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ASTÉRISQUE

2009

FROM PROBABILITY TO GEOMETRY (I)

VOLUME IN HONOR OF THE 60th BIRTHDAY OF JEAN-MICHEL BISMUT

Xianzhe DAI, Rémi LÉANDRE, Xiaonan MA and Weiping ZHANG, editors

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Astérisque
327, 2009, p. 1–16

SEMI-CLASSICAL LIMIT OF THE LOWEST EIGENVALUE OF A SCHRÖDINGER OPERATOR ON A WIENER SPACE: I. UNBOUNDED ONE PARTICLE HAMILTONIANS

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60^{th} birthday

Abstract. — We study a semi-classical limit of the lowest eigenvalue of a Schrödinger operator on a Wiener space. The Schrödinger operator is a perturbation of the second quantization operator of an unbounded self-adjoint operator by a C^3-potential function. This result is an extension of [**1**].

Résumé **(Limite semi-classique de la plus petite valeur propre d'un opérateur de Schrödinger sur l'espace de Wiener: cas d'un Hamiltonien non borné à une particule.)**
Nous étudions le comportement semi-classique de la plus petite valeur propre d'un opérateur de Schrödinger sur l'espace de Wiener. L'opérateur de Schrödinger est obtenu par perturbation de l'opérateur de seconde quantification associé à un opérateur non-borné autoadjoint donné par un potentiel C^3. Ce résultat est une extension de [**1**].

1. Introduction

In [**1**], we studied the semi-classical limit of the lowest eigenvalue of Schrödinger operators which are perturbations of the number operator. In that case, one particle Hamiltonian (the coefficient operator of the second order differential operator) is identity operator. However, we need to study the case where the coefficient operator is unbounded to study $P(\phi)$-type Hamiltonians. For example, the typical coefficient operator is $\sqrt{m^2 - \Delta}$, where $m > 0$ and Δ is the Laplace-Bertlami operator on $\mathbb{R}$. In this paper, we study the asymptotics of the lowest eigenvalue of a Schrödinger operator in the case where the coefficient operator is unbounded linear operator and the potential function is C^3. In $P(\phi)$-type model cases, the potential functions are defined by using a renormalization and they are not continuous. In [**2**], we studied

2010 Mathematics Subject Classification. — 81Q20, 35J10, 35P15, 60H07.
Key words and phrases. — Semi-classical limit, Quantum field theory, Schrödinger operator, $P(\phi)$-type Hamiltonian.

Schrödinger operators on path spaces over Riemannian manifolds. In that case, the differential operators are variable coefficient ones and the coefficient operators are not bounded linear because they contain stochastic integrals. Moreover, the dependence on the path of the coefficients are discontinuous in the natural topology. The discontinuity comes from the discontinuity of solutions of stochastic differential equations as a functional of Brownian motion. Thus, we need to consider two kind of discontinuity for potential functions and coefficient operators in that case. But, the difficulties are different from that of the $P(\phi)$-type potentials. We will study semi-classical limit of the lowest eigenvalue of a $P(\phi)_2$-Hamiltonian on a finite interval in [**3**].

2. Preliminaries

Let (W, H, μ) be an abstract Wiener space. That is,

(i) H is a separable Hilbert space and W is a separable Banach space. Moreover H is continuously and densely embedded into W,

(ii) μ is the unique Gaussian measure on W such that for any $\varphi \in W^*$,

$$\int_W e^{\sqrt{-1}\varphi(w)} d\mu(w) = e^{-\frac{1}{2}\|\varphi\|_H^2}.$$

Here we use the natural inclusion and the identification by the Riesz theorem $W^* \subset H^* \simeq H$.

In this paper, we assume that W is a Hilbert space. This is equivalent to that there exists a positive self-adjoint trace class operator S such that W is a completion of H with respect to the Hilbert norm $\|\sqrt{S}h\|_H$. That is, $\|h\|_W = \|\sqrt{S}h\|_H$ for all $h \in H$. We denote the sets of bounded linear operators, Hilbert-Schmidt operators, trace class operators on H by $L(H), L_1(H), L_2(H)$. Also we denote their operator norms, trace norms, Hilbert-Schmidt norms by $\| \|, \| \|_1, \| \|_2$, respectively. For $\lambda > 0$, we define the new measure μ_λ on W by $\mu_\lambda(E) = \mu\left(\sqrt{\lambda}E\right)$ $(E \subset W)$. Now we define our Schrödinger operators.

Definition 2.1. — *Let A be a strictly positive self-adjoint operator on H. That is, we assume that $\inf \sigma(A) > 0$, where $\sigma(A)$ denotes the spectral set of A. We denote $c_A = \inf \sigma(A^2)$. We denote by $\mathfrak{F}C_A^\infty(W)$ the space of all smooth cylindrical functions $f(w) = F(\varphi_1(w), \ldots, \varphi_n(w))$ $(F \in C_b^\infty(\mathbb{R}^n), \varphi_i \in W^* \cap_{n\in\mathbb{N}} \mathrm{D}(A^n))$. For such a f, we define $Df(w) = \sum_{i=1}^n \partial_i F(w)\varphi_i \in H$. Here we use the identification $\varphi_i \in W^* \subset H^* \simeq H$ and $\partial_i F(w)$ denotes the partial derivative with respect to the i-th variable. Moreover we define $D_A f(w) = \sum_{i=1}^n \partial_i F(w) A\varphi_i$. We define a Dirichlet form on $L^2(W, d\mu_\lambda)$ by $\mathcal{E}_{\lambda,A}(f,f) = \int_W \|D_A f(w)\|_H^2 d\mu_\lambda(w)$. $-L_{\lambda,A}$ denotes the generator. Let V be a real-valued measurable function on W such that $V \in \cap_{\lambda>0} L^1(W, \mu_\lambda)$. Under the assumption that for all $\lambda > 0$, $\mathcal{E}_{\lambda,A,V}(f,f) =$*

$\mathcal{E}_{\lambda,A}(f,f) + \int_W \lambda^2 V(w) f(w)^2 d\mu_\lambda(w)$ $(f \in \mathfrak{F}C_A^\infty(W))$ *is a lower bounded symmetric form, we denote the generator of the smallest closed extension by* $-L_{\lambda,A,V}$. *Also let* $E_0(\lambda, A, V) = \inf \sigma(-L_{\lambda,A,V})$.

Remark 2.2. — (1) $-L_{\lambda,A}$ can be viewed as the second quantization of A^2 on H. Let $H = H^{1/2}(\mathbb{R})$ be the Hilbert space with the norm $\|h\|_H^2 = \int_{\mathbb{R}} |(m^2-\Delta)^{1/4}h(x)|^2 dx$, where $m > 0$. Consider $A = (m^2-\Delta)^{1/4}$ on H. In this case, $-L_{1,A}$ is the time 0 field free Hamiltonian in $P(\phi)_2$-model. However note that $-L_{1,A}$ is usually identified with the second quantization of $\sqrt{m^2-\Delta}$ on $H^* = H^{-1/2}(\mathbb{R})$. See also Example 3.3.
(2) In [**1, 5**], the Schrödinger operator with semi-classical parameter λ is defined in a different way. Let $V_\lambda(w) = \lambda V\left(\frac{w}{\sqrt{\lambda}}\right)$. The semi-classical limit of $-L_{1,A} + V_\lambda$ on $L^2(W, d\mu)$ is studied in the above papers. However note that this operator is unitarily equivalent to $-L_{\lambda,A,V}/\lambda$ on $L^2(W, \mu_\lambda)$. We adopt the similar definition to $-L_{\lambda,A,V}$ in the case of Schrödinger operators on path spaces over Riemannian manifolds because the scaling $w/\sqrt{\lambda}$ can not defined on the curved spaces but the measure corresponding to μ_λ can be defined on curves spaces too. See Remark 5.3 in [**1**] and [**2**].

Let us introduce the following assumptions on potential functions of Schrödinger operators.

Assumption 2.3. — The following assumptions (A1), (A2) are standard in semi-classical analysis. (A4) assures that the symmetric form $\mathcal{E}_{\lambda,A,V}$ is bounded from below by Corollary 2.8 (2). Note that (A5) implies that A is an unbounded operator.

(A1) V is a C^2-function on H. Let $U(h) = \frac{1}{4}\|Ah\|_H^2 + V(h)$ $(h \in \mathrm{D}(A))$. Then $\min_{h\in \mathrm{D}(A)} U(h) = 0$ and the zero point set is a finite set $N = \{h_1, \ldots, h_n\}$.
(A2) $\frac{1}{2}D^2U(h_i) = \frac{1}{4}A^2 + K_i$ is a strictly positive self-adjoint operator on H, where $K_i = \frac{1}{2}D^2V(h_i) \in L(H,H)$.
(A3) V can be extended to a C^3-function on W such that for any $R > 0$ and $0 \le k \le 3$

$$\sup\left\{\|D^kV(w)\|_{L(W\times\cdots\times W,\mathbb{R})} \mid \|w\|_W \le R\right\} \le C(R) < \infty.$$

(A4) V can be extended to a continuous function on W and there exists $p > 1$ such that

$$\limsup_{\lambda\to\infty} \lambda^{-1}\log\int_W e^{-\frac{2p\lambda}{c_A}V(w)} d\mu_\lambda(w) < \infty,$$

(A5) There exists $\gamma_0 > 1$ such that $A^{-\gamma_0} \in L_2(H)$.

For $r > 0$ and $z \in W, k \in H$, we denote $B_r(z) = \{w \in W \mid \|w-z\|_W \le r\}$ and $B_{r,H}(k) = \{h \in H \mid \|h-k\|_H \le r\}$.

Lemma 2.4. — (1) *Suppose that* (A4) *holds or* $\inf\{V(h) \mid h \in H\} > -\infty$. *Then we have* $\displaystyle\lim_{\|h\|_H\to\infty}\left(\frac{c_A}{4}\|h\|_H^2 + V(h)\right) = +\infty$.

(2) *Assume* (A1), *the same assumptions in* (1) *and for any* $L > 0$, $\sup\{|V(h)| \mid \|h\|_H \leq L\} < \infty$. *Then for any* $\varepsilon > 0$,

$$\kappa(\varepsilon) := \inf\left\{U(h) \mid h \in \{\cup_{i=1}^n B_\varepsilon(h_i)\}^c\right\} > 0.$$

Proof. — (1) If $\inf\{V(h) \mid h \in H\} > -\infty$, the statement is trivial. We assume (A4). Let C be a positive number such that $\limsup_{\lambda\to\infty} \lambda^{-1}\log\int_W e^{-\frac{2p\lambda}{c_A}V} d\mu_\lambda < C$. Take $R > 0$. Then for sufficiently large λ, we have

$$\begin{aligned}
&\frac{1}{\lambda}\log\int_W \exp\left(-\frac{2p\lambda}{c_A}\left(R\wedge V(w)\vee(-R)\right)\right) d\mu_\lambda(w)\\
&\quad\leq \frac{1}{\lambda}\log\left(\int_W\left(e^{-\frac{2p\lambda}{c_A}R} + \exp\left(-\frac{2p\lambda}{c_A}(V(w)\vee(-R))\right) d\mu_\lambda(w)\right)\right)\\
&\quad\leq \frac{1}{\lambda}\log\left(e^{\lambda C} + e^{-\frac{2p\lambda}{c_A}R}\right) \leq C + \frac{\log 2}{\lambda}.
\end{aligned}$$

By the Large deviation estimate, we have

$$\sup_h\left(-\frac{1}{2}\|h\|_H^2 - \frac{2p}{c_A}\left((-R)\vee V(h)\wedge R\right)\right) \leq C.$$

Since R is an arbitrary number, we get

$$-\frac{c_A}{4}\|h\|_H^2 - pV(h) \leq \frac{C\cdot c_A}{2} \quad \text{for all } h \in H.$$

Suppose that there exists $\{h_n\}$ such that $\|h_n\|_H \to \infty$ and $\sup_n\left(\frac{c_A}{4}\|h_n\|_H^2 + V(h_n)\right) =: l < +\infty$. Then $\lim_{n\to\infty} V(h_n) = -\infty$. Hence

$$\frac{c_A}{4}\|h_n\|_H^2 + pV(h_n) = \frac{c_A}{4}\|h_n\|_H^2 + V(h_n) + (p-1)V(h_n) \leq l + (p-1)V(h_n) \to -\infty.$$

This is a contradiction. So we are done.

(2) By the result in (1), we need to prove that for sufficiently large positive number L,

$$\inf\{U(h) \mid h \in B_{L,H}(0)\cap(\cup_{i=1}^n B_\varepsilon(h_i))^c\} > 0.$$

Suppose that there exists $\{\varphi_l\} \subset B_{L,H}(0)\cap(\cup_{i=1}^n B_\varepsilon(h_i))^c$ such that $\lim_{l\to\infty} U(\varphi_l) = 0$. By the assumption, there exists a subsequence $\{\varphi_{l(i)}\}$ which converges to a certain element $\varphi_\infty \in H$ weakly. Since $\frac{1}{4}\|A\varphi_{l(i)}\|_H^2 = U(\varphi_{l(i)}) - V(\varphi_{l(i)})$, $\sup_i \|A\varphi_{l(i)}\|_H < \infty$ holds. Hence again by choosing a subsequence $\{\varphi_{p(i)}\}$, $A\varphi_{p(i)}$ also converges to some ϕ_∞ weakly. By the Banach-Saks theorem, we see that $\varphi_\infty \in \mathrm{D}(A)$ and $A\varphi_\infty = \phi_\infty$. On the other hand, since the embedding $H \subset W$ is compact, $\lim_{i\to\infty}\|\varphi_{p(i)} - \varphi_\infty\|_W = 0$ which implies $\lim_{i\to\infty} V(\varphi_{p(i)}) = V(\varphi_\infty)$. Since $\|A\varphi_\infty\|_H^2 \leq \liminf_{i\to\infty}\|A\varphi_{p(i)}\|_H^2$, we obtain $U(\varphi_\infty) \leq \liminf_{i\to\infty} U(\varphi_{p(i)}) = 0$. This implies $\varphi_\infty \in N$ and $\varphi_{p(i)} \in B_\varepsilon(h_j)$ for some large i and $1 \leq j \leq n$. This is a contradiction. □

Lemma 2.5. — *Let A be a strictly positive self-adjoint operator and K be a trace class self-adjoint operator on H. Assume that A^2+K is also a strictly positive operator. Then $\sqrt{A^2+K}-A \in L_1(H)$ and*

$$\left\|\sqrt{A^2+K}-A\right\|_1 \leq \frac{\|K\|_1}{\min\left\{\inf\sigma(\sqrt{A^2+K}), \inf\sigma(A)\right\}}.$$

Proof. — We prove this in three steps: (i) $A = I+T$ and T is a trace class operator, (ii) A is a bounded linear operator, (iii) General cases.

(i) We denote $S_1 = \sqrt{A^2+K}$ and $S_0 = A$. Note that $S_1 - S_0 = \sqrt{A^2+K}-A$ is a trace class operator. We denote the all eigenvalues and corresponding complete orthonormal system of S_1-S_0 by $\{\alpha_n\}$ and $\{e_n\}$. Then

$$\begin{aligned} |(Ke_n,e_n)| &= |\left((S_1^2-S_0^2)e_n,e_n\right)| \\ &= \left|\left((S_1(S_1-S_0)+(S_1-S_0)S_1-(S_1-S_0)^2\right)e_n,e_n\right)\right| \\ &= |\alpha_n\left((S_1+S_0)e_n,e_n\right)| \\ &\geq |\alpha_n|\inf\sigma(S_1+S_0). \end{aligned}$$

This implies that

$$\|\sqrt{A^2+K}-A\|_1 = \sum_{n=1}^{\infty}|\alpha_n| \leq \frac{\|K\|_1}{\inf\sigma(\sqrt{A^2+K}+A)}.$$

(ii) Let $\{u_m\}$ be all eigenvectors of K which is a c.o.n.s. of H. Set $P_m h = \sum_{i=1}^m (h,u_i)u_i$ and $A_m = \sqrt{P_mA^2P_m + P_m^\perp}$. Then $A_m^2 \to A^2$, $A_m \to A$ converge strongly. On the other hand, $A_m^2+K = P_m(A^2+K)P_m + P_m^\perp(I_H + P_m^\perp K P_m^\perp)P_m^\perp$. Hence for sufficiently large m, we have

$$\min\left\{\inf\sigma(\sqrt{A_m^2+K}), \inf\sigma(A_m)\right) \geq \min\left(\inf\sigma(\sqrt{A^2+K}), 1/2, \inf\sigma(A)\right).$$

Since $A_m - I_H$ is a trace class operator, by (i),

$$\|\sqrt{A_m^2+K}-A_m\|_1 \leq \frac{\|K\|_1}{\min\left(\inf\sigma(A^2+K), \inf\sigma(A), 1/2\right)}.$$

By taking the limit $m\to\infty$, we see that $\sqrt{A^2+K}-A \in L_1(H)$. Therefore again by the same argument as in (i), we can prove (ii).

(iii) Let $\chi_n(x)$ be a function such that $\chi_n(x) = 1$ for $x \leq n$ and $\chi_n(x) = 0$ for $x > n$. Then $\chi_n(A)$ is a projection operator which commutes with A. Let $A_n = A\chi_n(A) + (1-\chi_n(A))$ and $K_n = \chi_n(A)K\chi_n(A)$. Then

$$\begin{aligned} \sqrt{A^2+K_n}-A &= \sqrt{A^2\chi_n(A)+\chi_n(A)K\chi_n(A)} - A\chi_n(A) \\ &= \sqrt{A_n^2+K_n} - A_n \in L(\mathrm{Im}(\chi_n(A))) \end{aligned}$$

By (ii), we have

$$\text{(2.1)}\quad \|\sqrt{A^2+K_n}-A\|_1 \le \frac{\|K_n\|_1}{\inf\sigma\left(\sqrt{A^2\chi_n(A)+\chi_n(A)K\chi_n(A)}+A\chi_n(A)\right)} \le \frac{\|K_n\|_1}{\min\left(\inf\sigma(\sqrt{A^2+K}),\inf\sigma(A)\right)}.$$

For $l>n>m$,

$$\begin{aligned}\left(\sqrt{A_n^2+K_n}-A_n\right)-\left(\sqrt{A_m^2+K_m}-A_m\right) &= \sqrt{A^2+K_n}-\sqrt{A^2+K_m}\\ &= \sqrt{A_l^2+K_n}-\sqrt{A_l^2+K_m}.\end{aligned}$$

This and (ii) implies that $\sqrt{A_n^2+K_n}-A_n$ converges in the trace norm. It is not difficult to check that the strong limit is equal to $\sqrt{A^2+K}-A$. Therefore, (2.1) implies the conclusion. □

Proposition 2.6. — *Let A be a strictly positive self-adjoint operator. For a trace class self-adjoint operator K on H and $h\in \mathrm{D}(A^2)$, we set*

$$V_{K,h}(w)=\frac{1}{4}\|Ah\|_H^2-\frac{1}{2}(A^2h,w)+(K(w-h),w-h)\,.$$

We assume that A^2+4K is a strictly positive self-adjoint operator and AKA can be extended to a trace class operator. Then $\mathcal{E}_{\lambda,A,V_{K,h}}$ is a symmetric form bounded from below and $E_0(\lambda,A,V_{K,h})=\lambda e(A,K)$ holds, where

$$\text{(2.2)}\qquad e(A,K)=\frac{1}{2}\mathrm{tr}\left(\sqrt{A^4+4AKA}-A^2\right).$$

Moreover it is the lowest eigenvalue of $-L_{\lambda,A,V_{K,h}}$ and the corresponding normalized positive eigenfunction is

$$\begin{aligned}\Omega_{\lambda,A,V_{K,h}}(w) &= \det\left(I_H+T_K\right)^{1/4}\\ &\quad\times\exp\left\{-\frac{\lambda}{4}\left(\left(A^{-1}\{A^4+4AKA\}^{1/2}A^{-1}-I_H\right)(w-h),(w-h)\right)\right\}\\ &\quad\times\exp\left(\frac{\lambda}{2}(h,w)-\frac{\lambda}{4}\|h\|_H^2\right),\end{aligned}$$

where $T_K=A^{-1}(\sqrt{A^4+4AKA}-A^2)A^{-1}$.

Proof. — If A is bounded linear operator, the proof is a straightforward calculation. Suppose that A is unbounded. Let A_n and K_n be the operators which are defined in the proof of (iii) in Lemma 2.5. Then $AK_nA=A_nK_nA_n$. Thus $\left(A^{-1}\{A^4+4AK_nA\}^{1/2}A^{-1}-I_H\right)\in L_1(H)\cap_k \mathrm{D}(A^k)$. Therefore for sufficiently large n, $\Omega_{\lambda,A,V_{K_n,h}}\in L^2(\mu_\lambda)$ and the simple calculation shows that

$$-L_{\lambda,A,V_{K_n,h}}\Omega_{\lambda,A,V_{K_n,h}}=\lambda e(A,K_n)\Omega_{\lambda,A,V_{K_n,h}}.$$

Letting $n \to \infty$, we have

$$-L_{\lambda,A,V_{K,h}}\Omega_{\lambda,A,V_{K,h}} = \lambda e(A,K)\Omega_{\lambda,A,V_{K,h}}.$$

To prove that $\lambda e(A,K) = \inf \sigma\left(-L_{\lambda,A,V_{K,h}}\right)$, we note that for any $f \in \mathfrak{F}C_A^\infty(W)$, it holds that

$$\mathscr{E}_{\lambda,A,V_{K,h}}(f,f) = \int_W \|D_A(f\Omega^{-1}_{\lambda,A,V_{K,h}})\|_H^2 \Omega_{\lambda,A,V_{K,h}}(w)^2 d\mu_\lambda(w) + \lambda e(A,K)\|f\|^2_{L^2(\mu_\lambda)}. \quad \square$$

We use the following estimate to prove a lower bound in Lemma 3.4. We refer the reader to [**7, 12, 14**] for this estimate.

***Theorem 2.7* (NGS estimate).** — *Let $\mathscr{E}(f,f)$ be a closed form on $L^2(X,m)$, where $(X,\mathscr{F},m)$ is a probability space. Assume that there exists $\alpha > 0$ such that for any $f \in \mathrm{D}(\mathscr{E})$,*

$$\int_X f(x)^2 \log\left(f(x)^2/\|f\|^2_{L^2(X,m)}\right) dm(x) \le \alpha\mathscr{E}(f,f).$$

Then for any bounded measurable function V, it holds that

$$\mathscr{E}(f,f) + \int_X V(x)f(x)^2 dm(x) \ge -\frac{1}{\alpha}\log\left(\int_X e^{-\alpha V(x)}dm(x)\right)\|f\|^2_{L^2(X,m)}. \tag{2.3}$$

The following follows from the above estimate and Gross's logarithmic Sobolev inequality [**7**]: For any $f \in \mathfrak{F}C_I^\infty(W)$,

$$\int_W f(w)^2 \log\left(f(w)^2/\|f\|^2_{L^2(\mu_\lambda)}\right) d\mu_\lambda(w) \quad \le \quad \frac{2}{\lambda}\int_W \|Df(w)\|_H^2 d\mu_\lambda(w).$$

Originally NGS(=Nelson, Glimm, Segal) estimate (2.3) was proved by the hypercontractivity of the corresponding semigroup. See [**14**]. Corollary 2.8 (2) is proved by Lemma 4.5 in [**2**] which follows from Gross's log-Sobolev inequalities and finite dimensional approximations.

***Corollary 2.8*.** — (1) *It holds that*

$$E_0(\lambda,A,V) \ge -\frac{\lambda c_A}{2}\log\left(\int_W \exp\left(-\frac{2\lambda}{c_A}V\right)d\mu_\lambda(w)\right).$$

(2) *Suppose that there exists a Hilbert-Schmidt operator T such that $A = I + T$. Then*

$$\begin{aligned} &E_0(\lambda,A,V) \\ &\quad \ge -\frac{\lambda}{2}\log\left\{\int_W \exp\left(-2\lambda V(w) - \lambda : (Tw,w) :_{\mu_\lambda} - \frac{\lambda}{2}\|Tw\|_H^2\right)d\mu_\lambda(w)\right\} \\ &\qquad + \frac{\lambda}{2}\log\det{}_{(2)}(I_H + T) - \frac{\lambda}{2}\mathrm{tr}\left(T^2\right). \end{aligned} \tag{2.4}$$

In (2.4), $:(Tw,w):_{\mu_\lambda}$ is defined by the limit $\lim_{n\to\infty}\left\{(P_nTP_nw,w)-\frac{1}{\lambda}\mathrm{tr}P_nTP_n\right\}$, where P_n is a projection on to a finite dimensional subspace of H such that $P_n\uparrow I_H$. $\det_{(2)}$ denotes the Carleman-Fredholm determinant.

3. Results

***Theorem 3.1* (Bounded case).** — *We assume that A is a bounded linear operator and satisfies the assumptions* (A1), (A2), (A3), (A4). *Then we have*

$$\lim_{\lambda\to\infty}\frac{E_0(\lambda,A,V)}{\lambda}=\min_{1\le i\le n}e(A,K_i). \tag{3.1}$$

In the unbounded case, we can prove the following. The assumption is too strong to cover the $P(\phi)$-type Hamiltonian. We will relax the assumptions and discuss such a case in a separate paper.

***Theorem 3.2* (Unbounded case).** — *Assume* (A5). *Let $\gamma\ge 1+\gamma_0$ and $S=A^{-2\gamma}$. Then AK_iA is a trace class operator and* (2.2) *is well-defined. Furthermore, we assume that* (A1), (A2), (A3), (A4) *hold. Then the asymptotics* (3.1) *holds.*

Example 3.3. — Let $I=[-\frac{l}{2},\frac{l}{2}]$ $(l>0)$ be an interval of $\mathbb{R}$. Let $-\Delta$ be the Laplacian with periodic boundary condition on $X=L^2(I\to\mathbb{R},dx)$. Let $m>0$. For $\alpha\in\mathbb{R}$, let $H^\alpha=D((m^2-\Delta)^{\alpha/2})$ and $\|h\|_{H^\alpha}=\|(m^2-\Delta)^{\alpha/2}h\|_X$.
(1) Let $H=H^{1/2}$. Then for any $\varepsilon>0$, we can take $W=H^{-\varepsilon}$. Let $0<\varepsilon<1/2$. Then using the inclusion and the identification $H^{1/2}\subset H^\varepsilon=(H^{-\varepsilon})^*$, we can see that μ satisfies that $\int_{W\ H^{-\varepsilon}}(w,h)^2_{H^\varepsilon}\,d\mu(w)=\|(m^2-\Delta)^{-1/4}h\|_X^2$ for $h\in H$. Let $U:X\to H^{1/2}$ be the natural isometry operator and define $A=U(m^2-\Delta)^{1/4}U^{-1}$. This is a standard example in $P(\phi)_2$-model on finite interval. Let $P(u)=\sum_{k=0}^{2M}a_ku^k$ be a polynomial with real coefficients with $a_{2M}>0$. For $h\in H$, $\tilde V(h)=\int_I P(h(x))dx$ is well-defined by the Sobolev embedding theorem. However $H^{-\varepsilon}$ is the space of distribution and $P(w(x))$ is not defined for $w\in H^{-\varepsilon}$. Actually, it should be defined by $\int_I:P(w(x)):_{\mu_\lambda}dx$ where $:P(w(x)):$ denotes the Wick product. However this is not a smooth function on $W=H^{-\varepsilon}$ and cannot be covered by Theorem 3.2. This will be studied in [**3**].
(2) Let $H=H^2$. Then μ can be defined on $W=H^1$. For $0<\delta<1/2$, let $A=U(m^2-\Delta)^{\frac{1}{2}(\frac{1}{2}-\delta)}U^{-1}$, where U is the natural isometry from X to H. Let $Q(u)=\frac{1}{4}m^{1-2\delta}u^2+P(u)$, where $P(u)$ is the polynomial defined in (1). Let $\{c_1,\dots,c_n\}$ be the minimum points of Q and asssume that $Q''(c_i)>0$ $(1\le i\le n)$. Again let $\tilde V(h)=\int_I P(h(x))dx$ for $h\in H$. Then we see that $\tilde V(h)-l\min Q$ can be extended to a smooth function $V(w)$ on W. Then the zero point set of $U(h)=\frac{1}{4}\|Ah\|_H^2+V(h)$ is the set of the constant functions $\{c_1,\dots,c_n\}$. For this V and A, all assumptions in Theorem 3.2 hold with $\gamma_0=1+\frac{4\delta}{1-2\delta}$ and $\gamma=1+\gamma_0$.

We prove these theorems after preparations. Here we just prove $AK_iA \in L_1(H)$ under (A5). Since $V \in C^2(W)$, there exists a bounded linear operator $\hat{K}_i$ on W such that $D^2V(h_i)(u,v) = \left(\hat{K}_i u, v\right)_W$ for any $u,v \in W$. By the definition of the norm of W, there exists $\tilde{K}_i \in L(H)$ such that $\hat{K}_i = A^{\gamma}\tilde{K}_i A^{-\gamma}$. Thus for any $u,v \in H \subset W$,

$$D^2V(h_i)(u,v) = \left(\hat{K}_i u, v\right)_W = \left(A^{-\gamma}A^{\gamma}\tilde{K}_i A^{-\gamma}u, A^{-\gamma}v\right)_H = \left(A^{-\gamma}\tilde{K}_i A^{-\gamma}u, v\right)_H.$$

This shows $K_i = A^{-\gamma}\tilde{K}_i A^{-\gamma}$ and $AK_iA = A^{1-\gamma}\tilde{K}_i A^{1-\gamma}$. Because $\gamma - 1 \geq \gamma_0$, $A^{1-\gamma}$ is a Hilbert-Schmidt operator and this implies AK_iA is a trace class operator on H.

In our main theorems, we may assume that $c_A = 1$. Because, if Theorems hold in the case where $c_A = 1$, then it implies that $E_0\left(\lambda, \frac{A}{\sqrt{c_A}}, \frac{V}{c_A}\right) = e\left(\frac{A}{\sqrt{c_A}}, \frac{V}{c_A}\right)$. This shows the general cases.

The proof of upper bound is standard. Let χ be a smooth function on $\mathbb{R}$ satisfying $0 \leq \chi(x) \leq 1$, $\chi(x) = 1$ for $x \in [-1,1]$ and $\chi(x) = 0$ for $|x| \geq 2$. For $2/3 < \delta < 1$, set

$$\tilde{\Omega}_{\lambda,A,V_{K_i,h_i}}(w) = Z_\lambda \Omega_{\lambda,A,V_{K_i,h_i}}(w)\chi\left(\lambda^{\delta}\|w-h_i\|_W^2\right).$$

Here Z_λ is a normalization constant which makes the L^2-norm to be equal to 1. It holds that $\lim_{\lambda\to\infty} Z_\lambda = 1$. Since h_i is a minimizer of U, for any $k \in \mathrm{D}(A)$, $\frac{1}{2}(Ah_i, Ak)_H + DV(h_i)(k) = 0$. The fact $DV(h_i) \in H^*$ implies that $h_i \in \mathrm{D}(A^2)$ and $DV(h_i) = -\frac{1}{2}A^2h_i$. Using this and by the Taylor expansion, we have

$$\begin{aligned}
(3.2) \qquad V(w) &= V(h_i) + DV(h_i)(w-h_i) + (K_i(w-h_i), w-h_i) \\
&\quad + \frac{1}{3!}DV^3(w+\theta(w-h_i))((w-h_i)^{\otimes 3}) \\
&= \frac{1}{4}\|Ah_i\|_H^2 - \frac{1}{2}\left(A^2h_i, w\right) + (K_i(w-h_i), w-h_i) + R_{h_i}(w) \\
&= V_{K_i,h_i}(w) + R_{h_i}(w).
\end{aligned}$$

Here we denote the remainder term by $R_{h_i}(w)$. If $\chi(\lambda^{\delta}\|w-h_i\|_W^2) \neq 0$, then $|R_{h_i}(w)| \leq C\lambda^{-3\delta/2}$. This and the tail estimate of the Gaussian measure shows that

$$\mathscr{E}_{\lambda,A,V}\left(\tilde{\Omega}_{\lambda,A,V_{K_i,h_i}}, \tilde{\Omega}_{\lambda,A,V_{K_i,h_i}}\right) = E_0(\lambda, A, K_i) + O(\lambda^{2-\frac{3}{2}\delta}).$$

This proves the upper bound.

To prove the lower bound estimates, it suffices to prove the following Lemma 3.4. Let R be a sufficiently large positive number. Set $\chi_{i,R}(w) = \chi\left(R\|w-h_i\|_W^2\right)$ $(1 \leq i \leq n)$ and $\chi_{0,R}(w) = \sqrt{1-\sum_{i=1}^n \chi_{i,R}(w)^2}$.

Lemma 3.4. — *Let us assume that the conditions of either Theorem* 3.1 *or Theorem* 3.2 *hold.*

(1) *There exists a constant* $C > 0$ *such that for all* i, $\chi_{i,R} \in \mathrm{D}(D_A)$ *and* $\|D_A\chi_{i,R}(w)\|_H^2 \le CR$ μ_λ*-a.e.* w. *Moreover it holds that*

$$\begin{aligned}(3.3)\quad & \mathscr{E}_{\lambda,A,V}(f,f)\\ &= \sum_{i=0}^{n} \mathscr{E}_{\lambda,A,V}(f\chi_{i,R}, f\chi_{i,R}) - \sum_{i=0}^{n}\int_W \|D_A\chi_{i,R}(w)\|_H^2 f(w)^2 d\mu_\lambda(w).\end{aligned}$$

(2) *For* $1 \le i \le n$,

$$\mathscr{E}_{\lambda,A,V}(f\chi_{i,R}, f\chi_{i,R}) \quad \ge \quad \lambda(1+g(\lambda))e(A,K_i)\|f\chi_{i,R}\|_{L^2(\mu_\lambda)}^2,$$

where $\lim_{\lambda\to\infty} g(\lambda) = 0$.

(3) *There exists a constant* $C > 0$ *such that*

$$\mathscr{E}_{\lambda,A,V}(f\chi_{0,R}, f\chi_{0,R}) \quad \ge \quad C\lambda^2\|f\chi_{0,R}\|_{L^2(\mu_\lambda)}^2.$$

The essential part of this lemma is in (3). In the case where $A = I +$ Hilbert-Schmidt operator, we can apply the same method as in **[1]** without any modification by using Corollary 2.8 (2) to prove (3). In general cases, we need to approximate A by such kind of operators.

Lemma 3.5. — *Assume that* A *is a bounded linear operator and* (A1), (A3), (A4) *hold. Also we assume that* $c_A = 1$. *Let* R *be a sufficiently large positive number such that*

$$\inf\left\{\frac{1}{4}\|h\|_H^2 + V(h) \mid \|h\|_W \ge R\right\} \ge 1$$

and ε *be a small positive number. Set* $D_{\varepsilon,R} = B_R(0) \cap \left(\cup_{i=1}^n B_{3\varepsilon}(h_i)\right)^c$. *Then there exists a self-adjoint operator* $T_\varepsilon \in L_1(H)$ *and a positive number* $\delta(\varepsilon)$ *such that*

(1) *it holds that for any* $h \in \mathrm{D}(A)$, $\|Ah\|_H^2 \ge \|(I_H + T_\varepsilon)h\|_H^2$,

(2)
$$\inf\left\{\frac{1}{4}\|(I_H+T_\varepsilon)h\|_H^2 + V(h) \;\middle|\; h \in D_{\varepsilon,R}\cap H\right\} \quad \ge \quad \delta(\varepsilon).$$

Proof. — It holds that for a large positive number L,

$$\inf\left\{\frac{1}{4}\|h\|_H^2 + V(h) \;\middle|\; h \in D_{\varepsilon,R}\cap B_{L,H}(0)^c\right\} \ge 1.$$

Hence we prove the lemma on $D_{\varepsilon,R}\cap B_{L,H}(0)$. For a natural number k, we define $A_k = \sum_{i=2^k}^{\infty} \frac{i}{2^k} 1_{I_{k,i}}(A)$, where $I_{k,i} = \left\{x \in \mathbb{R} \mid \frac{i}{2^k} \le x < \frac{i+1}{2^k}\right\}$. Then

$$0 \le \|Ah\|_H^2 - \|A_k h\|_H^2 \le \frac{3}{2^k}\|Ah\|_H^2 \le \frac{3}{2^k}\|A\|^2\|h\|_H^2.$$

By Lemma 2.4 (2), for sufficiently large k_0,

$$\inf\left\{\frac{1}{4}\|A_{k_0}h\|_H^2 + V(h) \;\middle|\; h \in D_{\varepsilon,R}\cap B_{L,H}(0)\right\} \ge \frac{1}{2}\kappa(\varepsilon),\ \frac{3}{2^{k_0}}\|A\|^2 L^2 \le \frac{1}{4}\varepsilon^2\|\sqrt{S}\|^{-2}.$$

Note that there exists a family of finite dimensional projection operators on H such that $P_n \uparrow I_H$ and $A_{k_0}P_n = P_nA_{k_0}$ for all $n \geq 1$. Hence, it holds that for any $h \in H$ and n

$$\|A_{k_0}h\|_H^2 = \|A_{k_0}P_nh\|_H^2 + \|A_{k_0}P_n^{\perp}h\|_H^2 \geq \|A_{k_0}P_nh\|_H^2 + \|P_n^{\perp}h\|_H^2.$$

Let $h \in B_{L,H}(0)$. Then $|V(h) - V(P_nh)| \leq \|DV(P_nh + \theta P_n^{\perp}h)\|_{W^*}\|P_n^{\perp}h\|_W$ $(0 < \theta < 1)$. Noting

$$\begin{aligned} \|P_nh + \theta P_n^{\perp}h\|_W &\leq L\|\sqrt{S}\|, \\ \|P_n^{\perp}h\|_W &= \|\sqrt{S}P_n^{\perp}h\|_H \leq \|\sqrt{S}P_n^{\perp}\|_2\|h\|_H, \\ \lim_{n\to\infty}\|\sqrt{S}P_n^{\perp}\|_2 &= 0, \end{aligned}$$

by (A2),

$$\lim_{n\to\infty} \sup\{|V(h) - V(P_nh)| \mid h \in B_{L,H}(0)\} = 0.$$

Now we take a natural number n_0 such that

$$\sup\{|V(h) - V(P_{n_0}h)| \mid h \in B_{L,H}(0)\} \leq \frac{1}{4}\min\left(\kappa(\varepsilon), 1, \varepsilon^2\|\sqrt{S}\|^{-2}\right).$$

Let $h \in D_{\varepsilon,R} \cap B_{L,H}(0)$. Then three cases are possible for $P_{n_0}h$ such that (i) $P_{n_0}h \in D_{\varepsilon/3,R} \cap B_{L,H}(0)$, (ii) $P_{n_0}h \in B_R(0)^c$, (iii) $P_{n_0}h \in \cup_{i=1}^n B_\varepsilon(h_i)$.

In the case of (i),

$$\frac{1}{4}\|A_{k_0}P_{n_0}h\|_H^2 + V(h) = \frac{1}{4}\|A_{k_0}P_{n_0}h\|_H^2 + V(P_{n_0}h) + (V(h) - V(P_{n_0}h)) \geq \frac{1}{4}\kappa(\varepsilon).$$

If (ii) happens, then

$$\frac{1}{4}\|A_{k_0}P_{n_0}h\|_H^2 + V(h) = \frac{1}{4}\|A_{k_0}P_{n_0}h\|_H^2 + V(P_{n_0}h) + (V(h) - V(P_{n_0}h)) \geq 3/4.$$

In the case where $P_{n_0}h \in B_\varepsilon(h_i)$ for some i,

$$\|P_{n_0}^{\perp}h\|_W = \|h - P_{n_0}h\|_W = \|h - h_i\|_W - \|h_i - P_{n_0}h\|_W \geq 2\varepsilon.$$

Thus $\|P_{n_0}^{\perp}h\|_H \geq \|\sqrt{S}\|^{-1}\|P_{n_0}^{\perp}h\|_W \geq 2\varepsilon\|\sqrt{S}\|^{-1}$. Therefore, we have for $h \in D_{\varepsilon,R} \cap B_{L,H}(0)$ satisfying (iii),

$$\begin{aligned} &\frac{1}{4}\|A_{k_0}P_{n_0}h\|_H^2 + \frac{1}{4}\|P_{n_0}^{\perp}h\|_H^2 + V(h) \\ &\quad = \frac{1}{4}\|P_{n_0}^{\perp}h\|_H^2 + \frac{1}{4}\|AP_{n_0}h\|_H^2 + V(P_{n_0}h) - \frac{1}{4}\left(\|AP_{n_0}h\|_H^2 - \|A_{k_0}P_{n_0}h\|_H^2\right) \\ &\qquad + (V(h) - V(P_{n_0}h)) \\ &\quad \geq \varepsilon^2\|\sqrt{S}\|^{-2} - \frac{3}{2^{k_0}}\|A\|^2L^2 - \frac{1}{4}\varepsilon^2\|\sqrt{S}\|^{-2} \geq \frac{1}{2}\varepsilon^2\|\sqrt{S}\|^{-2}. \end{aligned}$$

Consequently,

$$\inf\left\{\frac{1}{4}\|A_{k_0}P_{n_0}h\|_H^2 + \frac{1}{4}\|P_{n_0}^{\perp}h\|_H^2 + V(h) \;\middle|\; h \in D_{\varepsilon,R}\right\} \geq \delta(\varepsilon).$$

This implies that the operator $T_\varepsilon = (A_{k_0} - I_H)P_{n_0}$ satisfies the desired properties. $\square$

In Theorem 3.2, we assume $\gamma \geq 1 + \gamma_0$. But $\gamma \geq \gamma_0$ is sufficient for $\chi_{i,R} \in \mathrm{D}(D_A)$.

Lemma 3.6. — (1) *Assume that A is bounded. Then $\|w\|_W \in \mathrm{D}(D_A)$ and $\|D_A\|w\|_W\|_H \leq \|A\sqrt{S}\|$.*
(2) *Assume* (A5) *and let $S = A^{-2\gamma}$, where $\gamma \geq \gamma_0$. Then $\|w\|_W \in \mathrm{D}(D_A)$ and $\|D_A\|w\|_W\|_H \leq \|A^{1-\gamma}\|$.*

Proof. — (1) We have $D\|w\|_W = \frac{Sw}{\|w\|_W}$. So $D_A\|w\|_W = \frac{ASw}{\|w\|_W}$ and $\|D_A\|w\|_W\|_H \leq \|A\sqrt{S}\|$.
(2) This is proved in the same way as in (1). $\square$

Lemma 3.7. — *Assume* (A1), (A3), (A4), (A5) *and $c_A = 1$. Let $\gamma \geq \gamma_0$ and $S = A^{-2\gamma}$. Then the same results as in Lemma* 3.5 *hold .*

Proof. — For $a > 0$ let $\psi_a(x)$ be the positive function such that $\psi_a(x) = 1$ for $x \leq a$ and $\psi_a(x) = a/x$ for $x \geq a$. Then for $h \in H$

$$\begin{aligned} \|\psi_a(A)h\|_W^2 &= \|\psi_a(A)A^{-\gamma}h\|_H^2 \leq \|A^{-\gamma}h\|_H^2 = \|h\|_W^2, \\ \|\psi_a(A)h - h\|_W^2 &\leq \|(\psi_a(A) - 1)A^{-\gamma}h\|_H^2 \leq \frac{1}{a^{2\gamma}}\|h\|_H^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\psi_a(A)h - h_i\|_W &= \|\psi_a(A)h - h + h - h_i\|_W \\ &\geq \|h - h_i\|_W - \frac{1}{a^\gamma}\|h\|_H \end{aligned}$$

Thus, if $\|h - h_i\|_W \geq 3\varepsilon$ and $\|h\|_H \leq \frac{3a^\gamma}{2}\varepsilon$, hold, then $\|\psi_a(A)h - h_i\|_W \geq \frac{3\varepsilon}{2}$. Let $A^{(a)} = A\psi_a(A)$. Let L be a positive number such that for h with $\|h\|_H \geq L$, $\frac{1}{4}\|h\|_H^2 + V(h) \geq \frac{1}{2}\kappa(\varepsilon)$. Now let a be a positive number satisfying that

$$\frac{C(L)}{a^\gamma}L \leq \min\left(\frac{1}{2}\kappa(\varepsilon), \frac{1}{4}\varepsilon^2\|\sqrt{S}\|^{-2}\right), \qquad \frac{3a^\gamma}{2}\varepsilon \geq L.$$

Here $C(L)$ is the number which appeared in (A3). Then for such an a, for h with $\|h\|_H \leq L$, by the above estimates, we have

$$\begin{aligned} |V(h) - V(\psi_a(A)h)| &\leq \frac{C(L)}{a^\gamma}L \leq \frac{1}{2}\kappa(\varepsilon), \\ \frac{1}{4}\|A\psi_a(A)h\|_H^2 + V(\psi_a(A)h) &\geq \kappa(\varepsilon). \end{aligned}$$

Consequently, we have, for sufficiently large a,

$$\inf\left\{\frac{1}{4}\|A^{(a)}h\|_H^2 + V(h) \mid h \in D_{\varepsilon,R} \cap H\right\} \geq \frac{1}{2}\kappa(\varepsilon).$$

Therefore, it suffices for us to do the same calculation as in the bounded case replacing A by $A^{(a)}$. But of course, the norm of W is still defined by $S = A^{-2\gamma}$. Note that

$(A^{(a)})_{k_0}$ is defined first and next P_{n_0} is defined by $(A^{(a)})_{k_0}$. Case (iii) requires some additional care. That is, we use the following estimate:

$$\begin{aligned}
&\frac{1}{4}\|(A^{(a)})_{k_0}P_{n_0}h\|_H^2+\frac{1}{4}\|P_{n_0}^{\perp}h\|_H^2+V(h)\\
&=\frac{1}{4}\|P_{n_0}^{\perp}h\|_H^2+\frac{1}{4}\|A^{(a)}P_{n_0}h\|_H^2+V(P_{n_0}h)-\frac{1}{4}\left(\|A^{(a)}P_{n_0}h\|_H^2-\|(A^{(a)})_{k_0}P_{n_0}h\|_H^2\right)\\
&\qquad+(V(h)-V(P_{n_0}h))\\
&\geq \varepsilon^2\|\sqrt{S}\|^{-2}+\frac{1}{4}\|A\psi_a(A)P_{n_0}h\|_H^2+V(\psi_a(A)P_{n_0}h)+(V(P_{n_0}h)-V(\psi_a(A)P_{n_0}h))\\
&\qquad-\frac{3}{2^{k_0}}\|A^{(a)}\|^2L^2-\frac{1}{4}\varepsilon^2\|\sqrt{S}\|^{-2}\\
&\geq\frac{1}{4}\varepsilon^2\|\sqrt{S}\|^{-2}.
\end{aligned}$$

Therefore, it suffices to put $T_\varepsilon=\left(\left(A^{(a)}\right)_{k_0}-I_H\right)P_{n_0}$. □

Proof of Lemma 3.4. — (1) The first assertion is proved in Lemma 3.6. (3.3) can be proved by a simple calculation

(2) In the Taylor expansion (3.2) when $\chi_{i,R}(w)\neq 0$, we have $|R_{h_i}(w)|\leq C\|w-h_i\|_B^3\leq CR^{-1/2}\|w-h_i\|_W^2$. This implies

$$\mathcal{E}_{\lambda,A,V}(f\chi_{i,R},f\chi_{i,R})\geq\lambda e(A,K_i-CR^{-1/2}S)\|f\chi_{i,R}\|_{L^2(\mu_\lambda)}^2.$$

Here S is the trace class operator which defines the norm of W. Using the fact that

$$\lim_{R\to\infty}e(A,K_i-CR^{-1/2}S)=e(A,K_i)$$

which follows from Lemma 2.5, we complete the proof of (2).

(3) Let ρ be a continuous function on W such that (i) $0\leq\rho(w)\leq 1$, (ii) ρ is 0 near the neighborhood $U(N)$ of the zero point set N, (iii) ρ is 1 in $V(N)^c$, where $V(N)$ is a neighborhood of N such that $U(N)\subset V(N)$. Moreover assume that $\{w\mid\chi_{0,R}(w)\neq 0\}\subset\{w\mid\rho(w)=1\}$. Let r be a small positive number. Then

$$\begin{aligned}
\mathcal{E}_{\lambda,A,V}(f\chi_{0,R},f\chi_{0,R}) &= \mathcal{E}_{\lambda,A,V-r\rho}(f\chi_{0,R},f\chi_{0,R})+\int_W r\lambda^2\rho f^2\chi_{0,R}^2d\mu_\lambda\\
&= \mathcal{E}_{\lambda,A,(V-r\rho)\rho}(f\chi_{0,R},f\chi_{0,R})+\int_W r\lambda^2 f^2\chi_{0,R}^2d\mu_\lambda.
\end{aligned}$$

L^2-norm of the second term on the right-hand side is $r\lambda^2\|f\chi_{0,R}\|^2$. To estimate the first term, we use again IMS localization formula. We write $g_0=f\chi_{0,R}$. Let $\varphi_0(w)=\chi\left(\frac{\|w\|_W^2}{R^2}\right)$ and $\varphi_1(w)=\sqrt{1-\varphi_0(w)^2}$. Then

$$\mathcal{E}_{\lambda,A,(V-r\rho)\rho}(g_0,g_0) = \sum_{i=0,1}\mathcal{E}_{\lambda,A,(V-r\rho)\rho}(g_0\varphi_i,g_0\varphi_i)-\sum_{i=0,1}\int_W\|D_A\varphi_i\|_H^2g_0^2d\mu_\lambda.$$

We use Corollary 2.8 (2) to estimate the term containing $g_0\varphi_0$. Let $\tilde{\varphi}_0(w) = \chi\left(\frac{\|w\|_W^2}{3R^2}\right)$. We can find a positive number ε' and R' such that $\{w \in W \mid \rho(w)\tilde{\varphi}_0(w) \neq 0\} \subset D_{\varepsilon',R'}$. Let $T_{\varepsilon'}$ be a trace class operator which satisfies the property in Lemma 3.5 for $D_{\varepsilon',R'}$. Then

$$\begin{aligned}
\mathscr{E}_{\lambda,A,(V-r\rho)\rho}(g_0\varphi_0, g_0\varphi_0) &\geq \mathscr{E}_{\lambda,I_H+T_{\varepsilon'},(V-r\rho)\rho\tilde{\varphi}_0}(g_0\varphi_0, g_0\varphi_0)\\
&\geq -\frac{\lambda}{2}\log I(\lambda)\|g_0\varphi_0\|^2_{L^2(\mu_\lambda)}\\
&\quad + \left(\frac{\lambda}{2}\log\det{}_{(2)}(I_H+T_{\varepsilon'}) - \frac{\lambda}{2}\mathrm{tr}\left(T^2_{\varepsilon'}\right)\right)\|g_0\varphi_0\|^2_{L^2(\mu_\lambda)},
\end{aligned}$$

where

$$\begin{aligned}
I(\lambda) = \int_W \exp\Big(&-2\lambda\left(\left(V(w) - r\rho(w)\right)\rho(w)\tilde{\varphi}_0(w)\right.\\
&\qquad - \lambda : (T_{\varepsilon'}w, w) :_\lambda - \frac{\lambda}{2}\|T_{\varepsilon'}w\|_H^2\Big)d\mu_\lambda(w).
\end{aligned}$$

Let $U_{\varepsilon'}(h) = \frac{1}{4}\|(I_H+T_{\varepsilon'})h\|_H^2 + (V(h) - r\rho(h))\rho(h)\tilde{\varphi}_0(h)$. Then

$$\begin{aligned}
U_{\varepsilon'}(h) &= \frac{1}{4}\|(I_H+T_{\varepsilon'})h\|_H^2(1-\rho(h)\tilde{\varphi}_0(h))\\
&\quad + \left\{\frac{1}{4}\|(I_H+T_{\varepsilon'})h\|_H^2 + (V(h) - r\rho(h))\right\}\rho(h)\tilde{\varphi}_0(h).
\end{aligned}$$

By the property of $T_{\varepsilon'}$, by taking r to be sufficiently small, we have

$$\left\{\frac{1}{4}\|(I_H+T_{\varepsilon'})h\|_H^2 + (V(h) - r\rho(h))\right\}\rho(h)\tilde{\varphi}_0(h) \geq 0 \qquad \text{for all } h \in H.$$

Therefore by the Large deviation estimate, for such an r, $\lim_\lambda \frac{1}{\lambda}\log I(\lambda) \leq 0$. This shows that for any $c > 0$ it holds that for large λ

$$\mathscr{E}_{\lambda,A,(V-r\rho)\rho}(g_0\varphi_0, g_0\varphi_0) \geq -c\lambda^2\|g_0\varphi_0\|^2_{L^2(\mu_\lambda)}.$$

Next, we give a lower bound estimate for the another term. Let $\tilde{\varphi}_1(w) = \sqrt{1-\chi\left(\frac{3\|w\|_W^2}{R^2}\right)^2}$. Then $\{w \mid g_0(w)\varphi_1(w) \neq 0\} \subset \{w \mid \tilde{\varphi}_1(w) = 1\}$. By using Corollary 2.8 (1),

$$\mathscr{E}_{\lambda,A,(V-r\rho)\rho}(g_0\varphi_1, g_0\varphi_1) \geq -\frac{\lambda}{2}\log\left(\int_W \exp\left(-2\lambda(V-r\rho)\rho\tilde{\varphi}_1\right)d\mu_\lambda\right)\|g_0\varphi_1\|^2_{L^2(\mu_\lambda)}.$$

If R is sufficiently large and r is small, then

$$\inf\left\{\frac{1}{4}\|h\|_H^2 + \left(V(h) - r\rho(h)\right)\rho(h)\tilde{\varphi}_1(h) \;\middle|\; \tilde{\varphi}_1(h) \neq 0, h \in H\right\} > 0.$$

Thus, by the Large deviation results, for any $c > 0$ it holds that for large λ

$$\mathscr{E}_{\lambda,A,(V-r\rho)\rho}(g_0\varphi_1, g_0\varphi_1) \geq -c\lambda^2\|g_0\varphi_1\|^2_{L^2(\mu_\lambda)}.$$

These prove (3). □

Remark 3.8. — Let $\tilde{V}$ be a bounded measurable function on W. Assume that $A^4 + 4AKA$ is strictly positive and AKA is a trace class operator. Let

$$c_{A,K} = \inf \sigma\left(\sqrt{A^4 + 4AKA}\right).$$

Then it holds that for any $f \in \mathfrak{F}C_A^\infty(W)$,

$$\begin{aligned}&\mathscr{E}_{\lambda,A,V_{k,h}+\tilde{V}}(f,f)\\ &\quad\geq E_0(\lambda,A,V_K)\|f\|^2_{L^2(\mu_\lambda)}\\ &\quad -\frac{\lambda c_{A,K}}{2}\log\left(\int_W \exp\left(-\frac{2\lambda}{c_{A,K}}\tilde{V}(w)\right)\Omega_{\lambda,A,V_{K,h}}(w)^2 d\mu_\lambda(w)\right)\|f\|^2_{L^2(\mu_\lambda)}.\end{aligned}$$

By this estimate, we can prove local estimates near N in Lemma 3.4 (2) using the Laplace method. This proof could be extended to the case of Schrödinger operators with more general potential functions.

