\varphi : Z \dashrightarrow Y$ une application dominante entre deux variétés algébriques affines lisses et irréductibles sur $\mathbb{C}$ et $\varphi_\star : \mathrm{Aut}(\varphi) \dashrightarrow J^*(Y)$ l'application induite. Un groupoïde de Lie $G$ sur $Z$ inclus dans $\mathrm{Aut}(\varphi)$ est dit $\varphi$-projetable si $\overline{\varphi_\star(G)}$ est un groupoïde de Lie sur $Y$.*

***Lemme 4.4***. — *Soient $\varphi : Z \dashrightarrow Y$ une application dominante à fibres connexes et $G$ un $\mathcal{D}$-groupoïde de Lie sur $Z$ contenu dans $\mathrm{Aut}(\varphi)$. Le $\mathcal{D}$-groupoïde de Lie $G$ est $\varphi$-projetable.*

*Le $\mathcal{D}$-groupoïde de Lie image sera noté $\varphi_\star G$. Pour tout $\mathcal{D}$-groupoïde de Lie $H$ sur $Y$, on a $\varphi_\star \varphi_\star^{-1} H = H$.*

*Démonstration.* — Considérons un sous-$\mathcal{D}$-groupoïde de Lie $G$ de $\mathrm{Aut}(\varphi)|_{V\times\overline{V}}$. Pour tout entier $q$ on a un groupoïde algébrique $G_q$ de $\mathrm{Aut}_q(\varphi)|_{V\times\overline{V}}$ et $EG_q$ la relation d'équivalence correspondante sur $R_q(Z)$.

Notons $R_q(\varphi)$ l'espace des repères d'ordre $q$, $r : (\mathbb{C}^{\dim Z}, 0) \to Z$ conjuguant $\varphi$ à la projection sur les $\dim Y$ premières coordonnées. La relation d'équivalence $EG_q$ induit une relation d'équivalence sur $R_q(\varphi)$ que nous noterons encore $EG_q$.

Notons encore $T_q\varphi$ la distribution verticale du prolongement $\varphi^R : R_q(\varphi) \dashrightarrow R_q(Y)$. Les fibres de $\varphi$ étant connexes, celles de $\varphi^R$ le sont aussi. Ceci permet d'identifier $R_q(Y)$ au quotient de $R_q(\varphi)$ par les feuilles de la distribution $T_q\varphi$.

Soit $T(EG_q)$ la distribution tangente aux orbites de $EG_q$ sur $R_q(\varphi)$. Elle est localement (en topologie transcendante) engendrée par les prolongements de champs de vecteurs solutions de $\mathcal{L}G$. Ceux-ci préservant la fibration $\varphi$, la distribution $T(EG_q)+T_q\varphi$ est intégrable. Par construction, les feuilles de $T(EG_q) + T_q\varphi$ sont algébriques. Pour récupérer une relation d'équivalence sur $R_q(Y)$ qui soit la projection de $EG_q$, commençons par regarder le quotient générique de $R_q(\varphi)$ par $EG_q$ suivant **[6]** :

$$\pi : R_q(\varphi) \dashrightarrow V.$$

La distribution $T(EG_q) + T_q\varphi$ se projette sur une distribution $D$ sur $V$ et l'image des feuilles de $T(EG_q) + T_q\varphi$ passant par une orbite de $EG_q$ est la projection d'une feuille de $T_q\varphi$, c'est donc un ensemble connexe. D'après **[8]**, la distribution $D$ admet une intégrale première rationnelle $h : V \dashrightarrow W$ telle que pour $w$ dans un ouvert Zariski dense de $W$, $h^{-1}(w)$ soit l'adhérence d'une feuille de $D$. Pour $w$ dans un ouvert Zariski dense de $W$, la fibre $(h\circ\pi)^{-1}(w)$ est la réunion de feuilles de $T(EG_q) + T_q\varphi$ passant par une orbite de $EG_q$. Cette application étant constante sur les fibres de $\varphi^R$, elle se projette en une application $\underline{h} : R_q(Y) \to W$ décrivant la relation d'équivalence cherchée sur un ouvert Zariski dense de $R_q(Y)$.

Cette construction, pour des valeurs de $q$ assez grandes, permet de définir le $\mathcal{D}$-groupoïde de Lie image $\varphi_\star G$. □

***Lemme 4.5.*** — *Soient $Z$ une variété algébrique affine lisse et irréductible sur $\mathbb{C}$, $\delta$ un champ de vecteurs sur $Z$ et $G$ un $\mathcal{D}$-groupoïde de Lie sur $Z$ inclus dans $\mathrm{Aut}(\mathcal{F}_\delta)$ tel que sa $\mathcal{D}$-algèbre de Lie $\mathcal{L}G$ contienne $\delta$. Il existe un $\mathcal{D}$-groupoïde de Lie $G^\delta$ sur $Z$ contenant $G$, dont la $\mathcal{D}$-algèbre de Lie $\mathcal{L}G^\delta$ contient tous les champs colinéaires à $\delta$ et ne diffère de $\mathcal{L}G$ que par ceux-ci.*

*Démonstration.* — Nous allons construire la relation d'équivalence sur $R_q(Z)$ donnant $G_q^\delta$ à partir de la relation de $G_q$ et du feuilletage donné par les prolongements des multiples de $\delta$. Notons $T_q\delta$ la distribution engendrée par les champs $(f\delta)^R$ pour tout $f \in \mathcal{O}_Z$ sur $R_q(Z)$.

Commençons par montrer que la distribution $T_q\delta \cap \ker dt$ verticale pour la projection *but* $R_q(Z) \xrightarrow{t} Z$ a toutes ses feuilles algébriques. Fixons un point $p \in Z$ au voisinage duquel le champ $\delta$ est formellement redressable par $\widehat{\phi}$ sur le champ $\frac{\partial}{\partial z_1}$ de $\mathbb{A}^{\dim Z}$. Le changement de variables $\widehat{\phi}$ induit un isomorphisme algébrique $\phi^R$ entre la fibre de $t$ en $p$ : $R_q(Z)_p$ et l'espace des repères d'ordre $q$ en 0 sur $\mathbb{A}^{\dim Z}$ : $R_q(\mathbb{A}^{\dim Z})_0$. Pour tout $f \in \mathcal{O}(\widehat{Z,p})$ on a

$$(\phi^R)^* \left((f\delta)^R|_p\right) = \left(f \circ \widehat{\phi}\frac{\partial}{\partial z_1}\right)^R \Big|_0.$$

Les champs verticaux de $T_q\delta$ étant engendrés par ceux de la forme $(f\delta)^R - f\delta^R$, une feuille de $T_q\delta \cap \ker dt$ au-dessus de $p$ a pour image par $\phi$ la sous-variété de $R_q(\mathbb{C}^{\dim Z})_0$ d'équation $z_i^\alpha = 0$ pour $2 \leq i \leq \dim Z$ et $1 \leq |\alpha| \leq q$. Elles sont donc bien algébriques.

Notons $T(EG_q)$ la distribution tangente aux orbites de $EG_q$. Comme le champ de vecteurs $\delta^R$ est tangent aux orbites de $EG_q$, la distribution $T(EG_q) + T_q\delta$ est engendrée par $T(EG_q)$ et $T_q\delta \cap \ker dt$. L'hypothèse $G \subset \mathrm{Aut}(\mathcal{F}_\delta)$ assure que cette distribution est involutive. Les feuilles des deux distributions $T(EG_q)$ et $T_q\delta \cap \ker dt$ étant algébriques, les feuilles de $T(EG_q)+T_q\delta$ le sont aussi. Le feuilletage ainsi construit est $\Gamma_q^{\dim Z}$-invariant et ses feuilles sont algébriques, il donne un sous-groupoïde de $J_q^*(Z)$. Ceci prouve le lemme. □

***Lemme 4.6***. — *Si* $\varphi : (Z, \delta_Z) \dashrightarrow (Y, \delta_Y)$ *est une application différentielle dominante entre variétés algébriques affines lisses et irréductibles sur* $\mathbb{C}$ *munies de champs de vecteurs alors* $\mathrm{Gal}(\mathcal{F}_Z) \cap \mathrm{Aut}(\varphi)$ *est* $\varphi$*-projetable et*

$$\varphi_\star\left(\mathrm{Gal}(\mathcal{F}_Z) \cap \mathrm{Aut}(\varphi)\right) = \mathrm{Gal}(\mathcal{F}_Y).$$

*Démonstration*. — Quitte à remplacer $Z$ et $Y$ par des ouverts, nous pouvons supposez que $\varphi$ est un morphisme qui admet la factorisation de Stein [**9**] :

$$Z \xrightarrow{\widetilde{\varphi}} \widetilde{Y} \xrightarrow{\pi} Y$$

où $\widetilde{\varphi}$ est un morphisme à fibres connexes et $\widetilde{Y}$ est étale sur $Y$.

Commençons par projeter $\mathrm{Gal}(\mathcal{F}_Z)$ suivant $\widetilde{\varphi}$. Le $\mathcal{D}$-groupoïde de Lie $\left[\widetilde{\varphi}_\star^{-1}\mathrm{Gal}(\mathcal{F}_{\widetilde{Y}}) \cap \mathrm{Aut}(\mathcal{F}_Z)\right]^{\delta_Z}$ contient $\mathrm{Gal}(\mathcal{F}_Z)$. En prenant l'intersection avec $\mathrm{Aut}(\widetilde{\varphi})$, on obtient l'inclusion

$$\mathrm{Gal}(\mathcal{F}_Z) \cap \mathrm{Aut}(\widetilde{\varphi}) \subset \widetilde{\varphi}_\star^{-1}\mathrm{Gal}(\mathcal{F}_{\widetilde{Y}}).$$

D'après le lemme 4.4, $\mathrm{Gal}(\mathcal{F}_Z)$ est $\widetilde{\varphi}$-projetable. En projetant suivant $\widetilde{\varphi}$ on obtient

$$\widetilde{\varphi}_\star\left(\mathrm{Gal}(\mathcal{F}_Z) \cap \mathrm{Aut}(\widetilde{\varphi})\right) \subset \mathrm{Gal}(\mathcal{F}_{\widetilde{Y}}).$$

Comme l'algèbre de Lie de $\mathrm{Gal}(\mathcal{F}_Z)$ contient tous les multiples $\widetilde{\varphi}$-projetables de $\delta_Z$, celle de $\widetilde{\varphi}_\star\left(\mathrm{Gal}(\mathcal{F}_Z) \cap \mathrm{Aut}(\widetilde{\varphi})\right)$ contient tous les multiples de $\delta_{\widetilde{Y}}$. La minimalité de $\mathrm{Gal}(\mathcal{F}_{\widetilde{Y}})$ pour cette propriété donne l'égalité

$$\widetilde{\varphi}_\star\left(\mathrm{Gal}(\mathcal{F}_Z) \cap \mathrm{Aut}(\widetilde{\varphi})\right) = \mathrm{Gal}(\mathcal{F}_{\widetilde{Y}}).$$

Occupons-nous maintenant de projeter $\mathrm{Gal}(\mathcal{F}_{\widetilde{Y}})$ sur $\mathrm{Gal}(\mathcal{F}_Y)$. Commençons par le cas où $\pi$ est un revêtement galoisien. Le revêtement induit $\pi^R : R_q(\widetilde{Y}) \to R_q(Y)$ est aussi galoisien de même groupe. Notons $\widetilde{E}$ la relation d'équivalence sur $R_q(\widetilde{Y})$ donnée par $\mathrm{Gal}(\mathcal{F}_{\widetilde{Y}})$. D'après [**8**], il existe des fonctions rationnelles sur $R_q(\widetilde{Y})$ constantes sur les composantes connexes des orbites de $\widetilde{E}$ en dehors d'une hypersurface et séparant ces composantes connexes. Le champ $\delta^R_{\widetilde{Y}}$ étant tangent aux orbites de $\widetilde{E}$, il préserve ces fonctions. Par minimalité de $\mathrm{Gal}(\mathcal{F}_{\widetilde{Y}})$, presque toute orbite de $\widetilde{E}$ est connexe. Pour prouver que $\mathrm{Gal}(\mathcal{F}_{\widetilde{Y}})$ est $\pi$-projetable, il suffit de prouver que les images des orbites de $\widetilde{E}$ par $\pi^R$ feuillettent $R_q(Y)$.

Le champ $\delta_{\widetilde{Y}}$ étant $\pi$-projetable, on a $g^*\delta_{\widetilde{Y}} = \delta_{\widetilde{Y}}$ pour tout $g \in \mathrm{Gal}(\pi)$. On a donc l'égalité $\mathrm{Gal}(\mathcal{F}_{\widetilde{Y}}) = g^*\mathrm{Gal}(\mathcal{F}_{\widetilde{Y}})$ pour tout $g \in \mathrm{Gal}(\pi)$. Cette égalité implique que $\pi^R$ envoie le feuilletage donné par les orbites de $\widetilde{E}$ sur un feuilletage de $R_q(Y)$. Ces feuilles sont algébriques et le feuilletage est $\Gamma^n_q$-invariant, ce sont donc les orbites d'une relation d'équivalence. Ceci implique que $\mathrm{Gal}(\mathcal{F}_{\widetilde{Y}})$ est $\pi$-projetable.

Le $\mathcal{D}$-groupoïde de Lie $\pi_\star^{-1}\mathrm{Gal}(\mathcal{F}_Y)$ contient $\mathrm{Gal}(\mathcal{F}_{\widetilde{Y}})$, la projection suivant $\pi$ prouve l'égalité

$$\pi_\star\mathrm{Gal}(\mathcal{F}_{\widetilde{Y}}) = \mathrm{Gal}(\mathcal{F}_Y).$$

Dans le cas où $\pi$ n'est pas galoisien, on considère

$$\begin{tikzcd} & \widetilde{\widetilde{Y}} \arrow[dl, "\widetilde{\pi}"'] \arrow[dd, "\widetilde{\widetilde{\pi}}"] \\ \widetilde{Y} \arrow[dr, "\pi"'] & \\ & Y \end{tikzcd}$$

telle que $\widetilde{\widetilde{\pi}}$ et $\widetilde{\pi}$ soient galoisiens. Projetons $\mathrm{Gal}(\mathcal{F}_{\widetilde{\widetilde{Y}}})$ suivant $\widetilde{\widetilde{\pi}}$ et $\widetilde{\pi}$. D'après ce que nous venons de montrer, les images sont respectivement $\mathrm{Gal}(\mathcal{F}_Y)$ et $\mathrm{Gal}(\mathcal{F}_{\widetilde{Y}})$. Le diagramme ci-dessus étant commutatif, on obtient l'égalité annoncée. □

***Remarque 4.7.*** — 1°) *En tant qu'application rationnelle entre pro-variétés algébriques,* $\pi_\star$ *de* $\mathrm{Gal}(\mathcal{F}_{\widetilde{Y}})$ *sur* $\mathrm{Gal}(\mathcal{F}_Y)$ *n'est pas un isomorphisme mais seulement une application finie.*

2°) *Ce type de résultat a été obtenu par H. Umemura dans le contexte du groupe de Galois infinitésimal d'une équation différentielle* ([**24**] *thm 5.14*).

***Corollaire 4.8.*** — *Soit* $\pi : (\widetilde{Y}, \delta_{\widetilde{Y}}) \to (Y, \delta_Y)$ *une application finie et différentielle entre variétés algébriques affines lisses et irréductibles sur* $\mathbb{C}$ *munies de champs de vecteurs. Soit* $p$ *un point de* $\widetilde{Y}$ *tel que* $\pi$ *soit étale,* $\mathrm{Gal}(\mathcal{F}_{\widetilde{Y}})$ *soit lisse en* $\mathrm{id}(p)$ *(le jet de l'identité en* $p$*) et* $\mathrm{Gal}(\mathcal{F}_Y)$ *soit lisse en* $\mathrm{id}(\pi(p))$*. La projection des champs de vecteurs formels sur* $\widehat{\widetilde{Y}, p}$ *sur les champs de vecteurs formels sur* $\widehat{Y, \pi(p)}$ *donne un isomorphisme de* $\mathbb{C}$*-algèbre de Lie entre* $\mathfrak{gal}(\mathcal{F}_{\widetilde{Y}})_p$ *et* $\mathfrak{gal}(\mathcal{F}_Y)_{\pi(p)}$.

*Démonstration.* — Si $\pi$ est étale en $p$ alors la projection des champs de vecteurs formels suivant $\pi$ en $p$ est un isomorphisme. De plus cet isomorphisme est exactement la différentielle de $\pi_\star$ en l'identité au point $p$ : $\mathrm{id}(p) \in \mathrm{Gal}(\mathcal{F}_{\widetilde{Y}})$. Il envoie donc $\mathfrak{gal}(\mathcal{F}_{\widetilde{Y}})_p$ sur $\mathfrak{gal}(\mathcal{F}_Y)_{\pi(p)}$. □

## 5. Quelques rappels sur les algèbres de Lie de champs de vecteurs

Nous aurons besoin dans la preuve du théorème d'irréductibilité de quelques faits sur les algèbres de Lie de champs de vecteurs. L'algèbre des champs formels sur $\widehat{\mathbb{C}^n,0}$ sera notée $\chi^n$ et l'espace des champs s'annulant à l'ordre $k$ sera $\chi^n_k$. Pour une algèbre de Lie de champs de vecteurs formels $A$, on notera $A_k = A \cap \chi^n_k$ la filtration induite.

***Définition 5.1.*** — *Soit $A$ un algèbre de Lie de champs formels. La croissance de $A$ est la suite d'entier $n_k(A) = \dim_{\mathbb{C}} A/A_k$. Le plus petit entier $p$ tel que la suite $\frac{n_k(A)}{k^p}$ soit bornée est le type de l'algèbre $A$.*

Une algèbre de type 0 sera aussi appelée de type fini, celles de types 1 seront dites de type linéaire etc.

***Définition 5.2.*** — *Soient $A$ une algèbre de Lie de champs formels et $\mathcal{F}$ un feuilletage de codimension $q$ régulier $A$-invariant de $\widehat{\mathbb{C}^n,0}$. L'algèbre quotient $A/(\mathcal{F} \cap A)$ est une algèbre de Lie de champs formels sur l'espace quotient $\widehat{\mathbb{C}^n,0}/\mathcal{F} = \widehat{\mathbb{C}^q,0}$.*

***Exemple 5.3.*** — *Soit $\gamma$ une 2-forme fermée régulière sur $\widehat{\mathbb{C}^3,0}$. On note $\mathfrak{inv}(\gamma)$ l'algèbre de Lie des champs préservant $\gamma$ (i.e. $X$ tel que $L_X\gamma = 0$) et $\mathcal{F}_\gamma$ le feuilletage donné par les champs $X$ tels que $\iota_X\gamma = 0$. On a l'inclusion $\mathcal{F}_\gamma \subset \mathfrak{inv}(\gamma)$ et l'algèbre quotient $\mathfrak{inv}(\gamma)/\mathcal{F}_\gamma$ est l'algèbre de Lie des champs de vecteurs sur $\widehat{\mathbb{C}^2,0}$ préservant une 2-forme $\underline{\gamma}$.*

*Cette algèbre de Lie est simple, on en déduit que les idéaux de $\mathfrak{inv}(\gamma)$ sont inclus dans $\mathcal{F}_\gamma$*

On obtient facilement les lemmes suivants.

***Lemmes 5.4.*** — 1°) *Si $B \subset A$ sont deux algèbres de Lie de champs formels, le type de $B$ est inférieur ou égal au type de $A$.*

2°) *Si $\pi : \widehat{\mathbb{C}^n,0} \to \widehat{\mathbb{C}^p,0}$ est une surjection induisant un morphisme surjectif d'algèbres de Lie entre $A \subset \chi^n$ et $B \subset \chi^p$ alors le type de $B$ est inférieur ou égal au type de $A$.*

***Exemple 5.5.*** — *Soit $\mathfrak{inv}(\Theta)$ l'algèbre des champs formels préservant une matrice de formes $\Theta$ vérifiant $d\Theta = \Theta \wedge \Theta$. Notons $\mathcal{F}$ le feuilletage définie par $\Theta$ que nous supposerons régulier et de codimension $q$. En passant au quotient par $\mathcal{F}$, on obtient l'algèbre de Lie des champs de vecteurs formels sur $\widehat{\mathbb{C}^q,0}$ préservant au moins un* *co-parallélisme donné par certains coefficients de $\Theta$. Ce type d'algèbre est de type fini* [**10**].

***Exemple 5.6***. — *Soient $h$ et $k$ des coordonnées sur $\widehat{\mathbb{C}^2, 0}$. L'algèbre* $\mathfrak{inv}(dh \wedge dk)$ *des champs formels préservant cette forme est l'ensemble des champs formels de divergence nulle, son type est quadratique.*

## 6. Irréductibilité de la première équation de Painlevé

***Théorème 6.1***. — *Soit $(Z, \delta_Z) \dashrightarrow (\mathbb{A}^1, \frac{\partial}{\partial x})$ une extention différentielle de degré de transcendance 2. S'il existe une 2-forme fermée $\gamma$ sur $Z$ telle que* $\mathrm{Gal}(\mathcal{F}_Z) = Inv(\gamma)$ *alors aucune application différentielle d'une extension réductrice de la droite affine dans $Z$ n'est dominante.*

*Autrement dit, si une trajectoire de $\delta_Z$ est réductible (i.e. a ses coordonnées dans une extension réductrice de $\mathbb{C}$) alors elle est algébrique ou appartient à une hypersurface $\delta_Z$-invariante de $Z$.*

*Démonstration.* — Supposons qu'il existe

$$(Y_m, \delta_m) \overset{\pi^m_{m-1}}{\dashrightarrow} (Y_{m-1}, \delta_{m-1}) \overset{\pi^{m-1}_{m-2}}{\dashrightarrow} \dots \overset{\pi^2_1}{\dashrightarrow} (Y_1, \delta_1) \overset{\pi^1_0}{\dashrightarrow} (\mathbb{A}_1, \frac{\partial}{\partial x})$$

une extension réductrice de la droite affine et $\varphi : (Y_m, \delta_m) \dashrightarrow (Z, \delta_Z)$ une application différentielle dominante. Nous noterons $\pi^m_i$ l'application de $Y_m$ dans $Y_i$ obtenue en composant les applications précédentes et $\mathcal{F}_1 \supset \cdots \supset \mathcal{F}_m$ les feuilletages sur $Y_m$ obtenus en ramenant ceux de $Y_i$ *i.e.* $\mathcal{F}_i = (\pi^m_i)^* \mathcal{F}_{Y_m}$. Soient $U$ un ouvert Zariski dense de $Y_m$ sur lequel $\mathrm{Gal}(\mathcal{F}_m)$ est un groupoïde et $p$ un point de $Y_m$ où les feuilletages sont réguliers et les projections régulières. Si $\mathcal{F}$ est un feuilletage sur $Y_m$, nous noterons $\mathfrak{gal}(\mathcal{F})$ l'algèbre de Lie des champs formels en $p$ solutions de $\mathcal{L}\mathrm{Gal}(\mathcal{F})$, $\mathcal{F}$ l'algèbre des champs formels en $p$ tangents au feuilletage et $\mathfrak{gal}(\mathcal{F})_\varphi$ l'algèbre $\mathfrak{gal}(\mathcal{F}) \cap \mathfrak{aut}(\varphi)$.

L'absence d'intégrale première pour $\mathcal{F}_Z$ et le lemme 4.6 assurent que la projection $\widehat{\varphi} : \widehat{Y_m, p} \to \widehat{Z, \varphi(p)}$ est surjective et induit un morphisme d'algèbre de Lie $\widehat{\varphi}_\star$ : $\mathfrak{gal}(\mathcal{F}_m)_\varphi \to \mathfrak{aut}(\mathcal{F}_Z)$ dont l'image est $\mathfrak{gal}(\mathcal{F}_Z)$. L'algèbre de Lie $\mathfrak{h}_{m-1} = \mathfrak{gal}(\mathcal{F}_m)_\varphi \cap \mathcal{F}_1$ est un idéal de $\mathfrak{gal}(\mathcal{F}_m)_\varphi$ contenant $\delta_{Y_m}$ . Son image est donc un idéal de $\mathfrak{gal}(\mathcal{F}_Z)$ contenant $\delta_Z$. Le passage au quotient donne un morphisme surjectif

$$\mathfrak{gal}(\mathcal{F}_m)_\varphi/\mathfrak{h}_{m-1} \to \mathfrak{gal}(\mathcal{F}_Z)/\widehat{\varphi}_\star(\mathfrak{h}_{m-1}).$$

Le type de $\mathfrak{gal}(\mathcal{F}_Z)/\widehat{\varphi}_\star(\mathfrak{h}_{m-1})$ doit être inférieur à celui de $\mathfrak{gal}(\mathcal{F}_m)_\varphi/\mathfrak{h}_{m-1}$, il est donc au plus linéaire. L'algèbre de Lie $\mathfrak{gal}(\mathcal{F}_Z)/\mathcal{F}_Z$ étant de type quadratique et simple, on a $\mathfrak{gal}(\mathcal{F}_Z) = \widehat{\varphi}_\star(\mathfrak{h}_{m-1})$.

Considérons maintenant $Z_{m-1}$ la feuille formelle de $\mathcal{F}_1$ passant par $p$. On a une application que nous appellerons encore $\widehat{\varphi} : Z_{m-1} \to \widehat{\mathbb{C}^3, \varphi(p)}$, surjective d'après les résultats précédents, donnée par le morphisme surjectif $\widehat{\varphi}_\star : \mathfrak{h}_{m-1} \to \mathfrak{gal}(\mathcal{F}_Z)$ d'algèbre de Lie. Notons $\mathfrak{h}_{m-2}$ l'idéal $\mathfrak{h}_{m-1} \cap \mathcal{F}_2$ de $\mathfrak{h}_{m-1}$. Le raisonnement précédent implique que $\mathfrak{gal}(\mathcal{F}_Z) = \widehat{\varphi}_\star(\mathfrak{h}_{m-2})$.

Par induction on en déduit que $\mathscr{F}_Z = \widehat{\varphi}_\star(\mathscr{F}_m)$ devrait être égale à $\mathfrak{gal}(\mathscr{F}_Z)$ ce qui n'est pas vrai. L'application ne peut donc pas être dominante. □

La première équation de Painlevé est modélisé par le champ de vecteurs sur $\mathbb{A}^3(\mathbb{C})$ :

$$\delta_P = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + (6y^2 + x)\frac{\partial}{\partial z}.$$

***Théorème 6.2***. — *La première équation de Painlevé est irréductible au sens de Nishioka-Umemura.*

Le théorème 6.1 réduit la preuve du théorème ci-dessus au lemme suivant.

***Lemme 6.3* (Painlevé [17])**. — *Une application différentielle à valeurs dans* $(\mathbb{C}^3, \delta_P)$ *est dominante.*

Vocabulaire mis à part, ce lemme est classique. Il affirme qu'aucune solution de la première équation de Painlevé n'est algébrique [**17, 22**] et qu'il n'existe pas d'hypersurface de $\mathbb{A}^3$ invariante sous $\delta_P$ [**3, 17**]. Ce dernier résultat admet une version plus générale appelée lemme de Kolchin-Kovacic [**16, 11, 22**].

## Références


[1] É. Cartan – Sur la structure des groupes infinis de transformation, *Ann. Sci. École Norm. Sup.* **22** (1905), p. 219–308.

[2] ———, Les sous-groupes des groupes continus de transformations, *Ann. Sci. École Norm. Sup.* **25** (1908), p. 57–194.

[3] G. Casale – Le groupoïde de Galois de $P_1$ et son irréductibilité, *Comment. Math. Helv.* **83** (2008), p. 471–519.

[4] J. Drach – Essai sur la théorie générale de l'itération et sur la classification des transcendantes, *Ann. Sci. École Norm. Sup.* **15** (1898), p. 243–384.

[5] ———, Sur le groupe de rationalité des équations du second ordre de M. Painlevé, *Bull. Sci. Math.* **39** (1915), p. 149–166.

[6] P. Gabriel – Construction de préschémas quotients, in *Schémas en groupes (SGA 63-64), fasc. 2a*, Lecture Notes in Math., vol. 151, Springer, 1970, p. 250–283.

[7] C. Godbillon – *Feuilletages. Études géométriques*, Progress in Mathematics, vol. 98, Birkhäuser, 1991.

[8] X. Gómez-Mont – Integrals for holomorphic foliations with singularities having all leaves compact, *Ann. Inst. Fourier (Grenoble)* **39** (1989), p. 451–458.

[9] R. Hartshorne – *Algebraic geometry*, Springer, 1977, Graduate Texts in Mathematics, No. 52.

[10] S. Kobayashi – *Transformation groups in differential geometry*, Classics in Mathematics, Springer, 1995, Reprint of the 1972 edition.

[11] E. Kolchin – *Selected works of Ellis Kolchin with commentary*, Amer. Math. Soc., 1999, edited and with a preface by Hyman Bass, Buium and Cassidy.

[12] S. Lie – Theorie der Transformationsgruppen I, *Math. Ann.* **16** (1880), p. 441–528.

[13] K. Mackenzie – *Lie groupoids and Lie algebroids in differential geometry*, London Mathematical Society Lecture Note Series, vol. 124, Cambridge University Press, 1987.

[14] B. Malgrange – La variété caractéristique d'un système différentiel analytique, *Ann. Inst. Fourier (Grenoble)* **50** (2000), p. 491–518.

[15] ———, Le groupoïde de Galois d'un feuilletage, in *Essays on geometry and related topics, Vol. 1, 2*, Monogr. Enseign. Math., vol. 38, Enseignement Math., 2001, p. 465–501.

[16] K. Nishioka – A note on the transcendency of Painlevé's first transcendent, *Nagoya Math. J.* **109** (1988), p. 63–67.

[17] P. Painlevé – Mémoire sur les équations différentielles dont l'intégrale générale est uniforme, *Bull. Soc. Math. France* **28** (1900), p. 201–261.

[18] ———, Démonstration de l'irréductibilité absolue de l'équation $y_{xx} = 6y^2 + x$, *C. R. Acad. Sci. Paris* **135** (1902), p. 641–647.

[19] ———, Leçons de Stokholm, in *Œuvres complètes*, vol. 1, Éditions du CNRS, 1972.

[20] J. F. Ritt – *Differential Algebra*, American Mathematical Society Colloquium Publications, Vol. XXXIII, Amer. Math. Soc., 1950.

[21] M.-H. Saito & H. Terajima – Nodal curves and Riccati solutions of Painlevé equations, *J. Math. Kyoto Univ.* **44** (2004), p. 529–568.

[22] H. Umemura – On the irreducibility of the first differential equation of Painlevé, in *Algebraic geometry and commutative algebra, Vol. II*, Kinokuniya, 1988, p. 771–789.

[23] ———, Second proof of the irreducibility of the first differential equation of Painlevé, *Nagoya Math. J.* **117** (1990), p. 125–171.

[24] ———, Differential Galois theory of infinite dimension, *Nagoya Math. J.* **144** (1996), p. 59–135.

[25] ———, Galois theory of algebraic and differential equations, *Nagoya Math. J.* **144** (1996), p. 1–58.

[26] E. Vessiot – Sur la théorie de Galois et ses diverses généralisations, *Ann. Sci. École Norm. Sup.* **21** (1904), p. 9–85.

[27] ———, Sur la réductibilité et l'intégration des systèmes complets, *Ann. Sci. École Norm. Sup.* **29** (1912), p. 209–278.

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# FEUILLETAGES EN DROITES, ÉQUATIONS DES EIKONALES ET AUTRES ÉQUATIONS DIFFÉRENTIELLES

*par*

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*À José Mañuel Aroca, épicurien et grand connaisseur de la géométrie classique*

***Résumé.*** — Nous donnons des résultats qui lient les équations classiques des eikonales et les feuilletages en droites de l'espace affine.

***Abstract* (Foliations by straight lines, equations of the eikonales and other differential equations)**
We give several results relating the classical eikonal equations and foliations by straight lines of the affine space.

## Introduction

Les solutions globales de certaines équations aux dérivées partielles satisfont souvent à des principes de rigidité étonnants. Ainsi un célèbre théorème de S. Bernstein affirme que si $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ est de classe $C^2$ et a son graphe minimal, i.e. $f$ satisfait à l'équation des surfaces minimales, alors $f$ est une fonction affine. Dans cet article on s'intéresse en particulier aux solutions rationnelles $f : \mathbb{C}^n \dashrightarrow \mathbb{C}$ de l'équation des eikonales $E(f) = c^2$ où $c \in \mathbb{C}$ et $E$ désigne l'opérateur

$$E(f) = \sum \left(\frac{\partial f}{\partial z_i}\right)^2 . \tag{1}$$

Cette équation est un cas spécial de l'équation différentielle :

$$\text{dét Hess } (f) = 0 \tag{2}$$

où Hess $f$ désigne la matrice Hessienne de $f$.

Si $f$ satisfait à l'équation (2) l'application de Gauss :

$$G_f : z \rightsquigarrow \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right) \tag{3}$$

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est dégénérée, i.e. n'est pas de rang maximal. Si $f$ est rationnelle l'adhérence de l'image de $G_f$ est contenue dans une hypersurface algébrique. En particulier il existe un polynôme $P$ irréductible tel que

$$P\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right) = 0 \tag{4}$$

et $f$ est solution d'une équation différentielle ne mettant en jeu que les dérivées partielles du premier ordre. Par exemple toute solution rationnelle $f$ de l'équation de Monge–Ampère :

$$\frac{\partial^2 f}{\partial z_1^2} \cdot \frac{\partial^2 f}{\partial z_2^2} - \left(\frac{\partial^2 f}{\partial z_1 \partial z_2}\right)^2 = 1 \tag{5}$$

satisfait à une équation de type 4.

Dans le cas réel il existe aussi un résultat de rigidité concernant les solutions globales de (5) ; plus précisément si

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

de classe $C^2$ est solution de l'équation de Monge-Ampère (5), alors $f$ est un polynôme de degré inférieur ou égal à 2. Cet énoncé est attribué à Jörgen ; la démonstration qui s'appuie sur le théorème de Picard (une fonction entière qui évite deux points est constante) permet de donner une preuve alternative à l'énoncé de Bernstein. Cette idée est attribuée à E. Bombieri.

On a longtemps cru avec Hesse qu'un polynôme homogène $f_\nu$ sur $\mathbb{C}^n$ satisfaisant à l'équation (2) dépendait en fait de moins de $n$ variables, cas où l'on peut trouver un polynôme $P$ satisfaisant à (4) de degré 1. Cet énoncé est correct en petite dimension $n \leq 4$ mais ne l'est plus dès la dimension 5. L'exemple qui suit dû à Gordan et Noether **[4]** est très populaire car il intervient dans différents contextes **[3]**. Il s'agit de :

$$\varphi(z) = z_1^2 z_3 + z_1 z_2 z_4 + z_2^2 z_5. \tag{6}$$

On remarque d'enblée que $\varphi$ est linéaire dans les variables $z_3, z_4, z_5$. Un calcul élémentaire montre que $\varphi$ satisfait à l'équation différentielle :

$$\frac{\partial \varphi}{\partial z_3} \cdot \frac{\partial \varphi}{\partial z_5} - \left(\frac{\partial \varphi}{\partial z_4}\right)^2 = 0. \tag{7}$$

Comme l'application de Gauss $G_\varphi$ est de rang maximal, l'adhérence de l'image de $G_\varphi$ est la quadrique de $\mathbb{C}^5$ définie par $X_3 X_5 - X_4^2 = 0$. Par suite il n'existe pas de polynôme $P$ non constant de degré 1 tel que l'on ait (4) : $P\left(\frac{\partial \varphi}{\partial z}\right) = 0$.

Revenons à l'équation des eikonales ; elle trouve son importance dans les fondements de l'optique géométrique et aussi en mécanique (Malus, Fresnel, Hamilton...). Si $f$ rationnelle satisfait à $E(f) = c^2$ alors l'image de l'application de Gauss $G_f$ est contenue dans la quadrique $X_1^2 + \cdots + X_n^2 = c^2$. Considérons, les coordonnées $z_1, \ldots, z_n$ étant fixées (à l'action près du groupe engendré par les translations et le groupe orthogonal complexe $O(n,\mathbb{C})$) le champ de vecteur rationnel $X = \text{grad } f$ défini par :

$$X = \text{grad } f = \sum \frac{\partial f}{\partial z_i}\frac{\partial}{\partial z_i}. \tag{8}$$

Comme on le sait au moins depuis Hamilton [**5**], si $\exp tX$ est le flot (local, là où il a sens) du champ de vecteur $X$ alors

$$f \circ \exp tX(x) = f(x) + c^2 t \tag{9}$$

(10) et les trajectoires de $X$ sont contenues dans des droites (en fait sont d'adhérence des droites).

On doit imaginer, tout du moins en réel, les niveau de $f$ comme un front d'onde (fig. 1) se déplaçant à la vitesse $c^2$ le long des droites paramétrées par :

$$t \rightsquigarrow z + tX(z). \tag{11}$$

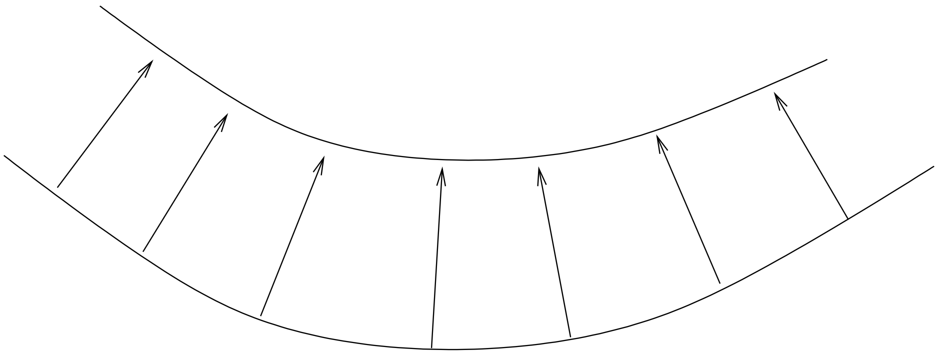

FIGURE 1.

On constate ainsi l'apparition naturelle de feuilletages algébriques de $\mathbb{C}^n$, et par suite de $\mathbb{CP}(n)$, dont les feuilles sont d'adhérence des droites. Ce sujet était très populaire chez les anciens géomètres en particulier pour l'étude des surfaces et l'intégration géométrique de certaines équations aux dérivées partielles. Dans ses « Leçons sur la théorie générale des surfaces », Gaston Darboux y consacre un volume entier « les congruences de droites et les équations aux dérivées partielles ». Il attribue à Malus le fait d'avoir le premier considéré « de tels assemblages de droites ».

En 1988 dans une courte note aux *C.R.A.S.* [**2**] j'ai proposé la classification des feuilletages (singuliers) en « droites » de $\mathbb{C}^3$, répondant en celà à des préocupations de René Thom et de l'astronome Pecker. J.V. Pereira m'a indiqué que cet énoncé était connu de Kummer. On trouvera en particulier dans cet article des résultats annoncés dans cette note ainsi qu'une application à l'étude des solutions rationnelles de l'équations des eikonales. Les énoncés précis sont dans le chapitre qui suit.

## 1. Champs de droites ou feuilletages en droites. Exemples et premiers résultats

Un feuilletage $\mathcal{D}$ en droites de l'espace $\mathbb{CP}(n)$, ou de l'espace affine $\mathbb{C}^n$, est par définition un feuilletage algébrique (singulier) de dimension 1, tel qu'en tout point régulier $m$ la feuille $\mathcal{L}_m$ passant par $m$ soit contenue dans une droite $D_m$. Si $\operatorname{Sing}\mathcal{D}$ désigne l'ensemble singulier de $\mathcal{D}$, $\operatorname{Sing}\mathcal{D}$ est un ensemble de codimension supérieure ou égale à deux ; visiblement $\mathcal{L}_m = D_m - \operatorname{Sing}\mathcal{D}$ pour tout point régulier $m$. Si $\mathbb{C}^n \subset \mathbb{CP}(n)$ est une carte affine la restriction $\mathcal{D}_{/\mathbb{C}^n}$ est donnée par un champ de vecteurs polynomial :

$$X = \sum_{i=1}^{n} X_i(z)\frac{\partial}{\partial z_i}$$

où les $X_i$ sont des polynômes tels que p.g.c.d $(X_1, \dots, X_n) = 1$. On a :

$$\operatorname{Sing}\mathcal{D} \cap \mathbb{C}^n = \{z \in \mathbb{C}^n \mid X_1(z) = \cdots = X_n(z)) = 0\}.$$

Si $m$ est un point régulier de $\mathcal{D}$, par $m$ passe évidemment une seule droite $D_m$ tangente à $\mathcal{D}$. A l'inverse, et nous le préciserons plus loin, si $m$ est un point singulier de $\mathcal{D}$ par $m$ passent une infinité de droites qui sont, en dehors de $\operatorname{Sing}\mathcal{D}$, des feuilles de $\mathcal{D}$.

Le flot du champ $X$ satisfait à l'équation différentielle

$$\dot{z}_i(t) = X_i(z(t)), \quad z(0) = m. \tag{12}$$

Dire que la trajectoire de $m$, paramétrée par $t \rightsquigarrow z(t)$, est rectiligne revient à dire que $\dot{z}(t)$ et $\ddot{z}(t)$ sont colinéaires. Comme

$$\begin{aligned} \ddot{z}_i(t) &= \sum_{k=1}^{n} \frac{\partial X_i}{\partial z_k}(z(t)).\dot{z}_k(t) = \sum_{k=1}^{n} X_k(z(t)) \cdot \frac{\partial X_k}{\partial z_k}(z(t)) \\ &= X(X_i)(z(t)) \end{aligned} \tag{13}$$

on constate que $X$ définit un feuilletage en droites si et seulement si :

$$\sum X(X_i)\frac{\partial}{\partial z_i} = \mu \cdot \sum_{i=1}^{n} X_i \frac{\partial}{\partial z_i} \tag{14}$$

pour un certain polynôme $\mu$, où $X(X_i)$ est la dérivée de $X_i$ le long de $X$.

En particulier les quotients $X_i/X_j$ sont des intégrales premières rationnelles du champ $X$.

À titre d'exemple considérons dans $\mathbb{C}^2$ un feuilletage en droites défini par le champ de vecteur local :

$$X = X_1\frac{\partial}{\partial z_1} + X_2\frac{\partial}{\partial z_2}.$$

Alors $a = X_2/X_1$ est solution de l'équation de Burger :

$$\frac{\partial a}{\partial z_1} + a \cdot \frac{\partial a}{\partial z_2} = 0.$$

Cet exemple d'équation différentielle non linéaire est bien connu des physiciens.

En dimension deux, dans $\mathbb{CP}(2)$, les pinceaux de droites sont des exemples de champs de droites. Nous verrons plus loin que ce sont les seuls.

En dimension trois c'est un peu plus compliqué. Dans $\mathbb{CP}(3)$, considérons un pinceau de plans donné dans la carte affine $\mathbb{C}^3 = \{(z_1, z_2, z_3)\}$ par $z_1/z_2 = t$. On se donne une application rationnelle du type :

$$t \rightsquigarrow (tz_2(t), z_2(t), z_3(t)) = m(t) \tag{15}$$

Une telle application revient à se donner dans chaque plan $z_1/z_2 = t$ un point $m(t)$. Maintenant dans chacun de ces plans on considère le pinceau de droites de point de base $m(t)$. On feuillette ainsi $\mathbb{CP}(3)$ (ou $\mathbb{C}^3$) en droites ; l'ensemble singulier est constitué de l'union de l'axe des $z_3$ et de l'image de l'application (15), (fig 2) :

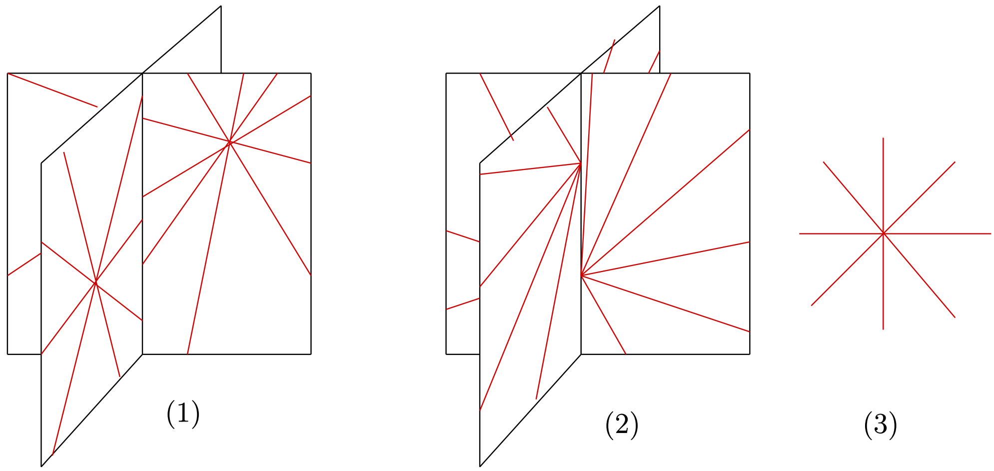


FIGURE 2.

Le premier cas est l'exemple générique. Dans le second, l'image de l'application (15) coincide avec l'axe des $z_3$. Dans ces deux éventualités nous dirons que nous avons un feuilletage radial dans les pages d'un livre ouvert. Dans le dernier l'application (15) est constante ; le champ de droites est « radial » au point $m = m(t)$.

Enfin introduisons la cubique gauche $\Sigma$ paramétrée dans une carte affine par

$$t \rightsquigarrow (t, t^2, t^3) = \gamma(t). \tag{16}$$

Pour chaque couple de points $\gamma(t_1)$ et $\gamma(t_2)$ de $\Sigma$ on mène la sécante à $\Sigma$ par ces deux points, convenant que si $t_1 = t_2$ il s'agit de la tangente à $\Sigma$ en $\gamma(t_1)$. On obtient ici encore un feuilletage de $\mathbb{CP}(3)$ (ou $\mathbb{C}^3$) que nous appellerons feuilletage associé à la cubique gauche $\Sigma$ (fig 3).

Parmi nos résultats en voici deux qu'il est aisé d'énoncer. Bien que leur preuves ne fassent intervenir que des arguments anciens et classiques, nous n'en avons pas trouvé trace.

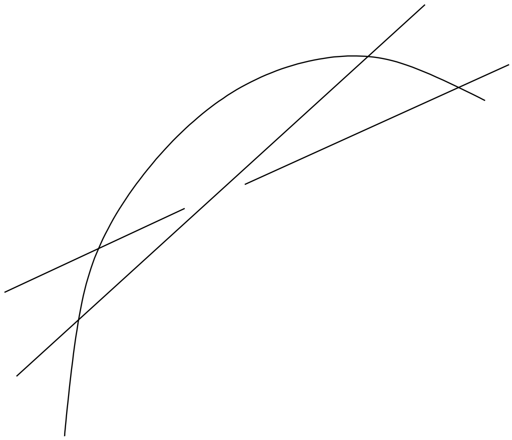

FIGURE 3.

***Théorème 1.1***. — *Soit $\mathcal{D}$ un feuilletage en droites de $\mathbb{CP}(3)$. Alors $\mathcal{D}$ est linéairement conjugué à l'un des exemples précédents. Plus précisément $\mathcal{D}$ est de l'un des trois types suivants :*

1. *feuilletage radial en un point ;*
2. *feuilletage radial dans les pages d'un livre ouvert ;*
3. *feuilletage associé aux cordes d'une cubique gauche.*

Nous étudierons quelques équations différentielles de type (2). En particulier en adaptant le théorème 1.1 on obtiendra le :

***Théorème 1.2***. — *Soit $f : \mathbb{C}^3 \dashrightarrow \mathbb{C}$ une solution rationnelle de l'équation aux eikonales $E(f) = c^2$, $c \in \mathbb{C}^*$. Il existe deux formes linéaires $L_1 = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3$, $L_2 = \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3$ et $\ell \in \mathbb{C}(t)$ tels que :*

$$f(z) = L_1 + \ell(L_2)$$

*avec $\|\alpha\|^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = c^2$, $\|\beta\|^2 = 0$, $< \alpha|\beta >= \sum \alpha_i \beta_i = 0$. Si $c = 0$, alors $f$ est affine, $f = L_1$ avec $\|\alpha\|^2 = 0$.*

On en déduit facilement le :

***Corollaire 1.3***. — *Soit $f : \mathbb{R}^3 \dashrightarrow \mathbb{R}$ une solution rationnelle réelle de l'équation aux eikonales. Alors $f$ est une fonction affine*

## 2. Généralités sur les feuilletages en droites sur les espaces projectifs

Tous les champs de vecteurs holomorphes rencontrés seront supposés satisfaire à $\operatorname{cod} \operatorname{Sing} X \geq 2$, condition à laquelle on se ramène en divisant par le p.g.c.d des composantes de $X$.

Commençons par un énoncé facile en dimension 2 dont on retrouvera l'argument naïf plusieurs fois.

***Proposition 2.1***. — *Soit $\mathcal{D}$ un feuilletage en droites sur l'espace projectif $\mathbb{CP}(2)$. Alors $\mathcal{D}$ est radial, i.e. correspond à un pinceau de droites concourantes.*

*Démonstration*. — Soit $\mathbb{C}^2 \subset \mathbb{CP}(2)$. Si $\mathcal{D}_{/\mathbb{C}^2}$ n'a pas de singularité alors les feuilles de $\mathcal{D}_{/\mathbb{C}^2}$ sont des droites parallèles. En particulier si $\mathcal{D}$ a une seule singularité il s'agit d'un pinceau de droites. En général si $D_1$ et $D_2$ sont deux droites tangentes à $\mathcal{D}$, elles se coupent en un point $M$. Un argument combinatoire immédiat montre que toutes les autres droites passent par $M$. □

Nous donnons quelques énoncés précisant le comportement des feuilles d'un champ de droites aux points singuliers.

***Lemme 2.2***. — *Soit $X$ un champ de vecteurs holomorphes définissant un feuilletage (singulier) d'une boule $B(0,\rho)$ dans $\mathbb{C}^n$. On suppose que chaque feuille régulière $\mathcal{L}_m$ est contenue dans une droite $D_m$. Si $M$ est un point singulier de $X$, il existe au moins une droite $D$ passant par $M$ et tangente à $X$ ; éventuellement $D$ est contenue dans l'ensemble singulier de $X$.*

*Démonstration*. — On considère la variété d'incidence :

$I := \{(x,D), x \in D \subset \mathbb{CP}(n),\ D \text{ droite de } \mathbb{CP}(n)\}$. Soit $m_i \in B(0,p) - \operatorname{Sing} X$ une suite de points réguliers convergeant vers le point $M$. Comme la variété $I$ est compacte, la suite $(m_i, D_{mi})$ possède une sous suite convergente vers $(M,D)$. Par continuité $X$ est tangent à $D$. □

Nous précisons le lemme précédent :

***Lemme 2.3***. — *Soit $X$ comme dans le lemme 2.2. On note $\mathcal{C}_M$ l'ensemble des droites $D$ passant par $M$ tel que $X$ soit tangent à $D$ (sur $B(0,p)$). Si le nombre de droites de $\mathcal{C}_M$ est fini alors $M$ est non singulier.*

*Démonstration*. — On choisit des coordonnées $z_1, \dots, z_n$ en $M$ telles que les droites $D_1 \cdots, D_s$ de $\mathcal{C}_M$ ne soient pas contenues dans l'hyperplan horizontal $z_n = 0$ et telles que $\operatorname{Sing} X \cap (z_n = 0)$ soit de codimension $\geq 3$. C'est possible.

Écrivons

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial z_i}.$$

Supposons que $X$ s'annule en $M = 0$ ; alors l'ensemble $\sum = \{(X_n = 0) \cap (z_n = 0)\}$ est non trivial. On choisit une suite $m_i \in \sum$ et convergeant vers $M$. Alors les $D_{m_i}$ sont des droites horizontales et produisent des droites limites passant par $M$, horizontales et donc différentes des $D_1, \dots, D_s$. Ceci est absurde et par conséquent $X$ est non singulier en $M$. □

***Corollaire 2.4***. — *Si $M$ est un point singulier de $X$ l'ensemble $\mathcal{C}_M$ est un cône algébrique de dimension $\geq 2$.*

*Démonstration.* — $\mathscr{C}_M$ est un ensemble analytique ; comme c'est un cône, le théorème de Chow assure qu'il est algébrique. □

Voici encore une précision qui sera utile pour décrire les singularités de feuilletages en droites.

***Lemme 2.5.*** — *Soit $X$ holomorphe sur $B(0,\rho)$ et non singulier sur $B(0,\rho)-\{0\}$. On suppose que les feuilles régulières $\mathscr{L}_m$ de $X$ sont contenues dans des droites $D_m$, $m \in B(0,\rho)-\{0\}$. Soit $D$ une droite tangente à $X$ passant par $0$. Si $D$ est isolée dans le cône $\mathscr{C}_0$ (i.e. $D$ est une composante irréductible de $\mathscr{C}_0$) alors $0$ est un point régulier de $X$.*

*Démonstration.* — Sans perdre de généralité on suppose que $D$ est l'axe des $z_n$. On peut supposer aussi que $\rho > 1$. Soit $H$ l'hyperplan $z_n = 1$. Alors au voisinage du point $M_0 = (0,\ldots,0,1)$, le champ $X$ est transverse à $H$. Par suite au voisinage de $M_0$ les feuilles sont paramétrées par les applications :

$$z'_n \rightsquigarrow (z_1 + z'_n\eta_1, \ldots, z_{n-1} + z'_n\eta_{n-1}, 1 + z'_n)$$

où $z_n = 1 + z'_n$, les $\eta_i$ sont holomorphes sur un voisinage de $M_0$ dans $H$ et satisfont à $\eta_i(M_0) = 0$.

Evidemment les applications précédentes sont globales en $z_n$. Comme la singularité éventuelle de $X$ en $0$ est isolée dans $B(0,\rho)$ ainsi que dans $\mathscr{C}_0$, les droites ci-dessus ne se coupent pas dans $B(0,\rho)$ et forment un voisinage de $0$. D'autre part elles définissent visiblement un feuilletage régulier au voisinage de $0$. D'où le lemme. □

L'énoncé qui suit décrit les feuilletages en droites locaux à singularité isolée.

***Proposition 2.6.*** — *Soit $X$ un champ de vecteur holomorphe sur la boule $B(0,\rho)$ dans $\mathbb{C}^n$ à singularité isolée en $0$. On suppose que les trajectoires de $X$ sont contenues dans des droites. Alors $X$ définit le feuilletage radial en $0$, i.e. à unité holomorphe multiplicative près $X = \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}$.*

*Démonstration.* — Nous voulons montrer que chaque droite $D$ passant par $0$ est tangente à $X$. Soit $D$ une telle droite que l'on suppose être l'axe des $z_1$. Écrivons :

$$X = \sum_{i=1}^n X_i \frac{\partial}{\partial z_i}$$

les $X_i$ étant holomorphes, $X_i(0) = 0$.

Plaçons nous sur l'ensemble analytique $\gamma$ :

$$\gamma := \{X_2 = \cdots = X_n = 0\}.$$

Comme $X$ est à singularité isolée, $\gamma$ est une courbe passant par $0$ sur laquelle $X_1$ ne s'annule qu'en $0$ quitte à restreindre. Soit $m_i \in \gamma$ une suite tendant vers $0$ ; visiblement avec les notations habituelles $D_{m_i}$ est parallèle à $D$ et nécessairement $(m_i, D_{m_i})$ converge vers $(0, D)$. Par suite $X$ est tangent à $D$. □

***Remarques***. — 1. *on retrouve ainsi la preuve de la proposition 2.1.*

2. *Évidemment l'énoncé se globalise. Un feuilletage en droites de* $\mathbb{C}^n$ *ou* $\mathbb{CP}(n)$ *ayant un point singulier isolé est radial.*

## 3. Classification des feuilletages en droites dans $\mathbb{CP}(3)$

Soit $\mathcal{D}$ un feuilletage en droite dans $\mathbb{CP}(3)$. On peut supposer que $\mathcal{D}$ n'a pas de singularité isolée. Comme tout feuilletage de $\mathbb{CP}(3)$ a des singularités, l'ensemble singulier $\operatorname{Sing}\mathcal{D}$ est de dimension pure 1 et est donc composé de l'union $\Gamma_1 \cup \cdots \cup \Gamma_s$ de courbes irréductibles. Si $M \in \operatorname{Sing}\mathcal{D}$, on note encore $\mathcal{C}_M$ l'union des droites $D$ tangentes à $\mathcal{D}$ et qui passent par $M$. D'après le corollaire 2.4, $\mathcal{C}_M$ est un cône algébrique de dimension 2, avec éventuellement des branches de dimension 1. Mais le lemme 2.5 indique que l'ensemble :

$$\mathcal{C}^*_M = \overline{\mathcal{C}_M - \operatorname{Sing}\mathcal{D}}$$

est une surface algébrique conique en $M$ ; visiblement l'union :

$$\bigcup_{M\in\Gamma_i} \mathcal{C}^*_M \tag{17}$$

est l'espace $\mathbb{CP}(3)$ tout entier pour $i = 1, \ldots, s$

Cette remarque implique que toute droite $D$ tangente à $\mathcal{D}$ coupe en au moins un point chaque composante $\Gamma_i$ du lieu singulier $\operatorname{Sing}\mathcal{D}$. En particulier on obtient comme conséquence la

***Proposition 3.1***. — *Soit $\mathcal{D}$ un feuilletage en droites de* $\mathbb{CP}(3)$ *dont l'ensemble singulier contient une droite $D$. Alors $\mathcal{D}$ est radial dans les pages d'un livre ouvert.*

*Démonstration*. — On considère le pinceau $P$ des plans contenant $D$. Si $m \in \mathbb{CP}(3) - \operatorname{Sing}\mathcal{D}$ la droite $D_m$ coupe $D$. En particulier chaque plan $\pi$ de $P$ est $\mathcal{D}$ invariant et $\mathcal{D}_{/\pi}$ est un pinceau linéaire de droites. □

***Remarque***. — *La description précise de ce type de feuilletages, en particulier des singularités, est donnée en 1.*

Pour terminer la classification, on utilise avec les notations habituelles le :

***Lemme 3.2***. — *L'ensemble singulier* $\operatorname{Sing}\mathcal{D}$ *a au plus 2 composantes.*

*Démonstration*. — Soit $m$ un point régulier et $D = D_m$ la droite tangente à $\mathcal{D}$ passant par $m$. Choisissons des coordonnées affines $(z_1, z_2, z_3)$ telles que $m = (0,0,0)$ et $D$ soit l'axe des $z_3$. Comme dans le lemme 2.5 nous paramétrons les feuilles de $\mathcal{D}$ par les applications :

$$z_3 \rightsquigarrow (z_1 + z_3\eta_1, z_2 + z_3\eta_2, z_3) = F(z_1, z_2, z_3)$$

où les $\eta_i$ sont des fonctions rationnelles en $(z_1, z_2)$ régulières en $(0,0)$.

Puisque $\mathcal{D}$ est régulier en 0, l'application $F$ est un difféomorphisme local en 0; en particulier le déterminant Jacobien :

$$\text{dé}JF = 1 + z_3 \left( \frac{\partial \eta_1}{\partial z_1} + \frac{\partial \eta_2}{\partial z_2} \right)(z_1, z_2) + z_3^2 \left( \frac{\partial \eta_1}{\partial z_1} \cdot \frac{\partial \eta_2}{\partial z_2} - \frac{\partial \eta_1}{\partial z_2} \cdot \frac{\partial \eta_2}{\partial z_1} \right)(z_1, z_2)$$

est non nul en $(0,0,0)$. Ceci implique qu'à $(z_1, z_2)$ fixés dét$JF$ ne s'annule qu'en deux valeurs de $z_3$ au plus. Comme toute droite de $\mathcal{D}$ coupe chaque composante du lieu singulier et qu'en chaque point singulier passent une infinité de droites de $\mathcal{D}$, on en déduit que Sing $\mathcal{D}$ a au plus deux composantes irréductibles. □

Supposons que Sing $\mathcal{D} = \Gamma_1 \cup \Gamma_2$. Soit $L \subset \mathbb{CP}(3)$ une droite évitant Sing $\mathcal{D}$ et donc non tangente à $\mathcal{D}$. En particulier, pour tout point $m \in L$ la droite $D_m$ est transverse à $L$ et coupe $\Gamma_i$ en un seul point $M_i(m)$. Ceci démontre que $\Gamma_1$ et $\Gamma_2$ sont rationnelles.

D'après (17) l'ensemble des droites constituant $\mathcal{D}$ est précisément l'ensemble des droites joignant $\Gamma_1$ à $\Gamma_2$. En particulier les courbes $\Gamma_1$ et $\Gamma_2$ ne sont pas situées dans un même plan. On remarque aussi que si $M_1$ est un point générique de $\Gamma_1$, le cône $\mathcal{C}_{M_1}$ contient $\Gamma_2$ : en effet, toute droite $D$ de $\mathcal{C}_{M_1}$ doit couper $\Gamma_2$.
Choisissons un 2-plan général $\pi$ et soient $\{M_1, \dots, M_t\} = \Gamma_1 \cap \pi$ , $\{m_1, \dots, m_s\} = \Gamma_2 \cap \pi$.

Les droites $[m_i M_k] \subset \pi$ joignant $m_i$ à $M_k$ sont dans $\mathcal{D}$. Mais sur chaque droite régulière, il y a au plus 2 points singuliers. Par suite $s$ ou $t$ vaut 1, i.e. l'une des composantes $\Gamma_i$ est une droite; cas décrit par la proposition 3.1.

Dans la suite on suppose que Sing $\mathcal{D}$ se réduit à une seule courbe irréductible $\Gamma$; toujours d'après la proposition 3.1 on suppose encore que $\Gamma$ n'est pas une droite. Considérons deux points distincts $M_1$ et $M_2$ de $\Gamma$.

Les cônes $\mathcal{C}_{M_1}$ et $\mathcal{C}_{M_2}$ se coupent le long d'une courbe qui est donc nécessairement $\Gamma$. Par suite une droite générique de $\mathcal{C}_{M_1}$ coupe $\Gamma$ en $M_1$ et en un autre point de $\Gamma$. Les feuille de $\mathcal{D}$ sont donc exactement les cordes de $\Gamma$ (privées des singularités) et les tangentes à $\Gamma$. Évidemment celà implique que $\Gamma$ est une courbe gauche. Nous allons montrer que $\Gamma$ est une cubique gauche. Comme toujours choisissons $\pi$ un plan général. Alors $\Gamma \cap \pi = \{M_1, \dots, M_s\}$ avec $s \geq 3$ puisque $\Gamma$ est irréductible non plane. Les droites $D_{ij}$ joignant les $M_i$ à $M_j$ sont des droites de $\mathcal{D}$. Par connexité de $\Gamma$, en chaque $M_i$ on a la même configuration pour les droites $D_{ij}$ ; d'autre part les droites $D_{ij}$ se coupent en des points de $\Gamma \cap M$. Comme sur chaque $D_{ij}$ on ne peut avoir que deux $M_k$, c'est-à-dire $M_i$ et $M_j$, nécessairement $s = 3$.

Maintenant à transformation linéaire près il n'y a qu'une courbe gauche de degré 3 dans $\mathbb{CP}^3$ : la cubique gauche rationnelle $\sum$ paramétrée par (16). Nous avons ainsi démontré le théorème 1.1.

## 4. L'équation dét Hess $f \equiv 0$

Considérons une solution rationnelle $f : \mathbb{C}^n \dashrightarrow \mathbb{C}$ non triviale de l'équation (2) dét Hess $f \equiv 0$. Lorsque l'application de Gauss $G_f$ de $f$ est de rang générique $n-1$ nous dirons que $f$ est une solution maximale de (2). L'idéal $H(f)$ des polynômes

$Q \in \mathbb{C}[z_1, \ldots, z_n]$ tels que $Q\left(\frac{\partial f}{\partial z}\right)$ est alors engendré par un polynôme irréductible $P$ ; c'est le cas comme nous l'avons vu pour le polynôme de Gordan et Noether.

Les polynômes $P$ que l'on peut ainsi obtenir ne sont certainement pas quelconques. Soient $f$ solution maximale de (2) et $P$ un polynôme générateur de $H(f)$. La restriction de l'application de Gauss $G_f$ à un hyperplan général paramètre les zéros de $P$ : ainsi l'ensemble $(P = 0)$ est une hypersurface unirationnelle.

Examinons plus précisément le cas de la dimension deux. Il existe alors une application rationnelle :

$$\begin{aligned} r : \mathbb{C} &\dashrightarrow \mathbb{C}^2 \\ t &\rightsquigarrow r(t) = (r_1(t), r_2(t))\ ,\ r_i \in \mathbb{C}(t) \end{aligned}$$

génériquement injective telle que $\overline{r(\mathbb{CP}(1)} = \overline{(P = 0)} \subset \mathbb{CP}(2)$.

Par suite on dispose d'une factorisation :

$$\begin{cases} \frac{\partial f}{\partial z_1}(z_1, z_2) & = r_1(\tau(z_1, z_2)) \\ \frac{\partial f}{\partial z_2}(z_1, z_2) & = r_2(\tau(z_1, z_2)) \end{cases} \tag{18}$$

où $\tau : \mathbb{C}^2 \dashrightarrow \mathbb{C}$ est rationnelle.

Un calcul élémentaire montre que :

$$\frac{\partial r_1}{\partial t}(\tau(z)) \cdot \frac{\partial \tau}{\partial z_2} = \frac{\partial r_2}{\partial t}(\tau(z)) \cdot \frac{\partial \tau}{\partial z_1}. \tag{19}$$

En particulier le long d'un niveau $\tau = \text{cste}$, la pente $\frac{\partial \tau}{\partial z_2} / \frac{\partial \tau}{\partial z_1}$ est constante. Ainsi les niveaux de $\tau$ sont des droites, ce qui produit un feuilletage en droite de $\mathbb{C}^2 \subset \mathbb{CP}(2)$. Il y a deux cas, suivant que le point base de ce feuilletage soit ou non à distance finie, qui conduisent à $\tau$ de l'un des deux types :

$$\begin{cases} \tau(z) = & \delta_1(a_1 z_1 + a_2 z_2) \\ \tau(z) = & \delta_2\left(\frac{z_1 - b_1}{z_2 - b_2}\right) \end{cases} \tag{20}$$

où les $a, b \in \mathbb{C}$ et $\delta_i \in \mathbb{C}(t)$.

Par une intégration élémentaire on obtient la :

***Proposition 4.1***. — *Soit $f : \mathbb{C}^2 \dashrightarrow \mathbb{C}$ une solution rationnelle maximale de dét Hess $f = 0$. Alors $f$ est de l'un des types suivants :*

1. $f = \ell_1(a_1 z_1 + a_2 z_2) + c_1 z_1 + c_2 z_2 + c_3$ ;
2. $f = \ell_2\left(\frac{z_1 - b_1}{z_2 - b_2}\right).(z_2 - b_2) + c_1 z_1 + c_2 z_2 + c_3$,

*où les $a_i, b_i, c_i$ sont des constantes et $\ell_i \in \mathbb{C}(t)$.*

On note que les solutions polynomiales de (2) sont de type 1. Dans le cas 1 le polynôme $P$ générateur de l'idéal $H(f)$ est affine. Dans le cas 2 on peut tirer explicitement en fonction de $f$ ce même polynôme $P$. Il n'y a pas d'énoncé général dès la dimension 3 décrivant les solutions de (2) ; toutefois dans la situation spéciale où l'on recherche les solutions de (2) sous forme polynomiales on dispose du :

***Théorème 4.2* ([1]).** — *Soit $f \in \mathbb{C}[z_1, z_2, z_3]$ un polynôme solution de dét Hess $f = 0$. Alors à conjugaison linéaire près $f$ est de l'un des types suivants :*

1. $\varphi_1 = \varepsilon z_1 + \varphi(z_2, z_3)$ , $\varphi \in \mathbb{C}[z_2, z_3]$ , $\varepsilon \in \{0, 1\}$ ;
2. $\varphi_2 = a_1(z_1) + z_2 a_2(z_1) + z_3 a_3(z_1), a_i \in \mathbb{C}[z_1]$.

Faisons quelques commentaires ; dans les deux cas $f$ est affine dans une ou deux variables. Le graphe de l'application de Gauss de l'application $\varphi_1$ est contenu dans $z_1 = \varepsilon$ ; si $\varphi_1$ est maximale le polynôme $P$ est alors $z_1 - \varepsilon$. Si $\varphi_1$ est non maximale il en est de même pour $\varphi$ ; dans ce cas $\varphi_1$ est du type $\varepsilon z_1 + \varphi(z_2)$ à conjugaison linéaire près. Le graphe de l'application de Gauss est une droite. Dans le cas 2 en général $\varphi_2$ est maximale. Si $P(z_2, z_3) = 0$ est l'équation de la courbe paramétrée par $t \rightsquigarrow (a_2(t), a_3(t))$ on a visiblement $P\left(\frac{\partial \varphi_2}{\partial z_2}, \frac{\partial \varphi_2}{\partial z_3}\right) = 0$.

## 5. L'équation des eikonales

Soit $f$ une fonction holomorphe définie sur un ouvert $V$ de $\mathbb{C}^n$ satisfaisant à l'équation aux eikonales $E(f) = c^2, c \in \mathbb{C}$. Considérons le champ de gradient de $f$ :

$$X = \operatorname{grad} f = \sum \frac{\partial f}{\partial z_i} \frac{\partial}{\partial z_i}, \tag{8}$$

les coordonnées $z_i$ étant fixées.

Soit $t \rightsquigarrow z(t)$ une trajectoire de $X$ :

$$\dot{z}_i(t) = \frac{\partial f}{\partial z_i}(z(t)). \tag{21}$$

Suivant Hamilton considérons l'accélération le long d'une trajectoire :

$$\ddot{z}_i(t) = \sum_{k=1}^{n} \frac{\partial^2 f}{\partial z_i \partial z_k}(z(t)).\dot{z}_k(t) = \sum_{k=1}^{n} \frac{\partial^2 f}{\partial z_i \partial z_k}(z(t)) \cdot \frac{\partial f}{\partial z_k}(z(t)). \tag{22}$$

Si maintenant on dérive par rapport à $z_i$ l'équation $E(f) = c^2$, on obtient :

$$2 \sum_{k=1}^{n} \frac{\partial^2 f}{\partial z_i \partial z_k} \cdot \frac{\partial f}{\partial z_i} = 0. \tag{23}$$

Ce qui implique la nullité des $\ddot{z}_i$ ; en résulte que, chaque fois qu'il est défini, le flot $\varphi_t$ de $X$ est affine en $t$ ce qui indique en particulier que le feuilletage associé à $X$ est un feuilletage en droites.

On remarque que, chaque fois que celà a un sens :

$$f \circ \varphi_t(z) = f(z) + c^2 t \tag{24}$$

et :

$$\varphi_t(z) = z + t.X(z) = z + t \frac{\partial f}{\partial z}(z) \tag{25}$$

avec des notations évidentes. Ainsi les fonctions $\frac{\partial f}{\partial z_i} : V \to \mathbb{C}$ sont intégrales premières du champ $X$ :

$$\frac{\partial f}{\partial z_i}\left(z + t\frac{\partial f}{\partial z}(z)\right) = \frac{\partial f}{\partial z_i}(z) \tag{26}$$

autre traduction de l'égalité (23).

Remarquons de suite que tout champ de droites n'est pas associé à une solution de l'équation aux eikonales. Considérons en effet le champ de droites dans $\mathbb{C}^3$ possédant les intégrales premières $z_3$ et $\frac{z_1 - r_1(z_3)}{z_2 - r_2(z_3)}$ où $r_1$ et $r_2$ sont rationnelles. Alors si $X = \operatorname{grad} f$ correspond à ce feuilletage on aura nécessairement $\frac{\partial f}{\partial z_3} \equiv 0$ et par suite $r_1$ et $r_2$ constants.

Nous allons maintenant, dans le cas de la dimension 3, utiliser le théorème 3 pour classifier les solutions rationnelles de l'équation aux eikonales.

Soit $f = \frac{P}{Q}$, solution de $E(f) = c^2$ où $P$ et $Q$ sont des polynômes sans facteur commun.

***Lemme 5.1***. — *Si $c \neq 0$, le feuilletage $\mathcal{F}$ en droites produit par $X = \operatorname{grad} f$ n'est pas associé aux cordes d'une cubique gauche.*

*Démonstration*. — Elle se fait par l'absurde. Remarquons que l'application de Gauss associée au feuilletage $\mathcal{F}$ par les cordes d'une cubique gauche est génériquement de rang $\geq 2$, et ceci pour tout choix de champ de vecteurs $Z$ rationnel définissant $\mathcal{F}$. En particulier l'image de l'application de Gauss :

$$G_f : z \leadsto \left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3}\right)$$

est dense dans la quadrique de $\mathbb{CP}(3)$ donnée par :

$$z_1^2 + z_2^2 + z_3^2 = c^2 \tag{27}$$

Soit $z_0$ un point générique de $\mathbb{C}^3$ où $f = \frac{P}{Q}$ est holomorphe. On a l'égalité entre fonctions rationnelles de $t$ :

$$\frac{P}{Q}\left(z_0 + t \cdot \frac{\partial f}{\partial z}(z_0)\right) = \frac{P}{Q}(z_0) + c^2 t. \tag{28}$$

Écrivons $P$ et $Q$ sous forme d'une somme de polynômes homogènes :

$$\begin{aligned} P = &\quad P_0 + \cdots + P_\nu \\ Q = &\quad Q_0 + \cdots + Q_\mu \end{aligned} \tag{29}$$

les polynômes $P_\nu$ et $Q_\mu$ étant non identiquement nuls. Remarquons, puisque $c \neq 0$, qu'un polynôme homogène non trivial ne peut s'annuler identiquement sur la quadrique (27) ; il existe donc un dense de $z_0$ pour lesquels on a $P_\nu\left(\frac{\partial f}{\partial z}(z_0)\right)$ et $Q_\mu\left(\frac{\partial f}{\partial z}(z_0)\right)$ non nuls. L'égalité (28) se traduit alors au niveau des termes de plus haut degré par :

$$\cdots + t^\nu P_\nu\left(\frac{\partial f}{\partial z}(z_0)\right) = \cdots + c^2 t^{\mu+1} Q_\mu\left(\frac{\partial f}{\partial z}(z_0)\right)$$

qui implique $\nu = \mu + 1$.

Puisque $G_f$ est dominante, le polynôme $P_\nu - c^2 Q_{\nu-1}$ s'annule sur la quadrique (27). D'où l'existence d'un polynôme $K \in \mathbb{C}[z_1, z_2, z_3]$ tel que :

$$P_\nu - c^2 Q_{\nu-1} = (z_1^2 + z_2^2 + z_3^2 - c^2).K \tag{30}$$

Une fois encore on développe $K$ en somme de polynômes homogènes :

$$K = K_\alpha + \cdots + K_\beta \quad , \quad \alpha \leq \beta.$$

et l'on observe en calculant les termes de plus haut et plus bas degré de (30) que $\alpha = \nu - 1$ et $\nu = \beta + 2$. Ce qui est absurde. □

Dans le lemme qui suit on traite le cas où $c = 0$ en utilisant une approche plus géométrique.

***Lemme 5.2***. — *Le feuilletage en droites associé à une solution rationnelle de l'équation aux eikonales $E(f) = 0$ n'est pas du type cordes d'une cubique gauche.*

*Démonstration.* — Sous les hypothèses du lemme $f$ est intégrale première de son gradient $X = \operatorname{grad} f$. Supposons que le feuilletage associé à $X$ ait pour trajectoires génériques les cordes d'une cubique gauche $\Gamma$. Comme on l'a vu les $\frac{\partial f}{\partial z_i}$ sont aussi intégrales premières du champ $X$. Par suite les fibres $f^{-1}(c)$ de $f$ sont des surfaces réglées par les trajectoires de $X$ et le long de ces trajectoires le plan tangent à $f^{-1}(c)$ est « constant ». En résulte que les $f^{-1}(c)$ sont des cône invariants par $X$, et donc les cônes $\mathscr{C}_M, M \in \Gamma$. Ceci vient du fait que les cordes de la cubique $\Gamma$ feuillettent $\mathbb{CP}(3) - \Gamma$ et ne peuvent donc se rencontrer en dehors de $\Gamma$. Mais la famille des cônes $\mathscr{C}_M$ ne feuillettent pas $\mathbb{CP}(3) - \Gamma$ ; il s'agit en fait d'un bi-feuilletage et par chaque point de $\mathbb{CP}(3) - \Gamma$ passent deux tels cônes. D'où une contradiction. □

Supposons que le champ $X = \operatorname{grad} f$ soit tangent à un pinceau de plans $L_1/L_2 = cst$ où les $L_i$ sont affines non constants. Quitte à faire une translation on peut supposer les $L_i$ linéaires et l'on écrira :

$$\begin{aligned} L_1 = &\quad \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 \\ L_2 = &\quad \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3. \end{aligned} \tag{31}$$

On pose $< \alpha \,|\, \beta >= \sum \alpha_i \beta_i$, $\|\alpha\|^2 = \sum \alpha_i^2$ et $\|\beta\|^2 = \sum \beta_i^2$, En écrivant explicitement que $L_1/L_2$ est intégrale première du champ $X$, on obtient :

(32)
$$\left\{(\beta_1 L_1 - \alpha_1 L_2)\frac{\partial}{\partial z_1} + (\beta_2 L_1 - \alpha_2 L_2)\frac{\partial}{\partial z_2} + (\beta_3 L_1 - \alpha_3 L_2)\frac{\partial}{\partial z_3}\right\}(f) = Y(f) = 0.$$

Ainsi le champ linéaire $Y$ s'annule sur la droite $L_1 = L_2 = 0$ et possède $f$ pour intégrale première. Le fait que ses composantes soient liées produit une forme linéaire non triviale :

$$L_3 = \gamma_1 z_1 + \gamma_2 z_2 + \gamma_3 z_3 \tag{33}$$

telle que $Y(L_3) = 0$. Remarquant que

$$Y(L_3) =< \beta \mid \gamma > L_1 + < \alpha \mid \gamma > L_2$$

on en déduit que :

$$< \alpha \mid \gamma >=< \beta \mid \gamma >= 0.$$

Remarquons aussi que :

$$\left\{ \begin{array}{ll} Y(L_1) = & < \alpha \mid \beta > L_1 - \|\alpha\|^2 L_2 \\ Y(L_2) = & \|\beta\|^2 L_1 - < \alpha \mid \beta > L_2. \end{array} \right. \tag{34}$$

Ainsi $Y$ agit linéairement sur l'espace vectoriel $\mathrm{Vect}_{\mathbb{C}}(L_1, L_2)$ ; comme cette action est de trace nulle on peut choisir $L_1$ et $L_2$ de sorte que :

$$\left\{ \begin{array}{ll} Y(L_1) = & \lambda L_2 \\ Y(L_2) = & -\lambda L_2. \end{array} \right. \tag{35}$$

où $\lambda \in \mathbb{C}$. Ceci revient donc à supposer que $\|\alpha\|^2 = \|\beta\|^2 = 0$ et $< \alpha \mid \beta >= \lambda$. On note alors, puisque $L_1$ et $L_2$ sont indépendants, que $\lambda$ est non nul. En particulier le corps des intégrales premières rationnelles de $Y$ est engendré par $L_3$ et la forme quadratique $L_1L_2$ ; on note aussi que $L_1, L_2$ et $L_3$ sont indépendants ($\lambda \neq 0$). Par conséquent les trajectoires de $Y$ sont les coniques :

$$\left\{ \begin{array}{ll} L_3 & = \text{cste} \\ L_1L_2 & = \text{cste}\,. \end{array} \right. \tag{36}$$

Le feuilletage $\mathscr{G}$ associé à $f$ (ses feuilles sont les fibres de $f$) est invariant par $Y$ ; s'il est défini par la 1-forme polynomiale

$$\omega = A_1 dz_1 + A_2 dz_2 + A_3 dz_3 \tag{37}$$

son lieu singulier :

$$\mathrm{Sing}\,\mathscr{G} = \{A_1 = A_2 = A_3 = 0\} \tag{38}$$

est aussi invariant par $Y$, et donc formé de trajectoires de $Y$. Le feuilletage associé à $X = \mathrm{grad}\, f$ est donné par le champ $Z$

$$Z = A_1 \frac{\partial}{\partial z_1} + A_2 \frac{\partial}{\partial z_2} + A_3 \frac{\partial}{\partial z_3}. \tag{39}$$

Mais d'après le théorème 1.1 son lieu singulier est constitué de la droite $L_1 = L_2 = 0$ et éventuellement d'une courbe rationnelle $\Gamma$ coupant le plan générique $L_1 = tL_2$ en un seul point en dehors de l'axe $L_1 = L_2 = 0$. Visiblement $\Gamma$ ne peut-être une fibre générique de $(L_1L_2, L_3)$ qui coupe deux fois chaque $L_1 = tL_2$ en dehors de l'axe $L_1 = L_2 = 0$ ; ni une fibre spéciale (contenue dans $L_1L_2 = 0$) qui n'apparaît pas comme lieu singulier de feuilletages en droites. Ne reste que la possibilité où le lieu singulier du champ de droites $Z$ est précisément l'axe $L_1 = L_2 = 0$

On sait que dans cette situation le champ $X = \mathrm{grad}\, f$ possède les deux intégrales premières $L_1/L_2$ et $\frac{L_3 - r(L_1/L_2)}{L_1}$ où $r \in \mathbb{C}(t)$ est une certaine fonction rationnelle.

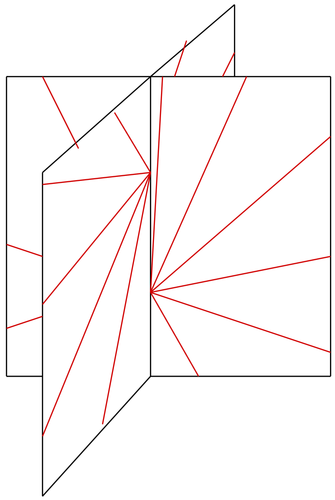

FIGURE 4.

Comme $f$ est intégrale première du champ $Y$, il existe une fonction rationnelle $\varphi \in \mathbb{C}(t_1, t_2)$ telle que :

$$f = \varphi(L_1 L_2, L_3). \tag{41}$$

De sorte que $X = \operatorname{grad} f$ s'écrit :

$$X = \frac{\partial \varphi}{\partial t_1}\{L_1 \operatorname{grad} L_2 + L_2 \operatorname{grad} L_1\} + \frac{\partial \varphi}{\partial t_2} \operatorname{grad} L_3. \tag{42}$$

En écrivant explicitement que $\frac{L_3 - r(L_1/L_2)}{L_1}$ est intégrale première du champ $X$ on obtient :

$$< \alpha \mid \beta > \cdot \frac{\partial \varphi}{\partial u}(L_1 L_2, L_3) \cdot \left(L_3 - r\left(\frac{L_1}{L_2}\right)\right) - \|\gamma\|^2 \frac{\partial \varphi}{\partial v}(L_1 L_2, L_3) = 0.$$

Comme $< \alpha \mid \beta >= \gamma$ est non nul et comme $L_1, L_2, L_3$ sont indépendants nécessairement $\|\gamma\|^2 \neq 0$ (utiliser $< \gamma \mid \beta >=< \gamma \mid \alpha >= 0$). Finalement $L_3 - r(\frac{L_1}{L_2})$ apparaît comme fonction de $L_3$ et de la forme quadratique $L_1 L_2$ ; ce qui est absurde sauf si $r$ est constante. Mais dans ce cas le champ $\operatorname{grad} f$ est radial.

Lorsque $\operatorname{grad} f$ est radial :

$$\operatorname{grad} f = h. \sum_{i=1}^{3} z_i \frac{\partial}{\partial z_i}, h \in \mathbb{C}(z_1, z_2, z_3)$$

on a :

$$z_i \frac{\partial h}{\partial z_j} - z_j \frac{\partial h}{\partial z_i} = 0. \tag{43}$$

En particulier il existe $\ell \in \mathbb{C}(t)$ telle que :

$$h = \ell(z_1^2 + z_2^2 + z_3^2)$$

Par suite $f = L(z_1^2 + z_2^2 + z_3^2)$. L'équation des eikonales se traduit par

$$4tL'(t)^2 = c^2$$

qui s'intègre en $L(t) = \pm c\sqrt{t} + \text{cste}$. La solution produite n'est pas rationnelle comme on le voit et comme le savait Malus...

Il reste finalement un cas à examiner, celui où le champ de droite $X = \operatorname{grad} f$ est tangent à un pinceaux d'hyperplans parallèles $L_3 = \text{cste}$ ; on pose

$$L_3 = \gamma_1 z_1 + \gamma_2 z_2 + \gamma_3 z_3.$$

Les formes linéaires $a_1 z_1 + a_2 z_2 + a_3 z_3$ avec $\|a\|^2 = c^2$ sont solutions de $E(f) = c^2$ et font partie du cas précédent.

Puisque :

$$\gamma_1 \frac{\partial f}{\partial z_1} + \gamma_2 \frac{\partial f}{\partial z_2} + \gamma_3 \frac{\partial f}{\partial z_3} = 0$$

le champ de vecteur $Y = \sum \gamma_i \frac{\partial}{\partial z_i}$ annule $f$. Considérons deux formes linéaires indépendantes

$$\begin{aligned} L_1 &= \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 \\ L_2 &= \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3 \end{aligned}$$

telles que $< \alpha \mid \gamma > = < \beta \mid \gamma > = 0$. Ces deux formes linéaires engendrent le corps des intégrales premières de $Y$ si bien que

(44) $$f = \varphi(L_1, L_2) \quad \text{où} \quad \varphi \in \mathbb{C}(t_1, t_2).$$

L'équation aux eikonales $E(f) = c^2$ se traduit par :

(45) $$\|\alpha\|^2 \left(\frac{\partial \varphi}{\partial t_1}\right)^2 + \|\beta\|^2 \left(\frac{\partial \varphi}{\partial t_2}\right)^2 + 2 < \alpha \mid \beta > \frac{\partial \varphi}{\partial t_1}\, \frac{\partial \varphi}{\partial t_2} = c^2.$$

Remarquons que $\|\alpha\|^2, \|\beta\|^2$ et $< \alpha, \beta >$ ne peuvent être simultanément nuls puisque $L_1$ et $L_2$ sont indépendantes. Par suite $\varphi \in \mathbb{C}(t_1, t_2)$ satisfait à :

$$\text{dét Hess}\, \varphi \equiv 0.$$

La proposition 4.1 assure qu'il existe deux formes linéaires $u_1$ et $u_2$ en deux variables telles que $\varphi$ soit de l'un des deux types :

$\varphi_1 = \varepsilon u_1 + \ell(u_2)$

$\varphi_2 = \varepsilon u_1 + \ell\left(\frac{u_1}{u_2}\right) u_2 \quad , \quad \ell \in \mathbb{C}(t), \varepsilon \in \{0, 1\}.$

Quitte à effectuer un changement de notation on peut donc supposer que $f$ est de l'un des deux types :

(46) $$f_1 = \varepsilon L_1 + \ell(L_2)$$

(47) $$f_2 = \varepsilon L_1 + \ell\left(\frac{L_1}{L_2}\right) . L_2 \qquad \varepsilon \in \{0, 1\}.$$

Dans le cas (46) l'équation des eikonales se traduit par :

$$(48) \qquad \varepsilon\|\alpha\|^2 + \ell'(L_2)^2\|\beta\|^2 + 2\ell'(L_2) < \alpha \mid \beta >= c^2.$$

En particulier si $\ell'$ est non constante on aura $< \alpha \mid \beta >= \|\beta\|^2 = 0$ et $\varepsilon\|\alpha\|^2 = c^2$ ; on peut alors, si $c \neq 0$, supposer que $\varepsilon = 1$. Si $c = 0$ alors $f$ est affine. Dans le cas (47) l'équation des eikonales conduit à :

$$(48) \quad \|\alpha\|^2(\varepsilon+\ell'(t))^2 + \|\beta\|^2(\ell(t) - t\ell'(t))^2 + 2 < \alpha \mid \beta > (\varepsilon+\ell'(t))(\ell(t) - t\ell'(t)) = c^2.$$

Elle ne produit pas de nouvelles solutions rationnelles comme nous allons le voir en l'intégrant explicitement. En dérivant (48) on obtient :

$$(49) \qquad \ell''(t)\{\ell'(t)(\|\alpha\|^2 - 2 < \alpha \mid \beta > t - t^2\|\beta\|^2)+$$
$$\ell(t)(t\|\beta\|^2 + < \alpha \mid \beta >) + \|\alpha\|^2\varepsilon - \varepsilon t < \alpha \mid \beta >\} = 0.$$

Évidemment (49) possède des solutions affines mais qui sont de type (46). Après simplification par $\ell''$ on obtient une équation différentielle linéaire avec second membre. On remarque que $t \rightsquigarrow -\varepsilon t$ en est une solution particulière. L'équation sans second membre :

$$(50) \qquad y'(\|\alpha\|^2 - 2 < \alpha \mid \beta > t - t^2\|\beta\|^2) + y(t\|\beta\|^2 + < \alpha \mid \beta >= 0$$

s'intègre en :

$$(51) \qquad y(t) = \mathrm{cste}\,(t^2\|\beta\|^2 + 2 < \alpha \mid \beta > t - \|\alpha\|^2)^{\frac{1}{2}}$$

qui produit donc des solutions explicites de (48). Ces solutions ne sont pas rationnelles sauf lorsque $t^2\|\beta\|^2 + 2 < \alpha \mid \beta > t - \|\alpha\|^2$ est un carré ; ce cas conduit encore à une solution de type 46.

On obtient in fine le :

***Théorème 5.3***. — *Les solutions rationnelles de $E(f) = c^2$, $c \neq 0$ sont de type $L_1 + \ell(L_2)$ où les $L_i$ sont des formes linéaires, $L_1 = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3, L_2 = \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3$, satisfaisant $\|\alpha\|^2 = c^2$, $\|\beta\|^2 = 0$, $< \alpha \mid \beta >= 0$ ; $\ell$ est une fonction rationnelle. Les solutions rationnelles de $E(f) = 0$ sont affines.*

## 6. Automorphismes des feuilletages en droites

Soit $\mathcal{F}$ un feuilletage en droites de $\mathbb{CP}(3)$ ; nous allons décrire quelques groupes $Aut\mathcal{F}$ où

$$\mathrm{Aut}\,\mathcal{F} := \{\varphi \in \mathrm{Aut}\ \mathbb{CP}(3) \ \mid\ \varphi^*\mathcal{F} = \mathcal{F}\}.$$

Rappelons que $\mathrm{Aut}\,\mathbb{CP}(3) \cong PGL(4,\mathbb{C})$ ; visiblement $\mathrm{Aut}\,\mathcal{F}$ est un sous groupe algébrique de $\mathrm{Aut}\,\mathcal{F}$. Si $\varphi$ est un élément de $\mathrm{Aut}\,\mathcal{F}$ alors $\varphi(\mathrm{Sing}\,\mathcal{F}) = \mathrm{Sing}\,\mathcal{F}$ ; dit autrement $\mathrm{Aut}\,\mathcal{F}$ est un sous groupe de $\mathrm{Aut}\,(\mathrm{Sing}\,\mathcal{F})$ le groupe des automorphismes de $\mathbb{CP}(3)$ qui préservent $\mathrm{Sing}\,\mathcal{F}$. Le groupe $Aut\mathcal{F}$ décrit les mouvements permis qui respectent la géométrie imposée par le feuilletage.

***Proposition 6.1***. — *Soit $\mathcal{F}$ un feuilletage en droites associé à une cubique gauche $\Gamma$ ; alors :*

$$\operatorname{Aut}\mathcal{F} = \operatorname{Aut}(\operatorname{Sing}\mathcal{F}) = \operatorname{Aut}\Gamma \cong PGL(2,\mathbb{C}).$$

*Démonstration.* — Si $\sigma : \mathbb{CP}(1) \longrightarrow \mathbb{CP}(3)$ est une paramétrisation de $\Gamma$, alors chaque élément de $\operatorname{Aut}\mathbb{CP}(1) \cong PGL(2,\mathbb{C})$ se relève à $\Gamma$ et s'étend en un automorphisme de $\mathbb{CP}(3)$. Comme un élément $\varphi \in \operatorname{Aut}\mathbb{CP}(3)$ dont la restriction à $\Gamma$ est l'identité en lui même, on a $\operatorname{Aut}\Gamma \cong PGL(2,\mathbb{C})$. Maintenant si $\varphi \in \operatorname{Aut}\Gamma$ il est clair que l'image par $\varphi$ d'une corde de $\Gamma$ est une corde de $\Gamma$ ; ainsi $\varphi \in \operatorname{Aut}\mathcal{F}$. □

On note que $\operatorname{Aut}\mathcal{F}$ agit transitivement sur $\mathbb{CP}(3) - \Gamma$ et sur $\Gamma$.

Lorsque $\mathcal{F}$ est le feuilletage radial en un point 0, il est clair que $\operatorname{Aut}\mathcal{F}$ est exactement le sous groupe des automorphismes de $\mathbb{CP}(3)$ qui fixent 0.

Examinons le cas où $\mathcal{F}$ est radial dans les pages d'un livre ouvert $z_1/z_2 = \text{cste}$. Commençons par le cas dégénéré où le lieu singulier de $\mathcal{F}$ se réduit à l'axe $z_1 = z_2 = 0$. On sait qu'alors $\mathcal{F}$ a deux intégrales premières de base : $z_2/z_1$ et $\frac{z_3 - r(\frac{z_2}{z_1})}{z_1}$ où $r$ est une certaine fonction rationnelle non constante. Si l'on éclate l'axe $z_1 = z_2 = 0$ dans $\mathbb{CP}(3)$, on obtient un feuilletage $\widetilde{\mathcal{F}}$ sur la variété éclatée $\widetilde{\mathbb{CP}(3)}$ dont le lieu singulier est donné dans la carte $(z_1, t = \frac{z_2}{z_1}, z_3)$ par :

$$\operatorname{Sing}\widetilde{\mathcal{F}} = \{z_1 = 0, z_3 = r(t)\}$$

Notons que si $\varphi \in \operatorname{Aut}\mathcal{F}$, alors $\varphi$ préserve l'axe $z_1 = z_2 = 0$ ; on peut donc relever $\varphi$ à $\widetilde{\mathbb{CP}(3)}$ et obtenir un biholomorphisme $\widetilde{\varphi} \in \operatorname{Aut}\widetilde{\mathcal{F}}$. Visiblement $\widetilde{\varphi}$ préserve le diviseur exceptionnel $(z_1 = 0)$ et laisse invariant $\operatorname{Sing}\widetilde{\mathcal{F}}$. De la même façon si $X$ est un champ de vecteurs sur $\mathbb{CP}(3)$ dont le flot $\varphi_s$ est dans $\operatorname{Aut}\mathcal{F}$ alors $X$ s'écrit :

$$\begin{aligned}(52)\qquad &(a_1z_1 + a_2z_2)\frac{\partial}{\partial z_1} + (b_1z_1 + b_2z_2)\frac{\partial}{\partial z_2} + (c_0 + c_1z_1 + c_2z_2 + c_3z_3)\frac{\partial}{\partial z_3}\\ &+(A_1z_1 + A_2z_2 + A_3z_3)\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3}\right).\end{aligned}$$

La restriction de l'éclaté $\widetilde{X}$ de $X$ au diviseur exceptionnel $z_1 = 0$ s'écrit :

$$(53)\qquad (b_1 + (b_2 - a_1)t - a_2t^2)\frac{\partial}{\partial t} + (c_0 + c_3z_3 + A_3z_3^2)\frac{\partial}{\partial z_3}$$

et doit être tangent à $\operatorname{Sing}\widetilde{\mathcal{F}}$. Ce sera évidemment le cas si (53) est identiquement nul. Dans cette éventualité $X$ est du type :

$$\begin{aligned}(54)\qquad &a_1\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}\right) + (c_1z_1 + c_2z_2)\frac{\partial}{\partial z_3} +\\ &(A_1z_1 + A_2z_2).\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3}\right),\end{aligned}$$

et son éclaté $\widetilde{X}$ s'écrit :

$$(55)\qquad \widetilde{X} = z_1\left\{a_1 + A_1z_1 + A_2z_1t)\frac{\partial}{\partial z_1} + (A_1 + c_1 + t(c_2 + A_2z_3))\frac{\partial}{\partial z_3}\right\}.$$

Comme $\widetilde{X}$ doit respecter $\widetilde{\mathcal{F}}$, dans chaque plan générique on a :

$$[R_1, \widetilde{X}] = h.R_1 \quad \text{où} \quad R_1 = z_1 \frac{\partial}{\partial z_1} + (z_3 - r(t)) \frac{\partial}{\partial z_3} \tag{55}$$

définit $\widetilde{\mathcal{F}}$ et $h$ est un polynôme en $z_1, z_2$ à paramètre $t$. En écrivant explicitement (55) on constate que $A_1 = A_2 = 0$.

Par contre tous les flots des champs :

$$a_1 \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \right) + (c_1 z_1 + c_2 z_2) \frac{\partial}{\partial z_3}$$

laissent invariant $\mathcal{F}$.

Lorsque la fonction rationnelle $r$ est suffisamment générique la composante neutre $\operatorname{Aut}_0 \mathcal{F}$ de $\operatorname{Aut} \mathcal{F}$ se limite au groupe résoluble de dimension 3 :

$$\exp \left\{ a_1 \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \right) + (c_1 z_1 + c_2 z_2) \frac{\partial}{\partial z_3} \right\} \tag{56}$$

c'est ce que nous allons voir maintenant tout en classifiant les cas exceptionnels. Considérons un champ $X$ (52) dont l'éclaté $\widetilde{X}$ est non identiquement nul sur le diviseur exceptionnel ; alors $\operatorname{Sing} \widetilde{\mathcal{F}}$ est invariant par le champ donné par (53). Dit autrement la fonction rationnelle $r$ est solution de l'équation différentielle de Riccati :

$$(b_1 + (b_2 - a_1)t - a_2 t^2) y' - (c_0 + c_3 y + A_3 y^2) = 0. \tag{57}$$

Ce qui prouve l'affirmation ci-dessus. L'équation (57) possède les solutions constantes $r_1$ et $r_2$, racines du trinôme

$$c_0 + c_0 y + A_3 y^2 = 0. \tag{58}$$

Notons que le champ (53) ne peut s'annuler sur $\operatorname{Sing} \mathcal{F}$ sans être identiquement nul, cas traité précédemment. Si (58) a deux racines distinctes on peut supposer qu'elles sont en 0 et $\infty$. A changement de coordonnées linéaires près (57) prend alors l'une des deux formes suivantes :

$$t^2 y' + y = 0 \tag{59}$$

$$t y' + \lambda y = 0. \tag{60}$$

Mais (59) n'a d'autre solutions rationnelles que $y = 0$, cas exclus tandis que (60) a des solutions rationnelles $y = r(t)$ non constantes si et seulement si $\lambda \in \mathbb{Z}$, auquel cas toujours à conjugaison près, de telles solutions sont de type $r(t) = t^n, \;\; n \in \mathbb{N}$. Si (58) a une racine double (le cas $A_3 = 0$ se ramène au précédent) (57) équivaut à

$$a(t) y' + y^2 = 0, \quad \text{où } a \text{ est un polynôme de degré deux.} \tag{61}$$

Le seul cas où (61) possède une solution rationnelle $r(t)$ non constante est celui où $a(t)$ à une racine double ; dans cette éventualité (61) est conjuguée à :

$$t^2 y' - y^2 = 0. \tag{62}$$

qui s'intègre en $y(t) = \frac{t}{1+\mu t}, \quad \mu \in \mathbb{C}$.

La fonction rationnelle $r$ est ici l'un des $\frac{t}{1+\mu t}$ et quitte à composer par $\frac{t}{1-\mu t}$ on supposera que $r(t) = t$. Nous étudions donc le groupe $\operatorname{Aut} \mathscr{F}_n$ où $\mathscr{F}_n$ est le feuilletage en droites associé à la fonction $t^n$. En écrivant l'invariance de $\operatorname{Sing} \widetilde{\mathscr{F}}$ par (53) on obtient :

$$\begin{aligned}\operatorname{Aut}_0 \mathscr{F}_1 = \exp \Big\{ &(a_1 z_1 + a_2 z_2) \frac{\partial}{\partial z_1} + (b_1 z_1 + b_2 z_2) \frac{\partial}{\partial z_2} \\ &\quad + (b_1 + c_1 z_1 + c_2 z_2 + (b_2 - a_1) z_3) \frac{\partial}{\partial z_3} \\ &+ (A_1 z_1 + A_2 z_2 - a_2 z_3) \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3} \right) a_i, b_i, c_i, A_i \in \mathbb{C} \Big\}\end{aligned}$$

et pour $n \geq 2$ :

$$\begin{aligned}\operatorname{Aut}_0 \mathscr{F}_n = \exp \Big\{ &(a_1 z_1 + a_2 z_2) \frac{\partial}{\partial z_1} \\ &+ \left( b_2 z_2 \frac{\partial}{\partial z_2} + c_0 + c_1 z_1 + c_2 z_2 + n(b_2 - a_1) z_3 \right) \frac{\partial}{\partial z_3} \\ &\quad + (A_1 z_1 + A_2 z_2 + A_3 z_3) \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3} \right) \Big\}.\end{aligned}$$

Les $\operatorname{Aut}_0 \mathscr{F}_n$ sont tous de dimension 9 et agissent transitivement sur $\mathbb{CP}(3) - \overline{(z_1 = z_2 = 0)}$.

Venons en au cas générique d'un feuilletage en droites $\mathscr{F}$ dans les pages d'un livre ouvert, cas où le lieu singulier $\operatorname{Sing} \mathscr{F}$ se compose de deux courbes rationnelles $(z_1 = z_2 = 0)$ et la courbe $t \rightsquigarrow (r_1(t), t r_1(t), r_3(t)) = R(t)$. Bien sûr pour $R$ générique $\operatorname{Aut}_0 \mathscr{F}$ est trivial puisque l'image de $R$ en général n'est pas trajectoire d'un champ de vecteur sur $\mathbb{CP}(3)$. La description des $\operatorname{Aut}_0 \mathscr{F}$ dans le cas contraire est « zoologique ». L'exemple le plus simple est le suivant.

Supposons que $t \rightsquigarrow R(t)$ soit affine, i.e. paramètre une droite générale $D$. On peut supposer que $D$ est donnée par $z_1 = 1, z_3 = 0$. Alors $\operatorname{Aut} \mathscr{F}$ coincide avec le sous-groupe de $\operatorname{Aut} \mathbb{CP}(3)$ qui laisse invariant $\operatorname{Sing} \mathscr{F}$. Il est isomorphe au projectivisé du groupe $G \subset GL(4, \mathbb{C})$, où $G$ est engendré par les matrices :

$$\left( \begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right) = A, B \in GL(2, \mathbb{C}) \quad \text{et} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Il agit encore transitivement sur $\mathbb{CP}(3) - \operatorname{Sing} \mathscr{F}$.

## 7. Remarques et problèmes

La description des solutions rationnelles de l'équation aux eikonales en dimension 3 donne bien sûr celle des équations différentielles de type :

$$Q\left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3}\right) = 0,$$

où $Q$ est une forme quadratique de rang maximum. Ces solutions ne sont jamais maximales.

On peut s'interroger sur les solutions Liouvilliennes de l'équation aux eikonales. C'est tout à fait naturel puisque les $c.\sqrt{z_1^2 + z_2^2 + z_3^2}$ en sont des solutions. De même on peut s'intéresser aux solutions globales réelles de classe $C^2$ ; on conjecture qu'elles sont affines. La classification des champs de droites en dimension supérieure à 4 et ses conséquences sur l'équation des eikonales restent ouvertes ; soit $P$ un polynôme irréductible tel qu'il existe une solution maximale $f$ rationnelle de $P\left(\frac{\partial f}{\partial z}\right) = 0$. On sait que $P = 0$ est unirationnelle ; il s'agit de classifier de tels polynômes $P$.

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## Références

[1] M. d. Bondt & A. v. d. Essen – Singular Hessians, *J. Algebra* **282** (2004), p. 195–204.

[2] D. Cerveau – Feuilletages réglés, *C. R. Acad. Sci. Paris Sér. I Math.* **307** (1988), p. 33–36.

[3] J. Déserti & D. Cerveau – Feuilletages et actions de groupes sur les espaces projectifs, *Mém. Soc. Math. Fr. (N.S.)* **103** (2005).

[4] P. Gordan & M. Nöther – Über die algebraischen Formen, deren Hesse'sche Determinante identisch verschwindet, *Math. Ann.* **10** (1876), p. 547–568.

[5] W. R. Hamilton – Mathematical papers, vol. I, Geometrical optics, Cambridge University Press (1931) and *Transactions of the Royal Irish Academy* **15** (1828), pp. 69-174.

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# SOME REGULARITIES AND SINGULARITIES APPEARING IN THE STUDY OF POLYNOMIALS AND OPERATORS

*by*

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***Abstract.*** — We apply the viewpoint of singularity theory to the following problems: how does the decomposition of a polynomial $P$ as the product of polynomials behave under perturbations of $P$? How do the eigenvalues, eigenspaces and more generally invariant subspaces of an operator $A$ behave under perturbations of $A$? We give a characterization of the regular situations and describe completely the singular ones in some moderately degenerate situations.

***Résumé*** **(Quelques regularités et singularités apparaissant dans l'étude des polynômes et des opérateurs )**

Nous appliquons le point de vue de la théorie des singularités aux deux problèmes suivants : comment la décomposition d'un polynôme $P$ comme produit de polynômes se comporte-t-elle quand on perturbe $P$ ? Comment les valeurs propres, vecteurs propres et plus généralement sous-espaces invariants d'un opérateur $A$ se comportent-ils quand on perturbe $A$ ? Nous caractérisons les situations régulières et décrivons complètement celles qui sont singulières mais pas trop dégénérées.

## Introduction

In the study of bifurcations of dynamical systems one has to deal frequently with the following situation: as a parameter varies one considers the variation of an eigenvalue or of the invariant line generated by the corresponding eigenvector of the linearization of the dynamical system at a certain point. It often happens that those elements vary smoothly with the parameter, which is known to be the case if the eigenvalue is simple. But nevertheless the system undergoes a bifurcation if the eigenvalue crosses a certain subset of the plane (the unit circle, the imaginary axis, etc.). A second, more complex, situation happens when the eigenvalue becomes multiple, since then its variation with the parameter ceases to be smooth. The same situations occur when instead of an

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invariant line one needs to consider an invariant subspace of dimension greater than one.

During the years we have meditated on these questions and have arrived at various forms of expressing the (essentially known) conditions for the smooth variation of those elements (see for example [**5, 4**] for recent versions). One of those forms seems especially suited for studying, in terms of singularities of mappings, the situations where that variation ceases to be smooth. In this article we describe the simplest of those singularities.

*The results.* — We begin by a study of the simplest singularities of the polynomial multiplication map:

$$\mathrm{Mult} : \mathrm{MP}(n) \times \mathrm{MP}(m) \to \mathrm{MP}(n+m)$$

where $\mathrm{MP}(n)$ will denote the space of monic polynomials of degree $n$ over $\mathbf{K}$, which will be either the real or the complex field. The rank of this map at a point $(f, g)$ can be expressed in terms of the degree of the greatest common divisor $\gcd(f, g)$ so that it is a local diffeomorphism precisely when this degree is 0, i.e. when the factors are relatively prime. And we can describe completely the singularities of Mult when this degree is 1 (Theorem 1). Then we proceed to study the higher corank singularities of Mult; here our results are not as sharp, but we have a complete geometric description of many cases and an algebraic description of the rest.

As a byproduct of Theorem 1 we give an interesting description of the classical resultant of two polynomials and we obtain the relation between the singularities of Mult we describe and the resultant set $\mathrm{Res}(f, g) = 0$.

Then we apply Theorem 1 (and its corollary, Theorem 3, which generalizes it to the multiplication of an arbitrary number of factors) to study the singularities of the (monic) characteristic polynomial map

$$\chi : \mathrm{M}(n \times n) \to \mathrm{MP}(n)$$

where $\mathrm{M}(n \times n)$ denotes the space of $n \times n$ matrices with entries in $\mathbf{K}$. We will view each $M \in \mathrm{M}(n \times n)$ as a linear mapping $\mathbb{C}^n \to \mathbb{C}^n$ and always take into account all its *complex* eigenvalues. We determine the matrices at which $\chi$ is a submersion and give a description of its simplest singularities (Theorem 5).

All the above is used to study the singularities of the eigenvalues of operators. For that, we introduce the set of all *proper elements* of a Banach space $E$ over $\mathbf{K}$ to be the space of triples consisting of a linear operator on $E$, an invariant line and the corresponding eigenvalue:

$$\mathrm{Eig}(E) := \{(\lambda, L, A) \in \mathbf{K} \times \mathbf{P}(E) \times \mathrm{End}(E) \ : \ A(L) \subseteq L \ \textit{and} \ A|_L = \lambda\}.$$

Here $\mathbf{P}(E)$ denotes the projectivization of $E$ and $\mathrm{End}(E)$ the space of continuous linear endomorphisms of $E$. There is a natural projection $\Pi : \mathrm{Eig}(E) \mapsto \mathrm{End}(E)$ on the third factor.

The basic fact here (Theorem 6) is that $\mathrm{Eig}(E)$ is a *smooth object*, actually an analytic manifold modelled on $\mathrm{End}(E)$, provided with a projection $\Pi$ onto $\mathrm{End}(E)$.

Therefore it is a kind a *resolution* of all the singularities associated to eigenvalue problems.

We show that, not surprisingly, $\Pi$ is a local diffeomorphism precisely at those points where $\lambda$ is a simple eigenvalue of $A$. And we can describe completely the singularities of $\Pi$ when $\lambda$ is a geometrically simple eigenvalue of $A$ of finite multiplicity (Theorem 8). In the finite dimensional case this means simply that it has only one corresponding invariant line, while in infinite dimensions there are some technical additional conditions. We also show that the mapping that forgets the invariant line is regular (in this case an immersion) precisely when the eigenvalue is geometrically simple, a fact that is useful in the proof of the singularity part of Theorem 8.

In section D, we generalize this to invariant subspaces of dimension greater than one. In fact, this was the starting point of the whole story: in **[2]**, we explained that the theory of formal normal forms for dynamical systems is an easy consequence of the Jordan decomposition of endomorphisms. Thinking about the generalization of this approach to *families*, we came to the conclusion that each characteristic space $F_0$ of an endomorphism $A_0$ of $\mathbf{C}^n$ must have the following stability property: every nearby endomorphism $A$ has a unique invariant subspace $F(A)$ of the same dimension as $F_0 = F(A_0)$ and close to it, depending analytically on $A$. This is an easy result but it is not so well-known[(1)], and in section D we consider (and extend) it in the spirit of singularity theory.

*The singularities.* — The singularities found in Theorems 1, 8, 15, 16 are a certain type of Morin singularities which we will call *swallowtails*:

The *standard k–swallowtail* is the map

$$\mathrm{SW}_k : \mathbf{K}^{k-1} \to \mathbf{K}^{k-1}$$

defined by

$$\mathrm{SW}_k(a_1, \dots, a_{k-2}, u) := (a_1, \dots, a_{k-2}, u^k + a_{k-2}u^{k-2} + \dots + a_1 u)$$

For us a *k–swallowtail* will be any map germ between two Banach spaces which is diffeomorphic to the germ at 0 of a map of the form

$$\mathrm{SW}_k \times \mathrm{Id} : \mathbf{K}^{k-1} \times E \to \mathbf{K}^{k-1} \times E$$

for some Banach space $E$. When $\mathbf{K} = \mathbf{C}$ but $E$ is *real*—a situation occurring whenever a real polynomial or endomorphism has nonreal roots or eigenvalues—we call such a map a *complex swallowtail.*

Interesting examples of $k$-swallowtails are the evaluation map

$$\begin{array}{rcl} \mathrm{ev} : \mathrm{MP}(k) \times \mathbf{K} & \to & \mathrm{MP}(k) \times \mathbf{K} \\ (P, a) & \mapsto & \big(P, P(a)\big) \end{array}$$

and the mapping

$$(a_1, \dots, a_{k-1}, a) \mapsto (aa_1, a_1 + aa_2, a_2 + aa_3, \dots, a_{k-2} + aa_{k-1}, a_{k-1} + a)$$

(1) The finite dimensional case led us to Theorem 1...

The second example shows that all swallowtails can be given by maps all of whose coordinate functions are polynomials of degree at most 2, a fact that we have not seen in the literature.

These examples, and some of their variants, will play an important role in the proofs of the theorems.

The singularities in Theorem 5 will be *$k$–swallowtail deformations*, by which we mean any map germ between two Banach spaces which is diffeomorphic to the germ at 0 of a map

$$G : E \times E' \to E$$

such that $G(x, 0)$ is a $k$–swallowtail, where $E, E'$ are Banach spaces.

There are many $k$–swallowtail deformations between spaces of the same dimension, so this term does not describe a precise singularity type. And though it is possible, in principle, to describe them all, there remains to do so specifically for the singularities of $\chi$.

We will show by examples that in all cases the singularities that are not swallowtails are more complicated than those one could expect from the classification results of singularities of mappings.

We hope to give soon some applications of these results to bifurcation problems of dynamical systems.

In the Appendices we recall the main properties of the singularities we will use in the text and describe completely the main examples. We also provide an introduction to the results on continuous linear maps between Banach spaces needed in the text, referring to Rudin's beautiful book **[10]** for more details about this magnificent theory.

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## A. Singularities of Polynomial Multiplication

Polynomial multiplication defines a map

$$\mathrm{Mult} : \mathrm{MP}(n) \times \mathrm{MP}(m) \to \mathrm{MP}(n+m)$$

We are interested in describing the regular points and the singularities of the map Mult. We will denote by $\gcd(P, Q)$ the monic greatest common divisor of the monic polynomials $P$ and $Q$.

***Theorem 1.*** — *For* $(P_0, Q_0)$ *in* $\mathrm{MP}(n) \times \mathrm{MP}(m)$,

(i) *The corank of the differential* $D\,\mathrm{Mult}(P_0, Q_0)$ *is the degree of* $\gcd(P_0, Q_0)$.

(ii) *In particular,* Mult *is a local diffeomorphism at* $(P_0, Q_0)$ *if and only if* $\gcd(P_0, Q_0) = 1$.

(iii) *The mapping* Mult *is a* $(k+1)$*–swallowtail at* $(P_0, Q_0)$ *for some positive integer* $k$ *if, and only if,* $\deg \gcd(P_0, Q_0) = 1$, *the integer* $k$ *being the maximum of the multiplicities in* $P_0$ *and* $Q_0$ *of their common root.*

(iv) *If* $\mathbf{K} = \mathbf{R}$, *the mapping* Mult *is a complex* $(k+1)$*–swallowtail at* $(P_0, Q_0)$ *for some positive integer* $k$ *if, and only if,* $\gcd(P_0, Q_0)$ *is an irreducible polynomial of degree* 2, *the integer* $k$ *being the maximum of the multiplicities in* $P_0$ *and* $Q_0$ *of their complex conjugate common roots*[2].

*Proof.* — The tangent space of $\mathrm{MP}(n)$ at any point is the set of polynomials of degree less that $n$. The derivative of Mult at $(P_0, Q_0)$ is then given by

$$(P, Q) \mapsto P_0 Q + P Q_0.$$

Therefore its image, being the set of multiples of $\gcd(P_0, Q_0)$ by polynomials of degree less than $n + m - \deg\gcd(P_0, Q_0)$, has this dimension. This proves (i) and therefore (ii).

If $\gcd(P_0, Q_0) = x - \alpha$ then $x - \alpha$ must divide one of $P_0, Q_0$ with multiplicity 1 and the other one with multiplicity $k$. By changing the variable in the polynomials (which induces a diffeomorphism of $\mathrm{MP}(n)$) we can assume $\alpha = 0$.

Consider first the case $P_0 = x$, $Q_0 = x^k$. Then Mult is given by

$$\mathrm{Mult}\left(x + a, x^k + \sum_{i=0}^{k-1} a_i x^i\right) = \sum_{i=0}^{k+1} (a_{i-1} + a a_i) x^i$$

(putting $a_k = 1$, $a_{k+1} = a_{-1} = 0$) or, in coordinates $(a, a_{k-1}, \dots, a_0)$, by

$$\mathrm{Mult}(a_0, \dots, a_{k-1}, a) = (a a_0, a_0 + a a_1, a_1 + a a_2, \dots, a_{k-1} + a),$$

which is a $(k+1)$–swallowtail by example 2 in Appendix A.

In general, let $P_0 = x P_1$, $Q_0 = x^k Q_1$, where $P_1, Q_1$ are not divisible by $x$. Then, setting $m_1 := m - k$ and $n_1 := n - 1$, we have a commutative diagram:

$$\begin{array}{ccccc}
\mathrm{MP}(n) & \times & \mathrm{MP}(m) & \to & \mathrm{MP}(m+n) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{MP}(1) \times \mathrm{MP}(n_1) & \times & \mathrm{MP}(k) \times \mathrm{MP}(m_1) & \to & \mathrm{MP}(k+1) \times \mathrm{MP}(m_1 + n_1)
\end{array}$$

where all maps are given by multiplication. By Theorem 1, the vertical arrows are local diffeomorphisms at $(x, P_1)$, $(x^k, Q_1)$ and $(x^{k+1}, P_1 Q_1)$ respectively. The lower map is the product of the multiplication $\mathrm{MP}(1) \times \mathrm{MP}(k) \to \mathrm{MP}(k+1)$, which we have just seen to be a $(k+1)$–swallowtail at $(x, x^k)$, and the multiplication $\mathrm{MP}(n_1) \times \mathrm{MP}(m_1) \to \mathrm{MP}(m_1 + n_1)$, a local diffeomorphism at $(P_1, Q_1)$ b Theorem 1. Therefore the upper multiplication map is diffeomorphic to the lower one, which is a $(k+1)$–swallowtail. This proves the "if" in (i). As for the "only if", just notice that, when the degree of $\gcd(P_0, Q_0)$ is greater than 1, the corank of $D\,\mathrm{Mult}$ is greater than 1 and Mult cannot be a swallowtail at that point. This proves (iii).

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[2] Or, in other words, the greatest integer $k$ such that $\gcd(P_0, Q_0)^k$ divides $P_0$ or $Q_0$.

Let us prove (iv). In the "if", the same diagram as for (iii) reduces the problem to the case where $P_0 = (x-\alpha)(x-\bar{\alpha})$ and $Q_0 = (x-\alpha)^k(x-\bar{\alpha})^k$, $\alpha \in \mathbf{C} \setminus \mathbf{R}$. Applying (ii) with $m = n = 1$ (resp. $m = n = k$), we see that every complex polynomial $P$ (resp. $Q$) of degree 2 (resp. $2k$) close enough to $P_0$ (resp. $Q_0$) writes in a (locally) unique fashion $P = P_1P_2$, $P_j \in \mathrm{MP}(1)$ (resp. $Q = Q_1Q_2$, $Q_j \in \mathrm{MP}(k)$), where $(P_1, P_2)$ (resp. $(Q_1, Q_2)$) is the image of $P$ (resp. $Q$) by the local inverse of Mult at $(x-\alpha, x-\bar{\alpha})$ resp. $\big((x-\alpha)^k, (x-\bar{\alpha})^k\big)$). This uniqueness property implies that, for real $P$ and $Q$, we must have $P_2 = \bar{P}_1$ and $Q_2 = \bar{Q}_1$. Thus, in that case, the mappings $P \mapsto P_1$ and $Q \mapsto Q_1$ are real analytic local diffeomorphisms, identifying $(P, Q) \mapsto PQ$ to $(P_1, Q_1) \mapsto P_1Q_1$, which is a complex $(k+1)$–swallowtail by (iii).

For the "only if", we observe that any other corang 2 singularity $(P_0, Q_0)$ has a common real root. In any neighborhood of $(P_0, Q_0)$ we can find a pair $(P_1, Q_1)$ with a single common *simple* real root and by (iii) the singular set of Mult is a codimension 1 smooth manifold near $(P_1, Q_1)$. Therefore, at a complex swallowtail, as the singular set of Mult is locally of real codimension 2, it cannot be diffeomorphic to the singularity at $(P_0, Q_0)$. □

*Remarks.* — Theorem 1 (ii), which can be found in **[3]** (Exercice 1, p. 234), extends the well-known result that a simple root varies smoothly with the coefficients of the polynomial (consider the local inverse of Mult).

If one writes down the Jacobian matrix of Mult at a point (f,g) with respect to the standard bases of the vector spaces of (non-necessarily monic) polynomials involved, one obtains the transpose of the usual Sylvester matrix, whose determinant is one of the definitions of the resultant of the polynomials $(g, f)$. Therefore we have an interesting equality:

$$\mathrm{Res}(f, g) = J\,\mathrm{Mult}(g, f) \quad \text{(Jacobian determinant).}$$

which is natural since both sides of the equality vanish precisely when $f, g$ have a common complex root. One can take this as a definition of the resultant and use it to prove its basic properties. The change from $(f, g)$ to $(g, f)$ in the right-hand side is only a sign convention, as in fact the definitions of the resultant by different authors only coincide up to sign: see for example **[11, 6]**.

The regular points of the variety $\mathrm{Res}(f, g) = 0$ are precisely the pairs with gcd of degree 1, since the points of higher corank have to be singularities of $J\,\mathrm{Mult}$ (see Appendix 1). By (iii), this regular set can be still stratified according to the singularity type of Mult, i.e. according to the order of the swallowtails. Thus, the singularity type of Mult gives more information than the singularities of the resultant variety. This is a first answer to a question by Bernard Teissier about the relation between those two singularities.

We now turn to the singularities of corank $\geq 2$ involving only simple common roots:

***Proposition 1***. — *Given* $P_0 \in \mathrm{MP}(n)$ *and* $Q_0 \in \mathrm{MP}(m)$, *assume that all the (complex) roots of* $\gcd(P_0, Q_0)$ *are simple.*

(i) *If* $\mathbf{K} = \mathbf{C}$, *then, denoting by* $\alpha_1, \ldots, \alpha_d$ *the roots of* $\gcd(P_0, Q_0)$, *the map* Mult *is the product of* $d$ *swallowtails of respective orders* $k_1 + 1, \ldots, k_d + 1$, *where* $k_j$ *denotes the maximum of the multiplicities of the root* $\alpha_j$ *in* $P_0$ *and* $Q_0$.

(ii) *If* $\mathbf{K} = \mathbf{R}$, *then, denoting by* $\alpha_1, \ldots, \alpha_r$ *the real roots of* $\gcd(P_0, Q_0)$ *and by* $\alpha_{r+1}, \bar{\alpha}_{r+1}, \ldots, \alpha_d, \bar{\alpha}_d$ *its other roots, the map* Mult *is the product of* $r$ *real swallowtails of respective orders* $k_1 + 1, \ldots, k_r + 1$ *and* $c$ *complex swallowtails of respective orders* $k_{r+1} + 1, \ldots, k_d + 1$, *where* $k_j$ *denotes the maximum of the multiplicities of the root* $\alpha_j$ *in* $P_0$ *and* $Q_0$.

*Proof.* — We establish (i) by induction on $d$. Theorem 1 (ii) tells us that (i) is true if $d = 1$. Given $d > 1$, assume (i) true for $d - 1$. Then, in the situation of (i), exchanging $P_0$ and $Q_0$ if necessary, we have $P_0 = (x - \alpha_d)P_1$, $Q_0 = (x - \alpha_d)^{k_d} Q_1$ and $\gcd(P_0, Q_0) = (x - \alpha_d)\gcd(P_1, Q_1)$. By the induction hypothesis, Mult is at $(P_1, Q_1)$ the product of $d-1$ swallowtails of respective orders $k_1+1, \ldots, k_{d-1}+1$. Now, setting $m_1 := m - k_d$ and $n_1 := n - 1$, we have a commutative diagram:

$$\begin{array}{ccccc}
\mathrm{MP}(n) & \times & \mathrm{MP}(m) & \to & \mathrm{MP}(m+n) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{MP}(1) \times \mathrm{MP}(n_1) & \times & \mathrm{MP}(k_d) \times \mathrm{MP}(m_1) & \to & \mathrm{MP}(k_d+1) \times \mathrm{MP}(m_1+n_1)
\end{array}$$

where all maps are given by multiplication, and we conclude as in the proof of Theorem 1 (iii).

This also proves (ii) if $c = 0$. Otherwise, exchanging $P_0$ and $Q_0$ if necessary, we have that $P_0 = (x - \alpha_{k_d})(x - \bar{\alpha}_{k_d})P_1$, $Q_0 = (x - \alpha_{k_d})^{k_d}(x - \bar{\alpha}_{k_d})^{k_d} Q_1$ and $\gcd(P_0, Q_0) = (x - \alpha_{k_d})(x - \bar{\alpha}_{k_d})\gcd(P_1, Q_1)$. Using Theorem 1 (iv), we conclude as for (i). □

*Remark.* — In the situation of (ii) with $r = 2$, $c = 0$ and $k_1 = k_2 = 1$, we get (a suspension of) the "twice folded handkerchief", product of two one-dimensional folds. More generally in the situation of (ii) with $r = 2$, $c = 0$ and $k_1 + k_2 = k$, the critical set of Mult is locally the (singular) union of two smooth hypersurfaces intersecting at $(P_0, Q_0)$. In particular, the critical set of Mult has codimension 1, making more precise the end of the proof of Theorem 1 (iv) in this case.

Let us now consider all the singularities of corank $\nu \geq 2$ with a single common root:

***Proposition 2***. — *Given* $P_0 \in \mathrm{MP}(n)$ *and* $Q_0 \in \mathrm{MP}(m)$,

(i) *Assume* $\gcd(P_0, Q_0) = (x - \alpha)^\nu$, $\alpha \in \mathbf{K}$, $\nu \geq 2$. *Then, denoting by* $k \geq \nu$ *the maximum of the multiplicities of the root* $\alpha$ *in* $P_0$ *and* $Q_0$, *the map* Mult *has a singularity at* $(P_0, Q_0)$ *which is diffeomorphic to the singularity at* $0$ *of the map*

$$\begin{array}{ccc}
(a, b, u) & \longmapsto & \big(a, b, f_{\nu,k}(b, u),\big) \\
\mathbf{K}^{m+n-k-\nu} \times \mathbf{K}^k \times \mathbf{K}^\nu & \longrightarrow & \mathbf{K}^{m+n-k-\nu} \times \mathbf{K}^k \times \mathbf{K}^\nu
\end{array}$$

*given by* $f_{\nu,k} = (f_{\nu,k,1}, \dots, f_{\nu,k,\nu})$ *and, if* $b = (b_1, \dots, b_k)$,

$$f_{\nu,k,\ell}(b,u) \quad := \quad \sum_{\|m\|=k+\ell} \frac{|m|!}{m!}\, u^m + \sum_{j=1}^{k} b_j \sum_{\|m\|=k+\ell-j} \frac{|m|!}{m!}\, u^m \,,\ 1 \le \ell \le \nu,$$

*where* $m = (m_1, \dots, m_\nu) \in \mathbf{N}^\nu$, $m! = m_1! \cdots m_\nu!$, $|m| := m_1 + \cdots + m_\nu$, $\|m\| := m_1 + 2m_2 + \cdots + \nu m_\nu$ *and* $u^m := u_1^{m_1} \cdots u_\nu^{m_\nu}$.

(ii) *If* $\mathbf{K} = \mathbf{R}$ *and* $\gcd(P_0, Q_0) = (x-\alpha)^\nu (x-\bar\alpha)^\nu$, $\alpha \in \mathbf{C} \setminus \mathbf{R}$, $\nu \ge 2$, *then, denoting by* $k \ge \nu$ *the maximum of the multiplicities of the root* $\alpha$ *in* $P_0$ *and* $Q_0$, *the map* Mult *has a singularity at* $(P_0, Q_0)$ *which is diffeomorphic to the singularity at* 0 *of the map*

$$\begin{array}{ccc} (a,b,u) & \longmapsto & \big(a, b, f_{\nu,k}(b,u),\big) \\ \mathbf{R}^{m+n-2k-2\nu} \times \mathbf{C}^k \times \mathbf{C}^\nu & \longrightarrow & \mathbf{R}^{m+n-2k-2\nu} \times \mathbf{C}^k \times \mathbf{C}^\nu, \end{array}$$

*where* $f_{\nu,k}$ *is as in (i).*

*Proof.* — (i) If $\gcd(P_0, Q_0) = (x-\alpha)^\nu$ then $x - \alpha$ must divide one of the two polynomials $P_0, Q_0$ with multiplicity $\nu$ and the other one with multiplicity $k \ge \nu$. As in the proof of Proposition 1, we can assume $\alpha = 0$ and reduce the general case to the case $P_0 = x^\nu$, $Q_0 = x^k$. Then, Mult is given by

$$\operatorname{Mult}\Big(x^\nu - \sum_{i=1}^{\nu} u_i x^{\nu-i}, x^k + \sum_{j=1}^{k} v_j x^{k-j}\Big) = \sum_{j=0}^{k+\nu} \Big(v_j - \sum_{i=1}^{\nu} v_{j-i} u_i\Big) x^{k+\nu-j}$$

where

$$(1) \qquad \begin{cases} v_0 = 1 \\ v_j = 0 \quad \text{for } j < 0 \text{ and for } j > k \end{cases}$$

or, taking $u_1, \dots, u_\nu, v_1, \dots, v_k$ as coordinates, by

$$\operatorname{Mult}(v_1, \dots, v_k, u_1, \dots, u_\nu) = \Big(v_j - \sum_{i=1}^{\nu} v_{j-i} u_i\Big)_{1 \le j \le k+\nu}.$$

Denoting by $b_1, \dots, b_{k+\nu}$ the components of the right-hand side, we shall express the variables $v_1, \dots, v_k$ as functions of $b_1, \dots, b_k$ and $u_1, \dots, u_\nu$ by solving the equations

$$(2) \qquad b_j = v_j - \sum_{i=1}^{\nu} v_{j-i} u_i$$

for $1 \le j \le k$, clearly an invertible linear system with respect to $v_1, \dots, v_k$. Then, the equations (2) with $k + 1 \le j \le k + \mu$ will yield the required expression

$$(3) \qquad b_j = -f_{\nu,k,j-k}(b_1, \dots, b_k, u_1, \dots, u_\nu)\,,\ k + 1 \le j \le k + \mu.$$

To do all this at once, we consider (2) for all $j \in \mathbf{Z}$ and, using (1), rewrite it as

$$\begin{cases} b_j = 0 & \text{for } j < 0 \\ b_0 = 1 & \\ v_j = b_j + \sum_{i=1}^{\nu} v_{j-i}\, u_i & \text{for } j > 0 \end{cases} \tag{4}$$

We claim that these conditions imply that

$$v_j \quad = \quad \sum_{\|m\| \le j} \frac{|m|!}{m!}\, b_{j-\|m\|}\, u^m\ ,\ j \in \mathbf{Z}, \tag{5}$$

hence (3) because of (1).

Indeed, by (4), we know that (5) is true for all $j < 0$. Given $j \ge 0$, we can therefore make the induction hypothesis that (5) is true for all $j - i$, $1 \le i \le \mu$, hence, by (4),

$$v_j \quad = \quad b_j + \sum_{i=1}^{\nu} \sum_{\|n\| \le j-i} \frac{|n|!}{n!}\, b_{j-i-\|n\|}\, u^{n+\delta_i}$$

where $n$ lies in $\mathbf{N}^\nu$ and $(\delta_1, \ldots, \delta_\nu)$ denotes the canonical basis of $\mathbf{K}^\nu$. Now, for each $m \in \mathbf{N}^\nu$, we have $m = n + \delta_i$ with $n \in \mathbf{N}^\nu$ if and only if $m_i$ is positive, in which case $\|m\| = \|n\| + i$, hence

$$\begin{aligned} v_j \quad &= \quad b_j + \sum_{1 \le \|m\| \le j} \sum_{m_i \ne 0} \frac{|m - \delta_i|!}{(m - \delta_i)!}\, b_{j-\|m\|}\, u^m \\ &= \quad b_j + \sum_{1 \le \|m\| \le j} \frac{(|m| - 1)!}{m!} \Big( \sum_{m_i \ne 0} m_i \Big)\, b_{j-\|m\|}\, u^m \\ &= \quad b_j + \sum_{1 \le \|m\| \le j} \frac{|m|!}{m!}\, b_{j-\|m\|}\, u^m\ , \end{aligned}$$

proving (5). From this particular case, we deduce (i) in general as in the proof of Theorem 1 (iii). The proof of (ii) is that of Theorem 1 (iv). □

*Remarks.* — Of course, for $\nu = 1$, the proof of Proposition 2 applies and is nothing but the proof of Theorem 1 (iii), the function $f_{1,k}$ being essentially the evaluation map of $\mathrm{MP}(k+1)$.

In the situation of (i) with $\nu = 2$ and $\mathbf{K} = \mathbf{R}$, the germ of Mult at $(P_0, Q_0)$ is analytically diffeomorphic to the germ at 0 of $(a, b, u) \mapsto \big(a, b, f_{2,k}(b, u)\big)$, $a \in \mathbf{R}^{m+n-k-2}$, $b = (b_1, \ldots, b_k) \in \mathbf{R}^k$, whose critical set is the set of zeros of the determinant $J_u f_{2,k}(b, u) := \big(\partial_{u_1} f_{2,k,1} \partial_{u_2} f_{2,k,2} - \partial_{u_1} f_{2,k,2} \partial_{u_2} f_{2,k,1}\big)(b, u)$. Since the latter is a quadratic form in the variable $b$ satisfying $J_u f_{2,k}(b, 0) = b_k^2 - b_{k-1}^2$, this gives a more precise idea of the shape of the critical set of Mult, which is a singular hypersurface, as shown at the end of the proof of Theorem 1 (iv).

We can now glue together Theorem 1 (iv) and Proposition 2 as in the proof of Proposition 1 to obtain an algebraic description of all the singularities of Mult:

***Theorem 2***. — *Given $P_0 \in \mathrm{MP}(n)$ and $Q_0 \in \mathrm{MP}(m)$,*

(i) *If $\mathbf{K} = \mathbf{C}$, then, denoting by $\alpha_1, \ldots, \alpha_d$ the roots of* $\gcd(P_0, Q_0)$ *and by $\nu_1, \ldots, \nu_d$ their respective multiplicities, the germ of* Mult *at $(P_0, Q_0)$ is diffeomorphic to the germ at 0 of the map*

$$\begin{array}{ccc} \mathbf{K}^p \times \prod_1^d \left(\mathbf{K}^{k_i} \times \mathbf{K}^{\nu_i}\right) & \longrightarrow & \mathbf{K}^p \times \prod_1^d \left(\mathbf{K}^{k_i} \times \mathbf{K}^{\nu_i}\right) \\ \big(a, (b_1, x_1), \ldots, (b_d, x_d)\big) & \longmapsto & \big(a, f_{\nu_1,k_1}(b_1, x_1), \ldots, f_{\nu_d,k_d}(b_d, x_d)\big)\,, \end{array}$$

*where $k_i \geq \nu_i$ denotes the maximum of the multiplicities of the root $\alpha_i$ in $P_0$ and $Q_0$, and $p = m + n - |k| - |\nu|$.*

(ii) *If $\mathbf{K} = \mathbf{R}$, then, denoting by $\alpha_1, \ldots, \alpha_r$ the real roots of* $\gcd(P_0, Q_0)$*, by $\alpha_{r+1}, \bar{\alpha}_{r+1}, \ldots, \alpha_d, \bar{\alpha}_d$ its other roots, by $\nu_j$ the multiplicity of the root $\alpha_j$ and setting $d := r + c$, the germ of* Mult *at $(P_0, Q_0)$ is diffeomorphic to the germ at $0$ of the map*

$$\big(a, (b_1, x_1), \ldots, (b_d, x_d)\big) \longmapsto \big(a, f_{\nu_1,k_1}(b_1, x_1), \ldots, f_{\nu_d,k_d}(b_d, x_d)\big)\,,$$

*of $\mathbf{R}^p \times \prod_1^r \left(\mathbf{R}^{k_i} \times \mathbf{R}^{\nu_i}\right) \times \prod_{r+1}^d \left(\mathbf{C}^{k_i} \times \mathbf{C}^{\nu_i}\right)$ into itself, where $k_i \geq \nu_i$ denotes the maximum of the multiplicities of the root $\alpha_i$ in $P_0$ and $Q_0$, and $p = m+n-\sum_1^r (k_i + \nu_i) - 2\sum_{r+1}^d (k_i + \nu_i)$.*

*Products of $p$ monic polynomials.* — Theorem 1 has the following obvious generalisation:

***Theorem 3***. — *Given integers $m_1, \ldots, m_p$, $p > 1$, denote again the multiplication map by* $\mathrm{Mult} : \mathrm{MP}(m_1) \times \cdots \times \mathrm{MP}(m_p) \to \mathrm{MP}(m_1 + \cdots + m_p)$*. Then, for each $(P_1, \ldots, P_p) \in \mathrm{MP}(m_1) \times \cdots \times \mathrm{MP}(m_p)$:*

(i) *The corank of $D\,\mathrm{Mult}(P_1, \ldots, P_p)$ is the degree of* $\gcd(P_1 \cdots P_p / P_j)_{1 \leq j \leq p}$*.*

(ii) *In particular,* Mult *is a local diffeomorphism at $(P_1, \ldots, P_p)$ if, and only if* $\gcd(P_i, P_j) = 1$ *for $1 \leq i < j \leq p$.*

(iii) *The map* Mult *is a $(k+1)$–swallowtail at $(P_1, \ldots, P_p)$ for some positive integer $k$ if, and only if, it has corank one. If this is the case, there exist $i, j \in \{1, \ldots, p\}$ and $\alpha \in \mathbf{K}$ such that*

$$\gcd\big(P_\ell, P_m\big) = \begin{cases} x - \alpha & \textit{if } \{\ell, m\} = \{i, j\} \\ 1 & \textit{otherwise,} \end{cases}$$

*and $k$ is the maximum of the multiplicities of the root $\alpha$ in $P_i$ and $P_j$.*

(iv) *If $\mathbf{K} = \mathbf{R}$, the map* Mult *is a complex $(k+1)$–swallowtail at $(P_1, \ldots, P_p)$ for some positive integer $k$ if, and only if, there exist $i, j \in \{1, \ldots, p\}$ and $\alpha \in \mathbf{C} \setminus \mathbf{R}$ such that*

$$\gcd\big(P_\ell, P_m\big) = \begin{cases} (x - \alpha)(x - \bar{\alpha}) & \textit{if } \{\ell, m\} = \{i, j\} \\ 1 & \textit{otherwise,} \end{cases}$$

*and $k$ is the maximum of the multiplicities of the root $\alpha$ in $P_i$ and $P_j$.*

The following result, whose proof is that of Proposition 2, describes the simplest singularities of higher corank:

***Theorem 4.*** — *Given* $(P_1, \dots, P_p) \in \mathrm{MP}(m_1) \times \cdots \times \mathrm{MP}(m_p)$, *assume that all the (complex) roots of* $\gcd(P_1 \cdots P_p/P_j)_{1 \le j \le p}$ *are simple.*

(i) *If* $\mathbf{K} = \mathbf{C}$, *then, denoting the roots by* $\alpha_1, \dots, \alpha_d$, *the map* Mult *is the product of* $d$ *swallowtails of respective orders* $k_1 + 1, \dots, k_d + 1$, *where* $k_j$ *denotes the maximum of the multiplicities of the root* $\alpha_j$ *in* $P_1, \dots, P_p$.

(ii) *If* $\mathbf{K} = \mathbf{R}$, *then, denoting the real roots of* $\gcd(P_1 \cdots P_p/P_j)_{1 \le j \le p}$ *by* $\alpha_1, \dots, \alpha_r$ *and its other roots by* $\alpha_{r+1}, \bar{\alpha}_{r+1}, \dots, \alpha_d, \bar{\alpha}_d$, *the map* Mult *is the product of* $r$ *real swallowtails of respective orders* $k_1 + 1, \dots, k_r + 1$ *and* $c$ *complex swallowtails of respective orders* $k_{r+1} + 1, \dots, k_d + 1$, *where* $k_j$ *denotes the maximum of the multiplicities of the root* $\alpha_j$ *in* $P_1, \dots, P_p$.

*Remark.* — For $p > 2$, when $\gcd(P_1 \cdots P_p/P_j)_{1 \le j \le p}$ has multiple roots, they can be common to three or more of the $P_j$'s, yielding other singularities which deserve a better study. For example, when $p = 3$, $\gcd(P_1P_2P_3/P_j)_{1 \le j \le 3} = x - \alpha$ and $\alpha$ is a simple root of all three polynomials, the singularity we get is a suspension of the germ at $0 \in \mathbf{C}$ of $z \mapsto \big(|z|^2, \Im(z^3)\big)$.

## B. Singularities of the characteristic polynomial function

Let $\mathrm{M}(n \times n)$ be the space of $n \times n$ matrices with entries in $\mathbf{K}$. We will view each $M \in \mathrm{M}(n \times n)$ as a linear mapping $\mathbb{C}^n \to \mathbb{C}^n$ and always take into account all its *complex* eigenvalues. We will denote by

$$\chi : \mathrm{M}(n \times n) \to \mathrm{MP}(n)$$

the mapping sending $M$ to its monic characteristic polynomial:

$$\chi(M) := \det(xI - M).$$

We are interested in the regular points and the simplest singularities of the mapping $\chi$. Recall that an eigenvalue of $M \in \mathrm{M}(n \times n)$ is called *simple* if it is a simple root of $\chi(M)$.

We will call the eigenvalue $\lambda$ of $M \in \mathrm{M}(n \times n)$ *geometrically simple* if the corresponding eigenspace is a line.

***Theorem 5.*** — *Let* $M_0 \in \mathrm{M}(n \times n)$. *Then*

(i) *The map* $\chi$ *is regular at* $M_0$ *if, and only if, all the eigenvalues of* $M_0$ *are geometrically simple.*

(ii) *The rank of* $D\chi(M_0)$ *is the degree* $m(M_0)$ *of the minimal polynomial of* $M_0$. *In other words, the corank of* $D\chi(M_0)$ *equals*

$$\sum_{\lambda \in \sigma(M_0)} (d_\lambda - m_\lambda),$$

*where $d_\lambda$ is the dimension of the characteristic space $E_\lambda$ of $M_0$ associated to $\lambda$, and $m_\lambda$ is the smallest integer $m$ such that $E_\lambda = \mathrm{Ker}(\lambda I - M_0)^m$.*[3]

(iii) *The map $\chi$ is a $(k+1)$–swallowtail deformation at $M_0$ if, and only if, it has corank 1, the integer $k$ being as follows: all eigenvalues of $M_0$ are geometrically simple except one, for which $m_\lambda = k$ and $d_\lambda = k+1$.*

*Proof.* — We can assume $\mathbf{K} = \mathbf{C}$ since even in the real case all definitions involve the complex numbers and the regularity of $\chi$ does not depend on the field. We will denote the Jordan block of order $n$ and eigenvalue $\lambda$ by $J_n(\lambda)$ :

$$J_n(\lambda) = \begin{cases} \lambda & \text{if } n = 1 \\ \lambda I_n + \begin{pmatrix} 0 & I_{n-1} \\ 0 & 0 \end{pmatrix} & \text{for } n > 1. \end{cases}$$

Let us prove the "if" part of (i), first in the case where $M_0 = J_n(\lambda)$ . To see that $\chi$ is a submersion at $M_0$ we can also assume $\lambda = 0$, since we can compose with a translation in the space of matrices and with a change of variable in the space of polynomials. In this case $\chi$ admits a section, sending a polynomial to its *companion matrix*, defined as follows: we let

$$\mathrm{Comp} : \mathrm{MP}(n) \to \mathrm{M}(n \times n),$$

be given by

$$\mathrm{Comp}\Big(x^n + \sum_{i=0}^{n-1} a_i x^i\Big) := J_n(0) + \begin{pmatrix} 0 & \cdots & 0 \\ -a_0 & \cdots & -a_{n-1} \end{pmatrix}.$$

Then $\chi(\mathrm{Comp}(P)) = P$ and in particular $\chi$ is a submersion.

If $M_0$ has only geometrically simple eigenvalues then, changing coordinates, we may assume that it is block-diagonal, of the form (as a map)

$$M_0 = J_{m_1}(\lambda_1) \times \cdots \times J_{m_p}(\lambda_p),$$

where the $\lambda_j$'s are all different. Now, the restriction of $\chi$ to the vector subspace of $\mathrm{M}(n \times n)$ consisting of block-diagonal matrices (linear maps) $M_1 \times \cdots \times M_p$ with $M_j \in \mathrm{M}(m_j \times m_j)$ is already a submersion at $M_0$: indeed, it is the composed map of

— the map $M_1 \times \cdots \times M_p \overset{\chi^p}{\longmapsto} \big(\chi(M_1), \ldots, \chi(M_p)\big)$, which is a submersion at $M_0$ by what we have just done, and

— the product map

$$\mathrm{MP}(m_1) \times \cdots \times \mathrm{MP}(m_p) \ni (P_1, \ldots, P_p) \mapsto P_1 \cdots P_p \in \mathrm{MP}(n),$$

which is a local diffeomorphism at $\chi^p(M_0) = \big((x-\lambda_1)^{m_1}, \ldots, (x-\lambda_p)^{m_p}\big)$ by Theorem 3 (i) since the $\lambda_j$'s are all different.

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[3] This is the size of the largest Jordan block with eigenvalue $\lambda$ or, equivalently, the multiplicity of $\lambda$ as a root of the minimal polynomial of $M_0$.

To prove (ii) and therefore the "only if" part of (i), we can again assume that $M_0 = J_{m_1}(\lambda_1) \times \cdots \times J_{m_p}(\lambda_p)$, where the $\lambda_j$'s may not be all different. Then, writing each $A \in \mathrm{M}(n \times n)$ in block form

$$A = \begin{pmatrix} a_1^1 & \cdots & a_p^1 \\ \vdots & \ddots & \vdots \\ a_1^p & \cdots & a_p^p \end{pmatrix}, \quad a_i^j : \mathbf{K}^{m_i} \to \mathbf{K}^{m_j},$$

we notice[4] that $D\chi(M_0)A = D\chi(M_0)(a_1^1 \times \cdots \times a_p^p)$. Therefore, the corank of $D\chi(M_0)$ is the corank of the differential at $M_0$ of the the restriction of $\chi$ to the space of all $M_1 \times \cdots \times M_p$ with $M_j \in \mathrm{M}(m_j \times m_j)$. Now, we have seen that this restriction is the composed map of the submersion $\chi^p$ and the map $\mathrm{Mult} : \mathrm{MP}(m_1) \times \cdots \times \mathrm{MP}(m_p) \to \mathrm{MP}(m_1 + \cdots + m_p)$. Thus, the corank of $D\chi(M_0)$ is that of $D\,\mathrm{Mult}\left(\chi^p(M_0)\right) = D\,\mathrm{Mult}\left((x-\lambda_1)^{m_1}, \dots, (x-\lambda_p)^{m_p}\right)$. By Theorem 3 (ii), this is indeed the degree of $\gcd\left(\prod_{i\neq j}(x-\lambda_i)^{m_i}\right)_{1\leq j\leq p} = \prod_{\lambda\in\sigma(M_0)}(x-\lambda)^{d_\lambda - m_\lambda}$.

To prove (iii), still assuming that $M_0 = J_{m_1}(\lambda_1) \times \cdots \times J_{m_p}(\lambda_p)$, just notice the following two facts:

— If the corank of $D\chi(M_0)$ is greater than 1, then $\chi$ is not a $(k+1)$–swallowtail deformation at $M_0$.
— If $D\chi(M_0)$ has corank 1, then, by (ii) and Theorem 3 (iii), the map Mult is a $(k+1)$–swallowtail at $\chi^p(M_0)$ with just the right $k$. Therefore $\chi$, being the composed map of Mult with a local submersion, is a $(k+1)$–swallowtail deformation at $M_0$. □

*Remarks on the real case.* — If $\mathbf{K} = \mathbf{R}$, it follows from Theorem 3 (iv) that $\chi$ is a $(k+1)$–*complex* swallowtail deformation at $M_0$ when all eigenvalues of $M_0$ are geometrically simple except one pair $\{\lambda, \bar{\lambda}\}$, for which $\lambda \in \mathbf{C} \setminus \mathbf{R}$, $m_\lambda = k$ and $d_\lambda = k+1$.

This is coherent with the following (maybe not so well-known) Jordan normal form theorem in the real case: every endomorphism $A$ of a real vector space $E$ of finite dimension $n$ is conjugate to a block-diagonal endomorphism of $\prod_1^r \mathbf{R}^{m_j} \times \prod_{r+1}^p \mathbf{C}^{m_j}$ of the form $J_{m_1}(\lambda_1) \times \cdots \times J_{m_p}(\lambda_p)$, where the $\lambda_j$'s and the $\bar{\lambda}_j$'s are the eigenvalues of $A$, real for $j \leq r$ and nonreal for $j > r$.

*Higher singularites.* — It follows from the above arguments that all the singularities of $\chi$ are deformations of singularities of Mult for 2 or more factors. We shall not dwell on this fact for the time being.

[4] Using the fact that $\det A$ is an $n$–linear function of the columns of $A$, implying that $D\chi(M_0)A$ is the sum of the determinants of the $n$ matrices obtained each by replacing one column of $xI - M_0$ by the corresponding column of $-A$.

## C. Singularities of eigenvalues of linear operators

First we will describe several linear spaces, manifolds and maps related to the Banach space $E$.

All closed hyperplanes $H \ni 0$ are isomorphic as Banach spaces. We will use the notation $E_0$ for their common type(5).

Let $\mathbf{P}(E)$ be the projective space associated to $E$, that is, the space of all one-dimensional linear subspaces of $E$. Then $\mathbf{P}(E)$ has a natural analytic (algebraic) Banach manifold structure modelled on $E_0$, defined in the usual way—it is a connected component of the Grassmannian $\mathbf{G}(E)$ described in Section D.

Let $\mathrm{End}(E)$ be the space of bounded linear operators from $E$ to $E$. We will denote the identity operator by 1 and its multiple by a scalar $k$ also by $k$. If $A \in \mathrm{End}(E)$ we denote by $\sigma(A)$ its spectrum.

More generally, if we have two Banach spaces $E_1, E_2$ over $\mathbf{K}$ we will denote by $\mathscr{B}(E_1, E_2)$ the space of continuous linear maps from $E_1$ to $E_2$.

**C1. The manifold of proper elements.** — The *manifold of proper elements* of $E$ is the space

$$\mathrm{Eig}(E) := \{(\lambda, L, A) \in \mathbf{K} \times \mathbf{P}(E) \times \mathrm{End}(E) \,:\, A(L) \subseteq L \text{ and } A|_L = \lambda\}.$$

That is, the space of triples consisting of a linear operator, an invariant line and the corresponding eigenvalue. The specification of the eigenvalue $\lambda$ is redundant but useful, as we shall see.

***Theorem 6.*** — *The set* $\mathrm{Eig}(E)$ *is an analytic (algebraic) Banach submanifold of the manifold* $\mathbf{K} \times \mathbf{P}(E) \times \mathrm{End}(E)$*, modelled on* $\mathrm{End}(E)$*.*

*Proof.* — Given $(\lambda_0, L_0, A_0) \in \mathrm{Eig}(E)$, choose $x \in L_0 \setminus \{0\}$ and a complementary subspace $H$ of $L_0$. Identifying $E = \mathbf{K}x \oplus H$ to $\mathbf{K} \times H$ we can identify each line $L \in \mathbf{P}(E)$ transversal to $H$ to the unique $h \in H$ satisfying $(1, h) \in L$ (6) and write every operator $A \in \mathrm{End}(E)$ in matrix form

$$A \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a \in \mathbb{K}, b \in H^*, c \in H, d \in \mathrm{End}(H)$$

Hence in particular

$$A_0 = \begin{pmatrix} \lambda_0 & b_0 \\ 0 & d_0 \end{pmatrix}.$$

(5) For $E$ the Hilbert space $l^2$ (or any of the well known Banach spaces from Functional Analysis) $E_0$ is isomorphic to $E$. However, there are examples of infinite dimensional Banach spaces where this is not the case [**7**].
(6) Therefore, $L_0$ corresponds to $h = 0$.

In these identifications, the relation $(\lambda, L, A) \in \mathrm{Eig}(E)$ reads

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ h \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ h \end{pmatrix},$$

that is

$$\begin{aligned} a &= \lambda - bh \\ c &= (\lambda - d)h. \end{aligned} \tag{6}$$

In other words, the open subset of $\mathrm{Eig}(E)$ consisting of those $(\lambda, L, A)$ such that $L$ is not contained in $H$ admits the parametrisation

$$\mathrm{End}(E) \ni \begin{pmatrix} \lambda & b \\ h & d \end{pmatrix} \mapsto \left(\lambda, h, \begin{pmatrix} \lambda - bh & b \\ (\lambda - d)h & d \end{pmatrix}\right)$$

as the graph of the polynomial map defined by (6). □

***Corollary***. — *For $(\lambda_0, L_0, A_0) \in \mathrm{Eig}(E)$, there is an analytic function $A(\lambda, L)$ defined in a neighborhood of $(\lambda_0, L_0)$ such that $A(\lambda_0, L_0) = A_0$ and that the nonzero elements of $L$ are eigenvectors of $A(\lambda, L)$ with eigenvalue $\lambda$.*

*Proof*. — Just take $A(\lambda, h) = \begin{pmatrix} \lambda - b_0 h & b_0 \\ (\lambda - d_0)h & d_0 \end{pmatrix}$ modulo the identifications of the previous proof. □

This function is clearly not unique, and although it is possible from the proof of the theorem to describe all of them, much more interesting is the question whether $\lambda, L$ are analytic functions of the operator $A$. To present our version of the (classical) answer, we consider the following geometric reformulation:

*Question*. — Let $\Pi$ be the projection from $\mathrm{Eig}(E)$ to $\mathrm{End}(E)$ which forgets the first two components. When is it a local diffeomorphism? What are its simplest singularities?

### C2. The Immersion Theorem

***Definition***. — If $\lambda$ is an eigenvalue of $A \in \mathrm{End}(E)$ with eigenvector $x$ and $L$ is the line generated by $x$, then $A$ induces an operator $\dot{A}$ from the quotient $E/L$ to itself. We will say that $\lambda$ is a *simple* eigenvalue of $A$ if we have $\lambda \notin \sigma(\dot{A})$.
We call $\lambda$ a *geometrically simple* eigenvalue of $A$ if
(i) the corresponding eigenspace is a line (i.e. $\dim \mathrm{Ker}(A - \lambda) = 1$) and,
(ii) the image of the operator $\lambda - A$ is a direct factor[7].
We call $\lambda$ a geometrically simple eigenvalue of $A$ *of (finite) multiplicity* $k \geq 1$ if

[7] Meaning that it is closed and admits a closed complementary subspace.

$\dim \operatorname{Ker}(A-\lambda)^k = k$, and $\lambda$ is not in the spectrum of the endomorphism of the quotient $E/\operatorname{Ker}(A-\lambda)^k$ induced by $A$.

Let

$$\operatorname{Val}(E) := \{(\lambda, A) \in \mathbf{K} \times \operatorname{End}(E) : \lambda \text{ is an eigenvalue of } A\}.$$

***Theorem 7.*** — *Let $j : \operatorname{Eig}(E) \to \mathbf{K} \times \operatorname{End}(E)$ be defined by $j(\lambda, L, A) := (\lambda, A)$, hence $j\big(\operatorname{Eig}(E)\big) = \operatorname{Val}(E)$.*

(i) *The map $j$ is an immersion in the neighbourhood of $(\lambda_0, L_0, A_0) \in \operatorname{Eig}(E)$ if, and only if, $\lambda_0$ is a geometrically simple eigenvalue of $A_0$.*

(ii) *The set of those $(\lambda, A) \in \operatorname{Val}(E)$ where $\lambda$ is a geometrically simple eigenvalue of $A$ with finite multiplicity is a manifold modelled on* $\operatorname{End}(E)$.

*Proof.* — (i) In terms of the parametrization of $\operatorname{Eig}(E)$ introduced in the proof of Theorem 6, $j$ is the map

$$j : \begin{pmatrix} \lambda & b \\ h & d \end{pmatrix} \longmapsto \left(\lambda, \begin{pmatrix} \lambda - bh & b \\ (\lambda - d)h & d \end{pmatrix}\right),$$

$(\lambda_0, L_0, A_0)$ and $A_0$ being identified to the same matrix

$$A_0 = \begin{pmatrix} \lambda_0 & b_0 \\ 0 & d_0 \end{pmatrix}. \tag{7}$$

Therefore, the derivative of $j$ at $(\lambda_0, L_0, A_0)$ is the map

$$Dj(\lambda_0, L_0, A_0) : \begin{pmatrix} \lambda & b \\ h & d \end{pmatrix} \longmapsto \left(\lambda, \begin{pmatrix} \lambda - b_0 h & b \\ (\lambda_0 - d_0)h & d \end{pmatrix}\right), \tag{8}$$

which vanishes if and only if $\lambda = 0$, $b = 0$, $d = 0$ and

$$b_0 h = 0\,, \quad (\lambda_0 - d_0)h = 0. \tag{9}$$

As (9) writes

$$\begin{pmatrix} \lambda_0 & b_0 \\ 0 & d_0 \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} = \lambda_0 \begin{pmatrix} 0 \\ h \end{pmatrix},$$

we have[8] $\operatorname{Ker} Dj(\lambda_0, L_0, A_0) \neq \{0\}$ if and only if $A_0 v = \lambda_0 v$ for some $v \notin L_0$. From this (i) follows for $\dim E < \infty$. In infinite dimensions, we have to check that, moreover, the image of $Dj(\lambda_0, L_0, A_0)$ is a direct factor if and only if so is the image of $\lambda_0 - A_0$. Now, by (7)–(8), this amounts to proving that the image of

$$\begin{pmatrix} \lambda \\ h \end{pmatrix} \longmapsto \left(\lambda, \begin{pmatrix} \lambda - b_0 h \\ (\lambda_0 - d_0)h \end{pmatrix}\right)$$

---

[8] Recall that, in our parametrization, $L_0$ is generated by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

is a direct factor in $\mathbf{K} \times E$ if and only if the image of

$$\begin{pmatrix} \lambda \\ h \end{pmatrix} \longmapsto \begin{pmatrix} -b_0 h \\ (\lambda_0 - d_0) h \end{pmatrix}$$

is a direct factor in $E$. In other words, setting $e := \left(1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$, we should prove that $\mathbf{K}e \oplus \big(\{0\} \times \operatorname{Im}(\lambda_0 - A_0)\big)$ has a closed complement $S$ in $\mathbf{K} \times E$ if and only if $\operatorname{Im}(\lambda_0 - A_0)$ has a closed complement $S_0$ in $E$, which is obvious: take $S = \{0\} \times S_0$ to get the "if" part and $\{0\} \times S_0 = (\{0\} \times E) \cap (\mathbf{K}e \oplus S)$ to obtain the converse.

(ii) Given $(\lambda_0, L_0, A_0) \in \operatorname{Eig}(E)$, we should prove that if $\lambda_0$ is a simple eigenvalue of $A_0$ of finite multiplicity $k$ then, near $(\lambda_0, A_0)$, the subset $\operatorname{Val}(E)$ consists solely of the image by $j$ of a neighbourhood of $(\lambda_0, L_0, A_0)$, a consequence of the following

***Lemma***. — *For each sequence $(\lambda_n, L_n, A_n)_{n \geq 1}$ in* $\operatorname{Eig}(E)$ *such that $(\lambda_n, A_n)$ converges to $(\lambda_0, A_0)$, the line $L_n$ tends to $L_0$ when $n \to \infty$.*

*Proof*. — For $\dim E < \infty$, the projective space $\mathbf{P}(E)$ is compact and every convergent subsequence of $(L_n)$ must tend to a line $L$ invariant by $A_0 = \lim A_n$ and such that $A_0|_L = \lim \lambda_n = \lambda_0$, hence $L = L_0$, proving the lemma.

When $E$ is infinite dimensional, the subspace $K_0 := \operatorname{Ker}(\lambda_0 - A_0)^k$ admits an $A_0$–invariant closed complement $S$ by the Hahn-Banach theorem. Identifying $E = \mathbf{K}_0 \oplus S$ to $K_0 \times S$, we can write $A_n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}$ with $\alpha_n \in \operatorname{End}(K_0)$, $\beta_n \in \mathscr{B}(S, K_0)$, $\gamma_n \in \mathscr{B}(K_0, S)$, $\delta_n \in \operatorname{End}(S)$ for all $n \in \mathbf{N}$ and, as $K_0$ is $A_0$–invariant, $\gamma_0 = 0$ and $\lambda_0 \notin \sigma(\delta_0)$. For $n \geq 1$, choose a generator $u_n = (v_n, w_n)$ of $L_n$ with $|u_n| = 1$. The $S$–component of the relation $(\lambda_n - A_n)u_n = 0$ reads

$$-\gamma_n v_n + (\lambda_n - \delta_n) w_n = 0. \tag{10}$$

Now, as $\lambda_0 - \delta_0$ is invertible, so is $\lambda_n - \delta_n$ for large enough $n$. Therefore, (10) can be written

$$w_n = (\lambda_n - \delta_n)^{-1} \gamma_n v_n, \tag{11}$$

hence in particular

$$\lim_{n \to \infty} w_n = 0 \tag{12}$$

since $v_n$ is bounded, $\lim(\lambda_n - \delta_n)^{-1} = (\lambda_0 - \delta_0)^{-1}$ and $\lim \gamma_n = \gamma_0 = 0$. By (11), for large enough $n$, the $K_0$–component of the relation $(\lambda_n - A_n)u_n = 0$ becomes

$$\big((\lambda_n - \alpha_n) + \beta_n (\lambda_n - \delta_n)^{-1} \gamma_n\big) v_n = 0. \tag{13}$$

By (12), $|v_n|$ tends to 1 when $n \to \infty$. Therefore, for large enough $n$, the vector $v_n$ generates a line $L'_n \subset K_0$ and, by (13), the hypotheses of the finite dimensional case are satisfied by the sequence $(0, L'_n, A'_n) \in \operatorname{Eig}(K_0)$ defined by the formula $A'_n := (\lambda_n - \alpha_n) + \beta_n (\lambda_n - \delta_n)^{-1} \gamma_n$, as $\lim A'_n = \lambda_0 - \alpha_0$. It follows that the line $L'_n$ tends to the one-dimensional kernel of $\lambda_0 - \alpha_0$, namely $L_0$. Therefore, by (12), we do have $\lim L_n = \lim L'_n = L_0$. $\square$

*Remark.* — Let $\pi : \mathrm{Val}(E) \to \mathrm{End}(E)$ be the natural map, induced by the projection $\mathbf{K} \times \mathrm{End}(E) \to \mathrm{End}(E)$. When $\lambda$ is geometrically simple this map is diffeomorphic to the natural map $\Pi : \mathrm{Eig}(E) \to \mathrm{End}(E)$ via the local diffeomorphism $\mathrm{Eig}(E) \approx \mathrm{Val}(E)$ defined by $j$.

The singularities of $\mathrm{Val}(E)$ might be of some interest. In the finite dimensional case, $\mathrm{Val}(E)$ is an algebraic subset, given by the equation $\chi(A)(\lambda) = 0$, whose regular part contains the points with geometrically simple eigenvalue. Near each such regular point, it can be proven that $\tilde{\chi} : (\lambda, A) \mapsto \big(\lambda, \chi(A)\big)$ is a submersion of $\mathrm{Val}(E)$ into $\mathrm{Root}(n)$ (see Example 1c).

### C3. Singularities of eigenvalues

***Theorem 8.*** — *Let $\Pi : \mathrm{Eig}(E) \to \mathrm{End}(E)$ be the natural map $(\lambda, L, A) \mapsto A$.*

(i) *The map $\Pi$ is a local diffeomorphism near $(\lambda_0, L_0, A_0) \in \mathrm{Eig}(E)$ if, and only if, $\lambda_0$ is a simple eigenvalue of $A_0$.*

(ii) *The dimension of the kernel of $D\Pi(\lambda_0, L_0, A_0)$ equals the dimension of the kernel of $\lambda_0 - \dot{A}_0$.*

(iii) *The map $\Pi$ is a $k$–swallowtail at $(\lambda_0, L_0, A_0)$ if (and, for $\dim E < \infty$, only if) $\lambda_0$ is a geometrically simple eigenvalue of $A_0$ with mutiplicity $k$.*

*Proof.* — In terms of the parametrization of $\mathrm{Eig}(E)$ introduced in the proof of Theorem 6, $\Pi$ is the map

$$\Pi : \begin{pmatrix} \lambda & b \\ h & d \end{pmatrix} \longmapsto \begin{pmatrix} \lambda - bh & b \\ (\lambda - d)h & d \end{pmatrix}$$

and $(\lambda_0, L_0, A_0)$ identifies to the matrix

$$A_0 = \begin{pmatrix} \lambda_0 & b_0 \\ 0 & d_0 \end{pmatrix}$$

Therefore, the derivative of $\Pi$ at $(\lambda_0, L_0, A_0)$ is

$$D\Pi(\lambda_0, L_0, A_0) : \begin{pmatrix} \lambda & b \\ h & d \end{pmatrix} \longmapsto \begin{pmatrix} \lambda - b_0 h & b \\ (\lambda_0 - d_0)h & d \end{pmatrix},$$

which is an isomorphism if, and only if, $\lambda_0 - d_0$ is an automorphism of $H$, i.e. $\lambda_0 \notin \sigma(d_0)$ or, equivalently, $\lambda_0 \notin \sigma(\dot{A}_0)$. By the Inverse Function Theorem in Banach spaces this condition is equivalent to $\Pi$ being a local diffeomorphism in the neighborhood of $(\lambda_0, L_0, A_0)$.

The same argument proves (ii).

*Proof of the "if" part of* (iii) *when* $\dim E = k$. — We can assume that $\lambda_0 = 0$ and $A_0 = J_k(0)$. Then, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Val}(\mathbf{K}^k) & \overset{\tilde{\chi}}{\to} & \mathrm{Root}(k) \\ \downarrow \pi & & \downarrow \bar{\pi} \\ \mathrm{End}(\mathbf{K}^k) & \overset{\chi}{\to} & \mathrm{MP}(k) \end{array}$$

where $\pi : (\lambda, A) \mapsto A$ equals $\Pi$ up to the local diffeomorphism $j$ near $\big(0, J_k(0)\big)$ and $\bar{\pi} : (\lambda, P) \mapsto P$ is a $k$–swallowtail by example 1c of section A. The map $\chi$ is a local submersion since it admits the local section Comp (see the proof of Theorem 3), and so is $\tilde{\chi} : (\lambda, A) \mapsto \big(\lambda, \chi(A)\big)$ since it admits the local section $(\lambda, P) \mapsto \big(\lambda, \mathrm{Comp}(P)\big)$. In particular, the (algebraic) fiber

$$F := \chi^{-1}(x^k) \subset \mathrm{End}(\mathbf{K}^k)$$

is a submanifold near $J_k(0)$ and so is

$$\tilde{\chi}^{-1}(0, x^k) = \{0\} \times F \approx F$$

near $\big(0, J_k(0)\big) \in \mathrm{Val}(\mathbf{K}^k)$. As $\chi$ is a submersion, there exists a local diffeomorphism defined near $J_k(0)$ and of the form

$$\begin{array}{ccc} \mathrm{End}(\mathbf{K}^k) & \overset{g}{\longrightarrow} & \mathrm{MP}(k) \times F \\ A & \longmapsto & \big(\chi(A), f(A)\big) \end{array}$$

such that

$$f(A) = A \quad \text{for} \quad A \in F.$$

It follows at once that, near $\big(0, J_k(0)\big)$, the map

$$\begin{array}{ccc} \mathrm{Val}(\mathbf{K}^k) & \overset{\tilde{g}}{\longrightarrow} & \mathrm{Root}(k) \times F \\ (\lambda, A) & \longmapsto & \big(\tilde{\chi}(A), f(A)\big) \end{array}$$

is a local diffeomorphism. As the diagram

$$\begin{array}{ccc} \mathrm{Val}(\mathbf{K}^k) & \overset{\tilde{g}}{\to} & \mathrm{Root}(k) \times F \\ \downarrow \pi & & \downarrow \bar{\pi} \times \mathrm{Id} \\ \mathrm{End}(\mathbf{K}^k) & \overset{g}{\to} & \mathrm{MP}(k) \times F \end{array}$$

is commutative and $\bar{\pi}$ is a $k$–swallowtail, so is $\pi$.

*Proof of the "if" part of* (iii) *in general.* — If $\lambda_0$ is a geometrically simple eigenvalue of $A_0$ with multiplicity $k$, we can choose a closed complement $F$ of $\mathrm{Ker}(\lambda_0 - A_0)^k$ and identify $\mathrm{Ker}(\lambda_0 - A_0)^k$ to $\mathbf{K}^k$ so that $E = \mathrm{Ker}(\lambda_0 - A_0)^k \oplus F$ identifies to $\mathbf{K}^k \times F$ and

$$A_0 = \begin{pmatrix} J_k(\lambda_0) & b_0 \\ 0 & d_0 \end{pmatrix} \quad \text{with} \quad b_0 \in \mathscr{B}(F, \mathbf{K}^k),\ d_0 \in \mathrm{End}(F),\ \lambda_0 \notin \sigma(d_0).$$

More generally, every $A \in \mathrm{End}(E)$ writes

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; a \in \mathrm{End}(\mathbf{K}^k),\; b \in \mathscr{B}(F, \mathbf{K}^k),\; c \in \mathscr{B}(\mathbf{K}^k, F),\; d \in \mathrm{End}(F).$$

Note that the graph of $h \in \mathscr{B}(\mathbf{K}^k, F)$ is invariant by $A$ if and only if

$$c + dh = h(a + bh)$$

or, equivalently, if and only if

$$\begin{array}{rcl} a & = & \alpha - bh \\ c & = & h\alpha - dh \end{array}$$

for some $\alpha \in \mathrm{End}(\mathbf{K}^k)$. The following crucial observation is a particular case of the proof of Theorem 10 hereafter:

***Lemma 1***. — *The polynomial map*

$$\begin{pmatrix} \alpha & b \\ h & d \end{pmatrix} \longmapsto \begin{pmatrix} \alpha - bh & b \\ h\alpha - dh & d \end{pmatrix}$$

*is a local diffeomorphism* $\big(\mathrm{End}(E), A_0\big) \to \big(\mathrm{End}(E), A_0\big)$.

*Proof of Lemma 1* As its derivative at $A_0$ is

$$\begin{pmatrix} \alpha & b \\ h & d \end{pmatrix} \longmapsto \begin{pmatrix} \alpha - b_0 h & b \\ hJ_k(\lambda_0) - dh & d \end{pmatrix},$$

we should show that the continuous linear map $h \mapsto hJ_k(\lambda_0) - d_0 h$ of $\mathscr{B}(\mathbf{K}^k, F)$ into itself is an isomorphism, i.e. that, for each $c \in \mathscr{B}(\mathbf{K}^k, F)$, the equation $hJ_k(\lambda_0) - d_0 h = c$ or, equivalently, $hJ_k(0) + (\lambda_0 - d_0)h = c$, which (as we have $\lambda_0 \notin \sigma(d_0)$) can be written

$$h = (\lambda_0 - d_0)^{-1}\big(c - hJ_k(0)\big), \tag{14}$$

has a unique solution. Now, this is obvious since, denoting the canonical basis of $\mathbf{K}^k$ by $(e_1, \dots, e_k)$, (14) is equivalent to the triangular system

$$he_j = \begin{cases} (\lambda_0 - d_0)^{-1} ce_j & \text{if } j = k \\ (\lambda_0 - d_0)^{-1}(ce_j - he_{j+1}) & \text{for } 1 \le j < k. \end{cases}$$

The following result concludes the proof of the "if" part of Theorem 8 (iii):

***Lemma 2.*** — *We have a commutative diagram*

$$\begin{array}{ccc}
\mathrm{Val}(\mathbf{K}^k) \times \mathscr{B}(F, \mathbf{K}^k) \times \mathscr{B}(\mathbf{K}^k, F) \times \mathrm{End}(F) & \to & \mathrm{Val}(E) \\
\downarrow \pi \times \mathrm{Id} \times \mathrm{Id} \times \mathrm{Id} & & \downarrow \pi \\
\mathrm{End}(\mathbf{K}^k) \times \mathscr{B}(F, \mathbf{K}^k) \times \mathscr{B}(\mathbf{K}^k, F) \times \mathrm{End}(F) & \to & \mathrm{End}(E)
\end{array}$$

$$\begin{array}{ccc}
\big((\lambda, \alpha), b, h, d\big) & \mapsto & \left(\lambda, \begin{pmatrix} \alpha - bh & b \\ h\alpha - dh & d \end{pmatrix}\right) \\
\downarrow & & \downarrow \\
(\alpha, b, h, d)) & \mapsto & \begin{pmatrix} \alpha - bh & b \\ h\alpha - dh & d \end{pmatrix},
\end{array}$$

*where the horizontal arrows are local diffeomorphisms at* $\big((\lambda_0, J_k(\lambda_0)), b_0, 0, d_0\big)$ *and* $\big(J_k(\lambda_0), b_0, 0, d_0\big)$ *respectively, the right vertical one is diffeomorphic to* $\Pi$ *(since* $\lambda$ *is geometrically simple) and the left one is a swallowtail by the particular case* $E = \mathbf{K}^k$ *already treated.*

*Proof of Lemma 2* The two things we do not already know are the following:

(a) the upper arrow does send $\mathrm{Val}(\mathbf{K}^k) \times \mathscr{B}(F, \mathbf{K}^k) \times \mathscr{B}(\mathbf{K}^k, F) \times \mathrm{End}(F)$ into $\mathrm{Val}(E)$

(b) it is a local diffeomorphism.

To obtain (a), just notice that $\begin{pmatrix} \alpha - bh & b \\ h\alpha - dh & d \end{pmatrix} \begin{pmatrix} x \\ hx \end{pmatrix} = \lambda \begin{pmatrix} x \\ hx \end{pmatrix}$ if and only if $\alpha x = \lambda x$.

By Lemma 1, to get (b), we should show that, for $(\lambda, \alpha, b, h, d)$ close enough to $\big(\lambda_0, J_k(\lambda_0), b_0, 0, d_0\big)$, we have $\begin{pmatrix} \alpha - bh & b \\ h\alpha - dh & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$ if and only if $\alpha x = \lambda x$ and $y = hx$. Now, setting $z := y - hx$, the first relation reads

$$\begin{cases} (\alpha - \lambda)x = bz \\ h(\alpha - \lambda)x = (\lambda - d)z \end{cases}$$

or, equivalently,

$$\begin{cases} (\alpha - \lambda)x = bz \\ (\lambda - d - hb)z = 0. \end{cases}$$

As $\lambda_0 - d_0$ is invertible, so is $\lambda - d - hb$ for $(\lambda, b, h, d)$ close enough to $(\lambda_0, b_0, 0, d_0)$, in which case our system is equivalent to $z = 0$ and $(\alpha - \lambda)x = 0$, proving Lemma 2.

*Proof of the "only if" part of* (iii). When the eigenspace associated to $\lambda$ is not a line there is (at least) a circle of invariant lines with the same eigenvalue mapping to the same operator. This cannot happen in a swallowtail, proving our result since we assume[9] $\dim E < \infty$. □

[9] Though the result might well be true in general.

**C4. Simple eigenvalues.** — Theorem 8 (i) is a version of the following well-known result:

***Corollary.*** — *For $(\lambda_0, L_0, A_0) \in \mathrm{Eig}(E)$ the following assertions are equivalent:*

*(i) There are (necessarily unique) analytic functions $\lambda(A), \mathcal{B}(A)$ defined near $A_0$ such that $\lambda(A_0) = \lambda_0, \mathcal{B}(A_0) = L_0$ and $\big(\lambda(A), \mathcal{B}(A), A\big) \in \mathrm{Eig}(E)$.*

*(ii) $\lambda_0$ is a simple eigenvalue of $A_0$.*

*Proof.* — Just observe that such functions provide a local inverse of $\Pi$. □

*Remarks.* — Actually, under the conditions of the corollary there is also an analytic function $v(A)$ (which is not unique) defined in a neighborhood of $A$ such that $v(A)$ is an eigenvector of $A$ with eigenvalue $\lambda(A)$. As we said above, this result is classical and many proofs have been given of it. For a proof that uses, as we do here, the Implicit Function Theorem in Banach spaces see for example [**3**], exercice 14 p. 268. Proofs using this method have been known for a long time, see for example [**9**].

Part (iii) of Theorem 8 is related to the usual expansion of the eigenvalue as a power series on roots of the parameters [**8**].

In the finite dimensional case, for $(\lambda, L, M) \in \mathrm{Eig}(\mathbf{K}^n)$ and $P = \chi(M)$ we always have $P(\lambda) = 0$, and $\lambda$ is simple if and only if the derivative $P'(\lambda)$ is nonzero. It can be checked that $\mathrm{Simp}(\lambda, L, M) := P'(\lambda)$ equals the Jacobian of $\Pi$ at $(\lambda, L, M)$ in the coordinates introduced in the proof of Theorem 6.

Since at the singularities with higher corank of $\Pi$ the jacobian determinant is a singular function (see Appendix A1), it follows that the regular points of the variety $\mathrm{Simp} = 0$ are precisely the points with a geometrically simple eigenvalue. This regular set can be still stratified according to the singularity type of $\Pi$, i.e. according to the order of the swallowtails.

In the case of geometrically multiple eigenvalues, we have observed that there is (at least) a circle of invariant lines with the same eigenvalue mapping to the same operator. This means that the singularity type is not finite, and is therefore very degenerate from the viewpoint of singularity theory. Nevertheless, it is a kind of blow-down map that could be described combining the blow-down singularities of the map $j$ and some simpler singularities.

The group $\mathrm{GL}(E)$ of invertible endomorphisms acts naturally by conjugation on both $\mathrm{Eig}(E)$ and $\mathrm{End}(E)$ and the mapping $\Pi$ is equivariant. Therefore $\Pi$ maps the stratification by orbits of the first space into the second one and is some kind of (partial) resolution of the latter's singularities.

In the finite dimensional case, Arnold [**1**] has given a complete description of the stratification by orbits of $M(n \times n)$ which is possible to lift to those of $\mathrm{Eig}(\mathbf{K}^n)$. The images of our swallowtail singularities can be observed in the slice around the corresponding orbit. Although no explicit statement appears it is probable that some version of our results was known to Arnold.

## D. Singularities of linear operators and invariant subspaces

**D1. Grassmannians.** — We will denote by $\mathbf{G}(E)$ the set of closed linear subspaces $S$ of $E$ with a closed complement. For each such $S$ and each pair $(V, W)$ of complementary closed subspaces of $E$ such that $E = S \oplus W$, the subspace $S$ is (in the identification of $E = V \oplus W$ to $V \times W$) the graph of a unique continuous linear map $h = h_{V,W}(S)$ of $V$ into $W$.

***Proposition.*** — *The charts $h_{V,W}$ make $\mathbf{G}(E)$ into an analytic Banach manifold, which is impure*[10] *but Hausdorff.*

*Proof.* — The intersection of the domains of two such maps $h_{V,W}$ and $h_{V_1,W_1}$ can be nonempty only if $V_1$ is isomorphic to $V$ and $W_1$ to $W$. When this is the case, each $v \in E$ writes $(x, y) \in V \times W$ in one identification and $(x_1, y_1) \in V_1 \times W_1$ in the other, and there exists a unique invertible transition matrix

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; a \in \mathscr{B}(V_1, V)\,,\; b \in \mathscr{B}(W_1, V)\,,\; c \in \mathscr{B}(V_1, W)\,,\; d \in \mathscr{B}(W_1, W)$$

such that the first expression of $v$ is obtained from the second by the formula

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

If $S \in \mathbf{G}(E)$ lies in the domain of both $h_{V,W}$ and $h_{V_1,W_1}$, then $h := h_{V,W}(S)$ and $h_1 := h_{V_1,W_1}(S)$ satisfy

$$c + dh_1 = h\,(a + bh_1)\,.$$

Now, $a + bh_1$ is an isomorphism of $V_1$ onto $V$, as it is obtained by composing the isomorphism $x_1 \mapsto (x_1, h_1 x_1)$ of $V_1$ onto $S$ and the isomorphism $(x, y) \mapsto x$ of $S$ onto $V$ in the identification of $E$ to $V \times W$. In other words, the domain of the transition map $h_{V,W} \circ h_{V_1,W_1}^{-1}$ is the open subset of $\mathscr{B}(V_1, W_1)$ consisting of those $h_1$ such that $a + bh_1$ is an isomorphism, and

$$h_{V,W} \circ h_{V_1,W_1}^{-1}(h_1) = (c + dh_1)\,(a + bh_1)^{-1}\,.$$

To see that $\mathbf{G}(E)$ is Hausdorff (which we do not care much about), one can proceeed as follows:

- If $E$ is a Hilbert space, each $S \in \mathbf{G}(E)$ can be identified to the orthogonal projector onto $S$, yielding a natural embedding of $\mathbf{G}(E)$ into $\operatorname{End}(E)$. The image is the smooth *real*[11] algebraic subset of $\operatorname{End}(E)$ defined by the equations $P^2 =$

[10] Meaning the following: each $h_{V,W}$ applies only to subspaces $S$ isomorphic to $V$ and such that $E/S$ is isomorphic to $W$. It follows for example that two subspaces $S$ which do not have the same dimension or the same codimension lie in different connected components—which have various dimensions when $E$ is finite dimensional. In particular, the projective space $\mathbf{P}(E)$ is a connected component of $\mathbf{G}(E)$.
[11] If $\mathbf{K} = \mathbf{C}$, the holomorphic manifold $\mathbf{G}(E)$ is compact for $\dim E < \infty$ and therefore cannot be embedded as a holomorphic submanifold of the complex linear space $\operatorname{End}(E)$.

$P$ and $(1-P^*)P + P^*(1-P) = 0$, where $P^*$ denotes the adjoint of $P$. See [**3**], exercice 31 p. 271.

– Otherwise, for each $p \in E$, it is quite easy to prove (using the charts $h_{V,W}$) that the function $\mathrm{dist}_p : \mathbf{G}(E) \to \mathbf{R}$ defined by

$$\mathrm{dist}_p(S) = \mathrm{dist}(p, S) := \inf_{q \in S} |q - p|$$

is continuous. Given distinct elements $S, S_1$ of $\mathbf{G}(E)$, exchanging them if necessary, there exists $p \in S_1 \setminus S$ and we have $\mathrm{dist}(p, S_1) = 0 < D := \mathrm{dist}(p, S)$, hence two disjoint open subsets $\mathrm{dist}_p^{-1}(-\infty, D/2) \supset S_1$ and $\mathrm{dist}_p^{-1}(D/2, +\infty) \supset S$. □

**D2. The manifold of invariant subspaces.** — The manifold of invariant subspaces of $E$ is the space

$$\mathrm{Inv}(E) := \{(S, A) \in \mathbf{G}(E) \times \mathrm{End}(E) \ : \ A(S) \subseteq S\}.$$

That is, the space of pairs consisting of a linear operator and an invariant subspace with a closed complement. Compare with the definition of $\mathrm{Eig}(E)$. This manifold resembles more the manifold $\mathrm{Eig}(E)$ (which it contains), than the more complicated Grassmannian:

***Theorem 9.*** — *The subset* $\mathrm{Inv}(E)$ *is an analytic submanifold modelled on* $\mathrm{End}(E)$.

*Proof.* — For each chart $h_{V,W}$, every $S \in \mathbf{G}(E)$ such that $E = S \oplus W$ identifies to the linear map $h = h_{V,W}(S)$ of which it is the graph in the identification of $E = V \oplus W$ to $V \times W$. In this identification, every $A \in \mathrm{End}(E)$ reads as usual

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a \in \mathrm{End}(V), b \in \mathscr{B}(W, V), c \in \mathscr{B}(V, W), d \in \mathrm{End}(W)$$

and we have $(S, A) \in \mathrm{Inv}(E)$ if and only if

$$c + dh = h(a + bh) \tag{15}$$

or, equivalently, if and only if

$$\begin{array}{rcl} a & = & \alpha - bh \\ c & = & h\alpha - dh \end{array}$$

for some $\alpha \in \mathrm{End}(\mathbf{K}^k)$. In other words:

– The image of the restriction of $h_{V,W} \times \mathrm{Id}_{\mathrm{End}(E)}$ to $\mathrm{Inv}(E)$ is the smooth algebraic submanifold of $\mathscr{B}(V, W) \times \mathrm{End}(E)$ defined by (15), which is the graph $c = h(a + bh) - dh$.

– This graph admits the *global* parametrization

$$\Phi_{V,W} : \begin{pmatrix} \alpha & b \\ h & d \end{pmatrix} \longmapsto \left(h, \begin{pmatrix} \alpha - bh & b \\ h\alpha - dh & d \end{pmatrix}\right) \tag{16}$$

by $\mathrm{End}(E)$. □

***Corollary***. — *For all $(S_0, A_0) \in \mathrm{Inv}(E)$, there is an analytic function $A(S)$ defined in a neighborhood of $(S_0)$ such that $A(S_0) = A_0$ and $A(S)$ is an operator with invariant subspace $S$.*

*Proof*. — Given any complementary subspace $W$ of $V := S_0$, with the notation of the above proof, we have

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ 0 & d_0 \end{pmatrix}$$

and we can take $A\big(h^{-1}_{V,W}(h)\big) := \varphi_{V,W} \begin{pmatrix} a_0 & b_0 \\ h & d_0 \end{pmatrix}$, where $\varphi_{V,W}$ is the second component of the parametrization $\Phi_{V,W}$ defined by (16). $\square$

Again, a more interesting question is whether $S$ is an analytic function of the operator $A$. Or, in our terms, whether the map $\Pi$ of $\mathrm{Inv}(E)$ into $\mathrm{End}(E)$ which forgets the first component is a local diffeomorphism.

**D3. Simple invariant subspaces.** — For $(S, A) \in \mathrm{Inv}(E)$, let $a$ denote the restriction of $A$ to $S$ and $\dot{A}$ the induced endomorphism of the quotient $E/S$.

***Proposition and definition***. — *The following three conditions are equivalent:*

(a) *We have that $\sigma(a) \cap \sigma(\dot{A}) = \varnothing$.*
(b) *The mapping $h \mapsto ha - \dot{A}h$ is an automorphism of $\mathscr{B}(S, E/S)$.*
(c) *The mapping $h \mapsto h\dot{A} - ah$ is an automorphism of $\mathscr{B}(E/S, S)$.*

*When they are satisfied, we call $S$ a* simple *invariant subspace of $A$ (compare with the definition of a simple eigenvalue).*

Equivalence between (a), (b), (c) follows from Proposition B.6 in Appendix B.

***Theorem 10***. — *The restriction $\Pi : \mathrm{Inv}(E) \to \mathrm{End}(E)$ of the canonical projection $\mathbf{G}(E) \times \mathrm{End}(E) \to \mathrm{End}(E)$ is a local diffeomorphism in the neighborhood of $(S_0, A_0) \in \mathrm{Inv}(E)$ if, and only if, $S_0$ is a simple invariant subspace of $A_0$.*

*Proof*. — Given any complementary subspace $W$ of $V := S_0$, we can read everything in the chart $(S, A) \mapsto \Phi^{-1}_{V,W}\big(h_{V,W}(S), A\big)$. Then, $\Pi$ is the map

$$\begin{pmatrix} \alpha & b \\ h & d \end{pmatrix} \longmapsto \begin{pmatrix} \alpha - bh & b \\ h\alpha - dh & d \end{pmatrix}$$

of $\mathrm{End}(E)$ into itself and $(S_0, A_0)$ equals

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ 0 & d_0 \end{pmatrix}$$

It follows that

$$d\Pi(S_0, A_0) : \begin{pmatrix} \alpha & b \\ h & d \end{pmatrix} \longmapsto \begin{pmatrix} \alpha - b_0 h & b \\ ha_0 - d_0 h & d \end{pmatrix}$$

is an isomorphism if, and only if, the mapping $h \to ha_0 - d_0 h$ is an automorphism of $\mathscr{B}(V, W)$. As $d_0$ identifies to $\dot{A}_0$, the theorem follows from the Inverse Function Theorem in Banach spaces and the characterization (b) of simple invariant subspaces. □

***Theorem 11***. — *Given $(S_0, A_0) \in \mathrm{Inv}(E)$, if $S_0$ is a simple invariant subspace of $A_0$, then:*

(i) *The subspace $S_0$ admits a unique $A_0$–invariant closed complement $S_1$, which is a simple invariant subspace of $A_0$.*

(ii) *Such a pair of complementary invariant subspaces exists for all $A \in \mathrm{End}(E)$ close enough to $A_0$. More precisely, there exists a unique analytic map germ $(V_0, V_1) : \big(\mathrm{End}(E), A_0\big) \to \big(\mathbf{G}(E)^2, (S_0, S_1)\big)$ such that $V_0(A)$ and $V_1(A)$ are complementary $A$–invariant subspaces.*

*Proof*. — Assertion (ii) clearly follows from (i) and Theorem 10, as $V_0$ and $V_1$ are the first components of the maps obtained by inverting the local diffeomorphisms $\big(\mathrm{Inv}(E), (S_0, A_0)\big) \to \mathrm{End}(E)$ and $\big(\mathrm{Inv}(E), (S_1, A_0)\big) \to \mathrm{End}(E)$ induced by $\Pi$.

To prove (i), denote by $W$ any closed complementary subspace of $S_0$. In the identification of $E = S_0 \oplus W$ to $S_0 \times W$, we can as usual write

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ 0 & d_0 \end{pmatrix}$$

and notice that $S_1$, if it exists, must be the "graph" $\{(x, y) \in S_0 \times W \,:\, x = hy\}$ of a map $h \in \mathscr{B}(W, S_0)$, whose invariance is expressed by the equation

$$a_0 h + b_0 = h d_0.$$

Now, as $d_0$ identifies to $\dot{A}_0$, the characterization (c) of simple invariant subspaces implies that this equation has a unique solution $h$, namely the inverse image of $b_0$ by the automorphism $h \mapsto hd_0 - a_0 h$ of $\mathscr{B}(W, S_0)$. □

**D4. Existence of simple invariant subsets.** — Under the hypothesis of Theorem 11, in the identification of $E = S_0 \oplus S_1$ to $S_0 \times S_1$, the operator $A_0$ writes

$$A_0 = \begin{pmatrix} a_0 & 0 \\ 0 & a_1 \end{pmatrix}$$

and therefore $\sigma(A_0)$ is the disjoint union of $\sigma(a_0)$ and $\sigma(a_1) = \sigma(\dot{A}_0)$. In fact, each decomposition of $\sigma(A)$ as the disjoint union of two compact subsets yields a unique decomposition of $E$ as the direct sum of two closed invariant subspaces:

***Theorem 12.*** — *Assume that* $\mathbf{K} = \mathbf{C}$. *If the spectrum of* $A_0 \in \mathrm{End}(E)$ *is the union of two disjoint nonempty compact subsets* $\sigma_0$ *and* $\sigma_1$, *then:*

(i) *For* $j = 0, 1$, *there exists a unique invariant subspace* $S_j$ *of* $A_0$ *such that the spectra of the maps* $a_j \in \mathrm{End}(S_j)$ *and* $\dot{A}_j \in \mathrm{End}(E/S_j)$ *induced by* $A_0$ *are* $\sigma_j$ *and* $\sigma_{j\pm1}$ *respectively (in particular,* $S_j$ *is simple).*

(ii) *The* $A_0$*–invariant subspaces* $S_0$ *and* $S_1$ *are complementary. Thus, in the identification of* $E = S_0 \oplus S_1$ *to* $S_0 \times S_1$,

$$A_0 = \begin{pmatrix} a_0 & 0 \\ 0 & a_1 \end{pmatrix}.$$

(iii) *Therefore, by Theorem 11, there exist uniquely determined analytic germ* $V_j : \big(\mathrm{End}(E), A_0\big) \to \big(\mathbf{G}(E), S_j\big)$, $j = 0, 1$, *such that* $V_0(A)$ *and* $V_1(A)$ *are complementary* $A$*–invariant subspaces for each* $A$ *in their domains.*

**Definition** Under the hypotheses of Theorem 12, we call $S_j$ the *invariant subspace of* $A_0$ *associated to* $\sigma_j$.

