Astérisque

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Astérisque, tome 322 (2008), p. 241-254
<http://www.numdam.org/item?id=AST_2008__322__241_0>

Astérisque
322, 2008, p. 241–254

THE PROJECTIVE HULL OF CERTAIN CURVES IN $\mathbb{C}^2$

by

Reese Harvey, Blaine Lawson & John Wermer

Dedicated to Jean Pierre Bourguignon on the occasion of his sixtieth birthday

Abstract. — The projective hull $\widehat{X}$ of a compact set $X \subset \mathbb{P}^n$ is an analogue of the classical polynomial hull of a set in $\mathbb{C}^n$. In the special case that $X \subset \mathbb{C}^n \subset \mathbb{P}^n$, the affine part $\widehat{X} \cap \mathbb{C}^n$ can be defined as the set of points $x \in \mathbb{C}^n$ for which there exists a constant M_x so that

$$\left|p(x)\right| \leq M_x^d \sup_X |p|$$

for all polynomials p of degree $\leq d$, and any $d \geq 1$. Let $\widehat{X}(M)$ be the set of points x where M_x can be chosen $\leq M$. Using an argument of E. Bishop, we show that if $\gamma \subset \mathbb{C}^2$ is a compact real analytic curve (not necessarily connected), then for any linear projection $\pi : \mathbb{C}^2 \to \mathbb{C}$, the set $\widehat{\gamma}(M) \cap \pi^{-1}(z)$ *is finite for almost all* $z \in \mathbb{C}$. It is then shown that for any compact stable real-analytic curve $\gamma \subset \mathbb{P}^n$, the set $\widehat{\gamma} - \gamma$ *is a* 1*-dimensional complex analytic subvariety of* $\mathbb{P}^n - \gamma$. Boundary regularity for $\widehat{\gamma}$ is also discussed in detail.

***Résumé* (L'enveloppe projective de certaines courbes dans $\mathbb{C}^2$).** — L'enveloppe projective $\widehat{X}$ d'un compact $X \subset \mathbb{P}^n$ est l'analogue de l'enveloppe polynomiale classique d'un sous-ensemble de $\mathbb{C}^n$. Dans le cas particulier où $X \subset \mathbb{C}^n \subset \mathbb{P}^n$, la partie affine $\widehat{X} \cap \mathbb{C}^n$ peut être définie en tant qu'ensemble de points $x \in \mathbb{C}^n$ pour lesquels il existe une constante M_x telle que

$$\left|p(x)\right| \leq M_x^d \sup_X |p|$$

pour tous les polynômes p de degré $\leq d$, et tout $d \geq 1$. Soit $\widehat{X}(M)$ l'ensemble de points x où M_x peut être choisi $\leq M$. En utilisant un argument d'E. Bishop, nous montrons que si $\gamma \subset \mathbb{C}^2$ est une courbe analytique réelle compacte (non nécessairement connexe), alors pour toute projection linéaire $\pi : \mathbb{C}^2 \to \mathbb{C}$, l'ensemble $\widehat{\gamma}(M) \cap \pi^{-1}(z)$ *est fini pour presque tout* $z \in \mathbb{C}$. Nous montrons alors que pour toute courbe analytique réelle compacte stable $\gamma \subset \mathbb{P}^n$, l'ensemble $\widehat{\gamma} - \gamma$ *est une sous-variété de* $\mathbb{P}^n - \gamma$ *analytique complexe de dimension* 1. Nous discutons également en détail la régularité de la frontière de $\widehat{\gamma}$.

2000 Mathematics Subject Classification. — 30H05, 32Q99.
Key words and phrases. — Projective hull, complex analytic curve.

The second author is partially supported by the N.S.F.

1. Introduction

The classical *polynomial hull* of a compact subset X of $\mathbb{C}^n$ is the set of points $x \in \mathbb{C}^n$ such that

$$|p(x)| \leq \sup_X |p| \quad \text{for all polynomials } p. \tag{1.1}$$

In [4] the first two authors introduced an analogue for compact subsets of projective space. Given $X \subset \mathbb{P}^n$, the *projective hull of* X is the set $\widehat{X}$ of points $x \in \mathbb{P}^n$ for which there exists a constant $C = C_x$ such that

$$\|P(x)\| \leq C_x^d \sup_X \|P\| \quad \text{for all sections } P \in H^0\big(\mathbb{P}^n, \mathcal{O}(d)\big) \tag{1.2}$$

and all $d \geq 1$. Here $\mathcal{O}(d)$ is the d-th power of the hyperplane bundle with its standard metric. Recall that $H^0(\mathbb{P}^n, \mathcal{O}(d))$ is given naturally as the set of homogeneous polynomials of degree d in homogeneous coordinates. If X is contained in an affine chart $X \subset \mathbb{C}^n \subset \mathbb{P}^n$ and $x \in \mathbb{C}^n$, then condition (1.2) is equivalent to

$$|p(x)| \leq M_x^d \sup_X |p| \quad \text{for all polynomials } p \text{ of degree } d \tag{1.3}$$

and all $d \geq 1$ where $M_x = \rho\sqrt{1 + \|x\|^2} C_x$ and ρ depends only on X. Therefore the set $\widehat{X} \cap \mathbb{C}^n$ consists exactly of those points $x \in \mathbb{C}^n$ for which there exists an M_x satisfying condition (1.3).

This paper is concerned with the case where $X = \gamma$ is a real analytic curve. In [4] evidence was given for the following conjecture.

