JAMES SIMONS
DENNIS SULLIVAN

Structured bundles define differential *K*-theory

Astérisque, tome 321 (2008), p. 1-3
<http://www.numdam.org/item?id=AST_2008__321__1_0>

Astérisque
321, 2008, p. 1–3

STRUCTURED BUNDLES DEFINE DIFFERENTIAL K-THEORY

by

James Simons & Dennis Sullivan

Abstract. — Complex bundles with connection up to isomorphism form a semigroup under Whitney sum which is far from being a group. We define a new equivalence relation (structured equivalence) so that stable isomorphism classes up to structured equivalence form a group which is describable in terms of the Chern character form plus some finite type invariants from algebraic topology. The elements in this group also satisfy two further somewhat contradictory properties: a locality or gluing property and an integrality property. There is interest in using these objects as pre-quantum fields in gauge theory and M-theory.

***Résumé* (Les fibrés structurés définissent la K-théorie différentielle).** — Les fibrés complexes à connexion forment, à isomorphisme près, un semi-groupe sous la somme de Whitney qui est loin d'être un groupe. Nous définissons une nouvelle relation d'équivalence (l'équivalence structurée) de manière à ce que les classes d'isomorphismes stable, à équivalence structurée près, forment un groupe qui puisse être décrit en termes de forme de caractère de Chern et de quelques invariants de type fini de la topologie algébrique. Les éléments de ce groupe satisfont également à deux propriétés en quelque sorte contradictoires : une propriété de localité ou de *gluing* et une propriété d'intégralité. Il semble intéressant d'utiliser ces objets en tant que champs pré-quantiques en théorie de gauge et en M-théorie.

Let M be the category whose objects are smooth manifolds and whose morphisms are smooth maps. We assume the manifolds are either compact manifolds possibly with boundary or diffeomorphic to those obtained from these by deleting some or all of the boundary components.

Let $K^\wedge$ denote the contravariant functor on M to abelian groups defined by equivalence classes of pairs (V, A) where V is a complex vector bundle and A is a connection on V. The equivalence relation is generated by stable isomorphisms of bundles with connection and by structured equivalence: namely any deformation of a connection A on a fixed bundle along any smooth path of connections so that the associated odd Chern Simons form is exact. Recall the exterior d of the Chern Simons form of a path

2000 Mathematics Subject Classification. — 57N65, 55N15, 19E08, 58J28, 14D21.
Key words and phrases. — Structured bundles, connections, Chern Simons form, differential K-theory.

of connections measures the change in the Chern Character form at the endpoints of the path.

We want to compute $K^\wedge$. Note an equivalence class in $K^\wedge$ has a precise total even form representing the Chern character plus some information related to a total odd cohomology class represented by Chern Simons forms which are closed but not exact.

Let $\mathrm{ch}(K)$ denote the contravariant functor on M to abelian groups defined by considering pairs $([V], C)$ where V represents an element of the K-theory of complex vector bundles and C is a total closed complex valued even dimensional form so that C represents the Chern character of V in rational cohomology.

We have the obvious map from $K^\wedge$ to $\mathrm{ch}(K)$ which assigns to the pair (V, A) of bundle with connection the pair $([V], C)$ where $[V]$ is the class of the stable vector bundle V in Atiyah's K-theory and C is the differential form defined by the Chern Weil curvature construction representing the total Chern character.

Let Torus be the functor on M to abelian complex Lie groups given by the odd cohomology with complex coefficients modulo the sublattice defined by considering all maps into G , the union over n of the n-dimensional complex linear groups, and by pulling back the desuspended Chern character class. This class is defined universally at level n defining G by desuspending the Chern character class of the bundle on suspension G defined using the identity map of G as a gluing function.