*Proof of Theorem 12* By Theorem 11, the subspace $S_j$ exists and is unique if and only if the pair $(S_0, S_1)$ exists and is unique. Therefore, our problem is the following:

– find a projector[12] $P \in \mathrm{End}(E)$ such that $S_0 := \mathrm{Im}\, P$ and $S_1 := \mathrm{Ker}\, P$ have the required properties
– prove that $P$ is unique.

***Lemma.*** — *Given a bounded open subset* $U$ *of* $\mathbf{C}$ *with smooth boundary, containing* $\sigma_0$ *and such that* $\sigma_1$ *lies in its exterior, we have the following:*

(a) *The map* $P \in \mathrm{End}(E)$ *defined by*

$$P := \frac{1}{2\pi i} \int_{\partial U} (z - A_0)^{-1} dz$$

*is a projector.*

(b) *As* $P$ *commutes with* $A_0$, *the complementary subspaces* $S_0 := \mathrm{Im}\, P$ *and* $S_1 := \mathrm{Ker}\, P$ *are invariant by* $A_0$.

(c) *Moreover, the maps* $a_0 \in \mathrm{End}(S_0)$ *and* $a_1 \in \mathrm{End}(S_1)$ *induced by* $A_0$ *satisfy* $\sigma(a_0) = \sigma_0$ *and* $\sigma(a_1) = \sigma_1$.

*Proof of the lemma.* — We can enlarge $U$ into a bounded open subset $U_1$ of $\mathbf{C}$ containing $\partial U$ and $\sigma_0$ and such that $\sigma_1$ lies in its exterior. Then[13] $P = f(A_0)$, where the holomorphic function $f : \mathbf{C} \setminus \partial U_1 \to \mathbf{C}$ is given by

$$f(z) := \begin{cases} 1 & \text{for } z \in U_1 \\ 0 & \text{otherwise.} \end{cases}$$

(12) Endomorphism $P$ such that $P^2 = P$.
(13) See **[10]**, chapter 10 from 10.21 to 10.29, for an account of the beautiful theory sometimes called *holomorphic functional calculus*.

Since $f(z)^2 = f(z)$ for all $z \in \mathbf{C} \setminus \partial U_1$, we have $f(A_0)^2 = f(A_0)$, hence (a).

As (b) is obvious, let us prove (c). Given $\lambda \in \mathbf{C} \setminus \sigma(A_0)$ observe that $(A_0 - \lambda)P = P(A_0 - \lambda)$ equals $g(A_0)$, where the holomorphic function $g : \mathbf{C} \setminus \partial U_1 \to \mathbf{C}$ is given by $g(z) = (z - \lambda) f(z)$. It follows that

(A) the spectrum of $(A_0 - \lambda)P$ is $g\big(\sigma(A_0)\big) = (\sigma_0 - \lambda) \cup \{0\}$

(B) similarly, the spectrum of $(A_0 - \lambda)(1 - P)$ is $(\sigma_1 - \lambda) \cup \{0\}$.

Now, in the identification $v \mapsto (Pv, v - Pv)$ of $E$ to $S_0 \times S_1$, we have

$$(A_0 - \lambda)P = \begin{pmatrix} a_0 - \lambda & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_0 - \lambda = \begin{pmatrix} a_0 - \lambda & 0 \\ 0 & a_1 - \lambda \end{pmatrix},$$

hence, by (A)–(B),

$$\begin{aligned} (\sigma(a_0) - \lambda) \cup (\sigma(a_1) - \lambda) &= \sigma(A_0 - \lambda) = (\sigma_0 - \lambda) \cup (\sigma_1 - \lambda) \\ (\sigma(a_0) - \lambda) \cup \{0\} &= \sigma\big((A_0 - \lambda)P\big) = (\sigma_0 - \lambda) \cup \{0\} \\ (\sigma(a_1) - \lambda) \cup \{0\} &= \sigma\big((A_0 - \lambda)(1 - P)\big) = (\sigma_1 - \lambda) \cup \{0\}, \end{aligned}$$

implying (c) since $\lambda$ belongs neither to $\sigma_0$, nor to $\sigma_1$. □

*Proof that $P$ is unique.* — Let $Q \in \mathrm{End}(E)$ be a projector with the required properties. As $\mathrm{Im}\, Q$ and $\mathrm{Ker}\, Q = \mathrm{Im}(1 - Q)$ are complementary subspaces invariant by $A_0$, we have, for all $v \in E$,

$$QA_0v + (1 - Q)A_0v = A_0v = A_0\big(Qv + (1 - Q)v\big) = A_0Qv + A_0(1 - Q)v\,,$$

hence $A_0Qv = QA_0v$ and therefore

$$A_0Q = QA_0.$$

Identifying $E = S_0 \oplus S_1$ to $S_0 \times S_1$, we can write

$$A_0 = \begin{pmatrix} a_0 & 0 \\ 0 & a_1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} q_0 & b \\ c & q_1 \end{pmatrix}.$$

The commutation relation implies that $a_0 b = b a_1$ and $c a_0 = a_1 c$, hence $b = 0$ and $c = 0$ by the characterizations (b)–(c) of a simple invariant subspace. It follows that

$$Q = \begin{pmatrix} q_0 & 0 \\ 0 & q_1 \end{pmatrix},$$

where $q_j \in \mathrm{End}(S_j)$ is a projector which commutes with $a_j$ for $j = 0, 1$. For every $\lambda \in \mathbf{C}$, we have

$$\sigma\big((A_0 - \lambda)Q\big) = (\sigma_0 - \lambda) \cup \{0\}$$

since the spectrum of $A_0 - \lambda$ restricted to $\mathrm{Im}\, Q$ is $\sigma_0 - \lambda$ and $Q$ is not the identity (otherwise $\sigma_1$ would be empty). Now, we also have

$$\sigma\big((A_0 - \lambda)Q\big) = \sigma\big((a_0 - \lambda)q_0\big) \cup \sigma\big((a_1 - \lambda)q_1\big)$$

and[14], as $a_j - \lambda$ commutes with $q_j$,

$$\sigma\big((a_j - \lambda)q_j\big) \subset \sigma(a_j - \lambda)\sigma(q_j) = (\sigma_j - \lambda)\sigma(q_j) \subset (\sigma_j - \lambda) \cup \{0\}.$$

It follows that we must have $\sigma\big((a_1 - \lambda)q_1\big) = \{0\}$ for every $\lambda$ and therefore $\sigma(q_1) = 0$, hence $q_1 = 0$ since $q_1$ is a projector, yielding

$$\operatorname{Im} Q \subset S_0.$$

Replacing $Q$ by $1 - Q$ in this argument, we obtain the inclusion

$$\operatorname{Im}(1 - Q) \subset S_1.$$

As $E = S_0 \oplus S_1 = \operatorname{Im} Q \oplus \operatorname{Im}(1 - Q)$, it follows that $\operatorname{Im} Q = S_0$ and $\operatorname{Im}(1 - Q) = S_1$, hence $Q = P$. □

*Remarks*. — Instead of deducing part (iii) of the theorem from the inverse function theorem, one can use directly the observation that, for $A$ close enough to $A_0$, the formula

$$P(A) := \frac{1}{2\pi i} \int_{\partial U} (z - A)^{-1} dz$$

defines a projector. As it depends analytically on $A$, so do its image $V_0(A)$ and its kernel $V_1(A)$, which are invariant by $A$ since $P(A)A = AP(A)$. This type of proof (and the result) are well-known, although it must be said that in the standard reference on the subject ([**8**], Chapter IV, Section 4, Theorem 3.16) the result is obscured by unnecessary "additional" hypotheses.

Of course, Theorem 12 enables us to associate to every decomposition of $\sigma(A_0)$ as the union of finitely many mutually disjoint nonempty compact subsets $\sigma_1, \dots, \sigma_p$ the decomposition $E = S_1 \oplus \cdots \oplus S_p$, where $S_j$ denotes the $A_0$–invariant subspace associated to $\sigma_j$.

In finite dimensions, we can consider the maximal decomposition of $\sigma(A_0)$ defined by $\sigma_j = \{\lambda_j\}$, where the $\lambda_j$'s are the eigenvalues of $A_0$. The subspace $S_j$ is then called the *characteristic subspace* of $A_0$ associated to $\lambda_j$. In that case, as mentioned in the introduction, Theorem 12 (iii) (or, rather, Theorem 10) tells us something which deserves being better known: every $A$ close enough to $A_0$ admits an invariant subspace $V_j(A)$ of the same dimension as the characteristic subspace $S_j$, unique in a suitable neighborhood of $S_j$ and depending analytically on $A$ even though the eigenvalues do not and the eigenspaces may explode—for generic $A$, the subspace $V_j(A)$ is the direct sum of one-dimensional eigenspaces corresponding to mutually distinct eigenvalues of $A$ close to $\lambda_j$.

***Theorem 13***. — *Theorem 12 holds if* $\mathbf{K} = \mathbf{R}$, *provided* $\sigma_0$ *and (therefore)* $\sigma_1$ *are invariant by complex conjugation.*

[14] See [**10**], Theorem 11.23.

*Proof.* — On the complex Banach space $E_{\mathbf{C}} := E \oplus iE$ obtained from $E$ by complexification, there is a conjugation

$$v + iw \mapsto \overline{v + iw} := v - iw, \; v, w \in E.$$

Denoting again by $A_0$ the complexified endomorphism $v + iw \mapsto A_0 v + iA_0 w$, the identity

$$A_0 \bar{z} = \overline{A_0 z}$$

implies that the complementary $A_0$–invariant subspaces $S_0, S_1 \subset E_{\mathbf{C}}$ obtained from Theorem 12 satisfy

$$\overline{S}_j = S_j. \tag{17}$$

Indeed, $\overline{S}_j$ is an $A_0$–invariant subspace of which $A_0$ induces an endomorphism with spectrum $\overline{\sigma}_j = \sigma_j$, hence (17) since $S_j$ is unique.

It follows that $S_j$ is the complexification of the real $A_0$–invariant subspace $S_j \cap E$, that $E = (S_0 \cap E) \oplus (S_1 \cap E)$ and, of course, that the spectrum of the endomorphism of $S_j \cap E$ induced by $A_0$ is $\sigma_j$. □

***Theorem 14.*** — *If $\mathbf{K} = \mathbf{R}$, then, given $A_0 \in \operatorname{End}(E)$:*

(i) *If $\sigma(A_0) \cap \mathbf{R} = \varnothing$, there exist a* complex *Banach space $F$ and an analytic local map $I : \big(\operatorname{End}(E), A_0\big) \to \operatorname{Iso}(E, F)$ (space of continuous isomorphisms of $E$ onto $F$) such that every $I(A)_* A := I(A) \circ A \circ I(A)^{-1}$ is a $\mathbf{C}$–linear operator whose spectrum is the intersection $\sigma^+(A)$ of $\sigma(A)$ with the upper half-plane $\Im z > 0$.*

(ii) *More generally, if $\sigma(A_0) \cap \mathbf{C} \setminus \mathbf{R}$ is compact[15], there exist Banach spaces $S, F$ with $F$* complex *and an analytic $I : \big(\operatorname{End}(E), A_0\big) \to \operatorname{Iso}(E, S \times F)$ such that every $I(A)_* A$ is block diagonal $a(A) \times d(A)$, the endomorphism $d(A)$ of $F$ is $\mathbf{C}$–linear, $\sigma\big(a(A)\big) = \sigma(A) \cap \mathbf{R}$ and $\sigma\big(d(A)\big) = \sigma^+(A)$.*

*Proof.* — (i) By Theorem 12, applied with $\sigma_0 := \sigma^+(A_0)$ to the complexified map $A_{0\mathbf{C}}$ of $A_0$, there is a unique analytic local map $P : \big(\operatorname{End}(E), A_0\big) \to \operatorname{End}(E_{\mathbf{C}})$ such that

– each $P(A)$ is a projector whose kernel and image are invariant by $A_{\mathbf{C}}$
– the map $A_{\mathbf{C}}$ induces endomorphisms of $\ker P(A)$ and $\operatorname{Im} P(A)$ whose spectra are $\overline{\sigma^+(A)}$ and $\sigma^+(A)$ respectively.

The projection $v \mapsto \Re v$ of $E_{\mathbf{C}}$ onto $E$, restricted to $\operatorname{Im} P(A)$, is an isomorphism and, denoting the inverse map by $r(A)$, the map $r(A)_* A$ clearly is the $\mathbf{C}$–linear endomorphism of $\operatorname{Im} P(A)$ induced by $A_{\mathbf{C}}$. The image of $r(A)$ depends on $A$ but we can make it constant by composing $r(A)$ with $P(A_0)$, as the latter induces an isomorphism of $\operatorname{Im} P(A)$ onto $\operatorname{Im} P(A_0)$ for $A$ close enough to $A_0$. This proves (i) with $F := \operatorname{Im} P(A_0)$ and $P(A) := P(A_0) \circ r(A)$.

(ii) By Theorem 13, applied to $A_0$ with $\sigma_0 := \sigma(A_0) \cap \mathbf{R}$, the same argument as in the proof of (i) shows that there exist real Banach spaces $S, V$ and an analytic $J : \big(\operatorname{End}(E), A_0\big) \to \operatorname{Iso}(E, S \times V)$ such that every $J(A)_* A$ has the form $a(A) \times D(A)$

[15] Which is automatically the case for $\dim E < \infty$.

with $\sigma\big(a(A)\big) = \sigma(A) \cap \mathbf{R}$ and $\sigma\big(D(A)\big) = \sigma(A) \cap \mathbf{C} \setminus \mathbf{R}$. Applying (i) to $D(A)$, we get what we want. □

*Remark.* — This easy complexification result is extremely useful in the theory of normal forms of dynamical systems.

**D5. Singularities of invariant subspaces.** — In infinite dimensions, the map $\Pi : \mathrm{Inv}(E) \to \mathrm{End}(E)$ can have a *very* wild singularity at $(S_0, A_0)$ when $S_0$ is not a simple invariant subspace of $A_0$. We shall only consider the simplest cases, which do occur naturally, at least for compact or Fredholm operators and in particular in finite dimensions.

***Theorem 15.*** — *Assume that $(S_0, A_0) \in \mathrm{Inv}(E)$ satisfies $\sigma(a_0) \cap \sigma(\dot{A}_0) = \{\lambda_0\}$, where $\lambda_0$ is a geometrically simple eigenvalue of $A_0$ of multiplicity $k$* and *a simple eigenvalue of $a_0$ or $\dot{A}_0$. Then $\Pi$ is a $k$–swallowtail at $(S_0, A_0)$.*

*Proof.* — Our hypothesis implies that $\sigma(a_0) = \tau_0 \cup \{\lambda_0\}$, $\sigma(\dot{A}_0) = \tau_1 \cup \{\lambda_0\}$ and $\sigma(A_0) = \tau_0 \cup \{\lambda_0\} \cup \tau_1$, where $\tau_0, \tau_1$ are disjoint compact subsets, which may be empty and do not contain $\lambda_0$.

First assume that $\lambda_0$ is a simple eigenvalue of $a_0$. If $\tau_0 = \varnothing$, our theorem is the "if" part of Theorem 8 (iii). Otherwise, we proceed as in the proof of Theorem 8 (iii) to show that the contribution of $\tau_0$ (and $\tau_1$) to the singularity is trivial.

If $\lambda_0$ is a simple eigenvalue of $\dot{A}_0$, then the contribution of $\tau_0$ and $\tau_1$ to the singularity is trivial, which reduces the problem to the case where $E = \mathbf{K}^k$. Then, the hypotheses of Theorem 8 (iii) are satisfied by the transposed map $A_0^* \in \mathrm{End}(E^*)$ and the line $S_0^\perp$, hence our theorem since $(A, S) \mapsto (A^*, S^\perp)$ is a diffeomorphism of the open subset of $\mathrm{Inv}(\mathbf{K}^k)$ associated to hyperplanes $S$ onto $\mathrm{Eig}(\mathbf{K}^{k*})$, fibered over the isomorphism $A \mapsto A^*$. □

*Remark.* — For $k > 1$, the hypotheses of Theorem 15 imply that $S_0$ *admits no $A_0$–invariant complement.*

***Theorem 16.*** — *Assume that $\mathbf{K} = \mathbf{R}$ and that $(S_0, A_0) \in \mathrm{Inv}(E)$ satisfies $\sigma(a_0) \cap \sigma(\dot{A}_0) = \{\lambda_0, \bar{\lambda}_0\}$, where $\lambda_0 \in \mathbf{C} \setminus \mathbf{R}$ is a geometrically simple eigenvalue of $A_0$ of multiplicity $k$* and *a simple eigenvalue of $a_0$ or $\dot{A}_0$. Then $\Pi$ is a* complex *$k$–swallowtail at $(S_0, A_0)$.*

*Proof.* — As in the proof of Theorem 15, we have that $\sigma(a_0) = \tau_0 \cup \{\lambda_0, \bar{\lambda}_0\}$, $\sigma(\dot{A}_0) = \tau_1 \cup \{\lambda_0, \bar{\lambda}_0\}$ and $\sigma(A_0) = \tau_0 \cup \{\lambda_0, \bar{\lambda}_0\} \cup \tau_1$, where $\tau_0, \tau_1$ are disjoint compact subsets, invariant by complex conjugation, which may be empty and do not contain $\lambda_0$. The contributions of $\tau_0$ and $\tau_1$ to the singularity are trivial, reducing us to the case where $E = \mathbf{R}^{2k}$ and $\lambda_0, \bar{\lambda}_0$ are the only (geometrically simple) eigenvalues of $A_0$. Therefore, our problem is to prove a real version of Theorem 8 (iii) in its simplest case, assuming that $S_0$ has dimension 2 (if it has codimension 2, the same duality argument as in the proof of Theorem 15 applies).

By Theorem 14, there exists an isomorphism $I(A)$ of $\mathbf{R}^{2k}$ onto $\mathbf{C}^k$, depending analytically on $A$ such that $I(A)_*A := I(A)\circ A\circ I(A)^{-1}$ is a *complex* endomorphism of $\mathbf{C}^k$, that $\lambda_0$ is the sole, geometrically simple, eigenvalue of $P(A_0)_*A_0$ and that $P(A_0)S_0$ is the corresponding *complex* one-dimensional eigenspace. Applying Theorem 8 (iii) to $I(A)_*A$, we do get what we want for $A$. □

## APPENDICES

### Appendix A
### Some Useful Singularities

We will recall in this appendix some simple properties of singularities, especially of the swallowtail type, and we describe some examples of them that are used in the main text.

Two function germs $f, g : E_1, 0 \to E_2, 0$ are called *diffeomorphic* [16] if there are local diffeomorphisms $\varphi$ of $E_1, 0$ and $\psi$ of $E_2, 0$ such that $g \circ \varphi = \psi \circ f$.

A singular function germ $f : \mathbf{K}^n, 0 \to \mathbf{K}^n, 0$ is called *good* (in the sense of Whitney) if its jacobian determinant $Jf : \mathbf{K}^n, 0 \to \mathbf{K}, 0$ is regular at 0.

It is clear that a good map is of corank 1, because if two rows of the jacobian matrix of $f$ vanish at 0, then the jacobian determinant is at least of order 2 and so is singular at the origin. In fact, the good function germs are exactly those such that $j^1 f$ is transversal to the stratum $\Sigma^1$ of mappings of corank 1.

*Swallowtails.* — The *standard $k$–swallowtail* is the mapping

$$\begin{aligned} SW_k : \mathbf{K}^{k-1} &\to \mathbf{K}^{k-1} \\ SW_k(a_1, \ldots, a_{k-2}, u) &:= (a_1, \ldots, a_{k-2}, u^k + a_{k-2}u^{k-2} + \cdots + a_1 u). \end{aligned}$$

In other words, it is the universal unfolding of the map $u \to u^k$.

For us a *$k$–swallowtail* will be any map germ between two Banach spaces which is diffeomorphic to the germ at 0 of a suspension of the standard one, that is, a map of the form

$$SW_k \times \mathrm{Id} : \mathbf{K}^{k-1} \times E \to \mathbf{K}^{k-1} \times E$$

for some Banach space $E$. In other words, it is a versal unfolding of the map $u \to u^k$.

A $k$–swallowtail has the following properties:

(i) It is a stable map germ of corank 1.

(ii) It is a Morin singularity of type $\Sigma^{1_k}$.

(iii) It is a *good* function in the sense of Whitney.

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[16] Usually called *left-right equivalent.*

*Example 1.* — The evaluation map

$$\begin{array}{rcl} \mathrm{ev} : \mathrm{MP}(k) \times \mathbf{K} & \to & \mathrm{MP}(k) \times \mathbf{K} \\ (P, a) & \mapsto & \big(P, P(a)\big) \end{array}$$

is a $k$–swallowtail. Indeed, if we restrict (in the souce and target) to the subspace of polynomials without terms of degree $k-1$ and $0$ we get the standard $k-$swallowtail. For the whole space of monic polynomials we need only put aside those coefficients by the standard translation procedures: Let

$$\begin{array}{rcl} P(x) & = & x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0 \\ Q(x) & := & P(x - a_{k-1}/k) \\ Q_0(x) & := & Q(x) - Q(0). \end{array}$$

Then, the map ev factors as:

$$\begin{array}{ccc} (P, a) & \to & (Q_0, a + a_{k-1}/k, a_{k-1}, Q(0)) \\ \downarrow & & \downarrow \\ (P, P(a)) & \leftarrow & (Q_0, Q_0(a + a_{k-1}/k), a_{k-1}, Q(0)) \end{array}$$

and the second vertical arrow is of the form $\mathrm{SW}_k \times \mathrm{Id}$ while the two horizontal ones are diffeomorphisms.

*Example 1a.* — The evaluation map restricted to polynomials $P$ with $a_{k-1} = 0$ is also a $k$–swallowtail.

This is because in the above factorization of ev one can eliminate the third component from the right-hand spaces.

*Example 1b.* — The evaluation map restricted to polynomials $P$ with $a_0 = 0$ is also a $k$–swallowtail.

This is because in the above factorization of ev one can eliminate the fourth component $Q(0)$ from the right-hand spaces since it is determined from the other ones by the relation $0 = Q_0(a + a_{k-1}/k) + Q(0)$ (the last expression equals $P(0) = a_0$).

*Example 1c.* — Let $\mathrm{Root}(n) = \{(a, P) \in \mathbf{K} \times \mathrm{MP}(n)$ such that $P(a) = 0\}$, which is diffeomorphic to $\mathbf{K}^n$ since the defining equation can be solved for $a_0$. The map $\mathrm{Root}(n) \to \mathrm{MP}(n)$ which sends $(a, P)$ to $P$ is also a $k$–swallowtail.

This is because, in terms of the natural parametrizations of $\mathrm{Root}(n)$ and $\mathrm{MP}(n)$ the above map is given by

$$(a, Q) \mapsto \big(a, Q - Q(a)\big) \mapsto Q - Q(a) \cong \big(Q, -Q(a)\big),$$

where $Q(0) = 0$, which is diffeomorphic to the evaluation map for polynomials with $a_0 = 0$ (Example 1b).

*Example 2.* — The map

$$(a_1, \ldots, a_{k-1}, a) \to (aa_1, a_1 + aa_2, a_2 + aa_3, \ldots, a_{k-2} + aa_{k-1}, a_{k-1} + a)$$

is a $k$–swallowtail.

To see this, take as new coordinates $u = -a$ and the last $k-1$ components of the map:

$$b_i := a_i - ua_{i+1})$$

for $i = 1, \dots k-1$, where we take $a_k = 1$ (so $b_{k-1} := a_{k-1} - u)$). This yields inductively

$$a_{k-i} = u^i + \sum_{j=1}^{i} b_{k-j} u^{i-j}$$

and therefore

$$aa_1 = -u\Big(u^{k-1} + \sum_{j=1}^{k-1} b_{k-j} u^{k-1-j}\Big) = -\Big(u^k + \sum_{j=1}^{k-1} b_{k-j} u^{k-j}\Big)$$

so the mapping is equivalent to

$$(u, b_1, \dots, b_{k-1}) \to (u^k + b_{k-1}u^{k-1} + \cdots + b_1 u, b_1, \dots, b_{k-1})$$

which is essentially the evaluation map for polynomials with null constant term. As we have seen in example 1b above, it is a $k$–swallowtail.

If we put $a_{k-1} = -a$ on example 2 we get essentially the map:

$$(a_1, \dots, a_{k-2}, a) \to (aa_1, a_1 + aa_2, a_2 + aa_3, \dots, a_{k-2} - a^2).$$

As this corresponds to making $b_{k-1} = 0$ in the new coordinates, it is diffeomorphic to the standard $k$–swallowtail. This shows that every swallowtail can be given by polynomial functions of degree 2.

*Complex Swallowtails.* — The complex swallowtail

$$\mathrm{SW}_k : \mathbf{C}^{k-1} \to \mathbf{C}^{k-1}$$

can be considered as a real mapping

$$\mathrm{SW}_k : \mathbf{R}^{2k-2} \to \mathbf{R}^{2k-2}$$

which we will call the *standard complex swallowtail* and by a *complex swallowtail* we will mean any map diffeomorphic to one of its *real* suspensions. For example, the standard 2-swallowtail is the real map $(x, y) \mapsto (x^2 - y^2, 2xy)$.

The complex swallowtails are not stable as real maps. In fact they are very degenerate since their singular set is of codimension 2 and can explode into a subset of codimension 1 (for $k > 2$, in uncountably many ways).

*Swallowtail deformations.* — A *$k$–swallowtail deformation* is any map germ between two Banach spaces which is diffeomorphic to the germ at 0 of a map

$$G : E \times E' \to E$$

such that $G(x, 0)$ is a $k$–swallowtail, where $E, E'$ are Banach spaces. The stability of the swallowtail implies that any $k$–swallowtail deformation is diffeomorphic to a simple form, essentially a $k$–swallowtail with coefficients that depend on the parameters.

A $k$-swallowtail has always corank 1 at its singular points. However, observe that, in a $k$–swallowtail deformation, the derivative with respect to the parameters may add the missing direction in the image of the derivative, in which case it is a submersion (this is due to the fact that we are thinking of it as a mapping where variables and parameters are not to be distinguished). Thus a $k$–swallowtail deformation can have corank 0 or 1.

The first case of a $k$–swallowtail deformation in our work corresponds to a double eigenvalue. In this case the map $\chi$ is

$$(a, b, c, d) \to (a + d, ad - bc)$$

which is easily seen to be diffeomorphic to

$$(a, b, c, d) \to (a, d^2 + bc)$$

This is a stable map, being the suspension of a Morse funcion.

## Appendix B
## Some useful facts about Banach spaces

We denote by $E, F$ two Banach spaces over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ and by $\mathscr{B}(E, F)$ the space of continuous linear maps of $E$ into $F$.

***Proposition B.1***. — *For $A \in \mathscr{B}(E, F)$, the following properties are equivalent:*

(i) *The map $A$ is injective and its image* $\operatorname{Im} A$ *is closed, in which case we call $A$ an* embedding.

(ii) *There exists $c > 0$ such that $c|Ax| \geq |x|$ for all $x \in E$.*

(iii) *There does not exist any sequence $(x_n)$ in $E$ such that $|x_n| = 1$ for all $n$ and* $\lim_{n\to\infty} Ax_n = 0$.

*Proof.* — By the open mapping theorem, if (i) holds, then $A$ induces an isomorphism $A_1$ of $E$ onto $\operatorname{Im} A$ and $A_1^{-1} \circ A = \mathrm{Id}_E$, hence (iii) since $\lim_{n\to\infty} Ax_n = 0$ implies $\lim_{n\to\infty} x_n = A_1^{-1} \lim_{n\to\infty} Ax_n = 0$.

If (ii) does not hold, there is a sequence $(y_n)$ in $E$ satisfying $|y_n| > n|Ay_n|$ for all $n$ and therefore the sequence $x_n := y_n/|y_n|$ is such that $|x_n| = 1$ for all $n$ and $\lim_{n\to\infty} Ax_n = 0$, proving that (iii) implies (ii).

Finally, assuming (ii), the linear map $A$ is clearly injective. Moreover, any sequence $(x_n)$ in $E$ such that $Ax_n$ converges to some $y$ satisfies $|x_n - x_p| \leq c|A(x_n - x_p)| = c|Ax_n - Ax_p|$ and therefore is Cauchy. It follows that $(x_n)$ converges to some $x \in E$, which must satisfy $Ax = A \lim x_n = \lim Ax_n = y$, proving that $\operatorname{Im} A$ is closed and therefore that (ii) implies (i). □

***Corollary B.2***. — *The set of all embeddings $A \in \mathscr{B}(E, F)$ is open.*

*Proof.* — By Proposition B.1 (ii), the embeddings are exactly those $A$ such that $|Ax|/|x|$ is bounded below by a positive constant on $E \setminus \{0\}$. □

***Proposition B.3***. — *Given $A \in \mathscr{B}(E, F)$, let $A^* \in \mathscr{B}(F^*, E^*)$ denote the* adjoint map $q \mapsto q \circ A$ *(also called* transposed map*).*

(i) *The map $A^*$ is injective if and only if* $\operatorname{Im} A$ *is dense.*
(ii) *The subspace $A^*F^*$ is closed if and only if* $\operatorname{Im} A$ *is closed.*
(iii) *In particular, $A^*$ is an embedding if and only if $A$ is onto.*
(iv) *The map $A^*$ is onto if and only if $A$ is an embedding.*

*Proof.* — As $A^*q(x) = q(Ax)$ for all $x \in E$, the kernel of $A^*$ is the set $(\operatorname{Im} A)^\perp$ of those $q$ which vanish on the image of $A$, hence (i). Assertion (ii) is Theorem 4.14 of **[10]**, and (iii) follows at once from (i)–(ii).

Let us prove (iv). Clearly, for each $q \in F^*$, the function $A^*q : v \mapsto q(Av)$ vanishes on $\operatorname{Ker} A$, hence the inclusion $A^*F^* \subset (\operatorname{Ker} A)^\perp$. Therefore, if $A^*$ is onto, then $(\operatorname{Ker} A)^\perp = E^*$, hence (Hahn-Banach) $\operatorname{Ker} A = \{0\}$, proving that $A$—which has closed image by (ii)—is an embedding. Conversely, if $A$ is an embedding, then it induces an isomorphism $A_1$ of $E$ onto $\operatorname{Im} A$. For each $p \in E^*$, the map $q := p \circ A_1^{-1} \in (\operatorname{Im} A)^*$ can be extended (Hahn-Banach) to a map $Q \in F^*$, which satisfies $A^*Q = Q \circ A = q \circ A = p$, proving that $A^*$ is onto. □

***Corollary B.4***. — *The set of all surjective $A \in \mathscr{B}(E, F)$ is open.*

*Proof.* — As the linear map $A \mapsto A^*$ of $\mathscr{B}(E, F)$ into $\mathscr{B}(F^*, E^*)$ is continuous (isometric), this follows at once from Proposition B.3 and Corollary B.2. □

***Proposition B.5***. — *The subset of $\mathscr{B}(E, F)$ consisting of those $A$ which are onto but not embeddings is open, and so is the subset consisting of those embeddings which are not onto.*

*Proof.* — By Proposition B.3 (iii)–(iv) and the fact that $A \mapsto A^*$ is continuous, we just have to prove the second assertion. Given an embedding $A$ which is not onto and $y \in F \setminus \operatorname{Im} A$, the distance $2D$ from $y$ to the closed subset $\operatorname{Im} A$ is positive, that is

$$|Ax - y| \geq 2D > 0 \text{ for all } x \in E \tag{18}$$

and, by Proposition B.1, there exists $C > 0$ such that

$$|Ax| > C|x| \text{ for all } x \in E. \tag{19}$$

We claim that every $B = A + u \in \mathscr{B}(E, F)$ close enough to $A$ is an embedding and satisfies

$$|Bx - y| \geq D > 0 \text{ for all } x \in E, \tag{20}$$

proving our result. Indeed, (19) implies the inequality

$$|Bx| = |Ax + ux| \geq |Ax| - |ux| \geq (C - |u|)|x|,$$

proving that $B$ is an embedding for $|u| < C$ and implying

$$|Bx - y| \geq |Bx| - |y| \geq (C - |u|)|x| - |y|,$$

which shows that (20) holds for

$$|u| < C \text{ and } |x| \geq \frac{D + |y|}{C - |u|}.$$

Therefore, all we have to prove is that it holds for small enough $|u| < C$ when $x$ satisfies $|x| \le \frac{D+|y|}{C-|u|}$. Now, this is clear as we then have

$$|Bx - y| = |Ax + ux - y| \ge |Ax - y| - |ux| \ge 2D - |u|\frac{D+|y|}{C-|u|}$$

by (18). □

***Proposition B.6***. — *Given two Banach spaces $S, F$ and endomorphisms $a, d$ of $S$ and $F$ respectively, the spectrum of the endomorphism $a^* - d_* : h \mapsto ha - dh$ of $\mathscr{B}(S,F)$ is $\sigma(a) - \sigma(d) := \{\lambda - \mu \,:\, \lambda \in \sigma(a), \mu \in \sigma(d)\}$.*

*Proof*. — Our endomorphism is the sum of the two commuting endomorphisms $a^* : u \mapsto ua$ and $-d_* : u \mapsto -du$. Now, a classical application of the Gel'fand transform ([**10**], Theorem 11.23) is that if two elements of a Banach algebra with unit(17) commute, the spectrum of their sum is included in the sum of their spectra. As, clearly, $\sigma(a^*) = \sigma(a)$ and $\sigma(d_*) = \sigma(d)$, we get the inclusion

$$\sigma(a^* - d_*) \subset \sigma(a) - \sigma(d).$$

Alternatively, one can use the holomorphic functional calculus to verify that, if $\sigma(a) \cap \sigma(d) = \varnothing$ then the mapping

$$k \longmapsto \frac{1}{2\pi i}\int_{\partial U} (\zeta I - d)^{-1} k (\zeta I - a)^{-1} d\zeta$$

(where $U$ is a bounded open subset of $\mathbf{C}$ with smooth boundary, containing $\sigma(a)$ only, cf. subsection D4) is the inverse of $a^* - d_*$ and then use this particular case to prove the above inclusion.

There remains to prove that we have $\lambda - \mu \in \sigma(a^* - d_*)$ for all $\lambda \in \sigma(a)$ and $\mu \in \sigma(d)$. Replacing $a$ by $a - \lambda$ and $d$ by $d - \mu$, this amounts to proving the following

***Lemma***. — *If neither $a$, nor $d$ is invertible, then $a^* - d_*$ is not invertible.*

Indeed(18), there are four possible situations:

- If $a$ is not onto and $d$ is not an embedding, then by Propositions B.1 and B.3 (iii), there exist a sequence $y_n$ in $F$ and a sequence $p_n$ in $S^*$ with $|y_n| = |p_n| = 1$ such that $p_n a$ and $dy_n$ converge to 0. The sequence $u_n$ in $\mathscr{B}(S,F)$ defined by $u_n(v) = p_n(v) y_n$ satisfies $|u_n| = 1$ and $\lim (a^* - d_*) u_n = 0$, proving that $a^* - d_*$ is not an embedding.
- Similarly, if $a$ is not an embedding and $d$ is not onto, there exist a sequence $x_n$ in $S$ and a sequence $q_n$ in $F^*$ with $|x_n| = |q_n| = 1$ such that $q_n d$ and $a x_n$ converge to 0. It follows that the sequence $\varphi_n$ in $\mathscr{B}(S,F)^*$ defined by $\varphi_n(u) := q_n u x_n$ satisfies $|\varphi_n| = 1$ and $\lim \varphi_n (a^* - d_*) = 0$, proving that $a^* - d_*$ is not onto.

(17) In our case, End $(\mathscr{B}(S,F))$.
(18) We are indebted to Georges Skandalis for what follows.

– If both $a$ and $d$ are non-surjective embeddings, it follows from Proposition B.5 that the set of those $\lambda \in \mathbf{C}$ such that both $a - \lambda$ and $b - \lambda$ are non-surjective embeddings is open. Going to the boundary, one of the maps $a - \lambda$ and $b - \lambda$ is not an embedding, and the other one is not onto by Corollary B.4. Thus, replacing $a, d$ by $a - \lambda, d - \lambda$, we are in one of the previous two cases.
– Similarly, if both $a$ and $d$ are non-injective but onto, the set of those $\lambda \in \mathbf{C}$ such that both $a - \lambda$ and $b - \lambda$ are non-injective but onto is open by Proposition B.5. Going to the boundary, we are again in one of the previous first two cases: indeed, one of the maps $a - \lambda$ and $b - \lambda$ is not onto and the other one is not an embedding by Corollary B.2. $\square$

## References

[1] V. I. Arnol′d – "Matrices depending on parameters", *Uspehi Mat. Nauk* **26** (1971), p. 101–114.

[2] M. Chaperon – "Géométrie différentielle et singularités de systèmes dynamiques", *Astérisque* **138-139** (1986).

[3] ———, *Calcul différentiel et calcul intégral, 3^e^ année*, Dunod, corrected and expanded second printing, 2008.

[4] M. Chaperon & S. López de Medrano – "On the Hopf bifurcation for flows", *C. R. Math. Acad. Sci. Paris* **340** (2005), p. 833–838.

[5] M. Chaperon, S. López de Medrano & J. L. Samaniego – "On sub-harmonic bifurcations", *C. R. Math. Acad. Sci. Paris* **340** (2005), p. 827–832.

[6] I. M. Gel′fand, M. M. Kapranov & A. V. Zelevinsky – *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications, Birkhäuser, 1994.

[7] W. T. Gowers – "A solution to Banach's hyperplane problem", *Bull. London Math. Soc.* **26** (1994), p. 523–530.

[8] T. Kato – *Perturbation theory for linear operators*, Corrected printing of the second edition, Springer, 1976, Grundlehren der Mathematischen Wissenschaften, Band 132.

[9] P. Rosenbloom – "Perturbation of linear operators in Banach spaces", *Arch. Math. (Basel)* **6** (1955), p. 89–101.

[10] W. Rudin – *Functional analysis*, McGraw-Hill Book Co., 1973.

[11] van der Waerden – *Modern algebra, in part a development from lectures by E. Artin and E. Noether*, vol. 1, F. Ungar, 1949.

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# POLAR PENCIL OF CURVES AND FOLIATIONS

*by*

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***Abstract.*** — The polar pencil $\Lambda_{\mathcal{F}}$ of a singular foliation $\mathcal{F}$ is the pencil of curves formed by the polar curves of $\mathcal{F}$. We study the relationship between the behaviour of $\Lambda_{\mathcal{F}}$ under blowing-up and the invariants associated to $\mathcal{F}$. The main result here describes a resolution of singularities of $\Lambda_{\mathcal{F}}$ in terms of the equireduction invariants of $\mathcal{F}$, for a Zariski-general foliation $\mathcal{F}$.

***Résumé* (Pinceau polaire de courbes et feuilletages).** — Le pinceau polaire $\Lambda_{\mathcal{F}}$ d'un feuilletage singulier $\mathcal{F}$ est le pinceau de courbes composé par les courbes polaires de $\mathcal{F}$. Nous allons étudier la relation entre le comportement de $\Lambda_{\mathcal{F}}$ par éclatement et les invariants associés à $\mathcal{F}$. Le résultat principal ici donne une description d'une résolution de singularités de $\Lambda_{\mathcal{F}}$ en termes des invariants d'équiréduction de $\mathcal{F}$ lorsque $\mathcal{F}$ est un feuilletage général de Zariski.

## 1. Introduction

Let $A, B$ be two germs of holomorphic functions at $(\mathbb{C}^2, 0)$ with no common component and consider the pencil of curves $\Lambda = \{aA + bB = 0 \ : \ a, b \in \mathbb{C}\}$. Classically, these pencils of curves have been studied in relation to the reduction of singularities of $A = 0$ and $B = 0$ (see for instance [**4, 8, 14**]). Here we propose a different approach: we consider $\Lambda$ as the *polar pencil* $\Lambda_{\mathcal{F}}$ associated to a singular foliation $\mathcal{F}$ defined by the 1-form $\omega = A(x, y)dx + B(x, y)dy$. Our objective is to describe properties of $\Lambda_{\mathcal{F}}$ in terms of the invariants associated to $\mathcal{F}$.

Let $\mathcal{G}_\omega$ be the *Gauss map* associated to $\mathcal{F}$ which is given by

$$\begin{array}{rccc} \mathcal{G}_\omega : & (\mathbb{C}^2, 0) \smallsetminus \{0\} & \longrightarrow & \mathbb{P}^1_{\mathbb{C}} \\ & (x, y) & \longmapsto & [-B(x, y) : A(x, y)]. \end{array}$$

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A curve $\Gamma_{[a:b]}$ of $\Lambda_{\mathcal{G}}$ is the closure in $(\mathbb{C}^2,0)$ of the fiber $\mathscr{G}_\omega^{-1}([a:b])$ for $[a:b]\in\mathbb{P}^1_{\mathbb{C}}$. There is a maximal non-empty Zariski open set of $\Omega\subset\mathbb{P}^1_{\mathbb{C}}$ such that all the curves $\Gamma_{[a:b]}$ with $[a:b]\in\Omega$ are equisingular: they are the *generic curves* of $\Lambda_{\mathcal{G}}$.

Let $\sigma: X\to(\mathbb{C}^2,0)$ be a finite sequence of punctual blow-ups. We say that $\sigma$ is an *elimination of indeterminations* of $\mathscr{G}_\omega$ (or *a resolution of singularities* of $\Lambda_{\mathcal{G}}$) iff the map $\tilde{\mathscr{G}}_\omega=\mathscr{G}_\omega\circ\sigma: X\to\mathbb{P}^1_{\mathbb{C}}$ is well-defined. Such $\sigma$ gives an embedded reduction of singularities of the union $\Gamma\cup\Gamma'$ of two different generic fibers, then $\sigma$ is a resolution of singularities of $\Lambda_{\mathcal{G}}$ (see **[14]**).

An irreducible component $D$ of $\sigma^{-1}(0)$ is called *dicritical* if the restriction $\tilde{\mathscr{G}}_\omega|_D: D\to\mathbb{P}^1_{\mathbb{C}}$ is not constant. The *degree* of a dicritical component $D$ is the degree of the map $\tilde{\mathscr{G}}_\omega|_D: D\to\mathbb{P}^1_{\mathbb{C}}$; this number coincides with the number of intersection points between $D$ and the strict transform $\sigma^*\Gamma$ of $\Gamma$ by $\sigma$, for any generic fiber $\Gamma$.

The curves of the polar pencil $\Lambda_{\mathcal{G}}$ can also be seen as the separatrices of a singular foliation: the *polar foliation* $\mathscr{P}_{\mathcal{G}}$ defined by $d(A/B)=0$. The minimal resolution $\sigma_\Lambda: X\to(\mathbb{C}^2,0)$ of $\Lambda_{\mathcal{G}}$ gives a *partial reduction* **[12]** of $\mathscr{P}_{\mathcal{G}}$ in the sense that the minimal reduction of singularities $\pi_{\mathscr{P}}:\mathfrak{X}\to(\mathbb{C}^2,0)$ of $\mathscr{P}_{\mathcal{G}}$ factorizes as $\pi_{\mathscr{P}}=\sigma_\Lambda\circ\tau$, where $\tau:\mathfrak{X}\to X$ is a finite sequence of punctual blow-ups which are non-dicritical for $\mathscr{P}_{\mathcal{G}}$.

Let $C\subset(\mathbb{C}^2,0)$ be a plane curve. We shall work in the space of foliations $\mathbb{G}_C$ of non-dicritical generalized curves over $C$ (see **[2]**). It is known that the minimal reduction of singularities $\pi_C: M_C\to(\mathbb{C}^2,0)$ of $C$ gives a reduction of singularities of any $\mathcal{G}\in\mathbb{G}_C$. But in general $\pi_C$ does not give a desingularization of a generic fiber $\Gamma$ of $\Lambda_{\mathcal{G}}$. This occurs essentially in the case that $C$ has a *kind equisingularity type* and $\mathcal{G}$ is Zariski-general (in the sense of the exponents of the logarithmic model) as we have shown in **[6, 7]**.

Take $\mathcal{G}\in\mathbb{G}_C$ and let $\sigma_{\Lambda,C}: M_{\Lambda,C}\to(\mathbb{C}^2,0)$ be the minimal reduction of singularities of $\Lambda=\Lambda_{\mathcal{G}}$ that factorizes through $\pi_C$. The main result of this paper provides a precise description of $\sigma_{\Lambda,C}$ for kind singularities and Zariski-general foliations. Let us state it.

Let $G(C)$ be the dual graph of $C$ oriented by the first divisor $E_1$. For each divisor $E$, let $m(E)$ be the multiplicity of any $E$-"curvette" and $v(E)$ be the coincidence of two $E$-curvettes. Denote by $b_E$ the number of edges and arrows which leave from $E$. Thus $E$ is a *bifurcation divisor* if $b_E\geq 2$ and a *terminal divisor* if $b_E=0$. A *dead arc* joins a bifurcation divisor with a terminal divisor, with no other bifurcations. We say that the equisingularity type $\epsilon(C)$ of $C$ is *kind* if $m(E_b)=2m(E_t)$, for each dead arc of $G(C)$ starting at $E_b$ and ending at $E_t$.

The main result here can be stated as

***Theorem 1.*** — *Let $C\subset(\mathbb{C}^2,0)$ be a plane curve with kind equisingularity type. Consider a Zariski-general foliation $\mathcal{G}\in\mathbb{G}^*_C$ and take any generic curve $\Gamma$ of $\Lambda_{\mathcal{G}}$. Then $\sigma_{\Lambda,C}$ is obtained from $\pi_C$ by blowing-up $\alpha_E$ times in a free way at each point $\pi^*_C\Gamma\cap E$*

*with*

$$(1) \qquad \alpha_E = \begin{cases} m(E)(v(E)-1), & \textit{if } E \textit{ is a bifurcation divisor;} \\ m(E)(v(E)-1)-1, & \textit{if } E \textit{ is the terminal divisor of a dead arc,} \end{cases}$$

*for each irreducible component $E$ of $\pi_C^{-1}(0)$. Moreover, the first divisor $E_1$ is dicritical for $\Lambda_{\mathcal{G}}$ if and only if $b_{E_1} > 1$, and the degree of $E_1$ as a dicritical component of $\Lambda_{\mathcal{G}}$ is equal to $b_{E_1} - 1$. The degree of the other dicritical components of $\Lambda_{\mathcal{G}}$ is equal to one.*

Observe that, under the hypothesis of theorem above, the points of the set $\pi_C^*\Gamma \cap \pi_C^{-1}(0)$ belong either to a bifurcation divisor or to the terminal divisor of a dead arc ([**6**]). Moreover, the points of $\pi_C^*\Gamma \cap \pi_C^{-1}(0)$ are non-singular points of $\pi_C^*\mathcal{G}$ and $\pi_C^*\Gamma$ cuts transversally $\pi_C^{-1}(0)$. Consequently $\sigma_{\Lambda,C} = \pi_C \circ \sigma_1$ where $\sigma_1$ is obtained by blowing-up free infinitely near points of $\pi_C^*\Gamma$, i.e., the centers of the blow-ups to obtain $\sigma_1$ are not corners of the corresponding exceptional divisor. Hence $\sigma_{\Lambda,C}$ is obtained from $\pi_C$ by "blowing-up in a free way" as it is stated in the theorem above.

The paper is organized as follows. Section 2 is devoted to introduce notations relative to the dual graph and the equisingularity data of a plane curve. In section 3 we remind some results concerning the generic fiber of the polar pencil and we also prove some technical lemmas. Section 4 deals with the base points of the pencil $\Lambda_{\mathcal{G}}$. In section 5 we state some results describing the dicritical components of a resolution of $\Lambda_{\mathcal{G}}$. The proof of the main result is given in section 6. We finish the paper with a list of examples showing different behaviours in the non Zariski-general cases.

## 2. Notation

In this section we introduce some notation concerning the dual graph and the equisingularity data of a plane curve $C = \cup_{i=1}^r C_i \subset (\mathbb{C}^2, 0)$ that will be used from now on. For each irreducible component $C_i$ of $C$, denote by $n^i = m_0(C_i)$ the multiplicity of $C_i$ at the origin. Let $y^i(x) = \sum_{j \geq n^i} a_j^i x^{j/n^i}$ be a Puiseux series of $C_i$, for $i = 1, \ldots, r$. The *characteristic exponents* $\{\beta_0^i, \beta_1^i, \ldots, \beta_{g_i}^i\}$ of $C_i$ are given by

$$\begin{aligned} \beta_0^i &= m_0(C_i) = n^i \\ \beta_q^i &= \min\{j \ : \ a_j^i \neq 0 \text{ and } j \not\equiv 0 \mod (\gcd(\beta_0^i, \ldots, \beta_{q-1}^i))\} \end{aligned}$$

for $q = 1, \ldots, g_i$, where $g_i$ is the first integer such that $\gcd(\beta_0^i, \ldots, \beta_{g_i}^i) = 1$. Data equivalent to the characteristic exponents of $C_i$ are the *Puiseux pairs* $\{(m_k^i, n_k^i)\}_{k=1}^{g_i}$ of $C_i$ defined by

$$\gcd(m_k^i, n_k^i) = 1 \quad \text{and} \quad \frac{\beta_k^i}{n^i} = \frac{m_k^i}{n_1^i \cdots n_k^i} \quad \text{for} \quad k = 1, \ldots, g_i.$$

In particular, we have that $n^i = n_1^i \cdots n_{g_i}^i$ and $\beta_k^i = m_k^i n_{k+1}^i \cdots n_{g_i}^i$ for $k = 1, \ldots, g_i$.

Let us denote by $\pi_C : M_C \to (\mathbb{C}^2, 0)$ the minimal reduction of singularities of $C$. We recall briefly the construction of the *dual graph* $G(C) = G(\pi_C)$ of $C$. Each irreducible component $E$ of $\pi_C^{-1}(0)$ is represented by a vertex in $G(C)$. Two vertices are joined by an edge if their associated divisors intersect. An irreducible component of $C$ is represented by an arrow attached to the only divisor that it meets. The dual graph weighted with the self-intersection of each divisor $E \subset M_C$ determines the equisingularity type $\epsilon(C)$ of the curve $C$.

It is also possible to construct in a similar way the dual graph of a resolution of singularities of a pencil or a dicritical foliation by marking the dicritical components. If $\sigma$ is any finite sequence of blow-ups, we denote by $G(\sigma, \Lambda)$ the graph constructed from the transform of a pencil $\Lambda$ by $\sigma$.

Denote by $E_1$ the irreducible component of $\pi_C^{-1}(0)$ obtained after blowing-up the origin. The dual graph $G(C)$ is oriented by the first divisor $E_1$. The *geodesic* of a divisor $E$ is the path which joins $E_1$ with $E$ and the geodesic of a curve $C_i$ is the geodesic of the divisor that meets the strict transform $\pi_C^* C_i$ of $C_i$. Thus, there is a partial order in the set of vertices of $G(C)$ given by $E < E'$ if, and only if, the geodesic of $E'$ goes through $E$. Given a divisor $E$ of $G(C)$, we denote by $I_E$ the set of indices $i \in \{1, \ldots, r\}$ such that $E$ belongs to the geodesic of $C_i$.

A *curvette* $\tilde{\gamma}$ of a divisor $E$ is a non-singular curve transversal to $E$ at a non-singular point of $\pi_C^{-1}(0)$. The projection $\gamma = \pi_C(\tilde{\gamma})$ is a germ of plane curve in $(\mathbb{C}^2, 0)$ and $\gamma$ is called an $E$-curvette. We denote by $m(E)$ the multiplicity at the origin of any $E$-curvette and by $v(E)$ the coincidence $\mathscr{C}(\gamma_E, \gamma'_E)$ of two $E$-curvettes $\gamma_E, \gamma'_E$ which cut $E$ in different points; observe that $v(E) < v(E')$ if $E < E'$. Recall that the *coincidence* $\mathscr{C}(\gamma, \delta)$ between two irreducible curves $\gamma$ and $\delta$ is defined as

$$\mathscr{C}(\gamma, \delta) = \sup_{\substack{1 \le i \le m_0(\gamma) \\ 1 \le j \le m_0(\delta)}} \{ \, ord_x(y_i^{\gamma}(x) - y_j^{\delta}(x)) \, \}$$

where $\{y_i^{\gamma}(x)\}_{i=1}^{m_0(\gamma)}$, $\{y_j^{\delta}(x)\}_{j=1}^{m_0(\delta)}$ are the Puiseux series of $\gamma$ and $\delta$ respectively.

Denote by $b_E$ the number of edges and arrows which leave from a divisor $E$ in $G(C)$. We say that $E$ is a *bifurcation divisor* if $b_E \geq 2$ and a *terminal divisor* if $b_E = 0$. A *dead arc* is a path which joins a bifurcation divisor with a terminal one, without passing through other bifurcation divisors. We denote by $B(C)$ the set of bifurcation vertices of $G(C)$.

Let $E$ be an irreducible component of the exceptional divisor $\pi_C^{-1}(0)$. The *reduction* $\pi_E : M_E \to (\mathbb{C}^2, 0)$ *of* $\pi_C$ *to* $E$ is the morphism satisfying that

- there is a factorization $\pi_C = \pi'_E \circ \pi_E$ where $\pi'_E$ and $\pi_E$ are composition of punctual blow-ups;
- the divisor $E$ is the strict transform by $\pi'_E$ of an irreducible component $E_{red}$ of $\pi_E^{-1}(0)$ and $E_{red} \subset M_E$ is the only component of $\pi_E^{-1}(0)$ with self-intersection equal to $-1$.

The morphism $\pi_E$ is obtained from $\pi_C$ by blowing-down successively the divisors different from $E$ with self-intersection is equal to $-1$. Given any curvette $\tilde{\gamma}_E$ of $E$, the curve $\pi'_E(\tilde{\gamma}_E)$ is also a curvette of $E_{red} \subset M_E$. Let $\{\beta_0^E, \beta_1^E, \ldots, \beta_{g(E)}^E\}$ be the characteristic exponents of $\gamma_E = \pi_C(\tilde{\gamma}_E)$. It is clear that $m(E) = m_0(\gamma_E) = \beta_0^E$. If $E$ is a bifurcation divisor of $G(C)$, there are two possibilities for the value $v(E)$:

1. either $\pi_E$ is the minimal reduction of singularities of $\gamma_E$ and then $v(E) = \beta^E_{g(E)}/\beta^E_0$. We say that $E$ is a *Puiseux divisor* for $\pi_C$.
2. or $\pi_E$ is obtained by blowing-up $q \geq 1$ times after the minimal reduction of singularities of $\gamma_E$ and in this situation $v(E) = (\beta^E_{g(E)} + q)/\beta^E_0$. We say that $E$ is a *contact divisor* for $\pi_C$.

Observe that $m(E) = m(E_{red})$ and $v(E) = v(E_{red})$. Moreover, a bifurcation divisor $E$ can belong to a dead arc only if it is a Puiseux divisor.

Consider a bifurcation divisor $E$ of $G(C)$ and let $\{(m^E_1, n^E_1), (m^E_2, n^E_2), \dots, (m^E_{g(E)}, n^E_{g(E)})\}$ be the Puiseux pairs of an $E$-curvette $\gamma_E$, we denote

$$n_E = \begin{cases} n_{g(E)}, & \text{if } E \text{ is a Puiseux divisor;} \\ 1, & \text{otherwise,} \end{cases}$$

and $\underline{n}_E = m(E)/n_E$. Observe that, if $E$ is a bifurcation divisor which belongs to a dead arc with terminal divisor $F$, then $m(F) = \underline{n}_E$. We define $k_E$ to be

$$k_E = \begin{cases} g(E) - 1, & \text{if } E \text{ is a Puiseux divisor;} \\ g(E), & \text{if } E \text{ is a contact divisor.} \end{cases}$$

Thus, we have that $\underline{n}_E = n^E_1 \cdots n^E_{k_E}$.

To finish this section, we recall a lemma which gives the relationship between the intersection multiplicity $(\gamma, \delta)_0$ and the coincidence $\mathscr{C}(\gamma, \delta)$ (see Zariski [**15**], prop. 6.1 or Merle [**11**], prop. 2.4):

***Lemma 2***. — *Let $\gamma$ and $\delta$ be two germs of irreducible plane curves of $(\mathbb{C}^2, 0)$. If $\{\beta_0, \beta_1, \dots, \beta_g\}$ are the characteristic exponents of $\gamma$ and $\alpha$ is a rational number such that $\beta_q \leq \alpha < \beta_{q+1}$ ($\beta_{g+1} = \infty$), then the following statements are equivalent:*

1. $\mathscr{C}(\gamma, \delta) = \dfrac{\alpha}{m_0(\gamma)}$,
2. $\dfrac{(\gamma, \delta)_0}{m_0(\delta)} = \dfrac{\bar\beta_q}{n_1 \cdots n_{q-1}} + \dfrac{\alpha - \beta_q}{n_1 \cdots n_q}$,

*where $\{(m_i, n_i)\}_{i=1}^g$ are the Puiseux pairs of $\gamma$ ($n_0 = 1$) and $\{\bar\beta_0, \bar\beta_1, \dots, \bar\beta_q\}$ is a minimal system of generators of the semigroup $S(\gamma)$ of $\gamma$.*

Recall that the *semigroup* $S(\gamma)$ of $\gamma$ is defined as

$$S(\gamma) = \{(\gamma, \delta)_0 : \gamma \text{ is not an irreducible component of } \delta\}.$$

There is a minimal system of generators $\{\bar\beta_0, \bar\beta_1, \dots, \bar\beta_g\}$ of $S(\gamma)$ whose elements are defined by

$$\text{(2)} \quad \bar\beta_0 = \beta_0 = m_0(\gamma), \quad \bar\beta_1 = \beta_1, \quad \bar\beta_l = n_{l-1}\bar\beta_{l-1} + \beta_l - \beta_{l-1}, \quad \text{for } l = 2, \dots, g,$$

where $\{\beta_0, \beta_1, \dots, \beta_g\}$ are the characteristic exponents of $\gamma$ (see [**1**] or [**16**]). It is clear that $S(\gamma)$ is determined by the equisingularity type of $\gamma$ and reciprocally.

## 3. Generic curves of the pencil

This section is devoted to describe some properties of a generic curve of the polar pencil $\Lambda_{\mathcal{F}}$ of a singular foliation $\mathcal{F}$. The reader may refer to **[5, 7]** for a more detailed description.

Consider a plane curve $C = \cup_{i=1}^{r} C_i \subset (\mathbb{C}^2, 0)$. Let $f = f_1 \cdots f_r$ be a reduced equation of $C$ and $\pi_C : M_C \to (\mathbb{C}^2, 0)$ be the minimal reduction of singularities of $C$. Denote by $\mathbb{G}_C$ the space of generalized curve foliations **[2]** having $C$ as curve of separatrices. Let $\mathbb{G}_C^*$ be the sub-space of $\mathbb{G}_C$ defined as follows: a foliation $\mathcal{F}$ is in $\mathbb{G}_C^*$ iff the logarithmic model $\mathcal{L}_\lambda$ of $\mathcal{F}$ avoids a finite set of resonances $R_{\epsilon(C)} \subset (\mathbb{Z}_{\geq 0})^r$. More precisely, each foliation $\mathcal{F} \in \mathbb{G}_C$ has a unique logarithmic model $\mathcal{L}_\lambda$ given by $f_1 \cdots f_r \sum_{i=1}^{r} \lambda_i df_i / f_i = 0$ with $\lambda = \lambda(\mathcal{F}) = (\lambda_1, \ldots, \lambda_r) \in \mathbb{P}_{\mathbb{C}}^{r-1}$ (see **[5]**). The logarithmic foliation $\mathcal{L}_\lambda$ has the same reduction of singularities as $\mathcal{F}$ and the same Camacho-Sad indices **[3]** at the final points of the reduction. Thus, a foliation $\mathcal{F}$ belongs to $\mathbb{G}_C^*$ iff $\sum_{i=1}^{r} k_i \lambda_i \neq 0$ for each $k = (k_1, \ldots, k_r) \in R_{\epsilon(C)}$ where $R_{\epsilon(C)} \subset (\mathbb{Z}_{\geq 0})^r$ is a finite set which depends only on the equisingularity type $\epsilon(C)$ of $C$ (see **[5, 7]** for a detailed construction of it).

***Remark 3***. — Note that a foliation $\mathcal{F}$ avoids the resonances of the set $R_{\epsilon(C)}$ if and only if there is no corner in the reduction of singularities of $\rho^* \mathcal{F}$ with Camacho-Sad equal to $-1$, where $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is any ramification transversal to $C$ such that $\rho^{-1} C$ has only non-singular irreducible components (see **[5]**).

Consider a generic fiber $\Gamma$ of the pencil $\Lambda_{\mathcal{F}}$. A first result describing some properties of the equisingularity type $\epsilon(\Gamma)$ of $\Gamma$ in terms of the equisingularity type $\epsilon(C)$ of $C$ is the following one:

***Theorem 4* (of decomposition [5, 9, 10, 11])**. — *Consider a foliation $\mathcal{F} \in \mathbb{G}_C^*$ and a generic curve $\Gamma$ of $\Lambda_{\mathcal{F}}$. There is a decomposition $\Gamma = \cup_{E \in B(C)} \Gamma^E$ such that:*

(i) $m_0(\Gamma^E) = \begin{cases} \underline{n}_E n_E (b_E - 1), & \textit{if } E \textit{ does not belong to a dead arc;} \\ \underline{n}_E n_E (b_E - 1) - \underline{n}_E, & \textit{otherwise.} \end{cases}$

(ii) *For each irreducible component $\gamma$ of $\Gamma^E$ we have that*
- $\mathcal{C}(C_i, \gamma) = v(E)$ *if $E$ belongs to the geodesic of $C_i$;*
- $\mathcal{C}(C_j, \gamma) = \mathcal{C}(C_j, C_i)$ *if $E$ belongs to the geodesic of $C_i$ but not to the one of $C_j$.*

It is clear that the result above does not determine $\epsilon(\Gamma)$. However, there is a Zariski-open set $U_C \subset \mathbb{P}_{\mathbb{C}}^{r-1}$ such that $\epsilon(\Gamma)$ is completely determined by $\epsilon(C)$ if $\lambda(\mathcal{F}) \in U_C$. The set $U_C$ depends on the analytic type of $C$ and it is a non-empty set if, and only if, the curve $C$ has a kind equisingularity type. We say that a curve $C$ has *kind equisingularity type* if $m(E_b) = 2m(E_t)$ for each dead arc of $G(C)$ with bifurcation divisor $E_b$ and terminal divisor $E_t$. Using the notation introduced in section 2, the curve $C$ has a kind equisingularity type if and only if $n_{E_b} = 2$ for each bifurcation divisor $E_b$ of $G(C)$ which belongs to a dead arc since $m(E_b) = n_{E_b} m(E_t)$. In particular, this implies that each dead arc in $G(C)$ has only two vertices: the bifurcation divisor and the terminal divisor.

A foliation $\mathcal{F}$ is called *Zariski-general* when $\lambda(\mathcal{F}) \in U_C$ and in this case $\epsilon(\Gamma)$ is described as follows:

***Theorem 5.*** — **[6, 7]** *Let $C$ be a curve with kind equisingularity type and consider a Zariski-general foliation $\mathcal{F} \in \mathbb{G}_C^*$. If $\Gamma$ is a generic curve of the pencil $\Lambda_{\mathcal{F}}$, then $\pi_C$ gives a reduction of singularities of $\Gamma \cup C$. Moreover, the branches of $\Gamma$ intersect an irreducible component $E$ of the exceptional divisor $\pi_C^{-1}(0)$ as follows:*

- *If $E$ is a bifurcation divisor of $G(C)$, the number of branches of $\Gamma$ cutting $E$ equals to $b_E - 2$ if $E$ belongs to a dead arc and to $b_E - 1$ otherwise.*
- *If $E$ is a terminal divisor of a dead arc of $G(C)$, there is exactly one branch of $\Gamma$ through $E$.*
- *Otherwise, no branches of $\Gamma$ intersect $E$.*

In particular, the characteristic exponents of the branches of $\Gamma$ can be completely determined in terms of the equisingularity data of $C$. Denote by $\{\beta_0^i, \beta_1^i, \ldots, \beta_{g_i}^i\}$ the characteristic exponents of an irreducible component $C_i$ of $C$. Given a bifurcation divisor $E$ of $G(C)$, let $I_E^*$ be the set of indices $i \in I_E$ such that $v(E) = \beta_{k_E+1}^i/\beta_0^i$; note that if $i \in I_E \setminus I_E^*$ then there exists $j \in I_E$ such that $v(E) = \mathcal{C}(C_i, C_j)$. Hence, if $E$ is a contact divisor $I_E^* = \varnothing$. Moreover, if $C$ has a kind equisingularity type and $E$ is a bifurcation divisor belonging to a dear arc of $G(C)$, then the corresponding Puiseux pair $(m_{k_E+1}^i, n_{k_E+1}^i)$ satisfies $n_{k_E+1}^i = 2$ for each $i \in I_E = I_E^*$.

***Lemma 6.*** — **[7]** *Consider a curve $C$ with kind equisingularity type and a Zariski general foliation $\mathcal{F} \in \mathbb{G}_C^*$. Let $\Gamma$ be a generic curve of $\Lambda_{\mathcal{F}}$ with decomposition $\Gamma = \cup_{E \in B(C)} \Gamma^E$. Then, for each $E \in B(C)$, we have that*

(i) *If $E$ is a contact divisor, the curve $\Gamma^E$ has $b_E - 1$ irreducible components. Each branch $\gamma$ of $\Gamma^E$ with characteristic exponents $\{\nu_0^\gamma, \nu_1^\gamma, \ldots, \nu_{k_E}^\gamma\}$ given by*

$$\nu_0^\gamma = m_0(\gamma) = \underline{n}_E, \quad \nu_l^\gamma = \underline{n}_E \beta_l^i/\beta_0^i, \quad l = 1, 2, \ldots, k_E,$$

*for any $i \in I_E$.*

(ii) *If $E$ is a Puiseux divisor which belongs to a dead arc, the curve $\Gamma^E$ has one irreducible component $\gamma_0$ with characteristic exponents $\{\nu_0^{\gamma_0}, \nu_1^{\gamma_0}, \ldots, \nu_{k_E}^{\gamma_0}\}$ given by*

$$\nu_0^{\gamma_0} = m_0(\gamma_0) = \underline{n}_E, \quad \nu_l^{\gamma_0} = \underline{n}_E \beta_l^i/\beta_0^i, \quad l = 1, 2, \ldots, k_E,$$

*and $b_E - 2$ irreducible components such that each branch $\zeta \subset \Gamma^E \setminus \gamma_0$ has characteristic exponents $\{\nu_0^\gamma, \nu_1^\gamma, \ldots, \nu_{k_E}^\gamma, \nu_{k_E+1}^\gamma\}$ given by*

$$\nu_0^\gamma = m_0(\gamma) = \underline{n}_E n_E, \quad \nu_l^\gamma = \underline{n}_E n_E \beta_l^i/\beta_0^i, \quad l = 1, 2, \ldots, k_E + 1,$$

*for any $i \in I_E^*$.*

(iii) *If $E$ is a Puiseux divisor which does not belong to a dead arc, then $\Gamma^E$ has $b_E - 1$ irreducible components. Each irreducible component $\gamma$ of $\Gamma^E$ with characteristic exponents $\{\nu_0^\gamma, \nu_1^\gamma, \ldots, \nu_{k_E}^\gamma, \nu_{k_E+1}^\gamma\}$ given by*

$$\nu_0^\gamma = m_0(\gamma) = \underline{n}_E n_E, \quad \nu_l^\gamma = \underline{n}_E n_E \beta_l^i / \beta_0^i, \quad l = 1, 2, \ldots, k_E + 1,$$

*for any $i \in I_E^*$.*

The last part of the section is devoted to prove some technical lemmas which will be useful in the sequel. The first one is a general result concerning intersection multiplicities of polar curves:

***Lemma** 7.* — *Consider a foliation $\mathcal{F} \in \mathbb{G}_C$ and let $\Gamma, \Gamma'$ be any two generic curves of $\Lambda_{\mathcal{F}}$. For any irreducible component $\gamma$ of $\Gamma$, we have that*

$$(\Gamma', \gamma)_0 + m_0(\gamma) = (C, \gamma)_0. \tag{3}$$

*Proof.* — Consider a 1-form $\omega = A(x,y)dx + B(x,y)dy$ which defines $\mathcal{F}$ and assume that $\Gamma = \Gamma_{[a:b]}$, $\Gamma' = \Gamma_{[a':b']}$. Take an irreducible component $\gamma$ of $\Gamma_{[a:b]}$ and let $\phi_\gamma(t) = (x_\gamma(t), y_\gamma(t))$ be a parametrization of $\gamma$. Since $\mathcal{F}$ is a generalized curve foliation, then

$$(C, \gamma)_0 = \mathrm{ord}_t(\phi_\gamma^* \omega) + 1$$

(see **[13]**, lemma 3.7). The intersection multiplicity $(\Gamma_{[a':b']}, \gamma)_0$ is given by

$$(\Gamma_{[a':b']}, \gamma)_0 = \mathrm{ord}_t\{a' A(\phi_\gamma(t)) + b' B(\phi_\gamma(t))\}.$$

Moreover, since $\gamma$ is an irreducible component of $\Gamma_{[a:b]}$, then $aA(\phi_\gamma(t)) + bB(\phi_\gamma(t)) \equiv 0$. Assume that $a \neq 0$, a similar argument holds if $b \neq 0$. In this case, we have that either $\mathrm{ord}_t(A(\phi_\gamma(t))) = \mathrm{ord}_t(B(\phi_\gamma(t)))$ when $b \neq 0$ or $A(\phi_\gamma(t)) \equiv 0$ otherwise. In both situations, the following equalities to compute $\mathrm{ord}_t(\phi_\gamma^* \omega)$ hold:

$$\begin{aligned}
\mathrm{ord}_t(\phi_\gamma^* \omega) &= \mathrm{ord}_t\{A(\phi_\gamma(t))\, \dot{x}_\gamma(t) + B(\phi_\gamma(t))\, \dot{y}_\gamma(t)\} \\
&= \mathrm{ord}_t \left\{ -\frac{b}{a} B(\phi_\gamma(t))\, \dot{x}_\gamma(t) + B(\phi_\gamma(t))\, \dot{y}_\gamma(t) \right\} \\
&= \mathrm{ord}_t(B(\phi_\gamma(t))) + \mathrm{ord}_t(-b\dot{x}_\gamma(t) + a\dot{y}_\gamma(t)) \\
&= \mathrm{ord}_t(a' A(\phi_\gamma(t)) + b' B(\phi_\gamma(t))) + (\gamma, -bx + ay = 0)_0 - 1 \\
&= (\Gamma_{[a':b']}, \gamma)_0 + (\gamma, \ell_{[a:b]})_0 - 1,
\end{aligned}$$

where $\ell_{[a:b]}$ is the line given by $-bx + ay = 0$. In particular, this implies that the formula (3) holds for all $[a:b]$ such that $\ell_{[a:b]}$ is not tangent to $\Gamma_{[a:b]}$ which is the case when $\Gamma_{[a:b]}$ is a generic curve of $\Lambda_{\mathcal{F}}$. $\square$

Let us introduce some notation in order to simplify the proofs of the following lemmas. Given a bifurcation divisor $E$ of $G(C)$, we denote

$$d_E^1 = \begin{cases} b_E & \text{if } E \text{ is a contact divisor;} \\ 1, & \text{if } E \text{ is a Puiseux divisor which does not belong to a dead arc;} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$d_E^2 = \begin{cases} 0, & \text{if } E \text{ is a contact divisor;} \\ b_E - 1, & \text{otherwise.} \end{cases}$$

Hence, if $\Gamma$ is a generic curve of $\Lambda_{\mathcal{G}}$ with decomposition $\Gamma = \cup_{E\in B(C)}\Gamma^E$, then $m_0(\Gamma^E) = \underline{n}_E(d_E^1 + d_E^2 n_E - 1)$.

***Lemma 8***. — *Consider a foliation $\mathcal{G} \in \mathbb{G}_C^*$ and a generic curve $\Gamma = \cup_{E\in B(C)}\Gamma^E$ of $\Lambda_{\mathcal{G}}$. Then, for each bifurcation divisor $E$ of $G(C)$, we have that*

$$m_0(\bigcup_{i\in I_E} C_i) - m_0(\bigcup_{E'>E} \Gamma^{E'}) = \underline{n}_E(d_E^1 + n_E d_E^2). \tag{4}$$

*Proof*. — Let $\ell_E$ be the size of the largest chain of divisors in $B(C)$ starting at $E$. We prove the lemma by induction on $\ell_E$. If $\ell_E = 1$, then $E$ is a maximal bifurcation divisor of $G(C)$. In this case, the equality (4) turns into

$$m_0(\cup_{i\in I_E} C_i) = \underline{n}_E(d_E^1 + n_E d_E^2)$$

and it can be directly deduced from the properties of $G(C)$. Assume now that $\ell_E > 1$ and let $E_1, \dots, E_s$ be the bifurcation vertices of $G(C)$ which are consecutive to $E$, that is, $E < E_i$ without any other bifurcation divisor between $E$ and $E_i$. Put $J_E = I_E \setminus \cup_{i=1}^s I_{E_i}$ and $t = \sharp J_E$. Note that $t + s = d_E^1 + d_E^2$. Then we have the following equalities

$$\begin{aligned} m_0(\bigcup_{i\in I_E} C_i) - m_0(\bigcup_{E'>E} \Gamma^{E'}) = \\ &= \sum_{j\in J_E} m_0(C_j) + \sum_{i=1}^s m_0(\bigcup_{j\in I_{E_i}} C_j) - \left[\sum_{i=1}^s m_0(\bigcup_{E'>E_i} \Gamma^{E'}) + \sum_{i=1}^s m_0(\Gamma^{E_i})\right] \\ &= \sum_{i\in J_E} m_0(C_i) + \sum_{i=1}^s \left[m_0(\bigcup_{j\in I_{E_i}} C_j) - m_0(\bigcup_{E'>E_i} \Gamma^{E'})\right] - \sum_{i=1}^s m_0(\Gamma^{E_i}). \end{aligned}$$

For each $i = 1, \dots, s$, we have that $m_0(\cup_{j\in I_{E_i}} C_j) - m_0(\cup_{E'>E_i}\Gamma^{E'}) = \underline{n}_{E_i}(d_{E_i}^1 + d_{E_i}^2 n_{E_i})$ by the induction hypothesis and $m_0(\Gamma^{E_i}) = \underline{n}_{E_i}(d_{E_i}^1 + d_{E_i}^2 n_{E_i} - 1)$ by theorem 4. Hence, we deduce that

$$m_0(\bigcup_{i\in I_E} C_i) - m_0(\bigcup_{E'>E} \Gamma^{E'}) = \sum_{j\in J_E} m_0(C_j) + \sum_{i=1}^s \underline{n}_{E_i}.$$

Now three situations may happen:

- If $E$ is a contact divisor, then $n_E = 1$, $\underline{n}_{E_i} = \underline{n}_E$ for $i = 1, \dots, s$ and $m_0(C_j) = \underline{n}_E$ for $j \in J_E$. Moreover, $d_E^2 = 0$ and $t + s = d_E^1$.
- If $E$ is a Puiseux divisor which belongs to a dead arc, then $\underline{n}_{E_i} = \underline{n}_E n_E$ with $n_E > 1$ for each $i = 1, \dots, s$ and $m_0(C_j) = \underline{n}_E n_E$ for $j \in J_E$. In this case, $d_E^1 = 0$ and $t + s = d_E^2$.

– If $E$ is a Puiseux divisor without dead arc, then $d_E^1 = 1$ and $t + s - 1 = d_E^2$. Moreover $n_E > 1$ and there is:
  - either a divisor $E_{i_0}$, with $i_0 \in \{1, \dots, s\}$, such that $\underline{n}_{E_{i_0}} = \underline{n}_E$ and $\underline{n}_{E_i} = \underline{n}_E n_E$ for $i \neq i_0$; in this situation $m_0(C_j) = \underline{n}_E n_E$ for all $j \in J_E$.
  - or a curve $C_{j_0}$ with $j_0 \in J_E$ such that $m_0(C_{j_0}) = \underline{n}_E$ and $m_0(C_j) = \underline{n}_E n_E$ if $j \neq j_0$; in this case $\underline{n}_{E_i} = \underline{n}_E n_E$ for all $i \in \{1, \dots, s\}$.

It follows that $\sum_{j \in J_E} m_0(C_j) + \sum_{i=1}^{s} \underline{n}_{E_i} = \underline{n}_E(d_E^1 + d_E^2 n_E)$ and the result is straightforward. □

Take a bifurcation divisor $E$ of $G(C)$. Let $F_1 < F_2 < \cdots < F_m < F_{m+1} = E$ be the bifurcation vertices in the geodesic of $E$ in $G(C)$ and denote $\mathscr{B}_i = \{E' \in B(C) : E' \geq F_i\}$. Then we have the following result

***Lemma 9***. — *Consider a foliation $\mathcal{F} \in \mathbb{G}_C^*$ and let $\Gamma, \Upsilon$ be two generic curves of $\Lambda_{\mathcal{F}}$ with decompositions $\Gamma = \cup_{E \in B(C)} \Gamma^E$ and $\Upsilon = \cup_{E \in B(C)} \Upsilon^E$. Let $\gamma$ be an irreducible component of $\Gamma^E \subset \Gamma$. Denote by $\{\nu_0^\gamma, \nu_1^\gamma, \dots, \nu_{g_\gamma}^\gamma\}$ the characteristic exponents of $\gamma$, by $\{(m_i^\gamma, n_i^\gamma)\}_{i=1}^{g_\gamma}$ the Puiseux pairs of $\gamma$ and by $\{\bar{\nu}_0^\gamma, \bar{\nu}_1^\gamma, \dots, \bar{\nu}_{g_\gamma}^\gamma\}$ the minimal set of generators of the semigroup of $\gamma$. Then we have that*

$$\sum_{l=1}^{m} \left[ \sum_{i \in I_{F_l} \smallsetminus I_{F_{l+1}}} (C_i, \gamma)_0 - \sum_{E' \in \mathscr{B}_l \smallsetminus \mathscr{B}_{l+1}} (\Upsilon^{E'}, \gamma)_0 \right] = \nu_{k_E}^\gamma - n_{k_E}^\gamma \bar{\nu}_{k_E}^\gamma. \tag{5}$$

*Proof*. — By the properties of the decomposition of a generic curve of $\Lambda_{\mathcal{F}}$ given in theorem 4, we have that:

- $\mathscr{C}(C_i, \gamma) = v(F_l)$ if $i \in I_{F_l} \smallsetminus I_{F_{l+1}}$;
- $\mathscr{C}(\zeta^{E'}, \gamma) = v(F_l)$ if $\zeta^{E'}$ is an irreducible component of $\Upsilon^{E'}$ with $E' \in \mathscr{B}_l \smallsetminus \mathscr{B}_{l+1}$.

For each $l \in \{1, \dots, m\}$, let $t(l)$ be the integer in $\{0, 1, \dots, g_\gamma\}$ such that

$$\nu_{t(l)}^\gamma \leq m_0(\gamma) v(F_l) < \nu_{t(l)+1}^\gamma$$

($\nu_{g_\gamma+1}^\gamma = +\infty$). Note that $t(l) \leq k_E \leq g_\gamma$ for $l = 1, \dots, m$ and $t(m) = k_E$. We use now the relationship between the coincidence and the intersection multiplicity given in lemma 2 to compute $(C_i, \gamma)_0$ and $(\zeta^{E'}, \gamma)_0$. We have that

$$\frac{(C_i, \gamma)_0}{m_0(C_i)} = \frac{\bar{\nu}_{t(l)}^\gamma \cdot n_{t(l)}^\gamma + m_0(\gamma) v(F_l) - \nu_{t(l)}^\gamma}{n_1^\gamma \cdots n_{t(l)}^\gamma}, \text{ for } i \in I_{F_l} \smallsetminus I_{F_{l+1}},$$

and

$$\frac{(\zeta^{E'}, \gamma)_0}{m_0(\zeta^{E'})} = \frac{\bar{\nu}_{t(l)}^\gamma \cdot n_{t(l)}^\gamma + m_0(\gamma) v(F_l) - \nu_{t(l)}^\gamma}{n_1^\gamma \cdots n_{t(l)}^\gamma},$$

for each irreducible component $\zeta^{E'}$ of $\Upsilon^{E'}$ with $E' \in \mathscr{B}_l \setminus \mathscr{B}_{l+1}$, Consequently, we obtain that

$$\sum_{i \in I_{F_l} \setminus I_{F_{l+1}}} (C_i, \gamma)_0 - \sum_{E' \in \mathscr{B}_l \setminus \mathscr{B}_{l+1}} (\Upsilon^{E'}, \gamma)_0 =$$

$$\left( \sum_{i \in I_{F_l} \setminus I_{F_{l+1}}} m_0(C_i) - \sum_{E' \in \mathscr{B}_l \setminus \mathscr{B}_{l+1}} m_0(\Upsilon^{E'}) \right) \frac{\bar{\nu}^\gamma_{t(l)} \cdot n^\gamma_{t(l)} + m_0(\gamma) v(F_l) - \nu^\gamma_{t(l)}}{n^\gamma_1 \cdots n^\gamma_{t(l)}}.$$

By lemma 8, we have that

$$\sum_{i \in I_{F_l}} m_0(C_i) - \sum_{E' \in \mathscr{B}_l} m_0(\Upsilon^{E'}) = \underline{n}_{F_l}(d^1_{F_l} + d^2_{F_l} n_{F_l}) - m_0(\Upsilon^{F_l}) = \underline{n}_{F_l},$$

and hence it follows that

$$\sum_{i \in I_{F_l} \setminus I_{F_{l+1}}} m_0(C_i) - \sum_{E' \in \mathscr{B}_l \setminus \mathscr{B}_{l+1}} m_0(\Upsilon^{E'}) = \underline{n}_{F_l} - \underline{n}_{F_{l+1}} = \underline{n}_{F_l}(1 - n_{F_l}).$$

By definition $n_{F_l}$ is given by

$$n_{F_l} = \begin{cases} 1, & \text{if } F_l \text{ is a contact divisor;} \\ n^\gamma_{t(l)}, & \text{if } F_l \text{ is a Puiseux divisor.} \end{cases}$$

Moreover, $m_0(\gamma)v(F_l) = \nu^\gamma_{t(l)}$ and $\underline{n}_{F_l} = n_1 \cdots n_{t(l)-1}$ if $F_l$ is a Puiseux divisor. Therefore, we deduce that

$$\sum_{i \in I_{F_l} \setminus I_{F_{l+1}}} (C_i, \gamma)_0 - \sum_{E' \in \mathscr{B}_l \setminus \mathscr{B}_{l+1}} (\Upsilon^{E'}, \gamma)_0 = \begin{cases} 0, & \text{if } F_l \text{ is a contact divisor;} \\ (1 - n^\gamma_{t(l)})\bar{\nu}^\gamma_{t(l)}, & \text{if } F_l \text{ is a Puiseux divisor.} \end{cases}$$

To finish the proof we use the relationship between the characteristic exponents of $\gamma$ and the minimal system of generators of the semigroup $S(\gamma)$ given in equation (2). The following computations complete the proof:

$$\begin{aligned} \sum_{i=1}^{m} \left[ \sum_{i \in I_{F_l} \setminus I_{F_{l+1}}} (C_i, \gamma)_0 - \sum_{E' \in \mathscr{B}_l \setminus \mathscr{B}_{l+1}} (\Upsilon^{E'}, \gamma)_0 \right] &= \sum_{j=1}^{k_E} (1 - n^\gamma_j) \bar{\nu}^\gamma_j \\ = \bar{\nu}^\gamma_1 - n^\gamma_{k_E} \bar{\nu}^\gamma_{k_E} + \sum_{j=1}^{k_E - 1} (\bar{\nu}^\gamma_{j+1} - n^\gamma_j \bar{\nu}^\gamma_j) &= \bar{\nu}^\gamma_1 - n^\gamma_{k_E} \bar{\nu}^\gamma_{k_E} + \sum_{j=1}^{k_E - 1} (\nu^\gamma_{j+1} - \nu^\gamma_j) \\ = \bar{\nu}^\gamma_1 - n^\gamma_{k_E} \bar{\nu}^\gamma_{k_E} + \nu^\gamma_{k_E} - \nu^\gamma_1 &= \nu^\gamma_{k_E} - n^\gamma_{k_E} \bar{\nu}^\gamma_{k_E}. \end{aligned}$$

□

## 4. Base points of the polar pencil

Consider a morphism $\sigma : N \to (\mathbb{C}^2, 0)$ composition of a finite number of punctual blow-ups. A point $p \in \sigma^{-1}(0)$ is a *base point* of the pencil $\Lambda_{\mathscr{G}}$ if $p$ is an infinitely near point of each generic curve of $\Lambda_{\mathscr{G}}$. More precisely, $p$ is a base point of $\Lambda_{\mathscr{G}}$ if and only if, there is an irreducible component $\gamma$ of $\Gamma$ such that $\sigma^*\gamma \cap \sigma^{-1}(0) = \{p\}$, for each generic fiber $\Gamma$ of $\Lambda_{\mathcal{G}}$.

A first property concerning the resolution of singularities of the polar foliation, and hence of the polar pencil, is the property of "separation of the separatrices" (see [**12**]). Let $\Pi$ be a morphism which is a partial reduction of $\mathcal{P}_{\mathcal{G}}$ and also a reduction of singularities of $\mathcal{G}$. We say that $\mathcal{G}$ satisfies the *property of separation of the separatrices* if the geodesic in $G(\Pi)$ of any separatrix of $\mathcal{G}$ does not go through a dicritical component of $\mathcal{P}_{\mathcal{G}}$, except maybe $E_1$. We proved [**5**] that the foliations in $\mathbb{G}_C^*$ satisfy the property of separation of the separatrices. From this property we can deduce the following result:

***Lemma 10***. — *Consider a foliation $\mathcal{G} \in \mathbb{G}_C^*$ and take any generic curve $\Gamma$ of $\Lambda_{\mathcal{G}}$. If $E$ is a bifurcation divisor of $G(C)$, $E \neq E_1$, then the points $\pi_C^*\Gamma \cap E$ are base points of the polar pencil $\Lambda_{\mathcal{G}}$.*

*Proof*. — The result is a direct consequence of the property of separation of the separatrices since $E$ cannot be a dicritical component and hence the points of the set $\pi_C^*\Gamma \cap E$ are base points of $\Lambda_{\mathcal{G}}$. □

***Remark 11***. — Note that, if $E_1$ is a bifurcation divisor, the points $\pi_C^*\Gamma \cap E_1$ are not base points of the polar pencil. In fact, if $\Gamma = \Gamma_{[a:b]}$, then the set $\pi_C^*\Gamma_{[a:b]} \cap E_1$ has exactly $b_{E_1} - 1$ points which depend on $[a : b]$ (see [**7**]).

Let $\sigma_{\Lambda,C} : M_{\Lambda,C} \to (\mathbb{C}^2, 0)$ be the minimal reduction of singularities of $\Lambda_{\mathcal{G}}$ that factorizes by $\pi_C$. The next result describes how to construct $\sigma_{\Lambda,C}$ from $\pi_C$.

***Proposition 12***. — *Assume that $C$ is a curve with kind equisingularity type and let $\mathcal{G} \in \mathbb{G}_C^*$ be a Zariski-general foliation. There is a morphism $\sigma_1 : M_{\Lambda,C} \to M_C$ composition of a finite number of punctual blow-ups such that $\sigma_{\Lambda,C} = \pi_C \circ \sigma_1$. Moreover, the centers of the blow-ups to obtain $\sigma_1$ are not singular points of $\pi_C^*\mathcal{G}$.*

*Proof*. — Let $\Gamma, \Gamma'$ be two generic curves of $\Lambda_{\mathcal{G}}$. If the morphism $\pi_C$ is also a reduction of singularities of $\Gamma \cup \Gamma'$, we take $\sigma_1 : M_C \to M_C$ to be the identity map $id_{M_C}$ on $M_C$ and hence $\sigma_{\Lambda,C} = \pi_C$. Otherwise, let $\{R_1, \ldots, R_s\}$ be the points of the set $\pi_C^*\Gamma \cap \pi_C^{-1}(0)$; observe that these points are not singular points of $\pi_C^*\mathcal{G}$ since $\pi_C$ is a reduction of singularities of $C \cup \Gamma$. By theorem 5, there is a unique irreducible component $\gamma_i$ of $\Gamma$ such that $\pi_C^*\gamma_i$ cuts transversally $\pi_C^{-1}(0)$ at $R_i$ for $i = 1, \ldots, s$. Moreover, a point $R_i$ belongs either to a bifurcation divisor of $G(C)$ or to the terminal divisor of a dead arc in $G(C)$. There are three possible situations:

- If $R_i$ belongs to $E_1$, then $R_i$ is not a base point of $\Lambda_{\mathcal{G}}$ by remark 11.
- If $R_i$ belongs to a bifurcation divisor $E$, $E \neq E_1$, then $R_i$ is a base point of $\Lambda_{\mathcal{G}}$ by lemma 10. Hence, there is a unique irreducible component $\gamma_i'$ of $\Gamma'$ such that $\pi_C^*\gamma_i' \cap E = \{R_i\}$ by theorem 5.
- If $R_i$ belongs to the terminal divisor $E$ of a dead arc, then there is a unique irreducible component $\gamma_i'$ of $\Gamma'$ such that $\pi_C^*\gamma_i' \cap E \neq \varnothing$. In this case, the point

$R_i$ can be either a base point or not. If it is a base point, then $\pi_C^*\gamma_i' \cap E = \pi_C^*\gamma_i \cap E = \{R_i\}$. Otherwise, $\pi_C^*\gamma_i' \cap E \neq \{R_i\}$ and $E$ is a dicritical component for $\Lambda_{\mathcal{G}}$.

Put $X_1 = M_C$ and consider the morphism $\tau_i : (X_{i+1}, R_{i+1}) \to (X_i, R_i)$, for $i = 1, \dots, s$, defined by

- $\tau_i = id_{X_i}$ if $R_i$ is not a base point of $\Lambda_{\mathcal{G}}$;
- $\tau_i$ is the minimal reduction of singularities of the strict transform of $\pi_C^*\gamma_i \cup \pi_C^*\gamma_i'$ by $\tau_1 \circ \tau_2 \circ \cdots \circ \tau_{i-1}$ when $R_i$ is a base point of $\Lambda_{\mathcal{G}}$.

The morphism $\sigma_1 : X_{s+1} \to M_C$ with $\sigma_1 = \tau_1 \circ \cdots \circ \tau_s$ fulfils the requirements of the statement because $\pi_C \circ \sigma_1$ is a reduction of singularities of $\Gamma \cup \Gamma'$. Moreover, it is clear by construction that $\pi_C \circ \sigma_1$ is the minimal resolution of $\Lambda_{\mathcal{G}}$ which factorizes by $\pi_C$; hence $\sigma_{\Lambda,C} = \pi_C \circ \sigma_1 : M_{\Lambda,C} \to (\mathbb{C}^2, 0)$ with $M_{\Lambda,C} = X_{s+1}$. □

## 5. Dicritical components

In this section we give some characteristics of the dicritical components which appear in a resolution of singularities of $\Lambda_{\mathcal{G}}$. Note that the degree and the valence $v(D)$ of a dicritical component $D$ do not depend on the choice of the resolution. Hence to determine these values it is enough to consider the morphism $\sigma_{\Lambda,C} : M_{\Lambda,C} \to (\mathbb{C}^2, 0)$. Next lemma gives the degree of the dicritical components

***Lemma 13***. — *Consider a foliation $\mathcal{G} \in \mathbb{G}_C^*$ and let $\sigma : X \to (\mathbb{C}^2, 0)$ be any resolution of singularities of $\Lambda_{\mathcal{G}}$. Then*

1. *The divisor $E_1$ of $G(C)$ is dicritical for $\Lambda_{\mathcal{G}}$ if and only if $b_{E_1} \geq 2$. Moreover, in that case, the degree of $E_1$ as a dicritical component of $\Lambda_{\mathcal{G}}$ is equal to $b_{E_1} - 1$.*
2. *If $\mathcal{G}$ is a Zariski-general foliation, each dicritical component $D$ of $\sigma^{-1}(0)$, $D \neq E_1$, has degree equal to 1.*

*Proof*. — The first assertion is a direct consequence of remark 11. The second one follows straightforward from the construction of the morphism $\sigma_{\Lambda,C}$ given in proposition 12. □

Next result determines the valence $v(D)$ of a dicritical component $D$ of $\Lambda_{\mathcal{G}}$ in terms of the data in $G(C)$. It is a key result in the proof of theorem 1.

***Theorem 14***. — *Let $\mathcal{G} \in \mathbb{G}_C^*$ be a Zariski-general foliation and let $\sigma : X \to (\mathbb{C}^2, 0)$ be any resolution of singularities of the polar pencil $\Lambda_{\mathcal{G}}$. Given any dicritical component $D$ of $\sigma^{-1}(0)$ and any $D$-curvette $\gamma$, we have that*

$$v(D) = 2 \sup_{1 \leq i \leq r} \{\mathscr{C}(C_i, \gamma)\} - 1. \tag{6}$$

If $\Gamma, \Upsilon$ are two generic curves of $\Lambda_{\mathcal{G}}$, then $v(D)$ is equal to $\mathscr{C}(\gamma_D, \zeta_D)$ where $\gamma_D, \zeta_D$ are irreducible components of $\Gamma$ and $\Upsilon$ respectively such that $\sigma^*\gamma_D \cap D \neq \varnothing$ and $\sigma^*\zeta_D \cap D \neq \varnothing$. Moreover, if we denote by $E_D$ the bifurcation divisor of $G(C)$ such that $\gamma_D$ is a branch of the curve $\Gamma^{E_D}$ of the decomposition of $\Gamma$ (and also $\zeta_D \subset \Upsilon^{E_D}$), then $\sup_{1\le i\le r}\{\mathscr{C}(C_i,\gamma_D)\} = v(E_D)$. Consequently, equation (6) can be written as follows

$$v(D) = 2v(E_D) - 1. \tag{7}$$

*Proof of theorem 14.* — Consider two generic curves $\Gamma, \Upsilon$ of $\Lambda_{\mathscr{G}}$ with decompositions given by $\Gamma = \cup_{E\in B(C)}\Gamma^E$ and $\Upsilon = \cup_{E\in B(C)}\Upsilon^E$. Let $D$ be a dicritical component of $\sigma^{-1}(0)$. If $D$ is equal to the first divisor $E_1$ of $G(C)$, then $E_D = E_1$ and equation (6) is held. Assume now that $D \neq E_1$. Let $\gamma, \zeta$ be irreducible components of $\Gamma$ and $\Upsilon$ respectively, with $\sigma^*\gamma \cap D \neq \varnothing$ and $\sigma^*\zeta \cap D \neq \varnothing$; note that they are unique by lemma 13 and $m_0(\gamma) = m_0(\zeta)$. Let us compute $(\gamma,\zeta)_0$. By lemma 7, we have that

$$(\Upsilon^{E_D},\gamma)_0 + \sum_{\substack{E\in B(C)\\ E\neq E_D}} (\Upsilon^E,\gamma)_0 + m_0(\gamma) = \sum_{i=1}^{r} (C_i,\gamma)_0. \tag{8}$$

The intersection multiplicity $(\Upsilon^{E_D},\gamma)_0$ can be computed using the decomposition of $\Upsilon^{E_D}$ into irreducible components:

$$(\Upsilon^{E_D},\gamma)_0 = (\gamma,\zeta)_0 + \sum_{\substack{\zeta'\subset\Upsilon^{E_D}\\ \zeta'\neq\zeta}} (\zeta',\gamma)_0. \tag{9}$$

From equalities (8) and (9) we deduce that $(\gamma,\zeta)_0$ is given by

$$\begin{aligned}(\gamma,\zeta)_0 &= \sum_{i=1}^{r}(C_i,\gamma)_0 - \sum_{\substack{\zeta'\subset\Upsilon^{E_D}\\ \zeta'\neq\zeta}} (\zeta',\gamma)_0 - \sum_{\substack{E\in B(C)\\ E\neq E_D}} (\Upsilon^E,\gamma)_0 - m_0(\gamma)\\ &= \sum_{i\in I_{E_D}} (C_i,\gamma)_0 - \sum_{\substack{\zeta'\subset\Upsilon^{E_D}\\ \zeta'\neq\zeta}} (\zeta',\gamma)_0 + \sum_{i\notin I_{E_D}} (C_i,\gamma)_0 - \sum_{\substack{E\in B(C)\\ E\neq E_D}} (\Upsilon^E,\gamma)_0 - m_0(\gamma).\end{aligned}$$

Denote by $F_1 < F_2 < \cdots < F_m < F_{m+1} = E_D$ the bifurcation vertices in the geodesic of $E_D$ in $G(C)$ and put $\mathscr{B}_i = \{E' \in B(C) \,:\, E' \geq F_i\}$ for $i = 1,\ldots,m$. Thus we have that

$$\begin{aligned}(\gamma,\zeta)_0 = &\sum_{i\in I_{E_D}} (C_i,\gamma)_0 - \sum_{\substack{\zeta'\subset\Upsilon^{E_D}\\ \zeta'\neq\zeta}} (\zeta',\gamma)_0 - \sum_{\substack{E\in B(C)\\ E> E_D}} (\Upsilon^E,\gamma)_0\\ &+ \sum_{i=1}^{m}\left[\sum_{i\in I_{F_l}\setminus I_{F_{l+1}}} (C_i,\gamma)_0 - \sum_{E\in\mathscr{B}_l\setminus\mathscr{B}_{l+1}} (\Upsilon^E,\gamma)_0\right] - m_0(\gamma).\end{aligned}$$

We shall use lemmas 8 and 9 to compute the right side of the equality above. Note that

- $\mathscr{C}(C_i,\gamma) = v(E_D)$ for each $i \in I_{E_D}$, by theorem 4.
- $\mathscr{C}(\zeta',\gamma) = v(E_D)$ for each branch $\zeta'$ of $\Upsilon^E$, with $E > E_D$.
- $\mathscr{C}(\zeta',\gamma) = v(E_D)$ for each branch $\zeta'$ of $\Upsilon^{E_D}$, with $\zeta' \neq \zeta$, by theorem 5, since $\mathscr{G}$ is a Zariski-general foliation.

Let $\{\nu_0^\gamma, \nu_1^\gamma, \dots, \nu_{g_\gamma}^\gamma\}$ be the characteristic exponents of $\gamma$, $\{(m_i^\gamma, n_i^\gamma)\}_{i=1}^{g_\gamma}$ the Puiseux pairs of $\gamma$ and $\{\bar\nu_0^\gamma, \bar\nu_1^\gamma, \dots, \bar\nu_{g_\gamma}^\gamma\}$ the minimal set of generators of the semigroup $S(\gamma)$ of $\gamma$. From lemma 6, we deduce that $\nu_{g_\gamma}^\gamma \leq m_0(\gamma)v(E_D)$. Consequently, applying lemmas 2 and 8, we get that

$$\begin{aligned}\sum_{i\in I_{E_D}}(C_i,\gamma)_0 - \sum_{\substack{\zeta'\subset\Upsilon^{E_D}\\ \zeta'\neq\zeta}}(\zeta',\gamma)_0 - \sum_{\substack{E\in B(C)\\ E>E_D}}(\Upsilon^E,\gamma)_0 &=\\ &= \left(\sum_{i\in I_{E_D}} m_0(C_i) - \sum_{E>E_D} m_0(\Upsilon^E) - m_0(\Upsilon^{E_D}\setminus\zeta)\right)\frac{\bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)v(E_D) - \nu_{g_\gamma}^\gamma}{m_0(\gamma)}\\ &= (\underline{n}_{E_D} + m_0(\zeta))\frac{\bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)v(E_D) - \nu_{g_\gamma}^\gamma}{m_0(\gamma)}.\end{aligned}$$

We use now the equality above and the result given in lemma 9 to compute $(\gamma,\zeta)_0$. We obtain that

$$(\gamma,\zeta)_0 = ((\underline{n}_{E_D} + m_0(\zeta))\frac{\bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)v(E_D) - \nu_{g_\gamma}^\gamma}{m_0(\gamma)} + \nu_{k_{E_D}}^\gamma - n_{k_{E_D}}^\gamma \bar\nu_{k_{E_D}}^\gamma - m_0(\gamma).$$

To finish the computation of $(\gamma,\zeta)_0$ we consider the different possibilities for the bifurcation divisor $E_D$ and we use the expression of the characteristic exponents of the irreducible components of the generic curves of $\Lambda_{\mathcal{G}}$ given in lemma 6.

- If $E$ is a contact divisor, then $m_0(\gamma) = m_0(\zeta) = \underline{n}_{E_D} = n_1^\gamma \cdots n_{g_\gamma}^\gamma$ with $g_\gamma = k_{E_D}$. Then
$$\begin{aligned}(\gamma,\zeta)_0 &= 2[\bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)v(E_D) - \nu_{g_\gamma}^\gamma] + \nu_{g_\gamma}^\gamma - \bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma - m_0(\gamma)\\ &= 2m_0(\gamma)v(E_D) + \bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma - \nu_{g_\gamma}^\gamma - m_0(\gamma).\end{aligned}$$
Moreover, by lemma 2, the relationship between $(\gamma,\zeta)_0$ and $\mathcal{C}(\gamma,\zeta)_0$ is given by $(\gamma,\zeta)_0 = \bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)\mathcal{C}(\gamma,\zeta) - \nu_{g_\gamma}^\gamma$. Taking into account that $\mathcal{C}(\gamma,\zeta) = v(D)$, we conclude that
$$v(D) = 2v(E_D) - 1.$$
- Assume now that $E_D$ is a Puiseux divisor which belongs to a dead arc. By lemma 6, the multiplicity $m_0(\gamma)$ can be either $\underline{n}_{E_D}$ or $\underline{n}_{E_D} n_{E_D}$ with $n_{E_D} > 1$. If $m_0(\gamma) = \underline{n}_{E_D}$, the same computations as in the previous case give the result. Consider now the case $m_0(\gamma) = \underline{n}_{E_D} n_{E_D}$. Thus we have that $m_0(\gamma)v(E_D) = \nu_{g_\gamma}^\gamma$, $g_\gamma = k_{E_D} + 1$ and $n_{E_D} = n_{g_\gamma}^\gamma$. Hence we get that
$$\begin{aligned}(\gamma,\zeta)_0 &= [\underline{n}_{E_D} + \underline{n}_{E_D} n_{E_D}]\frac{\bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma}{\underline{n}_{E_D} n_{E_D}} + \nu_{g_\gamma-1}^\gamma - \bar\nu_{g_\gamma-1}^\gamma n_{g_\gamma-1}^\gamma - m_0(\gamma)\\ &= (1+n_{g_\gamma})\bar\nu_{g_\gamma}^\gamma + \nu_{g_\gamma}^\gamma - \bar\nu_{g_\gamma}^\gamma - m_0(\gamma) = n_{g_\gamma}^\gamma \bar\nu_{g_\gamma}^\gamma + \nu_{g_\gamma}^\gamma - m_0(\gamma).\end{aligned}$$
By lemma 2, we have that $(\gamma,\zeta)_0 = \bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)\mathcal{C}(\gamma,\zeta) - \nu_{g_\gamma}^\gamma$. We obtain that
$$\mathcal{C}(\gamma,\zeta) = 2\frac{\nu_{g_\gamma}^\gamma}{m_0(\gamma)} - 1 = 2v(E_D) - 1.$$

– If $E_D$ is a Puiseux divisor which does not belong to a dead arc, then $m_0(\gamma)v(E_D) = \nu^{\gamma}_{g_\gamma}$, $g_\gamma = k_{E_D} + 1$ and $n_{E_D} = n^{\gamma}_{g_\gamma}$. Hence the computations in the previous case give the result. $\square$

## 6. Resolution of singularities

In this section we give the proof of the main result of the paper and some consequences than can be deduced from it.

*Proof of theorem 1.* — In proposition 12 we have shown that $\sigma_{\Lambda,C}$ is obtained from $\pi_C$ by a finite number of punctual blow-ups with centers at non-singular points of $\pi_C^*\mathcal{F}$. Recall that $\sigma_{\Lambda,C} = \pi_C \circ \sigma_1$, where $\sigma_1$ is obtained by blowing-up following the infinitely near points of the irreducible components of a generic curve $\Gamma$ of $\Lambda_{\mathcal{G}}$. Moreover, since $\pi_C^*\Gamma$ is non-singular, then the centers of the blow-ups to get $\sigma_1$ are free infinitely near points of $\Gamma$.

Let $\{R_1, \dots, R_s\}$ be the points of the set $\pi_C^*\Gamma \cap \pi_C^{-1}(0)$. By theorem 5, there is a unique irreducible component $\gamma_i$ of $\Gamma$ such that $\pi_C^*\gamma_i$ cuts transversally $\pi_C^{-1}(0)$ at $R_i$ for $i = 1, \dots, s$. Let $D_i$ be the dicritical component of $\sigma_{\Lambda,C}^{-1}(0)$ such that $\sigma_{\Lambda,C}^*\gamma_i \cap D_i \neq \varnothing$ and denote by $E_{R_i}$ the irreducible component of $\pi_C^{-1}(0)$ such that $\pi_C^*\gamma_i \cap E_{R_i} = \{R_i\}$. Note that it is possible that $E_{R_i} = E_{R_j}$ for $i \neq j$. Moreover, $E_{R_i}$ is either a bifurcation divisor of $G(C)$ or the terminal divisor of a dead arc in $G(C)$.

Let $\alpha_i = \alpha_{E_{R_i}}$ be the number of blow-ups needed to obtain $D_i$ from $E_{R_i}$. Let us show that the value of $\alpha_i$ is given by equation (1). We consider separately the different possibilities for $E_{R_i}$:

– $E_{R_i}$ is the first divisor $E_1$ of $\pi_C^{-1}(0)$, then it is a dicritical component for $\Lambda_{\mathcal{G}}$. Hence, $\alpha_i = 0$ and the equality $\alpha_i = m(E_{R_i})(v(E_{R_i}) - 1)$ holds since $v(E_1) = 1$.

– $E_{R_i}$ is a bifurcation divisor different from $E_1$, then $R_i$ is a base point of $\Lambda_{\mathcal{G}}$. The valuation $v(D_i)$ is equal to

$$v(D_i) = \frac{m(E_{R_i})v(E_{R_i}) + \alpha_i}{m(E_{R_i})}.$$

By theorem 14, we have that $v(D_i) = 2v(E_{R_i}) - 1$. Hence, we deduce that $\alpha_i = m(E_{R_i})(v(E_{R_i}) - 1)$.

– $E_{R_i}$ is the terminal divisor of a dead arc with bifurcation divisor $E$. Using the fact that $C$ has a kind equisingularity type, we get that

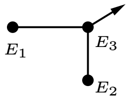
$$m(E_{R_i}) = m(E)/2; \quad v(E_{R_i}) = (m(E)v(E) + 1)/m(E). \tag{10}$$

By theorem 14, we have that $v(D_i) = 2v(E) - 1$. Thus we obtain the following equality

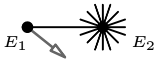
$$\frac{m(E_{R_i})v(E_{R_i}) + \alpha_i}{m(E_{R_i})} = \frac{2m(E_{R_i})v(E_{R_i}) - 1}{m(E_{R_i})} - 1,$$

and we conclude that $\alpha_i = m(E_{R_i})(v(E_{R_i}) - 1) - 1$. $\square$

Note that, in general, the minimal resolution of singularities $\sigma_\Lambda$ of $\Lambda_{\mathcal{F}}$ is not a reduction of singularities of the foliation $\mathcal{F}$. Consider, for instance, the foliation $\mathcal{F}$ given by $d(y^2-x^3)=0$. The generic curves of $\Lambda_{\mathcal{F}}$ are the parabolas $\{2by-3ax^2=0\}$; the minimal resolution of singularities $\sigma_\Lambda$ of $\Lambda_{\mathcal{F}}$ is a composition of two blow-ups whereas the separatrix of $\mathcal{F}$ is a $(3,2)$-cusp. The dual graphs $G(C)$ and $G(\sigma_\Lambda,\Lambda_{\mathcal{F}})$ are given by




$G(C)$




$G(\sigma_\Lambda,\Lambda_{\mathcal{F}})$

Next result characterizes the curves $C$ such that $\sigma_{\Lambda,C}$ coincides with the minimal reduction of singularities of $\Lambda_{\mathcal{F}}$.

***Corollary 15***. — *Let $C$ be a curve with kind equisingularity type and consider a Zariski-general foliation $\mathcal{F}\in\mathbb{G}_C^*$. The following statements are equivalent:*

1. *The morphism $\sigma_{\Lambda,C}$ is the minimal resolution of singularities of $\Lambda_{\mathcal{F}}$.*
2. *There is no maximal bifurcation divisor of $G(C)$ which belongs to the geodesic of only one irreducible component of $C$.*

*Proof*. — Let $\Gamma=\cup_{E\in B(C)}\Gamma^E$ be a generic curve of $\Lambda_{\mathcal{F}}$. Assume that $\sigma_{\Lambda,C}$ is the minimal resolution of singularities of $\Lambda_{\mathcal{F}}$. If there is a maximal bifurcation vertex $E$ of $G(C)$ which belongs to a dead arc and with $b_E=2$, then $\Gamma^E$ is irreducible and $\Gamma^E$ cuts the terminal divisor $F$ of the dead arc starting at $E$ (by theorem 5). Hence, $\pi_C$ is not the minimal reduction of singularities of $\Gamma$ and consequently $\sigma_{\Lambda,C}$ cannot be the minimal resolution of $\Lambda_{\mathcal{F}}$.

Assume now that $G(C)$ satisfies the conditions in the second statement. This implies that, for each maximal bifurcation divisor $E$ of $G(C)$, there is an irreducible component $\gamma$ of $\Gamma$ with $\pi_C^*\gamma\cap E\neq\varnothing$. If $E\neq E_1$, then $\pi_C^*\gamma\cap E$ is a base point of $\Lambda_{\mathcal{F}}$ and hence the minimal resolution of singularities of $\Lambda_{\mathcal{F}}$ factorizes by $\pi_C$. If $E=E_1$, then $\pi_C$ is a resolution of $\Lambda_{\mathcal{F}}$. We conclude that $\sigma_{\Lambda,C}$ is the minimal resolution of $\Lambda_{\mathcal{F}}$. □

Finally we characterize when a terminal divisor of a dead arc is a dicritical component for the pencil $\Lambda_{\mathcal{F}}$.

***Corollary 16***. — *Let $C$ be a curve with kind equisingularity type and consider a Zariski-general foliation $\mathcal{F}\in\mathbb{G}_C^*$. Let $F$ be terminal divisor of a dead arc in $G(C)$ starting at the bifurcation divisor $E$. The divisor $F$ is dicritical for $\Lambda_{\mathcal{F}}$ if and only if $v(E)=3/2$.*

*Proof*. — If $v(E)=3/2$, then $v(F)=2$ and $m(F)=1$ because $C$ has kind equisingularity type. Thus, by theorem 1, $\alpha(F)=0$ and hence $F$ is a dicritical component for $\Lambda_{\mathcal{F}}$.

Conversely, assume that $F$ is a dicritical divisor for $\Lambda_{\mathcal{G}}$ and then $v(F) = 1+1/m(F)$ by theorem 1. Since $C$ has a kind equisingularity type, the relationship between $v(F)$ and $v(E)$ is given by equation (10), thus $v(E) = 1 + 1/m(E)$.

Let $\{(m_l^i, n_l^i)\}_{l=1}^{g_i}$ be the Puiseux pairs of an irreducible component $C_i$ of $C$. We have that $m(E) = n_1^i \cdots n_{k_E}^i n_{k_E+1}^i$ and $v(E) = m_{k_E+1}^i / m(E)$ for $i \in I_E$ because $E$ is a Puiseux divisor. Consequently, the dicriticalness of $F$ implies that $m_{k_E+1}^i = 1 + n_1^i \cdots n_{k_E}^i n_{k_E+1}^i$. But

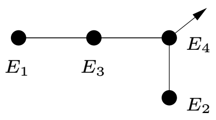
$$1 < \frac{m_{k_E}^i}{n_1^i \cdots n_{k_E}^i} < \frac{m_{k_E+1}^i}{n_1^i \cdots n_{k_E}^i n_{k_E+1}^i}$$

by the properties of the Puiseux pairs. This implies that $n_1^i \cdots n_{k_E}^i n_{k_E+1}^i < m_{k_E}^i n_{k_E+1}^i < m_{k_E+1}^i = 1 + n_1^i \cdots n_{k_E}^i n_{k_E+1}^i$. The previous inequalities hold only if $k_E = 0$, i.e., $m_{k_E}^i = 0$. Consequently $v(E) = (1 + n_1^i)/n_1^i$ and the result follows since $n_E = n_1^i = 2$. $\square$

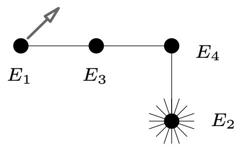
## 7. Examples

We illustrate here some different behaviours of a polar pencil $\Lambda_{\mathcal{G}}$ when $\mathcal{G}$ is not a Zariski-general foliation.

***Example 1***. — There can be dicritical components of $\Lambda_{\mathcal{G}}$ with degree $\geq 2$, which are different from $E_1$. Consider the foliation $\mathcal{G}$ given by $d(y^3 - x^5) = 0$; note that $C$ has not a kind equisingularity type. The pencil $\Lambda_{\mathcal{G}}$ has a dicritical component of degree 2 which corresponds to the terminal divisor $E_2$ of the unique dead arc in $G(C)$. In this case, $\pi_C$ gives a resolution of singularities of $\Lambda_{\mathcal{G}}$ but it is not the minimal resolution of $\Lambda_{\mathcal{G}}$.

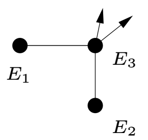



$G(C)$

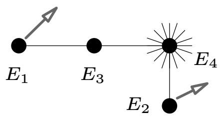



$G(\pi_C, \Lambda_{\mathcal{F}})$

***Example 2***. — Consider the foliation $\mathcal{G}$ given by $\omega = x^5 dx - y^3 dy = 0$. The minimal reduction of singularities $\pi_C$ of $\mathcal{G}$ is not a reduction of singularities of a generic fiber $\Gamma_{[a:b]} = \{ax^5 - by^3 = 0\}$. It is necessary to blow-up the corner $E_3 \cap E_2$ of $\pi_C^{-1}(0)$ to obtain an elimination of indeterminations $\sigma_\Lambda$ of $\Lambda_{\mathcal{G}}$; hence we need to blow-up a singular point of $\pi_C^* \mathcal{G}$.




$G(\mathcal{F})$




$G(\sigma_\Lambda, \Lambda_{\mathcal{F}})$

Notice that $v(E_4) = 5/3$ and $v(E_3) = 3/2$, thus equation (7) is not true for this foliation. In this example, the curve of separatrices $C$ has a kind equisingularity type but the foliation $\mathscr{F}$ is not Zariski-general.

## References

[1] H. Bresinsky – "Semigroups corresponding to algebroid branches in the plane", *Proc. Amer. Math. Soc.* **32** (1972), p. 381–384.

[2] C. Camacho, A. Lins Neto & P. Sad – "Topological invariants and equidesingularization for holomorphic vector fields", *J. Differential Geom.* **20** (1984), p. 143–174.

[3] C. Camacho & P. Sad – "Invariant varieties through singularities of holomorphic vector fields", *Ann. of Math.* **115** (1982), p. 579–595.

[4] E. Casas-Alvero – *Singularities of plane curves*, London Mathematical Society Lecture Note Series, vol. 276, Cambridge University Press, 2000.

[5] N. Corral – "Sur la topologie des courbes polaires de certains feuilletages singuliers", *Ann. Inst. Fourier (Grenoble)* **53** (2003), p. 787–814.

[6] ______, "Détermination du type d'équisingularité polaire", *C. R. Math. Acad. Sci. Paris* **344** (2007), p. 33–36.

[7] ______, "Infinitesimal adjunction and polar curves", submitted.

[8] F. Delgado & H. Maugendre – "Special fibres and critical locus for a pencil of plane curve singularities", *Compositio Math.* **136** (2003), p. 69–87.

[9] E. R. García Barroso – "Sur les courbes polaires d'une courbe plane réduite", *Proc. London Math. Soc.* **81** (2000), p. 1–28.

[10] T. C. Kuo & Y. C. Lu – "On analytic function germs of two complex variables", *Topology* **16** (1977), p. 299–310.

[11] M. Merle – "Invariants polaires des courbes planes", *Invent. Math.* **41** (1977), p. 103–111.

[12] P. Rouillé – "Courbes polaires et courbure", Thèse de doctorat, Université de Bourgogne, 1996.

[13] ______, "Théorème de Merle: cas des 1-formes de type courbes généralisées", *Bol. Soc. Brasil. Mat. (N.S.)* **30** (1999), p. 293–314.

[14] L. D. Trang & C. Weber – "Équisingularité dans les pinceaux de germes de courbes planes et $C^0$-suffisance", *Enseign. Math.* **43** (1997), p. 355–380.

[15] O. Zariski – "General theory of saturation and of saturated local rings. II. Saturated local rings of dimension 1", *Amer. J. Math.* **93** (1971), p. 872–964.

[16] ______, *Le problème des modules pour les branches planes*, École polytechnique, 1973, Cours donné au Centre de Mathématiques de l'École polytechnique, Paris, Octobre-Novembre 1973.

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# POLAR PENCIL OF CURVES AND FOLIATIONS

*by*

Nuria Corral

***Abstract.*** — The polar pencil $\Lambda_{\mathcal{F}}$ of a singular foliation $\mathcal{F}$ is the pencil of curves formed by the polar curves of $\mathcal{F}$. We study the relationship between the behaviour of $\Lambda_{\mathcal{F}}$ under blowing-up and the invariants associated to $\mathcal{F}$. The main result here describes a resolution of singularities of $\Lambda_{\mathcal{F}}$ in terms of the equireduction invariants of $\mathcal{F}$, for a Zariski-general foliation $\mathcal{F}$.

***Résumé* (Pinceau polaire de courbes et feuilletages).** — Le pinceau polaire $\Lambda_{\mathcal{F}}$ d'un feuilletage singulier $\mathcal{F}$ est le pinceau de courbes composé par les courbes polaires de $\mathcal{F}$. Nous allons étudier la relation entre le comportement de $\Lambda_{\mathcal{F}}$ par éclatement et les invariants associés à $\mathcal{F}$. Le résultat principal ici donne une description d'une résolution de singularités de $\Lambda_{\mathcal{F}}$ en termes des invariants d'équiréduction de $\mathcal{F}$ lorsque $\mathcal{F}$ est un feuilletage général de Zariski.

## 1. Introduction

Let $A, B$ be two germs of holomorphic functions at $(\mathbb{C}^2, 0)$ with no common component and consider the pencil of curves $\Lambda = \{aA + bB = 0 \ : \ a, b \in \mathbb{C}\}$. Classically, these pencils of curves have been studied in relation to the reduction of singularities of $A = 0$ and $B = 0$ (see for instance [**4, 8, 14**]). Here we propose a different approach: we consider $\Lambda$ as the *polar pencil* $\Lambda_{\mathcal{F}}$ associated to a singular foliation $\mathcal{F}$ defined by the 1-form $\omega = A(x, y)dx + B(x, y)dy$. Our objective is to describe properties of $\Lambda_{\mathcal{F}}$ in terms of the invariants associated to $\mathcal{F}$.

Let $\mathcal{G}_\omega$ be the *Gauss map* associated to $\mathcal{F}$ which is given by

$$\begin{array}{rccc} \mathcal{G}_\omega : & (\mathbb{C}^2, 0) \smallsetminus \{0\} & \longrightarrow & \mathbb{P}^1_{\mathbb{C}} \\ & (x, y) & \longmapsto & [-B(x, y) : A(x, y)]. \end{array}$$

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A curve $\Gamma_{[a:b]}$ of $\Lambda_{\mathcal{G}}$ is the closure in $(\mathbb{C}^2,0)$ of the fiber $\mathcal{G}_\omega^{-1}([a:b])$ for $[a:b]\in\mathbb{P}^1_{\mathbb{C}}$. There is a maximal non-empty Zariski open set of $\Omega\subset\mathbb{P}^1_{\mathbb{C}}$ such that all the curves $\Gamma_{[a:b]}$ with $[a:b]\in\Omega$ are equisingular: they are the *generic curves* of $\Lambda_{\mathcal{G}}$.

Let $\sigma: X\to(\mathbb{C}^2,0)$ be a finite sequence of punctual blow-ups. We say that $\sigma$ is an *elimination of indeterminations* of $\mathcal{G}_\omega$ (or *a resolution of singularities* of $\Lambda_{\mathcal{G}}$) iff the map $\tilde{\mathcal{G}}_\omega=\mathcal{G}_\omega\circ\sigma: X\to\mathbb{P}^1_{\mathbb{C}}$ is well-defined. Such $\sigma$ gives an embedded reduction of singularities of the union $\Gamma\cup\Gamma'$ of two different generic fibers, then $\sigma$ is a resolution of singularities of $\Lambda_{\mathcal{G}}$ (see [**14**]).

An irreducible component $D$ of $\sigma^{-1}(0)$ is called *dicritical* if the restriction $\tilde{\mathcal{G}}_\omega|_D: D\to\mathbb{P}^1_{\mathbb{C}}$ is not constant. The *degree* of a dicritical component $D$ is the degree of the map $\tilde{\mathcal{G}}_\omega|_D: D\to\mathbb{P}^1_{\mathbb{C}}$; this number coincides with the number of intersection points between $D$ and the strict transform $\sigma^*\Gamma$ of $\Gamma$ by $\sigma$, for any generic fiber $\Gamma$.

The curves of the polar pencil $\Lambda_{\mathcal{G}}$ can also be seen as the separatrices of a singular foliation: the *polar foliation* $\mathcal{P}_{\mathcal{G}}$ defined by $d(A/B)=0$. The minimal resolution $\sigma_\Lambda: X\to(\mathbb{C}^2,0)$ of $\Lambda_{\mathcal{G}}$ gives a *partial reduction* [**12**] of $\mathcal{P}_{\mathcal{G}}$ in the sense that the minimal reduction of singularities $\pi_{\mathcal{P}}:\mathfrak{X}\to(\mathbb{C}^2,0)$ of $\mathcal{P}_{\mathcal{G}}$ factorizes as $\pi_{\mathcal{P}}=\sigma_\Lambda\circ\tau$, where $\tau:\mathfrak{X}\to X$ is a finite sequence of punctual blow-ups which are non-dicritical for $\mathcal{P}_{\mathcal{G}}$.

Let $C\subset(\mathbb{C}^2,0)$ be a plane curve. We shall work in the space of foliations $\mathbb{G}_C$ of non-dicritical generalized curves over $C$ (see [**2**]). It is known that the minimal reduction of singularities $\pi_C: M_C\to(\mathbb{C}^2,0)$ of $C$ gives a reduction of singularities of any $\mathcal{G}\in\mathbb{G}_C$. But in general $\pi_C$ does not give a desingularization of a generic fiber $\Gamma$ of $\Lambda_{\mathcal{G}}$. This occurs essentially in the case that $C$ has a *kind equisingularity type* and $\mathcal{G}$ is Zariski-general (in the sense of the exponents of the logarithmic model) as we have shown in [**6, 7**].

Take $\mathcal{G}\in\mathbb{G}_C$ and let $\sigma_{\Lambda,C}: M_{\Lambda,C}\to(\mathbb{C}^2,0)$ be the minimal reduction of singularities of $\Lambda=\Lambda_{\mathcal{G}}$ that factorizes through $\pi_C$. The main result of this paper provides a precise description of $\sigma_{\Lambda,C}$ for kind singularities and Zariski-general foliations. Let us state it.

Let $G(C)$ be the dual graph of $C$ oriented by the first divisor $E_1$. For each divisor $E$, let $m(E)$ be the multiplicity of any $E$-"curvette" and $v(E)$ be the coincidence of two $E$-curvettes. Denote by $b_E$ the number of edges and arrows which leave from $E$. Thus $E$ is a *bifurcation divisor* if $b_E\geq 2$ and a *terminal divisor* if $b_E=0$. A *dead arc* joins a bifurcation divisor with a terminal divisor, with no other bifurcations. We say that the equisingularity type $\epsilon(C)$ of $C$ is *kind* if $m(E_b)=2m(E_t)$, for each dead arc of $G(C)$ starting at $E_b$ and ending at $E_t$.

The main result here can be stated as

***Theorem 1***. — *Let $C\subset(\mathbb{C}^2,0)$ be a plane curve with kind equisingularity type. Consider a Zariski-general foliation $\mathcal{G}\in\mathbb{G}^*_C$ and take any generic curve $\Gamma$ of $\Lambda_{\mathcal{G}}$. Then $\sigma_{\Lambda,C}$ is obtained from $\pi_C$ by blowing-up $\alpha_E$ times in a free way at each point $\pi_C^*\Gamma\cap E$*

*with*

$$(1) \qquad \alpha_E = \begin{cases} m(E)(v(E)-1), & \textit{if } E \textit{ is a bifurcation divisor;} \\ m(E)(v(E)-1)-1, & \textit{if } E \textit{ is the terminal divisor of a dead arc,} \end{cases}$$

*for each irreducible component $E$ of $\pi_C^{-1}(0)$. Moreover, the first divisor $E_1$ is dicritical for $\Lambda_{\mathcal{G}}$ if and only if $b_{E_1} > 1$, and the degree of $E_1$ as a dicritical component of $\Lambda_{\mathcal{G}}$ is equal to $b_{E_1} - 1$. The degree of the other dicritical components of $\Lambda_{\mathcal{G}}$ is equal to one.*

Observe that, under the hypothesis of theorem above, the points of the set $\pi_C^*\Gamma \cap \pi_C^{-1}(0)$ belong either to a bifurcation divisor or to the terminal divisor of a dead arc ([**6**]). Moreover, the points of $\pi_C^*\Gamma \cap \pi_C^{-1}(0)$ are non-singular points of $\pi_C^*\mathcal{G}$ and $\pi_C^*\Gamma$ cuts transversally $\pi_C^{-1}(0)$. Consequently $\sigma_{\Lambda,C} = \pi_C \circ \sigma_1$ where $\sigma_1$ is obtained by blowing-up free infinitely near points of $\pi_C^*\Gamma$, i.e., the centers of the blow-ups to obtain $\sigma_1$ are not corners of the corresponding exceptional divisor. Hence $\sigma_{\Lambda,C}$ is obtained from $\pi_C$ by "blowing-up in a free way" as it is stated in the theorem above.

The paper is organized as follows. Section 2 is devoted to introduce notations relative to the dual graph and the equisingularity data of a plane curve. In section 3 we remind some results concerning the generic fiber of the polar pencil and we also prove some technical lemmas. Section 4 deals with the base points of the pencil $\Lambda_{\mathcal{G}}$. In section 5 we state some results describing the dicritical components of a resolution of $\Lambda_{\mathcal{G}}$. The proof of the main result is given in section 6. We finish the paper with a list of examples showing different behaviours in the non Zariski-general cases.

## 2. Notation

In this section we introduce some notation concerning the dual graph and the equisingularity data of a plane curve $C = \cup_{i=1}^r C_i \subset (\mathbb{C}^2, 0)$ that will be used from now on. For each irreducible component $C_i$ of $C$, denote by $n^i = m_0(C_i)$ the multiplicity of $C_i$ at the origin. Let $y^i(x) = \sum_{j \geq n^i} a_j^i x^{j/n^i}$ be a Puiseux series of $C_i$, for $i = 1, \ldots, r$. The *characteristic exponents* $\{\beta_0^i, \beta_1^i, \ldots, \beta_{g_i}^i\}$ of $C_i$ are given by

$$\beta_0^i = m_0(C_i) = n^i$$
$$\beta_q^i = \min\{j \ : \ a_j^i \neq 0 \text{ and } j \not\equiv 0 \mod (\gcd(\beta_0^i, \ldots, \beta_{q-1}^i))\}$$

for $q = 1, \ldots, g_i$, where $g_i$ is the first integer such that $\gcd(\beta_0^i, \ldots, \beta_{g_i}^i) = 1$. Data equivalent to the characteristic exponents of $C_i$ are the *Puiseux pairs* $\{(m_k^i, n_k^i)\}_{k=1}^{g_i}$ of $C_i$ defined by

$$\gcd(m_k^i, n_k^i) = 1 \quad \text{and} \quad \frac{\beta_k^i}{n^i} = \frac{m_k^i}{n_1^i \cdots n_k^i} \quad \text{for} \quad k = 1, \ldots, g_i.$$

In particular, we have that $n^i = n_1^i \cdots n_{g_i}^i$ and $\beta_k^i = m_k^i n_{k+1}^i \cdots n_{g_i}^i$ for $k = 1, \ldots, g_i$.

Let us denote by $\pi_C : M_C \to (\mathbb{C}^2, 0)$ the minimal reduction of singularities of $C$. We recall briefly the construction of the *dual graph* $G(C) = G(\pi_C)$ of $C$. Each irreducible component $E$ of $\pi_C^{-1}(0)$ is represented by a vertex in $G(C)$. Two vertices are joined by an edge if their associated divisors intersect. An irreducible component of $C$ is represented by an arrow attached to the only divisor that it meets. The dual graph weighted with the self-intersection of each divisor $E \subset M_C$ determines the equisingularity type $\epsilon(C)$ of the curve $C$.

It is also possible to construct in a similar way the dual graph of a resolution of singularities of a pencil or a dicritical foliation by marking the dicritical components. If $\sigma$ is any finite sequence of blow-ups, we denote by $G(\sigma, \Lambda)$ the graph constructed from the transform of a pencil $\Lambda$ by $\sigma$.

Denote by $E_1$ the irreducible component of $\pi_C^{-1}(0)$ obtained after blowing-up the origin. The dual graph $G(C)$ is oriented by the first divisor $E_1$. The *geodesic* of a divisor $E$ is the path which joins $E_1$ with $E$ and the geodesic of a curve $C_i$ is the geodesic of the divisor that meets the strict transform $\pi_C^* C_i$ of $C_i$. Thus, there is a partial order in the set of vertices of $G(C)$ given by $E < E'$ if, and only if, the geodesic of $E'$ goes through $E$. Given a divisor $E$ of $G(C)$, we denote by $I_E$ the set of indices $i \in \{1, \dots, r\}$ such that $E$ belongs to the geodesic of $C_i$.

A *curvette* $\tilde{\gamma}$ of a divisor $E$ is a non-singular curve transversal to $E$ at a non-singular point of $\pi_C^{-1}(0)$. The projection $\gamma = \pi_C(\tilde{\gamma})$ is a germ of plane curve in $(\mathbb{C}^2, 0)$ and $\gamma$ is called an $E$-curvette. We denote by $m(E)$ the multiplicity at the origin of any $E$-curvette and by $v(E)$ the coincidence $\mathscr{C}(\gamma_E, \gamma'_E)$ of two $E$-curvettes $\gamma_E, \gamma'_E$ which cut $E$ in different points; observe that $v(E) < v(E')$ if $E < E'$. Recall that the *coincidence* $\mathscr{C}(\gamma, \delta)$ between two irreducible curves $\gamma$ and $\delta$ is defined as

$$\mathscr{C}(\gamma, \delta) = \sup_{\substack{1 \le i \le m_0(\gamma) \\ 1 \le j \le m_0(\delta)}} \{ \, ord_x(y_i^\gamma(x) - y_j^\delta(x)) \, \}$$

where $\{y_i^\gamma(x)\}_{i=1}^{m_0(\gamma)}$, $\{y_j^\delta(x)\}_{j=1}^{m_0(\delta)}$ are the Puiseux series of $\gamma$ and $\delta$ respectively.

Denote by $b_E$ the number of edges and arrows which leave from a divisor $E$ in $G(C)$. We say that $E$ is a *bifurcation divisor* if $b_E \ge 2$ and a *terminal divisor* if $b_E = 0$. A *dead arc* is a path which joins a bifurcation divisor with a terminal one, without passing through other bifurcation divisors. We denote by $B(C)$ the set of bifurcation vertices of $G(C)$.

Let $E$ be an irreducible component of the exceptional divisor $\pi_C^{-1}(0)$. The *reduction* $\pi_E : M_E \to (\mathbb{C}^2, 0)$ *of* $\pi_C$ *to* $E$ is the morphism satisfying that

- there is a factorization $\pi_C = \pi'_E \circ \pi_E$ where $\pi'_E$ and $\pi_E$ are composition of punctual blow-ups;
- the divisor $E$ is the strict transform by $\pi'_E$ of an irreducible component $E_{red}$ of $\pi_E^{-1}(0)$ and $E_{red} \subset M_E$ is the only component of $\pi_E^{-1}(0)$ with self-intersection equal to $-1$.

The morphism $\pi_E$ is obtained from $\pi_C$ by blowing-down successively the divisors different from $E$ with self-intersection is equal to $-1$. Given any curvette $\tilde{\gamma}_E$ of $E$, the curve $\pi'_E(\tilde{\gamma}_E)$ is also a curvette of $E_{red} \subset M_E$. Let $\{\beta_0^E, \beta_1^E, \dots, \beta_{g(E)}^E\}$ be the characteristic exponents of $\gamma_E = \pi_C(\tilde{\gamma}_E)$. It is clear that $m(E) = m_0(\gamma_E) = \beta_0^E$. If $E$ is a bifurcation divisor of $G(C)$, there are two possibilities for the value $v(E)$:

1. either $\pi_E$ is the minimal reduction of singularities of $\gamma_E$ and then $v(E) = \beta^E_{g(E)}/\beta^E_0$. We say that $E$ is a *Puiseux divisor* for $\pi_C$.
2. or $\pi_E$ is obtained by blowing-up $q \geq 1$ times after the minimal reduction of singularities of $\gamma_E$ and in this situation $v(E) = (\beta^E_{g(E)} + q)/\beta^E_0$. We say that $E$ is a *contact divisor* for $\pi_C$.

Observe that $m(E) = m(E_{red})$ and $v(E) = v(E_{red})$. Moreover, a bifurcation divisor $E$ can belong to a dead arc only if it is a Puiseux divisor.

Consider a bifurcation divisor $E$ of $G(C)$ and let $\{(m^E_1, n^E_1), (m^E_2, n^E_2), \dots, (m^E_{g(E)}, n^E_{g(E)})\}$ be the Puiseux pairs of an $E$-curvette $\gamma_E$, we denote

$$n_E = \begin{cases} n_{g(E)}, & \text{if } E \text{ is a Puiseux divisor;} \\ 1, & \text{otherwise,} \end{cases}$$

and $\underline{n}_E = m(E)/n_E$. Observe that, if $E$ is a bifurcation divisor which belongs to a dead arc with terminal divisor $F$, then $m(F) = \underline{n}_E$. We define $k_E$ to be

$$k_E = \begin{cases} g(E) - 1, & \text{if } E \text{ is a Puiseux divisor;} \\ g(E), & \text{if } E \text{ is a contact divisor.} \end{cases}$$

Thus, we have that $\underline{n}_E = n^E_1 \cdots n^E_{k_E}$.

To finish this section, we recall a lemma which gives the relationship between the intersection multiplicity $(\gamma, \delta)_0$ and the coincidence $\mathscr{C}(\gamma, \delta)$ (see Zariski [**15**], prop. 6.1 or Merle [**11**], prop. 2.4):

***Lemma 2***. — *Let $\gamma$ and $\delta$ be two germs of irreducible plane curves of $(\mathbb{C}^2, 0)$. If $\{\beta_0, \beta_1, \dots, \beta_g\}$ are the characteristic exponents of $\gamma$ and $\alpha$ is a rational number such that $\beta_q \leq \alpha < \beta_{q+1}$ ($\beta_{g+1} = \infty$), then the following statements are equivalent:*

1. $\mathscr{C}(\gamma, \delta) = \dfrac{\alpha}{m_0(\gamma)}$,
2. $\dfrac{(\gamma, \delta)_0}{m_0(\delta)} = \dfrac{\bar{\beta}_q}{n_1 \cdots n_{q-1}} + \dfrac{\alpha - \beta_q}{n_1 \cdots n_q}$,

*where $\{(m_i, n_i)\}_{i=1}^g$ are the Puiseux pairs of $\gamma$ ($n_0 = 1$) and $\{\bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_q\}$ is a minimal system of generators of the semigroup $S(\gamma)$ of $\gamma$.*

Recall that the *semigroup* $S(\gamma)$ of $\gamma$ is defined as

$$S(\gamma) = \{(\gamma, \delta)_0 : \gamma \text{ is not an irreducible component of } \delta\}.$$

There is a minimal system of generators $\{\bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_g\}$ of $S(\gamma)$ whose elements are defined by

$$\bar{\beta}_0 = \beta_0 = m_0(\gamma), \quad \bar{\beta}_1 = \beta_1, \quad \bar{\beta}_l = n_{l-1}\bar{\beta}_{l-1} + \beta_l - \beta_{l-1}, \quad \text{for } l = 2, \dots, g, \tag{2}$$

where $\{\beta_0, \beta_1, \dots, \beta_g\}$ are the characteristic exponents of $\gamma$ (see [**1**] or [**16**]). It is clear that $S(\gamma)$ is determined by the equisingularity type of $\gamma$ and reciprocally.

## 3. Generic curves of the pencil

This section is devoted to describe some properties of a generic curve of the polar pencil $\Lambda_{\mathcal{F}}$ of a singular foliation $\mathcal{F}$. The reader may refer to [**5, 7**] for a more detailed description.

Consider a plane curve $C = \cup_{i=1}^{r} C_i \subset (\mathbb{C}^2, 0)$. Let $f = f_1 \cdots f_r$ be a reduced equation of $C$ and $\pi_C : M_C \to (\mathbb{C}^2, 0)$ be the minimal reduction of singularities of $C$. Denote by $\mathbb{G}_C$ the space of generalized curve foliations [**2**] having $C$ as curve of separatrices. Let $\mathbb{G}_C^*$ be the sub-space of $\mathbb{G}_C$ defined as follows: a foliation $\mathcal{F}$ is in $\mathbb{G}_C^*$ iff the logarithmic model $\mathcal{L}_\lambda$ of $\mathcal{F}$ avoids a finite set of resonances $R_{\epsilon(C)} \subset (\mathbb{Z}_{\geq 0})^r$. More precisely, each foliation $\mathcal{F} \in \mathbb{G}_C$ has a unique logarithmic model $\mathcal{L}_\lambda$ given by $f_1 \cdots f_r \sum_{i=1}^{r} \lambda_i df_i/f_i = 0$ with $\lambda = \lambda(\mathcal{F}) = (\lambda_1, \ldots, \lambda_r) \in \mathbb{P}_{\mathbb{C}}^{r-1}$ (see [**5**]). The logarithmic foliation $\mathcal{L}_\lambda$ has the same reduction of singularities as $\mathcal{F}$ and the same Camacho-Sad indices [**3**] at the final points of the reduction. Thus, a foliation $\mathcal{F}$ belongs to $\mathbb{G}_C^*$ iff $\sum_{i=1}^{r} k_i \lambda_i \neq 0$ for each $k = (k_1, \ldots, k_r) \in R_{\epsilon(C)}$ where $R_{\epsilon(C)} \subset (\mathbb{Z}_{\geq 0})^r$ is a finite set which depends only on the equisingularity type $\epsilon(C)$ of $C$ (see [**5, 7**] for a detailed construction of it).

***Remark 3***. — Note that a foliation $\mathcal{F}$ avoids the resonances of the set $R_{\epsilon(C)}$ if and only if there is no corner in the reduction of singularities of $\rho^*\mathcal{F}$ with Camacho-Sad equal to $-1$, where $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is any ramification transversal to $C$ such that $\rho^{-1}C$ has only non-singular irreducible components (see [**5**]).

Consider a generic fiber $\Gamma$ of the pencil $\Lambda_{\mathcal{F}}$. A first result describing some properties of the equisingularity type $\epsilon(\Gamma)$ of $\Gamma$ in terms of the equisingularity type $\epsilon(C)$ of $C$ is the following one:

***Theorem 4* (of decomposition [5, 9, 10, 11])**. — *Consider a foliation $\mathcal{F} \in \mathbb{G}_C^*$ and a generic curve $\Gamma$ of $\Lambda_{\mathcal{F}}$. There is a decomposition $\Gamma = \cup_{E \in B(C)} \Gamma^E$ such that:*

(i) $m_0(\Gamma^E) = \begin{cases} \underline{n}_E n_E (b_E - 1), & \textit{if } E \textit{ does not belong to a dead arc;} \\ \underline{n}_E n_E (b_E - 1) - \underline{n}_E, & \textit{otherwise.} \end{cases}$

(ii) *For each irreducible component $\gamma$ of $\Gamma^E$ we have that*
- $\mathcal{C}(C_i, \gamma) = v(E)$ *if $E$ belongs to the geodesic of $C_i$;*
- $\mathcal{C}(C_j, \gamma) = \mathcal{C}(C_j, C_i)$ *if $E$ belongs to the geodesic of $C_i$ but not to the one of $C_j$.*

It is clear that the result above does not determine $\epsilon(\Gamma)$. However, there is a Zariski-open set $U_C \subset \mathbb{P}_{\mathbb{C}}^{r-1}$ such that $\epsilon(\Gamma)$ is completely determined by $\epsilon(C)$ if $\lambda(\mathcal{F}) \in U_C$. The set $U_C$ depends on the analytic type of $C$ and it is a non-empty set if, and only if, the curve $C$ has a kind equisingularity type. We say that a curve $C$ has *kind equisingularity type* if $m(E_b) = 2m(E_t)$ for each dead arc of $G(C)$ with bifurcation divisor $E_b$ and terminal divisor $E_t$. Using the notation introduced in section 2, the curve $C$ has a kind equisingularity type if and only if $n_{E_b} = 2$ for each bifurcation divisor $E_b$ of $G(C)$ which belongs to a dead arc since $m(E_b) = n_{E_b} m(E_t)$. In particular, this implies that each dead arc in $G(C)$ has only two vertices: the bifurcation divisor and the terminal divisor.

A foliation $\mathcal{F}$ is called *Zariski-general* when $\lambda(\mathcal{F}) \in U_C$ and in this case $\epsilon(\Gamma)$ is described as follows:

***Theorem* 5.** — **[6, 7]** *Let $C$ be a curve with kind equisingularity type and consider a Zariski-general foliation $\mathcal{F} \in \mathbb{G}_C^*$. If $\Gamma$ is a generic curve of the pencil $\Lambda_{\mathcal{F}}$, then $\pi_C$ gives a reduction of singularities of $\Gamma \cup C$. Moreover, the branches of $\Gamma$ intersect an irreducible component $E$ of the exceptional divisor $\pi_C^{-1}(0)$ as follows:*

- *If $E$ is a bifurcation divisor of $G(C)$, the number of branches of $\Gamma$ cutting $E$ equals to $b_E - 2$ if $E$ belongs to a dead arc and to $b_E - 1$ otherwise.*
- *If $E$ is a terminal divisor of a dead arc of $G(C)$, there is exactly one branch of $\Gamma$ through $E$.*
- *Otherwise, no branches of $\Gamma$ intersect $E$.*

In particular, the characteristic exponents of the branches of $\Gamma$ can be completely determined in terms of the equisingularity data of $C$. Denote by $\{\beta_0^i, \beta_1^i, \ldots, \beta_{g_i}^i\}$ the characteristic exponents of an irreducible component $C_i$ of $C$. Given a bifurcation divisor $E$ of $G(C)$, let $I_E^*$ be the set of indices $i \in I_E$ such that $v(E) = \beta_{k_E+1}^i / \beta_0^i$; note that if $i \in I_E \setminus I_E^*$ then there exists $j \in I_E$ such that $v(E) = \mathcal{C}(C_i, C_j)$. Hence, if $E$ is a contact divisor $I_E^* = \varnothing$. Moreover, if $C$ has a kind equisingularity type and $E$ is a bifurcation divisor belonging to a dear arc of $G(C)$, then the corresponding Puiseux pair $(m_{k_E+1}^i, n_{k_E+1}^i)$ satisfies $n_{k_E+1}^i = 2$ for each $i \in I_E = I_E^*$.

***Lemma* 6.** — **[7]** *Consider a curve $C$ with kind equisingularity type and a Zariski general foliation $\mathcal{F} \in \mathbb{G}_C^*$. Let $\Gamma$ be a generic curve of $\Lambda_{\mathcal{F}}$ with decomposition $\Gamma = \cup_{E \in B(C)} \Gamma^E$. Then, for each $E \in B(C)$, we have that*

(i) *If $E$ is a contact divisor, the curve $\Gamma^E$ has $b_E - 1$ irreducible components. Each branch $\gamma$ of $\Gamma^E$ with characteristic exponents $\{\nu_0^\gamma, \nu_1^\gamma, \ldots, \nu_{k_E}^\gamma\}$ given by*

$$\nu_0^\gamma = m_0(\gamma) = \underline{n}_E, \quad \nu_l^\gamma = \underline{n}_E \beta_l^i / \beta_0^i, \quad l = 1, 2, \ldots, k_E,$$

*for any $i \in I_E$.*

(ii) *If $E$ is a Puiseux divisor which belongs to a dead arc, the curve $\Gamma^E$ has one irreducible component $\gamma_0$ with characteristic exponents $\{\nu_0^{\gamma_0}, \nu_1^{\gamma_0}, \ldots, \nu_{k_E}^{\gamma_0}\}$ given by*

$$\nu_0^{\gamma_0} = m_0(\gamma_0) = \underline{n}_E, \quad \nu_l^{\gamma_0} = \underline{n}_E \beta_l^i / \beta_0^i, \quad l = 1, 2, \ldots, k_E,$$

*and $b_E - 2$ irreducible components such that each branch $\zeta \subset \Gamma^E \setminus \gamma_0$ has characteristic exponents $\{\nu_0^\gamma, \nu_1^\gamma, \ldots, \nu_{k_E}^\gamma, \nu_{k_E+1}^\gamma\}$ given by*

$$\nu_0^\gamma = m_0(\gamma) = \underline{n}_E n_E, \quad \nu_l^\gamma = \underline{n}_E n_E \beta_l^i / \beta_0^i, \quad l = 1, 2, \ldots, k_E + 1,$$

*for any $i \in I_E^*$.*

(iii) *If $E$ is a Puiseux divisor which does not belong to a dead arc, then $\Gamma^E$ has $b_E - 1$ irreducible components. Each irreducible component $\gamma$ of $\Gamma^E$ with characteristic exponents $\{\nu_0^\gamma, \nu_1^\gamma, \ldots, \nu_{k_E}^\gamma, \nu_{k_E+1}^\gamma\}$ given by*

$$\nu_0^\gamma = m_0(\gamma) = \underline{n}_E n_E, \quad \nu_l^\gamma = \underline{n}_E n_E \beta_l^i / \beta_0^i, \quad l = 1, 2, \ldots, k_E + 1,$$

*for any $i \in I_E^*$.*

The last part of the section is devoted to prove some technical lemmas which will be useful in the sequel. The first one is a general result concerning intersection multiplicities of polar curves:

***Lemma** 7.* — *Consider a foliation $\mathcal{F} \in \mathbb{G}_C$ and let $\Gamma, \Gamma'$ be any two generic curves of $\Lambda_{\mathcal{F}}$. For any irreducible component $\gamma$ of $\Gamma$, we have that*

$$(\Gamma', \gamma)_0 + m_0(\gamma) = (C, \gamma)_0. \tag{3}$$

*Proof.* — Consider a 1-form $\omega = A(x, y)dx + B(x, y)dy$ which defines $\mathcal{F}$ and assume that $\Gamma = \Gamma_{[a:b]}$, $\Gamma' = \Gamma_{[a':b']}$. Take an irreducible component $\gamma$ of $\Gamma_{[a:b]}$ and let $\phi_\gamma(t) = (x_\gamma(t), y_\gamma(t))$ be a parametrization of $\gamma$. Since $\mathcal{F}$ is a generalized curve foliation, then

$$(C, \gamma)_0 = \operatorname{ord}_t(\phi_\gamma^* \omega) + 1$$

(see **[13]**, lemma 3.7). The intersection multiplicity $(\Gamma_{[a':b']}, \gamma)_0$ is given by

$$(\Gamma_{[a':b']}, \gamma)_0 = \operatorname{ord}_t\{a' A(\phi_\gamma(t)) + b' B(\phi_\gamma(t))\}.$$

Moreover, since $\gamma$ is an irreducible component of $\Gamma_{[a:b]}$, then $aA(\phi_\gamma(t)) + bB(\phi_\gamma(t)) \equiv 0$. Assume that $a \neq 0$, a similar argument holds if $b \neq 0$. In this case, we have that either $\operatorname{ord}_t(A(\phi_\gamma(t))) = \operatorname{ord}_t(B(\phi_\gamma(t)))$ when $b \neq 0$ or $A(\phi_\gamma(t)) \equiv 0$ otherwise. In both situations, the following equalities to compute $\operatorname{ord}_t(\phi_\gamma^* \omega)$ hold:

$$\begin{aligned}
\operatorname{ord}_t(\phi_\gamma^* \omega) &= \operatorname{ord}_t\{A(\phi_\gamma(t))\, \dot{x}_\gamma(t) + B(\phi_\gamma(t))\, \dot{y}_\gamma(t)\} \\
&= \operatorname{ord}_t \left\{ -\frac{b}{a} B(\phi_\gamma(t))\, \dot{x}_\gamma(t) + B(\phi_\gamma(t))\, \dot{y}_\gamma(t) \right\} \\
&= \operatorname{ord}_t(B(\phi_\gamma(t))) + \operatorname{ord}_t(-b\dot{x}_\gamma(t) + a\dot{y}_\gamma(t)) \\
&= \operatorname{ord}_t(a' A(\phi_\gamma(t)) + b' B(\phi_\gamma(t))) + (\gamma, -bx + ay = 0)_0 - 1 \\
&= (\Gamma_{[a':b']}, \gamma)_0 + (\gamma, \ell_{[a:b]})_0 - 1,
\end{aligned}$$

where $\ell_{[a:b]}$ is the line given by $-bx + ay = 0$. In particular, this implies that the formula (3) holds for all $[a : b]$ such that $\ell_{[a:b]}$ is not tangent to $\Gamma_{[a:b]}$ which is the case when $\Gamma_{[a:b]}$ is a generic curve of $\Lambda_{\mathcal{F}}$. □

Let us introduce some notation in order to simplify the proofs of the following lemmas. Given a bifurcation divisor $E$ of $G(C)$, we denote

$$d_E^1 = \begin{cases} b_E & \text{if } E \text{ is a contact divisor;} \\ 1, & \text{if } E \text{ is a Puiseux divisor which does not belong to a dead arc;} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$d_E^2 = \begin{cases} 0, & \text{if } E \text{ is a contact divisor;} \\ b_E - 1, & \text{otherwise.} \end{cases}$$

Hence, if $\Gamma$ is a generic curve of $\Lambda_{\mathcal{G}}$ with decomposition $\Gamma = \cup_{E \in B(C)} \Gamma^E$, then $m_0(\Gamma^E) = \underline{n}_E(d_E^1 + d_E^2 n_E - 1)$.

***Lemma 8.*** — *Consider a foliation $\mathcal{G} \in \mathbb{G}_C^*$ and a generic curve $\Gamma = \cup_{E \in B(C)} \Gamma^E$ of $\Lambda_{\mathcal{G}}$. Then, for each bifurcation divisor $E$ of $G(C)$, we have that*

$$m_0(\bigcup_{i \in I_E} C_i) - m_0(\bigcup_{E' > E} \Gamma^{E'}) = \underline{n}_E(d_E^1 + n_E d_E^2). \tag{4}$$

*Proof.* — Let $\ell_E$ be the size of the largest chain of divisors in $B(C)$ starting at $E$. We prove the lemma by induction on $\ell_E$. If $\ell_E = 1$, then $E$ is a maximal bifurcation divisor of $G(C)$. In this case, the equality (4) turns into

$$m_0(\cup_{i \in I_E} C_i) = \underline{n}_E(d_E^1 + n_E d_E^2)$$

and it can be directly deduced from the properties of $G(C)$. Assume now that $\ell_E > 1$ and let $E_1, \ldots, E_s$ be the bifurcation vertices of $G(C)$ which are consecutive to $E$, that is, $E < E_i$ without any other bifurcation divisor between $E$ and $E_i$. Put $J_E = I_E \setminus \cup_{i=1}^s I_{E_i}$ and $t = \sharp J_E$. Note that $t + s = d_E^1 + d_E^2$. Then we have the following equalities

$$\begin{aligned}
m_0(\bigcup_{i \in I_E} C_i) - m_0(\bigcup_{E' > E} \Gamma^{E'}) &= \\
&= \sum_{j \in J_E} m_0(C_j) + \sum_{i=1}^{s} m_0(\bigcup_{j \in I_{E_i}} C_j) - \left[\sum_{i=1}^{s} m_0(\bigcup_{E' > E_i} \Gamma^{E'}) + \sum_{i=1}^{s} m_0(\Gamma^{E_i})\right] \\
&= \sum_{i \in J_E} m_0(C_i) + \sum_{i=1}^{s} \left[m_0(\bigcup_{j \in I_{E_i}} C_j) - m_0(\bigcup_{E' > E_i} \Gamma^{E'})\right] - \sum_{i=1}^{s} m_0(\Gamma^{E_i}).
\end{aligned}$$

For each $i = 1, \ldots, s$, we have that $m_0(\cup_{j \in I_{E_i}} C_j) - m_0(\cup_{E' > E_i} \Gamma^{E'}) = \underline{n}_{E_i}(d_{E_i}^1 + d_{E_i}^2 n_{E_i})$ by the induction hypothesis and $m_0(\Gamma^{E_i}) = \underline{n}_{E_i}(d_{E_i}^1 + d_{E_i}^2 n_{E_i} - 1)$ by theorem 4. Hence, we deduce that

$$m_0(\bigcup_{i \in I_E} C_i) - m_0(\bigcup_{E' > E} \Gamma^{E'}) = \sum_{j \in J_E} m_0(C_j) + \sum_{i=1}^{s} \underline{n}_{E_i}.$$

Now three situations may happen:

- If $E$ is a contact divisor, then $n_E = 1$, $\underline{n}_{E_i} = \underline{n}_E$ for $i = 1, \ldots, s$ and $m_0(C_j) = \underline{n}_E$ for $j \in J_E$. Moreover, $d_E^2 = 0$ and $t + s = d_E^1$.
- If $E$ is a Puiseux divisor which belongs to a dead arc, then $\underline{n}_{E_i} = \underline{n}_E n_E$ with $n_E > 1$ for each $i = 1, \ldots, s$ and $m_0(C_j) = \underline{n}_E n_E$ for $j \in J_E$. In this case, $d_E^1 = 0$ and $t + s = d_E^2$.

– If $E$ is a Puiseux divisor without dead arc, then $d_E^1 = 1$ and $t + s - 1 = d_E^2$. Moreover $n_E > 1$ and there is:
  - either a divisor $E_{i_0}$, with $i_0 \in \{1, \dots, s\}$, such that $\underline{n}_{E_{i_0}} = \underline{n}_E$ and $\underline{n}_{E_i} = \underline{n}_E n_E$ for $i \neq i_0$; in this situation $m_0(C_j) = \underline{n}_E n_E$ for all $j \in J_E$.
  - or a curve $C_{j_0}$ with $j_0 \in J_E$ such that $m_0(C_{j_0}) = \underline{n}_E$ and $m_0(C_j) = \underline{n}_E n_E$ if $j \neq j_0$; in this case $\underline{n}_{E_i} = \underline{n}_E n_E$ for all $i \in \{1, \dots, s\}$.

It follows that $\sum_{j \in J_E} m_0(C_j) + \sum_{i=1}^{s} \underline{n}_{E_i} = \underline{n}_E (d_E^1 + d_E^2 n_E)$ and the result is straightforward. □

Take a bifurcation divisor $E$ of $G(C)$. Let $F_1 < F_2 < \cdots < F_m < F_{m+1} = E$ be the bifurcation vertices in the geodesic of $E$ in $G(C)$ and denote $\mathscr{B}_i = \{E' \in B(C) : E' \geq F_i\}$. Then we have the following result

***Lemma 9.*** — *Consider a foliation $\mathscr{F} \in \mathbb{G}_C^*$ and let $\Gamma, \Upsilon$ be two generic curves of $\Lambda_{\mathscr{F}}$ with decompositions $\Gamma = \cup_{E \in B(C)} \Gamma^E$ and $\Upsilon = \cup_{E \in B(C)} \Upsilon^E$. Let $\gamma$ be an irreducible component of $\Gamma^E \subset \Gamma$. Denote by $\{\nu_0^\gamma, \nu_1^\gamma, \dots, \nu_{g_\gamma}^\gamma\}$ the characteristic exponents of $\gamma$, by $\{(m_i^\gamma, n_i^\gamma)\}_{i=1}^{g_\gamma}$ the Puiseux pairs of $\gamma$ and by $\{\bar{\nu}_0^\gamma, \bar{\nu}_1^\gamma, \dots, \bar{\nu}_{g_\gamma}^\gamma\}$ the minimal set of generators of the semigroup of $\gamma$. Then we have that*

$$\sum_{l=1}^{m} \left[ \sum_{i \in I_{F_l} \setminus I_{F_{l+1}}} (C_i, \gamma)_0 - \sum_{E' \in \mathscr{B}_l \setminus \mathscr{B}_{l+1}} (\Upsilon^{E'}, \gamma)_0 \right] = \nu_{k_E}^\gamma - n_{k_E}^\gamma \bar{\nu}_{k_E}^\gamma. \tag{5}$$

*Proof.* — By the properties of the decomposition of a generic curve of $\Lambda_{\mathscr{F}}$ given in theorem 4, we have that:

- $\mathscr{C}(C_i, \gamma) = v(F_l)$ if $i \in I_{F_l} \setminus I_{F_{l+1}}$;
- $\mathscr{C}(\zeta^{E'}, \gamma) = v(F_l)$ if $\zeta^{E'}$ is an irreducible component of $\Upsilon^{E'}$ with $E' \in \mathscr{B}_l \setminus \mathscr{B}_{l+1}$.

For each $l \in \{1, \dots, m\}$, let $t(l)$ be the integer in $\{0, 1, \dots, g_\gamma\}$ such that

$$\nu_{t(l)}^\gamma \leq m_0(\gamma) v(F_l) < \nu_{t(l)+1}^\gamma$$

($\nu_{g_\gamma+1}^\gamma = +\infty$). Note that $t(l) \leq k_E \leq g_\gamma$ for $l = 1, \dots, m$ and $t(m) = k_E$. We use now the relationship between the coincidence and the intersection multiplicity given in lemma 2 to compute $(C_i, \gamma)_0$ and $(\zeta^{E'}, \gamma)_0$. We have that

$$\frac{(C_i, \gamma)_0}{m_0(C_i)} = \frac{\bar{\nu}_{t(l)}^\gamma \cdot n_{t(l)}^\gamma + m_0(\gamma) v(F_l) - \nu_{t(l)}^\gamma}{n_1^\gamma \cdots n_{t(l)}^\gamma}, \text{ for } i \in I_{F_l} \setminus I_{F_{l+1}},$$

and

$$\frac{(\zeta^{E'}, \gamma)_0}{m_0(\zeta^{E'})} = \frac{\bar{\nu}_{t(l)}^\gamma \cdot n_{t(l)}^\gamma + m_0(\gamma) v(F_l) - \nu_{t(l)}^\gamma}{n_1^\gamma \cdots n_{t(l)}^\gamma},$$

for each irreducible component $\zeta^{E'}$ of $\Upsilon^{E'}$ with $E' \in \mathscr{B}_l \smallsetminus \mathscr{B}_{l+1}$, Consequently, we obtain that

$$\sum_{i\in I_{F_l}\smallsetminus I_{F_{l+1}}} (C_i,\gamma)_0 - \sum_{E'\in\mathscr{B}_l\smallsetminus\mathscr{B}_{l+1}} (\Upsilon^{E'},\gamma)_0 =$$
$$\left(\sum_{i\in I_{F_l}\smallsetminus I_{F_{l+1}}} m_0(C_i) - \sum_{E'\in\mathscr{B}_l\smallsetminus\mathscr{B}_{l+1}} m_0(\Upsilon^{E'})\right)\frac{\bar\nu^\gamma_{t(l)}\cdot n^\gamma_{t(l)} + m_0(\gamma)v(F_l) - \nu^\gamma_{t(l)}}{n^\gamma_1\cdots n^\gamma_{t(l)}}.$$

By lemma 8, we have that

$$\sum_{i\in I_{F_l}} m_0(C_i) - \sum_{E'\in\mathscr{B}_l} m_0(\Upsilon^{E'}) = \underline{n}_{F_l}(d^1_{F_l} + d^2_{F_l}n_{F_l}) - m_0(\Upsilon^{F_l}) = \underline{n}_{F_l},$$

and hence it follows that

$$\sum_{i\in I_{F_l}\smallsetminus I_{F_{l+1}}} m_0(C_i) - \sum_{E'\in\mathscr{B}_l\smallsetminus\mathscr{B}_{l+1}} m_0(\Upsilon^{E'}) = \underline{n}_{F_l} - \underline{n}_{F_{l+1}} = \underline{n}_{F_l}(1-n_{F_l}).$$

By definition $n_{F_l}$ is given by

$$n_{F_l} = \begin{cases} 1, & \text{if } F_l \text{ is a contact divisor;}\\ n^\gamma_{t(l)}, & \text{if } F_l \text{ is a Puiseux divisor.}\end{cases}$$

Moreover, $m_0(\gamma)v(F_l) = \nu^\gamma_{t(l)}$ and $\underline{n}_{F_l} = n_1\cdots n_{t(l)-1}$ if $F_l$ is a Puiseux divisor. Therefore, we deduce that

$$\sum_{i\in I_{F_l}\smallsetminus I_{F_{l+1}}} (C_i,\gamma)_0 - \sum_{E'\in\mathscr{B}_l\smallsetminus\mathscr{B}_{l+1}} (\Upsilon^{E'},\gamma)_0 = \begin{cases} 0, & \text{if } F_l \text{ is a contact divisor;}\\ (1-n^\gamma_{t(l)})\bar\nu^\gamma_{t(l)}, & \text{if } F_l \text{ is a Puiseux divisor.}\end{cases}$$

To finish the proof we use the relationship between the characteristic exponents of $\gamma$ and the minimal system of generators of the semigroup $S(\gamma)$ given in equation (2). The following computations complete the proof:

$$\begin{aligned}\sum_{i=1}^{m}\left[\sum_{i\in I_{F_l}\smallsetminus I_{F_{l+1}}} (C_i,\gamma)_0 - \sum_{E'\in\mathscr{B}_l\smallsetminus\mathscr{B}_{l+1}} (\Upsilon^{E'},\gamma)_0\right] &= \sum_{j=1}^{k_E}(1-n^\gamma_j)\bar\nu^\gamma_j\\ = \bar\nu^\gamma_1 - n^\gamma_{k_E}\bar\nu^\gamma_{k_E} + \sum_{j=1}^{k_E-1}(\bar\nu^\gamma_{j+1} - n^\gamma_j\bar\nu^\gamma_j) &= \bar\nu^\gamma_1 - n^\gamma_{k_E}\bar\nu^\gamma_{k_E} + \sum_{j=1}^{k_E-1}(\nu^\gamma_{j+1}-\nu^\gamma_j)\\ = \bar\nu^\gamma_1 - n^\gamma_{k_E}\bar\nu^\gamma_{k_E} + \nu^\gamma_{k_E} - \nu^\gamma_1 &= \nu^\gamma_{k_E} - n^\gamma_{k_E}\bar\nu^\gamma_{k_E}.\end{aligned}$$ □

## 4. Base points of the polar pencil

Consider a morphism $\sigma : N \to (\mathbb{C}^2, 0)$ composition of a finite number of punctual blow-ups. A point $p \in \sigma^{-1}(0)$ is a *base point* of the pencil $\Lambda_{\mathscr{G}}$ if $p$ is an infinitely near point of each generic curve of $\Lambda_{\mathscr{G}}$. More precisely, $p$ is a base point of $\Lambda_{\mathscr{G}}$ if and only if, there is an irreducible component $\gamma$ of $\Gamma$ such that $\sigma^*\gamma \cap \sigma^{-1}(0) = \{p\}$, for each generic fiber $\Gamma$ of $\Lambda_{\mathcal{F}}$.

A first property concerning the resolution of singularities of the polar foliation, and hence of the polar pencil, is the property of "separation of the separatrices" (see [**12**]). Let $\Pi$ be a morphism which is a partial reduction of $\mathcal{P}_{\mathcal{F}}$ and also a reduction of singularities of $\mathcal{F}$. We say that $\mathcal{F}$ satisfies the *property of separation of the separatrices* if the geodesic in $G(\Pi)$ of any separatrix of $\mathcal{F}$ does not go through a dicritical component of $\mathcal{P}_{\mathcal{F}}$, except maybe $E_1$. We proved [**5**] that the foliations in $\mathbb{G}_C^*$ satisfy the property of separation of the separatrices. From this property we can deduce the following result:

***Lemma 10***. — *Consider a foliation $\mathcal{F} \in \mathbb{G}_C^*$ and take any generic curve $\Gamma$ of $\Lambda_{\mathcal{F}}$. If $E$ is a bifurcation divisor of $G(C)$, $E \neq E_1$, then the points $\pi_C^*\Gamma \cap E$ are base points of the polar pencil $\Lambda_{\mathcal{F}}$.*

*Proof*. — The result is a direct consequence of the property of separation of the separatrices since $E$ cannot be a dicritical component and hence the points of the set $\pi_C^*\Gamma \cap E$ are base points of $\Lambda_{\mathcal{F}}$. □

***Remark 11***. — Note that, if $E_1$ is a bifurcation divisor, the points $\pi_C^*\Gamma \cap E_1$ are not base points of the polar pencil. In fact, if $\Gamma = \Gamma_{[a:b]}$, then the set $\pi_C^*\Gamma_{[a:b]} \cap E_1$ has exactly $b_{E_1} - 1$ points which depend on $[a : b]$ (see [**7**]).

Let $\sigma_{\Lambda,C} : M_{\Lambda,C} \to (\mathbb{C}^2, 0)$ be the minimal reduction of singularities of $\Lambda_{\mathcal{F}}$ that factorizes by $\pi_C$. The next result describes how to construct $\sigma_{\Lambda,C}$ from $\pi_C$.

***Proposition 12***. — *Assume that $C$ is a curve with kind equisingularity type and let $\mathcal{F} \in \mathbb{G}_C^*$ be a Zariski-general foliation. There is a morphism $\sigma_1 : M_{\Lambda,C} \to M_C$ composition of a finite number of punctual blow-ups such that $\sigma_{\Lambda,C} = \pi_C \circ \sigma_1$. Moreover, the centers of the blow-ups to obtain $\sigma_1$ are not singular points of $\pi_C^*\mathcal{F}$.*

*Proof*. — Let $\Gamma, \Gamma'$ be two generic curves of $\Lambda_{\mathcal{F}}$. If the morphism $\pi_C$ is also a reduction of singularities of $\Gamma \cup \Gamma'$, we take $\sigma_1 : M_C \to M_C$ to be the identity map $id_{M_C}$ on $M_C$ and hence $\sigma_{\Lambda,C} = \pi_C$. Otherwise, let $\{R_1, \ldots, R_s\}$ be the points of the set $\pi_C^*\Gamma \cap \pi_C^{-1}(0)$; observe that these points are not singular points of $\pi_C^*\mathcal{F}$ since $\pi_C$ is a reduction of singularities of $C \cup \Gamma$. By theorem 5, there is a unique irreducible component $\gamma_i$ of $\Gamma$ such that $\pi_C^*\gamma_i$ cuts transversally $\pi_C^{-1}(0)$ at $R_i$ for $i = 1, \ldots, s$. Moreover, a point $R_i$ belongs either to a bifurcation divisor of $G(C)$ or to the terminal divisor of a dead arc in $G(C)$. There are three possible situations:

- If $R_i$ belongs to $E_1$, then $R_i$ is not a base point of $\Lambda_{\mathcal{F}}$ by remark 11.
- If $R_i$ belongs to a bifurcation divisor $E$, $E \neq E_1$, then $R_i$ is a base point of $\Lambda_{\mathcal{F}}$ by lemma 10. Hence, there is a unique irreducible component $\gamma_i'$ of $\Gamma'$ such that $\pi_C^*\gamma_i' \cap E = \{R_i\}$ by theorem 5.
- If $R_i$ belongs to the terminal divisor $E$ of a dead arc, then there is a unique irreducible component $\gamma_i'$ of $\Gamma'$ such that $\pi_C^*\gamma_i' \cap E \neq \varnothing$. In this case, the point

$R_i$ can be either a base point or not. If it is a base point, then $\pi_C^*\gamma_i' \cap E = \pi_C^*\gamma_i \cap E = \{R_i\}$. Otherwise, $\pi_C^*\gamma_i' \cap E \neq \{R_i\}$ and $E$ is a dicritical component for $\Lambda_{\mathcal{F}}$.

Put $X_1 = M_C$ and consider the morphism $\tau_i : (X_{i+1}, R_{i+1}) \to (X_i, R_i)$, for $i = 1, \dots, s$, defined by

- $\tau_i = id_{X_i}$ if $R_i$ is not a base point of $\Lambda_{\mathcal{F}}$;
- $\tau_i$ is the minimal reduction of singularities of the strict transform of $\pi_C^*\gamma_i \cup \pi_C^*\gamma_i'$ by $\tau_1 \circ \tau_2 \circ \cdots \circ \tau_{i-1}$ when $R_i$ is a base point of $\Lambda_{\mathcal{F}}$.

The morphism $\sigma_1 : X_{s+1} \to M_C$ with $\sigma_1 = \tau_1 \circ \cdots \circ \tau_s$ fulfils the requirements of the statement because $\pi_C \circ \sigma_1$ is a reduction of singularities of $\Gamma \cup \Gamma'$. Moreover, it is clear by construction that $\pi_C \circ \sigma_1$ is the minimal resolution of $\Lambda_{\mathcal{F}}$ which factorizes by $\pi_C$; hence $\sigma_{\Lambda,C} = \pi_C \circ \sigma_1 : M_{\Lambda,C} \to (\mathbb{C}^2, 0)$ with $M_{\Lambda,C} = X_{s+1}$. □

## 5. Dicritical components

In this section we give some characteristics of the dicritical components which appear in a resolution of singularities of $\Lambda_{\mathcal{F}}$. Note that the degree and the valence $v(D)$ of a dicritical component $D$ do not depend on the choice of the resolution. Hence to determine these values it is enough to consider the morphism $\sigma_{\Lambda,C} : M_{\Lambda,C} \to (\mathbb{C}^2, 0)$. Next lemma gives the degree of the dicritical components

***Lemma 13***. — *Consider a foliation $\mathcal{F} \in \mathbb{G}_C^*$ and let $\sigma : X \to (\mathbb{C}^2, 0)$ be any resolution of singularities of $\Lambda_{\mathcal{F}}$. Then*

1. *The divisor $E_1$ of $G(C)$ is dicritical for $\Lambda_{\mathcal{F}}$ if and only if $b_{E_1} \geq 2$. Moreover, in that case, the degree of $E_1$ as a dicritical component of $\Lambda_{\mathcal{F}}$ is equal to $b_{E_1} - 1$.*
2. *If $\mathcal{F}$ is a Zariski-general foliation, each dicritical component $D$ of $\sigma^{-1}(0)$, $D \neq E_1$, has degree equal to 1.*

*Proof.* — The first assertion is a direct consequence of remark 11. The second one follows straightforward from the construction of the morphism $\sigma_{\Lambda,C}$ given in proposition 12. □

Next result determines the valence $v(D)$ of a dicritical component $D$ of $\Lambda_{\mathcal{F}}$ in terms of the data in $G(C)$. It is a key result in the proof of theorem 1.

***Theorem 14***. — *Let $\mathcal{F} \in \mathbb{G}_C^*$ be a Zariski-general foliation and let $\sigma : X \to (\mathbb{C}^2, 0)$ be any resolution of singularities of the polar pencil $\Lambda_{\mathcal{F}}$. Given any dicritical component $D$ of $\sigma^{-1}(0)$ and any $D$-curvette $\gamma$, we have that*

$$v(D) = 2 \sup_{1 \leq i \leq r} \{\mathcal{C}(C_i, \gamma)\} - 1. \tag{6}$$

If $\Gamma, \Upsilon$ are two generic curves of $\Lambda_{\mathcal{F}}$, then $v(D)$ is equal to $\mathcal{C}(\gamma_D, \zeta_D)$ where $\gamma_D, \zeta_D$ are irreducible components of $\Gamma$ and $\Upsilon$ respectively such that $\sigma^*\gamma_D \cap D \neq \varnothing$ and $\sigma^*\zeta_D \cap D \neq \varnothing$. Moreover, if we denote by $E_D$ the bifurcation divisor of $G(C)$ such that $\gamma_D$ is a branch of the curve $\Gamma^{E_D}$ of the decomposition of $\Gamma$ (and also $\zeta_D \subset \Upsilon^{E_D}$), then $\sup_{1\le i\le r}\{\mathscr{C}(C_i,\gamma_D)\} = v(E_D)$. Consequently, equation (6) can be written as follows

$$v(D) = 2v(E_D) - 1. \tag{7}$$

*Proof of theorem 14.* — Consider two generic curves $\Gamma, \Upsilon$ of $\Lambda_{\mathscr{G}}$ with decompositions given by $\Gamma = \cup_{E\in B(C)}\Gamma^E$ and $\Upsilon = \cup_{E\in B(C)}\Upsilon^E$. Let $D$ be a dicritical component of $\sigma^{-1}(0)$. If $D$ is equal to the first divisor $E_1$ of $G(C)$, then $E_D = E_1$ and equation (6) is held. Assume now that $D \neq E_1$. Let $\gamma, \zeta$ be irreducible components of $\Gamma$ and $\Upsilon$ respectively, with $\sigma^*\gamma \cap D \neq \varnothing$ and $\sigma^*\zeta \cap D \neq \varnothing$; note that they are unique by lemma 13 and $m_0(\gamma) = m_0(\zeta)$. Let us compute $(\gamma,\zeta)_0$. By lemma 7, we have that

$$(\Upsilon^{E_D},\gamma)_0 + \sum_{\substack{E\in B(C)\\ E\neq E_D}} (\Upsilon^E,\gamma)_0 + m_0(\gamma) = \sum_{i=1}^{r}(C_i,\gamma)_0. \tag{8}$$

The intersection multiplicity $(\Upsilon^{E_D},\gamma)_0$ can be computed using the decomposition of $\Upsilon^{E_D}$ into irreducible components:

$$(\Upsilon^{E_D},\gamma)_0 = (\gamma,\zeta)_0 + \sum_{\substack{\zeta'\subset\Upsilon^{E_D}\\ \zeta'\neq\zeta}} (\zeta',\gamma)_0. \tag{9}$$

From equalities (8) and (9) we deduce that $(\gamma,\zeta)_0$ is given by

$$\begin{aligned}(\gamma,\zeta)_0 &= \sum_{i=1}^{r}(C_i,\gamma)_0 - \sum_{\substack{\zeta'\subset\Upsilon^{E_D}\\ \zeta'\neq\zeta}} (\zeta',\gamma)_0 - \sum_{\substack{E\in B(C)\\ E\neq E_D}} (\Upsilon^E,\gamma)_0 - m_0(\gamma)\\ &= \sum_{i\in I_{E_D}} (C_i,\gamma)_0 - \sum_{\substack{\zeta'\subset\Upsilon^{E_D}\\ \zeta'\neq\zeta}} (\zeta',\gamma)_0 + \sum_{i\notin I_{E_D}} (C_i,\gamma)_0 - \sum_{\substack{E\in B(C)\\ E\neq E_D}} (\Upsilon^E,\gamma)_0 - m_0(\gamma).\end{aligned}$$

Denote by $F_1 < F_2 < \cdots < F_m < F_{m+1} = E_D$ the bifurcation vertices in the geodesic of $E_D$ in $G(C)$ and put $\mathscr{B}_i = \{E' \in B(C) \ : \ E' \geq F_i\}$ for $i = 1,\ldots,m$. Thus we have that

$$\begin{aligned}(\gamma,\zeta)_0 = &\sum_{i\in I_{E_D}} (C_i,\gamma)_0 - \sum_{\substack{\zeta'\subset\Upsilon^{E_D}\\ \zeta'\neq\zeta}} (\zeta',\gamma)_0 - \sum_{\substack{E\in B(C)\\ E> E_D}} (\Upsilon^E,\gamma)_0\\ &+ \sum_{i=1}^{m}\left[\sum_{i\in I_{F_l}\setminus I_{F_{l+1}}} (C_i,\gamma)_0 - \sum_{E\in\mathscr{B}_l\setminus\mathscr{B}_{l+1}} (\Upsilon^E,\gamma)_0\right] - m_0(\gamma).\end{aligned}$$

We shall use lemmas 8 and 9 to compute the right side of the equality above. Note that

- $\mathscr{C}(C_i,\gamma) = v(E_D)$ for each $i \in I_{E_D}$, by theorem 4.
- $\mathscr{C}(\zeta',\gamma) = v(E_D)$ for each branch $\zeta'$ of $\Upsilon^E$, with $E > E_D$.
- $\mathscr{C}(\zeta',\gamma) = v(E_D)$ for each branch $\zeta'$ of $\Upsilon^{E_D}$, with $\zeta' \neq \zeta$, by theorem 5, since $\mathscr{G}$ is a Zariski-general foliation.

Let $\{\nu_0^\gamma, \nu_1^\gamma, \ldots, \nu_{g_\gamma}^\gamma\}$ be the characteristic exponents of $\gamma$, $\{(m_i^\gamma, n_i^\gamma)\}_{i=1}^{g_\gamma}$ the Puiseux pairs of $\gamma$ and $\{\bar{\nu}_0^\gamma, \bar{\nu}_1^\gamma, \ldots, \bar{\nu}_{g_\gamma}^\gamma\}$ the minimal set of generators of the semigroup $S(\gamma)$ of $\gamma$. From lemma 6, we deduce that $\nu_{g_\gamma}^\gamma \leq m_0(\gamma)v(E_D)$. Consequently, applying lemmas 2 and 8, we get that

$$\begin{aligned}\sum_{i\in I_{E_D}} (C_i,\gamma)_0 - \sum_{\substack{\zeta'\subset\Upsilon^{E_D}\\ \zeta'\neq\zeta}} (\zeta',\gamma)_0 - \sum_{\substack{E\in B(C)\\ E>E_D}} (\Upsilon^E,\gamma)_0 = \\ = \left(\sum_{i\in I_{E_D}} m_0(C_i) - \sum_{E>E_D} m_0(\Upsilon^E) - m_0(\Upsilon^{E_D}\setminus\zeta)\right)\frac{\bar{\nu}_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)v(E_D) - \nu_{g_\gamma}^\gamma}{m_0(\gamma)} \\ = (\underline{n}_{E_D} + m_0(\zeta))\frac{\bar{\nu}_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)v(E_D) - \nu_{g_\gamma}^\gamma}{m_0(\gamma)}.\end{aligned}$$

We use now the equality above and the result given in lemma 9 to compute $(\gamma,\zeta)_0$. We obtain that

$$(\gamma,\zeta)_0 = ((\underline{n}_{E_D} + m_0(\zeta))\frac{\bar{\nu}_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)v(E_D) - \nu_{g_\gamma}^\gamma}{m_0(\gamma)} + \nu_{k_{E_D}}^\gamma - n_{k_{E_D}}^\gamma \bar{\nu}_{k_{E_D}}^\gamma - m_0(\gamma).$$

To finish the computation of $(\gamma,\zeta)_0$ we consider the different possibilities for the bifurcation divisor $E_D$ and we use the expression of the characteristic exponents of the irreducible components of the generic curves of $\Lambda_{\mathcal{G}}$ given in lemma 6.

- If $E$ is a contact divisor, then $m_0(\gamma) = m_0(\zeta) = \underline{n}_{E_D} = n_1^\gamma \cdots n_{g_\gamma}^\gamma$ with $g_\gamma = k_{E_D}$. Then

$$\begin{aligned}(\gamma,\zeta)_0 &= 2[\bar{\nu}_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)v(E_D) - \nu_{g_\gamma}^\gamma] + \nu_{g_\gamma}^\gamma - \bar{\nu}_{g_\gamma}^\gamma n_{g_\gamma}^\gamma - m_0(\gamma)\\ &= 2m_0(\gamma)v(E_D) + \bar{\nu}_{g_\gamma}^\gamma n_{g_\gamma}^\gamma - \nu_{g_\gamma}^\gamma - m_0(\gamma).\end{aligned}$$

  Moreover, by lemma 2, the relationship between $(\gamma,\zeta)_0$ and $\mathscr{C}(\gamma,\zeta)_0$ is given by $(\gamma,\zeta)_0 = \bar{\nu}_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)\mathscr{C}(\gamma,\zeta) - \nu_{g_\gamma}^\gamma$. Taking into account that $\mathscr{C}(\gamma,\zeta) = v(D)$, we conclude that

$$v(D) = 2v(E_D) - 1.$$

- Assume now that $E_D$ is a Puiseux divisor which belongs to a dead arc. By lemma 6, the multiplicity $m_0(\gamma)$ can be either $\underline{n}_{E_D}$ or $\underline{n}_{E_D} n_{E_D}$ with $n_{E_D} > 1$. If $m_0(\gamma) = \underline{n}_{E_D}$, the same computations as in the previous case give the result. Consider now the case $m_0(\gamma) = \underline{n}_{E_D} n_{E_D}$. Thus we have that $m_0(\gamma)v(E_D) = \nu_{g_\gamma}^\gamma$, $g_\gamma = k_{E_D} + 1$ and $n_{E_D} = n_{g_\gamma}^\gamma$. Hence we get that

$$\begin{aligned}(\gamma,\zeta)_0 &= [\underline{n}_{E_D} + \underline{n}_{E_D} n_{E_D}]\frac{\bar{\nu}_{g_\gamma}^\gamma n_{g_\gamma}^\gamma}{\underline{n}_{E_D} n_{E_D}} + \nu_{g_\gamma-1}^\gamma - \bar{\nu}_{g_\gamma-1}^\gamma n_{g_\gamma-1}^\gamma - m_0(\gamma)\\ &= (1+n_{g_\gamma})\bar{\nu}_{g_\gamma}^\gamma + \nu_{g_\gamma}^\gamma - \bar{\nu}_{g_\gamma}^\gamma - m_0(\gamma) = n_{g_\gamma}^\gamma \bar{\nu}_{g_\gamma}^\gamma + \nu_{g_\gamma}^\gamma - m_0(\gamma).\end{aligned}$$

  By lemma 2, we have that $(\gamma,\zeta)_0 = \bar{\nu}_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)\mathscr{C}(\gamma,\zeta) - \nu_{g_\gamma}^\gamma$. We obtain that

$$\mathscr{C}(\gamma,\zeta) = 2\frac{\nu_{g_\gamma}^\gamma}{m_0(\gamma)} - 1 = 2v(E_D) - 1.$$

– If $E_D$ is a Puiseux divisor which does not belong to a dead arc, then $m_0(\gamma)v(E_D) = \nu^\gamma_{g_\gamma}$, $g_\gamma = k_{E_D} + 1$ and $n_{E_D} = n^\gamma_{g_\gamma}$. Hence the computations in the previous case give the result. □

## 6. Resolution of singularities

In this section we give the proof of the main result of the paper and some consequences than can be deduced from it.

*Proof of theorem 1.* — In proposition 12 we have shown that $\sigma_{\Lambda,C}$ is obtained from $\pi_C$ by a finite number of punctual blow-ups with centers at non-singular points of $\pi_C^*\mathcal{F}$. Recall that $\sigma_{\Lambda,C} = \pi_C \circ \sigma_1$, where $\sigma_1$ is obtained by blowing-up following the infinitely near points of the irreducible components of a generic curve $\Gamma$ of $\Lambda_{\mathcal{F}}$. Moreover, since $\pi_C^*\Gamma$ is non-singular, then the centers of the blow-ups to get $\sigma_1$ are free infinitely near points of $\Gamma$.

Let $\{R_1, \ldots, R_s\}$ be the points of the set $\pi_C^*\Gamma \cap \pi_C^{-1}(0)$. By theorem 5, there is a unique irreducible component $\gamma_i$ of $\Gamma$ such that $\pi_C^*\gamma_i$ cuts transversally $\pi_C^{-1}(0)$ at $R_i$ for $i = 1, \ldots, s$. Let $D_i$ be the dicritical component of $\sigma_{\Lambda,C}^{-1}(0)$ such that $\sigma_{\Lambda,C}^*\gamma_i \cap D_i \neq \varnothing$ and denote by $E_{R_i}$ the irreducible component of $\pi_C^{-1}(0)$ such that $\pi_C^*\gamma_i \cap E_{R_i} = \{R_i\}$. Note that it is possible that $E_{R_i} = E_{R_j}$ for $i \neq j$. Moreover, $E_{R_i}$ is either a bifurcation divisor of $G(C)$ or the terminal divisor of a dead arc in $G(C)$.

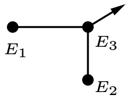
Let $\alpha_i = \alpha_{E_{R_i}}$ be the number of blow-ups needed to obtain $D_i$ from $E_{R_i}$. Let us show that the value of $\alpha_i$ is given by equation (1). We consider separately the different possibilities for $E_{R_i}$:

– $E_{R_i}$ is the first divisor $E_1$ of $\pi_C^{-1}(0)$, then it is a dicritical component for $\Lambda_{\mathcal{F}}$. Hence, $\alpha_i = 0$ and the equality $\alpha_i = m(E_{R_i})(v(E_{R_i}) - 1)$ holds since $v(E_1) = 1$.

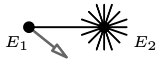
– $E_{R_i}$ is a bifurcation divisor different from $E_1$, then $R_i$ is a base point of $\Lambda_{\mathcal{F}}$. The valuation $v(D_i)$ is equal to
$$v(D_i) = \frac{m(E_{R_i})v(E_{R_i}) + \alpha_i}{m(E_{R_i})}.$$
By theorem 14, we have that $v(D_i) = 2v(E_{R_i}) - 1$. Hence, we deduce that $\alpha_i = m(E_{R_i})(v(E_{R_i}) - 1)$.

– $E_{R_i}$ is the terminal divisor of a dead arc with bifurcation divisor $E$. Using the fact that $C$ has a kind equisingularity type, we get that
$$m(E_{R_i}) = m(E)/2; \quad v(E_{R_i}) = (m(E)v(E) + 1)/m(E). \tag{10}$$
By theorem 14, we have that $v(D_i) = 2v(E) - 1$. Thus we obtain the following equality
$$\frac{m(E_{R_i})v(E_{R_i}) + \alpha_i}{m(E_{R_i})} = \frac{2m(E_{R_i})v(E_{R_i}) - 1}{m(E_{R_i})} - 1,$$
and we conclude that $\alpha_i = m(E_{R_i})(v(E_{R_i}) - 1) - 1$. □

Note that, in general, the minimal resolution of singularities $\sigma_\Lambda$ of $\Lambda_{\mathcal{F}}$ is not a reduction of singularities of the foliation $\mathcal{F}$. Consider, for instance, the foliation $\mathcal{F}$ given by $d(y^2 - x^3) = 0$. The generic curves of $\Lambda_{\mathcal{F}}$ are the parabolas $\{2by - 3ax^2 = 0\}$; the minimal resolution of singularities $\sigma_\Lambda$ of $\Lambda_{\mathcal{F}}$ is a composition of two blow-ups whereas the separatrix of $\mathcal{F}$ is a $(3,2)$-cusp. The dual graphs $G(C)$ and $G(\sigma_\Lambda, \Lambda_{\mathcal{F}})$ are given by




$G(C)$




$G(\sigma_\Lambda, \Lambda_{\mathcal{F}})$

Next result characterizes the curves $C$ such that $\sigma_{\Lambda,C}$ coincides with the minimal reduction of singularities of $\Lambda_{\mathcal{F}}$.

***Corollary 15.*** — *Let $C$ be a curve with kind equisingularity type and consider a Zariski-general foliation $\mathcal{F} \in \mathbb{G}^*_C$. The following statements are equivalent:*

1. *The morphism $\sigma_{\Lambda,C}$ is the minimal resolution of singularities of $\Lambda_{\mathcal{F}}$.*
2. *There is no maximal bifurcation divisor of $G(C)$ which belongs to the geodesic of only one irreducible component of $C$.*

*Proof.* — Let $\Gamma = \cup_{E \in B(C)} \Gamma^E$ be a generic curve of $\Lambda_{\mathcal{F}}$. Assume that $\sigma_{\Lambda,C}$ is the minimal resolution of singularities of $\Lambda_{\mathcal{F}}$. If there is a maximal bifurcation vertex $E$ of $G(C)$ which belongs to a dead arc and with $b_E = 2$, then $\Gamma^E$ is irreducible and $\Gamma^E$ cuts the terminal divisor $F$ of the dead arc starting at $E$ (by theorem 5). Hence, $\pi_C$ is not the minimal reduction of singularities of $\Gamma$ and consequently $\sigma_{\Lambda,C}$ cannot be the minimal resolution of $\Lambda_{\mathcal{F}}$.

Assume now that $G(C)$ satisfies the conditions in the second statement. This implies that, for each maximal bifurcation divisor $E$ of $G(C)$, there is an irreducible component $\gamma$ of $\Gamma$ with $\pi_C^*\gamma \cap E \neq \varnothing$. If $E \neq E_1$, then $\pi_C^*\gamma \cap E$ is a base point of $\Lambda_{\mathcal{F}}$ and hence the minimal resolution of singularities of $\Lambda_{\mathcal{F}}$ factorizes by $\pi_C$. If $E = E_1$, then $\pi_C$ is a resolution of $\Lambda_{\mathcal{F}}$. We conclude that $\sigma_{\Lambda,C}$ is the minimal resolution of $\Lambda_{\mathcal{F}}$. □

Finally we characterize when a terminal divisor of a dead arc is a dicritical component for the pencil $\Lambda_{\mathcal{F}}$.