Conjecture 1.1. — *Let $\gamma \subset \mathbb{P}^n$ be a finite union of simple closed real analytic curves. Then $\widehat{\gamma} - \gamma$ is a 1-dimensional complex analytic suvariety of $\mathbb{P}^n - \gamma$.*

This conjecture has many interesting geometric consequences (see [7], [5], and [6]). The assumption of real analyticity is important. The conjecture does not hold for all smooth curves. In particular, it does not hold for curves which are not pluripolar.

One point of this paper is to prove Conjecture 1.1 under the hypothesis that the function C_x is bounded on $\widehat{\gamma}$. We begin by adapting arguments of E. Bishop [2] to prove the following finiteness theorem.

Theorem 1.1. — *Let $\gamma \subset \mathbb{C}^2$ be a finite union of simple closed real analytic curves. Set*

$$\widehat{\gamma}_M \equiv \{x \in \widehat{\gamma} \cap \mathbb{C}^2 : M_x \leq M\}$$

where M_x is the function appearing in condition (1.3). *Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be a linear projection. Then*

$$\widehat{\gamma}_M \cap \pi^{-1}(z) \quad \textit{is finite for almost all } z \in \mathbb{C}.$$

Consequently, $\widehat{\gamma} \cap \pi^{-1}(z)$ is countable for almost all $z \in \mathbb{C}$.

In Section 3 this theorem is combined with results from [**4**] and the theorems concerning maximum modulus algebras to prove the following.

A set $X \subset \mathbb{P}^n$ is called *stable* if the function C_x in (1.2) is bounded on $\widehat{X}$.

Note that if X is stable and $X \subset \mathbb{C}^n \subset \mathbb{P}^n$, then the function M_x is bounded on $\mathbb{C}^n$ by $\rho\sqrt{1+\|x\|^2}$.

Theorem 1.2. — *Let $\gamma \subset \mathbb{P}^n$ be a finite union of simple closed real analytic curves. Assume γ is stable. Then $\widehat{\gamma}-\gamma$ is a 1-dimensional complex analytic subvariety of $\mathbb{P}^n - \gamma$.*

2. The finiteness theorem

Let X be a compact set in $\mathbb{C}^n$ and denote by $\mathcal{P}_d$ the space of polynomials of degree $\leq d$ on $\mathbb{C}^n$.

Definition 2.1. — Denote by $\widehat{X} \cap \mathbb{C}^n$ the set of all $x \in \mathbb{C}^n$ such that there exists a constant M_x with

$$|P(x)| \leq M_x^d \sup_X |P| \tag{2.1}$$

for every $P \in \mathcal{P}_d$ and $d \geq 1$. The set $\widehat{X} \cap \mathbb{C}^n$ is called *the projective hull of X in $\mathbb{C}^n$*.

As noted above, the projective hull, defined in [**4**], is a subset of projective space $\mathbb{P}^n$, and the set $\widehat{X} \cap \mathbb{C}^n$ is exactly that part of the projective hull which lies in the affine chart $\mathbb{C}^n \subset \mathbb{P}^n$. Closely related to Definition 2.1 is the following.

Definition 2.2. — Fix a number $M \geq 1$ and a point $z \in \mathbb{C}^{n-1}$. Then we set

$$\widehat{X}_M(z) = \big\{w \in \mathbb{C} : |P(z,w)| \leq M^d \sup_X |P|, \ \forall P \in \mathcal{P}_d \text{ and } \forall d \geq 1\big\}$$

and let $\widehat{X}(z) = \bigcup_{M\geq 1} \widehat{X}_M(z) = \{w \in \mathbb{C} : (z,w) \in \widehat{X}\}$.

We consider a special case of these definitions. We fix $n = 2$ and consider a simple closed real-analytic curve X in $\mathbb{C}^2$. Let Δ denote the unit disk in $\mathbb{C}$.

Theorem 2.1. — *Fix $M \geq 1$. For almost all $z \in \Delta$, $\widehat{X}_M(z)$ is a finite set.*

Corollary 2.1. — *For almost all $z \in \mathbb{C}$ the set $\widehat{X}(z)$ is countable.*

We shall prove Theorem 2.1 by adapting an argument, for the case of polynomially convex hulls, by Errett Bishop in [**2**]. We shall follow the exposition of Bishop's argument in [**10**, Chap. 12].

Definition 2.3. — The polynomial $Q(z,w) = \sum_{n,m} c_{nm} z^n w^m$ is called a *unit polynomial* if $\max_{n,m} |c_{nm}| = 1$.

Definition 2.4. — The polynomial $Q(z,w) = \sum_{n,m} c_{nm} z^n w^m$ is said to have *bidegree* (d,e), for non-negative integers d and e, if $c_{nm} = 0$ unless $n \le d$ and $m \le e$, and d, e are minimal with this property.

Note that $\deg Q \le d + e \le 2 \deg Q$.

Definition 2.5. — Fix $M \ge 1$. For each $z \in \mathbb{C}$ set

$$S_M(z) = \{w \in \mathbb{C} : |Q(z,w)| \le (M^{d+e}) \sup_X |Q|,\\ \forall Q \in \mathbb{C}[z,w] \text{ of bidegree } (d,e) \text{ for } d,e \ge 1\big\}.$$

We now fix a number $M \ge 1$ and keep it fixed throughout what follows.