Theorem 1. — *The homomorphism from $K^\wedge$ to $\mathrm{ch}(K)$ is onto.*

The kernel of the homomorphism is the abelian complex Lie group Torus. We have the natural short exact sequence:

$$0 \to \mathrm{Torus} \to K^\wedge \to \mathrm{ch}(K) \to 0. \tag{1}$$

Let k denote the kernel of the natural map from $K^\wedge$ to K, namely $(V, C) \to [V]$. Then from sequence (1), k maps with kernel Torus onto exact total even forms.

Let $O =$ total odd forms modulo all closed forms in the cohomology classes of the sublattice above defining Torus. Then O maps via d with kernel Torus onto exact total even forms. The construction of $K^\wedge$ shows the kernel k is naturally isomorphic to O. In the detailed paper [1], O is denoted $\wedge^{odd}(X)/\wedge_G(X)$.

Theorem 2. — *There is the natural short exact sequence*

$$0 \to O \to K^\wedge \to K \to 0. \tag{2}$$

One may also show from the construction:

Theorem 3. — *$K^\wedge$ satisfies the Mayer-Vietoris property: if X is A union B with intersection C then given two elements a in $K^\wedge(A)$ and b in $K^\wedge(B)$ which restrict to the same element c in $K^\wedge(C)$, then there is an element x in $K^\wedge(X)$ which restricts to a and to b respectively.*

Consider $E =$ all total even forms in the cohomology classes of the Chern characters of complex vector bundles. By Theorem 1 the map $K^\wedge \to E$ is surjective. Now consider k', the kernel of this map from $K^\wedge$ to E. Since $K^\wedge$ satisfies the Mayer-Vietoris property so does this kernel k'. One can show also that k' is a homotopy functor. Thus by Brown's representability theorem k' is represented by homotopy classes of maps into some space. Using this , the sequences above and side condition 1 in Remark 1 below leads to

Theorem 4. — *The kernel of the surjection of $K^\wedge$ onto E is naturally isomorphic to K-theory with coefficients in C/Z. Let us denote the latter by $K(C/Z)$. Then we have the natural short exact sequence:*

$$0 \to K(C/Z) \to K^\wedge \to E \to 0. \tag{3}$$

Now $K^\wedge$ is not a homotopy functor, but the change produced by an infinitesimal deformation v of a map can be computed. This change u is in $O =$ the kernel of $(K^\wedge \to K)$ because K is a homotopy functor. We know that du is the lie derivative of the Chern character form. So the following is natural and indeed true for $K^\wedge$:

Theorem 5. — *The change in $f^*(x)$ for x in $K^\wedge$ by an infinitesimal deformation v of a map f is obtained by contracting the Chern form of x byv and projecting it to O inside $K^\wedge$.*

Remark 1. — *We have omitted two natural side conditions in the statements of Theorems 2 and 4 which should be noted.*

1. *The composition $K(C/Z) \to K^\wedge \to K$ using* (2) *and* (3) *is the Bockstein map in the Bockstein exact sequence for K-theory.*
2. *The composition $O \to K^\wedge \to E$ using* (2) *and* (3) *is exterior d.*

Conjecture. — *There is at most one functor $K^\wedge$ up to natural equivalence satisfying Theorems 1, 2, 3, 4 and 5 and the side conditions 1 and 2 in Remark 1.*

The presence of the homotopy property expressed by Theorem 5 in the conjecture above was inspired by conversations with Moritz Wiethaup. This homotopy property was not needed in our axioms characterizing ordinary differential cohomology [**2**]. The details of the proofs of the results here will appear soon [**1**]. We close by expressing on this occasion our appreciation of and admiration for the geometer Jean Pierre Bourguignon.

References

[1] J. SIMONS & D. SULLIVAN – "Axiomatic characterization of ordinary differential cohomology", *J. Topol.* **1** (2008), p. 45–56.

[2] ———, "Structured bundles and differential K-theory", preprint arXiv:0810.4935 (submitted to *J. Topol.*).

J. SIMONS, Renaissance Technologies, New York, USA

D. SULLIVAN, Mathematics Department, Stony Brook University, Stony Brook, NY 11790, USA
E-mail : dennis@math.sunysb.edu