SUN-YUNG ALICE CHANG
PAUL C. YANG

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THE Q-CURVATURE EQUATION IN CONFORMAL GEOMETRY

by

Sun-Yung Alice Chang & Paul C. Yang

Dedicated to J.P. Bourguignon on his 60th birthday

Abstract. — In this paper we survey some analytic results concerned with the top order Q-curvature equation in conformal geometry. Q-curvature is the natural generalization of the Gauss curvature to even dimensional manifolds. Its close relation to the Pfaffian, the integrand in the Gauss-Bonnet formula, provides a direct relation between curvature and topology.

Résumé **(L'équation de Q-courbure en géométrie conforme).** — Dans cet article nous examinons certains résultats analytiques autour de l'équation de Q-courbure d'ordre maximal en géométrie conforme. La Q-courbure est la généralisation naturelle de la courbure de Gauss aux variétés de dimension paire. Sa proximité avec le pfaffien (l'intégrande de la formule de Gauss-Bonnet) nous fournit une relation directe entre géométrie et topologie.

1. Introduction

Recently, there is a lot of interest in the study of higher order Q-curvature invariant. This notion arises naturally in conformal geometry in the context of conformally covariant operators. Paneitz ([**23**], see also [**6**]) gave the first construction of the fourth order conformally covariant Paneitz operator in the context of Lorentzian geometry in dimension four. Based on the ambient metric construction introduced by Fefferman and Graham ([**14**],[**15**]), Graham-Jenne-Mason and Sparling [**18**] systematically constructed conformally covariant operators of higher orders. Each such operator gives rise to a semi-linear elliptic equation analogous to the Yamabe equations which

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we shall call the Q-curvature equation. These equations share a number of common features. Among these we mention the following:

(i) the lack of compactness: the nonlinearity always occur at the critical exponent, for which the Sobolev embedding is not compact;

(ii) the lack of maximum principle: for example, it is not known whether the solution of the fourth order Q-curvature equation on manifolds of dimensions greater than four may touch zero.

In spite of these difficulty, there has been significant progress on questions of existence, regularity and classification of entire solutions for these equations in the recent work of Djadli-Malchiodi [**13**], Adimurthi-Robert-Struwe [**1**] and X.Xu [**25**]. On the other hand, in the case when the dimension is even $n = 2k$, the Branson-Paneitz operator and its associated Q-curvature equation is more accessible. In this article, we will give a brief survey of two results for the Q-curvature equation, each of which makes use of its close relation to the Pfaffian; both of these results are joint works with Jie Qing. The first [**10**] is a generalization of the Cohn-Vossen-Huber inequality ([**22**]) to complete conformal metrics on domains in $\mathbb{R}^4$. The second gives a Gauss-Bonnet type formula for Poincaré-Einstein metrics in which the renormalized volume plays a role. As the original article [**12**] of the second result appeared in Russian, we provide an exposition with some details. In section two, we review the notion of conformally covariant equations, their associated Q-curvatures and the associated boundary operators for manifold with boundary. We then provide an outline for these two results in sections three to five.

2. Conformally covariant operators and the Q-curvature equation

In general, we call a metrically defined operator A defind on a Riemannian manifold (M^n, g) conformally covariant of bidegree (a, b), if under the conformal change of metric $g_w = e^{2w}g$, the pair of corresponding operators A_w and A are related by

$$A_w(\varphi) = e^{-bw}A(e^{aw}\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^n).$$

A basic example is the conformal Laplacian $L \equiv -\Delta + \frac{n-2}{4(n-1)}R$ where R is the scalar curvature of the metric. The conformal Laplacian is conformally covariant of bidegree $(\frac{n-2}{2}, \frac{n+2}{2})$, and the associated curvature equation is the equation for prescribing scalar curvature: writing $e^w = u^{\frac{2}{n-2}}$ we have

$$Lu = \frac{n-2}{4(n-1)} R_u u^{\frac{n+2}{n-2}}, \tag{1}$$

where R_u is the scalar curvature of the metric $g_w = g^{2w}g = u^{\frac{4}{n-2}}g$. In case of surfaces, the corresponding Q-curvature equation becomes the equation for prescribing Gauss curvature:

$$-\Delta w + K = K_w e^{2w}, \tag{2}$$

where K_w is the Gaussian curvature for the metric g_w, and we have the Gauss-Bonnet formula:

$$2\pi\chi(M) = \int_M K dA. \tag{3}$$

In dimension four, S. Paneitz found the fourth order conformally covariant operator:

$$P_4\varphi = P\varphi \equiv \Delta^2\varphi + \delta\left[\left(\frac{2}{3}Rg - 2\mathrm{Ric}\right)d\varphi\right] \tag{4}$$

where δ denotes the divergence, d the deRham differential and Ric the Ricci tensor.

For example:

- On $(R^4, |dx|^2)$, $P = \Delta^2$,
- On (S^4, g_c), $P = \Delta^2 - 2\Delta$,
- On (M^4, g), g Einstein, $P = (-\Delta) \circ (L)$.

The Paneitz operator P has bidegree $(0,4)$ on 4-manifolds, i.e.

$$P_{g_w}(\phi) = e^{-4\omega}P_g(\phi) \ \ \forall\phi \in \mathcal{C}^\infty(M^4)\,. \tag{5}$$

The fourth order Q-curvature is given by

$$Q \;=\; \frac{1}{6}(-\Delta R + R^2 - 3|\mathrm{Ric}|^2). \tag{6}$$

Under the conformal change of metric $g_w = e^{2w}g$, the Q-curvature equation (see [**6**], also [**8**]) takes the form

$$Pw + Q \;=\; Q_w e^{4w}, \tag{7}$$

where Q_w is the Q curvature for the metric g_w.