Theorem 2.2. — *For almost all $z \in \Delta$, $S_M(z)$ is a finite set.*

Theorem 2.1 is an immediate consequence of Theorem 2.2. To see this, fix $z \in \Delta$ and choose $w \in \widehat{X}_M(z)$. Choose next a polynomial Q of bidegree (d,e) and let $\delta = \deg Q$. Then

$$\big|Q(z,w)\big| \le M^\delta \|Q\|_X \le M^{d+e} \|Q\|_X$$

and so $w \in S_M(z)$. Since this holds for all such w, $\widehat{X}_M(z) \subseteq S_M(z)$. By Theorem 2.2 $S_M(z)$ is a finite set for a. a. $z \in \Delta$, so $\widehat{X}_M(z)$ is a finite set for almost all $z \in \Delta$. Thus Theorem 2.1 holds.

We now go to the proof of Theorem 2.2.

Lemma 2.1. — *Let Ω be a plane domain, let K be a compact set in Ω, and fix $z_0 \in \Omega$. Then there exists a constant r, $0 < r < 1$, so that if f is holomorphic on Ω and $|f| < 1$ on Ω and if f vanishes to order λ at z_0, then $|f| \le r^\lambda$ on K.*

Proof. — We construct a bounded and smoothly bounded subdomain Ω_0 of Ω with $\overline{\Omega}_0 \subset \Omega$, $z_0 \in \Omega_0$ and $K \subset \Omega_0$. Denote by $G(z_0, z)$ the Green's function of Ω_0 with pole at z_0.

Then $\mathrm{e}^{-(G+iH)}$ is a multiple-valued holomorphic function on Ω_0 with a single-valued modulus e^{-G}, and this modulus is $= 1$ on $\partial\Omega_0$ (H is the harmonic conjugate of G). Consequently,

$$f / \mathrm{e}^{-\lambda(G+iH)}$$

is multiple-valued and holomorphic on Ω_0, and its modulus is single-valued and < 1 on $\partial\Omega_0$. By the maximum principle for holomorphic functions, for each $z \in K$, we have $|f/\mathrm{e}^{-\lambda(G+iH)}| < 1$ at z and so

$$|f(z)| \le \left[\mathrm{e}^{-G(z_0,z)}\right]^\lambda.$$

Putting $r = \sup_K \mathrm{e}^{-G}$, we get our desired inequality. □

Lemma 2.2. — *Let Ω be a bounded plane domain and K a compact subset of Ω. Let $\mathcal{L}$ be an algebra of holomorphic functions on Ω. Put $\|\phi\| = \sup_K |\phi|$ for all $\phi \in \mathcal{L}$. Fix $f, g \in \mathcal{L}$. Then there exist r, $0 < r < 1$ and $C > 0$ such that for each pair of positive integers (d, e) we can find a unit polynomial $F_{d,e}$ of bidegree (d, e) such that*

$$\left\|F_{d,e}(f,g)\right\| \leq C^{d+e} r^{de}. \tag{2.2}$$

Proof. — Choose a subdomain Ω_1 of Ω with $K \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega$. Choose $C_0 > 1$ with $|f| < C_0$, $|g| < C_0$ on $\overline{\Omega}_1$. Consider an arbitrary polynomial

$$F(z,w) = \sum_{n=0}^{d} \sum_{m=0}^{e} c_{nm} z^n w^m$$

and let h be the function $F(f,g)$ in $\mathcal{L}$. Fix a positive integer λ. The requirement that h should vanish at z_0 to order λ imposes λ linear homogeneous conditions on the c_{nm}, and hence has a non-trivial solution if $\lambda < (d+1)(e+1)$. We may assume that the corresponding polynomial F is a unit polynomial. Since

$$\frac{\mathrm{d}^\nu h}{\mathrm{d}z^\nu}(z_0) = 0, \quad \nu = 0, 1, \ldots, \lambda - 1,$$

Lemma 2.1 gives us some r, $0 < r < 1$, such that

$$|h| \leq \left(\sup_{\overline{\Omega}_1} |h|\right) r^\lambda \quad \text{on } K.$$

Since F is a unit polynomial,

$$|h| \leq \sum_{n=0}^{d} \sum_{m=0}^{e} |c_{nm}| \cdot |f|^n \cdot |g|^m \leq (d+1)(e+1) C_0^{d+e} \quad \text{on } \overline{\Omega}_1.$$

Hence for large C,

$$\|h\| \leq (d+1)(e+1) C_0^{d+e} \leq C^{d+e} r^\lambda.$$

We choose $\lambda = de$. Since $de < (d+1)(e+1)$, we get

$$\|F(f,g)\| = \|h\| < C^{d+e} r^{de}$$

as desired. □

Note. — We shall apply this result to the case when K is the unit circle, Ω is an annulus containing K, and $\mathcal{L}$ is the algebra of functions holomorphic on Ω.

The curve X in our Theorem 2.2 is real analytic by hypothesis, and hence can be represented parametrically:

$$z = f(\zeta), \quad w = g(\zeta) \quad \zeta \in \Omega$$

where f, g are functions in $\mathcal{L}$.

Lemma 2.3. — *Let r, C and $F_{d,e}$ be as in Lemma 2.2. Fix r_0, $r < r_0 < 1$. Then there exists d_0 such that*

$$(MC)^{d+e} r^{de} \le r_0^{de} \quad \text{for } d, e > d_0. \tag{2.3}$$

Proof. — We write $\sim$ for "is equivalent to".

$$\begin{aligned}(2.3) \sim (MC)^{d+e} \le \left(\frac{r_0}{r}\right)^{de} &\sim (d+e)\log(MC) \le de \log\left(\frac{r_0}{r}\right) \\ &\sim \left(\frac{1}{e} + \frac{1}{d}\right)\log(MC) \le \log\left(\frac{r_0}{r}\right).\end{aligned}$$

The last inequality is true for $d, e > d_0$ for some suitable d_0. We are done. □

With M, r, r_0 fixed, we choose d_0 as in (2.3). Henceforth, we tacitly assume $d, e > d_0$.