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Astérisque
327, 2009, p. 17–45

INFINITE DIMENSIONAL OSCILLATORY INTEGRALS WITH POLYNOMIAL PHASE FUNCTION AND THE TRACE FORMULA FOR THE HEAT SEMIGROUP

by

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It is a special honour and pleasure to dedicate this work to Jean-Michel Bismut, as a small sign of gratitude for all he has taught us by his inspiring work

Abstract. — Infinite dimensional oscillatory integrals with a polynomially growing phase function with a small parameter $\epsilon \in \mathbb{R}^+$ are studied by means of an analytic continuation technique, as well as their asymptotic expansion in the limit $\epsilon \downarrow 0$. The results are applied to the study of the semiclassical behavior of the trace of the heat semigroup with a polynomial potential.

Résumé **(Intégrales oscillantes en dimension infinie avec une phase polynomiale et formule de la trace pour le semigroupe de la chaleur)**

Nous étudions les intégrales oscillantes en dimension infinie avec une phase de croissance polynomiale à petit paramètre $\epsilon \in \mathbb{R}^+$ au moyen d'une technique de prolongement analytique. Nous donnons aussi leur développement asymptotique en ϵ lorsque $\epsilon \downarrow 0$. Nous présentons une application de ces résultats à l'étude du comportement semiclassique de la trace du noyau de la chaleur avec un potentiel polynomial.

1. Introduction

Oscillatory integrals on finite dimensional Hilbert spaces, i.e. expressions of the form

$$\int_{\mathbb{R}^n} e^{-\frac{i}{\epsilon}\Phi(x)} g(x)dx, \tag{1}$$

2010 Mathematics Subject Classification. — 28C20, 34E05, 35K05, 35C15, 35C20.
Key words and phrases. — Infinite dimensional oscillatory integrals, asymptotics, stationary phase method, Laplace method, degenerate phase, heat kernels, polynomial potential, semiclassical limit.

The authors are very grateful to the Mathematics Departments at Bonn and Trento for their hospitality. The second author gratefully acknowledges the financial support of the Alexander Von Humboldt Stiftung and of the F. Severi fellowship of I.N.d.A.M..

(where $\Phi : \mathbb{R}^n \to \mathbb{R}$ is the phase function and $\epsilon \in \mathbb{R}^+$ a real positive parameter) are a classical topic of investigation, having several applications, e.g. in electromagnetism, optics and acoustics. They are part of the general theory of Fourier integral operators **[27, 35]**. Particularly interesting is the study of the asymptotic behavior of these integrals in the limit $\epsilon \downarrow 0$. The generalization of the definition of oscillatory integrals to the case where the integration is performed on an infinite dimensional space, in particular a space of continuous functions, presents a particular interest in connection with applications to quantum theory such as the mathematical realization of Feynman path integrals **[1, 7]** (see also, e.g. **[26, 36]** and references therein; applications include—besides quantum mechanics—quantum field theory and low dimensional geometry, see, e.g. **[10]** and references therein). In the case where the integration is performed on such spaces and on general real separable Hilbert spaces, the theory was for a long time restricted to oscillatory integrals with phase functions Φ which can be written as sums of a quadratic form and a bounded function belonging to the class of Fourier transforms of complex measures. In **[8, 9]** these results have been generalized to phase functions with quartic polynomial growth. In this paper we consider a generalization of the oscillatory integral (1) and its infinite dimensional analogue, in the case where the imaginary unity i in the exponent is replaced by a complex parameter $s \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} \ : \ \mathrm{Re}(z) \geq 0\}$:

$$I(s) \equiv \int e^{-\frac{s}{\epsilon}\Phi(x)} g(x) dx. \tag{2}$$

Strictly speaking $I(s)$ has an oscillatory behavior only for s being a pure imaginary number. By generalizing the results of **[8]**, we prove (in section 2) a representation formula which allows us to compute an infinite dimensional oscillatory integral of the form (2), with a phase function Φ having an arbitrary even polynomial growth, in terms of a Gaussian integral. In the non degenerate case (i.e. when the Hessian of the phase function is non degenerate), we compute (in section 3) the asymptotic expansion of the integral as $\epsilon \downarrow 0$ in powers of ϵ. In the degenerate case the situation is more involved. In section 4 we handle in detail a particular example and apply this result to the study of the asymptotic behavior of the trace of the heat semigroup $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$, $t > 0$, in the case where H is the essentially self-adjoint operator on $C_0^\infty \equiv C_0^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ given on the functions $\phi \in C_0^\infty$ by

$$H\phi(x) = \Big(-\frac{\hbar^2}{2}\Delta_x + V(x)\Big)\phi(x), \tag{3}$$

where $\hbar > 0$ and V is a polynomially growing potential of the form $V(x) = |x|^{2N}$, $x \in \mathbb{R}^d$, $N \in \mathbb{N}$. This corresponds to exhibiting the detailed behavior of $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$, $t > 0$, "near the classical limit". Indeed H can be interpreted as a Schrödinger Hamiltonian (in which case $\hbar$ is the reduced Planck's constant), and consequently $e^{-\frac{t}{\hbar}H}$

as a Schrödinger semigroup with imaginary time, i.e. the heat semigroup. In recent years a particular interest has been devoted to the study of the trace of the heat semigroup and of the corresponding Schrödinger group $e^{-\frac{it}{\hbar}H}$, $t \in \mathbb{R}$, (related to the heat semigroup by analytic continuation in the "time variable" t) and their asymptotics in the "semiclassical limit $\hbar \downarrow 0$" (see, e.g., **[46]**, **[1, 4, 12]** and also **[16, 17, 18, 20]** for related problems). In particular one is interested in the proof of a trace formula of Gutzwiller's type, relating the asymptotics of the trace of the Schrödinger group and the spectrum of the quantum mechanical energy operator H with the classical periodic orbits of the system. Gutzwiller's heuristic trace formula, which is a basis of the theory of quantum chaotic systems, is the quantum mechanical analogue of Selberg's trace formula, relating the spectrum of the Laplace-Beltrami operator on manifolds with constant negative curvature with the periodic geodesics (see, e.g., **[25]** and **[3, 4, 12]**).
In the case where the potential V is the sum of an harmonic oscillator part and a bounded perturbation V_0 that is the Fourier transform of a complex (bounded variation) measure on $\mathbb{R}^d$, rigorous results on the asymptotics of the trace of the Schrödinger group and the heat semigroup have been obtained in **[4, 12]** by means of an infinite dimensional version of the stationary phase method for infinite dimensional oscillatory integrals (see **[7]** for a review of this topic).
The paper is organized as follows. In section 2 we give the definition and the main results on infinite dimensional oscillatory integrals of the form (2) with a polynomial phase function Φ, in section 3 we study the asymptotic expansion of the integral in the case where the origin is a non degenerate critical point of Φ, while in section 4 we study a degenerate case and apply these results to the asymptotics of $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$, $t > 0$, as $\hbar \downarrow 0$.

2. Infinite dimensional oscillatory integrals

The present section is devoted to the study of the *oscillatory integrals with complex parameter s*. In the following we shall denote by $(\mathcal{H}, \langle\ ,\ \rangle, \|\ \|)$ a real separable infinite dimensional Hilbert space, s will be a complex number such that $\mathrm{Re}(s) \geq 0$, $g : \mathcal{H} \to \mathbb{C}$ a Borel function.
Let us consider the generalization of the oscillatory integral (1) to the case (2) where the imaginary unity i in the exponent is replaced by a complex parameter $s \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} \ :\ \mathrm{Re}(z) \geq 0\}$:

$$I(s) \equiv \int_{\mathbb{R}^n} e^{-\frac{s}{\epsilon}\Phi(x)} g(x) dx. \tag{4}$$

In the case where s is a pure imaginary number, by exploiting the oscillatory behavior of the integrand, the oscillatory integral (4) can still be defined as an improper Riemann integral even if the (continuous) function g is not summable. In the case where the phase function Φ is a quadratic form, the integral (4) is called *Fresnel integral*. We propose here for the general case (4) a modification of the Hörmander's definition [**27**], also considered in [**5, 23**] in connection to the generalization to the infinite dimensional case. This modification is as follows:

Definition 2.1. — Let $f : \mathbb{R}^n \to \mathbb{C}$ be a Borel function, $s \in \mathbb{C}^+$ a complex parameter. Let $\mathcal{S}$ be a subset of the space of the Schwartz test functions $S(\mathbb{R}^n)$. If for each $\phi \in \mathcal{S}$ such that $\phi(0) = 1$ the integrals

$$I_\delta(f, \phi) := \int_{\mathbb{R}^n} (2\pi s^{-1})^{-n/2} e^{-\frac{s}{2}|x|^2} f(x)\phi(\delta x)dx$$

exist for all $\delta > 0$ and $\lim_{\delta\to 0} I_\delta(f, \phi)$ exist and is independent of ϕ, then this limit is called the Fresnel integral of f with parameter s (with respect to the space $\mathcal{S}$ of regularizing functions) and denoted by

$$\mathcal{F}^s(f) \equiv \widetilde{\int_{\mathbb{R}^n}^{s}} e^{-\frac{s}{2}|x|^2} f(x)dx. \tag{5}$$

By an adaptation of the definition of infinite dimensional oscillatory integrals given in [**23**] it is possible to define the *oscillatory integral with parameter s* on the Hilbert space $\mathcal{H}$, namely

$$I(s) = \widetilde{\int_{\mathcal{H}}^{s}} e^{-\frac{s}{2}\|x\|^2} g(x)dx \tag{6}$$

as the limit of a sequence of (suitably normalized) finite dimensional approximations [**12**].

Definition 2.2. — A Borel measurable function $f : \mathcal{H} \to \mathbb{C}$ is called $\mathcal{F}^s$ integrable if for each sequence $\{P_n\}_{n\in\mathbb{N}}$ of projectors onto n-dimensional subspaces of $\mathcal{H}$, such that $P_n \leq P_{n+1}$ and $P_n \to I$ strongly as $n \to \infty$ (I being the identity operator in $\mathcal{H}$), the finite dimensional approximations of the Fresnel integral of f, with parameter s,

$$\mathcal{F}^s_{P_n}(f) \equiv \widetilde{\int_{P_n\mathcal{H}}^{s}} e^{-\frac{s}{2}\|P_n x\|^2} f(P_n x)d(P_n x) \tag{7}$$

exist (in the sense of definition 2.1) and the limit $\lim_{n\to\infty} \mathcal{F}^s_{P_n}(g)$ exists and is independent of the sequence $\{P_n\}$.
In this case the limit is called the infinite dimensional Fresnel integral of f with parameter s and is denoted by

$$\widetilde{\int_{\mathcal{H}}^{s}} e^{-\frac{s}{2}\|x\|^2} f(x)dx.$$

f is then said to be integrable (in the sense of Fresnel integrals with parameter s).

The description of the largest class of functions which are integrable in this sense is an open problem, even in the finite dimensional case. Clearly it depends on the class $\mathcal{S}$ of the regularizations. The common choice is $\mathcal{S} \equiv S(\mathbb{R}^n)$, **[5, 23]**. In this case **[5, 7, 23]** the space of integrable functions includes (in finite as well as in infinite dimensions) the Fresnel class $\mathcal{F}(\mathcal{H})$, that is the set of functions $f : \mathcal{H} \to \mathbb{C}$ that are Fourier transforms of complex bounded variation measures on $\mathcal{H}$:

$$f(x) = \int_{\mathcal{H}} e^{i\langle y,x\rangle} d\mu_f(y) \equiv \hat{\mu}_f(x), \qquad x \in \mathcal{H}$$

$$\sup \sum_i |\mu_f(E_i)| < \infty,$$

where the supremum is taken over all sequences $\{E_i\}$ of pairwise disjoint Borel subsets of $\mathcal{H}$, such that $\cup_i E_i = \mathcal{H}$.
In fact for any $f \in \mathcal{F}(\mathcal{H})$ it is possible to prove a Parseval type equality that allows to compute the infinite dimensional oscillatory integral of f (with purely imaginary parameter s) in terms of an absolutely convergent integral with respect to the associated complex-valued measure μ_f **[5, 23]**. Indeed given a self-adjoint trace-class operator $B : \mathcal{H} \to \mathcal{H}$, such that $(I - B)$ is invertible, a function $f \in \mathcal{F}(\mathcal{H})$, $f = \hat{\mu}_f$ and a positive parameter $\hbar \in \mathbb{R}^+$, it is possible to prove that the function $e^{-\frac{i}{2\hbar}\langle x,Bx\rangle} f(x)$ is Fresnel integrable and the corresponding Fresnel integral with parameter $s = -i/\hbar$ is given by

$$\text{(8)} \quad \widetilde{\int_{\mathcal{H}}^{-i/\hbar}} e^{\frac{i}{2\hbar}\|x\|^2} e^{-\frac{i}{2\hbar}\langle x,Bx\rangle} e^{i\langle x,y\rangle} f(x)dx$$
$$= (\det(I-B))^{-1/2} \int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle \alpha+y,(I-B)^{-1}(\alpha+y)\rangle} \mu_f(d\alpha)$$

where $\det(I - B) = |\det(I - B)| e^{-\pi i \text{ Ind } (I-B)}$ is the Fredholm determinant of the operator $(I - B)$, $|\det(I - B)|$ its absolute value and $\text{Ind}((I - B))$ is the number of negative eigenvalues of the operator $(I - B)$, counted with their multiplicities.

Let us also recall, for later use, a known result on infinite dimensional oscillatory integrals.

Let $\mathcal{H}$ be a Hilbert space with norm $|\cdot|$ and scalar product $(\cdot,\cdot)$. Let also $\|\cdot\|$ be an equivalent norm on $\mathcal{H}$ with scalar product denoted by $\langle\cdot,\cdot\rangle$. Let us denote the new Hilbert space by $\tilde{\mathcal{H}}$. Let us assume moreover that

$$\langle x_1, x_2\rangle = (x_1, x_2) + (x_1, Tx_2), \qquad x_1, x_2 \in \tilde{\mathcal{H}}$$

$$\|x\|^2 = |x|^2 + (x, Tx), \qquad x \in \tilde{\mathcal{H}},$$

where T is a self-adjoint trace class operator on $\mathcal{H}$. The following holds (see [**11, 12**]):

Theorem 2.3. — *Let $f : \mathcal{H} \to \mathbb{C}$ be a Borel function. f is integrable on $\mathcal{H}$ (in the sense of definition 2.2) if and only if f is integrable on $\tilde{\mathcal{H}}$ and in this case*

$$\widetilde{\int_{\tilde{\mathcal{H}}}} e^{-\frac{s}{2}|x|^2} f(x)dx = \det(I+T)^{1/2} \widetilde{\int_{\mathcal{H}}} e^{-\frac{s}{2}|x|^2} f(x)dx \tag{9}$$

Recently the class of "Fresnel integrable functions" in the sense of definition 2.2 has been further enlarged. In particular in [**9**] the Parseval type equality (8) has been generalized to the case where $\mathcal{H}$ is finite dimensional but the phase function is an even degree (not necessarily second order) polynomial, while in [**8**] a corresponding result has been proved for infinite dimensional Hilbert spaces and phase functions which are the sum of a quadratic and a quartic term.
Let us also remark that definition 2.2 can be seen as an extension of a line of development relating infinite dimensional integrals of probabilistic and oscillatory type, going back to Cameron, see, e.g., [**19**], [**37**] and corresponding references under "analytic approach" in [**1, 7**].

In the following we shall extend these results to infinite dimensional Hilbert spaces and suitable polynomial phase functions of higher degrees. The main idea is a generalization of a Parseval-type equality, obtained by modifying the definition 2.1 by restricting the class of regularizing functions to a class $\mathcal{S}$ of analytic functions.
Let $\alpha \in \mathbb{R}$, in the following I_α will denote the open interval $(0, \alpha)$ if $\alpha > 0$ and $(\alpha, 0)$ if $\alpha < 0$; D_α will denote the sector of the complex $z-$ plane

$$D_\alpha := \{z = |z|e^{i\varphi} \in \mathbb{C} \ : \ |z| > 0, \, \varphi \in I_\alpha\},$$

and $\mathcal{S}_\alpha(\mathbb{R}^n)$ will denote the space of functions $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying the following assumptions:

1. for any $x \in \mathbb{R}^n$ the function

   $$z \mapsto \phi(zx), \qquad z \in \mathbb{R}, \quad x \in \mathbb{R}^n$$

   can be extended to an analytic function in D_α, which is continuous in the closure $\bar{D}_\alpha$ of D_α.
2. for any $z \in \bar{D}_\alpha$ the map

   $$x \mapsto \phi^z(x) := \phi(zx), \qquad z \in \mathbb{C}, \quad x \in \mathbb{R}^n$$

   is bounded.

Clearly $\mathcal{S}_\beta(\mathbb{R}^n) \subset \mathcal{S}_\alpha(\mathbb{R}^n)$ if $\alpha < \beta$. As an example the function $x \in \mathbb{R}^n \mapsto e^{-\|x\|^2}$ is an element of $\mathcal{S}_{\pi/4}(\mathbb{R}^n)$.

Given a real separable Hilbert space of $\mathcal{H}$, with inner product $\langle\ ,\ \rangle$ and norm $\|\ \|$, let us consider the *abstract Wiener space* $(\mathcal{H},\mathcal{B})$ built on $\mathcal{H}$, where $(\mathcal{B},|\ |)$ is the Banach space completion of $\mathcal{H}$ with respect to the measurable norm $|\ |$ and let μ be the standard Gaussian measure on $\mathcal{B}$ associate with $\mathcal{H}$ (see **[24, 32]** and the Appendix of the present paper). $\mathcal{H}$ is sometimes called the reproducing kernel Hilbert space of $\mathcal{B}$. Let us denote by c the norm of the continuous inclusion of $\mathcal{H}$ in $\mathcal{B}$.

Theorem 2.4. — *Let $s,r\in\mathbb{C}$, $s=|s|e^{i\alpha}$ and $r=|r|e^{i\beta}$, with $\alpha,\beta\in[-\pi/2,\pi/2]$. Let us assume that for any φ belonging to the closure $\bar{I}_{-\alpha/2}$ of $I_{-\alpha/2}$, the angle $\beta+2N\varphi$ is included in the interval $[-\pi/2,\pi/2]$.*
Let $B:\mathcal{H}\to\mathcal{H}$ be a trace class symmetric operator such that $(I-B)$ is strictly positive. Let $V_{2N}:\mathcal{H}\to\mathbb{R}$ be a positive, continuous in the $|\ |$-norm and homogeneous function of order $2N$, i.e. $V_{2N}(\lambda x)=\lambda^{2N}V_{2N}(x)$, for any $\lambda\in\mathbb{R}$, $x\in\mathcal{H}$. Let $g:\mathcal{H}\to\mathbb{C}$ satisfy the following assumptions:

– *for any $x\in\mathcal{H}$ the map*

$$z\mapsto g(zx),\qquad z\in\mathbb{R},\quad x\in\mathcal{H}$$

can be extended to a function which is analytic on $D_{-\alpha/2}$ and continuous in $\bar{D}_{-\alpha/2}$.

– $\exists K_1>0$, $\exists K_2\in(0,1/c^2)$, $\forall x\in\mathcal{H}$

$$|g(zx)|\leq K_1|e^{\frac{sz^2}{2}(K_2|x|^2-\langle x,Bx\rangle)}|,\qquad \forall z\in\bar{D}_{-\alpha/2}\tag{10}$$

– *the function $x\mapsto g^\alpha(x)\equiv g(e^{-i\alpha/2}x)$, $x\in\mathcal{H}$, is continuous in the $|\cdot|$-norm.*

Then the infinite dimensional oscillatory integral with parameter s and regularizing class $\mathcal{S}_{-\alpha/2}$ of the function $f:\mathcal{H}\to\mathbb{C}$

$$f(x)=e^{\frac{s}{2}\langle x,Bx\rangle-rV_{2N}(x)}g(x),\qquad x\in\mathcal{H},\tag{11}$$

is well defined and it is given by

$$\widetilde{\int_{\mathcal{H}}^{s}}e^{-\frac{s}{2}\langle x,(I-B)x\rangle-rV_{2N}(x)}g(x)dx=\int_{\mathcal{B}}e^{\frac{1}{2}\langle\omega,B\omega\rangle-rs^{-N}\tilde{V}_{2N}(\omega)}\tilde{g}^\alpha(|s|^{-1/2}\omega)d\mu(\omega),\tag{12}$$

$\tilde{V}_{2N}$ resp. $\tilde{g}^\alpha$ being the stochastic extensions of V_{2N} resp. g^α to $\mathcal{B}$.

Proof. — The right hand side of (12) is well defined, indeed under the assumption of $|\ |$-norm continuity, the functions V_{2N} and g^α can be extended by continuity to random variables $\bar{V}_{2N}$ and $\bar{g}^\alpha$ on $\mathcal{B}$, which coincide with the stochastic extensions of $\tilde{V}_{2N}$ and $\tilde{g}^\alpha$ of V_{2N} and g^α μ-a.e. (cfr. Appendix, which is based on **[24]**). Moreover for any $\lambda\in\mathbb{C}^+$ and for any increasing sequence of $n-$dimensional projectors P_n in $\mathcal{H}$, the family of bounded random variables $e^{-\lambda V_{2N}\circ\tilde{P}_n(\cdot)}\equiv e^{-\lambda V_{2N}^n(\cdot)}$ ($\tilde{P}_n$ being the

stochastic extension of P_n to $\mathcal{B}$)converges $\mu-$a.e. to $e^{-\lambda\tilde{V}_{2N}(\,\cdot\,)}$.
As B is symmetric trace class, the quadratic form on $\mathcal{H}\times\mathcal{H}$:

$$x\in\mathcal{H}\mapsto\langle x,Bx\rangle$$

can be extended to a random variable on $\mathcal{B}$, denoted again by $\langle\,\cdot\,,B\,\cdot\,\rangle$. Moreover the random variable $e^{\frac{1}{2}\langle\,\cdot\,,B\,\cdot\,\rangle}$ is in $L^1(\mu)$ (see appendix). The bound (10) for $z=s^{-1/2}$ extends by continuity to $\tilde{g}^\alpha:\mathcal{B}\to\mathbb{C}$ and by Fernique's theorem the integral on the right hand side of (12) is convergent.
Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of finite dimensional projection operators on $\mathcal{H}$ converging strongly to the identity as $n\to\infty$. Let $\phi\in\mathcal{S}_{-\alpha/2}(\mathbb{R}^n)$ be a regularizing function. For any $\delta>0$ let us consider the regularized finite dimensional approximations

$$(13)\qquad (2\pi s^{-1})^{-n/2}\int_{P_n\mathcal{H}}e^{-\frac{s}{2}\langle P_nx,(I-B)P_nx\rangle-rV_{2N}(P_nx)}g(P_nx)\phi(\delta P_nx)d(P_nx).$$

For any $z\in\mathbb{R}^+$ the integral (13) is equal to

$$(14)\qquad (\frac{z^2s}{2\pi})^{n/2}\int_{P_n\mathcal{H}}e^{-\frac{sz^2}{2}\langle P_nx,(I-B)P_nx\rangle-rz^{2N}V_{2N}(P_nx)}g(zP_nx)\phi(z\delta P_nx)d(P_nx).$$

By the assumptions on the functions ϕ,g, as well as on the parameters s and r, and by Fubini and Morera theorems, the integral (14) is a function of the variable z which is analytic in the sector $D_{-\alpha/2}$ and continuous on $\bar{D}_{-\alpha/2}$, and coincides with the value of the integral (13) on $\mathbb{R}^+$. By a straightforward application of the reflection principle **[33]** it is a constant function on the whole closed sector $\bar{D}_{-\alpha/2}$. In particular for $z=s^{-1/2}:=|s|^{-1/2}e^{-i\alpha/2}$, we conclude that

$$\begin{aligned}&(2\pi s^{-1})^{-n/2}\int_{P_n\mathcal{H}}e^{-\frac{s}{2}\langle P_nx,(I-B)P_nx\rangle-rV_{2N}(P_nx)}g(P_nx)\phi(\delta P_nx)d(P_nx)\\&=(2\pi)^{-n/2}\int_{P_n\mathcal{H}}e^{-\frac{1}{2}\langle P_nx,(I-B)P_nx\rangle-rs^{-N}V_{2N}(P_nx)}g(s^{-1/2}P_nx)\phi(s^{-1/2}\delta P_nx)d(P_nx)\end{aligned}$$

By letting $\delta\downarrow 0$ and using again the dominated convergence theorem the latter is equal to

$$\begin{aligned}&\int_{P_n\mathcal{H}}e^{\frac{1}{2}\langle P_nx,BP_nx\rangle-rs^{-N}V_{2N}(P_nx)}g(s^{-1/2}P_nx)\frac{e^{-\frac{1}{2}\|P_nx\|^2}}{(2\pi)^{n/2}}d(P_nx)\\&\qquad\qquad=\int_{\mathcal{B}}e^{\frac{1}{2}\langle P_nx,BP_nx\rangle-rs^{-N}\tilde{V}_{2N}(P_nx)}\tilde{g}^\alpha(|s|^{-1/2}P_nx)d\mu(x)\end{aligned}$$

By letting $n\to\infty$ and by the dominated convergence theorem the latter converges to the right hand side of (12) $\square$

Remark 2.5. — Theorem 2.4 generalizes the results obtained in [**8**] concerning the oscillatory integrals of the form

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2}\langle x,x\rangle} e^{i\lambda V_4(x)} g(x)dx. \tag{15}$$

Indeed the Parseval type equality (12) allows one to compute explicitly infinite dimensional oscillatory integrals with polynomial phase of higher degree, provided that the parameter s has a non vanishing real part. For instance one can compute infinite dimensional oscillatory integrals of the form

$$\widetilde{\int_{\mathcal{H}}^{s}} e^{-\frac{|s|e^{i\alpha}}{2}\langle x,x\rangle} e^{i\lambda V_{2N}(x)} g(x)dx$$

with $\lambda \in \mathbb{R}^+$ and $\alpha \in [-\pi/N, 0]$.

Remark 2.6. — In the case $s \in \mathbb{R}^+$, theorem 2.4 relates a Gaussian integral on the Banach space B with an integral on its reproducing kernel Hilbert space $\mathcal{H}$.

If the operator $(I - B) : \mathcal{H} \to \mathcal{H}$ is not strictly positive, formula (12) does not hold. In the following we shall generalize the results of theorem 2.4 to the case where $(I - B)$ has non positive eigenvalues, by restricting the class of polynomial phase functions V_{2N}.
Given a trace class symmetric operator $B : \mathcal{H} \to \mathcal{H}$, the number of non positive eigenvalues of $(I - B)$ (counted with their multiplicity) is finite. We shall denote by $\mathcal{H}_0$ the kernel of $I - B$, by $\mathcal{H}_-$ the subspace of $\mathcal{H}$ where $I - B$ is negative definite, and by $\mathcal{H}_+$ the subspace of $\mathcal{H}$ where $I - B$ is positive definite. We have $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_0 \oplus \mathcal{H}_+$. Let us introduce the notation $\mathcal{H}_1 \equiv \mathcal{H}_- \oplus \mathcal{H}_0$, $\mathcal{H}_2 \equiv \mathcal{H}_+$ and $x \in \mathcal{H} = x_1 + x_2$, with $x_i \in \mathcal{H}_i$, $i = 1, 2$. Clearly $\dim(\mathcal{H}_1) < +\infty$ and this fact will be used in the following. Let us denote by $(\mathcal{H}_2, \mathcal{B}_2)$ the abstract Wiener space associated with $\mathcal{H}_2$ and by μ_2 the Gaussian measure on $\mathcal{B}_2$ associated with $\mathcal{H}_2$.

Theorem 2.7. — *Let $s, r \in \mathbb{C}$, $s = |s|e^{i\alpha}$ and $r = |r|e^{i\beta}$, with $\alpha, \beta \in [-\pi/2, \pi/2]$. Let us assume that for any $\varphi \in \bar{I}_{-\alpha/2}$, the angle $\beta + 2N\varphi$ is included in the interval $(-\pi/2, \pi/2)$.*
Let $B : \mathcal{H} \to \mathcal{H}$ be a trace class symmetric operator. Let $V_{2N} : \mathcal{H} \to \mathbb{R}$ satisfy the assumptions of theorem 2.4. Let us assume moreover that there exists a constant K_3 such that for any $x_1 \in \mathcal{H}_1, x_2 \in \mathcal{H}_2$ one has $V_{2N}(x_1 + x_2) - V_{2N}(x_1) \geq K_3$. Let $g : \mathcal{H} \to \mathbb{C}$ satisfy the following assumptions:

– *for any $x \in \mathcal{H}$ the map*

$$z \mapsto g(zx), \qquad z \in \mathbb{R}, \quad x \in \mathcal{H}$$

can be extended to a function which is analytic in $D_{-\alpha/2}$ *and continuous in* $\bar{D}_{-\alpha/2}$.

– $\exists K_4, K_5, \delta > 0$, $\exists K_6 \in (0, 1/c^2)$, $\forall x_1 \in \mathcal{H}_1, x_2 \in \mathcal{H}_2$, $\forall z \in \bar{D}_{-\alpha/2}$:

$$|g(z(x_1+x_2))| \leq K_4 |e^{K_5|zx_1|^{2N-\delta} + \frac{sz^2}{2}(K_6|x_2|^2_{\mathcal{B}_2} - \langle x_2, Bx_2\rangle)}|, \tag{16}$$

– *the function* $x \mapsto g^\alpha(x) \equiv g(e^{-i\alpha/2}x)$, $x \in \mathcal{H}$, *is continuous in the* $|\ |$*-norm.*

Then the infinite dimensional oscillatory integral with parameter s *and regularizing class* $\mathcal{S}_{-\alpha/2}$ *of the function* (11) *is well defined and it is given by*

$$\begin{aligned}\widetilde{\int_{\mathcal{H}}^{s}} e^{-\frac{s}{2}\langle x,(I-B)x\rangle - rV_{2N}(x)} g(x)dx = (2\pi)^{-dim(\mathcal{H}_1)/2} \int_{\mathcal{H}_1\times\mathcal{B}_2} e^{-\frac{1}{2}\langle x_1,(I-B)x_1\rangle} \\ e^{\frac{1}{2}\langle \omega_2, B\omega_2\rangle - rs^{-N}\tilde{V}_{2N}(x_1+\omega_2)} \tilde{g}^\alpha(|s|^{-1/2}(x_1+\omega_2)) d\mu(\omega_2)\times dx_1\end{aligned} \tag{17}$$

Proof. — The proof is completely analogous to the proof of theorem 2.4. Let us consider a sequence $\{P_n\}_{n\in\mathbb{N}}$ of finite dimensional projection operators on $\mathcal{H}_2$ converging strongly to the identity as $n \to \infty$. Because of the conditions on the parameters $s, r \in \mathbb{C}$, the regularized finite dimensional approximations of the integral

$$\begin{aligned}(2\pi s^{-1})^{-(n+dim(\mathcal{H}_1))/2} \int_{\mathcal{H}_1\times P_n\mathcal{H}_2} e^{-\frac{s}{2}\langle x_1,(I-B)x_1\rangle} e^{-\frac{s}{2}\langle P_nx_2,(I-B)P_nx_2\rangle - rV_{2N}(P_nx_2+x_1)} \\ g(P_nx_2+x_1)\phi(\delta(P_nx_2+x_1))dx_1\times d(P_nx_2)\end{aligned}$$

are equal to

$$\begin{aligned}(2\pi)^{-dim(\mathcal{H}_1)/2} &\int_{\mathcal{H}_1\times P_n\mathcal{H}_2} e^{-\frac{1}{2}\langle x_1,(I-B)x_1\rangle} e^{\frac{1}{2}\langle P_nx_2,BP_nx_2\rangle - rs^{-N}V_{2N}(P_nx_2+x_1)} \\ &g(s^{-1/2}(P_nx_2+x_1))\frac{e^{-\frac{1}{2}\|P_nx_2\|^2}}{(2\pi)^{n/2}} dx_1\times d(P_nx_2) \\ = (2\pi)^{-dim(\mathcal{H}_1)/2} &\int_{\mathcal{H}_1\times \mathcal{B}_2} e^{-\frac{1}{2}\langle x_1,(I-B)x_1\rangle} e^{\frac{1}{2}\langle P_nx_2,BP_nx_2\rangle - rs^{-N}\tilde{V}_{2N}(P_nx_2+x_1)} \\ &\tilde{g}^\alpha(|s|^{-1/2}(P_nx_2+x_1))dx_1\times d\mu(\omega_2).\end{aligned} \tag{18}$$

As by our hypothesis we have the inequality

$$\begin{aligned}|e^{-\frac{1}{2}\langle x_1,(I-B)x_1\rangle} e^{\frac{1}{2}\langle P_nx_2,BP_nx_2\rangle - rs^{-N}\tilde{V}_{2N}(P_nx_2+x_1)} \tilde{g}^\alpha(|s|^{-1/2}(P_nx_2+x_1))| \\ \leq K_4 e^{K_5|s^{-1/2}x_1|^{2N-\delta}} e^{-\frac{1}{2}\langle x_1,(I-B)x_1\rangle - |r||s|^{-N}\cos(\beta-N\alpha)V_{2N}(x_1)} \\ e^{-|r||s|^{-N}\cos(\beta-N\alpha)K_3} e^{\frac{K_6}{2}|P_nx_2|^2_{\mathcal{B}_2}},\end{aligned}$$

the dominated convergence theorem can be applied and by letting $n \to \infty$ the integral (18) converges to the right hand side of (17). □

Remark 2.8. — In theorem 2.7 the convergence of the integral on the subspace $\mathscr{H}_1$ is due to the fast decreasing behavior of the function $e^{-rs^{-N}V_{2N}}$ instead of $e^{-\frac{1}{2}\langle\cdot,(I-B)\cdot\rangle}$, as the latter has an exponential growth on $\mathscr{H}_1$. For this reason the assumptions of theorem 2.7 include the condition that for any $\varphi \in \bar{I}_{-\alpha/2}$, the angle $\beta+2N\varphi$ is included in the *open* interval $(-\pi/2,\pi/2)$, instead of the closed one (as in theorem 2.4). On the other hand this restriction allows us to admit a stronger growth of the function g on the subspace $\mathscr{H}_1$ and to replace condition (10) of theorem 2.4 by condition (16).

3. The asymptotic expansion

In the following we shall put $s := \frac{s'}{\epsilon}$, $s' = |s'|e^{i\alpha}$, $r := \frac{r'}{\epsilon}$, with $\epsilon \in \mathbb{R}^+$ and s', r' satisfying the assumptions of theorem 2.4, and we shall study the asymptotic behavior of the integral

$$I(\epsilon) := \widetilde{\int_{\mathscr{H}}^{s}} e^{-\frac{s'}{2\epsilon}\langle x,(I-B)x\rangle - \frac{r'}{\epsilon}V_{2N}(x)} g(x)dx \tag{19}$$

in the limit $\epsilon \downarrow 0$. Let us assume the operator $B : \mathscr{H} \to \mathscr{H}$ be of trace class, symmetric and such that $I - B > 0$ and the functions V_{2N}, g satisfy the assumptions of the theorem 2.4. Let us denote by $g_{s'} : \mathscr{B} \to \mathbb{C}$ the function given by $g_{s'}(\omega) := \tilde{g}^\alpha(|s'|^{-1/2}\omega)$ ($\tilde{g}^\alpha$ being the stochastic extension of $x \mapsto g(e^{-i\alpha/2}x)$, $x \in \mathscr{H}$). Assume that g_s satisfies the following hypothesis:

1. $\forall\omega \in \mathscr{B}$, the function $\lambda \mapsto g_{s'}(\lambda\omega)$ is $2m-$times continuously differentiable in $\lambda \in \mathbb{R}$.
2. $\forall k = 1,\dots,2m$, $\exists$ a polynomial Q_k in the variables $|\bar{\lambda}|$ and $|\omega|$ such that $\forall\omega \in \mathscr{B}$, $\forall\bar{\lambda} \in \mathbb{R}$

$$\left|\frac{d^k}{d\lambda^k} g_{s'}(\lambda\omega)_{|\lambda=\bar{\lambda}}\right| \leq Q_k(|\bar{\lambda}|, |\omega|)$$

For notational simplicity in the following we shall adopt the short writing

$$g^{(k)}(\bar{\lambda},\omega) := \frac{d^k}{d\lambda^k} g_{s'}(\lambda\omega)_{|\lambda=\bar{\lambda}}.$$

The following holds:

Theorem 3.1. — *Under the assumptions above the integral $I(\epsilon)$ admits the following asymptotic expansion*

$$I(\epsilon) = \sum_{n=0}^{m-1} \epsilon^n C_n + O(\epsilon^m) \tag{20}$$

and the leading term is $C_0 = \det(I-B)^{-1/2}\tilde{g}(0)$.

Proof. — By equation (12) the integral $I(\epsilon)$ is equal to

$$\int_{\mathcal{B}} e^{\frac{1}{2}\langle\omega,B\omega\rangle - r's'^{-N}\epsilon^{N-1}\tilde{V}_{2N}(\omega)} g_{s'}(\sqrt{\epsilon}\omega)d\mu(\omega) \tag{21}$$

For any $\omega \in \mathcal{B}$, let us consider the function $f : \mathbb{R}^+ \to \mathbb{C}$, given by

$$f(\epsilon) := e^{-r's'^{-N}\epsilon^{N-1}\tilde{V}_{2N}(\omega)} g_{s'}(\sqrt{\epsilon}\omega), \qquad \epsilon \in \mathbb{R}^+$$

By expanding $f(\epsilon)$ in power series of $\sqrt{\epsilon}$ we get

$$f(\epsilon) = \sum_{n}^{2m-1} c_n \sqrt{\epsilon}^n + R_{2m}(\sqrt{\epsilon})$$

where

$$c_n = \sum_{k,l\,:\,k+(2N-2)l=n} \frac{1}{l!k!} g^{(k)}(0,\omega)(-r's'^{-N})^l \tilde{V}_{2N}(\omega)^l$$

and $R_{2m} = \frac{\epsilon^m}{(2m-1)!}\int_0^1 f^{(2m)}(t\sqrt{\epsilon})(1-t)^{2m-1}dt$, with

$$f^{(2m)}(\lambda) = \sum_{k=0}^{2m} \frac{2m!}{k!(2m-k)!} g^{(k)}(\lambda,\omega) P_{2m-k}(\lambda,\omega) e^{-r's'^{-N}\lambda^{2N-2}\tilde{V}_{2N}(\omega)},$$

and $P_k(\bar{\lambda},\omega)$ are polynomials (in λ and $V(\omega)$) defined by $\frac{d^k}{d\lambda^k}_{|\lambda=\bar{\lambda}} e^{-r's'^{-N}\lambda^{2N-2}\tilde{V}_{2N}(\omega)} = P_k(\bar{\lambda},\omega)e^{-r's'^{-N}\lambda^{2N-2}\tilde{V}_{2N}(\omega)}$. By substituting into (21) we get

$$I(\epsilon) = \sum_{n=0}^{m-1} C_n \epsilon^n + \mathcal{R}_m(\epsilon)$$

$$C_n = \sum_{k,l\,:\,k+(2N-2)l=2n} \frac{(-r')^l s'^{-Nl}}{l!k!} \int_{\mathcal{B}} e^{\frac{1}{2}\langle\omega,B\omega\rangle} g^{(k)}(0,\omega)\tilde{V}_{2N}(\omega)^l d\mu(\omega) \tag{22}$$

and

$$\mathcal{R}_m(\epsilon) = \frac{\epsilon^m}{(2m-1)!}\int_{\mathcal{B}}\int_0^1 e^{\frac{1}{2}\langle\omega,B\omega\rangle} f^{(2m)}(t\sqrt{\epsilon})(1-t)^{2m-1}dt\,d\mu(\omega).$$

By the assumptions on the function g, the integrals in the formula (22) are well defined, as well as the remainder that satisfies the following estimate

$$\begin{aligned}
|\mathcal{R}_m(\epsilon)| &\le \frac{\epsilon^m}{(2m-1)!}\int_{\mathcal{B}}\int_0^1 e^{\frac{1}{2}\langle\omega,B\omega\rangle}|f^{(2m)}(t\sqrt{\epsilon})|(1-t)^{2m-1}dt\,d\mu(\omega) \\
&= \frac{\epsilon^m}{(2m-1)!}\int_{\mathcal{B}}\int_0^1 e^{\frac{1}{2}\langle\omega,B\omega\rangle}\sum_{k=0}^{2m}\frac{2m!}{k!(2m-k)!}|g^{(k)}(t\sqrt{\epsilon},\omega)| \\
&\quad |P_{2m-k}(t\sqrt{\epsilon},\omega)|e^{-r's'^{-N}t\epsilon^{N-1}\tilde{V}_{2N}(\omega)}(1-t)^{2m-1}dt\,d\mu(\omega) \\
&\le \epsilon^m \int_{\mathcal{B}}\int_0^1 e^{\frac{1}{2}\langle\omega,B\omega\rangle}\mathcal{P}_m(t\sqrt{\epsilon},|\omega|)(1-t)^{2m-1}dt\,d\mu(\omega),
\end{aligned} \tag{23}$$

where $\mathscr{P}_m(\lambda, |\omega|)$ denotes a polynomial in the variables $\lambda, |\omega|$ and

$$\lim_{\epsilon\downarrow 0}\int_{\mathscr{B}}\int_0^1 e^{\frac{1}{2}\langle\omega,B\omega\rangle}\mathscr{P}_m(t\sqrt{\epsilon},|\omega|)(1-t)^{2m-1}dt\,d\mu(\omega)<\infty.$$

The leading term is given by

$$C_0=\tilde{g}(0)\int_{\mathscr{B}}e^{\frac{1}{2}\langle\omega,B\omega\rangle}d\mu(\omega)=\tilde{g}(0)\det(I-B)^{-1/2},$$

with $\det(I-B)$ being the Fredholm determinant of the operator $I-B$ (see Appendix). $\square$

Remark 3.2. — Theorem 3.1 allows one to handle the asymptotic behavior of infinite dimensional integrals with a complex phase function Φ of the form

$$\Phi(x):=-\frac{s'}{2}\langle x,(I-B)x\rangle-r'V_{2N}(x),\qquad x\in\mathscr{B}.$$

It generalizes both the Laplace method (for the study of the asymptotics of integrals with real phase functions) and the stationary phase method (for the study of the asymptotics of integrals with purely imaginary phase functions). According to theorem 3.1, the only critical point contributing to the asymptotic behavior is the origin $x=0$. Indeed one can easily verify that the only *real* stationary point of the phase functional is $x=0$ and formula (20) is the asymptotic expansion around this critical point.

If the operator $(I-B):\mathscr{H}\to\mathscr{H}$ is not strictly positive, the results of theorem 3.1 are no longer valid. For instance, in the case where $(I-B)$ has a non trivial kernel, the phase function $\Phi:\mathscr{H}\to\mathbb{C}$,

$$\Phi(x):=-\frac{s'}{2}\langle x,(I-B)x\rangle-r'V_{2N}(x)$$

has a degenerate critical point in $x=0$, i.e. $\Phi'(0)=0$ and $Ker\Phi''(0)\neq\{0\}$. In the case where the negative eigenspace of the operator $I-B$ is not empty, the phase function Φ could have critical points $x_c\in\mathscr{H}$ different from 0 and the asymptotic behavior of the integral should be determined by these critical points. Let us consider for instance a factorisable integral of the following form:

$$I(\epsilon):=\widetilde{\int_{\mathscr{H}_1\times\mathscr{H}_2}^{s}}e^{-\frac{s'}{2\epsilon}\langle x_1,(I-B)x_1\rangle-\frac{s'}{2\epsilon}\langle x_2,(I-B)x_2\rangle-\frac{r'}{\epsilon}V_{2N}(x_1)-\frac{r'}{\epsilon}V_{2N}(x_2)}dx_1dx_2\tag{24}$$

where $\dim\mathscr{H}_1=1$. By theorem 2.7 $I(\epsilon)=I_1(\epsilon)I_2(\epsilon)$, with $I_2(\epsilon)=\int_{\mathscr{B}_2}e^{\frac{1}{2}\langle\omega_2,B\omega_2\rangle}e^{-r'(s')^{-N}\epsilon^{N-1}\tilde{V}_{2N}(\omega_2)}d\mu_2(\omega_2)$ satisfies the assumptions of theorem 3.1, and I_1 is of the form $I_1(\epsilon)=\int_{\mathbb{R}}e^{\frac{a^2}{2\epsilon}y^2-\frac{\lambda}{\epsilon}y^{2N}}dy$, with $a\geq 0$ and $\lambda\in\mathbb{C}^+$. In particular if $a=0,\lambda=1$, then $I_1(\epsilon)=\epsilon^{1/2N}\frac{\Gamma(1/2N)}{N}$, while if $a=1,\lambda=1/2N$, then $I_1(\epsilon)\sim e^{\frac{N-1}{2N\epsilon}}$ (where $\sim$ means that the quotient of both sides tends to 1 as $\epsilon\downarrow 0$). In the non factorisable case the situation is more involved. Indeed in principle one should apply an infinite dimensional version of the saddle point method and analyze the behavior of the integral around *non real* stationary points. Actually a detailed treatment of the saddle point method in the case where the dimension of the space on which the integral is performed is greater than 1 is still lacking (see however **[31]**). In the following we give an example of the study of the asymptotics of the integral in a degenerate (non factorisable case) and apply this result to the study of the trace of the heat semigroup with a polynomial potential.

4. A degenerate case

Let $(\mathcal{H}_{p,t}, \langle\ ,\ \rangle, \|\|)$ be the Hilbert space

$$\mathcal{H}_{p,t} := \{\gamma \in H^1([0,t];\mathbb{R}^d) :\ \gamma(0) = \gamma(t)\}$$

with inner product

$$\langle \gamma_1, \gamma_2 \rangle = \int_0^t \dot\gamma_1(\tau)\dot\gamma_2(\tau)d\tau + \int_0^t \gamma_1(\tau)\gamma_2(\tau)d\tau.$$

The present section is devoted to the study of the asymptotic behavior as $\epsilon \downarrow 0$ of an infinite dimensional Fresnel integral (with parameter s/ϵ) of the form

$$I(\epsilon) := \widetilde{\int_{\mathcal{H}_{p,t}}^{s/\epsilon}} e^{-\frac{s}{2\epsilon}\int_0^t \dot\gamma(\tau)^2 d\tau - \frac{r}{\epsilon}\int_0^t |\gamma(\tau)|^{2N} d\tau} d\gamma \tag{25}$$

with $N \in \mathbb{N}$, $N \geq 2$, and $s, r \in \mathbb{C}^+$ satisfying the assumptions of theorem 2.7. Heuristically the integral (25) can be written as "$\widetilde{\int}_{\mathcal{H}_{p,t}}^{1/\epsilon} e^{\frac{1}{\epsilon}\Phi(\gamma)} d\gamma$", where the phase function $\Phi : \mathcal{H}_{p,t} \to \mathbb{R}$ is given by

$$\Phi(\gamma) = -\frac{s}{2}\int_0^t \dot\gamma(\tau)^2 d\tau - r\int_0^t |\gamma(\tau)|^{2N} d\tau \tag{26}$$

and the asymptotic behavior of $I(\epsilon)$ should be determined by the stationary points of the phase functional Φ, i.e. the points such that

$$\Phi'(\gamma)(\phi) = 0, \qquad \forall \phi \in \mathcal{H}_{p,t},$$

Φ' being the Fréchet derivative. One can easily verify that the null path $\gamma = 0$ is a stationary point of Φ and it is degenerate, namely $\operatorname{Ker}(\Phi''(0))$ is not trivial. Indeed

$$\langle \Phi''(0)(\phi), \psi \rangle = -s\int_0^t \dot\phi(\tau)\dot\psi(\tau)d\tau := -s\langle \phi, (I+L)\psi \rangle, \tag{27}$$

where L is the unique self-adjoint operator on $\mathcal{H}_{p,t}$ defined by the quadratic form

$$\langle \phi, L\psi \rangle = -\int_0^t \phi(\tau)\psi(\tau)d\tau.$$

We easily see that L for any $\psi \in \mathcal{H}_{p,t}$ is given by:

$$(28)\quad L\psi(\tau) = \int_0^\tau \sinh(\tau-u)\psi(u)du - \frac{1}{(1-e^t)(1-e^{-t})}\int_0^t \sinh(\tau-u)\psi(u)du + \\ + \frac{1}{(1-e^t)(1-e^{-t})}\int_0^t \sinh(t+\tau-u)\psi(u)du,$$

The kernel of $I+L$ is given by the solution of the equation

$$(29)\quad \psi(\tau) + \frac{1}{(1-e^t)(1-e^{-t})}\int_0^t (\sinh(t+\tau-u) - \sinh(\tau-u))\psi(u)du + \\ + \int_0^\tau \sinh(\tau-u)\psi(u)du = 0$$

with the periodic condition $\psi(0) = \psi(t)$. By differentiating (29) twice, it is easy to see that if ψ satisfies (29) then

$$\ddot{\psi}(\tau) = 0, \qquad \forall \tau \in [0,t],$$

so that the only solutions of (29) satisfying the periodic condition $\psi(0) = \psi(t)$ are the constant paths. From (27) the kernel of $\Phi''(0)$ is the $d-$ dimensional subspace:

$$Ker[\Phi''(0)] = \{\gamma \in \mathcal{H}_{p,t} :\ \gamma(\tau) = x\ \forall \tau \in [0,t],\ x \in \mathbb{R}^d\}.$$

Let us decompose the Hilbert space $\mathcal{H}_{p,t}$ into the direct sum $\mathcal{H}_{p,t} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1 = Ker[\Phi''(0)]$ and $\mathcal{H}_2 = Ker[\Phi''(0)]^\perp$, $\gamma(\tau) = \gamma_1(\tau) + \gamma_2(\tau)$, $\gamma_1(\tau) = t^{-1}\int_0^t \gamma(u)du$, $\gamma_2(\tau) = \gamma(\tau) - \gamma_1(\tau)$. In particular

$$\mathcal{H}_2 = \{\gamma \in \mathcal{H}_{p,t}\ :\ \int_0^t \gamma(\tau)d\tau = 0\}.$$

As one can easily verify that for any $\gamma_2 \in \mathcal{H}_2, \gamma_1 \in \mathcal{H}_1$ one has

$$V_{2N}(\gamma_1+\gamma_2) - V_{2N}(\gamma_1) = \int_0^t |\gamma_1(\tau)+\gamma_2(\tau)|^{2N}d\tau - \int_0^t |\gamma_1(\tau)|^{2N}d\tau \geq 0,$$

the assumptions of theorem 2.7 (with $g=1$) are satisfied and

$$(30)\quad I(\epsilon) = (2\pi)^{-d/2}\int_{\mathcal{B}_2\times\mathcal{H}_1} e^{-\frac{1}{2}\langle\omega_2, L\omega_2\rangle - \lambda\epsilon^{N-1}\int_0^t |\gamma_1(\tau)+\omega_2(\tau)|^{2N}d\tau} d\mu_2(\omega_2)\times d\gamma_1 \\ = (2\pi)^{-d/2}\int_{\mathcal{B}_2\times\mathbb{R}^d} e^{-\frac{1}{2}\langle\omega_2, L\omega_2\rangle - \lambda\epsilon^{N-1}\int_0^t |\frac{y}{t}+\omega_2(\tau)|^{2N}d\tau} d\mu_2(\omega_2)\times dy,$$

where $\lambda = rs^{-N}$ and $(\mathcal{H}_2, \mathcal{B}_2)$ is the abstract Wiener space built on $\mathcal{H}_2$.
By putting $x := \sqrt{\epsilon}y/t$ and expanding the term $|\sqrt{\epsilon}\omega_2(\tau)+x|^{2N}$ we have

$$I(\epsilon) = \left(\frac{2\pi\epsilon}{t^2}\right)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{t\lambda}{\epsilon}|x|^{2N}} f(x,\epsilon)dx,$$

where

$$f(x,\epsilon)=\int_{\mathcal{B}_2} e^{-(\frac{1}{2}\langle\omega_2,L\omega_2\rangle+\frac{\lambda}{\epsilon}\int_0^t|\sqrt{\epsilon}\omega_2(\tau)+x|^{2N}d\tau-\frac{\lambda t}{\epsilon}|x|^{2N})}d\mu(\omega_2)$$
$$=\widetilde{\int_{\mathcal{H}_2}} e^{-(\frac{1}{2}\int_0^t\dot\gamma_2(\tau)^2d\tau+\frac{\lambda}{\epsilon}\int_0^t|\sqrt{\epsilon}\gamma_2(\tau)+x|^{2N}d\tau-\frac{\lambda t}{\epsilon}|x|^{2N})}d\gamma_2.$$

The asymptotic behavior of $f(x,\epsilon)$ as $\epsilon\downarrow 0$ can be simply determined by expanding the integrand in powers of ϵ. Indeed

$$f(x,\epsilon)=\widetilde{\int_{\mathcal{H}_2}} e^{-\frac{1}{2}(\langle\gamma_2,(I+L_x)\gamma_2\rangle}e^{-\frac{\lambda}{\epsilon}P_{2N}(x,\sqrt{\epsilon}\gamma_2)}d\gamma_2,$$

where $L_x:\mathcal{H}_2\to\mathcal{H}_2$ is the unique bounded self adjoint operator determined by the quadratic form

$$(31)\quad \langle\phi,(I+L_x)\psi\rangle=\int_0^t\dot\phi(\tau)\dot\psi(\tau)d\tau+2N\lambda|x|^{2N-2}\int_0^t\phi(\tau)\psi(\tau)d\tau$$
$$+4N(N-1)\lambda|x|^{2N-4}\int_0^t x\phi(\tau)\,x\psi(\tau)d\tau,\qquad \phi,\psi\in\mathcal{H}_2,$$

and one can easily see that L_x is given by

(32)
$$L_x\psi(\tau)=B\int_0^\tau\sinh(u-\tau)\psi(u)du+\frac{B}{(1-e^t)(1-e^{-t})}\int_0^t\sinh(\tau-u)\psi(u)du+$$
$$-\frac{B}{(1-e^t)(1-e^{-t})}\int_0^t\sinh(t+\tau-u)\psi(u)du.$$

B is the $d\times d$ matrix defined by $B:=A^2(x)-1_{d\times d}$ and

$$A^2(x)_{i,j}=2N\lambda|x|^{2N-2}\delta_i^j+4N(N-1)\lambda|x|^{2N-4}x_ix_j,\qquad i,j=1,\dots,d.$$

Moreover

$$(33)\quad P_{2N}(x,\sqrt{\epsilon}\gamma_2)=\int_0^t|\sqrt{\epsilon}\gamma_2(\tau)+x|^{2N}d\tau-t|x|^{2N}-2N|x|^{2N-2}\int_0^t\sqrt{\epsilon}x\gamma_2(\tau)d\tau$$
$$-\epsilon N|x|^{2N-2}\int_0^t|\gamma(\tau)|^2d\tau-2N(N-1)\epsilon|x|^{2N-4}\int_0^t(x\gamma(\tau))^2d\tau=:\epsilon^{3/2}g(x,\epsilon,\gamma_2)$$

(where we have used the fact that $\int_0^t\gamma_2(\tau)d\tau=0$ as $\gamma_2\in\mathcal{H}_2$), and for any x,γ_2 we have

$$(34)\quad \lim_{\epsilon\downarrow 0}g(x,\epsilon,\gamma_2)=\frac{N!}{(N-3)!3!}8|x|^{2N-6}\int_0^t(x\gamma_2(s))^3ds+$$
$$+2N(N-1)|x|^{2N-4}\int_0^t x\gamma_2(s)|\gamma_2(s)|^2ds.$$

By expanding $e^{-\lambda\epsilon^{1/2}g(x,\epsilon,\gamma_2)}$ around $\epsilon = 0$:

$$(35)\quad f(x,\epsilon) = \widetilde{\int_{\mathcal{H}_2}} e^{-\frac{1}{2}(\langle\gamma_2,(I+L_x)\gamma_2\rangle} e^{-\lambda\epsilon^{1/2}g(x,\epsilon,\gamma_2)}d\gamma_2 = f_1(x,\epsilon) - \lambda\epsilon^{1/2}f_2(x,\epsilon),$$

where

$$f_1(x,\epsilon) = \widetilde{\int_{\mathcal{H}_2}} e^{-\frac{1}{2}(\langle\gamma_2,(I+L_x)\gamma_2\rangle}d\gamma_2 = \det(I+L_x)^{-1/2}$$

and

$$(36)\qquad f_2(x,\epsilon) = \widetilde{\int_{\mathcal{H}_2}} g(x,\epsilon,\gamma_2)e^{-\frac{1}{2}(\langle\gamma_2,(I+L_x)\gamma_2\rangle}e^{-u\lambda\epsilon^{1/2}g(x,\epsilon,\gamma_2)}d\gamma_2,$$

with $u \in (0,1)$.
For the calculation of the spectrum $\sigma(L_x)$ of L_x, it is convenient to replace the standard basis of $\mathbb{R}^d$ with an orthonormal basis which diagonalizes the symmetric matrix $A^2(x)$. By denoting its eigenvalues by a_i^2, $i = 1,\dots,d$, it is easy to verify that the spectrum of L_x is given by $\sigma(L_x) = \{\lambda_{i,n}, i = 1,\dots,d,\ n = 1,2,\dots\}$, where

$$\lambda_{i,n} = \frac{a_i^2 - 1}{1 + \frac{4\pi^2n^2}{t^2}}, \qquad i = 1,\dots,d, \quad n = 1,2,\dots$$

are eigenvalues of multiplicity 2. By applying Lidskij's theorem [**45**] and the Hadamard factorization theorem (see [**47**], theorem 8.24) one gets

$$\det(I+L_x) = \begin{cases} \det\left(\frac{\cosh(A(x)t)-1}{A^2(x)(\cosh t-1)}\right), & \text{for } x \neq 0 \\ (2\cosh t - 2)^{-d}, & \text{for } x = 0 \end{cases}$$

The next result follows easily by the integral representation (36) of the function f_2.