The Gauss-Bonnet formula in dimension four may be written as

$$8\pi^2\chi(M) = \int_M (|W|^2 + Q)dV, \tag{8}$$

where W is the Weyl tensor. Since $|W_g|_g = e^{-2w}|W_{g_w}|_{g_w}$, on manifold of dimension four, $|W|^2\,dV$ is a pointwise conformal invariant, thus it follows from the Gauss-Bonnet formula that the Q-curvature integral is a global conformal invariant.

For 4-manifold X^4 with boundary M^3 and a Riemannian metric g defined on closure of X^4, Chang-Qing [**9**] derived the matching boundary operator

$$P_3 = -\frac{1}{2}\frac{\partial}{\partial n}\Delta - \tilde{\Delta}\frac{\partial}{\partial n} - \frac{2}{3}H\tilde{\Delta} + L_{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta + \left(\frac{1}{3}R - R_{\alpha N\alpha N}\right)\frac{\partial}{\partial n} + \frac{1}{3}\tilde{\nabla}H\cdot\tilde{\nabla}. \tag{9}$$

with the associated third order curvature invariant

$$T = \frac{1}{12}\frac{\partial}{\partial n}R + \frac{1}{6}RH - R_{\alpha N\beta N}L_{\alpha\beta} + \frac{1}{9}H^3 - \frac{1}{3}TrL^3 - \frac{1}{3}\tilde{\Delta}H, \tag{10}$$

where where $\frac{\partial}{\partial n}$ is the outer normal derivative, $\tilde{\Delta}$ is the trace of the Hessian of the metric on the boundary, $\tilde{\nabla}$ is the derivative in the boundary, L is the second fundamental form of boundary, $H = \mathrm{Tr}L$, N denotes the inner normal direction. We have used an orthonormal frame and let the latin indices run through the ambient indices and the Greek indices only run through the boundary directions, and all curvature are taken with respect to the metric g.

In particular, via the conformal change of metrics $g_w = e^{2w}g$, P_3 and T satisfy the equation:

$$P_3 w + T = T_w e^{3w} \quad \text{on} \quad M, \tag{11}$$

and

$$(P_3)_w = e^{-3w} P_3 \quad on \quad M. \tag{12}$$

The Chern-Gauss-Bonnet formula for 4-manifolds with boundary is then modified with a boundary term:

$$8\pi^2\chi(X) = \int_X (|W|^2 + Q)dv + 2\oint_M (T - \mathcal{L}_4 - \mathcal{L}_5)d\sigma. \tag{13}$$

In the boundary integral above the invariants $\mathcal{L}_4$ and $\mathcal{L}_5$ involve the ambient curvature tensor and the second fundamental form L_{ab}, and their expressions are

$$\mathcal{L}_4 = -\frac{RH}{3} + R_{\alpha N\alpha N}H - R_{\alpha N\beta N}L_{\alpha\beta} + R_{\gamma\alpha\gamma\beta}L_{\alpha\beta},$$

and

$$\mathcal{L}_5 = -\frac{2}{9}L_{\alpha\alpha}L_{\beta\beta}L_{\gamma\gamma} + L_{\alpha\alpha}L_{\beta\gamma}L_{\beta\gamma} - L_{\alpha\beta}L_{\beta\gamma}L_{\gamma\alpha}.$$

Analogous to the Weyl term, $\mathcal{L}_4$ and $\mathcal{L}_5$ are boundary invariant of order three which are pointwise invariant under conformal change of metrics. Hence

$$\int_X Qdv \; + \; 2\oint_M Td\sigma \tag{14}$$

is a global conformal invariant.

In dimension four, an important result is the following criteria for positivity of the Paneitz operator due to Gursky-Viaclovsky:

Theorem 1 **([21]).** — *Let (M^4, g) be a metric with positive Yamabe constant $Y(M,g) = inf_{u\neq 0}\frac{\int Lu\cdot u}{||u||_4^2}$, and satisfying* .

$$\int_M Qdv + \frac{1}{6}(Y(M,g))^2 \geq 0,$$

then the Paneitz operator is positive except for constants.

It is an open question whether there is an analogous result in higher dimensions.

3. A Gauss-Bonnet formula for noncompact 4-manifolds

On a four dimensional manifold, the conformal Laplacian and the Paneitz operator together give strong control of the geometry and topology. A particular example in the study of non-compact manifolds is the following:

Theorem 2 **([10])**. — *Let $(\Omega \subset S^4, g = e^{2w} g_0)$ be a complete conformal metric satisfying*

(a) *The scalar curvature is bounded between two positive constants, and $|\nabla_g R|$ is bounded,*

(b) *The Ricci curvature of the metric g has a lower bound,*

(c) *the Paneitz/Branson curvature is absolutely integrable, i.e.*

$$\int_\Omega |Q_g| dv_g < \infty; \tag{15}$$

then $\Omega = S^4 \setminus \{p_1, ..., p_k\}$ and

$$8\pi^2 \chi(\Omega) = \int_\Omega Q_g dv_g + \sum_1^k I_k, \tag{16}$$

where I_k is the local isoperimetric constant

$$I_k = \lim_{r \to 0} \frac{\text{Area}(\{r = |x - x_k|\})}{\text{Vol}(\{r < |x - x_k| < r_0\})}.$$

An essential idea in the above finiteness result is to view the Q-curvature integral as measuring the growth of volume. The finiteness of the Q integral implies a control on the growth of volume, which can only accommodate the growth of a finite number of puncture ends. We outline the main arguments to show how to use the fourth order curvature equation in such a situation.