Definition 2.6. — Fix d, e and put $F = F_{d,e}$ as above. Then

$$F(z,w) = \sum_{j=0}^{e} G_j(z) w^j$$

where for some $j = j_0$, G_{j_0} is a unit polynomial of degree $\le d$. We define

$$T(d,e) = \{z \in \Delta : |G_{j_0}(z)| \le r_0^{\frac{1}{2}de}\}.$$

Lemma 2.4. — *Let F be a unit polynomial in z, of degree k, and let α be a positive number. Put $\Lambda = \{z \in \Delta : |F(z)| \le \alpha^k\}$. Then*

$$m(\Lambda) \le 48\alpha,$$

where m is 2-dimensional measure.

Proof. — This is Lemma 12.3 in [**10**], and a proof of it is given there. □

Lemma 2.5. — *Fix d, e. Fix a point $z_1 \in \Delta - T(d,e)$. Then there exists a unit polynomial B in one variable, of degree $\le e$, such that for every $w_0 \in S_M(z_1)$, we have*

$$|B(w_0)| \le r_0^{\frac{1}{2}de}.$$

Proof. — Define the polynomial A in one variable by $A(w) = F(z_1, w)$, where $F = F_{d,e}$. As in Definition 2.6 then

$$A(w) = \sum_{j=0}^{e} G_j(z_1) w^j$$

and G_{j_0} is a unit polynomial in z_1. Since $z_1 \notin T(d,e)$, we have

$$|G_{j_0}(z_1)| > r_0^{\frac{1}{2}de}. \tag{2.4}$$

Fix $w_0 \in S_M(z_1)$. Then

$$\begin{aligned}\left|F(z_1, w_0)\right| &\leq M^{d+e} \cdot \|F\|_X \\ &\leq M^{d+e} C^{d+e} r^{de} && \text{by (2.2)} \\ &\leq r_0^{de} && \text{by (2.3).}\end{aligned}$$

We shall divide A by its largest coefficient K. Note that

$$|K| \geq |G_{j_0}(z_1)| > r_0^{\frac{1}{2}de}$$

by (2.4). Put $B(w) = A(w)/K$. Then $\deg B \leq e$ and

$$\left|B(w_0)\right| = \frac{|A(w_0)|}{|K|} = \frac{|F(z_1, w_0)|}{|K|} \leq \frac{r_0^{de}}{r_0^{\frac{1}{2}de}} = r_0^{\frac{1}{2}de}.$$

We are done. □

Lemma 2.6. — *For each d,*

$$m\big(T(d,e)\big) \leq 48 r_0^{\frac{1}{2}e}.$$

Proof. — Fix e and fix d. With G_{j_0} as above, write $G = G_{j_0}$. Then $\deg G \leq d$. By definition of $T(d,e)$, if $z \in T(d,e)$, then

$$\left|G(z)\right| \leq r_0^{\frac{1}{2}de} = \big(r_0^{\frac{1}{2}e}\big)^d \leq \big(r_0^{\frac{1}{2}e}\big)^{\deg G},$$

and so

$$T(d,e) \subseteq \big\{z \in \Delta : |G(z)| \leq (r_0^{\frac{1}{2}e})^{\deg G}\big\}.$$

Therefore,

$$m\big[T(d,e)\big] \leq m\big\{z \in \Delta : |G(z)| \leq \alpha^k\big\}$$

where $\alpha = r_0^{\frac{1}{2}e}$ and $k = \deg G$. By Lemma 2.4, $m\{z \in \Delta : |G(z)| \leq \alpha^k\} \leq 48\alpha$, and so $m[T(d,e)] \leq 48 r_0^{\frac{1}{2}e}$, as was to be shown. □

Definition 2.7. — Fix e and and set

$$H_e = \big\{z : z \in \Delta - T(d,e) \text{ for infinitely many } d\big\}.$$

Lemma 2.7. — *If $z^* \in H_e$, then $S_M(z^*)$ has at most e elements.*

Proof. — Fix $z^* \in H_e$. Then there exists a sequence $\{d_j\}$ such that $z^* \in \Delta - T(d_j, e)$ for each j. By Lemma 2.5, for each j there is a unit polynomial B_j with $\deg B_j \leq e$ such that

$$\left|B_j(w_0)\right| \leq r_0^{\frac{1}{2}(d_j e)} \quad \text{for each } w_0 \in S_M(z^*). \tag{2.5}$$

Since $\deg B_j \leq e$ for all j, and each B_j is a unit polynomial, there exists a subsequence of the sequence $\{B_j\}$ converging uniformly to a unit polynomial B^* on compact sets in the w-plane. Because of (2.5), $B^*(w_0) = 0$ for each $w_0 \in S_M(z^*)$. Also, $\deg B^* \leq e$. Hence the cardinality of $S_M(z^*)$ is $\leq e$. We are done. □

