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MONODROMY OF A FAMILY OF HYPERSURFACES

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ABSTRACT. – Let Y be an $(m+1)$-dimensional irreducible smooth complex projective variety embedded in a projective space. Let Z be a closed subscheme of Y, and δ be a positive integer such that $\mathcal{I}_{Z,Y}(\delta)$ is generated by global sections. Fix an integer $d \geq \delta + 1$, and assume the general divisor $X \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ is smooth. Denote by $H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp Z}$ the quotient of $H^m(X;\mathbb{Q})$ by the cohomology of Y and also by the cycle classes of the irreducible components of dimension m of Z. In the present paper we prove that the monodromy representation on $H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp Z}$ for the family of smooth divisors $X \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ is irreducible.

RÉSUMÉ. – Soit Y une variété projective complexe lisse irréductible de dimension $m+1$, plongée dans un espace projectif. Soit Z un sous-schéma fermé de Y, et soit δ un entier positif tel que $\mathcal{I}_{Z,Y}(\delta)$ soit engendré par ses sections globales. Fixons un entier $d \geq \delta+1$, et supposons que le diviseur général $X \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ soit lisse. Désignons par $H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp Z}$ le quotient de $H^m(X;\mathbb{Q})$ par la cohomologie de Y et par les classes des composantes irréductibles de Z de dimension m. Dans cet article, nous prouvons que la représentation de monodromie sur $H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp Z}$ pour la famille des diviseurs lisses $X \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ est irréductible.

1. Introduction

In this paper we provide an affirmative answer to a question formulated in [9].

Let $Y \subseteq \mathbb{P}^N$ ($\dim\, Y = m+1$) be an irreducible smooth complex projective variety embedded in a projective space $\mathbb{P}^N$, Z be a closed subscheme of Y, and δ be a positive integer such that $\mathcal{I}_{Z,Y}(\delta)$ is generated by global sections. Assume that for $d \gg 0$ the general divisor $X \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ is smooth. In the paper [9] it is proved that this is equivalent to the fact that the strata $Z_{\{j\}} = \{x \in Z \,:\, \dim\, T_x Z = j\}$, where $T_x Z$ denotes the Zariski tangent space, satisfy the following inequality:

$$\dim\, Z_{\{j\}} + j \leq \dim\, Y - 1 \quad \text{for any} \quad j \leq \dim\, Y. \tag{1}$$

This property implies that, for any $d \geq \delta$, there exists a smooth hypersurface of degree d which contains Z ([9], 1.2. Theorem).

It is generally expected that, for $d \gg 0$, the Hodge cycles of the general hypersurface $X \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ depend only on Z and on the ambient variety Y. A very precise conjecture in this direction was made in [9]:

Conjecture 1 (Otwinowska - Saito). – Assume *deg* $X \geq \delta + 1$. Then the monodromy representation on $H^m(X; \mathbb{Q})^{\mathrm{van}}_{\perp Z}$ for the family of smooth divisors $X \in |H^0(Y, \mathcal{O}_Y(d))|$ containing Z as above is irreducible.

We denote by $H^m(X; \mathbb{Q})^{\mathrm{van}}_Z$ the subspace of $H^m(X; \mathbb{Q})^{\mathrm{van}}$ generated by the cycle classes of the maximal dimensional irreducible components of Z modulo the image of $H^m(Y; \mathbb{Q})$ (using the orthogonal decomposition $H^m(X; \mathbb{Q}) = H^m(Y; \mathbb{Q}) \perp H^m(X; \mathbb{Q})^{\mathrm{van}}$) if $m = 2 \dim Z$, and $H^m(X; \mathbb{Q})^{\mathrm{van}}_Z = 0$ otherwise, and we denote by $H^m(X; \mathbb{Q})^{\mathrm{van}}_{\perp Z}$ the orthogonal complement of $H^m(X; \mathbb{Q})^{\mathrm{van}}_Z$ in $H^m(X; \mathbb{Q})^{\mathrm{van}}$. The conjecture above cannot be strengthened because, even in $Y = \mathbb{P}^3$, there exist examples for which $\dim H^m(X; \mathbb{Q})^{\mathrm{van}}_{\perp Z}$ is arbitrarily large and the monodromy representation associated to the linear system $|H^0(Y, \mathcal{I}_{Z,Y}(\delta))|$ is diagonalizable.

The authors of [9] observed that a proof for such a conjecture would confirm the expectation above and would reduce the Hodge conjecture for the general hypersurface $X_t \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ to the Hodge conjecture for Y. More precisely, by a standard argument, from Conjecture 1 it follows that when $m = 2 \dim Z$ and the vanishing cohomology of the general $X_t \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ ($d \geq \delta + 1$) is not of pure Hodge type $(m/2, m/2)$, then the Hodge cycles in the middle cohomology of X_t are generated by the image of the Hodge cycles on Y together with the cycle classes of the irreducible components of Z. So, the Hodge conjecture for X_t is reduced to that for Y (compare with [9], Corollary 0.5). They also proved that the conjecture is satisfied in the range $d \geq \delta + 2$, or for $d = \delta + 1$ if hyperplane sections of Y have non trivial top degree holomorphic forms ([9], 0.4. Theorem). Their proof relies on Deligne's semisimplicity Theorem and on Steenbrink's Theory for semistable degenerations.

Arguing in a different way, we prove in this paper Conjecture 1 in full. More precisely, avoiding degeneration arguments, in Section 2 we will deduce Conjecture 1 from the following:

Theorem 1.1. – *Fix integers* $1 \leq k < d$, *and let* $W = G \cap X \subset Y$ *be a complete intersection of smooth divisors* $G \in |H^0(Y, \mathcal{O}_Y(k))|$ *and* $X \in |H^0(Y, \mathcal{O}_Y(d))|$. *Then the monodromy representation on* $H^m(X; \mathbb{Q})^{\mathrm{van}}_{\perp W}$ *for the family of smooth divisors* $X_t \in |H^0(Y, \mathcal{O}_Y(d))|$ *containing* W *is irreducible.*

Here we define $H^m(X; \mathbb{Q})^{\mathrm{van}}_{\perp W}$ in a similar way as before, i.e. as the orthogonal complement in $H^m(X; \mathbb{Q})^{\mathrm{van}}$ of the image $H^m(X; \mathbb{Q})^{\mathrm{van}}_W$ of the map obtained by composing the natural maps $H_m(W; \mathbb{Q}) \to H_m(X; \mathbb{Q}) \cong H^m(X; \mathbb{Q}) \to H^m(X; \mathbb{Q})^{\mathrm{van}}$.

The proof of Theorem 1.1 will be given in Section 4 and consists in a Lefschetz type argument applied to the image of the rational map on Y associated to the linear system $|H^0(Y, \mathcal{I}_{W,Y}(d))|$, which turns out to have at worst isolated singularities. This approach was started in our paper [2] where we proved a particular case of Theorem 1.1, but the proof given here is independent and much simpler.

We begin by proving Conjecture 1 as a consequence of Theorem 1.1, and next we prove Theorem 1.1.

2. Proof of Conjecture 1 as a consequence of Theorem 1.1.

We keep the same notation we introduced before, and need further preliminaries.