Lemma 4.1. — *$f_2(x,\epsilon)$ is a C^∞ function of both $x \in \mathbb{R}^d$ and $\epsilon := \sqrt{\epsilon} \in \mathbb{R}^+$. Moreover for any $x \in \mathbb{R}^d$, $f_2(x,0) = 0$ and $\lim_{\epsilon\downarrow 0}\frac{f_2(x,\epsilon)-f_2(x,0)}{\epsilon^{1/2}} = C(x)$, where C is a positive function of $x \in \mathbb{R}^d$.*

Proof. — First of all we have

$$(37)\quad f_2(x,\epsilon) = \widetilde{\int_{\mathcal{H}_2}} e^{\frac{u\lambda t|x|^{2N}}{\epsilon}} g(x,\epsilon,\gamma_2)e^{-\frac{1}{2}\int_0^t\dot\gamma_2^2(s)ds}e^{-\frac{u\lambda}{\epsilon}\int_0^t|\sqrt{\epsilon}\gamma_2(s)+x|^{2N}ds}$$
$$e^{-\frac{1-u}{2}\left(2N|x|^{2N-2}\int_0^t|\gamma(s)|^2ds+4N(N-1)|x|^{2N-4}\int_0^t(x\gamma(s))^2ds\right)}d\gamma_2.$$

By expressing the infinite dimensional integral on the Hilbert space $\mathcal{H}_2$ as an integral on the abstract Wiener space $(i, \mathcal{H}_2, \mathcal{B}_2)$ associated with $\mathcal{H}_2$ one gets:

$$f_2(x,\epsilon) = e^{\frac{u\lambda t|x|^{2N}}{\epsilon}} \int_{\mathcal{B}_2} \tilde{g}(x,\epsilon,\omega_2) e^{\frac{1}{2}\langle \omega_2, L_0\omega_2\rangle} e^{-\frac{u\lambda}{\epsilon}\int_0^t |\sqrt{\epsilon}\omega_2(s)+x|^{2N} ds}$$
$$e^{-\frac{1-u}{2}\left(2N|x|^{2N-2}\int_0^t |\omega_2(s)|^2 ds + 4N(N-1)|x|^{2N-4}\int_0^t (x\omega_2(s))^2 ds\right)} d\mu(\omega_2), \tag{38}$$

where the functions

$$\omega_2 \mapsto \tilde{g}(x,\epsilon,\omega_2)$$

$$\omega_2 \mapsto \langle \omega_2, L_0\omega_2\rangle$$

$$\omega_2 \mapsto \int_0^t |\sqrt{\epsilon}\omega_2(s) + x|^{2N} ds$$

$$\omega_2 \mapsto 2N|x|^{2N-2}\int_0^t |\omega_2(s)|^2 ds + 4N(N-1)|x|^{2N-4}\int_0^t (x\omega_2(s))^2 ds$$

represent the stochastic extensions to $\mathcal{B}_2$ of the corresponding functions on $\mathcal{H}_2$. The stochastic extensions are well defined because of the regularity of the functions involved. Analogously

$$f_2(x,\epsilon) = \int_{\mathcal{B}_2} \tilde{g}(x,\epsilon,\omega_2) e^{-\frac{1}{2}(\langle \omega_2, L_x\omega_2\rangle} e^{-u\lambda\epsilon^{1/2}\tilde{g}(x,\epsilon,\omega_2)} d\mu(\omega_2). \tag{39}$$

Representation (38) shows the absolute convergence of the integrals involved, while representation (39) shows the regularity of f_2 as a function of $\sqrt{\epsilon}$.
By a direct computation we obtain

$$f_2(x,0) = \int_{\mathcal{B}_2} \tilde{g}(x,0,\omega_2) e^{-\frac{1}{2}(\langle \omega_2, L_x\omega_2\rangle} d\mu(\omega_2),$$

where

$$\tilde{g}(x,0,\omega_2) = \begin{cases} \frac{N!}{(N-3)!3!} 8|x|^{2N-6}\int_0^t (x\omega_2(s))^3 ds + \\ +2N(N-1)|x|^{2N-4}\int_0^t x\omega_2(s)|\omega_2(s)|^2 ds, & 2N \geq 6 \\ \\ 4\int_0^t x\omega_2(s)|\omega_2(s)|^2 ds, & 2N = 4 \end{cases} \tag{40}$$

and

$$\lim_{\epsilon\downarrow 0} \frac{f_2(x,\epsilon) - f_2(x,0)}{\epsilon^{1/2}} = \int_{\mathcal{B}_2} g_4(\omega_2, x) e^{-\frac{1}{2}(\langle \omega_2, L_x\omega_2\rangle} d\mu(\omega_2) < \infty, \tag{41}$$

where

$$g_4(\omega_2,x)=\begin{cases}\int_0^t|\omega_2(s)|^4ds, & 2N=4\\ 3|x|^2\int_0^t|\omega_2(s)|^4ds+12\int_0^t(x\omega_2(s))^2|\omega_2(s)|^2ds, & 2N=6\\ \binom{N}{2}|x|^{2N-4}\int_0^t|\omega_2(s)|^4ds & \\ \quad+4\binom{N}{2}\binom{N-2}{1}|x|^{2N-6}\int_0^t(x\omega_2(s))^2|\omega_2(s)|^2ds & \\ \quad+16\binom{N}{4}|x|^{2N-8}\int_0^t(x\omega_2(s))^4ds, & 2N\geq 8.\end{cases}\tag{42}$$
□

By equation (35), the integral $I(\epsilon)$ can be represented as the sum $I(\epsilon)=I_1(\epsilon)+I_2(\epsilon)$, where

$$I_1(\epsilon)=\Big(\frac{2\pi\epsilon}{t^2}\Big)^{-d/2}\int_{\mathbb{R}^d}e^{-\frac{t\lambda}{\epsilon}|x|^{2N}}f_1(x,\epsilon)dx,$$

$$I_2(\epsilon)=-\lambda\epsilon^{1/2}\Big(\frac{2\pi\epsilon}{t^2}\Big)^{-d/2}\int_{\mathbb{R}^d}e^{-\frac{t\lambda}{\epsilon}|x|^{2N}}f_2(x,\epsilon)dx$$

Lemma 4.2. — *$I_2(\epsilon)=O(\epsilon^{\frac{4-d}{2}-\frac{4-d}{2N}})$, as $\epsilon\downarrow 0$.*

Proof. — By scaling

$$\begin{aligned}I_2(\epsilon)&=-\lambda\epsilon^{1/2}t^d(2\pi)^{-d/2}\epsilon^{d/2N-d/2}\int_{\mathbb{R}^d}e^{-t\lambda|x|^{2N}}f_2(\epsilon^{1/2N}x,\epsilon)dx\\ &=-\lambda t^d(2\pi)^{-d/2}\epsilon^{d/2N-d/2+1/2}\int_{\mathbb{R}^d}e^{-t\lambda(1-u)|x|^{2N}}\int_{\mathcal{B}_2}\tilde g(\epsilon^{1/2N}x,\epsilon,\omega_2)\\ &\quad e^{-\frac{1-u}{2}\left(2N|\epsilon^{1/2N}x|^{2N-2}\int_0^t|\omega_2(s)|^2ds+4N(N-1)|\epsilon^{1/2N}x|^{2N-4}\int_0^t(\epsilon^{1/2N}x\omega_2(s))^2ds\right)}\\ &\quad e^{-\frac{u\lambda}{\epsilon}\int_0^t|\sqrt{\epsilon}\omega_2(s)+\epsilon^{1/2N}x|^{2N}ds}e^{\frac12\langle\omega_2,L_0\omega_2\rangle}d\mu(\omega_2)dx\end{aligned}\tag{43}$$

By the dominated convergence theorem, the definition (33) of the function g, lemma 4.1 and equation 41 we get:

$$\lim_{\epsilon\downarrow 0}\frac{I_2(\epsilon)}{\epsilon^{\frac{3-d}{2}-\frac{3-d}{2N}}}=-\lambda t^d(2\pi)^{-d/2}\int_{\mathbb{R}^d}e^{-t\lambda(1-u)|x|^{2N}}\int_{\mathcal{B}_2}\tilde g(x,0,\omega_2)e^{\frac12\langle\omega_2,L_0\omega_2\rangle}d\mu(\omega_2)dx=0,\tag{44}$$

where $\tilde g(x,0,\omega_2)$ is given by (40), and

$$\lim_{\epsilon\downarrow 0}\frac{I_2(\epsilon)}{\epsilon^{\frac{4-d}{2}-\frac{4-d}{2N}}}=-\lambda t^d(2\pi)^{-d/2}\int_{\mathbb{R}^d}e^{-t\lambda(1-u)|x|^{2N}}\int_{\mathcal{B}_2}g_4(\omega_2,x)e^{\frac12\langle\omega_2,L_0\omega_2\rangle}d\mu(\omega_2)dx<\infty,\tag{45}$$

$g_4(\omega_2,x)$ being given by (42). □

Lemma 4.3. — $I_1(\epsilon)=\epsilon^{-d\frac{N-1}{2N}}(\cosh t-1)^{d/2}2^{d/2}t^{-d/2N}\lambda^{-d/2N}\frac{\Gamma(d/2N)}{N\Gamma(d/2)}+O(\epsilon^{(2-d)\frac{N-1}{2N}})$ *as* $\epsilon\downarrow 0$.

Proof. —

$$
\begin{aligned}
(46)\quad I_1(\epsilon)&=\Big(\frac{2\pi\epsilon}{t^2}\Big)^{-d/2}\int_{\mathbb{R}^d}e^{-\frac{\lambda t}{\epsilon}|x|^{2N}}\det(I+L_x)^{-1/2}dx\\
&=\Big(\frac{2\pi\epsilon}{t^2}\Big)^{-d/2}\int_{\mathbb{R}^d}e^{-\frac{\lambda t}{\epsilon}|x|^{2N}}\det\Big(\frac{\cosh(A(x)t)-1}{A^2(x)(\cosh t-1)}\Big)^{-1/2}dx\\
&=t^d\Big(\frac{\cosh t-1}{2\pi\epsilon}\Big)^{d/2}\int_{\mathbb{R}^d}e^{-\frac{\lambda t}{\epsilon}|x|^{2N}}\det\Big(\frac{\cosh(A(x)t)-1}{A^2(x)}\Big)^{-1/2}dx
\end{aligned}
$$

By scaling

$$
\begin{aligned}
I_1(\epsilon)&=C_t\epsilon^{\frac{d}{2N}-\frac{d}{2}}\int_{\mathbb{R}^d}e^{-\lambda t|x|^{2N}}\det\Big(\frac{\cosh(A(\epsilon^{1/2N}x)t)-1}{A^2(\epsilon^{1/2N}x)}\Big)^{-1/2}dx\\
&=C_t\epsilon^{\frac{d}{2N}-\frac{d}{2}}\int_{\mathbb{R}^d}e^{-\lambda t|x|^{2N}}\det\Big(\frac{\cosh(\epsilon^{(N-1)/2N}A(x)t)-1}{\epsilon^{(N-1)/N}A^2(x)}\Big)^{-1/2}dx
\end{aligned}
$$

with $C_t=t^d\Big(\frac{\cosh t-1}{2\pi}\Big)^{d/2}$. Let $a_i^2(x)$, $i=1,...,d$ be the eigenvalues of the matrix $A^2(x)$. Then

$$
\begin{aligned}
I_1(\epsilon)&=C_t\epsilon^{\frac{d}{2N}-\frac{d}{2}}\int_{\mathbb{R}^d}e^{-\lambda t|x|^{2N}}\frac{\epsilon^{\frac{d(N-1)}{2N}}\prod_i a_i(x)}{\prod_i\sqrt{\cosh(\epsilon^{(N-1)/2N}a_i(x)t)-1}}dx\\
&=C_t\epsilon^{\frac{d}{2N}-\frac{d}{2}}\int_{\mathbb{R}^d}e^{-\lambda t|x|^{2N}}\frac{2^{d/2}t^{-d}}{\prod_i\sqrt{1+\frac{\cosh(\theta_i)}{12}\epsilon^{(N-1)/N}a_i^2(x)t^2}}dx\\
&=C_t\epsilon^{\frac{d}{2N}-\frac{d}{2}}2^{d/2}t^{-d}\int_{\mathbb{R}^d}e^{-\lambda t|x|^{2N}}\prod_i\Big(1-\frac{\frac{\cosh(\theta_i)}{24}\epsilon^{(N-1)/N}a_i^2(x)t^2}{(1+\frac{\xi_i\cosh(\theta_i)}{12}\epsilon^{(N-1)/N}a_i^2(x)t^2)^{3/2}}\Big)dx
\end{aligned}
$$

with $\theta_i\in(0,\epsilon^{(N-1)/2N}a_i(x)t)$ and $\xi_i\in(0,1)$. We have

$$I_1(\epsilon)=I_{1,1}(\epsilon)+I_{1,2}(\epsilon),$$

where the first term is equal to

$$
\begin{aligned}
I_{1,1}(\epsilon)&=\epsilon^{-d\frac{N-1}{2N}}\Big(\frac{\cosh t-1}{2\pi}\Big)^{d/2}2^{d/2}\int_{\mathbb{R}^d}e^{-\lambda t|x|^{2N}}dx\\
&=\epsilon^{-d\frac{N-1}{2N}}\Big(\frac{\cosh t-1}{2\pi}\Big)^{d/2}2^{d/2}t^{-d/2N}\lambda^{-d/2N}\int_{\mathbb{R}^d}e^{-|x|^{2N}}dx\\
&=\epsilon^{-d\frac{N-1}{2N}}(\cosh t-1)^{d/2}2^{d/2}t^{-d/2N}\lambda^{-d/2N}\frac{\Gamma(d/2N)}{N\Gamma(d/2)},
\end{aligned}
$$

and the second term is equal to

$$I_{1,2}(\epsilon)=\Big(\frac{\cosh t-1}{2\pi\epsilon}\Big)^{d/2}\epsilon^{d/2N}2^{d/2}\int_{\mathbb{R}^d}e^{-\lambda t|x|^{2N}}$$

$$\Big(\prod_i\Big(1-\frac{\frac{\cosh(\theta_i\epsilon^{(N-1)/2N}a_i(x)t)}{24}\epsilon^{(N-1)/N}a_i^2(x)t^2}{(1+\frac{\xi_i\cosh(\theta_i\epsilon^{(N-1)/2N}a_i(x)t)}{12}\epsilon^{(N-1)/N}a_i^2(x)t^2)^{3/2}}\Big)-1\Big)dx$$

and it satisfies the following relation

$$\lim_{\epsilon\downarrow 0}\frac{I_{1,2}(\epsilon)}{\epsilon^{-d\frac{N-1}{2N}+\frac{N-1}{N}}}=-\frac{t^2}{24}\Big(\frac{\cosh t-1}{2\pi}\Big)^{d/2}2^{d/2}\int_{\mathbb{R}^d}e^{-\lambda t|x|^{2N}}\sum_i a_i^2(x)dx<\infty.\qquad\square$$

By combining lemma 4.2 and 4.3 we get:

Theorem 4.4. — *As $\epsilon\downarrow 0$ the infinite dimensional oscillatory integral $I(\epsilon)$ (25) has the following asymptotic behavior:*

$$I(\epsilon)=\epsilon^{-d\frac{N-1}{2N}}(\cosh t-1)^{d/2}2^{d/2}t^{-d/2N}\lambda^{-d/2N}\frac{\Gamma(d/2N)}{N\Gamma(d/2)}+O(\epsilon^{(2-d)\frac{N-1}{2N}})\tag{47}$$

The latter result can be applied to the study of the asymptotic behavior of the trace $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$, $t>0$ of the heat semigroup, where $H:D(H)\subset L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ is the quantum mechanical Hamiltonian given on the dense set of vectors $\psi\in S(\mathbb{R}^d)$ by

$$H\psi(x)=-\frac{\hbar^2}{2}\Delta_x\psi(x)+V(x)\psi(x),\tag{48}$$

with $V(x)=\lambda|x|^{2N}$, $N\in\mathbb{N}$, $N\geq 2$, $\lambda>0$, $x\in\mathbb{R}^d$, $N\in\mathbb{N}$.
It is well known that H is an essentially self adjoint operator on $C_0^\infty(\mathbb{R}^d)$ (see **[42]**, theorem X.28). H is a positive operator and is the generator of an analytic semigroup, denoted by $e^{-\frac{t}{\hbar}H}$, $t\geq 0$ (the "heat semigroup" with potential V). Its trace $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$ is well defined as $V(x)$ is smooth and increases at least quadratically at infinity, hence the spectrum of H consists of (real positive) eigenvalues $\lambda_{\bar n}$, $\bar n\in\mathbb{N}^d$. By a standard WKB argument one can deduce that there exists a positive constant α (depending on N) with

$$\liminf_{|\bar n|\to\infty}\frac{\lambda_{\bar n}}{|\bar n|^\alpha}>0.$$

Theorem 4.5. — *The trace of the heat semigroup $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$, $t>0$, for H as in equation* (48)*, is given by the infinite dimensional Fresnel integral (with parameter $s=1/\hbar$, in the sense of definition 2.2)*

$$\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]=(2\cosh t-2)^{-d/2}\widetilde{\int_{\mathcal{H}_{p,t}}}e^{-\frac{1}{2\hbar}\int_0^t\dot\gamma(s)^2ds-\frac{\lambda}{\hbar}\int_0^t\gamma(s)^{2N}ds}d\gamma\tag{49}$$

For $\hbar\downarrow 0$ the following asymptotics holds:

$$\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]=\hbar^{-d\frac{N-1}{2N}}t^{-d/2N}\lambda^{-d/2N}\frac{\Gamma(d/2N)}{2^{d/2}N\Gamma(d/2)}+O(\hbar^{(2-d)\frac{N-1}{2N}})\tag{50}$$

Proof. — The proof of (49) is divided into 3 steps.
1_{st} *Step:* By Feynman-Kac formula (see e.g.[**45, 46**]) $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$ is given, for $t>0$ by:

$$\text{(51)}\quad \mathrm{Tr}[e^{-\frac{t}{\hbar}H}] = \int_{\mathbb{R}^d}\frac{dx}{(2\pi t)^{d/2}}\int_{C_{[0,t]}} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\alpha(s)+\sqrt{\hbar}x)ds}d\mu(\alpha)$$
$$= \int_{\mathbb{R}^d}\frac{dx}{(2\pi t)^{d/2}}\int_{C_{[0,t]}} e^{-\lambda\hbar^{N-1}\int_0^t|\alpha(s)+x|^{2N}ds}d\mu(\alpha)$$

where $C_{[0,t]}$ is the space of continuous paths $\alpha:[0,t]\to\mathbb{R}^d$ such that $\alpha(0)=\alpha(t)$ and μ is the Brownian bridge probability measure on it (see, e.g. [**46**] for this concept). Let us introduce the Hilbert spaces $Y_{0,t}$ and $Y_{p,t}$ of paths,

$$\{\gamma\in H^1(0,t;\mathbb{R}^d):\ \gamma(0)=\gamma(t)=0\}$$

with norms

$$\|\gamma\|^2_{Y_{0,t}}\equiv|\gamma| = \int_0^t\dot\gamma(s)^2ds.$$

$$\|\gamma\|^2_{Y_{p,t}}\equiv\|\gamma\| = \int_0^t\dot\gamma(s)^2ds+\int_0^t\gamma(s)^2ds.$$

It is well known that $(i,Y_{0,t},C_{[0,t]})$ is an abstract Wiener space.
First of all (see remark 2.6) the integral in (51) on $C_{[0,t]}$ with respect to the Brownian bridge measure can also be written in terms on an infinite dimensional integral (with parameter $s=1$) on the Hilbert space $Y_{0,t}$ (in the sense of definition 2.2):

$$\int_{C_{[0,t]}} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\alpha(s)+\sqrt{\hbar}x)ds}d\mu(\alpha) = \widetilde{\int_{Y_{0,t}}} e^{-\frac{1}{2}|\gamma|^2}e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\gamma(s)+\sqrt{\hbar}x)ds}d\gamma,$$

so that

$$\text{(52)}\qquad \mathrm{Tr}[e^{-\frac{t}{\hbar}H}] = \int_{\mathbb{R}^d}\frac{dx}{(2\pi t)^{d/2}}\widetilde{\int_{Y_{0,t}}} e^{-\frac{1}{2}|\gamma|^2}e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\gamma(s)+\sqrt{\hbar}x)ds}d\gamma$$

2_{nd} *Step:* By the transformation formula relating infinite dimensional integrals on Hilbert spaces with varying norms (theorem 2.3), we get a relation between the integral on $Y_{0,t}$ and the integral on $Y_{p,t}$. Indeed

$$\|\gamma\|^2 = |\gamma|^2+(\gamma,T\gamma)$$

where T is the unique self-adjoint trace class operator on $Y_{0,t}$ defined by the quadratic form

$$(\gamma_1,T\gamma_2) = \int_0^t\gamma_1(s)\gamma_2(s)ds.$$

Indeed (see [**12**] for details) $\eta = T\gamma$, $\gamma \in Y_{0,t}$ if and only if

$$\left\{\begin{array}{ll} \ddot{\eta}(s) + \gamma(s) = 0, & s \in [0,t] \\ \dot{\eta}(0) = 0 & \\ \dot{\eta}(t) = 0 & \end{array}\right. \tag{53}$$

and $\det(I+T) = \left(\frac{\sinh t}{t}\right)^d$. By inserting this into equation (9) we obtain:

$$\widetilde{\int_{Y_{0,t}}} e^{-\frac{1}{2}|\gamma|^2} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\gamma(s)+\sqrt{\hbar}x)ds} d\gamma$$
$$= \left(\frac{t}{\sinh t}\right)^{d/2} \widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}|\gamma|^2} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\gamma(s)+\sqrt{\hbar}x)ds} d\gamma$$

and by equation (52)

$$\mathrm{Tr}[e^{-\frac{t}{\hbar}H}] = \int_{\mathbb{R}^d} \frac{dx}{(2\pi \sinh t)^{d/2}} \widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}|\gamma|^2} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\gamma(s)+\sqrt{\hbar}x)ds} d\gamma \tag{54}$$

3_{rd} *Step:* The final step is a transformation of variable formula for integrals on the Hilbert space $\mathscr{H}_{p,t}$. $Y_{p,t}$ can be regarded as a subspace of $\mathscr{H}_{p,t}$ and any vector $\gamma \in \mathscr{H}_{p,t}$ can be written as a sum of a vector $\eta \in Y_{p,t}$ and a constant in the following way:

$$\gamma(s) = \eta(s) + x, \qquad s \in [0,t],\ \gamma \in \mathscr{H}_{p,t},\ \eta \in Y_{p,t},\ x = \gamma(0).$$

We have to compute a constant C_t such that for integrable functions f

$$\widetilde{\int_{\mathscr{H}_{p,t}}} e^{-\frac{1}{2}\|\gamma\|^2} f(\gamma) d\gamma = C_t \int_{\mathbb{R}^d} dx \widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta+x\|^2} f(\eta+x) d\eta.$$

By Fubini theorem

$$\widetilde{\int_{\mathscr{H}_{p,t}}} e^{-\frac{1}{2}\|\gamma\|^2} f(\gamma) d\gamma = \widetilde{\int_{Y_{p,t}^\perp}} \Big(\widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta+\xi\|^2} f(\eta+\xi) d\eta\Big) d\xi, \tag{55}$$

where $Y_{p,t}^\perp$ is the space orthogonal to $Y_{p,t}$ in $\mathscr{H}_{p,t}$. One can easily verify that $Y_{p,t}^\perp$ is $d-$dimensional and it is generated by the vectors $\{v_i\}_{i=1,\dots,d}$, with $v_i(s) = \hat{e}_i\left(\frac{e^s(1-e^{-t})+e^{-s}(e^t-1)}{2\sqrt{2}\sqrt{\sinh t(\cosh t-1)}}\right)$, $s \in [0,t]$, $\hat{e}_i$ being the i_{th} vector of the canonical basis in $\mathbb{R}^d$. The right hand side of (55) is equal to

$$\int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \Big(\widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta+\sum_i y_i v_i\|^2} f(\eta + \sum_i y_i v_i) d\eta\Big) dy,$$

where $\xi(s) = \sum_i y_i v_i(s)$, $i = 1, \dots, d$. By writing the finite dimensional approximation of $\widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta+\sum_i y_i v_i\|^2} f(\eta + \sum_i y_i v_1) d\eta$, by the formula for the change of variables in finite dimensional integrals and by noting that

$$\langle u_j, v_i\rangle_{\mathcal{H}_{p,t}} = \delta_i^j \frac{\sqrt{2\cosh t - 2}}{\sqrt{\sinh t}},$$

where $u_j \in \mathcal{H}_{p,t}$ is the vector given by $u_j(s) = \hat{e}_j$, $s \in [0,t]$, we get

$$\int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \Big(\widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta + \sum_i y_i v_i\|^2} f(\eta + \sum_i y_i v_i) d\eta \Big) dy$$
$$= \Big(\frac{\sqrt{2\cosh t - 2}}{\sqrt{\sinh t}} \Big)^d \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \Big(\widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta + \sum_i x_i u_i\|^2} f(\eta + \sum_i x_i u_i) d\eta \Big) dx,$$

so that the constant C_t is equal to $\left(\frac{\sqrt{2\cosh t - 2}}{\sqrt{2\pi \sinh t}} \right)^d$.
By combining these results we get equation (49).
The asymptotic behavior of the trace $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$ as $\hbar \downarrow 0$ follows by equation (49) and theorem 4.4. □

Remark 4.6. — In [**6, 12**] the representation (49) is proved for the case where V is a quadratic function plus a bounded perturbation (of the type of a Fourier transform of a complex measure) by means of a different technique (a Fubini theorem for infinite dimensional oscillatory integrals with respect to non-degenerate quadratic forms), that cannot be applied in our present case. Indeed the quadratic part of the phase function appearing in the integral on the right hand side of (49) can be written as

$$\int_0^t \dot{\gamma}^2(s) ds = -\langle \gamma, L\gamma \rangle,$$

with $L : \mathcal{H}_{p,t} \to \mathcal{H}_{p,t}$ is the operator (28). As we have seen, L is not invertible and $\det L = 0$. This fact forbids the application of the Fubini theorem as stated in [**6, 12**] and a direct application of the methods of [**6, 12**].

Remark 4.7. — A representation equivalent to (51) is discussed in [**46**] for other continuous potential V with $e^{-V} \in L^1$. However the limit $\hbar \downarrow 0$ discussed in [**46**] is not the semiclassical limit we discuss here. To the best of our knowledge our limit for our type of polynomially growing potentials has not been rigorously discussed before. In addition our result on this problem, besides coming as a direct application of a study concerning oscillatory integrals, also provides a method to derive an explicit expansion formula in fractional powers of $\hbar$ in terms of classical orbits (we shall however not provide here details on this, our main point was to indicate the method which permits us to obtain them).

Appendix
Abstract Wiener spaces

Let $(\mathcal{H}, \langle\ ,\ \rangle, \|\ \|)$ be a real real separable Hilbert space. Let ν be the finitely additive cylinder measure on $\mathcal{H}$, defined by its characteristic functional $\hat{\nu}(x) = e^{-\frac{1}{2}\|x\|^2}$. Let $|\ |$ be a "measurable" norm on $\mathcal{H}$, that is $|\ |$ is such that for every $\epsilon > 0$ there exist a finite-dimensional projection $P_\epsilon : \mathcal{H} \to \mathcal{H}$, such that for all $P \perp P_\epsilon$ one has

$$\nu(\{x \in \mathcal{H} |\ |P(x)| > \epsilon\}) < \epsilon,$$

where P and P_ϵ are called orthogonal ($P \perp P_\epsilon$) if their ranges are orthogonal in $(\mathcal{H}, \langle\ ,\ \rangle)$. One can easily verify that $|\ |$ is weaker than $\|\ \|$. Denoting by $\mathcal{B}$ the completion of $\mathcal{H}$ in the $|\ |$-norm and by i the continuous inclusion of $\mathcal{H}$ in $\mathcal{B}$, one can prove that $\mu \equiv \nu \circ i^{-1}$ is a countably additive Gaussian measure on the Borel subsets of $\mathcal{B}$. The triple $(i, \mathcal{H}, \mathcal{B})$ is called an *abstract Wiener space* (see, e.g., [**24, 32**]). Given $y \in \mathcal{B}^*$ one can easily verify that the restriction of y to $\mathcal{H}$ is continuous on $\mathcal{H}$, so that one can identify $\mathcal{B}^*$ as a subset of $\mathcal{H}$. Moreover $\mathcal{B}^*$ is dense in $\mathcal{H}$ and we have the dense continuous inclusions $\mathcal{B}^* \subset \mathcal{H} \subset \mathcal{B}$. Each element $y \in \mathcal{B}^*$ can be regarded as a random variable $n(y)$ on $(\mathcal{B}, \mu)$. A direct computation shows that $n(y)$ is normally distributed, with covariance $\|y\|^2$. More generally, given $y_1, y_2 \in \mathcal{B}^*$, one has

$$\int_{\mathcal{B}} n(y_1) n(y_2) d\mu = \langle y_1, y_2 \rangle.$$

The latter result allows the extension to the map $n : \mathcal{H} \to L^2(\mathcal{B}, \mu)$, because $\mathcal{B}^*$ is dense in $\mathcal{H}$. Given an orthogonal projection P in $\mathcal{H}$, with

$$P(x) = \sum_{i=1}^{n} \langle e_i, x \rangle e_i$$

for some orthonormal $e_1, \dots, e_n \in \mathcal{H}$, the stochastic extension $\tilde{P}$ of P on $\mathcal{B}$ is well defined by

$$\tilde{P}(\,\cdot\,) = \sum_{i=1}^{n} n(e_i)(\,\cdot\,) e_i.$$

Given a function $f : \mathcal{H} \to \mathcal{B}_1$, where $(\mathcal{B}_1, \|\ \|_{\mathcal{B}_1})$ is another real separable Banach space, the stochastic extension $\tilde{f}$ of f to $\mathcal{B}$ exists if the functions $f \circ \tilde{P} : \mathcal{B} \to \mathcal{B}_1$ converge to $\tilde{f}$ in probability with respect to μ as P converges strongly to the identity in $\mathcal{H}$. If $g : \mathcal{B} \to \mathcal{B}_1$ is continuous and $f := g|_{\mathcal{H}}$, then one can prove [**24**] that the stochastic extension of f is well defined and it is equal to g $\mu-$a.e. Moreover for any $h \in \mathcal{H}$ the sequence of random variables

$$\sum_{i=1}^{n} h_i n(e_i), \qquad h_i = \langle e_i, h \rangle$$

converges in $L^2(\mathscr{B},\mu)$, and by subsequences μ a.e., to the random variable $n(h)$.
Given a self-adjoint trace class operator $B:\mathscr{H}\to\mathscr{H}$, the quadratic form on $\mathscr{H}\times\mathscr{H}$:

$$x\in\mathscr{H}\mapsto\langle x,Bx\rangle$$

can be extended to a random variable on $\mathscr{B}$, denoted again by $\langle\,\cdot\,,B\,\cdot\,\rangle$. Indeed for each increasing sequence of finite dimensional projectors P_n converging strongly to the identity, $P_n(x)=\sum_{i=1}^n e_i\langle e_i,x\rangle$ ($\{e_i\}$ being a CONS in $\mathscr{H}$), the sequence of random variables

$$\omega\in\mathscr{B}\mapsto\sum_{i,j=1}^n\langle e_i,Be_j\rangle n(e_i)(\omega)n(e_j)(\omega)$$

is a Cauchy sequence in $L^1(\mathscr{B},\mu)$. By passing if necessary to a subsequence, it converges to $\langle\,\cdot\,,B\,\cdot\,\rangle$ $\mu-$a.e..
Let us assume that the largest eigenvalue of B is strictly less than 1 (or, in other words, that $(I-B)$ is strictly positive). Then one can prove that the random variable $g(\,\cdot\,):=e^{\frac{1}{2}\langle\,\cdot\,,B\,\cdot\,\rangle}$ is μ-summable. Indeed by considering a CONS $\{e_i\}$ made of eigenvectors of the operator B, b_i being the corresponding eigenvalues, the sequence of random variables

$$g_n:\mathscr{B}\to\mathbb{C},\qquad\omega\mapsto g_n(\omega)=e^{\frac{1}{2}\sum_{i=1}^n b_i([n(e_i)(\omega)]^2},$$

converges to $g(\omega)$ μ-a.e., as $n\to\infty$.
On the other hand one has

$$\int_{\mathscr{B}}g_n(\omega)d\mu(\omega)=\prod_{i=1}^n\int\frac{e^{-\frac{1}{2}(1-b_i)x_i^2}}{\sqrt{2\pi}}dx_i=(\prod_{i=1}^n(1-b_i))^{-1/2}$$

so that $\int g_n d\mu$ converges, as $n\to\infty$, to $(\det(I-B))^{-1/2}$, where $\det(I-B)$ denotes the Fredholm determinant of $(I-B)$, which is well defined as B is trace class. Moreover $0\leq g_n\leq g_{n+1}$ for each n . It follows that, as $n\to\infty$, $\int g_n d\mu\to\int g d\mu=(\det(I-B))^{-1/2}$.
By an analogous reasoning one can prove that, for any $y\in\mathscr{H}$, the sequence of random variables f_n:

$$\omega\mapsto f_n(\omega)=e^{\sum_{i=1}^n y_i n(e_i)(\omega)}e^{\frac{1}{2}\sum_{i=1}^n b_i([n(e_i)(\omega)]^2},$$

where $y_i=\langle y,e_i\rangle$, converges $\mu-$a.e. as n goes to ∞ to the random variable $f(\,\cdot\,)=e^{n(y)(\,\cdot\,)}e^{\frac{1}{2}\langle\,\cdot\,,B\,\cdot\,\rangle}$ and that

$$\int f_n d\mu\to\int f d\mu=(\det(I-B))^{-1/2}e^{\frac{1}{2}\langle y,(I-B)^{-1}y\rangle}\tag{56}$$

(see **[29, 32]**).

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Astérisque
327, 2009, p. 47–53

A NEW TECHNIQUE FOR PROVING UNIQUENESS FOR MARTINGALE PROBLEMS

by

Richard F. Bass & Edwin Perkins

Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — A new technique for proving uniqueness of martingale problems is introduced. The method is illustrated in the context of elliptic diffusions in $\mathbb{R}^d$.

***Résumé* (Une nouvelle technique pour démontrer l'unicité de la solution de problèmes de martingales)**
Une nouvelle technique est introduite pour démontrer l'unicité de la solution de problèmes de martingales. On applique les résultats aux diffusions elliptiques dans $\mathbb{R}^d$.

1. Introduction

When trying to prove uniqueness of a stochastic process corresponding to an operator, one of the most useful approaches is to consider the associated martingale problem. If $\mathscr{L}$ is an operator and w is a point in the state space $\mathscr{S}$, a probability $\mathbb{P}$ on the set of paths $t \to X_t$ taking values in $\mathscr{S}$ is a solution of the martingale problem for $\mathscr{L}$ started at w if $\mathbb{P}(X_0 = w) = 1$ and $f(X_t) - f(X_0) - \int_0^t \mathscr{L}f(X_s)\,ds$ is a martingale with respect to $\mathbb{P}$ for every f in an appropriate class $\mathscr{C}$ of functions.

The archetypical example is to let

$$\mathscr{L}f(x) = \sum_{i,j=1}^{d} a_{ij}(x) D_{ij} f(x). \tag{1.1}$$

Here, and for the rest of this paper, the state space $\mathscr{S}$ is $\mathbb{R}^d$, the probability measure is on the set of functions that are continuous maps from $[0,\infty)$ into $\mathbb{R}^d$ with the σ-field generated by the cylindrical sets, $D_{ij}f = \partial^2 f/\partial x_i \partial x_j$, and the class $\mathscr{C}$ of functions

2010 Mathematics Subject Classification. — Primary: 60J60; Secondary: 60H10.
Key words and phrases. — Martingale problem, uniqueness, diffusions, elliptic operators.

Research partially supported by NSF grant DMS-0244737.
Research partially supported by NSERC.

is the collection C_b^2 of C^2 functions which are bounded and whose first and second partial derivatives are bounded.

Stroock and Varadhan introduced the notion of martingale problem and proved in the case above that there was existence and uniqueness of the solution to the martingale problem provided the a_{ij} were bounded and continuous in x and the matrix $a(x)$ was strictly positive definite for each x. See **[2]** or **[4]** for an account of this result.

In this paper we present a new method of proving uniqueness for martingale problems. We illustrate it for the operator $\mathscr{L}$ given in (1.1) under the assumption that the a_{ij} are Hölder continuous in x. Our proof does not give as strong a result as that of Stroock and Varadhan in that we require Hölder continuity. (Actually, we only require a Dini-like condition, but this is still more than just requiring continuity.) In fact, when the a_{ij} are Hölder continuous, an older method using Schauder estimates can be applied.

Nevertheless our technique is applicable to situations for which no other known method seems to work. A precursor of our method, much disguised, was used in **[1]** to prove uniqueness for pure jump processes which were of variable order, i.e., the operator can not be viewed as a perturbation of a symmetric stable process of any fixed order. The result of **[1]** was improved in **[5]** to allow more general jump processes. Moreover our technique is useful in problems arising from certain infinite dimensional situations in the theory of stochastic partial differential equations and the theory of superprocesses; see **[3]**. Finally, even in the elliptic diffusion case considered here, the proof is elementary and short.

Stroock and Varadhan's method was essentially to view $\mathscr{L}$ given in (1.1) as a perturbation of the Laplacian with respect to the space L^p for appropriate p. The method using Schauder estimates views $\mathscr{L}$ as a perturbation of the Laplacian with respect to the Hölder space C^α for appropriate α. We use a quite different approach. We view $\mathscr{L}$ as a mixture of constant coefficient operators and use a mixture of the corresponding semigroups as an approximation of the semigroup for $\mathscr{L}$.

We use our method to prove the following theorem.

Theorem 1.1. — *Suppose $\mathscr{L}$ is given by (1.1), the matrices $a(x)$ are bounded and uniformly positive definite, and there exist c_1 and α such that*

$$|a_{ij}(x) - a_{ij}(y)| \le c_1(1 \wedge |x-y|^\alpha) \tag{1.2}$$

for all $i, j = 1, \ldots, d$ and all $x, y \in \mathbb{R}^d$. Then for each $w \in \mathbb{R}^d$ the solution to the martingale problem for $\mathscr{L}$ started at w is unique.

We do not consider existence, since that is much easier, and we have nothing to add to the existing proofs. The same comment applies to the inclusion of drift terms. In Section 2 we give some easy estimates and in Section 3 we prove Theorem 1.1. The letter c denotes constants whose exact value is unimportant and may change from occurrence to occurrence.

2. Some estimates

All the matrices we consider will be d by d, bounded, symmetric, and uniformly elliptic, that is, there exist constants Λ_m and Λ_M such that

$$\Lambda_m \sum_{i=1}^d z_i^2 \le \sum_{i,j=1}^d a_{ij} z_i z_j \le \Lambda_M \sum_{i=1}^d z_i^2, \qquad (z_1, \dots, z_d) \in \mathbb{R}^d. \tag{2.1}$$

Given any such matrix a, we use A for a^{-1}. It follows easily that

$$\sup_j \Big(\sum_{i=1}^d a_{ij}^2 \Big)^{1/2} \le \Lambda_M, \quad \sup_j \Big(\sum_{i=1}^d A_{ij}^2 \Big)^{1/2} \le \Lambda_m^{-1} \tag{2.2}$$

Define

$$p^a(t,x,y) = (2\pi t)^{-d/2} (\det a)^{-1/2} e^{-(y-x)^T A (y-x)/(2t)}, \tag{2.3}$$

and let

$$P_t^a f(x) = \int p^a(t,x,y) f(y)\, dy \tag{2.4}$$

be the corresponding transition operator. We assume throughout that the matrix valued function $a(y)$ satisfies the hypotheses of Theorem 1.1 and (2.1). Note that for a fixed, $p^a(t,x,y)\, dy$ is a Gaussian distribution for each x, but that $p^{a(y)}(t,x,y)\, dy$ need not be a probability measure. All numbered constants will depend only Λ_m, Λ_M and d.

We have the following.

Proposition 2.1. — *There exist c_1, c_2 and a function $c_3(p), p > 0$, depending only on Λ_M and Λ_m, such that for all $t, N, p > 0$ and $x \in \mathbb{R}^d$,*

(a) $\int p^{a(y)}(t,x,y)\, dy \le c_1$.

(b)

$$\int_{|y-x| > N/\sqrt{t}} p^{a(y)}(t,x,y)\, dy \le c_1 e^{-c_2 N^2}.$$

(c) For each $i \le d$,

$$\int \Big(\frac{|x_i - y_i|^2}{t} \Big)^p p^{a(y)}(t,x,y)\, dy \le c_3(p).$$

Proof. — For (a), after a change of variables $z = (y-x)/\sqrt{t}$, we need to bound

$$\int (2\pi)^{-d/2} (\det a(x + z\sqrt{t}))^{-1/2} e^{-z^T A(x+z\sqrt{t}) z/2}\, dz$$
$$\le \Big(\frac{\Lambda_M}{\Lambda_m} \Big)^{d/2} \int (2\pi \Lambda_M)^{-d/2} e^{-z^T z / 2\Lambda_M}\, dz \le \Big(\frac{\Lambda_M}{\Lambda_m} \Big)^{d/2}.$$

(b) and (c) are similar. □

Let $\|f\|$ be the C_0 norm of f.

Proposition 2.2. — *Let $g \in C^2$ with compact support and let*

$$F_\varepsilon(x) = \int g(y) p^{a(y)}(\varepsilon^2, x, y)\, dy.$$

Then $F_\varepsilon(x)$ converges to $g(x)$ boundedly and pointwise as $\varepsilon \to 0$.

Proof. — Because g is bounded, using Proposition 2.1(a) we see that the quantity $\sup_{\varepsilon>0} \|F_\varepsilon\|$ is finite. We next consider pointwise convergence. After a change of variables, we have

$$F_\varepsilon(x) = \int g(x+\varepsilon z)(2\pi)^{-d/2}(\det a(x+\varepsilon z))^{-1/2} e^{-z^T A(x+\varepsilon z)z/2}\, dz.$$

Since $|g(x+\varepsilon z) - g(x)| \le \varepsilon |z|\, \|\nabla g\|$, F_ε differs from

$$g(x)\int (2\pi)^{-d/2}(\det(a(x+\varepsilon z))^{-1/2} e^{-z^T A(x+\varepsilon z)z/2}\, dz$$

by at most

$$\|\nabla g\| \int (2\pi)^{-d/2}(\det(a(x+\varepsilon z)))^{-1/2} \varepsilon |z| e^{-z^T A(x+\varepsilon z)z/2}\, dz,$$

and this goes to 0 as $\varepsilon \to 0$ by a change of variables and Proposition 2.1(c) with $p = 1/2$. Let

$$V(\varepsilon, x, z) = (2\pi)^{-d/2}(\det(a(x+\varepsilon z)))^{-1/2} e^{-z^T A(x+\varepsilon z)z/2}.$$

It therefore suffices to show

$$\int V(\varepsilon, x, z)\, dz \to \int V(0, x, z)\, dz,$$

where we note this right-hand side is 1. Using Proposition 2.1(b) and the same change of variables, it suffices to show

$$\int_{|z|\le N} V(\varepsilon, x, z)\, dz \to \int_{|z|\le N} V(0, x, z)\, dz.$$

But this last follows by dominated convergence. □

Proposition 2.3. — *There exists a constant c_4 such that*

$$\int |a_{ij}(y) - a_{ij}(x)|\, |D_{ij} p^{a(y)}(t, x, y)|\, dy \le \begin{cases} c_4 t^{\frac{\alpha}{2}-1}, & t \le 1, \\ c_4 t^{-1}, & t \ge 1. \end{cases}$$

Proof. — A computation shows that

$$\begin{aligned} (2.5) \qquad & D_{ij} p^{a(y)}(t, x, y) \\ & = t^{-1} p^{a(y)}(t, x, y) \Big[\sum_k \sum_l \frac{(y_k - x_k) A_{ki}(y) A_{lj}(y)(y_l - x_l)}{t} - A_{ij}(y) \Big]. \end{aligned}$$

By (2.2) and Cauchy-Schwarz we have

$$\begin{aligned}
&\int |a_{ij}(y) - a_{ij}(x)|\, |D_{ij} p^{a(y)}(t,x,y)|\, dy \\
(2.6) \qquad &\leq \Big[\int |a_{ij}(y) - a_{ij}(x)| t^{-1} p^{a(y)}(t,x,y)[|x-y|^2 t^{-1} \Lambda_m^{-2} + \Lambda_m^{-1}]\, dy.
\end{aligned}$$

Suppose first that $t \leq 1$. By the Hölder condition on a the above is at most

$$\begin{aligned}
&c \int \frac{|y-x|^\alpha}{t^{\alpha/2}} \Big[\frac{|x-y|^2}{t} + 1\Big] p^{a(y)}(t,x,y)\, dy\, t^{\alpha/2-1} \\
&\leq ct^{\alpha/2-1},
\end{aligned}$$

where we have used Proposition 2.1(c) in the last inequality.

For the case $t > 1$ simply use the boundedness of a in (2.6) and Proposition 2.1 again to bound it by ct^{-1}. □

3. Proof of Theorem 1.1

For $f \in C_b^2$ and a a matrix with constant coefficients define

$$\mathcal{M}^a f(x) = \sum_{i,j=1}^{d} a_{ij} D_{ij} f(x).$$

Define the corresponding semigroup by (2.4), and let

$$R_\lambda^a f = \int_0^\infty e^{-\lambda t} P_t^a f\, dt.$$

For $f \in C_b^2$ we have

$$\mathcal{L} f(x) = \mathcal{M}^{a(x)} f(x).$$

Note that

$$(\lambda - \mathcal{M}^{a(y)}) R_\lambda^{a(y)} P_\varepsilon^{a(y)} f(x) = P_\varepsilon^{a(y)} f(x). \tag{3.1}$$

One way to verify that the superscript $a(y)$ does not cause any difficulty here is to check that

$$\sum_{i,j=1}^{d} a_{ij}(y) \frac{\partial^2}{\partial x_i \partial x_j} p^{a(y)}(s,x,y) = \frac{\partial}{\partial s} p^{a(y)}(s,x,y),$$

and then in the definition of $R_\lambda^{a(y)}$ use integration by parts in the time variable. By replacing ε with $\varepsilon/2$, setting $f(z) = p^{a(y)}(\varepsilon/2, z, y)$ and using Chapman-Kolmogorov, we see that (3.1) implies

$$(\lambda - \mathcal{M}^{a(y)})(R_\lambda^{a(y)} p^{a(y)}(\varepsilon, \cdot, y))(x) = p^{a(y)}(\varepsilon, x, y). \tag{3.2}$$

We are now ready to prove Theorem 1.1.

Proof. — Suppose $\mathbb{P}_1, \mathbb{P}_2$ are two solutions to the martingale problem for $\mathscr{L}$ started at a point w. Define

$$S_\lambda^i f = \mathbb{E}_i \int_0^\infty e^{-\lambda t} f(X_t)\, dt, \qquad i = 1, 2,$$

and

$$S_\lambda^\Delta f = S_\lambda^1 f - S_\lambda^2.$$

We make two observations. First, because $\mathbb{P}_i$ need not come from a Markov process, $S_\lambda^i f$ is not a function, and so S_λ^Δ is a linear functional. Second, if

$$\Theta = \sup_{\|f\| \le 1} |S_\lambda^\Delta f|,$$

then $\Theta < \infty$.

If $f \in C_b^2$, then by the definition of the martingale problem

$$\mathbb{E}_i f(X_t) - f(w) = \mathbb{E}_i \int_0^t \mathscr{L} f(X_s)\, ds, \qquad i = 1, 2.$$

Multiply both sides by $\lambda e^{-\lambda t}$, integrate over t from 0 to ∞, and use Fubini to obtain

$$f(w) = S_\lambda^i(\lambda f - \mathscr{L} f), \qquad i = 1, 2,$$

or

$$S_\lambda^\Delta(\lambda f - \mathscr{L} f) = 0. \tag{3.3}$$

Let $g \in C^2$ with compact support and set

$$f_\varepsilon(x) = \int R_\lambda^{a(y)}(p^{a(y)}(\varepsilon, \cdot, y))(x) g(y)\, dy.$$

Since this is the same as

$$e^{-\lambda\varepsilon} \int \int_\varepsilon^\infty e^{-\lambda t} p^{a(y)}(t, x, y)\, dt\, g(y)\, dy,$$

we see that f_ε is in C_b^2 in x by dominated convergence.

To calculate $(\lambda - \mathscr{L}) f_\varepsilon$ it is easy to differentiate under the dy integral and so we may write

$$\begin{aligned}(\lambda - \mathscr{L}) f_\varepsilon(x) &= (\lambda - \mathscr{M}^{a(x)}) f_\varepsilon(x) \\ &= \int (\lambda - \mathscr{M}^{a(y)}) R_\lambda^{a(y)}(p^{a(y)}(\varepsilon, \cdot, y))(x) g(y)\, dy \\ &\quad + \int (\mathscr{M}^{a(y)} - \mathscr{M}^{a(x)}) R_\lambda^{a(y)}(p^{a(y)}(\varepsilon, \cdot, y))(x) g(y)\, dy \\ &:= I_\varepsilon(x) + J_\varepsilon(x).\end{aligned}$$

By Proposition 2.3,

$$\begin{aligned}|J_\varepsilon(x)| &\leq \sum_{i,j=1}^{d}\int_0^\infty e^{-\lambda t}\int |a_{ij}(y)-a_{ij}(x)| \\ &\qquad\qquad \times |D_{ij}p^{a(y)}(\varepsilon+t,x,y)|\,|g(y)|\,dy\,dt \\ &\leq d^2\|g\|\int_0^\infty e^{-\lambda t}c_4t^{-1}(t^{\alpha/2}\wedge 1)\,dt \\ &\leq \frac{1}{2}\|g\|,\end{aligned}$$

for $\lambda \geq \lambda_0(\alpha,d,c_4)$. By (3.2), $I_\varepsilon(x) = \int p^{a(y)}(\varepsilon,x,y)g(y)\,dy$, and so by Proposition 2.2, $I_\varepsilon(x)$ converges to g boundedly and pointwise. Since $S^\Delta_\lambda(\lambda-\mathscr{L})f_\varepsilon = 0$ by (3.3), we have $|S^\Delta_\lambda I_\varepsilon| = |S^\Delta_\lambda J_\varepsilon|$. Letting $\varepsilon \to 0$,

$$|S^\Delta_\lambda g| = \lim_{\varepsilon\to 0}|S^\Delta_\lambda I_\varepsilon| = \lim_{\varepsilon\to 0}|S^\Delta_\lambda J_\varepsilon| \leq \Theta\limsup_{\varepsilon\to 0}\|J_\varepsilon\| \leq \tfrac{1}{2}\Theta\|g\|.$$

Using a monotone class argument, the above inequality holds for all bounded g, and then taking the supremum over g such that $\|g\|\leq 1$, we have $\Theta \leq \frac{1}{2}\Theta$. Since $\Theta<\infty$, this implies that $\Theta = 0$.

