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CLAIRE VOISIN

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RATIONALLY CONNECTED 3-FOLDS AND SYMPLECTIC GEOMETRY

by

Claire Voisin

Pour Jean Pierre Bourguignon, à l'occasion de ses 60 ans.

Abstract. — We study the following question asked by Kollár: Let X be a rationally connected 3-fold, and Y be a compact Kähler 3-fold symplectically equivalent to it. Is Y rationally connected? We show that the answer is positive if X is Fano or $b_2(X) \leq 2$.

***Résumé* (3-variétés rationnellement connexes et géométrie symplectique).** — Nous étudions la question suivante posée par Kollár: soient X et Y des variétés kählériennes compactes de dimension 3 symplectiquement équivalentes. On suppose que X est rationnellement connexe. Y est-elle aussi rationnellement connexe? Nous montrons que la réponse est positive si X est une variété de Fano ou $b_2(X) \leq 2$.

0. Introduction

Let X be a compact Kähler manifold. Denoting by J the operator of complex structure acting on T_X, Kähler forms on X are symplectic forms which satisfy the compatibility conditions

$$\omega(Ju, Jv) = \omega(u, v),\ u,\ v \in T_{X,x},\ \omega(u, Ju) > 0,\ 0 \neq u \in T_{X,x}.$$

The first condition tells that ω is of type $(1,1)$. The last condition is called the taming condition. The set of Kähler forms is a convex cone, in particular connected, and thus determines a deformation class of symplectic forms on X.

Let X and Y be two complex projective or compact Kähler manifolds. We will say that X and Y are symplectically equivalent if for some symplectic forms α on X, resp. β on Y, which are in the deformation class of a Kähler form on X, resp. Y, there is a diffeomorphism

$$\psi : X \cong Y,$$

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such that $\psi^*\beta = \alpha$. Notice that ψ^* induces a bijection between the sets of symplectic forms which are in the deformation class of a symplectic form on Y and X, and thus we may assume that α is a taming form, or even a Kähler form on X.

In the sequel, the compact Kähler manifolds X we will consider are *uniruled* manifolds, which means the following (cf [**8**]):

Definition 0.1. — *A projective complex manifold (or compact Kähler) is uniruled if there exist compact complex manifolds Z and B, and dominating morphisms*

$$f : Z \to X,\ g : Z \to B,$$

where f is non constant on the fibers of g and the generic fiber of g is isomorphic to $\mathbb{P}^1$.

In other words, there is a (maybe singular) rational curve in X passing through any point of X, where a (singular) rational curve is defined as a connected curve whose normalization has only rational components.

The starting point of this work is the following result, due independently to Kollár [**9**] and Ruan [**19**] (we refer to [**6**], [**13**], [**14**] for purely symplectic characterizations and studies of uniruledness) :

Theorem 0.2. — *Let X and Y be two symplectically equivalent compact Kähler manifolds. Then if X is uniruled, Y is also uniruled.*

We sketch later on the proof of this result, in order to point out why the proof does not extend to cover the rational connectedness property, which we will consider in this paper. Let us recall the definition (cf [**2**], [**10**], [**8**]).

Definition 0.3. — *A compact Kähler manifold X is rationally connected if for any two points $x,\ y \in X$, there exists a (maybe singular) rational curve $C \subset X$ with the property that $x \in C$, $y \in C$.*

Examples of rationally connected varieties are given by smooth Fano varieties, i.e. smooth projective varieties X satisfying the condition that $-K_X$ is ample. (This is the main result of [**2**], and [**10**].)

The following conjecture appears in [**9**]. It was asked to me by Pandharipande and Starr :

Conjecture 0.4 **(Kollár)**. — *Assume X is rationally connected. Let Y be a compact Kähler manifold symplectically equivalent to X. Then Y is also rationally connected.*

Remark 0.5. — A compact Kähler manifold X whis is rationally connected satisfies $H^2(X, \mathcal{O}_X) = 0$, hence is projective. Thus, under the assumption above, X is projective, and if the answer to conjecture 0.4 is positive, Y is also projective.

This conjecture has an easy positive answer in the case of surfaces, as an immediate consequence of theorem 0.2. Indeed, let X be rationally connected of dimension 2, and let Y be symplectically equivalent to X. Then Y is uniruled, as X is. On the other hand $b_1(Y) = 0$, because $b_1(X) = 0$ and Y is diffeomorphic to X. Thus Y is a rational surface, hence rationally connected.

In this note, we prove the following partial results concerning conjecture 0.4 in dimension 3. I should mention here that in these form the results are partly due to Jason Starr. Indeed, in the original version of this paper, I had worked with a more restricted notion of symplectic equivalence between compact Kähler manifolds, where I considered only symplectic diffeomeorphisms $(X, \alpha) \cong (Y, \beta)$ where α *and* β were taming for the complex structure. Jason Starr showed me how to make the proof of proposition 0.6 work as well when only α is taming, and β is any symplectic form which is a deformation (through a family of symplectic forms) of a Kähler form on Y.

Proposition 0.6. — *Let X be rationally connected of dimension 3, and let Y be compact Kähler symplectically equivalent to X. If Y is not rationally connected, X and Y admit almost holomorphic rational maps*

$$\phi : X \dashrightarrow \Sigma,\ \phi' : Y \dashrightarrow \Sigma'$$

to a surface, with rational fibers C, resp. D, of the same homology class (where we use the symplectomorphism $\psi : X \cong Y$ giving symplectic equivalence to identify $H_2(X, \mathbb{Z})$ and $H_2(Y, \mathbb{Z})$).

Here *almost holomorphic* means that the map is well-defined near a generic fiber. We then consider the case where the above map ϕ is well-defined.

Proposition 0.7. — *Under the same assumptions as in proposition 0.6, assume that the rational map ϕ above is well-defined and that either Σ is smooth, or ϕ does not contract a divisor to a point. Then Y is also rationally connected.*

We will use this result together with some birational geometry arguments to prove the following:

Theorem 0.8. — *Let X, Y be compact Kähler 3-folds. Assume that X and Y are symplectically equivalent and that one of the two following assumptions hold:*

1. *X is Fano.*
2. *X is rationally connected, and $b_2(X) \leq 2$.*

Then Y is rationally connected.

This anwers conjecture 0.4 when X is a Fano threefold or satisfies $b_2 \leq 2$. The two considered cases have a small overlap. In the class where $b_2(X) \leq 2$, one has all the blow-ups of Fano manifolds with $b_2 = 1$ along a connected submanifold. Thus this is not a bounded family. It is known on the contrary that Fano manifolds form a bounded family (see [**2**], [**10**], or [**17**] for the 3-dimensional case). However the bound for b_2 of a Fano threefold is 10 (cf [**17**]), showing that the Fano case is far from being included in the second case.

Remark 0.9. — Note that for varieties with $b_2 = 1$, conjecture 0.4 obviously has an affirmative answer. Indeed a uniruled projective manifold with $b_2 = 1$ is necessarily Fano. Hence if X is rationally connected with $b_2 = 1$, by theorem 0.2 any projective manifold which is symplectomorphic to it is also uniruled with $b_2 = 1$, hence Fano, hence rationally connected.

Remark 0.10. — The results presented here have a partial overlap with [**3**], where the authors show that for rigid and "primitive" Fano threefolds with $b_2 = 2$ and $b_3 = 0$, the projective (equivalently Kähler) complex structure is unique. I thank the referee for bringing this reference to my attention.

To conclude this introduction, let us sketch the proof of theorem 0.2, and explain on an example the difficulty one meets to extend it to the rational connectedness question.

Proof of theorem 0.2. — Let α be a taming symplectic form on X (one can take here a Kähler form). We will denote in the sequel the degree of curves C in X with respect to α (that is the integrals $\int_C \alpha$) by $\deg_\alpha(C)$. Let $\mu_\alpha(X)$ be the minimum of the following set:

$$S_X := \{\deg_\alpha(C),\ C \text{ moving rational curve in } X\}.$$

Here by "moving", we mean that the deformations of C sweep-out X. Note that the minimum of the set S_X is well defined, because there are finitely many families of curves of bounded degree in X and the $(1,1)$-part $\alpha^{1,1}$ of α is $> \epsilon\omega$ where ω is any Kähler form on X. Let now C be a moving rational curve on X, which satisfies $\deg_\alpha(C) = \mu_\alpha(X)$ and let $[C] \in H_2(X,\mathbb{Z})$ be its homology class. We claim that for $x \in X$, and for adequate cohomology classes $A_1, \ldots, A_r \in H^4(X,\mathbb{Z})$, the Gromov-Witten invariant $GW_{0,[C]}([x], A_1, \ldots, A_r)$ counting genus 0 curves passing through x and meeting representatives B_i of the homology classes Poincaré dual to A_i, is non zero. To see this, we observe that by minimality of $\deg_\alpha(C)$, any genus 0 curve of degree $< \deg_\alpha(C)$ is not moving, that is, its deformations do not sweep-out X. It follows that for a general point $x \in X$, any genus 0 curve of class $[C]$ and passing through x is irreducible, with normal bundle generated by sections. This implies that the set $Z_{x,[C]}$ of rational curves of classes $[C]$ passing through x has the expected dimension and it is nonempty by assumption. Let r be its dimension, and choose for A_i, $1 \leq i \leq r$, a class h^2, where h is a Kähler class on X. It is then clear that $GW^X_{0,[C]}([x], A_1, \ldots, A_r) \neq 0$, as this number is the volume of a semi-positive generically positive $(1,1)$-form on $Z_{x,[C]}$.

As Y is symplectically isomorphic to X, (for some symplectic structures on X, resp. Y, in the deformation class determined by Kähler forms,) we conclude that $GW^Y_{0,\psi_*[C]}([y], A'_1, \ldots, A'_r) \neq 0$, where $A'_i = \psi_* A_i \in H^4(Y, \mathbb{Z})$. But in turn, because Gromov-Witten invariants can be computed using rational curves on Y by excess formulas (see [**12**], [**1**], [**20**]), this implies that there is through any point $y \in Y$ a rational curve of class $\psi_*[C]$. Thus Y is uniruled. □

Remark 0.11. — The proof above shows in fact a strongest statement, namely the fact that a uniruled compact Kähler manifold X admits non-zero Gromov-Witten invariants in genus 0 passing through one point:

$$(*) \qquad GW^X_{0,[C]}([x], A_1, \ldots, A_r) \neq 0.$$

From this point of view, the proof of Theorem 0.8 is somewhat different. Indeed we do not prove that a projective rationally connected 3-fold X admits non-zero Gromov-Witten in genus 0 passing through two points: $GW^X_{0,[C]}([x], [x], A_1, \ldots, A_r) \neq 0$, which would be the natural symplectic analogue of rational connectedness.

Our argument uses Gromov-Witten invariants *in higher genus*, which of course works in the symplectic setting as well. What we show essentially is that there is a covering family of rational curves of class $[C]$ with a non zero 1 point Gromov-Witten invariant: $GW^X_{0,[C]}([x], A_1, \ldots, A_r) \neq 0$, and that there is a non zero Gromov-Witten invariant of the following shape

$$(**) \qquad GW^X_{g,[C']}(\underbrace{[C], \ldots, [C]}_{r}, A_1, \ldots, A_N) \neq 0,$$

for some $r > g$ and curve class $[C']$ not proportional to C. We have the same non vanishings for Y.

The second ingredient is the notion of maximal rationally connected fibration due to Kollár-Miyaoka-Mori and Campana in the Kähler context. This last notion does not seem to extend well to the symplectic geometry context. The argument consists roughly in proving that the basis of the maximal rationally connected fibration of Y cannot be a 3-fold by the non vanishing (*), and cannot be a surface, which would be uniruled by the non-vanishing (**). Finally it cannot be a curve by elementary topological considerations.

Remark 0.12. — We used in this sketch of proof the terminology "rational curve in X" to mean "stable n-pointed genus 0 maps $f : C \to X$", which are the correct objects to count in order to compute the Gromov-Witten invariants (cf [**4**]). However, note that if f is as above, $f(C)$ is a rational curve in the previous sense.

If we want to apply the reasoning to study rational connectedness, we are faced to the following problem: we could as before introduce the minimal degree for which there are rational curves in X passing through any two points of X. On the other hand, it might be that curves of this degree are all reducible, with one component which is highly obstructed, so that one cannot conclude that the corresponding Gromov-Witten invariant is non zero. In fact, consider the case of a Hirzebruch surface $p : F \to \mathbb{P}^1$ which is a deformation (hence symplectically equivalent to) of a quadric $\mathbb{P}^1 \times \mathbb{P}^1$: Let C_0 be a rational curve which is a section of p with sufficiently negative self-intersection : $C_0^2 < -4$. Then one has in F rational curves consisting of the union of two fibers with the section C_0. Such curves C can be chosen so as to pass through any two points of F, and we may assume they are, among the rational curves satisfying this property, of minimal degree with respect to an adequate polarization. On the other hand, we have $C^2 < 0$ and it is clear that these curves disappear under a deformation from F to $\mathbb{P}^1 \times \mathbb{P}^1$. The corresponding 2-points Gromov-Witten invariant is 0 in this case.

The paper is organized as follows. In section 1, we prove proposition 0.6. In section 2, we study the remaining case, where X is an almost conic bundle (we mean by this that X admits a rational map f to a projective surface Σ, with generic fiber isomorphic to $\mathbb{P}^1$, and that the rational map f is well-defined along the generic fiber). We show that ϕ is actually a morphism (for an adequate choice of birational model of Σ) when $b_2(X) \leq 2$ or X is Fano, unless there are some non trivial genus 0 Gromov-Witten invariants of the form $GW^X_{0,[C']}([C], A_1, \dots, A_r)$, with $[C']$ not proportional to $[C]$. These Gromov-Witten invariants will be used in the last section to conclude that in this last case, Y is also rationally connected. We also show that when ϕ is well-defined, there are many non zero Gromov-Witten invariants on X, maybe not in genus 0 however.

The proof of theorem 0.8 uses in turn these non zero Gromov-Witten invariants on Y. It is completed in section 3.

Thanks. — It is a pleasure to acknowledge discussions with Jason Starr and Rahul Pandharipande, which started me thinking to this question. I thank Dusa McDuff, Yongbin Ruan and Johan de Jong for comments on various versions of the paper. I am mostly indebted to Jason Starr for showing me how to modify my original work to get the present version of the result.

1. Study of the rationally connected fibration of Y

This section has been much simplified and improved thanks to the help of Jason Starr. In the proof of proposition 1.1 below, he showed me how to work with general symplectic equivalence, instead of restricted symplectic equivalence as I did originally.

We will assume that X is a projective rationally connected complex manifold, that Y is compact Kähler and that X and Y are symplectomorphic with respect to some symplectic forms α, β on X, Y respectively, with α a taming form for the complex structure on X and β in the deformation class (as a symplectic form) of a Kähler form on Y. We will denote as before $\psi : X \cong Y$, $\psi^*\beta = \alpha$ such a symplectomorphism. The theory of Gromov-Witten invariants shows that the map ψ identifies the Gromov-Witten invariants of X and Y, computed using holomorphic curves on X and Y.

We start now as in the proof of Theorem 0.2. Introducing as before moving rational curves (or rather genus 0 stable maps) C on X, of minimal degree with respect to α, we concluded that there is a covering family of rational curves (genus 0 stable maps) in Y in the class $\psi_*([C])$.

Our goal in this section is to show the following, (which implies proposition 0.6):

Proposition 1.1. — *If Y is not rationally connected, then the covering family of curves C in X is given by an almost holomorphic rational map*

$$\phi : X \dashrightarrow \Sigma$$

to a surface, with rational fibers of class $[C]$. Furthermore, Y also admits an almost holomorphic rational map

$$\phi' : Y \dashrightarrow \Sigma'$$

with rational fiber of class $[D] = \psi_*[C]$.

Here almost holomorphic means that the rational map ϕ is well-defined along the generic fiber of ϕ. Equivalently, choosing a desingularization

$$\tilde{\phi} : \widetilde{X} \to \Sigma,\ \tau : \widetilde{X} \to X$$

of ϕ, where τ is a composition of blow-ups along smooth centers, this means that the exceptional divisors of τ do not dominate Σ. As the fibers of this fibration are rational curves, but X is not necessarily ruled (as it may not exist a line bundle with intersection -1 with fibers), we will say that X is an *almost conic bundle.*

The proof of the proposition is based on the following lemma (here we do not distinguish the image curve and the map, as we know that the map is generically the normalization map):

Lemma 1.2. — *Y is rationally connected, unless possibly if the curve D above satisfies $c_1(K_Y) \cdot [D] = -2$ and $GW_{0,[D]}([y]) = 1$.*

Proof. — We study the maximal rationally connected fibration of Y, which exists even if Y is only Kähler by [**2**], and is an almost holomorphic rational map

$$Y \dashrightarrow B.$$

Notice that dim $B \leq 2$ because Y is covered by rational curves D of class $[D] = \psi_*[C]$. We use now the following elementary lemma.

Lemma 1.3. — *Let X, Y be compact Kähler manifolds which are symplectically equivalent. Assume X is rationally connected. If the basis B of the rationally connected fibration of Y has dimension ≤ 1, Y is rationally connected.*

Proof. — We know that $H^1(X, \mathbb{C}) = 0$ because X is rationally connected and this obviously implies $H^0(X, \Omega_X) = 0$, hence $H^1(X, \mathbb{C}) = 0$ by Hodge theory. As Y is diffeomorphic to X, $H^1(Y, \mathbb{C}) = 0$ as well. It follows that if the basis B of the rationally connected fibration of Y has dimension 1, it is isomorphic to $\mathbb{P}^1$. This contradicts [**5**], which implies that the basis of the rationally connected fibration is not uniruled. $\square$

Thus we conclude that if Y is not rationally connected, the basis B of the maximal rationally connected fibration of Y is a surface Σ'. Furthermore the map $\phi' : Y \dashrightarrow \Sigma'$ is almost holomorphic. The surface Σ' is not uniruled by [**5**], and thus any (connected) rational curve (or rather genus 0 map) $f : \Gamma \to Y$ passing through a general point y of Y (where we may assume, because ϕ' is almost holomorphic that ϕ' is well-defined everywhere along the smooth connected curve $D' := \phi'^{-1}(\phi'(y)))$ must have image supported on D'. It follows that $[f_*\Gamma] = m[D']$, for some $m \geq 1$.

We apply this to our covering family of rational curves D (genus 0 stable maps) in Y in the class $[D] = \psi_*([C])$ and we conclude that $\psi_*[C] = m[D']$. Next we observe that $GW^Y_{0,[D']}([y]) = 1$, because the only rational curve of class $[D']$ passing through y is D, which is smooth with trivial normal bundle, so that there is fact exactly one genus 0 map f of class $[D']$ passing through y, and as $H^1(N_f(-y)) = 0$, this stable map is computed with multiplicity 1 in $GW^Y_{0,[D']}([y])$.

This implies that $m = 1$, because we find that

$$GW^X_{0,\frac{1}{m}[C]}([x]) \neq 0,$$

hence that X admits a covering by a family of rational curves of class $\frac{1}{m}[C]$, so that $m > 1$ would contradict the minimality of $\deg_\alpha(C)$. Hence we proved that

$$[D] = [D'],\ GW^Y_{0,[D']}([y]) = 1.$$

Finally, as ϕ' is well-defined along the generic fiber D', we conclude that $N_{D'/Y}$ is trivial, which implies by adjunction that $K_Y \cdot D' = K_Y \cdot [D] = -2$. Thus lemma 1.2 is proved. $\square$

Proof of proposition 1.1. — Notice that, as ψ is a symplectomorphism with respect to symplectic forms α, β of X, resp. Y, which are respective deformations of Kähler forms on X resp. Y, $\psi^* c_1(K_Y) = c_1(K_X)$. This is indeed a standard fact of symplectic geometry: the canonical class of a symplectic manifold X is an invariant of the deformation class of the symplectic form ω on X. Indeed it can be computed using any almost complex structure on X which is tamed by ω or a deformation of ω, the set of such almost complex structures being connected. This almost complex structure makes the tangent bundle into a complex vector bundle and the canonical class is minus the first Chern class of this complex vector bundle.

Furthermore, we have by assumption $[D] = \psi_*([C])$. Thus we have

$$c_1(K_X) \cdot [C] = c_1(K_Y) \cdot [D] = -2,$$

$$GW^X_{0,[C]}([x]) = GW^Y_{0,[D]}([y]) = 1.$$

The first equality together with the fact that the general curve passing through the point X is irreducible, and thus has globally generated normal bundle, implies that for general $x \in X$, the normal bundle of a curve C of class $[C]$ passing through x is trivial, which shows that there are finitely many such curves through x, and that the set of such curves has the expected dimension 0. Thus the number of these curves is equal to $GW^X_{0,[C]}([x])$ and this is equal to 1 by the second equality above. In conclusion we proved that if Σ_0 is the set parameterizing rational curves in X of class $[C]$ and Σ is the union of components of Σ_0 parameterizing moving curves, then the universal curve

$$q' : \mathcal{C} \to \Sigma,\ \Phi' : \mathcal{C} \to X,$$

has the property that Φ' has degree 1. Thus Φ' is birational, and

$$\phi := q' \circ \Phi'^{-1} : X \dashrightarrow \Sigma$$

gives the desired fibration into rational curves.

In order to conclude the proof, it just remains to prove that the rational map $\phi : X \dashrightarrow \Sigma$ is almost holomorphic. Assume this is not the case: let $\tau : X' \to X$ be a composition of blow-ups along smooth centers, such that $\tilde{\phi} := \phi \circ \tau$ is well-defined. Assume there is an exceptional divisor $E \subset X'$ which dominates Σ and is contracted to a curve Z (or a point) in X. Then if C' is the general fiber of $\tilde{\phi}$, C' meets E. On the other hand, $K_{X'} = \tau^* K_X + F$ where F is an effective divisor supported on the exceptional locus, and the multiplicity of E in F is > 0. Thus we find that

$$c_1(K_{X'}) \cdot [C'] = -2 = (\tau^* c_1(K_X) + F) \cdot [C'] > \tau^* c_1(K_X) \cdot [C'] = c_1(K_X) \cdot [C] = -2,$$

which is a contradiction. □

2. The case where X is an almost conic bundle

We now study almost conic bundles $\phi : X \dashrightarrow \Sigma$ with generic fiber C. When X is rationally connected, Σ is a rational surface, and thus we may assume to begin that $\Sigma = \mathbb{P}^2$. (Indeed, the fact that ϕ is almost a morphism does not depend on the birational model of the target.) Notice that, because ϕ is almost holomorphic, we have $H \cdot C = 0$, where the line bundle H on X is defined by

$$H := \phi^* \mathcal{O}_{\mathbb{P}^2}(1).$$

The first result is the following:

Proposition 2.1. — *Assume that either X is Fano, or $b_2(X) = 2$. Then H is numerically effective, unless we are not in the Fano case, and there exists a curve class $[C']$ not proportional to $[C]$ such that for some cohomology classes $A_1, \ldots, A_r \in H^*(X)$,*

$$GW^X_{0,[C']}([C], A_1, \ldots, A_r) \neq 0.$$

Proof. — Suppose first that X is Fano. Then any irreducible curve $Z \subset X$ satisfies $K_X \cdot Z < 0$, hence the Chow variety of its cycle is at least one dimensional because $\dim X = 3$ (cf [**8**], theorem 1.15). (This can also be formulated by evaluating the dimension of the space of deformations of the composed map $\widetilde{Z} \to Z \to X$, where $\widetilde{Z} \to Z$ is the normalization.) Thus, Z being irreducible, its cycle can be moved so as to be not contained in the indeterminacy locus of ϕ. Thus $\phi^* H \cdot Z \geq 0$.

Suppose now that $b_2(X) = 2$ but X is not Fano. We have to show that either H is numerically effective, or there exists a curve class $[C']$ not proportional to $[C]$ such that for some cohomology classes $A_1, \ldots, A_r \in H^*(X)$,

$$GW^X_{0,[C']}([C], A_1, \ldots, A_r) \neq 0.$$

As K_X is not nef, there exists a Mori contraction $c : X \to X'$, with $(Pic\, X') \otimes \mathbb{Q} = \mathbb{Q}$ and $-K_{X/X'}$ relatively ample. We consider the three possible dimensions of X' (cf [**16**]).

1) $\dim X' = 1$, that is $X' = \mathbb{P}^1$. In this case, the contraction is given by a pencil whose fibers are Del Pezzo surfaces. Let $L = c^* \mathcal{O}_{\mathbb{P}^1}(1)$. If $L \cdot C = 0$, then L is proportional to H (because $b_2(X) = 2$), and this contradicts the fact that the Iitaka dimension of H is at least 2. In the other case, we observe that the fibers of c are uniruled. Fix a polarization h on X and introduce the minimal degree with respect to h of rational curves contained in the fibers of c and sweeping-out X. Let $[C']$ be a class curve such that $L.[C'] = 0$ and achieving this minimal degree. All curves of class $[C']$ are supported on fibers of c. Exactly as in the proof of theorem 0.2, one then shows that for a covering family of rational curves C' of this minimal degree, the generic member is irreducible with semipositive normal bundle. Using now the fact

that C intersects non trivially the generic fiber of c, one concludes immediately that there is a non zero Gromov-Witten invariant

$$GW^X_{0,[C']}([C], A_1, \dots, A_r).$$

2) dim $X' = 2$. We have $c^* Pic\, X' = \mathbb{Z}L$, where L is ample on X', and if $C \cdot L = 0$ we conclude as before that L is proportional to H. In this case H is numerically effective. In the other case, the map $c : X \to X'$ has for generic fiber a rational curve C' with trivial normal bundle and satisfying $K_X \cdot C' = -2$. Furthermore there are only finitely many 2-dimensional fibers of c. If C is generic, there is thus exactly a 1-dimensional family of fibers C' meeting C, and this is exactly the expected dimension. It thus follows that there is a non trivial Gromov-Witten invariant

$$GW^X_{0,[C']}([C], A_1),$$

where $A_i = h^2 \in H^4(X, \mathbb{Z})$ for some ample class $h \in H^2(X, \mathbb{Z})$.

3) dim $X' = 3$. In this case c is a divisorial contraction. Note that C is not proportional to the contracted extremal ray, because C is a moving curve. A look at the list of divisorial contractions (cf [**15**]) shows the following (see [**18**]): Let E be the exceptional divisor of the contraction, so that E is either a ruled surface contracted to a smooth curve, or $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ contracted to a point. Let $[C']$ be the class of the fiber of the contracting ruling in the first case, or the class of a line in the second case, or the class of one of the two rulings in the third case. Then for any curve class γ such that $\gamma \cdot E \neq 0$, one has $GW^X_{0,[C']}(\gamma, A_1, \dots, A_r) \neq 0$, for an adequate number r, which will be in fact 0 or 1.

On the other hand $E \cdot C = 0$ is impossible, because in this case E and H would be proportional in $Pic\, X$, and E is contractible while the Iitaka dimension of H is at least 2. We deduce from this that one has $GW^X_{0,[C']}([C], A_1, \dots, A_r) \neq 0$, where $A_i = h^2 \in H^4(X, \mathbb{Z})$ for some ample class $h \in H^2(X, \mathbb{Z})$. □

From this, we get the following result:

Corollary 2.2. — *Assume that either X is Fano, or $b_2(X) = 2$. Then there exists a well-defined morphism $\phi : X \to \Sigma$ with fiber C, where Σ is a normal surface, unless we are not in the Fano case and there exists a curve class $[C']$ not proportional to $[C]$ such that for some cohomology classes $A_1, \dots, A_r \in H^*(X)$,*

$$GW^X_{0,[C']}([C], A_1, \dots, A_r) \neq 0.$$

Proof. — We use the contraction theorem (cf. [**7**], or [**15**], p 162) which tells that such a morphism exists if and only if H is numerically effective and the curves $Z \subset X$ satisfying $Z.H = 0$ also satisfy $Z \cdot K_X < 0$.

Indeed, by the previous theorem, we know that H is numerically effective, unless there exists a curve class $[C']$ not proportional to $[C]$ such that for some cohomology classes $A_1, \ldots, A_r \in H^*(X)$,

$$GW^X_{0,[C']}([C], A_1, \ldots, A_r) \neq 0.$$

Thus, in order to apply the contraction theorem, we just have to show that for any curve $Z \subset X$ satisfying the condition $Z \cdot H = 0$, one has $K_X \cdot Z < 0$.

In the Fano case, this is obvious. When $b_2(X) = 2$, the orthogonal of H in $H_2(X, \mathbb{Q})$ is generated by the class of C, which satisfies the condition $C \cdot K_X = -2$. □

We will use the following observation:

Lemma 2.3. — *Assuming ϕ is well defined and either $b_2(X) \leq 2$ or X is Fano, we may furthermore assume (by changing Σ if necessary) that ϕ does not contract a divisor to a point of Σ.*

Proof. — First of all, note that if $b_2(X) \leq 2$, ϕ cannot contract a divisor D to a point of Σ. Indeed, such a divisor would satisfy $D \cdot C = 0$, hence would be proportional to H. But the Iitaka dimension of H is 2, while no multiple of D moves, which is a contradiction.

Consider now the Fano case. Let x be a point of Σ, and let E be the pure 2-dimensional part of $\phi^{-1}(x)$, (counted with multiplicities). We claim that $-E$ is numerically effective on the fibers of ϕ and non trivial on $\phi^{-1}(x)$.

Assuming the claim, $H - \epsilon E$ remains numerically effective for a sufficiently small ϵ. On the other hand, curves Z satisfying $Z \cdot (H - \epsilon E) = 0$ satisfy the condition $K_X \cdot Z < 0$ for the same reasons as before, hence we can apply the contraction theorem to $H - \epsilon E$, which does not contract E anymore. This leads eventually to a morphism ϕ' which does not contract any divisor to a point.

To see the claim, we observe that $-E \cdot F = 0$ for any irreducible curve F contained in a fiber of ϕ but not contained in $\phi^{-1}(x)$. Furthermore $-E_{|E}$ is effective and non trivial on each component of E. This implies that $-E \cdot F \geq 0$ for any irreducible curve $F \subset E$ whose deformations cover a 2-dimensional component of $\phi^{-1}(x)$. Consider now any irreducible curve $F \subset X$ contained in $\phi^{-1}(x)$. As X is Fano of dimension 3, the cycle of any such F deforms to cover at least a divisor in X (cf [**8**], Theorem 1.15). On the other hand, all such deformations remain contained in a fiber of ϕ. It follows that either the cycle of F deforms to cover a 2-dimensional component of $\phi^{-1}(x)$, so that $-E \cdot F \geq 0$ as shown previously, or the cycle of F can be moved to be supported in another fiber, in which case we have $-E \cdot F = 0$. □

We consider now the case where ϕ is well defined (but Σ may be singular). Our main result is the following:

Theorem 2.4. — *Let X be a rationally connected 3-fold which admits a morphism $\phi : X \to \Sigma$ to a normal surface Σ, with generic fiber a rational curve C. Assume that either Σ is smooth, or ϕ does not contract a divisor to a point of Σ. Then there exist integers g, r with $g < r$, cohomology classes $A_1, \dots, A_N \in H^4(X, \mathbb{Z})$ and a homology class $[C'] \in H_2(X, \mathbb{Z})$ not proportional to $[C]$ such that*

$$GW^X_{g,[C']}(\underbrace{[C], \dots, [C]}_{r}, A_1, \dots, A_N) \neq 0.$$

Before giving the proof, let us establish a few lemmas.

Lemma 2.5. — *Σ contains a complete linear system of generically smooth curves Z of genus g, which do not meet generically the singular locus of Σ, and satisfy*

$$r = h^0(\Sigma, \mathcal{O}_\Sigma(Z)) - 1 = h^0(Z, \mathcal{O}_Z(Z)) > g. \tag{2.1}$$

Proof. — If Σ is smooth, Σ is rational and the result is obvious (we can even take $g = 0$). In general, we start from a "very moving" generic smooth rational curve $\Gamma_0 \subset X$. Recall that "very moving" means that the normal bundle $N_{\Gamma_0/X}$ is ample. Using the assumption that no divisor is contracted to a point by ϕ or that Σ is smooth, one concludes that for Γ_0 generic, $\phi(\Gamma_0) =: \Gamma'_0$ avoids the singular locus of Σ.

Let $\mathcal{L} := \mathcal{O}_\Sigma(\Gamma'_0)$. Observe that $H^1(\Sigma, \mathcal{O}_\Sigma) = 0$, because Σ admits a desingularization which is rationally connected. It follows that the restriction map:

$$H^0(\Sigma, \mathcal{L}) \to H^0(\Gamma'_0, N_{\Gamma'_0/\Sigma})$$

is surjective. Observe now that because the equisingular deformations of Γ'_0 in Σ (which are singular rational curves) cover Σ, one has $K_\Sigma \cdot \Gamma'_0 < 0$.

In fact we may even assume $K_\Sigma \cdot \Gamma'_0 < -1$, replacing if necessary Γ_0 by a ramified cover of it, which by ampleness of the normal bundle can be deformed to an embedding.

It thus follows that

$$\deg N_{\Gamma'_0/\Sigma} = \deg K_{\Gamma'_0} \otimes K_\Sigma^{-1}{}_{|\Gamma'_0} \geq \deg K_{\Gamma'_0} + 2.$$

This inequality implies that the linear system $H^0(\Gamma'_0, N_{\Gamma'_0/\Sigma})$ has no base-point on Γ'_0 so that a generic deformation Z of Γ'_0 is smooth. Letting g be the arithmetic genus of Γ'_0, that is the genus of a generic deformation Z of Γ'_0 in Σ, we now find that Z satisfies the desired property

$$r = h^0(\Sigma, \mathcal{O}_\Sigma(Z)) - 1 = h^0(Z, \mathcal{O}_Z(Z)) > g = h^0(Z, K_Z),$$

because $\deg \mathcal{O}_Z(Z) \geq \deg K_Z + 2$ by adjunction and because $K_\Sigma \cdot Z < -1$. □

Remark 2.6. — The inequality $\deg \mathcal{O}_Z(Z) \geq \deg K_Z + 2$ also implies that $h^1(Z, \mathcal{O}_Z(Z)) = 0$, a fact which will be used later on.

Let $x_1, \dots, x_r$ be r generic points of Σ. Then there is a unique curve $Z \subset \Sigma$ belonging to the linear system $\mid \mathcal{L} \mid$ and passing through $x_1, \dots, x_r$. This curve is smooth and by Bertini the surface $X_Z := \phi^{-1}(Z)$ is smooth. Choose now a section $\Gamma \subset X_Z$ of the morphism $\phi_Z := \phi_{|X_Z} : X_Z \to Z$. Let $C_i := \phi^{-1}(x_i)$. Let us prove now the following:

Lemma 2.7. — *$\mathcal{L}$, $x_1, \dots, x_r$, Γ being as above, for any $k > 0$, any stable map $f : \Gamma_1 \to X$ of class*

$$[\Gamma] + k[C]$$

meeting the r generic fibers $C_1, \dots, C_r$ of ϕ has the property that $\phi \circ f(\Gamma_1) = Z$.

Proof. — This is almost obvious. We just have to be a little careful with the singularities of Σ. Let us thus introduce a desingularization $\tau : \Sigma' \to \Sigma$ of Σ. Let $\mathcal{L}' := \tau^*\mathcal{L}$ and $\tilde{x}_1, \dots, \tilde{x}_r$ the points of Σ' over the generic points $x_1, \dots, x_r$ of Σ.

Then if $f : \Gamma_1 \to X$ is a curve as above, denote by $\widetilde{\Gamma}'_1 \subset \Sigma'$ the proper transform of $\Gamma'_1 := \phi \circ f(\Gamma_1) \subset \Sigma$ (counted with multiplicities) in Σ'. We observe that because the class of $f(\Gamma_1)$ is $[\Gamma] + k[C]$ and $\phi(C)$ is a point, $\widetilde{\Gamma}'_1$ belongs to one of the linear systems

$$\mid \tau^*\mathcal{L} - E \mid$$

on Σ', where E is an effective divisor supported on the exceptional locus of the desingularization. The linear system above has dimension $\leq r$, with equality if and only if E is empty. As $\widetilde{\Gamma}'_1$ passes through r generic points of Σ', it follows that the linear system $\mid \tau^*\mathcal{L} - E \mid$ has dimension r. Thus E is empty, and the curve Γ'_1 does not meet the singular locus of Σ. Hence $\Gamma'_1 \in \mid \mathcal{L} \mid$, and as it passes through $x_1, \dots, x_r$, it must be equal to Z. $\square$

Consider the morphism $\phi_Z : X_Z \to Z$. The smooth fiber of ϕ_Z is a $\mathbb{P}^1$, and the singular fibers are chains of $\mathbb{P}^1$'s. Note that by successive contractions of -1-curves not meeting Γ, one can construct from F_Z a geometrically ruled surface X_Z^0. The curve Γ is then the inverse image of a curve (still denoted Γ) in X_Z^0. Γ is a section of the structural morphism $p : X_Z^0 = \mathbb{P}(\mathcal{E}) \to Z$, where $\mathcal{E} := p_*\mathcal{O}_{X_Z^0}(\Gamma)$ is a rank 2 vector bundle on Z. We shall denote by $\sigma : X_Z \to X_Z^0$ such a contraction morphism. It will be convenient to choose the following basis E_i of the lattice

$$H^2(X_Z, \mathbb{Z})/\sigma^* H^2(X_Z^0, \mathbb{Z}) = \sigma^* H^2(X_Z^0, \mathbb{Z})^{\perp}.$$

We factor $\sigma : X_Z \to X_Z^0$ as a sequence of m blow-ups at one point. Let $\sigma_i : X_Z \to X_Z^i$ be the successive surfaces appearing in this factorization. Then we define for $i \geq 1$, $[E_i] := \sigma_i^*[E]$, where E is the exceptional curve of the blow-up $X_Z^i \to X_Z^{i-1}$. The classes $[E_i]$ are effective, and they satisfy

$$[E_i]^2 = -1,\ [E_i] \cdot K_{X_Z} = -1.$$

Proof of theorem 2.4. — We will denote by $j : X_Z \hookrightarrow X$ the inclusion. For a curve Γ contained in X_Z, we will denote by $[\Gamma]_{X_Z} \in H^2(X_Z,\mathbb{Z})$ its cohomology class in X_Z and $[\Gamma] \in H^4(X,\mathbb{Z})$ its cohomology class in X. Hence $[\Gamma] = j_*[\Gamma]_{X_Z}$.

Let g, r, $x_1,\dots,x_r$, Z, $\Gamma \subset X_Z$ be as in lemmas 2.5, 2.7. Let $C_1,\dots,C_r$ be the generic fibers $\phi^{-1}(x_i)$ of ϕ. We now consider curves (stable maps) of genus g and class $[\Gamma]+k[C]$ in X, where k will be chosen sufficiently large.

The expected dimension of the family of such curves is equal to

$$\begin{aligned}-K_X\cdot([\Gamma]+k[C]) &= 2k - K_X\cdot[\Gamma] = 2k+\chi(\Gamma,N_{\Gamma/X})\\ &= 2k+\chi(\Gamma,N_{\Gamma/X_Z})+\chi(Z,N_{Z/\Sigma}) = 2k+\chi(\Gamma,N_{\Gamma/X_Z^0})+r\\ &= 2k+r+\chi(Z,\mathcal{E})+g-1 = 2k+r+\deg\mathcal{E}+1-g.\end{aligned}$$

If we consider the family of such curves meeting $C_1,\dots,C_r$, its expected dimension is $N := 2k+\deg\mathcal{E}+1-g$, and by lemma 2.7, we know that these curves are all contained in a given surface X_Z, where Z is a generic member of the linear system $\mid\mathcal{L}\mid$ on Σ. Note that N is the expected dimension of the space of deformations of a smooth curve of class $[\Gamma]+k[C]$ in X_Z. If k satisfies the condition $\Gamma^2+2k>2g$, choose a section Γ_k of $X_Z\to Z$ of class $[\Gamma]+k[C]$ in X_Z.

Then as N_{Γ_k/X_Z} has degree $>2g-2$, it satisfies

$$H^1(\Gamma_k,N_{\Gamma_k/X_Z})=0.$$

As furthermore $H^1(\Gamma_k,(N_{X_Z/X})_{|\Gamma_k}) = H^1(N_{Z/\Sigma})=0$ by remark 2.6, one concludes that $H^1(\Gamma_k,N_{\Gamma_k/X})=0$, so that the deformation space of Γ_k in X is locally smooth of the right dimension $N+r$. Furthermore, if $y_1,\dots,y_N\in\Gamma_k$ are generic, and $D:=\{y_1,\dots,y_N\}$, the restriction map:

$$H^0(\Gamma_k,N_{\Gamma_k/X_Z})\to H^0(D,(N_{\Gamma_k/X_Z})_{|D})$$

is an isomorphism. Choosing N curves $B_1,\dots,B_N\subset X$ meeting X_Z in $y_1,\dots,y_N$ respectively, we find that Γ_k is an isolated point in the family of curves of genus g meeting $C_1,\dots,C_r$ and $B_1,\dots,B_N$. This gives at least one positive contribution to $GW^X_{g,[\Gamma_k]}(\underbrace{[C],\dots,[C]}_{r},[B_1],\dots,[B_N])$.

However, in order to compute the Gromov-Witten invariant above, we need to control all curves in X_Z whose class in X is equal to $[\Gamma_k]=j_*[\Gamma_k]_{X_Z}$.

From lemma 2.7, we know that any curve in X of class $[\Gamma_k]$ which meets $C_1,\dots,C_r$ is contained in X_Z. In order to conclude the proof, we thus have to compute the contribution to $GW^X_{g,[\Gamma_k]}(\underbrace{[C],\dots,[C]}_{r},[B_1],\dots,[B_N])$ of all the families of curves $f:\Gamma_1\to X_Z$, where Γ_1 is (maybe nodal) of arithmetic genus g, such that the class in X of $f(\Gamma_1)$ (counted with multiplicities) is equal to $[\Gamma_k]$, with k large.

For this, we need the following lemma

Lemma 2.8. — *Classes in the kernel of $j_* : H_2(X_Z, \mathbb{Z}) \to H_2(X, \mathbb{Z})$ are integral combinations of the classes $\frac{1}{2}[C] - [E_i]$.*

Proof. — $H_2(X_Z, \mathbb{Z})$ is generated over $\mathbb{Z}$ by the classes $[C]$ of the fiber of ϕ_Z, the class $[\Gamma]$ of a section of ϕ_Z and the classes $[E_i]$.

If $\alpha \in \operatorname{Ker} j_*$, write

$$\alpha = n[C] + m[\Gamma] + \sum_i n_i \left(\frac{1}{2}[C] - [E_i]\right),\ n,\ m,\ n_i \in \mathbb{Z}.$$

Then we must have $m = 0$ because $\phi_*(j_*\alpha) = 0 = m[Z]$. Next we have $K_X \cdot [E_i] = -1$, because $K_{X_Z} \cdot [E_i] = -1$ and K_X has the same restriction as K_{X_Z} on the fibres of ϕ_Z. Furthermore $K_X \cdot [C] = -2 \neq 0$, and $K_X \cdot (\frac{1}{2}[C] - [E_i]) = 0$. Thus

$$j_*\alpha = 0 \Rightarrow K_X \cdot \alpha = 0 \Rightarrow n = 0.$$

Hence we proved that α is a combination of the $\frac{1}{2}[C] - [E_i]$ with integral coefficients n_i. Note that if such a combination belongs to $H_2(X_Z, \mathbb{Z})$, the $n_i \in \mathbb{Z}$ satisfy the condition that 2 divides $\sum_i n_i$. □

We need thus to study maps $f : \Gamma_1 \to X_Z$ where Γ_1 is a nodal curve of genus g, $f_*[\Gamma_1]_{fund} = \gamma := [\Gamma_k] + \sum_i n_i(\frac{1}{2}[C] - [E_i])$. Note that for each such map, $\phi_Z \circ f : \Gamma_1 \to Z$ is an isomorphism on the (unique) genus g component of Γ_1 and contracts all the other components of Γ_1, which must be rational. As deg $N_{Z/\Sigma} > 2g-2$, it follows that $H^1(\Gamma_1, f^*N_{Z/\Sigma}) = 0$, and as an easy consequence, for fixed γ, the contribution of this family to $GW^X_{g,[\Gamma_k]}(\underbrace{[C], \ldots, [C]}_{r}, [B_1], \ldots, [B_N])$ is equal to

$$GW^{X_Z}_{g,\gamma}([B_1]_{|X_Z}, \ldots, [B_N]_{|X_Z}).$$

Of course $[B_i]_{|X_Z}$ is a multiple of the class of a point of X_Z. It thus remains to prove that for k large enough and any $\gamma = [\Gamma_k] + \sum_i n_i(\frac{1}{2}[C] - [E_i])$,

$$GW^{X_Z}_{g,\gamma}(\underbrace{[pt], \ldots, [pt]}_{N}) \geq 0.$$

Note that by deforming X_Z, we may assume the successive blow-ups starting from X_Z^0 are at m distinct points $z_1, \ldots, z_m \in X_Z^0 = \mathbb{P}(\mathcal{E})$.

We have the following:

Lemma 2.9. — *m being fixed, for k sufficiently large, for a fixed choice of distinct points $z_1, \ldots, z_m \in \mathbb{P}(\mathcal{E})$, for any choice of integers $n_1, \ldots, n_m \in \mathbb{Z}$, any linear system L on the surface X'_Z which is $\mathbb{P}(\mathcal{E})$ blown-up at $z_1, \ldots, z_m$, of class*

$$c_1(L) = \gamma = [\Gamma_k] + l[C] - \sum_i n_i[E_i],\ l = \frac{1}{2}\sum_i n_i$$

satisfies $h^0(X'_Z, L) \leq N+1-g$, *and when equality holds, the generic member of this linear system is smooth.*

Assuming this lemma, it follows that for each γ as above, the dimension of the space of divisors in X'_Z of class γ has dimension $\leq N$. Thus the dimension of the space of divisors of class γ passing through N generically chosen points is 0. Furthermore, when equality holds, the finitely many divisors of class γ passing through N generically chosen points are smooth. It follows that the stable maps $f : \Gamma_1 \to X'_Z$ of class $f_*[\Gamma_1]_{fund} = \gamma$ passing through N generically chosen points have finitely many possible images which are smooth curves of genus g. Thus each of these f's must be an isomorphism, and there are also finitely many such stable maps f. It follows that $GW^{X'_Z}_{g,\gamma}(\underbrace{[pt], \dots, [pt]}_{N}) \geq 0$. The proof of Theorem 2.4 is thus finished, modulo the proof of lemma 2.9. □

Proof of lemma 2.9. — Note that if $n_i \leq 0$, n_iE_i is contained in the fixed part of $\mid L \mid$. Thus it suffices to prove the result assuming $n_i \geq 0$, and $l \leq \frac{1}{2}\sum_i n_i$. Next, note that because $\gamma \cdot [C] = 1$, any section of L vanishing to order n_i at z_i vanishes to order $n_i - 1$ along the fiber C_{z_i} passing through z_i. This way, we are now reduced to the case where $n_i = 0$ or $n_i = 1$, and $l \leq \frac{1}{2}\sum_i n_i$. Notice that, in both reduction steps, if either one of the $n_i < 0$ or $n_i \geq 2$, the inequality becomes a strict inequality.

We have thus to show that for k large enough, for any choice of s points $z_{i_1}, \dots, z_{i_s}$ among $z_1, \dots, z_m$, for $L \in Pic\, X'_Z$, with

$$c_1(L) = [\Gamma] + k[C] + l[C] - \sum_{j \leq s}[E_{i_j}],\ l \leq \frac{s}{2},$$

we have $h^0(X'_Z, L) \leq N+1-g$, while for $l < \frac{s}{2}$, we have $h^0(X'_Z, L) < N+1-g$. Note that for $l = 0$, $s = 0$, we can take for L the line bundle $\mathcal{O}_{X_Z}(\Gamma_k)$ which has $N+1-g$ sections.

The points $z_{i_j} \in \mathbb{P}(\mathcal{E})$ determine a vector bundle $\mathcal{E}'$ on Z, defined as the kernel of the evaluation map $p_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{E} \to \oplus \mathcal{O}(1)_{|z_{i_j}}$. Then sections of L on X'_Z identify via p_* to sections of $\mathcal{E}'(D)$ on Z, for some $D \in Pic^{k+l}(Z)$. There are finitely many bundles $\mathcal{E}'$, and thus for k large enough, and any $l \geq 0$, deg $D = k + l$, we have $H^1(Z, \mathcal{E}'(D)) = 0$. As deg $\mathcal{E}'$ = deg $\mathcal{E} - s$, it follows that

$$h^0(Z, \mathcal{E}'(D)) = \chi(Z, \mathcal{E}'(D)) = \deg \mathcal{E}'(D) + 2 - 2g$$

$$= \deg \mathcal{E} - s + 2k + 2l + 2 - 2g \leq \deg \mathcal{E} + 2 - 2g + 2k = h^0(X_Z, \Gamma_k) = N + 1 - g,$$

with equality only when $2l = s$.

When equality holds, we have seen that all the n_i must be equal to 0 or 1, and the fact that the generic curve of class γ is smooth is deduced from the fact that with the notation above, the bundle $\mathcal{E}'(D)$ is generated by sections, for $D \in Pic^{k+l}(Z)$. □

3. Proofs of the main results

Proof of Proposition 0.7. — Here $\psi : X \cong Y$ is a symplectomorphism with respect to some symplectic forms α, β on X, resp. Y, where α tames the complex structure on X and β is a deformation (as a symplectic form) of a Kähler form on Y. We assume that the conclusion of proposition 0.6 holds, but furthermore the rational map $\phi : X \dashrightarrow \Sigma$ is well-defined, and that either ϕ does not contract a divisor, or Σ is smooth. We can thus apply the conclusion of Theorem 2.4. This tells us that there exist integers $g < r$, cohomology classes $A_1, \dots, A_N \in H^4(X, \mathbb{Z})$ and a homology class $[C'] \in H_2(X, \mathbb{Z})$ not proportional to $[C]$ such that

$$GW^X_{g,[C']}(\underbrace{[C], \dots, [C]}_{r}, A_1, \dots, A_N) \neq 0.$$

It follows that the curve class $[D'] = \psi_*[C']$ and the cohomology classes $A_i := \psi_* A_i \in H^*(Y)$ satisfy:

$$GW^Y_{g,[D']}(\underbrace{[D], \dots, [D]}_{r}, A'_1, \dots, A'_N) \neq 0.$$

But then this means that there exist a curve D' of genus g in Y, of class not proportional to $[D]$, meeting r generic fibers $D_1, \dots, D_r$ of ϕ'. This implies that the surface Σ' contains genus g curves $D'' := \phi'(D')$ passing through r generic points, with $r > g$. In fact we will rather consider in the following lemma these curves as stable maps from a nodal curve to Σ. The normal bundle should be thought as $N_{\phi'}$.

Lemma 3.1. — *If Σ' satisfies this property, the Kodaira dimension of Σ' is $-\infty$.*

Proof. — Indeed, the generic curve D'' above has genus g and satisfies

$$h^0(N_{D''/\Sigma'}/\mathrm{Tors}) \geq r > g,$$

where Tors is the torsion of $N_{D''/\Sigma'}$. It follows that D'' contains at least one moving irreducible component D''_0 which has genus g_0, and satisfies

$$h^0(D''_0, N_{D''_0/\Sigma'}/\mathrm{Tors}) > g_0.$$

We claim that this implies $\deg(N_{D''_0/\Sigma'}/\mathrm{Tors}) > 2g_0 - 2$. Assuming the claim, it follows that $\deg(N_{D''_0/\Sigma'}) > 2g_0 - 2$, hence by adjunction that $K_{\Sigma'} \cdot D''_0 < 0$. This implies that $h^0(\Sigma', K^{\otimes l}_{\Sigma'}{}_{|D''_0}) = 0, \forall l > 0$, and as D''_0 is moving, this implies that $h^0(\Sigma', K^{\otimes l}_{\Sigma'}) = 0, \forall l > 0$.

To see the claim, observe that Riemann-Roch gives

$$\chi(D''_0, N_{D''_0/\Sigma'}/\mathrm{Tors}) = \deg(N_{D''_0/\Sigma'}/\mathrm{Tors}) + 1 - g_0.$$

Thus, if $\deg(N_{D_0''/\Sigma'}/\mathrm{Tors}) \leq 2g_0 - 2$ and $h^0(D_0'', N_{D_0''/\Sigma'}/\mathrm{Tors}) > g_0$, we find that $h^1(D_0'', N_{D_0''/\Sigma'}/\mathrm{Tors}) \neq 0$. Thus by Serre duality,

$$h^0(D_0'', (N_{D_0''/\Sigma'}/\mathrm{Tors})^* \otimes K_{D_0''}) \neq 0.$$

But then this implies, because D_0'' is irreducible, that

$$h^0(D_0'', N_{D_0''/\Sigma'}/\mathrm{Tors}) \leq h^0(D_0'', K_{D_0''}) = g_0,$$

which is a contradiction. □

Thus we conclude in this case that Σ' is (birationally) a ruled surface, and it follows that the basis of the rationally connected fibration of Y has dimension ≤ 1. By lemma 1.3, Y is rationally connected. □

Proof of theorem 0.8. — We assume that X and Y are symplectically equivalent and that, either X is Fano, or X is rationally connected with $b_2(X) \leq 2$. Thus there is a symplectomorphism $\psi : X \cong Y$ between X endowed with a Kähler form α and Y endowed with a symplectic form β which is a deformation of Kähler form.

We want to show that Y is rationally connected. We argue by contradiction, and assume that Y is not rationally connected. Applying lemma 1.2, we find that there are curve classes $[C]$, $[D]$ on X resp. Y, satisfying the following properties:

1. $c_1(K_Y) \cdot [D] = -2 = c_1(K_X) \cdot [C]$.
2. $GW^Y_{0,[D]}([y]) = 1 = GW^X_{0,[C]}([x])$.
3. The class $[C]$ is of minimal degree with respect to α, among those class curves satisfying the property $GW^X_{0,[C]}([x]) \neq 0$.

Furthermore, as proved in proposition 1.1, the manifolds X and Y are in this case almost conic bundles with fiber D, resp. C of class $[D]$, resp. $[C]$ where $[D] = \psi_*[C]$. Let us denote by $\phi : X \dashrightarrow \Sigma$, and $\phi' : Y \dashrightarrow \Sigma'$ the almost conic bundle structures on X and Y respectively.

Our assumption is that $b_2(X) \leq 2$ or X is Fano. Hence we can apply to X the corollary 2.2, because X is an almost conic bundle with fiber C. Thus we conclude, with the notations of this section, that the morphism $\phi : X \dashrightarrow \Sigma$ with fiber C is well-defined, unless there exists a curve class $[C']$ not proportional to $[C]$ such that for some cohomology classes $A_1, \ldots, A_r \in H^*(X'_{n-1})$,

$$GW^X_{0,[C']}([C], A_1, \ldots, A_r) \neq 0.$$

However, in the later case, we conclude, by denoting $[D'] = \psi_*[C']$, $A_i' = \psi_* A_i$, that

$$GW^Y_{0,[D']}([D], A_1', \ldots, A_r') \neq 0.$$

It follows that there exists a rational curve of class $[D']$ which meets a generic curve $D \subset Y$ and as $[D']$ is not proportional to D, we conclude that $\phi'(D')$ is not a point. Hence it follows that the surface Σ' is swept-out by rational curves and the basis of the rationally connected fibration of Y has dimension ≤ 1, which implies by lemma 1.3 that Y is rationally connected, a contradiction.

Thus the morphism $\phi : X \to \Sigma$ with fiber C is well-defined. Furthermore, by lemma 2.3, we may assume that ϕ does not contract a divisor to a point. By proposition 0.7, Y is then rationally connected, which is a contradiction. □

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C. Voisin, IHÉS, 35 route de Chartres, 91440 Bures-sur-Yvette, France

Sun-Yung Alice Chang
Paul C. Yang

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THE Q-CURVATURE EQUATION IN CONFORMAL GEOMETRY

by

Sun-Yung Alice Chang & Paul C. Yang

Dedicated to J.P. Bourguignon on his 60th birthday

Abstract. — In this paper we survey some analytic results concerned with the top order Q-curvature equation in conformal geometry. Q-curvature is the natural generalization of the Gauss curvature to even dimensional manifolds. Its close relation to the Pfaffian, the integrand in the Gauss-Bonnet formula, provides a direct relation between curvature and topology.

Résumé **(L'équation de Q-courbure en géométrie conforme).** — Dans cet article nous examinons certains résultats analytiques autour de l'équation de Q-courbure d'ordre maximal en géométrie conforme. La Q-courbure est la généralisation naturelle de la courbure de Gauss aux variétés de dimension paire. Sa proximité avec le pfaffien (l'intégrande de la formule de Gauss-Bonnet) nous fournit une relation directe entre géométrie et topologie.

1. Introduction

Recently, there is a lot of interest in the study of higher order Q-curvature invariant. This notion arises naturally in conformal geometry in the context of conformally covariant operators. Paneitz ([**23**], see also [**6**]) gave the first construction of the fourth order conformally covariant Paneitz operator in the context of Lorentzian geometry in dimension four. Based on the ambient metric construction introduced by Fefferman and Graham ([**14**],[**15**]), Graham-Jenne-Mason and Sparling [**18**] systematically constructed conformally covariant operators of higher orders. Each such operator gives rise to a semi-linear elliptic equation analogous to the Yamabe equations which

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we shall call the Q-curvature equation. These equations share a number of common features. Among these we mention the following:

(i) the lack of compactness: the nonlinearity always occur at the critical exponent, for which the Sobolev embedding is not compact;

(ii) the lack of maximum principle: for example, it is not known whether the solution of the fourth order Q-curvature equation on manifolds of dimensions greater than four may touch zero.

In spite of these difficulty, there has been significant progress on questions of existence, regularity and classification of entire solutions for these equations in the recent work of Djadli-Malchiodi [**13**], Adimurthi-Robert-Struwe [**1**] and X.Xu [**25**]. On the other hand, in the case when the dimension is even $n = 2k$, the Branson-Paneitz operator and its associated Q-curvature equation is more accessible. In this article, we will give a brief survey of two results for the Q-curvature equation, each of which makes use of its close relation to the Pfaffian; both of these results are joint works with Jie Qing. The first [**10**] is a generalization of the Cohn-Vossen-Huber inequality ([**22**]) to complete conformal metrics on domains in $\mathbb{R}^4$. The second gives a Gauss-Bonnet type formula for Poincaré-Einstein metrics in which the renormalized volume plays a role. As the original article [**12**] of the second result appeared in Russian, we provide an exposition with some details. In section two, we review the notion of conformally covariant equations, their associated Q-curvatures and the associated boundary operators for manifold with boundary. We then provide an outline for these two results in sections three to five.

2. Conformally covariant operators and the Q-curvature equation

In general, we call a metrically defined operator A defind on a Riemannian manifold (M^n, g) conformally covariant of bidegree (a, b), if under the conformal change of metric $g_w = e^{2w}g$, the pair of corresponding operators A_w and A are related by

$$A_w(\varphi) = e^{-bw}A(e^{aw}\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^n).$$

A basic example is the conformal Laplacian $L \equiv -\Delta + \frac{n-2}{4(n-1)}R$ where R is the scalar curvature of the metric. The conformal Laplacian is conformally covariant of bidegree $(\frac{n-2}{2}, \frac{n+2}{2})$, and the associated curvature equation is the equation for prescribing scalar curvature: writing $e^w = u^{\frac{2}{n-2}}$ we have

$$Lu = \frac{n-2}{4(n-1)} R_u u^{\frac{n+2}{n-2}}, \tag{1}$$

where R_u is the scalar curvature of the metric $g_w = g^{2w}g = u^{\frac{4}{n-2}}g$. In case of surfaces, the corresponding Q-curvature equation becomes the equation for prescribing Gauss curvature:

$$(2) \qquad -\Delta w + K = K_w e^{2w},$$

where K_w is the Gaussian curvature for the metric g_w, and we have the Gauss-Bonnet formula:

$$(3) \qquad 2\pi\chi(M) = \int_M K dA.$$

In dimension four, S. Paneitz found the fourth order conformally covariant operator:

$$(4) \qquad P_4\varphi = P\varphi \equiv \Delta^2\varphi + \delta\left[\left(\frac{2}{3}Rg - 2\mathrm{Ric}\right)d\varphi\right]$$

where δ denotes the divergence, d the deRham differential and Ric the Ricci tensor.

For example:

- On $(R^4, |dx|^2)$, $P = \Delta^2$,
- On (S^4, g_c), $P = \Delta^2 - 2\Delta$,
- On (M^4, g), g Einstein, $P = (-\Delta) \circ (L)$.

The Paneitz operator P has bidegree $(0,4)$ on 4-manifolds, i.e.

$$(5) \qquad P_{g_w}(\phi) = e^{-4\omega}P_g(\phi) \ \ \forall\phi \in \mathcal{C}^\infty(M^4).$$

The fourth order Q-curvature is given by

$$(6) \qquad Q \; = \; \frac{1}{6}(-\Delta R + R^2 - 3|\mathrm{Ric}|^2).$$

Under the conformal change of metric $g_w \; = \; e^{2w}g$, the Q-curvature equation (see [**6**], also [**8**]) takes the form

$$(7) \qquad Pw + Q \; = \; Q_w e^{4w},$$

where Q_w is the Q curvature for the metric g_w.

The Gauss-Bonnet formula in dimension four may be written as

$$(8) \qquad 8\pi^2\chi(M) = \int_M (|W|^2 + Q)dV,$$

where W is the Weyl tensor. Since $|W_g|_g = e^{-2w}|W_{g_w}|_{g_w}$, on manifold of dimension four, $|W|^2\, dV$ is a pointwise conformal invariant, thus it follows from the Gauss-Bonnet formula that the Q-curvature integral is a global conformal invariant.

For 4-manifold X^4 with boundary M^3 and a Riemannian metric g defined on closure of X^4, Chang-Qing [**9**] derived the matching boundary operator

$$(9) \ P_3 = -\frac{1}{2}\frac{\partial}{\partial n}\Delta - \tilde{\Delta}\frac{\partial}{\partial n} - \frac{2}{3}H\tilde{\Delta} + L_{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta + \left(\frac{1}{3}R - R_{\alpha N\alpha N}\right)\frac{\partial}{\partial n} + \frac{1}{3}\tilde{\nabla}H\cdot\tilde{\nabla}.$$

with the associated third order curvature invariant

$$(10) \qquad T = \frac{1}{12}\frac{\partial}{\partial n}R + \frac{1}{6}RH - R_{\alpha N\beta N}L_{\alpha\beta} + \frac{1}{9}H^3 - \frac{1}{3}TrL^3 - \frac{1}{3}\tilde{\Delta}H,$$

where where $\frac{\partial}{\partial n}$ is the outer normal derivative, $\tilde{\Delta}$ is the trace of the Hessian of the metric on the boundary, $\tilde{\nabla}$ is the derivative in the boundary, L is the second fundamental form of boundary, $H = \mathrm{Tr}L$, N denotes the inner normal direction. We have used an orthonormal frame and let the latin indices run through the ambient indices and the Greek indices only run through the boundary directions, and all curvature are taken with respect to the metric g.

In particular, via the conformal change of metrics $g_w = e^{2w}g$, P_3 and T satisfy the equation:

$$P_3 w + T = T_w e^{3w} \quad \text{on} \quad M, \tag{11}$$

and

$$(P_3)_w = e^{-3w} P_3 \quad on \quad M. \tag{12}$$

The Chern-Gauss-Bonnet formula for 4-manifolds with boundary is then modified with a boundary term:

$$8\pi^2\chi(X) = \int_X (|W|^2 + Q)dv + 2\oint_M (T - \mathcal{L}_4 - \mathcal{L}_5)d\sigma. \tag{13}$$

In the boundary integral above the invariants $\mathcal{L}_4$ and $\mathcal{L}_5$ involve the ambient curvature tensor and the second fundamental form L_{ab}, and their expressions are

$$\mathcal{L}_4 = -\frac{RH}{3} + R_{\alpha N\alpha N}H - R_{\alpha N\beta N}L_{\alpha\beta} + R_{\gamma\alpha\gamma\beta}L_{\alpha\beta},$$

and

$$\mathcal{L}_5 = -\frac{2}{9}L_{\alpha\alpha}L_{\beta\beta}L_{\gamma\gamma} + L_{\alpha\alpha}L_{\beta\gamma}L_{\beta\gamma} - L_{\alpha\beta}L_{\beta\gamma}L_{\gamma\alpha}.$$

Analogous to the Weyl term, $\mathcal{L}_4$ and $\mathcal{L}_5$ are boundary invariant of order three which are pointwise invariant under conformal change of metrics. Hence

$$\int_X Qdv \; + \; 2\oint_M Td\sigma \tag{14}$$

is a global conformal invariant.

In dimension four, an important result is the following criteria for positivity of the Paneitz operator due to Gursky-Viaclovsky:

Theorem 1 **([21]).** — *Let (M^4, g) be a metric with positive Yamabe constant $Y(M,g) = inf_{u\neq 0}\frac{\int Lu\cdot u}{||u||_4^2}$, and satisfying*

$$\int_M Qdv + \frac{1}{6}(Y(M,g))^2 \geq 0,$$

then the Paneitz operator is positive except for constants.

It is an open question whether there is an analogous result in higher dimensions.

3. A Gauss-Bonnet formula for noncompact 4-manifolds

On a four dimensional manifold, the conformal Laplacian and the Paneitz operator together give strong control of the geometry and topology. A particular example in the study of non-compact manifolds is the following:

***Theorem 2** ([10]).* — *Let $(\Omega \subset S^4, g = e^{2w} g_0)$ be a complete conformal metric satisfying*

(a) *The scalar curvature is bounded between two positive constants, and $|\nabla_g R|$ is bounded,*
(b) *The Ricci curvature of the metric g has a lower bound,*
(c) *the Paneitz/Branson curvature is absolutely integrable, i.e.*

$$\int_\Omega |Q_g| dv_g < \infty; \tag{15}$$

then $\Omega = S^4 \setminus \{p_1, ..., p_k\}$ and

$$8\pi^2 \chi(\Omega) = \int_\Omega Q_g dv_g + \sum_1^k I_k, \tag{16}$$

where I_k is the local isoperimetric constant

$$I_k = \lim_{r \to 0} \frac{\text{Area}(\{r = |x - x_k|\})}{\text{Vol}(\{r < |x - x_k| < r_0\})}.$$

An essential idea in the above finiteness result is to view the Q-curvature integral as measuring the growth of volume. The finiteness of the Q integral implies a control on the growth of volume, which can only accommodate the growth of a finite number of puncture ends. We outline the main arguments to show how to use the fourth order curvature equation in such a situation.

Let us denote $\Lambda = S^4 \setminus \Omega$.

Step I. — $e^{w(x)} \approx \text{dist}(x, \Lambda)^{-1}$.

This is the main analytic work.

The lower bound follows from a Harnack estimate for the gradient of a conformal harmonic function.

The upper bound is based on a delicate blowup argument. Assuming on the contrary that on a sequence of points $\{x_k\}$ we have $a_k = e^{w(x_k)} d(x_k, \Lambda) \to \infty$. Take a subsequence so that the balls $B(x_k, (1/2) d(x_k, \Lambda))$ are disjoint. A careful rescaling of the domain for the conformal metrics over suitably dilated balls will converge to a conformal metric on $\mathbb{R}^4$ with vanishing Q-curvature but having scalar curvature bounded from below by a positive constant. Such a metric cannot exist. This argument differs from the usual blowup argument in that the conformal factor only satisfies a differential inequality.

This assertion gives control of the level set of the function e^w in terms of the distance to the complement.

Step II. — An integration by parts computation yields an inequality using assumption (a):

$$-\int_{\{e^w \geq \lambda\}} Q dv \geq C\lambda \frac{d}{d\lambda} \int_{\{e^w \geq \lambda\}} dv + \text{positive terms}.$$

Step III. — To estimate the first term on the right hand side of the previous inequality, we use the coarea formula to find

$$\begin{aligned}\int_{\{e^w \geq \lambda\}} e^{4w} dx &\geq \int_{C_2/\lambda}^{C_1} \int_{\{d(x,\Lambda)=s\}} e^{4w} d\sigma \, ds \\ &\geq \int_{C_2/\lambda}^{C_1} |\{d(x,\Lambda) = s\}| s^{-4} ds.\end{aligned}$$

An elementary computation using a covering argument yields that

$$|\{d(x,\Lambda) = s\}| \geq \begin{cases} Ns^3 & \text{if } \dim(\Lambda) = 0 \text{ and } |\Lambda| \geq N \\ Cs^{3-\frac{3}{4\beta}} & \text{if } \dim(\Lambda) = \beta > 0. \end{cases}$$

In either case, we reach a contradiction if the complement Λ is more than a finite number of points.

A closely related result to Theorem 2 above is the recent work of Bonk-Heinonen-Saksman [4]: To state their result, we first observe that for a metric $g_w = e^{2w} dx^2$ conformal to the flat metric dx^2 on domains in $\mathbb{R}^4$, the equation (7) takes the form

$$\Delta^2 w = Q_w e^{4w}. \tag{17}$$

Thus the integrability condition (15) of Q_w is equivalent to the condition that $\Delta^2 w$ being integrable.

Theorem 3 **([4]).** — *Suppose $g = e^{2w}|dx|^2$ is a complete conformal metric on $\mathbb{R}^4$ where w is given as a potential*

$$w(x) = \int_{\mathbb{R}^4} \log(\frac{|x-y|}{|y|}) \Delta^2 w(y) dy,$$

and

$$\int_{\mathbb{R}^4} |\Delta^2 w(y)| dy \leq C, \tag{18}$$

then there is a bilipshitz equivalence $\Phi : (\mathbb{R}^4, |dx|^2) \to (\mathbb{R}^4, g)$.

Remarks

1. The theorem does not assert the boundedness of the conformal factor.
2. This result holds in all even dimensions.
3. In the case of dimension two, the best constant in the inequality in (18) for the theorem to hold is 2π, which is given by the example of the half infinite cylinder as shown in the work of Bonk-Lang [**5**]. It is a natural open question whether the same example gives the sharp constant in all higher dimensions.

4. Poincaré-Einstein structure and renormalized volume

Given a smooth manifold X^{n+1} of dimension $n+1$ with smooth boundary $\partial X = M^n$. Let x be a defining function for M^n in X^{n+1} as follows:

$$\begin{aligned} x > 0 &\ \text{ in } X^{n+1}; \\ x = 0 &\ \text{ on } M^n; \\ dx \neq 0 &\ \text{ on } M^n. \end{aligned}$$

A Riemannian metric g on X^{n+1} is conformally compact if $(X^{n+1}, x^2 g)$ is said to be a compact Riemannian manifold with boundary. A conformally compact manifold (X^{n+1}, g) carries a well-defined conformal structure $[\hat{g}]$ on the boundary M^n, where each $\hat{g}$ is the restriction of $x^2 g$ for a defining function x. We call $(M^n, [\hat{g}])$ the conformal infinity of the conformally compact manifold (X^{n+1}, g). If, in addition, g satisfies $\mathrm{Ric}_g = -ng$, where Ric_g denotes the Ricci tensor of the metric g, then we call (X^{n+1}, g) a conformally compact Einstein manifold.

A conformally compact metric is said to be asymptotically hyperbolic if its sectional curvature approach -1 at $\partial X = M$. If g is an asymptotically hyperbolic metric on X, then a choice of metric $\hat{g}$ in $[\hat{g}]$ on M uniquely determines a defining function x near the boundary M and an identification of a neighborhood of M in X with $M \times (0, \epsilon)$ such that g has the normal form

$$g = x^{-2}(dx^2 + g_x) \tag{19}$$

where g_x is a 1-parameter family of metrics on M. In addition (see for example [**17**])

$$g_x = \hat{g} + g^{(2)} x^2 + (\text{even powers of } x) + g^{(n-1)} x^{n-1} + g^{(n)} x^n + \cdots, \tag{20}$$

when n is odd, and

$$g_x = \hat{g} + g^{(2)} x^2 + (\text{even powers of } x) + g^{(n)} x^n + h x^n \log x + \cdots, \tag{21}$$

when n is even. Here $\hat{g} = x^2 g|_{x=0}$, $g^{(2i)}$ are determined by $\hat{g}$ for $2i < n$. The trace part of $g^{(n)}$ is zero when n is odd; the trace part of $g^{(n)}$ is determined by $\hat{g}$ and h is traceless and determined by $\hat{g}$ too when n is even.

To introduce the renormalized volume, we follow Graham [**17**] to consider the asymptotics of the volume of a conformally compact Einstein manifold (X^{n+1}, g). Namely, denoting by x the defining function associated with a choice of a metric $\hat{g} \in [\hat{g}]$, we have

$$\mathrm{Vol}_g(\{x > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots + c_{n-1}\epsilon^{-1} + V + o(1)$$

for n odd, and

$$\mathrm{Vol}_g(\{x > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots + c_{n-2}\epsilon^{-2} + L\log\frac{1}{\epsilon} + V + o(1)$$

for n even. We call the constant term V in all dimensions the renormalized volume for (X^{n+1}, g). We recall that V in odd dimension and L in even dimension are independent of the choice $\hat{g}$ in the class $[\hat{g}]$.

In this section, we will give an alternative proof of the following result of M. Anderson [**3**]. The main point of our proof is to explore the relationship between the renormalized volume and the Q curvature.

Theorem 4 **([3], [12])**. — *Suppose that (X^4, g) is a conformally compact Einstein manifold. Then*

$$8\pi^2\chi(X^4) = \int_{X^4} |\mathcal{W}|_g^2 dv_g + 6V(X^4, g) \tag{22}$$

First we recall that motivated by the recent work of Graham-Zworski [**20**], Fefferman and Graham [**15**] introduced the following procedure to calculate the renormalized volume V for a conformally compact Einstein manifold. Here we will quote a special case of their result. For odd n, upon a choice of a special defining function x, we solve for

$$-\Delta v = n \quad \text{in } X^{n+1}, \tag{23}$$

with the asymptotics

$$v = \log x + A + Bx^n \tag{24}$$

in a neighborhood of M^n, where A, B are functions even in x, and $A|_{x=0} = 0$.

Lemma 1 **([15])**. — *When n is odd,*

$$V(X^{n+1}, g) = \int_M B dv_{\hat{g}}. \tag{25}$$

In addition, we have

Lemma 2. — *When n is odd, $(Q_{n+1})_{e^{2v}g} = 0$.*

Proof of Lemma 2. — The proof is a computation based on an observation made by Graham ([**17**], see also [**6**]) that the Paneitz operator $P_{\frac{n+1}{2}}$ on an Einstein manifold is a polynomial of the Laplacian $\mathcal{P}(\Delta)$ and the polynomial $\mathcal{P}$ on the Einstein manifold is the same as the one on the constant curvature space with the same constant as the constant of the scalar curvature of the Einstein manifold. In addition, the Q-curvature Q_{n+1} of an Einstein manifold is the same as the one on the constant curvature space. Therefore $(P_{n+1})_g = \mathcal{P}(\Delta_g)$ if $(P_{n+1})_{g_H} = \mathcal{P}(\Delta_{g_H})$, and $(Q_{n+1})_g = (Q_{n+1})_{g_H}$, where (H^{n+1}, g_H) is the hyperbolic space.

$$(P_{n+1})_{g_{H^{n+1}}} = \prod_{l=1}^{\frac{n+1}{2}} (-\Delta_{H^{n+1}} - C_l) \tag{26}$$

where $C_l = (\frac{n+1}{2} + l - 1)(\frac{n+1}{2} - l)$. Therefore

$$(P_{n+1})_g = \sum_{l=2}^{\frac{n+1}{2}} (-1)^{\frac{n+1}{2}-l} B_l (\Delta_g)^l - (-1)^{\frac{n-1}{2}} (n-1)! \Delta_g, \tag{27}$$

for some coefficient B_l depending on $C_j's$. Meanwhile $(Q_{n+1})_{H^{n+1}} = (-1)^{\frac{n+1}{2}} n!$. Thus

$$(Q_{n+1})_g = (-1)^{\frac{n+1}{2}} n!. \tag{28}$$

Thus if v satisfies the equation (23), we have

$$(P_{n+1})_g v + (Q_{n+1})_g = 0. \tag{29}$$

It thus follows from the prescribing Q curvature equation (7) that $(Q_{n+1})_{e^{2v}g} = 0$. □

We will now combine the results in the above lemmas to give an alternative proof of the result of Anderson [**3**] in Theorem 4 for conformal compact Einstein manifold (X^4, g). We first relate our curvature T to the boundary term B in Lemma 1.

Lemma 3. — *We have*

$$T_{e^{2v}g} = 3B|_{x=0}. \tag{30}$$

Proof. — According to the scalar curvature equation we have

$$\frac{1}{12} R_{e^{2v}g} = \frac{1}{2}(-\Delta_g e^v + \frac{1}{6} R_g e^v) e^{-3v}.$$

Therefore for v satisfies equation (23), we have

$$\frac{1}{12} R_{e^{2v}g} = \frac{1}{2}((e^{-v})^2 - |\nabla e^{-v}|^2).$$

We now apply the asymptotic expansion of v in (24) and write

$$e^{-2v} = \frac{1}{x^2} - 2A_2 - 2B_0 x + O(x^2)$$

$$|\nabla e^{-v}|^2 = \frac{1}{x^2} + 2A_2 + 4B_0 x + O(x^2),$$

where A_2 is the coefficient of x^2 of A and $B_0 = B|_{x=0}$. We get

$$T_{e^{2v}g} = -\frac{1}{12}\frac{\partial}{\partial x} R_{e^{2v}g}|_{x=0} = 3B_0.$$

This finishes the proof of the lemma. □

Proof of Theorem 4. — Applying Lemma 2 to the Gauss-Bonnet formula (13), we have

$$8\pi^2\chi(X^4) = \int_{X^4} |\mathcal{W}|^2_{e^{2v}g} dv_{e^{2v}g} + 2\int_M (\mathcal{L} + T)_{(e^{2v}g,\hat{g})}\, dv_{\hat{g}}.$$

We now observe that as the boundary of M of X^4 is umbilical, the second fundamental form $L_{\alpha,\beta}$ vanishes along M; hence $\mathcal{L} = -\mathcal{L}_4 - \mathcal{L}_5 = 0$. We then apply Lemma 1 and Lemma 3 to identify the area element in the integral $\int_M T$ with the renormalized volume to establish the formula (22) for the metric $e^{2v}g$. The last observation is that once the formula (22) holds for the metric $e^{2v}g$, it holds for any metric $\tilde{g} \in [g]$ with $(X^{n+1}, \tilde{g})$ a conformally compact manifold as the term of the renormalized volume V is conformally invariant. □

5. Renormalized volume in higher dimensions

In this section, we will continue to explore the relation between the Q curvature and the renormalized volume, and to extend the result of Theorem 4 above to all conformally compact Einstein manifolds (X^{n+1}, g) when n is odd. The main result is:

Theorem 5. — *When n is odd, we have*

$$\int_{X^{n+1}} (\mathcal{W}_{n+1})_g dv_g + (-1)^{\frac{n+1}{2}} \frac{\Gamma(\frac{n+2}{2})}{\pi^{\frac{n+2}{2}}} V(X^{n+1}, g) = \chi(X^{n+1}) \tag{31}$$

for some curvature invariant $\mathcal{W}_{n+1}$, which is a sum of contractions of Weyl curvatures and/or its covariant derivatives in an Einstein metric.

In the case of conformally compact manifolds of dimension $3+1$, one advantage we have taken is a precise formula of the Q on X^4, which enables us to do the explicit computations in Lemma 2 and Lemma 3 above. In the case when the dimension m of the manifold X^m is even but greater than four, it has been established in ([**18**], [**6**]) the existence of some Q curvature satisfying the following properties:

(i) It is a curvature invariant of weight $-m$. That is under the re-scale of metric $g \to t^2 g$, $Q_g = t^{-m} Q_{t^2 g}$.
(ii) $\int Q$ is a global conformal invariant.

(iii) There exists a m order linear differential operator P_m defined on X^m which prescribes the changing of Q under conformal change of metric $g_w = e^{2w}g$.

$$P_m w + Q = Q_w e^{mw} \tag{32}$$

One should remark that although the existence of Q is known, the explicit formula of the curvature is in general quite complicated and only known in dimensions six ([**17**]) and eight. Only in the recent work of Graham-Juhl [**19**], there is an inductive formula to compute Q curvature in high dimensions. Thus it is remarkable that one knows (Theorem 6 below) the "leading" order term (in terms of the order of the differentiation on the metric) of the Q curvature for all dimensions and it is even more remarkable that under the assumptions (i) and (ii) above, S. Alexakis [**2**] has recently established a structure theorem of the Q curvature (Theorem 7 below) which is known in the field as the answer to the Deser-Schwimmer Conjecture.

***Theorem 6* (Branson [7]).** — *On any compact m-dimensional manifold for m even,*

$$Q_m = b_m \Delta^{\frac{m-2}{2}} R + \textit{lower order terms}, \tag{33}$$

where

$$b_m = (-1)^{\frac{m-2}{2}} \frac{2^{m-1}(\frac{m}{2})!\Gamma(\frac{m+1}{2})}{\sqrt{\pi}(m-1)m!}.$$

***Theorem 7* (S. Alexakis [2]).** — *On any compact closed m-dimensional manifold with m even, we have*

$$Q_m = a_m e + \mathcal{J} + \mathrm{Div}(T_m). \tag{34}$$

where e is the Euler class density, $\mathcal{J}$ is a pointwise conformal invariant, and $\mathrm{Div}(T_m)$ *is a divergence term and a_m is some dimensional constant.*

Proof of Theorem 5. — Let (X^{n+1}, g) be a conformally compact Einstein manifold, where $n = 2k+1 > 3$, we wish to determine the analogous formula for the renormalized volume. We continue to consider the metric $(X^{n+1}, e^{2v}g)$ where v satisfies the equations (23) and (24). We will find that the parity conditions imposed in (24) makes it possible to determine the local boundary invariants of order n for the compact manifold $(X^{n+1}, e^{2v}g)$. According to (19) and (20) we have the expansion of the metric $e^{2v}g$.

$$\begin{aligned} e^{2v}g = {} & H^2 dx^2 + \hat{g} + c^{(2)}x^2 + \text{even powers in } x \\ & + c^{(n-1)}x^{n-1} + (2B_0\hat{g} + g^{(n)})x^n + \cdots \end{aligned} \tag{35}$$

where

$$H = e^{A+Bx^n} = 1 + e_2x^2 + \text{even powers in } x + e_{n-1}x^{n-1} + B_0x^n + \cdots$$

and $c^{(2i)}$ for $1 \le i \le (n-1)/2$ are local invariants of $\hat{g}$. We remark that it is easy to see that the boundary of $(X^{n+1}, e^{2v}g)$ is totally geodesic.

Lemma 4. — *We have*

$$(\partial_x \Delta^{\frac{n-3}{2}} R)_{e^{2v}g}|_{x=0} = -2nn!B_0. \tag{36}$$

Proof of Lemma 4. — We have

$$\begin{aligned}\Delta_{e^{2v}g} &= \frac{1}{H\sqrt{\det g_x}}\partial_\alpha(H\sqrt{\det g_x}g_x^{\alpha\beta}\partial_\beta)\\ &= Q_2^{(2)}\partial_x^2 + Q_2^{(1)}\partial_x + Q_2^{(0)}\end{aligned} \tag{37}$$

where the coefficients $Q^{(i)}$ have the following properties: $Q_2^{(2)}$ is a zeroth order differential operator, having an asymptotic expansion in powers of x in which the first nonzero odd power term is x^n. $Q_2^{(1)}$ is a zeroth order differential operator, having an expansion in which the first nonzero even degree term is x^{n-1}. $Q_2^{(0)}$ is differential operator of order 2 of purely tangential differentiations with coefficients which have expansion in powers of x in which the first nonzero odd term is x^n. Inductively, we see that, for $k \leq \frac{n-3}{2}$,

$$\Delta^k = Q_{2k}^{(2k)}\partial_x^{2k} + Q_{2k}^{(2k-1)}\partial_x^{2k-1} + \cdots + Q_{2k}^{(1)}\partial_x + Q_{2k}^{(0)} \tag{38}$$

where $Q_{2k}^{(i)}$ $(i \neq 0)$ is a differential operator of order $2k - i$ of purely tangential differentiations with coefficients having expansions in powers of x in which the first nonzero even terms are $x^{n-(2k-i)}$ if i is odd, and the first nonzero odd terms are $x^{n-(2k-i)}$ if i is even, and $Q_{2k}^{(0)}$ is a differential operator of order $2k$ of purely tangential differentiations with coefficients whose expansions in x have the first nonzero odd terms x^{n-2k+2}. Thus

$$\partial_x\Delta^k = F^{(2k+1)}\partial_x^{2k+1} + F^{(2k)}\partial_x^{2k} + \cdots + F^{(1)}\partial_x + F^{(0)} \tag{39}$$

where $F^{(2k+1)} = Q^{(2k)}$, $F^{(i)}$ $(0 < i < 2k+1)$ is a differential operator of order $2k - i + 1$ of purely tangential differentiations with coefficients whose expansions in x have the first nonzero even terms are $x^{n-(2k-i)-1}$ if i is even, and the first nonzero odd terms are $x^{n-(2k-i)-1}$ if i is odd, and $F^{(0)}$ is a differential operator of order $2k$ of purely tangential differentiations with coefficients whose expansions in x have the first nonzero even terms x^{n-2k+1}.

On the other hand, we have

$$R_{e^{2v}g} = -2n^2(n-1)B_0x^{n-2} + \text{even powers of } x \text{ terms} + o(x^{n-2}). \tag{40}$$

Keeping track of the parity, we obtain (36) in Lemma 4. □

Next we deal with all other boundary terms which may appear in integrating the Q curvature over X. These are contractions of one or more factors consisting of curvatures, covariant derivatives of curvatures, except $\partial_x^{n-2}R$ which is accounted in the above term $\partial_x\Delta^{\frac{n-3}{2}}R$. Since n is odd, and ∂x is the normal direction, each such term must contain at least one x index. In fact, the total number of x indices appearing in each of such terms must be odd. Thus one finds that each of such terms always contains a factor which is a covariant derivatives of curvature and in which x index appears odd number of times. Such factors, if we insist on taking ∇_x first, must appear as one of the following three different types

$$\nabla_\spadesuit \cdots \nabla_\spadesuit \nabla_x^{2k+1} R_{\spadesuit\spadesuit\spadesuit\spadesuit}$$

where $\spadesuit$ stands for indices other than x, in other words, tangential.

$$\nabla_\spadesuit \cdots \nabla_\spadesuit \nabla_x^{2k} R_{x\spadesuit\spadesuit\spadesuit}$$

and

$$\nabla_\spadesuit \cdots \nabla_\spadesuit \nabla_x^{2k-1} R_{x\spadesuit x\spadesuit}.$$

Note that in all three types $1 \leq 2k+1 \leq n-2$. Since the boundary is totally geodesic, we only need to verify

Lemma 5. — *All three types of boundary terms*

$$\nabla_x^{2k+1} R_{\spadesuit\spadesuit\spadesuit\spadesuit}, \quad \nabla_x^{2k} R_{x\spadesuit\spadesuit\spadesuit}, \quad \nabla_x^{2k-1} R_{x\spadesuit x\spadesuit} \tag{41}$$

vanish at the boundary for $1 \leq 2k+1 \leq n-2$.

Proof of Lemma 5. — We consider a point at the boundary and choose a normal coordinate on the boundary M^n in the special coordinates for X^{n+1}. Recall that

$$\begin{aligned} R_{\alpha\beta\gamma\delta} = & \frac{1}{2}(-\partial_\beta\partial_\delta g_{\alpha\gamma} - \partial_\alpha\partial_\gamma g_{\beta\delta} + \partial_\beta\partial_\gamma g_{\alpha\delta} + \partial_\alpha\partial_\delta g_{\beta\gamma}) \\ & - g^{\eta\lambda}([\alpha\gamma,\eta][\beta\delta,\lambda] - [\beta\gamma,\eta][\alpha\delta,\lambda]), \end{aligned}$$

and

$$\nabla_x T_{\alpha\beta\cdots\delta} = \partial_x T_{\alpha\beta\cdots\delta} - \Gamma_{\alpha\ x}^{\lambda} T_{\lambda\beta\cdots\delta} - \Gamma_{\beta\ x}^{\lambda} T_{\alpha\lambda\cdots\delta} - \cdots - \Gamma_{\delta\ x}^{\lambda} T_{\alpha\beta\cdots\lambda}$$

where

$$\Gamma_{\beta\gamma}^{\alpha} = g^{\alpha\delta}[\beta\gamma,\delta]$$

and

$$[\alpha\beta,\gamma] = \frac{1}{2}(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}).$$

For simplicity of notation we will use g to stand for $e^{2v}g$ if no confusion can arise. Each of the three types is a sum of products of factors that are of the form:

$$\partial_\alpha\partial_\beta \cdots \partial_\gamma g_{\lambda\mu}$$

or

$$\partial_\alpha\partial_\beta \cdots \partial_\gamma g^{\lambda\mu}.$$

We claim that each summand must has a factor that is one of the following

$$\partial_\spadesuit \cdots \partial_\spadesuit \partial_x^{2k+1} g_{\spadesuit\spadesuit},$$

$$\partial_\spadesuit \cdots \partial_\spadesuit \partial_x^{2k-1} g_{xx},$$

$$\partial_\spadesuit \cdots \partial_\spadesuit \partial_x^{2k+1} g^{\spadesuit\spadesuit}.$$

and

$$\partial_\spadesuit \cdots \partial_\spadesuit \partial_x^{2k-1} g^{xx},$$

where $1 \leq 2k+1 \leq n-2$. To verify the claim, one needs to observe that, in writing the three types in local coordinates, the number of times the index x appears in each summand increases only when one sees

$$\Gamma^x_{\spadesuit\ x} T_{\alpha\beta\cdots\ x\cdots\delta},$$

where the number of x increases by 2. Thus, in the end, the total number of index x in each summand is still odd. Therefore one of the factors must have an odd number of x. Finally one observes that for any individual factor arising here the number of x can not exceed $n-1$. So the proof of Lemma 5 is complete. □

We now finish the proof of Theorem 5 for all n odd based on the results in Theorem 6 and Theorem 7.

Proof of Theorem 5. — We first establish that equation (34) remains valid on a conformally Einstein manifold (X^{n+1}, g). Let $g_w = e^{2w}g$ be such a metric, then it follows from the Paneitz equation that for $m = n+1$,

$$\begin{aligned}
(Q_m)_{g_w} e^{mv} &= (P_m)_g v \ + \ (Q_m)_g \\
&= a_m e_g + \mathcal{J}_g + \mathrm{Div}(T') \\
&= a_m e_{g_w} + \mathcal{J}_{g_w} + \mathrm{Div}(T'')
\end{aligned} \tag{42}$$

where the second equation follows from the fact that the Paneitz operator P_m is a divergence and Theorem 7. The third equation follows from the fact that the Pfaffians of any two Riemannian metrics on the same manifold differs by a divergence term and $\mathcal{J}$ is a conformal invariant.

In order to apply this formula, we need to observe that the leading order term $\Delta^{\frac{m-2}{2}} R$ in formula (33) cannot appear in the conformally invariant term $\mathcal{J}$. In order to see this, we first recall that $\mathcal{J}$ is a linear combination of terms of the form $Tr(\nabla^{I_1}\mathcal{R} \bigotimes \nabla^{I_2}\mathcal{R} ... \bigotimes \nabla^{I_k}\mathcal{R})$ of weight m where Tr denotes a suitably chosen pairwise contraction over all the indices. Observe that the conformal variation $\delta_w(\Delta^{\frac{m-2}{2}})R$, where δ_w denotes the variation of the metric g to g_w is of the form $\Delta^{\frac{m}{2}} w +$ lower order terms. Thus if $\Delta^{\frac{m-2}{2}} R$ does appear as a term in $\mathcal{J}$, its conformal variation must be cancelled by the conformal variations of the other terms in the linear combination, but it is clear that the conformal variations of the other possibilities of the curvature $\mathcal{R}$ other than the scalar curvature R cannot have order m in the number of derivatives of w and of the form $\Delta^{\frac{m}{2}} w$.

We can now apply the formula (42) to the metric $g_v = e^{2v}g$ where v is as in (23). Thus by Lemma 2 the left hand side of (42) is identically zero, and we find

$$a_m \chi(X^{n+1}) = \int_{X^{n+1}} (\mathcal{J}_{g_v} - \mathrm{Div}(T''))dv_{g_v}.$$

Among the divergence terms in $\mathrm{Div}(T'')$, only the leading order term $b_m \Delta^{\frac{m-2}{2}} R$ has a non-zero contribution according to Lemma 5. The computation in Lemma 5 determines the precise contribution of this term as a multiple of the renormalized volume. We also note that as g is an Einstein metric, we may assume that the terms which appear in the conformal invariant $\mathcal{J}$ are contractions of the Weyl curvature together with its covariant derivatives. We have thus finished the proof of Theorem 5. □

Corollary 1. — *When (X^{n+1}, g) is conformally compact hyperbolic, we have*

$$V(X^{n+1}, g) = \frac{(-1)^{\frac{n+1}{2}} \pi^{\frac{n+2}{2}}}{\Gamma(\frac{n+2}{2})} \chi(X). \tag{43}$$

One may compare (43) to a formula for renormalized volume given by Epstein in [**24**], where he has

$$V(X^{n+1}, g) = \frac{(-1)^m 2^{2m} m!}{(2m)!} \chi(X) \tag{44}$$

for $n = 2m - 1$ and our answers agree!.

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S.-Y. A. CHANG, Department of Mathematics, Princeton University, Princeton, NJ 08544, USA
E-mail : chang@math.princeton.edu

P. C. YANG, Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA; School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA

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A SURVEY OF THE HYPOELLIPTIC LAPLACIAN

by

Jean-Michel Bismut

À Jean Pierre Bourguignon pour son soixantième anniversaire

Abstract. — The purpose of this paper is to review the construction of the hypoelliptic Laplacian, in the context of de Rham theory for smooth manifolds, and also the construction of the hypoelliptic Dirac operator in the context of complex Kähler manifolds.

***Résumé* (Compte-rendu sur le laplacien hypoelliptique).** — Le but de cet article est d'établir un compte-rendu de la construction du laplacien hypoelliptique dans le contexte de la théorie de de Rham des variétés lisses, ainsi que de la construction de l'opérateur de Dirac hypoelliptique dans le contexte des variétés kähleriennes complexes.

Introduction

The purpose of this survey is to review certain aspects of the construction of the hypoelliptic Laplacian, in de Rham and in Dolbeault theory. The hypoelliptic Laplacian was introduced in [**3**] in de Rham theory, and in [**5**] for Dirac operators. The crucial analytic foundations for the theory were developed by Lebeau and ourselves in [**8**].

One motivation given in [**3**] is to interpret the hypoelliptic Laplacian in de Rham theory as a semiclassical limit of the Witten deformation of the Hodge theory of the loop space of a Riemannian manifold, which is associated with the energy functional. This point of view remains formal, since the Hodge theory of the loop space of a manifold is not analytically well defined. The motivation for the construction of the hypoelliptic Dirac operator of [**5**] is to understand the effect of replacing the standard

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L_2 metric on the loop space of a manifold by a H^1 metric. Again these considerations remain formal, although ultimately the hypoelliptic Dirac operator is well defined.

Whatever the motivations, and there are many others, some of which are explained in [**4, 6**], the conclusion is that a geometric Laplacian can be deformed into a family of hypoelliptic second order differential operators acting on the total space of the tangent or the cotangent bundle of the given manifold, which interpolates in the proper sense between the Laplacian and the generator of the geodesic flow. The existence of this deformation is counter-intuitive, since ellipticity is a stable property. However, the fact that the hypoelliptic Laplacian acts on a bigger space than the original elliptic Laplacian explains why ultimately it can be made to 'collapse' on the elliptic Laplacian.

Let us finally mention that up to lower order terms, the hypoelliptic Laplacian is the sum of a harmonic oscillator acting in the directions of the fibre, and of the vector field which generates the geodesic flow, these two operators being adequately scaled.

In this paper, first, we fully develop the theory in the case where the base manifold is the circle. The main point is that while in this case, the geometry is trivial, a complete understanding of the hypoelliptic Laplacian and of the interpolation property can be easily obtained via Fourier analysis on the circle and the spectral theory of the harmonic oscillator. The case of the circle is also useful, because the objects which appear there turn out to be at the same time the principal symbols of the geometric hypoelliptic operators, and because the circle is the model of a closed geodesic. The fact that the hypoelliptic Laplacian is self-adjoint with respect to a symmetric form of signature (∞, ∞) appears also naturally in that context.

The basic difference between the case of the circle and the geometric case is that the analysis of the hypoelliptic Laplacian is no longer explicit, and also that the convergence arguments, which are easy for the circle, are built on a functional analytic machinery described in detail in our work with Lebeau [**8**].