***Corollary 16.*** — *Let $C$ be a curve with kind equisingularity type and consider a Zariski-general foliation $\mathcal{F} \in \mathbb{G}^*_C$. Let $F$ be terminal divisor of a dead arc in $G(C)$ starting at the bifurcation divisor $E$. The divisor $F$ is dicritical for $\Lambda_{\mathcal{F}}$ if and only if $v(E) = 3/2$.*

*Proof.* — If $v(E) = 3/2$, then $v(F) = 2$ and $m(F) = 1$ because $C$ has kind equisingularity type. Thus, by theorem 1, $\alpha(F) = 0$ and hence $F$ is a dicritical component for $\Lambda_{\mathcal{F}}$.

Conversely, assume that $F$ is a dicritical divisor for $\Lambda_{\mathcal{G}}$ and then $v(F)=1+1/m(F)$ by theorem 1. Since $C$ has a kind equisingularity type, the relationship between $v(F)$ and $v(E)$ is given by equation (10), thus $v(E)=1+1/m(E)$.

Let $\{(m_l^i, n_l^i)\}_{l=1}^{g_i}$ be the Puiseux pairs of an irreducible component $C_i$ of $C$. We have that $m(E)=n_1^i\cdots n_{k_E}^i n_{k_E+1}^i$ and $v(E)=m_{k_E+1}^i/m(E)$ for $i\in I_E$ because $E$ is a Puiseux divisor. Consequently, the dicriticalness of $F$ implies that $m_{k_E+1}^i = 1+n_1^i\cdots n_{k_E}^i n_{k_E+1}^i$. But

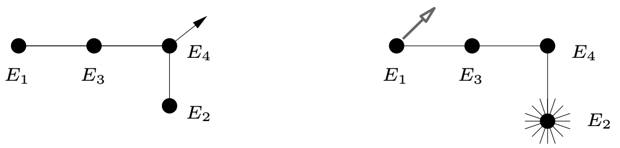
$$1<\frac{m_{k_E}^i}{n_1^i\cdots n_{k_E}^i}<\frac{m_{k_E+1}^i}{n_1^i\cdots n_{k_E}^i n_{k_E+1}^i}$$

by the properties of the Puiseux pairs. This implies that $n_1^i\cdots n_{k_E}^i n_{k_E+1}^i < m_{k_E}^i n_{k_E+1}^i < m_{k_E+1}^i = 1+n_1^i\cdots n_{k_E}^i n_{k_E+1}^i$. The previous inequalities hold only if $k_E=0$, i.e., $m_{k_E}^i=0$. Consequently $v(E)=(1+n_1^i)/n_1^i$ and the result follows since $n_E=n_1^i=2$. □

## 7. Examples

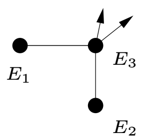
We illustrate here some different behaviours of a polar pencil $\Lambda_{\mathcal{G}}$ when $\mathcal{G}$ is not a Zariski-general foliation.

***Example 1.*** — There can be dicritical components of $\Lambda_{\mathcal{G}}$ with degree $\geq 2$, which are different from $E_1$. Consider the foliation $\mathcal{G}$ given by $d(y^3-x^5)=0$; note that $C$ has not a kind equisingularity type. The pencil $\Lambda_{\mathcal{G}}$ has a dicritical component of degree 2 which corresponds to the terminal divisor $E_2$ of the unique dead arc in $G(C)$. In this case, $\pi_C$ gives a resolution of singularities of $\Lambda_{\mathcal{G}}$ but it is not the minimal resolution of $\Lambda_{\mathcal{G}}$.

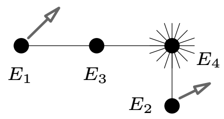



$G(C)$ $\qquad$ $G(\pi_C, \Lambda_{\mathcal{F}})$

***Example 2.*** — Consider the foliation $\mathcal{G}$ given by $\omega=x^5dx-y^3dy=0$. The minimal reduction of singularities $\pi_C$ of $\mathcal{G}$ is not a reduction of singularities of a generic fiber $\Gamma_{[a:b]}=\{ax^5-by^3=0\}$. It is necessary to blow-up the corner $E_3\cap E_2$ of $\pi_C^{-1}(0)$ to obtain an elimination of indeterminations $\sigma_\Lambda$ of $\Lambda_{\mathcal{G}}$; hence we need to blow-up a singular point of $\pi_C^*\mathcal{G}$.




$G(\mathcal{F})$




$G(\sigma_\Lambda, \Lambda_{\mathcal{F}})$

Notice that $v(E_4) = 5/3$ and $v(E_3) = 3/2$, thus equation (7) is not true for this foliation. In this example, the curve of separatrices $C$ has a kind equisingularity type but the foliation $\mathscr{F}$ is not Zariski-general.

## References

[1] H. Bresinsky – "Semigroups corresponding to algebroid branches in the plane", *Proc. Amer. Math. Soc.* **32** (1972), p. 381–384.

[2] C. Camacho, A. Lins Neto & P. Sad – "Topological invariants and equidesingularization for holomorphic vector fields", *J. Differential Geom.* **20** (1984), p. 143–174.

[3] C. Camacho & P. Sad – "Invariant varieties through singularities of holomorphic vector fields", *Ann. of Math.* **115** (1982), p. 579–595.

[4] E. Casas-Alvero – *Singularities of plane curves*, London Mathematical Society Lecture Note Series, vol. 276, Cambridge University Press, 2000.

[5] N. Corral – "Sur la topologie des courbes polaires de certains feuilletages singuliers", *Ann. Inst. Fourier (Grenoble)* **53** (2003), p. 787–814.

[6] ———, "Détermination du type d'équisingularité polaire", *C. R. Math. Acad. Sci. Paris* **344** (2007), p. 33–36.

[7] ———, "Infinitesimal adjunction and polar curves", submitted.

[8] F. Delgado & H. Maugendre – "Special fibres and critical locus for a pencil of plane curve singularities", *Compositio Math.* **136** (2003), p. 69–87.

[9] E. R. García Barroso – "Sur les courbes polaires d'une courbe plane réduite", *Proc. London Math. Soc.* **81** (2000), p. 1–28.

[10] T. C. Kuo & Y. C. Lu – "On analytic function germs of two complex variables", *Topology* **16** (1977), p. 299–310.

[11] M. Merle – "Invariants polaires des courbes planes", *Invent. Math.* **41** (1977), p. 103–111.

[12] P. Rouillé – "Courbes polaires et courbure", Thèse de doctorat, Université de Bourgogne, 1996.

[13] ———, "Théorème de Merle: cas des 1-formes de type courbes généralisées", *Bol. Soc. Brasil. Mat. (N.S.)* **30** (1999), p. 293–314.

[14] L. D. Trang & C. Weber – "Équisingularité dans les pinceaux de germes de courbes planes et $C^0$-suffisance", *Enseign. Math.* **43** (1997), p. 355–380.

[15] O. Zariski – "General theory of saturation and of saturated local rings. II. Saturated local rings of dimension 1", *Amer. J. Math.* **93** (1971), p. 872–964.

[16] ———, *Le problème des modules pour les branches planes*, École polytechnique, 1973, Cours donné au Centre de Mathématiques de l'École polytechnique, Paris, Octobre-Novembre 1973.

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# HOMOGENEOUS COMMUTING VECTOR FIELDS ON $\mathbb{C}^2$

*by*

Alcides Lins Neto

***Abstract.*** — In the main result of this paper we give a method to construct all pairs of homogeneous commuting vector fields on $\mathbb{C}^2$ of the same degree $d \geq 2$ (Theorem 1). As an application, we classify, up to linear transformations of $\mathbb{C}^2$, all pairs of commuting homogeneous vector fields on $\mathbb{C}^2$, when $d = 2$ and $d = 3$ (corollaries 1 and 2). We obtain also necessary conditions in the cases of quasi-homogeneous vector fields and when the degrees are different (theorem 2).

***Résumé* (Champs de vecteurs homogènes commutants dans $\mathbb{C}^2$).** — Dans le résultat principal de ce papier on donne une méthode de construction de tous les paires de champs de vecteurs homogènes de même degré $d \geq 2$ qui commutent (théorème 1). Comme application, on classifie les paires de champs de vecteurs homogènes commutantes dans $\mathbb{C}^2$ de degrés $d = 2$ et $d = 3$ (corollaires 1 et 2). Nous obtenons aussi des conditions nécessaires dans les cas quasi-homogènes et quand les degrés sont différents (théorème 2).

## 1. Introduction

A. Guillot in his thesis and in [**3**], gave a non-trivial example of a pair of commuting homogeneous vector fields of degree two on $\mathbb{C}^3$. The example is non-trivial in the sense that it cannot be reduced to two vector fields in separated variables, like in the pair $X := P(x,y)\partial_x + Q(x,y)\partial_y$ and $Y := R(z)\partial_z$. This suggested me the problem of classification of pairs of polynomial commuting vector fields on $\mathbb{C}^n$. This problem, in this generality, seems very difficult, even for $n = 2$. Even the restricted problem of classification of pairs of commuting vector fields, homogeneous of degree $d$, seems very difficult for $n \geq 3$ and $d \geq 2$ (see problem 3). However, for $n = 2$ and $d \geq 2$ it is possible to give a complete classification, as we will see in this paper.

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Let $X$ and $Y$ be two homogeneous commuting vector fields on $\mathbb{C}^2$, where $dg(X) = k$ and $dg(Y) = \ell$, and $R = x\,\partial_x + y\,\partial_y$ be the radial vector field.

***Definition 1.1***. — *We will say that $X$ and $Y$ are* colinear *if $X \wedge Y = 0$. In this case, we will use the notation $X//Y$. When $dg(X) = dg(Y)$, we will consider the 1-parameter family $(Z_\lambda)_{\lambda\in\mathbb{P}^1}$ given by $Z_\lambda = X + \lambda.Y$ if $\lambda \in \mathbb{C}$ and $Z_\infty = Y$. It will be called the* pencil generated *by $X$ and $Y$. The pencil will be called* trivial, *if $Y = \lambda.X$ for some $\lambda \in \mathbb{C}$. Otherwise, it will be called* non-trivial.

From now on, we will set:

$$\begin{cases} X \wedge Y = f\,\partial_x \wedge \partial_y \\ R \wedge X = g\,\partial_x \wedge \partial_y \\ R \wedge Y = h\,\partial_x \wedge \partial_y \end{cases} . \tag{1}$$

Since $dg(X) = k$ and $dg(Y) = \ell$, the polynomials $f$, $g$ and $h$ are homogeneous and $dg(f) = k + \ell$, $dg(g) = k + 1$, $dg(h) = \ell + 1$. Moreover, $f \not\equiv 0$ iff $X$ and $Y$ are non-colinear.

Our main result concerns the case where $k = \ell \geq 2$. In this case, if $g, h \not\equiv 0$, we will consider the meromorphic function $\phi = g/h$ as a holomorphic function $\phi\colon \mathbf{P}^1 \to \mathbf{P}^1$:

$$\phi[x : y] = \frac{g(x,y)}{h(x,y)} .$$

***Theorem 1***. — *Let $(Z_\lambda)_\lambda$ be a non-trivial pencil of homogeneous commuting vector fields of degree $d \geq 2$ on $\mathbb{C}^2$. Let $X$ and $Y$ be two generators of the pencil and $f, g, h$ and $\phi$ be as before. If the pencil is colinear then $X = \alpha.R$ and $Y = \beta.R$, where $\alpha$ and $\beta$ are homogeneous polynomials of degree $d - 1$. If the pencil is non-colinear then:*

(a) *$f, g, h \not\equiv 0$.*
(b) *$f/g$ (resp. $f/h$) is a non-constant meromorphic first integral of $X$ (resp. $Y$).*
(c) *Let $s$ be the (topological) degree of $\phi\colon \mathbf{P}^1 \to \mathbf{P}^1$. Then $1 \leq s \leq d - 1$.*
(d) *The decompositions of $f$, $g$ and $h$ into irreducible linear factors are of the form:*

$$\begin{cases} f = \Pi_{j=1}^r f_j^{2k_j+m_j} \\ g = \Pi_{j=1}^r f_j^{k_j}.\Pi_{i=1}^s g_i \\ h = \Pi_{j=1}^r f_j^{k_j}.\Pi_{i=1}^s h_i \end{cases} \tag{2}$$

*where $s + \sum_{j=1}^r k_j = d + 1$ and $\sum_{j=1}^r m_j = 2s - 2$. Moreover, we can choose the generators $X$ and $Y$ in such a way that $g_1, \ldots, g_s, h_1, \ldots, h_s$ are two by two relatively prime.*

(e) *Considering the direction $(f_j = 0) \subset \mathbb{C}^2$ as a point $p_j \in \mathbf{P}^1$, then*

$$m_j = \mathrm{mult}(\phi, p_j) - 1 \;,\; j = 1, \ldots, r \;, \tag{3}$$

*where $\mathrm{mult}(\phi, p)$ denotes the ramification index of $\phi$ at $p \in \mathbf{P}^1$.*

(f) *The generators $X$ and $Y$ can be chosen as:*

$$(4)\qquad \begin{cases} X = g.[\sum_{j=1}^{r}(k_j+m_j)\frac{1}{f_j}(f_{jx}\partial_y - f_{jy}\partial_x) - \sum_{i=1}^{s}\frac{1}{g_i}(g_{ix}\partial_y - g_{iy}\partial_x)] \\ Y = h.[\sum_{j=1}^{r}(k_j+m_j)\frac{1}{f_j}(f_{jx}\partial_y - f_{jy}\partial_x) - \sum_{i=1}^{t}\frac{1}{h_i}(h_{ix}\partial_y - h_{iy}\partial_x)] \end{cases}$$

*Conversely, given a non-constant map $\phi\colon \mathbf{P}^1 \to \mathbf{P}^1$ of degree $s \geq 1$ and a divisor $D$ on $\mathbf{P}^1$ of the form*

$$(5)\qquad D = \sum_{p\in\mathbf{P}^1} (2k(p) + \mathrm{mult}(\phi,p) - 1).[p]\ ,$$

*where $k(p) \geq min(1, \mathrm{mult}(\phi,p)-1)$ and $\sum_p k(p) < +\infty$, there exists a unique pencil $(Z_\lambda)_\lambda$ of homogeneous commuting vector fields of degree $d = \sum_p k(p) + s - 1$ with generators $X$ and $Y$ given by (4), and the $f_j$'s, $g_i$'s and $h_i$'s given in the following way: let $\{p_1 = [a_1 : b_1], \dots, p_r = [a_r : b_r]\} = \{p \in \mathbf{P}^1 \mid 2k(p) + \mathrm{mult}(\phi,p) - 1 > 0\}$. Set $k_j = k(p_j)$, $m_j = \mathrm{mult}(\phi,p_j) - 1$ and $f_j(x,y) = a_j\,y - b_j\,x$. Set $\phi[x:y] = G_1(x,y)/H_1(x,y)$, where $G_1$ and $H_1$ are homogeneous polynomials of degree $s$. Then the $g_i$'s and $h_i$'s are the linear factors of $G_1$ and $H_1$, respectively.*

***Definition 1.2***. — *Let $X$, $Y$, $g = \Pi_{j=1}^{r} f_j^{k_j}.\Pi_{i=1}^{s} g_i$ and $h = \Pi_{j=1}^{r} f_j^{k_j}.\Pi_{i=1}^{s} h_i$ be as in theorem 1. We call $(f_j = 0)$, $j = 1, \dots, r$, the fixed directions of the pencil.*

Given $\lambda \in \mathbb{C}$, the polynomial $g_\lambda = g + \lambda.h$ plays the same role for the vector field $Z_\lambda = X + \lambda.Y$ as $g$ and $h$ for $X$ and $Y$. Its decomposition into irreducible factors is of the form

$$g_\lambda = \Pi_{j=1}^{r} f_j^{k_j}.\Pi_{i=1}^{s} g_{i,\lambda}\ .$$

***Definition 1.3***. — *The directions given by $(g_{i,\lambda} = 0)$ are called the movable directions of the pencil.*

In particular, the number $s$ of movable directions coincides with the degree of the map $\phi = g/h\colon \mathbf{P}^1 \to \mathbf{P}^1$.

As an application of Theorem 1, we obtain the classification of the pencils of homogeneous commuting vector fields of degrees two and three.

***Corollary 1***. — *Let $(Z_\lambda)_\lambda$ be a pencil of commuting homogeneous of degree two vector fields on $\mathbb{C}^2$. Then, after a linear change of variables on $\mathbb{C}^2$, the generators $X$ and $Y$ of the pencil can be written as:*

(a) *$X = g.R$ and $Y = h.R$, where $g$ and $h$ are homogeneous polynomials of degree one and $R = x.\partial_x + y.\partial_y$.*
(b) *$X = x^2\partial_x$ and $Y = y^2\partial_y$. In this case, the pencil has two fixed directions.*
(c) *$X = y^2\partial_x$ and $Y = 2xy\partial_x + y^2\partial_y$. In this case, the pencil has one fixed direction.*

***Corollary 2***. — *Let $(Z_\lambda)_\lambda$ be a pencil of commuting homogeneous of degree three vector fields on $\mathbb{C}^2$. Then, after a linear change of variables on $\mathbb{C}^2$, the generators $X$ and $Y$ of the pencil can be written as:*

(a) $X = g.R$ *and* $Y = h.R$, *where* $g$ *and* $h$ *are homogeneous polynomials of degree two and* $R = x.\partial_x + y.\partial_y$.
(b) $X = y^3\partial_x$ *and* $Y = 3xy^2\partial_x + y^3\partial_y$. *In this case, the pencil has one movable and one fixed direction.*
(c) $X = x^2y\partial_x$ *and* $Y = xy^2\partial_x - y^3\partial_y$. *In this case, the pencil has one movable and two fixed directions.*
(d) $X = (2x^2y + x^3)\partial_x - x^2y\partial_y$ *and* $Y = -xy^2\partial_x + (2xy^2 + y^3)\partial_y$. *In this case, the pencil has one movable and three fixed directions.*
(e) $X = x^3\partial_x$ *and* $Y = y^3\partial_y$. *In this case, the pencil has two movable and two fixed directions.*

Some of the preliminary results that we will use in the proof of Theorem 1 are also valid for quasi-homogeneous vector fields.

***Definition 1.4***. — *Let* $S$ *be a linear diagonalizable vector field on* $\mathbb{C}^n$ *such that all eigenvalues of* $S$ *are relatively prime natural numbers. We say that a holomorphic vector field* $X \not\equiv 0$ *is quasi-homogeneous with respect to* $S$ *if* $[S, X] = m\,X$, $m \in \mathbb{C}$.

It is not difficult to prove that, in this case, we have the following:

(I) $m \in \mathbb{N} \cup \{0\}$.
(II) $X$ is a polynomial vector field.

Our next result concerns two commuting vector fields which are quasi-homogeneous with respect to the same linear vector field $S$. Let $X$ and $Y$ be two commuting vector fields on $\mathbb{C}^2$, quasi-homogeneous with respect to the same vector field $S$ with eigenvalues $p, q \in \mathbb{N}$ (relatively prime), where $[S, X] = m\,X$ and $[S, Y] = n\,Y$. Since $S$ is diagonalizable, after a linear change of variables, we can assume that $S = p\,x\partial_x + q\,y\partial_y$. Set $X \wedge Y = f\,\partial_x \wedge \partial_y$, $S \wedge X = g\,\partial_x \wedge \partial_y$ and $S \wedge Y = h\,\partial_x \wedge \partial_y$. We will always assume that $X, Y \not\equiv 0$

***Remark 1.1***. — *We would like to observe that* $f, g$ *and* $h$ *are quasi-homogeneous with respect to* $S$, *that is, we have* $S(f) = (m + n + tr(S))f$, $S(g) = (m + tr(S))g$ *and* $S(h) = (n + tr(S))h$, *where* $tr(S) = p + q$. *It is known that in this case, any irreducible factor of* $f$, $g$ *or* $h$, *is the equation of an orbit of* $S$, *that is,* $x$, $y$ *or a polynomial of the form* $y^p - c\,x^q$, *where* $c \neq 0$ .

***Theorem 2***. — *In the above situation, suppose that* $f, h \not\equiv 0$ *and* $n \neq 0$. *Then:*

(a) $g \not\equiv 0$ *and* $f/g$ *is a non-constant meromorphic first integral of* $X$.
(b) *Suppose that* $m, n \neq 0$. *Then* $f$, $g$ *and* $h$ *satisfy the two equivalent relations below:*

$$mn\,f^2\,dx \wedge dy = f\,dg \wedge dh + g\,dh \wedge df + h\,df \wedge dg \tag{6}$$

$$(m - n)\frac{df}{f} + n\frac{dh}{h} - m\frac{dg}{g} = \frac{m\,n\,f}{gh}(qy\,dx - px\,dy) \tag{7}$$

(c) *Suppose that $m, n \neq 0$. Then any irreducible factor of $f$ divides $g$ and $h$. Conversely, if $p = gcd(g, h)$ then any irreducible factor of $p$ divides $f$. Moreover, the decompositions of $f$, $g$ and $h$ into irreducible factors, are of the form*

$$\begin{cases} f = \Pi_{j=1}^r f_j^{\ell_j} \\ g = \Pi_{j=1}^r f_j^{m_j} . \Pi_{i=1}^s g_i^{a_i} \\ h = \Pi_{j=1}^r f_j^{n_j} . \Pi_{i=1}^t h_i^{b_i} \end{cases} \tag{8}$$

*where $r > 0$, $m_j, n_j > 0$, $\ell_j \geq m_j + n_j - 1$, for all $j$, and any two polynomials in the set $\{f_1, \dots, f_r, g_1, \dots, g_s, h_1, \dots, h_t\}$ are relatively prime.*

(d) *Suppose that $f$, $g$ and $h$ are as in (8). Then vector fields $X$ and $Y$ can be written as*

$$\begin{cases} X = \frac{1}{n} g . [\sum_{j=1}^r (\ell_j - m_j) \frac{1}{f_j} (f_{jx} \partial_y - f_{jy} \partial_x) - \sum_{i=1}^s a_i \frac{1}{g_i} (g_{ix} \partial_y - g_{iy} \partial_x)] \\ Y = \frac{1}{m} h . [\sum_{j=1}^r (\ell_j - n_j) \frac{1}{f_j} (f_{jx} \partial_y - f_{jy} \partial_x) - \sum_{i=1}^t b_i \frac{1}{h_i} (h_{ix} \partial_y - h_{iy} \partial_x)] \end{cases} \tag{9}$$

As an application, we have the following result:

***Corollary 3***. — *Let $X$ and $Y$ be germs of holomorphic commuting vector fields at $0 \in \mathbb{C}^2$. Let*

$$X = \sum_{j=d}^{\infty} X_j$$

*be the Taylor series of $X$ at $0 \in \mathbb{C}^2$, where $X_j$ is homogeneous of degree $j \geq d$. Assume that $d \geq 2$ and that the vector field $X_d$ has no meromorphic first integral and that $0$ is an isolated singularity of $X_d$. Then $Y = \lambda . X$, where $\lambda \in \mathbb{C}$.*

We would like to recall a well-known criterion for a homogeneous vector field of degree $d$ on $\mathbb{C}^2$, say $X_d$, to have a meromorphic first integral (see [**1**]). Since the radial vector field $R = x\,\partial_x + y\,\partial_y$ has the meromorphic first integral $y/x$, we can assume that $R \wedge X_d = g\,\partial_x \wedge \partial_y \not\equiv 0$. Let $\omega = i_{X_d}(dx \wedge dy)$, where $i$ denotes the interior product. Then the form $\omega_1 = \omega / g$ is closed. In this case, if $g = \Pi_{j=1}^r g_j^{k_j}$ is the decomposition of $g$ into linear irreducible factors, then we have

$$\omega_1 = \sum_{j=1}^r \lambda_j \frac{dg_j}{g_j} + d(h / g_1^{k_1 - 1} \cdots g_r^{k_r - 1}) \ ,$$

where $\lambda_j \in \mathbb{C}$, for all $1 \leq j \leq r$ and $h$ is homogeneous of degree $d + 1 - r = dg(X_d) + 1 - r = dg(g / g_1 \cdots g_r)$. In this case, $X_d$ has a meromorphic first integral if, and only if, either $\lambda_1 = \cdots = \lambda_r = 0$, or $\lambda_j \neq 0$ for some $j \in \{1, \dots, r\}$, $h \equiv 0$ and $[\lambda_1 : \cdots : \lambda_r] = [m_1 : \cdots : m_r]$, where $m_1, \dots, m_r \in \mathbb{Z}$. In particular, we obtain that the set of homogeneous vector fields of degree $d \geq 1$ with a meromorphic first integral is a countable union of Zariski closed sets.

Let us state some natural problems related to the above results.

***Problem 1***. — *Classify the pencils of commuting homogeneous vector fields of degree $d \geq 2$ on $\mathbb{C}^n$, $n \geq 3$.*

Problem 1 seems dificult even in dimension three.

***Problem 2***. — *Let $\mathcal{X}_2$ be the set of germs at $0 \in \mathbb{C}^2$ of holomorphic vector fields. Given $X \in \mathcal{X}_2$, $X \neq 0$, to determine the set*

$$C(X) = \{ Y \mid [X,Y] = 0 \} .$$

*Under which conditions is $C(X)$ of finite dimension?*

***Problem 3***. — *Classify all pairs of commuting polynomial vector fields on $\mathbb{C}^2$.*

Observe that problem 3 has the following relation with the so called Jacobian conjecture: let $f$ and $g$ be two polynomials on $\mathbb{C}^2$ such that $f_x.g_y - f_y.g_x \equiv 1$. Then their hamiltonians $X = f_y\,\partial_x - f_x\,\partial_y$ and $Y = g_y\,\partial_x - g_x\partial_y$ commute. By this reason, problem 3 seems very difficult.

## 2. Preliminary results

In this section we prove some general results that will be used in the next sections. Let $S$, $X$ and $Y$ be holomorphic vector fields defined in some domain $U$ of $\mathbb{C}^2$. Assume that:

(I) $[S,X] = m.X$, $[S,Y] = n.Y$ and $[X,Y] = 0$, where $m, n \in \mathbb{C}$.

(II) $X \wedge Y = f.\partial_x \wedge \partial_y$, $S \wedge X = g.\partial_x \wedge \partial_y$ and $S \wedge Y = h.\partial_x \wedge \partial_y$, where $f, g, h \not\equiv 0$.

We consider also the holomorphic 1-forms $\omega = i_X(dx \wedge dy)$ and $\eta = i_Y(dx \wedge dy)$, where $i$ denotes the interior product.

***Lemma 2.1***. — *In the above situation we have:*

(a) *The meromorphic functions $f/g$ and $f/h$ are first integrals of $X$ and $Y$, respectively. Moreover, $f/g$ (resp. $f/h$) is constant if, and only if, $n = 0$ (resp. $m = 0$).*

(b) *If $n \neq 0$ (resp. $m \neq 0$) then*

$$\omega = \frac{g}{n}\left[\frac{dg}{g} - \frac{df}{f}\right] \ (resp.\ \eta = \frac{h}{m}\left[\frac{dh}{h} - \frac{df}{f}\right]) . \tag{10}$$

(c) *The polynomials $f$, $g$ and $h$ satisfy the relation:*

$$mn\, f^2\, dx \wedge dy = f\, dg \wedge dh + g\, dh \wedge df + h\, df \wedge dg . \tag{11}$$

*Proof*. — Let us prove (a). Assume that $n \neq 0$. First of all, note that

$$L_X(S \wedge X) = [X,S] \wedge X + S \wedge [X,X] = -m.X \wedge X = 0$$

and simillarly $L_X(X \wedge Y) = 0$, where $L$ denotes the Lie derivative. Since $X \wedge Y = (f/g).S \wedge Y$, we get

$$\begin{aligned} 0 &= L_X(X \wedge Y) = L_X((f/g).S \wedge X) \\ &= X(f/g).S \wedge X + (f/g).L_X(S \wedge X) = X(f/g).S \wedge X \implies \\ &\implies X(f/g) = 0 . \end{aligned}$$

Therefore, $f/g$ is a first integral of $X$. It remains to prove that $f/g$ is a constant if, and only if $n = 0$. Since $L_S(X \wedge Y) = (m+n)\, X \wedge Y$ and $L_S(S \wedge X) = m\, S \wedge X$, we get

$$\begin{aligned}(m+n)\, X \wedge Y &= L_S((f/g).S \wedge X) \\ &= S(f/g).S \wedge X + (f/g).L_S(S \wedge X) \\ &= (S(f/g) + m.(f/g))\, S \wedge X\end{aligned}$$

which implies that $S(f/g) = n.(f/g)$. Hence, if $f/g$ is a constant then $n = 0$.

Conversely, if $n = 0$ then $S(f/g) = 0$ and $f/g$ is a first integral of $S$ and $X$ simultaniously. If $f/g$ was not constant then the vector fields $X$ and $S$ would be colinear in the non-empty open subset of $U$ defined by $d(f/g) \neq 0$. This would imply that $S \wedge X \equiv 0$, and so $g \equiv 0$, a contradiction. Therefore, $f/g$ is a constant.

Now, let $\omega = i_X(dx \wedge dy)$ and suppose that $n \neq 0$. Since $f/g$ is a non-constant first integral of $X$, we get $\omega \wedge d(f/g) = 0$, which implies that

$$\omega = k \left( \frac{dg}{g} - \frac{df}{f} \right) ,$$

where $k$ is meromorphic on $U$. On the other hand, we have

$$\begin{aligned}g &= -i_S(i_X(dx \wedge dy)) = -i_S(\omega) \\ &= k \left( \frac{S(f)}{f} - \frac{S(g)}{g} \right) = k\, \frac{S(f/g)}{f/g} = n.k \implies k = g/n .\end{aligned}$$

This proves (10).

Let us prove (c). Note first that $\omega \wedge \eta = f.dx \wedge dy$. We leave the proof of this fact to the reader. If $n = 0$ (or $m = 0$) then (11) follows from $f/g = c \neq 0$ (or $f/h = c \neq 0$), where $c$ is a constant. We leave the proof to the reader in this case. On the other hand, if $m, n \neq 0$ then

$$\begin{aligned}f.dx \wedge dy &= \omega \wedge \eta \\ &= \frac{g}{n} \left[ \frac{dg}{g} - \frac{df}{f} \right] \wedge \frac{h}{m} \left[ \frac{dh}{h} - \frac{df}{f} \right] = \frac{g.h}{m.n} \left[ \frac{dh \wedge df}{h.f} + \frac{df \wedge dg}{f.g} + \frac{dg \wedge dh}{g.h} \right] ,\end{aligned}$$

which implies (11). □

In the next result we prove a kind of converse of (11).

***Lemma 2.2***. — *Let $f$, $g$ and $h$ be holomorphic functions on a domain $U \subset \mathbb{C}^2$. Suppose that $f/g$ and $f/h$ are non-constant meromorphic functions on $U$. Define meromorphic vector fields $X$ and $Y$ by $i_X(dx \wedge dy) = g[\frac{dg}{g} - \frac{df}{f}]$ and $i_Y(dx \wedge dy) = h[\frac{dh}{h} - \frac{df}{f}]$. Suppose that*

$$f\, dg \wedge dh + g\, dh \wedge df + h\, df \wedge dg = \lambda\, f^2\, dx \wedge dy ,$$

*where $\lambda \neq 0$. Then $[X, Y] = 0$.*

*Proof*. — The idea is to prove that $d(f/g) \wedge d(f/h) \not\equiv 0$ and $[X,Y](f/g) = [X,Y](f/h) = 0$. This will imply that $f/g$ and $f/h$ are two independent meromorphic first integrals of $[X,Y]$, and so $[X,Y] = 0$. □

*Proof of* $d(f/g) \wedge d(f/h) \not\equiv 0$. — Note that

$$d(f/g) \wedge d(f/h) = \frac{f}{g^2h^2}[f\,dg \wedge dh + h\,df \wedge dg + g\,dh \wedge df] = \lambda.\frac{f^3}{g^2h^2}\,dx \wedge dy \neq 0 \implies$$

$$\implies d(f/g) \wedge d(f/h) \neq 0.$$ □

*Proof of* $[X,Y] = 0$. — We have

$$[X,Y](f/g) = X(Y(f/g)) - Y(X(f/g)) = X(Y(f/g)) ,$$

because $X(f/g) = 0$. On the other hand, a straightforward computation shows that

$$Y(f/g)\,dx \wedge dy = d(f/g) \wedge \eta , \tag{12}$$

where $\eta = i_Y(dx \wedge dy)$. Since $\eta = h[\frac{dh}{h} - \frac{df}{f}] = -\frac{h^2}{f}d(f/h)$, we get from (12) that

$$d(f/g) \wedge \eta = -\frac{h^2}{f}d(f/g) \wedge d(f/h) = -\frac{\lambda f^2}{g^2}\,dx \wedge dy \implies Y(f/g) = -\lambda\,(f/g)^2 \implies$$

$\implies X(Y(f/g)) = 0$. In a similar way, we get $[X,Y](f/h) = 0$. □

## 3. Proofs

*Proof of Theorem 2*. — Assume that $n \neq 0$, $f, h \not\equiv 0$ and $g \equiv 0$. Since $S$ has an isolated singularity at $0 \in \mathbb{C}^2$ and $S \wedge X = g.\partial_x \wedge \partial_y = 0$, we get $X = \psi.S$, where $\psi \neq 0$ is a polynomial. It follows that

$$0 = [Y,X] = [Y,\psi.S] = Y(\psi).S - \psi.[S,Y] = Y(\psi).S - n.\psi.Y \implies Y(\psi) \not\equiv 0$$

and $S \wedge Y = 0$, which implies $h \equiv 0$, a contradiction. Hence, $g \not\equiv 0$. It follows from lemma 2.1 that $f/g$ is a non-constant meromorphic first integral of $X$. This proves (a) of theorem 2.

Lemma 2.1 implies also that $f$, $g$ and $h$ satisfy relation (6). Let us prove that (6) is equivalent to (7). We will use the following fact: let $\mu$ be a 2-form in $\mathbb{C}^2$ such that $L_S(\mu) = \lambda.\mu$, where $\lambda \in \mathbb{C}$. Then

$$d(i_S(\mu)) = L_S(\mu) = \lambda.\mu \tag{13}$$

Set $\mu = f\,dg \wedge dh + g\,dh \wedge df + h\,df \wedge dg$ and $\mu_1 = mn\,f^2\,dx \wedge dy$. We have seen in remark 1.1 that $S(f) = (m+n+tr(S)).f$, $S(g) = (m+tr(S)).g$ and $S(h) = (n+tr(S)).h$. As the reader can check, this implies that $L_S(\mu) = \lambda.\mu$ and $L_S(\mu_1) = \lambda.\mu_1$, where $\lambda = 2m+2n+3tr(S) \neq 0$.

On the other hand, we have

$$\begin{cases} i_S(\mu_1) = mn\,f^2(px\,dy - qy\,dx) \\ i_S(\mu) = -n\,fg\,dh + m\,fh\,dg + (n-m)\,gh\,df \end{cases}$$

as the reader can check. If we assume (6), we have $\mu_1 = \mu$, so that $i_S(\mu) = i_S(\mu_1)$ and

$$mn\, f^2(px\,dy - qy\,dx) = -n\, fg\,dh + m\, fh\,dg + (n-m)\, gh\,df \implies (7)\ .$$

If we assume (7), then we have

$$(7) \implies i_S(\mu_1 - \mu) = 0 \overset{(13)}{\implies} \lambda\,(\mu_1 - \mu) = d(i_S(\mu_1 - \mu)) = 0 \implies (6)\ .$$

This proves (b) of theorem 2.

Let us prove (c). We will use (7) in the form

$$(14) \qquad (m-n)\, g.h\, df + n\, f.g\, dh - m\, f.h\, dg = m\, n\, f^2\, (q\, y\, dx - p\, x\, dy)\ .$$

It follows from (14) that, if $k$ is an irreducible factor of both polynomials $g$ and $h$, then $k$ divides $f^2$, and so it divides $f$.

Let us prove that any factor of $f$ is a factor of both polynomials $g$ and $h$. Here we use that $f/g$ is a first integral of $X$. This implies that

$$(15) \qquad f.X(g) = g.X(f)\ .$$

Recall that any irreducible factor of $f$ or $g$ is the equation of an orbit of $S$ (remark 1.1). Let $f = \Pi_{j=1}^r f_j^{\ell_j}$ $(r, \ell_j > 0)$, be the decomposition of $f$ into irreducible factors and set $F = \Pi_j\, f_j$. It follows from (15) that

(16)
$$F.X(g) = F\, \frac{X(f)}{f}\, g = g.k \text{ ,where } k = F\, \frac{X(f)}{f} = \sum_{j=1}^{r} \ell_j.f_1 \cdots f_{j-1}.X(f_j).f_{j+1} \cdots f_r\ .$$

On the other hand, (16) implies that for any $j = 1, \ldots, r$, $f_j$ divides $g$ or $X(f_j)$. If $f_j$ divides $g$, we are done. If $f_j$ divides $X(f_j)$ then $(f_j = 0)$ is invariant for $X$. Since $(f_j = 0)$ is also invariant for $S$, it is a common orbit of $X$ and $S$. This implies that $f_j$ divides $S \wedge X$, and so it divides $g$. Similarly, any irreducible factor of $f$ divides $h$.

Now, we can assume that the decompositions of $f$, $g$ and $h$ into irreducible factors are as in (8):

$$\begin{cases} f = \Pi_{j=1}^r\, f_j^{\ell_j} \\ g = \Pi_{j=1}^r\, f_j^{m_j}.\Pi_{i=1}^s\, g_i^{a_i} \\ h = \Pi_{j=1}^r\, f_j^{n_j}.\Pi_{i=1}^t\, h_i^{b_i} \end{cases}$$

where $\ell_j, m_j, n_j > 0$ and any two polynomials in the set

$$\{f_1, \ldots, f_r, g_1, \ldots, g_s, h_1, \ldots, h_t\}$$

are relatively prime. Let us prove that $\ell_j \geq m_j + n_j - 1$. As the reader can check, it follows from (14) that $f_j^{m_j+n_j+\ell_j-1}$ divides $f^2$. This implies that $m_j+n_j+\ell_j-1 \leq 2\ell_j$, and we are done.

It remains to prove (d). Let $\omega = i_X(dx \wedge dy)$. We have seen in lemma 2.1 that

$$\omega = \frac{g}{n}\left[\frac{dg}{g} - \frac{df}{f}\right] = \frac{g}{n}\left[\sum_{i=1}^{s} a_i\, \frac{dg_i}{g_i} - \sum_{j=1}^{r} (\ell_j - m_j)\, \frac{df_j}{f_j}\right]$$

As the reader can check, this implies that $X$ is like in (9). Similarly, $Y$ is also as in (9). □

*Proof of Corollary 3.* — Let $X = \sum_{j=d}^{\infty} X_j$ and $Y \not\equiv 0$ be germs of holomorphic vector fields at $0 \in \mathbb{C}^2$ such that $[X, Y] = 0$. Assume that $d \geq 2$ and $X_d$ has an isolated singularity at $0 \in \mathbb{C}^2$ and no meromorphic first integral. Set $Y = \sum_{i=r}^{\infty} Y_j$, where $Y_j$ is homogeneous of degree $j$, $r \geq 0$, and $Y_r \neq 0$. We have $[R, X_d] = m\, X_d$, $[R, Y_r] = n\, Y_r$, where $m = d - 1 \neq 0$ and $n = r - 1$. Note also that $[X_d, Y_r] = 0$.

***Claim 3.1***. — *We have $r = d$ and $Y_d = \lambda.X_d$, where $\lambda \neq 0$.*

*Proof.* — As before, set $X_d \wedge Y_r = f.\partial_x \wedge \partial_y$, $R \wedge X_d = g.\partial_x \wedge \partial_y$ and $R \wedge Y_r = h.\partial_x \wedge \partial_y$. Observe that $g \not\equiv 0$. Indeed, if $g \equiv 0$ then $R \wedge X_d = 0$. Since 0 is an isolated singularity of $R$, it follows from De Rham's division theorem (cf. [**4**]) that $X_d = \phi.R$, where $\phi$ is a homogeneous polynomial of degree $d - 1 > 0$. But, this implies that $sing(X_d) \supset (\phi = 0)$, and so 0 is not an isolated singularity of $X_d$.

Suppose by contradiction that $r \neq d$. Let us prove that in this case we have $f, h \not\equiv 0$. Suppose by contradiction that $f \equiv 0$. This implies that $X_d \wedge Y_r \equiv 0$. Since $X_d$ has an isolated singularity at $0 \in \mathbb{C}^2$, it follows from De Rham's division theorem that $Y_r = \phi.X_d$, where $\phi$ is a homogeneous polynomial of degree $r - d > 0$. Therefore,

$$0 = [X_d, Y_r] = [X_d, \phi.X_d] = X_d(\phi).X_d \implies X_d(\phi) = 0 \implies$$

that $\phi$ is a non-constant first integral of $X_d$, a contradiction. Hence, $f \not\equiv 0$. Suppose by contradiction that $h \equiv 0$. This implies that $R \wedge Y_r \equiv 0$, so that $Y_r = \phi.R$, where $\phi \neq 0$ is a homogeneous polynomial of degree $k = r - 1$. From this we get

$$0 = [X_d, Y_r] = [X_d, \phi.R] = X_d(\phi).R + \phi.[X_d, R] = X_d(\phi).R - (d-1).\phi.X_d \implies$$

$$X_d(\phi).R = (d-1).\phi.X_d \ .$$

If $\phi \neq 0$ is a constant then $d = 1$, a contradiction. If $\phi$ is not a constant then $X_d(\phi) \neq 0$, for otherwise $\phi$ would be a non-constant first integral of $X_d$. In this case, we get $R \wedge X_d = 0$, and so $g \equiv 0$, a contradiction. Hence, $f, g, h \not\equiv 0$. Now, we can apply (a) of lemma 2.1.

If $r \neq 1$ then $n = r - 1 \neq 0$ and $f/g$ is a non-constant meromorphic first integral of $X_d$, a contradiction. If $r = 1$ then $n = 0$ and (a) of lemma 2.1 implies that $f = c.g$, where $c \in \mathbb{C}$. Therefore,

$$0 = (f - cg)\, \partial_x \wedge \partial_y = X_d \wedge (Y_1 + c.R) \implies Y_1 = -c.R \neq 0 \ ,$$

by the division theorem and the fact that $d = dg(X_d) > 1$. But, this implies that $0 = [X_d, Y_1] = c(d-1).X_d \neq 0$, a contradiction. Hence, $r = d$.

Now, $r = d$ implies that $n = m = d - 1 > 0$ and $f \equiv 0$, for otherwise, $f/g$ would be a non-constant meromorphic first integral of $X_d$. It follows that $X_d \wedge Y_d = 0$, and so $Y_d = \lambda.X_d$, where $\lambda \neq 0$ is a constant. This proves the claim. □

Let us finish the proof of corollary 3. Let $Z = Y - \lambda.X$. Then $[X, Z] = 0$. If $Z \not\equiv 0$, then we could write $Z = \sum_{j=r}^{\infty} Z_j$, where $r > d$, $Z_j$ is homogeneous of degree $j$ and $Z_r \neq 0$. But, this contradicts claim 3.1 and proves the corollary. □

*Proof of Theorem 1.* — Let $(Z_\lambda)_{\lambda\in\mathbf{P}^1}$ be a non-trivial pencil of homogeneous of degree $d \geq 2$ commuting vector fields on $\mathbb{C}^2$. Fix two generators of the pencil, $X$ and $Y$, and set as before $X \wedge Y = f.\partial_x \wedge \partial_y$, $R \wedge X = g.\partial_x \wedge \partial_y$ and $R \wedge Y = h.\partial_x \wedge \partial_y$.

Suppose first that the pencil is colinear, that is, $f \equiv 0$. In this case, we can write $X = \alpha.Z$, where $\alpha$ is the greatest common divisor of the components of $X$ and $Z$ has an isolated singularity at $0 \in \mathbb{C}^2$. Since $Y \wedge X = 0$, we get $Y \wedge Z = 0$, and so $Y = \beta.Z$, where $\beta$ is a homogeneous polynomial with $dg(\beta) = dg(\alpha)$, by De Rham's division theorem. Now,

$$0 = [X, Y] = [\alpha.Z, \beta.Z] = (\alpha\, Z(\beta) - \beta\, Z(\alpha)).Z \implies Z(\beta/\alpha) = 0 .$$

Since the pencil is non-trivial, $\beta/\alpha$ is non-constant. On the other hand, we can write $\frac{\beta(x,y)}{\alpha(x,y)} = \phi(y/x)$, where $\phi(t) = \frac{\beta(1,t)}{\alpha(1,t)}$, because $\alpha$ and $\beta$ are homogeneous of the same degree. Therefore,

$$0 = Z(\phi(y/x)) = \phi'(y/x).Z(y/x) \implies Z(y/x) = 0 ,$$

because $\phi' \not\equiv 0$. This implies that $y\, Z(x) = x\, Z(y)$. If we set $Z = A\, \partial_x + B\, \partial_y$, then we get $y\, A = x\, B$, and so $A = \lambda.x$ and $B = \lambda.y$, where $\lambda$ is a homogeneous polynomial. Since 0 is an isolated singularity of $Z$, it follows that $\lambda$ is a constant. Hence, $X = \alpha_1.R$ and $Y = \beta_1.R$, where $\alpha_1 = \lambda.\alpha$ and $\beta_1 = \lambda.\beta$ are homogeneous polynomials of degree $d - 1$. This proves the first part of theorem 1.

Suppose now that the pencil is non-colinear. In this case, we have $f \not\equiv 0$. Let us prove that $g, h \not\equiv 0$. If $g \equiv 0$, for instance, then $X = \phi.R$, where $\phi \neq 0$ is a homogeneous polynomial of degree $m = n = d - 1 > 0$, by the division theorem. Therefore,

$$0 = [Y, \phi.R] = Y(\phi).R - m.\phi.Y .$$

Since $m.\phi.Y \neq 0$, the above relation implies that $Y$ and $R$ are colinear. Hence, $X//Y$, a contradiction. This proves (a) of theorem 1.

Since $m = n \neq 0$, it follows from (a) of theorem 2 that $f/g$ and $f/h$ are non-constant meromorphic first integrals of $X$ and $Y$, respectively, which proves (b) of theorem 1. Recall that $f$, $g$ and $h$ are homogeneous polynomials, where $dg(f) = 2d$, $dg(g) = dg(h) = d + 1$.

It follows from (c) of theorem 2 that we can write the decomposition of $f$, $g$ and $h$ into irreducible linear factors as $f = \Pi_{j=1}^{r} f_j^{\ell_j}$, $g = \Pi_{j=1}^{r} f_j^{m_j}.\Pi_{i=1}^{a} g_i^{a_i}$ and $h = \Pi_{j=1}^{r} f_j^{n_j}.\Pi_{i=1}^{b} h_i^{b_i}$, where $r > 0$, $m_j, n_j > 0$, $\ell_j \geq m_j + n_j - 1$ and any two polynomials of the set $\{f_1, \ldots, f_r, g_1, \ldots, g_a, h_1, \ldots, h_b\}$ are relatively prime. Set $k_j = min(m_j, n_j)$.

***Claim 3.2***. — *The generators of the pencil can be chosen in such a way that:*

(a) $m_j = n_j = k_j$ *for all* $j = 1, \ldots, r$.

(b) $a = b$ *and* $a_i = b_i = 1$ *for all* $i = 1, \ldots, a$.

*Proof.* — Set $X_\lambda = X + \lambda.Y$ and $R \wedge X_\lambda = g_\lambda.\partial_x \wedge \partial_y$,where $g_\lambda = g + \lambda.h$. It follows from Bertini's theorem that for a generic set of $\lambda \in \mathbb{C}$ the decomposition of $g_\lambda$ into linear irreducible factors is of the form:

$$g_\lambda = \Pi_{j=1}^r f_j^{k_j}.\Pi_{i=1}^s g_{i\lambda} \ , \tag{17}$$

where $s + \sum_j k_j = d+1$ and any two polynomials in the set $\{f_1, \dots, f_r, g_{1\lambda}, \dots, g_{s\lambda}\}$ are relatively prime. Now, it is sufficient to take $\lambda_1 \neq \lambda_2 \in \mathbb{C}$ such that $g_{\lambda_1}$ and $g_{\lambda_2}$ are as in (17). Set $X_1 = X_{\lambda_1}$, $Y_1 = X_{\lambda_2}$, $g = g_{\lambda_1}$ and $h = g_{\lambda_2}$. Then $X_1$ and $Y_1$ are generators of the pencil with the properties required in claim 3.2. □

From now on, we will suppose that the generators $X$ and $Y$ of the pencil satisfy claim 3.2. Let us prove that the decomposition of $f$ into irreducible linear factors is of the form

$$f = \Pi_{j=1}^r f_j^{2k_j+m_j} \ , \text{ where } m_j \geq 0. \tag{18}$$

Since $m = n = d-1 > 0$, relation (14) implies that

$$g\,dh - h\,dg = m\,f(y\,dx - x\,dy) \ , \ m \neq 0.$$

Set $g = \psi.G_1$ and $h = \psi.H_1$, where $\psi = \Pi_{j=1}^r f_j^{k_j}$. As the reader can check, we have

$$g\,dh - h\,dg = \psi^2.(G_1\,dH_1 - H_1\,dG_1) = m\,f(y\,dx - x\,dy) \implies \psi^2 \,|\, f \ .$$

Hence, the decomposition of $f$ is like in (18) and we get

$$G_1\,dH_1 - H_1\,dG_1 = m\,\Pi_{j=1}^r f_j^{m_j}(y\,dx - x\,dy) \ .$$

Now, consider the map $\phi\colon \mathbf{P}^1 \to \mathbf{P}^1$ given by

$$\phi[x:y] = \frac{g(x,y)}{h(x,y)} = \frac{G_1(x,y)}{H_1(x,y)} \ .$$

Since $G_1$ and $H_1$ are relatively prime, the degree of $\phi$ is $s = dg(G_1) = dg(H_1)$. Let $\{p_1, \dots, p_t\} \subset \mathbf{P}^1$ be the critical set of $\phi$ and $\phi(p_j) = c_j \in \mathbf{P}^1$. If $c_j \neq \infty$ set $K_j = G_1 - c_j.H_1$, and if $c_j = \infty$ set $K_j = H_1$. Suppose that $p_j$ is a critical point with $\operatorname{mult}(\phi, p_j) = \ell_j \geq 2$. This implies that we can write $K_j = \psi_j^{\ell_j}.A$, where $\psi_j$ is a linear polynomial, $A$ a homogeneous polynomial and $\psi_j$ does not divide $A$. We claim that $\psi_j^{\ell_j - 1} \,|\, \Pi_i f_i^{m_i}$. Indeed, if $c_j \neq \infty$, we get

$$K_j\,dH_1 - H_1\,dK_j = G_1\,dH_1 - H_1\,dG_1 = m\,\Pi_{i=1}^r f_i^{m_i}(y\,dx - x\,dy) \ . \tag{19}$$

Since $\psi_j^{\ell_j-1}$ divides $K_j\,dH_1 - H_1\,dK_j$, relation (19) implies the claim. If $c_j = \infty$ then $\psi_j^{\ell_j - 1}$ divides $G_1\,dH_1 - H_1\,dG_1$ and we get also the claim. Therefore, $\psi_j = \lambda_j.f_{i(j)}$, $\lambda_j \in \mathbb{C}^*$, for some $i(j) \in \{1, \dots, r\}$ and $\ell_j - 1 \leq m_{i(j)}$. In particular, we get $t \leq r$. By reordering the $f_{i's}$, if necessary, we can suppose without lost of generality that $i(j) = j$, $j = 1, \dots, t$. Set $\ell_j = 1$ for $t < j \leq r$. With these conventions, we have $m_j - (\ell_j - 1) \geq 0$ for all $j = 1, \dots, r$.

Let us prove that $m_j = \ell_j - 1$ for all $j = 1, \ldots, r$. Recall that $s + \sum_i k_i = d + 1$. Since $f = \Pi_i f_i^{2k_i+m_i}$ and $dg(f) = 2\,d$, we get

$$\sum_i m_i = dg(\Pi_i f_i^{m_i}) = 2\,d - 2\sum_i k_i = 2\,d - 2\,(d + 1 - s) = 2\,s - 2\ .$$

On the other hand, it follows from Riemann-Hurwitz formula (cf. [**2**]) and $m_i - (\ell_i - 1) \geq 0$ that

$$\sum_i (\ell_i - 1) = 2\,s - 2 = \sum_i m_i \implies 0 \leq \sum_{i=1}^{m} [m_i - (\ell_i - 1)] = 0 \implies m_i = \ell_i - 1\ ,\ \forall\, i\ .$$

This proves (d) and (e) of theorem 1. Note that (f) follows from (d) of theorem 2.

Let us prove that $1 \leq s \leq d - 1$ and $1 \leq r \leq d$. First of all note that

$$k_j \geq 1 \implies 2r \leq \sum_{j=1}^{r} (2k_j + m_j) = 2d \implies 1 \leq r \leq d\ .$$

Moreover,

$$s = d + 1 - \sum_{j=1}^{r} k_j \implies s \leq d + 1 - r \leq d \implies 0 \leq s \leq d\ .$$

Suppose by contradiction that $s = 0$. This implies that the map $\phi$ is constant, and so $g = \lambda.h$, where $\lambda \in \mathbb{C}^*$. It follows that

$$R \wedge (X - \lambda.Y) = 0 \implies X - \lambda.Y = \psi.R\ ,$$

where $\psi$ is homogeneous of degree $d - 1$. Therefore, the first part of theorem implies that $X$ and $Y$ are colinear with the radial vector field, a contradiction. Hence, $s \geq 1$. It remains to prove that $s \leq d - 1$. Suppose by contradiction that $s = d$. In this case, we get $g = f_1.g_1 \cdots g_d$, $h = f_1.h_1 \cdots h_d$ and $f = f_1^{2d}$. It follows that the map $\phi = (g_1 \cdots g_d)/(h_1 \cdots h_d)$ has degree $d \geq 2$ and just one ramification point, $(f_1 = 0)$, with multiplicity $2d - 1$. However, this is not possible, because this would imply that

$$\mathrm{mult}(\phi, (f_1 = 0)) = 2d - 1 > d\ .$$

It remains to prove that in the converse construction the vector fields $X$ and $Y$ defined by (9) in theorem 1 commute. But, this is a consequence of lemma 2.2 and the fact that $f$, $g$ and $h$ satisfy (b) of Theorem 2. This finishes the proof of Theorem 1. □

*Proof of Corollary 1.* — Let $X_1$ and $Y_1$ be generators of a pencil of commuting of degree two homogeneous vector fields on $\mathbb{C}^2$. As before, define $f_1$, $g_1$ and $h_1$ by $X_1 \wedge Y_1 = f_1\,\partial_x \wedge \partial_y$, $R \wedge X_1 = g_1\,\partial_x \wedge \partial_y$ and $R \wedge Y_1 = h_1\,\partial_x \wedge \partial_y$, respectively. If $g_1 \equiv h_1 \equiv 0$ then $X_1$ and $Y_1$ are multiple of the radial vector field, and so we are in case (a) of corollary 1. If not, then $f_1, g_1, h_1 \not\equiv 0$, by (a) of theorem 1. Moreover, the rational map $\phi = g_1/h_1$ has degree $s = 1$, by (c) of theorem 1. Therefore, the pencil has one movable direction and one or two fixed directions, because $g_1$ has degree $d + 1 = 3$.

Suppose that it has two fixed directions. In this case, we can suppose that they are $(x=0)$ and $(y=0)$. This implies that $g_1 = x.y.g_2$, $h_1 = x.y.h_2$ and $f_1 = x^2.y^2$, where $g_2$ and $h_2$ correspond to the movable direction. Since $g_2$ and $h_2$ are relatively prime, there exist $(a,b),(c,d)$ such that $a\,g_2 + b\,h_2 = x$ and $c\,g_2 + d\,h_2 = y$. If we set $g := x^2.y = x.y(a\,g_2 + b\,h_2)$ and $h := x.y^2 = x.y(c\,g_2 + d\,h_2)$, then we can apply lemma 2.2 to $f = x^2.y^2$, $g$ and $h$. We get the first integrals $f/g = (x^2.y^2)/(x^2.y) = y$, $f/h = (x^2.y^2)/(x.y^2) = x$, the forms $\omega := g\,\frac{d(f/g)}{f/g} = x^2\,dy$, $\eta := h\,\frac{d(f/h)}{f/h} = y^2\,dx$, and the vector fields $X = x^2\,\partial_x$, $Y = y^2\,\partial_y$. So, we are in case (b) of corollary 1.

Suppose that it has one fixed direction. We can suppose that it is $(y=0)$. In this case, we have $g_1 = y^2.g_2$, $h_1 = y^2.h_2$ and $f = y^4$. Consider linear combinations $a\,g_2 + b\,h_2 = x$ and $c\,g_2 + d\,h_2 = y$. So, we have just to apply lemma 2.2 to the polynomials $f = y^4$, $g = x.y^2$ and $h = y^3$. By doing this, we obtain case (c) of corollary 1, as the reader can check. $\square$

*Proof of Corollary 2.* — Let $f$, $g$ and $h$ be as in theorem 1. If $g \equiv h \equiv 0$ then we are in case (a) of corollary 2. If not, then $f,g,h \not\equiv 0$ and $\phi = g/h$ has degree $s$, where $s \in \{1,2\}$.

Let us consider the case where $s = 2$. Let $\phi\colon \mathbf{P}^1 \to \mathbf{P}^1$ be a map of degree two. It follows from Riemann-Hurwitz formula that $\sum_p(\mathrm{mult}(\phi,p) - 1) = 2\,s - 2 = 2$, and so the map must have two ramification points, both of multiplicity two. After composing the map in both sides with Möbius transformations, we can suppose that $\phi[x:y] = y^2/x^2$. This implies that $(x=0)$ and $(y=0)$ are fixed directions of the pencil, so that $x.y$ divides $g$ and $h$. Since $dg(g) = dg(h) = 4$ and $s = 2$, we get $g = x.y.g_1.g_2$ and $h = x.y.h_1.h_2$, and so $k_1 = k_2 = 1$ in (2) of theorem 1. Since $dg(f) = 6$ and $\mathrm{mult}(\phi,(x=0)) = \mathrm{mult}(\phi,(y=0)) = 2$, we must have $m_1 = m_2 = 1$ and $f = x^3.y^3$. In this case, we have

$$\phi = \frac{g}{h} = \frac{(g/x.y)}{(h/x.y)} = \frac{y^2}{x^2} \implies g = x.y^3 \text{ and } h = x^3.y\ .$$

So, when we apply lemma 2.2, we get $f/g = x^2$, $f/h = y^2$, $\omega = 2y^3\,dx$ and $\eta = 2x^3\,dy$. Hence, we can set $X = x^3\,\partial_x$ and $Y = y^3\,\partial_y$. In this case we get case (e) of corollary 2.

Suppose now that $s = 1$. In this case, we have just one movable direction and the map $\phi$ has no ramification points, which implies that $m_j = 0$ for all $j = 1,\ldots,r$. This implies that $f = \Pi_{j=1}^r f_j^{2k_j}$. Since $dg(f) = 6$, we have three possibilities: (1). $r = 1$ and $k_1 = 3$. (2). $r = 2$, $k_1 = 1$ and $k_2 = 2$. (3). $r = 3$ and $k_1 = k_2 = k_3 = 1$.

Case (1). In this case, we have just one fixed direction $f_1$. After a linear change of variables in $\mathbb{C}^2$, we can suppose that it is $f_1 = y$. This implies that $f = y^6$, $g = y^3.g_1$ and $h = y^3.h_1$. Since $g_1$ and $h_1$ are relatively prime, there exist $a,b,c,d \in \mathbb{C}$ such that $a.d - b.c \neq 0$ and $a.g_1 + b.h_1 = x$ and $c.g_1 + d.h_1 = y$. Therefore, we can apply the construction of lemma 2.2 to $f = y^6$, $g = y^4$ and $h = x.y^3$. This gives the first integrals $f/g = y^2$ and $f/h = y^3/x$. Moreover,

$$\begin{cases} \omega = i_X(dx \wedge dy) = 2\,y^4\,\frac{dy}{y} = 2\,y^3\,dy \implies X = 2\,y^3\,\partial_x \\ \eta = i_Y(dx \wedge dy) = x.y^3\,(3\,\frac{dy}{y} - \frac{dx}{x}) = 3\,xy^2\,dy - y^3\,dx \implies Y = 3\,xy^2\partial_x + y^3\,\partial_y. \end{cases}$$

Therefore, we get case (b) of corollary 2.

Case (2). In this case, we have two fixed directions, that we can suppose to be $f_1 = x$ and $f_2 = y$. Since $k_1 = 1$ and $k_2 = 2$, we get $g = x.y^2.g_1$, $h = x.y^2.h_1$ and $f = x^2.y^4$. After taking linear combinations, we can suppose that $g = x^2.y^2$ and $h = x.y^3$. This gives the first integrals $y^2$ and $x.y$ and so $\omega = 2\,x^2\,y\,dy$ and $\eta = xy^2\,dy + y^3\,dx$ and we are in case (c).

Case (3). In this case, we have three fixed directions. After a linear change of variables we can suppose that they are $f_1 = x$, $f_2 = y$ and $f_3 = x + y$. This gives $g = x\,y\,(x{+}y).g_1$, $h = x\,y\,(x{+}y).h_1$ and $f = x^2\,y^2\,(x{+}y)^2$. After taking linear combinations of $g_1$ and $h_1$, we can suppose that $g = x^2\,y\,(x+y)$ and $h = x\,y^2\,(x+y)$. Therefore we get the first integrals are $f/g = y\,(x+y)$, $f/h = x\,(x+y)$ and

$$\begin{cases} \omega = x^2\,y\,(x+y)\,[\frac{dy}{y} + \frac{dx+dy}{x+y}] = x^2\,y\,dx + (2\,x^2\,y + x^3)\,dy \\ \quad \implies X = (2\,x\,y^2 + x^3)\,\partial_x - x^2\,y\,\partial_y \\ \eta = x\,y^2\,(x+y)\,[\frac{dx}{x} + \frac{dx+dy}{x+y}] = (2\,x\,y^2 + y^3)\,dx + x\,y^2\,dy \\ \quad \implies Y = -x\,y^2\,\partial_x + (2\,x\,y^2 + y^3)\,\partial_y. \end{cases}$$

Therefore, we are in case (d) of corollary 2. □

## References

[1] D. Cerveau & J.-F. Mattei – "Formes intégrables holomorphes singulières", *Astérisque* **97** (1982).

[2] H. M. Farkas & I. Kra – *Riemann surfaces*, Springer, 1980.

[3] A. Guillot – "Sur les exemples de Lins Neto de feuilletages algébriques", *C. R. Math. Acad. Sci. Paris* **334** (2002), p. 747–750.

[4] G. d. Rham – "Sur la division de formes et de courants par une forme linéaire", *Comment. Math. Helv.* **28** (1954), p. 346–352.

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# UN THÉORÈME DE TYPE HAEFLIGER DÉFINISSABLE

*par*

Jean-Marie Lion & Patrick Speissegger

*À José Manuel Aroca, pour ses soixante ans*

***Résumé.*** — Soit $M \subset \mathbf{R}^n$ une sous-variété définissable dans une structure o-minimale $\mathcal{A}$ et soit $\omega \in \Lambda(M)$ une 1-forme différentielle $\mathcal{A}$-définissable. Nous montrons que si $\omega$ définit un feuilletage de codimension un sur $M$ alors il existe un recouvrement fini de $M$ par des ouverts $\mathcal{A}$-définissables $M_1, \ldots, M_r$ qui vérifient la propriété suivante : pour chaque $i$, tout lacet $C^1$ inclus dans $M_i$ est tangent à $\ker(\omega)$ en un point.

***Abstract* (Definisable Haefliger's Type Theorem).** — Let $\mathcal{A}$ be an o-minimal expansion of the real field, $M$ a submanifold of $R^n$ and $\omega$ a differentiable 1-form on $M$. We assume that $M$ and $\omega$ are definable in $\mathcal{A}$ and $\omega$ defines a foliation on $M$ of codimension one. Then there are definable, open subsets $M_i$ of $M$, for $i = 1, \ldots, r$, such that every $C^1$ loop contained in $M_i$ is tangent to $\ker(\omega)$ at some point.

## Introduction

Soit $M \subset \mathbf{R}^n$ une sous-variété différentielle de classe $C^k, k \geq 2$, de dimension $m$, connexe et $\omega \in \Lambda^k(M)$ une 1-forme différentielle de clase $C^k$ définie sur $M$. On suppose que $\omega$ est *non singulière et intégrable* : en tout point $x$ de $M$

$$\omega(x) \neq 0 \text{ et } \omega \wedge d\omega(x) = 0.$$

D'après le théorème de Frobenius, pour tout point $x$ de $M$ il existe une carte locale de $M$ centrée en $x$ dans laquelle le champ d'hyperplans $\ker(\omega)$ est le champ $\ker(dx_m)$. Par conséquent, la forme $\omega$ définit sur $M$ un *feuilletage de codimension un* noté $\mathcal{F}$. Par tout point $x$ de $M$ passe une unique *feuille* $V$ de $\mathcal{F}$. C'est une hypersurface de classe $C^k$, connexe, immergée injectivement dans $M$, tangente au champ d'hyperplans $x \in M \mapsto \ker(\omega)(x)$ et maximale pour ces propriétés (voir **[4]**, **[14]** ou **[16]** pour une introduction à la théorie des feuilletages).

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***Mots clefs.*** — Structure o-minimale, feuilletage.

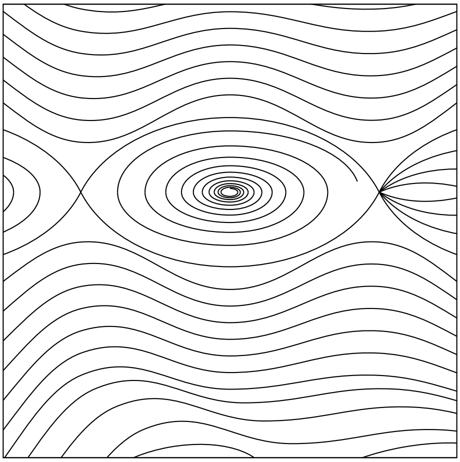

FIGURE 1. Un feuilletage du carré privé de trois points

Observons que si la condition d'intégrabilité de Frobenius n'est pas vérifiée alors d'une part il n'y aurait pas de feuilletage et d'autre part il existerait nécessairement des lacets transverses au champ $\ker(\omega)$. Ce dernier point est une conséquence immédiate du théorème de Darboux sur les modèles locaux des formes différentielles (voir par exemple [**13**]).

Les feuilles du feuilletage $\mathcal{F}$ ne sont pas toujours fermées ou plongées proprement dans $M$ et elles peuvent être denses. Cependant si $M$ est une variété analytique simplement connexe et $\omega$ est analytique, alors, d'après un théorème d'A. Haefliger [**15**] (voir aussi [**23**]), le feuilletage *n'admet pas de transversale fermée* (ceci signifie qu'il n'existe pas dans $M$ de lacet de classe $C^1$ et transverse au feuilletage) et toute feuille est une hypersurface analytique fermée de $M$ qui sépare $M$ en deux composantes connexes. En particulier toute feuille est de *Rolle* : tout lacet différentiable qui la rencontre est tangent au feuilletage en un point (voir [**17**] et [**23**]). Si la simple connexité de $M$ joue un rôle important dans la preuve du théorème de A. Haefliger, l'analyticité de $M$ et celle de $\omega$ sont aussi essentielles. Elles garantissent l'analyticité des holonomies. Un exemple de G. Reeb [**11**] montre qu'on ne peut s'affranchir totalement des hypothèses d'analyticité et de simple connexité (voir [**4**] ou [**14**]).

Dans [**26**], C.A. Roche conjecture qu'il est possible de recouvrir $M$ par un nombre fini d'ouverts $M_1, \ldots, M_s$ tels que sur chaque $M_i$ la forme $\omega$ induit un feuilletage $\mathcal{F}_i$ dynamiquement simple : il n'admet pas de transversale fermée et toute feuille est une hypersurface fermée de $M_i$ qui sépare $M_i$ en deux composantes connexes.

L'objet de cet article est de donner une réponse positive à cette conjecture dans un cadre assez général, le cadre o-minimal [**8**] (voir aussi [**10**] ou [**31**]).

***Théorème 1***. — *Si $M$ et $\omega$ sont définissables dans une structure o-minimale $\mathcal{A}$, il existe un recouvrement fini $M_1, \ldots, M_s$ de $M$ par des ouverts $\mathcal{A}$-définissables tel que* *la forme $\omega$ induit sur chaque $M_i$ un feuilletage $\mathcal{F}_i$ dynamiquement simple : il n'admet pas de transversale fermée et toute feuille est une hypersurface fermée de $M_i$ qui sépare $M_i$ en deux composantes connexes.*

Cette réponse positive à la conjecture de Roche a la conséquence dynamique suivante : un espace feuilleté par un champ d'hyperplans définissable dans une structure o-minimale se décompose en *un ensemble fini de régions avec un comportement dynamique uniforme* [**26**].

D'après les résultats de [**6**] adaptés au cadre o-minimal, pour chaque $i$ le feuilletage $\mathcal{F}_i$ est *presque* une $C^k$-fibration triviale : c'est le cas dans les composantes connexes de $M_i \setminus Z_i$ où $Z_i$ est la réunion d'un nombre fini de feuilles de $\mathcal{F}_i$. De plus, d'après [**30**], les feuilles du feuilletage $\mathcal{F}_i$ sont définissables dans une structure o-minimale, la clôture pfaffienne de $\mathcal{A}$. Ainsi le théorème, combiné à [**30**], pourrait être une source de nouveaux exemples de structures o-minimales.

Notre théorème est une généralisation du théorème 1 de [**20**] dans laquelle on s'affranchit de toute condition d'analyticité. On sait que les techniques de désingularisation des ensembles analytiques (présentées par exemple dans l'article de J.M. Aroca, H. Hironaka et J.L. Vicente [**1**]) appliquées à l'étude des feuilletages analytiques singuliers de codimension un et à celle des champs de vecteurs analytiques peuvent se révéler très fructueuses (voir par exemple [**5**] ou [**25**]). Ici, il faudra se résoudre à des méthodes plus naïves de nature essentiellement topologique et différentielle : l'idée principale, déja présente dans [**20**], est de construire un recouvrement fini de $M$ par des ouverts $M_i$ sur chacun desquels la forme $\omega$ induit un feuilletage dont les feuilles sont des graphes d'applications.

Après une présentation des structures o-minimales (partie 1) nous donnerons trois observations topologiques et une proposition utiles à la preuve du théorème (partie 2). Ensuite (partie 3) nous énoncerons une proposition qui permet de recouvrir l'espace en ouverts sur lesquels la dynamique du feuilletage se révèlera simple ou au moins contrôlée par la dynamique de feuilletages induits sur certaines parties de leurs bords. La preuve de cette proposition, purement technique, sera donnée en annexe (partie 5). Elle se conclut par un argument de transversalité de Thom [**32**]. Dans la partie 3 on donnera aussi un exemple qui illustre la nécessité du contrôle de la dynamique au bord. La partie 4 sera consacrée à la preuve du théorème.

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## 1. Les structures o-minimales (voir [8] ou [10])

***Définition 1.1***. — On appelle *structure* une famille $\mathcal{A} = \cup_{n\in\mathbf{N}} \mathcal{A}_n$ de sous-ensembles des espaces euclidiens $\mathbf{R}^n, n \in \mathbf{N}$ qui vérifie les propriétés suivantes :

– si $X, Y \in \mathcal{A}_n$ alors $X \cap Y$, $X \cup Y$ et $X \setminus Y$ appartiennent à $\mathcal{A}_n$

- si $X \in \mathcal{A}_n$ et $Y \in \mathcal{A}_m$ alors $X \times Y \in \mathcal{A}_{n+m}$
- si $Z \in \mathcal{A}_{n+m}$ alors $\{x \in \mathbf{R}^n | \exists y \in \mathbf{R}^m, (x,y) \in Z\} \in \mathcal{A}_n$
- si $P \in \mathbf{R}[x_1, \ldots, x_n]$ alors $\{P > 0\} \in \mathcal{A}_n$.

***Définition 1.2.*** — Suivant la terminologie introduite par L. van den Dries [**8**] la structure $\mathcal{A}$ est dite *o-minimale* si les éléments de $\mathcal{A}_1$ sont les unions finies d'intervalles et de points.

D'après le théorème de Tarski-Seidenberg (voir par exemple [**3**] ou [**2**]) les semi-algébriques forment une structure o-minimale. S. Łojasiewicz montre que les semi-analytiques possèdent de nombreuses propriétés de régularité (stratifications de Whitney, triangulations, ordre de contact, voir [**21**]) mais ils ne forment pas une structure o-minimale. Cependant d'après un théorème de A. Gabrielov [**12**] les sous-ensembles semi-analytiques relativement compacts en engendrent une. On connaît depuis la fin des années 80 de nombreux autres exemples de structures o-minimales (voir par exemple [**9**], [**35**], [**29**], [**28**]). L'article [**29**] montre en particulier qu'il n'existe pas une structure o-minimale maximale qui engloberait toutes les autres et que certaines structures o-minimales sont très éloignés des ensembles analytiques.

Considérons une structure o-minimale $\mathcal{A}$. Les éléments des $\mathcal{A}_n$ sont appelés *ensembles $\mathcal{A}$-définissables.* Une fonction ou une application est dite *$\mathcal{A}$-définissable* si son graphe est $\mathcal{A}$-définissable. Une sous-variété de $\mathbf{R}^n$ est dite *$\mathcal{A}$-définissable* si c'est un élément de $\mathcal{A}_n$. Une forme différentielle est dite *$\mathcal{A}$-définissable* si son graphe est $\mathcal{A}$-définissable. Les grassmanniennes $\mathcal{G}_n^p$ des $p$-plans de $\mathbf{R}^n$ et leur réunion $\mathcal{G}_n$ sont des sous-ensembles semi algébriques. Elles sont donc $\mathcal{A}$-définissables. Les trois propositions suivantes récapitulent des propriétés élémentaires mais fondamentales des structures o-minimales.

***Proposition 1* (Propriétés ensemblistes)**

- *La composée d'applications $\mathcal{A}$-définissables l'est aussi.*
- *Si $f : X \to \mathbf{R}$ est $\mathcal{A}$-définissable alors $\{f = 0\}$, $\{f > 0\}$ et $\{f < 0\}$ le sont aussi.*
- *Si $f : X \to \mathbf{R}^m$ et $Y \subset \mathbf{R}^m$ sont $\mathcal{A}$-définissables alors $f^{-1}(Y)$ l'est aussi.*

***Proposition 2* (Propriétés topologiques)**

- *Si $X$ est $\mathcal{A}$-définissable alors son adhérence $\overline{X}$ et son intérieur $\operatorname{Int}(X)$ le sont aussi.*

***Proposition 3* (Propriétés différentielles)**

- *Les dérivées partielles d'une application différentiable et $\mathcal{A}$-définissable sont $\mathcal{A}$-définissables.*
- *Si $X$ est une sous-variété de dimension $p$ de $\mathbf{R}^n$, de classe $C^k$ et $\mathcal{A}$-définissable alors son fibré tangent est aussi $\mathcal{A}$-définissable.*

– *Si* $\alpha : X \to \mathcal{G}_n^p$ *et* $\beta : X \to \mathcal{G}_n^q$ *sont deux applications* $\mathcal{A}$*-définissables alors les ensembles* $\{\alpha \subset \beta\}$ *et* $\{\dim(\alpha \cap \beta) = d\}, d = 1, \ldots, n$ *ainsi que l'application* $\alpha \cap \beta$ *à valeurs dans* $\mathcal{G}_n$ *le sont aussi.*
– *Si* $X$ *est une sous-variété de* $\mathbf{R}^n$ $\mathcal{A}$*-définissable et* $\omega \in \Lambda(X)$ *est une 1-forme différentielle* $\mathcal{A}$*-définissable alors l'application* $\ker(\omega)$ *l'est aussi.*

Les structures o-minimales possèdent la propriété de finitude uniforme suivante (voir [**8**], [**10**]).

***Proposition 4***. — *Si* $X \subset \mathbf{R}^n$ *est* $\mathcal{A}$*-définissable il existe un entier* $N$ *qui majore le nombre de composantes connexes de* $X \cap E$ *pour tout sous-espace affine* $E$ *de* $\mathbf{R}^n$.

***Définition 1.3***. — Soit $k \in \mathbf{N} \setminus \{0\}$. On définit par récurrence sur $n$ les *cylindres de* $\mathbf{R}^n$ *de classe* $C^k$ *et* $\mathcal{A}$*-définissables.* L'unique cylindre de $\mathbf{R}^0 = \{0\}$ de classe $C^k$ et $\mathcal{A}$-définissable est $\mathbf{R}^0$ lui même. Soit $n \in \mathbf{N}\setminus\{0\}$ et supposons avoir défini les cylindres de $\mathbf{R}^{n-1}$ de classe $C^k$ et $\mathcal{A}$-définissables. Un sous-ensemble $C$ de $\mathbf{R}^n$ est un *cylindre de* $\mathbf{R}^n$ *de classe* $C^k$ *et* $\mathcal{A}$*-définissable* si les conditions suivantes sont vérifiées.

– L'ensemble $C$ est une sous-variété différentielle de classe $C^k$ et $\mathcal{A}$-définissable.
– Il existe un cylindre $D$ de $\mathbf{R}^{n-1}$ de classe $C^k$ et $\mathcal{A}$-définissable tel que soit $C$ est le graphe d'une fonction de $D$ dans $\mathbf{R}$, de classe $C^k$ et $\mathcal{A}$-définissable, soit

$$C = D \times \mathbf{R}$$

ou
$$C = \{(x, y) : x \in D, \phi(x) < y < \psi(x)\}$$
ou
$$C = \{(x, y) : x \in D, \phi(x) < y\}$$
ou
$$C = \{(x, y) : x \in D, y < \psi(x)\}$$

avec $\phi$ et $\psi$ des fonctions définies sur $D$ à valeurs dans $\mathbf{R}$ de classe $C^k$, $\mathcal{A}$-définissables et telles que $\phi < \psi$. Le cylindre $D$ est appelé *base* de $C$.

On déduit de cette définition les propositions suivantes.

***Proposition 5***. — *Si* $C$ *est un cylindre de* $\mathbf{R}^n$*, de classe* $C^k$*,* $\mathcal{A}$*-définissable et de base* $D$ *et si* $D'$ *est un cylindre de* $\mathbf{R}^{n-1}$ *de classe* $C^k$*,* $\mathcal{A}$*-définissable et inclus dans* $D$ *alors* $C \cap (D' \times \mathbf{R})$ *est un cylindre de* $\mathbf{R}^n$*, de classe* $C^k$ *et* $\mathcal{A}$*-définissable.*

***Proposition 6***. — *Si* $C$ *est un cylindre de dimension* $d$ *de* $\mathbf{R}^n$ *de classe* $C^k$ *et* $\mathcal{A}$*-définissable alors quitte à permuter les coordonnées, le cylindre* $C$ *est le graphe d'une application définie sur un cylindre ouvert* $D'$ *de* $\mathbf{R}^d$*, à valeurs dans* $\mathbf{R}^{n-d}$*, de classe* $C^k$ *et* $\mathcal{A}$*-définissable. De plus il existe un difféomorphisme de* $\mathbf{R}^d$ *dans* $D'$*, de classe* $C^k$ *et* $\mathcal{A}$*-définissable. Ainsi* $C$ *est connexe et il existe un difféomorphisme de* $\mathbf{R}^d$ *dans* $C$*, de classe* $C^k$ *et* $\mathcal{A}$*-définissable.*

***Définition 1.4***. — Soit $k \in \mathbf{N} \setminus \{0\}$. On définit par récurrence sur $n$ les *décompositions cylindriques de classe* $C^k$ *et* $\mathcal{A}$*-définissables.* L'unique *décomposition cylindrique de* $\mathbf{R}^0 = \{0\}$ *de classe* $C^k$ *et* $\mathcal{A}$*-définissable* est $\mathbf{R}^0$ lui même. Soit $n \in \mathbf{N} \setminus \{0\}$ et supposons avoir défini les décompositions cylindriques de $\mathbf{R}^{n-1}$ de classe $C^k$ et

$\mathcal{A}$-définissables. Une famille finie $C_1, \ldots, C_r$ de sous-ensembles de $\mathbf{R}^n$ est une *décomposition cylindrique de* $\mathbf{R}^n$ *de classe* $C^k$ *et* $\mathcal{A}$*-définissable* si les conditions suivantes sont vérifiées.

- La famille $C_1, \ldots, C_r$ est une partition de $\mathbf{R}^n$ en cylindres de $\mathbf{R}^n$ de classe $C^k$ et $\mathcal{A}$-définissables.
- Les bases des $C_i$ forment une décomposition cylindrique de $\mathbf{R}^{n-1}$ de classe $C^k$ et $\mathcal{A}$-définissable.

Les structures o-minimales possèdent la propriété de décomposition cylindrique suivante (voir [**8**], [**10**]).

***Proposition 7***. — *Soient $X_1, \ldots, X_d$ des sous-ensembles de $\mathbf{R}^n$, $\mathcal{A}$-définissables et pour chaque $i = 1, \ldots, d$ une application $\phi_i : X_i \to \mathbf{R}^{m_i}$ $\mathcal{A}$-définissable. Soit $k \in \mathbf{N} \setminus \{0\}$. Il existe une décomposition cylindrique $C_1, \ldots, C_r$ de $\mathbf{R}^n$ de classe $C^k$ et $\mathcal{A}$-définissable qui vérifie les propriétés suivantes.*

- *Chaque $X_i$ est une réunion de $C_j$.*
- *Si $C_j \subset X_i$ alors la restriction de $\phi_i$ à $C_j$ est une application différentiable de classe $C^k$ et de rang constant.*

***Définition 1.5***. — La décomposition cylindrique obtenue dans la proposition 7 est dite *adaptée aux $X_i$ et aux $\phi_i$*. Pour chaque $i$, la partition de $X_i$ donnée par la proposition s'appelle *décomposition cylindrique de $X_i$*.

La proposition 7 permet de faire des stratifications de Whitney [**34**] adaptées à une famille finie d'ensembles définissables (voir [**8**], [**10**]) et d'expliquer plus finement l'adhérence d'un ensemble définissable que ne le fait la proposition 2.

Toute sous-variété de $\mathbf{R}^n$ qui est $\mathcal{A}$-définissable admet un recouvrement fini par des cartes $\mathcal{A}$-définissables. Plus précisément, on démontre à partir des propositions précédentes :

***Proposition 8***. — *Soit $M \subset \mathbf{R}^n$ une sous-variété différentielle de classe $C^k$, de dimension $m$ et $\mathcal{A}$-définissable. Il existe des sous-ensembles $M_1, \ldots, M_r \subset M$, $\mathcal{A}$-définissables et tels que*

- *$M = M_1 \cup \cdots \cup M_r$*
- *pour chaque $i \in \{1, \ldots, r\}$, quitte à faire une permutation $\sigma_i$ des coordonnées $(x_1, \ldots, x_n)$, $M_i$ est un cylindre de $\mathbf{R}^n$ de dimension $m$, de classe $C^k$ et $\mathcal{A}$-définissable.*

Le principe de la preuve de cette proposition sera repris pour établir la proposition 10.

*Démonstration*. — Si $i_1 < \cdots < i_m$ avec $i_1, \ldots, i_m \in \{1, \ldots, n\}$ on note $M_{i_1,\ldots,i_m}$ l'ensemble des $x \in M$ où la restriction à $M$ de la projection $\pi_{i_1,\ldots,i_m}$ définie par

$$\pi(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_m})$$

est de rang $m$. Les $M_{i_1,\dots,i_m}$ sont des sous-ensembles de $M$, ouverts pour la topologie induite et $\mathscr{A}$-définissables d'après la proposition 3. Leur réunion est égale à $M$. On peut donc supposer que $M$ est l'un d'eux, $M = M_{1,\dots,m}$ par exemple. On pose $\pi = \pi_{1,\dots,m}$. La restriction de $\pi$ à $M$ est de rang $m$.

D'après la proposition 7 on sait que $M$ admet une décomposition cylindrique de classe $C^k$ et $\mathscr{A}$-définissable. Considérons donc un cylindre $C$ de dimension $d \leq m$ de $\mathbf{R}^n$, de classe $C^k$, $\mathscr{A}$-définissable et contenu dans $M$. Il suffit de montrer qu'il existe $Z \subset C$, de dimension au plus $d-1$, $\mathscr{A}$-définissable et des sous-ensembles $M_1, \dots, M_r$ de $M$, $\mathscr{A}$-définissables et tels que

- $C \setminus Z \subset M_1 \cup \cdots \cup M_r$
- pour chaque $i \in \{1, \dots, r\}$, $M_i$ est un cylindre de $\mathbf{R}^n$ de dimension $m$, de classe $C^k$ et $\mathscr{A}$-définissable.

Si $d = m$ la conclusion est immédiate. On suppose $d < m$. Puisque la restriction de $\pi$ à $M$ est de rang $m$, d'après la proposition 6, quitte à permuter les coordonnées $(x_1, \dots, x_m)$ le cylindre $C$ est le graphe d'une application de classe $C^k$, $\mathscr{A}$-définissable et définie sur un cylindre de classe $C^k$ et $\mathscr{A}$-définissable de $\mathbf{R}^d$.

Si $x = (x'; x_{d+1}, \dots, x_n) \in C$ on pose

$$\begin{aligned}\delta(x) = \max\{\delta : \forall(\delta_{d+1}, \dots, \delta_m),\\ &\sup|\delta_i| < \delta \Rightarrow \exists!(\varepsilon_{m+1}, \dots, \varepsilon_n), \sup|\varepsilon_j| < \sqrt{\delta},\\ &(x'; x_{d+1} + \delta_{d+1}, \dots, x_m + \delta_m, x_{m+1} + \varepsilon_{m+1}, \dots, x_n + \varepsilon_n) \in M\}.\end{aligned}$$

D'après la proposition 1 la fonction $\delta$ est $\mathscr{A}$-définissable. Puisque la restriction de $\pi$ à $M$ est de rang $m$, la fonction $\delta$ est strictement positive en tout point de $C$. D'après la proposition 7 il existe des cylindres $C_1, \dots, C_r \subset C$, de dimension $d$, de classe $C^k$ et $\mathscr{A}$-définissables tels que

- $C \setminus (C_1 \cup \cdots \cup C_r)$ est de dimension au plus $d-1$
- la restriction de $\delta$ à chaque $C_i$ est de classe $C^k$ et $\mathscr{A}$-définisssable.

Si $i = 1, \dots, r$ on pose

$$\begin{aligned}M_i = \{(x'; x_{d+1} + \delta_{d+1}, \dots, x_m + \delta_m, x_{m+1} + \varepsilon_{m+1}, \dots, x_n + \varepsilon_n) \in M :\\ x = (x'; x_{d+1}, \dots, x_n) \in C_i, \sup|\delta_i| < \delta(x), \sup|\varepsilon_j| < \sqrt{\delta(x)}\}.\end{aligned}$$

Par construction les $M_i$ sont des cylindres de dimension $m$, de classe $C^k$, inclus dans $M$ et $\mathscr{A}$-définissables. L'ensemble

$$Z = C \setminus (M_1 \cup \cdots \cup M_r)$$

est inclus dans $C$, $\mathscr{A}$-définissable et de dimension au plus $d-1$. □

## 2. Observations et proposition topologiques

Soit $\rho : \tilde{V} \to M'$ un revêtement de base $M'$ simplement connexe et avec l'espace total $\tilde{V}$ connexe. Alors $\rho$ est un homéomorphisme (voir **[7]** ou **[24]**).

Dans $\mathbf{R}^m$ une hypersurface fermée sépare $\mathbf{R}^m$ en deux composantes connexes exactement (voir [**7**] ou [**24**]).

Soit $\mathscr{F}$ un feuilletage de codimension un d'une variété $M$ associé à une 1-forme différentielle $\omega$, non singulière et intégrable. Si une feuille $V$ de $\mathscr{F}$ est une sous-variété fermée de $M$ qui disconnecte $M$ alors $V$ est une feuille de Rolle. La réciproque est fausse : si $V$ est une feuille de Rolle alors c'est une sous-variété fermée de $M$ mais elle ne disconnecte pas toujours $M$. Par exemple, comme le mentionnent Moussu et Roche dans [**23**] les courbes $C_t = \{(x, t\exp(\frac{-1}{x})\}, t \in \mathbf{R}$ sont des feuilles de Rolle du feuilletage $\mathscr{F}$ de $\mathbf{R}^2 \setminus \{0\}$ associé à la forme $ydx - x^2dy$ mais elles ne disconnectent pas $\mathbf{R}^2 \setminus \{0\}$. En revanche, si $M$ est difféomorphe à $\mathbf{R}^m$ il est équivalent de dire que $V$ est de Rolle et que $V$ sépare en deux composantes connexes (voir [**17**] ou [**23**]).

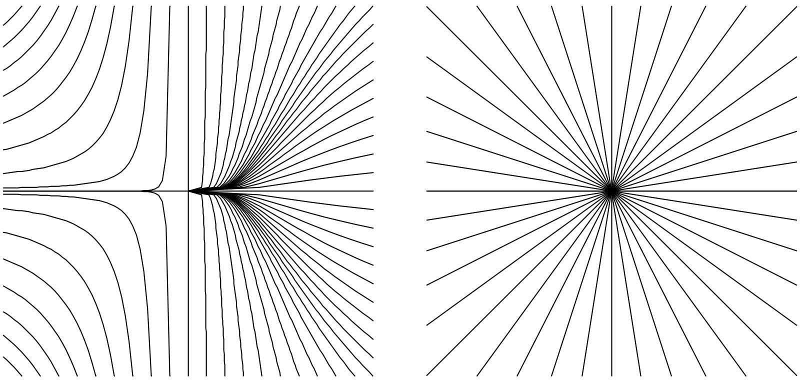


FIGURE 2. Rolle, être ou ne pas être : feuilletages associés aux formes $ydx - x^2dy$ et $ydx - xdy$

Outre ces trois observations topologiques la proposition générale qui suit est utile à la preuve du théorème. C'est un corollaire d'un lemme de Morse à paramètres en classe $C^k$ qui est dû à S. López de Medrano [**22**] et Kuiper [**18**]. Ici, comme dans tout le papier $k \geq 2$.

***Proposition 9***. — *Soit $f = x_1^2 + \cdots + x_n^2$ et soit $g$ une fonction de classe $C^k$ définie au voisinage de l'origine de $\mathbf{R}^n$ et telle que $g(0) = 0$ et $dg(0) \neq 0$. Il existe alors une fonction $G$ d'une variable, de classe $C^{k-1}$, telle que $G(0) = 0$ et $dG(0) \neq 0$ ainsi que des coordonnées $y = (y_1, \ldots, y_n)$ de classe $C^{k-1}$ au voisinage de l'origine de $\mathbf{R}^n$ telles que $f = y_1^2 + \cdots + y_n^2$ et $g = G \circ y_n$.*

*Démonstration*. — Quitte à faire un changement de coordonnées orthogonales (donc qui laisse invariant l'expression de $f$) on peut supposer que $dg(0) = \lambda dx_n$. Posons $z_n = \frac{1}{\lambda} g$. D'après le théorème des fonctions implicites il existe une fonction $h$ de classe $C^k$ définie au voisinage de l'origine telle que $x_n = h(x_1, \ldots, x_{n-1}, z_n)$ et $dh(0) = dz_n$. Par conséquent il existe une fonction $\varepsilon$ de classe $C^k$ et dont le 2-jet à l'origine est nul telle que $f = x_1^2 + \cdots + x_{n-1}^2 + z_n^2 + \varepsilon(x_1, \ldots, x_{n-1}, z_n)$. D'après le lemme de

Morse à paramètre en classe $C^k$ de López de Medrano [**22**] appliqué à la fonction $F = x_1^2 + \cdots + x_{n-1}^2 + \varepsilon(x_1, \ldots, x_{n-1}, z_n)$ il existe des coordonnées $y' = (y_1, \ldots, y_{n-1})$ de classe $C^{k-1}$ définies au voisinage de l'origine de $\mathbf{R}^{n-1}$ et une fonction $\alpha$ de classe $C^k$ définie au voisinage de l'origine de $\mathbf{R}$ et dont le 2-jet à l'origine est nul telles que $F = y_1^2 + \cdots + y_{n-1}^2 + \alpha(z_n)$. D'après le lemme de Morse (version de Kuiper [**18**]) en classe $C^k$ appliqué à $z_n^2 + \alpha(z_n)$ il existe une fonction $G$ d'une variable, de classe $C^{k-1}$, telle que $G(0) = 0$ et $dG(0) \neq 0$ vérifiant $z_n^2 + \alpha(z_n) = y_n^2$ et $g = \lambda z_n = G \circ y_n$. □

Cette proposition implique que si $\varepsilon > 0$ est petit alors pour tout $t \in \mathbf{R}$ l'ensemble $\{f < \varepsilon, g = t\}$, s'il n'est pas vide, est connexe, coupe l'axe vertical $\{y_1 = \cdots = y_{n-1} = 0\}$ transversalement en un point exactement et sépare la boule ouverte $\{f < \varepsilon\}$ en deux composantes connexes exactement.

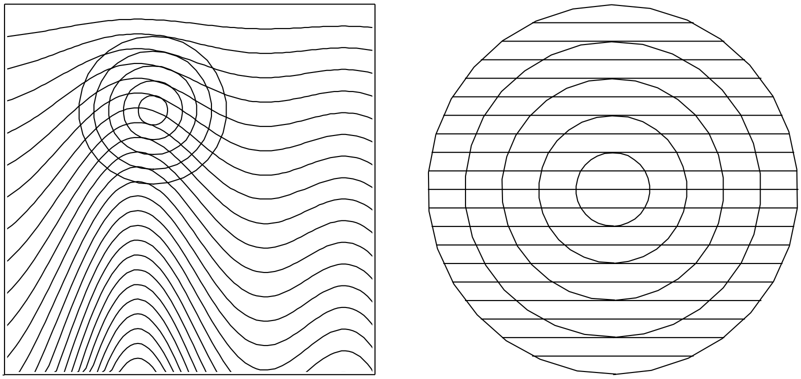


FIGURE 3. On redresse le feuilletage localement dans un système de coordonnées qui respectent les petites sphères

Dans le cas où $g$ est l'intégrale première locale d'un feuilletage $\mathcal{F}$ de codimension un défini au voisinage de l'origine, ceci implique que si $V$ est une feuille de Rolle de $\mathcal{F}$ alors $V \cap \{f < \varepsilon\}$ est vide ou connexe.

## 3. Recouvrement d'un ouvert relativement compact $M$ de $\mathbf{R}^m$ adapté à une forme différentielle définie au voisinage de $\overline{M} \setminus \{0\}$

On s'intéresse dans la proposition qui suit au cas où la 1-forme $\omega$ est définie sur un ouvert $M$ relativement compact de $\mathbf{R}^m$ et admet un prolongement intégrable, non singulier et $\mathcal{A}$-définissable au voisinage de $\overline{M}$ sauf peut-être en un point. La proposition affirme qu'on peut alors recouvrir l'espace $M$ en ouverts sur lesquels la dynamique du feuilletage se révélera simple. Sa démonstration, purement technique, sera donnée en annexe

***Proposition 10***. — *Soit $m \geq 2$, $M$ un ouvert relativement compact de $\mathbf{R}^m$, $\mathcal{A}$-définissable, de classe $C^k$ et $\omega$ une 1-forme différentielle à coefficients $\mathcal{A}$-définissables,* *de classe $C^k$, définie sur $M$ et au voisinage de $\overline{M} \setminus \{0\}$, non singulière et intégrable. Il existe une constante $K > 0$ et des ouverts $U_1, \ldots, U_s$ dits* adaptés à $\omega$ *et vérifiant les propriétés suivantes :*

(i) *l'ouvert $M$ est la réunion des ouverts $U_1, \ldots, U_s$ ;*

(ii) *pour chaque $i \in \{1, \ldots, s\}$ il existe des coordonnées linéaires dans lesquelles $U_i$ est un cylindre $\mathcal{A}$-définissable, de classe $C^k$,*
$$U_i = \{(x, y) : x \in U_i', \phi_i(x) < y < \psi_i(x)\}\,;$$

(iii) *pour tout $x \in U_i$ l'hyperplan* $\ker(\omega)(x)$ *est transverse à l'axe vertical et c'est le graphe d'une application linéaire $K$-lipschitzienne ;*

(iv) *le champ $x \mapsto$* $\ker(\omega)(x)$ *est soit partout transverse soit partout tangent au bas $B_i = \{(x, \phi_i(x)) : x \in U_i'\}$ (respectivement au haut $H_i = \{(x, \psi_i(x)) : x \in U_i'\}$) du cylindre $U_i$ et $0 \notin B_i$ (respectivement $0 \notin H_i$).*

Si $i \in \{1, \ldots, s\}$ la dynamique du feuilletage $\mathcal{F}_{U_i}$ induit par $\omega$ sur $U_i$ n'est pas nécessairement simple comme le montre l'exemple suivant construit à partir du feuilletage de Reeb. Cependant on verra dans la preuve du théorème principal que la dynamique de $\mathcal{F}_{U_i}$ est simple dès que celles des feuilletages induits par $\omega$ sur $B_i$ et $H_i$ le sont.

On peut trouver des énoncés voisins en particulier dans des articles qui abordent des questions de géométrie sous-analytique, de singularités ou de stratifications. C'est le cas par exemple dans un travail récent de Valette [**33**] mais aussi dans un papier plus ancien de Kurdyka et Raby [**19**] ou encore chez Roche [**27**]. Seulement dans ce dernier les conditions de Lipschitz et les feuilletages sont mêlés comme ici.

**Exemple** Partons de l'exemple de Reeb [**11**] (voir aussi [**4**] ou[**14**]) dont on donne une construction *o-minimale* à l'aide d'une 1-forme $\omega$ de classe $C^\infty$ et dont les coefficients sont définissables dans la structure o-minimale $\mathbf{R}_{\text{an,exp}}$ engendrée par les fonctions analytiques restreintes et l'exponentielle [**9**].

Soit $\mathbf{S}^3 = \{|z_1|^2 + |z_2|^2 = 1\}$ la sphère unité de $\mathbf{C}^2$. On pose $z_1 = \rho_1 \exp(i\theta_1)$, $z_2 = \rho_2 \exp(i\theta_2)$. On a donc $\mathbf{S}^3 = \{\rho_1^2 + \rho_2^2 = 1\}$ et donc $d\rho_1^2 + d\rho_2^2 = 0$ en restriction à $\mathbf{S}^3$ et $\mathbf{S}^3 \subset \{\rho_1^2 \leq 1/2 \text{ ou } \rho_2^2 \leq 1/2\}$. On considère la fonction de recollement $\mu$ définie par $\mu(t) = \mathbf{1}_{]0,+\infty)}(t) \exp(-1/t)$. Elle est définissable dans $\mathbf{R}_{\text{an,exp}}$. Soit $\tilde{\omega}$ la 1-forme différentielle définie sur $\mathbf{S}^3$ par
$$\tilde{\omega} = \mu(1/2 - \rho_2^2) d\theta_1 + \mu(1/2 - \rho_1^2) d\theta_2 + d\rho_2^2.$$

Elle est aussi définissable dans $\mathbf{R}_{\text{an,exp}}$. De plus

- si $\rho_2^2 \leq 1/2$ alors $\rho_1^2 \geq 1/2$ et $\tilde{\omega} = \mu(1/2 - \rho_2^2) d\theta_1 + d\rho_2^2$
- si $\rho_1^2 \leq 1/2$ alors $\rho_2^2 \geq 1/2$ et $\tilde{\omega} = \mu(1/2 - \rho_1^2) d\theta_2 - d\rho_1^2$.

On a donc bien $\tilde{\omega} \wedge d\tilde{\omega} \equiv 0$ et $\tilde{\omega}$ définit un feuilletage $\mathcal{F}_{\tilde{\omega}}$ sur $\mathbf{S}^3$. On remarque que le lacet $\tilde{C} = \{\rho_2^2 = 0\}$ est transverse à ce feuilletage. Le feuilletage $\mathcal{F}_{\tilde{\omega}}$ est donc un feuilletage de $\mathbf{R}^3$ qui admet des transversales fermées. On considère le champ de vecteurs $\tilde{X}$ *orthogonal à* $\tilde{\omega}$ défini sur $\mathbf{S}^3$ par

$$\tilde{X} = \mu(1/2 - \rho_2^2)\frac{\partial}{\partial\theta_1} + \mu(1/2 - \rho_1^2)\frac{\partial}{\partial\theta_2} + 2\rho_2\frac{\partial}{\partial\rho_2}.$$

Il est aussi égal à

$$\tilde{X} = \mu(1/2 - \rho_2^2)\frac{\partial}{\partial\theta_1} + \mu(\rho_2^2 - 1/2)\frac{\partial}{\partial\theta_2} + 2\rho_2\frac{\partial}{\partial\rho_2}.$$

Il est donc définissable dans $\mathbf{R}_{\mathrm{an,exp}}$, il est sans zéro et puisque ce champ ne dépend que de $\rho_2$, d'après **[30]**, son flot $\tilde{\phi} : \mathbf{S}^3 \times \mathbf{R} \to \mathbf{S}^3$ est une application définissable dans une structure o-minimale $\widehat{\mathbf{R}_{\mathrm{an,exp}}}$ appelée clôture pfaffienne de $\mathbf{R}_{\mathrm{an,exp}}$.

Par projection stéréographique de pôle nord on obtient un feuilletage $\mathscr{F}_{\overline{\omega}}$ de $\mathbf{R}^3$ défini à partir d'une forme $\overline{\omega}$ de classe $C^\infty$ et définissable dans la strucure o-minimale $\mathbf{R}_{\mathrm{an,exp}}$. Le feuilletage admet des tranversales fermées car le cercle $C$ image par la projection stéréographique du lacet $\tilde{C}$ est transverse au feuilletage $\mathscr{F}_{\overline{\omega}}$. Décrivons un peu la géométrie de ce feuilletage. Il est invariant par rotation autour de l'axe $\{x_1 = x_2 = 0\}$. L'une des feuilles du feuilletage $\mathscr{F}_{\overline{\omega}}$ est un tore $\mathbf{T}^2$. Il est dans l'adhérence de toutes les autres qui ne sont donc pas fermées dans $\mathbf{R}^3$. Le complémentaire de ce tore $\mathbf{T}^2$ se décompose en deux composantes connexes appelées *composantes de Reeb* du feuilletage.

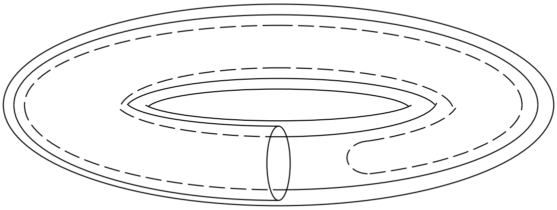


FIGURE 4. Morceau d'une feuille s'enroulant sur le tore $\mathbf{T}^2$, bord commun aux deux composantes de Reeb

On note $\overline{X}$ l'image de $\tilde{X}$ par la projection stéréographique. Il est définissable dans $\mathbf{R}_{\mathrm{an,exp}}$ et il est sans zéro. Son flot $\overline{\phi}$ est l'image par la projection stéréographique du flot $\tilde{\phi}$. Il est donc définissable dans la clôture pfaffienne de $\mathbf{R}_{\mathrm{an,exp}}$. Contrairement au champ $\tilde{X}$, le champ $\overline{X}$ n'est pas complet puisque par la projection stéréographique le pôle nord est envoyé à l'infini.

On étend trivialement la forme $\overline{\omega}$ en une forme $\breve{\omega}$ de $\mathbf{R}^4$ définissable dans la clôture pfaffienne $\widehat{\mathbf{R}_{\mathrm{an,exp}}}$ et qui définit un feuilletage $\mathscr{F}_{\breve{\omega}}$ de $\mathbf{R}^4$. Les feuilles de $\mathscr{F}_{\breve{\omega}}$ sont les produits $V \times \mathbf{R}$ des feuilles de $\mathscr{F}_{\overline{\omega}}$ par $\mathbf{R}$. On note $\breve{X}$ le champ $\breve{X} = \overline{X} + \dfrac{\partial}{\partial x_4}$. Le champ $\breve{X}$ est transverse à $\mathscr{F}_{\breve{\omega}}$, il est définissable dans la clôture pfaffienne $\widehat{\mathbf{R}_{\mathrm{an,exp}}}$, jamais vertical et son flot $\breve{\phi}$ est définissable dans la clôture pfaffienne $\widehat{\mathbf{R}_{\mathrm{an,exp}}}$. De plus le flot de $\breve{X}$ échange les plans horizontaux $\{x = cst\}$ et ceux-ci sont transverses aux feuilles de $\breve{\omega}$.

Pour faire la représentation planaire suivante de ces objets de $\mathbf{R}^4$ on a fait des choix : le feuilletage de Reeb est symbolisé par un feuilletage en points de l'intervalle, son extension à $\mathbf{R}^4$ est symbolisé par le feuilletage en intervalles verticaux du carré $]-1,1[\times]-1,1[$, le champ $\overline{X}$ par le champ $\dfrac{\partial}{\partial x}$, le tore $\mathbf{T}^2$ bordant les deux composantes de Reeb est symbolisé par l'origine et les orbites du champ $\breve{X}$ sont en pointillés.

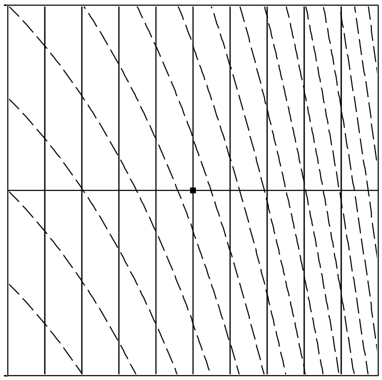


FIGURE 5.

Soit $\Delta = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : |x_1|, |x_2|, |x_3| < D\}$ avec $D$ assez grand pour que le tore $\mathbf{T}^2$ appartienne à $\Delta$ après avoir identifié $\mathbf{R}^3$ et $\mathbf{R}^3 \times \{0\}$. Soit $\varepsilon > 0$ assez petit pour que pour tout $t \in ]-2\varepsilon, 2\varepsilon[$ et pour tout point $x \in 2\Delta$, l'image $\breve{\phi}(x,t)$ de $x$ par le flot de $\breve{X}$ au temps $t$ existe.

On pose $M = \breve{\phi}(\Delta \times ]-\varepsilon, \varepsilon[)$, $\Omega = \breve{\phi}(2\Delta \times ]-2\varepsilon, 2\varepsilon[)$ et on note $\omega$ l'image de la forme $\breve{\omega}$ par $\breve{\phi}$. Les ensembles $M$ et $\Omega$ sont des ouverts de classe $C^\infty$ et définissables dans la clôture pfaffienne $\widehat{\mathbf{R}_{\mathrm{an,exp}}}$ et la forme $\omega$ est de classe $C^\infty$ et définissable dans la clôture pfaffienne $\widehat{\mathbf{R}_{\mathrm{an,exp}}}$. L'ouvert $M$ est un ouvert relativement compact de $\Omega$.

On munit l'ouvert $\Omega$ du système de coordonnées $y = (y_1, y_2, y_3, y_4)$ suivant qui trivialise le flot : les coordonnées de l'image $\breve{\phi}(x,t)$ du point $(x,0) \in 2\Delta \times \{0\}$ sont

$$y = (y_1, y_2, y_3, y_4)$$

avec

$$y_1 = x_1, y_2 = x_2, y_3 = x_3, y_4 = t.$$

C'est un système de coordonnées de classe $C^\infty$ et définissable dans la clôture pfaffienne $\widehat{\mathbf{R}_{\mathrm{an,exp}}}$. Dans ce système l'image $X$ du champ $\breve{X}$ est le champ vertical

$$X = \frac{\partial}{\partial y_4}.$$

Dans ces coordonnées on a

$$M = \{y = (y_1, y_2, y_3, y_4) : |y_1|, |y_2|, |y_3| < D, |y_4| < \varepsilon\}$$

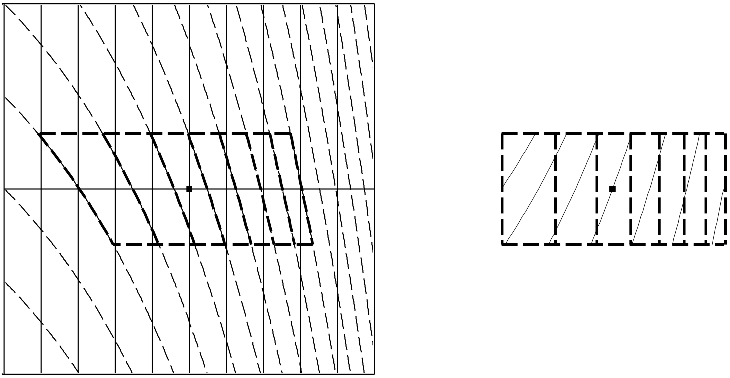


FIGURE 6. L'ouvert $M$ représenté dans les coordonnées $(x_1, x_2, x_3, x_4)$ puis dans les coordonnées $(y_1, y_2, y_3, y_4)$.

et

$$\Omega = \{y = (y_1, y_2, y_3, y_4) : |y_1|, |y_2|, |y_3| < 2D, |y_4| < 2\varepsilon\}.$$

La forme $\omega$ est définie sur $\Omega$, elle est de classe $C^\infty$, intégrable, non singulière et définissable dans la clôture pfaffienne $\widehat{\mathbf{R}_{\mathrm{an,exp}}}$. De plus pour tout $y \in \Omega$ l'hyperplan $\ker(\omega)(y)$ est transverse à l'axe vertical et il n'est jamais horizontal. Ainsi il existe $K > 0$ tel que pour tout $y \in \overline{M}$ l'hyperplan $\ker(\omega)(y)$ est transverse à l'axe vertical, jamais horizontal et c'est le graphe d'une application linéaire $K$-lipschitzienne.

Par conséquent l'ouvert $M = U_1$ vérifie les points i, ii, iii et iv de la proposition 10 mais la dynamique du feuilletage $\mathcal{F}$ associé à $\omega$ n'est pas simple. Il admet des transversales fermées et il possède des feuilles qui ne sont pas fermées dans $M = U_1$.

## 4. Preuve du théorème 1

La preuve du théorème se fait par récurrence sur la dimension $m$ de $M$.

1. Si $m = 0$ ou 1 c'est immédiat. Soit $m > 1$. On suppose le résultat prouvé jusqu'au rang $m - 1$.

L'objet des étapes 2, 3 et 4 et de se ramener au cas ou $M$ est un ouvert adapté à la forme $\omega$ comme les ouverts $U_i$ de la proposition 10.

2. Puisque $M$ admet un recouvrement fini par des cylindres ouverts pour la topologie induite et $\mathcal{A}$-définissables (proposition 8) on peut supposer que $M = \mathbf{R}^m$ (proposition 6).

3. Maintenant la forme $\omega$ est défini sur $\mathbf{R}^m$ tout entier. On la contrôle mal à l'infini. Soit $U_1 = \{||x|| > 1\}, V_1 = \{||x|| > \frac{1}{2}\}$, $\phi_1$ l'inversion de pôle 0 et qui fixe $\{||x|| = 1\}$. Soit $U_2 = \{||x - (3, 0, \dots, 0)|| > 1\}, V_2 = \{||x - (3, 0, \dots, 0)|| > \frac{1}{2}\}$, $\phi_2$ la translation de vecteur $-(3, 0, \dots, 0)$ composée à droite avec $\phi_1$. On a $\phi_i(U_i) = \{0 < ||x|| < 1\}$,

$\phi_i(V_i) = \{0 < ||x|| < 2\}$ et $U_1 \cup U_2 = \mathbf{R}^m = M$. Par conséquent quitte à transporter $\omega_{|U_i}$ et $\omega_{|V_i}$ par $\phi_i$ on peut supposer que $M = \{0 < ||x|| < 1\}$ et que $\omega$ se prolonge en une forme intégrable et non singulière toujours notée $\omega$ définie sur un voisinage de $\overline{M} \setminus \{0\} = \{0 < ||x|| \leq 1\}$.

4. En décomposant $M$ suivant la proposition 10 on peut supposer que $M$ est l'un des ouverts $U_i$ du recouvrement obtenu. On est donc dans la configuration suivante :

(i) l'ouvert $M$ est un cylindre ouvert de $\mathbf{R}^m$ de classe $C^k$ et $\mathscr{A}$-définissable de la forme
$$M = \{(x, y) : x \in M', \phi(x) < y < \psi(x)\}$$
où $M'$ est un cylindre ouvert de $\mathbf{R}^{m-1}$ de classe $C^k$ et $\mathscr{A}$-définissable ;

(ii) la forme $\omega$ est définie non singulière et intégrable et $\mathscr{A}$-définisssable sur un voisinage $\mathscr{A}$-définissable $\Omega$ de
$$\{(x, y) : x \in M', \phi(x) \leq y \leq \psi(x)\} ;$$

(iii) il existe $K > 0$ tel que pour tout $x \in \Omega$ l'hyperplan $\ker(\omega)(x)$ est transverse à l'axe vertical et c'est le graphe d'une application linéaire $K$-lipschitzienne ;

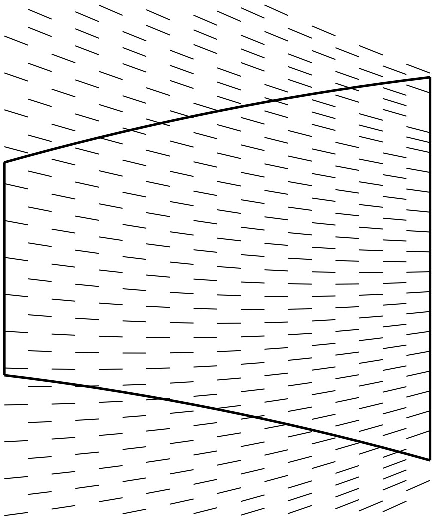
(iv) le champ $x \mapsto \ker(\omega)(x)$ est soit partout transverse soit partout tangent au bas $B = \{(x, \phi(x)) : x \in U'\}$ (respectivement au haut $H = \{(x, \psi(x)) : x \in U'\}$) du cylindre $M$.

Remarquons qu'on a $\overline{M} \cap (M' \times \mathbf{R}) = M \cup B \cup H$.

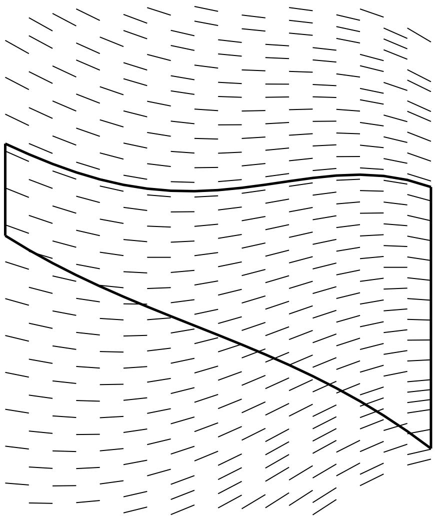
5. Le feuilletage $\mathscr{F}$ induit par $\omega$ sur $M$ se prolonge en un feuilletage de codimension un $\mathscr{F}_\Omega$ sur $\Omega$. Si $V$ est une feuille de $\mathscr{F}$ ou de $\mathscr{F}_\Omega$ alors c'est localement le graphe d'une fonction de $\mathbf{R}^{m-1}$ dans $\mathbf{R}$. Plus précisément, si $a \in V$ il existe un voisinage $\mathscr{O}$ de $a$ tel que la composante connexe de $V \cap \mathscr{O}$ qui contient $a$ est le graphe d'une fonction $K$-lipschitzienne de $\mathbf{R}^{m-1}$ dans $\mathbf{R}$. Si le champ $x \mapsto \ker(\omega)(x)$ est partout transverse à $B$ (respectivement $H$) il induit sur $B$ (respectivement $H$) un feuilletage de codimension un $\mathscr{F}_{|B}$ sur $B$ (respectivement $\mathscr{F}_{|H}$ sur $H$). Si le champ $x \mapsto \ker(\omega)(x)$ est partout tangent à $B$ (respectivement $H$) alors $B$ (respectivement $H$) est un morceau de feuille de $\mathscr{F}_\Omega$.

6. Il n'est pas certain que la dynamique du feuilletage $\mathscr{F}_{|B}$ (respectivement $f_{|H}$) soit simple lorsque le champ $x \mapsto \ker(\omega)(x)$ est partout transverse à $B$ (respectivement $H$). Cependant, d'après l'hypothèse de récurrence et la proposition 5 on peut supposer que si le champ $x \mapsto \ker(\omega)(x)$ est partout transverse à $B$ (respectivement $H$) alors le feuilletage $\mathscr{F}_{|B}$ (respectivement $\mathscr{F}_{|H}$) induit sur $B$ (respectivement $H$) est dynamiquement simple : il n'admet pas de transversale fermée et toute feuille est une hypersurface fermée de $B$ (respectivement $H$) qui sépare $B$ (respectivement $H$) en deux composantes connexes.

7. Fixons $x_0 \in M'$. Il existe $K_0 > K$, et $y_0 < y_0' \in \mathbf{R}$ tels que pour tout $\varepsilon_0 > 0$ petit les propriétés suivantes sont vérifiées. On note $U_0$ la boule $\{x \in \mathbf{R}^{m-1} : ||x - x_0|| < \varepsilon_0\}$ et $\Omega_0$ le cylindre $U_0 \times ]y_0, y_0'[$. Alors $\overline{U_0} \subset M'$, $\overline{M} \cap (U_0 \times \mathbf{R}) \subset \Omega_0 \subset \Omega$ et $B \cap (U_0 \times \mathbf{R})$, $H \cap (U_0 \times \mathbf{R})$ ainsi que les feuilles du feuilletage $\mathscr{F}_{\Omega_0}$ qui rencontrent $\overline{M} \cap (U_0 \times \mathbf{R})$



Le champ $x \mapsto \ker(\omega)(x)$ est transverse à $B$ et $H$



Le champ $x \mapsto \ker(\omega)(x)$ est transverse à $B$ et tangent à $H$

FIGURE 7. Deux cas où $M$ est adapté a $\omega$ : transversalité et tangence

sont des graphes de fonctions $K_0$-lipschitziennes définies sur $U_0$. Ce sont donc des hypersurfaces fermées de $\Omega_0$. De même, les feuilles de $\mathcal{F}_{|B\cap\Omega_0}$ ou de $\mathcal{F}_{|H\cap\Omega_0}$ (en cas de transversalité) sont des hypersurfaces fermées de $B \cap \Omega_0$ ou de $H \cap \Omega_0$. Quitte à réduire un peu $\varepsilon_0$ toute feuille $\mathcal{V}$ du feuilletage $\mathcal{F}_{\Omega_0}$ vérifie l'une des conditions suivantes.

- La feuille $\mathcal{V}$ ne rencontre ni $B$ ni $H$.
- La feuille $\mathcal{V}$ est égale à $B \cap \Omega_0$ ou à $H \cap \Omega_0$.
- La feuille $\mathcal{V}$ coupe $B \cap \Omega_0$ transversalement le long d'une unique feuille de $\mathcal{F}_{|B\cap\Omega_0}$ et elle ne rencontre pas $H$.
- La feuille $\mathcal{V}$ coupe $H \cap \Omega_0$ transversalement le long d'une unique feuille de $\mathcal{F}_{|H\cap\Omega_0}$ et elle ne rencontre pas $B$.

Les deux dernières conditions résultent de la proposition 9 appliquée avec $n = m - 1$, en choisissant comme origine l'intersection de $B$ (ou de $H$) avec la verticale issue de $x_0$, en paramétrant $B$ (ou $H$) par $M'$ et en prenant comme fonction $g$ la restriction à $B$ (ou $H$) d'une intégrale première locale du feuilletage $\mathcal{F}_{\Omega_0}$ définie au voisinage de l'origine considérée.

8. Vu les hypothèses faites sur $\mathcal{F}_{|B}$ ou $\mathcal{F}_{|H}$ en cas de transversalité, d'après le modèle des feuilletages $\mathcal{F}_{|B\cap\Omega_0}$ ou de $\mathcal{F}_{|H\cap\Omega_0}$ donné par la proposition 9 on peut aussi supposer en cas de transversalité que toute feuille $W$ du feuilletage $\mathcal{F}_{|B}$ (ou du feuilletage $\mathcal{F}_{|H}$) qui rencontre $\Omega_0$ est telle que $W \cap (B \cap \Omega_0)$ (ou $W \cap (H \cap \Omega_0)$) est connexe et sépare $B \cap \Omega_0$ (ou $H \cap \Omega_0$) en deux composantes connexes exactement.

Dorénavant on note $\pi$ la projection définie par $\pi(x_1, \dots, x_m) = (x_1, \dots, x_{m-1})$.

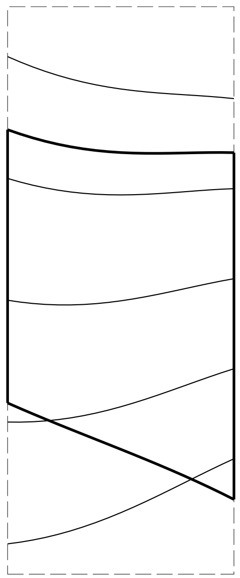


FIGURE 8. $\Omega_0 = U_0 \times ]y_0, y_0'[$, $M \cap \Omega_0$ et des feuilles de $\mathcal{F}_{\Omega_0}$

9. Décrivons ce qui se passe au voisinage du point $a_0$ de $B$ (ou de $H$) qui se projette sur $x_0$. Plaçons nous sur $B$ par exemple. Si $B$ est une feuille du feuilletage $\mathcal{F}_\Omega$ induit par $\omega$ sur $\Omega$ alors il existe un voisinage $\mathcal{W}_0$ de $B \cap \Omega_0$ et un changement de coordonnées (pas nécessairement $\mathcal{A}$-définissable) dans lesquelles $\mathcal{W}_0 = \mathbf{R}^m$, $B \cap \Omega_0 = \{x_m = 0\}$, $M \cap \mathcal{W}_0 = \{x_m > 0\}$ et le feuilletage induit par $\omega$ sur $M \cap \mathcal{W}_0$ est le feuilletage par hyperplans horizontaux. Supposons $B$ transverse au champ $x \mapsto \ker(\omega)(x)$. Soit $a \in B \cap \Omega_0$. D'après l'étape 8, la feuille $\mathcal{W}_a$ du feuilletage $\mathcal{F}_{|B \cap \Omega_0}$ qui passe par $a$ est exactement la trace sur $B \cap \Omega_0$ de la feuille $W_a$ de $\mathcal{F}_{|B}$ qui passe par $a$. De plus $\mathcal{W}_a$ sépare $B \cap \Omega_0$ en deux composantes connexe $C_+(a)$ et $C_-(a)$ qui sont incluses dans $B \setminus W_a$. On pose $C'_+(a) = \pi(C_+(a))$ et $C'_-(a) = \pi(C_-(a))$. Les ensembles $C_+(a)$ et $C_-(a)$ sont des graphes d'applications définies au dessus de $C'_+(a)$ et $C'_-(a)$. Il existe une et une seule feuille $\mathcal{V}_a$ de $\mathcal{F}_{|M \cap \Omega_0}$ qui contient $a$ (ou un quelconque point de $\mathcal{W}_a$) dans son adhérence. Quitte à permuter $C_+(a)$ et $C_-(a)$ cette feuille $\mathcal{V}_a$ est le graphe d'une application définie sur $C'_+(a)$. La remarque cruciale est la suivante. La réunion de $\mathcal{V}_a$, de $\mathcal{W}_a$ et de $C_-(a)$ est le graphe d'une application continue définie au dessus de la projection $U_0$ de $\Omega_0$ sur $M'$. On a bien sûr les mêmes conclusions si $a \in H \cap \Omega_0$.

10. Finalement si $\mathcal{V}$ est une feuille de $\mathcal{F}_{|M \cap \Omega_0}$ alors elle vérifie l'une des conditions suivantes.

- C'est une feuille de $\mathcal{F}_{|\Omega_0}$ et c'est le graphe d'une application définie sur $U_0$.
- Il existe un point $a$ de $B \cap \Omega_0$ qui est dans l'adhérence de $\mathcal{V}$ et alors $\mathcal{V} = \mathcal{V}_a$ et $\mathcal{V}_a \cup \mathcal{W}_a \cup C_-(a)$ est le graphe d'une application définie sur $U_0$.

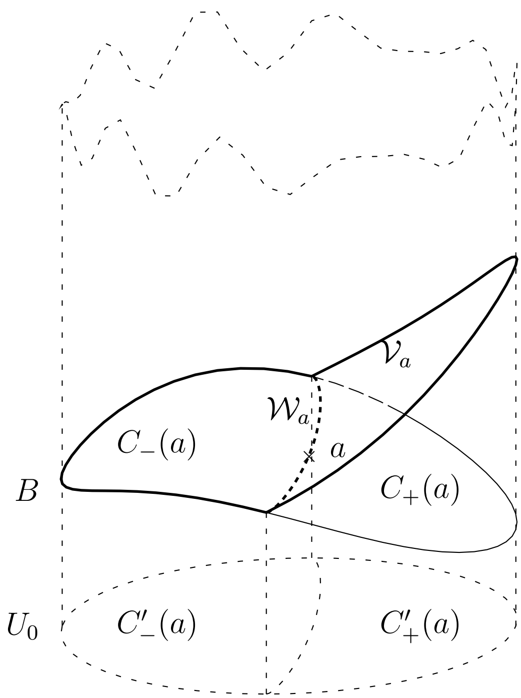


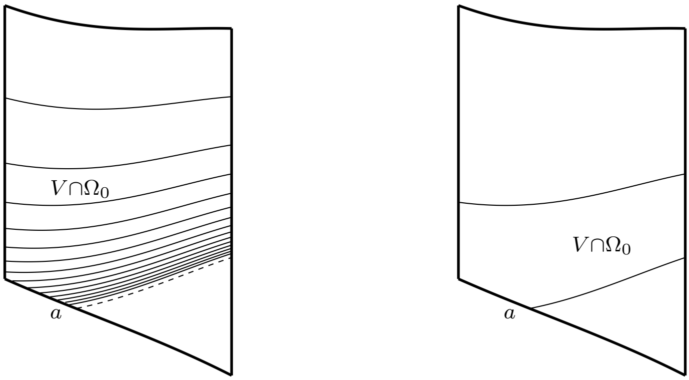

FIGURE 9. $a$, $\mathcal{W}_a$, $\mathcal{V}_a$, $C_+(a)$, $C_-(a)$, $C'_+(a)$, $C'_-(a)$ et $U_0$ avec $B$ transverse au champ $x \mapsto \ker(\omega)(x)$

– Il existe un point $a$ de $H \cap \Omega_0$ qui est dans l'adhérence de $V$ et alors $\mathcal{V} = \mathcal{V}_a$ et $\mathcal{V}_a \cup \mathcal{W}_a \cup C_-(a)$ est le graphe d'une application définie sur $U_0$.

11. Soit $V$ une feuille de $\mathcal{F}$ et soit $a \in B$. On dit que $a$ est dans l'adhérence directe de $V$ s'il existe $x_0$, $\Omega_0$ et $a_0$ comme précédemment tels que la feuille $\mathcal{V}_a$ précédente est dans $V$.

On suppose que $a$ est dans l'adhérence directe de $V$. Soit $W$ la feuille de $\mathcal{F}_{|B}$ qui contient $\mathcal{W}_a$. Puisque le feuilletage $\mathcal{F}_{|B}$ est dynamiquement simple, $W$ est une hypersurface fermée de $B$ qui sépare $B$ en deux composantes connexes. D'après l'étape 8, l'une d'elles, notée $C_+(W)$, contient $C_+(a)$ et l'autre, notée $C_-(W)$, contient $C_-(a)$. Alors tout point $a'$ de $W$ est dans l'adhérence directe de $V$ et $C_+(a')$ est inclus dans $C_+(W)$ alors que $C_-(a')$ est inclus dans $C_-(W)$. On dit que $W$ adhère directement à $V$. On a des définitions et des conclusions analogues pour $H$ au lieu de $B$.

12. Considérons maintenant $V$ comme une variété abstraite et non plus comme une feuille de $M$. On note $j : V \to M$ l'immersion de $V$ dans $M$. On va prolonger cette immersion différentiable et injective en une immersion continue et non nécessairement injective $\tilde{j}$ définie sur $\tilde{V}$ variété topologique connexe à valeurs dans $\overline{M} \cap (M' \times \mathbf{R})$ telle que la composée de $\tilde{j}$ avec la projection $\pi$ soit un revêtement de base $M'$. Les $W$




$a \in \overline{V \cap \Omega_0}$ mais $a$ n'est pas dans l'adhérence directe de $V$ ($V \cap \Omega_0$ a une infinité de composantes connexes)

$a$ est dans l'adhérence directe de $V$ ($V \cap \Omega_0$ a un nombre fini de composantes connexes, ici deux)

FIGURE 10.

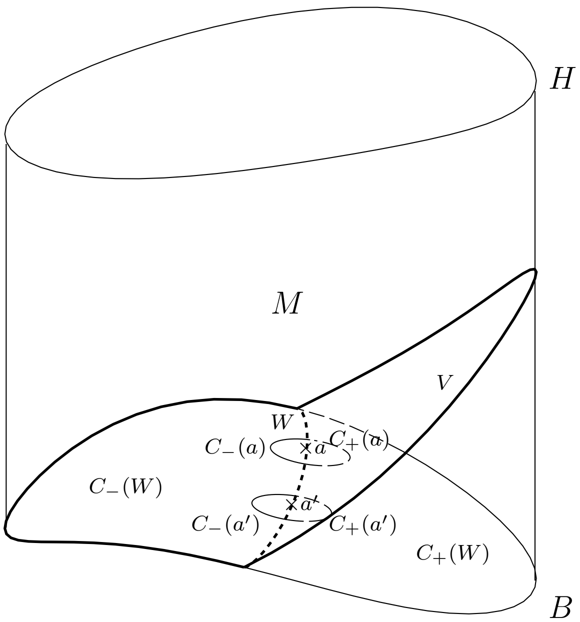



FIGURE 11. $W$ adhère directement à $V$

de $B$ ou $H$ qui adhèrent directement à $V$ sont en nombre au plus dénombrable. Pour tout $W$ on colle $C_-(W)$ à $V$ le long de $W$. C'est bien sur un collage abstrait : il faut dans cette opération considérer les $C_-(W)$ disjoints, c'est à dire oublier un instant qu'ils sont dans $\overline{M} \cap (M' \times \mathbf{R})$ et les voir comme des variétés topologiques abstraites. Par ces collages en nombre au plus dénombrable on obtient une variété abstraite $\tilde{V}$.

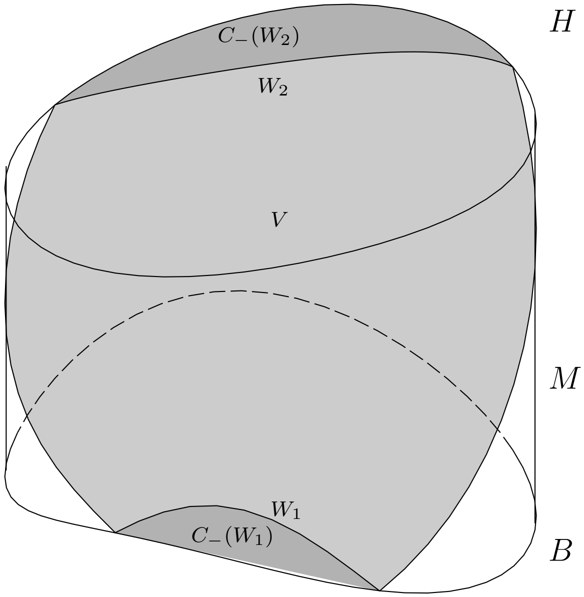



FIGURE 12. $\tilde{V}$ est la réunion de $V$, $W_1$, $C_-(W_1)$, $W_2$ et $C_-(W_2)$

13. Nous voulons démontrer que la projection $\pi \circ \tilde{j}$ de $\tilde{V}$ sur $M'$ est un revêtement. Pour cela décrivons d'abord plus précisément $\tilde{V}$ et sa topologie. Indexons par un sous-ensemble $I$ de $\mathbf{N}$ les $W$ de $B$ et $H$ qui sont dans l'adhérence directe de $V$ : $(W_i)_{i \in I}$. D'un point de vue ensembliste, $\tilde{V}$ est l'union disjointe de $V$, des $W_i, i \in I$ et des $C_-(W_i), i \in I$. Soit $x_0 \in M'$ et $\varepsilon_0 > 0$, $\Omega_0$ et $U_0$ comme dans l'étape 7. Soit $\tilde{\alpha}$ un point de $\tilde{V}$ tel que $\tilde{j}(\tilde{\alpha}) = \alpha \in \Omega_0$. Trois cas sont à considérer.

- Le point $\alpha$ est dans $M$ et la composante connexe $V_0$ de $V \cap \Omega_0$ qui contient $\alpha$ est un graphe au dessus de $U_0$. Alors $\tilde{j}^{-1}(V_0)$ est un voisinage ouvert de $\tilde{\alpha}$ dans $\tilde{V}$ et la restriction de $\pi \circ \tilde{j}$ à $\tilde{j}^{-1}(V_0)$ sur $U_0$ est un homéomorphisme.
- Il existe $i \in I$ et $a_i \in W_i \cap \Omega_0$ tels que $\tilde{\alpha}$ provient du collage de $W_i$. Dans ce cas $\alpha$ peut-être un élément de $V$, de $W_i$ ou de $C_-(W_i)$, l'ensemble $(\mathcal{V}_{a_i} \cup \mathcal{W}_{a_i} \cup C_-(a_i))$ vu comme sous-ensemble de $\tilde{V}$ est un voisinage ouvert de $\tilde{\alpha}$ dans $\tilde{V}$ et la restriction à cet ouvert de $\pi \circ \tilde{j}$ est un homéomorphisme sur $U_0$.
- Il existe $i \in I$ tel que $\tilde{\alpha}$ provient du collage de $C_-(W_i)$ et $W_i \cap \Omega_0 = \varnothing$. Dans ce cas $\alpha \in C_-(W_i)$, $C_-(W_i) \cap \Omega_0$ est égal à $B \cap \Omega_0$ ou $H \cap \Omega_0$, $U_0 = \pi(\Omega_0 \cap C_-(W_i))$ et $\Omega_0 \cap C_-(W_i)$ vu comme sous-ensemble de $\tilde{V}$ est un voisinage ouvert de $\tilde{\alpha}$ dans $\tilde{V}$. De plus la restriction à cet ouvert de $\pi \circ \tilde{j}$ est un homéomorphisme sur son image $U_0$.

Ces cas donnent une famille d'ouverts connexes disjoints qui recouvrent $\tilde{j}^{-1}(\Omega_0)$.

À la fin de la démonstration on pourra affirmer que l'application $\pi \circ \tilde{j}$ est une un homéorphisme de $\tilde{V}$ sur $M'$ et que $\tilde{j}$ est injective. Pour l'instant nous ne l'affirmons pas encore. Ainsi la figure suivante illustre une configuration qui n'est a priori pas à exclure pour l'intersection $V \cap \Omega_0$ et l'intersection $\tilde{j}(\tilde{C}) \cap \overline{\Omega_0}$.

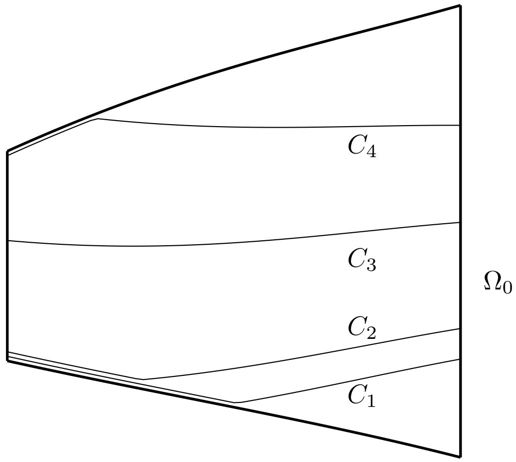



FIGURE 13. $V \cap \Omega_0$ a quatre composantes connexes, $C_1, C_2, C_3$ et $C_4$ $\tilde{V} \cap (\tilde{j})^{-1}(\overline{\Omega_0})$ a quatre composantes connexes mais $\tilde{j}(\tilde{V}) \cap \overline{\Omega_0}$ a trois composantes connexes

14. L'application $\tilde{j}$ envoie continument $\tilde{V}$ dans $\overline{M} \cap (M' \times \mathbf{R})$ et a priori elle n'est pas nécessairement injective. Par construction l'application $\pi \circ \tilde{j}$ est un revêtement de base $M'$. En effet si $x_0$ est dans $M'$ alors $(\pi \circ \tilde{j})^{-1}(U_0)$ (avec les notations précédentes) est l'union disjointe et dénombrable d'ouverts qui sont des feuilles du feuilletage $\mathcal{F}_{|\Omega_0}$, des ensembles du type $(\mathcal{V}_{a_i} \cup \mathcal{W}_{a_i} \cup C_-(a_i))$ et des ensembles du type $\Omega_0 \cap C_-(W_i)$. De plus la restriction de $\pi \circ \tilde{j}$ à chacun de ces ouverts est un homéomorphisme sur $U_0$.

15. Puisque $\tilde{V}$ est connexe alors que $M'$ est simplement connexe le revêtement $\pi \circ \tilde{j}$ est un homémorphisme (voir la première observation topologique de la partie 2). Ceci implique que la restriction à $V$ est injective et que $\tilde{j}(\tilde{V})$ est fermé relativement à $M \cup B \cup H$. Ainsi $V$ vue à nouveau comme feuille est le graphe d'une application continue définie sur un ouvert $V'$ de $M'$ et $V$ est fermé relativement à $M$. C'est donc une sous-variété fermée de $M$. Puisque $M$ est difféomorphe à $\mathbf{R}^m$, la feuille $V$ sépare $M$ en deux composantes connexes et elle est de Rolle (voir la deuxième et la troisième observations topologiques de la partie 2). Puisque ces conclusions valent pour toute feuille, le feuilletage induit par $\omega$ sur $M$ est donc un feuilletage dynamiquement simple : il n'admet pas de transversale fermée et toute feuille est une hypersurface fermée de $M$ qui sépare $M$ en deux composantes connexes. □

## 5. Annexe : preuve de la proposition 10

On va montrer par une récurrence descendante sur la dimension $d$ que si $Z$ est un sous-ensemble inclus dans $M$, $\mathcal{A}$-définissable de dimension $d$ alors il existe $K > 0$ et $Z' \subset Z$ $\mathcal{A}$-définissable et de dimension au plus $d-1$ tel que $Z \setminus Z'$ peut être recouvert d'ouverts qui vérifient les points ii, iii et iv de la proposition 10.

Si $d = m \geq 2$, on peut supposer que $Z = M$. Soit $K_1 > 0$. Si $i = 1, \ldots, m$ on note $M_i$ l'ensemble des $x \in M$ tels que $\ker(\omega)(x)$ soit le graphe d'une application linéaire strictement $K_1$-lipschitzienne des variables $(x_1, \ldots, \hat{x_i}, \ldots, x_m)$. Les $M_i$ sont des ouverts $\mathcal{A}$-définissables inclus dans $M$. Quitte à choisir $K_1$ assez grand, on a $M = M_1 \cup \cdots \cup M_m$. Fixons $i \in \{1, \ldots, m\}$. Permutons les coordonnées $x_i$ et $x_m$. On fait une décomposition cylindrique de $\mathbf{R}^m$ adaptée à $M_i$ et à $\{0\}$ et on ne retient que les cylindres ouverts $M_{i,1}, \ldots, M_{i,r_i}$. Soit $M_{i,j}$ l'un d'eux et $M'_{i,j}$ sa base :

$$M_{i,j} = \{(x, y) : x \in M'_{i,j}, \phi_{i,j}(x) < y < \psi_{i,j}(x)\}.$$

On pose

$$B_{i,j} = \{(x, y) : x \in M'_{i,j}, y = \phi_{i,j}(x)\}$$

et

$$H_{i,j} = \{(x, y) : x \in M'_{i,j}, y = \psi_{i,j}(x)\}.$$

Les ensembles $B_{i,j}$ et $H_{i,j}$ sont dans l'adhérence $\overline{M}$ de $M$. Par conséquent la forme $\omega$ est définie sur $B_{i,j}$ et $H_{i,j}$ car elle est définie sur $\overline{M} \setminus \{0\}$ et $B_{i,j}$ et $H_{i,j}$ sont des cylindres de dimension $m - 1 \geq 1$ d'une décomposition de $\mathbf{R}^m$ adaptée à 0 et à $M$. On note $TB_{i,j}$ et $TH_{i,j}$ leurs fibrés tangents. On fait une décomposition cylindrique de $M'_{i,j}$ adapté à

$$\{x \in M'_{i,j} : T_{(x,\phi_{i,j}(x))} B_{i,j} \subset \ker(\omega)(x, \phi_{i,j}(x))\}$$

et à

$$\{x \in M'_{i,j} : T_{(x,\psi_{i,j}(x))} H_{i,j} \subset \ker(\omega)(x, \psi_{i,j}(x))\}$$

et on ne retient que les cylindres ouverts $M'_{i,j,1}, \ldots, M'_{i,j,r_{i,j}}$. Pour chaque $k = 1, \ldots, r_{i,j}$ on pose

$$M_{i,j,k} = \{(x, y) \in M_{i,j} : x \in M'_{i,j,k}\}.$$

Les $M_{i,j,k}$ ainsi construits vérifient ii, iii et iv de la proposition avec tout $K > K_1$). De plus la réunion des $M_{i,j,k}$ est dense dans $M$ et $M \setminus (\bigcup_{i,j,k} M_{i,j,k})$ est un sous-ensemble $\mathcal{A}$-définissable de $M$ de dimension au plus $m - 1$. Le cas $d = m$ est résolu.

Soit $0 < d < m$ et considérons $Z \subset M$, $\mathcal{A}$-définissable de dimension $d$. D'après la proposition 7 on peut supposer que $Z$ est une sous-variété différentielle de dimension $d$, de classe $C^k$ et $\mathcal{A}$-définissable. Soit $K_2 > 0$. Si $\tilde{\mathcal{X}} = (\mathcal{X}_1, \ldots, \mathcal{X}_m)$ est un système de coordonnées orthogonales de $\mathbf{R}^m$ on note $Z_{\tilde{\mathcal{X}}}$ l'ensemble des $x \in Z$ tels que $\ker(\omega)(x)$ soit le graphe d'une application linéaire strictement $K_2$-lipschitzienne des variables $(\mathcal{X}_1, \ldots, \mathcal{X}_{m-1})$ et le plan tangent et tels que $T_x Z$ soit le graphe d'une application linéaire strictement $K_2$-lipschitzienne des variables $(\mathcal{X}_1, \ldots, \mathcal{X}_d)$. S'il n'est pas vide, l'ensemble $Z_{\tilde{\mathcal{X}}}$ est $\mathcal{A}$-définissable et ouvert dans $Z$. C'est donc une sous-variété différentielle de dimension $d \leq m-1$, de classe $C^k$ et $\mathcal{A}$-définissable. On note aussi $\tilde{M}_{\tilde{\mathcal{X}}}$ l'ensemble des $x \in M$ tels que $\ker(\omega)(x)$ soit le graphe d'une application linéaire strictement $K_2$-lipschitzienne des variables $(\mathcal{X}_1, \ldots, \mathcal{X}_{m-1})$. C'est un ouvert $\mathcal{A}$-définissable de $M$ qui contient $Z_{\tilde{\mathcal{X}}}$. Si $K_2$ est assez grand on peut choisir un nombre fini de ces systèmes de coordonnées $\tilde{\mathcal{X}}_1, \ldots, \tilde{\mathcal{X}}_l$ de telle sorte que $Z_1 = Z_{\tilde{\mathcal{X}}_1}, \ldots, Z_l = Z_{\tilde{\mathcal{X}}_l}$ recouvrent $Z$ et $\tilde{M}_1 = \tilde{M}_{\tilde{\mathcal{X}}_1}, \ldots, \tilde{M}_l = \tilde{M}_{\tilde{\mathcal{X}}_l}$ recouvrent $M$. On peut supposer que $Z = Z_1$, $M = \tilde{M}_1$ et que $\tilde{\mathcal{X}}_1 = (x_1, \ldots, x_m)$. D'après la proposition 7 on peut supposer que $Z$ est un cylindre $\mathcal{A}$-définissable de classe $C^k$ de la forme

$$Z = \{(x', F(x')) : x' \in D'\}$$

où $D' \subset \mathbf{R}^d$ est un cylindre de classe $C^k$, $\mathcal{A}$-définissable et $F = (F_{d+1}, \ldots, F_m)$ est une application définie sur $D'$, de classe $C^k$, strictement $K_2$-lipschitzienne et $\mathcal{A}$-définissable.

Si $x' = (x_1, \ldots, x_d) \in D'$ on pose

$$\delta(x') = \max\{\delta : \forall h = (h_{d+1}, \ldots, h_m),\ \sup|h_i| < \delta \Rightarrow (x', F(x') + h) \in M\}.$$

La fonction $\delta$ est $\mathcal{A}$-définissable. D'après la proposition 7 il existe des cylindres $D'_1, \ldots, D'_r \subset D'$, de dimension $d$, de classe $C^k$ et $\mathcal{A}$-définissables tels que

– $D' \setminus (D'_1 \cup \cdots \cup D'_r)$ est de dimension au plus $d-1$

– la restriction de $\delta$ à chaque $D'_i$ est de classe $C^k$ et définisssable dans $\mathcal{A}$.

On peut donc supposer que $D' = D'_1$ et

$$Z = \{(x', F(x')) : x' \in D' = D'_1\}.$$

On pose

$$\Gamma' = \{(x', x_{d+1}, \ldots, x_{m-1}) : x' \in D',\ \sup_{j=d+1,\ldots,m-1} |x_j - F_j(x')| < \delta(x')\}$$

et

$$\Gamma = \{x \in M : (x_1, \ldots, x_{m-1}) \in \Gamma', x_m = F_m(x_1, \ldots, x_d)\}.$$

Ce sont des cylindres de dimension $m-1$, de classe $C^k$, $\mathcal{A}$-définissables et $\Gamma$ est inclus dans $M$ et contient $Z$. On considère aussi

$$\Delta = \{x + (0, \ldots, 0, h) : x = (x', x'') \in \Gamma, x' \in D', |h| < \delta(x')\}.$$

C'est un cylindre de dimension $m$, de classe $C^k$, $\mathcal{A}$-définissable et inclus dans $M$. Si $\lambda > 0$ et $\mu = (\mu_1, \ldots, \mu_m) \in \mathbf{R}^m$ avec $||\mu|| < 1$ on définit la fonction $\Theta_{\lambda,\mu}$ sur $\Gamma$ en posant pour $x = (x', x'') \in \Gamma$ avec $x' \in D'$, $\Theta_{\lambda,\mu}(x) = \lambda(1 + ||x - \mu||^2)\delta(x')$. C'est une fonction $\mathcal{A}$-définissable de classe $C^k$. On pose

$$\begin{aligned} \Delta_{\lambda,\mu} &= \{x + (0, \ldots, 0, h) : x \in \Gamma, |h| < \Theta_{\lambda,\mu}(x)\}, \\ B_{\lambda,\mu} &= \{x - (0, \ldots, 0, \Theta_{\lambda,\mu}(x)) : x \in \Gamma\}, \\ H_{\lambda,\mu} &= \{x + (0, \ldots, 0, \Theta_{\lambda,\mu}(x)) : x \in \Gamma\}. \end{aligned}$$

Il existe $\lambda_0$ tel que si $0 < \lambda < \lambda_0$ alors $\Theta_{\lambda,\mu}(x) < \delta(x)$ si $x \in \Gamma$. Les ensembles $\Delta_{\lambda,\mu}, B_{\lambda,\mu}$ et $H_{\lambda,\mu}$ sont alors des cylindres $\mathscr{A}$-définissables de classe $C^k$ inclus dans $M$. On déduit du théorème de transversalité [**32**] (voir [**7**]) et de la o-minimalité de la structure $\mathscr{A}$ (proposition 4) que pour presque tout $(\lambda, \mu)$ et pour tous les points $x$ de $\Gamma$ sauf peut-être pour un nombre fini d'entre eux formant un ensemble $F_{\lambda,\mu}$, le plan tangent

$$T_{x-(0,\ldots,0,\Theta_{\lambda,\mu}(x))}B_{\lambda,\mu}$$

et le noyau

$$\ker(\omega)(x - (0, \ldots, 0, \Theta_{\lambda,\mu}(x)))$$

ainsi que le plan tangent

$$T_{x+(0,\ldots,0,\Theta_{\lambda,\mu}(x))}H_{\lambda,\mu}$$

et le noyau

$$\ker(\omega)(x + (0, \ldots, 0, \Theta_{\lambda,\mu}(x)))$$

sont transverses. Fixons $\lambda$ et $\mu$. D'après la proposition 8 la différence $\Gamma \setminus F_{\lambda,\mu}$ est la réunion d'un nombre fini de cylindres ouverts de classe $C^k$ et $\mathscr{A}$-définissables $\Gamma_1, \ldots, \Gamma_s$. Pour chaque $i \in \{1, \ldots, s\}$ on pose

$$\Delta_{\lambda,\mu,i} = \{x + (0, \ldots, 0, h) : x \in \Gamma_i, |h| < \Theta_{\lambda,\mu}(x)\}.$$

Les ensembles $\Delta_{\lambda,\mu,i}$ sont inclus dans $M$, recouvrent $Z \setminus F_{\lambda,\mu}$ et ce sont des cylindres qui vérifient les points ii, iii et iv de la proposition avec tout $K > K_2$. De plus l'ensemble $Z' = F_{\lambda,\mu} \cap Z$ est fini donc de dimension strictement plus petite que celle de $Z$.

Il reste à considérer le cas où $Z$ est de dimension 0. C'est un ensemble fini de points. On peut supposer que c'est un unique point $x^0$. Il existe un système de coordonnées orthogonales dans lesquelles $x^0 = 0$ et $\ker(\omega)(0)$ est transverse à l'axe vertical et au plan horizontal $dx_m$. Dès que $\varepsilon > 0$ est petit le polydisque $\{x \in \mathbf{R}^m : |x_1|, \ldots, |x_m| < \varepsilon\}$ vérifie les points ii, iii et iv de la proposition 10 avec tout $K > 0$ assez grand. □

## Références

[1] J. M. Aroca, H. Hironaka & J. L. Vicente – *Introduction to the theory of infinitely near singular points. The theory of the maximal contact. Desingularization theorems*, Memorias de Matemática del Instituto « Jorge Juan », vol. 28, 29, 30, Consejo Superior de Investigaciones Científicas, 1977.

[2] R. Benedetti & J.-J. Risler – *Real algebraic and semi-algebraic sets*, Actualités Mathématiques, Hermann, 1990.

[3] J. Bochnak, M. Coste & F. Roy – *Géométrie algébrique réelle*, Springer, 1986.

[4] C. Camacho & A. Lins Neto – *Teoria geométrica das folheaçoes*, Projeto Euclides, vol. 9, Instituto de Matemática Pura e Aplicada, 1979.

[5] F. Cano – Reduction of the singularities of codimension one singular foliations in dimension three, *Ann. of Math.* **160** (2004), p. 907–1011.

[6] F. Chazal – Structure locale et globale des feuilletages de Rolle, un théorème de fibration, *Ann. Inst. Fourier (Grenoble)* **48** (1998), p. 553–592.

[7] B. Doubrovine, S. Novikov & A. Fomenko – *Géométrie contemporaine*, Éditions Mir, 1982.

[8] L. v. d. Dries – *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, 1998.

[9] L. v. d. Dries, A. Macintyre & D. Marker – The elementary theory of restricted analytic fields with exponentiation, *Ann. of Math.* **140** (1994), p. 183–205.

[10] L. v. d. Dries & C. Miller – Geometric categories and o-minimal structures, *Duke Math. J.* **84** (1996), p. 497–540.

[11] C. Ehresmann & G. Reeb – Sur les champs d'éléments de contact de dimension $p$ complètement intégrables dans une variété continuement différentiable $V_n$, *C. R. Acad. Sci. Paris* **218** (1944), p. 955–957.

[12] A. M. Gabrièlov – Projections of semianalytic sets, *Funkcional. Anal. i Priložen.* **2** (1968), p. 18–30.

[13] C. Godbillon – *Géométrie différentielle et mécanique analytique*, Hermann, 1969.

[14] ______, *Feuilletages*, Progress in Mathematics, vol. 98, Birkhäuser, 1991, Études géométriques.

[15] A. Haefliger – Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes, *Comment. Math. Helv.* **32** (1958), p. 248–329.

[16] ______, Naissance des feuilletages, d'Ehresmann-Reeb à Novikov, in *Géométrie au* XX$^{e}$ *siècle, histoire et horizons*, Hermann, 2005.

[17] A. G. Khovanskiĭ – Real analytic manifolds with the property of finiteness, and complex abelian integrals, *Funktsional. Anal. i Prilozhen.* **18** (1984), p. 40–50.

[18] N. H. Kuiper – $C^1$-equivalence of functions near isolated critical points, in *Symposium on Infinite-Dimensional Topology (Louisiana State Univ., Baton Rouge, La., 1967)*, Princeton Univ. Press, 1972, p. 199–218. Ann. of Math. Studies, No. 69.

[19] K. Kurdyka & G. Raby – Densité des ensembles sous-analytiques, *Ann. Inst. Fourier (Grenoble)* **39** (1989), p. 753–771.

[20] J.-M. Lion – Partitions normales de Łojasiewicz et hypersurfaces pfaffiennes, *C. R. Acad. Sci. Paris Sér. I Math.* **311** (1990), p. 453–456.

[21] S. Łojasiewicz – Ensembles semi-analytiques, prépublication IHES, 1965.

[22] S. López de Medrano – A splitting lemma for $C^r$ functions, $r \geq 2$, in *Singularity theory (Trieste, 1991)*, World Sci. Publ., River Edge, NJ, 1995, p. 444–450.

[23] R. Moussu & C. Roche – Théorèmes de finitude pour les variétés pfaffiennes, *Ann. Inst. Fourier (Grenoble)* **42** (1992), p. 393–420.

[24] J. R. Munkres – *Elements of algebraic topology*, Addison-Wesley Publishing Company, 1984.

[25] D. Panazzolo – Resolution of singularities of real-analytic vector fields in dimension three, *Acta Math.* **197** (2006), p. 167–289.

[26] C. Roche – Sur les variétés pfaffiennes de Rolle, Habilitation, 1993, Université de Bourgogne.

[27] ______, Densities for certain leaves of real analytic foliations, *Astérisque* **222** (1994), p. 7, 373–387, Complex analytic methods in dynamical systems (Rio de Janeiro, 1992).

[28] J.-P. Rolin, F. Sanz & R. Schäfker – Quasianalytic solutions of differential equations and o-minimal structures, prépublication, 2005.

[29] J.-P. Rolin, P. Speissegger & A. J. Wilkie – Quasianalytic Denjoy-Carleman classes and o-minimality, *J. Amer. Math. Soc.* **16** (2003), p. 751–777.

[30] P. Speissegger – The Pfaffian closure of an o-minimal structure, *J. reine angew. Math.* **508** (1999), p. 189–211.

[31] B. Teissier – Tame and stratified objects, in *Geometric Galois actions, 1*, London Math. Soc. Lecture Note Ser., vol. 242, Cambridge Univ. Press, 1997, p. 231–242.

[32] R. Thom – Les singularités des applications différentiables, *Ann. Inst. Fourier, Grenoble* **6** (1955–1956), p. 43–87.

[33] G. Valette – Lipschitz triangulations, *Illinois J. Math.* **49** (2005), p. 953–979.

[34] H. Whitney – Tangents to an analytic variety, *Ann. of Math.* **81** (1965), p. 496–549.

[35] A. J. Wilkie – A theorem of the complement and some new o-minimal structures, *Selecta Math. (N.S.)* **5** (1999), p. 397–421.

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# PROJECTIVE STRUCTURES AND PROJECTIVE BUNDLES OVER COMPACT RIEMANN SURFACES

*by*

Frank Loray & David Marín Pérez

*To José Manuel Aroca for his $60^{th}$ birthday*

***Abstract.*** — A projective structure on a compact Riemann surface $C$ of genus $g$ is given by an atlas with transition functions in $\mathrm{PGL}(2,\mathbb{C})$. Equivalently, a projective structure is given by a $\mathbb{P}^1$-bundle over $C$ equipped with a section $\sigma$ and a foliation $\mathcal{F}$ which is both transversal to the $\mathbb{P}^1$-fibers and the section $\sigma$. From this latter geometric bundle picture, we survey on classical problems and results on projective structures. By the way, we will recall some basic facts about $\mathbb{P}^1$-bundles. We will give a complete description of projective (actually affine) structures on the torus with an explicit versal family of foliated bundle picture.

***Résumé* (Structures projectives et fibrés projectifs sur les surfaces de Riemann compactes)**
Une structure projective sur une surface de Riemann $C$ de genre $g$ est donnée par un atlas dont les applications de transition sont à valeurs dans $\mathrm{PGL}(2,\mathbb{C})$. De manière équivalente, une structure projective est donnée par un fibré en $\mathbb{P}^1$ sur $C$ équipé d'une section $\sigma$ et d'un feuilletage $\mathcal{F}$ transverse à la fois aux fibres $\mathbb{P}^1$ et à la section $\sigma$. À partir de cette dernière description géométrique, nous survolons quelques problèmes et résultats classiques sur les structures projectives. Nous rappelons quelques propriétés de base sur les fibrés en $\mathbb{P}^1$. Nous donnons une description complète des structures projectives (qui sont en fait affines) sur le tore avec une famille verselle explicite de fibrés feuilletés.

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## 1. Projective structures

**1.1. Definition and examples.** — Denote by $\Sigma_g$ the orientable compact real surface of genus $g$. A *projective structure* on $\Sigma_g$ is given by an atlas $\{(U_i, f_i)\}$ of coordinate charts (local homeomorphisms) $f_i : U_i \to \mathbb{P}^1$ such that the transition functions $f_i = \varphi_{ij} \circ f_j$ are restrictions of Moebius transformations $\varphi_{ij} \in \mathrm{PGL}(2, \mathbb{C})$.

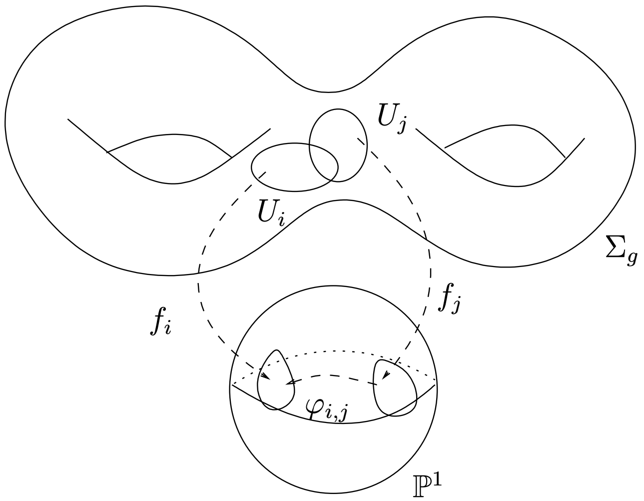


FIGURE 1. Projective atlas

There is a unique maximal atlas defining the projective structure above, obtained from the previous one by adding all charts $\{(U_i, \varphi \circ f_i)\}$ when $\varphi$ runs over $\mathrm{PGL}(2, \mathbb{C})$.

A projective structure induces a *complex structure* on $\Sigma_g$, just by pulling-back that of $\mathbb{P}^1$ by the projective charts. We will denote by $C$ the corresponding Riemann surface (complex curve).

***Example 1.1* (The Universal cover).** — Let $C$ be a compact Riemann surface having genus $g$ and consider its universal cover $\pi : U \to C$. By the Riemann Mapping Theorem, we can assume that $U$ is either the Riemann sphere $\mathbb{P}^1$, or the complex plane $\mathbb{C}$ or the unit disk $\Delta$ depending wether $g = 0$, 1 or $\geq 2$. We inherit a representation of the fundamental group $\rho : \pi_1(C) \to \mathrm{Aut}(U)$ whose image $\Lambda$ is actually a subgroup of $\mathrm{PGL}(2, \mathbb{C})$. All along the paper, by abuse of notation, we will identify elements $\gamma \in \pi_1(C)$ with their image $\rho(\gamma) \in \mathrm{PGL}(2, \mathbb{C})$. The atlas defined on $C$ by all local determinations of $\pi^{-1} : C \dashrightarrow \mathbb{P}^1$ defines a projective structure on $C$ compatible with the complex one. Indeed, any two determinations of $\pi^{-1}$ differ by left composition with an element of $\Lambda$.

We thus see that any complex structure on $\Sigma_g$ is subjacent to a projective one. In fact, for $g \geq 1$, we will see that there are many projective structures compatible to a given complex one (see Theorem 1.2). We will refer to the projective structure above as the *canonical projective structure* of the Riemann surface $C$: it does not depend on the choice of the uniformization of $U$. We now give other examples.

***Example 1.2* (Quotients by Kleinian groups).** — Let $\Lambda \subset \mathrm{PGL}(2,\mathbb{C})$ be a subgroup acting properly, freely and discontinuously on some connected open subset $U \subset \mathbb{P}^1$. Then, the quotient map $\pi : U \to C := U/\Lambda$ induces a projective structure on the quotient $C$, likely as in Example 1.1. There are many such examples where $U$ is neither a disk, nor the plane. For instance, *quasi-Fuchsian groups* are obtained as image of small perturbations of the representation $\rho$ of Example 1.1; following [**35**], such perturbations keep acting discontinuously on some *quasi-disk* (a topological disk whose boundary is a Jordan curve in $\mathbb{P}^1$).

***Example 1.3* (Schottky groups).** — Pick $2g$ disjoint discs $\Delta_1^-,\dots,\Delta_g^-$ and $\Delta_1^+,\dots,\Delta_g^+$ in $\mathbb{P}^1$, $g \geq 1$. For $i = 1,\dots,n$, let $\varphi_i \in \mathrm{PGL}(2,\mathbb{C})$ be a loxodromic map sending the disc $\Delta_i^-$ onto the complement $\mathbb{P}^1 - \Delta_i^+$.

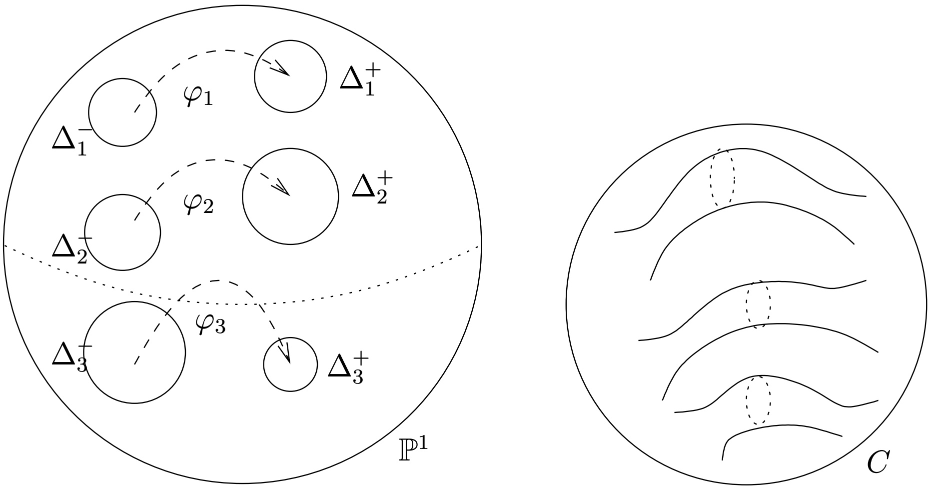


FIGURE 2. Schottky groups

The group $\Lambda \subset \mathrm{PGL}(2,\mathbb{C})$ generated by $\varphi_1,\dots,\varphi_g$ acts properly, freely, and discontinuously on the complement $U$ of some closed set contained inside the disks (a Cantor set whenever $g \geq 2$). The fundamental domain of this action on $U$ is given by the complement of the disks and the quotient $C = U/\Lambda$ is obtained by gluing together the boundaries of $\Delta_i^+$ and $\Delta_i^-$ by means of $\varphi_i$, $i = 1,\dots,g$. Therefore, $C$ is a compact Riemann surface of genus $g$. This picture is clearly stable under small deformation of the generators $\varphi_i$ and we thus obtain a complex $3g-3$ dimensional family of projective structures on the genus $g$ surface $\Sigma_g$ (we have divided here by the action of $\mathrm{PGL}(2,\mathbb{C})$ by conjugacy).

**1.2. Developping map and monodromy representation.** — Given a projective atlas and starting from any initial coordinate chart $(U_0, f_0)$, one can extend it analytically along any path $\gamma$ starting from $p_0 \in U_0$.

Indeed, after covering $\gamma$ by finitely many projective coordinate charts, say $(U_0, f_0)$, $(U_1, f_1)$, ... ,$(U_n, f_n)$, one can modify them step by step in the following way. First of all, since $f_0 = \varphi_{01} \circ f_1$ on $U_0 \cap U_1$, one can replace the chart $f_1$ by $\tilde{f}_1 := \varphi_{01} \circ f_1$ which is well-defined on $U_1$, extending $f_0$. Next, we replace $f_2$ by $\tilde{f}_2 := \varphi_{01} \circ \varphi_{12} \circ f_2$ which, on $U_1 \cap U_2$, coincide with $\tilde{f}_1$. Step by step, we finally arrive at the chart $\tilde{f}_n := \varphi_{01} \circ \cdots \circ \varphi_{n-1\ n} \circ f_n$ which, by construction, is the analytic continuation of $f_0$ along $\gamma$.

Therefore, the local chart $(U_0, f_0)$ extends (after lifting on the universal covering) as a global submersion on the universal cover

$$f : U \to \mathbb{P}^1$$

which is called the *developping map* of the projective structure. The developping map is moreover holomorphic with respect to the complex structure subjacent to the projective one. By construction, the monodromy of $f$ along loops takes the form

$$f(\gamma.u) = \varphi_\gamma \circ f, \quad \varphi_\gamma \in \mathrm{PGL}(2, \mathbb{C}) \quad \forall \gamma \in \pi_1(\Sigma_g, p_0) \tag{1}$$

($u$ is the coordinate on $U$ and $\gamma.u$, the canonical action of $\pi_1(\Sigma_g, p_0)$ on $U$). In fact, $\varphi_\gamma$ is the composition of all transition maps $\varphi_{i,j}$ encoutered along $\gamma$ for a given finite covering of projective charts: with notations above, setting $(U_n, f_n) = (U_0, f_0)$, we have $\varphi_\gamma = \varphi_{01} \circ \cdots \circ \varphi_{n-1\ n}$. It turns out that $\varphi_\gamma$ only depends on the homotopy class of $\gamma$ and we inherit a *monodromy representation*

$$\rho : \pi_1(\Sigma_g, p_0) \to \mathrm{PGL}(2, \mathbb{C}) \ ; \ \gamma \mapsto \varphi_\gamma. \tag{2}$$

The image $\Lambda$ of $\rho$ will be called monodromy group. The developping map $f$ is defined by the projective structure up to the choice of the initial chart $(U_0, f_0)$ above: it is unique up to left composition $\varphi \circ f$, $\varphi \in \mathrm{PGL}(2, \mathbb{C})$. Therefore, the monodromy representation is defined by the projective structure up to conjugacy: the monodromy of $\varphi \circ f$ is $\gamma \mapsto \varphi \circ \varphi_\gamma \circ \varphi^{-1}$.

Conversely, any global submersion $f : U \to \mathbb{P}^1$ on the universal covering $\pi : U \to \Sigma_g$ satisfying (1) is the developping map of a unique projective structure on $\Sigma_g$. We note that condition (1) forces the map $\gamma \to \varphi_\gamma$ to be a morphism.

***Example 1.4***. — The developping map of the canonical projective structure (see example 1.1) is the inclusion map $U \hookrightarrow \mathbb{P}^1$ of the universal cover of $C$. More generally, when the projective structure is induced by a quotient map $\pi : U \to C = U/\Lambda$ like in example 1.2, then the developping map $f$ is the universal cover $\tilde{U} \to U$ of $U$ and the monodromy group is $\Lambda$. In example 1.3, the open set $U$ is not simply connected (the complement of a Cantor set) and the developping map is a non trivial covering. Thus the corresponding projective structure is not the canonical one. Similarly, the developping map of a quasi-Fuchsian group is the uniformization map of the corresponding quasi-disk and is not trivial; the projective structure is neither the canonical one, nor of Schottky type.

***Example 1.5* (The Sphere)**. — Given a projective structure of the Riemann sphere $\mathbb{P}^1$, we see that the developping map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is uniform (no monodromy since $\pi_1(\mathbb{P}^1)$ is trivial). Therefore, $f$ is a global holomorphic submersion (once we have fixed the complex structure) and thus $f \in \mathrm{PGL}(2, \mathbb{C})$. Consequently, the projective structure is the canonical one and it is the unique projective structure on $\mathbb{P}^1$.

For similar reason, we remark that the monodromy group of a projective structure on a surface of genus $g \geq 1$ is never trivial.

***Example 1.6* (The Torus)**. — Let $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ be a lattice in $\mathbb{C}$, $\tau \in \mathbb{H}$, and consider the elliptic curve $C := \mathbb{C}/\Lambda$. The monodromy of a projective structure on $C$ is abelian; therefore, after conjugacy, it is in one of the following abelian groups:

– the linear group $\{\varphi(z) = az \ ; \ a \in \mathbb{C}^*\}$,
– the translation group $\{\varphi(z) = z + b \ ; \ b \in \mathbb{C}\}$,
– the finite abelian dihedral group generated by $-z$ and $1/z$.

The canonical projective structure on $C$ has translation monodromy group $\Lambda$. On the other hand, for any $c \in \mathbb{C}^*$ the map

$$f_c : \mathbb{C} \to \mathbb{P}^1 \ ; \ u \mapsto \exp(c.u) \tag{3}$$

is the developping map of a projective structure on $C$ whose monodromy is linear, given by

$$f_c(u+1) = e^c \cdot f(u) \quad \text{and} \quad f_c(u+\tau) = e^{c\tau} \cdot f(u). \tag{4}$$

We inherit a 1-parameter family of projective structures parametrized by $c \in \mathbb{C}^*$ (note that $f_0 \equiv 1$ is not a submersion). We will see latter that this list is exhaustive. In particular, all projective structures on the torus are actually affine (transition maps in the affine group).

The projective structures listed in example 1.6 are actually *affine structures*: the developping map takes values in $\mathbb{C}$ with affine monodromy.

***Theorem 1.1* (Gunning [12])**. — *All projective structures on the elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, are actually affine and listed in example 1.6 above. There is no projective structure having affine monodromy on surfaces $\Sigma_g$ of genus $g \geq 2$.*

In particular, the dihedral group is not the holonomy group of a projective structure on the torus.

*Partial proof*. — Here, we only prove that the list of example 1.6 exhausts all affine structures on compact Riemann surfaces. In example 1.7, we will see that there are no other projective structure on tori than the affine ones.

Let $f : U \to \mathbb{P}^1$ be the developping map of a projective structure with affine monodromy on the compact Riemann surface $C \neq \mathbb{P}^1$: $f$ is a holomorphic local homeomorphism satisfying

$$f(\gamma \cdot u) = a_\gamma \cdot f(u) + b_\gamma,\ a_\gamma \in \mathbb{C}^*,\ b_\gamma \in \mathbb{C}, \quad \forall \gamma \in \pi_1(C, p_0)$$

Choose a holomorphic 1-form $\omega_0$ on $C$ and write $\omega_0 = \phi \cdot df$. Here, we identify $\omega_0$ with its lifting on the universal covering. Since $f$ is a local diffeomorphism, $df$ has no zeroes and $\phi$ is holomorphic on $U$, vanishing exactly on zeroes of $\omega_0$ and poles of $f$. Moreover, the monodromy of $\phi$ is that of $df$, given by $\phi(\gamma \cdot u) = a_\gamma^{-1} \cdot \phi(u)$. Therefore, the meromorphic 1-form $\omega_1 = \frac{d\phi}{\phi}$ has no monodromy: it defines a meromorphic 1-form on $C$ having only simple poles, located at the zeroes of $\omega_0$ and poles of $f$, the residues of which are positive integers. Following Residue Theorem, $\omega_1$ has actually no poles: $f$ is holomorphic, $\omega_0$ does not vanish and thus genus $g = 1$. This proves the second assertion of the statement.

Now, assume $g = 1$, $U = \mathbb{C}$ and $C = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. The 1-form $\omega_1$ above is holomorphic and thus takes the form $\omega_1 = -c \cdot du$ for some constant $c \in \mathbb{C}$. In other words, we have $f''/f' = c$ and we obtain after integration

- $f(u) = a \cdot e^{cu} + b$ when $c \neq 0$,
- $f(u) = a \cdot u + b$ when $c = 0$

for constants $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. After left composition by an affine map, which does not affect the affine structure, we can set $a = 1$ and $b = 0$ and $f$ belongs to the list of example 1.6. □

***Remark 1.1***. — We see from the proof that the projective structures on $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ are naturally parametrized by $\mathbb{C}$, namely the constant map $\phi = f''/f' \equiv c$, which is not clear from the description of example 1.6 (we see $\mathbb{C}^*$ plus one point ). One can recover this by choosing conveniently the integration constants $a$ and $b$ in the proof above. Indeed, consider the alternate family of developping maps given by

$$F : \mathbb{C}^2 \to \mathbb{C}\ ;\ (c, u) \mapsto f_c(u) := \begin{cases} \frac{e^{cu}-1}{c}, & c \neq 0 \\ u, & c = 0 \end{cases} \tag{5}$$

The map $F$ is clearly holomorphic on $\mathbb{C}^2$ and makes the developping maps $f_c$ into a holomorphic family parametrized by $c \in \mathbb{C}$. Moreover, the corresponding holonomy representations are given by

$$f_c(u + \gamma) = \begin{cases} e^{c\gamma} f_c(u) + \frac{e^{c\gamma}-1}{c}, & c \neq 0 \\ u + \gamma, & c = 0 \end{cases} \quad \forall \gamma \in \mathbb{Z} + \tau\mathbb{Z}$$

and we see the affine motions with common fixed point $-1/c$ converging to translations while $c \to 0$.

***Remark 1.2***. — When we set $g = 1$ in example 1.3, we have $U = \mathbb{C}^*$ and $\Lambda$ is generated by a single map $\varphi(z) = e^{2i\pi\lambda} z$. The quotient $C = U/\Lambda$ is the elliptic curve with lattice $\mathbb{Z} + \lambda\mathbb{Z}$. The complex structure varies with $\lambda$ and very few projective structures on the torus are obtained by this way. In fact, we see in example 1.6 that, for generic values of $c$, the monodromy group of the corresponding projective structure is not discrete ($c$ is not $\mathbb{Z}$-commensurable with 1 and $\tau$).

**1.3. Quadratic differentials.** — In order to generalize the arguments involved in the proof of Theorem 1.1 for genus $g \geq 2$ Riemann surfaces, we have to replace $f''/f'$ by the *Schwartzian derivative* of $f$

$$\mathcal{S}(f) := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{6}$$

Recall that, for any holomorphic functions $f$ and $g$, we have

$$\mathcal{S}(f \circ g) = \mathcal{S}(f) \circ g \cdot (g')^2 + \mathcal{S}(g). \tag{7}$$

Given a projective structure on a Riemann surface $C$, consider the Schwartzian derivative of the corresponding developping map $\phi := \mathcal{S}(f)$. For any $\gamma \in \Lambda = \pi_1(C)$, we deduce from property (1) of $f$ that

$$\phi \circ \gamma \cdot (\gamma')^2 = \mathcal{S}(f \circ \gamma) = \mathcal{S}(\varphi_\gamma \circ f) = \phi.$$

In other words, the *quadratic differential* $\omega = \phi(u) \cdot du^2$ is invariant under $\Lambda$ and gives rise to a quadratic differential on the Riemann surface $C$. We note that $\omega$ is holomorphic. Indeed, outside the poles of $f$, $\phi_1 := f'$ is not vanishing, thus $\phi_2 := \phi_1'/\phi_1$ is holomorphic and $\phi = \phi_2' - (\phi_2)^2/2$ well. On the other hand, at a pole of $f$, one can replace $f$ for instance by $1/f$, which is not relevant for the Schwartzian derivative, and go back to the previous argument. By this way, we canonically associate to any projective structure on $C$ a holomorphic quadratic differential $\omega$ on $C$, i.e., a global section of $K_C^{\otimes 2}$, where $K_C$ is the canonical line bundle over $C$.

Conversely, given any holomorphic quadratic differential $\omega = \phi(u) \cdot du^2$ on the Riemann surface $C$, one can solve locally the differential equation $\mathcal{S}(f) = \phi$ in $f$ and recover the coordinate charts of a projective structure on $C$ (compatible with the complex one): the fact is that any two (local) solutions of $\mathcal{S}(f) = \phi$ differ by left composition by a Moebius transformation.

***Example 1.7.*** — In genus 1 case, any holomorphic quadratic differential takes the form $\omega = c \cdot du^2$ for a constant $c \in \mathbb{C}$ ($K_C^{\otimes 2} = K_C$ is still the trivial bundle). In fact, $\omega = \tilde{\omega}^2$, where $\tilde{\omega} = \sqrt{c}du$. On the other hand, any solution of $f''/f' = \tilde{c}$ gives rise to a solution of $\mathcal{S}(f) = -\tilde{c}^2/2 = c$; therefore, the projective structure defined by $\omega$ is actually subjacent to the affine structure defined by $\tilde{c}$. This concludes the proof of Theorem 1.1. We note that the space of affine structures forms a two fold covering of the space of projective structures (the choice of the square root $\tilde{c}$). Of course, this comes from the fact that the 2 affine structures given by $f_c$ and $1/f_c$ (with notations of example 1.6) do not define distinct projective structures.

For genus $g \geq 2$ Riemann surfaces, the dimension of $H^0(C, K_C^{\otimes 2})$ can be computed by Riemann-Roch Formula, and we obtain

***Theorem 1.2* (Gunning [12]).** — *The set of projective structures on a complex Riemann surface $C$ of genus $g \geq 2$ is parametrized by the $3g-3$-dimensional complex vector space $H^0(C, K_C^{\otimes 2})$.*

In this vector space, 0 stands for the canonical structure of example 1.1.

**1.4. The monodromy mapping.** — A natural question arising while studying projective structures is to understand, for a given surface $\Sigma_g$, the nature of the Monodromy (or Riemann-Hilbert) Mapping

$$\mathscr{P}_g \longrightarrow \mathscr{R}_g.$$

On the left-hand side, $\mathscr{P}_g$ denotes the set of all projective structure on $\Sigma_g$ up to isomorphism; on the right-hand side, $\mathscr{R}_g$ is the set of representations of the fundamental group in $\mathrm{PGL}(2,\mathbb{C})$ up to conjugacy:

$$\mathscr{R}_g = \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PGL}(2,\mathbb{C}))/_{\mathrm{PGL}(2,\mathbb{C})}.$$

Let us first consider the genus $g=1$ case. From Gunning's Theorem 1.1, the left-hand side can be viewed as a $\mathbb{C}$-bundle over the modular orbifold $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ where $\mathbb{H}$ denotes the upper-half plane whose fiber at a given complex structure is the affine line of holomorphic differentials. Nevertheless, to avoid dealing with orbifold points, we prefer to deal with the parametrization of affine structures by $\mathbb{H}\times\mathbb{C}$ given by the map

$$(\tau, c) \mapsto (C, \omega) \text{ where } \left\{ \begin{array}{ll} C = & \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) \\ \omega = & c\cdot du \end{array} \right.$$

Here, the base $\mathbb{H}$ is the space of marked complex structures on the torus, up to isomorphism, and the fiber over $\tau$ is the affine line of differentials $\mathbb{C}\cdot du$, $u$ the variable of $\mathbb{C}$. Since all projective structures are actually affine, we can replace $\mathscr{R}_1$ by $\mathscr{A}_1 := \mathrm{Hom}(\pi_1(C), \mathrm{Aff}(\mathbb{C}))/_{\mathrm{Aff}(\mathbb{C})}$ where

$$\mathrm{Aff}(\mathbb{C}) := \{\varphi(z) = az+b,\ a\in\mathbb{C}^*,\ b\in\mathbb{C}\}$$

is the group of affine transformations. Once we have fixed generators 1 and $\tau$ for the fundamental group of $C=\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, the set $\mathrm{Hom}(\pi_1(C), \mathrm{Aff}(\mathbb{C}))$ identifies with the complex 3-dimensional subvariety

$$\{(a_1z+b_1, a_\tau z+b_\tau)\ ;\ (a_1-1)b_\tau = (a_\tau-1)b_1\} \subset (\mathbb{C}^*\times\mathbb{C})^2$$

(here, we see the condition for the commutativity). The $\mathrm{Aff}(\mathbb{C})$)-action by conjugacy commutes with the projection on the linear part $\mathbb{C}^*\times\mathbb{C}^*$. For any $(a_1,a_\tau)\neq(1,1)$, the action is transitive on the fibre: it is the usual action of $\mathrm{Aff}(\mathbb{C})$) on the line $\{(b_1,b_\tau)\ ;\ (a_1-1)b_\tau=(a_\tau-1)b_1\}\subset\mathbb{C}^2$. The fibre over $(1,1)$ is $\mathbb{C}^2$ and $\mathrm{Aff}(\mathbb{C})$) acts by homothecy: after deleting the point $(b_1,b_\tau)=(0,0)$ corresponding to the trivial representation, the quotient by the action is $\mathbb{P}^1$. Since the trivial representation does not occur as monodromy representation of an affine structure on the torus, we consider the quotient $\mathscr{B}_1\subset\mathscr{A}_1$ of its complement in $\mathrm{Hom}(\pi_1(C), \mathrm{Aff}(\mathbb{C}))$. By the previous remarks, $\mathscr{B}_1$ is the blowing-up of $\mathbb{C}^* \times \mathbb{C}^*$ at the point $(1,1)$ and can be identified with the 2-dimensional complex manifold

$$\mathscr{B}_1 = \{(a_1, a_\tau, [b_1 : b_\tau])\ ;\ (a_1 - 1)b_\tau = (a_\tau - 1)b_1\} \subset \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{P}^1$$

where $[z : w]$ denotes homogeneous coordinates on $\mathbb{P}^1$.

The projection $\mathscr{B}_1 \to \mathbb{C}^* \times \mathbb{C}^*$ is just the blow-up of the point $(1,1)$ and the exceptional divisor consists in euclidean representation. Finally, the monodromy map is described by

$$\mathbb{H} \times \mathbb{C} \to \mathscr{B}_1\ ;\ (\tau, c) \mapsto \begin{cases} (e^c, e^{c\tau}, [\frac{e^c - 1}{c} : \frac{e^{c\tau} - 1}{c}]), & c \neq 0 \\ (1, 1, [1 : \tau]), & c = 0 \end{cases}$$

Looking at the differential of the Monodromy Map above, we see that it has always rank 2 and *the Monodromy Map is a holomorphic local diffeomorphism*; it is moreover *injective and proper in restriction to each fiber* $\tau \times \mathbb{C}$. Its image is the complement of the preimage on $\mathscr{B}_1$ of the real torus $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C}^* \times \mathbb{C}^*$ plus the complement of $\mathbb{P}^1(\mathbb{R})$ inside the exceptional divisor $\mathbb{P}^1(\mathbb{C})$ over the point $(1,1) \in \mathbb{C}^* \times \mathbb{C}^*$.

*But* the Monodromy Map *is neither injective, nor a covering map onto its image*: for instance, for any $\tau, \tau' \in \mathbb{H}$, $\tau' \neq \tau$, and for any $(m,n) \in \mathbb{Z}^2 - \{(0,0)\}$, the two affine structures

$$(\tau, 2i\pi \frac{m\tau' + n}{\tau' - \tau}) \quad \text{and} \quad (\tau', 2i\pi \frac{m\tau + n}{\tau' - \tau})$$

have the same monodromy representation. In particular, the injectivity is violated for arbitrarily close complex structures. On the other hand, the monodromy of the canonical structure $(\tau, 0)$ occurs only for this structure.

Consider now the genus $g \geq 2$ case. It follows from Gunning's Theorem 1.2 above that the set $\mathscr{P}_g$ of projective structures on the genus $g \geq 2$ surface $\Sigma_g$ can be viewed as a complex $6g - 6$-dimensional space. Indeed, if we denote by $\mathscr{T}_g$ the Teichmüller space of complex marked structures on $\Sigma_g$ viewed as an open subset of $\mathbb{C}^{3g-3}$, then $\mathscr{P}_g$ is parametrized by the rank $3g - 3$-vector bundle $\tilde{\mathscr{P}}_g$ over $\mathscr{T}_g$ whose fiber over a given complex structure $C$ is the space of quadratic differentials $H^0(C, K_C^{\otimes 2})$.

By Theorem 1.1, the monodromy representation cannot be affine in the case $g \geq 2$. The image of $\mathscr{P}_g$ by the Monodromy Map is thus included in the subset of *irreducible* representations

$$\mathscr{R}_g^{\mathrm{irr}} := \mathscr{R}_g - \mathscr{A}_g$$

where $\mathscr{A}_g = \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{Aff}(\mathbb{C}))/_{\mathrm{PGL}(2,\mathbb{C})}$ is the set of affine representations *up to* $\mathrm{PGL}(2,\mathbb{C})$*-conjugacy*. One can check (see [**13**]) that $\mathscr{R}_g^{\mathrm{irr}}$ forms a non-singular complex manifold of dimension $6g - 6$. Thus, the Monodromy Map can locally be described as a holomorphic map between open subsets of $\mathbb{C}^{6g-6}$ and the following result makes sense (see proof in section 1.5).

***Theorem 1.3* (Hejhal [7, 16, 18])**. — *The Monodromy Map is a local diffeomorphism.*

In [20], it is moreover proved that the Monodromy Map is symplectic with respect to symplectic structures that can be respectively canonically defined on both spaces (see [10]).

The restriction of the Monodromy Map to each fiber $H^0(C, K_C^{\otimes 2})$ of $\tilde{\mathscr{P}}_g$ over $C \in \mathscr{T}_g$ is injective. In other words, we have the following result whose proof will be given in section 2.2.

***Theorem 1.4* (Poincaré [28]).** — *Given a compact Riemann surface $C$, any two projective structures are the same if, and only if, they have the same monodromy representation (up to* $\mathrm{PGL}(2,\mathbb{C})$*).*

It is clear that the Monodromy Map is not surjective. First of all, by Theorem 1.1, its image is contained in $\mathscr{R}_g^{\mathrm{irr}} \subset \mathscr{R}_g$. On the other hand, the space of representations $\mathrm{Hom}(\pi_1(C), \mathrm{PGL}(2,\mathbb{C}))$ falls into 2 connected components, namely the component of those that can be lifted as $\mathrm{Hom}(\pi_1(C), \mathrm{SL}(2,\mathbb{C}))$ and the other ones. Since the Monodromy Map is continuous (actually holomorphic) and since the monodromy of canonical projective structures can be lifted to $\mathrm{SL}(2,\mathbb{R})$, it becomes clear that the image of the Monodromy Map will be in the former component. Finally, notice that the monodromy representation cannot be in $\mathrm{PSU}(2,\mathbb{C})$, i.e., conjugated to a group of rotations of the sphere, otherwise we could pull-back the invariant spherical metric of $\mathbb{P}^1$ by the developping map, giving rise to a curvature $+1$ metric on the surface, impossible except in the trivial case $g = 0$ (see Example 1.5). The main result in the field, which has been conjectural for decades, is the following.

***Theorem 1.5* (Gallo-Kapovich-Marden [9]).** — *Consider the genus $g$ surface $\Sigma_g$, $g \geq 2$. A homomorphism $\rho \in \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PGL}(2,\mathbb{C}))$ is the monodromy representation of a projective structure on the $\Sigma_g$ if, and only if, $\rho$ can be lifted as $\tilde{\rho} \in \mathrm{Hom}(\pi_1(C), \mathrm{SL}(2,\mathbb{C}))$ and the image of $\rho$ is, up to $\mathrm{PGL}(2,\mathbb{C})$-conjugacy, neither in the affine group $\mathrm{Aff}(\mathbb{C})$, nor in the rotation group $\mathrm{PSU}(2,\mathbb{C})$.*

**1.5. The fibre bundle picture.** — Let $f : U \to \mathbb{P}^1$ be the developping map of a projective structure on $C$ (here we fix the underlying complex structure) and consider its graph $\{(u, f(u))\ ;\ u \in U\} \subset U \times \mathbb{P}^1$. The fundamental group $\pi_1(C)$ acts on the product $U \times \mathbb{P}^1$ as follows: for any $\gamma \in \pi_1(C)$, set

$$\gamma : (u, y) \mapsto (\gamma \cdot u, \varphi_\gamma(y))$$

where $u \mapsto \gamma \cdot u$ is the canonical action of $\pi_1(C)$ on the universal cover and $\varphi_\gamma$ is the monodromy of the projective structure along $\gamma$. This action of $\pi_1(C)$ is proper, free and discontinuous since its projection on $U$ is so. By consequence, we can consider the quotient:

$$P := U \times \mathbb{P}^1/_{\pi_1(C)}.$$

The projection $U \times \mathbb{P}^1 \to C$ defined by $(u, y) \mapsto \pi(u)$, where $\pi : U \to C$ is the universal cover, is preserved by the action and induces a global submersion

$$\pi : P \to C$$

making $P$ *into a* $\mathbb{P}^1$*-bundle over* $C$*. The graph of* $f$ *also is invariant under the action (consequence of (1)) thus defining a section*

$$\sigma : C \to P.$$

Finally, the horizontal foliation defined by $\{y = \text{constant}\}$ is also preserved and defines a foliation $\mathcal{F}$ transversal to all $\mathbb{P}^1$-fibres on $P$. Since the developping map $f$ is regular, its graph is transversal to the horizontal foliation and $\sigma$ is transversal to $\mathcal{F}$. In this situation, we say that the $\mathbb{P}^1$-bundle $P$ is flat. The triple $(\pi : P \to C, \mathcal{F}, \sigma)$ is well-defined by the projective structure up to analytic isomorphism of $\mathbb{P}^1$-bundles.