From this point on, we use standard arguments. By the uniqueness of the Laplace transform together with continuity in t, $\mathbb{E}_1 f(X_t) = \mathbb{E}_2 f(X_t)$ for all t if f is continuous and bounded. Using regular conditional probabilities, one shows as usual that the finite dimensional distributions under $\mathbb{P}_1$ and $\mathbb{P}_2$ agree. This suffices to prove uniqueness; see **[2]** or **[4]** for details. □

Note that no localization argument is needed in the above proof.

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Astérisque
327, 2009, p. 55–68

FEYNMAN INTEGRALS AS HIDA DISTRIBUTIONS: THE CASE OF NON-PERTURBATIVE POTENTIALS

by

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Dedicated to Jean-Michel Bismut as a small token of appreciation

Abstract. — In this note the concepts of path integrals as generalized expectations of White Noise distributions is presented. Combining White Noise techniques with a generalized time-dependent Doss' formula Feynman integrands are constructed as Hida distributions beyond perturbation theory.

***Résumé* (Les intégrales de chemins comme distributions de Hida: le cas de potentiel non-perturbatif)**

Dans cette note, on introduit les intégrales de chemins comme étant des espérances de bruits blancs généralisés. On combine les techniques de bruits blancs avec une généralisation de la méthode de Doss pour construire les « intégrales » de Feynman comme distributions de Hida, au-delà de la théorie perturbative.

1. Introduction

Feynman "integrals", such as

$$J = \int d^{\infty}x \exp\left(i\int_0^t \left(T(\dot{x}(s)) - V\left(x(s)\right)\right) ds\right) f(x(\cdot))$$

are commonplace in physics and meaningless mathematically as they stand. Within white noise analysis [**1, 2, 9, 10, 12, 14, 15, 16, 17**] the concept of integral has a natural extension in the dual pairing of generalized and test functions and allows for the construction of generalized functions (the "Feynman integrands") for various classes of interaction potentials V, see e.g. [**5, 6, 7, 10, 11, 13, 17**], all of them by perturbative methods. This work extends this framework to the case where these fail, using complex scaling as in [**4**], see also [**3**].

In Section 2 we characterize Hida distributions. In Section 3 the U-functional is constructed, see Theorem 3.3. We prove in Section 4 that we obtain a solution of the

2010 Mathematics Subject Classification. — 60H40, 81S40.
Key words and phrases. — Feynman path integrals, white noise analysis.

Schroedinger equation, see Theorem 4.4. The strategy for a general construction of the Feynman integrand is provided in Section 5. Examples are given in Section 6.

2. White Noise Analysis

The white noise measure μ on Schwartz distribution space arises from the characteristic function

$$C(f) := \exp\Big(-\tfrac{1}{2}\|f\|_2^2\Big), \quad f \in S(\mathbb{R}),$$

via Minlos' theorem, see e.g. [**1, 9, 10**]:

$$C(f) = \int_{S'} \exp\big(i\langle \omega, f\rangle\big)\, d\mu(\omega).$$

Here $\langle \cdot, \cdot \rangle$ denotes the dual pairing of $S'(\mathbb{R})$ and $S(\mathbb{R})$. We define the space

$$\big(L^2\big) := L^2(S'(\mathbb{R}), \mathscr{B}, \mu).$$

In the sense of an L^2-limit to indicator functions $\mathbf{1}_{[0,t)}, t > 0$, a version of Wiener's Brownian motion is given by:

$$B(t,\omega) := \langle \omega, \mathbf{1}_{[0,t)}\rangle = \int_0^t \omega(s)\, ds, \quad t > 0.$$

One then constructs a Gel'fand triple:

$$(S) \subset L^2(\mu) \subset (S)'$$

of Hida test functions and distributions, see e.g. [**10**]. We introduce the T-transform of $\Phi \in (S)'$ by

$$(T\Phi)(g) := \big\langle\!\big\langle \Phi, \exp\big(i\langle \cdot, g\rangle\big)\big\rangle\!\big\rangle, \quad g \in S(\mathbb{R}),$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes the bilinear dual pairing between $(S)'$ and (S). Expectation extends to Hida distributions Φ by

$$E_\mu(\Phi) := \langle\!\langle \Phi, 1 \rangle\!\rangle.$$

Definition 2.1. — *A function $F : S(\mathbb{R}) \to \mathbb{C}$ is called U-functional if*

(i): *F is "ray-analytic": for all $g, h \in S(\mathbb{R})$ the mapping*

$$\mathbb{R} \ni y \mapsto F(g + yh) \in \mathbb{C}$$

has an analytic continuation to $\mathbb{C}$ as an entire function.

(ii): *F is uniformly bounded of order 2, i.e., there exist some constants $0 < K, D < \infty$ and a continuous norm $\|\cdot\|$ on $S(\mathbb{R})$ such that for all $w \in \mathbb{C}$, $g \in S(\mathbb{R})$*

$$|F(wg)| \leq K \exp(D|w|^2 \|g\|^2).$$

Theorem 2.2. — *The following statements are equivalent:*

(i): *$F : S(\mathbb{R}) \to \mathbb{C}$ is a U-functional.*

(ii): *F is the T-transform of a unique Hida distribution $\Phi \in (S)'$.*

For the proof and more see e.g. [**10**].

3. Hida distributions as candidates for Feynman Integrands

In this section we construct Hida distributions as candidates for the Feynman integrands. First we list which properties potentials must fulfill.

Assumption 3.1. — *For $\mathcal{O} \subset \mathbb{R}$ open, where $\mathbb{R} \setminus \mathcal{O}$ is a set of Lebesgue measure zero, we define the set $\mathcal{D} \subset \mathbb{C}$ by*

$$\mathcal{D} := \Big\{ x + \sqrt{i} y \;\Big|\; x \in \mathcal{O} \text{ and } y \in \mathbb{R} \Big\},$$

and consider analytic functions $V_0 : \mathcal{D} \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$. Let $0 \leq t \leq T < \infty$. We require that there exists an $0 < \varepsilon < 1$ and a function $I : \mathcal{D} \to \mathbb{R}$ such that its restriction to $\mathcal{O}$ is measurable and locally bounded and

$$E\left[\left|\exp\left(-i\int_0^t V_0\Big(z+\sqrt{i}B_s\Big)ds\right) f\Big(z+\sqrt{i}B_t\Big)\right| \exp\left(\frac{\varepsilon \|B\|^2_{\sup,T}}{2}\right)\right] \leq I(z), \quad z \in \mathcal{D}, \tag{3.1}$$

uniformly in $0 \leq t \leq T$. Here E denotes the expectation w.r.t. a Brownian motion B starting at 0. $\|\cdot\|_{\sup,T}$ denotes the supremum norm over $[0,T]$.

We shall consider time-dependent potentials of the form

$$\begin{aligned} V_{\dot g} : [0,T] \times \mathcal{D} &\to \mathbb{C} \\ (t,z) &\mapsto V_0(z) + \dot g(t) z \end{aligned} \tag{3.2}$$

for $g \in S(\mathbb{R})$.

Remark 3.2. — *One can show that (3.1) implies that*

$$E\left[\exp\left(-i\int_0^{t-t_0} V_{\dot g}\Big(t-s, z+\sqrt{i}B_s\Big)ds\right) f\Big(z+\sqrt{i}B_{t-t_0}\Big)\right],$$

is well-defined for all $g \in S(\mathbb{R})$, $0 \leq t_0 \leq t \leq T$ and $z \in \mathcal{D}$.

Theorem 3.3. — *Let $0 < T < \infty$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be Borel measurable, bounded with compact support. Moreover we assume that V_0 and f fulfill Assumption 3.1. Then for all $0 \leq t_0 \leq t \leq T$, the mapping*

$$\begin{aligned} &F_{\varphi,t,f,t_0} : S(\mathbb{R}) \to \mathbb{C} \\ &g \mapsto \exp\left(-\frac{1}{2}\int_{[t_0,t]^c} g^2(s)\,ds\right)\int_{\mathbb{R}} \exp(-ig(t_0)x)\varphi(x)\Big(G(g,t,t_0)\exp(ig(t)\cdot)f\Big)(x)\,dx \end{aligned} \tag{3.3}$$

is a U-functional where for $x \in \mathcal{O}$

$$(3.4)\quad \Big(G(g,t,t_0)\exp(ig(t)\cdot)f\Big)(x) := E\Bigg[\exp\Bigg(-i\int_0^{t-t_0} V_{\dot g}\Big(t-s, x+\sqrt{i}B_s\Big)\,ds\Bigg) \times \exp\Big(ig(t)\Big(x+\sqrt{i}B_{t-t_0}\Big)\Big) f\Big(x+\sqrt{i}B_{t-t_0}\Big)\Bigg].$$

Proof. — F_{φ,t,f,t_0} is well-defined: (3.4) is finite because of (3.1), and the integral in (3.3) exists since φ is bounded with compact support.

To show that F_{φ,t,f,t_0} is a U-functional we must verify two properties, see Definition 2.1.

First F_{φ,t,f,t_0} must have a "ray-analytic" continuation to $\mathbb{C}$ as an entire function. I.e., for all $g,h \in S(\mathbb{R})$ the mapping

$$\mathbb{R} \ni y \mapsto F_{\varphi,t,f,t_0}(g+yh) \in \mathbb{C}$$

has an entire extension to $\mathbb{C}$.

We note first that this is true for the expression

$$(3.5)\quad u(y) := \exp\Bigg(-i\int_0^{t-t_0} V_{\dot g + y\dot h}\Big(t-s, x+\sqrt{i}B_s\Big)\,ds\Bigg) \times \exp\Big(i\,(g+yh)\,(t)\Big(x+\sqrt{i}B_{t-t_0}\Big)\Big) f\Big(x+\sqrt{i}B_{t-t_0}\Big)$$

inside the expectation in (3.4). Hence the integral of u over any closed curve in $\mathbb{C}$ is zero. By Lebesgue dominated convergence the expectation $E[u(w)]$ is continuous in w. With Fubini

$$\oint E\left[u\left(w\right)\right] dw = E\left[\oint u\left(w\right) dw\right] = 0,$$

for all closed paths, hence by Morera $E(u(w))$ is entire. This extends to (3.3) since φ is bounded with compact support. Thus

$$\mathbb{C} \ni w \mapsto F_{\varphi,t,f,t_0}(g+wh) \in \mathbb{C}$$

is entire for all $0 \le t_0 \le t \le T$ and all $g,h \in S(\mathbb{R})$.

Verification is straightforward that F_{φ,t,f,t_0} is of 2nd order exponential growth, F_{φ,t,f,t_0} is a U-functional. □

One can show the same result by choosing the delta distribution $\delta_x,\ x \in \mathcal{O}$, instead of a test function φ:

Corollary 3.4. — *Let V_0 and f fulfill Assumption 3.1 and let $x \in \mathcal{O}$. Then for all $0 \le t_0 \le t \le T$ the mapping*

$$F_{\delta_x,t,f,t_0} : S(\mathbb{R}) \to \mathbb{C}$$

$$g \mapsto \exp\Bigg(-\frac{1}{2}\int_{[t_0,t]^c} g^2(s)\,ds\Bigg)\exp(-ig(t_0)x)\Big(G(g,t,t_0)\exp(ig(t)\cdot)\Big)f(x)$$

is a U-functional, where $\Big(G(g,t,t_0)\exp(ig(t)\cdot)\Big)f(x)$ is defined as in Theorem 3.3.

4. Solution to time-dependent Schrödinger equation

Assumption 4.1. — *Let $V_0 : \mathcal{D} \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ such that Assumption 3.1 is fulfilled and $V_{\dot g},\ g \in S(\mathbb{R})$, as in (3.2).*

(i): *For all $u, v, r, l \in [0,T]$ and all $z \in \mathcal{D}$ we require that*

$$E^1\Bigg[\Bigg|\exp\Bigg(-i\int_0^u V_{\dot g}\Big(v-s, z+\sqrt{i}B_s^1\Big)\,ds\Bigg)$$

$$\times E^2\Bigg[\exp\Bigg(-i\int_0^r V_{\dot g}\Big(l-s, z+\sqrt{i}B_u^1+\sqrt{i}B_s^2\Big)\,ds\Bigg)f\Big(z+\sqrt{i}B_u^1+\sqrt{i}B_r^2\Big)\Bigg]\Bigg|\Bigg] < \infty. \tag{4.1}$$

(ii): *For all $z \in \mathcal{D}$, $0 \le t_0 \le t \le T$ and some $0 < \varepsilon \le T$ the functions*

$$\omega \mapsto \sup_{0\le h\le\varepsilon}\Bigg|\Bigg(V_{\dot g}\Big(t, z+\sqrt{i}B_h(\omega)\Big)+\int_0^h \frac{\partial}{\partial t}V_{\dot g}\Big(t+h-s, z+\sqrt{i}B_s(\omega)\Big)\,ds\Bigg)$$

$$\times\exp\Bigg(-i\int_0^h V_{\dot g}\Big(t+h-s, z+\sqrt{i}B_s(\omega)\Big)\,ds\Bigg)f\Big(z+\sqrt{i}B_h(\omega)\Big)\Bigg| \tag{4.2}$$

and

$$\omega \mapsto \sup_{h\in[0,T]}\Bigg|\Delta E^2\Bigg[\exp\Bigg(-i\int_0^{t-t_0} V_{\dot g}\Big(t-s, z+\sqrt{i}B_h^1(\omega)+\sqrt{i}B_s^2\Big)\,ds\Bigg)$$

$$\times f\Big(z+\sqrt{i}B_h^1(\omega)+\sqrt{i}B_{t-t_0}^2\Big)\Bigg]\Bigg| \tag{4.3}$$

are integrable.

Here B^1 and B^2 are Brownian motions starting at 0 with corresponding expectations E^1 and E^2, respectively. Moreover Δ denotes $\frac{\partial^2}{\partial z^2}$ and $\frac{\partial}{\partial t}$ the derivative w.r.t. the first variable.

We define $H(\mathcal{D})$ to be the set of holomorphic functions from $\mathcal{D}$ to $\mathbb{C}$. As pointed out by H. Doss, see [**4**], under specified assumptions (similar to Assumption 3.1 and Assumption 4.1 (ii)) there is a solution $\psi : [0,T]\times\mathcal{D} \to \mathbb{C}$ to the time-independent Schrödinger equation, i.e., for all $t \in [0,T]$ and $x \in \mathcal{O}$

$$\begin{cases} i\frac{\partial}{\partial t}\psi(t,x) = -\frac{1}{2}\Delta\psi(t,x) + V_0(x)\psi(t,x) \\ \qquad\qquad \psi(0,x) = f(x), \end{cases}$$

which is given by

$$\psi(t,x) = E\Bigg[\exp\Bigg(-i\int_0^t V_0\Big(x+\sqrt{i}B_s\Big)ds\Bigg)f\Big(x+\sqrt{i}B_t\Big)\Bigg].$$

Remark 4.2. — *Let us consider the case of the free motion, i.e.,* $V_0 \equiv 0$. *We assume that* $f : \mathcal{D} \to \mathbb{C}$ *is an analytic function, such that* $E\Big[f\Big(z+\sqrt{i}B_t\Big)\Big]$, $z \in \mathcal{D}$, $0 \le t \le T$, *exists and is uniformly bounded on* $[0,T]$. *Moreover let*

$$\omega \mapsto \sup_{h\in[0,T]} \Big|\Delta f\Big(z+\sqrt{i}B_h(\omega)\Big)\Big|$$

be integrable, then

$$\frac{\partial}{\partial t}E\Big[f\Big(x+\sqrt{i}B_t\Big)\Big] = -i\frac{1}{2}\Delta E\Big[f\Big(x+\sqrt{i}B_t\Big)\Big],$$

for $x \in \mathcal{O}$, $0 \le t \le T$, *see* [**4**].

For our purpose a generalization to the time-dependent case

$$\begin{cases} i\frac{\partial}{\partial t}\big(U(t,t_0)f\big)(x) = \big(H(t)U(t,t_0)f\big)(x) \\ \qquad \big(U(t_0,t_0)f\big)(x) = f(x), \end{cases} \qquad x \in \mathcal{O},\ 0 \le t_0 \le t \le T, \tag{4.4}$$

where $H(t) := -\frac{1}{2}\Delta + V_{\dot g}(t,\cdot)$ for $g \in S(\mathbb{R})$ and $0 \le t \le T$, is necessary. In the following we show that the operator $U(t,t_0) : D(t,t_0) \subset H(\mathcal{D}) \to H(\mathcal{D})$, $0 \le t_0 \le t \le T$, given by

(4.5)

$$U(t,t_0)f(z) := E\left[\exp\left(-i\int_0^{t-t_0} V_{\dot g}\Big(t-s, z+\sqrt{i}B_s\Big)ds\right)f\Big(z+\sqrt{i}B_{t-t_0}\Big)\right],\ z \in \mathcal{D},$$

provides us with a solution to (4.4). Here by $D(t,t_0)$ we denote the set of functions $f \in H(\mathcal{D})$ such that the expectation in (4.5) is a well-defined object in $H(\mathcal{D})$.

Lemma 4.3. — *Let* V_0 *and* f *fulfill the Assumptions 3.1 and 4.1 then the operator* $U(t,t_0)$, $0 \le t_0 \le t \le T$, *as in (4.5), maps from* $D(t,t_0)$ *to* $H(\mathcal{D})$. *Moreover* $U(r,t_0)f \in D(t,r)$ *and one gets that*

$$U(t,t_0)f(z) = U(t,r)(U(r,t_0)f)(z),$$

for all $0 \le t_0 \le r \le t \le T$ *and* $z \in \mathcal{D}$.

Proof. — The property that $U(t,t_0)$, $0 \le t_0 \le t \le T$, as in (4.5), maps from $D(t,t_0)$ to $H(\mathcal{D})$ follows by using Morera and Assumption 3.1. The fact that $U(r,t_0)f \in D(t,r)$ follows by Assumption 4.1 (i). Let $0 \le t_0 \le r \le t \le T$ and $z \in D$, then one gets with the Markov property and the time-reversibility of Brownian motion that

$$\begin{aligned} U(t,t_0)f(z) &= E\left[\exp\left(-i\int_0^{t-t_0} V_{\dot g}\Big(t-s, z+\sqrt{i}B_s\Big)ds\right)f\Big(z+\sqrt{i}B_{t-t_0}\Big)\right] \\ &= E\left[\exp\left(-i\int_0^{t-r} V_{\dot g}\Big(t-s, z+\sqrt{i}B_s\Big)ds\right)\right. \\ &\quad\left.\times\exp\left(-i\int_{t-r}^{t-r+r-t_0} V_{\dot g}\Big(t-s, z+\sqrt{i}B_s\Big)ds\right)f\Big(z+\sqrt{i}B_{t-t_0}\Big)\right] \end{aligned} \tag{4.6}$$

$$= E\bigg[\exp\bigg(-i\int_0^{t-r} V_{\dot g}\Big(t-s, z+\sqrt{i}B_s\Big)ds\bigg)$$

$$\times \exp\bigg(-i\int_0^{r-t_0} V_{\dot g}\Big(r-s+t-r, z+\sqrt{i}B_{s+t-r}\Big)ds\bigg) f\Big(z+\sqrt{i}B_{t-r+r-t_0}\Big)\bigg]$$

$$= E^1\bigg[\exp\bigg(-i\int_0^{t-r} V_{\dot g}\Big(t-s, z+\sqrt{i}B^1_s\Big)ds\bigg)$$

$$\times E^2\bigg[\exp\bigg(-i\int_0^{r-t_0} V_{\dot g}\Big(r-s, z+\sqrt{i}B^1_{t-r}+\sqrt{i}B^2_s\Big)ds\bigg) f\Big(z+\sqrt{i}B^1_{t-r}+\sqrt{i}B^2_{r-t_0}\Big)\bigg]\bigg]$$

$$= U(t,r)\big(U(r,t_0)f\big)(z). \quad \square$$

One can show that by $U(t,t_0)$, $0 \le t_0 \le t \le T$, a pointwise-defined (unbounded) evolution system is given.

Theorem 4.4. — *Let $0 < T < \infty$, V_0, $V_{\dot g}$, $g \in S(\mathbb{R})$, as in (3.2), and f such that Assumption 3.1 and 4.1 are fulfilled. Then $U(t,t_0)f(x)$, $0 \le t_0 < t \le T$, $x \in \mathcal{O}$, given in (4.5) solves the Schrödinger equation (4.4).*

Proof. — Let $0 \le t_0 < t \le T$, $x \in \mathcal{O}$ and $g \in S(\mathbb{R})$. If we have a look at the difference quotient from the right side, we get with Lemma 4.3 that

$$\frac{\partial}{\partial t}^{+} U(t,t_0)f(x) = \lim_{h\searrow 0} \frac{U(t+h,t_0)-U(t,t_0)}{h} f(x)$$
$$= \lim_{h\searrow 0} \frac{U(t+h,t)-U(t,t)}{h} U(t,t_0)f(x).$$

Hence it is left to show that

$$\lim_{h\searrow 0} \frac{U(t+h,t)k(x)-U(t,t)k(x)}{h} = H(t)k(x),$$

for $k = U(t,t_0)f$. Note that

(4.7)

$$\lim_{h\searrow 0}\frac{1}{h}E\bigg[\exp\bigg(-i\int_0^{t+h-t} V_{\dot g}\Big(t+h-s, x+\sqrt{i}B_s\Big)ds\bigg)k\Big(x+\sqrt{i}B_h\Big)-k\Big(x+\sqrt{i}B_0\Big)\bigg]$$

$$= \lim_{h\searrow 0} E\bigg[\frac{1}{h}\exp\bigg(-i\int_0^{h} V_{\dot g}\Big(t+h-s, x+\sqrt{i}B_s\Big)ds\bigg)k\Big(x+\sqrt{i}B_h\Big)-\frac{1}{h}k\Big(x+\sqrt{i}B_h\Big)\bigg]$$

$$+ \lim_{h\searrow 0} E\bigg[\frac{1}{h}k\Big(x+\sqrt{i}B_h\Big)-\frac{1}{h}k\Big(x+\sqrt{i}B_0\Big)\bigg].$$

The integrand of the first summand yields

$$\lim_{h\searrow 0}\frac{1}{h}\bigg(\exp\bigg(-i\int_0^{h} V_{\dot g}\Big(t+h-s, x+\sqrt{i}B_s\Big)ds\bigg)-1\bigg)k\Big(x+\sqrt{i}B_h\Big)$$

$$= -\, iV_{\dot g}\Big(t, x + \sqrt{i}B_0\Big)k\Big(x + \sqrt{i}B_0\Big) = -iV_{\dot g}(t,x)k(x).$$

Hence by Assumption 4.1 (ii), the mean value theorem and Lebesgue dominated convergence

$$\lim_{h\searrow 0} E\left[\frac{1}{h}\left(\exp\left(-i\int_0^h V_{\dot g}\Big(t+h-s, x+\sqrt{i}B_s\Big)ds\right)-1\right)k\Big(x+\sqrt{i}B_h\Big)\right] = -iV_{\dot g}(t,x)k(x).$$

Moreover we know by Remark 4.2 and Assumption 4.1 (ii) that $E\Big[k(x+\sqrt{i}B_t)\Big]$ solves the free Schrödinger equation, hence

$$\lim_{h\searrow 0} E\Big[k\Big(x+\sqrt{i}B_h\Big) - k\Big(x+\sqrt{i}B_0\Big)\Big] = -\frac{i}{2}\Delta k(x).$$

Similar with

$$\begin{aligned}\frac{\partial}{\partial t}^{-} U(t,t_0)f(x) &= \lim_{h\searrow 0}\frac{U(t-h,t_0)-U(t,t_0)}{h}f(x)\\ &= \lim_{h\searrow 0}\frac{U(t-h,t-h)-U(t,t-h)}{h}U(t-h,t_0)f(x)\end{aligned}$$

one can show the same for the difference quotient from the left side. □

5. General construction of the Feynman integrand

Of course one is interested in the Feynman integrand I_{V_0} for a general class of potentials $V_0 : \mathcal{O} \to \mathbb{C}$, where $\mathbb{R}\setminus\mathcal{O}$ is of measure zero, having an analytic continuation to $\mathcal{D}$. I.e., we are interested in the Feynman integrand corresponding to the Hamiltonian

$$H = -\frac{1}{2}\Delta + V_0(q),$$

where q is the position operator, i.e.,

$$H\varphi(x) = -\frac{1}{2}\Delta\varphi(x) + V_0(x)\varphi(x), \qquad x \in \mathcal{O},$$

for suitable $\varphi : \mathbb{R} \to \mathbb{R}$ (see the introduction for a comprehensive list of references). In all cases it turned out that for a test function $g \in S(\mathbb{R})$ and $0 \le t_0 < t \le T$ we have that

$$(TI_{V_0})(g) = \exp\left(-\frac{1}{2}\left\|g\mathbf{1}_{[t_0,t]^c}\right\|^2 + ig(t)x - ig(t_0)x_0\right)K_{V_0}^{(\dot g)}(x,t|x_0,t_0), \tag{5.1}$$

where $K_{V_0}^{(\dot g)}(x,t|x_0,t_0)$ denotes the Green's function corresponding to the potential $V_{\dot g}$ (see [**8**] for a justification of (5.1) under natural assumptions on I_{V_0}). This leads us to the following definition (see e.g. [**6**]).

Definition 5.1. — *Let $V_0 : \mathcal{D} \to \mathbb{C}$ be an analytic potential, $f : \mathbb{C} \to \mathbb{C}$ an analytic initial state, $V_{\dot g}$, $g \in S(\mathbb{R})$, as in (3.2), and $\varphi : \mathbb{R} \to \mathbb{R}$, Borel measurable, bounded with compact support. Assume that V_0, $V_{\dot g}$ and f fulfill Assumption 3.1 and Assumption 4.1. Then by Theorem 3.3 one has that for all $0 \le t_0 \le t \le T$, the function F_{φ,t,f,t_0} exists and forms a U-functional. Moreover by Theorem 4.4 it follows that for all $x \in \mathcal{D}$ and all $0 \le t_0 \le t \le T$*

$$U_{\dot g}(t,t_0)f(x) = E\left[\exp\left(-i\int_0^{t-t_0} V_{\dot g}\left(s, x+\sqrt{i}B_s\right)ds\right) f\left(x+\sqrt{i}B_{t-t_0}\right)\right]$$

exists and solves the Schrödinger equation (4.4) corresponding to the Hamiltonian

$$H(t) = -\frac{1}{2}\Delta + V_0(q) + \dot g(t)q,$$

for all $g \in S(\mathbb{R})$. Then by Theorem 2.2 we define the Feynman integrand

$$I_{V_0,\varphi,f} := T^{-1}F_{\varphi,t,f,t_0} \in (S)'.$$

Definition 5.2. — *Again let $V_0 : \mathcal{D} \to \mathbb{C}$ be an analytic potential, $f : \mathbb{C} \to \mathbb{C}$ an analytic initial state, $V_{\dot g}$, $g \in S(\mathbb{R})$, as in (3.2) and $x \in \mathcal{O}$. Analogously with Theorem 2.2, Corollary 3.4 and Theorem 4.4 we define the Feynman integrand*

$$I_{V_0,\delta_x,f} := T^{-1}F_{\delta_x,t,f,t_0} \in (S)'.$$

Remark 5.3. — *Note that the Green's function $K_{V_0}^{(\dot g)}(x,t|x_0,t_0)$, if it exists, is the integral kernel of the operator $U_{\dot g}(t,t_0)$.*

6. Examples

To show the existence of the Feynman integrand for concrete examples one only has to verify Assumption 3.1 and 4.1. In this section we look at analytic potentials V_0 which are already considered in [**4**]. First we introduce the set of initial states f. For $m \in \mathbb{N}$ we choose the function

$$\begin{aligned} f_m &: \mathbb{C} \to \mathbb{C} \\ z &\mapsto (2^m m!)^{-\frac{1}{2}}(-1)^m \pi^{-\frac{1}{4}} e^{\frac{1}{2}z^2}\left(\frac{\partial}{\partial z}\right)^m e^{-z^2}. \end{aligned} \tag{6.1}$$

Note that the set of functions given by the restrictions of f_m, $m \in \mathbb{N}$, to $\mathbb{R}$ are the Hermite functions, whose span is a dense subset of $L^2(\mathbb{R})$.

Lemma 6.1. — *Let $k : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a measurable function and B a real-valued Brownian motion, then*

$$E\left[k(\|B\|_{\sup,T})\right] \le 2\left(\frac{2}{\pi T}\right)^{1/2} \int_0^\infty k(u)e^{-\frac{u^2}{2T}}du.$$

For the proof see [**4**, Sec.1, Lem.1].

Lemma 6.2. — *Let f_m, $m \in \mathbb{N}$, be as in (6.1). Then for all $l \in \mathbb{N}_0$ and $\varepsilon > 0$ there exists a locally bounded measurable function $c_{m,l} : \mathbb{C} \to \mathbb{R}^+$ such that*

$$\left|f_m^{(l)}\left(z+\sqrt{i}y\right)\right| \leq c_{m,l}(z)|y|^{m+l}\exp\left(\left(\frac{1}{2}+\frac{1}{\sqrt{2\varepsilon}}\right)|z|^2\right)\exp\left(\frac{\varepsilon}{2}|y|^2\right) \quad \text{for all } z\in\mathbb{C}, y\in\mathbb{R},$$

where $f_m^{(l)}$ denotes the l-th derivative of f.

6.1. The Feynman integrand for polynomial potentials. — Here for $n \in \mathbb{N}_0$ we have a look at the potential

$$\begin{aligned} V_0 : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto (-1)^{n+1}a_{4n+2}z^{4n+2} + \sum_{j=1}^{4n+1} a_j z^j, \end{aligned} \tag{6.2}$$

for $a_0, \ldots, a_{4n+1} \in \mathbb{C}$ and $a_{4n+2} > 0$. If we have a look at the function

$$y \mapsto -iV_{\dot{g}}\left(t, x+\sqrt{i}y\right)$$

for $g \in S(\mathbb{R})$, $x \in \mathbb{C}$ and $t \in [0,T]$, then it is easy to see that the term of highest order of the real part is given by $-a_{4n+2}y^{4n+2}$. So it follows that for all compact sets $K \subset \mathbb{C}$ there exists a constant $C_K > 0$ such that

$$\sup_{z\in K}\sup_{t\in[0,T]}\sup_{y\in\mathbb{R}}\left|\exp\left(|g(t)|\ (|z|+|y|) - iV_0\left(z+\sqrt{i}y\right)\right)\right| \leq C_K. \tag{6.3}$$

Hence the function

$$\omega \mapsto \exp\left(-i\int_0^t V_{\dot{g}}\left(s, z+\sqrt{i}B_s(\omega)\right)ds\right) \tag{6.4}$$

is bounded uniformly in $0 \leq t \leq T$ and locally uniformly in $z \in \mathbb{C}$.

Theorem 6.3. — *Let $0 < T < \infty$, V_0 as in (6.2) and f_m, $m \in \mathbb{N}$, as in (6.1). Then it is possible to define the corresponding Feynman integrand I_{V_0,φ,f_m}, φ Borel measurable, bounded with compact support, and I_{V_0,δ_x,f_m}, $x \in \mathbb{R}$, as in Definition 5.1 and Definition 5.2, respectively.*

Proof. — As discussed above V_0 and f_m are analytic. Moreover with Lemma 6.1 and

$$\begin{aligned} k_{z,l} : \mathbb{R}_0^+ &\to \mathbb{R}_0^+ \\ u &\mapsto c_{m,l}(z)u^{m+l}\exp\left(\left(\frac{1}{2}+\frac{1}{\sqrt{2\varepsilon}}\right)|z|^2\right)\exp\left(\frac{\varepsilon}{2}u^2\right), \end{aligned} \tag{6.5}$$

$l \in \mathbb{N}_0$, we get that

(6.6)
$$E\left[\exp\left(\frac{\varepsilon\|B\|^2_{\sup,T}}{2}\right)\left|f_m\left(z+\sqrt{i}B_t\right)\right|\right] \leq E\left[\exp\left(\frac{\varepsilon\|B\|^2_{\sup,T}}{2}\right)k_{z,0}(\|B\|_{\sup,T})\right] < \infty,$$

for $0 < \varepsilon < \frac{1}{2T}$, $z \in \mathbb{C}$ and $c_{m,l}$ as in Lemma 6.2. If we multiply the integrand in (6.6) with the bounded function in (6.4) we still have an integrable function for all $z \in \mathbb{C}$ and all $0 \leq t \leq T$. So for showing Assumption 3.1 one has to check whether there exists a function $I : \mathbb{C} \to \mathbb{R}^+$ whose restriction to $\mathbb{R}$ is locally bounded and measurable, such that relation (3.1) holds. It is easy to see that this is true for the function

$$\begin{aligned} &I : \mathbb{C} \to \mathbb{R}^+ \\ &\quad z \mapsto E\left[\left| \exp\left(Re\left(-i \int_0^t V_0\left(z + \sqrt{i} B_s\right) ds \right)\right)\right| \exp\left(\frac{\varepsilon \|B\|^2_{\sup,T}}{2} \right) k_{z,0}(\|B\|_{\sup,T}) \right]. \end{aligned}$$

The locally boundedness of the restriction of I to $\mathbb{R}$ follows from (6.3) and the fact that $c_{m,l}$ is locally bounded. Since $\mathcal{O} = \mathbb{R}$ one can choose an arbitrary φ, Borel measurable, bounded with compact support, to apply Theorem 3.3. Moreover if we omit the integration the assumptions of Corollary 3.4 are also fulfilled.

So it is only left to check whether Assumption 4.1 is fulfilled. To show (4.1) again by the boundedness of (6.4) one only has to show that

$$E^1\left[\left| E^2\left[f_m\left(z + \sqrt{i} B^1_{t-r} + \sqrt{i} B^2_{r-t_0}\right)\right]\right|\right] < \infty, \quad z \in \mathbb{C}, \quad 0 \leq t_0 \leq r \leq t \leq T.$$

But this follows directly by Lemma 6.1 and Lemma 6.2. To show Assumption 4.1 (ii) note first that differentiation and integration in (4.3) can be interchanged since the integrand is analytic and its derivatives are integrable. Since V is polynomial, using the functions $k_{z,0}$, $k_{z,1}$ and $k_{z,2}$, see (6.5), Lemma 6.1 and Lemma 6.2 one can show a estimate similar to (6.6) for (4.2) and (4.3), respectively. Hence they are integrable. $\square$

Remark 6.4. — *For $n = 0$ we are not dealing with the harmonic oscillator. Nevertheless it is possible to handle a potential of the form*

$$x \mapsto a_0 + a_1 x + a_2 x^2,$$

for $a_0, a_1 \in \mathbb{C}$ and $a_2 \in \mathbb{R}$ such that $a_2 < \frac{1}{2T^2}$. In this case the function in (6.4) might be unbounded. So one has to estimate the potential as in Lemma 6.2, separately.

6.2. Non-perturbative accessible potentials. — In this section $\mathcal{O} = \mathbb{R} \setminus \{b\}$, $b \in \mathbb{R}$. We first consider analytic potentials of the form

$$\begin{aligned} V_0 &: \mathcal{D} \to \mathbb{C} \\ z &\mapsto \exp\left(\log(a) - \frac{n}{2} \log\left((z-b)^2\right)\right), \end{aligned} \tag{6.7}$$

where $n \in \mathbb{N}$, $a \in \mathbb{C}$ and $b \in \mathbb{R}$. Note that for $x \in \mathcal{O}$ one has that $V(x) = \frac{a}{|x-b|^n}$.

Lemma 6.5. — *Let V_0 be defined as in (6.7). Then V_0 is analytic on $\mathcal{D}$ and for all $z \in \mathcal{D}$, $z = x + \sqrt{i}y$, $x \in \mathcal{O}$, $y \in \mathbb{R}$, and all $0 \le t \le T$ we get that*

$$\left|V_0\left(z+\sqrt{i}B_t\right)\right| = |a|\,\left|\exp\left(-\frac{n}{2}\log\left(\left(z-b+\sqrt{i}B_t\right)^2\right)\right| \le |a|\,\exp\left(-\frac{n}{2}\log\left(\frac{(x-b)^2}{2}\right)\right).$$

For the proof see [**4**].

Theorem 6.6. — *Let $0 < T < \infty$, $\mathcal{O} = \mathbb{R}\setminus\{b\}$, V_0 as in Lemma 6.5 and f_m, $m \in \mathbb{N}$, as in (6.1). Then it is possible to define the corresponding Feynman integrand I_{V_0,φ,f_m}, $\varphi : \mathbb{R}\setminus\{b\} \to \mathbb{C}$, Borel measurable, bounded with compact support and I_{V_0,δ_x,f_m}, $x \in \mathbb{R}\setminus\{b\}$, as in Definition 5.1 and Definition 5.2, respectively.*

Proof. — W.l.o.g. we set $a = 1$ and $b = 0$. Then $\mathcal{O} = \mathbb{R}\setminus\{0\}$. So let $z \in \mathcal{D}$, $z = x + \sqrt{i}y$, $x \in \mathcal{O}$, $y \in \mathbb{R}$, and $0 \le t \le T$. Again we have to check Assumption 3.1 and 4.1. From Lemma 6.5 know that V_0 is analytic on $\mathcal{D}$.

Now we check whether relation (3.1) is true. From Lemma 6.5 we know that

$$\begin{aligned}&\left|\exp\left(-i\int_0^t V_0\left(z+\sqrt{i}B_s\right)ds\right)\exp\left(\frac{\varepsilon\|B\|^2_{\sup,T}}{2}\right)f_m\left(z+\sqrt{i}B_t\right)\right|\\ &\le \exp\left(t\exp\left(-\frac{n}{2}\log\left(\frac{x^2}{2}\right)\right)\right)\exp\left(\frac{\varepsilon\|B\|^2_{\sup,T}}{2}\right)\left|f_m\left(z+\sqrt{i}B_t\right)\right|.\end{aligned}$$

So with

$$k_{z,l} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$$

$$u \mapsto \exp\left(T\exp\left(-\frac{n}{2}\log\left(\frac{x^2}{2}\right)\right)\right)c_{m,l}(z)u^{m+l}\exp\left(\left(\frac{1}{2}+\frac{1}{\sqrt{2}\varepsilon}\right)|z|^2\right)\exp\left(\frac{\varepsilon}{2}u^2\right), \tag{6.8}$$

$l \in \mathbb{N}_0$, Lemma 6.1 and Lemma 6.2 we get that

$$E\left[\left|\exp\left(-i\int_0^t V_0\left(z+\sqrt{i}B_s\right)ds\right)\right|\exp\left(\frac{\varepsilon\|B\|^2_{\sup,T}}{2}\right)\left|f_m\left(z+\sqrt{i}B_t\right)\right|\right]$$

$$\le 2\left(\frac{2}{\pi T}\right)^{1/2}\exp\left(T\exp\left(\frac{n}{2}\log\left(\frac{x^2}{2}\right)\right)\right) \tag{6.9}$$

$$\times\int_0^\infty c_{m,l}(z)u^{m+l}\exp\left(\left(\frac{1}{2}+\frac{1}{\sqrt{2}\varepsilon}\right)|z|^2\right)\exp\left(\frac{\varepsilon}{2}u^2\right)e^{-\frac{u^2}{2T}}\,du =: I(z), \tag{6.10}$$

for all $z \in \mathcal{D}$, $0 < \varepsilon < \frac{1}{4T}$ and $c_{m,l}$ as in Lemma 6.2. Again since $c_{m,l}$ is measurable and locally bounded it follows that the restriction of I to $\mathcal{O}$ is also measurable and locally bounded. Now we check whether Assumption 4.1 is true. Relation (4.1) follows by Lemma 6.1, Lemma 6.2 and Lemma 6.5. Again with Lemma 6.1, Lemma 6.2 and Lemma 6.5 and the functions $k_{z,0}$, $k_{z,1}$ and $k_{z,2}$ one can show integrability for (4.2) and (4.3), respectively. □

Corollary 6.7. — *In the same way one can also show the existence of the Feynman integrand for potentials of the form*

$$
\begin{aligned}
V_0 : \mathcal{D} &\to \mathbb{C} \\
z &\mapsto \frac{a}{(z-b)^n},
\end{aligned}
\tag{6.11}
$$

for $a \in \mathbb{C}$, $b \in \mathbb{R}$ and $n \in \mathbb{N}$. Moreover one can choose linear combination of the potentials given in (6.2),(6.7) and (6.11).

Acknowledgments

We would like to thank Michael Röckner for valuable discussions. Financial support of Project PTDC/MAT/67965/2006 and FCT, POCTI-219, FEDER is gratefully acknowledged.

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March 23, 2009

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Astérisque
327, 2009, p. 69–91

SMOOTH DENSITY OF CANONICAL STOCHASTIC DIFFERENTIAL EQUATION WITH JUMPS

by

Hiroshi Kunita

Dedicated to Professor J.-M. Bismut for his sixtieth birthday

Abstract. — We consider jump diffusion process ξ_t on $\mathbf{R}^d$ determined by a canonical SDE:

$$d\xi_t = \sum_{i=1}^m V_i(\xi_t) \diamond dZ_t^i + V_0(\xi_t)dt,$$

where $Z_t = (Z_t^1, ..., Z_t^m)$ is an m-dimensional Lévy process and $V_0, ..., V_m$ are smooth vector fields. We prove that the law of the solution ξ_t has a C^∞-density under the following two conditions. (1) The Lévy process Z_t is nondegenerate. (2) $\{V_0, V_1, ..., V_m\}$ can be degenerate but satisfies a uniform Hörmander condition (H). For the proof we make use of the Malliavin calculus on the Wiener-Poisson space studied by Ishikawa-Kunita.

Résumé **(Densité lisse pour les solutions d'équations différentielles stochastiques avec sauts)**

Nous considérons un processus de diffusion à sauts ξ_t dans $\mathbf{R}^d$ déterminé par une EDS canonique:

$$d\xi_t = \sum_{i=1}^m V_i(\xi_t) \diamond dZ_t^i + V_0(\xi_t)dt,$$

où $Z_t = (Z_t^1, ..., Z_t^m)$ est un processus de Lévy m-dimensionnel et $V_0, ..., V_m$ sont des champs de vecteurs. Nous montrons que la loi de ξ_t a une densité C^∞ si les conditions suivantes sont satisfaites. (1) Le processus de Lévy Z_t est non dégénéré. (2) La distribution $\{V_0, V_1, ..., V_m\}$ peut être dégénérée mais elle satisfait à une condition de Hörmander uniforme (H). Pour la démonstration, nous utilisons le calcul de Malliavin sur l'espace de Wiener-Poisson étudié par Ishikawa-Kunita.

1. Introduction and main results

Let $V_0, V_1, \cdots V_m$ be smooth vector fields on $\mathbf{R}^d$ whose derivatives (including higher orders) are all bounded. Let $Z_t = (Z_t^1, ..., Z_t^m), t \geq 0$ be an m-dimensional nondegenerate Lévy process. In this paper, we consider a jump diffusion determined by a

2010 Mathematics Subject Classification. — 60H07; 60J75.
Key words and phrases. — Malliavin calculus, jump process, canonical process, density function.

canonical SDE based on $\{V_0, V_1, \cdots, V_m\}$ and Z_t;

$$d\xi_t = \sum_{i=1}^{m} V_i(\xi_t) \diamond dZ_t^i + V_0(\xi_t)dt. \tag{1.1}$$

Canonical SDE's are studied in mathematical finance. Let Z_t be a one dimensional Lévy process. We consider a one dimensional linear canonical SDE.

$$dS_t = S_t \diamond dZ_t.$$

The solution starting from S_0 at time 0 is unique and it is written as $S_t := S_0 \exp Z_t$ (See Section 2). It is called a *geometric Lévy process*. The solution S_t describes the movement of a stock. If Z_t is a Lévy process with finite Lévy measure (a compound Poisson process), the process S_t is the *Merton model* or the *Kou model*, according as the normalized Lévy measure is a Gaussian distribution or a double exponential distribution, respectively. See **[16]**,**[8]**. The precise definition of the canonical SDE will be given at Section 2.

The main purpose of this paper is to show the existence of the smooth density for the law of the random variable ξ_t that is a solution of equation (1.1). For this purpose we need to assume suitable nondegenerate conditions both for the Lévy process Z_t and the family of vector fields $\{V_0, ..., V_m\}$.

We first consider the Lévy process. The Lévy process Z_t is represented for arbitrary $\delta > 0$, by

$$Z_t = \sigma W_t + \int_0^t \int_{0<|z|\le\delta} z\tilde{N}(drdz) + \int_0^t \int_{|z|>\delta} zN(drdz) + b_\delta t,$$

where σ is an $m \times m$-matrix, W_t is an m-dimensional standard Brownian motion. $N(dtdz)$ is a Poisson random measure which is independent of W_t with intensity $\hat{N}(dtdz) = dt\nu(dz)$, ν being the Lévy measure. Further, $\tilde{N}(dtdz) = N(dtdz) - \hat{N}(dtdz)$ and $b_\delta = (b_\delta^1, ..., b_\delta^m)$ is a drift vector. Set $A = (a_{ij}) = \sigma\sigma^T$. It is a covariance of the Gaussian part σW_1 (Lévy-Itô decomposition). Throughout this paper, we assume that the Lévy measure ν has finite moments of any order. Set $v(\rho) := \int_{|z|<\rho} |z|^2 \nu(dz)$. If there exists $\alpha \in (0, 2)$ such that

$$\liminf_{\rho \to 0} \frac{v(\rho)}{\rho^\alpha} > 0,$$

then the Lévy measure is said to satisfy an *order condition*. Note that the Lévy measure ν satisfying an order condition is an infinite measure: Indeed, we have $\nu(\{z; 0 < |z| < \delta\}) = \infty$ for any $\delta > 0$. In case of one dimensional Lévy process, the above order condition is known as a sufficient condition for the existence of the smooth density of the law of the Lévy process (Orey's theorem. See Sato **[20]**, Proposition 28.3). Then the law of the geometric Lévy process S_t has a smooth density if the order condition is satisfied.

Now we set $b_{ij}(\rho) = \int_{|z|\leq\rho} z^i z^j \nu(dz)/v(\rho)$ and $B(\rho) = (b_{ij}(\rho))$. The *infinitesimal covariance* B is a symmetric and nonnegative definite matrix, which coincides with the greatest lower bound of the matrix $B(\rho)$ as $\rho \to 0$. If the Lévy measure satisfies an order condition and the matrix $A + B$ is nondegenerate (invertible), then we say that the Lévy process is *nondegenerate*. In this paper, *we assume that the Lévy process Z_t is nondegenerate.*

We will next consider nondegenete properties for the family of vector fields $\{V_0, ..., V_m\}$. In Ishikawa-Kunita **[6]**, we studied the case where the family of vector fields $\{V_1, ..., V_m\}$ is *uniformly nondegenerate*, i.e., there exists a positive constant C such that the inequality

$$\sum_{i=1}^{m} |l^T V_i(x)|^2 \geq C|l|^2, \quad \forall x \in \mathbf{R}^d, \quad \forall l \in \mathbf{R}^d$$

holds valid, where l^T is the transpose of l and $l^T V(x)$ denotes the inner product of two vectors l and $V(x)$. We showed the existence of the smooth density of its law by applying Malliavin calculus on the Wiener-Poisson space.

In this paper we want to relax the above uniformly nondegenerate condition. Let $V_0, ..., V_m$ be C^∞-vector fields such that their derivatives (including higher orders) are all bounded. Then Lie brackets $[V_{i_1}[\cdots[V_{i_{n-1}}, X_{i_n}]\cdots]$, $i_1, ..., i_n \in \{0, 1, ..., m\}$ are bounded vector fields. We introduce families of vector fields. Let $\Sigma_0 = \{V_1, ..., V_m\}$ be a linear space of vector fields spanned by $V_1, ..., V_m$. Given $\delta > 0$, we set

$$\hat{V}_0^\delta = V_0 + \sum_{i=1}^{m} b_\delta^i V_i.$$

Set $\Sigma_0^\delta = \Sigma_0$ and define for $k = 1, 2, ...$

$$\Sigma_k^\delta = \left\{ [\hat{V}_0^\delta, V] + \frac{1}{2} \sum_{i,j=1}^{m} a_{ij}[V_i, [V_j, V]],\ [V_i, V], i = 1, ..., m, V \in \Sigma_{k-1}^\delta \right\}.$$

Theorem 1.1. — *Assume that for the family of vector fields $\{V_0, ..., V_m\}$ there exist a positive integer N_0 and a positive number δ_0 such that for any $0 < \delta < \delta_0$ the inequality*

$$\sum_{k=0}^{N_0} \sum_{V \in \Sigma_k^\delta} |l^T V(x)|^2 \geq C(\delta)|l|^2, \quad \forall x \in \mathbf{R}^d, \quad \forall l \in \mathbf{R}^d \tag{1.2}$$

holds valid, where $C(\delta)$ are positive numbers satisfying

$$\liminf_{\delta \to 0} C(\delta)/v(\delta)^2 = \infty.$$

Then for any initial random variable ξ_0 and $0 < T_0 < \infty$, the law of the solution ξ_{T_0} of the canonical SDE (1.1) has a C^∞-density.

The condition required for vector fields in the above theorem is complicated, since δ's are involved. We can replace it by a simpler one if we restrict the Lévy process Z_t to a simpler one, namely if we assume

$$b_0 = \lim_{\delta \to 0} b_\delta \quad \text{exists and is finite.} \tag{1.3}$$

The existence of b_0 is equivalent to that of $\lim_{\delta\to 0}\int_{\delta<|z|\le 1} z\nu(dz)$. In this case, it holds $b_0 = b_1 - \lim_{\delta\to 0}\int_{\delta<|z|\le 1} z\nu(dz)$. In particular, if the integral $\int_{0<|z|\le 1}|z|\nu(dz)$ is finite, b_0 exists and is finite. Hence for any stable process whose exponent is less than 1, b_0 exists. Further, if the Lévy measure ν is symmetric, b_0 exists and is equal to b_1 even if $\int_{0<|z|\le 1}|z|\nu(dz)$ is infinite. Hence for any symmetric stable process, b_0 exists and is equal to b_1.

Now, assume (1.3) and let $\delta \to 0$ in the Lévy-Itô decomposition of Z_t. Then we obtain

$$Z_t = \sigma W_t + \int_0^t \int_{|z|>0} zN(drdz) + b_0 t.$$

Hence b_0 can be regarded as the drift vector of the Lévy process Z_t. We define a new drift vector field $\hat{V}_0$ by

$$\hat{V}_0 = V_0 + \sum_{i=1}^m b_0^i V_i,$$

and introduce families of vector fields by $\Sigma_0 = \{V_1, ..., V_m\}$ and for $k = 1, ...$

$$\Sigma_k = \Big\{[\hat{V}_0, V] + \frac{1}{2}\sum_{i,j=1}^m a_{ij}[V_i,[V_j,V]],\ [V_i, V], i = 1, ..., m, V \in \Sigma_{k-1}\Big\}.$$

Theorem 1.2. — *Assume (1.3) for the Lévy process Z_t. Assume further that the family of vector fields $\{\hat{V}_0, V_1, ..., V_m\}$ satisfy the uniform Hörmander condition (H), i.e., there exists a positive integer N_0 and a positive constant C such that*

$$\sum_{k=0}^{N_0}\sum_{V\in\Sigma_k} |l^T V(x)|^2 \ge C|l|^2, \quad \forall x \in \mathbf{R}^d, \quad \forall l \in \mathbf{R}^d \tag{1.4}$$

holds valid. Then for any initial random variable ξ_0 and $0 < T_0 < \infty$, the law of the solution ξ_{T_0} of the canonical SDE (1.1) has a C^∞-density.

Observe that Theorem 1.2 indicates that both the canonical SDE with jumps and Stratonovich SDE (diffusion) have the common local criterion (Hörmander' condition) for the existence of the smooth density of their laws. This is partly because that we restrict our attention to small jumps of the SDE, ignoring the effect of big jumps. Loosely speaking, under an order condition, the solution of equation (1.1) could behave like a diffusion if sizes of jumps are small.

Perhaps, Bismut [**2**] is the first work toward the smooth density of the law of the solution of SDE with jumps, where he developed the Malliavin calculus for jump processes. After this fundamental work, the similar problem has been discussed in some different contexts by Léandre [**13**],[**14**],[**15**], Bichteler-Gravreau-Jacod [**1**], Komatsu-Takeuchi [**7**] and others. A common feature in the above works might be that they assumed for the Lévy measure ν the existence of a smooth density and an asymptotic of the density as $z \to 0$. Furthermore, a formula of integration by parts holds valid in these cases, which are shown through Girsanov's theorem for jump diffusion.

In our discussion any Lévy measure (singular or not) is allowed, as far as it satisfies an order condition. Then no formula of integration by parts is known. We take another approach to the Malliavin calculus, developed in Ishikawa-Kunita [**6**]. It will be presented in the next section.