NOTATIONS 2.1. – (i) Let $V_\delta \subseteq H^0(Y,\mathcal{I}_{Z,Y}(\delta))$ be a subspace generating $\mathcal{I}_{Z,Y}(\delta)$, and $V_d \subseteq H^0(Y,\mathcal{I}_{Z,Y}(d))$ $(d \geq \delta+1)$ be a subspace containing the image of $V_\delta \otimes H^0(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(d-\delta))$ in $H^0(Y,\mathcal{I}_{Z,Y}(d))$. Let $G \in |V_\delta|$ and $X \in |V_d|$ be divisors. Put $W := G \cap X$. From condition (1), and [9], 1.2. Theorem, we know that if G and X are general then they are smooth. Moreover, by ([4], p. 133, Proposition 4.2.6. and proof), we know that if G and X are smooth then W has only isolated singularities.

(ii) In the case $m > 2$, fix a smooth $G \in |V_\delta|$. Let $H \in |H^0(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(l))|$ be a general hypersurface of degree $l \gg 0$, and put $Z' := Z \cap H$ and $G' := G \cap H$. Denote by $V'_d \subseteq H^0(G',\mathcal{I}_{Z',G'}(d))$ the restriction of V_d on G', and by $V''_d \subseteq H^0(G,\mathcal{I}_{Z,G}(d))$ the restriction of V_d on G. Since $H^0(G,\mathcal{I}_{Z,G}(d)) \subseteq H^0(G',\mathcal{I}_{Z',G'}(d))$, we may identify $V''_d = V'_d$. Put $W' := W \cap H \in |V'_d|$. Similarly as we did for the triple (Y,X,Z), using the orthogonal decomposition $H^{m-2}(W';\mathbb{Q}) = H^{m-2}(G';\mathbb{Q}) \perp H^{m-2}(W';\mathbb{Q})^{\mathrm{van}}$, we define the subspaces $H^{m-2}(W';\mathbb{Q})^{\mathrm{van}}_{Z'}$ and $H^{m-2}(W';\mathbb{Q})^{\mathrm{van}}_{\perp Z'}$ of $H^{m-2}(W';\mathbb{Q})$ with respect to the triple (G',W',Z'). Passing from (Y,X,Z) to (G',W',Z') will allow us to prove Conjecture 1 arguing by induction on m (see the proof of Proposition 2.4 below).

(iii) Let $\varphi : \mathcal{W} \to |V''_d|$ $(\mathcal{W} \subseteq G \times |V''_d|)$ be the universal family parametrizing the divisors $W = G \cap X \in |V''_d|$. Denote by $\sigma : \widetilde{\mathcal{W}} \to \mathcal{W}$ a desingularization of $\mathcal{W}$, and by $U_\varphi \subseteq |V''_d|$ a nonempty open set such that the restriction $(\varphi\circ\sigma)_{|U_\varphi} : (\varphi\circ\sigma)^{-1}(U_\varphi) \to U_\varphi$ is smooth. Next, let $\psi : \mathcal{W}' \to |V'_d|$ $(\mathcal{W}' \subseteq G \times |V'_d|)$ be the universal family parametrizing the divisors $W' = W \cap H \in |V'_d|$, and denote by $U_\psi \subseteq |V'_d|$ a nonempty open set such that the restriction $\psi_{|U_\psi} : \psi^{-1}(U_\psi) \to U_\psi$ is smooth. Shrinking U_φ and U_ψ if necessary, we may assume $U := U_\varphi = U_\psi \subseteq |V''_d| = |V'_d|$. For any $t \in U$ put $W_t := \varphi^{-1}(t)$, $\widetilde{W}_t := \sigma^{-1}(W_t)$, and $W'_t := \psi^{-1}(t)$. Observe that $W_t \cap \mathrm{Sing}(\mathcal{W}) \subseteq \mathrm{Sing}(W_t)$, so we may assume $W'_t = W_t \cap H \subseteq W_t \backslash \mathrm{Sing}(W_t) \subseteq \widetilde{W}_t$. Denote by ι_t and $\tilde{\iota}_t$ the inclusion maps $W'_t \to W_t$ and $W'_t \to \widetilde{W}_t$. The pull-back maps $\tilde{\iota}^*_t : H^{m-2}(\widetilde{W}_t;\mathbb{Q}) \to H^{m-2}(W'_t;\mathbb{Q})$ give rise to a natural map $\tilde{\iota}^* : R^{m-2}((\varphi\circ\sigma)_{|U})_*\mathbb{Q} \to R^{m-2}(\psi_{|U})_*\mathbb{Q}$ between local systems on U, showing that $\Im(\tilde{\iota}^*_t)$ is globally invariant under the monodromy action on the cohomology of the smooth fibers of ψ. Finally, we recall that the inclusion map ι_t defines a Gysin map $\iota^\star_t : H_m(W_t;\mathbb{Q}) \to H_{m-2}(W'_t;\mathbb{Q})$ (see [5], p. 382, Example 19.2.1).

REMARK 2.2. – Fix a smooth $G \in |V_\delta|$, and assume $m \geq 2$. The linear system $|V_d|$ induces an embedding of $G\backslash Z$ in some projective space: denote by Γ the image of $G\backslash Z$ through this embedding. Since $G\backslash Z$ is irreducible, then also Γ is, and so is its general hyperplane section, which is isomorphic to $(G\cap X)\backslash Z$ via $|V_d|$. So we see that, when $m \geq 2$, for any smooth $G \in |V_\delta|$ and any general $X \in |V_d|$, one has that $W\backslash Z$ is irreducible. In particular, when $m > 2$, then also W is irreducible.

Lemma 2.3. – *Fix a smooth $G \in |V_\delta|$, and assume $m > 2$. Then, for a general $t \in U$, one has $\Im(\tilde{\iota}_t^*) = \Im(PD \circ \iota_t^\star)$, and the map $PD \circ \iota_t^\star$ is injective (PD means "Poincaré duality": $H_{m-2}(W_t';\mathbb{Q}) \cong H^{m-2}(W_t';\mathbb{Q})$).*

Proof. – By ([13], p. 385, Proposition 16.23) we know that $\Im(\tilde{\iota}_t^*)$ is equal to the image of the pull-back $H^{m-2}(W_t \backslash \mathrm{Sing}(W_t);\mathbb{Q}) \to H^{m-2}(W_t';\mathbb{Q})$. On the other hand, by ([3], p. 157 Proposition 5.4.4., and p. 158 (PD)) we have natural isomorphisms involving intersection cohomology groups:

$$H^{m-2}(W_t \backslash \mathrm{Sing}(W_t);\mathbb{Q}) \cong IH^{m-2}(W_t) \cong IH^m(W_t)^\vee \cong H^m(W_t;\mathbb{Q})^\vee \cong H_m(W_t;\mathbb{Q}). \tag{2}$$

So we may identify the pull-back $H^{m-2}(W_t \backslash \mathrm{Sing}(W_t);\mathbb{Q}) \to H^{m-2}(W_t';\mathbb{Q})$ with $PD \circ \iota_t^\star$. This proves that $\Im(\tilde{\iota}_t^*) = \Im(PD \circ \iota_t^\star)$. Moreover, since W_t' is smooth, then $IH^{m-2}(W_t') \cong H^{m-2}(W_t';\mathbb{Q})$ ([3], p. 157). So, from (2), we may identify $PD \circ \iota_t^\star$ with the natural map $IH^{m-2}(W_t) \to IH^{m-2}(W_t \cap H)$, which is injective in view of Lefschetz Hyperplane Theorem for intersection cohomology ([3], p. 158 (I), and p. 159, Theorem 5.4.6) (recall that $W_t' = W_t \cap H$). □

We are in position to prove Conjecture 1.