Let us denote $\Lambda = S^4 \setminus \Omega$.

Step I. — $e^{w(x)} \approx \text{dist}(x, \Lambda)^{-1}$.

This is the main analytic work.

The lower bound follows from a Harnack estimate for the gradient of a conformal harmonic function.

The upper bound is based on a delicate blowup argument. Assuming on the contrary that on a sequence of points $\{x_k\}$ we have $a_k = e^{w(x_k)} d(x_k, \Lambda) \to \infty$. Take a subsequence so that the balls $B(x_k, (1/2)d(x_k, \Lambda))$ are disjoint. A careful rescaling of the domain for the conformal metrics over suitably dilated balls will converge to a conformal metric on $\mathbb{R}^4$ with vanishing Q-curvature but having scalar curvature bounded from below by a positive constant. Such a metric cannot exist. This argument differs from the usual blowup argument in that the conformal factor only satisfies a differential inequality.

This assertion gives control of the level set of the function e^w in terms of the distance to the complement.

Step II. — An integration by parts computation yields an inequality using assumption (a):

$$-\int_{\{e^w \geq \lambda\}} Q dv \geq C\lambda \frac{d}{d\lambda} \int_{\{e^w \geq \lambda\}} dv + \textit{positive terms}.$$

Step III. — To estimate the first term on the right hand side of the previous inequality, we use the coarea formula to find

$$\begin{aligned}\int_{\{e^w \geq \lambda\}} e^{4w} dx &\geq \int_{C_2/\lambda}^{C_1} \int_{\{d(x,\Lambda)=s\}} e^{4w} d\sigma\, ds \\ &\geq \int_{C_2/\lambda}^{C_1} |\{d(x,\Lambda) = s\}| s^{-4} ds.\end{aligned}$$

An elementary computation using a covering argument yields that

$$|\{d(x,\Lambda) = s\}| \geq \begin{cases} N s^3 & \text{if } \dim(\Lambda) = 0 \text{ and } |\Lambda| \geq N \\ C s^{3-\frac{3}{4\beta}} & \text{if } \dim(\Lambda) = \beta > 0. \end{cases}$$

In either case, we reach a contradiction if the complement Λ is more than a finite number of points.

A closely related result to Theorem 2 above is the recent work of Bonk-Heinonen-Saksman [**4**]: To state their result, we first observe that for a metric $g_w = e^{2w} dx^2$ conformal to the flat metric dx^2 on domains in $\mathbb{R}^4$, the equation (7) takes the form

$$\Delta^2 w = Q_w e^{4w}. \tag{17}$$

Thus the integrability condition (15) of Q_w is equivalent to the condition that $\Delta^2 w$ being integrable.

Theorem 3 **([4])**. — *Suppose $g = e^{2w}|dx|^2$ is a complete conformal metric on $\mathbb{R}^4$ where w is given as a potential*

$$w(x) = \int_{\mathbb{R}^4} \log(\frac{|x-y|}{|y|}) \Delta^2 w(y) dy,$$

and

$$\int_{\mathbb{R}^4} |\Delta^2 w(y)| dy \leq C, \tag{18}$$

then there is a bilipshitz equivalence $\Phi : (\mathbb{R}^4, |dx|^2) \to (\mathbb{R}^4, g)$.

Remarks

1. The theorem does not assert the boundedness of the conformal factor.
2. This result holds in all even dimensions.
3. In the case of dimension two, the best constant in the inequality in (18) for the theorem to hold is 2π, which is given by the example of the half infinite cylinder as shown in the work of Bonk-Lang [**5**]. It is a natural open question whether the same example gives the sharp constant in all higher dimensions.

4. Poincaré-Einstein structure and renormalized volume

Given a smooth manifold X^{n+1} of dimension $n+1$ with smooth boundary $\partial X = M^n$. Let x be a defining function for M^n in X^{n+1} as follows:

$$\begin{aligned} x > 0 &\;\text{ in } X^{n+1};\\ x = 0 &\;\text{ on } M^n;\\ dx \neq 0 &\;\text{ on } M^n. \end{aligned}$$

A Riemannian metric g on X^{n+1} is conformally compact if $(X^{n+1}, x^2 g)$ is said to be a compact Riemannian manifold with boundary. A conformally compact manifold (X^{n+1}, g) carries a well-defined conformal structure $[\hat{g}]$ on the boundary M^n, where each $\hat{g}$ is the restriction of $x^2 g$ for a defining function x. We call $(M^n, [\hat{g}])$ the conformal infinity of the conformally compact manifold (X^{n+1}, g). If, in addition, g satisfies $\mathrm{Ric}_g = -ng$, where Ric_g denotes the Ricci tensor of the metric g, then we call (X^{n+1}, g) a conformally compact Einstein manifold.