Also we describe the construction of the hypoelliptic Laplacian, in the de Rham case, and also for Kähler manifolds. We emphasize the role of the symmetric bilinear forms, at least in the de Rham case, because of the important spectral theoretic consequences which are derived in [**8**].

This paper is organized as follows. In section 1, we consider the case of the circle. Since the hypoelliptic Laplacian is ultimately obtained as a Hodge Laplacian with respect to an exotic bilinear form on the de Rham or the Dolbeault complex, this point of view is systematically emphasized in this simple case too.

In section 2, we recall classical results on the Hodge theory of a compact manifold, and on the Witten deformation of classical Hodge theory which is associated with a smooth function. Also we show that if (M, ω) is a symplectic manifold, there is a symplectic Witten Laplacian, which turns out to be the Lie derivative operator associated with the corresponding Hamiltonian vector field. This point of view is further developed in [**3**], where the hypoelliptic Laplacian in de Rham theory is obtained by linearly interpolating between the Riemannian metric of the base manifold, and the symplectic form of its cotangent bundle.

In section 3, we explain the construction of the hypoelliptic Laplacian in de Rham theory. We also give the main arguments of [**3**] in favour of the fact that the hypoelliptic Laplacian interpolates between the Hodge Laplacian and the geodesic flow.

In section 4, we give the construction of the hypoelliptic Dirac operator of [**5**] in the context of Kähler manifolds, and we give the arguments showing that this operator should indeed be a deformation of the classical elliptic Dirac operator.

As we already said, the analytic justifications which make that the whole construction ultimately exists as a mathematical theory are developed in detail in our work with Lebeau [**8**]. Also applications to Ray-Singer torsion [**19**] and Quillen metrics [**17**] are given in [**8**] and [**5**].

1. The case of the circle

The purpose of this section is to construct the hypoelliptic Laplacian in the case where the base manifold X is just S^1. In this case, all the objects are simple and natural. Besides, the operators which are obtained in this case can be viewed as the symbols of the operators which are obtained later in the geometric case.

This section is organized as follows. In subsection 1.1, we recall elementary properties of elliptic and hypoelliptic operators.

In subsection 1.2, we introduce the Kolmogorov operator on $S^1 \times \mathbf{R}$, which is a simple case of an operator verifying Hörmander's hypoellipticity theorem [**14**], and at the same time, coincides, up to important lower order terms, with the hypoelliptic Laplacian. Formal conjugation arguments are used to relate the hypoelliptic Laplacian to the elliptic Laplacian on S^1. The fact that the hypoelliptic Laplacian interpolates in the proper sense between the Laplacian and the generator of the geodesic flow can be exhibited by hand. One obtains this way a proof of Poisson's formula by interpolation.

In subsection 1.3, we show that our hypoelliptic Laplacian is a Hodge Laplacian with respect to an exotic bilinear form on the space of compactly supported differential forms on $S^1 \times \mathbf{R}$. This result will be used in section 3 to construct the geometric hypoelliptic Laplacian in the context of de Rham theory.

1.1. Elliptic and hypoelliptic operators. — Let X be a compact manifold. Let $\mathcal{X}^*$ be the total space of T^*X. Then X embeds in $\mathcal{X}^*$ as the zero section of T^*X.

Let E and F be two complex vector bundles on X. If P is a pseudodifferential operator of order m mapping $C^{\infty}(X,E)$ into $C^{\infty}(X,F)$, its principal symbol $\sigma_P(x,\xi)$ is a smooth map on $\mathcal{X}^* \setminus X$ with values in $\operatorname{Hom}(E,F)$, which is homogeneous of order m in the variable ξ. The operator P is said to be elliptic if $\sigma_P(x,\xi)$ is invertible on $\mathcal{X}^* \setminus X$.

If X is equipped with a Riemannian metric, if Δ^X is the Laplace-Beltrami operator acting on $C^{\infty}(X,\mathbf{R})$, then $-\Delta^X$ is an elliptic operator of order 2, and its principal symbol is $|\xi|^2$. The standard example is the operator $-\frac{\partial^2}{\partial x^2}$ acting on S^1.

Ellipticity is a stable property. Indeed a small deformation of an elliptic operator is still elliptic. This should make all the more surprising the fact that certain elliptic operators can be deformed into hypoelliptic operators. This is only possible because the deformed operators act on a different space than the original operator. Besides elliptic operators of order m act on Sobolev spaces, and decrease the Sobolev index by m. As an example, the operator $-\Delta^X$ decreases the Sobolev index by 2, and any pseudoinverse of $-\Delta^X$ (an inverse up to regularizing operators) increases the Sobolev index by 2. In particular if u is a scalar distribution on X such that $-\Delta^X u \in H^s$, then $u \in H^{s+2}$.

Hypoellipticity is a weaker property. A pseudodifferential operator P is said to be hypoelliptic if when u is a distribution such that Pu is C^{∞} on some open set, then u is also C^{∞} on this open set. For example the parabolic operator $\frac{\partial}{\partial t} - \frac{1}{2}\Delta^X$ on $\mathbf{R} \times X$ is hypoelliptic.

1.2. The Kolmogorov operator and Hörmander's theorem. — Consider the operator A on $\mathbf{R} \times \mathbf{R}^2$ introduced by Kolmogorov [**15**],

$$A = \frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial x}. \tag{1.1}$$

In [**15**], Kolmogorov computed the fundamental solution of (1.1), as a time dependent Gaussian kernel in the variables (x,y), from which the hypoellipticity of A follows.

The hypoellipticity of A prompted Hörmander [**14**] to develop his theory of hypoelliptic second order differential operators which we now briefly describe. Indeed if $X_0,\ldots,X_m$ are smooth vector fields on $\mathbf{R}^n$, consider the differential operator

$$M = -\frac{1}{2}\sum_{i=1}^{m} X_i^2 + X_0. \tag{1.2}$$

Let $\mathcal{E}(x) \subset \mathbf{R}^n$ be the vector space spanned at x by $X_0,\ldots,X_m$ and their Lie brackets. Hörmander's theorem asserts that if at each x, $\mathcal{E}(x) = \mathbf{R}^n$, then M is a hypoelliptic operator.

The fact that A is hypoelliptic is a consequence of Hörmander's theorem since the Lie bracket $\left[\frac{\partial}{\partial y}, \frac{\partial}{\partial t} - y\frac{\partial}{\partial x}\right] = -\frac{\partial}{\partial x}$ is enough to make the Hörmander distribution associated with $\frac{\partial}{\partial y}, \frac{\partial}{\partial t} - y\frac{\partial}{\partial x}$ span $\mathbf{R}^3$.

More generally, consider the operator A_n on $\mathbf{R}^{2n+1}$ which is given by

$$A_n = \frac{\partial}{\partial t} - \frac{1}{2}\Delta^V - \nabla_y. \tag{1.3}$$

In (1.3), Δ^V denotes the Laplacian in the variables $y_1, \dots, y_n$, and ∇_y denotes differentiation on the variables $x^1, \dots, x^n$ in the direction y, i.e., $\nabla_y = \sum_1^n y^i \frac{\partial}{\partial x^i}$. In this case, the n Lie brackets $\left[\frac{\partial}{\partial y^i}, \nabla_y\right] = \frac{\partial}{\partial x^i}$ are necessary to make the Hörmander distribution span $\mathbf{R}^{2n+1}$.

The parabolic operator $\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}$ is the model of the geometric parabolic operator $\frac{\partial}{\partial t} - \frac{1}{2}\Delta^X$. Let us now describe the model of its hypoelliptic deformation.

Let L be the operator on $\mathbf{R}^3$,

$$L = \frac{\partial}{\partial t} + \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) - y\frac{\partial}{\partial x}. \tag{1.4}$$

Clearly,

$$L = A + \frac{1}{2}\left(y^2 - 1\right). \tag{1.5}$$

The term which is added to A in the right-hand side of (1.5) has no effect on hypoellipticity, which is by definition a local property. On the other hand, the operator H given by

$$H = \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) \tag{1.6}$$

is the harmonic oscillator, which has discrete spectrum and compact resolvent. From this point of view, the operator L is significantly different from the operator A in (1.1).

As in (1.3), we may as well define the operator L_n on $\mathbf{R}^{2n+1}$, which is given by

$$L_n = \frac{\partial}{\partial t} + \frac{1}{2}\left(-\Delta + |y|^2 - n\right) - \nabla_y. \tag{1.7}$$

To make the notation simpler, we now proceed with the case $n = 1$. Also we disregard for the moment the variable t, which can be included in everything which follows. For $b > 0$, set

$$\mathcal{L}_b = \frac{1}{2b^2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) - \frac{1}{b}y\frac{\partial}{\partial x}. \tag{1.8}$$

Clearly,

$$\mathcal{L}_b = \frac{1}{2b^2}\left(-\frac{\partial^2}{\partial y^2} + \left(y - b\frac{\partial}{\partial x}\right)^2 - 1\right) - \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{1.9}$$

In the sequel, it will be convenient to assume that $x \in S^1 = \mathbf{R}/\mathbf{Z}$, and that $y \in \mathbf{R}$ lies in TS^1 or T^*S^1.

Let us formally make the translation $y \to y + b\frac{\partial}{\partial x}$. Equivalently let U_b be the formal operator,

$$U_b = \exp\left(b\frac{\partial^2}{\partial x \partial y}\right). \tag{1.10}$$

Set

$$\mathcal{M}_b = U_b \mathcal{L}_b U_b^{-1}. \tag{1.11}$$

Then $\mathcal{M}_b$ is given by the operator,

$$\mathcal{M}_b = \frac{1}{2b^2}\left(-\Delta^V + y^2 - 1\right) - \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{1.12}$$

We can write the operator $\mathcal{M}_b$ in the form,

$$\mathcal{M}_b = \frac{H}{b^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{1.13}$$

Before we proceed, let us observe that conjugation by U_b has transformed the hypoelliptic operator $\mathcal{L}_b$ into the elliptic operator $\mathcal{M}_b$, in which the variables x, y have been uncoupled.

Since the spectrum of H is equal to $\mathbf{N}$, the spectrum of $\mathcal{M}_b$ is given by

$$\operatorname{Sp}(\mathcal{M}_b) = \frac{\mathbf{N}}{b^2} + \left\{2\pi^2 k^2, k \in \mathbf{Z}\right\}. \tag{1.14}$$

Therefore when $b \to 0$, the finite part of the spectrum of $\mathcal{M}_b$ converges to the spectrum of $-\frac{1}{2}\frac{\partial^2}{\partial x^2}$, and as $b \to +\infty$, $\operatorname{Sp}(\mathcal{M}_b)$ while staying real, accumulates near 0. Also 0 is a simple eigenvalue of $\mathcal{M}_b$.

Before we explain how the spectrum of $\mathcal{M}_b$ relates to the spectrum of $\mathcal{L}_b$, let us first explain how to eliminate the nonzero eigenvalues of H. Let $\Lambda^{\cdot}(\mathbf{R}^*)$ be the exterior algebra of $\mathbf{R}$, which is spanned by $1, dy$. Let N be the number operator on $\Lambda^{\cdot}(\mathbf{R}^*)$, which acts like 0 on 0-forms, and 1 on 1-forms. Set

$$\mathcal{O} = H + N. \tag{1.15}$$

Let $\operatorname{Tr_s}$ be our notation for the supertrace. Indeed let $V = V_+ \oplus V_-$ be a $\mathbf{Z}_2$-graded Hilbert space, and let $\tau = \pm 1$ be the endomorphism of V which defines the $\mathbf{Z}_2$-grading. If $A \in \mathcal{L}(V)$ is trace class, then

$$\operatorname{Tr_s}[A] = \operatorname{Tr}[\tau A]. \tag{1.16}$$

Here we use the $\mathbf{Z}_2$-grading associated with the grading of $\Lambda^{\cdot}(\mathbf{R}^*)$. Then one has the easy identity,

$$\operatorname{Tr_s}[\exp(-t\mathcal{O})] = 1. \tag{1.17}$$

Put

$$\mathcal{M}'_b = \mathcal{M}_b + \frac{N}{b^2}. \tag{1.18}$$

Of course (1.14) remains valid for $\mathcal{M}'_b$, and 0 is still a simple eigenvalue of $\mathcal{M}'_b$. From (1.17), (1.18), we get

$$\mathrm{Tr_s}\left[\exp\left(-t\mathcal{M}'_b\right)\right] = \mathrm{Tr}\left[\exp\left(\frac{t}{2}\frac{\partial^2}{\partial x^2}\right)\right]. \tag{1.19}$$

The remarkable fact in (1.19) is that it does not depend on $b > 0$. We already saw that as $b \to 0$, the spectrum of $\mathcal{M}'_b$ converges to the spectrum of $-\frac{1}{2}\frac{\partial^2}{\partial x^2}$. The question is now to know how to use (1.19) with $b \to +\infty$.

Using hypoellipticity, it is not difficult to show that $\mathcal{L}_b$ has a smooth heat kernel, and that for $t > 0$, $\exp\left(-t\mathcal{L}_b\right)$ is trace class.

We claim that

$$\mathrm{Tr}\left[\exp\left(-t\mathcal{L}_b\right)\right] = \mathrm{Tr}\left[\exp\left(-t\mathcal{M}_b\right)\right]. \tag{1.20}$$

One could try using the conjugation by the operator U_b which was described above to get (1.20). However, the operator U_b is poorly defined, and does not act on any natural function space.

However, we can use Fourier series to diagonalize the operator $\frac{\partial}{\partial x}$, and try obtaining an analogue of (1.20) for each eigenvalue $2i\pi k, k \in \mathbf{Z}$, from which (1.20) would follow by summation. This can indeed be done. In fact the eigenvectors of the harmonic oscillator H are given by $P_n(y)\exp\left(-y^2/2\right), n \in \mathbf{N}$, where the P_n are the Hermite polynomials. Now the complex translations $y \to y + 2i\pi bk, k \in \mathbf{Z}$ maps these eigenvectors into well defined elements of L_2. It is not difficult to conclude that the consequences of the above conjugation by U_b are correct, and that (1.20) holds.

Set

$$\mathcal{L}'_b = \mathcal{L}_b + \frac{N}{b^2}. \tag{1.21}$$

By (1.8) and (1.21), we get

$$\mathcal{L}'_b = \frac{1}{2b^2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) + \frac{N}{b^2} - \frac{1}{b}y\frac{\partial}{\partial x}. \tag{1.22}$$

Using (1.18)-(1.21), we obtain,

$$\mathrm{Tr_s}\left[\exp\left(-t\mathcal{L}'_b\right)\right] = \mathrm{Tr}\left[\exp\left(\frac{t}{2}\frac{\partial^2}{\partial x^2}\right)\right]. \tag{1.23}$$

Instead of (1.23), one can replace (1.23) by a pointwise equality in the x variable of the integral of the corresponding kernels in the y variable, simply by using the Fourier series argument we just gave. However, this will not be used in the sequel.

Now we will make $b \to +\infty$ in equation (1.23). For $b > 0$, let K_b be the map

$$K_b s(x,y) = s(x, by). \tag{1.24}$$

Set

$$\mathfrak{L}_b = K_b \mathcal{L}'_b K_b^{-1}. \tag{1.25}$$

By (1.22), we get

$$\mathfrak{L}_b = \frac{y^2}{2} - y\frac{\partial}{\partial x} - \frac{1}{2b^4}\frac{\partial^2}{\partial y^2} + \frac{1}{b^2}\left(-\frac{1}{2} + N\right). \tag{1.26}$$

By (1.26), we find that as $b \to +\infty$,

$$\mathfrak{L}_b = \frac{y^2}{2} - y\frac{\partial}{\partial x} + \mathcal{O}\left(1/b^2\right). \tag{1.27}$$

Equation (1.27) indicates that up to the translation by $\frac{y^2}{2}$, the leading term in the asymptotics of $\mathfrak{L}_b$ is the generator of the geodesic flow.

We briefly show how the above can be used to give a proof of the Poisson formula. Indeed (1.26), (1.27) already indicates that $\mathrm{Tr_s}\left[\exp\left(-t\mathfrak{L}_b\right)\right]$ concentrates along the closed geodesics in S^1 parametrized by $[0,t]$, which start and end at x and have speed y. This means that $y = k/t, k \in \mathbf{Z}$. Let $R_{k,b}$ be the map

$$R_{k,b} s(x,y) = s\left(x, k/t + y/b^2\right). \tag{1.28}$$

Set

$$\mathfrak{L}'_{k,b} = R_{k,b}\mathfrak{L}_b R_{k,b}^{-1}. \tag{1.29}$$

By (1.26), we get

$$\mathfrak{L}'_{k,b} = \frac{1}{2}\left(k/t + y/b^2\right)^2 - \left(k/t + y/b^2\right)\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{1}{b^2}\left(-\frac{1}{2} + N\right). \tag{1.30}$$

Now observe that the term $k/t\frac{\partial}{\partial x}$ can be disregarded, because, once it is multiplied by t, it exponentiates to the identity. We still use the notation $\mathfrak{L}'_{k,b}$ for the operator in which this term has been deleted. Let S_b be the map $s(x,y) \to s\left(b^2 x, y\right)$. Note that this map is only defined for $x \in \mathbf{R}$. Put

$$\mathfrak{L}''_{k,b} = S_b^{-1}\mathfrak{L}'_{k,b} S_b. \tag{1.31}$$

By (1.30), we obtain,

$$\mathfrak{L}''_{k,b} = \frac{1}{2}\left(k/t + y/b^2\right)^2 - y\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{1}{b^2}\left(-\frac{1}{2} + N\right). \tag{1.32}$$

The effect of the above change of variables is that for every $k \in \mathbf{Z}$, we should evaluate the asymptotics as $b \to +\infty$ of $I_{k,b,t}$ given by

$$I_{k,b,t} = b^2 \int_{\mathbf{R}} \mathrm{Tr_s}\left[\exp\left(-t\mathfrak{L}''_{k,b}\right)\left((0,y),(0,y)\right)\right] dy. \tag{1.33}$$

In (1.33), $\exp\left(-t\mathfrak{L}''_{k,b}\right)((x,y),(x',y'))$ denotes the smooth kernel on $\mathbf{R}^2$ which is associated with the operator $\exp\left(-t\mathfrak{L}''_{k,b}\right)$. As to the factor b^2, it appears because of S_b.

Clearly,

$$b^2\mathrm{Tr_s}\left[\exp\left(-tN/b^2\right)\right] = b^2\left(1-e^{-t/b^2}\right), \tag{1.34}$$

so that as $b \to +\infty$,

$$b^2\mathrm{Tr_s}\left[\exp\left(-tN/b^2\right)\right] \to t. \tag{1.35}$$

Put

$$\mathfrak{N} = -y\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial^2}{\partial y^2}. \tag{1.36}$$

By (1.33)-(1.36), we find that as $b \to +\infty$,

$$I_{k,b,t} \to I_{k,+\infty,t} = t\exp\left(-k^2/2t\right)\int_{\mathbf{R}} \mathrm{Tr}\left[\exp\left(-t\mathfrak{N}\right)((0,y),(0,y))\right]dy. \tag{1.37}$$

Now one verifies easily that

$$\int_{\mathbf{R}} \mathrm{Tr}\left[\exp\left(-t\mathfrak{N}\right)((0,y),(0,y))\right]dy = \frac{t^{-3/2}}{\sqrt{2\pi}}. \tag{1.38}$$

By (1.37), (1.38), we obtain,

$$I_{k,+\infty,t} = \frac{\exp\left(-k^2/2t\right)}{\sqrt{2\pi t}}, \tag{1.39}$$

which is exactly the contribution of $k \in \mathbf{Z}$ to $\mathrm{Tr}\left[\exp\left(\frac{t}{2}\frac{\partial^2}{\partial x^2}\right)\right]$.

The same sort of argument can also be used to evaluate the full heat kernel for $\exp\left(\frac{t}{2}\frac{\partial^2}{\partial x^2}\right)$ on S^1.

The operator $\mathcal{L}'_b$ is the prototype of a hypoelliptic Laplacian. We have shown by elementary arguments how and in what sense it interpolates between the standard Laplacian and the generator of the geodesic flow. The remarkable fact is that the full spectrum of the Laplacian can be recovered from the spectrum of its hypoelliptic deformation, and the heat kernel on S^1 can also be obtained by this procedure.

Later, we will describe the deformation of the Laplacian of a manifold to a hypoelliptic Laplacian, that is in a geometric context. However, when taking the obvious n-dimensional extension of what we just did, the above exactly describes the deformation of the associated principal symbols. Needless to say, the proper geometric context cannot be described just via the principal symbol, the full symbol is obviously needed. This ultimately means that there is not only one hypoelliptic Laplacian, there are as many as possible geometric deformations which one can possibly envision. This will be illustrated in the sequel in the two main classes of examples, which correspond to deformations of de Rham Hodge theory, and of Dolbeault Hodge theory. Moreover it will not be possible to make geometric sense of a conjugation like the one in (1.11), because the considered vector fields will not commute.

Finally, it is instructive to observe that we made two kinds of translations on the variable y. One type of translations has been to replace y by $y + 2i\pi bk$ for $k \in \mathbf{Z}$, or equivalently to change y into $y + b\frac{\partial}{\partial x}$. This imaginary translation has allowed us to relate the hypoelliptic operator $\mathcal{L}'_b$ to the elliptic operator $\mathcal{M}_b$. The other kind of translation has been the real translation $y \to y + bk/t$, to connect the operator $\mathcal{L}'_b$ with the geodesic flow. It should then be clear that the possibility to make at the same time translations on y in the imaginary and in the real directions is critical in explaining the fact that $\mathcal{L}'_b$ interpolates between the Laplacian of S^1 and the geodesic flow of S^1.

1.3. The hypoelliptic Laplacian as a Hodge Laplacian. — Now we will explain in what sense the operator $\mathcal{L}'_b$ is a Laplacian of Hodge type.

Let $d^{S^1\times\mathbf{R}}$ be the de Rham operator on $S^1 \times \mathbf{R}$. Then

$$d^{S^1\times\mathbf{R}} = dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}. \tag{1.40}$$

The standard adjoint $d^{S^1\times\mathbf{R}*}$ of $d^{S^1\times\mathbf{R}}$ is given by

$$d^{S^1\times\mathbf{R}*} = -i_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} - i_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}. \tag{1.41}$$

Set

$$\mathcal{H}(y) = \frac{y^2}{2}. \tag{1.42}$$

Let $d_T^{S^1\times\mathbf{R}}$ be the Witten twist of $d^{S^1\times\mathbf{R}}$, i.e.

$$d_T^{S^1\times\mathbf{R}} = e^{-T\mathcal{H}}d^{S^1\times\mathbf{R}}e^{T\mathcal{H}}. \tag{1.43}$$

Then

$$d_T^{S^1\times\mathbf{R}} = d^{S^1\times\mathbf{R}} + Tydy\wedge. \tag{1.44}$$

Let $d_T^{S^1\times\mathbf{R}*}$ be the usual adjoint of $d_T^{S^1\times\mathbf{R}}$, i.e.,

$$d_T^{S^1\times\mathbf{R}*} = e^{T\mathcal{H}}d^{S^1\times\mathbf{R}*}e^{-T\mathcal{H}}. \tag{1.45}$$

Equivalently,

$$d_T^{S^1\times\mathbf{R}*} = d^{S^1\times\mathbf{R}*} + Tyi_{\frac{\partial}{\partial y}}. \tag{1.46}$$

Let $\square_T^{S^1\times\mathbf{R}}$ be the corresponding Witten Laplacian [**20**], i.e.,

$$\square_T^{S^1\times\mathbf{R}} = \left[d_T^{S^1\times\mathbf{R}}, d_T^{S^1\times\mathbf{R}*}\right]. \tag{1.47}$$

In (1.47), [] is our notation for a supercommutator, which, in this case, is an anti-commutator. Then

$$\frac{1}{2}\square_T^{S^1\times\mathbf{R}} = \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + T^2y^2 - T\right) + TN - \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{1.48}$$

By (1.48), we get

$$K_{\sqrt{T}^{-1}}\frac{1}{2}\square_T^{S^1\times\mathbf{R}}K_{\sqrt{T}} = \frac{T}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) + TN - \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{1.49}$$

If $T = 1/b^2$, by comparing (1.12), (1.18) and (1.49), we get

$$K_{\sqrt{T}}^{-1}\frac{1}{2}\square_T^{S^1\times\mathbf{R}}K_{\sqrt{T}} = \mathcal{M}'_b. \tag{1.50}$$

Equation (1.50) suggests what should be done to write $\mathcal{L}'_b$ as a Hodge like Laplacian. Set

$$\mathcal{L}''_b = K_b^{-1}\mathcal{L}'_b K_b. \tag{1.51}$$

By (1.22), we obtain,

$$\mathcal{L}''_b = \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + \frac{y^2}{b^4} - \frac{1}{b^2}\right) + \frac{N}{b^2} - \frac{y}{b^2}\frac{\partial}{\partial x}. \tag{1.52}$$

Recall that U_b has been defined in (1.10). Set

$$\underline{d}_b^{S^1\times\mathbf{R}} = U_{b^2}^{-1} d_{1/b^2}^{S^1\times\mathbf{R}} U_{b^2}, \qquad \underline{d}_b^{S^1\times\mathbf{R}*} = U_{b^2}^{-1} d_{1/b^2}^{S^1\times\mathbf{R}*} U_{b^2}. \tag{1.53}$$

By (1.40), (1.41), (1.44), (1.45), we get

$$\begin{aligned} \underline{d}_b^{S^1\times\mathbf{R}} &= (dx - dy)\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + \frac{1}{b^2}y\,dy, \\ \underline{d}_b^{S^1\times\mathbf{R}*} &= -i_{\frac{\partial}{\partial x}+\frac{\partial}{\partial y}}\frac{\partial}{\partial x} - i_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} + \frac{1}{b^2}y\,i_{\frac{\partial}{\partial y}}. \end{aligned} \tag{1.54}$$

Then (1.11), (1.18), (1.21), (1.47), (1.50) or an easy direct computation show that

$$\frac{1}{2}\left[\underline{d}_b^{S^1\times\mathbf{R}}, \underline{d}_b^{S^1\times\mathbf{R}*}\right] = \mathcal{L}''_b. \tag{1.55}$$

Let $\Omega^{\cdot c}\left(S^1\times\mathbf{R}\right)$ be the vector space of smooth forms on $S^1\times\mathbf{R}$ with compact support. Let r be the map $(x,y)\to(x,-y)$. Let h be the symmetric bilinear form on $\Omega^{\cdot c}\left(S^1\times\mathbf{R}\right)$,

$$h(s,s') = \int_{S^1\times\mathbf{R}} \langle r^*s, s'\rangle\, dxdy. \tag{1.56}$$

In (1.56), $\langle\rangle$ is the obvious scalar product on $\Lambda^{\cdot}\left(T^*\left(S^1\times\mathbf{R}\right)\right)$. Then (1.54) shows that $\underline{d}_b^{S^1\times\mathbf{R}*}$ is the formal adjoint of $\underline{d}_b^{S^1\times\mathbf{R}}$ with respect to h.

Still $\underline{d}_b^{S^1\times\mathbf{R}}$ has no obvious relation to the de Rham operator. However, observe that

$$\exp\left(dyi_{\frac{\partial}{\partial x}}\right)\underline{d}_b^{S^1\times\mathbf{R}}\exp\left(-dyi_{\frac{\partial}{\partial x}}\right)=dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}+\frac{1}{b^2}ydy, \tag{1.57}$$

which we can rewrite in the form,

$$\exp\left(dyi_{\frac{\partial}{\partial x}}\right)\underline{d}_b^{S^1\times\mathbf{R}}\exp\left(-dyi_{\frac{\partial}{\partial x}}\right)=d_{1/b^2}^{S^1\times\mathbf{R}}. \tag{1.58}$$

Set

$$\overline{d}_b^{S^1\times\mathbf{R}*}=\exp\left(dyi_{\frac{\partial}{\partial x}}\right)\underline{d}_b^{S^1\times\mathbf{R}*}\exp\left(-dyi_{\frac{\partial}{\partial x}}\right). \tag{1.59}$$

By (1.54),

$$\overline{d}_b^{S^1\times\mathbf{R}*}=-i_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x}-i_{\frac{\partial}{\partial y}-\frac{\partial}{\partial x}}\frac{\partial}{\partial y}+\frac{y}{b^2}i_{\frac{\partial}{\partial y}-\frac{\partial}{\partial x}}. \tag{1.60}$$

By (1.52), (1.55), and (1.58)-(1.60), we get

$$\frac{1}{2}\left[d_{1/b^2}^{S^1\times\mathbf{R}},\overline{d}_b^{S^1\times\mathbf{R}*}\right]=\frac{1}{2}\left(-\frac{\partial^2}{\partial y^2}+\frac{y^2}{b^4}-\frac{1}{b^2}\right)+\frac{1}{b^2}\left(N-dyi_{\frac{\partial}{\partial x}}\right)-\frac{y}{b^2}\frac{\partial}{\partial x}. \tag{1.61}$$

Equations (1.54) and (1.61) should give ample matter to think about. First we consider (1.54). Observe that

$$\left(dx-i_{\frac{\partial}{\partial x}}\right)^2=-1,\qquad\qquad \left(dy+i_{\frac{\partial}{\partial y}}\right)^2=1, \tag{1.62}$$

which shows that

$$\left(dx-dy-i_{\frac{\partial}{\partial x}+\frac{\partial}{\partial y}}\right)^2=0. \tag{1.63}$$

Equation (1.63) exactly says the operator $dx-dy-i_{\frac{\partial}{\partial x}+\frac{\partial}{\partial y}}$ is nilpotent. This in turn explains why there is no term $\frac{\partial^2}{\partial x^2}$ in the right-hand side of (1.55).

Now we concentrate on the pair $\left(d_{1/b^2}^{S^1\times\mathbf{R}},\overline{d}_b^{S^1\times\mathbf{R}*}\right)$. From (1.57), (1.58), we will obtain an analogue of the bilinear form h. Indeed let $\mathfrak{h}^{T(S^1\times\mathbf{R})}$ be the bilinear form on $T\left(S^1\times\mathbf{R}\right)=\mathbf{R}\oplus\mathbf{R}$ which is given by the matrix,

$$\mathfrak{h}^{T(S^1\times\mathbf{R})}=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}. \tag{1.64}$$

The corresponding bilinear form on $T^*\left(S^1\times\mathbf{R}\right)$, which we denote by $\mathfrak{h}^{T^*(S^1\times\mathbf{R})}$, is given by

$$\mathfrak{h}^{T^*(S^1\times\mathbf{R})}=\begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}. \tag{1.65}$$

Then $\mathfrak{h}^{T^*(S^1\times\mathbf{R})}$ induces a corresponding symmetric bilinear form $\mathfrak{h}^{\Lambda^\cdot(T^*(S^1\times\mathbf{R}))}$ on $\Lambda^\cdot\left(T^*\left(S^1\times\mathbf{R}\right)\right)$.

Let $\mathfrak{h}$ be the symmetric bilinear form on $\Omega^{\cdot c}\left(S^1 \times \mathbf{R}\right)$ which is given by

$$\mathfrak{h}\left(s, s'\right) = \int_{S^1 \times \mathbf{R}} \mathfrak{h}^{\Lambda^{\cdot}\left(T^*\left(S^1 \times \mathbf{R}\right)\right)}\left(s\left(x, -y\right), s'\left(x, y\right)\right) dx dy. \tag{1.66}$$

Observe that in (1.66), the map r is made only to act on the function $s\left(x, y\right)$ without acting on the form part of s. Then $\overline{d}_b^{S^1 \times \mathbf{R}*}$ is the formal adjoint of $d_{1/b^2}^{S^1 \times \mathbf{R}}$ with respect to $\mathfrak{h}$.

The bilinear forms h and $\mathfrak{h}$ are symmetric, but they are non local, in the sense their construction involves the antipodal map r. Consider instead the matrix ϕ acting on $T\left(S^1 \times \mathbf{R}\right)$,

$$\phi = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.67}$$

and the corresponding bilinear form η on $T\left(S^1 \times \mathbf{R}\right)$,

$$\eta\left(U, V\right) = \left\langle U, \phi V \right\rangle. \tag{1.68}$$

Then

$$\phi^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}. \tag{1.69}$$

If we identify $T\left(S^1 \times \mathbf{R}\right)$ and $T^*\left(S^1 \times \mathbf{R}\right)$ by ϕ, the corresponding bilinear form η^* on $T^*\left(S^1 \times \mathbf{R}\right)$ is given by

$$\eta^*\left(s, s'\right) = \left\langle \phi^{-1} s, s' \right\rangle. \tag{1.70}$$

Then η^* induces a nondegenerate bilinear form on $\Lambda^{\cdot}\left(T^*\left(S^1 \times \mathbf{R}\right)\right)$. If $s, s' \in \Omega^{\cdot c}\left(S^1 \times \mathbf{R}\right)$, set

$$\eta\left(s, s'\right) = \int_{S^1 \times \mathbf{R}} \eta^*\left(s, s'\right) dx dy. \tag{1.71}$$

Then one verifies that

$$\eta\left(s, d_{1/b^2}^{S^1 \times \mathbf{R}} s'\right) = \eta\left(\overline{d}_b^{S^1 \times \mathbf{R}*} s, s'\right). \tag{1.72}$$

Let ω be the symplectic form on $S^1 \times \mathbf{R}$,

$$\omega = dy \wedge dx. \tag{1.73}$$

Then observe that if $U, V \in T\left(S^1 \times \mathbf{R}\right)$,

$$\eta\left(U, V\right) = \left\langle \pi_* U, \pi_* V \right\rangle + \omega\left(U, V\right). \tag{1.74}$$

2. Hodge theory and the Witten Laplacian

In this section, we briefly recall elementary results in Hodge theory. Also we describe the Witten deformation of the classical Hodge Laplacian. Finally, we show that on a symplectic manifold, up to a constant, the symplectic Witten Laplacian is the Lie derivative operator associated with the corresponding Hamiltonian vector field.

This section is organized as follows. In subsection 2.1, we recall known results on Hodge theory and on the Witten Laplacian.

In subsection 2.2, we give a formula for the symplectic Witten Laplacian.

2.1. Classical Hodge theory and the Witten Laplacian. — Let X be a compact Riemannian manifold of dimension n, let g^{TX} be the metric on TX, and let dv_X be the associated volume form. The metric g^{TX} induces a corresponding scalar product $\langle\rangle_{\Lambda^{\cdot}(T^*X)}$ on $\Lambda^{\cdot}(T^*X)$.

Let $\left(\Omega^{\cdot}(X), d^X\right)$ be the de Rham complex on X. Let $\langle\rangle_{\Omega^{\cdot}(X)}$ be the scalar product on $\Omega^{\cdot}(X)$ associated with g^{TX}, i.e.,

$$\langle s, s'\rangle_{\Omega^{\cdot}(X)} = \int_X \langle s, s'\rangle_{\Lambda^{\cdot}(T^*X)}\, dv_X. \tag{2.1}$$

Let d^{X*} be the formal adjoint of d^X with respect to $\langle\rangle_{\Omega^{\cdot}(X)}$.

The Hodge Laplacian $\square^X$ is given by

$$\square^X = \left[d^X, d^{X*}\right]. \tag{2.2}$$

The Hodge Laplacian $\square^X$ is a second order elliptic self-adjoint nonnegative operator, whose principal symbol is $|\xi|^2$. If Δ^X is the Laplace-Beltrami operator, the restriction of $\square^X$ to smooth functions coincides with $-\Delta^X$.

Let $\mathcal{H} = \ker \square^X$ be the finite dimensional vector space of the harmonic forms. The basic result of Hodge theory asserts that

$$\mathcal{H} \simeq H^{\cdot}(X, \mathbf{R}). \tag{2.3}$$

Now we briefly describe the Witten deformation [**20**] of the above Hodge Laplacian. Its purpose is to provide an interpolation between classical Hodge theory and Morse theory. Let $f : X \to \mathbf{R}$ be a smooth function. For $T \geq 0$, as in (1.43), set

$$d_T^X = e^{-Tf} d^X e^{Tf}. \tag{2.4}$$

The complex $\left(\Omega^{\cdot}(X), d_T^X\right)$ is canonically isomorphic to the complex $\left(\Omega^{\cdot}(X), d^X\right)$.

Let d_T^{X*} be the formal adjoint of d_T^X with respect to $\langle\rangle_{\Omega^{\cdot}(X)}$, so that

$$d_T^{X*} = e^{Tf} d^{X*} e^{-Tf}. \tag{2.5}$$

The corresponding Witten Laplacian $\square_T^X$ is given by

$$\square_T^X = \left[d_T^X, d_T^{X*}\right]. \tag{2.6}$$

The Laplacian $\square_T^X$ has exactly the same properties as $\square^X$. In particular if

$$\mathcal{H}_T = \ker \square_T^X, \tag{2.7}$$

then

$$\mathcal{H}_T \simeq H^{\cdot}(X, \mathbf{R}). \tag{2.8}$$

Of course, for $T = 0$, $\square_T^X$ coincides with $\square^X$, so that $\square_T^X$ is a deformation of $\square^X$.

Clearly,

$$d_T^X = d^X + T df \wedge, \qquad d_T^{X*} = d^{X*} + T i_{\nabla f}. \tag{2.9}$$

Let $e_1, \ldots, e_n$ be an orthonormal basis of TX, let $e^1, \ldots, e^n$ be the corresponding dual basis of T^*X. From (2.9) we deduce that

$$\square_T^X = \square^X + T^2 |df|^2 + T\left(2\left\langle \nabla_{e_i}^{TX} \nabla f, e_j\right\rangle e^i i_{e_j} - \Delta^{TX} f\right). \tag{2.10}$$

An essentially equivalent construction is to keep d^X fixed, and instead to consider the adjoint of d^X with respect to the L_2 scalar product in (2.1), in which the volume form dv_X has been replaced by $e^{-2Tf} dv_X$. The adjoint of d^X is just d_{2T}^{X*} and the associated Laplacian is given by $e^{Tf} \square_T^X e^{-Tf}$.

Assume f to be a Morse function. Using (2.10), Witten showed in [**20**] that as $T \to +\infty$, most of the spectrum of $\square_T^X$ tends to $+\infty$, and the remaining finite eigenvalues tend to 0. Some of these are exactly 0, and correspond to the harmonic forms, and others are asymptotically small, decaying to 0 like $e^{-cT}, c > 0$. Let $F_T^{\cdot}$ be the direct sum of eigenforms of $\square_T^X$ for eigenvalues ≤ 1. Witten showed that as $T \to +\infty$, $F_T^{\cdot}$ localizes near the critical points of f. More precisely, F_T^i localizes near the critical points of index i. Also Witten conjectured that as $T \to +\infty$, the complex $\left(F_T^{\cdot}, d_T^X\right)$ approximates in the proper sense a complex constructed from the 'instantons' which connect the critical points. These instantons are integral curves of the gradient field ∇f. When ∇f is Morse-Smale, this complex was identified to be the Morse-Smale complex associated with ∇f. In [**12**], Helffer and Sjöstrand proved the conjecture of Witten. For another proof we refer to [**10**].

The Witten deformation was used in Bismut-Zhang [**9, 10**] to give a new proof of the Cheeger-Müller theorem [**11, 16**].

One of the main motivations given in [**3**] for the introduction of the hypoelliptic Laplacian has been an attempt to extend the construction of the Witten Laplacian to the loop space LX of X. On LX there are many natural functionals like the energy. If the Witten Laplacian associated with the energy existed, it would interpolate between the Hodge Laplacian $\square^{LX}$ of LX and the Morse theory of the energy functional, whose critical points are precisely the closed geodesics. The hypoelliptic Laplacian provides a semiclassical version of this interpolation. For a review of these aspects of the hypoelliptic Laplacian, we refer the reader to [**4, 6**].

2.2. The symplectic Witten Laplacian. — Let (M,ω) be a symplectic manifold of dimension n. The nondegenerate bilinear form ω on TM induces an isomorphism $\phi : TM \to T^*M$, so that

$$\omega(U,V) = \langle U, \phi V\rangle . \tag{2.11}$$

Let ω^* be the nondegenerate bilinear form on T^*M which corresponds to ω via the canonical isomorphism ϕ. We still denote by ω^* the associated bilinear form on $\Lambda^{\cdot}(T^*M)$. The form ω determines a volume form dv_M on M.

If $\alpha \in \Lambda^{\cdot}(T^*M)$, set

$$L\alpha = \omega \wedge \alpha. \tag{2.12}$$

Let Λ be the adjoint of L with respect to ω^*, so that

$$\omega^*(\Lambda s, s') = \omega^*(s, Ls'). \tag{2.13}$$

The operators L, Λ are the well-known Lefschetz operators. Let N be the number operator of $\Lambda^{\cdot}(T^*M)$, i.e. the operator which acts by multiplication by k on $\Lambda^k(T^*M)$. Set

$$H = \frac{1}{2}(N - n/2) \tag{2.14}$$

Then we have the well-known commutation relations

$$[H,L] = L, \qquad [H,\Lambda] = -\Lambda, \qquad [L,\Lambda] = 2H. \tag{2.15}$$

Let $\overline{d}^M$ be the formal adjoint of d^M with respect to the bilinear form associated with the symplectic form ω as in (1.71), (1.72), with η^* replaced by ω^* in (1.71).

Now we state the simple result in [**3**, Theorem 2.2].

Proposition 2.1. — *The following identities hold:*

$$\overline{d}^M = -\left[d^M, \Lambda\right], \qquad d^M = -\left[\overline{d}^M, L\right], \qquad \left[d^M, \overline{d}^M\right] = 0. \tag{2.16}$$

Proof. — Using Darboux's theorem, we may as well assume that locally, the form ω has constant coefficients. Then (2.16) is elementary linear algebra. In particular the last identity is just a reflection of the fact that ω vanishes on the diagonal. □

Let $\mathcal{H} : M \to \mathbf{R}$ be a smooth function. Let $d^M_{\mathcal{H}}$ be the twisted de Rham operator

$$d^M_{\mathcal{H}} = e^{-\mathcal{H}} d^M e^{-\mathcal{H}}, \tag{2.17}$$

and let $\overline{d}^M_{\mathcal{H}}$ be its symplectic adjoint, i.e.,

$$\overline{d}^M_{\mathcal{H}} = e^{\mathcal{H}} \overline{d}^M e^{-\mathcal{H}}. \tag{2.18}$$

Then $\left[d^M_{\mathcal{H}}, \overline{d}^M_{\mathcal{H}}\right]$ is the symplectic Witten Laplacian associated with $\mathcal{H}$.

Let $Y^{\mathcal{H}}$ be the Hamiltonian vector field associated with $\mathcal{H}$, so that

$$d\mathcal{H} + i_{Y^{\mathcal{H}}}\omega = 0. \tag{2.19}$$

Let $L_{Y^{\mathcal{H}}}$ be the Lie derivative operator associated with $Y^{\mathcal{H}}$.

Now we state a simple formula established in [**6**, eq. (2.34)].

Proposition 2.2. — *The following identity holds:*

$$\left[d^M_{\mathcal{H}}, \overline{d}^M_{\mathcal{H}}\right] = -2L_{Y^{\mathcal{H}}}. \tag{2.20}$$

Proof. — One verifies easily that

$$\overline{d}^M_{2\mathcal{H}} = \overline{d}^M - 2i_{Y^{\mathcal{H}}}, \tag{2.21}$$

so that using (2.16), we get

$$\left[d^M, \overline{d}^M_{2\mathcal{H}}\right] = -2L_{Y^{\mathcal{H}}}. \tag{2.22}$$

By conjugating (2.22) by $e^{-\mathcal{H}}$ and using the fact that $Y^{\mathcal{H}}$ preserves $\mathcal{H}$, we get (2.22). □

Proposition 2.2 is quite interesting. Indeed remember that our ultimate goal is to interpolate between the Hodge Laplacian $\square^X$ of the Riemannian manifold X and the generator $L_{Y^{\mathcal{H}}}$ of the geodesic flow on the total space $\mathcal{X}^*$ of the cotangent bundle of X. However, (2.20) indicates that $L_{Y^{\mathcal{H}}}$ is itself a symplectic Witten Laplacian. One possible construction of the hypoelliptic Laplacian consists in linearly interpolating between the scalar product of TX and the symplectic form of $\mathcal{X}^*$. This point of view is explained in detail in [**3**, section 2.12].We also refer to equations (1.74) and (3.5) for a hint on how to do this.

3. The hypoelliptic Laplacian in de Rham theory

The purpose of this section is to construct the hypoelliptic Laplacian in de Rham theory. This operator, which acts on the total space $\mathcal{X}^*$ of the cotangent bundle of a Riemannian manifold X, depends on a parameter $b > 0$. Also we give arguments showing that it should interpolate between the standard Hodge Laplacian of X and the generator of the geodesic flow on $\mathcal{X}^*$.

This section is organized as follows. In subsection 3.1, we give a formula for the operator $d^{\mathcal{X}^*}$.

In subsection 3.2, we introduce a symmetric bilinear form on $\Omega^{\cdot c}(\mathcal{X}^*)$, and we obtain the formal adjoint $\overline{d}^{\mathcal{X}^*}$ of $d^{\mathcal{X}^*}$ with respect to this form.

In subsection 3.3, given a Hamiltonian function $\mathcal{H}$ on $\mathcal{X}^*$, we obtain corresponding symmetric bilinear forms, and we construct the adjoint of the Witten twist $d^{\mathcal{X}^*}_{\mathcal{H}}$.

In subsection 3.4, we discuss the self-adjointness of our first order differential operators.

In subsection 3.5, we give Weitzenböck formulas for our new Hodge like Laplacians.

In subsection 3.6, when the function $\mathcal{H}$ is proportional to $|p|^2/2$, we show that our new Laplacians are hypoelliptic.

In subsection 3.7, we show that $b \to 0$, the hypoelliptic Laplacian should converge in the proper sense to the classical Hodge Laplacian of X.

Finally, in subsection 3.8, we give arguments showing that as $b \to +\infty$, the hypoelliptic Laplacian converges to the generator of the geodesic flow.

3.1. The de Rham operator on $\mathcal{X}^*$. — Let X be a compact Riemannian manifold of dimension n, let $\mathcal{X}, \mathcal{X}^*$ be the total spaces of the vector bundles TX, T^*X over X, and let π denote the projection from $\mathcal{X}$ or $\mathcal{X}^*$ on X. The metric g^{TX} induces an identification of the fibres TX and T^*X, and a corresponding isomorphism of $\mathcal{X}$ and $\mathcal{X}^*$.

Let ∇^{TX} be the Levi-Civita connection on TX, and let R^{TX} be its curvature. Let ∇^{T^*X} be the corresponding connection on T^*X, and let R^{T^*X} be its curvature.

Let

$$T\mathcal{X}^* = \pi^* \left(TX \oplus T^*X\right) \tag{3.1}$$

be the splitting of $\mathcal{X}^*$ which is associated with the connection ∇^{T^*X}. In (3.1), TX corresponds to the horizontal part of $T\mathcal{X}^*$, and T^*X to the tangent bundle to the fibres T^*X.

From (3.1), we get the isomorphism,

$$\Lambda^\cdot \left(T^*\mathcal{X}^*\right) = \pi^* \left(\Lambda^\cdot \left(T^*X\right) \widehat{\otimes} \widehat{\Lambda}^\cdot \left(TX\right)\right). \tag{3.2}$$

In (3.2), $\widehat{\Lambda}^\cdot (TX)$ is our notation for the exterior algebra of the fibre, the hat permitting us to distinguish $\widehat{\Lambda}^\cdot (TX)$ from the exterior algebra $\Lambda^\cdot (TX)$. Of course $\Lambda^\cdot (TX)$ and $\widehat{\Lambda}^\cdot (TX)$ are canonically isomorphic. Let $\nabla^{\Lambda^\cdot (T^*\mathcal{X}^*)}$ be the connection on $\Lambda^\cdot (T^*\mathcal{X}^*)$ induced by ∇^{TX}.

Let $\left(\Omega^\cdot (\mathcal{X}^*), d^{\mathcal{X}^*}\right)$ be the de Rham complex of $\mathcal{X}^*$. Let $\mathbf{I}$ be the vector bundle on X of smooth sections of $\Lambda^\cdot (TX)$ along the fibre T^*X. By (3.2), we get

$$\Omega^\cdot (\mathcal{X}^*) = \Omega^\cdot (X, \mathbf{I}). \tag{3.3}$$

Let p be the tautological section of the fibre π^*T^*X over $\mathcal{X}^*$. Using (3.2), we may write $d^{\mathcal{X}^*}$ in the form,

$$d^{\mathcal{X}^*} = d^V + \nabla^{\mathbf{I}} + i_{\widehat{R^{T^*X}p}}. \tag{3.4}$$

In (3.4), d^V is the de Rham operator along the fibre T^*X, $\nabla^{\mathbf{I}}$ is the obvious connection on $\mathbf{I}$, and $i_{\widehat{R^{T^*X}p}}$ is the interior multiplication by the vertical vector $\widehat{R^{T^*X}p}$. Of course

R^{T^*X} is viewed as a 2-form on X, so that ultimately $i_{\widehat{R^{T^*X}p}}$ increases the total degree by 1.

3.2. A bilinear form on $\Omega^{\cdot c}(\mathcal{X}^*)$. — Now we inspire ourselves from the arguments which were given in subsection 1.3. Let $\Omega^{\cdot c}(\mathcal{X}^*)$ be the vector space of smooth forms on $\mathcal{X}^*$ which have compact support. Let ω be the symplectic form of $\mathcal{X}^*$. On $T\mathcal{X}^*$, let η be the nondegenerate bilinear form,

$$\eta(U,V) = \langle \pi_* U, \pi_* V\rangle + \omega(U,V). \tag{3.5}$$

The isomorphism $\phi : T\mathcal{X}^* \to T^*\mathcal{X}^*$ associated to η is given by

$$\phi = \begin{pmatrix} g^{TX} & -1|_{T^*X} \\ 1|_{TX} & 0 \end{pmatrix}. \tag{3.6}$$

Equation (3.5) should be compared with equation (1.74), and equation (3.6) should be compared with equation (1.67).

The volume form on $\mathcal{X}^*$ associated to η is exactly the symplectic volume form $dv_{\mathcal{X}^*}$. Let $\overline{d}^{\mathcal{X}^*}$ be the formal adjoint of $d^{\mathcal{X}^*}$ with respect to the bilinear form η on $\Omega^{\cdot c}(\mathcal{X}^*)$, which one obtains as in (1.71) from (3.5), (3.6). Of course, we use the same conventions as in subsection 1.3 to define the formal adjoint, and we use in particular equation (1.72).

Let $e_1, \dots, e_n$ be an orthonormal basis of TX, let $e^1, \dots, e^n$ be the corresponding dual basis of T^*X. Recall that $T\mathcal{X}^* = \pi^*(TX \oplus T^*X)$. We denote by $\widehat{e}^1, \dots, \widehat{e}^n$ the basis of the vertical fibre T^*X in $T\mathcal{X}^*$, and by $\widehat{e}_1, \dots, \widehat{e}_n$ the corresponding dual basis.

Set

$$R^{T^*X}p\wedge = \frac{1}{2} i_{\widehat{e^i}} i_{\widehat{e^j}} R^{T^*X}(e_i, e_j)\, p \wedge. \tag{3.7}$$

In (3.7), $R^{T^*X}(e_i, e_j)\,p$ is viewed as a section of T^*X, which lifts to a 1-form on $\mathcal{X}^*$. Therefore $R^{T^*X}p$ decreases the total degree by 1.

We now have the result established in [**3**, Proposition 2.10].

Proposition 3.1. — *The following identity holds:*

$$\overline{d}^{\mathcal{X}^*} = -i_{\widehat{e^i}} \nabla^{T\mathcal{X}^*}_{e_i} + i_{e_i} \nabla_{\widehat{e^i}} + R^{T^*X} p \wedge - i_{\widehat{e^i}} \nabla_{\widehat{e^i}}. \tag{3.8}$$

3.3. A Hamiltonian function. — Let $\mathcal{H} : \mathcal{X}^* \to \mathbf{R}$ be a smooth function. Let $Y^{\mathcal{H}}$ be the associated Hamiltonian vector field, so that $d\mathcal{H} + i_{Y^{\mathcal{H}}}\omega = 0$.We denote by $\widehat{\nabla^V \mathcal{H}}$ the fibrewise gradient field of $\mathcal{H}$.

Definition 3.2. — Set

$$d^{\mathcal{X}^*}_{\mathcal{H}} = e^{-\mathcal{H}} d^{\mathcal{X}^*} e^{\mathcal{H}}, \qquad \overline{d}^{\mathcal{X}^*}_{\mathcal{H}} = e^{\mathcal{H}} \overline{d}^{\mathcal{X}^*} e^{-\mathcal{H}}. \tag{3.9}$$

Observe that $\overline{d}_{\mathcal{H}}^{\mathcal{X}^*}$ is the adjoint of $d_{\mathcal{H}}^{\mathcal{X}^*}$ with respect to η. Also, if $s, s' \in \Omega^{\cdot c}(\mathcal{X}^*)$, put

$$\eta_{\mathcal{H}}(s, s') = \int_{\mathcal{X}^*} \eta^*(s, s')\, e^{-2\mathcal{H}} dv_{\mathcal{X}^*}. \tag{3.10}$$

Then $\overline{d}_{2\mathcal{H}}^{\mathcal{X}^*}$ is the adjoint of $d^{\mathcal{X}^*}$ with respect to $\eta_{\mathcal{H}}$.

Definition 3.3. — Set

$$A_{\mathcal{H}} = \frac{1}{2}\left(\overline{d}_{2\mathcal{H}}^{\mathcal{X}^*} + d^{\mathcal{X}^*}\right), \qquad \mathfrak{A}_{\mathcal{H}} = \frac{1}{2}\left(\overline{d}_{\mathcal{H}}^{\mathcal{X}^*} + d_{\mathcal{H}}^{\mathcal{X}^*}\right). \tag{3.11}$$

Then

$$\mathfrak{A}_{\mathcal{H}} = e^{-\mathcal{H}} A_{\mathcal{H}} e^{\mathcal{H}}. \tag{3.12}$$

Moreover,

$$A_{\mathcal{H}}^2 = \frac{1}{4}\left[d^{\mathcal{X}^*}, \overline{d}_{2\mathcal{H}}^{\mathcal{X}^*}\right]. \tag{3.13}$$

We have the result established in [**3**, Proposition 2.18].

Proposition 3.4. — *The following identities hold:*

$$\begin{aligned} A_{\mathcal{H}} &= \frac{1}{2}\left(e^i - i_{\widehat{e^i}}\right)\nabla_{e_i}^{\Lambda^{\cdot}(T^*\mathcal{X}^*)} + \frac{1}{2}\left(\widehat{e}_i + i_{e_i - \widehat{e^i}}\right)\nabla_{\widehat{e^i}} + \frac{1}{2}\left(R^{T^*X}p\wedge + i_{\widehat{R^{T^*X}p}}\right) \\ &\quad + i_{\widehat{e^i}}\nabla_{e_i}\mathcal{H} + i_{\widehat{e^i} - e_i}\nabla_{\widehat{e^i}}\mathcal{H}, \\ \mathfrak{A}_{\mathcal{H}} &= \frac{1}{2}\left(e^i - i_{\widehat{e^i}}\right)\nabla_{e_i}^{\Lambda^{\cdot}(T^*\mathcal{X}^*)} + \frac{1}{2}\left(\widehat{e}_i + i_{e_i - \widehat{e^i}}\right)\nabla_{\widehat{e^i}} + \frac{1}{2}\left(R^{T^*X}p\wedge + i_{\widehat{R^{T^*X}p}}\right) \\ &\quad + \frac{1}{2}\left(e^i + i_{\widehat{e^i}}\right)\nabla_{e_i}\mathcal{H} + \frac{1}{2}\left(\widehat{e}_i + i_{\widehat{e^i} - e_i}\right)\nabla_{\widehat{e^i}}\mathcal{H}. \end{aligned} \tag{3.14}$$

Set

$$\mu_0 = \widehat{e}_i \wedge i_{e_i}. \tag{3.15}$$

Put

$$\mathfrak{A}'_{\mathcal{H}} = e^{-\mu_0}\mathfrak{A}_{\mathcal{H}} e^{\mu_0}. \tag{3.16}$$

The proper interpretation for (3.16) can be guessed from (1.57)-(1.59). The operator $\mathfrak{A}'_{\mathcal{H}}$ will also be considered in the sequel.

3.4. A self-adjointness property. — The bilinear form $\eta_{\mathcal{H}}$ on $\Omega^{\cdot}(\mathcal{X}^*)$ is in general not symmetric. However, we will here follow the arguments in (1.64)-(1.67).

Let $\mathfrak{h}^{T\mathcal{X}^*}$ be the bilinear form on $T\mathcal{X}^* = \pi^*(TX \oplus T^*X)$ which is given by

$$\mathfrak{h}^{T\mathcal{X}^*} = \begin{pmatrix} g^{TX} & 1|_{T^*X} \\ 1|_{TX} & 0 \end{pmatrix}. \tag{3.17}$$

Let $\mathfrak{p} : T\mathcal{X}^* \to T^*X$ be the projection with respect to the above splitting of $T\mathcal{X}^*$. If $U \in T\mathcal{X}^*$, then

$$\mathfrak{h}^{T\mathcal{X}^*}(U,U) = \langle \pi_* U, \pi_* U\rangle + 2\langle \pi_* U, \mathfrak{p}U\rangle. \tag{3.18}$$

Then the volume form on $\mathcal{X}^*$ which is attached to $\mathfrak{h}^{T\mathcal{X}^*}$ is the symplectic volume form $dv_{\mathcal{X}^*}$. Let $\mathfrak{h}^{\Lambda^{\cdot}(T^*\mathcal{X})}$ be the corresponding symmetric form on $\Lambda^{\cdot}(T^*\mathcal{X}^*)$.

Let $r : (x,p) \to (x,-p)$ be the obvious involution of $\mathcal{X}^*$.

Definition 3.5. — Let $\mathfrak{h}$ be the symmetric form on $\Omega^{\cdot c}(\mathcal{X}^*)$,

$$\mathfrak{h}(s,s') = \int_{\mathcal{X}^*} \mathfrak{h}^{\Lambda^{\cdot}(T^*\mathcal{X}^*)}(s \circ r, s')\, dv_{\mathcal{X}^*}. \tag{3.19}$$

As in (1.66), in (3.19), the change of variable $p \to -p$ is not made on the form part of s. Set

$$\mathfrak{h}_{\mathcal{H}}(s,s') = \mathfrak{h}\left(e^{-2\mathcal{H}}s, s'\right). \tag{3.20}$$

If $\mathcal{H}$ is r-invariant, then $\mathfrak{h}_{\mathcal{H}}$ is a symmetric form.

Let g^{TT^*X} be the natural metric on $T\mathcal{X}^*$ which is associated with the splitting of $T\mathcal{X}^*$, and let g be the scalar product on $\Omega^{\cdot c}(\mathcal{X}^*)$ associated to g^{TT^*X}. Let h be the symmetric form on $\Omega^{\cdot c}(\mathcal{X}^*)$,

$$h(s,s') = g(r^*s, s'). \tag{3.21}$$

The symmetric forms in (3.19) and (3.21) have signature (∞,∞). If $\mathcal{H}$ is r-invariant, the same property holds for the symmetric form in (3.20) .

We state a result established in [**3**, Theorems 2.21 and 2.30].

Theorem 3.6. — *If $\mathcal{H}$ is r-invariant, $A_{\mathcal{H}}$ is $\mathfrak{h}_{\mathcal{H}}$-symmetric, $\mathfrak{A}_{\mathcal{H}}$ is $\mathfrak{h}$-symmetric, and $\mathfrak{A}'_{\mathcal{H}}$ is h-symmetric.*

3.5. The Weitzenböck formula. — We give the Weitzenböck formula established in [**3**, Theorem 3.3].

Theorem 3.7. — *The following identities hold:*

$$(3.22)\qquad A^2_{\mathcal{H}} = \frac{1}{4}\Big(-\Delta^V - \frac{1}{2}\left\langle R^{TX}(e_i,e_j)e_k,e_l\right\rangle e^ie^j i_{\widehat{e^k}} i_{\widehat{e^\ell}} + 2L_{\widehat{\nabla^V\mathcal{H}}}\Big) - \frac{1}{2}L_{Y^{\mathcal{H}}},$$

$$\mathfrak{A}^2_{\mathcal{H}} = \frac{1}{4}\Bigg(-\Delta^V - \frac{1}{2}\left\langle R^{TX}(e_i,e_j)e_k,e_l\right\rangle e^ie^j i_{\widehat{e^k}} i_{\widehat{e^\ell}} + \left|\nabla^V\mathcal{H}\right|^2$$
$$- \Delta^V\mathcal{H} + 2\left(\nabla_{\widehat{e^i}}\nabla_{\widehat{e^j}}\mathcal{H}\right)\widehat{e}_i i_{\widehat{e^j}} + 2\left(\nabla_{\widehat{e^i}}\nabla_{e_j}\mathcal{H}\right)e^j i_{\widehat{e^i}}\Bigg) - \frac{1}{2}L_{Y^{\mathcal{H}}}.$$

3.6. The hypoelliptic Laplacian. — Let N the operator counting the degree in $\widehat{\Lambda}^{\cdot}(TX)$. For $c \in \mathbf{R}$, set

$$(3.23)\qquad \mathcal{H}^c = c\frac{|p|^2}{2}.$$

Let $u \in \mathbf{R}$ be an extra variable. The following result was established in [**3**, Theorems 3.4 and 3.6].

Theorem 3.8. — *The following identities hold:*

$$A^2_{\mathcal{H}^c} = \frac{1}{4}\Big(-\Delta^V + 2cL_{\widehat{p}} - \frac{1}{2}\left\langle R^{TX}(e_i,e_j)e_k,e_l\right\rangle e^ie^j i_{\widehat{e^k}} i_{\widehat{e^\ell}}\Big) - \frac{1}{2}L_{Y^{\mathcal{H}^c}},$$

$$(3.24)\qquad \mathfrak{A}^2_{\mathcal{H}^c} = \frac{1}{4}\Big(-\Delta^V + c^2|p|^2 + c(2N-n) - \frac{1}{2}\left\langle R^{TX}(e_i,e_j)e_k,e_l\right\rangle e^ie^j i_{\widehat{e^k}} i_{\widehat{e^\ell}}\Big)$$
$$- \frac{1}{2}L_{Y^{\mathcal{H}^c}}.$$

For $c \neq 0$, the operators $\frac{\partial}{\partial u} - A^2_{\mathcal{H}^c}$, $\frac{\partial}{\partial u} - \mathfrak{A}^2_{\mathcal{H}^c}$ are hypoelliptic.

Remark 3.9. — Of course (3.24) follows from theorem 3.7. Hypoellipticity follows from Hörmander [**14**]. Any of the operators in theorem 3.8 is called a hypoelliptic Laplacian.

3.7. An interpolation property: the limit $b \to 0$ and classical Hodge theory. — In the sequel, we take $b > 0$, and we set $\mathcal{H} = |p|^2/2, c = 1/b^2$.

For $b > 0$, let K_b be the map $s(x,p) \to s(x,bp)$. By [**3**, Theorem 3.8], we get

$$(3.25)\qquad K_b 2\mathfrak{A}'^2_{\mathcal{H}^c} K_b^{-1} = \frac{\alpha}{b^2} + \frac{\beta}{b} + \gamma,$$

with α, β given by

$$(3.26)\qquad \alpha = \frac{1}{2}\left(-\Delta^V + |p|^2 - n\right) + N, \qquad\qquad \beta = -\nabla^{\Lambda^{\cdot}(T^*\mathcal{X}^*)}_{Y^{\mathcal{H}}}.$$

Observe that α is a standard self-adjoint harmonic oscillator. Also $\ker\alpha$ is spanned by the function $\exp\left(-|p|^2/2\right)$.

We identify $\Omega^{\cdot}(X)$ to $\ker \alpha$ by the map $s \to \pi^* s \exp\left(-|p|^2/2\right)/\pi^{n/4}$. Let P be the standard L_2 -projector from $\Omega^{\cdot}(\mathcal{X}^*)$ on $\ker\alpha$. Note that β maps $\ker\alpha$ into its L_2 orthogonal.

Assume for the moment that α, β are endomorphisms of a finite dimensional vector space E, that α is semisimple, so that

$$E = \ker \alpha \oplus \operatorname{Im}\alpha. \tag{3.27}$$

Let Q be the projector from E on $\ker\alpha$ with respect to the splitting (3.27). We also assume that β maps $\ker\alpha$ into $\operatorname{Im}\alpha$.

Let $u \in \operatorname{End}(E)$. We write u as a matrix with respect to the splitting (3.27).

$$u = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{3.28}$$

Suppose u to be invertible. Now we give a matrix expression for the inverse u^{-1} of u under the assumption that D is invertible. We will assume implicitly that other matrix expressions are invertible as well. We have the formula,

$$u^{-1} = \begin{bmatrix} \left(A - BD^{-1}C\right)^{-1} & -\left(A - BD^{-1}C\right)^{-1} BD^{-1} \\ -D^{-1}C\left(A - BD^{-1}C\right)^{-1} & D^{-1} + D^{-1}C\left(A - BD^{-1}C\right)^{-1} BD^{-1} \end{bmatrix}. \tag{3.29}$$

Let α^{-1} be the inverse of α restricted to $\operatorname{Im}\alpha$. By (3.29), when $\lambda \in \mathbf{C}$, we get

$$\begin{aligned}\left(\lambda - \frac{\alpha}{b^2} - \frac{\beta}{b} - \gamma\right)^{-1} \\ = \begin{bmatrix} \left(\lambda - Q\gamma Q + Q\beta\alpha^{-1}\beta Q\right)^{-1} + \mathcal{O}(b) & \mathcal{O}(b) \\ \mathcal{O}(b) & \mathcal{O}\left(b^2\right) \end{bmatrix}.\end{aligned} \tag{3.30}$$

By (3.30) we find that as $b \to 0$,

$$\left(\lambda - \frac{\alpha}{b^2} - \frac{\beta}{b} - \gamma\right)^{-1} = Q\left(\lambda - Q\left(\gamma - \beta\alpha^{-1}\beta\right)Q\right)^{-1} Q + \mathcal{O}(b). \tag{3.31}$$

The operator appearing in the limit $b \to 0$ is $Q\left(\gamma - \beta\alpha^{-1}\beta\right)Q$ acting on $\ker\alpha$.

Passing from the above finite dimensional argument to an infinite dimensional considered in (3.25) is a wild jump. However, this is the sort of situation one encounters typically in adiabatic limit problems in the theory of Quillen metrics [**1**, **7**]. The major difference is that the operators considered in these references are elliptic and self-adjoint, which is not the case here.

We have given enough motivation for studying the operator $P\left(\gamma - \beta\alpha^{-1}\beta\right)P$ in the context of (3.25).

In [**3**, Theorem 3.14], the following result is established.

Theorem 3.10. — *The following identity holds:*

$$P\left(\gamma-\beta\alpha^{-1}\beta\right)P=\frac{1}{2}\square^{X}. \tag{3.32}$$

Remark 3.11. — Theorem 3.10 gives an argument in favour of the fact that $A^2_{\mathcal{H}^c}$ is a deformation of $\square^X/4$.

In joint work with Lebeau [**8**], the hard analysis involved in the convergence as $b\to 0$ of the resolvent of $2\mathfrak{A}'^2_{\mathcal{H}^c}$ to the resolvent of $\frac{1}{2}\square^X$ is carried out in detail. The convergence is also valid for the traces of the corresponding heat kernels, as well as for the spectrum of these operators.

3.8. An interpolation property: the limit $b\to+\infty$ and the geodesic flow. — We still take $\mathcal{H}=|p|^2/2, c=1/b^2$. Let r_b be the map $(x,p)\to(x,bp)$. Using (3.22), we get

$$r^*_{b^2}2\mathfrak{A}^2_{\mathcal{H}^c}r^*_{1/b^2}=\frac{1}{2}|p|^2-L_{Y^{\mathcal{H}}}+\mathcal{O}(1/b). \tag{3.33}$$

The dynamics associated to the right-hand side of (3.33) is the geodesic flow.

From (3.33), we deduce that when $b\to+\infty$, the trace of an operator like $\exp\left(-A^2_{\mathcal{H}^c}\right)$ should localize around closed geodesics.

4. The hypoelliptic Dirac operator

The purpose of this section is to briefly develop the construction of the hypoelliptic Dirac operator obtained in [**5**] in the case of Kähler manifolds. This deformation of the classical elliptic Dirac operator is not a generalization of what was done in section 3.

This section is organized as follows. In subsection 4.1, we discuss another method to obtain a Laplacian which looks like the hypoelliptic Laplacian of section 3.

In subsection 4.2, we construct the hypoelliptic Dirac operator, which depends again on a parameter $b>0$.

In subsection 4.3, by squaring our Dirac operator, we get our new hypoelliptic Laplacian.

In subsection 4.4, we give arguments in favour of the fact that as $b\to 0$, our hypoelliptic Laplacian converges in the proper sense to the classical elliptic Hodge Dolbeault Laplacian of X.

4.1. Another approach to hypoellipticity. — Let $\left(X,g^{TX}\right)$ be a compact Riemannian manifold, let $\mathcal{X}$ be the total space of TX. The generic element of $\mathcal{X}$ will be denoted (x,Y). We will now try to give another approach to the construction of a second order hypoelliptic operator on $\mathcal{X}$.

Let $Y^{\mathcal{H}}$ be the generator of the geodesic flow over $\mathcal{X}$, and let $L_{Y^{\mathcal{H}}}$ be the corresponding Lie derivative operator. Then

$$L_{Y^{\mathcal{H}}} = \left[d^{\mathcal{X}}, i_{Y^{\mathcal{H}}}\right]. \tag{4.1}$$

On the other hand, one would would like to obtain as a square of a Dirac operator an operator $\mathcal{L}$ looking like the sum of a harmonic oscillator in the variable Y and of $\nabla_{Y^{\mathcal{H}}}$, i.e.,

$$\mathcal{L} = \frac{1}{2}\left(-\Delta^{V} + |Y|^{2} - n\right) + \nabla_{Y^{\mathcal{H}}}. \tag{4.2}$$

We still write $d^{\mathcal{X}}$ as in (3.4), i.e.,

$$d^{\mathcal{X}} = d^{V} + \nabla^{\mathbf{I}} + i_{\widehat{R^{TX}Y}}. \tag{4.3}$$

Equation (4.3) expresses $d^{\mathcal{X}}$ as a superconnection on $\mathbf{I}$ in the sense of Quillen [**18**].

For $\nabla_{Y^{\mathcal{H}}}$ to appear in (4.2), one should think of replacing $d^{\mathcal{X}}$ by $d^{\mathcal{X}} + i_{Y^{\mathcal{H}}}$. However, how to obtain the full operator $\mathcal{L}$ is not clear, not to speak of the possibility of producing a deformation of the classical elliptic Dirac operator or of its square.

4.2. The hypoelliptic Dirac operator. — To explain the construction of the hypoelliptic deformation of the Dirac operator which is carried out in [**5**], we will work in the context of complex Kähler manifolds.

Let $\left(X, g^{TX}\right)$ be a compact complex Kähler manifold of real dimension n. Let TX be the holomorphic tangent bundle to X, and let $T_{\mathbf{R}}X$ be the corresponding real tangent bundle. Let $\left(E, g^{E}\right)$ be a holomorphic Hermitian vector bundle on X. We denote by ∇^{TX}, ∇^{E} the holomorphic Hermitian connections on TX, E, and by R^{TX}, R^{E} their curvatures. Let $\nabla^{\Lambda^{\cdot}\left(\overline{T^{*}X}\otimes E\right)}$ be the corresponding connection on $\Lambda^{\cdot}\left(\overline{T^{*}X}\right)\otimes E$.

Let $\left(\Omega^{(0,\cdot)}(X,E), \overline{\partial}^{X}\right)$ be the Dolbeault complex of smooth antiholomorphic forms on X with coefficients in E. The cohomology of this complex is denoted $H^{(0,\cdot)}(X,E)$.

Let $\langle\rangle$ be the L_2 Hermitian product on $\Omega^{(0,\cdot)}(X,E)$ which is associated with g^{TX}, g^{E}. Let $\overline{\partial}^{X*}$ be the formal adjoint of $\overline{\partial}^{X}$ with respect to $\langle\rangle$. Set

$$D^{X} = \overline{\partial}^{X} + \overline{\partial}^{X*}. \tag{4.4}$$

If $u \in TX$, let $u^{*} \in \overline{T^{*}X}$ corresponding to u by g^{TX}. Recall that $\Lambda^{\cdot}\left(\overline{T^{*}X}\right)$ is a $\left(T_{\mathbf{R}}X, g^{T_{\mathbf{R}}X}\right)$ Clifford algebra. Namely if $u \in TX$, set

$$c(u) = \sqrt{2}u^{*}\wedge, \qquad c(\overline{u}) = -\sqrt{2}i_{\overline{u}}. \tag{4.5}$$

We extend the definition of c to $T_{\mathbf{R}}X \otimes_{\mathbf{R}} \mathbf{C}$ by linearity. If $U, V \in T_{\mathbf{R}}X$, then

$$c(U)c(V) + c(V)c(U) = -2\langle U, V\rangle. \tag{4.6}$$

By [**13**], $\sqrt{2}D^X$ is a Dirac operator. Namely, if $e_1, \dots, e_n$ is an orthonormal basis of $T_{\mathbf{R}}X$, then

$$\sqrt{2}D^X = c(e_i)\nabla_{e_i}^{\Lambda^\cdot(\overline{T^*X})\otimes E}. \tag{4.7}$$

Let $\pi : \mathcal{X} \to X$ be the total space of TX, with fibre $\widehat{TX}$. The hat on $\widehat{TX}$ will allow us to distinguish the fibre $\widehat{TX}$ from the tangent bundle to X. Then $\mathcal{X}$ is a also a complex manifold. Let $i : X \to \mathcal{X}$ be the embedding of X into $\mathcal{X}$ as the zero section of $\widehat{TX}$. Using the connection ∇^{TX}, we have the identification of smooth vector bundles,

$$T\mathcal{X} \simeq \pi^*\left(TX \oplus \widehat{TX}\right). \tag{4.8}$$

From (4.8), we get the smooth identification,

$$\Lambda^\cdot\left(\overline{T^*\mathcal{X}}\right) = \pi^*\left(\Lambda^\cdot\left(\overline{T^*X}\right)\widehat{\otimes}\Lambda^\cdot\left(\overline{\widehat{T^*X}}\right)\right). \tag{4.9}$$

Set

$$F = \pi^*\left(\Lambda^\cdot(T^*X) \otimes E\right). \tag{4.10}$$

In (4.10), $\Lambda^\cdot(T^*X)$ is the holomorphic exterior algebra of the base X. However, since TX and $\widehat{TX}$ are isomorphic, $\Lambda^\cdot(T^*X)$ will also be considered as the holomorphic exterior algebra of the fibre $\widehat{TX}$.

Let $\left(\Omega^{(0,\cdot)}(\mathcal{X}, F), \overline{\partial}^{\mathcal{X}}\right)$ be the Dolbeault complex of smooth antiholomorphic forms on $\mathcal{X}$ with coefficients in F.

Let $\mathbf{I}$ be the vector bundle on X of the smooth sections of $\pi^*\left(\Lambda^\cdot\left(\overline{\widehat{T^*X}}\right) \otimes E\right)$ along the fibre $\widehat{TX}$. By proceeding as in (3.4) and using (4.9), we get

$$\overline{\partial}^{\mathcal{X}} = \nabla^{\mathbf{I}''} + \overline{\partial}^V. \tag{4.11}$$

In (4.11), $\overline{\partial}^V$ is the Dolbeault operator along the fibre $\widehat{TX}$, and $\nabla^{\mathbf{I}''}$ is the horizontal part of $\overline{\partial}^{\mathcal{X}}$. Note that contrary to what happens in (3.4), there is no extra term in (4.11). Writing $\overline{\partial}^{\mathcal{X}}$ in the form (4.11) emphasizes the fact that $\overline{\partial}^{\mathcal{X}}$ can also be viewed as a holomorphic superconnection on $\mathbf{I}$.

Let y be the tautological holomorphic section of $\pi^*\widehat{TX}$ over $\mathcal{X}$, and let $Y = y + \overline{y}$ be the corresponding section of $\pi^*\widehat{T_{\mathbf{R}}X}$. Of course $\widehat{TX}$ and TX are canonically isomorphic. In particular the operator i_y acts on $\pi^*\Lambda^\cdot(T^*X)$. The Koszul complex $(\mathcal{O}_{\mathcal{X}}\pi^*\Lambda^\cdot(T^*X), i_y)$ provides a resolution of the sheaf $i_*\mathcal{O}_X$. More generally the Koszul complex $(\mathcal{O}_{\mathcal{X}}(F), i_y)$ provides a resolution of $i_*\mathcal{O}_X(E)$.

Observe that

$$\left(\overline{\partial}^{\mathcal{X}} + i_y\right)^2 = 0. \tag{4.12}$$

Equation (4.12) reflects the fact that $\left(\Omega^\cdot(\mathcal{X}), \overline{\partial}^{\mathcal{X}} + i_y\right)$ is the Dolbeault resolution of the Koszul complex we just considered.

For $b > 0$, set

$$A''_b = \overline{\partial}^{\mathcal{X}} + i_y/b^2. \tag{4.13}$$

By (4.11), (4.13), we get

$$A''_b = \nabla^{\mathbf{I}''} + \overline{\partial}^V + i_y/b^2. \tag{4.14}$$

Then A''_b can be viewed as an operator acting on $\Omega^{(0,\cdot)}(\mathcal{X}, F)$. By (4.12),

$$A''^2_b = 0. \tag{4.15}$$

Let $\overline{\partial}^{V*}$ be the fibrewise formal adjoint of $\overline{\partial}^V$. Now we will take the 'adjoint' of A''_b partly in the sense of superconnections. Namely set

$$A'_b = \nabla^{\mathbf{I}'} + \overline{\partial}^{V*} + i_{\overline{y}}/b^2. \tag{4.16}$$

Then A'_b also acts on $\Omega^{(0,\cdot)}(\mathcal{X}, F)$. Indeed $\nabla^{\mathbf{I}'}$ increases the degree in $\Lambda^{\cdot}(T^*X)$ by 1, and $i_{\overline{y}}$ decreases the degree in $\Lambda^{\cdot}\left(\overline{T^*X}\right)$ by 1. Moreover,

$$A'^2_b = 0. \tag{4.17}$$

Set

$$A_b = A''_b + A'_b. \tag{4.18}$$

When making instead $y = 0$, we will denote by A'', A', A the corresponding operators. In particular, when identifying $Y \in \widehat{T_{\mathbf{R}}X}$ to the corresponding section of $T_{\mathbf{R}}X$, we get

$$A_b = A + i_Y/b^2. \tag{4.19}$$

Also A is a superconnection on $\mathbf{I}$.

Observe that the principal symbol of A or of A_b is exactly $i\xi^H \wedge +ic\left(\xi^V\right)/\sqrt{2}$, where ξ^H, ξ^V are the horizontal and vertical components of $\xi \in T^*_{\mathbf{R}}\mathcal{X}$. In particular the principal symbol of A^2_b is just $\left|\xi^V\right|^2/2$. Adding i_Y has no effect on the principle symbol of A^2_b. However,

$$A^2_b = A^2 + \left[A, i_Y/b^2\right]. \tag{4.20}$$

Now in $[A, i_Y]$ appears the critical operator $\nabla^{\mathbf{I}}_Y$, which makes the operator A^2_b hypoelliptic.

The operator A^2_b is still not the right one, since it does not contain a positive multiple of $|Y|^2/2$, which is needed to produce a harmonic oscillator in the fibre direction.

So we slightly modify the above construction. Let ω^X be the Kähler form associated with the metric g^{TX}. If J is the complex structure of $T_{\mathbf{R}}X$, if $U, V \in T_{\mathbf{R}}X$, then

$$\omega^X(U, V) = \langle U, JV\rangle. \tag{4.21}$$

We will view ω^X as a section of $\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}\left(\overline{T^*X}\right)$.

Put

$$B_b'' = A_b'', \qquad B_b' = e^{i\omega^X} A_b' e^{-i\omega^X}, \qquad B_b = B_b'' + B_b'. \tag{4.22}$$

Since ω^X is closed, we get the formula,

$$B_b' = A_b' + \overline{y}^* \wedge /b^2. \tag{4.23}$$

Of course, we still have

$$B_b''^2 = 0, B_b'^2 = 0. \tag{4.24}$$

However, the effect of the addition of $\overline{y}^* \wedge /b^2$ in (4.23) is precisely to produce the desired $|Y|^2/2b^4$ in B_b^2. We will give a formula for a conjugate of the operator B_b^2.

4.3. The hypoelliptic Laplacian in Dolbeault theory. — If $\widehat{U} \in \widehat{T_{\mathbf{R}}X}$, we define $c\left(\widehat{U}\right)$ as in (4.5). Then $c\left(\widehat{U}\right)$ acts on $\Lambda^{\cdot}\left(\widehat{T^*X}\right)$. If $u \in TX$, set

$$\widehat{c}'(u) = \sqrt{2} i_u, \qquad \widehat{c}'(\overline{u}) = \sqrt{2}\left(\overline{u}^* \wedge + i_{\overline{u}}\right). \tag{4.25}$$

We extend $\widehat{c}'$ by linearity into a linear map from $T_{\mathbf{R}}X \otimes_{\mathbf{R}} \mathbf{C}$ into $\operatorname{End}\left(\Lambda^{\cdot}\left(T_{\mathbf{R}}^*X\right)\right) \otimes_{\mathbf{R}} \mathbf{C}$, which is such that if $U, V \in T_{\mathbf{R}}X$,

$$\widehat{c}'(U)\widehat{c}'(V) + \widehat{c}'(V)\widehat{c}'(U) = 2\langle U, V\rangle. \tag{4.26}$$

Of course, if $\widehat{U} \in \widehat{T_{\mathbf{R}}X}, V \in T_{\mathbf{R}}X$,

$$\left[c\left(\widehat{U}\right), \widehat{c}'(V)\right] = 0. \tag{4.27}$$

The curvature R^E is a section of $\Lambda^2\left(T_{\mathbf{R}}^*X\right) \otimes \operatorname{End}(E)$, and R^{TX} a section of $\Lambda^2\left(T_{\mathbf{R}}^*X\right) \otimes \operatorname{End}(TX)$. The following result was established in [**5**, Theorem 3.8].

Theorem 4.1. — *The following identity holds:*

(4.28)

$$A_b^2 = \frac{1}{2}\left(-\Delta^V + \frac{|Y|^2}{b^4} + \frac{1}{b^2} c(\widehat{e}_i)\widehat{c}'(e_i)\right) - \nabla_{\widehat{R^{TX}Y}} + \frac{1}{4}\left\langle R^{TX} e_i, e_j\right\rangle c(\widehat{e}_i) c(\widehat{e}_j) + \frac{1}{2}\operatorname{Tr}\left[R^{TX}\right] + \frac{1}{b^2}\nabla_Y^F + R^E.$$

Let L be the operator $\alpha \to \omega^X \wedge \alpha$, and let Λ be its adjoint as in subsection 2.2. Set

$$C_b = \exp(i\Lambda) B_b \exp(-i\Lambda). \tag{4.29}$$

The operators $\nabla^{\mathbf{I}''}, \nabla^{\mathbf{I}'}$ increase the horizontal degree by 1. Let $\nabla^{\mathbf{I}''*}, \nabla^{\mathbf{I}'*}$ be their formal adjoints in the classical L_2 sense. These operators decrease the horizontal degree by 1.

Now we have the result of [**5**, Theorem 3.6].

Theorem 4.2. — *The following identity holds:*

$$C_b = \nabla^{\mathbf{I}''} + \nabla^{\mathbf{I}'} + \nabla^{\mathbf{I}''*} - \nabla^{\mathbf{I}'*} + \overline{\partial}^V + i_y/b^2 + \overline{\partial}^{V*} + \overline{y}^* \wedge /b^2. \tag{4.30}$$

Remark 4.3. — Using (4.30), the fact that the horizontal part of the principal symbol of C_b^2 is nilpotent follows from well-known identities in Kähler geometry.

4.4. The limit as $b \to 0$. — Let K_b be the map $s(x,Y) \to s(x,bY)$. Set

$$D_b = K_b C_b K_b^{-1}. \tag{4.31}$$

By (4.30), we get

$$D_b = \nabla^{\mathbf{I}''} + \nabla^{\mathbf{I}'} + \nabla^{\mathbf{I}''*} - \nabla^{\mathbf{I}'*} + \frac{1}{b}\left(\overline{\partial}^V + i_y + \overline{\partial}^{V*} + \overline{y}^* \wedge\right). \tag{4.32}$$

Let $\widehat{\omega}^{\mathcal{X},V}$ be the Kähler form of the fibre $\widehat{TX}$. Since $\Lambda^{\cdot}\left(\widehat{T^*X}\right)$ has been identified to $\Lambda^{\cdot}(T^*X)$, $\widehat{\omega}^{\mathcal{X},V}$ will be viewed as a section of $\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}\left(\widehat{T^*X}\right)$.

By [**2**, Proposition 1.5 and Theorem 1.6], the fibrewise kernel of the operator $\overline{\partial}^V + i_y + \overline{\partial}^{V*} + \overline{y}^* \wedge$ is 1-dimensional and spanned by $\beta = \exp\left(i\widehat{\omega}^{\mathcal{X},V} - |Y|^2/2\right)$.

We will embed $\Omega^{(0,\cdot)}(X,E)$ into $\Omega^{(0,\cdot)}(\mathcal{X},F)$ by the embedding $\alpha \to \pi^*\alpha \wedge \beta$. Let P be the orthogonal projection operator from $\Omega^{(0,\cdot)}(\mathcal{X},F)$ on $\Omega^{(0,\cdot)}(X,E)$.

Set

$$E = \nabla^{\mathbf{I}''} + \nabla^{\mathbf{I}'} + \nabla^{\mathbf{I}''*} - \nabla^{\mathbf{I}'*}. \tag{4.33}$$

Let us pretend for the moment $\Omega^{(0,\cdot)}(\mathcal{X},F)$ to be finite dimensional. Elementary linear algebra shows that under the proper conditions, as $b \to 0$,

$$\left(\lambda - D_b^{-1}\right)^{-1} \to P\left(\lambda - PEP\right)^{-1} P. \tag{4.34}$$

The critical result which was established in [**5**, Theorem 3.12] is as follows.

Theorem 4.4. — *The following identity holds:*

$$PEP = \overline{\partial}^X + \overline{\partial}^{X*}. \tag{4.35}$$

Proof. — Let $N^{\Lambda^{\cdot}(T^*X)}, N^{\Lambda^{\cdot}\left(\widehat{\overline{T^*X}}\right)}$ be the number operators of $\Lambda^{\cdot}(T^*X), \Lambda^{\cdot}\left(\widehat{\overline{T^*X}}\right)$. Set

$$\mathcal{N} = N^{\Lambda^{\cdot}(T^*X)} - N^{\Lambda^{\cdot}\left(\widehat{\overline{T^*X}}\right)}. \tag{4.36}$$

Then $\Omega^{(0,\cdot)}(X,E)$ is of degree 0 with respect to $\mathcal{N}$. The operators $\nabla^{\mathbf{I}'}, \nabla^{\mathbf{I}'*}$ are of degree $+1$ and -1 with respect to $\mathcal{N}$, so that they disappear under the compression by P. The proof of our theorem is completed. □

Remark 4.5. — Theorem 4.4 is the main algebraic argument which justifies that when $b \to 0$, the operator D_b is indeed a deformation of the Dirac operator D^X. This result is intimately related with theorem 3.10. Indeed as explained in [**3**, Proposition 2.41] there is a corresponding version of theorem 4.4 in the context of de Rham theory. Conversely, by squaring (4.32), we see that the operator D_b^2 can be written in the form (3.25). In [**5**, Theorem 1.14], an analogue of Theorem 3.10 is proved. One of the proofs consists simply into squaring (4.32) and identifying properly the various terms.

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J.-M. BISMUT, Département de Mathématique, Université Paris-Sud, Bâtiment 425, 91405 Orsay, France • *E-mail :* Jean-Michel.Bismut@math.u-psud.fr

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GANG TIAN

New results and problems on Kähler-Ricci flow

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NEW RESULTS AND PROBLEMS ON KÄHLER-RICCI FLOW

by

Gang Tian

Abstract. — In this paper, I give a brief tour on a program of studying the Kähler-Ricci flow with surgery and its interaction with the classification of projective manifolds. The Kähler-Ricci flow may develops singularity at finite time. It is important to understand how to extend the Kähler-Ricci flow across the singular time, that is, construct solution of the Kähler-Ricci flow with surgery. The first task of this paper is to describe a procedure of constructing global solutions for the Kähler-Ricci flow with surgery. This procedure is rather canonical. I will discuss properties of such solutions with surgery and their geometric implications. I will also discuss their asymptotic limits at time infinity. The results discussed here were mainly from my joint works with Z. Zhang, J. Song *et al.* Some open problems will be also discussed. The paper is mostly expository.

***Résumé* (Nouveaux problèmes et résultats sur le flot de Kähler-Ricci).** — Dans cet article, nous donnons un aperçu rapide d'un programme d'études sur le flot de Kähler-Ricci avec chirurgie et son interaction avec la classification des variétés projectives. Le flot de Kähler-Ricci peut développer des singularités en un temps fini. Il est important de comprendre comment étendre le flot de Kähler-Ricci à travers le temps singulier, c'est-à-dire, comment construire une solution du flot de Kähler-Ricci avec chirurgie. La première tâche de cette article consiste à décrire une procédure de construction de solutions globales pour le flot de Kähler-Ricci avec chirurgie. Cette procédure est plutôt canonique. Nous allons discuter les propriétés de telles solutions avec chirurgie et leurs implications géométriques. Nous allons également discuter leurs limites asymptotiques au temps infini. Les résultats présentés ici proviennent principalement de travaux communs avec Z. Zhang, J. Song *et al.* Nous allons également présenter certains problèmes ouverts. L'article est plutôt explicatif.

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1. Introduction

Let X be an n-dimensional compact Kähler manifold. We denote a Kähler metric by its Kähler form ω, in local complex coordinates $z^1, \dots, z^n$,

$$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

where we use the standard convention for summation and $(g_{i\bar{j}})$ is the positive Hermitian matrix valued function given by

$$g_{i\bar{j}} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right).$$

The Ricci flow was introduced by R. Hamilton. It has a nice property: If the initial metric is Kählerian, so do any metrics which evolve along the Ricci flow. This can be proved by either using the uniqueness of its local solutions or applying the maximum principle in an appropriate way. Thus we can consider the following Kähler-Ricci flow

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\tilde{\omega}_t), \qquad \tilde{\omega}_0 = \omega_0, \tag{1.1}$$

where ω_0 is any given Kähler metric and $\mathrm{Ric}(\omega)$ denotes the Ricci form of ω, i.e., in the complex coordinates above,

$$\mathrm{Ric}(\omega) = \sqrt{-1} R_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

where $(R_{i\bar{j}})$ is the Ricci tensor of ω.

This paper is essentially expository. In this paper, I will discuss some new results and open problems in recent study of the Kähler-Ricci flow. They were mainly from my joint works with Z. Zhang, J. Song et al. I will also describe briefly a program of studying the singularity formation of the Kähler-Ricci flow and how it interacts with the classification of projective manifolds. The results and problems discussed here arise from our long efforts in pursuing this program (cf. **[28]**, **[30]**, **[20]**, **[22]**, **[31]**, **[6]** etc.).

2. A sharp local existence for Kähler-Ricci flow

By the local existence of Ricci flow, given any initial Kähler metric ω_0, there is a unique solution $\tilde{\omega}_t$ of (1.1) ($t \in [0, T)$) for some $T > 0$. The following theorem was proved in **[30]** (also see **[2]** [(1)]) and characterizes the maximal T for which the solution $\tilde{\omega}_t$ exists for $t < T$.

(1) In this cited paper, the authors claimed a proof of a related result under certain extra technical assumptions.

Theorem 2.1. — *Let X be a compact Kähler manifold. Then for any initial Kähler metric ω_0, the flow* (1.1) *has a maximal solution $\tilde{\omega}_t$ on $X \times [0, T_{\max})$, where*

$$T_{\max} = \sup\{t \mid [\omega_0] - t\, c_1(X) > 0\,^{(2)}\}.$$

In particular, if the canonical class K_X is numerically effective, then (1.1) *has a global solution $\tilde{\omega}_t$ for all $t > 0$. Here, $c_1(X)$ denotes the 2π mutiple of the first Chern class.*

In [**1**], Cao proved this theorem in the case that $c_1(X)$ is definite and proportional to the initial Kähler class. In the case that K_X is nef, i.e., numerically effective, and the initial metric ω_0 is sufficiently positive, H. Tsuji proved in [**32**] the above theorem, that is, (1.1) has a global solution $\tilde{\omega}_t$.

Now let us sketch a proof of the above theorem following the arguments in the proof of Proposition 1.1 in [**30**]. (3)

For any small $\epsilon > 0$, we can choose $T_\epsilon > 0$ such that $T_\epsilon + \epsilon < T_{\max}$ and a real closed $(1,1)$ form ψ_ϵ such that $[\psi_\epsilon] = c_1(X)$ and $\omega_0 - (T_\epsilon + \epsilon)\,\psi_\epsilon \geq 0$. Choose a smooth volume form Ω_ϵ such that $\mathrm{Ric}(\Omega_\epsilon) = \psi_\epsilon$. This Ω_ϵ is unique up to multiplication by a positive constant.

Set $\omega_t = \omega_0 - t\psi_\epsilon$ for $t \in [0, T_\epsilon]$. One can easily show that $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$ satisfies (1.1) if u satisfies

$$\frac{\partial u}{\partial t} = \log \frac{\tilde{\omega}_t^n}{\Omega_\epsilon}, \qquad u(0,\cdot) = 0. \tag{2.1}$$

We shall show the solution for (2.1) exists for $t \in [0, T_\epsilon]$.

First observe that ω_t is a Kähler metric for $t \in [0, T_\epsilon]$ with uniformly bounded geometry.

By the standard theory, u exists for small $t > 0$. In order to prove that u exists for $t \in [0, T_\epsilon]$, we only need to get uniform estimates of u whenever it exists for $t \in [0, T_\epsilon]$.

Applying the Maximum Principle to (2.1), we can easily have $|u| \leq C_\epsilon$. (4) In fact, the upper bound is independent of ϵ.

Taking derivative of (2.1) with respect to t, we get

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \Delta_{\tilde{\omega}_t}\left(\frac{\partial u}{\partial t}\right) - \langle \tilde{\omega}_t, \psi_\epsilon \rangle,$$

where Δ_ω denotes the Laplacian of a Kähler metric ω and $\langle \omega, F \rangle$ means the trace of F with respect to ω for a real $(1,1)$-form F.

It follows

$$\frac{\partial}{\partial t}\left(t\frac{\partial u}{\partial t} - u\right) = \Delta_{\tilde{\omega}_t}\left(t\frac{\partial u}{\partial t} - u\right) + n - \langle \tilde{\omega}_t, \omega_0 \rangle. \tag{2.2}$$

---

(2) This means that $[\omega_0] - tc_1(X) > 0$ represents a Kähler class.

(3) The flow equation in [**30**] is not the same as, but equivalent to (1.1).

(4) The constant C, C_ϵ may differ at various places. A subscript indicates the dependence on another constant.

Noticing $\langle\tilde{\omega}_t, \omega_0\rangle > 0$ and applying the Maximum Principle, we see that the maximum of $t\frac{\partial u}{\partial t} - u - nt$ is non-increasing, so we have that

$$t\frac{\partial u}{\partial t} - u - nt \leqslant 0.$$

Now we combine it with local existence for small time and the uniform upper bound for u to conclude that

$$\frac{\partial u}{\partial t} \leq C.$$

On the other hand, we have

$$\begin{aligned}(2.3)\quad & \frac{\partial}{\partial t}\left((T_\epsilon + \epsilon - t)\frac{\partial u}{\partial t} + u\right) \\ &= \Delta_{\tilde{\omega}_t}\left((T_\epsilon + \epsilon - t)\frac{\partial u}{\partial t} + u\right) - n + \langle\tilde{\omega}_t, \omega_0 - (T_\epsilon + \epsilon)\psi_\epsilon\rangle.\end{aligned}$$

Since $\langle\tilde{\omega}_t, \omega_0 - (T_\epsilon + \epsilon)\psi_\epsilon\rangle \geq 0$, by the Maximum Principle, we see that minimum of $(T_\epsilon + \epsilon - t)\frac{\partial u}{\partial t} + u + nt$ is non-decreasing. It follows

$$(T_\epsilon + \epsilon - t)\frac{\partial u}{\partial t} + u + nt \geqslant (T_\epsilon + \epsilon)\mathrm{min}_{t=0}\frac{\partial u}{\partial t} = -C_\epsilon,$$

from this we can conclude

$$\frac{\partial u}{\partial t} > -C_\epsilon.$$

Now we have gotten all the C^0-estimates needed. By using the Maximum principle and the standard arguments, one can derive the second and higher order estimates for u (cf. [**30**] for more details). Then one obtains the existence of solution for (2.1) for $t \in [0, T_\epsilon]$.