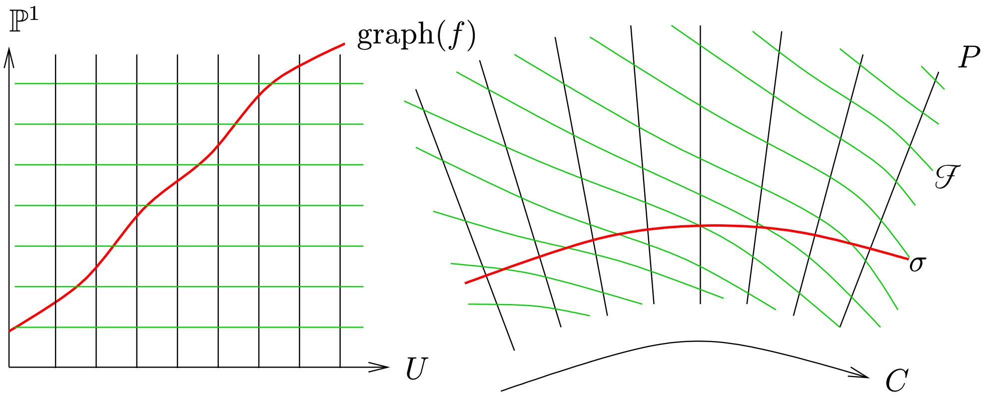


FIGURE 3. From projective structure to bundle picture

Conversely, given a $\mathbb{P}^1$-bundle $\pi : P \to C$, a foliation $\mathcal{F}$ on $P$ transversal to $\pi$ and a section $\sigma : C \to P$ transversal to $\mathcal{F}$, then the (unique) projective structure on $\mathbb{P}^1$-fibres can be transported, transversely to the foliation $\mathcal{F}$, inducing a projective structure on the section $\sigma(C)$, and thus on its $\pi$-projection $C$.

In the recent terminologoly of [**3**], such triple $(\pi : P \to C, \mathcal{F}, \sigma)$ are called $\mathfrak{sl}(2, \mathbb{C})$-opers.

***Remark 1.3***. — Given a homomorphism $\rho \in \mathrm{Hom}(\pi_1(C), \mathrm{PGL}(2, \mathbb{C}))$, one can at least construct the pair $(\pi : P \to C, \mathcal{F})$ as above. This foliated surface is called the *suspension* of the representation $\rho$, also known as the *flat* $\mathbb{P}^1$*-bundle* associated to $\rho$. Conversely, consider a flat $\mathbb{P}^1$-bundle, i.e., a pair $(\pi : P \to C, \mathcal{F})$ where $\pi : P \to C$ is a $\mathbb{P}^1$-bundle and $\mathcal{F}$ is a foliation transversal to $\pi$. Then one can associate to it a representation $\rho$ in the following way.

Over any sufficiently small open subset $U_i \subset C$, one can construct a trivializing coordinate $F_i : \pi^{-1}(U_i) \to \mathbb{P}^1$ for the flat bundle, that is to say inducing an isomorphism in restriction to each fibre and such that the level curves $F_i^{-1}(y_0)$ are local leaves of the foliation $\mathcal{F}$. In fact, $F_i$ is uniquely determined after choosing the local $\mathcal{F}$-invariant sections $\sigma_0, \sigma_1, \sigma_\infty : U_i \to P$ along which $F_i$ takes values 0, 1 and $\infty$

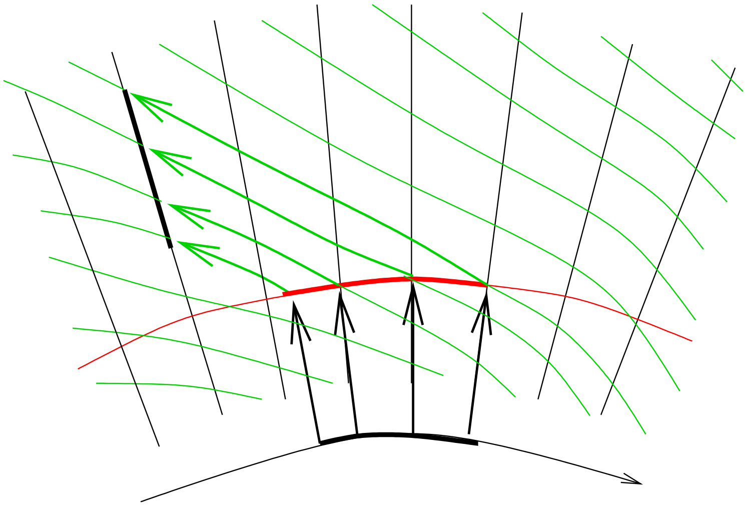

FIGURE 4. From bundle picture to projective structure

respectively. Such flat coordinate is well defined up to left-composition by a Moebius transformation; likely as in section 1.2, after fixing a flat local coordinate $F$ over some neighborhood of the base point $x_0 \in C$, we inherit a monodromy representation $\rho : \pi_1(C, x_0) \to \mathrm{PGL}(2, \mathbb{C})$ where the analytic continuation of $F$ along any loop $\gamma$ satisfies $F(\gamma \cdot u) = \rho(\gamma) \circ F(u)$.

It turns out that any flat $\mathbb{P}^1$-bundle is isomorphic to the suspension of its monodromy representation just defined. In fact, *any two flat $\mathbb{P}^1$-bundles are isomorphic if, and only if, they have the same monodromy representation up to* $\mathrm{PGL}(2, \mathbb{C})$ *conjugacy*. Indeed, let $(\pi : P \to C, \mathcal{F})$ and $(\pi' : P' \to C, \mathcal{F}')$ be flat $\mathbb{P}^1$-bundles having flat coordinates $F$ and $F'$ over $U_0 \subset C$ giving rise to the same monodromy representation; then the local isomorphism $\Phi : \pi^{-1}(U_0) \to \pi'^{-1}(U_0)$ sending any point $p$ to the unique point $p'$ satisfying $(\pi(p), F(p)) = (\pi'(p'), F'(p'))$ extends uniformly as a global isomorphism of flat $\mathbb{P}^1$-bundles $\Phi : P \to P'$, i.e., conjugating $\mathcal{F}$ to $\mathcal{F}'$ and satisfying $\pi' \circ \Phi = \pi$.

*Proof of Hejal's Theorem 1.3.* — In fact, since the Monodromy Map is clearly holomorphic, it is enough to prove that it is locally bijective.

Let $(\pi : P \to C, \mathcal{F}, \sigma)$ be the triple associated to a projective structure having monodromy representation $\rho \in \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PGL}(2, \mathbb{C}))$. For any perturbation $\rho' \in \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PGL}(2, \mathbb{C}))$ of $\rho$, the corresponding suspension $(\pi' : P' \to C, \mathcal{F}')$ is close to the foliated bundle $(\pi : P \to C, \mathcal{F})$; if the perturbation is small enough, one can find a real $C^\infty$ section $\sigma' : C \to P'$ close to $\sigma : C \to P$ and still transversal to $\mathcal{F}'$ (all of this makes sense and can be checked on the neighborhood of a fundamental domain of the universal cover $U \times \mathbb{P}^1$). The foliation $\mathcal{F}'$ still induces a projective structure on the real surface $\sigma'(C)$ that, by construction, has the required monodromy. This proves the surjectivity.

Let $(\pi : P \to C, \mathcal{F}, \sigma)$ be the triple associated to a projective structure $\mathcal{P}$ and consider another projective structure $\mathcal{P}'$ close to this $\mathcal{P}$ having the same monodromy representation. The fibre bundle construction can be done in the real $C^\infty$ setting so that one can associate to $\mathcal{P}'$ a triple $(\pi : P \to C, \mathcal{F}, \sigma')$ where $C$ is still the complex curve attached to $\mathcal{P}$ and $\sigma' : C \to P$ is now a real $C^\infty$ section transversal to $\mathcal{F}$; we note that the pair $(\pi : P \to C, \mathcal{F})$ is the same for $\mathcal{P}$ and $\mathcal{P}'$ since they have the same monodromy representation. If $\mathcal{P}'$ is close enough to $\mathcal{P}$, say in the $C^\infty$ category, then $\sigma'$ is close to $\sigma$; one can therefore unambiguously define a $C^\infty$ diffeomorphism $\phi : \sigma'(C) \to \sigma(C)$ by following the leaves of the foliation from one section to the other one. By construction, the projective structures induced by $\mathcal{F}$ on both sections are conjugated by $\phi$. The diffeomorphism $\pi_*\phi := \pi \circ \phi \circ \sigma'$ actually integrates the quasi-conformal structure induced by $\mathcal{P}'$ on $C$; it is close to the identity. $\square$

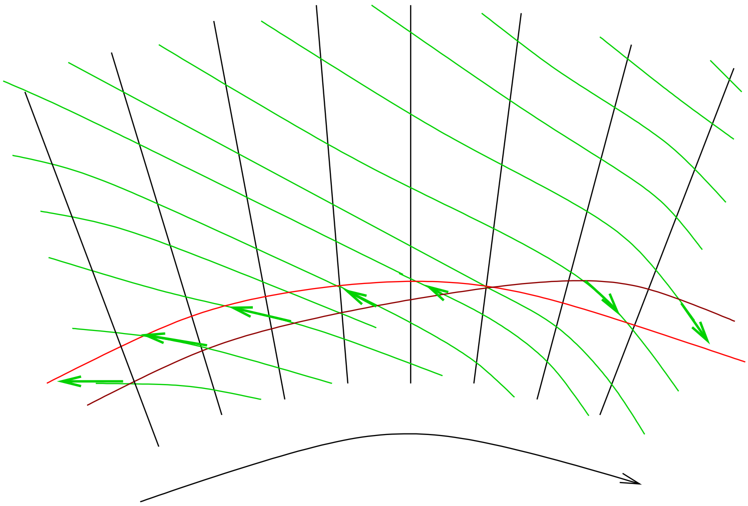

FIGURE 5. Local injectivity of the Monodromy Map

***Remark 1.4***. — Since the Monodromy Map is not globally injective, the injectivity argument of the previous proof cannot be carried out for sections $\sigma$ and $\sigma'$ that are not close enough: the set of $C^\infty$ sections transversal to $\mathcal{F}$ may have infinitely many connected components as it so happens in the case of affine structures on the torus. Similarly, the surjectivity argument of the proof cannot be globalized: when the monodromy representation $\rho'$ eventually becomes reducible for instance, there does not exist $C^\infty$ section transversal to $\mathcal{F}$ anymore. Following Theorem 1.5, the existence of a $C^\infty$ section transversal to $\mathcal{F}$ is possible if, and only if, $\mathcal{F}$ is the suspension of a non-elementary representation $\rho$ (lifting to $\mathrm{SL}(2,\mathbb{C})$)! From this point of view, Theorem 1.5 looks like a very subtle transversality result.

## 2. $\mathbb{P}^1$-bundles and Riccati foliations

Motivated by the fibre bundle picture of section 1.5, we developp here the study of Riccati foliations on $\mathbb{P}^1$-bundles over compact Riemann surfaces.

**2.1. Classification of $\mathbb{P}^1$-bundles.** — Let $\pi : P \to C$ be a $\mathbb{P}^1$-bundle over a compact Riemann surface $C$: $P$ is a smooth surface and the fibers of $\pi$ are rational, isomorphic to $\mathbb{P}^1$. We also say that $P$ is a *ruled surface*. Another $\mathbb{P}^1$-bundle $\pi' : P' \to C$ is *analytically equivalent* to the previous one if there is a holomorphic diffeomorphism $\phi : P \to P'$ such that $\pi' \circ \phi = \pi$. We recall some basic facts (see [**15, 24**]).

On open charts $u_i : U_i \to \mathbb{C}$ on $C$, the bundle becomes analytically trivial (see [**8**]): we have holomorphic diffeomorphisms (trivializing coordinates)

$$\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{P}^1 \ ; \ p \mapsto (\pi(p), \varphi_i(p)).$$

On overlapping charts $U_i \cap U_j$, the transition maps take the form $\phi_i = \phi_{i,j} \circ \phi_j$ where $\phi_{i,j}(u,y) = (u, \varphi_{i,j}(u,y))$ and

$$\varphi_{i,j} \in \mathrm{PGL}(2, \mathcal{O}(U_{i,j})).$$

The $\mathbb{P}^1$-bundle is equivalently defined by the collection

$$(\varphi_{i,j})_{i,j} \in H^1(C, \mathrm{PGL}(2, \mathcal{O})).$$

By lifting conveniently the transition maps into $H^1(C, \mathrm{GL}(2,\mathcal{O}))$, we may view a $\mathbb{P}^1$-bundle as the projectivization $P = \mathbb{P}V$ of a *rank* 2 *vector bundle* $V$ over $C$. Moreover, another vector bundle $V'$ will give rise to the same $\mathbb{P}^1$-bundle if, and only if, $V' = L \otimes V$ for a line bundle $L$ over $C$. The classification of $\mathbb{P}^1$-bundles is thus equivalent to the classification of rank 2 vector bundles up to tensor product by a line bundle.

From the *topological* point of view, due to the fact that $\pi_1(\mathrm{PGL}(2,\mathbb{C})) = \mathbb{Z}/2\mathbb{Z}$, there are exactly 2 distinct $\mathbb{S}^2$-bundles over a compact real surface.

From the *birational* point of view, any $\mathbb{P}^1$-bundle is equivalent to the trivial bundle: there are infinitely many holomorphic sections $\sigma : C \to P$; after choosing 3 distinct ones $\sigma_0$, $\sigma_1$ and $\sigma_\infty$, one defines a birational transformation $\phi : P \dashrightarrow C \times \mathbb{P}^1$ commuting with $\pi$ by sending those sections respectively to $\{y = 0\}$, $\{y = 1\}$ and $\{y = \infty\}$. When the 3 sections are *disjoint*, the transformation $\phi$ is actually biregular and $P$ is the trivial bundle $C \times \mathbb{P}^1$.

The *analytic classification* is a much more subtle problem. If $P$ admits 2 disjoint sections, say $\sigma_0, \sigma_\infty : C \to P$, we then say that the bundle is *decomposable*: one can choose trivialization charts sending those two sections respectively onto $\{y = 0\}$ and $\{y = \infty\}$, so that $P$ may be viewed as the compactification $\overline{L}$ of a line bundle $L$. Recall that line bundles are analytically classified by the Picard group $\mathrm{Pic}(C)$. Any two elements $L, L' \in \mathrm{Pic}(C)$ have the same compactification $P$ if, and only if, $L' = L$ or $L^{\otimes(-1)}$: we just exchange the role of $\sigma_0$ and $\sigma_\infty$ (see proof of Proposition 3.1).

For instance, on $C = \mathbb{P}^1$, $\mathrm{Pic}(\mathbb{P}^1) \simeq \mathbb{Z}$ and the compactification of $\mathcal{O}(e)$ (or $\mathcal{O}(-e)$), $e \in \mathbb{N}$, gives rise to the Hirzebruck surface $\mathbb{F}_e$. It follows from Birkhoff's Theorem **[5]** that all $\mathbb{P}^1$-bundle is decomposable on $\mathbb{P}^1$ and is thus one of the $\mathbb{F}_e$ above.

An important analytic invariant of a $\mathbb{P}^1$-bundle over a curve $C$ is the minimal self-intersection number of a section

$$e(P) := -\min\{\sigma.\sigma \ ; \ \sigma : C \to P\} \in \mathbb{Z}.$$

For a decomposable bundle $P = \overline{L}$, $L \in \mathrm{Pic}(C)$, we have $e(\overline{L}) = |\deg(L)| \geq 0$. For an undecomposable bundle, Nagata proved in **[25]** that $-g \leq e \leq 2g-2$ and all those possibilities occur.

From the *homological* point of view, $H^2(P, \mathbb{Z})$ is generated by the homology class of $\sigma_0$ and $f$ where $\sigma_0$ is any holomorphic section and $f$ any fibre. Let us choose $\sigma_0$ with minimal self-intersection:

$$\sigma_0 \cdot \sigma_0 = -e, \quad f \cdot f = 0 \quad \text{and} \quad \sigma_0 \cdot f = 1.$$

The homology class of any other holomorphic section is $\sigma = \sigma_0 + n \cdot f$ with $n \in \mathbb{N}$: it has self-intersection

$$\sigma \cdot \sigma = \sigma_0 \cdot \sigma_0 + 2n \cdot \sigma_0 \cdot f + n \cdot f \cdot f = -e + 2n \geq -e.$$

In particular, the intersection number of holomorphic sections are either all even, either all odd: $e$ mod 2 is the *topological invariant* of the bundle.

On the other hand, if $\sigma_0$ and $\sigma$ are not homologous then the intersection number $\sigma_0 \cdot \sigma = n - e$ must be non negative and we deduce that $\sigma \cdot \sigma \geq e$: when $e > 0$, this implies that $\sigma_0$ *is the unique holomorphic section having negative self-intersection*; there is a gap between $-e$ and $e$.

***Theorem 2.1* (Atiyah [1]).** — *Beside compactifications of line bundles, there are exactly 2 undecomposable $\mathbb{P}^1$-bundles over an elliptic curve, $P_0$ and $P_1$, with invariant $e = 0$ and $-1$ respectively.*

A $\mathbb{P}^1$-bundle $P$ is *flat* (in the sense of Steenrod **[29]**) when a trivializing atlas can be choosen with constant transition maps $\varphi_{i,j} \in \mathrm{PGL}(2, \mathbb{C})$ (not depending on $u$). This means that this atlas defines by the same time a foliation $\mathcal{F}$ transversal to the fibres on $P$, namely the horizontal foliation defined by $\{y = \text{constant}\}$ in trivializing coordinates, see Remark 1.3.

***Theorem 2.2* (Weil [34]).** — *The flat $\mathbb{P}^1$-bundles over $C$ are all the undecomposable bundles and all those arising as compactification of elements of $Pic_0(C)$.*

The pairs $(\pi : P \to C, \mathcal{F})$ are classified by $H^1(C, \mathrm{PGL}(2, \mathbb{C}))$. All triples $(\pi : P \to C, \mathcal{F}, \sigma)$ associated to a projective structure on $\mathcal{P}$ are actually supported on the same flat $\mathbb{P}^1$-bundle $P$, namely

- the Hirzebruch surface $\mathbb{F}_1$ when $g = 0$,
- the trivial bundle $C \times \mathbb{P}^1$ when $g = 1$,

– the unique maximally unstable ($e = 2g - 2$) and undecomposable bundle when $g > 1$ (see [**14**]).

A birational description of the bundle will be given in Remark 2.2.

**2.2. Riccati foliations on $\mathbb{P}^1$-bundles.** — A Riccati foliation on the bundle $\pi : P \to C$ is a singular foliation (see definition in [**6**]) $\mathcal{F}$ on $P$ which is transversal to a generic fibre. In trivialization charts $(u_i, y)$, it is defined by a Riccati differential equation $\frac{dy}{du_i} = a(u_i)y^2 + b(u_i)y + c(u_i)$, $a, b, c$ meromorphic in $u$, whence the name. The poles of the coefficients correspond to vertical invariant fibres for the foliation. Outside of those poles, the leaves of the foliation are graphs of solutions for the Riccati equation. The foliation $\mathcal{F}$ arising in the fibre bundle picture of section 1.5 is a *regular* Riccati foliation. Nevertheless, we will need to deal with singular foliations later.

One can define the monodromy representation of a Riccati foliation as

$$\rho : \pi_1(C - \{\text{projection of invariant fibres}\}) \to \mathrm{PGL}(2, \mathbb{C}).$$

A classical Theorem due to Poincaré asserts that, in the regular case, the monodromy representation characterizes the Riccati foliation as well as the $\mathbb{P}^1$-bundle supporting it up to analytic equivalence.

***Remark 2.1.*** — One can view a Riccati foliation $\mathcal{F}$ on the $\mathbb{P}^1$-bundle $P = \mathbb{P}V$ as the projectivization of a meromorphic linear connection $\nabla$ on the vector bundle $V$. In fact, given a (meromorphic linear) connection $\zeta$ on the determinant bundle $\det V = \bigwedge^2 V \to C$, there is a unique connection $\nabla$ on $V$ lifting $\mathcal{F}$ and such that $\mathrm{trace}(\nabla) = \zeta$. Indeed, over a local coordinate $u_i : U_i \to \mathbb{C}$, the bundle $V$ is trivial and a connection $\nabla$ is just a meromorphic system

$$\nabla : \quad \frac{d}{du_i}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha(u_i) & \beta(u_i) \\ \gamma(u_i) & \delta(u_i) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

and the trace of $\nabla$ is the rank 1 connection defined by

$$\zeta := \mathrm{trace}(\nabla) : \quad \frac{d\lambda}{du_i} = (\alpha(u_i) + \delta(u_i))\,\lambda.$$

The projection of $\nabla$ on $\mathbb{P}V$ is therefore the Riccati equation defined in affine coordinate $(y : 1) = (y_1 : y_2)$ by

$$\mathcal{F} := \mathbb{P}\nabla : \quad \frac{dy}{du_i} = -\gamma(u_i)y^2 + (\alpha(u_i) - \delta(u_i))y + \beta(u_i).$$

Clearly, $\nabla$ is uniquely defined by $\mathcal{F}$ and $\zeta$. Notice finally that the line bundle $\det V$ admits a linear connection $\zeta$ without poles if and only if it belongs to $Pic_0(C)$.

We start recalling some usefull homological formulae from [**6**]. First of all, let us introduce $T_{\mathcal{F}}$, the *tangent bundle* of $\mathcal{F}$, which is a line bundle on the total space $P$ defined as follows. In trivializing charts $(u_i, y)$, the Riccati foliation is also defined by the meromorphic vector field $V_i := \partial_{u_i} + (a(u_i)y^2 + b(u_i)y + c(u_i))\partial_y$. The leaves of

$\mathcal{F}$ are just complex trajectories of the vector field $V_i$. After choosing a global (meromorphic) vector field $v$ on $C$, one can write $v = f_i \cdot \partial_{u_i}$ for a meromorphic function $f_i$ on the chart $U_i$ so that the new meromorphic vector fields $f_i \cdot V_i$ glue together into a global meromorphic vector field $V$ on $P$ still defining $\mathcal{F}$ at a generic point. One can think of $V$ as the lifting of $v$ by the (meromorphic) projective connection defined $\mathcal{F}$ on the bundle. Then, $T_{\mathcal{F}}$ is the line bundle defined by the divisor of $V$, i.e., $T_{\mathcal{F}} = \mathcal{O}((V)_0 - (V)_\infty)$. If $d$ denotes the number of invariant fibres (counted with the multiplicity of the corresponding pole for $V_i$), then the homology class of $T_{\mathcal{F}}$ is given by $T_{\mathcal{F}} = (2-2g-d)\cdot f$.

Given a curve $\sigma$ on $P$, each component of which is not invariant by $\mathcal{F}$, then the number of *tangencies* $\mathrm{Tang}(\mathcal{F},\sigma)$ counted with multiplicities is given by (see [**6**], p. 23)

$$\mathrm{Tang}(\mathcal{F},\sigma) = \sigma\cdot\sigma - T_{\mathcal{F}}\cdot\sigma. \tag{8}$$

For instance, if $\sigma = \sigma_0 + n\cdot f$ is a section, we immediately deduce that $\mathrm{Tang}(\mathcal{F},\sigma) = 2n - e - 2 + 2g + d$.

*Proof of Poincaré Theorem 1.4*. — Consider two projective structures on $C$ (compatible with the complex structure of $C$) having the same monodromy representation: by the construction given in section 1.5, they correspond to triples $(\pi : P \to C, \mathcal{F}, \sigma)$ and $(\pi : P \to C, \mathcal{F}, \sigma')$ with common $\mathbb{P}^1$-bundle and Riccati foliation. Since $\mathcal{F}$ is regular and the section $\sigma$ defining the first projective structure is transversal to $\mathcal{F}$, we have $d = 0$, $\mathrm{Tang}(\mathcal{F},\sigma) = 0$, and we deduce that $e = 2n + 2g - 2$. On the other hand, $\mathrm{Tang}(\mathcal{F},\sigma_0) = 2g - 2 - e$ should be non negative and we obtain $e = 2g-2$ and $n = 0$: in the genus $g \geq 2$ case, $\sigma = \sigma_0$ is the unique section having negative self-intersection in $P$, and by the way $\sigma' = \sigma$. In genus 0 case, there is nothing to show; in genus 1 case, the result follows directly from formula (4) and Theorem 1.1. □

Another important formula is the Camacho-Sad Index Theorem (see [**6**]). Given a curve $\sigma$ on $P$ invariant by $\mathcal{F}$, the self-intersection number of $\sigma$ equals the sum of Camacho-Sad index of $\mathcal{F}$ along this curve. When $\mathcal{F}$ is regular, all invariant curves are smooth and all Camacho-Sad index vanish: when $\mathcal{F}$ is regular, any invariant curve $\sigma$ has zero self-intersection.

For instance, if $\mathcal{F}$ has affine monodromy, then the fixed point gives rise to an invariant section $\sigma_\infty : C \to P$. We deduce that $e = 0$ and $\sigma_\infty$ realizes this minimal self-intersection number. In particular, we recover the fact that a projective structure on a genus $g \geq 2$ curve cannot have affine monodromy since the corresponding bundle has invariant $e = 2g - 2 > 0$.

More generally, if the monodromy of $\mathcal{F}$ has a finite orbit (e.g., a finite group of the infinite dihedral group), then $\mathcal{F}$ has an invariant curve $\sigma = m\cdot\sigma_0 + n\cdot f$ and formulae $\sigma\cdot\sigma = m(2n - em) = 0$ together with $\sigma\cdot f = m \geq 0$ and $\sigma\cdot\sigma_0 = n - em \geq 0$ show that $e \leq 0$. Again, this is not the monodromy of a projective structure whenever $g \geq 2$.

In the particular case where the monodromy of $\mathscr{F}$ is linear, we have 2 invariant disjoint sections $\sigma_0$ and $\sigma_\infty$ showing that the bundle $P$ is actually a compactification of a line bundle $L \in \mathrm{Pic}_0(C)$.

We should emphasize that any two line bundles $L$ and $L'$ have the same compactification if, and only if, $L' = L$ or $L^{\otimes(-1)}$. Indeed, we first note that, for a bundle $P$ satisfying $e(P) = 0$, any two sections $\sigma$ and $\sigma'$ are disjoint if, and only if, they have 0 self-intersection (and are distinct). The compactification $\overline{L}$ of a line bundle $L$ always has the two canonical disjoint sections $\sigma_0$ and $\sigma_\infty$. Now, a diffeomorphism $\phi : \overline{L} \to \overline{L}'$ between the compactifications of 2 non trivial line bundles has to preserve or permute the two canonical sections; in the former case, $\phi$ is actually an equivalence of line bundles; in the latter case, $\phi$ restricts to the fibres as $1/z$ and invert the monodromy. Of course, $\overline{L}$ is trivial if, and only if, $L$ is trivial as a line bundle. It follows that when $C$ has genus 1 the corresponding set of equivalence classes of $\mathbb{P}^1$-bundles with $e = 0$ may be thought as $C/\{\pm 1\} \simeq \mathbb{P}^1$.

A bundle $P$ obtained by suspension of a representation $\rho : \pi_1(C) \to \mathrm{PGL}(2,\mathbb{C})$ is topologically trivial ($e$ even) if, and only if, $\rho$ can be lifted as a representation $\tilde{\rho} : \pi_1(C) \to \mathrm{SL}(2,\mathbb{C})$.

There is an algebraic and somewhat technical notion of (semi-) stability of vector bundles of arbitrary rank on Riemann surfaces due to Mumford, see for instance [**27**]. We can define the (semi-) stability of a $\mathbb{P}^1$-bundle $\mathbb{P}V$ by the same requirement to the rank 2 vector bundle $V$. It turns out that a $\mathbb{P}^1$-bundle is *stable* (resp. *semi-stable*) when $e < 0$ (resp. $e \leq 0$). It is known that if such a bundle occur along an algebraic (resp. analytic) family, it occurs for a Zariski open subset of the family. There is a theorem of Narashimhan and Seshadri characterizing stable bundles on a compact Riemann surface $C$ by means of a precise, but some technical, construction in terms of unitary representations of the fundamental group of $C$. We present here a more comprehensible consequence, see Corollaries 1 and 2 of [**27**]:

***Theorem 2.3* (Narasimhan-Seshadri [27]).** — *Let $C$ be a compact Riemann surface of genus $g \geq 2$. Then a holomorphic vector bundle of degree zero is stable if and only if it arises from an irreducible unitary representation of the fundamental group $\pi_1(C)$ of $C$. A holomorphic vector bundle on $C$ arises from a unitary representation of the fundamental group if and only if each of its undecomposable components is of degree zero and stable.*

Applying this general result to our situation we obtain that the map $\rho \mapsto (\pi : P \to C, \mathscr{F}) \mapsto (\pi : P \to C)$ which to a representation $\rho \in \mathrm{Hom}(\pi_1(C), \mathrm{PGL}(2,\mathbb{C}))$ associate the $\mathbb{P}^1$-bundle obtained by suspension (forgetting the flat structure) induces a bijection from the set of irreducible representations $\rho : \pi_1(C) \to \mathrm{PSU}(2,\mathbb{C})$ up to $\mathrm{PSU}(2,\mathbb{C})$ conjugacy onto the set of isomorphism class of $\mathbb{P}^1$-bundles with invariant $e < 0$ and even (not fixed).

The complete analytic classification of $\mathbb{P}^1$-bundles (including unstable ones) over curves of genus 2 has been achieved by the works of Atiyah [**1**] and Maruyama [**24**].

The analytic classification of rank 2 stable vector bundles over curves of arbitrary genus from the algebraic point of view (in contrast with Narasimhan-Seshadri's approach) was done by Tyurin in [**31**] (see also [**32**] for a survey in arbirtrary rank).

**2.3. Birational geometry of $\mathbb{P}^1$-bundles.** — Given a point $p$ on (the total space of) a $\mathbb{P}^1$-bundle $\pi : P \to C$, we will denote by $\mathrm{elm}_p P$ the new $\mathbb{P}^1$-bundle obtained after elementary transformation centered at $p$: after blowing-up the point $p$, $\mathrm{elm}_p P$ is obtained by contracting the strict transform of the fiber passing through $p$. The strict transform of a section $\sigma$ passing through $p$ (resp. not passing through $p$) is a section of the new bundle having self-intersection $\sigma \cdot \sigma - 1$ (resp. $\sigma \cdot \sigma + 1$). All birational transformations between $\mathbb{P}^1$-bundles over curves are obtained by composing finitely many elementary transformations. On the other hand, any $\mathbb{P}^1$-bundle over a curve is birational to the trivial bundle.

***Example 2.1.*** — For instance, let $D$ be a divisor on $C$ and let $p_0$ be the point on the zero section of the (total space of the) line bunde $\mathcal{O}(D)$ over $x \in C$. Denote by $\overline{\mathcal{O}(D)}$ the $\mathbb{P}^1$-bundle obtained after compactification (adding a section at infinity). Then

$$\mathrm{elm}_{p_0} \overline{\mathcal{O}(D)} = \overline{\mathcal{O}(D - [x])}.$$

Similarly, if $p_\infty$ lies on the infinity section of $\overline{\mathcal{O}(D)}$ over $x$, then

$$\mathrm{elm}_{p_\infty} \overline{\mathcal{O}(D)} = \overline{\mathcal{O}(D + [x])}.$$

Now, recall that, as a consequence of Abel Theorem, the map

$$C^g \to \mathrm{Pic}_0(C) \ ; \ (x_1, \ldots, x_g) \mapsto \mathcal{O}(g[x_0] - [x_1] - \cdots - [x_g])$$

is surjective for any $x_0 \in C$: it follows that (compactification of) line bundles of degree 0 can be obtained after applying at most $2g$ elementary transformations to the trivial bundle.

In [**26**], Maruyama and Nagata proved that an undecomposable bundle can be obtained from the trivial one after at most $2g+1$ elementary transformations. On the other hand, we note that the minimal number of elementary transformations needed to trivialize all decomposable bundle is unbounded: for a line bundle of large degree $d >> 0$, one need at least $d$ elementary transformations.

Next section, we will give an explicit birational trivialization of the bundle $P$ supporting all triples $(P, \mathcal{F}, \sigma)$ associated to projective structures. After birational trivialization, the Riccati foliation becomes singular, and the section $\sigma$ no more transversal.

**2.4. Riccati equation, schwartzian derivative and the $2^{\mathrm{nd}}$ order linear differential equation.** — First, we would like to make explicit the correspondance between the point of view of quadratic differentials, and that one of bundle triples.

Consider the triple $(\pi : P \to C, \mathcal{F}, \sigma)$ associated to a projective structure on the curve $C$. One can reduce $P$ to the trivial bundle and $\sigma$ to the infinity section $\{y = \infty\}$ either locally, by a fibre bundle isomorphism, or globally on $C$, by birational transformation. Here below, we adopt the later point of view; everything can be carried out *mutatis mutandis* in the local regular setting. After a birational trivialization like above, $\mathcal{F}$ becomes possibly singular, but is now defined by a global Riccati equation

$$dy + \alpha \cdot y^2 + \beta \cdot y + \gamma = 0 \tag{9}$$

where $\alpha, \beta, \gamma$ are meromorphic 1-forms on $C$. This trivialization is unique up to birational transformation of the form $y = a\tilde{y} + b$ where $a$ and $b$ are meromorphic function on $C$, $a \not\equiv 0$. Let us see how such change of coordinate acts on the equation. A change of coordinate of the form $y = a\tilde{y}$ transforms the Riccati equation into

$$d\tilde{y} + a\alpha\tilde{y}^2 + (\beta + \frac{da}{a})\tilde{y} + \frac{\gamma}{a} = 0 \tag{10}$$

although a change of coordinate $y = \tilde{y} + b$ yields

$$d\tilde{y} + \alpha\tilde{y}^2 + (\beta + 2b\alpha)\tilde{y} + (db + b^2\alpha + b\beta + \gamma) = 0; \tag{11}$$

after a combination of those two transformations, we can choose $\alpha$ and $\beta$ arbitrary (with $\alpha \not\equiv 0$) and then $\gamma$ is uniquely determined by the projective structure. Let us show how to compute it from the developping map $f$ of the projective structure.

Let us go back to the universal cover where the Riccati foliation is given by $dy_0 = 0$ and $\sigma$ is the graph of $f$ (see section 1.5). By a preliminary change of coordinate $y_0 = y_1 + f(u)$, we have now $\sigma = \{y_1 = 0\}$ and the equation becomes $\mathcal{F} : dy_1 + df = 0$. A second change of coordinate $y_1 = f'(u) \cdot y_2$ yields $\mathcal{F} : dy_2 + (1 + \frac{f''}{f'}y_2)du = 0$, and $\sigma$ is still the zero section $y_2 = 0$. In the case of an affine structure on a torus, the later Riccati equation is well-defined: the corresponding triple $(\pi : P \to C, \mathcal{F}, \sigma)$ is then given by:

$$P = C \times \mathbb{P}^1 \ni (u, y), \quad \mathcal{F} : dy + (1 + cy)du = 0 \quad \text{and} \quad \sigma(u) \equiv 0$$

for some $c \in \mathbb{C}$.

In the general projective case, it is more convenient to send the section $\sigma$ to infinity: in the coordinate $\tilde{y}_2 = -1/y_2$, $\mathcal{F}$ is defined by $d\tilde{y}_2 + (\tilde{y}_2^2 - \frac{f''}{f'}\tilde{y}_2)du = 0$ and $\sigma$, by $\tilde{y}_2 = \infty$. We finally apply the change of coordinate $\tilde{y}_2 = y + \frac{1}{2}\frac{f''}{f'}$ and obtain

$$\mathcal{F} : dy + (y^2 + \frac{1}{2}\mathcal{S}_u(f))du = 0 \tag{12}$$

where $\mathcal{S}_u(f)$ is the schwartzian derivative of $f$ with respect to the variable $u$. Unfortunately, $du$ is not a global 1-form. Moreover, $u$ is a transcendental variable that we do not want to deal with when we are considering a triple $(\pi : P \to C, \mathcal{F}, \sigma)$. In general, by birational trivialization of the bundle, one can assume $\sigma$ at infinity and, after choosing a global holomorphic 1-form $\alpha$ on $C$, reduce the Riccati foliation to the special form

$$\mathcal{F} : d\tilde{y} + \alpha\tilde{y}^2 + \gamma = 0 \tag{13}$$

with $\gamma$ meromorphic on $C$. Here, $\alpha$ plays the role of $du$, that is $u$ is replaced by a variable $v$ such that $\alpha = dv$; this makes sense at least at a generic point of $C$ where everything is regular. Setting $u = \psi(v)$, the change of coordinate $y =$

$\frac{1}{\psi'}\left(\tilde{y}+\frac{1}{2}\frac{\psi''}{\psi'}\right)$ transforms equation (12) into (13); after computation we find $\gamma = \frac{1}{2}\left(\mathcal{S}_u(f)\circ\psi\cdot(\psi')^2+\mathcal{S}_v(\psi)\right)dv$. Using (7), one finally obtains $\gamma = \mathcal{S}_v(f\circ\psi)dv$ where $\mathcal{S}_v$ is the schwarzian derivative with respect to $v$ and deduce

***Proposition 2.1***. — *Let $(\pi : P \to C, \mathcal{F}, \sigma)$ be a triple defining a projective structure on $C$. Let $(v, y) \in U \times \mathbb{P}^1$ be bundle coordinates over $\pi^{-1}(U)$, $U \subset C$, such that*

$$\sigma : y = \infty \quad \textit{and} \quad \mathcal{F} : dy + (y^2 + \frac{\phi(v)}{2})dv = 0.$$

*Then the projective coordinates $f$ on $U$ are the solutions of $\mathcal{S}_v f = \phi$.*

***Remark 2.2***. — Following [**26**], the maximally unstable undecomposable $\mathbb{P}^1$-bundle $P$ corresponding to projective triples $(\mathcal{P}, \mathcal{F}, \sigma)$ can be trivialized after $2g$ elementary transformations (here $e = 2g-2$ is even). The birational tranformation constructed above to put $\mathcal{F}$ into the normal form (13) however needs many more elementary transformations.

Indeed, at a point where $\alpha = dv \sim u^\nu du$ has a zero of order $\nu$, i.e., $v \sim u^{\nu+1}$, the expression

$$\alpha\otimes\gamma = \frac{1}{2}\mathcal{S}_v(f)dv^{\otimes 2} \sim \left(\frac{dv}{v}\right)^{\otimes 2} \sim \left(\frac{du}{u}\right)^{\otimes 2}$$

has a pole of order 2 and thus $\gamma \sim \frac{du}{u^{\nu+2}}$ has a pole of order $\nu+2$. In fact, $\psi' \sim \frac{1}{u^\nu}$ and the birational change of coordinate takes the form

$$y \sim u^\nu\left(\tilde{y} - \frac{\nu}{2}\frac{1}{u^{\nu+1}}\right) = \frac{1}{u^{\nu+1}}\left(u^{2\nu+1}\tilde{y} - \frac{\nu}{2}\right);$$

$3\nu+2$ elementary transformations are needed at this point.

Now, we look for a sharp birational trivialization of $P$, that is to say with exactly $2g$ elementary transformations. For any choice of global meromorphic 1-forms $\alpha$ and $\beta$, there is a unique birational transformation of the form $y = a(\tilde{y}+b)$ putting the initial Riccati equation (12) into the form (9) and we have

$$\begin{cases} \alpha &= adu \\ \beta &= \frac{da}{a} + 2ab \\ \gamma &= db + ab^2du + \frac{da}{a}b + \frac{\mathcal{S}_u(f)}{2a}du \end{cases}$$

Each zero or pole of $\alpha$ (or $a$) gives rise to an elementary transformation: if we choose $\alpha$ holomorphic, we already get $2g-2$ elementary transformations with the first change of coordinate. We would like now $ab = \frac{1}{2}(\beta - \frac{da}{a})$ be holomorphic (as much as possible). The sum of residues of $\frac{da}{a}$ is $2g-2$: we can construct a meromorphic 1-form $\beta$ having the same principal part as $\frac{da}{a}$, plus one extra simple pole (at, say, $p$) with residue $2-2g$. The final change of coordinate $y = a\tilde{y}+ab$ is therefore a combination of $2g$ elementary transformations: the change of coordinate $y = a\tilde{y}$ goes from the trivial bundle to $\overline{K}$ with $2g-2$ elementary transformations; the ultimate transformation $y = \tilde{y}+ab$ has one simple pole corresponding to a succession of 2 generic elementary transformations of the same fibre (compare [**24**]).

Setting $y = z'/z$, $z' = \frac{dz}{dv}$, the differential equation $dy + (y^2 + \frac{\phi(v)}{2})dv = 0$ is transformed into

$$z'' + \frac{\phi(v)}{2} z = 0. \tag{14}$$

Then the following goes back to Schwarz:

***Proposition 2.2***. — *Any solution $f$ to the differential equation $\mathcal{S}_v(f) = \phi(v)$ takes the form $f = z_1/z_2$ where $z_1$ and $z_2$ are independent solutions of (14).*

*Proof*. — A straightforward computation shows that $\mathcal{S}(z_1/z_2) = -2\frac{z_2''}{z_2}$ provided that $z_1$ and $z_2$ are solutions of (14). Any other solution $\mathcal{S}_v(f) = \phi(v)$ takes the form $f = \frac{az_1+bz_2}{cz_1+dz_2}$, a quotient of two other solutions. In fact, one can take $f = z_1/z_2$ with $z_1 = \frac{f}{\sqrt{f'}}$ and $z_2 = \frac{1}{\sqrt{f'}}$. $\square$

***Remark 2.3***. — One can easily generalize the notion of projective structure to the branching case by considering triples $(\pi : P \to C, \mathcal{F}, \sigma)$ with $\sigma$ generically transversal to $\mathcal{F}$: branching points of the structure are those points $x \in C$ over which $\sigma$ has a contact with $\mathcal{F}$. The local projective chart then takes the form $f \sim u^{\nu+1}$ where $\nu \in \mathbb{N}$ is the order of contact. More generally, one can consider a singular Riccati foliation $\mathcal{F}$ generically transversal to $\sigma$, or equivalently linear equation $dy + (y^2 + \frac{\phi(v)}{2})dv = 0$ with $\phi$ meromorphic. By the way, projective structures on the 3-punctured sphere (3 simple poles) correspond to the Gauss Hypergeometric equation, on the 4-punctured sphere, to the Heun equation and on the punctured torus, to the Lamé equation.

***Remark 2.4***. — Let $D = \sum_{i=1}^{k} \nu_i p_i$ be an effective divisor on $\Sigma_g$. Consider the set $\mathcal{P}_g(D)$ consisting in all the projective structures on $\Sigma_g$ branched over the points $p_i$ with ramification order $\nu_i \geq 0$ (see **[21, 22]**). Notice that the case $D = 0$ corresponds to genuine projective structures on $\Sigma_g$. As before we can describe the elements of $\mathcal{P}_g(D)$ as triples $(P, \mathcal{F}, \sigma)$, where $P \to \Sigma_g$ is a $\mathbb{P}^1$-bundle with structural group $\mathrm{PSL}(2, \mathbb{C})$, $\mathcal{F}$ is a transversely projective foliation transverse to the fibres, $\sigma$ is a section such that $\sigma(p_i)$ is a tangency point with $\mathcal{F}$ of order $\nu_i$ for each $i = 1, \ldots, k$ and outside these points $\sigma$ is transverse to $\mathcal{F}$. Projecting to $\Sigma_g$ the branched projective structure induced by $\mathcal{F}$ on $\sigma$ we obtain an orbifold complex structure $C$ over $(\Sigma_g, D)$. We can make a finite number of elementary transformations centered at the tangency points $\sigma(p_i)$ in order to obtain a birationally equivalent triple $(P', \mathcal{F}', \sigma')$, where $\mathcal{F}'$ is a singular Riccati foliation and $\sigma' : C \to P'$ is a holomorphic section everywhere transverse to $\mathcal{F}'$. Applying the same transversality arguments of the proof of Hejal's theorem to $(P', \mathcal{F}', \sigma')$ one shows that the monodromy mapping $\mathcal{M} : \mathcal{P}_g(D) \to \mathcal{R}_g$ is also a local diffeomorphism.

## 3. The genus 1 case

### 3.1. Monodromy and bundles

***Proposition 3.1.*** — *Let $C = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ be an elliptic curve, $\rho : \pi_1(C) \to PGL(2, \mathbb{C})$ be any representation and $(\pi : P \to C, \mathscr{F})$ be the associated suspension. Then we are, up to conjugacy, in one of the following cases:*

- *$\rho : \pi_1(C) \to \mathbb{C}^*$ is linear and $P \in Pic_0(C)$ is the compactification of a line bundle; $P$ is trivial if, and only if, $\rho(1, \tau) = (e^c, e^{\tau c})$.*
- *$\rho : \pi_1(C) \to \mathbb{C}$ is euclidean and either $P = P_0$ is the semi-stable undecomposable bundle, or $P$ is the trivial bundle; we are in the latter case if, and only if, $\rho(1, \tau) = (c, \tau c)$.*
- *$\rho(1, \tau) = (-z, \frac{1}{z})$ and $P = P_{-1}$ is the stable undecomposable bundle.*

*Proof.* — It is easy to verify that all representations $\rho : \mathbb{Z}^2 \to \mathrm{PGL}(2, \mathbb{C})$ appear in the statement. We have already noticed that a linear representation gives rise to the compactification of a line bundle (this is almost the definition). In fact, for linear representations, we have the exact sequence of sheaves

$$0 \to \mathbb{C}^* \to \mathscr{O}^* \to \Omega \to 0$$

where $\mathscr{O}^*$ is the sheaf of invertible holomorphic functions and the morphism $\mathscr{O}^* \to \Omega$ is given by $f \mapsto \frac{df}{f}$. From the corresponding exact sequence of cohomology groups, we deduce the following one

$$0 \to H^0(C, \Omega) \to \mathrm{Hom}(\pi_1(C), \mathbb{C}^*) \to \mathrm{Pic}_0(C) \to 0.$$

The first non-trivial morphism associates to a holomorphic 1-form $\omega$ the homomorphism $\gamma \to \exp(\int_\gamma \omega)$ while the second one is the suspension. In our particular case where $C$ is an elliptic curve, we finally deduce

$$0 \to \mathbb{C}du \to \mathrm{Hom}(\pi_1(C), \mathbb{C}^*) \to C \to 0$$

and the first alternative of the statement follows.

The suspension of an euclidean representation gives rise to a bundle with a section $\sigma_\infty$ having 0 self-intersection. If there is another section $\sigma_0$ disjoint from $\sigma_\infty$, then it should be either transversal, or invariant by $\mathscr{F}$ from (8): in the first case, $\sigma_0$ provides a projective structure on $C$ and the monodromy satisfies $\rho(1, \tau) = (c, \tau c)$ by Guning Theorem 1.1; in the second case, the monodromy has two fixed points and is trivial, so is the bundle. In the remaining case where there is no disjoint section from $\sigma_\infty$, the bundle is undecomposable with invariant $e = 0$ and we conclude with Atiyah Theorem 2.1 that $P = P_0$.

Finally, if $\rho(1, \tau) = (-z, \frac{1}{z})$ is the irreducible representation, we note that $\rho$ cannot be lifted to $\mathrm{SL}(2, \mathbb{C})$ and thus $e$ is odd. On the other hand, Weil Theorem 2.2 tells us that $P$ must be undecomposable (being flat with $e \neq 0$). From Atiyah Theorem 2.1, the only possibility is $P = P_{-1}$. □

**3.2. Algebraic families of bundles and Riccati foliations.** — It follows from [**24**] that all degree 0 line bundles as well as $P_0$ can be obtained after 2 elementary transformations of the trivial bundle. In order to obtain $P_{-1}$, a third one is needed. We use this approach to provide an algebraic family of flat bundles and Riccati foliations.

Let $p \in \overline{\mathcal{O}}$, $q \in \mathrm{elm}_p(C \times \mathbb{P}^1)$ and consider $P = \mathrm{elm}_q \mathrm{elm}_p \overline{\mathcal{O}}$. Fix trivializing coordinates $(u, z) \in C \times \mathbb{P}^1$ and, for simplicity, set $p = (0, \infty)$. This is irrelevant since all flat $\mathbb{P}^1$-bundles over $C$ admit a one parameter group of automorphism lifting the action of $\partial_u$ (see description of Proposition 3.1). After elementary transformation at the point $p$, we obtain the bundle $\overline{\mathcal{O}(-[0])}$ having one section $\sigma_\infty$ with $-1$ self-intersection and a special point $\tilde{p}$, on the fiber over $u = 0$ but not on $\sigma_\infty$, through which all sections having $+1$ self-intersection intersect. Indeed, $+1$-sections come from horizontal sections of the trivial bundle. Here, we use the fact that there is no holomorphic section of homology type $\sigma_0 + f$ on $\overline{\mathcal{O}}$, otherwise it would be the graph of a regular covering $C \to \mathbb{P}^1$.

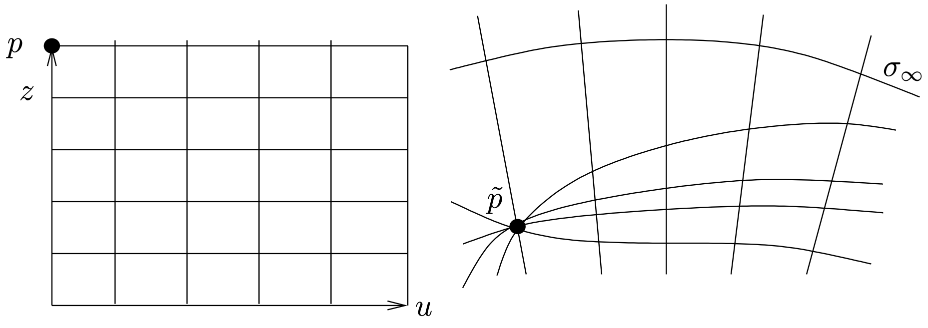


FIGURE 6. The bundle $\overline{\mathcal{O}(-[0])}$

Case 0: $q = \tilde{p}$. The elementary transformation centered at $\tilde{p}$ goes back to the trivial bundle: $P = \overline{\mathcal{O}}$.

Case 1: $q = (u_0, z_0)$ with $u_0 \neq 0$ and $z_0 \neq \infty$. After vertical automorphism, one may assume $z = 0$. The sections $\{z = 0\}$ and $\{z = \infty\}$ respectively give rise to disjoint sections $\sigma_0$ and $\sigma_\infty$ on $P$ having 0 self-intersection. We are in $\mathrm{Pic}_0(C)$ case: $P = \overline{L}$.

The generic horizontal section $\{z = c\}$ gives rise to a section $\sigma$ on $P$ intersecting $\sigma_0$ at $u = 0$ and $\sigma_\infty$ at $u = u_0$; in other words, $\sigma$ is a meromorphic section of $L$ with divisor $\mathrm{Div}(\sigma) = [0] - [u_0]$ on $C$: $L = \mathcal{O}([u_0] - [0])$ corresponds to $u_0 \in \mathrm{Pic}_0(C) \simeq C$.

Case 2: $q$ is on the fibre over $u = 0$ but is neither $\tilde{p}$, nor on $\sigma_\infty$. Then, $P = P_0$ is the indecomposable bundle. Indeed, assume that there exists a section $\sigma$ on $P$ disjoint from $\sigma_\infty$. It then comes from a section of $\overline{\mathcal{O}(-[0])}$ disjoint from $\sigma_\infty$ and passing through $q$, itself coming from a section of $\overline{\mathcal{O}}$ intersecting $\sigma_\infty$ only at $u = 0$, without multiplicity. We have already seen that this cannot happen.

Case 3: $q$ is on $\sigma_\infty$, over $u_0$. We obtain the bundle $\overline{\mathcal{O}(-[0] - [u_0])}$.

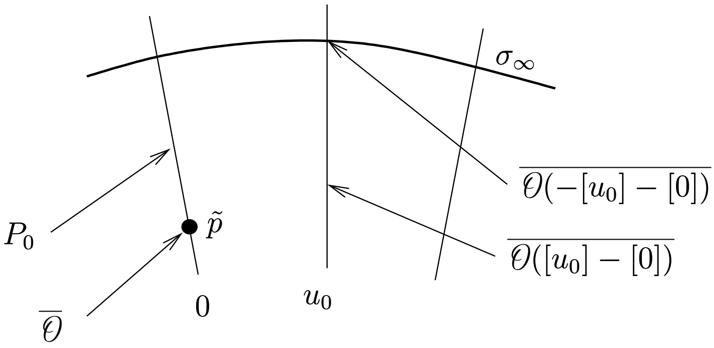


FIGURE 7. An algebraic family of topologically trivial bundles

Here, we have parametrized all topologically trivial flat bundles by the line bundle $\mathcal{O}(-[0])$, see **[30]**. We now want to parametrize all regular Riccati foliations on topologically trivial bundles. The natural way to do this is to provide an explicit family of Riccati equations on the trivial bundle $\overline{\mathcal{O}}$ having apparent singular fibres whose desingularization span all regular Riccati foliations. For instance, consider a linear Riccati foliation defined on the bundle $\overline{\mathcal{O}([u_0]-[0])}$, $u_0 \neq 0$. Apart from the invariant sections $\sigma_0$ and $\sigma_\infty$, the leaves are multivalued sections without zero or pole; after trivialisation of the bundle, those multivalued sections $z(u)$ have now a simple pole over $u=0$ and a simple zero over $u=u_0$ (and still have linear monodromy): their logarithmic derivative $\frac{dz(u)}{z(u)}$ is a meromorphic 1-form on $C$ having exactly 2 simple poles, one at 0 with residue $-1$ and one at $u_0$ with residue $+1$. In other words, the Riccati equation defining the singular foliation after trivialization of the bundle is

$$\frac{dz}{du} = \left(\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))}+c\right)\cdot z.$$

Indeed, the 1-form $\left(\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))}+c\right)du$ has a simple pole at $u=0$ with residue $-1$ since its principal part is given by $\frac{1}{2}\frac{\wp'(u)}{\wp(u)}du$ and $\wp$ has a double pole at $u=0$; the other poles may come from the two zeroes of $\wp(u)-\wp(u_0)$, namely $u=\pm u_0$, but $u=-u_0$ is actually regular since the numerator $\wp'(u)+\wp'(u_0)$ also vanishes at this point: by Residue Theorem, $u=u_0$ is a simple pole with residue $+1$. Of course, any other 1-form having the same principal part must differ by a holomorphic 1-form, namely $c\cdot du$, $c\in\mathbb{C}$. We have omitted from our discussion the case $u_0=-u_0$ is an order 2 point which can be treated like $u=0$.

After two elementary transformations of $\overline{\mathcal{O}}$ centered at the points $(u,z)=(0,\infty)$ and $(u_0,0)$, we obtain by this way all (linear) foliations on the bundle $\mathcal{O}([u_0]-[0])$ while $c$ runs over $\mathbb{C}$. This does not provide yet a universal family for linear connections on $C$ since the limit of the Riccati foliation while $u_0\to 0$ is the vertical fibration: for

$(u,z)$ in a compact set not intersecting $\{u=0\}$, $\{z=0\}$ and $\{z=\infty\}$, we have

$$\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))}+c\sim-\frac{1}{2}\frac{\wp'(u_0)}{\wp(u_0)}\sim\frac{1}{u_0}\quad\text{while}\quad u_0\sim 0.$$

In other words, the 1-form $u_0dz-u_0\left(\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))}+c\right)\cdot zdu$ tends uniformly to $-zdu$ on the compact set, so does the foliation. We would like to complete this $\mathbb{C}$-bundle over $u_0\neq 0$ with the family $\frac{dz}{du}=c_0\cdot z$, $c_0\in\mathbb{C}$, of linear connections on the trivial bundle $\mathcal{O}$ $(u_0=0)$. A way to obtain it from our large family is obviously to set $c=c(u_0)=c_0-\frac{1}{u_0}$ and take the limit while $u_0\to 0$ with $c_0\in\mathbb{C}$ fixed. In other words, in the parameter space $(u_0,c)$ we consider only the limit at $(0,\infty)$ while $u_0\to 0$ with a special direction. The good global parameter space is obtained after separating the germs of curves $c+\frac{1}{u_0}=$ constant. This is done after 2 elementary transformations on $\overline{\mathcal{O}}$: first we blow-up $(u_0,c)=(0,\infty)$ by setting $c=t/u_0$, and then we blow-up $(u_0,t)=(0,-1)$ by setting $t+1=su_0$, so that $s=\frac{1}{u_0}+c$ coincides with the expected parameter $c_0$. The resulting parameter space is the *affine bundle* $A_0:=P_0-\sigma_\infty$ where $\sigma_\infty$ is the unique 0-section of the undecomposable bundle $P_0$.

Now, we construct a fine moduli space as follows. Consider the product $\overline{\mathcal{O}}\times\overline{\mathcal{O}}$ with global coordinates $((u_0,c),(u,z))$, and equipp the bundle over $(u_0,c)$ with the Riccati foliation

$$\frac{dz}{du}=\left(\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))}+c\right)\cdot z.$$

This can be seen as an algebraic foliation on the total space. Now, apply the elementary transformations with center along the surfaces $\{u=0,z=\infty\}$ and $\{u=u_0,z=0\}$. Then, we modify the base $(u_0,c)\in\overline{\mathcal{O}}$ by two elementary transformations so that we obtain $P_0$ as a base and the foliation extends as a linear connections all along $u_0=0$.

The euclidean connections on $P_0$ are given by:

$$\frac{dz}{du}=\wp(u)+\gamma.$$

Indeed, one can check that the reduction of the singularity over $u=0$ yields $P_0$; on the other hand, it is clear that monodromy is given by translations. This can be obtained also as a limit of our previous family of connections, or better from

$$\frac{dz}{du}=\left(\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))}+c\right)\cdot(z-c)$$

which is equivalent to the previous one by the change of coordinate $z\mapsto z+c$. Now, instead of taking limit along curves $c=c(u_0)=c_0-\frac{1}{u_0}$ as $u_0\to 0$ with $c_0$ constant, we take limit along $c=c(u_0)=\gamma u_0-\frac{1}{u_0}$, $\gamma\in\mathbb{C}$ constant, i.e., $c_0=\gamma u_0$. We have on convenient compact sets:

$$\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))}-\frac{1}{u_0}\sim u_0\wp(u)\quad\text{while}\quad u_0\sim 0$$

so that

$$\frac{dz}{du} = \left(\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))}+c\right)\cdot(z-c) \sim (\wp(u)+\gamma)\cdot(u_0z+1-\gamma u_0^2) \sim \wp(u)+\gamma.$$

**3.3. The Riemann-Hilbert Mapping.** — For a given elliptic curve $C=\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, the Riemann-Hilbert Mapping provides an analytic isomorphism

$$\mathscr{M} : A_0 \to \mathbb{C}^* \times \mathbb{C}^*$$

between two spaces of algebraic nature.

The space of linear connections on $C$ is an affine $\mathbb{C}$-bundle over $\mathrm{Pic}_0(C) \simeq C$ that we have identified with $A_0$: it is defined by gluing the chart $(u_0,c) \in (C-\{0\})\times\mathbb{C}$ with the chart $(u_0,c_0)\in(C,0)\times\mathbb{C}$ by the transition map

$$(u_0,c)\mapsto(u_0,c_0):=(u_0,c+\frac{1}{u_0}).$$

The space of linear representations of $\pi_1(C)$ is $\mathbb{C}^*\times\mathbb{C}^*$. In the main chart $(u_0,c)$, the analytic connection is given by

$$\frac{dz}{z} = \left(\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))}+c\right)\cdot du.$$

Introducing Weierstrass Zeta Function $\zeta(u)=-\int_0^u\wp(\xi)d\xi$, one can write $\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))} = \zeta(u-u_0)-\zeta(u)+\zeta(u_0)$ and integrate the differential equation above by means of the Weierstrass Sigma Function: the general solution[(1)] is therefore given by $z(u)=a\frac{\sigma(u-u_0)}{\sigma(u)}e^{\zeta(u_0)\cdot u}$, $a\in\mathbb{C}^*$, and the monodromy is given by the homomorphism

$$\Lambda=\mathbb{Z}+\tau\mathbb{Z}\to\mathbb{C}^* \ ;\ \gamma\mapsto\exp(-u_0\zeta(\gamma)+\zeta(u_0)\gamma+c\gamma).$$

Finally we obtain the full monodromy mapping

$$\mathscr{M} : A_0\to\mathbb{C}^*\times\mathbb{C}^* \ ;\ \begin{cases} (u_0,c) & \mapsto & (e^{-u_0\zeta(1)+\zeta(u_0)+c}, e^{-u_0\zeta(\tau)+\zeta(u_0)\tau+c\tau}) \\ (0,c_0) & \mapsto & (e^{c_0}, e^{c_0\tau}) \end{cases}$$

The image by the monodromy map of the algebraic fibration defined on $A_0$ is the holomorphic foliation defined on $\mathbb{C}^*\times\mathbb{C}^*$ by the linear vector field

$$x\partial_x+\tau y\partial_y.$$

As a particular case of Narasimhan-Seshadri Theorem, the unitary representations $\mathbb{S}^1\times\mathbb{S}^1$ form a smooth real 2-dimensional torus transversal to the foliation and cutting each leaf once. It is the space of the leaves. It inherits, from the transversal complex foliation, a complex structure, namely the structure of $C$. The euclidean foliations defined on $A_0$ by $\frac{dc}{du_0}=\wp(u_0)+\gamma$, $\gamma\in\mathbb{C}$, are sent to the linear foliations $x\partial x+\lambda y\partial y$, $\lambda\in\mathbb{P}^1\setminus\{\tau\}$. The space of linear connections is equipped with a group law given by tensor product; it is just the pull-back of the natural group law on $\mathbb{C}^*\times\mathbb{C}^*$. We thus

(1) This computation was communicated to the first author by Frits Beukers; a similar computation but with a slightly different presentation was done in [**17**].

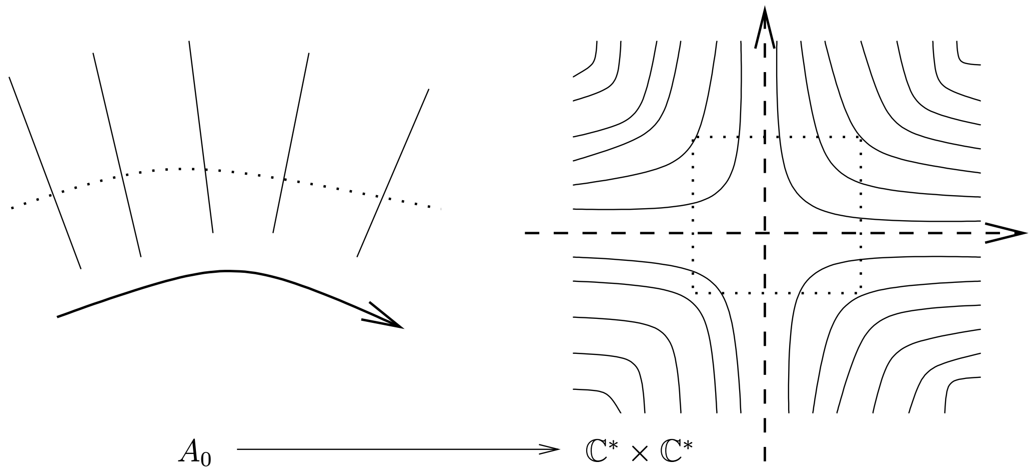


FIGURE 8. The Riemann-Hilbert Mapping

get an analytic isomorphism between two algebraic groups that are not algebraically equivalent.

One can compute the group law on $A_0$ as follows. Given two connections

$$\frac{dz}{z} = \left(\frac{\wp'(u)+\wp'(u_1)}{2(\wp(u)-\wp(u_1))} + c_1\right)\cdot du \quad \text{and} \quad \frac{dz}{z} = \left(\frac{\wp'(u)+\wp'(u_2)}{2(\wp(u)-\wp(u_2))} + c_2\right)\cdot du,$$

the tensor product is a connection of the form

$$\frac{dz}{z} = \left(\frac{\wp'(u)+\wp'(u_3)}{2(\wp(u)-\wp(u_3))} + c_3\right)\cdot du$$

with $u_3 = u_1 + u_2$ (group law on $\mathrm{Pic}_0(C)$). Then, $c_3$ is determined by the fact that

$$\frac{\wp'(u)+\wp'(u_1)}{2(\wp(u)-\wp(u_1))} + \frac{\wp'(u)+\wp'(u_2)}{2(\wp(u)-\ wp(u_2))} - \frac{\wp'(u)+\wp'(u_3)}{2(\wp(u)-\wp(u_3))} + c_1 + c_2 - c_3 = \frac{df}{f}$$

for a meromorphic function $f$ on $C$. Looking at the principal part of the left hand side, one see that $f$ must have divisor $\mathrm{Div}(f) = [u_1] + [u_2] - [u_3] - [0]$ so that, up to a scalar, we have

$$f = \frac{\wp'(u) - \wp'(u_1) - \frac{\wp'(u_2)-\wp'(u_1)}{\wp(u_2)-\wp(u-1)}(\wp(u)-\wp(u_1))}{\wp(u)-\wp(u_3)};$$

after computations, one finds that

$$c_3 = c_1 + c_2 - \frac{\wp'(u_2)-\wp'(u_1)}{2(\wp(u_2)-\wp(u_1))}.$$

## References


[1] M. F. Atiyah – "Complex fibre bundles and ruled surfaces", *Proc. London Math. Soc.* **5** (1955), p. 407–434.

[2] W. Barth, C. Peters & A. Van de Ven – *Compact complex surfaces*, Ergebnisse Math. Grenzg. (3), vol. 4, Springer, 1984.

[3] A. Beilinson & V. Drinfeld – "Opers", preprint arXiv:math/0501398, 2005.

[4] F. Beukers – "Solutions of a Lamé system", private notes, 2004.

[5] G. D. Birkhoff – "A theorem on matrices of analytic functions", *Math. Ann.* **74** (1913), p. 122–133.

[6] M. Brunella – *Birational geometry of foliations*, Monografías de Matemática, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2000, `http://www.impa.br/Publicacoes/Monografias/Abstracts/brunella.ps`.

[7] C. J. Earle – "On variation of projective structures", in *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, Ann. of Math. Stud., vol. 97, Princeton Univ. Press, 1981, p. 87–99.

[8] O. Forster – *Lectures on Riemann surfaces*, Graduate Texts in Math., vol. 81, Springer, 1981.

[9] D. Gallo, M. Kapovich & A. Marden – "The monodromy groups of Schwarzian equations on closed Riemann surfaces", *Ann. of Math.* **151** (2000), p. 625–704.

[10] W. M. Goldman – "The symplectic nature of fundamental groups of surfaces", *Adv. in Math.* **54** (1984), p. 200–225.

[11] X. Gómez-Mont – "Holomorphic foliations in ruled surfaces", *Trans. Amer. Math. Soc.* **312** (1989), p. 179–201.

[12] R. C. Gunning – "Special coordinate coverings of Riemann surfaces", *Math. Ann.* **170** (1967), p. 67–86.

[13] ______, "Analytic structures on the space of flat vector bundles over a compact Riemann surface", in *Several complex variables, II (Proc. Internat. Conf., Univ. Maryland, College Park, Md., 1970)*, Springer, 1971, p. 47–62. Lecture Notes in Math., Vol. 185.

[14] ______, "Affine and projective structures on Riemann surfaces", in *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, Ann. of Math. Stud., vol. 97, Princeton Univ. Press, 1981, p. 225–244.

[15] R. Hartshorne – *Algebraic geometry*, Springer, 1977, Graduate Texts in Mathematics, No. 52.

[16] D. A. Hejhal – "Monodromy groups and linearly polymorphic functions", *Acta Math.* **135** (1975), p. 1–55.

[17] N. J. Hitchin – "Twistor spaces, Einstein metrics and isomonodromic deformations", *J. Differential Geom.* **42** (1995), p. 30–112.

[18] J. H. Hubbard – "The monodromy of projective structures", in *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, Ann. of Math. Stud., vol. 97, Princeton Univ. Press, 1981, p. 257–275.

[19] N. M. Katz – "An overview of Deligne's work on Hilbert's twenty-first problem", in *Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Vol. XXVIII, Northern Illinois Univ., De Kalb, Ill., 1974)*, Amer. Math. Soc., 1976, p. 537–557.

[20] S. Kawai – "The symplectic nature of the space of projective connections on Riemann surfaces", *Math. Ann.* **305** (1996), p. 161–182.

[21] R. Mandelbaum – "Branched structures on Riemann surfaces", *Trans. Amer. Math. Soc.* **163** (1972), p. 261–275.

[22] ———, "Branched structures and affine and projective bundles on Riemann surfaces", *Trans. Amer. Math. Soc.* **183** (1973), p. 37–58.

[23] ———, "Unstable bundles and branched structures on Riemann surfaces", *Math. Ann.* **214** (1975), p. 49–59.

[24] M. Maruyama – *On classification of ruled surfaces*, Lectures in Mathematics, Department of Mathematics, Kyoto University, vol. 3, Kinokuniya Book-Store Co. Ltd., 1970.

[25] M. Nagata – "On self-intersection number of a section on a ruled surface", *Nagoya Math. J.* **37** (1970), p. 191–196.

[26] M. Nagata & M. Maruyama – "Note on the structure of a ruled surface", *J. reine angew. Math.* **239/240** (1969), p. 68–73.

[27] M. S. Narasimhan & C. S. Seshadri – "Stable and unitary vector bundles on a compact Riemann surface", *Ann. of Math.* **82** (1965), p. 540–567.

[28] H. Poincaré – "Sur les groupes des équations linéaires", *Acta Math.* **4** (1884), p. 201–312.

[29] N. Steenrod – *The Topology of Fibre Bundles*, Princeton Mathematical Series, vol. 14, Princeton University Press, 1951.

[30] T. Suwa – "On ruled surfaces of genus 1", *J. Math. Soc. Japan* **21** (1969), p. 291–311.

[31] A. N. Tjurin – "On the classification of two-dimensional fibre bundles over an algebraic curve of arbitrary genus", *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), p. 21–52.

[32] ———, "Geometry of moduli of vector bundles", *Uspehi Mat. Nauk* **29** (1974), p. 59–88.

[33] ———, "Periods of quadratic differentials", *Uspekhi Mat. Nauk* **33** (1978), p. 149–195, 272.

[34] A. Weil – "Généralisation des fonctions abéliennes", *J. Math. Pures Appl.* **17** (1938), p. 47–87.

[35] ———, "On discrete subgroups of Lie groups", *Ann. of Math.* **72** (1960), p. 369–384.

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# ON GENERALIZED SURFACES IN $(\mathbb{C}^3, 0)$

*by*

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*Dedicated to J.M. Aroca for his sixtieth birthday*

En esto fueron razonando los dos, hasta que llegaron a un pueblo
donde fue ventura hallar un algebrista, con quien se curó el Sansón desgraciado.

*El Ingenioso Hidalgo Don Quijote de la Mancha*

***Abstract.*** — In this paper we study germs of codimension one holomorphic, non-dicritical, singular foliations in $(\mathbb{C}^3, 0)$ having no saddle-nodes in their reduction of singularities. These are called *generalized surfaces*. The main result says that the reduction of the singularities of a generalized surface agrees with the reduction if its separatrix set.

***Résumé* (Sur les surfaces généralisées dans $(\mathbb{C}^3, 0)$).** — Dans cet article, on étudie les germes de feuilletages holomorphes de codimension un, non dicritiques et singuliers en $(\mathbb{C}^3, 0)$, qui n'ont pas de selles-nœuds dans la réduction des leurs singularités. Ces feuilletages s'appellent *surfaces généralisées*. Le résultat principal affirme que la réduction des singularités d'une surface généralisée coïncide avec la réduction de son ensemble de séparatrices.

## 1. Introduction

The main objective of this paper is to characterize a class of holomorphic codimension one foliations in $(\mathbb{C}^3, 0)$, whose reduction of singularities agrees with the reduction of their separatrices. We will call *generalized surfaces* to these foliations, as they are a generalization of the notion of generalized curves introduced by C. Camacho, A. Lins

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Neto and P. Sad in [**2**]. Following that paper, a codimension one singular foliation $\mathcal{F}$ defined by a 1-form $\omega$ in $(\mathbb{C}^2, 0)$ will be called a generalized curve if:

1. $\mathcal{F}$ is not dicritical, i.e., it has a finite number of separatrices, or equivalently the exceptional divisor obtained after reduction of singularities is invariant by $\mathcal{F}$.
2. There are no saddle-nodes in the reduction of the singularities of $\mathcal{F}$.

Recall that a germ of singular foliation $\mathcal{F}$ defined by $\omega$ is called reduced, or simple, if there are coordinates $(x, y)$ such that

$$\omega = (\lambda x + h.o.t.)dy + (\mu y + h.o.t.), \quad \mu \neq 0, \text{ and } \frac{\lambda}{\mu} \notin \mathbb{Q}_{<0}.$$

If $\lambda = 0$, the singularity is called a saddle-node.

Generalized curves have a number of good properties, as shown in [**2**]. If $(f = 0)$ is an analytic equation of their set $S$ of separatrices, then:

1. $\nu(\omega) = \nu(df)$, where $\nu$ denotes the order at 0.
2. The reduction of the singularities of $\mathcal{F}$ is the same as the reduction of $(f = 0)$.
3. $\mu(\omega) = \mu(f)$, where $\mu$ denotes the Milnor number of $\omega$, $f$, at the origin. Note that for every foliation we always have the inequality $\mu(\omega) \geq \mu(f)$, and the equality only holds for generalized curves.

Another properties of generalized curves have been established more recently. For instance, let $BB(\mathcal{F}, P)$ denote the Baum-Bott index of $\mathcal{F}$ at $P$, $CS(\mathcal{F}, S, P)$ the Camacho-Sad index of $\mathcal{F}$ at $P$ relative to the invariant curve $S$, and $GSV(\mathcal{F}, S, P)$ the index of Gómez-Mont, Seade and Verjovsky. Then, M. Brunella shows in [**1**] that, for a generalized curve, $GSV(\mathcal{F}, S, P) = 0$, and that this implies $BB(\mathcal{F}, P) = CS(\mathcal{F}, S, P)$.

In our previous paper [**12**], we classified analytically a class of singular foliations in $(\mathbb{C}^3, 0)$, which we called quasi-ordinary cuspidal foliations. These are foliations with one separatrix, that is a cuspidal, quasi-ordinary surface. In particular it can be seen in that paper that, after resolving the singularities of the surface, the singularities of the foliation are automatically reduced, and the analytic classification of these foliations agrees with the analytic classification of the projective holonomy of a certain component of the exceptional divisor. This property is similar to the property of generalized curves we saw before, and it motivates the generalization of this notion to dimension three. In fact, in dimension three it exists a theorem of reduction of singularities for holomorphic foliations [**6, 5**], and a theorem of existence of separatrices (in the non-dicritical case) [**6**] that allows the generalization.

The plan of the paper is as follows: in section 2, we shall recall some basic facts about holomorphic foliations, mainly about the final forms (simple singularities). In section three, we will give the definition of generalized surface, and will show its main properties. The main result of this paper will then be as follows.

***Theorem 1.1***. — *If $\mathcal{F}$ is a generalized surface in $(\mathbb{C}^3, 0)$, and $S \equiv (f = 0)$ is the union of their separatrices, then the reduction of singularities of $\mathcal{F}$ agrees with the reduction of the surface $S$.*

Of course, from a careful study of [**6**] the main result of this paper can be deduced, but it is not explicit. The purpose of this paper is to offer an independent proof more in the spirit of [**2**]. It is also worth to mention here related works about foliations in dimension three that admit particular reduction of singularities, for instance foliations that can be desingularized with only punctual blow-ups [**7**].

There is work in progress about the analytic classification of generalized curves. For instance, several papers have been written about the classification of quasi-homogeneous singularities of foliations (see [**17**] for definitions and statements). Let us mention the works of D. Marín [**15, 16**], and more recently, Y. Genzmer [**13**], who shows that two quasi-homogeneous generalized curves with analytically equivalent separatrices are analytically equivalent if and only if the projective holonomies of the component of the exceptional divisor where the separatrices lie are also analytically conjugated. We expect to extend this result to dimension three, thanks to the characterization given in this paper.

## 2. Pre-simple and simple singularities

Consider a germ of codimension one holomorphic foliation $\mathcal{F}$ defined in a neighbourhood of a point $P$ on a complex manifold $M$ of dimension $n$ by an integrable 1-form $\omega$, and let $E$ be a germ of normal crossings divisor through $P$, invariant by $\mathcal{F}$.

We will call the *dimensional type* of $\mathcal{F}$ at $P$, $t = t(\mathcal{F}, P)$, the minimum number of variables needed to write a generator of $\mathcal{F}$. If $\Theta_{M,P}$ denotes the set of germs of holomorphic vector fields in a neighbourhood of $P$, and

$$\mathcal{D}(\omega)(P) = \{\mathcal{D}(P) \mid \mathcal{D} \in \Omega_{M,P},\ \omega(\mathcal{D}) \equiv 0\}$$

then $t(\mathcal{F}, P)$ equals the codimension of the complex vector space $\mathcal{D}(\omega)(P)$. Note that the integrability of $\omega$ implies that the set of vector fields in the kernel of $\omega$ define an integrable distribution, and Frobenius' Theorem allows to assume that they are coordinates.

Denote $e = e(E, P)$ the number of components of $E$ through $P$. Assuming that $E = \prod_{i=1}^{e} x_i$, we can write

$$\omega = \prod_{i=1}^{e} x_i \cdot \left[\sum_{i=1}^{e} a_i \frac{dx_i}{x_i} + \sum_{i=e+1}^{t} a_i dx_i\right].$$

The adapted order of $\omega$ at $P$, relative to $E$, is then

$$\nu(\mathcal{F}, E; P) = \min\{\nu_P(a_i) \mid 1 \leq i \leq t\},$$

where $\nu_P$ denotes the order at $P$. The adapted multiplicity is

$$\mu(\mathcal{F}, E; P) = \min\{\nu_P(a_i)\}_{1 \leq i \leq e} \cup \{\nu_P(a_i) + 1\}_{i > e}.$$

Then F. Cano and D. Cerveau [**6**] propose the following definition:

***Definition 2.1.*** — *A singularity $P \in \operatorname{Sing}(\mathcal{F})$ is called a pre-simple singularity adapted to $E$ if one of the following holds:*

1. $\nu(\mathcal{F}, E; P) = 0$.
2. $\nu(\mathcal{F}, E; P) = \mu(\mathcal{F}, E; P) = 1$, *and perhaps after performing a change of variables, there exists $s > e$ such that, if $i \leq e$, $a_i = \lambda_i x_s + h.o.t.$, with some $\lambda_i \neq 0$.*

***Remark 2.2.*** — Due to the integrability of $\omega$, in 2. it is sufficient to assume that, for some $i \leq e$, $a_i = x_s + h.o.t.$. The condition above follows.

Pre-simple singularities of dimensional type $t$ are formally conjugated to one of the following meromorphic models, according to [**5, 6, 4**]:

a. $\omega = \sum_{i=1}^{t} \lambda_i \frac{dx_i}{x_i}$, $\lambda_i \in \mathbb{C}^*$.

b. $\omega = \sum_{i=1}^{k} p_i \frac{dx_i}{x_i} + \varphi(\mathbf{x}^{\mathbf{P}}) \cdot \sum_{i=2}^{t} \lambda_i \frac{dx_i}{x_i}$, where $p_i \in \mathbb{Z}_{>0}$, $\varphi \in \mathbb{C}[[T]]$, $\varphi(0) = 0$, $\alpha_i \in \mathbb{C}$, $\sum_{i=2}^{t} |\alpha_i| \neq 0$, $\alpha_i \neq 0$ if $i > k$, $\mathbf{x}^{\mathbf{P}} := x_i^{p_1} \cdots x_k^{p_k}$.

c. $\omega = dx_1 - x_1 \sum_{i=2}^{k} p_i \frac{dx_i}{x_i} + x_2^{p_2} \cdots x_n^{p_n} \sum_{i=2}^{t} \lambda_i \frac{dx_i}{x_i}$.

A pre-simple singularity is called *simple* or *reduced* if it is formally conjugated to the types (a) and (b) above, and moreover, the vector $\lambda = (\lambda_1, \ldots, \lambda_t)$ in case (a) or $\lambda = (\lambda_{k+1}, \ldots, \lambda_t)$ in case (b) is strongly non-resonant, i.e., for any non zero function $\Phi : \{1, \ldots, t\} \to \mathbb{Z}_{\geq 0}$, we have that $\sum \Phi(i) \lambda_i \neq 0$. This last condition, that does not appear in [**6**], is imposed in order to avoid dicriticalness.

If $P$ is a pre-simple singularity of dimensional type $t$, consider the set

$$\{Q \in U \setminus \{P\} \mid Q \in \operatorname{Sing}(\mathcal{F}),\ U \text{ a neighbourhood of } P\}.$$

All elements of this set are singularities of dimensional type 2 (see [**6**] for a proof). Then, $P$ is simple if and only if all elements in the previous set are simple. For instance, in dimension three we have the following result.

***Theorem 2.3* (Cano-Cerveau [6]).** — *Let $P$ be a pre-simple singularity of a singular foliation $\mathcal{F}$ adapted to a normal crossings divisor $E$ over an ambient space of dimension three.*

1. *If $e(E, P) = 2$, take an inmersion $i : (\mathbb{C}^2, 0) \hookrightarrow (\mathbb{C}^3, 0)$ through $P$, tranversal to $E$. Then $P$ is simple if and only if $i^*\mathcal{F}$ is simple.*
2. *If $e(E, P) = 3$, and there exists an open set $U \ni P$, such that if $Q \neq P$ is a singular point for $\mathcal{F}$ ($e(E, Q) = 2$), the restriction at $\mathcal{F}$ over a transversal plane through $Q$ is a simple foliation, then $\mathcal{F}$ is simple.*

At this point, let us recall the following result:

***Theorem 2.4* (Cerveau-Mattei [11]).** — *Let $\mathcal{F}$ be a foliation in $(\mathbb{C}^n, 0)$ defined by an integrable 1-form $\omega$. Suppose we have an inmersion $i : (\mathbb{C}^2, 0) \hookrightarrow (\mathbb{C}^n, 0)$, such that 0 is an isolated singularity of $i^*\omega$, reduced. Then, we have the following possibilities:*

1. *If $\operatorname{cod}(\operatorname{Sing}(\omega)) \geq 3$, then $\omega$ has an holomorphic first integral.*

2. *If* $\operatorname{cod}(\operatorname{Sing}(\omega)) = 2$*, then* $\mathcal{F}$ *is a cylinder over the foliation induced by* $i^*\omega$*, i.e. if in appropriate coordinates* $i$ *is defined by* $i(x_1, x_2) = (x_1, x_2, \dots, x_n)$*,* $\mathcal{F}$ *is the pull-back of* $i^*\mathcal{F}$ *by the projection* $\pi : \mathbb{C}^n \to \mathbb{C}^2$ *given by* $\pi(x_1, x_2, \dots, x_n) = (x_1, x_2)$*. In particular, the dimensional type of* $\omega$ *is 2.*

Applying this result to a simple singularity $P$ in $\mathbb{C}^3$, with $e(E, P) = 3$, we obtain that, around nearby singularities, the foliation is simple of dimensional type 2, and trivial over a transversal.

Recall that, in dimension two, there are two types of simple singularities, one of them called saddle-nodes, namely the ones that have a null eigenvalue. This is also valid for foliations of dimensional type two in greater dimensions. If we are in an ambient space of dimension three, and $P$ is a simple singular point for $\mathcal{F}$ with $t(\mathcal{F}, P) = 3$, we say that $P$ is a saddle-node if some of nearby singularities are saddle-nodes. This occurs if the foliation is formally conjugated to the form (b) above, with $k < 3$.

## 3. Generalized surfaces

From now on, we will work in dimension three. At present, it is not known a result of reduction of singularities for foliations in dimension greater that three, but with that result in hand, our conclusions could be extended to upper dimensions.

***Definition 3.1***. — *Let* $\mathcal{F}$ *be a holomorphic foliation of codimension one, defined by an integrable 1-form* $\omega$ *in* $(\mathbb{C}^3, 0)$*. We will say that* $\mathcal{F}$ *is a generalized surface if in the reduction of singularities of* $\mathcal{F}$*, no saddle-nodes appear, and moreover, it is not dicritical.*

The main objective of this work is the proof of the theorem 1.1.

We have two possible situations. If $\operatorname{cod}(\operatorname{Sing}(\mathcal{F})) = 3$, then $\mathcal{F}$ has a first integral, according to Malgrange's singular Frobenius Theorem [**14**] and the result is obvious. So, in the sequel, we asume that $\operatorname{cod}(\operatorname{Sing}(\mathcal{F})) = 2$.

D. Cerveau introduces in [**10**] a notion of quasi-regular foliation. As a consequence of the results that will follow, our notion of generalized surface agrees with the notion introduced by Cerveau.

***Remark 3.2***. — In dimension two, a foliation admits only one minimal reduction of singularities. The algorithm is obvious: if one point is singular, blow-up. In dimension three, as in the case of surface, different reduction processes may give as a result desingularized foliations. So, the process is not canonical and we must check that the definition above is independent of the reduction.

The independence of the dicriticalness condition may be seen in [**3**]. Let us also observe that saddle-nodes cannot be destroyed by subsequent blow-ups. So, if we have two different reduction processes over a neighbourhood of 0, such that a saddle-node for $\mathcal{F}_1 = \pi_1^*\mathcal{F}$ appears, take a tranversal germ of surface $S_1 \subseteq \tilde{M}_1$, such that $\mathcal{F}_1|_{S_1}$

has a saddle-node, and $S_2$ its image in $\tilde{M}_2$ by the birational transformation $\pi_2^{-1} \circ \pi_1$. The theorem of factorization of birational maps between surfaces allows us to find a surface $S$ that dominates $S_1$ and $S_2$.

The maps $\rho_1$ and $\rho_2$ are obtained blowing-up points in $S_1$, $S_2$ respectively, so that we have the following picture:

$$\begin{tikzcd}
 & M \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\
M_1 \arrow[dr, "\pi_1"'] & & M_2 \arrow[dl, "\pi_2"] \\
 & (\mathbb{C}^3, 0) &
\end{tikzcd}$$

The foliation $\rho_1^* \pi_1^* \mathcal{F}$ restricted to $S$ must have a saddle-node, while the foliation $\rho_2^* \pi_2^* \mathcal{F}$ must not have, and this is a contradiction.

***Definition 3.3***. — *Let $\mathcal{F}$ be a foliation in $(\mathbb{C}^n, 0)$, defined by $\omega$. An inmersion $i : (\mathbb{C}^2, 0) \hookrightarrow (\mathbb{C}^n, 0)$ is transversal to $\mathcal{F}$ if:*

1. $\mathrm{Sing}(i^*\omega) = i^{-1}(\mathrm{Sing}(\omega)) = \{0\}$.
2. $\nu_0(\mathcal{F}) = \nu_0(i^*\mathcal{F})$.

J.-F. Mattei and R. Moussu **[18]** show that inmersions satisfying (1) and (2) are generic. Recall their construction: first, as $\mathrm{cod}(\mathrm{Sing}(\omega)) = 2$, take an inmersion such that $i^{-1}(\mathrm{Sing}(\omega)) = \{0\}$. Then, a generic deformation of order two verifies the result. In general, multiplicity may increase if there are tangencies, and it may drop if a common factor appears in the coefficients of $i^*\omega$. That this does not happen is a consequence of (1).

In order to prove the main theorem, we need some lemmas,that will allow us to reduce the problem to dimension two.

***Lemma 3.4***. — *Let $\mathcal{F}_0$ be a generalized surface defined by an integrable 1-form $\omega$. If $i : (\mathbb{C}^2, 0) \hookrightarrow (\mathbb{C}^3, 0)$ is a transversal inmersion, then $i^*\mathcal{F}_0$ is a generalized curve.*

*Proof.* — Consider a reduction of $\mathcal{F}$:

$$\pi : (M_k, \mathcal{F}_k) \to \cdots \to (M_0 = U, \mathcal{F}_0),$$

$U$ a neighbourhood of $0 \in \mathbb{C}^3$, and lift $i$ to $\tilde{i} : (\mathbb{C}^2, 0) \hookrightarrow (M_k, P)$, transversal, with $P \in \mathrm{Sing}(\mathcal{F}_k)$. Let $E_k$ be the exceptional divisor $\pi^{-1}(\mathrm{Sing}(\mathcal{F}_0))$.

If $e(E_k, P) = 2$, $\mathcal{F}_k$ is a cylinder over the foliation $\tilde{i}^*\mathcal{F}_k$, so $P$ is not a saddle-node for $i^*\mathcal{F}_k$.

If $e(E, P) = 3$, blow up again $P$. The strict transform of $\tilde{i}$ after this new blow-up will be transversal to the transform of $\mathcal{F}_k$, in three points corresponding to the three axis of $\mathrm{Sing}(\mathcal{F}_k)$. These are now points of dimensional type 2, and the result follows. $\square$

***Lemma 3.5.*** — *Let $\mathcal{F}$ be a generalized surface, defined by $\omega$, and $f=0$ a reduced equation of its set $S$ of separatrices. Then $\nu_0(\omega)=\nu_0(df)$.*

*Proof.* — Take an inmersion $i:(\mathbb{C}^2,0)\hookrightarrow(\mathbb{C}^3,0)$ transversal to $\mathcal{F}$ and to $\mathcal{F}_S$ $(=df)$. By previous lemma, $i^*\mathcal{F}$ is a generalized curve, having $i^{-1}S$ as separatrix. As $i$ is transversal to $\mathcal{F}$, $i^{-1}S$ is a separatrix of $i^*\mathcal{F}$; as a consequence of [**6**], we have that for $\mathcal{F}$ non dicritical, every separatrix of $i^*\mathcal{F}$ extends to a separatrix of $\mathcal{F}$, see also [**10**, Section 5.6]. By transversality, $\nu_0(\omega)=\nu_0(i^*\mathcal{F})=\nu_0(d(f\circ i))=\nu_0(df)$. □

***Lemma 3.6.*** — *Let $\mathcal{F}$ be a generalized surface in $(\mathbb{C}^3,0)$. If $\mathcal{F}$ has precisely three transversal smooth separatrices at $0$, then $\mathcal{F}$ is simple.*

*Proof.* — Choose local coordinates such that the set of separatrices of $\mathcal{F}$ is $(x_1x_2x_3=0)$. Then $\mathcal{F}$ is given by a 1-form

$$\omega=x_1x_2x_3\left(a_1\frac{dx_1}{x_1}+a_2\frac{dx_2}{x_2}+a_3\frac{dx_3}{x_3}\right),$$

where $a_i\in\mathbb{C}\{x_1,x_2,x_3\}$. Any transversal inmersion $i:(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$, through a point $P$, with $P\neq 0$, $P\in\operatorname{Sing}(\mathcal{F})$ is a generalized curve and it has two separatrices, then, by [**2**, Lemma II.3.1], $i^*\mathcal{F}$ is simple.

Take now a transversal section $i$ through the origin. The foliation $i^*\mathcal{F}$ has three transversal smooth separatrices, and it is a generalized curve, so $\nu_0(i^*\omega)=2$.

By Lemma 3.5, $\nu_0(\omega)=2$, so $\nu_0(a_i)=0$ for some $i$. Then the origin is a pre-simple singularity, and all the separatrices in a neighbourhood of 0 are simple, so 0 itself is simple. □

*Proof of Theorem 1.1.* — Desingularize the separatrix set $S$ of $\mathcal{F}$, of reduced equation $f=0$:

$$\pi: M_n\xrightarrow{\pi_n}M_{n-1}\longrightarrow\cdots\xrightarrow{\pi_1}M_0=U\subseteq\mathbb{C}^3.$$

After this, we have a foliation $\mathcal{F}_n$ on $M_n$. Every $P\in\operatorname{Sing}(\mathcal{F}_n)$ is in the crossing of 2 or 3 smooth surfaces, invariant by $\mathcal{F}_n$ (namely, the total transform of $S$). In fact, according to Lemma 3.5, a regular point of the total transform of $S$ cannot be a singular point for $\mathcal{F}$. If $P$ is a point in the intersection of two components, take a transversal at $P$, it is a generalized curve with two separatrices, so it is simple. Then $\mathcal{F}_n$ is simple of dimensional type two.

If $P$ is in the intersection of three components, Lemma 3.6 shows that $\mathcal{F}_n$ is simple at $P$. □

***Example 3.7.*** — As told in the introduction, in a previous paper [**12**], we studied quasi-ordinary cuspidal foliations in $(\mathbb{C}^3,0)$, i.e., foliations generated by a 1-form

$$\omega=d(z^2+x^py^q)+A(x,y)dz.$$

satisfying the integrability condition $d(x^py^q)\wedge A(x,y)=0$. In that paper the reduction of singularities is analyzed carefully, according with the parity of $p$, $q$. It can be seen that reduction of the singularities of the separatrices for these foliations agrees with the reduction of the singularities of the foliation. So, they provide an example of the generalized surfaces we have studied in this paper.

## References


[1] M. Brunella – "Some remarks on indices of holomorphic vector fields", *Publ. Mat.* **41** (1997), p. 527–544.

[2] C. Camacho, A. Lins Neto & P. Sad – "Topological invariants and equidesingularization for holomorphic vector fields", *J. Differential Geom.* **20** (1984), p. 143–174.

[3] F. Cano – "Foliaciones singulares dicriticas", in *Memorias de la Real Academia de Ciencias Exactas Fisicas y Naturales de Madrid*, 1989.

[4] ———, "Reduction of the singularities of foliations and applications", in *Singularities Symposium—Łojasiewicz 70 (Kraków, 1996; Warsaw, 1996)*, Banach Center Publ., vol. 44, Polish Acad. Sci., 1998, p. 51–71.

[5] ———, "Reduction of the singularities of codimension one singular foliations in dimension three", *Ann. of Math. (2)* **160** (2004), p. 907–1011.

[6] F. Cano & D. Cerveau – "Desingularization of nondicritical holomorphic foliations and existence of separatrices", *Acta Math.* **169** (1992), p. 1–103.

[7] F. Cano, D. Cerveau & B. Scárdua – "Holomorphic foliations desingularized by punctual blow-ups", *J. Pure Appl. Algebra* **178** (2003), p. 235–243.

[8] F. Cano, J.-M. Lion & R. Moussu – "Frontière d'une hypersurface pfaffienne", *Ann. Sci. École Norm. Sup. (4)* **28** (1995), p. 591–646.

[9] F. Cano & J.-F. Mattei – "Hypersurfaces intégrales des feuilletages holomorphes", *Ann. Inst. Fourier (Grenoble)* **42** (1992), p. 49–72.

[10] D. Cerveau – "Déploiements non dicritiques des formes différentielles holomorphes $A\,dx + B\,dy$", *Bol. Soc. Brasil. Mat. (N.S.)* **22** (1991), p. 39–62.

[11] D. Cerveau & J.-F. Mattei – *Formes intégrables holomorphes singulières*, Astérisque, vol. 97, Société Mathématique de France, 1982.

[12] P. Fernández-Sánchez & J. Mozo-Fernández – "On the quasi-ordinary cuspidal foliations in $(\mathbb{C}^3, 0)$", *J. Differential Equations* **226** (2006), p. 250–268.

[13] Y. Genzmer – "Déformations isoholonomiques de feuilletages et modules de germes de courbes", Ph.D. Thesis, Université Paul Sabatier, Toulouse, 2006.

[14] B. Malgrange – "Frobenius avec singularités. I. Codimension un", *Publ. Math. I.H.É.S.* **46** (1976), p. 163–173.

[15] D. Marín – "Moduli spaces of germs of holomorphic foliations in the plane", *Comment. Math. Helv.* **78** (2003), p. 518–539.

[16] ———, "Sur les notions de quasi-homogénéité de feuilletages holomorphes en dimension deux", *Bull. Braz. Math. Soc. (N.S.)* **36** (2005), p. 177–185.

[17] J.-F. Mattei – "Quasi-homogénéité et équiréductibilité de feuilletages holomorphes en dimension deux", *Astérisque* **261** (2000), p. 253–276, Géométrie complexe et systèmes dynamiques (Orsay, 1995).

[18] J.-F. Mattei & R. Moussu – "Holonomie et intégrales premières", *Ann. Sci. École Norm. Sup. (4)* **13** (1980), p. 469–523.



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# THE GALOISIAN ENVELOPE OF A GERM OF FOLIATION: THE QUASI-HOMOGENEOUS CASE

*by*

Emmanuel Paul

*À José-Manuel, pour ses 60 ans*

***Abstract.*** — We give geometric and algorithmic criterions in order to have a proper Galois envelope for a germ of quasi-homogeneous foliation in an ambient space of dimension two. We recall this notion recently introduced by B. Malgrange, and describe the Galois envelope of a group of germs of analytic diffeomorphisms. The geometric criterions are obtained from transverse analytic invariants, whereas the algorithmic ones make use of formal normal forms.

***Résumé*** **(L'enveloppe galoisienne d'un germe de feuilletage : le cas quasi-homogène)**
Nous donnons des critères géométriques et algorithmiques pour qu'un feuilletage quasi-homogène en dimension deux possède une enveloppe galoisienne propre. Nous rappelons cette notion récemment introduite par B. Malgrange et nous décrivons l'enveloppe galoisienne d'un groupe de germes de difféomorphismes analytiques. Les critères géométriques sont obtenus à partir d'invariants analytiques transverses, tandis que les critères algorithmiques utilisent les formes normales.

## Introduction

There are several notions of integrability for a system of differential equations. Most of them are related to the existence of a sufficient number of first integrals for the solutions of the system. These definitions differ each other on the additional properties required for this family of invariants functions. We can separate them into two types:
— conditions between the first integrals: one may ask commutativity conditions for the Poisson bracket, or relax such a condition;
— conditions on the nature of these functions: rational, meromorphic or multivalued

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functions in some "reasonable" class of transcendency.
The main methods for proving non integrability (analytical methods, Ziglin method or Morales-Ramis method) are based on the linearization of the system around a particular solution. Therefore they only deliver sufficient criterions on non integrability, using for the last mentioned method *linear* differential Galois theory.

In order to investigate the second type of condition, and –in the future– to get necessary and sufficient conditions for integrability, we have to consider the system in the whole, which suggests to consider a *non linear* differential Galois theory. The first attempts in this direction was done by J. Drach and E. Vessiot. More recently, B. Malgrange introduced in [**12**] (see also the introductive version [**13**]) a "Galois envelope" for any dynamical system, namely the smallest D-groupoid which contains the solutions of the system. Roughly speaking, a D-groupoid is a system of partial differential equations whose local solutions satisfy groupoid conditions outside an analytic codimension one set. They are not strict Lie groupoid, in order to deal with singular systems. As a matter of introduction to this notion, we shall describe in the first section the Galois envelope of a group of germs of analytic diffeomorphisms at the origin of $\mathbb{C}$.

Each D-groupoid admits a D-algebra obtained by the linearization of its equations along the identity solutions. The local solutions of this linear differential system are stable under the Lie bracket outside of a codimension one analytic set. The Galois envelope of a singular analytic foliation $\mathcal{F}$ is the smallest D-groupoid $\mathrm{Gal}(\mathcal{F})$ whose D-algebra contains the germs of tangent vector fields to $\mathcal{F}$. It is a proper one if it doesn't coincide with the whole groupoid $\mathrm{Aut}(\mathcal{F})$ obtained by writing the equations of invariance of the foliation under a local diffeomorphism. In this case –which is not the general case–, its solutions satisfy an additional differential relation, and we shall say that the foliation is Galois reducible.

For a local codimension one singular foliation defined by a holomorphic one-form $\omega$, this reducibility property is equivalent to the existence of a Godbillon-Vey sequence of finite length for $\omega$ (at most three): there exists a finite sequence of meromorphic one forms $\omega_0$, $\omega_1$, and $\omega_2$ such that $\omega_0$ is an equation of the foliation and

$$d\omega_0 = \omega_0 \wedge \omega_1,\ d\omega_1 = \omega_0 \wedge \omega_2,\ d\omega_2 = \omega_1 \wedge \omega_2.$$

This fact was described in a manuscript of B. Malgrange [**14**], and then has been extensively proved by G. Casale in [**5**] with some different arguments. In particular, the transverse rank of $\mathrm{Gal}\mathcal{F}$ (i.e. the order of its transverse local expression) is also the minimal lenght of a Godbillon-Vey sequence for $\mathcal{F}$. Finally, G. Casale proved in [**2**] that this Godbillon-Vey condition is also equivalent to the existence of first integrals for the foliation with a particular type of transcendency which belongs to a Darboux or Liouville or Riccati type differential extension, according to the transverse rank of the Galois envelope. These different points of view on the Galois reducibility admit a generalization for higher codimension foliations: see [**6**] for the Painlevé 1 foliation.

In the present paper we shall only deal with codimension one foliations in ambient spaces of dimension two. Therefore, we expect the existence of at most one first integral, and we only have to discuss the second type of integrability condition: the existence of such a first integral in a given class of transcendency. The previous discussion allows us to reformulate the integrability problem as following: *give necessary and sufficient criterions for the Galois reducibility of a germ of codimension one foliation.* We present an answer to this problem in the following context: $\mathcal{F}$ is defined by a vector field $X = X_h + \cdots$ where the "initial" hamiltonian vector field

$$X_h = \frac{\partial h}{\partial y}\frac{\partial}{\partial x} - \frac{\partial h}{\partial x}\frac{\partial}{\partial y}$$

is quasi-homogeneous with to respect to $R = p_1 x\frac{\partial}{\partial x} + p_2 y\frac{\partial}{\partial y}$ ($p_1$, $p_2$ positive integers): $R(h) = \delta h$, $\delta = \deg_R(h)$. The dots means terms of higher quasihomogeneous degree. We furthermore require that $h$ has an isolated singularity (with Milnor number $\mu$) and that $X$ still keep invariant the analytic set $h = 0$. Therefore, $X$ is a logarithmic vector field for the polar set $h = 0$, and we have:

$$X = aX_h + bR,\ a \in \mathcal{O}_2, b \in \mathcal{O}_2,\ a(0) = 1$$

with $deg_R(bR) > \deg_R(X_h)$. The restriction to this class of foliation is motivated by the two following reasons:
— the desingularization of these foliations by blowing up's is "simple": it is similar to the one of the quasi-homogeneous function $h$: the exceptional divisor is only a chain of projective lines and all the pull-back of the irreducible components of $h$ –excepted the axis if they appear in $h$– meet the same "principal" projective line $C$.
— in this class of foliations, we have at our disposal *formal normal forms* which give us complete formal invariants: see [**21**].

This will allow us to give two different types of criterions for the Galois reducibility of $\mathcal{F}$ : a geometric one which is related to the holonomy of the principal component $C$ of the desingularized foliation, and an algorithmic one which directly holds on the normalized formal equation of the foliation. For the first one, let us denote $\mathrm{Hol}(\mathcal{F})$ the holonomy group of the principal component $C$ for the desingularized foliation. This is an analytic invariant of $\mathcal{F}$ (in fact, this "transverse invariant" is also a complete invariant in this quasi-homogeneous context: see [**8**]). We prove in theorem (2.4) the following result :

**Theorem 1.** *The Galois groupoid of the germ of quasi-homogeneous foliation $\mathcal{F}$ is a proper one if and only if the Galois envelope of* $\mathrm{Hol}(\mathcal{F})$ *is a proper one.*

This theorem reduces the initial problem to the determination of the Galois envelope of a subgroup $G$ of $\mathrm{Diff}(\mathbb{C},0)$, which is described in the first section (theorem 1.8). The main argument in the proof of this theorem is an extension of the equation which define the Galois envelope of $\mathrm{Hol}(\mathcal{F})$ to the whole exceptional divisor. This is possible, since the elements of the holonomy group of $C$ are solutions of this equation and therefore keep it invariant. This proof suggests that even in non quasi-homogeneous cases, these criterions for the Galois reducibility will only depend on the transverse structure of the foliation.

Theorem 1 is not an explicit criterion since in general, we can't compute the invariant $\mathrm{Hol}(\mathcal{F})$. In order to get an algorithmic criterion, we recall in section 3 the formal normal forms for this class of foliations. Notice that in general these models are divergent models. The radial component of these normal forms make appear a collection $\mathcal{L}(\mathcal{F})$ of $\mu$ formal one-variable vector fields, and it turns out that this collection (up to a commun conjugacy) is a complete invariant for the formal class of $\mathcal{F}$. It must be surprising to try to charaterize the Galois reducibility of $\mathcal{F}$ using only formal invariants. Nevertheless, we can perform it according to the two following facts:
— if a foliation is Galois reducible, then its formal normal form is a convergent one;
— if the foliation $\mathcal{F}$ is a "non exceptional" one (see [**7**]), then there exists a convergent conjugacy between $\mathcal{F}$ and its model.
Clearly, for exceptional foliations, we need an additional condition on the analytic class of $\mathcal{F}$, which is not yet an algorithmic one. The central result of this work is the following theorem which summarize theorem 3.5, corollary 3.7 and theorem 3.8:

**Theorem 2.** *If the quasi-homogeneous foliation $\mathcal{F}$ is a non exceptional one, the Galois envelope of $\mathcal{F}$ is proper if and only if the explicit invariant $\mathcal{L}(\mathcal{F})$ generates a finite dimensional Lie algebra. In this case, this one is always of dimension one, and the foliation is at most Liouvillian.*

*If the quasi-homogeneous foliation $\mathcal{F}$ is an exceptional one, the Galois envelope of $\mathcal{F}$ is proper if and only if the explicit invariant $\mathcal{L}(\mathcal{F})$ is a finite dimensional Lie algebra, and the analytic invariants of $\mathcal{F}$ are of "unitary" or "binary" type. In this case, the foliation will be a Liouvillian one (for unitary invariants), or of Riccati type, (for binary invariants).*

We shall recall in the first section the definition of unitary or binary invariants which is a terminology introduced by J. Ecalle. The first part of the theorem is an extension of a result of F. Loray and R. Meziani for nilpotent singularities [**11**], while the second one is an extension of a theorem of G. Casale for reduced singularities [**5**]. Notice that in the local context, the Galois reducible foliations which are not Liouvillian are very rare.

Clearly, the relationship between the algorithmic invariant $\mathcal{L}(\mathcal{F})$ and the geometric one $\mathrm{Hol}(\mathcal{F})$ has a transcendental nature since the first one is directly obtained from the differential equation whereas the second one is related to the solutions of this equation. Nevertheless, for Galois reducible foliations we can describe this relationship: it reduces to the exponential map of the one-variable vector fields of $\mathcal{L}(\mathcal{F})$. In order to check this fact it is more convenient to consider an equivalent data to $\mathrm{Hol}(\mathcal{F})$: the relative holonomy of $\mathcal{F}$ with respect to its initial part defined by $X_h$ (see section 4).

Finally, we conclude this paper with a list of open questions related to the present results.

## 1. The Galois envelope of a subgroup of Diff($\mathbb{C}$,0)

Let $\Delta$ be a disc around 0 in $\mathbb{C}$. We first recall the list of all the D-groupoids on $\Delta$ (see [**14**] and [**3**]). We denote $(x, y, y_1, y_2, \ldots, y_k)$ the coordinates for the space of $k$-jets of maps from $\Delta$ to itself.

***Theorem 1.1***. — *The differential ideal of a D-groupoid on $\Delta$ is generated by a meromorphic equation of one of the five types:*

1. *D-groupoids of order zero: they are generated by an equation of the form: $h(x) - h(y) = 0$ where $h$ is a holomorphic function on $\Delta$. We denote them: $G_0(h)$.*
2. *D-groupoids of order one: they are generated by an equation of the form: $\eta(y)(y_1)^n - \eta(x) = 0$ where $n$ is an integer, and $\eta$ a meromorphic function on $\Delta$. We denote them $G_1^n(\eta)$.*
3. *D-groupoids of order two: they are generated by an equation of the form: $\mu(y)y_1 + \frac{y_2}{y_1} - \mu(x) = 0$ where $\mu$ is meromorphic on $\Delta$. We denote them $G_2(\mu)$.*
4. *D-groupoids of order three: they are generated by an equation of the form: $\nu(y){y_1}^2 + 2\frac{y_3}{y_1} - 3\left(\frac{y_2}{y_1}\right)^2 - \nu(x) = 0$ where $\nu$ is meromorphic on $\Delta$. We denote them $G_3(\nu)$.*
5. *The D-groupoid of infinite order $G_\infty$ defined by the trivial equation $0 = 0$, whose solutions are the whole sheaf $\mathrm{Aut}(\Delta)$.*

The Galois envelope of a subgroup $G$ of Diff(C,0) is the smallest D-groupoid in the previous list which admits all the elements $g$ of $G$ as solutions. Clearly, the existence of a proper Galois envelope of finite order $k$, only depends on the analytic class of $G$. The Galois envelope for a monogeneous subgroup generated by $g$ is the Galois envelope of $g$ itself, since all the iterates of $g$ will also satisfy the same equation, by composition or inversion stability. The Galois envelope $\mathrm{Gal}(g)$ of $g$ is given by the two following results, see B. Malgrange [**14**], and G. Casale ([**3**]). Let $\alpha = g'(0)$. If $\alpha$ is an irrational number, then $g$ is formally linearizable. We have:

***Proposition 1.2***. — *A formally linearizable diffeomorphism has a proper Galois envelope if and only if it is an analytically linearizable diffeomorphism. In this case, its Galois envelope is a rank one D-groupoid.*

If $\alpha$ is a rational number, $g$ is a resonant diffeomorphism, and there exists an integer $q$ such that $g^q$ is tangent to the identity. The following lemma

***Lemma 1.3* ([3])**. — *For all non vanishing integer $q$, $\mathrm{Gal}(g) = \mathrm{Gal}(g^q)$.*

reduces the study to the case $\alpha = 1$. Any diffeomorphism tangent to the identity to an order $k$ is conjugated via a formal series to a normal form $g_N$ which is the exponential of the vector field $\frac{x^{k+1}}{1+\lambda x^k}\frac{d}{dx}$. Following the description of J. Martinet and J.P. Ramis, we obtain a complete analytic invariant $\mathrm{Inv}(g)$ of $g$ by the following construction (see [**15**]). Using $2k$ sectorial normalizations, one can prove that the space of the orbits of $g$ is obtained by gluing $2k$ bipunctured Riemann spheres $(S_i, 0, \infty)$ with local diffeomorphisms $\varphi_i^0 : (S_{i-1}, 0) \to (S_i, 0)$ and $\varphi_i^\infty : (S_i, \infty) \to (S_{i+1}, \infty)$. The collection $\mathrm{Inv}(g)$ of these gluing maps up to global automorphisms on each $(S_i, 0, \infty)$ is a complete analytic invariant of $g$. This invariant Inv(g) is *unitary* if there exists a positive integer $p$ such that the gluing maps $\varphi_i^0$ are of the form $z \mapsto z/(1 + a_i z^p)^{1/p}$ and $\varphi_i^\infty$: $u \mapsto u$ at infinity $(u = 1/z)$. It is a *binary* one when the gluing maps are alternatively of the form $z \mapsto z/(1+a_i z^p)^{1/p}$ in 0 and $u \mapsto u/(1+b_i u^p)^{1/p}$ at infinity. We have:

***Theorem 1.4* (see [3])**. — *Let $g$ be an element of* Diff *($\mathbb{C}$,0) tangent to the identity. The Galois envelope* $\mathrm{Gal}(g)$ *is proper of rank two (resp. three) if and only if its analytic invariant* $\mathrm{Inv}(g)$ *is a unitary one (resp. a binary one).*

***Remark 1.5***. — *The proof of this theorem make use of the following result (see* [**3**]*): Let $\theta_g$ be the formal vector field such that $g = \exp\theta_g$ (there is existence and unicity of such vector field, and its multiplicity at 0 is greater or equal to 2). The diffeomorphism $g$ is a solution of a D-groupoid if and only the vector field $\theta_g$ is a formal solution of its D-Lie algebra.*

We now discuss the Galois envelope of a subgroup $G$ of Diff($\mathbb{C}$,0) generated by $g_1, \cdots g_\mu$. Let $\Theta$ (resp. $\widehat{\Theta}$) be the Lie algebra of one variable holomorphic (resp. formal) vector fields which vanish at the origin: $\theta = (a_k z^k + \cdots)d/dz$. Recall that (see for example [**9**]):

***Lemma 1.6***. — *A subalgebra $\mathscr{G}$ of $\Theta$ (resp. $\widehat{\Theta}$) is a finite dimensional one if and only if $\mathscr{G}$ is at most of dimension two. Furthermore, such a Lie algebra is always a solvable one, and if the multiplicity $k$ of each element of $\mathscr{G}$ is greater or equal to two, then its dimension is at most one.*

Notice that such a result, and thus the following proposition, cannot be generalized in a global situation, in which there exist three dimensional Lie algebras of one variable vector fields which are not solvable ones.

***Proposition 1.7***. — *If the subgroup $G$ of* Diff *($\mathbb{C}$,0) has a proper Galois envelope, then $G$ is a solvable group.*

*Proof*. — Let $G_1$ be the subgroup of $G$ whose elements are tangent to the identity map. If $G_1$ is trivial, then $G$ is abelian since the first derivative group $[G, G]$ of $G$ is contained in $G_1$, and we are done. Therefore, we suppose that $G_1$ is non trivial. For each element $g$ of $G_1$, let $\theta_g$ be the element of $\widehat{\Theta}$ such that $g = \exp\theta_g$. From remark 1.5, the Lie algebra $\mathscr{L}(G_1)$ generated by these vector fields is included in the solutions of the D-Lie algebra of the Galois envelope of $G$, and is a finite dimensional one. Therefore, from the previous lemma, its dimension is one, and there exist a vector field $\theta$ and constants $c_g$ such that for all $g$ in $G_1$, $g = \exp c_g\theta$. This proves that $G_1$ is an abelian group. Since $[G,G]$ is contained in $G_1$, the group $G$ is a solvable one. □

The converse of Proposition (1.7) is false: all the monogeneous subgroups are abelian, but from (1.4), outside of the unitary or binary cases, they don't have a proper Galois envelope. The Galois reducibility is not only an algebraic property of the group $G$.

We shall recall the formal classification of the solvable subgroups of Diff(ℂ,0) (see [**21**] or [**7**]). We denote $G_1$ the subgroup of $G$ of its elements tangent to the identity map. We have:

— the group $G$ is formally linearizable if and only if $G_1$ is the trivial group;

— every solvable non linearizable group $G$ is formally conjugated to a group $G_N$ of the following type:

$$G_N = \left\{ g_{\lambda,t} = \lambda \exp t \frac{z^{k+1}}{1+\alpha z^k}\frac{d}{dz},\ \lambda \in \Lambda,\ t \in T \right\}$$

where $\Lambda$ is a multiplicative subgroup of $\mathbb{C}^*$ and $T$ is an additive subgroup of $\mathbb{C}$. Furthermore, $G_N$ is abelian if and only if $\Lambda$ is a group of $k$-roots of 1. If $G_N$ is not an abelian group, the residue $\alpha$ vanishes, and the elements of $G_N$ are obtained by lifting homographies fixing 0 with the ramification $z \mapsto z^k$.

— Following the terminology of D. Cerveau and R. Moussu [**7**], $G$ is an *exceptional* subgroup of Diff(ℂ,O) if $G_1$ is monogeneous. In particular, they are solvable groups. These authors prove that, among the non linearizable groups, the non exceptional groups are exactly the rigid ones: the formal classification coincides with the analytic one. One should say that an exceptional group is a unitary or binary one when $G_1$ is generated by a unitary or binary element.

***Theorem 1.8.*** — *The only subgroups of* Diff (ℂ,0) *which have a proper Galois envelope are:*

(1) *the analytically linearizable groups;*
(2) *the non exceptional solvable groups;*
(3) *the exceptional unitary groups;*
(4) *the exceptional binary groups.*

*Furthermore, the rank of their D-envelope is at most one in case* (1)*, at most two in cases* (2) *and* (3)*, and at most three in case* (4).

We call *Liouvillian* group every subgroup of Diff(ℂ,0) whose Galois envelope is at most of rank two, and *Riccatitian* group every subgroup of Diff(ℂ,0) whose Galois envelope is at most of rank three. In the present local situation, the Riccatitian non Liouvillian groups are very rare: their class is restricted to the (non empty!) set defined by $(4) \setminus (3)$.

*Proof.* — We first check that these groups have a proper Galois envelope:

(1) Since the existence of a proper Galois envelope of finite order $k$ only depends on the analytic class of $G$, it suffices to consider a group of linear diffeomorphisms. They keep invariant the differential form $dx/x$ and therefore satisfy the differential equation $xy_1 - y = 0$ which is, according to the notations of (1.1), the equation of the $D$-groupoid $G_1^1(1/x)$. Remark that this is only an upper bound of $\mathrm{Gal}(G)$: for example, if $G$ is a group of periodic rotations, they keep invariant an holomorphic function $h$ and $\mathrm{Gal}(G) = G_0(h)$.

(2) The formal model $G_N$ of a solvable group is Liouvillian. Indeed, the differential form $\omega = (1 + \alpha x^k)/x^{k+1}dx$ is invariant by each element $f_{\lambda,t}$ of $G_N$ up to a multiplicative constant $c_{\lambda,t}$. Therefore, each element of $G_N$ satisfies

$$a(y)y_1 = c_{\lambda,t}a(x),$$

where $a$ is the coefficient of $\omega$. Derivating these equations, each element of $G$ is a solution of the same equation

$$a(x)a(y)y_2 + a'(y)a(x)y_1^2 - a'(x)a(y)y_1 = 0$$

where $a'$ is the derivative of $a$ with respect to $x$. This is the equation of the rank two D-groupoid $G_2(a'/a)$. The same previous remark holds: this is only an upper bound of the Galois envelope of $G_N$: if $G_N$ is abelian, its elements all satisfy the rank one equation $a(y)y_1 - a(x) = 0$ of $G_1^1(a)$. Now, if $G$ is a non exceptional group, by rigidity, it is analytically conjugated to $G_N$, and still have a proper Galoisian envelope of rank at most two.

(3) and (4): Let $G$ be an exceptional group and let $g_1$ be a generator of the monogeneous group $G_1$, which is supposed to be unitary or binary. From (1.4), $G_1$ has a proper envelope of rank two or three with equation $E = 0$. If $G$ is not equal to $G_1$, we know from proposition 2 of **[7]** that $G$ is generated by $g_1$ and a second resonant element $g_2$. If $g_1$ is tangent to the identity at order $k$, the normal form of $G$ described by **[7]** shows that $g_2^{2k}$ belongs to $G_1$ and therefore $g_2^{2k} = g_1^l$ for some integer $l$. With lemma (1.3), we conclude that $g_2$ also belongs to the Galois envelope of $G_1$, and finally, $\mathrm{Gal}(G) = \mathrm{Gal}(G_1)$.

On the converse, we now suppose that $G$ has a proper Galois envelope. If $G_1$ is a trivial group, then $G$ is formally linearizable and from proposition (1.2) we conclude that $G$ is of type (1). If $G_1$ is non trivial, we know from (1.7) that $G$ is a solvable group. Either it is a non exceptional one, and $G$ is of type (2), or it is an exceptional one: $G_1$ is generated by an element $g_1$. Since the Galois envelope of this one is non trivial, we know from theorem (1.4) that $g_1$ and thus $G$ is of type (3) or (4). □

## 2. A geometric criterion for Galois reducibility

We first recall general facts on the Galois reducibility for singular holomorphic foliations. Let $\mathcal{F}$ be a singular holomorphic foliation of codimension $k$ on a $n$-dimensional holomorphic manifold of $M$. Following the definition of B. Malgrange [**12**], the Galois groupoid of $\mathcal{F}$ is its D-envelope, i.e. the smallest D-groupoid Gal($\mathcal{F}$) wich is "admissible" for the foliation: its D-Lie algebra contains the tangent vector fields. The Galois groupoid of $\mathcal{F}$ is always contained in the D-groupoid Aut($\mathcal{F}$) of the germs of diffeomorphisms which keep invariant the foliation. We shall say that $\mathcal{F}$ is Galois reducible if its Galois envelope is proper: $\mathrm{Gal}(\mathcal{F}) \neq \mathrm{Aut}(\mathcal{F})$. This property only depends on the analytic class of the foliation, and is invariant by blowing up or blowing down transformations.

If $U$ is an open set in $M$ on which the foliation is trivializable by tangent-transverse coordinates $(s,t)$, $s = (s_1, \cdots s_{n-k})$, $t = (t_1, \cdots t_k)$, the local ideal of Gal($\mathcal{F}$) can be generated by equations (see [**5**]):

$$\frac{\partial T_j}{\partial z_i} = 0, \; E_i\left(t, T, \cdots \frac{\partial^{|\alpha|} T}{\partial t^\alpha}\right) \tag{1}$$

where $E_i$ are the equations of a D-groupoid on the $k$-dimensional polydisc $t(U)$. The rank of this local transverse groupoid doesn't depend on the local chart [**5**]: this is the *transverse rank* of Gal($\mathcal{F}$).

We now suppose that $\mathcal{F}$ is a codimension one foliation on a polydisc $\Delta$ in $(\mathbb{C}^n, 0)$, defined by a one-form $\omega$ which satisfies the Frobenius condition. We may suppose that the singular locus is at least a codimension two analytic set. From (1.1), the transverse rank of $\mathcal{F}$ can only get the values 0, 1, 2, 3 or $\infty$, the finite values corresponding to the proper cases. A Godbillon-Vey sequence for $\omega$ is a sequence of meromorphic one-forms $\omega_n$ such that

$$d\omega = \omega \wedge \omega_1, \quad d\omega_1 = \omega \wedge \omega_2, \cdots$$

$$d\omega_i = \omega \wedge \omega_{i+1} + \sum_{j=1}^{i} \binom{i}{j} \omega_j \wedge \omega_{i-j+1}$$

A Godbillon-Vey sequence of lenght $l > 1$ is a Godbillon-Vey sequence such that $\omega_i = 0$, $i \geq l$. A Godbillon-Vey sequence of lenght 1, is a Godbillon-Vey sequence of lenght 2, such that $\omega_1 = p^{-1} df/f$ for an integer $p$: $f^{1/p}$ is an integrating factor of $\omega$. The existence of a Godbillon-Vey sequence of lenght $l$ only depends on the foliation defined by $\omega$. We have (see [**14**] and [**5**]):

***Theorem 2.1***. — *The foliation $\mathcal{F}$ has a Godbillon-Vey sequence of lenght $l$ with $l \leq 3$ if and only if the transverse rank of its Galois groupoid is at most $l$.*

Furthermore, G. Casale has proved in [**2**] that the existence of a proper Galois envelope for $\mathcal{F}$ is also equivalent to the existence of a transcendental first integral which belongs to a particular type of extension, namely a meromorphic, Darboux, Liouvillian or Riccatician type, according to the values $l = 0, 1, 2$ or 3 of the transverse rank of Gal($\mathcal{F}$). Therefore, in each case, one should call the foliation with the same terminology.

If $L$ is a leaf of $\mathcal{F}$, and if $\operatorname{Hol}(L)$ is the image of its holonomy representation, then all its elements are solutions of the local ideal of $\operatorname{Gal}(\mathcal{F})$. Indeed, for any loop $\gamma$ which represents an element of $\pi_1(L, m)$, we can cover $\gamma$ by trivializing open sets $U_1, \cdots U_p$ such that the transverse coordinate on $U_i$ is an analytic extension of the previous one. With this choice, the change of local coordinates are tangent to the foliation and therefore are solutions of $\operatorname{Gal}(\mathcal{F})$. By the stability under composition, the change of coordinates between $U_p$ and $U_1$ is a solution of $\operatorname{Gal}(\mathcal{F})$. In particular, its transverse component –which is the holonomy representation of $\gamma$– is a solution of the local expression of $\operatorname{Gal}(\mathcal{F})$. From this remark, and since the existence of a proper Galois envelope is an invariant property under birational maps, we obtain

***Proposition 2.2***. — *If $\mathcal{F}$ has a proper Galois envelope, then any holonomy group of $\mathcal{F}$ or of any foliation $\widetilde{\mathcal{F}}$ obtained from $\mathcal{F}$ by blowing up's has a proper Galois envelope whose rank is at most the transverse rank of* $\operatorname{Gal}(\mathcal{F})$.

We shall prove that for the present class of quasi-homogeneous germs of foliations, we have a converse of this statement. In order to do this, we consider the desingularization process of $\mathcal{F}$: see **[22]** or **[17]**. For a quasi-homogeneous foliation which is a perturbation of the foliation defined by $h = 0$, extending an argument of **[7]**, one can prove that the desingularization process is the same as the one of $dh$, namely: the exceptional divisor is a chain of projective lines which are invariant for the desingularized foliation; all the strict pull back of each component of $h = 0$ different from the axis are transverse to the same projective line $C$: we call it the principal one. One can check that $C$ is also the space of the values for the meromorphic first integral $x^{p_2}/y^{p_1}$ of the quasi-radial vector field $R$. The singularities on $C$ are the different values corresponding to each branch of $X$, and $0$, $\infty$, which are the intersections with other components. If $x$ or $y$ occurs in the decomposition of $h$, their pullback by the composition of blowing up's is a line transverse to the end components of the chain. All the reduced singularities are resonant saddles (not necessarily linearizable), since their linear part is obtained by the local expression of the desingularization of $dh/h$. The projective holonomy of $\mathcal{F}$ is the holonomy of the principal component $C$ of the desingularized foliation $\widetilde{\mathcal{F}}$. We denote $\operatorname{Hol}(\mathcal{F})$ the image of this representation: this is a subgroup of $\operatorname{Diff}(\mathbb{C}, 0)$ defined up to a conjugacy (the choice of a transverse on which we realize the holonomy group). The following result is announced in **[8]**, and proved for cuspidal singularities in **[18]**:

***Theorem 2.3***. — *Two quasi-homogeneous germs of foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are analytically equivalent if and only if* $\operatorname{Hol}(\mathcal{F}_1)$ *is conjugated to* $\operatorname{Hol}(\mathcal{F}_2)$.

The easier following result can be proved independently:

***Theorem 2.4***. — *The Galois envelope of the germ of quasi-homogeneous foliation $\mathcal{F}$ is a proper one if and only if the Galois envelope of* $\operatorname{Hol}(\mathcal{F})$ *is a proper one.*

*Proof.* — If the Galois envelope of $\mathcal{F}$ is a proper one, the same holds for $\mathrm{Hol}(\mathcal{F})$ from proposition (2.2). We now suppose that $\mathrm{Hol}(\mathcal{F})$ has a Galois envelope of finite rank given by an equation $E=0$ of type (0), (1), (2) or (3) in the list given by (1.1). Let $(s_0,t_0)$ be a local system of tangent-transverse coordinates on an open set $U_0$ around a regular point $m$ in the principal component of $\widetilde{\mathcal{F}}$, and let $T$ be the transversal $s_0=s_0(m)$. As above, we can extend $E=0$ to a local equation $E_0$ of a D-groupoid on $U_0$ whose transverse expression is $E=0$ and is admissible for the foliation setting:

$$\frac{\partial T}{\partial s}=0,\ E\left(t,T,\cdots\frac{\partial^k T}{\partial t^k}\right)=0.$$

We can extend this D-groupoid along a path $\gamma$ by covering this path with open sets $U_\alpha$, $\alpha=0,\cdots n$ with local systems $(s_\alpha,t_\alpha)$: the first equation is preserved by a foliated change of coordinates, and the second one $E_\alpha=0$ is extended on $U_\beta$ by $\psi^*_{\alpha\beta}E_\alpha$ where $t_\beta=\psi_{\alpha\beta}(t_\alpha)$. If $\gamma$ is a loop, this analytic extension coincides at the end of $\gamma$ with the initial groupoid: indeed, the composition of the transition maps $\psi_{\alpha\beta}$ is the holonomy map of $\gamma$ and we know that this one is a solution of the Galois envelope, and therefore keep invariant $E_\alpha=0$. By this way, we get an extension of the D-groupoid $E_0=0$ to the smooth part of the principal component $C$. Now, we can extend this groupoid to a neibourhood of each reduced singularity on $C$, from a result of Guy Casale: see proposition (5.2) in [**5**]. Let $C'$ be an adjacent component to $C$ and $p$ a regular point near from $C\cap C'$. One can choose local generators of the groupoid in $p$ which are still written under the previous adapted form. Furthermore, the local holonomy of $C'$ around $C\cap C'$ is a solution of this groupoid. From the previous description of the exceptional divisor, $C'$ gets at most two singularities, and the fundamental group of the complement of its singularities is generated by one element. Therefore we can extend the groupoid along $C'$ and inductively to the whole divisor. □

Such a type of argument can be used to prove that if $\mathrm{Hol}(\mathcal{F}_1)$ is conjugated to $\mathrm{Hol}(\mathcal{F}_2)$, then this conjugacy gives a local conjucagy around $m$ for the desingularized corresponding foliations, whose *transverse* expression can be extended to the whole divisor. The main difficulty in theorem (2.3) is to prove that for quasihomogeneous foliations there is no *tangent* obstruction to construct a global conjugacy along the divisor. Here, the existence of a proper Galois envelope -or of a Godbillon-Vey sequence: see [**21**] for the Liouvillian case- only involves transverse obstructions, and thus are easier to obtain.

## 3. An algorithmic criterion for Galois reducibility

We want to test the Galois reducibility by making use of formal normal forms for the germs of quasi-homogeneous foliations $\mathcal{F}_X$ defined by the vector fields:

$$X=aX_h+bR,\ a\in\mathcal{O}_2,b\in\mathcal{O}_2,\ a(0)=1,\deg(bR)>\deg(X_h).$$

In the general situation, both normal forms and conjugacies are formal objects. This will only give a criterion of *formal* Galois reducibility. We can consider two definitions for the *formal* Galois reducibility of an analytic foliation:

(i) There exists an analytic foliation $\mathcal{F}'$ which is formally conjugated to $\mathcal{F}$ and Galois reducible;

(ii) The foliation $\mathcal{F}$ admits a formal finite Godbillon-Vey sequence.

Clearly the first one implies the second one, by taking with the formal conjugacy the pull back of the Godbillon-Vey sequence of $\mathcal{F}'$ given by (2.1). We first choose the second definition here, since we deal with formal models. But finally, it turns out that, for our class of foliations, both definitions coincide (see remark (3.6) below): when this criterion of formal Galois reducibility holds, we shall obtain convergent final normal forms. Therefore, if we are in a non exceptional (or "rigid") case, the conjugacy will also converge, and we shall obtain an algorithmic criterion for analytic Galois reducibility.

We first recall the construction of these normal forms and introduce the related complete formal invariant, obtained in [**21**]. They generalize the normal forms of the cuspidal case ($h = y^2 - x^3$) described in [**23**] and [**10**]. We split it into two steps:

*First step: prenormalization.* It is based on the following general lemma. Let $M$ be a submodule of the $\widehat{\mathcal{O}}_n$-module of formal vector fields at the origin of $\mathbb{C}^n$, endowed with a graduation, and stable under the Lie bracket (in the present case, $M$ is the module of logarithmic vector fields, endowed with the quasi-homogeneous degree induced by $R$). Let $X = X_0 + \cdots$ be a perturbation of the initial quasi-homogeneous vector field $X_0$ of degree $\delta_0$ by higher order terms.

***Lemma 3.1***. — [**21**] *Let $B$ be the image of the operator $[X_0, \cdot]$ in $M$, and $W$ a complement space of $A = B + \widehat{\mathcal{O}}_n X_0$ in $M$. There exist a vector field $Y$ in $W$, a formal diffeomorphism $\Phi$ and a formal unity $u$ such that $\Phi^* X = u(X_0 + Y)$.*

Notice that if we want to classify the vector fields instead of the foliations (i.e. if we don't work up to a unity) the same statement holds with a complement of $B$ instead of $A$. This lemma reduces the first step to an appropriate choice of a submodule $W$ isomorphic to the quotient space $M/A$. Denote by $\mathcal{I}$ the ring of first integrals of the initial vector field $X_0$. The rich cases for normal forms occur when $\mathcal{I}$ doesn't reduce to the constants. In our case, $\mathcal{I} = \mathbb{C}[[h]]$. Clearly, the quotient $M/A$ is a $\mathcal{I}$-module, and one should naturally require the same property in our choice for $W$. In our present situation ($X_0 = X_h$ with a quasi-homogeneous function $h$ which has an isolated singularity), we can compute the quotient $M/A$ (see [**21**] for details): this a free $\mathcal{I}$-module generated by the $\mu$ classes of vector fields $a_k R$, where $a_1, \cdots a_\mu$ is a monomial basis of the $\mathbb{C}$-vector space $\mathcal{O}_2/J(h)$, and $J(h)$ is the jacobian ideal of $h$. This allows us to choose $W = X_h \oplus_{k=1}^{\mu} \mathcal{I} a_k\ R$, and from lemma (3.1), we have

***Theorem 3.2.*** — *Let $X = aX_h + bR$ be a perturbation of $X_h$. There exist an element $(d_1, \cdots d_\mu)$ of $\mathbb{C}[[h]]^\mu$, a formal diffeomorphism $\Phi$ which conjugates the foliation $\mathcal{F}_X$ to the foliation defined by the vector field*

$$Y = X_h + \sum_{k=1}^{\mu} d_k(h) a_k R.$$

*Furthermore, we can require that this conjucagy is "fibered" with respect to $R$, i.e. is formally the exponential of a vector field proportional to $R$. Such a conjugacy keeps invariant each trajectory of $R$.*

*Second step: final reduction.* In the previous step, for a fixed complement space, there is no unicity of the prenormal form $Y$. One can prove that the set of prenormal forms for $\mathcal{F}_X$ is the orbit of one of them under the action of a final reduction group of transformations of the following type: $\Phi = \exp b \cdot R$, with a formal coefficient $b$ in $\mathcal{I}$: $b = b(h)$. Such transformations satisfy the relation $h \circ \Phi = \varphi \circ h$ for a one variable formal diffeomorphism $\varphi$. In order to study the action of this final reduction group on the prenormal forms, it is convenient to introduce a modified expression of them. We shall make use of the two following remarks:

i- Setting $\alpha = h^{-\delta_0/\delta}$, he have $[\alpha X_h, R] = 0$. The introduction of this multivalued coefficient will allow us to work with an abelian basis of logarithmic vector fields.

ii- Setting $r_i = \frac{\deg(\alpha a_i)}{\delta}$ we have $R(\alpha a_i h^{-r_i}) = 0$. This will allow us to work with coefficients which are constants for $R$.

Multiplying $Y$ with $\alpha$, and grouping coefficients in order to transform coefficients $a_i$ in constants $f_i$ for $R$ we obtain the following "adapted" prenormal forms:

$$X_\alpha + \sum_{i=1}^{\mu} f_i \delta_i(h) R \tag{2}$$

with $X_\alpha = \alpha X_h$, $f_i = \alpha a_i h^{-r_i}$ and $\delta_i = d_i(h) h^{r_i}$. By these two tricks, any element $\Phi$ of the final reduction group keep invariant $X_\alpha$ and the coefficients $f_i$. Therefore we have

$$\Phi^*(X_\alpha + \sum_{i=1}^{\mu} f_i \delta_i R) = X_\alpha + \sum_{i=1}^{\mu} f_i \Phi^*(\delta_i(h) R).$$

The action of $\Phi$ over $\delta_i(h)R$ is given by

$$\Phi^*(\delta_i(h)R) = d_i \circ \varphi(h) \frac{\varphi(h))^{r_i+1}}{\varphi'(h)} \frac{R}{h}$$

where $\varphi$ is defined by $h \circ \Phi = \varphi \circ h$. This is the lifted action by $h$ of the action of $\varphi$ on the one-variable vector fields

$$\theta_i(z) = d_i(z) z^{r_i+1} \frac{d}{dz}.$$

Since $r_i = p_i/\delta$ for a positive integer $p_i$, we can uniformize these vector fields setting $t = z^{1/\delta}$ in

$$\theta_i(t) = \delta^{-1} d_i(t^\delta) t^{p_i+1} \frac{d}{dt}.$$

We may choose $\varphi$ -and therefore $\Phi$- in such a way that one of the vector fields $\theta_i$ is normalized under its usual normal form

$$\delta^{-1}\frac{t^{q_i+1}}{1+\lambda t^{q_i}}\frac{d}{dt}, \text{ with } q_i = \delta k_i + p_i$$

where $k_i$ is the multiplicity of each series $d_i$. Going back to the non adapted prenormal forms, we obtain the following final normal forms:

***Theorem 3.3***. — *Let $Y = X_h + \sum_{k=1}^{\mu} d_k(h)a_k R$ be a prenormal form of $X$ and $i$ an indice arbitrary chosen among $1,\cdots\mu$. There exists a diffeomorphism in the final reduction group which conjugate $Y$ to a normal form $Y_N$ in which the coefficient of indice $i$ is a rational function of $h$ of the following type:*

$$d_i^N(h) = \frac{h^m}{1+\lambda h^{m+n}}$$

*where $\lambda$ is a complex number, and $m$, $n$ are positive integers.*

In fact, the previous argument gives rise to the following explicit formal invariant:

***Proposition 3.4***. — *The family of the $\mu$ formal vector fields $\theta_i(t)$ up to a common conjugacy is a complete formal invariant for the foliation defined by $X$. We denote it $\mathcal{L}(\mathcal{F})$.*

Notice that as soon as $\mu$ is greater than two, we can't normalize simultaneously all the coefficients $d_i$ under a rational form. The final normal form is still a formal object. A result of M. Canalis and R. Schafke in the cuspidal situation ($h = y^2 - x^3$) suggests that these final normal forms are defined by $k$-summable series in $t$: see [**1**]. Nevertheless, the generalization of this fact, and the computation of the order $k$ is still an opened question. Furthermore, even if they are of the same nature (conjugacy class of $\mu$ one variable objects) the relationship between this algorithmic invariant $\mathcal{L}(\mathcal{F})$ and the geometric one $\mathrm{Hol}(\mathcal{F})$ is not clear (it is of transcendental nature), excepted in the Galois reductible situations, in which we shall be able to specify it in the next section.

We now give a criterion of formal Galois reducibility, for the class of quasi-homogeneous foliations described in the introduction.

***Theorem 3.5***. — *The following propositions are equivalent:*

(1) *The foliation $\mathcal{F}$ is formally Galois reducible;*
(2) *The Lie algebra generated by the elements of $\mathcal{L}(\mathcal{F})$ is a finite dimensional one;*
(3) *The Lie algebra generated by the elements of $\mathcal{L}(\mathcal{F})$ is one dimensional;*
(4) *$\mathcal{F}$ is a formally Liouvillian foliation.*

*Proof*. — The equivalence between propositions (2) and (3) comes from Lemma (1.6), since one can check that the multiplicity of each vector field $\theta_i$ is greater than one: this is a consequence of $\deg(bR) > \deg(X_h)$.

We now prove the implication (3) $\Rightarrow$ (4). Let $\theta$ and $c_k$ be a vector field and $\mu$ constants such that $\theta_k = c_k\theta$. The adapted normal form obtained in (2) is here of the following type:

$$X_\alpha + (\sum_{i=1}^{\mu} c_i f_i)\delta(h)R.$$

The final reduction step normalize $\theta$ -and therefore here all the $\theta_i$- under its usual rational normal form. We obtain a *convergent* normal form $X_N$ in the formal class of $\mathcal{F}_X$. In order to prove that the foliation $\mathcal{F}_N$ defined by $X_N$ is Galois reducible of order two, by theorem (2.1) we have to prove that there exist two logarithmic one-forms $\omega_N$ and $\omega_1$ such that $\omega_N$ define the foliation $\mathcal{F}_N$ and $\omega_1$ is a closed form such that $d\omega_N = \omega_N \wedge \omega_1$. We consider the logarithmic one forms (for details on this dual point of view, see [**21**]):

$$\omega_h = \delta^{-1}\frac{dh}{h} = \frac{dx \wedge dy}{h}(\delta^{-1}X_h, \cdot), \quad \omega_R = \frac{p_2ydx - p_1xdy}{h} = \frac{dx \wedge dy}{h}(\cdot, R).$$

Since $dx \wedge dy/h(\delta^{-1}X_h, R) = \delta^{-1}R(h)/h = 1$, $\{\omega_R, \omega_h\}$ is a dual basis of $\{X_h, R\}$ for the pairing $(\omega, X) = \omega(X)$. Therefore, for any function $f$, we have $df = R(f)\omega_h + X_h(f)\omega_R$, and the one-form $a\omega_h - b\omega_R$ define the same foliation as $X = aX_h + bR$. Notice that $\omega_R$ is not a closed form, but dividing it with $\alpha = h^{-\delta_0/\delta}$, we have $d(\alpha^{-1}\omega_R) = 0$: this is similar to the trick (i-). The foliation $\mathcal{F}_N$ is defined by

$$\omega_N = \omega_h - \sum_{i=1}^{\mu} a_i d_i(h)\omega_R = \omega_h - \sum_{i=1}^{\mu} f_i\delta_i(h)\frac{\omega_R}{\alpha} = \omega_h - f_c\delta(h)\frac{\omega_R}{\alpha}$$

where $f_c = \sum_{i=1}^{\mu} c_i f_i$ only depends on $c = (c_1, \cdots c_\mu)$. Since $R(f) = 0$, we have $d(\omega_N/\delta(h)) = 0$, and the logarithmic derivative $\omega_1$ of $\delta(h)$ is a closed form which satisfies the Godbillon-Vey relation.

We now prove the main implication (4) $\Rightarrow$ (3). We shall give another proof of it in the next section. If $\mathcal{F}$ is formally Liouvillian then $\mathcal{F}_N$ have a (formal) Godbillon-Vey sequence of lenght two given by $\omega_N$, $\omega_1$, and it suffices to prove that $\mathcal{L}(\mathcal{F}_N)$ is one dimensional. We can check that $\omega_1$ also keep invariant $X : \; h = 0$, with simple poles along $X$ (for this last point, which is only formally true, see [**20**]). Therefore, $\omega_1$ is a closed logarithmic form and there exist two formal coefficients $\lambda$ and $\mu$ such that $\omega_1 = \lambda\omega_h + \mu\omega_R$. We may suppose that $\omega_1 = \lambda_0\omega_h$, where $\lambda_0$ in the residue of $\omega_1$ along $h = 0$ , even if it means replacing $h$ with $h \circ \Phi$ and replacing the logarithmic basis with its pull back by $\Phi$: indeed, $\omega_1 - \lambda_0\omega_h$ is a closed logarithmic form with vanishing residue and therefore, there exists a formal coefficient $g$ such that $\omega_1 - \lambda_0\omega_h = \lambda_0 dg$. Setting $u = \exp g$ we obtain: $\omega_1 = \lambda_0 d(uh)/(uh)$. Since $h$ is a quasihomogeneous function there exists a change of variable $\Phi$ such that $h \circ \Phi = uh$. Conjugating the Godbillon-Vey relation by $\Phi$ we normalize $\omega_1$ under the previous form, and $\Phi^*\omega_N$ is still normalized relatively to the new logarithmic basis.

Using the relations $df = R(f)\omega_h + X_h(f)\omega_R$ and $d(\omega_R/\alpha) = 0$, we obtain

$$d\omega_N = d(\omega_h - \sum_{i=1}^{\mu} f_i\delta_i(h)\frac{\omega_R}{\alpha}) = -\sum_{i=1}^{\mu} f_i R(\delta_i(h))\omega_h \wedge \frac{\omega_R}{\alpha}.$$

Therefore, the Godbillon-Vey relation $d\omega_N = \omega_N \wedge \omega_1$ is equivalent to

$$\lambda_0 \sum_{i=1}^{\mu} f_i\delta_i(h) = -\sum_{i=1}^{\mu} f_i\delta_i'(h)\delta h.$$

where $\delta'$ is the derivative of this one-variable function. The decomposition of any element $b$ under a sum $\sum_{i=1}^{\mu} f_i\delta_i(h)$ or equivalently under a sum $\sum_{i=1}^{\mu} a_i d_i(h)$ is unique. Indeed, the space of prenormal forms $W$ is isomorphic to the $\mathscr{I}$-module $\mathrm{coker}(X_h)$, and this one is a free module over the basis given by the classes of $a_1, \cdots a_\mu$ (see **[21]**). Therefore the Godbillon-Vey equation is equivalent to the $\mu$ linear differential equations

$$\delta h\delta_i'(h) = \lambda_0\delta_i(h), \quad i = 1\cdots\mu.$$

Since the functions $\delta_i(h)$ are solutions of the same one-dimensional first order linear differential equation, we have $\delta_i(h) = c_i\delta(h)$ for all $i$ in $\{1, \cdots \mu\}$.

Finally, we have to prove the non trivial implication of (1) $\Leftrightarrow$ (4), i.e.: any formally Galois reducible foliation is a formally Liouvillian one. This is essentially a consequence of (1.7). Indeed, if $\mathscr{F}$ has a proper Galois envelope, we know that the same holds for $\mathrm{Hol}(\mathscr{F})$. From theorem (1.8), $\mathrm{Hol}(\mathscr{F})$ is a solvable group. According to **[20]**, this allows us to construct a formal Godbillon-Vey sequence of lenght two for the foliation. We summarize this construction: from Theorem (1.7) of **[20]**, a solvable subgroup of $\mathrm{Diff}(\mathbb{C},0)$ admits a formal symmetry i.e. a formal one variable vector field which is invariant up to a multiplicative constant by each element of the group. Evaluating $\omega$ on this symmetry, we obtain a local integrating factor whose logarithmic derivative $\omega_1$ satisfies the Godbillon-Vey relation. We can extend $\omega_1$ on the regular part of the principal component the exceptional divisor, since it is invariant by the holonomy of this component. Then, we extend it along the whole exceptional divisor with similar arguments as in the proof of (2.4). $\square$

***Remark 3.6***. — *If the foliation $\mathscr{F}$ is formally Galois reducible, then its final normal form is a convergent one. Indeed, if the Lie algebra generated by the elements of $\mathscr{L}(\mathscr{F})$ is one dimensional, then the action of the final reduction group will simultaneously normalize each coefficient $d_i(h)$ under a rational form. Therefore the final normal form of $\mathscr{F}$ has a convergent expression.*

Following and extending the definition of D. Cerveau and R. Moussu in **[7]**, a quasi-homogeneous foliation $\mathscr{F}$ is a non exceptional foliation if and only if its invariant $\mathrm{Hol}(\mathscr{F})$ is a non exceptional group. Two non exceptional holomorphic foliations which are formally conjugated are analytically equivalents: indeed, by **[21]**, we know that we can construct a conjugacy which is fibered with respect to $R$, and which is only a transversally formal one. Therefore, the restriction of such a transformation to any fiber of $R$ will define a conjugacy between the holonomy groups. Since they are non exceptional this conjugacy is a convergent one.

***Corollary 3.7.*** — *We suppose that the foliation $\mathcal{F}$ is a non exceptional one. The following propositions are equivalent:*

(1) *The foliation $\mathcal{F}$ is Galois reducible;*
(2) *The Lie algebra generated by the elements of $\mathcal{L}(\mathcal{F})$ is a finite dimensional one;*
(3) *The Lie algebra generated by the elements of $\mathcal{L}(\mathcal{F})$ is one dimensional;*
(4) *$\mathcal{F}$ is a Liouvillian foliation.*

*Proof.* — The first implication (1) $\Rightarrow$ (2) (or (3)) comes from the corresponding implication in Theorem (3.5). Since (4) $\Rightarrow$ (1) is trivial, we only have to prove (3) $\Rightarrow$ (4). Let $\mathcal{F}$ be a holomorphic foliation such that $\mathcal{L}(\mathcal{F})$ is one dimensional. Following the previous remark (3.6), its final normal form is a convergent one and defines a holomorphic foliation $\mathcal{F}_N$, which is Liouvillian. Since $\mathcal{F}$ is a non exceptional foliation, the conjucacy between $\mathcal{F}$ and $\mathcal{F}_N$ is a convergent one and $\mathcal{F}$ is also a Liouvillian foliation. □

Notice that for non exceptional germs of foliations, there doesn't exist Riccatitian foliations which are not Liouvillian. Clearly, for an exceptional foliation, we need an additional criterion on the analytic class itself inside the formal one (they are all formally Liouvillian). This one is given by theorem (2.4), and by the classification of the groups of diffeomorphisms with proper envelope (1.8), and therefore is not yet an algorithmic one:

***Theorem 3.8.*** — *An exceptional foliation $\mathcal{F}$ has a proper Galois envelope if and only if the group* Hol*($\mathcal{F}$) is an exceptional unitary or binary one.*

## 4. Relationship between geometric and algorithmic invariants for Galois reducible foliations

We introduced in section 2 the notion of projective holonomy, namely the holonomy of the principal component $C$ in the desingularization of the foliation. For explicit computations, the following notion of "relative holonomy" is more efficient. Let $m$ be a regular point of the desingularized foliation on $C$ and $T$ the pull back of the fiber of $R$ corresponding to this value $m$. Any element of $\pi_1(C, m)$ can be lifted into a path from a point in $T$ in a leaf of the initial foliation $\mathcal{F}_h$ defined by $dh = 0$. We consider the normal subgroup $\pi_1'(C, m)$ of $\pi_1(C, m)$ corresponding to the elements which can be lifted in *loops* in the initial foliation: this is the kernel of the representation of the projective holonomy of $\mathcal{F}_h$. The *relative representation of holonomy of $\mathcal{F}$* is the restriction of the projective holonomy to $\pi_1'(C, m)$. We denote by $\mathrm{Hol}(\mathcal{F}, \mathcal{F}_h)$ its image. This is a subgroup of the group $\mathrm{Diff}_1(\mathbb{C}, 0)$ of germs of diffeomorphisms tangent to the identity. We have:

(i) The Galois envelope of $\mathrm{Hol}(\mathcal{F}, \mathcal{F}_h)$ is identical to the one of $\mathrm{Hol}(\mathcal{F})$;
(ii) The class of conjugacy of $\mathrm{Hol}(\mathcal{F}, \mathcal{F}_h)$ is still an analytic complete invariant for $\mathcal{F}$.

These two facts only hold for foliations $\mathcal{F}$ which are a perturbation of $\mathcal{F}_h$. Indeed, in this case any element of $\mathrm{Hol}(\mathcal{F})$ is a resonant one, and the statement (i-) is a consequence of proposition (1.3). The second one can be deduced from theorem (2.3) by proving that the relative holonomy groups are conjugated if and only if the projective holonomy groups are conjugated. We don't give the details since we shall not make use of this result.

The main interest of this holonomy is a more efficient presentation of $\pi_1'(C, m)$ interpreting its elements as *horizontal classes of evanescent cycles.* Let us develop this point of view. We first remark that the quasi-radial vector field $R$ is a basic vector field for the initial foliation $\mathcal{F}_h$: from $R(h) = \delta h$, we deduce that its flow $\exp[\tau]R$ sends the fiber $F_{z_0}\ :\ h = z_0$ on the fiber $F_z$ with the formula $z = z_0 e^{\tau\delta}$. This implies that the flow of the vector field $\delta^{-1}R$ commutes via $h$ with the flow $zd/dz$ on the disc image of $h$. In particular, one can lift the circle with base point $m_0$: $\exp[\sigma]zd/dz \cdot m_0, \sigma \in [0, 2i\pi]$. For $\sigma = 2i\pi$, we obtain a diffeomorphism

$$\rho = \exp[2i\pi]\delta^{-1}R$$

which keep invariant each fiber of $h$. This is the *geometric monodromy* of $\mathcal{F}_h$. The diffeomorphism $\rho$ is periodic with period $\delta$. The orbit of a point $p$ on $F_z$ under the action of $\rho$ is a set of $\delta$ points, intersection of $F_z$ with the trajectory $T_p$ of $R$ through $p$. The meromorphic first integral of $R$ defines a projection $\pi_R$ onto $C$. From the previous description, for any loop $\gamma$ in $F_z$ the $\delta$ elements of its orbit via $\rho$ have the same projection by $\pi_R$ onto a loop which represents an element of $\pi_1'(C, m)$. Finally, the elements of $\pi_1'(C, m)$ can be identified to the classes of evanescent loops in a fiber $F_z$ modulo the action of $\rho$, or also to the horizontal family of evanescent loops, obtained by the action of the flow of $R$ on $\gamma$ (the previous description is only the intersection of this family with $F_z$).

Let $\gamma_1, \cdots \gamma_\mu$ be a basis of the free group $\pi_1(F_z, p)$, and let $\Gamma_1, \cdots \Gamma_\mu$ be their projection in $\pi_1'(C, m)$ or their class modulo $\rho$. We want to compute their image $h_{\Gamma_i}$ in $\mathrm{Hol}(\mathcal{F}, \mathcal{F}_h)$ when $\mathcal{F}$ has a proper Galois envelope. In the non exceptional case, the probleme reduces to the computation of $\mathrm{Hol}(\mathcal{F}_N, \mathcal{F}_h)$ where $\mathcal{F}_N$ may be defined by the following one-form written under its adapted form

$$\omega_N = \omega_h - \left(\sum_{i=1}^{\mu} c_i f_i\right) \delta(h) \frac{\omega_R}{\alpha}.$$

Notice that the $\mu$ one-forms $\eta_i = f_i \frac{\omega_R}{\alpha}$ are horizontal, i.e. invariant under the action of $R$. Indeed, we have

$$L_R(\eta_i) = (i_R d + d i_R)\left(f_i \frac{\omega_R}{\alpha}\right) = 0$$

since $R(f_i) = 0$ and $d(\frac{\omega_R}{\alpha}) = 0$. Let $\eta_c = \sum_{i=1}^{\mu} c_i\eta_i$. The choice of $c = (c_1, \cdots c_\mu)$ completely determines the class of $\mathcal{F}_N$ . Since $\eta_c$ is a horizontal form, its integration $T_i = \int_{\gamma_i} \eta_c$ only depends on the horizontal classe $\Gamma_i$.

The vector field $\delta(h)R$ is a vertical vector field, i.e. a vector field tangent to the foliation defined by $R$, and its restriction on each fiber doesn't depend on this fiber. If we introduce local coordinates around a point $p$ outside $X$ defined by $s = \int \eta_c$ and $t = h^{1/\delta}$ this vector field is a one variable holomorphic vector field $\theta$ in $t$. In the final final form, we have: $\theta = \frac{t^{q+1}}{1+\lambda t^q}\frac{d}{dt}$.

***Theorem 4.1***. — *The generators of* $\mathrm{Hol}(\mathcal{F}_N, \mathcal{F}_h)$ *are given by*

$$h_{\Gamma_i}(p) = \exp[T_i]\theta \cdot p$$

*where the* $T_i$*'s are the periods of* $\eta_c$ *on the horizontal cycles* $\Gamma_i$*.*

We can remark that this holonomy is an abelian group. This is coherent, since for any solvable group $G$ of germs of diffeomorphisms, the subgroup of its elements which are tangent to the identity is always an abelian one (see [**7**] or [**20**]).

*Proof*. — The foliation $\mathcal{F}_N$ is also defined by the vector field

$$\alpha X_h + \left(\sum_{i=1}^{\mu} c_i f_i\right)\delta(h)R$$

or by $\frac{\alpha X_h}{f_c} + \delta(h)R$ with $f_c = \sum_{i=1}^{\mu} c_i f_i$. The key point here is that the vector field $\frac{\alpha X_h}{f_c}$ commutes with $\delta(h)R$. Indeed, $[\alpha X_h, R] = 0$, $f_c$ is a first integral for $R$ and $\delta(h)$ a first integral for $X_h$. Therefore we have:

$$\exp[\sigma]\left(\frac{\alpha X_h}{f_c} + \delta(h)R\right)\cdot p = \exp[\sigma]\delta(h)R \,\circ\, \exp[\sigma]\frac{\alpha X_h}{f_c}\cdot p.$$

If $\sigma$ runs over a segment $[0, T]$ in $\mathbb{C}$, the first member is a path with origin $p$ into the leaf of $\mathcal{F}_N$ through $p$. Likewise, the term $\exp[\sigma]\frac{\alpha X_h}{f_c}\cdot p$ describes a path of origin $p$ into a leaf of the initial foliation $\mathcal{F}_h$ and $\exp[\sigma]\delta(h)R \cdot q$ is a path into a fiber of the vertical foliation defined by $R$. Therefore, the first member defines a lift in $\mathcal{F}_N$ of a path in the initial foliation by the projection defined by $R$. If this path is closed for $\sigma = T$, $\exp[T]\delta(h)R \cdot p$ is its relative holonomy. Since $\eta_c(\frac{\alpha X_h}{f_c}) = 1$, in the (multivalued) coordinate $s = \int \eta_c$ such a flow is a translation, and for the periods $T_i$ of $\eta_c$, the segments $[0, T_i]$ are covering of a basis $\gamma_i$. This proves the theorem. □

This allows us to characterize the exceptional foliations (i.e. those which have a monogeneous relative holonomy group) on their normal form:

***Corollary 4.2***. — $\mathcal{F}_N$ *is an exceptional foliation if and only if the quotients of the periods* $T_i$ *are rational numbers.*

Finally, we can deduce from theorem (4.1) the following realization theorem:

***Theorem 4.3.*** — *We fix the quasi-homogeneous curve $X : h = 0$ with Milnor number $\mu$. Given a non exceptional abelian sub-group $H$ of* $\mathrm{Diff}_1(\mathbb{C}, 0)$ *generated by $h_1, \cdots h_\mu$, there exists a germ of quasi-homogeneous foliation whose relative holonomy group* $\mathrm{Hol}(\mathcal{F}, \mathcal{F}_h)$ *is $H$.*

*Proof.* — In order to construct the class of $\mathcal{F}$ we have to choose $\alpha X_h + (\sum_{i=1}^{\mu} c_i f_i)\delta(h)R$. Since $H$ is a non exceptional abelian subgroup of $\mathrm{Diff}_1(\mathbb{C}, 0)$, there exists an analytic vector field $\theta$ and a constants $T_i$ such that $h_i = \exp T_i \theta$ (see [**20**]). The vector field $\theta$ induces a unique vector field $\delta(h)R$ whose expression on each fiber of $R$ is $\theta$. We only have to choose the constants $c_i$ which will induce the given relative holonmy. The relationship between the constants $T_i$ and the $c_i$'s is given by

$$T_i = \int_{\Gamma_i} \eta_c = \sum_{j=1}^{\mu} c_j \int_{\Gamma_i} \eta_j.$$

It follows that one should have the matricial equality $T = M \cdot C$ where $T$ is the column of the $T_i$'s, $C$ is the column of the $c_i$'s and $M = (m_{i,j})$ with $m_{i,j} = \int_{\Gamma_i} \eta_j$. The coefficients of this matrix are constants since $\eta_j$ and $\Gamma_i$ are horizontal. Since the loops $\gamma_i$ generate a basis of the homology of the Milnor fiber and the $\eta_j$ a basis of its cohomology, it is a well known fact that this matrix is an inversible one. Therefore, we may compute $C$ from $T$. □

**Remark.** If we admit that the relative holonomy group is a complete invariant of the foliation, the previous result gives us another proof of the main implication (4) $\Rightarrow$ (3) in (3.7). Indeed, if $\mathcal{F}$ is a Liouvillian foliation, its relative holonomy group is a non exceptional abelian subgroup $H$ of $\mathrm{Diff}_1(\mathbb{C}, 0)$. We can realize it by another foliation given by a normalized vector field $X_N$, whose algoritmic invariant $\mathcal{L}(\mathcal{F}_N)$ is one dimensional. Since the two foliations are analytically equivalent, we have $\mathcal{L}(\mathcal{F}_N) = \mathcal{L}(\mathcal{F})$ up to a conjugacy, and $\mathcal{L}(\mathcal{F})$ is also one-dimensional.

## 5. Open problems

In the present class of quasi-homogeneous foliations, there remain the following questions:

- Find the relation between the algorithmic invariant $\mathcal{L}(\mathcal{F})$ and the geometric one (relative holonomy) outside the Galois reducible case. In the general case, this transcendental relation will not reduce to the exponential of one variable vector field. Probably, we shall have to consider Campbell-Hausdorff type formulae;
- Prove the $k$-summability for the final normal forms and find the geometric meaning of this order $k$.

One can try to extend such a study to any germ of foliation in $\mathbb{C}^2$:

- In the non dicritical case (i.e. when the exceptional divisor is an invariant set of the foliation), outside the quasi-homogeneous context, we have no normal forms. We would like to construct them, having in mind the present motivations: a good representative of an holomorphic foliation may allow us to compute its Galois envelope, and its geometric invariants. Of course, we agree divergent models in order to get the previous conditions, and we expect their summability.
- In the generic dicritical case (i.e. when the foliation is desingularized after one blowing up such that the projective line is not an invariant set), we have formal normal forms: see [**19**]. Can we make use of these models to compute their Galois envelope?

We can also consider the following developments:

- (suggested by B. Malgrange) study the Galois envelope for any local codimension one foliations: can we reduce it to the previous dimension two cases?
- study the Galois envelope of vector fields in $(\mathbb{C}^2, 0)$. This means that we first have to classify vector fields not only up to a unity, and to construct formal normal forms with respect to this classification.
- develop a similar study for an algebraic foliation on the projective plane near an algebraic invariant set.

## References

[1] M. Canalis-Durand & R. Schäfke – "Divergence and summability of normal forms of systems of differential equations with nilpotent linear part", *Ann. Fac. Sci. Toulouse Math. (6)* **13** (2004), p. 493–513.

[2] G. Casale – "Suites de Godbillon-Vey et intégrales premières", *C. R. Math. Acad. Sci. Paris* **335** (2002), p. 1003–1006.

[3] ———, "D-enveloppe d'un difféomorphisme de $(\mathbb{C}, 0)$", *Ann. Fac. Sci. Toulouse Math. (6)* **13** (2004), p. 515–538.

[4] ———, "Enveloppe galoisienne d'une application rationnelle de $\mathbb{P}^1$", *Publ. Mat.* **50** (2006), p. 191–202.

[5] ———, "Feuilletages singuliers de codimension un, groupoïde de Galois et intégrales premières", *Ann. Inst. Fourier (Grenoble)* **56** (2006), p. 735–779.

[6] ———, "Le groupoïde de Galois de $P_1$ et son irréductibilité", *Comment. Math. Helv.* **83** (2008), p. 471–519.

[7] D. Cerveau & R. Moussu – "Groupes d'automorphismes de $(\mathbf{C}, 0)$ et équations différentielles $ydy + \cdots = 0$", *Bull. Soc. Math. France* **116** (1988), p. 459–488 (1989).

[8] Y. Genzmer – "Déformations isoholonomiques de feuilletages et modules de germes de courbes", Ph.D. Thesis, Université Paul Sabatier, Toulouse, 2006.

[9] F. Loray – "Feuilletages holomorphes à holonomie résoluble", Ph.D. Thesis, Université de Rennes, 1994.

[10] ———, "Réduction formelle des singularités cuspidales de champs de vecteurs analytiques", *J. Differential Equations* **158** (1999), p. 152–173.

[11] F. Loray & R. Meziani – "Classification de certains feuilletages associés à un cusp", *Bol. Soc. Brasil. Mat. (N.S.)* **25** (1994), p. 93–106.

[12] B. Malgrange – "Le groupoïde de Galois d'un feuilletage", in *Essays on geometry and related topics, Vol. 1, 2*, Monogr. Enseign. Math., vol. 38, Enseignement Math., 2001, p. 465–501.

[13] ______, "On nonlinear differential Galois theory", *Chinese Ann. Math. Ser. B* **23** (2002), p. 219–226, Dedicated to the memory of Jacques-Louis Lions.

[14] ______, Notes manuscrites, 2000.

[15] J. Martinet & J.-P. Ramis – "Classification analytique des équations différentielles non linéaires résonnantes du premier ordre", *Ann. Sci. École Norm. Sup. (4)* **16** (1983), p. 571–621 (1984).

[16] J.-F. Mattei – "Quasi-homogénéité et équiréductibilité de feuilletages holomorphes en dimension deux", *Astérisque* **261** (2000), p. 253–276.

[17] J.-F. Mattei & R. Moussu – "Holonomie et intégrales premières", *Ann. Sci. École Norm. Sup. (4)* **13** (1980), p. 469–523.

[18] R. Moussu – "Holonomie évenescente des équations différentielles dégénerées transverses", in *Singularities and dynamical systems (Iráklion, 1983)*, North-Holland Math. Stud., vol. 103, North-Holland, 1985, p. 161–173.

[19] L. Ortiz-Bobadilla, E. Rosales-González & S. M. Voronin – "Rigidity theorems for generic holomorphic germs of dicritic foliations and vector fields in $(\mathbb{C}^2, 0)$", *Mosc. Math. J.* **5** (2005), p. 171–206.

[20] E. Paul – "Feuilletages holomorphes singuliers à holonomie résoluble", *J. reine angew. Math.* **514** (1999), p. 9–70.

[21] ______, "Formal normal forms for the perturbations of a quasi-homogeneous Hamiltonian vector field", *J. Dynam. Control Systems* **10** (2004), p. 545–575.

[22] A. Seidenberg – "Reduction of singularities of the differential equation $A\,dy = B\,dx$", *Amer. J. Math.* **90** (1968), p. 248–269.

[23] E. Stróżyna & H. Żoładek – "The analytic and formal normal form for the nilpotent singularity", *J. Differential Equations* **179** (2002), p. 479–537.

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# RIGIDITY OF FIBRATIONS

*by*

Jorge Vitório Pereira & Paulo Sad

*To José Manuel Aroca in occasion of his 60$^{th}$ birthday*

***Abstract.*** — We consider a set $\Gamma$ of points in the projective plane obtained as the intersection of two curves of the same degree. We blow-up the projective plane at that points to get $S_\Gamma$. We consider the foliation $\mathcal{F}_\gamma$ in $S_\Gamma$ obtained from the pencil of the two curves above. Under generic conditions $\mathcal{F}_\gamma$ is isolated in the space of foliations of $S_\Gamma$.

***Résumé* (Rigidité des fibrations).** — Nous considérons l'ensemble $\Gamma$ des points du plan projectif obtenu comme intersection de deux courbes du même degré. Nous éclatons cet ensemble pour obtenir la surface $S_\Gamma$ et nous considérons sur $S_\Gamma$ le feuilletage $\mathcal{F}_\gamma$ obtenu à partir du pinceau de deux courbes précédentes. Sous des conditions de généricité $\mathcal{F}_\gamma$ est isolé dans l'espace des feuilletages de $S_\Gamma$.

## 1. Introduction

Let $\Gamma$ be a finite set of points in the projective plane $\mathbb{P}^2$ defined as the intersection of two transverse curves of the same degree (we say that $\Gamma$ is a *complete intersection set*); let also $\pi : S_\Gamma \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the points of $\Gamma$. The surface $S_\Gamma$ admits a natural foliation $\mathcal{G}_\Gamma$: the strict transform of the pencil $\mathcal{F}_\Gamma : F dG - G dF = 0$ generated by the curves $\{F = 0\}$ and $\{G = 0\}$ that define $\Gamma$.

A natural problem is to understand the families of reduced foliations of surfaces (in the sense of [**2**]) containing $(S_\Gamma, \mathcal{G}_\Gamma)$; this is related to studying the foliations of $\mathbb{P}^2$, in a neighborhood of $\mathcal{F}_\Gamma$, that have radial singularities close to the points of $\Gamma$.

We consider in this paper the particular situation where the surface $S_\Gamma$ does not change in the family (or, equivalently, we look at the foliations of $\mathbb{P}^2$ with radial

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singularities at the points of $\Gamma$). The leaves of $\mathcal{G}_\Gamma$ are fibers of the holomorphic fibration $(F/G)\circ\pi \to \mathbb{P}^1$. In order to study a deformation $\mathcal{G}$ of this fibration (in the space of foliations of $S_\Gamma$) we analyze how a generic fiber $\widetilde{C}$ of $\mathcal{G}_\Gamma$ is intersected by t he leaves of $\mathcal{G}$. If $\widetilde{C}$ is not $\mathcal{G}$-invariant then $\mathcal{N}_{\mathcal{G}}.\widetilde{C} = \operatorname{tang}(\mathcal{G},\widetilde{C}) + \chi(\widetilde{C})$, where $\mathcal{N}_{\mathcal{G}}$ is the normal bundle of $\mathcal{G}$, $\chi(\widetilde{C})$ is the Euler characteristic of $\widetilde{C}$ and $\operatorname{tang}(\mathcal{G},\widetilde{C})$ is the number of tangency points between $\mathcal{G}$ and $\widetilde{C}$. Notice that $\operatorname{tang}(\mathcal{G},\widetilde{C}) \geq 0$ and also $\mathcal{N}_{\mathcal{G}}.\widetilde{C} = \mathcal{N}_{\mathcal{G}_\Gamma}.\widetilde{C}$ by continuity. Since $\widetilde{C}$ is $\mathcal{G}_\Gamma$-invariant, $\mathcal{N}_{\mathcal{G}_\Gamma}.\widetilde{C} = Z(\mathcal{G},\widetilde{C}) + \widetilde{C}.\widetilde{C}$, where $Z(\mathcal{G},\widetilde{C})$ denotes the number of singularities of $\mathcal{G}_\Gamma$ along $\widetilde{C}$, and we get that $\operatorname{tang}(\mathcal{G},\widetilde{C}) = -\chi(\widetilde{C})$.

Let $c \in \mathbb{N}$ be the common degree of the polynomials $F$ and $G$. When $c = 1$ or $c = 2$ we have $\chi(\widetilde{C}) = 2$ and we get a contradiction unless $\mathcal{G} = \mathcal{G}_\Gamma$ ([**12**]). When $c = 3$ we have $\operatorname{tang}(\mathcal{G},\widetilde{C}) = -\chi(\widetilde{C}) = 0$ and therefore $\mathcal{G}$ is transverse to the generic fiber of $\mathcal{G}_\Gamma$, implying that the regular fibers are all isomorphic; this is not possible for a generic choice of $F$ and $G$, and we conclude again that $\mathcal{G} = \mathcal{G}_\Gamma$ in this case (see [**11**] for a related result). When $c \geq 4$ this type of argument fails since $\chi(\widetilde{C}) < 0$. Nevertheless we are able to prove for $c \geq 3$:

***Theorem 1***. — *If $\Gamma$ is a generic complete intersection set then $\mathcal{G}_\Gamma$ is an isolated point of the space of foliations of $S_\Gamma$, i.e., $\mathcal{G}_\Gamma$ is* rigid.

In the statement *generic complete intersection set* refers to the set of base points of a generic element of the space of lines of $\mathbb{P}\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(c))$; in other words, the couple $(F, G)$ of polynomials of degree $c \in \mathbb{N}$ is generically chosen in order to define $\Gamma$. In §3.2 we exhibit some examples of non-rigidity to show that the hypothesis of genericity is necessary.

We have no result when the surface $S_\Gamma$ changes in the family of reduced foliations; but still we should mention that for $c = 3$ we can only deform $\mathcal{G}_\Gamma$ as a fibration (starting with a generic choice of $\Gamma$). In fact, $\mathcal{G}_\Gamma$ has Kodaira dimension equal to 1 and this dimension is constant along the family ([**2**]). We then apply the Classification Theorem ([**1**]) to conclude that any foliation in the family is an elliptic fibration.

The proof of Theorem 1 relies on the analysis of the indexes of a plane foliation along a smooth invariant algebraic curve. Let $\{F = 0\}$ be such a curve, of degree $c \in \mathbb{N}$, containing singularities of the foliation at the intersection points with another curve $\{G = 0\}$ of degree $k \leq c$. We prove then that if $(F, G)$ is generically chosen the set of indexes is sufficient to identify completely the foliation (Theorem 2.2). Application of this result in order to prove Theorem 1 is not immediate; we have to show first that the defining curves for the set $\Gamma$ are invariant curves of the foliation.

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## 2. Variation of Indexes

**2.1. Division Lemma.** — All foliations, unless stated otherwise, are supposed to have isolated singularities.

Let $C \subset \mathbb{P}^2$ be a smooth curve of degree $c \in \mathbb{N}$, invariant by a plane projective foliation $\mathcal{F} \in Fol(d)$ of degree $d \in \mathbb{N}$. The Lemma below can be implicitly found in [**4**, Proof of Proposition 3]; we assume that $\mathcal{F}$ is defined by $\omega = 0$, $\omega$ a homogeneous 1-form of $\mathbb{C}^3$ of degree $d+1$ (or by a homogeneous vector field of $\mathbb{C}^3$ of degree $d \in \mathbb{N}$), and that $C$ is defined by $F = 0$, $F$ a homogeneous polynomial of degree $c \in \mathbb{N}$. Let us denote by $R$ the radial vector field of $\mathbb{C}^3$.

***Lemma 2.1.*** — *There exist a polynomial $G$ of degree $d-c+2$ and a 1-form $\eta$ of degree $d-c+1$, both homogeneous, such that*

$$\omega = GdF - \frac{\deg(F)}{\deg(G)} FdG + F\eta \quad \textit{and} \quad i_R(\eta) = 0.$$

*Furthermore, the foliation $\mathcal{F}_\eta$ defined by $\eta = 0$ depends only on $\mathcal{F}$ and $C$ when $d \leq 2c-2$.*

*Proof.* — It follows from ([**4**, Proposition 1]) that there exist a homogeneous polynomial $G$ of degree $d-c+2$ and a homogeneous 1-form $\alpha$ of degree $d-c+1$ such that

$$\omega = GdF + F\alpha\,. \tag{1}$$

After contracting the above expression with the radial vector field we obtain

$$\deg(F)FG + Fi_R\alpha = 0\,,$$

We rewrite (1) as

$$\omega = GdF - \frac{\deg(F)}{\deg(G)} FdG + F\left(\alpha + \frac{\deg(F)}{\deg(G)} dG\right)\,.$$

and define $\eta := \alpha + \frac{\deg(F)}{\deg(G)} dG$; it follows that $i_R(\eta) = 0$.

Let us replace (1) by $\omega' = G'dF' + F'\alpha'$, where $\omega' = \lambda\omega$ and $F' = \mu F$ for $\lambda, \mu \in \mathbb{C}$. Consequently:

$$\omega = \left(\frac{\mu}{\lambda} G'\right) dF + F\left(\frac{\mu}{\lambda}\alpha'\right) = GdF + F\alpha$$

and

$$\left(\frac{\mu}{\lambda} G' - G\right) dF = F\left(\alpha - \frac{\mu}{\lambda}\alpha'\right)\,.$$

From $\left(\frac{\mu}{\lambda} G' - G\right)_{|C} \equiv 0$ we have $\frac{\mu}{\lambda} G' - G = P.F$ for some homogeneous polynomial $P$; two possibilities arise:

- $d < 2c-2$; therefore $\frac{\mu}{\lambda} G' = G$ , $\frac{\mu}{\lambda}\alpha' = \alpha$ and we get

$$\eta' = \alpha' + \frac{\deg(F)}{\deg(G)} dG' = \frac{\mu}{\lambda}\eta.$$

- $d = 2c - 2$, so that $\frac{\mu}{\lambda}G' - G = aF$ , $\alpha - \frac{\mu}{\lambda}\alpha' = adF$ for $a \in \mathbb{C}$. It follows that $\alpha - \frac{\mu}{\lambda}\alpha' = \frac{\mu}{\lambda}dG' - dG$ and again $\eta' = \frac{\mu}{\lambda}\eta$.

$\square$

We observe that $\mathcal{F}_\eta$ *may have a curve of singularities.*

Our results follow from the analysis of the behavior of $\mathcal{F}_\eta$ with respect to $C$ when $d \leq 2c - 2$. For the moment we remark that:

- the singularities of $\mathcal{F}$ contained in $C$ are the points of $\{G = 0\} \cap C$.
- $C$ is not contained in the singular set of $\mathcal{F}_\eta$ (because $\deg(\eta) < \deg(F)$).
- $C$ is not $\mathcal{F}_\eta$-invariant (because otherwise $\deg(C) \leq \deg(\mathcal{F}_\eta) + 1$, see [**4**], or $c \leq d - c + 1$). Let us write $k = \deg(G) = d - c + 2$ for simplicity, so that $\deg(\mathcal{F}_\eta) = k-2$. Since $\operatorname{tang}(\mathcal{F}_\eta, C) = \mathcal{N}_{\mathcal{F}_\eta}.C - \chi(C) = k.c - \left(2 - 2\frac{(c-1)(c-2)}{2}\right)$, we find $\operatorname{tang}(\mathcal{F}_\eta, C) = c(k + c - 3)$; the tangency points between $C$ and $\mathcal{F}_\eta$ are given by the common solutions of $F = 0$ and $dF(Z_\eta) = 0$ ($Z_\eta$ is the homogeneous vector field of $\mathbb{C}^3$ of degree $k - 2$ which defines $\mathcal{F}_\eta$).

**2.2. Indexes and Foliations.** — In [**13**] we have proved the existence of foliations of sufficiently high degree with prescribed linear holonomy group with respect to a given curve. Here we will consider the opposite situation when the degree of the curve is comparable to the degree of the foliation. More precisely we will consider foliations of degree $d \in \mathbb{N}$ which have an invariant smooth curve of degree $c \in \mathbb{N}$ such that $d \leq 2c - 2$ (remark that in all cases $c \leq d + 1$). This inequality is equivalent to $Z(\mathcal{F}, C) \leq c^2$. As already pointed out it implies that the decomposition given by Lemma 2.1 is essentially unique.

Let us take a pair of transverse algebraic curves $C = \{F = 0\}$ and $E$ defined by polynomials of degree $c \in \mathbb{N}$ and $k \in \mathbb{N}$ respectively; $C$ is supposed to be a smooth curve and $F$ a reduced polynomial. Denote by $Fol_{C,C\cap E}(d)$ the space of foliations of degree $d = c+k-2$ which leave $C$ invariant and have $C \cap E$ as the singular set along $C$. We define the Index Map $\mathcal{I}(C, E) = \mathcal{I}$ as

$$\begin{array}{rcl} \mathcal{I} : \mathbb{F}ol_{C,C\cap E}(d) & \rightarrow & \mathcal{A}(C \cap E, \mathbb{C}) \\ \mathcal{F} & \mapsto & (p \mapsto i(\mathcal{F}, C, p)) \end{array}$$

where $\mathcal{A}(C \cap E, \mathbb{C})$ is the space of maps from $\Gamma$ to $\mathbb{C}$ and $i(\mathcal{F}, C, p)$ is in the index of $\mathcal{F}$ with respect to $C$ at the point $p$ (cf. [**3**]).

According to Lemma 2.1 , a foliation $\mathcal{F} \in Fol_{C,C\cap E}(d)$ is defined by a 1-form $\omega = GdF - (c/k)FdG + F\eta = 0$; we may assume that $E = \{G = 0\}$. A simple computation shows that

$$i(\mathcal{F}, C, p) = \frac{c}{k} - \operatorname{Res}\left(\left(\frac{\eta}{G}\right)_{|C}, p\right) , \tag{2}$$

where $\left(\frac{\eta}{G}\right)_{|C}$ means $i^*\left(\frac{\eta}{G}\right)$ for the inclusion $i : C \to \mathbb{P}^2$.

When $C$ and $E$ are transverse to each other at $p \in C \cap E$, we have

$$i(\mathcal{F}, C, p) = \frac{c}{k} \Leftrightarrow i^*\eta\,(p) = 0 \tag{3}$$

The equality $i^*\eta\,(p) = 0$ means that $\mathcal{F}_\eta$ is tangent to $C$ at $p$.

If the foliation is the pencil $\mathcal{F}_\Gamma : GdF - (c/k)FdG = 0$, all $k.c$ indexes at the points of $C \cap E$ are equal to $c/k$; a natural question to ask is whether the converse is true. This is not always the case (see **[14]** for a counterexample). Before stating the main result of this Section, we need a lemma; set $S_l = \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(l))$ and $\mathbb{S}_l = \mathbb{P}\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(l)))$ for $l > 0$.

***Lemma 2.2***. — *Let $c \geq k$. There exists a Zariski open subset $\mathcal{U}_0(c, k) \subset \mathbb{S}_c \times \mathbb{S}_k$ such that if $(C, E) \in \mathcal{U}_0(c, k)$ then $C$ and $E$ are transverse to each other and no foliation of degree $k - 2$ is tangent to $C$ at the points of $C \cap E$.*

*Proof*. — Let $X_h(n)$ be the set of homogeneous vector fields of $\mathbb{C}^3$ of degree $n$, and $H$ the set

$$\{(F, G) \in S_c \times S_k; \exists (Z, A, B) \in X_h(k-2) \times S_{k-3} \times S_{c-3}; dF(Z) = A.F + B.G\}$$

Then $H$ is an algebraic subvariety of $S_c \times S_k$. Let us show that $H$ is a strict subvariety. For that we take $F_0 \in S_c$ as the equation of a plane rational curve of degree $c$ with nodal singularities and $G_0$ defining a plane curve of degree $k$ which is transverse to $\{F_0 = 0\}$. We know from the genus formula that $\{F_0 = 0\}$ has $\frac{(c-1).(c-2)}{2}$ nodal singularities. If $(F_0, G_0) \in H$, one has $Df_0(Z_0) = A_0.F_0 + B_0.G_0$ for a $(Z_0, A_0, B_0) \in X_h(k-2) \times S_{c-3} \times S_{k-3}$. Let us compute the number of intersection points between $\{dF_0(Z_0) = 0\}$ and $\{F_0 = 0\}$:

- $k.c$ points of $\{F_0 = 0\} \cap \{G_0 = 0\}$, which are smooth points of $\{F_0 = 0\}$.
- $(c-1)(c-2)$ points corresponding to the nodal singularities of $\{F_0 = 0\}$

We have then $k \cdot c + (c-1) \cdot (c-2) = (k + c - 3) \cdot c$, contradiction.

Let now $\mathcal{U}(c, k)$ be the open subset of $S_c \times S_k$ of pairs of curves $(C, E)$ such that $C$ and $E$ are transverse to each other; finally we set $\mathcal{U}_0(c, k) = \mathcal{U}(c, k) \cap (S_c \times S_k \setminus H)$. Consider $(\bar{C}, \bar{E}) = (\{\bar{F} = 0\}, \{\bar{G} = 0\}) \in \mathcal{U}_0(c, k)$ and suppose that $d\bar{F}(\bar{Z})(p) = 0$ at all points in $\bar{C} \cap \bar{E}$ for some $\bar{Z} \in X_h(k-2)$. By Noether's Theorem (**[6]**), $d\bar{F}(\bar{Z}) = \bar{A}.\bar{F} + \bar{B}.\bar{G}$ for some $(\bar{A}, \bar{B}) \in S_{c-3} \times S_{k-3}$, so that $(\bar{C}, \bar{E}) \in H$, contradiction unless $\bar{Z}$=0. □

**Remark.** — The argument above is inspired in Severi's idea to prove the Brill-Noether Theorem (**[8]**, p. 240–244).

We have as a consequence:

***Theorem 2***. — *Let $c \geq k$. There exists a Zariski open subset $\mathcal{U}_1(c, k) \subset \mathbb{S}_c \times \mathbb{S}_k$ such that if $(C, E) \in \mathcal{U}_1(c, k)$ then $C$ is smooth, $C$ and $E$ are transverse to each other and $\mathcal{I}(C, E)$ is injective.*

*Proof.* — It is enough to define $\mathscr{U}_1(c,k) \subset \mathscr{U}(c,k)$ (obtained in Lemma 2.2) as the set of pairs $(C,E) \in \mathscr{U}(c,k)$ such that $C$ is a smooth curve, and use (3). □

Before proceeding let us take a closer look at the case $E$ is a *conic*.

***Example 2.1.*** — When $E$ is a conic then $\eta$ induces a degree 0 foliation of $\mathbb{P}^2$. These foliations are pencils of lines and, as such, are completely determined by the base point of the pencil.

If $\eta_p$ is the degree 0 foliation corresponding to the pencil of lines through $p = [a : b : c] \in \mathbb{P}^2$ then the tangency points between $\mathscr{F}_{\eta_p}$ and $C$ are the points of intersection of $C$ with its polar curve centered at $p$, i.e.,

$$T_qC \subset \ker \eta_p(q) \iff \left( a\frac{\partial F}{\partial x} + b\frac{\partial F}{\partial y} + c\frac{\partial F}{\partial z} \right)(q) = 0\,,$$

where $F$ is an irreducible polynomial defining $C$. It follows from Noether's Theorem that the map $\mathscr{I}(C,E)$ is not injective if, and only if, there exists $[a : b : c] \in \mathbb{P}^2$ such that

$$G \text{ divides } \left( a\frac{\partial F}{\partial x} + b\frac{\partial F}{\partial y} + c\frac{\partial F}{\partial z} \right)\,,$$

where $G$ is quadratic polynomial which defines $E$.

When $C$ is a conic then this never happens since a polar curve of $C$ has degree 1. Thus for any $C$ and any $E$ the map $\mathscr{I}(C,E)$ is always injective.

When $C$ is a cubic then the map $\mathscr{I}(C,E)$ is not injective if, and only if, $E$ is a polar curve of $C$.

Suppose now that $C$ is a quartic; let us identify the set of polar curves of $C$ with a projective plane $\Lambda$, linearly embedded in the projective space $\mathbb{S}_3$. If $\mathscr{I}(C,E)$ is not injective then there exists a point in $\Lambda$ intersecting $W_{1,2}$, the image of the multiplication map $\mathbb{S}_1 \times \mathbb{S}_2 \to \mathbb{S}_3$. Since $W_{1,2}$ has codimension 2 in $\mathbb{S}_3$ then a generic $\Lambda$ will intersect $W_{1,2}$ is a finite set of points. Moreover it can be easily verified that $W_{1,2}$ is a linear projection of the Segre Variety $S_{2,5} \subset \mathbb{P}(\mathbb{C}^3 \otimes \mathbb{C}^6) \cong \mathbb{P}(S_1 \otimes S_2)$ to $\mathbb{S}_3$ from a center that does not intersect $S_{2,5}$. Since the degree of $S_{2,5}$ (cf. [**7**, page 233]) is $\binom{5+2}{2} = 21$ then the degree of $W_{1,2}$ is also 21. Thus a generic $\Lambda$ will intersect $W_{1,2}$ in 21 points counted with multiplicity. Translating to our situation we obtain that for a generic $C \in \mathbb{S}_4$ the cardinality of

$$\{E \in \mathbb{S}_2 \mid \mathscr{I}(C,E) \text{ is not injective}\}$$

is 21.

For $C$ of degree at least 5 we can argue as follows. Let $\Delta = \mathbb{P}(\mathbb{C}\frac{\partial}{\partial x} \oplus \mathbb{C}\frac{\partial}{\partial y} \oplus \mathbb{C}\frac{\partial}{\partial z})$ and $\Sigma \subset \mathbb{S}_{c-3} \times \mathbb{S}_2 \times \Delta \times \mathbb{S}_c$ be defined by the relation

$$([B],[G],[\partial],[F]) \in \Sigma \iff [\partial(F)] = [B \cdot G]$$

Notice that every $\partial \in \mathbb{C}\frac{\partial}{\partial x} \oplus \mathbb{C}\frac{\partial}{\partial y} \oplus \mathbb{C}\frac{\partial}{\partial z}$ acting as a derivation induces a surjective linear map $\partial : S_c \to S_{c-1}$. Thus if $\pi_1 : \Sigma \to \mathbb{S}_{c-3} \times \mathbb{S}_2 \times \Delta$ is the natural projection to $\mathbb{S}_{c-3} \times \mathbb{S}_2 \times \Delta$ then $\pi_1$ induces a structure of $\mathbb{P}^{c+1}$-bundle on $\Sigma$. In particular

$$\dim \Sigma = c + 8 + \frac{c(c-3)}{2}.$$

Since $c \geq 5$, $\dim \Sigma < \dim \mathbb{S}_c$ and consequently $U = \mathbb{S}_c \setminus \pi_2(\Sigma) \neq \varnothing$ where $\pi_2 : \Sigma \to \mathbb{S}_c$ is the natural projection to $\mathbb{S}_c$.

We conclude that for every $C \in U$ and every $E \in \mathbb{S}_2$ the map $\mathscr{I}(C, E)$ is injective.

We summarize the discussion above in the following table.

| *degree of* **C** | *type of* **C** | $\{E \in \mathbb{S}_k \mid \mathscr{I}(C,E)$ is not injective $\}$ |
|---|---|---|
| 2 | arbitrary | empty |
| 3 | arbitrary | $\{aF_x + bF_y + cF_z = 0\}_{[a:b:c]\in\mathbb{P}^2}$ |
| 4 | generic | finite with 21 elements |
| $\geq 5$ | generic | empty |

## 3. The Rigidity of a generic $\mathscr{F}_\Gamma$: Proof of Theorem 1

The proof of Theorem 1 will follow from the above results. We start with a simple lemma:

***Lemma 3.1***. — *Let $\Gamma$ be the intersection of two transversal curves $C = \{F = 0\}$ and $E = \{G = 0\}$ of degree $c$ and $k$ respectively and let $\mathscr{F}$ be a holomorphic foliation of degree $c + k - 2$ with singular set containing $\Gamma$. Then $\mathscr{F}$ is induced by a 1-form*

$$\omega = GdF - \frac{\deg(F)}{\deg(G)} FdG + F\alpha + G\beta,$$

*where $\alpha$ and $\beta$ are homogenous 1-form satisying $i_R\alpha = i_R\beta \equiv 0$. In particular $\alpha$ and $\beta$ define foliations of $\mathbb{P}^2$ of degrees $k - 2$ and $c - 2$ respectively.*

*Proof*. — A direct application of Noether's Theorem ([**6**]) gives that $\mathscr{F}$ is induced by a 1-form $\omega = F\alpha_0 + G\beta_0$. Thus $i_R\omega = 0$ implies that $Fi_R\alpha_0 = -Gi_R\beta_0$. To conclude it is sufficient to take $\alpha = \alpha_0 + \frac{\deg(F)}{\deg(G)} FdG$ and $\beta = \beta_0 + (-GdF)$. □

Let $C = \{F = 0\}$ and $E = \{G = 0\}$ be transverse curves of degree $c$ in $\mathbb{P}^2$ and let $\Gamma = C \cap E$. Recall from the introduction that we denote by $\pi : S_\Gamma \to \mathbb{P}^2$ the blow-up of $\mathbb{P}^2$ at the points of $\Gamma$ and by $\mathscr{G}_\Gamma$ the strict transform of the foliation $\mathscr{F}_\Gamma$ induced by $FdG - GdF = 0$. If $\mathscr{G}$ is a foliation *close* to $\mathscr{G}_\Gamma$ then both $\mathscr{G}$ and $\mathscr{G}_\Gamma$ are transversal to the exceptional divisor of $S_\Gamma$. Thus $\mathscr{F} = \pi_*\mathscr{G}$ is a foliation of $\mathbb{P}^2$ with radial singularities on $\Gamma$. Let now $\mathscr{U}_2(c, c) \subset \mathscr{U}_1(c, c) \subset \mathscr{U}_0(c, c)$ be the Zariski open subset of $\mathbb{S}_c \times \mathbb{S}_c$ with the property that if $(C, E) \in \mathscr{U}_1(c, c)$ then both $C$ and $E$ are smooth curves.

Using Lemma 3.1 we see that $\mathcal{F}$ is induced by

$$\omega = GdF - FdG + F\alpha + G\beta$$

where both $\alpha$ and $\beta$ induce foliations of degree $c-2$. We write this equality using homogeneous vector fields in $\mathbb{C}^3$:

$$Z = Z_\Gamma + F.Z_\alpha + G.Z_\beta$$

where $Z$ defines $\mathcal{F}$, $Z_\Gamma$ defines the pencil $\mathcal{F}_\Gamma$ and $Z_\alpha$, $Z_\beta$ define the foliations associated to $\alpha = 0$ and $\beta = 0$, respectively.

We claim that $\beta = 0$, that is, $C$ is $\mathcal{F}$-invariant. If not, we observe that $\operatorname{tang}(\mathcal{F}, C, p) \geq 2$ at any point $p \in \Gamma$; therefore the points of $\Gamma$ contribute at least $2c^2$ to $\operatorname{tang}(\mathcal{F}, C)$. The points of tangency between $\mathcal{F}$ and $C$ are the common solutions to $dF(Z) = 0$ and $F = 0$, or $G.dF(Z_\beta) = 0$ and $F = 0$. Since we have already $c^2$ solutions to $G = 0$ and $F = 0$, it follows that the points of $\Gamma$ are also solutions to $dF(Z_\beta) = 0$ and $F = 0$. Consequently $\mathcal{F}_\beta$ is tangent to $C$ along the points of $\Gamma$, which is impossible since $(C, E) \in \mathcal{U}_2(c,c) \subset \mathcal{U}_0(c,c)$ (Lemma 2.2). Therefore $\beta = 0$.