A conformally compact metric is said to be asymptotically hyperbolic if its sectional curvature approach -1 at $\partial X = M$. If g is an asymptotically hyperbolic metric on X, then a choice of metric $\hat{g}$ in $[\hat{g}]$ on M uniquely determines a defining function x near the boundary M and an identification of a neighborhood of M in X with $M \times (0, \epsilon)$ such that g has the normal form

$$g = x^{-2}(dx^2 + g_x) \tag{19}$$

where g_x is a 1-parameter family of metrics on M. In addition (see for example [**17**])

$$g_x = \hat{g} + g^{(2)}x^2 + (\text{even powers of } x) + g^{(n-1)}x^{n-1} + g^{(n)}x^n + \cdots, \tag{20}$$

when n is odd, and

$$g_x = \hat{g} + g^{(2)}x^2 + (\text{even powers of } x) + g^{(n)}x^n + hx^n \log x + \cdots, \tag{21}$$

when n is even. Here $\hat{g} = x^2 g|_{x=0}$, $g^{(2i)}$ are determined by $\hat{g}$ for $2i < n$. The trace part of $g^{(n)}$ is zero when n is odd; the trace part of $g^{(n)}$ is determined by $\hat{g}$ and h is traceless and determined by $\hat{g}$ too when n is even.

To introduce the renormalized volume, we follow Graham [**17**] to consider the asymptotics of the volume of a conformally compact Einstein manifold (X^{n+1}, g). Namely, denoting by x the defining function associated with a choice of a metric $\hat{g} \in [\hat{g}]$, we have

$$\mathrm{Vol}_g(\{x > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots + c_{n-1}\epsilon^{-1} + V + o(1)$$

for n odd, and

$$\mathrm{Vol}_g(\{x > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots + c_{n-2}\epsilon^{-2} + L\log\frac{1}{\epsilon} + V + o(1)$$

for n even. We call the constant term V in all dimensions the renormalized volume for (X^{n+1}, g). We recall that V in odd dimension and L in even dimension are independent of the choice $\hat{g}$ in the class $[\hat{g}]$.

In this section, we will give an alternative proof of the following result of M. Anderson [**3**]. The main point of our proof is to explore the relationship between the renormalized volume and the Q curvature.

Theorem 4 **([3], [12])**. — *Suppose that (X^4, g) is a conformally compact Einstein manifold. Then*

$$8\pi^2\chi(X^4) = \int_{X^4} |\mathcal{W}|_g^2 dv_g + 6V(X^4, g) \tag{22}$$

First we recall that motivated by the recent work of Graham-Zworski [**20**], Fefferman and Graham [**15**] introduced the following procedure to calculate the renormalized volume V for a conformally compact Einstein manifold. Here we will quote a special case of their result. For odd n, upon a choice of a special defining function x, we solve for

$$-\Delta v = n \quad \text{in } X^{n+1}, \tag{23}$$

with the asymptotics

$$v = \log x + A + Bx^n \tag{24}$$

in a neighborhood of M^n, where A, B are functions even in x, and $A|_{x=0} = 0$.

Lemma 1 **([15])**. — *When n is odd,*

$$V(X^{n+1}, g) = \int_M B dv_{\hat{g}}. \tag{25}$$

In addition, we have

Lemma 2. — *When n is odd, $(Q_{n+1})_{e^{2v}g} = 0$.*

Proof of Lemma 2. — The proof is a computation based on an observation made by Graham ([**17**], see also [**6**]) that the Paneitz operator $P_{\frac{n+1}{2}}$ on an Einstein manifold is a polynomial of the Laplacian $\mathcal{P}(\Delta)$ and the polynomial $\mathcal{P}$ on the Einstein manifold is the same as the one on the constant curvature space with the same constant as the constant of the scalar curvature of the Einstein manifold. In addition, the Q-curvature Q_{n+1} of an Einstein manifold is the same as the one on the constant curvature space. Therefore $(P_{n+1})_g = \mathcal{P}(\Delta_g)$ if $(P_{n+1})_{g_H} = \mathcal{P}(\Delta_{g_H})$, and $(Q_{n+1})_g = (Q_{n+1})_{g_H}$, where (H^{n+1}, g_H) is the hyperbolic space.

$$(P_{n+1})_{g_{H^{n+1}}} = \prod_{l=1}^{\frac{n+1}{2}} (-\Delta_{H^{n+1}} - C_l) \tag{26}$$

where $C_l = (\frac{n+1}{2} + l - 1)(\frac{n+1}{2} - l)$. Therefore

$$(P_{n+1})_g = \sum_{l=2}^{\frac{n+1}{2}} (-1)^{\frac{n+1}{2}-l} B_l (\Delta_g)^l - (-1)^{\frac{n-1}{2}} (n-1)! \Delta_g, \tag{27}$$

for some coefficient B_l depending on C_j's. Meanwhile $(Q_{n+1})_{H^{n+1}} = (-1)^{\frac{n+1}{2}} n!$. Thus

$$(Q_{n+1})_g = (-1)^{\frac{n+1}{2}} n!. \tag{28}$$

Thus if v satisfies the equation (23), we have

$$(P_{n+1})_g v + (Q_{n+1})_g = 0. \tag{29}$$

It thus follows from the prescribing Q curvature equation (7) that $(Q_{n+1})_{e^{2v}g} = 0$. □

We will now combine the results in the above lemmas to give an alternative proof of the result of Anderson [**3**] in Theorem 4 for conformal compact Einstein manifold (X^4, g). We first relate our curvature T to the boundary term B in Lemma 1.