The desired existence of the solution for (1.1) can be proved by considering the relations between all the equations as (2.1) for different ϵ's as follows:

Consider (2.1) for some $\delta > 0$. Assume $\psi_\delta = \psi_\epsilon + \sqrt{-1}\partial\bar{\partial}f$ for some smooth real function f over X. Since $\mathrm{Ric}_{\Omega_\epsilon} = \psi_\epsilon$, we have $\mathrm{Ric}_{e^{-f}\Omega_\epsilon} = \psi_\delta$. Thus we can take $\Omega_\delta = e^{-f}\Omega_\epsilon$. Now the new "$\omega_t$" is

$$\eta_t = \omega_0 - t\psi_\delta = \omega_t - t\sqrt{-1}\partial\bar{\partial}f.$$

The equation (2.1) for δ is

$$\frac{\partial v}{\partial t} = \log\frac{(\eta_t + \sqrt{-1}\partial\bar{\partial}v)^n}{e^f\Omega_\epsilon}, \qquad v(0,\cdot) = 0.$$

Define $\tilde{u} = v - tf$. Then

$$\begin{aligned}(2.4)\qquad \frac{\partial\tilde{u}}{\partial t} = \frac{\partial v}{\partial t} - f &= \log\frac{(\eta_t + \sqrt{-1}\partial\bar{\partial}v)^n}{e^{-f}\Omega_\epsilon} + f \\ &= \log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\tilde{u})^n}{\Omega_\epsilon}.\end{aligned}$$

Noticing that $\tilde{u}(0,\cdot) = v(0,\cdot) = 0$, from the uniqueness of the solution for (2.1), we conclude that $\tilde{u}$ coincides with u.

This actually gives the explicit relation between solutions of (2.1) associated to different ϵ's and would allow us to glue together all these solutions for (2.1) to get a maximal solution of (1.1) until the time $T_{\max}$. Thus Theorem 2.1 is proved.

Remark 2.2. — Note that ω_t depends on ϵ and may not be a Kähler metric for t sufficiently close to $T_{\max}$ The above arguments also show that the solution u of (2.1) extends to all $t < T_{\max}$ even if ω_t is a Kähler metric when t is sufficiently close to $T_{\max}$.

Next we need to examine behavior of $\tilde{\omega}_t$ as t tends to $T_{\max}$.

3. Finite-time singularity

In this section, we assume that $T = T_{\max} < \infty$, that is, the Kähler-Ricci flow develops singularity at finite time T. We want to examine the limiting behavior of $\tilde{\omega}_t$ as t tends to T. We shall adopt the notations in the last section.

First we observe

Lemma 3.1. — *Let ψ be any smooth (1,1)-form ψ representing $c_1(X)$. Then there is a smooth solution, say $\tilde{u}_t$, for* (2.1) *with $\psi = \psi_\epsilon$ satisfying:*

(1) $\tilde{\omega}_t = \omega_0 - t\psi - \sqrt{-1}\partial\bar{\partial}\tilde{u}_t$;

(2) *For any sequence $t_i \to T$, a subsequence of $\tilde{\omega}_{t_i}$ converges to a positive current $\tilde{\omega}_T$ weakly.* (5)

(3) *If $\overline{\lim}_{t\to T}\sup_X \tilde{u}_t$ is not $-\infty$, then $\tilde{u}_t$ converges to a unique $\tilde{u}_T$ in any L^p-topology as t tends to T for any $p > 1$. In particular, $\tilde{\omega}_t$ converges to a unique positive current $\tilde{\omega}_T$ weakly as t tends to T in this case.*

Proof. — (1) follows directly from the remark at the end of last section.

For (2), we notice that $\tilde{\omega}_t \geq 0$ and

$$\int_X \tilde{\omega}_t \wedge \omega_0^{n-1} = ([\omega_0] - tc_1(X))[\omega_0]^{n-1}(X),$$

so there is a $\alpha > 0$ such that (cf. [**23**])

$$\int_X e^{-\alpha(\tilde{u}_t - \sup_X \tilde{u}_t)}\omega_0^n \leq C'.$$

In particular, for any $p \geq 1$, $v_t = \tilde{u}_t - \sup_X \tilde{u}_t - 1$ has uniformly bounded L^p norm.

(5) $\tilde{\omega}_T$ can be 0.

Furthermore, for any $\delta \in (0,1)$, we have

$$C' \geq \int_X (-v_t)^{-\delta}(\omega_t - \tilde{\omega}_t) \wedge \omega_0^{n-1} \geq \frac{4\delta}{(1-\delta)^2}\int_X |\nabla(-v_t)^{\frac{1-\delta}{2}}|^2 \omega_0^n.$$

Choose $\delta = 1/3$. By the Sobolev embedding theorem, for any sequence t_i with $\lim t_i = T$, there is a subsequence, again denoted by t_i for simplicity, such that $(1 + \sup_X \tilde{u}_t - \tilde{u}_{t_i})^{\frac{1}{3}}$ converges to some function $(-v)^{\frac{1}{3}}$ in L^2-norm. Since v_t have uniform L^p-norm for any $p > 1$, v_i converges to v in the L^p-topology. Then (2) follows.

Now we prove (3). First recall that by the Maximum Principle, we have proved in last section

$$\tilde{u}_t \leq C \quad \text{and} \quad t\,\frac{\partial \tilde{u}_t}{\partial t} - \tilde{u}_t - nt \leq 0.$$

Here C is a uniform constant. It follows that $t^{-1}\tilde{u}_t - n\,\log t$ is non-increasing, consequently $\tilde{u}_t$ converges to a unique function $\tilde{u}_T$, which may take $-\infty$ as values. as t tends to T. By our assumption, $\tilde{u}_T$ is not identically $-\infty$, so $\sup_X \tilde{u}_t$ is uniformly bounded. It follows that the above v coincides with $\tilde{u}_T - \sup_X \tilde{u}_T$. So we have proved (3). □

Let $\tilde{\omega}_T$ be a limiting positive current at the finite-time singularity from the above lemma. A natural question is: *How regular is this limiting $\tilde{\omega}_T$?* It is reasonable to expect that $\tilde{\omega}_T$ is bounded and smooth on a Zariski open subset of X. We also expect that it has controlled behavior along its subvariety of singularity in a suitable sense.

We conjecture that *the limiting current $\tilde{\omega}_T$ is independent of the choice of the sequence $\{t_i\}$*. But we can not prove it in full generality yet. The following lemma gives a sufficient condition for this to be true.

Lemma 3.2. — *If there is a representative ψ of $c_1(X)$ such that $\omega_0 - T\psi \geq 0$ as a (1,1)-form. Then the limiting potential $\tilde{u}_T$ is unique and bounded. If, in addition, $\int_X(\omega_0 - T\psi)^n > 0$, then $\tilde{u}_T$ is continuous.*

Proof. — Set $\omega_t = \omega_0 - t\psi$, then $\omega_t \geq 0$ for any $t \in [0,T]$. We have derived in last section

$$\frac{\partial}{\partial t}\left((T-t)\frac{\partial \tilde{u}_t}{\partial t} + \tilde{u}_t\right) = \Delta_{\tilde{\omega}_t}\left((T-t)\frac{\partial \tilde{u}_t}{\partial t} + \tilde{u}_t\right) - n + \langle \tilde{\omega}_t, \omega_T\rangle. \tag{3.1}$$

By using the Maximum principle, we can deduce from this equation that the auxiliary function

$$(T-t)\frac{\partial \tilde{u}_t}{\partial t} + \tilde{u}_t + nt$$

is non-decreasing. Since $\frac{\partial \tilde{u}_t}{\partial t}$ is bounded form above, $\tilde{u}_t$ is bounded from below. Then the uniqueness follows from Lemma 3.1.

The continuity follows from the extension of S. Kolodziej's work [**17**] by Z. Zhang in [**35**] or [**9**] (also see [**11**], [**8**]) since

$$\lim_{t\to T}\int_X \tilde\omega_t^n = \lim_{t\to T}\int_X \omega_t^n = \int_X \omega_T^n > 0. \qquad \square$$

Now let me discuss some special cases:

First we assume that $\tilde\omega_T = 0$, that is, $c_1(X) = \frac{1}{T}[\omega_0]$ is positive.[(6)] Set

$$\omega(s) = e^{\frac{s}{T}}\tilde\omega_{T(1-e^{-\frac{s}{T}})}.$$

Here s goes from 0 to ∞. Then we have

$$\frac{\partial \omega(s)}{\partial s} = -\left(\operatorname{Ric}(\omega(s)) - \frac{1}{T}\omega(s)\right). \tag{3.2}$$

This $\omega(s)$ is a global solution for the renormalized Ricci flow. A challenging problem is to show the convergence of $\omega(s)$ as s goes to ∞. A folklore conjecture claims that *there is a family of diffeomorphisms $\phi(s) : X \mapsto X$ such that $\phi(s)^*\omega(s)$ converges to a Kähler-Ricci soliton on a variety with possible singularity of codimension 2* (cf. [**14**], [**27**], [**18**]). In the case that $X = S^2$, $\omega(s)$ converges to the standard metric on S^2 as shown in [**13**] and [**7**] (also see [**3**]). In the case that ω_0 has non-negative bisectional curvature, it was proved in [**4**], [**5**] that $\omega(s)$ converges to the unique Kähler-Einstein metric on X. Perelman proved that the scalar curvature and the diameter of $\omega(s)$ are uniformly bounded along (3.2) (cf. [**19**]). It follows that the above conjecture holds if one can bound the Ricci curvature of (3.2) [**19**]. The following theorem was first claimed by Perelman and proved in [**31**].

Theorem 3.3. — *Assume that X has no non-trivial holomorphic fields. If X admits a Kähler-Einstein metric and $c_1(X) = \frac{1}{T}[\omega_0]$, then $\omega(s)$ converges to a Kähler-Einstein metric.*

In [**31**], the above theorem was also extended to the case that X admits only a Kähler-Ricci soliton. The proof of the above theorem was proved by using one of Perelman's estimates and exploring the properness of the K-energy.

Next we consider $X = X_1 \times X_2$ with both $c_1(X_1)$ and $c_1(X_2)$ definite. For simplicity, we assume that $H_2(X_i, \mathbb{Z}) = \mathbb{Z}$ with generator represented by a Kähler form β_i for $i = 1, 2$. Then the initial Kähler class $[\omega_0] = \mu_1[\beta_1] + \mu_2[\beta_2]$ with $\mu_1, \mu_2 > 0$. We further assume that $c_1(X_i) = m_i\beta_i$ with $m_1 > \max(0, m_2)$. Then the flow (1.1) develops singularity at $T = \mu_1/m_1$. First we assume that ω_0 is a product metric $\omega_{01} + \omega_{02}$, where ω_{0i} is a Kähler metric on X_i, then the flow becomes a product flow $\tilde\omega_t = \tilde\omega_{t1} + \tilde\omega_{t2}$, where $\tilde\omega_{ti}$ solves (1.1) on X_i with initial metric ω_{0i}. Then $\tilde\omega_{t1}$ converges

(6) One can easily show that the limiting current is unique in this case, in fact, it is always zero.

to 0 as t tends to T, while $\tilde{\omega}_{t2}$ exists on $X_2 \times [0, T+\epsilon]$ for some $\epsilon > 0$. Hence, the flow $\tilde{\omega}_t$ collapses to $\tilde{\omega}_{t2}$ on X_2 at T and continues beyond T.

Now if ω_0 is a general Kähler metric in $\mu_1[\beta_1] + \mu_2[\beta_2]$, then there is a smooth function θ such that $\omega_0 = \omega_{01} + \omega_{02} + \sqrt{-1}\partial\bar{\partial}\theta$. By the Maximum Principle, the solution $\tilde{\omega}_t$ of (1.1) with initial metric ω_0 is equal to $\tilde{\omega}_{t1} + \tilde{\omega}_{t2} + \sqrt{-1}\partial\bar{\partial}\theta_t$ with θ_t uniformly bounded. This implies that modulo a bounded potential function, $\tilde{\omega}_t$ collapses to a current on X_2 at T. I believe that this collapsing occurs in the L^∞-topology.

In our next example, we assume that X is a projective manifold with Kodaira dimension ≥ 0 and ω_0 is rational. Then $T = T_{\max}$ is rational and consequently, $m[\omega_0]$ is the first Chern class of a line bundle L and $a = mT$ is an integer for some $m > 0$. Clearly, $L + aK_X$ is nef. Since the Kodaira dimension is non-negative, for m sufficiently large, aK_X admits a holomorphic section S. It follows that $S^k S'$ is a global section of $k(L+aK_X)$ for any section S' of kL, so $\dim H^0(X, k(L+aK_X)) \geq ck^n$ for some $c > 0$. It follows that $(L+aK_X)^n > 0$, i.e., it is big. By a result of Kawamata [**16**], $L + aK_X$ is semi-positive, i.e., there is a $k > 0$ such that any basis of $H^0(X, k(L+aK_X))$ maps X onto a subvariety in some $\mathbb{C}P^N$. In particular, there is a ψ representing $c_1(X)$ such that $\omega_0 - T\psi$ is a semi-positive smooth form. In this case, we can say more about the limiting behavior of $\tilde{\omega}_t$ as $t \to T$.

The following lemma can be found in [**15**].

Lemma 3.4. — *Let E be a divisor in a projective manifold X. If E is nef. and big, then there is an effective divisor D such that $E - \epsilon D > 0$ for any sufficiently small $\epsilon > 0$.*

The proof follows essentially from the openness of the big cone of X which clearly contains the positive cone and the fact that E is in the closure of the positive cone. In fact one can choose D to be big. (7)

Applying the above lemma to $L + aK_X$, there is a Hermitian metric h_ϵ on D such that for any small $\epsilon > 0$,

$$\omega_T + \epsilon\sqrt{-1}\partial\bar{\partial}\log h_\epsilon > 0.$$

Let σ be a defining holomorphic section for D. Then we have

$$\omega_T + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2 > 0,$$

where $|\cdot|$ denotes the norm induced by h_ϵ. (8)

The following theorem was essentially proved in [**30**]. (9)

(7) Even if $[\omega_0]$ is irrational, the arguments for proving the above lemma still work.

(8) For simplicity, if there is no possible confusion, we will drop the subscript ϵ in the norm later.

(9) In [**30**], K_X is assumed to be big. It is clear from the arguments in the proof that this assumption was not used.

Theorem 3.5. — *Let X, $L+aK_X$ be as above. Then the solution $\tilde{\omega}_t$ of* (1.1) *converges to a unique current $\tilde{\omega}_T$ as $t \to T$ satisfying:*

(1) *$\tilde{\omega}_T$ represents the cohomology class of $L + aK_X$;*
(2) *$\tilde{\omega}_T$ is a smooth Kähler metric outside a subvariety $B_T \subset X$ along which $c_1(L + aK_X)$ vanishes;*
(3) *$\tilde{\omega}_t$ converges to $\tilde{\omega}_T$ on any compact subset outside B_T in the C^∞-topology.*

Proof. — We will outline a proof of this theorem following [**30**].

Since $L + aK_X$ is semi-positive and big, by Lemma 3.2, we know that the limiting current $\tilde{\omega}_T$ exists with locally continuous potential and satisfies (1). It suffices to prove (2).

Let σ be a defining section of D. Then $\log|\sigma|^2$ is a well-defined function outside $D \subset X$.

First we need a second order estimate. Set

$$\omega_{t,\epsilon} = \omega_t + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2.$$

Then for any $t \in [0, T + \delta]$, where $\delta = \delta(\epsilon)$ may depend on ϵ,[10] $\omega_{t,\epsilon}$ is a smooth Kähler metric, in particular, there is a bound on their curvature which is uniform in $t \in [0, T + \delta]$ but may depend on ϵ.

In order to derive the second order estimate, we need a lower bound on $\frac{\partial \tilde{u}_t}{\partial t}$ for any $t \in [0, T]$. Using the same arguments in deriving (3.1), we get

$$\frac{\partial w_t}{\partial t} = \Delta_{\tilde{\omega}_t} w_t - n + \langle \tilde{\omega}_t, \omega_{T+\delta,\epsilon} \rangle, \tag{3.3}$$

where

$$w_t = (T + \delta - t)\frac{\partial \tilde{u}_t}{\partial t} + \tilde{u}_t - \epsilon \log|\sigma|^2.$$

Since $\langle \tilde{\omega}_t, \omega_{T+\delta,\epsilon} \rangle \geq 0$, by the Maximum Principle, we can show that the minimum of $w_t + nt$ is non-decreasing. Since $\tilde{u}_t$ is bounded for $t \in [0, T]$, we conclude from this

$$\frac{\partial \tilde{u}_t}{\partial t} \geq \frac{\epsilon}{\delta} \log|\sigma|^2 - C_\delta, \tag{3.4}$$

where C_δ is a uniform constant which may depend on δ.

Now we write

$$\tilde{\omega}_t = \omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}(\tilde{u}_t - \epsilon\log|\sigma|^2).$$

Note that the function $v_t = \tilde{u}_t - \epsilon \log|\sigma|^2$ is defined only outside D.

On $X \setminus D$, we can rewrite (2.1) as

$$(\omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}v_t)^n = e^{\frac{\partial \tilde{u}_t}{\partial t}}\Omega.$$

Note that $\mathrm{Ric}(\Omega) = \psi$.

[10] One can show that $\delta \geq b\epsilon$ for some $b > 0$.

As in [34], [1] and [32], using the bound on $\frac{\partial \tilde{u}_t}{\partial t}$ and the curvature of $\omega_{t,\epsilon}$, one can deduce

$$
\begin{aligned}
&e^{Cv_t}(\Delta_{\tilde{\omega}_t} - \frac{\partial}{\partial t})\left(e^{-Cv_t}\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle\right)\\
&> -C' + (C\frac{\partial u}{\partial t} - C')\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle + C'\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle^{\frac{n}{n-1}}\\
&> -C' + (\frac{C\epsilon}{\delta}\log|\sigma|^2 - C')\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle + C'\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle^{\frac{n}{n-1}}.
\end{aligned}
\tag{3.5}
$$

Here C, C' etc. are constants which may depend on ϵ. For instance, we need to choose C such that $C + \inf_M Rm(\omega_{t,\epsilon}) \geq 1$ for $t \in [0, T]$, where $Rm(\omega')$ denotes the bisectional curvature tensor of ω'.

Clearly, $e^{-C(u-\epsilon\log|\sigma|^2)}\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle$ attains its maximum in $X \setminus \{\sigma = 0\}$. At such a maximum point, we have

$$
\begin{aligned}
0 &> -C' + (C''\log|\sigma|^2 - C')\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle + C'\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle^{\frac{n}{n-1}}\\
&= -C' + C'\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle\left(\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle^{\frac{1}{n-1}} + C''\log|\sigma|^2 - C'\right).
\end{aligned}
$$

Here $C'' = C\epsilon/\delta$. Since $|\sigma|$ is bounded, it follows from this

$$\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle \leq (C' - C''\log|\sigma|^2)^{n-1}.$$

Hence, at this maximum point,

$$e^{-Cv_t}\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle \leq (C' - C''\log|\sigma|^2)^{n-1}e^{-Cv_t} \leq C_1(1 - \log|\sigma|^2)|\sigma|^{C\epsilon}.$$

Here we have used that fact that $\tilde{u}_t$ is uniformly bounded and C_1 is a constant which depends on ϵ.

Then we can easily deduce the second order estimate:

$$\langle \omega_0, \tilde{\omega}_t\rangle \leqslant C_2|\sigma|^{-C\epsilon}. \tag{3.6}$$

Observe that our lower bound estimate on $\frac{\partial \tilde{u}_t}{\partial t}$ implies the volume estimate:

$$\tilde{\omega}_t^n > C_3|\sigma|^{2\epsilon}{\omega_0}^n.$$

It follows that $\tilde{\omega}_t$ defines a Kähler metric on $X \setminus \{\sigma = 0\}$. Furthermore, we have a uniform bound on $\frac{\partial \tilde{u}_t}{\partial t}$ on any given compact subset outside D.

The higher order derivative estimates for $\tilde{u}_T$ outside $\{\sigma = 0\}$ follow from the standard theory on Monge-Ampere equations ([10] etc.) or Calabi's third order estimates as shown in [34].

We have shown that $\tilde{u}_T = \lim_{t\to T}\tilde{u}_t$ exists. The above shows that $\tilde{u}_T$ is smooth and defines a smooth Kähler metric $\tilde{\omega}_T$ outside D. Moreover, we have

$$(\omega_T + \sqrt{-1}\partial\bar{\partial}\tilde{u}_T)^n = e^{\frac{\partial \tilde{u}_t}{\partial t}|_T}\Omega, \quad \text{on } X \setminus \{\sigma = 0\}. \tag{3.7}$$

Notice that D may not be unique. We can choose any D's in the above discussions so long as it satisfies Lemma 3.4. Since the limit $\tilde{u}_T$ is unique, $\tilde{u}_T$ is smooth and gives rise to a Kähler metric outside the intersection B_T of all such D's. Thus this theorem follows. $\square$

Theorem 3.5 tells us that the solution $\tilde{\omega}_t$ extends to a Kähler metric $\tilde{\omega}_T$ outside the subvariety $B_T \subset X$. However, this limiting $\tilde{\omega}_T$ does have singularity along B_T. This singular behavior can be caused by the metric's either blowing-up or failing to be non-degenerate along B_T. In order to extend the Ricci flow across T, we need to study how $\tilde{\omega}_T$ behaves along B_T. Here is what we expect (also see [**20**])

Conjecture 3.6. — *Let X_1 be the metric completion of $X \setminus B_T$ with respect to the distance d_T on $X \setminus B_T$ induced by $\tilde{\omega}_T$. Then X_1 is a projective variety which can be obtained from X by flips or algebraic surgeries of certain "standard" type. Moreover, $(L_0 + aK_X)|_{X\setminus B_T}$ extends to an ample line bundle over X_1.*

If X has the Kodaira dimension $-\infty$ and $[\omega_0]$ is again rational, then $[\omega_0] - Tc_1(M)$ is still rational and nef, but it is not big anymore. If $[\omega_0] - Tc_1(X) \neq 0$ and the well-known Abundance Conjecture holds, then for k sufficiently large, any basis of $H^0(X, k(L_0 + aK_X))$ maps to a subvariety $Y \subset \mathbb{C}P^N$ for some $N > 0$. By Lemma 3.2, the limit $\tilde{u}_T$ exists and clearly descends to a bounded function on Y. It follows that $\tilde{\omega}_T$ descends to a positive current on Y, denoted by $\tilde{\omega}_T$ again for simplicity. We expect

Conjecture 3.7. — *The limit $\tilde{u}_T$ is continuous and $\tilde{\omega}_T$ is a smooth Kähler metric outside a subvariety B'_T of Y. If Y_1 denotes the metric completion of $Y \setminus B'_T$ with respect to the distance induced by $\tilde{\omega}_T$, then Y_1 is a projective variety and $(L_0 + aK_X)|_{Y\setminus B'_T}$ extends to an ample line bundle over Y_1.*

More generally, I believe that even if X is only a Kähler manifold (not necessarily projective) or ω_0 may be irrational, what we have shown and conjectured in the above still hold with slight modification. But it is harder to prove them.

4. Extending Kähler-Ricci flow across singular time

In this section, we discuss how to extend the Kähler-Ricci flow $\tilde{\omega}_t$ across the singular time T, assuming that we have solved the two conjectures proposed at the end of last section. Then we have a projective variety X_T, which can be either X_1 or Y_1 as above, and a limit $\tilde{\omega}_T$ on X_T which is smooth outside a subvariety B. A natural question is *how to continue the Kähler-Ricci flow on X_T starting at $\tilde{\omega}_T$*. There are two difficulties:

1. X_T may not be smooth;
2. Even if X_T is smooth, $\tilde{\omega}_T$ or its potential $\tilde{u}_T$ may not be smooth.

Hence, we need a local existence theorem for (1.1) when the underlying space may be singular or initial Kähler potential is non-smooth.

First we assume that X_T is smooth. We have shown that the limiting current $\tilde{\omega}_T$ has a bounded Kähler potential $\tilde{u}_T$. Then, it follows from the theory of complex Monge-Ampere equations that $\tilde{\omega}_T^k$, where $k = \dim_{\mathbb{C}} X_T$, is well-defined as a measure. So it makes sense to consider the Kähler-Ricci flow (2.1) with a weak initial value $\tilde{u}_T$. *Is there a smooth solution* $\varphi(t)$ *of* (2.1) *for* $t > 0$ *such that* $\lim_{t\to 0} \varphi(t) = \tilde{u}_T$? A partial answer to this question was provided in the following theorem.

Theorem 4.1. — [**6**] *Let* X *be a compact Kähler manifold and* ω_t *be a smooth family of Kähler metrics* ($t \in [0, t_0]$)*. Assume that* ψ_0 *is any bounded function satisfying: There are smooth functions* ψ_ϵ ($\epsilon > 0$) *such that*

(1) $\omega_0 + \sqrt{-1}\partial\overline{\partial}\psi_\epsilon > 0$;

(2) $\lim_{\epsilon\to 0} \psi_\epsilon = \psi_0$;

(3) *The volume form* $\omega_0 + \sqrt{-1}\partial\overline{\partial}\psi_0$ *is* $L^p(M, \omega)$ *for some* $p \geq 3$.

Then there is a unique smooth solution $\varphi(t)$ *of* (2.1)*, and consequently, a solution* $\omega(t)$ *of* (1.1)*, for* $t \in (0, t_0]$ *such that* $\lim_{t\to 0} \varphi(t) = \psi_0$ *and* $\omega(t)^n$ *converges to* $(\omega_0 + \sqrt{-1}\partial\overline{\partial}\psi_0)^n$ *strongly in the* L^2*-topology.*

If the Kodaira dimension of X is non-negative, then $L_0 + aK_X$ is nef and big on X_T and $\dim_{\mathbb{C}} X_T = n$. According to Conjecture 3.6, if X_T is smooth, then $\tilde{\omega}_T$ extends to be a Kähler class on X_T. Since $\frac{\partial \tilde{u}_t}{\partial t}$ is uniformly bounded from above for $t \in (0, T)$, we can show that the assumptions in the above theorem are satisfied. Then one can extend (1.1) across T and continue the flow on X_T until $T_2 > T$ when $[\omega_T] - (t - T)c_1(X_T)$ fails to be a Kähler class. If T_2 is finite, one can proceed as we did for $\tilde{\omega}_t$ at T.

However, in general, the resulting variety X_T from the surgery at T may not be smooth.(11) Nevertheless, we expect

Conjecture 4.2. — *The algebraic variety* X_T *given above has only mild singularity on which we can still run the Kähler-Ricci flow.*

There is an approach in [**21**] to this conjecture: One can try to run the Kähler-Ricci flow on a resolution $\tilde{X}_T$ of X_T with the initial value being the the pull-back of ω_T to $\tilde{X}_T$, which may be a degenerate Kähler metric vanishing along the exceptional divisor E.

Assuming that one can affirm the above three conjectures. When (1.1) runs into a finite-time singularity at T, one can apply the solutions to the above conjectures to

(11) It will be interesting to construct an explicit example of such a singular X_T, even though I believe it does exist.

extend (1.1) across T and evolve the Kähler metrics along the flow on X_T until we run into another finite-time singularity at $T_2 > T$. So we can get a solution $(X_t, \tilde{\omega}_t)$ with surgery for (1.1) for $t \in [0, T_2)$ satisfying:

(1) For $t \in [0, T)$, $X_t = X$ and $\tilde{\omega}_t$ is a standard solution of (1.1) with initial Kähler metric ω_0;

(2) For $t \in [T, T_2)$, $X_t = X_T$ and $\tilde{\omega}_t$ is a solution of (1.1) on X_T such that the potential $\tilde{u}_t$ of $\tilde{\omega}_t$ converges to the potential $\tilde{u}_T$ of $\tilde{\omega}_T$ in the L^∞-topology as t tends to T.

As usual, we call T a surgery time. One repeats the above process to continue the flow beyond T_2 and so on. Thus one can construct a global solution $(X_t, \tilde{\omega}_t)$ with surgery of (1.1)$(t \geq 0)$. We expect that this process ends after finitely many finite-time singularities, that is,

Conjecture 4.3. — *There are only finitely many surgery times $T_0 = 0 < T_1 < T_2 < \cdots < T_N < \infty$ such that $X_t = X_{T_i}$ and $\tilde{\omega}_t$ is a solution of* (1.1) *on X_{T_i} for $t \in [T_i, T_{i+1})$ $(i = 0, 1, \ldots, N-1)$ or $t \in [T_N, \infty)$. Furthermore, for $t \geq T_N$, either $X_t = \varnothing$ or K_{X_t} is nef and consequently,* (1.1) *has a global solution.*

There are two possibilities for $t > T_N$. In the first case, $X_t = \varnothing$, i.e., (1.1) becomes extinct at T_N. At each T_i $(i = 1, \ldots, N)$, we do surgery along some "rational" components along which $c_1(X)$ integrates positively. In particular, X_{T_i} is birational to X_t for $t < T_i$. Thus we have

Conjecture 4.4. — *The Kähler-Ricci flow* (1.1) *becomes extinct at finite time if and only if X is birational to a Fano manifold.* (12)

We will leave the second case to the next section. Note that X_{T_N} has nef canonical bundle if it is non-empty.

5. Asymptotic behavior of Kähler-Ricci flow

In last two sections, we have discussed results and speculations on singularity formation of the Kähler-Ricci flow at finite time. We also conjectured that there is always a global solution $(X_t, \tilde{\omega}_t)$ with surgery of (1.1) with only finitely many surgery times. This generalized solution with surgery becomes an usual solution $\tilde{\omega}_t$ of (1.1) on a variety with nef canonical bundle when t is sufficiently large. In this section, we study the asymptotic behavior of $\tilde{\omega}_t$ as t goes to ∞. For simplicity, we assume that X is a compact Kähler manifold with K_X nef. The general case can be dealt with in the same approach as we did for Conjecture 4.3 in case of possible singular varieties.

(12) To be safer, we may need to include some algebraic manifolds which are Fano-like if such manifolds ever exist.

It is known that (1.1) has a global solution $\tilde\omega_t$ for any given initial metric. Set $t = e^s - 1$ and $\tilde\omega(s) = e^{-s}\tilde\omega_t$, then $\tilde\omega(s)$ is a solution of the following normalized Kähler-Ricci flow:

$$\frac{\partial\tilde\omega(s)}{\partial s} = -\mathrm{Ric}(\tilde\omega(s)) - \tilde\omega(s), \qquad \tilde\omega(0) = \omega_0. \tag{5.1}$$

The advantage of doing this is that $[\tilde\omega(s)] = e^{-s}[\omega_0] - (1-e^{-s})c_1(X)$, which converges to $-c_1(X)$ as $s \to \infty$.

We also assume that there is a (1,1)-form $\psi \geq 0$ representing $-c_1(X)$. This is of course the case if K_X is semi-positive or equivalently, for m sufficiently large, $H^0(X, K_X^m)$ is free of base points. The Abundance conjecture in algebraic geometry claims that it is true for any X with K_X nef.

Since $H^0(X, K_X^m)$ is base-point free, any basis of it induces a holomorphic map $\phi : X \mapsto \mathbb{C}P^N$ for some $N > 0$ so that $\phi^*\mathcal{O}_{\mathbb{C}P^N}(1) = K_X^m$. The dimension of ϕ's image is just the Kodaira dimension $\kappa = \kappa(X)$ of X.

If $\kappa(X) = 0$, then $c_1(X) = 0$ and by the result in [**1**], the global solution $\tilde\omega_t$ of (1.1) converges to a Calabi-Yau metric on X.

If $\kappa(X) = \dim X = n$, then X is minimal and of general type. It follows from [**32**] and [**30**] that $\tilde\omega(s)$ converges to the unique (possibly singular along a subvariety) Kähler-Einstein metric with scalar curvature $-n$ on X as s tends to ∞.

The more tricky cases are for those X with $1 \leq \kappa(X) \leq n-1$. If X is such a manifold, one can not expect the existence of any Kähler-Einstein metrics (even with possibly singular along a subvariety) on X since $K_X^n = 0$. Hence, the first problem is to find what limiting metrics for $\tilde\omega(s)$ one supposes to have as s tends to ∞. To solve this problem, we introduced a class of new canonical metrics which we call generalized Kähler-Einstein metrics in [**20**] (13) and [**22**]. Let us briefly describe them.

Since we assume that K_X is semi-ample, the canonical ring

$$R(X) = \oplus_{m\geq 0} H^0(X, K_X^m)$$

is finitely generated, so there is a canonical model X_{can} of X (possibly singular). Let $\pi : X \mapsto X_{\mathrm{can}}$ be the natural map from X onto its canonical model X_{can}. Then generic fibers of π are Calabi-Yau manifolds of dimension $n - \kappa$, and consequently, there is a holomorphic map $f : X^0_{\mathrm{can}} \mapsto \mathcal{M}_{CY}$ which assigns $p \in X^0_{\mathrm{can}}$ to the fiber $\pi^{-1}(p)$ in the moduli $\mathcal{M}_{CY}$, where X^0_{can} consists of all p such that $\pi^{-1}(p)$ is smooth.

The moduli $\mathcal{M}_{CY}$ admits a canonical metric, the Weil-Petersson metric. Let us recall its definition. Let $\mathcal{X} \to \mathcal{M}_{CY}$ be a universal family of Calabi-Yau manifolds. Let $(U; t_1, \ldots, t_\ell)$ be a local holomorphic coordinate chart of $\mathcal{M}_{CY}$, where $\ell = \dim \mathcal{M}$.

(13) [**20**] is mainly for complex surfaces, but the part of introducing limiting metrics works for any dimensions.

Then each $\frac{\partial}{\partial t_i}$ corresponds to an element $\iota(\frac{\partial}{\partial t_i}) \in H^1(\mathcal{X}_t, T_{\mathcal{X}_t})$ through the Kodaira-Spencer map ι. The Weil-Petersson metric is defined by the L^2-inner product of harmonic forms representing classes in $H^1(\mathcal{X}_t, T_{\mathcal{X}_t})$. In the case of Calabi-Yau manifolds, as shown in [**24**], it has the following simple expression: Let Ψ be a nonzero holomorphic $(n-\kappa, 0)$-form on the fibre $\mathcal{X}_t$ and $\Psi \lrcorner \iota(\frac{\partial}{\partial t_i})$ be the contraction of Ψ and $\frac{\partial}{\partial t_i}$. Then the Weil-Petersson metric is given by

$$\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial \bar{t_j}}\right)_{\omega_{WP}} = \frac{\int_{\mathcal{X}_t} \Psi \lrcorner \iota(\frac{\partial}{\partial t_i}) \wedge \overline{\Psi \lrcorner \iota(\frac{\partial}{\partial t_i})}}{\int_{\mathcal{X}_t} \Psi \wedge \overline{\Psi}}. \tag{5.2}$$

Now we can introduce the generalized Kähler-Einstein metrics.

Definition 5.1. — Let X, X_{can} etc. be as above. A closed positive $(1,1)$-current ω on X_{can} is called a generalized Kähler-Einstein metric if it satisfies the following.

1. $f^*\omega \in -c_1(X)$;
2. ω is smooth on X^0_{can}; (14)
3. $\mathrm{Ric}(\omega) = -\sqrt{-1}\,\partial\bar\partial \log \omega^\kappa$ lifts to a well-defined current on X and on X^0_{can}

$$\mathrm{Ric}(\omega) = -\omega + f^*\omega_{WP}. \tag{5.3}$$

If $\kappa = n$, then it is just the equation for Kähler-Einstein metrics with negative scalar curvature.

Remark 5.2. — More generally, one can consider the generalized Kähler-Einstein equation:

$$\mathrm{Ric}(\omega) = -\lambda\,\omega + f^*\omega_{WP},$$

where λ is a constant.

In [**22**], the following theorem was proved.

Theorem 5.3. — *Let X be an n-dimensional projective manifold with semi-ample canonical bundle K_X. Suppose that $0 < \kappa(X) \leq n$. There exists a unique generalized Kähler-Einstein metric on X_{can}.*

To prove this theorem, we reduce (5.3) to a complex Monge-Ampere equation as in the proof of the Aubin-Yau theorem.

First we introduce a function which will appear in such a complex Monge-Ampere equation.

(14) One can establish an extra property: $(\pi^*\omega)^\kappa \wedge \Theta$ extends to a continuous function on X, where Θ is the (n-κ, n-κ)-form which restricts to polarized flat volume form on each smooth fiber (see [**22**], p15).

Since K_X is semi-ample, there is a semi-ample form $\pi^*\chi$ representing $-c_1(X)$, where χ is defined in the following way: X_{can} can be embedded into some projective space $\mathbb{C}P^N$ by using any basis of $H^0(X, K_X^m)$ for a sufficiently large m, then

$$\chi = \frac{1}{m}\omega_{FS}|_{X_{\mathrm{can}}}.$$

Let Ω be a volume form on X satisfying:

$$\sqrt{-1}\,\partial\bar\partial\log\Omega = \chi.$$

We push forward Ω to get a current $\pi_*\Omega$, where $\pi : X \to X_{\mathrm{can}}$ as above, as follows: For any continuous function ψ on X_{can}

$$\int_{X_{\mathrm{can}}} \psi\,\pi_*\Omega = \int_X (\pi^*\psi)\,\Omega.$$

It is easy to see that for any $x \in X^0_{\mathrm{can}}$, we have

$$\pi_*\Omega(x) = \int_{\pi^{-1}(x)} \Omega.$$

Definition 5.4. — We define a function F on X_{can} by

$$F\,\chi^\kappa = \pi_*\Omega. \tag{5.4}$$

There is another way of defining F: Choose any Kähler class β on X, by using the Hodge theory, one can find a flat relative volume form Θ on $X^0 = \pi^{-1}(X^0_{\mathrm{can}})$ in the cohomology class $\beta^{n-\kappa}$, this means a $(n-\kappa, n-\kappa)$-form Θ in $\beta^{n-\kappa}$ whose restriction to each fiber $\pi^{-1}(x)$ for $x \in X^0_{\mathrm{can}}$ is flat, that is,

$$\partial\bar\partial\log\Theta|_{\pi^{-1}(x)} = 0.$$

This is possible because $c_1(X)$ vanishes along each smooth fiber. One can show

$$c\,\pi^*F = \left(\frac{\Omega}{\Theta\wedge\pi^*\chi^\kappa}\right), \tag{5.5}$$

where c is a constant determined by

$$c\int_{\pi^{-1}(x)} \beta^{n-\kappa} = 1,$$

where x is any point in X^0_{can}. For simplicity, assume that $c = 1$. In particular, it follows that $\Theta\wedge\pi^*\chi^\kappa$ can be extended to X as a current. Furthermore, one can show (see [**24**])

$$f^*\omega_{WP} = \sqrt{-1}\partial\bar\partial\log(\Theta\wedge\chi^\kappa) - \sqrt{-1}\partial\bar\partial\log\chi^\kappa.$$

The function F may not extend smoothly to X_{can}, but we have some controls on it along the subvariety $X_{\mathrm{can}}\setminus X^0_{\mathrm{can}}$.

Lemma 5.5. — *F is smooth on X^0_{can} and is in $L^{1+\epsilon}(X_{\mathrm{can}})$ for some $\epsilon > 0$, where the L^p-norm is defined by using the metric corresponding to χ.*

To prove it, we notice

$$\int_{X_{\mathrm{can}}} F^{1+\epsilon}\chi^\kappa = \int_X \pi^* F^{1+\epsilon}\pi^*\chi^\kappa \wedge \Theta = \int_X \pi^* F^\epsilon \Omega.$$

Furthermore, one can show that if $\iota : Y \to X_{\mathrm{can}}$ is any resolution of X_{can}, then $\iota^* F$ has at worst pole singularities on Y. The proof is a bit technical and we refer the readers to [**22**] for details. Consequently, $\pi^* F^\epsilon$ is integrable for sufficiently small $\epsilon > 0$ (see [**22**], Proposition 3.2).

Consider

$$(\chi + \sqrt{-1}\partial\overline{\partial}\varphi)^\kappa = F e^\varphi \chi^\kappa. \tag{5.6}$$

If φ is a bounded solution for (5.6), then $\omega = \chi + \sqrt{-1}\partial\overline{\partial}\varphi$ is a generalized Kähler-Einstein metric. To see this, we first observe that $[\pi^*\omega] = [\pi^*\chi] = -c_1(X)$. Next we observe

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\,\partial\overline{\partial}\log\omega^\kappa = -\sqrt{-1}\partial\overline{\partial}\log\chi^\kappa - \sqrt{-1}\partial\overline{\partial}\log F - \sqrt{-1}\partial\overline{\partial}\varphi$$

is a well-defined current on X_{can}. A direct computation shows

$$\begin{aligned}
& \sqrt{-1}\partial\overline{\partial}\log\chi^\kappa + \sqrt{-1}\partial\overline{\partial}\log F + \sqrt{-1}\partial\overline{\partial}\varphi \\
= \;& \sqrt{-1}\partial\overline{\partial}\log\chi^\kappa + \sqrt{-1}\partial\overline{\partial}\log\left(\frac{\Omega}{\Theta\wedge\chi^\kappa}\right) + \omega - \chi \\
= \;& \omega + \sqrt{-1}\left(-\partial\overline{\partial}\log(\Theta\wedge\chi^\kappa) + \partial\overline{\partial}\log\chi^\kappa\right) \\
= \;& \omega - f^*\omega_{WP}.
\end{aligned}$$

Therefore

$$\mathrm{Ric}(\omega) = -\omega + f^*\omega_{WP}.$$

Thus, in order to prove Theorem 5.3, we only need to prove the following

Theorem 5.6. — *There exists a unique solution $\varphi \in C^0(X_{\mathrm{can}}) \cap C^\infty(X^0_{\mathrm{can}})$ for* (5.6) *with $\chi + \sqrt{-1}\partial\overline{\partial}\varphi \geq 0$.*

This is proved by using the continuity method and establishing an a priori C^3-estimate for solutions of (5.6). We refer the readers to [**22**] for its proof.

We would like to point out that $\pi^*\omega^\kappa \wedge \Theta = \Omega\, e^{f^*\varphi}$ is continuous since both $\pi^*\varphi$ and Ω are continuous on X.

Now we can discuss the limit of $\tilde{\omega}(s)$ in (5.1) as s tends to ∞. The following theorem was proved in [**22**] (also see [**20**] for complex surfaces).

Theorem 5.7. — *Let X be a projective manifold with semi-ample canonical bundle K_X. So X admits an algebraic fibration $\pi : X \to X_{\mathrm{can}}$ over its canonical model X_{can}. Suppose $0 < \dim X_{\mathrm{can}} = \kappa < \dim X = n$. Then for any initial Kähler metric ω_0, the solution $\tilde{\omega}(s)$ for* (5.1) *converges to $\pi^*\omega_{\mathrm{can}}$ as currents, where ω_{can} is the* *unique generalized Kähler-Einstein metric on* X_{can}. *Moreover, for any compact subset* $K \subset X^0_{\mathrm{can}}$, *there is a constant* C_K *such that*

$$(5.7) \qquad ||R(\tilde\omega(s))||_{L^\infty(\pi^{-1}(K))} + e^{(n-\kappa)s} \sup_{x\in K} ||\tilde\omega(s)^{n-\kappa}|_{\pi^{-1}(x)}||_{L^\infty(\pi^{-1}(x))} \le C_K,$$

where $R(\tilde\omega(s))$ *denotes the scalar curvature of* $\tilde\omega(s)$.

If $n = 2$, then the above implies the convergence in the $C^{1,\alpha}$-topology for any $\alpha \in (0,1)$ on any compact subset in X^0_{can}. We believe that *the same can be proved in any dimensions.* Moreover, we also expect

Conjecture 5.8. — *The solution* $\tilde\omega(s)$ *converges to the unique limit* $\pi^*\omega_{GKE}$ *in the Gromov-Hausdorff topology and the convergence is in the smooth topology in* $\pi^{-1}(X^0_{\mathrm{can}})$.

This is even open for complex surfaces.

In the above, we assume that X has semi-ample K_X. This is indeed true if the Abundance conjecture holds. If K_X is nef, (5.1) still has a global solution $\tilde\omega(s)$. Clearly, it will be extremely interesting to study the asymptotic behavior of $\tilde\omega(s)$ without assuming the Abundance Conjecture, namely, give a differential geometric proof of the convergence of $\tilde\omega(s)$. The success of such a direct approach will yield many deep applications to studying the structures of Kähler manifolds.

To solve the above conjecture or succeed in the above direct approach, we may need to develop a theory of compactness for Kähler metrics with bounded scalar curvature. For Kähler surfaces, a compactness theorem of this sort was proved in [**29**]. Also note that the scalar curvature is uniformly bounded along (5.1) on any compact projective manifold with big and nef canonical bundle (see [**36**]).

6. The case of algebraic surfaces

In this section, we will carry out the program described above for complex surfaces. Basically, all the results in this section are taken from [**30**] (for surfaces of general type) and [**20**] (for elliptic surfaces). We just make a few simple observations in order to deduce the program from those previous works.

Let X be a compact algebraic surface.

As before, let $\tilde\omega_t$ be a maximal solution of (1.1) on $X \times [0,T]$. If $T < \infty$, then $[\omega_0] - Tc_1(X)$ is nef. There are three possibilities:

1. If $[\omega_0] - Tc_1(X) = 0$, then X is a Del-Pezzo surface and $\tilde\omega(s) = (1 - \frac{t}{T})^{-1}\tilde\omega_t$, where $s = -T\log(1 - \frac{t}{T})$, converges to a Kähler-Ricci soliton as $s \to \infty$ or equivalently, $t \to T$ (cf. [**26**], [**31**], [**33**]).

2. If $[\omega_0]-Tc_1(X)\neq 0$ but $([\omega_0]-Tc_1(X))^2=0$, then there is a fibration $\pi: X\mapsto\Sigma$ with rational curves as fibers (possibly with finitely many singular fibers) such that $[\omega_0]-Tc_1(X)=\pi^*[\omega_\Sigma]$ for some Kähler metric ω_Σ on Σ. It follows that as $t\to T$, $\tilde\omega_t$ converges to a positive current of the form $\pi^*(\omega_\Sigma+\sqrt{-1}\partial\bar\partial u_T)$ for some bounded function u_T on Σ. To extend (1.1) across T, one needs to solve (2.1) on Σ with u_T as the initial value. This is the same as solving the following for $t\geq T$,
$$\frac{\partial u}{\partial t}=\log\left(\frac{\omega_\Sigma-(t-T)\psi_\Sigma+\sqrt{-1}\partial\bar\partial u}{\Omega_\Sigma}\right),\quad u(T,\cdot)=u_T, \tag{6.1}$$
where Ω_Σ is a volume form on Σ with $\mathrm{Ric}(\Omega_\Sigma)=\psi_\Sigma$. One can solve this flow by using the standard potential theory in complex dimension 1. Let $\tilde\omega_t$ be the resulting maximal solution of (6.1) ($t\geq T$). If the genus $g(\Sigma)$ of Σ is zero, then $\tilde\omega_t$ becomes extinct at some finite time $T_2>T$ or after appropriate scaling, these metrics converge to the standard round metric on $\Sigma=S^2$ as $t\to T_2$. Hence, it verifies Conjecture 4.4 in case of algebraic surfaces. If $g(\Sigma)=1$, then $\tilde\omega_t$ exists for all $t\geq T$ and converges to a flat metric as $t\to\infty$. If $g(\Sigma)>1$, then $\tilde\omega_t$ exists for all $t\geq T$ and after scaling, converges to a hyperbolic metric as $t\to\infty$.
3. If $([\omega_0]-Tc_1(X))^2>0$, then $[\omega_0]-Tc_1(X)$ is semi-ample, so it can vanish only along a divisor. It is easy to see that for each irreducible component D of this divisor, $K_X\cdot D<0$. Moreover, $D^2<0$. By the Adjunction Formula, D is a rational curve of self-intersection -1, so the divisor is made of finite disjoint (-1) rational curves and consequently, we can blow down them to get a new algebraic surface X_T. Moreover, the limit $\tilde\omega_T$ descends to a positive current with continuous potential and well-defined bounded volume form. By Theorem 4.1, one can extend (1.1) across T.

Notice that the extension $\tilde\omega_t$ for $t>T$ is smooth. Either K_{X_T} is nef and there is a global solution on X_T,or $\tilde\omega_t$ develops finite-time singularity at some $T_2>T$. In the later case, one can repeat the above steps 1, 2 and 3. Since $H_2(X,\mathbb{Z})$ is finite, after finitely many surgeries, we will arrive at a minimal algebraic surface X_N, that is, K_{X_N} is nef. Then (1.1) has a global solution, denoted again by $\tilde\omega_t$, on X_N. Let us study its asymptotic behavior.

There are 3 possibilities according to the Kodaira dimension $\kappa(X)$ of X:

1. If $\kappa(X)=0$, then $c_1(X)_{\mathbb{R}}=0$ or a finite cover of X is either a K3 surface or an Abelian surface. In this case, the solution $\tilde\omega_t$ on X_N converges to a Ricci flat Kähler metric.

   In other two cases, we better use the normalized Kähler-Ricci flow (5.1) on X_N:
$$\frac{\partial\tilde\omega(s)}{\partial s}=-\mathrm{Ric}(\tilde\omega(s))-\tilde\omega(s),\qquad \tilde\omega(0)=\omega_0,$$

where $t = e^s - 1$ and $\tilde{\omega}(s) = e^{-s}\tilde{\omega}_t$.

2. If $\kappa(X) = 1$, then X_N is a minimal elliptic surface: $\pi : X_N \mapsto \Sigma$. It was proved in [**20**] that as $s \to \infty$, $\tilde{\omega}(s)$ converges to a positive current of the form $\pi^*(\tilde{\omega}_\infty)$ and the convergence is in the $C^{1,1}$-topology on any compact subset outside singular fibers $F_{p_1}, \ldots, F_{p_k}$, where $p_1, \ldots, p_k \in \Sigma$. Furthermore, $\tilde{\omega}_\infty$ satisfies the generalized Kähler-Einstein equation:
$$\mathrm{Ric}(\tilde{\omega}_\infty) = -\tilde{\omega}_\infty + f^*\omega_{WP}, \qquad \text{on } \Sigma \setminus \{p_1, \ldots .p_k\},$$
where f is the induced holomorphic map from $\Sigma \setminus \{p_1, \ldots .p_k\}$ into the moduli of elliptic curves.
3. If $\kappa(X) = 2$, then X_N is a surface of general type and its canonical model X_{can} is a Kähler orbifold with possibly finitely many rational double points and ample canonical bundle. By the version of the Aubin-Yau Theorem for orbifolds, there is an unique Kähler-Einstein metric $\tilde{\omega}_\infty$ on X_{can} with scalar curvature -2. It was proved in [**30**] that as $s \to \infty$, $\tilde{\omega}(s)$ converges to $\tilde{\omega}_\infty$ and converges in the C^∞-topology outside those rational curves over the rational double points.

This verifies that our program indeed works for algebraic surfaces except that we did not check if the blown-down surfaces coincide with the metric completions described in Conjecture 3.6.

Furthermore, it should be possible to extend all the above discussions to compact Kähler surfaces which may not be projective.

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G. TIAN, Department of Mathematics, Princeton University and Peking University

VESTISLAV APOSTOLOV
DAVID M. J. CALDERBANK
PAUL GAUDUCHON
CHRISTINA W. TØNNESEN-FRIEDMAN

Extremal Kähler metrics on ruled manifolds and stability

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EXTREMAL KÄHLER METRICS ON RULED MANIFOLDS AND STABILITY

by

Vestislav Apostolov, David M. J. Calderbank, Paul Gauduchon & Christina W. Tønnesen-Friedman

Abstract. — This article gives a detailed account and a new presentation of a part of our recent work [**3**] in the case of admissible ruled manifolds without blow-downs. It also provides additional results and pieces of information that have been omitted or only sketched in [**3**].

Résumé **(Métriques kähleriennes extrêmes).** — Cet article fournit une étude détaillée et une nouvelle présentations d'une partie de notre travail récent [**3**] dans le cas des variétés admissibles réglées sans *blow-down*. Il fournit également des résultats supplémentaires et des informations qui ont été omis ou simplement esquissés dans [**3**].

Introduction

Compact complex manifolds which admit hamiltonian 2-forms of order 1 in the sense of [**1, 2**]—cf. Section 1.8 for a formal definition—have been classified in [**2**] and extensively studied in [**3**]. The main motivation in [**3**] for studying this class of Kähler manifolds is the fact that they provide a fertile testing ground for the conjectures relating extremal and CSC Kähler metrics to stability. In particular, by using recent results of X. Chen–G. Tian, here quoted as Theorem 2.1, we were able to solve in [**3**] a long pending open question since [**42**], namely the non-existence of extremal Kähler metrics in "large" Kähler classes on "pseudo-Hirzebruch surfaces", which was the last missing step towards the full resolution of the existence problem of extremal Kähler metrics on geometrically ruled complex surfaces [**5**].

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The main goal of this paper is to present some salient results of our joint work [**3**]. To simplify the exposition, we here only consider the simple case of $\mathbb{P}^1$-bundles over a product of compact Kähler manifolds of constant scalar curvature, which in the terminology in [**3**] is referred to as the case *without blow-downs*. This allows us for a specific treatment, somewhat simpler than the general case worked out in [**3**], to which we refer the reader for more information and details.

For the comfort of the reader, we tried to make this paper as self-contained and easy to read as possible. With regard to [**3**], we introduce in places slightly different notation and terminology, that seem to be more adapted to the specific situations worked out in this paper. Similarly, some computations and arguments taken from [**3**] here appear in a slightly different and/or a more detailed presentation. The paper also includes new pieces of information, which were omitted or only sketched in [**3**], like Proposition 1.5 in Section 1.9, Proposition A.1 in Appendix A, a specific account of the deformation to the normal cone of the infinity section in admissible ruled manifolds, etc.

The paper is organized as follows.

In Sections 1.1 to 1.7, we set the general framework of the paper by introducing the class of *admissible ruled manifolds*, the cone of *admissible Kähler classes*, the set of *admissible momenta* and the associated set of of *admissible Kähler metrics*, and by recalling the main geometric features of these metrics (isometry groups, Ricci form, scalar curvature, etc.). In Section 1.8, we briefly explain how hamiltonian 2-forms of order 1 arise in this setting. In Section 1.9, we use a variant of the Calabi method in [**8**], also used in [**42**], to construct extremal admissible Kähler metrics in a given admissible Kähler class Ω; as in [**42**], we show that this method works successfully if and only if the *extremal polynomial* F_Ω, canonically attached to Ω, is positive on its interval of definition. Section 1.10 is devoted to the special case of admissible ruled *surfaces*, here called *Hirzebruch-like ruled surfaces.*

In Section 2.1, we review some well-known general facts concerning the space of Kähler metrics in a given Kähler class on a compact complex manifold. In Section 2.2, we recall some basic results recently obtained by X. X. Chen and G. Tian, here stated as Theorem 2.1, which play an important role in several parts of the paper. In Section 2.3, we compute the relative Mabuchi K-energy on the space of admissible Kähler metrics in any admissible Kähler class Ω and we show that Ω admits an extremal Kähler metric, which is then admissible up to automorphism, if and only if F_Ω is positive on its interval of definition (Theorem 2.2). Proposition A.1 established in Appendix A is used to complete the proof of Theorem 2.2 in the *borderline case*, when F_Ω is non-negative but has zeros, possibly irrational, in its interval of definition.

In Section 3.1, we recall the interpretation given by Donaldson and adapted by Székelyhidi to the relative case of the Futaki invariant of an S^1-action on a general polarized projective manifold. In Section 3.2, we construct the *deformation to the normal cone*, $\mathcal{D}(M)$, of the infinity section Σ_∞ of an admissible ruled manifold M. In Section 3.3, for any admissible polarization Ω on M, we turn $\mathcal{D}(M)$ into a *test configuration* in the sense of Tian [**41**] and Donaldson [**15**], by constructing a family of relative polarizations, parametrized by rational numbers in the interval of definition of the extremal polynomial F_Ω. In Section 3.4, we extend to admissible ruled manifolds a beautiful computation done by G. Székelyhidi [**39**] for ruled surfaces, and we show that, for any rational number x in $(-1,1)$, $F_\Omega(x)$ is equal, up to a constant (negative) factor, to the relative Futaki invariant of the test configuration $\mathcal{D}(M)$ equipped with the relative polarization determined by x, see Theorem 3.1. Together with Theorem 2.2, this striking—and still mysterious—fact has the following consequence: for admissible ruled manifolds and admissible Kähler classes, the relative slope K-stability, as defined by J. Ross and R. Thomas [**35, 34**], implies the existence of extremal Kähler metrics, cf. [**3**, Theorem 2]. For a more detailed discussion on this matter, including the role of the examples of Section 2.4 in the current refined definitions of the slope stability, the reader is referred to [**3**, Theorem 2].

Notation and convention. — For any Kähler structure (g, J, ω), the riemannian metric g, the complex structure J and the Kähler form ω are linked together by $\omega = g(J\cdot,\cdot)$. The Levi-Civita connection of g, as a covariant derivative acting on any sorts of tensor fields, will be denoted by D^g, or simply D when the metric is understood. The twisted differential d^c acting on exterior forms is defined by $d^c = JdJ^{-1}$, where J acts on a p-form φ by $(J\varphi)(X_1,\ldots,X_p) = \varphi(J^{-1}X_1,\ldots,J^{-1}X_p)$; in terms of the operators ∂ and $\bar\partial$ we then have $d^c = i(\bar\partial - \partial)$ and $dd^c = 2i\partial\bar\partial$. Our overall convention for the curvature of a linear connection ∇ is $R^\nabla_{X,Y} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$.

1. Extremal metrics on admissible ruled manifolds

1.1. Admissible ruled manifolds. — Unless otherwise specified, M will denote a connected, compact, complex manifold of complex dimension $m \geq 2$, of the form

$$M = \mathbb{P}(1 \oplus L), \tag{1.1}$$

where L denotes a holomorphic line bundle over some (connected, compact) complex manifold S of complex dimension $(m-1)$. Here, 1 stands for the trivial holomorphic line bundle $S \times \mathbb{C}$ and $\mathbb{P}(1 \oplus L)$ then denotes the projective line bundle associated to the holomorphic rank 2 vector bundle $E = 1 \oplus L$: an element ξ of M over a point y of S is then a complex line through the origin in the complex 2-plane $E_y = \mathbb{C} \oplus L_y$, where E_y, L_y denote the fibres of E, L at y; if ξ is generated by the pair (z, u) in

$\mathbb{C} \oplus L_y$, we write $\xi = (z : u)$. The natural (holomorphic) projection $\pi : M \to S$ admits two natural (holomorphic) sections: the *zero section* $\sigma_0 : y \mapsto \mathbb{C} \subset \mathbb{C} \oplus L_y$, and the *infinity section* $\sigma_\infty : y \mapsto L_y \subset \mathbb{C} \oplus L_y$. We denote by Σ_0, Σ_∞ the images of σ_0, σ_∞ in M, still called zero section and infinity section, both identified with S via π. Each element of $M \setminus \Sigma_\infty$ over y has a unique generator of the form $(1, u)$, with u in L_y: we thus get a natural identification of $M \setminus \Sigma_\infty$ with L and M can therefore be regarded as a compactification of (the total space of) L obtained by adding a *point at infinity* to each fiber. The open set $M_0 = M \setminus (\Sigma_0 \cup \Sigma_\infty)$ is similarly identified with the set of non-zero elements of L.

The natural $\mathbb{C}^*$-action on L extends to a holomorphic $\mathbb{C}^*$-action on M defined by: $\zeta \cdot (z : u) = (z : \zeta\, u)$. This action pointwise fixes Σ_0 and Σ_∞. The vector field on M generating the induced S^1-action is denoted by T.

We furthermore assume that $S = \prod_{i=1}^N S_i$ is the product of $N \geq 1$ (connected, compact) complex manifolds S_i, of complex dimensions d_i, and that L comes equipped with a (fiberwise) hermitian inner product, h, such that the curvature, R^∇, of the corresponding Chern connection, ∇, is of the form: $R^\nabla = -i \sum_{i=1}^N \epsilon_i\, \omega_{S_i}$, where each ω_{S_i} is the Kähler form of a Kähler metric, g_{S_i}, on S_i (viewed as a 2-form on S. i.e. identified with $p_i^* \omega_i$, if p_i denotes the natural projection from S to S_i), and ϵ_i is equal to 1 or to -1. In particular, $\sum_{i=1}^N \epsilon_i\, [\omega_{S_i}] = 2\pi\, c_1(L^*)$, where $c_1(L^*)$ denotes the first Chern class of the dual complex line bundle L^* and $[\omega_{S_i}]$ the class of ω_{S_i} in $H^2(S, \mathbb{R})$.

Moreover, for $i = 1, \ldots, N$, we assume that $R^{\nabla_i} = -i\epsilon_i \omega_{S_i}$ is the Chern curvature of a hermitian holomorphic line bundle, L_i, on S_i—so that (S_i, ω_{S_i}) is polarized by $\tilde{L}_i = L_i^{-\epsilon_i}$—and that $L = \otimes_{i=1}^N p_i^* L_i$, equipped with the induced (fiberwise) hermitian metric.

On M_0, identified with $L \setminus \Sigma_0$ as above, define t by

$$t = \log r, \tag{1.2}$$

where $r = |\cdot|_h$ denotes the norm relative to h, viewed as a function on $L = M \setminus \Sigma_\infty$. We then have

$$d^c t(T) = 1, \qquad dd^c t = \pi^*(\sum_{i=1}^N \epsilon_i\, \omega_{S_i}), \tag{1.3}$$

where the twisted differential operator d^c, as defined above, is relative to the natural complex structure of M. The latter, as well as the complex structures of S and of each factor S_i, will be uniformly denoted by J and will be kept unchanged throughout the paper.

Definition 1.1. — Ruled manifolds of the above kind, with the additional pieces of structure described in this section, will be referred to as *admissible ruled manifolds.*

Later on in this paper, we shall assume that the scalar curvature of each factor (S_i, g_{S_i}) of S is *constant*, but this assumption is not needed until Section 1.9.

1.2. Admissible Kähler classes. — We denote by e_0, resp. e_∞, the Poincaré dual of (the homology class of) Σ_0, resp. Σ_∞, in $H^2(M,\mathbb{R})$ and we set:

$$\Xi = 2\pi(e_0 + e_\infty). \tag{1.4}$$

The class $e_0 + e_\infty$ can be regarded as a projective version of the Thom class of L, whereas

$$\pi^* c_1(L) = e_0 - e_\infty, \tag{1.5}$$

where $c_1(L)$ denotes the first Chern class of L (cf. Remark 1.1 below). Any element, γ, of $H^2(M,\mathbb{R})$ can be written in a unique way as $\gamma = a\,\Xi + \pi^*\alpha$, with a in $\mathbb{R}$ and α in $H^2(S,\mathbb{R})$. Moreover, in order that γ belong to the Kähler cone of M, it certainly must satisfy the following two conditions: (i) its value on each fiber of π is positive, hence $a > 0$; (ii) $\gamma_{|\Sigma_0}$ and $\gamma_{|\Sigma_\infty}$ both belong to the Kähler cone of S, via the natural identification of Σ_0 and Σ_∞ with S. Now, $(e_0 + e_\infty)_{|\Sigma_0} = e_{0|\Sigma_0} = -\frac{1}{2\pi}\sum_{i=1}^N \epsilon_i\,[\omega_{S_i}]$ and $(e_0+e_\infty)_{|\Sigma_\infty} = e_{\infty|\Sigma_\infty} = \frac{1}{2\pi}\sum_{i=1}^N \epsilon_i\,[\omega_{S_i}]$, via the natural identification of Σ_0, Σ_∞ with S (recall that $e_{0|\Sigma_0}$ is the first Chern class of the normal bundle of Σ_0 in M, identified with L on S; similarly, $e_{\infty|\Sigma_\infty}$ is the first Chern class of the normal bundle of Σ_∞ in M, identified with L^* on S). It follows that Ξ does *not* belong to the Kähler cone of M, whereas

$$\Omega_{\boldsymbol{\lambda}} = \sum_{i=1}^N \lambda_i\,\pi^*[\omega_{S_i}] + \Xi \tag{1.6}$$

clearly satisfies the above conditions (i)-(ii) whenever all λ_i's are real numbers greater than 1. In fact, as will be checked in the next section (cf. Remark 1.2), $\Omega_{\boldsymbol{\lambda}}$ *is* a Kähler class on M for any N-tuple $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ of real numbers such that $\lambda_i > 1$, $i = 1, \ldots, N$. Such N-tuples of real numbers will be called *admissible.*

Definition 1.2. — A *normalized admissible Kähler class* is a Kähler class of the form (1.6), where $\boldsymbol{\lambda}$ is an admissible N-tuple of real numbers. The *characteristic polynomial*, $p_{\Omega_{\boldsymbol{\lambda}}}$, of a normalized admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$ is defined by

$$p_{\Omega_\lambda}(x) = \prod_{i=1}^N (\lambda_i + \epsilon_i\,x)^{d_i}. \tag{1.7}$$

An *admissible* Kähler class is a multiple of a normalized one by a positive real number. The *admissible Kähler cone* is the set of all admissible Kähler classes.

Remark 1.1. — Denote by $\mathcal{O}_M(-1)$ the *tautological line bundle* on M and by $\mathcal{O}_M(1)$ its complex dual: for any $\xi = (z : u)$ in M, the fiber of $\mathcal{O}_M(-1)$ at ξ is the complex line ξ itself, whereas the fiber of $\mathcal{O}_M(1)$ at ξ is $\xi^* = \mathrm{Hom}(\xi, \mathbb{C})$. The natural projection of $\mathbb{C} \oplus L$ on $\mathbb{C}$ determines a holomorphic section of $\mathcal{O}_M(1)$, whose zero locus is Σ_∞; similarly, the natural projection of $\mathbb{C} \oplus L$ on L determines a holomorphic section of $\mathcal{O}_M(1) \otimes L$, whose zero locus is Σ_0. We then have

$$e_\infty = c_1(\mathcal{O}_M(1)), \qquad e_0 = c_1(\mathcal{O}_M(1)) + c_1(\pi^* L), \tag{1.8}$$

hence

$$\Xi = 2\pi\big(2\, c_1(\mathcal{O}_M(1)) + \pi^* c_1(L)\big), \tag{1.9}$$

and

$$\Omega_{\boldsymbol{\lambda}} = 2\pi \left(2 c_1(\mathcal{O}_M(1) + \sum_{i=1}^{N} (\lambda_i - \epsilon_i)\, c_1(\pi^* L_i^{-\epsilon_i})\right). \tag{1.10}$$

It follows that $\Omega_{\boldsymbol{\lambda}}/2\pi$ belongs to the image of $H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{R})$ if and only if all λ_i's are (positive) integers. If so, $\Omega_{\boldsymbol{\lambda}}/2\pi = c_1(\mathcal{F}_{\boldsymbol{\lambda}})$, with

$$\mathcal{F}_{\boldsymbol{\lambda}} = \mathcal{O}_M(2) \otimes \pi^* \big(\otimes_{i=1}^N L_i^{1-\epsilon_i \lambda_i} \big). \tag{1.11}$$

1.3. Admissible momenta and Kähler metrics. — For each admissible Kähler class we construct a distinguished family of Kähler metrics called *admissible*. For convenience, we restrict our attention to *normalized* admissible Kähler classes, i.e. to Kähler classes which are of the form (1.6). The other ones are obtained by homothety.

Let $z = z(t)$ be any smooth *increasing* function of t which, as a function on M_0, smoothy extends to M, with $z_{|\Sigma_0} \equiv -1$ and $z_{|\Sigma_\infty} \equiv +1$. Equivalently, we demand that z, as a function of t, satisfies the following boundary conditions:

$B_{-\infty}$: Near $t = -\infty$, $z(t) = \Phi_{-\infty}(e^{2t})$, where $\Phi_{-\infty}$ is smoothly defined on $[0, \varepsilon)$, for some $\varepsilon > 0$, with $\Phi_{-\infty}(0) = -1$ and $\Phi'_{-\infty}(0) > 0$.

$B_{+\infty}$: Near $t = +\infty$, $z(t) = \Phi_{+\infty}(e^{-2t})$, where $\Phi_{+\infty}$ is smoothly defined on $[0, \varepsilon)$, for some $\varepsilon > 0$, with $\Phi_{+\infty}(0) = +1$ and $\Phi'_{+\infty}(0) < 0$.

Definition 1.3. — A smooth, increasing function $z : \mathbb{R} \to (-1, 1)$, satisfying the boundary conditions $B_{-\infty}$ and $B_{+\infty}$ is called an *admissible momentum.*

For any admissible momentum z, the 2-form $\psi_z = z \sum_{i=1}^N \pi^* \epsilon_i\, \omega_{S_i} + dz \wedge d^c t$ on M_0 smoothly extends to M. Because of (1.3), ψ_z is closed. Moreover, $\psi_{z|\Sigma_0} = -\sum_{i=1}^N \epsilon_i\, \omega_{S_i}$, $\psi_{z|\Sigma_\infty} = \sum_{i=1}^N \epsilon_i\, \omega_{S_i}$, and, for any fiber $\pi^{-1}(y)$, $\int_{\pi^{-1}(y)} \psi = 4\pi$, meaning that $[\psi_z] = \Xi$ for any admissible momentum z. For any admissible Kähler class and for any admissible momentum z, we then define

$$
\begin{aligned}
\omega = \omega_{\boldsymbol{\lambda},z} &= \sum_{i=1}^{N} \lambda_i \, \pi^* \omega_{S_i} + \psi_z \\
&= \sum_{i=1}^{N} (\lambda_i + \epsilon_i \, z) \, \pi^* \omega_{S_i} + dz \wedge d^c t.
\end{aligned} \tag{1.12}
$$

Then, ω is closed, with $[\omega] = \Omega_{\boldsymbol{\lambda}}$, and is positive definite with respect to J, as $z'(t)$, the derivative of z with respect to t, is positive; it is then the Kähler form of a Kähler metric, $g = g_{\boldsymbol{\lambda},z}$, in $\Omega_{\boldsymbol{\lambda}}$. Moreover, by (1.3), $\iota_T \omega = -z'(t)dt = -dz$, meaning that z is a *momentum* of T with respect to ω.

Definition 1.4. — A Kähler metric is called *admissible* if its Kähler form is of the form (1.12) (for some admissible momentum z) or is a multiple of such metric by a positive real number.

Remark 1.2. — The above construction shows that $\Omega_{\boldsymbol{\lambda}}$ actually belongs to the Kähler cône of M, as claimed in Section 1.2. This also shows that the necessary conditions (i) and (ii) in Section 1.2 are also sufficient.

Remark 1.3. — In each admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$, admissible Kähler metrics are, by their very definition, in one-to-one correspondence with the space, $\mathcal{A}$ say, of admissible momenta. Notice however that $\mathcal{A}$ is independent of $\Omega_{\boldsymbol{\lambda}}$.

Remark 1.4. — For any admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$, the space of admissible Kähler metrics in $\Omega_{\boldsymbol{\lambda}}$ is preserved by the natural $\mathbb{C}^*$-action on M: each admissible Kähler metric is S^1-invariant whereas, for any real number c and any admissible Kähler metric $g_{\boldsymbol{\lambda},z}$, we have that $e^c \cdot g_{\boldsymbol{\lambda},z} = g_{\boldsymbol{\lambda},z^c}$, where z^c denotes the translated admissible momentum defined by $z^c(t) = z(t+c)$.

Proposition 1.1. — *Let $\Omega_{\boldsymbol{\lambda}_k}$ be a sequence of (normalized) admissible Kähler classes converging to a (normalized) admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$, meaning that $\boldsymbol{\lambda}_k$ converges to $\boldsymbol{\lambda}$ in $\mathbb{R}^N$ for the usual topology. For each k, let g_k be an admissible Kähler metric in $\Omega_{\boldsymbol{\lambda}_k}$, determined by the admissible momentum z_k in $\mathcal{A}$. Suppose that g_k tends to a (smooth) riemannian metric g in the C^1-topology. Then, g is an admissible Kähler metric in $\Omega_{\boldsymbol{\lambda}}$.*

Proof. — Since the g_k tend to g in the C^1-topology, the limit, ω, of the corresponding Kähler forms $\omega_k = g_k(J\cdot,\cdot)$ is closed: g is then a Kähler metric in Ω. On the other hand, ω_k is of the form (1.12) for a well-defined z_k in $\mathcal{A}$. Since the $|z_k|$ are bounded and the sequence dz_k converges to $-\iota_T\omega$, the sequence z_k converges in the C^0-topology to a smooth function z, which is the momentum of T with respect to ω. This function z still factors through t, satisfies the boundary conditions $B_{-\infty} - B_{+\infty}$ and is still increasing, since $z' = dz(T) = g(T,T)$; it then belongs to $\mathcal{A}$ and g is then the associated admissible Kähler metric in Ω. □

1.4. Admissible momentum profiles. — It is convenient to consider an alternative parametrization of the space of admissible Kähler metrics by introducing, for any admissible momentum map $z : \mathbb{R} \to (-1,1)$, the *momentum profile* Θ defined by

$$\Theta(x) = z'(z^{-1}(x)), \tag{1.13}$$

for any x in the open interval $(-1,1)$, where, $z^{-1} : (-1,1) \to \mathbb{R}$ denote the inverse of z, cf. [**26**]. Alternatively, for any x in $(-1,1)$, $\Theta(x)$ is the square norm of T at any point of M_0 in the level set $z^{-1}(x)$ with respect to the admissible Kähler metric determined by z. In particular, Θ is positive on $(-1,1)$ and smoothly extends to the closed interval $[-1,1]$, with

$$\Theta(-1) = \Theta(1) = 0. \tag{1.14}$$

Moreover, it easily follows from the boundary conditions $B_{-\infty} and B_{+\infty}$ for z that Θ satisfies the following additional boundary conditions:

$$\Theta'(-1) = 2, \qquad \Theta'(1) = -2, \tag{1.15}$$

where Θ' denotes the derivative of Θ with respect to x.

Definition 1.5. — A positive function $\Theta : (-1,1) \to \mathbb{R}^{>0}$ is called an *admissible momentum profile* if it smoothy extends to a function $\Theta : [-1,1] \to \mathbb{R}^{\geq 0}$ and satisfies the boundary conditions (1.14) and (1.15).

Proposition 1.2. — *For any (normalized) admissible Kähler class $\Omega_{\not{\lambda}}$ there is a natural 1–1 correspondence between the space of admissible momentum profiles and the space of admissible Kähler metrics in $\Omega_{\not{\lambda}}$ up to the natural $\mathbb{C}^*$-action on M.*

Proof. — We recover z from Θ by firstly defining $t : (-1,1) \to \mathbb{R}$ by means of the differential equation $\frac{dt}{dx} = \frac{1}{\Theta(x)}$, then $z : \mathbb{R} \to (-1,1)$ as the inverse function of t (notice that $t = t(x)$ is increasing, as Θ is positive on $(-1,1)$). It is then easily checked that $z = z(t)$ defined that way is an admissible momentum, i.e. satisfies the boundary conditions $B_{-\infty}$–$B_{+\infty}$. Finally, $t = t(x)$ is only defined up to an additive constant; we already saw that the corresponding admissible Kähler metric is only defined up to the natural $\mathbb{C}^*$-action on M. □

1.5. Standard admissible metrics. — Each admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$ contains a *standard* $\mathbb{C}^*$-orbit of admissible Kähler metrics, namely admissible Kähler metrics determined by the admissible momentum $z_0 = \tanh t$ or the translated momenta $z_0^c = \tanh(t + c)$. For all of them, the momentum profile, Θ_0 is given by

$$\Theta_0(x) = 1 - x^2. \tag{1.16}$$

When restricted to the affine open set $L_y \setminus \{0\}$ of each fiber $\pi^{-1}(y)$, the Kähler form $\omega_{\boldsymbol{\lambda},z}$ (corresponding to admissible momentum $z = z(t)$) is $z'(t)\,dt \wedge d^c t$, or equivalently, is equal to $dd^c\Phi(t)$, where the Kähler potential $\Phi(t)$ is a primitive of $z(t)$, defined up to an affine function of t. (Notice that the restriction of $dd^c t$ on $\pi^{-1}(y)$ vanishes.) In the standard case, when the admissible momentum is $z_0(t) = \tanh t$, we can choose as Kähler potential $\Phi_0(t) = \log(1 + e^{2t}) = \log(1 + r^2)$, which is the Kähler potential of the Fubini-Study of $\mathbb{P}^1$ of sectional curvature $+1$. The resulting toric Kähler manifold is then the standard unit sphere $S^2 = \{u = (x_1, x_2, x_3) \mid \sum_{i=1}^3 x_i^2 = 1\}$ in $\mathbb{R}^3$, equipped with: (i) the standard S^1-action $e^{i\theta} \cdot (x_1, x_2, x_3) = (\cos\theta\, x_1 + \sin\theta\, x_2, -\sin\theta\, x_1 + \cos\theta\, x_2, x_3)$; (ii) the standard symplectic form $\omega_0 = dx_3 \wedge d\theta$; (iii) the standard complex structure $JX = u \times X$ for any X in T_uS^2, where $\times$ stands for the cross product in $\mathbb{R}^3$ with respect to the natural orientation; (iv) the riemannian metric g_0 induced by the standard flat metric of $\mathbb{R}^3$. The momentum of the S^1-action with respect to ω_0 is then the map $z_0 : u = (x_1, x_2, x_3) \mapsto x_3$.

For a general admissible Kähler metric in a normalized admissible Kähler class, the induced toric Kähler structure on the fibres of π are again isomorphic to S^2, equipped with the same S^1-action and the same complex structure J, and with symplectic form $\omega = f\,\omega_0$ and the metric $g = f\,g_0$, where $f = f(x_3)$ denotes an S^1-invariant invariant positive function on S^2, submitted to the only constraint that $\int_{S^2} f\,\omega_0 = \int_{S^2} \omega_0$ or, equivalently, $\int_{-1}^{1} f(x)\,dx = 2$; the corresponding admissible momentum is then

$$z(t) = -1 + \int_{-1}^{\tanh t} f(x)\,dx. \tag{1.17}$$

1.6. Symmetries of admissible Kähler metrics. — In general, for any (connected) compact complex manifold (M, J), we denote by $\mathrm{H}(M, J)$ the identity component of the group of complex automorphisms of (M, J) and by $\mathfrak{h} = \mathfrak{h}(M, J)$ its Lie algebra, which we regard as the Lie algebra of real vector fields X such that $\mathcal{L}_X J = 0$, where $\mathcal{L}_X$ denotes the Lie derivative along X; X is then called a (real) holomorphic vector field. Equivalently, X is the real part of a complex vector field of type $(1,0)$, $X^{1,0}$, which is a holomorphic section of the holomorphic tangent bundle $T^{1,0}M$.

For any riemannian metric g which is Kähler with respect to J, a (real) vector field X is holomorphic if and only if $D^-X^\flat = 0$—where $X^\flat$ denotes the riemannian dual 1-form of X and $D^-X^\flat$ denotes the J-anti-invariant part of $DX^\flat$—and X can then be written in a unique way as

$$X = X_H + \mathrm{grad}_g f_g^X + J\mathrm{grad}_g h_g^X, \tag{1.18}$$

where X_H is the dual of a g-harmonic (real) 1-form and f_g^X, h_g^X are real functions normalized by $\int_M f_g^X\, v_g = \int_M h_g^X\, v_g = 0$; f_g^X, called the (real) *potential* of X, is determined by $\mathcal{L}_X\omega = dd^c f_g^X$, where $\omega = g(J\cdot,\cdot)$ is the Kähler form of the pair (g, J), cf. *e.g.* [**27**].

A (real) vector field X is called a Killing vector field with respect to g if $\mathcal{L}_X g = 0$. The Lie algebra, denoted by $\mathfrak{k}$, of Killing vector fields is the Lie algebra of the identity component, $\mathrm{K}(M, g)$, of the group of isometries of (M, g). It is well-known [(1)] that $\mathrm{K}(M, g)$ is a (compact) subgroup of $\mathrm{H}(M, J)$. In view of the above, $\mathfrak{k}$ then coincides with the space of those (real) holomorphic vector fields whose (real) potential is identically zero.

The space, $\mathfrak{h}_0$, of (real) holomorphic vector fields such that $X_H \equiv 0$ in the decomposition (1.18) is the Lie algebra of a closed subgroup, $\mathrm{H}_0(M, J)$, of $\mathrm{H}(M, J)$, namely the kernel of the *Albanese map* from $\mathrm{H}(M, J)$ to the *Albanese torus* of (M, J): $\mathfrak{h}_0$ is then the space of (real) vector fields of the form $X = \mathrm{grad}_g f + J\mathrm{grad}_g h$, with $D^-(df + d^c h) = 0$. It can be alternatively described as the space of (real) holomorphic vector fields whose zero set is not empty [**30**]. The space $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{h}_0$ is the Lie algebra of *hamiltonian Killing* vector fields, i.e. the space of Killing vector fields of the form $X = J\mathrm{grad} h_g^X = \mathrm{grad}_\omega h_g^X$; it is the Lie algebra of a closed subgroup of $\mathrm{K}(M, g)$ denoted $\mathrm{K}_0(M, g)$.

We denote by P_g the space of *Killing potential with respect to* g, i.e. the space of a real functions, h, on M such that $X = J\mathrm{grad}_g h$ is a hamiltonian Killing vector field (notice that constants are included in P_g). This space is the kernel [(2)] of the *Lichnerowicz fourth order differential operator* $(D^-d)^*D^-d$.

The group $\mathrm{H}_0(M, J)$ and its subgroup $\mathrm{K}_0(M, g)$ will be referred to as the *reduced automorphism group* of (M, J) and the *reduced isometry group* of (M, g) respectively. We then have (cf. [**3**, Proposition 2]):

Proposition 1.3. — (i) *For any admissible ruled manifold* $M = \mathbb{P}(1 \oplus L)$, $\mathrm{H}_0(M, J)$ *projects surjectively to* $\mathrm{H}_0(S, J) = \prod_{i=1}^N \mathrm{H}_0(S_i, J)$, *with kernel the semi-direct product* $\mathbb{C}^* \ltimes H^0(S, L^{\pm})$, *where* $H^0(S, L^{\pm})$ *stands for the space of holomorphic sections of* L *or* $L^* = L^{-1}$ *according as* $H^0(S, L^*)$ *or* $H^0(S, L)$ *is reduced to zero* [(3)].

(ii) *For any admissible Kähler metric* g *on* M, $\mathrm{K}_0(M, g)$ *projects surjectively to* $\mathrm{K}_0(S, g_S) = \prod \mathrm{K}_0(S_i, g_{S_i})$, *with kernel* S^1, *which is contained in the center of*

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(1) The easy argument goes as follows: for any γ in $\mathrm{K}(M, g)$, $\gamma \cdot \omega$ is g-harmonic, as γ is an isometry, and it belongs to the de Rham class $[\omega]$, as γ is homotopic to the identity; since M is compact, this implies that $\gamma \cdot \omega = \omega$, hence also $\gamma \cdot J = 0$.

(2) Since M is compact, f belongs to the kernel of $(D^-d)^*D^-d$ if and only if the Hessian Ddf is J-invariant, which amounts to saying that the hamiltonian vector field $J\mathrm{grad}_g f$ is Killing.

(3) For any non-trivial holomorphic line bundle over a connected compact complex manifold M, either $H^0(M, L)$ or $H^0(M, L^*)$ is reduced to $\{0\}$: if σ belongs to $H^0(M, L)$ and α belongs to $H^0(M, L^*)$, the holomorphic function $\langle \sigma, \alpha \rangle$ is constant, as M is compact, hence identically zero, as L is non-trivial; since M is connected, it follows that either σ or α is identically zero.

$\mathrm{K}(M,J)$. *In particular,* $\mathrm{K}_0(M,g)$ *is independent of the chosen admissible Kähler metric.*

Proof. — For any X in $\mathfrak{h}(M,J)$ and for any y in S, the projection of $X_{|\pi^{-1}(y)}$ to T_yS can be viewed as a holomorphic map from the fiber $\pi^{-1}(y)$ to $T^{1,0}_yS$, which is then constant: each X in $\mathfrak{h}$ is then projectable and we thus get a Lie algebra homomorphism from $\mathfrak{h}(M,J)$ to $\mathfrak{h}(S,J)$. This implies that any element of $\mathrm{H}(M,J)$ maps fiber to fiber, hence that the above Lie algebra homomorphism is induced by a homomorphism from $\mathrm{H}(M,J)$ to $\mathrm{H}(S,J)$. Moreover, if X belongs to $\mathfrak{h}_0(M,J)$, its projection on S belongs to $\mathfrak{h}_0(S,J)$, as each zero of X is mapped to a zero of its projection. We denote by τ the resulting homomorphism from $\mathfrak{h}_0(M,J)$ to $\mathfrak{h}_0(S,J)$ and by $\tilde{\tau}$ the corresponding Lie group homomorphism from $\mathrm{H}_0(M,J)$ to $\mathrm{H}_0(S,J)$. We show that τ is surjective by constructing a right inverse. Any element V of $\mathfrak{h}_0(S,J)$ splits as $V=\sum_{i=1}^N V_i$, with V_i in $\mathfrak{h}_0(S_i,J)$; we can then assume that $V=\mathrm{grad}_{g_{S_i}} f + J\mathrm{grad}_{g_{S_i}} h$ belongs to $\mathfrak{h}_0(S_i,J)$ for some i. Define $\hat{V}$ by

$$\hat{V}=\tilde{V}+\epsilon_i(\pi^*h)\,T-\epsilon_i\,(\pi^*f)\,JT \tag{1.19}$$

on M, where $\tilde{V}$ denotes the horizontal lift of V on M_0 with respect to the Chern connection of L. In general, for any vector field X on any almost-complex manifold (M,J), the Lie derivative of J along X is given by $\mathcal{L}_XJ=[X,J\cdot]-J[X,\cdot]$; in particular, for any function f on M, we have:

$$\mathcal{L}_{fX}J=f\,\mathcal{L}_XJ+d^cf\otimes X+df\otimes JX. \tag{1.20}$$

We thus get:

$$\begin{aligned}\mathcal{L}_{\hat{V}}J=\mathcal{L}_{\tilde{V}}J+\epsilon_i\,df\otimes T+\epsilon_i\,d^ch\otimes T\\ -\epsilon_i\,d^cf\otimes JT+\epsilon_i\,dh\otimes JT.\end{aligned} \tag{1.21}$$

In particular, $(\mathcal{L}_{\tilde{V}}J)(T)=0$, as $\tilde{V}$ commutes with T and JT for *any* vector field V on S. For any vector field Z on S, the horizontal component of $(\mathcal{L}_{\tilde{V}}J)(\tilde{Z})=[\tilde{V},\tilde{JZ}]-J[\tilde{V},\tilde{Z}]$ is zero, as V is (real) holomorphic, whereas its vertical component is equal to $-\epsilon_i\,\omega_i(V,JZ)\,T+\epsilon_i\,\omega_i(V,Z)\,JT$, hence to

$$-\epsilon_i\,df(Z)-\epsilon_i\,d^c(Z)\,T+\epsilon_i\,d^cf(Z)-\epsilon_i\,dh(Z)\,JT.$$

By substituting in (1.21), we get $\mathcal{L}_{\hat{V}}J=0$. The map $\hat{\tau}:V\mapsto\hat{V}$ is then a linear map—in fact a Lie algebra homomorphism (easy verification)—from $\mathfrak{h}_0(S,J)$ to $\mathfrak{h}_0(M,J)$, hence is right inverse of τ. The kernel of τ is the Lie algebra of those holomorphic vector fields on M which are tangent to the fibers of π, hence restrict to holomorphic vector fields on the projective lines $\mathbb{P}(\mathbb{C}\oplus L_y)$, for all y on S: $\ker\tau$ is then identified with the space $H^0(S,\mathrm{End}_0(1\oplus L))$ of holomorphic sections of the holomorphic vector bundle $\mathrm{End}_0(E)$ of trace-free endomorphisms of $E=1\oplus L$, which is isomorphic to

$\mathbb{C} \oplus H^0(S, L^{\pm})$, cf. footnote 3 of page 102. The kernel of $\tilde{\tau}$ in $\mathrm{H}_0(M, J)$ is therefore identified with $\mathbb{C}^* \ltimes H^0(S, L^{\pm})$ [4]. This proves (i). For any admissible metric $g = g_{\boldsymbol{\lambda},z}$, (1.19) can be re-written as

$$\hat{V} = \mathrm{grad}_g((\lambda_i + \epsilon_i\, z)\, \pi^* f) + J\, \mathrm{grad}_g((\lambda_i + \epsilon_i\, z)\, \pi^* h) \tag{1.22}$$

In particular, $\hat{V}$ is Killing with respect to g if and only if V is Killing with respect to g_{S_i}. Moreover, all admissible Kähler metrics are invariant under the natural S^1-action; since S^1 is a maximal subgroup of $\mathbb{C}^* \ltimes H^0(S.L^{\pm})$, we get (ii). □

In the sequel, the common reduced isometry group $\mathrm{K}_0(M, g)$ for all admissible Kähler metrics will be simply denoted by G. The Lie algebra, $\mathfrak{g}$, of G splits as

$$\mathfrak{g} = \mathbb{R}T \oplus \oplus_{i=1}^{N} \mathfrak{k}_0(S_i, g_{S_i}), \tag{1.23}$$

which is a Lie algebra direct sum; in particular, T belongs to the center of $\mathfrak{g}$. For any $X = a\,T + \sum_{i=1}^{N} X_i$ in $\mathfrak{g}$ and for any admissible metric $g = g_{\boldsymbol{\lambda},z}$ in the (normalized) admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$, a Killing potential of X with respect of g—cf. Section 1.6—is $h_g^X = a\,z + \sum_{i=1}(\lambda_i + \epsilon_i\, z)\, \pi^* h_i$, where h_i is a Killing potential of X_i with respect to g_{S_i}.

1.7. Ricci form and scalar curvature. — Throughout this section we fix a (normalized) admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$. For any admissible momentum z, $p_{\Omega_{\boldsymbol{\lambda}}}(z)$ then denotes the function on M obtained by substituting $z = x$ in the characteristic polynomial; $p'_{\Omega_{\boldsymbol{\lambda}}}(z)$, $p''_{\Omega_{\boldsymbol{\lambda}}}(z), \ldots$, etc. are defined similarly, by substituting $z = x$ in the derivatives of $p_{\Omega_{\boldsymbol{\lambda}}}$. We then have (cf. [**1**, Section 5.1]):

Lemma 1.1. — *For any admissible metric $g_{\lambda,z}$ in Ω_{λ} the Ricci form, ρ, and the scalar curvature, s, of $g_{\lambda,z}$, on M_0, are given by*

$$\rho = \sum_{i=1}^{N} \pi^* \rho_i - \frac{1}{2} dd^c \log\,(p_{\Omega_\lambda} \Theta)(z), \tag{1.24}$$

and

$$s = \sum_{i=1}^{N} \frac{\pi^* s_i}{(\lambda_i + \epsilon_i\, z)} - \frac{(p_{\Omega_\lambda} \Theta)''(z)}{p_{\Omega_\lambda}(z)}, \tag{1.25}$$

where ρ_i and s_i denote the Ricci form and the scalar curvature of the Kähler structure (g_{S_i}, ω_{S_i}) on S_i.

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[4] An element α of $H^0(S, L^*)$ acts on $M = \mathbb{P}(1 \oplus L)$ as follows: for any element $\xi = (z : u)$ of M over y in S, $\alpha \cdot \xi = (z + \langle \alpha(y), u\rangle : u)$; similarly, any σ of $H^0(S, L)$ acts on M by: $\sigma \cdot \xi = (z : u + z\, \sigma(y))$. In the former case, $\mathbb{C}^*$ acts on $H^0(S, L^*)$ by $\zeta \cdot \alpha = \zeta^{-1}\, \alpha$, in the latter case $\mathbb{C}^*$ acts on $H^0(S, L)$ by $\zeta \cdot \sigma = \zeta\, \sigma$.

Proof. — In general, the Ricci form of a Kähler structure (g, ω) of complex dimension m is defined by $\rho(\cdot, \cdot) = r(J\cdot, \cdot)$, where r denotes the Ricci tensor of g, and has the following local expression on the domain of any system of holomorphic coordinates

$$\rho =_{\rm loc} -\frac{1}{2} dd^c \log \frac{v_g}{v_0}, \tag{1.26}$$

where $v_g = \frac{\omega^m}{m!}$ denotes the volume form of g and v_0 stands for the volume form of the standard flat Kähler metric determined by the chosen holomorphic coordinates. (If these are denoted $z_1, \dots, z_m$, we then have $v_0 = \prod_{j=1}^m \frac{i}{2} dz_j \wedge d\bar{z}_j$, but the rhs of (1.26) is independent of this choice.)

For any admissible Kähler metric g, whose Kähler form is given by (1.12), we clearly have

$$v_g = p_{\Omega_\lambda}(z) \prod_{i=1}^N v_{g_{S_i}} \wedge dz \wedge d^c t = p_{\Omega_\lambda}(z)\, \Theta(z) \prod_{i=1}^N v_{g_{S_i}} \wedge dt \wedge d^c t. \tag{1.27}$$

To compute v_0, we use holomorphic coordinates on each factor S_i, viewed as holomorphic coordinates on M, and complete them to a system of holomorphic coordinates on an appropriate open subset of M_0, by choosing any local non-vanishing holomorphic section σ of L and adding the associated holomorphic coordinate λ determined by $u = \lambda(u)\, \sigma(\pi(u))$ for any u in L (viewed as an element of M_0). We then have $\frac{i}{2} d\lambda \wedge d\bar{\lambda} = |\lambda|^2\, dt \wedge d^c t$ up to terms which only involve the differential of holomorphic coordinates coming from the base S, hence contribute nothing to v_0. We thus get

$$v_0 = |\lambda|^2 \prod_{i=1}^N v_{i,0} \wedge dt \wedge d^c t, \tag{1.28}$$

where $v_{i,0}$ denotes the volume form of the flat Kähler metric determined by the chosen local holomorphic coordinates on S_i. By comparing (1.27) and (1.28) and by using (1.26), we get (1.24). The scalar curvature s is deduced from the Ricci form ρ via the general identity:

$$\rho \wedge *\omega = \rho \wedge \frac{\omega^{m-1}}{(m-1)!} = \frac{s}{2} v_g. \tag{1.29}$$

From (1.12), we infer [5]

$$\begin{aligned}\frac{\omega^{m-1}}{(m-1)!} &= p_{\Omega_\lambda}(z)\prod_{i=1}^N \pi^* v_{g_{S_i}} \\ &+ p_{\Omega_\lambda}(z)\sum_{i=1}^N \frac{1}{(\lambda_i+\epsilon_i\, z)}\frac{\pi^*\omega_i^{d_i-1}}{(d_i-1)!}\wedge\prod_{k\neq i}\pi^* v_{g_{S_k}}\wedge dz\wedge d^c t.\end{aligned} \tag{1.31}$$

The contribution of $\pi^*\rho_i$ in $\rho\wedge\frac{\omega^{m-1}}{(m-1)!}$ only involves the second term in the rhs of (1.31); by using (1.29) for each factor S_i, this contribution is found to be equal to $\frac{1}{2}(\sum_{i=1}^N \frac{\pi^* s_i}{(\lambda_i+\epsilon_i)})\, v_g$. On the other hand, $d^c\log\Theta(z) = \frac{\Theta'(z)}{\Theta(z)}\, d^c z = \Theta'(z)\, d^c t$ and $d^c\log p_{\Omega_\lambda}(z) = \frac{p'_{\Omega_\lambda}(z)}{p_{\Omega_\lambda}(z)}\, d^c z = \frac{p'_{\Omega_\lambda}(z)\,\Theta(z)}{p_{\Omega_\lambda}}\, d^c t$, so that $d^c\log(p_{\Omega_\lambda}(z)\Theta(z)) = \frac{(p_{\Omega_\lambda}\Theta)'(z)}{p_{\Omega_\lambda}(z)}\, d^c t$; it follows that:

$$\begin{aligned}-\frac{1}{2}dd^c\log(p_{\Omega_\lambda}(z)\Theta(z)) &= -\frac{1}{2}\frac{(p_{\Omega_\lambda}\Theta)''(z)}{p_{\Omega_\lambda}(z)}\, dz\wedge d^c t \\ &+\frac{1}{2}\frac{(p_{\Omega_\lambda}\Theta)'(z)}{p_{\Omega_\lambda}(z)}\left(\frac{p'_{\Omega_\lambda}(z)}{p_{\Omega_\lambda}(z)}\, dz\wedge d^c t-\sum_{i=1}^N\epsilon_i\,\omega_{S_i}\right).\end{aligned} \tag{1.32}$$

In the wedge product with $\frac{\omega^{m-1}}{(m-1)!}$, $dz\wedge d^c t$ contributes via the first term in the rhs of (1.31) only, whereas $\sum_{i=1}^N\epsilon_i\,\omega_{S_i}$ contributes via the second term only, giving $\sum_{i=1}^N\frac{d_i\epsilon_i}{(\lambda_i+\epsilon_i\, z)}\, v_g = \frac{p'_{\Omega_\lambda}(z)}{p_{\Omega_\lambda}(z)}\, v_g$; the second term of (1.32) then contributes to zero. □

1.8. Hamiltonian 2-forms. — In general, a *hamiltonian 2-form* on a (connected) Kähler manifold (M,g,J,ω) of complex dimension m is a J-invariant real 2-form ϕ such that

$$D_X\phi = \frac{1}{2}(d\,\mathrm{tr}\phi\wedge JX^\flat - d^c\,\mathrm{tr}\phi\wedge X^\flat) \tag{1.33}$$

for any vector field X, where $X^\flat$ denotes the dual 1-form of X with respect to g and $\mathrm{tr}\varphi = (\phi,\omega)$ denotes the *trace* of ϕ with respect to g, defined as follows: If ϕ is viewed as a skew-hermitian $\mathbb{C}$-linear endomorphism of (TM,J) via the metric g, so that $\phi(X,Y) = g(\phi(X),Y)$, and if $\lambda_1\leq\cdots\leq\lambda_m$ denote the (real) eigenvalues of the corresponding hermitian operator $-J\circ\phi$, then $\mathrm{tr}\phi := \sum_{i=1}^m\lambda_i$ (for simplicity, the λ_i's will be referred to as the eigenfunctions of ϕ). Hamiltonian 2-forms in Kähler geometry have nice properties, extensively studied in [**2, 3, 4, 1**]. In particular, for any hamiltonian 2-form ϕ, the elementary symmetric functions of its eigenfunctions

[5] In this and the above computation we use the general combinatorial identity

$$\frac{\left(\sum_{i=1}^d a_i\right)^k}{k!} = \sum_{\substack{k_1,\dots,k_d\\ \Sigma\, k_i=k}}\prod_{i=1}^d\frac{a_i^{k_i}}{k_i!}. \tag{1.30}$$

$\sigma_1 = \mathrm{tr}\phi = \lambda_1 + \cdots + \lambda_m, \sigma_2 = \sum_{i<j} \lambda_i\lambda_j, \ldots, \sigma_m = \lambda_1 \ldots \lambda_m$ are Poisson commuting Killing potentials. Moreover, if $K_r = J\,\mathrm{grad}_g \sigma_r$, $r = 1, \ldots, m$, denote the corresponding hamiltonian vector fields, there exists an integer $0 \leq \ell \leq m$, called the *order* of ϕ, and an open dense subset M_0 of M such that $K_1, \ldots, K_\ell$ are linearly independent, whereas K_r linearly depends of $K_1, \ldots, K_\ell$ for any $r > \ell$. If $\ell = 1$, the case of main interest in this paper, $K = K_1 = J\mathrm{grad}_g \mathrm{tr}\phi$ is called the *hamiltonian Killing vector field* of ϕ.