Proof of Theorem 2.2. — Our task is to show that $m\{z \in \Delta : S_M(z) \text{ is infinite }\} = 0$. Fix e. Fix $z \in \Delta - H_e$. Since $z \notin H_e$, we have $z \in \Delta - T(d,e)$ for only finitely many d, so $z \in T(d,e)$ for all d from some $d = k$ on. Therefore,

$$z \in \bigcap_{d=k}^{\infty} T(d,e)$$

and so

$$\Delta - H_e \subseteq \bigcup_{k=k_0}^{\infty} \Big[\bigcap_{d=k}^{\infty} T(d,e)\Big]. \tag{2.6}$$

By Lemma 2.6, $m(T(d,e)) \leq 48 r_0^{\frac{1}{2}e}$ for each d. Therefore,

$$m\Big(\bigcap_{k=1}^{\infty} T(d,e)\Big) \leq 48 r_0^{\frac{1}{2}e}$$

for each k. So the right hand side of (2.6) is the union of an increasing family of sets each of which has m-measure $\leq 48 r_0^{\frac{1}{2}e}$. Thus (2.6) gives

$$m\big(\Delta - H_e\big) \leq 48 r_0^{\frac{1}{2}e}. \tag{2.7}$$

Also, by Lemma 2.7, we have

$$\text{If } z^* \in H_e, \text{ then } \#\big\{S_M(z^*)\big\} \leq e. \tag{2.8}$$

Fix $z \in \Delta$ such that the set $S_M(z)$ is infinite. Then $z \notin H_e$ for each e, that is, $z \in \Delta - H_e$ for all e. Hence, $\{z \in \Delta : S_M(z) \text{ is infinite }\} \subset \Delta - H_e$. Therefore

$$m\big\{z \in \Delta : S_M(z) \text{ is infinite }\big\} \leq m(\Delta - H_e) \leq 48 r_0^{\frac{1}{2}e}$$

by (2.7). We now let $e \to \infty$ and conclude that $m\{z \in \Delta : S_M(z) \text{ is infinite }\} = 0$. Theorem 2.2 is proved. □

Proof of Corollary 2.1. — Fix $r > 0$ and apply Theorem 2.1 to the curve $\rho_r(X)$ where $\rho_r : \mathbb{C}^2 \to \mathbb{C}^2$ is given by $\rho_r(z) = rz$. Since $\rho_r(\widehat{X} \cap \mathbb{C}^2) = \widehat{(\rho_r X)} \cap \mathbb{C}^2$, we conclude that Theorem 2.1 holds with Δ replaced by $\frac{1}{r}\Delta$. □

Theorem 2.3. — *Theorem* 2.1 *remains valid without the assumption that X is connected, that is, it is valid when X is a finite union of real analytic simple closed curves in $\mathbb{C}^2$.*

Proof. — Write $X = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_N$ where each $\gamma_k \subset \mathbb{C}^2$ is a simple closed real analytic curve. Choose a neighborhood Ω of the unit circle K in $\mathbb{C}$ and complex analytic maps $(f_k, g_k) : \Omega_k \to \mathbb{C}^2$, $k = 1, \ldots, N$ whose restriction to K is a parameterization of γ_k. We now apply the following.

Lemma 2.8. — *Let Ω be a plane domain and K a compact subset of Ω. Let $\mathcal{L}$ be an algebra of holomorphic functions on Ω. Put $\|\phi\| = \sup_K |\phi|$ for all $\phi \in \mathcal{L}$. Fix $f_k, g_k \in \mathcal{L}$ for $k = 1, \dots, N$. Then there exist r, $0 < r < 1$ and $C > 0$ such that for each pair of positive integers (d, e) with $d+e > N$, we can find a unit polynomial $F_{d,e}$ of bidegree (d, e) such that*

$$\big\|F_{d,e}(f_k, g_k)\big\| \le C^{d+e} r^{\frac{de}{N}} \quad \text{for } k = 1, \dots, N. \tag{2.9}$$

Proof. — We fix a point $z_0 \in \Omega$ and choose $F_{d,e}$ so that $F_{d,e}(f_k, g_k)$ vanishes to order de/N at z_0 for all k. This is possible if $d + e > N$. We then proceed as in the proof of Lemma 2.2. □

One can now carry out the arguments given above for the case of one component. The only difference is that in the estimates, r_0^e will be replaced by $r_0^{e/N}$. □

3. The analyticity theorem

Let $\mathcal{O}(1) \to \mathbb{P}^n$ denote the holomorphic line bundle of Chern class 1 over complex projective n-space, endowed with its standard U(n+1)-invariant metric $\|\,.\,\|$. Following [**4**], we define the *projective hull* of a compact subset $X \subset \mathbb{P}^n$ to be the set $\widehat{X}$ of points $x \in \mathbb{P}^n$ for which there exists a constant $C = C_x$ such that

$$\|P(x)\| \le C_x^d \sup_X \|P\|. \tag{3.1}$$

for all holomorphic sections $P \in H^0(\mathbb{P}^n, \mathcal{O}(d))$ and all $d \ge 1$.

Note. — Recall that the holomorphic sections $H^0(\mathbb{P}^n, \mathcal{O}(d))$ correspond naturally to the homogeneous polynomials of degree d in homogeneous coordinates $[Z_0, \dots, Z_n]$ for $\mathbb{P}^n$. From this one can see (cf. [**4**, §6]) that if X is contained in an affine chart $\mathbb{C}^n \subset \mathbb{P}^n$, then $\widehat{X} \cap \mathbb{C}^n$ is exactly the "projective hull of X in $\mathbb{C}^n$" introduced in § 2. Moreover, the function M_ζ appearing in (2.1) can be taken to be $M_\zeta = \rho\sqrt{1 + \|\zeta\|^2} C_\zeta$ for $\zeta \in \widehat{X} \cap \mathbb{C}^n$, where ρ is a constant depending only on X.