Lemma 3. — *We have*

$$T_{e^{2v}g} = 3B|_{x=0}. \tag{30}$$

Proof. — According to the scalar curvature equation we have

$$\frac{1}{12} R_{e^{2v}g} = \frac{1}{2}(-\Delta_g e^v + \frac{1}{6} R_g e^v) e^{-3v}.$$

Therefore for v satisfies equation (23), we have

$$\frac{1}{12} R_{e^{2v}g} = \frac{1}{2}((e^{-v})^2 - |\nabla e^{-v}|^2).$$

We now apply the asymptotic expansion of v in (24) and write

$$e^{-2v} = \frac{1}{x^2} - 2A_2 - 2B_0 x + O(x^2)$$
$$|\nabla e^{-v}|^2 = \frac{1}{x^2} + 2A_2 + 4B_0 x + O(x^2),$$

where A_2 is the coefficient of x^2 of A and $B_0 = B|_{x=0}$. We get

$$T_{e^{2v}g} = -\frac{1}{12}\frac{\partial}{\partial x} R_{e^{2v}g}|_{x=0} = 3B_0.$$

This finishes the proof of the lemma. □

Proof of Theorem 4. — Applying Lemma 2 to the Gauss-Bonnet formula (13), we have

$$8\pi^2\chi(X^4) = \int_{X^4} |\mathcal{W}|^2_{e^{2v}g} dv_{e^{2v}g} + 2\int_M (\mathcal{L}+T)_{(e^{2v}g,\hat{g})}\, dv_{\hat{g}}.$$

We now observe that as the boundary of M of X^4 is umbilical, the second fundamental form $L_{\alpha,\beta}$ vanishes along M; hence $\mathcal{L} = -\mathcal{L}_4 - \mathcal{L}_5 = 0$. We then apply Lemma 1 and Lemma 3 to identify the area element in the integral $\int_M T$ with the renormalized volume to establish the formula (22) for the metric $e^{2v}g$. The last observation is that once the formula (22) holds for the metric $e^{2v}g$, it holds for any metric $\tilde{g} \in [g]$ with $(X^{n+1}, \tilde{g})$ a conformally compact manifold as the term of the renormalized volume V is conformally invariant. □

5. Renormalized volume in higher dimensions

In this section, we will continue to explore the relation between the Q curvature and the renormalized volume, and to extend the result of Theorem 4 above to all conformally compact Einstein manifolds (X^{n+1}, g) when n is odd. The main result is:

Theorem 5. — *When n is odd, we have*

$$\int_{X^{n+1}} (\mathcal{W}_{n+1})_g dv_g + (-1)^{\frac{n+1}{2}} \frac{\Gamma(\frac{n+2}{2})}{\pi^{\frac{n+2}{2}}} V(X^{n+1}, g) = \chi(X^{n+1}) \tag{31}$$

for some curvature invariant $\mathcal{W}_{n+1}$, which is a sum of contractions of Weyl curvatures and/or its covariant derivatives in an Einstein metric.

In the case of conformally compact manifolds of dimension $3+1$, one advantage we have taken is a precise formula of the Q on X^4, which enables us to do the explicit computations in Lemma 2 and Lemma 3 above. In the case when the dimension m of the manifold X^m is even but greater than four, it has been established in ([**18**], [**6**]) the existence of some Q curvature satisfying the following properties:

(i) It is a curvature invariant of weight $-m$. That is under the re-scale of metric $g \to t^2 g$, $Q_g = t^{-m} Q_{t^2 g}$.
(ii) $\int Q$ is a global conformal invariant.

(iii) There exists a m order linear differential operator P_m defined on X^m which prescribes the changing of Q under conformal change of metric $g_w = e^{2w}g$.

$$P_m w + Q = Q_w e^{mw} \tag{32}$$

One should remark that although the existence of Q is known, the explicit formula of the curvature is in general quite complicated and only known in dimensions six ([**17**]) and eight. Only in the recent work of Graham-Juhl [**19**], there is an inductive formula to compute Q curvature in high dimensions. Thus it is remarkable that one knows (Theorem 6 below) the "leading" order term (in terms of the order of the differentiation on the metric) of the Q curvature for all dimensions and it is even more remarkable that under the assumptions (i) and (ii) above, S. Alexakis [**2**] has recently established a structure theorem of the Q curvature (Theorem 7 below) which is known in the field as the answer to the Deser-Schwimmer Conjecture.

***Theorem 6* (Branson [7]).** — *On any compact m-dimensional manifold for m even,*

$$Q_m = b_m \Delta^{\frac{m-2}{2}} R + \textit{lower order terms}, \tag{33}$$

where

$$b_m = (-1)^{\frac{m-2}{2}} \frac{2^{m-1}(\frac{m}{2})!\Gamma(\frac{m+1}{2})}{\sqrt{\pi}(m-1)m!}.$$

***Theorem 7* (S. Alexakis [2]).** — *On any compact closed m-dimensional manifold with m even, we have*

$$Q_m = a_m e + \mathcal{J} + \mathrm{Div}(T_m). \tag{34}$$

where e is the Euler class density, $\mathcal{J}$ is a pointwise conformal invariant, and $\mathrm{Div}(T_m)$ *is a divergence term and a_m is some dimensional constant.*

Proof of Theorem 5. — Let (X^{n+1}, g) be a conformally compact Einstein manifold, where $n = 2k+1 > 3$, we wish to determine the analogous formula for the renormalized volume. We continue to consider the metric $(X^{n+1}, e^{2v}g)$ where v satisfies the equations (23) and (24). We will find that the parity conditions imposed in (24) makes it possible to determine the local boundary invariants of order n for the compact manifold $(X^{n+1}, e^{2v}g)$. According to (19) and (20) we have the expansion of the metric $e^{2v}g$.

$$\begin{aligned} e^{2v}g = {} & H^2dx^2 + \hat{g} + c^{(2)}x^2 + \text{even powers in } x \\ & + c^{(n-1)}x^{n-1} + (2B_0\hat{g} + g^{(n)})x^n + \cdots \end{aligned} \tag{35}$$

where

$$H = e^{A+Bx^n} = 1 + e_2x^2 + \text{even powers in } x + e_{n-1}x^{n-1} + B_0x^n + \cdots$$

and $c^{(2i)}$ for $1 \le i \le (n-1)/2$ are local invariants of $\hat{g}$. We remark that it is easy to see that the boundary of $(X^{n+1}, e^{2v}g)$ is totally geodesic.