Proposition 1.4. — *Let M be an admissible ruled manifold and let Ω_ℓ be a normalized admissible Kähler class on M. Then, any admissible Kähler metric $g = g_{\ell,z}$ in Ω_ℓ admits a hamiltonian 2-form of order 1, whose hamiltonian Killing vector field is T, namely the 2-form φ defined by*

$$\phi = -\sum_{i=1}^{N} \epsilon_i\lambda_i(\lambda_i + \epsilon_i z)\,\pi^*\omega_{S_i} + z\,dz \wedge d^c t. \tag{1.34}$$

Proof. — We fist observe that the eigenfunctions of the J-invariant 2-form φ defined by (1.34) with respect to g are the admissible momentum z, of multiplicity 1, and the constant functions $\xi_i = -\epsilon_i\lambda_i$, each of multiplicity d_i. In particular,

$$\mathrm{tr}\phi = z - \sum_{i=1}^{N} d_i\,\epsilon_i\lambda_i. \tag{1.35}$$

The fact that φ is hamiltonian with respect to g is a straightforward consequence of the following two lemmas, whose easy verification is left to the reader:

Lemma 1.2. — *The covariant derivative of T with respect to the Levi-Civita connection of g is given by*

$$\begin{aligned} D_T T &= \frac{1}{2}\Theta'(z)\,JT, \qquad D_{JT}T = -\frac{1}{2}\Theta'(z)\,T,\\ D_{\tilde{X}}T &= \frac{1}{2}\Theta(z)\sum_{i=1}^{N}\frac{\epsilon_i\,J\tilde{X}_i}{(\lambda_i + \epsilon_i z)} \end{aligned} \tag{1.36}$$

for any vector field $X = \sum_{i=1}^{N} X_i$ on S, where X_i sits in TS_i, and $\tilde{X} = \sum_{i=1}^{N}\tilde{X}_i$ denotes its horizontal lift on M with respect to the Chern connection ∇.

Lemma 1.3. — *With the same notation, for $i = 1, \ldots, N$, the covariant derivative of $\pi^*\omega_{S_i}$ is given by:*

$$\begin{aligned} D_T(\pi^*\omega_{S_i}) &= 0, \qquad D_{JT}(\pi^*\omega_{S_i}) = \Theta(z)\sum_{i=1}^{N}\frac{\epsilon_i\pi^*\omega_{S_i}}{(\lambda_i + \epsilon_i z)},\\ D_{\tilde{X}}(\pi^*\omega_{S_i}) &= \frac{1}{2}\sum_{i=1}^{N}\frac{\epsilon_i}{(\lambda_i + \epsilon_i z)}\left(d^c z \wedge \pi^*(X_i^\flat) - dz \wedge \pi^*(JX_i^\flat)\right), \end{aligned} \tag{1.37}$$

where $X_i^\flat$ stands for the dual 1-form of X_i with respect to g_{S_i}. □

1.9. Extremal admissible Kähler class. — In general, a Kähler structure (g, J, ω) is called *extremal* if the scalar curvature $s = s_g$ is a Killing potential with respect to g, i.e. if the hamiltonian vector field $K = \mathrm{grad}_\omega s = J\mathrm{grad}_g s$, is Killing or, equivalently, (real) holomorphic, cf. Section 1.6 and Section 2.1.

Proposition 1.5. — *Let g be an admissible Kähler metric in a (normalized) admissible Kähler class $\Omega_{\not\lambda}$ determined by an admissible momentum z. Then, g is extremal if and only if its scalar curvature s is an affine function of z. In this case the scalar curvatures of (S_i, g_{S_i}) are constant.*

Proof. — For any $i = 1, \dots, N$, the dual vector field of $d^c\pi^* s_i$ with respect to the chosen admissible Kähler metric on M is $\frac{1}{(\lambda_i+\epsilon_i z)} \tilde{K}_i$, where K_i denotes the dual vector field of $d^c s_i$ on S_i with respect to g_{S_i}, and $\tilde{K}_i$ denotes the horizontal lift of K_i on M_0. Notice that for any vector field, X, on S, the horizontal lift $\tilde{X}$ commutes with T and JT; we thus have $[\tilde{K}_i, T] = [\tilde{K}_i, JT] = 0$ for all i. On the other hand, for any admissible Kähler metric, T is the symplectic gradient of z. We thus infer from (1.25) the following expression of K:

$$K = \sum_{i=1}^{N} \frac{1}{(\lambda_i + \epsilon_i z)^2} (\tilde{K}_i - \epsilon_i (\pi^* s_i) T) - \left(\frac{(p_{\Omega_\lambda} \Theta)''}{p_{\Omega_\lambda}}\right)'(z) T. \tag{1.38}$$

By using (1.20), we infer:

(1.39)
$$\begin{aligned}\mathcal{L}_K J = &\sum_{i=1}^{N} \frac{1}{(\lambda_i + \epsilon_i z)^2} \mathcal{L}_{(\tilde{K}_i - \epsilon_i (\pi^* s_i) T)} J \\ &+ \sum_{i=1}^{N} \left(\frac{\epsilon_i}{(\lambda_i + \epsilon_i z)^2}\right)'(z) \left(d^c z \otimes (\tilde{K}_i - \epsilon_i (\pi^* s_i) T) + dz \otimes J(\tilde{K}_i - \epsilon_i (\pi^* s_i) T)\right) \\ &- \left(\frac{(p_{\Omega_\lambda} \Theta)''}{p_{\Omega_\lambda}}\right)''(z) (d^c z \otimes T + dz \otimes JT).\end{aligned}$$

Since the $\tilde{K}_i$'s commute with T and JT, we have that $(\mathcal{L}_{(\tilde{K}_i - \epsilon_i (\pi^* s_i) T)} J)(T) = 0$, whereas $d^c z(T) = \Theta(z)$, $dz(T) = 0$; we thus get

$$(\mathcal{L}_K J)(T) = \Theta(z) \Big(\Big(\sum_{i=1}^{N} \Big(\frac{\epsilon_i}{(\lambda_i + \epsilon_i z)^2}\Big)'(\tilde{K}_i - \epsilon_i (\pi^* s_i) T) - \Big(\frac{(p_{\Omega_\lambda} \Theta)''}{p_{\Omega_\lambda}}\Big)'' T\Big). \tag{1.40}$$

Assume that the chosen admissible Kähler metric is extremal; then $(\mathcal{L}_K J)(T)$ is identically zero. Since T and the $\tilde{K}_i$'s sit in separate spaces, we infer that the $\tilde{K}_i$'s, hence the K_i's are all identically zero; the scalar curvatures s_i are then constant, so that s is a function of z. Moreover, the coefficient of T in the rhs of (1.40), which is identically zero, is then equal to $\frac{d^2 s}{dz^2}$, cf. (1.25); it follows that s is an affine function of z. Conversely, if s is an affine function of z, then K is a constant multiple of T, hence a Killing vector field, meaning that g is extremal. □

In view of Proposition 1.5, we henceforth assume without further comment that the s_i*'s are* constant.

This assumption has in particular the following consequence, cf. [**3**, Proposition 5]:

Proposition 1.6. — *The common reduced isometry group G of all admissible Kähler metrics—cf.* Proposition 1.3—*is a* maximal *compact subgroup of the reduced automorphism group* $\mathrm{H}_0(M, J)$.

Proof. — It is a well-known fact that for any compact Kähler manifold (M, J) of constant scalar curvature the reduced isometry group $\mathrm{K}_0(M, J)$ is a maximal compact subgroup of the reduced automorphism group $\mathrm{H}_0(M, J)$. Proposition 1.6 is then a direct consequence of Proposition 1.3. □

For any (normalized) admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$, we infer from (1.25) and Proposition 1.5 that an admissible Kähler metric $g = g_{\boldsymbol{\lambda},z}$ of momentum profile Θ is extremal if and only if

$$(p_{\Omega_{\boldsymbol{\lambda}}}\Theta)''(x) = R(x), \tag{1.41}$$

by setting

$$R(x) = p_{\Omega_{\boldsymbol{\lambda}}}(x) \sum_{i=1}^{N} \frac{s_i}{(\lambda_i + \epsilon_i\, x)} - p_{\Omega_{\boldsymbol{\lambda}}}(x)(\alpha x + \beta), \tag{1.42}$$

for some (unknown) real constants α, β. All functions appearing in (1.41)–(1.42) are defined on the open interval $(-1, 1)$. Because of the boundary conditions (1.14)–(1.15) for Θ, the polynomial R is subjected to the following two constraints:

$$\int_{-1}^{1} R(x)\, dx = -2\, p_{\Omega_{\boldsymbol{\lambda}}}(-1) - 2\, p_{\Omega_{\boldsymbol{\lambda}}}(1), \tag{1.43}$$

$$\int_{-1}^{1} R(x)\, x\, dx = 2\, p_{\Omega_{\boldsymbol{\lambda}}}(-1) - 2\, p_{\Omega_{\boldsymbol{\lambda}}}(1). \tag{1.44}$$

These constraints in turn determine α, β, hence the polynomial R in terms of the (constant) scalar curvatures s_i, and the characteristic polynomial $p_{\Omega_{\boldsymbol{\lambda}}}(x)$. In particular, R is entirely determined by the chosen admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$, as are the constants α, β.

In view of the extremality equation (1.41), we *define* $F = F(x)$—a polynomial of degree at most $(m + 2)$—by

$$F''(x) = R(x) \tag{1.45}$$

and

$$F(-1) = F(1) = 0, \tag{1.46}$$

cf. [**3**, Proposition 8]. The constraints (1.43)–(1.44) then insure that F also satisfies:

$$F'(-1) = 2\,p_{\Omega_{\boldsymbol{\lambda}}}(-1), \qquad F'(1) = -2\,p_{\Omega_{\boldsymbol{\lambda}}}(1). \tag{1.47}$$

Like $R(x)$, the polynomial $F(x)$ determined that way only depends of the admissible Kähler $\Omega_{\boldsymbol{\lambda}}$.

Definition 1.6. — For any (normalized) admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$ on M, the polynomial F of degree at most $(m+2)$ determined by (1.45)–(1.46) is called the *extremal polynomial* of $\Omega_{\boldsymbol{\lambda}}$, henceforth denoted by $F_{\Omega_{\boldsymbol{\lambda}}}$.

From the above discussion, we readily infer:

$$F_{\Omega_{\boldsymbol{\lambda}}}(x) = 2p_{\Omega_{\boldsymbol{\lambda}}}(-1)(1+x) + \int_{-1}^{x} R(s)(x-s)\,ds. \tag{1.48}$$

Remark 1.5. — It readily follows from (1.42) and from the above definition of the extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$ that for each $i = 1, \ldots, N$, the scalar curvature s_i can be expressed by

$$s_i = \frac{F''_{\Omega_{\boldsymbol{\lambda}}}(-\epsilon_i\lambda_i)}{\prod_{k\neq i}(\lambda_k - \epsilon_k\epsilon_i\lambda_i)} \tag{1.49}$$

provided that $\epsilon_i\lambda_i \neq \epsilon_k\lambda_k$ for $k \neq i$.

Proposition 1.7. — *A (normalized) admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$ on M admits an extremal admissible Kähler metric, $g = g_{\boldsymbol{\lambda},z}$, for some admissible momentum z, if and only if its extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$ is positive on the open interval $(-1,1)$. The momentum profile of g is then given by*

$$\Theta(x) = \frac{F_{\Omega_{\boldsymbol{\lambda}}}(x)}{p_{\Omega_{\boldsymbol{\lambda}}}(x)}. \tag{1.50}$$

In particular, g is then uniquely defined up to the natural $\mathbb{C}^*$-action on M. Moreover, the scalar curvature s of g is given by

$$s = \alpha\, z + \beta, \tag{1.51}$$

where α, β are the real constants determined by (1.42)–(1.43)–(1.44). In particular, s is constant if and only if the leading coefficient of $F_{\Omega_{\boldsymbol{\lambda}}}$ is equal to zero; it is identically zero if and only if the leading and the sub-leading coefficients of $F_{\Omega_{\boldsymbol{\lambda}}}$ are both equal to zero.

Proof. — In view of the above discussion, g is extremal if and only if its momentum profile is given by (1.50). From (1.46)–(1.47), we deduce that the function Θ *defined* by (1.50) is smoothy defined on the closed interval $[-1,1]$ and satisfies the boundary conditions (1.14)–(1.15). It is then an admissible momentum profile if and only if it is positive on $(-1,1)$. Since $p_{\Omega_{\boldsymbol\lambda}}(x)$ is positive on $[-1,1]$, this is equivalent to $F_{\Omega_{\boldsymbol\lambda}}$ being positive on $(-1,1)$. In view of Proposition 1.2, Θ is then the momentum profile of an extremal admissible Kähler metric, uniquely defined up to the natural $\mathbb{C}^*$-action. For a general admissible Kähler metric in $\Omega_{\boldsymbol\lambda}$, the scalar curvature is given by (1.25), or equivalently,

$$s = \alpha\, z + \beta + \frac{R(z) - (p_{\Omega_{\boldsymbol\lambda}}\Theta)''(z)}{p_{\Omega_{\boldsymbol\lambda}}(z)}, \tag{1.52}$$

where the constants α, β are determined by (1.42)–(1.43)–(1.44). If g is extremal, this reduces to (1.51), because of (1.50) and (1.45). Moreover, from (1.42) and (1.45), we readily infer that the extremal polynomial $F_{\Omega_{\boldsymbol\lambda}}$ is of the form $F_{\Omega_{\boldsymbol\lambda}}(x) = \sum_{j=0}^{m+2} a_j x^{m+2-j}$, where the leading and the sub-leading coefficients are given by

$$a_0 = \pm\frac{\alpha}{(m+1)(m+2)}, \qquad a_1 = \pm\frac{\beta + (\sum_{k=1}^N d_k\lambda_k\epsilon_k)\,\alpha}{m(m+1)}, \tag{1.53}$$

with $\pm = -\prod_{i=1}^N \epsilon_i^{d_i}$. The last statement of Proposition 1.7 follows immediately. □

In view of of (1.53), the constants α, β will be referred to as the *renormalized leading coefficients* of the extremal polynomial.

Definition 1.7. — An admissible Kähler class Ω is said to be *far from the boundary* if Ω is a positive multiple of a normalized admissible Kähler class $\Omega_{\boldsymbol\lambda}$, with $\lambda_i \gg 1$, $i = 1, \dots, N$.

Lemma 1.4. — *The extremal polynomial $F_{\Omega_{\boldsymbol\lambda}}$ of a normalized admissible Kähler class $\Omega_{\boldsymbol\lambda}$ far from the boundary has the following asymptotic behavior:*

$$F_{\Omega_\lambda}(x) = \left(\prod_{i=1}^N \lambda_i^{d_i}\right)(1-x^2) + o(\lambda), \tag{1.54}$$

meaning that each coefficients of the polynomial $\frac{F_{\Omega_\lambda}(x)}{\prod_{i=1}^N \lambda_i^{d_i}} - (1-x^2)$ *tends to* 0 *when all* λ_i*'s tend to* $+\infty$.

Proof. — By dividing both sides of (1.43)–(1.44) by $\prod_{i=1}^N \lambda_i^{d_i}$ and observing that $\left(\prod_{i=1}^N \lambda_i^{d_i}\right)^{-1} p_{\Omega_\lambda}(x)$ tends to the constant polynomial 1 on $[-1,1]$ when the λ_i's tend to $+\infty$, we get the following limits for $\alpha = \alpha(\lambda_1, \dots, \lambda_N)$ and $\beta = \beta(\lambda_1, \dots, \lambda_N)$:

$$\lim_{\lambda_1\to+\infty,\dots,\lambda_N\to+\infty} \alpha = 0, \qquad \lim_{\lambda_1\to+\infty,\dots,\lambda_N\to+\infty} \beta = 2. \tag{1.55}$$

This, in turn, implies that the polynomial R in (1.42) tends to the constant polynomial -2; since $R = F''_{\Omega_\lambda}$ and $F_{\Omega_\lambda}(-1) = F_{\Omega_\lambda}(1) = 0$ for all λ_i's, we infer that F_{Ω_λ} tends to the polynomial $1 - x^2$ when all λ_i's tend to 0. □

Proposition 1.8. — *Each admissible Kähler class far enough from the boundary admits an extremal admissible Kähler metric.*

Proof. — We can assume that Ω is a normalized admissible Kähler class $\Omega_{\boldsymbol\lambda}$. It follows from (1.54) that, when the λ_i's go to infinity, all roots of the extremal polynomial $F_{\Omega_{\boldsymbol\lambda}}$ other than ± 1 go to infinity. In particular, $F_{\Omega_{\boldsymbol\lambda}}$ has no root in the open interval $(-1, 1)$ when Ω_λ is far enough from the boundary; because of the boundary conditions (1.46)–(1.47) and the fact that $p_{\Omega_{\boldsymbol\lambda}}(-1) = \prod_{i=1}^N (\lambda_i - \epsilon_i)^{d_i}$ and $p_{\Omega_{\boldsymbol\lambda}}(1) = \prod_{i=1}^N (\lambda_i + \epsilon_i)^{d_i}$ are both positive, $F_{\Omega_{\boldsymbol\lambda}}$ is positive on $(-1, 1)$. Proposition 1.8 then follows from Proposition 1.7. □

A further consequence of Proposition 1.7 is the following result ([**3**, Proposition 11]):

Proposition 1.9. — *In the case when all s_i are non-negative, any admissible Kähler class admits an admissible extremal Kähler metric.*

Proof. — By Proposition 1.7, it is sufficient to check that $F_{\Omega_{\boldsymbol\lambda}}$ is positive on $(-1, 1)$ for any (normalized) admissible Kähler class $\Omega_{\boldsymbol\lambda}$. Assume, for a contradiction, that $F_{\Omega_{\boldsymbol\lambda}}$ has zeros on $(-1, 1)$. Because of the boundary conditions (1.46)-(1.47), where $p_{\Omega_{\boldsymbol\lambda}}(-1)$ and $p_{\Omega_{\boldsymbol\lambda}}(1)$ are both positive, $F_{\Omega_{\boldsymbol\lambda}}$ must have at least two maxima and two inflection point on $(-1, 1)$. Denote respectively by $x_m < x_M$ the smallest and greatest point of maxima in $(-1, 1)$. Note also that $F''_{\Omega_{\boldsymbol\lambda}} = R(x)$ has at least two zeros in $(-1, 1)$.

By (1.42), $R(x)$ can be re-written as $R(x) = \left(\prod_{a=1}^N (\lambda_a + \epsilon_a x)^{d_a - 1}\right) q(x)$, where q is the polynomial defined by

$$q(x) = \sum_{a=1}^N s_a \prod_{b \neq a} (\lambda_b + \epsilon_b x) - (\alpha x + \beta) \prod_{a=1}^N (\lambda_a + \epsilon_a x). \tag{1.56}$$

In this expressions and in the sequel of the argument, we (temporarily) change our overall notation in the following manner: N denotes the number of *distinct* $\epsilon_i \lambda_i$—that is to say the number of distinct constant values of the hamiltonian 2-form ϕ, cf. Section 1.8—and the latter are labeled by $a, b = 1, \ldots, N$ in such a way that

$$\epsilon_K \lambda_K < \cdots < \epsilon_1 \lambda_1 < -1 < 1 < \epsilon_N \lambda_N < \cdots < \epsilon_{K+1} \lambda_{K+1} \tag{1.57}$$

where K is the number of *negative* ϵ_a's. For each label a, we put $d_a = \sum_{i \,|\, \epsilon_i \lambda_i = \epsilon_a \lambda_a} d_i$ so that $p_{\Omega_{\boldsymbol\lambda}}(x) = \prod_{a=1}^N (\lambda_a + \epsilon_a x)^{d_a}$ and $s_a = \sum_{i \,|\, \epsilon_i \lambda_i = \epsilon_a \lambda_a} s_i$.

With this notation, the roots of $R(x)$ are counted as follows: (1) the N real numbers $-\epsilon_a \lambda_a$—each with multiplicity $d_a - 1$—which all sit outside $[-1, 1]$, and (2) the roots of q. With our assumption, q has at least two roots, r_1, r_2 say, in $(-1,1)$, in fact in the subinterval (x_m, x_M). Moreover, $F''_{\Omega_\lambda}(x_m)$ and $F''_{\Omega_{x_M}}$ are both non-positive; since $\prod_{a=1}^{N}(\lambda_a+\epsilon_a x)^{d_a-1}$ is positive on $(-1,1)$, we then have $q(x_m)\leq 0$ and $q(x_M)\leq 0$.

Denote by n_-, resp. n_+, the number of real roots of q in the interval $(-\infty,-1]$, resp. in the interval $[1,+\infty)$ (counted with multiplicity). From (1.56), we infer

$$q(-\epsilon_a\lambda_a) = s_a\prod_{b\neq a}(\lambda_b-\epsilon_b\epsilon_a\lambda_a). \tag{1.58}$$

Since all s_i's, hence all s_a's in the new notation, are non-negative, we infer that $q(-\epsilon_a\lambda_a)\,q(-\epsilon_b\lambda_b)\leq 0$ for any pair a,b, such that $a,b\leq K$ or $a,b>K$ and $|a-b|=1$. There is then at least one real root of q between any two consecutive $-\epsilon_a\lambda_a$, $-\epsilon_b\lambda_b$, with $a,b\leq K$ or $a,b>K$. It follows that

$$n_+ + 1\geq K, \qquad n_- + 1\geq N-K, \tag{1.59}$$

hence

$$n_+ + n_- + 2\geq N. \tag{1.60}$$

On the other hand,

$$n_+ + n_- + 2\leq N+1, \tag{1.61}$$

as the degree of q is at most equal to $N+1$ and q has at least $n_+ + n_- + 2$ real roots: the 2 roots r_1, r_2 in $(-1,1)$ and $n_+ + n_-$ roots outside this interval. From (1.60) and (1.61), we infer that $n_+ + 1 = K$ or $n_- + 1 = N-K$.

First assume that $n_+ + 1 = K$; there is then exactly one root of q between any two consecutive $-\epsilon_a\lambda_a, -\epsilon_b\lambda_b$, with $i,j\leq K$ and no roots in $[1,+\infty)$. In particular, there is no root in the interval $[1,-\epsilon_1\lambda_1)$. From (1.58) we easily infer $q(-\epsilon_1\lambda_1)\geq 0$, whereas $q(x_M)\leq 0$; then, there exists a root, r_3 say, of q in the interval $[x_M,1)$, hence distinct from r_1, r_2; we thus get at least *three* roots of q in $(-1,1)$ and (1.61) can then be replaced by $n_+ + n_- + 2\leq N$; this, together with (1.60), implies $n_+ + n_- + 2 = N$, hence $n_- + 1 = N-K$; as above, we infer that there is no root of q in the interval $(-\epsilon_N\lambda_N,-1]$; by (1.58) again, $q(-\epsilon_N\lambda_N)\geq 0$, whereas $q(x_m)\leq 0$; there then exists a root of q, r_4 say, in the interval $(-1,x_m]$, hence distinct from r_1, r_2 and r_3; we thus obtain (at least) *four* roots, r_1, r_2, r_3, r_4, of q in $(-1,1)$. It follows that (1.61) can be improved by $n_+ + n_- + 2\leq N-1$, which contradicts (1.60). Same reasoning and same conclusion apply if we assume $n_- + 1 = N-K$. □

Remark 1.6. — Proposition 1.8 is a part of [**3**, Proposition 9]. Proposition 1.9 is [**3**, Proposition 10]; similar results have previously appeared in the literature, in particular in [**25**] and [**21**], cf. [**3**] for more details.

1.10. Hirzebruch-like surfaces. — In this section, we consider the particular case when $N = 1$ and the base $S = S_1$ is a compact Riemann surface of genus g. The resulting complex surface $M = \mathbb{P}(1 \oplus L)$ will be called a *Hirzebruch-like surface* of genus g: it is a genuine *Hirzebruch surface* [**23**] when $g = 0$, a *pseudo-Hirzebruch surface* in the sense of [**42**] if $g > 1$. We assume that the *degree* $\deg(L) = \langle c_1(L), [S] \rangle$ is *negative*—meaning that $\epsilon_1 = 1$—equal to $-\ell$, and that g_S is of constant scalar curvature $s = 2\kappa$. It then follows from the Gauss-Bonnet formula that

$$\kappa = \frac{2(1-g)}{\ell}. \tag{1.62}$$

With the above assumption, for any real number $\lambda > 1$, the characteristic polynomial of the (normalized) admissible Kähler Ω_λ is simply

$$p_{\Omega_\lambda}(x) = \lambda + x. \tag{1.63}$$

In view of (1.5), $\Omega_{\boldsymbol{\lambda}}$ can also be written:

$$\Omega_\lambda = 2\pi \left(-(\lambda - 1)\, e_0 + (\lambda + 1)\, e_\infty\right) \tag{1.64}$$

for $\lambda > 1$. In the notation of Section 1.9, we have

$$R(x) = -\alpha\, x^2 - (\lambda\, \alpha + \beta)\, x + 2\kappa - \lambda\, \beta. \tag{1.65}$$

The constraints (1.43)–(1.44) then read:

$$\begin{aligned} \int_{-1}^{1} R(x)\, dx &= -\frac{2\,\alpha}{3} + 4\,\kappa - 2\lambda\,\beta = -4\lambda, \\ \int_{-1}^{1} R(x)\, x\, dx &= -\frac{2\lambda\alpha}{3} - \frac{2\beta}{3} = -4, \end{aligned} \tag{1.66}$$

so that

$$\alpha = \frac{12\lambda - 6\kappa}{3\lambda^2 - 1}, \qquad \beta = \frac{6\lambda^2 + 6\lambda\,\kappa - 6}{3\lambda^2 - 1}. \tag{1.67}$$

The extremal polynomial is then $F_{\Omega_\lambda} = (1 - x^2)\, Q(x)$, with

$$Q(x) = A(x^2 - 1) + x + \lambda, \tag{1.68}$$

by setting

$$A = A(\lambda) = \frac{\lambda - \kappa/2}{3\lambda^2 - 1} \tag{1.69}$$

(because of (1.62), A is positive; moreover, $\lim_{\lambda \to +\infty} A = 0$). By Proposition 1.7, $\Omega_{\boldsymbol{\lambda}}$ admits an admissible extremal Kähler metric if and only if $Q(x)$ is positive on the open interval $(-1, 1)$. Notice that $Q(-1) = \lambda - 1$ and $Q(1) = \lambda + 1$ are both positive, whereas $Q'(-1) = 1 - 2A = \frac{(\lambda-1)(3\lambda+1)+\kappa}{3\lambda^2-1}$ and $Q'(1) = 1 + 2A > 0$. If $\kappa \geq 0$, i.e. if the genus g of S is 0 or 1, then $Q'(-1) > 0$ and $Q(x)$ is then positive on $(-1, 1)$ for any $\lambda > 1$. If $\kappa < 0$, i.e. $g > 1$, $Q'(-1)$ is positive for large values of λ—hence $Q(x)$ is positive on $(-1, 1)$—but it takes negative values when λ is small, namely for any

λ such that $(\lambda-1)(3\lambda+1) < -\kappa$. For these values of λ, Q achieves its minimum at $x_0 = -\frac{1}{2A}$; this belongs to the open interval $(-1, 0)$, as $Q'(-1) = 1 - 2A < 0$, and $Q(x_0) = -\frac{D(\lambda)}{4(3\lambda^2-1)(\lambda-\kappa/2)}$, where

$$D(\lambda) = -3\lambda^4 + 6\kappa\lambda^3 + 2\lambda^2 - 6\kappa\lambda + 1 + \kappa^2 \tag{1.70}$$
$$= (\lambda^2-1)(-3\lambda^2 + 6\kappa\,\lambda - 1) + \kappa^2.$$

It is easyto check that, for any negative value of κ, the rhs of (1.70) decreases from $\Delta(1) = \kappa^2 > 0$ up to $-\infty$, when λ runs from 1 to $+\infty$; it follows that the equation $D(\lambda) = 0$ has a unique root greater than 1, called λ_0. From this and from Proposition 1.7 we infer:

Theorem 1.1. — *Let M be a Hirzebruch-like surface of genus g. Then, each Kähler class Ω is admissible, hence a positive multiple of a normalized admissible Kähler class Ω_λ for some $\lambda > 1$.*

Denote by λ_0 the unique root greater than 1 *of the equation $D(\lambda) = 0$, where $D(\lambda)$ is defined by* (1.70). *Then:*

(i) *If $g \leq 1$ or if $g > 1$ and $\lambda > \lambda_0$, then Ω_λ admits an extremal admissible metric, unique up to the natural action of $\mathbb{C}^*$.*

(ii) *If $g > 1$ and $\lambda \leq \lambda_0$, then Ω_λ admits no extremal admissible Kähler metric.*

Remark 1.7. — The case when $g = 0$ in Theorem 1.1, and, more generally, the case when S is a complex projective space of any dimensions, are due to E. Calabi [**8**] and constitute the first examples of (compact) extremal Kähler manifolds of non-constant scalar curvature (cf. also [**37**] for an alternative approach). As mentioned earlier, our general approach can be viewed as a generalization of Calabi's method. The case when $g = 1$ was worked out by A. Hwang in [**25**] and D. Guan in [**21**]. The case when $g > 1$ is due to the fourth author [**42**] and constitute the first known family of examples of (compact) Kähler manifolds where the extremal Kähler cone is non-empty but does not fill the Kähler cone. Notice that in the latter case, Theorem 1.1 does not imply the non-existence of—non-admissible—extremal Kähler metric if $\lambda \geq \lambda_0$ (more on this point in [**42**]). This question will be settled in Section 2.3 (an alternative treatment can be found in [**40**]).

2. Relative K-energy and extremal metrics

2.1. The space of Kähler metrics. — In this section, we briefly review some general facts concerning the space $\mathcal{M}_\Omega$ of Kähler metrics on a compact complex manifold (M, J) of (complex) dimension m, in a fixed Kähler class Ω. The presentation and the notations are taken from [**19**].

An element of $\mathcal{M}_\Omega$ will be indifferently designated by a Kähler riemannian metric g or by its Kähler form $\omega = g(J\cdot,\cdot)$, with $[\omega] = \Omega$, or by the pair (g,ω). As a consequence of the dd^c-lemma in Kähler geometry, cf. *e.g.*, **[20]**, $\mathcal{M}_\Omega$ is essentially a space of (real) functions. More precisely, for any fixed reference element ω_0 in $\mathcal{M}_\Omega$, we have that

$$\mathcal{M}_\Omega = \{\varphi \,|\, \omega := \omega_0 + dd^c\varphi > 0\}, \tag{2.1}$$

where φ, the *relative Kähler potential* of ω relative to ω_0, is well-defined up to a (real) additive constant (here, $\omega > 0$ means that $g = \omega(\cdot, J\cdot)$ is a riemannian metric). The relative potential can be normalized, cf. **[14]**, in such a way that, for any g in $\mathcal{M}_\Omega$, the tangent space $T_g\mathcal{M}_\Omega$ be identified with the space of real functions f on M such that $\int_M f\, v_g = 0$. The L^2-norm on this space then gives $\mathcal{M}_\Omega$ a structure of riemannian Fréchet manifold, first introduced and studied by T. Mabuchi **[32]**.

The Mabuchi metric on $\mathcal{M}_\Omega$ admits a Levi-Civita connection, denoted by $\mathcal{D}$. For any real function f on M, let $\hat{f}$ be the *constant vector field* on $\mathcal{M}_\Omega$ defined by $g \mapsto f - \bar{f}$, where $\bar{f} = \frac{\int_M f v_g}{V_\Omega}$ denotes the mean value of f. The covariant derivative $\mathcal{D}$ is entirely determined by the $\mathcal{D}\hat{f}$'s, which are given by

$$\mathcal{D}_{f_1}\hat{f}_2 = -(df_1, df_2)_g + \frac{\int_M (df_1, df_2)\, v_g}{V_\Omega} \tag{2.2}$$

for any g in $\mathcal{M}_\Omega$ and any f_1 in $T_g\mathcal{M}_\Omega$. In particular, a curve $\omega_t = \omega_0 + dd^c\varphi_t$, $t \in [0,1]$, in $\mathcal{M}_\Omega$ is a geodesic if and only if

$$\ddot{\varphi}_t - (d\dot{\varphi}_t, d\dot{\varphi}_t)_{g_t} = 0. \tag{2.3}$$

As observed by S. Semmes **[36]**, the geodesic equation (2.3) can be re-written as a degenerate homogeneous Monge-Ampère equation my considering φ_t as a function, Φ say, defined on the product $\hat{M} := M \times \Sigma$, where Σ here stands for the cylinder $[0,1] \times S^1$, equipped with the complex structure determined by $J\partial/\partial t = \partial/\partial s$, where s denotes the natural parameter of the additional circle factor S^1. By still denote by ω the pull-back of ω on $\hat{M}$, the geodesic equation can be rewritten as

$$(\omega + dd^c\Phi)^{m+1} = 0 \tag{2.4}$$

for S^1-invariant functions Φ defined on $M \times \Sigma$ such that $\Phi(\cdot, t)$ is a relative Kähler potential on M with respect to ω_0.

Remark 2.1. — The Monge-Ampère equation (2.4) makes sense when Σ is replaced by any Riemann surface with boundary. Let Φ be a (smooth) solution of (2.4), such that $\Phi(\cdot,\tau)$ is a relative Kähler potential on M with respect to ω_0 for any τ in Σ. Choose a local holomorphic coordinate $z = t + is$ on Σ: Φ then appears as a family of relative Kähler potentials parametrized by s, t, $\varphi = \varphi(t, s)$, and the Monge-Ampère equation (2.4) is then equivalent to

$$\ddot{\varphi}_{tt} + \ddot{\varphi}_{ss} - |d\dot{\varphi}_t - d^c\dot{\varphi}_s|^2_{g_{t,s}} = 0, \tag{2.5}$$

where $g_{s,t}$ denotes the Kähler metric of relative Kähler potential $\varphi(s,t)$, cf. [**14**].

The Monge-Ampère equation (2.4) makes sense in particular when Σ is the (closed) unit disk D in $\mathbb{C}$. In this case, it has a nice interpretation in terms of holomorphic disks [**31**], [**36**], [**13**], which plays a crucial rôle in the theory, in particular in the proof given by Chen-Tian of Theorem 2.1 below.

The group $\mathrm{H}(M,J)$—cf. Section 1.6—acts on $\mathcal{M}_\Omega$ and preserves its riemannian structure. For any (real) vector field X in its Lie algebra $\mathfrak{h}$ and any (g,ω) in $\mathcal{M}_\Omega$, the induced vector field $\hat{X}$ on $\mathcal{M}_\Omega$ is $g \mapsto f_g^X$, where f_g^X denotes the real potential of X with respect to g, as defined in Section 1.6.

The scalar curvature determines a vector field, $\hat{s}$, on $\mathcal{M}_\Omega$ via the assignment $g \mapsto (s_g - \bar{s})$, with $\bar{s} = \frac{\int_M s_g\, v_g}{V_\Omega}$ (notice that $\int_M s_g\, v_g = 2\pi\, (c_1(M) \cup \frac{\Omega^{m-1}}{(m-1)!})[M] =: S_\Omega$ is independent of g in $\mathcal{M}_\Omega$). The dual 1-form, σ, is $\sigma(g) = s_g\, v_g$, via the duality relation $\langle \sigma, f\rangle = \int_M s_g f v_g$, for any f in $T_g\mathcal{M}_\Omega$. Both $\hat{s}$ and σ are left invariant by $\mathrm{H}(M,J)$. The covariant derivative of σ is given by

$$\mathcal{D}_f\sigma = -2\,(D^-d)^*D^-df\, v_g, \tag{2.6}$$

for any g in $\mathcal{M}_\Omega$ and any f in $T_g\mathcal{M}_\Omega$, cf. *e.g.* [**19**, Chapter 4] and Section 1.6 for the notation. Recall, cf. Section 1.6, that the kernel of the operator $(D^-d)^*D^-d$ is the space P_g of Killing potentials for g. It then follows from (2.6) that the critical point of the *Calabi functional* $\mathcal{C}(g) = \int_M (s_g - \bar{s})^2\, v_g = \sigma_g(\hat{s})$ on $\mathcal{M}_\Omega$ are those metrics g in $\mathcal{M}_\Omega$ whose scalar curvature is a Killing potential.

Since $(D^-d)^*D^-d$ is self-adjoint, a further direct consequence of (2.6) is that the 1-form σ is *closed*. Since σ is $\mathrm{H}(M,J)$-invariant, by using the Cartan formula $0 = \mathcal{L}_{\hat{X}}\sigma = \iota_{\hat{X}}d\sigma + d(\iota_{\hat{\sigma}}\sigma)$, we infer that $\sigma(\hat{X})$ is *constant* on $\mathcal{M}_\Omega$ for any X in $\mathfrak{h}$, cf. [**7**]. We thus obtain an $\mathbb{R}$-linear form $\mathcal{F}_\Omega : \mathfrak{h} \to \mathbb{R}$, defined by

$$\mathcal{F}_\Omega(X) = \sigma(\hat{X}) = \int_M f_g^X s_g\, v_g. \tag{2.7}$$

By the above discussion, the rhs of (2.7) is independent of the choice of the metric g in $\mathcal{M}_\Omega$. This linear form has been first introduced by A. Futaki in [**17**] for Fano manifolds, then extended to general Kähler manifolds by E. Calabi in [**9**]. It will be referred to as the *Futaki invariant* or the *Futaki character*(6) of Ω.

(6) It easily follows from its definition that $\mathcal{F}_\Omega$ is a character of the Lie algebra $\mathfrak{h}$, i.e. vanishes on the derived ideal $[\mathfrak{h},\mathfrak{h}]$.

We also consider the *Futaki-Mabuchi bilinear form*, B_Ω, defined on $\mathfrak{h}_0$, the Lie algebra of the reduced group of automorphisms $\mathrm{H}_0(M,J)$, cf. Section 1.6, by

$$B_\Omega(X,Y) = \int_M f_g^X f_g^Y \, v_g - \int_M h_g^X h_g^Y \, v_g, \tag{2.8}$$

for any $X = \mathrm{grad}_g f^X + J\mathrm{grad}_g h^X$, $Y = \mathrm{grad}_g f^Y + J\mathrm{grad}_g h^Y$ in $\mathfrak{h}_0$. It can be checked that the rhs of (2.8) is independent of the metric g in $\mathcal{M}_\Omega$, cf. [**18**]. Notice that $B_\Omega(JX,JY) = -B_\Omega(X,Y)$, for any X,Y in $\mathfrak{h}_0$ and that B_Ω is negative definite on the space, $\mathfrak{k}_0$, of hamiltonian Killing vector fields, and positive definite on $J\mathfrak{k}_0 \subset \mathfrak{h}_0$. For any two elements X,Y in $\mathfrak{h}_0$, with $B_\Omega(Y,Y) \neq 0$, we define the *relative Futaki invariant of X with respect to Y* by

$$\mathcal{F}_\Omega(X \mathrm{mod}\, Y) = \mathcal{F}_\Omega(X) - \frac{B_\Omega(X,Y)}{B_\Omega(Y,Y)} \mathcal{F}_\Omega(Y). \tag{2.9}$$

The *Mabuch K-energy*, $\mathcal{E}$, is defined on $\mathcal{M}_\Omega$ by

$$\sigma = -d\mathcal{E}, \tag{2.10}$$

i.e.

$$d\mathcal{E}_g(f) = -\int_M f \, s_g \, v_g, \tag{2.11}$$

for any g in $\mathcal{M}_\Omega$ and any f in $T_g\mathcal{M}_\Omega$. Since σ is closed and $\mathcal{M}_\Omega$ is contractible, $\mathcal{E}$ exists and is well-defined up to an additive constant; we denote by $\mathcal{E}_{\omega_0}$ the unique determination of $\mathcal{E}$ which vanishes at the chosen base element ω_0 on $\mathcal{M}_\Omega$. It follows from (2.6) that $\mathcal{E}$ is $\mathcal{D}$-convex on $\mathcal{M}_\Omega$, meaning that its Hessian $\mathcal{D}d\mathcal{E}$ is non-negative; moreover, for any g in $\mathcal{M}_\Omega$, its kernel in $T_g\mathcal{M}_\Omega$ is the space of Killing potentials of mean value zero.

Because of (2.10), the critical points of $\mathcal{E}$ are the zeros of σ, hence the metrics of constant scalar curvature in $\mathcal{M}_\Omega$. To generalize the setting to include extremal metrics of non-constant scalar curvature—the case of main interest in this paper—it is convenient to substitute a relative version introduced by D. Guan in [**22**] and S. Simanca in [**38**]. This is done as follows.

Let G be a maximal compact subgroup of $\mathrm{H}_0(M,J)$ and denote by $\mathcal{M}_\Omega^G$ the space of G-invariant Kähler metrics in Ω. $\mathcal{M}_\Omega^G$ is a totally geodesic submanifold of $\mathcal{M}_\Omega$. In virtue of a celebrated theorem of Calabi [**9**], any extremal Kähler metric in $\mathcal{M}_\Omega$—if any—belongs to the $\mathrm{H}_0(M,J)$-orbit of an element of $\mathcal{M}_\Omega^G$. Since G is maximal in $\mathrm{H}_0(M,J)$, its Lie algebra, $\mathfrak{g}$, is the Lie algebra of *all* hamiltonian Killing vector fields for each element, g, of $\mathcal{M}_\Omega^G$. Notice that, while the latter is independent of g, the space, P_g, of Killing potentials with respect to g does depend of g.

For any g in $\mathcal{M}_\Omega^G$, of scalar curvature s_g, the *Killing part*, $\Pi_g^G(s_g)$, of s_g is defined as the L^2-projection of s relative to g in P_g. The *reduced scalar curvature* of g, denoted by s_g^G, is defined by

$$s_g^G = s_g - \Pi_g^G(s_g). \tag{2.12}$$

Then, g is extremal if and only if its reduced scalar curvature s_g^G is identically zero.

The vector field $Z_\Omega^G = \mathrm{grad}(\Pi_g^G(s))$—called the *extremal vector field* of the pair (Ω, G)—is independent of g in $\mathcal{M}_\Omega^G$ and can be alternatively determined by

$$\mathcal{F}_\Omega(JX) = B_\Omega(JX, Z_\Omega^G), \tag{2.13}$$

for any X in $\mathfrak{g}$. Notice that Z_Ω^G belongs to the center $\mathfrak{z}$ of $\mathfrak{g}$. Its lift, $\hat{Z}_\Omega^G$, on $\mathcal{M}_\Omega^G$ is the vector field $g \mapsto \Pi_g^G(s_g)$. It turns out that $\hat{Z}_\Omega^G$ is $\mathcal{D}$-parallel, and so is its dual 1-form ζ_Ω^G, cf. [**19**]. We now consider the 1-form on $\mathcal{M}_\Omega^G$ defined by

$$\sigma^G = \sigma_{|\mathcal{M}_\Omega^G} - \zeta_\Omega^G. \tag{2.14}$$

Since ζ_Ω^G is $\mathcal{D}$-parallel, we infer from (2.6)

$$\mathcal{D}_f \sigma^G = -2\,(D^- d)^* D^- df, \tag{2.15}$$

for any f in $T_g\mathcal{M}_\Omega^G$. In particular, σ^G is closed.

Denote by $\mathrm{H}_G(M, J)$ the normalizer of G in $\mathrm{H}_0(M, G)$ and by $\mathfrak{h}_G$ the Lie algebra of $\mathrm{H}_G(M, J)$. The group $\mathrm{H}_G(M, J)$ acts on $\mathcal{M}_\Omega^G$ and we define as above the *relative Futaki character* $\mathcal{F}_\Omega^G : \mathfrak{h}_G \to \mathbb{R}$ by

$$\mathcal{F}_\Omega^G(X) = \sigma^G(\hat{X}) = \int_M f_g^X\, s_g^G v_g. \tag{2.16}$$

As before, $\Omega : \mathfrak{h}_G \to \mathbb{R}$ is independent of g in $\mathcal{M}_\Omega^G$.

The *relative K-energy* $\mathcal{E}^G$ is defined by

$$\sigma^G = -d\mathcal{E}^G, \tag{2.17}$$

i.e. by

$$d\mathcal{E}_g^G(f) = -\int_M f\, s_g^G\, v_g, \tag{2.18}$$

for any g in $\mathcal{M}_\Omega^G$ and any f in $T_g\mathcal{M}_\Omega^G$. Since σ^G is closed and $\mathcal{M}_\Omega^G$ is contractible, $\mathcal{E}^G$ is well-defined up to an additive real constant. As before, we denote by $\mathcal{E}_{\omega_0}^G$ the determination of $\mathcal{E}^G$ which is zero at the chosen base point ω_0. By (2.17), the critical points of $\mathcal{E}^G$ are the zeros of σ^G, hence the extremal metrics in $\mathcal{M}_\Omega^G$. Moreover, since $\mathcal{D}\sigma^G = \mathcal{D}\sigma_{|\mathcal{M}_\Omega^G}$, $\mathcal{E}^G$ is $\mathcal{D}$-convex and, at each g in $\mathcal{M}_\Omega^G$, the kernel of the Hessian $\mathcal{D}d\mathcal{E}^G$ is the space of G-invariant Killing potentials relative to g.

2.2. The Chen-Tian Theorem. — The K-energy $\mathcal{E}$ and the relative K-energy $\mathcal{E}^G$ defined in Section 2.1 play an important role in the theory of extremal Kähler metrics, due in particular to the following observation.

***Proposition 2.1* (S. Donaldson [14]).** — *Let ω_0, ω be any two elements of $\mathcal{M}_\Omega^G$. Assume that ω_0 is extremal. Assume, moreover, that there exists a geodesic $\omega_t = \omega_0 + dd^c\varphi_t$, $t \in [0,1]$, between ω_0 and $\omega = \omega_1$. Then*

$$\mathcal{E}^G(\omega) \geq \mathcal{E}^G(\omega_0) \tag{2.19}$$

and equality holds if and only if ω is extremal. If so, ω belongs to the $\mathrm{H}_0(M, J)$*-orbit of ω_0.*

Proof. — (Sketch) To simplify notation, let's write $f(t)$ for $\mathcal{E}^G(\omega_t)$; we can assume $f(0) = 0$. By (2.17), we have that $f'(t) = -\sigma^G(T)$, where T denotes the tangent vector field along the geodesic ω_t, given by the assignment $t \mapsto \dot{\varphi}_t \in T_{\omega_t}\mathcal{M}_\Omega^G$. In particular, $f'(0) = 0$, since ω_0 is extremal. By using (2.15), we get:

$$\begin{aligned} f''(t) &= -(\mathcal{D}_T\sigma^G)(T) - \sigma^G(\mathcal{D}_T T) \\ &= -(\mathcal{D}_T\sigma^G)(T) = 2\int_M ((D^-d)D^-d\dot{\varphi}_t, \dot{\varphi}_t)\, v_{g_t} \\ &= 2\int_M |D^-d\dot{\varphi}_t|^2\, v_{g_t} \end{aligned} \tag{2.20}$$

The last term is non-negative and is zero if and only if $\dot{\varphi}_t$ is a Killing potential with respect to g_t for each t in $[0,1]$, cf. Section 1.6. Proposition 2.1 follows easily. □

This argument has been extended by X. X. Chen and G. Tian in the following way (cf. also Remark 2.1 for the notation):

***Proposition 2.2* (X. Chen–G. Tian [13]).** — *Let ω_0 be a fixed element of $\mathcal{M}_\Omega^G$ and let Φ be a smooth G-invariant solution of the Monge-Ampère equation (2.4) defined on $M \times \Sigma$ for any Riemann surface with boundary Σ. Suppose that, for any τ in Σ, $\Phi(\cdot, p)$ is the relative Kähler potential of an element $\omega^{(\tau)} = \omega_0 + dd^c\Phi(\cdot,\tau)$ in $\mathcal{M}_\Omega^G$, so that the relative energy $\mathcal{E}^G(\tau) := \mathcal{E}^G(\omega^{(\tau)})$ can be regarded as a function defined on Σ. Let $z = t + is$ be a local holomorphic coordinate on Σ. Then, with the notation of* Remark 2.1, *$\mathcal{E}^G(\tau)$ satisfies the following equality*

$$\frac{d^2\mathcal{E}^G}{dt^2} + \frac{d^2\mathcal{E}^G}{ds^2} = 2\int_M |D^-(d\dot{\varphi}_t - d^c\dot{\varphi}_s)|^2_{\omega_{(\tau)}}\, v_{g_{(\tau)}}. \tag{2.21}$$

In particular, $\frac{d^2\mathcal{E}^G}{dt^2} + \frac{d^2\mathcal{E}^G}{ds^2} \geq 0$, with equality if and only if $Z :=$ $\mathrm{grad}_{g_{t,s}}\dot{\varphi}_t - J\mathrm{grad}_{g_{t,s}}\dot{\varphi}_s$ is a (real) holomorphic vector field on M for any τ in Σ.

Proof. — From (2.18), we infer $\frac{d\mathcal{E}^G}{dt} = -\int_M s^G_{g(\tau)} \dot{\varphi}_t \, v_{g(\tau)}$. It is easily deduced from (2.15) that, in general, the first variation of the reduced scalar curvature at g in $\mathcal{M}^G_\Omega$ in the direction of f is given by

$$\dot{s^G}(f) = -2(D^-d)^* D^- df + (ds^G_g, f), \tag{2.22}$$

whereas the first variation of the volume form is given by $\dot{v_g}(f) = -\Delta_g f \, v_g$. The second derivative of $\mathcal{E}^G$ with respect to t is then given by

$$\frac{d^2\mathcal{E}^G}{dt^2} = 2\int_M |D^- d\dot{\varphi}_t|^2 \, v_g - \int_M (\ddot{\varphi}_{tt} - (d\dot{\varphi}_t, d\dot{\varphi}_t)_{g(\tau)}) \, s^G_{g(\tau)} v_{g(\tau)}. \tag{2.23}$$

We get a similar formula by replacing t by s, hence, by using (2.5):

$$\begin{aligned} \frac{d^2\mathcal{E}^G}{dt^2} + \frac{d^2\mathcal{E}^G}{ds^2} &= 2\int_M |D^-(d\dot{\varphi}_t - d^c\dot{\varphi}_s)|^2 \, v_{g_\tau} \\ &+ 2\int_M \left(2(D^- d\dot{\varphi}_t, D^- d^c\dot{\varphi}_s) + (d\dot{\varphi}_t, d^c\dot{\varphi}_s) \, s^G_{g_\tau}\right) v_{g_\tau} \end{aligned} \tag{2.24}$$

where the second term in the rhs is actually zero [7]. The last assertion of Proposition 2.2 follows easily (see Section 1.6). □

The argument in Proposition 2.1 only holds for metrics which are linked to extremal metrics by a geodesic. On the other hand, the existence issue for geodesics in $\mathcal{M}_\Omega$ has remained an open question, principally because of the lack of regularity for solutions of the Monge-Ampère equation (2.4). In [**13**], X. X. Chen and G. Tian established a (weak) regularity theorem for solutions of (2.4), improving a previous regularity result by X. X. Chen [**10**] which asserts the existence of solutions in the class $C^{1,1}$. From this, and by using the above Proposition 2.2, they were able to deduce the following fudamental results:

***Theorem 2.1* (X. X. Chen–G. Tian [12], [11], [13]).** — (i) *All extremal metrics in $\mathcal{M}_\Omega$, if any, belong to a unique* $\mathrm{H}_0(M, J)$*-orbit.*

(ii) *Let ω_0 be an extremal metric in $\mathcal{M}_\Omega$. Without loss of generality, assume that ω_0 belongs to $\mathcal{M}^G_\Omega$. Then,*

$$\mathcal{E}^G(\omega) \geq \mathcal{E}^G(\omega_0), \tag{2.26}$$

with equality if and only if ω is extremal.

(7) This is an easy consequence of the following general formula (see Section 1.6 for the notation):

$$(D^-d)^* D^- d^c f = -\tfrac{1}{2}\mathcal{L}_K f, \tag{2.25}$$

for any function f on a Kähler manifold of scalar curvature s_g, with $K := J\mathrm{grad}_g s_g$. Here, (2.25) is applied to $f = \dot{\varphi}_s$. Moreover, since $\dot{\varphi}_s$ is G-invariant, K can be replaced by $K^G := J\mathrm{grad}_g s^G_g$.

2.3. The relative energy of admissible metrics. — Denote by $\mathcal{M}^{\mathrm{adm}}_{\Omega_\lambda}$ the space of admissible Kähler metrics in a given (normalized) admissible Kähler class Ω_λ. Then, $\mathcal{M}^{\mathrm{adm}}_{\Omega_\lambda} \subset \mathcal{M}^G_{\Omega_\lambda}$, where G is the maximal compact subgroup of $\mathrm{H}_0(M,J)$ given by Propositions 1.3–1.6, and the reduced scalar curvature is given by the following proposition (cf. [**3**, Proposition 6]):

Proposition 2.3. — *For any (normalized) admissible Kähler class Ω_λ and for any admissible Kähler metric $g = g_{\lambda,z}$ in Ω_λ, of scalar curvature s_g, the Killing part of s_g is given by*

$$\Pi^G_g(s_g) = \alpha\, z + \beta, \tag{2.27}$$

where α, β denote the renormalized leading coefficients of the extremal polynomial F_{Ω_λ} defined by (1.53)*, whereas the reduced scalar curvature has the following expression:*

$$s^G_g = \frac{(F_{\Omega_\lambda} - p_{\Omega_\lambda}\Theta)''(z)}{p_{\Omega_\lambda}(z)}. \tag{2.28}$$

Proof. — For any admissible Kähler metric in a (normalized) Kähler class, it follows from (1.23) that the space P_g of Killing potentials relative to g splits as

$$P_g = \mathbb{R} \oplus \mathbb{R}\, z \oplus \left(\oplus_{i=1}^N P^0_{g_{S_i}}\right), \tag{2.29}$$

where: $\mathbb{R}$ denotes the space of constant functions; $\mathbb{R}\, z$ the space generated by z; $P^0_{g_{S_i}}$ denotes the space of Killing potentials of mean value zero on (S_i, g_{S_i}). By (1.52), the scalar curvature s is a function of z only; by (1.27), s is then L^2-orthogonal to all Killing potentials in $\oplus_{i=1}^N P^0_{g_{S_i}}$. In order to prove (2.27), it is sufficient to check that $\frac{R(x)-(p_{\Omega_\lambda}\Theta)''(x)}{p_{\Omega_\lambda}(x)}$ is orthogonal to 1 and to z. In view of (1.27), this amounts to checking that $\int_{-1}^1 (R(x) - (p_{\Omega_\lambda}\Theta)''(x))\, dx = 0$ and $\int_{-1}^1 (R(x) - (p_{\Omega_\lambda}\Theta)''(x))\, x\, dx = 0$; in view of the boundary conditions (1.14)–(1.15) for Θ, these two conditions are equivalent to (1.43)–(1.44); since $R = F''_{\Omega_\lambda}$, (2.28) follows from (2.27) and (1.52). □

Corollary 2.1. — *For any admissible Kähler class Ω_λ, denote by $Z^G_{\Omega_\lambda}$ the extremal vector field relative to the pair (G, Ω_λ), see* Section 2.1. *Then*

$$JZ^G_{\Omega_\lambda} = \alpha\, T. \tag{2.30}$$

Proof. — By definition, $Z^G_{\Omega_\lambda} = \mathrm{grad}_g(\Pi^G_g(s_g))$, for any g in $\mathcal{M}^G_{\Omega_\lambda}$, hence for $g_{\lambda,z}$. Since, $-JT = \mathrm{grad}_{g_{\lambda,z}} z$, (2.30) readily follows from (2.27). □

Corollary 2.2. — *For any admissible Kähler class Ω_λ, we have*

$$\begin{aligned} \mathcal{F}_{\Omega_\lambda}(-JT) = {} & \frac{2\pi V(S)}{\int_{-1}^1 p_{\Omega_\lambda}(s)ds} \times \\ & \alpha\Big(\int_{-1}^1 s^2 p_{\Omega_\lambda}(s)ds \int_{-1}^1 p_{\Omega_\lambda}(s)ds - \int_{-1}^1 s\, p_{\Omega_\lambda}(s)ds \int_{-1}^1 s\, p_{\Omega_\lambda}(s)ds\Big) \end{aligned} \tag{2.31}$$

and

(2.32)

$$B_{\Omega_\lambda}(-JT,-JT)=\frac{2\pi V(S)}{\int_{-1}^{1}p_{\Omega_\lambda}(s)ds}\times \Big(\int_{-1}^{1}s^2p_{\Omega_\lambda}(s)ds\int_{-1}^{1}p_{\Omega_\lambda}(s)ds-\int_{-1}^{1}s\,p_{\Omega_\lambda}(s)ds\int_{-1}^{1}s\,p_{\Omega_\lambda}(s)ds\Big),$$

where $V(S)=\prod V(S_i,g_{S_i})$ *denotes the volume of* S. *In particular,*

$$\mathcal{F}_{\Omega_\lambda}(-JT)=\alpha\,B_{\Omega_\lambda}(-JT,-JT). \tag{2.33}$$

Proof. — Since T is a hamiltonian Killing vector field of momentum z, $-JT$ belongs to $J\mathfrak{g}$ and its real holomorphic potential is $z-\bar{z}$, where $\bar{z}=\frac{\int_M zv_g}{\int_M v_g}$ is the mean value of z. Since $z-\bar{z}$ belongs to P_g, in (2.7) only the Killing part $\Pi_g^G(s_g)=\alpha z+\beta$ contributes: we then get $\mathcal{F}_\Omega(-JT)=\alpha\int_M(z-\bar{z})zv_g$ and $B_{\Omega_\lambda}(-JT,-JT)=\int_M(z-\bar{z})^2\,v_g$. By using the expression (1.27) of v_g, we readily get (2.31) and (2.32); (2.33) follows readily; alternatively, (2.33) follows from (2.32) and Corollary 2.1, via (2.13). □

Choose a reference element in $\mathcal{M}^{\mathrm{adm}}_{\Omega_\lambda}$, e.g. the standard admissible metric ω_0 corresponding to the admissible momentum $z_0(t)=\tanh t$, cf. Section 1.5. Any other element ω of $\mathcal{M}^{\mathrm{adm}}_{\Omega_\lambda}$ can be written $\omega=\omega_0+dd^c\phi$, where $\phi=\phi(t)$, called the *relative potential* of ω, is uniquely determined by ω up to an additive constant. Notice that

$$z=z_0+\frac{d\phi}{dt}. \tag{2.34}$$

For any curve ω_s in $\mathcal{M}^{\mathrm{adm}}_{\Omega_\lambda}$ we set $\dot{\omega}=\frac{d\omega_s}{ds}{}_{|s=0}$ and we denote similarly the first variations of all objects determined by ω; we thus have: $\dot{\omega}=dd^c\dot{\phi}$, $\dot{z}=\frac{d\dot{\phi}}{dt}$, etc. By identifying $\dot{\omega}$ with $\dot{\phi}$ we identify each tangent space $T_g\mathcal{M}^{\mathrm{adm}}_{\Omega_\lambda}$ of $\mathcal{M}^{\mathrm{adm}}_{\Omega_\lambda}$ with the space of all smooth real functions of t mod constant functions.

Although it is a hard task to get an explicit expression of the relative energy $\mathcal{E}^G(g)$ for a general element of $\mathcal{M}^G_{\Omega_\lambda}$, it turns out that the restriction of $\mathcal{E}^G$ to $\mathcal{M}^{\mathrm{adm}}_{\Omega_\lambda}$ admits a simple explicit expression in terms of the extremal polynomial F_{Ω_λ}, given by the following proposition (cf. [**3**, Proposition 7]):

Proposition 2.4. — *For any admissible metric* g *in* Ω_λ *of momentum profile* Θ, *we have*

$$\mathcal{E}^G(g)=C\int_{-1}^{1}\Big(\frac{F_{\Omega_\lambda}(x)}{\Theta(x)}+p_{\Omega_\lambda}(x)\,\log\Theta(x)\Big)\,dx \mod \mathbb{R}, \tag{2.35}$$

with $C=2\pi\prod_{i=1}^N V_i$, *where* V_i *denotes the volume of* (S_i,g_{S_i}).

Proof. — The restriction of $\mathcal{E}^G$ to $\mathcal{M}^{\mathrm{adm}}_{\Omega_{\boldsymbol{\lambda}}}$ is determined by

$$d\mathcal{E}^G_g(\dot\phi) = -\int_M s^G_g\, \dot\phi\, v_g \tag{2.36}$$

for any $g = g_{\boldsymbol{\lambda},z}$ in $\mathcal{M}^{\mathrm{adm}}_{\Omega_{\boldsymbol{\lambda}}}$ and for $\dot\phi = \dot\phi(t)$, any function of $t \mod \mathbb{R}$, where, we recall, s^G_g denotes the reduced scalar curvature of g with respect to G. By using (2.28) and (1.27), we get

$$(d\mathcal{E}^G)_g(\dot\phi) = -C\int_{-1}^{1} (F_{\Omega_{\boldsymbol{\lambda}}} - p_{\Omega_{\boldsymbol{\lambda}}}\Theta)''(x) f(x)\, dx, \tag{2.37}$$

where C is as above and by setting

$$f(x) = \dot\phi(z^{-1}(x)). \tag{2.38}$$

By integrating by part twice and by observing that at each step the intregrated terms vanish because of (1.14)–(1.15)–(1.46)–(1.47), we get

$$(d\mathcal{E}^G)_g(\dot\phi) = -C\int_{-1}^{1} (F_{\Omega_{\boldsymbol{\lambda}}} - p_{\Omega_{\boldsymbol{\lambda}}}\Theta)(x) f''(x)\, dx. \tag{2.39}$$

From (2.38) we get $f'(x) = \frac{\dot z(z^{-1}(x))}{\Theta(x)}$, hence

$$f''(x) = \frac{1}{\Theta^2(x)}\left(\frac{d\dot z}{dt}(z^{-1}(x)) - \Theta'(x)(\dot z(z^{-1}(x))\right).$$

On the other hand, from (1.13), we get $\dot\Theta(x) = \frac{d\dot z}{dt}(z^{-1}(x)) - \Theta'(x)\dot z(z^{-1}(x))$. We thus end up with

$$f''(x) = \frac{\dot\Theta(x)}{\Theta^2(x)}. \tag{2.40}$$

By substituting in (2.39), we eventually obtain

$$(d\mathcal{E}^G)_g(\dot\phi) = C\int_{-1}^{1}\left(-\frac{F_{\Omega_{\boldsymbol{\lambda}}}(x)}{\Theta^2(x)} + \frac{p_{\Omega_{\boldsymbol{\lambda}}}(x)}{\Theta(x)}\right)\dot\Theta(x)\, dx, \tag{2.41}$$

for any $\dot\phi = \dot\phi(t)$ in $T_g\mathcal{M}^{\mathrm{adm}}_{\Omega_{\boldsymbol{\lambda}}}$, where the extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$ and the characteristic polynomial $p_{\Omega_{\boldsymbol{\lambda}}}$ are both independent of g in $\mathcal{M}_{\Omega_{\boldsymbol{\lambda}}}$. The rhs of (2.41) is then the first derivative in $\mathcal{M}^{\mathrm{adm}}_{\Omega_{\boldsymbol{\lambda}}}$ of the rhs of (2.35). □

Proposition 2.5. — *Let $\Omega_{\boldsymbol{\lambda}}$ be any admissible Kähler class on M.*

(i) *Assume that $F_{\Omega_{\boldsymbol{\lambda}}}$ is positive on $(-1,1)$ and denote by g_0 an admissible extremal Kähler metric in $\Omega_{\boldsymbol{\lambda}}$ of momentum profile $\Theta_0 = \frac{F_{\Omega_{\boldsymbol{\lambda}}}}{p_{\Omega_{\boldsymbol{\lambda}}}}$* (cf. Proposition 1.7). *Then, for any admissible Kähler metric in $\Omega_{\boldsymbol{\lambda}}$ we have*

$$\mathcal{E}^G(g) \geq \mathcal{E}^G(g_0), \tag{2.42}$$

with equality if and only if g is extremal, hence equal to g_0 up to the natural $\mathbb{R}^{>0}$-action on M.

(ii) *Assume that $F_{\Omega_{\lambda}}$ is negative on a non-empty open subinterval I of $(-1,1)$. Then, for any admissible Kähler metric g in Ω_{λ} there exists a half-line g_s of admissible Kähler metrics in Ω_{λ} with s in $[0,+\infty)$ and $g_0 = g$, such that $\mathcal{E}^G(g_s)$ tends to $-\infty$ when s tends to $+\infty$.*

Proof. — (i) From (2.35) we infer

$$\mathcal{E}^G(g_0) = C\int_{-1}^{1}(1+\log\frac{F_{\Omega_{\lambda}}(x)}{p_{\Omega_{\lambda}}}(x))\,p_{\Omega_{\lambda}}(x)\,dx, \tag{2.43}$$

whereas

$$\mathcal{E}^G(g) = C\int_{-1}^{1}(\frac{F_{\Omega_{\lambda}}(x)}{p_{\Omega_{\lambda}}(x)\,\Theta(x)} + \log\Theta(x))\,p_{\Omega_{\lambda}}(x)\,dx. \tag{2.44}$$

We thus get

$$\mathcal{E}^G(g) - \mathcal{E}^G(g_0) = C\int_{-1}^{1}(A(x) - 1 - \log A(x))\,p_{\Omega_{\lambda}}(x)\,dx \tag{2.45}$$

by setting

$$A(x) = \frac{F_{\Omega_{\lambda}}(x)}{p_{\Omega_{\lambda}}(x)\,\Theta(x)}. \tag{2.46}$$

Now, $A(x)$ is positive for any x in $(-1,1)$ by hypothesis and, by Proposition 1.7, is identically equal to 1 if and only if g is extremal. It is easy to check that the function $\phi(t) := t - 1 - \log t$ defined on $(0,+\infty)$ is convex, tends to $+\infty$ when t tends to 0 or to $+\infty$, and reaches its unique minimum 0 at $t = 1$. It follows that the rhs of (2.45) is positive except when $A = A(x)$ is identically equal to 1, i.e. when g is extremal.

(ii) Let Θ be the momentum profile of any admissible Kähler metric g in $\Omega_{\boldsymbol{\lambda}}$. Let φ be a non-negative, non-constant smooth function on $(-1,1)$ which is compactly supported in the interval I, and set

$$\Theta_s(x) = \frac{\Theta(x)}{1 + s\,\varphi(x)\Theta(x)}, \tag{2.47}$$

for any non-negative real number s. By Proposition 1.2, Θ_s is the momentum profile of an admissible Kähler metric, g_s, in $\Omega_{\boldsymbol{\lambda}}$ for any $s \geq 0$, with $g_0 = g$. Moreover

$$\begin{aligned}\mathcal{E}^G(g_s) =& \mathcal{E}^G(g) + C\int_{-1}^{1} s\,\varphi(x)F_{\Omega_{\lambda}}(x)\,dx\\ &- C\int_{-1}^{1}\log(1 + s\,\varphi(x)\Theta(x))\,dx\end{aligned} \tag{2.48}$$

where, $\int_{-1}^{1} s\,\varphi(x)F_{\Omega_{\lambda}}(x))\,dx = \int_I \varphi(x)F_{\Omega_{\lambda}}(x))\,dx$ is a *negative* multiple of s. It follows that the rhs of (2.48) tends to $-\infty$ when s tends to $+\infty$. □

Remark 2.2. — The expression (2.35) of the (relative) energy of admissible metrics, as well as the argument in Proposition 2.5, are quite reminiscent to Donaldson's paper [**15**] for toric manifolds.

Now we are ready to state and prove the main result of [**3**]:

Theorem 2.2. — *Let $M = \mathbb{P}(1 \oplus L)$ be any admissible ruled manifold and let $\Omega_{\boldsymbol{\lambda}}$ be any (normalized) admissible Kähler class on M. Then, $\Omega_{\boldsymbol{\lambda}}$ contains an extremal Kähler metric—which is then admissible up to the action of* $\mathrm{H}_0(M,J)$*—if and only if the extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$ is (strictly) positive on $(-1,1)$.*

Proof. — By Proposition 1.7, if $F_{\Omega_{\boldsymbol{\lambda}}}$ is positive on $(-1,1)$, $\Omega_{\boldsymbol{\lambda}}$ contains an admissible extremal Kähler metrics. By Proposition 2.5, if $F_{\Omega_{\boldsymbol{\lambda}}}$ is negative on some open subinterval of $(-1,1)$, the relative K-energy $\mathcal{E}^G$ is not bounded from below: by Theorem 2.1 (ii), $\Omega_{\boldsymbol{\lambda}}$ contains no extremal Kähler metric.

It remains to consider the limiting case, when $F_{\Omega_{\boldsymbol{\lambda}}}$ is non-negative but has (repeated) zeros on $(-1,1)$. Suppose that $F_{\Omega_{\boldsymbol{\lambda}}}$ is of this form and assume, for a contradiction, that $\Omega = \Omega_{\boldsymbol{\lambda}}$ contains an extremal Kähler metric, (g,ω) say. In view of the already mentioned Calabi theorem, we can assume that the pair (g,ω) is G-invariant (cf. Proposition 1.6). By LeBrun-Simanca's openness theorem [**29, 30**], any (normalized) admissible Kähler class $\Omega_{\boldsymbol{\lambda}'}$, with $\boldsymbol{\lambda}'$ close to $\boldsymbol{\lambda}$ in $\mathbb{R}^N$, contains an extremal Kähler metric. More precisely, LeBrun-Simanca's theorem asserts the existence of a sequence of extremal Kähler metrics $(\tilde{g}_k, \tilde{\omega}_k)$, with $[\tilde{\omega}_k] = \Omega_k$, which converges to (g,ω) in the Fréchet topology and the $(\tilde{g}_k, \tilde{\omega}_k)$ can be again chosen G-invariant.

Two cases then may *a priori* occur: (i) either, $F_{\Omega_{\boldsymbol{\lambda}'}}$ has repeated roots on $(-1,1)$ for all $\boldsymbol{\lambda}'$ in some open neighborhood of $\boldsymbol{\lambda}$ in $\mathbb{R}^N$, or else: (ii) there exists a sequence of (normalized) admissible Kähler classes $\Omega_k = \Omega_{\boldsymbol{\lambda}_k}$ converging to Ω—meaning that $\boldsymbol{\lambda}_k$ converges to $\boldsymbol{\lambda}$ in the usual sense—such that F_{Ω_k} is positive on $(-1,1)$ for each k.

Case (i) would imply that the discriminant of $F_{\Omega_{\boldsymbol{\lambda}}}$ is zero as a polynomial with coefficients in the field $R(\lambda_1,\ldots,\lambda_N)$ of rational fractions in $\{\lambda_1,\ldots,\lambda_N\}$: this would contradict Proposition A.1 in Appendix A (by substituting $\lambda_i = \epsilon_i \lambda$ in $F_{\Omega_{\boldsymbol{\lambda}}}$, regarded as a polynomial with coefficients in $R(\lambda_1,\ldots,\lambda_N)$, up to a factor $\prod_{i=1}^N \epsilon_i^{d_i}$, we get the extremal polynomial of an admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$, as a polynomial with coefficient in $R(\lambda)$, on an admissible ruled manifold with $N=1$, $d = \sum_{i=1} d_i$ and $s = \sum_{i=1}^N s_i$). Case (i) is thus discarded.

Now assume, again for a contradiction, that Case (ii) occurs. LeBrun-Simanca openness theorem actually guarantees the existence of a sequence, $(\tilde{g}_k, \tilde{\omega}_k)$, of G-invariant extremal Kähler metrics, with $[\tilde{\omega}_k] = \Omega_k$ for each k, which converges to (g,ω) in the Fréchet topology. On the other hand, since F_{Ω_k} is positive on $(-1,1)$, Proposition 1.7 guarantees the existence of an *admissible* extremal Kähler metric,

(g_k, ω_k) say, in each Ω_k, unique up to the natural $\mathbb{C}^*$-action, with $\omega_k = \sum_{i=1}^N ((\boldsymbol{\lambda}_k)_i + \epsilon_i z_k)\pi^*\omega_i + dz_k \wedge d^c t$, cf. Section 1.3.

By Theorem 2.1, for any k the extremal Kähler metrics (g_k, ω_k) and $(\tilde{g}_k, \tilde{\omega}_k)$ in Ω_k are linked together by $\tilde{g}_k = \Psi_k \cdot g_k$, for some Ψ_k in $\mathrm{H}_0(M, J)$. Moreover, from the invariance of the extremal vector field $Z^G_{\Omega_k}$ of each pair (Ω_k, G)—see Sections 2.1 and 2.3—we get $Z^G_{\Omega_k} = \mathrm{grad}_{g_k} s_{g_k} = \mathrm{grad}_{\tilde{g}_k} s_{\tilde{g}_k} = \Psi_k \cdot \mathrm{grad}_{g_k} s_{g_k}$, meaning that $Z^G_{\Omega_k}$, hence also T by Corollary 2.1, are preserved by Ψ_k for any k. We infer that the Ψ_k's all belong to the subgroup of elements of $\mathrm{H}_0(M, J)$ which commute with $\mathbb{C}^*$, hence, by Proposition 1.3, to the extension of $\mathrm{H}_0(S, J)$ by $\mathbb{C}^*$. Moreover, since the (g_k, ω_k) are only defined up to the natural $\mathbb{C}^*$-action, we can actually arrange that the Ψ_k's all belong to a lift of $\mathrm{H}_0(S, J)$ in $\mathrm{H}_0(M, J)$, meaning that each Ψ_k is induced by a linear lift on L of an element, Φ_k say, of $\mathrm{H}_0(S, J)$. Each $\tilde{\omega}_k$ is then of the form $\tilde{\omega}_k = \sum_{i=1}^N ((\boldsymbol{\lambda}_k)_i + \epsilon_i \Psi_k \cdot z_k)\pi^*(\Phi_k \cdot \omega_i) + d(\Psi_k \cdot z_k) \wedge d^c(\Psi_k \cdot t)$, hence the Kähler form of an (extremal) admissible Kähler metrics on the admissible ruled manifold obtained by simply substituting the hermitian inner product $\tilde{h}_k = \Psi_k \cdot h$ on L. Since any two hermitian inner products on L are conformal, $\tilde{h}_k$ can be alternatively written as $\tilde{h}_k = e^{2F_k} h$ for some well-defined (real) smooth function F_k on S and we then have $\tilde{t}_k = \Psi_k \cdot t = t + \pi^* F_k$. Since $\Psi_k \cdot T = T$, we also have that $\tilde{z}_k = \Psi_k \cdot z_k$ is a momentum of T with respect to $\tilde{\omega}_k$.

By assumption, the sequence $\tilde{\omega}_k$ converges to ω in the Fréchet topology: it follows that $\tilde{z}_k$ converges to a momentum of T with respect to ω; similarly, since $\iota_{JT}\tilde{\omega}_k = -\tilde{g}_k(T,T)\, d^c \tilde{t}_k = -\tilde{g}_k(T,T)\, d^c(t + \pi^* F_k)$, the sequence F_k converges to a smooth function F on S, meaning that the sequence $\tilde{h}_k$ converges to the hermitian inner product $\tilde{h} = e^{2F} h$, whereas each $\Psi_k \cdot \omega_i$ converges to $\tilde{\omega}_i$, which is the curvature form of $L^{-\epsilon_i}$ equipped with the hermitian inner product induced by $\tilde{h}$.

It follows that ω is the Kähler form of an extremal admissible Kähler metric on M with respect to $(L, \tilde{h})$. Since the extremal polynomial F_Ω of Ω only depends of the N-tuple $\boldsymbol{\lambda}$ and of the ϵ_i's, F_Ω should then be positive on $(-1,1)$ by Proposition 1.7 again. Case (ii) is then discarded as well. □

2.4. A borderline case example. — In this section, we present a family of examples of (normalized) admissible Kähler classes on an admissible rules manifold $M = \mathbb{P}(1 \oplus L) \to S$ whose extremal polynomials are non-negative but have a repeated root, which can be chosen irrational, on $(-1,1)$.

The simplest examples are obtained by considering (complex) four-dimensional admissible ruled manifolds for which $S = \prod_{i=1}^3 S_i$, where each S_i is a Riemann surfaces of genus g_i greater than one. For $i = 1,2,3$, the (constant) scalar curvatures s_i of S_i

is then negative; more precisely, by the Gauss-Bonnet formula,

$$s_i = \frac{4(1-g_i)}{k_i}, \tag{2.49}$$

where k_i denotes the degree of the polarizing line bundle $\tilde{L}_i = L_i^{-\epsilon_i}$ (cf. Section 1.1 and formula (1.62) in Section 1.10). In particular, each s_i can be made equal to any negative rational number by an appropriate choice of the genus g_i and of the degree k_i.

Our aim is to construct a family of (normalized) admissible Kähler classes $\Omega_{\boldsymbol{\lambda}}$ on M, for an appropriate choice of the scalar curvatures s_i—hence of the line bundles L_i on S_i by (2.49)—in such a way that the extremal polynomials be of the form

$$F_{\Omega_{\boldsymbol{\lambda}}}(x) = C(1-x^2)(x^2+rx-1)^2, \tag{2.50}$$

for some positive constants C and r. The polynomial in the rhs of (2.50) satisfies the first boundary condition (1.46) for extremal polynomials and is non-negative on $(-1,1)$. It has two repeated roots: a positive one, $r_+ = \frac{-r+\sqrt{r^2+4}}{2}$, in the open interval $(0,1)$; a negative one, $r_- = \frac{-r-\sqrt{r^2+4}}{2}$, in $(-\infty,-1)$. The first and second derivatives of $F_{\Omega_{\boldsymbol{\lambda}}}$ are given by:

$$F'_{\Omega_{\boldsymbol{\lambda}}}(x) = C(-6x^5 - 10rx^4 + 4(3-r^2)x^3 + 12rx^2 + 2(r^2-3)x - 2r) \tag{2.51}$$

and

$$F''_{\Omega_{\boldsymbol{\lambda}}}(x) = C(-30x^4 - 40rx^3 + 12(3-r^2)x^2 + 24rx + 2(r^2-3)). \tag{2.52}$$

In particular, $F'_{\Omega_{\boldsymbol{\lambda}}}(-1) = 2Cr^2$ and $F'_{\Omega_{\boldsymbol{\lambda}}} = -2Cr^2$. It follows that $F_{\Omega_{\boldsymbol{\lambda}}}$ satisfies the second boundary condition (1.47) for extremal polynomials if and only if

$$p_{\Omega_{\boldsymbol{\lambda}}}(-1) = p_{\Omega_{\boldsymbol{\lambda}}}(1) = Cr^2, \tag{2.53}$$

where $p_{\Omega_{\boldsymbol{\lambda}}}(x) = \prod_{i=1}^{3}(\lambda_i + \epsilon_i x)$ denotes the characteristic polynomial[8] of $\Omega_{\boldsymbol{\lambda}}$, cf. (1.7). If we write $p_{\Omega_{\boldsymbol{\lambda}}}(x) = \sum_{j=0}^{3} p_j x^{3-j}$, with $p_0 = \epsilon_1\epsilon_2\epsilon_3$, $p_1 = \sum_{ijk}\epsilon_i\epsilon_j\lambda_k$, $p_2 = \sum_{ijk}\epsilon_i\lambda_j\lambda_k$, $p_3 = \lambda_1\lambda_2\lambda_3$ (summation over the circular permutation of $(1,2,3)$), (2.53) is equivalent to the two conditions:

$$p_0 + p_2 = 0, \tag{2.54}$$

$$p_1 + p_3 = Cr^2. \tag{2.55}$$

The condition (2.54) cannot be satisfied if all ϵ_i are equal to 1 or -1: We then assume

$$\epsilon_1 = \epsilon_2 = 1, \qquad \epsilon_3 = -1, \tag{2.56}$$

[8] As long as the s_i and the ϵ_i—hence the S_i and the polarizing line bundles $L_i^{-\epsilon_i}$ over S_i—have not been fixed, $\Omega_{\boldsymbol{\lambda}}$ is only a "virtual" admissible Kähler class encoded by an admissible triple $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \lambda_3\}$.

and (2.54) then reads:

$$\lambda_3 = \frac{1+\lambda_1\lambda_2}{\lambda_1+\lambda_2}. \tag{2.57}$$

Notice that $\frac{1+\lambda_1\lambda_2}{\lambda_1+\lambda_2} = 1 + \frac{(\lambda_1-1)(\lambda_2-1)}{\lambda_1+\lambda_2} > 1$. The condition (2.55) determines the constant C as follows:

$$C = \frac{p_1+p_3}{r^2} = \frac{\lambda_1\lambda_2\lambda_3 + \lambda_3 - \lambda_1 - \lambda_2}{r^2}. \tag{2.58}$$

Notice, by using (2.57), that $\lambda_1\lambda_2\lambda_3 + \lambda_3 - \lambda_1 - \lambda_2 = \frac{(1+\lambda_1\lambda_2)^2-(\lambda_1+\lambda_2)^2}{\lambda_1+\lambda_2} = \frac{(\lambda_1^2-1)(\lambda_2^2-1)}{\lambda_1+\lambda_2} > 0$. Also notice that

$$\lambda_1 - \lambda_3 = \frac{\lambda_1^2-1}{\lambda_1+\lambda_2} > 0, \qquad \lambda_2 - \lambda_3 = \frac{\lambda_2^2-1}{\lambda_1+\lambda_2} > 0. \tag{2.59}$$

Now, for any positive real number r and for any admissible triple $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \lambda_3\}$ satisfying (2.57), the polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$ defined by (2.50), where C is defined by (2.58), is actually the extremal polynomial of the (normalized) admissible Kähler class Ω_λ if and only if $F''_{\Omega_{\boldsymbol{\lambda}}}(x) = R(x)$, where, in general, $R(x)$ is defined by (1.42) in Section 1.9. In the present situation, this condition is then:

$$\begin{aligned} F''_{\Omega_{\boldsymbol{\lambda}}}(x) &= s_1(\lambda_2+x)(\lambda_3-x) + s_2(\lambda_3-x)(\lambda_1+x) + s_3(\lambda_1+x)(\lambda_2+x) \\ &\quad - (\alpha x+\beta)(\lambda_1+x)(\lambda_2+x)(\lambda_3-x), \end{aligned} \tag{2.60}$$

where α, β are real constants. In view of (2.60), we now assume that λ_1 and λ_2 are distinct, hence $\lambda_1 > \lambda_2$ say. This implies that the s_i's are uniquely determined by

$$s_1 = \frac{F''_{\Omega_{\boldsymbol{\lambda}}}(-\lambda_1)}{(\lambda_2-\lambda_1)(\lambda_3+\lambda_1)}, \quad s_2 = \frac{F''_{\Omega_{\boldsymbol{\lambda}}}(-\lambda_2)}{(\lambda_1-\lambda_2)(\lambda_3+\lambda_2)}, \quad s_3 = \frac{F''_{\Omega_{\boldsymbol{\lambda}}}(\lambda_3)}{(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}, \tag{2.61}$$

a special case of the general formula (1.49). By using (2.52), this can be re-written as

$$s_1 = \frac{2C}{(\lambda_1-\lambda_2)(\lambda_1+\lambda_3)}\big((6\lambda_1^2-1)\,r^2 - 4\lambda_1(5\lambda_1^2-3)\,r + 15\lambda_1^4 - 18\lambda_1^2 + 3\big), \tag{2.62}$$

$$s_2 = \frac{2C}{(\lambda_1-\lambda_2)(\lambda_2+\lambda_3)}\big(-(6\lambda_2^2-1)\,r^2 + 4\lambda_2(5\lambda_2^2-3)\,r - 15\lambda_2^4 + 18\lambda_2^2 - 3\big), \tag{2.63}$$

$$s_3 = \frac{2C}{(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}\big(-(6\lambda_3^2-1)\,r^2 - 4\lambda_3(5\lambda_3^2-3)\,r - 15\lambda_3^4 + 18\lambda_3^2 - 3\big). \tag{2.64}$$

Conversely, if s_1, s_2, s_3 are given these values, then $F''_{\Omega_{\boldsymbol{\lambda}}}(x)$ is of the form (2.60)—as $F''_{\Omega_{\boldsymbol{\lambda}}}(x) - s_1(\lambda_2+x)(\lambda_3-x) + s_2(\lambda_3-x)(\lambda_1+x) + s_3(\lambda_1+x)(\lambda_2+x)$ is then divisible by $(\lambda_1+x)(\lambda_2+x)(\lambda_3-x)$—so that $F_{\Omega_{\boldsymbol{\lambda}}}$ is indeed an extremal polynomial provided however that the real numbers s_i defined by (2.62)-(2.63)-(2.64) can be realized as the scalar curvatures of Riemann surfaces S_i of genus greater than 1, polarized by a holomorphic line bundle $L_i^{-\epsilon_i}$. According to (2.49), this can be done whenever s_i are (arbitrary) negative rational numbers. This forces us to assume that λ_1, λ_2—hence also λ_3 by (2.57)—are rational, as well as the parameter r.

By (2.64), s_3 is negative for any $r > 0$ and any admissible triple $\{\lambda_1, \lambda_2, \lambda_3\}$. By (2.63)-(2.64), s_1 is negative if and only if

$$\psi_-(\lambda_1) < r < \psi_+(\lambda_1), \tag{2.65}$$

and s_2 is negative if and only if

$$r < \psi_-(\lambda_2) \qquad or \qquad r > \psi_+(\lambda_2), \tag{2.66}$$

by setting

$$\psi_\pm(\lambda) = \frac{2\lambda(5\lambda^2 - 3) \pm \sqrt{10\lambda^6 + 3\lambda^4 + 3}}{6\lambda^2 - 1}. \tag{2.67}$$

It is easy to check that ψ_- is increasing from $\psi_-(1) = 0$ to $+\infty$ and that ψ_+ is increasing from $\psi_+(1) = 8/5$ to ∞ when λ runs from 1 to $+\infty$. We readily infer: For any (rational) admissible triple $\{\lambda_1, \lambda_2, \lambda_3\}$ satisfying (2.57) and $\lambda_1 > \lambda_2$, the (rational) numbers s_1, s_2, s_3 given by (2.62)-(2.63)-(2.64) are all negative if and only if

$$\psi_-(\lambda_1) < r < \psi_+(\lambda_1) \tag{2.68}$$

if $\psi_-(\lambda_1) \geq \psi_+(\lambda_2)$, or

$$\psi_+(\lambda_2) < r < \psi_+(\lambda_1), \tag{2.69}$$

if $\psi_-(\lambda_1) \leq \psi_+(\lambda_2)$. The above discussion can be summarized by the following statement ([**3**, Example 1]):

Proposition 2.6. — *For any admissible triple $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \lambda_3\}$ of rational numbers satisfying* (2.57) *and* $\lambda_1 > \lambda_2$, *denote by $I_{\boldsymbol{\lambda}}$ the open interval in* $(8/5, +\infty)$ *defined by* (2.68)-(2.69). *Then, for any rational number r in $I_{\boldsymbol{\lambda}}$, there exists a* (*complex*) *four-dimensional ruled manifold $M = \mathbb{P}(1 \oplus L) \to S = \prod_{i=1}^{3} S_i$, where each S_i is a Riemann surface of hyperbolic type, such that $\Omega_{\boldsymbol{\lambda}}$ is a* (*normalized*) *admissible Kähler class on M whose extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$ is of the form* (2.50), *with C defined by* (2.58).

Remark 2.3. — In view of the current conjectures concerning the link between the existence of extremal Kähler metrics and stability questions considered in the next chapter, the case of particular interest in Proposition 2.6 is when r is chosen so that the repeated root $r_+ = \frac{-1+\sqrt{r^2+4}}{2}$ of $F_{\Omega_{\boldsymbol{\lambda}}}$ in $(0, 1)$ is *irrational*. If r is written as $r = p/q$, for two (relatively prime) positive integers, this happens if and only if the integer $p^2 + 4q^2$ is *not* a square, hence for "most" rational numbers in $I_{\boldsymbol{\lambda}}$.

3. Extremal metrics and stability

3.1. The Futaki character on polarized manifolds. — In this section, $M = (M, J, g, \omega)$ denotes a general compact Kähler manifold of complex dimension m, polarized by a hermitian holomorphic line bundle L, meaning that $R^\nabla = i\omega$, i.e. that the Kähler form ω is the curvature form of the Chern connection ∇ of L. In particular, $\Omega = [\omega] = 2\pi\, c_1(L)$. We denote by π the projection of L on M. As usual, L is viewed as a complex manifold of complex dimension $m+1$.

We consider an S^1-action on M which preserves the whole Kähler structure. Denote by X the generator of this action, *i.e.* the (real) vector field X defined by $X(x) = \frac{d}{dt}_{|t=0} e^{it} \cdot x$, for any x in M. We assume that the action is hamiltonian, *i.e.* that $X = \mathrm{grad}_\omega f^X = J\mathrm{grad}_g f^X$, for some real function f^X well-defined up to an additive constant.

For any choice of f^X, X lifts to a vector field $\hat{X}$ on L, preserving the natural complex structure of L, defined by $\hat{X} = \tilde{X} - (\pi^* f^X)\, T$, where $\tilde{X}$ denotes the horizontal lift of X on L determined by ∇ and T the generator of the standard S^1-action on L (= the usual multiplication by S^1 on each fiber). Moreover, for an appropriate choice of f^X, $\hat{X}$ is the generator of a holomorphic S^1-action on L which lifts the given S^1-action on M, cf. e.g. [**19**, Proposition 7.5.1]. Such a *distinguished momentum* is well-defined up to an additive integer. We henceforth assume that $\hat{X}$ is the generator of a lifted S^1-action on L, corresponding to the distinguished momentum f^X. Notice that the lifted action on L determines a lifted S^1-action on all tensor powers L^k of L.

The lifted action induces a $\mathbb{C}$-linear S^1-action on the space, $\Gamma(L)$, of smooth sections of L, defined by

$$(\zeta \cdot s)(x) = \zeta \cdot (s\,(\zeta^{-1} \cdot x)), \tag{3.1}$$

for any s in $\Gamma(L)$, any ζ in S^1 and any x in M. According to the general definition of the Lie derivative, we then define:

$$\mathcal{L}_X s = -\frac{d}{dt}_{|t=0} e^{it} \cdot s, \tag{3.2}$$

for any s in $\Gamma(L)$ and any x in M. In terms of covariant derivative, this can be rewritten as

$$\mathcal{L}_X s = \nabla_X s + i f^X\, s. \tag{3.3}$$

The Lie derivative $\mathcal{L}_X$ preserves the subspace $H^0(M, L)$ of holomorphic sections of L and thus induces a $\mathbb{C}$-linear, skew-symmetric action on $H^0(M, L)$ and, more generally, on $H^0(M, L^k)$ for any positive integer k.

Definition 3.1. — The *infinitesimal weight* of the lifted S^1-action on L is the trace of the hermitian operator $-i\mathcal{L}_X$ on $H^0(M, L)$.

Example 3.1. — Let $(V, \langle\cdot,\cdot\rangle)$ be any hermitian $(m+1)$-dimensional complex vector space and denote by $\mathbb{P}(V)$ the corresponding complex projective space, equipped with the induced Fubini-Study Kähler metric of holomorphic sectional curvature equal to 2: the Kähler form ω is then the curvature form $-iR^{\nabla}$ of the Chern connection of the dual tautological line bundle $\mathcal{O}(1)$, equipped with the induced hermitian inner product, cf. Section 1.1. Any hermitian endomorphism A of V with integer eigenvalues $a_0, a_1, \ldots, a_m$ determines an S^1-action on $\mathbb{P}(V)$ by: $e^{it} \cdot x = e^{itA}(x)$, for any x in $\mathbb{P}(V)$. This action preserves the whole Kähler metric. The generator of this action is the (real) Hamiltonian Killing vector field X^A defined by $X^A(x) : u \in x \mapsto iA(u)$ mod x (we here the natural identification $T_x\mathbb{P}(V) = \mathrm{Hom}(x, V/x)$). This action has a natural, tautological, lift on the tautological bundle $\mathcal{O}(-1)$, namely $e^{it} \cdot u = e^{itA}(u)$, for any x in $\mathbb{P}(V)$ and any u in the complex line x. The dual S^1-action on $\mathcal{O}(1)$ is then $(e^{it} \cdot \alpha)(u) = \alpha(e^{-itA}(u))$, for any α in $\mathcal{O}(1)_x = x^*$. This is a lift of the above S^1-action on $\mathcal{O}(1)$, corresponding to the distinguished momentum defined by $f^{X^A}(x) = \langle Au, u\rangle$, for any unit generator u of x. The space $H^0(\mathbb{P}(V), \mathcal{O}(1))$ is naturally identified with the dual space V^*: each element α of V^* can be viewed as a holomorphic section of $\mathcal{O}(1)$ by setting $\alpha(x) = \alpha_{|x}$. From the above discussion, we readily infer $\mathcal{L}_{X^A}\alpha = \alpha \circ A$. In particular, the infinitesimal weight of X^A is the trace of A, *i.e.* $\sum_{i=0}^m a_i$.

It is a far reaching observation by S. Donaldson [**15**] that $\mathcal{F}_\Omega(-JX)$ can be computed by using the asymptotic expansions of the infinitesimal weights, $w_k(X)$, of the lifted S^1-action on L^k, when k tends to infinity. More precisely, denote by d_k the (complex) dimension of $H^0(M, L^k)$; then

$$\frac{w_k(X)}{k\, d_k} = \frac{\int_M f^X v_g}{V_\Omega} + \frac{1}{4}\frac{\mathcal{F}_\Omega(-JX)}{V_\Omega} k^{-1} + O(k^{-2}), \tag{3.4}$$

where f^X denotes the distinguished momentum of X determined by the chosen lifted S^1-action on L and V_Ω the volume of (M, Ω).