2. Malliavin calculus for canonical SDE

Let $Z_t, t \geq 0$ be an m-dimensional Lévy process admitting the Lévy-Itô decomposition and let ξ_0 be an $\mathbf{R}^d$-valued random variable independent of Z_t. By the solution of equation (1.1) starting from ξ_0 at time 0, we mean a cadlag $\mathbf{R}^d$-valued semimartingale $\{\xi_t; t \geq 0\}$ adapted to $\mathscr{F}_t = \sigma(\xi_0, Z_r; r \leq t)$ satisfying

$$
\begin{aligned}
\xi_t &= \xi_0 + \sum_{i=1}^m \int_0^t V_i(\xi_r) \diamond dZ_r^i + \int_0^t V_0(\xi_r)dr \\
&= \xi_0 + \sum_{i,k=1}^m \int_0^t V_i(\xi_r)\sigma_{ik} \circ dW_r^k + \int_0^t \hat{V}_0^\delta(\xi_r)dr. \\
&\quad + \int_0^t \int_{|z|<\delta} \{\phi_1^z(\xi_{r-}) - \xi_{r-}\}\tilde{N}(drdz) \\
&\quad + \int_0^t \int_{|z|\geq\delta} \{\phi_1^z(\xi_{r-}) - \xi_{r-}\}N(drdz) \\
&\quad + \int_0^t \int_{|z|<\delta} \{\phi_1^z(\xi_r) - \xi_r - \sum_{i=1}^m z^i V_i(\xi_r)\}\hat{N}(drdz).
\end{aligned} \tag{2.1}
$$

Here " $\circ$ " denotes the Stratonovitch integral. Using Itô integral, it holds

$$
\begin{aligned}
&\sum_{k=1}^m \int_0^t V_i(\xi_r)\sigma_{ik} \circ dW_r^k \\
&= \sum_{k=1}^m \int_0^t V_i(\xi_{r-})\sigma_{ik} dW_r^k + \frac{1}{2}\sum_{j=1}^m a_{ij} \int_0^t \Big(\sum_{l=1}^d \frac{\partial V_i}{\partial x^l} V_j^l\Big)(\xi_{r-})dr.
\end{aligned}
$$

Further, for $z = (z^1, ..., z^m) \in \mathbf{R}^m$ $\phi_s^z, s \in \mathbf{R}$ is the one parameter group of diffeomorphisms generated by the vector field $\sum_{i=1}^m z^i V_i$, i.e., $\phi_s^z = \exp s(\sum_i z^i V_i)$.

The equation has a unique solution ξ_t^δ. It holds $\xi_t^\delta = \xi_t^{\delta'}$ for any $\delta > 0$ and $\delta' > 0$. Hence the common solution is denoted by ξ_t. In the case where $\xi_0 = x$, we denote the solution by $\xi_{0,t}(x)$. Then it has a modification such that the maps $\xi_{0,t}; \mathbf{R}^d \to \mathbf{R}^d$ are onto diffeomorphisms a.s. and further the Jacobian matrix $\nabla\xi_{0,t}(x)$ is invertible for any x a.s. It defines a stochastic flow of diffeomorphisms (Fujiwara-Kunita [**3**]). We have $\xi_t = \xi_{0,t}(\xi_0)$.

We will consider a one dimensional linear SDE $dS_t = S_t \diamond dZ_t$. In this case we have $V_1(x) = x$. Then it holds $(\exp szV_1)(x) = e^{sz}x$. Hence equation (2.1) is written by

$$\begin{aligned} S_t &= S_0 + \sigma\int_0^t S_{r-}dW_r + \frac{1}{2}\sigma^2\int_0^t S_{r-}dr + b_\delta\int_0^t S_{r-}dr \\ &\quad + \int_0^t\int_{0<|z|\le\delta}(e^z - 1)S_{r-}\tilde{N}(drdz) + \int_0^t\int_{|z|>\delta}(e^z - 1)S_{r-}N(drdz) \\ &\quad + \int_0^t\int_{0<|z|\le\delta}(e^z - 1 - z)S_{r-}dr\nu(dz). \end{aligned}$$

The solution is given by $S_t = S_0 \exp Z_t$. Indeed apply Itô's formula to the function $F(x) = e^x$ and the semimartingale Z_t (Theorem 2.5 in [**10**]). Then we find that $S_t := \exp Z_t$ satisfies the above equation.

Now, for the proof of theorems stated in Section 1, we need the Malliavin calculus on the Wiener-Poisson space studied in Ishikawa-Kunita [**6**]. We will quickly recall it. Let T_0 be an arbitrarily fixed positive number and let $U = [0, T_0] \times \mathbf{R}^m$. Elements of U are denoted by $u = (t, z)$. Let ε_u^+ be a perturbation of the Poisson random measure N such that $N(A) \circ \varepsilon_u^+ = N(A \cap \{u\}^c) + 1_A(u)$. If we apply $\varepsilon_{(t_1,z_1)}^+$ to the solution ξ_t of SDE (2.1), we have $\xi_t \circ \varepsilon_{(t_1,z_1)}^+ = \xi_t$ if $t_1 > t$ and $\xi_t \circ \varepsilon_{(t_1,z_1)}^+ = \xi_{t_1,t} \circ \phi_1^{z_1} \circ \xi_{t_1-}$ if $t_1 \le t$, where $\xi_{s,t} := \xi_{0,t} \circ \xi_{0,s}^{-1}$ are diffeomorphisms of $\mathbf{R}^d$, a.s.

For $\mathbf{u} = (u_1, ..., u_n)$, we set $\varepsilon_{\mathbf{u}}^+ = \varepsilon_{u_1}^+ \circ \cdots \circ \varepsilon_{u_n}^+$. Let $\mathbf{u} = ((t_1, z_1), ..., (t_n, z_n))$ where $t_1 < t_2 < \cdots < t_n$. Then $\xi_t^{\mathbf{u}} := \xi_t \circ \varepsilon_{\mathbf{u}}^+$ is represented by

$$\xi_t^{\mathbf{u}} = \xi_{t_i,t} \circ \phi_1^{z_i} \circ \xi_{t_{i-1},t_i-} \circ \cdots \circ \phi_1^{z_1} \circ \xi_{t_1-}, \quad \text{if} \quad t_i \le t < t_{i+1}.$$

Malliavin covariances R and $\tilde{K}$ of the random variable ξ_{T_0} with respect to the Wiener space and the Poisson space are defined by

$$\begin{aligned} R &= \int_0^{T_0} \nabla\xi_{t,T_0}(\xi_{t-})C(\xi_{t-})AC(\xi_{t-})^T\nabla\xi_{t,T_0}(\xi_{t-})^T dt, \\ \tilde{K} &= \int_0^{T_0} \nabla\xi_{t,T_0}(\xi_{t-})C(\xi_{t-})BC(\xi_{t-})^T\nabla\xi_{t,T_0}(\xi_{t-})^T dt, \end{aligned}$$

respectively. Here $\nabla\xi_{t,T_0}(x)$ is the Jacobian matrix of the map $\xi_{t,T_0}(x)$. The $d \times m$ matrix $C(x)$ is given by $C(x) = (V_1(x), ..., V_m(x))$.

We set $Q = R + \tilde{K}$ and call it as the *Malliavin covariance of* ξ_{T_0}. Set $Q^{\mathbf{u}} = Q \circ \varepsilon_{\mathbf{u}}^+$. Then $Q^{\mathbf{u}}$ is the Malliavin covariance of $\xi_{T_0}^{\mathbf{u}}$.

Now consider $\hat{Q} = \nabla\xi_{0,T_0}(\xi_0)^{-1}Q(\nabla\xi_{0,T_0}(\xi_0)^T)^{-1}$ (*modified Malliavin covariance* of ξ_{T_0}). It is written as

$$\hat{Q} = \int_0^{T_0} (\nabla\xi_{t-})^{-1}C(\xi_{t-})(A+B)C(\xi_{t-})^T(\nabla\xi_{t-}^T)^{-1}dt,$$

where $\nabla\xi_t = \nabla\xi_{0,t}(\xi_0)$. Then the modified Malliavin covariance $\hat{Q}^{\mathbf{u}}$ of $\xi_{T_0}^{\mathbf{u}}$ equals $\hat{Q}\circ\varepsilon_{\mathbf{u}}^+$.

A criterion for the existence of the smooth density of the law of ξ_{T_0} is given by the following.

Lemma 2.1. — *Assume that*

$$\sup_{\mathbf{u}\in A(1)^n}\sup_{l\in S_{d-1}} E[(l^T\hat{Q}^{\mathbf{u}}l)^{-p}] < \infty \tag{2.2}$$

holds for any positive integer n and $p>1$. Then the law of ξ_{T_0} has a C^∞-density.

Proof. — It is shown in **[6]**, Proposition 6.1 that if $Q^{\mathbf{u}}$ is invertible a.s. and

$$\sup_{\mathbf{u}\in A(1)^n}\sup_{l\in S_{d-1}} E[(l^TQ^{\mathbf{u}}l)^{-p}] < \infty \tag{2.3}$$

is satisfied for any positive integer n and $p>1$, then the law of ξ_{T_0} has a C^∞-density. Here, we set $A(1)=\{(t,z); t\in(0,T_0), |z|\le 1\}$ and $S_{d-1}=\{l\in\mathbf{R}^d; |l|=1\}$.

We will show that condition (2.2) implies condition (2.3). Note that (2.2) implies $\sup_{\mathbf{u}\in A(1)^n} E[\sup_{l\in S_{d-1}}(l^T\hat{Q}^{\mathbf{u}}l)^{-p}] < \infty$. Then the minimum eigenvalue $\Lambda_1^{\mathbf{u}}$ of the matrix $\hat{Q}^{\mathbf{u}}$ satisfies $\sup_{\mathbf{u}\in A(1)^n} E[(\Lambda_1^{\mathbf{u}})^{-p}] < \infty$ for any $p>1$. Since the equality $(Q^{\mathbf{u}})^{-1} = \nabla\xi_{0,T_0}^T(\hat{Q}^{\mathbf{u}})^{-1}\nabla\xi_{0,T_0}$ holds and $\nabla\xi_{0,T_0}\in L^p$ holds for any $p>1$,

$$\{(l^TQ^{\mathbf{u}}l)^{-1}, l\in S_{d-1}, \mathbf{u}\in A(1)^n\}$$

is also L^p bounded for any $p>1$. Thus we have (2.3). □

Theorem 2.2. — *Assume that for any $l\in S_{d-1}$ and $\mathbf{u}\in A(1)^n$, the random variable*

$$\sum_{V\in\Sigma_0}\int_0^{T_0}\left|l^T(\nabla\xi_t^{\mathbf{u}})^{-1}V(\xi_t^{\mathbf{u}})\right|^2dt$$

is strictly positive a.s. Assume further that for any $p>1$ and positive integer n there exists a positive constant $C_{n,p}$ such that

$$E\left[\left(\sum_{V\in\Sigma_0}\int_0^{T_0}\left|l^T(\nabla\xi_t^{\mathbf{u}})^{-1}V(\xi_t^{\mathbf{u}})\right|^2dt\right)^{-p}\right] < C_{n,p}, \tag{2.4}$$

for any $l\in S_{d-1}$ and $\mathbf{u}\in A(1)^n$. Then the law of ξ_{T_0} has a C^∞-density.

Proof. — Let $\lambda_1 > 0$ be the minimum eigen value of the matrix $A+B$. Then we have

$$l^T \hat{Q}^{\mathbf{u}} l \geq \lambda_1 \sum_{V \in \Sigma_0} \int_0^{T_0} |l^T (\nabla \xi_t^{\mathbf{u}})^{-1} V(\xi_t^{\mathbf{u}})|^2 dt.$$

Therefore the assertion follows from Lemma 2.1. □

The proof of our main theorem will be completed by checking the above criterion (2.4). However its process will be quite long. Our program for the proof is as follows. In Section 4, instead of the uniform Hörmander condition (H), we will present another criterion that ensures the existence of the smooth density of the law of ξ_{T_0} (Theorem 4.1). Sections 3,4 and 6 are devoted to the proof of Theorem 4.1. Section 3 is a preliminary part. We will discuss SDE governed by semimartingales $(\nabla \xi_t)^{-1} V(\xi_t)$, where V is a vector field. In Theorem 6.1 (Appendix), we obtain an estimate for probabilities of events concerned with these semimartingales, where "Komatsu-Takeuchi's key lemma" plays an important role. The estimate is analogous to the one obtained by Kusuoka-Stroock [**12**] or Norris [**17**] in case of diffusion process. The proof of Theorem 4.1 will be completed by proving criterion (2.4) through these estimates.

In Section 5 we show that the uniform Hörmander condition fulfills the criterion of Theorem 4.1 and then we give the proof of our main theorems (Theorems 1.1-1.2).

3. SDE's for derivatives of stochastic flow

Let $V(x)$ be a vector field. We begin by studying the SDE which governs $(\nabla \xi_t)^{-1} V(\xi_t)$.

Lemma 3.1. — *We have a.s.*

$$
\begin{aligned}
(\nabla\xi_t)^{-1}V(\xi_t) &= V(\xi_0) + \sum_{i,j=1}^{m}\int_0^t (\nabla\xi_{s-})^{-1}[V_i,V](\xi_{s-})\sigma_{ij}dW^j(s)\\
&+\frac{1}{2}\sum_{i,j=1}^{m} a_{ij}\int_0^t (\nabla\xi_{s-})^{-1}[V_i,[V_j,V]](\xi_{s-})ds\\
&+\int_0^t (\nabla\xi_{s-})^{-1}[\tilde{V}_0^{\delta},V](\xi_{s-})ds\\
&+\int_0^t\int_{|z|<\delta} (\nabla\xi_{s-})^{-1}\{\nabla\phi_1^z(\xi_{s-})^{-1}V(\phi_1^z\circ\xi_{s-}) - V(\xi_{s-})\}\tilde{N}(dsdz)\\
&+\int_0^t\int_{|z|\geq\delta} (\nabla\xi_{s-})^{-1}\{\nabla\phi_1^z(\xi_{s-})^{-1}V(\phi_1^z\circ\xi_{s-}) - V(\xi_{s-})\}N(dsdz)\\
&+\int_0^t\int_{|z|<\delta} (\nabla\xi_{s-})^{-1}\Big\{\nabla\phi_1^z(\xi_{s-})^{-1}V(\phi_1^z\circ\xi_{s-}) - V(\xi_{s-})\\
&\qquad -\sum_i z^i[V_i,V](\xi_{s-})\Big\}\hat{N}(dsdz),
\end{aligned}
$$

where $\nabla\phi_1^z(x)$ is the Jacobian matrix of $\phi_1^z(x); \mathbf{R}^d \to \mathbf{R}^d$ and $\nabla\phi_1^z(x)^{-1}$ is its inverse matrix.

Proof. — It is shown in Ishikawa-Kunita [**6**] that the inverse matrix $(\nabla\xi_t)^{-1}$ satisfies a.s.

$$
\begin{aligned}
(\nabla\xi_t)^{-1} &= I - \sum_{i,j}\int_0^t (\nabla\xi_{s-})^{-1}\nabla V_i(\xi_{s-})\sigma_{i,j}\circ dW^j(s)\\
&-\int_0^t (\nabla\xi_{s-})^{-1}\nabla\tilde{V}_0^{\delta}(\xi_{s-})ds\\
&+\int_0^t\int_{|z|<\delta} (\nabla\xi_{s-})^{-1}\{\nabla\phi_1^z(\xi_{s-})^{-1} - I\}\tilde{N}(dsdz)\\
&+\int_0^t\int_{|z|\geq\delta} (\nabla\xi_{s-})^{-1}\{\nabla\phi_1^z(\xi_{s-})^{-1} - I\}N(drdz)\\
&+\int_0^t\int_{|z|<\delta} (\nabla\xi_{s-})^{-1}\{\nabla\phi_1^z(\xi_{s-})^{-1} - I\\
&\qquad +\sum_i z^i\nabla V_i(\xi_{s-})\}\hat{N}(dsdz).
\end{aligned}
$$

On the other hand, in view of Itô's formula for semimartingale with jumps, we have

$$
\begin{aligned}
V(\xi_t) = V(\xi_0) &+ \sum_{i,j} \int_0^t \nabla V(\xi_{s-}) V_i(\xi_{s-}) \sigma_{ij} \circ dW^j(s) \\
&+ \int_0^t \nabla V(\xi_{s-}) \tilde{V}_0^\delta(\xi_{s-}) ds \\
&+ \int_0^t \int_{|z|<\delta} (V(\phi_1^z \circ \xi_{s-}) - V(\xi_{s-})) \tilde{N}(dsdz) \\
&+ \int_0^t \int_{|z|\geq\delta} (V(\phi_1^z \circ \xi_{s-}) - V(\xi_{s-})) N(dsdz) \\
&+ \int_0^t \int_{|z|<\delta} \{V(\phi_1^z \circ \xi_{s-}) - V(\xi_{s-}) \\
&\qquad - \sum_i z^i \nabla V(\xi_{s-}) V_i(\xi_{s-})\} \hat{N}(dsdz).
\end{aligned}
$$

For the product of two semimartingales $X_t = (\nabla \xi_t)^{-1}$ and $Y_t = V(\xi_t)$, we have the formula

$$
\begin{aligned}
X_t Y_t &= X_0 Y_0 + \int_0^t X_s \circ dY_s^c + \int_0^t (\circ dX_s^c) Y_s \\
&\quad + \int_0^t X_{s-} dY_s^d + \int_0^t dX_s^d Y_{s-} + [X^d, Y^d]_t,
\end{aligned}
$$

where X_t^c, Y_t^c are continuous parts of semimartingales X_t, Y_t, respectively and X_t^d, Y_t^d are discontinuous parts of X_t, Y_t, respectively. A direct application of the above formula implies the equation of the lemma. □

Now define

$$
\begin{aligned}
\Psi_0^\delta V(x) &= \frac{1}{2} \sum_{i,j=1}^m a_{ij} [V_i, [V_j, V]](x) \\
&+ [\hat{V}_0^\delta, V](x) + \int_{0<|z|\leq\delta} \Big(\nabla \phi_1^z(x)^{-1} V(\phi_1^z(x)) - V(x) \\
&- \sum_{i=1}^m [V_i, V](x) z^i \Big) \nu(dz),
\end{aligned} \tag{3.1}
$$

and set

$$
\Phi_s(z) V(x) := \nabla \phi_s^z(x)^{-1} V(\phi_s^z(x)) - V(x). \tag{3.2}
$$

To simplify notations, we introduce the following. We set $\mathcal{S} = \hat{\mathbf{R}}^m \cup \mathbf{R}^m \cup \{\Delta\}$, where $\hat{\mathbf{R}}^m$ is an m-dimensional Euclidean space. Elements of $\hat{\mathbf{R}}^m$ and $\mathbf{R}^m$ are denoted by $y=(y^1,...,y^m)$ and $z=(z^1,...,z^m)$, respectively. We define stochastic process $Y^{(1)}_{l,V}(t,v)$ with parameter $l\in S_{d-1}$, vector field V and $v\in\mathcal{S}$ by

$$\begin{aligned}
Y^{(1)}_{l,V}(t,\Delta) &= l^T(\nabla\xi_t)^{-1}\Psi_0^\delta V(\xi_t),\\
Y^{(1)}_{l,V}(t,y) &= \sum_{i=1}^m l^T(\nabla\xi_t)^{-1}[V_i,V](\xi_t)\frac{y^i}{|y|},\\
Y^{(1)}_{l,V}(t,z) &= l^T(\nabla\xi_t)^{-1}\frac{\Phi_1(z)}{|z|}V(\xi_t).
\end{aligned}$$

Let $W(dsdy)$ be a Gaussian orthogonal random measure on $[0,T_0]\times\hat{\mathbf{R}}^m$ such that $E[W(dsdy)]=0$ and $\sigma W_t=\int_0^t\int_{\hat{\mathbf{R}}^m} yW(dsdy)$. Then the intensity measure $E(W(dsdy)^2)=dsw(dy)$ satisfies $(\int_{\hat{\mathbf{R}}^m} y^iy^jw(dy))=A$. We set $\hat{w}(dy)=|y|^2w(dy)$.

Then, setting $Y_{l,V}(t)=l^T\nabla\xi_tV(\xi_t)$, the equation of Lemma 3.1 is written as

$$\begin{aligned}
Y_{l,V}(t) &= l^TV(\xi_0)+\int_0^t Y^{(1)}_{l,V}(s-,\Delta)ds\\
&\quad+\int_0^t\int_{\hat{\mathbf{R}}^m} Y^{(1)}_{l,V}(s-,y)|y|dW\\
&\quad+\int_0^t\int_{|z|\le\delta} Y^{(1)}_{l,V}(s-,z)|z|d\tilde{N}\\
&\quad+\int_0^t\int_{|z|>\delta} Y^{(1)}_{l,V}(s-,z)|z|dN.
\end{aligned}\tag{3.3}$$

We will continue the above argument inductively. Let $k\ge 1$. We will define a family of k-th step semimartingales with spatial parameter associated with a given vector field V. We set $\Psi(\Delta)V=\Psi_0^\delta V$, $\Psi(y)V=\sum_k[V_k,V]y^k/|y|$ and $\Psi(z)V=\Phi_1(z)V/|z|$. Define for $v_k,...,v_1\in\mathcal{S}$

$$\Psi(v_k,...,v_1)V=\Psi(v_k)\circ\cdots\circ\Psi(v_1)V.\tag{3.4}$$

Apply equality (3.3) to the vector field $\Psi(v_k,...,v_1)V$ in place of V. Then, setting

$$Y^{(k)}_{l,V}(t,v_k,...,v_1)=l^T(\nabla\xi_t)^{-1}\Psi(v_k,...,v_1)V(\xi_t),$$

equality (3.3) is written as

$$
\begin{aligned}
(3.5)\qquad Y^{(k)}_{l,V}(t,v_k,...,v_1) &= Y^{(k)}_{l,V}(0,v_k,...,v_1)\\
&+\int_0^t Y^{(k+1)}_{l,V}(s-,\Delta,v_k,...,v_1)ds\\
&+\int_0^t\int Y^{(k+1)}_{l,V}(s-,y_{k+1},v_k,...,v_1)|y_{k+1}|W(dsdy_{k+1})\\
&+\int_0^t\int_{|z_{k+1}|\le\delta} Y^{(k+1)}_{l,V}(s-,z_{k+1},v_k,...,v_1)|z_{k+1}|\tilde N(dsdz_{k+1})\\
&+\int_0^t\int_{|z_{k+1}|>\delta} Y^{(k+1)}_{l,V}(s-,z_{k+1},v_k,...,v_1)|z_{k+1}|N(dsdz_{k+1}).
\end{aligned}
$$

4. Alternative criterion for the smooth density

We will now study the existence of the smooth density of the law of ξ_t. In this section we present an alternative criterion which ensures the existence of the smooth density. The condition will be given at Theorem 4.1. In the next section we will study how the condition given in this section is related to Hörmander's condition in Theorem 1.1.

Let $\epsilon > 0$. Associated with the Lévy measure ν, we define a probability measure $\hat\mu_\epsilon$ on $\mathbf{R}^m$ by

$$\hat\mu_\epsilon(dz) = \frac{1}{v(\epsilon)}|z|^2 1_{[0,\epsilon]}(|z|)\nu(dz),$$

where $v(\rho) = \int_{|z|<\rho}|z|^2\nu(dz)$. We denote by μ_ϵ the measure on $\mathcal{S}$ such that it is equal to $\hat\mu_\epsilon$ on $\mathbf{R}^m$, equals to $\hat\omega$ on $\hat{\mathbf{R}}^m$ and equals to $\delta_{\{\Delta\}}$ on Δ.

Keeping Theorem 6.1 (in Appendix) in mind, we introduce some positive constants. Let α be the exponent of the order condition of ν and let β and r be positive numbers such that $\frac{3}{2} < \alpha(1+\beta) < 2$ and $r > (2-\alpha(1+\beta))^{-1}$. Let $q > 4r$ and $q(k) = (1+\beta)rq^{-k}$. For a positive integer N_0 and $\varepsilon,\delta > 0$, define $L^{N_0}_{\epsilon,\delta}(w,x), w,x\in\mathbf{R}^d$ by

$$
\begin{aligned}
L^{N_0}_{\epsilon,\delta}(w,x) = \sum_{V\in\Sigma_0}\Big\{&|w^TV(x)|^2+\\
&+\sum_{k=1}^{N_0}\int\cdots\int |w^T\Psi(v_k,...,v_1)V(x)|^2\mu_{\varepsilon^{q(k)}}(dv_k)\cdots\mu_{\varepsilon^{q(1)}}(dv_1)\Big\}.
\end{aligned}
$$

($\Psi(v_k,...,v_1)$ may depend on δ).

Theorem 4.1. — *For the canonical SDE (1.1), assume that there exists a positive integer N_0, a nonnegative integer n_0, $\delta_0,\epsilon_0 > 0$ and a positive number C such that*

$$L^{N_0}_{\varepsilon,\delta_0}(w,x) \ge \frac{C|w|^2}{(1+|x|)^{n_0}} \tag{4.1}$$

holds for any $0 < \epsilon < \epsilon_0$ *and* $w, x \in \mathbf{R}^d$. *Then for any initial random variable* ξ_0 *and* $0 < T_0 < \infty$, *the law of the solution* ξ_{T_0} *has a* C^∞*-density.*

For the proof of the above theorem, we need Norris' type estimate stated in Theorem 6.1 in Appendix. We fix δ_0 satisfying (4.1). We define events (with parameter $l \in S_{d-1}$ and $\varepsilon > 0$) by

$$E = \left\{ \sum_{V \in \Sigma_0} \int_0^{T_0} |Y_{l,V}(t-)|^2 dt < \varepsilon \right\}.$$

We want to prove that for any $p > 1$ there exists $C_p > 0$ such that $P(E) \leq C_p \epsilon^p$ holds for any $0 < \epsilon < \epsilon_0$ and $l \in S_{d-1}$. In order to prove this, associated with the vector field V we introduce a sequence of events $E_V^{(k)}$ (with parameter $l \in S_{d-1}$ and ε) by

$$\left\{ \int_0^{T_0} \left(\int |Y_{l,V}^{(k)}(t-, v_k, ..., v_1)|^2 \mu_{\varepsilon^{q(k)}}(dv_k) \cdots \mu_{\varepsilon^{q(1)}}(dv_1) \right) dt < \varepsilon^{q^{-k}} \right\},$$

for $k = 0, 1, 2, ...$, where $Y_{l,V}^{(0)} = Y_{l,V}$. Then we have $E \subset \cap_{V \in \Sigma^0} E_V^{(0)}$ and the set $E_V^{(0)}$ is included in

$$(E_V^{(0)} \cap (E_V^{(1)})^c) \cup (E_V^{(1)} \cap (E_V^{(2)})^c) \cup \cdots \cup (E_V^{(N_0-1)} \cap E_V^{(N_0)})^c) \cup G_V,$$

where

$$G_V = E_V^{(0)} \cap E_V^{(1)} \cap \cdots \cap E_V^{(N_0)}.$$

Consequently, in order to prove that $P(E)$ is small, it is sufficient to prove that both $P(E_V^{(k)} \cap (E_V^{(k+1)})^c)$ and $P(\cap_{V \in \Sigma_0} G_V)$ are small. These two assertions will be shown in the following two lemmas.

Lemma 4.2. — *For any* $p > 1$ *there exists a positive constant* C_p *such that*

$$P(E_V^{(k)} \cap (E_V^{(k+1)})^c) \leq C_p \varepsilon^p, \quad k = 0, 1, .., N^0 - 1 \tag{4.2}$$

holds for all $0 < \varepsilon < \varepsilon_0$ *and* $l \in S_{d-1}$.

Proof. — We first consider the case $k = 0$. We want to apply Theorem 6.1 in Appendix to the semimartingale $Y_{l,V}(t)$. The integrand functions of the right hand side of (3.3) have finite moments of any order ([**3**]), i.e.,

$$E[\sup_t |Y_{l,V}^{(1)}(t)|^{p'} + \sup_{t,y} |Y_{l,V}^{(1)}(t, y)|^{p'} + \sup_{t,z} |Y_{l,V}^{(1)}(t, z)|^{p'}] < \infty.$$

Therefore the functional θ^γ defined by (6.2) satisfies $E[(\sup_\gamma \theta^\gamma)^{p'}] < \infty$ for any p'. Then we can apply Theorem 6.1 and we get

$$P(E_V^{(0)} \cap (E_V^{(1)})^c) \leq C_{p'} \varepsilon^{p'}$$

for all $0 < \varepsilon < \varepsilon_0$ and $l \in S_{d-1}$.

We want to apply Theorem 6.1 again to $Y_{l,V}^{(k)}(t, v_k, ..., v_1)$, which is written by (3.5). Set $\gamma = (v_k, ..., v_1)$ and $\pi(d\gamma) = \mu_{\varepsilon^{q(k)}}(dv_k) \cdots \mu_{\varepsilon^{q(1)}}(dv_1)$. It can be shown that for any $p' > 1$, $E[(\sup_\gamma \theta^\gamma)^{p'}] < \infty$ holds. Then the inequality

$$P(E_V^{(k)} \cap (E_V^{(k+1)})^c) \le C_{p'} \varepsilon^{p' q^{-(k+1)}}, \quad k = 1, 2, ...$$

holds for all $0 < \varepsilon < \varepsilon_0$ and $l \in S_{d-1}$ by Theorem 6.1. Set $p = p' q^{-N_0}$. Then (4.2) holds valid for any k. □

Lemma 4.3. — *Assume (4.1). Then for any $p > 1$ there exists a positive constant C'_p such that*

$$P(\cap_{V \in \Sigma_0} G_V) < C'_p \varepsilon^p, \tag{4.3}$$

for all $0 < \varepsilon < 1$ and $l \in S_{d-1}$.

Proof. — Set

$$K_\epsilon = \sum_{k=0}^{N_0} \sum_{V \in \Sigma_0} \int_0^{T_0} \left(\int |Y_{l,V}^{(k)}(t-, v_k, ..., v_1)|^2 \mu_{\varepsilon^{q(k)}}(dv_k) \cdots \mu_{\varepsilon^{q(1)}}(dv_1) \right) dt.$$

Then, if $\omega \in G := \cap_{V \in \Sigma_0} G_V$, we have the inequality

$$K_\varepsilon(\omega) < m \sum_{k=0}^{N_0} \varepsilon^{q^{-k}} < m(N_0+1)\varepsilon^{q^{-N_0}}$$

if $\varepsilon^{1/q} < 1$. Therefore, we have $G \subset \{K_\varepsilon < m(N_0+1)\varepsilon^{q^{-N_0}}\}$. Thus, the problem is reduced to getting the estimate of $P(K_\varepsilon < m(N_0+1)\varepsilon^{q^{-N_0}})$.

Observe that K_ε is written as

$$K_\varepsilon = \int_0^{T_0} L_{\varepsilon,\delta_0}^{N_0}((\nabla \xi_{t-})^{-1} l, \xi_{t-}) dt.$$

Inequality (4.1) implies

$$K_\varepsilon \ge C \int_0^{T_0} \frac{|(\nabla \xi_{t-})^{-1} l|^2}{(1 + |\xi_{t-}|)^{n_0}} dt.$$

Further, for any $l \in S_{d-1}$, we have the inequality

$$\left(\int_0^{T_0} \frac{|(\nabla \xi_{t-})^{-1} l|^2}{(1 + |\xi_{t-}|)^{n_0}} dt \right)^{-1} \le \frac{1}{T_0^2} \int_0^{T_0} |\nabla \xi_{t-}|^2 (1 + |\xi_{t-}|)^{n_0} dt,$$

by using Jensen's inequality. Therefore

$$G \subset \left\{ \int_0^{T_0} |\nabla \xi_{t-}|^2 (1 + |\xi_{t-}|)^{n_0} dt > \frac{C T_0^2}{m(N_0+1)\varepsilon^{q^{-N_0}}} \right\}.$$

Then we get by Chebyschev's inequality, $P(G) \le C'_p \varepsilon^p$ where

$$C'_p = \left(\frac{m(N_0+1)}{C T_0^2} \right)^{\frac{p}{q^{-N_0}}} E\left[\left(\int_0^{T_0} |\nabla \xi_{t-}|^2 (1 + |\xi_{t-}|)^{n_0} dt \right)^{\frac{p}{q^{-N_0}}} \right].$$

We have thus obtained the estimate (4.3) for all $0 < \varepsilon < 1$ and $l \in S_{d-1}$. $\square$

Proof of Theorem 4.1. — It suffices to prove (2.4). Inequalities of Lemmas 4.1 and 4.2 imply

$$P\left(\sum_{V\in\Sigma_0}\int_0^{T_0}|Y_{l,V}(t-)|^2dt < \epsilon\right) < C_p''\epsilon^p$$

for all $0 < \varepsilon < \varepsilon_0$ and $l \in S_{d-1}$. Consequently we obtain

$$\sup_{l\in S_{d-1}} E\left[\left(\sum_{V\in\Sigma_0}\int_0^{T_0}|l^T(\nabla\xi_t)^{-1}V(\xi_t)|^2dt\right)^{-p}\right] \le C_p$$

for any $p > 1$.

Consider next the case where $\mathbf{u} \neq 0$. Let $\mathbf{u} = \{(t_1, z_1), ..., (t_n, z_n)\}$, where we have $0 < t_1 < \cdots < t_n < T_0$. We set $\xi_t^{\mathbf{u}} = \xi_t \circ \varepsilon_{\mathbf{u}}^+$ and $Y_{l,V}^{\mathbf{u}}(t) = l^T(\nabla\xi_t^{\mathbf{u}})^{-1}V(\xi_t^{\mathbf{u}})$. Then there exists an interval $[t_i, t_{i+1}]$ such that its length is greater than or equal to $T_0/(n+1)$. Choose $t_i' < t_{i+1}'$ such that $[t_i', t_{i+1}'] \subset [t_i, t_{i+1}]$ and $t_{i+1}' - t_i' = T_0/(n+1)$. Then $\xi_t^{\mathbf{u}}, t \in [t_i', t_{i+1}']$ is a solution of SDE (1.1) with the initial data $\xi_{t_i}^{\mathbf{u}}$. We can apply the argument of this section to the process $Y_{l,V}^{\mathbf{u}}(t), t \in [t_i', t_{i+1}']$. Then we have

$$\sup_{l\in S_{d-1}} E\left[\left(\sum_{V\in\Sigma_0}\int_{t_i'}^{t_{i+1}'}|l^T(\nabla\xi_t^{\mathbf{u}})^{-1}V(\xi_t^{\mathbf{u}})|^2dt\right)^{-p}\right] < C_{p,\mathbf{u}}.$$

Note that the family of initial data satisfies

$$\sup_{\mathbf{u}\in A(1)^n} E[|\xi_{t_i}^{\mathbf{u}}|^p] \le c(n, \rho_0, p) < \infty.$$

Then we can choose a positive constant $C_{n,p}$ such that it dominates all $C_{p,\mathbf{u}}$. Therefore,

$$\sup_{\mathbf{u}\in A(1)^n}\sup_{l\in S_{d-1}} E\left[\left(\sum_{V\in\Sigma_0}\int_0^T|l^T(\nabla\xi_t^{\mathbf{u}})^{-1}V(\xi_t^{\mathbf{u}})|^2dt\right)^{-p}\right] < C_{n,p}$$

for any n and p. $\square$

5. Relation with Lie algebra

In this section we want to prove the following.

Theorem 5.1. — *Under the same condition as in Theorem 1.1, there exists $\delta_0', \epsilon_0' > 0$ and $C' > 0$ such that the inequality*

$$L_{\varepsilon,\delta_0'}^{N_0}(w, x) \ge C'|w|^2, \quad \forall w, x \in \mathbf{R}^d \tag{5.1}$$

holds for all $0 < \varepsilon < \varepsilon_0'$.

If the above theorem is established, Theorem 1.1 follows from Theorem 4.1 and Theorem 5.1, immediately. Theorem 1.2 is an easy consequence of Theorem 4.1. Indeed, it is verified as follows.

Proof of Theorem 1.2. — Since b_0 exists by the assumption of the theorem, there exists $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$, the inequality

$$\sum_{k=0}^{N_0} \sum_{V \in \Sigma_k^\delta} |l^T V(x)|^2 \geq \frac{1}{2} \sum_{k=0}^{N_0} \sum_{V \in \Sigma_k} |l^T V(x)|^2 \geq \frac{C}{2}$$

holds. Then Theorem 1.2 follows from Theorem 1.1. □

Before we proceed to the proof of Theorem 5.1, we shall approximate the vector field $\Psi(v_k, ..., v_1)V$ given by (3.4) by a linear sum of vector fields of the form $\Psi_{k_k}\Psi_{k_{k-1}} \cdots \Psi_{k_1} V$ where Ψ_{k_i} are such that $\Psi_0 = \Psi_0^\delta V$ or $\Psi_i V = [V_i, V]$, $i = 1, ..., m$, in the case where $v_1, ..., v_k \in \hat{\mathbf{R}}^m \cup \mathbf{R}^m$ are small.

We first consider the case $k = 1$. We have $\Psi(\Delta)V = \Psi_0^\delta V$, $\Psi(y)V = \sum_i [V_i, V] y^i / |y|$ and $\Psi(z)V = \Phi_1(z)V/|z|$. Set $z = (z^1, ..., z^m)$. Then $\Phi_s(z)$ given by (3.2) satisfies the differential equation

$$\frac{\Phi_s(z)V(x)}{ds} = (\nabla \phi_s^z(x))^{-1} \Big(\sum_{i=1}^m [V_i, V](\phi_s^z(x)) z^i \Big).$$

Hence $\Phi_1(z)V(x)$ is written as

$$\begin{aligned} \Phi_1(z)V(x) &- \sum_{i=1}^m [V_i, V](x) z^i \\ &= \frac{1}{2} (\nabla \phi_\theta^z(x))^{-1} \sum_{i,j} [V_j, [V_i, V]](\phi_\theta^z(x)) z^i z^j, \end{aligned}$$

where $0 \leq \theta \leq 1$, by the mean value theorem. Consequently we obtain

$$\Big| \Phi_1(z)V(x) - \sum_{i=1}^m [V_i, V](x) z^i \Big| \leq c_1 |z|^2.$$

Since $\Psi(z) = \Phi_1(z)/|z|$, we get

$$\Big| \Psi(z)V(x) - \sum_{i=1}^m [V_i, V](x) \frac{z^i}{|z|} \Big| \leq c_1 |z|$$

for sufficiently small z.

We next consider the case $k \geq 2$. Suppose $v_k = z_k, ..., v_1 = z_1$. We can show similarly that there exists $\delta_0 > 0$ such that the inequality

$$\Big| \Psi(z_k, ..., z_1)V(x) - \sum_{i_k, ..., i_1 = 1}^m \Psi_{i_k} \cdots \Psi_{i_1} V(x) \frac{z_k^{i_k}}{|z_k|} \cdots \frac{z_1^{i_1}}{|z_1|} \Big| \leq c_2 \sum_{i=1}^k |z_i|$$

holds for $|z_i| \le \delta_0, i = 1, ..., k$, where $z_i = (z_i^1, ..., z_i^m)$. For the general $v_k, ..., v_1$, we have

$$(5.2) \qquad \Big|\Psi(v_k, ..., v_1)V(x) - \sum_{i_k,...,i_1=0}^{m} \Psi_{i_k} \cdots \Psi_{i_1} V(x)\varphi_{i_k}(v_k) \cdots \varphi_{i_1}(v_1)\Big| \le c_2\Big(\sum_{i\in\{k; v_k=z_k\}} |z_i|\Big).$$

Here $\varphi_0(\Delta) = 1, \varphi_k(\Delta) = 0, k = 1, ..., m$ and $\varphi_0(y) = \varphi_0(z) = 0, \varphi_k(z) = \frac{z^k}{|z|}$ and $\varphi_k(y) = \frac{y^k}{|y|}, k = 1, ..., m$.

We claim:

Lemma 5.2. — *For any $\delta > 0$ and $c > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta, c) > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $l \in S_{d-1}$, we have*

$$(5.3) \qquad \int \cdots \int |l^T \Psi(v_k, ..., v_1)V(x)|^2 \mu_{\varepsilon^{q(k)}}(dv_k) \cdots \mu_{\varepsilon^{q(1)}}(dv_1) \ge \frac{\hat{\lambda}_1^k}{2} \left(\sum_{i_k,...,i_1=0}^{m} |l^T \Psi_{i_k} \cdots \Psi_{i_1} V(x)|^2 \right) - c,$$

where λ_1 is the minimal eigen value of the matrix $A + B$ and $\hat{\lambda}_1 = \lambda \wedge 1$.

Proof. — Let us consider F_ϵ given by

$$\int\int\Big(\sum_{i_k,...,i_1=0}^{m} l^T \Psi_{i_k} \cdots \Psi_{i_1} V \varphi_{i_k}(v_k) \cdots \varphi_{i_1}(v_1)\Big)^2 \mu_{\varepsilon^{q(k)}}(dv_k) \cdots \mu_{\varepsilon^{q(1)}}(dv_1).$$

Since

$$\lim_{\varepsilon\to 0} \int\int \varphi_{i_k}(v_k) \cdots \varphi_{i_1}(v_1)\varphi_{i'_k}(v_k) \cdots \varphi_{i'_1}(v_1)\mu_{\varepsilon^{q(k)}}(dv_k) \cdots \mu_{\varepsilon^{q(1)}}(dv_1) \ge \prod_{j=1}^{k}(a_{i_j i'_j} + b_{i_j i'_j} + c_{i_j i'_j}),$$

(where $c_{ij} = 1$ if $i = j = 0$ and $= 0$ otherwise), the inferior limit of F_ϵ is greater than or equal to

$$\sum_{i_k,...,i_1=0}^{m} \sum_{i'_k,...,i'_1=0}^{m} l^T \Psi_{i_k} \cdots \Psi_{i_1} V l^T \Psi_{i'_k} \cdots \Psi_{i'_1} V \times (a_{i_k,i'_k} + b_{i_k,i'_k} + c_{i_k i'_k}) \cdots (a_{i_1 i'_1} + b_{i_1,i'_1} + c_{i_1 i'_1}).$$

The above has the lower bound $\hat{\lambda}_1^k \sum_{i_k,...,i_1=0}^{m} |l^T \Psi_{i_k} \cdots \Psi_{i_1} V|^2$. Therefore, we have

$$F_\epsilon \ge \hat{\lambda}_1^k \left(\sum_{i_k,...,i_1=0}^{m} |l^T \Psi_{i_k} \cdots \Psi_{i_1} V(x)|^2 \right) - \frac{c}{2},$$

for sufficiently small ε.

On the other hand, we have from (5.2) the inequality

$$\int\int\Big(l^T\Psi(v_k,...,v_1)V-\textstyle\sum_{i_k,...,i_1=0}^m l^T\Psi_{i_k}\cdots\Psi_{i_1}V\varphi_{i_k}(v_k)\cdots\varphi_{i_1}(v_1)\Big)^2$$
$$\times\mu_{\varepsilon^{q(k)}}(dv_k)\cdots\mu_{\varepsilon^{q(1)}}(dv_1)\le c_2\sum{}'\varepsilon^{2q(i)}\le\frac{1}{2}c$$

for sufficiently small ε, where $\sum'$ is the summation for $i\in\{k;v_k=z_k\}$. Consequently we get the inequality (5.3). □

Proof of Theorem 5.1. — We shall first introduce another family of vector fields. Given $\delta>0$, we define a linear transformation Ψ_0^δ of vector fields by (3.1). We may consider $\Psi_0^\delta V$ as a modification of the vector field $[\hat{V}_0^\delta,V]$. We define

$$\Gamma_0^\delta=\Sigma_0,\cdots,\Gamma_k^\delta=\{\Psi_0^\delta V,[V_i,V],i=1,...,m,V\in\Gamma_{k-1}^\delta\}.$$

These can be regarded as a modification of Σ_k^δ of Section 1.

Now, apply (5.3) to each term of $L_{\varepsilon,\delta}(l,x)$. Then for any $0<\varepsilon<\varepsilon_0(\delta,c)$ and $l\in S_{d-1}$, $L_{\varepsilon,\delta}(l,x)$ is greater than or equal to

$$\sum_{V\in\Sigma_0}|l^TV(x)|^2+\frac{\hat{\lambda}_1^{N_0}}{2}\sum_{k=1}^{N_0}\sum_{V\in\Sigma_0}\sum_{i_k=0}^m\cdots\sum_{i_1=0}^m|l^T\Psi_{i_k}\cdots\Psi_{i_1}V(x)|^2$$
$$-(m+1)^{N_0}N_0c\ge\frac{\hat{\lambda}_1^{N_0}}{2}\left\{\sum_{V\in\cup_{k=0}^{N_0}\Gamma_k^\delta}|l^TV(x)|^2\right\}-(m+1)^{N_0}N_0c. \tag{5.4}$$

We want to rewrite the right hand side of the above by using vector fields in Σ_k^δ. We set

$$\Phi_0^\delta V=[\hat{V}_0^\delta,V]+\frac{1}{2}\sum_{i,j=1}^m a_{ij}[V_i,[V_j,V]].$$

Then we have

$$\begin{aligned}&\Big|l^T\Psi_0^\delta V(x)-l^T\Phi_0^\delta V(x)\Big|^2\\ =\;&\left|\int_{|z|\le\delta}\Big(\Phi_1(z)V(x)-\sum_i[V_i,V](x)z^i\Big)\nu(dz)\right|^2\\ \le\;&c_1v(\delta)^2.\end{aligned}$$

We can show by induction

$$\Big|l^T(\Psi_0^\delta)^kV(x)-l^T(\Phi_0^\delta)^kV(x)\Big|^2\le2^kc_1v(\delta)^2.$$

Therefore,

$$|l^T(\Psi_0^\delta)^kV(x)|^2\ge\frac{1}{2}|l^T(\Phi_0^\delta)^kV(x)|^2-2^{k+1}c_1v(\delta)^2.$$

Summing up these inequalities, we obtain

$$\sum_{k=0}^{N_0} \sum_{V\in\Gamma_k^\delta} |l^T V(x)|^2 \geq \frac{1}{2}\sum_{k=0}^{N_0} \sum_{V\in\Sigma_k^\delta} |l^T V(x)|^2 - N2^{N_0+1}c_1 v(\delta)^2,$$

where N is the number of terms of the sum $\sum_{k=0}^{N_0}\sum_{V\in\Sigma_k^\delta}$. Therefore, assuming (1.2), the right hand side of (5.4) dominates

$$C' := \Big\{\frac{\hat\lambda_1^{N_0}}{4}\Big(\frac{C(\delta)}{2} - N2^{N_0+1}c_1 v(\delta)^2\Big) - (m+1)^{N_0}N_0 c\Big\}.$$

The above constant C' becomes positive if we choose δ, c sufficiently small, say $\delta = \delta_0'$ and $c = c_0'$. Set $\varepsilon_0' = \varepsilon_0(\delta_0', c_0')$. Then we get the inequality (5.1) for $l \in S_{d-1}$ and $x \in \mathbf{R^d}$. The inequality is extended to any $w, x \in \mathbf{R}^d$. □

6. Appendix. An analogue of Norris' estimate

In this section, we will consider semimartingales with parameter γ, which is directly related to the solution of an SDE. We consider a semimartingale $Y_t^\gamma, 0 \leq t \leq T_0$ defined by

$$\begin{aligned}(6.1)\qquad Y_t^\gamma &= y^\gamma + \int_0^t a^\gamma(s)ds + \sum_i \int_0^t f_i^\gamma(s)dW_s^i \\ &\quad + \int_0^t \int_{|z|\leq\delta} g^\gamma(s,z)d\tilde N + \int_0^t \int_{|z|>\delta} g^\gamma(s,z)dN,\end{aligned}$$

where $a^\gamma(s), f^\gamma(s), g^\gamma(s,z)$ are left continuous predictable processes, continuous with respect to parameters $z \in \mathbf{R}^m, \gamma \in \Gamma$. Here Γ is a compact space. We assume further that $a^\gamma(t)$ is a semimartingale represented by

$$\begin{aligned}a^\gamma(t+) &= a^\gamma + \int_0^t b^\gamma(s)ds + \sum_i \int_0^t e_i^\gamma(s)dW_s^i \\ &\quad + \int_0^t \int_{|z|\leq\delta} h^\gamma(s,z)d\tilde N + \int_0^t \int_{|z|>\delta} h^\gamma(s,z)dN,\end{aligned}$$

where $b^\gamma(s), e_i^\gamma(s), h^\gamma(s,z), s \geq 0$ are left continuous predictable processes continuous with respect to z and γ. We set

$$\begin{aligned}(6.2)\qquad \theta^\gamma &= \|(a^\gamma)^2 + (b^\gamma)^2\| + \sum_i \|(f_i^\gamma)^2 + (e_i^\gamma)^2\| \\ &\quad + \int_{|z|\leq\delta} \|g^\gamma(z)^2 + h^\gamma(z)^2\|\nu(dz) + \sup_{|z|>\delta}\|h^\gamma(z)^2\|,\end{aligned}$$

where $\|F\| = \sup_{0\leq t\leq T_0}|F(t)|$. Set further

$$\hat g^\gamma(t,z) = \frac{g^\gamma(t,z)}{|z|}, \quad \hat\mu_\epsilon(dz) = \frac{1}{v(\epsilon)}|z|^2 1_{[0,\epsilon]}(|z|)\nu(dz).$$

We shall consider two events for given $r > 0, q > 4r, \beta > 0$ and $\epsilon > 0$

$$A(\epsilon) = \Big\{ \int_\Gamma \Big(\int_0^{T_0} |Y_{t-}^\gamma|^2 dt \Big) \pi(d\gamma) < \varepsilon^q \Big\},$$

$$B(\epsilon) =$$
$$\Big\{ \int_\Gamma \int_0^{T_0} \Big\{ a^\gamma(t)^2 + \sum_i |f_i^\gamma(t)|^2 + \int \hat{g}^\gamma(t,z)^2 \hat{\mu}_{\varepsilon^{(1+\beta)r}}(dz) \Big\} \pi(d\gamma) dt > \varepsilon \Big\}.$$

We will show that the probability where both $A(\epsilon)$ and $B(\epsilon)$ occur simultaneously is small if ϵ is small.

Theorem 6.1. — *Let α be the exponent of the order condition of the Lévy measure ν. Let $\beta > 0$ be a number such that $3/2 < \alpha(1+\beta) < 2$. Let $r > \frac{1}{2-\alpha(1+\beta)}$ and $q > 4r$. Assume $E[(\sup_\gamma \theta^\gamma)^p] < \infty$ holds for any $p > 1$. Then for any $p > 1$, there exists a positive constant C_p such that the inequality*

$$P(A(\epsilon) \cap B(\epsilon)) < C_p \epsilon^p \tag{6.3}$$

holds for any semimartingale Y_t^γ represented by (6.1), any probability measure π on Γ and any $0 < \varepsilon < \varepsilon_0$, where $0 < \varepsilon_0 < 1$ is a positive number independent of p.

In order to prove the above theorem, we need the following. Let Y_t^γ be the process of (6.1) and let λ be an arbitrary positive number.

Komatsu-Takeuchi's estimate. *([7], Theorem 3) For any $0 < \upsilon < \frac{1}{4}$, there exist a positive random variable $\mathcal{E}(\lambda,\gamma)$ with $E[\mathcal{E}(\lambda,\gamma)] \le 1$ and positive constants C, C_0, C_1, C_2 such that the inequality*

$$\begin{aligned} \lambda^4 \int_0^{T_0} |Y_t^\gamma|^2 \wedge \frac{1}{\lambda^2} dt + \lambda^{-\upsilon} \log \mathcal{E}(\lambda,\gamma) + C \ge \\ C_0 \lambda^{1-4\upsilon} \int_0^{T_0} |a^\gamma(t)|^2 dt + C_1 \lambda^{2-2\upsilon} \sum_i \int_0^{T_0} |f_i^\gamma(t)|^2 dt \\ + C_2 \lambda^{2-2\upsilon} \int_0^{T_0} \int_{\mathbf{R}^m} |g^\gamma(t,z)|^2 \wedge \frac{1}{\lambda^2} dt \nu(dz) \end{aligned} \tag{6.4}$$

holds on the set $\{\theta^\gamma \le \lambda^{2\upsilon}\}$ for all $\lambda > 1$ and Y^γ.

Remark 6.2. — In Theorem 3 in [7], the assertion is stated in the case where $Y_t^\gamma, a^\gamma(t)$ etc. do not depend on the parameter γ. Further the Lévy measure is assumed to be of the form $\nu(dz) = |z|^{-m-\alpha} dz$. However their result can be applied to the present case.

Proof of Theorem 6.1. — By the choice of β and r, it holds $0 < 2-\alpha(1+\beta)-\frac{1}{r}$. We will choose υ such that $0<\upsilon<(2-\alpha(1+\beta)-\frac{1}{r})\wedge\frac{1}{8}$. We want to rewrite inequality (6.4) in order to apply it for the estimate (6.3). Our aim is to get (6.5) below on the set $\{\sup_\gamma \theta^\gamma \le e^{-\upsilon r}\}$. We first consider the last term of (6.4). It holds for any $0<\kappa<\lambda$

$$\begin{aligned}\int_{\mathbf{R}^m}\Big(|g^\gamma(t,z)|^2\wedge\frac{1}{\lambda^2}\Big)\nu(dz) &\ge \int_{|z|<\frac{\kappa}{\lambda}}\Big(|\hat g^\gamma(t,z)|^2\wedge\frac{1}{|z|^2\lambda^2}\Big)|z|^2\nu(dz)\\ &\ge v\Big(\frac{\kappa}{\lambda}\Big)\int_{|z|<\frac{\kappa}{\lambda}}\Big(|\hat g^\gamma(t,z)|^2\wedge\frac{1}{\kappa^2}\Big)\hat\mu_{\frac{\kappa}{\lambda}}(dz).\end{aligned}$$

Now set $\lambda=\varepsilon^{-r}$ and $\kappa=\varepsilon^{\beta r}$. Then $\frac{\kappa}{\lambda}=\varepsilon^{(1+\beta)r}$ and $v(\frac{\kappa}{\lambda})\ge C_4\varepsilon^{\alpha(1+\beta)r}$ by the order condition for $v(\rho)$. Therefore, (6.4) is rewritten by

$$\begin{aligned}&\varepsilon^{-4r}\int_0^{T_0}|Y_t^\gamma|^2\wedge\varepsilon^{2r}dt+\varepsilon^{\upsilon r}\log\mathscr{E}(\varepsilon^{-r},\gamma)+C\\ &\ge C_0\varepsilon^{-r(1-4\upsilon)}\int_0^{T_0}|a^\gamma(t)|^2dt+C_1\varepsilon^{-r(2-2\upsilon)}\sum_i\int_0^{T_0}|f_i^\gamma(t)|^2dt\\ &\quad+C_2C_4\varepsilon^{-r(2-2\upsilon)+\alpha(1+\beta)r}\int_0^{T_0}\int_{\mathbf{R}^m}|\hat g^\gamma(t,z)|^2\wedge\varepsilon^{-2\beta r}dt\hat\mu_{\varepsilon^{(1+\beta)r}}(dz).\end{aligned}$$

Now set $\rho=\min\{r(1-4\upsilon),r(2-2\upsilon)-\alpha(1+\beta)\}-1$. In view of the choice of υ, we have $\rho>0$. Set $C_5=\min\{C_0,C_2,C_4\}$. Then the above inequality yields

$$\begin{aligned}\varepsilon^{-4r}\int_0^{T_0}&|Y_t^\gamma|^2\wedge\varepsilon^{2r}dt+\varepsilon^{\upsilon r}\log\mathscr{E}(\varepsilon^{-r},\gamma)+C\ge\\ &C_5\varepsilon^{-(\rho+1)}\int_0^{T_0}\Big\{|a^\gamma(t)|^2+\sum_i|f_i^\gamma(t)|^2\\ &\qquad+\int|\hat g^\gamma(t,z)|^2\wedge\varepsilon^{-2\beta r}\hat\mu_{\varepsilon^{(1+\beta)r}}(dz)\Big\}dt,\end{aligned}$$

on the set $\{\theta^\gamma\le\epsilon^{-\upsilon r}\}$.