Lemma 4. — *We have*

$$(\partial_x \Delta^{\frac{n-3}{2}} R)_{e^{2v}g}|_{x=0} = -2nn!B_0. \tag{36}$$

Proof of Lemma 4. — We have

$$\begin{aligned}\Delta_{e^{2v}g} &= \frac{1}{H\sqrt{\det g_x}}\partial_\alpha(H\sqrt{\det g_x}g_x^{\alpha\beta}\partial_\beta) \\ &= Q_2^{(2)}\partial_x^2 + Q_2^{(1)}\partial_x + Q_2^{(0)}\end{aligned} \tag{37}$$

where the coefficients $Q^{(i)}$ have the following properties: $Q_2^{(2)}$ is a zeroth order differential operator, having an asymptotic expansion in powers of x in which the first nonzero odd power term is x^n. $Q_2^{(1)}$ is a zeroth order differential operator, having an expansion in which the first nonzero even degree term is x^{n-1}. $Q_2^{(0)}$ is differential operator of order 2 of purely tangential differentiations with coefficients which have expansion in powers of x in which the first nonzero odd term is x^n. Inductively, we see that, for $k \le \frac{n-3}{2}$,

$$\Delta^k = Q_{2k}^{(2k)}\partial_x^{2k} + Q_{2k}^{(2k-1)}\partial_x^{2k-1} + \cdots + Q_{2k}^{(1)}\partial_x + Q_{2k}^{(0)} \tag{38}$$

where $Q_{2k}^{(i)}$ $(i \neq 0)$ is a differential operator of order $2k - i$ of purely tangential differentiations with coefficients having expansions in powers of x in which the first nonzero even terms are $x^{n-(2k-i)}$ if i is odd, and the first nonzero odd terms are $x^{n-(2k-i)}$ if i is even, and $Q_{2k}^{(0)}$ is a differential operator of order $2k$ of purely tangential differentiations with coefficients whose expansions in x have the first nonzero odd terms x^{n-2k+2}. Thus

$$\partial_x\Delta^k = F^{(2k+1)}\partial_x^{2k+1} + F^{(2k)}\partial_x^{2k} + \cdots + F^{(1)}\partial_x + F^{(0)} \tag{39}$$

where $F^{(2k+1)} = Q^{(2k)}$, $F^{(i)}$ $(0 < i < 2k+1)$ is a differential operator of order $2k - i + 1$ of purely tangential differentiations with coefficients whose expansions in x have the first nonzero even terms are $x^{n-(2k-i)-1}$ if i is even, and the first nonzero odd terms are $x^{n-(2k-i)-1}$ if i is odd, and $F^{(0)}$ is a differential operator of order $2k$ of purely tangential differentiations with coefficients whose expansions in x have the first nonzero even terms x^{n-2k+1}.

On the other hand, we have

$$R_{e^{2v}g} = -2n^2(n-1)B_0x^{n-2} + \text{even powers of } x \text{ terms} + o(x^{n-2}). \tag{40}$$

Keeping track of the parity, we obtain (36) in Lemma 4. □

Next we deal with all other boundary terms which may appear in integrating the Q curvature over X. These are contractions of one or more factors consisting of curvatures, covariant derivatives of curvatures, except $\partial_x^{n-2}R$ which is accounted in the above term $\partial_x\Delta^{\frac{n-3}{2}}R$. Since n is odd, and ∂x is the normal direction, each such term must contain at least one x index. In fact, the total number of x indices appearing in each of such terms must be odd. Thus one finds that each of such terms always contains a factor which is a covariant derivatives of curvature and in which x index appears odd number of times. Such factors, if we insist on taking ∇_x first, must appear as one of the following three different types

$$\nabla_\spadesuit \cdots \nabla_\spadesuit \nabla_x^{2k+1} R_{\spadesuit\spadesuit\spadesuit\spadesuit}$$

where $\spadesuit$ stands for indices other than x, in other words, tangential.

$$\nabla_\spadesuit \cdots \nabla_\spadesuit \nabla_x^{2k} R_{x\spadesuit\spadesuit\spadesuit}$$

and

$$\nabla_\spadesuit \cdots \nabla_\spadesuit \nabla_x^{2k-1} R_{x\spadesuit x\spadesuit}.$$

Note that in all three types $1 \leq 2k+1 \leq n-2$. Since the boundary is totally geodesic, we only need to verify

Lemma 5. — *All three types of boundary terms*

$$\nabla_x^{2k+1} R_{\spadesuit\spadesuit\spadesuit\spadesuit}, \quad \nabla_x^{2k} R_{x\spadesuit\spadesuit\spadesuit}, \quad \nabla_x^{2k-1} R_{x\spadesuit x\spadesuit} \tag{41}$$

vanish at the boundary for $1 \leq 2k+1 \leq n-2$.