Finally we may apply Theorem 2 to $(C, E) \in \mathcal{U}_2(c,c) \subset \mathcal{U}_1(c,c)$ to conclude that $\alpha = 0$. This concludes the proof of Theorem 1.

**Remarks.** — When $c > 3$ we do not really understand for which pair of curves the conclusion of the Theorem holds. For instance we do not know if the conclusion holds if we suppose that the pencil is a Lefschetz pencil, i.e., all singularities have multiplicity one and every element of the pencil has at most one singularity.

Theorem 1 is also true for generic complete intersection sets defined as the the intersection of curves $\{F = 0\}$ and $\{G = 0\}$ of degrees $k$ and $c$ with $k < c$; the fibration which is the desingularisation of $GdF - \frac{c}{k}FdG = 0$ is rigid. The same proof as above applies with minor modifications.

### 3.1. Fermat Curves and Non-Rigid Foliations.

— In order to conclude we exhibit below a family of examples showing that Theorem 1 does not hold for arbitrary $\Gamma$ when $c \geq 3$.

***Example 3.1.*** — *For every $c \geq 3$ there exists a complete intersection $\Gamma \subset \mathbb{P}^2$ of degree $c^2$ such that $\mathcal{F}_\Gamma$ is not rigid.*

*Proof.* — If $C = \{x^c - y^c = 0\}$ and $E = \{y^c - z^c = 0\}$ then the pencil generated by $C$ and $E$ is a pencil whose generic element is isomorphic to the Fermat curve of degree $c$ and three singular elements: $C$, $E$ and $\{x^c - z^c = 0\}$. Let $\omega_{2c-2}$ be a 1-form which defines the associated foliation.

The pencil generated by $\{x^c(y^c - z^c) = 0\}$ and $\{y^c(x^c - z^c) = 0\}$ defines a foliation of degree

$$c + 1 = \underbrace{4c - 2}_{\text{is a pencil of degree } 2c \text{ curves}} - \underbrace{3(c - 1)}_{\text{with 3 singular fibers of degree 1 and multiplicity } c}$$

and has radial singularities at $\Gamma = C \cap E$. Denote by $\eta_{c+1}$ the 1-form which induces this foliation. Thus for arbitrary $P_{c-3} \in S_{c-3}$ the strict transform of the foliation associated to the 1-form

$$\omega_{2c-2} + P_{c-3}\eta_{c+1}$$

is a deformation of $\mathscr{F}_\Gamma$. □

## References


[1] M. Brunella – *Birational geometry of foliations*, Monografías de Matemática, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2000.

[2] ______, "Invariance par déformations de la dimension de Kodaira d'un feuilletage sur une surface", in *Essays on geometry and related topics, Vol. 1, 2*, Monogr. Enseign. Math., vol. 38, Enseignement Math., 2001, p. 113–132.

[3] C. Camacho & P. Sad – "Invariant varieties through singularities of holomorphic vector fields", *Ann. of Math. (2)* **115** (1982), p. 579–595.

[4] D. Cerveau & A. Lins Neto – "Holomorphic foliations in **C**P(2) having an invariant algebraic curve", *Ann. Inst. Fourier (Grenoble)* **41** (1991), p. 883–903.

[5] S. C. Coutinho & J. V. Pereira – "On the density of foliations without algebraic invariant sets", *J. reine angew. Math.* **557** (2003), p. 117–135.

[6] W. Fulton – *Algebraic curves*, Mathematics Lecture Note Series, Benjamin, 1969.

[7] J. Harris – *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer, 1992.

[8] J. Harris & I. Morrison – *Moduli of curves*, Graduate Texts in Mathematics, vol. 187, Springer, 1998.

[9] J. P. Jouanolou – *Équations de Pfaff algébriques*, Lecture Notes in Mathematics, vol. 708, Springer, 1979.

[10] A. Lins Neto – "Foliations on $\mathbb{C}$P(2) with invariant algebraic curve and radial singularities", *Rev. Semin. Iberoam. Mat. Singul. Tordesillas* **2** (1998), p. 29–35.

[11] L. G. Mendes & P. Sad – "On dicritical foliations and Halphen pencils", *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **1** (2002), p. 93–109.

[12] L. G. Mendes & M. Sebastiani – "Sur la densité des systèmes de Pfaff sans solution algébrique", *Ann. Inst. Fourier (Grenoble)* **44** (1994), p. 271–276.

[13] J. V. Pereira & P. Sad – "On the holonomy group of algebraic curves invariant by holomorphic foliations", *Ann. Mat. Pura Appl. (4)* **185** (2006), p. 257–271.

[14] P. Sad – "Regular foliations along curves", *Ann. Fac. Sci. Toulouse Math. (6)* **8** (1999), p. 661–675.



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# THE $q$-ANALOGUE OF THE WILD FUNDAMENTAL GROUP (II)

*by*

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***Abstract.*** — In a previous paper, we defined $q$-analogues of alien derivations and stated their basic properties. In this paper, we prove the density theorem and the freeness theorem announced there.

***Résumé* (Le $q$-analogue du groupe fondamental sauvage (II)).** — Dans un article précédent nous avons défini les $q$-analogues des dérivations étrangères et leurs propriétés de base. Dans cet article nous démontrons le théorème de densité et d'indépendance que nous y avions annoncé.

## 1. Introduction

**1.1. The problem.** — In this paper we shall continue the study of the local meromorphic classification of $q$-difference modules. In [**10**] we gave such a classification in Birkhoff style, using normal forms and index theorems; this classification is complete in the "integral slope case". (One could extend it to the general case using some results of [**3**].)

In [**6**] we introduced a new approach of the classification, using a "fundamental group" and its finite dimensional representations, in the style of the Riemann-Hilbert correspondence for linear differential equations. At some abstract level, such a classification is well known: the fundamental group is the tannakian Galois group of the tannakian category of local meromorphic $q$-modules. But we wanted more information: our essential aim was to get a *smaller* fundamental group which is Zariski dense in the tannakian Galois group and to describe it *explicitly*, in the spirit of what was done by the first author for the differential case [**5**].

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In [**6**] we built a family of elements of the Lie algebra of the tannakian group, the $q$-alien derivations, we achieved our program for the one-level case and we announced the main results in the general case. The aim of the present paper is to give some proofs omitted in [**6**] for the multi-level case. We will finally give a more precise algebraic formulation of our results in [**7**], which will end the series.

**1.2. Contents of the paper.** — General notations and conventions are explained in the next paragraph 1.3. In section 2, we recall basic properties of the category $\mathcal{E}_1^{(0)}$ of linear analytic $q$-difference equations with integral slopes, and the structure and action of its Galois group $G_1^{(0)}$. In section 3, we recall the unipotent structure of the Stokes subgroup $\mathfrak{St}$ of $G_1^{(0)}$, and the construction (taken from [**6**]) of elements of the Lie algebra $\mathfrak{st}$ of $\mathfrak{St}$, the so-called *$q$-alien derivations*. Our "$q$-analogue of the wild fundamental group" is the Lie subalgebra they generate. We then prove in 3.2 and 3.3 our main results: density and a freeness property of the $q$-alien derivations. In section 4, we summarize what remains to be solved, and will be the contents of [**7**].

The paper is written so as to be read widely independently of [**6**] - granted the reader is willing to take on faith some key points. The principle of the proofs is almost purely tannakian, but we have stated explicitly the underlying methods and prerequisites. Moreover, they are described in a concrete, computational form (with a systematic use of matrices). Since neither $q$-difference equations, nor even tannakian methods are so popular, this may help the reader to get familiarized with either domain. Note that, since we heavily rely on transcendental tools, the methods here are, to a large extent, independent of those of M. van der Put and his coauthors.

**1.3. General notations.** — The notations are the same as in [**6**]. Here are the most useful ones.

We let $q \in \mathbf{C}$ be a complex number with modulus $|q| > 1$. We write $\sigma_q$ the $q$-dilatation operator, so that, for any map $f$ on an adequate domain in $\mathbf{C}$, one has: $\sigma_q f(z) = f(qz)$. Thus, $\sigma_q$ defines a ring automorphism in each of the following rings: $\mathbf{C}\{z\}$ (convergent power series), $\mathbf{C}[[z]]$ (formal power series), $\mathcal{O}(\mathbf{C}^*)$ (holomorphic functions over $\mathbf{C}^*$), $\mathcal{O}(\mathbf{C}^*, 0)$ (germs at 0 of elements of $\mathcal{O}(\mathbf{C}^*)$). Likewise, $\sigma_q$ defines a field automorphism in each of their fields of fractions: $\mathbf{C}(\{z\})$ (convergent Laurent series), $\mathbf{C}((z))$ (formal Laurent series over), $\mathcal{M}(\mathbf{C}^*)$ (meromorphic functions over $\mathbf{C}^*$), $\mathcal{M}(\mathbf{C}^*, 0)$ (germs at 0 of elements of $\mathcal{M}(\mathbf{C}^*)$)

The $\sigma_q$-invariants elements of $\mathcal{M}(\mathbf{C}^*)$ can be considered as meromorphic functions on the quotient Riemann surface $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$. Through the mapping $x \mapsto z = e^{2i\pi x}$, the latter is identified to the complex torus $\mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau)$, where $q = e^{2i\pi\tau}$. Accordingly, we shall identify the fields $\mathcal{M}(\mathbf{C}^*)^{\sigma_q}$ and $\mathcal{M}(\mathbf{E}_q)$. We shall write $a \mapsto \overline{a}$ the canonical projection map $\pi : \mathbf{C}^* \to \mathbf{E}_q$ and $[c; q] = \pi^{-1}(\overline{c}) = cq^{\mathbf{Z}}$ (a discrete logarithmic $q$-spiral).

Last, we shall have use for the function $\theta \in \mathcal{O}(\mathbf{C}^*)$, a Jacobi Theta function such that $\sigma_q\theta = z\theta$ and $\theta$ has simple zeroes along $[-1; q]$. One then puts $\theta_c(z) = \theta(z/c)$, so that $\theta_c \in \mathcal{O}(\mathbf{C}^*)$ satisfies $\sigma_q\theta_c = (z/c)\theta_c$ and $\theta_c$ has simple zeroes along $[-c; q]$.

## 2. Linear analytic $q$-difference equations

A linear analytic (resp. formal) $q$-difference equation (implicitly: at $0 \in \mathbf{C}$) is an equation:

$$\sigma_q X = AX, \tag{1}$$

where $A \in GL_n(\mathbf{C}(\{z\}))$ (resp. $A \in GL_n(\mathbf{C}((z)))$). There is an intrinsic description as a "$q$-difference module $M_A$", which runs as follows. We consider the operator $\Phi_A$ on $\mathbf{C}(\{z\})^n$ which maps a column vector $X$ to $A^{-1}\sigma_q X$. This can be abstracted as a finite dimensional $\mathbf{C}(\{z\})$-vector space $V$ endowed with a so-called "$\sigma_q$-linear automorphism" $\Phi$. A $q$-difference module is such a pair $M = (V, \Phi)$. Here, we have $M_A = (\mathbf{C}(\{z\})^n, \Phi_A)$.

We shall here stick to the matrix model and, for all practical purposes, the reader may identify the equation (1), the matrix $A$ and the $q$-difference module $M_A$ with each other. For instance, we call *solution of $A$, or of (1), or of $M_A$* in some extension $K$ of $\mathbf{C}(\{z\})$, on which $\sigma_q$ operates, a column vector $X \in K^n$ such that $\sigma_q X = AX$. The *underlying space* of $A \in GL_n(\mathbf{C}(\{z\}))$ is $\mathbf{C}(\{z\})^n$.

**2.1. Description of the tannakian structure.** — We now proceed to describe the *tannakian category of analytic $q$-difference equations* $\mathcal{E}^{(0)}$. There is a similar description for the corresponding formal category $\hat{\mathcal{E}}^{(0)}$. The objects of $\mathcal{E}^{(0)}$ are linear analytic $q$-difference equations (1). A morphism from $A \in GL_n(\mathbf{C}(\{z\}))$ to $B \in GL_p(\mathbf{C}(\{z\}))$ is a matrix $F \in M_{p,n}(\mathbf{C}(\{z\}))$ such that:

$$(\sigma_q F)A = BF. \tag{2}$$

This just means that $F$ sends any solution $X$ of $A$ to a solution $FX$ of $B$. One can check that $\mathcal{E}^{(0)}$ is an abelian category. Indeed, it is the category of finite length left modules over the euclidean non-commutative ring $\mathcal{D}_{q,K}$ of $q$-difference operators over $K = \mathbf{C}(\{z\})$.

The abelian category $\mathcal{E}^{(0)}$ is endowed with a tensor structure. The tensor product $A_1 \otimes A_2$ of two objects (resp. the tensor product $F_1 \otimes F_2$ of two morphisms) is just the Kronecker product of the matrices; of course, we must define a consistent way of identifying $\mathbf{C}^n \otimes \mathbf{C}^p$ with $\mathbf{C}^{np}$, or $\mathbf{C}(\{z\})^n \otimes \mathbf{C}(\{z\})^p$ with $\mathbf{C}(\{z\})^{np}$ (see, for instance **[11]**).

The unit object $\underline{1}$ (which is neutral for the tensor product) is the matrix $(1) \in GL_1(\mathbf{C}(\{z\})) = \mathbf{C}(\{z\})^*$, with underlying space $\mathbf{C}(\{z\})$. The object $\underline{1}$ of course corresponds to the "trivial" equation [(1)] $\sigma x = x$. One easily checks that the space $\mathrm{Hom}(\underline{1}, A)$ of morphisms from $\underline{1}$ to $A$ is exactly the space of solutions of $A$ in $\mathbf{C}(\{z\})$, or, equivalently, the space of fixed points of $\Phi_A$ in $\mathbf{C}(\{z\})^n$. We shall write $\Gamma(A)$ or $\Gamma(M_A)$

[(1)] In differential Galois theory, the matrix $A$ of a system is in $M_n(\mathbf{C}(\{z\}))$ (rather than in $GL_n$), the trivial equation is $x' = 0$, etc. The theory of $q$-difference equations rather has a multiplicative character that space, as it is similar to a space of global sections (and, indeed, can be realised as such, see [**14**]).

The characterization (2) of morphisms can itself be seen as a $q$-difference equation $\sigma_q F = BFA^{-1}$. This means that there is an "internal Hom" object, which can be described in the following way. Consider the linear operator $F \mapsto BFA^{-1}$ on the vector space $M_{p,n}(\mathbf{C}(\{z\}))$. Through identification of $M_{p,n}(\mathbf{C}(\{z\}))$ with $\mathbf{C}(\{z\})^{np}$, this operator is described by a matrix in $GL_{np}(\mathbf{C}(\{z\}))$, which yields the desired equation. We shall write $\underline{\mathrm{Hom}}(A, B)$ the corresponding object. Thus, one gets:

$$\Gamma(\underline{\mathrm{Hom}}(A, B)) \simeq \mathrm{Hom}(\underline{1}, \underline{\mathrm{Hom}}(A, B)) \simeq \mathrm{Hom}(A, B). \tag{3}$$

Actually, this is a special case of the following canonical isomorphism::

$$\mathrm{Hom}(A, \underline{\mathrm{Hom}}(B, C)) \simeq \mathrm{Hom}(A \otimes B, C). \tag{4}$$

The reader will check that the object $\underline{\mathrm{Hom}}(A, \underline{1})$ has the following description. The underlying space is $M_{1,n}(\mathbf{C}(\{z\}))$, which we identify with $\mathbf{C}(\{z\})^n$. The corresponding matrix for the linear operator $F \mapsto FA^{-1}$ is the *contragredient matrix* $A^\vee = {}^tA^{-1}$. We call the object $A^\vee$ the *dual* of the object $A$. From this, we get yet another construction of the internal Hom:

$$\underline{\mathrm{Hom}}(A, B) \simeq A^\vee \otimes B. \tag{5}$$

We summarize these properties by saying that $\mathcal{E}^{(0)}$ is a tannakian category. This is halfway to showing that it is (isomorphic to) the category of representations of a proalgebraic group, our hoped for Galois group. To get further, one needs a *fiber functor on* $\mathcal{E}^{(0)}$. This was defined and, to some extent, studied in full generality in [**13**], [**12**] and [**6**]. However, for our strongest results, we need to restrict to the case of *integral slopes.*

**2.2. Equations with integral slopes.** — In [**13**], one defined the Newton polygon of a $q$-difference module (analytic or formal). This consists in slopes (2) $\mu_1 < \cdots < \mu_k$ (rational numbers) together with ranks (or multipicities) $r_1, \ldots, r_k$ (positive integers). We shall say that a module is *pure isoclinic* if it has only one slope and that it is *pure* (3) if it is a direct sum of pure isoclinic modules. We call *fuchsian* a pure isoclinic module with slope 0. The Galois theory of fuchsian modules was studied in [**11**]. Pure modules are irregular objects without *wild monodromy*, as follows from [**10**], [**12**] and [**6**].

The tannakian subcategory of $\mathcal{E}^{(0)}$ made up of pure modules is called $\mathcal{E}_p^{(0)}$. Modules with integral slopes also form tannakian subcategories, which we write $\mathcal{E}_1^{(0)}$ and $\mathcal{E}_{p,1}^{(0)}$. *From now on, we restrict to the case of integral slopes*. Our category of interest is therefore $\mathcal{E}_1^{(0)}$ and we shall now start its description.

(2) Note that here, as in [**6**], we have changed the definitions of slopes. Those used here are the opposites of those used in [**13**], [**8**]and [**12**].

(3) Here again, starting with [**6**], we changed our terminology: we now call pure isoclinic (resp. pure) what was previously called pure (resp. tamely irregular). The latter are called *split modules* in [**3**].

Any equation in $\mathcal{E}_1^{(0)}$ can be written in the following *standard form*:

$$A = \begin{pmatrix} z^{\mu_1}A_1 & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & U_{i,j} & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & z^{\mu_k}A_k \end{pmatrix}, \tag{6}$$

where the slopes $\mu_1 < \cdots < \mu_k$ are integers, $r_i \in \mathbf{N}^*$, $A_i \in GL_{r_i}(\mathbf{C})$ $(i = 1, \ldots, k)$ (those $\mu_i$ and $r_i$ make up the Newton polygon of $A$) and:

$$\forall i, j \text{ s.t. } 1 \leq i < j \leq k\ ,\ U_{i,j} \in \mathrm{Mat}_{r_i,r_j}(\mathbf{C}(\{z\})).$$

We actually can, and will, require the blocks $U_{i,j}$ to have all their coefficients in $\mathbf{C}[z, z^{-1}]$. Then any morphism $F : A \to B$ between two matrices in standard form is easily seen to be meromorphic at 0 (by definition) and holomorphic all over $\mathbf{C}^*$; this is because the equation $\sigma_q F = BFA^{-1}$ allows one to propagate the regularity near 0 to increasing neighborhoods.

We moreover say that $A$ is in *polynomial* standard form if each block $U_{i,j}$ with $1 \leq i < j \leq k$ has coefficients in $\sum\limits_{\mu_i \leq d < \mu_j} \mathbf{C}z^d$. It was proved in [**10**] that any object in $\mathcal{E}_1^{(0)}$ is analytically equivalent to one written in polynomial standard form (in essence, this is due to Birkhoff and Guenther). Last, we say that $A$ is in *normalized* standard form is if all the eigenvalues of all the blocks $A_i$ are in the fundamental annulus $\{z \in \mathbf{C}^* \mid 1 \leq |z| < |q|\}$. Any standard form can be normalized through shearing transformations. Note that polynomial standard form is stable under tensor product, while normalized standard form is not.

The standard form (6) above expresses the existence of a *filtration by the slopes* ([**13**]). The functoriality of the filtration moreover entails that a morphism $F : A \to B$ is also upper triangular (by blocks) in the following sense: if the slopes of $B \in GL_p(\mathbf{C}(\{z\}))$ are $\nu_1 < \cdots < \nu_l$, with ranks $s_1 < \cdots < s_l$, then the morphism $F \in M_{p,n}(\mathbf{C}(\{z\}))$ from $A$ to $B$ has only non null blocks $F_{i,j} \in M_{s_j,r_i}(\mathbf{C}(\{z\}))$, $1 \leq i \leq k$, $1 \leq j \leq l$ for $\nu_j \leq \mu_i$.

To the matrix $A$ and module $M = M_A$ is associated the graded module $\mathrm{gr}M = M_0 = M_{A_0}$ with block diagonal matrix:

$$A_0 = \begin{pmatrix} z^{\mu_1}A_1 & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & 0 & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & z^{\mu_k}A_k \end{pmatrix}, \tag{7}$$

The graded module $M_0$ is the direct sum $P_1 \oplus \cdots \oplus P_k$, where each module $P_i$ is pure of rank $r_i$ and slope $\mu_i$ and corresponds to the matrix $z^{\mu_i}A_i$. The functor $M \rightsquigarrow \mathrm{gr}M$

also acts on morphisms. To $F$, it associates $F_0$ which has the same diagonal blocks as $F$, that is, $(F_0)_{i,j} = F_{i,j}$ if $\mu_i = \nu_j$. But all the $(F_0)_{i,j}$ such that $\mu_i \neq \nu_j$ are null.

By formalisation, *i.e.* base change $\mathbf{C}(\{z\}) \to \mathbf{C}((z))$, the slope filtration splits and the functor gr becomes isomorphic to the identity functor. In matrix terms, this translates as follows. There is a unique isomorphism $F : A_0 \to A$ with formal components $F_{i,j} \in M_{r_i,r_j}(\mathbf{C}((z)))$ (for $1 \leq i,j \leq k$) and the following shape:

$$F = \begin{pmatrix} I_{r_1} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & F_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & I_{r_k} \end{pmatrix}. \tag{8}$$

To express that a matrix has such a shape and coefficients in some domain $K$, we shall write $F \in \mathfrak{G}_{A_0}(K)$. Thus, $\mathfrak{G}_{A_0}$ is a unipotent algebraic subgroup of the linear group and it can be realised above any field $K$: in the above case, one has $F \in \mathfrak{G}_{A_0}(\mathbf{C}((z)))$. For further use, we also give a notation for the corresponding Lie algebra $\mathfrak{g}_{A_0}$. An element $f \in \mathfrak{g}_{A_0}(K)$ has the shape:

$$\begin{pmatrix} 0_{r_1} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & f_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & 0_{r_k} \end{pmatrix}, \tag{9}$$

where $0_r$ is the null $r \times r$ matrix and where each $f_{i,j} \in M_{r_i,r_j}(K)$.

We shall denote $\hat{F}_A$ the unique $F$ mentioned above. Its blocks can be characterised as the unique formal solutions to the following recursive equations:

$$\forall 1 \leq i < j \leq k \ , \ \sigma_q F_{i,j} z^{\mu_j} A_j - z^{\mu_i} A_i F_{i,j} = \sum_{i<l<j} U_{i,l} F_{l,j} + U_{i,j}. \tag{10}$$

There are usually no analytic solutions (that is, with coefficients in $\mathbf{C}(\{z\})$) for equations (10). (The existence of analytic solutions is equivalent to $M_A$ being pure.) There are, however, meromorphic solutions, to be considered as *resummations* of the formal solution $\hat{F}_A$ (section 3.1).

The graded counterpart $F_0$ of $F = \hat{F}_A$ satisfies simpler equations. From the above description, we know that $(F_0)_{i,j} = 0$ for any $i,j$ such that $\mu_i \neq \mu_j$, that is, if $i \neq j$; if $i = j$:

$$\sigma_q (F_0)_{i,i} z^{\mu_i} A_i = z^{\mu_i} A_i (F_0)_{i,i}.$$

This implies that $\sigma_q (F_0)_{i,i} A_i = A_i (F_0)_{i,i}$, and it then follows from **[11]** that the coefficients of $F_0$ are Laurent polynomials (elements of $\mathbf{C}[z, z^{-1}]$); if moreover $A$ is in normalized standard form, then these coefficients are in $\mathbf{C}$.

**2.3. Description of the fiber functor.** — In Tannaka theory, the Galois group is defined as the group of tensor automorphisms of a fiber functor. We now describe a fiber functor on $\mathcal{E}_1^{(0)}$. There is actually a whole family of these, indexed by $\mathbf{C}^*$, and one can therefore define a Galois groupoïd ([**6**]). Here, we shall first choose an arbitrary basepoint $a \in \mathbf{C}^*$. As a consequence, some constructions of 3.1 will be valid for most equations, but not all. This means that, to study a particular equation, one has to choose a basepoint compatible with it, which will be seen to be a generically true condition.

The fiber functor $\hat{\omega}_a^{(0)}$ goes from $\mathcal{E}_1^{(0)}$ to the category of finite dimensional $\mathbf{C}$-vector spaces. On the side of objects, to each matrix $A \in GL_n(\mathbf{C}(\{z\}))$ and module $M_A$, it associates the space $\hat{\omega}_a^{(0)}(A) = \mathbf{C}^n$. On the side of morphisms, to $F : A \to B \in GL_p(\mathbf{C}(\{z\}))$, it associates $F_0(a) : \mathbf{C}^n \to \mathbf{C}^p$. (The dimensions are right and it follows from the last remark in 2.2 that $F_0(a)$ is well defined).

Apart from functoriality, the properties of $\hat{\omega}_a^{(0)}$ which make it a fiber functor are the following: it is exact, faithful and $\otimes$-compatible. The latter means that, for any $A, B$, the natural map $t_{A,B} : \hat{\omega}_a^{(0)}(A) \otimes \hat{\omega}_a^{(0)}(B) \to \hat{\omega}_a^{(0)}(A \otimes B)$ is an isomorphism.

We now define the Galois group of $\mathcal{E}_1^{(0)}$ (at base point $a$) as $G_1^{(0)} = Aut^{\otimes}(\hat{\omega}_a^{(0)})$. It would be more rigorous to write explicitly the index $a$ indicating the basepoint, but this would make the notation heavier without true necessity. An element of the group $Aut^{\otimes}(\hat{\omega}_a^{(0)})$ is, by definition, a natural transformation $g : A \rightsquigarrow g(A) \in GL\left(\hat{\omega}_a^{(0)}(A)\right) = GL_n(\mathbf{C})$, subject to the following conditions:

1. Functoriality: for any morphism $F : A \to B$, one has $g(B) \circ F_0(a) = F_0(a) \circ g(A)$. Thus, the following diagram is commutative:

$$\begin{tikzcd} \hat{\omega}_a^{(0)}(A) \arrow[r, "F_0(a)"] \arrow[d, "g(A)"'] & \hat{\omega}_a^{(0)}(B) \arrow[d, "g(B)"] \\ \hat{\omega}_a^{(0)}(A) \arrow[r, "F_0(a)"] & \hat{\omega}_a^{(0)}(B) \end{tikzcd}$$

2. Tensor compatibility: for any objects $A, B$, up to the natural identifications, one has an equality $g(A \otimes B) = g(A) \otimes g(B)$. Thus, the following diagram is commutative:

$$\begin{tikzcd} \hat{\omega}_a^{(0)}(A) \otimes \hat{\omega}_a^{(0)}(B) \arrow[r, "t_{A,B}"] \arrow[d, "g(A)\otimes g(B)"'] & \hat{\omega}_a^{(0)}(A \otimes B) \arrow[d, "g(A\otimes B)"] \\ \hat{\omega}_a^{(0)}(A) \otimes \hat{\omega}_a^{(0)}(B) \arrow[r, "t_{A,B}"] & \hat{\omega}_a^{(0)}(A \otimes B) \end{tikzcd}$$

In [**11**] was completely described the Galois group $G_f^{(0)}$ of the subcategory $\mathcal{E}_f^{(0)}$ of $\mathcal{E}^{(0)}$ made up of fuchsian equations. From [**13**], one could (trivially) deduce the Galois group $G_{p,1}^{(0)}$ of the category $\mathcal{E}_{p,1}^{(0)}$ of pure objects with integral slopes. Here, we will describe the Galois group $G^{(0)}$ of $\mathcal{E}_1^{(0)}$. The extension of these results to the case of non integral slopes should not involve new ideas on the analytic side, but will have to take in account the results of van der Put and Reversat in [**3**].

## 2.4. Galois group and Galois action

***Theorem 2.1.*** — *The structure of the Galois group $G_1^{(0)}$ is as follows:*

$$
\begin{aligned}
G_1^{(0)} &= \mathfrak{St} \rtimes G_{p_1}^{(0)} && \textit{(total Galois group with integral slopes)},\\
G_{p,1}^{(0)} &= T_1^{(0)} \times G_f^{(0)} && \textit{(pure Galois group with integral slopes)},\\
T_1^{(0)} &= \mathbf{C}^* && \textit{(theta torus with integral slopes)},\\
G_f^{(0)} &= G_{f,s}^{(0)} \times G_{f,u}^{(0)} && \textit{(fuchsian Galois group)},\\
G_{f,u}^{(0)} &= \mathbf{C} && \textit{(unipotent component of the fuchsian Galois group)},\\
G_{f,s}^{(0)} &= \mathrm{Hom}_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*) && \textit{(semisimple component of the fuchsian Galois group)}.
\end{aligned}
$$

The structure and action of the prounipotent Stokes group $\mathfrak{St}$ are the subject matter of [**6**] and of section 3 of the present paper. We shall presently explain the structure and action of the pure group $G_{p,1}^{(0)}$. This means that we should associate to any object $A$ a representation of $G_{p,1}^{(0)}$ in the space $\hat{\omega}_a^{(0)}(A)$; thus, for any $g \in G_{p,1}^{(0)}$ and any matrix $A \in GL_n(\mathbf{C}(\{z\}))$, we should realize $g(A) \in GL_n(\mathbf{C})$.

We start from the standard form (6). For each of the block matrices $A_i$, we write:

$$A_i = A_{i,s} A_{i,u}$$

its multiplicative Dunford decomposition: $A_{i,s}$ is semisimple, $A_{i,u}$ is unipotent and they commute.

1. Let $g = \gamma \in G_{f,s}^{(0)} = \mathrm{Hom}_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*)$. The latter is here identified with the group of morphisms from the abstract group $\mathbf{C}^*$ to itself that send $q$ to 1. We let $\gamma$ act on each $A_{i,s}$ through its eigenvalues: if $A_{i,s} = P\mathrm{diag}(c_1, \ldots, c_r)P^{-1}$, then $\gamma(A_{i,s}) = P\mathrm{diag}(\gamma(c_1), \ldots, \gamma(c_r))P^{-1}$ (it does not depend on the choice of a particular diagonalisation). Then:

$$g(A) = \begin{pmatrix} \gamma(A_{1,s}) & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & 0 & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & \gamma(A_{k,s}) \end{pmatrix},$$

2. Let $g = \lambda \in G^{(0)}_{f,u} = \mathbf{C}$. Since the $A_{i,u}$ are unipotent matrices, the $A^{\lambda}_{i,u}$ are well defined and we put:

$$g(A) = \begin{pmatrix} A^{\lambda}_{1,u} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & A^{\lambda}_{k,u} \end{pmatrix},$$

3. Let $g = t \in T^{(0)}_1 = \mathbf{C}^*$. This *theta torus* is the analogue here of the *exponential torus* of the classical differential Galois theory. Then:

$$g(A) = \begin{pmatrix} t^{\mu_1} I_{r_1} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & t^{\mu_k} I_{r_k} \end{pmatrix},$$

Note that all these depend on $A_0$ only. This is because the category $\mathscr{E}^{(0)}_{p,1}$ of pure modules with integral slopes is equivalent to the category of representations of $G^{(0)}_{p,1}$, so that giving a representation of the latter group is the same as giving an object in the former category. We leave as an exercise for the reader the reconstruction of $A_0$ from the representation described above. For further use, we shall now prove two lemmas about the action of $G^{(0)}_{p,1}$ on $\hat{\omega}^{(0)}_a(A)$. These lemmas actually express the "duality" of $G^{(0)}_{p,1}$ and $\mathscr{E}^{(0)}_{p,1}$.

***Lemma 2.2***. — *Let $A$ be in normalized standard form (6). Let $X \in \hat{\omega}^{(0)}_a(A)$ be* covariant *under the action of $G^{(0)}_{p,1}$, that is, for all $g \in G^{(0)}_{p,1}$, the vectors $X$ and $g(A)X$ are colinear. Then there exists $i \in \{1, \dots, k\}$ and $\alpha \in Sp(A_i)$ such that: $A_0 X = \alpha z^{\mu_i} X$.*

*Proof*. — First note that the block decomposition of $A_0$ (or, equivalently, the action of the theta torus) entails a splitting of vector spaces:

$$\hat{\omega}^{(0)}_a(A) = \mathbf{C}^n = \mathbf{C}^{r_1} \oplus \cdots \oplus \mathbf{C}^{r_k},$$

each $A_i$ acting upon the corresponding $\mathbf{C}^{r_i}$. We can accordingly write $X = (X_1, \dots, X_k)$ (in row form, instead of column form, for economy of space). Covariance under the action of $T^{(0)}_1$ say that $(t^{\mu_1} X_1, \dots, t^{\mu_k} X_k)$ and $(X_1, \dots, X_k)$ are colinear for all $t \in \mathbf{C}^*$, which implies that at most one component $X_i$ is non trivial. Then, covariance under the action of $G^{(0)}_{f,u}$ says that $X_i$ is fixed by $A_{i,u}$ (since the latter is unipotent). Last, covariance under $G^{(0)}_{f,s}$ implies that $X_i$ is an eigenvector of $A_{i,s}$. Indeed, this comes from the fact that, if $\alpha \neq \alpha'$ are eigenvalues of $A_i$, then, by the normalization condition, $\alpha q^{\mathbf{Z}} \cap \alpha' q^{\mathbf{Z}} = \varnothing$; it is then easy to see that there exists

$\gamma \in \mathrm{Hom}_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*)$ such that $\gamma(\alpha) \neq \gamma(\alpha')$, so that $X_i$ cannot have nontrivial components in both eigenspaces of $A_i$. The conclusion follows. □

***Lemma 2.3.*** — *Let $A$ be in normalized standard form (6). Let $X \in \hat{\omega}_a^{(0)}(A)$ be* invariant *under the action of $G_{p,1}^{(0)}$, that is, for all $g \in G_{p,1}^{(0)}$, the vectors $X$ and $g(A)X$ are equal. Then $A_0 X = X$.*

*Proof.* — The proof is similar, with two adaptations. First, equality of $(t^{\mu_1}X_1, \ldots, t^{\mu_k}X_k)$ and $(X_1, \ldots, X_k)$ entails that at most one component $X_i$ is non trivial and the corresponding slope is $\mu_i = 0$; second, invariance under $G_{f,s}^{(0)}$ implies that at most one component of $X_i$ (in the eigenspace decomposition) is non trivial, that the corresponding $\alpha \in \mathrm{Sp}(A_i)$ is in the kernel of all elements of $\mathrm{Hom}_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*)$, so it is in $q^{\mathbf{Z}}$, so equal to 1 by the normalisation condition. □

Again because of the duality of $G_{p,1}^{(0)}$ and $\mathscr{E}_{p,1}^{(0)}$, the conclusions of these two lemmas have useful interpretations. The conclusion of lemma 2.2 says that the column matrix $X \in M_{n,1}(\mathbf{C})$ is a morphism from the rank one object $(\alpha z^{\mu_i}) \in GL_1(\mathbf{C}(\{z\}))$ into $A_0$. The conclusion of lemma 2.3 says that the column matrix $X \in M_{n,1}(\mathbf{C})$ is a morphism from the unit object $\underline{1} = (1) \in GL_1(\mathbf{C}(\{z\}))$ into $A_0$, *i.e.* a section $X \in \Gamma(A_0)$.

## 3. The wild fundamental group

**3.1. The action of the Stokes group.** — An element $s \in \mathfrak{St}$ is characterized by the following properties:

1. To each $A$ in standard form (6), it associates a matrix $s(A) \in \mathfrak{G}_{A_0}(\mathbf{C})$; recall that $\mathfrak{G}_{A_0}$ was described as the algebraic group of matrices of shape as in equation (8).
2. If $A = A_0$, that is, if $A$ is pure, then $s(A) = I_n$.
3. Functoriality and tensor compatibility are defined as in section 2.3.

Since $\mathfrak{St}$ is a prounipotent proalgebraic group, it is convenient to study it through its Lie algebra $\mathfrak{st}$. (The underlying formalism is expounded in the appendix of [**2**].) An element $D \in \mathfrak{st}$ is also a natural transformation of $\hat{\omega}_a^{(0)}$. It associates to each object $A$ an endomorphism $D(A) \in \mathscr{L}(\hat{\omega}_a^{(0)}(A)) = M_n(\mathbf{C})$, subject to the following conditions:

1. For each $A$ in standard form (6), the matrix $D(A) \in M_n(\mathbf{C})$ is in $\mathfrak{g}_{A_0}(\mathbf{C})$; recall that $\mathfrak{g}_{A_0}$ was described as the Lie algebra of matrices of shape as in equation (9).
2. If $A = A_0$, that is, if $A$ is pure, then $D(A) = 0_n$.
3. Functoriality is defined as in section 2.3.
4. Tensor compatibility is that of "Lie-like elements" (as in [**15**], §6): for any $A \in GL_n(\mathbf{C}(\{z\}))$ and $B \in GL_p(\mathbf{C}(\{z\}))$, one should have, up to natural identifications: $D(A \otimes B) = D(A) \otimes I_p + I_n \otimes D(B)$. Thus, $D$ behaves like a derivation.

In [**6**], we have produced many elements of $\mathfrak{St}$ and of $\mathfrak{st}$. However, for a given basepoint $a \in \mathbf{C}^*$, these do not operate on the whole of $\mathcal{E}_1^{(0)}$ but on a tannakian subcategory of it. Therefore, the way of using them is the following: given an equation $A$ of interest, proposition 4.2 of *loc. cit.* yields an explicit criterion to select adequate basepoints (these are generically adequate). Then all the constructions that follow make sense in the tannakian subcategory of $\mathcal{E}_1^{(0)}$ generated by $A$. This means that each time we shall evaluate a meromorphic function at $a$, this will be possible. Henceforth, we shall not anymore discuss this matter. *We assume that the basepoint has been chosen so that all the objects we deal with are compatible with it.*

In [**12**] and [**6**], we defined an explicit finite subset $\Sigma_{A_0}$ of $\mathbf{E}_q$ and proved:

***Theorem 3.1***. — *Let $\overline{c} \in \mathbf{E}_q \setminus \Sigma_{A_0}$. Then, there is a unique $F : A_0 \to A$ such that $F \in \mathfrak{G}_{A_0}(\mathcal{M}(\mathbf{C}^*))$, with poles only on $[-c; q] = -cq^{\mathbf{Z}}$ and such that, for $1 \leq i < j \leq k$, the poles of $F_{i,j}$ have multiplicity $\leq \mu_j - \mu_i$.*

We write this meromorphic isomorphism $S_{\overline{c}}\hat{F}_A$ and see it as some kind of *summation of $\hat{F}_A$ in the direction $\overline{c} \in \mathbf{E}_q$*. Therefore, changing direction of summation, we may define, for every $\overline{c}, \overline{d} \in \mathbf{E}_q \setminus \Sigma_{A_0}$:

$$S_{\overline{c},\overline{d}}\hat{F}_A = (S_{\overline{c}}\hat{F}_A)^{-1} S_{\overline{d}}\hat{F}_A,$$

some kind of "ambiguity of summation", that is, a Stokes operator. It is plainly a meromorphic automorphism of $A_0$. We also proved in *loc. cit.*:

***Proposition 3.2***. — *If moreover $\overline{a} \neq \overline{c}, \overline{d}$, then $A \rightsquigarrow S_{\overline{c},\overline{d}}\hat{F}_A(a)$ is an element of $\mathfrak{St}$. In particular, $S_{\overline{c},\overline{d}}\hat{F}_A(a) \in \mathfrak{St}(A)$. (Recall that we implicitly restrict ourselves to a subcategory of $\mathcal{E}_1^{(0)}$ where everything is defined.)*

For the following corollary, we fix an arbitrary direction of summation $\overline{c_0} \in \mathbf{E}_q$, again to be considered as a choice of basepoint (and inessential).

***Corollary 3.3***. — *Putting $LS_{\overline{c},a}(A) = \log(S_{\overline{c_0},\overline{c}}\hat{F}_A(a)) \in \mathfrak{st}(A)$ yields a family of elements of elements of $\mathfrak{st}(A)$. Moreover, $A \rightsquigarrow LS_{\overline{c},a}(A)$ is an element of $\mathfrak{st}$. (We omit $\overline{c_0}$ in the notation.)*

The above family is a meromorphic map from $\mathbf{E}_q$ to a vector space, hence one can take residues. Define the $q$-alien derivations by the formula:

$$\dot{\Delta}_{\overline{c}}(A) = Res_{\overline{d}=\overline{c}} LS_{\overline{d},a}(A).$$

(We do not mention the arbitrary basepoints $\overline{c_0}, a$ in the notation.) Of course, for $\overline{c} \notin \Sigma_{A_0}$, we have $\dot{\Delta}_{\overline{c}}(A) = 0$. Another result we need from [**6**] is:

***Theorem 3.4***. — *One has $\dot{\Delta}_{\overline{c}}(A) \in \mathfrak{st}(A)$. More precisely, $A \rightsquigarrow \dot{\Delta}_{\overline{c}}(A)$ is an element of $\mathfrak{st}$.*

Since $\mathfrak{St}$ is a normal subgroup of $G_1^{(0)}$, it admits a conjugation action by $G_{p,1}^{(0)}$. This can be transferred to the Lie algebra $\mathfrak{st}$. Because of the action by the theta torus $T_1^{(0)} = \mathbf{C}^*$, we thus have a spectral decomposition:

$$\mathfrak{st} = \bigoplus_{\delta \geq 1} \mathfrak{st}^{\delta},$$

and each alien derivation admits a canonical decomposition:

$$\dot{\Delta}_{\overline{c}} = \bigoplus_{\delta \geq 1} \dot{\Delta}_{\overline{c}}^{(\delta)},$$

where $\dot{\Delta}_{\overline{c}}^{(\delta)}(A) \in \mathfrak{st}^{\delta}(A)$ has only non null blocks for $\mu_j - \mu_i = \delta$. Each $t \in T_1^{(0)}$ acts on $\mathfrak{st}^{\delta}$ by multiplication by $t^{\delta}$, and carries $\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$ to $t^{\delta} \dot{\Delta}_{\overline{c}}^{(\delta)}(A)$.

*Remarks*

1. The theta torus actually operates on each $\hat{\omega}_a^{(0)}(A) = \hat{\omega}_a^{(0)}(A_0)$ and, being semi-simple, splits it into the direct sum of its eigenspaces: one for each slope $\mu$, with rank $r(\mu)$. The corresponding increasing filtration comes from the filtration by the slopes:

$$\hat{\omega}_a^{(0)}(A)_{\geq \mu} = \hat{\omega}_a^{(0)}(A_{\geq \mu}).$$

   The elements of the group $\mathfrak{G}_{A_0}(\mathbf{C})$ are the automorphisms of $\hat{\omega}_a^{(0)}(A)$ which respect that filtration and are trivial (*i.e.* the identity) on the associated graded space. The elements of the algebra $\mathfrak{G}_{A_0}(\mathbf{C})$ are the endomorphisms of $\hat{\omega}_a^{(0)}(A)$ which respect that filtration and are trivial (*i.e.* null) on the associated graded space.
2. From this, we deduce a spectral decomposition:

$$\mathfrak{g}_{A_0} = \bigoplus_{\delta \geq 1} {\mathfrak{g}_{A_0}}^{\delta},$$

   from which the decomposition of $\mathfrak{st}$ follows.
3. Putting ${\mathfrak{g}_{A_0}}^{\geq \delta} = \sum_{\delta' \geq \delta} {\mathfrak{g}_{A_0}}^{\delta'}$ defines a filtration of the Lie algebra $\mathfrak{g}_{A_0}$ by ideals. Putting ${\mathfrak{G}_{A_0}}^{\geq \delta} = I_n + {\mathfrak{g}_{A_0}}^{\geq \delta} = \exp {\mathfrak{g}_{A_0}}^{\geq \delta}$ then defines a filtration of $\mathfrak{G}_{A_0}$ by normal subgroups.
4. Similarly, we can decompose each $(i,j)$ block of $\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$ (where $\mu_j - \mu_i = \delta$) into subblocks indexed by pairs $(\alpha, \beta) \in \mathrm{Sp}(A_i) \times \mathrm{Sp}(A_j)$. Each $\gamma \in G_{f,s}^{(0)}$ then multiplies the corresponding subblock of $\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$ by $\dfrac{\gamma(\alpha)}{\gamma(\beta)}$. This may be deduced as above from the action of $G_{f,s}^{(0)}$ on $\hat{\omega}_a^{(0)}(A_0)$ and a corresponding splitting of each of the ${\mathfrak{g}_{A_0}}^{\delta}$.

### 3.2. The density theorem

3.2.1. *Plain density theorem.* — The wild monodromy group actually is the Lie subalgebra of $\mathfrak{st}$ generated by the $q$-alien derivations $\dot{\Delta}_{\overline{c}}$. The justification of the name is that its definition has a transcendental character and the following result.

***Theorem 3.5* (Density theorem).** — *(i) The subgroup of $\mathfrak{St}$ associated with the wild monodromy group (as defined above), together with the pure group $G_{p,1}^{(0)}$, generate a Zariski-dense subgroup of the whole Galois group $G_1^{(0)}$. (ii) The $\dot{\Delta}_{\overline{c}}$ together with all their conjugates under the action of $G_{p,1}^{(0)}$ generate a Zariski-dense Lie subalgebra of $\mathfrak{st}$.*

*Proof.* — Actually, (i) is but a rephrasing of (ii) and we shall prove the latter. We shall use Chevalley's criterion in the following form:
*For a subset $H \subset G_1^{(0)}$ to generate a Zariski-dense subgroup of $G_1^{(0)}$, it is sufficient that, for each object $A$ and each line $D \subset \hat{\omega}_a^{(0)}(A)$ which is invariant under the action of all elements of $H$, then $D$ is actually invariant under the action of $G_1^{(0)}$.* Our way of using it is similar to that in **[11]** (2.2.3.3 and 3.1.2.3).

We take $H = G_{p,1}^{(0)} \cup \exp\left(\{\dot{\Delta}_{\overline{c}} \mid c \in \mathbf{C}^*\}\right)$. If we choose a generator $X$ of the line $D$, the assumption is that $X$ is covariant under $G_{p,1}^{(0)}(A) = G_{p,1}^{(0)}(A_0)$ on the one hand, under all the $\dot{\Delta}_{\overline{c}}(A)$ on the other hand. Since the latter are nilpotent, this means that all $\dot{\Delta}_{\overline{c}}(A)X = 0$. Then, must prove that for all $D \in \mathfrak{st}$, one has $D(A)X = 0$.

Using lemma 2.2, along with its proof and notations, we may write (in row form) $X = (0, \ldots, X_i, \ldots, 0)$, where the components have sizes $r_1, \ldots, r_k$ and where $A_i X_i = cX_i$ for some $c \in \mathrm{Sp}(A_i)$, so that $A_0 X = cz^{\mu_i} X$.

Now, we note that components of slopes $> \mu_i$ are neither involved in the assumptions nor in the conclusion, so that one may as well assume from start that $i = k$. Indeed, write $n' = r_1 + \cdots + r_i$ the size of the components corresponding to slopes $\leq \mu_i$ (equivalently, the rank of the submodule $M'_{\leq \mu_i} \subset M = M_A$ of slopes $\leq \mu_i$ in the slope filtration), $A'$ the corresponding submatrix of $A$ (so that $M' = M_{A'}$) and $X' = (0, \ldots, X_i)$ the corresponding subvector of $X$. The matrix $\Phi = \begin{pmatrix} I_{n'} \\ 0 \end{pmatrix} \in M_{n,n'}(\mathbf{C})$ is a morphism $\Phi : A' \to A$ (it is the inclusion $M' \subset M$) and $\Phi X' = X$. For all $g \in G_1^{(0)}$, one has (by functoriality) $g(A)\Phi = \Phi g(A')$ (here, one has $\Phi(a) = \Phi$). The reader will check that $A'$ and $X'$ satisfy the same assumption as $A$ and $X$, and that it is enough to prove the conclusions for them.

So we assume from now on that $X = (0, \ldots, X_k)$, that $A_k X_k = cX_k$ for some $c \in \mathrm{Sp}(A_k)$, so that $A_0 X = cz^{\mu_k} X$. Then $X : (cz^{\mu_k}) \to A_0$ is an analytic morphism, and therefore $G = \hat{F}_A X : (cz^{\mu_k}) \to A$ is a formal morphism. We shall prove below (lemma 3.6) that it is actually an analytic morphism. Therefore, taking $D \in \mathfrak{st}$ and using functoriality, we get the commutative diagram:

$$\begin{tikzcd} \mathbf{C} \arrow[r, "G_0(a)"] \arrow[d, "D(cz^{\mu_k})"'] & \hat{\omega}_a^{(0)}(A) \arrow[d, "D(A)"] \\ \mathbf{C} \arrow[r, "G_0(a)"] & \hat{\omega}_a^{(0)}(A) \end{tikzcd}$$

The matrix $G_0$ is the graded part of the column matrix $\hat{F}_A X$, that is, in row notation, $X$ itself. Since $D \in \mathfrak{st}$ and since the source object $(cz^{\mu_k})$ is pure, one has $D(cz^{\mu_k}) = 0$. Hence we get $D(A)X = 0$ as wanted. □

***Lemma 3.6***. — *The matrix $\hat{F}_A X$ is analytic and the summations $S_{\overline{c}}\hat{F}_A$ do not depend on the direction $\overline{c} \in \mathbf{E}_q$, and they are all equal to $\hat{F}_A X$ (that is, its "classical summation", as a* convergent *power series).*

*Proof.* — First fix a direction $\overline{c} \in \mathbf{E}_q$ and write $F = S_{\overline{c}}\hat{F}_A$. Likewise, write $G = \hat{F}_A$ for short. The components $F_{i,j}$ and $G_{i,j}$ satisfy equations (10). We are interested in the $F_{i,k}X_k$ and the $G_{i,k}X_k$ for $1 \leq i < k$ (for $i = k$, both equal $X_k$).

We shall put: $Y_i = F_{i,k}X_k$, $Z_i = G_{i,k}X_k$ and $V_i = U_{i,k}X_k$. Then, for $1 \leq i < k$, multiplying (10) by $X_k$ on the right and taking in account the equalities $\sigma_q X_k = X_k$ and $A_k X_k = cX_k$, one gets:

$$cz^{\mu_k}(\sigma_q Y_i) - z^{\mu_i} A_i Y_i = \sum_{i<j<k} U_{i,j} Y_j + V_i \quad \text{and} \quad cz^{\mu_k}(\sigma_q Z_i) - z^{\mu_i} A_i Z_i = \sum_{i<j<k} U_{i,j} Z_j + V_i.$$

On the other hand, we shall have to use the assumptions: $\dot{\Delta}_{\overline{c}}(A)X = 0$. Since $X = (0, \ldots, X_k)$, this means that, for each $i < k$, one has $(\dot{\Delta}_{\overline{c}}(A))_{i,k} X_k = 0$. Writing for short $F_d = S_{\overline{d}}\hat{F}_A(a)$, $F_0 = S_{\overline{c_0}}\hat{F}_A(a)$ and $L = LS_{\overline{d},a}(A) = \log(S_{\overline{c_0},\overline{d}}\hat{F}_A(a))$ (see the notations of section 3.1), we shall see in lemma 3.7 below that, for $i < k$:

$$L_{i,k} = (F_d)_{i,k} - (F_0)_{i,k} + \sum_{i<j<k} M_{i,j,k}\big((F_d)_{j,k} - (F_0)_{j,k}\big),$$

where the $M_{i,j,k}$ are some arbitrary matrices (their values are inessential here).

We use a downward induction on $i$. For $i = k-1$:

$$cz^{\mu_k}(\sigma_q Y_{k-1}) - z^{\mu_{k-1}} A_{k-1} Y_{k-1} = V_{k-1} \quad \text{and} \quad cz^{\mu_k}(\sigma_q Z_{k-1}) - z^{\mu_{k-1}} A_{k-1} Z_{k-1} = V_{k-1}.$$

Thus, $Z_{k-1}$ is the formal solution and $Y_{k-1}$ the solution summed in direction $\overline{c}$ of the equation $cz^{\mu_k}(\sigma_q Y) - z^{\mu_{k-1}} A_{k-1} Y = V_{k-1}$. On the other hand, from the formula above, one has: $L_{k-1,k} = (F_d)_{k-1,k} - (F_0)_{k-1,k}$. Taking residue at $\overline{d} = \overline{c}$ and multiplying at right by the constant vector $X_k$, one gets:

$$(\dot{\Delta}_{\overline{c}}(A))_{k-1,k} X_k = Res_{\overline{d}=\overline{c}} LS_{k-1,k} X_k = Res_{\overline{d}=\overline{c}} (F_d)_{k-1,k} X_k = Res_{\overline{d}=\overline{c}} Y_{k-1}.$$

This means that the residues of resummed solutions of the equation just mentioned are all 0. According to the results of section 4.2 of **[6]**, this implies that $Y_{k-1}$ is analytic near 0 (it has no poles other than 0), that it does not depend on the direction of summation $\overline{d}$, and that it is equal to $Z_{k-1}$. This completes the first step of the induction.

Now take $i < k-1$ and assume that the property has been proved for all $j > i$. Consider the equation of which $Y_i$ is solution. Its right hand member $\sum\limits_{i<j<k} U_{i,j}Y_j + V_i$ is analytic, by the induction hypothesis (analyticity part). The residue at $\overline{d} = \overline{c}$ of $Y_i$ is equal to, $Res_{\overline{d}=\overline{c}}(F_d)_{i,k}X_k$, thus to:

$$Res_{\overline{d}=\overline{c}} LS_{i,k}X_k = (\dot{\Delta}_{\overline{c}}(A))_{i,k}X_k = 0.$$

This is because all other terms in the formula taken from lemma 3.7 have at right a factor $\big((F_d)_{j,k} - (F_0)_{j,k}\big)X_k = Y_j - Y_j$, since $Y_j$ does not depend on the direction of summation. Thus we have again a solution $Y_i$ with all residues null, so that it is analytic and independent of the direction of summation by *loc. cit.*. □

***Lemma 3.7***. — *With the notations of section 3.1, one has, for $i < k$:*

$$\begin{aligned}&\log(S_{\overline{c_0},\overline{d}}\hat{F}_A(a))_{i,k}\\ &\quad = (S_{\overline{d}}\hat{F}_A(a))_{i,k} - (S_{\overline{c_0}}\hat{F}_A(a))_{i,k} + \sum_{i<j<k} M_{i,j,k}\big((S_{\overline{d}}\hat{F}_A(a))_{j,k} - (S_{\overline{c_0}}\hat{F}_A(a))_{j,k}\big),\end{aligned}$$

*where the $M_{i,j,k}$ are some arbitrary matrices.*

*Proof*. — We write $A = S_{\overline{d}}\hat{F}_A(a)$, $B = S_{\overline{c_0}}\hat{F}_A(a)$ and $C = A - B$, which is strictly upper triangular by blocks. Then:

$$\log(B^{-1}A) = \log(I_n + B^{-1}C) = \sum_{p\geq 1} \frac{(-1)^{p-1}}{p}(B^{-1}C)^p,$$

from which the equality of blocks:

$$\big(\log(B^{-1}A)\big)_{i,k} = C_{i,k} + \sum_{i<j<k} M_{i,j,k}C_{j,k}$$

follows easily. □

3.2.2. *Functorial density theorem.* — In section 3.3, we shall describe how the Zariski generators $\dot{\Delta}_{\overline{c}}$ of $\mathfrak{st}$ (theorem 3.5) are related. For that, we shall first give a more functorial version of the density theorem.

Since $\mathcal{E}_{p,1}^{(0)}$ and $\mathcal{E}_1^{(0)}$ are respectively isomorphic to the category of (finite dimensional complex) representations of $G_{p,1}^{(0)}$ and $G_1^{(0)} = \mathfrak{St} \rtimes G_{p,1}^{(0)}$, and since finite dimensional representations of the prounipotent proalgebraic group $\mathfrak{St}$ are equivalent to finite dimensional representations of the pronilpotent proalgebraic Lie algebra $\mathfrak{st}$, the tannakian category $\mathcal{E}_1^{(0)}$ admits an alternative "mixed" description, which runs as follows:

1. Objects are pairs $A = (A_0, (D(A))_{D\in\mathfrak{st}})$, where $A_0$ is some object of $\mathcal{E}_{p,1}^{(0)}$, *e.g.* a matrix in pure standard form (7), and where each $D(A) \in \mathfrak{g}_{A_0}(\mathbf{C})$.

2. Morphisms from $A = (A_0, (D(A))_{D\in\mathfrak{st}})$ to $B = (B_0, (D(B))_{D\in\mathfrak{st}})$ are morphisms $F_0 : A_0 \to B_0$ in $\mathcal{E}_{p,1}^{(0)}$ such that, for each $D \in \mathfrak{st}$, one has $D(B)F_0(a) = F_0(a)D(A)$. (Recall that an arbitrary basepoint $a \in \mathbf{C}^*$ has been chosen once for all.)
3. The tensor product of $A = (A_0, (D(A))_{D\in\mathfrak{st}})$ and $B = (B_0, (D(B))_{D\in\mathfrak{st}})$ is the object $C = (C_0, (D(C))_{D\in\mathfrak{st}})$, with the previous rule $C_0 = A_0 \otimes B_0$ from $\mathcal{E}_{p,1}^{(0)}$, and with the "Lie-like element" rule $D(A \otimes B) = D(A) \otimes I_p + I_n \otimes D(B)$. The unit is $\underline{1} = ((1), (0)_{D\in\mathfrak{st}})$. The dual of $A$ is $A^\vee = (A_0^\vee, (-{}^tD(A))_{D\in\mathfrak{st}})$. The space of sections of $A$ is $\Gamma(A) = \mathrm{Hom}(\underline{1}, A) = \{X_0 \in \Gamma(A_0) \mid \forall D \in \mathfrak{st}\ ,\ D(A)X_0 = 0\}$. (Recall that $\Gamma(A_0) = \{X_0 \in \mathbf{C}(\{z\})^n \mid \sigma_q X_0 = A_0 X_0\}$.)
4. There is a a fiber functor $\hat{\omega}_a^{(0)}(A) \underset{def}{=} \hat{\omega}_a^{(0)}(A_0)$.

To be complete, such a description should take in account the adjoint action of $G_{p,1}^{(0)}$ on $\mathfrak{st}$, which is, for all $A$, the restriction of the action of $G_{p,1}^{(0)}$ on $\mathfrak{g}_{A_0}(\mathbf{C})$. For instance, from the action of the theta torus, one draws the graduation $\mathfrak{st} = \bigoplus\limits_{\delta\geq 1} \mathfrak{st}^\delta$, whence decompositions $D(A) = \sum\limits_{\delta\geq 1} D^\delta(A)$, where each $D^\delta(A) \in {\mathfrak{g}_{A_0}}^\delta(\mathbf{C})$. We shall take in account the adjoint action of $G_f^{(0)}$ later.

We would like to consider the $\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$ as encoding a Lie algebra representation from the free Lie algebra $L$ generated by the family of symbols $(\dot{\Delta}_{\overline{c}}^{(\delta)})_{\delta\geq 1, \overline{c}\in\mathbf{E}_q}$, and so describe $\mathcal{E}_1^{(0)}$ as the category of representations of $L \rtimes G_{p,1}^{(0)}$ in a way similar to that above. This would require some other tools (see the conclusion of the paper). As a substitute, we define a new tannakian category $\mathcal{E}'$ as follows:

1. Objects are pairs:
$$A = \left(A_0, (\dot{\Delta}_{\overline{c}}^{(\delta)}(A))_{\delta\geq 1, \overline{c}\in\mathbf{E}_q}\right),$$
where $A_0$ is in pure standard form (7), and where each $\dot{\Delta}_{\overline{c}}^{(\delta)}(A) \in {\mathfrak{g}_{A_0}}^\delta(\mathbf{C})$.
2. Morphisms from $A$ to $B = \left(B_0, (\dot{\Delta}_{\overline{c}}^{(\delta)}(B))_{\delta\geq 1, \overline{c}\in\mathbf{E}_q}\right)$ are morphisms $F_0 : A_0 \to B_0$ in $\mathcal{E}_{p,1}^{(0)}$ such that, for each $\delta \geq 1, \overline{c} \in \mathbf{E}_q$, one has $\dot{\Delta}_{\overline{c}}^{(\delta)}(B)F_0(a) = F_0(a)\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$.
3. The tensor product of $A$ and $B$ is the object $C = \left(C_0, (\dot{\Delta}_{\overline{c}}^{(\delta)}(B))_{\delta\geq 1, \overline{c}\in\mathbf{E}_q}\right)$, with $C_0 = A_0 \otimes B_0$ and $\dot{\Delta}_{\overline{c}}^{(\delta)}(A \otimes B) = \dot{\Delta}_{\overline{c}}^{(\delta)}(A) \otimes I_p + I_n \otimes \dot{\Delta}_{\overline{c}}^{(\delta)}(B)$. The unit and dual are described as before. The space of sections of $A$ is $\Gamma(A) = \mathrm{Hom}(\underline{1}, A) = \{X_0 \in \Gamma(A_0) \mid \forall \delta \geq 1, \overline{c} \in \mathbf{E}_q\ ,\ \dot{\Delta}_{\overline{c}}^{(\delta)}(A)X_0 = 0\}$.
4. There is a a fiber functor $\hat{\omega}_a^{(0)}(A) \underset{def}{=} \hat{\omega}_a^{(0)}(A_0)$.

For the time being, we do not take in account the action of $G_f^{(0)}$.

We now consider the functor $A \rightsquigarrow \mathcal{F}(A) = \left(A_0, (\dot{\Delta}_{\overline{c}}^{(\delta)}(A))_{\delta\geq 1, \overline{c}\in\mathbf{E}_q}\right)$ from $\mathcal{E}_1^{(0)}$ to $\mathcal{E}'$. It is plainly an exact faithful $\otimes$-functor.

***Theorem 3.8* (Functorial density theorem).** — *The functor $\mathcal{F}$ is fully faithful.*

*Proof.* — To prove that $\mathrm{Hom}(A,B) \to \mathrm{Hom}(\mathcal{F}(A),\mathcal{F}(B))$ is onto, we draw on the identifications $\mathrm{Hom}(A,B) = \Gamma(A^\vee \otimes B)$ and $\mathrm{Hom}(\mathcal{F}(A),\mathcal{F}(B)) = \Gamma(\mathcal{F}(A)^\vee \otimes \mathcal{F}(B))$. Since $\mathcal{F}$ is a $\otimes$-functor, the latter is identified with $\Gamma(\mathcal{F}(A^\vee \otimes B))$, so that we are left check that, for any $A$, the map $\Gamma(A) \to \Gamma(\mathcal{F}(A))$ is onto.

That map sends a vector $X \in \mathbf{C}(\{z\})^n$ such that $\sigma_q X = AX$ to its graded part $X_0 \in \mathbf{C}(\{z\})^n$. The vector $X_0$ has the same null slope component as $X$ and is zero elsewhere. It satisfies $\sigma_q X_0 = A_0 X_0$ and $\forall \delta \geq 1, \overline{c} \in \mathbf{E}_q$ , $\dot{\Delta}^{(\delta)}_{\overline{c}}(A) X_0 = 0$. If we start from such a vector $X_0$, lemma 2.3 tells us that it comes indeed from some $X \in \Gamma(A)$. □

**3.3. A freeness theorem.** — We now shall describe the (essential) image of the functor $\mathcal{F}$, or, what amounts to the same, which families $(\dot{\Delta}^{(\delta)}_{\overline{c}}(A))_{\delta \geq 1, \overline{c} \in \mathbf{E}_q}$ can be realized for a given $A_0$ in $\mathcal{E}^{(0)}_{p,1}$. To understand what is going on, we start with the first level, which is easier.

3.3.1. *The first level.* — In theorem 3.1, we obtained the $F_{i,j}$ blocks of $S_{\overline{c}} \hat{F}_A$ as solutions of the following equations:

$$(\sigma_q F_{i,j}) z^{\mu_j} A_j - z^{\mu_i} A_i F_{i,j} = \sum_{i<l<j} U_{i,l} F_{l,j} + U_{i,j}.$$

We consider the first non trivial level in the computation of $S_{\overline{c}} \hat{F}_A$, that is: $\delta_0 = \min\limits_{i<j} (\mu_j - \mu_i)$. For a block $F_{i,j}$ of level $\mu_j - \mu_i = \delta_0$, there is no $F_{l,j} \neq 0$ for $i < l < j$, so that the second hand member in the equation above is $U_{i,j}$, which is analytic near 0. In that case, there is a solution $F_{i,j}$ with poles on $[-c;q]$ and multiplicity $\leq \delta_0$ for any $\overline{c} \in \mathbf{E}_q$ which satisfies the non-resonancy condition:

$$\forall \alpha \in \mathrm{Sp}(A_i) \ , \ \forall \beta \in \mathrm{Sp}(A_j) \ , \ \alpha c^{\mu_i} \not\equiv \beta c^{\mu_j} \pmod{q^{\mathbf{Z}}}.$$

We recall briefly, from [**6**], how this was computed. One puts $F_{i,j} = \dfrac{G_{i,j}}{\theta_c^{\delta_0}}$, (the function $\theta_c$ has been defined in section 1.3). We thus look for $G_{i,j}$ holomorphic on $\mathbf{C}^*$ and satisfying:

$$c^{\delta_0} (\sigma_q G_{i,j}) A_j - A_i G_{i,j} = z^{-\mu_i} U_{i,j} \theta_c^{\delta_0} = \sum_{n \in \mathbf{Z}} v_n z^n.$$

Writing the Laurent series $G_{i,j} = \sum g_n z^n$, we are left to solve, for each $n \in \mathbf{Z}$:

$$c^{\delta_0} q^n g_n A_j - A_i g_n = v_n \in \mathrm{Mat}_{r_i, r_j}(\mathbf{C}).$$

If $\mathrm{Sp}(c^{\delta_0} q^n A_j) \cap \mathrm{Sp}(A_i) = \varnothing$, which is is just the non-resonancy condition above, then, for each $n$, this admits a unique solution.

Using the notations given at the end of 3.1, we see that, for any $\overline{d} \notin \Sigma_{A_0}$, one has $S_{\overline{d}} \hat{F}_A(a) \in \mathfrak{G}_{A_0}{}^{\geq \delta_0}(\mathbf{C})$. Provisionally call $f^{\delta}_{\overline{d}}$ its component at level $\delta$. A small computation shows that $LS_{\overline{d},a}(A)$ is in $\mathfrak{g}_{A_0}{}^{\geq \delta_0}(\mathbf{C})$ and that its component at level $\delta_0$ is $f^{\delta_0}_{\overline{d}} - f^{\delta}_{\overline{c_0}}$. Thus, $\dot{\Delta}^{(\delta_0)}_{\overline{c}}(A) = Res_{\overline{d} = \overline{c}} f^{\delta_0}_{\overline{d}}$.

From the previous computation, we now conclude that, for $\mu_j - \mu_i = \delta_0$, the $(i,j)$ block of $\dot{\Delta}_{\overline{c}}^{(\delta_0)}(A)$ is trivial for non-resonant directions, *i.e.* if $\mathrm{Sp}(c^{\delta_0} q^n A_j) \cap \mathrm{Sp}(A_i) = \varnothing$. This is the necessary conditions we were looking for. It is not hard to see (and it will come as a particular case of the following sections) that these are indeed the only conditions on the first level.

3.3.2. *Structure of the q-alien derivations at an arbitrary level.* — We are led to introduce some more notations. We first refine the spectral decomposition of $\hat{\omega}_a^{(0)}(A)$ under the action of the theta torus by taking in account the action of $G_{f,s}^{(0)}$, the semi-simple component of the fuchsian group. From the equivalence:

$$\alpha \equiv \beta \pmod{q^{\mathbf{Z}}} \Longleftrightarrow \forall \gamma \in G_{f,s}^{(0)}\,,\ \gamma(\alpha) = \gamma(\beta),$$

we see that the action of $G_{f,s}^{(0)}$ splits each eigenspace under $\mathbf{C}^*$ corresponding to the slope $\mu_i$ into a sum indexed by the $\overline{\alpha} \in \overline{\mathrm{Sp}(A_i)}$. Precisely, if $V = \hat{\omega}_a^{(0)}(A)$, then one may write:

$$V = \bigoplus V^{(\mu)},$$

where $\mu$ runs through the set of slopes of $A$, and, for each $\mu$:

$$V^{(\mu)} = \bigoplus V^{(\mu,\overline{\alpha})},$$

where $\alpha$ runs through $\mathrm{Sp}(A_i)$ if $\mu = \mu_i$ in our usual notations.

To be able to carry this splitting to matrices, we fix an arbitrary linear order on $\mathbf{E}_q$ and assume the order on indices is compatible with that arbitrary order. The corresponding adjoint action of $G_{f,s}^{(0)}$ on $\mathfrak{g}_{A_0}(A)$ then allows one to define the eigenspaces: ${\mathfrak{g}_{A_0}}^{(\delta,\overline{c})}(\mathbf{C}) = \{M \in \mathfrak{g}_{A_0}(\mathbf{C}) \mid M$ is trivial out of the $(\mu_i, \overline{\alpha}, \mu_j, \overline{\beta})$ components such that $\alpha c^{\mu_i} \equiv \beta c^{\mu_j} \pmod{q^{\mathbf{Z}}}\}$. This can be non-trivial only if $\overline{c} \in \Sigma_{A_0}^{\delta}$, where:

$$\Sigma_{A_0}^{\delta} = \{\overline{c} \in \mathbf{E}_q \mid \exists i < j \text{ such that } \mu_j - \mu_i = \delta \text{ and } \frac{\overline{\alpha}}{\overline{\beta}} = \overline{c^{\delta}}\}.$$

By definition, $\Sigma_{A_0} = \bigcup\limits_{\delta \geq 1} \Sigma_{A_0}^{\delta}$. then:

$${\mathfrak{g}_{A_0}}^{\delta}(\mathbf{C}) = \bigoplus_{\overline{c} \in \Sigma_{A_0}^{\delta}} {\mathfrak{g}_{A_0}}^{(\delta,\overline{c})}(\mathbf{C}).$$

**Example.** From the previous paragraph, it follows that on the first non trivial level, $\dot{\Delta}_{\overline{c}}^{(\delta)}(A) \in {\mathfrak{g}_{A_0}}^{\delta,\overline{c}}(\mathbf{C})$. The difficulty is to properly generalize this fact to upper levels.

**Remark.** The equality $\overline{\alpha c}^{\mu_i} = \overline{\beta}\overline{c}^{\mu_j}$ is equivalent to: $\forall \gamma \in G_{f,s}^{(0)}\,,\ \gamma(\alpha c^{\mu_i}) = \gamma(\beta c^{\mu_j})$. Thus, ${\mathfrak{g}_{A_0}}^{\delta,\overline{c}}(\mathbf{C})$ can be characterized as the common fixed space of all the $\big(\gamma(\overline{c})^{-1}, \gamma, 0\big) \in G_{p,1}^{(0)}$, where $\gamma$ runs through $G_{f,s}^{(0)}$.

Now let $A, A'$ be matrices in standard form with the same graded part $A_0$. From **[12]** and **[6]**, we have the following generalisation of theorem 3.1: for each $\overline{c} \in \mathbf{E}_q \backslash \Sigma_{A_0}$, there exists a unique meromorphic morphism $F : A \to A'$ in $\mathfrak{G}_{A_0}(\mathscr{M}(\mathbf{C}^*))$, with poles on $[-c;q]$ and with multiplicities prescribed as in the theorem. We write it $S_{\overline{d}}\hat{F}_{A,A'}$. One then has:

$$S_{\overline{d}}\hat{F}_{A,A'} = S_{\overline{d}}\hat{F}_{A'}\left(S_{\overline{d}}\hat{F}_A\right)^{-1}.$$

Assume now that $A \equiv A' \pmod{\mathfrak{g}_{A_0}{}^{\geq\delta}(\mathbf{C}(\{z\}))}$, that is, $A$ and $A'$ have the same over-diagonals at levels $< \delta$. The components $F_{i,j}$ of $S_{\overline{d}}\hat{F}_{A,A'}$ for $0 < \mu_j - \mu_i < \delta$ are solutions of the equations:

$$(\sigma_q F_{i,j})z^{\mu_j}A_j - z^{\mu_i}A_iF_{i,j} = 0.$$

Therefore, they are null (*cf. loc. cit.*). This implies:

$$S_{\overline{d}}\hat{F}_{A,A'} \in \mathfrak{g}_{A_0}{}^{\geq\delta}(\mathcal{M}(\mathbf{C}^*)).$$

From the equality: $S_{\overline{d}}\hat{F}_{A'} = S_{\overline{d}}\hat{F}_{A,A'}S_{\overline{d}}\hat{F}_A$, we deduce:

$$S_{\overline{d}}\hat{F}_A \equiv S_{\overline{d}}\hat{F}_{A'} \pmod{\mathfrak{g}_{A_0}{}^{\geq\delta}(\mathcal{M}(\mathbf{C}^*))}.$$

***Proposition 3.9***. — *Let $f_{A,A',\overline{d}}$ be the component at level $\delta$ of $S_{\overline{d}}\hat{F}_{A,A'}(a)$. Then:*

$$\dot{\Delta}_{\overline{c}}^{(\delta)}(A') = \dot{\Delta}_{\overline{c}}^{(\delta)}(A) + Res_{\overline{d}=\overline{c}}f_{A,A',\overline{d}}.$$

*Proof*. — To alleviate notations, we omit the evaluation at $a$ and the direction $\overline{d}$ in the notations; to indicate summation along the arbitrary fixed direction $\overline{c}_0$, we just add the index 0. Thus, we respectively write:

$$\begin{array}{rcl} F_A \text{ for } S_{\overline{d}}\hat{F}_A(a) & \text{and} & F_{A,0} \text{ for } S_{\overline{c_0}}\hat{F}_A(a) \\ F_{A'} \text{ for } S_{\overline{d}}\hat{F}_{A'}(a) & \text{and} & F_{A',0} \text{ for } S_{\overline{c_0}}\hat{F}_{A'}(a) \\ F_{A,A'} \text{ for } S_{\overline{d}}\hat{F}_{A,A'}(a) & \text{and} & F_{A,A',0} \text{ for } S_{\overline{c_0}}\hat{F}_{A,A'}(a) \\ f_{A,A'} \text{ for } f_{A,A',\overline{d}} & \text{and} & f_{A,A',0} \text{ for } f_{A,A',\overline{c_0}}. \end{array}$$

From the previous remark:

$$F_{A'} = F_{A,A'}F_A \equiv (I_n + f_{A,A'})F_A \pmod{\mathfrak{g}_{A_0}{}^{\geq\delta}(\mathbf{C})},$$

so that:

$$F_{A',0}^{-1}F_{A'} \equiv F_{A,0}^{-1}F_{A,0} + f_{A,A'} - f_{A,A',0} \pmod{\mathfrak{g}_{A_0}{}^{\geq\delta}(\mathbf{C})}.$$

The conclusion then comes by taking logarithms, applying the following lemma and then taking residues. □

***Lemma 3.10***. — *Let $M \in \mathfrak{G}_{A_0}(\mathbf{C})$ and $N \in \mathfrak{g}_{A_0}{}^{\geq\delta}(\mathbf{C})$. Then:*

$$\log(M+N) \equiv (\log M) + N \pmod{\mathfrak{g}_{A_0}{}^{\geq\delta}(\mathbf{C})}.$$

*Proof*. — Write $M = I_n + M'$. Then:

$$\begin{aligned} \log(M+N) &\equiv \sum_{m\geq 1}\frac{(-1)^{m-1}}{m}(M'+N)^m \pmod{\mathfrak{g}_{A_0}{}^{\geq\delta}(\mathbf{C})} \\ &\equiv \sum_{m\geq 1}\frac{(-1)^{m-1}}{m}M'^m + N \pmod{\mathfrak{g}_{A_0}{}^{\geq\delta}(\mathbf{C})}. \end{aligned}$$ □

***Corollary 3.11***. — *Under the assumptions of the proposition, we have:*

$$\dot\Delta^{(\delta)}_{\overline{c}}(A') - \dot\Delta^{(\delta)}_{\overline{c}}(A) \in {\mathfrak{g}_{A_0}}^{(\delta,\overline{c})}(\mathbf{C}).$$

We are going to prove that these are, in some sense, the only conditions on the $q$-alien derivations at a given level $\delta$.

3.3.3. *Interpolating categories.* — There are two equivalent ways of defining $\mathcal{E}_{p,1}^{(0)}$ from $\mathcal{E}_{1}^{(0)}$: the first is by restriction to a subclass of objects, the pure ones; the second is by formalisation, *i.e.* extension of the base field $\mathbf{C}(\{z\}) \to \mathbf{C}((z))$. The former way amounts to shrinking the Galois group $G_1^{(0)}$ to its *quotient* $G_{p,1}^{(0)}$. The latter way amounts to extending the class of morphisms (indeed, there are no really new objects), and therefore to shrinking the Galois group $G_1^{(0)}$ to its *subgroup* $G_{p,1}^{(0)}$.

The existence of a natural filtration on the Stokes group $\mathfrak{St}$ suggests that it should be possible to interpolate between $\mathcal{E}_{p,1}^{(0)}$ and $\mathcal{E}_{1}^{(0)}$. We shall presently do so by extending the class of morphisms; the interpretation by restriction to subobjects is a bit more complicated.

We first define intermediate fields between $\mathbf{C}(\{z\})$ and $\mathbf{C}((z))$, for all levels $\delta \in \mathbf{N}$:

$$\mathbf{C}((z))^{(\delta)} = \left\{\sum f_n z^n \in \mathbf{C}((z)) \mid \exists R > 0 \ : \ f_n = \mathrm{O}(R^n q^{n^2/2\delta})\right\}.$$

Thus $\mathbf{C}((z))^{(+\infty)} \underset{def}{=} \mathbf{C}(\{z\}) \subset \mathbf{C}((z))^{(\delta)} \subset \mathbf{C}((z))^{(\delta-1)} \subset \mathbf{C}((z))^{(0)} \underset{def}{=} \mathbf{C}((z))$ [4]. The following is standard ([**4**], [**16**],[**12**]):

***Lemma 3.12***. — *If $\nu - \mu = \delta \geq 1$, then, the following equation:*

$$(\sigma_q F)(z^\nu B) - (z^\mu A)F = U, \quad A \in GL_r(\mathbf{C})\ ,\ B \in GL_s(\mathbf{C})\ ,\ U \in Mat_{r,s}(\mathbf{C}(\{z\}))$$

*has a unique solution $F \in Mat_{r,s}(\mathbf{C}((z))^{(\delta)})$. If moreover $F \in Mat_{r,s}(\mathbf{C}((z))^{(\delta')})$ for some $\delta' > \delta$, then $F \in Mat_{r,s}(\mathbf{C}(\{z\}))$.*

Write $\mathbf{C}((z))^{>\delta} = \bigcup\limits_{\delta'>\delta} \mathbf{C}((z))^{(\delta')}$. Then we call $\mathcal{C}^\delta$ the category with the same objects as $\mathcal{E}_1^{(0)}$ (seen in matrix form) and with morphisms satisfying the same conditions, but with $F \in GL_n(\mathbf{C}((z))^{>\delta})$. (Actually, since we deal only with integral slopes, we could as well take $\mathbf{C}((z))^{(\delta+1)}$ instead of $\mathbf{C}((z))^{>\delta}$.)

It is then clear that the $\mathcal{C}^\delta$ are tannakian categories, and that embeddings are natural exact faithful $\otimes$-functors $\mathcal{C}^\delta \to \mathcal{C}^{\delta-1}$. Moreover, $\mathcal{C}^0 = \mathcal{E}_{p,1}^{(0)}$, because equations with integral slopes can be solved in $\mathbf{C}((z))^{(1)}$ by the lemma, so that $\hat{F}_A$ is an isomorphism from $A_0$ to $A$ in $\mathcal{C}^0$. In the opposite direction, we have $\mathcal{C}^\infty = \mathcal{E}_1^{(0)}$. Actually, if $A$ has slopes $\mu_1 < \cdots < \mu_k$, then it is entirely determined by its image in $\mathcal{C}^\delta$ for any $\delta \geq \mu_k - \mu_1$.

[4] This $\mathbf{C}((z))^{(\delta)}$ is the field of fraction of the algebra of $q$-Gevrey series of level $\delta$, which was introduced in [**1**] and denoted $\mathbf{C}[[z]]_{q,s}$ with $s = 1/\delta$ (the $q$-Gevrey order) in [**4**].

From the composite functor $\mathscr{C}^\delta \to \mathscr{C}^0 = \mathscr{E}_{p,1}^{(0)}$, we draw that objects in $\mathscr{C}^\delta$ have a well defined Newton polygon, that there is on $\mathscr{C}^\delta$ a "graded module" functor, and that $\hat{\omega}_a^{(0)}$ defines a fiber functor on $\mathscr{C}^\delta$.

3.3.3.1. *An alternative description of $\mathscr{C}^\delta$.* — We also see that, if two objects $A$ and $B$ in $\mathscr{E}_1^{(0)}$ have isomorphic images in $\mathscr{C}^\delta$, then they have isomorphic images in $\mathscr{C}^0 = \mathscr{E}_{p,1}^{(0)}$ and they can be written in standard form (6) with the same block diagonal $A_0$. Of course, we may moreover assume $A_0$ to be in *normalized* standard form and $A$, $B$ to be in *polynomial* standard form (section 2.2).

***Proposition 3.13***. — *Let $A$ and $B$ in $\mathscr{E}_1^{(0)}$ be in normalized polynomial standard form with the same block diagonal $A_0$. Then they have isomorphic images in $\mathscr{C}^\delta$ if, and only if, there exists $F_0 \in GL_n(\mathbf{C})$ such that $B \equiv F_0 A F_0^{-1} \pmod{{\mathfrak{g}_{A_0}}^{>\delta}(\mathbf{C}(\{z\}))}$.*

*Proof*. — Here, of course, we have put ${\mathfrak{g}_{A_0}}^{>\delta} = \sum\limits_{\delta'>\delta} {\mathfrak{g}_{A_0}}^{\delta'}$ (which is the same as ${\mathfrak{g}_{A_0}}^{\geq\delta+1}$ since we deal with integral slopes) and the condition just means that $B$ and $F_0 A F_0^{-1}$ have the same over-diagonals up to level $\delta$.

The diagonal part $F_0$ of any formal morphism $F$ from $A$ to $B$ is an automorphism of $A_0$, thus constant (because of normalisation). Up to composing $F$ with $F_0^{-1}$ and replacing $A$ by $F_0 A F_0^{-1}$, we may assume that $F_0 = I_n$, so that $F = \hat{F}_{A,B}$. The condition then means that $\hat{F}_{A,B}$ has its coefficients in $\mathbf{C}((z))^{(\delta)}$. From the lemma, we draw, by induction on the level, that all over-diagonals up to level $\delta$ are analytic, therefore 0 because of results in [**10**]. $\square$

***Corollary 3.14***. — *One can define $\mathscr{C}^\delta$ in the following alternative way:*

1. *Objects of $\mathscr{C}^\delta$ are matrices in $\mathscr{E}_1^{(0)}$ modulo the equivalence relation $A \equiv B \pmod{{\mathfrak{g}_{A_0}}^{>\delta}(\mathbf{C}(\{z\}))}$.*
2. *Morphisms from (the class of) $A$ to (the class of) $B$ are matrices $F \in Mat_{p,n}(\mathbf{C}(\{z\}))$ such that $(\sigma_q F)A$ and $BF$ differ only in levels $> \delta$.*

***Corollary 3.15***. — *The Galois group of $\mathscr{C}^\delta$ is $\mathfrak{St}(\delta) \rtimes G_{p,1}^{(0)}$ for some unipotent subgroup $\mathfrak{St}(\delta)$ of $\mathfrak{St}$. For $i \leq \delta$, the $\dot{\Delta}_{\overline{c}}^{(i)}$ are well defined on $\mathscr{C}^\delta$ and belong to the Lie algebra $\mathfrak{st}(\delta)$ of $\mathfrak{St}(\delta)$.*

3.3.4. *A freeness theorem.* — We now describe precisely the essential image of the functor $\mathscr{T}$, that is, given $A_0$ in $\mathscr{E}_{p,1}^{(0)}$, the exact conditions on $(\dot{\Delta}_{\overline{c}}^{(\delta)}(A))_{\delta\geq 1, \overline{c}\in\mathbf{E}_q}$ that allow the reconstruction of $A$. The reconstruction will be done inductively, using $q$-alien derivations of levels up to $\delta$ to reconstruct the over-diagonals of $A$ up to level $\delta$, that is (after the previous paragraph) an object in $\mathscr{C}^\delta$.

In the same spirit as the definition of isoformal analytic classes in [**10**], we consider classes of objects $A$ in $\mathscr{C}^\delta$ above an object $B$ of $\mathscr{C}^{\delta-1}$ under the equivalence induced by gauge transform $F \equiv I_n \pmod{{\mathfrak{g}_{A_0}}^{\leq\delta}}$. Using polynomial standard normal form and the results of *loc. cit.*, we see that these classes make up a vector space of dimension:

$$\mathrm{irr}^{\delta}(A_0) = \sum_{\mu_j - \mu_i = \delta} r_i r_j (\mu_j - \mu_i) = \delta \sum_{\mu_j - \mu_i = \delta} r_i r_j .$$

Moreover, to see if two objects $A, A'$are in the same class, one computes $\hat{F}_{A,A'} \in \mathfrak{G}_{A_0}{}^{\delta}(\mathbf{C}((z)))$; if its over-diagonal at level $\delta$ has null $q$-Borel invariants, then we have the same class.

***Theorem 3.16* (Freeness theorem).** — *Let $B$ be an object of $\mathscr{C}^{\delta-1}$. Then, there is an affine space $V_{\overline{c}}(B)$ of direction $\mathfrak{g}_{A_0}{}^{(\delta,\overline{c})}(\mathbf{C})$ such that:*
*(i) The $\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$ for $A$ an object of $\mathscr{C}^{\delta}$ above $B$ belong to $V_{\overline{c}}(B)$.*
*(ii) The mapping which sends an object $A$ of $\mathscr{C}^{\delta}$ above $B$ to the family of all $\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$ induces a one-to-one correspondance between classes of such objects (as defined above) and $\prod V_{\overline{c}}(B)$.*

*Proof.* — (i) It follows from paragraph 3.3.2 that all $\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$ (where $A$ is fixed and $\overline{c}$ varies) belong to a unique affine space of direction $\mathfrak{g}_{A_0}{}^{(\delta,\overline{c})}(\mathbf{C})$. Call it $V_{\overline{c}}(B)$. It is easily seen that the product space $\prod V_{\overline{c}}(B)$ has dimension $\mathrm{irr}^{\delta}(A_0)$.
(ii) The map from the set of polynomial representatives of a class, as described above, onto the above affine space, is affine. Up to the choice of an arbitrary basepoint, it is equivalent, after the results of **[7]** (section 3.2), to the parametrisation of the isoformal class by $q$-Borel transform, which is one-to-one after **[10]**. □

***Corollary 3.17.*** — *The following algorithm allows one to reconstruct $A$ in $\mathscr{E}_1^{(0)}$ from $A_0$ and the $\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$:*

1. *Reconstruct the first over-diagonal using the $q$-derivations of lowest level (this is the linear situation and it rests on* **[6]**.
2. *Having reconstructed the over-diagonals up to level $\delta - 1$ (using $q$-alien derivations up to level $\delta - 1$), call $A'$ the matrix with these over-diagonals and $0$ above; then compute the $\dot{\Delta}_{\overline{c}}^{(\delta)}(A')$.*
3. *Use the relation $\dot{\Delta}_{\overline{c}}^{(\delta)}(A') - \dot{\Delta}_{\overline{c}}^{(\delta)}(A) \in \mathfrak{g}_{A_0}{}^{(\delta,\overline{c})}(\mathbf{C})$ to find the level $\delta$ over-diagonal of $\hat{F}_{A,A'}$, then the level $\delta$ over-diagonal of $A$.*

In **[7]**, we shall give a representation-theoretic formulation of the theorem, and a description of the nonlinear part of $\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$ (that part which depends on the lower level $q$-derivations) in terms of convolution.

## 4. Conclusion

Write $H = \mathbf{C}^* \times \mathrm{Hom}_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*)$ and $\nu = (t, \gamma) \in H$. We saw how the group $H$ acts upon the diagonal of $A_0$: for $1 \leq i \leq k$ and $\alpha \in \mathrm{Sp}(A_i$, positions corresponding to slope $\mu_i$ and the eigenspace of $A_{i,s}$ for $\alpha$ are multiplied by $t^{\mu_i}\gamma(\alpha)$. Now let $i < j$

be indices of slopes $\mu_i < \mu_j$ and $\alpha \in \mathrm{Sp}(A_i,\ \beta \in \mathrm{Sp}(A_j)$ be corresponding exponents. The adjoint action of $\nu$ on the $(\mu_i, \overline{\alpha}, \mu_j, \overline{\beta})$ block is the multiplication by:

$$t^{\mu_i-\mu_j}\frac{\gamma(\alpha)}{\gamma(\beta)} = \left(t^{-1}\gamma(\overline{c})\right)^{\delta}$$

for each "resonant" $\overline{c}$, *i.e.* $\overline{c} \in \mathbf{E}_q$ such that $c^{\delta} \equiv \dfrac{\alpha}{\beta} \pmod{q^{\mathbf{Z}}}$. For any Galois derivation $D \in \mathfrak{st}$, we now put:

$$\Phi_{\nu}^{(\delta,\overline{c})}(D) = \nu(D) - \left(t^{-1}\gamma(\overline{c})\right)^{\delta} D \in \mathfrak{st},$$

where $\nu(D)$ comes from the adjoint action of $H$ on $\mathfrak{st}$. From the remark on page 318 and from paragraph 3.3.2, one draws that, for two objects $A, A'$ of $\mathscr{C}^{\delta}$ above the same object $B$ of $\mathscr{C}^{\delta-1}$, $\Phi_{\nu}^{(\delta,\overline{c})}(\dot{\Delta}_{\overline{c}}^{(\delta)})(A)\Phi_{\nu}^{(\delta,\overline{c})}(\dot{\Delta}_{\overline{c}}^{(\delta)})(A') = 0$. In other words, $\Phi_{\nu}^{(\delta,\overline{c})}(\dot{\Delta}_{\overline{c}}^{(\delta)})(A)$ depends only on the lower levels $\delta' < \delta$ of $A$. Moreover it is trivial on the first level. Actually, with methods similar to those used here, one can prove that $\Phi_{\nu}^{(\delta,\overline{c})}(\dot{\Delta}_{\overline{c}}^{(\delta)})(A)$ is in the Lie algebra generated by the $q$-alien derivations at lower levels. So it is natural to conjecture that $\Phi_{\nu}^{(\delta,\overline{c})}(\dot{\Delta}_{\overline{c}}^{(\delta)})$ belong to the free Lie algebra generated by the $\dot{\Delta}_{\overline{d}}^{(\delta')}$ ($\delta' < \delta$, $\overline{d} \in \mathbf{E}_q$), and even that there is a universal explicit formula. This would allow us to define a semi-direct product by a free Lie algebra, and to definitely "free" the $q$-alien derivations.

All the problems comes from the fact that points come from two distinct origins: elements of the dual of $H$ on the one hand, packs of points of $\mathbf{E}_q$ on the other hand, and from the interplay of the corresponding games of localisation. Comparing with the differential case, where one localizes geometrically on the circle of directions $S^1$, *then* one takes a Log, here, we take a Log, *then* we localise on $Eq$; whence an embroilment with plenty of Campbell-Haussdorff formulas between the two approaches [(5)].

We shall also give in **[7]** various applications, to the abelianisation of the tannakian $\pi_1$ and to the inverse problem for the local Galois group. For the latter problem, we shall state a list of necessary conditions; we don't know for the time being if they are sufficient.

Last, we built our alien $q$-derivations by tannakian methods. One can ask what happens for *solutions*. There, one meets the usual difficulty about constants, since one wishes operators defined over $\mathbf{C}$ and acting upon solutions (while constants are here elliptic functions). That problem maybe has no solution; however, one could perhaps, in analogy with the differential case, "unpoint" the $q$-alien derivations and build operators acting upon adequate spaces of formal power series. This seems related to a "$q$-convolution" mechanism presently studied by Changgui Zhang.

(5) Actually, we think that, in the end, we'll have a simpler description with a denumerable family of $q$-alien derivations, freed by the mere action of the theta torus.

## References


[1] J.-P. Bézivin – "Sur les équations fonctionnelles aux $q$-différences", *Aequationes Math.* **43** (1992), p. 159–176.

[2] P. Deligne & A. B. Goncharov – "Groupes fondamentaux motiviques de Tate mixte", *Ann. Sci. École Norm. Sup.* **38** (2005), p. 1–56.

[3] P. M. v. der) & R. M. – "Galois theory of $q$-difference equations", *Ann. Fac. Sci. de Toulouse* **26** (2007), p. 1–54.

[4] J.-P. Ramis – "About the growth of entire functions solutions of linear algebraic $q$-difference equations", *Ann. Fac. Sci. Toulouse Math.* **1** (1992), p. 53–94.

[5] ______, "About the inverse problem in differential Galois theory: the differential Abhyankar conjecture", in *The Stokes phenomenon and Hilbert's 16th problem (Groningen, 1995)*, World Sci. Publ., River Edge, NJ, 1996, p. 261–278.

[6] J.-P. Ramis & J. Sauloy – "The $q$-analogue of the wild fundamental group. I", in *Algebraic, analytic and geometric aspects of complex differential equations and their deformations. Painlevé hierarchies*, RIMS Kôkyûroku Bessatsu, B2, Res. Inst. Math. Sci. (RIMS), Kyoto, 2007, p. 167–193.

[7] ______, "The $q$-analogue of the wild fundamental group. III", in preparation.

[8] J.-P. Ramis, J. Sauloy & C. Zhang – "La variété des classes analytiques d'équations aux $q$-différences dans une classe formelle", *C. R. Math. Acad. Sci. Paris* **338** (2004), p. 277–280.

[9] ______, "Développement asymptotique et sommabilité des solutions des équations linéaires aux $q$-différences", *C. R. Math. Acad. Sci. Paris* **342** (2006), p. 515–518.

[10] ______, "Local analytic classification of irregular $q$-difference equations", in preparation.

[11] J. Sauloy – "Galois theory of Fuchsian $q$-difference equations", *Ann. Sci. École Norm. Sup.* **36** (2003), p. 925–968 (2004).

[12] ______, "Algebraic construction of the Stokes sheaf for irregular linear $q$-difference equations", *Astérisque* **296** (2004), p. 227–251.

[13] ______, "La filtration canonique par les pentes d'un module aux $q$-différences et le gradué associé", *Ann. Inst. Fourier (Grenoble)* **54** (2004), p. 181–210.

[14] ______, "Équations aux $q$-différences et fibrés vectoriels holomorphes sur la courbe elliptique $\mathbf{C}^*/q^{\mathbf{Z}}$", ce volume.

[15] C. T. Simpson – "Higgs bundles and local systems", *Publ. Math. I.H.É.S.* **75** (1992), p. 5–95.

[16] C. Zhang – "Une sommation discrète pour des équations aux $q$-différences linéaires et à coefficients analytiques: théorie générale et exemples", in *Differential equations and the Stokes phenomenon*, World Sci. Publ., River Edge, NJ, 2002, p. 309–329.



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# UNFOLDINGS OF TANGENT TO THE IDENTITY DIFFEOMORPHISMS

*by*

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***Abstract.*** — This paper is devoted to classify one dimensional unfoldings of tangent to the identity analytic diffeomorphisms, in other words elements $\varphi$ of $\mathrm{Diff}(\mathbb{C}^2, 0)$ of the form $(x, f(x, y))$ with $(\partial f/\partial y)(0, 0) = 1$. We provide the topological classification in absence of small divisors phenomena and an analytic classification of the finite codimension unfoldings. Such results are based on the study of the stable structures preserved by the diffeomorphisms. The main tool is the use of real flows. In both the topological and the analytic cases a non-wandering property is required, namely the Rolle property in the topological setting and infinitesimal stability in the analytic one.

We also prove that under generic hypotheses the analytic class of an unfolding $\varphi$ depends only on the analytic classes of the germs of 1-dimensional diffeomorphisms obtained by localizing along an irreducible component of the fixed points set of $\varphi$.

***Résumé* (Déploiements des difféomorphismes tangents à l'identité).** — Cet article est consacré à la classification des déploiements à un paramètre des difféomorphismes analytiques tangents à l'identité, en d'autres termes, les éléments $\varphi$ de $\mathrm{Diff}(\mathbb{C}^2, 0)$ qui sont de la forme $(x, f(x, y))$, avec $(\partial f/\partial y)(0, 0) = 1$. Nous fournissons une classification topologique en l'absence de phénomènes de petits diviseurs et la classification analytique des déploiements de codimension finie. Les preuves sont basées sur l'étude des structures stables qui sont invariants par l'action des difféomorphismes. L'outil principal est le recours aux flots réels. Une propriété de non-errance est nécessaire, à savoir la propriété de Rolle dans le cas topologique et la stabilité infinitésimale dans le cas analytique.

On prouve aussi que, sous des hypothèses génériques, la classe analytique d'un déploiement $\varphi$ ne dépend que des classes analytiques des germes de difféomorphismes en dimension 1 obtenus en localisant le long d'une composante irréductible de l'ensemble des points fixes de $\varphi$.

## 1. Introduction

We classify one dimensional unfoldings of tangent to the identity diffeomorphisms, i.e. elements $\varphi$ of $\mathrm{Diff}\,(\mathbb{C}^2, 0)$ of the form $(x, f(x, y))$ with $(\partial f/\partial y)(0, 0) = 1$. The

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set of diffeomorphisms of the previous form is denoted by $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. We provide a topological classification of the multi-parabolic elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ and an analytic classification of the non-degenerate ones (see section 2). Such results are based on the study of the stable structures preserved by the diffeomorphisms whose main tool is the use of real flows (section 3). This part of the paper is a survey of the papers **[10]** (topological classification) and **[12]** (analytic classification).

In section 6 we present a new result. We are interested on the local behavior of global objects. For instance in our setting we are interested on describing the nature of $\varphi = (x, f(x,y)) \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ in a neighborhood of $\gamma \setminus \{(0,0)\}$ for an irreducible component $\gamma$ of the fixed points set $\mathrm{Fix}(\varphi)$ of $\varphi$. Given $(x_0,y_0) \in \gamma \setminus \{(0,0)\}$ we denote by $\varphi_{(x_0,y_0)}$ the germ of $\varphi_{|x=x_0}$ in the neighborhood of $y = y_0$. Suppose $\gamma$ is parabolic, i.e. $(\partial f/\partial y)_{|\mathrm{Fix}(\varphi)} \equiv 1$. Then "part" of the Ecalle-Voronin invariants of $\varphi_{(x,y)}$, where $(x,y)$ belongs to $\gamma \setminus \{(0,0)\}$, can be extended continuously to $x = 0$ in good sectors $S$ in the parameter space. Under the proper hypothesis (see theorem 6.1) we can prove that the analytic class of $\varphi$ in the neighborhood of $\gamma \setminus \{(0,0)\}$ determines the analytic class of $\varphi$ in $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. By varying the parameter $x$ we show that the Ecalle Voronin invariants of $\varphi_{(x,y)}$ "turn" with respect to the Ecalle Voronin invariants of $\varphi$ (see subsection 6.2). Section 6 is a small glimpse of a more detailed work to be published.

## 2. Notations

We denote by $\mathcal{X}(\mathbb{C}^2,0)$ the group of germs of complex analytic vector fields in a neighborhood of $0 \in \mathbb{C}^2$. The elements of $\mathcal{X}(\mathbb{C}^2,0)$ which are singular at $0 \in \mathbb{C}^2$ can be interpreted as derivations of the maximal ideal of the ring $\mathbb{C}\{x,y\}$. We denote by $\hat{\mathcal{X}}(\mathbb{C}^2,0)$ the group of derivations of the maximal ideal $\hat{\mathfrak{m}}$ of the ring $\mathbb{C}[[x,y]]$. An element $\hat{X} \in \hat{\mathcal{X}}(\mathbb{C}^2,0)$ can be expressed in the more conventional form

$$\hat{X} = \hat{X}(x)\frac{\partial}{\partial x} + \hat{X}(y)\frac{\partial}{\partial y}.$$

Let $\mathrm{Diff}\,(\mathbb{C}^2,0)$ be the group of germs of complex analytic diffeomorphisms in a neighborhood of $0 \in \mathbb{C}^2$. We define $\mathrm{Fix}(\varphi)$ the *fixed points set* of $\varphi \in \mathrm{Diff}\,(\mathbb{C}^2,0)$. We denote by $\mathrm{Diff}_1(\mathbb{C}^2,0)$ the subgroup of $\mathrm{Diff}\,(\mathbb{C}^2,0)$ whose elements are tangent to the identity, i.e. given $\varphi \in \mathrm{Diff}\,(\mathbb{C}^2,0)$ then it belongs to $\mathrm{Diff}_1(\mathbb{C}^2,0)$ if $j^1\varphi \equiv Id$. Let $\widehat{\mathrm{Diff}}\,(\mathbb{C}^2,0)$ be the formal completion of $\mathrm{Diff}\,(\mathbb{C}^2,0)$.

We denote by $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ the subgroup of $\mathrm{Diff}\,(\mathbb{C}^2,0)$ of unfoldings of tangent to the identity diffeomorphisms. More precisely an element $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ is of the form

$$\varphi(x,y) = (x, f(x,y))$$

where $(\partial f/\partial y)(0,0) = 1$. We say that $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ is *non-degenerate* if $y \circ \varphi(0,y) \not\equiv y$. The linear part $j^1\varphi$ of an element $\varphi$ of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ is of the form $(x, y + ax)$ for some $a \in \mathbb{C}$. The linear unipotent isomorphism $(x, y + ax)$ is the exponential of the linear nilpotent vector field $ax\partial/\partial y$. Indeed $\varphi$ is the exponential of a unique nilpotent formal vector field $\hat{X}$. More precisely $\varphi$ can be interpreted as an operator $g \to g \circ \varphi$ acting on $\hat{\mathfrak{m}}$. The operator $\varphi$ is the exponential of an operator $\hat{X}$ acting on $\hat{\mathfrak{m}}$ as a derivation and such that $j^1 \hat{X} = ax\partial/\partial y$. We say that $\hat{X}$ is the *infinitesimal generator* of $\varphi$. We denote $\log \varphi = \hat{X}$.

Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$; we say that $\varphi$ is *multi-parabolic* if $(\partial f/\partial y)_{|\mathrm{Fix}(\varphi)} \equiv 1$. We denote by $\mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$ the set of multi-parabolic unfoldings and we call its elements MP-diffeomorphisms.

Given $\varphi, \eta \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ with $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\eta)$ we denote $\varphi \sim_{\mathrm{an}} \eta$ if $\varphi$ and $\eta$ are conjugated by some $\sigma \in \mathrm{Diff}(\mathbb{C}^2, 0)$ such that $x \circ \sigma = x$ and $\sigma_{|\mathrm{Fix}(\varphi) \setminus \{x=0\}} \equiv Id$. If we replace $\mathrm{Diff}(\mathbb{C}^2, 0)$ with the group of germs of homeomorphisms we obtain the equivalence $\varphi \sim_{\mathrm{top}} \eta$. By replacing $\mathrm{Diff}(\mathbb{C}^2, 0)$ with $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ we obtain $\varphi \sim_{for} \eta$. In the formal setting $\sigma_{|\mathrm{Fix}(\varphi) \setminus \{x=0\}} \equiv Id$ means that $y \circ \sigma - y$ belongs to the ideal of $\overline{\mathrm{Fix}(\varphi) \setminus \{x=0\}}$ in the ring $\mathbb{C}[[x, y]]$ (supposed $x \circ \sigma = x$).

## 3. Real flows

Our goal is describing the dynamics of $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Instead of trying a direct approach we consider a continuous dynamical system similar to $\varphi$. More precisely we choose a germ of holomorphic vector field $X_\varphi = g(x, y)\partial/\partial y$ satisfying the *proximity condition*, namely

$$y \circ \varphi - y \circ \exp(X_\varphi) \in (y \circ \varphi - y)^2.$$

Such a choice of $X_\varphi$ is possible **[11]** even if not unique. Supposed $\varphi = \exp(X_\varphi)$ then the orbits of $\varphi$ would be contained in the trajectories of the real flow $\Re(X_\varphi)$ of $X_\varphi$. Anyway $\Re(X_\varphi)$ provides a continuous "model" for the iterates of $\varphi$.

The proximity condition implies that $\mathrm{Fix}(\varphi) = Sing X_\varphi$ and that $\varphi$ is formally conjugated to $\exp(X_\varphi)$ **[11]**.

***Definition 3.1***. — *We say that $\varphi$ satisfies the $\epsilon$-property if there exist open neighborhoods $V \subset W$ of $(0,0)$ such that for all $(x, y) \in V$ and $j \in \mathbb{Z}$ satisfying*

$$\cup_{k \in [\min(j,0), \max(j,0)] \cap \mathbb{Z}} \{\exp(X_\varphi)^k(x, y)\} \subset V$$

*then* $\cup_{k \in [\min(j,0), \max(j,0)] \cap \mathbb{Z}} \{\varphi^k(x, y)\} \subset W$ *and*

$$\varphi^j(x, y) \in \exp(B(0, \epsilon)X_\varphi)(\exp(X_\varphi)^j(x, y)).$$

*We say that $\varphi$ satisfies the stability property if it satisfies the $\epsilon$-property for any $\epsilon > 0$ small enough.*

***Theorem 3.1* ($\epsilon$-theorem or stability theorem)**. — **[10]** *Let $\varphi \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$. Then $\varphi$ satisfies the stability property for every choice of $X_\varphi$ satisfying the proximity property.*

The $\epsilon$-theorem implies that in the multi-parabolic case the orbits of $\varphi$ and $\exp(X_\varphi)$ remain close independently of the number of iterations. Moreover, the dynamics of $\varphi$ is roughly speaking the dynamics of $\exp(X_\varphi)$ plus some small "noise". Analyzing the noise is not trivial since not every MP-diffeomorphism is topologically conjugated to the exponential of a holomorphic vector field as we will see later on. The stability property is crucial [**10**] to provide a complete system of topological invariants for the MP-diffeomorphisms. The situation in the general case is different since

***Theorem 3.2***. — *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0) \setminus \mathrm{Diff}_{MP}(\mathbb{C}^2,0)$. Then $\varphi$ holds the $\epsilon$-property for some choice of $X_\varphi$ if and only if $\log\varphi \in \mathcal{X}(\mathbb{C}^2,0)$.*

*Proof.* — Denote $\varphi(x,y) = (x, f(x,y))$. Since $\varphi \notin \mathrm{Diff}_{MP}(\mathbb{C}^2,0)$ there exists an irreducible component $\gamma_0$ of $\mathrm{Fix}(\varphi)$ such that $(\partial(y\circ\varphi)/\partial y)_{|\gamma_0} \not\equiv 1$. Up to replace $\varphi$ with $(x, f(x^k,y))$ and $\gamma_0$ with one of the irreducible components of $(x^k,y)^{-1}(\gamma_0)$ for some $k$ in $\mathbb{N}$ we can choose $\gamma_0$ of the form $y = h(x)$. Up to the change of coordinates $(x, y-h(x))$ we can suppose $\gamma_0 \equiv \{y=0\}$. Denote $L(w) = (\partial(y\circ\varphi)/\partial y)(w,0)$.

Let $w \in \mathbb{C}$; denote $\varphi_w$ the germ of $\varphi_{|x=w}$ in the neighborhood of $(w,0)$. Suppose $L(w) \in \mathbb{S}^1 \setminus \{1\}$, then the $\epsilon$-property implies that the sequence $\{\varphi_w^j\}$ is normal in some neighborhood of $(w,0)$. Therefore $\varphi_w$ is analytically linearizable for any $w \in L^{-1}(\mathbb{S}^1 \setminus \{1\})$ and then for any $w$ in a pointed neighborhood of $0$ in $\mathbb{C}$.

The infinitesimal generator $\log\varphi$ is of the form $\hat{f}(x,y)\partial/\partial y$ for some $\hat{f} \in \mathbb{C}[[x,y]]$. We have $\hat{f} = \sum_{j\geq 0} f_j(x)y^j$. Indeed $\hat{f}$ is transversally formal along $\gamma_0$, i.e. there exists a neighborhood $V \subset \mathbb{C}$ of $0$ such that $f_j \in \mathcal{O}(V)$ for any $j \geq 0$. This is a consequence of $\varphi_w$ being linearizable for any $w \in L^{-1}(e^{2\pi i\mathbb{Q}} \setminus \{1\})$ [**11**].

Consider a path $\eta \subset V \setminus \{0\}$ turning once around $0$ and transversal to $L^{-1}(\mathbb{S}^1)$. Moreover we can suppose that whenever $w \in \eta \cap L^{-1}(\mathbb{S}^1)$ then $L(w)$ is a Bruno number. Denote by $\sigma(w,y)$ the element of $\mathrm{Diff}_1(\mathbb{C},0)$ linearizing $\varphi_w$. By the choice of $\eta$ then $\sigma$ is continuous in $\eta \times W_0$ for some neighborhood $W_0$ of $0 \in \mathbb{C}$. As a consequence $\hat{f}$ is a continuous function in $\eta \times W$ for some neighborhood $W$ of $0 \in \mathbb{C}$. Then there exists $C \in \mathbb{R}^+$ such that

$$|f_j(w)| \leq C^j \quad \text{for all } (w,j) \in \eta \times \mathbb{N}.$$

The modulus maximum principle implies that $\log\varphi \in \mathcal{X}(\mathbb{C}^2,0)$. $\square$

The previous theorem implies that the dynamics of a generic $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ is unstable. Then the stable dynamics of a vector field is not a good model of the dynamics of $\varphi$. As a consequence the study of real flows of holomorphic vector fields is no good to classify topologically generic elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$.

In spite of the previous discussion real flows are useful to provide a complete system of analytic invariants for non-degenerate elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. Why this? This phenomenon is linked to the rigidity of analytic structures. Roughly speaking by doing cuts in the domain of definition of $\varphi$ we can find subsets $S$ such that the dynamics of $\varphi_{|S}$ is stable and close to the dynamics of $\exp(X_\varphi)_{|S}$. Moreover, the analytic class of $\varphi$ is determined by the analytic classes of $\varphi_{|S}$ for good choices of $S$ (here the rigidity in the analytic world plays a role). The cuts in the domain of definition allows us to avoid the instability related to resonances, small divisors and renormalized return maps. This point of view is developed in section 5 to obtain the theorem of analytic classification.

***Theorem 3.3.*** — *Consider non-degenerate elements* $\varphi, \eta$ *of* $\text{Diff}_{p1}(\mathbb{C}^2, 0)$. *Suppose that* $\text{Fix}(\varphi) = \text{Fix}(\eta)$. *Then* $\varphi \sim_{\text{an}} \eta$ *if and only if there exists* $r \in \mathbb{R}^+$ *such that for any* $w$ *in a pointed neighborhood of* 0 *the restrictions* $\varphi_{|x=w}$ *and* $\eta_{|x=w}$ *are conjugated by an injective holomorphic mapping defined in* $B(0, r)$ *and fixing the points in* $\text{Fix}(\varphi) \cap \{x = w\}$.

In the previous theorem the mappings conjugating the restrictions of $\varphi$ and $\eta$ to $x = w$ do not depend a priori continuously on $w$. Even so we can obtain an analytic conjugation because the spaces of orbits associated to the cuts $\varphi_{|S}$ are rigid (see section 5). The theorem is representative of a more general property: a non-degenerate element $\varphi$ of $\text{Diff}_{p1}(\mathbb{C}^2, 0)$ inherits the rigidity properties associated to $\varphi_{|x=0}$ and its space of orbits.

Resuming, we can use real flows of holomorphic vector fields to catch the stable structures contained in the dynamics of $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$. We will use such information to classify topologically stable elements of $\text{Diff}_{p1}(\mathbb{C}^2, 0)$ and to classify analytically non-degenerate elements of $\text{Diff}_{p1}(\mathbb{C}^2, 0)$.

$$\text{topological stability} \ \rightarrow \ \text{topological classification}$$

$$\text{analytic substability} \ + \text{rigidity of analytic structures} \ \rightarrow \ \text{analytic classification.}$$

## 4. Topological classification of MP-diffeomorphisms

Consider $\varphi \in \text{Diff}_{MP}(\mathbb{C}^2, 0)$. Because of the stability property its dynamics by iteration is close to the dynamics of $\exp(X_\varphi)$. The latter one is embedded in the dynamics of the vector field $\Re(X_\varphi)$. The subsections 4.1, 4.2 and 4.4 are intended to describe briefly the dynamics of $\Re(X_\varphi)_{|x=w}$ for $w$ in a neighborhood of $0 \in \mathbb{C}$.

In subsection 4.5 we introduce the tools to describe the instability phenomena attached to $\Re(X_\varphi)$, we also explain that the dynamics of $\Re(X_\varphi)_{|x=w}$ is simple for generic values of $w$. That could make us think that the dynamics of $\Re(X_\varphi)_{|x=w}$ does not depend on $w$. In subsection 4.6 we show that this is not the case if $\sharp(\text{Fix}(\varphi) \cap \{x = w\}) > 1$ for $w \neq 0$.

The subsection 4.7 is intended to describe the properties of the sets $Z$ corresponding to unstable parameters. The limit of the dynamics of $\Re(X_\varphi)_{|x=w}$ when $w \in Z$ and $w \to 0$ is more complicated than the dynamics of $\Re(X_\varphi)_{|x=0}$. In order to do this, given an analytic curve $\gamma \subset Z$ and a sequence of points $\gamma \times \mathbb{C} \ni (w_n, y_n) \to (0, y_0) \neq (0, 0)$, we study the limits of the trajectories of $\Re(X_\varphi)$ passing through points $(w_n, y_n)$ when $n \to \infty$. There are choices of $\gamma$ and $(w_n, y_n)$ such that the limit is bigger than the closure of the trajectory of $\Re(X_\varphi)$ passing through $(0, y_0)$. In subsection 4.8 we describe the evolution of the limits of trajectories with respect to $\gamma$. The existence of big limits (or long trajectories) is invariant by topological conjugation, their study provides the first topological invariants both for vector fields (subsection 4.9) and diffeomorphisms (subsection 4.10). These invariants are of formal type, they only depend on the formal class of the vector field or diffeomorphism. A second type of invariant, namely the analytic class of $\varphi_{|x=0}$, can be obtained by studying the evolution of the big limits with respect to curves $\gamma$. Finally the theorem of topological classification is presented in subsection 4.12.

**4.1. Flower Type Vector Fields.** — Consider the real flow $\Re(\xi)$ of a complex analytic vector field $\xi$ defined in an open subset $V$ of $\mathbb{C}$. Let $P \in V$ be a singular point of $\xi = a(y)\partial/\partial y$; the point $P$ is parabolic if $a'(P) = 0$. Throughout this subsection we will consider a complex analytic vector field $\xi$ defined in a neighborhood of $\overline{\mathbb{D}}$. By definition the vector field $\Re(\xi)$ is of *flower type* if $Sing\xi \cap \partial\mathbb{D} = \emptyset$ and all the singularities are parabolic.

We define $\Gamma_\xi[Q]$ the trajectory of $\Re(\xi)$ in $\mathbb{D}$ passing through $Q$. We also define the positive and negative trajectories $\Gamma_{\xi,+}[Q]$ and $\Gamma_{\xi,-}[Q]$ obtained by restraining $\Gamma_\xi[Q]$ to non-negative and non-positive times respectively. The trajectory $\Gamma_{\xi,+}[Q]$ is defined for times in some interval $[0,a)$ for some $a \in \mathbb{R}^+ \cup \{\infty\}$. Whenever $a < \infty$ we have $\Gamma_\xi[Q](a) \in \partial\mathbb{D}$; we denote $\omega_\xi(Q) = \infty$. Otherwise we denote by $\omega_\xi(Q)$ the omega limit of $\Gamma_\xi[Q]$. We can define the mapping $\alpha_\xi$ in an analogous way.

***Remark 4.1.*** — *If $\omega_\xi(Q)$ contains a singular point $P \in Sing\xi$ then $\omega_\xi(Q) = \{P\}$ since singular points are parabolic.*

**4.2. The dynamical Rolle property.** — We say that a flower type vector field $\Re(\xi)$ satisfies the *dynamical Rolle property* if there is no connected transversal $I$ such that $\Gamma_\xi[Q]$ cuts $I$ for two different values of time. Our definition implies that any vector field having cycles can not hold the Rolle condition. Anyway, the definition coincides with the usual one if all the cycles are isolated. We also call *no return property* the dynamical Rolle property.

***Proposition 4.1.*** — *Let $\Re(\xi)$ be a flower type vector field. Then $\Re(\xi)$ satisfies the dynamical Rolle property.*

*Proof.* — Suppose that $\Re(\xi)$ does not hold the dynamical Rolle property. There exist a trajectory $\gamma : [0,t] \to \mathbb{D}$ of $\Re(\xi)$ and a connected transversal $T \subset \mathbb{D}$ such that $\gamma^{-1}(T) = \{0,t\}$. We can suppose that $T$ is closed and $\partial T = \{\gamma(0), \gamma(t)\}$.

Consider the bounded connected component $B$ of $\mathbb{C} \setminus (\gamma[0,t] \cup T)$. We can suppose that $B$ is invariant by the positive iteration of $\Re(\xi)$ by changing $\xi$ with $-\xi$ if necessary. Choose a point $Q \in B$, the set $\omega_\xi(Q)$ is either a singular point or a cycle. In the latter case the cycle is limiting a bounded domain containing a singular point of $\xi$. Thus we obtain $Sing\xi \cap B \neq \emptyset$.

The mapping $\exp(s\xi)_{|B} : B \to B$ is well-defined for any $s \in \mathbb{R}^+$. Moreover $\exp(s\xi)_{|B}$ is tangent to the identity at the points in $Sing\xi \cap B$ by the flower type character. By Cartan's theorem we have $\exp(s\xi)_{|B} \equiv Id$ for any $s \in \mathbb{R}^+$. We obtain a contradiction since $B$ is not contained in $Sing\xi$. □

### 4.3. Multi-parabolic vector fields

***Definition 4.1.*** — *We denote by $\mathcal{X}_{p1}(\mathbb{C}^2,0)$ the set of vector fields of the form $g(x,y)\partial/\partial y \in \mathcal{X}(\mathbb{C}^2,0)$ with $g(0,0) = (\partial g/\partial y)(0,0) = 0$. We say that an element $g(x,y)\partial/\partial y$ of $\mathcal{X}_{p1}(\mathbb{C}^2,0)$ is non-degenerate if $g(0,y) \not\equiv 0$.*

***Definition 4.2.*** — *We say that $X = g(x,y)\partial/\partial y \in \mathcal{X}(\mathbb{C}^2,0)$ is a multi-parabolic vector field if $g(0,0) = 0$ and $(\partial g/\partial y)_{|SingX} \equiv 0$. We denote by $\mathcal{X}_{MP}(\mathbb{C}^2,0)$ the set of multi-parabolic vector fields.*

***Definition 4.3.*** — *A vector field $X \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$ is of the form $x^{m(X)}h(x,y)\partial/\partial y$ with $h(0,y) \not\equiv 0$. The radical ideal $\sqrt{(h)}$ is generated by some $k \in \mathbb{C}\{x,y\}$; we denote by $N(X)$ the order $\nu(k(0,y))$ at $y=0$. Indeed $N(X)$ is the cardinal of $\{x=w\} \cap SingX$ for $w \neq 0$. We define $\nu(X)$ as $\nu(h(0,y))-1$. We can define $m(\varphi) = m(X_\varphi)$, $N(\varphi) = N(X_\varphi)$ and $\nu(\varphi) = \nu(X_\varphi)$.*

Clearly a vector field $X \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$ is nilpotent. Moreover given a nilpotent germ of vector field $Y \in \mathcal{X}(\mathbb{C}^2,0)$ we have that $\exp(Y)$ belongs to $\mathrm{Diff}_{MP}(\mathbb{C}^2,0)$ (resp. $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$) if and only if $Y \in \mathcal{X}_{MP}(\mathbb{C}^2,0)$ (resp. $Y \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$).

***Remark 4.2.*** — *Let $X \in \mathcal{X}_{MP}(\mathbb{C}^2,0)$. The multi-parabolic character excludes unstable behavior. Indeed $\Re(X)_{|x=w}$ enjoys the no-return property for any $w$ in a neighborhood of 0 (prop. 4.1).*

In the case $N=1$, $m=0$ both multi-parabolic vector fields and diffeomorphisms are topological products. The problem of topological classification can be reduced to the setting of tangent to the identity diffeomorphisms in one variable.

***Proposition 4.2.*** — *Let $\varphi, \eta \in \mathrm{Diff}_{MP}(\mathbb{C}^2,0)$ with $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\eta)$. Suppose that $N(\varphi)=1$ and $m(\varphi)=0$. Then we have $\varphi \sim_{\mathrm{top}} \eta$.*

The case $N=1$, $m>0$ is a sort of singular "reparametrization" of the previous one since $X = x^{m(X)}Y$ with $m(Y)=0$ and $x$ is constant in $x=w$. We can use the one variable techniques to provide a complete system of topological invariants. Analogously the case $N=0$ (and then $m>0$) is also simple since it is a reparametrization of a regular case. Indeed proposition 4.2 is also valid for $N=0$. From now on we focus on the most interesting case, namely $N>1$. We will suppose the condition of non-degeneracy $m=0$, it is not necessary but the presentation is simpler.

**4.4. The graph.** — Consider $X \in \mathcal{X}_{MP}(\mathbb{C}^2,0)$. Let us fix a domain of definition $U_\epsilon = B(0,\delta) \times B(0,\epsilon)$ for some $\epsilon, \delta > 0$ small enough. We denote $\Gamma_X^\epsilon[Q]$ the trajectory of $\Re(X)$ in $U_\epsilon$ passing through $Q$. Analogously to subsection 4.2 we can define $\alpha_X^\epsilon$ and $\omega_X^\epsilon$.

We can associate an oriented graph $\mathcal{G}_{X,w}^\epsilon$ to $\Re(X)_{|\{w\}\times B(0,\epsilon)}$. The vertices are the elements of $SingX \cap \{x=w\}$. Consider the space $Sp(w)$ of trajectories $\gamma$ of $\Re(X)_{|\{w\}\times B(0,\epsilon)}$ such that $(\alpha,\omega)_X^\epsilon(\gamma) \in SingX \times SingX$. The edges of $\mathcal{G}_{X,w}^\epsilon$ are the isotopy classes of those trajectories in $Sp(w)$. We define the unoriented graph $\mathcal{N}G_{X,w}^\epsilon$

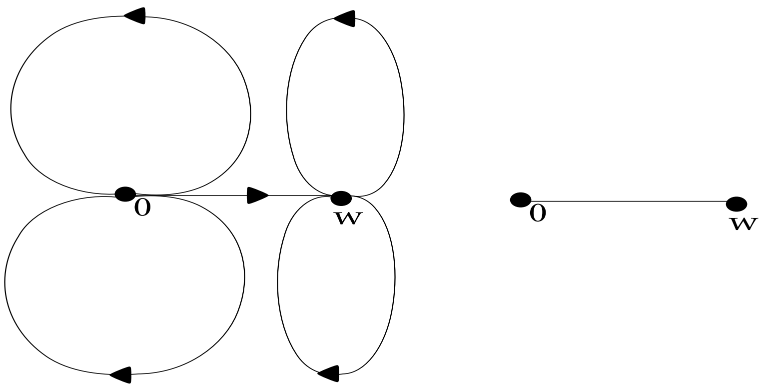


FIGURE 1. $\mathscr{G}_{X,w}$ and $\mathscr{N}G_{X,w}$ for $X = y^2(y-x)^2\partial/\partial y$ and $w \in \mathbb{R}^+$

obtained from $\mathscr{G}^{\epsilon}_{X,w}$ by removing the reflexive edges and the orientation.

***Proposition 4.3***. — *Let $X \in \mathcal{X}_{MP}(\mathbb{C}^2, 0)$. Then $\mathscr{N}G^{\epsilon}_{X,w}$ has no cycles.*

The proposition shares the spirit of the no-return property and they have analogous proofs. The lack of connectedness of $\mathscr{N}G^{\epsilon}_{X,w}$ is related to the existence of "long" trajectories.

***Proposition 4.4***. — *Let $X \in \mathcal{X}_{MP}(\mathbb{C}^2, 0)$. Then $(\alpha^{\epsilon}_X, \omega^{\epsilon}_X)^{-1}(\infty, \infty) \cap \{x = w\} = \emptyset$ implies that $\mathscr{N}G^{\epsilon}_{X,w}$ is connected.*

*Proof.* — Let $G_1$, …, $G_l$ be the set of connected components of $\mathscr{N}G^{\epsilon}_{X,w}$. Denote by $S_j$ the set of vertexes of $G_j$ for any $j \in \{1, \dots, l\}$. We define

$$F_j = ((\alpha^{\epsilon}_X)^{-1}(S_j) \cup (\omega^{\epsilon}_X)^{-1}(S_j)) \cap ((\{w\} \times B(0, \epsilon)) \setminus SingX) \quad \forall j \in \{1, \dots, l\}.$$

By the open character of parabolic fixed points we obtain that $F_j$ is an open set for $j \in \{1, \dots, l\}$. Moreover, we have $F_j \cap F_k = \emptyset$ for $j \neq k$ since $G_j$ and $G_k$ are different connected components of $\mathscr{N}G^{\epsilon}_{X,w}$. Then $(\alpha^{\epsilon}_X, \omega^{\epsilon}_X)^{-1}(\infty, \infty) \cap \{x = w\} = \emptyset$ implies

$$(\{w\} \times B(0, \epsilon)) \setminus SingX = \cup_{j=1}^{l} F_j.$$

The set $(\{w\} \times B(0, \epsilon)) \setminus SingX$ is connected, thus we get $l = 1$. □

***Remark 4.3***. — *It can be proved that $(\alpha^{\epsilon}_X, \omega^{\epsilon}_X)^{-1}(\infty, \infty) \cap \{x = w\} = \emptyset$ and the connectedness of $\mathscr{N}G^{\epsilon}_{X,w}$ are equivalent for any $w$ in some neighborhood of* 0.

**4.5. Quantitative analysis of trajectories.** — Let $X = g(x, y)\partial/\partial y \in \mathcal{X}_{MP}(\mathbb{C}^2, 0)$.

***Definition 4.4***. — *Let $\mathrm{Res}_X(x_0, y_0)$ be the residue of $dy/g(x_0, y)$ at $y = y_0$.*

Suppose that $(\alpha_X^\epsilon, \omega_X^\epsilon)^{-1}(\infty,\infty) \cap \{x = w\} \neq \emptyset$ for infinitely many points $w$ in every neighborhood of 0. Then there exist a sequence $\{w_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} w_n = 0$ and trajectories $\gamma_n : [0,t_n] \to \{w_n\} \times B(0,\epsilon)$ of $\Re(X)$ such that

$$\gamma_n[0,t_n] \cap (\{w_n\} \times \partial B(0,\epsilon)) = \{\gamma_n(0), \gamma_n(t_n)\}.$$

Let $\eta_n : [0,a_n] \to \{w_n\} \times \partial B(0,\epsilon)$ be the arc going from $\gamma_n(0)$ to $\gamma_n(t_n)$ in counter clock-wise sense. We denote by $C_n(w_n)$ the connected component of the set

$$(\{w_n\} \times B(0,\epsilon)) \setminus \gamma_n(0,t_n)$$

such that $\eta_n[0,a_n] \subset \overline{C_n(w)}$ . We define $E_n(w_n) = C_n(w_n) \cap SingX$. Given a set $E_n(w_n)$ we can define $E_n(w) \subset SingX \cap \{x = w\}$ for any $w$ in a neighborhood of 0 by continuous extension of $E_n(w_n)$. Any set build in this way is called a *continuous set of singular points.*

By taking a subsequence we can suppose that the limits $\lim_{n\to\infty} \gamma_n(0)$ and $\lim_{n\to\infty} \gamma_n(t_n)$ exist. We can also suppose that there exists a continuous set of singular points $E$ such that $E \equiv E_n$ for any $n \in \mathbb{N}$. Consider a holomorphic function $\psi_0$ defined in a neighborhood of $\lim_{n\to\infty} \gamma_n(0)$ and such that $X(\psi_0) \equiv 1$. Denote by $\psi_\partial$ the analytic continuation of $\psi_0$ along the arc going from $\lim_{n\to\infty} \gamma_n(0)$ to $\lim_{n\to\infty} \gamma_n(t_n)$ in counter clock-wise sense. Finally we define $\psi_I(w_n, y)$ the analytic continuation of $(\psi_0)_{|x=w_n}$ along $\gamma_n[0,t_n]$. We have $\psi_I(\gamma_n(t_n)) - \psi_0(\gamma_n(0)) = t_n$ for any $n \in \mathbb{N}$ by definition.

The theorem of the residues implies

$$\psi_I = \psi_\partial - 2\pi i \sum_{P\in E(x)} \mathrm{Res}_X(P).$$

Therefore we obtain

$$\psi_\partial(\gamma_n(t_n)) - \psi_0(\gamma_n(0)) - 2\pi i \sum_{P\in E(w_n)} \mathrm{Res}_X(P) = t_n.$$

The quantity $\psi_\partial(\gamma_n(t_n)) - \psi_0(\gamma_n(0))$ is uniformly bounded by a constant depending only on $X$. The function $x \to 2\pi i \sum_{P\in E(x^k)} \mathrm{Res}_X(P)$ is meromorphic for some $k \in \mathbb{N}$ [**11**]. In spite of that, it is not a holomorphic function since otherwise the sequence $\{t_n\}$ is bounded and this implies $(\alpha_X^\epsilon, \omega_X^\epsilon)^{-1}(\infty,\infty) \cap \{x = 0\} \neq \emptyset$.

***Proposition 4.5***. — *Let $X \in \mathcal{X}_{MP}(\mathbb{C}^2, 0)$. The graph $\mathcal{N}G_{X,w}^\epsilon$ is connected for generic values of $w$.*

*Proof.* — A point $w$ such that $(\alpha_X^\epsilon, \omega_X^\epsilon)^{-1}(\infty,\infty) \cap \{x = w\} \neq \emptyset$ is contained in some curve

$$\beta_{E,C} \equiv \{-2\pi i \sum_{P\in E(x)} \mathrm{Res}_X(P) \in \mathbb{R}^+ + iC\}$$

for some continuous set $E$ of singular points and some $C \in \mathbb{R}$. Moreover there exists a constant $D \in \mathbb{R}^+$ depending only on $X$ and such that $C \in [-D, D]$. Denote $\pi : (\mathbb{R}^+ \cup \{0\}) \times \mathbb{S}^1 \to \mathbb{C}$ the mapping defined by $\pi(r,\lambda) = r\lambda$. The set of tangent directions $\overline{\pi^{-1}(\beta_{E,C} \setminus \{0\})} \cap (\{0\} \times \mathbb{S}^1)$ of $\beta_{E,C}$ at 0 is finite and it does not depend on $C$. There are finitely many choices of continuous sets of singular points. Thus there exists a finite set $D_X \subset \mathbb{S}^1$ such that for any closed arc $ac \subset \mathbb{S}^1 \setminus D_X$ there exists $b > 0$ such that $(\alpha_X^\epsilon, \omega_X^\epsilon)^{-1}(\infty, \infty) \cap \{x = w\} = \varnothing$ for any $w \in (0, b)ac$. □

**4.6. Splitting of the dynamics.** — The dynamics of $\Re(X)_{|\{w\}\times B(0,\epsilon)}$ is not constant with respect to $w$ if $N(X) > 1$ (see def. 4.3). Otherwise the graph $\mathcal{NG}_{X,w}^\epsilon$ is connected for any $w$ in a neighborhood of 0 and it depends continuously on $w$. The next proposition states that this is not the case; indeed the graph can be disassembled.

***Proposition 4.6.*** — *Let $X \in \mathcal{X}_{MP}(\mathbb{C}^2, 0)$ with $N(X) > 1$. There are no permanent edges in $\mathcal{NG}_{X,w}^\epsilon$.*

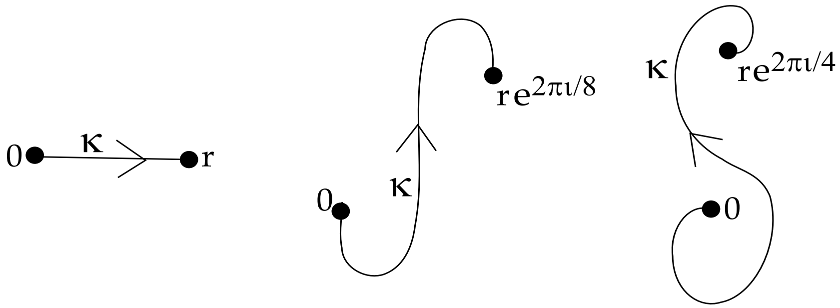


FIGURE 2. $X = y^2(y-x)^2\partial/\partial y$. Parameters $\theta = 0, 1/8, 1/4$

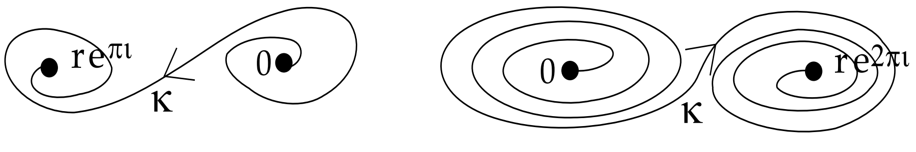


FIGURE 3. Parameters $\theta = 1/2$ and $\theta = 1$

Proposition 4.6 is proved by analyzing the dependence of the edge with respect to the parameter [**10**]. Let us consider the example $X = y^2(y-x)^2\partial/\partial y$. The edge $0 \to w$ belongs to $\mathcal{G}_{X,w}^\epsilon$ for any $w \in \mathbb{R}^+$. Suppose that $0 \to w$ is a permanent edge. Every trajectory $\gamma_w(-\infty, \infty)$ of $\Re(X)_{|\{w\}\times B(0,\epsilon)}$ representing the edge $0 \to w$ has well-defined tangents $\gamma_w(-\infty)$ at 0 and $\gamma_w(\infty)$ at $w$. Moreover the homotopy class $\kappa(w)$ of $\gamma_w[-\infty, \infty]$ in the space obtained by doing the real blow-up of $\mathbb{C}$ at the points 0 and $w$ does not depend on the choice of $\gamma_w$. Fix $r > 0$ small enough and let us consider $x = re^{2\pi i\theta}$ when $\theta$ goes from 0 to 1. The point $\gamma_{re^{2\pi i\theta}}(-\infty)$ is the direction given by $e^{-4\pi i\theta}\mathbb{R}^+$ at $y = 0$. The point $\gamma_{re^{2\pi i\theta}}(\infty)$ is the direction given by $re^{2\pi i\theta} + e^{\pi i - 4\pi i\theta}\mathbb{R}^+$ at $y = re^{2\pi i\theta}$. Then both $\gamma_{re^{2\pi i\theta}}(-\infty)$ and $\gamma_{re^{2\pi i\theta}}(\infty)$ are turning in clock-wise sense whereas the singular point $y = re^{2\pi i\theta}$ turns around $y = 0$ in counter clock-wise sense (see figures (2) and (3)). This phenomenon, i.e. tangents and singular points turning in opposite senses, forces the absurd inequality $\kappa(e^{2\pi i}r) \neq \kappa(r)$.

**4.7. Long Trajectories.** — The next two subsections are devoted to make clear that the limit of the dynamics of $\Re(X)_{|x=w}$ when $w \to 0$ is more complex than the dynamics of the limit $\Re(X)_{|x=0}$.

We already saw that the points $w$ in $(\alpha_X^\epsilon, \omega_X^\epsilon)^{-1}(\infty,\infty) \cap \{x = w\} \neq \emptyset$ are in curves of the form $-2\pi i \sum_{P \in E(x)} \mathrm{Res}_X(P) \in \mathbb{R}^+ + iC$ for some continuous set $E$ of singular points and some $C \in \mathbb{R}$ (subsection 4.5). Then it is interesting to consider limits of trajectories along branches of analytic curves contained in the parameter space.

Let $X \in \mathcal{X}_{MP}(\mathbb{C}^2, 0)$. Consider a branch of analytic curve $\beta$ and a point $y_0$ in $\overline{B(0,\epsilon)} \setminus \{0\}$ such that $\omega_X^\epsilon(0, y_0) = (0,0)$. We are interested on describing the limit of $\Gamma_{X,+}^\epsilon[w, y_0]$ when $w \in \beta$ and $w \to 0$. Let $y_1 \in \overline{B(0,\epsilon)} \setminus \{0\}$ be a point satisfying that there exists a mapping $y_1(w) : \beta \to \mathbb{C}$ such that

- $(0, y_1) = \lim_{w \in \beta, w \to 0} (w, y_1(w))$ and $(0, y_1) \notin \cap_{\eta > 0} \Gamma_{X,+}^{\epsilon+\eta}[0, y_0]$.
- For any $\eta > 0$ there exists $\upsilon(\eta) \in \mathbb{R}^+$ such that $(w, y_1(w)) \in \Gamma_{X,+}^{\epsilon+\eta}[w, y_0]$ for any $w \in B(0, \upsilon(\eta)) \cap \beta$.

The set of points satisfying the previous conditions will be denoted by $L_{\beta,y_0}^{+,\epsilon}(X)$ (or just $L_{\beta,y_0}^{+,\epsilon}$ if $X$ is implicit); it is the positive Long Limit (or just $L$-limit) associated to $y_0$, $\epsilon$ and $\beta$. We can define $L_{\beta,y_0}^{-,\epsilon}$ by replacing in the definition the positive trajectories with the negative ones.

***Remark 4.4.*** — *We have $L_{\beta,y_0}^{+,\epsilon}(X) = L_{\beta,y_1}^{+,\epsilon}(X)$ if $(0, y_0)$ and $(0, y_1)$ are in the same trajectory of $\Re(X)_{|\{0\} \times \overline{B(0,\epsilon)}}$. Every connected component of a L-limit $L_{\beta,y_0}^{+,\epsilon}(X)$ is a trajectory of $\Re(X)_{|\{0\} \times \overline{B(0,\epsilon)}}$.*

***Proposition 4.7* (Existence of the $L$-limit).** — *Let $X \in \mathcal{X}_{MP}(\mathbb{C}^2, 0)$. Suppose $N(X) > 1$ (def. 4.3). Consider $y_0 \in B(0,\epsilon) \setminus \{0\}$ such that $\omega_X^\epsilon(0, y_0) = (0,0)$. Then there exists a branch of analytic curve $\beta$ such that $L_{\beta,y_0}^{+,\epsilon}(X) \neq \emptyset$.*

The proof of the previous proposition is based on the splitting of the dynamics phenomenon. Analogously as in the proposition 4.6 it can be proved that the trajectory $\Gamma_{X,+}^\epsilon[x, y_0]$ does not depend continuously on $x$. In particular the set $S = \{x \in B(0,\delta) : \omega_X^\epsilon(x, y_0) \in SingX\}$ is not a neighborhood of the origin. The quantitative analysis in subsection 4.5 can be used to prove that $\partial S$ is a union of branches of analytic curves. It is enough to choose $\beta \subset \partial S$.

***Lemma 4.1.*** — *The number of connected components of $L_{\beta,y_0}^{+,\epsilon}(X)$ is finite.*

The lemma is a consequence of the dynamical Rolle property. Indeed a connected transversal to $\Re(X)_{|\{0\} \times B(0,\epsilon)}$ can not intersect two connected components of $L_{\beta,y_0}^{+,\epsilon}(X)$.

The Long limits admit a quantitative analysis. Given $y_1 \in L^{+,\epsilon}_{\beta,y_0}(X)$ there exist a continuous section $(x, y_1(x))$ for $x \in \beta \cup \{0\}$, a continuous set $E_1$ of singular points and a continuous function $T_1 : \beta \to \mathbb{R}^+$ such that

- $y_1(0) = y_1$ and $\lim_{x\in\beta,x\to 0} T_1(x) = \infty$.
- $(x, y_1(x)) = \exp(T_1(x)X)(x, y_0)$
- $T_1(x) = \psi_\partial(x, y_1(x)) - \psi_0(x, y_0) - 2\pi i \sum_{P\in E_1(x)} \mathrm{Res}_X(P)$ for any $x \in \beta$.

Last property is obtained as in subsection 4.5 for trajectories of $\Re(X)_{\{w\}\times B(0,\epsilon)}$ dividing $\{w\} \times B(0, \epsilon)$. Indeed Long Trajectories can be interpreted as dividing trajectories just by reducing the domain of definition.

The connected components of $L^{+,\epsilon}_{\beta,y_0}$ are ordered by the time of the flow. Let $(0, y_1), (0, y_2) \in L^{+,\epsilon}_{\beta,y_0}$. Denote by $\gamma_j$ the trajectory of $\Re(X)_{|\{0\}\times\overline{B(0,\epsilon)}}$ containing $(0, y_j)$ for $j \in \{1, 2\}$. Consider data $(x, y_j(x))$, $E_j$ and $T_j$ as above. By the non-oscillating property for analytic curves we have three possibilities:

- $\lim_{x\in\beta,x\to 0} T_2(x) - T_1(x) = \infty$. We define $\gamma_1 < \gamma_2$.
- $\lim_{x\in\beta,x\to 0} T_2(x) - T_1(x) = -\infty$. We define $\gamma_1 > \gamma_2$.
- $\lim_{x\in\beta,x\to 0} T_2(x) - T_1(x) \in \mathbb{R}$. We have $\gamma_1 = \gamma_2$.

**4.8. Evolution of the Long Trajectories.** — In this subsection we study the dependence of $L^{+,\epsilon}_{\eta,y_0}(X)$ with respect to $\eta$. Moreover we introduce the openness principle. Namely, if there exists a Long Trajectory joining $y_0$ and $y_1$, i.e. $y_1 \in L^{+,\epsilon}_{\eta,y_0}(X)$, then there exists a Long Trajectory joining $y_0$ and $y_2$ for any $y_2$ in a neighborhood of $y_1$.

Suppose $y_1 \in L^{+,\epsilon}_{\eta,y_0}(X)$. There exist a continuous section $(x, y_1(x))$ for $x$ in $\eta \cup \{0\}$, a continuous set $E$ of singular points and a function $T : \eta \to \mathbb{R}^+$ such that

$$T(x) = \psi_\partial(x, y_1(x)) - \psi_0(x, y_0) - 2\pi i \sum_{P\in E(x)} \mathrm{Res}_X(P)$$

for any $x \in \eta$ (see subsection 4.7). Let $\upsilon \in \mathbb{R}$; denote by $\eta(\upsilon)$ the curve given by

$$i\upsilon + \psi_\partial(0, y_1) - \psi_0(0, y_0) - 2\pi i \sum_{P\in E(x)} \mathrm{Res}_X(P) \in \mathbb{R}^+$$

and whose tangent at $x = 0$ coincides with the tangent of $\eta$. The point $y_1$ belongs to $L^{+,\epsilon}_{\eta(0),y_0}(X)$. Moreover, we have

$$\lim_{x\in\eta(\upsilon),x\to 0} \psi_\partial \circ \exp(i\upsilon X)(0, y_1) - \psi_0(0, y_0) - 2\pi i \sum_{P\in E(x)} \mathrm{Res}_X(P) = \infty + i0.$$

The previous equality is key to prove the openness principle.

***Proposition 4.8* (Openness principle).** — *Let $X \in \mathcal{X}_{MP}(\mathbb{C}^2, 0)$ and $y_1 \in L^{+,\epsilon}_{\eta,y_0}(X)$. Then $\exp((\rho + i\upsilon)X)(0, y_1)$ belongs to $L^{+,\epsilon(\rho,\upsilon)}_{\eta(\upsilon),y_0}(X)$ for all $\rho, \upsilon \in \mathbb{R}$ in a small neighborhood of 0 and some $\epsilon(\rho, \upsilon) > 0$ such that $\lim_{(\rho,\upsilon)\to 0} \epsilon(\rho, \upsilon) = \epsilon$.*

Let us remind the reader that we are studying multi-parabolic vector fields as models of MP-diffeomorphisms. Therefore we are interested on the limit of the discrete trajectories $\{\exp(X)^n(x,y_0)\}_{n\in\mathbb{N}}$ when $x\to 0$. Denote

$$F_{\rho,\upsilon}(x)=\rho+i\upsilon+\psi_\partial(0,y_1)-\psi_0(0,y_0)-2\pi i\sum_{P\in E(x)}\mathrm{Res}_X(P).$$

Consider $\rho,\upsilon\in\mathbb{R}$ in a small neighborhood of 0. We have

$$\exp((\rho+i\upsilon)X)(0,y_1)=\lim_{n\to\infty}\exp(F_{\rho,\upsilon}(x_n^{\rho,\upsilon})X)(x_n^{\rho,\upsilon},y_0)$$

where $\{x_n^{\rho,\upsilon}\}$ is a sequence of points in $\eta(\upsilon)\cap F_{\rho,\upsilon}^{-1}(\mathbb{N})$ with $\lim_{n\to\infty}x_n^{\rho,\upsilon}=0$. Thus the orbits $\{\exp(X)^n(x,y_0)\}$ accumulate on $\Gamma_X^\epsilon[\exp(i\upsilon X)(0,y_1)]$ when $x\in\eta(\upsilon)$ and $x\to 0$.

Resuming, the limit of the orbit $\{\exp(X)^n(x,y_0)\}_{n\in\mathbb{N}}$ when $x\in\eta$, $L_{\eta,y_0}^{+,\epsilon}(X)=\varnothing$ and $x\to 0$ is the discrete orbit $\{\exp(X)^n(0,y_0)\}_{n\in\mathbb{N}}$. Supposed $y_1\in L_{\eta(0),y_0}^{+,\epsilon}(X)$ we have that $\{\exp(X)^n(x,y_0)\}_{n\in\mathbb{N}}$ generates the real flow $\Re(X)$ of $X$ through $(0,y_1)$ when $x\in\eta(0)$ and $x\to 0$. Moreover $\{\exp(X)^n(x,y_0)\}_{n\in\mathbb{N}}$ generates the complex flow of $X$ through $(0,y_1)$ when $x\in\cup_{\upsilon\in\mathbb{R}}\eta(\upsilon)$ and $x\to 0$ by the openness principle.

**4.9. Formal type topological invariants.** — In this subsection we introduce topological invariants obtained by analyzing how much time a multi-parabolic vector field spends in travelling along a Long Trajectory.

Consider $X_1$ and $X_2$ in $\mathcal{X}_{MP}(\mathbb{C}^2,0)$ with $SingX_1=SingX_2$. Suppose that $\exp(X_1)\sim_{\mathrm{top}}\exp(X_2)$ by a germ of homeomorphism $\sigma$. We have

$$\sigma(B(0,\delta)\times B(0,\epsilon_1))\subset B(0,\delta)\times B(0,\epsilon_2)$$

for some $\epsilon_1,\epsilon_2>0$ small enough. Given $y_1\in L_{\beta,y_0}^{+,\epsilon_1}(X_1)$ consider a continuous section $y_1(x):\beta\cup\{0\}\to\mathbb{C}$, a continuous set of singular points $E$ and a function $T:\beta\to\mathbb{R}$ such that

$$T(x)=\psi_{\partial,1}(x,y_1(x))-\psi_{0,1}(x,y_0)-2\pi i\sum_{P\in E(x)}\mathrm{Res}_{X_1}(P)$$

for any $x\in\beta$ (see subsection 4.7). Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in $\beta\cap T^{-1}(\mathbb{N})$ such that $\lim_{n\to\infty}x_n=0$. Since $\exp(X_1)$ is topologically conjugated to $\exp(X_2)$ we obtain

$$T(x_n)=\psi_{\partial,2}\circ\sigma(x_n,y_1(x_n))-\psi_{0,2}(x_n,y_0)-2\pi i\sum_{P\in E(x_n)}\mathrm{Res}_{X_2}(P)$$

for any $n\in\mathbb{N}$. Denote $G(x)=\sum_{P\in E(x)}\mathrm{Res}_{X_1}(P)-\sum_{P\in E(x)}\mathrm{Res}_{X_2}(P)$. Suppose for simplicity that $X_1$ is non-degenerate. Hence the previous formula implies that the sequence $G(x_n)$ is bounded. Since $G(x)$ is a meromorphic function up to a ramification $x\to x^k$ then $G(x)$ is a bounded function in a neighborhood of $x=0$. Thus the principal parts of $\sum_{P\in E(x)}\mathrm{Res}_{X_1}(P)$ and $\sum_{P\in E(x)}\mathrm{Res}_{X_2}(P)$ coincide. By analogous techniques and the splitting of the dynamics phenomena we obtain the sufficient condition in the next theorem.

We say that the extended principal parts Ext.ppal.$(X_1)$ and Ext.ppal.$(X_2)$ of $X_1$ and $X_2$ respectively coincide if the function

$$x \to x^{m(X_1)k}(\mathrm{Res}_{X_1}(x^k, h(x)) - \mathrm{Res}_{X_2}(x^k, h(x)))$$

is holomorphic and vanishes at 0 for all continuous sections $(x^k, h(x))$ of $SingX_1$.

***Theorem 4.1.*** — *Let $X_1, X_2 \in \mathcal{X}_{MP}(\mathbb{C}^2, 0)$ with $SingX_1 = SingX_2$. Suppose that $(N(X_1), m(X_1)) \neq (1,0)$ (see def. 4.3). Then* $\exp(X_1) \sim_{\mathrm{top}} \exp(X_2)$ *if and only if* Ext.ppal.$(X_1) =$ Ext.ppal.$(X_2)$. *Moreover, we have*

$$\exp(X_1) \sim_{\mathrm{top}} \exp(X_2) \Leftrightarrow \Re(X_1) \sim_{\mathrm{top}} \Re(X_2).$$

The extended principal part is an invariant of formal type since $X_1 \sim_{for} X_2$ implies Ext.ppal.$(X_1) =$ Ext.ppal.$(X_2)$.

**4.10. Multi-parabolic diffeomorphisms.** — The stability theorem 3.1 implies that MP-vector fields provide good models for MP-diffeomorphisms. The proof is based on dividing a neighborhood of $(0,0)$ in regions in which we use different techniques to show that the orbits of $\varphi \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$ remain close to the orbits of $\exp(X_\varphi)$. A finiteness argument, based on the dynamical Rolle property, is required to prove that the trajectories of $\Re(X_\varphi)$ can not visit infinitely many regions.

***Definition 4.5.*** — *Let $\varphi \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$. We define* $\mathrm{Res}_\varphi(P) = \mathrm{Res}_{X_\varphi}(P)$ *(see def. 4.4) for $P \in$* $\mathrm{Fix}(\varphi)$. *Denote* Ext.ppal.$(\varphi) =$ Ext.ppal.$(X_\varphi)$. *The definitions do not depend on the choice of $X_\varphi$.*

Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$ with $\mathrm{Fix}(\varphi_1) = \mathrm{Fix}(\varphi_2)$. Suppose that $\varphi_1 \sim_{\mathrm{top}} \varphi_2$ by a homeomorphism $\sigma$. Given $y_1 \in L^{+,\epsilon}_{\beta, y_0}(X_{\varphi_1})$ we have a continuous section $y_1(x) : \beta \cup \{0\} \to \mathbb{C}$, a continuous set of singular points $E$ and a function $T : \beta \to \mathbb{R}^+$ such that

$$T(x) = \psi_\partial(x, y_1(x)) - \psi_0(x, y_0) - 2\pi i \sum_{P \in E(x)} \mathrm{Res}_{\varphi_1}(P).$$

The stability theorem implies that $\sigma$ almost conjugates $\exp(X_{\varphi_1})$ and $\exp(X_{\varphi_2})$. Then we can use arguments analogous to those in the previous section to prove that the principal parts of $\sum_{P \in E(x)} \mathrm{Res}_{\varphi_1}(P)$ and $\sum_{P \in E(x)} \mathrm{Res}_{\varphi_2}(P)$ coincide. Indeed we can push forward the analogy to obtain

***Theorem 4.2.*** — *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$ with* $\mathrm{Fix}(\varphi_1) = \mathrm{Fix}(\varphi_2)$. *Suppose that $(N(\varphi_1), m(\varphi_1)) \neq (1,0)$. Then $\varphi_1 \sim_{\mathrm{top}} \varphi_2$ implies* Ext.ppal.$(\varphi_1) =$ Ext.ppal.$(\varphi_2)$.

The extended principal part is, as for vector fields, an invariant of formal type since $\varphi_1 \sim_{for} \varphi_2$ implies Ext.ppal.$(\varphi_1) =$ Ext.ppal.$(\varphi_2)$.

**4.11. Analytic type topological invariants.** — The extended principal part is a complete system of topological invariants for MP-vector fields. In spite of this more topological invariants are required in order to classify MP-diffeomorphisms.

Let $\varphi \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$. Denote $X = X_\varphi$. Given $y_1 \in B(0,\epsilon) \setminus \{0\}$ such that $\alpha_X^\epsilon(0, y_1) = (0,0)$ (the situation where $\omega_X^\epsilon(0,y_1) = (0,0)$ is analogous) there exists $L_{\eta(0),y_1}^{-,\epsilon} \neq \emptyset$ by prop. 4.7. The diffeomorphism $\varphi_{|x=0}$ is embedded in a natural way in a complex flow $Y$ defined in the repelling petal $V \subset \{0\} \times B(0,\epsilon)$ containing $(0,y_1)$. The following theorem is the analogue of proposition 4.8 for MP-diffeomorphisms.

***Theorem 4.3* (Openness principle).** — *Let $\varphi \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$ with $N(\varphi) > 1$. Consider $y_1 \in B(0,\epsilon) \setminus \{0\}$ such that $\alpha_X^\epsilon(0,y_1) = (0,0)$. There exists $y_0 \in B(0,\epsilon) \setminus \{0\}$ such that for any $\rho + iv$ in a neighborhood of $0 \in \mathbb{C}$ there exist a branch of analytic curve $\eta(v)$ and sequences $\{T_n^{\rho,v}\}_{n\in\mathbb{N}}$ of natural numbers and $\{x_n^{\rho,v}\}_{n\in\mathbb{N}}$ of points in $\eta(v)$ satisfying*

- $\lim_{n\to\infty} T_n^{\rho,v} = \infty$ *and* $\lim_{n\to\infty} x_n^{\rho,v} = 0$.
- $\lim_{n\to\infty} \varphi^{T_n^{\rho,v}}(x_n^{\rho,v}, y_0) = \exp((\rho + iv)Y)(0, y_1)$.

*Moreover $\eta(v)$ depends continuously on $v$.*

Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$ with $\mathrm{Fix}(\varphi_1) = \mathrm{Fix}(\varphi_2)$. Suppose that $\varphi_1 \sim_{\mathrm{top}} \varphi_2$ by a homeomorphism $\sigma$. The openness theorem implies that given $y_1 \in B(0,\epsilon) \setminus \{0\}$ the restriction $\sigma_{|x=0}$ is determined in a neighborhood of $(0,y_1)$ by $\sigma(0,y_0)$ for some $y_0 \in B(0,\epsilon) \setminus \{0\}$. The germ of $\sigma_{|x=0}$ in the neighborhood of $(0,y_1)$ only depends on the set of analytic data $\varphi_1$, $\varphi_2$ and $\sigma(0,y_0)$. This property allows to prove that $\sigma_{|x=0}$ is analytic in the neighborhood of $(0,y_1)$ for any $y_1 \in B(0,\epsilon) \setminus \{0\}$. By Riemann's theorem we obtain

***Theorem 4.4.*** — *Let $\varphi, \eta \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$. Suppose that we have $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\eta)$ and $(N,m)(\varphi) \neq (1,0)$ (see def. 4.3). Then*

$$\varphi \sim_{\mathrm{top}} \eta \implies \varphi_{|x=0} \sim_{\mathrm{an}} \eta_{|x=0}.$$

This theorem provides topological invariants of analytic type.

**4.12. Theorem of topological classification.** — The formal and analytic type topological invariants compose a complete system of topological invariants.

***Theorem 4.5.*** — *Let $\varphi, \eta \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$. Suppose that we have $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\eta)$ and $(N,m)(\varphi) \neq (1,0)$. Then*

$$\varphi \sim_{\mathrm{top}} \eta \Leftrightarrow \begin{cases} \mathrm{Ext.ppal.}(\varphi) = \mathrm{Ext.ppal.}(\eta) \\ \varphi_{|x=0} \sim_{\mathrm{an}} \eta_{|x=0} \end{cases}$$

The last theorem and the proposition 4.2 provide a complete system of topological invariants for the multi-parabolic diffeomorphisms. The discussion so far has been intended to introduce the invariants and the ideas in the proof of the implication $\Rightarrow$ in the previous theorem.

***Remark 4.5.*** — *The topological invariants are related to the study of the unstable phenomena, more precisely the Long Trajectories. As a consequence there are no topological invariants related to the behavior of a MP-diffeomorphism in a neighborhood of its fixed points.*

***Remark 4.6.*** — *The theorem of topological classification relates formal, analytic and topological invariants.*

The proof of the implication $\Leftarrow$ in theorem 4.5 can not be reduced to the case of vector fields since $\varphi \sim_{\text{top}} \exp(Y)$ implies $\varphi_{|x=0} \sim_{\text{an}} \exp(Y)_{|x=0}$. Thus $\varphi$ is not topologically conjugated to the exponential of a vector field whenever $\varphi_{|x=0}$ is not embedded in an analytic flow, i.e in the generic situation.

We have that

$$\text{Ext.ppal.}(X_1) = \text{Ext.ppal.}(X_2) \implies \Re(X_1) \sim_{\text{top}} \Re(X_2)$$

for $X_1, X_2 \in \mathcal{X}_{MP}(\mathbb{C}^2, 0)$ with $Sing X_1 = Sing X_2$. What can we say for multi-parabolic diffeomorphisms? In other words how far are $\varphi, \eta \in \text{Diff}_{MP}(\mathbb{C}^2, 0)$ from being topologically conjugated if $\text{Ext.ppal.}(\varphi) = \text{Ext.ppal.}(\eta)$?

Let $\varphi, \eta \in \text{Diff}_{MP}(\mathbb{C}^2, 0)$ with $\text{Fix}(\varphi) = \text{Fix}(\eta)$. We say that $\sigma$ is a *tangential conjugation* if there exists $\epsilon > 0$ such that

- $x \circ \sigma \equiv x$ and $\sigma_{|\text{Fix}(\varphi) \setminus \{x=0\}} \equiv Id$.
- $\sigma, \sigma^{-1}$ are homeomorphisms defined in $((B(0,\delta) \setminus \{0\}) \times B(0,\epsilon)) \cup \{(0,0)\}$.

Roughly speaking a tangential conjugation is not very different from a germ of homeomorphism, but we allow some noise when $x \to 0$.

***Theorem 4.6.*** — *Let $\varphi, \eta \in \text{Diff}_{MP}(\mathbb{C}^2, 0)$ with $\text{Fix}(\varphi) = \text{Fix}(\eta)$. Then the equality $\text{Ext.ppal.}(\varphi) = \text{Ext.ppal.}(\eta)$ implies the existence of a tangential $\sigma_T$ conjugating $\varphi$ and $\eta$.*

The choice of $\sigma_T$ is not unique but it is possible to build $\sigma_T$ in such a way [**10**] that $\sigma_T$ is a germ of diffeomorphism if and only if $\varphi_{|x=0} \sim_{\text{an}} \eta_{|x=0}$. Thus the property $\varphi_{|x=0} \sim_{\text{an}} \eta_{|x=0}$ can be interpreted as a condition on elimination of noise for tangential conjugations.

***Remark 4.7.*** — *Let $\varphi, \eta \in \text{Diff}_{MP}(\mathbb{C}^2, 0)$ with $\text{Fix}(\varphi) = \text{Fix}(\eta)$. Supposed $\varphi \sim_{\text{top}} \eta$ then the topological conjugation $\sigma$ can be chosen to be $C^\infty$ outside $\text{Fix}(\varphi) \cup \{x = 0\}$. It is known (Martinet-Ramis [**6**], Ahern-Rosay [**1**] and Rey [**8**] for optimal results) that in general $\sigma$ can not be chosen $C^{\nu_0}$ for some $\nu_0 \in \mathbb{N}$. The MP-diffeomorphisms*

$$\varphi = \exp\left(\frac{y^3 (y-x)^2}{1 + x y^2 (y-x)^2} \frac{\partial}{\partial y}\right) \text{ and } \eta = \exp\left(y^3 (y-x)^2 \frac{\partial}{\partial y}\right)$$

*are topologically conjugated. It can be proved that the topological conjugation can not be chosen $C^1$ in $\{0\} \times (B(0,\epsilon) \setminus \{0\})$. The fixed points set is singular with respect to the $C^\nu$ conjugation as well as the dynamical singular set $x = 0$.*

Finally let us stress some of the key points in our approach of the dynamics of multi-parabolic diffeomorphisms:

- The limit of the dynamics is more complex than the dynamics of the limit in the dynamically interesting cases ($N > 1$). Indeed the limit of discrete dynamics in the generic lines $x = w$ with $w \neq 0$ generate a complex flow-like structure in the limit line $x = 0$. A consequence of this property is theorem 4.4.
- The topological classification theorem relies on a qualitative and quantitative description of the dynamics.
- A multi-parabolic diffeomorphism $\varphi$ supports non-trivial dynamics whenever $N(\varphi) > 1$. The existence of the $L$-limits is an example of this statement. Moreover, generically $\varphi$ is not topologically conjugated to the exponential of a vector field.

## 5. Analytic classification of elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$

We provide in this section an analytic classification of the elements of $\varphi$ in $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ (theorems 5.3 and 5.4). Moreover, we sketch the proof of theorem 3.3. Both goals are fulfilled in subsection 5.9. We introduce an extension of the Ecalle-Voronin invariants (subsection 5.8) based on an extension of the Fatou coordinates of $\varphi_{|x=0}$ to the nearby parameters (subsection 5.7). The philosophy of this approach is explained in subsection 5.2.

The subsections 5.3, 5.4, 5.5 and 5.6 can be considered as a setup for the construction of the Fatou coordinates. The proof of theorem 3.3 requires quantitative estimates for the Fatou coordinates in the neighborhood of the fixed points. We introduce the notion of infinitesimal stability (subsection 5.4) allowing to obtain the desired estimates. This is a stronger version of a qualitative version of stability, namely topological stability provided in subsection 5.5. The definition of infinitesimal stability is obtained after splitting conveniently a neighborhood of the origin (subsection 5.3). Finally in subsection 5.6 we introduce the sets in which the extensions of the Fatou coordinates are defined.

**5.1. Relation between the analytic and formal classifications.** — Generically the analytic classification of elements in $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ can be reduced to a formal classification problem. Anyway our approach has an advantage, it allows to take profit of the rigidity properties of elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$.

***Definition 5.1.*** — *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ be non-degenerate. Consider an irreducible component $\gamma$ of $\mathrm{Fix}(\varphi)$. We say that $\gamma$ is parabolic if $(\partial(y \circ \varphi)/\partial y)_{|\gamma} \equiv 1$.*

***Theorem 5.1.*** — *Let $\varphi, \eta$ be non-degenerate elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0) \setminus \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$ with $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\eta)$. Suppose that $\varphi \sim_{for} \eta$ by $\hat{\sigma} \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$. Suppose also that $y \circ \hat{\sigma}$ is transversally formal along a non-parabolic irreducible component $\gamma$ of $\mathrm{Fix}(\varphi)$. Then $\sigma$ is analytic.*

*Proof.* — Analogously to the proof of th. 3.2 we can suppose that $\gamma \equiv \{y = 0\}$. Denote $L(x) = (\partial(y \circ \varphi)/\partial y)(x, 0)$. Since $\varphi \sim_{for} \eta$ we have $L(x) \equiv (\partial(y \circ \eta)/\partial y)(x, 0)$ [**11**]. By hypothesis there exists a neighborhood $V$ of $0 \in \mathbb{C}$ such that the series $y \circ \hat{\sigma}$ is of the form

$$y \circ \hat{\sigma} = \sum_{j=1}^{\infty} \sigma_j(x) y^j \text{ with } \sigma_j \in \mathcal{O}(V) \text{ for all } j \in \mathbb{N}.$$

Consider a path $\kappa \subset V \setminus \{0\}$ turning once around 0 and transversal to $L^{-1}(\mathbb{S}^1)$. Moreover we can suppose that whenever $w \in \kappa \cap L^{-1}(\mathbb{S}^1)$ then $L(w)$ is a Bruno number. The choice of $\kappa$ implies that $\hat{\sigma}$ is continuous in $\kappa \times B(0, \upsilon)$ for some $\upsilon > 0$. Then there exists $C \in \mathbb{R}^+$ such that

$$|\sigma_j(x)| \leq C^j \quad \text{for all } (x, j) \in \kappa \times \mathbb{N}.$$

The modulus maximum principle implies that $\hat{\sigma} \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$. □

There is a downside in interpreting the analytic invariants of a non-degenerate $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ as formal invariants along a non-parabolic component of $\mathrm{Fix}(\varphi)$. Indeed this system of invariants does not provide as much information on the nature of $\varphi$ as the one we are going to introduce. For instance theorem 3.3 relies on the fact that $\varphi_{|\{w\} \times B(0,\epsilon)}$ inherits the rigidity properties of tangent to the identity diffeomorphisms in one variable for $w \neq 0$. This property is of very difficult translation in the formal setting.

**5.2. Analytic classification and extension of Fatou coordinates.** — We provide a complete system of analytic invariants for non-degenerate $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. The system of invariants is based on building extensions of the Fatou coordinates of $\varphi_{|x=0}$ to the nearby parameters. This topic is is a classical subject of study (Lavaurs [**3**], Shishikura [**13**], Oudkerk [**7**], Mardesic-Roussarie-Rousseau [**5**]).

***Definition 5.2.*** — *A Fatou coordinate for a diffeomorphism* $\varphi \in \mathrm{Diff}\,(\mathbb{C}^n, 0)$ *is a complex valued function* $\psi^\varphi$ *such that* $\psi^\varphi \circ \varphi = \psi^\varphi + 1$.

***Definition 5.3.*** — *Consider a non-degenerate element* $\varphi$ *of* $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. *Denote* $\nu(\varphi) = \nu(y \circ \varphi(0, y)) - 1$ *where* $\nu(y \circ \varphi(0, y))$ *is the order of* $y \circ \varphi(0, y)$ *at* 0.

The methods to obtain extensions of the Fatou coordinates are based on choosing a system of transversals to the dynamics of $\varphi_{|x=w}$ depending continuously on $w$. Even if the dynamics of $\varphi$ is discrete it is possible to make sense of the previous statement. The approach consists in considering coordinates $(x, z)$ in which $\varphi$ is very similar to $(x, z + 1)$. Then every real line not parallel to $\mathbb{R}$ provides a choice of transversal to $\varphi_{|x=w}$. Indeed we always choose transversals $T_w \subset \{x = w\}$ homeomorphic to $\mathbb{R}$. Given a parametrization $T_w(t) : (-\infty, \infty) \to \{x = w\}$ we demand

$$\lim_{t \to -\infty} T_w(t) \in \mathrm{Fix}(\varphi) \ni \lim_{t \to \infty} T_w(t).$$

The transversal $T_w$ and its image $\varphi(T_w)$ enclose a strip $S(T_w)$. The space of orbits of

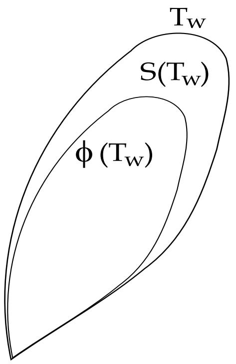


FIGURE 4.

$\varphi_{|S(T_w)}$ is biholomorphic to $\mathbb{C}^*$. Given a biholomorphism $\rho_{T_w}$ conjugating the space of orbits of $\varphi_{|S(T_w)}$ and $\mathbb{C}^*$ the function $\psi^{\varphi}_{T_w} = (1/2\pi i)\ln \rho_{T_w}$ is a Fatou coordinate of $\varphi$ in the fundamental domain $S(T_w)$. The Lavaurs vector field is by definition the unique holomorphic vector field $X^{\varphi}_{T_w}$ in $S(T_w)$ such that $X^{\varphi}_{T_w}(\psi^{\varphi}_{T_w}) \equiv 1$. Clearly it satisfies $\varphi \equiv \exp(X^{\varphi}_{T})$.

Suppose that $\varphi_{|x=0}$ is generic, then $\varphi_{|x=0}$ is of codimension 1, i.e. $\nu(\varphi) = 1$, and $N(\varphi) = 2$. In Mardesic-Roussarie-Rousseau's paper **[5]** the diffeomorphism $\varphi$ is "prepared". More precisely, up to an analytic change of coordinates, they suppose that the multipliers of the fixed points of $\varphi$ be equal to the multipliers of $\exp(\frac{y^2-x}{1+a(x)y}\frac{\partial}{\partial y})$ for some $a \in \mathbb{C}\{x\}$. The change of coordinates at $x = w$ approaching the dynamics of $\varphi$ to $(x, z+1)$ is obtained by lifting $\varphi_{|x=w}$ to the universal covering of $\mathbb{P}^1(\mathbb{C})\setminus(\mathrm{Fix}(\varphi)\cap\{x = w\})$. The covering transformation $\pi_w : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \setminus (\mathrm{Fix}(\varphi) \cap \{x = w\})$ can be explicitly written and be made to depend holomorphically on $w$. Therefore a continuous system of transversals can be build. This method is only valid if $\varphi_{|x=0}$ is of codimension 1.

The Oudkerk's approach is different. He obtains Fatou coordinates for unfoldings of every $\phi \in \mathrm{Diff}_1(\mathbb{C}, 0)$ independently of the codimension of $\phi$. Given a point $(w, y_0) \notin \mathrm{Fix}(\varphi)$ we consider the change of coordinates $\Delta(z) = \exp(zX)(w, y_0)$ defined from a neighborhood of 0 to a neighborhood of $(w, y_0)$ in $x = w$. We have

$$\Delta^* X = \partial/\partial z \ \text{ and } \ \Delta^{-1} \circ \exp(X) \circ \Delta = z + 1.$$

In order to find transversals to the dynamics of $\varphi$ we choose a vector field $X$ such that $\varphi \sim \exp(X)$. Then we fix $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$ and consider as transversals trajectories of the form $\Gamma^{\epsilon}_{\mu X}[x, y_0(x)]$ where $y_0$ is continuous. In order to obtain holomorphic extensions of the Fatou coordinates for $x$ in a domain $D \subset \mathbb{C}^*$ then $\omega^{\epsilon}_{\mu X}(x, y_0(x))$ and $\alpha^{\epsilon}_{\mu X}(x, y_0(x))$ have to be continuous sections of $\mathrm{Fix}(\varphi)$ for $x$ in $D$. In other words the trajectory $\Gamma^{\epsilon}_{\mu X}[x, y_0(x)]$ has a stable behavior for $x \in D$. Indeed given $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$

the trajectories of $\Re(\mu X)$ provide a good choice of system of transversals for the parameters in which $w \to \Re(\mu X)_{|x=w}$ is stable.

More precisely, consider a direction $e^{2\pi i\theta}\mathbb{R}^+$ ($\theta \in \mathbb{R}$) in the parameter space. We choose $\mu \in \mathbb{S}^1 \setminus \{-1,1\}$ such that $\exists \upsilon > 0$ satisfying that $\Re(\mu X)_{|\{w\}\times B(0,\epsilon)}$ is stable with respect to $w$ for $w \in e^{2\pi i(\theta-\upsilon,\theta+\upsilon)}(0,\delta)$. We can obtain a continuous extension of the Fatou coordinates in $x \in e^{2\pi i(\theta-\upsilon,\theta+\upsilon)}(0,\delta) \cup \{0\}$ which is holomorphic for $x \in e^{2\pi i(\theta-\upsilon,\theta+\upsilon)}(0,\delta)$.

Our approach is of analytic type and takes profit of the properties of the unfolding. We choose $X = X_\varphi$ as the vector field such that $\varphi \sim \exp(X)$, i.e. we ask $X$ to fulfill the proximity condition. In this way the model $\exp(X)$ reflects better the nature of $\varphi$ in the neighborhood of the fixed points. The approach of Lavaurs, Shishikura, Oudkerk, Mardesic-Roussarie-Rousseau is of topological type; they implicitly use a notion of continuous stability. We replace it with a concept of infinitesimal stability for $\Re(\mu X)_{|x=w}$. Our approach provides:

- Asymptotic developments of the Lavaurs vector field $X_T^\varphi$ until the first non-zero term in the neighborhood of the fixed points.
- Accurate estimates for the domains of definition of $\exp(cX_T^\varphi)$ for $c \in \mathbb{C}$.
- Canonical normalizing conditions for the Fatou coordinates.

These improvements allow us to:

- Identify the Taylor's series expansion of the analytic mappings conjugating $\varphi$, $\zeta \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$.
- Study the dependance of the domain of definition of a conjugation with respect to the parameter.
- Give a geometrical interpretation of our complete system of analytic invariants (theorem 3.3).

**5.3. Dynamical splitting.** — We provide a division of the domain of definition of a multi-parabolic vector field. The division is a key tool to handle the concept of infinitesimal stability that we will introduce later on.

***Remark 5.1.*** — *Let $X = g(x,y)\partial/\partial y$ be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2,0)$. For the sake of simplicity we suppose that all the irreducible components of $SingX$ are parameterized by $x$. In the non-degenerate case, this property can always be obtained by replacing $X$ with $g(x^k,y)\partial/\partial y$ for some $k \in \mathbb{N}$. Then $X$ is of the form*

$$X = u(x,y)(y-g_1(x))^{n_1}\dots(y-g_p(x))^{n_p}\partial/\partial y$$

*where $u$ is a unit in $\mathbb{C}\{x,y\}$ and $n_1+\dots+n_p \geq 2$.*

We say that $\overline{U}_\epsilon = B(0,\delta)\times\overline{B}(0,\epsilon)$ is a *seed*. If $p=1$ we do not divide $\overline{U}_\epsilon$ and we call $\overline{U}_\epsilon$ a terminal seed. We say that $\overline{U}_\epsilon$ is a product-like set (or also an exterior set). Denote by $\partial_e\overline{U}_\epsilon$ its exterior boundary $B(0,\delta)\times\partial B(0,\epsilon)$. Denote

$$L = \{(\partial g_1/\partial x)(0),\dots,(\partial g_p/\partial x)(0)\}.$$

Suppose $p > 1$. Consider a set $\overline{U}_\epsilon \cap \{(x,y) \in \mathbb{C}^2 : |y| \geq C|x|\}$ with $C >> 1$. Since $C >> \max L$ an observer in $|y| \geq C|x|$ can not distinguish the singular points. Thus we obtain

$$X = u(x,y)y^{\nu(X)+1}\left(1-\frac{g_1(x)}{y}\right)^{n_1} \dots \left(1-\frac{g_p(x)}{y}\right)^{n_p} \frac{\partial}{\partial y} \sim u(0,0)y^{\nu(X)+1}\frac{\partial}{\partial y}$$

in $\overline{U}_\epsilon \cap \{(x,y) \in \mathbb{C}^2 : |y| \geq C|x|\}$. Indeed it can be proved that the dynamics of $\Re(X)_{|\overline{U}_\epsilon \cap \{|y| \geq C|x|\}}$ is a topological product. The set $\overline{U}_\epsilon \cap \{|y| \geq C|x|\}$ is called exterior set or also product-like set associated to the seed $\overline{U}_\epsilon$.

Next we want to study the dynamics of $\Re(X)$ in $|y| \leq C|x|$. We consider the change of coordinates $x = x$, $y = xt$. The set $\{(x,y) \in \mathbb{C}^2 : |y| \leq C|x|\}$ and the vector field $X$ become $\{(x,t) \in \mathbb{C}^2 : |t| \leq C\}$ and

$$X = u(x,xt)x^{\nu(X)}\left(t-\frac{g_1(x)}{x}\right)^{n_1} \dots \left(t-\frac{g_p(x)}{x}\right)^{n_p} \frac{\partial}{\partial t}$$

respectively in the new coordinates. Consider the polynomial vector field

$$Y(\lambda) = u(0,0)\lambda^{\nu(X)}\left(t-\frac{\partial g_1}{\partial x}(0)\right)^{n_1} \dots \left(t-\frac{\partial g_p}{\partial x}(0)\right)^{n_p} \frac{\partial}{\partial t}$$

for $\lambda \in \mathbb{S}^1$. We say that $Y$ is the polynomial vector field associated to $X$ and the seed $\overline{U}_\epsilon$. Consider a domain

$$\mathscr{C} = \{(x,t) \in \mathbb{C}^2 : (x,t) \in B(0,\delta) \times [\overline{B(0,C)} \setminus \cup_{t_0 \in L} B(t_0, a(t_0))]\}$$

where $0 < a(t_0) << 1$ for any $t_0 \in L$. We say that $\mathscr{C}$ is a *compact-like set* associated to $X$ and the seed $\overline{U}_\epsilon$. We say also that $Y$ is associated to $\mathscr{C}$. We denote by $\partial_e \mathscr{C}$ and $\mathscr{C}^\circ$ the exterior boundary $|t| = C$ and the interior $\{|t| < C\} \setminus \cup_{t_0 \in L}\{|t-t_0| \leq a(t_0)\}$ of $\mathscr{C}$ respectively. A set of the form $\{(x,t) \in \mathbb{C}^2 : (x,t) \in B(0,\delta) \times \overline{B(t_0,a(t_0))}\}$ for some $t_0 \in L$ is called a seed that is a son of $\overline{U}_\epsilon$. We have

$$\frac{u(x,xt)x^{\nu(X)}(t-g_1(x)/x)^{n_1} \dots (t-g_p(x)/x)^{n_p}}{u(0,0)x^{\nu(X)}(t-(\partial g_1/\partial x)(0))^{n_1} \dots (t-(\partial g_p/\partial x)(0))^{n_p}} \to 1$$

uniformly in $\mathscr{C}$ when $x \to 0$. Therefore we obtain that

$$\Re(\mu X)_{|\mathscr{C} \cap \{x = r\lambda\}} \sim \Re(\mu Y(\lambda)) \ \text{ for } r \in \mathbb{R}^+ \text{ and } (\lambda,\mu) \in \mathbb{S}^1 \times \mathbb{S}^1.$$

The last property provides a justification for the term compact-like since the behavior of $\Re(\mu X)_{|\mathscr{C} \cap \{x = r\lambda\}}$ is determined by a polynomial vector field in one variable and a parameter $(\lambda,\mu)$ belonging to the compact set $\mathbb{S}^1 \times \mathbb{S}^1$. We can repeat the process with the seeds $|t-t_0| \leq a(t_0)$ for $j \in \{1,\dots,p\}$. We say that $\mathscr{B}$ is a *basic set* if $\mathscr{B}$ is either a product-like set or a compact-like set.

EXAMPLE: Consider $X = y(y-x^2)(y-x)\partial/\partial y$. Denote $t = y/x$. The vector field $X$ has the form $x^2 t(t-x)(t-1)\partial/\partial t$ in coordinates $(x,t)$. The polynomial vector field $Y(\lambda)$ associated to the seed $\mathscr{S} = \overline{U}_\epsilon$ is equal to $\lambda^2 t^2(t-1)\partial/\partial t$

The product and compact-like sets associated to $\mathscr{S}$ are $\mathscr{E} = \mathscr{S} \cap \{|y| \geq C|x|\}$ and $\mathscr{C} = \{|y| \leq C|x|\} \setminus (\{|t| < a_0\} \cup \{|t-1] < a_1\})$ respectively. The sons of $\mathscr{S}$ are the

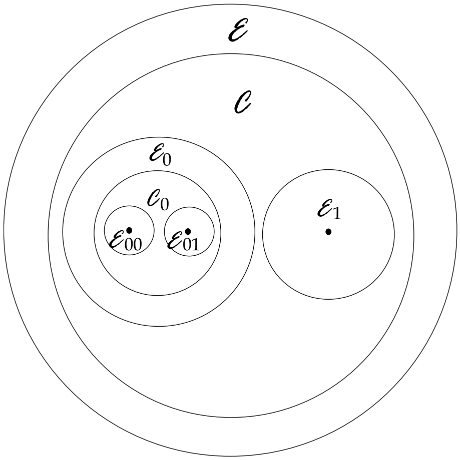


FIGURE 5. Splitting for $X = y(y-x^2)(y-x)\partial/\partial y$ in a line $x = x_0$

seeds $\mathscr{S}_0 = \{|t| \leq a_0\}$ and $\mathcal{E}_1 = \mathscr{S}_1 = \{|t-1| \leq a_1\}$. The seed $\mathscr{S}_1$ is terminal since it only contains one irreducible component of $SingX$.

Denote $w = t/x$. We have $X = x^3 w(w-1)(xw-1)\partial/\partial w$ in coordinates $(x, w)$. Thus $-\lambda^3 w(w-1)\partial/\partial w$ is the polynomial vector field $Y_0(\lambda)$ associated to $\mathscr{S}_0$. The seed $\mathscr{S}_0$ contains a product-like set $\mathcal{E}_0 = \mathscr{S}_0 \cap \{|t| \geq C_0|x|\}$ for $C_0 >> 1$, a compact-like set $\mathcal{C}_0 = \{|w| \leq C_0\} \setminus (\{|w| < b_0\} \cup \{|w-1| < b_1\})$ and two terminal seeds $\mathcal{E}_{00} = \mathscr{S}_{00} = \{|w| \leq b_0\}$ and $\mathcal{E}_{01} = \mathscr{S}_{01} = \{|w-1| \leq b_1\}$ for some $0 < b_0, b_1 << 1$.

**5.4. Infinitesimal stability.** — The dynamics of $\Re(\mu X)$ in a product-like set is a topological product. As a consequence it is stable with respect to the parameter $x$. Then the stability is a property depending of the behavior of $\Re(\mu X)$ in the compact-like sets.

Many of the results in this subsection on polynomial vector fields either come from [**2**] or have been found independently in [**2**].

***Definition 5.4.*** — *Given a polynomial vector field $Y = P(t)\partial/\partial t \in \mathcal{X}(\mathbb{C}, 0)$ we define $\nu(Y) = \deg(P(t)) - 1$. Let $t_0 \in SingY$, we define $\mathrm{Res}_Y(t_0)$ as the residue of the differential form $dt/P(t)$ at $t = t_0$.*

Consider a polynomial vector field $Y = P(t)\partial/\partial t \in \mathcal{X}(\mathbb{C}, 0)$ such that $\nu(Y) \geq 1$. We want to characterize the behavior of $Y$ in the neighborhood of $\infty$. We define the set $Tr_{\to\infty}(Y)$ of trajectories $\gamma : (c, d) \to \mathbb{C}$ of $\Re(Y)$ such that $c \in \mathbb{R} \cup \{-\infty\}$, $d \in \mathbb{R}$ and $\lim_{\zeta \to d} \gamma(\zeta) = \infty$. In an analogous way we define $Tr_{\leftarrow\infty}(Y) = Tr_{\to\infty}(-Y)$. We define $Tr_\infty(Y) = Tr_{\leftarrow\infty}(Y) \cup Tr_{\to\infty}(Y)$.

We consider a change of coordinates $z = 1/t$. We have

$$Y = \frac{-1}{z^{\nu(Y)-1}}(z^{\nu(Y)+1}P(1/z))\frac{\partial}{\partial z}$$

where $z^{\nu(Y)+1}P(1/z)$ is a unit in the neighborhood of $z=0$. Thus the meromorphic vector field $Y$ is analytically conjugated to $1/(\nu(Y)z^{\nu(Y)-1})\partial/\partial z = (z^{\nu(Y)})^{*}(\partial/\partial z)$ in a neighborhood of $t=\infty$. We have $Tr_{\to\infty}(\partial/\partial z)=\mathbb{R}^{-}$ and $Tr_{\leftarrow\infty}(\partial/\partial z)=\mathbb{R}^{+}$. Hence the set $Tr_{\infty}(Y)$ has $2\nu(Y)$ trajectories in the neighborhood of $\infty$, there is exactly one of them which is tangent to $\arg(w) = -\arg(C)/\nu(Y) + k\pi/\nu(Y)$ for $0 \leq k < 2\nu(Y)$ where $C=(z^{\nu(Y)+1}P(1/z))(0)$. The even values of $k$ correspond to $Tr_{\to\infty}(Y)$.

We say that $\Re(Y)$ has *homoclinic trajectories* if $Tr_{\to\infty}(Y)\cap Tr_{\leftarrow\infty}(Y)\neq\emptyset$. In other words there exists a trajectory $\gamma:(c_{-1},c_1)\to\mathbb{C}$ of $\Re(Y)$ such that $c_{-1},c_1\in\mathbb{R}$ and $\lim_{\zeta\to c_s}\gamma(\zeta)=\infty$ for any $s\in\{-1,1\}$. The notion of homoclinic trajectory has been introduced in **[2]** for the study of deformations of elements of $\mathrm{Diff}_1(\mathbb{C},0)$. We say that $\Re(Y)$ is stable if the dynamics of $\Re(\mu Y)$ is stable in a neighborhood of $\mu=1$ in $\mathbb{S}^1$.

***Theorem 5.2.*** — **[2]** *Let $Y\in\mathcal{X}(\mathbb{C},0)$ be a polynomial vector field such that $\nu(Y)\geq 1$. Then $\Re(Y)$ is stable if and only if $\Re(Y)$ has no homoclinic trajectories.*

***Definition 5.5.*** — *We denote by $\mathcal{X}_{\infty}(\mathbb{C},0)$ the set of polynomial vector fields $Y$ in $\mathcal{X}(\mathbb{C},0)$ such that $\nu(Y)\geq 1$ and $\sum_{P\in E}\mathrm{Res}_Y(P)\notin i\mathbb{R}^{*}$ for any subset $E$ of $SingY$.*

***Proposition 5.1.*** — *Let $Y\in\mathcal{X}_{\infty}(\mathbb{C},0)$. Then $\Re(Y)$ is stable. Moreover given $Q$ in $\mathbb{C}\setminus Tr_{\to\infty}(Y)$ we have that $\omega_Y(Q)$ is a finite singular point. In particular $\Re(Y)$ has no cycles.*

*Proof.* — Let $\gamma:(a,b)\to\mathbb{C}$ be a homoclinic trajectory for $\Re(Y)$. We suppose that $(a,b)$ is a maximal domain of definition of the trajectory. The set $\mathbb{P}^1(\mathbb{C})\setminus(\gamma\cup\{\infty\})$ has two connected components $C_1$ and $C_2$. Denote $E=C_1\cap SingY$. By the theorem of the residues we have

$$b-a=\mp 2\pi i\sum_{P\in E}\mathrm{Res}_Y(P).$$

We obtain a contradiction.

Given $Q\in\mathbb{C}\setminus Tr_{\to\infty}(Y)$ suppose that $P\in\omega_Y(Q)\cap(\mathbb{C}\setminus SingY)$. Consider a transversal $T$ to $\Re(Y)$ passing through $P$. Let $\{t_n\}_{n\in\mathbb{N}}$ be the increasing sequence of points in $\{t\in\mathbb{R}^{+}:\Gamma_{Y,+}[Q](t)\in T\}$. There exists $\eta>0$ such that $t_{n+1}-t_n>\eta$ for any $n\in\mathbb{N}$. Denote by $\gamma_n$ the curve composed by $\Gamma_{Y,+}[Q][t_n,t_{n+1}]$ and the segment $S_n$ of $T$ such that $\partial S_n=\{\Gamma_{Y,+}[Q](t_n),\Gamma_{Y,+}[Q](t_{n+1})\}$. Let $C_n$ be the bounded connected component of $\mathbb{C}\setminus\gamma_n$. Denote $E_n=C_n\cap SingY$. We obtain

$$\mp 2\pi i\sum_{P\in E_n}\mathrm{Res}_Y(P)=(t_{n+1}-t_n)+a_n$$

where $a_n\in\mathbb{C}$ for any $n\in\mathbb{N}$ and $\lim_{n\to\infty}a_n=0$. Thus there exists a set $E\subset SingY$ such that $\mp 2\pi i\sum_{P\in E}\mathrm{Res}_Y(P)\in[\eta,\infty)$. That is a contradiction. $\square$

Let $Y\in\mathcal{X}(\mathbb{C},0)$ be a polynomial vector field such that $\nu(Y)\geq 1$. We denote by $\mathcal{U}(Y)$ the complementary of the set $\{\kappa\in\mathbb{S}^1:\Re(\kappa Y)\text{ is stable}\}$. It is clear that $\kappa Y$ belongs to $\mathcal{X}_{\infty}(\mathbb{C},0)$ for any $\kappa\in\mathbb{S}^1$ outside a finite set. Thus we obtain

***Proposition 5.2***. — *Let $Y \in \mathcal{X}(\mathbb{C}, 0)$ be a polynomial vector field such that $\nu(Y) \geq 1$. Then $\mathcal{U}(Y)$ is a finite set.*

Let $Y_1(\lambda) = \lambda^{m_1} Y_1(1), \dots, Y_l(\lambda) = \lambda^{m_l} Y_l(1)$ the polynomial vector fields associated to a non-degenerate $X \in \mathcal{X}_{p1}(\mathbb{C}^2, 0)$. We define

$$\mathcal{U}(X) = \cup_{j=1}^{l} \{\lambda \in \mathbb{S}^1 : \lambda^{m_j} \in \mathcal{U}(Y_j(1))\}.$$

Clearly $\mathcal{U}(X)$ is a finite set. Denote by $\mathcal{S}_1, \dots, \mathcal{S}_l$ the seeds associated to the vector fields $Y_1, \dots, Y_l$ respectively. Consider the compact-like set $\mathcal{C}_j$ associated to $\mathcal{S}_j$. We say that $\Re(\mu X)$ is *infinitesimally stable* at the direction $x \in \lambda\mathbb{R}^+$ if $\Re(\mu Y_j(\lambda))$ is stable for any $j \in \{1, \dots, l\}$. This property is equivalent to $\lambda \notin \mathcal{U}(\mu X)$.