If Y is the generator of another hamiltonian S^1-action on M, preserving the whole Kähler structure, the *combined infinitesimal weight* $w(X,Y)$ on L is defined as the trace of the product operator $(-i\mathcal{L}_X) \circ (-i\mathcal{L}_Y)$ on $H^0(M, L)$. Denote by $w_k(X,Y)$ the combined infinitesimal weight on $H^0(M, L^k)$. We then have

$$\frac{w_k(X,Y)}{k^2 d_k} - \frac{w_k(X)}{k d_k}\frac{w_k(Y)}{k d_k} = \frac{B_\Omega(-JX, -JY)}{V_\Omega} + O(k^{-1}). \tag{3.5}$$

The key point is that formulae (3.4)-(3.5) can be used to *define* $\mathcal{F}_\Omega(-JX)$ and $B_\Omega(-JX, -JY)$ in the case when M is singular and these objects cannot be defined directly in geometric terms. Such situations occur in particular when considering *test configurations* introduced by G. Tian [**41**] and S. Donaldson [**15**] to check the stability of polarized projective manifolds.

3.2. Deformation to the normal cone. — In general, for any closed subscheme Σ of a complex variety M, the *deformation to the normal cone* of Σ in M is a classical construction in algebraic geometry, by which the embedding of Σ in M is connected to its embedding in its normal cone $C = C_\Sigma M$ as the zero section.

This is done by considering the blow-up—call it $\mathcal{D}(M)$—of the product $M \times \mathbb{P}^1$ along $\Sigma \times (1:0)$, where $(1:0)$ is the *point at infinity* of the standard complex projective line $\mathbb{P}$, and the induced projection $p : \mathcal{D}(M) \to \mathbb{P}^1$. Denote by $q : \mathcal{D}(M) \to M \times \mathbb{P}^1$ the blow-down mapping: the exceptional divisor $q^{-1}(\Sigma \times (1:0))$ is then the projectivized normal cone $\mathbb{P}(C \oplus 1)$ of $\Sigma \times (1:0)$ in $M \times \mathbb{P}^1$. For each $(\lambda : \mu) \neq (1:0)$ in $\mathbb{P}^1$, the fiber $p^{-1}((\lambda : \mu))$ is naturally identified with M, whereas the *central fiber* $p^{-1}((1:0))$ splits into two pieces:

(i) the exceptional divisor $\mathbb{P}(1 \oplus C)$, and
(ii) the blow-up $\hat{M}$ of M along Σ.

Notice that the two pieces $\hat{M}$ and $\mathbb{P}(1 \oplus C)$ of the central fiber intersect at the divisor at infinity $\mathbb{P}(C)$ in $\mathbb{P}(1 \oplus C)$, which is also the exceptional divisor of the blow-up of M along Σ.

Since the blow-up of $\Sigma \times \mathbb{P}^1$ along $\Sigma \times ((1:0))$ is $\Sigma \times \mathbb{P}^1$ again, $\Sigma \times \mathbb{P}^1$ is naturally embedded over $\mathbb{P}^1$ in $\mathcal{D}(M)$: For any $(\lambda : \mu) \neq (1:0)$ in $\mathbb{P}^1$, the induced embedding $\Sigma \hookrightarrow p^{-1}((\lambda : \mu)) \cong M$ is isomorphic to the initial embedding $\Sigma \hookrightarrow M$, whereas, over $(1:0)$, Σ is embedded in $p^{-1}(1:0)) = \mathbb{P}(1 \oplus C) \cup \hat{M}$ as the zero section in the normal cone $C \subset \mathbb{P}(1 \oplus C)$ (cf. [**16**, Chapter 5] for details).

In this paper, we consider this construction in the case when $M = \mathbb{P}(1 \oplus L)$ is an admissible ruled manifold and $\Sigma = \Sigma_\infty$ is the infinity section [9]. Since Σ_∞ is smooth, its normal cone C is simply the normal bundle $TM_{|\Sigma_\infty}/T\Sigma_\infty \cong (\pi^* L^*)_{|\Sigma_\infty}$. With the above notation, the central fiber $p^{-1}((1:0))$ is the union of

(i) $\hat{M}$, identified with M, as Σ_∞ is a divisor of M, and
(ii) the exceptional divisor $\mathbb{P}(C \oplus 1)$, identified with $\mathbb{P}(L^* \oplus 1)$ via the natural identification $\Sigma_\infty = S$.

Via the natural isomorphism $\mathbb{P}(L^* \oplus 1) = \mathbb{P}(1 \oplus L)$ obtained by tensoring $L^* \oplus 1$ by L, $\mathbb{P}(C \oplus 1)$ is naturally identified with M again and its intersection with $\hat{M} = M$ in $\mathcal{D}(M)$ is then the zero section Σ_0.

As observed in Remark 1.1 of Section 1.2, Σ_∞ is the zero divisor of the holomorphic section of $\mathcal{O}_M(1)$, s say, determined by the natural projection of $1 \oplus L$ to the trivial bundle $1 = S \times \mathbb{C}$. This allows for the following alternative description of $\mathcal{D}(M)$, which is a particular case of the general *MacPherson's graph construction* [**33**]. Let $\mathbb{P}(1 \oplus \mathcal{O}_M(-1))$ denote the natural compactification of $\mathcal{O}_M(-1)$ over M and consider

(9) The choice of Σ_∞ instead of the zero section Σ_0 is inessential.

the embedding $M \times (\mathbb{P}^1 \setminus (1:0)) \hookrightarrow \mathbb{P}(1 \oplus \mathcal{O}_M(-1)) \times \mathbb{P}^1$ defined by

$$(\xi = (z:u), (\lambda:\mu)) \to \big((\lambda z : \mu\,(z,u)), (\lambda:\mu)\big) \in \mathbb{P}(\mathbb{C} \oplus \xi) \times \mathbb{P}^1, \tag{3.6}$$

for any $\xi = (z:u) \in \mathbb{P}(\mathbb{C} \oplus L_y)$ in M—cf. Section 1.1 for the notation—and for any $(\lambda:\mu) \neq (1:0)$ in $\mathbb{P}^1$. In (3.6), λz has to be regarded as $\lambda\, s(\xi)((z,u))$. Then, $\mathcal{D}(M)$ is alternatively defined as the closure of the image of $M \times (\mathbb{P}^1 \setminus (1:0))$ in $\mathbb{P}(1 \oplus \mathcal{O}_M(-1)) \times \mathbb{P}^1$ by the embedding (3.6), hence as the (closed) complex submanifold of $\mathbb{P}(1 \oplus \mathcal{O}_M(-1)) \times \mathbb{P}^1$ whose elements are of the form $((\alpha : (\beta, u)), (\lambda:\mu))$, for any pair (α, β) of complex numbers such that $\lambda\beta - \mu\alpha = 0$, cf. Example 5.1.2 and Example 18.1.6 (d) in [**16**].

We denote by $\tilde{\pi} : \mathcal{D}(M) \to S$ the natural projection induced by $\pi : M \to S$; for any y in S, we set $\mathcal{D}(M)_y = \tilde{\pi}^{-1}(y)$.

In order to get a more concrete grasp on $\mathcal{D}(M)_y$, we write $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}_1 \oplus \mathbb{C}_2)$, where $\mathbb{C}_1$ and $\mathbb{C}_2$ stand for two copies of $\mathbb{C}$, we rewrite $M_y = \pi^{-1}(y) = \mathbb{P}(\mathbb{C}_2 \oplus L_y)$ and we introduce the complex projective plane $\mathbb{P}^2_y = \mathbb{P}(\mathbb{C}_1 \oplus \mathbb{C}_2 \oplus L_y)$: $\mathcal{D}(M)_y$ can then be viewed as a (compact) complex submanifold of the product $M_y \times \mathbb{P}^1 \times \mathbb{P}^2_y$, namely the space of $\big((z:u), (\lambda:\mu), (\alpha:\beta:v)\big)$ in $M_y \times \mathbb{P}^1 \times \mathbb{P}^2_y$ such that (α,β) belongs to the complex line $(\lambda:\mu)$ (in $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}_1 \oplus \mathbb{C}_2)$) and (β, v) belongs to the complex line $(z:u)$ (in $M_y = \mathbb{P}(\mathbb{C}_2 \oplus L_y)$), that is to say the 2-dimensional (compact, smooth) complex submanifold of $M_y \times \mathbb{P}^1 \times \mathbb{P}^2_y$ defined by the equations:

$$\mu\alpha - \lambda\beta = 0, \qquad zv - \beta u = 0. \tag{3.7}$$

For any y in S, denote by $p_{1,y} : \mathcal{D}(M)_y \to M_y$, $p_{2,y} : \mathcal{D}(M)_y \to \mathbb{P}^1$, $p_{3,y} : \mathcal{D}(M)_y \to \mathbb{P}^2_y$ the induced projections and by $C_{1,y}, C_{2,y}, C_{3,y}$ the (complex) curves in $\mathcal{D}(M)_y$ defined by

$$C_{1,y} = \{\big((z:u), (1:0), (1:0:0)\big), \quad (z:u) \in M_y = \mathbb{P}(\mathbb{C}_2 \oplus L_y)\}, \tag{3.8}$$

$$C_{2,y} = \{\big((0:u), (\lambda:\mu), (0:0:u)\big), \quad (\lambda:\mu) \in \mathbb{P}^1 = \mathbb{P}(\mathbb{C}_1 \oplus \mathbb{C}_2)\}, \tag{3.9}$$

$$C_{3,y} = \{\big((0:u), (1:0), (\alpha:0:v)\big), \quad (\alpha:v) \in \mathbb{P}(\mathbb{C}_1 \oplus L_y)\}. \tag{3.10}$$

The curves $C_{1,y}$ and $C_{2,y}$ are tautologically identified with M_y and $\mathbb{P}^1$ respectively, whereas $C_{3,y}$ will be identified with M_y via the the natural identification $\mathbb{C}_1 = \mathbb{C}_2$, i.e. via the map $(\alpha:v) \in \mathbb{P}(\mathbb{C}_1 \oplus L_y) = \mathbb{P}(\mathbb{C}_2 \oplus L_y) \mapsto \big((0:u), (1:0), (\alpha:0:v)\big)$. The curves $C_{1,y}$ and $C_{2,y}$ are disjoint; the intersection $C_{1,y} \cap C_{3,y}$ is $\sigma_\infty(y)$ in $C_{1,y} = M_y$ and $\sigma_0(y)$ in $C_{3,y} = M_y$; the intersection $C_{2,y} \cap C_{3,y}$ is $(1:0)$ in $C_{2,y} = \mathbb{P}^1$ and $\sigma_\infty(y)$ in $C_{3,y} = M_y$.

Each fiber $\mathcal{D}(M)_y$ of $\tilde{\pi} : \mathcal{D}(M) \to S$ is a blow-up of $\mathbb{P}^2_y$ at two points, via the map $p_{3,y}$, which contracts the curves $C_{1,y}$ and $C_{2,y}$ to the points $[\mathbb{C}_1]$ and $[L_y]$ of $\mathbb{P}^2_y$

respectively, *and* a blow-up of $M_y \times \mathbb{P}^1$ at one point, via the map $(p_{1,y}, p_{2,y})$, which contracts the curve $C_{3,y}$ to the point $([L_y] = \sigma_\infty(y), (1:0))$ of $M_y \times \mathbb{P}^1$.

Denote by $q : \mathcal{D}(M) \to M \times \mathbb{P}^1$, resp. $p : \mathcal{D}(M) \to \mathbb{P}^1$, the map whose restriction to each $\mathcal{D}(M)_y$ is $(p_{1,y}, p_{2,y})$, resp. $p_{2,y}$. Then, q realizes $\mathcal{D}(M)$ as a blow-up of $M \times \mathbb{P}^1$ along $\Sigma_\infty \times (1:0)$, hence as the deformation to the normal cone of Σ_∞, according to the general construction described at the beginning of this section, and p is the induced projection on $\mathbb{P}^1$. Accordingly, each $(p_{1,y}, p_{2,y})$, resp. $p_{2,y}$, will be renamed q_y, resp. p_y.

For any y in S and for any $(\lambda : \mu) \neq (1:0)$, $p_y^{-1}((\lambda : \mu))$ is isomorphic to M_y, via the embedding $M_y \hookrightarrow \mathcal{D}(M)_y$ defined by:

$$(z:u) \mapsto \big((z:u), (\lambda:\mu), (\lambda z : \mu z : \mu u)\big). \tag{3.11}$$

This family of embeddings parametrized by $\mathbb{P}^1 \setminus (1:0)$ can be viewed as a unique embedding of $M \times (\mathbb{P}^1 \setminus (1:0))$ in $\mathcal{D}(M)_y$. The restriction of this embedding to $\sigma_\infty(y) \times (\mathbb{P}^1 \setminus (1:0))$ then extends to an embedding of $\sigma_\infty(y) \times \mathbb{P}^1$ in $\mathcal{D}(M)_y)$, given by

$$((0:u), (\lambda:\mu)) \mapsto \big((0:u), (\lambda:\mu), (0:0:u)\big), \tag{3.12}$$

whose image is $C_{2,y}$.

The central fiber $p_y^{-1}((1:0))$ is $C_{1,y} \cup C_{3,y}$ over each y in S. By setting $C_1 = \cup_{y \in S} C_{1,y}$, $C_2 = \cup_{y\in S} C_{2,y}$ and $C_3 = \cup_{y \in S} C_{3,y}$, we then get

$$p^{-1}((1:0)) = C_1 \cup C_3, \tag{3.13}$$

where C_1 and C_3 are both identified with M as explained above. The intersection $C_1 \cap C_3$ is then identified with Σ_0 in $C_1 \cong M$ and with Σ_∞ in $C_3 \cong M$.

3.3. The space $\mathcal{D}(M)$ as a test configuration: Polarizations. — For any y in S, denote by $\Lambda_{1,y}, \Lambda_{2,y}, \Lambda_{3,y}$ the holomorphic line bundles on $\mathcal{D}(M)_y$ defined by $p_{1,y}^*(\mathcal{O}_{M_y}(1))$, $p_{2,y}^*(\mathcal{O}_{\mathbb{P}^1}(1))$, $p_{3,y}^*(\mathcal{O}_{\mathbb{P}^2_y}(1))$ respectively. Each $\Lambda_{1,y}, \Lambda_{2,y}, \Lambda_{3,y}$ admits a distinguished holomorphic section whose zero divisor is $C_{2,y} + C_{3,y}$, $C_{1,y} + C_{3,y}$, $C_{1,y} + C_{2,y} + C_{3,y}$ respectively. If $C_{1,y}, C_{2,y}, C_{3,y}$ are regarded a elements of $H^2(\mathcal{D}(M)_y, \mathbb{Z})$, by Poincaré duality, we then have

$$\begin{aligned} C_{1,y} &= c_1(\Lambda_{1,y}^{-1} \otimes \Lambda_{3,y}), \\ C_{2,y} &= c_1(\Lambda_{2,y}^{-1} \otimes \Lambda_{3,y}), \\ C_{3,y} &= c_1(\Lambda_{1,y} \otimes \Lambda_{2,y} \otimes \Lambda_{3,y}^{-1}), \end{aligned} \tag{3.14}$$

where $c_1(\cdot)$ stands for the (first) Chern class.

We now choose an admissible polarization on M, i.e. an admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$ on M in the image of $H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{R})$. By Remark 1.1, this means that the λ_i's are integers and that $\Omega/2\pi = c_1(\mathcal{F}_{\boldsymbol{\lambda}})$, where $\mathcal{F}_{\boldsymbol{\lambda}}$ is given by (1.11).

In order to turn $\mathcal{D}(M)$ into a test configuration compatible with this polarization, we need a hermitian holomorphic line bundle, $\mathcal{L}$, on $\mathcal{D}(M)$, whose restriction to $p^{-1}((\lambda : \mu))$ is the chosen polarization of $M = p^{-1}((\lambda : \mu))$ if $(\lambda : \mu) \neq (1 : 0)$ and which induces, in some sense, a polarization on the central fiber $p^{-1}((1 : 0))$ (however, $\mathcal{L}$ is not required to be a polarization on the whole space $\mathcal{D}(M)$).

For each $\mathcal{D}(M)_y$, this will be done by twisting the pull-back of $(\mathcal{F}_{\boldsymbol{\lambda}})_{|M_y}$ on $\mathcal{D}(M)_y$ by an appropriate multiple $-aC_{3,y}$ of the exceptional divisor, i.e. by tensoring the pull-back of $(\mathcal{F}_{\boldsymbol{\lambda}})_{|M_y}$ by $\Lambda_{1,y}^{-a} \otimes \Lambda_{2,y}^{-a} \otimes \Lambda_{3,y}^{a}$ for some positive rational number a (strictly speaking, a should be chosen an integer but, for our purposes, it will be sufficient that ka be an integer for k a positive integer growing to infinity). By using (1.11), we thus get:

$$\mathcal{L}_{|\mathcal{D}(M)_y} = \Lambda_{1,y}^{2-a} \otimes \Lambda_{2,y}^{-a} \otimes \Lambda_{3,y}^{a} \otimes (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i \lambda_i})_y. \tag{3.15}$$

We now show that the restriction of $\mathcal{L}$ to each fiber $p^{-1}((\lambda : \mu))$, is ample whenever $0 < a < 2$.

We first consider the case when $(\lambda : \mu) \neq (1; 0)$. From (3.11) we infer that the restriction of $\Lambda_{3,y}$ to $p_y^{-1}((\lambda : \mu))$ is naturally identified with the restriction of $\Lambda_{1,y} \otimes \Lambda_{2,y}$, so that:

$$\mathcal{L}_{|p_y^{-1}((\lambda:\mu))} = \Lambda_{1,y}^{2} \otimes (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i \lambda_i})_y = (\mathcal{F}_{\boldsymbol{\lambda}})_{|M_y}, \tag{3.16}$$

for any a.

We now consider the central fiber $p^{-1}((1 : 0))$, which is $C_{1,y} \cup C_{3,y}$ in each $\mathcal{D}(M)_y$. On $C_{1,y}$, we have $\Lambda_{2,y} = \Lambda_{3,y} = \mathbb{C}_1^*$, so that:

$$\begin{aligned} \mathcal{L}_{|C_{1,y}} &= \Lambda_{1,y}^{2-a} \otimes (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i \lambda_i})_y \\ &= (\mathcal{F}_{\boldsymbol{\lambda}}^{(1-\frac{a}{2})})_{|M_y} \otimes (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i \lambda_i})_y^{\frac{a}{2}}, \end{aligned} \tag{3.17}$$

whereas, on $C_{3,y}$, we have $\Lambda_{1,y} = L_y^*$, $\Lambda_{2,y} = \mathbb{C}_1^*$, $\Lambda_{3,y} = \Lambda_{1,y}$, so that:

$$\begin{aligned} \mathcal{L}_{|C_{3,y}} &= \Lambda_{1,y}^{a} \otimes \mathbb{C}_1^{a} \otimes L_y^{a-2} \otimes (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i \lambda_i})_y \\ &= (\mathcal{F}_{\boldsymbol{\lambda}}^{\frac{a}{2}})_{|M_y} \otimes (\bigotimes_{i=1}^{N} L_i^{-1-\epsilon_i \lambda_i})_y^{1-\frac{a}{2}}. \end{aligned} \tag{3.18}$$

By setting: $\Omega^{(0)} = 2\pi c_1(\mathcal{L}_{|C_1})$ and $\Omega^{(\infty)} = 2\pi c_1(\mathcal{L}_{|C_3})$, both regarded as defined on M, we thus get

$$\Omega^{(0)} = (1 - a/2)\big(\Xi + \sum_{i=1}^{N} \frac{\lambda_i - a/2\,\epsilon_i}{1 - a/2}\,\pi^*[\omega_{S_i}]\big), \tag{3.19}$$

and

$$\Omega^{(\infty)} = a/2\,\big(\Xi + \sum_{i=1}^{N} \frac{\lambda_i + (1 - a/2)\,\epsilon_i}{a/2}\,\pi^*[\omega_{S_i}]\big). \tag{3.20}$$

These evidently belong to the (admissible) Kähler cone of M if and only if $0 < a < 2$. Moreover, via the common identification $\Sigma_0 = \Sigma_\infty = S$, the restriction of $\Omega^{(0)}$ to Σ_∞ coincides with the restriction of $\Omega^{(\infty)}$ to Σ_0, as it must be. More precisely, by setting

$$a = 1 - x, \tag{3.21}$$

we have

$$\Omega^{(0)}_{|\Sigma_\infty} = \Omega^{(\infty)}_{|\Sigma_0} = \sum_{i=1}^{N} (\lambda_i + x\epsilon_i)\,[\omega_{S_i}], \tag{3.22}$$

which is the class of the Kähler form of the Kähler reduction of M, equipped with the admissible Kähler metric (1.12) in $\Omega_{\boldsymbol{\lambda}}$, for the level set $z = x$. We infer that the pair $(\Omega^{(0)}, \Omega^{(\infty)})$ determines a well-defined "polarization" on the (singular) central fiber $p^{-1}((1:0))$. This polarization depends on the parameter x in $(-1, 1)$ and will be therefore denoted by $\tilde{\Omega}^{(x)}$.

3.4. The space $\mathcal{D}(M)$ as a test configuration: $\mathbb{C}^*$-actions. — The $\mathbb{C}^*$-action on $\mathbb{P}^1$ defined by $\zeta \cdot (\lambda : \mu) = (\zeta^{-1}\lambda : \mu)$ determines a $\mathbb{C}^*$-action, denoted by $\boldsymbol{\alpha}$, on $\mathcal{D}(M)$, defined by:

$$\zeta \cdot_{\boldsymbol{\alpha}} \big((z : u), (\lambda : \mu), (\alpha : \beta : v)\big) = \big((z : u), (\zeta^{-1}\lambda : \mu), (\zeta^{-1}\,\alpha : \beta : v)\big). \tag{3.23}$$

This action moves the fibers of p. It fixes the fiber $p^{-1}((0:1))$ (this is smooth, identified with M, and plays no particular role in the story), and the central fiber $p^{-1}((1:0)) = C_1 \cup C_3$: the action $\boldsymbol{\alpha}$ is then trivial on C_1 and coincides with the natural $\mathbb{C}^*$-action on $C_3 = M$.

The natural $\mathbb{C}^*$-action on $M = \mathbb{P}(1 \oplus L)$—cf. Section 1.1—induces an $\mathbb{C}^*$-action on $\mathcal{D}(M)$, denoted by $\boldsymbol{\beta}$, defined by

$$\zeta \cdot_{\boldsymbol{\beta}} \big((z : u), (\lambda : \mu), (\alpha : \beta : v)\big) = \big((z : \zeta u), (\lambda : \mu), (\alpha : \beta : \zeta v)\big), \tag{3.24}$$

for ζ in $\mathbb{C}^*$. This action preserves the fibers of p and coincides with the natural $\mathbb{C}^*$-action on each fiber $p^{-1}((\lambda : \mu))$, $(\lambda : \mu) \neq (1:0)$, via the embedding (3.11). On the central fiber $p^{-1}((1:0) = C_1 \cup C_3$, where C_1 and C_3 are both identified with M as explained above, the action $\boldsymbol{\beta}$ coincides with the natural $\mathbb{C}^*$-action on M.

Notice that these actions preserve each fiber of $\tilde{\pi} : \mathcal{D}(M) \to S$ and are therefore entirely determined by their induced actions on $\mathcal{D}(M)_y$ for each y in S. Moreover, on each $\mathcal{D}(M)_y$, both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have natural lifts on the line bundles $\Lambda_{1,y}, \Lambda_{2,y}, \Lambda_{3,y}$. This determines an $\boldsymbol{\alpha}$- and a $\boldsymbol{\beta}$-action on $\mathcal{L}$ as well as on the vector space of its holomorphic sections.

For each fiber $p^{-1}((\lambda : \mu))$, and each positive integer k, the space of holomorphic sections of $\mathcal{L}^k_{|p^{-1}((\lambda:\mu))}$, coincides with the space of holomorphic sections of the holomorphic vector bundle, $\mathbb{E}^{k,(\lambda:\mu)}$, on S whose fiber $\mathbb{E}^{k,(\lambda:\mu)}_y$ at y is the space of holomorphic sections of $\mathcal{L}^k_{|p_y^{-1}((\lambda:\mu))}$.

If $(\lambda;\mu) \neq (1:0)$, we infer from (3.16):

$$\begin{aligned}\mathbb{E}^{k,(\lambda:\mu)}_y &= S_{2k}((\mathbb{C}_2 \oplus L_y)^*) \otimes (\bigotimes_{i=1}^{M} L_i^{1-\epsilon_i\lambda_i})^k_y \\ &= \sum_{j=0}^{2k} L_y^{-j} \otimes (\bigotimes_{i=1}^{M} L_i^{1-\epsilon_i\lambda_i})^k_y,\end{aligned} \tag{3.25}$$

where, in general, $S_\ell(V)$ denotes the ℓ-th symmetric tensor power of V. We thus have

$$H^0(p^{-1}((\lambda:\mu)), \mathcal{L}^k_{|p^{-1}((\lambda:\mu))}) = \sum_{j=0}^{2k} H^0(S, (\bigotimes_{i=1}^{M} L_i^{1-\epsilon_i\lambda_i})^k \otimes L^{-j}). \tag{3.26}$$

On the central fiber $p^{-1}((1:0))$, $\mathbb{E}^{k,(1:0)}_y$ is obtained by considering the direct sum of the spaces of holomorphic sections of $\mathcal{L}^k$ on C_1 and C_3 separately, then removing the common part on $C_1 \cap C_3$. From (3.17), we infer

$$\begin{aligned}H^0(C_{1,y}, \mathcal{L}^k_{|C_{1,y}}) &= (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})^k_y \otimes S_{k(2-a)}((\mathbb{C}_2 \oplus L_y)^*) \\ &= (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})^k_y \otimes \sum_{j=0}^{k(2-a)} L_y^{-j}.\end{aligned} \tag{3.27}$$

Moreover, the infinitesimal weight of $\boldsymbol{\alpha}$, as defined in Definition 3.1, is 0 on this space, whereas the infinitesimal weight of $\boldsymbol{\beta}$ is j on each factor $(\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})^k_y \otimes L^{-j}$ (for this computation and similar ones in the sequel, compare with Example 3.1 in Section 3.1).

From (3.18), we infer

$$\begin{aligned}H^0(C_{3,y}, \mathcal{L}^k_{|C_{1,y}}) &= (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})^k_y \otimes \mathbb{C}_1^{ka} \otimes L_y^{-k(2-a)} \otimes S_{ka}((\mathbb{C}_1 \oplus L_y)^*) \\ &= (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})^k_y \otimes \sum_{j=k(2-a)}^{2k} \mathbb{C}_1^{j-k(2-a)} \otimes L_y^{-j}.\end{aligned} \tag{3.28}$$

Moreover, on each factor $(\bigotimes_{i=1}^N L_i^{1-\epsilon_i\lambda_i})_y^k \otimes \mathbb{C}_1^{j-k(2-a)} \otimes L_y^{-j}$, the infinitesimal weight of $\boldsymbol{\alpha}$ is $j-k(2-a)$, whereas the infinitesimal weight of $\boldsymbol{\beta}$ is j.

Finally, $H^0(C_{1,y}\cap C_{3,y},\mathcal{L}^k) = (\bigotimes_{i=1}^N L_i^{1-\epsilon_i\lambda_i})_y^k \otimes L_y^{-k(2-a)}$, which appears in both expressions with weight 0 for $\boldsymbol{\alpha}$.

By removing this term from the rhs of (3.27) or (3.28), and by removing the factors $\mathbb{C}_1^{j-k(2-a)}$ appearing in the rhs of (3.28)—but keeping them in mind for weight issues—we eventually get

$$\mathbb{E}_y^{k,(1:0)} = (\bigotimes_{i=1}^N L_i^{1-\epsilon_i\lambda_i})_y^k \otimes \sum_{j=0}^{2k} L_y^{-j}, \tag{3.29}$$

hence

$$H^0(p^{-1}(1:0),\mathcal{L}^k_{|p^{-1}((1:0))}) = \sum_{j=0}^{2k} H^0(S,(\bigotimes_{i=1}^N L_i^{1-\epsilon_i\lambda_i})^k \otimes L^{-j}). \tag{3.30}$$

It is convenient to rewrite (3.30) as follows

$$H^0(p^{-1}((1:0)),\mathcal{L}^k_{|p^{-1}((1:0))}) = \sum_{\ell=-k}^{k} H^0(S,(\bigotimes_{i=1}^N \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k), \tag{3.31}$$

where each $\tilde{L}_i = L_i^{-\epsilon_i}$ is ample and polarizes (S_i,ω_{S_i})—cf. Section 1.1—and where we changed the index by setting

$$\ell = j-k. \tag{3.32}$$

Moreover, the infinitesimal weight of $\boldsymbol{\alpha}$ on $H^0(S,(\bigotimes_{i=1}^N \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k)$ is

$$\begin{aligned} &0 \qquad \text{if } \ell \le k(1-a) = kx \\ &\ell - kx \qquad \text{if } kx \le \ell \le k, \end{aligned} \tag{3.33}$$

whereas the infinitesimal weight of $\boldsymbol{\beta}$ on $H^0(S,(\bigotimes_{i=1}^N \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k)$ is

$$k+\ell, \qquad -k\le \ell \le k. \tag{3.34}$$

3.5. The relative Futaki invariant of $\mathcal{D}(M)$. — For any x in $(-1,1)\cap\mathbb{Q}$, the *Futaki invariant* of the $\mathbb{C}^*$-action $\boldsymbol{\alpha}$ on the central fiber $p^{-1}((1:0))$ with respect to the polarization $\tilde{\Omega}_{\lambda}^{(x)}$ is defined by $\mathcal{F}^{(x)}(\boldsymbol{\alpha}) = \mathcal{F}_{\tilde{\Omega}_{\lambda}^{(x)}}(-JX)$, where X denotes the generator of the S^1-action induced by $\boldsymbol{\alpha}$. We similarly define: $\mathcal{F}^{(x)}(\boldsymbol{\beta}) = \mathcal{F}_{\tilde{\Omega}_{\lambda}^{(x)}}(-JY)$, where Y denotes the generator of the S^1-action induced by $\boldsymbol{\beta}$, $B^{(x)}(\boldsymbol{\alpha},\boldsymbol{\beta}) = B_{\tilde{\Omega}_{\lambda}^{(x)}}(-JX,-JY)$ and $B(\boldsymbol{\beta},\boldsymbol{\beta}) = B_{\tilde{\Omega}_{\lambda}^{(x)}}(-JY,-JY)$ (as we shall see below, $\mathcal{F}(\boldsymbol{\beta})$ and $B(\boldsymbol{\beta},\boldsymbol{\beta})$ are independent of x). The *relative Futaki invariant* of $\boldsymbol{\alpha}$ with respect to $\boldsymbol{\beta}$, in the sense of (2.9), is then

$$\mathcal{F}_{\boldsymbol{\beta}}^{(x)}(\boldsymbol{\alpha}) = \mathcal{F}_{\boldsymbol{\beta}}^{(x)}(\boldsymbol{\alpha}) - \frac{B^{(x)}(\boldsymbol{\alpha},\boldsymbol{\beta})}{B(\boldsymbol{\beta},\boldsymbol{\beta})}\,\mathcal{F}(\boldsymbol{\beta}). \tag{3.35}$$

The aim of this section is to provide a self-contained computation of $\mathcal{F}^{(x)}_{\boldsymbol{\beta}}(\boldsymbol{\alpha})$ by using (3.4)-(3.5) and to prove the following theorem, first established by G. Székelyhidi in [**39**] in the case of *pseudo-Hirzebruch surfaces*, then extended to the general case in [**3**, Section 4.4]:

Theorem 3.1. — *For any x in (-1, 1), we have*

$$\mathcal{F}^{(x)}_{\beta}(\alpha) = -2\pi V(S)\,\frac{F_{\Omega}(x)}{\int_{-1}^{1} p_{\Omega}(s)ds}, \tag{3.36}$$

where $V(S) = \prod_{i=1}^{N} V(S_i, g_{S_i})$ denotes the volume of S and, we recall, p_{Ω} and F_{Ω} denote the characteristic and the extremal polynomial of Ω respectively.

Proof. — Denote by $d_k(\ell)$ the (complex) dimension of $H^0(S, (\bigotimes_{i=1}^{N} \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k)$ and by d_k the dimension of $H^0(p^{-1}((1:0)), \mathcal{L}^k_{|p^{-1}((1:0))})$; by (3.31), we then have

$$d_k = \sum_{\ell=-k}^{k} d_k(\ell). \tag{3.37}$$

We denote by $w_k(\boldsymbol{\alpha})$, resp. $w_k(\boldsymbol{\beta})$, the infinitesimal weight of $\boldsymbol{\alpha}$, resp. $\boldsymbol{\beta}$, and by $w_k(\boldsymbol{\alpha},\boldsymbol{\beta})$, resp. $w_k(\boldsymbol{\beta},\boldsymbol{\beta})$, the combined infinitesimal weight—as defined in Section 3.1—of $\boldsymbol{\alpha},\boldsymbol{\beta}$, resp. of $\boldsymbol{\beta},\boldsymbol{\beta}$, on the space $H^0(p^{-1}((1:0)), \mathcal{L}^k_{|p^{-1}((1:0))})$. From (3.33)-(3.34), we readily infer:

$$w_k(\boldsymbol{\alpha}) = \sum_{\ell=kx}^{k} (\ell - kx)\, d_k(\ell), \quad w_k(\boldsymbol{\beta}) = \sum_{\ell=-k}^{k} (\ell + k)\, d_k(\ell), \tag{3.38}$$

$$w_k(\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{\ell=kx}^{k} (\ell + k)(\ell - kx)\, d_k(\ell), \quad w_k(\boldsymbol{\beta},\boldsymbol{\beta}) = \sum_{\ell=-k}^{k} (\ell + k)^2\, d_k(\ell). \tag{3.39}$$

Lemma 3.1. — *When k tends to infinity, $d_k(\ell)$ has the asymptotic expansion*

$$\begin{aligned} d_k(\ell) = \frac{V(S)}{(2\pi)^d}\,\big(&k^d p_{\Omega}(\ell/k) \\ &+ \frac{k^{d-1}}{4}\big(R(\ell/k) + p_{\Omega}(\ell/k)(\alpha\,\ell/k + \beta)\big) + O(k^{d-2})\big) \end{aligned} \tag{3.40}$$

where, we recall, p_{Ω} denotes the characteristic polynomial of Ω, defined by (1.7)*; R is the polynomial defined in* (1.42)*; α, β are the normalized leading coefficients of the extremal polynomial F_{Ω}, i.e. the constant appearing in the rhs of* (1.42).

Proof. — Since $\tilde{L}_i$ is ample on S_i, and $0 < \lambda_i - 1 \leq \lambda_i + \ell/k\,\epsilon_i \leq \lambda_i + 1$ for each $-k \leq \ell \leq k$, for k large enough $d_k(\ell)$ is equal to $\chi((\bigotimes_{i=1}^{N} \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k)$, the *holomorphic*

Euler characteristic of $(\bigotimes_{i=1}^N \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k$. By the Riemann-Roch theorem, we have that

$$\chi((\bigotimes_{i=1}^N \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k) = \int_S \mathrm{ch}((\bigotimes_{i=1}^N \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k)\,\mathrm{td}(S), \tag{3.41}$$

where $\mathrm{ch}((\bigotimes_{i=1}^N \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k)$ denotes the *Chern character* of the complex line bundle $(\bigotimes_{i=1}^N \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k$ and $\mathrm{td}(S)$ the *Todd class* of the holomorphic tangent bundle of S. Recall that the Chern character of any complex line bundle $\mathcal{L}$ is defined by $\mathrm{ch}(\mathcal{L}) = e^{c_1(\mathcal{L})} = \sum_{r=0}^{\infty} \frac{c_1(\mathcal{L})^r}{r!}$, whereas the Todd class is the multiplicative characteristic class associated to the generating series $x/(1-e^{-x})$; in particular $\mathrm{td}(S) = 1+c_1(S)/2+\cdots$, cf. e.g. [**24**]. We thus get:

$$\begin{aligned}
d_k(\ell) &= \sum_{r=0}^{d} \frac{k^r}{(2\pi)^r} \int_S \frac{(\sum_{i=1}^N (\lambda_i + \ell/k\,\epsilon_i)\,[\omega_{S_i}])^r}{r!}(1 + c_1(S)/2 + \cdots) \\
&= \frac{k^d}{(2\pi)^d} \int_S \frac{(\sum_{i=1}^N (\lambda_i + \ell/k\,\epsilon_i)\,[\omega_{S_i}])^d}{d!} \\
&+ \frac{k^{d-1}}{(2\pi)^d} \int_S \frac{(\sum_{i=1}^N (\lambda_i + \ell/k\,\epsilon_i)\,[\omega_{S_i}])^{d-1}}{(d-1)!} \wedge \frac{c_1(S)}{2} + O(k^{d-2}) \\
&= \frac{V(S)}{(2\pi)^d}\,(k^d p_{\Omega_\lambda}(\ell/k) + k^{d-1} p_{\Omega_\lambda}(\ell/k) \sum_{i=1}^N \frac{s_i/4}{\lambda_i + \ell/k\,\epsilon_i} + O(k^{d-2})).
\end{aligned} \tag{3.42}$$

We conclude by using (1.42). □

In order to evaluate the asymptotic expansions of the sums in (3.37), (3.38) etc. we use the following asymptotic formula, known as the *trapezium rule*:

$$\sum_{\ell=ak}^{bk} f(\ell/k) = k \int_a^b f(t)\,dt + \frac{1}{2}(f(a) + f(b)) + O(k^{-1}) \tag{3.43}$$

for any polynomial f, where $ak \leq bk$ are integers, and ℓ runs over all integers between ka and kb.

For convenience, we assume, without loss of generality, that $V(S) = (2\pi)^d$ and we simply write $p(t)$ for $p_{\Omega_\lambda}(t)$.

Corollary 3.1. — *When k tends to infinity, d_k has the asymptotic expansion*

$$d_k = k^{d+1} \int_{-1}^{1} p(s)\,ds + \frac{k^d}{4} \int_{-1}^{1} (\alpha s + \beta)\,p(s)ds + O(k^{d-1}). \tag{3.44}$$

Proof. — Direct consequence of Lemma 3.1 and of the trapezium rule (3.43). □

Corollary 3.2. — *When k tends to infinity, $w_k(\mathscr{a})$ has the asymptotic expansion*

$$\begin{aligned} w_k(\mathscr{a}) &= -k^{d+2}\int_x^1 (s-x)\,p(s)ds \\ &\quad - \frac{k^{d+1}}{4}\Big(F_{\Omega_\lambda}(x) + \int_x^1 (s-x)(\alpha s+\beta)\,p(s)ds\Big) + O(k^d). \end{aligned} \tag{3.45}$$

In particular,

$$\begin{aligned} \frac{w_k(\mathscr{a})}{kd_k} &= -\frac{\int_x^1 (s-x)\,p(s)ds}{\int_{-1}^1 p(s)\,ds} - \frac{1}{4}\,\frac{F_{\Omega_\lambda}(x)}{\int_{-1}^1 p(s)ds}\,k^{-1} \\ &\quad - \frac{\alpha}{4}\,\frac{\int_x^1 s(s-x)\,p(s)ds \int_{-1}^1 p(s)ds - \int_x^1 (s-x)\,p(s)ds \int_{-1}^1 s\,p(s)ds}{\big(\int_{-1}^1 p(s)ds\big)^2}\,k^{-1} \\ &\quad + O(k^{-2}) \end{aligned} \tag{3.46}$$

Proof. — (3.45) is a direct consequence of Lemma 3.1 and of (3.43), by using the identity (1.43)-(1.44) and the expression (1.48) of the extremal polynomial F_{Ω_λ}; (3.46) readily follows from (3.45) and (3.44). □

Corollary 3.3. — *When k tends to infinity, $w_k(\mathscr{b})$ has the asymptotic expansion*

$$w_k(\mathscr{b}) = -k^{d+2}\int_{-1}^1 (s+1)\,p(s)ds - \frac{k^{d+1}}{4}\int_{-1}^1 (\alpha s+\beta)(s+1)\,p(s)ds + O(k^d). \tag{3.47}$$

In particular,

$$\begin{aligned} \frac{w_k(\mathscr{b})}{k\,d_k} &= \frac{\int_{-1}^1 (1+s)\,p(s)ds}{\int_{-1}^1 p(s)ds} \\ &\quad + \frac{\alpha}{4}\,\frac{\big(\int_{-1}^1 s^2\,p(s)ds \int_{-1}^1 p(s)ds - \int_{-1}^1 s\,p(s)ds \int_{-1}^1 s\,p(s)ds\big)}{\big(\int_{-1}^1 p(s)ds\big)^2}\,k^{-1} \\ &\quad + O(k^{-2}). \end{aligned} \tag{3.48}$$

Proof. — Direct consequence of Lemma 3.1 and of (3.43). □

Corollary 3.4. — *When k tends to infinity, $w_k(\mathscr{a},\mathscr{b})$ has the asymptotic expansion*

$$w_k(\mathscr{a},\mathscr{b}) = -k^{d+3}\int_x^1 (s-x)(s+1)\,p(s)ds + O(k^{d+2}). \tag{3.49}$$

In particular,

$$
\begin{aligned}
&\frac{w_k(\mathscr{a},\mathscr{b})}{k^2 d_k} - \frac{w_k(\mathscr{a})}{kd_k}\,\frac{w_k(\mathscr{b})}{kd_k} = \\
&\qquad -\frac{\int_x^1 s\,(s-x)\,p(s)ds\,\int_{-1}^1 p(s)ds - \int_x^1 (s-x)\,p(s)ds\,\int_{-1}^1 s\,p(s)ds}{\left(\int_{-1}^1 p(s)ds\right)^2} \\
&\qquad + O(k^{-1})
\end{aligned}
\tag{3.50}
$$

Proof. — Direct consequence of Lemma 3.1 and of (3.43). □

Corollary 3.5. — *When k tends to infinity, $w_k(\mathscr{b},\mathscr{b})$ has the following asymptotic expansion:*

$$
w_k(\mathscr{b},\mathscr{b}) = k^{d+3}\int_{-1}^1 (s+1)^2\,p(s)ds + O(k^{d+2}). \tag{3.51}
$$

In particular,

$$
\begin{aligned}
&\frac{w_k(\mathscr{b},\mathscr{b})}{k^2 d_k} - \frac{w_k(\mathscr{b})}{kd_k}\,\frac{w_k(\mathscr{b})}{kd_k} = \\
&\qquad \frac{\int_{-1}^1 s^2\,p(s)ds\,\int_{-1}^1 p(s)ds - \int_{-1}^1 t\,p(s)ds\,\int_{-1}^1 s\,p(s)ds}{\left(\int_{-1}^1 p(s)ds\right)^2} \\
&\qquad + O(k^{-1})
\end{aligned}
\tag{3.52}
$$

Proof. — Direct consequence of Lemma 3.1 and of (3.43). □

By using (3.4)-(3.5) and $V_{\Omega_\lambda} = 2\pi\,V(S)\int_{-1}^1 p(s)ds$ (deduced from (1.27)), we obtain (by temporarily omitting the overall factor $2\pi V(S)/\int_{-1}^1 p(s)ds$):

$$
\begin{aligned}
\mathcal{F}^{(x)}(\boldsymbol{\alpha}) &= -F_{\Omega_\lambda}(x) \\
&\quad - \alpha\,\Big(\int_x^1 s(s-x)\,p(s)ds\,\int_{-1}^1 p(s)ds \\
&\quad - \int_x^1 (s-x)\,p(s)ds\,\int_{-1}^1 s\,p(s)ds\Big),
\end{aligned}
\tag{3.53}
$$

$$
\mathcal{F}(\boldsymbol{\beta}) = \alpha\,\Big(\int_{-1}^1 s^2\,p(s)ds\int_{-1}^1 p(s)ds - \int_{-1}^1 s\,p(s)ds\int_{-1}^1 s\,p(s)ds\Big), \tag{3.54}
$$

$$
\begin{aligned}
B^{(x)}(\boldsymbol{\alpha},\boldsymbol{\beta}) &= -\int_x^1 s\,(s-x)\,p(s)ds\,\int_{-1}^1 p(s)ds \\
&\quad - \int_x^1 (s-x)\,p(s)ds\,\int_{-1}^1 s\,p(s)ds,
\end{aligned}
\tag{3.55}
$$

$$
B(\boldsymbol{\beta},\boldsymbol{\beta}) = \int_{-1}^1 s^2\,p(s)ds\,\int_{-1}^1 p(s)ds - \int_{-1}^1 s\,p(s)ds\,\int_{-1}^1 s\,p(s)ds. \tag{3.56}
$$

Notice that $\mathcal{F}(\boldsymbol{\beta}) = \alpha\,B(\boldsymbol{\beta},\boldsymbol{\beta})$—cf. Remark 3.1 below—whereas $\mathcal{F}^{(x)}(\boldsymbol{\alpha}) = -F_{\Omega_\lambda}^{(x)}(x) + \alpha\,B^{(x)}(\boldsymbol{\alpha},\boldsymbol{\beta})$. By restoring the missing factor $2\pi V(S)/\int_{-1}^1 p(s)ds$, we get (3.36). □

Remark 3.1. — By comparing (3.54) and (3.56) with (2.31) and (2.32) in Corollary 2.2, we get:

$$\mathcal{F}(\boldsymbol{\beta}) = \mathcal{F}_\Omega(-JT), \quad B(\boldsymbol{\beta},\boldsymbol{\beta}) = B_\Omega(-JT,-JT). \tag{3.57}$$

This was in fact quite expected as the $\boldsymbol{\beta}$ action is the same on any fiber $p^{-1}((\lambda : \mu))$ and coincides with the natural S^1-action on M.

Remark 3.2. — The extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$ is of degree less than $m+2$ if and only the normalized leading coefficient α is zero. In this case, $\mathcal{F}(\boldsymbol{\beta}) = 0$, by (3.54), and, by (3.53), (3.36) then reduces to

$$\mathcal{F}^{(x)}(\boldsymbol{\alpha}) = -2\pi V(S)\,\frac{F_{\Omega_{\boldsymbol{\lambda}}}(x)}{\int_{-1}^{1} p_{\Omega_{\boldsymbol{\lambda}}}(s)\,ds}. \tag{3.58}$$

Appendix A
The extremal polynomial for $N = 1$

We here compute the extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$ of any (admissible) Kähler class on an admissible ruled manifold $M : \mathbb{P}(1 \oplus L) \to S = \prod_{i=1}^N S_i$ in the case when $N = 1$. The Kähler class $\Omega_{\boldsymbol{\lambda}}$ is then determined by a unique real number $\lambda > 1$, the chosen (constant) scalar curvature s of $S = S_1$ and $\epsilon = \epsilon_1$ which, without loss of generality, will be chosen equal to 1, see Section 1.1. For convenience, we set

$$\kappa = \frac{s}{d(d+1)}, \tag{A.1}$$

where d denotes the complex dimension of S (we then have $\dim_{\mathbb{C}} M = d+1$ and $F_{\Omega_{\boldsymbol{\lambda}}}$ is of degree at most $d+3$) and we replace the variable x in $(-1,1)$ by

$$X := \lambda + x, \tag{A.2}$$

in the interval $(\lambda-1, \lambda+1)$ and we set $P(X) = F_{\Omega_\lambda}(x)$: $P = P(X)$ will be referred to as the *modified* extremal polynomial of $\Omega_{\boldsymbol{\lambda}}$; it will be occasionally denoted by $P_\kappa(X)$ or $P_\kappa(X,\lambda)$ to emphasize the dependence in κ and λ; it will be most often regarded as a polynomial in X with coefficients in the field $R(\lambda)$ of rational fractions in λ; in particular, except for poles, $P_\kappa(X,\lambda)$ is well-defined for any real (or complex) value of λ, not only for admissible $\lambda > 1$. In terms of the modified extremal polynomial $P(X)$, the boundary conditions (1.46)-(1.47) read as follows

$$\begin{aligned} &P(\lambda-1) = P(\lambda+1) = 0, \\ &P'(\lambda-1) = 2(\lambda-1)^d, \qquad P'(\lambda+1) = -2(\lambda+1)^2, \end{aligned} \tag{A.3}$$

whereas the second derivative of P has the form

$$P''(X) = -\alpha X^{d+1} + (\alpha\lambda - \beta)\,X^d + d(d+1)\kappa\,X^{d-1}, \tag{A.4}$$

where α, β are determined by (A.3), cf. Section 1.9. In particular, P is of the form

$$P_\kappa(X,\lambda) = a_0(\lambda)X^{d+3} + a_1(\lambda)X^{d+2} + \kappa\, X^{d+1} + a_3(\lambda)X + a_4(\lambda), \tag{A.5}$$

where a_0, a_1, a_3, a_4 are rational fractions in λ, which depend on κ in an affine way. For convenience, we introduce

$$S_k(\lambda) = (\lambda+1)^k + (\lambda-1)^k, \qquad A_k(\lambda) = (\lambda+1)^k - (\lambda-1)^k. \tag{A.6}$$

Then, a_0, a_1 are solutions of the linear system:

$$\begin{aligned}
&(d+3)A_{d+2}(\lambda)\, a_0 + (d+2)A_{d+1}(\lambda)\, a_1 = -(d+1)A_d(\lambda)\,\kappa - 2S_d(\lambda),\\
&((d+3)S_{d+2}(\lambda) - A_{d+3})(\lambda)\, a_0 + ((d+2)S_{d+1}(\lambda) - A_{d+2})(\lambda)\, a_1\\
&\qquad\qquad = (A_{d+1}(\lambda) - (d+1)S_d(\lambda))\,\kappa - 2A_d(\lambda),
\end{aligned} \tag{A.7}$$

whereas a_3, a_4 are deduced from a_0, a_1 by

$$\begin{aligned}
a_3 &= -\frac{1}{2}(A_{d+3}(\lambda)\, a_0 + A_{d+2}(\lambda)\, a_1 + \kappa\, A_{d+1}(\lambda)),\\
a_4 &= \frac{1}{2}(\lambda^2-1)(A_{d+2}(\lambda)\, a_0 + A_{d+1}(\lambda)\, a_1 + \kappa\, A_d(\lambda)),
\end{aligned} \tag{A.8}$$

We thus get (see also [**8**]):

$$\begin{aligned}
a_0 = {} & \frac{\kappa}{\Delta(\lambda)}\left(-S_{2d+2}(\lambda) + 2(\lambda^2-1)^{d+1} + 4(d+1)^2(\lambda^2-1)^d\right)\\
& + \frac{1}{\Delta(\lambda)}\left(2\, A_{2d+2}(\lambda) - 8(d+1)\lambda(\lambda^2-1)^d\right),
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
a_1 = {} & \frac{\kappa}{\Delta(\lambda)}\left(2S_{2d+3}(\lambda) - 4\lambda(\lambda^2-1)^{d+1} - 8(d+1)(d+2)\lambda(\lambda^2-1)^d\right)\\
& + \frac{1}{\Delta(\lambda)}\left(-2\, A_{2d+3}(\lambda) + 4(2d+3)(\lambda^2-1)^{d+1} + 16(d+2)(\lambda^2-1)^d\right),
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
a_3 = {} & \frac{(\lambda^2-1)^d\,\kappa}{\Delta(\lambda)}\Big(-\frac{1}{2}(\lambda^2-1)^3 A_{d-1}(\lambda) - 2(d+2)^2(\lambda^2-1)A_{d+1}(\lambda)\\
& + 2\lambda(\lambda^2-1)A_{d+2}(\lambda) + 4(d+1)(d+2)\lambda A_{d+2}(\lambda)\\
& - \frac{3}{2}(\lambda^2-1)A_{d+3}(\lambda) - 2(d+1)^2 A_{d+3}(\lambda)\Big)\\
& + \frac{(\lambda^2-1)^d}{\Delta(\lambda)}\Big(- 2(\lambda^2-1)^2 A_d - 2(2d+3)(\lambda^2-1)A_{d+2}(\lambda)\\
& - 8(d+2)A_{d+2}(\lambda) + 4(d+1)\lambda A_{d+3}(\lambda)\Big),
\end{aligned} \tag{A.11}$$

$$
\begin{aligned}
\text{(A.12)} \quad a_4 = \frac{(\lambda^2-1)^{d+1}\kappa}{\Delta(\lambda)}\Big(&\frac{3}{2}(\lambda^2-1)^2 A_d(\lambda) + 2(d+2)^2(\lambda^2-1)A_d(\lambda) \\
&- 2\lambda(\lambda^2-1)A_{d+1}(\lambda) - 4(d+1)(d+2)\lambda A_{d+1}(\lambda) \\
&+ 2(d+1)^2 A_{d+2}(\lambda) + \frac{1}{2}A_{d+4}(\lambda)\Big) \\
+ \frac{(\lambda^2-1)^{d+1}}{\Delta(\lambda)}&\Big(4(d+2)(\lambda^2+1)A_{d+1}(\lambda) - 4(d+1)\lambda A_{d+2}(\lambda)\Big),
\end{aligned}
$$

where we have set:

$$
\text{(A.13)} \qquad \Delta(\lambda) = -S_{2d+4}(\lambda) + 4(d+2)^2(\lambda^2-1)^{d+1} + 2(\lambda^2-1)^{d+2}.
$$

Proposition A.1. — *For any real number κ, the discriminant of $P_\kappa(X)$ is non-zero in $R(\lambda)$.*

Proof. — In general, for any polynomial $f(X) = \sum_{i=0}^n a_i X^{n-i} = a_0 \prod_{j=1}^n (X - t_j)$ with coefficients in some field K, with $a_0 \neq 0$ and $n \geq 1$, the discriminant (10), $D(f)$, of f is defined by

$$
\text{(A.14)} \qquad D(f) = a_0^{-1} R(f, f') = a_0^{2n-2} \prod_{j \neq k} (t_j - t_k) = a_0^{n-2} \prod_{j=1}^n f'(t_j),
$$

where $R(f, f')$ denotes the resultant (11) of f and its derivative f', and t_j, $j = 1, \ldots, n$, denote the n roots of f in a suitable field extension $\tilde{K}$ of K.

In the present case, we observe that $P_\kappa(X)$, defined by (A.5), can be written as

$$
\text{(A.16)} \qquad P_\kappa(X) = \Phi(X) + \big(X + \frac{a_4(\lambda)}{a_3(\lambda)}\big) P'_\kappa(X),
$$

by setting $\Phi(X) = -X^d Q(X)$ and

$$
\text{(A.17)} \qquad
\begin{aligned}
Q(X) = (d+2)\,a_0(\lambda)\,X^3 &+ \Big((d+3)\,a_0(\lambda)\,\frac{a_4(\lambda)}{a_3(\lambda)} + (d+1)\,a_1(\lambda)\Big)\,X^2 \\
&+ \Big((d+2)\,a_1(\lambda)\,\frac{a_4(\lambda)}{a_3(\lambda)} + d\,\kappa\Big)\,X + (d+1)\,\kappa\,\frac{a_4(\lambda)}{a_3(\lambda)}.
\end{aligned}
$$

We then have $R(P, P') = R(\Phi, P')$, hence

$$
\text{(A.18)} \qquad D(P) = (-1)^d (d+2)^{d+3} a_0(\lambda)^{d+3} a_3(\lambda)^d \prod_{i=1}^3 P'(\beta_i),
$$

(10) We here adopt the definition which appears in [28]. The definition in [6] differs by a factor $(-1)^{\frac{n(n-1)}{2}}$.

(11) Recall that the *resultant* $R(f, g)$ of two polynomials $f(X) = \sum_{i=0}^n a_i X^{n-i} = a_0 \prod_{j=1}^n (X - t_j)$ and $g = \sum_{i=0}^m b_i X^{m-i} = b_0 \prod_{r=1}^m (X - u_r)$, with $a_0 b_0 \neq 0$, has the following expressions:

$$
\text{(A.15)} \qquad R(f, g) = a_0^m b_0^n \prod_{j=1}^n \prod_{r=1}^m (t_j - u_r) = a_0^m \prod_{j=1}^n g(t_j) = (-1)^{mn} b_0^n \prod_{r=1}^m f(u_r).
$$

where $\beta_1, \beta_2, \beta_3$ denote the roots of Q in a suitable field extension, $\widetilde{R(\lambda)}$, of $R(\lambda)$. It follows that $D(P)$ is zero in $R(\lambda)$ if and only if $P'(\beta_i) = 0$ in $\widetilde{R(\lambda)}$ for some $i = 1, 2$ or 3. We show that this cannot happen by considering the behaviour of the product $\prod_{i=1}^{3} P'(\beta_i)$ near $\lambda = \pm 1$. Notice that

$$\begin{aligned} a_0(\lambda) &\cong \frac{1}{4}(\kappa \mp 2), \\ a_1(\lambda) &\cong -\kappa \pm 1, \\ a_3(\lambda) &\cong (-(d+1)\,\kappa) \pm 2)\,(\lambda \mp 1)^d, \\ a_4(\lambda) &\cong (d\,\kappa \mp 2)(\lambda \mp 1)^{d+1}, \end{aligned} \tag{A.19}$$

modulo terms of higher orders in $(\lambda \mp 1)$ near $\lambda = \pm 1$. We temporarily assume that $\kappa \neq \frac{2}{d}$ and $\kappa \neq \frac{2}{d+1}$, so that $\frac{a_4(\lambda)}{a_3(\lambda)}$ is exactly of order 1 in $(\lambda \mp 1)$ near $\lambda = \pm 1$. We also assume $\kappa \neq \pm 2$ and $\kappa \neq 0$. It then follows that one root, β_3 say, of Q is of order 1 as well, with

$$\beta_3 \cong \frac{\kappa \mp \frac{2}{d}}{\kappa \mp \frac{2}{d+1}}\,(\lambda \mp 1), \tag{A.20}$$

whereas the other two, β_1, β_2 tend to the roots, r_1, r_2 say, of the equation

$$\frac{(d+2)}{4}(\kappa \mp 2)\,X^2 + (d+1)(-\kappa \pm 1)\,X + d\kappa = 0, \tag{A.21}$$

which are both finite (as $\kappa \neq \pm 2$) and non zero (as $\kappa \neq 0$). It is easily checked that, for $i = 1, 2$, the limit of $P'(\beta_i)$ at $\lambda = \pm 1$, which is equal to $r_i^d\big(\frac{(d+3)}{4} r_i^2 + (d+2)(-\kappa \pm 1) r_i + (d+1)\kappa\big)$, is non-zero for any value of κ ; indeed, a common root, r, of (A.21) and of the equation

$$\frac{(d+3)}{4}(\kappa \mp 2)\,X^2 + (d+2)(-\kappa \pm 1)\,X + (d+1)\kappa = 0. \tag{A.22}$$

would satisfy $r = -\frac{2(-\kappa \pm 1)}{\kappa \mp 2} = -\frac{2\kappa}{(-\kappa \pm 1)}$, which is clearly impossible. In particular, $P'(\beta_1)$ and $P'(\beta_2)$ are both non zero in K. As for $P'(\beta_3)$, we have

$$\begin{aligned} P'(\beta_3) &\cong a_3(\lambda) + (d+1) a_2 \beta_3^d \\ &= -\frac{(d+1)}{(\kappa \mp \frac{2}{d+1})^d}\,\big((\kappa \mp \frac{2}{d+1})^{d+1} - \kappa(\kappa \mp \frac{2}{d})^d\big)\,(\lambda \mp 1)^d \end{aligned} \tag{A.23}$$

modulo terms of higher orders in $\lambda \mp 1$. If $P'(\beta_3)$ was zero in K, the rhs of (A.23) would be zero for $\lambda = -1$ and $\lambda = 1$, meaning that κ and $-\kappa$ would be both a root of the equation

$$(X + \frac{2}{d+1})^{d+1} - X(X + \frac{2}{d})^d = 0. \tag{A.24}$$

On the other hand, if $h(X) = \sum_{j=0}^{d+1} c_j X^{d+1-j}$ denotes the polynomial in the rhs of (A.24), we have that

$$c_j = \frac{2^j \binom{d}{j}}{(d+1-j)(d+1)^{j-1}d^j}\, \varphi_{j-1}(d), \tag{A.25}$$

for $j = 0, \dots, d+1$, by setting $\varphi_k(x) = x^{k+1} - (x-k)(x+1)^k$, for any integer k. It follows that $c_0 = c_1 = 0$, whereas $c_j > 0$ for any $j \geq 2$. To prove the last assertion, it is sufficient to check that $\varphi_k(x)$ is positive on $[1, +\infty)$ for all integers $k \geq 1$. Observe that $\varphi_k'(x) = (k+1)\varphi_{k-1}(x)$. We then conclude by a simple argument by induction: if φ_{k-1} is positive, then φ_k is increasing, hence positive on $[1, +\infty)$, as $\varphi_k(1) = 1$; the argument by induction is then completed by observing that $\varphi_1(x) \equiv 1$. We infer that κ and $-\kappa$ cannot be simultaneously roots of (A.24), proving that $P'(\beta_3)$ is non-zero in K. The case when κ is $\pm 2, \pm\frac{2}{d}, \pm\frac{2}{d+1}$ which were discarded in the argument, is solved by using the same argument at $\lambda = -1$ *or* at $\lambda = 1$ and by observing that none of these values is a root of the equation (A.24). If $\kappa = 0$, we observe that (A.16) holds with $\Phi = -X^{d+1}\,\tilde{Q}(X)$ and

$$\begin{aligned} \tilde{Q}(X) = (d+2)a_0(\lambda)X^2 + \Big((d+3)\,a_0(\lambda)\,\frac{a_4(\lambda)}{a_3(\lambda)} + (d+1)\,a_1(\lambda)\Big)\,X^2 \\ + (d+2)\,a_1(\lambda)\,\frac{a_4(\lambda)}{a_3(\lambda)}. \end{aligned} \tag{A.26}$$

This polynomial has two roots, α_1, α_2, in some extension of $R(\lambda)$ and, as before, the discriminant of P is zero if and only if $P'(\alpha_1)$ or $P'(\alpha_2)$ is 0 in this extension. One of these roots, α_2 say, is zero at $\lambda = \pm 1$, with $\alpha_2 \cong \frac{(d+2)}{(d+1)}\,(\lambda \mp 1)$, whereas $\alpha_1 = \frac{2(d+1)}{(d+2)}$. Then, $P'(\alpha_1) = \pm\frac{2^{d+1}(d+1)^{d+1}}{(d+2)^{d+2}} \neq 0$ at $\lambda = \pm 1$, whereas $P'(\alpha_1) \cong a_3(\lambda)$ is non-zero in $R(\lambda)$. This completes the proof of Proposition A.1. □

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V. Apostolov, Département de Mathématiques, Université du Québec à Montréal, Case postale 8888, Succursale Centre-ville, Montréal (Québec) H3C 3P8, Canada
E-mail : apostolov.vestislav@uqam.ca

D. Calderbank, Department of Mathematical Sciences, University of Bath, BA2 7AY, UK
E-mail : D.M.J.Calderbank@bath.ac.uk

P. Gauduchon, Centre de Mathématiques *Laurent Schwartz*, UMR 7640 du CNRS, École Polytechnique, 91128 Palaiseau, France • *E-mail :* pg@math.polytechnique.fr

C. Tønnesen-Friedman, Department of Mathematics, Union College, Schenectady, New York 12308, USA • *E-mail :* tonnesec@union.edu

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NGAIMING MOK

Geometric structures on uniruled projective manifolds defined by their varieties of minimal rational tangents

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GEOMETRIC STRUCTURES ON UNIRULED PROJECTIVE MANIFOLDS DEFINED BY THEIR VARIETIES OF MINIMAL RATIONAL TANGENTS

by

Ngaiming Mok

Abstract. — In a joint research programme with Jun-Muk Hwang we have been investigating geometric structures on uniruled projective manifolds, especially Fano manifolds of Picard number 1, defined by varieties of minimal rational tangents associated to moduli spaces of minimal rational curves. In this article we outline a heuristic picture of the geometry of Fano manifolds of Picard number 1 with non-linear varieties of minimal rational tangents, taking as hints from prototypical examples such as those from holomorphic conformal structures. On an open set in the complex topology the local geometric structure associated to varieties of minimal rational tangents is equivalently given by families of local holomorphic curves marked at a variable base point satisfying certain compatibility conditions. Differential-geometric notions such as (null) geodesics, curvature and parallel transport are a source of inspiration in our study. Formulation of problems suggested by this heuristic analogy and their solutions, sometimes in a very general context and at other times applicable only to special classes of Fano manifolds, have led to resolutions of a series of well-known problems in Algebraic Geometry.

***Résumé* (Structures géométriques sur des variétés projectives uniréglées définies par leurs variétés de tangentes rationnelles minimales)**

Dans un programme de recherche avec Jun-Muk Hwang nous avons étudié des structures géométriques sur les variétés projectives uniréglées, en particulier les variétés de Fano de nombres de Picard égaux à 1, definies par les variétés de tangentes rationnelles minimales associées aux espaces de modules de courbes rationnelles minimales. Dans cet article nous esquissons un dessin heuristique sur la géométrie des variétés de Fano de nombres de Picard égaux à 1 dont les variétés de tangentes rationnelles minimales sont non linéaires, en prenant comme prototypes les exemples tels ques les structures conformes holomorphes. Dans un ouvert par rapport à la topologie complexe, la structure géométrique associée aux variétés de tangentes rationnelles minimales équivaut aux données de familles de courbes holomorphes locales marquées à un point de base variable vérifiant des conditions de compatibilité. Des notions de la géometrie différentielle comme les géodésiques (nulles), la courbure et le transport parallèle constituent une source d'inspiration dans notre étude. Des formulations de problèmes suggérés par cette analogie heuristique et leurs solutions, parfois dans

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un contexte très générale et parfois applicables seulement aux classes de variétés de Fano spéciales, ont conduit a des résolutions d'une série de problèmes bien connus en géométrie algébrique.

1. Introduction

1.1. Background and motivation. — In 1979, Mori [**45**] established the fundamental existence result on rational curves on a projective manifold where the canonical line bundle is not numerically effective, thereby resolving the Hartshorne Conjecture. When the manifold is Fano, Miyaoka-Mori [**38**] (1986) proved that the manifold is uniruled. In a joint research programme undertaken with Jun-Muk Hwang, we have been studying algebro-geometric and complex-analytic problems on uniruled projective manifolds basing on geometric objects arising from special classes of rational curves, viz., minimal rational curves. In this article the author would like to highlight some geometric aspects of the underlying theory.

Given a uniruled projective manifold X and fixing an ample line bundle L, by a minimal rational curve we will mean a free rational curve of minimal degree with respect to L among all free rational curves. A connected component $\mathcal{K}$ of the space of minimal rational curves will be called a minimal rational component. In practice we will fix a minimal rational component $\mathcal{K}$ and consider only minimal rational curves belonging to $\mathcal{K}$. Associated to $\mathcal{K}$, there is the universal family $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$, where $\rho : \mathcal{U} \to \mathcal{K}$ is a holomorphic $\mathbb{P}^1$-bundle, and $\mu : \mathcal{U} \to X$ is the evaluation map. In connection with $\mathcal{U}$ there is the tangent map $\tau : \mathcal{U} \to \mathbb{P}T_X$. For a minimal rational curve C marked at $x \in X$ and immersed at the marking, and for α denoting a nonzero vector tangent to C at the marking, the tangent map associates to the marked point the element $[\alpha] \in \mathbb{P}T_x(X)$. For a general point $x \in X$ we define the variety of minimal rational tangents (VMRT) $\mathcal{C}_x$ at x to be the strict transform of the tangent map $\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$. The basic set-up of our study takes place on the total space of the double fibration given by the universal family $\rho : \mathcal{U} \to X$, $\mu : \mathcal{U} \to X$, equipped with the tangent map $\tau : \mathcal{U} \to \mathbb{P}T_x(X)$ and the fibered space $\pi : \mathcal{C} \to X$ of VMRTs. The overriding question is the extent to which a uniruled projective manifold X is determined by its VMRTs.

Given a uniruled projective manifold $(X, \mathcal{K})$ equipped with a minimal rational component $\mathcal{K}$, and a connected open subset $U \subset X$ in the complex topology, we consider $(U, \mathcal{C}|_U)$ as a complex manifold equipped with a geometric structure. Here the term 'geometric structure' is understood by analogy to standard examples. As a prototype in the context of smooth manifolds, a real m-dimensional Riemannian manifold (M, g) can be understood as one equipped with a reduction of the frame bundle from the structure group $\mathrm{GL}(m,\mathbb{R})$ to $\mathrm{O}(m)$. In the context of complex manifolds, a simplest example of a *holomorphic* geometric structure relevant to the study of uniruled projective manifolds is the case of holomorphic conformal structures, alias hyperquadric structures. A holomorphic conformal structure on an n-dimensional complex manifold X determines at every point $x \in X$ its null-cone, defining equivalently a holomorphic fiber subbundle $\mathcal{Q} \subset \mathbb{P}T_X$ consisting of fibers $\mathcal{Q}_x$ isomorphic to an $(n-2)$-dimensional hyperquadric. It corresponds to a reduction of the holomorphic frame bundle from $\mathrm{GL}(n;\mathbb{C})$ to $\mathbb{C}^* \cdot \mathrm{O}(n;\mathbb{C})$, and this reduction is completely determined by $\mathcal{Q} \subset \mathbb{P}T_X$. When $X = Q^n$, the n-dimensional hyperquadric, $\mathcal{Q}_x$ agrees with the VMRT $\mathcal{C}_x$, and by analogy we speak of the geometric structure on a uniruled projective manifold $(X,\mathcal{K})$ equipped with a minimal rational component as defined by its fibered space $\pi : \mathcal{C} \to X$ of VMRTs. As our geometric study of VMRTs are in many cases motivated by differential-geometric consideration, especially in relation to global properties that can be captured by local differential-geometric information, we will be considering a general point $x \in X$, and the local geometric structure defined by the germ of the fibered space $\pi : \mathcal{C} \to X$ at x, equivalently the restriction $\pi|_U : \mathcal{C}|_U :\to U$ to arbitrarily small Euclidean open neighborhoods U of x.

1.2. A heuristic picture. — While a substantial part of our programme applies generally to any uniruled projective manifold, our focus of investigation has been primarily on those of Picard number 1. These manifolds, which are necessarily Fano, are not amenable to further reduction by means of extremal rays in Mori theory, and as such they are called 'hard nuts' among Fano manifolds in Miyaoka [**36**]. Our geometric theory on uniruled projective manifolds based on VMRTs serve in particular as a basis for a systematic study of all Fano manifolds of Picard number 1. There emerges a dichotomy between those for which the VMRT at a general point is the union of finitely many projective linear subspaces and the rest. We will say that $(X,\mathcal{K})$ has linear VMRTs in the former case and non-linear VMRTs otherwise. The linear case includes those for which the VMRT at a general point is 0-dimensional, where the fibered space $\pi : \mathcal{C} \to X$ gives rise to a geometry on X resembling that of web geometry. We will discuss in this article exclusively the non-linear case and refer the reader to Hwang-Mok [**20**] (2003) for results in the case of 0-dimensional VMRTs, and to Hwang [**13**] (2007) for a problem which necessitates the study of the hypothetical case of linear VMRTs of higher dimensions.

At this stage of the investigation we have the following heuristic picture in the case of non-linear VMRTs. The universal $\mathbb{P}^1$-bundle $\rho : \mathcal{U} \to \mathcal{K}$ associated to the minimal rational component $\mathcal{K}$ gives rise via the tangent map to a tautological multi-foliation on the fibered space $\pi : \mathcal{C} \to X$ of VMRTs, and the 'local' geometric structure $(U,\mathcal{C}|_U)$ on open subsets $U \subset X$ in the complex topology corresponds to the data of families of local holomorphic curves marked at points $x \in U$. The local holomorphic curves are then solutions to a system of partial differential equations which in the case of holomorphic conformal structures correspond to the null geodesics. We may think of the local holomorphic curves as analogues of (null) geodesics. The fact that these 'geodesics' can be extended to minimal rational curves on $(X, \mathcal{K})$ should impose serious constraints on the underlying geometric structure. In the case of the holomorphic conformal structure on the hyperquadric, the splitting type of the tangent bundles on minimal rational curves is enough to force the vanishing of the holomorphic Bochner-Weyl tensor and thus to force flatness of the structure. In the general case of $(X, \mathcal{K})$, for a general $\mathcal{K}$-minimal rational curve the normal bundle has only direct summands of degree 1 or 0. Such a rational curve, to be called a standard rational curve, resembles minimal rational curves on a hyperquadric, and there ought to be partial 'flatness' of the geometric structure of $(X, \mathcal{C})$ along standard rational curves which places serious restrictions on geometric structures that can possibly arise from VMRTs. The heuristic analogy between minimal rational curves and (null) geodesics also goes further as the former should serve to propagate geometric information from a germ of geometric structure to the ambient Fano manifold X of Picard number 1. In this case, any two general points can be connected by a chain of minimal rational curves, and the bad set of 'inaccessible points' must be of codimension ≥ 2.

A further geometric concept that ought to play an important role in the study of geometric structures defined by VMRTs is the notion of parallel transport along a standard rational curve. In the special case of irreducible Hermitian symmetric spaces of the compact type the VMRTs are invariant under parallel transport with respect to any choice of a canonical Kähler-Einstein metric. For Fano manifolds of Picard number 1, endowed with geometric structures arising from VMRTs but without privileged local holomorphic connections, the only general source for the notion of parallel transport arises from splitting types over minimal rational curves. In this direction it is found that for the germ of families of VMRTs along the tautological lifting $\widehat{C}$ of a standard rational curve, the second fundamental in the fiber directions can be identified as a section of a flat bundle over C, and as such one can speak of the parallel transport of second fundamental forms along a standard rational curve.

Other than geometric structures defined by VMRTs, in important classes of Fano manifolds X of Picard number 1 there are additional underlying structures with differential-geometric meaning. These are the cases where the VMRTs are positive-dimensional, irreducible and linearly degenerate at a general point. They span distributions which give rise to differential systems by taking Lie brackets. The study of this class of manifolds, which is particularly important for questions on deformation rigidity, reveals an intimate link between issues of integrability and projective-geometric properties of the VMRT at a general point.

1.3. Summary and presentation of results. — While some aspects of the overall heuristic picture on geometric structures defined by VMRTs can be confirmed to a large extent, other aspects are only beginning to be explored. In the research programme emphasis has been placed on solutions of concrete problems, and in some cases confirmation of some conjectural properties on VMRTs in special cases can already lead to important consequences. Here we describe general results and highlights of applications that fall within the framework of the heuristic picture discussed.

For the prototypical examples of geometric structures on irreducible Hermitian symmetric spaces S of the compact type and of rank ≥ 2, Ochiai's result [**47**] (1970) can be interpreted as saying that a local VMRT-preserving holomorphic map necessarily extends to an automorphism of S. In Hwang-Mok [**17**] (1999), [**18**] (2001) we established the analogous phenomenon, which we call Cartan-Fubini extension, for Fano manifolds of Picard number 1 with positive-dimensional VMRTs under the additional assumption that the Gauss map of the VMRT is generically finite, proving at the same time that the tangent map at a general point is birational under the same assumption. In conjunction with the works of Kebekus [**26**] (2002) on the tangent map and Cho-Miyaoka-Shepherd-Barron [**3**] (2002) on a characterization of the projective space in terms of minimal rational curves we proved in Hwang-Mok [**21**] (2004) that the same results hold true for any Fano manifold of Picard number 1 with non-linear VMRTs at a general point, resulting in a new solution of the Lazarsfeld Problem in [**32**] (1984) regarding finite holomorphic maps on rational homogeneous spaces G/P of Picard number 1 (Hwang-Mok [**21**]). Cartan-Fubini extension has recently been extended to non-equidimensional VMRT-respecting local holomorphic maps between uniruled projective manifolds in Mok [**42**] and Hong-Mok [**9**] with applications to the characterization of certain submanifolds saturated with respect to minimal rational curves, in analogy to totally geodesic submanifolds in Riemannian geometry.

The idea of exploiting the splitting type of the tangent bundle over standard rational curves to prove vanishing theorems on curvature has given rise to a characterization of irreducible Hermitian symmetric spaces S of the compact type and of rank ≥ 2 as the unique uniruled projective manifolds admitting G-structures for reductive complex Lie groups G (Hwang-Mok [**14**], 1997), leading also to an analogous result of Hong [**6**] (2000) for geometric structures modeled after Fano homogeneous contact manifolds of Picard number 1. The idea of parallel transport of second fundamental forms was first used in relation to the Campana-Peternell Conjecture, leading to the characterization of Fano manifolds of Picard number 1 with 1-dimensional VMRTs and nef tangent bundle under the additional assumption that the fourth Betti number equals 1 (Mok [**41**], 2001), a condition that was removed in Hwang [**12**] (2007), resulting together with earlier works in the confirmation of the Campana-Peternell Conjecture for 4 dimensions. The same idea was further exploited to yield for rational homogeneous manifolds G/P of Picard number 1 defined by long simple roots a characterization of G/P by the VMRT at a general point (Mok [**43**] and Hong-Hwang [**8**]). The study of distributions spanned by irreducible linearly degenerate VMRTs has led to projective-geometric necessary conditions on such VMRTs (Hwang-Mok [**15**], 1998; [**17**], 1999), and applications of such results to deformation of complex structures are important in the final confirmation of rigidity of rational homogeneous manifolds G/P of Picard number 1 under Kähler deformation (Hwang-Mok [**23**] (2005) and references therein). Another important element in relation to deformation rigidity is the study of Lie algebras of holomorphic vector fields by means of prolongation theory for infinitesimal automorphisms of VMRTs.

In the current article results falling within the general geometric framework described revolving around the geometry of VMRTs will be stated and discussed, with (sketches of) proofs of special cases for the purpose of illustration, in an order different from the above that conforms more (but not strictly) to the chronology. The reader may consult Hwang-Mok [**17**], Hwang [**11**] (2000) for more systematic overviews at earlier stages of the programme, Mok [**40**] (1999) for aspects of the theory in relation to G-structures, Hwang-Mok [**21**] for general results on the tangent map, and Hwang [**12**] (2007) for an overview on rigidity of rational homogeneous manifolds. We have completely omitted the important role played by VMRTs on the geometry of moduli spaces of stable vector bundles on an algebraic curve, for which the reader is referred to Hwang-Ramanan [**24**] (2004) and the references contained therein.

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2. Varieties of minimal rational tangents

2.1. Minimal rational curves. — By a projective $\mathbb{P}^1$-fibered space $\nu : Z \to B$ we mean an irreducible reduced projective variety Z equipped with a surjective holomorphic map ν onto a projective variety B, such that the general fiber of ν is an algebraic curve of genus 0, i.e., isomorphic to the Riemann sphere $\mathbb{P}^1$. A projective manifold X is said to be uniruled if there exists a projective $\mathbb{P}^1$-fibered space $\nu : Z \to B$ and a dominant holomorphic map $\varphi : Z \to X$ onto X. By restricting ν to a properly chosen subvariety of B of dimension equal to $\dim(X) - 1$, without loss of generality we may assume that the dominant holomorphic map $\varphi : Z \to X$ is generically finite. Replacing Z by its normalization we may also assume that Z is a projective manifold. By Miyaoka-Mori [**38**] (1986) any Fano manifold is uniruled.

By a parametrized rational curve on a projective manifold X we mean a nonconstant holomorphic map $f : \mathbb{P}^1 \to X$ from the Riemann sphere $\mathbb{P}^1$ into X. We say that two parametrized rational curves f_1 and f_2 are equivalent if and only if they are the same up to a re-parametrization of $\mathbb{P}^1$, i.e., if and only if there exists $\gamma \in \mathrm{Aut}(\mathbb{P}^1)$ such that $f_2 = f_1 \circ \gamma$. By a rational curve we mean an equivalence class $[f]$ of parametrized rational curves $f : \mathbb{P}^1 \to X$ under this equivalence relation. We will sometimes also refer to the nontrivial image $f(\mathbb{P}^1) = C$ (as a cycle) as a rational curve.

Let X be a uniruled projective manifold and fix an ample line bundle L on X. By the degree of an algebraic curve C on X will will mean the degree of C with respect to L, i.e., the integral of a (positive) curvature form of L over C. Let $\varphi : Z \to X$ be a generically finite dominant holomorphic map from a projective $\mathbb{P}^1$-fibered space $\nu : Z \to X$ onto X where Z is nonsingular. From the surjectivity of $\varphi : Z \to X$ it follows that for a general $\mathbb{P}^1$-fiber E of $\nu : Z \to X$, $\lambda : \mathbb{P}^1 \cong E$, and for the parametrized rational curve $f : \mathbb{P}^1 \to X$ defined by $f := \varphi \circ \lambda$, the holomorphic vector bundle f^*T_X must be spanned by global sections at a general point. By the Grothendieck Splitting Theorem any holomorphic vector bundle over $\mathbb{P}^1$ splits into the direct sum of holomorphic line bundles, and it follows that f^*T_X is nonnegative in the sense that it is a direct sum of holomorphic line bundles of degree ≥ 0.

By a free rational curve on X we mean the equivalence class of a nonconstant holomorphic map $f : \mathbb{P}^1 \to X$ such that f^*T_X is nonnegative. From the above discussion it follows that any uniruled projective manifold admits a free rational curve. Conversely, if a projective manifold X admits a free rational curve parametrized as $f : \mathbb{P}^1 \to X$, then $H^0(\mathbb{P}^1, f^*T_X)$ is spanned by global sections, and $H^1(\mathbb{P}^1, f^*T_X) = 0$ since $H^1(\mathbb{P}^1, \mathcal{O}(k)) = 0$ whenever $k \geq -1$, so that there is no obstruction in the deformation of $f : \mathbb{P}^1 \to X$ as a parametrized rational curve. By deforming f and considering Chow spaces it follows readily that there exists a projective $\mathbb{P}^1$-fibered space $\nu : Z \to B$ such that Z dominates X. As a consequence, a projective manifold X is uniruled if and only if X admits a free rational curve.

By a minimal rational curve on X we will mean a free rational curve of minimal degree among all free rational curves on X. The set of minimal rational curves can be given naturally the structure of a complex manifold, a connected component of which will be called a minimal rational component $\mathcal{K}$. A rational curve belonging to

$\mathcal{K}$ will sometimes be called a $\mathcal{K}$-curve. The degree of $\mathcal{K}$, to be denoted by $\deg(\mathcal{K})$, is the degree of one and hence any $\mathcal{K}$-curve.

For a general reference on rational curves in Algebraic Geometry we refer the reader to Kollár [**29**]. The reader may also consult Hwang-Mok ([**15**], §2; [**17**], (1.1)) for basic facts on the deformation theory of rational curves relevant to our discussion.

2.2. The universal family of $\mathcal{K}$-curves and the canonical double fibration. — Associated to $(X, \mathcal{K})$ there is the universal family $\rho : \mathcal{U} \to \mathcal{K}$ of $\mathcal{K}$-curves, where $\mathcal{U}$ is smooth and $\rho : \mathcal{U} \to \mathcal{K}$ is a holomorphic $\mathbb{P}^1$-bundle, constructed as follows. Let $\mathcal{H}$ be the connected component of the space of all parametrized free rational curves $f : \mathbb{P}^1 \to X$ such that $\mathcal{K} = \mathcal{H}/\mathrm{Aut}(\mathbb{P}^1)$. Since f^*T_X is nonnegative, the obstruction group $H^1(\mathbb{P}^1, f^*T_X) = 0$, hence $\mathcal{H}$ carries naturally the structure of a complex manifold with tangent spaces $T_f(\mathcal{H}) = H^0(\mathbb{P}^1, f^*T_X)$. Recall that $\mathcal{K}$ is the quotient of $\mathcal{H}$ by the group $\mathrm{Aut}(\mathbb{P}^1)$, which acts on $\mathcal{H}$ by setting $\gamma(f) = f \circ \gamma$ for $\gamma \in \mathrm{Aut}(\mathbb{P}^1)$ and $f \in \mathcal{H}$. By the minimality of $\mathcal{K}$ any $f \in \mathcal{H}$ must be generically injective, from which it follows that $\mathrm{Aut}(\mathbb{P}^1)$ acts effectively on $\mathcal{H}$, so that $\mathcal{K}$ inherits the structure of a complex manifold with $T_{[f]}(\mathcal{K}) = H^0(\mathbb{P}^1, f^*T_X)/df(H^0(\mathbb{P}^1, T_{\mathbb{P}^1}))$. The canonical projection $p : \mathcal{H} \to \mathcal{K}$ realizes $\mathcal{H}$ as a principal $\mathrm{Aut}(\mathbb{P}^1)$-bundle over $\mathcal{K}$. $\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{SL}(2, \mathbb{C})/\{\pm I\}$ is a 3-dimensional complex Lie group which acts transitively on $\mathbb{P}^1$, and we can represent $\mathbb{P}^1 \cong \mathrm{Aut}(\mathbb{P}^1)/\mathrm{Aut}(\mathbb{P}^1; 0))$, as a homogeneous space, where $\mathrm{Aut}(\mathbb{P}^1; 0)) \subset \mathrm{Aut}(\mathbb{P}^1)$ is the (2-dimensional) isotropy subgroup at $0 \in \mathbb{P}^1$. Define $\mathcal{U} := \mathcal{H}/\mathrm{Aut}(\mathbb{P}^1; 0)$. Associated to the principal $\mathrm{Aut}(\mathbb{P}^1)$-bundle $p : \mathcal{H} \to \mathcal{K}$ we have thus a holomorphic bundle of homogeneous spaces $\rho : \mathcal{U} \to \mathcal{K}$ with fibers $\mathrm{Aut}(\mathbb{P}^1)/\mathrm{Aut}(\mathbb{P}^1; 0)) \cong \mathbb{P}^1$, which gives the universal family $\pi : \mathcal{U} \to \mathcal{K}$.

It can be proven that as a complex manifold $\mathcal{K}$ is biholomorphic to a quasi-projective manifold. In fact, there is a canonical injective holomorphic map from $\mathcal{K}$ into the Chow space of X whose image is a dense Zariski-open subset $\mathcal{K}_0$ of a projective subvariety $\mathcal{Q}$ of some irreducible component of the Chow space of X. Thus, $\mathcal{K}$ can be identified as the normalization of $\mathcal{K}_0$ and must itself be quasi-projective. From this identification the universal $\mathbb{P}^1$-bundle $\rho : \mathcal{U} \to \mathcal{K}$ can be compactified to a projective $\mathbb{P}^1$-fibered space. In particular, $\mathcal{U}$ is also quasi-projective.

The fiber $\rho^{-1}(\kappa) \cong \mathbb{P}^1$ of a point $\kappa \in \mathcal{K}$ gives a copy of the Riemann sphere $\mathbb{P}^1$ corresponding to the rational curve represented by κ. From any choice of parametrization $f : \mathbb{P}^1 \to X$ of κ, a point on $\rho^{-1}(\kappa)$ gives a point of the cycle $C = f(\mathbb{P}^1) \subset X$ determined by κ, and we have in fact a canonical holomorphic map $\mu : \mathcal{U} \to X$ which we call the evaluation map. From the nonnegativity of f^*T_X it follows readily that $\mu : \mathcal{U} \to X$ must be a holomorphic submersion. Thus, the universal family comes equipped with a canonical double fibration $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$ such that $\mu(\mathcal{U})$ must contain a dense Zariski-open subset of X. As X is of Picard number 1, any $\mathcal{K}$-curve must intersect any nontrivial divisor D, hence $\mathcal{K}$-curves must cover the complement of a subvariety $Z \subset X$ of codimension ≥ 2; i.e., $\mu(\mathcal{U}) \supset X - Z$.

2.3. $\mathcal{K}$-curves marked at a point. — Fix a point $x \in X$ and consider the set $\mathcal{H}_x$ of all holomorphic maps $f : \mathbb{P}^1 \to X$ belonging to $\mathcal{H}$ such that $f(0) = x$. As a space of free rational curves marked at x, $\mathcal{H}_x$ carries naturally the structure of a complex manifold, as follows. The infinitesimal deformation of $f \in \mathcal{H}_x$ as a parametrized rational curve marked at x is given by $H^0(\mathbb{P}^1, f^*T_X \otimes \mathcal{I}_0)$, while the obstruction group to the deformation of f fixing the marking at x is given by $H^1(\mathbb{P}^1, f^*T_X \otimes \mathcal{I}_0)$, where $\mathcal{I}_0$ stands for the ideal sheaf defined by the reduced point $0 \in \mathbb{P}^1$. Since f^*T_X is nonnegative, $f^*T_X \otimes \mathcal{I}_0 \cong f^*T_X \otimes \mathcal{O}(-1)$ is a direct sum of holomorphic line bundles of degree ≥ -1, and we still have $H^1(\mathbb{P}^1, f^*T_X \otimes \mathcal{I}_0) = 0$. Again $\mathrm{Aut}(\mathbb{P}^1; 0)$ acts effectively on $\mathcal{H}_x$, and we have a nonsingular quotient manifold $\mathcal{K}_x = \mathcal{H}_x/\mathrm{Aut}(\mathbb{P}^1; 0)$ serving as the base manifold of a holomorphic principal $\mathrm{Aut}(\mathbb{P}^1; 0)$-bundle $q_x : \mathcal{H}_x \to \mathcal{K}_x$. Through a general point $x \in X$ any rational curve of degree $\leq \deg(\mathcal{K})$ must be free. It follows that $\mathcal{K}$-curves marked at such a point x cannot be decomposed into two or more irreducible components under deformations fixing the base point x. Thus, $\mathcal{K}_x$ must be compact, hence projective for a general point $x \in X$.

For a point $x \in X$, although the complex structures on $\mathcal{H}_x$ and $\mathcal{H}$ arise from two distinct classification problems, set-theoretically $\mathcal{H}_x$ can still be identified with a subset of the complex manifold $\mathcal{H}$. For every $f \in \mathcal{H}_x$ the canonical inclusion $i : \mathcal{H}_x \subset \mathcal{H}$ identifies the tangent space $T_f(\mathcal{H}_x) = H^0(\mathbb{P}^1, f^*T_X \otimes \mathcal{I}_0)$ as a vector subspace of $H^0(\mathbb{P}^1, f^*T_X) = T_f(\mathcal{H})$ so that $i : \mathcal{H}_x \subset \mathcal{H}$ is a holomorphic immersion, hence an embedding. We can therefore identify $\mathcal{H}_x$ as a complex submanifold of $\mathcal{H}$. After this identification, in the construction of the universal family $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$ the μ-fiber $\mathcal{U}_x$ over any $x \in X$ is nothing other than $\mathcal{H}_x/\mathrm{Aut}(\mathbb{P}^1; 0)$, so that $\mathcal{K}_x$ can be identified with $\mathcal{U}_x$. On the other hand, $\rho|_{\mathcal{U}_x} : \mathcal{U}_x \to \mathcal{K}$ need not be an embedding. In fact, it need not be bijective as *a priori* the cycle $C = f(\mathbb{P}^1)$ underlying $f \in \mathcal{H}_x$ may be locally reducible at x. At the same time, a simple calculation also shows that $\rho|_{\mathcal{U}_x}$ is an immersion at $u \in \mathcal{U}$ precisely when the $\mathcal{K}$-curve $\kappa = \rho(u)$ is immersed at $x := \mu(u)$. Thus, it fails to be an immersion at a point $u \in \mathcal{U}_x$ corresponding to a cusp on the minimal rational curve $\kappa = \rho(u)$.

2.4. The tangent map and varieties of minimal rational tangents. — By Mori's Breaking-up Lemma, on a projective manifold X there does not exist any nontrivial algebraic family of rational curves fixing 2 distinct points. In fact, to each nontrivial algebraic 1-parameter family of rational curves fixing two distinct points one can associate a ruled surface $\pi : S \to B$ over an algebraic curve B equipped with two disjoint holomorphic sections Γ_0 and Γ_∞ corresponding to the two distinct fixed points. On the one hand, each of the two sections must have negative self-intersection number as it is an exceptional divisor on S. On the other hand, $\Gamma_0^2 = -\Gamma_\infty^2$ as disjoint sections of a ruled surface, thus leading to a contradiction.

Let now $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component. For $x \in X$ denote by $\mathcal{K}_x$ the moduli space of $\mathcal{K}$-curves marked at x. From Mori's Breaking-up Lemma one deduces (cf. Mok [**39**], Lemma (2.4.3), pp. 203ff.)

Lemma 1. — *For a general point $x \in X$, a general member $[f] \in \mathcal{K}_x$ is standard in the sense that $f^*T_X \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ for some nonnegative integers p and q.*

Proof. — Suppose otherwise. Then, a general $\mathcal{K}$-curve is not standard. Hence there exists a nonempty open subset $\mathcal{W} \subset \mathcal{H}$ and a holomorphic vector field $\mathcal{Z}$ on $\mathcal{W}$ such that for every $f \in \mathcal{W}$, $\mathcal{Z}(f)$ vanishes at $0, \infty \in \mathbb{P}^1$ and does not belong to $df(H^0(\mathbb{P}^1, T_{\mathbb{P}^1}))$. Integrating $\mathcal{Z}$ and descending from $\mathcal{H}$ to $\mathcal{K}$ we obtain some nontrivial holomorphic 1-parameter family $\{\Phi_t : t \in \Delta\}$ of $\mathcal{K}$-curves passing through two distinct points $x, y \in X$. Identifying $\mathcal{K}$ as the normalization of a Zariski-open subset $\mathcal{K}_0$ of a projective subvariety $\mathcal{Q}$ of the Chow space of X, the set of $\mathcal{K}$-curves passing through x and y is naturally endowed the structure of a quasi-projective variety. The existence of a nontrivial holomorphic 1-parameter family of such curves implies therefore that there also exists a nontrivial algebraic 1-parameter family $\{\Psi_t : t \in B\}$ of such curves. We may choose x such that any rational curve passing through x of degree $\leq \deg(\mathcal{K})$ must be free, in which case any $\mathcal{K}$-curve passing through x cannot decompose under deformation fixing x, and the base curve B can be taken to be projective, leading to a contradiction with Mori's Breaking-up Lemma. $\square$

We have the following important notion of the tangent map and the associated varieties of minimal rational tangents.

***Definition 1* (the tangent map & VMRTs).** — *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component $\mathcal{K}$. Over a general point $x \in X$ we have a rational map called the tangent map $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(X)$ defined by assigning each rational curve $[f]$ marked and immersed at x to the complex line $\mathbb{C}df(T_0(\mathbb{P}^1)) \subset T_x(X)$. The total transform $\mathcal{C}_x := \overline{\tau_x(\mathcal{K}_x)} \subset \mathbb{P}T_x(X)$ is called the variety of minimal rational tangents, alias* VMRT, *of $(X, \mathcal{K})$ at x.*

Note that a standard rational curve is immersed, since the natural map $\nu : \mathcal{O}(2) \cong T_{\mathbb{P}^1} \to f^*T_X \cong \mathcal{O}(2) \oplus \mathcal{O}(1)]^p \oplus \mathcal{O}^q$ is injective at every point. For $x \in X$ a general point and $[\alpha] \in \mathcal{C}_x$ a smooth point such that α is tangent to a standard $\mathcal{K}$-curve ℓ, assumed embedded for convenience, we write P_α for the positive part $(\mathcal{O}(2) \oplus \mathcal{O}(1)^p)_x \subset T_x(X)$ at x with respect to a splitting of $T_X|_\ell$. The following result highlights the role of standard rational curves in relation to the tangent map.

***Lemma* 2.** — *Let $(X,\mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component. Suppose $x \in X$, and $\lambda \in \mathcal{K}_x$ is a marked $\mathcal{K}$-curve which is immersed at its marking at x. Then, the tangent map $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(X)$ is a holomorphic immersion at λ if and only if the underlying $\mathcal{K}$-curve is standard. Moreover, writing $\tau_x(\lambda) = \mathbb{C}\alpha$, in the latter case we have $T_{[\alpha]}(\mathcal{C}_x) = P_\alpha/\mathbb{C}\alpha$.*

Proof. — Parametrize λ by $f : \mathbb{P}^1 \to X$ such that $f(0) = x$. A tangent vector in $T_\lambda(\mathcal{H}_x)$ is equivalently a holomorphic section $\sigma \in H^0(\mathbb{P}^1, f^*T_X \otimes \mathcal{I}_0)$. Write $\overline{\sigma} := \sigma \bmod df(T_0(\mathbb{P}^1) \otimes \mathcal{I}_0)$. Let $\eta \in T_0(\mathbb{P}^1)$ and write $\alpha := df(\eta) \in T_x(X)$. Let $\Gamma \subset X$ be a germ of holomorphic curve at $x \in X$ which is the image under f of the germ of $\mathbb{P}^1$ at 0. The germ of s at 0 corresponds to a section s in $H^0(\Gamma, T_X)$ vanishing at x. Extend s to a holomorphic vector field $\widetilde{s}$ on a neighborhood of x in X. Choose any holomorphic coordinate system at $x \in X$ and denote by ∇ the flat connection defined by it. $\nabla_\alpha(\widetilde{s})$ is independent of the extension $\widetilde{s}$, and it is further independent of the choice of holomorphic coordinates since $s(x) = 0$. The differential of the tangent map $d\tau_x$ at $s \in T_\lambda(\mathcal{K}_x)$ is an element of $\mathrm{Hom}\big(T_\lambda(\mathcal{K}_x), T_{[\alpha]}(\mathbb{P}T_x(X))\big)$. Now $T_{[\alpha]}(\mathbb{P}T_x(X)) \cong \mathrm{Hom}\big(\mathbb{C}\alpha, T_x(X)/\mathbb{C}\alpha\big)$, so that we can interpret $d\tau_x$ as an element of $\mathrm{Hom}\big(T_\lambda(\mathcal{K}_x) \otimes \mathbb{C}\alpha, T_{[\alpha]}(\mathbb{P}T_x(X))/\mathbb{C}_\alpha\big)$ canonically. In local coordinates we have

$$d\tau_x(\overline{s})(\alpha) = \nabla_\alpha(\widetilde{s}) \bmod \mathbb{C}\alpha.$$

Thus $\overline{s} \in \mathrm{Ker}(d\tau_x)$ if and only if $\nabla_\alpha(\widetilde{s}) \in \mathbb{C}\alpha$, which is the case if and only if s vanishes to the order ≥ 2 at x modulo $\mathbb{C}\alpha$. Hence $\mathrm{Ker}(d\tau_x) = 0$ if and only if $f \in \mathcal{H}_x \subset \mathcal{H}$ is standard. The last statement in Lemma 2 follows readily from the proof. □

By a line on a projective subvariety $S \subset \mathbb{P}^N$ we will mean a projective line lying on S. Regarding minimal rational components and their VMRTs on a projective submanifold $X \subset \mathbb{P}^N$ uniruled by lines we have

***Lemma* 3.** — *Let $X \subset \mathbb{P}^N$ be a projective submanifold equipped with the polarization inherited from the projective space, and $\mathcal{K}$ be a minimal rational component of X corresponding to a uniruling of X by lines. Then, at a general point $x \in X$, the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is nonsingular, and the tangent map $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(X)$ is a biholomorphism onto $\mathcal{C}_x$.*

Proof. — A $\mathcal{K}$-curve is a line ℓ on X, and we have $T_X|_\ell \subset T_{\mathbb{P}^n}|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{N-1}$. When ℓ is a free rational curve on X, $T_X|_\ell$ is a direct sum of holomorphic line bundles of degree ≥ 0. Since $\mathcal{O}(2) \cong T_\ell \subset T_X|_\ell$, we conclude that $T_X|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$ for some nonnegative integers p and q. Now every $\mathcal{K}$-curve passing through a general point x is free, and the moduli space $\mathcal{K}_x$ of $\mathcal{K}$-curves marked at x is projective. By Lemma 2 the tangent map $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(X)$ is a holomorphic immersion. On the other hand, for each nonzero vector $\xi \in T_x(X) \subset T_x(\mathbb{P}^N)$ there is at most one line ℓ

on X tangent to ξ, so that τ_x must be injective. In other words, $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(X)$ is a biholomorphism onto its image $\mathcal{C}_x$, the VMRT at x, as desired. $\square$

While for projective submanifolds $X \subset \mathbb{P}^N$ uniruled by lines the tangent map at a general point is an isomorphism, and the same remains true for all known examples, on a theoretical level the behavior of the tangent map on an abstract uniruled projective manifold $(X, \mathcal{K})$ is far from being fully understood. In Hwang-Mok [**17**] (1999) it was proven that the tangent map $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(X)$ is birational under an additional non-degeneracy assumption on the Gauss map of the VMRT. On the other hand, the tangent map τ_x is holomorphic whenever every $\mathcal{K}$-curve marked at x is immersed at the marking. In 2002, Kebekus [**26**] showed by studying cusps of rational curves on X that this is indeed the case at a general point $x \in X$. He proved in fact that the tangent map is a finite holomorphic map at a general point $x \in X$. In conjunction with [**26**] and Cho-Miyaoka-Shepherd-Barron [**3**], we proved

Theorem 1 **(Hwang-Mok [21]).** — *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component $\mathcal{K}$ and x be a general point on X. Then, ($\mathcal{K}_x$ is projective and) the tangent map $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ is a finite birational holomorphic map onto its image. In other words, $\mathcal{K}_x$ is the normalization of the variety of minimal rational tangents $\mathcal{C}_x$ at a general point $x \in X$.*

Remarks. — The results on the tangent map apply to a rational component $\mathcal{K}$ whenever the variety of $\mathcal{K}$-tangents is projective at a general point. In the literature $\mathcal{K}$ is referred to as a non-splitting family of rational curves on X. One may extend the notion in (2.1) of a minimal rational component to mean a rational component $\mathcal{K}$ such that the variety of $\mathcal{K}$-tangents at a general point is projective. In this article we use the term 'minimal' to mean minimality of degrees among free rational curves, but statements of results remain valid for the extended meaning of 'minimality'.

2.5. Examples. — As first examples we consider the n-dimensional Fermat hypersurface X of degree d in $\mathbb{P}^{n+1}$, where $1 \le d \le n$. Thus,

$$X := \left\{[z_0, z_1, \cdots, z_{n+1}] \in \mathbb{P}^{n+1} : z_0^d + z_1^d + \cdots + z_{n+1}^d = 0\right\}$$

To determine the VMRT at a general point $x = [z_0, z_1, \ldots, z_{n+1}] \in X$, it is equivalent to find all $(w_0, w_1, \ldots, w_{n+1})$ such that for every $t \in \mathbb{C}$, $[z_0 + tw_0, z_1 + tw_1, \ldots, z_{n+1} + tw_{n+1}] \in X$. In other words, we have

$$(z_0 + tw_0)^d + \cdots + (z_{n+1} + tw_{n+1})^d = 0, \quad \text{i.e.,}$$
$$(z_0^d + \cdots + z_n^d) + t(z_0^{d-1}w_0 + \cdots + z_{n+1}^{d-1}w_{n+1}) \cdot d$$
$$+\, t^2(z_0^{d-2}w_0^2 + \cdots + z_{n+1}^{d-2}w_{n+1}^2) \cdot \frac{d(d-1)}{2} + \cdots + t^d(w_0^d + w_1^d + \cdots + w_{n+1}^d) = 0.$$

When $(z_0, z_1, \ldots, z_{n+1})$ is fixed, we get $d+1$ homogeneous equations given by

$$(\flat)_k \qquad z_0^{d-k}w_0^k + \cdots + z_{n+1}^{d-k}w_{n+1}^2; \quad 0 \le k \le d$$

The equation $(\flat)_0$ says that $x = [z_0, z_1, \cdots, z_{n+1}]$ lies on X. The equation $(\flat)_1$ says that the vector $(w_0, w_1, \cdots, w_{n+1})$ mod $\mathbb{C}(z_0, z_1, \cdots, z_{n+1})$ is tangent to X at x. The $d-1$ other equations describe $\mathcal{C}_x$ as the intersection of $d-1$ hypersurfaces of degree $2, 3, \cdots, d$ in $\mathbb{P}T_x(X) \cong \mathbb{P}^{n-1}$. Geometrically the system of equations $(\flat)_k$, $0 \le k \le d$, says that a line ℓ touching X at x to the order $\ge d$ must necessarily lie on X. By Lemma 3, $\mathcal{C}_x$ is smooth for a general point $x \in X$. The anti-canonical line bundle of $\mathbb{P}^{n+1}$ is isomorphic to $\mathcal{O}(n+2)$. Since $X \subset \mathbb{P}^{n+1}$ is of degree d, the normal bundle $N_{X|\mathbb{P}^{n+1}}$ on X is isomorphic to the restriction of $\mathcal{O}(d)$ to X. By the Adjunction Formula, $\det(T_X) \cong \mathcal{O}(n+2-d)|_X$. Over a line $\ell \subset X \subset \mathbb{P}^{n+1}$ which is free as a rational curve we have $T_X|_\ell \cong \mathcal{O}(2) \oplus (\mathcal{O}(1))^{n-d} \oplus \mathcal{O}^{d-1}$ by the proof of Lemma 3, so that the VMRT at a general point of X is of dimension $n-d$. It follows that for $1 \le d \le n$, the degree-d Fermat hypersurface $X \subset \mathbb{P}^{n+1}$ is uniruled by lines such that the VMRT at a general point is the $(n-d)$-dimensional smooth complete intersection of $d-1$ hypersurfaces on $\mathbb{P}T_x(X) \cong \mathbb{P}^{n-1}$, which is necessarily connected whenever $n-d > 0$. With exactly the same argument the VMRT at a general point of *any* smooth Fano hypersurface of $\mathbb{P}^{n+1}$ of degree $d \le n-1$ must necessarily be a (connected) smooth complete intersection of dimension $n-d \ge 1$.

Note that in general for any smooth hypersurface $X \subset \mathbb{P}^{n+1}$, $K_X^{-1} \cong \mathcal{O}(n+2-d)$ is in fact ample for $1 \le d \le n+1$. In the case where $d = n+1$, the minimal rational curves are however no longer lines. They are quadric curves C of $\mathbb{P}^{n+1}$ which lie on X, and $T_X|_C \cong \mathcal{O}(2) \oplus \mathcal{O}^{n-1}$, so that VMRTs are 0-dimensional at a general point.

Table 1 gives a description of the (smooth) VMRT at a general point of a smooth Fano hypersurface of degree $\le n$ in $\mathbb{P}^{n+1}$ highlighting some examples of special interest. Here we denote by X_d^n a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$.

The first problem that we treated in our programme is the question of rigidity of irreducible Hermitian symmetric spaces under Kähler deformation (Hwang-Mok [**15**]) by a study of deformation of their VMRTs. Table 2, taken from [**15**] ((2.1), p. 440), gives their VMRTs. In this table G stands for the identity component of the isometry group of (S, g), where g is a canonical Kähler-Einstein metric on S, and $K \subset G$ denotes the isotropy subgroup at $0 \in S$. $G(p, q)$ stands for the Grassmannian of p-planes in $\mathbb{C}^{p+q}$, $G^{II}(n, n) \subset G(n, n)$ the complex submanifold of n-planes in $\mathbb{C}^{2n}$ isotropic with respect to a non-degenerate symmetric form, $G^{II}(n, n) \subset G(n, n)$ the complex submanifold of n-planes in $\mathbb{C}^{2n}$ isotropic with respect to a symplectic form. $\mathbb{O}$ stands for the octonions.

X	VMRT $\mathcal{C}_x$ at a general point
$\mathbb{P}^n$	$\mathbb{P}^{n-1}$
Q^n	$Q^{n-2} \subset \mathbb{P}^{n-1}$
smooth cubic $\subset \mathbb{P}^{n+1}$	quadric $\cap$ cubic in $\mathbb{P}^{n-1}$
$X_3^5 \subset \mathbb{P}^6$	K^3-surfaces
$X_n^n \subset \mathbb{P}^{n+1}$	$n!$ points
$X_d^n \subset \mathbb{P}^{n+1}$, $d \leq n$	codim$-(d-1)$ complete intersection $\subset \mathbb{P}^{n-1}$ of hypersurfaces of degrees $2, \ldots, d$

TABLE 1. VMRT at a general point for smooth hypersurfaces of degree $d \leq n$ in $\mathbb{P}^{n+1}$

Type	G	K	$G/K = S$	$\mathcal{C}_0$	Embedding
I	$SU(p+q)$	$S(U(p) \times U(q))$	$G(p,q)$	$\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$	Segre
II	$SO(2n)$	$U(n)$	$G^{II}(n,n)$	$G(2, n-2)$	Plücker
III	$Sp(n)$	$U(n)$	$G^{III}(n,n)$	$\mathbb{P}^{n-1}$	Veronese
IV	$SO(n+2)$	$SO(n) \times SO(2)$	Q^n	Q^{n-2}	by $\mathcal{O}(1)$
V	E_6	$\mathrm{Spin}(10) \times U(1)$	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	$G^{II}(5,5)$	by $\mathcal{O}(1)$
VI	E_7	$E_6 \times U(1)$	exceptional	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	Severi

TABLE 2. Table of irreducible Hermitian symmetric spaces S of the compact type and their VMRTs $\mathcal{C}_0$

3. Linearly degenerate VMRTs

3.1. Distributions and differential systems generated by VMRTs. — Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component. Suppose the VMRT $\mathcal{C}_x$ at a general point $x \in X$ is irreducible and linearly degenerate. Then, it spans a meromorphic distribution $W \subsetneq T_X$. The singularity set $\mathrm{Sing}(W)$ is of codimension ≥ 2 in X. Suppose W is integrable, then a leaf L of W is quasi-projective, and its compactification $\overline{L}$ can be obtained as follows. Pick a point $x \in X - \mathrm{Sing}(W)$. Consider the subvariety $\mathcal{V}_1(x)$ swept out by all $\mathcal{K}$-curves passing through x. Enlarge $\mathcal{V}_1(x)$ to obtain $\mathcal{V}_2(x)$ by adjoining all minimal rational curves passing through general points on $\mathcal{V}_1(x)$ and taking topological closure. Repeating this process a finite number of times, we obtain a compactification of the leaf L_x through x (Hwang-Mok [**15**], Proposition 11). By definition, any $\mathcal{K}$-curve ℓ_0 emanating from x lies on $\overline{L_x}$, and by the deformation theory of rational curves ℓ_0 can always be deformed to avoid the set Sing(W) which is of codimension ≥ 2 in X, yielding a $\mathcal{K}$-curve ℓ disjoint from a hypersurface $\mathcal{H} \subset X$ swept out by compactifications of leaves of W. This is possible only if X is of Picard number ≥ 2. We have in fact

Proposition 1. — *Let $(X, \mathcal{K})$ be a Fano manifold of Picard number 1. Suppose for a general point $x \in X$ the associated variety of minimal rational tangents $\mathcal{C}_x$ is irreducible and linearly degenerate. Then, the distribution W spanned at a general point by $\widetilde{\mathcal{C}}_x$ cannot be integrable. More generally, any proper distribution D on X containing W cannot be integrable.*

In general, from $W \subsetneq X$ one can derive a finite series of distributions $W = W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_k = W_{k+1} = \cdots$ where W_i is defined by induction by setting $\mathcal{W}_{j+1} = [\mathcal{W}_j, \mathcal{W}_j]$ as sheaves. We have thus the weak derived system generated by W. In case X is of Picard number 1, Proposition 1 applies to $D = W_k$ to show that the tangent bundle can be recovered from W by successively taking Lie brackets.

3.2. Integrability of distributions via projective geometry of VMRTs. — While [(3.1), Proposition 1] forces a distribution spanned by the VMRT at a general point to be non-integrable when the uniruled projective manifold X is of Picard number 1, we prove on the other hand that sufficient conditions for integrability of W can be deduced from projective-geometric properties of VMRTs. The argument goes as follows. The lack of integrability of W is encoded in the Frobenius form $\varphi : \Lambda^2 W \to T_X/W$, and integrability amounts to the vanishing of φ by the Frobenius Theorem. To prove that W is integrable it suffices to produce at a general point $x \in X$ enough elements of $\mathrm{Ker}(\varphi_x)$ to span $\Lambda^2 W_x$. In particular, if Σ is a germ of complex-analytic integral surface of W passing through x and $T_x(\Sigma)$ is spanned by η_1 and η_2, then $\eta_1 \wedge \eta_2 \in \mathrm{Ker}(\varphi_x)$. We consider a standard $\mathcal{K}$-curve ℓ passing through x and smooth at x, and take a smooth point $x_0 \in \ell$ distinct from x. Then, any pencil of rational curves emanating from x_0 including ℓ and smooth along ℓ produces a germ of surface Σ at x. Since the pencil fixes $y, T_x(\Sigma)$ is spanned by $T_x\ell = \mathbb{C}\alpha$ and a vector belonging to P_α. Thus $T_x(\Sigma) \subset P_\alpha \subset \mathrm{Span}(\widetilde{\mathcal{C}_x}) = W_x$. An analogous statement holds for any $y \in \Sigma$ sufficiently close to x, implying that Σ is a germ of integral surface of W at x. By linear algebra as explained in Hwang-Mok ([**14**], §2) we derived the following sufficient conditions for the integrability of W in terms of projective-geometric properties of VMRTs. For the formulation, given a finite-dimensional complex vector space V and any irreducible subvariety $Z \subset \mathbb{P}V$, its tangent variety $\mathcal{T} \subset \mathbb{P}(\Lambda^2 V)$ is by definition the closure of the set of elements $[\alpha \wedge \beta]$ where α is a smooth point of $\widetilde{Z}$ and $\beta \in T_\alpha(\widetilde{Z})$. We have

Proposition 2. — *The distribution W is integrable if the tangent variety $\mathcal{T}_x \subset \mathbb{P}(\Lambda^2 W_x)$ of $\mathcal{C}_x$ is linearly non-degenerate for a general point $x \in X$. The latter is in particular the case whenever the second fundamental form $\sigma_{[\alpha]} : T_{[\alpha]}(\mathcal{C}_x) \times T_{[\alpha]}(\mathcal{C}_x) \longrightarrow N_{\mathcal{C}_x|\mathbb{P}W_x,[\alpha]}$ at a general smooth point $[\alpha]$ of $\mathcal{C}_x$ is surjective.*

Proposition 3. — *Suppose at a general point $x \in X$ the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}W \subset \mathbb{P}T_x(X)$ is irreducible and smooth and $\dim(\mathcal{C}_x) > \frac{1}{2}\mathrm{rank}(W) - 1$. Then, W is integrable.*

3.3. Fano homogeneous contact manifolds. — From the perspective of geometric structures associated to VMRTs, after the irreducible Hermitian symmetric spaces of the compact type one naturally turns to rational homogeneous manifolds $S = G/P$ of Picard number 1. Here G is simple and $P \subset G$ is a maximal parabolic, corresponding to the choice of a simple root in the Dynkin diagram of the Lie algebra $\mathfrak{g}$ of G. For the background on rational homogeneous manifolds, especially root space decompositions, graded Lie algebras and G-invariant distributions we refer the reader to Hwang-Mok ([**16**], (3.3)-(3.4)). Among them, the Fano homogeneous contact manifolds were studied in relation to rigidity under Kähler deformation in Hwang [**10**] (1997). On a complex manifold X of dimension ≥ 2, a holomorphic distribution $W \subset T_X$ is said to be a contact distribution if and only if W is of co-rank 1 and the Frobenius form $\varphi : \Lambda^2 W \to T_X/W$ is non-degenerate at every point $x \in X$.

For the classification of Fano homogeneous contact manifolds we follow Boothby [**1**]. In the case of $\mathfrak{g} = A_k, k \geq 2$, S is of Picard number 2, $S \cong \mathbb{P}T^*_{\mathbb{P}^k}$. For the case of $\mathfrak{g} = C_k$ we have $S \cong \mathbb{P}^{2k-1}$ as a complex manifold. These cases will be excluded. For any other simple complex Lie algebra $\mathfrak{g}$ there is a unique choice of a long simple root in the Dynkin diagram of $\mathfrak{g}$, corresponding to a choice of a maximal parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$, such that the associated rational homogeneous manifold $S = G/P$ is of contact type. We write $S = K(\mathfrak{g})$. In Table 3 we list the relevant Fano homogeneous contact manifolds of Picard number 1 according to the classification of $\mathfrak{g}$, with information on the Levi factor $\mathfrak{q} \subset \mathfrak{p}$, and a description of the VMRT $\mathcal{C}_0 \subset \mathbb{P}W_0$ as given in Hwang ([**10**], Proposition 5).

As examples of Fano homogeneous contact manifolds described in geometric terms consider those arising from hyperquadrics as follows. For the hyperquadric Q^n of dimension $n \geq 5$ consider the minimal rational component $\mathcal{K}(Q^n)$, i.e., the moduli space of lines ℓ on Q^n, which is a rational homogeneous manifold. We have $T_{Q^n}|_\ell \cong \mathcal{O}(2) \oplus (\mathcal{O}(1))^{n-2} \oplus \mathcal{O}$ for every $\ell \in \mathcal{K}(Q^n)$. The normal bundle $N_{\ell|Q^n} \cong (\mathcal{O}(1))^{n-2} \oplus \mathcal{O}$. At any $\ell \in \mathcal{K}(Q^n)$ the tangent space $T_\ell(\mathcal{K}(Q^n))$ can be identified with the vector space $H^0(\ell, N_{\ell|Q^n})$ and it contains a vector subspace $H^0(\ell, (\mathcal{O}(1))^{n-2}))$ of codimension 1 which defines, as ℓ varies, a holomorphic distribution $\mathcal{D} \subset T_{\mathcal{K}(Q^n)}$ of co-rank 1. Since

$\mathfrak{g}$	$\mathfrak{q}$	$\mathcal{C}_0$	Embedding
B_k	$A_1 \times B_{k-2}$	$\mathbb{P}^1 \times Q^{2k-5}$	Segre*
D_k	$A_1 \times D_{k-2}$	$\mathbb{P}^1 \times Q^{2k-6}$	Segre*
G_2	A_1	$\mathbb{P}^1$	by $\mathcal{O}(3)$
F_4	C_3	$G^{II}(3,3)$	by $\mathcal{O}(1)$
E_6	A_5	$G(3,3)$	by $\mathcal{O}(1)$
E_7	D_6	$G^{II}(6,6)$	by $\mathcal{O}(1)$
E_8	E_7	exceptional**	by $\mathcal{O}(1)$

* Here $k \geq 3$ for $\mathfrak{g} = B_k, k \geq 4$ for $\mathfrak{g} = D_k$. The embedding arises from the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^m$ into $\mathbb{P}^{2m+1}$ and the canonical embedding $Q^{m-1} \subset \mathbb{P}^m$.

** In this case $\mathcal{C}_0$ is biholomorphic to the irreducible compact Hermitian symmetric space of type VI pertaining to E_7, of dimension 27.

TABLE 3. Table of Fano contact homogeneous spaces $S \not\cong \mathbb{P}^{2n-1}$ of Picard number 1 and their varieties of minimal rational tangents

$n \geq 5$, $\mathcal{C}_x \cong Q^{n-2}$ is of Picard number 1, and the base manifold $\mathcal{K}(Q^n)$ of the double fibration $\mu : \mathcal{U} \to Q^n$, $\rho : \mathcal{U} \to \mathcal{K}(Q^n)$ is also of Picard number 1. For any $x \in Q^n$ any vector α tangent to $\mathcal{C}_x$ arises from an element of $H^0(\ell, N_{\ell|Q^n})$ vanishing at x, thus taking values in $(\mathcal{O}(1))^{n-2}$, and $\mathcal{C}_x$ projects under the canonical map $\rho' : \mathcal{C} \to \mathcal{K}$ to a submanifold $\mathcal{Q}_x \in \mathcal{K}(Q^n)$ which is tangent to $\mathcal{D}$. The VMRT $\mathcal{C}_x$ is isomorphic to $Q^{n-2} \cong \mathbb{P}^{n-1}$, and it contains a projective line λ whose image under ρ' gives a minimal rational curve on $\mathcal{K}(Q^n)$. (For this $n \geq 4$ is enough.) Thus, any minimal rational curve on $\mathcal{K}(Q^n)$ is tangent to $\mathcal{D}$. From [(3.1), Proposition 1], $\mathcal{D} \subset T_{\mathcal{K}(Q^n)}$ is not integrable. Q^n is associated to the classical groups G of type B_k or D_k, for which every rational homogeneous manifold $S = G/P$ of Picard number 1 has at most 1 proper G-invariant distribution. Hence, denoting by $\varphi : \Lambda^2\mathcal{D} \to T_{\mathcal{K}(Q^n)}/\mathcal{D}$ the Frobenius form, the kernel $\mathrm{Ker}(\varphi) \subsetneq \mathcal{D}$ must be trivial, and we conclude that the Frobenius form φ defines a twisted symplectic form on the distribution $\mathcal{D}$, and $\mathcal{K}(Q^n)$ is a Fano homogeneous contact manifold of Picard number 1.

For any $\ell \in \mathcal{K}(Q^n)$, any $x \in \ell, T_x(\ell) = \mathbb{C}\alpha$, $\mathbb{P}T_\ell(\mathcal{Q}_x) \cap \mathbb{P}T_\ell(\mathcal{K}(Q^n))$ parametrizes the space of lines on $\mathcal{C}_x$ passing through $[\alpha]$, and it defines a hyperquadric in $\mathbb{P}T_{[\alpha]}(\mathcal{C}_x)$, of dimension $n-4$. As the point x varies over ℓ, we recover a $\mathbb{P}^1$-family of disjoint $(n-4)$-dimensional hyperquadrics which exhausts the VMRT $\mathcal{C}'_\ell \in \mathbb{P}T_\ell(\mathcal{K}(Q^n))$. This family is actually isomorphic to the product $\mathbb{P}^1 \times Q^{n-4}$. (This product structure can be explained in terms of the parallel transport of second fundamental forms along ℓ to be given in (6.2).) For $n = 2k-1$ with $k \geq 3$, $\mathcal{K}(Q^n) = K(B_k)$ and $\mathcal{C}'_\ell \cong \mathbb{P}^1 \times Q^{2k-5}$; for $n = 2k-2$ with $k \geq 4$ we have $\mathcal{K}(Q^n) = K(D_k)$ and $\mathcal{C}'_\ell \cong \mathbb{P}^1 \times Q^{2k-6}$.

Excepting for $\mathbb{P}^{2n-1}$ of dimension ≥ 3, which we exclude, for any Fano homogeneous contact manifold (S, D) of Picard number 1, $\dim(S) = 2s+1$, the line bundle $L := T_S/D$ is isomorphic to $\mathcal{O}(1)$, the positive generator of the Picard group $\mathrm{Pic}(S)$. Thus for any minimal rational curve ℓ on S, $T_\ell \cong \mathcal{O}(2)$ must project to 0 on $L = T_S/D$, so that ℓ is tangent to D. Over a minimal rational curve ℓ on S we have $D|_\ell = \mathcal{O}(2) \oplus (\mathcal{O}(1))^p \oplus \mathcal{O}^p \oplus \mathcal{O}(-1)$ by root space decomposition. All known Fano contact manifolds are homogeneous. The question of characterization of Fano contact manifolds (X, D) is known to be reducible to the essential case where X is of Picard number 1 and where $L := T_X/D \cong \mathcal{O}(1)$ (Kebekus-Peternell-Sommese-Wiśniewski [**27**] (2000)). Kebekus [**25**] (2001) proved in this case that X is uniruled by degree-1 curves. From elementary consideration involving splitting types and the non-degeneracy of the Frobenius form $\varphi : \Lambda^2 D \to L$ one deduces readily that all minimal rational curves ℓ passing through a general point x are standard. In [**25**] it was proven that ℓ is actually smooth. Thus, $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is a Lagrangian submanifold with respect to the symplectic form φ_x. It is tempting to believe that the complex structure of X can be recovered from its VMRTs.

Conjecture 1. — *Let X be a Fano contact manifold. Then, X is biholomorphic to a Fano homogeneous contact manifold.*

Confirmation of Conjecture 1 would imply the same for the LeBrun-Salamon Conjecture (LeBrun [**34**], 1995), according to which a compact quaternionic Kähler manifold (M, h) of positive scalar curvature is Riemannian symmetric. The link is given by the twistor construction, by which one obtains from (M, h) a twistor space X which admits the structure of a Fano contact manifold. We note that for a Fano contact manifold X of Picard number 1 other than $\mathbb{P}^{2n-1}$, the contact structure is unique since the contact distribution is spanned at a general point by the VMRT.

Among Fano homogeneous contact manifolds of Picard number 1 other than $\mathbb{P}^{2n-1}$, the one of smallest dimension is $K(G_2)$, of dimension 5, where the VMRT is the cubic rational curve in $\mathbb{P}D_x \cong \mathbb{P}^3$ for the contact distribution D of rank 4. Other than the projective plane $\mathbb{P}^2$ and the 3-dimensional hyperquadric Q^3, $K(G_2)$ is the only rational homogeneous manifold of Picard number 1 with 1-dimensional VMRTs.

3.4. Applications to rigidity under Kähler deformation. — Regarding the problem of rigidity of rational homogeneous manifolds $S = G/P$ of Picard number 1 under Kähler deformation, the first result was established for the special case of irreducible Hermitian symmetric spaces of the compact type in Hwang-Mok ([**15**], 1998). After a series of articles we have now settled the problem, as follows.

***Theorem 2* (Hwang-Mok [23]).** — *Let $S = G/P$ be a rational homogeneous manifold of Picard number 1. Let $\pi : \mathcal{X} \to \Delta := \{t \in \mathbb{C}, |t| < 1\}$ be a regular family of projective manifolds such that $X_t := \pi^{-1}(t)$ is biholomorphic to S for $t \neq 0$. Then, X_0 is also biholomorphic to S.*

$S = G/P$ is determined by the choice of a simple root in the Dynkin diagram. When it is a long root, considerations on integrability of distributions spanned by or derived from VMRTs enter in an essential way. In the case of irreducible Hermitian symmetric spaces S, excluding the obvious case of $\mathbb{P}^n$, we make use of S-structures (cf. (4.2)). An S-structure on a complex manifold M can be equivalently defined by the varieties of highest weight tangents $\pi : \mathcal{W}(M) \to M$, and in the case of $M = S$, the latter agrees with the fibered space $\pi : \mathcal{C} \to S$ of VMRTs. The idea is to consider the VMRT $\mathcal{C}_{x_0}(X_0)$ at a general point of X_0. Suppose $\mathcal{C}_{x_0}(X_0) \subset \mathbb{P}T_{x_0}(X_0)$ is congruent to the model $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$. From closedness of the flatness condition (cf. (4.3)) the S-structure at $x_0 \in X_0$ is flat. By Matsushima-Morimoto [**35**] the moduli space of projective submanifolds $\mathcal{A} \subset \mathbb{P}V$ congruent to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ is isomorphic to an affine algebraic variety. Let $E \subset X_0$ be the singularity set of the S-structure defined at general points of X_0. Since $E \subset \mathcal{X}$ is of codimension ≥ 2 we have by [**35**] Hartogs extension of S-structures on the relative tangent bundle of $\pi : \mathcal{X} \to \Delta$, and X_0 carries a flat S-structure, implying that X_0 is isomorphic to the model space S from Ochiai's Theorem [**47**] on S-structures (cf. (4.2) here) and the method of developing maps.

Thus it remains to identify the VMRT at a general point $x_0 \in X_0$ with that of the model space. For $t \in \Delta$, at $x_t \in X_t$ denote by $\mathcal{K}_{x_t}$ the moduli space of minimal rational curves marked at x_t. For a generic choice of holomorphic section $\sigma : \Delta \to \mathcal{X}$, as t varies over Δ, $\{\mathcal{K}_{\sigma(t)}\}$ constitutes a regular family of projective manifolds such that $\mathcal{K}_{\sigma(t)} \cong \mathcal{C}_0(S)$ for $t \neq 0$. Noting that $\mathcal{C}_0(S)$ is itself a Hermitian symmetric space (cf. (2.5)), irreducible except in the case of the Grassmannian, by an inductive argument coupled with cohomological considerations in the case of the Grassmannian, $\mathcal{K}_{\sigma(0)}$ remains biholomorphically equivalent to $\mathcal{C}_0(S)$. To reconstruct an S-structure on X_0 it remains to examine the tangent map $\tau_{\sigma(0)} : \mathcal{K}_{\sigma(0)} \to \mathbb{P}T_{\sigma(0)}(X_0)$. From the rigidity of $\mathcal{K}_{\sigma(t)}$ at $t = 0$, degeneration of VMRTs can only arise from a linear projection on the model $\mathcal{C}_0(S)$. If this happens at a general point of X_0, we obtain a distribution $W \subsetneq T_{X_0}$ generated at a general point by its VMRT. On the one hand, by [(3.1), Proposition 1] W is not integrable since X_0 is of Picard number 1. On the other hand, from the description of $\mathcal{C}_0(S)$ as the closure of the graph of a vector-valued quadratic polynomial, at any $[\alpha] \in \mathcal{C}_0(S)$ the second fundamental form σ is surjective. By linear projection the same remains true for $\mathcal{C}_{\sigma(0)}(X_0)$, and by [(3.2), Proposition

2] the distribution $\mathcal{W} \subset T_{X_0}$ is integrable, yielding a contradiction and proving that the VMRT is linearly non-degenerate at a general point of X_0, implying $X_0 \cong S$.

In the case where S is a Fano homogeneous contact manifold other than $\mathbb{P}^{2n-1}$, the VMRT $\mathcal{C}_0(S) \subset \mathbb{P}D_0$, where $D \subset T_S$ is the contact distribution. The kernel of the Frobenius form $\varphi_0 : \Lambda^2 D_0 \to T_0(S)/D_0 \cong \mathbb{C}$ is of codimension 1. Theorem 2 for the contact case was established in Hwang [**10**]. Following the same scheme as in the Hermitian symmetric case, the problem reduces to showing that for a generic choice of a holomorphic section $\sigma : \Delta \to \mathcal{X}$, the linear span $W_{\sigma(0)}$ of $\widetilde{\mathcal{C}}_{\sigma(0)}(X_0)$ is of codimension 1, and $\mathcal{C}_{\sigma(0)}(X_0) \subset \mathbb{P}W_{\sigma(0)}$ is congruent to the model $\mathcal{C}_0(S) \subset \mathbb{P}D_0$. In fact, granting this one can recover the structure of a Fano contact manifold on the central fiber X_0, and we have $X_0 \cong S$ by the local rigidity result of LeBrun [**34**] for Fano contact manifolds. It remains to rule out degeneration of VMRTs at a general point $x_0 \in X_0$ corresponding to a proper linear projection of $\mathcal{C}_0(S)$. Such a linear projection cannot occur, because the second fundamental form σ_0 of $\mathcal{C}_0 \subset \mathbb{P}D_0$ at $[\alpha] \in \mathcal{C}_0(S)$ has image of codimension 1, and any proper linear projection χ of $\mathcal{C}_0(S)$ renders the second fundamental form $\sigma_{[\beta]}$ surjective at a general point $[\beta]$ of the image $\chi(\mathcal{C}_0(S))$. In other words, if the VMRT at a general point on X_0 were more linearly degenerate than the model case, the distribution W on X_0 would become integrable, violating [(3.2), Proposition 2].

Given a distribution on a complex manifold, one can define a differential system by successively taking Lie brackets. On a uniruled projective manifold $(X, \mathcal{K})$ with an irreducible and linearly degenerate VMRT a general point, the distribution W spanned by VMRTs gives rise to such a differential system. When $S = G/P$ is defined by a long simple root but is neither of the symmetric nor of the contact type, Theorem 2 was solved by Hwang-Mok ([**19**], 2001). We make use of the work of Yamaguchi [**51**] on symbol algebras arising from differential systems on rational homogeneous manifolds. Following the same scheme of proof for Theorem 2 as above and making use of [**51**], the key issue is to prove that the differential system on the central fiber derived from the VMRTs is isomorphic to that of the model space. The VMRTs are tangents to minimal rational curves, and the argument using pencils of minimal rational curves in (3.2) produces elements in the kernel of the Frobenius form $\varphi_x : \Lambda^2 W_{x_0} \to T_{x_0}(X_0)/W_{x_0}$ at a general point $x_0 \in X_0$. We can consider the universal Lie algebra defined by taking elements of W_{x_0} as generators, and by taking the relations to be those generated by the argument of pencils of minimal rational curves in (3.2). Using Serre relations, we show that this universal Lie algebra is isomorphic to the symbol algebra at $0 \in S$ defined by $T_0(S)$ as a nilpotent algebra. In particular, proper linear projection of $\mathcal{C}_0(S)$ will yield a distribution such that the maximal distribution obtained by successively taking Lie brackets, which is by definition integrable, remains a proper distribution $W^\sharp \subsetneq T_{X_0}$. This violates [(3.2), Proposition 1] and solves the key difficulty of Theorem 2 for the long root case being considered.

The method of using distributions associated to VMRTs does not in general work for the short root case. In all remaining cases one imitates the same scheme of proof, but in a typical case defined by a short root the key difficulty occurs *after* we already know that the VMRT at a general point of the central fiber agrees with that of the model space. New ideas are needed to complete the proof of Theorem 2. In (4.4) we will examine the degeneration of the Lie algebras of holomorphic vector fields associated to $\pi : \mathcal{X} \to \Delta$ by resorting to a study of prolongation of algebras of infinitesimal automorphisms associated to VMRTs.

4. Holomorphic G-structures and prolongations associated to VMRTs

4.1. Holomorphic conformal structures. — By a holomorphic metric on a complex manifold M we mean a nowhere degenerate holomorphic symmetric 2-tensor. In local holomorphic coordinates (z_i), we have $g = \sum g_{ij}(z)dz^i \otimes dz^j$ such that $\det(g_{ij})(z)$ is nowhere zero. For $x \in M$, a tangent vector $\alpha \in T_x(M)$ is called a null vector if and only if $g(\alpha, \alpha) = 0$. The space $\mathcal{N}_x$ of null vectors at x is called the null cone at x. It corresponds to a hyperquadric $\mathcal{Q}_x \subset \mathbb{P}T_x(M)$ which we call the variety of null tangents. On (M, g) there is a unique holomorphic torsion-free connection ∇ such that $\nabla g = 0$ on M, analogous to the Levi-Civita connection in Riemannian geometry, given by the same formula

$$\Gamma^k_{ij} = \frac{1}{2} \sum_\ell g^{k\ell} \Big(\frac{\partial g_{i\ell}}{\partial z_j} + \frac{\partial g_{j\ell}}{\partial z_i} - \frac{\partial g_{ij}}{\partial z_\ell} \Big)$$

for the Riemann-Christoffel symbols (Γ^k_{ij}). On a complex manifold M two holomorphic metrics g and $\widetilde{g}$ on are said to be conformally equivalent to each other if and only if there exists a nowhere vanishing holomorphic function λ such that $\widetilde{g} = \lambda g$. The Riemann-Christoffel symbols $(\widetilde{\Gamma}^k_{ij})$ of $\widetilde{g}$ are related to those of g by

$$\begin{aligned}
\widetilde{\Gamma}^k_{ij} &= \sum_\ell \frac{1}{2\lambda} g^{k\ell} \Big(\frac{\partial}{\partial z_j}(\lambda g_{i\ell}) + \frac{\partial}{\partial z_i}(\lambda g_{j\ell}) - \frac{\partial}{\partial z_\ell}(\lambda g_{ij}) \Big) \\
&= \Gamma^k_{ij} + \frac{1}{2}\delta^k_i \frac{\partial}{\partial z_j} \log \lambda + \frac{1}{2}\delta^k_j \frac{\partial}{\partial z_i} \log \lambda - \frac{1}{2}\Big(\sum_\ell g^{k\ell} \frac{\partial}{\partial z_\ell} \log \lambda \Big) g_{ij} \ .
\end{aligned}$$

A (parametrized) complex geodesic on M is a nonconstant holomorphic map $\gamma : D \to M$ defined on some domain $D \subset \mathbb{C}$ satisfying in analogy to geodesics in Riemannian geometry the second order differential equation

$$\frac{\partial^2 \gamma}{\partial t^2} + \Gamma_{\dot\gamma\dot\gamma} = 0 \ .$$

Replacing g by $\widetilde{g} = \lambda g$ we have

$$\frac{\partial^2\gamma}{\partial t^2} + \widetilde{\Gamma}_{\dot\gamma\dot\gamma} = (\partial_{\dot\gamma}\log\lambda)\dot\gamma - \frac{1}{2}\Big(\sum_\ell g^{k\ell}\frac{\partial}{\partial z_\ell}\log\lambda\Big)g_{\dot\gamma\dot\gamma}\ .$$

where $\dot\gamma$ stands for $\frac{\partial\gamma}{\partial t}$. In invariant form the differential equation is given by $\nabla_{\dot\gamma}\dot\gamma = 0$. A complex geodesic γ is called a null geodesic if and only if $\dot\gamma(t)$ lies on the null cone $\mathcal{N}_{\gamma(t)}$ for every $t \in D$. Since $\nabla g = 0$, for a complex geodesic $g_{\dot\gamma\dot\gamma}$ is a constant. In particular, γ is a null geodesic if and only if $\dot\gamma$ is a null vector at one point. Suppose γ is a null geodesic on (M, g). Then, with respect to the holomorphic metric $\widetilde{g}$ we have

$$\frac{\partial^2\gamma}{\partial t^2} + \widetilde{\Gamma}_{\dot\gamma\dot\gamma} = (\partial_{\dot\gamma}\log\lambda)\dot\gamma\ .$$

Write $f(t) := \partial_{\dot\gamma}\log\lambda(t)$. At a point $t_0 \in D$, making a local holomorphic change of variable $s = s(t)$ at t_0 and writing $t = \varphi(s), \gamma(t) = \mu(s)$, we have

$$\frac{\partial^2\mu}{\partial s^2} + \widetilde{\Gamma}\Big(\frac{\partial\mu}{\partial s}, \frac{\partial\mu}{\partial s}\Big) = \varphi'(s)^2\frac{\partial^2\gamma}{\partial t^2} + \varphi''(s)\frac{\partial\gamma}{\partial t} + \varphi'(s)^2\widetilde{\Gamma}_{\dot\gamma\dot\gamma} = \Big(\varphi''(s) + \big(\varphi'(s)^2 f(\varphi(t))\big)\frac{\partial\gamma}{\partial t}\Big)\,.$$

Thus, making a change of variables by solving by means of power series the second order differential equation $\varphi''(s) + \big(\varphi'(s)^2 f(\varphi(s))\big) = 0$ admits a unique solution subject to a choice of $s_0 = \varphi^{-1}(t_0)$ and a choice of $\varphi'(s_0)$. In other words, a germ of null geodesic on (M, g) can be re-parametrized to give a germ of null geodesic on $(M, \widetilde{g})$. We will sometimes speak of a complex geodesic to mean the image of a parametrized complex geodesic. In this sense, the space of null geodesics on (M, g) is a property of the conformal equivalence class of g.

By a holomorphic conformal structure on M we will mean a holomorphic line subbundle $\Lambda \subset S^2T_M^*$, generated at each point by a non-degenerate holomorphic symmetric 2-tensor. Equivalently, it is given by the data $(U_\alpha, g_\alpha)_{\alpha\in A}$ consisting of holomorphic metrics g_α on open subsets U_α covering M such that over the non-empty overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$, g_α and g_β are conformally equivalent. A holomorphic conformal structure on M is equivalently defined by the fibered space of varieties of null tangents $\pi : \mathcal{Q} \to M$, and we will speak of $(M, \mathcal{Q})$ as a complex manifold equipped with a holomorphic conformal structure. Each null geodesic lifts to a local holomorphic curve on $\mathcal{Q}$ by sending a point $\gamma(t)$ to $[\dot\gamma(t)] \in \mathcal{Q}_{[\gamma(t)]}$, which we call the tautological lifting, and we have a 1-dimensional holomorphic foliation on $(M, \mathcal{Q})$ by liftings of null geodesics. In Riemannian geometry, for computations at a given base point one often makes use of local coordinates with respect to which the Riemann-Christoffel symbols (Γ^k_{ij}) vanish at the base point 0. The proof of existence of such coordinates works *verbatim* in the holomorphic situation. Starting with a given holomorphic local coordinate system (z_i) at a point $x \in M$, $z(x) = 0$, such that $g_{ij}(0) = \delta_{ij}$, we introduce a new holomorphic coordinate system (w_j) such that $w(0) = 0$ and $\frac{\partial w^k}{\partial z_i}(0) = \delta^k_i$.

Writing

$$\sum_{i,j} g_{ij} dz^i \otimes dz^j = \sum_{k,\ell} h_{k\ell} dw^k \otimes dw^\ell \ , \quad h_{k\ell} = \sum_{i,j} g_{ij} \frac{\partial z^i}{\partial w_k} \frac{\partial z^j}{\partial w_\ell} \ ;$$

$$\frac{\partial h_{k\ell}}{\partial w_s}(0) = \frac{\partial g_{k\ell}}{\partial z_s}(0) + \frac{\partial^2 z^\ell}{\partial w_s \partial w_k}(0) + \frac{\partial^2 z^k}{\partial w_s \partial w_\ell}(0) \ .$$

Now choose (w_k) such that $z^k = w^k + \sum_{s,\ell} c^k_{s\ell} w^s w^\ell$, where $c^k_{s\ell} = c^k_{\ell s}$. Then, setting

$$c^k_{s\ell} = -\frac{1}{2}\Big(\frac{\partial g_{k\ell}}{\partial z_s} + \frac{\partial g_{ks}}{\partial z_\ell} - \frac{\partial g_{s\ell}}{\partial z_k}\Big)(0) = -\Gamma^k_{s\ell}(0) \ .$$

we conclude that $\frac{\partial h_{k\ell}}{\partial w_s}(0) = 0$, and as a consequence $\widetilde{\Gamma}^k_{ij}(0) = 0$ in w coordinates.

In Riemannian geometry for a given base point x there is a privileged coordinate system adapted to x given by the geodesic normal coordinates in terms of which in particular the Riemann-Christoffel symbols vanish at x. The notion of geodesic normal coordinates generalizes in the setting of holomorphic metrics.

To start with we note that complex geodesics can be re-parametrized by a rescaling of the domain variable. Let $D \subset \mathbb{C}$ be a domain containing 0, $x \in M$, and $\gamma : D \to M$ be a parametrized complex geodesic such that $\gamma(0) = x$. Then, given any nonzero complex number $\lambda \in \mathbb{C}$, the function $\delta : \frac{1}{\lambda}D \to M$ defined by $\delta(t) = \gamma(\lambda t)$ is again a parametrized complex geodesic, as can be seen from the defining equation for a complex geodesic. On the total space $\pi : L \to \mathbb{P}T_x(M)$ of the tautological line bundle over $\mathbb{P}T_x(M)$, for a sufficiently small neighborhood U of $\mathbb{P}T_x(M)$ one can define a holomorphic map $\Phi_0 : U \to M$, as follows. For $[\alpha] \in \mathbb{P}T_x(M)$ and $\eta \in L_{[\alpha]} = \mathbb{C}\alpha$, $\eta = t\alpha$ sufficiently small, let $\Phi_0(\lambda)$ be $\gamma_\alpha(t)$, where γ is the unique germ of complex geodesic at $0 \in \mathbb{C}$ such that $\gamma(0) = x$ and $\frac{\partial \gamma}{\partial t}|_{t=0} = \alpha$. If we replace α by $\lambda\alpha$ for some nonzero λ, then $\gamma_{\lambda\alpha}(\frac{t}{\lambda}) = \gamma_\alpha(t)$ from uniqueness of geodesics with fixed initial value and initial first derivative. It follows that $\Phi(\eta)$ is well-defined, and we have a holomorphic map $\Phi_0 : U \to M$ which collapses $\mathbb{P}T_x(M)$ to x, from which it follows readily that Φ_0 descends to a holomorphic map $\Phi : \Omega \to M$, where Ω is a neighborhood of 0 in T_x. From the construction we have readily $d\Phi(0) = \mathrm{id}$. φ is the holomorphic exponential map, and it defines holomorphic geodesic normal coordinates at x. With respect to these coordinates, obviously the Riemann-Christoffel symbols vanish at 0. Moreover, by the same proof as in Riemannian geometry, the holomorphic metric admits a power series expansion at 0 in terms of the curvature tensor and its covariant derivatives at x. In particular, if the curvature vanishes identically, the holomorphic geodesic normal coordinates define a coordinate system with respect to which the holomorphic metric tensor (g_{ij}) is of constant coefficients. We may take g_{ij} to be δ_{ij}.

Exactly as in Riemannian geometry, the curvature tensor $R_{ijk}{}^\ell$ of (M, g) admits a decomposition $R_{ijk}{}^\ell = A_{ijk}{}^\ell + W_{ijk}{}^\ell$, where $W = (W_{ijk}{}^\ell) \in H^0(M, \Lambda^2 T^*_M \otimes \mathrm{End}(T_M))$

is the Bochner-Weyl tensor, which is unchanged when a holomorphic metric is modified by a conformal factor. A holomorphic metric is by definition conformally flat if and only if $W = 0$. A conformally flat holomorphic metric g is conformally equivalent to a holomorphic metric h with vanishing curvature, i.e., $R_h = 0$. Using holomorphic geodesic normal coordinates for h, we have seen that g is conformally flat if and only of it is given locally by $g_{ij} = \lambda\delta_{ij}$ for an appropriate choice of holomorphic coordinates and for some non-zero holomorphic function λ.

4.2. G-structures associated to irreducible Hermitian symmetric spaces of rank ≥ 2. — The model space of a holomorphic conformal structure is the hyperquadric $Q^n, n \geq 3$. In terms of Harish-Chandra coordinates on an open Schubert cell $\mathbb{U} \subset Q^n$, the Euclidean translations on $\mathbb{U}$ extend to automorphisms of Q^n, and the null-cones $\widetilde{\mathcal{N}}$ on Q^n form a constant family since they are invariant under automorphisms of Q^n, showing that the the holomorphic conformal structure on $\mathbb{U} \subset Q^n$ is defined by the equivalence class of a holomorphic metric of constant coefficients. Holomorphic conformal structures will also be referred to as hyperquadric structures, or Q^n-structures, in a sense that applies in general to Hermitian symmetric spaces S of the compact type and of rank ≥ 2. In this general context the hyperquadric structure on Q^n is said to be flat (or integrable) in the sense that there exists local holomorphic coordinates (the Harish-Chandra coordinates) with respect to which the null cones $\mathcal{N} \subset T_{Q^n}$ form constant families over the coordinate charts.

The notion of a hyperquadric structure generalizes to S-structures for any irreducible Hermitian symmetric space of rank ≥ 2. For the fibered space of null cones $\pi : \mathcal{N} \to M$ of a complex manifold M equipped with a holomorphic conformal structure, there is an underlying complex Lie group consisting of linear transformations preserving a model light cone $\mathcal{N}_0 \subset V := T_0(Q^n)$. The group is precisely the reductive complex Lie subgroup $\mathbb{C} \cdot \mathrm{O}(n;\mathbb{C}) \subset \mathrm{GL}(V)$. In general for any complex Lie subgroup G of $GL(V)$ for a finite-dimensional complex vector space we have the notion of a (holomorphic) G-structure. For its formulation let n be a positive integer, V be an n-dimensional complex vector space, and M be any n-dimensional complex manifold. In what follows all bundles are understood to be holomorphic. The frame bundle $\mathcal{F}(M)$ is a principal $\mathrm{GL}(V)$-bundle with the fiber at x defined as $\mathcal{F}(M)_x = \mathrm{Isom}(V, T_x(M))$.

***Definition 2* (G-structure).** — *Let $\mathrm{G} \subset GL(V)$ be any complex Lie subgroup. A holomorphic G-structure is a G-principal subbundle $\mathcal{G}(M)$ of $\mathcal{F}(M)$. An element of $\mathcal{G}_x(M)$ will be called a G-frame at x. For $\mathrm{G} \subsetneq \mathrm{GL}(V)$ we say that $\mathcal{G}(M)$ defines a holomorphic reduction of the tangent bundle to G.*

We have in general the notion of a flat G-structure, as follows.

***Definition 3* (flat G-structure).** — *In terms of Euclidean coordinates we identify $\mathcal{F}(U_\alpha)$ with the product* $\mathrm{GL}(V) \times U_\alpha$. *We say that a* G*-structure $\mathcal{G}(M)$ on M is flat if and only if there exists an atlas of charts $\{\varphi_\alpha : U_\alpha \to V\}$ such that the restriction $\mathcal{G}(U_\alpha)$ of $\mathcal{G}(M)$ to U_α is the product* $\mathrm{G} \times U_\alpha \subset \mathrm{GL}(V) \times U_\alpha$.

Let (S, g) be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2. Write G_c for the identity component of the isometry subgroup of (S, g), and $K \subset G_c$ be the isotropy subgroup at a reference point $0 \in S$. As a rational homogeneous manifold $S = G/P$, where G is a complexification of G_c and $P \subset S$ is a maximal parabolic subgroup. We have the Harish-Chandra decomposition of the Lie algebra $\mathfrak{g}$ of G, $\mathfrak{g} = \mathfrak{m}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^-$, in which $\mathfrak{k}^{\mathbb{C}}$ is the complexification of the Lie algebra $\mathfrak{k}$ of K. Regarding $\mathfrak{g}$ as the Lie algebra of holomorphic vector fields on S, $\mathfrak{m}^-$ stands for the vector space of holomorphic vector fields vanishing to the order ≥ 2 at 0. P admits a Levi decomposition $P = K^{\mathbb{C}} \cdot M^-$. Here $K^{\mathbb{C}} = \exp(\mathfrak{k}^{\mathbb{C}})$ is the reductive group consisting of automorphisms of S fixing 0, identified with a complex linear subgroup of $\mathrm{GL}(T_0(S))$ where $\gamma \in K^{\mathbb{C}}$ is mapped to $d\gamma(0)$, and $M^- = \exp(\mathfrak{m}^-)$. S then carries a G-structure with $\mathrm{G} = K^{\mathbb{C}}$. Regarding S-structures we have

***Theorem 3* (Ochiai [47]).** — *Let S be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2. Let X be a compact simply-connected complex manifold with a flat S-structure. Then, X is biholomorphic to S.*

$K^{\mathbb{C}}$ acts irreducibly on the model vector space $V = T_0(S)$, and its highest weight orbits define a rational homogeneous manifold $\mathcal{W}_0 \subset \mathbb{P}T_0(S)$, leading to a fibered space of highest weight tangents $\pi : \mathcal{W} \to M$ on any complex manifold equipped with a $K^{\mathbb{C}}$-structure. Let $(M_1, \mathcal{G}_1)$ resp. $(M_2, \mathcal{G}_2)$ be two complex manifolds equipped with G-structures, $\mathrm{G} = K^{\mathbb{C}}$, with fibered spaces of highest weight tangents $\pi_1 : \mathcal{W}_1 \to M_1$ resp. $\pi_2 : \mathcal{W}_2 \to M_2$. A biholomorphism $f : M_1 \to M_2$ preserves the G-structures if and only if it preserves the fibered spaces of highest weight tangents, i.e., $f_* \mathcal{W}_1 = \mathcal{W}_2$.

Denote by $\mathcal{O}(1)$ the ample line bundle on S which is the positive generator of the Picard group of S. S can be embedded into the projective space by $\mathcal{O}(1)$, e.g., the Grassmannian is embedded by the Plücker embedding. With respect to this embedding, S is uniruled by lines. When S itself is considered as the underlying space of an S-structure, the variety of highest weight tangents $\mathcal{W}_x$ agrees with the VMRT $\mathcal{C}_x$ at any $x \in S$. This follows from the construction of lines on S by means of $\mathrm{SL}(2, \mathbb{C})$ orbits highest weight vectors (cf. Mok ([**40**], (1.4)) for a verification in the case of Grassmannians). To give a proof of Ochiai's Theorem using VMRTs, the starting point is the following result on local VMRT-preserving holomorphic maps.

Lemma 4. — *Let S be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2. Let $D, D' \subset S$ be nonempty connected open subsets of S and*

$f : D \to D'$ be a VMRT*-preserving biholomorphic map. Then, for any line on S intersecting D, $f(L \cap D)$ is an open subset of some line L' of S*

Proof. — Denote by $\pi : \mathcal{C} \to S$ the fibered space of VMRTs over S and by $\mathcal{F}$ the tautological foliation on $\mathcal{C}$. By assumption $[df](\mathcal{C}|_D) = \mathcal{C}|_{D'}$. We have to show that for any line $L \subset S$ such that $L \cap D \neq \varnothing$, $[df](\widehat{L} \cap \mathcal{C}|_D)$ is an integral curve of the tautological foliation on S. This is the case if and only if $f^*\mathcal{F}$ agrees with $\mathcal{F}$ on $\mathcal{C}|_D$, i.e., if and only if the image under $[df]$ of each $\widehat{L} \cap \mathcal{C}|_D$ is tangent at every point to the tautological lifting $\widehat{L'}$ of some line L'. Equivalently this means that the image of each $L \cap D$ is tangent at every point to a line on S up to the second order. To prove Lemma 4 it suffices therefore to show that $\partial^2 f(\alpha, \alpha)$ is proportional to $df(\alpha)$ for any minimal rational tangent $[\alpha]$. In these coordinates $\pi : \mathcal{C} \to S$ is a constant family. Let α, β be vectors in $\widetilde{\mathcal{C}}_0$. (For a projective subvariety $\mathcal{A} \subset \mathbb{P}^N$ we denote by $\widetilde{\mathcal{A}} \subset \mathbb{C}^{N+1} - \{0\}$ its homogenization.) Then, $\partial^2 f(\alpha, \beta) = \partial_\alpha(df(\beta^\sharp))$, where $\beta^\sharp$ stands for the constant vector field on D such that $\beta^\sharp(0) = \beta$. Thus, $\partial^2 f(\alpha, \beta)$ is the tangent at β to some holomorphic curve on $\widetilde{\mathcal{C}}_0$, so that $\partial^2 f(\alpha, \beta) \in P_\beta = T_\beta(\widetilde{\mathcal{C}}_0)$. By symmetry we have $\partial^2 f(\alpha, \beta) \in P_\alpha \cap P_\beta$.

It remains to derive that for any $\alpha \in \widetilde{C}$, $\partial^2 f(\alpha, \alpha) = \lambda\alpha$ for some λ. On a non-linear projective submanifold, by Zak's Theorem (Zak [**52**]) the Gauss map is non-degenerate at a general point. Thus, the kernel of the second fundamental form σ is trivial at a general point. In the case of $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$, which is homogeneous as a projective submanifold, $\mathrm{Ker}(\widetilde{\sigma}) = 0$ everywhere. Equivalently, lifting to homogenizations, $\mathrm{Ker}(\widetilde{\sigma}_\alpha) = \mathbb{C}\alpha$ for the (Euclidean) second fundamental form $\widetilde{\sigma}_\alpha$ at any $\alpha \in \widetilde{\mathcal{C}}_0$, and it remains for the proof of Lemma 4 to show that $\partial^2 f(\alpha, \alpha) \in \mathrm{Ker}(\widetilde{\sigma}_\alpha)$ for any such α. Fix now $\alpha \in \widetilde{\mathcal{C}}_0$ and let $\beta = \alpha(t), \alpha(0) = \alpha$, vary holomorphically on $\widetilde{\mathcal{C}}_0$ in the complex parameter t. Writing $\xi = \partial_t(\alpha)(0)$, from $\partial^2 f(\alpha, \alpha(t)) \in P_\alpha$ it follows that $\partial^2 f(\alpha, \xi) \in P_\alpha$. On the other hand $\partial_t(\partial^2 f(\alpha(t), \alpha(t))|_{t=0} = 2\partial^2 f(\alpha, \xi)$, and hence $\nabla_\xi(\partial^2 f(\alpha(t), \alpha(t))|_{t=0} \in P_\alpha$ in terms of the Euclidean flat connection ∇ on $T_0(S)$. It follows that $\widetilde{\sigma}_\alpha(\xi, \partial^2 f(\alpha, \alpha)) = 0$. Since ξ can be chosen to be any tangent vector in $T_\alpha(\widetilde{\mathcal{C}}_0) = P_\alpha$, we conclude that $\partial^2 f(\alpha, \alpha) \in \mathrm{Ker}(\widetilde{\sigma}_\alpha)$, and we are done. □

By means of Lemma 4 the mapping $f : D \to D'$ can be analytically continued to give an automorphism of S. The idea is to pass to the moduli space $\mathcal{K}$ of lines. For each $x \in S$ denote by $\mathcal{Q}_x \subset \mathcal{K}$ the projective submanifold consisting of lines passing through x. We may assume D to be convex in Harish-Chandra coordinates. For any $\ell \in \mathcal{K}$ sufficiently close to $\mathcal{Q}_x$, $\ell \cap D$ is non-empty and connected, and $f(\ell \cap D)$ is an open subset of some line ℓ'. Thus, for a sufficiently small open neighborhood $\mathbb{U}$ of $\mathcal{Q}_x$ in $\mathcal{K}$, f induces a holomorphic map $f^\sharp : \mathbb{U} \to \mathcal{K}$. The problem of analytic continuation can be solved first by meromorphically extending $f^\sharp$ to $F^\sharp : \mathcal{K} \to \mathcal{K}$ and then by recovering $F : S \to S$ by considering a point $y \in S$ as the intersection of all lines passing through it, and by defining $f(y) := \bigcap\{f^\sharp(\ell) : y \in \ell\}$ for a general point $y \in S$. The meromorphic extension of $f^\sharp$ to $F^\sharp$ is plausible because $\mathbb{U}$ is a 'big' open set in an analytic sense, as it contains the projective subvarieties $\mathcal{Q}_y$ for y sufficiently close to x. This latter extension problem can be solved by methods of Hartogs extension as done in Mok-Tsai [**44**]. The extension $F : S \to S$ thus obtained may have singularities, but they are proven to be removable by arguments involving deformation theory of rational curves (cf. Mok [**40**], (2.4)).

4.3. Flatness of G-structures via VMRTs. — Let V be a fixed n-dimensional complex vector space and $\mathrm{G} \subset \mathrm{GL}(V)$ be a connected complex Lie subgroup. Let X be an n-dimensional complex manifold endowed with a G-structure $\mathcal{G} \subset \mathcal{F}(X)$. We examine necessary and sufficient conditions for the G-structure to be flat. Recall that the G-structure $\mathcal{G}$ is flat if local holomorphic trivializations of $\mathcal{G}$ can be realized by choices of local holomorphic coordinates on X. Flatness imposes therefore differential constraints on $(X, \mathcal{G})$. The problem of identifying flat G-structures was solved in terms of obstructions to prolongations of G-structures (cf. Guillemin [**4**]).

Given a G-structure $(X, \mathcal{G})$ and a biholomorphic map $f : X \to Y$ onto another complex manifold Y, we have an induced G-structure $(Y, f_*\mathcal{G})$. Let $(X, \mathcal{G})$ and $(X', \mathcal{G}')$ be two complex manifolds endowed with G-structures. For $x \in X$ denote by (X, x) the germ of complex manifolds defined by X at x, etc. A germ of local biholomorphism $f : (X, x) \to (X', x')$ is said to be (0-th order) structure-preserving if $(f_*\mathcal{G})_{x'} = \mathcal{G}'_{x'}$. For k a positive integer, f is said to be k-th order structure-preserving if furthermore $f_*\mathcal{G}$ is tangent to $\mathcal{G}'$ along $\mathcal{G}'_x$ to an order $\geq k$. This notion depends only on the $(k+1)$-jet of f. For $k \geq 0$ the G-structure $(X, \mathcal{G})$ is said to be k-flat at x if there exists a local biholomorphism $f : (X, x) \to (V, 0)$ which is k-th order structure-preserving, when V is endowed with the trivial G-structure $V \times G$.

When $(X, \mathcal{G})$ is uniformly k-flat, i.e., k-flat at every point $x \in X$, one can define in a canonical way some structure function c^k on some prolongation bundle over $\mathcal{G}$, such that $c^k \equiv 0$ if and only if $(X, \mathcal{G})$ is uniformly $(k+1)$-flat (Guillemin [**5**], Cor. to Theorem 4.1). By the Cartan-Kähler Theorem (Singer-Sternberg [**49**]) a G-structure is flat if and only if it is k-flat for every integer $k \geq 0$. In the case where G is reductive, the structure functions can be translated as obstruction tensors $\theta_k \in H^0\big(X, \mathrm{Hom}(\Lambda^2 T_X, T_X \otimes S^k T_X^*)\big)$. In the case of S-structures (cf. (4.2)) corresponding to $\mathrm{G} = K^{\mathbb{C}}$ it is known that $(X, \mathcal{G})$ is flat if and only if it is uniformly 2-flat. When S is Q^n, $n \geq 3$, given a point $x \in X$ the fibered space $\pi : \mathcal{Q} \to X$ of null tangents is always tangent at x to that of the flat Q^n-structure in terms of holomorphic normal coordinates at x. Thus, the only obstruction tensor is $\theta_1 \in H^0\big(X, \mathrm{Hom}(\Lambda^2 T_X, \mathrm{End}(T_X))\big)$, which agrees with the Bochner-Weyl tensor $(W_{ijk}{}^{\ell})$ of the holomorphic conformal structure.

***Theorem 4* (Hwang-Mok [14]).** — *Let X be a uniruled projective manifold admitting an irreducible reductive* G*-structure,* $\mathrm{G} \subsetneq \mathrm{GL}(V)$*. Then, X is biholomorphic to an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2.*

Outline of Proof. — Associated to a G-structure with $\mathrm{G} \subsetneq GL(V)$ reductive, there is on X the fibered space $\lambda : \mathcal{W} \to X$ of highest weight tangents. We show first of all that the latter agrees with the fibered space $\pi : \mathcal{C} \to X$ of VMRTs. The proof makes use of Grothendieck's classification of G-principal bundles on $\mathbb{P}^1$ in [**4**]. Then, we show that the G-structure is flat by proving successively the vanishing of the structure functions c^k. Finally, we identify the candidates of VMRTs on X to show that they correspond to S-structures in the Hermitian symmetric case, and conclude that $X \cong S$ by observing that X is rationally connected, hence simply connected.

To prove the vanishing of the structure functions c^k it suffices to prove the vanishing of the obstruction tensors $\theta = \theta_k$, which give in the reductive case sections in $H^0\big(X, \mathrm{Hom}(\Lambda^2 T_X, T_X \otimes S^k T_X^*)\big)$. Let ℓ be a standard rational curve, assumed embedded for convenience, so that $T_X|_\ell \cong \mathcal{O}(2) \oplus (\mathcal{O}(1))^p \oplus \mathcal{O}^q$. Each direct summand of $(T_X \otimes S^k T_X^*)|_\ell$ is of degree ≤ 2. If we fix $x \in X$, then $\theta_x(\alpha, \xi) = 0$ whenever $\alpha \in \widetilde{\mathcal{C}}_x$ and $\xi \in T_\alpha(\widetilde{\mathcal{C}}_x) = P_\alpha$, since $\alpha \wedge \xi$ belongs to a direct summand of degree 3. By [(3.2), Proposition 3], such elements generate $\Lambda^2 T_x(X)$, and we conclude that $\theta \equiv 0$. □

In the same vein Hong ([**6**], Proposition (3.1.4)) established the following characterization of Fano homogeneous contact manifolds of Picard number 1. The statement here is a slight modification of the original one which is implicit from the proof there.

***Theorem 5* (Hong [6]).** — *Let S be a Fano homogeneous contact manifold of Picard number 1 different from an odd-dimensional projective space. Let $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ be the* VMRT *of S at a reference point $0 \in S$. Let X be a Fano manifold of Picard number 1 whose* VMRT *$\mathcal{C}_x \subset \mathbb{P}T_x(S)$ at $x \in X$ is isomorphic to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ as a projective subvariety for x lying outside a subvariety $Z \subset X$ of codimension ≥ 2. Denoting by D the distribution on X spanned by* VMRTs*, assume that the Frobenius form $\varphi : \Lambda^2 D \to T_X/D$ is everywhere non-degenerate on $X - Z$. Suppose furthermore that at every point $x \in X - Z$, a general minimal rational curve passing through x lies on $X - Z$. Then, X is biholomorphic to S.*

4.4. Prolongation of linear subalgebras of infinitesimal automorphisms of VMRTs. — Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component with non-linear VMRTs, and $x \in X$ be a general point. Regarding the VMRTs in a neighborhood of x as defining a germ of geometric structure at x, we are interested to study its germs of infinitesimal automorphisms vanishing at x. By Cartan-Fubini extension, as to be explained in §5, this is the same as studying holomorphic vector fields on X vanishing at x. As a preparation we have

Lemma 5. — *Let X be a complex manifold, $x \in X$ be any point, $m \geq 1$ be a positive integer and Z be a holomorphic vector field vanishing at x to the order $\geq m$. Let $\{\varphi_t\}$ be the complex 1-parameter group of automorphisms on X generated by Z. Let $E \subset \mathbb{P}T_X$ be an irreducible subvariety invariant under the induced automorphisms $\{\Phi_t\}$ on $\mathbb{P}T_X$. Assume that $\pi|_E : E \to X$ is a holomorphic submersion at a general smooth point of $E_x := E \cap \mathbb{P}T_x$. In terms of local holomorphic coordinates (z_i) at x; $z_i(x) = 0$; write $Z = \sum_{i_1 \dots i_m;k} A^k_{i_1\cdots i_m} z^{i_1}\cdots z^{i_m}\frac{\partial}{\partial z_k} + O(|z|^{m+1})$, where the Taylor coefficients $A^k_{i_1,\cdots,i_m}$ are symmetric in $i_1,\cdots i_m$. Then, regarding the Taylor coefficients of m-th order terms as coefficients of a homomorphism $A : S^m T_x \to T_x$; for any choice of $m-1$ tangent vectors $\eta_1,\cdots,\eta_{m-1}$; the linear vector field $\sum_i w^i A(\eta_1,\cdots,\eta_{m-1},\frac{\partial}{\partial w_i})$ on T_x is tangent to $\widetilde{E}_x$ at its smooth points.*

Proof. — Write $\varphi^k_t(z) = z + \sum B^k_{i_1\cdots i_m}(t) z^{i_1}\cdots z^{i_m} + O(|z|^{m+1})$ for z lying on a small neighborhood of x and for t sufficiently small, where the summation is over $(i_1,\cdots,i_m)$. We have $\frac{\partial}{\partial t}B^k_{i_1\cdots i_m}(t)|_{t=0} = A^k_{i_1\cdots i_m}$. Writing (w_i) for fiber coordinates for T_X induced by (z_i), the induced automorphism Φ_t on T_X is given by

$$\Phi_t(z,w) = \big(\varphi_t(z); d\varphi_t(z)(w)\big) =$$

$$\sum\Big(B^k_{i_1\cdots i_m}(t)z^{i_1}\cdots z^{i_m}e_k + O(|z|^{m+1}); mB^k_{i_1\cdots i_m}(t)z^{i_1}\cdots z^{i_{m-1}}w^{i_m}\epsilon_k + O(|z|^m|w|)\Big)$$

Here $e_k = \frac{\partial}{\partial z_k}$ and $\epsilon_k = \frac{\partial}{\partial w_k}$. Since φ_t preserves the subvariety $\widetilde{E}$, the infinitesimal automorphism $\widetilde{Z} = \frac{\partial}{\partial t}\Phi_t|_{t=0}$ is tangent to $\widetilde{E}$ at smooth points. It is given by

$$\widetilde{Z} = \sum\Big(A^k_{i_1\cdots i_m}z^{i_1}\cdots z^{i_m}e_k + O(|z|^{m+1}); mA^k_{i_1\cdots i_m}z^{i_1}\cdots z^{i_{m-1}}w^{i_m}\epsilon_k + O(|z|^m|w|)\Big),$$

showing that the latter vanishes on T_x to the order $\geq m-1$. Taking partial derivatives $m-1$ times against horizontal constant vector fields $\eta_1,\cdots\eta_{m-1}$. we obtain $\sigma := \sum_{i,k} A^k_{\eta_1\cdots\eta_{m-1}i}w^i\frac{\partial}{\partial w_k} = \sum_i w^i A(\eta_1,\eta_2,\cdots,\eta_{m-1},\frac{\partial}{\partial w_i})$. When $m=1$ no differentiation is involved, and σ is simply the restriction of $\widetilde{Z}$ to T_x. Since at a smooth point of $\widetilde{E}_x$, σ is both tangent to $\widetilde{E}$ and to T_x, it must be tangent to $\widetilde{E}_x$, as desired. □

Lemma 6. — *Let X be an n-dimensional uniruled projective manifold admitting a minimal rational component whose* VMRT *$\mathcal{C}_x \subset \mathbb{P}T_x$ at a general point x is p-dimensional; $0 < p < n-1$; nonsingular and linearly non-degenerate. Given a general point $x \in X$, let Z be a holomorphic vector field vanishing at x to the order ≥ 2. In terms of local holomorphic coordinates (z_i) in a neighborhood of x; $z_i(x) = 0$; write $Z = \sum_{i,j,k} A^k_{ij}z^iz^j\frac{\partial}{\partial z_k} + O(|z|^3)$, where $A^k_{ij} = A^k_{ji}$. Then, regarding A^k_{ij} as coefficients of a linear homomorphism $A : S^2T_x \to T_x$ we have $A_{\alpha\alpha} \in \mathbb{C}\alpha$ for any $\alpha \in \widetilde{\mathcal{C}}_x$.*

Proof. — By Lemma 5, for any $\eta \in T_x$ and any nonzero $\alpha \in \widetilde{\mathcal{C}}_x$ we have $A_{\alpha\eta} \in T_\alpha(\widetilde{\mathcal{C}}_x) = P_\alpha$. In particular, if η is itself a nonzero vector in $\widetilde{\mathcal{C}}_x$, we have from the symmetry of A the property that $A_{\alpha\beta} \in P_\alpha \cap P_\beta$. The rest of the proof is the same as in Lemma 2. (Here $A_{\alpha\beta}$ plays the same role as $\partial^2 f(\alpha,\beta)$ there.) □

Proposition 4. — *Under the hypothesis in Lemma 6 and in the notations there, suppose for the holomorphic vector field Z vanishing at x to the order ≥ 2 we have $A_{\alpha\alpha} = 0$ for any $\alpha \in \mathcal{C}_x$, then $A \equiv 0$.*

Sketch of proof. — Fixing $\eta \in T_x(X)$, $A_{\alpha\eta} \in P_\alpha$. From $A_{\alpha\alpha} = 0$ for every $\alpha \in \mathcal{C}_x$, varying $\alpha = \alpha(t)$ holomorphically and differentiating against t we conclude that $A_{\alpha\xi} = 0$ for every $\xi \in P_\alpha$. Regarding A_α as an endomorphism of $T_x(X)$ given by $A_\alpha(\eta) = A_{\alpha\eta}$, we have $\mathrm{Im}(A_\alpha) \subset P_\alpha \subset \mathrm{Ker}(A_\alpha)$, so that $A_\alpha^2 = 0$. Thus, choosing sufficiently general points $\alpha, \beta \in \mathcal{C}_x$, the closure of the orbit of $[\alpha]$ under $\exp(tA_\beta)$ is a line joining $[\alpha]$ to $[\xi]$, where $\xi := A_{\alpha\beta} \neq \alpha, \beta$; likewise with α and β interchanged. Hence $\mathcal{C}_x$ is rationally 2-connected by lines. Proposition 4 is proven inductively. We denote by $\mathcal{K}'$ a minimal rational component consisting of lines on $\mathcal{C}_x$, and $\mathcal{C}'_{[\alpha]}$ the associated VMRT at $[\alpha]$. For induction we replace x by $[\alpha]$, X by $\mathcal{C}_x$, and consider the VMRT $\mathcal{C}'_{[\alpha]}$ at $[\alpha] \in \mathcal{C}_x$. Given a holomorphic vector field Z vanishing at x to the order ≥ 2 for which $A_{\alpha\alpha} = 0$ for every $\alpha \in \mathcal{C}_x$, we derive a holomorphic vector field $\mathcal{Z}$ on $\mathcal{C}_x$ vanishing at $[\alpha]$ to the order ≥ 2 such that $\mathcal{A}_{\mu\mu} = 0$ for every $\mu \in \mathcal{C}'_\alpha$.

Starting with the data $\big(X, \mathcal{K}, x, \mathcal{C}_x, Z, (A_{ij})\big)$ we derive $\big(\mathcal{C}_x, \mathcal{K}', [\alpha], \mathcal{C}'_{[\alpha]}, \mathcal{Z}, (\mathcal{A}_{k\ell})\big)$, noting that $\mathcal{C}'_{[\alpha]}$ is nonsingular at a general point $[\alpha] \in \mathcal{C}_x$, by Lemma 3. To be able to proceed by induction on the dimension, it remains to prove that $\mathcal{C}'_{[\alpha]} \subset \mathbb{P}T_{[\alpha]}(\mathcal{C}_x)$ is linearly non-degenerate. From the fact that $\mathcal{C}'_{[\alpha]}$ is rationally 2-connected by lines, it follows that $\dim(\mathcal{C}'_{[\alpha]}) \geq \frac{1}{2}\dim(\mathbb{P}T_{[\alpha]}(\mathcal{C}_x))$, and by [(3.2), Proposition 3] it would follow that $\mathcal{C}'_{[\alpha]}$ is linearly non-degenerate in $\mathbb{P}T_{[\alpha]}(\mathcal{C}_x)$, if we knew that $\mathcal{C}_{[\alpha]}$ is of Picard number 1. However, the latter need not be the case. Nonetheless, the proof of Proposition 3 still works since we know that the VMRT is rationally 2-connected by lines as explained, making it possible to prove Proposition 4 by induction.

Write $\mathfrak{f}$ for the germs of $\mathcal{C}$-preserving holomorphic vector fields at x. For $\ell \geq -1$, write $\mathfrak{f}^\ell$ for the vector subspace of all $Z \in \mathfrak{f}$ vanishing to the order $\geq \ell + 1$ at x. Then Proposition 4 says that, under the assumption that the VMRT $\mathcal{C}_x \subsetneq \mathbb{P}T_x(X)$ is irreducible, nonsingular and linearly non-degenerate, there is an injection of $\mathfrak{f}^1$ into $\Gamma(\mathcal{C}_x, \mathrm{Hom}(L^2, L)) = \Gamma(\mathcal{C}_x, L^*)$, where L stands for the tautological line bundle over $\mathbb{P}T_x(X)$. If furthermore $\mathcal{C}_x$ is linearly normal in $\mathbb{P}T_x(X)$, i.e., the embedding of $\mathcal{C}_x \subset \mathcal{P}T_x(X)$ is defined by a complete linear system, then $\dim(\mathfrak{f}^1) \leq n$. From the proof of Proposition 4 it follows readily that $\mathfrak{f}^\ell = 0$ for $\ell \geq 2$, i.e., there does not exist any nontrivial holomorphic vector field vanishing at x to the order ≥ 3. In fact, if a $\mathcal{C}$-preserving germ of holomorphic vector field Z vanishes at x to the order ≥ 2, and A_{ijk} are the coefficients of the third order terms of the Taylor expansion of Z at x, then for any $\gamma \in T_x(X)$, $B_{\alpha\beta} = A_{\alpha\beta\gamma}$ defines a 2-tensor for which the arguments apply, and from there the vanishing of A_{ijk} follows easily. The same arguments apply to the leading terms of any nontrivial holomorphic vector field Z vanishing at x to the order $s \geq 3$, and we have a contradiction unless $Z \equiv 0$. $\square$

Lemma 5 can be stated in the language of prolongation theory for Lie subalgebras of $\mathrm{End}(T_x(X))$, as follows. Let V complex vector space, $\dim V = n$, and $\mathfrak{g} \subset \mathrm{End}(V)$ be a Lie subalgebra. For $k \geq -1$ denote by $\mathfrak{g}^{(k)} \subset S^{k+1}V^* \otimes V$ the vector subspace consisting of all $\sigma \in S^{k+1}V^* \otimes V$ such that, writing $\sigma_{v_1,\cdots,v_k}(v) = \sigma(v; v_1, \ldots, v_k)$, we have $\sigma_{v_1,\ldots,v_k} \in \mathfrak{g}$. Let now $Y \subset \mathbb{P}V$ be a projective subvariety, and $\widetilde{Y} \subset V$ be its lifting to V. We write $\mathfrak{aut}(Y) := \{A \in \mathrm{End}(V) : \exp(tA)(\widetilde{Y}) \subset \widetilde{Y} \text{ for all } t \in \mathcal{C}\}$. Then for every $\ell \geq 0$, $\mathfrak{f}^\ell \subset \mathfrak{aut}(Y)^{(\ell)}$. The argument in the proof of Proposition 4 applies to elements of $\mathfrak{aut}(\mathcal{C}_x)^{(\ell)}$ to imply that $\dim\left(\mathfrak{aut}(\mathcal{C}_x)^{(1)}\right) \leq \dim \Gamma(\mathcal{C}_x, L^*)$, and hence that $\mathfrak{aut}(\mathcal{C}_x)^{(\ell)} = 0$ whenever $\ell \geq 2$. In relation to holomorphic vector fields on a Fano manifold of Picard number 1 there are the following conjectures and results.

Conjecture 2. — *Let X be a Fano manifold of Picard number 1. Then, at a general point $x \in X$ there does not exist any nontrivial holomorphic vector field Z vanishing at x to the order ≥ 3.*

Conjecture 3. — *Let X be an n-dimensional Fano manifold of Picard number 1. Then, $\dim(\mathrm{Aut}(X)) \leq n^2 + 2n$. Moreover, equality holds if and only if $X \cong \mathbb{P}^n$.*

***Theorem 6* (Hwang-Mok [23]).** — *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component. Suppose the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at a general point $x \in X$ is irreducible, nonsingular and linearly non-degenerate. Then, at a general point $x \in X$ there does not exist any nontrivial holomorphic vector field vanishing at x to the order ≥ 3. If furthermore $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is linearly normal, then $\dim\left(\mathfrak{aut}(\mathcal{C}_x)^{(1)}\right) \leq n$, and equality holds if and only if $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is congruent to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ for the variety of minimal rational tangents of an irreducible Hermitian symmetric space of the compact type. Furthermore, $\dim(\mathrm{Aut}(X)) \leq n^2 + 2n$, and equality holds if and only if $X \cong \mathbb{P}^n$.*

Remarks. — As will be seen in (6.3) the statement that $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is congruent to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ implies that X is biholomorphic to S.

Corollary 1. — *Let X be an n-dimensional Fano manifold of Picard number 1, and denote by $\mathcal{O}(1)$ the positive generator of $\mathrm{Pic}(X) \cong \mathbb{Z}$. Assume that $\mathcal{O}(1)$ is very ample. Suppose $c_1(X) > \frac{n+1}{2}$. Then, for a general point $x \in X$ there does not exist any nontrivial holomorphic vector field vanishing at x to the order ≥ 3. Suppose X satisfies the stronger condition $c_1(X) > \frac{2(n+2)}{3}$, then $\dim(\mathrm{Aut}(X)) \leq n^2 + 2n$, and equality holds if and only if $X \cong \mathbb{P}^n$.*

In relation to VMRTs in general the following conjecture summarizes what one can optimistically hope as compared to known results in [(2.4), Theorem 1].

Conjecture 4. — *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component, and $\pi : \mathcal{C} \to X$ be the fibered space of varieties of minimal rational tangents associated to $\mathcal{K}$. Then, at a general point $x \in X$, either $\mathcal{C}_x$ is finite, or it is irreducible, nonsingular and linearly normal in its linear span $\mathbb{P}W_x \subset \mathbb{P}T_x(X)$.*

Regarding Conjectures 2 and 3, the fundamental assumption in the partial result (Theorem 6) is the linear non-degeneracy of the VMRT $\mathcal{C}_x$ at a general point. At least the statement regarding vanishing orders of holomorphic vector field is accessible whenever an irreducible component of $\mathcal{C}_x$ is linearly non-degenerate.

4.5. Applications to rigidity under Kähler deformation. — We return to the question of rigidity of rational homogeneous manifolds $S = G/P$ of Picard number 1 under Kähler deformation, as given in [(3.4), Theorem 2]. In (3.4) we explained that for the case of $P \subset G$ defined by a long simple root, the problem is solved by studying the integrability of distributions spanned by or derived from VMRTs. In Hwang-Mok [**22**] we settled the problem for $G = F_4$ for the 20-dimensional F_4-homogeneous space associated to a short root. There we have the nilpotent graded algebra $\mathfrak{n} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \oplus \mathfrak{g}_4$. As opposed to the long root case the VMRT does not lie in $\mathbb{P}D_1$ for the minimal proper G-invariant distribution D_1, but it remains linearly degenerate, spanning the proper G-invariant distribution $D_2 \neq T_S$, and the method using distributions spanned by VMRTs and Yamaguchi [**51**] is still applicable.

What remain are the cases of $S = G/P$ defined by short simple roots in the cases of C_n, and the 15-dimensional case of type F_4. In both cases we have $\mathfrak{n} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, the VMRT $\mathcal{C}_x$ at any point $x \in S$ is almost homogeneous with two orbits corresponding to highest weight vectors in $\mathfrak{g}_1$ resp. $\mathfrak{g}_2$, and $\mathcal{C}_x \subset \mathbb{P}T_x(S)$ is linearly non-degenerate. The problem is solved in Hwang-Mok [**23**] (2005). To proceed we showed that the VMRT at a general point of the central fiber X_0 of $\pi : \mathcal{X} \to \Delta$ remains isomorphic to that of the model space $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$. On X_0 we still have a 2-step filtration $0 \subset D^1 \subset D^2 = T_{X_0}$, but $\mathcal{C}_{x_0} \cap \mathbb{P}D^1_{x_0}$ does not have an algebro-geometric meaning, and the methods involving distributions spanned by VMRTs do not apply.

To solve the problem we examine the Lie algebra of holomorphic vector fields on X_0 which occur as limits of those on $X_t, t \neq 0$, with an aim to recuperating the Lie algebra $\mathfrak{g}$ on X_0. For illustration we consider the Hermitian symmetric case and sketch a proof in the last step using holomorphic vector fields in place of Ochiai's Theorem on S-structures. We assume already known that, over a suitably chosen holomorphic section $\sigma : \Delta \to \mathcal{X}$, the VMRTs of $\mathcal{C}_{\sigma(t)} \subset \mathbb{P}T_{\sigma(t)}(X_t)$ on $X_{\sigma(t)}$ form a holomorphically trivial family of projective submanifolds all congruent to $\mathcal{C}_0 \subset T_0(S)$

on the model space S. Writing $\mathcal{T}$ for the relative tangent sheaf of $\pi : \mathcal{X} \to \Delta$, the direct image $\mathcal{V} = \pi_* \mathcal{T}$ is the sheaf of germs of sections of a holomorphic vector bundle V on Δ, where for $t \neq 0, \mathfrak{g}^t := V_t$ carries naturally the structure of a Lie algebra isomorphic to the Lie algebra $\mathfrak{g}$ of $G = \mathrm{Aut}_0(S)$, and our aim is to prove that this remains true at $t = 0$. The idea is to reconstruct the Lie algebra structure from data that can be recovered along $\sigma : \Delta \to \mathcal{X}$. For the model space $S = G/P$ we have the decomposition of the Lie algebra $\mathfrak{g}$ of G as a graded Lie algebra, and equivalently the Harish-Chandra decomposition (in the notations of (4.2)) given by

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathfrak{m}^- \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^+;$$
$$[\mathfrak{m}^-, \mathfrak{m}^-] = [\mathfrak{m}^+, \mathfrak{m}^+] = 0, \quad \text{where} \quad \mathfrak{m}^- = \{Z \in \Gamma(S, T_S) : \mathrm{ord}_0 Z \geq 2\}.$$

For $k, k' \in \mathfrak{k}^{\mathbb{C}}$, $m^+ \in \mathfrak{m}^+$ and $m^- \in \mathfrak{m}^-$ the Lie brackets $[k, m^+] \in \mathfrak{m}^+, [k, m^-] \in \mathfrak{m}^-, [k, k'], [m^-, m^+] \in \mathfrak{k}^{\mathbb{C}}$ are completely determined by the leading terms of the Lie algebra elements at 0. Here the leading term stands for the 0-th order term for m^+, the first-order term for k and k', and the second-order term for m^-. For a holomorphic vector field Z on X_t vanishing at $\sigma(t)$ we denote by A_Z the coefficient matrix for the linear term of Z, which defines an element of $\mathrm{End}(T_{\sigma(t)}(X_t))$. Define

$$J_t^{(k)} = \{Z \in \mathfrak{g}^t : \mathrm{ord}_{\sigma(t)}(Z) \geq k\}; \quad I_t = \{Z \in \mathfrak{g}^t : Z(\sigma(t)) = 0 \ , \ A_Z \in \mathbb{C} \cdot \mathrm{id}\} \ .$$

For $t \neq 0$ we have $\dim J_t^{(2)} = n; \dim J_t^{(k)} = 0$ for any $k \geq 3$, and $\dim I_t = n + 1$, and any $Z \in I_t$, $A_Z \not\equiv 0$ determines a $\mathbb{C}^*$-action. Since $\mathcal{C}_{\sigma(0)} \subset \mathbb{P}T_{\sigma(0)}(I_t)$ is conjugate to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$, by [(4.4), Theorem 6] we have

$$\dim J_0^{(2)} \leq n \ , \ J_0^{(k)} = 0 \text{ for } k \geq 3 \ .$$

Thus, $\dim I_0 \leq n+1$ while $\dim I_0 \geq n+1$ by upper semicontinuity of $\dim I_t$ in $t \in \Delta$. Therefore, $\dim I_0 = n + 1$, so that there exists $Z \in I_0$ such that $A_Z \not\equiv 0$ and such that $e^{\lambda Z}$ defines a $\mathbb{C}^*$-action on X_0 of period $2\pi i$ in λ. This $\mathbb{C}^*$-action on X_0 can be extended to a holomorphic family T_t of $\mathbb{C}^*$-actions on X_t, of period $2\pi i$ in λ, given by $T_t(\lambda) = e^{\lambda E_t}$, $E_0 = Z$. Finally, defining

$$\mathfrak{g}_i^t := \{Z \in \mathfrak{g}^t : [E_t, Z] = iZ\}; \quad \text{we have} \quad \mathfrak{g}^t = \mathfrak{g}_{-1}^t \oplus \mathfrak{g}_0^t \oplus \mathfrak{g}_1^t \ .$$

For $t \neq 0$,

$$\mathfrak{g}_0^t \cong \{A \in \mathrm{End}_{\sigma(t)}(T_{\sigma(t)}) : A|_{\widetilde{\mathcal{C}}_{\sigma(t)}} \text{is tangent to } \widetilde{\mathcal{C}}_{\sigma(t)}\}.$$

Dimension count forces the same for $t = 0$. The Lie algebra structure on $\mathfrak{g}^0$ is determined by leading terms at $\sigma(0)$ of elements in $\mathfrak{g}_{-1}^0$, $\mathfrak{g}_0^0$ and $\mathfrak{g}_1^0$. Clearly, the rules for taking Lie brackets by means of the leading terms at σ_0 agrees with those at $0 \in S$ for the model space, and we have shown that $X_0 = G/P \cong S$.

Let $n \geq 2$ and W be a $2n$-dimensional complex vector space equipped with a symplectic form ν. For $1 < k < n$ we denote by $S_{k,n}$ the symplectic Grassmannian of

k-planes V in W isotropic with respect to ν. The symplectic Grassmannian $S = S_{k,n}$ is clearly homogeneous under the group G of symplectic transformations of W, $G \cong \mathrm{Sp}(n, \mathbb{C})$. It is a complex submanifold of the Grassmannian $\mathrm{Gr}(k, W)$ of k-planes in W. With respect to the Plücker embedding $p : \mathrm{Gr}(k, W) \to \mathbb{P}^N$, a line ℓ on S passing through the point $[V] \in S$, where $V = V^{(k)}$, is defined by the choices of a $(k-1)$-plane $E^{(k-1)}$ and a $(k+1)$-plane $F^{(k+1)}$ such that $E^{(k-1)} \subset V^{(k)} \subset F^{(k+1)}$. There are precisely two distinct isomorphism classes of lines with respect to the action of $\mathrm{Sp}(W)$, according to whether $\nu|_F$ is isotropic or otherwise. The VMRT $\mathcal{C}_0$ at $0 \in S$ is only *almost* homogeneous with precisely two orbits. Since $S \subset \mathrm{Gr}(k, W) \subset \mathbb{P}^N$ is uniruled by lines, $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ is non-singular. As a rational homogeneous manifold $S_{k,n}$ is of type C_n, corresponding to a short simple root α_k, $1 < k < n$. The tangent bundle of $T_{S_{k,n}}$ has exactly one proper invariant distribution, and we have a decomposition $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. From this description the $\mathrm{SL}(2, \mathbb{C})$-orbit of a highest weight vector of $\mathfrak{g}_1$ gives a highest weight line which is a minimal rational curve. Such a line corresponds to a line $\ell \subset S_{k,n}$ arising from the choice of some $F^{(k+1)} \supset E^{(k)}$ for which $\nu|_F \equiv 0$. When $S = G/P$ is defined by a short simple root, the $\mathrm{SL}(2, \mathbb{C})$-orbit C_s defined by a highest weight vector of $\mathfrak{g}_s$ need not be of degree s. In the case of $s = 2$ for $S = S_{k,n}$, $C = C_2$ is in fact a line, and it corresponds to the generic choice of $F^{(k+1)}$ so that $\nu|_F \not\equiv 0$. From this description the VMRT $\mathcal{C}_0 \subset \mathbb{P}T_0(S_{k,n})$ is linearly non-degenerate, and the question of rigidity under Kähler deformation of symplectic Grassmannians $S_{k,n}, 1 < k < n$ is therefore susceptible to be studied by means of the method of prolongation of infinitesimal automorphisms of VMRTs, as is the case of irreducible Hermitian symmetric spaces of rank ≥ 2.

The proof of deformation rigidity for $S_{k,n}$ and also for the remaining 15-dimensional case of type F_4 were settled along the line of arguments as sketched for the Hermitian symmetric case. For the graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of the model space, the summands $\mathfrak{g}_i$ can be described in terms of conditions on vanishing orders and leading terms of holomorphic vector fields, and the multiplication table of $\mathfrak{g}$ as a Lie algebra can be determined to a good extent from the leading terms. For instance, denoting by $D \subset T_S$ the proper invariant distribution $D \subset T_S$, $\mathfrak{g}_{-1}$ consists of holomorphic vector fields Z vanishing at $0 \in S$ to the order ≥ 1 with leading terms corresponding to $A_Z \in \mathrm{End}(T_0(S))$ satisfying $A|_{D_1} \equiv 0$, and $\mathfrak{g}_{-2} \subset \mathfrak{g}$ is the subspace consisting of holomorphic vector fields vanishing to the order ≥ 2. Nonetheless, as opposed to the Hermitian symmetric case, the structure of the Lie algebra $\mathfrak{g}$ thus obtained is incomplete. In the case of the symplectic Grassmannian $S = S_{k,n}$ the missing element is some symplectic form appearing implicitly in the Frobenius form $\varphi : \Lambda^2 D \to T_S/D$. From $\pi : \mathcal{X} \to \Delta$ we are able to identify the structure of the Lie algebra $\mathfrak{g}^0$ of limit holomorphic vector fields at the central fiber X_0, thereby showing that X_0 is obtained by blowing down some holomorphic fiber bundle, and the final step is achieved by showing that, in the event that there is actually a degeneration of the Lie algebra structure, singularities must occur in the blown-down space, contradicting the starting point that $\pi : \mathcal{X} \to \Delta$ is a regular family.

5. Analytic continuation of VMRT-preserving maps

5.1. Characterization of the tautological foliation under a non-degeneracy condition on the Gauss map. — Let $x \in X$ be a general point and $u \in \mathcal{U}_x$ be a point such that $\kappa := \rho(u) \in \mathcal{K}$ is a standard rational curve. Then, the tangent map τ is a holomorphic immersion at u, and it maps some open neighborhood $\mathbb{W}$ of u in $\mathcal{U}$ biholomorphically onto some locally closed complex submanifold Ω of $\mathbb{P}T_X$. Ω gives the germ of some irreducible branch of $\mathcal{C}$ at $[\alpha]$. Choosing x and $u \in \mathcal{U}_x$ sufficiently general and $\mathbb{W}$ sufficiently small we assume furthermore that $[\alpha] \in \mathcal{C}$ is a smooth point and that Ω is a neighborhood of $[\alpha]$ in $\mathcal{C}$.

On Ω we define a distribution $\mathcal{P}$, as follows. Let $f : \mathbb{P}^1 \to \mathcal{U}$ be a parametrization of κ. The base point $x \in X$ is a smooth point of the support $C := \mu(\rho^{-1}(\kappa))$ of the standard rational curve κ. The decomposition $f^*T_X \cong \mathcal{O}(2) \oplus (\mathcal{O}(1))^p \oplus \mathcal{O}^q$ over $\mathbb{P}^1$ gives a filtration $T_{\mathbb{P}^1} \subset Q \subset f^*T_X$ of f^*T_X over $\mathbb{P}^1$, where $Q = \mathcal{O}(2) \oplus (\mathcal{O}(1))^p$ is the positive part of f^*T_X, which is well-defined since $Q \otimes \mathcal{O}(-1) \subset f^*T_X \otimes \mathcal{O}(-1)$ is the vector subbundle spanned by global sections. At the point $x = f(0)$ we have correspondingly a filtration $T_x(C) \subset P_x \subset T_x(X)$, where $P_x = df(Q_0)$. Define now $\mathcal{P}_{[\alpha]} \subset T_{[\alpha]}(\mathcal{C})$ to be the vector subspace consisting of all tangent vectors η such that $d\pi(\eta) \in P_x$. The tangent vector η is equivalently the image under $d\tau$ of some $\overline{\sigma}$, where $\sigma \in H^0(\mathbb{P}^1, f^*T_X)$, and $\overline{\sigma} := \sigma \bmod df(H^0(\mathbb{P}^1, T_{\mathbb{P}^1} \otimes \mathcal{I}_0))$. For the universal family $\rho : \mathcal{U} \to \mathcal{K}$ we have $d\rho(\overline{\sigma}) = \sigma \bmod df(H^0(\mathbb{P}^1, T_{\mathbb{P}^1})$. Equivalently, writing $\widehat{\rho} := \rho \circ \tau^{-1}$ over Ω, where $\tau^{-1} : \Omega \to \mathcal{W}$, we have $d\widehat{\rho}(\eta) = \sigma \bmod df(H^0(\mathbb{P}^1, T_{\mathbb{P}^1}))$. The assumption that $d\pi(\eta) \in P_x$ means precisely that $\sigma(0) \in Q_0$, thus $\sigma' := \sigma \bmod Q \in H^0(\mathbb{P}^1, \mathcal{O}^q)$ must vanish at 0 and hence on all of $\mathbb{P}^1$, showing that $\sigma \in H^0(\mathbb{P}^1, Q)$. On an open neighborhood $\mathbb{U}$ of κ in $\mathcal{K}$ consisting solely of standard rational curves we define a distribution $\mathcal{D} \subset T_{\mathcal{K}}|_{\mathbb{U}}$ by setting $\mathcal{D}_\kappa := H^0(\mathbb{P}^1, Q) \bmod df(H^0(\mathbb{P}^1, T_{\mathbb{P}^1})) \cong \mathbb{C}^{2p}$. Then, for $\xi \in T_{[\alpha]}$ we have $d\widehat{\rho}(\xi) \in \mathcal{D}_\kappa$ if and only if $d\pi(\xi) \in P_x$, hence $\mathcal{P}_{[\alpha]} = (d\widehat{\rho})^{-1}(\mathcal{D}_\kappa)$. Finally there is a 1-dimensional distribution underlying the tautological foliation $\mathcal{F}$ on Ω which will be denoted by the same symbol $\mathcal{F}$. Thus, $\mathcal{F}_{[\alpha]} := T_{[\alpha]}(\widehat{\rho}^{\,-1}(\kappa))$.

To relate the distributions $\mathcal{F}, \mathcal{P}$ on Ω and the distribution $\mathcal{D}$ on $\mathcal{G}$ we recall the notion of the Cauchy characteristic of a distribution. Given a complex manifold M and a holomorphic distribution $E \subset T_M$ and denoting by $\mathcal{E}$ the corresponding locally free sheaf of germs of holomorphic sections of E, then $Ch(\mathcal{E}) \subset \mathcal{E}$ is the subsheaf consisting of germs of holomorphic sections ζ such that $[\zeta, \mathcal{E}] \subset \mathcal{E}$. Thus, the Cauchy characteristic $Ch(\mathcal{E}) = \mathcal{E}$ if and only if $E \subset T_M$ is integrable. Outside an analytic subvariety of codimension ≥ 2 the Cauchy characteristic is locally free, and from now on we will make no distinction between a distribution and its associated locally free sheaf, and think of the Cauchy characteristic as a distribution defined outside an analytic subvariety of codimension ≥ 2. To proceed we note

***Lemma** 7.* — *Let $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$ be Euclidean domains, and $\lambda : U \times V \to V$ be the canonical projection. Let $S \subset T_V$ be a holomorphic distribution and $G := (d\lambda)^{-1}(S)$. Write $H \subset T_{U\times V}$ for the distribution corresponding to the foliation by fibers of λ, i.e., $H = (d\lambda)^{-1}(0)$. Then, $H \subset Ch(G)$.*

At a general point of the fibered space $\pi : \mathcal{C} \to X$ of VMRTs, *a priori* there can be more than one tautological foliation coming from different sets of families of local holomorphic curves. The question whether the tangent map τ_x is birational at a general point $x \in X$ has to do with uniqueness of the tautological foliation. Such a uniqueness result would follow if the tautological foliation $\mathcal{F}$ can be characterized as in fact the Cauchy characteristic of $\mathcal{P}$ at a general point of $\mathcal{C}$. We have proven that $\mathcal{F} \subset Ch(\mathcal{P})$. For the inverse inclusion we impose an additional assumption on the Gauss map on the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at a general point, a condition that is always satisfied whenever the $\mathcal{C}_x$ is nonsingular and non-linear.

***Proposition** 5.* — *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component, and $\pi : \mathcal{C} \to \mathcal{X}$ be the associated fibered space of* VMRTs. *Let $\Omega \subset \mathcal{C}$ be a connected nonempty open subset consisting of nonsingular points on which both a tautological foliation $\mathcal{F}$ by standard $\mathcal{K}$-curves and hence the corresponding distribution $\mathcal{P}$ are defined. Suppose at a general point $[\alpha] \in \Omega$, $\pi([\alpha]) := x$, the Gauss map of $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is a holomorphic immersion at $[\alpha]$. Then, $\mathcal{F} = Ch(\mathcal{P})$.*

Proof. — In what follows we denote by $\widetilde{\Omega} = \pi^{-1}(\Omega) \subset \widetilde{\mathcal{C}}$, $\widetilde{\mathcal{P}} = (d\pi)^{-1}(\mathcal{P})$, etc., by lifting to homogenizations. At a general point $\alpha \in \widetilde{\Omega}$ choose local holomorphic coordinates $(z_1, \cdots, z_n)$ at $x = \widetilde{\pi}(\alpha)$ and corresponding fiber coordinates $(w_1, \cdots, w_n)$ on T_X in a neighborhood of u. Suppose $s := \sum g^i \frac{\partial}{\partial z_i} + \sum h^j \frac{\partial}{\partial w_j}$ is a germ of holomorphic section of $\widetilde{\mathcal{P}}$ at u such that $[s, \eta]$ is a germ of $\widetilde{\mathcal{P}}$ at α. Denote by $\mathcal{V} \subset \mathcal{P}$ the subbundle of vertical vectors, i.e., of vectors tangent to the fibers $\mathcal{C}_y$ of $\pi|_\Omega$. Now for η an arbitrary germ of vertical holomorphic vector field at α we have

$$\left[\sum g^i \frac{\partial}{\partial z_i} + \sum h^j \frac{\partial}{\partial w_j}, \sum \eta^k \frac{\partial}{\partial w_k}\right] = \sum \eta^k \frac{\partial g^i}{\partial w_k} \frac{\partial}{\partial z_i} \bmod \widetilde{\mathcal{V}}\ . \qquad (*)$$

The condition that $[s, \eta]$ takes values in $\widetilde{\mathcal{P}}$ implies that $\sum \eta^k \frac{\partial g^i}{\partial w_k} \frac{\partial}{\partial w_i} \in P_\alpha$. Since the germ of vertical vector field η is arbitrary, it follows that $\sum g^i \frac{\partial}{\partial w_i}(\alpha) \in \mathrm{Ker}(\widetilde{\sigma}_\alpha) = \mathbb{C}\alpha$. Thus, $s = \lambda \sum w^i \frac{\partial}{\partial z_i} + \sum h^j \frac{\partial}{\partial w_j}$ for some λ holomorphic. Suppose the holomorphic vector field $\sum w^i \frac{\partial}{\partial z_i} + \sum r^j \frac{\partial}{\partial w_j}$ takes values in $\widetilde{\mathcal{F}}$. Since $\widetilde{\mathcal{F}} \subset Ch(\widetilde{\mathcal{P}})$, comparing s

with $\widetilde{\mathcal{F}}$ we conclude that for $\xi := \sum(h^j - \lambda r^j)\frac{\partial}{\partial w_j} \in Ch(\widetilde{\mathcal{P}})$, and to prove Proposition 5 it remains to show that ξ is pointwise a multiple of the Euler vector field $\sum w^j \frac{\partial}{\partial w_j}$ (which descends to 0 when we project from $\widetilde{\mathcal{C}}$ to $\mathcal{C}$). Write $\xi^j := h^j - \lambda r^j$. By the same formula $(*)$ above for Lie bracket, replacing η^k by ξ^k and letting $\sum g^i \frac{\partial}{\partial z_i} + \sum h^j \frac{\partial}{\partial w_j}$ now stand for an arbitrary germ of $\widetilde{\mathcal{P}}$-valued holomorphic vector field at α we conclude that $\sum \xi^k \frac{\partial g^i}{\partial w_k}\frac{\partial}{\partial w_i} \in P_\alpha$ for any choice of (g^i) such that $\sum g^i \frac{\partial}{\partial w_i}$ is a $\widetilde{\mathcal{V}}$-valued germ of holomorphic vector field at α. Hence $\xi \in \mathrm{Ker}(\sigma_\alpha) = \mathbb{C}\alpha$, as desired. □

5.2. Birationality of the tangent map and Cartan-Fubini extension. — The characterization of the tautological foliation under the Gauss map condition (†) in [(4.1), Propoition 5] implies the birationality of the tangent map $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ under the same condition (Hwang-Mok [**17**], 1999). Kebekus [**26**] (2002) proved that any $\mathcal{K}$-curve marked at a general point x is immersed at the marking, and deduced

***Theorem 7* (Kebekus [26]).** — *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component. Then, at a general point $x \in X$, the tangent map $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(X)$ is a finite holomorphic map.*

Together with Theorem 7 one obtains a proof of [(2.4), Theorem 1], the structure theorem on the tangent map and VMRTs stating that the tangent map is a birational finite holomorphic map at a general point, under the additional Gauss map condition (†). To remove (†) the first question is to characterize the case where $\mathcal{C}_x = \mathbb{P}T_x(X)$. This was obtained by Cho-Miyaoka-Shepherd-Barron ([**2**], 2002) by a method involving the holomorphicity of the tangent map made possible by Kebekus [**26**].

***Theorem 8* (Cho-Miyaoka-Shepherd-Barron [3]).** — *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component,* $\dim(X) := n$. *Suppose at a general point $x \in X$ the associated variety of minimal rational tangents $\mathcal{C}_x$ is the same as $\mathbb{P}T_x(X)$. Then, X is biholomorphic to $\mathbb{P}^n$.*

To prove [(2.4), Theorem 1] in its full generality, we considered in Hwang-Mok [**21**] (2004) the integrable distribution $Ch(\mathcal{P})$ for the distribution $\mathcal{P}$ defined in (4.1). We showed using [**26**] and [**3**] that a local leaf of $Ch(\mathcal{P})$ is the projectivized tangent bundle of a locally closed complex submanifold on X which extends to an immersed projective space, and deduce from there the birationality of the tangent map at a general point, leading to a proof of Theorem 1.

The statement of birationality of the tangent map leads to a method of analytic continuation, which we call Cartan-Fubini extension, for local VMRT-preserving biholomorphic maps. In 2004 we proved

***Theorem 9* (Hwang-Mok [21]).** — *Let X and X' be Fano manifolds of Picard number 1 with minimal rational components. Assume that at a general point $x \in X$ the variety of minimal rational tangents $\mathcal{C}_x(X)$ of X is non-linear and of positive dimension. Let $f : U \to U'$ be a biholomorphic map from an open connected subset $U \subset X$ onto $U' \subset X'$. Suppose the differential df sends each irreducible component of $\mathcal{C}(X)|_U$ to an irreducible component of $\mathcal{C}(X')|_{U'}$ biholomorphically. Then, f extends to a biholomorphic map $F : X \to X'$.*

Sketch of proof. — In the case of an irreducible Hermitian symmetric space S of the compact type and of rank ≥ 2, Cartan-Fubini extension is equivalent to Ochiai's Theorem, and in (4.2) we sketched a proof using VMRTs. The analogue of [(4.2), Lemma 4] for Theorem 9 under the additional Gauss map condition (†) is given by [(5.1), Proposition 5]. In Hwang-Mok [**18**] we proved Theorem 9 under the condition (†), and in [**21**] the latter condition was removed starting with an extension of the birationality result for non-linear VMRTs. To explain the special case in [**18**], along the line of argument of (4.2) for a proof of Ochiai's Theorem we can likewise pass to the moduli space $\mathcal{K}$ resp. $\mathcal{K}'$ of minimal rational curves on X resp. X'. Picking a base point $x \in X$, and denoting by $\mathcal{Q}_x \subset \mathcal{K}$ the subspace of minimal rational curves passing through x, $f : U \cong U'$ extends by Proposition 5 to some holomorphic map $f^\sharp$ on some neighborhood $\mathbb{U}$ of $\mathcal{Q}_x$ in $\mathcal{K}$ as in (4.2). In the general case we do not however have the Hartogs-type extension theorem as used in Mok-Tsai [MT] to extend $f^\sharp$ meromorphically to $\mathcal{K}$. Instead, we developed in [**18**] a method of parametrized analytic continuation along minimal rational curves. Let $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$ be the universal family of $(X, \mathcal{K})$. Fix a standard $\mathcal{K}$-curve $\ell \in \mathcal{K}$ passing through $x \in U$. We have a map $\lambda := f^\sharp \circ \rho \circ \tau^{-1}$ which is defined on some arbitrarily small neighborhood Ω of the tautological lifting $\widehat{\ell}$ of ℓ in $\mathcal{C}$. To extend f meromorphically on a neighborhood of $\ell \in X$ by the argument in (4.2) in which a point y is regarded as the intersection of minimal rational curves passing through y, it is not necessary to have λ defined on all of $\mathcal{C}|_\ell$. It suffices to have λ defined on the arbitrarily small neighborhood Ω of $\widehat{\ell}$, and the upshot is that we can do meromorphic extension of f and $f^\sharp$ simultaneously along a standard $\mathcal{K}$-curve issuing from U. Each general point of X is accessible from U by a finite chain of standard $\mathcal{K}$-curves. Since X is of Picard number 1, the inaccessible points can be cut down to codimension ≥ 2. A major difficulty in completing the proof after meromorphic extension along standard $\mathcal{K}$-curves lies in the lack of univalence, and, after proving univalence, there remains the difficulty due to singularities of the extended map. Overcoming these difficulties necessitates the use of the deformation theory of rational curves, and for the latter difficulty we need to further use the Fano property of both X and X', which gives rise to projective embeddings using positive powers of the anti-canonical line bundle.

The proof of Theorem 9 in the general case requires a combination of [**18**] and the use of integral manifolds of $Ch(\mathcal{P})$ as mentioned in relation to Theorem 8. □

The method of analytic continuation on VMRT-preserving maps makes explicit use of the geometry arising from minimal rational curves. From the perspective of Several Complex Variables, it would be of interest to prove an extension result solely basing on the neighborhood structure of the cycles $\mathcal{Q}_x \subset \mathcal{K}$. Examination of the Hermitian symmetric case suggests that in general one can hope for constructing a fundamental system of pseudoconcave neighborhoods $\mathcal{Q}_x$, thereby guaranteeing meromorphic extension of $f^\sharp$ and hence of f from methods in Several Complex Variables. In this direction the following formulation in a special case is of independent interest.

***Conjecture** 5.* — *Let $(X, \mathcal{K})$ be a Fano manifold of Picard number 1 equipped with a minimal rational component. Assume that at a general point $x \in X$ the moduli space $\mathcal{K}_x$ of $\mathcal{K}$-curves marked at x is irreducible and non-linear, and that the tangent map $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ is a biholomorphism onto $\mathcal{C}_x$, so that, denoting by $p : \mathcal{K}_x \to \mathcal{K}$ the canonical map, the image $\mathcal{Q}_x = p(\mathcal{K}_x)$ is nonsingular. Let $\mathbb{U} \supset \mathcal{Q}_x$ be any connected open neighborhood of $\mathcal{Q}_x$ in $\mathcal{K}$. Then, any meromorphic function on $\mathbb{U}$ extends to a meromorphic function on $\mathcal{K}$.*

5.3. The Lazarsfeld Problem and other applications of Cartan-Fubini extension. — As an application of the Cartan-Fubini extension on uniruled projective manifolds with non-linear VMRTs ([(5.2), Theorem 9]) we have the following result on the local rigidity of generically finite surjective holomorphic maps of a fixed projective manifold X' onto a Fano manifold $(X, \mathcal{K})$ of Picard number 1 equipped with a minimal rational component with non-linear VMRTs. We have

***Theorem 10* (Hwang-Mok [21]).** — *Let $\pi : \mathcal{X} \to \Delta := \{t \in \mathbb{C}, |t| < 1\}$ be a regular family of Fano manifolds of Picard number 1 so that $X_0 = \pi^{-1}(0)$ has a minimal rational component with non-linear varieties of minimal rational tangents. For a given projective manifold Y, suppose there exists a surjective holomorphic map $f : \mathcal{Y} = Y \times \Delta \to \mathcal{X}$ respecting the projections to Δ so that $f_t : Y \to X_t$ is a generically finite for each $t \in \Delta$. Then, there exists $\epsilon > 0$ and a holomorphic family of biholomorphic maps $\Phi_t : X_0 \to X_t$ for $|t| < \epsilon$, satisfying $\Phi_0 = \mathrm{id}$ and $f_t = \Phi_t \circ f_0$.*

Sketch of proof. — Fix a minimal rational component $\mathcal{K}_0$ on X_0 with non-linear VMRTs. To simplify notations we assume minimal rational curves to be embedded. Let $\ell_0 \subset X_0$ be a $\mathcal{K}_0$-curve. ℓ_0 is also free on $\mathcal{X}$ since $T_{\mathcal{X}}|_{\ell_0} = T_{X_0}|_{\ell_0} \oplus \mathcal{O}$. Consider the space $\mathcal{K}$ of free rational curves on $\mathcal{X}$ obtained by deforming some ℓ_0 in $\mathcal{X}$. Any $\ell \in \mathcal{K}$ must lie on some $X_t, t \in \Delta$. We may think of $(\mathcal{X}, \mathcal{K})$ as a holomorphic family of $(X_t, \mathcal{K}_t)$ fibered over Δ. To simplify the discussion we assume that the VMRTs are irreducible at a general point of $X = X_0$. Shrinking Δ around 0 if necessary we may assume that the VMRT at a general point of X_t remains irreducible.

In Hwang-Mok [**16**] we introduced the notion of varieties of distinguished tangents on a projective manifold Y (cf. Hwang-Mok [**17**], §5) which generalizes the notion of VMRTs. Let $y \in Y$ be a very general point, i.e., a point outside some countable union of proper subvarieties. Consider an irreducible component $\mathcal{M}$ of the Chow space of curves on Y, and denote by $\mathcal{M}_y \subset \mathcal{M}$ the subvariety corresponding to curves through y. For curves belonging to $\mathcal{M}_y$ and smooth at y we have the notion of the tangent map. The rank on the tangent map leads to stratifications of $\mathcal{M}_y$ such that the tangent map is of constant rank on each stratum. Fix a uniruled projective manifold $(X, \mathcal{K})$ equipped with a minimal rational component and denote by $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ the VMRT at a general point $x \in X$. For a generically finite surjective holomorphic map $h : Y \to X$ and for a very general point $y \in Y$ such that $dh(y)$ is of maximal rank at y, $[dh(y)]^{-1}(\mathcal{C}_{h(y)})$ is a variety of distinguished tangents at y.

If we take $y \in Y$ to be a very general point of Y, a variety of distinguished tangents at y is the closure under the tangent map of a stratum of $\mathcal{M}_y$. Since there are only countably many irreducible components of the Chow space of curves on Y, from the construction by stratification there are at most countably many varieties of distinguished tangents passing through y. In the context of Theorem 10, choose a connected open subset $U \subset Y$ such that f_t is a biholomorphism of U onto $V_t \subset X_t$. Let $y \in Y$ be a very general point lying on U. We have a holomorphic family of VMRTs $\mathcal{C}_{f_t(y)}(X_t)$. Then, for each $t \in \Delta$, $f_t^{-1}(\mathcal{C}_{f_t(y))}(X_t)) := \mathcal{D}_t \subset \mathbb{P}T_y(Y)$ is a variety of distinguished tangent at y. By the countability of the space of varieties of distinguished tangents at y it follows that $\mathcal{D}_t$ is actually *independent* of t. There is an obvious identification $\varphi_t : V_t \cong V_0$ given by $\varphi_t = f_t \circ f_0^{-1}$, and we have $[d\varphi_t](\mathcal{C}_{f(t)}(X_t)) = \mathcal{C}_{f(0)}(X_0)$. Thus φ_t is VMRT-preserving, and by Cartan-Fubini extension as given in Theorem 9, φ_t extends to a biholomorphism $\Phi_t : X_0 \cong X_t$ such that $f_0 = \Phi_t \circ f_t$. □

In relation to finite holomorphic maps on rational homogeneous manifolds $S = G/P$, Lazarsfeld [**32**] proved that for any finite holomorphic map $f : \mathbb{P}^n \to X$ from the complex projective space onto a projective manifold X, X must itself be biholomorphic to $\mathbb{P}^n$. He raised the question of characterizing finite holomorphic maps $f : S \to X$ from a rational homogeneous manifold S of Picard number 1 onto a projective manifold. Hwang-Mok [**16**] solved the problem in 1999, and obtained [**21**] (2004) a new proof using Cartan-Fubini extension as given in Theorem 10.

Theorem 11 **(Hwang-Mok [16], [21]).** — *Let $S = G/P$ be a rational homogeneous manifold of Picard number 1. Let $f : S \to X$ be a nonconstant surjective holomorphic map onto a projective manifold X. Then either $X \cong \mathbb{P}^n$, where $n = \dim(S)$; or f is a biholomorphism.*

In the first proof in [**16**] we considered intertwining maps of $f : S \to X$, as follows. Suppose $f : S \to X$ is not a biholomorphism and $X \not\cong \mathbb{P}^n$, and write s for the sheeting number of the map. The image manifold X is necessarily Fano. Equip X with a minimal rational component $\mathcal{K}$. Denote by $\mathcal{C}_x$ the variety of $\mathcal{K}$-tangents at x and assume known in the ensuing discussion that $\mathcal{C}_x \neq \mathbb{P}T_x(X)$ at a general point. Let $x \in X$ be outside the branch locus of f, and let V be a sufficiently small connected open neighborhood of x in X such that $f^{-1}(V)$ decomposes into a union of s open subsets $U_i, 1 \leq i \leq s,$, where $f_i := f|_{U_i} : U_i \to V$ is a biholomorphism for each i. For $i \neq j$ let $\varphi : U_i \to U_j$ be defined by $\varphi(z) = f_j^{-1} \circ f_i$. Consider the pull-back $\mathcal{D} := [df]^{-1}(\mathcal{C}|_V)$. We have tautologically $[d\varphi] : \mathcal{D}|_{U_i} \cong \mathcal{D}|_{U_j}$. For a general point $s \in S$, $\mathcal{D}_s$ is a variety of distinguished tangents. At any such point $\mathcal{D}_s$ is shown to be invariant under the isotropy subgroup $P_s \subset G$ at s. For instance, in the Hermitian symmetric case this implies that $\mathcal{D}_s$ must be one of the finitely many proper P_s-invariant subsets defined in terms of ranks of tangent vectors, $\mathcal{D}$ is actually G-invariant, and the condition $[d\varphi] : \mathcal{D}|_{U_i} \cong \mathcal{D}|_{U_j}$ forces $[d\varphi]$ to be VMRT-preserving, since at $s \in S$ the variety of minimal rational tangents $\mathcal{C}_s(S)$ is the most singular P_s-invariant stratum of $\mathcal{D}_s$. In this case by Ochiai's Theorem [**47**] the intertwining map must extend to an automorphism of S, and that is enough to force a contradiction. In the general case there may be continuous families of P_s-orbits, but using the fact that there are at most countably many distinct varieties of distinguished tangents at $s \in S$, it remains true that $\mathcal{D}$ is G-invariant. This leads to the conclusion that either $\mathcal{D}_s \subset \mathbb{P}T_s(S)$ is linearly non-degenerate, in which case we proved using Hwang-Mok [**14**] that S must be Hermitian symmetric, or $\mathcal{D}_s \subset \mathbb{P}T_s(S)$ is linearly degenerate, and the intertwining map φ must preserve some proper G-invariant distribution, after which we can work with results of Yamaguchi [**51**] to show that φ extends to $\Phi \in \mathrm{Aut}(S)$ to reach a contradiction. This line of argument has been recently generalized to the case of rational homogeneous spaces of Picard number ≥ 2, leading to a solution to a generalized Lazarsfeld Problem.

***Theorem 12* (Lau [31]).** — *Let G be a simple complex Lie group and $Q \subset G$ be a parabolic subgroup. Denote by $S = G/Q$ the corresponding rational homogeneous manifold,* $\dim(S) = n$. *Let $f : S \to X$ be a surjective holomorphic map from S onto a projective manifold X. Then one of the following holds:* (1) *f is a biholomorphism;* (2) *$f : S \to X$ is a finite map and X is the projective space $\mathbb{P}^n$;* (3) *there exists a parabolic subgroup Q' of G containing Q as a proper subgroup such that f factors through a finite map $g : G/Q' \to X$.*

The generalized Lazarsfeld Problem for $S = G/Q$ of Picard number ≥ 2 leads to a Fano manifold $(X, \mathcal{K})$ equipped with a minimal rational component and admitting the structure of a holomorphically fibered space $\lambda : X \to B$ such that the $\mathcal{K}$-curves lie on the fibers of λ. The principal algebro-geometric difficulty, solved in [**31**], is to produce a minimal rational component $\mathcal{K}'$ such that the $\mathcal{K}'$-curves are transversal to the fibration λ. After that Lau made use of multi-graded differential systems using Yamaguchi [**51**]. As in [**16**] the proof involves a substantial amount of Lie theory.

As far as the original Lazarsfeld Problem is concerned, Hwang-Mok [**21**] gave a new proof which frees the solution from Lie theory, deriving Theorem 11 as a consequence of Theorem 10, as follows. Let $S = G/P$ be an n-dimensional rational homogeneous manifold of Picard number 1 and $f : S \to X$ be a generically finite surjective holomorphic map onto a projective manifold X, which is necessarily Fano, such that $X \not\cong \mathbb{P}^n$ and f is not a biholomorphism. Equip X with a minimal rational component $\mathcal{K}$ and suppose that the associated VMRT at a general point is non-linear. Let θ be a holomorphic vector field on S and $\Theta_t = \exp(t\theta)$ be a holomorphic 1-parameter group of automorphism of S. Write $f_t = f \circ \Theta_t$. Then, applying the local rigidity result Theorem 11 we have $f_t = \Phi_t \circ f$. Thus $df_t(\eta) = 0$ whenever $df(\eta) = 0$. Thus the non-empty ramification divisor R of $f = f_0$ remains the ramification divisor of f_t for $t \neq 0$. On the other hand from the definition $f_t = f \circ \Theta_t$ it follows that the ramification divisor of f_t is $\Theta_{-t}(R)$, and a contradiction is obtained when we choose the vector field θ not to vanish identically on R. Finally, it remains to rule out the possibility that the VMRT of $(X, \mathcal{K})$ is linear at a general point $x \in X$. Choose a general point $x \in X$ lying outside the branching locus of f, $s \in S$ such that $f(s) = x$. An irreducible component of $[df]^{-1}(\mathcal{C}_x)$ then gives a P_s-invariant projective linear subspace of $\mathbb{P}T_s(S)$, giving rise to one of the finitely many G-invariant holomorphic distributions on S. D is non-integrable since S is of Picard number 1. On the other hand in the case of linear VMRTs on X an irreducible component of $\mathcal{C}$ over a sufficiently small open subset corresponds to an integrable distribution, a contradiction.

It would be interesting to give a proof of Theorem 12 along the line of Cartan-Fubini extension for special classes of Fano manifolds of Picard number ≥ 2.

6. Parallel transport of the second fundamental form

6.1. VMRTs in a differential-geometric context-parallel transport in the solution of the Generalized Frankel Conjecture. — In Algebraic Geometry Hartshorne conjectured that over an algebraically closed field a projective manifold with ample tangent bundle is isomorphic to the projective space. The conjecture was solved by Mori [**45**] (1979) by proving an existence theorem on rational curves using methods of characteristic $p > 0$, and the deformation theory of rational curves. In the context of Kähler Geometry, Frankel conjectured that a compact Kähler manifold of positive holomorphic bisectional curvature is biholomorphic to the complex projective space. The conjecture was resolved in the affirmative by the method of stable harmonic maps by Siu-Yau [**50**] (1980) who further formulated the conjecture that a compact Kähler manifold of nonnegative holomorphic bisectional curvature is locally symmetric. The latter conjecture, commonly called the Generalized Frankel Conjecture, was resolved in the affirmative by Mok [**39**] (1988).

Mok [**39**] made use of the Kähler Ricci flow, proving that nonnegativity of holomorphic bisectional curvature is preserved under the flow for the evolved metric $g_t, t > 0$. From earlier reduction of the problem, to confirm the Generalized Frankel Conjecture it suffices to consider the case where we have a compact Kähler manifold (X, g) of nonnegative holomorphic bisectional curvature and of positive Ricci curvature at some point such that furthermore $b_2(X) = 1$. For the latter class of (X, g), the evolved Kähler metric (X, g_t) is shown to be of positive Ricci curvature. Thus, X is Fano and hence uniruled by Miyaoka-Mori [**38**]. Since (X, g) is of nonnegative holomorphic bisectional curvature, the pull-back of its tangent bundle by any $f : \mathbb{P}^1 \to X$ is nonnegative, hence every rational curve on X is free. In [**39**] we studied minimal rational curves on X and the associated varieties of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ (although the terminology was not used there). We proved that there are the following alternatives on the evolved metrics g_t defined for $t > 0$ sufficiently small. For such $t > 0$, either (X, g_t) is of positive holomorphic bisectional curvature, or (X, g_t) admits non-trivial zeros of holomorphic bisectional curvature at any point of X. Write n for $\dim(X)$. If the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is of dimension $p < n - 1$ at a general point, we showed that $\mathcal{C}$ is invariant under parallel transport of (X, g_t). If however $\mathcal{C}_x$ agrees with $\mathbb{P}T_x(X)$, we showed that there exists a hypersurface $\mathcal{S} \subset \mathbb{P}T_X$ such that $\mathcal{S}$ is invariant under parallel transport of (X, g_t). In either case we applied Berger's Theorem which characterizes Riemannian locally symmetric spaces by the fact that at any point there exists some proper subset of the unit sphere invariant under parallel transport. Thus (X, g_t) is an irreducible Hermitian symmetric space of the compact type for $t > 0$ and hence for $t = 0; g_0 = g$. More precisely we have proved

Theorem 13 **(Mok [39]).** — *Let (X, g) be a compact Kähler manifold of nonnegative holomorphic bisectional curvature and of positive Ricci curvature at some point. Assume that X is of Picard number 1. Then, either X is biholomorphically equivalent to the complex projective space, or (X, g) is biholomorphically isometric to an irreducible Hermitian symmetric space S of rank ≥ 2.*

On an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2, the fibered space $\pi : \mathcal{C} \to S$ is invariant under parallel transport with respect to *any* choice of a canonical Kähler-Einstein metric, an elementary fact that follows from the parallelism of the Riemannian curvature tensor. Theorem 13 says in particular that on S this basic fact can be derived from curvature properties. In the negative direction, Berger's Theorem implies that for a rational homogeneous manifold $S = G/P$ of Picard number 1 which is not isomorphic to a Hermitian symmetric space, the VMRTs are not invariant under parallel transport. In an algebro-geometric context it remains interesting to introduce some algebraic notion of parallel transport applicable to any uniruled projective manifold $(X, \mathcal{K})$ equipped with a minimal rational component. A related problem is the Campana-Peternell Conjecture, which is a form of Generalized Hartshorne Conjecture (cf. (6.4)). Here the principal geometric problem is whether the notion of invariance of VMRTs under some restricted form of parallel transport is sufficient to characterize rational homogeneous manifolds $S = G/P$ of Picard number 1 by means of some algebro-geometric condition of nonnegativity on the tangent bundle. Such an approach in a very special situation has been established for Fano manifolds of Picard number 1 with nef tangent bundle and 1-dimensional VMRTs by Mok [**41**] (2001) and Hwang [**13**] (2007).

6.2. Propagation of the second fundamental form along a standard rational curve. — Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component, $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$ be the universal family of $\mathcal{K}$-curves and $\pi : \mathcal{C} \to X$ be the fibered space of varieties of minimal rational tangents. Let $B \subset X$ be the largest subvariety, necessarily of codimension ≥ 2, such that $\pi|_{X-B} : \mathcal{C}|_{X-B} \to X - B$ is flat. Let $f : \mathbb{P}^1 \to X$ be a parametrized standard rational curve, $f(\mathbb{P}^1) := C$, such that $C \subset X - B$. C lifts canonically to $\widetilde{C} \subset \mathcal{U}$, whose image under the tangent map gives the tautological lifting $\widehat{C} \subset \mathcal{C}$. At each of the finitely many points $\widetilde{x}_k$ of $\widetilde{C} \cap \mu^{-1}(x)$ there is an open neighborhood U_k such that τ_x embeds U_k holomorphically onto a smooth submanifold $\mathcal{C}_x^k$, which is the germ of some irreducible component of $\mathcal{C}_x$ at $[\alpha_k] = \tau_x(\widetilde{x}_k)$. In what follows $\widehat{C}$ will mean the pull-back of the tautological lifting of C to $f^*\mathcal{C}$, so that $\widehat{C}$ is smooth. For $t \in \mathbb{P}^1$ we write $\mathcal{C}_t$ for $(f^*\mathcal{C})_t$, $[\alpha(t)]$ for $\widehat{C} \cap \mathcal{C}_t$, and V_t for $f^*T_{f(t)}(X)$. We have $\mathcal{C}_t \subset \mathbb{P}V_t$. For every $t \in \mathbb{P}^1$ we have a germ of smooth projective submanifold $\mathcal{C}_t^o \subset \mathcal{C}_t \subset \mathbb{P}V_t$ at $[\alpha(t)]$ corresponding to one of the germs $\mathcal{C}_x^k, x = f(t)$, chosen in such a way that the union of $\mathcal{C}_t^o$ is a germ of complex submanifold along the smooth curve $\widetilde{C} \subset \mathbb{P}V$. Write $T_{[\alpha(t)]}$ for $T_{[\alpha(t)]}(\mathcal{C}_t^o)$. In Mok [**14**] (§3.2, p. 2651ff.) we introduced implicitly the notion of parallel transport of the second fundamental form along the tautological lifting $\widehat{C}$ of a standard rational curve C. By this we mean that the second fundamental form can be interpreted in a natural way as a holomorphic section of a vector bundle which is trivial over $\widehat{C}$. We formulate the notion of isomorphisms of second fundamental forms and the result on parallel transport, as follows.

Definition 4. — *Let V and V' be two complex Euclidean spaces of the same dimension, and $A \subset \mathbb{P}V$, $A' \subset \mathbb{P}V'$ be two local complex submanifolds of the same dimension. Let $a \in A$, $a = [\alpha]$; $a' \in A'$, $a' = [\alpha']$. Write $T_a(A) = \mathbb{P}E/\mathbb{C}\alpha$ (resp. $T_a(A') = \mathbb{P}E'/\mathbb{C}\alpha'$*

where $E \subset V$ (resp. $E' \subset V'$) is a vector subspace containing α (resp. α'). We say that the second fundamental form σ_a of $A \subset \mathbb{P}V$ at $a \in A$ is isomorphic to the second fundamental form $\sigma_{a'}$ of $A' \subset \mathbb{P}V'$ at $a' \in A'$ if and only if there exists a linear isomorphism $\varphi : V \cong V'$ such that $\varphi(\alpha) = \alpha'$, $\varphi(E) = E'$, and such that φ satisfies the following additional property ($\sharp$)

($\sharp$) *Let $\overline{\varphi} : V/E \to V'/E'$ be the linear map induced by φ, $\varphi(E) = E'$, and denote by $\widetilde{\sigma}_\alpha$ (resp. $\widetilde{\sigma}_{\alpha'}$) the second fundamental form of $\widetilde{A}$ at α (resp. of $\widetilde{A}'$ at α'). Then, for any $\xi, \eta \in E$ we have $\widetilde{\sigma}_{\alpha'}\big(\varphi(\xi), \varphi(\eta)\big) = \overline{\varphi}\big(\widetilde{\sigma}_\alpha(\xi, \eta)\big)$.*

Proposition 6. — *For every $t \in \mathbb{P}^1$, denote by $\sigma_{[\alpha(t)]} : S^2T_{[\alpha(t)]} \to N_{\mathcal{C}_t^o|\mathbb{P}V_t,[\alpha(t)]}$ the second fundamental form of $\mathcal{C}_t^o \subset \mathbb{P}V_t$ at $[\alpha(t)]$. Then, for $t_1, t_2 \in \mathbb{P}^1$, $\sigma_{[\alpha(t_1)]}$ is isomorphic to $\sigma_{[\alpha(t_2)]}$.*

Proof. — Write $\nu : \mathbb{P}V \to \mathbb{P}^1$ for the canonical projection, where $V = f^*T_X$, and T_ν for its relative tangent bundle. Write $\lambda = \nu|_{f^*\mathcal{C}}$, and recall that $T_{[\alpha(t)]} = T_{[\alpha(t)]}(\mathcal{C}_t^o)$. Write $N_{[\alpha(t)]} = T_{\nu,[\alpha(t)]}/T_{[\alpha(t)]}$. Putting together $T_{[\alpha(t)]}$, $t \in \mathbb{P}^1$, we obtain a holomorphic vector bundle $T_\lambda|_{\widehat{C}}$ on $\widehat{C}$. Likewise, putting together $N_{[\alpha(t)]}$, $t \in \mathbb{P}^1$, we obtain a holomorphic vector bundle $N_\lambda|_{\widehat{C}}$ on $\widehat{C}$. For a nonzero vector $\alpha(t) \in V_t$ we have the canonical isomorphism $T_{[\alpha(t)]}(\mathbb{P}V_t) \otimes L_{[\alpha(t)]} \cong \nu^*V_t/L_{[\alpha(t)]}$, where $L_{[\alpha(t)]} = \mathbb{C}\alpha(t)$ is the tautological line at $[\alpha(t)]$. Varying over $\widehat{C}$ we obtain a canonical isomorphism $T_\nu \otimes L \cong \nu^*V_t/L$ over $\widehat{C}$. Since $L|_{\widehat{C}} \cong T_{\widehat{C}}$ canonically, and C is a standard rational curve, we have $\nu^*V|_{\widehat{C}} \cong \mathcal{O}(2) \oplus \big(\mathcal{O}(1)\big)^p \oplus \mathcal{O}^q$, so that

$$T_\nu|_{\widehat{C}} \cong \nu^*V|_{\widehat{C}}/T_{\widehat{C}} \otimes T^*_{\widehat{C}} \cong \big((\mathcal{O}(1))^p \oplus \mathcal{O}^q\big) \otimes \mathcal{O}(-2) \cong \big(\mathcal{O}(-1)\big)^p \oplus \big(\mathcal{O}(-2)\big)^q.$$

Since at $[\alpha(t)], T_{[\alpha(t)]} \otimes L_{[\alpha(t)]} \cong P_{\alpha(t)}/\mathbb{C}\alpha(t)$, where $P_{\alpha(t)} \subset V_t$ is the positive part of V_t at $[\alpha(t)]$, over $\widehat{C}$ we have $T_\lambda|_{\widehat{C}} \cong \big(\mathcal{O}(1)\big)^p \otimes \mathcal{O}(-2) \cong \big(\mathcal{O}(-1)\big)^p$ and $N_\lambda|_{\widehat{C}} \cong \mathcal{O}^q \otimes \mathcal{O}(-2) \cong \big(\mathcal{O}(-2)\big)^q$. Thus, over $\widehat{C}$

$$\mathrm{Hom}\big(S^2T_\lambda|_{\widehat{C}}, N_\lambda|_{\widehat{C}}\big) \cong \mathrm{Hom}\Big(\big(\mathcal{O}(-2)\big)^{\frac{p(p+1)}{2}}, \big(\mathcal{O}(-2)\big)^q\Big) \cong \mathcal{O}^{\frac{qp(p+1)}{2}}$$

is holomorphically trivial. Hence, at $t_1, t_2 \in \mathbb{P}^1$ the second fundamental forms $\sigma_{[\alpha(t_i)]} : S^2T_{[\alpha(t)]} \to N_{[\alpha(t)]}$; $i = 1, 2$; must be isomorphic to each other, as desired. □

Taking $\sigma_{[\alpha(t)]}$ as defining a holomorphic section of a holomorphically trivial vector bundle $E := S^2T_\lambda|_{\widehat{C}} \otimes N_\lambda|_{\widehat{C}}$ over $\mathbb{P}^1$, parallel transport of the second fundamental form from $t_1 \in \mathbb{P}^1$ to $t_2 \in \mathbb{P}^1$ can be understood as sending an element of $\epsilon_{t_1} \in E_{t_1}$ to the unique element $\epsilon_{t_2} \in E_{t_2}$ for which there exists $\epsilon \in \Gamma(\mathbb{P}^1, E)$ such that $\epsilon(t_1) = \epsilon_{t_1}$, $\epsilon(t_2) = \epsilon_{t_2}$. Fixing a decomposition of $V = f^*T_X$ over $\mathbb{P}^1$ given by $V = \mathcal{O}(2) \oplus (\mathcal{O}(1))^p \oplus \mathcal{O}^q$, there is a linear isomorphism $\varphi : V_{t_1} \to V_{t_2}$ which respects the decomposition of V and which induces parallel transport from $\sigma_{[\alpha(t_1)]}$ and $\sigma_{[\alpha(t_2)]}$.

6.3. Recognition of certain rational homogeneous manifolds from VMRTs at a general point. — We consider the question of characterizing certain rational homogeneous manifolds of Picard number 1 by their VMRTs at general points. Let S be an irreducible Hermitian symmetric space of Picard number 1, and denote its VMRT at $0 \in S$ by $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$. Suppose $(X, \mathcal{K})$ is a uniruled projective manifold equipped with a minimal rational component such that at a general point $x \in X$ the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is congruent to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ of the model space. Let $B \subset X$ be a proper subvariety such that $\pi|_{X-B} : \mathcal{C} \to X$ is a locally trivial holomorphic fiber bundle with fibers $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ being congruent to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ as a projective submanifold. By [(4.3), Theorem 4], which in particular characterizes irreducible Hermitian symmetric spaces S of rank ≥ 2 by means of S-structures, to prove $X \cong S$ it suffices to show that B can be reduced to the empty set by methods of holomorphic extension. By Hartogs extension of S-structures (cf. (3.4)) it is enough to show that for every irreducible component $E_i \subset B$ of codimension 1 in X and any general point $y \in E_i$, there exists a neighborhood U_y of y such that $\pi|_{U_y - E_i} : \mathcal{C}|_{U_y - E_i} \to U_y - E_i$ extends holomorphically across $U_y \cap E_i$ as a holomorphic fiber subbundle of $\pi : \mathbb{P}T_{U_y} \to U_y$. Since X is of Picard number 1, for $y \in E_i$ sufficiently general there exists a standard parametrized rational curve $f : \mathbb{P}^1 \to X$ such that $f(0) \notin B$ and $f(\infty) = y$. The idea is to consider the tautological lifting of $C = f(\mathbb{P}^1)$ to $\widehat{C} \subset f^*\mathcal{C}$, and to recapture $\mathcal{C}_\infty$ which corresponds to $\mathcal{C}_y$ by knowing its second fundamental form at the point $[\alpha(\infty)] \in \mathcal{C}_\infty$ corresponding to $[df(\infty)] \in \mathcal{C}_y$.

The simplest case for this to work is the case of the n-dimensional hyperquadric $Q^n, n \geq 3$. For the family $f^*\mathcal{C} \subset \mathbb{P}(f^*T_X)$, the general fiber is isomorphic to a hyperquadric in $\mathbb{P}^{n-1}$. Degeneration of the hyperquadrics can occur at $t = \infty$, to give a degenerate hyperquadric defined by a degenerate symmetric bilinear form. However, this is precisely the case if and only if the second fundamental form σ at a general point of $\mathcal{C}_\infty$ is degenerate. The method of parallel transport of second fundamental forms then rules out the latter possibility, showing that $\mathcal{C}_y \subset \mathbb{P}T_y(X)$ is congruent to the VMRT of the model space for a general point y of the hypersurface E_i. With this holomorphic extension result of VMRTs across general points of hypersurfaces and Hartogs extension for bad sets of codimension ≥ 2 we have shown that X is biholomorphically isomorphic to the hyperquadric whenever the VMRT at a general point is congruent to $Q^{n-2} \subset \mathbb{P}^{n-1}$.

As seen from the table in (2.4) in the general symmetric case the VMRT $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ is itself a Hermitian symmetric space, either of rank 2 and embedded by the minimal canonical embedding, or of rank 1 and embedded by the second canonical embedding. In some sense they are quadratic objects. In fact, $\mathcal{C}_0$ is the closure of the graph of a vector-valued quadratic function Q on the tangent space $T_{[\alpha]}(\mathcal{C}_0)$. Q is essentially the second fundamental form. To illustrate how the argument of parallel transport of second fundamental forms works in the other cases, we consider the cases where $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ is an irreducible Hermitian symmetric space of rank 2, so that it carries a canonical G-structure for some reductive Lie subgroup of the general linear group. In the notations analogous to those in the preceding discussion, $\mathcal{C}_\infty \subset \mathbb{P}(f^*T_X)$ has the same second fundamental form at $[\alpha(\infty)]$ as that of the model space. $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ is uniruled by lines. Denoting by $\mathcal{K}'$ the minimal rational component on $\mathcal{C}_0$ consisting of lines, the G-structure of $\mathcal{C}_0$ is completely determined by VMRTs $\mathcal{C}'_{[\alpha]}$ associated to $(\mathcal{C}_0, \mathcal{K}')$, where $\mathcal{C}'_{[\alpha]}$ is defined by the set of non-zero tangent vectors $\eta \in T_{[\alpha]}(\mathcal{C}_0)$ such that $\sigma_{[\alpha]}(\eta, \eta) = 0$. Parallel transport of second fundamental forms then implies that $\mathcal{C}_\infty$ inherits a G-structure. By making use of developing maps $\mathcal{C}_\infty \subset \mathbb{P}(f^*T_y(X)$ can be shown to be congruent to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$. Here one has to exclude the possibility of linear degeneration of $\mathcal{C}_\infty \subset \mathbb{P}(f^*T_y(X)$, a possibility that is ruled out by the surjectivity of the second fundamental $\sigma_{[\alpha]}$ on the model space, and hence of $\sigma_{[\alpha(\infty)]}$ at $y = f(\infty)$ on X by parallel transport.

The preceding line of argumentation can be strengthened to yield

***Theorem 14* (Mok [43], Hong-Hwang [8]).** — *Let G be a simple complex Lie group, $P \subset G$ be a maximal parabolic subgroup corresponding to a long simple root, and by $S := G/P$ be the corresponding rational homogeneous manifold of Picard number 1. Denote by $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ the variety of minimal rational tangents at a reference point $0 \in S$ associated to the minimal rational component of lines on S. Let X be a Fano manifold of Picard number 1 and $\mathcal{K}$ be a minimal rational component on X. Suppose the variety of $\mathcal{K}$-tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at a general point $x \in X$ is congruent to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ as a projective submanifold. Then, X is biholomorphic to S.*

For the case where S is the projective space Theorem 14 follows from [**3**]. A sketch of the proof for S Hermitian symmetric and of rank ≥ 2 has been given in the above. When $P \subset G$ corresponds to a long simple root, the VMRT $\mathcal{C}_0 \subset \mathbb{P}D_0$ for the minimal nontrivial G-invariant distribution D on S. $\mathcal{C}_0$ is the highest weight orbit in $\mathbb{P}D_0$, and it is itself a Hermitian symmetric space. $D \neq T_S$ unless S is Hermitian symmetric. When S is non-symmetric and $\mathcal{C}_0$ is irreducible as a Hermitian symmetric space, it is of rank 3, embedded by the minimal canonical embedding. In general $\mathcal{C}_0 \subset \mathbb{P}D_0$ is of rank 3 as an embedded Hermitian symmetric space, when the degree for the embedding on each irreducible factor of $\mathcal{C}_0$ is taken into account in the obvious way. In fact, $\mathcal{C}_0 \subset \mathbb{P}D_0$ is a cubic object, being the closure of the graph of a vector-valued cubic polynomial on the tangent space $T_{[\alpha]}(\mathcal{C}_0)$ (cf. Hwang-Mok [**17**], p. 377). The cubic nature of the VMRT is reflected in the table for Fano contact homogeneous manifolds of Picard number 1 in (3.1), and applies in general to the long-root case.

For non-symmetric S there is an additional notion of the third fundamental form for $\mathcal{C}_0 \subset \mathbb{P}D_0$, defined as follows. The image of the second fundamental form $\sigma_{[\alpha]}$:

$S^2T_{[\alpha]} \to T_0(S)/P_\alpha$ is not surjective. For $\alpha \in \widetilde{\mathcal{C}}_0$ one can define a filtration $\mathbb{C}\alpha \subset P_\alpha \subset Q_\alpha \subset T_0(S)$, where Q_α is obtained by adjoining the image of the second fundamental form at α. This filtration corresponds to the splitting $D|_\ell \cong \mathcal{O}(2) \oplus (\mathcal{O}(1))^p \oplus \mathcal{O}^q \oplus (\mathcal{O}(-1))^r$ for the minimal proper distribution $D \subset T_S$. At every point $[\alpha] \subset \mathcal{C}_0$ one can define the third fundamental form $\kappa_{[\alpha]} : S^3T_{[\alpha]} \to T_0(S)/Q_\alpha$. In the case of a Fano manifold X of Picard number 1 satisfying the hypothesis of Theorem 14 for a non-symmetric S defined by a long simple root, Proposition 6 generalizes to show that over a standard parametrized rational curve $f : \mathbb{P}^1 \to X$, the corresponding third fundamental form on its tautological lifting $\widehat{C}$ defines a holomorphic section of a holomorphically trivial vector bundle over $\mathbb{P}^1$. Using this we have a version of parallel transport of the third fundamental form κ, with which one can prove extension results of VMRTs across a general point of a hypersurface as in the Hermitian symmetric case. In the contact case Theorem 14 is proved in Mok [**43**] by resorting to Hong's characterization of Fano contact homogeneous manifolds of Picard number 1 in [Ho]. In the remaining cases Theorem 14 was established in Hong-Hwang [**8**].

In view of Theorem 14, one may raise the following conjecture.

Conjecture 6. — *Let $S = G/P$ be any Fano homogeneous manifold of Picard number 1 and denote by $\mathcal{C}_0(S) \subset \mathbb{P}T_0(S)$ its variety of minimal rational tangents at a reference point $0 \in S$. Let $(X, \mathcal{K})$ be a Fano manifold of Picard number 1 equipped with a minimal rational component such that the associated* VMRT *at a general point is congruent to $\mathcal{C}_0(S) \subset \mathbb{P}T_0(S)$. Then, X is biholomorphic to S.*

To resolve Conjecture 6 it remains to consider the short-root case. Confirmation of the conjecture would provide a unified proof of rigidity of Fano homogeneous manifolds of Picard number 1 under Kähler deformation [(3.4), Theorem 2].

6.4. Projective manifolds with nef tangent bundles and 1-dimensional VMRTs. — In analogy with the Generalized Frankel Conjecture in Kähler Geometry one can formulate a Generalized Hartshorne Conjecture in Algebraic Geometry. This is given by the Campana-Peternell Conjecture [**2**] (1991). In particular, restricting to Fano manifolds X of Picard number 1, the Campana-Peternell Conjecture asserts that X is biholomorphic to a rational homogeneous manifold $S = G/P$ whenever the tangent bundle of X is nef, i.e., numerically effective. The latter assumption implies that the deformation of any rational curve on X is unobstructed. As a consequence, for any choice of a minimal rational component $\mathcal{K}$ on X, the evaluation map $\mu : \mathcal{U} \to X$ associated to the universal family for $\mathcal{K}$ gives a regular family of projective manifolds. This imposes some restrictions on possible complex structures of moduli spaces $\mathcal{K}_x \cong \mathcal{U}_x$ of $\mathcal{K}$-curves marked at x by restricting $\mathcal{U}$ over minimal rational curves. While there is so far no strong evidence why the Campana-Peternell

Conjecture should hold, with the latter fact in mind Mok [**41**] considered a special case of the conjecture, under the restrictive assumption that the VMRT at a general point is 1-dimensional. In [**41**] we considered Fano manifolds whose second and fourth Betti numbers are equal to 1. The condition on the fourth Betti number was removed recently by Hwang [**12**], and we have now

Theorem 15. — (Mok [**43**], Hwang [**12**]) *Let X be a Fano manifold of Picard number 1 with nef tangent bundle. Suppose X is equipped with a minimal rational component for which the variety of minimal rational tangents at a general point $x \in X$ is 1-dimensional. Then, X is biholomorphic to the projective plane $\mathbb{P}^2$, the 3-dimensional hyperquadric Q^3, or the 5-dimensional Fano contact homogeneous manifold $K(G_2)$ of type G_2. In particular, X is a rational homogeneous manifold.*

We note that the only algebro-geometric property used which arises from the nefness of the tangent bundle is the fact that the restriction of the tangent bundle to any $\mathcal{K}$-curve is nonnegative. In particular, the nefness assumption in Theorem 15 can be replaced by the assumption that any rational curve on X is free. The approach of [**41**] was to reconstruct X under the given assumptions from its VMRTs by making use of the canonical double fibration $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$ associated to $\mathcal{K}$. We note that no *a priori* assumption is placed on $\dim(X)$.

To start with, restricting $\mu : \mathcal{U} \to X$ to a minimal rational curve we obtain an algebraic surface Σ holomorphically fibered over $\mathbb{P}^1$ which admits a holomorphic section Γ corresponding to the tautological lifting of the minimal rational curve. Thus, $\Gamma \subset \Sigma$ is an exceptional curve. Since the base is $\mathbb{P}^1$, if the fibers are of genus ≥ 1 the family must be holomorphically trivial, and the existence of the exceptional curve $\Gamma \subset \Sigma$ forces a contradiction. Thus, any $\mathcal{U}_x$ is isomorphic to $\mathbb{P}^1$. At a general point $x \in X$ the tangent map $\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$ is a holomorphic map. To determine the VMRT at a general point the next step is to bound $d := \deg(\tau_x^*(\mathcal{O}(1))$. For this purpose we introduce the use of Chern class inequalities. First, the universal $\mathbb{P}^1$-bundle $\rho : \mathcal{U} \to \mathcal{K}$ gives rise to a holomorphic rank-2 vector bundle $\nu : V \to \mathcal{K}$ such that $\mathbb{P}V \cong \mathcal{U}$. We prove that V is stable and deduce that $d \leq 4$ from the Bogomolov inequality $c_1^2(V) \cdot [\omega]^{n-2} \leq 4c_2(V) \cdot [\omega]^{n-2}$ for stable rank-2 vector bundles V over an n-dimensional projective manifold, where ω stands for the first Chern form of a positive line bundle on X, and $[\omega]$ for its cohomology class. It is here that we make use of the assumption $b_4(X) = 1$ when applying Chern class inequalities. Using the existence of Hermitian-Einstein metrics due to Uhlenbeck-Yau the equality case in the Bogomolov inequality can be ruled out, and we end up with $d = 1, 2, 3$, which we eventually prove to correspond to the three examples in the statement of Theorem 15. To proceed we make use of results from (2.3) on the integrability of differential systems generated by VMRTs to show that in each of the three cases $d = 1, 2, 3$ the

VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is congruent to $\mathcal{C}_0 \subset S$ of the model space, and the proof is completed by invoking special cases of Theorem 14. The condition $b_4(X) = 1$ is removed in Hwang [**10**] by resorting to the determination of a certain Chow group pertinent to the problem in the application of the Bogomolov inequality.

Finally, from Theorem 15, together with earlier works of Campana-Peternell [**2**] and Zheng [**53**], and Miyaoka's characterization of the hyperquadric [**37**], one confirms the Campana-Peternell Conjecture up to 4 dimensions. More precisely, we have

Theorem 16. — *Let X be a Fano manifold of dimension ≤ 4 on which all rational curves are free. Then, X is biholomorphic to a rational homogeneous manifold.*

7. Privileged subvarieties of uniruled projective manifolds

7.1. Subvarieties saturated with minimal rational curves. — In analogy to totally geodesic submanifolds in Riemannian geometry we introduce for uniruled projective manifolds $(X\mathcal{K})$ endowed with minimal rational components the notion of $\mathcal{K}$-saturated subvarieties, as follows.

Definition 5. — *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component, $\pi : \mathcal{C}_X \to X$ be the associated fibered space of varieties $\mathcal{K}$-tangents. Let $\Sigma \subset X$ be an irreducible analytic subvariety of some connected open subset $U \subset X$ and $\mathcal{E} \subset \mathcal{C}_X\big|_\Sigma$ be an analytic subvariety. For $y \in \Sigma$ denote by $\mathcal{E}_y$ the fiber of $\mathcal{E}$ over y. We say that $(\Sigma, \mathcal{E}) \hookrightarrow (X, \mathcal{C}_X)$ is $\mathcal{K}$-saturated if and only if*

(a) *$\mathcal{E}_y = \mathbb{P}T_y(\Sigma) \cap \mathcal{C}_X \neq \varnothing$ for a smooth point $y \in \Sigma$, and*

(b) *for a general smooth point y on Σ, and for the germ C of an irreducible branch of a standard $\mathcal{K}$-curve passing through y, C must lie on Σ whenever $\big[T_y(C)\big] \in \mathcal{E}_y$.*

When the choice of $\mathcal{K}$ is understood, we simply say that Σ is saturated with respect to minimal rational curves. If we take a minimal rational curve on $(X, \mathcal{K})$ to play the role of a geodesic, a $\mathcal{K}$-saturated subvariety is the analogue of a totally geodesic subspace in Riemannian geometry, except that the 'geodesics' are now only defined for tangent directions corresponding to varieties of minimal rational tangents.

7.2. A relative version of the Gauss map condition for linear sections of VMRTs. — In (5.1) we have introduced a non-degeneracy condition (†) on the Gauss map of the variety of minimal rational tangents $\mathcal{C}_x$ at a general point x of a uniruled projective manifold $(X, \mathcal{K})$ equipped with a minimal rational component, viz., we require that the Gauss map is generically finite on $\mathcal{C}_x$. Equivalently (†) is satisfied if and only if at a general smooth point $[\alpha]$ of $\mathcal{C}_x$, the kernel $\mathrm{Ker}\, \sigma_{[\alpha]} = 0$ for the second fundamental form $\sigma_{[\alpha]}$ at $[\alpha] \in \mathrm{Reg}(\mathcal{C}_x)$. We extend this to the situation of a linear section of $\mathcal{C}_x$ and define a non-degeneracy condition (††) which reduces to (†) when the linear section is $\mathcal{C}_x$ itself. Recall that a variety is said to be of pure dimension n if and only if each irreducible component is of the same dimension n.

Definition 6. — *Let $m \geq 2$, $\mathcal{A} \subset \mathbb{P}^m$ be a projective subvariety of pure dimension $a \geq 1$. Let $\Pi \subset \mathbb{P}^m$ be a projective linear subspace, and $\mathcal{B} := \Pi \cap \mathcal{A}$ be a non-empty projective subvariety of pure dimension $b \geq 1$. We say that the pair $(\mathcal{B}, \mathcal{A})$ satisfies the non-degeneracy condition (††) if and only if for every general smooth point $[\beta] \in \mathcal{B}$, $[\beta]$ is also a smooth point of $\mathcal{A}$ and* $\mathrm{Ker}\ \sigma_{[\beta]}\big(T_{[\beta]}(\mathcal{B}), \cdot\big) = 0$.

By an adaptation of the proof of Cartan-Fubini extension in the equidimensional case under the non-degeneracy assumption (†) as explained in (5.2) we have the following non-equidimensional analogue of Cartan-Fubini extension under some non-degeneracy assumption involving (††) on second fundamental forms. For the formulation a point $x \in X$ is said to be a good point if and only if every minimal rational curve passing through x is free, and a general element of every irreducible component of $\mathcal{K}_x$ represents a standard rational curve, otherwise x is called a bad point. The bad locus of $(X, \mathcal{K})$ is the set of bad points on X, which is a subvariety of X.

Theorem 17 **(Hong-Mok [9]).** — *Let $(Z, \mathcal{H})$ and $(X, \mathcal{K})$ be two uniruled projective manifolds equipped with minimal rational components. Assume that Z is of Picard number 1 and that $\mathcal{C}_z(Z)$ is of positive dimension at a general point $z \in Z$. Let $U \subset Z$ be a connected open subset and $f : U \to X$ be a holomorphic embedding onto a locally closed complex submanifold $S \subset X$ lying outside the bad locus of $(X, \mathcal{K})$. Suppose f respects varieties of minimal rational tangents in the sense that $df(\widetilde{\mathcal{C}}_z(Z)) = df(T_z(Z)) \cap \widetilde{\mathcal{C}}_{f(z)}(X)$. Assume furthermore that at a general point $x \in S$, the non-degeneracy condition (††) on second fundamental forms is satisfied for the pair $(\mathcal{C}_x \cap \mathbb{P}T_x(S), \mathcal{C}_x)$. Then, f extends to a rational map $F : Z \to X$.*

In terms of the holomorphic map f, the non-degeneracy condition on second fundamental forms translate into

$$\mathrm{Ker}\ \widetilde{\sigma}_{df(\alpha)}\big(T_{df(\alpha)}(df(\widetilde{\mathcal{C}}_z(Z))), \cdot\big) = \mathbb{C}df(\alpha).$$

As an important intermediate step in the proof of Theorem 17, Hong-Mok established under the assumption there the following result.

Proposition 7. — *Under the assumptions of Theorem 17 and in the notations there, f sends germs of standard $\mathcal{H}$-curves into germs of standard $\mathcal{K}$-curves. In particular, $(S, \mathcal{C} \cap \mathbb{P}T_S) \subset (X, \mathcal{C})$ is saturated with respect to $\mathcal{K}$-curves.*

7.3. Parallel transport of VMRTs along minimal rational curves. — As an application of non-equidimensional Cartan-Fubini extension, Mok [**42**] gave a characterization of standard embeddings between Grassmannians of rank ≥ 2. The result by itself had been known and proven by different methods by Neretin [**46**] and Hong [**7**]. Our proof started with non-equidimensional Cartan-Fubini extension in the Hermitian symmetric case with a proof relying on the use of Harish-Chandra coordinates. More recently, Hong-Mok [**9**] have established the general form of Proposition 7, obtaining at the same time a characterization of a general class of standard embeddings between rational homogeneous manifolds of Picard number 1. On a rational homogeneous manifold Y of Picard number 1 we consider the minimal rational component consisting of lines on Y and denote by $\mathcal{C}_y(Y)$ the associated VMRT at $y \in Y$.

Theorem 18 **(Hong-Mok [9]).** — *Let $X = G/P$ be a rational homogeneous manifold of Picard number 1 associated to a long simple root and let $Z = G_0/P_0$ be a rational homogeneous space associated to a subdiagram of the marked Dynkin diagram of G/P. Assume that Z is not linear. If $f : U \to X$ is a holomorphic embedding from a connected open subset U of Z into X satisfying $df_z(\widetilde{\mathcal{C}}_z(Z)) = df_z(T_z(Z)) \cap \widetilde{\mathcal{C}}_{f(z)}(X)$ for a general point $z \in \mathcal{U}$, then f extends to a standard embedding of Z into X.*

Sketch of proof. — A marked Dynkin subdiagram defines naturally an embedding λ from $Z = G_0/P_0$ into $X = G/P$. By a standard embedding from Z into X we mean $\varphi \circ \lambda$ for some $\varphi \in \mathrm{Aut}(X)$. For the proof of Theorem 18, first of all the method of non-equidimensional Cartan-Fubini extension as given in [(7.2), Theorem 17] can be implemented by checking the validity of the non-degeneracy condition (††) on the Gauss map yielding therefore a rational extension $F : Z \to X$. Write $S = F(Z)$ for the total transform of F. By Proposition 7, $S \subset X$ is $\mathcal{K}$-saturated. The condition $df_z(\widetilde{\mathcal{C}}_z(Z)) = df_z(T_z(Z)) \cap \widetilde{\mathcal{C}}_{f(z)}(X)$ says that S is tangent at a general point $s \in f(U)$ to a (unique) copy Z_s of a standard embedding of Z into X. Extending $f : U \to X$ to $F : Z \to X$ the same applies for a general point $s \in S$.

Start with a base point $0 \in Z$, $f(0) = 0$. Z_0 and S are tangent to each other at 0 and they share the same VMRTs at 0. Let $\mathcal{A}$ be the subvariety on Z_0 swept out by lines ℓ on Z_0 passing through 0. Since $S \subset X$ is $\mathcal{K}$-saturated, $\ell \in \mathcal{A} \subset Z_0 \cap S$. At a general point $s \in \ell$, write $\mathcal{E}_s := \mathbb{P}T_s(S) \cap \mathcal{C}_s(X) = \mathbb{P}T_s(Z_s) \cap \mathcal{C}_s(X)$. We argue that Z_0 and S are tangent at $s \in \ell$, i.e., $\mathcal{C}_s(Z_0) = \mathcal{E}_s$. Write $T_s(\ell) = \mathbb{C}\alpha$. From the deformation theory of rational curves $T_{[\alpha]}(\mathcal{C}_s(Z_0)) = T_s(\mathcal{A})/T_s(\ell)$ while also $T_{[\alpha]}(\mathcal{E}_s) = T_s(\mathcal{A})/T_s(\ell)$. This means that $\mathcal{E}_s$ and $\mathcal{C}_s(Z_0)$ are tangent to each other at $[\alpha]$. In general the tangency property does not imply identity of the two VMRTs, but we have found that this is the case for pairs (Z, X) of rational homogeneous manifolds of Picard number 1 as given in Theorem 18. We may think of this as a form of parallel transport of VMRTs for $\mathcal{K}$-saturated subvarieties along a minimal rational curve in special situations. Thus, $Z_s = Z_0$ for any line ℓ on Z_0 passing through 0 and for a general point $s \in \ell$. It follows that $Z_s = Z_0$ for a general point $s \in \mathcal{A}$, and s can now play the same role as the initial base point 0. Finally, $S = F(Z)$ can be recovered from the single point $0 \in Z$ in a finite number of steps by the procedure of adjoining minimal rational curves (cf. (3.1)), and we have proven that $S = Z_0$, as desired. □

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N. MOK, Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong
E-mail : `nmok@hkucc.hku.hk`

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DAVID HOFFMAN
BRIAN WHITE

On the number of minimal surfaces with a given boundary

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ON THE NUMBER OF MINIMAL SURFACES WITH A GIVEN BOUNDARY

by

David Hoffman & Brian White

Dedicated to Jean Pierre Bourguignon on the occasion of his 60th birthday

Abstract. — We prove results allowing us to count, mod 2, the number of embedded minimal surfaces of a specified topological type bounded by a curve $\Gamma \subset \partial N$, where N is a weakly mean convex 3-manifold with piecewise smooth boundary. These results are extended to curves and minimal surfaces with prescribed symmetries. The parity theorems are used in an essential manner to prove the existence of embedded genus-g helicoids in $\mathbf{S}^2 \times \mathbf{R}$, and we give an outline of this application.

***Résumé* (Sur le nombre de surfaces minimales avec une frontière donnée).** — Nous démontrons des résultats qui nous permettent de compter, modulo 2, le nombre de surfaces minimales plongées d'un type topologique donné, borné par une courbe $\Gamma \subset \partial N$, où N est une 3-variété convexe faiblement moyenne munie d'une frontière lisse par morceaux. Ces résultats sont étendus aux courbes et aux surfaces minimales à symétries préscrites. Les théorèmes de parité sont utilisés de manière essentielle pour prouver l'existence d'hélicoïdes de genre imbriqué g dans $\mathbf{S}^2 \times \mathbf{R}$, et nous donnons un aperçu de cette application.

1. Introduction

In [4], Tomi and Tromba used degree theory to solve a longstanding problem about the existence of minimal surfaces with a prescribed boundary: they proved that every smooth, embedded curve on the boundary of a convex subset of $\mathbf{R}^3$ must bound an embedded minimal disk. Indeed, they proved that a generic such curve must bound an odd number of minimal embedded disks. White [8] generalized their result by proving the following parity theorem. Suppose N is a compact, strictly convex domain in $\mathbf{R}^3$

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with smooth boundary. Let Σ be a compact 2-manifold with boundary. Then a generic smooth curve $\Gamma \cong \partial\Sigma$ in ∂N bounds an odd or even number of embedded minimal surfaces diffeomorphic to Σ according to whether Σ is or is not a union of disks.

In this paper, we generalize the parity theorem in several ways. First, we prove (Theorem 2.1) that the parity theorem holds for any compact riemannian 3-manifold N such that N is strictly mean convex, N is homeomorphic to a ball, ∂N is smooth, and N contains no closed minimal surfaces. We then further relax the hypotheses by allowing N to be mean convex rather than strictly mean convex, and to have piecewise smooth boundary. Note that if N is mean convex but not strictly mean convex, then Γ might bound minimal surfaces that lie in ∂N. We prove (Theorem 2.4) that the parity theorem remains true for such N provided (1) unstable surfaces lying in ∂N are not counted, and (2) no two contiguous regions of $(\partial N) \setminus \Gamma$ are both smooth minimal surfaces. We give examples showing that the theorem is false without these provisos.

We extend the parity theorem yet further (see Theorem 2.7) by showing that, under an additional hypothesis, it remains true for minimal surfaces with prescribed symmetries.

The parity theorems described above are all mod 2 versions of stronger results that describe integer invariants. The stronger results are given in section 3.

The parity theorems are used in an essential way to prove the the existence of embedded genus-g helicoids in $\mathbf{S}^2 \times \mathbf{R}$. In Sections 4 and 5 we give a very brief outline of this application. (The full argument will appear in [**3**].)

2. Counting minimal surfaces

Throughout the paper, N will be a compact riemannian 3-manifold and Σ will be a fixed compact 2 manifold. If Γ is an embedded curve in N diffeomorphic to $\partial\Sigma$, we let $\mathcal{M}(N,\Gamma)$ denote the set of embedded minimal surfaces in N that are diffeomorphic to Σ and that have boundary Γ. We let $|\mathcal{M}(N,\Gamma)|$ denote the number of surfaces in $\mathcal{M}(N,\Gamma)$.

In case N has smooth boundary, we say that N is strictly mean convex provided the mean curvature is a (strictly) positive multiple of the inward unit normal on a dense subset of ∂N.

2.1. Theorem. — *Let N be a smooth, compact, strictly mean convex riemannian 3-manifold that is homeomorphic to a ball and that has smooth boundary. Suppose also that N contains no closed minimal surfaces. Let $\Gamma \subset \partial N$ be a smooth curve diffeomorphic to $\partial\Sigma$. Assume that Γ is bumpy in the sense that no surface in $\mathcal{M}(N,\Gamma)$ supports a nontrivial normal Jacobi field with zero boundary values.*

Then $|\mathcal{M}(N,\Gamma)|$ *is even unless* Σ *is a union of disks, in which case* $|\mathcal{M}(N,\Gamma)|$ *is odd.*

We remark that generic smooth curves $\Gamma \subset \partial N$ are bumpy [**7**].

Proof. — Theorems 2.1 and 2.3 of [**8**] are special cases of the theorem. The proofs given there establish the more general result here provided one makes the following observations:

1. There N was assumed to be strictly convex, but exactly the same proof works assuming strict mean convexity.
2. There Σ was assumed to be connected, but the same proof works for disconnected Σ.
3. In the proofs of Theorems 2.1 and 2.3 of [**8**], the assumption that N is a subset of $\mathbf{R}^3$ was used in order to invoke an isoperimetric inequality, i.e., an inequality bounding the area of a minimal surface in N in terms of the length of its boundary. There are compact mean convex 3-manifolds for which no such isoperimetric inequality holds. However, if (as we are assuming here) N contains no closed minimal surfaces, then N does admit such an isoperimetric inequality [**9**].
4. In the proofs in [**8**], one needs to isotope any specified component of Γ to a curve C that bounds exactly one minimal surface, namely an embedded disk. This was achieved by choosing C to be a planar curve. For a general ambient manifold N, "planar" makes no sense. However, any sufficiently small, nearly circular curve $C \subset \partial N$ bounds exactly one embedded minimal disk and no other minimal surfaces. (This property of such a curve C is proved in the last paragraph of §3 in [**8**].) □

2.2. Mean convex ambient manifolds N with piecewise smooth boundary. — For the remainder of the paper, we allow ∂N to be piecewise smooth. For simplicity, let us take this to mean that ∂N is a union of smooth 2-manifolds with boundary ("faces" of N), any two of which are either disjoint or meet along a common edge with interior angle everywhere strictly between 0 and 2π. (More generally, one could allow the faces of N to have corners.) We say that such an N is mean convex provided (1) at each interior point of each face of N, the mean curvature vector is a nonnegative multiple of the inward-pointing unit normal, and (2) where two faces meet along an edge, the interior angle is everywhere at most π.

The following example shows what can go wrong in Theorem 2.1 if N is mean convex but not strictly mean convex.

Example 1. Let N be a region in $\mathbf{R}^3$ whose boundary consists of an unstable catenoid C bounded by two circles, together with the two disks bounded by those circles. Note that N is mean convex with piecewise smooth boundary. Let Γ be a pair of horizontal circles in C that are bumpy (in the sense of Theorem 2.1). Theorem 2.1 suggests that Γ should bound an even number of embedded minimal annuli in N. First consider the case when Γ consists of two circles in C very close to the waist circle. Then Γ bounds precisely two minimal annuli. One of them is the component of C bounded by Γ. Because the circles in Γ are close, this annulus is strictly stable. The other annulus bounded by Γ is a strictly unstable catenoid lying in the interior of N. In order to get an even number of examples, we must count the stable catenoid lying on C. Now suppose the two components of Γ are the two components of ∂C. Then again Γ bounds exactly two minimal annuli: the unstable catenoid C, which is part of ∂N, and a strictly stable catenoid that lies outside N. Here, of course, we do not count the stable catenoid since it does not lie in N. Thus to get an even number, we also must not count the unstable catenoid that lies in ∂N.

This example motivates the following definition:

2.3. Definition. — $\mathcal{M}^*(N,\Gamma)$ is the set of embedded minimal surfaces $M \subset N$ such that

i.) $\partial M = \Gamma$,
ii.) M is diffeomorphic to Σ, and
iii.) each connected component of M lying in ∂N is stable.

Example 1 suggests that in order to generalize Theorem 2.1 to mean convex N with piecewise smooth boundary, we should replace $\mathcal{M}(N,\Gamma)$ by $\mathcal{M}^*(N,\Gamma)$. However, even if one makes that replacement, the following example shows that an additional hypothesis is required.

Example 2. Let N be a compact, convex region in $\mathbf{R}^3$ such that ∂N is smooth and contains a planar disk D. Let Γ be a pair of concentric circles lying in D. Then Γ bounds exactly one minimal annulus: the region in D between the two components of Γ. That annulus is strictly stable and lies in ∂N. Thus Γ is bumpy (in the sense of Theorem 2.1) and $|\mathcal{M}^*(N,\Gamma)| = 1$. Consequently, if we wish $|\mathcal{M}^*(N,\Gamma)|$ to be even (as Theorem 2.1 suggests it should be), then we need an additional hypothesis on N and Γ.

Note that in example 2, $(\partial N)\setminus\Gamma$ contains two contiguous connected components (a planar annulus and a planar disk) both of which are minimal surfaces. The additional hypothesis we require is that $(\partial N)\setminus\Gamma$ contains no two such components.

2.4. Theorem. — *Let N be a smooth, compact, mean convex riemannian 3-manifold that is homeomorphic to a ball, that has piecewise smooth boundary, and that contains* *no closed minimal surfaces. Let $\Gamma \subset \partial N$ be a smooth, embedded bumpy curve diffeomorphic to $\partial\Sigma$. Suppose that no two contiguous connected components of $(\partial N) \setminus \Gamma$ are both smooth minimal surfaces.*

Then $|\mathcal{M}^*(N,\Gamma)|$ is even unless Σ is a union of disks, in which case $|\mathcal{M}^*(N,\Gamma)|$ is odd.

Proof. — Since N is compact, mean convex, and contains no closed minimal surfaces, the areas of minimal surfaces in N are bounded in terms of the lengths of their boundaries [**9**].

If ∂N is smooth and has nowhere-vanishing mean curvature, the result follows immediately from Theorem 2.1. We reduce the general case to this special case as follows. Note that we can find a one-parameter family N_t, $0 \le t < \epsilon$, of mean convex subregions of N such that

i.) $N_0 = N$,
ii.) the boundaries ∂N_t foliate a relatively open subset of N containing ∂N.
iii.) for $t > 0$ small, ∂N_t is smooth and the mean curvature of ∂N_t is nowhere zero and points into N_t.

For example, we can let ∂N_t be the result of letting ∂N flow for time t by the mean curvature flow.

Claim. — *Suppose M_i are smooth embedded minimal surfaces in N diffeomorphic to Σ and that $\partial M_i \to \Gamma$ smoothly. Then a subsequence of the M_i converges smoothly to a limit $M \in \mathcal{M}^*(N,\Gamma)$.*

Proof of claim. — By Theorem 3 in [**6**] a subsequence converges smoothly away from a finite set S to a limit surface M. The surface M is smooth and embedded, though portions of it may have multiplicity > 1. Indeed, the proof of Theorem 3 in [**6**] shows that the multiplicity is 1 and the convergence $M_i \to M$ is smooth everywhere unless an interior point of M touches Γ.

In fact, no interior point of M can touch Γ. For suppose to the contrary that the interior of M touches Γ at a point p. Let C be the connected component of Γ containing p. By the strong maximum principle, M must contains a whole neighborhood of $p \in \partial N$. Indeed, by the strong maximum principle (or by unique continuation), M must contain the two connected components of $(\partial N) \setminus \Gamma$ on either side of C. But by hypothesis, at most one of those components is a minimal surface, a contradiction. This proves that no interior point of M touches Γ.

Consequently, as noted above, M has multiplicity 1 and the convergence $M_i \to M$ is smooth everywhere. Thus $M \in \mathcal{M}(N,\Gamma)$.

Now suppose some connected component M' of M lies in ∂N. Then the corresponding component M_i' of M_i converges smoothly to M' from one side of M. This one-sided convergence implies that M' is stable. Thus $M \in \mathcal{M}^*(N, \Gamma)$. This completes the proof of the claim. $\square$

Continuing with the proof of Theorem 2.4, note that $\mathcal{M}^*(N, \Gamma)$ is finite. For if it contained an infinite sequence of surfaces then by the claim, it would contain a smoothly convergent subsequence. The limit of that subsequence would be an element of $\mathcal{M}^*(N, \Gamma)$. But by bumpiness of Γ, the elements of $\mathcal{M}^*(N, \Gamma)$ are isolated. The contradiction proves that $\mathcal{M}^*(N, \Gamma)$ is finite.

Let Γ_t, $0 \leq t < \epsilon$, be a smooth one-parameter family of embedded curves such that $\Gamma_0 = \Gamma$ and such that $\Gamma_t \subset \partial N_t$. Let $M_0^1, \ldots M_0^k$ be the set of surfaces in $\mathcal{M}^*(N, \Gamma)$. By the implicit function theorem, we can (if ϵ is sufficiently small) extend these to one-parameter families

$$M_t^i \in \mathcal{M}^*(\widehat{N}, \Gamma_t) \qquad (i = 1, 2, \ldots, k;\ 0 \leq t < \epsilon)$$

where $\widehat{N}$ is a riemannian 3-manifold containing N in its interior.

In fact, M_t^i must lie in N provided $\epsilon > 0$ is chosen sufficiently small. To see this, assume for simplicity that Σ is connected. If M_0^i does not lie in ∂N, then by the strong maximum principle, it is never tangent to ∂N, so by continuity, $M_t^i \subset N$ for all sufficiently small t. Now suppose that M_0^i does lie in ∂N. Then (by definition of $\mathcal{M}^*(N, \Gamma)$) it is strictly stable. The strict stability implies that in fact M_t^i lies in N for sufficiently small t.

Indeed, M_t^i must lie not only in N but also in $N_t \subset N$, for all sufficiently small t. For let $T = T(t) \in [0, t]$ be the largest number such that $M_t^i \subset N_T$. If $T < t$, then M_t^i would touch ∂N_T at an interior point, violating the maximum principle. Hence $T = t$ and therefore $M_t^i \subset N_t$.

The claim implies that if ϵ is sufficiently small, then each surface in $\mathcal{M}^*(N_t, \Gamma_t)$ will be one of the surfaces in $M_t^1, \ldots, M_t^k$. We may also choose ϵ sufficiently small that the M_t^i all have zero nullity. Then

$$|\mathcal{M}^*(N, \Gamma)| = k = |\mathcal{M}(N_t, \Gamma_t)|$$

which must have the asserted parity by Theorem 2.1 (applied to N_t and Γ_t.) $\square$

2.5. Counting in the presence of symmetry. — In some situations, it is important to be able to say something about the number of minimal surfaces that are diffeomorphic to a specified surface Σ and that possess specified symmetries. Suppose G is a group of isometries of N.

2.6. Definition. — If Γ is a G-invariant curve in N, we let $\mathcal{M}_G^*(N, \Gamma) \subset \mathcal{M}^*(N, \Gamma)$ denote the set of surfaces in $\mathcal{M}^*(N, \Gamma)$ that are invariant under G. A boundary $\Gamma \subset \partial N$ is called G-bumpy if no surface in $\mathcal{M}_G^*(N, \Gamma)$ has a nontrivial G-invariant normal Jacobi field that vanishes on ∂M.

Theorem 2.4 has a natural extension to G-invariant surfaces:

2.7. Theorem. — *Let N be a smooth, compact, mean convex riemannian 3-manifold that is homeomorphic to a ball, that has piecewise smooth boundary, and that contains no closed minimal surfaces. Let G be a group of isometries of N. Let $\Gamma \subset \partial N$ be a smooth curve that is G-invariant and G-bumpy. Suppose that no two contiguous components of $(\partial N) \setminus \Gamma$ are both minimal surfaces.*
Suppose also that

(*) $\Gamma = \partial\Omega$ *for some G-invariant region $\Omega \subset \partial N$.*

Then $|\mathcal{M}_G^*(N,\Gamma)|$ is even unless Σ is a union of disks, in which case $|\mathcal{M}_G^*(N,\Gamma)|$ is odd.

2.8. Remark. — In Theorem 2.7, the hypothesis that N contains no closed minimal surfaces is equivalent to the hypothesis that N contains no closed G-invariant minimal surfaces. See [**9**], Theorem 2.5.

Proof. — In general, the proof is exactly the same as the proof in the non-invariant case. However (see Observation (4) in the proof of Theorem 2.1), to carry out the proof, one must be able to isotope the connected components of Γ in a G-invariant way to arbitrarily small, nearly circular curves in ∂N. The hypothesis that $\Gamma = \partial\Omega$ for a G-invariant region $\Omega \subset \partial N$ ensures that such isotopy is possible. (Indeed, it is equivalent to the existence of such G-invariant isotopies.) ☐

We do not know whether Theorem 2.7 remains true without the hypothesis (*).

3. An Integer Invariant

Suppose $N \subset \mathbf{R}^3$ is a compact, strictly convex set with smooth boundary. In the introduction, we quoted Theorems 2.1 and 2.3 of [**8**] as asserting that if $\Gamma \subset \partial N$ is a smooth, bumpy curve diffeomorphic to $\partial\Sigma$, then

$$|\mathcal{M}(N,\Gamma)| \cong \begin{cases} 1 & \text{if } \Sigma \text{ is a union of disks, and} \\ 0 & \text{if not} \end{cases} \tag{1}$$

where $\cong$ denotes congruence modulo 2.

In fact, the conclusion in [**8**] is actually much stronger than (1). To state that conclusion, we need some terminology.

3.1. Definition. — Let $\delta(\Sigma) = 1$ if Σ is a union of disks and 0 if not. If $\mathcal{M}$ is a collection of smooth minimal surfaces, let

$$d(\mathcal{M}) = |\mathcal{M}_{\text{even}}| - |\mathcal{M}_{\text{odd}}|$$

where $\mathcal{M}_{\text{even}}$ is the set of surfaces in $\mathcal{M}$ with even index of instability and $\mathcal{M}_{\text{odd}}$ is the set of surfaces in $\mathcal{M}$ with odd index of instability.

With this terminology, the conclusion of Theorem 2.1 in [**8**] is

$$d(\mathcal{M}(N,\Gamma)) = \delta(\Sigma). \tag{2}$$

Note that (2) is stronger than (1). Indeed, (1) merely asserts that the two sides of (2) are congruent modulo 2. (See [**5**] for a similar result for immersed minimal disks in $\mathbf{R}^n$.)

If we start with the stronger conclusion (2), then the arguments in §2 produce stronger versions of Theorems 2.1, 2.4, , and 2.7:

3.2. Theorem. — *Under the hypotheses of Theorem 2.1,*

$$d(\mathcal{M}(N,\Gamma) = \delta(\Sigma).$$

Under the hypotheses of Theorem 2.4,

$$d(\mathcal{M}^*(N,\Gamma)) = \delta(\Sigma).$$

Under the hypotheses of Theorem 2.7,

$$d_G(\mathcal{M}^*_G(N,\Gamma)) = \delta(\Sigma)$$

where $d_G(\cdot)$ is defined exactly like $d(\cdot)$, except that in determining index of instability, we only count eigenfunctions that are G-invariant.

The proofs are exactly as before.

4. Counting the number of handles on a surface invariant under an involution

Consider a minimal surface that has an axis of orientation preserving, 180° rotational symmetry. In many examples of interest, the handles of the surface are in some sense aligned along the axis. In this section, we make this notion precise, and we observe that our parity theorems apply to such surfaces.

Recall, for example, that Sherk constructed a singly periodic, properly embedded minimal surface $M \subset \mathbf{R}^3$ that is asymptotic to the planes $x = 0$ and $z = 0$ away from the y-axis, Y. By scaling, we may assume that M intersects Y precisely at the lattice points $(0, n, 0)$, $n \in \mathbf{Z}$. Now M has various lines of orientation preserving, 180° rotational symmetry. For example, Y is one such a line, and the line L given by $x = z$, $y = 1/2$ is another. Intuitively, the handles of M are lined up along Y but not along L. (The surface M is also invariant under 180° rotation about the x and z axes, but those rotations reverse orientation on M.) We make the intuition into a precise notion by observing that the rotation about Y acts on the first homology group $H_1(M, \mathbf{Z})$ by multiplication by -1, whereas rotation about L acts on $H_1(M, \mathbf{Z})$ in a more complicated way.

4.1. Proposition. — *Suppose S is a noncompact 2-dimensional riemannian manifold of finite topology. Suppose that $\rho : S \to S$ is an orientation preserving isometry of order two, and that S/ρ is connected. Then the following are equivalent:*

1. *ρ acts by multiplication by -1 on the first homology group $H_1(S, \mathbf{Z})$.*
2. *the quotient S/ρ is topologically a disk.*
3. *S has exactly $2 - \chi(S)$ fixed points of ρ, where $\chi(S)$ is the Euler characteristic of S.*

4.2. Corollary. — *If the equivalent conditions (1)-(3) hold, then the surface S has either one or two ends, according to whether ρ has an odd or even number of fixed points in S.*

4.3. Remark. — To apply Proposition 4.1 and its corollary to a compact manifold M with non-empty boundary, one lets $S = M \setminus \partial M$. Of course the number of ends of S is equal to the number of boundary components of M.

Proof of Proposition 4.1. — Suppose that (1) holds. Let $\pi : S \to S/\rho$ be the projection and let C be a closed curve in S/ρ. Then $C' = \pi^{-1}(C)$ is a ρ-invariant cycle in S and thus (by (1)) it bounds a 2-chain in S. Consequently $\pi(C') = 2C$ bounds a 2-chain in S/ρ. Thus $2C$ is homologically trivial in S/ρ. But S/ρ is orientable, so $H_1(S, \mathbf{Z})$ has no torsion. Thus C is homologically trivial in S/ρ. Since S/ρ is noncompact and connected with trivial first homology group, it must be a disk. Hence (1) implies (2).

To see that (2) implies (1), suppose that (2) holds. It suffices to show that any ρ-invariant 1-cycle in S is a boundary. (For if C_0 is any cycle in S, then $C_0 + \rho(C_0)$ forms a ρ-invariant cycle.) Since S is oriented, $H_1(S, \mathbf{Z})$ has no torsion, so it suffices to show that any ρ-invariant cycle 1-cycle in S must be a boundary mod 2. Let $C \subset S$ be any ρ-invariant closed curve, not necessarily connected. We may assume that C is smooth and in general position, i.e., that the self-intersections are transverse. By doing the obvious surgeries at the intersections, we may assume in fact that C is embedded.

Now $\pi(C)$ is a smooth, embedded, not necessarily connected, closed curve in S/ρ. Since S/ρ is topologically a disk, $\pi(C)$ bounds a region Ω. It follows that C bounds the region $\pi^{-1}(\Omega)$. Thus C is homologically trivial mod 2. This completes the proof that (2) implies (1).

Finally we show that (2) and (3) are equivalent. Let P be the number of fixed points of ρ. Consider a triangulation of S/ρ such the fixed points of ρ are vertices in the triangulation, and consider the corresponding triangulation of S. Then from Euler's formula one sees that

$$\chi(S) = 2\chi(S/\rho) - P$$

or

$$P = 2\chi(S/\rho) - \chi(S).$$

Thus $P = 2 - \chi(S)$ if and only if $\chi(S/\rho) = 1$. Since S/ρ is orientable and connected, its Euler characteristic is 1 if and only if it is a disk. This proves that (2) and (3) are equivalent. □

Proof of Corollary 4.2. — Since S/ρ is a disk, it has exactly one end. Since S is a double cover of S/ρ, it must have either one or two ends. Since S is oriented,

$$\chi(S) = 2c - 2g - e, \tag{3}$$

where c is the number of connected components, g is the sum of the genera of the connected components, and e is the number of ends. Thus e is congruent mod 2 to $\chi(S)$, which by Proposition 4.1 is congruent, mod 2, to the number of fixed points of ρ. □

4.4. Counting Y-surfaces. — Let N be a riemannian 3-manifold. We suppose that N has a geodesic Y and an orientation preserving, order two isometry $\rho = \rho_Y : N \to N$ for which the set of fixed points is Y.

4.5. Definition. — Suppose $M \subset N$ is an orientable, non-closed ρ-invariant surface such that $\rho : M \to M$ preserves orientation and such that $(M \setminus \partial M)/\rho$ is connected. We will say that M *is a Y-surface* if $S := M \setminus \partial M$ satisfies the equivalent conditions in Proposition 4.1.

Suppose for example that $N = \mathbf{R}^3$ and that Y is a line. Then $\rho = \rho_Y$ is 180° rotation about Y. If M is a ρ_Y-invariant catenoid, then either Y is the axis of rotational symmetry of M, or else Y intersects M orthogonally at two points on the waist of M. In the first case, ρ acts trivially on the first homology of M, so M is *not* a Y-surface. In the second case, ρ acts by multiplication by -1 on the first homology of M, so M is a Y-surface.

4.6. Definition. — We let

$$\mathcal{M}^*_Y(N,\Gamma) = \{M \in \mathcal{M}^*(N,\Gamma) : M \text{ is a } Y\text{-surface}\}.$$

We say that a curve $\Gamma \subset \partial N$ is *Y-bumpy* if no surface in $\mathcal{M}^*_Y(N,\Gamma)$ carries a nontrivial, ρ_Y-invariant, normal Jacobi field that vanishes on Γ.

The following result is a version of Theorem 2.7:

4.7. Theorem. — *Let N be a smooth, compact, mean convex riemannian 3-manifold that is homeomorphic to a ball, that has piecewise smooth boundary, and that contains no closed minimal surfaces. Suppose that Y is a geodesic in N and that $\rho = \rho_Y : N \to N$ is an orientation preserving, order two isometry of N with fixed point set Y.*

Let $\Gamma \subset \partial N$ be a smooth, embedded, ρ-invariant, Y-bumpy curve that carries a ρ-invariant orientation.

Suppose that no two contiguous components of $(\partial N) \setminus \Gamma$ are both minimal surfaces.

Then $|\mathcal{M}_Y^*(N,\Gamma)|$ is even unless Σ is a union of disks, in which case $|\mathcal{M}_Y^*(N,\Gamma)|$ is odd.

Proof. — The proof is almost identical to the proof of Theorem 2.7. One lets the group G in Theorem 2.7 be the group generated by ρ. The hypothesis (*) there follows from the hypothesis here that Γ carries a ρ_Y-invariant orientation. □

5. Higher genus helicoids in $\mathbf{S}^2 \times \mathbf{R}$

5.1. A boundary value problem for minimal Y-surfaces. — Our motivation in formulating Proposition 4.1 and Theorem 4.7 comes from the desire to construct embedded minimal surfaces in $\mathbf{S}^2 \times \mathbf{R}$, each of whose ends is asymptotic to a helicoid in $\mathbf{S}^2 \times \mathbf{R}$. Take as a model of $\mathbf{S}^2 \times \mathbf{R}$ the space $\mathbf{R}^2 \times \mathbf{R}$ on which each $\mathbf{R}^2 \times \{z\}$ has the metric of the sphere pulled back by inverse stereographic projection. (The radius of that sphere is fixed but arbitrary.) This model is missing a line, $Z^* = \{\infty\} \times \mathbf{R}$, which we append in a natural way to $\mathbf{R}^2 \times \mathbf{R}$ with the aforementioned product metric. It is easy to verify that a standard helicoid $H \subset \mathbf{R}^3$ with axis $Z = \{(0,0,z) : z \in \mathbf{R}\}$, an embedded and ruled surface, is also a minimal surface in $\mathbf{S}^2 \times \mathbf{R}$. Here, it has two axes, Z and Z^*. By a slight abuse of notation, we will use H to refer to this minimal surface in $\mathbf{S}^2 \times \mathbf{R}$.

The horizontal lines on the euclidean helicoid are great circles in the totally geodesic level-spheres of $\mathbf{S}^2 \times \mathbf{R}$, the circle at height z passing through the antipodal points $(\mathbf{0}, z) \in Z$ and $(\infty, z) \in Z^*$. Let

$$X = (\mathbf{S}^2 \times \{0\}) \cap H,$$

and denote by Y the great circle at height 0 passing through $O = (\mathbf{0}, 0)$, $O^* = (\infty, 0)$, and orthogonal to the great circle X. Just as on the Euclidean helicoid, ρ_Y, order-two rotation about Y, is an orientation preserving involution of H. Note that under our identification of $\mathbf{S}^2 \times \mathbf{R}$ with $\mathbf{R}^3$, each of the great circles on H corresponds to a horizontal line passing throught the z-axis, and the great circles X and Y are identified with the x- and y-axes of $\mathbf{R}^3$.

Denote by H^+ the component of the complement of H that contains $Y^+ := \{(0, y, 0) \mid y > 0\}$. Then for any $c > 0$, ρ_Y is an orientation preserving involution of the domain

$$N_c = H^+ \cap \{|z| < c\}. \tag{4}$$

Note that ∂N_c is mean convex, consisting of three minimal surfaces: $H \cap \{|z| < c\}$, and two totally geodesic hemispheres, $H^+ \cap \{z = \pm c\}$. We will label these minimal surfaces H_c and $S_{\pm c}$, respectively.

The set $H_c \setminus (Z \cup Z^* \cup X)$ has four components. Let Q be the component whose boundary contains the three geodesics $X^+ = \{(x, 0, 0) \mid x \geq 0\}$, $Z \cap \{0 \leq z \leq c\}$, and $Z^* \cap \{0 \leq z \leq c\}$. The "quadrant" Q has a fourth boundary curve, which is one of the two semicircular components of $\partial S_c \setminus (Z \cup Z^*)$. We label this semicircle T_c. Note that $T_{-c} := \rho_Y(T_c)$ lies in $\partial(\rho_Y(Q))$.

Fix a value of c and let $N = N_c$. Consider the union $Q \cup \rho_Y(Q)$, and define $\Gamma \subset \partial N$ to be the boundary of $Q \cup \rho_Y(Q)$. Then

$$\Gamma = (Z \cap H_c) \cup T_c \cup (Z^* \cap H_c) \cup T_{-c} \cup X. \tag{5}$$

See Figure 1. The first four segments of Γ form a piecewise smooth curve with four corners. Adding the great circle X produces a curve that is singular at $O = (\mathbf{0}, 0)$ and at $O^* = (\infty, 0)$, where there are right-angle crossings. Note that Γ is ρ_Y-invariant.

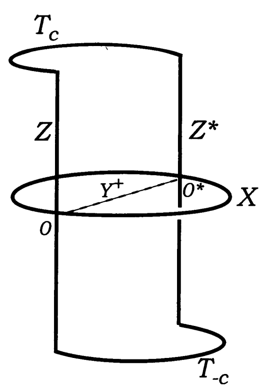

FIGURE 1. The curve Γ. In the figure, we have taken $\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$ as our model for $\mathbf{S}^2 \times \mathbf{R}$, with the metric on $\mathbf{R}^2$ given by the pullback of the metric on $\mathbf{S}^2$ via inverse stereographic projection. In this case, the pole of $\mathbf{S}^2$ is placed at the center of the semicircle Y^-.

If Γ defined in (5) is not Y-bumpy, we can make arbitrarily small perturbations of the curves $T_{\pm c}$ to make it so, while keeping the resulting curve in ∂N, and also ρ_Y-invariant. *We will assume from now on that* Γ *is* Y*-bumpy.*

Suppose for the moment that we could produce a connected Y-surface $M \subset N$ with boundary Γ. We will show in the next paragraph how this will enable us to construct a higher-genus helicoid.

Since $\rho_Y|_M$ is orientation preserving, Y must intersect M orthogonally in a discrete set of points, precisely the fixed points of $\rho_Y|_M$. We will consider M without its boundary, allowing us to apply Proposition 4.1. Namely, if $k = |Y \cap M|$, the number of points in $Y \cap M$, then

$$k = 2 - \chi(M).$$

Extend M by ρ_Z, Schwarz reflection in Z (or equivalently in Z^*), and let

$$\tilde{M} = \text{interior}(\overline{M \cup \rho_Z(M)}). \tag{6}$$

The surface $\tilde{M}$ is smooth because M is ρ_Y-symmetric, and

$$|Y \cap \tilde{M}| = 2k + 2$$

because the points $O = (\mathbf{0}, 0)$ and at $O^* = (\infty, 0)$, which lie on Y, are in $\tilde{M}$. The surface $\tilde{M}$ is bounded by two great circles at levels $\pm c$. It is embedded because $\rho_Z(M)$ lies in H^-. Furthermore it is ρ_Y-invariant by construction and satisfies the condition that ρ_Y acts by multiplication by -1 on $H_1(M, Z)$. Therefore, $2k + 2 = 2 - \chi(\tilde{M})$ by Proposition 4.1. Since $\tilde{M}$ has two ends, we have

$$2k + 2 = 2 - (2 - 2\,\text{genus}(\tilde{M}) - 2),$$

or

$$\text{genus}(\tilde{M}) = k.$$

If we can produce $\tilde{M} = \tilde{M}_c$ for any cutoff height c, it is reasonable to expect that as $c \to \infty$, the $\tilde{M}_c$ converge subsequentially to an embedded genus-k minimal surface each of whose ends is asymptotic to H or a rotation of H. In [**3**], we prove that this is the case.

5.2. Existence of a suitable $M \in \mathcal{M}_Y^*(N, \Gamma)$ with $|Y \cap M| = k$. — How are we going to produce, for each positive integer k, a connected, embedded, minimal Y-surface $M \subset N$ with boundary Γ? The answer is: by induction on k, using Theorem 4.7. The details, carried out in [**3**] are somewhat intricate. We describe here the main idea and the intuition behind the proof.

First of all, it would seem that Theorem 4.7 is not suited to prove existence of the desired surfaces because in most cases it asserts that the number of surfaces in a given class is even. This could mean that there are *zero* surfaces in the class. We begin to address this problem by dividing the class of surfaces according to their geometric behavior near O. Why this helps will be made clear below.

Since we are working with one fixed domain, namely $N = N_c$ as defined in (4), we will suppress the reference to N and write $\mathcal{M}_Y^*(\Gamma)$ instead of $\mathcal{M}_Y^*(N, \Gamma)$. We can decompose $\mathcal{M}_Y^*(\Gamma)$ into two sets by looking at how a surface $S \in \mathcal{M}_Y^*(\Gamma)$ attaches to Γ at the crossing O, the intersection of the vertical line Z and the great circle X. The geodesics X, Z, and Z^* divide H into four "quadrants". A quadrant whose boundary contains $Z^+ \cup X^+$ or $Z^- \cup X^-$will be called a *positive* quadrant. The other two quadrants will be called *negative* quadrants.

5.3. Definition. — Given a nonnegative integer k,

$\mathcal{M}_Y^*(\Gamma, k) \subset \mathcal{M}_Y^*(\Gamma)$ is the collection of embedded minimal Y-surfaces M with the property that $|M \cap Y| = k$.

$\mathcal{M}_Y^*(\Gamma, k, +) \subset \mathcal{M}_Y^*(\Gamma, k)$ is the subset of surfaces tangent to the positive quadrants at O.

$\mathcal{M}_Y^*(\Gamma, k, -) \subset \mathcal{M}_Y^*(\Gamma, k)$ is the subset of surfaces tangent to the negative quadrants at O.

Now we approximate Γ by smooth embedded curves $\Gamma(t) \subset \partial N$. We have to do this in order to apply any of our parity theorems. We want the four corners to be rounded and the two crossings to be resolved. At O, we modify Γ in a small neighborhood of radius $t > 0$ by connecting Z^+ to X^+ and Z^- to X^-. Given this choice at O, we resolve the crossing at O^* according to whether k is even or odd as follows: connect positively if k is even (i.e. Z^+ to X^+ and Z^- to X^-) and negatively (i.e. Z^+ to X^- and Z^- to X^+) if k is odd. Again we modify in a manner that preserves ρ_Y-invariance, and we choose t small enough so that the neighborhoods of the corners and the crossings are pairwise-disjoint. We will refer to such a rounding as an *adapted positive rounding of* Γ. Note that when k is odd, an adapted positive rounding of Γ is connected, while when k is even, such a rounding has two components. See Figure 2.

Our motivation for the choice of desingularization at O^* is given by the following

5.4. Proposition. — *A surface $S \in \mathcal{M}_Y^*(\Gamma, k, +)$ is tangent at O^* to the positive quadrants if k is even, and to the negative quadrants if k is odd.*

Proof. — For any oriented surface S, we have (3)

$$\chi(S) = 2c(S) - 2\ \mathrm{genus}(S) - e(S),$$

where $e(s)$ is the number of ends of S, $c(S)$ is the number of components of S, and genus(S) is the sum of the genera of the components of S. If $S \in \mathcal{M}_Y^*(\Gamma, k)$, then using Proposition 4.1 we have

$$k = |Y \cap S| = 2 - \chi(S) \cong e(S), \tag{7}$$

where $\cong$ denotes equivalence mod 2.

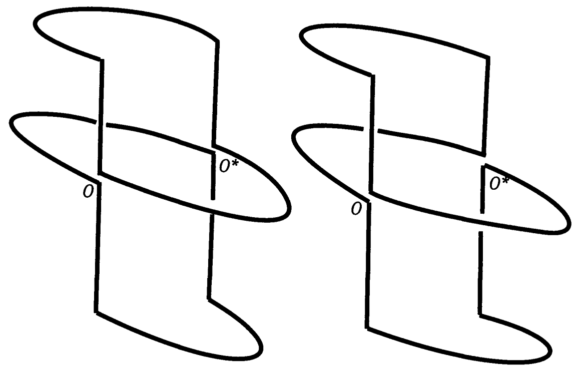

FIGURE 2. The two adapted positive roundings of Γ. On the left, the rounding at O^* is the same as at the point O, resulting in a curve with two components. On the right the rounding at O^* is positive to negative, resulting in a connected curve.

Claim. — *If $S \in \mathcal{M}(\Gamma, k, +)$, then* $e(S) = \begin{cases} 2 & \textit{if } S \textit{ is positive at } O^*, \\ 1 & \textit{if } S \textit{ is negative at } O^*. \end{cases}$

The proposition follows from the claim and the congruence (7).

Proof of Claim. — Let $B(O)$ be a geodesic ball of radius $r > 0$ centered at O, and let $B(O^*)$ be the corresponding ball centered at O^* with the same radius. We may choose r small enough so that the surface $S' = S \setminus (B(O) \cup B(O^*))$ has the same number of ends as S: i.e., $e(S') = e(S)$. We may make r smaller if necessary so that near O (say in a geodesic ball of radius $2r$ centered at O), the boundary curve $\Gamma' = \partial S'$ consists of a segment of X^+ joined to a segment of Z^+ by a single curve in $\partial B(O)$ together with a segment of X^- joined to a segment of Z^- by a single curve in $\partial B(O)$. It is precisely here that we have used the fact that $S \in \mathcal{M}^*_Y(\Gamma, k, +)$ and not just in $\mathcal{M}^*_Y(\Gamma, k)$. Making r smaller if necessary, we may assert that if S is tangent to the positive quadrants at O^*, then near O^* the curve Γ' connects positively, just as it does near O. This implies that Γ' has two components. Therefore $e(S') = 2$. If S is tangent to the negative quadrants at O^*, then near O^* the curve Γ' will connect X^+ to Z^- and X^- to Z^+. In this case, Γ' is connected and $e(S') = 1$. Since we chose r small enough so that $e(S') = e(S)$, we have proved the claim. $\square$

Let $\Gamma(t)$, $t > 0$ small, be a smooth family of adapted positive roundings of Γ. We will round in such a way that for each corner and crossing q,

$$\lim_{t \to 0} (1/t)(\Gamma(t) - q)$$

is a smooth embedded curve, and such that $\Gamma(t)$ converges smoothly to Γ except perhaps at the corners and crossings of Γ. It is now reasonable to expect that if we specify a surface $M \in \mathcal{M}^*_Y(\Gamma_t, k)$ as a sort of initial data at $\Gamma = \Gamma(0)$ we can deform it to a family of embedded minimal Y-surfaces $S_t \subset N$ with $\partial S_t = \Gamma(t)$. In fact we can do this in a unique manner.

5.5. Definition. — For any nonegative integer j, the set $\mathcal{M}^*_Y(\Gamma(t), j)$ is the collection of embedded minimal Y-surfaces $S \subset N$ with $\partial S = \Gamma(t)$ and $|S \cap Y| = j$

5.6. Theorem. — *Let $N = N_c \subset \mathbf{S}^2 \times \mathbf{R}$ be a domain of the form given in (4) for some fixed positive constant c. Let Γ be the curve specified in (5), perturbed if necessary to become Y-bumpy.*

Let $\Gamma(t)$, $t > 0$ small, be a smooth family of adapted positive roundings of Γ. Suppose for some nonnegative integer j, that there exists a surface $M \in \mathcal{M}^*_Y(\Gamma, j)$. Then there exists a constant $a = a(\Gamma, M) > 0$ such that for $t < a$, each approximating curve $\Gamma(t)$ bounds an embedded minimal Y-surface S_t with the following properties:

1. *Each S_t is the normal graph over a region $\Omega_t \subset \tilde{M}$ that is bounded by the projection of $\Gamma(t)$ onto $\tilde{M}$;*
2. *The family of surfaces S_t is smooth in t and converges smoothly to M as $t \to 0$;*
3. *If $M \in \mathcal{M}^*_Y(\Gamma, j, +)$, then $S_t \in \mathcal{M}^*_Y(\Gamma(t), j)$, i.e. $|S_t \cap Y| = j$;*
4. *If $M \in \mathcal{M}^*_Y(\Gamma, j, -)$, then $S_t \in \mathcal{M}^*_Y(\Gamma(t), j+2)$, i.e. $|S_t \cap Y| = j+2$.*

Furthermore, if $\hat{S} \in \mathcal{M}^*_Y(\Gamma(t_0), j)$, $t_0 < a$, then it lies in a smooth one-parameter family of surfaces $S_t \in \mathcal{M}^*_Y(\Gamma(t), j)$, $t \le t_0$, with the property that the family has, as a smooth limit as $t \to 0$, an embedded minimal Y-surface $M \subset N$ that lies either in $\mathcal{M}^*_Y(\Gamma, j)$ or in $\mathcal{M}^*_Y(\Gamma, j-2)$.

Statements (3) and (4) have a simple geometric interpretation. Suppose we have a family of surfaces in $S_t \in \mathcal{M}^*_Y(\Gamma(t), k)$ for some smooth family $\Gamma(t)$ of adapted positive roundings of Γ. They will limit to an embedded minimal Y-surface $M \subset N$ with boundary Γ. If they limit to an $M \in \mathcal{M}^*_Y(\Gamma, j, +)$, then the points $S_t \cap Y$ stay bounded away from the crossings $\{O, O^*\}$. Hence the S_t have the property that $|S_t \cap Y| = |M \cap Y| = j$. However, if they limit to an $M \in \mathcal{M}^*_Y(\Gamma, j, -)$, then each of the S_t is a graph over a region Ω_t that contains both O and O^*. Two points are lost. Hence $j = |S_t \cap Y| = |M \cap Y| + 2$.

Since the theorem above tells us that there is a correspondence between every surface in $\mathcal{M}(\Gamma(t), k)$ and some embedded minimal Y-surface in N bounded by Γ, we have

5.7. Corollary. — *We have*

$$|\mathcal{M}^*_Y(\Gamma(t), k)| = |\mathcal{M}^*_Y(\Gamma, k, +)| + |\mathcal{M}^*_Y(\Gamma, k-2, -)|.$$

We can now carry out the induction. We use $\cong$ to denote congruence modulo 2. In our situation, the number of ends of a surface $S \in \mathcal{M}_Y^*(\Gamma, k)$ is one or two, so the number of components of S is at most two. Since S is a Y-surface we know, by Proposition 4.1, that $k = |S \cap Y| = 2 - \chi(S)$. It is easy to see that when $k = 1$ (or $k = 0$), S is a disk (or the union of two disks). Corollary 5.7 and Theorem 4.7 yield in this situation that

$$1 \cong |\mathcal{M}_Y^*(\Gamma(t), k)| = |\mathcal{M}_Y^*(\Gamma, k, +)| + |\mathcal{M}_Y^*(\Gamma, k-2, -)| = |\mathcal{M}_Y^*(\Gamma, k, +)|,$$

the last equality being simply the fact that it is impossible for a surface to intersect Y in a negative number of points. Therefore we have established the existence of the desired surface for $k = 0$ or $k = 1$. In fact we get existence of a surface in $\mathcal{M}_Y^*(\Gamma, k, +)$. However there is nothing special in this context about being in $\mathcal{M}_Y^*(\Gamma, k, +)$ as opposed to being in $\mathcal{M}_Y^*(\Gamma, k, -)$. If we redid the entire construction by starting out by requiring our smoothing to be negative at O, we would wind up with an odd number of surfaces in $\mathcal{M}_Y^*(\Gamma, k, -)$, for $k = 0$ and $k = 1$.

Now assume $k \geq 2$, and suppose that for any $j < k$, that $|\mathcal{M}_Y^*(\Gamma, j, +)| \cong \mathcal{M}_Y^*(\Gamma, j, -) \cong 1$. Corollary 5.7 together with Theorem 4.7 yield in our situation that

$$0 \cong |\mathcal{M}_Y^*(\Gamma(t), k)| = |\mathcal{M}_Y^*(\Gamma, k, +)| + |\mathcal{M}_Y^*(\Gamma, k-2, -)|.$$

But $|\mathcal{M}_Y^*(\Gamma, k-2, -)| \cong 1$, by assumption. Therefore $0 \cong |\mathcal{M}_Y^*(\Gamma, k, +)| + 1$, or

$$|\mathcal{M}_Y^*(\Gamma, k, +)| \cong 1.$$

Hence, this class of surfaces is not empty for any value nonnegative integer k. As indicated above the same is true for $\mathcal{M}_Y^*(\Gamma, k, -)$. Whether or not we have produced two geometrically *different* (i.e. non-congruent) solutions to our problem turns out to depend on whether k is even or odd—but that is another story.

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D. HOFFMAN, Department of Mathematics, Stanford University, Stanford, CA 94305, USA
E-mail : `hoffman@math.stanford.edu`

BRIAN WHITE, Department of Mathematics, Stanford University, Stanford, CA 94305, USA
E-mail : `white@math.stanford.edu`

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PETER SARNAK

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EQUIDISTRIBUTION AND PRIMES

by

Peter Sarnak

*To Jean Pierre Bourguignon**

Abstract. — We begin by reviewing various classical problems concerning the existence of primes or numbers with few prime factors as well as some of the key developments towards resolving these long standing questions. Then we put the theory in a natural and general geometric context of actions on affine n-space and indicate what can be established there. The methods used to develop a combinational sieve in this context involve automorphic forms, expander graphs and unexpectedly arithmetic combinatorics. Applications to classical problems such as the divisibility of the areas of Pythagorean triangles and of the curvatures of the circles in an integral Apollonian packing, are given.

***Résumé* (Équidistribution des nombres premiers).** — Nous commençons par l'examen de divers problèmes classiques concernant l'existence de nombres premiers ou de nombres avec peu de facteurs premiers, ainsi que quelques-uns des développéments clés vers la résolution de ces questions posées il y a bien longtemps. Ensuite, nous plaçons la théorie dans un contexte géométrique naturel et général d'actions sur le n-espace affine et nous indiquons ce qui peut être établi dans ce contexte. Les méthodes utilisées pour développer un crible combinatoire dans ce contexte impliquent les formes automorphes, les graphes d'expansion et, de manière inattendue, les combinatoires arithmétiques. Nous fournissons des applications aux problèmes classiques, tels que la divisibilité des aires des triangles pythagoriens et les courbures des circles dans un paquetage apollonien entier.

I have chosen to talk on this topic because I believe it has a wide appeal and also there have been some interesting developments in recent years on some of these classical problems. The questions that we discuss are generalizations of the twin prime conjecture; that there are infinitely many primes p such that $p+2$ is also a prime. I

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* This is an expanded version of the lecture that I had intended to give at the conference honoring Bourguignon on the occasion of his 60^{th} birthday and which I gave at the Pacific Institute of Mathematical Sciences in 2007.

am not sure who first asked this question but it is ancient and it is a question that occurs to anyone who looks, even superficially, at a list of the first few primes. Like Fermat's Last Theorem there appears to be nothing fundamental about this problem. We ask it simply out of curiosity. On the other hand the techniques, theories and generalizations that have been developed in order to understand such problems are perhaps more fundamental.

Dirichlet's Theorem. — In many ways this theorem is still the center piece of the subject. Like many landmark papers in mathematics, Dirichlet's paper proving the theorem below, initiated a number of fields: abelian groups and their characters, L-functions, class number formulae... The theorem asserts that an arithmetic progression $c, c+q, c+2q, c+3q, \dots$ contains infinitely primes if and only if there is no obvious congruence obstruction. An obvious such obstruction would be say that c and q are both even or more generally that the greatest common divisor (c, q) of c and q is bigger than 1. Stated somewhat differently, let $L \neq 0$ be a subgroup of $\mathbb{Z}$, so $L = q\mathbb{Z}$ for some $q \geq 1$, and let $\mathcal{O} = c + L$ be the corresponding orbit of c under L, then $\mathcal{O}$ contains infinitely many primes iff $(c, q) = 1$ (strictly speaking this statement is slightly weaker since Dirichlet considers one-sided progressions and here and elsewhere we allow negative numbers and call $-p$ a prime if p is a positive prime).

Initial Generalizations. — There are at least two well known generalizations of Dirichlet's Theorem that have been investigated. The first is the generalization of his L-functions to ones associated with general automorphic forms on linear groups. This topic is one of the central themes of modern number theory but other than pointing out that these are used indirectly in proving some of the results mentioned below, I will not discuss them in this lecture. The second generalization is to consider other polynomials besides linear ones. Let $f \in \mathbb{Z}[x]$ be a polynomial with integer coefficients and let $\mathcal{O} = c + L$ as above. Does $f(\mathcal{O})$ contain infinitely many primes? For example if $\mathcal{O} = \mathbb{Z}$; is $f(x) = x^2 + 1$ a prime number for infinitely many x (a question going back at least to Euler). Is $f(x) = x(x+2)$ a product of two primes infinitely often? (this is a reformulation of the twin prime question). Neither of these questions have been answered and the answer to both is surely, yes. We will mention later what progress has been made towards them. In his interesting and provocative article "Logical Dreams" [**35**], Shelah puts forth the dream, that this question of Euler "cannot be decided". This is rather far fetched but for the more general questions about primes and saturation on very sparse orbits associated with tori that are discussed below, such a possibility should be taken seriously. We turn first in the next paragraph to several variables, that being the setting in which some problems of this type have been resolved.

Two Variables. — Let $\mathcal{O} = \mathbb{Z}^2$ and let f be a nonconstant polynomial in $\mathbb{Z}[x_1, x_2]$. If f is irreducible in $\mathbb{Q}[x_1, x_2]$ and the greatest common divisor of the numbers $f(x)$ with $x \in \mathcal{O}$ is 1, then it is conjectured that f takes on infinitely many prime values. In this higher dimensional setting we have found it more intrinsic and natural from many points of view to ask for more. That is the set of $x \in \mathcal{O}$ at which $f(x)$ is prime should not only produce an infinite set of primes for the values $f(x)$ but these (infinitely) many points should not satisfy any nontrivial algebraic relation. In the language of algebraic geometry, these points should be Zariski dense in the affine plane A^2. The Zariski topology on affine n-space A^n is gotten by declaring the closed sets to be the zero sets (over $\mathbb{C}$) of a system of polynomial equations. Thus a subset S of A^n is Zariski dense in A^n iff S is not contained in the zero set of a nontrivial polynomial $g(x_1, \ldots, x_n)$. In A^1 a set is the zero set of a nontrivial polynomial iff the set is finite. So the Zariski dense subsets of A^1 are simply the infinite sets. We denote the operation of taking the Zariski closure of a set in A^n by Zcl.

All the approaches to the conjecture that we are discussing involve giving lower bounds for the number of points in finite subsets of $\mathcal{O}$ at which $f(x)$ is prime. Usually one defines these sets by ordering by size of the numbers (so a big box in A^2) but in some variations of these problems that I discuss later quite different orderings are employed. A measure of the quality of the process is whether in the end the lower bound is strong enough to ensure the Zariski density of the points produced. As far as the conjecture that under the assumptions on f at the beginning of (4), the set of $x \in \mathcal{O}$ at which $f(x)$ is prime, is Zariski dense in A^2, the following is known:

(i) For f linear it follows from Dirichlet's theorem.

(ii) For f of degree two and f non-degenerate (in the sense of not reducing to a polynomial in one variable) it follows from Iwaniec [**23**]. His method uses the combinatorial sieve which we will discuss a bit further on, as well as the Bombieri-A. Vinogradov theorem which is a sharp quantitative version of Dirichlet's theorem (when counting primes p of size at most x and which are congruent to varying c modulo q, with q as large as $x^{1/2}$).

(iii) A striking breakthrough was made by Friedlander and Iwaniec [**10**]. It follows from their main result that the conjecture is true for $f(x_1, x_2) = x_1^2 + x_2^4$. They exploit the structure of this form in that it can be approached by examining primes $\alpha = a + b\sqrt{-1}$ in $\mathbb{Z}[\sqrt{-1}]$ with $b = z^2$. This was followed by work of Heath-Brown and the results in [**19**] imply that the conjecture is true for any homogeneous binary cubic form. They exploit a similar structure, in that such an $f(x_1, x_2)$ is of the form $N(x_1, x_2, 0)$ where $N(x_1, x_2, x_3)$ is the norm form of cubic extension of $\mathbb{Q}$, so that the problem is to produce prime ideals in the latter with one coordinate set to 0.

(iv) If $f(x_1, x_2)$ is reducible then we seek a Zariski dense set of points $x \in \mathbb{Z}^2$ at which $f(x)$ has as few as possible prime factors. For polynomials f of the special form $f(x) = f_1(x)f_2(x)\cdots f_t(x)$, with $f_j(x) = x_1 + g_j(x_2)$ where $g_j \in \mathbb{Z}[x]$ and $g_j(0) = 0$, it follows from the results in the recent paper of Tao and Ziegler [**36**] that the set of $x \in \mathbb{Z}^2$ at which $f(x)$ is a product of t primes, is Zariski dense in A^2. Equivalently the set of x at which $f_1(x), \dots, f_t(x)$ are simultaneously prime, is dense. This impressive result is based on the breakthrough in Green and Tao [**16**] in particular their transference principle, which is a tool for replacing sets of positive density in the usual setting of Szémeredi type theorems with a set of positive density in the primes. The corresponding positive density theorem is that of Bergelson and Leibman [**2**]. Note that for these f_j's there is no local obstruction to $x_1 + g_j(x_2)$ being simultaneously prime since for a given $q \geq 1$ we can choose $x_1 \equiv 1(q)$ and $x_2 \equiv 0(q)$ $(g_j(0) = 0)$. Apparently this is a feature of these positive density Szemerédi type theorems in that they don't allow for congruence obstructions.(*) The above theorem with $g_j(x_2) = (j-1)x_2$, $j = 1, \dots, t$ recovers the Green-Tao theorem, that the primes contain arbitrary long arithmetic progressions. From our point of view in paragraph (8) the amusing difference between the "existence of primes in an arithmetic progression" and that of an "arithmetic progression in the primes", will be minimized as they both fall under the same umbrella.

Hardy-Littlewood n-tuple Conjecture. — This is concerned with $\mathbb{Z}^n$ and subgroups L of $\mathbb{Z}^n$ acting by translations. If L is such a group denote by $r(L)$ its rank. We assume $L \neq 0$ so that $1 \leq r \leq n$ and also that for each j the coordinate function x_j restricted to L is not identically zero. For $c \in \mathbb{Z}^n$ and $\mathcal{O} = c + L$ the conjecture is about finding points in $\mathcal{O}$ all of whose coordinates are simultaneously prime. We state it as the following local to global conjecture:

HLC. — *If $\mathcal{O} = c + L$ as above then the set of $x = (x_1, \dots, x_n) \in \mathcal{O}$ for which the x_j's are simultaneously prime, is Zariski dense in $Zcl(\mathcal{O})$ iff for each $q \geq 1$ there is an $x \in \mathcal{O}$ such that $x_1x_2\dots x_n \in (\mathbb{Z}/q\mathbb{Z})^*$.*

Note that the condition on q, which is obviously necessary for the Zariski density, involves only finitely many q (for each given $\mathcal{O}$). Also to be more accurate, the conjecture in [**18**] concerns only the case of $r(L) = 1$ (which in fact implies the general case). In this case $Zcl(\mathcal{O})$ is a line and the conjecture asserts that there are infinitely many points in $x \in \mathcal{O}$ for which the n-tuples $(x_1, x_2, \dots x_n)$ are all prime

(*) Though the paper "Intersective polynomials and polynomial Szemerédi theorem" by V. Bergelson, A. Leibman and E. Lesigne posted on ArXiv Oct/25/07, begins to address this issue.

iff there is no local obstruction. We observe that for any r the $Zcl(\mathcal{O})$ is simply a translate of a linear subspace.

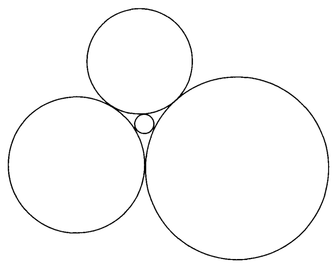
The main breakthrough on the HLC as stated above is due to I. Vinogradov (1937) in his proof of his celebrated "ternary Goldbach theorem", that every sufficiently large positive odd number is a sum of 3 positive prime numbers. His approach was based on Hardy and Littlewood's circle method, a novel sieve and the technique of bilinear estimates, see Vaughan [**37**]. It can be used to prove HLC for a non-degenerate L in $\mathbb{Z}^3$ of rank at least 2. Special cases of HLC in higher dimensions are established by Balog in [**1**] and recently Green and Tao [**15**] made a striking advance. Their result implies HLC for $L \leq \mathbb{Z}^4$ and $r(L) \geq 2$ and L non-degenerate in a suitable sense. Their approach combines Vinogradov's methods with their transference principle. It makes crucial use of Gowers' techniques from his proof of Szemerédi's theorem, and it has close analogies with the ergodic theoretic proofs of Szémeredi's theorem due to Furstenberg and in particular the work of Host and Kra [**22**]. These ideas have potential to establish HLC for $L \leq \mathbb{Z}^n$ of rank at least two (and non-degenerate), which would be quite remarkable.

Pythagorean Triples. — We turn to examples of orbits $\mathcal{O}$ in $\mathbb{Z}^n$ of groups acting by matrix multiplication rather than by translations (i.e. addition). By a Pythagorean triple we mean a point $x \in \mathbb{Z}^3$ lying on the affine cone C given as $\{x : F(x) = x_1^2 + x_2^2 - x_3^2 = 0\}$ and for which $gcd(x_1, x_2, x_3) = 1$. We are allowing x_j to be negative though in this example we could stick to all $x_j > 0$, so that such triples correspond exactly to primitive integral right triangles. Let O_F denote the orthogonal group of F, that is the set of 3×3 matrices g for which $F(xg) = F(x)$ for all x. Let $O_F(\mathbb{Z})$ be the group of all such transformations with entries in $\mathbb{Z}$. Some elements of $O_F(\mathbb{Z})$ are

$$A_1 = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -2 \\ 2 & 2 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}, A_3 = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

In fact $O_F(\mathbb{Z})$ is generated by A_1, A_2 and A_3. It is a big group and one can show that the set of all Pythagorean triples P is the orbit of $(3,4,5)$ under $O_F(\mathbb{Z})$, i.e. $P = (3,4,5)O_F(\mathbb{Z})$. Following the lead of Dirichlet, let L be a subgroup of $O_F(\mathbb{Z})$ and let $\mathcal{O} = (3,4,5)L$ be the corresponding orbit of Pythagorean triples. The area $A(x) = x_1x_2/2$ of the corresponding triangle is in $\mathbb{Q}[x_1, x_2, x_3]$. We seek triangles in $\mathcal{O}$ for which the area has few prime factors. What is the minimal divisibility of the areas of a Zariski dense (in $Zcl(\mathcal{O})$, which for us will be equal to C) set of triples in $\mathcal{O}$? We return to this later on. As a side comment, a similar problem asks which numbers are the square free parts of the areas of Pythagorean triangles in P? This is the ancient "congruent number problem" about which much has been written especially because of its connection to the question of the ranks of a certain family of elliptic curves. Heegner [**20**] using his precious method for producing rational points on elliptic curves shows that any prime $p \equiv 5$ or $7 \bmod 8$ is a congruent number. For a given such p the set of triangles realizing p is very sparse but never-the-less is Zariski dense in C. Via the same relation the congruent number problem is connected to automorphic L-functions through the Birch and Swinnerton-Dyer Conjecture (see [**38**]).

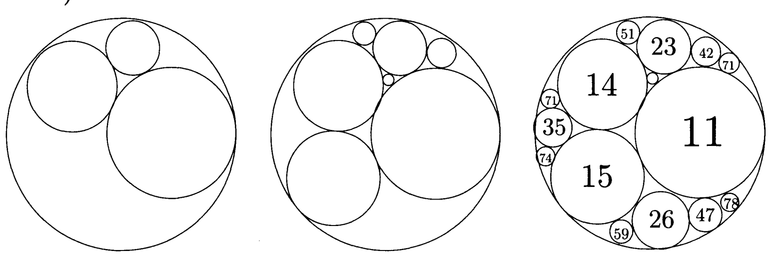
Integral Apollonian Packings. — As a final example before putting forth the general theory we discuss some Diophantine aspects of integral Apollonian packings. Descartes is well known among other things for his describing various geometric facts in terms of his Cartesian coordinates. One such example is the following relation between four mutually tangent circles:



If the radius of the j^{th} circle is R_j then its curvature a_j is equal to $1/R_j$, $j = 1, 2, 3, 4$. The relation is that

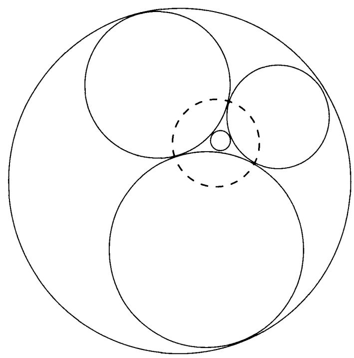
$$F(a_1, a_2, a_3, a_4) := 2\left(a_1^2 + a_2^2 + a_3^2 + a_4^2\right) - \left(a_1 + a_2 + a_3 + a_4\right)^2 = 0\,.$$

Consider now an Apollonian packing which is defined as follows; starting with 4 tangent circles of the first generation in Figure 2 (in this configuration the outer circle has all the other circles in its interior so by convention its curvature is $-1/R$ where R is its radius).



Génération 1 Génération 2 $a = (-6,\ 11,\ 14,\ 23)$

Now place a circle in each of the 4 lune regions in generation 1 so that these are tangent to the three circles that bound the lune. The placement is possible and is unique according to a theorem of Apollonius. At generation 2, there are now 12 new lunes and we repeat the process ad-infinitum. The resulting packing by circles is called an Apollonian packing. The complement of all the open disks in the packing is a closed fractal set whose Hausdorff dimension δ is approximately 1.30. Boyd [**7**] has shown that if $N(T)$ is the number circles in the packing whose curvature is at most T, then $\log N(T)/\log T \to \delta$ as $T \longrightarrow \infty$. The interesting Diophantine fact is that if the initial 4 circles have integral curvatures then so do all the rest of the circles in the packing. This is apparent in the example in Figure 2 where the initial 4 circles have curvatures (-6, 11, 14, 23) and where the curvatures of each circle is displayed in the circle. It is customary in any lecture to offer at least one proof. Ours is the demonstration of this integrality of curvatures.



In this figure the inversion S in the dotted circle, which is the unique circle orthogonal to the inside circles, takes the outermost circle to the innermost one and fixes the other three. It takes the 4-tuple (a_1, a_2, a_3, a_4) representing the curvatures of the 4 outer circles to (a_1', a_2, a_3, a_4) where a_1' is the curvature of the inner most circle. From the Descartes relation it follows that a_1 and a_1' are roots of the same quadratic equation and a simple calculation yields that $a_1' = -a_1 + 2a_2 + 2a_3 + 2a_4$. This inversion is also the step which places the circle in the corresponding lune, that being a single step in the Apollonian packing. It follows that if the 4×4 integral involutions S_1, S_2, S_3, S_4 are given by

$$S_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}, S_2 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}, S_3 = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}, S_4 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and A is the group generated by S_1, S_2, S_3, S_4 then the orbit $\mathcal{O} = (a_1, a_2, a_3, a_4)A$, corresponds precisely to the configurations of 4 mutually tangent circles in the packing. Hence if $a \in \mathbb{Z}^4$ and is primitive then so is every member of the orbit and in particular every curvature is an integer. The Diophantine properties of the numbers that are curvatures of an integral packing are quite subtle and are investigated in [**13**]. The reason for the subtlety is that the Apollonian group A is clearly a subgroup of $O_F(\mathbb{Z})$ but it is of infinite index in the latter (corresponding to the fractal dimension $\delta = 1.30\ldots$). Still, the $Zcl(A)$ is all of O_F, which is important for our investigation below. From the point of view of our theme in this lecture the immediate question is whether there are infinitely many circles in an integral packing with curvature a prime number. Or on looking at Figure 2, are there infinitely many "twin primes" that is pairs of tangent circles with curvatures that are both prime?

Affine Orbits and Saturation. — There is a simple and uniform formulation of all the questions above which is as follows: Let L be a group of morphisms (that is polynomial maps) of affine n-space which preserves $\mathbb{Z}^n$. Let $c \in \mathbb{Z}^n$ and $\mathcal{O} = cL$ the corresponding orbit. If $f \in \mathbb{Q}[x_1, \ldots, x_n]$ for which $f(\mathcal{O})$ is integral and is infinite, we seek points $x \in \mathcal{O}$ at which $f(x)$ has few (or fewest) prime factors. We assume that f when restricted to $\mathcal{O}$ is primitive, that is $\gcd(f(\mathcal{O})) = 1$ (otherwise divide f by the gcd). The key definition is the *saturation number* $r_0(\mathcal{O}, f)$, of the pair $(\mathcal{O}, f)$, which is the least r such the set of $x \in \mathcal{O}$ for which $f(x)$ has at most r prime factors, is Zariski dense in $Zcl(\mathcal{O})$. This number is by no means easy to determine and it is far from clear that it is even finite. Knowing it however answers all our questions. For example the following are easy to check

(i) $r_0(c + q\mathbb{Z}, x) = 1$ is Dirichlet's Theorem.
(ii) $r_0(\mathbb{Z}, x(x+2)) = 2$ is the twin prime conjecture.
(iii) If $f \in \mathbb{Z}[x]$ and f factors into t irreducible factors over $\mathbb{Q}[x]$, then $r_0(\mathbb{Z}, f) = t$ is equivalent to Schinzel's hypothesis H [**34**] concerning simultaneous primality of t distinct irreducible integral polynomials in one variable.
(iv) Let $\mathcal{O} = c + L$ as in the HLC in (5). Then $r_0(\mathcal{O}, x_1 x_2 \ldots x_n) = n$ is equivalent to the HLC as stated in (5).

The fundamental general tool to study r_0 is the Brun combinatorial sieve. He used his ingenious invention to show that $r_0(\mathbb{Z}, x(x+2))$ is finite and his arguments can be easily extended to show that $r_0(\mathbb{Z}, f) < \infty$ for any $f \in \mathbb{Z}[x]$. In fact the combinatorial sieve in any of its axiomatic modern formulations can be used to show that $r_0(\mathcal{O}, f) < \infty$ for any orbit $\mathcal{O}$ of L which is a subgroup of $\mathbb{Z}^n$ acting by additive translations. As pointed out at the end of paragraph (2) above we insist on not restricting $f(x)$ to be positive when looking for primes or numbers with few prime factors. The reason is that in this several variable context the condition that $f(x) > 0$, $f \in \mathbb{Z}[x_1, \ldots, x_n]$ can encode the general diophantine equation (for example if $f(x) = 1 - g^2(x)$ then $f(x) > 0$ is equivalent to $g(x) = 0$). The work of Matiyasevich et al [**26**] on Hilbert's 10^{th} problem shows that given any recursively enumerable subset S of the positive integers, there is a $g \in \mathbb{Z}[x_1, \ldots, x_{10}]$ such that S is exactly the set of positive values assumed by g. From this it is straight forward to construct an $f \in \mathbb{Z}[x_1, \dots, x_{10}]$ such that for any $r < \infty$, $\{x \in \mathbb{Z}^{10} : f(x) > 0$ and $f(x)$ is a product of at most r primes$\}$ is not Zariski dense in $Zcl\{x \in \mathbb{Z}^{10} : f(x) > 0\}$. That is if we insist on positive values for f we may lose the basic finiteness of saturation property.

Returning to one variable the theory of the sieve has been developed and refined in far-reaching ways to give good bounds for r_0. For example:

$$r_0(\mathbb{Z}, x(x+2)) \leq 3 \qquad \text{(Chen 1973)}$$
$$r_0(\mathbb{Z}, x^2+1) \leq 2 \qquad \text{(Iwaniec 1978)}$$
$$r_0(\mathbb{Z}, f) \leq d+1, \quad \text{if } f \text{ is irreducible over } \mathbb{Q}[x] \text{ and has degree } d \text{ [17]}.$$

The first two are especially striking as they come as close as possible to the twin prime and Euler problems, without solving them.

While there are interesting examples of groups L acting nonlinearly and morphically on A^n and preserving $\mathbb{Z}^n$, that come from the actions of mapping class groups on representation varieties [**11**], the understanding of anything about saturation numbers in such cases is very difficult and is at its infancy. For L acting linearly (as in paragraphs (6) and (7)) a theory can be developed.

An Affine Linear Sieve. — The classical setting is concerned with motions of n-space of the form $x \longrightarrow x + b$. In this affine linear setting we allow multiplication as well, that is transformations of the form $x \longrightarrow xa + b$ with $a \in GL_n(\mathbb{Z})$ and $b \in \mathbb{Z}^n$ (such as the orthogonal group examples (6) and (7)). Note that it is only for $n \geq 2$ that this group of motions is significantly larger than translations (since $GL_1(\mathbb{Z}) = \pm 1$). For the purpose of developing a Brun combinatorial sieve, apparently multiplication is quite a bit more difficult than addition. The basic problems for our pair $(\mathcal{O}, f)$ are

(i) Is $r_0(\mathcal{O}, f)$ finite?

(ii) If it is, then to give good upper bounds for $r_0(\mathcal{O}, f)$. Ideally these should be in terms of the degree of f and its factorization in the coordinate ring of $Zcl(\mathcal{O})$, as has been done in the setting of one variable [**17**].

(iii) To determine $r_0(\mathcal{O}, f)$ for some interesting pairs and to give an algorithm to predict its exact value in general, that is a generalized local to global conjecture for which HLC and Schinzel's Hypothesis H, are special cases.

When L is a group of affine linear transformations we now have a theory that comes close to answering these questions, there being the caveat of tori (see below) and some other nontrivial technical issues that still need to be resolved in general. In Bourgain-Gamburd-Sarnak ([**4**], [**5**]) the finiteness of $r_0(\mathcal{O}, f)$ is proven in many cases. The new tools needed to address these questions, as well as the general setup that we have been discussing are introduced in these papers. The proof given there of the finiteness does not yield any feasible values for $r_0(\mathcal{O}, f)$. In [**28**] the problem is studied in the case that L is a congruence subgroup of the $\mathbb{Q}$ points of a semi-simple linear algebraic group defined over $\mathbb{Q}$, such as the group $O_F(\mathbb{Z})$ in paragraphs (6) and (7) above (an affine linear action can be linearized by doubling the number of variables). For such congruence L's we develop the combinatorial sieve using tools from the general theory of automorphic forms on such groups and in particular make use of the strong bounds towards the general Ramanujan Conjectures that are now known (see [**31**], [**9**]). With this we get, in this congruence subgroup case, effective bounds for $r_0(\mathcal{O}, f)$ which in many such cases are of the same quality as what is known in one variable.

There is a lacuna in this affine linear sieve theory coming from tori. As we mentioned, allowing multiplication as well as addition, is what makes the problem hard and in fact pure multiplication is simply too hard and even the finiteness is questionable in that case. Consider the example of $L = \left\{ \left[\begin{smallmatrix} 3 & 1 \\ -1 & 0 \end{smallmatrix} \right]^n, n \in \mathbb{Z} \right\} \leq \mathrm{SL}_2(\mathbb{Z})$. L is infinite cyclic, $Zcl(L)$ is a torus and if $\mathcal{O} = (1,0) \cdot L$ then $Zcl(\mathcal{O})$ is the hyperbola $\{(x_1, x_2) = x_1^2 - 3x_1x_2 + x_2^2 = 1\}$. The orbit consists of pairs (F_{2n}, F_{2n-2}) $n \in \mathbb{Z}$ where F_m is the m^{th} Fibonacci number. This kind of sequence is too sparse both from the analytic and algebraic points of view to do any kind of (finite) sieve. While it is conjectured that F_m is prime for infinitely many m, as was pointed out to me by Lagarias, standard heuristic probabilistic considerations suggest a very different behavior for F_{2n}. Indeed $F_{2n} = F_n \cdot L_n$ where L_n is the n^{th} Lucas number and assuming a probabilistic model for the number of prime factors of a large integer in terms of its size and that F_n and L_n are independent leads to F_{2n} having an unbounded number of prime factors as $n \to \infty$. A precise conjecture along these lines is put forth in [**8**] (see Conjecture 5.1). In our language this asserts that if $\mathcal{O}$ is as above and $f(x_1, x_2) = x_1$ then $r_0(\mathcal{O}, f) = \infty$. It would be very interesting to produce an example of a pair $(\mathcal{O}, f)$ for which one can prove that $r_0(\mathcal{O}, f)$ is infinite. In view of the above we must steer clear of tori and the precise setting in which the affine linear sieve is developed (see [**29**]) is for linear L's for which the radical (the largest normal solvable subgroup) of the $\mathbb{Q}$ linear algebraic group $G := Zcl(L)$, contains no tori (the unipotent radical causes no difficulties). Applying this theory to the examples of orthogonal groups in (6) and (7) we obtain the following. Let $F(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2$ and $L \leq O_F(\mathbb{Z})$. Assume that L is not an elementary group (in particular not finite or abelian, in fact precisely that $Zcl(L)$ is either of the linear algebraic groups O_F or SO_F). If $\mathcal{O} = (3,4,5)L$, then $Zcl(\mathcal{O}) = C$ the affine cone; $F = 0$. For $f \in \mathbb{Z}[x_1, x_2, x_3]$ the results in [**5**] imply that $r_0(\mathcal{O}, f) < \infty$. In particular this applies to $f(x) = A(x) = x_1x_2/2$, the area. This says that given such an orbit of Pythagorean triangles (which may be very sparse!) there is an $r < \infty$ such that the set of triangles in $\mathcal{O}$ whose areas have at most r prime factors is Zariski dense in C. It is elementary that $\gcd(A(O)) = 6$. From the ancient parametrization of all the Pythagorean triples P (i.e. the $\mathbb{Q}$ morphism of A^2 into C) these are all of the form $(x_1, x_2, x_3) = (a^2 - b^2, 2ab, a^2 + b^2)$ with $a, b \in \mathbb{Z}$, $(a, b) = 1$ and not both odd, one sees that $A/6 = (a - b)(a + b)(ab)/6$. Now the last has at most two prime factors for only finitely many pairs (a, b). The set of (a, b) for which it has at most 3 prime factors lie in a finite union of curves in C (and if HLC is true for $\mathcal{O} = (2, 2, 0) + (3, 3, 1)\mathbb{Z}$, i.e. this rank one orbit in $\mathbb{Z}^3$, then these curves contain infinitely many points with $A/6$ having 3 prime factors). Hence for any $\mathcal{O}$ as above $r_0(\mathcal{O}, A/6) \geq 4$. The general local to global conjectures [**5**] then assert that $r_0(\mathcal{O}, A/6) = 4$ for any such orbit. Interestingly the recent advance in [**15**] mentioned in (5) above just suffices to prove that for the full set of Pythagorean triples P, $r_0(P, A/6) = 4$. Put another way the minimal divisibility of the areas of a Zariski dense set of Pythagorean triangles is 6 (here we include the forced factors 3 and 2). The deduction is immediate, set $a = 2x$ and $b = 3y$ in the ancient parametrization. Then $A/6 = xy(2x + 3y)\ (2x - 3y)$ and apply [**15**] to $\mathcal{O} = L = (1, 0, 2, 2)\mathbb{Z} + (0, 1, 3, -3)\mathbb{Z}$. For some other applications of [**16**] see Granville [**14**].

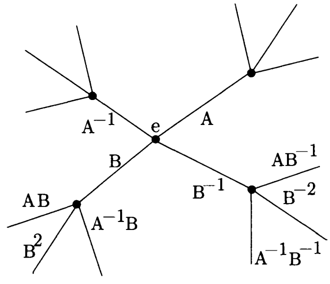
As an example of an application of the affine linear sieve in the context of an L which is a congruence group, consider an integral quadratic form $F(x)$ in 3-variables. That is $F(x) = x^t A x$ where A is 3×3 symmetric and is integral on the diagonal and half integral on the off-diagonal. We assume that F is indefinite over the reals but that it is anisotropic over $\mathbb{Q}$ (so $F(x) = 0$ for $x \in \mathbb{Z}^3$ implies that $x = 0$) and that $\det A$ is square free (so $F(x_1, x_2, x_3) = x_1^2 + x_2^2 - 7x_3^2$ is an example, the anisotropy following from looking at $F(x) \equiv 0 \bmod 8$). Let $0 \neq t \in \mathbb{Z}$ for which $V_t(\mathbb{Z}) = \{x \in \mathbb{Z}^3 : F(x) = t\}$ is nonempty, which according to the work of Hasse and Siegel will happen iff there are no local congruence obstructions to solving $F(x) \equiv t (\bmod q)$ for $q \geq 1$. In this case $V_t(\mathbb{Z})$ is a finite union of $O_F(\mathbb{Z})$ orbits and $Zcl(V_t(\mathbb{Z})) = V_t$, the affine quadric $\{x : F(x) = t\}$. We seek points in $V_t(\mathbb{Z})$ whose coordinates have few prime factors, i.e. to estimate $r_0(V_t(\mathbb{Z}), x_1x_2x_3)$. By the general finiteness theorem, $r_0(V_t(\mathbb{Z}), x_1x_2x_3)$ is finite. However by developing optimal weighted counting results on such quadrics and also exploiting the best bounds known towards the Ramanujan-Selberg Conjecture, it is shown in [**24**] that $r_0(V_t(\mathbb{Z}), x_1x_2x_3) \leq 26$.

We turn to the Apollonian packing. An extension of the $(\mathcal{O}, f)$ finiteness theorem in [**5**] applies to the orbit $\mathcal{O} = aL$ for any L which is Zariski dense in O_F, where F is the quadratic form in 4-variables in paragraph (7). In particular it applies to the Apollonian group A with $f(x) = x_1x_2x_3x_4$. This asserts that in any given integral packing there is an $r < \infty$ such that the set of 4 mutually tangent circles in the packing for which all 4 curvatures have at most r prime factors is Zariski dense in $Zcl(\mathcal{O}) = C = \{x : F(x) = 0\}$. One can determine r_0 for $\mathcal{O} = a.A$ and some special f's using some ad hoc and elementary methods together with (ii) of paragraph (4).

In [**32**] it is shown that $r_0(\mathcal{O}, x_1) = 1$ and $r_0(\mathcal{O}, x_1x_2) = 2$, from which it follows that in any such packing there are infinitely many circles whose curvatures are prime and better still there are infinitely many pairs of tangent circles both of whose curvatures are prime.

As a final example of an interesting pair $(\mathcal{O}, f)$ for which we can determine r_0, consider the variety V_t in affine n^2-space given by $V_t = \{X = (x_{ij})_{\substack{i=1,\dots n \\ j=1,\dots n}} : \det X = t\}$. For t a nonzero integer $V_t(\mathbb{Z})$ consists of a finite union of $L = \mathrm{SL}_n(\mathbb{Z})$ orbits where the action of g is by $X \longrightarrow X.g$. In [**28**] we show using Vinogradov's methods mentioned in (5), that if $n \geq 3$ then $r_0(V_t(\mathbb{Z}), \prod_{i,j} x_{ij}) = n^2$ if $\prod_{i,j} x_{ij}$ is primitive on $V_t(\mathbb{Z})$. Examining in detail when this happens, we deduce that the set of $n \times n$ integral matrices of determinant t all of whose entries are prime, is Zariski dense in V_t iff $t \equiv 0($ mod $2^{n-1})$. This should of course, also hold for $n = 2$ where it is concerned with the equation $x_{11}x_{22} - x_{12}x_{21} = t$ and the x_{ij}'s are to be primes. The best that appears to be known concerning this is the recent development by [**12**] from which it follows that for this $n = 2$ case, $r_0(V_t(\mathbb{Z}),\ x_{11}x_{12}x_{21}x_{22}) = 4$, for at least one t in $\{2,4,6\}$.

Comments about Proofs. — I end the lecture with a very brief hint as to what is involved in developing a combinatorial sieve in the affine linear context. This entails getting a little more technical. Let $\mathcal{O} = cL$ be our orbit and $f \in \mathbb{Z}[x_1, \dots, x_n]$. After some algebro-geometric reductions of the problems (using the $\mathbb{Q}$ dominant morphisms from $G = Zcl(L)$ to $V = Zcl(\mathcal{O})$ and $\widetilde{G}$ to G where $\widetilde{G}$ is the simply connected cover of G) we can assume that $\mathcal{O}$ is the group L itself (as a group of matrices in the affine space of $n \times n$ matrices) and $Zcl(\mathcal{O}) = Zcl(L) = G$ is a simply connected $\mathbb{Q}$-group. To do any kind of sieving we need to order the elements of L, so as to carry out some truncated inclusion-exclusion procedure, this being at the heart of Brun's method. Usually one orders by archimedian size perhaps with positive weights, however in this general setting we don't know how to do this, so we order L combinatatorially instead. For the groups that we are considering and for the purpose of proving that $r_0(\mathcal{O}, f)$ is finite, we can (according to a theorem of Tits) assume that L is free on two generators A and B. We use the tree structure of the Cayley graph $T = (L, S)$ of L with respect to the generators $S = \{A, A^{-1}, B, B^{-1}\}$. T is a 4-regular tree;



For $x, y \in T$ let $d(x,y)$ denote the distance from x to y in the tree. The key sums that arise in sieving on L for divisibility of f are:

For $d \geq 1$ square-free and $x_0 \in T$,

$$S(Y,d) := \sum_{\substack{x \in L \\ d(x,x_0) \leq Y \\ f(x) \equiv 0(d)}} 1 ,$$

or perhaps with 1 replaced by positive weights.

We are interested in $S(Y,d)$ when Y is large and d as large as $e^{\alpha Y}$ for some $\alpha > 0$. The larger the α for which S can be understood the better. To study such sums a couple of key features intervene:

(i) Algebraic stabilization: This is the analogue of the Chinese remainder theorem. We state it for the basic case of $G = \mathrm{SL}_n$, it is valid for G semisimple and simply connected. It is due (originally) to Matthews-Vaserstein and Weisfeiler [**27**] who employ the classification of finite simple groups in the proof. Let $L \leq \mathrm{SL}_n(\mathbb{Z})$ be Zariski dense in SL_n. Then there is a positive integer $\nu = \nu(L)$ such that for d with $(d,\nu) = 1$ the reduction $L \longrightarrow SL_n(\mathbb{Z}/d\mathbb{Z})$ is onto.

This eventually allows us to bring in more standard tools from arithmetic algebraic geometry, in order to identify the main term in the form

$$S(Y,d) = \beta(d)\, S(Y,1) + R(Y,d) .$$

Here $\beta(d)$ is a multiplicative arithmetical function associated with counting points $\bmod d$ on the variety $G \cap \{f = 0\}$ and R is the remainder which is expected to be smaller. The demonstration of the latter for the purpose of sieving far enough to get the finiteness of $r_0(\mathcal{O}, f)$, is essentially equivalent to the second feature.

(ii) The (finite) Cayley graphs $(\mathrm{SL}_n(\mathbb{Z}/d\mathbb{Z}), S)$ are an expander family as $d \longrightarrow \infty$ (see [**30**] for a definition of expanders and [**25**] where this is conjectured). As yet, this expander property has not been established in general and this is the main reason that the finiteness of $r_0(\mathcal{O}, f)$ has not been established in general for the affine linear sieve. It is proven for SL_2 and related groups for d square free, in [**5**]. The proof uses a variety of inputs some of which were to me at least, quite unexpected. We list them for the simpler case that $d = p$ is prime:

   (a) The dichotomy that an irreducible complex representation of $G(\mathbb{Z}/p\mathbb{Z})$ is either 1-dimensional or is of very large dimension (here $p \to \infty$) coupled with a "softer" *upper* bound density theorem for multiplicities of exceptional eigenvalues of the Cayley graphs, leads to a proof of the key spectral gap defining an expander [**33**]. For the soft upper bound we use techniques from arithmetic combinatorics.

(b) Sum-Product Theorem [**6**]: This is an elementary and very useful theorem concerning mixing the additive and multiplicative structures of a finite field. Let $\epsilon > 0$ be given, there is a $\delta > 0$, $\delta = \delta(\epsilon)$, such that if $A \subset \mathbb{F}_p$ and $|A| \leq p^{1-\epsilon}$ then $|A + A| + |A \cdot A| \geq |A|^{1+\delta}$ (here p is sufficiently large).

(c) Helfgott's $\mathrm{SL}_2(\mathbb{F}_p)$ Theorem [**21**]: Let $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that if $A \subset \mathrm{SL}_2(\mathbb{F}_p)$, A is not contained in a proper subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$ and $|A| \leq |\mathrm{SL}_2(\mathbb{F}_p)|^{1-\epsilon}$, then $|A \cdot A \cdot A| \geq |A|^{1+\delta}$.

(d) Balog-Szemerédi, Gowers Theorem: This is a purely combinatorial theorem from graph theory which is used in [**3**] to give the required upper bounds on counting closed circuits in the graph, and leads to a proof that $(\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}), S)$ is an expander family.

A point worth noting is that once the affine sieve is set up and gives lower bounds in our combinatorial group theoretic ordering, for points in $\mathcal{O}$ for which f has at most r prime factors, the expander property is used again and in a different way to demonstrate the Zariski density of these points.

To end let me highlight the fundamental difference between the additive translational counting and the affine linear counting which necessitates the introduction of expanders. In $\mathbb{Z}$ the boundary of a large interval is small compared with the size of the interval and the same is true uniformly for an arithmetic progression of common difference q in the interval, even for q almost as large as the interval length. On the other hand on a k-regular tree ($k \geq 3$) this is not true. Given a big ball B (or any large finite set), the size of the boundary ∂B is of the same order of magnitude as B. It is exactly the expander property that allows one to draw an effective approximation for the number of points in B lying in the orbit with a congruence condition.

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P. SARNAK, Princeton University & Institute for Advanced Study, Princeton, NJ, USA

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BLAINE LAWSON
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THE PROJECTIVE HULL OF CERTAIN CURVES IN $\mathbb{C}^2$

by

Reese Harvey, Blaine Lawson & John Wermer

Dedicated to Jean Pierre Bourguignon on the occasion of his sixtieth birthday

Abstract. — The projective hull $\widehat{X}$ of a compact set $X \subset \mathbb{P}^n$ is an analogue of the classical polynomial hull of a set in $\mathbb{C}^n$. In the special case that $X \subset \mathbb{C}^n \subset \mathbb{P}^n$, the affine part $\widehat{X} \cap \mathbb{C}^n$ can be defined as the set of points $x \in \mathbb{C}^n$ for which there exists a constant M_x so that

$$\left|p(x)\right| \leq M_x^d \sup_X |p|$$

for all polynomials p of degree $\leq d$, and any $d \geq 1$. Let $\widehat{X}(M)$ be the set of points x where M_x can be chosen $\leq M$. Using an argument of E. Bishop, we show that if $\gamma \subset \mathbb{C}^2$ is a compact real analytic curve (not necessarily connected), then for any linear projection $\pi : \mathbb{C}^2 \to \mathbb{C}$, the set $\widehat{\gamma}(M) \cap \pi^{-1}(z)$ *is finite for almost all* $z \in \mathbb{C}$. It is then shown that for any compact stable real-analytic curve $\gamma \subset \mathbb{P}^n$, the set $\widehat{\gamma} - \gamma$ *is a* 1*-dimensional complex analytic subvariety of* $\mathbb{P}^n - \gamma$. Boundary regularity for $\widehat{\gamma}$ is also discussed in detail.

***Résumé* (L'enveloppe projective de certaines courbes dans $\mathbb{C}^2$).** — L'enveloppe projective $\widehat{X}$ d'un compact $X \subset \mathbb{P}^n$ est l'analogue de l'enveloppe polynomiale classique d'un sous-ensemble de $\mathbb{C}^n$. Dans le cas particulier où $X \subset \mathbb{C}^n \subset \mathbb{P}^n$, la partie affine $\widehat{X} \cap \mathbb{C}^n$ peut être définie en tant qu'ensemble de points $x \in \mathbb{C}^n$ pour lesquels il existe une constante M_x telle que

$$\left|p(x)\right| \leq M_x^d \sup_X |p|$$

pour tous les polynômes p de degré $\leq d$, et tout $d \geq 1$. Soit $\widehat{X}(M)$ l'ensemble de points x où M_x peut être choisi $\leq M$. En utilisant un argument d'E. Bishop, nous montrons que si $\gamma \subset \mathbb{C}^2$ est une courbe analytique réelle compacte (non nécessairement connexe), alors pour toute projection linéaire $\pi : \mathbb{C}^2 \to \mathbb{C}$, l'ensemble $\widehat{\gamma}(M) \cap \pi^{-1}(z)$ *est fini pour presque tout* $z \in \mathbb{C}$. Nous montrons alors que pour toute courbe analytique réelle compacte stable $\gamma \subset \mathbb{P}^n$, l'ensemble $\widehat{\gamma} - \gamma$ *est une sous-variété de* $\mathbb{P}^n - \gamma$ *analytique complexe de dimension* 1. Nous discutons également en détail la régularité de la frontière de $\widehat{\gamma}$.

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1. Introduction

The classical *polynomial hull* of a compact subset X of $\mathbb{C}^n$ is the set of points $x \in \mathbb{C}^n$ such that

$$|p(x)| \leq \sup_X |p| \quad \text{for all polynomials } p. \tag{1.1}$$

In [**4**] the first two authors introduced an analogue for compact subsets of projective space. Given $X \subset \mathbb{P}^n$, the *projective hull of* X is the set $\widehat{X}$ of points $x \in \mathbb{P}^n$ for which there exists a constant $C = C_x$ such that

$$\|P(x)\| \leq C_x^d \sup_X \|P\| \quad \text{for all sections } P \in H^0(\mathbb{P}^n, \mathcal{O}(d)) \tag{1.2}$$

and all $d \geq 1$. Here $\mathcal{O}(d)$ is the d-th power of the hyperplane bundle with its standard metric. Recall that $H^0(\mathbb{P}^n, \mathcal{O}(d))$ is given naturally as the set of homogeneous polynomials of degree d in homogeneous coordinates. If X is contained in an affine chart $X \subset \mathbb{C}^n \subset \mathbb{P}^n$ and $x \in \mathbb{C}^n$, then condition (1.2) is equivalent to

$$|p(x)| \leq M_x^d \sup_X |p| \quad \text{for all polynomials } p \text{ of degree } d \tag{1.3}$$

and all $d \geq 1$ where $M_x = \rho\sqrt{1 + \|x\|^2} C_x$ and ρ depends only on X. Therefore the set $\widehat{X} \cap \mathbb{C}^n$ consists exactly of those points $x \in \mathbb{C}^n$ for which there exists an M_x satisfying condition (1.3).

This paper is concerned with the case where $X = \gamma$ is a real analytic curve. In [**4**] evidence was given for the following conjecture.

Conjecture 1.1. — *Let $\gamma \subset \mathbb{P}^n$ be a finite union of simple closed real analytic curves. Then $\widehat{\gamma} - \gamma$ is a 1-dimensional complex analytic suvariety of $\mathbb{P}^n - \gamma$.*

This conjecture has many interesting geometric consequences (see [**7**], [**5**], and [**6**]). The assumption of real analyticity is important. The conjecture does not hold for all smooth curves. In particular, it does not hold for curves which are not pluripolar.

One point of this paper is to prove Conjecture 1.1 under the hypothesis that the function C_x is bounded on $\widehat{\gamma}$. We begin by adapting arguments of E. Bishop [**2**] to prove the following finiteness theorem.

Theorem 1.1. — *Let $\gamma \subset \mathbb{C}^2$ be a finite union of simple closed real analytic curves. Set*

$$\widehat{\gamma}_M \equiv \{x \in \widehat{\gamma} \cap \mathbb{C}^2 : M_x \leq M\}$$

where M_x is the function appearing in condition (1.3). *Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be a linear projection. Then*

$$\widehat{\gamma}_M \cap \pi^{-1}(z) \quad \textit{is finite for almost all } z \in \mathbb{C}.$$

Consequently, $\widehat{\gamma} \cap \pi^{-1}(z)$ is countable for almost all $z \in \mathbb{C}$.

In Section 3 this theorem is combined with results from [**4**] and the theorems concerning maximum modulus algebras to prove the following.

A set $X \subset \mathbb{P}^n$ is called *stable* if the function C_x in (1.2) is bounded on $\widehat{X}$.

Note that if X is stable and $X \subset \mathbb{C}^n \subset \mathbb{P}^n$, then the function M_x is bounded on $\mathbb{C}^n$ by $\rho\sqrt{1+\|x\|^2}$.

Theorem 1.2. — *Let $\gamma \subset \mathbb{P}^n$ be a finite union of simple closed real analytic curves. Assume γ is stable. Then $\widehat{\gamma}-\gamma$ is a 1-dimensional complex analytic subvariety of $\mathbb{P}^n - \gamma$.*

2. The finiteness theorem

Let X be a compact set in $\mathbb{C}^n$ and denote by $\mathcal{P}_d$ the space of polynomials of degree $\leq d$ on $\mathbb{C}^n$.

Definition 2.1. — Denote by $\widehat{X} \cap \mathbb{C}^n$ the set of all $x \in \mathbb{C}^n$ such that there exists a constant M_x with

$$|P(x)| \leq M_x^d \sup_X |P| \tag{2.1}$$

for every $P \in \mathcal{P}_d$ and $d \geq 1$. The set $\widehat{X} \cap \mathbb{C}^n$ is called *the projective hull of X in* $\mathbb{C}^n$.

As noted above, the projective hull, defined in [**4**], is a subset of projective space $\mathbb{P}^n$, and the set $\widehat{X} \cap \mathbb{C}^n$ is exactly that part of the projective hull which lies in the affine chart $\mathbb{C}^n \subset \mathbb{P}^n$. Closely related to Definition 2.1 is the following.

Definition 2.2. — Fix a number $M \geq 1$ and a point $z \in \mathbb{C}^{n-1}$. Then we set

$$\widehat{X}_M(z) = \bigl\{w \in \mathbb{C} : |P(z,w)| \leq M^d \sup_X |P|, \ \forall P \in \mathcal{P}_d \text{ and } \forall d \geq 1\bigr\}$$

and let $\widehat{X}(z) = \bigcup_{M \geq 1} \widehat{X}_M(z) = \{w \in \mathbb{C} : (z,w) \in \widehat{X}\}$.

We consider a special case of these definitions. We fix $n = 2$ and consider a simple closed real-analytic curve X in $\mathbb{C}^2$. Let Δ denote the unit disk in $\mathbb{C}$.

Theorem 2.1. — *Fix $M \geq 1$. For almost all $z \in \Delta$, $\widehat{X}_M(z)$ is a finite set.*

Corollary 2.1. — *For almost all $z \in \mathbb{C}$ the set $\widehat{X}(z)$ is countable.*

We shall prove Theorem 2.1 by adapting an argument, for the case of polynomially convex hulls, by Errett Bishop in [**2**]. We shall follow the exposition of Bishop's argument in [**10**, Chap. 12].

Definition 2.3. — The polynomial $Q(z,w) = \sum_{n,m} c_{nm} z^n w^m$ is called a *unit polynomial* if $\max_{n,m} |c_{nm}| = 1$.

Definition 2.4. — The polynomial $Q(z,w) = \sum_{n,m} c_{nm} z^n w^m$ is said to have *bidegree* (d,e), for non-negative integers d and e, if $c_{nm} = 0$ unless $n \le d$ and $m \le e$, and d, e are minimal with this property.

Note that $\deg Q \le d + e \le 2 \deg Q$.

Definition 2.5. — Fix $M \ge 1$. For each $z \in \mathbb{C}$ set

$$S_M(z) = \{w \in \mathbb{C} : |Q(z,w)| \le (M^{d+e}) \sup_X |Q|,\quad \forall Q \in \mathbb{C}[z,w] \text{ of bidegree } (d,e) \text{ for } d, e \ge 1\}.$$

We now fix a number $M \ge 1$ and keep it fixed throughout what follows.

Theorem 2.2. — *For almost all $z \in \Delta$, $S_M(z)$ is a finite set.*

Theorem 2.1 is an immediate consequence of Theorem 2.2. To see this, fix $z \in \Delta$ and choose $w \in \widehat{X}_M(z)$. Choose next a polynomial Q of bidegree (d,e) and let $\delta = \deg Q$. Then

$$\big|Q(z,w)\big| \le M^\delta \|Q\|_X \le M^{d+e} \|Q\|_X$$

and so $w \in S_M(z)$. Since this holds for all such w, $\widehat{X}_M(z) \subseteq S_M(z)$. By Theorem 2.2 $S_M(z)$ is a finite set for a. a. $z \in \Delta$, so $\widehat{X}_M(z)$ is a finite set for almost all $z \in \Delta$. Thus Theorem 2.1 holds.

We now go to the proof of Theorem 2.2.

Lemma 2.1. — *Let Ω be a plane domain, let K be a compact set in Ω, and fix $z_0 \in \Omega$. Then there exists a constant r, $0 < r < 1$, so that if f is holomorphic on Ω and $|f| < 1$ on Ω and if f vanishes to order λ at z_0, then $|f| \le r^\lambda$ on K.*

Proof. — We construct a bounded and smoothly bounded subdomain Ω_0 of Ω with $\overline{\Omega}_0 \subset \Omega$, $z_0 \in \Omega_0$ and $K \subset \Omega_0$. Denote by $G(z_0, z)$ the Green's function of Ω_0 with pole at z_0.

Then $\mathrm{e}^{-(G+iH)}$ is a multiple-valued holomorphic function on Ω_0 with a single-valued modulus e^{-G}, and this modulus is $= 1$ on $\partial\Omega_0$ (H is the harmonic conjugate of G). Consequently,

$$f/\mathrm{e}^{-\lambda(G+iH)}$$

is multiple-valued and holomorphic on Ω_0, and its modulus is single-valued and < 1 on $\partial\Omega_0$. By the maximum principle for holomorphic functions, for each $z \in K$, we have $|f/\mathrm{e}^{-\lambda(G+iH)}| < 1$ at z and so

$$|f(z)| \le \big[\mathrm{e}^{-G(z_0,z)}\big]^\lambda.$$

Putting $r = \sup_K \mathrm{e}^{-G}$, we get our desired inequality. $\square$

Lemma 2.2. — *Let Ω be a bounded plane domain and K a compact subset of Ω. Let $\mathcal{L}$ be an algebra of holomorphic functions on Ω. Put $\|\phi\| = \sup_K |\phi|$ for all $\phi \in \mathcal{L}$. Fix $f, g \in \mathcal{L}$. Then there exist r, $0 < r < 1$ and $C > 0$ such that for each pair of positive integers (d, e) we can find a unit polynomial $F_{d,e}$ of bidegree (d, e) such that*

$$\left\|F_{d,e}(f, g)\right\| \leq C^{d+e} r^{de}. \tag{2.2}$$

Proof. — Choose a subdomain Ω_1 of Ω with $K \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega$. Choose $C_0 > 1$ with $|f| < C_0$, $|g| < C_0$ on $\overline{\Omega}_1$. Consider an arbitrary polynomial

$$F(z, w) = \sum_{n=0}^{d} \sum_{m=0}^{e} c_{nm} z^n w^m$$

and let h be the function $F(f, g)$ in $\mathcal{L}$. Fix a positive integer λ. The requirement that h should vanish at z_0 to order λ imposes λ linear homogeneous conditions on the c_{nm}, and hence has a non-trivial solution if $\lambda < (d + 1)(e + 1)$. We may assume that the corresponding polynomial F is a unit polynomial. Since

$$\frac{\mathrm{d}^\nu h}{\mathrm{d}z^\nu}(z_0) = 0, \quad \nu = 0, 1, \ldots, \lambda - 1,$$

Lemma 2.1 gives us some r, $0 < r < 1$, such that

$$|h| \leq \Bigl(\sup_{\overline{\Omega}_1} |h|\Bigr) r^\lambda \quad \text{on } K.$$

Since F is a unit polynomial,

$$|h| \leq \sum_{n=0}^{d} \sum_{m=0}^{e} |c_{nm}| \cdot |f|^n \cdot |g|^m \leq (d + 1)(e + 1) C_0^{d+e} \quad \text{on } \overline{\Omega}_1.$$

Hence for large C,

$$\|h\| \leq (d + 1)(e + 1) C_0^{d+e} \leq C^{d+e} r^\lambda.$$

We choose $\lambda = de$. Since $de < (d + 1)(e + 1)$, we get

$$\|F(f, g)\| = \|h\| < C^{d+e} r^{de}$$

as desired. □

Note. — We shall apply this result to the case when K is the unit circle, Ω is an annulus containing K, and $\mathcal{L}$ is the algebra of functions holomorphic on Ω.

The curve X in our Theorem 2.2 is real analytic by hypothesis, and hence can be represented parametrically:

$$z = f(\zeta), \quad w = g(\zeta) \quad \zeta \in \Omega$$

where f, g are functions in $\mathcal{L}$.

Lemma 2.3. — *Let r, C and $F_{d,e}$ be as in Lemma 2.2. Fix r_0, $r < r_0 < 1$. Then there exists d_0 such that*

$$(MC)^{d+e} r^{de} \leq r_0^{de} \quad \text{for } d, e > d_0. \tag{2.3}$$

Proof. — We write $\sim$ for "is equivalent to".

$$(2.3) \sim (MC)^{d+e} \leq \left(\frac{r_0}{r}\right)^{de} \sim (d+e)\log(MC) \leq de \log\left(\frac{r_0}{r}\right)$$
$$\sim \left(\frac{1}{e} + \frac{1}{d}\right) \log(MC) \leq \log\left(\frac{r_0}{r}\right).$$

The last inequality is true for $d, e > d_0$ for some suitable d_0. We are done. □

With M, r, r_0 fixed, we choose d_0 as in (2.3). Henceforth, we tacitly assume $d, e > d_0$.

Definition 2.6. — Fix d, e and put $F = F_{d,e}$ as above. Then

$$F(z, w) = \sum_{j=0}^{e} G_j(z) w^j$$

where for some $j = j_0$, G_{j_0} is a unit polynomial of degree $\leq d$. We define

$$T(d, e) = \left\{z \in \Delta : \left|G_{j_0}(z)\right| \leq r_0^{\frac{1}{2}de}\right\}.$$

Lemma 2.4. — *Let F be a unit polynomial in z, of degree k, and let α be a positive number. Put $\Lambda = \{z \in \Delta : |F(z)| \leq \alpha^k\}$. Then*

$$m(\Lambda) \leq 48\alpha,$$

where m is 2-dimensional measure.

Proof. — This is Lemma 12.3 in [**10**], and a proof of it is given there. □

Lemma 2.5. — *Fix d, e. Fix a point $z_1 \in \Delta - T(d, e)$. Then there exists a unit polynomial B in one variable, of degree $\leq e$, such that for every $w_0 \in S_M(z_1)$, we have*

$$\left|B(w_0)\right| \leq r_0^{\frac{1}{2}de}.$$

Proof. — Define the polynomial A in one variable by $A(w) = F(z_1, w)$, where $F = F_{d,e}$. As in Definition 2.6 then

$$A(w) = \sum_{j=0}^{e} G_j(z_1) w^j$$

and G_{j_0} is a unit polynomial in z_1. Since $z_1 \notin T(d, e)$, we have

$$\left|G_{j_0}(z_1)\right| > r_0^{\frac{1}{2}de}. \tag{2.4}$$

Fix $w_0 \in S_M(z_1)$. Then

$$\begin{aligned}|F(z_1,w_0)| &\le M^{d+e}\cdot\|F\|_X\\ &\le M^{d+e}C^{d+e}r^{de} &&\text{by (2.2)}\\ &\le r_0^{de} &&\text{by (2.3)}.\end{aligned}$$

We shall divide A by its largest coefficient K. Note that

$$|K| \ge |G_{j_0}(z_1)| > r_0^{\frac{1}{2}de}$$

by (2.4). Put $B(w) = A(w)/K$. Then $\deg B \le e$ and

$$|B(w_0)| = \frac{|A(w_0)|}{|K|} = \frac{|F(z_1,w_0)|}{|K|} \le \frac{r_0^{de}}{r_0^{\frac{1}{2}de}} = r_0^{\frac{1}{2}de}.$$

We are done. □

Lemma 2.6. — *For each d,*

$$m\big(T(d,e)\big) \le 48 r_0^{\frac{1}{2}e}.$$

Proof. — Fix e and fix d. With G_{j_0} as above, write $G = G_{j_0}$. Then $\deg G \le d$. By definition of $T(d,e)$, if $z \in T(d,e)$, then

$$|G(z)| \le r_0^{\frac{1}{2}de} = \big(r_0^{\frac{1}{2}e}\big)^d \le \big(r_0^{\frac{1}{2}e}\big)^{\deg G},$$

and so

$$T(d,e) \subseteq \big\{z \in \Delta : |G(z)| \le (r_0^{\frac{1}{2}e})^{\deg G}\big\}.$$

Therefore,

$$m\big[T(d,e)\big] \le m\big\{z \in \Delta : |G(z)| \le \alpha^k\big\}$$

where $\alpha = r_0^{\frac{1}{2}e}$ and $k = \deg G$. By Lemma 2.4, $m\{z \in \Delta : |G(z)| \le \alpha^k\} \le 48\alpha$, and so $m[T(d,e)] \le 48 r_0^{\frac{1}{2}e}$, as was to be shown. □

Definition 2.7. — Fix e and and set

$$H_e = \big\{z : z \in \Delta - T(d,e) \text{ for infinitely many } d\big\}.$$

Lemma 2.7. — *If $z^* \in H_e$, then $S_M(z^*)$ has at most e elements.*

Proof. — Fix $z^* \in H_e$. Then there exists a sequence $\{d_j\}$ such that $z^* \in \Delta - T(d_j,e)$ for each j. By Lemma 2.5, for each j there is a unit polynomial B_j with $\deg B_j \le e$ such that

$$|B_j(w_0)| \le r_0^{\frac{1}{2}(d_j e)} \quad \text{for each } w_0 \in S_M(z^*). \tag{2.5}$$

Since $\deg B_j \le e$ for all j, and each B_j is a unit polynomial, there exists a subsequence of the sequence $\{B_j\}$ converging uniformly to a unit polynomial B^* on compact sets in the w-plane. Because of (2.5), $B^*(w_0) = 0$ for each $w_0 \in S_M(z^*)$. Also, $\deg B^* \le e$. Hence the cardinality of $S_M(z^*)$ is $\le e$. We are done. □

Proof of Theorem 2.2. — Our task is to show that $m\{z \in \Delta : S_M(z) \text{ is infinite }\} = 0$. Fix e. Fix $z \in \Delta - H_e$. Since $z \notin H_e$, we have $z \in \Delta - T(d,e)$ for only finitely many d, so $z \in T(d,e)$ for all d from some $d = k$ on. Therefore,

$$z \in \bigcap_{d=k}^{\infty} T(d,e)$$

and so

$$\Delta - H_e \subseteq \bigcup_{k=k_0}^{\infty} \Big[\bigcap_{d=k}^{\infty} T(d,e)\Big]. \tag{2.6}$$

By Lemma 2.6, $m(T(d,e)) \leq 48 r_0^{\frac{1}{2}e}$ for each d. Therefore,

$$m\Big(\bigcap_{k=1}^{\infty} T(d,e)\Big) \leq 48 r_0^{\frac{1}{2}e}$$

for each k. So the right hand side of (2.6) is the union of an increasing family of sets each of which has m-measure $\leq 48 r_0^{\frac{1}{2}e}$. Thus (2.6) gives

$$m\big(\Delta - H_e\big) \leq 48 r_0^{\frac{1}{2}e}. \tag{2.7}$$

Also, by Lemma 2.7, we have

$$\text{If } z^* \in H_e, \text{ then } \#\big\{S_M(z^*)\big\} \leq e. \tag{2.8}$$

Fix $z \in \Delta$ such that the set $S_M(z)$ is infinite. Then $z \notin H_e$ for each e, that is, $z \in \Delta - H_e$ for all e. Hence, $\{z \in \Delta : S_M(z) \text{ is infinite }\} \subset \Delta - H_e$. Therefore

$$m\big\{z \in \Delta : S_M(z) \text{ is infinite }\big\} \leq m(\Delta - H_e) \leq 48 r_0^{\frac{1}{2}e}$$

by (2.7). We now let $e \to \infty$ and conclude that $m\{z \in \Delta : S_M(z) \text{ is infinite }\} = 0$. Theorem 2.2 is proved. □

Proof of Corollary 2.1. — Fix $r > 0$ and apply Theorem 2.1 to the curve $\rho_r(X)$ where $\rho_r : \mathbb{C}^2 \to \mathbb{C}^2$ is given by $\rho_r(z) = rz$. Since $\rho_r(\widehat{X} \cap \mathbb{C}^2) = \widehat{(\rho_r X)} \cap \mathbb{C}^2$, we conclude that Theorem 2.1 holds with Δ replaced by $\frac{1}{r}\Delta$. □

Theorem 2.3. — *Theorem* 2.1 *remains valid without the assumption that X is connected, that is, it is valid when X is a finite union of real analytic simple closed curves in $\mathbb{C}^2$.*

Proof. — Write $X = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_N$ where each $\gamma_k \subset \mathbb{C}^2$ is a simple closed real analytic curve. Choose a neighborhood Ω of the unit circle K in $\mathbb{C}$ and complex analytic maps $(f_k, g_k) : \Omega_k \to \mathbb{C}^2$, $k = 1, \ldots, N$ whose restriction to K is a parameterization of γ_k. We now apply the following.

Lemma 2.8. — *Let Ω be a plane domain and K a compact subset of Ω. Let $\mathcal{L}$ be an algebra of holomorphic functions on Ω. Put $\|\phi\| = \sup_K |\phi|$ for all $\phi \in \mathcal{L}$. Fix $f_k, g_k \in \mathcal{L}$ for $k = 1, \ldots, N$. Then there exist r, $0 < r < 1$ and $C > 0$ such that for each pair of positive integers (d, e) with $d+e > N$, we can find a unit polynomial $F_{d,e}$ of bidegree (d, e) such that*

$$\big\|F_{d,e}(f_k, g_k)\big\| \leq C^{d+e} r^{\frac{de}{N}} \quad \text{for } k = 1, \ldots, N. \tag{2.9}$$

Proof. — We fix a point $z_0 \in \Omega$ and choose $F_{d,e}$ so that $F_{d,e}(f_k, g_k)$ vanishes to order de/N at z_0 for all k. This is possible if $d + e > N$. We then proceed as in the proof of Lemma 2.2. □

One can now carry out the arguments given above for the case of one component. The only difference is that in the estimates, r_0^e will be replaced by $r_0^{e/N}$. □

3. The analyticity theorem

Let $\mathcal{O}(1) \to \mathbb{P}^n$ denote the holomorphic line bundle of Chern class 1 over complex projective n-space, endowed with its standard U(n+1)-invariant metric $\|\,.\,\|$. Following [**4**], we define the *projective hull* of a compact subset $X \subset \mathbb{P}^n$ to be the set $\widehat{X}$ of points $x \in \mathbb{P}^n$ for which there exists a constant $C = C_x$ such that

$$\|P(x)\| \leq C_x^d \sup_X \|P\|. \tag{3.1}$$

for all holomorphic sections $P \in H^0(\mathbb{P}^n, \mathcal{O}(d))$ and all $d \geq 1$.

Note. — Recall that the holomorphic sections $H^0(\mathbb{P}^n, \mathcal{O}(d))$ correspond naturally to the homogeneous polynomials of degree d in homogeneous coordinates $[Z_0, \ldots, Z_n]$ for $\mathbb{P}^n$. From this one can see (cf. [**4**, §6]) that if X is contained in an affine chart $\mathbb{C}^n \subset \mathbb{P}^n$, then $\widehat{X} \cap \mathbb{C}^n$ is exactly the "projective hull of X in $\mathbb{C}^n$" introduced in § 2. Moreover, the function M_ζ appearing in (2.1) can be taken to be $M_\zeta = \rho\sqrt{1 + \|\zeta\|^2} C_\zeta$ for $\zeta \in \widehat{X} \cap \mathbb{C}^n$, where ρ is a constant depending only on X.

For each $x \in \widehat{X}$ there is a *best constant* $C(x) \equiv \min\{C_x : (3.1) \text{ holds } \forall P\}$. The set X is called *stable* if the best constant function C is bounded on $\widehat{X}$. We know from [**4**, Prop. 10.2] that if X is stable, then $\widehat{X}$ is compact.

The point of this section is to prove the following projective version of the main theorem in [**9**].

Theorem 3.1. — *Let $\gamma \subset \mathbb{P}^n$ be a finite union of real analytic closed curves and assume γ is stable. Then $\widehat{\gamma} - \gamma$ is a one-dimensional complex analytic subvariety of $\mathbb{P}^n - \gamma$.*

Note. — When this conclusion holds, one can show that, in fact, $\widehat{\gamma}$ is the image of a compact Riemann surface with analytic boundary under a holomorphic map to $\mathbb{P}^n$. We will prove this in § 4.

Proof. — Assume to begin that $n = 2$. Since γ is real analytic, it is pluripolar, i.e., locally contained in the $\{-\infty\}$-set of a plurisubharmonic function (which is $\not\equiv -\infty$). Therefore, by [**4**, Cor. 4.4] we know that $\widehat{\gamma}$ is also pluripolar. In particular, it is nowhere dense. As noted above, $\widehat{\gamma}$ is closed by stability. Hence, we may choose a point $x \in \mathbb{P}^2$ and a ball B centered at x such that $\widehat{\gamma} \subset \mathbb{P}^2 - \overline{B}$. Let

$$\mathbb{P}^2 - \{x\} \xrightarrow{\pi} \mathbb{P}^1 \tag{3.2}$$

be linear projection with center x. This projection (3.2) is naturally a holomorphic line bundle $\cong \mathcal{O}(1)$, and

$$\mathbb{P}^2 - \overline{B} \xrightarrow{\pi} \mathbb{P}^1 \tag{3.3}$$

can be identified, after scalar multiplication by some constant $r > 0$, with its open unit disk bundle.

Cover $\mathbb{P}^1$ with two affine charts: $V_0 = \mathbb{P}^1 - \{0\}$ and $V_\infty = \mathbb{P}^1 - \{\infty\}$, and assume that $\gamma \cap \pi^{-1}(0) = \gamma \cap \pi^{-1}(\infty) = \varnothing$. By symmetry we may restrict attention to $\pi^{-1}(V_\infty)$. This chart has an identification

$$\pi^{-1}(V_\infty) \cong \mathbb{C}^2 = \{(z, w) : z, w \in \mathbb{C}\}$$

with the property that V_∞ maps linearly to the z-axis and π can be written as $\pi(z, w) = z$. The subset $\mathbb{P}^2 - \overline{B}$, intersected with this chart, is represented by

$$\left(\mathbb{P}^2 - \overline{B}\right) \cap \mathbb{C}^2 = \{(z, w) : |w|^2 \leq |z|^2 + 1\}. \tag{3.4}$$

Set

$$\Omega \equiv \mathbb{C} - \pi(\gamma) \quad \text{and} \quad U \equiv \pi^{-1}(\Omega) = \mathbb{C}^2 - \pi^{-1}\big(\pi(\gamma)\big).$$

Proposition 3.1. — *Let $\gamma \subset \mathbb{C}^2$ be a stable real analytic curve with the property that*

$$\widehat{\gamma} \cap \mathbb{C}^2 \subset \{(z, w) : |w|^2 \leq |z|^2 + 1\}. \tag{3.5}$$

Then $\widehat{\gamma} \cap U$ is a 1-dimensional complex analytic subvariety of U.

Proof. — Note to begin that since $\widehat{\gamma}$ is compact, condition (3.5) implies that

$$\pi : \widehat{\gamma} \cap U \longrightarrow \Omega \quad \text{is a proper map.} \tag{3.6}$$

Consider now the algebra A of functions on $\widehat{\gamma} \cap U$ given by restriction of the holomorphic functions on U, i.e.,

$$A \equiv \{f|_{\widehat{\gamma} \cap U} : f \in \mathcal{O}(U)\}.$$

We now claim that $(A, \widehat{\gamma} \cap U, \Omega, \pi)$ is a *maximum modulus algebra*, as defined in [**1**, p. 64]. Given (3.6) this means that we need only prove the following.

Lemma 3.1. — *For each $z_0 \in \Omega$ and each closed disk $D \subset \Omega$ centered at z_0, the equality*

$$|f(z_0, w_0)| \leq \sup_{\widehat{\gamma} \cap \pi^{-1}(\partial D)} |f| \tag{3.7}$$

holds for all $f \in A$.

Proof. — By hypothesis (3.5) there exists an $R > 0$ such that

$$\widehat{\gamma} \cap \pi^{-1}(D) \subset D \times \Delta_{\frac{1}{2}R}$$

where $\Delta_r \equiv \{w : |w| \leq r\}$. In particular, we have that

$$\widehat{\gamma} \cap \partial(D \times \Delta_R) = \widehat{\gamma} \cap (\partial D \times \Delta_R) = \widehat{\gamma} \cap \pi^{-1}(\partial D). \tag{3.8}$$

Now Theorem 12.8 in [**4**] states that

$$\widehat{\gamma} \cap \pi^{-1}(D) = \widehat{\gamma} \cap (D \times \Delta_R) \subset \text{polynomial hull of } \widehat{\gamma} \cap \partial(D \times \Delta_R).$$

Applying (3.8) gives

$$\widehat{\gamma} \cap \pi^{-1}(D) \subset \text{polynomial hull of } \widehat{\gamma} \cap \pi^{-1}(\partial D),$$

and Lemma 3.1 follows immediately. □

We have now shown that $(A, \widehat{\gamma} \cap U, \Omega, \pi)$ is a maximum modulus algebra. Furthermore, since $\widehat{\gamma}$ is stable, we know from Theorem 2.1 that there exists an $N > 0$ such that

$$\Omega(N) \equiv \{z \in \Omega : \#(\pi^{-1}(z) \cap \widehat{\gamma}) \leq N\}$$

has positive measure. (Since $\Omega - \bigcup_N \Omega(N)$ has measure zero.) It now follows from Theorem 11.8 in [**1**] that:

(i) $\Omega = \Omega(N)$, and

(ii) there exists a discrete subset $\Lambda \subset \Omega$ such that $\widehat{\gamma} \cap \pi^{-1}(\Omega - \Lambda)$ has the structure of a Riemann surface on which every function in A is analytic.

Since A is the restriction of holomorphic functions on U to $\widehat{\gamma}$, condition (ii) implies that $\widehat{\gamma} \cap \pi^{-1}(\Omega - \Lambda)$ is a 1-dimensional complex analytic subvariety of $\pi^{-1}(\Omega - \Lambda) = U - \pi^{-1}(\Lambda)$.

It now follows that $\widehat{\gamma} \cap U$ is a 1-dimensional complex analytic subvariety of U. To see this, fix $z_0 \in \Lambda$ and choose a small closed disk $D \subset \Omega$ centered at z_0 with $D \cap \Lambda = \{z_0\}$. The arguments above show that $\widehat{\gamma} \cap \pi^{-1}(D)$ is contained in the polynomial hull of the real analytic curve $\widehat{\gamma} \cap \pi^{-1}(\partial D)$. Applying standard results [**1**, §12] proves Proposition 3.1. □

Proposition 3.1 together with the discussion preceding it, give the following.

Corollary 3.1. — *The set $\widehat{\gamma} - \pi^{-1}(\pi\gamma)$ is a complex analytic subvariety of dimension* 1 *in $\mathbb{P}^2 - \pi^{-1}(\pi\gamma)$.*

Observe that for every point $y \in \mathbb{P}^2 - \widehat{\gamma}$ there is a point $x \in \mathbb{P}^2 - \widehat{\gamma}$ such that $\pi(y) \notin \pi(\gamma)$ where π is the projection (3.2) with center x. Consequently, Corollary 3.1 proves Theorem 3.1 for the case $n = 2$.

Suppose now that $n = 3$ and choose $x \in \mathbb{P}^3 - \widehat{\gamma}$. The set of such x is open and dense since $\widehat{\gamma}$ is a compact pluripolar set of Hausdorff dimension 2 (cf. [**4**, Cor. 4.4 and Thm. 12.5]). Let $\Pi : \mathbb{P}^3 - \{x\} \to \mathbb{P}^2$ be the projection with center x. One sees easily that

$$\Pi(\widehat{\gamma}) \subseteq \widehat{\Pi\gamma},$$

and by the above $\widehat{\Pi\gamma} - \Pi\gamma$ is a complex analytic curve in $\mathbb{P}^2 - \Pi\gamma$. Standard arguments now show that $\widehat{\gamma} - \gamma$ is a complex analytic curve in $\mathbb{P}^3 - \gamma$. Proceeding by induction on n completes the proof of Theorem 3.1. □

4. Boundary Regularity

The conclusion of Theorem 3.1 implies a strong regularity at the boundary. For future reference we include a discussion of this regularity.

Theorem 4.1. — *Let $\gamma \subset \mathbb{P}^n$ be a finite disjoint union of real analytic regular closed curves, and suppose V is a 1-dimensional complex analytic subvariety of the complement $\mathbb{P}^n - \gamma$. Then the closure*

$$\overline{V} = \bigcup_{j=1}^{m} \overline{V}_j \cup \bigcup_{k=m+1}^{\ell} \overline{V}'_k$$

where:

1) *Each V_j is a 1-dimensional complex analytic subvariety of finite area in $\mathbb{P}^n - \gamma$ whose closure $\overline{V}_j$ is an immersed variety in $\mathbb{P}^n$ with non-empty boundary $\partial\overline{V}_j = \gamma_j$ consisting of a union of components of γ. In particular, there exists a connected Riemann surface S_j, a compact subdomain $\overline{W}_j \subset S_j$ with real analytic boundary, and a generically injective holomorphic map*

$$\rho_j : S_j \longrightarrow \mathbb{P}^n \quad \textit{with} \quad \rho_j(\overline{W}_j) = \overline{V}_j$$

which is an embedding on a neighborhood of $\partial\overline{W}_j$ and has $\rho_j(\partial\overline{W}_j) = \gamma_j$.

2) *The closure of each V'_k is an irreducible algebraic curve in $\mathbb{P}^n$ with $\gamma_k \subset \mathrm{Reg}(\overline{V}'_k)$ where γ_k is a (possibly empty) finite union of components of γ.*

Note. — When γ is stable and $V = \widehat{\gamma}$, each γ_k is non-empty for $m < k \leq \ell$.

Theorem 4.1 can be put into a more succinct form.

Theorem 4.2. — *Let γ and V be as above. Then there exists a Riemann surface S (not necessarily connected), a compact subdomain $\overline{W} \subset S$ with real analytic boundary, and a holomorphic map $\rho : S \to \mathbb{P}^n$ which is generically injective and satisfies*

1) $\rho(\overline{W}) = \overline{V}$,
2) *ρ is an embedding on a tubular neighborhood of $\partial\overline{W}$ in S and*
3) *$\rho(\partial\overline{W})$ is a union of components of γ.*

Proof of Theorem 4.1. — We assume $n = 2$. The case of general n is similar.

We first note that V has finite area and finitely many irreducible components $V_1, \dots, V_\ell$. This follows from work of Shiffman, but can be seen directly as follows. Choose any $p \in \mathbb{P}^2 - \overline{V}$ and let $\pi : \mathbb{P}^2 - \{p\} \to \mathbb{P}^1$ be projection. Then $\pi_{|V}$ is finitely sheeted over $\mathbb{P}^1 - \pi(\gamma)$, and therefore V has finitely many components. In fact $\pi_{|V}$ must also be finitely sheeted over all of $\mathbb{P}^1$. To see this note that V can contain no fibre of π since $p \notin \overline{V} = V \cup \gamma$. Hence, the intersection $\pi^{-1}(x) \cap V$ for $x \in \pi(\gamma)$ is at most countable. If it were infinite, one easily sees that the sheeting number in contiguous domains of $\mathbb{P}^1 - \pi(\gamma)$ is unbounded. Choosing two distinct such projections and an easy estimate shows that the integral of the projective Kähler form on V is finite.

Now each irreducible component V_j of V defines a current $[V_j]$ by integration whose boundary is supported in γ. By the Federer Flat Support Theorem [**3**, 4.1.15],

$$\partial[V_j] = n_j[\gamma_j]$$

where $\gamma_j \equiv \mathrm{supp}\partial[V_j]$ is a union of connected components of γ (appropriately oriented) and $n_j \geq 0$ is a locally constant integer-valued function on γ_j. Order the V_j so that $\partial[V_j] \neq 0$ for $j = 1, \dots, m$ and $\partial[V_j] = 0$ for $j > m$.

Since γ is a regularly embedded real analytic curve, it has a complexification $\Sigma \supset \gamma$ which is a union of regularly embedded closed complex analytic annuli. Let Σ_j denote that part of Σ which is the complexification of γ_j for $j \leq m$. Write $\Sigma_j = \Sigma_j^+ \cup \gamma_j \cup \Sigma_j^-$ where $\Sigma_j^\pm$ are the components of $\Sigma_j - \gamma_j$ with signs chosen so that Σ^+ is the "outer strip", that is, so that

$$\partial\Sigma_j^+ = \gamma_j^+ - \gamma_j.$$

Consider the current $[V_j^*] \equiv [V_j] + n_j[\Sigma_j^+]$ which has

$$\partial[V_j^*] = n_j[\gamma_j^+].$$

The structure theorem of King [**8**] implies that $\mathrm{supp}[V_j^*]$ is a 1-dimensional subvariety of $\mathbb{P}^2 - \gamma_j^+$. It follows that V_j^* is an analytic continuation of V_j and in particular

$$n_j \equiv 1 \quad \text{and} \quad \Sigma_j^- \subset V_j.$$

Defining $\rho_j : S_j \to V_j^*$ to be the normalization of V_j^* and setting $\overline{W}_j = \rho^{-1}(\overline{V}_j)$ completes part 1) unless there exist $V_i \neq V_j$ which share some common boundary components. In this case $\overline{V}_i$ and $\overline{V}_j$ are analytic continuations of each other and can be combined into a single component of $\overline{V}$. Eliminating all common boundaries in this manner completes part 1).

Note that after fusing components, one may obtain algebraic curves which contain a non-empty union of components of γ in their regular locus. These will be listed in part 2). The remaining components of V (whose current boundaries are zero) are algebraic curves by King [**8**]. □

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R. Harvey, Mathematics Department, 6100 South Main Street, Houston, TX 77005-1892
E-mail : `harvey@rice.edu`

B. Lawson, Mathematics Department, Stony Brook Univeristy, Stony Brook, NY 11794
E-mail : `blaine@math.sunysb.edu`

J. Wermer, Mathematics Department, Brown University, 151 Thayer Street, Providence, RI 02912
E-mail : `wermer@math.brown.edu`