Fix a smooth $G \in |V_\delta|$, and a general $X \in |V_d|$. Put $W = G \cap X$. Since the monodromy group of the family of smooth divisors $X \in |H^0(Y, \mathcal{O}_Y(d))|$ containing W is a subgroup of the monodromy group of the family of smooth divisors $X \in |H^0(Y, \mathcal{O}_Y(d))|$ containing Z, in order to deduce Conjecture 1 from Theorem 1.1, it suffices to prove that $H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp Z} = H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp W}$. Equivalently, it suffices to prove that $H^m(X;\mathbb{Q})^{\mathrm{van}}_Z = H^m(X;\mathbb{Q})^{\mathrm{van}}_W$. This is the content of the following:

Proposition 2.4. – *For any smooth $G \in |V_\delta|$ and any general $X \in |V_d|$, one has $H^m(X;\mathbb{Q})^{\mathrm{van}}_Z = H^m(X;\mathbb{Q})^{\mathrm{van}}_W$.*

Proof. – First we analyze the cases $m = 1$ and $m = 2$, and next we argue by induction on $m > 2$ (recall that $\dim Y = m + 1$).

The case $m = 1$ is trivial because in this case $\dim Z \leq \dim W = 0$.

Next assume $m = 2$. In this case $\dim Y = 3$ and $\dim Z \leq 1$. Denote by $Z_1, \dots, Z_h$ ($h \geq 0$) the irreducible components of Z of dimension 1 (if there are). Fix a smooth $G \in |V_\delta|$ and a general $X \in |V_d|$, and put $W = G \cap X = Z_1 \cup \dots \cup Z_h \cup C$, where C is the residual curve, with respect to $Z_1 \cup \dots \cup Z_h$, in the complete intersection W. By Remark 2.2 we know that C is irreducible. Then, as (co)cycle classes, $Z_1, \dots, Z_h, C$ generate $H^2(X;\mathbb{Q})^{\mathrm{van}}_W$, and $Z_1, \dots, Z_h$ generate $H^2(X;\mathbb{Q})^{\mathrm{van}}_Z$. Since $Z_1 + \dots + Z_h + C = \delta H_X$ in $H^2(X;\mathbb{Q})$ (H_X= general hyperplane section of X in $\mathbb{P}^N$), and this cycle comes from $H^2(Y;\mathbb{Q})$, then $Z_1 + \dots + Z_h + C = 0$ in $H^2(X;\mathbb{Q})^{\mathrm{van}}$, and so $H^2(X;\mathbb{Q})^{\mathrm{van}}_Z = H^2(X;\mathbb{Q})^{\mathrm{van}}_W$. This concludes the proof of Proposition 2.4 in the case $m = 2$.

Now assume $m > 2$ and argue by induction on m. First we observe that the intersection pairing on $H^{m-2}(W';\mathbb{Q})^{\mathrm{van}}_{Z'}$ is non-degenerate: this follows from Hodge Index Theorem, because the cycles in $H^{m-2}(W';\mathbb{Q})^{\mathrm{van}}_{Z'}$ are primitive and algebraic. So we have the following orthogonal decomposition:

(3) $$H^{m-2}(W';\mathbb{Q}) = H^{m-2}(G';\mathbb{Q}) \perp H^{m-2}(W';\mathbb{Q})^{\mathrm{van}}_{Z'} \perp H^{m-2}(W';\mathbb{Q})^{\mathrm{van}}_{\perp Z'}.$$

Let $\mathcal{J}$ be the local system on U with fibre given by $H^{m-2}(G';\mathbb{Q}) \perp H^{m-2}(W';\mathbb{Q})^{\mathrm{van}}_{Z'}$. We claim that:

(4) $$\Im(\tilde{\iota}^*) = \mathcal{J}.$$

We will prove (4) shortly after. From (4) and Lemma 2.3 we get an isomorphism: $H_m(W;\mathbb{Q}) \cong H^{m-2}(G';\mathbb{Q}) \perp H^{m-2}(W';\mathbb{Q})^{\mathrm{van}}_{Z'}$. Taking into account that by Lefschetz Hyperplane Theorem we have $H^{m-2}(Y;\mathbb{Q}) \cong H^{m-2}(G;\mathbb{Q}) \cong H^{m-2}(G';\mathbb{Q})$, and that the Gysin map $H_m(Z;\mathbb{Q}) \to H_{m-2}(Z';\mathbb{Q})$ is bijective (because $H_m(Z;\mathbb{Q})$ and $H_{m-2}(Z';\mathbb{Q})$ are simply generated by the components which are of dimension m or $m-2$ of Z and Z' (if there are)), one sees that the natural map $H_m(W;\mathbb{Q}) \to H_m(X;\mathbb{Q}) \cong H^m(X;\mathbb{Q})$ sends $H^{m-2}(G';\mathbb{Q})$ in $H^m(Y;\mathbb{Q})$, and $H^{m-2}(W';\mathbb{Q})^{\mathrm{van}}_{Z'}$ in $H^m(X;\mathbb{Q})^{\mathrm{van}}_{Z}$. This proves $H^m(X;\mathbb{Q})^{\mathrm{van}}_{Z} \supseteq H^m(X;\mathbb{Q})^{\mathrm{van}}_{W}$. Since the reverse inclusion is obvious, it follows that $H^m(X;\mathbb{Q})^{\mathrm{van}}_{Z} = H^m(X;\mathbb{Q})^{\mathrm{van}}_{W}$.

So, to conclude the proof of Proposition 2.4, it remains to prove claim (4). To this purpose first notice that $\Im(\tilde{\iota}^*_t)$ contains $H^{m-2}(W'_t;\mathbb{Q})^{\mathrm{van}}_{Z'}$, because, by Lemma 2.3, we have $\Im(\tilde{\iota}^*_t) = \Im(PD \circ \iota^*_t)$, and $\Im(PD \circ \iota^*_t) \supseteq H^{m-2}(W'_t;\mathbb{Q})^{\mathrm{van}}_{Z'}$ in view of the quoted isomorphism $H_m(Z;\mathbb{Q}) \cong H_{m-2}(Z';\mathbb{Q})$. Moreover $\Im(\tilde{\iota}^*_t)$ contains $H^{m-2}(G';\mathbb{Q})$ because $H^{m-2}(G';\mathbb{Q}) \cong H^{m-2}(G;\mathbb{Q})$, and $H^{m-2}(G;\mathbb{Q})$ is contained in $\Im(\tilde{\iota}^*_t)$. Therefore we obtain $\Im(\tilde{\iota}^*) \supseteq \mathcal{J}$, from which we deduce that $\Im(\tilde{\iota}^*) = \mathcal{J}$. In fact, otherwise, since by induction $H^{m-2}(W'_t;\mathbb{Q})^{\mathrm{van}}_{\perp Z'}$ is irreducible, from (3) it would follow that $\Im(\tilde{\iota}^*) = R^{m-2}(\psi_{|U})_*\mathbb{Q}$. This is impossible because for $l \gg 0$ the dimension of $H^{m-2}(W'_t;\mathbb{Q})$ is arbitrarily large (by the way, we notice that the same argument proves that $\mathcal{J}$ is nothing but the invariant part of $R^{m-2}(\psi_{|U})_*\mathbb{Q}$). □

3. A monodromy theorem

In this section we prove a monodromy theorem (see Theorem 3.1 below), which we will use in next section for proving Theorem 1.1, and that we think of independent interest.