The set $\mathcal{C}_j$ is of the form $\{(x, t) \in B(0, \delta) \times [\overline{B(0, C)} \setminus \cup_{j=1}^{k} B(t_j, a_j)]\}$ for some coordinates $(x, t)$. Denote $x = r\lambda$ ($r \in \mathbb{R}^+ \cup \{0\}$ and $\lambda \in \mathbb{S}^1$). We have

$$\Re(\mu X)(r\lambda, y) \sim \Re(\mu Y_j(\lambda)) = \Re(\lambda^{m_j} \mu Y_j(1))$$

in $\mathcal{C}_j$ for $j \in \{1, \dots, l\}$. Suppose $\lambda \notin \mathcal{U}(\mu X)$. Since $\Re(\lambda^{m_j} \mu Y_j(1))$ is stable in the neighborhood of $\{\infty\} \cup Sing Y_j(1)$ and $C >> 1$, $a_j << 1$ then the stability of $\Re(\lambda^{m_j} \mu Y_j(1))$ implies the stability of $\Re(\mu X)_{|\mathcal{C}_j}$ in the neighborhood of the direction $x \in \lambda\mathbb{R}^+$. This discussion leads us to

***Proposition 5.3***. — *Let $X$ be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2, 0)$ and let $\mu \in \mathbb{S}^1$. Suppose that $\lambda \in \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. Then $\Re(\mu X)_{|\mathcal{C}}$ is stable in the neighborhood of $x \in \lambda\mathbb{R}^+$ for any compact-like set $\mathcal{C}$ associated to $X$.*

**5.5. Topological stability.** — The infinitesimal stability is a pretty strong property. For most of the discussion in the paper a weaker property, namely topological stability, is sufficient. Anyway, the quantitative estimations on the constructions rely on the use of infinitesimal stability as well as the theorems depending on those estimations (for instance theorem 3.3). In this subsection we define topological stability and show that it is implied by its infinitesimal counterpart.

Infinitesimal stability is incompatible with the existence of centers.

***Proposition 5.4***. — *Let $X$ be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2, 0)$ and let $\mu \in \mathbb{S}^1$. Consider a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. The vector field $X_{|x=x_0}$ has no center at $y_0$ for any $(x_0, y_0) \in ([0, \delta)K \times B(0, \epsilon)) \cap Sing X$.*

*Proof*. — We have to prove that the multiplicator $\iota$ of $X_{|x=x_0}$ at $y = y_0$ does not belong to $i\mathbb{R}^*$. We can suppose that $\iota \neq 0$. The point $(x_0, y_0)$ belongs to a curve $y = h(x)$ of $Sing X$. Consider the non-terminal seed $\mathcal{S}$ such that $(x_0, y_0)$ belongs to a terminal seed $\mathcal{S}_0 = \{(x, t) \in B(0, \delta) \times \overline{B(t_0, a(t_0))}\}$ which is a son of $\mathcal{S}$. Let $\lambda^m Y(1)$ be the polynomial vector field associated to $\mathcal{S}$. The infinitesimal stability implies $\text{Res}(Y(1), t_0)/(\lambda^m \mu) \notin i\mathbb{R}^*$ for $\lambda \in K$. The multiplicator of $X$ at $(x, h(x))$ is of the form $x^m \mu / \text{Res}(Y(1), t_0) + O(x^{m+1})$. Since we have

$$\frac{x^m \mu}{\text{Res}(Y(1), t_0)} + O(x^{m+1}) = |x|^m \left( \frac{\lambda^m \mu}{\text{Res}(Y(1), t_0)} + O(x) \right)$$

with $\lambda = x/|x|$ then $(x_0, y_0)$ is not a center if $x_0$ is in a neighborhood of 0. □

Infinitesimal stability also implies that the limit of the dynamics is not more complicated than the dynamics of the limit. More precisely, the limit of a sequence of trajectories of $\Re(\mu X)_{|x=x_n}$ when $x_n \to 0$ is a priori only invariant by the flow and it can contain several trajectories of $\Re(\mu X)_{|x=0}$. Next proposition shows that this is not the case.

***Proposition 5.5***. — *Let $X$ be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2, 0)$ and let $\mu \in \mathbb{S}^1$. Consider a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. There do not exist sequences $(x_n, y_n^0)$, $(x_n, y_n^1)$ of points in $(0, \delta)K \times B(0, \epsilon)$ such that*

- $\lim_{n\to\infty}(x_n, y_n^k)$ *exists and belongs to* $\{0\} \times (B(0, \epsilon) \setminus \{0\})$ $\forall k \in \{0, 1\}$.
- $(x_n, y_n^1)$ *is in the positive trajectory of* $\Re(\mu X)_{|\{x_n\}\times B(0,\epsilon)}$ *passing through* $(x_n, y_n^0)$ *for any* $n \in \mathbb{N}$.
- $\lim_{n\to\infty}(x_n, y_n^1)$ *is not in the positive trajectory of* $\Re(\mu X)_{|\{0\}\times \overline{B(0,\epsilon)}}$ *passing through* $\lim_{n\to\infty}(x_n, y_n^0)$.

*Sketch of proof.* — Suppose that the result is not true. Consider $t_n \in \mathbb{R}^+$ such that $(x_n, y_n^1) = \Gamma^{\epsilon}_{\mu X,+}[x_n, y_n^0](t_n)$ for $n \in \mathbb{N}$. The points $(x_n, y_n^0)$ and $(x_n, y_n^1)$ are in the product-like set $\mathcal{E}$ associated to the seed $\overline{U}_\epsilon$ for $n >> 0$. The trajectory $\Gamma^{\epsilon}_{\mu X,+}[x_n, y_n^0][0, t_n]$ is not contained in $\mathcal{E}$, otherwise $\lim_{n\to\infty}(x_n, y_n^1)$ is in the positive trajectory of $\Re(\mu X)_{|\{0\}\times\overline{B}(0,\epsilon)}$ passing through $\lim_{n\to\infty}(x_n, y_n^0)$. Therefore there exists a sequence $0 < t_n^1 < \ldots < t_n^k < t_n$ such that given $t \in [0, t_n]$ the point $\Gamma^{\epsilon}_{\mu X,+}[x_n, y_n^0](t)$ belongs to the boundary of a basic set if $t \in \{t_n^1, \ldots, t_n^k\}$. The property

$$Pr(\mathcal{B}, n) = \left(\exists\ 0 < s < k \ s.t.\ \Gamma^{\epsilon}_{\mu X,+}[x_n, y_n^0]\{t_n^s, t_n^{s+1}\} \subset \partial_e \mathcal{B}\right)$$

holds true for some basic set $\mathcal{B}$ and any $n >> 0$. The set $\mathcal{B}$ can not be product-like, it is necessarily a compact-like set

$$\{(x,t) \in B(0,\delta) \times [\overline{B}(0,C) \setminus \cup_{t_0 \in L} B(t_0, a(t_0))]\}.$$

Let $Y$ be the polynomial vector field associated to $\mathcal{B}$. If the property $Pr(\mathcal{B}, n)$ holds true for all $C > 0$ and $n > n(C)$ then there exists a homoclinic trajectory of $\Re Y(\lambda)$ for any $\lambda \in K$ being an accumulation point of the sequence $x_n/|x_n|$. Then the absence of homoclinic trajectories implies that there exists a choice of the dynamical splitting leading us to a contradiction. □

***Definition 5.6***. — *Let $X$ be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2, 0)$ and let $\mu$ in $\mathbb{S}^1$. Consider a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. We say that $\Re(\mu X)$ is topologically stable at $K$ if it satisfies the theses in propositions 5.4 and 5.5.*

The result on next corollary is a consequence of the topological stability property.

***Corollary 5.1***. — *Let $X$ be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2, 0)$ and let $\mu \in \mathbb{S}^1$. Consider a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. Then $\mathcal{N}G^{\epsilon}_{\mu X, w}$ is connected and has no cycles for any $w \in (0, \delta)K$.*

*Proof.* — The proposition 5.5 implies that $(\alpha_{\mu X}^{\epsilon}, \omega_{\mu X}^{\epsilon})^{-1}(\infty,\infty) \cap \{x = w\} = \emptyset$ for any $w \in (0,\delta)K$. We can argue as in proposition 4.4 to prove that $\mathcal{N}G_{\mu X,w}^{\epsilon}$ is connected since the proof is based on the openness of basins of attraction and repulsion of singular points. This is guaranteed in this context by prop. 5.4.

Suppose there exists a cycle in $\mathcal{N}G_{\mu X,w}^{\epsilon}$. Then there exists a bounded region $T$ in $x = w$ bounded by singular points and trajectories of $\Re(\mu X)$ representing the edges of the cycle. We claim that the mapping $\omega_{\mu X}^{\epsilon} : T \setminus SingX \to SingX$ is well-defined. The absence of centers (prop. 5.4) implies that there are no cycles. We obtain that every $\omega$ limit contains a singular point and then it is a singular point since the no-center property implies that a pointed neighborhood of a singular point is a union of open basins of attraction and repulsion. The mapping $\omega_{\mu X}^{\epsilon} : T \setminus SingX \to SingX$ is not constant, otherwise all the trajectories of $\Re(\mu X)$ in $\partial T$ represent the same edge. This provides a contradiction since the topological stability implies that $\omega_{\mu X}^{\epsilon} : T \setminus SingX \to SingX$ is locally constant. □

The corollary 5.1 is a "topological" regularity property. The infinitesimal stability is a deeper condition and provides regularity in infinitesimal neighborhoods of the singular points. For instance we can associate graphs to every seed $\mathcal{S}$ just by considering the dynamics of $\Re(\mu X)_{|\mathcal{S}}$. The graphs $\mathcal{N}G_{\mu X,w}^{\mathcal{S}}$ obtained are connected and with no cycles for compact connected sets $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$ and $w \in (0,\delta(K))K$.

**5.6. Defining regions.** — Let $X$ be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2,0)$ and let $\mu \in \mathbb{S}^1$. Consider a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$.

***Definition 5.7.*** — *We define* $\mathrm{Reg}^{\epsilon}(\mu X, K)$ *as the set of connected components of*

$$(\alpha_X^{\epsilon}, \omega_X^{\epsilon})^{-1}(SingX \times SingX) \cap ([[0,\delta)K \times B(0,\epsilon)] \setminus SingX).$$

*We call regions the elements of* $\mathrm{Reg}^{\epsilon}(\mu X, K)$. *Given a region* $R \in \mathrm{Reg}^{\epsilon}(\mu X, K)$ *and a point* $w \in B(0,\delta)$ *we denote by* $R(w)$ *the set* $R \cap (\{w\} \times B(0,\epsilon))$.

Given a non-degenerate element $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ and $X = X_{\varphi}$ the regions in $\mathrm{Reg}^{\epsilon}(\mu X, K)$ are the sets where we define extensions of the Fatou coordinates of $\varphi_{|x=0}$. We take profit that the orbit space of $\varphi_{|R^{\circ}}$ is biholomorphic to the set $(0,\delta)K^{\circ} \times \mathbb{C}^*$ for any $R \in \mathrm{Reg}^{\epsilon}(\mu X, K)$.

***Definition 5.8.*** — *Denote by* $\mathrm{orb}(\varphi, E)$ *the space of orbits of* $\varphi_{|E}$.

***Definition 5.9.*** — *Given* $R \in \mathrm{Reg}^{\epsilon}(\mu X, K)$ *we denote* $R \in \mathrm{Reg}_1^{\epsilon}(\mu X, K)$ *if we have* $(\alpha_X^{\epsilon})_{|R} \equiv (\omega_X^{\epsilon})_{|R}$, *otherwise we denote* $R \in \mathrm{Reg}_2^{\epsilon}(\mu X, K)$.

As a consequence of the stability a region $R \in \mathrm{Reg}_1^{\epsilon}(\mu X, K)$ is homeomorphic to $[0,\delta)K \times R(0)$ by a mapping $(x, h(x,y))$. Therefore $\mathrm{orb}(\varphi, R)$ is homeomorphic to $[0,\delta)K \times \mathbb{C}^*$ and $\mathrm{orb}(\varphi, R^{\circ})$ is biholomorphic to $(0,\delta)K^{\circ} \times \mathbb{C}^*$ (see the regions $R_2$ and $R_5$ in picture (6)). A region $R \in \mathrm{Reg}_2^{\epsilon}(\mu X, K)$ satisfies that $R(x)$ has one connected component for any $x$ in $(0,\delta)K$ whereas $R(0)$ has two connected components. Moreover $\mathrm{orb}(\varphi, R^{\circ})$ is biholomorphic to $(0,\delta)K^{\circ} \times \mathbb{C}^*$ by stability and $\mathrm{orb}(\varphi, R(0))$

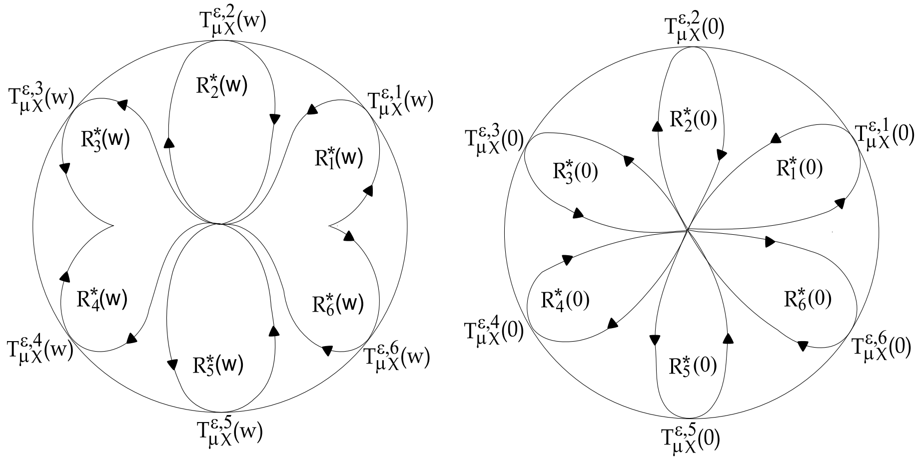


FIGURE 6. Regions for $\mu X = y^2(y-x)(y+x)\partial/\partial y$ and $w \in \mathbb{R}^+$

is biholomorphic to the union of two disjoint copies of $\mathbb{C}^*$ (see the regions $R_1 = R_6$ and $R_3 = R_4$ in picture (6)).

Denote by $T^{\epsilon}_{\mu X}$ the set of tangent points between $\Re(\mu X)$ and $\partial_e U_\epsilon$. Denote $T^{\epsilon}_{\mu X} \cap (\{w\} \times \mathbb{C})$ by $T^{\epsilon}_{\mu X}(w)$. As the picture (6) suggests we have

***Lemma 5.1***. — *Let $X$ be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2,0)$ and let $\mu \in \mathbb{S}^1$. Consider a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. Given $R \in \mathrm{Reg}^{\epsilon}_k(\mu X, K)$ we have $\sharp(\overline{R(x)} \cap T^{\epsilon}_{\mu X}(x)) = k$ for any $x \in [0,\delta)K$.*

The cardinal of $\mathrm{Reg}^{\epsilon}_2(\mu X, K)$ coincides with the number of edges in $\mathcal{N}G^{\epsilon}_{\mu X,w}$ for $w \in (0,\delta)K$. The corollary 5.1 implies

***Lemma 5.2***. — *Let $X \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$ be non-degenerate and let $\mu \in \mathbb{S}^1$. Consider a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. Then we have $\sharp\mathrm{Reg}^{\epsilon}_2(\mu X, K) = N(X) - 1$.*

Let $X = u(x,y)(y-g_1(x))^{n_1}\dots(y-g_p(x))^{n_p}\partial/\partial y$ where $u \in \mathbb{C}\{x,y\}$ is a unit. The number of regions of $\mathrm{Reg}^{\epsilon}(\mu X, K)$ in $(\alpha^{\epsilon}_X, \omega^{\epsilon}_X)^{-1}(\{y = g_j(x)\} \times \{y = g_j(x)\})$ is equal to $2(n_j - 1)$ for any $j \in \{1,\dots,l\}$. This leads us to

***Lemma 5.3***. — *Let $X$ be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2,0)$ and let $\mu \in \mathbb{S}^1$. Consider a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. Then we have*

$$\sharp\mathrm{Reg}^{\epsilon}_1(\mu X, K) + \sharp\mathrm{Reg}^{\epsilon}_2(\mu X, K) = 2\nu(X) - N(X) + 1$$

*and $\sharp\mathrm{Reg}^{\epsilon}_1(\mu X, K) + 2\sharp\mathrm{Reg}^{\epsilon}_2(\mu X, K) = 2\nu(X)$.*

There are $2\nu(X)$ continuous sections $T^{\epsilon,1}_X, \dots, T^{\epsilon,2\nu(X)}_X$ of the set $T^{\epsilon}_X$. We will always suppose that $T^{\epsilon,1}_X, \dots, T^{\epsilon,2\nu(X)}_X, T^{\epsilon,2\nu(X)+1}_X = T^{\epsilon,1}_X$ are ordered in counter clock-wise sense. Fix $\mu \in \mathbb{S}^1 \setminus \{-1,1\}$. There exists a unique $T^{\epsilon,j}_{\mu X}(x) \in T^{\epsilon}_{\mu X}(x)$ in the arc of

$\partial U_\epsilon(x)$ going in counter clock-wise sense from $T_X^{\epsilon,j}(x)$ to $T_X^{\epsilon,j+1}(x)$. We define $R_j^{\mu X,K}$ (or just $R_j$ when $\mu X$ and $K$ are implicit) as the element of $\mathrm{Reg}^\epsilon(\mu X, K)$ such that $T_{\mu X}^{\epsilon,j}(x) \in \partial R_j(x)$ for any $x \in [0,\delta)K$.

***Definition 5.10.*** — *We define $R_j^*$ as the union of $R_j \setminus R_j(0)$ and the connected component of $R_j(0)$ whose closure contains $T_{\mu X}^{\epsilon,j}(0)$. Given $w \in B(0,\delta)$ we denote by $R_j^*(w)$ the set $R_j^* \cap (\{w\} \times \mathbb{C})$.*

The set $\mathrm{orb}(\varphi, R_j^*)$ is homeomorphic to $[0,\delta)K \times \mathbb{C}^*$ for any $j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$. We denote by $\mathrm{Reg}_*^\epsilon(\mu X, K)$ the set $\{R_1^*, \dots, R_{2\nu(X)}^*\}$ (see picture (6) where we have $R_1 = R_6$ and $R_3 = R_4$).

We want to define an integral $\psi_j^X$ of the time form of $X$ (i.e. $X(\psi_j^X) = 1$) in an element $R_j^*$ of $\mathrm{Reg}_*^\epsilon(\mu X, K)$ for $j \in \mathbb{Z}$. An integral of the time form then transforms the vector field $X$ in the regular vector field $\partial/\partial y$. The set $R_j^*(x)$ is simply connected for any $x \in [0,\delta)K$. We fix a holomorphic integral $\psi_1^X$ of the time form of $X$ in a neighborhood of $T_X^{\epsilon,1}(0)$. We can extend $\psi_1^X$ to a neighborhood of $T_{\mu X}^{\epsilon,1}(0)$ by doing analytic continuation along the arc going from $T_X^{\epsilon,1}(0)$ to $T_{\mu X}^{\epsilon,1}(0)$ in counter clock-wise sense. We want to define a holomorphic integral $\psi_j^X$ of the time form of $X$ in a neighborhood of $T_{\mu X}^{\epsilon,j}(0)$ for $j \in \mathbb{Z}$; then we extend analytically $\psi_j^X$ to $R_j^*(x)$ for any $x \in [0,\delta)K$. We obtain that $\psi_j^X$ is continuous in $R_j^*$ and holomorphic in $(R_j^*)^\circ$.

Given $\psi_j^X$ we denote by $\psi_j^X(T_{\mu X}^{\epsilon,j+1}(x))$ the result of evaluating at $T_{\mu X}^{\epsilon,j+1}(x)$ the analytic extension of $\psi_j^X$ along the arc joining $T_{\mu X}^{\epsilon,j}(0)$ and $T_{\mu X}^{\epsilon,j+1}(0)$ in $\partial_e U_\epsilon(0)$ in counter clock-wise sense. We require two conditions to the sequence $\{\psi_j^X\}_{j\in\mathbb{Z}}$, namely

- $\psi_{j+2\nu(X)}^X \equiv \psi_j^X$ for any $j \in \mathbb{Z}$.
- $c(x) \equiv \psi_{j+1}^X(T_{\mu X}^{\epsilon,j+1}(x)) - \psi_j^X(T_{\mu X}^{\epsilon,j+1}(x))$ is independent of $j \in \mathbb{Z}$.

We define the function $\zeta_X(x) = -\pi i \nu(X)^{-1} \sum_{P \in Sing X \cap (\{x\} \times B(0,\epsilon))} \mathrm{Res}_X(P)$. It is holomorphic in a neighborhood of 0. It turns out that the previous conditions imply $c \equiv \zeta_X$. Moreover the choice of $\psi_1^X$ determines $\{\psi_j^X\}_{j\in\mathbb{Z}}$.

**5.7. Extension of the Fatou coordinates.** — Fix a non-degenerate element $\varphi$ of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Denote $X_\varphi = f(x,y)\partial/\partial y$ and $X = X_\varphi$. There exists a unique $\Delta_\varphi$ in the ideal $(f)$ of the ring $\mathbb{C}\{x,y\}$ such that

$$\varphi(x,y) = (\exp(tX)(t,x,y)) \circ (1 + \Delta_\varphi(x,y), x, y).$$

The previous formula is a consequence of the implicit function theorem.

Consider $\mu \in \mathbb{S}^1$ and a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. We obtain

$$\psi_j^X \circ \varphi = \psi_j^X + 1 + \Delta_\varphi \ \text{ in } \ R_j^* \in \mathrm{Reg}_*^\epsilon(\mu X, K).$$

Hence $\psi_j^X$ is "almost" a Fatou coordinate of $\varphi$ in $R_j^*$. In this subsection we sketch how to obtain a Fatou coordinate $\psi_j^\varphi$ of $\varphi$ in $R_j^*$ by slight deformation of $\psi_j^X$.

Fix $j \in \mathbb{Z}/(2\nu(X)\mathbb{Z})$. Consider a trajectory $\gamma$ of $\Re(\mu X)$ passing through a point $T_{\mu X}^{\epsilon,j}(w)$ for some $w \in [0,\delta)K$. The strip $\exp([0,1]X)(\gamma)$ is a fundamental domain of

$\exp(X)$ in $R_j^*(w)$. Denote by $S_j(w)$ the strip enclosed by $\gamma$ and $\varphi(\gamma)$. Then $S_j(w)$ is a fundamental domain of $\varphi$ in $R_j^*(w)$. We can build a $C^\infty$ diffeomorphism $\sigma_\gamma$ from a

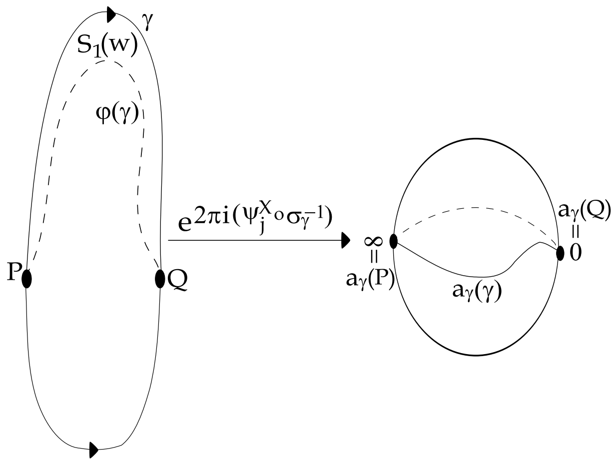


FIGURE 7. Constructing Fatou coordinates

neighborhood of $\exp([0,1]X)(\gamma)$ in $x = w$ to a neighborhood of $S_j(w)$ in $x = w$ and such that

- $\sigma_\gamma \circ \exp(X) = \varphi \circ \sigma_\gamma$ and $\sigma(\exp([0,1]X)(\gamma)) = S_j(w)$.
- $\psi_j^X \circ \sigma_\gamma - \psi_j^X = O(f)$.

The function $\psi_j^X \circ \sigma_\gamma^{-1}$ is a $C^\infty$ Fatou coordinate of $\varphi$ in the neighborhood of $S_j(w)$, i.e. $(\psi_j^X \circ \sigma_\gamma^{-1}) \circ \varphi = \psi_j^X \circ \sigma_\gamma^{-1} + 1$. We want to deform $\psi_j^X \circ \sigma_\gamma^{-1}$ slightly in order to obtain a holomorphic Fatou coordinate. Consider the diffeomorphism

$$a_\gamma : \mathrm{orb}(\varphi, S_j(w)) \overset{e^{2\pi i(\psi_j^X \circ \sigma_\gamma^{-1})}}{\longrightarrow} \mathbb{C}^*.$$

The complex dilatation $\kappa_\gamma(z) = \kappa_{a_\gamma^{-1}}(z) = (\partial a_\gamma^{-1}/\partial \overline{z})/(\partial a_\gamma^{-1}/\partial z)$ of $a_\gamma^{-1}$ satisfies $|\kappa_\gamma|(z) = O(f(w,y))$ where $\psi_j^X \circ \sigma_\gamma^{-1}(w,y) = (\ln z)/(2\pi i)$. We can suppose that $\sup_{\mathbb{C}^*} |\kappa| \leq 1/2$ since $f(0,0) = 0$. Thus there exists a unique quasi-conformal mapping $d : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that $\kappa_d = \kappa_\gamma$, $d(0) = 0$, $d(\infty) = \infty$ and $d(1) = 1$. By choice $d \circ a_\gamma$ is a biholomorphism from $\mathrm{orb}(\varphi, S_j(w))$ to $\mathbb{C}^*$; there exists a Fatou coordinate $\tilde{\psi}_j^\varphi$ in $S_j(w)$ given by the formula $\tilde{\psi}_j^\varphi = (\ln(d \circ a_\gamma))/(2\pi i)$. The Fatou coordinate can be extended to a holomorphic function defined in $R_j^*(w)$ by using the formula $\tilde{\psi}_j^\varphi \circ \varphi \equiv \tilde{\psi}_j^\varphi + 1$. The function $\tilde{\psi}_j^\varphi + b$ is also a Fatou coordinate of $\varphi$ in $R_j^*(w)$ for any $b \in \mathbb{C}$. The set $\{\tilde{\psi}_j^\varphi + b\}_{b \in \mathbb{C}}$ does not depend on the choices of $\psi_j^X$ and $\sigma_\gamma$.

We would like to have

$$\lim_{P \in S_j(w),\ \mathrm{Img}(\psi_j^X(P)) \to \pm\infty} \tilde{\psi}_j^\varphi(P) - \psi_j^X(P) = c_\pm^\gamma \in \mathbb{C}. \tag{1}$$

In other words we want $\tilde{\psi}_j^{\varphi} - \psi_j^X$ to have a continuous extension to $\overline{S_j(w)} \cap \mathrm{Fix}(\varphi)$. The importance of the previous condition can be understood in terms of the Lavaurs vector fields.

***Definition 5.11***. — *There exists a unique vector field $X_j^{\varphi}$ defined in $R_j^*(w)$ and such that $X_j^{\varphi}(\tilde{\psi}_j^{\varphi}) \equiv 1$. Therefore we have $\varphi \equiv \exp(X_j^{\varphi})$ in $R_j^*(w)$. Moreover, since $X_j^{\varphi}$ is unique then it is continuous in $R_j$ and holomorphic in $R_j^{\circ}$. The vector field $X_j^{\varphi}$ is the Lavaurs vector field associated to $\varphi$ in $R_j$.*

The vector field $X_{|x=w}$ is of the form $a(y-y_0)^r(1+O(y-y_0))\partial/\partial y$ in the neighborhood of a point $(w, y_0)$ in $\overline{S_j(w)} \cap \mathrm{Fix}(\varphi)$ for some $a \in \mathbb{C}^*$ and $r \in \mathbb{N}$. The equation (1) implies that $(X_j^{\varphi})_{|x=w}$ is of the form $a(y-y_0)^r(1+h(y))\partial/\partial y$ where $h$ is a continuous function defined in $R_j^*(w) \cup (\overline{S_j(w)} \cap \mathrm{Fix}(\varphi))$ such that $h(y_0) = 0$. The Lavaurs vector field coincides with $X$ until the first non-zero term. In absence of the equation (1) we can only say that $X_j^{\varphi}$ can be extended continuously to $\overline{S_j(w)} \cap \mathrm{Fix}(\varphi)$ as a singular vector field. The analytic invariants can be expressed in terms of the Lavaurs vector fields; as a consequence a better knowledge of their behavior provides more accurate results. An example is given by the "quantitative" theorem 3.3.

Equation (1) holds if and only if the quasi-conformal mapping $d : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is conformal at 0 and $\infty$. To get the conformality at 0 is enough to prove that

$$\frac{1}{\pi} \int_{|z|<r} \frac{1}{1 - |\kappa_\gamma(z)|} \frac{|\kappa_\gamma(z)|}{|z|^2} d\sigma < \infty$$

for any $r \in \mathbb{R}^+$ (theorem 6.1 in page 232 of [**4**]). Such a property can be obtained if

$$(2) \qquad |\kappa_\gamma| = O\left(\frac{1}{(1 + |\ln z|)^{1+1/\nu(\varphi)}}\right) \Leftarrow f_{|S_j(w)} = O\left(\frac{1}{(1 + |\psi_j^X|)^{1+1/\nu(\varphi)}}\right).$$

Denote $S_j = \cup_{x \in [0,\delta)K} S_j(x)$. Now suppose that there exists $y_1 \in B(0, \epsilon) \setminus \{0\}$ such that $(0, y_1) \notin S_j(0)$ but $(0, y_1)$ belongs to $\overline{S_j}$ (of course such a behavior is ruled out because of stability but that is precisely the point, to justify the choice of infinitesimal stability). Then typically $f(0, y_1) \neq 0$ whereas

$$\lim_{(x,y) \in S_j,\ (x,y) \to (0,y_1)} \psi_j^X(x, y) = \infty$$

since $(0, y_1) \notin S_j(0)$. In this context the estimate in the right hand side of condition (2) does not hold and then we can not get the good estimates for Fatou coordinates and Lavaurs vector fields in the neighborhood of the fixed points. The problem is associated to the fact that $\lim_{x \in (0,\delta)K, x \to 0} \Gamma_{\mu X}[T_{\mu X}^{\epsilon,j}(x)]$ is bigger than $\Gamma_{\mu X}[T_{\mu X}^{\epsilon,j}(0)]$.

In order to obtain condition (2) is not enough to ask for $\overline{S_j} \cap \{x = 0\} \subset \overline{S_j(0)}$ (see picture (8)). The condition (2) is of quantitative type and it can be broken if we have instable phenomena even in an infinitesimal scale, for instance if $\Gamma_{\mu X}[T_{\mu X}^{\epsilon,j}(x)]$ does not converge "fast enough" to $\Gamma_{\mu X}[T_{\mu X}^{\epsilon,j}(0)]$ when $x \in (0, \delta)K$ and $x \to 0$. An erratic behavior of $\Gamma_{\mu X}[T_{\mu X}^{\epsilon,j}(x)]$ makes difficult to obtain similar Fatou coordinates for $\varphi$ and $X$. An example of bad convergence would be provided by the existence

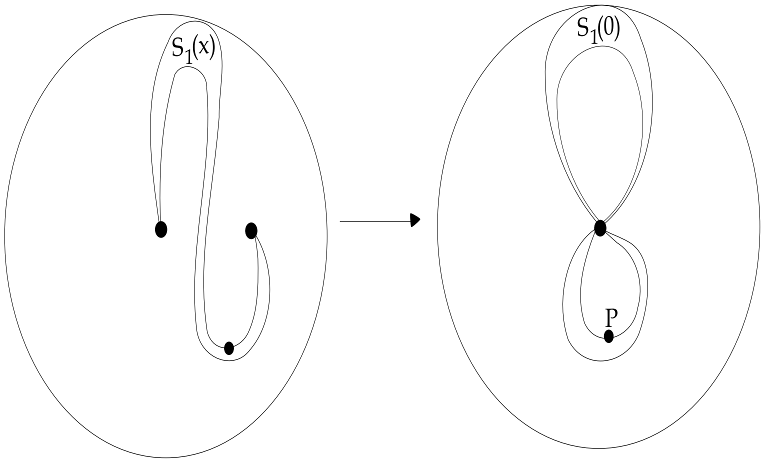


FIGURE 8. The point $P$ is in $\overline{S_j} \setminus \overline{S_j(0)}$

of a sequence $(w_n, y_n) \in S_j$ tending to $(0,0)$ and with $1/(1+|\psi_j^X(w_n, y_n)|)^{1+1/\nu(\varphi)}$ tending faster to 0 than $|f(w_n, y_n)|$. (see picture (9), it is somehow an infinitesimal version of picture (8)). All these pathologies are excluded as a consequence of the infinitesimal stability. The infinitesimal stability plays here a somehow analogous role to the Rolle property in the multi-parabolic case. It is a property of non-wandering type, forcing the trajectories of $\Re(\mu X)$ to go "fast" towards the fixed points. We bound the "complexity" of trajectories of $\Re(\mu X)$. Our quantitative study makes all these ideas rigorous.

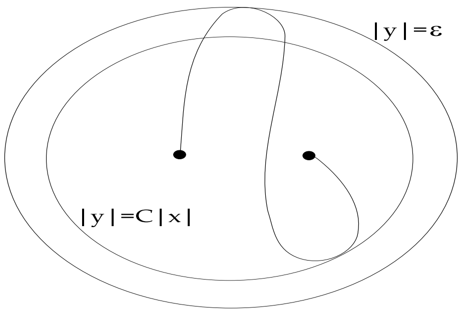


FIGURE 9. There is no trajectory splitting $y = |C|x$

***Proposition 5.6***. — *Let $\varphi$ be a non-degenerate element of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Consider $X_\varphi = f\partial/\partial y$, $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$ and a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X_\varphi)$. Then* *we have*

$$\Delta_\varphi = O(f) = O(y \circ \varphi - y) = O\left(\frac{1}{(1+|\psi_j^X|)^{1+\frac{1}{\nu(\varphi)}}}\right)$$

*in* $S_j$ *for any* $R_j^* \in \mathrm{Reg}_*^\epsilon(\mu X, K)$.

Fix $w \in [0,\delta)K$. By the previous discussion there exists a Fatou coordinate $\tilde{\psi}_j^\varphi$ of $\varphi$ in $R_j^* \in \mathrm{Reg}_*^\epsilon(\mu X, K)$ satisfying the condition (1). It is unique if we require

$$\lim_{P \in S_j(w),\ \mathrm{Img}(\psi_j^X(P)) \to +\infty} \tilde{\psi}_j^\varphi(P) - \psi_j^X(P) = 0. \tag{3}$$

The uniqueness implies that the Fatou coordinate $\tilde{\psi}_j^\varphi$ is continuous in $R_j^*$ and holomorphic in $R_j^\circ$. In other words the uniqueness forces the holomorphic dependance with respect to $w$. A stronger property can be proved, namely the function $\tilde{\psi}_j^\varphi - \psi_j^X$ admits a continuous extension to the fixed points.

***Proposition 5.7***. — *Let* $\varphi$ *be a non-degenerate element of* $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. *Consider* $\mu \in \mathbb{S}^1 \setminus \{-1,1\}$ *and a compact connected set* $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X_\varphi)$. *Then* $\tilde{\psi}_j^\varphi - \psi_j^X$ *is continuous in* $R_j \cup (\mathrm{Fix}(\varphi) \cap \overline{R_j})$ *and holomorphic in its interior for any* $R_j$ *in* $\mathrm{Reg}^\epsilon(\mu X, K)$.

With respect to the previous proposition let us remark that a priori $\tilde{\psi}_j^\varphi - \psi_j^X$ is defined only in $R_j^*$ and not in the whole $R_j$ when $R_j \in \mathrm{Reg}_2(\mu X, K)$. But then there exists a unique $k \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z}) \setminus \{j\}$ such that $R_j \setminus R_j(0) = R_k \setminus R_k(0)$. The function $(\tilde{\psi}_j^\varphi - \psi_j^X) - (\tilde{\psi}_k^\varphi - \psi_k^X)$ depends only on $x$ and by condition (3) is identically 0. As a consequence the function $\tilde{\psi}_j^\varphi - \psi_j^X$ is well-defined in $R_j = R_j^* \cup R_k^*$. The next result is the analogous of proposition 5.7 for Lavaurs vector fields.

***Corollary 5.2***. — *Let* $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ *be non-degenerate. Consider* $X_\varphi = f\partial/\partial y$. *Let* $\mu \in \mathbb{S}^1 \setminus \{-1,1\}$ *and consider a compact connected set* $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X_\varphi)$. *Then* $X_j^\varphi$ *is of the form* $f(x,y)(1+q(x,y))\partial/\partial y$ *where* $q$ *is continuous in* $R_j \cup (\mathrm{Fix}(\varphi) \cap \overline{R_j})$ *and holomorphic in its interior for any* $R_j \in \mathrm{Reg}^\epsilon(\mu X, K)$. *Moreover we have* $q_{|\mathrm{Fix}(\varphi)\cap\overline{R_j}} \equiv 0$.

Fix an irreducible component $\tau = \{y = h(x)\}$ of $\mathrm{Fix}(\varphi)$. Consider a system of integrals of the time form $\{\psi_j^X\}_{j \in \mathbb{Z}/(2\nu(X)\mathbb{Z})}$ of $X$ constructed as described in subsection 5.6. There exists a unique family $\{b_j\}_{j \in \mathbb{Z}/(2\nu(X)\mathbb{Z})}$ of functions being continuous in $[0,\delta)K$ and holomorphic in $(0,\delta)K^\circ$ such that the family

$$\{\psi_j^\varphi\}_{j \in \mathbb{Z}/(2\nu(X)\mathbb{Z})} \stackrel{\mathrm{def}}{=} \{\tilde{\psi}_j^\varphi + b_j\}_{j \in \mathbb{Z}/(2\nu(X)\mathbb{Z})}$$

satisfies

- $(\psi_{j+1}^\varphi - \psi_j^\varphi)_{|\overline{R_j}\cap\overline{R_{j+1}}\cap\mathrm{Fix}(\varphi)} \equiv \zeta_X(x)$ for any $j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$.
- $(\psi_j^\varphi - \psi_j^X)_{|\tau} \equiv 0$ for any $j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$.

We say that $\tau$ is a *privileged curve* and that $\{\psi_j^\varphi\}_{j\in\mathbb{Z}/(2\nu(\varphi)\mathbb{Z})}$ is a *privileged system of Fatou coordinates* with respect to $\{\psi_j^X\}_{j\in\mathbb{Z}/(2\nu(X)\mathbb{Z})}$ and $\tau$.

**5.8. Extension of the Ecalle-Voronin invariants.** — Fix a non-degenerate element $\varphi$ of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. Denote $X = X_\varphi$. Consider $\mu \in \mathbb{S}^1 \setminus \{-1,1\}$ and a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. We denote $D(\varphi) = \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$. We define

$$D_{-1}(\varphi) = \{j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z}) : \Re(X) \text{ points at } T_{\mu X}^{\epsilon,j}(0) \text{ towards } R_j\}.$$

We denote $D_1(\varphi) = D(\varphi) \setminus D_{-1}(\varphi)$.

The mapping $(x,\psi_j^\varphi)$ conjugates $\varphi$ and $(x,z+1)$ in $R_j^*$ for any $j \in D(\varphi)$. The set $R_j^*(0)$ is contained in an attracting petal (resp. an repelling petal) $P_j$ of $\varphi_{|x=0}$ if $j \in D_{-1}(\varphi)$ (resp. $j \in D_1(\varphi)$). For instance in the repelling case the set $P_j$ is composed by the points $y \in B(0,\epsilon)$ satisfying that there exists $k \in \mathbb{N}$ such that $\varphi^{-k}(0,y) \in R_j^*(0)$ and $\varphi^{-l}(0,y) \in \{0\} \times B(0,\epsilon)$ for any $0 < l < k$. We have that $P_j \cap P_k \neq \emptyset$ implies $k \in \{j-1, j, j+1\}$.

We can interpret $\varphi_{|x=0}$ as the diffeomorphism $z+1$ defined in the manifold with charts $\psi_j^\varphi(P_j)$ for $j \in D(\varphi)$ and whose changes of charts commute with $z+1$ and are of the form $\psi_{j+1}^\varphi(0,y) \circ (\psi_j^\varphi(0,y))^{-1}$ for $j \in D(\varphi)$. Moreover, the function $\psi_{j+1}^\varphi(0,y) \circ (\psi_j^\varphi(0,y))^{-1}$ is defined in $\psi_j^\varphi(\{0\} \times (P_j \cap P_{j+1})) \sim \{s\mathrm{Img} z > I\}$ for all $s \in \{-1,1\}$, $j \in D_s(\varphi)$ and some $I \in \mathbb{R}^+$.

The map $e^{2\pi i \psi_j^\varphi}(0,y)$ is a biholomorphism from $\mathrm{orb}(\varphi, R_j^*(0)) = \mathrm{orb}(\varphi_{|x=0}, P_j)$ to $\mathbb{C}^*$. Thus $\mathrm{orb}(\varphi, x=0)$ is the union of $\mathrm{orb}(\varphi, R_j^*(0)) \sim \mathbb{C}^*$ for $j \in D(\varphi)$. We have that $\mathrm{orb}(\varphi, R_j^*(0)) \cap \mathrm{orb}(\varphi, R_{j+1}^*(0)) \neq \emptyset$ in $\mathrm{orb}(\varphi, x=0)$ since $P_j \cap P_{j+1} \neq \emptyset$ for $j \in D(\varphi)$. Suppose $j \in D_1(\varphi)$; then the germs of $\mathrm{orb}(\varphi, R_j^*(0)) \sim \mathbb{C}^*$ and $\mathrm{orb}(\varphi, R_{j+1}^*(0)) \sim \mathbb{C}^*$ at 0 are identified by a germ $\Upsilon_j \in \mathrm{Diff}\,(\mathbb{C},0)$ defined in a domain $B(0, e^{-2\pi I})$. For $j \in D_{-1}(\varphi)$ the germs of $\mathrm{orb}(\varphi, R_j^*(0)) \sim \mathbb{C}^*$ and $\mathrm{orb}(\varphi, R_{j+1}^*(0)) \sim \mathbb{C}^*$ at $\infty$ are identified by a germ $\Upsilon_j \in \mathrm{Diff}(\mathbb{C},\infty)$ defined in a domain $\{|z| > e^{2\pi I}\}$. The space of orbits $\mathrm{orb}(\varphi, x=0)$ is a string of spheres glued by the system of changes of charts $\{\Upsilon_j\}_{j\in D(\varphi)}$. Indeed the string of spheres provides a complete system of analytic

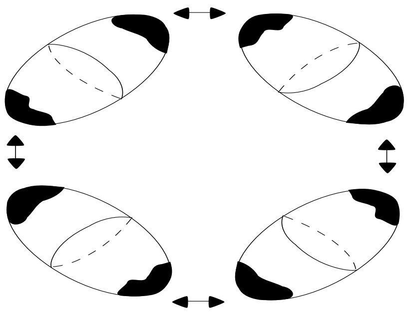

FIGURE 10. Space of orbits of $\varphi_{|x=0}$ with identifications in black

invariants for the classification of elements of $\mathrm{Diff}_1(\mathbb{C},0)$. This is the Martinet-Ramis [**6**] presentation of the Ecalle-Voronin invariants. We will apply the same program to non-degenerate elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$.

Let $R_j \in \mathrm{Reg}_2^{\epsilon}(\mu X, K)$. There exists a seed $\mathcal{S}_j$ containing $\{\alpha_X^{\epsilon}(R_j), \omega_X^{\epsilon}(R_j)\}$ but such that $\alpha_X^{\epsilon}(R_j(x))$ and $\omega_X^{\epsilon}(R_j(x))$ are in different seeds among the sons of $\mathcal{S}_j$ for any $x \in (0,\delta)K$. Consider the compact-like set $\mathcal{C}_j$ associated to $\mathcal{S}_j$. We define $R_j^{\natural} = \{Q \in R_j : \Gamma_{\mu X}^{\epsilon}[Q] \subset \mathcal{C}_j\}$ and $R_j^{\flat}$ as the connected component of $R_j^* \setminus R_j^{\natural}$ whose closure contains $T_{\mu X}^{\epsilon,j}(x)$ for any $x \in [0,\delta)K$. We define $R_j^{\flat} = R_j$ for $R_j \in \mathrm{Reg}_1^{\epsilon}(\mu X, K)$. Denote by $R_{j,j+1}$ the connected component of the set $([0,\delta)K \times B(0,\epsilon)) \setminus (R_j \cup R_{j+1} \cup \mathrm{Fix}(\varphi))$ such that $\overline{R_{j,j+1}(x)}$ contains the arc in $\partial U_{\epsilon}(x)$ going from $T_{\mu X}^{\epsilon,j}(x)$ to $T_{\mu X}^{\epsilon,j+1}(x)$ in counter clock-wise sense for any point $x \in [0,\delta)K$.

The restriction of $\Re(X)$ to $R_j^* \cup R_{j+1}^* \cup R_{j,j+1}$ satisfies the no-return property (see subsection 4.2) if and only if $R_j \neq R_{j+1}$. This is the reason because we introduced the sets $R_j^{\flat}$ since the restriction of $\Re(X)$ to $R_j^{\flat} \cup R_{j+1}^{\flat} \cup R_{j,j+1}$ satisfies the no-return property for any $j \in D(\varphi)$. Thus $\mathrm{orb}(\varphi, R_j^*)$ and $\mathrm{orb}(\varphi, R_{j+1}^*)$ are embedded in $\mathrm{orb}(\varphi, R_j^{\flat} \cup R_{j+1}^{\flat} \cup R_{j,j+1})$ and their intersection is not empty. Define the change of charts $\varpi_{\varphi,\mu,K}^{j}(x,z) = e^{2\pi i \psi_{j+1}^{\varphi}} \circ (x, e^{2\pi i \psi_j^{\varphi}})^{\circ(-1)}$ between $\mathrm{orb}(\varphi, R_j^*)$ and $\mathrm{orb}(\varphi, R_{j+1}^*)$ for any $j \in D(\varphi)$. We have

$$\mathrm{orb}(\varphi, R_j^*) \cap \mathrm{orb}(\varphi, R_{j+1}^*) \sim [0,\delta)K \times \{0 < |z|^s < h\}$$

or more precisely $\varpi_{\varphi,\mu,K}^{j}$ is defined in $[0,\delta)K \times \{0 < |z|^s < h\}$ for all $s \in \{-1,1\}$, $j \in D_s(\varphi)$ and some $h > 0$.

The value of $\psi_{j+1}^{\varphi} - \psi_j^{\varphi}$ at the fixed point $(\overline{R_{j,j+1}} \cap \mathrm{Fix}(\varphi)) \cap \{x = w\}$ is equal to $\zeta_X(w)$ for any $w \in [0,\delta)K$. The previous discussion implies:

***Proposition 5.8***. — *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ be non-degenerate. Fix $\mu \in \mathbb{S}^1 \setminus \{-1,1\}$ and a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X_{\varphi})$. Then there exists $h \in \mathbb{R}^+$ such that for all $s \in \{-1,1\}$ and $j \in D_s(\varphi)$ we have*

- $\varpi_{\varphi,\mu,K}^{j} \in C^0([0,\delta)K \times \{|z|^s < h\}) \cap \mathcal{O}((0,\delta)K^{\circ} \times \{|z|^s < h\})$.
- $\varpi_{\varphi,\mu,K}^{j}(0,z)$ *does not depend on $\mu$ and $K$.*
- $\varpi_{\varphi,\mu,K}^{j}$ *is of the form* $e^{2\pi i \zeta_{X_{\varphi}}(x)} z \left(1 + \sum_{l=1}^{\infty} b_{j,l,\mu,K}^{\varphi}(x) z^{sl}\right)$.
- $b_{j,l,\mu,K}^{\varphi} \in C^0([0,\delta)K) \cap \mathcal{O}((0,\delta)K^{\circ})$ *for any $l \in \mathbb{N}$.*

We define the *$\mu$-space of orbits* of $\varphi$ at $K$ as the variety obtained by taking an atlas composed of charts $\mathrm{orb}(\varphi, R_j^*) \sim [0,\delta)K \times \mathbb{C}^*$ for $j \in D(\varphi)$ and the changes of charts $(x, \varpi_{\varphi,\mu,K}^{j})$ identifying subsets of $\mathrm{orb}(\varphi, R_j^*)$ and $\mathrm{orb}(\varphi, R_{j+1}^*)$ for any $j$ in $D(\varphi)$. The $\mu$-space and the space of orbits of $\varphi$ coincide for $x = 0$. They are different for $x = w \in (0,\delta)K$ since the $\mu$-space of orbits does not contain the identifications

$$\mathrm{orb}(\varphi, R_j^* \setminus R_j^*(0)) \sim \mathrm{orb}(\varphi, R_k^* \setminus R_k^*(0))$$

for $R_j = R_k \in \mathrm{Reg}_2^\epsilon(\mu X, K)$ with $R_j^* \neq R_k^*$. We have that $\psi_k^\varphi - \psi_j^\varphi \equiv \psi_k^X - \psi_j^X$ is a pure meromorphic function in $\mathbb{C}\{x\}[x^{-1}]$. As a consequence we obtain

$$(x, e^{2\pi i \psi_k^\varphi}) \circ (x, e^{2\pi i \psi_j^\varphi})^{\circ(-1)} = (x, e^{2\pi i (\psi_k^X - \psi_j^X)(x)} z).$$

The $\mu$-space of orbits of $\varphi$ at $K$ is the space of orbits of $\varphi$ restricted to

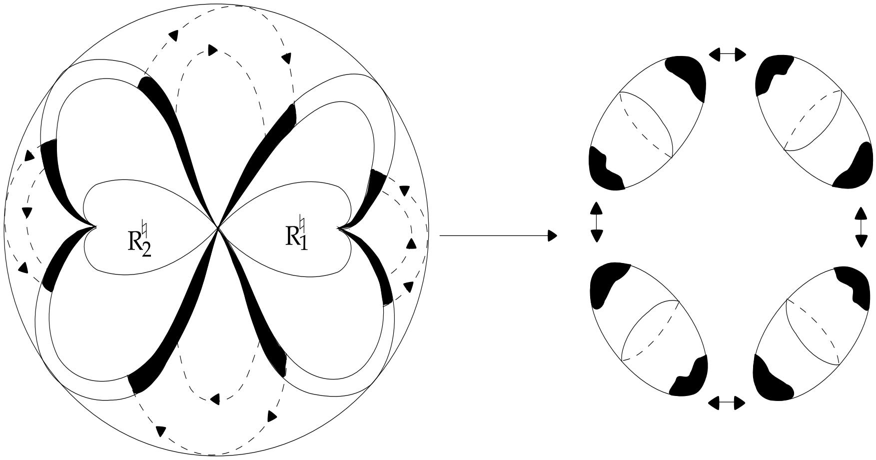


FIGURE 11. The $\mu$-space of orbits restricted to $x = w$

$$B_{\varphi,\mu,K} \stackrel{\mathrm{def}}{=} ([0,\delta)K \times B(0,\epsilon)) \setminus ((\cup_{R_j \in \mathrm{Reg}_2^\epsilon(\mu X, K)} R_j^\natural) \cup \mathrm{Fix}(\varphi)).$$

Moreover the restriction of $\varphi$ to $B_{\varphi,\mu,K}$ satisfies the no-return property. Thus the $\mu$-space of orbits depend nicely on the parameter $x$. By removing $\cup_{R_j \in \mathrm{Reg}_2^\epsilon(\mu X, K)} R_j^\natural$ we dismiss the complexity related to small divisors problems. Of course we can recover any dynamical information attached to the system by adding the "singular" identifications $(x, e^{2\pi i (\psi_k^X - \psi_j^X)(x)} z)$ for any $R_j \in \mathrm{Reg}_2^\epsilon(\mu X, K)$ where $R_k = R_j$ and $R_k^* \neq R_j^*$.

***Remark 5.2.*** — *Given a direction $x \in \lambda \mathbb{R}^+$ with $\lambda \in \mathbb{S}^1$ the set*

$$\{\mu \in \mathbb{S}^1 : \lambda \in \mathcal{U}(\mu X)\}$$

*is finite. The choices of $\mu$ in the same connected component of*

$$e^{(0,\pi i)} \setminus \{\iota \in \mathbb{S}^1 : \lambda \in \mathcal{U}(\iota X)\}$$

*provide equivalent definitions of the $\mu$-space of orbits in the neighborhood of the direction $x \in \lambda \mathbb{R}^+$. Thus there are finitely many choices of the $\mu$-space of orbits.*

The diffeomorphism $\varphi$ is of the form $(x, z+1)$ in a manifold with changes of charts

$$(x, \xi_{\varphi,\mu,K}^j) = (x, \psi_{j+1}^\varphi) \circ (x, \psi_j^\varphi)^{-1}$$

for $j \in D(\varphi)$. Since

$$\varpi_{\varphi,\mu,K}^j = e^{2\pi i z} \circ \xi_{\varphi,\mu,K}^j \circ \left(x, \frac{\ln z}{2\pi i}\right)$$

then proposition 5.8 implies

***Proposition 5.9.*** — *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ be non-degenerate. Fix $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$ and a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X_\varphi)$. Then there exists $I \in \mathbb{R}^+$ such that for all $s \in \{-1, 1\}$ and $j \in D_s(\varphi)$ we have*
- $\xi^j_{\varphi,\mu,K}(x, z + 1) = \xi^j_{\varphi,\mu,K}(x, z) + 1$.
- $\xi^j_{\varphi,\mu,K} \in C^0([0, \delta)K \times \{s\mathrm{Im}z > I\}) \cap \mathcal{O}((0, \delta)K^\circ \times \{s\mathrm{Im}z > I\})$.
- $\xi^j_{\varphi,\mu,K}(0, z)$ *does not depend on* $\mu$ *and* $K$.
- $\xi^j_{\varphi,\mu,K}$ *is of the form* $z + \zeta_{X_\varphi}(x) + \sum_{l=1}^{\infty} a^{\varphi}_{j,l,\mu,K}(x) e^{2\pi i s l z}$.
- $a^{\varphi}_{j,l,\mu,K} \in C^0([0, \delta)K) \cap \mathcal{O}((0, \delta)K^\circ)$ *for any* $l \in \mathbb{N}$.

**5.9. Analytic conjugacy.** — Fix a non-degenerate element $\varphi$ of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Denote $X = X_\varphi$. Consider $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$ and a compact connected $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. The $\mu$-space of orbits of $\varphi$ at $K$ is a rigid object since it is composed of copies of $[0, \delta)K \times \mathbb{C}^*$ and the set of biholomorphisms of $\mathbb{C}^*$ fixing 0 and $\infty$ is isomorphic to $\mathbb{C}^*$. The rigidity implies that the $\mu$-space of orbits determines the analytic class of conjugacy of $\varphi$. We obtain a complete system of analytic invariants for non-degenerate elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Roughly speaking it is a fibered version of the Ecalle-Voronin system of invariants.

Let $\varphi, \eta$ be non-degenerate elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Suppose that $\varphi$ and $\eta$ are formally conjugated. Then there exists $\hat{\sigma} \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ satisfying $\hat{\sigma} \circ \varphi = \eta \circ \hat{\sigma}$ and whose action on $\mathrm{Fix}(\varphi)$ is analytic **[9]** **[11]**. More precisely there exists an element $\sigma$ of $\mathrm{Diff}(\mathbb{C}^2, 0)$ such that

(a) $x \circ \hat{\sigma} \equiv x \circ \sigma$.
(b) $y \circ \hat{\sigma} - y \circ \sigma$ belongs to the ideal associated to the analytic set $\mathrm{Fix}(\varphi)$.

Indeed the mapping $\sigma$ satisfies **[9]** **[11]**

(1) $x \circ \sigma \in \mathbb{C}\{x\}$
(2) $(y \circ \eta - y) \circ \sigma = (y \circ \varphi - y)$
(3) $\mathrm{Res}(X_\eta, \sigma(P)) = \mathrm{Res}(X_\varphi, P)$ for any $P \in \mathrm{Fix}(\varphi)$.

Moreover, given $\sigma$ holding conditions (1), (2) and (3) there exists $\hat{\sigma} \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ conjugating $\varphi$ and $\eta$ and such that $\sigma$ and $\hat{\sigma}$ satisfy the conditions (a) and (b). Then up to identify such $\sigma$ and replacing $\eta$ with $\sigma^{-1} \circ \eta \circ \sigma$ we can suppose

(A) $(y \circ \eta - y) = (y \circ \varphi - y)$
(B) $\mathrm{Res}(X_\eta, P) = \mathrm{Res}(X_\varphi, P)$ for any $P \in \mathrm{Fix}(\varphi)$ (see def. 4.4).

The ideal $(X_\varphi(y)) = (y \circ \varphi - y)$ and the function $\mathrm{Res}(X_\varphi) : \mathrm{Fix}(\varphi) \to \mathbb{C}$ compose a complete system of analytic invariants of $X_\varphi$. As a consequence if the conditions (A) and (B) are satisfied there exists $\tilde{\sigma} = (x, \sigma_1(x, y)) \in \mathrm{Diff}(\mathbb{C}^2, 0)$ such that $\tilde{\sigma}_{|\mathrm{Fix}(\varphi)} \equiv Id$ and and $\tilde{\sigma}_* X_\varphi = X_\eta$. Then up to replace $\eta$ with $\tilde{\sigma}^{-1} \circ \eta \circ \tilde{\sigma}$ we can suppose that $X_\varphi \equiv X_\eta$.

***Definition 5.12.*** — *Let $\varphi, \eta \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ be non-degenerate. We say that $\varphi$ and $\eta$ have common normal form* $\exp(X)$ *if there exists a vector field $X \in \mathcal{X}_{p1}(\mathbb{C}^2, 0)$*

*satisfying the proximity condition for both $\varphi$ and $\eta$. We say that a formal conjugation $\hat{\sigma} \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^2,0)$ between $\varphi$ and $\eta$ is normalized if $\hat{\sigma}_{|\mathrm{Fix}(\varphi)} \equiv Id$. We denote by $\hat{Z}(\varphi)$ the group of normalized elements of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^2,0)$ commuting with $\varphi$.*

The previous discussion implies that we can suppose that when dealing with formal conjugacy problems we can reduce the problem to diffeomorphisms $\varphi, \eta$ with common normal form and the equivalence $\sim_{\mathrm{an}}$ instead of analytic classification.

Let $\varphi, \eta$ be non-degenerate elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ with common normal form. Suppose that $\varphi \sim_{for} \eta$ by $\hat{\sigma} \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^2,0)$. The set of normalized elements of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^2,0)$ conjugating $\varphi$ and $\eta$ is equal to $\hat{Z}(\eta) \circ \hat{\sigma}$. Then it makes sense to analyze the nature of $\hat{Z}(\eta)$.

***Proposition 5.10***. — *Let $\varphi$ be a non-degenerate element of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. Suppose $N(\varphi)=1$. Then we obtain that $\hat{Z}(\varphi)$ is the semi-direct product of a finite group of order $\nu(\varphi)$ and $\{\exp(\hat{c}(x)\log\varphi) : \hat{c} \in \mathbb{C}[[x]]\}$. Moreover $\hat{Z}(\varphi)$ is commutative.*

***Proposition 5.11***. — *Let $\varphi$ be a non-degenerate element of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. Suppose $N(\varphi)>1$. Then we obtain $\hat{Z}(\varphi) = \{\exp(\hat{c}(x)\log\varphi) : \hat{c} \in \mathbb{C}[[x]]\}$. In particular $\hat{Z}(\varphi)$ is commutative.*

Let $\varphi, \eta$ be non-degenerate elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ with common normal form $\exp(X)$. Fix $\mu \in \mathbb{S}^1 \setminus \{-1,1\}$ and a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. We choose a system $\{\psi_j^X\}_{j \in D(\varphi)}$ as described in subsection 5.6. By fixing a privileged curve $\tau$ in $\mathrm{Fix}(\varphi)$ we can construct $\{\psi_j^\varphi\}_{j \in D(\varphi)}$ and $\{\psi_j^\eta\}_{j \in D(\varphi)}$ (subsection 5.7). We can define the mapping

$$\sigma_j(\varphi,\eta) = (x,\psi_j^\eta)^{-1} \circ (x,\psi_j^\varphi)$$

defined in $R_j \in \mathrm{Reg}^\epsilon(\mu X, K)$ for any $j \in D(\varphi)$ and some $\epsilon > 0$ independent of $j$. As usual the diffeomorphism $\sigma_j(\varphi,\eta)$ admits an extension to a bigger domain by using the formula $\sigma_j(\varphi,\eta) \circ \varphi = \eta \circ \sigma_j(\varphi,\eta)$. Suppose that $\varphi \sim_{\mathrm{an}} \eta$ by $\sigma \in \mathrm{Diff}\,(\mathbb{C}^2,0)$. Supposed $N(\varphi) > 1$, it satisfies

$$\sigma_{|R_j} = \exp(c(x)X_j^\eta) \circ \sigma_j(\varphi,\eta)$$

for any $j \in D(\varphi)$ and some $c \in \mathbb{C}\{x\}$ independent of $j$ and $K$. In the case $N(\varphi)=1$ the group $\hat{Z}(\eta)$ contains a finite group; thus $\sigma$ satisfies

$$\sigma_{|R_j} = \exp(c(x)X_j^\eta) \circ [(x,\psi_{j+l}^\eta)^{-1} \circ (x,\psi_j^\eta)] \circ \sigma_j(\varphi,\eta)$$

for any $j \in D(\varphi)$ and some $c \in \mathbb{C}\{x\}$ and $l \in \mathbb{Z}/(\nu(\varphi)\mathbb{Z})$ independent of $j$ and $K$. The latter equation can be rewritten in the form

$$\sigma_{|R_j} = (x,\psi_{j+l}^\eta)^{-1} \circ (x,\psi_j^\varphi + c(x))$$

since $\exp(c(x)X_j^\eta) = (x,\psi_j^\eta)^{-1} \circ (x,\psi_j^\eta + c(x))$. Then we obtain $\varphi \sim_{\mathrm{an}} \eta$ if the equations

$$(4) \qquad (x,\psi_{j+l}^\eta)^{-1} \circ (x,\psi_j^\varphi + c(x)) = (x,\psi_{j+1+l}^\eta)^{-1} \circ (x,\psi_{j+1}^\varphi + c(x)) \ \ \forall j \in D(\varphi)$$

hold true in the intersections of their domains of definition for some $c \in \mathbb{C}\{x\}$ and $l$ in $\mathbb{Z}/(\nu(\varphi)\mathbb{Z})$ (note that $l = 0$ if $N(\varphi) > 1$). The equation (4) is equivalent to

$$\xi^{j+l}_{\eta,\mu,K}(x, z + c(x)) \equiv \xi^{j}_{\varphi,\mu,K}(x, z) + c(x) \quad \forall j \in D(\varphi).$$

As a consequence we obtain

***Theorem 5.3***. — *Let $\varphi, \eta \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ be non-degenerate with common normal form* $\exp(X)$. *Suppose $N(\varphi) = 1$. Then $\varphi \sim_{\mathrm{an}} \eta$ if and only if*

$$\xi^{j+l}_{\eta,i,\mathbb{S}^1}(x, z + c(x)) \equiv \xi^{j}_{\varphi,i,\mathbb{S}^1}(x, z) + c(x) \quad \forall j \in D(\varphi)$$

*for some $c \in \mathbb{C}\{x\}$ and $l \in \mathbb{Z}/(\nu(\varphi)\mathbb{Z})$.*

***Theorem 5.4***. — *Let $\varphi, \eta \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ be non-degenerate with common normal form* $\exp(X)$. *Fix $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$ and a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. Suppose $N(\varphi) > 1$. Then $\varphi \sim_{\mathrm{an}} \eta$ if and only if there exists $c \in \mathbb{C}\{x\}$ such that*

$$\xi^{j}_{\eta,\mu,K}(x, z + c(x)) \equiv \xi^{j}_{\varphi,\mu,K}(x, z) + c(x)$$

*for any $j \in D(\varphi)$.*

***Corollary 5.3***. — *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ be non-degenerate. Fix $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$ and a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X_\varphi)$. Then $\log \varphi \in \mathcal{X}(\mathbb{C}^2, 0)$ if and only if $\xi^{j}_{\varphi,\mu,K}(x, z) \equiv z + \zeta_{X_\varphi}(x)$ for any $j \in D(\varphi)$.*

*Proof of the corollary.* — The infinitesimal generator $\log \varphi$ of $\varphi$ is analytic if and only if $\varphi \sim_{\mathrm{an}} \exp(X_\varphi)$. Indeed if $\log \varphi$ is analytic then $\log \varphi$ and $X_\varphi$ satisfy conditions (A) and (B). Thus we obtain $\log \varphi \sim_{\mathrm{an}} X_\varphi$ and then $\varphi \sim_{\mathrm{an}} \exp(X_\varphi)$. The corollary is now a consequence of $\xi^{j}_{\exp(X_\varphi),\mu,K}(x, z) \equiv z + \zeta_{X_\varphi}(x)$ for any $j \in D(\varphi)$ and theorems 5.3 and 5.4. $\square$

The theorem 5.3 is a parametrized version of the Ecalle-Voronin theorem of analytic classification. The proof of the implication $\Rightarrow$ in theorem 5.4 is simple. The implication $\Leftarrow$ is trickier since a priori the conjugating mapping $\sigma$ is defined in a set $[0, \delta)K \times B(0, \epsilon)$. Anyway, given a compact connected $J \subset \mathcal{U}(\mu' X)$ the existence of $\sigma$ implies that

$$\xi^{j}_{\eta,\mu',J}(x, z + c(x)) \equiv \xi^{j}_{\varphi,\mu',J} + c(x)$$

in $\{x \in [0, \delta)(K \cap J)\}$ for any $j \in D(\varphi)$. By analytic continuation we can extend the previous equalities to $\{x \in [0, \delta)J\}$. We then construct a conjugation $\sigma_J$ between $\varphi$ and $\eta$ and defined in a set $[0, \delta)J \times B(0, \epsilon)$. The mappings $\sigma$ and $\sigma_J$ coincide, we have extended $\sigma$ to $[0, \delta)(K \cup J) \times B(0, \epsilon)$. By iterating this process we obtain that $\sigma$ belongs to $\mathrm{Diff}\,(\mathbb{C}^2, 0)$.

We discuss next why theorem 3.3 holds true. Let $r \in \mathbb{R}^+$ such that there exists an injective $\kappa_w$ defined in $B(0, r)$, with $(\kappa_w)_{|\mathrm{Fix}(\varphi) \cap \{x=w\}} \equiv Id$ and conjugating $\varphi_{|x=w}$ and $\eta_{|x=w}$ for any $w \neq 0$. The existence of $\kappa_w$ for $w \neq 0$ implies that $\varphi$ and $\eta$ satisfy conditions (A) and (B). As a consequence we can suppose that $\varphi$ and $\eta$ have a common normal form $\exp(X)$. It can be proved that the mappings $\kappa_w$ have a moderated behavior. More precisely, by considering a smaller $r>0$ if necessary we can suppose that there exists $R>0$ such that $\kappa_w(B(0,r)) \subset B(0,R)$ for any $w \neq 0$. Fix $\mu \in \mathbb{S}^1 \setminus \{-1,1\}$ and a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X_\varphi)$. Given $w \in (0,\delta)K$ then $\kappa_w$ satisfies

$$(\kappa_w)_{|R_j(w)}(y) = \exp(c(w)X_j^\eta) \circ \left([(x,\psi_{j+l(w)}^\eta)^{-1} \circ (x,\psi_j^\eta)] \circ \sigma_j(\varphi,\eta)(w,y)\right)$$

for some $c(w) \in \mathbb{C}$, $l(w) \in \mathbb{Z}/(\nu(\varphi)\mathbb{Z})$ and any $j \in D(\varphi)$. Indeed we have

$$\xi_{\eta,\mu,K}^{j+l(w)}(w,z+c(w)) \equiv \xi_{\varphi,\mu,K}^{j}(w,z) + c(w) \ \ \forall (w,j) \in (0,\delta)K \times D(\varphi).$$

We can suppose that both $\log\varphi$ and $\log\eta$ are non-analytic (divergent), otherwise both of them are analytic by corollary 5.3 and theorems 5.3 and 5.4. In such a case $\log\varphi$ and $\log\eta$ satisfy conditions (A) and (B) and then $\varphi \sim_{\mathrm{an}} \eta$. The mappings

$$[(x,\psi_{j+l(w)}^\eta)^{-1} \circ (x,\psi_j^\eta)] \circ \sigma_j(\varphi,\eta)(w,y)$$

are moderated, i.e. their images are contained in $B(0,R)$ for all $w \in (0,\delta)K$ and $j \in D(\varphi)$ (maybe by considering a smaller $r>0$). Moreover since $X$ is very similar to $X_j^\varphi$ in $R_j$ (prop. 5.7 and cor. 5.2) then

$$\exp(c(w)X)(\{w\} \times B(0,r)) \subset \{w\} \times B(0,R) \tag{5}$$

for any $w \in (0,\delta)K$. We obtain the expression (5) for any $w \neq 0$ by considering other compact sets. Now it is not complicated to prove that the previous condition implies that $c$ is bounded in a pointed neighborhood of 0. The existence of $\kappa_w$ for $w \neq 0$ also implies the existence of a function $d(x)$ and $l \in \mathbb{Z}/(\nu(\varphi)\mathbb{Z})$ such that

$$\xi_{\eta,\mu,K}^{j+l}(x,z+d(x)) \equiv \xi_{\varphi,\mu,K}^{j}(x,z) + d(x) \ \ \forall j \in D(\varphi).$$

The function $d$ is defined in the universal covering of $B(0,\delta) \setminus T$ where $T$ is a closed set such that $T \setminus \{0\}$ is discrete. It turns out that since $\eta$ is not analytically trivial the function $\mathrm{Im}(d)$ is well-defined in $B(0,\delta) \setminus T$. Moreover the boundness of $c$ implies that $\mathrm{Im}(d)$ is bounded in $B(0,\delta) \setminus T$. By using the last property we deduce that $d$ can be extended to a neighborhood of the origin to obtain an element of $\mathbb{C}\{x\}$. Theorems 5.3 and 5.4 imply that $\varphi \sim_{\mathrm{an}} \eta$.

## 6. Compactification of the Ecalle-Voronin invariants

Let $\varphi$ be a non-degenerate element of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. Suppose that there exists a parabolic component $\gamma = \{y = h(x)\}$ of $\mathrm{Fix}(\varphi)$ (see def. 5.1). Given $w \in B(0,\delta)$ we can consider the germ $\varphi_{(w,h(w))}$ of $\varphi_{|x=w}$ in the neighborhood of $y = h(w)$. This section is devoted to sketch the proof of the following theorem:

***Theorem 6.1***. — *Let $\varphi,\eta \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ be non-degenerate with common normal form $\exp(X)$. Suppose that there exists a parabolic irreducible component $\gamma$ of $\mathrm{Fix}(\varphi)$. Suppose also that $\varphi_0$ is highly non-trivial and $N(\varphi)=2$. Then $\varphi \sim_{\mathrm{an}} \eta$ if and only if $\varphi_Q \overset{an_1}{\sim} \eta_Q$ for any $Q$ in a pointed neighborhood of 0 in $\gamma$.*

***Definition 6.1.*** — *We denote* $\phi \overset{an_1}{\sim} \tau$ *for* $\phi, \tau \in \mathrm{Diff}\,(\mathbb{C}, 0)$ *if there exists a mapping in* $\mathrm{Diff}_1(\mathbb{C}, 0)$ *conjugating* $\phi$ *and* $\tau$.

***Definition 6.2.*** — *We say that* $\phi \in \mathrm{Diff}_1(\mathbb{C}, 0)$ *is highly non-trivial if no change of charts associated to* $\phi$ *is a translation. Given* $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ *the highly non-triviality of* $\varphi_{|x=0}$ *is equivalent to* $\xi^j_{\varphi,\mu,K}(0, z) \not\equiv z + \zeta_{X_\varphi}(0)$ *for any* $j \in D(\varphi)$. *This condition is generic and independent of the choices of* $\mu \in \mathbb{S}^1$ *and* $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$.

***Definition 6.3.*** — *Let* $\varphi$ *be a non-degenerate element of* $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. *Given an irreducible component* $\gamma$ *of* $\mathrm{Fix}(\varphi)$ *we define* $\nu_\varphi(\gamma)$ *as the only natural number such that* $y \circ \varphi - y \in I(\gamma)^{\nu_\varphi(\gamma)+1} \setminus I(\gamma)^{\nu_\varphi(\gamma)+2}$ *where* $I(\gamma)$ *is the ideal of* $\gamma$.

***Remark 6.1.*** — *There is a version of theorem 6.1 for the case* $N(\varphi) = 1$. *It is much less interesting since in that case the number of changes of charts associated to* $\varphi_Q$ *for* $Q \in \gamma \setminus \{(0, 0)\}$ *and to* $\varphi$ *coincide. Roughly speaking, the analytic invariants of* $\varphi$ *are the union of the Ecalle-Voroning invariants of* $\varphi_Q$ *for* $Q \in \gamma \setminus \{(0, 0)\}$.

***Remark 6.2.*** — *The case* $N(\varphi) = 2$ *is generic among the diffeomorphisms satisfying* $N(\varphi) > 1$.

The implication $\Rightarrow$ in theorem 6.1 is trivial. We focus on the implication $\Leftarrow$. Suppose without lack of generality that $\gamma = \{y = 0\}$. Consider a compact connected set $K' \subset \mathbb{S}^1 \setminus \mathcal{U}(iX)$. Consider also a compact connected set $K' \subset K \subset \mathbb{S}^1$ such that there exists a continuous function $\mu_K : K \to \mathbb{S}^1 \setminus \{-1, 1\}$ satisfying that $\lambda \notin \mathcal{U}(\mu_K(\lambda)X)$ for any $\lambda \in K$. We also require the condition $\mu_K^{-1}(i) \cap K' \neq \emptyset$.

We can order the petals $P_1^\varphi(w), \ldots, P^\varphi_{2\nu_\varphi(\gamma)}(w)$ of $\varphi_{(w,h(w))}$ in counter clock-wise sense. These petals are open sets and they depend continuously on $w$. More precisely the sets $P_1^\varphi, \ldots, P^\varphi_{2\nu_\varphi(\gamma)}$ are open in $\mathbb{C}^2$. Each petal $P_j^\varphi \cap ([0, \delta)K' \times B(0, \epsilon))$ with $j \in \mathbb{Z}/(2\nu_\varphi(\gamma)\mathbb{Z})$ contains a region $R_{\upsilon(K,j)}$ of $\mathrm{Reg}_1^\epsilon(iX, K')$. We choose $\psi^\varphi_{\upsilon(K,j)}$ as a Fatou coordinate of $\varphi$ in $R_{\upsilon(K,j)}$, it can be extended to $P_j^\varphi \cap ([0, \delta)K' \times B(0, \epsilon))$ by using $\psi^\varphi_{\upsilon(K,j)} \circ \varphi = \psi^\varphi_{\upsilon(K,j)} + 1$. Moreover, the Fatou coordinate $\psi^\varphi_{\upsilon(K,j)}$ can be extended continuously to $P_j^\varphi \cap ([0, \delta)K \times B(0, \epsilon))$ and is holomorphic in the set $P_j^\varphi \cap ((0, \delta)K^\circ \times B(0, \epsilon))$. This is a consequence of the fact that $\mu$-spaces of orbits and Fatou coordinates in $[0, \delta)\lambda \times B(0, \epsilon)$ for $\lambda \in \mathbb{S}^1$ do not depend on $\mu$ but on the connected component of $\{\mu_0 \in \mathbb{S}^1 : \lambda \notin \mathcal{U}(\mu_0 X)\}$ containing $\mu$ (see remark 5.2). Define $d(K, j)$ as the smallest natural number such that

$$\upsilon(K, j + 1) - \upsilon(K, j) = d(K, j) + 2\nu(\varphi)\mathbb{Z}.$$

Suppose that $\upsilon(K, j) \in D_s(\varphi)$; we have that the change of charts

$$\xi^j_{\varphi,\gamma,K}(x, z) = \psi^\varphi_{\upsilon(K,j+1)} \circ (x, \psi^\varphi_{\upsilon(K,j)})^{-1}(x, z)$$

associated to the petals $P_j^\varphi$ and $P_{j+1}^\varphi$ is of the form

$$\xi^j_{\varphi,\gamma,K}(x, z) = z + \zeta_{K,j}(x) + \sum_{l=1}^{\infty} a^\varphi_{j,l,\gamma,K}(x) e^{2\pi i s l z} \tag{6}$$

where $\zeta_{K,j}(x)$ belongs to $\mathbb{C}\{x\}[x^{-1}]$ and does not depend on $\varphi$ but only on $X_\varphi$ (indeed it does not exactly depend on $X_\varphi$ but on $\mathrm{Fix}(\varphi)$ and $\mathrm{Res}_\varphi$). In the case $d(K,j)=1$ we have $\zeta_{K,j}(x) \equiv \zeta_{X_\varphi}(x)$ and $\xi_{\varphi,i,K}^{\upsilon(K,j)} \equiv \xi_{\varphi,\gamma,K}^{j}$.

**6.1. Strategy of the proof of theorem 6.1.** — . It is easy to construct a conjugation $\sigma$ defined in a neighborhood of $\gamma \setminus (T \cup \{(0,0)\})$ where $T$ is a discrete set in $\gamma \setminus \{(0,0)\}$. The mapping $\sigma$ is of the form

$$\sigma_{|P_j^\varphi \cap([0,\delta)K \times B(0,\epsilon))} = \exp(c(x)X_{\upsilon(K,j)}^{\eta}) \circ [(x, \psi_{\upsilon(K,j)}^{\eta})^{-1} \circ (x, \psi_{\upsilon(K,j)}^{\varphi})]$$

The function $c$ is holomorphic in the universal covering of $\gamma \setminus (T \cup \{(0,0)\})$ and it does not depend on the choices of $j$ and $K$. Suppose that we can prove that $c(x)$ is a holomorphic function defined in the neighborhood of 0, this is the key step. The map $\sigma_{|P_j^\varphi}$ has an asymptotic development $\hat{\sigma}$ at $\gamma$. Moreover $\hat{\sigma}$ is the only formal diffeomorphism conjugating $\varphi$ and $\eta$ of the form

$$\exp(c(x)\log\eta) \circ (x, y+\hat{a}(x,y))$$

where $\hat{a} \in (y)^{\nu_\varphi(\gamma)+2} \subset \mathbb{C}[[x,y]]$ and $\log\eta$ is the infinitesimal generator of $\eta$. Now $\sigma$ is defined in a neighborhood of $\gamma \setminus \{(0,0)\}$ and $\hat{\sigma}$ is its asymptotic development at $\gamma$. We deduce that $\hat{\sigma} \equiv \sigma$. As a consequence $y \circ \hat{\sigma}$ converges in a neighborhood of $\gamma \setminus \{(0,0)\}$. By the modulus maximum principle (see proof of theorem 3.2) we obtain that $\hat{\sigma}$ belongs to $\mathrm{Diff}\,(\mathbb{C}^2,0)$.

A priori controlling $c$ when $x \to 0$ is difficult since $c$ is expected to behave as an essential singularity. We will use the extension of the Ecalle-Voronin invariants provided in subsection 5.8 in sectors $[0,\delta)K \times B(0,\epsilon)$ to extend to $x=0$ some of the changes of charts $\xi_{\varphi,\gamma,K}^{j}$ (namely the ones satisfying $\upsilon(K,j+1)=\upsilon(K,j)+1$). Then, the highly non-triviality of $\varphi_0$ implies that $c(x)$ is a holomorphic function in the neighborhood of 0.

**6.2. Geometrical motivation.** — The vector field $X$ is of the form

$$X = u(x,y)y^{\nu_\varphi(\gamma)+1}(y-q(x))^n \partial/\partial y$$

where $u$ is a unit of $\mathbb{C}\{x,y\}$. Denote by $\kappa$ the order of the series $q$ at 0. The vector field $X$ has associated $\kappa$ polynomial vector fields $\lambda^{m_1}Y_1(1)$, …, $\lambda^{m_\kappa}Y_\kappa(1)$. We have

$$Y_k(\lambda) = \lambda^{k(\nu_\varphi(\gamma)+n)}u(0,0)t^{\nu_\varphi(\gamma)+n+1}\partial/\partial t$$

for $1 \leq k < \kappa$. The vector field $Y_k(1)$ has a unique singular point at 0 such that $\mathrm{Res}(Y_k(1),0)=0$ for any $1 \leq k < \kappa$. As a consequence we obtain that

$$\mathcal{U}_k(\mu X) = \{\lambda_0 \in \mathbb{S}^1 : \mu \in \mathcal{U}(Y_k(\lambda_0))\} \text{ and } \mathcal{U}_k^\lambda(X) = \{\mu_0 \in \mathbb{S}^1 : \lambda \in \mathcal{U}_k(\mu_0 X)\}$$

are empty for all $(\lambda,\mu) \in \mathbb{S}^1 \times \mathbb{S}^1$ and $1 \leq k < \kappa$. The dynamics of $\Re(\mu Y_k(\lambda))$ is the dynamics of a Fatou flower for all $1 \leq k < \kappa$ and $(\lambda,\mu) \in \mathbb{S}^1 \times \mathbb{S}^1$. Then the compact-like set to which $Y_k(\lambda)$ is associated behaves as a product-like set when dealing with the dynamics of $\Re(\mu X)$ for $\mu \in \mathbb{S}^1$ and $k < \kappa$. The dynamics of $\Re(\mu X)_{|[0,\delta)\lambda \times B(0,\epsilon)}$ is determined by the dynamics of $\Re(\mu Y_\kappa(\lambda))$. The vector field $Y_\kappa(\lambda)$ is of the form

$$Y_\kappa(\lambda) = \lambda^{\kappa(\nu_\varphi(\gamma)+n)} u(0,0) t^{\nu_\varphi(\gamma)+1}(t-t_0)^n \partial/\partial t$$

where $t_0$ is the coefficient of $x^\kappa$ in $q(x)$. The set $\mathcal{U}_\kappa(\mu X)$ has $2m_\kappa = 2\kappa(\nu_\varphi(\gamma)+n)$ points whereas $\mathcal{U}_\kappa^\lambda(X)$ has 2 points for any $(\lambda,\mu) \in \mathbb{S}^1 \times \mathbb{S}^1$.

Consider a direction $\lambda_0 \notin \mathcal{U}(iX)$ and $\lambda_1 = e^{i\pi/m_\kappa}\lambda_0$. We have $\lambda_1 \notin \mathcal{U}(iX)$ and $i\lambda_0^{m_\kappa} Y_\kappa(1) = -i\lambda_1^{m_\kappa} Y_\kappa(1)$. As a consequence the vector fields $\Re(iX)_{|[0,\delta)\lambda_0 \times B(0,\epsilon)}$ and $\Re(iX)_{|[0,\delta)\lambda_1 \times B(0,\epsilon)}$ are expected to be qualitatively analogous. Let us be more precise. Denote $\lambda_s = \lambda_0 e^{is\pi/m_\kappa}$ for $s \in [0,1]$. The set of tangent points between $\Re(iY_\kappa(\lambda_s))$ and $B(0,\rho)$ for $\rho >> 0$ is composed of $2(\nu_\varphi(\gamma)+n)$ points. They are very close to the points of the set

$$T_s \stackrel{\text{def}}{=} \begin{cases} \operatorname{Im}(\lambda_s^{\kappa(\nu_\varphi(\gamma)+n)} u(0,0) t^{\nu_\varphi(\gamma)+n}) = 0 \\ |t| = \rho. \end{cases}$$

Moreover, we obtain $T_s = e^{-i\pi s/(\nu_\varphi(\gamma)+n)} T_0$. A point $t^0 \in T_0$ corresponds by continuous extension to the point $e^{-i\pi s/(\nu_\varphi(\gamma)+n)} t^0 \in T_s$. We have $T_1 = -T_0 = T_0$ but it turns out that $e^{-i\pi/(\nu_\varphi(\gamma)+n)} t^0$ is not equal to $t^0$, indeed $t^0$ is the point following $e^{-i\pi/(\nu_\varphi(\gamma)+n)} t^0$ in $T_0$ when we consider the points of $T_0$ ordered in counter clock-wise sense. The dynamics of $\Re(iX)_{|\{\delta_0\lambda_0\}\times\overline{B(0,\epsilon)}}$ is topologically equivalent by a mapping $H_{\delta_0}$ to the dynamics of $\Re(iX)_{|\{\delta_0\lambda_1\}\times\overline{B(0,\epsilon)}}$ for $\delta_0 \in (0,\delta)$ but the role of the points of $T_{iX}^\epsilon$ is not preserved by $H_{\delta_0}$. The previous discussion implies that $H_{\delta_0}(T_{iX}^{\epsilon,j}(\delta_0\lambda_0)) = T_{iX}^{\epsilon,j+1}(\delta_0\lambda_1)$ for $j \in D(\varphi)$. Every petal of $\exp(Y_\kappa(\lambda_s))$ at $t=0$ contains the germ of a line $\mathbb{R}^+\omega$ for some $\omega \in \mathbb{S}^1$ in the set

$$L_s \stackrel{\text{def}}{=} \{\omega \in \mathbb{S}^1 : \lambda_s^{\kappa(\nu_\varphi(\gamma)+n)} u(0,0)(-t_0)^n \omega^{\nu_\varphi(\gamma)} \in \mathbb{R}\}.$$

We get $L_s = e^{-i\pi s/\nu_\varphi(\gamma)} L_0$ and $L_1 = e^{-i\pi/\nu_\varphi(\gamma)} L_0$. Therefore we obtain

$$H_{\delta_0}(P_j^\varphi(\delta_0\lambda_0)) = P_{j+1}^\varphi(\delta_0\lambda_1) \ \ \forall j \in \mathbb{Z}/(2\nu_\varphi(\gamma)\mathbb{Z}).$$

The dynamics of $\Re(iX)_{|\{\delta_0\lambda_0\}\times\overline{B(0,\epsilon)}}$ and $\Re(iX)_{|\{\delta_0\lambda_1\}\times\overline{B(0,\epsilon)}}$ in an example are represented in picture (12).

Denote $\tau = i\tau_0$ for $\tau_0 = \operatorname{Res}(Y_\kappa(1),0)/|\operatorname{Res}(Y_\kappa(1),0)|$. We obtain

$$\{(\lambda,\mu) \in \mathbb{S}^1 \times \mathbb{S}^1 : \lambda \in \mathcal{U}(\mu X)\} = \{(\lambda,\mu) \in \mathbb{S}^1 \times \mathbb{S}^1 : \lambda^{m_\kappa}\mu \in \{-\tau,\tau\}\}.$$

We choose $\lambda_0 \in \mathcal{U}_\kappa(X)$. We define $K_j' = \{\lambda_0 e^{i\pi j/m_\kappa}\}$ for $j \in \mathbb{Z}/(2m_\kappa\mathbb{Z})$. We have $(\cup_{j=0}^{2m_\kappa-1} K_j') \cap \mathcal{U}(iX) = \emptyset$. We can define $K_j$ to be any compact connected set containing $K_j'$ and contained in the open set $V_j = \lambda_0 e^{i\pi j/m_\kappa} e^{i\pi(-1,1)/m_\kappa}$ of $\mathbb{S}^1$. The continuous function $\mu_j = \mu_{K_j}$ is given by

$$\mu_j(\lambda_0 e^{i\pi j/m_\kappa} e^{i\pi s/m_k}) = e^{i\pi(1-s)/2} \text{ for } s \in (-1,1)$$

for $j \in \mathbb{Z}/(2m_\kappa\mathbb{Z})$. We have $\mu_j(V_j) \subset \mathbb{S}^1 \setminus \{-1,1\}$ and $\lambda \notin \mathcal{U}(\mu_j(\lambda)X)$ for any $\lambda \in V_j$ by construction. Every point $\lambda$ in $\mathbb{S}^1 \setminus \cup_{l=0}^{2m_\kappa-1} K_l' = \mathbb{S}^1 \setminus \mathcal{U}(X)$ is contained exactly in two open sets $V_k$ and $V_{k+1}$ (among all the sets of the form $V_l$) for some $k$ in

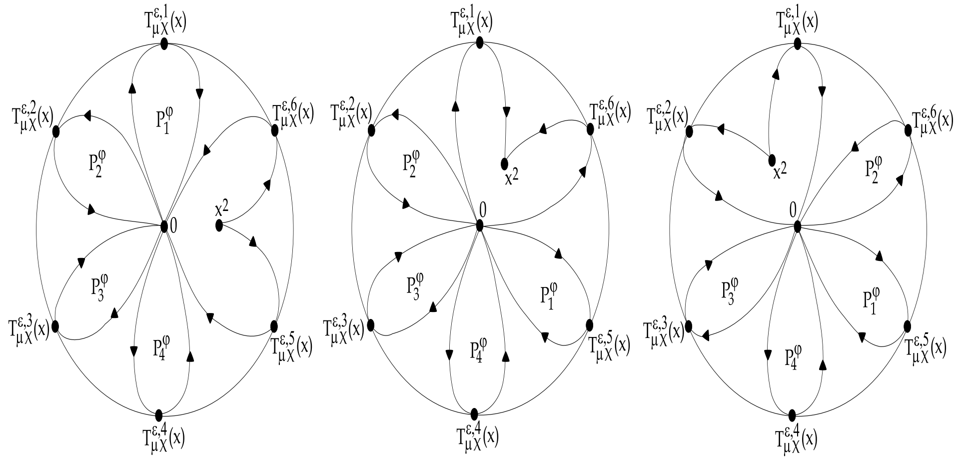


FIGURE 12. $\Re(iX)_{|\{\delta_0\lambda\}\times\overline{B(0,\epsilon)}}$ for $X=-iy^3(y-x^2)\partial/\partial y$ and $\lambda\in\{1,e^{i\pi/6},e^{i\pi/3}\}$

$\mathbb{Z}/(2m_\kappa\mathbb{Z})$. Since $\mathcal{U}^{\lambda}_{\kappa}(X)$ contains exactly one point in $e^{i(0,\pi)}$ then $\{\xi^j_{\varphi,i,K_k}\}_{j\in D(\varphi)}$ and $\{\xi^j_{\varphi,i,K_{k+1}}\}_{j\in D(\varphi)}$ are the two systems of changes of charts associated to the direction $x\in\lambda\mathbb{R}^+$. A point $\lambda_0 e^{i\pi k/m_\kappa}$ belongs to $V_k$ and does not belong to $V_l$ for $l\in\mathbb{Z}/(2m_\kappa\mathbb{Z})\setminus\{k\}$. The set $\mathcal{U}^{\lambda_0 e^{i\pi k/m_\kappa}}_{\kappa}(X)$ does not contain points $e^{i(0,\pi)}$. Indeed $\{\xi^j_{\varphi,i,K_k}\}_{j\in D(\varphi)}$ is the unique system of changes of charts associated to the direction $x\in\lambda_0 e^{i\pi k/m_\kappa}\mathbb{R}^+$.

Let us focus on the example $X=-iy^3(y-x^2)\partial/\partial y$ with $\gamma=\{y=0\}$. We have

$$\{(\lambda,\mu)\in\mathbb{S}^1\times\mathbb{S}^1:\lambda\in\mathcal{U}(\mu X)\}=\{(\lambda,\mu)\in\mathbb{S}^1\times\mathbb{S}^1:\lambda^6\mu\in\{-1,1\}\}.$$

We define $\lambda_0=1$. Consider the notations in the picture (12). Given $k\in\{3,7,11\}$ there exists a function $\xi^1_{\varphi,\gamma,k}$ defined in the union of the domains of definition of $\xi^1_{\varphi,\gamma,K_{k-1}}$, $\xi^1_{\varphi,\gamma,K_k}$ and $\xi^1_{\varphi,\gamma,K_{k+1}}$. We obtain

- $(\xi^1_{\varphi,\gamma,11})_{|\{x\in[0,\delta)K_k\}}=\xi^1_{\varphi,\gamma,K_k}\equiv\xi^1_{\varphi,i,K_k}$ for $k\in\{10,11,0\}$.
- $(\xi^1_{\varphi,\gamma,3})_{|\{x\in[0,\delta)K_k\}}=\xi^1_{\varphi,\gamma,K_k}\equiv\xi^5_{\varphi,i,K_k}$ for $k\in\{2,3,4\}$.
- $(\xi^1_{\varphi,\gamma,7})_{|\{x\in[0,\delta)K_k\}}=\xi^1_{\varphi,\gamma,K_k}\equiv\xi^3_{\varphi,i,K_k}$ for $k\in\{6,7,8\}$.

The union of the domains of definition of $\xi^1_{\varphi,\gamma,11}$, $\xi^1_{\varphi,\gamma,3}$ and $\xi^1_{\varphi,\gamma,7}$ is of the form

$$\{x\in[0,\delta)\,([K_2\cup K_3\cup K_4]\cup[K_6\cup K_7\cup K_8]\cup[K_{10}\cup K_{11}\cup K_0])\}\times\{\mathrm{Im} z<-I\}$$

for some $I\in\mathbb{R}^+$. Analogously given $k\in\{0,4,8\}$ there exists a function $\xi^2_{\varphi,\gamma,k}$ defined in the union of the domains of definition of $\xi^1_{\varphi,\gamma,K_{k-1}}$, $\xi^1_{\varphi,\gamma,K_k}$ and $\xi^1_{\varphi,\gamma,K_{k+1}}$. We have

- $(\xi^2_{\varphi,\gamma,0})_{|\{x\in[0,\delta)K_k\}}=\xi^2_{\varphi,\gamma,K_k}\equiv\xi^2_{\varphi,i,K_k}$ for $k\in\{11,0,1\}$.
- $(\xi^2_{\varphi,\gamma,4})_{|\{x\in[0,\delta)K_k\}}=\xi^2_{\varphi,\gamma,K_k}\equiv\xi^6_{\varphi,i,K_k}$ for $k\in\{3,4,5\}$.
- $(\xi^2_{\varphi,\gamma,8})_{|\{x\in[0,\delta)K_k\}}=\xi^2_{\varphi,\gamma,K_k}\equiv\xi^4_{\varphi,i,K_k}$ for $k\in\{7,8,9\}$.

The union of the domains of definition of $\xi^1_{\varphi,\gamma,0}$, $\xi^1_{\varphi,\gamma,4}$ and $\xi^1_{\varphi,\gamma,8}$ is of the form

$$\{x \in [0,\delta)\,([K_3 \cup K_4 \cup K_5] \cup [K_7 \cup K_8 \cup K_9] \cup [K_{11} \cup K_0 \cup K_1])\} \times \{\mathrm{Im} z > I\}$$

for some $I \in \mathbb{R}^+$. Let us remark that $\cup_{k=0}^{11} K_k = \mathbb{S}^1$. Moreover, in every sector $\{x \in [0,\delta)K_k\}$ either the change of charts between the Fatou coordinates of $P_1^{\varphi}$ and $P_2^{\varphi}$ or the change of charts between the Fatou coordinates of $P_2^{\varphi}$ and $P_3^{\varphi}$ admits a continuous extension to $x = 0$. This kind of argument fact is key to prove theorem 6.1.

**6.3. Proof of theorem 6.1.** — The first part of the proof is a technical lemma.

***Lemma 6.1.*** — *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ be non-degenerate. Suppose that there exists a parabolic irreducible component $\gamma$ of $\mathrm{Fix}(\varphi)$. Then $\log\varphi$ belongs to $\mathcal{X}(\mathbb{C}^2,0)$ if and only if $\log\varphi_Q$ is analytic for any $Q \in \gamma$.*

*Proof.* — The implication $\Rightarrow$ is trivial. Let us prove the implication $\Leftarrow$. We can suppose that $\gamma = \{y = 0\}$ without lack of generality. The infinitesimal generator $\log\varphi$ is of the form $\hat{f}(x,y)\partial/\partial y$ for some $\hat{f} \in \mathbb{C}[[x,y]]$. By hypothesis the function $\hat{f}$ is analytic in a neighborhood of $\gamma \setminus \{(0,0)\}$. This implies $\hat{f} \in \mathbb{C}\{x,y\}$ by the modulus maximum principle (see proof of theorem 3.2). □

Consider the compact sets $K_k$ for $k \in \mathbb{Z}/(2m_\kappa\mathbb{Z})$ defined in subsection 6.2. We can define (see equation 6)

$$E_k^{\varphi} = \{l \in \mathbb{N} : \exists j \in \mathbb{Z}/(2\nu_\varphi(\gamma)\mathbb{Z}) : a^{\varphi}_{j,l,\gamma,K_k} \not\equiv 0\}.$$

We claim that $E_k^{\varphi}$ does not depend on $k$. We have

$$\xi^j_{\varphi,\gamma,K_{k+1}}(x, z + c_{j,k}(x)) \equiv \xi^j_{\varphi,\gamma,K_k}(x,z) + d_{j,k}(x)$$

for all $j \in \mathbb{Z}/(2\nu_\varphi(\gamma)\mathbb{Z})$ and $k \in \mathbb{Z}/(2m_\kappa\mathbb{Z})$ and some functions $c_{j,k}$ and $d_{j,k}$ defined in $(0,\delta)(K_k \cap K_{k+1})$. Thus we obtain $E_k^{\varphi} = E_{k+1}^{\varphi}$ for any $k \in \mathbb{Z}/(2m_\kappa\mathbb{Z})$. The highly non-triviality of $\varphi_0$ implies $\log\varphi \notin \mathcal{X}(\mathbb{C}^2,0)$. The sets $E_k^{\varphi}$ are not empty. We define $g(\varphi) = \gcd(E_1^{\varphi}) = \ldots = \gcd(E_{2m_\kappa}^{\varphi})$.

Since $\varphi_Q \overset{an_1}{\sim} \eta_Q$ for $Q \in \gamma \setminus \{(0,0)\}$ there exists a function $c' : B(0,\delta) \setminus \{0\} \to \mathbb{C}$ such that

$$\xi^j_{\eta,\gamma,K_k}(x, z + c'(x)) = \xi^j_{\varphi,\gamma,K_k}(x,z) + c'(x) \tag{7}$$

for all $j \in \mathbb{Z}/(2\nu_\varphi(\gamma)\mathbb{Z})$ and $k \in \mathbb{Z}/(2m_\kappa\mathbb{Z})$. A priori the function $c'$ is not even continuous.

Fix $k \in \mathbb{Z}/(2m_\kappa\mathbb{Z})$. Each $j \in \mathbb{Z}/(2\nu_\varphi(\gamma)\mathbb{Z})$ except one satisfies that $d(K_k, j) = 1$. Choose $j_0 \in \mathbb{Z}/(2\nu_\varphi(\gamma)\mathbb{Z})$ such that $d(K_k, j_0) = 1$. Since $\varphi_0$ is highly non-trivial then we obtain $\xi^{\upsilon(K_k,j_0)}_{\varphi,i,K_k}(0,z) \not\equiv z + \zeta_X(0)$. We deduce the existence of $l \in \mathbb{N}$ such that $a^{\varphi}_{j_0,l,\gamma,K_k}(0) \neq 0$. Denote by $s$ the point of $\{-1,1\}$ such that $\upsilon(K_k,j_0) \in D_s(\varphi)$. By equation (7) we obtain

$$e^{-2\pi i s l c'(x)} = \frac{a^{\eta}_{j_0,l,\gamma,K_k}}{a^{\varphi}_{j_0,l,\gamma,K_k}}(x)$$

for any $x \in (0,\delta)K_k$. As a consequence the function $e^{-2\pi i s l c'(x)}$ is continuous in $[0,\delta)K_k$, it is holomorphic in $(0,\delta)K_k^{\circ}$ and $e^{-2\pi i s l c'(x)}((0,\delta)K_k) \subset \mathbb{C}^*$. By definition of $g(\varphi)$ there exists $l_1, \ldots, l_q$ in $E_k^{\varphi}$ and integer numbers $p_1, \ldots, p_q$ such that $g(\varphi) = p_1 l_1 + \ldots p_q l_q$. Choose $j_e$ such that $a^{\varphi}_{j_e,l_e,\gamma,K_k} \not\equiv 0$ for $e \in \{1,\ldots,q\}$. Denote by $s_e$ the element of $\{-1,1\}$ such that $v(K_k,j_e) \in D_{s_e}(\varphi)$. We obtain

$$e^{-2\pi i s g(\varphi)c'(x)} = (e^{-2\pi i s_1 l_1 c'(x)})^{s_1 s p_1} \ldots (e^{-2\pi i s_q l_q c'(x)})^{s_q s p_q}$$

and then $e^{-2\pi i s g(\varphi)c'(x)}$ is well-defined in $(0,\delta)K_k$ since we have

$$e^{-2\pi i s g(\varphi)c'(x)} = \left(\frac{a^{\eta}_{j_1,l_1,\gamma,K_k}}{a^{\varphi}_{j_1,l_1,\gamma,K_k}}(x)\right)^{s_1 s p_1} \ldots \left(\frac{a^{\eta}_{j_q,l_q,\gamma,K_k}}{a^{\varphi}_{j_q,l_q,\gamma,K_k}}(x)\right)^{s_q s p_q}.$$

The equality $(e^{-2\pi i s g(\varphi)c'(x)})^l = (e^{-2\pi i s l c'(x)})^{g(\varphi)}$ implies that $e^{-2\pi i s g(\varphi)c'(x)}$ is a complex-valued function that is continuous in $[0,\delta)K_k$ and holomorphic in $(0,\delta)K_k^{\circ}$. Moreover, it satisfies $e^{-2\pi i s g(\varphi)c'(x)}((0,\delta)K_k) \subset \mathbb{C}^*$.

The previous argument can be repeated for every $K_k$ with $k \in \mathbb{Z}/(2m_\kappa\mathbb{Z})$. As a consequence $e^{2\pi i g(\varphi)c'(x)}$ is a meromorphic function in a neighborhood of 0 such that $e^{-2\pi i s g(\varphi)c'(0)} \in \mathbb{C}$. Since we can choose $j_0' \in \mathbb{Z}/(2\nu_\varphi(\gamma)\mathbb{Z})$ such that $d(K_{k+1}, j_0') = 1$ and $v(K_{k+1}, j_0') \in D_{-s}(\varphi)$ then we obtain also $e^{2\pi i s g(\varphi)c'(0)} \in \mathbb{C}$. As a consequence $e^{2\pi i g(\varphi)c'(x)}$ is a unit of $\mathbb{C}\{x,y\}$ and there exists a function $c \in \mathbb{C}\{x\}$ such that

$$e^{2\pi i g(\varphi)c(x)} \equiv e^{2\pi i g(\varphi)c'(x)}.$$

The function $c$ is the function we are looking for. □

## References


[1] P. Ahern & J.-P. Rosay – "Entire functions, in the classification of differentiable germs tangent to the identity, in one or two variables", *Trans. Amer. Math. Soc.* **347** (1995), p. 543–572.

[2] A. Douady, F. Estrada & P. Sentenac – "Champs de vecteurs polynômiaux sur $\mathbb{C}$", *To appear in the Proceedings of Boldifest.*

[3] P. Lavaurs – "Systèmes dynamiques holomorphes: explosion de points périodiques paraboliques", Ph.D. Thesis, Université de Paris-Sud, 1989.

[4] O. Lehto & K. I. Virtanen – *Quasiconformal mappings in the plane*, second ed., Die Grund. Math. Wiss., vol. 126, Springer, 1973.

[5] P. Mardešić, R. Roussarie & C. Rousseau – "Modulus of analytic classification for unfoldings of generic parabolic diffeomorphisms", *Mosc. Math. J.* **4** (2004), p. 455–502, 535.

[6] J. Martinet & J.-P. Ramis – "Problèmes de modules pour des équations différentielles non linéaires du premier ordre", *Publ. Math. I.H.É.S.* **55** (1982), p. 63–164.

[7] R. Oudkerk – "The parabolic implosion for $f_0(z) = z + z^{\nu+1} + O(z^{\nu+2})$", Ph.D. Thesis, University of Warwick, 1999.

[8] J. Rey – "Difféomorphismes résonnants de $(\mathbb{C}^1, 0)$", Ph.D. Thesis, Université Paul Sabatier, 1996.

[9] J. Ribón – "Germes de difféomorphisme de $(\mathbb{C}^2, 0)$ tangents à l'identité qui préservent la fibration de hopf", Ph.D. Thesis, Universidad de Valladolid, 2001.

[10] ———, "Topological classification of families of diffeomorphisms without small divisors", *To appear in Memoirs of the AMS* .

[11] J. Ribón – "Formal classification of unfoldings of parabolic diffeomorphisms", *Ergodic Theory Dynam. Systems* **28** (2008), p. 1323–1365.

[12] ———, "Modulus of analytic classification for unfoldings of resonant diffeomorphisms", *Mosc. Math. J.* **8** (2008).

[13] M. Shishikura – "Bifurcation of parabolic fixed points", in *The Mandelbrot set, theme and variations*, London Math. Soc. Lecture Note Ser., vol. 274, Cambridge Univ. Press, 2000, p. 325–363.

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# SÉRIES DE POINCARÉ MOTIVIQUES D'UN GERME D'HYPERSURFACE IRRÉDUCTIBLE QUASI-ORDINAIRE

*par*

Guillaume Rond

***Résumé.*** — Nous donnons ici une description combinatoire, faisant intervenir les exposants caractéristiques de la singularité, des arcs tronqués tracés sur un germe d'hypersurface quasi-ordinaire. Cela nous permet d'obtenir une expression inductive des séries de Poincaré de ce type de singularité.

***Abstract* (Motivic Poincaré series of a quasi-ordinary irreducible germ of hypersurface)**
We give here a combinatorial description, using caracteristic exponents of the singularity, of the troncated arcs on a quasi-ordinary hypersurface germ. This allows us to give an inductive expression of the Poincaré series of this kind of singularity.

## 1. Séries de Poincaré motiviques

Soit $(X, 0)$ un germe d'espace analytique sur un corps $\mathbb{k}$ de caractéristique nulle. Soit $p$ un entier naturel. Nous définissons l'espace des jets d'ordre $p$, noté $X_p$, comme étant la variété algébrique sur $\mathbb{k}$ dont les points $\mathbb{K}$-rationnels, pour toute extension de corps $\mathbb{K}$ de $\mathbb{k}$, sont les $\mathbb{K}[t]/t^{p+1}$-points de $(X, 0)$. C'est-à-dire que nous avons $X_p = \{\varphi : \operatorname{Spec} \mathbb{k}[[t]]/t^{p+1} \longrightarrow (X, 0)\}$. Dans le cas particulier où $(X, 0)$ est un germe d'espace analytique défini par les équations $f_i(x) = 0$, pour $i = 1, \ldots, r$ et $x = (x_1, \ldots, x_s)$, alors $X_p$ est la variété affine définie par les équations en les variables $x_{j,k}$ pour $k = 1, \ldots, p$ et $j = 1, \ldots, s$, provenant du fait que $f_i(x_{j,1}t + \cdots + x_{j,p}t^p) = 0$ mod $t^{p+1}$ pour tout $i$.

La limite projective de ces variétés, appelée espace des arcs sur $X$, est notée $X_\infty$ et n'est en général pas de type fini sur $\mathbb{k}$.

Nous avons les morphismes naturels de troncations

$$\pi_p \ : \ X_\infty \longrightarrow X_p \ \text{ et } \ \pi_{p,q} \ : \ X_p \longrightarrow X_q \text{ pour } p \geq q \ .$$

Nous nous intéressons ici au comportement des arcs tronqués, c'est-à-dire aux $\pi_p(X_\infty)$ quand $p$ varie. Nous savons, d'après le théorème de Greenberg [**13**], que ce sont des

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ensembles constructibles. On peut donc considérer leur image dans l'anneau de Grothendieck $K_0(Var_{\Bbbk})$ des variétés sur $\Bbbk$ [**6**]. Plus précisément nous nous intéressons à la série de Poincaré géométrique $P_{\text{géom},X,0}(T) := \sum_{p\geq 0}[\pi_p(X_\infty)]T^p$ où $[Y]$ représente la classe de la variété $Y$ dans $K_0(Var_{\Bbbk})$. Denef et Loeser ont montré que cette série est rationnelle avec un dénominateur qui s'écrit sous forme d'un produit de termes de la forme $1-\mathbb{L}^aT^b$ où $\mathbb{L} := [\mathbb{A}^1_{\Bbbk}]$ et $a\in\mathbb{Z}$ et $b\in\mathbb{N}\backslash\{0\}$ (cf. [**6**]). Cependant la preuve utilise à la fois la résolution des singularités du germe singulier et un théorème d'élimination des quantificateurs dû à Pas [**23**], et n'apporte aucune information quantitative, en particulier sur les pôles. Cette série a, jusqu'à présent, été calculée pour les branches planes (cf. [**7**]) et les singularités de surfaces toriques normales (cf. [**17**] et [**22**]).

Par ailleurs, on peut aussi considérer $\varphi_p$, la formule dans le langage de premier ordre de $\Bbbk[[T]]$, dû à Pas [**23**], qui définit $\pi_p(X_\infty)$, qui est un ensemble constructible, et regarder sa mesure arithmétique $\chi_c(\varphi_p)$ dans l'anneau de Grothendieck $K_0^v(Mot_{\Bbbk,\overline{\mathbb{Q}}})_{\mathbb{Q}}$, c'est-à-dire l'anneau de Grothendieck des motifs de Chow sur $\Bbbk$ à coefficients dans $\overline{\mathbb{Q}}$ tensorisé avec $\mathbb{Q}$ (cf. [**7**] ou [**14**] pour une introduction). Une autre série intéressante est alors la série définie par $\sum_{p\geq 0}\chi_c(\varphi_p)T^p$. Cette série se spécialise pour tout $q$ premier, sauf un nombre fini, en la série $\sum_{p\geq 0}N_{q^p}(X,0)T^p$, où $N_{q^p}(X,0)$ est le cardinal des $\mathbb{Z}/q^p\mathbb{Z}$-points de $(X,0)$ qui se relèvent en des $\mathbb{Z}_q$-points de $(X,\,0)$. Denef et Loeser ont montré le même résultat de rationnalité pour cette série que pour la série géométrique (cf. [**7**]). Cette série a été calculée pour les branches planes (cf. [**7**]) et les singularités de surfaces toriques normales (cf. [**22**]). Pour les branches planes ces deux séries diffèrent et pour les surfaces normales toriques, J. Nicaise montre l'égalité.

Nous avons effectué ici le calcul des séries géométrique et arithmétique d'un germe d'hypersurface irréductible quasi-ordinaire. Ce type de singularité généralise les singularités de courbes planes dans le sens où il existe un paramétrage de ces singularités à l'aide de séries fractionnaires à plusieurs variables dont le dénominateur est borné. L'ensemble des exposants, apparaissant dans l'écriture des ces séries, appartient au groupe engendré par un nombre fini d'exposants, appelés *exposants caractéristiques*, qui généralisent les exposants caractéristiques d'une courbe plane.

Pour calculer la mesure motivique de l'ensemble des arcs tronqués à l'ordre $p$, nous décomposons cet ensemble en deux ensembles constructibles : l'ensemble des arcs tronqués qui ne se relèvent pas en arcs inclus dans le complémentaire du tore et son complémentaire. Nous donnons d'abord une caractérisation combinatoire des arcs tronqués qui ne se relèvent pas en arcs inclus dans le complémentaire du tore. Cela nous permet de calculer la mesure motivique de l'ensemble de ces arcs tronqués. Enfin nous donnons une formule de récurrence sur la dimension du germe pour la mesure de son complémentaire. Nous obtenons alors des formules générales, inductives sur la dimension de l'hypersurface, de ces deux séries. Malheureusement ces formules font intervenir des sommes géométriques sur des cônes rationnels assez difficiles à calculer en général. Nous donnons enfin une formule explicite de ces séries dans le cas où les coordonnées des exposants caractéristiques de la singularité sont supérieures à 1, c'est-à-dire quand la projection de celle-ci est « très » transverse (théorème 9.3).

Nous faisons remarquer que ces séries sont différentes des séries d'Igusa étudiées dans [**2**], séries qui sont essentiellement les séries génératrices des espaces de jets tracés sur un germe singulier. Nous pouvons par ailleurs citer le travail en cours [**3**] dû à Helena Cobo Pablos et Pedro Gonzalez-Perez où le calcul des séries de Poincaré motiviques pour les germes de surfaces irréductibles à singularité quasi-ordinaire est fait à l'aide de méthodes de géométrie torique qui s'inspirent du travail effectué dans [**17**].

Je tiens à remercier ici M. Lejeune-Jalabert pour avoir fait preuve de patience à l'écoute de ces résultats et pour ses précieux commentaires. Je remercie aussi J. Nicaise pour m'avoir, le premier, parlé des séries de Poincaré motivique lors du GAEL XII, et Helena Cobo Pablos et Pedro Gonzalez-Perez pour m'avoir indiqué une erreur dans la première version de ce travail.

## 2. Singularités quasi-ordinaires

**2.1. Exposants caractéristiques.** — Nous rappelons ici la définition de singularité quasi-ordinaire et les propriétés de ces singularités dont nous aurons besoin.

***Définition 2.1*** **(cf. [11] par exemple)**. — Soit $f \in \mathbb{C}\{X_1, \dots, X_m\}[Y]$ un polynôme distingué. On dit que $f$ est quasi-ordinaire si son discriminant $\Delta_Y(f)$ a un terme dominant, c'est-à-dire si il s'écrit $X^\alpha u$ où $\alpha \in \mathbb{Z}^m_{\geq 0}$ et $u$ est inversible. Géométriquement, cela revient à dire que le discriminant de la projection du germe $(X, 0) \subset \mathbb{C}^{m+1} \longrightarrow \mathbb{C}^m$ qui envoie le point de coordonnées $(x_1, \dots, x_m, y)$ sur le point de coordonnées $(x_1, \dots, x_m)$ est à croisements normaux.

Nous avons alors le théorème

***Théorème 2.2*** **([10], [20])**. — *Soit $f$ irréductible et quasi-ordinaire. Alors nous avons :*

1. *Si $\deg_Y(f) = n$ alors $f$ a $n$ racines distinctes dans $\mathbb{C}\{X^{\frac{1}{n}}\}$.*
2. *Si $\xi$ est une racine de $f$ dans $\mathbb{C}\{X^{\frac{1}{n}}\}$, alors il existe des élements de $(\frac{1}{n}\mathbb{Z})^m$, strictement ordonnés, $a(1) < \cdots < a(g)$ (i.e. $a_k(1) \leq \cdots \leq a_k(g)$ pour tout $k$ et $a(i) \neq a(j)$ pour $i \neq j$) tels que l'on puisse écrire*

$$\xi = \xi_0 + \xi_1 + \cdots + \xi_g$$

$$\textit{avec} \;\; \xi_0 \in \mathbb{C}\{X\},$$

$$X^c \textit{ apparaît dans } \xi \Longrightarrow c \in \mathbb{Z}^m + \sum_{a(i) \leq c} a(i)\mathbb{Z},$$

$$X^c \textit{ apparaît dans } \xi_i \Longrightarrow c \in \mathbb{Z}^m + \sum_{j \leq i} a(j)\mathbb{Z},$$

$$\textit{et} \;\; \nu_X(\xi_k) = a(k) \;\; \textit{pour tout } k.$$

3. *Si $\xi$ est racine de $f$ dans $\mathbb{C}\{X^{\frac{1}{n}}\}$, alors l'ensemble des racines de $f$ est l'ensemble formé des $\xi(w_1 X_1^{\frac{1}{n}}, \dots, w_m X_m^{\frac{1}{n}})$ où les $w_k$ parcourent l'ensemble des racines $n$-ièmes de l'unité.*

***Remarque 2.3***. — Quitte à faire le changement de variables

$$X_k \longmapsto X_k \ \forall k, \quad \text{et} \quad Y \longmapsto Y + \xi_0$$

nous pouvons supposer que $\xi_0 = 0$ dans le théorème précédent.

***Définition 2.4***. — Les $a(k)$ du lemme précédent sont appelés les exposants caractéristiques de $f$.

Nous pouvons définir les réseaux $M = M_0 := \mathbb{Z}^m$ et $M_k := \mathbb{Z}^m + \sum_{l \leq k} a(l)\mathbb{Z}^m$ pour $1 \leq k \leq g$ et les réseaux duaux $N_i := \check{M}_i$ et $N = N_0$. Nous définissons les entiers caractéristiques $n_k$ du germe d'hypersurface comme étant les indices des $M_{k-1}$ dans $M_k$ :

$$n_k = [M_k : M_{k-1}].$$

Nous posons aussi $n_0 = 1$ et $n_{-1} = 0$. Nous notons

$$e_{k-1} = n_k \dots n_g \text{ pour } k = 1, \dots, g.$$

En particulier nous avons $e_0 = n = n_1 \dots n_g$.

Nous pouvons aussi définir les vecteurs $\gamma(k)$ par :

$$\gamma(1) := a(1)$$

$$\gamma(k+1) := n_k \gamma(k) + a(k+1) - a(k)$$

Dans le cas $m = 1$, ce sont les $n\gamma_i$ sont des générateurs du semi-groupe de l'ensemble des multiplicités d'intersection $(C, X)_0$ où $C$ parcourt l'ensemble des germes en 0 de courbes planes non contenues dans $X$ (cf. [**24**]).

**2.2. Remarques sur l'écriture en coordonnées d'un arcs tracé sur un germe d'hypersurface irréductible à singularité quasi-ordinaire.** — Soit $\varphi(t) := (x_1(t), \dots, x_m(t), y(t))$ un arc tracé sur un germe $(X, 0)$ d'hypersurface irréductible à singularité quasi-ordinaire d'exposants caractéristiques $a(1), \dots, a(g)$ défini par un polynôme de Weierstrass $f \in \mathbb{C}\{X_1, \dots, X_m\}[Y]$. Alors $f(x_1(t), \dots, x_m(t), y(t)) = 0$, donc $y(t) = \xi(x^{1/n}(t))$ où $\xi = \xi_1 + \dots + \xi_g$ est une racine de $f$ (avec les notations du théorème 2.2) et les $x_i^{1/n}(t)$ sont des racines $n$-ièmes de $x_i(t)$ dans $\mathbb{C}[[t^{1/n}]]$. Nous noterons souvent $\xi$ au lieu de $\xi(x^{1/n}(t))$ quand les racines $n$-ièmes de $x_i(t)$ seront fixées.

Nous pouvons alors faire les deux remarques suivantes :

***Remarque 2.5***. — Soit $X^a = X_1^{a_1} \dots X_m^{a_m}$ un monôme de $\mathbb{C}[X_1^{1/n}, \dots, X_m^{1/n}]$. Considérons $m$ séries $x_i(t) = \sum_k x_{i,k} t^k$ de $\mathbb{C}[[t]]$, et notons $l_i = \mathrm{ord}(x_i(t))$. Le choix d'une racine $n$-ième de $x_i(t)$ dans $\mathbb{C}[[t^{1/n}]]$ dépend uniquement du choix d'une racine $n$-ième de $x_{i,l_i}$ dans $\mathbb{C}$. Pour tout $i$ fixons une racine $n$-ième de $x_{i,l_i}$ et notons la $x_{i,l_i}^{1/n}$. Nous noterons alors sans équivoque $x_{i,l_i}^{a_i} = (x_{i,l_i}^{1/n})^{na_i}$. Dans ce cas nous avons

$$x_i(t)^{a_i} = x_{i,l_i}^{a_i} t^{a_i l_i} \left(1 + \sum_{k \geq l_i+1} \frac{x_{i,k}}{x_{i,l_i}} t^{k-l_i}\right)^{a_1} = x_{i,l_i}^{a_i} t^{a_i l_i} \left(1 + \sum_{k \geq 1} \frac{x_{i,k+l_i}}{x_{i,l_i}} t^k\right)^{a_1}$$

$$= x_{i,l_i}^{a_i} t^{a_i l_i} \left(1 + a_i \sum_{k \geq 1} \frac{x_{i,k+l_i}}{x_{i,l_i}} t^k + \cdots + \binom{a_i}{j} \left(\sum_{k \geq 1} \frac{x_{i,k+l_i}}{x_{i,l_i}} t^k\right)^j + \cdots\right)$$

avec $\binom{a_i}{j} := \frac{a_i(a_i-1)\ldots(a_i-j+1)}{j!}$. Nous voyons que $x_1^{a_1}(t)\ldots x_m^{a_m}(t)$ est dans $\mathbb{C}[[t]]$ si et seulement si $\sum_i a_i l_i \in \mathbb{N}$. Si tel est le cas, pour tout $c \in \mathbb{N}$ avec $c > \sum_i a_i l_i$, le coefficient de $t^c$ dans l'expression de la série $x_1^{a_1}(t)\ldots x_m^{a_m}(t)$ est un polynôme de la forme suivante :

$$\prod_{i=1}^{m} x_{i,l_i}^{a_i} P\left(x_{i,k}/x_{i,l_i};\, l_i + 1 \leq k \leq c - \sum_j a_j l_j + l_i \text{ et } 1 \leq i \leq m\right)$$

où $P$ est un polynôme quasi-homogène de poids $c - \sum_j a_j l_j$ et où $x_{i,k}/x_{i,l_i}$ est de poids $k - l_i$.

***Définition 2.6***. — Notons $b_k : \mathbb{Z}^m \longrightarrow \mathbb{Q}$ la forme linéaire

$$b_k(\underline{l}) := \sum_{i=1}^{m} a_i(k) l_i, \ \forall k \in \{0, \ldots, g\}$$

et $M$ l'application linéaire

$$\begin{array}{rcl} M \ : \ \mathbb{Z}^m & \longrightarrow & (\mathbb{Z}/n\mathbb{Z})^g \\ (l_1, \ldots, l_m) & \longmapsto & (n \sum a_i(1) l_i, \ldots, n \sum a_i(g) l_i) \end{array}$$

***Remarque 2.7***. — Si $l_i$ est l'ordre de $x_i(t)$, alors d'après le théorème 2.2, nous voyons que nécessairement $b_1(\underline{l}) \in \mathbb{N}$ et donc $\xi_1(x^{1/n}(t)) \in \mathbb{C}[[t]]$. En retranchant $\xi_1(x^{1/n}(t))$ à $y(t)$, nous voyons alors que $b_2(\underline{l}) \in \mathbb{N}$ et donc que $\xi_2(x^{1/n}(t)) \in \mathbb{C}[[t]]$. Par induction nous voyons que

$$(\mathrm{ord}(x_1(t)), \ldots, \mathrm{ord}(x_m(t))) \in \mathrm{Ker}\, M.$$

Inversement, si l'on se fixe $m$ séries formelles en $t$, notées $x_i(t)$ pour $1 \leq i \leq m$, qui vérifient $(\mathrm{ord}(x_1(t)), \ldots, \mathrm{ord}(x_m(t))) \in \mathrm{Ker}\, M \cap (\mathbb{N}^*)^m$, alors pour toute solution $\xi$ de $f$, nous avons $\xi(x^{1/n}(t)) \in \mathbb{C}[[t]]$, et $(x_1(t), \ldots, x_m(t), \xi(x^{1/n}(t)))$ définit un arc tracé sur $(X, 0)$.

**2.3. Squelette de la singularité.** — Nous allons maintenant relier $\operatorname{Ker} M$ aux réseaux apparaissant dans une résolution plongée torique de $(X, 0)$. Nous rappelons ici la construction de la résolution plongée torique construite par P. Gonzalez Perez [**12**]. Dans la suite, nous noterons $Z_\Sigma$ la variété torique d'éventail $\Sigma$.

Soit $R = \mathbb{C}\{X_1, \dots, X_m\}[Y]/(f)$ l'anneau des fonctions de $(X, 0)$. L'anneau $R$ est une $\mathbb{C}\{X_1, \dots, X_m\}$-algèbre de finie. Notons $\Delta$ le cône $\mathbb{R}_{\geq 0}^{m+d} \subset (N_\Delta)_{\mathbb{R}}$ où $N_\Delta$ est le réseau $N \oplus \mathbb{Z}^g$. Soit $u_1, \dots, u_g$ la base canonique de $\{0\} \oplus \mathbb{Z}^g$. Les éléments de $\check{\Delta} \cap \check{N}_\Delta$ sont de la forme $(v, w)$ où $v \in \mathbb{R}_{\geq 0}^m \cap M$ et $w = w_1 u_1^* + \cdots + w_g u_g^*$ avec $w_i \in \mathbb{Z}_{\geq 0}$. Soient $Z_\Delta$ est la variété torique associée au cône $\Delta$ et $o_\Delta$ l'orbite de dimension 0 de la variété torique $Z_\Delta$. La variété $Z_\Delta$ est l'espace affine de dimension $m + g$ et $o_\Delta$ est exactement l'origine de celui-ci. Nous allons considérer le plongement $(X, 0) \subset (Z_\Delta, o_\Delta)$, défini par le morphisme de $\mathbb{C}\{X_1, \dots, X_m\}$-algèbres :

$$\Psi : \mathbb{C}\{X_1, \dots, X_m\}[U_1, \dots, U_{g-1}] \longrightarrow R$$

qui à $U_j$ associe $q_{j-1}(\xi)$ où $(q_0, q_1, \dots, q_{g-1})$ est un système de racines approchées de $f$ (c'est-à-dire que $q_j$ est le polynôme quasi-ordinaire minimal de $\mathbb{C}\{X_1, \dots, X_m\}[Y]$ associé à la branche $\xi_1 + \cdots + \xi_j$ ; en particulier $q_0 = Y$).

Soit

$$\begin{array}{cccc} \psi & : & M_\Delta & \longrightarrow & M_g \\ & & (v, w) & \longmapsto & v + \sum_{k=1}^g w_k \gamma(k) \end{array}$$

Soit $L \subset (N_\Delta)_{\mathbb{R}}$ l'espace linéaire défini comme étant l'orthogonal de $\operatorname{Ker}(\psi)$ et soit $\Xi$ le cône $\Delta \cap L$.

***Définition 2.8*** **([18])**. — Nous appellerons *squelette* de la singularité $(X, 0)$ le cône $\Xi$.

Nous avons alors le

***Théorème 2.9*** **([12])**. — *Soit $\Sigma$ une division de $\Delta$ contenant $\Xi$. Alors pour toute division régulière $\Sigma'$ de $\Sigma$ contenant tous les cônes réguliers de $\Sigma$, la composition du morphisme torique induit $\varphi : Z_{\Sigma'} \longrightarrow Z_\Sigma$ avec $\pi_\Sigma : Z_\Sigma \longrightarrow Z_\Delta$ est une résolution plongée de $X$.*

Nous pouvons alors donner une description de $\operatorname{Ker} M$. Tout d'abord nous appelerons *vecteur caractéristique* de $h$ (où $h$ est un arc de $(\mathbb{A}_{\mathbb{C}}^m, 0)$ défini par $(x_1(t), \dots, x_m(t))$ le vecteur $(x_1(t), \dots, x_m(t), q_0(\xi), \dots, q_g(\xi))$. Nous appellerons *vecteur des ordres* du vecteur caractéristique $h$ le vecteur de coordonnées $(\operatorname{ord}(x_1(t)), \dots, \operatorname{ord}(x_m(t)), \operatorname{ord}(q_0(\xi)), \dots, \operatorname{ord}(q_g(\xi)))$.

***Proposition 2.10***. — *Le squelette $\Xi$ correspond exactement à l'ensemble des vecteurs des ordres des vecteurs caractéristiques des arcs tracés sur $(X, 0)$. En particulier le sous-réseau $\operatorname{Ker} M \subset N$ est égal à la projection de $\Xi$ sur $N$.*

*Démonstration.* — L'application linéaire $\psi$ est définie par la matrice

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \gamma_1(1) & \cdots & \gamma_1(g) \\ 0 & 1 & 0 & \cdots & 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \gamma_m(1) & \cdots & \gamma_m(g) \end{pmatrix}$$

Le noyau de $\psi$ est donc l'image de l'application linéaire de $N_g$ dans $N_\Delta$ définie par la transposée de cette matrice. Or $N_g = \operatorname{Ker} M$. Donc $\Xi$ est exactement l'ensemble des éléments de la forme $(\underline{l}, \underline{p})$ avec $\underline{l} \in \operatorname{Ker} M$ et $p_k = \sum_i \gamma_i(k) l_i$ pour $1 \leq k \leq g$. Or, pour tout vecteur caractéristique $(x_1(t), \ldots, x_m(t), q_0(\xi), \ldots, q_g(\xi))$, nous avons $\operatorname{ord}(q_k(\xi)) = \sum_i \gamma_i(k) \operatorname{ord}(x_i(t))$. Donc $\Xi$ correspond exactement à l'ensemble des vecteurs caractéristiques. La dernière assertion découle alors de la remarque 2.7. □

## 3. Définitions

Soit $(X(a(1), \ldots, a(g)), 0)$ est un germe d'hypersurface irréductible, mais non nécessairement réduit, défini par $f$ un polynôme de Weierstrass quasi-ordinaire de $\mathbb{C}\{X_1, \ldots, X_m\}[Y]$ de degré $n$ comme précédemment. Nous noterons $h(Z) = h(Z_1, \ldots, Z_m)$ un élément de $\mathbb{C}\{Z_1, \ldots, Z_m\}$ tel que $f(Z_1^n, \ldots, Z_m^n, h(Z)) = 0$. Nous noterons $X(a(1), \ldots, a(g))_\infty$ l'ensemble des arcs tracés sur ce germe et centrés en l'origine, et $\varphi = \varphi(a(1), \ldots, a(g))$ la formule qui définit $X(a(1), \ldots, a(g))_\infty$ dans le langage de Pas (cf. [**23**] et [**7**]). Nous noterons $\pi_p(X(a(1), \ldots, a(g))_\infty)$ l'ensemble des arcs tronqués à l'ordre $p+1$ et $\varphi_p$ la formule dans le langage de Pas qui définit cet ensemble constructible.

Nous allons étudier ici les séries de Poincaré géométrique et arithmétique :

$$P_{\text{géom}, X(a(1), \ldots, a(g)), 0}(T) := \sum_{p \geq 0} [\pi_p(X(a(1), \ldots, a(g))_\infty)] T^p,$$

$$P_{\text{arit}, X(a(1), \ldots, a(g)), 0}(T) := \sum_{p \geq 0} \chi_c(\varphi_p) T^p.$$

L'espace des arcs tracés sur le germe d'hypersurface défini par $f$ et celui des arcs tracés sur le germe d'hypersurface réduit associé sont les mêmes. Les espaces d'arcs tronqués sont donc aussi les mêmes. Nous pouvons donc supposer que le germe considéré est réduit.

Notons

$$\chi_{p, l_1, \ldots, l_m} := \pi_p(\{(x_1(t), \ldots, x_m(t), y(t)) \in X(a(1), \ldots, a(g))_\infty \,/\, \operatorname{ord}(x_i) = l_i\})$$

et $\varphi_{p, l_1, \ldots, l_m}$ la formule qui définit $\chi_{p, l_1, \ldots, l_m}$.

Nous noterons $X(a(1),\ldots, a(g))_{p,cn}$ le sous-ensemble constructible de l'ensemble constructible $\pi_p(X(a(1),\ldots, a(g))_\infty)$ qui correspond à la troncation d'arcs pour lesquels un des $x_i(t)$ est nul et nous noterons $X(a(1),\ldots, a(g))_{p,to}$ le sous-ensemble de $\pi_p(X(a(1),\ldots, a(g))_\infty)$ qui correspond à la troncation d'arcs qui ne peuvent pas se relever en arcs pour lesquels un des $x_i(t)$ est nul. De même nous noterons $\varphi_{p,cn} = \varphi_{p,cn}(a(1),\ldots, a(m))$ et $\varphi_{p,to} = \varphi_{p,to}(a(1),\ldots, a(m))$ les formules qui définissent ces deux ensembles constructibles. L'idée de découper ainsi l'ensemble des arcs tronqués à été introduite par J. Denef et F. Loeser dans **[7]** pour calculer ces deux séries dans le cas des courbes planes.

Nous avons évidemment

$$[\pi_p(X(a(1),\ldots, a(g))_\infty)] = [X(a(1),\ldots, a(g))_{p,cn}] + [X(a(1),\ldots, a(g))_{p,to}]$$

$$\text{et } \chi_c(\varphi_p(a(1),\ldots, a(g))) = \chi_c(\varphi_{p,to}(a(1),\ldots, a(m))) + \chi_c(\varphi_{p,cn}(a(1),\ldots, a(g))) \ .$$

Notons aussi

$$P_{\text{géom}}^{\text{tore}^c}(a(1),\ldots, a(g))(T) := \sum_{p\geq 0}[X(a(1),\ldots, a(g))_{p,cn}]T^p \ ,$$

$$P_{\text{géom}}^{\text{tore}}(a(1),\ldots, a(g))(T) := \sum_{p\geq 0}[X(a(1),\ldots, a(g))_{p,to}]T^p \ ,$$

$$P_{\text{arit}}^{\text{tore}^c}(a(1),\ldots, a(g))(T) := \sum_{p\geq 0}\chi_c(\varphi_{p,cn}(a(1),\ldots, a(m)))T^p \ ,$$

$$\text{et } P_{\text{arit}}^{\text{tore}}(a(1),\ldots, a(g))(T) := \sum_{p\geq 0}\chi_c(\varphi_{p,to}(a(1),\ldots, a(g)))T^p \ .$$

***Remarque 3.1***. — Si pour tout $i$ nous avons $l_i < +\infty$, alors $[\chi_{p,\,l_1,\ldots,\,l_m}] = 0$ si et seulement si $\underline{l} \notin \operatorname{Ker} M$ (et de même $\chi_c(\varphi_{p,\,l_1,\ldots,\,l_m}) = 0$ si et seulement si $\underline{l} \notin \operatorname{Ker} M$).

## 4. Étude des troncations d'arcs ne se relevant pas en arcs pour lesquels un des $x_i(t)$ est nul

Le terme $[X(a(1),\ldots, a(g))_{p,to}]$ correspond aux arcs tronqués qui ne peuvent pas se relever en arcs pour lesquels un des $x_i(t)$ est nul. Si $(x_1(t),\ldots, x_m(t), y(t))$ est un arc dont la $p$-troncation ne peut pas se relever en un arc pour lequel un des $x_i(t)$ est nul, nous notons $l_i = \operatorname{ord}(x_i(t))$ pour tout $i$. Nous donnons tout d'abord la définition suivante :

***Définition 4.1***. — Soit $X$ un germe irréductible à singularité quasi-ordinaire d'exposants caractéristiques $(a(1),\ldots, a(g))$. Pour tout $i$ compris entre 1 et $m$ notons $k_i$ le plus petit entier qui vérifie $a_i(k_i) \neq 0$. Quitte à effectuer une permutation des variables $X_i$ (ce que nous ferons à partir de maintenant), nous pouvons supposer que nous avons

$$1 = k_1 = k_2 = \cdots = k_{i_0} < k_{i_0+1} \leq k_{i_0+2} \leq \cdots \leq k_m \leq g.$$

Nous avons alors la proposition suivante :

***Proposition 4.2***. — *Soit $h(t) := (x_1(t), \ldots, x_m(t), y(t))$ un arc de $(X, 0)$. Alors la $p$-troncation de $h$ ne peut pas se relever en un arc pour lequel un des $x_i(t)$ est nul, si et seulement si l'une des deux conditions suivantes est vérifiée :*

C1) *Soit $l_i \leq p$ pour tout $i$.*

C2) *Soit il existe $i$ tel que $+\infty > l_i \geq p+1$, et*

i) $p - l_j \geq b_{k_i}(\underline{l}) - b_{k_j}(\underline{l})$ *pour tout $j$ tel que $k_j < k_i$,*

ii) $b_{k_i}(\underline{l}) \leq p$.

***Remarque 4.3***. — En particulier, si $k_i \neq k_j$, alors $l_i$ et $l_j$ ne peuvent être supérieurs strictement à $p$ en même temps. En effet, supposons $l_i > p$ et $l_j > p$ avec $i \neq j$ et $k_i > k_j$. Alors nous avons selon la seconde condition : $p - l_j \geq b_{k_i}(\underline{l}) - b_{k_j}(\underline{l}) > 0$, ce qui est contradictoire.

***Définition 4.4***. — Soit $h(t) := (x_1(t), \ldots, x_m(t), y(t))$ un arc de $(X, 0)$. Si il existe $i$ tel que $l_i > p$, nous noterons $r_{\underline{l}}$ l'entier $k_i$.

*Démonstration*. — Montrons tout d'abord qu'un arc tronqué qui ne peut pas se relever en un arc pour lesquel un des $x_i(t)$ est nul vérifie nécessairement l'une des deux conditions. Tout d'abord, si $l_i \geq p+1$ et si $b_{k_i}(\underline{l}) \geq p+1$, alors on peut relever cet arc tronqué en un arc pour lequel $x_i(t) = 0$, en considérant un relevé quelconque $h$ de cet arc tronqué, et en choisissant l'arc $h'$ dont toutes les coordonnées, sauf la $i$-ième, sont égales à celles de $h$, et la $i$-ème est nulle.

Maintenant supposons que $l_i \geq p+1$ et que $p - l_j < b_{k_i}(\underline{l}) - b_{k_j}(\underline{l})$ pour un $j$ tel que $k_j < k_i$. En particulier, d'après la remarque 4.3, pour $j$ tel que $k_j \neq k_i$, $l_j \leq p$ et donc les $x_{j,k}$ sont fixés pour $k \leq p$ car les $x_j(t)$ sont fixés modulo $t^{p+1}$. Considérons alors les coefficients de $t^{b_{k_i}(\underline{l})}$ dans l'écriture de $\xi_{k_j}(x^{1/n})$ et dans l'écriture de $\xi_{k_i}(x^{1/n})$. Dans $\xi_{k_j}(x^{1/n})$, celui-ci est de la forme

$$\prod_{r \neq j} x_{r,l_r}^{a_r(k_j)} . \left( \frac{x_{j, b_{k_i}(\underline{l}) - b_{k_j}(\underline{l}) + l_j}}{x_{j,l_j}} + P\left( \frac{x_{r,k}}{x_{r,l_r}} ; \, k \leq b_{k_i}(\underline{l}) - b_{k_j}(\underline{l}) + l_r \text{ et } 1 \leq r \leq m \right) \right)$$

où $P$ ne dépend pas de $\frac{x_{j, b_{k_i}(\underline{l}) - b_{k_j}(\underline{l}) + l_j}}{x_{j,l_j}}$. Comme $b_{k_i}(\underline{l}) - b_{k_j}(\underline{l}) + l_j > p$, le terme $\frac{x_{j, b_{k_i}(\underline{l}) - b_{k_j}(\underline{l}) + l_j}}{x_{j,l_j}}$ n'apparait pas dans l'écriture de $x_j(t)$ modulo $t^{p+1}$. Il n'apparait pas non plus dans l'expression des coefficients de $t^d$, pour $d < b_{k_i}(\underline{l})$, dans l'écriture de $\xi(x^{1/n})$. D'autre part, le coefficient de $t^{b_{k_i}(\underline{l})}$ dans l'écriture de $\xi_{k_i}(x^{1/n})$ est de la forme :

$$\prod_{r \neq i} x_{r,l_r}^{a_r(k_i)} . x_{i,l_i}^{a_r(k_i)}$$

En particulier, nous pouvons choisir $x_{i,l_i}$ égal à zéro en donnant la bonne valeur au terme $\frac{x_{j, b_{k_i}(\underline{l}) - b_{k_j}(\underline{l}) + l_j}}{x_{j,l_j}}$. Ceci n'affecte alors ni la valeur de $x_j(t)$ modulo $t^{p+1}$, ni la valeur des coefficients de $t^d$, pour $d < b_{k_i}(\underline{l})$, dans l'écriture de $\xi(x^{1/n})$. Nous pouvons continuer ainsi par récurrence croissante sur $c$ en regardant le coefficient de $t^c$ dans l'écriture de $\xi(x^{1/n})$ et annuler le terme $x_{i,c-\sum_{r\neq i} a_r(k_i)l_r}$ en modifiant éventuellement le terme $\frac{x_{j,c-b_{k_j}(\underline{l})+l_j}}{x_{j,l_j}}$.

Montrons maintenant la suffisance de ces deux conditions. La première condition implique clairement que l'arc tronqué ne peut pas se relever en un arc dont l'une des coordonnées est nulle. Montrons que la seconde est aussi suffisante.

Soit $h$ un arc qui vérifie la seconde condition. Tout d'abord, comme $l_j \leq p$ pour $j$ tel que $k_j \neq k_i$, les coordonnées $x_j(t)$ d'un relèvement d'une $p$-troncation de $h$ sont non nulles. Plus particulièrement, pour $j$ tel que $k_j \neq k_i$, $x_{j,k}$ est fixé pour $l_j \leq k \leq p$. Le coefficient de $t^{b_1(\underline{l})}$ fixe $\prod_j x_{j,l_j}^{a_j(1)}$. Par récurrence croissante sur les coefficients de $t^c$, pour $c < b_{k_i}(\underline{l})$, $\prod_j x_{j,l_j}^{a_j(r)}$ est fixé pour $r < k_i$ car seuls les $x_{j,k}$ pour $j \neq i$ et $k \leq c - b_{k_j}(\underline{l}) + l_j < p$ apparaissent dans son expression. Considérons alors le coefficient de $t^{b_{k_i}(\underline{l})}$ dans l'expression de la coordonnée $y(t)$ de $h$. Nous avons $y(t) = \xi_1(x^{1/n}(t)) + \cdots + \xi_{k_i}(x^{1/n}(t)) + \cdots + \xi_g(x^{1/n}(t))$ pour un choix d'une racine $n$-ième des $x_j(t)$. Le coefficient de $t^{b_{k_i}(\underline{l})}$ dans l'expression de $\xi_r(x^{1/n}(t))$ est de la forme (pour $r < k_i$) :

$$\prod_{j\neq i} x_{j,l_j}^{a_j(r)} P_r(x_{j,k}/x_{j,l_j}, k \leq b_{k_i}(\underline{l}) - b_{k_r}(\underline{l}) + l_j)$$

d'après la remarque 2.5. Or nous avons $p - l_j \geq b_{k_i}(\underline{l}) - b_{k_j}(\underline{l}) \geq b_r(\underline{l}) - b_{k_j}(\underline{l})$, et donc $P_r(x_{j,k}/x_{j,l_j}, k \leq b_{k_i}(\underline{l}) - b_{k_r}(\underline{l}) + l_j)$ est fixé du fait que les $x_j(t)$ sont fixés modulo $t^{p+1}$. D'après la récurrence, le coefficient considéré est donc fixé. Le coefficient de $t^{b_{k_i}(\underline{l})}$ dans l'expression de $\xi_{k_i}(x^{1/n}(t))$ est de la forme :

$$\prod_{j\neq i} x_{j,l_j}^{a_j(k_i)} . x_{i,l_i}^{a_i(k_i)}$$

Donc le coefficient de $t^{b_{k_i}(\underline{l})}$ dans l'expression de $y(t)$ est de la forme $a x_{i,l_i}^{a_i(k_i)} + b$ où $b$ est fixé et $a$ est non nul par hypothèse sur $h$. Comme $x_{i,l_i} \neq 0$ par hypothèse sur $h$, $b \neq 0$ et nécessairement tout relevé de la $p$-troncation de $h$ a sa $i$-ième coordonnée non nulle. Plus précisément, nous pouvons remarquer que nécessairement $b_{k_i}(\underline{l}) = \sum_j a_j(k_i) l_j$ est fixé. $\square$

Nous avons donc

$$X(a(1), \ldots, a(g))_{p,to} = \bigcup_{(l_1,\ldots,l_m)\in D(m)_p} \chi_{p,l_1,\ldots,l_m} \tag{1}$$

où

$$\Sigma_{m,p} := \{(l_1, \ldots, l_m) \,/\, l_i \leq p\} \bigcup$$
$$\left( \bigcup_{i=1}^{m} \{(l_1, \ldots, l_m) \,/\, l_i > p,\, p - l_j \geq b_{k_i}(\underline{l}) - b_{k_j}(\underline{l}),\, k_j < k_i,\, b_{k_i}(\underline{l}) \leq p\} \right)$$
$$\text{et } D(m)_p := \Sigma_{m,p} \cap \operatorname{Ker} M \cap (\mathbb{N}^*)^m.$$

Il nous suffit donc de calculer $\chi_{p,l_1,\ldots,l_m}$ et $\varphi_{p,l_1,\ldots,l_m}$, et d'étudier l'union (1).

## 5. Calcul de $\chi_{p,\,l_1,\ldots,\,l_m}$ et $\varphi_{p,\,l_1,\ldots,\,l_m}$

Pour calculer $\chi_{p,\,l_1,\ldots,\,l_m}$ et $\varphi_{p,\,l_1,\ldots,\,l_m}$, nous allons voir que $\chi_{p,\,l_1,\ldots,\,l_m}$ peut s'écrire sous la forme $\overline{\chi}_{p,\,l_1,\ldots,\,l_m} \times \mathbb{A}_{\mathbb{C}}^{n(p,\underline{l})}$, où $n(p,\underline{l})$ est un entier, et nous allons trouver un revêtement galoisien $W \longrightarrow \overline{\chi}_{p,\,l_1,\ldots,\,l_m}$ surjectif où $[W]$ est facile à calculer et l'action du groupe de Galois du revêtement est facile à décrire.

Pour cela, notons $x_j = \sum_{r\geq l_j} x_{j,r}t^r$ et $z_j = z_{j,0}t^{\frac{l_j}{n}}(1+\sum_{r\geq 1} z_{j,r}t^r)$. Les termes $z_{j,r}$ pour $r > p - l_j$ n'apparaissent pas dans l'expression de $z_j^n$ modulo $t^{p+1}$ mais $z_{j,p-l_j}$ y apparait. De même les termes $z_{j,r}$ pour $r > p - b_{k_j}(\underline{l})$ n'apparaissent pas dans l'expression de $\xi(z)$ modulo $t^{p+1}$ si $p \geq b_{k_j}(\underline{l})$ mais $z_{j,p-b_{k_j}(\underline{l})}$ y apparait. Si $p < b_{k_j}(\underline{l})$, alors aucun terme $z_{j,r}$ n'apparait dans l'expression de $\xi(z)$ modulo $t^{p+1}$.

Soit

$$W_{\underline{l}} := \mathbb{G}_{m,\mathbb{C}}^m \times \mathbb{A}_{\mathbb{C}}^{\max\{p-l_1,\,p-b_{k_1}(\underline{l})\}} \times \cdots \times \mathbb{A}_{\mathbb{C}}^{\max\{p-l_m,\,p-b_{k_m}(\underline{l})\}}.$$

On considère sur $W_{\underline{l}}$ les coordonnées $(z_{1,0},\ldots,\,z_{m,0})$ sur le premier facteur et les coordonnées $(z_{j,1},\ldots,\,z_{j,\max\{p-l_j,\,p-b_{k_j}(\underline{l})\}})$ sur le $(j+1)$-ième facteur.

Soit $V$ la variété $\mathbb{A}^{p(m+1)}$ ; considérons le morphisme de $\mathbb{C}$-schémas $h_{\underline{l}}\,:\,W_{\underline{l}} \longmapsto V$ qui envoie le point de coordonnées précédentes sur les $p$ premiers coefficients de $z_1^n,\ldots,$ $z_m^n$ et $\xi(z)$ (où $z_j(t) := z_{j,0}t^{\frac{l_j}{n}}(1+\sum_{r=1}^{\max(p-l_j,\,p-b_{k_j}(\underline{l}))} z_{j,r}t^r))$ :

$$h_{\underline{l}}\,:\,W_{\underline{l}} \longrightarrow V$$

$$z_i = z_{i,0}t^{\frac{l_i}{n}}(1+\sum_{j=1}^{\max(p-l_i,\,p-b_{k_i}(\underline{l}))} z_{i,j}t^j) \longmapsto z_1^n,\ldots,\,z_m^n,\,\xi(z)$$

L'image de $h_{\underline{l}}$ coïncide clairement avec $\chi_{p,\,l_1,\ldots,\,l_m}$, mais ce morphisme n'est pas fini en général.

***Définition 5.1***. — Nous définissons les ensembles suivants :

$$\begin{aligned}
\Sigma_{m,p} &:= \{(l_1,\ldots,\,l_m)\,/\,l_i \leq p\} \bigcup \\
&\left(\bigcup_{i=1}^m \{(l_1,\ldots,\,l_m)\,/\,l_i > p,\, p-l_j \geq b_{k_i}(\underline{l}) - b_{k_j}(\underline{l}),\, k_j < k_i,\, b_{k_i}(\underline{l}) \leq p\}\right), \\
D(m)_p &:= \Sigma_{m,p} \cap \operatorname{Ker} M \cap (\mathbb{N}^*)^m, \\
I_{p,\,\underline{l}} &:= \{j \in \{1,\,m\}\ /\ l_j > p\}, \\
D_{0,i}(m)_p &:= \big\{\underline{l} \in D(m)_p \setminus l_j \leq p\ \ \forall j,\ l_i - b_{k_i}(\underline{l}) \geq l_j - b_{k_j}(\underline{l})\ \ \forall j \neq i \\
&\qquad \text{et } l_i - b_{k_i}(\underline{l}) > l_j - b_{k_j}(\underline{l})\ \ \forall j < i\big\}, \\
D_{q,i}(m)_p &:= \big\{\underline{l} \in D(m)_p \setminus l_i > p,\ \operatorname{Card}(I_{p,\,\underline{l}}) = q,\ l_i - b_{k_i}(\underline{l}) \geq l_j - b_{k_j}(\underline{l})\ \ \forall j \neq i \\
&\qquad \text{et } l_i - b_{k_i}(\underline{l}) > l_j - b_{k_j}(\underline{l})\ \ \forall j < i\big\}, \\
D_{q,i,I}(m)_p &:= \big\{\underline{l} \in D(m)_{q,i}(m)_p \setminus I_{p,\,\underline{l}} = I\big\}.
\end{aligned}$$

***Remarque 5.2.*** — Si $l_i > p$ et si $j$ est tel que $k_j \neq k_i$, alors $l_i - b_{k_i}(\underline{l}) > l_j - b_{k_j}(\underline{l})$. En effet, si $k_i > k_j$, alors par hypothèse nous avons $p - l_j \geq b_{k_i}(\underline{l}) - b_{k_j}(\underline{l})$. Or $p - l_j < l_i - l_j$ et le résultat s'ensuit. Si $k_i < k_j$, alors $0 < l_i - p \leq l_i - l_j$. Or $b_{k_i}(\underline{l}) - b_{k_j}(\underline{l}) < 0$ et le résultat s'ensuit là encore. En particulier nous avons

$$D(m)_p = \coprod_{0 \leq i,\, q \leq m} D_{i,q}(m)_p.$$

**5.1. Cas C1.** — Soit $\underline{l} \in D_{0,i}(m)_p$. Nous notons alors

$$W'_{\underline{l}} := \mathbb{G}^m_{m,\mathbb{C}} \times \mathbb{A}^{p-l_1}_{\mathbb{C}} \times \cdots \times \mathbb{A}^{\max\{p-l_i,\, p-b_{k_i}(\underline{l})\}}_{\mathbb{C}} \times \cdots \times \mathbb{A}^{p-l_m}_{\mathbb{C}}$$

et

$$W''_{\underline{l}} := \mathbb{G}^m_{m,\mathbb{C}} \times \mathbb{A}^{p-l_1}_{\mathbb{C}} \times \cdots \times \mathbb{A}^{p-l_m}_{\mathbb{C}}.$$

On considère sur $W'_{\underline{l}}$ les coordonnées $(z_{1,0}, \ldots, z_{m,0})$ sur le premier facteur, les coordonnées $(z_{j,1}, \ldots, z_{j,p-l_j})$ sur le $(j+1)$-ième facteur pour $j \neq i$ et les coordonnées $(z_{i,1}, \ldots, z_{i,\max\{p-l_i,\, p-b_{k_i}(\underline{l})\}})$ sur $(i+1)$-ième facteur. On considère sur $W''_{\underline{l}}$ les coordonnées $(z_{1,0}, \ldots, z_{m,0})$ sur le premier facteur, et les coordonnées $(z_{j,1}, \ldots, z_{j,p-l_j})$ sur le $(j+1)$-ième facteur pour tout $j$.

Nous avons clairement

$$W''_{\underline{l}} \subset W'_{\underline{l}} \subset W_{\underline{l}}.$$

***Définition 5.3.*** — Nous notons

$$e := \max_j \{l_j - b_{k_j}(\underline{l}),\ 0\} = \max\{l_i - b_{k_i}(\underline{l}),\ 0\}.$$

Par ailleurs, nous notons $k_{\underline{l}}$ l'unique entier qui vérifie les inégalités suivantes :

$$b_{k_{\underline{l}}}(\underline{l}) = \sum_j a_j(k_{\underline{l}}) l_j \leq p - e < \sum_j a_j(k_{\underline{l}}+1) l_j = b_{k_{\underline{l}}+1}(\underline{l}). \tag{2}$$

Nous avons alors le résultat suivant :

***Lemme 5.4.*** — *Le morphisme $h_{\underline{l}}$ restreint à $W'_{\underline{l}}$ est fini et d'image $\chi_{p,l_1,\ldots,l_m}$. Si nous notons $\overline{\chi}_{p,l_1,\ldots,l_m} := h_{\underline{l}}(W''_{\underline{l}})$, nous avons $\chi_{p,\,l_1,\ldots,\,l_m} = \overline{\chi}_{p,l_1,\ldots,l_m} \times \mathbb{A}^{l_i - b_{k_i}(\underline{l})}_{\mathbb{C}}$ et $h_{\underline{l}}$ restreint à $W''_{\underline{l}}$ est un revêtement galoisien de $\overline{\chi}_{p,\,l_1,\ldots,\,l_m}$. Son groupe de Galois est le groupe $G_{k_{\underline{l}}}$, sous-groupe commutatif de $\mathbb{U}^m_n$ formé des éléments $(\varepsilon_1, \ldots, \varepsilon_m)$ qui vérifient les équations $\prod_i \varepsilon_i^{na_r(i)} = 1$ pour $r = 1, .., k_{\underline{l}}$ où $\mathbb{U}_n$ est le groupe des racines $n$-ièmes de l'unité, qui agit par multiplication terme à terme sur $\mathbb{G}^m_{m,\mathbb{C}}$.*

*Démonstration.* — Nous allons d'abord montrer que l'image de $W'_{\underline{l}}$ par $h_{\underline{l}}$ est égale à $\chi_{p,l_1,\ldots,l_m}$.

Soit $z = (z_1, \ldots, z_m) \in W_{\underline{l}}$, et montrons que l'on peut trouver $w \in W'_{\underline{l}}$ ayant même image par $h_{\underline{l}}$ que $z$. Nécessairement nous devons avoir

$$z^n_{j,0} = w^n_{j,0} \text{ si } \ l_j \leq p,$$

$$\left(1+\sum_{k=1}^{p-l_j} z_{j,k}t^k\right)^n = \left(1+\sum_{k=1}^{p-l_j} w_{j,k}t^k\right)^n \mod t^{p-l_j+1} \text{ si } \ l_j \leq p.$$

Comme $l_j \leq p$ pour tout $j$, nécessairement $z_{j,0}^n = w_{j,0}^n$ et $z_{j,k} = w_{j,k}$, pour tout $j$ et $1 \leq k \leq p - l_j$. Nous posons alors $w_{j,k} = z_{j,k}$, pour tout $j$ et $0 \leq k \leq p - l_j$. Les coefficients de $t^c$ dans l'écriture de $\xi(z)$, pour $c \leq p - e$, ne dépendent que des $z_{j,k}$ pour $k \leq p - l_j$. Le coefficient de $t^{p-e+1}$ est de la forme

$$\prod_j z_{j,l_j}^{a_j(k_j)}.z_{i,p-l_i+1} + P$$

où $P$ ne dépend que des $z_{i,k}$ pour $k \leq p - l_i$ et des $z_{j,k}$ pour $j \neq i$ et $k \leq p - l_j + 1$. Comme $\prod_j z_{j,l_j}^{a_j(k_j)} = \prod_j w_{j,l_j}^{a_j(k_j)} \neq 0$, nous pouvons trouver $w_{i,p-l_i+1}$ de telle manière à choisir (s'il apparaissent ici) les $w_{j,p-l_j+1}$ égaux à zéro pour $j \neq i$ sans changer la valeur de ce coefficient. Nous pouvons continuer ainsi par récurrence croissante sur les coefficients de $t^c$ dans l'écriture de $\xi(z)$, pour voir que l'on peut trouver $w \in W'_{\underline{l}}$ tel que $h_{\underline{l}}(z) = h_{\underline{l}}(w)$, le système d'équations apparaissant ici étant triangulaire. Le morphisme $h_{\underline{l}}$ restreint à $W'_{\underline{l}}$ est donc fini.

Nous remarquons au passage, que le système d'équations obtenu des coefficients de $t^c$ dans l'écriture de $\xi(z)$ est triangulaire en les variables $z_{i,k}$ pour $p - l_i + 1 \leq k \leq p - b_{k_i}(\underline{l})$, pour $p - e + 1 \leq c \leq p$. Donc nous pouvons écrire $\chi_{p,l_1,\ldots,l_m}$ sous la forme $\overline{\chi}_{p,l_1,\ldots,l_m} \times \mathbb{A}_{\mathbb{C}}^{l_i - b_{k_i}(\underline{l})}$ où $\overline{\chi}_{p,l_1,\ldots,l_m} = h_{\underline{l}}(W''_{\underline{l}})$.

Déterminons la fibre au-dessus d'un point de $\overline{\chi}_{p,l_1,\ldots,l_m}$. Remarquons que $k_i \leq k_{\underline{l}}$ car $b_{k_{\underline{l}}+1}(\underline{l}) > p - e = p - l_i + b_{k_i}(\underline{l}) \geq b_{k_i}(\underline{l})$. Soient $z$ et $w$ dans $W''_{\underline{l}}$ tels que $h_{\underline{l}}(z) = h_{\underline{l}}(w)$. Dans ce cas $z_{j,0}^n = w_{j,0}^n$ pour tout $j$ et $z_{j,k} = w_{j,k}$ pour tout $j$ et pour tout $1 \leq k \leq p - l_j$. Considérons alors le coefficient de $t^c$ dans l'écriture de $\xi(z) = \xi(w)$. Pour $c = b_{k_1}(\underline{l})$, ce coefficient nous permet de dire que $(z_{1,0}, \ldots, z_{m,0}) = \varepsilon(w_{1,0}, \ldots, w_{m,0})$ pour un $\varepsilon \in G_{k_1}$. Pour $b_{k_1}(\underline{l}) < c < b_{k_2}(\underline{l})$, ces coefficients n'apportent aucune information supplémentaire. Pour $c = b_{k_2}(\underline{l})$, coefficient nous permet de dire que $(z_{1,0}, \ldots, z_{m,0}) = \varepsilon(w_{1,0}, \ldots, w_{m,0})$ pour un $\varepsilon \in G_{k_2}$. Par récurrence sur $c$, nous voyons alors que $(z_{1,0}, \ldots, z_{m,0}) = \varepsilon(w_{1,0}, \ldots, w_{m,0})$ pour un $\varepsilon \in G_{k_{\underline{l}}}$. Comme $l_j < +\infty$ pour tout $j$, le revêtement est étale et donc galoisien de groupe de Galois $G_{k_{\underline{l}}}$. □

Nous avons alors

$$[\chi_{p,\, l_1,\ldots,\, l_m}] = [\mathbb{G}_{m,\mathbb{C}}^m / G_{k_{\underline{l}}}]\mathbb{L}^{pm+\max\{l_i - b_{k_i}(\underline{l}),\, 0\} - \sum_{j=1}^m l_j} \ .$$

**5.2. Cas C2.** — Soit $\underline{l} \in D_{q,i}(m)_p$ avec $q \geq 1$. Nous notons alors

$$W'_{\underline{l}} := \mathbb{G}_{m,\mathbb{C}}^{m-q+1} \times \left(\prod_{j \notin I_{\underline{l},p}} \mathbb{A}_{\mathbb{C}}^{p-l_j}\right) \times \mathbb{A}_{\mathbb{C}}^{p-b_{k_i}(\underline{l})}$$

et

$$W''_{\underline{l}} := \mathbb{G}_{m,\mathbb{C}}^{m-q+1} \times \left( \prod_{j \notin I_{\underline{l},p}} \mathbb{A}_{\mathbb{C}}^{p-l_j} \right).$$

On considère sur $W'_{\underline{l}}$ les coordonnées $(z_{j_1,0}, \ldots, z_{j_{m-q},0})$ sur le premier facteur (où $\{j_1, \ldots, j_{m-q+1}\} = \{1, \ldots, m\} \setminus \left\{I_{\underline{l},p} \setminus \{i\}\right\}$), les coordonnées $(z_{j,1}, \ldots, z_{j,p-l_j})$ sur le $(j+1)$-ième facteur pour $j \notin I_{\underline{l},p}$ et les coordonnées $(z_{i,1}, \ldots, z_{i,p-b_{k_i}(\underline{l})})$ sur $(i+1)$-ième facteur. On considère ces deux variétés plongées dans $W_{\underline{l}}$ en identifiant un élément de coordonnées

$$(z_{j_1,0}, \ldots, z_{j_{m-q+1},0},\ z_{j,1}, \ldots, z_{j,p-l_j}, j \notin I_{\underline{l},p})$$

avec l'élément de $W_{\underline{l}}$ de coordonnées

$$(z_{1,0}, \ldots, z_{m,0},\ z_{j,1}, \ldots, z_{j,p-l_j}, j \neq i, z_{i,1}, \ldots, z_{i,p-b_{k_i}(\underline{l})})$$

en posant $z_{j,0} = 1$ et $z_{j,k} = 0$ si $j \in I_{p,\underline{l}}$ et $k \leq p - l_j$. Nous avons alors le résultat suivant :

***Lemme 5.5***. — *Le morphisme $h_{\underline{l}}$ restreint à $W'_{\underline{l}}$ est fini et d'image $\chi_{p,l_1,\ldots,l_m}$. Si nous notons $\overline{\chi}_{p,l_1,\ldots,l_m} := h_{\underline{l}}(W''_{\underline{l}})$, nous avons $\chi_{p,l_1,\ldots,l_m} = \overline{\chi}_{p,l_1,\ldots,l_m} \times \mathbb{A}_{\mathbb{C}}^{l_i - b_{k_i}(\underline{l})}$ et $h_{\underline{l}}$ restreint à $W''_{\underline{l}}$ est un revêtement galoisien. Son groupe de Galois est le groupe $G_{k_i} \cap \mathbb{U}_n^{m-q+1}$, où $\mathbb{U}_n^{m-q+1}$ est le sous-groupe de $\mathbb{U}_n^m$ dont les éléments ont les coordonnées d'indice dans $I_{\underline{l},p} \setminus \{i\}$ égales à 0, et $G_{k_i}$ est le sous-groupe de $\mathbb{U}_n^m$ formé des éléments $(\varepsilon_1, \ldots, \varepsilon_{m-q+1})$ qui vérifient les équations $\prod_{j \notin I_{p,\underline{l}} \setminus \{i\}} \varepsilon_j^{na_r(j)} = 1$ pour $r = 1, .., k_i$ où $\mathbb{U}_n$ est le groupe des racines $n$-ièmes de l'unité, qui agit par multiplication terme à terme sur $\mathbb{G}_{m,\mathbb{C}}^{m-q}$.*

*Démonstration*. — Nous allons d'abord montrer que l'image de $W'_{\underline{l}}$ par $h_{\underline{l}}$ est égale à $\chi_{p,l_1,\ldots,l_m}$.

Soit $z = (z_1, \ldots, z_m) \in W_{\underline{l}}$, et montrons que l'on peut trouver $w \in W'_{\underline{l}}$ ayant même image par $h_{\underline{l}}$ que $z$. En particulier nous avons

$$z_{j,0}^n \text{ est fixé si } \ l_j \leq p,$$

$$\left(1 + \sum_{k=1}^{p-l_j} z_{j,k} t^k\right)^n \mod t^{p-l_j+1} \text{ est fixé si } \ l_j \leq p.$$

Les coefficients de $t^c$ dans l'écriture de $\xi(z)$, pour $c < b_{k_i}(\underline{l})$, ne dépendent que des $z_{j,k}$ pour $j \notin I_{p,\underline{l}}$ et $k \leq p$, d'après la remarque 5.2. Si $c = b_{k_i}(\underline{l})$, ce coefficient est de la forme

$$\prod_{j \neq i} z_{j,0}^{a_j(k_i)} . z_{i,0}^{a_i(k_i)} + P$$

où $P$ ne dépend que des $z_{j,k}$ pour $j \notin I_{\underline{l},p}$, $k \leq p - l_j + 1$. Si nous posons $w_{j,0} = 1$ pour $j \in I_{p,\underline{l}} \setminus \{i\}$ et $w_{j,p-l_j+1} = 0$, il existe toujours un $w_{i,0}$ qui permet de garder la valeur de ce coefficient inchangée. Nous pouvons continuer ainsi par récurrence croissante sur les coefficients de $t^c$ dans l'écriture de $\xi(z)$, pour voir que l'on peut trouver $w \in W'_{\underline{l}}$ tel que $h_{\underline{l}}(z) = h_{\underline{l}}(w)$, le système d'équations apparaissant ici étant triangulaire. Le morphisme $h_{\underline{l}}$ restreint à $W'_{\underline{l}}$ est donc fini.

Nous remarquons au passage, que le système d'équations obtenu des coefficients de $t^c$ dans l'écriture de $\xi(z)$ est triangulaire en les variables $z_{i,k}$, pour $0 \leq k \leq p - b_{k_i}(\underline{l})$ et $b_{k_i}(\underline{l}) \leq c \leq p$. Donc nous pouvons écrire $\chi_{p,l_1,\dots,l_m}$ sous la forme $\overline{\chi}_{p,l_1,\dots,l_m} \times \mathbb{A}_{\mathbb{C}}^{l_i - b_{k_i}(\underline{l})}$ où $\overline{\chi}_{p,l_1,\dots,l_m} = h_{\underline{l}}(W''_{\underline{l}})$.

Comme précédemment nous voyons que deux éléments $z$ et $w$ de $W''_{\underline{l}}$ ont même image par $h_{\underline{l}}$ si et seulement si $z = \varepsilon w$ où $\varepsilon \in G_{k_i} \cap \mathbb{U}_n^{m-q+1}$. □

Nous avons alors

$$[\chi_{p,\,l_1,\dots,\,l_m}] = [\mathbb{G}_{m,\mathbb{C}}^{m-q+1}/G'_{k_i}]\mathbb{L}^{p(m-q+1) - \sum_{j \notin I_{p,\,\underline{l}}} l_j - b_{k_i}(\underline{l})}.$$

**5.3. Calcul de $\chi_{p,\,l_1,\dots,\,l_m}$ et $\varphi_{p,\,l_1,\dots,\,l_m}$.** — Nous allons maintenant énoncer un lemme utile pour achever le calcul (la partie *ii*) étant un analogue du lemme 1.4.3 de [**5**]). Pour les définitions se rapportant à la mesure d'une variété munie d'une action de groupe nous renvoyons le lecteur à [**5**] et [**7**].

***Lemme 5.6***. — *Soit $G$ un sous-groupe de $\prod_{i=1}^m \mathbb{U}_{n_i}$ agissant sur $\mathbb{G}_{m,\mathbb{C}}^m$ par multiplication sur chaque terme. Alors :*

i) *La variété quotient $\mathbb{G}_{m,\mathbb{C}}^m/G$ est isomorphe à $\mathbb{G}_{m,\mathbb{C}}^m$.*
ii) *Pour pour tout caractère irréductible non trivial $\alpha$ de $G$, nous avons $\chi_c(\mathbb{G}_{m,\mathbb{C}}^m,\, \alpha) = 0$, et $\chi_c(\mathbb{G}_{m,\mathbb{C}}^m,\, 1) = (\mathbb{L}-1)^m$ où $1$ est le caractère trivial de $G$.*

*Démonstration*. — Montrons d'abord i). D'un point de vue torique, le cône associé à $\mathbb{G}_{m,\mathbb{C}}^m$ est l'origine de $\mathbb{Z}^m$. Le semi-groupe de cette variété est donc $\mathbb{Z}^m$ en entier. L'ensemble des puissances des monômes invariants par l'action de $G$ forment un semi-groupe $N$ de $\mathbb{Z}^m$ qui est en fait un groupe car $X_1^{k_1} \dots X_m^{k_m}$ est invariant sous l'action de $G$ si et seulement si $X_1^{-k_1} \dots X_m^{-k_m}$ l'est aussi. Donc $N$ est isomorphe à $\mathbb{Z}^l$ pour $l \leq m$ et $N \otimes_{\mathbb{Z}} \mathbb{R} \equiv \mathbb{R}^l$. Or pour tout $i$, $X_i^{n_i}$ est invariant par $G$ donc $N \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^m$ et $l = m$. Donc $\mathbb{G}_{m,\mathbb{C}}^m/G$ est isomorphe à $\mathbb{G}_{m,\mathbb{C}}^m$.

Montrons maintenant ii). L'action de $G$ sur $\mathbb{G}_{m,\mathbb{C}}^m$ s'étend en une action sur $\left(\mathbb{P}_{\Bbbk}^1\right)^m$ qui laisse fixe les coordonnées $0$ et $\infty$. Pour chaque $\varepsilon \in (\Bbbk^*)^m$, la classe, dans le groupe de Chow $A^m\left(\left(\mathbb{P}_{\Bbbk}^1\right)^m \times \left(\mathbb{P}_{\Bbbk}^1\right)^m\right)$, du graphe de la multiplication par $\varepsilon$ sur $\left(\mathbb{P}_{\Bbbk}^1\right)^m$ est la même que la classe de la diagonale : en effet, la classe, dans le groupe de Chow $A^1\left(\mathbb{P}_{\Bbbk}^1 \times \mathbb{P}_{\Bbbk}^1\right)$, de la multiplication par la $i$-ème coordonnée de $\varepsilon$ sur la $i$-ème composante de $\left(\mathbb{P}_{\Bbbk}^1\right)^m$ est la même que la classe de la diagonale. Pour tout caractère $\alpha$ de $G$, notons $f_\alpha := |G|^{-1} \sum_{g \in G} \alpha^{-1}(g)[g]$ où $[g]$ est la correspondance donnée par le graphe de la multiplication par $g$ (voir [**5**] partie 1.3 ou [**7**] partie 3.1). Nous voyons donc que $f_\alpha = 0$ si $\alpha$ est un caractère non trivial, et donc que $\chi_c(\mathbb{G}_{m,\mathbb{C}}^m,\, \alpha) = 0$ d'après le théorème 1.3.1 de [**5**].

Enfin $\chi_c(\mathbb{G}_{m,\mathbb{C}}^m,\, 1) = (\mathbb{L}-1)^m$ par définition (voir [**5**] 1.3 par exemple). □

D'après le point $i$) du lemme précédent, nous avons $[\mathbb{G}^r_{m,\mathbb{C}}/G_k] = (\mathbb{L}-1)^r$ et nous obtenons :

$$\text{pour le cas C1 : } [\chi_{p,\,l_1,\dots,\,l_m}] = (\mathbb{L}-1)^m \mathbb{L}^{pm} \mathbb{L}^{-\sum_{j=1}^m l_j + \max\{l_i - b_{k_i}(\underline{l})),\,0\}}.$$

$$\text{pour le cas C2 : } [\chi_{p,\,l_1,\dots,\,l_m}] = (\mathbb{L}-1)^{m-q+1} \mathbb{L}^{p(m-q+1) - \sum_{j\notin I_{\underline{l},p}} l_j - b_{k_i}(\underline{l})}.$$

Nous utilisons le même morphisme $h_{\underline{l}}$ pour calculer $\chi_c(\varphi_{p,\,l_1,\dots,\,l_m})$. En effet, dans le cas C1, la mesure arithmétique de l'image de $h_{\underline{l}}$ coïncide avec $\chi_c(\varphi_{p,\,l_1,\dots,\,l_m})$. Or $\varphi_{p,\,l_1,\dots,\,l_m} = \overline{\varphi}_{p,\,l_1,\dots,\,l_m} \wedge \varphi'$ où $\overline{\varphi}_{p,\,l_1,\dots,\,l_m}$ est la formule qui définit $\overline{\chi}_{p,\,l_1,\dots,l_m}$ et $\varphi'$ est la formule qui définit $\mathbb{A}_{\mathbb{C}}^{\max\{l_i - b_{k_i}(\underline{l}),\,0\}}$. La mesure arithmétique de $W''_{\underline{l}}$ est égale à $\mathbb{L}^{p(m-q) - \sum_{j\notin I_{p,\,\underline{l}}} l_j}$ car la formule qui définit $W''_{\underline{l}}$ est sans quantificateurs. Le morphisme $h_{\underline{l}}$ restreint à $W''_{\underline{l}}$ est un revêtement galoisien de groupe de Galois fini, de cardinal $n/n_{k_{\underline{l}}}$ dans le cas C1 et de cardinal $n(i,\,I_{p,\underline{l}})$ dans le cas C2 ($n(i,\,I_{p,\underline{l}})$ ne dépend que de $i$ et de $I_{p,\underline{l}}$)). La mesure $\chi_c(\overline{\varphi}_{p,\,l_1,\dots,\,l_m})$ est par construction égale à la mesure $\chi_c(W''_{\underline{l}},\delta)$ où $\delta$ est la fonction centrale qui vaut 1 en l'identité et 0 ailleurs car $G$ est abélien (cf. [**7**] 3.2. et 3.4). Cette fonction $\delta$ est une combinaison linéaire sur $\mathbb{Q}$ de caractères irreductibles de $G$, et le coefficient du caratère trivial est égal à $1/|G|$. Finalement, la formule $\varphi'$ est sans quantificateurs, donc sa mesure vaut $\mathbb{L}^{l_i - b_{k_i}(\underline{l})}$. Donc nous avons

$$\text{cas C1 : } \chi_c(\varphi_{p,\,l_1,\dots,\,l_m}) = \frac{n_{k_{\underline{l}}}}{n} (\mathbb{L}-1)^m \mathbb{L}^{pm} \mathbb{L}^{-\sum_{j=1}^m l_j + \max\{l_i - b_{k_i}(\underline{l})),\,0\}}.$$

$$\text{cas C2 : } \chi_c(\varphi_{p,\,l_1,\dots,\,l_m}) = \frac{1}{n(i,\,I_{p,\underline{l}})} (\mathbb{L}-1)^{m-q+1} \mathbb{L}^{p(m-q+1) - \sum_{j\notin I_{p,\,\underline{l}}} l_j - b_{k_i}(\underline{l})}.$$

## 6. Étude de l'union (1)

L'union (1) n'est cependant pas toujours disjointe comme le montre l'exemple ci-dessous :

***Exemple 6.1***. — Soit $f = Z^3 - XY$. Notons $h_1(t)$, $h_2(t)$ et $h_3(t)$ les arcs définis par $h_1(t) = (t^5,\,t^7,\,t^4)$, $h_2(t) = (t^6,\,t^6,\,t^4)$ et $h_3(t) = (t^7,\,t^5,\,t^4)$. Modulo $t^5$, ces trois arcs ont la même troncation mais les ordres des différentes coordonnées ne coïncident pas. C'est-à-dire que $\chi_{4,\,5,\,7} \cap \chi_{4,\,6,\,6} \cap \chi_{4,\,7,\,5} \neq \varnothing$.

Nénamoins nous pouvons énoncer le résultat suivant :

***Lemme 6.2***. — *Nous avons*

$$\chi_{p,\,l_1,\dots,\,l_m} = \chi_{p,\,l'_1,\dots,\,l'_m} \iff \chi_{p,\,l_1,\dots,\,l_m} \cap \chi_{p,\,l'_1,\dots,\,l'_m} \neq \varnothing$$
$$\iff \begin{cases} l_i \leq p \Rightarrow l_i = l'_i \\ l_i > p \Leftrightarrow l'_i > p \\ b_{r_{\underline{l}}}(\underline{l} - \underline{l}') = 0 \end{cases}$$

*où $r_{\underline{l}}$ est l'entier $k_i$ si $l_i > p$ (voir définition 4.4).*

*Démonstration.* — Il est clair que

$$\chi_{p,\,l_1,\ldots,\,l_m} = \chi_{p,\,l'_1,\ldots,\,l'_m} \Longrightarrow \chi_{p,\,l_1,\ldots,\,l_m} \cap \chi_{p,\,l'_1,\ldots,\,l'_m} \neq \varnothing.$$

L'implication

$$\chi_{p,\,l_1,\ldots,\,l_m} \cap \chi_{p,\,l'_1,\ldots,\,l'_m} \neq \varnothing \Longrightarrow \begin{cases} b_{r_{\underline{l}}}(\underline{l}-\underline{l}')=0, \\ l_i \leq p \Rightarrow l_i = l'_i \\ l_i > p \Leftrightarrow l'_i > p \end{cases}$$

découle de la remarque faite à la fin de la preuve de la proposition 4.2.
Fixons $(l_1,\ldots,\,l_m) \in \operatorname{Ker} M$ et $(l'_1,\ldots,\,l'_m) \in \operatorname{Ker} M$. Supposons que nous ayons $b_{r_{\underline{l}}}(\underline{l}-\underline{l}')=0$, $l_i \leq p \Rightarrow l_i = l'_i$ et que $l_i > p \Leftrightarrow l'_i > p$. Supposons qu'il existe un entier $i$ pour lequel $l_i > p$. Dans le cas contraire les équivalences sont triviales. Soient $h \in \chi_{p,\,l_1,\ldots,\,l_m}$ et $\overline{h}$ un arc égal à $h$ modulo $(t)^{p+1}$ et dont chaque composante $x_i$ est d'ordre $l_i$ pour $1 \leq i \leq m$. Nous avons donc $\overline{h}(t) = (x_1(t),\ldots,\,x_m(t),\,z(t))$ et nous pouvons supposer, quitte à changer les variables, que $x_1(t) = \cdots = x_k(t) = 0$ modulo $(t)^{p+1}$, et que $x_{k+1}(t),\ldots,\,x_m(t)$ sont non nuls modulo $(t)^{p+1}$ (c'est-à-dire que $l_1,\ldots,\,l_k > p$ et $l_{k+1},\ldots,\,l_m \leq p$). Nous allons chercher des $x'_i(t)$, avec $\operatorname{ord}(x'_i(t)) = l'_i$ pour $1 \leq i \leq m$, tels que

$$\xi(x^{1/n}(t)) = \xi(x'^{1/n}(t)).$$

Pour cela, il nous suffit de poser $x'_i(t) = x_i(t)$ pour tout $i$ compris entre $k+1$ et $m$, et de voir que cela revient alors à trouver des $x'_i(t)$ tels que

$$\xi_{k_i}(x^{1/n}(t)) + \xi_{k_i+1}(x^{1/n}(t)) + \cdots = \xi_{k_i}(x'^{1/n}(t)) + \cdots. \tag{3}$$

Il suffit pour cela de trouver $x'_{i,l'_i}$ pour $k+1 \leq i \leq m$ tels que $\prod_i x_{i,l_i}^{a_i(k_i)} = \prod_i x'^{a_i(k_i)}_{i,l'_i}$ ce qui est toujours possible. Ensuite il suffit de choisir les $x'_{i,k}$ pour $k > l'_i$ ce qui est toujours possible car les équations en les $x'_{i,k}$, qui découlent de l'annulation des différents termes de l'equation 3, forment un système triangulaire. Donc $h \in \chi_{p,\,l'_1,\ldots,\,l'_m}$. □

***Exemple 6.3***. — Si pour tous $i$ et $j$ tels que $i \neq j$ nous avons $a_i(1) + a_j(1) \geq 1$, alors l'union (1) précédente est disjointe.

***Définition 6.4***. — Nous allons alors noter :

$$N_{p,\,l_1,\ldots,\,l_m} := \operatorname{Card}\left\{\operatorname{Ker}(\,b_{p,\,l_1,\ldots,\,l_m}) \bigcap \prod_{i \in I_{p,\,\underline{l}}} (\mathbb{Z} \cap ]-\infty,\, l_i - p[)\right\}$$

où $I_{p,\,\underline{l}} = \{i_1,\ldots,\,i_q\}$ est l'ensemble des indices $i$ pour lesquels $l_i > p$ et $b_{p,\,l_1,\ldots,\,l_m}$ est l'application linéaire

$$\begin{aligned} b_{p,\,l_1,\ldots,\,l_m} \;:\; & \mathbb{Z}^q \longrightarrow \mathbb{Z} \\ & (u_1,\ldots,\,u_q) \longmapsto na_{i_1}(r_{\underline{l}})u_1 + \cdots + na_{i_q}(r_{\underline{l}})u_q \end{aligned}$$

D'après le lemme précédent, nous voyons que $\chi_{p,\,l_1,\dots,\,l_m} = \chi_{p,\,l'_1,\dots,\,l'_m}$ si seulement si $(l_1 - l'_1, \dots,\, l_m - l'_m) \in \mathrm{Ker}(\,b_{p,\,l_1,\dots,\,l_m}) \bigcap \prod_{i\in I_{p,\,\underline{l}}} (\mathbb{Z}\cap] - \infty,\, l_i - p[)$.

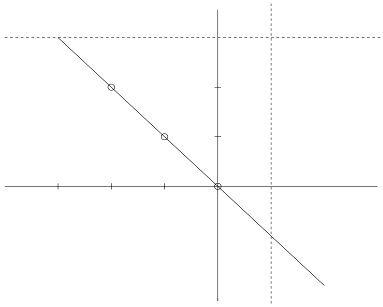
La mesure motivique des $p$-troncations d'arcs qui ne se relèvent pas en arcs dont l'une des coordonnées $x_i$ est nulle est alors égale à :

$$\left[\bigcup_{(l_1,\dots,\,l_m)\in D(m)_p} \chi_{p,\,l_1,\dots,\,l_m}\right] = \sum_{(l_1,\dots,\,l_m)\in D(m)_p} \frac{[\chi_{p,\,l_1,\dots,\,l_m}]}{N_{p,\,l_1,\dots,\,l_m}} \tag{4}$$

***Exemple 6.5***. — Dans le cas $Z^3 = XY$, nous avons

$$N_{4,\,5,\,7} = \mathrm{Card}\left\{\mathrm{Ker}\,(1,\,1) \bigcap ((\mathbb{Z}\cap] - \infty,\, 1[) \times (\mathbb{Z}\cap] - \infty,\, 3[))\right\}.$$

Cet ensemble est formé de trois points : le point $(0,\,0)$ correspond à $\chi_{4,\,5,\,7}$, le point $(1,\,-1)$ correspond à $\chi_{4,\,6,\,6}$ et le point $(2,\,-2)$ correspond à $\chi_{4,\,7,\,5}$. Et donc le cardinal de cet ensemble vaut 3 :



## 7. Étude des troncations d'arcs pour lesquels un des $x_i(t)$ est nul

Ce terme correspond aux troncations d'arcs pour lesquels un des $x_i(t)$ est nul.

Nous allons exprimer le terme $[X(a(1), \dots,\, a(g))_{p,cn}]$ en fonction de différents termes de la forme $[X(b(1), \dots,\, b(i))_{p,to}]$ pour $i < g$.

Rappelons que pour tout $i$ compris entre 1 et $m$, $k_i$ est le plus petit entier qui vérifie $a_i(k_i) \neq 0$ et que nous avons

$$1 = k_1 = k_2 = \dots = k_{i_0} < k_{i_0+1} \leq k_{i_0+2} \leq \dots \leq k_m \leq g.$$

Soit $i \in \{1, \dots,\, m\}$. Que vaut $[\chi_{p,\,l_1,\dots,\,l_m}]$ où $l_i = +\infty$ et $l_j < +\infty$ pour $j < i$ ?

Si $i > i_0$ les $x_j$ pour $j > i$ peuvent être choisis quelconques et

$$[\chi_{p,\,l_1,\dots,\,l_m}] = \mathbb{L}^{p(m-i)}[\chi^i_{p,\,l_1,\dots,\,l_{i-1}}]$$

où $\chi^i_{p,\,l_1,\dots,\,l_{i-1}}$ est l'ensemble des troncations à l'ordre $p$ d'arcs tracés sur $X(a^i(1), \dots,$ $a^i(k_i - 1))$ et d'ordre $l_1, \dots,\, l_{i-1}$, où $X(a^i(1), \dots,\, a^i(k_i - 1))$ est le germe d'hypersurface à singularité quasi-ordinaire d'exposants caractéristiques $a^i(j) = (a_1(j), \dots,\, a_{i-1}(j))$ pour $1 \leq j \leq k_i - 1$. Le germe de variété $X(a^i(1), \dots,\, a^i(k_i - 1))$

n'est peut-être pas réduit mais il reste irréductible car ses exposants caractéristiques sont conjugués.

Si $i \leq i_0$, les $x_j$ pour $j \neq i$ peuvent être choisis quelconques mais les $x_j$ pour $j < i$ doivent être non nuls donc $[\chi_{p,\, l_1,\dots,\, l_m}] = \mathbb{L}^{p(m-i)}\left(\mathbb{L}^p - 1\right)^{i-1}$.

Nous avons donc

$$\begin{aligned}[X(a(1),\dots,\, a(g))_{p,cn}] = {} & \sum_{i=1}^{i_0} \mathbb{L}^{p(m-i)}\left(\mathbb{L}^p - 1\right)^{i-1} \\ & + \sum_{i=i_0+1}^{m} \mathbb{L}^{p(m-i)}[X(a^i(1),\dots,\, a^i(k_i - 1))_{p,to}],\end{aligned} \tag{5}$$

et

$$\begin{aligned}P_{\text{géom}}^{\text{tore}^c}(a(1),\dots,\, a(g))(T) = {} & \sum_{i=0}^{i_0-1}\sum_{j=0}^{i} \frac{\binom{i}{i-j}(-1)^j}{1-\mathbb{L}^{m-j-1}T} \\ & + \sum_{i=i_0+1}^{m} P^{\text{tore}}_{\text{géom},X(a^i(1),\dots,\, a^i(k_i-1)),0}(\mathbb{L}^{m-i}T).\end{aligned} \tag{6}$$

En suivant le même raisonnement nous obtenons

$$\begin{aligned}\chi_c(\varphi_{p,cn}(a(1),\dots,\, a(g))) = {} & \sum_{i=1}^{i_0} \mathbb{L}^{p(m-i)}\left(\mathbb{L}^p - 1\right)^{i-1} \\ & + \sum_{i=i_0+1}^{m} \mathbb{L}^{p(m-i)}\chi_c(\varphi_{p,to}(a^i(1),\dots,\, a^i(k_i - 1))),\end{aligned} \tag{7}$$

et

$$\begin{aligned}P_{\text{arit}}^{\text{tore}^c}(a(1),\dots,\, a(g))(T) = {} & \sum_{i=0}^{i_0-1}\sum_{j=0}^{i} \frac{\binom{i}{i-j}(-1)^j}{1-\mathbb{L}^{m-j-1}T} \\ & + \sum_{i=i_0+1}^{m} P^{\text{tore}}_{\text{arit},X(a^i(1),\dots,\, a^i(k_i-1)),0}(\mathbb{L}^{m-i}T).\end{aligned} \tag{8}$$

## 8. Résultat principal

Nous pouvons alors donner la forme générale des deux séries étudiées :

***Théorème 8.1***. — *Soit $(X,0)$ un germe d'hypersurface irréductible quasi-ordinaire d'exposants caractéristiques $a(1),\dots,a(g)$. Nous avons alors les relations de récurrence suivantes*

$$P^{\text{tore}}_{\text{géom},X(a(1),\dots,a(g)),0}(T)=$$

$$(\mathbb{L}-1)^m\sum_{i=1}^{m}\sum_{p\geq 0}\mathbb{L}^{pm}T^p\times\sum_{(l_1,\dots,l_m)\in D_{0,i}(m)_p}\mathbb{L}^{-\sum_{j=1}^m l_j+\max\{l_i-b_{k_i}(\underline{l}),0\}}+ \tag{9}$$

$$+\sum_{q=1}^{m}\sum_{i=1}^{m}(\mathbb{L}-1)^{m-q+1}\sum_{p\geq 0}(\mathbb{L}^{m-q+1}T)^p\times\sum_{(l_1,\dots,l_m)\in D_{q,i}(m)_p}\frac{\mathbb{L}^{-\sum_{j\notin I_{p,\underline{l}}}l_j-b_{k_i}(\underline{l})}}{N_{p,l_1,\dots,l_m}},$$

$$P^{\text{cône}}_{\text{géom},X(a(1),\dots,a(g)),0}(T)=$$

$$\sum_{i=0}^{i_0-1}\sum_{j=0}^{i}\frac{\binom{i}{i-j}(-1)^j}{1-\mathbb{L}^{m-j-1}T}+\sum_{i=i_0+1}^{m}P^{\text{tore}}_{\text{géom},X(a^i(1),\dots,a^i(k_i-1)),0}(\mathbb{L}^{m-i}T) \tag{10}$$

(11)

$$P^{\text{géom}}_{\text{arit},X(a(1),\dots,a(g)),0}(T)=$$

$$(\mathbb{L}-1)^m\sum_{k=0}^{g}\frac{n_{k_l}}{n}\sum_{i=1}^{m}\sum_{p\geq 0}\mathbb{L}^{pm}T^p\times\sum_{(l_1,\dots,l_m)\in D_{0,i}(m)_p}\mathbb{L}^{-\sum_{j=1}^m l_j+\max\{l_i-b_{k_i}(\underline{l}),0\}}+$$

$$+\sum_{k=0}^{g}\sum_{q=1}^{m}\sum_{i=1}^{m}\sum_{\substack{I\subset\{1,\dots,m\}\\ i\in I,\#I=q}}(\mathbb{L}-1)^{m-q+1}\sum_{p\geq 0}(\mathbb{L}^{m-q+1}T)^p\times$$

$$\times\sum_{(l_1,\dots,l_m)\in D_{q,i,I}(m)_p}\frac{1}{n(i,I_{p,\underline{l}})}\frac{\mathbb{L}^{-\sum_{j\notin I_{p,\underline{l}}}l_j-b_{k_i}(\underline{l})}}{N_{p,l_1,\dots,l_m}},$$

*et*

$$P^{\text{cône}}_{\text{arit},X(a(1),\dots,a(g)),0}(T)=$$

$$\sum_{i=0}^{i_0-1}\sum_{j=0}^{i}\frac{\binom{i}{i-j}(-1)^j}{1-\mathbb{L}^{m-j-1}T}+\sum_{i=i_0+1}^{m}P^{\text{tore}}_{\text{arit},X(a^i(1),\dots,a^i(k_i-1)),0}(\mathbb{L}^{m-i}T) \tag{12}$$

*où $b_k$ et $M$ sont donnés définition 2.6, $k_i$ et $i_0$ sont donnés définition 4.1, $D(m)_p$, $D_{0,i}(m)_p$, $D_{q,i}(m)_p$, $I_{p,\underline{l}}$ et $D_{q,i,I}(m)_p$ sont donnés définition 5.1, $k_{\underline{l}}$ est donné définition 5.3, $N_{p,l_1,\dots,l_m}$ est donné définition 6.4 et $X(a^i(1),\dots,a^i(k_i-1))$ est le germe d'hypersurface à singularité quasi-ordinaire d'exposants caractéristiques $a^i(j)=(a_1(j),\dots,a_{i-1}(j))$ pour $1\leq j\leq k_i-1$.*

***Remarque 8.2***. — Ce résultat nous donne une expression des deux séries de Poincaré motiviques d'un germe d'hypersurface irréductible à singularitè quasi-ordinaire de dimension $m$ en termes d'autres séries de Poincaré motiviques de germes de dimensions strictement inférieures à $m$. Malheureusement la description des cônes $D_{q,\,i}(m)_p$ (pour $0 \leq q \leq m$) est trop compliquée pour pouvoir donner une expression plus effective de ces séries dans des cas qui pourraient sembler simples (par exemple dans le cas d'un germe défini par une équation de la forme $Y^n - X_1^{a_1} \ldots X_m^{a_m} = 0$).