Proof of Lemma 5. — We consider a point at the boundary and choose a normal coordinate on the boundary M^n in the special coordinates for X^{n+1}. Recall that

$$\begin{aligned} R_{\alpha\beta\gamma\delta} = & \frac{1}{2}(-\partial_\beta\partial_\delta g_{\alpha\gamma} - \partial_\alpha\partial_\gamma g_{\beta\delta} + \partial_\beta\partial_\gamma g_{\alpha\delta} + \partial_\alpha\partial_\delta g_{\beta\gamma}) \\ & - g^{\eta\lambda}([\alpha\gamma,\eta][\beta\delta,\lambda] - [\beta\gamma,\eta][\alpha\delta,\lambda]), \end{aligned}$$

and

$$\nabla_x T_{\alpha\beta\cdots\delta} = \partial_x T_{\alpha\beta\cdots\delta} - \Gamma_{\alpha\ x}^{\lambda} T_{\lambda\beta\cdots\delta} - \Gamma_{\beta\ x}^{\lambda} T_{\alpha\lambda\cdots\delta} - \cdots - \Gamma_{\delta\ x}^{\lambda} T_{\alpha\beta\cdots\lambda}$$

where

$$\Gamma_{\beta\gamma}^{\alpha} = g^{\alpha\delta}[\beta\gamma,\delta]$$

and

$$[\alpha\beta,\gamma] = \frac{1}{2}(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}).$$

For simplicity of notation we will use g to stand for $e^{2v}g$ if no confusion can arise. Each of the three types is a sum of products of factors that are of the form:

$$\partial_\alpha\partial_\beta \cdots \partial_\gamma g_{\lambda\mu}$$

or

$$\partial_\alpha\partial_\beta \cdots \partial_\gamma g^{\lambda\mu}.$$

We claim that each summand must has a factor that is one of the following

$$\partial_\spadesuit \cdots \partial_\spadesuit \partial_x^{2k+1} g_{\spadesuit\spadesuit},$$

$$\partial_\spadesuit \cdots \partial_\spadesuit \partial_x^{2k-1} g_{xx},$$

$$\partial_{\spadesuit} \cdots \partial_{\spadesuit} \partial_x^{2k+1} g^{\spadesuit\spadesuit}.$$

and

$$\partial_{\spadesuit} \cdots \partial_{\spadesuit} \partial_x^{2k-1} g^{xx},$$

where $1 \leq 2k+1 \leq n-2$. To verify the claim, one needs to observe that, in writing the three types in local coordinates, the number of times the index x appears in each summand increases only when one sees

$$\Gamma^{x}_{\spadesuit\ x} T_{\alpha\beta\cdots\ x\cdots\delta},$$

where the number of x increases by 2. Thus, in the end, the total number of index x in each summand is still odd. Therefore one of the factors must have an odd number of x. Finally one observes that for any individual factor arising here the number of x can not exceed $n-1$. So the proof of Lemma 5 is complete. □

We now finish the proof of Theorem 5 for all n odd based on the results in Theorem 6 and Theorem 7.

Proof of Theorem 5. — We first establish that equation (34) remains valid on a conformally Einstein manifold (X^{n+1}, g). Let $g_w = e^{2w} g$ be such a metric, then it follows from the Paneitz equation that for $m = n+1$,

$$\begin{aligned} (Q_m)_{g_w} e^{mv} &= (P_m)_g v + (Q_m)_g \\ &= a_m e_g + \mathcal{J}_g + \mathrm{Div}(T') \\ &= a_m e_{g_w} + \mathcal{J}_{g_w} + \mathrm{Div}(T'') \end{aligned} \tag{42}$$

where the second equation follows from the fact that the Paneitz operator P_m is a divergence and Theorem 7. The third equation follows from the fact that the Pfaffians of any two Riemannian metrics on the same manifold differs by a divergence term and $\mathcal{J}$ is a conformal invariant.

In order to apply this formula, we need to observe that the leading order term $\Delta^{\frac{m-2}{2}} R$ in formula (33) cannot appear in the conformally invariant term $\mathcal{J}$. In order to see this, we first recall that $\mathcal{J}$ is a linear combination of terms of the form $Tr(\nabla^{I_1}\mathcal{R} \bigotimes \nabla^{I_2}\mathcal{R} ... \bigotimes \nabla^{I_k}\mathcal{R})$ of weight m where Tr denotes a suitably chosen pairwise contraction over all the indices. Observe that the conformal variation $\delta_w(\Delta^{\frac{m-2}{2}})R$, where δ_w denotes the variation of the metric g to g_w is of the form $\Delta^{\frac{m}{2}} w +$ lower order terms. Thus if $\Delta^{\frac{m-2}{2}} R$ does appear as a term in $\mathcal{J}$, its conformal variation must be cancelled by the conformal variations of the other terms in the linear combination, but it is clear that the conformal variations of the other possibilities of the curvature $\mathcal{R}$ other than the scalar curvature R cannot have order m in the number of derivatives of w and of the form $\Delta^{\frac{m}{2}} w$.