Next, integrate each term of the above by the measure π with respect to the parameter γ. We have by Jensen's inequality $\int\log\mathscr{E}(\lambda,\gamma)\pi(d\gamma)\le\log\mathscr{E}(\lambda)$, where $\mathscr{E}(\lambda)=\int\mathscr{E}(\lambda,\gamma)\pi(d\gamma)$ is a positive random variable such that $E[\mathscr{E}(\lambda)]\le 1$. Therefore we have

$$\begin{aligned}\varepsilon^{-4r}\int_\Gamma\Big(\int_0^{T_0}&|Y_t^\gamma|^2\wedge\varepsilon^{2r}dt\Big)\pi(d\gamma)+\varepsilon^{\upsilon r}\log\mathscr{E}(\varepsilon^{-r})+C\ge\\ &C_5\varepsilon^{-(\rho+1)}\int_\Gamma\int_0^{T_0}\Big\{|a^\gamma(t)|^2+\sum_i|f_i^\gamma(t)|^2\\ &\qquad+\int|\hat g^\gamma(t,z)|^2\wedge\varepsilon^{-2\beta r}\hat\mu_{\varepsilon^{(1+\beta)r}}(dz)\Big\}dt\pi(d\gamma)\end{aligned}\tag{6.5}$$

on the set $\{\sup_\gamma \theta^\gamma \le \epsilon^{-vr}\}$.

We can now give the proof of (6.3). We define three events by

$$\begin{aligned} A_1(\varepsilon) &= \Big\{\sup_\gamma \theta^\gamma > \epsilon^{-vr}\Big\}, \\ A_2(\varepsilon) &= \Big\{\sup_\gamma \theta^\gamma \le \epsilon^{-vr}\Big\} \bigcap \Big\{\int_\Gamma \int_0^{T_0} |Y_{t-}^\gamma|^2 \wedge \varepsilon^{2r} dt \pi(d\gamma) < \varepsilon^q\Big\} \\ &\quad \bigcap \Big\{\sup_\gamma \|\hat{g}^\gamma\| \le \varepsilon^{-2\beta r}\Big\} \bigcap \Bigg\{\int_\Gamma \int_0^{T_0} \Big(a^\gamma(t)^2 + \sum_i |f_i^\gamma(t)|^2 + \\ &\quad + \int \hat{g}^\gamma(t,z)^2 \wedge \varepsilon^{-2\beta r} \hat{\mu}_{\varepsilon^{(1+\beta)r}}(dz)\Big) dt \pi(d\gamma) > \varepsilon\Bigg\}, \\ A_3(\varepsilon) &= \Big\{\sup_\gamma \|\hat{g}^\gamma\| > \varepsilon^{-2\beta r}\Big\}. \end{aligned}$$

Then it holds $A(\varepsilon) \cup B(\varepsilon) \subset A_1(\varepsilon) \cup A_2(\varepsilon) \cup A_3(\varepsilon)$ for any γ. Therefore, the probability of (6.3) is dominated by $P(A_1(\varepsilon)) + P(A_2(\varepsilon)) + P(A_3(\varepsilon))$. We shall get estimates of $P(A_i(\varepsilon)), i = 1, 2, 3$. In view of our assumption of the theorem, the first one is estimated as

$$P(A_1(\varepsilon)) \le \varepsilon^p E\Big[(\sup_\gamma \theta^\gamma)^{p/r}\Big] \le c_p \varepsilon^p.$$

A similar estimate is valid for $P(A_3(\varepsilon))$. For the estimate of $P(A_2(\varepsilon))$, we remark that (6.5) implies

$$A_2(\varepsilon) \subset \Big\{\mathscr{E}(\varepsilon^{-r})^{\varepsilon^{vr}} \ge \exp\big(-\varepsilon^{q-4r} + C_5 \varepsilon^{-\rho} - C\big)\Big\}.$$

Therefore, by Chebyschev's inequality

$$P(A_2(\varepsilon)) \le e^C \exp\big(\varepsilon^{q-4r} - C_5 \varepsilon^{-\rho}\big) E\Big[\mathscr{E}(\varepsilon^{-r})^{\varepsilon^{vr}}\Big].$$

Further $\varepsilon^{q-4r} < \frac{C_5}{2} \varepsilon^{-\rho}$ holds for $\varepsilon < \varepsilon_0$, where $\varepsilon_0^{q-4r} = C_5/2$. Therefore,

$$P(A_2(\varepsilon)) \le e^C \exp\Big(-\frac{C_5}{2} \varepsilon^{-\rho}\Big) \le c_p' \varepsilon^p$$

for $\varepsilon < \varepsilon_0$. □

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Astérisque
327, 2009, p. 93–114

TWO-PARAMETER STOCHASTIC CALCULUS AND MALLIAVIN'S INTEGRATION-BY-PARTS FORMULA ON WIENER SPACE

by

James R. Norris

Dedicated to Jean-Michel Bismut on the occasion of his 60^{th} birthday

Abstract. — The integration-by-parts formula discovered by Malliavin for the Itô map on Wiener space is proved using the two-parameter stochastic calculus. It is also shown that the solution of a one-parameter stochastic differential equation driven by a two-parameter semimartingale is itself a two-parameter semimartingale.

***Résumé* (Calcul stochastique à deux paramètres et formule d'intégration par parties de Malliavin sur l'espace de Wiener)**

La formule d'intégration par parties, qui a été établie par Malliavin pour l'application d'Itô sur l'espace de Wiener, est démontrée en utilisant le calcul stochastique à deux paramètres. On montre aussi que la solution d'une équation différentielle stochastique à un paramètre, guidée par une semimartingale à deux paramètres, est elle-même une semimartingale à deux paramètres.

1. Introduction

The stochastic calculus of variations was conceived by Malliavin [**6, 7, 8**] as follows. Let $(z_t)_{t\geqslant 0}$ denote the Ornstein–Uhlenbeck process on Wiener space $(W, \mathcal{W}, \mu)$ and let $\Phi : W \to \mathbb{R}^d$ denote the (almost-everywhere unique) Itô map obtained by solving a stochastic differential equation in $\mathbb{R}^d$ up to time 1. Then $(z_t)_{t\geqslant 0}$ is stationary and reversible, so, for functions f, g on $\mathbb{R}^d$, setting $F = f \circ \Phi, G = g \circ \Phi$,

$$\mathbb{E}\left[\{F(z_t) - F(z_0)\}\{G(z_t) - G(z_0)\}\right] = -2\mathbb{E}\left[F(z_0)\{G(z_t) - G(z_0)\}\right]. \tag{1}$$

Once certain terms of mean zero are subtracted, a differentiation of this identity with respect to t *inside the expectation* is possible, and leads to the integration-by-parts

2010 Mathematics Subject Classification. — 60H07; 60H15.
Key words and phrases. — Malliavin calculus, two-parameter stochastic calculus, integration by parts formula.

formula on Wiener space

$$\int_W \nabla_i f(\Phi)\Gamma^{ij}\nabla_j g(\Phi)d\mu = -\int_W f(\Phi)LGd\mu, \tag{2}$$

where LG and the *covariance matrix* Γ will be defined below. As is now well known, this formula and its generalizations hold the key to many deep results of stochastic analysis.

Malliavin's proof of the integration-by-parts formula was based on a *transfer principle*, allowing some calculations for two-parameter random processes to be made using classical differential calculus. Stroock [**11, 12, 13**] and Shigekawa [**10**] gave alternative derivations having a a more functional-analytic flavour. Bismut [**1**] gave another derivation based on the Cameron–Martin–Girsanov formula. Elliott and Kohlmann [**3**] and Elworthy and Li [**4**] found further elementary approaches to the formula. The alternative proofs are relatively straightforward. Nevertheless, we have found it interesting to go back to Malliavin's original approach in [**8**] and to review the calculations needed, especially since this can be done now in a more explicit way using the two-parameter stochastic calculus, as formulated in [**9**].

In Section 2 we review in greater detail the various mathematical objects mentioned above. Then, in Section 3, we review some points of two-parameter stochastic calculus from [**9**]. Section 4 contains the main technical result of the paper, which is a regularity property for two-parameter stochastic differential equations. We consider equations in which some components are given by two-parameter integrals and others by one-parameter integrals. It is shown, under suitable hypotheses, that the components which are presented as one-parameter integrals are in fact two-parameter semimartingales. This is useful because one can then compute martingale properties for both parameters by stochastic calculus. The sorts of differential equation to which this theory applies are just one way to realise continuous random processes indexed by the plane. See the survey [**5**] by Léandre for a wider discussion. But this regularity property makes our processes more tractable to analyse than some others. This is illustrated in Section 5, where we do the calculations needed to obtain the integration-by-parts formula.

2. Integration-by-parts formula

The Wiener space $(W, \mathcal{W}, \mu)$ over $\mathbb{R}^m$ is a probability space with underlying set $W = C([0,\infty), \mathbb{R}^m)$, the set of continuous paths in $\mathbb{R}^m$. Let $\mathcal{W}^o$ denote the σ-algebra on W generated by the family of coordinate functions $w \mapsto w_s : W \to \mathbb{R}^m$, $s \geqslant 0$, and let μ^o be Wiener measure on $\mathcal{W}^o$, that is to say, the law of a Brownian motion in $\mathbb{R}^m$ starting from 0. Then $(W, \mathcal{W}, \mu)$ is the completion of the probability space $(W, \mathcal{W}^o, \mu^o)$. Write $\mathcal{W}_s$ for the μ-completion of $\sigma(w \mapsto w_r : r \leqslant s)$. Let $X_0, X_1, \dots, X_m$ be vector fields on $\mathbb{R}^d$, with bounded derivatives of all orders. Fix $x_0 \in \mathbb{R}^d$ and consider the stochastic differential equation

$$\partial x_s = X_i(x_s)\partial w_s^i + X_0(x_s)\partial s.$$

Here and below, the index i is summed from 1 to m, and ∂ denotes the Stratonovich differential. There exists a map $x : [0, \infty) \times W \to \mathbb{R}^d$ with the following properties:

- x is a continuous semimartingale on $(W, \mathcal{W}, (\mathcal{W}_s)_{s \geqslant 0}, \mu)$,
- for μ-almost all $w \in W$, for all $s \geqslant 0$ we have

$$x_s(w) = x_0 + \int_0^s X_i(x_r(w))\partial w_r^i + \int_0^s X_0(x_r(w))dr.$$

The first integral in this equation is the Stratonovich stochastic integral. Moreover, for any other such map x', we have $x_s(w) = x'_s(w)$ for all $s \geqslant 0$, for μ-almost all w. We have chosen here a Stratonovich rather than an Itô formulation to be consistent with later sections, where we have made this choice in order to take advantage of the simpler calculations which the Stratonovich calculus allows. The Itô map referred to above is the map $\Phi(w) = x_1(w)$.

We can define on some complete probability space, $(\Omega, \mathcal{F}, \mathbb{P})$ say, a two-parameter, continuous, zero-mean Gaussian field $(z_{st} : s, t \geqslant 0)$ with values in $\mathbb{R}^m$, and with covariances given by

$$\mathbb{E}(z_{st}^i z_{s't'}^j) = \delta^{ij}(s \wedge s')e^{-|t-t'|/2}.$$

Such a field is called an Ornstein–Uhlenbeck sheet. Set $z_t = (z_{st} : s \geqslant 0)$. Then, for $t > 0$, both z_0 and z_t are Brownian motions in $\mathbb{R}^m$ and (z_0, z_t) and (z_t, z_0) have the same distribution. We have now defined all the terms in, and have justified, the identity (1).

Consider the following stochastic differential equation for an unknown process $(U_s : s \geqslant 0)$ in the space of $d \times d$ matrices

$$\partial U_s = \nabla X_i(x_s)U_s\partial w_s^i + \nabla X_0(x_s)U_s\partial s, \quad U_0 = I.$$

This equation may be solved, jointly with the equation for x, in exactly the same sense as the equation for x alone. Thus we obtain a map $U : [0, \infty) \times W \to \mathbb{R}^d \otimes (\mathbb{R}^d)^*$, with properties analogous to those of x. Moreover, by solving an equation for the inverse, we can see that $U_s(w)$ remains invertible for all $s \geqslant 0$, for almost all w. Write U_s^* for the transpose matrix and set $\Gamma_s = U_s C_s U_s^*$, where

$$C_s = \int_0^s U_r^{-1}X_i(x_r) \otimes U_r^{-1}X_i(x_r)dr.$$

Set also

$$\begin{aligned} L_s = -U_s \int_0^s U_r^{-1}X_i(x_r)\partial w_r^i + U_s \int_0^s U_r^{-1}\{\nabla^2 X_i(x_r)\partial w_r^i + \nabla^2 X_0(x_r)dr\}\Gamma_r, \\ + U_s \int_0^s U_r^{-1}\nabla X_i(x_r)X_i(x_r)dr \end{aligned}$$

and define for $G = g \circ \Phi$

$$LG = L_1^i \nabla_i g(x_1) + \Gamma_1^{ij}\nabla_i\nabla_j g(x_1).$$

We have now defined all the terms appearing in the integration-by-parts formula (2). We will give a proof in Section 5.

3. Review of two-parameter stochastic calculus

In [**9**], building on the fundamental works of Cairoli and Walsh [**2**] and Wong and Zakai [**14, 15**], we gave an account of two-parameter stochastic calculus, suitable for the development of a general theory of two-parameter hyperbolic stochastic differential equations. We recall here, for the reader's convenience, the main features of this account.

We take as our probability space $(\Omega, \mathcal{F}, \mathbb{P})$ the canonical complete probability space of an m-dimensional Brownian sheet $(w_{st} : s,t \geqslant 0)$, extended to a process $(w_{st} : s,t \in \mathbb{R})$ by independent copies in the other three quadrants. Thus $w_{st} = (w^1_{st}, \dots, w^m_{st})$ is a continuous, zero-mean Gaussian process, with covariances given by

$$\mathbb{E}(w^i_{st} w^j_{s't'}) = \delta^{ij}(s \wedge s')(t \wedge t'), \quad i,j = 1,\dots,m, \quad s,t \geqslant 0, \quad s',t' \geqslant 0.$$

It will be convenient to define also $w^0_{st} = st$ for all $s,t \in \mathbb{R}$. For $s,t \geqslant 0$, write $\mathcal{F}_{st}$ for the completion with respect to $\mathbb{P}$ of the σ-algebra generated by w_{ru} for $r \in (-\infty, s]$ and $u \in (-\infty, t]$. We say that a two-parameter process $(x_{st} : s,t \geqslant 0)$ is *adapted* if x_{st} is $\mathcal{F}_{st}$-measurable for all $s,t \geqslant 0$, and is *continuous* if $(s,t) \mapsto x_{st}(\omega)$ is continuous on $(\mathbb{R}^+)^2$ for all $\omega \in \Omega$. The *previsible* σ-algebra on $\Omega \times (\mathbb{R}^+)^2$ is that generated by sets of the form $A \times (s,s'] \times (t,t']$ with $A \in \mathcal{F}_{st}$. If we allow $A \in \mathcal{F}_{s\infty}$ in this definition, we get the *s-previsible* σ-algebra.

The classical approach to defining stochastic integrals, by means of an isometry of Hilbert spaces, adapts in a straightforward way from one-dimensional times to two, allowing the construction of stochastic integrals with respect to certain two-parameter processes, in particular with respect to the Brownian sheet. Given an s-previsible process[(1)] $(a_s(t) : s,t \geqslant 0)$, such that

$$\mathbb{E} \int_0^s \int_0^t a_r(u)^2 dr du < \infty$$

for all $s,t \geqslant 0$, we can define, for $i = 1,\dots,m$ and all $t_1, t_2 \geqslant 0$ with $t_1 \leqslant t_2$, one-parameter processes M and A by

$$M_s = \int_0^s \int_{t_1}^{t_2} a_r(t) d_r d_t w^i_{rt}, \quad A_s = \int_0^s \int_{t_1}^{t_2} a_r(t)^2 dr dt. \tag{3}$$

Then M is a continuous $(\mathcal{F}_{s\infty})_{s \geqslant 0}$-martingale, with quadratic variation process $[M] = A$. A localization argument by adapted initial open sets (see below) allows an extension of the integral under weaker integrability conditions. By the Burkholder–Davis–Gundy inequalities, for all $\alpha \in [2, \infty)$, there is a constant $C(\alpha) < \infty$ such that

$$\mathbb{E}\left(\left| \int_{s_1}^{s_2} \int_{t_1}^{t_2} a_s(t) d_s d_t w^i_{st} \right|^\alpha \right) \leqslant C(\alpha) \mathbb{E}\left(\left| \int_{s_1}^{s_2} \int_{t_1}^{t_2} a_s(t)^2 ds dt \right|^{\alpha/2} \right). \tag{4}$$

(1) We write any time parameter with respect to which a process is previsible, here s, as a subscript. Where previsibility is not assumed, here in t, we write the parameter in parentheses.

By an (s,t)-*semimartingale*, s-*semimartingale*, t-*semimartingale*, we mean, respectively, previsible processes $(x_{st} : s,t \geqslant 0)$, $(p_{st} : s,t \geqslant 0)$, $(q_{st} : s,t \geqslant 0)$ for which we may write

$$\begin{aligned} &x_{st} - x_{s0} - x_{0t} + x_{00} \\ &= \sum_{i=0}^{m} \int_0^s \int_0^t (x''_{ru})_i d_r d_u w^i_{ru} + \sum_{i,j=0}^{m} \int_0^s \int_{-1}^t \left(\int_{-1}^s \int_0^t (x''_{ru}(r',u'))_{ij} d_{r'} d_u w^j_{r'u} \right) d_r d_{u'} w^i_{ru'} \end{aligned}$$

and

$$p_{st} - p_{0t} = \sum_{i=0}^{m} \int_0^s \int_{-1}^t (p'_{rt}(u'))_i d_r d_{u'} w^i_{ru'}, \quad q_{st} - q_{s0} = \sum_{i=0}^{m} \int_{-1}^s \int_0^t (q'_{su}(r'))_i d_{r'} d_u w^i_{r'u}.$$

Here, $(x''_{st} : s,t \geqslant 0)$ is a previsible process, having components $(x''_{st})_i$, subject to certain local integrability conditions, which are implied, in particular, by almost sure local boundedness. The process $(x''_{st}(r,u) : s,t \geqslant 0, r,u \in \mathbb{R})$ is required to be previsible in (ω,s,t) and (Borel) measurable in (r,u), with $x''_{st}(r,u) = 0$ for $r > s$ or $u > t$, and is subject to similar local integrability conditions. The inner and outer parts of the second integral are both cases of the stochastic integral at (3), or its t-analogue, or of the usual Lebesgue integral, and the value of the iterated integral is unchanged if we reverse the order in which the integrals are taken. The integrals appearing in the expression for x_{st} are called stochastic integrals of the *first* and *second kind*. The processes $(p'_{st}(u) : s,t \geqslant 0, u \in \mathbb{R})$ and $(q'_{st}(r) : s,t \geqslant 0, r \in \mathbb{R})$ are required to be previsible in (ω,s,t) and measurable in u and r, respectively, with $p'_{st}(u) = 0$ for $u > t$ and $q'_{st}(r) = 0$ for $r > s$, and are subject to similar local integrability conditions. For fixed $t \geqslant 0$, if $(x_{s0} : s \geqslant 0)$ is a continuous $(\mathscr{F}_{s0})_{s\geqslant 0}$-semimartingale, then $(x_{st} : s \geqslant 0)$ is a continuous $(\mathscr{F}_{st})_{s\geqslant 0}$-semimartingale, in the usual one-parameter sense. Also $(p_{st} : s \geqslant 0)$ is a continuous $(\mathscr{F}_{st})_{s\geqslant 0}$-semimartingale, for all $t \geqslant 0$.

The heuristic formulae

$$\begin{aligned} d_s d_t x_{st} &= \sum_{i=0}^{m} (x''_{st})_i d_s d_t w^i_{st} + \sum_{i,j=0}^{m} \int_{-1}^s \int_{-1}^t (x''_{st}(r,u))_{ij} d_s d_u w^i_{su} d_r d_t w^j_{rt}, \\ d_s p_{st} &= \sum_{i=0}^{m} \int_{-1}^t (p'_{st}(u))_i d_s d_u w^i_{su}, \\ d_t q_{st} &= \sum_{i=0}^{m} \int_{-1}^s (q'_{st}(r))_i d_r d_t w^i_{rt} \end{aligned}$$

provide a good intuition in representing the two-parameter increment

$$d_s d_t x_{st} = x_{s+ds,t+dt} - x_{s,t+dt} - x_{s+ds,t} + x_{st}$$

and the one-parameter increments $d_s p_{st} = p_{s+ds,t} - p_{st}$ and $d_t q_{st} = q_{s,t+dt} - q_{st}$ in terms of a linear combinations of increments, and of products of increments of the Brownian sheet.

By a (*two-parameter*) *semimartingale*, we mean a process which is at the same time an (s,t)-semimartingale, an s-semimartingale and a t-semimartingale. Such processes are necessarily continuous. An (s,t)-semimartingale which is constant on the s-axis and t-axis is a semimartingale. By an obvious choice of integrands, the process $(w_{st} : s,t \geqslant 0)$ is itself a semimartingale. The choice of lower limit -1 is useful to us in allowing as semimartingales a pair of independent $\mathbb{R}^m$-valued Brownian motions $(z_{s0} : s \geqslant 0)$ and $(b_{0t} : t \geqslant 0)$, given by

$$z_{s0} = \int_0^s \int_{-1}^0 d_r d_u w_{ru}, \quad b_{0t} = \int_{-1}^0 \int_0^t d_r d_u w_{ru},$$

which are moreover independent of $(w_{st} : s,t \geqslant 0)$. Here and below, we bring one-parameter processes defined on the s or t axes into the class of two-parameter processes by extending them as constant in the second parameter.

We say that a subset $\mathcal{D} \subseteq (\mathbb{R}^+)^2$ is an *initial open set* if it is non-empty and is a union of rectangles of the form $[0,s) \times [0,t)$, where $s,t \geqslant 0$. A random subset $\mathcal{D} \subseteq \Omega \times (\mathbb{R}^+)^2$ is *adapted* if the event $\{(s,t) \in \mathcal{D}\}$ is $\mathcal{F}_{st}$-measurable for all $s,t \geqslant 0$. For an adapted initial open set $\mathcal{D}$, a process $(x_{st} : (s,t) \in \mathcal{D})$ is a *semimartingale in* $\mathcal{D}$ if there exists a sequence of adapted initial open sets $\mathcal{D}_n \uparrow \mathcal{D}$, almost surely, and a sequence of semimartingales $(x^n_{st} : s,t \geqslant 0)$, such that $x_{st} = x^n_{st}$ for all $(s,t) \in \mathcal{D}_n$ for all n. The notion of an *s-semimartingale in* $\mathcal{D}$ is defined analogously. We write $\zeta(\mathcal{D})$ for the boundary of $\mathcal{D}$ as a subset of $(\mathbb{R}^+)^2$. In particular, if $\mathcal{D} = (\mathbb{R}^+)^2$, then $\zeta(\mathcal{D}) = \varnothing$.

The theory which we now describe is symmetrical in s and t. Where a statement is made for s, there is also a corresponding statement for t, which we shall often omit. Let $(x_{st} : s,t \geqslant 0)$ and $(x'_{st} : s,t \geqslant 0)$ be s-semimartingales and let $(a_{st} : s,t \geqslant 0)$ be a locally bounded previsible process, for example, a continuous adapted process. There exist s-semimartingales which, for each $t \geqslant 0$, provide versions of the one-parameter stochastic integral and the one-parameter covariation process

$$\zeta^1_{st} = \int_0^s a_{rt} d_r x_{rt}, \quad \zeta^2_{st} = \int_0^s d_r x_{rt} d_r x'_{rt}.$$

From now on, when we write these integrals, we assume that such a version has been chosen. We define also four types of two-parameter integral, each of which is a (two-parameter) semimartingale. These are written

$$\zeta^3_{st} = \int_0^s \int_0^t a_{ru} d_r d_u x_{ru}, \quad \zeta^4_{st} = \int_0^s \int_0^t d_r x_{ru} d_u y_{ru},$$

$$\zeta^5_{st} = \int_0^s \int_0^t d_r x_{ru} d_r d_u y_{ru}, \quad \zeta^6_{st} = \int_0^s \int_0^t d_r d_u x_{ru} d_r d_u y_{ru}.$$

In the first and last integral, we require x to be an (s,t)-semimartingale, whereas, in the second and third, x should be an s-semimartingale. We require that y be a t-semimartingale in the second integral and an (s,t)-semimartingale in the third and fourth. All these integrals are defined as sums of certain integrals of the first and second kind with respect to the Brownian sheet. We refer to [**9**] for the details. We use the following differential notations:

$$
\begin{array}{lcr}
d_s z_{st} = a_{st} d_s x_{st} & \text{means} & z_{st} - z_{0t} = \zeta^1_{st}, \\
d_s z_{st} = d_s x_{st} d_s x'_{st} & \text{means} & z_{st} - z_{0t} = \zeta^2_{st}, \\
d_s d_t z_{st} = a_{st} d_s d_t x_{st} & \text{means} & z_{st} - z_{s0} - z_{0t} + z_{00} = \zeta^3_{st}, \\
d_s d_t z_{st} = d_s x_{st} d_t y_{st} & \text{means} & z_{st} - z_{s0} - z_{0t} + z_{00} = \zeta^4_{st}, \\
d_s d_t z_{st} = d_s x_{st} d_s d_t y_{st} & \text{means} & z_{st} - z_{s0} - z_{0t} + z_{00} = \zeta^5_{st}, \\
d_s d_t z_{st} = d_s d_t x_{st} d_s d_t y_{st} & \text{means} & z_{st} - z_{s0} - z_{0t} + z_{00} = \zeta^6_{st}.
\end{array}
$$

The integrals ζ^2_{st}, ζ^5_{st} and ζ^6_{st} all vanish if $d_s x_{st} = a_{st} ds$. It is shown in [**9**] that a series of identities hold among the various types of integral, which can be expressed conveniently in terms of this differential notation. Some identities assert the associativity of products involving a combination of three differentials or processes, the others are written as the following three rules

$$
\begin{aligned}
d_s(f(x_{st})) &= f'(x_{st}) d_s x_{st} + \tfrac{1}{2} f''(x_{st}) d_s x_{st} d_s x_{st}, \\
d_s(a_{st} d_t x_{st}) &= d_s a_{st} d_t x_{st} + a_{st} d_s d_t x_{st} + d_s a_{st} d_s d_t x_{st}, \\
d_s(d_t x_{st} d_t y_{st}) &= d_s d_t x_{st} d_t y_{st} + d_t x_{st} d_s d_t y_{st} + d_s d_t x_{st} d_s d_t y_{st}.
\end{aligned}
$$

These rules combine the usual calculus of partial differentials with Itô calculus in an obvious way. As a consequence, we can obtain a geometrically simpler Stratonovich-type calculus by defining, for processes $(x_{st} : s, t \geqslant 0)$ and $(y_{st} : s, t \geqslant 0)$, some further integrals, corresponding to the following differential rules

$$
X_{st} \partial_s X_{st} = X_{st} dY_{st} + \tfrac{1}{2} d_s X_{st} d_s Y_{st}, \quad \partial_s X_{st} \partial_s Y_{st} = \partial_s X_{st} d_s Y_{st} = d_s X_{st} d_s Y_{st},
$$

where X_{st} may stand for any one of $x_{st}, d_t x_{st}, \partial_t x_{st}$ and Y_{st} may stand for any one of $y_{st}, d_t y_{st}, \partial_t y_{st}$. Then we have

$$
\begin{aligned}
\partial_s(f(x_{st})) &= f'(x_{st}) \partial_s x_{st}, \\
\partial_s(a_{st} \partial_t x_{st}) &= \partial_s a_{st} \partial_t x_{st} + a_{st} \partial_s \partial_t x_{st}, \\
\partial_s(\partial_t x_{st} \partial_t y_{st}) &= \partial_s \partial_t x_{st} \partial_t y_{st} + \partial_t x_{st} \partial_s \partial_t y_{st}.
\end{aligned}
$$

The Brownian sheet $(w_{st} : s, t \geqslant 0)$ and the boundary Brownian motions $(z_{s0} : s \geqslant 0)$ and $(b_{0t} : t \geqslant 0)$ have some special properties, which are reflected in the following differential formulae, for $1 \leqslant i, j \leqslant m$,

$$
d_s d_t w^i_{st} d_s d_t w^j_{st} = \delta^{ij} ds dt, \quad d_s z^i_{s0} d_s z^j_{s0} = \delta^{ij} ds, \quad d_t b^i_{0t} d_t b^j_{0t} = \delta^{ij} dt,
$$

and, for any semimartingale $(x_{st} : s, t \geqslant 0)$,

$$
d_s x_{st} d_s d_t w^i_{st} = d_t x_{st} d_s d_t w^i_{st} = 0.
$$

4. A regularity result for two-parameter stochastic differential equations

We discussed in [**9**] a class of two-parameter hyperbolic stochastic differential equations, in which there is given, for a system of processes $(x_{st}, p_{st}, q_{st} : s, t \geqslant 0)$, one equation for the mixed second-order differential $d_s d_t x_{st}$, together with two further equations for the one-parameter differentials $d_s p_{st}$ and $d_t q_{st}$. We review briefly the details below, and then give a new regularity result, which we need for our application to Malliavin's integration-by-parts formula, but which may be of independent interest. This result concerns the process $(p_{st} : s, t \geqslant 0)$ (and analogously also $(q_{st} : s, t \geqslant 0)$), which, since integrated in s, has naturally the regularity of an s-semimartingale. The point at issue is whether $(p_{st} : s, t \geqslant 0)$ is a full (two-parameter) semimartingale. A method to establish this is stated in [**9**, pp. 299, 315-316], but the argument given is incomplete. A full proof is given below in Theorem 4.2. As an illustrative example, we note that, if $(w_{st} : s, t \geqslant 0)$ is a Brownian sheet with values in $\mathbb{R}^m$, then the result will show that there is a two-parameter semimartingale $(x_{st} : s, t \geqslant 0)$ such that, for all $t \geqslant 0$, the process $(x_{st} : s \geqslant 0)$ satisfies the one-parameter stochastic differential equation

$$\partial_s x_{st} = X_i(x_{st}) \partial_s w^i_{st} + X_0(x_{st}) \partial s,$$

with given initial values $x_{0t} = x_0$, say. This is useful because, now, despite the irregular dependence of the Brownian sheet on t, we can use a differential calculus in t as well as in s.

Consider the class of hyperbolic stochastic differential equations in $(\mathbb{R}^+)^2$ of the form

$$d_s d_t x_{st} = a(d_s d_t w_{st}) + b(d_s x_{st}, d_t x_{st}), \tag{5}$$

$$d_s p_{st} = c(d_s x_{st}), \tag{6}$$

$$d_t q_{st} = e(d_t x_{st}). \tag{7}$$

Here $w_{st} = (w^1_{st}, \ldots, w^m_{st})$,with $(w^i_{st} : s, t \geqslant 0)$, $i = 1, \ldots, m$, independent Brownian sheets, as above. The unknown processes $(x_{st} : s, t \geqslant 0)$, $(p_{st} : s, t \geqslant 0)$ and $(q_{st} : s, t \geqslant 0)$ take values in $\mathbb{R}^d$, $\mathbb{R}^n$ and $\mathbb{R}^n$, respectively, and are subject to given boundary values $(x_{s0} : s \geqslant 0)$, $(x_{0t} : t \geqslant 0)$, both assumed to be semimartingales, and $(p_{0t} : t \geqslant 0)$, $(q_{s0} : s \geqslant 0)$, both assumed continuous and adapted. The coefficients a, b, c, e are allowed to have a locally Lipschitz dependence on the unknown processes, with the restriction that b depends only on x. Thus, for example, we would write $a(x_{st}, p_{st}, q_{st}, d_s d_t w_{st})$ and $b(x_{st}, d_s x_{st}, d_t x_{st})$, but have not done so in order to keep the notation compact. Moreover, we allow a dependence on the differentials which is a sum of linear and quadratic terms. Thus, in an expanded notation, we would write

$$\begin{aligned} d_s d_t x_{st} = {} & a_1(d_s d_t w_{st}) + a_2(d_s d_t w_{st}, d_s d_t w_{st}) \\ & + b_{11}(d_s x_{st}, d_t x_{st}) + b_{12}(d_s x_{st}, d_t x_{st}, d_t x_{st}), \\ & + b_{21}(d_s x_{st}, d_s x_{st}, d_t x_{st}) + b_{22}(d_s x_{st}, d_s x_{st}, d_t x_{st}, d_t x_{st}), \end{aligned}$$

$$d_s p_{st} = c_1(d_s x_{st}) + c_2(d_s x_{st}, d_s x_{st}),$$
$$d_t q_{st} = e_1(d_t x_{st}) + e_2(d_t x_{st}, d_t x_{st}),$$

where, for $i, j, k = 1, 2$,

$$a_i : \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d \otimes ((\mathbb{R}^m)^*)^{\otimes i},$$
$$b_{jk} : \mathbb{R}^d \to \mathbb{R}^d \otimes ((\mathbb{R}^d)^*)^{\otimes j+k},$$
$$c_j : \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \otimes ((\mathbb{R}^d)^*)^{\otimes j},$$
$$e_k : \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \otimes ((\mathbb{R}^d)^*)^{\otimes k}.$$

We may and do assume with loss that $a_2, b_{12}, b_{21}, b_{22}, c_2, e_2$ are symmetric in any pair of repeated differential arguments.

By a *local solution of* (5–7) *with domain* $\mathcal{D}$ we mean an adapted initial open set $\mathcal{D}$, together with a semimartingale $(x_{st} : (s,t) \in \mathcal{D})$, an s-semimartingale $(p_{st} : (s,t) \in \mathcal{D})$, and a t-semimartingale $(q_{st} : (s,t) \in \mathcal{D})$, all continuous on $\mathcal{D}$, such that, for all $(s,t) \in \mathcal{D}$,

$$x_{st} = x_{s0} + x_{0t} - x_{00} + \int_0^s \int_0^t a(d_r d_u w_{ru}) + \int_0^s \int_0^t b(d_r x_{ru}, d_u x_{ru}),$$
$$p_{st} = p_{0t} + \int_0^s c(d_r x_{rt}),$$
$$q_{st} = q_{s0} + \int_0^t e(d_u x_{su}).$$

Given such a solution, for each $t \geqslant 0$, we can define processes $(u_{st} : (s,t) \in \mathcal{D})$ and $(u^*_{st} : (s,t) \in \mathcal{D})$, taking values in $\mathbb{R}^d \times (\mathbb{R}^d)^*$ and $\mathbb{R}^d \times (\mathbb{R}^d)^* \times (\mathbb{R}^d)^*$ respectively, by solving the linear one-parameter stochastic differential equations

$$d_s u_{st} = b_{11}(d_s x_{st}, \cdot) u_{st} + b_{12}(d_s x_{st}, d_s x_{st}, \cdot) u_{st}, \tag{8}$$
$$\begin{aligned} d_s u^*_{st} = u^{-1}_{st}\{&b_{12}(d_s x_{st}, u_{st}\cdot, u_{st}\cdot) \\ &+ b_{22}(d_s x_{st}, d_s x_{st}, u_{st}\cdot, u_{st}\cdot) - b_{11}(d_s x_{st}, b_{12}(d_s x_{st}, u_{st}\cdot, u_{st}\cdot))\}. \end{aligned} \tag{9}$$

Here u^{-1}_{st} denotes the inverse of the linear map u_{st}. For fixed $t \geqslant 0$, almost surely, u_{st} remains in the set of invertible maps while $(s,t) \in \mathcal{D}$. To see this, one can obtain formally a linear equation for the process $(u^{-1}_{st} : (s,t) \in \mathcal{D})$, and then check that its solution is indeed an inverse for u_{st}. Similarly, for each $s \geqslant 0$, we can define processes $(v_{st} : (s,t) \in \mathcal{D})$ and $(v^*_{st} : (s,t) \in \mathcal{D})$, taking values in $\mathbb{R}^d \times (\mathbb{R}^d)^*$ and $\mathbb{R}^d \times (\mathbb{R}^d)^* \times (\mathbb{R}^d)^*$, by solving the analogous equations

$$d_t v_{st} = b_{11}(\cdot, d_t x_{st}) v_{st} + b_{21}(\cdot, d_t x_{st}, d_t x_{st}) v_{st}. \tag{10}$$
$$\begin{aligned} d_t v^*_{st} = v^{-1}_{st}\{&b_{21}(v_{st}\cdot, v_{st}\cdot, d_t x_{st}) \\ &+ b_{22}(v_{st}\cdot, v_{st}\cdot, d_t x_{st}, d_t x_{st}) - b_{11}(b_{21}(v_{st}\cdot, v_{st}\cdot, d_t x_{st}), d_t x_{st})\}. \end{aligned} \tag{11}$$

We specify initial conditions $u_{00} = v_{00} = I$, so determining completely $(u_{0s} : s \geqslant 0)$ and $(v_{0t} : t \geqslant 0)$. Then we complete the determination of the above processes by specifying that $u_{0t} = v_{0t}$, $u^*_{0t} = 0$, $v_{s0} = u_{s0}$, and $v^*_{s0} = 0$ for all $s, t \geqslant 0$. Let us say that $(x_{st}, p_{st}, q_{st} : (s,t) \in \mathcal{D})$ is a *regular* local solution(2) if there exist continuous s-semimartingales $(u_{st} : (s,t) \in \mathcal{D})$ and $(u^*_{st} : (s,t) \in \mathcal{D})$ satisfying, for each $t \geqslant 0$, the equations (8–9), and if there exist also continuous t-semimartingales $(v_{st} : (s,t) \in \mathcal{D})$ and $(v^*_{st} : (s,t) \in \mathcal{D})$ satisfying, for each $s \geqslant 0$, the equations (10–11). A local solution is *maximal* if it is not the restriction of any local solution with larger domain. The notion of a maximal regular local solution is defined analogously. We assume that the boundary semimartingales $(x_{s0} : s \geqslant 0)$, $(x_{0t} : t \geqslant 0)$, $(p_{0t} : t \geqslant 0)$ and $(q_{s0} : s \geqslant 0)$ are *regular*(3). By this we mean that the Lebesgue–Stieltjes measures defined by their quadratic variation processes and by the total variation processes of their finite variation parts are all dominated by Kds, or Kdt as appropriate, for some constant $K < \infty$. We give a result first for the case where $b = 0$.

Lemma 4.1. — *Assume that $b = 0$. Let U be an open subset of $\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n$ and let $m : U \to [0, \infty)$ be a continuous function with $m(x,p,q) \to \infty$ as $(x,p,q) \to \partial U$. Assume that, for all $M \geqslant 1$, the coefficients a, c, e are bounded and Lipschitz on the set $U_M = \{(x,p,q) \in U : m(x,p,q) < M\}$. Then, for any set of regular boundary semimartingales $(x_{s0} : s \geqslant 0)$, $(x_{0t} : t \geqslant 0)$, $(p_{0t} : t \geqslant 0)$ and $(q_{s0} : s \geqslant 0)$, with $(x_{00}, p_{00}, q_{00}) \in U$, the equations (5–7) have a unique maximal local solution $(x_{st}, p_{st}, q_{st} : (s,t) \in \mathcal{D})$ with values in U. Moreover, we have, almost surely(4)*

$$\sup_{r \leqslant s, u \leqslant t} m(x_{ru}, p_{ru}, q_{ru}) \to \infty \quad \textit{as} \quad (s,t) \uparrow \zeta(\mathcal{D}).$$

Proof. — In the case where m is bounded (so $U_M = U = \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n$ for large M), the existence of a (global) solution is proved in [**9**, Theorem 3.2.2]. The proof is of a standard type, using Picard iteration, Gronwall's lemma and Kolmogorov's continuity criterion, and gives also the uniqueness of local solutions on the intersections of their domains. When m is unbounded, we can find, for each $M \geqslant 1$, bounded Lipschitz coefficients a_M, c_M, e_M on $\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n$, which agree with a, c, e on U_M. For each $M_0 \geqslant 1$, the corresponding global solutions $(x^M_{st}, p^M_{st}, q^M_{st} : s, t \geqslant 0)$ agree, for all integers $M \geqslant M_0$, almost surely, on $\mathcal{D}_{M_0}$, where

$$\mathcal{D}_M = \{(s,t) \in (\mathbb{R}^+)^2 : \sup_{r \leqslant s, u \leqslant t} m(x^M_{ru}, p^M_{ru}, q^M_{ru}) \leqslant M\}.$$

Hence, we obtain a local solution with all the claimed properties by setting $\mathcal{D} = \cup_M \mathcal{D}_M$ and by setting, for all $M \geqslant 1$, $(x_{st}, p_{st}, q_{st}) = (x^M_{st}, p^M_{st}, q^M_{st})$ for all $(s,t) \in \mathcal{D}_M \setminus \mathcal{D}_{M-1}$. □

Our main result deals with the case when b is non-zero.

(2) It is not hard to see that, for any local solution, the processes just defined have previsible versions, which are then s-semimartingales or t-semimartingales, depending on the variable of integration. However, we have not determined whether they have a continuous version in general.
(3) No connection with the notion of regular local solution is intended.
(4) To clarify, we mean that, for all $(s^*, t^*) \in \zeta(\mathcal{D})$, the given limit holds whenever $(s,t) \uparrow (s^*, t^*)$. In particular, in the case where $\mathcal{D} = (\mathbb{R}^+)^2$, there are no such points (s^*, t^*) and nothing is claimed.

Theorem 4.2. — *Assume that the coefficients a, b, c, e are uniformly bounded and Lipschitz. Then, for each set of regular semimartingale boundary values $(x_{s0} : s \geqslant 0)$, $(x_{0t} : t \geqslant 0)$, $(p_{0t} : t \geqslant 0)$, $(q_{s0} : s \geqslant 0)$, the system of equations (5–7) has a unique maximal regular solution, with domain $\mathcal{D}$ say. As $(s,t) \uparrow \zeta(\mathcal{D})$, we have*

$$m_{st} = \sup_{s' \leqslant s, t' \leqslant t} |(u_{s't'}, u^{-1}_{s't'}, v_{s't'}, v^{-1}_{s't'})| \to \infty. \tag{12}$$

Moreover, if c has Lipschitz first and second derivatives and has no dependence on q, then $(p_{st} : s,t \in \mathcal{D})$ is a semimartingale in $\mathcal{D}$.

Proof. — We consider first the question of existence. We follow, to begin, the strategy used in the proof of [**9**, Theorem 3.2.3]. Consider the following system of differential equations, for unknown processes $y_{st}, z_{st}, x'_{st}, u_{st}, u^*_{st}, p_{st}, x''_{st}, v_{st}, v^*_{st}, q_{st}$, taking values in $\mathbb{R}^d, \mathbb{R}^d, \mathbb{R}^d, \mathbb{R}^d \otimes (\mathbb{R}^d)^*, \mathbb{R}^d \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^d)^*, \mathbb{R}^n, \mathbb{R}^d, \mathbb{R}^d \otimes (\mathbb{R}^d)^*, \mathbb{R}^d \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^d)^*, \mathbb{R}^n$ respectively:

$$d_s d_t y_{st} = u^{-1}_{st} a(d_s d_t w_{st}) - u^*_{st}(u^{-1}_{st} a(d_s d_t w_{st}) \otimes u^{-1}_{st} a(d_s d_t w_{st})), \tag{13}$$

$$d_s d_t z_{st} = v^{-1}_{st} a(d_s d_t w_{st}) - v^*_{st}(v^{-1}_{st} a(d_s d_t w_{st}) \otimes v^{-1}_{st} a(d_s d_t w_{st})), \tag{14}$$

$$d_s x'_{st} = v_{st}(d_s z_{st} + v^*_{st} d_s z_{st} \otimes d_s z_{st}), \tag{15}$$

$$d_s u_{st} = b_{11}(v_{st}(d_s z_{st} + v^*_{st} d_s z_{st} \otimes d_s z_{st}), \cdot) u_{st} + b_{21}(v_{st} d_s z_{st}, v_{st} d_s z_{st}, \cdot) u_{st}, \tag{16}$$

$$\begin{aligned} d_s u^*_{st} = u^{-1}_{st}\{ & b_{12}(v_{st}(d_s z_{st} + v^*_{st} d_s z_{st} \otimes d_s z_{st}), u_{st}\cdot, u_{st}\cdot) \\ & + b_{22}(v_{st} d_s z_{st}, v_{st} d_s z_{st}, u_{st}\cdot, u_{st}\cdot) && (17) \\ & - b_{11}(v_{st} d_s z_{st}, b_{12}(v_{st} d_s z_{st}, u_{st}\cdot, u_{st}\cdot))\}, && (18) \end{aligned}$$

$$d_s p_{st} = c(v_{st}(d_s z_{st} + v^*_{st} d_s z_{st} \otimes d_s z_{st})), \tag{19}$$

$$d_t x''_{st} = u_{st}(d_t y_{st} + u^*_{st} d_t y_{st} \otimes d_t y_{st}), \tag{20}$$

$$d_t v_{st} = b_{11}(\cdot, u_{st}(d_t y_{st} + u^*_{st} d_t y_{st} \otimes d_t y_{st})) v_{st} + b_{12}(\cdot, u_{st} d_t y_{st}, u_{st} d_t y_{st}) v_{st}, \tag{21}$$

$$\begin{aligned} d_t v^*_{st} = v^{-1}_{st}\{ & b_{21}(v_{st}\cdot, v_{st}\cdot, u_{st}(d_t y_{st} + u^*_{st} d_t y_{st} \otimes d_t y_{st})) \\ & + b_{22}(v_{st}\cdot, v_{st}\cdot, u_{st} d_t y_{st}, u_{st} d_t y_{st}) && (22) \\ & - b_{11}(b_{21}(v_{st}\cdot, v_{st}\cdot, u_{st} d_t y_{st}), u_{st} d_t y_{st})\}, && (23) \end{aligned}$$

$$d_t q_{st} = e(u_{st}(d_t y_{st} + u^*_{st} d_t y_{st} \otimes d_t y_{st})). \tag{24}$$

We evaluate the coefficients a, b, c and e here at $(x'_{st}, p_{st}, q_{st})$ (rather than at x''_{st}). Note that this system has the same form as the system (5–7) with $b = 0$. We use the boundary conditions given above for $u_{st}, p_{st}, v_{st}, q_{st}$. Define boundary values for y_{st} and z_{st} by

$$d_s y_{s0} = d_s z_{s0} = v^{-1}_{s0} d_s x_{s0}, \quad d_t y_{0t} = d_t z_{0t} = u^{-1}_{0t} d_t x_{0t}, \quad y_{00} = z_{00} = 0. \tag{25}$$

Set $u^*_{0t} = v^*_{s0} = 0$ and use the given boundary values $(x_{0t} : t \geqslant 0)$ for x'_{st} and $(x_{s0} : s \geqslant 0)$ for x''_{st}. Define, on the set U where u and v are invertible,

$$m(y, z, x', u, u^*, p, x'', v, v^*, q) = |(u, u^{-1}, v, v^{-1})| + |(u^*, v^*)|.$$

Then the preceding lemma applies, to show that (13–24) has a unique maximal local solution with the given boundary values, with domain $\mathcal{D}$ say, such that u_{st} and v_{st} are invertible for all $(s,t) \in \mathcal{D}$, and such that, almost surely, as $t \uparrow \zeta(\mathcal{D})$, either

$$m_{st} = \sup_{s' \leqslant s, t' \leqslant t} |(u_{s't'}, u_{s't'}^{-1}, v_{s't'}, v_{s't'}^{-1})| \uparrow \infty, \tag{26}$$

or

$$n_{st} = \sup_{s' \leqslant s, t' \leqslant t} |(u^*_{s't'}, v^*_{s't'})| \uparrow \infty. \tag{27}$$

Now v_{st} and v^*_{st} are continuous t-semimartingales (in $\mathcal{D}$) and z_{st} is a semimartingale. Moreover $d_t a_{st} d_s d_t z_{st} = 0$ for any t-semimartingale a_{st}. Hence, by [**9**, Theorem 2.3.1], x'_{st} is a semimartingale and we may take the t-differential in (15) to obtain

$$\begin{aligned} d_s d_t x'_{st} &= d_t v_{st}(d_s z_{st} + v^*_{st} d_s z_{st} \otimes d_s z_{st}) \\ &\quad + v_{st}(d_s d_t z_{st} + d_t v^*_{st} d_s z_{st} \otimes d_s z_{st} + v^*_{st} d_s d_t z_{st} \otimes d_s d_t z_{st}) \\ &\quad + d_t v_{st}(d_t v^*_{st} d_s z_{st} \otimes d_s z_{st}) \\ &= a(d_s d_t w_{st}) + b(d_s x'_{st}, d_t x''_{st}). \end{aligned}$$

Similarly, by taking the s-differential in (20), we obtain

$$d_s d_t x''_{st} = a(d_s d_t w_{st}) + b(d_s x'_{st}, d_t x''_{st}).$$

We also have $x'_{00} = x''_{00}$ and

$$d_s x'_{s0} = v_{s0} d_s z_{s0} = d_s x''_{s0}, \quad d_t x'_{0t} = u_{0t} d_t y_{0t} = d_t x''_{0t},$$

so $x'_{st} = x''_{st}$ for all $(s,t) \in \mathcal{D}$, almost surely. Denote the common value of these processes by x_{st}. Then $(x_{st} : (s,t) \in \mathcal{D})$ satisfies (5). On using (15) and (20) to substitute(5) for $d_s z_{st}$ and $d_t y_{st}$ in (16, 19, 21, 24), we see also that p_{st}, q_{st}, u_{st}, u^*_{st}, v_{st}, v^*_{st} satisfy (6–11) respectively. Hence $(x_{st}, p_{st}, q_{st} : (s,t) \in \mathcal{D})$ is a regular local solution to (5–7), which is moreover maximal by virtue of (26–27).

We turn to the question of uniqueness. Suppose that $(\tilde{x}_{st}, \tilde{p}_{st}, \tilde{q}_{st} : (s,t) \in \tilde{\mathcal{D}})$ is any regular local solution to (5–7). Write $(\tilde{u}_{st}, \tilde{u}^*_{st}, \tilde{v}_{st}, \tilde{v}^*_{st} : (s,t) \in \tilde{\mathcal{D}})$ for the associated processes, satisfying (8–11). Define semimartingales $(\tilde{y}_{st} : (s,t) \in \tilde{\mathcal{D}})$ and $(\tilde{z}_{st} : (s,t) \in \tilde{\mathcal{D}})$ by

$$d_s d_t \tilde{y}_{st} = \tilde{u}_{st}^{-1} a(d_s d_t w_{st}) - \tilde{u}^*_{st}(\tilde{u}_{st}^{-1} a(d_s d_t w_{st}) \otimes \tilde{u}_{st}^{-1} a(d_s d_t w_{st})), \tag{28}$$

$$d_s d_t \tilde{z}_{st} = \tilde{v}_{st}^{-1} a(d_s d_t w_{st}) - \tilde{v}^*_{st}(\tilde{v}_{st}^{-1} a(d_s d_t w_{st}) \otimes \tilde{v}_{st}^{-1} a(d_s d_t w_{st})), \tag{29}$$

with boundary values (25). The following equations may be verified by checking that the initial values and differentials of left and right hand sides agree

$$d_s \tilde{x}_{st} = \tilde{v}_{st}(d_s \tilde{z}_{st} + \tilde{v}^*_{st} d_s \tilde{z}_{st} \otimes d_s \tilde{z}_{st}), \quad d_t \tilde{x}_{st} = \tilde{u}_{st}(d_t \tilde{y}_{st} + \tilde{u}^*_{st} d_t \tilde{y}_{st} \otimes d_t \tilde{y}_{st}). \tag{30}$$

(5) Such substitutions result in differential formulae corresponding to valid identities between processes. This is because the two-parameter stochastic differential calculus is associative, as mentioned above, and as discussed in [**9**, pp. 290–291].

Then, using these equations to substitute for $d_s\tilde{x}_{st}$ and $d_t\tilde{x}_{st}$ in (6–11), we see that $(\tilde{y}_{st}, \tilde{z}_{st}, \tilde{x}_{st}, \tilde{u}_{st}, \tilde{u}^*_{st}, \tilde{p}_{st}, \tilde{x}_{st}, \tilde{v}_{st}, \tilde{v}^*_{st}, \tilde{q}_{st} : (s,t) \in \tilde{\mathcal{D}})$ is a local solution to (13–24). By local uniqueness for this system, $\tilde{\mathcal{D}} \subseteq \mathcal{D}$ and $(\tilde{x}_{st}, \tilde{p}_{st}, \tilde{q}_{st}) = (x_{st}, p_{st}, q_{st})$ for all $(s,t) \in \tilde{\mathcal{D}}$, almost surely. Thus $(x_{st}, p_{st}, q_{st} : (s,t) \in \mathcal{D})$ is the unique maximal regular local solution to (5–7).