For each $x \in \widehat{X}$ there is a *best constant* $C(x) \equiv \min\{C_x : (3.1) \text{ holds } \forall P\}$. The set X is called *stable* if the best constant function C is bounded on $\widehat{X}$. We know from [**4**, Prop. 10.2] that if X is stable, then $\widehat{X}$ is compact.

The point of this section is to prove the following projective version of the main theorem in [**9**].

Theorem 3.1. — *Let $\gamma \subset \mathbb{P}^n$ be a finite union of real analytic closed curves and assume γ is stable. Then $\widehat{\gamma} - \gamma$ is a one-dimensional complex analytic subvariety of $\mathbb{P}^n - \gamma$.*

Note. — When this conclusion holds, one can show that, in fact, $\widehat{\gamma}$ is the image of a compact Riemann surface with analytic boundary under a holomorphic map to $\mathbb{P}^n$. We will prove this in § 4.

Proof. — Assume to begin that $n = 2$. Since γ is real analytic, it is pluripolar, i.e., locally contained in the $\{-\infty\}$-set of a plurisubharmonic function (which is $\not\equiv -\infty$). Therefore, by [**4**, Cor. 4.4] we know that $\widehat{\gamma}$ is also pluripolar. In particular, it is nowhere dense. As noted above, $\widehat{\gamma}$ is closed by stability. Hence, we may choose a point $x \in \mathbb{P}^2$ and a ball B centered at x such that $\widehat{\gamma} \subset \mathbb{P}^2 - \overline{B}$. Let

$$\mathbb{P}^2 - \{x\} \xrightarrow{\pi} \mathbb{P}^1 \tag{3.2}$$

be linear projection with center x. This projection (3.2) is naturally a holomorphic line bundle $\cong \mathcal{O}(1)$, and

$$\mathbb{P}^2 - \overline{B} \xrightarrow{\pi} \mathbb{P}^1 \tag{3.3}$$

can be identified, after scalar multiplication by some constant $r > 0$, with its open unit disk bundle.

Cover $\mathbb{P}^1$ with two affine charts: $V_0 = \mathbb{P}^1 - \{0\}$ and $V_\infty = \mathbb{P}^1 - \{\infty\}$, and assume that $\gamma \cap \pi^{-1}(0) = \gamma \cap \pi^{-1}(\infty) = \varnothing$. By symmetry we may restrict attention to $\pi^{-1}(V_\infty)$. This chart has an identification

$$\pi^{-1}(V_\infty) \cong \mathbb{C}^2 = \{(z, w) : z, w \in \mathbb{C}\}$$

with the property that V_∞ maps linearly to the z-axis and π can be written as $\pi(z, w) = z$. The subset $\mathbb{P}^2 - \overline{B}$, intersected with this chart, is represented by

$$(\mathbb{P}^2 - \overline{B}) \cap \mathbb{C}^2 = \{(z, w) : |w|^2 \le |z|^2 + 1\}. \tag{3.4}$$

Set

$$\Omega \equiv \mathbb{C} - \pi(\gamma) \quad \text{and} \quad U \equiv \pi^{-1}(\Omega) = \mathbb{C}^2 - \pi^{-1}(\pi(\gamma)).$$

Proposition 3.1. — *Let $\gamma \subset \mathbb{C}^2$ be a stable real analytic curve with the property that*

$$\widehat{\gamma} \cap \mathbb{C}^2 \subset \{(z, w) : |w|^2 \le |z|^2 + 1\}. \tag{3.5}$$

Then $\widehat{\gamma} \cap U$ is a 1-dimensional complex analytic subvariety of U.

Proof. — Note to begin that since $\widehat{\gamma}$ is compact, condition (3.5) implies that

$$\pi : \widehat{\gamma} \cap U \longrightarrow \Omega \quad \text{is a proper map.} \tag{3.6}$$

Consider now the algebra A of functions on $\widehat{\gamma} \cap U$ given by restriction of the holomorphic functions on U, i.e.,

$$A \equiv \{f|_{\widehat{\gamma} \cap U} : f \in \mathcal{O}(U)\}.$$

We now claim that $(A, \widehat{\gamma} \cap U, \Omega, \pi)$ is a *maximum modulus algebra*, as defined in [**1**, p. 64]. Given (3.6) this means that we need only prove the following.

Lemma 3.1. — *For each $z_0 \in \Omega$ and each closed disk $D \subset \Omega$ centered at z_0, the equality*

$$|f(z_0, w_0)| \le \sup_{\widehat{\gamma}\cap\pi^{-1}(\partial D)} |f| \tag{3.7}$$

holds for all $f \in A$.