We can now apply the formula (42) to the metric $g_v = e^{2v}g$ where v is as in (23). Thus by Lemma 2 the left hand side of (42) is identically zero, and we find

$$a_m \chi(X^{n+1}) = \int_{X^{n+1}} (\mathcal{J}_{g_v} - \mathrm{Div}(T''))dv_{g_v}.$$

Among the divergence terms in $\mathrm{Div}(T'')$, only the leading order term $b_m \Delta^{\frac{m-2}{2}} R$ has a non-zero contribution according to Lemma 5. The computation in Lemma 5 determines the precise contribution of this term as a multiple of the renormalized volume. We also note that as g is an Einstein metric, we may assume that the terms which appear in the conformal invariant $\mathcal{J}$ are contractions of the Weyl curvature together with its covariant derivatives. We have thus finished the proof of Theorem 5.

□

Corollary 1. — *When (X^{n+1}, g) is conformally compact hyperbolic, we have*

$$V(X^{n+1}, g) = \frac{(-1)^{\frac{n+1}{2}} \pi^{\frac{n+2}{2}}}{\Gamma(\frac{n+2}{2})} \chi(X). \tag{43}$$

One may compare (43) to a formula for renormalized volume given by Epstein in [**24**], where he has

$$V(X^{n+1}, g) = \frac{(-1)^m 2^{2m} m!}{(2m)!} \chi(X) \tag{44}$$

for $n = 2m - 1$ and our answers agree!.

References

[1] Adimurthi, F. Robert & M. Struwe – "Concentration phenomena for Liouville's equation in dimension four", *J. Eur. Math. Soc. (JEMS)* **8** (2006), p. 171–180.

[2] S. Alexakis – "The decomposition of global conformal invariants: On a conjecture of Deser and Schwimmer", preprint arXiv:0711.1685.

[3] M. T. Anderson – "L^2 curvature and volume renormalization of AHE metrics on 4-manifolds", *Math. Res. Lett.* **8** (2001), p. 171–188.

[4] M. Bonk, J. Heinonen & E. Saksman – "Logarithmic potentials, quasiconformal flows, and Q-curvature", *Duke Math. J.* **142** (2008), p. 197–239.

[5] M. Bonk & U. Lang – "Bi-Lipschitz parameterization of surfaces", *Math. Ann.* **327** (2003), p. 135–169.

[6] T. P. Branson – "Differential operators canonically associated to a conformal structure", *Math. Scand.* **57** (1985), p. 293–345.

[7] ______, *The functional determinant*, Lecture Notes Series, vol. 4, Seoul National University Research Institute of Mathematics Global Analysis Research Center, 1993.

[8] T. P. Branson & B. Ørsted – "Explicit functional determinants in four dimensions", *Proc. Amer. Math. Soc.* **113** (1991), p. 669–682.

[9] S.-Y. A. Chang & J. Qing – "The zeta functional determinants on manifolds with boundary. I. The formula", *J. Funct. Anal.* **147** (1997), p. 327–362.

[10] S.-Y. A. Chang, J. Qing & P. C. Yang – "Compactification of a class of conformally flat 4-manifold", *Invent. Math.* **142** (2000), p. 65–93.

[11] ———, "On the topology of conformally compact Einstein 4-manifolds", in *Noncompact problems at the intersection of geometry, analysis, and topology*, Contemp. Math., vol. 350, Amer. Math. Soc., 2004, p. 49–61.

[12] ———, "On renormalized volume on conformally compact Einstein manifolds", in *Proceedings DFDE-2005*, Contemporary Mathematics, Fundamental Directions, 2005 (Russian).

[13] Z. Djadli & A. Malchiodi – "Existence of conformal metrics with constant Q-curvature", preprint arXiv:math/0410141, to appear in *Annals of Math.*.

[14] C. Fefferman & C. R. Graham – "Conformal invariants", *Astérisque* numéro hors série "The mathematical heritage of Élie Cartan (Lyon, 1984)" (1985), p. 95–116.

[15] ———, "Q-curvature and Poincaré metrics", *Math. Res. Lett.* **9** (2002), p. 139–151.

[16] ———, "The ambient metric", preprint, 2007.

[17] C. R. Graham – "Volume and area renormalizations for conformally compact Einstein metrics", in *The Proceedings of the 19th Winter School "Geometry and Physics" (Srní, 1999)*, vol. 63, 2000, p. 31–42.

[18] C. R. Graham, R. Jenne, L. J. Mason & G. A. J. Sparling – "Conformally invariant powers of the Laplacian. I. Existence", *J. London Math. Soc.* **46** (1992), p. 557–565.

[19] C. R. Graham & A. Juhl – "Holographic formula for Q-curvature", preprint, to appear in *Advances in Math.*, 2006.

[20] C. R. Graham & M. Zworski – "Scattering matrix in conformal geometry", *Invent. Math.* **152** (2003), p. 89–118.

[21] M. J. Gursky & J. A. Viaclovsky – "A fully nonlinear equation on four-manifolds with positive scalar curvature", *J. Differential Geom.* **63** (2003), p. 131–154.

[22] A. Huber – "On subharmonic functions and differential geometry in the large", *Comment. Math. Helv.* **32** (1957), p. 13–72.

[23] S. Paneitz – "A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds", preprint, 1993.

[24] S. J. Patterson & P. A. Perry – "The divisor of Selberg's zeta function for Kleinian groups", *Duke Math. J.* **106** (2001), p. 321–390.

[25] X. Xu – "Uniqueness theorem for integral equations and its application", *J. Funct. Anal.* **247** (2007), p. 95–109.

S.-Y. A. Chang, Department of Mathematics, Princeton University, Princeton, NJ 08544, USA
E-mail : chang@math.princeton.edu

P. C. Yang, Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA; School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA