

Astérisque

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Astérisque, tome 322 (2008), p. 207-224

[<http://www.numdam.org/item?id=AST_2008__322__207_0>](http://www.numdam.org/item?id=AST_2008__322__207_0)

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ON THE NUMBER OF MINIMAL SURFACES WITH A GIVEN BOUNDARY

by

David Hoffman & Brian White

Dedicated to Jean Pierre Bourguignon on the occasion of his 60th birthday

Abstract. — We prove results allowing us to count, mod 2, the number of embedded minimal surfaces of a specified topological type bounded by a curve $\Gamma \subset \partial N$, where N is a weakly mean convex 3-manifold with piecewise smooth boundary. These results are extended to curves and minimal surfaces with prescribed symmetries. The parity theorems are used in an essential manner to prove the existence of embedded genus- g helicoids in $\mathbf{S}^2 \times \mathbf{R}$, and we give an outline of this application.

Résumé (Sur le nombre de surfaces minimales avec une frontière donnée). — Nous démontrons des résultats qui nous permettent de compter, modulo 2, le nombre de surfaces minimales plongées d'un type topologique donné, borné par une courbe $\Gamma \subset \partial N$, où N est une 3-variété convexe faiblement moyenne munie d'une frontière lisse par morceaux. Ces résultats sont étendus aux courbes et aux surfaces minimales à symétries prescrites. Les théorèmes de parité sont utilisés de manière essentielle pour prouver l'existence d'hélicoïdes de genre imbriqué g dans $\mathbf{S}^2 \times \mathbf{R}$, et nous donnons un aperçu de cette application.

1. Introduction

In [4], Tomi and Tromba used degree theory to solve a longstanding problem about the existence of minimal surfaces with a prescribed boundary: they proved that every smooth, embedded curve on the boundary of a convex subset of $\mathbf{R}^3$ must bound an embedded minimal disk. Indeed, they proved that a generic such curve must bound an odd number of minimal embedded disks. White [8] generalized their result by proving the following parity theorem. Suppose N is a compact, strictly convex domain in $\mathbf{R}^3$

2000 Mathematics Subject Classification. — 53A10; 49Q05, 58E12.

Key words and phrases. — Properly embedded minimal surface, Plateau problem, degree theory, helicoid.

The research of the second author was supported by the NSF under grants DMS-0406209 and DMS 0707126.

with smooth boundary. Let Σ be a compact 2-manifold with boundary. Then a generic smooth curve $\Gamma \cong \partial\Sigma$ in ∂N bounds an odd or even number of embedded minimal surfaces diffeomorphic to Σ according to whether Σ is or is not a union of disks.

In this paper, we generalize the parity theorem in several ways. First, we prove (Theorem 2.1) that the parity theorem holds for any compact riemannian 3-manifold N such that N is strictly mean convex, N is homeomorphic to a ball, ∂N is smooth, and N contains no closed minimal surfaces. We then further relax the hypotheses by allowing N to be mean convex rather than strictly mean convex, and to have piecewise smooth boundary. Note that if N is mean convex but not strictly mean convex, then Γ might bound minimal surfaces that lie in ∂N . We prove (Theorem 2.4) that the parity theorem remains true for such N provided (1) unstable surfaces lying in ∂N are not counted, and (2) no two contiguous regions of $(\partial N) \setminus \Gamma$ are both smooth minimal surfaces. We give examples showing that the theorem is false without these provisos.

We extend the parity theorem yet further (see Theorem 2.7) by showing that, under an additional hypothesis, it remains true for minimal surfaces with prescribed symmetries.

The parity theorems described above are all mod 2 versions of stronger results that describe integer invariants. The stronger results are given in section 3.

The parity theorems are used in an essential way to prove the existence of embedded genus- g helicoids in $\mathbf{S}^2 \times \mathbf{R}$. In Sections 4 and 5 we give a very brief outline of this application. (The full argument will appear in [3].)

2. Counting minimal surfaces

Throughout the paper, N will be a compact riemannian 3-manifold and Σ will be a fixed compact 2 manifold. If Γ is an embedded curve in N diffeomorphic to $\partial\Sigma$, we let $\mathcal{M}(N, \Gamma)$ denote the set of embedded minimal surfaces in N that are diffeomorphic to Σ and that have boundary Γ . We let $|\mathcal{M}(N, \Gamma)|$ denote the number of surfaces in $\mathcal{M}(N, \Gamma)$.

In case N has smooth boundary, we say that N is strictly mean convex provided the mean curvature is a (strictly) positive multiple of the inward unit normal on a dense subset of ∂N .

2.1. Theorem. — *Let N be a smooth, compact, strictly mean convex riemannian 3-manifold that is homeomorphic to a ball and that has smooth boundary. Suppose also that N contains no closed minimal surfaces. Let $\Gamma \subset \partial N$ be a smooth curve diffeomorphic to $\partial\Sigma$. Assume that Γ is bumpy in the sense that no surface in $\mathcal{M}(N, \Gamma)$ supports a nontrivial normal Jacobi field with zero boundary values.*

Then $|\mathcal{M}(N, \Gamma)|$ is even unless Σ is a union of disks, in which case $|\mathcal{M}(N, \Gamma)|$ is odd.

We remark that generic smooth curves $\Gamma \subset \partial N$ are bumpy [7].

Proof. — Theorems 2.1 and 2.3 of [8] are special cases of the theorem. The proofs given there establish the more general result here provided one makes the following observations:

1. There N was assumed to be strictly convex, but exactly the same proof works assuming strict mean convexity.
2. There Σ was assumed to be connected, but the same proof works for disconnected Σ .
3. In the proofs of Theorems 2.1 and 2.3 of [8], the assumption that N is a subset of $\mathbf{R}^3$ was used in order to invoke an isoperimetric inequality, i.e., an inequality bounding the area of a minimal surface in N in terms of the length of its boundary. There are compact mean convex 3-manifolds for which no such isoperimetric inequality holds. However, if (as we are assuming here) N contains no closed minimal surfaces, then N does admit such an isoperimetric inequality [9].
4. In the proofs in [8], one needs to isotope any specified component of Γ to a curve C that bounds exactly one minimal surface, namely an embedded disk. This was achieved by choosing C to be a planar curve. For a general ambient manifold N , “planar” makes no sense. However, any sufficiently small, nearly circular curve $C \subset \partial N$ bounds exactly one embedded minimal disk and no other minimal surfaces. (This property of such a curve C is proved in the last paragraph of §3 in [8].) $\square$

2.2. Mean convex ambient manifolds N with piecewise smooth boundary.

— For the remainder of the paper, we allow ∂N to be piecewise smooth. For simplicity, let us take this to mean that ∂N is a union of smooth 2-manifolds with boundary (“faces” of N), any two of which are either disjoint or meet along a common edge with interior angle everywhere strictly between 0 and 2π . (More generally, one could allow the faces of N to have corners.) We say that such an N is mean convex provided (1) at each interior point of each face of N , the mean curvature vector is a nonnegative multiple of the inward-pointing unit normal, and (2) where two faces meet along an edge, the interior angle is everywhere at most π .

The following example shows what can go wrong in Theorem 2.1 if N is mean convex but not strictly mean convex.

Example 1. Let N be a region in $\mathbf{R}^3$ whose boundary consists of an unstable catenoid C bounded by two circles, together with the two disks bounded by those

circles. Note that N is mean convex with piecewise smooth boundary. Let Γ be a pair of horizontal circles in C that are bumpy (in the sense of Theorem 2.1). Theorem 2.1 suggests that Γ should bound an even number of embedded minimal annuli in N . First consider the case when Γ consists of two circles in C very close to the waist circle. Then Γ bounds precisely two minimal annuli. One of them is the component of C bounded by Γ . Because the circles in Γ are close, this annulus is strictly stable. The other annulus bounded by Γ is a strictly unstable catenoid lying in the interior of N . In order to get an even number of examples, we must count the stable catenoid lying on C . Now suppose the two components of Γ are the two components of ∂C . Then again Γ bounds exactly two minimal annuli: the unstable catenoid C , which is part of ∂N , and a strictly stable catenoid that lies outside N . Here, of course, we do not count the stable catenoid since it does not lie in N . Thus to get an even number, we also must not count the unstable catenoid that lies in ∂N .

This example motivates the following definition:

2.3. Definition. — $\mathcal{M}^*(N, \Gamma)$ is the set of embedded minimal surfaces $M \subset N$ such that

- i.) $\partial M = \Gamma$,
- ii.) M is diffeomorphic to Σ , and
- iii.) each connected component of M lying in ∂N is stable.

Example 1 suggests that in order to generalize Theorem 2.1 to mean convex N with piecewise smooth boundary, we should replace $\mathcal{M}(N, \Gamma)$ by $\mathcal{M}^*(N, \Gamma)$. However, even if one makes that replacement, the following example shows that an additional hypothesis is required.

Example 2. Let N be a compact, convex region in $\mathbf{R}^3$ such that ∂N is smooth and contains a planar disk D . Let Γ be a pair of concentric circles lying in D . Then Γ bounds exactly one minimal annulus: the region in D between the two components of Γ . That annulus is strictly stable and lies in ∂N . Thus Γ is bumpy (in the sense of Theorem 2.1) and $|\mathcal{M}^*(N, \Gamma)| = 1$. Consequently, if we wish $|\mathcal{M}^*(N, \Gamma)|$ to be even (as Theorem 2.1 suggests it should be), then we need an additional hypothesis on N and Γ .

Note that in example 2, $(\partial N) \setminus \Gamma$ contains two contiguous connected components (a planar annulus and a planar disk) both of which are minimal surfaces. The additional hypothesis we require is that $(\partial N) \setminus \Gamma$ contains no two such components.

2.4. Theorem. — *Let N be a smooth, compact, mean convex riemannian 3-manifold that is homeomorphic to a ball, that has piecewise smooth boundary, and that contains*

no closed minimal surfaces. Let $\Gamma \subset \partial N$ be a smooth, embedded bumpy curve diffeomorphic to $\partial \Sigma$. Suppose that no two contiguous connected components of $(\partial N) \setminus \Gamma$ are both smooth minimal surfaces.

Then $|\mathcal{M}^*(N, \Gamma)|$ is even unless Σ is a union of disks, in which case $|\mathcal{M}^*(N, \Gamma)|$ is odd.

Proof. — Since N is compact, mean convex, and contains no closed minimal surfaces, the areas of minimal surfaces in N are bounded in terms of the lengths of their boundaries [9].

If ∂N is smooth and has nowhere-vanishing mean curvature, the result follows immediately from Theorem 2.1. We reduce the general case to this special case as follows. Note that we can find a one-parameter family N_t , $0 \leq t < \epsilon$, of mean convex subregions of N such that

- i.) $N_0 = N$,
- ii.) the boundaries ∂N_t foliate a relatively open subset of N containing ∂N .
- iii.) for $t > 0$ small, ∂N_t is smooth and the mean curvature of ∂N_t is nowhere zero and points into N_t .

For example, we can let ∂N_t be the result of letting ∂N flow for time t by the mean curvature flow.

Claim. — Suppose M_i are smooth embedded minimal surfaces in N diffeomorphic to Σ and that $\partial M_i \rightarrow \Gamma$ smoothly. Then a subsequence of the M_i converges smoothly to a limit $M \in \mathcal{M}^*(N, \Gamma)$.

Proof of claim. — By Theorem 3 in [6] a subsequence converges smoothly away from a finite set S to a limit surface M . The surface M is smooth and embedded, though portions of it may have multiplicity > 1 . Indeed, the proof of Theorem 3 in [6] shows that the multiplicity is 1 and the convergence $M_i \rightarrow M$ is smooth everywhere unless an interior point of M touches Γ .

In fact, no interior point of M can touch Γ . For suppose to the contrary that the interior of M touches Γ at a point p . Let C be the connected component of Γ containing p . By the strong maximum principle, M must contain a whole neighborhood of $p \in \partial N$. Indeed, by the strong maximum principle (or by unique continuation), M must contain the two connected components of $(\partial N) \setminus \Gamma$ on either side of C . But by hypothesis, at most one of those components is a minimal surface, a contradiction. This proves that no interior point of M touches Γ .

Consequently, as noted above, M has multiplicity 1 and the convergence $M_i \rightarrow M$ is smooth everywhere. Thus $M \in \mathcal{M}(N, \Gamma)$.

Now suppose some connected component M' of M lies in ∂N . Then the corresponding component M'_i of M_i converges smoothly to M' from one side of M . This

one-sided convergence implies that M' is stable. Thus $M \in \mathcal{M}^*(N, \Gamma)$. This completes the proof of the claim. $\square$

Continuing with the proof of Theorem 2.4, note that $\mathcal{M}^*(N, \Gamma)$ is finite. For if it contained an infinite sequence of surfaces then by the claim, it would contain a smoothly convergent subsequence. The limit of that subsequence would be an element of $\mathcal{M}^*(N, \Gamma)$. But by bumpiness of Γ , the elements of $\mathcal{M}^*(N, \Gamma)$ are isolated. The contradiction proves that $\mathcal{M}^*(N, \Gamma)$ is finite.

Let Γ_t , $0 \leq t < \epsilon$, be a smooth one-parameter family of embedded curves such that $\Gamma_0 = \Gamma$ and such that $\Gamma_t \subset \partial N_t$. Let $M_0^1, \dots, M_0^k$ be the set of surfaces in $\mathcal{M}^*(N, \Gamma)$. By the implicit function theorem, we can (if ϵ is sufficiently small) extend these to one-parameter families

$$M_t^i \in \mathcal{M}^*(\widehat{N}, \Gamma_t) \quad (i = 1, 2, \dots, k; 0 \leq t < \epsilon)$$

where $\widehat{N}$ is a riemannian 3-manifold containing N in its interior.

In fact, M_t^i must lie in N provided $\epsilon > 0$ is chosen sufficiently small. To see this, assume for simplicity that Σ is connected. If M_0^i does not lie in ∂N , then by the strong maximum principle, it is never tangent to ∂N , so by continuity, $M_t^i \subset N$ for all sufficiently small t . Now suppose that M_0^i does lie in ∂N . Then (by definition of $\mathcal{M}^*(N, \Gamma)$) it is strictly stable. The strict stability implies that in fact M_t^i lies in N for sufficiently small t .

Indeed, M_t^i must lie not only in N but also in $N_t \subset N$, for all sufficiently small t . For let $T = T(t) \in [0, t]$ be the largest number such that $M_t^i \subset N_T$. If $T < t$, then M_t^i would touch ∂N_T at an interior point, violating the maximum principle. Hence $T = t$ and therefore $M_t^i \subset N_t$.

The claim implies that if ϵ is sufficiently small, then each surface in $\mathcal{M}^*(N_t, \Gamma_t)$ will be one of the surfaces in $M_t^1, \dots, M_t^k$. We may also choose ϵ sufficiently small that the M_t^i all have zero nullity. Then

$$|\mathcal{M}^*(N, \Gamma)| = k = |\mathcal{M}(N_t, \Gamma_t)|$$

which must have the asserted parity by Theorem 2.1 (applied to N_t and Γ_t .) $\square$

2.5. Counting in the presence of symmetry. — In some situations, it is important to be able to say something about the number of minimal surfaces that are diffeomorphic to a specified surface Σ and that possess specified symmetries. Suppose G is a group of isometries of N .

2.6. Definition. — If Γ is a G -invariant curve in N , we let $\mathcal{M}_G^*(N, \Gamma) \subset \mathcal{M}^*(N, \Gamma)$ denote the set of surfaces in $\mathcal{M}^*(N, \Gamma)$ that are invariant under G . A boundary $\Gamma \subset \partial N$ is called G -bumpy if no surface in $\mathcal{M}_G^*(N, \Gamma)$ has a nontrivial G -invariant normal Jacobi field that vanishes on ∂M .

Theorem 2.4 has a natural extension to G -invariant surfaces:

2.7. Theorem. — *Let N be a smooth, compact, mean convex riemannian 3-manifold that is homeomorphic to a ball, that has piecewise smooth boundary, and that contains no closed minimal surfaces. Let G be a group of isometries of N . Let $\Gamma \subset \partial N$ be a smooth curve that is G -invariant and G -bumpy. Suppose that no two contiguous components of $(\partial N) \setminus \Gamma$ are both minimal surfaces.*

Suppose also that

(*) $\Gamma = \partial\Omega$ for some G -invariant region $\Omega \subset \partial N$.

Then $|\mathcal{M}_G^(N, \Gamma)|$ is even unless Σ is a union of disks, in which case $|\mathcal{M}_G^*(N, \Gamma)|$ is odd.*

2.8. Remark. — In Theorem 2.7, the hypothesis that N contains no closed minimal surfaces is equivalent to the hypothesis that N contains no closed G -invariant minimal surfaces. See [9], Theorem 2.5.

Proof. — In general, the proof is exactly the same as the proof in the non-invariant case. However (see Observation (4) in the proof of Theorem 2.1), to carry out the proof, one must be able to isotope the connected components of Γ in a G -invariant way to arbitrarily small, nearly circular curves in ∂N . The hypothesis that $\Gamma = \partial\Omega$ for a G -invariant region $\Omega \subset \partial N$ ensures that such isotopy is possible. (Indeed, it is equivalent to the existence of such G -invariant isotopies.) $\square$

We do not know whether Theorem 2.7 remains true without the hypothesis (*).

3. An Integer Invariant

Suppose $N \subset \mathbf{R}^3$ is a compact, strictly convex set with smooth boundary. In the introduction, we quoted Theorems 2.1 and 2.3 of [8] as asserting that if $\Gamma \subset \partial N$ is a smooth, bumpy curve diffeomorphic to $\partial\Sigma$, then

$$(1) \quad |\mathcal{M}(N, \Gamma)| \cong \begin{cases} 1 & \text{if } \Sigma \text{ is a union of disks, and} \\ 0 & \text{if not} \end{cases}$$

where $\cong$ denotes congruence modulo 2.

In fact, the conclusion in [8] is actually much stronger than (1). To state that conclusion, we need some terminology.

3.1. Definition. — Let $\delta(\Sigma) = 1$ if Σ is a union of disks and 0 if not. If $\mathcal{M}$ is a collection of smooth minimal surfaces, let

$$d(\mathcal{M}) = |\mathcal{M}_{\text{even}}| - |\mathcal{M}_{\text{odd}}|$$

where $\mathcal{M}_{\text{even}}$ is the set of surfaces in $\mathcal{M}$ with even index of instability and $\mathcal{M}_{\text{odd}}$ is the set of surfaces in $\mathcal{M}$ with odd index of instability.

With this terminology, the conclusion of Theorem 2.1 in [8] is

$$(2) \quad d(\mathcal{M}(N, \Gamma)) = \delta(\Sigma).$$

Note that (2) is stronger than (1). Indeed, (1) merely asserts that the two sides of (2) are congruent modulo 2. (See [5] for a similar result for immersed minimal disks in $\mathbf{R}^n$.)

If we start with the stronger conclusion (2), then the arguments in §2 produce stronger versions of Theorems 2.1, 2.4, , and 2.7:

3.2. Theorem. — *Under the hypotheses of Theorem 2.1,*

$$d(\mathcal{M}(N, \Gamma)) = \delta(\Sigma).$$

Under the hypotheses of Theorem 2.4,

$$d(\mathcal{M}^*(N, \Gamma)) = \delta(\Sigma).$$

Under the hypotheses of Theorem 2.7,

$$d_G(\mathcal{M}_G^*(N, \Gamma)) = \delta(\Sigma)$$

where $d_G(\cdot)$ is defined exactly like $d(\cdot)$, except that in determining index of instability, we only count eigenfunctions that are G -invariant.

The proofs are exactly as before.

4. Counting the number of handles on a surface invariant under an involution

Consider a minimal surface that has an axis of orientation preserving, 180° rotational symmetry. In many examples of interest, the handles of the surface are in some sense aligned along the axis. In this section, we make this notion precise, and we observe that our parity theorems apply to such surfaces.

Recall, for example, that Sherk constructed a singly periodic, properly embedded minimal surface $M \subset \mathbf{R}^3$ that is asymptotic to the planes $x = 0$ and $z = 0$ away from the y -axis, Y . By scaling, we may assume that M intersects Y precisely at the lattice points $(0, n, 0)$, $n \in \mathbf{Z}$. Now M has various lines of orientation preserving, 180° rotational symmetry. For example, Y is one such a line, and the line L given by $x = z$, $y = 1/2$ is another. Intuitively, the handles of M are lined up along Y but not along L . (The surface M is also invariant under 180° rotation about the x and z axes, but those rotations reverse orientation on M .) We make the intuition into a precise notion by observing that the rotation about Y acts on the first homology

group $H_1(M, \mathbf{Z})$ by multiplication by -1 , whereas rotation about L acts on $H_1(M, \mathbf{Z})$ in a more complicated way.

4.1. Proposition. — *Suppose S is a noncompact 2-dimensional riemannian manifold of finite topology. Suppose that $\rho : S \rightarrow S$ is an orientation preserving isometry of order two, and that S/ρ is connected. Then the following are equivalent:*

1. ρ acts by multiplication by -1 on the first homology group $H_1(S, \mathbf{Z})$.
2. the quotient S/ρ is topologically a disk.
3. S has exactly $2 - \chi(S)$ fixed points of ρ , where $\chi(S)$ is the Euler characteristic of S .

4.2. Corollary. — *If the equivalent conditions (1)-(3) hold, then the surface S has either one or two ends, according to whether ρ has an odd or even number of fixed points in S .*

4.3. Remark. — To apply Proposition 4.1 and its corollary to a compact manifold M with non-empty boundary, one lets $S = M \setminus \partial M$. Of course the number of ends of S is equal to the number of boundary components of M .

Proof of Proposition 4.1. — Suppose that (1) holds. Let $\pi : S \rightarrow S/\rho$ be the projection and let C be a closed curve in S/ρ . Then $C' = \pi^{-1}(C)$ is a ρ -invariant cycle in S and thus (by (1)) it bounds a 2-chain in S . Consequently $\pi(C') = 2C$ bounds a 2-chain in S/ρ . Thus $2C$ is homologically trivial in S/ρ . But S/ρ is orientable, so $H_1(S, \mathbf{Z})$ has no torsion. Thus C is homologically trivial in S/ρ . Since S/ρ is non-compact and connected with trivial first homology group, it must be a disk. Hence (1) implies (2).

To see that (2) implies (1), suppose that (2) holds. It suffices to show that any ρ -invariant 1-cycle in S is a boundary. (For if C_0 is any cycle in S , then $C_0 + \rho(C_0)$ forms a ρ -invariant cycle.) Since S is oriented, $H_1(S, \mathbf{Z})$ has no torsion, so it suffices to show that any ρ -invariant cycle 1-cycle in S must be a boundary mod 2. Let $C \subset S$ be any ρ -invariant closed curve, not necessarily connected. We may assume that C is smooth and in general position, i.e., that the self-intersections are transverse. By doing the obvious surgeries at the intersections, we may assume in fact that C is embedded.

Now $\pi(C)$ is a smooth, embedded, not necessarily connected, closed curve in S/ρ . Since S/ρ is topologically a disk, $\pi(C)$ bounds a region Ω . It follows that C bounds the region $\pi^{-1}(\Omega)$. Thus C is homologically trivial mod 2. This completes the proof that (2) implies (1).

Finally we show that (2) and (3) are equivalent. Let P be the number of fixed points of ρ . Consider a triangulation of S/ρ such the fixed points of ρ are vertices

in the triangulation, and consider the corresponding triangulation of S . Then from Euler's formula one sees that

$$\chi(S) = 2\chi(S/\rho) - P$$

or

$$P = 2\chi(S/\rho) - \chi(S).$$

Thus $P = 2 - \chi(S)$ if and only if $\chi(S/\rho) = 1$. Since S/ρ is orientable and connected, its Euler characteristic is 1 if and only if it is a disk. This proves that (2) and (3) are equivalent. $\square$

Proof of Corollary 4.2. — Since S/ρ is a disk, it has exactly one end. Since S is a double cover of S/ρ , it must have either one or two ends. Since S is oriented,

$$(3) \quad \chi(S) = 2c - 2g - e,$$

where c is the number of connected components, g is the sum of the genera of the connected components, and e is the number of ends. Thus e is congruent mod 2 to $\chi(S)$, which by Proposition 4.1 is congruent, mod 2, to the number of fixed points of ρ . $\square$

4.4. Counting Y -surfaces. — Let N be a riemannian 3-manifold. We suppose that N has a geodesic Y and an orientation preserving, order two isometry $\rho = \rho_Y : N \rightarrow N$ for which the set of fixed points is Y .

4.5. Definition. — Suppose $M \subset N$ is an orientable, non-closed ρ -invariant surface such that $\rho : M \rightarrow M$ preserves orientation and such that $(M \setminus \partial M)/\rho$ is connected. We will say that M is a Y -surface if $S := M \setminus \partial M$ satisfies the equivalent conditions in Proposition 4.1.

Suppose for example that $N = \mathbf{R}^3$ and that Y is a line. Then $\rho = \rho_Y$ is 180° rotation about Y . If M is a ρ_Y -invariant catenoid, then either Y is the axis of rotational symmetry of M , or else Y intersects M orthogonally at two points on the waist of M . In the first case, ρ acts trivially on the first homology of M , so M is *not* a Y -surface. In the second case, ρ acts by multiplication by -1 on the first homology of M , so M is a Y -surface.

4.6. Definition. — We let

$$\mathcal{M}_Y^*(N, \Gamma) = \{M \in \mathcal{M}^*(N, \Gamma) : M \text{ is a } Y\text{-surface}\}.$$

We say that a curve $\Gamma \subset \partial N$ is Y -bumpy if no surface in $\mathcal{M}_Y^*(N, \Gamma)$ carries a nontrivial, ρ_Y -invariant, normal Jacobi field that vanishes on Γ .

The following result is a version of Theorem 2.7:

4.7. Theorem. — *Let N be a smooth, compact, mean convex riemannian 3-manifold that is homeomorphic to a ball, that has piecewise smooth boundary, and that contains no closed minimal surfaces. Suppose that Y is a geodesic in N and that $\rho = \rho_Y : N \rightarrow N$ is an orientation preserving, order two isometry of N with fixed point set Y .*

Let $\Gamma \subset \partial N$ be a smooth, embedded, ρ -invariant, Y -bumpy curve that carries a ρ -invariant orientation.

Suppose that no two contiguous components of $(\partial N) \setminus \Gamma$ are both minimal surfaces.

Then $|\mathcal{M}_Y^(N, \Gamma)|$ is even unless Σ is a union of disks, in which case $|\mathcal{M}_Y^*(N, \Gamma)|$ is odd.*

Proof. — The proof is almost identical to the proof of Theorem 2.7. One lets the group G in Theorem 2.7 be the group generated by ρ . The hypothesis (*) there follows from the hypothesis here that Γ carries a ρ_Y -invariant orientation. $\square$

5. Higher genus helicoids in $S^2 \times \mathbf{R}$

5.1. A boundary value problem for minimal Y -surfaces. — Our motivation in formulating Proposition 4.1 and Theorem 4.7 comes from the desire to construct embedded minimal surfaces in $S^2 \times \mathbf{R}$, each of whose ends is asymptotic to a helicoid in $S^2 \times \mathbf{R}$. Take as a model of $S^2 \times \mathbf{R}$ the space $\mathbf{R}^2 \times \mathbf{R}$ on which each $\mathbf{R}^2 \times \{z\}$ has the metric of the sphere pulled back by inverse stereographic projection. (The radius of that sphere is fixed but arbitrary.) This model is missing a line, $Z^* = \{\infty\} \times \mathbf{R}$, which we append in a natural way to $\mathbf{R}^2 \times \mathbf{R}$ with the aforementioned product metric. It is easy to verify that a standard helicoid $H \subset \mathbf{R}^3$ with axis $Z = \{(0, 0, z) : z \in \mathbf{R}\}$, an embedded and ruled surface, is also a minimal surface in $S^2 \times \mathbf{R}$. Here, it has two axes, Z and Z^* . By a slight abuse of notation, we will use H to refer to this minimal surface in $S^2 \times \mathbf{R}$.

The horizontal lines on the euclidean helicoid are great circles in the totally geodesic level-spheres of $S^2 \times \mathbf{R}$, the circle at height z passing through the antipodal points $(0, z) \in Z$ and $(\infty, z) \in Z^*$. Let

$$X = (S^2 \times \{0\}) \cap H,$$

and denote by Y the great circle at height 0 passing through $O = (0, 0)$, $O^* = (\infty, 0)$, and orthogonal to the great circle X . Just as on the Euclidean helicoid, ρ_Y , order-two rotation about Y , is an orientation preserving involution of H . Note that under our identification of $S^2 \times \mathbf{R}$ with $\mathbf{R}^3$, each of the great circles on H corresponds to a horizontal line passing through the z -axis, and the great circles X and Y are identified with the x - and y -axes of $\mathbf{R}^3$.

Denote by H^+ the component of the complement of H that contains $Y^+ := \{(0, y, 0) \mid y > 0\}$. Then for any $c > 0$, ρ_Y is an orientation preserving involution of

the domain

$$(4) \quad N_c = H^+ \cap \{|z| < c\}.$$

Note that ∂N_c is mean convex, consisting of three minimal surfaces: $H \cap \{|z| < c\}$, and two totally geodesic hemispheres, $H^+ \cap \{z = \pm c\}$. We will label these minimal surfaces H_c and $S_{\pm c}$, respectively.

The set $H_c \setminus (Z \cup Z^* \cup X)$ has four components. Let Q be the component whose boundary contains the three geodesics $X^+ = \{(x, 0, 0) \mid x \geq 0\}$, $Z \cap \{0 \leq z \leq c\}$, and $Z^* \cap \{0 \leq z \leq c\}$. The “quadrant” Q has a fourth boundary curve, which is one of the two semicircular components of $\partial S_c \setminus (Z \cup Z^*)$. We label this semicircle T_c . Note that $T_{-c} := \rho_Y(T_c)$ lies in $\partial(\rho_Y(Q))$.

Fix a value of c and let $N = N_c$. Consider the union $Q \cup \rho_Y(Q)$, and define $\Gamma \subset \partial N$ to be the boundary of $Q \cup \rho_Y(Q)$. Then

$$(5) \quad \Gamma = (Z \cap H_c) \cup T_c \cup (Z^* \cap H_c) \cup T_{-c} \cup X.$$

See Figure 1. The first four segments of Γ form a piecewise smooth curve with four corners. Adding the great circle X produces a curve that is singular at $O = (0, 0)$ and at $O^* = (\infty, 0)$, where there are right-angle crossings. Note that Γ is ρ_Y -invariant.

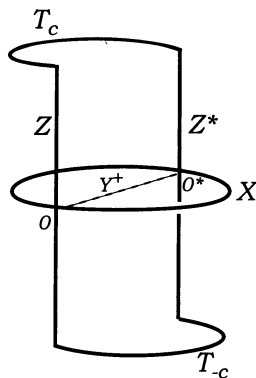


FIGURE 1. The curve Γ . In the figure, we have taken $\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$ as our model for $\mathbf{S}^2 \times \mathbf{R}$, with the metric on $\mathbf{R}^2$ given by the pullback of the metric on $\mathbf{S}^2$ via inverse stereographic projection. In this case, the pole of $\mathbf{S}^2$ is placed at the center of the semicircle Y^- .

If Γ defined in (5) is not Y -bumpy, we can make arbitrarily small perturbations of the curves $T_{\pm c}$ to make it so, while keeping the resulting curve in ∂N , and also ρ_Y -invariant. We will assume from now on that Γ is Y -bumpy.

Suppose for the moment that we could produce a connected Y -surface $M \subset N$ with boundary Γ . We will show in the next paragraph how this will enable us to construct a higher-genus helicoid.

Since $\rho_Y|_M$ is orientation preserving, Y must intersect M orthogonally in a discrete set of points, precisely the fixed points of $\rho_Y|_M$. We will consider M without its boundary, allowing us to apply Proposition 4.1. Namely, if $k = |Y \cap M|$, the number of points in $Y \cap M$, then

$$k = 2 - \chi(M).$$

Extend M by ρ_Z , Schwarz reflection in Z (or equivalently in Z^*), and let

$$(6) \quad \tilde{M} = \text{interior}(\overline{M \cup \rho_Z(M)}).$$

The surface $\tilde{M}$ is smooth because M is ρ_Y -symmetric, and

$$|Y \cap \tilde{M}| = 2k + 2$$

because the points $O = (0, 0)$ and at $O^* = (\infty, 0)$, which lie on Y , are in $\tilde{M}$. The surface $\tilde{M}$ is bounded by two great circles at levels $\pm c$. It is embedded because $\rho_Z(M)$ lies in H^- . Furthermore it is ρ_Y -invariant by construction and satisfies the condition that ρ_Y acts by multiplication by -1 on $H_1(M, Z)$. Therefore, $2k + 2 = 2 - \chi(\tilde{M})$ by Proposition 4.1. Since $\tilde{M}$ has two ends, we have

$$2k + 2 = 2 - (2 - 2 \text{genus}(\tilde{M}) - 2),$$

or

$$\text{genus}(\tilde{M}) = k.$$

If we can produce $\tilde{M} = \tilde{M}_c$ for any cutoff height c , it is reasonable to expect that as $c \rightarrow \infty$, the $\tilde{M}_c$ converge subsequentially to an embedded genus- k minimal surface each of whose ends is asymptotic to H or a rotation of H . In [3], we prove that this is the case.

5.2. Existence of a suitable $M \in \mathcal{M}_Y^*(N, \Gamma)$ with $|Y \cap M| = k$. — How are we going to produce, for each positive integer k , a connected, embedded, minimal Y -surface $M \subset N$ with boundary Γ ? The answer is: by induction on k , using Theorem 4.7. The details, carried out in [3] are somewhat intricate. We describe here the main idea and the intuition behind the proof.

First of all, it would seem that Theorem 4.7 is not suited to prove existence of the desired surfaces because in most cases it asserts that the number of surfaces in a given class is even. This could mean that there are *zero* surfaces in the class. We begin to address this problem by dividing the class of surfaces according to their geometric behavior near O . Why this helps will be made clear below.

Since we are working with one fixed domain, namely $N = N_c$ as defined in (4), we will suppress the reference to N and write $\mathcal{M}_Y^*(\Gamma)$ instead of $\mathcal{M}_Y^*(N, \Gamma)$. We can

decompose $\mathcal{M}_Y^*(\Gamma)$ into two sets by looking at how a surface $S \in \mathcal{M}_Y^*(\Gamma)$ attaches to Γ at the crossing O , the intersection of the vertical line Z and the great circle X . The geodesics X , Z , and Z^* divide H into four “quadrants”. A quadrant whose boundary contains $Z^+ \cup X^+$ or $Z^- \cup X^-$ will be called a *positive* quadrant. The other two quadrants will be called *negative* quadrants.

5.3. Definition. — Given a nonnegative integer k ,

$\mathcal{M}_Y^*(\Gamma, k) \subset \mathcal{M}_Y^*(\Gamma)$ is the collection of embedded minimal Y -surfaces M with the property that $|M \cap Y| = k$.

$\mathcal{M}_Y^*(\Gamma, k, +) \subset \mathcal{M}_Y^*(\Gamma, k)$ is the subset of surfaces tangent to the positive quadrants at O .

$\mathcal{M}_Y^*(\Gamma, k, -) \subset \mathcal{M}_Y^*(\Gamma, k)$ is the subset of surfaces tangent to the negative quadrants at O .

Now we approximate Γ by smooth embedded curves $\Gamma(t) \subset \partial N$. We have to do this in order to apply any of our parity theorems. We want the four corners to be rounded and the two crossings to be resolved. At O , we modify Γ in a small neighborhood of radius $t > 0$ by connecting Z^+ to X^+ and Z^- to X^- . Given this choice at O , we resolve the crossing at O^* according to whether k is even or odd as follows: connect positively if k is even (i.e. Z^+ to X^+ and Z^- to X^-) and negatively (i.e. Z^+ to X^- and Z^- to X^+) if k is odd. Again we modify in a manner that preserves ρ_Y -invariance, and we choose t small enough so that the neighborhoods of the corners and the crossings are pairwise-disjoint. We will refer to such a rounding as an *adapted positive rounding* of Γ . Note that when k is odd, an adapted positive rounding of Γ is connected, while when k is even, such a rounding has two components. See Figure 2.

Our motivation for the choice of desingularization at O^* is given by the following

5.4. Proposition. — *A surface $S \in \mathcal{M}_Y^*(\Gamma, k, +)$ is tangent at O^* to the positive quadrants if k is even, and to the negative quadrants if k is odd.*

Proof. — For any oriented surface S , we have (3)

$$\chi(S) = 2c(S) - 2 \text{ genus}(S) - e(S),$$

where $e(s)$ is the number of ends of S , $c(S)$ is the number of components of S , and $\text{genus}(S)$ is the sum of the genera of the components of S . If $S \in \mathcal{M}_Y^*(\Gamma, k)$, then using Proposition 4.1 we have

$$(7) \quad k = |Y \cap S| = 2 - \chi(S) \cong e(S),$$

where $\cong$ denotes equivalence mod 2.

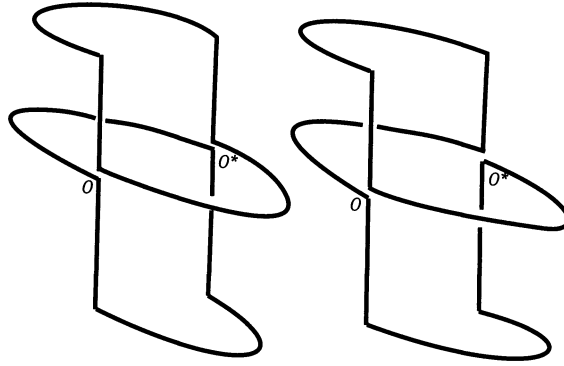


FIGURE 2. The two adapted positive roundings of Γ . On the left, the rounding at O^* is the same as at the point O , resulting in a curve with two components. On the right the rounding at O^* is positive to negative, resulting in a connected curve.

Claim. — If $S \in \mathcal{M}(\Gamma, k, +)$, then $e(S) = \begin{cases} 2 & \text{if } S \text{ is positive at } O^*, \\ 1 & \text{if } S \text{ is negative at } O^*. \end{cases}$

The proposition follows from the claim and the congruence (7).

Proof of Claim. — Let $B(O)$ be a geodesic ball of radius $r > 0$ centered at O , and let $B(O^*)$ be the corresponding ball centered at O^* with the same radius. We may choose r small enough so that the surface $S' = S \setminus (B(O) \cup B(O^*))$ has the same number of ends as S : i.e., $e(S') = e(S)$. We may make r smaller if necessary so that near O (say in a geodesic ball of radius $2r$ centered at O), the boundary curve $\Gamma' = \partial S'$ consists of a segment of X^+ joined to a segment of Z^+ by a single curve in $\partial B(O)$ together with a segment of X^- joined to a segment of Z^- by a single curve in $\partial B(O)$. It is precisely here that we have used the fact that $S \in \mathcal{M}_Y^*(\Gamma, k, +)$ and not just in $\mathcal{M}_Y^*(\Gamma, k)$. Making r smaller if necessary, we may assert that if S is tangent to the positive quadrants at O^* , then near O^* the curve Γ' connects positively, just as it does near O . This implies that Γ' has two components. Therefore $e(S') = 2$. If S is tangent to the negative quadrants at O^* , then near O^* the curve Γ' will connect X^+ to Z^- and X^- to Z^+ . In this case, Γ' is connected and $e(S') = 1$. Since we chose r small enough so that $e(S') = e(S)$, we have proved the claim. $\square$

Let $\Gamma(t)$, $t > 0$ small, be a smooth family of adapted positive roundings of Γ . We will round in such a way that for each corner and crossing q ,

$$\lim_{t \rightarrow 0} (1/t)(\Gamma(t) - q)$$

is a smooth embedded curve, and such that $\Gamma(t)$ converges smoothly to Γ except perhaps at the corners and crossings of Γ . It is now reasonable to expect that if we specify a surface $M \in \mathcal{M}_Y^*(\Gamma, k)$ as a sort of initial data at $\Gamma = \Gamma(0)$ we can deform it to a family of embedded minimal Y -surfaces $S_t \subset N$ with $\partial S_t = \Gamma(t)$. In fact we can do this in a unique manner.

5.5. Definition. — For any nonnegative integer j , the set $\mathcal{M}_Y^*(\Gamma(t), j)$ is the collection of embedded minimal Y -surfaces $S \subset N$ with $\partial S = \Gamma(t)$ and $|S \cap Y| = j$

5.6. Theorem. — Let $N = N_c \subset \mathbf{S}^2 \times \mathbf{R}$ be a domain of the form given in (4) for some fixed positive constant c . Let Γ be the curve specified in (5), perturbed if necessary to become Y -bumpy.

Let $\Gamma(t)$, $t > 0$ small, be a smooth family of adapted positive roundings of Γ . Suppose for some nonnegative integer j , that there exists a surface $M \in \mathcal{M}_Y^*(\Gamma, j)$. Then there exists a constant $a = a(\Gamma, M) > 0$ such that for $t < a$, each approximating curve $\Gamma(t)$ bounds an embedded minimal Y -surface S_t with the following properties:

1. Each S_t is the normal graph over a region $\Omega_t \subset \tilde{M}$ that is bounded by the projection of $\Gamma(t)$ onto $\tilde{M}$;
2. The family of surfaces S_t is smooth in t and converges smoothly to M as $t \rightarrow 0$;
3. If $M \in \mathcal{M}_Y^*(\Gamma, j, +)$, then $S_t \in \mathcal{M}_Y^*(\Gamma(t), j)$, i.e. $|S_t \cap Y| = j$;
4. If $M \in \mathcal{M}_Y^*(\Gamma, j, -)$, then $S_t \in \mathcal{M}_Y^*(\Gamma(t), j + 2)$, i.e. $|S_t \cap Y| = j + 2$.

Furthermore, if $\hat{S} \in \mathcal{M}_Y^*(\Gamma(t_0), j)$, $t_0 < a$, then it lies in a smooth one-parameter family of surfaces $S_t \in \mathcal{M}_Y^*(\Gamma(t), j)$, $t \leq t_0$, with the property that the family has, as a smooth limit as $t \rightarrow 0$, an embedded minimal Y -surface $M \subset N$ that lies either in $\mathcal{M}_Y^*(\Gamma, j)$ or in $\mathcal{M}_Y^*(\Gamma, j - 2)$.

Statements (3) and (4) have a simple geometric interpretation. Suppose we have a family of surfaces in $S_t \in \mathcal{M}_Y^*(\Gamma(t), k)$ for some smooth family $\Gamma(t)$ of adapted positive roundings of Γ . They will limit to an embedded minimal Y -surface $M \subset N$ with boundary Γ . If they limit to an $M \in \mathcal{M}_Y^*(\Gamma, j, +)$, then the points $S_t \cap Y$ stay bounded away from the crossings $\{O, O^*\}$. Hence the S_t have the property that $|S_t \cap Y| = |M \cap Y| = j$. However, if they limit to an $M \in \mathcal{M}_Y^*(\Gamma, j, -)$, then each of the S_t is a graph over a region Ω_t that contains both O and O^* . Two points are lost. Hence $j = |S_t \cap Y| = |M \cap Y| + 2$.

Since the theorem above tells us that there is a correspondence between every surface in $\mathcal{M}(\Gamma(t), k)$ and some embedded minimal Y -surface in N bounded by Γ , we have

5.7. Corollary. — We have

$$|\mathcal{M}_Y^*(\Gamma(t), k)| = |\mathcal{M}_Y^*(\Gamma, k, +)| + |\mathcal{M}_Y^*(\Gamma, k - 2, -)|.$$

We can now carry out the induction. We use $\cong$ to denote congruence modulo 2. In our situation, the number of ends of a surface $S \in \mathcal{M}_Y^*(\Gamma, k)$ is one or two, so the number of components of S is at most two. Since S is a Y -surface we know, by Proposition 4.1, that $k = |S \cap Y| = 2 - \chi(S)$. It is easy to see that when $k = 1$ (or $k = 0$), S is a disk (or the union of two disks). Corollary 5.7 and Theorem 4.7 yield in this situation that

$$1 \cong |\mathcal{M}_Y^*(\Gamma(t), k)| = |\mathcal{M}_Y^*(\Gamma, k, +)| + |\mathcal{M}_Y^*(\Gamma, k - 2, -)| = |\mathcal{M}_Y^*(\Gamma, k, +)|,$$

the last equality being simply the fact that it is impossible for a surface to intersect Y in a negative number of points. Therefore we have established the existence of the desired surface for $k = 0$ or $k = 1$. In fact we get existence of a surface in $\mathcal{M}_Y^*(\Gamma, k, +)$. However there is nothing special in this context about being in $\mathcal{M}_Y^*(\Gamma, k, +)$ as opposed to being in $\mathcal{M}_Y^*(\Gamma, k, -)$. If we redid the entire construction by starting out by requiring our smoothing to be negative at O , we would wind up with an odd number of surfaces in $\mathcal{M}_Y^*(\Gamma, k, -)$, for $k = 0$ and $k = 1$.

Now assume $k \geq 2$, and suppose that for any $j < k$, that $|\mathcal{M}_Y^*(\Gamma, j, +)| \cong |\mathcal{M}_Y^*(\Gamma, j, -)| \cong 1$. Corollary 5.7 together with Theorem 4.7 yield in our situation that

$$0 \cong |\mathcal{M}_Y^*(\Gamma(t), k)| = |\mathcal{M}_Y^*(\Gamma, k, +)| + |\mathcal{M}_Y^*(\Gamma, k - 2, -)|.$$

But $|\mathcal{M}_Y^*(\Gamma, k - 2, -)| \cong 1$, by assumption. Therefore $0 \cong |\mathcal{M}_Y^*(\Gamma, k, +)| + 1$, or

$$|\mathcal{M}_Y^*(\Gamma, k, +)| \cong 1.$$

Hence, this class of surfaces is not empty for any value nonnegative integer k . As indicated above the same is true for $\mathcal{M}_Y^*(\Gamma, k, -)$. Whether or not we have produced two geometrically *different* (i.e. non-congruent) solutions to our problem turns out to depend on whether k is even or odd—but that is another story.

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