Let $Q \subseteq \mathbb{P}$ be an irreducible, reduced, non-degenerate projective variety of dimension $m+1$ ($m \geq 0$), with isolated singular points $q_1, \dots, q_r$. Let $L \in \mathbb{G}(1, \mathbb{P}^*)$ be a general pencil of hyperplane sections of Q, and denote by Q_L the blowing-up of Q along the base locus of L, and by $f : Q_L \to L$ the natural map. The ramification locus of f is a finite set $\{q_1, \dots, q_s\} := \mathrm{Sing}(Q) \cup \{q_{r+1}, \dots, q_s\}$, where $\{q_{r+1}, \dots, q_s\}$ denotes the set of tangencies of the pencil. Set $a_i := f(q_i)$, $1 \leq i \leq s$ (compare with [12], p. 304). The restriction map $f : Q_L \backslash f^{-1}(\{a_1, \dots, a_s\}) \to L \backslash \{a_1, \dots, a_s\}$ is a smooth proper map. Hence the fundamental group $\pi_1(L \backslash \{a_1, \dots, a_s\}, t)$ (t = general point of L) acts by monodromy on $Q_t := f^{-1}(t)$, and so on $H^m(Q_t;\mathbb{Q})$. By [10], p. 165-167, we know that $f : Q_L \backslash f^{-1}(\{a_1, \dots, a_s\}) \to L \backslash \{a_1, \dots, a_s\}$ induces an orthogonal decomposition: $H^m(Q_t;\mathbb{Q}) = I \perp V$, where I is the subspace of the invariant cocycles, and V is its orthogonal complement.

In the case Q is smooth, a classical basic result in Lefschetz Theory states that V is generated by "standard vanishing cycles" (i.e. by vanishing cycles corresponding to the tangencies of the pencil). This implies the irreducibility of V by standard classical reasonings ([7], [13]). Now we are going to prove that it holds true also when Q has isolated singularities. This is the content of the following Theorem 3.1, for which we did not succeed in finding an appropriate reference (for a related and somewhat more precise statement, see Proposition 3.4 below).

THEOREM 3.1. – *Let $Q \subseteq \mathbb{P}$ be an irreducible, reduced, non-degenerate projective variety of dimension $m+1 \geq 1$, with isolated singularities, and Q_t be a general hyperplane section of Q. Let $H^m(Q_t;\mathbb{Q}) = I \perp V$ be the orthogonal decomposition given by the monodromy action on the cohomology of Q_t, where I denotes the invariant subspace. Then V is generated, via monodromy, by standard vanishing cycles.*

REMARK 3.2. – (i) For a particular case of Theorem 3.1, see [12], Theorem (2.2).

(ii) When Q is a curve, i.e. when $m = 0$, then Theorem 3.1 follows from the well-known fact that the monodromy group is the full symmetric group (see [1], p. 111). So we assume from now on that $m \geq 1$.

(iii) When Q is a cone over a degenerate and necessarily smooth subvariety of $\mathbb{P}$, then $f : Q_L \to L$ has only one singular fiber $f^{-1}(a_1)$ (i.e. $s = 1$). In this case $\pi_1(L\backslash\{a_1\}, t)$ is trivial. Therefore we have that $H^m(Q_t;\mathbb{Q}) = I$, $V = 0$, and Theorem 3.1 follows.

Before proving Theorem 3.1, we need some preliminaries. We keep the same notation we introduced before.

NOTATIONS 3.3. – (i) Let $R_L \to Q_L$ be a desingularization of Q_L. The decomposition $H^m(Q_t;\mathbb{Q}) = I \perp V$ can be interpreted via R_L as $I = j^*(H^m(R_L;\mathbb{Q}))$ and $V = \mathrm{Ker}(H^m(Q_t;\mathbb{Q}) \to H^{m+2}(R_L;\mathbb{Q})) \cong \mathrm{Ker}(H_m(Q_t;\mathbb{Q}) \to H_m(R_L;\mathbb{Q}))$, where j denotes the inclusion $Q_t \subset R_L$. Using standard arguments (compare with [13], p. 325, Corollaire 14.23) one deduces a natural isomorphism:

$$V \cong \Im(H_{m+1}(R_L - g^{-1}(t_1), Q_t;\mathbb{Q}) \to H_m(Q_t;\mathbb{Q})), \tag{5}$$

where $g : R_L \to L$ denotes the composition of $R_L \to Q_L$ with $f : Q_L \to L$, and $t_1 \neq t$ another regular value of g.

(ii) For any critical value a_i of L fix a closed disk $\Delta_i \subset L\backslash\{t_1\} \cong \mathbb{C}$ with center a_i and radius $0 < \rho \ll 1$. As in [7], (5.3.1) and (5.3.2), one may prove that $H_{m+1}(R_L - g^{-1}(t_1), Q_t;\mathbb{Q}) \cong \oplus_{i=1}^{s} H_{m+1}(g^{-1}(\Delta_i), g^{-1}(a_i+\rho);\mathbb{Q})$. By (5) we have:

$$V = V_1 + \cdots + V_s, \tag{6}$$

where we denote by V_i the image in $H^m(Q_t;\mathbb{Q}) \cong H_m(g^{-1}(a_i+\rho);\mathbb{Q})$ of each $H_{m+1}(g^{-1}(\Delta_i), g^{-1}(a_i+\rho);\mathbb{Q})$. When $r+1 \leq i \leq s$, we recognize in $V_i \subseteq H^m(Q_t;\mathbb{Q})$ the subspace generated by the standard vanishing cocycle δ_i corresponding to a tangent hyperplane section of Q (see [7], [13], [12]).

(iii) Consider again the pencil $f : Q_L \to L$, and let $\mathbb{P}_L$ be the blowing-up of $\mathbb{P}$ along the base locus B_L. For any $i \in \{1, \ldots, s\}$, denote by $D_i \subset \mathbb{P}_L$ a closed ball with center q_i and small radius ϵ. Define $M_i := \Im(H_m(f^{-1}(a_i+\rho) \cap D_i;\mathbb{Q}) \to H_m(f^{-1}(a_i+\rho);\mathbb{Q}))$, with $0 < \rho \ll \epsilon \ll 1$. Since $H_m(f^{-1}(a_i+\rho);\mathbb{Q}) \cong H_m(Q_t;\mathbb{Q}) \cong H^m(Q_t;\mathbb{Q})$, we may regard

$M_i \subseteq H^m(Q_t;\mathbb{Q})$. When $1 \leq i \leq r$, M_i represents the subspace spanned by the cocycles "coming" from the singularities of Q, and lying in the Milnor fibre $f^{-1}(a_i+\rho)\cap D_i$. When $r+1 \leq i \leq s$, i.e. when a_i corresponds to a tangent hyperplane section of Q, then $V_i = M_i$. In general we have:

$$V_i \subseteq M_i \quad \textit{for any } i = 1,\dots,s. \tag{7}$$

This is a standard fact, that one may prove as in ([8], (7.13) Proposition). For Reader's convenience, we give the proof of property (7) in the appendix, at the end of the paper.