Proof. — By hypothesis (3.5) there exists an $R > 0$ such that

$$\widehat{\gamma} \cap \pi^{-1}(D) \subset D \times \Delta_{\frac{1}{2}R}$$

where $\Delta_r \equiv \{w : |w| \le r\}$. In particular, we have that

$$\widehat{\gamma} \cap \partial(D \times \Delta_R) = \widehat{\gamma} \cap (\partial D \times \Delta_R) = \widehat{\gamma} \cap \pi^{-1}(\partial D). \tag{3.8}$$

Now Theorem 12.8 in [**4**] states that

$$\widehat{\gamma} \cap \pi^{-1}(D) = \widehat{\gamma} \cap (D \times \Delta_R) \subset \text{polynomial hull of } \widehat{\gamma} \cap \partial(D \times \Delta_R).$$

Applying (3.8) gives

$$\widehat{\gamma} \cap \pi^{-1}(D) \subset \text{polynomial hull of } \widehat{\gamma} \cap \pi^{-1}(\partial D),$$

and Lemma 3.1 follows immediately. □

We have now shown that $(A, \widehat{\gamma} \cap U, \Omega, \pi)$ is a maximum modulus algebra. Furthermore, since $\widehat{\gamma}$ is stable, we know from Theorem 2.1 that there exists an $N > 0$ such that

$$\Omega(N) \equiv \{z \in \Omega : \#(\pi^{-1}(z) \cap \widehat{\gamma}) \le N\}$$

has positive measure. (Since $\Omega - \bigcup_N \Omega(N)$ has measure zero.) It now follows from Theorem 11.8 in [**1**] that:

(i) $\Omega = \Omega(N)$, and

(ii) there exists a discrete subset $\Lambda \subset \Omega$ such that $\widehat{\gamma} \cap \pi^{-1}(\Omega - \Lambda)$ has the structure of a Riemann surface on which every function in A is analytic.

Since A is the restriction of holomorphic functions on U to $\widehat{\gamma}$, condition (ii) implies that $\widehat{\gamma} \cap \pi^{-1}(\Omega - \Lambda)$ is a 1-dimensional complex analytic subvariety of $\pi^{-1}(\Omega - \Lambda) = U - \pi^{-1}(\Lambda)$.

It now follows that $\widehat{\gamma} \cap U$ is a 1-dimensional complex analytic subvariety of U. To see this, fix $z_0 \in \Lambda$ and choose a small closed disk $D \subset \Omega$ centered at z_0 with $D \cap \Lambda = \{z_0\}$. The arguments above show that $\widehat{\gamma} \cap \pi^{-1}(D)$ is contained in the polynomial hull of the real analytic curve $\widehat{\gamma} \cap \pi^{-1}(\partial D)$. Applying standard results [**1**, §12] proves Proposition 3.1. □

Proposition 3.1 together with the discussion preceding it, give the following.

Corollary 3.1. — *The set $\widehat{\gamma} - \pi^{-1}(\pi\gamma)$ is a complex analytic subvariety of dimension* 1 *in $\mathbb{P}^2 - \pi^{-1}(\pi\gamma)$.*

Observe that for every point $y \in \mathbb{P}^2 - \widehat{\gamma}$ there is a point $x \in \mathbb{P}^2 - \widehat{\gamma}$ such that $\pi(y) \notin \pi(\gamma)$ where π is the projection (3.2) with center x. Consequently, Corollary 3.1 proves Theorem 3.1 for the case $n = 2$.

Suppose now that $n = 3$ and choose $x \in \mathbb{P}^3 - \widehat{\gamma}$. The set of such x is open and dense since $\widehat{\gamma}$ is a compact pluripolar set of Hausdorff dimension 2 (cf. [**4**, Cor. 4.4 and Thm. 12.5]). Let $\Pi : \mathbb{P}^3 - \{x\} \to \mathbb{P}^2$ be the projection with center x. One sees easily that

$$\Pi(\widehat{\gamma}) \subseteq \widehat{\Pi\gamma},$$

and by the above $\widehat{\Pi\gamma} - \Pi\gamma$ is a complex analytic curve in $\mathbb{P}^2 - \Pi\gamma$. Standard arguments now show that $\widehat{\gamma} - \gamma$ is a complex analytic curve in $\mathbb{P}^3 - \gamma$. Proceeding by induction on n completes the proof of Theorem 3.1. $\square$

4. Boundary Regularity

The conclusion of Theorem 3.1 implies a strong regularity at the boundary. For future reference we include a discussion of this regularity.

Theorem 4.1. — *Let $\gamma \subset \mathbb{P}^n$ be a finite disjoint union of real analytic regular closed curves, and suppose V is a 1-dimensional complex analytic subvariety of the complement $\mathbb{P}^n - \gamma$. Then the closure*

$$\overline{V} = \bigcup_{j=1}^{m} \overline{V}_j \cup \bigcup_{k=m+1}^{\ell} \overline{V}'_k$$

where:

1) *Each V_j is a 1-dimensional complex analytic subvariety of finite area in $\mathbb{P}^n - \gamma$ whose closure $\overline{V}_j$ is an immersed variety in $\mathbb{P}^n$ with non-empty boundary $\partial\overline{V}_j = \gamma_j$ consisting of a union of components of γ. In particular, there exists a connected Riemann surface S_j, a compact subdomain $\overline{W}_j \subset S_j$ with real analytic boundary, and a generically injective holomorphic map*

$$\rho_j : S_j \longrightarrow \mathbb{P}^n \quad \textit{with} \quad \rho_j(\overline{W}_j) = \overline{V}_j$$

which is an embedding on a neighborhood of $\partial\overline{W}_j$ and has $\rho_j(\partial\overline{W}_j) = \gamma_j$.

2) *The closure of each V'_k is an irreducible algebraic curve in $\mathbb{P}^n$ with $\gamma_k \subset \mathrm{Reg}(\overline{V}'_k)$ where γ_k is a (possibly empty) finite union of components of γ.*

Note. — When γ is stable and $V = \widehat{\gamma}$, each γ_k is non-empty for $m < k \leq \ell$.

Theorem 4.1 can be put into a more succinct form.