Our next goal is to obtain αth-moment and L^α-Hölder estimates on the process $(x_{st}, p_{st}, q_{st}, u_{st}, u^*_{st}, v_{st}, v^*_{st} : (s,t) \in \mathcal{D})$, for $\alpha \in [2,\infty)$. Write K for a uniform bound on a, b, c, e which is also a Lipschitz constant for b. Fix $M, N, T \geqslant 1$ and set

$$\mathcal{D}_M = \{(s,t) \in \mathcal{D} : s,t \leqslant T \text{ and } m_{st} \leqslant M\},$$
$$\mathcal{D}_{M,N} = \{(s,t) \in \mathcal{D} : s,t \leqslant T, m_{st} \leqslant M \text{ and } n_{st} \leqslant N\}.$$

Fix α and define

$$g(s,t) = \sup_{s' \leqslant s, t' \leqslant t} \mathbb{E}(|(u^*_{s't'}, v^*_{s't'})|^\alpha 1_{\{(s',t') \in \mathcal{D}_{M,N}\}}).$$

Let $(a_s : s \geqslant 0)$ be a locally bounded, $(\mathcal{F}_{s\infty})_{s\geqslant 0}$-previsible process. The following identities follow from equations (29) and (30): for $(s,t) \in \mathcal{D}$, respectively in $\mathbb{R}^d$ and $\mathbb{R}^d \otimes \mathbb{R}^d$,

(31)
$$\int_0^s a_r d_r x_{rt} = \int_0^s a_r d_r x_{r0} + \int_0^s \int_0^t a_r v_{rt} \left\{ v_{ru}^{-1} a(d_r d_u w_{ru}) + (v^*_{rt} - v^*_{ru})(v_{ru}^{-1} a(d_r d_u w_{ru}))^{\otimes 2} \right\}$$

and

$$\int_0^s a_r d_r x_{rt} \otimes d_r x_{rt} = \int_0^s a_r d_r x_{r0} \otimes d_r x_{r0} + \int_0^s \int_0^t a_r (v_{rt} v_{ru}^{-1} a(d_r d_u w_{ru}))^{\otimes 2}. \tag{32}$$

Hence, using the estimate (4), we obtain a constant $C = C(\alpha, K, M, T) < \infty$ such that, for all $s,t \geqslant 0$,

$$\begin{aligned} &\mathbb{E}\left(\left| \int_0^s a_r d_r x_{rt} \right|^\alpha 1_{\{(s,t)\in\mathcal{D}_{M,N}\}} \right) \\ &\qquad \leqslant C\mathbb{E}\left(\left| \left(\int_0^s a_r^2 dr \right)^{1/2} + \int_0^s \int_0^t |a_r|(|v^*_{rt}| + |v^*_{ru}|) dr du \right|^\alpha 1_{\{(s,t)\in\mathcal{D}_{M,N}\}} \right) \end{aligned} \tag{33}$$

and

$$\mathbb{E}\left(\left| \int_0^s a_r d_r x_{rt} \otimes d_r x_{rt} \right|^\alpha 1_{\{(s,t)\in\mathcal{D}_{M,N}\}} \right) \leqslant C\mathbb{E}\left(\left| \int_0^s |a_r| dr \right|^\alpha 1_{\{(s,t)\in\mathcal{D}_{M,N}\}} \right). \tag{34}$$

Here and below, we suppress any dependence of constants on the dimensions d, n, m. If we allow C to depend also on N, then (33) may be simplified to

$$\mathbb{E}\left(\left| \int_0^s a_r d_r x_{rt} \right|^\alpha 1_{\{(s,t)\in\mathcal{D}_{M,N}\}} \right) \leqslant C\mathbb{E}\left(\left| \int_0^s a_r^2 dr \right|^{\alpha/2} 1_{\{(s,t)\in\mathcal{D}_{M,N}\}} \right) \tag{35}$$

We use these estimates, along with analogous estimates for integrals $d_t x_{st}$, in the equations (9) and (11), to arrive at the inequality

$$g(s,t) \leqslant C\left(1+\int_0^s g(s',t)ds' + \int_0^t g(s,t')dt'\right),$$

for a constant $C = C(\alpha, K, M, T) < \infty$. Since $N < \infty$, we know that $g(s,t) < \infty$ for all s, t, so this inequality implies that $g(s,t) \leqslant C$ for another constant $C < \infty$ of the same dependence. Similar arguments yield a further constant $C < \infty$ of the same dependence such that, for all $s, s' \geqslant 0$ and all $t, t' \geqslant 0$,

$$\mathbb{E}(|(x_{st}, u_{st}, u^*_{st}, p_{st}) - (x_{s't}, u_{s't}, u^*_{s't}, p_{s't})|^\alpha 1_{\{(s,t),(s',t)\in\mathscr{D}_{M,N}\}}) \leqslant C|s-s'|^{\alpha/2} \tag{36}$$

and

$$\mathbb{E}(|(x_{st}, v_{st}, v^*_{st}, q_{st}) - (x_{st'}, v_{st'}, v^*_{st'}, q_{st'})|^\alpha 1_{\{(s,t),(s,t')\in\mathscr{D}_{M,N}\}}) \leqslant C|t-t'|^{\alpha/2}. \tag{37}$$

Here, we have used Cauchy–Schwarz to obtain in an intermediate step

$$\int_s^{s'}\int_0^t |v^*_{ru}|drdu \leqslant |s-s'|^{1/2}\left(\int_s^{s'}\int_0^t |v^*_{ru}|^2 drdu\right)^{1/2}.$$

On going back to (31) and (32) with these Hölder estimates, we obtain, using (4) again, a constant $C < \infty$ of the same dependence such that

$$\mathbb{E}\left(\left|\int_0^s a_r(d_r x_{rt} - d_r x_{rt'})\right|^\alpha 1_{\{(s,t),(s,t')\in\mathscr{D}_{M,N}\}}\right) \leqslant C|t-t'|^{\alpha/2}\left(\mathbb{E}\left|\int_0^s a_r^2 ds\right|^\alpha\right)^{1/2} \tag{38}$$

and

$$\mathbb{E}\left(\left|\int_0^s a_r d_r x_{rt}\otimes(d_r x_{rt} - d_r x_{rt'})\right|^\alpha 1_{\{(s,t),(s,t')\in\mathscr{D}_{M,N}\}}\right) \leqslant C|t-t'|^{\alpha/2}\left(\mathbb{E}\left|\int_0^s a_r^2 ds\right|^\alpha\right)^{1/2}. \tag{39}$$

Now

$$\begin{aligned} d_s(u_{st}^{-1}u_{st'}) = {}& u_{st}^{-1}\{b(x_{st'}, d_s x_{st'}, \cdot) - b(x_{st}, d_s x_{st}, \cdot)\}u_{st'} \\ & - u_{st}^{-1} b_{11}(x_{st}, d_s x_{st}, \cdot)\{b_{11}(x_{st'}, d_s x_{st'}, \cdot) - b_{11}(x_{st}, d_s x_{st}, \cdot)\}u_{st'}. \end{aligned}$$

We have made explicit the dependence of b and b_{11} on x_{st} or $x_{st'}$. We use the estimates (33), (34), (37–39) to find a constant $C = C(\alpha, K, M, T) < \infty$ such that

$$\mathbb{E}(|u_{st} - u_{st'}|^\alpha 1_{\{(s,t),(s,t')\in\mathscr{D}_{M,N}\}}) \leqslant C|t-t'|^{\alpha/2}. \tag{40}$$

Moreover, the same estimates, applied to the difference of (9) at t and at t', show that C may be chosen such that

$$\mathbb{E}(|u^*_{st} - u^*_{st'}|^\alpha 1_{\{(s,t),(s,t')\in\mathscr{D}_{M,N}\}}) \leqslant C|t-t'|^{\alpha/2}. \tag{41}$$

Since C does not depend on N, by monotone convergence, we can replace $\mathcal{D}_{M,N}$ by $\mathcal{D}_M$ in these estimates By symmetry, there are analogous estimates for v_{st} and v^*_{st}. Hence, using [**9**, Theorem 3.2.1], almost surely, for all $M \geqslant 1$, n_{st} remains bounded on $\mathcal{D}_M$. Thus (27) implies (26) so, in any case, (12) holds.

It remains to consider the case where c has Lipschitz first and second derivatives and has no dependence on q, and to show then that $(p_{st} : (s,t) \in \mathcal{D})$ is a semimartingale. For ease of writing, we shall assume that c has no dependence on x either. This is done without loss of generality, by the device of adding to our system the equation $d_s x_{st} = d_s x_{st}$, thus making x_{st} a component of p_{st}.

We seek to find a solution in a smaller class of processes, in which p_{st} is a semimartingale. Recall that

$$d_s p_{st} = c(d_s x_{st}) = c_1(p_{st})(d_s x_{st}) + c_2(p_{st})(d_s x_{st}, d_s x_{st}). \tag{42}$$

By Itô's formula, if p_{st} is a semimartingale, then

$$\begin{aligned} d_s d_t p_{st} &= c'(d_t p_{st}, d_s x_{st}) + \tfrac{1}{2} c''(d_t p_{st}, d_t p_{st}, d_s x_{st}) + c(d_s d_t x_{st}) + c'(d_t p_{st}, d_s d_t x_{st}) \\ &\qquad + 2c_2(d_s x_{st}, d_s d_t x_{st}) + 2c_2'(d_t p_{st}, d_s x_{st}, d_s d_t x_{st}) \\ &= c'(d_t p_{st}, d_s x_{st}) + \tfrac{1}{2} c''(d_t p_{st}, d_t p_{st}, d_s x_{st}) + c(a(d_s d_t w_{st})) + c(b(d_s x_{st}, d_t x_{st})) \\ &\qquad + c'(d_t p_{st}, b(d_s x_{st}, d_t x_{st})) + 2c_2(d_s x_{st}, b(d_s x_{st}, d_t x_{st})) \\ &\qquad + 2c_2'(d_t p_{st}, d_s x_{st}, b(d_s x_{st}, d_t x_{st})). \end{aligned}$$

Here we are writing c', c'' for the derivatives with respect to p. We set $\tilde{d} = d + n$ and combine this equation with the equation (5) to obtain a two-parameter equation for the $\mathbb{R}^{\tilde{d}}$-valued process $\tilde{x}_{st} = \binom{x_{st}}{p_{st}}$, which we can write in the form

$$d_s d_t \tilde{x}_{st} = \tilde{a}(d_s d_t w_{st}) + \tilde{b}(d_s \tilde{x}_{st}, d_t \tilde{x}_{st}). \tag{43}$$

(The $\sim$ notation in this paragraph has nothing to do with that used in the paragraph on uniqueness above.) We impose regular semimartingale initial values $\tilde{x}_{s0} = \binom{x_{s0}}{p_{s0}}$ and $\tilde{x}_{0t} = \binom{x_{0t}}{p_{0t}}$, where $(p_{s0} : s \geqslant 0)$ is obtained by solving the one-parameter equation (42) along x_{s0}. Introduce the two companion equations for $\tilde{d} \times \tilde{d}$ matrix-valued processes $\tilde{u}_{st}$ and $\tilde{v}_{st}$

$$d_s \tilde{u}_{st} = \tilde{b}_{11}(d_s \tilde{x}_{st}, \cdot)\tilde{u}_{st} + \tilde{b}_{12}(d_s \tilde{x}_{st}, d_s \tilde{x}_{st}, \cdot)\tilde{u}_{st}, \tag{44}$$

$$d_t \tilde{v}_{st} = \tilde{b}_{11}(\cdot, d_t \tilde{x}_{st})\tilde{v}_{st} + \tilde{b}_{21}(\cdot, d_t \tilde{x}_{st}, d_t \tilde{x}_{st})\tilde{v}_{st}. \tag{45}$$

Impose boundary conditions for $\tilde{u}_{st}$ and $\tilde{v}_{st}$ analogous to those for u_{st} and v_{st}. Write (7) in the form

$$d_t \tilde{q}_{st} = \tilde{e}(d_t \tilde{x}_{st}). \tag{46}$$

By assumption, there exists a $K' < \infty$ which is both a uniform bound for a, b, c, e and is also a Lipschitz constant for b, c, c', c''. We can then find a uniform bound $\tilde{K} < \infty$ on $\tilde{a}, \tilde{b}, \tilde{e}$, which is also a Lipschitz constant for $\tilde{b}$, and which depends only on K'. The above argument shows that the system of equations (43–46) has a unique maximal regular solution $(\tilde{x}_{st}, \tilde{q}_{st}, \tilde{u}_{st}, \tilde{v}_{st} : (s,t) \in \tilde{\mathcal{D}})$, with the property that, as $(s,t) \uparrow \zeta(\tilde{\mathcal{D}})$, almost surely,

$$\tilde{m}_{st} := \sup_{r \leq s, u \leq t} |(\tilde{u}_{ru}, \tilde{u}_{ru}^{-1}, \tilde{v}_{ru}, \tilde{v}_{ru}^{-1})| \uparrow \infty.$$

Write

$$\tilde{x}_{st} = \begin{pmatrix} x_{st}^1 \\ x_{st}^2 \end{pmatrix}, \quad \tilde{u}_{st} = \begin{pmatrix} u_{st}^{11} & u_{st}^{12} \\ u_{st}^{21} & u_{st}^{22} \end{pmatrix}, \quad \tilde{v}_{st} = \begin{pmatrix} v_{st}^{11} & v_{st}^{12} \\ v_{st}^{21} & v_{st}^{22} \end{pmatrix},$$

and use analogous block notation for the tensors $\tilde{u}_{st}^*$ and $\tilde{v}_{st}^*$. Note that

$$\tilde{b}(d_s\tilde{x}_{st}, \cdot) = \begin{pmatrix} b(d_s x_{st}^1, \cdot) & 0 \\ f(d_s x_{st}^1) & c'(\cdot, d_s x_{st}^1) \end{pmatrix}, \quad \tilde{b}(\cdot, d_t\tilde{x}_{st}) = \begin{pmatrix} b(\cdot, d_t x_{st}^1) & 0 \\ g(d_t\tilde{x}_{st}) & 0 \end{pmatrix},$$

where

$$\begin{aligned} f(d_s x_{st}^1) &= c(b(d_s x_{st}^1, \cdot)) + 2c_2(d_s x_{st}^1, b(d_s x_{st}^1, \cdot)), \\ g(d_t\tilde{x}_{st}) &= c'(d_t x_{st}^2, \cdot) + \tfrac{1}{2}c''(d_t x_{st}^2, d_t x_{st}^2, \cdot) + c(b(\cdot, d_t x_{st}^1)) + c'(d_t x_{st}^2, b(\cdot, d_t x_{st}^1)). \end{aligned}$$

Here, we have written $b(d_s x_{st}, \cdot)$ as a short form of $b_{11}(d_s x_{st}, \cdot) + b_{12}(d_s x_{st}, d_s x_{st}, \cdot)$, and analogously for $b(\cdot, d_t x_{st})$ and $\tilde{b}(d_s\tilde{x}_{st}, \cdot)$. On multiplying out in blocks, we see that the process $(x_{st}^1, x_{st}^2, \tilde{q}_{st}, u_{st}^{11}, (u_{st}^*)^{111}, v_{st}^{11}, (v_{st}^*)^{111} : (s,t) \in \tilde{\mathcal{D}})$ satisfies equations (5–11). Hence, we must have $\tilde{\mathcal{D}} \subseteq \mathcal{D}$ and $(x_{st}^1, x_{st}^2, \tilde{q}_{st}, u_{st}^{11}, v_{st}^{11}) = (x_{st}, p_{st}, q_{st}, u_{st}, v_{st})$ for all $(s,t) \in \tilde{\mathcal{D}}$. In particular, $(p_{st} : (s,t) \in \tilde{\mathcal{D}})$ is a semimartingale.

It remains to show that $\tilde{\mathcal{D}} = \mathcal{D}$, which we can do by showing that, almost surely, $\tilde{m}_{st}$ remains bounded on $\tilde{\mathcal{D}}_{M,N} = \tilde{\mathcal{D}} \cap \mathcal{D}_{M,N}$, for all $M, N \geqslant 1$. We first obtain a Hölder estimate in t for p_{st}. We have

$$d_s(p_{st} - p_{st'}) = c(p_{st}, d_s x_{st}) - c(p_{st'}, d_s x_{st'}),$$

where we have now made the dependence of c on p explicit. Set

$$f(s) = \mathbb{E}\left(|p_{st} - p_{st'}|^\alpha 1_{\{(s,t),(s,t') \in \tilde{\mathcal{D}}_{M,N}\}}\right).$$

We use the estimates (34) and (35) to obtain a constant $C = C(\alpha, K', M, N, T) < \infty$ such that

$$f(s) \leqslant C\left(|t - t'|^{\alpha/2} + \int_0^s f(r)dr\right).$$

This implies that $f(s) \leqslant C|t - t'|^{\alpha/2}$ for all $s \geqslant 0$ for a constant $C < \infty$ of the same dependence. We now know that, for such a constant $C < \infty$, we have

$$\mathbb{E}\left(|p_{s't'} - p_{st}|^\alpha 1_{\{(s,t),(s',t') \in \tilde{\mathcal{D}}_{M,N}\}}\right) \leqslant C(|s - s'|^{\alpha/2} + |t - t'|^{\alpha/2}). \tag{47}$$

We turn to $\tilde{u}_{st}$ and $\tilde{v}_{st}$. The following equations hold

$$d_s u_{st}^{12} = b(d_s x_{st}, \cdot)u_{st}^{12}, \quad d_t v_{st}^{12} = b(\cdot, d_t x_{st})v_{st}^{12}, \quad d_t v_{st}^{22} = g(d_t\tilde{x}_{st})v_{st}^{12}.$$

By uniqueness of solutions, we obtain $u_{st}^{12} = u_{st}u_{0t}^{-1}u_{0t}^{12}$ so, in particular, $u_{s0}^{12} = 0$. Similarly, $v_{st}^{12} = v_{st}v_{s0}^{-1}v_{s0}^{12}$, so $v_{0t}^{12} = 0$. Since $\tilde{u}_{0t} = \tilde{v}_{0t}$ and $\tilde{u}_{s0} = \tilde{v}_{s0}$, we deduce that $u_{st}^{12} = v_{st}^{12} = 0$. Then $d_t v_{st}^{22} = 0$, so $v_{st}^{22} = v_{s0}^{22} = u_{s0}^{22}$. We also have the equations

$$d_s u_{st}^{21} = f(d_s x_{st})u_{st} + c'(., d_s x_{st})u_{st}^{21}, \quad d_s u_{st}^{22} = c'(., d_s x_{st})u_{st}^{22}, \quad d_t v_{st}^{21} = g(d_t\tilde{x}_{st})v_{st}$$

and we note that

$$\tilde{u}_{st}^{-1} = \begin{pmatrix} u_{st}^{-1} & 0 \\ -(u_{st}^{22})^{-1}u_{st}^{21}u_{st}^{-1} & (u_{st}^{22})^{-1} \end{pmatrix}, \quad \tilde{v}_{st}^{-1} = \begin{pmatrix} v_{st}^{-1} & 0 \\ -(v_{st}^{22})^{-1}v_{st}^{21}v_{st}^{-1} & (v_{st}^{22})^{-1} \end{pmatrix},$$

and

$$d_s(u_{st}^{22})^{-1} = -(u_{st}^{22})^{-1}c'(.,d_s x_{st}) + (u_{st}^{22})^{-1}c'(.,d_s x_{st})c'(.,d_s x_{st}).$$

We use the inequalities (34), (35) and (47), and an easy variation of the argument leading to (36) and (40) to obtain a constant $C = C(\alpha, K', M, N, T) < \infty$ such that

$$\mathbb{E}\left(\left|(\tilde{u}_{s't'}, \tilde{u}_{s't'}^{-1}) - (\tilde{u}_{st}, \tilde{u}_{st}^{-1})\right|^{\alpha} 1_{\{(s,t),(s',t')\in\tilde{\mathcal{D}}_{M,N}\}}\right) \leqslant C(|s-s'|^{\alpha/2} + |t-t'|^{\alpha/2}). \tag{48}$$

Then, using [**9**, Theorem 3.2.1] as above, we can conclude that, almost surely, $(\tilde{u}_{st}, \tilde{u}_{st}^{-1})$ remains bounded on $\tilde{\mathcal{D}}_{M,N}$. It remains to show that the same is true for $(\tilde{v}_{st}, \tilde{v}_{st}^{-1})$ and, given the relations already noted, it will suffice to show this for v_{st}^{21}. We have

$$\begin{aligned} d_s\tilde{u}_{st}^* &= \tilde{u}_{st}^{-1}\{\tilde{b}_{12}(d_s\tilde{x}_{st}, \tilde{u}_{st}\cdot, \tilde{u}_{st}\cdot) \\ &\qquad + \tilde{b}_{22}(d_s\tilde{x}_{st}, d_s\tilde{x}_{st}, \tilde{u}_{st}\cdot, \tilde{u}_{st}\cdot) - \tilde{b}_{11}(d_s\tilde{x}_{st}, \tilde{b}_{12}(d_s\tilde{x}_{st}, \tilde{u}_{st}\cdot, \tilde{u}_{st}\cdot))\} \\ &= h(x_{st}, p_{st}, \tilde{u}_{st}, \tilde{u}_{st}^{-1}, d_s x_{st}), \end{aligned}$$

where h is defined by the final equality and where we have used (6) to write $d_s\tilde{x}_{st}$ in terms of $d_s x_{st}$. A variation of the argument used for $\tilde{u}_{st}$ shows that, almost surely, $\tilde{u}_{st}^*$ remains bounded on $\tilde{\mathcal{D}}_{M,N}$. Then, we can use the $\sim$ and t-analogue of equations (31) and (32) to express v_{st}^{21} as a sum of integrals with respect to $(x_{0t}, p_{0t} : t \geqslant 0)$ and $(w_{st} : s, t \geqslant 0)$. This leads, as above, to L^α-Hölder estimates which allow us to conclude that, almost surely, v_{st}^{21} remains bounded on $\tilde{\mathcal{D}}_{M,N}$, as required. □

5. Derivation of the formula

Let $(w_{st} : s, t \geqslant 0)$ be an $\mathbb{R}^m$-valued Brownian sheet and let $(z_{s0} : s \geqslant 0)$ be an independent $\mathbb{R}^m$-valued Brownian motion. Thus $w_{st} = (w_{st}^1, \dots, w_{st}^m)$ and $z_{s0} = (z_{s0}^1, \dots, z_{s0}^m)$, and each component process is an independent scalar Brownian sheet, or Brownian motion, respectively. The two-parameter hyperbolic stochastic differential equation

$$d_s d_t z_{st} = d_s d_t w_{st} - \tfrac{1}{2} d_s z_{st} dt, \quad s, t \geqslant 0, \tag{49}$$

with given boundary values $(z_{s0} : s \geqslant 0)$ and $z_{0t} = 0$, for $t \geqslant 0$, has a unique solution $(z_{st} : s, t \geqslant 0)$. Set $z_t = (z_{st} : s \geqslant 0)$, then $(z_t)_{t\geqslant 0}$ is a realization of the Ornstein-Uhlenbeck process on the m-dimensional Wiener space. See [**8**] or [**9**]. The Stratonovich form of (49) is given by

$$\partial_s\partial_t z_{st} = \partial_s\partial_t w_{st} - \tfrac{1}{2}\partial_s z_{st}\partial t, \quad s, t \geqslant 0.$$

Fix $x \in \mathbb{R}^d$ and consider for each $t \geqslant 0$ the Stratonovich stochastic differential equation

$$\partial_s x_{st} = X_i(x_{st})\partial_s z_{st}^i + X_0(x_{st})\partial s, \quad s \geqslant 0,$$

with initial value $x_{0t} = x$. This can be written in Itô form as

$$(50) \qquad d_s x_{st} = X_i(x_{st}) d_s z^i_{st} + \tilde{X}_0(x_{st}) ds, \quad s \geqslant 0,$$

where $\tilde{X}_0 = X_0 + \frac{1}{2}\sum_{i=1}^d \nabla X_i . X_i$. Consider also, for each $t \geqslant 0$, the stochastic differential equation

$$\partial_s U_{st} = \nabla X_i(x_{st}) U_{st} \partial_s z^i_{st} + \nabla X_0(x_{st}) U_{st} \partial s, \quad s \geqslant 0,$$

with initial value $U_{0t} = I$, and its Itô form

$$(51) \qquad d_s U_{st} = \nabla X_i(x_{st}) U_{st} d_s z^i_{st} + \nabla \tilde{X}_0(x_{st}) U_{st} ds, \quad s \geqslant 0.$$

Proposition 5.1. — *There exist (two-parameter) semimartingales $(z_{st} : s, t \geqslant 0)$, $(x_{st} : s, t \geqslant 0)$ and $(U_{st} : s, t \geqslant 0)$ such that $(z_{st} : s, t \geqslant 0)$ satisfies (49) and, for all $t \geqslant 0$, $(x_{st} : s \geqslant 0)$ and $(U_{st} : s \geqslant 0)$ satisfy (50) and (51), with the boundary conditions given above. Moreover, almost surely, U_{st} is invertible for all $s, t \geqslant 0$.*

Proof. — We seek to apply Theorem 4.2. There are three minor obstacles: firstly to deal with the ds and dt differentials appearing in the equations, secondly, to show that the domain of the solutions is the whole of $(\mathbb{R}^+)^2$ and, thirdly, to deal with the fact that the coefficients in (51) do not have the required boundedness of derivatives.

Let us introduce a further equation

$$d_s d_t z^0_{st} = 0,$$

with boundary conditions $z^0_{s0} = s$ and $z_{0t} = t$ for all $s, t \geqslant 0$. We then replace dt and ds in (49) and (50), respectively, by $d_t z^0_{st}$ and $d_s z^0_{st}$. When we obtain a solution, it will follow that $z^0_{st} = s + t$, so $d_t z^0_{st} = dt$ and $d_s z^0_{st} = ds$, as required.

In order to show that $\mathcal{D} = (\mathbb{R}^+)^2$, it will suffice to show that the companion processes u_{st} and v_{st} associated with the equations

$$d_s d_t z^0_{st} = 0, \quad d_s d_t z_{st} = d_s d_t w_{st} - \frac{1}{2} d_s z_{st} d_t z^0_{st},$$

according to equations (8) and (10), along with their inverses, remain bounded on compacts in s and t. We leave this to the reader.

Finally, choose for each $M \in \mathbb{N}$ a smooth and compactly supported function ψ_M on $\mathbb{R}^d \otimes (\mathbb{R}^d)^*$, such that $\psi_M(U) = U$ whenever $|U| \leqslant M$. We can apply Theorem 4.2 to the system (49), (50), together with the modified equation

$$d_s U^M_{st} = \nabla X_i(x_{st}) \psi_M(U^M_{st}) d_s z^i_{st} + \nabla \tilde{X}_0(x_{st}) \psi_M(U^M_{st}) ds.$$

Define

$$\mathcal{D}_M = \{(s,t) : |U^M_{s't'}| \leqslant M \text{ for all } s' \leqslant s, t' \leqslant t\}.$$

By local uniqueness, we can define consistently U on $\mathcal{D} = \cup_M \mathcal{D}_M$ by $U_{st} = U^M_{st}$ for $(s,t) \in \mathcal{D}_M$. By some straightforward estimation using the one-parameter equations (51), we obtain, for all $T < \infty$ and all $p \in [1, \infty)$, a constant $C < \infty$ such that

$$\sup_{s,s',t,t' \leqslant T} \mathbb{E}(|U_{st} - U_{s't'}|^p 1_{\{(s,t),(s't') \in \mathcal{D}\}}) \leqslant C(|s - s'|^{p/2} + |t - t'|^{p/2}).$$

Then, by [**9**, Theorem 3.2.1], almost surely, U is bounded uniformly on $\mathscr{D} \cap [0,T]^2$. Hence $\mathscr{D} = (\mathbb{R}^+)^2$, and we have obtained the desired semimartingale U. The invertibility of U can be proved by applying the same argument to the usual equation for the inverse. □

By the Stratonovich chain rule,

$$\partial_s\partial_t x_{st} = \nabla X_i(x_{st})\partial_s z^i_{st}\partial_t x_{st} + \nabla X_0(x_{st})\partial s\partial_t x_{st} + X_i(x_{st})\partial_s\partial_t z^i_{st}.$$

Now

$$\begin{aligned}\partial_s\partial_t U_{st} &= \nabla X_i(x_{st})\partial_s z^i_{st}\partial_t U_{st} + \nabla X_0(x_{st})\partial s\partial_t U_{st}\\ &\quad+(\nabla^2 X_i(x_{st})\partial_t x_{st})U_{st}\partial_s z^i_{st} + (\nabla^2 X_0(x_{st})\partial_t x_{st})U_{st}\partial s + \nabla X_i(x_{st})U_{st}\partial_s\partial_t z^i_{st},\end{aligned}$$

so

$$\partial_t U_{st}\partial_s\partial_t z^i_{st} = \tfrac{1}{2}\partial_s\partial_t U_{st}\partial_s\partial_t w^i_{st} = \tfrac{1}{2}\nabla X_i(x_{st})U_{st}\partial s\partial t$$

and

$$\partial_s(U_{st}^{-1}\partial_t U_{st}) = U_{st}^{-1}\left\{\nabla^2 X_i(x_{st})\partial_s z^i_{st}\partial_t x_{st} + \nabla^2 X_0(x_{st})\partial s\partial_t x_{st} + \nabla X_i(x_{st})\partial_s\partial_t z^i_{st}\right\}U_{st}.$$

Define also a two-parameter, $\mathbb{R}^d$-valued, semimartingale $(y_{st} : s,t \geqslant 0)$ by

$$\partial_t y_{st} = U_{st}^{-1}\partial_t x_{st}, \quad y_{s0} = 0.$$

Then

$$\partial_s\partial_t y_{st} = U_{st}^{-1}X_i(x_{st})\partial_s\partial_t z^i_{st}.$$

Note that

$$\partial_t y_{st}\partial_s\partial_t z^i_{st} = \partial_t y_{st}\partial_s\partial_t w^i_{st} = \tfrac{1}{2}\partial_s\partial_t y_{st}\partial_s\partial_t w^i_{st} = \tfrac{1}{2}U_{st}^{-1}X_i(x_{st})\partial s\partial t.$$

So

$$\partial_s(\partial_t y_{st}\otimes\partial_t y_{st}) = \partial_s\partial_t y_{st}\otimes\partial_t y_{st} + \partial_t y_{st}\otimes\partial_s\partial_t y_{st} = U_{st}^{-1}X_i(x_{st})\otimes U_{st}^{-1}X_i(x_{st})\partial s\partial t.$$

Note also that

$$\partial_s(U_{st}^{-1}X_i(x_{st})) = U_{st}^{-1}[X_i,X_j](x_{st})\partial_s z^j_{st} + U_{st}^{-1}[X_i,X_0](x_{st})\partial s.$$

So

$$\partial_s(U_{st}^{-1}X_i(x_{st}))\partial_s\partial_t z^i_{st} = U_{st}^{-1}[X_i,X_j](x_{st})\partial_s z^j_{st}(\partial_s\partial_t w^i_{st} - \tfrac{1}{2}\partial_s z^i_{st}\partial t) = 0.$$

Moreover

$$\partial_t(U_{st}^{-1}X_i(x_{st}))d_s\partial_t z^i_{st} = \partial_t(U_{st}^{-1}X_i(x_{st}))d_s\partial_t w^i_{st} = 0.$$

Hence, we have

$$d_s d_t y_{st} = U_{st}^{-1}X_i(x_{st})d_s d_t z^i_{st} = U_{st}^{-1}X_i(x_{st})(\partial_s\partial_t w^i_{st} - \tfrac{1}{2}\partial_s z^i_{st}\partial t).$$

We compute

$$\begin{aligned}&\partial_s(U_{st}^{-1}\partial_t U_{st}\partial_t y_{st})\\ &\quad= U_{st}^{-1}\left\{\nabla^2 X_i(x_{st})\partial_s z^i_{st} + \nabla^2 X_0(x_{st})\partial s\right\}\partial_t x_{st}\otimes\partial_t x_{st} + U_{st}^{-1}\nabla X_i(x_{st})X_i(x_{st})\partial s\partial t.\end{aligned}$$

Define

$$R_{st} = -\int_0^s U_{rt}^{-1} X_i(x_{rt}) d_r z_{rt}^i, \quad C_{st} = \int_0^s U_{rt}^{-1} X_i(x_{rt}) \otimes U_{rt}^{-1} X_i(x_{rt}) dr.$$

Our calculations show that the $(\mathcal{F}_{st} : t \geqslant 0)$-semimartingale $(y_{st} : t \geqslant 0)$ has finite-variation part $(\bar{y}_{st} : t \geqslant 0)$ and quadratic variation given by

$$d_t \bar{y}_{st} = \tfrac{1}{2} R_{st} dt, \quad \partial_t y_{st} \otimes \partial_t y_{st} = C_{st} dt.$$

Moreover

$$d_t x_{st} = U_{st} d_t y_{st} + \tfrac{1}{2} \partial_t U_{st} \partial_t y_{st},$$

so $(x_{st} : t \geqslant 0)$ has finite-variation part $(\bar{x}_{st} : t \geqslant 0)$ and quadratic variation given by

$$d_t \bar{x}_{st} = \tfrac{1}{2} L_{st} dt, \quad \partial_t x_{st} \otimes \partial_t x_{st} = \Gamma_{st} dt,$$

where

$$\begin{aligned} L_{st} = U_{st} R_{st} + U_{st} &\int_0^s U_{rt}^{-1} \{\nabla^2 X_i(x_{rt}) \partial_r z_{rt}^i + \nabla^2 X_0(x_{rt}) \partial r\} \Gamma_{rt} \\ &+ U_{st} \int_0^s U_{rt}^{-1} \nabla X_i(x_{rt}) X_i(x_{rt}) \partial r \end{aligned}$$

and where $\Gamma_{st} = U_{st} C_{st} U_{st}^*$.

Note that both $(\Gamma_{st} : t \geqslant 0)$ and $(L_{st} : t \geqslant 0)$ are stationary processes and that, by standard one-parameter estimates, Γ_{s0} and L_{s0} have finite moments of all orders. By Itô's formula, for any C^2 function f, setting $f_{st} = f(x_{st})$, the process $(f_{st} : t \geqslant 0)$ is an $(\mathcal{F}_{st} : t \geqslant 0)$-semimartingale with finite-variation part $(\bar{f}_{st} : t \geqslant 0)$ and quadratic variation given by

$$d_t \bar{f}_{st} = \tfrac{1}{2} \left(L_{st}^i \nabla_i f(x_{st}) + \Gamma_{st}^{ij} \nabla_i \nabla_j f(x_{st}) \right) dt, \quad \partial_t f_{st} \partial_t f_{st} = \nabla_i f(x_{st}) \Gamma_{st}^{ij} \nabla_j f(x_{st}) dt.$$

In particular, if $m_{st} = f_{st} - f_{s0} - \bar{f}_{st}$, then $(m_{st} : t \geqslant 0)$ is a (true) martingale. Hence, for $f, g \in C_b^2(\mathbb{R}^d)$, we obtain the integration-by-parts formula

$$\begin{aligned} \mathbb{E}[\nabla_i f(x_{s0}) \Gamma_{s0}^{ij} \nabla_j g(x_{s0})] &= \lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}\left[\{f(x_{st}) - f(x_{s0})\}\{g(x_{st}) - g(x_{s0})\} \right] \\ = -2 \lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}[f(x_{s0})\{g(x_{st}) - g(x_{s0})\}] &= -\mathbb{E}[f(x_{s0})\{L_{s0}^i \nabla_i g(x_{s0}) + \Gamma_{s0}^{ij} \nabla_i \nabla_j g(x_{s0})\}]. \end{aligned}$$

An obvious limit argument allows us to deduce the following simple formula, corresponding to the case $g(x) = x^j$. For all $f \in C_b^2(\mathbb{R}^d)$ and for $j = 1, \dots, d$, we have

$$\mathbb{E}[\nabla_i f(x_{s0}) \Gamma_{s0}^{ij}] = -\mathbb{E}[f(x_{s0}) L_{s0}^j].$$

The general formula can then be recovered by replacing f by $f \nabla_j g$ and summing over j.

The basic observation underlying this formula is that the distributions of (z_0, z_t) and (z_t, z_0) are identical, and hence that the same is true for (x_{s0}, x_{st}) and (x_{st}, x_{s0}), when $(x_{st} : s \geqslant 0)$ is obtained by solving a stochastic differential equation driven by $(z_{st} : s \geqslant 0)$, with initial condition independent of t. In fact a stronger notion of reversibility is true. The distributions of $(z_{su} : s \geqslant 0, u \in [0, t])$ and $(z_{s,t-u} : s \geqslant$

$0, u \in [0,t])$ are identical, and hence the same is true for $(x_{su} : s \geqslant 0, u \in [0,t])$ and $(x_{s,t-u} : s \geqslant 0, u \in [0,t])$. This may be combined with the fact that the Stratonovich integral is invariant under time-reversal to see that

$$\mathbb{E}\left[\{f(x_{st}) - f(x_{s0})\} \int_0^t U_{su}^{-1} \partial_u x_{su}\right] = -2\mathbb{E}\left[f(x_{s0}) \int_0^t U_{su}^{-1} \partial_u x_{su}\right].$$

From this identity, by a similar argument, we obtain the following alternative integration-by-parts formula. For all $f \in C_b^2(\mathbb{R}^d)$, we have

$$\mathbb{E}[\nabla f(x_{s0}) U_{s0} C_{s0}] = -\mathbb{E}[f(x_{s0}) R_{s0}].$$

This formula is the variant discovered by Bismut, which is closely related to the Clark–Haussmann formula.

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Astérisque
327, 2009, p. 115–129

WITTEN LAPLACIAN ON A LATTICE SPIN SYSTEM

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — We consider an unbounded lattice spin system with a Gibbs measure. We introduce the Hodge-Kodaira operator acting on differential forms and give a sufficient condition for the positivity of the lowest eigenvalue.

***Résumé* (Laplacien de Witten sur un système de spin sur réseau).** — Nous considérons un réseau de spin muni d'une mesure de Gibbs. Nous introduisons l'opérateur de Hodge-Kodaira agissant sur les formes différentielles, et nous donnons une condition suffisante pour la positivité de la plus petite valeur propre.

1. Introduction

In this paper, we consider the spectral gap problem for a lattice spin system. Here, in our case, the single spin space is $\mathbb{R}$ and so it is non-compact. This is sometimes called an unbounded spin system.

We consider a model that each spin sits on the lattice $\mathbb{Z}^d$, and so the configuration space is $\mathbb{R}^{\mathbb{Z}^d}$. We suppose that a Gibbs measure is given in $\mathbb{R}^{\mathbb{Z}^d}$, which has the following formal expression:

$$\nu = Z^{-1} \exp\Big\{-2\mathscr{J} \sum_{\substack{i,j\in\mathbb{Z}^d \\ i\sim j}} (x^i - x^j)^2 - 2\sum_{i\in\mathbb{Z}^d} U(x^i)\Big\} \prod_{i\in\mathbb{Z}^d} dx^i. \tag{1.1}$$

Here U is a function of $\mathbb{R}$, called a self potential and $i \sim j$ means that $\|i-j\|_1 = \sum_k |i_k - j_k| = 1$. Under this measure we define the Hodge-Kodaira operator and discuss the positivity of the lowest eigenvalue of the operator. For unbounded spin

2010 Mathematics Subject Classification. — 60H07, 47D08, 47A10.
Key words and phrases. — Lattice spin system, Witten Laplacian, spectral gap.

systems, the Poincaré inequality, the logarithmic Sobolev inequality and other properties are well discussed, e.g., Zegarlinski [**11**], Yoshida [**10**], etc. In particular, Helffer [**5, 6, 7, 8**] dealt with this problem in connection to the Witten Laplacian. In fact, he proved the positivity of the lowest eigenvalue of the Hodge-Kodaira operator acting on 1-forms. From this point of view, we generalize his result to any p-forms ($p \geq 1$), i.e., we will prove that the lowest eigenvalue of the Hodge-Kodaira operator acting on p-forms is positive.

The organization of the paper is as follows. In Section 2, we discuss the Witten Laplacian on a finite dimensional space and in Section 3, we summarize differential forms, the Hodge-Kodaira operator and the Weitzenböck formula, which is crucial in the later argument. In Section 4, we give an estimate of spectral gap for 1-dimensional case. Last in Section 5, we prove the positivity of the lowest eigenvalue of the Hodge-Kodaira operator. We only consider the finite region case but we give a uniform estimate. In fact, it is independent of the choice of region and the boundary condition. So the result is valid for the infinite volume case as well.

2. Witten Laplacian in finite dimension

We give a quick review of the Witten Laplacian, which we need later. Details and related topics can be found in Hellfer [**8**], Albeverio-Daletskii-Kondratiev [**1**], Elworthy-Rosenberg [**4**], etc. Simon et al [**3**] is also a good reference for the supersymmetry.

Our interest is in the infinite dimensional case, but we start with the finite dimensional case. Suppose we are given a C^2 function Φ on $\mathbb{R}^N$ and define a measure ν by

$$\nu(dx) = Z^{-1} e^{-2\Phi} dx. \tag{2.1}$$

Here $Z = \int_{\mathbb{R}^N} e^{-2\Phi} dx$ so that ν is a probability measure. Define a Dirichlet form $\mathscr{E}$ by

$$\mathscr{E}(f,g) = \int_{\mathbb{R}^N} (\nabla f, \nabla g) e^{-2\Phi} dx, \tag{2.2}$$

where $\nabla = (\partial_1, \ldots, \partial_N)$, $\partial_k = \frac{d}{dx_k}$. $(\nabla f, \nabla g)$ stands for the Euclidean inner product. We must specify the domain of $\mathscr{E}$. (2.2) is well-defined for $f, g \in C_0^\infty(\mathbb{R}^N)$. So at first, $\mathscr{E}$ is defined on $C_0^\infty(\mathbb{R}^N)$. Let us give an explicit form of the dual operator ∂_j^* of ∂_j in $L^2(\nu)$. To do this, note that

$$\int_{\mathbb{R}^N} \partial_j f g e^{-2\Phi} dx = -\int_{\mathbb{R}^N} f \partial_j (g e^{-2\Phi}) dx = -\int_{\mathbb{R}^N} f(\partial_j g - 2\partial_j \Phi g) e^{-2\Phi} dx,$$

which means

$$\partial_j^* = -\partial_j + 2\partial_j \Phi. \tag{2.3}$$

Here ∂_j^* is the dual operator of ∂_j in $L^2(\nu)$.

From this, we can see that the dual operator of ∇ has dense domain and so ∇ is closable. Moreover the generator $\mathfrak{A}$ is given by

$$\mathfrak{A}f = -\sum_j \partial_j^*\partial_j = \sum_j (\partial_j^2 f - 2\partial_j\Phi\partial_j f) = \triangle f - 2(\nabla\Phi, \nabla f). \tag{2.4}$$

This is valid for $f \in C_0^\infty(\mathbb{R}^N)$. We can show that $\mathfrak{A}$ is essentially self-adjoint and so, by taking closure, we may regard $\mathfrak{A}$ as self-adjoint operator. The domain of $\mathscr{E}$ is a set of all functions $f \in L^2(\nu)$ with $\nabla f \in L^2(\nu; \mathbb{R}^N)$.

We now define a Witten Laplacian. Let $I\colon L^2(dx) \longrightarrow L^2(\nu)$ be a unitary operator defined by

$$If(x) = e^\Phi f. \tag{2.5}$$

Let us obtain a operator X_j which satisfies the following commutative diagram:

$$\begin{tikzcd} L^2(dx) \arrow[r, "I"] \arrow[d, "X_j"'] & L^2(\nu) \arrow[d, "\partial_j"] \\ L^2(dx) \arrow[r, "I"] & L^2(\nu) \end{tikzcd} \tag{2.6}$$

It is not hard to see that

$$X_j = e^{-\Phi}\partial_j e^\Phi = \partial_j + \partial_j\Phi.$$

We denote the dual operator of X_j in $L^2(dx)$ by $\tilde{X}_j$. Here we use the following convention. $*$ stands for the dual operator in $L^2(\nu)$ and $\tilde{}$ stands for the dual operator in $L^2(dx)$, dx being the Lebesgue measure in $\mathbb{R}^N$. $\tilde{X}_j$ has the following form:

$$\tilde{X}_j = -\partial_j + \partial_j\Phi.$$

This is also equal to $e^{-\Phi}\partial_j^* e^\Phi$. The operator A associated with the generator $\mathfrak{A} = -\sum_j \partial_j^*\partial_j$ is computed by

$$\begin{aligned} A = e^{-\Phi}\mathfrak{A}e^\Phi &= -e^{-\Phi}(\sum_j \partial_j^*\partial_j)e^\Phi = -\sum_j \tilde{X}_j X_j \\ &= -\sum_j(-\partial_j + \partial_j\Phi)(\partial_j + \partial_j\Phi) = \sum_j(\partial_j^2 + \partial_j^2\Phi - (\partial_j\Phi)^2) \\ &= \triangle + \triangle\Phi - |\nabla\Phi|^2. \end{aligned}$$

Definition 2.1. — $A = \triangle + \triangle\Phi - |\nabla\Phi|^2$ in $L^2(dx)$ is called a Witten Laplacian.

$\mathfrak{A}$ and A are unitarily equivalent to each other but we distinguish them and call A as the Witten Laplacian, which is an operator in $L^2(dx)$.

The following commutation relation is easily checked.

Proposition 2.1. — In $L^2(\nu)$, we have

$$[\partial_i, \partial_j] = 0, \tag{2.7}$$

$$[\partial_i, \partial_j^*] = 2\partial_i\partial_j\Phi, \tag{2.8}$$

$$[\partial_j^*, \partial_k^*] = 0. \tag{2.9}$$

Further, in $L^2(dx)$, we have

$$[X_i, X_j] = 0, \tag{2.10}$$

$$[X_i, \tilde{X}_j] = 2\partial_i\partial_j\Phi, \tag{2.11}$$

$$[\tilde{X}_j, \tilde{X}_j] = 0. \tag{2.12}$$

3. Witten Laplacian acting on differential forms

In Section 2, we have introduced the Witten Laplacian. We now proceed to the Witten Laplacian acting on differential forms.

Let us quickly review the exterior algebra. In the sequel, we will deal with multilinear functionals on $\mathbb{R}^N$. Let t be a p-linear functional and s be a q-linear functional, e.g., t is a functional from $\underbrace{\mathbb{R}^N \times \cdots \times \mathbb{R}^N}_{p}$ into $\mathbb{R}$ which is linear in each coordinate. We define $p+q$-linear functional $t \otimes s$ by

$$t \otimes s(v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q}) = t(v_1, \ldots, v_p)s(v_{p+1}, \ldots, v_{p+q}). \tag{3.1}$$

$t \otimes s$ is called a tensor product. We also define the alternation mapping A_p by

$$A_p t(v_1, \ldots, v_p) = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} (\operatorname{sgn} \sigma)\, t(v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \tag{3.2}$$

for p-linear functional t. Here $\mathfrak{S}_p$ is the symmetric group of degree p and $\operatorname{sgn}\sigma$ stands for the signature. If p-linear functional θ satisfies $A_p\theta = \theta$, θ is called alternating. We denote the set of all alternating functionals of degree p by $\bigwedge^p(\mathbb{R}^N)^*$. For $\theta \in \bigwedge^p(\mathbb{R}^N)^*$ and $\eta \in \bigwedge^q(\mathbb{R}^N)^*$, we define their exterior product $\theta \wedge \eta$ by

$$\theta \wedge \eta = \frac{(p+q)!}{p!q!} A_{p+q}(\theta \otimes \eta). \tag{3.3}$$

Taking an orthonormal basis $\theta_1, \ldots, \theta_N$ in $(\mathbb{R}^N)^*$, any element of $\bigwedge^p(\mathbb{R}^N)^*$ is represented as a unique linear combination of the following elements

$$\theta_{i_1} \wedge \cdots \wedge \theta_{i_p}. \tag{3.4}$$

We define an inner product in $\bigwedge^p(\mathbb{R}^N)^*$ so that all elements of the form (3.4) become an orthonormal basis in $\bigwedge^p(\mathbb{R}^N)^*$.

$A^p(\mathbb{R}^N) = \mathbb{R}^N \times \bigwedge^p(\mathbb{R}^N)^*$ has a structure of vector bundle and a section of $A^p(\mathbb{R}^N)$ is called a differential form of degree p. The set of all sections is denoted by $\Gamma(A^p(\mathbb{R}^N))$.

Since the vector bundle $A^p(\mathbb{R}^N)$ is trivial, any section can be identified with a mapping from $\mathbb{R}^N$ into $\bigwedge^p(\mathbb{R}^N)^*$. In the sequel, we use this convention. $\Gamma^\infty(A^p(\mathbb{R}^N))$ denotes the set of all smooth differential forms and $\Gamma_0^\infty(A^p(\mathbb{R}^N))$ denotes the set of all smooth differential forms with compact support.

We introduce some operators in $\bigwedge^p(\mathbb{R}^N)^*$ as follows. For $\theta \in (\mathbb{R}^N)^*$, we define $\mathrm{ext}(\theta)\colon \bigwedge^p(\mathbb{R}^N)^* \longrightarrow \bigwedge^{p+1}(\mathbb{R}^N)^*$ by

$$\mathrm{ext}(\theta)\omega = \theta \wedge \omega \tag{3.5}$$

and for $v \in \mathbb{R}^N$, we define $\mathrm{int}(\theta)\colon \bigwedge^p(\mathbb{R}^N)^* \longrightarrow \bigwedge^{p-1}(\mathbb{R}^N)^*$ by

$$(\mathrm{int}(v)\omega)(v_1, \dots, v_{p-1}) = \omega(v, v_1, \dots, v_{p-1}). \tag{3.6}$$

Taking a standard basis $\{e_1, \dots, e_N\}$ of $\mathbb{R}^N$ and its dual basis $\{\theta^1, \dots, \theta^N\}$, we define operators a^i, $(a^i)^*$ by

$$a^i = \mathrm{int}(e_i), \tag{3.7}$$

$$(a^i)^* = \mathrm{ext}(\theta^i). \tag{3.8}$$

Here we regard a^i, $(a^i)^*$ as operators on an exterior algebra $\mathbb{R} \oplus (\mathbb{R}^N)^* \oplus \bigwedge^2(\mathbb{R}^N)^* \oplus \cdots \oplus \bigwedge^N(\mathbb{R}^N)^*$. They satisfy the following commutation relation:

$$[a^i, a^j]_+ = 0, \tag{3.9}$$

$$[a^i, (a^j)^*]_+ = \delta_{ij}, \tag{3.10}$$

$$[(a^i)^*, (a^j)^*]_+ = 0. \tag{3.11}$$

Here $[\ ,\ ]_+$ stands for an anti-commutator, i.e., $[a^i, a^j]_+ = a^i a^j + a^j a^i$.

For differential forms, the covariant differentiation ∇ can be defined. More generally, the covariant differentiation ∇ is defined for tensor fields as follows:

$$\nabla t = \sum_i \theta^i \otimes \partial_i t.$$

Here we remark that the operator is considered in $L^2(\nu)$, i.e., the reference measure is ν. The dual operator of ∇ in $L^2(\nu)$ is given by

$$\nabla^*(\sum_i \theta^i \otimes t_i) = \sum_i \partial_i^* t_i$$

and so we have

$$\nabla^*\nabla t = \sum_i \partial_i^* \partial_i t = -\sum_i (\partial_i^2 - 2\partial_i \Phi \partial_i) t.$$

For differential forms, we can define the exterior differentiation as follows. Let ω be a differential form of degree p. Then its exterior derivative is defined by $d\omega = (p+1)A_{p+1}\nabla\omega$ and it is written as

$$d = \sum_i \mathrm{ext}(\theta^i)\partial_i = \sum_i (a^i)^* \partial_i. \tag{3.12}$$

Hence, its dual operator is expressed as

$$d^* = \sum_i a^i \partial_i^*. \tag{3.13}$$

Using these operators, the Hodge-Kodaira Laplacian is defined as $-(dd^* + d^*d)$. The following formula is called the Weitzenböck formula:

Theorem 3.1. — We have the following identity.

$$dd^* + d^*d = \nabla^*\nabla + 2\sum_{i,j} \partial_i\partial_j\Phi\,(a^i)^* a^j. \tag{3.14}$$

Proof. — By (3.12) and (3.13), we have

$$\begin{aligned}
dd^* + d^*d &= \sum_{i,j}\{(a^i)^*\partial_i\, a^j\partial_j^* + a^j\partial_j^*\,(a^i)^*\partial_i\} \\
&= \sum_{i,j}\{(a^i)^* a^j\,\partial_i\partial_j^* - (a^i)^* a^j\,\partial_j^*\partial_i + (a^i)^* a^j\,\partial_j^*\partial_i + a^j (a^i)^*\,\partial_j^*\partial_i\} \\
&= \sum_{i,j}\{(a^i)^* a^j[\partial_i, \partial_j^*] + [(a^i)^*, a^j]_+ \partial_j^*\partial_j\} \\
&= \sum_{i,j}\{2(a^i)^* a^j \partial_i\partial_j\Phi + \delta_{ij}\partial_j^*\partial_i\} \\
&= \sum_{i,j} 2\partial_i\partial_j\Phi\,(a^i)^* a^j + \sum_i \partial_i^*\partial_i \\
&= \nabla^*\nabla + \sum_{i,j} 2\partial_i\partial_j\Phi\,(a^i)^* a^j.
\end{aligned}$$

This is what we wanted. □

So far, the reference measure has been ν. The isomorphism $I\colon L^2(dx) \longrightarrow L^2(\nu)$ can be extended to differential forms. Under the Lebesgue measure, the corresponding exterior differentiation and its dual operator are given by

$$D = e^{-\Phi} d e^{\Phi}, \tag{3.15}$$

$$\tilde{D} = e^{-\Phi} d^* e^{\Phi}. \tag{3.16}$$

So the operator $\tilde{D}D + D\tilde{D}$ can be defined similarly and it has the following expression:

Theorem 3.2. — We have the following identities:

$$\tilde{D}D + D\tilde{D} = \sum_i \tilde{X}_i X_i + 2\sum_{i,j} \partial_i\partial_j\Phi\,(a^i)^* a^j. \tag{3.17}$$

We call the operator $\tilde{D}D + D\tilde{D}$ in $L^2(dx)$ as the Witten Laplacian and distinguish from $dd^* + d^*d$, which is defined in $L^2(d\nu)$ and is called the Hodge-Kodaira Laplacian.

We remark that the operators a^j, $(a^i)^*$ are independent of the underlying measure and so we used the same notation.

Proof. — We easily have

$$\begin{aligned}\tilde{D}D + D\tilde{D} &= e^{-\Phi}(dd^* + d^*d)e^{\Phi} \\ &= e^{-\Phi}\{\nabla^*\nabla + 2\sum_{i,j}\partial_i\partial_j\Phi\,(a^i)^*a^j\}e^{\Phi} \\ &= \sum_i \tilde{X}_i X_i + 2\sum_{i,j}\partial_i\partial_j\Phi\,(a^i)^*a^j.\end{aligned}$$

This is the desired result. □

Due to this unitary equivalence, the following theorem is well-known (see, e.g., [**2**]).