Now we are going to prove Theorem 3.1

Proof of Theorem 3.1. – Let $\pi : \mathcal{F} \to \mathbb{P}^*$ ($\mathcal{F} \subseteq \mathbb{P}^*\times\mathbb{P}$) be the universal family parametrizing the hyperplane sections of $Q \subseteq \mathbb{P}$, and denote by $\mathcal{D} \subseteq \mathbb{P}^*$ the discriminant locus of π, i.e. the set of hyperplanes $H \in \mathbb{P}^*$ such that $Q\cap H$ is singular. At least set-theoretically, we have $\mathcal{D} = Q^* \cup \mathcal{H}_1 \cup \dots \cup \mathcal{H}_r$, where Q^* denotes the dual variety of Q, and $\mathcal{H}_j$ denotes the dual hyperplane of q_j (compare with [12], p. 303).

When the codimension of Q^* in $\mathbb{P}^*$ is 1, denote by T_t the stalk at $t \in \mathbb{P}^*\backslash\mathcal{D}$ of the local subsystem of $R^m(\pi|_{\pi^{-1}(\mathbb{P}^*\backslash\mathcal{D})})_*\mathbb{Q}$ generated by the vanishing cocycle at general point of Q^* (compare with [9], p. 373, or [12], p. 306). If the codimension of Q^* in $\mathbb{P}^*$ is ≥ 2, put $T_t := \{0\}$. In order to prove Theorem 3.1 it suffices to prove that $V = T$ ($T := T_t$). By Deligne Complete Reducibility Theorem ([10], p. 167), we may write $H^m(Q_t;\mathbb{Q}) = W \oplus T$, for a suitable invariant subspace W. Now we claim the following proposition, which we will prove below:

Proposition 3.4. – *The monodromy representation on the quotient local system with stalk $H^m(Q_t;\mathbb{Q})/T_t$ at $t \in \mathbb{P}^*\backslash\mathcal{D}$ is trivial.*

By previous Proposition 3.4 it follows that for any $g \in \pi_1(L\backslash\{a_1,\dots,a_s\},t)$ and any $w \in W$ there exists $\tau \in T$ such that $w^g = w+\tau$. Then $\tau = w^g - w \in T\cap W = \{0\}$, and so $w^g = w$. Therefore W is invariant, i.e. $W \subseteq I$, and since $T \subseteq V$ and $H^m(Q_t;\mathbb{Q}) = I \oplus V = W \oplus T$, then we have $T = V$. □

It remains to prove Proposition 3.4. To this aim, we need some preliminaries. We keep the same notation we introduced before.

Consider again the universal family $\pi : \mathcal{F} \to \mathbb{P}^*$ parametrizing the hyperplane sections of $Q \subseteq \mathbb{P}$. We will denote by H_x the hyperplane parametrized by $x \in \mathbb{P}^*$. Fix a point $q_i \in \mathrm{Sing}(Q)$ (hence $i \in \{1,\dots,r\}$). For general L, q_i is not a base point of the pencil defined by L, hence $Q_L \cong Q$ over q_i. Combined with the inclusion $Q_L \subseteq \mathcal{F}$, we thus have a natural lift of q_i to a point of $\mathcal{F}$, still denoted by q_i.

Remark 3.5. – If Q^* is contained in $\mathcal{H}_j$ for some $j \in \{1,\dots,r\}$, then Q^* is degenerate in $\mathbb{P}^*$, and so $Q = Q^{**}$ is a cone in $\mathbb{P}$. Therefore, if Q is not a cone, then Q^* is not contained in $\mathcal{H}_j$ for any $j \in \{1,\dots,r\}$. In this case, for a general line $\ell \subseteq \mathcal{H}_i$, the set $\ell\cap Q^*$ is finite, and for any $x \in \ell$, $H_x \cap Q$ has an isolated singularity at q_i.

Notations 3.6. – (i) Let $\ell \subseteq \mathcal{H}_i$ be a general line. For any $u \in \ell \cap Q^*$, denote by Δ_u° an open disk of ℓ with center u and small radius. Consider the compact $K := \ell \backslash (\bigcup_{u\in\ell\cap Q^*} \Delta_u^\circ)$. In the appendix below (see Lemma 5.1) we prove that *there is a closed ball $D_{q_i} \subseteq \mathbb{P}^* \times \mathbb{P}$, with positive radius and centered at q_i, such that for any $x \in K$ the distance function $p \in H_x \cap Q \cap D_{q_i} \to ||p - q_i|| \in \mathbb{R}$ has no critical points $p \neq q_i$* (we already proved a similar result in [2], Lemma 3.4, (v)). By ([8], pp. 21-28) it follows that for any $x \in K$ there is a closed ball $C_x \subseteq \mathbb{P}^*$ centered at x, for which the induced map $z \in \pi^{-1}(C_x) \cap D_{q_i} \to \pi(z) \in C_x$ is a Milnor fibration, with discriminant locus given by $\mathcal{H}_i \cap C_x$. Since K is compact, we may cover it with finitely many of such C_x's. So we deduce the existence of a connected closed tubular neighborhood $\mathcal{K}$ of K in $\mathbb{P}^*$, such that the map:

$$\pi_{\mathcal{K}} : z \in \pi^{-1}(\mathcal{K}) \cap D_{q_i} \to \pi(z) \in \mathcal{K} \tag{8}$$

defines a C^∞-fiber bundle on $\mathcal{K}\backslash\mathcal{H}_i$, and whose fibre $\pi_{\mathcal{K}}^{-1}(t) = H_t \cap Q \cap D_{q_i}, t \in \mathcal{K}\backslash\mathcal{H}_i$, may be identified with the Milnor fibre.

(ii) Let $\mathcal{M}_i$ be the local system with fibre $\mathcal{M}_{i,t}$ at $t \in \mathcal{K}\backslash\mathcal{D}$ given by the image of $H_m(H_t \cap Q \cap D_{q_i}; \mathbb{Q})$ in $H_m(H_t \cap Q; \mathbb{Q}) \cong H^m(Q_t; \mathbb{Q})$. Notice that, for any general pencil $L \in \mathbb{G}(1, \mathbb{P}^*)$, the local system $\mathcal{M}_i$ extends, as a local system, M_i on all $L \cap (\mathcal{K}\backslash\mathcal{D})$ (compare with Notations 3.3 (iii)). In particular we may assume $M_i = \mathcal{M}_{i,t}$.

We are in position to prove Proposition 3.4. We keep the same notation we introduced before.

Proof of Proposition 3.4. – As in ([12], proof of Theorem (2.2)), we need to consider only the action of $\pi_1(\mathbb{P}^*\backslash(\bigcup_{1\leq j\leq r} \mathcal{H}_j), t)$.

Consider the finite set $A := \ell \cap (\bigcup_{j\neq i} \mathcal{H}_j)$, and let $a \in A$ be a point. In view of Remark 3.2 (iii), and Remark 3.5, we may assume that $H_a \cap Q$ has an isolated singularity at q_i. Notice that, a priori, it may happen that $a \in \ell \cap Q^*$ and so $a \notin K$. But in any case, since $H_a \cap Q$ has an isolated singularity at q_i, as before, for any $a \in A$ we may construct a closed ball $D_{q_i}^{(a)} \subseteq \mathbb{P}^* \times \mathbb{P}$, with positive radius and centered at q_i, and a closed ball $C_a \subseteq \mathbb{P}^*$ centered at a, for which the induced map

$$z \in \pi^{-1}(C_a) \cap D_{q_i}^{(a)} \to \pi(z) \in C_a \tag{9}$$

is a Milnor fibration with discriminant locus contained in $\mathcal{H}_i \cup Q^*$. We may assume $D_{q_i} \subseteq D_{q_i}^{(a)}$ for any $a \in A$, and, shrinking the disks Δ_u° ($u \in \ell \cap Q^*$) if necessary, we may also assume that the interior $\mathcal{K}^\circ$ of $\mathcal{K}$ meets the interior C_a° of each C_a. Therefore, in $(\mathcal{K}^\circ \cap C_a^\circ)\backslash(\mathcal{H}_i \cup Q^*)$, the bundle (8) appears as a subbundle of (9).

Observe that the image in $H^m(Q_t; \mathbb{Q})/T_t$ of the cohomology of (9) coincides with $(\mathcal{M}_{i,t} + T_t)/T_t$ on $(\mathcal{K}^\circ \cap C_a^\circ)\backslash(\mathcal{H}_i \cup Q^*)$. This implies that, in a suitable small analytic neighborhood $\mathcal{L}$ of ℓ in $\mathbb{P}^*$, the quotient local system $(\mathcal{M}_{i,t} + T_t)/T_t$ extends on all $\mathcal{L}\backslash\mathcal{D}$. Taking into account Picard-Lefschetz formula, and that the discriminant locus of (9) is contained in $\mathcal{H}_i \cup Q^*$, we have that $\pi_1(\mathbb{P}^*\backslash\mathcal{D}, t)$ acts trivially on $(\mathcal{M}_{i,t} + T_t)/T_t$. This holds true for any $i \in \{1, \cdots, r\}$. Hence, in view of (6) and (7), it follows that the monodromy action is trivial on $H^m(Q_t; \mathbb{Q})/T_t$. This concludes the proof of Proposition 3.4. □

By standard classical reasonings as in [7] or [13], from Theorem 3.1 we deduce the following:

Corollary 3.7. – *V is irreducible.*

Proof. – Let $\{0\} \neq V' \subset V$ be an invariant subspace. As before, we may write $H^m(Q_t;\mathbb{Q}) = U \oplus V'$, for a suitable invariant subspace U. Hence we have $V = (V \cap U) \oplus V'$. On the other hand, one knows that V is nondegenerate with respect to the intersection form $\langle \cdot, \cdot \rangle$ on Q_t ([10], p.167). Therefore, for some $i \in \{r+1, \dots, s\}$, there exists $\tau \in (V \cap U) \cup V'$ such that $\langle \tau, \delta_i \rangle \neq 0$ ($Span(\delta_i) := V_i$). From the Picard-Lefschetz formula it follows that the tangential vanishing cycle δ_i lies in $(V \cap U) \cup V'$. If $\delta_i \in V \cap U$, then by Theorem 3.1 we deduce $V = V \cap U$ (compare with [7], [8], [12], [13]), and this is in contrast with the fact that $\{0\} \neq V'$. Hence $\delta_i \in V'$, and by the same reason $V' = V$. This proves that V is irreducible. □

4. Proof of Theorem 1.1

4.1. The set-up

Consider the rational map $Y \dashrightarrow \mathbb{P} := \mathbb{P}(H^0(Y, \mathcal{I}_{W,Y}(d))^*)$ defined by the linear system $|H^0(Y, \mathcal{I}_{W,Y}(d))|$. By [5], 4.4, such a rational map defines a morphism $Bl_W(Y) \to \mathbb{P}$. We denote by Q the image of this morphism, i.e.:

$$Q := \Im(Bl_W(Y) \to \mathbb{P}). \tag{10}$$

Set $E := \mathbb{P}(\mathcal{O}_Y(k) \oplus \mathcal{O}_Y(d))$. The surjections $\mathcal{O}_Y(k) \oplus \mathcal{O}_Y(d) \to \mathcal{O}_Y(d)$ and $\mathcal{O}_Y(k) \oplus \mathcal{O}_Y(d) \to \mathcal{O}_Y(k)$ give rise to divisors $\Theta \cong Y \subseteq E$ and $\Gamma \cong Y \subseteq E$, with $\Theta \cap \Gamma = \varnothing$. The line bundle $\mathcal{O}_E(\Theta)$ is base point free and the corresponding morphism $E \to \mathbb{P}(H^0(E, \mathcal{O}_E(\Theta))^*)$ sends E to a cone over the Veronese variety of Y (i.e. over Y embedded via $|H^0(Y, \mathcal{O}_Y(d-k))|$) in such a way that Γ is contracted to the vertex v_∞ and Θ to a general hyperplane section. In other words, we may view E, via $E \to \mathbb{P}(H^0(E, \mathcal{O}_E(\Theta))^*)$, as the blowing-up of the cone over the Veronese variety at the vertex, and Γ as the exceptional divisor ([6], p. 374, Example 2.11.4).

From the natural resolution of $\mathcal{I}_{W,Y}$: $0 \to \mathcal{O}_Y(-k-d) \to \mathcal{O}_Y(-k) \oplus \mathcal{O}_Y(-d) \to \mathcal{I}_{W,Y} \to 0$, we find that $Bl_W(Y) = \mathbf{Proj}(\oplus_{i \geq 0} \mathcal{I}^i_{W,Y})$ is contained in E, and that $\mathcal{O}_E(\Theta - d\Lambda) \mid_{Bl_W(Y)} \cong \mathcal{O}_{Bl_W(Y)}(1)$ ($\Lambda :=$ pull-back of the hyperplane section of $Y \subseteq \mathbb{P}^N$ through $E \to Y$). Therefore:

(i) we have natural isomorphisms: $H^0(Y, \mathcal{I}_{W,Y}(d)) \cong H^0(Y, \mathcal{O}_Y \oplus \mathcal{O}_Y(d-k)) \cong H^0(E, \mathcal{O}_E(\Theta))$;

(ii) the linear series $|\Theta|$ cut on $Bl_W(Y)$ the linear series spanned by the strict transforms $\tilde{X}$ of the divisors $X \in |H^0(Y, \mathcal{I}_{W,Y}(d))|$, and, sending E to a cone in $\mathbb{P}$ over a Veronese variety, restricts to $Bl_W(Y)$ to the map $Bl_W(Y) \to Q$ defined above. Hence we have a natural commutative diagram:

$$\begin{array}{ccccc} Bl_W(Y) & \hookrightarrow & E & & \\ \downarrow & \searrow & & \searrow & \\ Y & \dashrightarrow & Q & \hookrightarrow & \mathbb{P}. \end{array}$$

By the same reason $\Gamma \cap Bl_W(Y) = \tilde{G}$ ($\tilde{G} :=$ the strict transform of G in $Bl_W(Y)$). Notice that $\tilde{G} \cong G$ since W is a Cartier divisor in G. Similarly $\tilde{X} \cong X$ when G is not contained in X;

(iii) since $|\Theta|$ contracts Γ to the vertex v_∞, the map $Bl_W(Y) \to Q$ contracts $\tilde{G}$ to $v_\infty \in Q$. Furthermore we have $Bl_W(Y)\backslash\tilde{G} \cong Q\backslash\{v_\infty\}$ and so the hyperplane sections of Q not containing the vertex are isomorphic, via $Bl_W(Y) \to Q$, to the corresponding divisors $X \in |H^0(Y, \mathcal{I}_{W,Y}(d))|$;

(iv) by (ii) above, $\tilde{G}$ is a smooth Cartier divisor in $Bl_W(Y)$, hence $\tilde{G}$ is disjoint with $\mathrm{Sing}(Bl_W(Y))$. On the other hand, from ([4], p. 133, Proposition 4.2.6. and proof) we know that $\mathrm{Sing}(W)$ is a finite set. The singularities of $Bl_W(Y)$ must be contained in the inverse image of $\mathrm{Sing}(W)$ via $Bl_W(Y) \to Y$: this is a finite set of lines none of which lying in $\mathrm{Sing}(Bl_W(Y))$ because $\tilde{G}$ meets all such lines. Therefore $\mathrm{Sing}(Bl_W(Y))$ must be a finite set, and so also $\mathrm{Sing}(Q)$ is. Observe also that $\tilde{G}$ is isomorphic to the tangent cone to Q at v_∞, and its degree is $k(d-k)^m deg\, Y$. Hence Q is nonsingular at v_∞ only when $Y = \mathbb{P}^{m+1}$, $k = 1$ and $d = 2$. In this case X is a smooth quadric, therefore $\dim\, H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp W} \leq 1$, and Theorem 1.1 is trivial. So we may assume $v_\infty \in \mathrm{Sing}(Q)$.

4.2. The proof

We are going to prove Theorem 1.1, that is the irreducibility of the monodromy action on $H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp W}$. The proof consists in an application of previous Corollary 3.7 to the variety $Q \subseteq \mathbb{P}$ defined in (10). We keep the same notation we introduced in 4.1.

Proof of Theorem 1.1. – Consider the variety $Q \subseteq \mathbb{P}$ defined in (10). By the description of it given in 4.1, we know that Q is an irreducible, reduced, non-degenerate projective variety of dimension $m + 1 \geq 2$, with isolated singularities.

Let $L \in \mathbb{G}(1, \mathbb{P}^*)$ be a general pencil of hyperplane sections of Q, and denote by Q_L the blowing-up of Q along the base locus of L, and by $f : Q_L \to L$ the natural map (compare with Section 3). Denote by $\{a_1, \dots, a_s\} \subseteq L$ the set of the critical values of f. The fundamental group $\pi_1(L\backslash\{a_1, \dots, a_s\}, t)$ (t = general point of L) acts by monodromy on $f^{-1}(t)$, and so on $H^m(f^{-1}(t);\mathbb{Q})$, and this action induces an orthogonal decomposition: $H^m(f^{-1}(t);\mathbb{Q}) = I \perp V$, where I is the subspace of the invariant cocycles, and V is its orthogonal complement. By Corollary 3.7 we know that V is irreducible.

On the other hand, in view of 4.1, we may identify $f^{-1}(t)$ with a general $X_t \in |H^0(Y, \mathcal{I}_{W,Y}(d))|$, and the action of $\pi_1(L\backslash\{a_1, \dots, a_s\}, t)$ with the action induced on X_t by a general pencil of divisors in $|H^0(Y, \mathcal{I}_{W,Y}(d))|$. So, in order to prove Theorem 1.1, it suffices to prove that $H^m(X_t;\mathbb{Q})^{\mathrm{van}}_{\perp W} = V$. This is equivalent to prove that $I = H^m(Y;\mathbb{Q}) + H^m(X_t;\mathbb{Q})^{\mathrm{van}}_{W}$. Since the inclusion $H^m(Y;\mathbb{Q}) + H^m(X_t;\mathbb{Q})^{\mathrm{van}}_{W} \subseteq I$ is obvious, to prove Theorem 1.1 it suffices to prove that:

$$I \subseteq H^m(Y;\mathbb{Q}) + H^m(X_t;\mathbb{Q})^{\mathrm{van}}_{W}. \tag{11}$$

To this purpose, let $B_L \subseteq Q$ be the base locus of L. Since $v_\infty \notin B_L$, then we may regard $B_L \subseteq Bl_W(Y)$ via $Bl_W(Y) \to Q$. Notice that $B_L \cong X_t \cap M_L$, for a suitable general $M_L \in |H^0(Y, \mathcal{O}_Y(d-k))|$. Let $Bl_W(Y)_L$ be the blowing-up of $Bl_W(Y)$ along B_L, and consider the pencil $f_1 : Bl_W(Y)_L \to L$ induced from the natural map $Bl_W(Y)_L \to Q_L$.

We have $Q_L \backslash f^{-1}(\{a_1, \dots, a_s\}) \cong Bl_W(Y)_L \backslash f_1^{-1}(\{a_1, \dots, a_s\})$. So, if $R_L \to Bl_W(Y)_L$ denotes a desingularization of $Bl_W(Y)_L$, then the subspace I of the invariant cocycles can be interpreted via R_L as $I = j^*(H^m(R_L; \mathbb{Q}))$, where j denotes the inclusion $X_t \subseteq R_L$.

Denote by $\widetilde{W}$ and $\widetilde{B_L}$ the inverse images of $W \subseteq Y$ and $B_L \subseteq Bl_W(Y)$ in R_L. The map $R_L \to Y$ induces an isomorphism $\alpha_1 : R_L \backslash (\widetilde{W} \cup \widetilde{B_L}) \to Y \backslash (W \cup (X_t \cap M_L))$. Consider the following natural commutative diagram:

$$\begin{array}{ccc} H^m(R_L; \mathbb{Q}) & \stackrel{\rho_1}{\to} & H^m(R_L \backslash (\widetilde{W} \cup \widetilde{B_L}); \mathbb{Q}) \\ \alpha \downarrow & & \| \, \alpha_1 \\ H^m(Y; \mathbb{Q}) & \stackrel{\rho_2}{\to} & H^m(Y \backslash (W \cup (X_t \cap M_L)); \mathbb{Q}) \\ \beta \downarrow & & \downarrow \beta_1 \\ H^m(X_t; \mathbb{Q}) & \stackrel{\rho_3}{\to} & H^m(X_t \backslash (W \cup (X_t \cap M_L)); \mathbb{Q}) \end{array}$$