## 9. Cas où les $a_i(1) \geq 1$ pour tout $i$

Dans ce cas beaucoup de choses peuvent être simplifiées, et nous pouvons donner une expression explicite de ces deux séries. Tous les $k_j$ sont égaux à 1, et $b_1(\underline{l}) \geq l_j$ pour tout $j$, si $l_j > 0$ pour tout $j$. En particulier, si $l_i > p$, alors $b_{k_i}(\underline{l}) > p$ et seul le cas C1 est à considérer ici. D'autre part $e = 0$. Nous avons

$$D(m)_p = \{(l_1, \ldots, l_m) \in \operatorname{Ker} M \,/\, 0 < l_i \leq p\}\,.$$

Notons

$$\mathbb{E}_p := \sum_{\substack{(l_1,\ldots,l_m)\in \operatorname{Ker} M\\ 0<l_i\leq p}} \mathbb{L}^{-\sum_{i=1}^m l_i}$$

$$\text{et } \mathbb{F}_{k,p} := \sum_{\substack{(l_1,\ldots,l_m)\in \operatorname{Ker} M\\ 0<l_i\leq p\\ b_k(\underline{l})\leq p}} \mathbb{L}^{-\sum_{i=1}^m l_i}\,.$$

Nous avons alors

$$P^{\text{tore}}_{\text{géom}}(a(1), \ldots, a(g))(T) = (\mathbb{L}-1)^m \sum_{p\geq 0} \mathbb{E}_p (\mathbb{L}^m T)^p$$

et

(13)

$$\begin{aligned} P^{\text{tore}}_{\text{arit}}(a(1), \ldots, a(g))(T) &= \sum_{k=1}^{g-1} (\mathbb{L}-1)^m \sum_{p\geq 0} (\mathbb{L}^m T)^p \,(\mathbb{F}_{k,\,p} - \mathbb{F}_{k+1,\,p}) \\ &\quad + (\mathbb{L}-1)^m \sum_{p\geq 0} (\mathbb{L}^m T)^p \mathbb{F}_{g,\,p} + (\mathbb{L}-1)^m \sum_{p\geq 0} (\mathbb{E}_p - \mathbb{F}_{1,\,p})(\mathbb{L}^m T)^p \\ &= (\mathbb{L}-1)^m \sum_{p\geq 0} \mathbb{E}_p (\mathbb{L}^m T)^p + (\mathbb{L}-1)^m \sum_{k=1}^{g} \left(\frac{n_k - n_{k-1}}{n}\right) \sum_{p\geq 0} \mathbb{F}_{k,p} (\mathbb{L}^m T)^p\,. \end{aligned}$$

Pour calculer ces séries, notons

$$\mathbb{E}_{\underline{p}} = \mathbb{E}_{p_1,\ldots,\,p_m} = \sum_{\substack{(l_1,\ldots,\,l_m)\in \operatorname{Ker} M\\ 0<l_i\leq p_i}} \mathbb{L}^{-\sum_{i=1}^m l_i}.$$

Nous avons alors le lemme suivant :

***Lemme 9.1***. — *Nous avons pour tous $n \geq p_k > 0$, $p > 0$, $k \geq 0$ :*

1. $\mathbb{E}_p = \mathbb{E}_{p,\ldots,p}$ ,
2. $\mathbb{E}_{p_1,\ldots,\,p_j+kn,\ldots,\,p_m} = \mathbb{E}_{p_1,\ldots,\,kn,\ldots,\,p_m} + \mathbb{L}^{-kn}\mathbb{E}_{p_1,\ldots,\,p_j,\ldots,\,p_m}$ ,

3. $\mathbb{E}_{p_1,\ldots,kn,\ldots,p_m} = \frac{1-\mathbb{L}^{-kn}}{1-\mathbb{L}^{-n}}\mathbb{E}_{p_1,\ldots,n,\ldots,p_m}$ ,
4. $\mathbb{E}_{p_1,\ldots,p_j+kn,\ldots,p_m} = \frac{1-\mathbb{L}^{-kn}}{1-\mathbb{L}^{-n}}\mathbb{E}_{p_1,\ldots,n,\ldots,p_m} + \mathbb{L}^{-kn}\mathbb{E}_{p_1,\ldots,p_j,\ldots,p_m}$ .

*Notons* $\mathcal{S}_m := \{s : \{1,\ldots,m\} \longrightarrow \{0,1\}^m\}$*. Pour tout* $s \in \mathcal{S}_m$*, nous noterons* $|s|$ *le nombre d'éléments de* $\{1,\ldots,m\}$ *d'image* $1$ *par* $s$*. Nous avons alors*

$$\mathbb{E}_{\underline{p}+n\underline{k}} = \sum_{s\in\mathcal{S}_m} \mathbb{L}^{\sum_i -k_i n(1-s(i))}\mathbb{E}_{p_1(1-s(1))+k_1 ns(1),\ldots,p_m(1-s(m))+k_m ns(m)} \ .$$

*En notant* $\mathbb{E}_{p,s} = \mathbb{E}_{p(1-s(1))+ns(1),\ldots,p(1-s(m))+ns(m)}$*, nous avons*

$$\mathbb{E}_{p+nk} = \sum_{s\in\mathcal{S}_m}\left(\frac{\mathbb{L}^{-kn}-1}{\mathbb{L}^{-n}-1}\right)^{|s|}\mathbb{L}^{-kn(m-|s|)}\mathbb{E}_{p,s} \ .$$

*Démonstration.* — Le 1. découle des définitions.

Il faut tout d'abord remarquer que nous avons $M(l_1,\ldots,l_m) = 0$ si et seulement si $M(l_1,\ldots,l_i+ln,\ldots,l_m) = 0$ pour tout $i$ et tout $l$ entier. Le 2. découle alors de l'égalité suivante :

$$\mathbb{E}_{p_1,\ldots,p_j+kn,\ldots,p_m} = \sum_{\substack{(l_1,\ldots,l_m)\in\operatorname{Ker} M\\ 0<l_i\leq p_i \ \ i\neq j\\ 0<l_j\leq p_j+kn}} \mathbb{L}^{\sum_{i=1}^m -l_i}$$

$$= \sum_{\substack{(l_1,\ldots,l_m)\in\operatorname{Ker} M\\ 0<l_i\leq p_i \ \ i\neq j\\ 0<l_j\leq kn}} \mathbb{L}^{\sum_{i=1}^m -l_i} + \mathbb{L}^{-kn}\sum_{\substack{(l_1,\ldots,l_m)\in\operatorname{Ker} M\\ 0<l_i\leq p_i}} \mathbb{L}^{\sum_{i=1}^m -l_i} \ .$$

Le 3. s'en déduit par récurrence et le 4. en compilant 2. et 3. □

***Corollaire 9.2.*** — *Nous avons*

$$\sum_{p\geq 0}\mathbb{E}_p(\mathbb{L}^m T)^p = \sum_{p=1}^n\sum_{s\in\mathcal{S}_m}(\mathbb{L}^m T)^p\frac{\mathbb{E}_{p,s}}{(\mathbb{L}^{-n}-1)^{|s|}}\sum_{j=0}^{|s|}\frac{\binom{|s|}{j}(-1)^j}{1-(\mathbb{L}^j T)^n} \ .$$

*Démonstration.* — Nous avons

$$\sum_{p\geq 0}\mathbb{E}_p(\mathbb{L}^m T)^p = \sum_{p=1}^n\sum_{k\geq 0}(\mathbb{L}^m T)^{kn+p}\mathbb{E}_{p+kn}$$

$$= \sum_{p=1}^n(\mathbb{L}^m T)^p\sum_{s\in\mathcal{S}_m}\sum_{k\geq 0}(\mathbb{L}^m T)^{nk}\left(\frac{\mathbb{L}^{-kn}-1}{\mathbb{L}^{-n}-1}\right)^{|s|}\mathbb{L}^{-kn(m-|s|)}\mathbb{E}_{p,s} \ .$$

Le résultat est alors direct en développant $\left(\frac{\mathbb{L}^{-kn}-1}{\mathbb{L}^{-n}-1}\right)^{|s|}$ et sommant sur $k$. □

Pour calculer les séries où apparaissent les $\mathbb{F}_{k,p}$ nous pouvons remarquer, comme tous les $a_i(k) \geq 1$, que

$$\mathbb{F}_{k,p} = \sum_{\substack{(l_1,\ldots,l_m)\in \operatorname{Ker} M \\ 0<l_i \\ b_k(\underline{l})\leq p}} \mathbb{L}^{-\sum_{i=1}^m l_i} .$$

Donc nous avons

$$\begin{aligned}\sum_{p\geq 0} \mathbb{F}_{k,p}(\mathbb{L}^m T)^p &= \sum_{p\geq 0} \sum_{\substack{(l_1,\ldots,l_m)\in \operatorname{Ker} M \\ 0<l_i \\ b_k(\underline{l})\leq p}} \mathbb{L}^{-\sum_{i=1}^m l_i} (\mathbb{L}^m T)^p \\
&= \sum_{\substack{(l_1,\ldots,l_m)\in \operatorname{Ker} M \\ 0<l_i}} \sum_{p\geq b_k(\underline{l})} \mathbb{L}^{-\sum_{i=1}^m l_i} (\mathbb{L}^m T)^p \\
&= \sum_{\substack{(l_1,\ldots,l_m)\in \operatorname{Ker} M \\ 0<l_i}} \frac{\mathbb{L}^{-\sum_{i=1}^m l_i} (\mathbb{L}^m T)^{b_k(\underline{l})}}{1-\mathbb{L}^m T} \\
&= \sum_{r_i\geq 0} \sum_{\substack{(l_1,\ldots,l_m)\in \operatorname{Ker} M \\ 0<l_i\leq n}} \frac{\mathbb{L}^{-\sum_{i=1}^m l_i} (\mathbb{L}^m T)^{b_k(\underline{l})}}{1-\mathbb{L}^m T} \mathbb{L}^{-n\sum_{i=1}^m r_i} (\mathbb{L}^m T)^{n b_k(\underline{r})} .\end{aligned}$$

Car $(l_1+nr_1,\ldots,l_m+nr_m) \in \operatorname{Ker} M \iff (l_1,\ldots,l_m) \in \operatorname{Ker} M$. Nous avons donc

$$\sum_{p\geq 0} \mathbb{F}_{k,p}(\mathbb{L}^m T)^p =$$

$$= \sum_{\substack{(l_1,\ldots,l_m)\in \operatorname{Ker} M \\ 0<l_i\leq n}} \frac{\mathbb{L}^{-\sum_{i=1}^m l_i} (\mathbb{L}^m T)^{b_k(\underline{l})}}{1-\mathbb{L}^m T} \prod_{i=1}^m \frac{1}{1-\mathbb{L}^{n(ma_i(k)-1)} T^{na_i(k)}} .$$

Nous pouvons alors en déduire le

***Théorème 9.3***. — *Soit $(X, 0)$ est un germe irréductible à singularité quasi-ordinaire tel que $a_i(1) \geq 1$ pour tout $i \in \{1,\ldots,m\}$. Alors nous avons*

$$\begin{aligned}P_{\text{géom},X,0}(T) = {} & \sum_{j=1}^m \binom{m}{j} \frac{(-1)^{j+1}}{1-\mathbb{L}^{m-j}T} \\
& + (\mathbb{L}-1)^m \sum_{p=1}^n \sum_{s\in \mathcal{S}_m} (\mathbb{L}^m T)^p \frac{\mathbb{E}_{p,s}}{(\mathbb{L}^{-n}-1)^{|s|}} \sum_{j=0}^{|s|} \frac{\binom{|s|}{j}(-1)^j}{1-(\mathbb{L}^j T)^n} ,\end{aligned} \tag{14}$$

$$
\begin{aligned}
P_{\mathrm{arit},X,0}(T) &= \sum_{j=1}^{m}\binom{m}{j}\frac{(-1)^{j+1}}{1-\mathbb{L}^{m-j}T}\\
&+\frac{1}{n}(\mathbb{L}-1)^m\sum_{p=1}^{n}\sum_{s\in\mathcal{S}_m}(\mathbb{L}^mT)^p\frac{\mathbb{E}_{p,s}}{(\mathbb{L}^{-n}-1)^{|s|}}\sum_{j=0}^{|s|}\frac{\binom{|s|}{j}(-1)^j}{1-(\mathbb{L}^jT)^n}\\
&+(\mathbb{L}-1)^m\sum_{k=1}^{g}\left(\frac{n_k-n_{k-1}}{n}\right)\frac{\mathbb{H}_k}{1-\mathbb{L}^mT}\prod_{i=1}^{m}\frac{1}{1-\mathbb{L}^{n(ma_i(k)-1)}T^{na_i(k)}}
\end{aligned}
\tag{15}
$$

*où $\mathbb{E}_{p,s}$ et $\mathbb{E}_{\underline{p},s}$ sont définis lemme 9.1 et*

$$\mathbb{H}_k := \sum_{\substack{(l_1,\ldots,l_m)\in Ker\, M\\ 0<l_i\leq n}} \mathbb{L}^{-\sum_{i=1}^{m} l_i}(\mathbb{L}^mT)^{b_k(\underline{l})}$$

***Remarque 9.4***. — En particulier, les pôles de la série géométrique sont tous de la forme $\mathbb{L}^{-j}$ pour un entier $j$ compris entre 0 et $m$, alors que certains pôles de la série arithmétique peuvent dépendre des exposants caractéristiques (voir par exemple le cas des courbes planes).

### 9.1. Exemples

9.1.1. *Courbes planes.* — Considérons un germe de courbe plane de caractéristique $(\beta_0 = n, \beta_1, \ldots, \beta_g)$ [**24**]. Nous avons alors $n = n$, $m = 1$ et $g = g$. De plus

$$l \in \operatorname{Ker} M \iff l \in n\mathbb{Z}\ .$$

Nous obtenons alors

$$P_{\mathrm{géom},X,0}(T) = \frac{1}{1-T} + \frac{\mathbb{L}-1}{1-\mathbb{L}T}\frac{T^n}{1-T^n}$$

$$\text{et } P_{\mathrm{arit},X,0}(T) = \frac{1}{1-T} + \frac{\mathbb{L}-1}{1-\mathbb{L}T}\sum_{k=0}^{g}\left(\frac{n_k-n_{k-1}}{n}\right)\frac{\mathbb{L}^{\beta_k-n}T^n}{1-\mathbb{L}^{\beta_k-n}T^n}.$$

On retrouve là le résultat de Denef et Loeser [**7**].

9.1.2. *L'hypersurface définie par $Z^2 - X^3Y^3 = 0$.* — Soit l'hypersurface de $\mathbb{C}^3$ définie par $Z^2 - X^3Y^3 = 0$. La série fractionnaire associée est $\xi = X^{3/2}Y^{3/2}$. Dans ce cas nous avons $n = 2$, $m = 2$, $g = 1$, $n_1 = 2$ et

$$
\begin{array}{rcl}
M\ :\ \mathbb{Z}^2 & \longrightarrow & \frac{\mathbb{Z}}{2\mathbb{Z}}\\
(l_1,\, l_2) & \longmapsto & 3l_1 + 3l_2
\end{array}
$$

Nous avons

$$(l_1,\, l_2) \in \operatorname{Ker}\, M \iff l_1 + l_2 \in \frac{\mathbb{Z}}{2\mathbb{Z}}\ .$$

Soit $p \in \{1,\, 2\}$, nous avons

$$\mathbb{E}_{1,(0,0)} = \mathbb{E}_{1,(1,0)} = \mathbb{E}_{1,(0,1)} = \mathbb{L}^{-2},$$

$$\mathbb{E}_{2,(0,0)} = \mathbb{E}_{2,(0,1)} = \mathbb{E}_{2,(1,0)} = \mathbb{E}_{2,(1,1)} = \mathbb{E}_{1,(1,1)} = \mathbb{L}^{-2} + \mathbb{L}^{-4} .$$

D'où

$$(1+\mathbb{L})^2 P_{\text{géom},X,0}(T) = \frac{-2\mathbb{L}}{1-T} + \frac{(1+\mathbb{L})^2}{1-\mathbb{L}T} - \frac{(1-\mathbb{L})^2}{1+\mathbb{L}T} + \frac{1+\mathbb{L}^2}{1-\mathbb{L}^2 T} .$$

Dans ce cas, la série géométrique a 4 pôles en $T$ : 1, $\mathbb{L}^{-1}$, $-\mathbb{L}^{-1}$ et $\mathbb{L}^{-2}$.

## Références


[1] S. Abhyankar – On the ramification of algebraic functions, *Amer. J. Math.* **77** (1955), p. 575–592.

[2] E. Artal Bartolo, P. Cassou-Noguès, I. Luengo & A. Melle Hernández – Quasi-ordinary power series and their zeta functions, *Mem. Amer. Math. Soc.* **178** (2005).

[3] H. Cobo Pablos & P. D. González-Pérez – en préparation.

[4] J. Denef – The rationality of the Poincaré series associated to the $p$-adic points on a variety, *Invent. Math.* **77** (1984), p. 1–23.

[5] J. Denef & F. Loeser – Motivic Igusa zeta functions, *J. Algebraic Geom.* **7** (1998), p. 505–537.

[6] ———, Germs of arcs on singular algebraic varieties and motivic integration, *Invent. Math.* **135** (1999), p. 201–232.

[7] ———, Definable sets, motives and $p$-adic integrals, *J. Amer. Math. Soc.* **14** (2001), p. 429–469.

[8] ———, Geometry on arc spaces of algebraic varieties, in *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, Progr. Math., vol. 201, Birkhäuser, 2001, p. 327–348.

[9] W. Fulton – *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, 1993, The William H. Roever Lectures in Geometry.

[10] Y.-N. Gau – Embedded topological classification of quasi-ordinary singularities, *Mem. Amer. Math. Soc.* **74** (1988), p. 109–129.

[11] P. D. González-Pérez – The semigroup of a quasi-ordinary hypersurface, *J. Inst. Math. Jussieu* **2** (2003), p. 383–399.

[12] ———, Toric embedded resolutions of quasi-ordinary hypersurface singularities, *Ann. Inst. Fourier (Grenoble)* **53** (2003), p. 1819–1881.

[13] M. J. Greenberg – Rational points in Henselian discrete valuation rings, *Publ. Math. I.H.É.S.* **31** (1966), p. 59–64.

[14] T. C. Hales – What is motivic measure ?, *Bull. Amer. Math. Soc. (N.S.)* **42** (2005), p. 119–135.

[15] M. Kontsevich – Cours à Orsay, 7 décembre 1995.

[16] M. Lejeune-Jalabert – La série de Denef-Loeser d'une branche plane, notes informelles, 2001.

[17] M. Lejeune-Jalabert & A. J. Reguera – The Denef-Loeser series for toric surface singularities, in *Proceedings of the International Conference on Algebraic Geometry and Singularities (Spanish) (Sevilla, 2001)*, vol. 19, 2003, p. 581–612.

[18] M. Lejeune-Jalabert & A. J. Reguera-López – Arcs and wedges on sandwiched surface singularities, *Amer. J. Math.* **121** (1999), p. 1191–1213.

[19] J. Lipman – Quasi-ordinairy singularities of embedded surfaces, Thèse, Université de Harvard, 1965.

[20] ———, Topological invariants of quasi-ordinary singularities, *Mem. Amer. Math. Soc.* **74** (1988), p. 1–107.

[21] J. Nicaise – Arcs and resolution of singularities, *Manuscripta Math.* **116** (2005), p. 297–322.

[22] ———, Motivic generating series for toric surface singularities, *Math. Proc. Cambridge Philos. Soc.* **138** (2005), p. 383–400.

[23] J. Pas – Uniform $p$-adic cell decomposition and local zeta functions, *J. reine angew. Math.* **399** (1989), p. 137–172.

[24] O. Zariski – *Le problème des modules pour les branches planes*, École Polytechnique, 1973, Cours donné au Centre de Mathématiques de l'École Polytechnique, Paris, Octobre-Novembre 1973.

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# ÉQUATIONS AUX $q$-DIFFÉRENCES ET FIBRÉS VECTORIELS HOLOMORPHES SUR LA COURBE ELLIPTIQUE $\mathbf{C}^*/q^{\mathbf{Z}}$

*par*

Jacques Sauloy

***Résumé.*** — Nous présentons diverses applications des fibrés vectoriels aux équations aux $q$-différences, dans la lignée de la correspondance de Weil.

***Abstract* (Equations in $q$-differences and holomorphic vector bundles over the elliptic curve $\mathbb{C}^*/q^{\mathbb{Z}}$)**

We present some applications of vector bundles to $q$-difference equations, in continuation of Weil's correspondence.

## 1. Introduction

Divers fils mathématiques et historiques relient les équations aux $q$-différences aux *fibrés vectoriels holomorphes sur une courbe elliptique* (1). Ces dernières années, ces derniers sont apparus à plusieurs reprises comme un cadre naturel pour des problèmes de classification et de théorie de Galois (problème de Riemann-Hilbert). Il est peut-être temps de survoler et de mettre en ordre des résultats épars, dont certains ont été énoncés dans diverses conférences (Groningen, Conférence Ramis, Lisbonne, Luminy, Kyoto, Tordesillas) mais n'ont jamais été publiés. Ces résultats ont été très largement motivés par les travaux de Ramis, Zhang et l'auteur et l'une des raisons de non publication est le blocage sur une question difficile, celle du « problème global » (section 4). Cependant les percées des dernières années sur le problème local ([**28**], [**24**] et [**25**]) nous encouragent.

L'article comprend peu de résultats extraordinaires mais permet un éclairage nouveau de la théorie. Il permet en particulier de proposer une énigme (apparition de la dualité de Serre) et un problème ouvert (le problème global mentionné ci-dessus).

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***Mots clefs.*** — Correspondance de Weil, équations aux $q$-différences, fibrés vectoriels, courbes elliptiques.

(1) Dans tout le texte, nous dirons « fibré » pour « fibré vectoriel holomorphe » (sur une surface de Riemann).

Nous n'évoquons pas deux autres pistes, celle de la *confluence* ([**33**], [**34**]) et celle des *déformations isomonodromiques* ([**37**]).

Nous nous occupons principalement d'équations aux $q$-différences et ne sommes venus aux fibrés vectoriels que par nécessité : nous ne prétendons à aucune expertise dans ce domaine, et espérons au contraire que les spécialistes nous apporteront leurs lumières.

Ce fut un plaisir tout particulier de parler de tout cela à la conférence en l'honneur de José-Manuel Aroca, Gran Jefe Capitán Pirata, en présence de tant d'amis de Valladolid et d'ailleurs. À Valladolid et à Tordesillas, on rit beaucoup avant, pendant et après les exposés (parfois, à la place) parce que le plaisir de faire des mathématiques s'y exprime plus librement qu'ailleurs. Merci pour tout cela à Jose-Manuel, l'âme du groupe.

J'avais préfacé mon exposé (en anglais) à Tordesillas de la dédicace suivante :

*With a special thought for Jean Giraud,*
*who, a long time ago, guided my first steps*
*into the wild world of singularities ...*

Jean Giraud, qui n'avait pu assister à la conférence, nous a quittés le 27 mars. Je partage ici ma tristesse avec nos amis espagnols.

**1.1. Apparition des fibrés dans la théorie des équations fonctionnelles.** — Le théorème clé dans la résolution par Birkhoff du problème de Riemann-Hilbert ([**3**]) est un théorème de factorisation de matrice holomorphe. Dans [**30**], [**31**], Röhrl a interprété ce théorème en termes de *trivialité de fibré vectoriel* (voir aussi [**10**]). Dans [**23**], van der Put et Singer donnent de cette factorisation une preuve moderne, qui s'appuie directement sur la cohomologie des fibrés vectoriels sur une surface de Riemann, et l'appliquent (dans la droite ligne de [**3**]) aux équations aux différences et aux $q$-différences. Auparavant, Praagman, un élève de van der Put, avait invoqué la trivialité méromorphe des fibrés pour démontrer l'existence d'un système fondamental de solutions méromorphes sur $\mathbf{C}^*$ pour les équations aux différences et aux $q$-différences ([**20**]). Cependant, dans tous ces cas, les fibrés n'interviennent qu'à travers leurs propriétés cohomologiques, et non en tant qu'objets géométriques.

Dans [**2**], Baranovsky et Ginzburg étudient la classification formelle des équations aux $q$-différences fuchsiennes (dans une autre terminologie, liée aux groupes de lacets). Ils caractérisent chaque classe *formelle* à l'aide d'un objet *analytique*, un fibré vectoriel sur la courbe elliptique $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$. Sur une suggestion de Kontsevitch, ils en déduisent le groupe de Galois local. Indépendamment, l'auteur a obtenu dans [**34**] la classification (formelle ou analytique, ce qui revient au même dans ce cas) des équations aux $q$-différences fuchsiennes par des fibrés plats, d'où se déduit la description complète du groupe de Galois local et celle moins détaillée du groupe de Galois global (cas abélien régulier).

Nous allons, dans cette introduction, suivre le chemin inverse et montrer comment la description des fibrés sur une courbe (resp. une courbe elliptique) se traduit naturellement en termes d'équations fonctionnelles (resp. d'équations aux $q$-différences).

1.1.1. *La correspondance de Weil.* — Dans [**41**], Weil propose, sous le nom de $G$-diviseurs, une généralisation non-abélienne de la notion de diviseur sur une surface de Riemann. Ces $G$-diviseurs ne sont autres que des fibrés vectoriels avant la lettre. Selon la présentation « moderne » de [**14**] (et sous une forme simplifiée), cela donne ce qui suit.

1.1.1.1. *Fibrés équivariants.* — Soit $E$ une surface de Riemann, et soit $\tilde{E}$ son revêtement universel, qui est donc également une surface de Riemann. Nous noterons $\pi : \tilde{E} \to E$ la projection canonique.

Soit $\mathcal{F}$ un fibré (vectoriel holomorphe) sur $E$. En relevant $\mathcal{F}$ à $\tilde{E}$ par $\pi$, on obtient un fibré $\tilde{\mathcal{F}} = \pi^*\mathcal{F}$, qui est trivial puisque $\tilde{E}$ est simplement connexe. On écrit donc $\tilde{\mathcal{F}} = \tilde{E} \times V$, où $V$ est un $\mathbf{C}$-espace vectoriel de dimension finie. Provenant de $E$, ce fibré trivial est muni d'une *action équivariante* du groupe $G = \mathrm{Aut}(\tilde{E}/E) = \pi_1(E)$ (nous ne précisons pas le point-base pour le groupe fondamental $\pi_1$, qui n'apparaitra qu'en tant que groupe des automorphismes du revêtement). Le mot « action équivariante » signifie ici « action sur $\tilde{E} \times V$ qui commute avec l'action sur $\tilde{E}$ » (on dit aussi que $\tilde{\mathcal{F}}$ est un fibré équivariant). Une telle action est complètement décrite par l'action naturelle $(\gamma, x) \mapsto \gamma.x$ de $G$ sur $\tilde{E}$ et par la donnée d'une application holomorphe (en la seconde variable) :

$$A : G \times \tilde{E} \longrightarrow \mathcal{GL}(V).$$

Tout $\gamma \in G$ opère alors sur $\tilde{\mathcal{F}} = \tilde{E} \times V$ par l'application :

$$(x, X) \mapsto (\gamma.x, A(\gamma, x)X)\,.$$

Pour que ce soit bien une opération de groupe, il faut, et il suffit, que soit réalisée une condition de cocycle :

$$\forall \gamma, \gamma' \in G\ ,\ \forall x \in \tilde{E}\ ,\ A(\gamma'\gamma, x) = A(\gamma', \gamma.x)A(\gamma, x).$$

On peut également exprimer, par une condition de cobord, la trivialité du fibré $\mathcal{F}$ de départ ou, plus généralement, à quelle condition deux cocycles représentent des fibrés isomorphes.

Un morphisme $\mathcal{F} \to \mathcal{F}'$ de fibrés sur $E$ se relève en un morphisme $\tilde{\mathcal{F}} = \tilde{E} \times V \to \tilde{\mathcal{F}}' = \tilde{E} \times V'$ de fibrés sur $\tilde{E}$ compatible avec la structure ci-dessus : si $\tilde{\mathcal{F}}$ et $\tilde{\mathcal{F}}'$ sont respectivement décrits par les cocycles $A$ et $A'$, le morphisme $\tilde{\mathcal{F}} \to \tilde{\mathcal{F}}'$ est de la forme $(x, X) \mapsto (x, F(x)X)$, où $F$ est une application holomorphe de $\tilde{E}$ dans $\mathcal{L}(V, V')$, qui satisfait à la condition suivante :

$$\forall \gamma \in G\ ,\ \forall x \in \tilde{E}\ ,\ F(\gamma.x)A(\gamma, x) = A'(\gamma, x)F(x).$$

1.1.1.2. *Description géométrique.* — Supposons réciproquement donné le cocycle holomorphe (en la seconde variable) $A : \pi_1(E) \times \tilde{E} \to \mathscr{GL}(V)$. On lui associe la relation d'équivalence $\sim_A$ sur le fibré trivial $\tilde{F} = \tilde{E} \times V$ engendrée par les relations : $(x, X) \sim_A (\gamma.x, A(\gamma, x)X)$ : la relation $\sim_A$ provient donc d'une action équivariante de $\pi_1(E)$ sur $\tilde{E}$. En un sens évident, cette relation est compatible avec la relation $\sim$ sur $\tilde{E}$ induite par l'action de $\pi_1(E)$. Le fibré sur $E$ associé, que nous noterons $\tilde{F}_A$, s'obtient par passage au quotient de la projection $\tilde{F} = \tilde{E} \times V \to \tilde{E}$ par ces relations d'équivalence :

$$\mathscr{F}_A = \frac{\tilde{E} \times V}{\sim_A} \longrightarrow E = \frac{\tilde{E}}{\sim}\cdot$$

On peut alors décrire le faisceau des sections de $\mathscr{F}_A$. Soit $V$ un ouvert de $E$. Alors l'espace des sections de $\mathscr{F}_A$ sur $V$ est :

$$\Gamma(V, \mathscr{F}_A) = \{X : \pi^{-1}(V) \to V \text{ holomorphes } \mid \\ \forall x \in \pi^{-1}(V)\ ,\ \forall \gamma \in \pi_1(E)\ ,\ X(\gamma.x) = A(\gamma, x)X(x)\}.$$

**Exemple.** — Prenons $E = \mathbf{C}^*$. Alors $\tilde{E} = \mathbf{C}$ sur lequel $\pi_1(E) = \mathbf{Z}$ agit par translations, et la projection canonique $\pi : \tilde{E} \to E$ est ici $x \mapsto e^{2\mathrm{i}\pi x}$. La condition de cocycle entraine que $A$ est entièrement déterminée par la matrice $A(1, x)$. Notons (abusivement) $A(x) = A(1, x)$. De même, la condition qui définit les sections peut se tester simplement en prenant $\gamma = 1$ :

$$\Gamma(V, \mathscr{F}_A) = \{X : \pi^{-1}(V) \to V \text{ holomorphes } \mid \forall x \in \pi^{-1}(V)\ ,\ X(x+1) = A(x)X(x)\}.$$

On voit bien la parenté avec les équations fonctionnelles.

Si l'on note $\underline{1} = \mathscr{F}_1$ (« objet unité ») le fibré en droites trivial sur $E$, associé au cocycle trivial $(\gamma, x) \mapsto 1 \in \mathscr{GL}(\mathbf{C})$, le lecteur pourra vérifier que les morphismes de $\underline{1}$ dans un fibré $\mathscr{F}_A$ quelconque s'identifient aux sections globales de $\mathscr{F}_A$.

1.1.1.3. *Fibrés plats et représentations de* $\pi_1(E)$. — Un cas important est celui où, à isomorphisme près, on peut supposer $A(\gamma, x)$ indépendant de $x \in \tilde{E}$ : on l'écrit donc $A(\gamma)$, et la condition de cocycle dit alors que $\gamma \mapsto A(\gamma)$ est une représentation de $\pi_1(E)$ dans $\mathscr{GL}(V)$. Un tel fibré est appelé *plat* ([**16**]). Les fibrés plats admettent une caractérisation topologique : les classes de Chern sur leurs facteurs indécomposables sont nulles ; et une caractérisation différentielle : on peut les munir d'une connexion holomorphe. *Nous n'aurons pas l'usage de ces caractérisations* (2). On obtient ainsi la célèbre *correspondance de Weil* entre fibrés plats et représentations du groupe fondamental.

Il faut cependant prendre garde que cette correspondance n'est pas une équivalence entre la catégorie des fibrés plats sur $E$ et celle des représentations de $\pi_1(E)$. Soient en effet $A : \pi_1(E) \to \mathscr{GL}(V)$ et $A' : \pi_1(E) \to \mathscr{GL}(V')$ deux telles représentations, et soient $\mathscr{F}_A$ et $\mathscr{F}_{A'}$ les fibrés plats qui leur correspondent respectivement. Un morphisme de

(2) Van der Put et Reversat utilisent la seconde dans [**22**], voir là-dessus la section 2.2.

$\mathcal{F}_A$ dans $\mathcal{F}_{A'}$ est décrit comme une application holomorphe $F : \tilde{E} \to \mathcal{L}(V, V')$, telle que :

$$\forall \gamma \in G \ , \ \forall x \in \tilde{E} \ , \ F(\gamma.x)A(\gamma) = A'(\gamma)F(x).$$

Si $F$ est constant sur $\tilde{E}$, c'est bien un morphisme de représentations, mais pas autrement. Nous en verrons un exemple à la section suivante, et des conséquences pour le groupe de Galois à la section 2.1.1.

1.1.2. *Le cas des fibrés sur une courbe elliptique*

1.1.2.1. *Fibrés sur* $\mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$. — Prenons pour $E$ la courbe elliptique [(3)] $\mathbf{C}/\Lambda_\tau$, où $\operatorname{Im} \tau < 0$ et $\Lambda_\tau = \mathbf{Z} + \mathbf{Z}\tau$. (Nous poserons plus loin $q = e^{2i\pi\tau}$ et voudrons avoir $|q| > 1$.) Ici, $\pi_1(E) = \Lambda_\tau$ agit sur $\tilde{E} = \mathbf{C}$ par translations. Notons encore $\pi : \mathbf{C} \to E$ la projection canonique. Pour tout cocycle $A$, notons $A_1(x) = A(1, x)$ et $A_\tau(x) = A(\tau, x)$. À cause de la relation de commutation $\tau + 1 = 1 + \tau$, la condition de cocycle entraine :

$$\forall x \in \mathbf{C} \ , \ A_\tau(x+1)A_1(x) = A_1(x+\tau)A_\tau(x).$$

Réciproquement, deux applications holomorphes de $\mathbf{C}$ dans $\mathcal{GL}(V)$ qui vérifient cette relation s'étendent de manière unique en un cocycle $A$ et définissent donc un fibré $\mathcal{F} = \mathcal{F}_A$ sur $E$. Les sections de ce fibré sur l'ouvert $V \subset E$ s'identifient aux solutions holomorphes sur $\pi^{-1}(V) \subset \mathbf{C}$ de l'équation fonctionnelle :

$$\forall x \in \pi^{-1}(V) \ , \ X(x+1) = A_1(x)X(x) \quad \text{et} \quad X(x+\tau) = A_\tau(x)X(x).$$

Si $\mathcal{F}' = \mathcal{F}_{A'}$ est le fibré défini par $A'_1$ et $A'_\tau$ (holomorphes de $\mathbf{C}$ dans $\mathcal{GL}(V')$), un morphisme de $\mathcal{F}$ dans $\mathcal{F}'$ est représenté par une application holomorphe de $\mathbf{C}$ dans $\mathcal{L}(V, V')$ telle que :

$$\forall x \in \mathbf{C} \ , \ F(x+1)A_1(x) = A'_1(x)F(x) \quad \text{et} \quad F(x+\tau)A'_\tau(x) = A_\tau(x)F(x).$$

Le fibré $\mathcal{F}_A$ est plat si, à isomorphisme près, on peut supposer que $A_1$ et $A_\tau$ ne dépendent pas de $x$ : $A_1, A_\tau \in \mathcal{GL}(V)$. La condition de cocycle dit alors que ces deux matrices commutent. La représentation de $\pi_1(E) = \Lambda_\tau$ associée à $\mathcal{F}_A$ par la correspondance de Weil est celle définie par $1 \mapsto A_1$ et $\tau \mapsto A_\tau$.

1.1.2.2. *Fibrés sur* $\mathbf{C}^*/q^{\mathbf{Z}}$. — Pour trivialiser le fibré $\mathcal{F}$ sur $E$, il n'est cependant pas nécessaire de le relever au revêtement universel $\mathbf{C}$. Ce revêtement se factorise en $\mathbf{C} \to \mathbf{C}/\mathbf{Z} \to \mathbf{C}/\Lambda_\tau$. Or, l'application $x \mapsto z = e^{2i\pi x}$ permet d'identifier $\mathbf{C}/\mathbf{Z}$ à la surface de Riemann ouverte $\mathbf{C}^*$. La même application permet d'identifier $E = \mathbf{C}/\Lambda_\tau$ à $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$, où $q = e^{2i\pi\tau}$ est un nombre complexe arbitraire de module $|q| > 1$. On peut alors relever le fibré $\mathcal{F}$ sur $\mathbf{E}_q$ en un fibré sur $\mathbf{C}^*$ par le revêtement $\mathbf{C}^* \to \mathbf{E}_q$. L'interêt de cette opération est que tout fibré vectoriel holomorphe sur une surface de Riemann ouverte (*i.e.* non compacte) est trivial ([**15**], théorème 3 p. 184).

Le formalisme des fibrés équivariants décrit à la section 1.1.1 s'applique alors tout aussi bien ici. Nous partirons donc maintenant de la description « de Jacobi » (ou

(3) *A priori*, la surface de riemann $E$ devrait être appelée « tore complexe », mais l'on sait que c'est essentiellement la même chose qu'une courbe elliptique.

« de Tate ») des courbes elliptiques pour fixer nos notations. Soit $q$ un complexe de module $|q| > 1$. Soit $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$. On note $\pi$ la projection canonique $\mathbf{C}^* \to \mathbf{E}_q$. C'est un revêtement, dont le groupe $\mathrm{Aut}(\mathbf{C}^*/\mathbf{E}_q)$ est $q^{\mathbf{Z}}$ agissant sur $\mathbf{C}^*$ en tant que sous-groupe.

Tout fibré $\mathcal{F}$ sur $\mathbf{E}_q$ se relève par $\pi$ en un fibré trivial $\tilde{\mathcal{F}} = \mathbf{C}^* \times V$ muni d'une action équivariante de $q^{\mathbf{Z}}$, autrement dit, d'une application holomorphe (en la seconde variable) $A : q^{\mathbf{Z}} \times \mathbf{C}^* \to \mathcal{GL}(V)$. Celle-ci satisfait la condition de cocycle suivante :

$$\forall m, n \in \mathbf{Z} \ , \ \forall z \in \mathbf{C}^* \ , \ A(q^{m+n}, z) = A(q^m, q^n z) A(q^n, z).$$

Il est aisé de voir que la donnée de $A(q, z)$ détermine $A$. Notant abusivement $A(z) = A(q, z)$, on trouve que l'on a, pour $n \geq 1$ : $A(q^n, z) = A(q^{n-1}z) \cdots A(z)$ ; et, pour $n \leq -1$ ... une formule laissée en exercice au lecteur ! Ainsi, il revient au même de se donner un cocycle $A$ ou une application holomorphe $A : \mathbf{C}^* \to \mathcal{GL}(V)$ ; et une telle fonction matricielle $A$ définit un fibré $\mathcal{F}_A$ sur $\mathbf{E}_q$. Ce dernier peut être construit géométriquement ainsi :

$$\mathcal{F}_A = \frac{\mathbf{C}^* \times V}{\sim_A} \longrightarrow \mathbf{E}_q = \frac{\mathbf{C}^*}{\sim},$$

où les relations d'équivalences sont définies par $(z, X) \sim_A (qz, A(z)X)$ et $z \sim qz$. Une section de $\mathcal{F}_A$ sur l'ouvert $V \subset \mathbf{E}_q$ s'identifie à une solution holomorphe sur $\pi^{-1}(V)$ de l'*équation aux q-différences* :

$$X(qz) = A(z)X(z).$$

Soient $A : \mathbf{C}^* \to \mathcal{GL}(V)$, $A' : \mathbf{C}^* \to \mathcal{GL}(V')$ deux telles applications holomorphes et $\mathcal{F} = \mathcal{F}_A$, $\mathcal{F}' = \mathcal{F}_{A'}$ les fibrés sur $\mathbf{E}_q$ associés. Un morphisme de $\mathcal{F}$ dans $\mathcal{F}'$ est représenté par une application holomorphe $F : \mathbf{C}^* \to \mathcal{L}(V, V')$ telle que :

$$\forall z \in \mathbf{C}^* \ , \ F(qz)A(z) = A'(z)F(z).$$

Par exemple, si l'on note $\underline{1}$ le fibré en droites trivial [(4)], provenant de la fonction constante 1 de $\mathbf{C}^*$ dans $\mathbf{C}^* = \mathcal{GL}(\mathbf{C})$, on voit que les morphismes de $\underline{1}$ dans $\mathcal{F}$ s'identifient aux sections de $\mathcal{F}$.

**Remarque.** — Si l'on relève le fibré $\mathcal{F}$ sur $\mathbf{E}_q$ d'abord à $\mathbf{C}^*$ puis à $\mathbf{C}$, on obtient successivement $\mathbf{C}^* \times V$, (muni d'une fonction matricielle $A(z)$ sur $\mathbf{C}^*$), et $\mathbf{C} \times V$. Ainsi, le fibré trivial équivariant sur $\mathbf{C}$ décrit plus haut à l'aide des fonctions matricielles $A_1$ et $A_\tau$ sur $\mathbf{C}$, peut-il toujours être réalisé en prenant $A_1(x) = Id_V$ et $A_\tau(x) = A(e^{2i\pi x})$. Pour être précis, parmi toutes les trivialisations de l'image réciproque de $\mathcal{F}$ sur $\mathbf{C}$, l'une au moins est munie d'une action équivariante de cette nature. Cette propriété, qui traduit la trivialité des fibrés holomorphes sur $\mathbf{C}^*$, équivaut à la suivante : la fonction matricielle $A_1$ étant donnée, l'équation fonctionnelle $X(x+1) = A_1(x)X(x)$ admet une solution fondamentale (c'est-à-dire une solution à valeurs dans $\mathcal{GL}(V)$) holomorphe.

(4) La notation $\underline{1}$ désigne l'objet unité, c'est à dire le neutre pour le produit tensoriel, dans une « catégorie tannakienne ».

Si l'on se restreint aux fibrés plats, on en déduit (correspondance de Weil) que toute représentation de $\mathbf{Z}^2 \simeq \mathbf{Z} + \mathbf{Z}\tau$ est équivalente à une représentation triviale sur le premier facteur. C'est évidemment faux pour l'équivalence habituelle des représentations, mais c'est vrai au sens de l'équivalence « équivariante » décrite à la fin de la section 1.1.1.

1.1.2.3. *Relations avec la théorie classique des équations fonctionnelles.* — Comme on l'a vu, es sections de $\mathcal{F}_A$ s'identifient aux solutions de l'*équation aux q-différences* : $X(qz) = A(z)X(z)$. Il y a cependant une différence notable avec la théorie classique des équations fonctionnelles ([**3**], [**6**], [**12**], [**23**], [**33**]) : ici, la matrice $A(z)$ est holomorphe sur $\mathbf{C}^*$, et même régulière, *i.e.* son inverse $A^{-1}$ est aussi holomorphe ; alors que dans la théorie classique, la matrice $A(z)$ est rationnelle (et inversible). Ainsi :

- Pour ramener la théorie classique à celle des fibrés, il faut se débarrasser des pôles de $A$ et de $A^{-1}$.
- Pour ramener la théorie des fibrés sur $\mathbf{E}_q$ à la théorie classique des équations aux $q$-différences, il faut dompter la sauvagerie des équations (et des solutions) en 0 et en $\infty$.

Comme on le verra (section 2.1.1), la théorie fuchsienne vient naturellement se placer à l'intersection des deux points de vue.

1.1.2.4. *Le cas des équations aux différences.* — Dans le cas des équations aux $q$-différences, le corps des constantes de la théorie (solutions méromorphes sur $\mathbf{C}^*$ de l'équation triviale $f(qz) = f(z)$) s'identifie au corps des fonctions elliptiques $\mathcal{M}(\mathbf{E}_q)$, corps des donctions méromorphes sur la surface de Riemann compacte $\mathbf{E}_q$ ; celle-ci s'identifie à une courbe algébrique (courbe elliptique) et $\mathcal{M}(\mathbf{E}_q)$ à un corps de fonctions algébriques ; plus généralement, les fibrés vectoriels holomorphes sont algébriques ([**38**]).

La théorie des *équations aux différences* $X(z+1) = A(z)X(z)$ se prête également au point de vue des fibrés, mais c'est plus compliqué. En effet, la surface de Riemann appropriée est ici $E = \mathbf{C}/\mathbf{Z} \simeq \mathbf{C}^*$, mais celle-ci n'est pas compacte. Les constantes de la théorie (solutions méromorphes sur $\mathbf{C}$ de l'équation triviale $f(z+1) = f(z)$) est « très gros ». Il faut donc artificiellement imposer des conditions de croissance aux solutions pour les maitriser. Au fond, le cas des équations aux différences est une dégénerescence du cas des équations aux $q$-différences. C'est parce que l'opérateur de translation $z \mapsto z+1$ n'a qu'un point fixe sur la sphère de Riemann, alors que l'opérateur de dilatation $z \mapsto qz$ en a deux. Anne Duval ([**7**], voir aussi [**8**]) a étudié la *confluence* de ces deux points fixes en un seul et ses conséquences sur les liens entre les deux types d'équations.

**1.2. Conventions générales.** — Dans tout l'article, nous fixerons un nombre complexe $q \in \mathbf{C}$ de module $|q| > 1$. Nous noterons $\mathbf{E}_q$ la courbe elliptique $\mathbf{C}^*/q^{\mathbf{Z}}$ et $\pi : \mathbf{C}^* \to \mathbf{E}_q$ la projection canonique. L'image dans $\mathbf{E}_q$ de $a \in \mathbf{C}^*$ sera notée $\overline{a}$. La *spirale logarithmique discrète* $\pi^{-1}(\overline{a}) = aq^{\mathbf{Z}}$ sera notée $[a;q]$. On écrira alors $[a,b;q] = [a;q] \cup [b;q]$, etc.

L'opérateur de dilatation $z \mapsto qz$ de la sphère de Riemann $\mathbf{S}$ induit un automorphisme $\sigma_q$ sur de nombreux anneaux ou corps de fonctions, par la formule $(\sigma_q f)(z) = f(qz)$ (cette notation s'étend naturellement à des vecteurs ou des matrices de fonctions). Les principaux corps d'interêt sont $\mathbf{C}(z)$ (fonctions rationnelles), $\mathbf{C}(\{z\})$ (germes méromorphes en 0), $\mathbf{C}((z))$ (séries de Laurent formelles) et $\mathscr{M}(\mathbf{C}^*)$ (fonctions méromorphes sur $\mathbf{C}^*$). Plus généralement, le corps des fonctions méromorphes (resp. l'anneau des fonctions holomorphes) sur une surface de Riemann $E$ est noté $\mathscr{M}(E)$ (resp. $\mathscr{O}(E)$).

1.2.1. *Fonctions.* — Les fonctions méromorphes sur $E = \mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau)$ s'identifient aux fonctions méromorphes sur $\mathbf{C}$ admettant le réseau de périodes $\mathbf{Z}+\mathbf{Z}\tau$ : c'est la description classique du corps $\mathscr{M}(E)$ des fonctions elliptiques. Les fonctions méromorphes sur $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$ s'identifient de même aux fonctions méromorphes sur $\mathbf{C}^*$ invariantes par $\sigma_q$, ce qui donne la description *loxodromique* du corps $\mathscr{M}(\mathbf{E}_q) = \mathscr{M}(\mathbf{C}^*)^{\sigma_q}$ des fonctions elliptiques : si $q = e^{2i\pi\tau}$, il s'agit des mêmes fonctions et des mêmes corps. Toute fonction elliptique $f \in \mathscr{M}(\mathbf{E}_q)$ non triviale admet un *diviseur des zéros et des pôles sur* $\mathbf{E}_q$, noté $\operatorname{div}_{\mathbf{E}_q}(f)$. En tant que fonction $q$-invariante sur $\mathbf{C}^*$, elle admet également un diviseur sur $\mathbf{C}^*$, noté $\operatorname{div}_{\mathbf{C}^*}(f)$.

La théorie classique des fibrés en droites (ou des diviseurs) sur $E = \mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau)$ est la théorie des fonctions Theta de la forme $\Theta(\tau, x)$ ([**19**]) : par trivialisation sur le revêtement universel $\mathbf{C}$, on identifie les sections d'un tel fibré comme des fonctions sur $\mathbf{C}$. La trivialisation sur $\mathbf{C}^*$ fait de même apparaitre les *fonctions Theta de Jacobi.* Nous utiliserons principalement la fonction :

$$\theta_q(z) = \sum_{n\in\mathbf{Z}} q^{-n(n+1)/2} z^n.$$

Cette fonction, qui est holomorphe sur $\mathbf{C}^*$ y admet la factorisation (*formule du triple produit de Jacobi*) :

$$\theta_q(z) = \prod_{n\geq 1}(1-q^{-n}) \prod_{n\geq 1}(1+q^{-n}z) \prod_{n\geq 0}(1+q^{-n}z^{-1}).$$

Ses zéros sont donc les points de $[-1;q]$, comptés avec multiplicité 1. Comme elle n'a pas de pôles, son diviseur sur $\mathbf{C}^*$ est :

$$\operatorname{div}_{\mathbf{C}^*}(\theta_q) = \sum_{a\in[-1;q]} [a].$$

La fonction $\theta_q$ vérifie l'équation fonctionnelle :

$$\sigma_q \theta_q = z\theta_q.$$

C'est donc une section du fibré en droite $\mathscr{F}_{(z)}$. En tant que section, elle admet un diviseur sur $\mathbf{E}_q$ :

$$\operatorname{div}_{\mathbf{E}_q}(\theta_q) = [\overline{-1}];$$

autrement dit, bien que ses valeurs sur $\mathbf{E}_q$ ne soient pas définies, elle y admet le zéro simple $\overline{-1}$ et pas de pôles. Nous noterons, pour $a \in \mathbf{C}^*$ :

$$\theta_{q,a}(z) = \theta_q(z/a).$$

C'est une fonction holomorphe sur $\mathbf{C}^*$, qui y vérifie l'équation aux $q$-différences $\sigma_q \theta_{q,a} = \frac{z}{a}\theta_{q,a}$ (c'est donc une section de $\mathscr{F}_{(z/a)}$) et l'on a : $\mathrm{div}_{\mathbf{E}_q}(\theta_q) = [\overline{-a}]$. Elle permet de construire les *$q$-caractères* :

$$e_{q,a} = \frac{\theta_q}{\theta_{q,a}}.$$

On a $\sigma_q e_{q,a} = e_{q,a}$ (c'est donc une section de $\mathscr{F}_{(a)}$) et $\mathrm{div}_{\mathbf{E}_q}(e_{q,a}) = [\overline{-1}] - [\overline{-a}]$.

1.2.2. *Modules aux q-différences.* — Soit $K$ l'un de nos corps de fonctions, muni de l'automorphisme $\sigma_q$. Notre objet est l'étude des *équations aux q-différences* :

$$\sigma_q X = AX, \quad A \in \mathscr{GL}_n(K). \tag{1}$$

Les cas d'intérêt sont ceux des corps $\mathbf{C}(z)$ et $\mathbf{C}(\{z\})$. Pour avoir un bon formalisme algébrique, on définit un anneau de polynômes de Laurent non commutatifs :

$$\mathscr{D}_{q,K} = K\left\langle \sigma, \sigma^{-1} \right\rangle,$$

par la règle de (non-)commutation : $\sigma z = qz\sigma$. Nous noterons $\mathrm{DiffMod}\,(K, \sigma_q)$ la catégorie des $\mathscr{D}_{q,K}$-modules à gauche de longueur finie.

Un objet de $\mathrm{DiffMod}\,(K, \sigma_q)$ peut se réaliser sous la forme $M = (V, \Phi)$ où $V$ est un $K$-espace vectoriel de dimension finie et $\Phi$ un automorphisme $\sigma_q$-linéaire, c'est-à-dire tel que $\Phi(\lambda x) = \sigma_q(\lambda)\Phi(x)$ (l'action de $\sigma$ sur $M$ est alors celle de $\Phi$). Après choix d'une base, on peut même écrire $M = M_A = (K^n, \Phi_A)$, où $\Phi_A(X) = A^{-1}(\sigma_q X)$ pour une matrice $A \in \mathscr{GL}_n(K)$. Si $A \in \mathscr{GL}_n(K)$ et $B \in \mathscr{GL}_p(K)$, un morphisme de $A$ dans $B$ est une matrice $F \in \mathrm{Mat}_{p,n}(K)$ telle que :

$$(\sigma_q F)A = BF. \tag{2}$$

Si par exemple $F$ est un isomorphisme, alors on a la formule de transformation de jauge :

$$B = F[A] = (\sigma_q F)AF^{-1}.$$

Lorsque par exemple $K = \mathbf{C}(z)$, on retrouve l'équivalence rationnelle des équations aux $q$-différences, étudiée par Birkhoff. Par ailleurs, $\mathrm{DiffMod}\,(K, \sigma_q)$ est une catégorie abélienne que l'on peut munir de constructions tensorielles (par exemple [**23**] ou [**25**]), et il n'est pas très difficile de vérifier que c'est une catégorie tannakienne ([**4**]). En particulier, outre le produit tensoriel, les constructions suivantes sont disponibles.

1. Hom interne : si $M = (V, \Phi)$ et $N = (W, \Psi)$ sont deux modules, alors $\mathscr{L}_K(V, W)$ muni de $f \mapsto \Psi \circ f \circ \Phi^{-1}$ est le module noté $\underline{\mathrm{Hom}}(M, N)$. On a une adjonction : $\mathrm{Hom}\,(M, \underline{\mathrm{Hom}}(M', M'')) = \mathrm{Hom}(M \otimes M', M'')$.
2. Objet unité : c'est le module $\underline{1} = M_{(1)} = (K, \sigma_q)$, qui modélise l'équation triviale $\sigma_q f = f$. Il est neutre pour le produit tensoriel et l'on a $\underline{\mathrm{Hom}}(\underline{1}, M) = M$.
3. Dual : c'est $M^\vee = \underline{\mathrm{Hom}}(M, \underline{1})$. Si $M = (K^n, \Phi_A)$, on peut le décrire comme $M^\vee = (K^n, \Phi_{A^\vee})$, où $A^\vee = {}^t A^{-1}$.

4. Foncteur des sections : $\Gamma(M) = \mathrm{Hom}(\underline{1}, M)$ s'identifie au $K^{\sigma_q}$-espace vectoriel des points fixes de $M = (V, \Phi)$ (les $X \in V$ tels que $\Phi(X) = X$). Par exemple $\Gamma(M_A)$ est l'espace des solutions de (1) dans $K$. Le foncteur $\Gamma$ est exact à gauche. Son premier foncteur dérivé est $\Gamma^1(M) = Ext(\underline{1}, M)$.

Notons, pour un usage futur, les identifications naturelles suivantes : $\underline{\mathrm{Hom}}(M, N) = M^\vee \otimes N$ et par conséquent : $\mathrm{Hom}(M, N) = \Gamma(M^\vee \otimes N)$. Par un argument d'algèbre homologique, on en déduit $Ext(M, N) = \Gamma^1(M^\vee \otimes N)$. Les $Ext^n(M, N)$ pour $n \geq 2$ sont nuls, car $\mathcal{D}_{q,K}$ est euclidien à gauche.

Les objets de $\mathrm{DiffMod}\,(\mathbf{C}(z), \sigma_q)$, $\mathrm{DiffMod}\,(\mathbf{C}(\{z\}), \sigma_q)$ et $\mathrm{DiffMod}\,(\mathbf{C}((z)), \sigma_q)$ sont appelés *modules aux $q$-différences.* Dans la pratique, on ne distingue pas toujours le module $M_A$, l'équation (1) et la matrice $A$. Nous étudierons de près des foncteurs fibres sur ces trois catégories. La première (« cas global ») est *a priori* notre catégorie d'intérêt, mais l'étude locale préliminaire conduit à examiner $\mathrm{DiffMod}\,(\mathbf{C}(\{z\}), \sigma_q)$ (« cas local analytique ») et $\mathrm{DiffMod}\,(\mathbf{C}((z)), \sigma_q)$ (« cas formel »).

Contrairement à ce qui se fait pour les équations différentielles, ni la classification ni la théorie de Galois ne reposent fortement sur l'étude des solutions. La raison est essentiellement celle-ci. Pour construire une solution matricielle fondamentale $\mathcal{X}$ de l'équation (1), il faut un assez gros corps de fonctions, mettons $\mathcal{M}(\mathbf{C}^*)$. Les solutions vectorielles sont alors les $X = \mathcal{X}C$, où le vecteur colonne $C$ a ses coefficients dans le corps des constantes $\mathcal{M}(\mathbf{C}^*)^{\sigma_q} = \mathcal{M}(\mathbf{E}_q)$ : c'est un trop gros corps des constantes (en théorie de Galois différentielle, le corps des constantes qui fournit les invariants de classification est $\mathbf{C}$). Ces raisons et la stratégie qui en découle ont été détaillées dans [**34**], [**28**] et [**24**]. Si l'on ne tient pas à une théorie qui fournisse des invariants transcendants, alors l'approche algébrique de van der Put et Singer dans [**23**] est appropriée.

## 2. Constructions locales

**2.1. Construction géométrique générale.** — Nous noterons désormais $\mathcal{E}^{(0)} = \mathrm{DiffMod}\,(\mathbf{C}(\{z\}), \sigma_q)$ la catégorie des modules (ou équations) aux $q$ différences sur $\mathbf{C}(\{z\})$. Soit $A(z) \in \mathcal{GL}_n(\mathbf{C}(\{z\}))$. Soit $D$ un disque épointé en son centre 0 tel que $A \in \mathcal{GL}_n(\mathcal{O}(D))$, *i.e.* $A$ et $A^{-1}$ sont holomorphes sur $D$. Sur le fibré trivial $D \times \mathbf{C}^n$ (resp., sur sa base $D$), on définit une action *partielle* de $q^{\mathbf{Z}}$ par l'action de son générateur : $(z, X) \mapsto (qz, A(z)X)$ (resp. $z \mapsto qz$). (Il reviendrait donc au même de considérer l'action du semi-groupe $q^{-\mathbf{N}}$.) Via la projection $D \times \mathbf{C}^n \to D$, ces actions sont compatibles. On a donc une relation d'équivalence $\sim_A$ sur $D \times \mathbf{C}^n$ (resp. $\sim$ sur $D$) engendrée par les relations $(z, X) \sim_A (qz, A(z)X)$ (resp. $z \sim qz$). On en déduit, par passage au quotient, un diagramme commutatif :

$$\begin{array}{ccc} D \times \mathbf{C}^n & \xrightarrow{pr_1} & D \\ \big\downarrow & & \big\downarrow \pi \\ \mathcal{F}_A = \frac{D \times \mathbf{C}^n}{\sim_A} & \longrightarrow & \mathbf{E}_q = \frac{D}{\sim} \end{array}$$

Il est en effet bien évident que le quotient de la surface de Riemann $D$ par $\sim$ est bien la courbe elliptique $\mathbf{E}_q$. La seule nouveauté ici, par rapport au formalisme général de l'introduction, est que la projection $\pi : D \to \mathbf{E}_q$ n'est plus un revêtement, c'est seulement un isomorphisme local. La ligne du bas décrit un *fibré vectoriel holomorphe* $\mathcal{F}_A$ sur $\mathbf{E}_q$, et la plus grande partie du discours des sections 1.1.1 et 1.1.2 se transpose ici. On peut décrire le fibré $\mathcal{F}_A$ en termes de cocycles, comme dans [**16**] : c'est fait dans [**13**]. Voici la description du faisceau des sections. Soit $V$ un ouvert de $\mathbf{E}_q$. Alors l'espace des sections de $\mathcal{F}_A$ sur $V$ est :

(3)
$$\Gamma(V, \mathcal{F}_A) = \{X \in \mathcal{O}\left(\pi^{-1}(V) \cap D\right)^n \mid \forall z \in \pi^{-1}(V) \cap q^{-1}D \ , \ X(qz) = A(z)X(z)\}.$$

(La condition $z \in q^{-1}D$ équivaut à $z \in D \wedge qz \in D$.) On peut vérifier directement que ce faisceau est localement libre sur $\mathbf{E}_q$ ([**20**], [**13**]).

Si l'on remplace $D$ par $D' \subset D$, on obtient un fibré $\mathcal{F}'_A$ et, pour tout ouvert $V$ de $\mathbf{E}_q$, un morphisme canonique de restriction $\Gamma(V, \mathcal{F}_A) \to \Gamma(V, \mathcal{F}'_A)$ dont il est facile de vérifier qu'il est bijectif. Ainsi, le fibré $\mathcal{F}_A$ ne dépend pas du choix particulier du disque $D$. Ce dernier peut d'ailleurs être remplacé par un disque épointé topologique. Dans (3), il faut alors remplacer la condition $z \in \pi^{-1}(V) \cap q^{-1}D$ par la condition $z \in \pi^{-1}(V) \cap q^{-1}D \cap D$. Puisque notre construction ne dépend que du *germe* $(D, 0)$ de disque épointé $D$ au voisinage de 0, nous noterons plus intrinsèquement :

$$\mathcal{F}_A = \frac{(D,0) \times \mathbf{C}^n}{\sim_A} \longrightarrow \mathbf{E}_q = \frac{(D,0)}{\sim}.$$

On obtient ainsi un foncteur $A \rightsquigarrow \mathcal{F}_A$ de $\mathcal{E}^{(0)} = \mathrm{DiffMod}\,(\mathbf{C}(\{z\}), \sigma_q)$ dans la catégorie des fibrés vectoriels holomorphes sur $\mathbf{E}_q$. Les propriétés (faciles) suivantes sont alors essentielles pour la théorie de Galois : *ce foncteur est exact, fidèle et $\otimes$-compatible.* Dans la terminologie de [**4**], on dit que c'est un foncteur fibre de $\mathcal{E}^{(0)}$ sur $\mathbf{E}_q$. Il permet de construire des familles de foncteurs fibres sur $\mathbf{C}$, respectivement indexées par $\mathbf{E}_q$ et par $\mathbf{C}^*$ : voir [**24**] et [**25**].

D'un point de vue purement fonctoriel, nous verrons que le fonteur $A \rightsquigarrow \mathcal{F}_A$ est essentiellement surjectif ; mais cela ne sert à rien, car il n'est pas pleinement fidèle. Par exemple, il peut associer des fibrés isomorphes à des modules qui ne le sont pas (voir un exemple section 2.3.3). Cette question sera abordée à la la section 2.3.2.

Pratiquement, le problème se présente ainsi. Soient $A \in \mathcal{GL}_n(\mathbf{C}(\{z\}))$ et $B \in \mathcal{GL}_p(\mathbf{C}(\{z\}))$ les matrices de deux objets $M_A, M_B$ de $\mathcal{E}^{(0)}$. Soit $\phi : \mathcal{F}_A \to \mathcal{F}_B$ un morphisme. D'après les généralités de l'introduction, on peut décrire $\phi$ comme une application holomorphe $F$ de $D$ dans $\mathrm{Mat}_{p,n}(\mathbf{C})$ qui satisfait l'équation (2). Pour en faire un morphisme dans $\mathcal{E}^{(0)}$, il faudrait le prolonger méromorphiquement en 0. *Mais on ne sait (en général) rien du mode de croissance de $F$ en* 0.

**Exemples.** — 1. Un morphisme de $\mathcal{F}_{(1)}$ dans $\mathcal{F}_{(z)}$ s'identifie à une fonction holomorphe $f : \mathbf{C}^* \to \mathbf{C}$ telle que $f(qz) = zf(z)$, autrement dit, à une section holomorphe de $\mathcal{F}_{(z)}$. (Ainsi, $\mathcal{F}_{(1)}$ se comporte comme l'objet unité $\underline{1}$ décrit en 1.2.2.) Par exemple,

$\theta_q$ réalise un tel morphisme, et il a une singularité essentielle en 0. D'ailleurs, il n'existe aucun morphisme non trivial de $\underline{1} = M_{(1)}$ dans $M_{(z)}$ : en effet, ce serait une série de Laurent $f \in \mathbf{C}(\{z\})$ telle que $\sigma_q f = zf$, ce qui implique $f = 0$. (Cela reste vrai dans DiffMod $(\mathbf{C}((z)), \sigma_q)$.)
2. Soient $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & z \end{smallmatrix}\right)$ et $F = \left(\begin{smallmatrix} 1 & 0 \\ \theta_q & 1 \end{smallmatrix}\right)$. Alors $F$ réalise un automorphisme de $\mathcal{F}_A$, qui ne provient pas d'un automorphisme de $M_A$.
3. Soit $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & z^2 \end{smallmatrix}\right)$. Nous verrons en 2.3.3 que $\mathcal{F}_A$ est somme de deux fibrés en droites de degré 1, alors que $M_A$ est indécomposable.

**Remarques.** — 1. Le fibré $\mathcal{F}_A$ défini à partir du quotient de $D \times \mathbf{C}^n$ se relève par $\pi : \mathbf{C}^* \to \mathbf{E}_q$ en un fibré trivial sur $\mathbf{C}^*$ (section 1.1.2). Il est donc de la forme $\mathcal{F}_{A'}$, où $A'$ est holomorphe de $\mathbf{C}^*$ dans $\mathcal{GL}_n(\mathbf{C})$. Le fait qu'il s'agisse de deux relèvements du même fibré signifie qu'il existe $F : D \to \mathcal{GL}_n(\mathbf{C})$ holomorphe tel que $(\sigma_q F)A = A'F$. Les matrices $A$ et $A'$ sont holomorphiquement équivalentes sur $D$, mais $A$ se prolonge méromorphiquement en 0 alors que $A'$ se prolonge holomorphiquement à $\mathbf{C}^*$.
2. Tout fibré sur une surface de Riemann compacte est méromorphiquement trivial ([**16**], p. 103) [(5)]. Il existe donc $F : D \to \mathcal{GL}_n(\mathbf{C})$ méromorphe tel que $A = F[I_n]$ : $F$ est donc une solution matricielle fondamentale *méromorphe* de (1). (C'est en substance l'argument de Praagman dans [**20**].)

2.1.1. *Équations fuchsiennes.* — Nous dirons que le module $M$ de $\mathcal{E}^{(0)}$ ou DiffMod $(\mathbf{C}((z)), \sigma_q)$ est *fuchsien* s'il est de la forme $M_A$, où $A(0) \in \mathcal{GL}_n(\mathbf{C})$. De même, si $K = \mathbf{C}(\{z\})$ ou $\mathbf{C}((z))$, l'équation (1) est dite *fuchsienne* si elle est équivalente à une équation de matrice $B$ telle que $B(0) \in \mathcal{GL}_n(\mathbf{C})$. Dans le cas d'un module de DiffMod $(\mathbf{C}(z), \sigma_q)$ ou d'une équation à coefficients dans $\mathbf{C}(z)$, considérés via le plongement $\mathbf{C}(z) \hookrightarrow \mathbf{C}(\{z\})$, on dit *fuchsien(ne) en* 0. Cette propriété équivaut à des conditions de croissance des solutions ([**33**]) ou de polygone de Newton (section 2.2.1). Pour une caractérisation plus intrinsèque, en termes de « réseau stable », voir [**23**], [**5**].

Un lemme clé dit alors que *toute équation fuchsienne est localement équivalente à une équation à coefficients constants*, autrement dit, il existe $A^{(0)} \in \mathcal{GL}_n(\mathbf{C})$ et $F \in \mathcal{GL}_n(\mathbf{C}(\{z\}))$ tels que $A = F\left[A^{(0)}\right]$. Comme dans le cas des équations différentielles, se lemme se prouve en deux étapes : élimination des *résonnances* à l'aide de transformations de shearing ; détermination d'un unique $F$ formel tangent à l'identité, et preuve de convergence.

La catégorie $\mathcal{E}_f^{(0)}$ des équations fuchsiennes en 0 est, par définition, la sous-catégorie pleine de $\mathcal{E}^{(0)}$ formée des objets fuchsiens. (En fait, on constate *a posteriori* que l'on peut aussi bien partir de DiffMod $(\mathbf{C}(z), \sigma_q)$ ou de DiffMod $(\mathbf{C}((z)), \sigma_q)$.) C'est une sous-catégorie tannakienne de $\mathcal{E}^{(0)}$ (elle est stable par toutes les opérations linéaires). La sous-catégorie pleine formée des équations à coefficients constants est également

(5) Gunning l'affirme en général, en faisant référence à la page 43 où c'est prouvé seulement pour la droite projective ; en fait, la démonstration s'adapte sans problème.

une sous-catégorie tannakienne, que nous noterons $\mathscr{P}$. Il découle du lemme-clé que *l'inclusion* $\mathscr{P} \hookrightarrow \mathscr{E}_f^{(0)}$ *est une équivalence de catégories.*

Pour tout objet $M_A$ de $\mathscr{P}$ défini par $A \in \mathscr{GL}_n(\mathbf{C})$, le fibré $\mathscr{F}_A$ admet la construction simplifiée :

$$\mathscr{F}_A = \frac{\mathbf{C}^* \times \mathbf{C}^n}{(z, X) \sim (qz, AX)}.$$

C'est donc un fibré plat.

***Théorème 2.1*** **([34])**. — *Le foncteur* $A \rightsquigarrow \mathscr{F}_A$ *de* $\mathscr{P}$ *dans la catégorie* $Fib_p(\mathbf{E}_q)$ *des fibrés plats sur* $\mathbf{E}_q$ *est une équivalence de catégories tannakiennes.*

*Démonstration.* — Soit $\phi : \mathscr{F}_A \to \mathscr{F}_B$, représenté par une matrice $F$ holomorphe sur $\mathbf{C}^*$ et telle que $(\sigma_q F)A = BF$. On a donc $F(qz) = BF(z)A^{-1}$, d'où l'on déduit facilement que $F$ a une croissance modérée en 0, donc un prolongement méromorphe, d'où la pleine fidélité. (En fait, il n'est pas très difficile de voir que $F$ est à coefficients dans $\mathbf{C}[z, z^{-1}]$, *cf.* **[34]**.)
Pour l'essentielle surjectivité, on part d'un fibré plat défini par la représentation telle que $1 \mapsto A_1$, $\tau \mapsto A_\tau$. Les matrices $A_1$ et $A_\tau$ commutent. Il existe donc un logarithme $2\mathrm{i}\pi B$ de $A_1$ qui commute avec $A_1$ et $A_\tau$. En posant $G(x) = e^{2\mathrm{i}\pi x B}$, on voit que $G(x+1) = A_1 G(x)$ et $G(x+\tau)A = A_\tau G(x)$, où $A = A_\tau A_1^{-\tau}$. Le fibré est donc isomorphe à $\mathscr{F}_A$. □

Au fibré plat $\mathscr{F}_A$ est associée par la correspondance de Weil la représentation de $\pi_1(\mathbf{E}_q) = \mathbf{Z} + \mathbf{Z}\tau$ définie par $1 \mapsto I_n$, $\tau \mapsto A$. Le théorème dit que toute représentation de $\pi_1(\mathbf{E}_q)$ est équivalente (dans ce sens étendu) à une représentation de cette forme. La catégorie $Fib_p(\mathbf{E}_q)$ (dont les morphismes sont *tous* les morphismes de fibrés vectoriels holomorphes) est donc équivalente à la catégorie dont les objets sont les représentations du groupe $\mathrm{Aut}(\mathbf{C}^*/\mathbf{E}_q) = q^{\mathbf{Z}} \simeq \mathbf{Z}$, et les morphismes sont les morphismes équivariants de la section 1.1.1. La catégorie $\mathscr{E}_f^{(0)}$ est donc équivalente à la catégorie $Rep_{\mathbf{E}_q}(\mathbf{Z})$ obtenue en épaississant la catégorie $Rep(\mathbf{Z})$ des représentations (complexes de dimension finie) de $\mathbf{Z}$. Dans **[34]** (et, par une autre voie, dans **[2]**) on en déduit la description du groupe de Galois de $\mathscr{E}_f^{(0)}$. L'action de ce groupe de Galois est explicitée dans **[24]** et **[25]**.

2.1.2. *Quelques exemples de rang* 1. — Un module de rang 1 dans $\mathscr{E}^{(0)}$ est de la forme $M_{(a)}$, où $a \in \mathbf{C}(\{z\})^*$. Écrivons $a = a_0 z^\mu u$, où $a_0 \in \mathbf{C}^*$, $\mu = v_0(a) \in \mathbf{Z}$ (valuation $z$-adique de $a$) et $u = 1 + u_1 z + \cdots$. Pour construire des solutions, on applique la section 1.2.1.

L'équation $\sigma_q f = uf$ admet la solution régulière $v(z) = \prod_{n \geq 1} u(q^{-n}z)$. Il revient au même de dire que $v : (1) \to (u)$ est une équivalence analytique, ou que $v : \underline{1} \to M_{(u)}$ est un isomorphisme. Le fibré $\mathscr{F}_{(u)}$ est donc isomorphe au fibré en droites trivial $\underline{1} = \mathscr{F}_{(1)}$.

L'équation $\sigma_q f = z^\mu f$ admet la solution $\theta_q^\mu$. Le module $M_{(z^\mu)}$ est isomorphe à $M_{(z)}^{\otimes \mu}$, puissance tensorielle $\mu^{\text{e}}$ de $M_{(z)}$. (Comme tout module de rang 1, il admet son dual comme inverse pour $\otimes$, ses puissances tensorielles négatives sont donc définies.) Le fibré associé est $\mathcal{F}_{(z^\mu)} \simeq \mathcal{F}_{(z)}^{\otimes \mu}$, dont une section est $\theta_q^\mu$, de diviseur $\operatorname{div}_{\mathbf{E}_q} \theta_q^\mu = \mu[\overline{-1}]$.

L'équation fuchsienne scalaire $\sigma_q f = a_0 f$ admet pour solution le $q$-caractère $e_{q,a_0}$, qui est une section du fibré plat $\mathcal{F}_{a_0}$. Comme $\operatorname{div}_{\mathbf{E}_q}(e_{q,a_0}) = [\overline{-1}] - [\overline{-a_0}]$, le degré de ce fibré est 0. (Le lecteur remarquera que, si $a_0 \in q^{\mathbf{Z}}$, le degré est nul et le module et le fibré sont en fait triviaux.)

Le module $M_{(a)}$ (resp. le fibré $\mathcal{F}_{(a)}$ associé) est isomorphe au produit tensoriel de ces trois modules (resp. de ces trois fibrés). Le degré de $\mathcal{F}_{(a)}$ est donc $\mu$. Sa pente (quotient du degré par le rang, *cf.* [**40**]) est donc $\mu$.

## 2.2. Équations irrégulières

2.2.1. *Polygone de Newton.* — On montre que tout module aux $q$-différences sur $K = \mathbf{C}(z)$, $\mathbf{C}(\{z\})$ ou $\mathbf{C}((z))$ peut se mettre sous la forme $M = \mathcal{D}_{q,K}/\mathcal{D}_{q,K}P$ (« lemme du vecteur cyclique » et euclidianité à gauche de $\mathcal{D}_{q,K}$). On peut prendre $P$ sous la forme $a_n\sigma^n + \cdots + a_0$, où $a_0 a_n \neq 0$. Notons $v_0(a)$ la valuation $z$-adique de $a \in K$. La frontière de l'enveloppe convexe de $\{(i,j) \in \mathbf{N} \times \mathbf{Z} \mid 0 \leq i \leq n \text{ et } j \geq v_0(a_j)\}$ est formée de deux demi-droites verticales et de $k$ vecteurs $(r_1, d_1), \ldots, (r_k, d_k) \in \mathbf{N}^* \times \mathbf{Z}$. Notant $\mu_i = \frac{d_i}{r_i} \in \mathbf{Q}$, on indexe ces vecteurs de gauche à droite, de sorte que $\mu_1 < \cdots < \mu_k$. On prouve que les *pentes* [(6)] $\mu_i$ et leurs *multiplicités* $r_i$ ne dépendent que de $M$.

**Exemples.** — 1. Soit $L = \sigma - a$ où $a \in \mathbf{C}(\{z\})^*$. L'équation $Lf = 0$, c'est-à-dire $\sigma_q f = af$, est modélisée par le module $M_{(a)}$, dont on peut démontrer qu'il est isomorphe à $\mathcal{D}_{q,K}/\mathcal{D}_{q,K}\mathcal{L}^\vee$, où $L^\vee = a\sigma - 1$. Les pentes de $M_{(a)}$ se calculent avec $L^\vee$ : l'unique pente est $\mu = v_0(a)$. Le fibré $\mathcal{F}_{(a)}$ est de rang 1 et de degré $\mu$ (section 2.1.2), donc de pente $\mu$.

2. Soit $L = qz\sigma^2 - (1+z)\sigma + 1 = (\sigma - 1)(z\sigma - 1)$. L'équation $Lf$ est intéressante, parce qu'elle est satisfaite par la *série de Tschakaloff* $\sum\limits_{n \geq 0} q^{n(n-1)/2} z^n$, qui est un $q$-analogue de la série d'Euler. On en fait une équation vectorielle de type (1) en posant (par exemple) $X = \begin{pmatrix} f \\ z\sigma_q f - f \end{pmatrix}$ et $A = \begin{pmatrix} z^{-1} & z^{-1} \\ 0 & 1 \end{pmatrix}$. Le module $M_A$ est isomorphe à à $\mathcal{D}_{q,K}/\mathcal{D}_{q,K}\mathcal{L}^\vee$, où $L^\vee = (\sigma - z)(\sigma - 1) = \sigma^2 - (1+z)\sigma + z$. Ses pentes sont $-1$ et 0. La forme triangulaire de la matrice indique qu'il y a une suite exacte : $0 \to M_{(z^{-1})} \longrightarrow M_A \longrightarrow M_{(1)} \to 0$, où l'inclusion a pour matrice $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ et la projection $\begin{pmatrix} 0 & 1 \end{pmatrix}$. Il y a donc également une suite exacte de fibrés : $0 \to \mathcal{F}_{(z^{-1})} \longrightarrow \mathcal{F}_A \longrightarrow \mathcal{F}_{(1)} \to 0$.

La donnée du *polygone de Newton* $N_M$ de $M$ équivaut à celle de la fonction $r_M$ de $\mathbf{Q}$ dans $\mathbf{N}$ telle que $\mu_i \mapsto r_i$ et qui est nulle par ailleurs. Selon [**36**], le polygone de Newton est additif pour les suites exactes, multiplicatif pour le produit tensoriel et

(6) Depuis [**24**], [**25**], nous avons adopté pour les pentes une convention *opposée* à celle qui prévalait dans [**36**], [**28**], [**35**].

tel que $r_{M^\vee}(\mu) = r_M(-\mu)$. Ces règles sont assez différentes de celles qui régissent les équations différentielles.

Nous dirons qu'un module est *pur isocline* s'il admet une seule pente et *pur* s'il est somme directe de modules purs isoclines [(7)]. Les modules fuchsiens sont les modules purs isoclines de pente 0.

2.2.2. *Filtration par les pentes*

***Théorème 2.2*** **([36])**. — *(i) Tout module de* $\mathscr{E}^{(0)}$ *ou* $\mathrm{DiffMod}\,(\mathbf{C}((z)), \sigma_q)$ *admet une unique filtration croissante* $(M_{\leq\mu})_{\mu\in\mathbf{Q}}$ *telle que les quotients* $M_{(\mu)} = M_{\leq\mu}/M_{<\mu}$ *sont purs de pente* $\mu$. *(Le rang de* $M_{(\mu)}$ *est donc* $r_M(\mu)$.)
*(ii) la filtration est strictement fonctorielle : le foncteur gradué associé :* $M \rightsquigarrow grM = \bigoplus_{\mu\in\mathbf{Q}} M_{(\mu)}$ *est exact. Il est de plus fidèle et* $\otimes$*-compatible.*
*(iii) Dans* $\mathrm{DiffMod}\,(\mathbf{C}((z)), \sigma_q)$, *le foncteur gr est isomorphe au foncteur identité.*

En termes d'opérateurs, ce théorème dit que, pour $K = \mathbf{C}(\{z\})$, tout $L \in \mathscr{D}_{q,K}$ admet une factorisation $L = L_1 \cdots L_k$ où les $L_i$ sont purs de pentes $\mu_1 < \cdots < \mu_k$. Pour $K = \mathbf{C}((z))$, une telle factorisation est possible avec un ordre des pentes arbitraires. En termes de matrices, toute $A \in \mathscr{GL}_n(\mathbf{C}(\{z\}))$ peut se mettre sous la forme triangulaire supérieure par blocs :

$$A = \begin{pmatrix} A_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & U_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & A_k \end{pmatrix},$$

où chaque $A_i$ est pure de pente $\mu_i$. Si $A \in \mathscr{GL}_n(\mathbf{C}((z)))$, on peut prendre les $U_{i,j} = 0$.

**Exemple.** — Soit $L = qz\sigma^2 - (1+z)\sigma + 1$, dont les pentes sont 0 et $-1$. L'existence analytique de la filtration vient de la factorisation analytique $L = (\sigma - 1)(z\sigma - 1)$ (qui remonte à Adams, Birkhoff et Guenther). L'opérateur dual se factorise : $L^\vee = (\sigma - z)(\sigma - 1)$, d'où une suite exacte pour le module associé $M = \mathscr{D}_{q,K}/\mathscr{D}_{q,K}\mathscr{L}^\vee$ :

$$0 \to \mathscr{D}_{q,K}/\mathscr{D}_{q,K}(\sigma - z) \longrightarrow M \longrightarrow \mathscr{D}_{q,K}/\mathscr{D}_{q,K}(\sigma - 1) \to 0.$$

C'est cette suite exacte qui permet de construire la forme triangulaire $A = \left(\begin{smallmatrix} z^{-1} & z^{-1} \\ 0 & 1 \end{smallmatrix}\right)$ obtenue page 410. En effet, le module $\mathscr{D}_{q,K}/\mathscr{D}_{q,K}(\sigma - z)$ se décrit également comme $(\mathbf{C}(\{z\}), \Phi_{z^{-1}})$, et correspond donc à la matrice $(z^{-1}) \in \mathscr{GL}_1(\mathbf{C}(\{z\}))$. De même, $\mathscr{D}_{q,K}/\mathscr{D}_{q,K}(\sigma - 1)$ correspond à $(1) \in \mathscr{GL}_1(\mathbf{C}(\{z\}))$.

***Corollaire 2.3***. — *Le fibré* $\mathscr{F}_A$ *est muni d'une filtration* $\mathscr{F}_{(1)} \subset \cdots \subset \mathscr{F}_k$ *telle que chaque* $\mathscr{F}_i/\mathscr{F}_{i-1}$ *est le fibré associé à une équation pure de pente* $\mu_i$.

(7) Nous avons adopté cette terminologie depuis **[24]**, **[25]** (anciens termes : « pur », et « modérément irrégulier ». Nos modules purs sont les « split modules » de **[22]**.

**Remarque.** — En fait, sous cette forme, l'énoncé ci-dessus est presque vide. Selon le théorème 10, p. 63 de [**16**], tout fibré sur $\mathbf{E}_q$ peut être décrit par une matrice triangulaire supérieure (sans blocs). Avec les calculs de la section 2.1.2, on peut obtenir des termes diagonaux de la forme $\alpha_i z^{d_i}$ ($\alpha_i \in \mathbf{C}^*$, $d_i \in \mathbf{Z}$). De plus, chaque fois que $i < j$ et $d_i > d_j$, on peut permuter les deux termes diagonaux correspondants. En effet, cela se ramène à la constatation que l'équation $\alpha_j z^{d_j} \sigma_q f - \alpha_i z^{d_i} f = u$ admet, pour tout $u \in \mathscr{O}(\mathbf{C}^*)$, une solution $f \in \mathscr{O}(\mathbf{C}^*)$ ; et ce point est immédiat par identification des séries de Laurent. On obtient donc un résultat apparemment analogue au théorème mais plus fort pour les fibrés. Cependant, les exposants $d_i$ qui apparaissent ici n'ont aucune signification intrinsèque en termes de fibrés. (Pour une mise en forme plus intrisèque, voir la section 2.3.3.)

À partir de ce théorème, deux voies distinctes ont été suivies. En supposant les pentes entières, on a une description simple des blocs purs $A_i$ (section 2.3). On en tire, *par voie transcendante*, des conséquences pour la classification ([**28**], [**35**], section 3.1) et pour la théorie de Galois ([**24**], [**25**], section 3.2).

Récemment, van der Put et Reversat ont élucidé la structure des modules purs dans le cas général ([**22**]). Dans le cas d'un module indécomposable de rang $r$ et pente $\mu = \frac{d}{r}$, on a $d \wedge r = 1$ et la description est similaire à celle des fibrés indécomposables sur $\mathbf{E}_q$ par Atiyah ([**1**], [**17**]), qu'elle permet de retrouver de manière plus simple. L'extension des résultats de Ramis, Sauloy et Zhang au cas des pentes rationnelles à l'aide de [**22**] ne semble pas avoir été faite. Elle devrait entrainer des complications algébriques, mais peut-être pas mettre en jeu d'idées analytiques nouvelles. Dans *loc. cit.* , van der Put et Reversat enrichissent de plus le fibré d'une connexion méromorphe, mais cela ne modifie pas le problème de la pleine fidélité. Nous proposerons une autre structure à la section 2.3.2.

## 2.3. Équations irrégulières à pentes entières

2.3.1. *Description des équations à pentes entières.* — Pour tout $d \in \mathbf{N}^*$, la sous-catégorie pleine de $\mathscr{E}^{(0)} = \mathrm{DiffMod}\,(\mathbf{C}(\{z\}), \sigma_q)$ formée des équations à pentes dans $\frac{1}{d}\mathbf{Z}$ est une sous-catégorie tannakienne. Pour $d = 1$, on obtient la catégorie $\mathscr{E}_1^{(0)}$ des équations à pentes entières. La sous-catégorie pleine de $\mathscr{E}_1^{(0)}$ formée des équations pures à pentes entières en est encore une sous-catégorie tannakienne $\mathscr{E}_{p,1}^{(0)}$.

Si $M$ est pur (isocline) de pente $\mu \in \mathbf{Z}$, alors $M_{(z^{-\mu})} \otimes M$ est fuchsien, donc de la forme $M_A$ avec $A \in \mathscr{GL}_n(\mathbf{C})$. On a donc $M \simeq M_{z^\mu A}$. On peut alors améliorer la forme triangulaire des matrices en une *forme standard* :

$$A = \begin{pmatrix} z^{\mu_1} A_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & U_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & z^{\mu_k} A_k \end{pmatrix}, \tag{4}$$

où les pentes $\mu_1 < \cdots < \mu_k$ sont entières, $r_i \in \mathbf{N}^*$, $A_i \in \mathscr{GL}_{r_i}(\mathbf{C})$ $(i = 1, \ldots, k)$ (ces $\mu_i$ et $r_i$ constituent le polygone de Newton de $A$) et :

$$U = (U_{i,j})_{1 \leq i < j \leq k} \in \prod_{1 \leq i < j \leq k} \mathrm{Mat}_{r_i, r_j}(\mathbf{C}(\{z\})).$$

On peut même imposer aux $U_{i,j}$ d'être polynomiaux ([**28**]).

Soit $B$ une matrice à pentes entières décrite de manière similaire : blocs diagonaux $z^{\nu_{i'}} B_{i'}$, où $B_{i'} \in \mathscr{GL}_{s_{i'}}(\mathbf{C})$ $(i' = 1, \ldots, k')$ et surdiagonaux $V_{i',j'} \in \mathrm{Mat}_{s_{i'}, s_{j'}}(\mathbf{C}(\{z\}))$. La filtration étant fonctorielle, tout morphisme $F : A \to B$ est triangulaire supérieur par blocs. Plus précisément, $F$ admet une décomposition en blocs $F_{i',i} \in \mathrm{Mat}_{s_{i'}, r_i}(\mathbf{C}(\{z\}))$, nuls pour $i < i'$ et tels que, pour $i' \geq i$ :

$$(\sigma_q F_{i',i}) z^{\mu_i} A_i + \sum_{i' \leq l < i} (\sigma_q F_{i',l}) U_{l,i} = z^{\nu_{i'}} B_{i'} F_{i',i} + \sum_{i' \leq l < i} V_{i',l} F_{l,i}.$$

2.3.2. *Fibrés associés aux équations à pentes entières.* — Pour tout module pur isocline $M \simeq M_{z^\mu A}$, le fibré $\mathscr{F}_M$ est isomorphe à $\mathscr{F}_{(z^\mu)} \otimes \mathscr{F}_A$. Disons qu'un fibré est *pur isocline de pente $\mu$* si c'est le produit tensoriel d'un fibré en droites de degré $\mu$ par un fibré plat. Il résulte du théorème 2.2 et de son corollaire que l'on peut associer à tout objet $M$ de $\mathscr{E}_1^{(0)}$ un fibré $\mathscr{F}_M$ muni d'une filtration à quotients purs isoclines $\mathscr{F}_{(1)} \subset \cdots \subset \mathscr{F}_k$.

***Théorème 2.4***. — *Le foncteur $M \rightsquigarrow (F_M, \mathscr{F}_{(1)} \subset \cdots \subset \mathscr{F}_k)$ est exact, pleinement fidèle et $\otimes$-compatible.*

*Démonstration.* — Les objets de la catégorie d'arrivée sont les couples $(\mathscr{F}, (\mathscr{F}_{\leq \mu})_{\mu \in \mathbf{Z}})$ formés d'un fibré et d'une filtration croissante par des sous-fibrés telle que les $\mathscr{F}_{(\mu)} = \mathscr{F}_{\leq \mu} / \mathscr{F}_{< \mu}$ sont des fibrés purs isoclines de pente $\mu$. Les morphismes de cette catégorie sont les morphismes de fibrés $\phi : \mathscr{F} \to \mathscr{F}'$ qui respectent la filtration : $\phi(\mathscr{F}_{\leq \mu}) \subset \mathscr{F}'_{\leq \mu}$. La structure tensorielle est définie en munissant $\mathscr{F} \otimes \mathscr{F}'$ de la filtration :

$$(\mathscr{F} \otimes \mathscr{F}')_{\leq \mu} = \mathrm{Im} \left( \sum_{\nu + \nu' \leq \mu} \mathscr{F}_{\leq \nu} \otimes \mathscr{F}'_{\leq \nu'} \to \mathscr{F} \otimes \mathscr{F}' \right).$$

Le seul point nouveau dans l'énoncé du théorème est que ce foncteur est *pleinement* fidèle. Soient $A, B$ des objets de $\mathscr{E}_1^{(0)}$ et $\phi : \mathscr{F}_A \to \mathscr{F}_B$ un morphisme. On écrit $\phi$ sous la forme $F \in \mathrm{Mat}_{p,n}(\mathscr{O}(\mathbf{C}^*))$ tel que $(\sigma_q F) A = BF$. L'hypothèse sur le respect des filtrations dit que $F$ est triangulaire par blocs au sens de la section précédente, et que ces blocs vérifient les mêmes équations que ceux des morphismes dans $\mathscr{E}_1^{(0)}$. Le lemme ci-dessous entraine alors, par récurrence, que $F$ admet un prolongement méromorphe en 0. (La récurrence est amorcée par les blocs diagonaux de $F$ qui se déduisent du cas fuchsien.) □

***Lemme 2.5***. — *Soient $\mu < \nu$, $C \in \mathscr{GL}_r(\mathbf{C})$, $D \in \mathscr{GL}_s(\mathbf{C})$ et $U \in Mat_{s,r}(\mathbf{C}(\{z\}))$. Soit $F \in Mat_{s,r}(\mathscr{O}(\mathbf{C}^*))$ tel que $(\sigma_q F)(z^\mu C) - (z^\nu D) F = U$. Alors $F$ est méromorphe en* 0.

*Démonstration.* — On écrit le développement en série de Laurent : $F = \sum_{n \in \mathbf{Z}} F_n z^n$. On a donc : $q^{n-\mu} F_{n-\mu} C - D F_{n-\nu} = U_n$ pour tout $n \in \mathbf{Z}$. Puisque $U$ est méromorphe en 0, il existe donc $n_0$ tel que, pour $n \geq n_0$, on ait : $q^n F_n = D F_{n-\delta} C^{-1}$, où $\delta = \nu - \mu \geq 1$. Si l'on n'a pas $F_n = 0$ pour $n \ll 0$, alors $F_n$ est de l'ordre de grandeur de $q^{n^2/2\delta}$ lorsque $n \to -\infty$, contredisant la convergence de la série de Laurent. $\square$

2.3.3. *Lien avec la filtration de Harder-Narasimhan.* — La filtration introduite à la section précésente présente des ressemblances formelles avec la filtration de Harder-Narasimhan ([**40**]), mais nous allons voir qu'elles ne sont pas liées. Cependant, il y a d'intrigantes questions de stabilité. Les calculs qui suivent sont en grande partie tirés de [**13**].

2.3.3.1. *Une famille génériquement stable.* — Soit $A_u = \left(\begin{smallmatrix} 1 & u \\ 0 & z \end{smallmatrix}\right)$, avec $u \in \mathbf{C}(\{z\})$. D'après [**28**], on peut ramener $A_u$ à la forme $A_v$ avec $v \in \mathbf{C}$ via un unique morphisme de la forme $\left(\begin{smallmatrix} 1 & f \\ 0 & 1 \end{smallmatrix}\right)$, où $f \in \mathbf{C}(\{z\})$, *i.e.* trivial sur les gradués associés.
Soient maintenant $u, v \in \mathbf{C}$. D'après la fonctorialité de la filtration et ce que l'on sait sur les morphismes entre objets fuchsiens, tout morphisme de $A_u$ dans $A_v$ est de la forme $\left(\begin{smallmatrix} 1 & f \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$, où $f \in \mathbf{C}(\{z\})$ et $\alpha, \beta \in \mathbf{C}^*$. Or, l'équation correspondante $z\sigma_q f - f = v - \frac{\alpha}{\beta} u$ admet pour seule solution formelle $f = \left(v - \frac{\alpha}{\beta} u\right) \hat{\phi}$, où $\hat{\phi}$ est la série de Tschakaloff, et $f$ n'est donc convergente que si $v = \frac{\alpha}{\beta} u\cdot$ Ainsi, $A_u$ n'est isomorphe à $A_v$ que si $u = v = 0$ ou bien si $u \neq 0, v \neq 0$. Les $A_u$, autrement dit, les extensions de $(z)$ par $(1)$ se répartissent en deux classes [(8)] : scindée ou non. Dans le cas non scindé, $A_u$ est de plus indécomposable (cela se déduit de la fonctorialité de la filtration par les pentes).

Discutons maintenant la *stabilité* du fibré $\mathcal{F}_{A_u}$ ([**16**], [**40**]). Le degré de $\mathcal{F}_{A_u}$ est 1 (extension de $\mathcal{F}_{(z)}$ par $\mathcal{F}_{(1)}$), son rang est 2, sa pente est donc $\frac{1}{2}\cdot$ Pour qu'il soit instable, il faut, et il suffit, qu'il admette un sous-fibré en droite de pente, donc de degré $> \frac{1}{2}$, donc $\geq 1$, autrement dit, qu'il admette une section méromorphe non triviale de degré $\geq 1$. Les sections méromorphes de $\mathcal{F}_{A_u}$ s'identifient aux solutions méromorphes $X = \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$ de l'équation $\sigma_q X = A_u X$, *i.e.* du système :

$$\begin{cases} \sigma_q f = f + ug, \\ \sigma_q g = zg. \end{cases}$$

Le degré $\deg X \underset{\text{déf}}{=} \deg \operatorname{div}_{\mathbf{E}_q} X$ de la section $X$ se calcule ainsi : si $h \in \mathcal{M}(\mathbf{C}^*)^*$ est telle que $h^{-1} X \in \mathcal{O}(\mathbf{C}^*)^2$ ne s'annule en aucun point, alors $\operatorname{div}_{\mathbf{E}_q} X \underset{\text{déf}}{=} \operatorname{div}_{\mathbf{E}_q} h$ et $\deg \operatorname{div}_{\mathbf{E}_q} X = \deg \operatorname{div}_{\mathbf{E}_q} h$. Il est clair que, si $\phi$ est elliptique et non triviale, alors $\deg(\phi X) = \deg X$.
Si $g = 0$, $f$ est elliptique et, par hypothèse, non triviale, donc $\deg X = \deg f = 0$ (ce cas correspond au sous-fibré $\mathcal{F}_{(1)}$).

---

(8) Il s'agit des classes d'isomorphie des modules qui sont des extensions de $(z)$ par $(1)$, et non des classes d'extension au sens habituel. Ces dernières forment l'espace $Ext((z), (1))$, qui est de dimension 1 ([**28**]) et paramétré par $u$.

Si $g \neq 0$, alors $g = h\theta_q$, où $h$ est elliptique et, quitte à diviser par $h$, on peut aussi bien supposer que $g = \theta_q$ et $\sigma_q f - f = u\theta_q$. Dire que $\deg X \geq 1$ c'est dire que $f$ et $g$ ont un zéro commun, donc que $f = \theta_q f_1$, où $f_1 \in \mathcal{O}(\mathbf{C}^*)$. Mais alors $z\sigma_q f_1 - f_1 = u$, qui n'a de solution dans $\mathcal{O}(\mathbf{C}^*)$ que si $u = 0$ (on le voit en examinant les séries de Laurent).
Nous avons donc une dichotomie : soit $u = 0$ et $A_u$ et $\mathcal{F}_{A_u}$ sont scindés ; soit $u \neq 0$, et $A_u$ est indécomposable et $\mathcal{F}_{A_u}$ est stable. Dans ce dernier cas, la filtration de Harder-Narasimhan de $\mathcal{F}_{A_u}$ est triviale, alors que celle induite par la filtration par les pentes de $A_u$ ne l'est pas.

2.3.3.2. *Une famille semi-stable.* — Soit $A_u = \begin{pmatrix} 1 & u \\ 0 & z^2 \end{pmatrix}$, avec $u \in \mathbf{C}(\{z\})$. D'après [**28**], on peut ramener $A_u$ à la forme forme $A_v$ avec $v \in \mathbf{C} + \mathbf{C}z$ via un unique morphisme trivial sur les gradués associés. Nous supposerons donc d'emblée que $u = u_0 + u_1 z$. Par un raisonnement similaire à celui du paragraphe précédent, on voit que la classe d'isomorphie de $A_u$ détermine $(u_0, u_1) \in \mathbf{C}^2$ à un facteur près dans $\mathbf{C}^*$. De plus, si $u \neq 0$, l'objet $A_u$ est indécomposable.
Le degré de $\mathcal{F}_{A_u}$ est 2, sa pente est 1. Si $u = 0$, ce fibré est scindé. On suppose désormais que $u \neq 0$. On va voir que, dans ce cas, $\mathcal{F}_{A_u}$ est semi-stable et non stable : autrement dit, il admet des sections méromorphes de degré 1, mais pas plus. Une section méromorphe de $\mathcal{F}_{A_u}$ est (par les identifications habituelles) de la forme $X = \begin{pmatrix} f \\ g \end{pmatrix}$, où :

$$\begin{cases} \sigma_q f = f + ug, \\ \sigma_q g = z^2 g. \end{cases}$$

Si $g = 0$ et $f \neq 0$, $\deg X = 0$ (cas du sous-fibré $\mathcal{F}_{(1)}$). Si $g \neq 0$, on se ramène au cas où $g = \theta_q^2$, on écrit $f = f_1 \theta_q^2$ et l'on doit avoir : $z^2 \sigma_q f_1 - f_1 = u$. Pour que $\deg X \geq 1$, il faut que $f_1$ ait au pire un pôle simple sur $\mathbf{E}_q$, soit $\overline{a}$, donc que l'on ait $f_1 = \frac{h}{\theta_a}$ avec $h \in \mathcal{O}(\mathbf{C}^*)$. On est donc ramené à chercher $a \in \mathbf{C}^*$ et $h \in \mathcal{O}(\mathbf{C}^*)$ tels que $az\sigma_q h - h = u\theta_a$. Par développement en série de Laurent $h = \sum h_n z^n$, cette équation équivaut à :

$$\forall n \in \mathbf{Z} \ , \ aq^{n-1} h_{n-1} - h_n = \left( u_0 q^{-n(n+1)/2} + u_1 a q^{-n(n-1)/2} \right) a^{-n},$$

soit encore à :

$$\forall n \in \mathbf{Z} \ , \ \frac{h_{n-1}}{q^{(n-1)(n-2)/2} a^{n-1}} - \frac{h_n}{q^{n(n-1)/2} a^n} = \left( u_0 q^{-n^2} + u_1 a q^{-n(n-1)} \right) a^{-2n}.$$

Il s'agit essentiellement d'une *transformation de q-Borel* (voir [**28**] et la section 3.2). Par sommation et annulation télescopique, on voit immésiatement qu'une condition *nécessaire* est la nullité de :

$$\phi_u(a) := \sum_{n \in \mathbf{Z}} \left( u_0 q^{-n^2} + u_1 a q^{-n(n-1)} \right) a^{-2n}.$$

Les mêmes majorations que dans [**35**], preuve du lemme 2.9, montrent que c'est une condition suffisante.

On a sans peine :

$$\phi(a) = u_0\theta_{q^2}(qa^{-2}) + u_1 a^{-1}\theta_{q^2}(a^{-2}).$$

Notons que la condition posée, $\phi(a) = 0$, est invariante par $a \leftarrow qa$, comme il se doit (c'est une condition sur le pôle $\overline{a} \in \mathbf{E}_q$).
Si $u_0 = 0 \neq u_1$, on doit résoudre $\theta_{q^2}(a^{-2}) = 0$, ce qui donne $a^{-2} \in [-1, q^2]$. Les deux solutions dans $\mathbf{E}_q$ sont $\overline{\mathrm{i}}$ et $\overline{-\mathrm{i}}$. De même, si $u_0 \neq 0 = u_1$, on résoud $\theta_{q^2}(qa^{-2}) = 0$, donc $a^{-2} \in [-q, q^2]$, donc les deux solutions dans $\mathbf{E}_q$ sont $\overline{\sqrt{-q}}$ et $\overline{-\sqrt{-q}}$ (avec un choix arbitraire de racine carrée).
Supposons $u_0u_1 \neq 0$. La fonction $\psi(a) = \frac{\theta_{q^2}(qa^{-2})}{a^{-1}\theta_{q^2}(a^{-2})}$ est $q$-elliptique [9] et admet, dans $\mathbf{E}_q$, deux pôles simples et deux zéros simples. Elle prend donc chaque valeur $\frac{u_1}{u_0}$ deux fois, et il y a encore deux pôles $\overline{a} \in \mathbf{E}_q$ possibles.
Si l'on a trouvé $\overline{a_1} \neq \overline{a_2}$, les sections $X_1$ et $X_2$ correspondantes sont non proportionnelles. On peut en déduire que $\mathcal{F}_{A_u}$ est scindé dans ce cas (qui est générique).

## 3. Le phénomène de Stokes

**3.1. Classification d'équations irrégulières.** — D'après la section 2.3.1, on peut représenter tout objet $M$ de $\mathcal{E}_1^{(0)}$ par une matrice $A$ de la forme (4) page 412. Le gradué $M_0 = \mathrm{gr}M$ est alors décrit par la matrice :

$$A_0 = \begin{pmatrix} z^{\mu_1}A_1 & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & 0 & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & z^{\mu_k}A_k \end{pmatrix}, \tag{5}$$

De la dernière assertion du théorème 2.2, il découle alors qu'il existe une unique matrice *formelle*, c'est-à-dire à coefficients dans $\mathbf{C}((z))$ :

$$F = \begin{pmatrix} I_{r_1} & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & F_{i,j} & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & I_{r_k} \end{pmatrix}, \tag{6}$$

telle que $F[A_0] = A$. Nous la noterons $\hat{F}_A \in \mathfrak{G}(\mathbf{C}((z)))$, la notation $\mathfrak{G}$ désignant le sous-groupe algébrique unipotent de $\mathcal{GL}_n$ correspondant au format ci-dessus. Plus généralement, si $A'$ est aussi de la forme (4), alors l'unique $F \in \mathfrak{G}(\mathbf{C}((z)))$ tel que $F[A] = A'$ est $\hat{F}_{A,A'} = \hat{F}_{A'}\left(\hat{F}_A\right)^{-1}$. Les isomorphismes formels $\hat{F}_A$, $\hat{F}_{A,A'}$ sont en général très divergents : les blocs $F_{i,j}$ tels que $\mu_j - \mu_i = \delta \in \mathbf{N}^*$ sont en général

[9] En fait, on a une factorisation : $\psi(a) = \frac{\theta_q(-\sqrt{-q}a)\,\theta_q(\sqrt{-q}a)}{a\,\theta_q(-\mathrm{i}a)\,\theta_q(\mathrm{i}a)}$.

de *niveau q-Gevrey* $\delta$, autrement dit, leurs coefficients sont de la forme $\sum f_n z^n$ avec $f_n = O\left(R^n q^{n^2/2\delta}\right)$ pour un certain $R > 0$.

3.1.0.3. *Classification analytique isoformelle.* — Il s'agit de la classification analytique à classe formelle fixée (ou à gradué fixé, ce qui revient au même). On fixe un module pur $M_0 = P_1 \oplus \cdots \oplus P_k$ dans $\mathcal{E}_{p,1}^{(0)}$ et l'on considère tous les couples $(M, u)$, où $M$ est un objet de $\mathcal{E}_1^{(0)}$ et $u : \mathrm{gr}M \to M_0$ un isomorphisme. On dit que $(M, u) \sim (N, v)$ s'il existe un isomorphisme $\phi : M \to N$ tel que $v \circ (\mathrm{gr}\phi) = u$ (isomorphismes triviaux sur le gradué). Avec les notations vues plus haut, il n'est pas très difficile de voir que, si $M_0$ est décrit par la matrice $A_0$ de la forme (5), alors un couple $(M, u)$ est la même chose qu'une matrice $A$ sous la forme (4) et que, si $(M, u)$ et $(M', u')$ correspondent à $A$ et $A'$, alors ils sont équivalents si, et seulement si $\hat{F}_{A,A'} \in \mathfrak{G}(\mathbf{C}(\{z\}))$.

Il a été démontré dans **[28]** que l'ensemble $\mathcal{F}(M_0)$ des classes pour cette relation est un espace affine de dimension $\mathrm{irr}(M_0) \underset{\text{déf}}{=} \sum\limits_{1 \leq i < j \leq k} r_i r_j (\mu_j - \mu_i)$, où les $\mu_i$, $r_i$ proviennent du polygone de Newton de $M_0$ (et de tous les $M$ de sa classe formelle) [10]. Il s'agit en fait d'un vrai schéma de modules pour ce problème. Une preuve consiste à vérifier que, dans chaque classe, il existe une unique matrice $A$ en *forme normale*, c'est-à-dire en forme standard (4) et telle que chaque $U_{i,j}$ est à coefficients dans $\sum\limits_{\mu_i \leq d < \mu_j} \mathbf{C} z^d$. Cette forme normale est inspirée de Birkhoff et Guenther. Cependant, l'objectif de **[28]** est d'obtenir des *invariants transcendants* sous une forme $q$-analogue aux théorèmes de Malgrange-Sibuya. Nous allons décrire l'un de ces résultats sous la forme moins puissante, mais plus simple, de **[35]**.

3.1.0.4. *Sommation de* $\hat{F}_A$. — Dans **[35]**, on définit un sous-ensemble fini explicite $\Sigma_{A_0}$ de $\mathbf{E}_q$, qui est, en général, de cardinal $\mathrm{irr}(A_0)$ (et moins dans certains cas « résonnants »). On prouve alors :

***Théorème 3.1***. — *Pour tout* $\overline{c} \in \mathbf{E}_q \setminus \Sigma_{A_0}$, *il existe un unique isomorphisme méromorphe* $F : A_0 \to A$ *tel que* $F \in \mathfrak{G}(\mathcal{M}(\mathbf{C}^*))$, *dont les pôles sont situés sur la q-spirale discrète* $[-c; q] = -cq^{\mathbf{Z}}$ *et tel que, pour* $1 \leq i < j \leq k$, *les pôles de* $F_{i,j}$ *ont une multiplicité* $\leq \mu_j - \mu_i$. *(Ce que l'on peut écrire :* $\mathrm{div}_{\mathbf{E}_q} F_{i,j} \geq -(\mu_j - \mu_i)[\overline{c}]$.*) Nous noterons* $S_{\overline{c}}\hat{F}_A$ *cette matrice* $F$.

La description de $\Sigma_{A_0}$ et le calcul de $S_{\overline{c}}\hat{F}_A$ seront explicités dans un cas crucial à la section 3.2, en vue de comparaison avec d'autres constructions intéressantes.

**Remarque.** — Le morphisme méromorphe $S_{\overline{c}}\hat{F}_A$ induit un isomorphisme holomorphe du faisceau $\mathcal{F}_{A_0}$ sur le faisceau $\mathcal{F}_A$ en restriction à l'ouvert $\mathbf{E}_q \setminus \{\overline{c}\}$. ces deux faisceaux sont donc localement isomorphes. Comme il est facile de prouver que $\mathcal{F}_{A_0}$ est un fibré, cela fournit une nouvelle preuve du fait que $\mathcal{F}_A$ l'est également. En

(10) Ce résultat est utilisé dans **[21]**, mais attribué de manière erronée à Ramis et Sauloy, et donné sans référence.

fait, cela montre que le fibré $\mathcal{F}_A$ s'obtient à partir du fibré $\mathcal{F}_{A_0}$ par l'opération de cohomologie non-abélienne « torsion par un cocycle » ([**11**]).

Nous considérons $S_{\overline{c}}\hat{F}_A$ comme une *sommation de la série divergente* $\hat{F}_A$ *dans la direction* $\overline{c} \in \mathbf{E}_q$. Ce point de vue admet diverses justifications, dont celle-ci : selon la théorie asymptotique $q$-Gevrey développée par Ramis et Zhang dans [**29**], $S_{\overline{c}}\hat{F}_A$ est asymptote à $\hat{F}_A$.

Il est facile de déduire du théorème que, si $A$ et $A'$ sont sous la forme (4) (avec même gradué $A_0$), alors $S_{\overline{c}}\hat{F}_{A'}\left(S_{\overline{c}}\hat{F}_A\right)^{-1}$ est l'unique isomorphisme méromorphe de $A$ dans $A'$ satisfaisant aux mêmes conditions de polarité : il est donc légitime de le noter $S_{\overline{c}}\hat{F}_{A,A'}$. On peut alors démontrer :

***Proposition 3.2***. — *Soit* $\overline{c} \in \mathbf{E}_q \setminus \Sigma_{A_0}$. *Pour que* $A$ *et* $A'$ *soient dans la même classe analytique, il faut, et il suffit, que* $S_{\overline{c}}\hat{F}_{A,A'}$ *n'ait pas de pôle sur* $[-c; q]$.

Naturellement, dans ce cas, $\hat{F}_{A,A'}$ est analytique et tous les $S_{\overline{c}}\hat{F}_{A,A'}$ lui sont égaux. Ce qui est remarquable dans ce résultat, c'est que l'absence des pôles sur $[-c; q]$, qui *a priori* ne devrait entrainer que l'holomorphie sur $\mathbf{C}^*$, suffit en fait à garantir la méromorphie en 0. L'argument est directement lié à celui qui a permis de prouver le théorème 2.1.

3.1.0.5. *Opérateurs de Stokes.* — Classiquement, l'ambiguïté dans la sommation d'une solution divergente d'équation fonctionnelle est le *phénomène de Stokes*. Il se réalise ici sous la forme suivante. On prend deux directions de sommation autorisées $\overline{c}, \overline{d} \in \mathbf{E}_q \setminus \Sigma_{A_0}$ et l'on pose :

$$S_{\overline{c},\overline{d}}\hat{F}_A = \left(S_{\overline{c}}\hat{F}_A\right)^{-1} S_{\overline{d}}\hat{F}_A.$$

C'est donc un automorphisme méromorphe de $A_0$. Il est holomorphe sur l'ouvert $U_{c,d} = \mathbf{C}^* \setminus [-c, -d; q]$ de $\mathbf{C}^*$. On peut aussi bien le considérer comme un automorphisme méromorphe du fibré $\mathcal{F}_{A_0}$, holomorphe sur l'ouvert $V_{\overline{c},\overline{d}} = \mathbf{E}_q \setminus \{\overline{c}, \overline{d}\}$ de $\mathbf{E}_q$. On a donc $U_{c,d} = \pi^{-1}\left(V_{\overline{c},\overline{d}}\right)$.

On définit le faisceau en groupes (non commutatifs) $\Lambda_I(M_0)$ comme suit ; pour tout ouvert $V$ de $\mathbf{E}_q$ :

$$\Gamma\left(V, \Lambda_I(M_0)\right) = \{F \in \mathfrak{G}\left(\mathcal{O}(\pi^{-1}(V))\right) \mid F[A_0] = A_0\}.$$

C'est le faisceau des *automorphismes de* $M_0$ *tangents à l'identité*. La terminologie (empruntée au cas analogue des équations différentielles) est justifiée par le fait que, si $F \in \Gamma\left(V, \Lambda_I(M_0)\right)$, alors $F - I_n$ est plat près de 0 sur $V$. En effet, tout bloc hors diagonal $F_{i,j}$ vérifie $\sigma_q F_{i,j} = z^{\mu_i - \mu_j} A_i F_{i,j} A_j^{-1}$, d'où l'on tire que $\theta_q^{\mu_j - \mu_i} F_{i,j}$ est à croissance modérée près de 0 sur son ouvert de définition. On dit que $F_{i,j}$ est *t-plat*, où $t = \mu_j - \mu_i$.

On a donc :

$$S_{\overline{c},\overline{d}}\hat{F}_A \in \Gamma\left(V_{\overline{c},\overline{d}}, \Lambda_I(M_0)\right).$$

Notant $V_{\overline{c}} = \mathbf{E}_q \setminus \{\overline{c}\}$, il est donc clair que l'on a défini un élément de l'ensemble $Z^1(\mathfrak{V}, \Lambda_I(M_0))$ des cocycles du faisceau en groupes $\Lambda_I(M_0)$ associés au recouvrement $\mathfrak{V} = (V_{\overline{c}})$ de $\mathbf{E}_q$ ($Z^1(\mathfrak{V}, \Lambda_I(M_0))$ est bien un ensemble pointé, pas un groupe). Voici le théorème de type Malgrange-Sibuya annoncé :

***Théorème 3.3***. — *On obtient ainsi une bijection :*

$$\mathscr{F}(M_0) \simeq H^1(\mathbf{E}_q, \Lambda_I(M_0)).$$

3.1.0.6. *Dévissage q-Gevrey.* — On peut plagier le *dévissage Gevrey* de Ramis en un dévissage $q$-Gevrey du faisceau $\Lambda_I(M_0)$ et de l'ensemble de cohomologie $H^1(\mathbf{E}_q, \Lambda_I(M_0))$. Tout d'abord, il s'agit d'un faisceau de groupes unipotents dont on peut facilement décrire le faisceau $\Lambda_I$ des algèbres de Lie :

$$\Gamma(V, \lambda_I(M_0)) = \{F \in \mathfrak{g}\left(\mathscr{O}(\pi^{-1}(V))\right) \mid (\sigma_q F)A_0 = A_0 F\}.$$

Naturellement, $\mathfrak{g}$ désigne l'algèbre de Lie de $\mathfrak{G}$, définie par :

$$F \in \mathfrak{g}(K) \Leftrightarrow I_n + F \in \mathfrak{G}(K) \Leftrightarrow \exp(F) \in \mathfrak{G}(K).$$

On note $\lambda_I^t(M_0)$ le sous-faisceau en algèbres de Lie nilpotentes de $\lambda_I(M_0)$ formé des éléments $t$-plats. On peut montrer qu'il s'agit des éléments dont les seuls blocs $F_{i,j}$ non nuls sont ceux tels que $\mu_j - \mu_i \geq t$ (donc triangulaires supérieurs par blocs, « assez loin » de la diagonale).

Si l'on note de même $\lambda_I^{(t)}(M_0)$ le sous-faisceau formé des éléments dont les seuls blocs $F_{i,j}$ non nuls sont ceux tels que $\mu_j - \mu_i = t$ (donc ne comportant qu'une « surdiagonale par blocs »), on conctate que $\lambda_I(M_0)$ est graduée :

$$\lambda_I(M_0) = \bigoplus_{t \in \mathbf{N}^*} \lambda_I^{(t)}(M_0).$$

Les $\lambda_I^t(M_0)$ forment la filtration correspondante par des idéaux :

$$\lambda_I^t(M_0) = \bigoplus_{t' \geq t} \lambda_I^{(t')}(M_0).$$

***Proposition 3.4***. — *(i) Le faisceau $\lambda_I^{(t)}(M_0)$ est localement libre. C'est le fibré associé au module pur isocline : $\underline{\mathrm{End}}_{(-t)}(M_0)$.*

*(ii) Le faisceau $\lambda_I(M_0)$ est localement libre. C'est le fibré associé au module pur : $\underline{\mathrm{End}}_{<0}(M_0)$.*

Rappelons que $\underline{\mathrm{Hom}}$ désigne le *Hom interne*, $\underline{\mathrm{End}}(M_0)$ n'est autre que le module $\underline{\mathrm{Hom}}(M_0, M_0) = \bigoplus \underline{\mathrm{Hom}}(P_i, P_j)$ (où $M_0 = \bigoplus P_i$, chaque $P_i$ étant pur isocline de pente $\mu_i$). On a alors : $\underline{\mathrm{End}}(M_0)_{(t)} = \bigoplus\limits_{\mu_j - \mu_i = t} \underline{\mathrm{Hom}}(P_i, P_j)$.

À la *graduation* de $\lambda_I(M_0)$ correspond une *filtration* de $\Lambda_I(M_0)$ par les (faisceaux en) sous-groupes distingués :

$$\Lambda_I^t(M_0) = I_n + \lambda_I^t(M_0) = \exp \lambda_I^t(M_0).$$

Le *dévissage q-Gevrey* du faisceau en groupes unipotents non commutatifs par des fibrés est donné par les suites exactes :

$$1 \to \Lambda_I^{t+1}(M_0) \to \Lambda_I^t(M_0) \to \lambda_I^{(t)}(M_0) \to 0. \tag{7}$$

Il y a aussi une suite d'extensions centrales :

$$0 \to \lambda_I^{(t)}(M_0) \to \frac{\Lambda_I(M_0)}{\Lambda_I^{t+1}(M_0)} \to \frac{\Lambda_I(M_0)}{\Lambda_I^t(M_0)} \to 1. \tag{8}$$

Ce sont ces suites qui permettent une preuve « élémentaire » du théorème 3.3.

**3.2. Calculs explicites d'invariants.** — Puisque l'espace de modules $\mathscr{F}(M_0)$ est de dimension finie, on doit pouvoir déterminer des coordonnées, *i.e.* un jeu complet d'invariants finis. Les $\operatorname{irr}(M_0)$ coefficients des polynômes $U_{i,j}$ dans l'écriture en forme normale conviennent, mais ne sont pas très intéressants.

On va détailler ici le cas d'un polygone de Newton à deux pentes. Trois types d'invariants se présentent. Le premier est fourni par la transformation de $q$-Borel (qui, en analyse, devrait être accompagnée de la ransformation de $q$-Laplace, [**42**]). En théorie de Galois apparaissent les $q$-dérivées étrangères ([**24**] et [**25**]). Enfin, nous allons rencontrer une application inattendue de la dualité de Serre. Actuellement, nous ne savons définir pour un nombre arbitraire de pentes que les $q$-dérivées étrangères (et les pentes doivent être entières).

***Lemme 3.5***. — *Pour tout module $M$, on a une identification naturelle :*

$$\Gamma^1(M) \simeq H^1(\mathbf{E}_q, \mathscr{F}_M).$$

*Démonstration.* — Le foncteur exact à gauche $M \rightsquigarrow \Gamma(M)$ s'identifie naturellement au foncteur composé du foncteur exact $M \rightsquigarrow \mathscr{F}_M$ et du foncteur exact à gauche $\mathscr{F} \rightsquigarrow \Gamma(\mathbf{E}_q, \mathscr{F})$. Leurs foncteurs dérivés à droite sont donc égaux. □

Soit $M_0 = P_1 \oplus P_2$, où $P_i = M_{z^{\mu_i} A_i}$, $A_i \in \mathscr{GL}_{r_i}(\mathbf{C})$ $(i = 1, 2)$, $\mu_1 < \mu_2 \in \mathbf{Z}$. L'espace $\mathscr{F}(M_0)$ s'identifie naturellement à $Ext(P_2, P_1)$ et ce dernier, d'après la section 1.2.2, à $\Gamma^1(P)$, où $P = P_2^\vee \otimes P_1$ est un module pur de pente $\mu = \mu_1 - \mu_2 < 0$ et de rang $r = r_1 r_2$. Pour l'étude du cas de deux pentes, on peut donc se restreindre aux calculs sur $M_0 = P \oplus \underline{1} = M_{A_0}$, où $P = M_{z^\mu} A$ :

$$A_0 = \begin{pmatrix} z^\mu A & 0 \\ 0 & 1 \end{pmatrix}, \quad \mu \in \mathbf{Z}, \mu < 0 \text{ et } A \in \mathscr{GL}_r(\mathbf{C}), r \in \mathbf{N}^*.$$

Nous poserons $d = -\mu$ (« niveau » $q$-Gevrey). Nous allons expliciter dans ce cas les isomorphismes $\mathscr{F}(M_0) = \Gamma^1(P) \simeq H^1(\mathbf{E}_q, \mathscr{F}_{M_0})$.

Les sections de $\Lambda_I(M_0)$ sont ici les matrices $\left(\begin{smallmatrix} I_r & F \\ 0 & 1 \end{smallmatrix}\right)$ telles que $\sigma_q F = z^\mu A F$, la multiplication dans le groupe $\Gamma(V, \Lambda_I(M_0))$ correspondant à l'addition de leurs composantes $F$. Le faisceau $\Lambda_I(M_0)$ est donc isomorphe au fibré $\mathscr{F}_P = \mathscr{F}_{z^\mu A}$. Puisque l'on a un faisceau en groupes commutatifs, le $H^1$ est ici un groupe (cohomologie des faisceaux abéliens).

3.2.0.7. *Calcul de cocycles.* — On va calculer des cocycles par « sommation ». Pour cela, on choisit un module $M_U$, de matrice $A_U = \begin{pmatrix} z^\mu A & U \\ 0 & 1 \end{pmatrix}$ de gradué $A_0$, avec $U \in \mathrm{Mat}_{r,1}(\mathbf{C}(\{z\}))$ ; on impose que $U$ soit polynomial (en fait, holomorphe sur $\mathbf{C}^*$ et méromorphe en 0 suffit pour les calculs qui suivent). On fixe une direction de sommation $\overline{c} \in \mathbf{E}_q$, et l'on cherche $\begin{pmatrix} I_r & F \\ 0 & 1 \end{pmatrix}$ qui soit un isomorphisme méromorphe de $A_0$ dans $A_U$, avec pour seuls pôles $[-c; q]$, de multiplicité $\geq d = -\mu$. L'équation satisfaite par $F$ est :

$$\sigma_q F - z^\mu AF = U.$$

On pose $F = \frac{G_c}{\theta_{q,c}^d}$. On est ramené à résoudre l'équation :

$$c^d \sigma_q G_c - AG_c = U\theta_{q,c}^d.$$

On développe en séries de Laurent $G_c = \sum G_n z^n$, $V = U\theta_{q,c}^d = \sum V_n z^n$, d'où :

$$\forall n \in \mathbf{Z}\ ,\ (c^d q^n I_r - A)G_n = V_n.$$

Supposons maintenant que toutes les matrices $c^d q^n I_r - A$ soient inversibles, *i.e.* éléments de $\mathcal{GL}_r(\mathbf{C})$. Cela équivaut à :

$$\overline{c} \in \mathbf{E}_q \setminus \Sigma_{A_0}, \quad \text{où } \Sigma_{A_0} = \overline{\sqrt[d]{\mathrm{Sp}(A)}} = \sqrt[d]{\mathrm{Sp}(A)} \pmod{q^{\mathbf{Z}}}.$$

Notons que, génériquement (c'est-à-dire si $\mathrm{Sp}(A)$ admet $r$ valeurs distinctes modulo $q^{\mathbf{Z}}$), l'ensemble interdit $\Sigma_{A_0}$ a $rd = \mathrm{irr}(M_0)$ éléments. Pour une direction autorisée, les équations ci-dessus admettent une unique solution :

$$\begin{aligned} G_c &= \sum_{n \in \mathbf{Z}} (c^d q^n I_r - A)^{-1} V_n z^n, \\ F = F_{\overline{c}} &= \frac{1}{\theta_{q,c}^d} \sum_{n \in \mathbf{Z}} (c^d q^n I_r - A)^{-1} V_n z^n. \end{aligned}$$

La notation $F_{\overline{c}}$ est légitime, car cette fonction ne dépend que de la classe de $c$ dans $\mathbf{E}_q$. Si l'on pose :

$$F_{\overline{c},\overline{d}} = F_{\overline{d}} - F_{\overline{c}},$$

on voit que $F_{\overline{c},\overline{d}}$ est holomorphe sur $\mathbf{C}^* \setminus [-c, -d; q]$ et vérifie $\sigma_q F_{\overline{c},\overline{d}} = z^\mu AF_{\overline{c},\overline{d}}$. Les fonctions $F_{\overline{c},\overline{d}}$ sont des sections du fibré $\mathcal{F}_P = \mathcal{F}_{z^\mu A}$ sur les ouverts $V_{\overline{c},\overline{d}}$, et constituent un cocycle. La classe de ce cocycle dans le groupe de cohomologie $H^1(\mathbf{E}_q, \mathcal{F}_{M_0})$ s'identifie à la fois à la classe de $A_U$ dans $\mathcal{F}(M_0)$ et à la classe de l'extension $M_U$ de $\underline{1}$ par $P$ dans $\Gamma^1(P) = Ext(\underline{1}, P)$. Nous noterons $\mathrm{cl}(M_U)$ cette classe (dans l'un quelconque des trois ensembles ainsi identifiés).

3.2.0.8. *Invariants à la q-Borel.* — Pour résoudre (ou pour étudier l'obtruction à résoudre) l'équation $\sigma_q F - z^\mu AF = U$, on développe en séries de Laurent : $F = \sum F_n z^n$, $U = \sum U_n z^n$, d'où :

$$\forall n \in \mathbf{Z}\ ,\ q^n F_n - AF_{n+d} = U_n.$$

On introduit des coefficients $t_n$ tels que :

$$\forall n \in \mathbf{Z} \ , \ q^n t_n = t_{n-d}.$$

Par exemple, le développement en série de Laurent $\theta_q^d = \sum t_n z^n$ fournit de tels coefficients, comme il découle de l'équation fonctionnelle $\sigma_q \theta_q^d = z^d \theta_q^d$. On appelle *transformées de q-Borel au niveau d* d'une série $f(z) = \sum\limits_{n \in \mathbf{Z}} f_n z^n$ la série :

$$\mathcal{B}_q^{(\delta)} f(\xi) = \sum_{n \in \mathbf{Z}} q^{-n} t_{-n} f_n \xi^n.$$

Un petit calcul montre que notre relation ci-dessus équivaut à :

$$(I_r - \xi^{-d} A) \mathcal{B}_q^{(\delta)} F(q\xi) = \mathcal{B}_q^{(\delta)} U(\xi).$$

Si $U$ (resp. $F$) est analytique, $\mathcal{B}_q^{(\delta)} U$ (resp. $\mathcal{B}_q^{(\delta)} F$) est entière, et l'on peut évaluer cette égalité partout. On choisit une matrice $B$ racine $d^{\text{e}}$ de $A$, et l'on voit qu'une condition nécessaire est l'annulation des $\mathcal{B}_q^{(\delta)}(j\mathcal{B})$ pour $j^d = 1$. C'est en fait une condition suffisante, et l'on peut démontrer que les $d$ vecteurs $\mathcal{B}_q^{(\delta)}(j\mathcal{B}) \in \mathbf{C}^r$ constituent un jeu complet d'invariants analytiques. Plus précisément, l'application qui, à $U$, associe le $d$-uplet des $\mathcal{B}_q^{(\delta)}(j\mathcal{B})$ induit un isomorphisme de l'espace vectoriel $\mathcal{F}(M_0)$ sur $\mathbf{C}^{rd}$.

3.2.0.9. *Invariants de* **[24]**, **[25]**. — On fixe un $a \in \mathbf{C}^*$ « générique » (pratiquement, loin de tous les points qui vont intervenir). Il jouera le rôle d'un point-base. On fixe une direction de sommation autorisée arbitraire $\overline{c_0} \in \mathbf{E}_q$. L'application $\overline{d} \mapsto F_{\overline{c_0}, \overline{d}}(a)$ est méromorphe sur $\mathbf{E}_q$, avec des pôles sur $\Sigma_{A_0}$. Elle est à valeurs dans *l'algèbre de Lie* $\mathfrak{st}(M_U)$ *du groupe de Stokes* $\mathfrak{St}(M_U)$. (C'est pour cela que l'on a dû introduire $\overline{c_0}$, qui n'intervient pas dans le résultat du calcul). La prise de résidu est une intégration, donc donne un résultat dans l'espace vectoriel $\mathfrak{st}(M_U)$. On pose, pour tout $\overline{c} \in \Sigma_{A_0}$ :

$$\dot{\Delta}_{\overline{c}}(A_U) = \operatorname{Res}_{\overline{d} = \overline{c}} F_{\overline{c_0}, \overline{d}}(a).$$

Les $\dot{\Delta}_{\overline{c}}(A_U)$ prennent leurs valeurs dans des espaces vectoriels de dimension totale $rd$ et constituent un jeu complet d'invariants analytiques. Dans **[24]**, on donne des formules de transformations entre ces « $q$-dérivées étrangères » et les invariants de $q$-Borel.

3.2.0.10. *Invariants provenant de la dualité de Serre.* — Il s'agit de la dualité de Serre pour les fibrés vectoriels holomorphes sur une surface de Riemann compacte (**[16]**). Le diviseur canonique de $\mathbf{E}_q$ est trivial. Il y a en effet une différentielle globale de diviseur nul : c'est, par exemple $\frac{dz}{z} = 2i\pi dx$, où $x$ et $z$ sont les uniformisantes globales provenant des revêtements $\mathbf{C} \to \mathbf{E}_q$ et $\mathbf{C}^* \to \mathbf{E}_q$. De la dualité de Serre, on déduit alors que, pour tout fibré sur $\mathbf{E}_q$, les espaces vectoriels $H^0(\mathbf{E}_q, \mathcal{F}^\vee)$ et $H^1(\mathbf{E}_q, \mathcal{F})$ sont duaux l'un de l'autre. Une base de $H^0(\mathbf{E}_q, \mathcal{F}_{M_0}^\vee)$ fournira donc un système de coordonnées sur $H^1(\mathbf{E}_q, \mathcal{F}_{M_0})$, c'est-à-dire un jeu complet d'invariants analytiques pour $\mathcal{F}(M_0)$.

Le mécanisme de dualité est le suivant. Soit $X$ une section globale de $\mathscr{F}^{\vee}_{M_0}$ : c'est donc une solution holomorphe sur $\mathbf{C}^*$ de l'équation :

$$\sigma_q X = z^{-\mu}\, {}^tA^{-1}X.$$

Alors $Y = {}^tX$ est holomorphe sur $\mathbf{C}^*$ et vérifie : $\sigma_q Y = z^{-\mu} Y A^{-1}$. On forme le produit scalaire

$$\phi_{\overline{c},\overline{d}} = <X, F_{\overline{c},\overline{d}}> = {}^tXF_{\overline{c},\overline{d}} = YF_{\overline{c},\overline{d}}.$$

C'est une fonction méromorphe sur $\mathbf{C}^*$ avec des pôles connus, et, des équations satisfaites par $Y$ et $F_{\overline{c},\overline{d}}$, on déduit :

$$\sigma_q\phi_{\overline{c},\overline{d}} = \sigma_q Y \sigma_q F_{\overline{c},\overline{d}} = z^{-\mu}\, YA^{-1}\, z^{\mu}AF_{\overline{c},\overline{d}} = \phi_{\overline{c},\overline{d}}.$$

On a donc une fonction elliptique $\phi_{\overline{c},\overline{d}}$ de pôles $\overline{c},\overline{d} \in \mathbf{E}_q$. La somme de ses résidus sur $\mathbf{E}_q$ est donc nulle. Par ailleurs, cette fonction est, par définition, une différence :

$$\phi_{\overline{c},\overline{d}} = <X, F_{\overline{d}} - F_{\overline{c}}> = <X, F_{\overline{d}}> - <X, F_{\overline{c}}> = \phi_{\overline{d}} - \phi_{\overline{c}},$$

où chaque section $\phi_{\overline{c}} = <X, F_{\overline{c}}>$ a un seul pôle sur $\mathbf{E}_q$ (une $q$-spirale sur $\mathbf{C}^*$), à savoir $\overline{-c}$. On a donc :

$$\mathrm{Res}_{\overline{-c}}\, \phi_{\overline{c}} = \mathrm{Res}_{\overline{-d}}\, \phi_{\overline{d}}.$$

(On prendra garde qu'il s'agit ici de résidus par rapport à la variable $z$ et non par rapport à la direction de sommation, comme dans le cas des $\dot{\Delta}_{\overline{c}}$!) Par définition, le nombre calculé ci-dessus, qui ne dépend pas de $\overline{c} \in \mathbf{E}_q$, est celui associé par la dualité de Serre à $X \in H^0(\mathbf{E}_q, \mathscr{F}^{\vee}_{M_0})$ et à $\mathrm{cl}(M_U) \in H^1(\mathbf{E}_q, \mathscr{F}_{M_0})$. Nous le noterons $<X, \mathrm{cl}(M_U)>$.

***Lemme 3.6***. — *On a les égalités :*

$$<X, \mathrm{cl}(M_U)> = [(\sigma_q X)U]_0 = [z^{d\,t}XA^{-1}U]_0 = [{}^tXA^{-1}U]_\mu.$$

*Démonstration.* — Nous employons la notation commode suivante : si $f = \sum f_n z^n$, alors $[f]_n = f_n$. Ici, $X = \sum X_n z^n$, $U = \sum U_n z^n$ (séries de Laurent convergentes sur $\mathbf{C}^*$) et l'on calcule le coefficient de degré 0 (resp. de degré $\mu$) de $z^{d\,t}XA^{-1}U$ (resp. de ${}^tXA^{-1}U$). La dernière égalité est donc triviale. La seconde l'est également, puisque $\sigma_q X = z^{d}\, {}^tXA^{-1}$.
Pour calculer le résidu en $\overline{-c} \in \mathbf{E}_q$, on choisit un représentant $c \in \mathbf{C}^*$ et une couronne fondamentale contenant le pôle $-c$,. On écrit la frontière orientée de cette couronne sous la forme $q\gamma - \gamma$, où $\gamma$ est un cercle de centre 0 parcouru positivement. Alors, par le théorème des résidus :

$$\begin{aligned} <X, \mathrm{cl}(M_U)> = \mathrm{Res}_{\overline{-c}}\, \phi_{\overline{c}} &= \int_{q\gamma} {}^tXF_{\overline{c}} \frac{dz}{2\mathrm{i}\pi z} - \int_{\gamma} {}^tXF_{\overline{c}} \frac{dz}{2\mathrm{i}\pi z} \\ &= \int_{\gamma} \left(\sigma_q({}^tXF_{\overline{c}}) - {}^tXF_{\overline{c}}\right) \frac{dz}{2\mathrm{i}\pi z} = \int_{\gamma} (\sigma_q X)U \frac{dz}{2\mathrm{i}\pi z} = [(\sigma_q X)U]_0. \quad \square \end{aligned}$$

Donc $<X, \mathrm{cl}(M_U)> = \sum q^{-n} X_n U_n$. Pour produire concrètement de tels nombres, il est naturel de construire la section $X$ à l'aide de fonctions Theta. On choisit une racine $d^{\mathrm{e}}$ $B$ de la matrice $A$, et l'on voit que :

$$T_B = \Theta_B^d(z) \underset{\text{déf}}{=} \theta^d(B^{-1}z) = \sum_{n \in \mathbf{Z}} t_n B^{-n} z^n$$

vérifie $\sigma_q T_B = T_B z^d A^{-1}$. On peut donc prendre pour ${}^t X$ l'une quelconque des $r$ lignes de $T_B$. En faisant le même calcul pour chacune des $d$ matrices $jB$ ($j^d = 1$), on trouve une base de $H^0(\mathbf{E}_q, \mathscr{F}_{M_0}^\vee)$ et les invariants correspondants sont les invariants de $q$-Borel calculés précédemment.

## 4. Constructions globales

Historiquement, les équations aux $q$-différences intéressantes sont définies sur la sphère de Riemann $\mathbf{S}$, c'est-à-dire à coefficients rationnels : matrice $A \in \mathscr{GL}_n(\mathbf{C}(z))$ codant un objet $M_A$ de $\mathscr{E} = \mathrm{DiffMod}\,(\mathbf{C}(z), \sigma_q)$ (voir [**3**]). Par exemple, l'équation $q$-hypergéométrique ([**12**], [**33**], [**6**]).

Pour étudier une équation globale, on la localise. Il n'y a *a priori* que deux points possibles pour localiser, 0 et $\infty$, car ce sont les seuls points de $\mathbf{S}$ fixés par la dilatation $z \mapsto qz$. En localisant en 0, c'est à dire par extension de base $\mathbf{C}(z) \hookrightarrow \mathbf{C}(\{z\})$, on obtient un plongement $\mathscr{E} \hookrightarrow \mathscr{E}^{(0)}$. Il y a similairement une localisation en $\infty$ : $\mathscr{E} \hookrightarrow \mathscr{E}^{(\infty)}$ (cette dernière catégorie est d'ailleurs isomorphe à $\mathscr{E}^{(0)}$).

En fait, il y aurait lieu de localiser aux *singularités intermédiaires* de $A$, *i.e.* les singularités dans $\mathbf{C}^*$. Mais elles bougent sous l'action de $z \mapsto qz$, et il faut en fait trouver une notion de localisation en une $q$-spirale ou en un point de $\mathbf{E}_q$. Le peu que nous savons faire en ce sens est l'objet de cette section.

### 4.1. Le cas des équations fuchsiennes.

— C'est celui où Birkhoff a posé et résolu le problème du recollement des données locales. Classiquement, pour les équations différentielles, c'est le problème des matrices de connexion. La corrsepondance de Riemann-Hilbert permet, à partir des monodromies locales et d'un nombre fini de matrices de connexion, de reconstruire la classe d'équivalence rationnelle d'une équation différentielle linéaire fuchsienne. Sous forme moderne, les données locales et de connexion sont traduites comme une représentation de monodromie.

Disons que $A \in \mathscr{GL}_n(\mathbf{C}(z))$ est *fuchsienne* si elle l'est en 0 et $\infty$ (*i.e.* dans $\mathscr{E}^{(0)}$ et dans $\mathscr{E}^{(\infty)}$). Il revient au même de dire que $A$ est rationnellement équivalente (c'est-à-dire via une transformation de jauge $F \in \mathscr{GL}_n(\mathbf{C}(z))$ à une matrice qui est non singulière en 0 et $\infty$. Nous supposerons donc, comme le fait Birkhoff, que $A(0), A(\infty) \in \mathscr{GL}_n(\mathbf{C})$.

Birkhoff construit alors des « solutions locales » fondamentales $\mathscr{X}^{(0)}$ et $\mathscr{X}^{(\infty)}$ de $\sigma_q X = AX$, qui sont méromorphes (mais multiformes) sur $\mathbf{C}^*$, et définit la *matrice* *de connexion de Birkhoff* :

$$P = \left(\mathcal{X}^{(\infty)}\right)^{-1} \mathcal{X}^{(0)}.$$

La matrice $P$ est méromorphe (mais multiforme) sur $\mathbf{C}^*$ et vérifie $P(qz) = P(z)$ par construction (elle connecte deux solutions d'une même équation, elle est donc « $q$-constante »). Elle est donc presque elliptique. Birkhoff démontre que la donnée d'un nombre fini d'invariants locaux en 0 et $\infty$ et de la matrice de connexion $P$ permet de retrouver la matrice $A$ à équivalence rationnelle près. C'est pour prouver l'*existence* d'une matrice $A$ qu'il a inventé le théorème de factorisation dont nous avons parlé au début.

Le travail a été repris dans **[33]**, en n'utilisant que des fonctions méromorphes uniformes. Par exemple, on résout $\sigma_q f = cf$ $(c \in \mathbf{C}^*)$ à l'aide de $e_{q,c} = \theta_q/\theta_{q,c}$, là où Birkhoff utilisait $z^{\log_q \gamma}$. Ainsi, $P \in \mathcal{GL}_n(\mathcal{M}(\mathbf{E}_q))$. Le résultat de Birkhoff prend la forme précise d'une équivalence de catégories. L'inconvénient de la matrice de Birkhoff (dans sa version d'origine ou dans celle de **[33]**) est qu'elle ne se comporte pas bien par produit tensoriel, d'où des difficultés pour la théorie de Galois (c'est-à-dire pour obtenir une représentation de groupes). Ce mauvais comportement a sa source dans le fait que $e_{q,c}e_{q,d} \neq e_{q,cd}$ et ce, quel que soit le choix fait des solutions méromorphes $e_{q,c}$.

En fait, pour des équations *régulières en* 0 *et* $\infty$, c'est-à-dire telles que $A(0) = A(\infty) = I_n$, on n'a pas besoin de fonctions spéciales pour résoudre l'équation, on peut le faire avec des séries convergentes. Etingof a montré dans ce cas (**[9]**) que les valeurs de la matrice de connexion engendrent le groupe de Galois. Sans cette hypothèse, van der Put et Singer ont obtenu un résultat similaire mais à l'aide de la résolution symbolique : les $e_{q,c}$ sont remplacés par des symboles $e_c$ tels que $e_c e_d = e_{cd}$. On n'obtient pas par cette voie de véritables invariants transcendants.

4.1.0.11. *Avec des fibrés, ça marche mieux.* — Dans **[34]**, on procède comme suit. D'après le lemme-clé de la section 2.1.1, on peut écrire :

$$\begin{aligned} A &= F^{(0)}\left[A^{(0)}\right], \quad A^{(0)} \in \mathcal{GL}_n(\mathbf{C}) \text{ et } F^{(0)}(z) \in \mathcal{GL}_n(\mathbf{C}(\{z\})) \\ &= F^{(\infty)}\left[A^{(\infty)}\right], \quad A^{(\infty)} \in \mathcal{GL}_n(\mathbf{C}) \text{ et } F^{(\infty)}(w) \in \mathcal{GL}_n(\mathbf{C}(\{z\})), \end{aligned}$$

où $w = z^{-1}$ (uniformisante en $\infty \in \mathbf{S}$). Des équations fonctionnelles satisfaites par $F^{(0)}$ et $F^{(\infty)}(w)$ découle d'ailleurs qu'ils admettent des prolongements méromorphes respectivement à $\mathbf{C}$ et à $\mathbf{S} \setminus \{0\}$.

La matrice $F = \left(F^{(\infty)}\right)^{-1} F^{(0)}$ vérifie alors : $F \in \mathcal{GL}_n(\mathcal{M}(\mathbf{C}^*))$ et $F\left[A^{(0)}\right] = A^{(\infty)}$. Elle code donc un *isomorphisme méromorphe* :

$$\phi : \mathcal{F}^{(0)} = \mathcal{F}_{A^{(0)}} \to \mathcal{F}^{(\infty)} = \mathcal{F}_{A^{(\infty)}}.$$

En fait, $\mathcal{F}^{(0)}$ est le fibré local étudié aux sections 2 et 3. On démontre alors que $A \rightsquigarrow (\mathcal{F}^{(0)}, \phi, \mathcal{F}^{(\infty)})$ est une *équivalence de catégories tannakiennes.* On en déduit que le groupe de Galois de $A$ est engendré par ses composantes locales $G_f^{(0)}$ et $G_f^{(\infty)}$, plus des données de recollement qui sont essentiellement les valeurs de $\phi$. Le résultat d'Etingof est un cas particulier.

**Remarque.** — Un premier résultat, beaucoup moins élégant, avait consisté à tordre la matrice de connexion, de manière assez compliquée, pour que ses valeurs contribuent au groupe de Galois. C'est cette version peu conceptuelle qui sert dans la pratique : pour la confluence des automorphismes galoisiens dans [**34**] ; et pour le calcul par Julien Roques de la plupart des groupes de Galois $q$-hypergéométriques dans [**32**].

4.1.0.12. *Localisation dans le cas abélien.* — Les constructions ci-dessus ont toutes le même défaut : il faut une quantité non-dénombrable de générateurs $P(a)$ ou $\phi(a)$ pour engendrer le groupe de Galois. Cela est à comparer avec la représentation de monodromie, qui est de type fini. Le groupe de Galois obtenu par voie algébrique n'a pas le caractère discret et transcendant de la « vraie » correspondance de Riemann-Hilbert.

Dans le cas régulier, ou seule la matrice de connexion compte, on a vu que le groupe de Galois de $A$ est paramétré par la fonction matricielle elliptique $P$. (En fait, par la fonction $a \mapsto P(a_0)^{-1}P(a)$, mais on peut supposer que $P(a_0) = I_n$). Comme une fonction méromorphe sur la surface de Riemann $\mathbf{E}_q$ est la même chose qu'une fonction rationnelle sur la courbe algébrique $\mathbf{E}_q$, on a donc un groupe algébrique rationnellement paramétré par une courbe algébrique. Dans le cas où le groupe est abélien, cette situation relève de la *théorie géométrique du corps de classes* ([**39**]). On a pu ainsi, dans [**34**] localiser la matrice de connexion et en déduire un équivalent raisonnable du groupe de monodromie. Cette description est trop lourde pour être reprise ici. Elle est surtout peu utile (sauf comme encouragement), car les équations abéliennes sont trop rares (elles sont presque la même chose que les équations de rang 1).

**4.2. Le cas général.** — La vraie généralisation attendue n'est pas tellement le passage du cas fuchsien au cas irrégulier (qui commence à être bien compris), mais le passage au cas non abélien : comment alors localiser l'effet des singularités ? On va proposer une construction intéressante qui réalise platoniquement cette localisation.

Soit $A \in \mathscr{GL}_n(\mathbf{C}(z))$. (Une bonne partie de ce qui suit garde un sens si $A \in \mathscr{GL}_n(\mathscr{M}(\mathbf{C}^*))$.) Notons $Sing(A)$ le lieu singulier de $A$, défini comme sui :

$$Sing(A) = \{\text{pôles de } A \text{ dans } \mathbf{C}^*\} \cup \{\text{pôles de } A^{-1} \text{ dans } \mathbf{C}^*\}.$$

Soit par ailleurs $U$ un ouvert connexe de $\mathbf{C}^*$ vérifiant les deux conditions suivantes :

1. $\pi(U) = \mathbf{E}_q$ ; la restriction à $U$ du revêtement $\pi : \mathbf{C}^* \to \mathbf{E}_q$ est donc un isomorphisme local. Un exemple de tel ouvert est toute couronne $\mathscr{C}(r, R) = \{z \in \mathbf{C}^* \mid r < |z| < R\}$, où $R > r|q|$, en particulier les disques épointés $\mathscr{C}(0, r)$ et $\mathscr{C}(r, \infty)$ pour $r \in \mathbf{R}_+^*$.
2. $U \cap q^{-1}U \cap Sing(A) = \varnothing$ ; l'ouvert $U$ ne contient donc aucune *paire singulière* $\{z, qz\} \subset Sing(A)$. (Une bonne partie de ce qui suit reste valable sans cette condition.)

On pose alors :

$$\mathcal{F}_{U,A} = \frac{U \times \mathbf{C}^n}{\sim_A},$$

avec la même définition de $\sim_A$ que précédemment. C'est un fibré vectoriel holomorphe sur $\mathbf{E}_q$. Le faisceau des sections se calcule ainsi :

$$\Gamma(V, \mathcal{F}_{U,A}) = \{X \in \mathcal{O}\left(U \cap \pi^{-1}(V)\right)^n \mid \sigma_q X = AX\}.$$

Une section $X \in \Gamma(V, \mathcal{F}_{U,A})$, vue comme fonction holomorphe $U \cap \pi^{-1}(V)$, admet automatiquement un prolongement méromorphe à $\pi^{-1}(V)$, en vertu de l'équation fonctionnelle $\sigma_q X = AX$. On peut décrire le fibré $\mathcal{F}_{U,A}$ en termes de diviseurs matriciels, proches de [**41**]. Pour cela on introduit une solution méromorphe fondamentale $\mathcal{X}_0 \in \mathcal{GL}_n(\mathcal{M}(\mathbf{C}^*))$ de $\sigma_q X = AX$. (Il en existe pour les mêmes raisons qu'auparavant.) Alors le faisceau $\mathcal{F}_{U,A}$ est isomorphe, via la transformation de jauge $X = \mathcal{X}_0 Y$, au faisceau $\mathcal{F}_{U,\mathcal{X}_0}$ défini par :

$$\Gamma(V, \mathcal{F}_{U,\mathcal{X}_0}) = \{\mathcal{O}_{\mathbf{E}_q}(V)^n \mid \mathcal{X}_0 Y \text{ est holomorphe sur } U \cap \pi^{-1}(V)\}.$$

On a noté ici $\mathcal{O}_{\mathbf{E}_q}$ le faisceau des fonctions holomorphes sur $\mathbf{E}_q$. Il s'identifie naturellement au faisceau $V \mapsto \mathcal{O}_{\mathbf{C}^*}(\pi^{-1}(V))^{\sigma_q}$ des fonctions holomorphes $\sigma_q$-invariantes sur $\mathbf{C}^*$, ce qui donne un sens au produit $\mathcal{X}_0 Y$.

4.2.0.13. *Application souhaitée à la localisation.* — Si $U$ et $U'$ sont deux ouverts vérifiant les conditions précédentes, il y a un isomorphisme méromorphe naturel $\phi_{U,U',A}$ de $\mathcal{F}_{U,A}$ sur $\mathcal{F}_{U',A}$. Par exemple, si $U$ et $U'$ sont des disques épointés respectivement centrés en 0 et en $\infty$, on obtient essentiellement la matrice de connexion (sous sa forme intrinsèque).

En général, on peut se ramener sans difficulté au cas où $Sing(A) = \{z_1, \ldots, z_l\}$, avec $|z_{i+1}| > |qz_i|$ pour $1 \leq i \leq l-1$ et $\overline{z_i} \neq \overline{z_j}$ pour $1 \leq i < j \leq l$. On peut de plus choisir des réels positifs $r_i$ tels que $|z_i| < r_i < |qz_i|$ pour $1 \leq i \leq l$. Posons de plus $r_0 = 0$, $r_{l+1} = +\infty$ et $U_i = \mathcal{C}(r_{i-1}, r_i)$ pour $1 \leq i \leq l+1$ et $\phi_i = \phi_{U_i,U_{i+1},A}$ pour $1 \leq i \leq l$. Alors chaque isomorphisme méromorphe $\phi_i$ admet pour seule singularité $\overline{z_i} \in \mathbf{E}_q$, et leur produit est la matrice de connexion $\phi = \phi_{U_0,U_{l+1},A}$. On a donc localisé [11] les singularités de celle-ci. De plus, chaque $U_i$ permet de construire des foncteurs fibres, et les valeurs des $\phi_i$ sur $\mathbf{E}_q$ fournissent des opérateurs galoisiens.

Comparons maintenant le problème à celui du groupe de Stokes. On avait également une quantité non dénombrable de générateurs, les $S_{\overline{c},\overline{d}}\hat{F}_A(a)$. Mais ceux-ci étaient tous dans un même groupe unipotent, que l'on a pu remplacer par son algèbre de Lie. Une fois le problème linéarisé (et abélianisé), on pouvait localiser l'effet des singulariés par prise de résidus. Nous ne savons rien faire de tel ici, parce que nous ne connaissons pas de forme normale maniable pour les $\phi_i$.

Il est à noter que Krichever a réussi dans [**18**] à traiter un problème analogue pour les équations aux différences.

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(11) On a choisi d'utiliser de vraies couronnes, à bords circulaires. En fait, on pourrait prendre des couronnes topologiques, n'imposer aucune condition aux $z_i$ et les parcourir dans n'importe quel ordre. Une géométrie et une combinatoire de $\mathbf{C}^*$ interviennent donc ici.

## Références

[1] M. F. Atiyah – Vector bundles over an elliptic curve, *Proc. London Math. Soc.* **7** (1957), p. 414–452.

[2] V. Baranovsky & V. Ginzburg – Conjugacy classes in loop groups and $G$-bundles on elliptic curves, International Mathematics Research Notes, 1996.

[3] G. Birkhoff – The generalized Riemann problem for linear differential equations and the allied problems for linear difference and $q$-difference equations, *Proc. Amer. Acad.* **49** (1913), p. 521–568.

[4] P. Deligne – Catégories tannakiennes, in *The Grothendieck Festschrift, Vol. II*, Progr. Math., vol. 87, Birkhäuser, 1990, p. 111–195.

[5] L. Di Vizio – Arithmetic theory of $q$-difference equations : the $q$-analogue of Grothendieck-Katz's conjecture on $p$-curvatures, *Invent. Math.* **150** (2002), p. 517–578.

[6] L. Di Vizio, J.-P. Ramis, J. Sauloy & C. Zhang – Équations aux $q$-différences, *Gaz. Math.* **96** (2003), p. 20–49.

[7] A. Duval – Confluence $q$-différence vers différence pour un système fuchsien, *Pacific J. Math.* **217** (2004), p. 221–245.

[8] A. Duval & J. Roques – Confluence $q$-différence vers différence, soumis à publication, 2006.

[9] P. I. Etingof – Galois groups and connection matrices of $q$-difference equations, *Electron. Res. Announc. Amer. Math. Soc.* **1** (1995), p. 1–9.

[10] O. Forster – *Lectures on Riemann surfaces*, Graduate Texts in Math., vol. 81, Springer, 1991.

[11] J. Frenkel – Cohomologie non abélienne et espaces fibrés, *Bull. Soc. Math. France* **85** (1957), p. 135–220.

[12] G. Gasper & M. Rahman – *Basic hypergeometric series*, Encyclopedia of Mathematics and its Applications, vol. 35, Cambridge University Press, 1990.

[13] A. Granier – Fibrés vectoriels et équations aux $q$-différences, mémoire de DEA, 2005.

[14] A. Grothendieck – Sur le mémoire de Weil : généralisation des fonctions abéliennes, in *Séminaire Bourbaki, 1956/57, exposé n° 141*, Réédition Soc. Math. France, 1995, p. 57–71.

[15] J. Guenot & R. Narasimhan – *Introduction à la théorie des surfaces de Riemann*, L'Enseignement Mathématique, 1976, Extrait de l'Enseignement Math. (2) **2**1 (1975), no. 2-4, 123–328.

[16] R. C. Gunning – *Lectures on vector bundles over Riemann surfaces*, University of Tokyo Press, 1967.

[17] J.-L. Koszul – Fibrés vectoriels sur les courbes elliptiques, in *Séminaire Bourbaki, 1957/58*, exposé n° 154, 1958.

[18] I. M. Krichever – Analytic theory of difference equations with rational and elliptic coefficients and the Riemann-Hilbert problem, *Uspekhi Mat. Nauk* **59** (2004), p. 111–150.

[19] D. Mumford – *Tata lectures on theta. I*, Progress in Mathematics, vol. 28, Birkhäuser, 1983.

[20] C. Praagman – Fundamental solutions for meromorphic linear difference equations in the complex plane, and related problems, *J. reine angew. Math.* **369** (1986), p. 101–109.

[21] M. v. d. Put – Skew differential fields, differential and difference equations, *Astérisque* **296** (2004), p. 191–205.

[22] M. v. d. Put & M. Reversat – Galois theory of $q$-difference equations, *Ann. Fac. Sci. de Toulouse* **16** (2005), p. 1–54.

[23] M. v. d. Put & M. F. Singer – *Galois theory of difference equations*, Lecture Notes in Math., vol. 1666, Springer, 1997.

[24] J.-P. Ramis & J. Sauloy – The $q$-analogue of the wild fundamental group. I, in *Algebraic, analytic and geometric aspects of complex differential equations and their deformations. Painlevé hierarchies*, RIMS Kôkyûroku Bessatsu, B2, Res. Inst. Math. Sci. (RIMS), Kyoto, 2007, p. 167–193.

[25] ———, The $q$-analogue of the wild fundamental group. II, ce volume.

[26] J.-P. Ramis, J. Sauloy & C. Zhang – La variété des classes analytiques d'équations aux $q$-différences dans une classe formelle, *C. R. Math. Acad. Sci. Paris* **338** (2004), p. 277–280.

[27] ———, Développement asymptotique et sommabilité des solutions des équations linéaires aux $q$-différences, *C. R. Math. Acad. Sci. Paris* **342** (2006), p. 515–518.

[28] ———, Local analytic classification of irregular $q$-difference equations, en préparation, 2007.

[29] J.-P. Ramis & C. Zhang – Développement asymptotique $q$-Gevrey et fonction thêta de Jacobi, *C. R. Math. Acad. Sci. Paris* **335** (2002), p. 899–902.

[30] H. Röhrl – Das Riemann-Hilbertsche Problem der Theorie der linearen Differentialgleichungen, *Math. Ann.* **133** (1957), p. 1–25.

[31] ———, Holomorphic fiber bundles over Riemann surfaces, *Bull. Amer. Math. Soc.* **68** (1962), p. 125–160.

[32] J. Roques – Groupe de Galois des équations $q$-hypergéométriques, soumis à publication, 2007.

[33] J. Sauloy – Systèmes aux $q$-différences singuliers réguliers : classification, matrice de connexion et monodromie, *Ann. Inst. Fourier (Grenoble)* **50** (2000), p. 1021–1071.

[34] ———, Galois theory of Fuchsian $q$-difference equations, *Ann. Sci. École Norm. Sup.* **36** (2003), p. 925–968.

[35] ———, Algebraic construction of the Stokes sheaf for irregular linear $q$-difference equations, *Astérisque* **296** (2004), p. 227–251.

[36] ———, La filtration canonique par les pentes d'un module aux $q$-différences et le gradué associé, *Ann. Inst. Fourier (Grenoble)* **54** (2004), p. 181–210.

[37] ———, Isomonodromy for $q$-difference equations, à paraître dans *Séminaires et congrés*, SMF, *Actes de la Conférence Painlevé, Angers, 2004.*

[38] J-P. Serre – Géométrie algébrique et géométrie analytique, *Ann. Inst. Fourier, Grenoble* **6** (1955–1956), p. 1–42.

[39] ———, *Groupes algébriques et corps de classes*, Publications de l'institut de mathématique de l'université de Nancago, VII. Hermann, Paris, 1959.

[40] C. S. Seshadri – *Fibrés vectoriels sur les courbes algébriques*, Astérisque, vol. 96, Soc. Math. France, 1982.

[41] A. Weil – Généralisation des fonctions abéliennes, *J. Math. Pures et Appl.* **17** (1938), p. 47–87.

[42] C. Zhang – Une sommation discrète pour des équations aux $q$-différences linéaires et à coefficients analytiques : théorie générale et exemples, in *Differential equations and the Stokes phenomenon*, World Sci. Publ., River Edge, NJ, 2002, p. 309–329.

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# SINGULARITIES OF LOGARITHMIC FOLIATIONS AND CONNECTEDNESS OF THE UNION OF LOGARITHMIC COMPONENTS

*by*

Márcio G. Soares

*To José Manuel Aroca on the occasion of his $60^{th}$ birthday*

***Abstract.*** — We show that the singular locus of a logarithmic foliation may have components of any dimension between 0 and $n-2$. Moreover, it is shown that the union of the logarithmic components of the space of codimension one foliations of $\mathbb{P}^n_{\mathbb{C}}$, of fixed degree, is connected.

***Résumé* (Singularités des feuilletages logarithmiques et connectivité de l'union des composantes logarithmiques)**

On montre que le lieu singulier d'un feuilletage logarithmique peut avoir des composantes de dimension 1 à $n-2$. On montre aussi que, dans l'espace des feuilletages de codimension un de $\mathbb{P}^n_{\mathbb{C}}$, de degré fixé, la réunion des composantes logarithmiques est connexe.

## 1. Introduction

In this note we treat two simple questions.

The first is to explore some propoerties of the singular locus of a projective logarithmic foliation. Our motivation for this is the result given in [**6**] which states that, for a generic logarithmic foliation of $\mathbb{P}^n_{\mathbb{C}}$, $n \geq 3$, the singular scheme only exhibits components of dimensions 0 and $n-2$. We exploit this by considering non-generic logarithmic foliations, and by examining the possible dimensions of the components of their singular loci. It turns out that, for a non-generic such foliation, the singular locus may have components of any dimension between 0 and $n-2$, with the only constraint that the dimension $n-2$ is compulsory. On the other hand, we do not know if the isolated singularities, away from the polar divisors of the logarithmic form, which

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necessarily appear in case the foliation is generic (see Theorem 1.5 below), are persistent under a deformation into a foliation whose singular set admits components of dimensions ranging from 1 to $n-3$. In this direction however, see Example 2.6 at the end of this article.

The second consists in showing that the union of the logarithmic components of the space of codimension one foliations of $\mathbb{P}^n_{\mathbb{C}}$, of fixed degree, is connected. This is true also for the rational components of fixed degree. It is perhaps worth remarking that this result resembles a foundational contribution, by R. Hartshorne [**8**], on the connectedness of the Hilbert scheme parametrizing the family of closed subschemes of $\mathbb{P}^n_{\mathbb{C}}$ with a fixed Hilbert polynomial.

We start by recalling some definitions and (known) results.

A codimension one holomorphic foliation $\mathcal{F}$ of $\mathbb{P}^n_{\mathbb{C}}$ is defined by an integrable polynomial 1-form $\omega = \sum_{i=0}^n A_i(z)\,dz_i$, where each $A_i$ is a homogeneous polynomial, say of degree $k-1$, and such that $\omega$ contracts to zero by the radial vector field $R = \sum_{i=0}^n z_i \frac{\partial}{\partial z_i}$, that is, $\iota_R\,\omega = \sum_{i=0}^n z_i A_i(z) \equiv 0$.

Such 1-forms define global sections of $\Omega^1_{\mathbb{P}^n_{\mathbb{C}}}(k)$ and, since $\omega \wedge d\omega \equiv 0$, they make up a Zariski closed subset of $\mathbb{P}(H^0(\Omega^1_{\mathbb{P}^n_{\mathbb{C}}}(k)))$.

The degree of a foliation $\mathcal{F}$, $\deg \mathcal{F}$, is the number of tangencies of the leaves of $\mathcal{F}$ with a generic one-dimensional linear subspace of $\mathbb{P}^n_{\mathbb{C}}$. A simple calculation shows that $\deg \mathcal{F} = k-2$ if the 1-form defining $\mathcal{F}$ has components of degree $k-1$. We denote by $Fol(\mathbb{P}^n_{\mathbb{C}}; k)$ the space of codimension one holomorphic foliations of degree $k-2$ of $\mathbb{P}^n_{\mathbb{C}}$.

A very interesting question is that of recognizing the irreducible components of $Fol(\mathbb{P}^n_{\mathbb{C}}; k)$ and to determine their dimensions and degrees. Results in this direction can be found in [**4**], [**5**] and [**3**].

Logarithmic foliations are defined as follows: given positive integers $d_0, \ldots, d_m$, set $\underline{\mathrm{d}} = d_0, \ldots, d_m$ and $\mathrm{d} = \sum_{i=0}^m d_i$. Consider the hyperplane

$$\mathbb{P}(m-1, \mathrm{d}) = \{(\lambda_0, \ldots, \lambda_m) \in \mathbb{P}^m_{\mathbb{C}} \ :\ \sum_{i=0}^m d_i\,\lambda_i = 0\}. \tag{1}$$

Define a rational map $\Psi$ by

$$\begin{aligned} \mathbb{P}(m-1, \mathrm{d}) \times \prod_{i=0}^m \mathbb{P}\left(H^0(\mathbb{P}^n_{\mathbb{C}}; \mathcal{O}(d_i))\right) &\xrightarrow{\ \Psi\ } Fol(\mathbb{P}^n_{\mathbb{C}}; \mathrm{d}) \\ ((\lambda_0, \ldots, \lambda_m), (F_0, \ldots, F_m)) &\longmapsto \left(\prod_{j=0}^m F_j\right) \sum_{i=0}^m \lambda_i \frac{dF_i}{F_i}. \end{aligned} \tag{2}$$

Note that $\iota_R \left(\prod_{j=0}^m F_j\right) \sum_{i=0}^m \lambda_i \frac{dF_i}{F_i} = \left(\prod_{j=0}^m F_j\right) \sum_{i=0}^m d_i\,\lambda_i \equiv 0$. Also, it's immediate that this form is closed (hence integrable).

The closure of the image of $\Psi$ is the set $\mathrm{Log}_n(d_0, \ldots, d_m)$ of logarithmic foliations of type $\underline{\mathrm{d}}$ and degree $\mathrm{d}-2$ of $\mathbb{P}^n_{\mathbb{C}}$.

Let us recall the following result, due to Calvo-Andrade [**2**]:

***Theorem 1.1.*** — *For fixed* $\underline{\mathrm{d}}$ *and* $n \geq 3$, $\mathrm{Log}_n(d_0, \dots, d_m)$ *is an irreducible component of* $Fol(\mathbb{P}^n_{\mathbb{C}}; \mathrm{d})$.

***Remark 1.2.*** — In case $m = 1$, $\mathrm{Log}_n(d_0, d_1)$ is a component of $Fol(\mathbb{P}^n_{\mathbb{C}}; d_0 + d_1)$, called a *rational* component. This is because such $\mathscr{F}$'s are necessarily given by $\varpi = d_1 F_1\, dF_0 - d_0 F_0\, dF_1$, and they all admit a rational first integral, namely, $\dfrac{F_0^{d_1}}{F_1^{d_0}}$.

Let $\mathscr{F}$ be a logarithmic foliation of $\mathbb{P}^n_{\mathbb{C}}$, $n \geq 3$, given, in $\mathbb{C}^{n+1}$, by the 1-form

$$\omega = \left(\prod_{j=0}^{m} F_j\right) \sum_{i=0}^{m} \lambda_i \frac{dF_i}{F_i} = \lambda_0 \widehat{F_0}\, dF_0 + \cdots + \lambda_m \widehat{F_m}\, dF_m. \tag{3}$$

where $F_i \in \mathbb{C}[z_0, \dots, z_n]$ is homogeneous of degree $d_i \geq 1$, $\sum_{i=0}^{m} d_i = \mathrm{d}$, $\sum_{i=0}^{m} \lambda_i\, d_i = 0$ and $\widehat{F_j} = F_1 \cdots F_{j-1} F_{j+1} \cdots F_m$.

The singular set of $\mathscr{F}$ is $S(\mathscr{F}) = \{z \in \mathbb{P}^n_{\mathbb{C}} : \omega(z) = 0\}$ and the fact that $\omega$ is integrable imposes the existence of codimension 2 components in $S(\mathscr{F})$ (see [**9**]).

It is immediate that $S(\mathscr{F})$ contains the union of all codimension two subsets $F_i = F_j = 0$.

Let $D_i = \{p \in \mathbb{P}^n_{\mathbb{C}} : F_i(p) = 0\}$ be the divisor associated to $F_i$.

***Definition 1.3.*** — *A logarithmic foliation given by* $\omega = \lambda_0 \widehat{F_0}\, dF_0 + \cdots + \lambda_m \widehat{F_m}\, dF_m$ *is said to be generic if*

$$\begin{cases} F_i,\ i = 0, \dots, m, & \textit{is irreducible} \\ \textit{the } D_i\text{'s},\ i = 0, \dots, m, & \textit{are smooth and in general position.} \\ \lambda_i \neq 0,\ i = 0, \dots, m. & \end{cases} \tag{4}$$

Remark that (4) defines a Zariski open subset of

$$\mathbb{P}(m-1, \mathrm{d}) \times \prod_{i=0}^{m} \mathbb{P}\left(H^0(\mathbb{P}^n_{\mathbb{C}}; \mathscr{O}(d_i))\right).$$

The following result was proven in [**6**]:

***Proposition 1.4.*** — *Let* $\mathscr{F} \in \mathrm{Log}_n(d_0, \dots, d_m)$ *be given by* $\omega = \left(\prod_{j=0}^{m} F_j\right) \sum_{i=0}^{m} \lambda_i \dfrac{dF_i}{F_i}$ *and assume* $\mathscr{F}$ *is generic. Then* $S(\mathscr{F})$ *has only codimension 2 and dimension 0 components.*

*Proof.* — The proof of this is a simple argument, which we present here for the sake of completness:

We will show that, if a point is non isolated in $S(\mathscr{F})$, then it lies in $D_i \cap D_j$ for some $i < j$. Indeed, let $C$ be an irreducible component of $S(\mathscr{F})$ of dimension $1 \leq \dim C \leq n-2$. By ampleness and general position, we may pick a point $p \in C$

lying in the intersection of precisely $k$ of the divisors $D_i$, $1 \le k \le \min\{n, m+1\}$. Let $f_i$ be a local equation for $D_i$ at $p$. Near $p$, the foliation $\mathscr{F}$ is given by the 1–form

$$\varpi = f_0 \dots f_m \sum_{i=0}^{m} \lambda_i \frac{df_i}{f_i}.$$

Renumbering the indices we may assume $p \in D_0 \cap \cdots \cap D_{k-1}$. The local defining equations $f_i = 0$ of the $D_i$'s, for $i = 0, \dots, k-1$, are part of a regular system of parameters, i.e., $df_0, \dots, df_{k-1}$ are linearly independent at $p$. Write $\tilde{g} = f_k \cdots f_m$. Since $p \notin D_j$, $k \le j \le m$, we may assume $\tilde{g}$ vanishes nowhere around $p$ and write $\varpi$ as

$$\varpi = f_0 \cdots f_{k-1}\, \tilde{g} \left[\sum_{j=0}^{k-1} \lambda_j \frac{df_j}{f_j} + \sum_{i=k}^{m} \lambda_i \frac{df_i}{f_i}\right] = f_0 \cdots f_{k-1}\, \tilde{g} \left[\sum_{j=0}^{k-1} \lambda_j \frac{df_j}{f_j} + \eta\right],$$

where $\eta = \sum_{i=k}^{m} \lambda_i \frac{df_i}{f_i}$ is a holomorphic closed form near $p$. Since $\eta$ is closed, it is exact near $p$, say $\eta = d\xi$. Set $\vartheta = \varpi/\tilde{g}$. Then $\mathscr{F}$ is defined around $p$ by

$$\vartheta = f_0 \cdots f_{k-1} \left[\sum_{j=0}^{k-1} \lambda_j \frac{df_j}{f_j} + d\xi\right] = f_0 \cdots f_{k-1} \left[\lambda_0 \frac{d(\exp[\xi/\lambda_0] f_0)}{\exp[\xi/\lambda_0] f_0} + \sum_{j=1}^{k-1} \lambda_j \frac{df_j}{f_j}\right].$$

Set $z_0 = \exp[\xi/\lambda_0] f_0$ and $z_1 = f_1, \dots, z_{k-1} = f_{k-1}$. Since $u = \exp[\xi/\lambda_0]$ is a unit, we have that also $z_0, \dots, z_{k-1}$ are part of a regular system of parameters at $p$. Now $\vartheta$ can be written as

$$\vartheta = \frac{z_0}{u}\, z_1 \cdots z_{k-1} \left[\lambda_0 \frac{dz_0}{z_0} + \sum_{j=1}^{k-1} \lambda_j \frac{dz_j}{z_j}\right].$$

Thus $\mathscr{F}$ is defined around $p$ by the 1-form

$$\widetilde{\vartheta} = z_0\, z_1 \cdots z_{k-1} \left[\lambda_0 \frac{dz_0}{z_0} + \sum_{j=1}^{k-1} \lambda_j \frac{dz_j}{z_j}\right] = \sum_{j=0}^{k-1} \lambda_j\, z_0 \cdots \widehat{z_j} \cdots z_{k-1}\, dz_j.$$

If $k = 1$, the above expression shows that the foliation is defined near $p$ by $dz_0$ and then is non-singular at $p$. Hence we necessarily have $k \ge 2$. Note that the ideal of the scheme of zeros of $\widetilde{\vartheta}$ (as well as of $\omega$) near $p$ is generated by the $k$ monomials $z_0 \cdots \widehat{z_j} \cdots z_{k-1}$ with $0 \le j \le k-1$. That is just the scheme union $\cup_{i,j} D_i \cap D_j$, for $0 \le i < j \le k-1$. Thus $C$ must be contained in $D_i \cap D_j$, for some $i < j$, and therefore $C$ is an irreducible component of $D_i \cap D_j$ and $\dim C = n-2$. □

The main result of [**6**] is:

***Theorem 1.5***. — *Let $\mathscr{F}$ be a generic logarithmic foliation of $\mathbb{P}^n_{\mathbb{C}}$ of type $\underline{\mathrm{d}} = d_0, \dots, d_m$, given by $\lambda_0 \widehat{F_0}\, dF_0 + \cdots + \lambda_m \widehat{F_m}\, dF_m$. Then the singular scheme $S(\mathscr{F})$ of $\mathscr{F}$ can be written as a disjoint union*

$$S(\mathscr{F}) = Z \bigcup R \tag{5}$$

*where* $Z = \bigcup_{i<j} D_{ij}$ *and* $R$ *is finite, consisting of*

$$N(n, \underline{\mathrm{d}}) = \textit{the coefficient of } h^n \textit{ in } \frac{(1-h)^{n+1}}{\prod_{i=0}^{m}(1-d_i h)} \tag{6}$$

*points counted with natural multiplicities.*

It turns out that

$$N(n, \underline{\mathrm{d}}) = \sum_{j=0}^{n} \binom{m-1}{j} \mathcal{W}_{n-j}(d_0 - 1, \ldots, d_m - 1), \tag{7}$$

where $\mathcal{W}_k(X_0, \ldots, X_m)$ is the complete symmetric function of degree $k$ in $m+1$ variables, that is, $\mathcal{W}_0 = 1$ and $\mathcal{W}_k(X_0, \ldots, X_m) = \sum_{i_0+\cdots+i_m=k} X_0^{i_0} \cdots X_m^{i_m}$.

It follows that, whenever at least one $d_i \geq 2$, we have $N(n, \underline{\mathrm{d}}) > 0$. Remark also that by (6) or (7), in case all $d_i = 1$, only for $n \geq m$ we have $N(n, \underline{\mathrm{d}}) = 0$.

*Proof.* — The proof of this theorem is based on the fact that, if $\mathcal{F}$ is a generic logarithmic foliation, then the codimension two part of its singular scheme is equal to the singular scheme of the normal crossings divisor $\bigcup_i D_i$. We can then use Aluffi's formula ([**1**]) for the Segre class of $\bigcup_i D_i$. □

## 2. Deformations of generic logarithmic foliations

We start by stating some simple remarks, in the form we need. The first one is given in Fulton ([**7**], 8.4.13):

***Remark 2.1.*** — *The hypersurfaces of degree $d_j$ in $\mathbb{P}^n_{\mathbb{C}}$ are parametrized by $\mathbb{P}^{M_j}_{\mathbb{C}}$, where $M_j = \binom{d_j+n}{n} - 1$. Let $X = (X_0 : \ldots : X_n)$ be homogeneous coordinates in $\mathbb{P}^n_{\mathbb{C}}$, $\mu^j_{(\ell)}$ homogeneous coordinates in $\mathbb{P}^{M_j}_{\mathbb{C}}$ and $\Phi_{d_j} = \sum \mu^j_{(\ell)} X^{(\ell)}$ the expression defining the universal hypersurface of degree $d_j$ in $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^{M_j}_{\mathbb{C}}$. The incidence variety $\mathcal{V} \subset \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^{M_j}_{\mathbb{C}}$ is defined by*

$$\mathcal{V} = \{(X, \mu^j) \,:\, \Phi_{d_j}(\mu^j, X) = 0\}.$$

*$\mathcal{V}$ is a smooth, irreducible (hence connected) subvariety of $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^{M_j}_{\mathbb{C}}$ of codimension 1. Let $\pi : \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^{M_j}_{\mathbb{C}} \longrightarrow \mathbb{P}^{M_j}_{\mathbb{C}}$ be the projection and consider its restriction to $\mathcal{V}$, $\pi_{|\mathcal{V}} : \mathcal{V} \longrightarrow \mathbb{P}^{M_j}_{\mathbb{C}}$. The fiber $D_{\mu^j}$ of $\pi_{|\mathcal{V}}$ over $\mu^j$ is the corresponding hypersurface. Notice that the generic fiber is an irreducible, non-singular hypersurface of degree $d_j$.*

Next we have

***Remark 2.2.*** — *An irreducible hypersurface of degree $d \geq 2$ in $\mathbb{P}^n_{\mathbb{C}}$, $n \geq 3$, may have a singular set of dimension $0, 1, \ldots, n-2$. Examples of such are given by:*

$$P_k(z_0, \ldots, z_n) = z_0^d + z_0^{d-1}(z_1 + \cdots + z_k) + z_{k+1}^d + \cdots + z_n^d$$

*where, for each* $1 \leq k \leq n-1$, *sing*$\{P_k = 0\} = \mathbb{L}^{k-1}$, *a linear subspace of* $\mathbb{P}^n_{\mathbb{C}}$ *of dimension* $k-1$.

### 2.1. Bifurcations of singular sets

***Theorem 2.3***. — *A generic foliation* $\mathcal{F} \in \mathrm{Log}_n(d_0, \ldots, d_m)$ *admits a one-parameter deformation, say* $\mathcal{F}_t$, *either real analytic or holomorphic, into an* $\mathcal{F}_0 \in \mathrm{Log}_n(d_0, \ldots, d_m)$, *such that* $S(\mathcal{F}_0)$ *has components, at our choice, of any dimension from* $0$ *up to* $n-2$, *provided at least one* $d_j \geq 2$. *Moreover, if* $t$ *is real then* $\mathcal{F}_t$ *may be taken generic for all* $t \neq 0$ *whereas, in case* $t$ *is complex,* $\mathcal{F}_t$ *may fail to be generic only for a finite set of non-zero parameters* $t$.

*Proof.* — Let $\mathcal{F}$ be a generic foliation in $\mathrm{Log}_n(d_0, \ldots, d_m)$, given by

$$\omega = \lambda_0 \widehat{F_0}\, dF_0 + \cdots + \lambda_m \widehat{F_m}\, dF_m$$

and reorder the indices in such a way that $d_0 \leq \cdots \leq d_m$. Let $j$ be the smallest index such that $d_j \geq 2$ and $(\pi_{|\mathcal{V}})^{-1}(\mu_j) = D_{\mu_j} = \{F_j = 0\} = D_j$(see Remark 2.1). By Remark 2.2, we can choose a $\tilde{\mu}^j_k$, in the discriminant locus $\Delta_{d_j}$ of $\pi_{|\mathcal{V}}$, such that the hypersurface $D_{\tilde{\mu}^j_k}$ is irreducible and has as singular set a linear subspace $\mathbb{L}^{k(d_j)}$ of dimension $k(d_j)$, at our choice, with $0 \leq k(d_j) \leq n-3$.

Choose a real analytic path, $\gamma_j : [0,1] \to \mathbb{P}^{M_j}_{\mathbb{C}}$, such that $D_{\gamma_j(1)} = D_{\mu^j}$, $D_{\gamma_j(0)} = D_{\tilde{\mu}^j_k}$, $D_{\gamma_j(s)}$ is irreducible and smooth for $s \neq 0$, $D_{\gamma_j(s)}$ is in general position with $D_0, \ldots, \widehat{D_j}, \ldots,$
$D_m$, for $s \neq 0$, and $\mathbb{L}^{k(d_j)}$ is in general position with $D_0, \ldots, \widehat{D_j}, \ldots, D_m$. The existence of such a path is assured by the following procedure: start by choosing a $\tilde{\mu}^j_k$ such that both $\mathbb{L}^{k(d_j)}$ and $D_{\tilde{\mu}^j_k}$ are in general position with $D_0, \ldots, \widehat{D_j}, \ldots, D_m$. Next, the set of hypersurfaces of degree $d_j$ which are in general position with $D_0, \ldots, \widehat{D_j}, \ldots, D_m$ correspond to a Zariski open set $\mathcal{O}_1$ in $\mathbb{P}^{M_j}_{\mathbb{C}}$, and so does the set of smooth irreducible hypersurfaces, call it $\mathcal{O}_2$, which is the complement of the discriminant of $\pi_{|\mathcal{V}}$. The complement of $\mathcal{O}_1 \cap \mathcal{O}_2$ has complex codimension $\geq 1$ and we have the real path $\gamma$. Along the path the foliation is given by

$$\omega_{\gamma(t)} = F^t_0 \ldots F^t_j \ldots F^t_m \left( \lambda_0 \frac{dF^t_0}{F^t_0} + \cdots + \lambda_j \frac{dF^t_j}{F^t_j} + \cdots \lambda_m \frac{dF^t_m}{F^t_m} \right) \text{ with } F^t_i \equiv F_i,\ i \neq j.$$

We now repeat the above argument for $d_{j+1}$ and so on. Of course, the $n-2$ dimensional components of $S(\mathcal{F}_0)$ are given by $\{F^0_i = F^0_j = 0\}$, $i < j$.

A holomorphic deformation can be obtained as follows: let $\mu_j$ and $\tilde{\mu}^j_k$ be as above and take the line $\ell$ in $\mathbb{P}^{M_j}_{\mathbb{C}}$ passing through them. Going from $\mu_j$ to $\tilde{\mu}^j_k$ along $\ell$ gives at most $\deg \Delta_{d_j}$ points which correspond to singular hypersurfaces, plus a finite number of points at which $D_{\tilde{\mu}^j_k}$ fails to be in general position with $D_0, \ldots, \widehat{D_j}, \ldots, D_m$. $\square$

## 2.2. Deformations into other logarithmic components of the same degree

Here we consider logarithmic components to which $\mathrm{Log}_n(d_0,\ldots,d_m)$ specializes.

From Remark 2.1 we deduce the following: let $H$ be an irreducible homogeneous polynomial of degree $d_j$ which defines a smooth projective hypersurface $\{H=0\}$ and $G=G_1\ldots G_\ell$ be a reducible homogeneous polynomial of degree $d_j=\deg G_1+\cdots+\deg G_\ell$. Then we can find a real analytic one-parameter family $\widetilde{D_s}$ of smooth irreducible hypersurfaces, whenever $s\neq 0$, such that $\widetilde{D_1}=\{H=0\}$ and $\widetilde{D_0}=\bigcup\limits_{i=1}^{\ell}\{G_i=0\}$.

Now, given a non-negative integer $d$, let $P(d)$ be the number of unrestricted partitions of $d$, for which we adopt the natural representation, that is, $\underline{\sigma_d}=i_1,\ldots,i_q$ means $d=i_1+\cdots+i_q$. It's well known (Euler's formula) that

$$\frac{1}{\prod\limits_{i=1}^{\infty}(1-\xi^i)}=\sum_{d=0}^{\infty}P(d)\ \xi^d.$$

With this at hand we have

***Theorem 2.4***. — *Let $\mathscr{F}\in\mathrm{Log}_n(d_0,\ldots,d_m)$ be a generic logarithmic foliation given by*

$$\omega=\left(\prod_{j=0}^{m}F_j\right)\sum_{i=0}^{m}\lambda_i\frac{dF_i}{F_i}=\lambda_0\widehat{F_0}\,dF_0+\cdots+\lambda_m\widehat{F_m}\,dF_m.$$

*For each degree $d_j$, let $\underline{\sigma_{d_j}}=i_1,\ldots,i_{q(d_j)}$ be any partition of $d_j$, $0\leq j\leq m$. Then, there is a real analytic one-parameter deformation $\mathscr{F}^s$ of $\mathscr{F}$ such that $\mathscr{F}^1=\mathscr{F}$ and $\mathscr{F}^0\in\mathrm{Log}_n(\underline{\sigma_{d_0}},\ldots,\underline{\sigma_{d_m}})$.*

*Proof*. — Let $\underline{\sigma_{d_j}}=i_1,\ldots,i_{q(d_j)}$ and choose a real analytic one-parameter deformation of $F_j$, $F_j^s$, such that $F_j^1=F_j$ and $F_j^0=G_{i_1}\cdots G_{i_{q(d_j)}}$. Since

$$\lambda_j\frac{dF_j^0}{F_j^0}=\lambda_j\sum_{i_1}^{i_{q(d_j)}}\frac{dG_r}{G_r}$$

and $\sum\limits_{i_1}^{i_{q(d_j)}}\deg G_r=d_j$, we have that $\mathscr{F}_j^0$ is given by

$$\omega_j^0=F_0\ldots G_{i_1}\cdots G_{i_{q(d_j)}}\ldots F_m\left(\lambda_0\frac{dF_0}{F_0}+\cdots+\lambda_j\sum_{i_1}^{i_{q(d_j)}}\frac{dG_r}{G_r}+\cdots\lambda_m\frac{dF_m}{F_m}\right),$$

hence $\mathscr{F}_j^0\in\mathrm{Log}_n(d_0,\ldots\underline{\sigma_{d_j}},\ldots,d_m)$. Remark that, by the argument used in the proof of Theorem 2.3, we may choose the $G_r$'s to be smooth and in general position with the $F_i$'s, $i\neq j$. Doing this for each $0\leq j\leq m$ we obtain the stated deformation $\square$

From this it follows

***Theorem 2.5.*** — *The union* $\bigcup_{\underline{\mathrm{d}}} \mathrm{Log}_n(d_0,\dots,d_m) \subset Fol(\mathbb{P}^n_{\mathbb{C}};\mathrm{d})$ *is connected.*

*Proof.* — Given $\underline{\mathrm{d}} = d_0,\dots,d_m$ and $\underline{\mathrm{d}}' = d'_0,\dots,d'_{m'}$, with $\sum_{i=0}^{m} d_i = \sum_{j=0}^{m'} d'_j = \mathrm{d}$, consider the complete partitions

$$\underline{\tau_{d_i}} = \underbrace{1,\dots\dots,1}_{d_i \text{ times}} \text{ of } d_i \text{ and } \underline{\tau_{d'_j}} = \underbrace{1,\dots\dots,1}_{d'_j \text{ times}} \text{ of } d'_j$$

for $0 \le i \le m$ and $0 \le j \le m'$.

Let $\mathcal{F} \in \mathrm{Log}_n(d_0,\dots,d_m)$ and $\mathcal{G} \in \mathrm{Log}_n(d'_0,\dots,d'_{m'})$ be, respectively, given by

$$\lambda_0 \widehat{F_0}\, dF_0 + \dots + \lambda_m \widehat{F_m}\, dF_m \quad \text{and} \quad \lambda'_0 \widehat{F'_0}\, dF'_0 + \dots + \lambda'_{m'} \widehat{F'_{m'}}\, dF'_{m'}.$$

By Theorem 2.4 these can be deformed into

$$[g_1^0 \cdots g_{d_0}^0] \dots [g_1^j \cdots g_{d_j}^j] \dots [g_1^m \cdots g_{d_m}^m] \left[\lambda_0 \sum_1^{d_0} \frac{dg_r^0}{g_r^0} + \dots + \lambda_j \sum_1^{d_j} \frac{dg_r^j}{g_r^j} + \dots + \lambda_m \sum_1^{d_m} \frac{dg_r^m}{g_r^m}\right]$$

and

$$[l_1^0 \cdots l_{d'_0}^0] \dots [l_1^j \cdots l_{d'_j}^j] \dots [l_1^{m'} \cdots l_{d'_{m'}}^{m'}] \left[\lambda'_0 \sum_1^{d'_0} \frac{dl_r^0}{l_r^0} + \dots + \lambda'_j \sum_1^{d'_j} \frac{dl_r^j}{l_r^j} + \dots + \lambda'_{m'} \sum_1^{d'_{m'}} \frac{dl_r^{m'}}{l_r^{m'}}\right]$$

where the $g_i^j$'s and the $l_i^j$'s are homogeneous of degree 1. Both these foliations lie in

$$\mathrm{Log}_n(\underbrace{1,1,1,1,1,\dots\dots,1,1,1,1,1}_{\sum_{i=0}^{m} d_i = \sum_{j=0}^{m'} d'_j = \mathrm{d} \text{ times}})$$

and, since an irreducible algebraic set is connected, the result follows. □

***Example 2.6.*** — Let $\mathcal{F} \in \mathrm{Log}_n(d_0,\dots,d_m)$, suppose $\sum_{i=0}^{m} d_i = \mathrm{d} \le n$ and at least one $d_i$ satisfies $d_i \ge 2$. By Theorem 1.5, the singular set $S(\mathcal{F})$ contains isolated points away from $\bigcup_{i<j} D_{ij}$. By Theorem 2.4, when we specialize

$$\mathrm{Log}_n(d_0,\dots,d_m) \ni \mathcal{F} = \mathcal{F}^1 \quad \xrightarrow{\mathcal{F}^t} \quad \mathcal{F}^0 \in \mathrm{Log}_n(\underbrace{1,1,1,\dots\dots,1,1,1}_{\sum_{i=0}^{m} d_i = \mathrm{d} \text{ times}}),$$

$\mathcal{F}^0$ is such that, by Theorem 1.5 again, $S(\mathcal{F}^0)$ has only components of dimension $n-2$. Hence, the isolated zeros of $\mathcal{F}^t$ are absorbed by the $n-2$ dimensional components of $S(\mathcal{F}^0)$.

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## References

[1] P. Aluffi – "Chern classes for singular hypersurfaces", *Trans. Amer. Math. Soc.* **351** (1999), p. 3989–4026.

[2] O. Calvo-Andrade – "Irreducible components of the space of holomorphic foliations", *Math. Ann.* **299** (1994), p. 751–767.

[3] O. Calvo-Andrade, D. Cerveau, L. Giraldo & A. Lins Neto – "Irreducible components of the space of foliations associated to the affine Lie algebra", *Ergodic Theory Dynam. Systems* **24** (2004), p. 987–1014.

[4] D. Cerveau & A. Lins Neto – "Irreducible components of the space of holomorphic foliations of degree two in $\mathbf{C}\mathrm{P}(n)$, $n \geq 3$", *Ann. of Math. (2)* **143** (1996), p. 577–612.

[5] D. Cerveau, A. Lins Neto & S. J. Edixhoven – "Pull-back components of the space of holomorphic foliations on $\mathbb{CP}(n)$, $n \geq 3$", *J. Algebraic Geom.* **10** (2001), p. 695–711.

[6] F. Cukierman, M. G. Soares & I. Vainsencher – "Singularities of logarithmic foliations", *Compos. Math.* **142** (2006), p. 131–142.

[7] W. Fulton – *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 2, Springer, 1984.

[8] R. Hartshorne – "Connectedness of the Hilbert scheme", *Publ. Math. I.H.É.S.* **29** (1966), p. 5–48.