Theorem 4.2. — *Let γ and V be as above. Then there exists a Riemann surface S (not necessarily connected), a compact subdomain $\overline{W} \subset S$ with real analytic boundary, and a holomorphic map $\rho : S \to \mathbb{P}^n$ which is generically injective and satisfies*

1) $\rho(\overline{W}) = \overline{V}$,
2) *ρ is an embedding on a tubular neighborhood of $\partial\overline{W}$ in S and*
3) *$\rho(\partial\overline{W})$ is a union of components of γ.*

Proof of Theorem 4.1. — We assume $n = 2$. The case of general n is similar.

We first note that V has finite area and finitely many irreducible components $V_1, \dots, V_\ell$. This follows from work of Shiffman, but can be seen directly as follows. Choose any $p \in \mathbb{P}^2 - \overline{V}$ and let $\pi : \mathbb{P}^2 - \{p\} \to \mathbb{P}^1$ be projection. Then $\pi_{|V}$ is finitely sheeted over $\mathbb{P}^1 - \pi(\gamma)$, and therefore V has finitely many components. In fact $\pi_{|V}$ must also be finitely sheeted over all of $\mathbb{P}^1$. To see this note that V can contain no fibre of π since $p \notin \overline{V} = V \cup \gamma$. Hence, the intersection $\pi^{-1}(x) \cap V$ for $x \in \pi(\gamma)$ is at most countable. If it were infinite, one easily sees that the sheeting number in contiguous domains of $\mathbb{P}^1 - \pi(\gamma)$ is unbounded. Choosing two distinct such projections and an easy estimate shows that the integral of the projective Kähler form on V is finite.

Now each irreducible component V_j of V defines a current $[V_j]$ by integration whose boundary is supported in γ. By the Federer Flat Support Theorem [**3**, 4.1.15],

$$\partial[V_j] = n_j[\gamma_j]$$

where $\gamma_j \equiv \operatorname{supp}\partial[V_j]$ is a union of connected components of γ (appropriately oriented) and $n_j \geq 0$ is a locally constant integer-valued function on γ_j. Order the V_j so that $\partial[V_j] \neq 0$ for $j = 1, \dots, m$ and $\partial[V_j] = 0$ for $j > m$.

Since γ is a regularly embedded real analytic curve, it has a complexification $\Sigma \supset \gamma$ which is a union of regularly embedded closed complex analytic annuli. Let Σ_j denote that part of Σ which is the complexification of γ_j for $j \leq m$. Write $\Sigma_j = \Sigma_j^+ \cup \gamma_j \cup \Sigma_j^-$ where $\Sigma_j^\pm$ are the components of $\Sigma_j - \gamma_j$ with signs chosen so that Σ^+ is the "outer strip", that is, so that

$$\partial\Sigma_j^+ = \gamma_j^+ - \gamma_j.$$

Consider the current $[V_j^*] \equiv [V_j] + n_j[\Sigma_j^+]$ which has

$$\partial[V_j^*] = n_j[\gamma_j^+].$$

The structure theorem of King [**8**] implies that $\operatorname{supp}[V_j^*]$ is a 1-dimensional subvariety of $\mathbb{P}^2 - \gamma_j^+$. It follows that V_j^* is an analytic continuation of V_j and in particular

$$n_j \equiv 1 \quad \text{and} \quad \Sigma_j^- \subset V_j.$$

Defining $\rho_j : S_j \to V_j^*$ to be the normalization of V_j^* and setting $\overline{W}_j = \rho^{-1}(\overline{V}_j)$ completes part 1) unless there exist $V_i \neq V_j$ which share some common boundary components. In this case $\overline{V}_i$ and $\overline{V}_j$ are analytic continuations of each other and can be combined into a single component of $\overline{V}$. Eliminating all common boundaries in this manner completes part 1).

Note that after fusing components, one may obtain algebraic curves which contain a non-empty union of components of γ in their regular locus. These will be listed in part 2). The remaining components of V (whose current boundaries are zero) are algebraic curves by King [**8**]. □

References

[1] H. ALEXANDER & J. WERMER – *Several complex variables and Banach algebras*, third ed., Graduate Texts in Math., vol. 35, Springer, 1998.

[2] E. BISHOP – "Holomorphic completions, analytic continuation, and the interpolation of semi-norms", *Ann. of Math.* **78** (1963), p. 468–500.

[3] H. FEDERER – *Geometric measure theory*, Die Grund. Math. Wiss., Band 153, Springer, New York, 1969.

[4] F. R. HARVEY & H. B. J. LAWSON – "Projective hulls and the projective Gelfand transform", *Asian J. Math.* **10** (2006), p. 607–646.

[5] ———, "Projective linking and boundaries of positive holomorphic chains in projective manifolds, part II", in *"Inspired by S. S. Chern", Nakai Tracts in Mathematics*, vol. 11, World Scientific Publishing Co., Singapore, 2006.

[6] ———, "Relative holomorphic cycles and duality", Stony Brook preprint, 2006.

[7] ———, "Projective linking and boundaries of positive holomorphic chains in projective manifolds, part I", in *The Many Facets of Geometry: a Tribute to Nigel Hitchin* (J. P. Bourguignon, O. Garcia-Prada & S. Salamon, eds.), Oxford University Press, 2008.

[8] J. R. KING – "The currents defined by analytic varieties", *Acta Math.* **127** (1971), p. 185–220.

[9] J. WERMER – "The hull of a curve in $\mathbb{C}^n$", *Ann. of Math.* **68** (1958), p. 550–561.

[10] ———, *Banach algebras and several complex variables*, Markham Publishing Co., Chicago, Ill., 1971.

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