where α is the Gysin map, and fix $c \in I = j^*(H^m(R_L; \mathbb{Q}))$. Let $c' \in H^m(R_L; \mathbb{Q})$ such that $j^*(c') = c$. Since $\beta_1 \circ \alpha_1 \circ \rho_1 = \rho_3 \circ j^*$, then we have: $\rho_3(c) = (\rho_3 \circ \beta \circ \alpha)(c')$. Hence we have $c - \beta(\alpha(c')) \in \operatorname{Ker} \rho_3 = \Im(H^m(X_t, X_t \backslash (W \cup (X_t \cap M_L)); \mathbb{Q}) \to H^m(X_t; \mathbb{Q}))$. Since $H^m(X_t, X_t \backslash (W \cup (X_t \cap M_L)); \mathbb{Q}) \cong H_m(W \cup (X_t \cap M_L); \mathbb{Q})$ ([5], (3), p. 371), we deduce $c - \beta(\alpha(c')) \in \Im(H_m(W \cup (X_t \cap M_L); \mathbb{Q}) \to H_m(X_t; \mathbb{Q}) \cong H^m(X_t; \mathbb{Q}))$. So to prove (11), it suffices to prove that $\Im(H_m(W \cup (X_t \cap M_L); \mathbb{Q}) \to H_m(X_t; \mathbb{Q}) \cong H^m(X_t; \mathbb{Q}))$ is contained in $H^m(Y; \mathbb{Q}) + \Im(H_m(W; \mathbb{Q}) \to H_m(X_t; \mathbb{Q}) \cong H^m(X_t; \mathbb{Q}))$.

Since W has only isolated singularities, and M_L is general, then $W \cap M_L$ and $X_t \cap M_L$ are smooth complete intersections. From Lefschetz Hyperplane Theorem and Hard Lefschetz Theorem it follows that the natural map $H_{m-1}(W \cap M_L; \mathbb{Q}) \to H_{m-1}(X_t \cap M_L; \mathbb{Q})$ is injective. Hence, from the Mayer-Vietoris sequence of the pair $(W, X_t \cap M_L)$ we deduce that the natural map $H_m(W; \mathbb{Q}) \oplus H_m(X_t \cap M_L; \mathbb{Q}) \to H_m(W \cup (X_t \cap M_L); \mathbb{Q})$ is surjective. So to prove (11) it suffices to prove that $\Im(H_m(X_t \cap M_L; \mathbb{Q}) \to H_m(X_t; \mathbb{Q}) \cong H^m(X_t; \mathbb{Q}))$ is contained in $H^m(Y; \mathbb{Q})$. And this follows from the natural commutative diagram:

$$\begin{array}{ccc} H_m(X_t \cap M_L; \mathbb{Q}) \cong H^{m-2}(X_t \cap M_L; \mathbb{Q}) & \stackrel{\rho}{\leftarrow} & H^{m-2}(Y; \mathbb{Q}) \cong H_{m+4}(Y; \mathbb{Q}) \\ \downarrow & & \downarrow \cap M_L \\ H_m(X_t; \mathbb{Q}) \cong H^m(X_t; \mathbb{Q}) & \leftarrow & H^m(Y; \mathbb{Q}) \cong H_{m+2}(Y; \mathbb{Q}), \end{array}$$

taking into account that ρ is an isomorphism by Lefschetz Hyperplane Theorem. This proves (11), and concludes the proof of Theorem 1.1. □

5. Appendix

Proof of property (7). – First notice that since $f^{-1}(\Delta_i) - D_i^\circ \to \Delta_i$ is a trivial fiber bundle (D_i°= interior of D_i), then the inclusion $(f^{-1}(a), f^{-1}(a) \cap D_i) \subseteq (f^{-1}(\Delta_i), f^{-1}(\Delta_i) \cap D_i)$ induces natural isomorphisms $H_m(f^{-1}(a), f^{-1}(a) \cap D_i; \mathbb{Q}) \cong H_m(f^{-1}(\Delta_i), f^{-1}(\Delta_i) \cap D_i; \mathbb{Q})$ for any $a \in \Delta_i$ (use [11], p. 200 and 258). So, from the natural commutative diagram:

$$\begin{array}{ccc} H_m(f^{-1}(a_i+\rho);\mathbb{Q}) & \xrightarrow{\beta} & H_m(f^{-1}(a_i+\rho), f^{-1}(a_i+\rho)\cap D_i;\mathbb{Q}) \\ {}^{\alpha}\downarrow & & \| \\ H_m(f^{-1}(\Delta_i);\mathbb{Q}) & \rightarrow & H_m(f^{-1}(\Delta_i), f^{-1}(\Delta_i)\cap D_i;\mathbb{Q}), \end{array}$$

we deduce that $\operatorname{Ker}\alpha \subseteq \operatorname{Ker}\beta = M_i$.

On the other hand, since the inclusion $f^{-1}(a_i+\rho) \subseteq f^{-1}(\Delta_i)$ is the composition of the isomorphism $f^{-1}(a_i+\rho) \cong g^{-1}(a_i+\rho)$ with $g^{-1}(a_i+\rho) \subseteq g^{-1}(\Delta_i)$, followed by the desingularization $g^{-1}(\Delta_i) \to f^{-1}(\Delta_i)$, we have: $V_i \subseteq \operatorname{Ker}\alpha$. □

LEMMA 5.1. – *Let $\ell \subseteq \mathcal{H}_i$ be a general line. For any $u \in \ell \cap Q^*$, denote by Δ_u° an open disk of ℓ with center u and small radius. Consider the compact $K := \ell \backslash (\bigcup_{u\in\ell\cap Q^*} \Delta_u^\circ)$. Then there is a closed ball $D_{q_i} \subseteq \mathbb{P}^* \times \mathbb{P}$, with positive radius and centered at q_i, such that for any $x \in K$ the distance function $p \in H_x \cap Q \cap D_{q_i} \to ||p - q_i|| \in \mathbb{R}$ has no critical points $p \neq q_i$.*

Proof. – We argue by contradiction. Suppose the claim is false. Then there is a sequence of hyperplanes $y_n \in K$, $n \in \mathbb{N}$, converging to some $x \in K$, and a sequence of critical points $p_n \neq q_i$ for the distance function on $H_{y_n} \cap Q$, converging to q_i (we may assume p_n is smooth for $H_{y_n} \cap Q$). Let $T_{p_n,Q}$, $T'_{p_n,H_{y_n}\cap Q}$ and s_{q_i,p_n} be the corresponding sequences of tangent spaces and secants, and denote by $r_{q_i,p_n} \subseteq s_{q_i,p_n}$ the real line meeting q_i and p_n. We may assume they converge, and we denote by T, T', s and r their limits ($r \subseteq s$). Since p_n is a critical point, then r_{q_i,p_n} is orthogonal to $T'_{p_n,H_{y_n}\cap Q}$, hence $r \nsubseteq T'$, and so T is spanned by $T' \cup s$ by dimension reasons. Since $T' \cup s \subseteq H_x$ then $T \subseteq H_x$, so H_x contains a limit of tangent spaces of Q, with tangencies converging to q_i. This implies that $x \in Q^*$, contradicting the fact that $x \in K$. □

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