[9] J. P. Jouanolou – *Équations de Pfaff algébriques*, Lecture Notes in Mathematics, vol. 708, Springer, 1979.

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# ON THE DEFINITION OF THE GALOIS GROUPOID

*by*

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*For José Manuel Aroca on the occasion of his $60^{th}$ birthday*

***Abstract.*** — We sketch a proof of equivalence of two general differential Galois theories, Malgrange's theoy and ours, if the base field consists only of constants.

***Résumé* (Sur la définition du groupoïde de Galois).** — Nous esquissons la démonstration du fait que deux théories de Galois, la théorie de Malgrange et la nôtre, sont équivalentes dans le cas absolu, i.e. quand le corps de base consiste uniquement en des constantes.

## 1. Introduction

Today we have two general differential Galois theories [**4**] and [**3**]. While the first published in 1996 is a Galois theory of differential field extensions, the latter proposed in 2001 is a Galois theory of foliations on varieties. They look somehow different but specialists observed coincidence in examples. The aim of this note is to sketch in fact they are equivalent in the absolute case, by which we mean the case where the base field $K$ of the differential field extension $L/K$ consists of only constants. For the relative case or for a general differential field extension $L/K$, there may be a similar result but there are subtle questions. First of all we must have an adequate definition of the Galois groupoid for the extension $L/K$ in terms of foliations in the spirit of [**3**].(1)

We show by analyzing a non-trivial interesting example, the equivalence. Given a differential field, it is an algebraic counter part of a dynamical system on a algebraic variety. If we observe this dynamical system closely by algebraic method, or if an algebraist observes the dynamical system, then we get as a natural object Galois groupoid of the dynamical system, or of the given differential field. This procedure of

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(1) Added on 25 August 2008, we can apply this method also to the reative case.

observation is done through the universal Taylor morphism and ties Malgrange's idea and ours.

## 2. Differential fields and dynamical systems

A differential field $(L, \delta)$ consists of a field $L$ and a derivation $\delta : L \to L$. So we have $\delta(a+b)) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for every $a, b \in L$. Similarly we define a differential ring $(R, \delta)$. An element $a$ of a differential field or a differential ring is called a constant if $\delta(a) = 0$. The set $C_L$ or $C_R$ of constants forms respectively a subfield or subring.

Now we consider a differential field that is finitely generated as an abstract field over the complex number field $\mathbb{C}$ in such a way that the complex number field $\mathbb{C}$ is a subfield of the field $C_L$ of constants.

***Remark 2.1.*** — *In the sequel, we work over the complex number field $\mathbb{C}$ so that the reader has a concrete image, we may replace, however, the complex number field $\mathbb{C}$ by any field of characteristic* 0.

We explain by examples that a differential field is an algebraic counter part of a differential dynamical system on an algebraic variety.

**Example** 2.1. Let us consider the differential field $(\mathbb{C}(x), d/dx)$, where $x$ is a variable over $\mathbb{C}$ and hence $\mathbb{C}(x)$ is the rational function field of one variable. A geometric model of the differential field $(\mathbb{C}(x), d/dx)$ is a dynamical system $(\mathbb{A}^1, d/dx) =$ (Spec $\mathbb{C}[x]$, $d/dx$). In other words, the field of rational functions of the affine line $\mathbb{A}^1$ with derivation $d/dx$ gives the differential field $(\mathbb{C}(x), d/dx)$.

***Remark 2.2.*** — *Since for any non-empty Zariski open subset $U$ of $\mathbb{A}^1$, $(U, d/dx)$ satisfies the condition required above, the general model of the differential field $(\mathbb{C}(x), d/dx)$ is ($\mathbb{A}^1-$ (a finite number of points ), $d/dx$). The model is determined up to birational equivalence.*

**Example** 2.2. Let $x, y$ be two independent variables over $\mathbb{C}$ so that $\mathbb{C}[x, y]$ is a polynomial ring over $\mathbb{C}$. Let us consider the differential field

$$(\mathbb{C}(x, y), \partial/\partial x + y\partial/\partial y).$$

A model of this differential field is the $(x, y)$-plane $\mathbb{A}^2$ or Spec $\mathbb{C}[x, y]$ with vector field $\partial/\partial x + y\partial/\partial y$. A general flow on the affine plane $\mathbb{A}^2$ is given by $(t, c\exp t)$, $t \in \mathbb{C}$ for a fixed $c \in \mathbb{C}$. In this Example we may replace the affine plane $\mathbb{A}^2$ by any non-empty Zariski open set of $\mathbb{A}^2$.

Generally we can prove the following proposition.

***Proposition 2.1.*** — *Let $(L, \delta)$ be a differential field such that the field $L$ is of finite type over the complex number field $\mathbb{C}$ and $\mathbb{C}$ is a subfield of the field $C_L$ of constants of $(L, \delta)$. Then there exists a smooth algebraic variety $V$ over $\mathbb{C}$, with regular algebraic vector field $X$ such that $(V, X)$ is a model of the differential field $(L, \delta)$. In other* *words , the rational function field $\mathbb{C}(V)$ of $V$ is isomorphic to the field $L$ and the vector field $X$ is identified with the derivation $\delta$ through this isomorphism.*

See Lemma (1.5), [**5**].

## 3. Groupoids

We need a seemingly abstract definition of groupoid but it is as concrete as vector space.

***Definition 3.1.*** — *A groupoid is a small category $G$ in which all morphisms are isomorphisms. An object of $G$ is called a vertex and a morphism in $G$ is called an element of $G$.*

The groupoid was introduced by Brandt in 1926. In 1950's Ehresmann used groupoids in theory of foliations. In 1960's Grothendieck studied quotients by groupoids in algebraic geometry. Here are examples of groupoids to have an image of groupoids.

**Example** 3.1. A group $G$ is a groupoid. We define a category $\mathscr{C}$ that is a groupoid. The object of the category $\mathscr{C}$ is one point $P$, i.e. $ob\,\mathscr{C} = \{P\}$ . We set

$$\mathrm{Hom}\ (P,\, P) = G$$

and compose two morphisms of $\mathrm{Hom}\ (P,\, P) = G$ according as the group law of $G$.

**Example** 3.2. Equivalence relation $\sim$ on a set X. The set $ob\,G$ of the objects of the groupoid $G$ is the set $X$. For $x,\, y \in ob\,G$, we define

$$\mathrm{Hom}\ (x,\, y) = \begin{cases} 1 \text{ morphism}, & \text{if } x \sim y, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Since every element $x$ is equivalent to itself, we have the identity $Id_x$. Since equivalence relation is reflexive, every morphism is an isomorphism. Since equivalence relation is transitive, we can compose two morphisms. So the above definition yields us a groupoid.

**Example** 3.3. Group operation $(G, X)$ of a group $G$ on a set $X$ is a groupoid. The set $ob\,C$ of the groupoid $C$ is the set $X$. For $x,\, y \in X = ob\,C$, we set $\mathrm{Hom}\ (x,\, y) = \{g \in G | gx = y\}$. If $g \in \mathrm{Hom}\ (x,\, y)$ and $h \in \mathrm{Hom}\ (y,\, z)$, then $gx = y$ and $hy = z$ by definition so that $z = hy = h(gx) = (hg)x$ and consequently $hg \in \mathrm{Hom}\ (x, z)$. So we can compose two morphisms. If $gx = y$, then $hy = x$, $h$ being $g^{-1}$ so that every morphism is an isomorphism.

**Example** 3.4. Poincaré groupoid. Let $X$ be a topological space. Let $ob\,G$ of the category be the set $X$. A path from a point $x \in X$ to another point $y \in X$ is a continuous map $\varphi : [0, 1] \to X$ from the interval $[0, 1]$ to the topological space $X$ such that $\varphi(0) = x$ and $\varphi(1) = y$. We set in the category $G$,

$$\text{Hom }(x, y) := \text{ the set of paths from } x \text{ to } y \text{ modulo homotopy equivalence.}$$

Then it is well-know that the category $G$ is a groupoid, which is called a Poincaré groupoid.

Now let $G$ be a groupoid . We set

$$Y := \{\text{morphisms in the category } G\}$$

and

$$X := ob\, G.$$

Let $\varphi \in Y$ so that $\varphi \in$ Hom $(A, B)$ for some $A, B \in ob\, G$. Let us denote the source $A$ of $\varphi$ by $s(\varphi)$ and the target $B$ of $\varphi$ by $t(\varphi)$. So we get two maps $s : Y \to X$ and $t : Y \to X$. Let $(Y, t) \times (Y, s)$ be the fiber product of $t : Y \to X$ and $s : Y \to X$ so that

$$(Y, t) \times (Y, s) = \{(\varphi, \psi) \in Y \times Y | s(\varphi) = t(\psi)\}.$$

The composition of morphisms defines a map

$$\Phi : (Y, t) \times (Y, s) \to Y, \qquad (\varphi, \psi) \mapsto \psi \circ \varphi.$$

The associativity of the composition is described by a commutative diagram that we do not make precise. See [**2**]. The existence of the identity map $Id_A$ for every $A \in ob\, G$ as well as the property called symmetry that every morphism is an isomorphism is also characterized in terms of maps and commutative diagrams.

Here is a summary of the above observation. Groupoid is described by two sets $Y$ and $X$, two maps $s : Y \to X$ and $t : Y \to X$ and the composition maps

$$\Phi : (Y, t) \times (Y, s) \to Y, \qquad (\varphi, \psi) \mapsto \psi \circ \varphi.$$

that satisfy certain commutative diagrams and so on.

This allows us to generalize the notion of groupoid in a category in which fiber product exists. This is exactly by the same way as we define an algebraic group $G$ requiring that, first of all, $G$ is an algebraic variety, the composition law $G \times G \to G$ is a morphism of algebraic varieties and so on.

***Definition** 3.2*. — *Let $C$ be a category in which fiber product exists. A groupoid in the category $C$ consists of two objects $Y, X \in ob\, C$, two morphisms $s : Y \to X$ and $t : Y \to X$ and a morphism*

$$\Phi : (Y, t) \times (Y, s) \to Y$$

*etc, satisfying the above conditions (cf. Grothendieck* [**2**]*)*

**Example** 3.5. Let $C$ be the category of algebraic varieties defined over a field $k$ and let $(G, V)$ be an operation of an algebraic group on an algebraic variety $V$ defined over $k$. We have two morphisms $p$, $h$ from $G \times V$ to $V$, namely the second projection

$p$ and the group operation $h(g, v) = gv$. Then $Y = G \times X$, $X = V$ $s = p$ and $t = h$ is a groupoid in the category $C$. Compare to Example 3.3.

We need a tool, an algebraic $D$-groupoid that generalizes Example 3.5.

## 4. Lie groupoids and D-groupoids

For a complex manifold $V$, we can attach its invertible jets $J^*(V \times V)$ that is a groupoid over $V \ \times V$ in the category of analytic spaces. We recall the definition for $V = \mathbb{C}$. The jet space $J(\mathbb{C} \times \mathbb{C})$ is an infinite dimensional analytic space $\mathbb{C} \times \mathbb{C}^{\mathbb{N}}$ with coordinate system $(x, , y_0, y_1, y_2, \dots)$, We have two morphisms $s : J(\mathbb{C} \times \mathbb{C}) \to \mathbb{C}$ and $t : J(\mathbb{C} \times \mathbb{C}) \to \mathbb{C}$ given by

$$s((x, , y_0, y_1, y_2, \dots)) = x \qquad \textit{and} \qquad t((x, , y_0, y_1, y_2, \dots)) = y_0.$$

So we have a morphism $(s, t) : J(\mathbb{C} \times \mathbb{C}) \to \mathbb{C} \times \mathbb{C}$ that makes $J(\mathbb{C} \times \mathbb{C})$ an infinite dimensional affine space over $\mathbb{C} \times \mathbb{C}$. The invertible jet space $J^*(\mathbb{C} \times \mathbb{C})$ is , by definition, the Zarisiki open set of $J(\mathbb{C} \times \mathbb{C})$. Namely,

$$J^*(\mathbb{C} \times \mathbb{C}) := \{(x, y_0, y_1, y_2, \dots) \in J(\mathbb{C} \times \mathbb{C}) |\, y_1 \neq 0\,\}.$$

We simply denote $J^*(\mathbb{C} \times \mathbb{C})$ by $J^*$ and we write the restrictions of the morphisms $s$, $t$ to the Zariski open set $J^*$ by the same letters. Now we explain $J^*$ with two morphisms $s : J^* \to \mathbb{C}$ and $t : J^* \to \mathbb{C}$ is a groupoid. To this end we must define the composite morphism $\Phi : (J^*, t) \times (J^*, s) \to J^*$. Let

$$\varphi = (x, y_0, y_1, \dots), \qquad \psi = (u, v_0, v_1, \dots),$$

be points of $J^*$ such that $y_0 = t(\varphi) = s(\psi) = u$, i.e. $(\varphi, \psi)$ is a point of $(J^*, t) \times (J^*, s)$. Then we set

$$\Phi(\psi, \varphi) := (x, v_0, y_1 v_1, y_2 v_1 + y_1^2 v_2, \dots). \tag{1}$$

The $n$-th component of $\Phi(\psi, \varphi)$ is given by the following rule. Imagine formally that $\varphi$ were a function of $x$ taking the value $y_0$ at $x$, or $\varphi(x) = y_0$, with $\varphi'(x) = y_1, \varphi''(x) = y_2 \dots$. Similarly consider as if $\psi$ were a function of $u$ with $\psi(u) = v_0$, $\psi'(u) = v_1$, $\psi''(u) = v_2, \dots$. Then $\Phi(\psi, \varphi)$ is the composite function $\psi \circ \varphi$, which is a function of $x$, so that its $n$-th component is the value of $d^n \psi \circ \varphi / dx^n$ at $x$. For example ,

$$d(\psi \circ \varphi)/dx = \psi_u \varphi_x = y_1 v_1, d^2(\psi \circ \varphi)/dx^2 = \varphi_{xx} \psi_u + \varphi_x^2 \psi_{uu} = y_2 v_1 + y_1^2 u_2, \dots$$

One can check this composition law is associative and the inverse of

$$\varphi = (x, y_0, y_1, \dots)$$

is given by the inverse function $x(y_0)$ and its derivatives $d^n x(y_0)/dy_0^n$ for $n \in \mathbb{N}$, namely by

$$(y_0, x, 1/y_1, -y_2/y_1^3, \dots).$$

We can very naturally extend this construction over a complex manifold of any dimension.

***Remark 4.1.*** — *We considered above the n-th coordinate to be $d^n y/dx^n$. But it is more natural to use $(1/n!)d^n y/dx^n$. In this way we can work over $\mathbb{Z}$.*

The above construction of Lie groupoids, in the category of analytic spaces, also works in the category of algebraic varieties, or to be more correct in the category of schemes over a field $C$. The most important ingredient in the algebraic constriction is the universal extension of derivations [**9**]. We do not go into the detail because it is technical and will be published elsewhere. So for a non-singular algebraic variety $V$ defined over the field $C$ of characteristic 0, we can define its invertible jet space $J^*(V \times V)$ that is an algebraic variety of infinite dimension, i.e. an affine scheme over $V \times V$.

***Definition 4.1.*** — *An algebraic D-groupoid is a sbugroupoid of $J^*$ defined by a differential ideal.*

We are going to show what the definition means by concrete Examples.

Let $V = \mathbb{A}^1$ and we consider the Lie groupoid $J^*(\mathbb{C} \times \mathbb{C})$. Recall that the construction above in this case is purely algebraic. The coordinate ring or the ring of (algebraic) regular function on $J^*(\mathbb{C} \times \mathbb{C})$ is $\mathbb{C}[x, y_0, y_1, \ldots, 1/y_0]$. The derivation $\delta = \partial/\partial x + \sum_{i=0}^{\infty} y_{i+1}\partial/\partial y_i)$ operates on the coordinate ring $\mathbb{C}[x, y_0, y_1, \ldots, 1/y_1]$ of $J^*(\mathbb{C} \times \mathbb{C})$. So $(\mathbb{C}[x, y_0, y_1, \ldots, 1/y_1], \delta)$ is a differential algebra. Consider the differential ideal $I$ of the coordinate ring generated by $y_1 - 1$ so that $y_{n+1} = \delta^n(y_1) = \delta^n(y_1 - 1) \in I$ for $n = 1, 2, 3, \ldots$. Hence the algebraic subvariety of $J^*(\mathbb{C} \times \mathbb{C})$ defined by the ideal $I$ is

$$Y = \{(a, b, 1, 0, \ldots) \in J^*(\mathbb{C} \times \mathbb{C})\}.$$

Let

$$((a, b, 1, 0, \ldots), (c, d, 1, 0, \ldots)) \in (J^*, t) \times (J^*, s)$$

so that $b = c$, then

$$\Phi((a, b, 1, 0, \ldots), (c, d, 1, 0, \ldots)) = (a, d, 1, 0, \ldots)$$

by (1). This shows that the subvariety $Y \subset J^*$ is closed by the composition. Since the inverse of $(a, b, 1, 0, \ldots)$ is $(b, a, 1, 0, \ldots)$, $Y$ is an algebraic groupoid. This groupoids is nothing but the strongest equivalence relation on $\mathbb{C}$ according which arbitrary two points of $\mathbb{C}$ are equivalent. See Example 3.2. Another interpretation is the operation of the additive group $\mathbb{C}$, which is an algebraic group, on it self. See Examples 3.3 and 3.5.

**Example** 4.2. Let $V = \mathbb{A}^1 - \{0\}$ and $n$ an integer . Then by construction $J^*(V \times V) = \{(x, y_0, y_1, \ldots) \in \mathbb{C} \times \mathbb{C}^{\mathbb{N}} | \, x, y_0, y_1 \neq 0\}$. Let $I_n$ be the differential ideal of

$$\mathbb{C}[x, y_0, y_1, \ldots, 1/x, 1/y_0, 1/y_1]$$

generated by $y_1 - (x/y_0)^n$. Let us assume that $\varphi = (x, y_0, y_1, \ldots) \in J^*(V \times V)$ satisfies the differential equation

$$\varphi_x = (x/\varphi)^n \tag{2}$$

and $\psi = (u, v_0, v_1, \ldots) \in J^*(V \times V)$ satisfies the differential equation

$$\psi_u = (u/\psi)^n \tag{3}$$

with $y_0 = u$, then by (2) and (3)

$$\frac{d(\psi \circ \varphi)}{dx} = \frac{d\psi}{du}\frac{d\varphi}{dx} = \left(\frac{u}{\psi}\right)^n \left(\frac{x}{\varphi}\right)^n = \left(\frac{x}{\psi}\right)^n.$$

So the ideal $I_n$ defines a groupoid $G_n$. By considering the automorphism $V \to V$, $u \mapsto u^{-1}$, the groupoid $G_n$ is isomorphic to $G_{-n+2}$. In fact this follows from

$$\frac{d(\varphi(x)^{-1})}{dz} = \left(\frac{z}{\varphi(x)^{-1}}\right)^{2-n}$$

if $\varphi$ satisfies (2), where $z = x^{-1}$.

**Example** 4.3. The Schwarzian defines a D-groupoid over $\mathbb{C}$. To be more precise let us recall the Schwarzian derivative

$$\{y;\, x\} := \left(\frac{d^3y}{dx^3}\right) / \left(\frac{dy}{dx}\right) - \frac{3}{2}\left[\frac{d^2y}{dx^2} / \frac{dy}{dx}\right]^2,$$

where $y$ is a function of $x$. We know that when $y$ is a function of $x$ and $z$ is a function of $y$ and consequently $z$ is a function of $x$, we have a formula

$$\{z;\, x\} = \left(\frac{dy}{dx}\right)^2 \{z;\, y\} + \{y;\, x\} \tag{4}$$

The formula (4) shows that the differential ideal $I$ of $\mathbb{C}[x, y_0, y_1, \ldots, 1/y_1]$ generated by $y_2/y_1 - (3/2)y_2/y_1)^2$ defines a D-groupoid that is a subgroupoid of $J^*(\mathbb{C} \times \mathbb{C})$.

## 5. Examples of Galois groupoids

**Example** 5.1. Let $t, x, y,$ be independent variables over $\mathbb{C}$. Let $\delta : \mathbb{C}(t, x, y) \to \mathbb{C}(t, x, y)$ be the $\mathbb{C}$-derivations of the rational function field $\mathbb{C}(t, x, y)$ such that

$$\delta(t) = 0, \qquad \delta(x) = 1, \qquad \delta(y) = \frac{t}{x}y, \tag{5}$$

In other words

$$\delta = \frac{\partial}{\partial x} + \frac{ty}{x}\frac{\partial}{\partial y}.$$

This is the differential field studied by Cassidy and Singer [**1**]. We analyzed this example in [**8**] . We present here a new point of view that ties our previous definition of Galois group *Inf-gal* and Galois groupoid of Malgrange.([**4**], [**3**]). The first is, as we briefly review in §6, an automorphism group of a certain differential field and the latter is a D-groupoid over an algebraic variety. See also §6.

We explain from the view point of dynamical system what is the Galois groupoid

$$\mathrm{Gal}\,(\mathbb{C}(t, x, y)/\mathbb{C})$$

of the differential field

$$(\mathbb{C}(t,\, x,\, y),\, \delta) \tag{6}$$

over $\mathbb{C}$. First of all, $R := \mathbb{C}[t,\, x,\, y,\, 1/x]$ is $\delta$-invariant so that

$$(\mathrm{Spec}\ R,\, \delta)$$

is a model of the differential field (6) (cf. §2). The Galois groupoid $\mathrm{Gal}((\mathbb{C}(t,\, x,\, y),\, \delta)/\mathbb{C})$ is a D-groupoid over the algebraic variety $V = \mathrm{Spec}\ R = \{(t,\, x,\, y) \in \mathbb{C}^3 \mid x \neq 0\}$. Let us recall the universal Taylor morphism

$$\iota : R \to R^{\natural}[[X]]$$

that sends an element $a \in R$ to its formal Taylor expansion

$$\sum_{n=0}^{\infty} \frac{1}{n!} \delta^n(a) X^n, \tag{7}$$

where $X$ is a variable. Recall according to our convention that $R^{\natural}$ denotes the ring $R$ without derivation $\delta$. So logically we must write Spec $R^{\natural}$. It follows from (7)

$$\iota(t) = t, \qquad \iota(x) = x + X, \qquad \iota(y) = y\left(1 + \frac{X}{x}\right)^t,$$

where

$$\left(1 + \frac{X}{x}\right)^t = \sum_{n=0}^{\infty} \frac{t(t-1)\cdots(t-n+1)}{n!} X^n.$$

See [**4**]. We set

$$\tilde{T}(t,\, x,\, y;\, X) := \iota(t), \qquad \tilde{X}(t,\, x,\, y;\, X) := \iota(x), \qquad \tilde{Y}(t,\, x,\, y;\, X) := \iota(y).$$

Since $\iota$ is an algebra morphism compatible with $\delta$ and $d/dX$ , we have by (5)

$$\frac{d\tilde{T}}{dX} = 0, \qquad \frac{d\tilde{X}}{dX} = 1, \qquad \frac{d\tilde{X}}{dX} = \frac{\tilde{T}}{\tilde{X}}\tilde{Y}. \tag{8}$$

In other words $\tilde{T}$, $\tilde{X}$, $\tilde{Y}$ is a solution to (5) with initial conditions

$$\tilde{T}(t,\, x,\, y; 0) = t, \qquad \tilde{X}(t,\, x,\, y; 0) = x, \qquad \tilde{Y}(t,\, x,\, y; 0) = y$$

at $X = 0$. We are interested in the dynamical system

$$(t,\, x,\, y) \mapsto (\tilde{T},\, \tilde{X},\, \tilde{Y}) \tag{9}$$

on the algebraic variety $V = \mathrm{Spec}\ R = \{(u,\, v,\, w) \in \mathbb{C}^3 \mid v \neq 0\}$.

Suppose now that an algebraist, who knows nothing about transcendental functions, lives on the variety Spec $R$ and he observes the dynamical system (9). As he can not recognize what is transcendental or analytic, he tries to understand the dynamical system(9) through algebraic differential equations. Above all, he will take the first derivatives to get

$$\begin{aligned}
&\tilde{T} = t, \\
&\partial_t \tilde{X} = 0, \qquad\qquad\qquad \partial_x \tilde{X} = 1, \qquad\qquad \partial_y \tilde{X} = 0, \\
&(\partial_t \tilde{Y})/\tilde{Y} = \log(1 + \tfrac{X}{x}), \quad (\partial_x \tilde{Y})/\tilde{Y} = t(\tfrac{1}{\tilde{X}} - \tfrac{1}{x}), \quad y\partial_y \tilde{Y} = \tilde{Y}.
\end{aligned} \tag{10}$$

The first equation of the last line of (10) contains the transcendental function log so that the poor algebraist can not understand it. So he will take the second derivative of the expression to conclude

$$\partial_t \left((\partial_t \tilde{Y})/\tilde{Y}\right) = 0, \qquad \partial_x \left((\partial_t \tilde{Y})/\tilde{Y}\right) = \frac{1}{\tilde{X}} - \frac{1}{x}, \qquad \partial_y \left((\partial_t \tilde{Y})/\tilde{Y}\right) = 0.$$

These are algebraic relations so that the algebraist can understand but these three equations except for the first are consequences of the other equations of (10) and hence they are superfluous. So all the necessary algebraic differential relations are

$$\begin{aligned}
&\tilde{T} = t, \\
&\partial_t \tilde{X} = 0, \qquad\qquad\qquad \partial_x \tilde{X} = 1, \qquad\qquad \partial_y \tilde{X} = 0, \\
&\partial_t \left((\partial_t \tilde{Y})/\tilde{Y}\right) = 0, \quad (\partial_x \tilde{Y})/\tilde{Y} = t(\tfrac{1}{\tilde{X}} - \tfrac{1}{x}), \quad y\partial_y \tilde{Y} = \tilde{Y}.
\end{aligned} \tag{11}$$

Let us summarize the analysis above. The algebraist observed the dynamical system (9) and arrived at the system (11) of algebraic partial differential equations. What is this system? The answer is this is the defining equation of the Galois groupoid $\mathrm{Gal}\left((\mathbb{C}(t, x, y), \delta)/\mathbb{C}\right)$ that is a D-groupoid on $V = \mathrm{Spec}\, R = \{(t, x, y) \in \mathbb{C}^3 \mid x \neq 0\}$.

To be more precise, we introduce variables $\mathcal{T}$, $\mathcal{X}$, $\mathcal{Y}$, which you might imagine functions of $t$, $x$, $y$, and their formal derivatives

$$\partial_t^l \partial_x^m \partial_y^n \mathcal{T}, \qquad \partial_t^l \partial_x^m \partial_y^n \mathcal{X}, \qquad \partial_t^l \partial_x^m \partial_y^n \mathcal{Y}, \qquad \text{for } (l, m, n) \in \mathbb{N}^3 \tag{12}$$

that are also variables. We identify

$$\partial_t^0 \partial_x^0 \partial_y^0 \mathcal{T} = \mathcal{T}, \qquad \partial_t^0 \partial_x^0 \partial_y^0 \mathcal{X} = \mathcal{X}, \qquad \partial_t^0 \partial_x^0 \partial_y^0 \mathcal{Y} = \mathcal{Y}.$$

The invertible jet space $J^*(V \times V)$ over $V = \mathrm{Spec}\ R$ is by definition

$$J(V \times V) := \mathrm{Spec}\, \mathbb{S},$$

where

$$\mathbb{S} := \mathbb{C}[t, x, y, \mathcal{T}, \mathcal{X}, \mathcal{Y}, \partial_t^l \partial_x^m \partial_y^n \mathcal{T}, \partial_t^l \partial_x^m \partial_y^n \mathcal{X}, \partial_t^l \partial_x^m \partial_y^n \mathcal{Y}, 1/x, 1/\mathcal{X}, 1/\mathrm{Jac}]_{(l, m, n) \in \mathbb{N}^3},$$

and where Jac is the Jacobian

$$\mathrm{Jac} = \begin{vmatrix} \partial_t \mathcal{T} & \partial_x \mathcal{T} & \partial_y \mathcal{T} \\ \partial_t \mathcal{X} & \partial_x \mathcal{X} & \partial_y \mathcal{X} \\ \partial_t \mathcal{Y} & \partial_x \mathcal{Y} & \partial_y \mathcal{Y} \end{vmatrix}.$$

The algebraic differential relations (11) gives us a differential ideal $I$ of the ring $\mathbb{S}$ generated by

$$
(13)\qquad
\begin{array}{lll}
\mathcal{T}-t, & & \\
\partial_t\mathcal{X}, & \partial_x\mathcal{X}-1, & \partial_y\mathcal{X}, \\
\mathcal{Y}^2\partial_t\left((\partial_t\mathcal{Y})/\mathcal{Y}\right), & \mathcal{Y}[(\partial_x\mathcal{Y})/\mathcal{Y}-t(\frac{1}{\mathcal{X}}-\frac{1}{x})], & y\partial_y\mathcal{Y}-\mathcal{Y}.
\end{array}
$$

Here the ring $\mathbb{S}$ is a partial differential algebra with respect to the three derivations $\partial_t$, $\partial_x$, $\partial_y$ that operate on the variables (12) just formally as in the one variable case treated in §3. Then the ideal $I$ defines a D-groupoid on $V$ that is the Galois groupoid $\mathrm{Gal}\,(\mathbb{C}(t,\,x,\,y)/\mathbb{C})$.

An easy calculation leads us to the following

***Proposition 5.1.*** — *For a general point $(t,\,x,\,y)\in V$, the solution $(\mathcal{T},\,\mathcal{X},\,\mathcal{Y})$ to the differential ideal $I$ or equivalently to the partial differential system (11) is*

$$
(14)\qquad \mathcal{T}=t,\qquad \mathcal{X}=x+c_1,\qquad \mathcal{Y}=y\left(\frac{x+c_1}{x}\right)^t\exp(c_2t+c_3)
$$

*where $c_1,\,c_2,c_3$ are constants.*

Differentiating the solution (14) with respect to the parameters $c_1,\,c_2,\,c_3$ at $(c_1,\,c_2,\,c_3)=(0,\,0,\,0)$, we get the vector fields

$$
(15)\qquad D_1=\frac{\partial}{\partial x}+\frac{ty}{x}\frac{\partial}{\partial y},\qquad D_2=ty\frac{\partial}{\partial y},\qquad D_3=y\frac{\partial}{\partial y}
$$

on the variety $V=\mathrm{Spec}\,R=\{(t,\,x,\,y)\in\mathbb{C}^3\,|\,x\neq 0\}$. The vector space spanned by these vector fields is closed under the bracket and forms a 3-dimensional commutative Lie algebra. This is the Lie algebra of the Galois groupoid $\mathrm{Gal}\,(\mathbb{C}(t,\,x,\,y)/\mathbb{C})$. We have thus proved

***Corollary 5.1.*** — *The Lie algebra of the Galois groupoid* $\mathrm{Gal}\,(\mathbb{C}(t,\,x,\,y)/\mathbb{C})$ *is a 3 dimensional Abelian Lie algebra spanned by $D_1,\,D_2,\,D_3$.*

**Example** 5.2. Let $n\geq$ be an integer and we consider a field extension $L:=\mathbb{C}(x,\,y)/\mathbb{C}(x)$ of the rational function field $\mathbb{C}(x)$ such that $y$ is algebraic over $\mathbb{C}(x)$ with minimal polynomial

$$
(16)\qquad y^n-x=0.
$$

With derivation $\delta=d/dx$, $L/\mathbb{C}(x)$ is a differential field extension. We have

$$
(\mathbb{C}(x,\,y),\delta)=\left(\mathbb{C}(x),\frac{1}{n}y^{1-n}\frac{d}{dy}\right).
$$

Then the Galois groupoid $\mathrm{Gal}\,((L,\delta)/\mathbb{C})$ is the groupoid $G_{n-1}$ of Example 4.2.

In fact let $\iota:\mathbb{C}[y,1/y]\to\mathbb{C}[y,\,1/y]^\natural[[X]]$ is the universal Taylor morphism. Applying the universal Taylor morphism $\iota$ to (16), we have

$$
\tilde{Y}^n-(x+X)=0,
$$

where $\tilde{Y} := \iota(y)$. Then apply the derivation $d/dy$ to this equation, we get

$$\frac{\tilde{Y}}{dy} = \left(\frac{y}{\tilde{Y}}\right)^{n-1}.$$

## 6. Relation with our previous definition

Using Example 5.1, we briefly explain the equivalence of the definition above of Galois groupoid through the dynamical system and our previous definition depending on the differential field automorphism group of a certain partial differential field extension.

We keeping the notation of Example 5.1, we apply the method of our previous paper([**4**]. We start from the ordinary differential field extension $(\mathbb{C}(t,\, x,\, y),\, \delta)/\mathbb{C}$ and construct a partial differential field extension $\mathscr{L}/\mathscr{K}$ and our infinitesimal Galois group $\mathit{Inf\text{-}gal}\,(\mathscr{L}/\mathscr{K})$ is the infinitesimal automorphism group $\mathit{Inf\text{-}aut}\,(\mathscr{L}/\mathscr{K})$.

In $R^\natural[[X]] of \S 5$, we have $R^\natural$ and $\iota(R)$. The derivations $\partial_t,\, \partial_x,\, \partial_y$ of $R^\natural$ operate on the power series ring $R^\natural[[X]]$ through coefficients. Now $\mathscr{R}$ is the differential algebra generated by $R^\natural$ and $\iota(R)$ in $R^\natural[[X]]$. It follows from the definition of $\mathscr{R}$, $\mathbb{S}$ and the ideal $I$ of $\mathbb{S}$

***Proposition 6.1***. — *The ring $\mathscr{R}$ is isomorphic to $\mathbb{S}/I$ as $\{\partial_t,\, \partial_x,\, \partial_y\}$-differential algebra.*

It follows from (11) that $\mathscr{R} = \mathbb{C}[t,\, x,\, y,\, 1/y][\tilde{X},\, \tilde{Y},\, \tilde{Y}_t]$ and $\tilde{X},\, \tilde{Y},\, \tilde{Y}_t]$ are transcendental over $\mathbb{C}[t,\, x,\, y,\, 1/y]$ The base ring is by definition $\mathbb{C}[t,\, x,\, y,\, 1/y]$. Our Galois group $\mathit{Inf\text{-}gal}\,((\mathbb{C}(t,\, x,\, y),\, \delta)/\mathbb{C})$ is the infinitesimal automorphism group of $\mathscr{R}/\mathbb{C}[t,\, x,\, y,\, 1/y]$. Propositions (5.1) and (6.1) give us 3 infinitesimal $\{\partial_t, \partial_x,\, \partial_y\}$-differential automorphisms $\sigma_i$, of $\mathscr{R}[\epsilon]/\mathbb{C}[t,\, x,\, y,\, 1/y]$ ( $1 \leq i \leq 3$ ) with $\epsilon^2 = 0$ such that

$$\begin{array}{llll}
\sigma_1(\tilde{T}) = \tilde{T}, & \sigma_1(\tilde{X}) = \tilde{X} + \epsilon, & \sigma_1(\tilde{Y}) = \tilde{Y} + \frac{t\tilde{Y}}{\tilde{X}}\epsilon, & \sigma_1(\tilde{Y}_t) = \tilde{Y}_t + \frac{\tilde{Y} - t\tilde{Y}_t}{\tilde{X}}\epsilon \\
\sigma_2(\tilde{T}) = \tilde{T}, & \sigma_2(\tilde{X}) = \tilde{X}, & \sigma_2(\tilde{Y}) = \tilde{Y} + t\tilde{Y}\epsilon, & \sigma_2(\tilde{Y}_t) = \tilde{Y}_t + (\tilde{Y} - t\tilde{Y}_t)\epsilon \\
\sigma_3(\tilde{T}) = \tilde{T}, & \sigma_3(\tilde{X}) = \tilde{X}, & \sigma_3(\tilde{Y}) = \tilde{Y} + \tilde{Y}\epsilon, & \sigma_3(\tilde{Y}_t) = \tilde{Y}_t + \tilde{Y}_t\epsilon.
\end{array}$$

These infinitesimal automorphisms define infinitesimal automorphisms of $\mathscr{L}[\epsilon]/\mathscr{K}$, where $\mathscr{L}$ and $\mathscr{K}$ are respectively the quotient field of $\mathscr{R}$ and $\mathbb{C}[t,\, x,\, y,\, 1/y]$.

## References


[1] P. J. Cassidy & M. F. Singer – “Galois theory of parameterized differential equations and linear differential algebraic groups”, in *Differential equations and quantum groups*, IRMA Lect. Math. Theor. Phys., vol. 9, Eur. Math. Soc., Zürich, 2007, p. 113–155.

[2] A. Grothendieck – “Techniques de constructions et théorème d’existence en géométrie algébrique. III. Préschémas quotients”, Séminaire Bourbaki, 1960/61, exposé n° 212.

[3] B. Malgrange – "Le groupoïde de Galois d'un feuilletage", in *Essays on geometry and related topics, Vol. 1, 2*, Monogr. Enseign. Math., vol. 38, Enseignement Math., 2001, p. 465–501.
[4] H. Umemura – "Differential Galois theory of infinite dimension", *Nagoya Math. J.* **144** (1996), p. 59–135.
[5] ———, "Galois theory of algebraic and differential equations", *Nagoya Math. J.* **144** (1996), p. 1–58.
[6] ———, "Monodromy preserving deformation and differential Galois group. I", *Astérisque* **296** (2004), p. 253–269.
[7] ———, "Galois theory and Painlevé equations", in *Théories asymptotiques et équations de Painlevé*, Sémin. Congr., vol. 14, Soc. Math. France, 2006, p. 299–339.
[8] ———, "Invitation to Galois theory", in *Differential equations and quantum groups*, IRMA Lect. Math. Theor. Phys., vol. 9, Eur. Math. Soc., Zürich, 2007, p. 269–289.
[9] ———, "Sur l'équivalence des théories de Galois différentielles générales", to appear in *C. R. Acad. Sci. Paris.*

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# TRANSSERIAL HARDY FIELDS

*by*

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***Abstract.*** — It is well known that Hardy fields can be extended with integrals, exponentials and solutions to Pfaffian first order differential equations $f' = P(f)/Q(f)$. From the formal point of view, the theory of transseries allows for the resolution of more general algebraic differential equations. However, until now, this theory did not admit a satisfactory analytic counterpart. In this paper, we will introduce the notion of a transserial Hardy field. Such fields combine the advantages of Hardy fields and transseries. In particular, we will prove that the field of differentially algebraic transseries over $\mathbb{R}\{\{x^{-1}\}\}$ carries a transserial Hardy field structure. Inversely, we will give a sufficient condition for the existence of a transserial Hardy field structure on a given Hardy field.

***Résumé* (Corps de Hardy transsériels).** — Il est bien connu que des corps de Hardy peuvent être étendus par des intégrales, des exponentielles et des solutions d'équations différentielles Pfaffiennes du type $f' = P(f)/Q(f)$. D'un point de vue formel, la théorie des transséries permet la résolution d'équations différentielles algébriques plus générales. Toutefois, cette théorie n'admettait pas encore de contre-partie analytique satisfaisante jusqu'à présent. Dans cet article, nous introduisons la notion de corps de transséries transsériel. Ces corps combinent les avantages des corps de Hardy et de la théorie des transséries. En particulier, nous démontrons que le corps des transséries vérifiant une équation différentiello-algébrique sur $\mathbb{R}\{\{x^{-1}\}\}$ possède une structure de corps de Hardy transsériel. Réciproquement, nous donnerons une condition suffisante pour l'existence d'une structure transsérielle sur un corps de Hardy donné.

## 1. Introduction

A Hardy field is a field of infinitely differentiable germs of real functions near infinity. Since any non-zero element in a Hardy field $\mathscr{H}$ is invertible, it admits no zeros in a suitable neighbourhood of infinity, whence its sign remains constant. It follows that Hardy fields both carry a total ordering and a valuation. The ordering and valuation can be shown to satisfy several natural compatibility axioms with the

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differentiation, so that Hardy fields are models of the so called theory of H-fields [**1, 3, 2**].

Other natural models of the theory of H-fields are fields of transseries [**23, 31, 15, 16, 27, 26**]. Contrary to Hardy fields, these models are purely formal, which makes them particularly useful for the automation of asymptotic calculus [**23**]. Furthermore, the so called field of grid-based transseries $\mathbb{T}$ (for instance) satisfies several remarkable closure properties. Namely, $\mathbb{T}$ is differentially Henselian [**26**, theorem 8.21] and it satisfies the differential intermediate value theorem [**26**, theorem 9.33].

Now the purely formal nature of the theory of transseries is also a drawback, since it is not *a priori* clear how to associate a genuine real function to a transseries $f$, even in the case when $f$ satisfies an algebraic differential equation over $\mathbb{R}\{\{x^{-1}\}\}$. One approach to this problem is to develop Écalle's accelero-summation theory [**17, 18, 19, 20, 11, 12**], which constitutes a more or less canonical way to associate analytic functions to formal transseries with a "natural origin". In this paper, we will introduce another approach, based on the concept of a *transserial Hardy field*.

Roughly speaking, a transserial Hardy field is a truncation-closed differential subfield $\mathscr{T}$ of $\mathbb{T}$, which is also a Hardy field. The main objectives of this paper are to show the following two things:

1. The differentially algebraic closure in $\mathbb{T}$ of a transserial Hardy field can be given the structure of a transserial Hardy field.
2. Any differentially algebraic Hardy field extension of a transserial Hardy field, which is both differentially Henselian and closed under exponentiation, admits a transserial Hardy field structure.

We have chosen to limit ourselves to the context of grid-based transseries. More generally, an interesting question is which H-fields can be embedded in fields of well-based transseries and which differential fields of well-based transseries admit Hardy field representations. We hope that work in progress [**5, 4**] on the model theory of H-fields and asymptotic fields will enable us to answer these questions in the future.

The theory of Hardy fields admits a long history. Hardy himself proved that the field of so called L-functions is a Hardy field [**21, 22**]. The definition of a Hardy field and the possibility to add integrals, exponentials and algebraic functions is due to Bourbaki [**10**]. More generally, Hardy fields can be extended by the solutions to Pfaffian first order differential equations [**32, 6**] and solutions to certain second order differential equations [**9**]. Further results on Hardy fields can be found in [**28, 29, 30, 7, 8**]. The theory of transserial Hardy fields can be thought of as a systematic way to deal with differentially algebraic extensions of any order.

The main idea behind the addition of solutions to higher order differential equations to a given transserial Hardy field $\mathscr{T}$ is to write such solutions in the form of "integral series" over $\mathscr{T}$ (see also [**25**]). For instance, consider a differential equations such as

$$f' = \mathrm{e}^{-2\mathrm{e}^{x}} + f^2,$$

for large $x \succ 1$. Such an equation may typically be written in integral form

$$f = \int \mathrm{e}^{-2\mathrm{e}^x} + \int f^2.$$

The recursive replacement of the left-hand side by the right-hand side then yields a "convergent" expansion for $f$ using iterated integrals

$$f = \int \mathrm{e}^{-2\mathrm{e}^x} + \int \left( \int \mathrm{e}^{-2\mathrm{e}^x} \right)^2 + 2 \left( \int \mathrm{e}^{-2\mathrm{e}^x} \right) \left( \int \left( \int \mathrm{e}^{-2\mathrm{e}^x} \right)^2 \right) + \cdots,$$

where we understand that each of the integrals in this expansion are taken from $+\infty$:

$$\left( \int g \right)(x) = \int_\infty^x g(t)\mathrm{d}t.$$

In order to make this idea work, one has to make sure that the extension of $\mathscr{T}$ with a solution $f$ of the above kind does not introduce any oscillatory behaviour. This is done using a combination of arguments from model theory and differential algebra.

More precisely, whenever a transseries solution $f$ to an algebraic differential equation over $\mathscr{T}$ is not yet in $\mathscr{T}$, then we may assume the equation to be of minimal "complexity" (a notion which refines Ritt rank). In section 2, we will show how to put the equation in normal form

$$Lf = P(f), \tag{1}$$

where $P \in \mathscr{T}\{F\}$ is "small" and $L \in \mathscr{T}[\partial]$ admits a factorization

$$L = (\partial - \varphi_1) \cdots (\partial - \varphi_r)$$

over $\mathscr{T}[\mathrm{i}]$. In section 4, it will be show how to solve (1) using iterated integrals, using the fact that the equation $(\partial - \varphi)f = g$ admits $\mathrm{e}^{\int \varphi} \int \mathrm{e}^{-\int \varphi} g$ as a solution. Special care will be taken to ensure that the constructed solution is again real and that the solution admits the same asymptotic expansion over $\mathscr{T}$ as the formal solution.

Section 3 contains some general results about transserial Hardy fields. In particular, we prove the basic extension lemma: given a transseries $f$ and a real germ $\hat{f}$ at infinity which behave similarly over $\mathscr{T}$ (both from the asymptotic and differentially algebraic points of view), there exists a transserial Hardy field extension of $\mathscr{T}$ in which $f$ and $\hat{f}$ may be identified. The differential equivalence of $f$ and $\hat{f}$ will be ensured by the fact that the equation (1) was chosen to be of minimal complexity. Using Zorn's lemma, it will finally be possible to close $\mathscr{T}$ under the resolution of real differentially algebraic equations. This will be the object of the last section 5. Throughout the paper, we will freely use notations from **[26]**. For the reader's convenience, some of the notations are recalled in section 2.1. We also included a glossary at the end.

It would be interesting to investigate whether the theory of transserial Hardy fields can be generalized so as to model some of the additional compositional structure on $\mathbb{T}$. A first step would be to replace all differential polynomials by restricted analytic functions [14]. A second step would be to consider postcompositions with operators $x+\delta$ for sufficiently flat transseries $f$ for which Taylor's formula holds:

$$f \circ (x+\delta) = f + f'\delta + \frac{1}{2} f''\delta^2 + \cdots .$$

This requires the existence of suitable analytic continuations of $f$ in the complex domain. Typically, if $f \in \mathbb{T}_{\underline{\lll} g}$ with $g \in \mathbb{T}^{>,\succ}$, then $f \circ g^{\mathrm{inv}}$ should be defined on some sector at infinity (notice that this can be forced for the constructions in this paper). Finally, more violent difference equations, such as

$$f(x) = \frac{1}{\mathrm{e}^{\mathrm{e}^{\mathrm{e}^x}}} + f(x+1),$$

generally give rise to quasi-analytic solutions. From the model theoretic point view, they can probably always be seen as convergent sums.

Finally, one may wonder about the respective merits of the theory of accelero-summation and the theory of transserial Hardy fields. Without doubt, the first theory is more canonical and therefore has a better behaviour with respect to composition. In particular, we expect it to be easier to prove o-minimality results [13]. On the other hand, many technical details still have to be worked out in full detail. This will require a certain effort, even though the resulting theory can be expected to have many other interesting applications. The advantage of the theory of transserial Hardy fields is that it is more direct (given the current state of art) and that it allows for the association of Hardy field elements to transseries which are not necessarily accelero-summable.

## 2. Preliminaries

**2.1. Notations.** — Let $\mathbb{T} = \mathbb{R}[[x]] = \mathbb{R}[[\mathfrak{T}]]$ be the totally ordered field of grid-based transseries, as in [26]. Any transseries is an infinite linear combination $f = \sum_{\mathfrak{m} \in \mathfrak{T}} f_{\mathfrak{m}} \mathfrak{m}$ of transmonomials, with grid-based support $\operatorname{supp} f \subseteq \mathfrak{T}$. Transmonomials $\mathfrak{m}, \mathfrak{n}, \ldots$ are systematically written using the fraktur font. Each transmonomial is an iterated logarithm $\log_l x$ of $x$ or the exponential of a transseries $g$ with $\mathfrak{n} \succ 1$ for each $\mathfrak{n} \in \operatorname{supp} g$. The asymptotic relations $\preccurlyeq, \prec, \asymp, \sim, \underline{\lll}, \lll, \asymp\!\!\!\!\asymp$ and $\approx$ on $\mathbb{T}$ are defined by

$$\begin{aligned}
f \preccurlyeq g &\iff f = O(g)\\
f \prec g &\iff f = o(g)\\
f \asymp g &\iff f \preccurlyeq g \preccurlyeq f\\
f \sim g &\iff f - g \prec g\\
f \underline{\lll} g &\iff \log|f| \preccurlyeq \log|g|\\
f \lll g &\iff \log|f| \prec \log|g|\\
f \asymp\!\!\!\!\asymp g &\iff \log|f| \asymp \log|g|\\
f \approx g &\iff \log|f| \sim \log|g|.
\end{aligned}$$

Given $\mathfrak{v} \neq 1$, one also defines variants of $\preccurlyeq, \prec$, etc. modulo flatness:

$$\begin{aligned} f \preccurlyeq_{\mathfrak{v}} g &\iff \exists \mathfrak{m} \prec\!\!\!\prec \mathfrak{v}, f \preccurlyeq g\mathfrak{m} \\ f \prec_{\mathfrak{v}} g &\iff \forall \mathfrak{m} \prec\!\!\!\prec \mathfrak{v}, f \prec g\mathfrak{m} \\ f \preccurlyeq^*_{\mathfrak{v}} g &\iff \exists \mathfrak{m} \preceq\!\!\!\preccurlyeq \mathfrak{v}, f \preccurlyeq g\mathfrak{m} \\ f \prec^*_{\mathfrak{v}} g &\iff \forall \mathfrak{m} \preceq\!\!\!\preccurlyeq \mathfrak{v}, f \prec g\mathfrak{m}. \end{aligned}$$

It is convenient to use relations as superscripts in order to filter elements, as in

$$\begin{aligned} \mathbb{T}^{>} &= \{f \in \mathbb{T} : f > 0\} \\ \mathbb{T}^{\neq} &= \{f \in \mathbb{T} : f \neq 0\} \\ \mathbb{T}^{\succ} &= \{f \in \mathbb{T} : f \succ 1\}. \end{aligned}$$

Similarly, we use subscripts for filtering on the support:

$$\begin{aligned} f_{\succ} &= \sum_{\mathfrak{m} \in \operatorname{supp} f, \mathfrak{m} \succ 1} f_{\mathfrak{m}} \mathfrak{m} \\ f_{\prec\!\!\!\prec \mathfrak{v}} &= \sum_{\mathfrak{m} \in \operatorname{supp} f, \mathfrak{m} \preceq\!\!\!\preccurlyeq \mathfrak{v}} f_{\mathfrak{m}} \mathfrak{m} \\ \mathbb{T}_{\succ} &= \{f_{\succ} : f \in \mathbb{T}\} \\ \mathbb{T}_{\prec\!\!\!\prec \mathfrak{v}} &= \{f_{\prec\!\!\!\prec \mathfrak{v}} : f \in \mathbb{T}\}. \end{aligned}$$

We denote the derivation on $\mathbb{T}$ w.r.t. $x$ by $\partial$ and the corresponding distinguished integration (with constant part zero) by $\int$. The logarithmic derivative of $f$ is denoted by $f^\dagger$. The operations $\uparrow$ and $\downarrow$ of upward and downward shifting correspond to post-composition with $\exp x$ resp. $\log x$. We finally write $f \trianglelefteq g$ if the transseries $f$ is a truncation of $g$, i.e. $\mathfrak{m} \prec \operatorname{supp} f$ for all $\mathfrak{m} \in \operatorname{supp}(g - f)$.

**2.2. Differential fields of transseries and cuts.** — Given $f \in \mathbb{T}$, we define the *canonical span* of $f$ by

$$\operatorname{span} f = \max_{\preceq\!\!\!\preccurlyeq} \{ \mathrm{e}^{-\partial(\log(\mathfrak{m}/\mathfrak{n}))} : \mathfrak{m}, \mathfrak{n} \in \operatorname{supp} f \}. \tag{2}$$

By convention, $\operatorname{span} f = 1$ if $\operatorname{supp} f$ contains less than two elements. We also define the *ultimate canonical span* of $f$ by

$$\operatorname{uspan} f = \min_{\preceq\!\!\!\preccurlyeq} \{ \operatorname{span} f_{\prec \mathfrak{v}} : \mathfrak{v} \in \operatorname{supp} f \}. \tag{3}$$

We notice that $\operatorname{uspan} f \neq 1$ if and only if $\operatorname{supp} f$ admits no minimal element for $\preccurlyeq$.

***Example 1.*** — We have

$$\begin{aligned} \operatorname{span}\left(1 + \frac{\mathrm{e}^{-x}}{1 - x^{-1}}\right) &= \mathrm{e}^{-x} \\ \operatorname{uspan}\left(1 + \frac{\mathrm{e}^{-x}}{1 - x^{-1}}\right) &= x^{-1} \end{aligned}$$

Consider a differential subfield $\mathscr{T}$ of $\mathbb{T}$ and let $\mathfrak{v} \in \mathfrak{T}^{\prec}$. We say that $\mathscr{T}$ has span $\mathfrak{v}$, if $\operatorname{span} f \preceq\!\!\!\prec \mathfrak{v}$ for all $f \in \mathscr{T}$ and $\operatorname{span} f \asymp\!\!\!\asymp \mathfrak{v}$ for at least one $f \in \mathscr{T}$ (notice that we do not require $\mathfrak{v} \in \mathrm{e}^{-\mathfrak{T}}$). Since $\mathscr{T}$ is stable under differentiation, we have $\mathfrak{v} \succeq\!\!\!\succ x^{-1}$ as soon as $\mathscr{T} \neq \mathbb{R}$. Notice also that we must have $\mathscr{T} \subseteq \mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}$ if $\mathscr{T}$ has span $\mathfrak{v}$.

A transseries $f \in \mathbb{T} \setminus \mathscr{T}$ is said to be a *serial cut* over $\mathscr{T}$, if $\varphi \in \mathscr{T}$ for every $\varphi \lhd f$ and $\operatorname{supp} f$ admits no minimal element for $\preccurlyeq$. In that case, let $\mathfrak{m} \in \operatorname{supp} f$ be maximal for $\preccurlyeq$ such that $\mathfrak{m}^{-1} \operatorname{supp} f_{\preccurlyeq \mathfrak{m}} \preceq\!\!\!\prec \operatorname{span} f$. Then $H_f = f_{\succ \mathfrak{m}}$ and $T_f = f_{\preccurlyeq \mathfrak{m}}$ are called the *head* and the *tail* of $f$. We say that $f$ is a *normal serial cut* if $f \in \mathbb{T}_{\preceq\!\!\!\prec \operatorname{span} f}$, which implies in particular that $H_f = 0$.

Assuming that $\mathscr{T}$ has span $\mathfrak{v}$, any serial cut over $\mathscr{T}$ is necessarily in $\mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}$. Conversely, any $f \in \mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}} \setminus \mathscr{T}$ with $\operatorname{uspan} f \asymp\!\!\!\asymp \mathfrak{v}$ is a serial cut over $\mathscr{T}$. We will denote by $\hat{\mathscr{T}}$ the set of all $f \in \mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}$ which are either in $\mathscr{T}$ or serial cuts over $\mathscr{T}$ with $\operatorname{uspan} f \asymp\!\!\!\asymp \mathfrak{v}$. Notice that $\hat{\mathscr{T}}$ is again a differential subfield of $\mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}$.

The above definitions naturally adapt to the complexifications $\mathbb{T}[\mathrm{i}]$ and $\mathscr{T}[\mathrm{i}]$ of $\mathbb{T}$ and differential subfields $\mathscr{T}$ of $\mathbb{T}$. If $\mathscr{T}$ has span $\mathfrak{v}$, then the set $\hat{\mathscr{T}}[\mathrm{i}]$ coincides with the set of all $f \in \mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}[\mathrm{i}] = \mathbb{T}[\mathrm{i}]_{\preceq\!\!\!\prec \mathfrak{v}}$ which are either in $\mathscr{T}[\mathrm{i}]$ or serial cuts over $\mathscr{T}[\mathrm{i}]$ with $\operatorname{uspan} f \asymp\!\!\!\asymp \mathfrak{v}$.

**2.3. Complements on differential algebra.** — Let $\mathscr{T}$ be a differential field. We denote by $\mathscr{T}\{F\}$ the ring of differential polynomials in $F$ over $\mathscr{T}$ and by $\mathscr{T}\langle F\rangle$ its quotient field. Given $P \in \mathscr{T}\{F\}$ and $i \in \mathbb{N}$, we recall that $P_i$ denotes the homogeneous part of degree $i$ of $P$. We will denote by $L_P$ the linear operator in $\mathscr{T}[\partial]$ with $L_P F = P_1(F)$. Assuming that $P \setminus \mathscr{T}$, we also denote the order of $P$ by $r_P$, the degree of $P$ in $F^{(r_P)}$ by $s_P$ and the total degree of $P$ by $t_P$. Thus, the Ritt rank of $P$ is given by the pair $(r_P, s_P)$. The triple $\chi_P = (r_P, s_P, t_P)$ will be called the *complexity* of $P$; likewise ranks, complexities are ordered lexicographically.

As usual, we will denote the initial and separator of $P$ by $I_P$ resp. $S_P$ and set $H_P = I_P S_P$. Given $P, Q \in \mathscr{T}\{F\}$ with $P \notin \mathscr{T}$, Ritt reduction of $Q$ by $P$ provides us with a relation

$$H_P^{\alpha} Q = \boldsymbol{A} P + R, \tag{4}$$

where $\boldsymbol{A} \in \mathscr{T}\{F\}[\partial]$ is a linear differential operator, $\alpha \in \mathbb{N}$ and the remainder $R \in \mathscr{T}\{F\}$ satisfies $\chi_R < \chi_P$.

Let $\mathscr{K}$ be a differential field extension of $\mathscr{T}$. An element $f \in \mathscr{K}$ is said to be *differentially algebraic* over $\mathscr{T}$ if there exists an annihilator $P \in \mathscr{T}\{F\} \setminus \mathscr{T}$ with $P(f) = 0$. An annihilator $P$ of minimal complexity $\chi_P$ will then be called a *minimal annihilator* and $\chi_f = \chi_P$ is also called the *complexity* of $f$ over $\mathscr{T}$. The order $r_f = r_P$ of such a minimal annihilator $P$ is called the *order* of $f$ over $\mathscr{T}$. We say that $\mathscr{K}$ is a *differentially algebraic extension* of $\mathscr{T}$ if each $f \in \mathscr{K}$ is differentially algebraic over $\mathscr{T}$.

We say that $\mathscr{T}$ is differentially closed in $\mathscr{K}$, if $\mathscr{K} \setminus \mathscr{T}$ contains no elements which are differentially algebraic over $\mathscr{T}$.. Given $\chi \in \mathbb{N}^3$ (resp. $r \in \mathbb{N}$), we say that $\mathscr{T}$ is $\chi$-*differentially closed* (resp. $r$-*differentially closed*) in $\mathscr{K}$ if $\chi_f > \chi$ (resp. $r_f > r$) for all $f \in \mathscr{K} \setminus \mathscr{T}$. We say that $\mathscr{T}$ is *weakly differentially closed* if every $P \in \mathscr{T}\{F\} \setminus \mathscr{T}$ admits a root in $\mathscr{T}$. We say that $\mathscr{T}$ is *weakly $r$-differentially closed* if every $P \in \mathscr{T}\{F\} \setminus \mathscr{T}$ of order $\leqslant r$ admits a root in $\mathscr{T}$.

Given a differential polynomial $P \in \mathscr{T}\{F\}$ and $\varphi \in \mathscr{T}$, we define the *additive* and *multiplicative conjugates* of $P$ by $\varphi$:

$$\begin{aligned} P_{+\varphi}(F) &= P(F+\varphi) \\ P_{\times\varphi}(F) &= P(\varphi F). \end{aligned}$$

We have $P_{+\varphi}, P_{\times\varphi} \in \mathscr{T}\{F\}$ and

$$\begin{aligned} \chi_{P_{+\varphi}} &= \chi_P \\ \chi_{P_{\times\varphi}} &= \chi_P \\ I_{P_{+\varphi}} &= I_{P,+\varphi} \\ I_{P_{\times\varphi}} &= I_{P,\times\varphi} \\ S_{P_{+\varphi}} &= S_{P,+\varphi} \\ S_{P_{\times\varphi}} &= S_{P,\times\varphi} \end{aligned}$$

We also notice that additive and multiplicative conjugation are compatible with Ritt reduction: given $\varphi \in \mathscr{T}$ and assuming (4), we have

$$\begin{aligned} H^{\alpha}_{P_{+\varphi}} Q_{+\varphi} &= \boldsymbol{A} P_{+\varphi} + R_{+\varphi} \\ H^{\alpha}_{P_{\times\varphi}} Q_{\times\varphi} &= \boldsymbol{A} P_{\times\varphi} + R_{\times\varphi}, \end{aligned}$$

***Remark 1.*** — The compatibility of Ritt's reduction theory with additive and multiplicative conjugation holds more generally for rings of differential polynomials in a finite number of commutative partial derivations (or with a finite dimensional Lie algebra of non-commutative derivations). Similar compatibility results hold for upward shiftings or changes of derivations (in the partial case, this requires the rankings to be order-preserving).

In the case when $\mathscr{T}$ is a differential subfield of $\mathbb{T} = \mathbb{R}[[\mathfrak{T}]]$, we recall that a differential polynomial $P \in \mathscr{T}\{F_1, \ldots, F_k\}$ may also be regarded as a series in $\mathbb{R}\{F_1, \ldots, F_k\}[[\mathfrak{T}]]$. Similarly, elements $P/Q$ of the fraction field $\mathscr{T}\langle F_1, \ldots, F_k\rangle$ of $\mathscr{T}\{F_1, \ldots, F_k\}$ may be regarded as series with coefficients in $\mathbb{R}\langle F_1, \ldots, F_k\rangle$. Indeed, writing $P = D_P \mathfrak{d}_P + R_P$ and $Q = D_Q \mathfrak{d}_Q + R_Q$, where $D_P \mathfrak{d}_P$ denotes the dominant term of $P$, we may expand

$$\frac{P}{Q} = \frac{D_P}{D_Q} \cdot \frac{\mathfrak{d}_P}{\mathfrak{d}_Q} \cdot \frac{1 + \frac{R_P}{D_P \mathfrak{d}_P}}{1 + \frac{R_Q}{D_Q + \mathfrak{d}_Q}}$$

In the case when $P, Q \in \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_n]]\{F_1, \ldots, F_k\}$ for some transbasis $\mathfrak{B} = \{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\}$, then $P$ and $P/Q$ may also be expanded lexicographically with respect to $\mathfrak{b}_n, \ldots, \mathfrak{b}_1$.

**2.4. Linear differential operators and factorization.** — Let $\mathscr{T}$ be a differential field and consider a linear differential operator $L \in \mathscr{T}[\partial]^{\neq}$. We will denote the order of $L$ by $r_L$. Given $\psi \in \mathscr{T}$, we define the *multiplicative conjugate* $L_{\times\psi}$ and the *twist* $L_{\ltimes\psi}$ by

$$\begin{aligned} L_{\times\psi} &= L\psi \\ L_{\ltimes\psi} &= \psi^{-1}L\psi \end{aligned}$$

We notice that $L_{\ltimes\psi}$ is also obtained by substitution of $\partial + \psi^\dagger$ for $\partial$ in $L$. We say that $L$ *splits* over $\mathscr{T}$, if it admits a complete factorization

$$L = c(\partial - \varphi_1)\cdots(\partial - \varphi_r) \tag{5}$$

with $c, \varphi_1, \ldots, \varphi_r \in \mathscr{T}$. In that case, each of the twists $L_{\ltimes\psi}$ of $L$ also splits:

$$L_{\ltimes\psi} = c(\partial + \psi^\dagger - \varphi_1)\cdots(\partial + \psi^\dagger - \varphi_r).$$

We say that $\mathscr{T}$ is $r$-*linearly closed* if any linear differential operator of order $\leqslant r$ splits over $\mathscr{T}$.

***Proposition 1***. — *If $\mathscr{T}$ is weakly $(r-1)$-differentially closed, then $\mathscr{T}$ s $r$-linearly closed.*

*Proof*. — The proof proceeds by induction over $r$. For $r = 0$, we have nothing to prove, so assume that $r > 0$ and let $L \in \mathscr{T}[\partial]$ be of order $r$. Then the differential Riccati polynomial $R_L$ has order $r-1$, so it admits a root $\varphi_r \in \mathscr{T}$. Division of $L$ by $\partial - \varphi_r$ in $\mathscr{T}[\partial]$ yields a factorization $L = \tilde{L}(\partial - \varphi_r)$ where $\tilde{L} \in \mathscr{T}[\partial]$ has order $r-1$. By the induction hypothesis, $\tilde{L}$ splits over $\mathscr{T}$, whence so does $L$. □

***Proposition 2***. — *Let $L \in \mathscr{T}[\partial]^{\neq}$ be an operator which splits over $\mathscr{T}$ and let $A, B \in \mathscr{T}[\partial]$ be such that $L = AB$. Then $A$ and $B$ split over $\mathscr{T}$.*

*Proof*. — Recall that greatest common divisors and least common multiples exist in the ring $\mathscr{T}[\partial]$. Given a splitting (5), consider the operators

$$\begin{aligned} \Lambda_i &= \operatorname{lcm}(B, (\partial - \varphi_{r+1-i})\cdots(\partial - \varphi_r)) \\ \Gamma_i &= \gcd(B, (\partial - \varphi_{r+1-i})\cdots(\partial - \varphi_r)) \end{aligned}$$

We have $B = \Lambda_0|\cdots|\Lambda_r = AB$ and $1 = \Gamma_0|\cdots|\Gamma_r = B$. Moreover, the orders of $\Lambda_i$ and $\Lambda_{i+1}$ (resp. $\Gamma_i$ and $\Gamma_{i+1}$) differ at most by one for each $i$. It follows that $A$ and $B$ split over $\mathscr{T}$. □

Assume now that $\mathscr{T}$ is a totally ordered differential field. A monic operator $L \in \mathscr{T}[\partial]^{\neq}$ is said to be an *atomic real operator* if $L$ has either one of the forms

$$\begin{aligned} L &= \partial - \varphi, && \varphi \in \mathscr{T} \\ L &= (\partial - (\varphi - \psi \mathrm{i} + \psi^\dagger))(\partial - (\varphi + \psi\mathrm{i})), && \varphi, \psi \in \mathscr{T} \end{aligned}$$

A *real splitting* of an operator $L \in \mathscr{T}[\partial]^{\neq}$ over $\mathscr{T}$ is a factorization of the form

$$L = K_1 \cdots K_s, \tag{6}$$

where each $K_i$ is an atomic real operator. A splitting (5) over $\mathscr{T}[\mathrm{i}]$ is said to *preserve realness*, if it gives rise to a real splitting (6) for $K_i = (\partial - \varphi_{i_j})$ or $K_i = (\partial - \varphi_{i_j})(\partial - \varphi_{i_j+1})$ and $i_1 < \cdots < i_s$.

***Proposition 3***. — *Let $L \in \mathscr{T}[\partial]^{\neq}$ be an operator which splits over $\mathscr{T}[\mathrm{i}]$. Then $L$ admits a real splitting over $\mathscr{T}$.*

*Proof.* — Assuming that $L \notin \mathscr{T}$, we claim that there exists an atomic real right factor $K \in \mathscr{T}[\partial]$ of $L$. Consider a splitting (5) over $\mathscr{T}[\mathrm{i}]$. If $\varphi_r \in \mathscr{T}$, then we may take $K = \partial - \varphi_r$. Otherwise, we write

$$L = \bar{c}(\partial - \bar{\varphi}_1) \cdots (\partial - \bar{\varphi}_r)$$

and take $K$ to be the least common multiple of $\partial - \varphi_r$ and $\partial - \bar{\varphi}_r$ in $\mathscr{T}[\mathrm{i}]$. Since $K = \bar{K}$, we indeed have $K \in \mathscr{T}[\partial]$. Since $\partial - \varphi_r | L$ and $\partial - \bar{\varphi}_r | L$, we also have $K|L$. In particular, proposition 2 implies that $K$ splits over $\mathscr{T}[\mathrm{i}]$. Such a splitting is necessarily of the form

$$K = (\partial - (\varphi - \psi\mathrm{i} + \psi^{\dagger}))(\partial - (\varphi + \psi\mathrm{i})), \quad \varphi, \psi \in \mathscr{T},$$

whence $K$ is atomic. Having proved our claim, the proposition follows by induction over $r$. Indeed, let $\tilde{L} \in \mathscr{T}[\partial]$ be such that $\tilde{L}K = L$. By proposition 2, $\tilde{L}$ splits over $\mathscr{T}[\mathrm{i}]$. By the induction hypothesis, $\tilde{L}$ therefore admits a real splitting $\tilde{L} = K_1 \cdots K_s$ over $\mathscr{T}$. But then $L = K_1 \cdots K_s K$ is a real splitting of $L$. □

***Corollary 1***. — *An operator $L \in \mathscr{T}[\partial]^{\neq}$ is atomic if and only if $L$ is irreducible over $\mathscr{T}$ and $L$ splits over $\mathscr{T}[\mathrm{i}]$.*

**2.5. Factorization at cuts.** — Let $\mathscr{T}$ be a differential subfield of $\mathbb{T}$ of span $\mathfrak{v}$. Given $P \in \mathscr{T}[\mathrm{i}]\{F\}$ and $f \in \hat{\mathscr{T}}[\mathrm{i}]$, we say that $P$ *splits* over $\hat{\mathscr{T}}[\mathrm{i}]$ at $f$, if $L_{P_{+f}}$ and $P$ have the same order $r$ and $L_{P_{+f}}$ splits over $\hat{\mathscr{T}}[\mathrm{i}]$.

***Lemma 1***. — *Let $\mathscr{T}$ be a differential subfield of $\mathbb{T}$ of span $\mathfrak{v}$. Let $P \in \mathscr{T}[\mathrm{i}]\{F\}$ be a minimal annihilator of a differentially algebraic cut $f \in \hat{\mathscr{T}}[\mathrm{i}]$ over $\mathscr{T}[\mathrm{i}]$, which splits over $\hat{\mathscr{T}}[\mathrm{i}]$ at $f$. Then any minimal annihilator $Q \in \mathscr{T}[\mathrm{i}]\langle f\rangle\{\bar{F}\}$ of $\bar{f}$ over $\mathscr{T}[\mathrm{i}]\langle f\rangle$ splits over $\hat{\mathscr{T}}[\mathrm{i}]$ at $\bar{f}$.*

*Proof.* — Since $\bar{P}(\bar{f}) = 0$, Ritt division of $\bar{P}$ by $Q$ yields

$$H_Q^{\alpha}\bar{P} = \boldsymbol{A}Q \tag{7}$$

for some $\alpha \in \mathbb{N}$ and $\boldsymbol{A} \in \mathscr{T}[\mathrm{i}]\langle f\rangle\{\bar{F}\}[\partial]$. Additive conjugation of (7) yields

$$H_{Q_{+\bar{f}}}^{\alpha}\bar{P}_{+\bar{f}} = \boldsymbol{A}Q_{+\bar{f}}. \tag{8}$$

By the minimality hypothesis for $Q$, we have $L_{Q_{+\bar{f}},r_Q} = S_Q(\bar{f}) \neq 0$ and $H_Q(\bar{f}) \neq 0$, so that $\operatorname{val} Q_{+\bar{f}} = 1$ and $\operatorname{val} H_{Q_{+\bar{f}}} = 0$. Similarly, we have $\operatorname{val} \bar{P}_{+\bar{f}} = 1$. Consequently, when considering the linear part of the equation (8), we obtain

$$H_{Q_{+\bar{f}},0}^{\alpha} L_{\bar{P}_{+\bar{f}}} = \boldsymbol{A}_0 L_{Q_{+\bar{f}}},$$

whence $L_{Q_{+\bar{f}}}$ divides $L_{\bar{P}_{+\bar{f}}}$ in $\mathscr{T}[\mathrm{i}]\langle f\rangle[\partial]$. Now $L_{P_{+f}}$ splits over $\hat{\mathscr{T}}[\mathrm{i}][\partial]$, whence so does $L_{\bar{P}_{+\bar{f}}}$. By proposition 2, we infer that $L_{Q_{+\bar{f}}}$ splits over $\hat{\mathscr{T}}[\mathrm{i}][\partial]$. Since $S_Q(\bar{f}) \neq 0$, we also have $r_{L_{Q_{+\bar{f}}}} = r_Q$ and we conclude that $Q$ splits over $\hat{\mathscr{T}}[\mathrm{i}]$ at $\bar{f}$. □

***Corollary 2.*** — *Let $\mathscr{T}$ be a differential subfield of $\mathbb{T}$ of span $\mathfrak{v}$. Let $P \in \mathscr{T}[\mathrm{i}]\{F\}$ be a minimal annihilator of a differentially algebraic cut $f \in \hat{\mathscr{T}}[\mathrm{i}]$ over $\mathscr{T}[\mathrm{i}]$, which splits over $\hat{\mathscr{T}}[\mathrm{i}]$ at $f$. Then any minimal annihilator $R \in \mathscr{T}[\mathrm{i}]\langle f\rangle\{G\}$ of $\operatorname{Re} f$ over $\mathscr{T}[\mathrm{i}]\langle f\rangle$ splits over $\hat{\mathscr{T}}[\mathrm{i}]$ at $\operatorname{Re} f$.*

*Proof.* — Applying the lemma to $Q = R_{/2,-f}$, we see that $L_{Q_{+\bar{f}}}$ splits over $\hat{\mathscr{T}}[\mathrm{i}]$. Now $Q_{+\bar{f}} = R_{+\operatorname{Re} f,/2}$, whence $L_{R_{+\operatorname{Re} f,/2}}$ and $L_{R_{\operatorname{Re} f}} = L_{R_{+\operatorname{Re} f,/2,\times 2}}$ also split over $\hat{\mathscr{T}}[\mathrm{i}]$. □

***Lemma 2.*** — *Let $\mathscr{T}$ be a differential subfield of $\mathbb{T}$ of span $\mathfrak{v}$, such that $\hat{\mathscr{T}}[\mathrm{i}]$ is $r$-linearly closed. Let $P \in \mathscr{T}[\mathrm{i}]\{F\}$ be a minimal annihilator of a differentially algebraic cut $f \in \hat{\mathscr{T}}[\mathrm{i}]$ over $\mathscr{T}[\mathrm{i}]$, such that $P$ has order $r$. Assume that $\operatorname{Re} f \notin \mathscr{T}$ and let $S \in \mathscr{T}\{G\}$ be a minimal annihilator of $\operatorname{Re} f$ over $\mathscr{T}$. Then $S$ splits over $\hat{\mathscr{T}}[\mathrm{i}]$ at $\operatorname{Re} f$.*

*Proof.* — Let $R$ be as in the above corollary, so that $R$ splits over $\hat{\mathscr{T}}[\mathrm{i}]$ at $\operatorname{Re} f$. Since $R$ has minimal complexity and $S(\operatorname{Re} f) = 0$, Ritt division of $S$ by $R$ yields

$$H_R^\alpha S = \boldsymbol{A} R$$

for some $\alpha \in \mathbb{N}$ and $\boldsymbol{A} \in \mathbb{T}[\mathrm{i}]\langle f\rangle\{G\}[\partial]$. Additive conjugation and extraction of the linear part yields

$$H^\alpha_{S_{+\operatorname{Re} f},0} L_{S_{+\operatorname{Re} f}} = \boldsymbol{A}_0 L_{R_{+\operatorname{Re} f}},$$

so $L_{R_{+\operatorname{Re} f}}$ divides $L_{S_{+\operatorname{Re} f}}$ in $\hat{\mathscr{T}}[\mathrm{i}][\partial]$. Since the separants of $R$ and $S$ don't vanish at $\operatorname{Re} f$, we have

$$\begin{array}{rcrcl}
r_{L_{R_{+\operatorname{Re} f}}} & = & r_R & = & \operatorname{tr deg}(\mathscr{T}[\mathrm{i}]\langle f, \operatorname{Re} f\rangle : \mathscr{T}[\mathrm{i}]\langle f\rangle) \\
 & & & = & \operatorname{tr deg}(\mathscr{T}[\mathrm{i}]\langle \operatorname{Re} f, \operatorname{Im} f\rangle : \mathscr{T}[\mathrm{i}]) - \operatorname{tr deg}(\mathscr{T}[\mathrm{i}]\langle f\rangle : \mathscr{T}[\mathrm{i}]) \\
 & & & = & \operatorname{tr deg}(\mathscr{T}\langle \operatorname{Re} f, \operatorname{Im} f\rangle : \mathscr{T}) - \operatorname{tr deg}(\mathscr{T}[\mathrm{i}]\langle f\rangle : \mathscr{T}[\mathrm{i}]) \\
r_{L_{S_{+\operatorname{Re} f}}} & = & r_S & = & \operatorname{tr deg}(\mathscr{T}\langle \operatorname{Re} f\rangle : \mathscr{T}) \\
 & & & = & \operatorname{trdeg}(\mathscr{T}\langle \operatorname{Re} f, \operatorname{Im} f\rangle : \mathscr{T}) - \\
 & & & & \quad \operatorname{tr deg}(\mathscr{T}\langle \operatorname{Re} f, \operatorname{Im} f\rangle : \mathscr{T}\langle \operatorname{Re} f\rangle)
\end{array}$$

and

$$r_S - r_R \;=\; \operatorname{tr deg}(\mathscr{T}[\mathrm{i}]\langle f\rangle : \mathscr{T}[\mathrm{i}]) - \operatorname{tr deg}(\mathscr{T}\langle \operatorname{Re} f, \operatorname{Im} f\rangle : \mathscr{T}\langle \operatorname{Re} f\rangle) \;\leqslant\; r.$$

Consequently, the quotient of $L_{S_{+\operatorname{Re} f}}$ and $L_{R_{+\operatorname{Re} f}}$ has order at most $r$, whence it splits over $\hat{\mathscr{T}}[\mathrm{i}]$. It follows that $L_{S_{+\operatorname{Re} f}}$ splits over $\hat{\mathscr{T}}[\mathrm{i}]$ and $S$ splits over $\hat{\mathscr{T}}[\mathrm{i}]$ at $\operatorname{Re} f$. □

**2.6. Normalization of linear operators.** — Let $\mathscr{T}$ be a differential subfield of $\mathbb{T}$ of span $\mathfrak{v} \ggcurly x$. Recall from [**26**, Section 7.7] that $Lh = 0$ with $L \in \mathscr{T}[\mathrm{i}][\partial]$ admits a canonical fundamental system of oscillatory transseries solutions $\Sigma_L = \{h_1, \ldots, h_r\} \subseteq \mathbb{O}$ with $\log h_1, \ldots, \log h_r \in \mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}[\mathrm{i}]$. We will denote by $\mathfrak{H}_L$ the set of dominant monomials of $h_1, \ldots, h_r$. The neglection relation on $\mathbb{T}$ is extended to $\mathbb{O}$ by $f \prec 1$ if and only if $f = f_{;\psi_1} \mathrm{e}^{\mathrm{i}\psi_1} + \cdots + f_{;\psi_p} \mathrm{e}^{\mathrm{i}\psi_p}$ with $f_{;\psi_1}, \ldots, f_{;\psi_p} \in \mathbb{T}[\mathrm{i}]^{\prec}$ and $\psi_1, \ldots, \psi_p \in \mathbb{T}$.

We say that $L$ is *normal*, if we have $h_i \succ_{\mathfrak{v}} 1$ or $\operatorname{Re} \log h_i \succ \log \mathfrak{v}$ for each $i$. In that case, any quasi-linear equation of the form

$$Lf = g, \qquad f \preccurlyeq_{\mathfrak{v}} 1$$

with $g \in \mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}[\mathrm{i}]$ admits $L^{-1} g$ as its only solution in $\mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}[\mathrm{i}]$. If $L$ is a first order operator of the form $L = \partial - \varphi$, then $L$ is normal if and only if $\operatorname{Re} \varphi \geqslant c \mathfrak{v}^{\dagger}$ for some $c > 0$ or $\operatorname{Re} \varphi \succ \mathfrak{v}^{\dagger}$. In particular, we must have $\varphi \succcurlyeq_{\mathfrak{v}} 1$ and $\operatorname{Re} \varphi \succcurlyeq \mathfrak{v}^{\dagger}$.

***Proposition 4.*** — *Let $L \in \mathscr{T}[\mathrm{i}][\partial] \setminus \mathscr{T}[\mathrm{i}]$.*

a) *There exists a $\lambda \in \mathbb{R}$ such that $L_{\ltimes \mathfrak{v}^{\lambda}}$ is normal.*
b) *If $L$ is normal and $\lambda \geqslant 0$, then $L_{\ltimes \mathfrak{v}^{\lambda}}$ is normal.*

*Proof.* — Let $\Sigma_L = \{h_1, \ldots, h_r\}$. For each $\lambda \in \mathbb{R}$, the operator $L_{\ltimes \mathfrak{v}^{\lambda}}$ admits $h_1 / \mathfrak{v}^{\lambda}, \ldots, h_r / \mathfrak{v}^{\lambda}$ as solutions, which implies in particular that $\mathfrak{H}_{L_{\ltimes \mathfrak{v}^{\lambda}}} = \mathfrak{v}^{-\lambda} \mathfrak{H}_L$. Now $\operatorname{Re} \log(h_i / \mathfrak{v}^{\lambda}) \preccurlyeq \log \mathfrak{v} \Leftrightarrow \operatorname{Re} \log h_i \preccurlyeq \log \mathfrak{v}$ for all $i$. Choosing $\lambda$ sufficiently large, it follows that $h_i / \mathfrak{v}^{\lambda} \succ_{\mathfrak{v}} 1$ for all $i$ with $\operatorname{Re} \log(h_i / \mathfrak{v}^{\lambda}) \preccurlyeq \log \mathfrak{v}$, so that $L_{\ltimes \mathfrak{v}^{\lambda}}$ is normal. Similarly, if $h_i \succ_{\mathfrak{v}} 1$ for some $i$ with $\operatorname{Re} \log(h_i / \mathfrak{v}^{\lambda}) \preccurlyeq \log \mathfrak{v}$, then $h_i \succ_{\mathfrak{v}} \mathfrak{v}^{\lambda}$ for all $\lambda \geqslant 0$. □

***Proposition 5.*** — *Consider a normal operator $L \in \mathscr{T}[\mathrm{i}][\partial]$, which admits a splitting*

$$L = (\partial - \varphi_1) \cdots (\partial - \varphi_r)$$

*with $\varphi_1, \ldots, \varphi_r \in \mathscr{T}[\mathrm{i}]$. Then each $\partial - \varphi_i$ is a normal operator.*

*Proof.* — We will call $h \in \mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}[\mathrm{i}] \mathrm{e}^{\mathrm{i} \mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}}$ normal, if $\partial - h^{\dagger}$ is normal. Let us first prove the following auxiliary result: given $\varphi \in \mathscr{T}[\mathrm{i}]$ and $h \in \mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}[\mathrm{i}] \mathrm{e}^{\mathrm{i} \mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}}$ such that $\partial - \varphi$ and $h$ are normal and $\mathfrak{h} = \mathfrak{d}_h \notin \mathfrak{H}_{\partial - \varphi}$, then $(\partial - \varphi) h$ is also normal. If $\operatorname{Re} \log h \succ \log \mathfrak{v}$, then $0 \neq (\partial - \varphi) h \asymp_{\mathfrak{v}}^{*} h$, whence $\operatorname{Re} \log(\partial - \varphi) h = \operatorname{Re} \log h + O(\log \mathfrak{v}) \succ \log \mathfrak{v}$. In the other case, we have $h \succ_{\mathfrak{v}} 1$. Now if $\mathfrak{h}^{\dagger} \not\sim \varphi$, then $(\partial - \varphi) h \succ_{\mathfrak{v}} 1$, since $\varphi \succcurlyeq_{\mathfrak{v}} 1$. If $\mathfrak{h}^{\dagger} \sim \varphi$, then $\mathfrak{h} \notin \mathfrak{H}_{\partial - \varphi}$ implies $1 \notin \mathfrak{H}_{(\partial - \varphi)_{\ltimes h}}$, whence $\varphi - h^{\dagger} \succ 1 / (x \log x \cdots)$. It again follows that $(\partial - \varphi) h \succcurlyeq_{\mathfrak{v}} h / (x \log x \cdots) \succ_{\mathfrak{v}} 1$.

Let us now prove the proposition by induction over $r$. For $r = 1$, we have nothing to do, so assume that $r > 1$. Since $\tilde{L} = (\partial - \varphi_2) \cdots (\partial - \varphi_r)$ is normal, the induction hypothesis implies that $\partial - \varphi_i$ is normal for all $i \geqslant 2$. Now let $h$ be the unique element in $\Sigma_L \setminus \Sigma_{\tilde{L}}$. Since $h$ is normal, $(\partial - \varphi_i) \cdots (\partial - \varphi_r) h$ is also normal for $i = r, \ldots, 2$, by the auxiliary result. We conclude that $\partial - \varphi_1$ is normal, since $\varphi_1 = (\tilde{L} h)^{\dagger}$. □

Let $L$ and $\Sigma_L = \{h_1, \ldots, h_r\}$ be as above. The smallest real number $\nu \geqslant 0$ with $\log h_i \preccurlyeq_{\mathfrak{v}} \mathfrak{v}^{-\nu}$ for all $i$ will be called the *growth rate* of $L$, and we denote $\sigma_L = \nu$. For all $\alpha \in \mathbb{R}$, we notice that $\sigma_{L_{\ltimes \mathfrak{v}^\alpha}} = \sigma_L$.

***Proposition 6***. — *Let $K, L \in \mathscr{T}[\mathrm{i}][\partial]$ be operators of the same order with*

$$K = L + o_{\mathfrak{v}}(\mathfrak{v}^{r_L \sigma_L} L).$$

*Then $\mathfrak{H}_K = \mathfrak{H}_L$.*

*Proof*. — Given $h \in \Sigma_L$, we have

$$K_{\ltimes h} = L_{\ltimes h} + o_{\mathfrak{v}}(L_{\ltimes h}),$$

since $h^\dagger \asymp_{\mathfrak{v}} \log h \preccurlyeq \mathfrak{v}^{-\sigma_L}$. In particular, $K_{\ltimes h,0} \prec_{\mathfrak{v}} K$, whence $1 \in \mathfrak{H}_{K_{\ltimes h}}$ and $\mathfrak{d}_h \in \mathfrak{H}_K$. □

***Proposition 7***. — *Given a splitting*

$$L = (\partial - \varphi_1) \cdots (\partial - \varphi_r)$$

*with $\varphi_1, \ldots, \varphi_r \in \mathbb{T}_{\underline{\lll}\mathfrak{v}}[\mathrm{i}]$, we have $\varphi_i \preccurlyeq_{\mathfrak{v}} \mathfrak{v}^{-\sigma_L}$ for all $i$.*

*Proof*. — Assume for contradiction that $\varphi_i \succ_{\mathfrak{v}} \mathfrak{v}^{-\sigma_L}$ for some $i$ and choose $i$ maximal with this property. Setting

$$K = (\partial - \varphi_{i+1}) \cdots (\partial - \varphi_r),$$

the transseries

$$h = K^{-1}(\mathrm{e}^{\int \varphi_i}) \in \mathbb{T}_{\underline{\lll}\mathfrak{v}}[\mathrm{i}]\mathrm{e}^{\int \varphi_i}$$

satisfies $Lh = 0$, as well as $\log h \asymp_{\mathfrak{v}} \varphi_i \succ_{\mathfrak{v}} \mathfrak{v}^{-\sigma_L}$. But such an $h$ cannot be a linear combination of the $h_i$ with $\log h_i \preccurlyeq_{\mathfrak{v}} \mathfrak{v}^{-\sigma_L}$. □

***Remark 2***. — It can be shown (although this will not be needed in what follows) that an operator $L \in \mathscr{T}[\mathrm{i}][\partial]$ splits over $\hat{\mathscr{T}}[\mathrm{i}]$ if and only if there exists an approximation $\tilde{L} \in \mathscr{T}[\mathrm{i}][\partial]$ with $\tilde{L} - L \preccurlyeq_{\mathfrak{v}} \mathfrak{v}^\lambda$ which splits over $\mathscr{T}[\mathrm{i}]$ for every $\lambda \in \mathbb{R}$. In particular, $\hat{\mathscr{T}}[\mathrm{i}]$ is $r$-linearly closed if and only if $\mathscr{T}[\mathrm{i}]$ is $r$-linearly closed over $\hat{\mathscr{T}}[\mathrm{i}]$.

**2.7. Normalization of quasi-linear equations.** — Assume now that $\mathscr{T}$ is a differential subfield of $\mathbb{T}$ of span $\mathfrak{v} \ggg x$. We say that $P$ is *normal* if $L_P$ is normal of order $r_P$ and $P_{\neq 1} \prec_{\mathfrak{v}} \mathfrak{v}^{r_P \sigma_{L_P}} L_P$. In that case, the equation

$$P(f) = 0, \qquad f \preccurlyeq_{\mathfrak{v}} 1 \tag{9}$$

is quasi-linear and it admits a unique solution in $\mathbb{T}_{\underline{\lll}\mathfrak{v}}$. Indeed, let $f \in \mathbb{T}_{\underline{\lll}\mathfrak{v}}$ be the distinguished solution to (9). By proposition 6, the operator $L_{P_{+f}}$ is normal. If $\tilde{f} \in \mathbb{T}_{\underline{\lll}\mathfrak{v}}$ were another solution to (9), then $\mathfrak{d}_{\tilde{f}-f}$ would be in $\mathfrak{H}_{L_{+f}}$, whence $\tilde{f} \succ 1$, which is impossible.

***Proposition 8***. — *Let $\mathscr{T}$ be a differential subfield of $\mathbb{T}$ of span $\mathfrak{v}$. Let $P \in \mathscr{T}[\mathrm{i}]\{F\}$ be a minimal annihilator of a differentially algebraic cut $f \in \hat{\mathscr{T}}[\mathrm{i}]$ over $\mathscr{T}[\mathrm{i}]$. Then there exists a truncation $\varphi \lhd f$ and $\lambda \in \mathbb{R}$ such that $P_{+\varphi, \times \mathfrak{v}^\lambda}$ is normal.*

*Proof.* — Let $\tilde{P} = P_{+f}$ and $\nu = r_{L_{\tilde{P}}} \sigma_{L_{\tilde{P}}}$. Modulo a multiplicative conjugation by $\mathfrak{v}^\alpha$ for some $\alpha \geqslant 0$, we may assume without loss of generality that $\tilde{P} \asymp L_{\tilde{P}}$. Modulo an additive conjugation by $f_{\succcurlyeq_\mathfrak{v} 1}$, we may also assume that $f \prec_\mathfrak{v} 1$. For any $\lambda, \mu \geqslant 0$ and $\varphi = f_{\succcurlyeq_\mathfrak{v} \mathfrak{v}^\mu} \lhd f$, we have

$$P_{+\varphi} = \tilde{P}_{+\varphi - f} = \tilde{P} + o_\mathfrak{v}(\mathfrak{v}^\mu \tilde{P}),$$

whence

$$P_{+\varphi, \times \mathfrak{v}^\lambda} = \tilde{P}_{1, \times \mathfrak{v}^\lambda} + O_\mathfrak{v}(\mathfrak{v}^{2\lambda} \tilde{P}) + o_\mathfrak{v}(\mathfrak{v}^\mu \tilde{P}). \tag{10}$$

Since $S_P(f) \neq 0$, we have $\tilde{P}_1 \neq 0$. By proposition 4, there exists a $\lambda > \nu$ for which $L_{\tilde{P}, \ltimes \mathfrak{v}^\lambda}$ is normal. Now take $\mu = \lambda + \nu$. Denoting $N = P_{+\varphi, \times \mathfrak{v}^\lambda}$, proposition 6 and (10) imply that $L_N$ is normal with $\sigma_{L_N} = \nu$ and $N_{\neq 1} \prec_\mathfrak{v} \mathfrak{v}^\nu \tilde{P}_{1, \times \mathfrak{v}^\lambda} \asymp \mathfrak{v}^\nu L_N$. □

We say that $P \in \mathscr{T}[\mathrm{i}]\{F\}$ is *split-normal*, if $P$ is normal and $L_P$ can be decomposed $L_P = L + K$ such that $L$ splits over $\mathscr{T}[\mathrm{i}]$ and $K \prec_\mathfrak{v} \mathfrak{v}^{r_L \sigma_L} L$. In that case, we may also decompose $P(F) = LF + R(F)$ for $R(F) = P_{\neq 1}(F) + KF$ with $R \prec_\mathfrak{v} \mathfrak{v}^{r_L \sigma_L} L$. If $L$ is monic, then we say that $P$ is *monic split-normal*. Any split-normal equation (9) is clearly equivalent to a monic split-normal equation of the same form.

***Proposition 9.*** — *Let $\mathscr{T}$ be a differential subfield of $\mathbb{T}$ of span $\mathfrak{v}$ such that $\hat{\mathscr{T}}[\mathrm{i}]$ is r-linearly closed. Let $P \in \mathscr{T}[\mathrm{i}]\{F\}$ be a minimal annihilator of a differentially algebraic cut $f \in \hat{\mathscr{T}}[\mathrm{i}]$ of order $r$ over $\mathscr{T}[\mathrm{i}]$. Let $S \in \mathscr{T}\{F\}$ be a minimal annihilator of* $\operatorname{Re} f$ *and assume that $r_S \geqslant r_P$. Then there exists a truncation $\varphi \lhd \operatorname{Re} f$ and $\lambda \in \mathbb{R}$ such that $S_{+\varphi, \times \mathfrak{v}^\lambda}$ is split-normal.*

*Proof.* — By proposition 8 and modulo a replacement of $f$ by $\mathfrak{v}^{-\lambda}(f - \varphi)$, we may assume without loss of generality that $S$ is normal. By lemma 2, $S$ splits over $\hat{\mathscr{T}}[\mathrm{i}]$ at $\operatorname{Re} f$. Let $c, \varphi_1, \ldots, \varphi_s \in \hat{\mathscr{T}}[\mathrm{i}]$ be such that

$$L_{S_{+f}} = c(\partial - \varphi_1) \cdots (\partial - \varphi_s).$$

Setting $\nu = s \sigma_{L_S}$, we notice that $L_S = L_{S_{+f}} + o_\mathfrak{v}(\mathfrak{v}^\nu L_S)$. Now take

$$L = c_{\succcurlyeq_\mathfrak{v} \mathfrak{v}^\nu \mathfrak{d}_c} (\partial - \varphi_{1, \succcurlyeq_\mathfrak{v} \mathfrak{v}^\nu}) \cdots (\partial - \varphi_{s, \succcurlyeq_\mathfrak{v} \mathfrak{v}^\nu}) \in \mathscr{T}[\mathrm{i}][\partial].$$

Then $L = L_S + o_\mathfrak{v}(\mathfrak{v}^\nu L_S)$ and proposition 6 implies that $L$ is normal, with $\sigma_L = \sigma_{L_S} = \sigma_{L_{S_{+f}}}$. Denoting $R(F) = S(F) - LF$, we finally have $R \prec_\mathfrak{v} \mathfrak{v}^{s \sigma_L} L$. □

## 3. Transserial Hardy fields

**3.1. Transserial Hardy fields.** — Let $\mathbb{T} = \mathbb{R}[[x]] = \mathbb{R}[[\mathfrak{T}]]$ be the field of grid-based transseries **[26]** and $\mathscr{G}$ the set of infinitely differentiable germs at infinity. A *transserial Hardy* field is a differential subfield $\mathscr{T}$ of $\mathbb{T}$, together with a monomorphism $\rho : \mathscr{T} \to \mathscr{G}$ of ordered differential $\mathbb{R}$-algebras, such that

**TH1:** For every $f \in \mathscr{T}$, we have $\operatorname{supp} f \subseteq \mathscr{T}$.

**TH2:** For every $f \in \mathscr{T}$, we have $f_{\prec} \in \mathscr{T}$.

**TH3:** There exists an $d \in \mathbb{Z}$, such that $\log \mathfrak{m} \in \mathscr{T} + \mathbb{R} \log_d x$ for all $\mathfrak{m} \in \mathfrak{T} \cap \mathscr{T}$.
**TH4:** The set $\mathfrak{T} \cap \mathscr{T}$ is stable under taking real powers.
**TH5:** We have $\rho(\log f) = \log \rho(f)$ for all $f \in \mathscr{T}^{>}$ with $\log f \in \mathscr{T}$.

In what follows, we will always identify $\mathscr{T}$ with its image under $\rho$, which is necessarily a Hardy field in the classical sense. The integer $d$ in **TH3** is called the *depth* of $\mathscr{T}$; if $\log \mathfrak{m} \in \mathscr{T}$ for all $\mathfrak{m} \in \mathfrak{T} \cap \mathscr{T}$, then the depth is defined to be $+\infty$. We always have $d \geqslant 0$, since $\mathscr{T}$ is stable under differentiation. If $d \neq \infty$, then $f \uparrow_d$ is exponential for all $f \in \mathscr{T}$ and $\mathscr{T}$ contains $\log_{d-1} x$. If $d = \infty$ and $\mathscr{T} \neq \mathbb{R}$, then $\mathscr{T}$ contains $\log_k x$ for all sufficiently large $k$.

***Example 2.*** — The field $\mathscr{T} = \mathbb{R}$ is clearly a transserial Hardy field. As will follow from theorem 2 below, other examples are

$$\begin{aligned} \mathbb{R}(x^{\mathbb{R}}) &= \bigcup_{\alpha_1, \ldots, \alpha_k \in \mathbb{R}} \mathbb{R}(x^{\alpha_1}, \ldots, x^{\alpha_k}) \\ \mathbb{R}(\mathrm{e}^{\mathbb{R}x}) &= \bigcup_{\alpha_1, \ldots, \alpha_k \in \mathbb{R}} \mathbb{R}(\mathrm{e}^{\alpha_1 x}, \ldots, \mathrm{e}^{\alpha_k x}). \end{aligned}$$

***Remark 3.*** — Although the axioms **TH4** and **TH5** are not really necessary, **TH4** allows for the simplification of several proofs, whereas it is natural to enforce **TH5**. Notice that **TH5** automatically holds for $f \in \mathscr{T}^{>}$ with $f \asymp 1$ since

$$\rho(\log f)' = \rho((\log f)') = \rho(f'/f) = \rho(f)'/\rho(f) = (\log \rho(f))',$$

whence $\rho(\log f) = \log \rho(f) + c$ for some $c \in \mathbb{R}$. Since both $\rho(\log f) - \log f_{\asymp}$ and $\log \rho(f) - \log f_{\asymp}$ are infinitesimal in $\mathscr{G}$, we have $c = 0$. Consequently, it suffices to check **TH5** for monomials $f \in \mathscr{T} \cap \mathfrak{T}$ with $\log f \in \mathscr{T}$.

***Proposition 10.*** — *Let $\mathscr{T}$ be a transserial Hardy field with $x \in \mathscr{T}$. Then the upward shift $\mathscr{T} \uparrow$ of $\mathscr{T}$ carries a natural transserial Hardy field structure with $\rho(f \uparrow) = \rho(f) \circ \mathrm{e}^x$.*

*Proof.* — The field $\mathscr{T} \uparrow$ is stable under differentiation, since $f \uparrow' = (xf') \uparrow$ for all $f \in \mathscr{T}$. □

***Corollary 3.*** — *If $\mathscr{T}$ has depth $d < \infty$, then $\mathscr{T} \uparrow_d$ is a transserial Hardy field of depth 0.*

We recall that a *transbasis* $\mathfrak{B}$ is a finite set of transmonomials $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\}$ with

**TB1:** $\mathfrak{b}_1, \ldots, \mathfrak{b}_n \succ 1$ and $\mathfrak{b}_1 \prec\!\!\prec \cdots \prec\!\!\prec \mathfrak{b}_n$.
**TB2:** $\mathfrak{b}_1 = \log_{d-1} x$ for some $d \in \mathbb{Z}$.
**TB3:** $\log \mathfrak{b}_i \in \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_{i-1}]]$ for all $1 < i \leqslant n$.

If $d = 0$, then $\mathfrak{B}$ is called a *plane* transbasis and $\mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_n]]$ is stable under differentiation. The incomplete transbasis theorem for $\mathbb{T}$ also holds for transserial Hardy fields:

***Proposition 11***. — *Let $\mathfrak{B} \subseteq \mathcal{T}$ be a transbasis and $f \in \mathcal{T}$. Then there exists an supertransbasis $\hat{\mathfrak{B}} \subseteq \mathcal{T}$ of $\mathfrak{B}$ with $f \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]]$. Moreover, if $\mathfrak{B}$ is plane and $f$ is exponential, then $\hat{\mathfrak{B}}$ may be taken to be plane.*

*Proof*. — The same proof as for [**26**, Theorem 4.15] may be used, since all field operations, logarithms and truncations used in the proof can be carried out in $\mathcal{T}$. □

Given a set $\mathcal{F}$ of exponential transseries in $\mathcal{T}$, the *transrank* of $\mathcal{F}$ is the minimal size of a plane transbasis $\mathfrak{B} = \{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\}$ with $\mathcal{F} \subseteq \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_n]]$. This notion may be extended to allow for differential polynomials $P$ in $\mathcal{T}$ (modulo the replacement of $P$ by its set of coefficients).

***Remark 4***. — The span and ultimate span of $f \in \mathcal{T}$ are not necessarily in $\mathcal{T}$. Nevertheless, if $\operatorname{span} f \neq 1$ and $\mathfrak{B} = \{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\} \subseteq \mathcal{T}$ is a transbasis for $f$, then we do have $\operatorname{span} f \asymp \mathfrak{b}_i$ for some $i$ (and similarly for the ultimate span of $f$).

**3.2. Cuts in transserial Hardy fields.** — Let $\mathcal{T}$ be a transserial Hardy field. Given $f \in \mathbb{T}$ and $\hat{f} \in \mathcal{G}$, we write $f \sim \hat{f}$ if there exists a $\varphi \in \mathcal{T}$ with

$$f \sim_{\mathbb{T}} \varphi \sim_{\mathcal{G}} \hat{f}.$$

We say that $f$ and $\hat{f}$ are *asymptotically equivalent* over $\mathcal{T}$ if for each $\varphi \in \mathcal{T}$ (or, equivalently, for each $\varphi \lhd f$), we have

$$f - \varphi \sim \hat{f} - \varphi.$$

We say that $f$ and $\hat{f}$ are *differentially equivalent* over $\mathcal{T}$ if

$$P(f) = 0 \Leftrightarrow P(\hat{f}) = 0$$

for all $P \in \mathcal{T}\{F\}$.

***Lemma 3***. — *Let $\mathcal{T}$ be a transserial Hardy field and let $f \in \mathbb{T} \setminus \mathcal{T}$ be differentially algebraic over $\mathcal{T}$. Let $\mathfrak{m} \in \operatorname{supp} f$ be maximal for $\succcurlyeq$, such that $\varphi = f_{\succ \mathfrak{m}} \notin \mathcal{T}$. Then $\varphi$ is differentially algebraic over $\mathcal{T}$ and $\chi_\varphi \leqslant \chi_f$.*

*Proof*. — Let $P \in \mathcal{T}\{F\}$ be a minimal annihilator of $f$. Modulo upward shifting, we may assume without loss of generality that $P$ and $f$ are exponential. Since $\varphi \in \hat{\mathcal{T}}$, all monomials in $\operatorname{supp} \varphi$ are in $\mathcal{T}$, whence there exists a plane transbasis $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\} \subseteq \mathcal{T}$ for $P$ and $\varphi$. Modulo subtraction of $H_\varphi$ from $f$ and $\varphi$, we may assume without loss of generality that $H_\varphi = 0$. Let $k$ be such that $\operatorname{uspan} \varphi \asymp \mathfrak{b}_k$ and let $\mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_n^{\alpha_n}$ be the dominant monomial of $\varphi$. Modulo division of $f$ and $\varphi$ by $\mathfrak{b}_{k+1}^{\alpha_{k+1}} \cdots \mathfrak{b}_n^{\alpha_n}$, we may also assume that $\varphi$ is a normal serial cut. But then the equation $P(f) = 0$ gives rise to the equation $P_{\preccurlyeq\!\!\!\prec \mathfrak{b}_k}(\varphi) = 0$ for $\varphi = f_{\preccurlyeq\!\!\!\prec \mathfrak{b}_k}$. The complexity of $P_{\preccurlyeq\!\!\!\prec \mathfrak{b}_k}$ is clearly bounded by $\chi_P = \chi_f$. □

***Lemma 4***. — *Let $\mathcal{T}$ be a transserial Hardy field and $\mathfrak{v} \in \mathcal{T} \cap \mathfrak{T}^{\prec}$. Let $f \in \widehat{\mathcal{T}_{\preccurlyeq\!\!\!\prec \mathfrak{v}}}$ and $\hat{f} \in \mathcal{G}$ be such that $f$ and $\hat{f}$ are both asymptotically and differentially equivalent over $\mathcal{T}_{\preccurlyeq\!\!\!\prec \mathfrak{v}}$. Then $f$ and $\hat{f}$ are both asymptotically and differentially equivalent over $\mathcal{T}$.*

*Proof.* — Given $\varphi \in \mathscr{T}$, we either have $\varphi \succ_{\mathfrak{v}}^{*} 1$ and

$$f - \varphi \ \sim_{\mathbb{T}} \ -\varphi \ \sim_{\mathscr{G}} \ \hat{f} - \varphi$$

or $\varphi \preccurlyeq_{\mathfrak{v}}^{*} 1$, in which case

$$f - \varphi \ \sim_{\mathbb{T}} \ f - \varphi_{\asymp_{\mathfrak{v}}^{*} 1} \ \sim \ \hat{f} - \varphi_{\asymp_{\mathfrak{v}}^{*} 1} \ \sim_{\mathscr{G}} \ \hat{f} - \varphi.$$

This proves that $f$ and $\hat{f}$ are asymptotically equivalent over $\mathscr{T}$.

As to their differential equivalence, let us first assume that $f$ is differentially transcendent over $\mathscr{T}_{\underline{\lll}\mathfrak{v}}$. Given $R \in \mathscr{T}\{F\}^{\neq}$, let us denote

$$D_R = \mathfrak{d}_R^{-1} Q_{\asymp_{\mathfrak{v}}^{*} \mathfrak{d}_R} \in \mathscr{T}_{\underline{\lll}\mathfrak{v}}.$$

We have $D_R(f) \neq 0$, $D_R(\hat{f}) \neq 0$ and

$$R(f) \ \sim_{\mathfrak{v}}^{*} \ D_R(f)\mathfrak{d}_R \tag{11}$$

$$R(\hat{f}) \ \sim_{\mathfrak{v}}^{*} \ D_R(\hat{f})\mathfrak{d}_R, \tag{12}$$

whence $R(f) \neq 0$ and $R(\hat{f}) \neq 0$.

Assume now that $f$ is differentially algebraic over $\mathscr{T}_{\underline{\lll}\mathfrak{v}}$ and let $P \in \mathscr{T}_{\underline{\lll}\mathfrak{v}}\{F\}$ be a minimal annihilator. Given $Q \in \mathscr{T}\{F\}$, Ritt reduction of $Q$ w.r.t. $P$ gives

$$H_P^k Q = \boldsymbol{A} P + R,$$

where $\boldsymbol{A} \in \mathscr{T}\{F\}[\partial]$ and $R \in \mathscr{T}\{F\}$ is such that $\chi_R < \chi_P$. Since $\chi_{H_P} < \chi_P$ and $H_P \in \mathscr{T}_{\underline{\lll}\mathfrak{v}}$, we both have $H_P(f) \neq 0$ and $H_P(\hat{f}) \neq 0$, whence

$$\begin{aligned} Q(f) &= \frac{R(f)}{H_P(f)^k} \\ Q(\hat{f}) &= \frac{R(\hat{f})}{H_P(\hat{f})^k}. \end{aligned}$$

If $R = 0$, this clearly implies $R(f) = R(\hat{f}) = 0$. Otherwise, $D_R$ vanishes neither at $f$ nor at $\hat{f}$ and the relations (11) and (12) again yield $R(f) \neq 0$ and $R(\hat{f}) \neq 0$. □

***Lemma 5.*** — *Let $\mathscr{T}$ be a transserial Hardy field and let $f \in \hat{\mathscr{T}} \setminus \mathscr{T}$ be a differentially algebraic cut over $\mathscr{T}$ with minimal annihilator $P$. Let $\hat{f} \in \mathscr{G}$ be a root of $P$ such that $f$ and $\hat{f}$ are asymptotically equivalent over $\mathscr{T}$. Then $f$ and $\hat{f}$ are differentially equivalent over $\mathscr{T}$.*

*Proof.* — Let $\mathfrak{v} \in \mathscr{T}$ be such that uspan $f \asymp\!\!\!\prec \mathfrak{v}$. Modulo some upward shiftings, we may assume without loss of generality that $f$ and $P$ are exponential. Modulo an additive conjugation by $H_f$ and a multiplicative conjugation by $\mathfrak{d}_f$, we may also assume that $f$ is a normal cut. Modulo a division of $P$ by $\mathfrak{d}_P$ and replacing $P$ by $P_{\underline{\lll}\mathfrak{v}}$, we may finally assume that $P \in \mathscr{T}_{\underline{\lll}\mathfrak{v}}\{F\}$.

Now consider $Q \in \mathscr{T}_{\underline{\lll}\mathfrak{v}}\{F\}^{\neq}$ with $\chi_Q < \chi_P$. Since $Q(f) \neq 0$, there exists a $\varphi \lhd f$ with $f - \varphi \prec_{\mathfrak{v}} 1$ and $Q_{+\varphi,\neq 0} \prec_{\mathfrak{v}} Q(\varphi)$. But then

$$Q(\hat{f}) = Q(\varphi) + Q_{+\varphi,\neq 0}(\hat{f} - \varphi) \sim Q(\varphi) \neq 0.$$

For general $Q \in \mathscr{T}\{F\}$, we use Ritt reduction of $Q$ w.r.t. $P$ and conclude in a similar way as in the proof of lemma 4. □

### 3.3. Elementary extensions

***Lemma 6.*** — *Let $f \in \mathbb{T} \setminus \mathscr{T}$ and $\hat{f} \in \mathscr{G} \setminus \mathscr{T}$ be such that*

i. *$f$ is a serial cut over $\mathscr{T}$.*
ii. *$f$ and $\hat{f}$ are asymptotically equivalent over $\mathscr{T}$.*
iii. *$f$ and $\hat{f}$ are differentially equivalent over $\mathscr{T}$.*

*Then $\mathscr{T}\langle f\rangle$ carries the structure of a transserial Hardy field for the unique differential morphism $\rho : \mathscr{T}\langle f\rangle \to \mathscr{G}$ over $\mathscr{T}$ with $\rho(f) = \hat{f}$.*

*Proof.* — Modulo upward shifting, an additive conjugation by $H_f$ and a multiplicative conjugation by $\mathfrak{d}_f$, we may assume without loss of generality that $f$ is an exponential normal serial cut. Let $\mathfrak{v} \in \mathscr{T}$ be such that $\operatorname{uspan} f \asymp \mathfrak{v}$. We have to show that $\mathscr{T}\langle f\rangle$ is closed under truncation and that $P(f) \sim P(\hat{f})$ for all $P \in \mathscr{T}\{F\}$ with $P(f) \neq 0$ (this implies in particular that $\rho$ is increasing). Notice that $\operatorname{supp} f \subseteq \mathscr{T}$ implies $\mathscr{T}\langle f\rangle \cap \mathfrak{T} = \mathscr{T} \cap \mathfrak{T}$.

**Truncation closedness.** Given $R \in \mathscr{T}\langle F\rangle$, let us prove by induction over the transrank $n$ of $\{R, f\}$ that $P(f)_{\succ} \in \mathscr{T}\langle f\rangle$. So let $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\}$ be a plane transbasis for $R$ and $f$. Assume first that $\mathfrak{b}_n \ggcurly \mathfrak{v}$. Writing

$$R = \sum_{\alpha \in \mathbb{R}} R_\alpha \mathfrak{b}_n^\alpha \in \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_{n-1}]]\langle F\rangle[[\mathfrak{b}_n]],$$

the sum

$$R_{\succ_{\mathfrak{b}_n}} = \sum_{\alpha > 0} R_\alpha \mathfrak{b}_n^\alpha$$

is finite, whence

$$R(f)_{\succ_{\mathfrak{b}_n}} = R_{\succ_{\mathfrak{b}_n}}(f) = \sum_{\alpha > 0} R_\alpha(f) \mathfrak{b}_n^\alpha \in \mathscr{T}\langle f\rangle.$$

By the induction hypothesis, we also have $R_0(f)_{\succ} \in \mathscr{T}\langle f\rangle$ and $R(f)_{\succ} \in \mathscr{T}\langle f\rangle$. If $\mathfrak{b}_n \asymp \mathfrak{v}$, then

$$R(f)_{\succ} = R(\varphi)_{\succ}$$

for a sufficiently large truncation $\varphi \lhd f$, whence $R(f)_{\succ} \in \mathscr{T}$.

**Preservation of dominant terms.** Given $P \in \mathscr{T}\{F\}$ with $P(f) \neq 0$, let us prove by induction over the transrank $n$ of $\{P, f\}$ that $P(f) \sim P(\hat{f})$. Let $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\}$ be a plane transbasis for $P$ and $f$ and assume first that $\mathfrak{v} \llcurly \mathfrak{b}_n$. Since $P(f) \neq 0$, there exists a maximal $\alpha$ with $P_\alpha(f) \neq 0$, when considering $P = \sum_{\alpha \in \mathbb{R}} P_\alpha \mathfrak{b}_n^\alpha$ as a series in $\mathfrak{b}_n$. But then

$$P(f) \sim P_\alpha(f) \mathfrak{b}_n^\alpha \sim P_\alpha(\hat{f}) \mathfrak{b}_n^\alpha \sim P(\hat{f}),$$

by the induction hypothesis. If $\mathfrak{b}_n \asymp \mathfrak{v}$, then there exists an $\alpha \in \mathbb{R}$ such that, for all sufficiently large truncations $\varphi \lhd f$, the Taylor series expansion of $P(\varphi + (f - \varphi))$ yields

$$\begin{aligned} P(f) &= P(\varphi) + O_{\mathfrak{v}}((f-\varphi)\mathfrak{v}^{\alpha}) \\ P(\hat{f}) &= P(\varphi) + O_{\mathfrak{v}}((\hat{f}-\varphi)\mathfrak{v}^{\alpha}). \end{aligned}$$

Taking $\varphi \lhd f$ such that $(f-\varphi)\mathfrak{v}^{\alpha} \prec_{\mathfrak{v}} P(f)$, we obtain

$$P(f) \sim P(\varphi) \sim P(\hat{f}).$$

This completes the proof. □

***Theorem 1***. — *Let $\mathscr{T}$ be a transserial Hardy field. Then its real closure $\mathscr{T}^{\mathrm{rcl}}$ admits a unique transserial Hardy field structure which extends the one of $\mathscr{T}$.*

*Proof.* — Assume that $\mathscr{T}^{\mathrm{rcl}} \neq \mathscr{T}$ and choose $f \in \mathscr{T}^{\mathrm{rcl}} \setminus \mathscr{T}$ of minimal complexity. By lemma 3, we may assume without loss of generality that $f$ is a serial cut. Consider the monic minimal polynomial $P \in \mathscr{T}[F]$ of $f$. Since $P'(f) \neq 0$, we have

$$\deg_{\preccurlyeq f-\varphi} P_{+\varphi} = 1$$

for a sufficiently large truncation $\varphi \lhd f$ of $f$ (we refer to [**26**, Section 8.3] for a definition of the Newton degrees $\deg_{\preccurlyeq\psi} P$). But then

$$P_{+\varphi}(g) = 0, \quad g \preccurlyeq f - \varphi \tag{13}$$

admits unique solutions $g$ and $\hat{g}$ in $\mathbb{T}$ resp. $\mathscr{G}$, by the implicit function theorem. It follows in particular that $f = \varphi + g$. Let $\hat{f} = \varphi + \hat{g}$ and consider $\psi$ with $\varphi \trianglelefteq \psi \lhd f$. Then

$$\begin{aligned} P(f) - P(\psi) &\sim P_{+\psi,1}(f-\psi) \\ P(\hat{f}) - P(\psi) &\sim P_{+\psi,1}(\hat{f}-\psi) \end{aligned}$$

Since $P(f) = P(\hat{f}) = 0$, we obtain $f - \psi \sim \hat{f} - \psi$, whence $f$ and $\hat{f}$ are asymptotically equivalent over $\mathscr{T}$. By lemmas 5 and 6, it follows that $\mathscr{T}\langle f\rangle$ carries a transserial Hardy field structure which extends the one on $\mathscr{T}$. Since (13) has a unique solution $\hat{g}$ in $\mathscr{G}$, this structure is unique. We conclude by Zorn's lemma. □

### 3.4. Exponential and logarithmic extensions

***Theorem 2***. — *Let $\mathscr{T}$ be a transserial Hardy field and let $\varphi \in \mathscr{T}_{\succ}$ be such that $\mathrm{e}^{\varphi} \notin \mathscr{T}$. Then the set $\mathscr{T}(\mathrm{e}^{\mathbb{R}\varphi})$ carries the structure of a transserial Hardy field for the unique differential morphism $\rho : \mathscr{T}(\mathrm{e}^{\mathbb{R}\varphi}) \to \mathscr{G}$ over $\mathscr{T}$ with $\rho(\mathrm{e}^{\lambda\varphi}) = \mathrm{e}^{\lambda\rho(\varphi)}$ for all $\lambda \in \mathbb{R}$.*

*Proof.* — Each element in $f = \mathscr{T}(\mathrm{e}^{\mathbb{R}\varphi})$ is of the form $f = R(\mathrm{e}^{\lambda_1\varphi}, \ldots, \mathrm{e}^{\lambda_k\varphi})$ for $R \in \mathscr{T}(F_1, \ldots, F_k)$ and $\mathbb{Q}$-linearly independent $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. Given $R \in \mathscr{T}(F_1, \ldots, F_k)$, let $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\}$ be a transbasis for $R$. We may write

$$\mathrm{e}^{\varphi} = \mathrm{e}^{\tilde{\varphi}} \mathfrak{b}_i^{\alpha_i} \cdots \mathfrak{b}_n^{\alpha_n}$$

with $\mathfrak{b}_{i-1} \prec\!\!\prec e^{\tilde{\varphi}} \prec\!\!\prec \mathfrak{b}_i$ (or the obvious adaptations if $i=1$ or $i=n+1$). Modulo the substitution of $\varphi$ by $\alpha_i \log \mathfrak{b}_i + \cdots + \alpha_n \log \mathfrak{b}_n + \tilde{\varphi}$, we may assume without loss of generality that $\alpha_i = \cdots = \alpha_n = 0$.

If $\mathfrak{b}_n \prec\!\!\prec e^{\varphi}$, then we may regard $f = \sum_{\mu \in \mathbb{R}} f_\mu e^{\mu\varphi}$ as a convergent grid-based series in $e^{\varphi}$ with coefficients in $\mathcal{T} \cap \mathbb{R}[\![\mathfrak{b}_1; \ldots; \mathfrak{b}_n]\!]$. In particular,

$$f_{\succ} = \left[ \sum_{\mu \operatorname{sign} \varphi > 0} f_\mu e^{\mu\varphi} \right] + f_{0,\succ} \in \mathcal{T}(e^{\mathbb{R}\varphi}).$$

Furthermore, if $f$ admits $\nu$ as its dominant exponent in $e^{\varphi}$, then $f \sim f_\nu e^{\nu\varphi}$ holds both in $\mathbb{T}$ and in $\mathcal{G}$.

If $e^{\varphi} \prec\!\!\prec \mathfrak{b}_n$, then we may consider $R$ as a series

$$R \in \mathcal{S} := (\mathcal{T} \cap \mathbb{R}[\![\mathfrak{b}_1; \ldots; \mathfrak{b}_{i-1}]\!])(F_1, \ldots, F_k)[\![\mathfrak{b}_i; \ldots; \mathfrak{b}_n]\!]$$

in $\mathfrak{b}_i, \ldots, \mathfrak{b}_n$. Since $\mathcal{T}$ is closed under truncation, both $R_{\succ \mathfrak{b}_i}$ and $R_{\asymp \mathfrak{b}_i}$ lie in $\mathcal{S}$, whence

$$f_{\succ} = R_{\succ \mathfrak{b}_i}(e^{\lambda_1\varphi}, \ldots, e^{\lambda_k\varphi}) + R_{\asymp \mathfrak{b}_i}(e^{\lambda_1\varphi}, \ldots, e^{\lambda_k\varphi})_{\succ} \in \mathcal{T}(e^{\mathbb{R}\varphi}),$$

by what precedes. Similarly, if $R_{\nu_i,\ldots,\nu_n} \mathfrak{b}_i^{\nu_i} \cdots \mathfrak{b}_n^{\nu_n}$ is the dominant term of $R$ as a series in $\mathfrak{b}_i, \ldots, \mathfrak{b}_n$ and $c e^{\nu\varphi}$ is the dominant term of $R_{\nu_i,\ldots,\nu_n}(e^{\lambda_1\varphi}, \ldots, e^{\lambda_k\varphi})$ as a series in $e^{\varphi}$ (with $c \in \mathcal{T} \cap \mathbb{R}[\![\mathfrak{b}_1; \ldots; \mathfrak{b}_{i-1}]\!]$), then $f \sim c e^{\nu\varphi} \mathfrak{b}_i^{\nu_i} \cdots \mathfrak{b}_n^{\nu_n}$ holds both in $\mathbb{T}$ and in $\mathcal{G}$.

This shows that $\mathcal{T}(e^{\mathbb{R}\varphi})$ is truncation closed and that the extension of $\rho$ to $\mathcal{T}(e^{\mathbb{R}\varphi})$ is increasing. We also have $\mathcal{T}(e^{\mathbb{R}\varphi}) \cap \mathfrak{T} = (\mathcal{T} \cap \mathfrak{T}) e^{\mathbb{R}\varphi}$. In other words, $\mathcal{T}(e^{\mathbb{R}\varphi})$ is a transserial Hardy field. □

***Theorem 3***. — *Let $\mathcal{T}$ be a transserial Hardy field of finite depth $d < \infty$. Then $\mathcal{T}((\log_d x)^{\mathbb{R}})$ carries the structure of a transserial Hardy field for the unique differential morphism $\rho : \mathcal{T}((\log_d x)^{\mathbb{R}}) \to \mathcal{G}$ over $\mathcal{T}$ with $\rho((\log_d x)^{\lambda}) = (\log_d x)^{\lambda}$ for all $\lambda \in \mathbb{R}$.*

*Proof.* — The proof is similar to the proof of theorem 2, when replacing $e^{\varphi}$ by $\log_l x$. □

**3.5. Complex transserial Hardy fields.** — Let $\mathcal{T}$ be a transserial Hardy field. Asymptotic and differential equivalence over $\mathcal{T}[\mathrm{i}]$ are defined in a similar way as over $\mathcal{T}$.

***Proposition 12***. — *Let $\mathcal{T}$ be a transserial Hardy field. Let $f \in \mathbb{T}[\mathrm{i}]$ be a serial cut over $\mathcal{T}[\mathrm{i}]$ and $\hat{f} \in \mathcal{G}[\mathrm{i}]$. Then $f$ and $\hat{f}$ are asymptotically equivalent over $\mathcal{T}[\mathrm{i}]$ if and only if $\operatorname{Re} f$ and $\operatorname{Re} \hat{f}$ as well as $\operatorname{Im} f$ and $\operatorname{Im} \hat{f}$ are asymptotically equivalent over $\mathcal{T}$.*

*Proof.* — Assume that $f$ and $\hat{f}$ are asymptotically equivalent over $\mathcal{T}[\mathrm{i}]$ and let $\varphi \vartriangleleft \operatorname{Re} f$. Consider $\psi = (\operatorname{Im} f)_{\succ \operatorname{Re} f - \varphi} \trianglelefteqslant \operatorname{Im} f$. We have $\varphi + \psi\mathrm{i} \vartriangleleft f$, so that $f - \varphi - \psi\mathrm{i} \sim \hat{f} - \varphi - \psi\mathrm{i}$. Moreover, $f - \varphi - \psi\mathrm{i} \asymp \operatorname{Re} f - \varphi$, whence $\operatorname{Re} f - \varphi \sim \operatorname{Re} \hat{f} - \varphi$ and $\operatorname{Re} f \sim \operatorname{Re} \hat{f}$. The relation $\operatorname{Im} f \sim \operatorname{Im} \hat{f}$ is proved similarly. Inversely, assume that $\operatorname{Re} f$ and $\operatorname{Re} \hat{f}$ as well as $\operatorname{Im} f$ and $\operatorname{Im} \hat{f}$ are asymptotically equivalent over $\mathcal{T}$. Given $\varphi \vartriangleleft f$, we have $\operatorname{Re} \varphi, \operatorname{Im} \varphi \in \mathcal{T}$, whence there exist $g, h \in \mathcal{T}$ with $\operatorname{Re} f - \operatorname{Re} \varphi \sim g \sim \operatorname{Re} \hat{f} - \operatorname{Re} \varphi$

and $\operatorname{Im} f - \operatorname{Im} \varphi \sim h \sim \operatorname{Im} \hat{f} - \operatorname{Im} \varphi$. It follows that $f - \varphi \sim g + h\mathrm{i} \sim \hat{f} - \varphi$, whence $f \sim \hat{f}$. □

***Proposition 13***. — *Let $\mathscr{T}$ be a transserial Hardy field, $f \in \mathbb{T}$ and $\hat{f} \in \mathscr{G}$. Then $f$ and $\hat{f}$ are differentially equivalent over $\mathscr{T}[\mathrm{i}]$ if and only if they are differentially equivalent over $\mathscr{T}$.*

*Proof*. — Differential equivalence over $\mathscr{T}[\mathrm{i}]$ clearly implies differential equivalence over $\mathscr{T}$. Assuming that $f$ and $\hat{f}$ are differentially equivalent over $\mathscr{T}$, we also have

$$\begin{aligned} P(f) = 0 \quad &\Leftrightarrow \quad (\operatorname{Re} P)(f) = 0 \wedge (\operatorname{Im} P)(f) = 0 \\ &\Leftrightarrow \quad (\operatorname{Re} P)(\hat{f}) = 0 \wedge (\operatorname{Im} P)(\hat{f}) = 0 \\ &\Leftrightarrow \quad P(\hat{f}) = 0 \end{aligned}$$

for every $P \in \mathscr{T}[\mathrm{i}]\{F\}$. □

***Remark 5***. — Given $f \in \mathbb{T}$ and $\hat{f} \in \mathscr{G}$, it can happen that $f$ and $\hat{f}$ are differentially equivalent over $\mathscr{T}[\mathrm{i}]$, without $\operatorname{Re} f$ and $\operatorname{Re} \hat{f}$ being differentially equivalent over $\mathscr{T}$. This is for instance the case for $\mathscr{T} = \mathbb{R}(x^{\mathbb{R}})$, $f = \mathrm{e}^x$ and $\hat{f} = \mathrm{i}\mathrm{e}^x$. Indeed, the differential ideals which annihilate $f$ resp. $\hat{f}$ are both $F' - F$.

Most results from the previous sections generalize to the complex setting in a straightforward way. In particular, lemmas 3, 4 and 5 also hold over $\mathscr{T}[\mathrm{i}]$. However, the fundamental extension lemma 6 admits no direct analogue: when taking $f \in \mathbb{T}[\mathrm{i}] \setminus \mathscr{T}[\mathrm{i}]$ and $\hat{f} \in \mathscr{G}[\mathrm{i}] \setminus \mathscr{T}[\mathrm{i}]$ such that the complexified conditions *i*, *ii* and *iii* hold, we cannot necessarily give $\mathscr{T}\langle \operatorname{Re} f \rangle$ the structure of a transserial Hardy field. This explains why some results such as lemmas 2 and 9 have to be proved over $\mathscr{T}$ instead of $\mathscr{T}[\mathrm{i}]$. Of course, theorem 1 does imply the following:

***Theorem 4***. — *Let $\mathscr{T}$ be a transserial Hardy field. Then there exists a unique algebraic transserial Hardy field extension $\mathscr{T}^{\mathrm{rcl}}$ of $\mathscr{T}$ such that $\mathscr{T}^{\mathrm{rcl}}[\mathrm{i}]$ is algebraically closed.*

## 4. Analytic resolution of differential equations

Recall that $\mathscr{G}$ stands for the differential algebra of infinitely differentiable germs of real functions at $+\infty$. Given $x_0 \in \mathbb{R}$, we will denote by $\mathscr{G}_{x_0}$ the differential subalgebra of infinitely differentiable functions on $[x_0, \infty)$. We define a norm on $\mathscr{G}_{x_0}^{\preccurlyeq} = \{f \in \mathscr{G}_{x_0} : f \preccurlyeq 1\}$ by

$$\|f\|_{x_0} = \sup_{x \geqslant x_0} |f(x)|$$

Given $r \in \mathbb{N}$, we also denote $\mathscr{G}_{x_0;r}^{\preccurlyeq} = \{f \in \mathscr{G}_{x_0} : f, \ldots, f^{(r)} \preccurlyeq 1\}$ and define a norm on $\mathscr{G}_{x_0;r}^{\preccurlyeq}$ by

$$\|f\|_{x_0;r} = \max\{\|f\|_{x_0}, \ldots, \|f^{(r)}\|_{x_0}\}.$$

Notice that

$$\|fg\|_{x_0;r} \leqslant 2^r \|f\|_{x_0;r} \|g\|_{x_0;r}.$$

An operator $K : \mathscr{G}_{x_0} \to \mathscr{G}_{x_0}$ (resp. $K : \mathscr{G}_{x_0} \to \mathscr{G}_{x_0;r}$) is said to be *continuous* if there exists an $M \in \mathbb{R}$ with $\|Kf\|_{x_0} \leqslant M\|f\|_{x_0}$ (resp. $\|Kf\|_{x_0;r} \leqslant M\|f\|_{x_0}$) for all $f \in \mathscr{G}_{x_0}$. The smallest such $M$ is called the *norm* of $K$ and denoted by $|||K|||_{x_0}$ (resp. $|||K|||_{x_0;r}$). The above definitions generalize in an obvious way to the complexifications $\mathscr{G}_{x_0}^{\preccurlyeq}[\mathrm{i}]$ and $\mathscr{G}_{x_0;r}^{\preccurlyeq}[\mathrm{i}]$.

**4.1. Continuous right-inverses of first order operators.** — Let $\mathscr{T}$ be a transserial Hardy field of span $\mathfrak{v} \succeq\!\!\succ \mathrm{e}^x$. Consider a normal operator $\partial - \varphi$ with $\varphi \in \mathscr{T}[\mathrm{i}]$ and let $x_0$ be sufficiently large such that $\operatorname{Re}\varphi$ does not change sign on $[x_0, \infty)$. We define a primitive $\Phi \in \mathscr{G}$ of $\varphi$ by

$$\Phi(x) = \begin{cases} \int_\infty^x \varphi(t)\mathrm{d}t & \text{if } \varphi \text{ is integrable at } \infty \\ \int_{x_0}^x \varphi(t)\mathrm{d}t & \text{otherwise} \end{cases}$$

Decomposing $\Phi = \Re + \Im\mathrm{i}$, we are either in one of the following two cases:

1. The repulsive case when $\mathrm{e}^{\Re} \succ_{\mathfrak{v}} 1$.
2. The attractive case when both $\mathrm{e}^{\Re} \prec_{\mathfrak{v}} 1$ and $\mathrm{e}^{\Re} \succ\!\!\succ \mathfrak{v}$.

Notice that the hypothesis $\mathfrak{v} \succeq\!\!\succ \mathrm{e}^x$ implies $\Re' = \operatorname{Re}\varphi \succcurlyeq \mathfrak{v}^\dagger \succcurlyeq 1$.

***Proposition 14.*** — *The operator $J = (\partial - \varphi)_{x_0}^{-1}$, defined by*

$$(Jf)(x) \;=\; \begin{cases} \mathrm{e}^{\Phi(x)} \int_\infty^x \mathrm{e}^{-\Phi(t)} f(t)\mathrm{d}t & \textit{(repulsive case)} \\ \mathrm{e}^{\Phi(x)} \int_{x_0}^x \mathrm{e}^{-\Phi(t)} f(t)\mathrm{d}t & \textit{(attractive case)} \end{cases} \tag{14}$$

*is a continuous right-inverse of $L = \partial - \varphi$ on $\mathscr{G}^{\preccurlyeq}[\mathrm{i}]$, with*

$$|||\, J|||_{x_0} \;\leqslant\; \left\| \frac{1}{\operatorname{Re}\varphi} \right\|_{x_0}. \tag{15}$$

*Proof.* — In the repulsive case, the change of variables $\Re(t) = u$ yields

$$(Jf)(x) \;=\; \mathrm{e}^{\Phi(x)} \int_\infty^{\Re(x)} \mathrm{e}^{-u - \Im(\Re^{\mathrm{inv}}(u))\mathrm{i}} \frac{f(\Re^{\mathrm{inv}}(u))}{\Re'(\Re^{\mathrm{inv}}(u))} \mathrm{d}u.$$

It follows that

$$|(Jf)(x)| \;\leqslant\; \mathrm{e}^{\Re(x)} \int_\infty^{\Re(x)} \mathrm{e}^{-u} \|f\|_x \left\| \frac{1}{\Re'} \right\|_x \mathrm{d}u \;=\; \|f\|_x \left\| \frac{1}{\Re'} \right\|_x$$

for all $x \geqslant x_0$, whence (15). In the attractive case, the change of variables $-\Re(t) = u$ leads in a similar way to the bound

$$\begin{aligned} |(Jf)(x)| \;&\leqslant\; \mathrm{e}^{\Re(x)} \int_{-\Re(x_0)}^{-\Re(x)} \mathrm{e}^{u} \|f\|_{x_0} \left\| \frac{1}{\Re'} \right\|_{x_0} \mathrm{d}u \\ &=\; \left[1 - \mathrm{e}^{\Re(x) - \Re(x_0)}\right] \|f\|_{x_0} \left\| \frac{1}{\Re'} \right\|_{x_0} \\ &\leqslant\; \|f\|_{x_0} \left\| \frac{1}{\Re'} \right\|_{x_0}, \end{aligned}$$

for all $x \geqslant x_0$, using the monotonicity of $\mathfrak{R}$. Again, we have (15). □

***Corollary 4***. — *In the attractive case, the operator*

$$J_\lambda : f \longmapsto (Jf)(x) + \lambda \mathrm{e}^{\Phi(x)} \|f\|_{x_0}$$

*is a continuous right-inverse of $L$ on $\mathscr{G}^{\preccurlyeq}[\mathrm{i}]$, for any $\lambda \in \mathbb{C}$.*

**4.2. Continuous right-inverses of higher order operators.** — Let $\mathscr{T}$ be a transserial Hardy field of span $\mathfrak{v} \not\succ \mathrm{e}^x$. A monic operator $L \in \mathscr{T}[\mathrm{i}][\partial]$ is said to be *split-normal*, if it is normal and if it admits a splitting

$$L = (\partial - \varphi_1) \cdots (\partial - \varphi_r) \tag{16}$$

with $\varphi_1, \ldots, \varphi_r \in \mathscr{T}[\mathrm{i}]$. In that case, proposition 5 implies that each $\partial - \varphi_i$ is a normal first order operator. For a sufficiently large $x_0$, it follows that $L$ admits a continuous "factorwise" right-inverse $J_r \cdots J_1$ on $\mathscr{G}[\mathrm{i}]^{\preccurlyeq}$, where $J_i = (\partial - \varphi_i)^{-1}_{x_0}$. We have

$$|||J_r \cdots J_1|||_{x_0} \;\leqslant\; |||J_r\,|||_{x_0} \cdots |||\,J_1\,|||_{x_0}\,.$$

***Proposition 15***. — $\mathfrak{v}^\nu J_r \cdots J_1 : \mathscr{G}^{\preccurlyeq}_{x_0}[\mathrm{i}] \to \mathscr{G}^{\preccurlyeq}_{x_0;r}[\mathrm{i}]$ *is a continuous operator for every $\nu > r\sigma_L$.*

*Proof*. — Given $f \in \mathscr{G}^{\preccurlyeq}[\mathrm{i}]$, the the first $r$ derivatives of $(\mathfrak{v}^\nu J_r \cdots J_1) f$ satisfy

$$[(\mathfrak{v}^\nu J_r \cdots J_1) f]^{(i)} \;=\; \sum_{j=r-i}^{r} c_{i,j} (\mathfrak{v}^\nu J_j \cdots J_1) f,$$

with

$$\begin{aligned} c_{0,r} &= 1 \\ c_{i+1,j} &= c'_{i,j} + \nu \mathfrak{v}^\dagger c_{i,j} + \varphi_j c_{i,j} + \frac{1}{\psi_{j+1}} c_{i,j+1}. \end{aligned}$$

By proposition 7 and induction over $i$, we have $c_{i,j} \preccurlyeq_{\mathfrak{v}} \mathfrak{v}^{-i\sigma_L}$ for all $i, j$. Since $\nu > r\sigma_L$, it follows that

$$\|[(\mathfrak{v}^\nu J_r \cdots J_1) f]^{(i)}\|_{x_0} \;\leqslant\; C_i \|f\|_{x_0}, \tag{17}$$

for all $f \in \mathscr{G}^{\preccurlyeq}[\mathrm{i}]$ and $i$, where

$$C_i \;=\; \sum_{j=r-1}^{r} \|\mathfrak{v}^\nu c_{i,j}\|_{x_0}\, |||\,J_j\,|||_{x_0} \cdots |||\,J_1\,|||_{x_0}\,.$$

We conclude that

$$|||\mathfrak{v}^\nu J_r \cdots J_1|||_{x_0;r} \leqslant \max\{C_0, \ldots, C_r\}.$$ □

***Proposition 16***. — *If $L \in \mathscr{T}[\partial]$ and the splitting* (16) *preserves realness, then $J_r \cdots J_1$ preserves realness in the sense that it maps $\mathscr{G}^{\preccurlyeq}_{x_0}$ into itself.*

*Proof.* — It clearly suffices to prove the proposition for an atomic real operator $L$. If $L$ has order 1, then the result is clear. Otherwise, we have

$$L = (\partial - (a - b\mathrm{i} + b^\dagger))(\partial - (a + b\mathrm{i}))$$

for certain $a, b \in \mathscr{T}$. In particular, we are in the same case (attractive or repulsive) for both factors of $L$. Setting $\varphi = a + b\mathrm{i}$, let $\Phi = \Re + \Im\mathrm{i}$ be as in the previous section. Consider $f \in \mathscr{G}_{x_0}^{\preccurlyeq}$ and $g = J_2 J_1 f$. In the repulsive case, we have

$$g(x) = b(x)\mathrm{e}^{\bar{\Phi}(x)} \int_{x_0}^{x} \frac{\mathrm{e}^{2\mathrm{i}\Im(t)}}{b(t)} \int_{x_0}^{t} \mathrm{e}^{-\Phi(u)} f(u)\mathrm{d}u\mathrm{d}t.$$

In particular, we have $g(x_0) = g'(x_0) = 0$, whence $g \in \mathscr{G}_{x_0}^{\preccurlyeq}$, since $g$ satisfies the differential equation $Lg = f$ of order 2 with real coefficients. In the attractive case, we have

$$g(x) = b(x)\mathrm{e}^{\bar{\Phi}(x)} \int_{\infty}^{x} \frac{\mathrm{e}^{2\mathrm{i}\Im(t)}}{b(t)} \int_{\infty}^{t} \mathrm{e}^{-\Phi(u)} f(u)\mathrm{d}u\mathrm{d}t,$$

so that $g, g' \preccurlyeq_{\mathfrak{v}} 1$. Since $Lg = L\bar{g} = f$, the difference $\bar{g} - g$ satisfies $L(\bar{g} - g) = 0$. Now 0 is the only solution with $h \preccurlyeq_{\mathfrak{v}} 1$ to the equation $Lh = 0$. This proves that $\bar{g} = g$. □

**4.3. The fixed point theorem.** — Let $\mathscr{T}$ be a transserial Hardy field of span $\mathfrak{v} \succcurlyeq\!\!\!\succ \mathrm{e}^x$ and consider a monic split-normal quasi-linear equation

$$Lf = P(f), \quad f \prec 1, \tag{18}$$

where $L \in \mathscr{T}[\mathrm{i}][\partial]$ has order $r$ and $P \in \mathscr{T}[\mathrm{i}]\{F\}$ has degree $d$. Of course, we understand that $L$ is a monic split-normal operator with $P \prec_{\mathfrak{v}} \mathfrak{v}^{r\sigma_L}$. We will denote by $v_P > r\sigma_L$ the valuation of $P$ in $\mathfrak{v}$ (i.e. $P \asymp_{\mathfrak{v}} \mathfrak{v}^{v_P}$ for $P \neq 0$ and $v_0 = \infty$). We will show how to construct a solution to (18) using the fixed-point technique.

***Proposition 17.*** — *Given $\nu$ with $r\sigma_L < \nu < v_P$, let $J_{r,\ltimes\mathfrak{v}^\nu} \cdots J_{1,\ltimes\mathfrak{v}^\nu}$ be a continuous factorwise right-inverse of $L_{\ltimes\mathfrak{v}^\nu}$ beyond $x_0$ and consider the operator*

$$\Xi : f \longmapsto (J_r \cdots J_1)(P(f)) \tag{19}$$

*on $\mathscr{G}_{x_0;r}^{\preccurlyeq}$. Then there exists a constant $C_{x_0}$ with*

$$\|\Xi(f + \delta) - \Xi(f)\|_{x_0;r} \leqslant C_{x_0}(1 + \cdots + \|f\|_{x_0;r}^d)(\|\delta\|_{x_0;r} + \cdots + \|\delta\|_{x_0;r}^d), \tag{20}$$

*for all $f, \delta \in \mathscr{G}_{x_0;r}^{\preccurlyeq}$.*

*Proof.* — Consider the Taylor series expansion

$$\begin{aligned} P(f+\delta) &= \sum_{\boldsymbol{i}} P^{(\boldsymbol{i})}(f)\delta^{(\boldsymbol{i})} \\ &= \sum_{\boldsymbol{i}} \left[\sum_{\boldsymbol{j}} P_{\boldsymbol{j}}^{(\boldsymbol{i})} f^{(\boldsymbol{j})}\right] \delta^{(\boldsymbol{i})} \end{aligned}$$

Since $P_{\boldsymbol{j}}^{(\boldsymbol{i})} \prec_{\mathfrak{v}} \mathfrak{v}^\nu$ for all $\boldsymbol{i}$ and $\boldsymbol{j}$, we may define $A_{x_0}$ by

$$A_{x_0} = \sum_{\boldsymbol{i},\boldsymbol{j}} \left\| \mathfrak{v}^{-\nu} P_{\boldsymbol{j}}^{(\boldsymbol{i})} \right\|_{x_0} \tag{21}$$

and obtain

$$\left\| \mathfrak{v}^{-\nu}(P(f+\delta) - P(f)) \right\|_{x_0} \leqslant A_{x_0}(1 + \cdots + \|f\|_{x_0;r}^d)(\|\delta\|_{x_0;r} + \cdots + \|\delta\|_{x_0;r}^d).$$

On the other hand, for each $g \in \mathscr{G}_{x_0}$ with $g \preccurlyeq \mathfrak{v}^\nu$, we have

$$\|(J_r \cdots J_1)(g)\|_{x_0;r} \;=\; \|(\mathfrak{v}^\nu J_{r,\ltimes \mathfrak{v}^\nu} \cdots J_{1,\ltimes \mathfrak{v}^\nu})(\mathfrak{v}^{-\nu} g)\|_{x_0;r} \;\leqslant\; B_{x_0} \|\mathfrak{v}^{-\nu} g\|_{x_0},$$

where

$$B_{x_0} = |||\mathfrak{v}^\nu J_{r,\ltimes \mathfrak{v}^\nu} \cdots J_{1,\ltimes \mathfrak{v}^\nu}|||_{x_0;r} \tag{22}$$

Consequently, the proposition holds for $C_{x_0} = A_{x_0} B_{x_0}$. □

***Theorem 5***. — *Let* (18) *be a monic split-normal equation and let $\nu$ be such that $r\sigma_L < \nu < v_P$. Then for any sufficiently large $x_0$, there exists a continuous factorwise right-inverse $J_{r,\ltimes \mathfrak{v}^\nu} \cdots J_{1,\ltimes \mathfrak{v}^\nu}$ of $L_{\ltimes \mathfrak{v}^\nu}$, such that the operator* (19) *satisfies*

$$\|\Xi(f+\delta) - \Xi(f)\|_{x_0;r} \leqslant \frac{1}{2}\|\delta\|_{x_0;r} \tag{23}$$

*for all*

$$f, \delta \in \mathscr{B}\left(\mathscr{G}_{x_0;r}^{\preccurlyeq}, \frac{1}{2}\right) = \left\{ f \in \mathscr{G}_{x_0;r}^{\preccurlyeq} : \|f\|_{x_0;r} \leqslant \frac{1}{2} \right\}.$$

*Moreover, taking $x_0$ such that $\|P_0\|_{x_0;r} \leqslant \frac{1}{4}$, the sequence $\Xi^{(n)}(0)$ tends to a unique fixed point $f \in \mathscr{B}(\mathscr{G}_{x_0;r}^{\preccurlyeq}, \frac{1}{2})$ for the operator $\Xi$.*

*Proof*. — Since $\mathfrak{v}^{-\nu} P_{\boldsymbol{j}}^{(\boldsymbol{i})} \prec 1$ for all $i, j$, the number $A_{x_0}$ from (21) tends to 0 for $x_0 \to \infty$. When constructing $J_{1,\ltimes \mathfrak{v}^\nu}, \ldots, J_{r,\ltimes \mathfrak{v}^\nu}$ using proposition 14, the number $B_{x_0}$ from (22) decreases as a function of $x_0$. Taking $x_0$ sufficiently large so that $C_{x_0} = A_{x_0} B_{x_0} \leqslant \frac{1}{4}$, we obtain (23). By induction over $n$, it follows that

$$\begin{aligned} \|\Xi^n(0) - \Xi^{n-1}(0)\|_{x_0;r} &\leqslant \frac{1}{2^{n+1}} \\ \|\Xi^n(0)\|_{x_0;r} &\leqslant \frac{1}{2} - \frac{1}{2^{n+1}}. \end{aligned}$$

Now let $\hat{\mathscr{G}}_{x_0;r}^{\preccurlyeq}$ be the space of $r$ times continuously differentiable functions $f$ on $[x_0, \infty)$, such that $f, \ldots, f^{(r)}$ are bounded. This space is complete, whence $\Xi^n(0)$ converges to a limit $f \in \mathscr{B}(\hat{\mathscr{G}}_{x_0;r}^{\preccurlyeq}, \frac{1}{2})$. Since this limit satisfies the equation (18), the function $f$ is actually infinitely differentiable, i.e. $f \in \mathscr{B}(\mathscr{G}_{x_0;r}^{\preccurlyeq}, \frac{1}{2})$. □

**4.4. Asymptotic analysis.** — With the notations from the previous section, assume now that $\mathscr{T}[\mathrm{i}]$ is $(1,1,1)$-differentially closed in $\mathbb{T}[\mathrm{i}]_{\preccurlyeq\mathfrak{v}}$, i.e. any solution $f \in \mathbb{T}[\mathrm{i}]_{\preccurlyeq\mathfrak{v}}$ to an equation $(\partial - \varphi)f = g$ with $\varphi, g \in \mathscr{T}[\mathrm{i}]$ is already in $\mathscr{T}[\mathrm{i}]$. Each $J_i$ is the right-inverse of an operator $\partial - \varphi_i$ with $\varphi_i \in \mathscr{T}[\mathrm{i}]$. Now $\partial - \varphi_i$ also admits a formal distinguished right-inverse $\tilde{J}_i$. Consequently, the operator $\Xi$ also admits a formal counterpart

$$\tilde{\Xi} : f \longmapsto (\tilde{J}_r \cdots \tilde{J}_1)(P(f)).$$

For each $n \in \mathbb{N}$, we have

$$\tilde{\Xi}^{n+1}(0) - \tilde{\Xi}^n(0) \prec_{\mathfrak{v}} \tilde{\Xi}^n(0)$$

so the sequence $\tilde{\Xi}^n(0)$ also admits a formal limit $\tilde{f}$ in $\hat{\mathscr{T}}[\mathrm{i}]$. In order to show that the fixed point $f$ from proposition 5 and $\tilde{f}$ are asymptotically equivalent over $\mathscr{T}[\mathrm{i}]$, we need some further notations. Given $f \in \mathscr{G}^{\preccurlyeq}[\mathrm{i}]$ and $\tilde{f} \in \mathscr{T}[\mathrm{i}]$, let us write $f \approx \tilde{f}$ if $f - \tilde{f} \prec \mathfrak{v}^{\mathbb{R}}$, i.e. $f - \tilde{f} \prec \mathfrak{v}^{\alpha}$ for all $\alpha \in \mathbb{R}$. We also write $f \approx_r \tilde{f}$ if $f \approx \tilde{f}, \ldots, f^{(r)} \approx \tilde{f}^{(r)}$.

***Proposition 18.*** — *For $f, g \in \mathscr{G}^{\preccurlyeq}[\mathrm{i}]$, $\tilde{f}, \tilde{g} \in \mathscr{T}[\mathrm{i}]$ and $r \in \mathbb{N}$, we have*

$$\begin{aligned} f \approx_r \tilde{f} \wedge g \approx_r \tilde{g} \quad &\Rightarrow \quad f + g \approx_r \tilde{f} + \tilde{g} \\ f \approx_r \tilde{f} \wedge g \approx_r \tilde{g} \quad &\Rightarrow \quad fg \approx_r \tilde{f}\tilde{g} \\ f \approx_{r+1} \tilde{f} \quad &\Rightarrow \quad f' \approx_r \tilde{f}' \end{aligned}$$

*Proof.* — Trivial. □

***Proposition 19.*** — *For $f \in \mathscr{G}^{\preccurlyeq}[\mathrm{i}]$, $\tilde{f} \in \mathscr{T}[\mathrm{i}]$ and $r \in \mathbb{N}$ with $f, \tilde{f} \prec_{\mathfrak{v}} \mathfrak{v}^{\nu}$, we have*

$$f \approx_r \tilde{f} \Rightarrow J_i f \approx_{r+1} \tilde{J}_i \tilde{f}.$$

*Proof.* — Let us first show that

$$f \approx 0 \Rightarrow J_i f \approx_1 0. \tag{24}$$

Given $\alpha \geqslant \nu$ with $f \preccurlyeq \mathfrak{v}^{\alpha}$, we have $J_{i,\ltimes\mathfrak{v}^{\alpha}}(\mathfrak{v}^{-\alpha} f) \preccurlyeq 1$, whence $J_i f \preccurlyeq \mathfrak{v}^{\alpha}$. Moreover,

$$(J_i f)' = \psi_i^{-1} f + \varphi(J_i f), \tag{25}$$

whence $f \preccurlyeq \mathfrak{v}^{\alpha} \Rightarrow (J_i f)' \preccurlyeq \mathfrak{v}^{\alpha+\beta}$ for some fixed $\beta$. This proves (24). More generally, $r$ additional applications of (25) yield

$$f \approx_r 0 \Rightarrow J_i f \approx_{r+1} 0.$$

Now assume that $f \approx_r \tilde{f}$ and write

$$J_i f - \tilde{J}_i \tilde{f} = J_i(f - \tilde{f}) + (J_i - \tilde{J}_i)(\tilde{f}).$$

By what precedes, we have $J_i(f - \tilde{f}) \approx_{r+1} 0$. On the other hand,

$$(J_i - \tilde{J}_i)(\tilde{f}) = c\mathrm{e}^{\int \varphi_i}$$

for some $c \in \mathbb{C}$. Since $\partial - \varphi_i$ is normal, we either have $\mathrm{e}^{\int \varphi_i} \prec \mathfrak{v}^{\mathbb{R}}$ (in which case $(\mathrm{e}^{\int \varphi_i})^{(i)} \prec \mathfrak{v}^{\mathbb{R}}$ for all $i \in \mathbb{N}$) or $c = 0$. In both cases, we get $(J_i - \tilde{J}_i)(\tilde{f}) \approx_{r+1} 0$, so that $J_i f \approx_{r+1} \tilde{J}_i \tilde{f}$. □

***Theorem* 6.** — *Let $\mathscr{T}$ be a transserial Hardy field of span $\mathfrak{v} \succcurlyeq\!\!\succ \mathrm{e}^x$ such that $\mathscr{T}[\mathrm{i}]$ is $(1,1,1)$-differentially closed in $\mathbb{T}_{\preccurlyeq\!\!\prec\mathfrak{v}}$. Consider a monic split-normal quasi-linear equation (18) without solutions in $\mathscr{T}$. Then there exist solutions $f \in \mathscr{G}[\mathrm{i}]$ and $\tilde{f} \in \hat{\mathscr{T}}[\mathrm{i}]$ to (18), such that $f$ and $\tilde{f}$ are asymptotically equivalent over $\mathscr{T}[\mathrm{i}]$.*

*Proof.* — With the above notations, let $f$ and $\tilde{f}$ be the limits in $\mathscr{G}[\mathrm{i}]$ resp. $\hat{\mathscr{T}}[\mathrm{i}]$ of the sequences $\Xi^n(0)$ resp. $\tilde{\Xi}^n(0)$. Given $g \in \mathscr{T}[\mathrm{i}]$, there exists an $n$ with

$$\Xi^{n+1}(0) - \Xi^n(0) \prec_{\mathfrak{v}} g.$$

At that point, we have

$$f - g \sim \Xi^n(0) - g \approx \tilde{\Xi}^n(0) - g \sim \tilde{f} - g$$

In other words, $f$ and $\tilde{f}$ are asymptotically equivalent over $\mathscr{T}[\mathrm{i}]$. □

***Theorem* 7.** — *Let $\mathscr{T}$ be a transserial Hardy field of span $\mathfrak{v} \succcurlyeq\!\!\succ \mathrm{e}^x$. Consider a monic split-normal quasi-linear equation (18) without solutions in $\mathscr{T}$ such that $L$ and $P$ have coefficients in $\mathscr{T}$. Assume that one of the following conditions holds:*

a) *$\mathscr{T}$ is $(1,1,1)$-differentially closed in $\mathbb{T}_{\preccurlyeq\!\!\prec\mathfrak{v}}$ and $r_L = r_P = 1$.*

b) *$\mathscr{T}[\mathrm{i}]$ is $(1,1,1)$-differentially closed in $\mathbb{T}[\mathrm{i}]_{\preccurlyeq\!\!\prec\mathfrak{v}}$.*

*Then there exist solutions $f \in \mathscr{G}$ and $\tilde{f} \in \hat{\mathscr{T}}$ to (18), such that $f$ and $\tilde{f}$ are asymptotically equivalent over $\mathscr{T}$.*

*Proof.* — In view of propositions 3 and 16, we may assume that $J_r \cdots J_1$ and $\Xi$ preserve realness in all results from sections 4.3 and 4.4. In particular, the solutions $f$ and $\tilde{f}$ in the conclusion of theorem 6 are both real. □

## 5. Differentially algebraic Hardy fields

### 5.1. First order extensions

***Lemma* 7.** — *Let $\mathscr{T}$ be a transserial Hardy field of span $\mathfrak{v} \succcurlyeq\!\!\succ \mathrm{e}^x$. Let $L = \partial - \varphi \in \mathscr{T}[\partial]$ be a normal operator. Let $\tilde{f} \in \hat{\mathscr{T}}^{\preccurlyeq}$ and $g \in \mathscr{T}^{\preccurlyeq}$ be such that $\tilde{f}$ is transcendental over $\mathscr{T}$ and $L\tilde{f} = g$. Then there exists an $f \in \mathscr{G}^{\preccurlyeq}$ with $Lf = g$, such that $f$ and $\tilde{f}$ are both differentially and asymptotically equivalent over $\mathscr{T}$.*

*Proof.* — With the notations of section 4.1, let $f = Jg$. Given a truncation $\psi \vartriangleleft \tilde{f}$, we claim that

$$f - \psi \approx J(g - (\psi' - \varphi\psi)).$$

Indeed, consider

$$\delta = \psi - J(\psi' - \varphi\psi) \in \mathbb{R}\mathrm{e}^{\Phi}.$$

In the attractive case, $\psi \prec_{\mathfrak{v}} \mathrm{e}^{\Phi}$ implies $\delta = 0$. In the repulsive case, we have $\mathrm{e}^{\Phi} \prec_{\mathfrak{v}}^{*} 1$ and again $\delta \approx 0$. By proposition 19, we also have

$$\tilde{f} - \psi = \tilde{J}(g - \psi' + \varphi\psi) \approx J(g - \psi' + \varphi\psi).$$

Since $\psi' - \varphi\psi \neq g$, it follows that $\tilde{f} - \psi \sim f - \psi$, whence $f$ and $\tilde{f}$ are asymptotically equivalent over $\mathcal{T}$. Furthermore, $LF - g$ is a minimal annihilator of $\tilde{f}$ over $\mathcal{T}$, since $\tilde{f}$ is transcendental over $\mathcal{T}$. Lemma 5 therefore implies that $f$ and $\tilde{f}$ are differentially equivalent over $\mathcal{T}$. □

***Theorem 8.*** — *Let $\mathcal{T}$ be a transserial Hardy field. Let $\mathcal{T}^{\mathrm{fo}} \supseteq \mathcal{T}$ be the smallest differential subfield of $\mathbb{T}$, such that for any $P \in \mathcal{T}^{\mathrm{fo}}\{F\}^{\neq}$ with $r_P \leqslant 1$ and $f \in \mathbb{T}$ we have $P(f) = 0 \Rightarrow f \in \mathcal{T}^{\mathrm{fo}}$. Then the transserial Hardy field structure of $\mathcal{T}$ can be extended to $\mathcal{T}^{\mathrm{fo}}$.*

*Proof.* — By theorems 1, 2 and 3, we may assume that $\mathcal{T}$ is closed under the resolution of real algebraic equations, exponentiation and logarithm. Assume that $\mathcal{T}^{\mathrm{fo}} \neq \mathcal{T}$ and let $P \in \mathcal{T}\{F\}^{\neq}$ be of minimal complexity $\chi_P = (1, s, t)$, such that $P(f) = 0$ for some $f \in \mathcal{T}^{\mathrm{fo}}$. Without loss of generality, we may make the following assumptions:

- $f$ and $P$ are exponential (modulo upward shifting).
- $f$ is a serial cut (by lemma 3).
- $f$ is a normal cut (modulo additive and multiplicative conjugations by $H_f$ resp. $\mathfrak{d}_f$).
- $P \in \mathcal{T}[\mathrm{i}]_{\preccurlyeq\mathfrak{v}}\{F\}$, where $\mathfrak{v} \in \mathcal{T} \cap \mathfrak{T}$ satisfies $\operatorname{uspan} f \asymp \mathfrak{v}$ (modulo replacing $P$ by $P_{\preccurlyeq\mathfrak{v}}$).
- $P$ is monic split-normal (modulo proposition 9, additive and multiplicative conjugations, and division by $\mathfrak{d}_P$).

By Zorn's lemma, it suffices to show that $\mathcal{T}\langle f\rangle$ carries the structure of a transserial Hardy field, which extends the structure of $\mathcal{T}$.

If $s = t = 1$, then lemma 7 implies the existence of an $\hat{f} \in \mathcal{G}^{\preccurlyeq}$ such that $f$ and $\hat{f}$ are both asymptotically and differentially equivalent over $\mathcal{T}_{\preccurlyeq\mathfrak{v}}$. Hence, the result follows from lemmas 4 and 6.

If $t > 1$, then $\mathcal{T}$ and $\mathcal{T}_{\preccurlyeq\mathfrak{v}}$ are $(1,1,1)$-differentially closed in $\mathbb{T}$ resp. $\mathbb{T}_{\preccurlyeq\mathfrak{v}}$. Now $\mathfrak{v} \succcurlyeq \mathrm{e}^x$, since $f$ is exponential. Therefore, theorem 7 provides us with an $\hat{f} \in \mathcal{G}^{\preccurlyeq}$ with $P(\hat{f}) = 0$, such that $f$ and $\hat{f}$ are asymptotically equivalent over $\mathcal{T}_{\preccurlyeq\mathfrak{v}}$. We conclude by lemmas 5, 4 and 6. □

### 5.2. Higher order extensions

***Lemma 8.*** — *Let $\mathcal{T}$ be a transserial Hardy field of span $\mathfrak{v} \succcurlyeq \mathrm{e}^x$. Let $L = \partial - \varphi \in \mathcal{T}[\mathrm{i}][\partial]$ be a normal operator. Let $\tilde{f} \in \hat{\mathcal{T}}[\mathrm{i}]^{\preccurlyeq}$ and $g \in \mathcal{T}[\mathrm{i}]^{\preccurlyeq}$ be such that* $\operatorname{Re}\tilde{f}$ *has order 2 over $\mathcal{T}$ and $L\tilde{f} = g$. Then there exists an $f \in \mathcal{G}^{\preccurlyeq}[\mathrm{i}]$ with $Lf = g$, such that* $\operatorname{Re} f$ *and* $\operatorname{Re}\tilde{f}$ *are both differentially and asymptotically equivalent over $\mathcal{T}$.*

*Proof.* — The fact that $f$ and $\tilde{f}$ are asymptotically equivalent over $\mathcal{T}$ is proved in a similar way as for lemma 7. It follows in particular that $\operatorname{Re} f$ and $\operatorname{Re}\tilde{f}$ are asymptotically equivalent. Since $\operatorname{lcm}(L, \bar{L})$ annihilates $f$, $\bar{f}$, $\tilde{f}$ and $\bar{\tilde{f}}$, it also annihilates both $\operatorname{Re} f$ and $\operatorname{Re}\tilde{f}$. The fact that $\operatorname{Re}\tilde{f}$ has complexity $(2,1,1)$ over $\mathcal{T}$ now guarantees that $\operatorname{lcm}(L, \bar{L})$ is a minimal annihilator of $\operatorname{Re}\tilde{f}$. We conclude by lemma 5. □

***Theorem 9.*** — *Let $\mathscr{T}$ be a transserial Hardy field. Let $\mathscr{T}^{\mathrm{dalg}} \supseteq \mathscr{T}$ be the smallest differential subfield of $\mathbb{T}$, such that for any $P \in \mathscr{T}^{\mathrm{dalg}}\{F\}^{\neq}$ and $f \in \mathbb{T}$ we have $P(f) = 0 \Rightarrow f \in \mathscr{T}^{\mathrm{dalg}}$. Then the transserial Hardy field structure of $\mathscr{T}$ can be extended to $\mathscr{T}^{\mathrm{dalg}}$.*

*Proof.* — By theorems 2, 3 and 8, we may assume that $\mathscr{T}$ is closed under exponentiation, logarithm and the resolution of first order differential equations. Assume that $\mathscr{T}^{\mathrm{dalg}} \neq \mathscr{T}$ and let $P \in \mathscr{T}[\mathrm{i}]\{F\}^{\neq}$ be of minimal complexity $\chi_P = (r, s, t)$, such that $P(f) = 0$ for some $f \in \mathscr{T}^{\mathrm{dalg}}[\mathrm{i}]$ with $\operatorname{Re} f \notin \mathscr{T}$. Let $Q \in \mathscr{T}\{F\}$ be a minimal annihilator of $\operatorname{Re} f$ and notice that $r_Q \geqslant r_P$, since $\operatorname{Re} f \notin \mathscr{T}$. Without loss of generality, we may make the following assumptions:

- $f$, $P$ and $Q$ are exponential (modulo upward shifting).
- $f$ is a serial cut (by the complexified version of lemma 3).
- $f$ is a normal cut (modulo additive and multiplicative conjugations by $H_f$ resp. $\mathfrak{d}_f$).
- $P \in \mathscr{T}[\mathrm{i}]_{\underline{\prec\!\!\prec}\mathfrak{v}}\{F\}$ and $Q \in \mathscr{T}_{\underline{\prec\!\!\prec}\mathfrak{v}}\{F\}$, where $\mathfrak{v} \in \mathscr{T} \cap \mathfrak{T}$ satisfies uspan $f \asymp\!\!\asymp \mathfrak{v}$ (modulo the replacement of $P$ and $Q$ by $P_{\underline{\prec\!\!\prec}\mathfrak{v}}$ resp. $Q_{\underline{\prec\!\!\prec}\mathfrak{v}}$).
- $Q$ is monic split-normal (modulo proposition 9, additive and multiplicative conjugations, and division by $\mathfrak{d}_Q$).

By Zorn's lemma, it now suffices to show that $\mathscr{T}\langle \operatorname{Re} f\rangle$ carries the structure of a transserial Hardy field, which extends the structure of $\mathscr{T}$.

If $r = s = t = 1$, then lemma 8 and the fact that $\mathscr{T}$ is 1-differentially closed imply the existence of an $\hat{f} \in \mathscr{G}^{\preccurlyeq}[\mathrm{i}]$ such that $\operatorname{Re} f$ and $\operatorname{Re} \hat{f}$ are both asymptotically and differentially equivalent over $\mathscr{T}_{\underline{\prec\!\!\prec}\mathfrak{v}}$. The result follows by lemmas 4 and 6.

If $\chi_P \neq (1, 1, 1)$, then $\mathscr{T}[\mathrm{i}]$ and $\mathscr{T}[\mathrm{i}]_{\underline{\prec\!\!\prec}\mathfrak{v}}$ are $(1, 1, 1)$-differentially closed in $\mathbb{T}[\mathrm{i}]$ resp. $\mathbb{T}[\mathrm{i}]_{\underline{\prec\!\!\prec}\mathfrak{v}}$. Now $\mathfrak{v} \not\succeq\!\!\succ \mathrm{e}^x$, since $f$ is exponential. Therefore, theorem 7 provides us with a $g \in \mathscr{G}^{\preccurlyeq}$ with $Q(g) = 0$, such that $\operatorname{Re} f$ and $g$ are asymptotically equivalent over $\mathscr{T}_{\underline{\prec\!\!\prec}\mathfrak{v}}$. We conclude by lemmas 5, 4 and 6. □

***Corollary 5.*** — *There exists a transserial Hardy field $\mathscr{T}$, such that for any $P \in \mathscr{T}\{F\}$ and $f, g \in \mathscr{T}$ with $f < g$ and $P(f)P(g) < 0$, there exists a $h \in \mathscr{T}$ with $f < h < g$ and $P(h) = 0$.*

*Proof.* — Take $\mathscr{T} = \mathbb{R}(x^{\mathbb{R}})^{\mathrm{dalg}}$ and endow it with a transserial Hardy field structure. Let $P \in \mathscr{T}\{F\}$ and $f, g \in \mathscr{T}$ with $f < g$ be such that $P(f)P(g) < 0$. By [**26**, Theorem 9.33], there exists a $h \in \mathbb{T}$ with $f < h < g$ and $P(h) = 0$. But $P(h) = 0$ implies $h \in \mathscr{T}$. □

***Corollary 6.*** — *There exists a transserial Hardy field $\mathscr{T}$, such that $\mathscr{T}[\mathrm{i}]$ is weakly differentially closed.*

*Proof.* — Take $\mathscr{T} = \mathbb{R}^{\mathrm{dalg}}$. By a straightforward adaptation of [**26**, Chapter 8] (see also [**24**, theorem 9.3]), it can be shown that any differential equation $P(f) = 0$ of degree $d$ with $P \in \mathscr{T}[\mathrm{i}]\{F\}$ admits $d$ distinguished solutions in $\mathbb{T}[\mathrm{i}]$ when counting with multiplicities. Let $f$ be such a solution. Since $P(f)=\bar{P}(\bar{f})=0$, both $\operatorname{Re} f$ and $\operatorname{Im} f$ are differentially algebraic over $\mathscr{T}$, whence $f\in\mathscr{T}[\mathrm{i}]$. □

***Corollary** 7.* — *There exists a differentially Henselian transserial Hardy field $\mathscr{T}$, i.e., such that any quasi-linear differential equation over $\mathscr{T}$ admits a solution in $\mathscr{T}$.*

**5.3. Differential Newton polynomials for Hardy fields.** — Let $\mathscr{H}$ be a differentially algebraic Hardy field extension of a transserial Hardy field $\mathscr{T}$.

***Proposition 20.*** — *Given $\varepsilon\in\mathscr{H}^{\prec}$, there exists an $l\in\mathbb{N}$ with $\varepsilon\prec(\log_l x)^{-1}$.*

*Proof.* — The functional inverse $|\varepsilon^{-1}|^{\mathrm{inv}}$ of $|\varepsilon^{-1}|$ satisfies an algebraic differential equation $P(|\varepsilon^{-1}|^{\mathrm{inv}})=0$ over $\mathscr{T}$. Let $P_{\langle\boldsymbol{i}\rangle}f^{\langle\boldsymbol{i}\rangle}$ be the leading term of $P$ for its logarithmic decomposition. As in [**26**, Section 8.1.4]. there exists an $l\in\mathbb{N}$ with $P(f)\sim P_{\langle\boldsymbol{i}\rangle}f^{\langle\boldsymbol{i}\rangle}$ for all $f\succcurlyeq\exp_l x$ . It follows that $|\varepsilon^{-1}|^{\mathrm{inv}}\prec\exp_l x$ and $\varepsilon\prec(\log_l x)^{-1}$. □

Given a differential polynomial $P\in\mathscr{H}\{F\}^{\neq}$, we define its *dominant part* to be the unique monic $D_P\in\mathbb{R}\{F\}$ such that $P=\ell_P(D_P+E_P)$ for some $\ell_P\in\mathscr{H}$ and $E_P\in\mathscr{H}\{F\}^{\prec}$. Here $D_P$ is said to be monic if its leading coefficient w.r.t. $F^{(r_P)},\ldots,F$ equals 1.

***Theorem 10.*** — *Given $P\in\mathscr{H}\{F\}^{\neq}$, there exists a polynomial $N_P\in\mathbb{R}[F](F')^{\mathbb{N}}$ with*

$$\begin{aligned} D_{P\uparrow_l} &= N_P\\ E_{P\uparrow_l} &= o_{\mathrm{e}^x}(1)\end{aligned}$$

*for all sufficiently large $l\in\mathbb{N}$.*

*Proof.* — As in the proof of [**26**, Theorem 8.6], we have

$$\operatorname{wt} D_P\geqslant\operatorname{wv} D_P\geqslant\operatorname{wt} D_{P\uparrow}\geqslant\operatorname{wv} D_{P\uparrow}\geqslant\cdots,$$

so we may assume without loss of generality that $\operatorname{wt} D_{P\uparrow_i}=\operatorname{wv} D_{P\uparrow_i}=w$ is constant for all $i\in\mathbb{N}$. Now

$$\begin{aligned} P\uparrow &= \ell_{P\uparrow}(D_{P\uparrow}+E_{P\uparrow})\\ &= \ell_P\uparrow(D_P\uparrow+E_P\uparrow)\\ &= \ell_P\uparrow(\mathrm{e}^{-wx}D_{P\uparrow}+E_P\uparrow),\end{aligned}$$

whence

$$\ell_{P\uparrow} = \ell_P\uparrow\mathrm{e}^{-wx} \tag{26}$$

$$D_{P\uparrow} = D_{D_P\uparrow} \tag{27}$$

$$E_{P\uparrow} = E_P\uparrow\mathrm{e}^{wx}. \tag{28}$$

Indeed, we must have

$$E_P\uparrow\mathrm{e}^{wx}=(E_{P_{[<w]}}\uparrow+E_{P_{[\geqslant w]}}\uparrow)\mathrm{e}^{wx}\prec 1,$$

because $E_{P_{[<w]}} \uparrow \mathrm{e}^{wx} \succcurlyeq 1$ would imply $\operatorname{wt} D_{P\uparrow} < w$. Applying [**26**, Lemma 8.5] to (27), and similarly for $P\uparrow, P\uparrow\uparrow, \ldots$, we get

$$D_{P\uparrow_l} = D_P \in \mathbb{R}[F](F')^w$$

for all $l \in \mathbb{N}$.

By proposition 20 and (28), we have $E_{P,[\geqslant v]} \prec_{\log_l x} 1$ and $E_{P\uparrow_{l+1},[\geqslant v]} \prec_{\mathrm{e}^x} 1$ for some $l \in \mathbb{N}$. Modulo upward shiftings, we may thus assume without loss of generality that $E_{P,[\geqslant v]} \prec_{\mathrm{e}^x} 1$. More generally, assume that $E_{P,[>v]} \prec_{\mathrm{e}^x} 1$ for some $v < w$. By (28), this implies $E_{P\uparrow_l,[>v]} \prec_{\mathrm{e}^x} 1$ for all $l \in \mathbb{N}$ and

$$\begin{aligned} E_{P\uparrow,[\boldsymbol{\omega}]} &= (E_{P,[v]} \uparrow_{[\boldsymbol{\omega}]} + E_{P,[>v]} \uparrow_{[\boldsymbol{\omega}]})\mathrm{e}^{wx} \\ &= \mathrm{e}^{(w-v)x}(E_{P,[\boldsymbol{\omega}]} \uparrow + o_{\mathrm{e}^x}(1)), \end{aligned} \tag{29}$$

for all $\boldsymbol{\omega}$ of weight $v$. We claim that there exists an $l \in \mathbb{N}$ with

$$E_{P,[v]} \prec [(\log_l^{-1} x)']^{w-v}. \tag{30}$$

Assume the contrary and consider a coefficient $E_{P,[\boldsymbol{\omega}]}$ of weight $v$ with

$$\psi = \sqrt[w-v]{E_{P,[\boldsymbol{\omega}]}} \succcurlyeq (\log_l^{-1} x)'$$

for all $l \in \mathbb{N}$. Without loss of generality, we may assume that $\psi$ and $\int \psi$ are in $\mathscr{H}$. Then proposition 20 implies $\int \psi \succcurlyeq 1$ and even $\int \psi \succ 1$ (by integrating from $+\infty$ when possible). Again by proposition 20, it follows that $\int \psi \succ \log_l x$ and $\psi \succ (\log_l x)'$ for some $l \in \mathbb{N}$. But then (29) yields

$$E_{P\uparrow_l,[\boldsymbol{\omega}]} = [(\log_l x)']^{v-w} \uparrow_l (E_{P,[\boldsymbol{\omega}]} \uparrow_l + o_{\mathrm{e}^x}(1)) \succ 1,$$

which contradicts the fact that $E_{P\uparrow_l} \prec 1$. The relations (30) and (29) imply the existence of an $l \in \mathbb{N}$ with $E_{P\uparrow_{l+1},[v]} \prec_{\mathrm{e}^x} 1$. By induction over $v = w, w-1, \ldots, 0$ and modulo upward shiftings, we may thus ensure that $E_{P,[\geqslant v]} \prec_{\mathrm{e}^x} 1$ for all $v \leqslant w$. □

The polynomial $N_P$ in theorem 10 is called the *differential Newton polynomial* of $P$. The generalization of this concept to $\mathscr{H}$ allows us to mimic a lot of the theory from [**26**, chapter 8] in $\mathscr{H}$. In what follows, we will mainly need a few more definitions. The *Newton degree* of an equation

$$P(f) = 0, \qquad f \prec \varphi \tag{31}$$

with $P \in \mathscr{H}\{F\}$ and $\varphi \in \mathscr{H}^{\neq}$ is defined by $\deg_{\prec\varphi} P = \deg N_{P_{\times\varphi}}$. Setting

$$\hat{\gamma} = \frac{1}{x \log x \log_2 x \cdots}$$

we also define

$$\deg_{\prec\hat{\gamma}} P = \min_{\varphi \succ \hat{\gamma}} \deg_{\prec\varphi} P.$$

We say that $f \prec \varphi$ is a solution to (31) modulo $o(\psi), \psi \in \mathscr{T} \cup \{\hat{\gamma}\}$ if $\deg_{\prec\psi} P_{+f} > 0$. We say that $\mathscr{H}$ is *differentially Henselian*, if every quasi-linear equation over $\mathscr{H}$ admits a solution. Given a solution $f$ to (31), we say that $f$ has *algebraic type* if $N_{P_{\times f}}$ is not homogeneous and *differential type* in the other case. The following proposition is proved along the same lines as [**26**, proposition 8.16]:

***Proposition 21***. — *Let $f$ be a solution to* (31) *of differential type and let $i$ be the degree of $N_{P_{\times f}}$. Then $f^\dagger$ is a solution modulo $o(\hat{\gamma})$ of $R_{P_i}$.*

***Remark 6***. — In this section, we assumed that $\mathscr{H}$ is a differentially algebraic Hardy field extension of a transserial Hardy field $\mathscr{T}$. We expect that the theory can be adapted to even more general H-field. This is one of the objectives of a current collaboration with Lou van den Dries and Matthias Aschenbrenner [**4**].

### 5.4. Transserial models of differentially algebraic Hardy fields

***Theorem 11***. — *Let $\mathscr{T}$ be a transserial Hardy field and $\mathscr{H}$ a differentially algebraic Hardy field extension of $\mathscr{T}$, such that $\mathscr{H}$ is differentially Henselian and stable under exponentiation. Then there exists a transserial Hardy field structure on $\mathscr{H}$ which extends the structure on $\mathscr{T}$.*

*Proof*. — By theorems 1, 2 and 8, we may assume that $\mathscr{T}$ is closed under the resolution of real algebraic equations, exponentiation and integration. Assume that $\mathscr{H} \neq \mathscr{T}$ and choose $P \in \mathscr{T}\{F\}$ of minimal complexity $\chi_P = (r, s, t)$, such that either

- **C1:** $P(f) = 0$ for some $f \in \mathscr{H}$.
- **C2:** $P(f) = 0$ modulo $o(\mathfrak{m}\hat{\gamma})$ for some $f \in \mathscr{H}$, $\mathfrak{m} \in \mathscr{T} \cap \mathfrak{T}$ and $P$ admits no roots in $\mathscr{T}$ modulo $o(\mathfrak{m}\hat{\gamma})$. Moreover, $\mathscr{T}$ is $\chi_P$-differentially closed in $\mathscr{H}$.

Modulo upward shifting, we may assume without loss of generality that $P$ is exponential. In view of Zorn's lemma, it suffices to show that there exists a transserial Hardy field structure on $\mathscr{T}\langle f\rangle$ which extends the structure on $\mathscr{T}$.

Let $\Phi$ be the set of $\tilde{f} \in \mathscr{T}$ such that $f - \tilde{f} \prec \operatorname{supp} \tilde{f}$. The set $\Phi$ is totally ordered for $\trianglelefteq$, so there exists a minimal well-based transseries $\tilde{f}$ with $\varphi \trianglelefteq \tilde{f}$ for all $\varphi \in \Phi$. We call $\tilde{f}$ the *initializer* of $f$ over $\mathscr{T}$. Assume first that $\tilde{f} \in \mathscr{T}$. Then we may assume without loss of generality that $\varphi = 0$, modulo an additive conjugation by $\varphi$. Now $f$ is of differential type, since $f \asymp \mathfrak{m}$ for no $\mathfrak{m} \in \mathscr{T} \cap \mathfrak{T}$. Let $i \in \mathbb{N}$ be such that $R_{P_i}(f^\dagger) = 0$ modulo $o(\hat{\gamma})$. Since $R_{P_i}$ has lower complexity than $P$, there exists a $g \in \mathscr{T}$ with $R_{P_i}(g) = 0$ modulo $o(\hat{\gamma})$. Since $\mathscr{T}$ is truncation closed we may take $g \in \mathscr{T}_{\succ \hat{\gamma}}$. But then $f \asymp \mathrm{e}^{\int g} \in \mathscr{T} \cap \mathfrak{T}$. This contradiction proves that we cannot have $\tilde{f} \in \mathscr{T}$.

Let us now consider the case when $\tilde{f} \notin \mathscr{T}$. Since $\deg_{\prec \operatorname{supp} \tilde{f}} P_{+\tilde{f}} > 0$, there exists a root $\varphi \trianglerighteq \tilde{f}$ of $P$ in the set of well-based transseries with complex coefficients. But $P$ admits only grid-based solutions, whence $\tilde{f} \in \mathbb{T}$. By construction, $f$ and $\tilde{f}$ are asymptotically equivalent over $\mathscr{T}$. Let $\mathfrak{v} \in \mathscr{T} \cap \mathfrak{T}$ be such that $\operatorname{uspan} \tilde{f} \asymp\!\!\!\asymp \mathfrak{v}$. Modulo an additive and a multiplicative conjugation we may assume without loss of generality that $\tilde{f}$ is a normal cut. In case **C2**, we notice that $\operatorname{supp} \tilde{f} \succ \mathfrak{m}\hat{\gamma}$, whence $\mathfrak{m} \prec_{\mathfrak{v}}^* 1$, since $\operatorname{uspan} \tilde{f} = \mathfrak{v}$. Consequently, we always have $P_{\underline{\preccurlyeq}\mathfrak{v}}(\tilde{f}) = 0$.

We claim that the cuts $f$ and $\tilde{f}$ are differentially equivalent over $\mathscr{T}$. Assume the contrary and let $Q \in \mathscr{T}_{\underline{\preccurlyeq}\mathfrak{v}}\{F\}$ be a minimal annihilator of $\tilde{f}$. By lemma 8 and modulo an additive and multiplicative conjugation, we may assume without loss of generality that $\tilde{f} \prec_{\mathfrak{v}} 1$ and that $Q$ is normal. Since $\mathscr{H}$ is differentially Henselian, it follows that $Q$ admits a root $g \prec_{\mathfrak{v}} 1$ in $\mathscr{H}$. Now $\chi_Q < \chi_P$ in case **C1** and $\chi_Q \leqslant \chi_P$ in case **C2**, so this root is already in $\mathscr{T}$, by the induction hypothesis. But $Q$ admits at most one solution in $\mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}$, whence $\tilde{f} = g_{\preceq\!\!\!\prec \mathfrak{v}} \in \mathscr{T}$. This contradiction completes the proof of our claim. By lemma 6, we conclude that $\mathscr{T}\langle f \rangle$ carries the structure of a transserial Hardy field extension of $\mathscr{T}$. □

***Corollary 8***. — *Let $\mathscr{T}$ be a transserial Hardy field and $\mathscr{H}$ a differentially algebraic Hardy field extension of $\mathscr{T}$, such that $\mathscr{H}$ is differentially Henselian. Assume that $\mathscr{H}$ admits no non-trivial algebraically differential Hardy field extensions. Then $\mathscr{H}$ satisfies the differential intermediate value property.*

*Proof.* — The fact that $\mathscr{H}$ admits no non-trivial algebraically differential Hardy field extensions implies that $\mathscr{H}$ is stable under exponentiation. By theorem 11, we may give $\mathscr{H}$ the structure of a transserial Hardy field. By theorem 9, we also have $\mathscr{T}^{\mathrm{dalg}} = \mathscr{T}$. We conclude in a similar way as in the proof of corollary 5. □

It is quite possible that there exist maximal Hardy fields whose differentially algebraic parts are not differentially Henselian, although we have not searched hard for such examples yet. The differentially algebraic part of the intersection of all maximal Hardy fields is definitely not differentially Henselian (and therefore does not satisfy the differential intermediate value property), due to the following result [**9**, Proposition 3.7]:

***Theorem 12***. — *Any solution of the equation*

$$f'' + f = \mathrm{e}^{x^2}$$

*is contained in a Hardy field. However, none of these solutions is contained in the intersection of all maximal Hardy fields.*

## Glossary

| | | |
|---|---|---|
| $f \preccurlyeq g$ | $f$ is dominated by $g$ | 456 |
| $f \prec g$ | $f$ is negligible w.r.t. $g$ | 456 |
| $f \asymp g$ | $f$ is asymptotic to $g$ | 456 |
| $f \sim g$ | $f$ is asymptotically similar to $g$ | 456 |
| $f \trianglelefteq\!\!\preccurlyeq g$ | $f$ is flatter than or as flat as $g$ | 456 |
| $f \prec\!\!\prec g$ | $f$ is flatter than $g$ | 456 |
| $f \asymp\!\!\asymp g$ | $f$ is as flat as $g$ | 456 |
| $f \approx g$ | $f$ and $g$ are similar modulo flatness | 456 |
| $f \preccurlyeq_{\mathfrak{v}} g$ | $f \preccurlyeq g$ modulo elements flatter than $\mathfrak{v}$ | 457 |
| $f \prec_{\mathfrak{v}} g$ | $f \prec g$ modulo elements flatter than $\mathfrak{v}$ | 457 |
| $f \preccurlyeq_{\mathfrak{v}}^{*} g$ | $f \preccurlyeq g$ modulo elements flatter than or as flat as $\mathfrak{v}$ | 457 |
| $f \prec_{\mathfrak{v}}^{*} g$ | $f \prec g$ modulo elements flatter than or as flat as $\mathfrak{v}$ | 457 |
| $\mathbb{T}^{>}$ | shorthand for $\{f \in \mathbb{T} : f > 0\}$ | 457 |
| $\mathbb{T}^{\neq}$ | shorthand for $\{f \in \mathbb{T} : f \neq 0\}$ | 457 |
| $\mathbb{T}^{\succ}$ | shorthand for $\{f \in \mathbb{T} : f \succ 1\}$ | 457 |
| $f_{\succ}$ | infinite part of $f$ | 457 |
| $f_{\prec\!\!\prec \mathfrak{v}}$ | part of $f$ which is flatter than $\mathfrak{v}$ | 457 |
| $\mathbb{T}_{\succ}$ | shorthand for $\{f_{\succ} : f \in \mathbb{T}\}$ | 457 |
| $\mathbb{T}_{\prec\!\!\prec \mathfrak{v}}$ | shorthand for $\{f_{\prec\!\!\prec \mathfrak{v}} : f \in \mathbb{T}\}$ | 457 |
| $\partial$ | derivation with respect to $x$ | 457 |
| $\int$ | integration with respect to $x$ | 457 |
| $f^{\dagger}$ | logarithmic derivative of $f$ | 457 |
| $\uparrow$ | upward shifting | 457 |
| $\downarrow$ | downward shifting | 457 |
| $f \trianglelefteq g$ | $f$ is a truncation of $g$ | 457 |
| span $f$ | canonical span of $f$ | 457 |
| uspan $f$ | ultimate canonical span of $f$ | 457 |
| $\hat{\mathscr{T}}$ | completion of $\mathscr{T}$ with serial cuts | 458 |
| $\mathscr{T}\{F\}$ | ring of differential polynomials in $F$ over $\mathscr{T}$ | 458 |
| $\mathscr{T}\langle F\rangle$ | quotient field of $\mathscr{T}\{F\}$ | 458 |
| $L_P$ | linear part of $P$ as an operator | 458 |
| $r_P$ | order of $P$ | 458 |
| $s_P$ | degree of $P$ in its leader | 458 |
| $t_P$ | total degree of $P$ | 458 |
| $\chi_P$ | complexity of $P$ | 458 |
| $I_P$ | initial of $P$ | 458 |
| $S_P$ | separant of $P$ | 458 |
| $H_P$ | the product $I_P S_P$ | 458 |
| $\chi_f$ | complexity of $f$ over $\mathscr{T}$ | 458 |
| $r_f$ | order of $f$ over $\mathscr{T}$ | 458 |
| $P_{+\varphi}$ | additive conjugation of $P$ by $\varphi$ | 459 |

## References


[1] M. Aschenbrenner & L. van den Dries – "$H$-fields and their Liouville extensions", *Math. Z.* **242** (2002), p. 543–588.

[2] ———, "Asymptotic differential algebra", in *Analyzable functions and applications*, Contemp. Math., vol. 373, Amer. Math. Soc., 2005, p. 49–85.

[3] ———, "Liouville closed $H$-fields", *J. Pure Appl. Algebra* **197** (2005), p. 83–139.

[4] M. Aschenbrenner, L. van den Dries & van der Hoeven – "Linear differential equations over $H$-fields", in preparation.

[5] M. Aschenbrenner, L. van den Dries & J. van der Hoeven – "Differentially algebraic gaps", *Selecta Math. (N.S.)* **11** (2005), p. 247–280.

[6] M. Boshernitzan – "An extension of Hardy's class $L$ of 'orders of infinity'", *J. Analyse Math.* **39** (1981), p. 235–255.

[7] ———, "New 'orders of infinity'", *J. Analyse Math.* **41** (1982), p. 130–167.

[8] ———, "Hardy fields and existence of transexponential functions", *Aequationes Math.* **30** (1986), p. 258–280.

[9] ———, "Second order differential equations over Hardy fields", *J. London Math. Soc. (2)* **35** (1987), p. 109–120.

[10] N. Bourbaki – *Fonctions d'une variable réelle. Éléments de mathématiques, chap. 5*, 2nd ed., Hermann, 1961.

[11] B. L. J. Braaksma – "Multisummability and Stokes multipliers of linear meromorphic differential equations", *J. Differential Equations* **92** (1991), p. 45–75.

[12] ———, "Multisummability of formal power series solutions of nonlinear meromorphic differential equations", *Ann. Inst. Fourier (Grenoble)* **42** (1992), p. 517–540.

[13] L. van den Dries – *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, 1998.

[14] L. van den Dries, A. Macintyre & D. Marker – "The elementary theory of restricted analytic fields with exponentiation", *Ann. of Math. (2)* **140** (1994), p. 183–205.

[15] ———, "Logarithmic-exponential power series", *J. London Math. Soc. (2)* **56** (1997), p. 417–434.
[16] ———, "Logarithmic-exponential series", in *Proceedings of the International Conference "Analyse & Logique" (Mons, 1997)*, vol. 111, 2001, p. 61–113.
[17] J. Écalle – "Les fonctions résurgentes I–III", *Publ. Math. d'Orsay* (1981, 1985).
[18] ———, "L'accélération des fonctions résurgentes (survol)", unpublished manuscript, 1987.
[19] J. Écalle – *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*, Actualités Mathématiques, Hermann, 1992.
[20] ———, "Six lectures on transseries, analysable functions and the constructive proof of Dulac's conjecture", in *Bifurcations and periodic orbits of vector fields (Montreal, PQ, 1992)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 408, Kluwer Acad. Publ., 1993, p. 75–184.
[21] G. Hardy – *Orders of infinity*, Cambridge Univ. Press, 1910.
[22] ———, "Properties of logarithmico-exponential functions", *Proceedings of the London Mathematical Society* **10** (1911), p. 54–90.
[23] J. van der Hoeven – "Asymptotique automatique", Ph.D. Thesis, École polytechnique, Palaiseau, France, 1997.
[24] ———, "Complex transseries solutions to algebraic differential equations", Technical Report 2001-34, Université Paris-Sud, Orsay, 2001.
[25] ———, "Integral transseries", Technical Report 2005-15, Université Paris-Sud, Orsay, 2005.
[26] ———, *Transseries and real differential algebra*, Lecture Notes in Mathematics, vol. 1888, Springer, 2006.
[27] S. Kuhlmann – *Ordered exponential fields*, Fields Institute Monographs, vol. 12, American Mathematical Society, 2000.
[28] M. Rosenlicht – "Hardy fields", *J. Math. Anal. Appl.* **93** (1983), p. 297–311.
[29] ———, "The rank of a Hardy field", *Trans. Amer. Math. Soc.* **280** (1983), p. 659–671.
[30] ———, "Growth properties of functions in Hardy fields", *Trans. Amer. Math. Soc.* **299** (1987), p. 261–272.
[31] M. Schmeling – "Corps de transséries", Ph.D. Thesis, Université Paris 7, 2001.
[32] M. Singer – "Asymptotic behavior of solutions of differential equations and Hardy fields: Preliminary report", `http://www4.ncsu.edu/~singer/papers/hardy.pdf`, 1975.

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