Theorem 3.3. — The Hodge-Kodaira operator $\tilde{D}D+D\tilde{D}$ with a domain $\Gamma_0^\infty(A^p(\mathbb{R}^N))$ is essentially self-adjoint in $L^2(dx;\bigwedge^p(\mathbb{R}^N)^*)$. Furthermore, $d^*d + d^*d$ with a domain $\Gamma_0^\infty(A^p(\mathbb{R}^N))$ is essentially self-adjoint in $L^2(\nu;\bigwedge^p(\mathbb{R}^N)^*)$.

4. Witten Laplacian in one-dimension

In this section, we give an estimate of the bottom of the spectrum in 1-dimensional case. In the sequel, the state space is $\mathbb{R}$ and the underlying measure is Lebesgue measure. We denote the Hamiltonian by ϕ instead of Φ to distinguish. We define an operator $X_\phi = \partial_t + \phi'$, $\partial_t = \frac{d}{dt}$. Using this operator, the Witten Laplacian can be written as

$$-\Box_0 = \tilde{X}_\phi X_\phi, \tag{4.1}$$

which acts on scalar functions, and

$$-\Box_1 = \tilde{X}_\phi X_\phi + 2\phi''(t) \tag{4.2}$$

which acts on 1-forms. Here we identify 1-forms with scalar functions. This is possible since the dimension of fiber space is 1-dimension. Our aim is to give an estimate of the lowest eigenvalue of $-\Box_1$, which we denote by $\lambda_1(\phi)$. From (4.2), we can see that $\lambda_1(\phi) \geq 2c$ if ϕ is convex and $\phi''(t) \geq c$. Noting that $\tilde{X}_\phi = -\partial_t + \partial_t\phi$, we have

$$\begin{aligned}\tilde{X}_\phi X_\phi - X_\phi\tilde{X}_\phi &= (-\partial_t + \phi')(\partial_t + \phi') - (\partial_t + \phi')(-\partial_t + \phi') \\ &= -\partial_t^2 - \phi'' - \phi'\partial_t + \phi'(\partial_t + \phi') + \partial_t^2 - \phi'' - \phi'\partial_t - \phi'(-\partial_t + \phi') \\ &= -2\phi'',\end{aligned}$$

which means

$$-\Box_1 = X_\phi\tilde{X}_\phi. \tag{4.3}$$

As in Definition 2.1, we have

$$-\Box_0 = \tilde{X}_\phi X_\phi = -\frac{d^2}{dt^2} + \phi'(t)^2 - \phi''(t) \tag{4.4}$$

and further

$$-\Box_1 = X_\phi \tilde{X}_\phi = -\frac{d^2}{dt^2} + \phi'(t)^2 + \phi''(t). \tag{4.5}$$

We note that the pair of $\Box_0$ and $\Box_1$ has a supersymmetric structure. In fact, in the space $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, we define

$$Q = \begin{pmatrix} 0 & \tilde{X}_\phi \\ X_\phi & 0 \end{pmatrix}. \tag{4.6}$$

Then Q is symmetric and satisfies

$$Q^2 = \begin{pmatrix} \tilde{X}_\phi X_\phi & 0 \\ 0 & X_\phi \tilde{X}_\phi \end{pmatrix} = \begin{pmatrix} -\Box_0 & 0 \\ 0 & -\Box_1 \end{pmatrix}.$$

By using the following well-known fact (see, e.g., [**3**, Theorem 6.3]), we can see that $-\Box_0$ and $-\Box_1$ have the same spectrum except for 0.

Proposition 4.1. — Let T be a closed operator in a Hilbert space H. Then T^*T and TT^* has the same spectrum except for 0.

By this special structure, we have that eigenvalues of $-\Box_0$ coincide with those of $-\Box_1$ excluding 0.

The next Lemma shows that a bounded perturbation preserves the positivity of the lowest eigenvalue.

Lemma 4.2. — Let χ be a bounded function. We denote by $\chi_{\sup}$, $\chi_{\inf}$ the infimum and the supremum of χ, respectively. Then we have

$$\lambda_1(\phi) \geq e^{-2(\chi_{\sup} - \chi_{\inf})} \lambda_1(\phi + \chi). \tag{4.7}$$

Proof. — Note that

$$\begin{aligned} e^{-\chi}(-\partial_t + \phi' + \chi')e^\chi &= -e^{-\chi}\partial_t e^\chi + \phi' + \chi' \\ &= -e^{-\chi}(e^\chi \chi' + e^\chi \partial_t) + \phi' + \chi' = -\partial_t + \phi' = \tilde{X}_\phi. \end{aligned}$$

Hence we have

$$\begin{aligned} (-\Box_1 u, u) &= (\tilde{X}_\phi u, \tilde{X}_\phi u) \\ &= ((-\partial_t + \phi')u, (-\partial_t + \phi')u) \\ &= (e^{-\chi}(-\partial_t + \phi' + \chi')e^\chi u, e^{-\chi}(-\partial_t + \phi' + \chi')e^\chi u) \\ &\geq e^{-2\chi_{\sup}}((-\partial_t + \phi' + \chi')e^\chi u, (-\partial_t + \phi' + \chi')e^\chi u) \\ &\geq e^{-2\chi_{\sup}} \lambda_1(\phi + \chi) \|e^\chi u\|_2^2 \end{aligned}$$

$$\geq e^{-2\chi_{\sup}}\lambda_1(\phi+\chi)e^{2\chi_{\inf}}\|u\|_2^2$$
$$= e^{-2(\chi_{\sup}-\chi_{\inf})}\lambda_1(\phi+\chi)\|u\|_2^2.$$

This means (4.7). $\square$

The equation (4.7) can be written as

$$\lambda_1(\phi+\chi) \geq e^{-2(\chi_{\sup}-\chi_{\inf})}\lambda_1(\phi).$$

This implies the following. If the function ϕ is a sum of a convex function and a bounded function, then the lowest eigenvalue of $-\square_1$ is positive, and further the operator $-\square_0$ has a spectral gap. To be precise, writing $\phi = V + W$ with $V'' \geq c$ and W being bounded, we have the following estimate:

$$\lambda_1(\phi) \geq 2ce^{-2(W_{\sup}-W_{\inf})}.$$

Lastly we give another type of estimate of the lowest eigenvalue for a double well potential of the form $at^4 - bt^2$. To do this, we recall the harmonic oscillator $-\mathfrak{A} = -\frac{d^2}{dt^2} + at^2$ on $L^2(\mathbb{R}, dx)$. It is well-known that the lowest eigenvalue of this operator is $\sqrt{a}$ with an eigenfunction $e^{-\sqrt{a}t^2/2}$. Using this, we have the following:

Proposition 4.3. — If $\phi(t) = at^4 - bt^2$, then we have

$$\lambda_1(\phi) \geq 2\sqrt{3a} - 2b. \tag{4.8}$$

Proof. — From (4.5),

$$(-\square_1 u, u) = ((-\frac{d^2}{dt^2} + \phi''(t) + \phi'(t)^2)u, u)$$
$$\geq ((-\frac{d^2}{dt^2} + \phi''(t))u, u)$$
$$= ((-\frac{d^2}{dt^2} + 12at^2 - 2b)u, u).$$

Here the operator $-\frac{d^2}{dt^2} + 12at^2$ is a harmonic oscillator and hence the lowest eigenvalue is $2\sqrt{3a}$. This yields that

$$(-\square_1 u, u) \geq ((2\sqrt{3a} - 2b)u, u),$$

which is the desired result. $\square$

5. Positivity of the lowest eigenvalue for the Witten Laplacian

Lattice spin systems are characterized by Gibbs measures on $X = \mathbb{R}^{\mathbb{Z}^d}$. To define a Gibbs measure, we have to introduce a Hamiltonian. Suppose we are given an potential

$U: \mathbb{R} \to \mathbb{R}$. Then the Hamiltonian is defined by

$$\Phi(x) = \sum_{\substack{i,j\in\mathbb{Z}^d \\ i\sim j}} \mathscr{J}(x^i - x^j)^2 + \sum_{i\in\mathbb{Z}^d} U(x^i). \tag{5.1}$$

Here $x = (x^i)_{i\in\mathbb{Z}^d}$ and $i \sim j$ means that $\|i-j\|_1 = |i_1 - j_1| + \cdots + |i_d - j_d| = 1$. $(x^i - x^j)^2$ stands for an interaction between particles. We only deal with this type of nearest neighbor interaction. We can generalize it to finite range interaction but we restrict ourselves to nearest neighbor interaction for the sake of simplicity. The expression of (5.1) involves an infinite sum and it is no more than a formal expression. The Gibbs measure is sometimes expressed as

$$\nu = Z^{-1} e^{-2\Phi(x)} dx. \tag{5.2}$$

But it does not make sense since $\Phi(x)$ diverges and the Lebesgue measure dx is nothing but a fictitious measure.

Precise characterization of Gibbs measures is given by the Dobrushin-Lanford-Ruelle equation. For a given finite region $\Lambda \subseteq \mathbb{Z}^d$ (we denote this fact by $\Lambda \Subset \mathbb{Z}^d$) and a boundary condition $\eta \in X$, we define a Hamiltonian on $\mathbb{R}^\Lambda$ by

$$\Phi_{\Lambda,\eta}(x) = \sum_{\substack{i,j\in\Lambda \\ i\sim j}} \mathscr{J}(x^i - x^j)^2 + \sum_{i\in\Lambda} U(x^i) + 2 \sum_{\substack{i\in\Lambda, j\in\Lambda^c \\ i\sim j}} \mathscr{J}(x^i - \eta^j)^2 \tag{5.3}$$

and introduce a measure on $\mathbb{R}^\Lambda$ by

$$\nu_{\Lambda,\eta} = Z^{-1} e^{-2\Phi_{\Lambda,\eta}(x)} dx_\Lambda. \tag{5.4}$$

Here dx_Λ denotes the Lebesgue measure on $\mathbb{R}^\Lambda$. Let $\mathscr{F}_{\Lambda^c} = \sigma\{x^i;\, i \in \Lambda^c\}$. We also denote $x_\Lambda = (x^i;\, i\in\Lambda)$ and $x_{\Lambda^c} = (x^i;\, i \in \Lambda^c)$. Then ν is called a Gibbs measure if the conditional probability with respect to $\mathscr{F}_{\Lambda^c}$ is given as

$$E^\nu[\,\cdot\,|x_{\Lambda^c} = \eta_{\Lambda^c}] = \nu_{\Lambda,\eta}(dx_\Lambda) \otimes \delta_{\eta_{\Lambda^c}}(dx_{\Lambda^c}) \tag{5.5}$$

for any $\Lambda \Subset \mathbb{Z}^d$. Here $\delta_{\eta_{\Lambda^c}}$ is the Dirac measure at a point $\eta_{\Lambda^c} \in \mathbb{R}^{\Lambda^c}$, η_{Λ^c} being the restriction of η to Λ^c. The existence and the uniqueness of such measures is a subtle problem. In this paper, we only consider finite region measure and will give uniform estimates. Then our result holds for the infinite system if it exists. In fact, suppose that estimates are uniform. Take any differential form θ which depends on finite variables. We can find a finite region Λ which contains these variables. Then, by the identity (5.5), we have

$$E^\nu[(d\theta, d\theta) + (d^*\theta, d^*\theta)\ |x_{\Lambda^c} = \eta_{\Lambda^c}] \geq kE^\nu[(\theta,\theta)\ |x_{\Lambda^c} = \eta_{\Lambda^c}].$$

Then, by integrating with respect to η_{Λ^c}, we have

$$E^\nu[(d\theta, d\theta) + (d^*\theta, d^*\theta)] \geq kE^\nu[(\theta,\theta)].$$

Now we fix $\Lambda \Subset \mathbb{Z}^d$ and $\eta \in X$ and the Hamiltonian is given by (5.3). As was discussed in the previous section, the Hodge-Kodaira operator dd^*+d^*d is well-defined on $\mathbb{R}^\Lambda$. Our aim is to show that the bottom of the spectrum $\sigma(dd^*+d^*d)$ is positive for p-forms ($p \geq 1$). Under the unitary operator $I\colon L^2(dx_\Lambda) \to L^2(\nu_{\Lambda,\eta})$, we consider the Witten Laplacian $D\tilde{D}+\tilde{D}D$ in $L^2(dx_\Lambda)$.

From now on, we fix $p \geq 1$. Indices $I, J, \ldots$ denote p distinct elements $i_1, i_2, \ldots, i_p$ of Λ. We denote $|I| = p$. When $I = \{i_1, \ldots, i_p\}$, we set $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$. So any p-form θ can be written uniquely as $\theta = \sum_I \theta_I dx^I$. From the Weitzenböck formula, we have

$$\begin{aligned}
(\tilde{D}D + D\tilde{D})\theta &= \sum_i \tilde{X}_i X_i \sum_I \theta_I dx^I + 2\sum_{i,j} \partial_i\partial_j\Phi\,(a^i)^* a^j \sum_I \theta_I dx^I \\
&= \sum_I \sum_i \tilde{X}_i X_i \theta_I dx^I + 2\sum_I \theta_I \sum_i \partial_i^2\Phi\,(a^i)^* a^i dx^I \\
&\quad + 2\sum_I \theta_I \sum_{i\neq j} \partial_i\partial_j\Phi\,(a^i)^* a^j dx^I \\
&= \sum_I \sum_i \tilde{X}_i X_i \theta_I dx^I + 2\sum_I \theta_I \sum_{i\in I} \partial_i^2\Phi dx^I \\
&\quad + 2\sum_I \theta_I \sum_{i\neq j} \partial_i\partial_j\Phi\,(a^i)^* a^j dx^I.
\end{aligned}$$

Therefore

$$\begin{aligned}
&((\tilde{D}D + D\tilde{D})\theta, \theta) \\
&= (\sum_I \sum_i \tilde{X}_i X_i \theta_I dx^I + 2\sum_I \theta_I \sum_{i\in I} \partial_i^2\Phi dx^I \\
&\quad + 2\sum_I \theta_I \sum_{i\neq j} \partial_i\partial_j\Phi\,(a^i)^* a^j dx^I, \sum_J \theta_J dx^J) \\
&= \sum_I (\sum_i \tilde{X}_i X_i \theta_I, \theta_I) + 2\sum_I (\theta_I \sum_{i\in I} \partial_i^2\Phi, \theta_I) \\
&\quad + 2\sum_{I,J} (\theta_I \sum_{i\neq j} \partial_i\partial_j\Phi\,(a^i)^* a^j dx^I, \sum_J \theta_J dx^J) \\
&= \sum_I (\sum_{i\in I} \tilde{X}_i X_i \theta_I, \theta_I) + \sum_I (\sum_{i\notin I} \tilde{X}_i X_i \theta_I, \theta_I) + 2\sum_I (\theta_I \sum_{i\in I} \partial_i^2\Phi, \theta_I) \\
&\quad + 2\sum_{I,J} (\theta_I \sum_{i\neq j} \partial_i\partial_j\Phi(a^i)^* a^j dx^I, \sum_J \theta_J dx^J) \\
&\geq \sum_I (\sum_{i\in I} \tilde{X}_i X_i \theta_I, \theta_I) + 2\sum_I (\theta_I \sum_{i\in I} \partial_i^2\Phi, \theta_I) \\
&\quad + 2\sum_{I,J} (\theta_I \sum_{i\neq j} \partial_i\partial_j\Phi\,(a^i)^* a^j dx^I, \sum_J \theta_J dx^J)
\end{aligned}$$

$$= \sum_I \sum_{i \in I} ((\tilde{X}_i X_i + 2\partial_i^2 \Phi)\theta_I, \theta_I) + 2 \sum_{I,J} (\theta_I \sum_{i \neq j} \partial_i \partial_j \Phi \, (a^i)^* a^j dx^I, \sum_J \theta_J dx^J).$$

To get positivity of the left hand side, we estimate the right hand side term by term. We state our result as a theorem.

Theorem 5.1. — Suppose U is decomposed as $U = V + W$ so that $V'' \geq c > 0$ and W is bounded. W_{sup} and W_{inf} denote the supremum and the infimum of W, respectively. If $2(c + 8d\mathscr{J})e^{-2(W_{\mathrm{sup}} - W_{\mathrm{inf}})} > 16d\mathscr{J}$, then the lowest eigenvalue of $\tilde{D}D + D\tilde{D}$ for p-forms is greater than $\{2(c + 8d\mathscr{J})e^{-2(W_{\mathrm{sup}} - W_{\mathrm{inf}})} - 16d\mathscr{J}\}p$. Therefore there is no harmonic p-forms for $p \geq 1$.

Proof. — We first estimate the second term. To do this, we first compute $\partial_j \partial_i \Phi$. For $i \neq j$, we have

$$\partial_i \partial_j \Phi(x) = \partial_i \partial_j \Big\{ \sum_{\substack{k,l \in \Lambda \\ k \sim l}} \mathscr{J}(x^k - x^l)^2 + \sum_{k \in \Lambda} U(x^k) + 2 \sum_{\substack{k \in \Lambda, l \in \Lambda^c \\ k \sim l}} \mathscr{J}(x^k - \eta^l)^2 \Big\}$$

$$= \begin{cases} -4\mathscr{J}, & i \sim j \\ 0, & \text{otherwise.} \end{cases}$$

For $i = j$, we have

$$\partial_i^2 \Phi(x) = U''(x^i) + 8\mathscr{J}d.$$

Hence

$$\begin{aligned} 2 \sum_{I,J} &(\theta_I \sum_{i \neq j} \partial_i \partial_j \Phi (a^i)^* a^j dx^I, \sum_J \theta_J dx^J) \\ &= -8\mathscr{J} \sum_{I,J} (\theta_I \sum_{i \sim j} (a^i)^* a^j dx^I, \theta_J dx^J) \\ &= -8\mathscr{J} \sum_{I,J} (\sum_{i \sim j} (a^i)^* a^j dx^I, dx^J)(\theta_I, \theta_J) \\ &= -8\mathscr{J} \sum_{I,J} c(I,J)(\theta_I, \theta_J). \end{aligned}$$

Here we set $c(I,J) = (\sum_{i \sim j} (a^i)^* a^j dx^I, dx^J)$. $c(I,J) = 1$ or -1 only when I and J differs by only one element and they are adjacent to each other. Otherwise $c(I,J) = 0$. For each fixed I, there are utmost $2dp$ J's with $c(I,J) \neq 0$. Therefore we have

$$\begin{aligned} -8\mathscr{J} \sum_{I,J} c(I,J)(\theta_I, \theta_J) &\geq -4\mathscr{J} \sum_{I,J} |c(I,J)| \{\|\theta_I\|_2^2 + \|\theta_J\|_2^2\} \\ &\geq -4\mathscr{J} \sum_I 2dp \|\theta_I\|_2^2 - 4\mathscr{J} \sum_J 2dp \|\theta_J\|_2^2 \\ &= -16dp\mathscr{J} \sum_I \|\theta_I\|_2^2. \end{aligned}$$

Eventually the second term is estimated as follows:

$$2\sum_{I,J}(\theta_I \sum_{i\neq j}\partial_i\partial_j\Phi(a^i)^* a^j dx^I, \sum_J \theta_J dx^J) \geq -16dp\mathscr{J}\sum_I |\theta_I|^2.$$

We will show that the first term is greater than the second term if $\mathscr{J}$ is sufficiently small. To estimate the first term, we need to compute $((\tilde{X}_i X_i + 2\partial_i^2\Phi)\theta_I, \theta_I)$ and we regards it as a function of x^i for a moment. So other variables are fixed. We denote other variables by y^i, i.e.,

$$y^i = \{x^j\}_{j\in\Lambda\setminus\{i\}} \in \mathbb{R}^{\Lambda\setminus\{i\}}.$$

The variables $\{x^j\}_{j\in\Lambda}$ are decomposed into x^i and y^i. Then

$$\Phi(x) = \Phi(x^i, y^i) = U(x^i) + 4d\mathscr{J}(x^i)^2 - x^i\Big\{\sum_{\substack{j\in\Lambda\\ j\sim i}} 4x^j + \sum_{\substack{j\in\Lambda^c\\ j\sim i}} 4\eta^j\Big\} + \hat{\Phi}_i(y^i).$$

It is enough to consider the 1-dimensional Hamiltonian of the form

$$\phi(t) = U(t) + 4d\mathscr{J}t^2 - \alpha t.$$

In this case, let us estimate the lowest eigenvalue of a operator $\tilde{X}_\phi X_\phi + 2\phi''(t)$. Here $X_\phi = \partial_t + \phi'$, $\partial_t = \frac{d}{dt}$. But we have already considered the 1-dimensional case in the previous section and so we are ready to estimate $((\tilde{X}_i X_i + 2\partial_i^2\Phi)\theta_I, \theta_I)$. In fact, by Lemma 4.2, the lowest eigenvalue of $\tilde{X}_i X_i + 2\partial_i^2\Phi$ is greater than $(c + 8d\mathscr{J})e^{-2(W_{\sup}-W_{\inf})}$.

$$\begin{aligned}
&((\tilde{D}D + D\tilde{D})\theta, \theta)\\
&\quad\geq \sum_I\sum_{i\in I}((\tilde{X}_i X_i + 2\partial_i^2\Phi)\theta_I, \theta_I) + 2\sum_{I,J}(\theta_I\sum_{i\neq j}\partial_i\partial_j\Phi\,(a^i)^* a^j dx^I, \sum_J \theta_J dx^J)\\
&\quad\geq \sum_I\sum_{i\in I} 2(c + 8d\mathscr{J})e^{-2(W_{\sup}-W_{\inf})}\|\theta_I\|_2^2 - 16dp\mathscr{J}\sum_I\|\theta_I\|_2^2\\
&\quad= \sum_I 2p(c + 8d\mathscr{J})e^{-2(W_{\sup}-W_{\inf})}\|\theta_I\|_2^2 - 16dp\mathscr{J}\sum_I\|\theta_I\|_2^2\\
&\quad= p\{2(c + 8d\mathscr{J})e^{-2(W_{\sup}-W_{\inf})} - 16d\mathscr{J}\}\|\theta\|_2^2.
\end{aligned}$$

This is what we wanted. □

When U is a double well potential, we can give another kind of estimate. This time, we use Proposition 4.3.

Theorem 5.2. — Assume that U is of the form $U(t) = at^4 - bt^2$. If $\sqrt{3a} - b - 4d\mathscr{J} > 0$, then the lowest eigenvalue of $\tilde{D}D + D\tilde{D}$ for p-forms is not smaller than $2(\sqrt{3a} - b - 4d\mathscr{J})p$. Therefore there is no harmonic p-forms ($p \geq 1$).

Proof. — A proof is almost same as the previous one. This time, one dimensional Hamiltonian is of the form

$$\phi(t) = at^4 - bt^2 + 4d\mathscr{J}t^2 - \alpha t.$$

By Proposition 4.3, the lowest eigenvalue of $\tilde{X}_i X_i + 2\phi''(t) + \phi'(t)^2$ is not smaller than $2\sqrt{3a} - 2b + 8d\mathscr{J}$. Hence we have

$$\begin{aligned}
&((\tilde{D}D + D\tilde{D})\theta, \theta) \\
&\quad \geq \sum_I \sum_{i\in I} ((\tilde{X}_i X_i + 2\partial_i^2 \Phi)\theta_I, \theta_I) + 2\sum_{I,J} (\theta_I \sum_{i\neq j} \partial_i \partial_j \Phi\,(a^i)^* a^j dx^I, \sum_J \theta_J dx^J) \\
&\quad \geq \sum_I \sum_{i\in I} (2\sqrt{3a} - 2b - 8d\mathscr{J})\|\theta_I\|_2^2 - 16dp\mathscr{J} \sum_I \|\theta_I\|_2^2 \\
&\quad = 2(\sqrt{3a} - b - 4d\mathscr{J})p\|\theta\|_2^2.
\end{aligned}$$

This completes the proof. □

Lastly we will show that any differential form can be decomposed into three parts; exact, coexact and harmonic, which is usually called the Hodge-Kodaira decomposition. We have seen the positivity of the lowest eigenvalue, the decomposition follows easily. We state it as a theorem.

Theorem 5.3. — Under the assumption of Theorem 5.1 or Theorem 5.2, the following Hodge-Kodaira decomposition holds: For $p = 0$,

$$L^2(\nu) = \{\text{ constant functions }\} \oplus \mathrm{Ran}(d^*) \tag{5.6}$$

and for $p \geq 1$,

$$L^2(\nu; \textstyle\bigwedge^p(\mathbb{R}^\Lambda)^*) = \mathrm{Ran}(d) \oplus \mathrm{Ran}(d^*). \tag{5.7}$$

Proof. — We only give a proof for $p \geq 1$. Set

$$T = (d, d^*) \colon L^2(\nu; \textstyle\bigwedge^p(\mathbb{R}^\Lambda)^*) \to L^2(\nu; \bigwedge^{p+1}(\mathbb{R}^\Lambda)^*) \oplus L^2(\nu; \bigwedge^{p-1}(\mathbb{R}^\Lambda)^*).$$

Here, by taking a closure, T is defined as a closed operator. Then we can have $-\Box_p = T^*T$. In fact, both operator coincides for smooth p-forms with compact support. So the identity follows from the essential self-adjointness of $\Box_p$. Thus we have

$$-\Box_p = dd^* + d^*d$$

on the domain of $\Box_p$. Since the lowest eigenvalue of $-\Box_p$ is positive, it has an inverse operator, which we denote by $-G$. Then for $\omega \in L^2(\nu; \bigwedge^p(\mathbb{R}^\Lambda)^*)$ we have

$$\omega = (dd^* + d^*d)G\omega = d(d^*G\omega) + d^*(dG\omega).$$

The orthogonality between $d(d^*G\omega)$ and $d^*(dG\omega)$ follows easily from the property $d^2 = 0$. This completes the proof. □

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Astérisque
327, 2009, p. 131–199

PURE SPINORS ON LIE GROUPS

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday.

Abstract. — For any manifold M, the direct sum $\mathbb{T}M = TM \oplus T^*M$ carries a natural inner product given by the pairing of vectors and covectors. Differential forms on M may be viewed as spinors for the corresponding Clifford bundle, and in particular there is a notion of *pure spinor*. In this paper, we study pure spinors and Dirac structures in the case when $M = G$ is a Lie group with a bi-invariant pseudo-Riemannian metric, e.g. G semi-simple. The applications of our theory include the construction of distinguished volume forms on conjugacy classes in G, and a new approach to the theory of quasi-Hamiltonian G-spaces.

***Résumé* (Spineurs purs sur les groupes de Lie).** — Pour toute variété lisse M, le fibré $\mathbb{T}M = TM \oplus T^*M$ est muni d'un produit scalaire naturel défini par la dualité entre vecteurs et co-vecteurs. Les formes différentielles sur M sont des spineurs pour le fibré de Clifford correspondant. On définit alors les *spineurs purs*. Dans cet article, nous étudions les spineurs purs et les structures de Dirac dans le cas où M est un groupe de Lie G muni d'une métrique pseudo-riemannienne bi-invariante, par exemple un groupe semi-simple. Comme applications de notre théorie, nous définissons une forme volume distinguée sur les classes de conjugaison de G, et nous proposons une nouvelle approche de la théorie des G-espaces quasi-hamiltoniens.

0. Introduction

For any manifold M, the direct sum $\mathbb{T}M = TM \oplus T^*M$ carries a non-degenerate symmetric bilinear form, extending the pairing between vectors and covectors. There is a natural Clifford action ϱ of the sections $\Gamma(\mathbb{T}M)$ on the space $\Omega(M) = \Gamma(\wedge T^*M)$ of differential forms, where vector fields act by contraction and 1-forms by exterior multiplication. That is, $\wedge T^*M$ is viewed as a spinor module over the Clifford bundle $\mathrm{Cl}(\mathbb{T}M)$. A form $\phi \in \Omega(M)$ is called a *pure spinor* if the solutions $w \in \Gamma(\mathbb{T}M)$ of

2010 Mathematics Subject Classification. — 53D17, 53D20, 15A66.
Key words and phrases. — Dirac structures, Courant algebroids, Clifford algebras, pure spinors, moment maps.

$\varrho(w)\phi = 0$ span a Lagrangian subbundle $E \subset \mathbb{T}M$. Given a closed 3-form $\eta \in \Omega^3(M)$, a pure spinor ϕ is called *integrable* (relative to η) [**9, 28**] if there exists a section $w \in \Gamma(\mathbb{T}M)$ with

$$(\mathrm{d} + \eta)\phi = \varrho(w)\phi.$$

In this case, there is a generalized foliation of M with tangent distribution the projection of E to TM. The subbundle E defines a *Dirac structure* [**20, 50**] on M, and the triple (M, E, η) is called a *Dirac manifold.*

The present paper is devoted to the study of Dirac structures and pure spinors on Lie groups G. We assume that the Lie algebra $\mathfrak{g}$ carries a non-degenerate invariant symmetric bilinear form B, and take $\eta \in \Omega^3(G)$ as the corresponding Cartan 3-form. Let $\overline{\mathfrak{g}}$ denote the Lie algebra $\mathfrak{g}$ with the opposite bilinear form $-B$. We will describe a trivialization

$$\mathbb{T}G \cong G \times (\mathfrak{g} \oplus \overline{\mathfrak{g}}),$$

under which any Lagrangian Lie subalgebra $\mathfrak{s} \subset \mathfrak{g} \oplus \overline{\mathfrak{g}}$ defines a Dirac structure on G. There is also a similar identification of spinor bundles

$$\mathscr{R}\colon G \times \mathrm{Cl}(\mathfrak{g}) \xrightarrow{\cong} \wedge T^*G,$$

taking the standard Clifford action of $\mathfrak{g} \oplus \overline{\mathfrak{g}}$ on $\mathrm{Cl}(\mathfrak{g})$, where the first summand acts by left (Clifford) multiplication and the second summand by right multiplication, to the Clifford action ϱ. This isomorphism takes the *Clifford differential* d_{Cl} on $\mathrm{Cl}(\mathfrak{g})$, given as Clifford commutator by a cubic element [**4, 38**], to the the differential $\mathrm{d} + \eta$ on $\Omega(G)$. As a result, pure spinors $x \in \mathrm{Cl}(\mathfrak{g})$ for the Clifford action of $\mathrm{Cl}(\mathfrak{g} \oplus \overline{\mathfrak{g}})$ on $\mathrm{Cl}(\mathfrak{g})$ define pure spinors $\phi = \mathscr{R}(x) \in \Omega(G)$, and the integrability condition for ϕ is equivalent to a similar condition for x. The simplest example $x = 1$ defines the *Cartan-Dirac structure* E_G [**14, 50**], introduced by Alekseev, Ševera and Strobl in the 1990's. In this case, the resulting foliation of G is just the foliation by conjugacy classes. We will study this Dirac structure in detail, and examine in particular its behavior under group multiplication and under the exponential map. When G is a complex semi-simple Lie group, it carries another interesting Dirac structure, which we call the *Gauss-Dirac structure.* The corresponding foliation of G has a dense open leaf which is the 'big cell' from the Gauss decomposition of G.

The main application of our study of pure spinors is to the theory of q-Hamiltonian actions [**2, 3**]. The original definition of a q-Hamiltonian G-space in [**3**] involves a G-manifold M together with an invariant 2-form ω and a G-equivariant map $\Phi\colon M \to G$ satisfying appropriate axioms. As observed in [**14, 15**], this definition is equivalent to saying that the 'G-valued moment map' Φ is a suitable morphism of Dirac manifolds (in analogy with classical moment maps, which are morphisms $M \to \mathfrak{g}^*$ of Poisson manifolds). In this paper, we will carry this observation further, and develop all the basic results of q-Hamiltonian geometry from this perspective. A conceptual advantage of this alternate viewpoint is that, while the arguments in [**3**] required G to be compact, the Dirac geometry approach needs no such assumption, and in fact works in the complex (holomorphic) category as well. This is relevant for applications: For instance, the symplectic form on a representation variety $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ (for Σ a closed surface) can be obtained by q-Hamiltonian reduction, and there are many interesting examples for noncompact G. (For instance, the case $G = \mathrm{PSL}(2, \mathbb{R})$ gives the symplectic form on Teichmüller space.) Complex q-Hamiltonian spaces appear e.g. in the work of Boalch [**13**] and Van den Bergh [**11**].

The organization of the paper is as follows. Sections 1 and 2 contain a review of Dirac geometry, first on vector spaces and then on manifolds. The main new results in these sections concern the geometry of Lagrangian splittings $\mathbb{T}M = E \oplus F$ of the bundle $\mathbb{T}M$. If $\phi, \psi \in \Omega(M)$ are pure spinors defining E, F, then, as shown in [**17, 19**], the top degree part of $\phi^\top \wedge \psi$ (where $\top$ denotes the standard anti-involution of the exterior algebra) is nonvanishing, and hence defines a volume form μ on M. Furthermore, there is a bivector field $\pi \in \mathfrak{X}^2(M)$ naturally associated with the splitting, which satisfies

$$\phi^\top \wedge \psi = e^{-\iota(\pi)} \mu.$$

We will discuss the properties of μ and π in detail, including their behavior under Dirac morphisms.

In Section 3 we specialize to the case $M = G$, where G carries a bi-invariant pseudo-Riemannian metric, and our main results concern the isomorphism $\mathbb{T}G \cong G \times (\mathfrak{g} \oplus \overline{\mathfrak{g}})$ and its properties. Under this identification, the Cartan-Dirac structure $E_G \subset \mathbb{T}G$ corresponds to the diagonal $\mathfrak{g}_\Delta \subset \mathfrak{g} \oplus \overline{\mathfrak{g}}$, and hence it has a natural Lagrangian complement $F_G \subset \mathbb{T}G$ defined by the anti-diagonal. We will show that the exponential map gives rise to a Dirac morphism $(\mathfrak{g}, E_\mathfrak{g}, 0) \to (G, E_G, \eta)$ (where $E_\mathfrak{g}$ is the graph of the linear Poisson structure on $\mathfrak{g} \cong \mathfrak{g}^*$), but this morphism does not relate the obvious complements $F_\mathfrak{g} = T\mathfrak{g}$ and F_G. The discrepancy is given by a 'twist', which is a solution of the *classical dynamical Yang-Baxter equation.* For G complex semi-simple, we will construct another Lagrangian complement of E_G, denoted by $\widehat{F}_G$, which (unlike F_G) is itself a Dirac structure. The bivector field corresponding to the splitting $E_G \oplus \widehat{F}_G$ is then a Poisson structure on G, which appeared earlier in the work of Semenov-Tian-Shansky [**49**].

In Section 4, we construct an isomorphism $\wedge T^*G \cong G \times \mathrm{Cl}(\mathfrak{g})$ of spinor modules, valid under a mild topological assumption on G (which is automatic if G is simply connected). This allows us to represent the Lagrangian subbundles E_G, F_G and $\widehat{F}_G$ by explicit pure spinors ϕ_G, ψ_G, and $\widehat{\psi}_G$, and to derive the differential equations controlling their integrability. We show in particular that the Cartan-Dirac spinor satisfies

$$(\mathrm{d} + \eta)\phi_G = 0.$$

Section 5 investigates the foundational properties of q-Hamiltonian G-spaces from the Dirac geometry perspective. Our results on the Cartan-Dirac structure give a direct construction of the fusion product of q-Hamiltonian spaces. On the other hand, we use the bilinear pairing of spinors to show that, for a q-Hamiltonian space (M, ω, Φ), the top degree part of $e^\omega \Phi^* \psi_G \in \Omega(M)$ defines a volume form μ_M. This volume form was discussed in [**8**] when G is compact, but the discussion here applies equally well to non-compact or complex Lie groups. Since conjugacy classes in G are examples of q-Hamiltonian G-spaces, we conclude that for any simply connected Lie group G with bi-invariant pseudo-Riemannian metric (e.g. G semi-simple), *any conjugacy class in G carries a distinguished invariant volume form.* If G is complex semi-simple, one obtains the same volume form μ_M if one replaces ψ_G with the Gauss-Dirac spinor $\widehat{\psi}_G$. However, the form $e^{\omega}\Phi^*\widehat{\psi}_G$ satisfies a nicer differential equation, which allows us to compute the volume of M, and more generally the measure $\Phi_*|\mu_M|$, by Berline-Vergne localization [**12**]. We also explain in this Section how to view the more general q-Hamiltonian q-Poisson spaces [**2**] in our framework.

Lastly, in Section 6, we revisit the theory of K^*-valued moment maps in the sense of Lu [**42**] and its connections with P-valued moment maps [**3**, Sec. 10] from the Dirac geometric standpoint.

Acknowledgments. — We would like to thank Marco Gualtieri and Shlomo Sternberg for helpful comments on an earlier version of this paper. The research of A.A. was supported in part by the Swiss National Science Foundation, while E.M. was supported by an NSERC Discovery Grant and a Steacie Fellowship. H.B. acknowledges financial support from CNPq, and thanks University of Toronto, the Fields Institute and Utrecht University for hospitality at various stages of this project. We also thank the Erwin Schrödinger Institut and the Forschungsinstitut Oberwolfach where part of this work was carried out.

Notation. — Our conventions for Lie group actions are as follows: Let G be a Lie group (not necessarily connected), and $\mathfrak{g}$ its Lie algebra. A G-action on a manifold M is a group homomorphism $\mathscr{A}\colon G \to \mathrm{Diff}(M)$ for which the action map $G\times M \to M,\ (g,m)\mapsto \mathscr{A}(g)(m)$ is smooth. Similarly, a $\mathfrak{g}$-action on M is a Lie algebra homomorphism $\mathscr{A}\colon \mathfrak{g} \to \mathfrak{X}(M)$ for which the map $\mathfrak{g}\times M \to TM,\ (\xi,m)\mapsto \mathscr{A}(\xi)_m$ is smooth. Given a G-action $\mathscr{A}$, one obtains a $\mathfrak{g}$-action by the formula $\mathscr{A}(\xi)(f) = \frac{\partial}{\partial t}\big|_{t=0}\mathscr{A}(\exp(-t\xi))^* f$, for $f\in C^\infty(M)$ (here vector fields are viewed as derivations of the algebra of smooth functions).

1. Linear Dirac geometry

The theory of Dirac manifolds was initiated by Courant and Weinstein in [**20, 21**]. We briefly review this theory, developing and expanding the approach via pure spinors advocated by Gualtieri [**28**] (see also Hitchin [**32**] and Alekseev-Xu [**9**]). All vector spaces in this section are over the ground field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We begin with some background material on Clifford algebras and spinors (see e.g. [**19**] or [**47**].)

1.1. Clifford algebras. — Suppose V is a vector space with a non-degenerate symmetric bilinear form B. We will sometimes refer to such a bilinear form B as an *inner product* on V. The *Clifford algebra* over V is the associative unital algebra generated by the elements of V, with relations

$$vv' + v'v = B(v, v')\,1.$$

It carries a compatible $\mathbb{Z}_2$-grading and $\mathbb{Z}$-filtration, such that the generators $v \in V$ are odd and have filtration degree 1. We will denote by $x \mapsto x^\top$ the canonical anti-automorphism of exterior and Clifford algebras, equal to the identity on V. For any $x \in \mathrm{Cl}(V)$, we denote by $l^{\mathrm{Cl}}(x), r^{\mathrm{Cl}}(x)$ the operators of graded left and right multiplication on $\mathrm{Cl}(V)$:

$$l^{\mathrm{Cl}}(x)x' = xx', \quad r^{\mathrm{Cl}}(x)x' = (-1)^{|x||x'|}x'x.$$

Thus $l^{\mathrm{Cl}}(x) - r^{\mathrm{Cl}}(x)$ is the operator of graded commutator $[x, \cdot]_{\mathrm{Cl}}$.

The *quantization map* $q\colon \wedge V \to \mathrm{Cl}(V)$ is the isomorphism of vector spaces defined by $q(v_1 \wedge \cdots \wedge v_r) = v_1 \cdots v_r$ for pairwise orthogonal elements $v_i \in V$. Let

$$\mathrm{str}\colon \mathrm{Cl}(V) \to \det(V) := \wedge^{\mathrm{top}}(V)$$

be the *super-trace*, given by q^{-1}, followed by taking the top degree part. It has the property $\mathrm{str}([x, x']_{\mathrm{Cl}}) = 0$.

A *Clifford module* is a vector space S together with an algebra homomorphism $\varrho\colon \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S})$. If S is a Clifford module, one has a dual Clifford module given by the dual space S^* with Clifford action $\varrho^*(x) = \varrho(x^\top)^*$.

Recall that $\mathrm{Pin}(V)$ is the subgroup of $\mathrm{Cl}(V)^\times$ generated by all $v \in V$ whose square in the Clifford algebra is $vv = \pm 1$. It is a double cover of the orthogonal group $\mathrm{O}(V)$, where $g \in \mathrm{Pin}(V)$ takes $v \in V$ to $(-1)^{|g|}gvg^{-1}$, using Clifford multiplication. The *norm homomorphism* for the Pin group is the group homomorphism

$$\mathsf{N}\colon \mathrm{Pin}(V) \to \{-1, +1\}, \quad \mathsf{N}(g) = g^\top g = \pm 1. \tag{1}$$

Let $\{\cdot, \cdot\}$ be the graded Poisson bracket on $\wedge V$, given on generators by $\{v_1, v_2\} = B(v_1, v_2)$. Then $\wedge^2 V$ is a Lie algebra under the Poisson bracket, isomorphic to $\mathfrak{o}(V)$ in such a way that $\varepsilon \in \wedge^2 V$ corresponds to the linear map $v \mapsto \{\varepsilon, v\}$. The Lie algebra $\mathfrak{pin}(V) \cong \mathfrak{o}(V)$ is realized as the Lie subalgebra $q(\wedge^2(V)) \subset \mathrm{Cl}(V)$.

A subspace $E \subset V$ is called *isotropic* if $E \subset E^\perp$ and *Lagrangian* if $E = E^\perp$. The set of Lagrangian subspaces is non-empty if and only if the bilinear form is *split.* If $\mathbb{K} = \mathbb{C}$, this just means that $\dim V$ is even, while for $\mathbb{K} = \mathbb{R}$ this requires that the bilinear form has signature (n, n). From now on, we will reserve the letter W for a vector space with split bilinear form $\langle \cdot, \cdot \rangle$. We denote by $\mathrm{Lag}(W)$ the Grassmann manifold of Lagrangian subspaces of W. It carries a transitive action of the orthogonal group $\mathrm{O}(W)$.

Remark 1.1. — Suppose $\mathbb{K} = \mathbb{R}$, and identify $W \cong \mathbb{R}^{2n}$ with the standard bilinear form of signature (n, n). The group $\mathrm{O}(W) \cong \mathrm{O}(n, n)$ has maximal compact subgroup $\mathrm{O}(n) \times \mathrm{O}(n)$. Already the subgroup $\mathrm{O}(n) \times \{1\}$ acts transitively on $\mathrm{Lag}(W)$, and in fact the action is free. It follows that $\mathrm{Lag}(W)$ is diffeomorphic to $\mathrm{O}(n)$. Further details may be found in [**46**].

1.2. Pure spinors. — An irreducible module S over the Clifford algebra $\mathrm{Cl}(W)$ is called a *spinor module.* Any $E \in \mathrm{Lag}(W)$ defines a spinor module $\mathsf{S} = \mathrm{Cl}(W)/\mathrm{Cl}(W)E$. The choice of a Lagrangian complement F to E identifies $\mathsf{S} = \wedge E^*$, where the generators in $E \subset W$ act by contraction and the generators in $F \subset W$ act by exterior multiplication. (Here F is identified with E^*, using the pairing defined by $\langle \cdot, \cdot \rangle$.) The dual spinor module is $\mathsf{S}^* = \wedge E$, with generators in E acting by exterior multiplication and those in F by contraction.

For any non-zero element $\phi \in \mathsf{S}$ of a spinor module, its null space

$$N_\phi = \{w \in W |\, \varrho(w)\phi = 0\}$$

is easily seen to be isotropic. The element $\phi \in \mathsf{S}$ is a *pure spinor* **[17]** provided N_ϕ is Lagrangian. One can show that any Lagrangian subspace $E \in \mathrm{Lag}(W)$ arises in this way: in fact, $\mathsf{S}^E = \{\phi \in \mathsf{S} |\, \varrho(E)\phi = 0\}$ is a one-dimensional subspace, with non-zero elements given by the pure spinors defining E. Any spinor module S admits a $\mathbb{Z}_2$-grading (unique up to parity inversion) compatible with the Clifford action. Pure spinors always have a definite parity, either even or odd.

Example 1.2. — Let V be a vector space with inner product B. We denote by $\overline{V}$ the same vector space with the opposite bilinear form $-B$. Then $W = V \oplus \overline{V}$ is a vector space with split bilinear form. The space $\mathsf{S} = \mathrm{Cl}(V)$ is a spinor module over $\mathrm{Cl}(W) = \mathrm{Cl}(V) \otimes \mathrm{Cl}(\overline{V})$, with Clifford action given on generators by $\varrho(v \oplus v') = l^{\mathrm{Cl}}(v) - r^{\mathrm{Cl}}(v')$. The element $1 \in \mathrm{Cl}(V)$ is a pure spinor, with corresponding Lagrangian subspace the diagonal $V_\Delta \subset V \oplus \overline{V}$.

1.3. The bilinear pairing of spinors. — For any two spinor modules $\mathsf{S}_1, \mathsf{S}_2$ over $\mathrm{Cl}(W)$, the space $\mathrm{Hom}_{\mathrm{Cl}(W)}(\mathsf{S}_1, \mathsf{S}_2)$ of intertwining operators is one-dimensional. Given a spinor module S, let

$$K_{\mathsf{S}} = \mathrm{Hom}_{\mathrm{Cl}(W)}(\mathsf{S}^*, \mathsf{S})$$

be the *canonical line.* There is a bilinear pairing **[17]**

$$\mathsf{S} \otimes \mathsf{S} \to K_{\mathsf{S}},\ \ \phi \otimes \psi \mapsto (\phi, \psi)_{\mathsf{s}},$$

defined by the isomorphism $\mathsf{S} \otimes \mathsf{S} \cong \mathsf{S} \otimes \mathsf{S}^* \otimes \mathrm{Hom}_{\mathrm{Cl}(W)}(\mathsf{S}^*, \mathsf{S})$ followed by the duality pairing $\mathsf{S} \otimes \mathsf{S}^* \to \mathbb{K}$. The pairing satisfies

$$(\varrho(x^\top)\phi, \psi)_{\mathsf{s}} = (\phi, \varrho(x)\psi)_{\mathsf{s}}, \quad x \in \mathrm{Cl}(W), \tag{2}$$

and is characterized by this property up to a scalar. (2) implies the following invariance property under the action of the group $\mathrm{Pin}(V)$, involving the norm homomorphism (1),

$$(g\phi, g\psi)_{\mathsf{s}} = \mathsf{N}(g)(\phi, \psi)_{\mathsf{s}}, \quad g \in \mathrm{Pin}(V).$$

***Theorem 1.3* (E. Cartan [17]).** — *Let S be a spinor modules over $\mathrm{Cl}(W)$, and let $\phi, \psi \in \mathsf{S}$ be pure spinors. Then the corresponding Lagrangian subspaces N_ϕ, N_ψ are transverse if and only if $(\phi, \psi)_{\mathsf{s}} \neq 0$.*

A simple proof of this result is given in Chevalley's book [**19**, III.2.4], see also [**47**, Section 3.5].

Example 1.4. — Suppose V is a space with inner product B, and take $\mathsf{S} = \mathrm{Cl}(V)$ as a spinor module over $\mathrm{Cl}(V \oplus \overline{V})$ (cf. Example 1.2). Then $K_{\mathsf{S}} = \det(V)$, with bilinear pairing on spinors given as

$$(x, x')_{\mathrm{Cl}(V)} = \mathrm{str}(x^\top x') \in \det(V). \tag{3}$$

Using the isomorphism $q : \wedge(V) \to \mathrm{Cl}(V)$ to identify $\mathsf{S} \cong \wedge(V)$, the bilinear pairing becomes

$$(y, y')_{\wedge(V)} = (y^\top \wedge y')^{[\mathrm{top}]} \in \det(V). \tag{4}$$

1.4. Contravariant spinors. — For any vector space V, the direct sum $\mathbb{V} := V \oplus V^*$ carries a split bilinear form given by the pairing between V and V^*:

$$\langle w_1, w_2 \rangle = \langle \alpha_1, v_2 \rangle + \langle \alpha_2, v_1 \rangle, \qquad w_i = v_i \oplus \alpha_i \in \mathbb{V}. \tag{5}$$

Every vector space W with split bilinear form is of this form, by choosing a pair of transverse Lagrangian subspaces V, V', and using the bilinear form to identify $V' = V^*$. Then $\mathsf{S} = \wedge V^*$, with Clifford action given on generators $w = v \oplus \alpha \in \mathbb{V}$ by

$$\varrho(w) = \epsilon(\alpha) + \iota(v)$$

(where $\epsilon(\alpha) = \alpha \wedge \cdot$), is a natural choice of spinor module for $\mathrm{Cl}(\mathbb{V})$. The restriction of ϱ to $\wedge V^* \subset \mathrm{Cl}(\mathbb{V})$ is given by exterior multiplication, while the restriction to $\wedge V \subset \mathrm{Cl}(\mathbb{V})$ is given by contraction [(1)]. The line $K_{\mathsf{S}} = \mathrm{Hom}_{\mathrm{Cl}(\mathbb{V})}(\mathsf{S}^*, \mathsf{S})$ is canonically isomorphic to $\det(V^*) = \wedge^{\mathrm{top}} V^*$, and the bilinear pairing on spinors is simply

$$(\phi, \psi)_{\wedge(V^*)} = (\phi^\top \wedge \psi)^{[\mathrm{top}]} \in \det(V^*),$$

similar to Example 1.4. Theorem 1.3 shows that if ϕ, ψ are pure spinors for transverse Lagrangian subspaces, the pairing $(\phi, \psi)_{\wedge(V^*)}$ defines a *volume form* on V.

Remarks 1.5. — We mention the following two facts for later reference.

a. We have the identity

$$(-1)^{|\phi|}(-(\varrho(w)\phi)^\top \wedge \psi + \phi^\top \wedge (\varrho(w)\psi)) = \iota(v)(\phi^\top \wedge \psi), \quad w = v \oplus \alpha \in \mathbb{V},$$

which refines property (2) of the bilinear pairing.

b. One can also consider the *covariant spinor module* $\wedge(V)$, obtained by reversing the roles of V and V^*. Suppose $\mu \in \det(V)$ is non-zero, and let $\star\colon \wedge(V^*) \to \wedge(V)$ be the corresponding star operator, defined by $\star\phi = \iota(\phi)\mu$. Let μ^* be the dual generator defined by $\star((\mu^*)^\top) = 1$. Then $\star$ is an isomorphism of $\mathrm{Cl}(\mathbb{V})$-modules. Furthermore, using μ, μ^* to trivialize $\det(V), \det(V^*)$, the isomorphism intertwines the bilinear pairings:

$$(\phi, \psi)_{\wedge(V^*)} = (\star\phi, \star\psi)_{\wedge(V)}, \quad \phi, \psi \in \wedge(V^*).$$

(1) We are using the convention that $\iota\colon \wedge(V) \to \mathrm{End}(\wedge V^*)$ is the extension of the map $v \mapsto \iota(v)$ as an *algebra homomorphism*. Note that some authors use the extension as an algebra anti-homomorphism.

Any 2-form $\omega \in \wedge^2 V^*$ defines a pure spinor $\phi = e^{-\omega}$, with N_ϕ the graph of ω:

$$\mathrm{Gr}_\omega = \{v \oplus \alpha |\ v \in V,\ \alpha = \iota(v)\omega\}.$$

Note that, in accordance with Theorem 1.3, $\mathrm{Gr}_\omega \cap V = \{0\}$ if and only if ω is non-degenerate, if and only if $(e^\omega)^{[\mathrm{top}]}$ is non-zero. The most general pure spinor $\phi \in \wedge V^*$ can be written in the form

$$\phi = e^{-\omega_Q} \wedge \theta, \tag{6}$$

where $\omega_Q \in \wedge^2 Q^*$ is a 2-form on a subspace $Q \subset V$ and $\theta \in \det(\mathrm{Ann}(Q))\backslash\{0\}$ is a volume form on V/Q. To write (6), we have chosen an extension of ω_Q to a 2-form on V. (Clearly, ϕ does not depend on this choice.) The corresponding Lagrangian subspace is

$$N_\phi = \{v \oplus \alpha |\ v \in Q,\ \alpha|_Q = \iota(v)\omega_Q\}.$$

The triple (Q, ω_Q, θ) is uniquely determined by ϕ, see e.g. [**19**, III.1.9]. A simple consequence is that any pure spinor has definite parity, that is, ϕ is either even or odd depending on the parity of $\dim(V/Q)$. For any $E \in \mathrm{Lag}(\mathbb{V})$ we define subspaces $\ker(E) \subset \mathrm{ran}(E) \subset V$ by

$$\ker(E) = E \cap V, \qquad \mathrm{ran}(E) = \mathrm{pr}_V(E),$$

where $\mathrm{pr}_V : \mathbb{V} \to V$ is the projection along V^*. For any pure spinor ϕ, written in the form (6), we have $\mathrm{ran}(E_\phi) = Q$ and $\ker(E_\phi) = \ker(\omega_Q)$. In particular, $\phi^{[\mathrm{top}]}$ is non-zero if and only if $\ker(E_\phi) = 0$. Similarly, $\mathrm{ran}(E_\phi) = V$ if and only if $\phi^{[0]}$ is non-zero, if and only if $\phi = e^{-\omega}$ for a global 2-form ω.

1.5. Action of the orthogonal group. — Recall the identification $\wedge^2(W) \cong \mathfrak{o}(W)$ (see Section 1.1). For any Lagrangian subspace $E \subset W$, the space $\wedge^2(E)$ is embedded as an Abelian subalgebra of $\mathfrak{o}(W)$. The inclusion map exponentiates to an injective group homomorphism,

$$\wedge^2(E) \to \mathrm{O}(W),\ \ \varepsilon \mapsto A^\varepsilon, \qquad A^\varepsilon(v \oplus \alpha) = v \oplus (\alpha - \iota(v)\varepsilon), \tag{7}$$

with image the orthogonal transformations fixing E pointwise. The subgroup $\wedge^2(E)$ acts freely and transitively on the subset of $\mathrm{Lag}(W)$ of Lagrangian subspaces transverse to E, which therefore becomes an affine space. Observe that A^ε has a distinguished lift $\widetilde{A}^\varepsilon = \exp(\varepsilon) \in \mathrm{Pin}(W)$ (exponential in the subalgebra $\wedge(E) \subset \mathrm{Cl}(W)$).

For any spinor module S over $\mathrm{Cl}(W)$, the induced representation of the group $\mathrm{Pin}(W) \subset \mathrm{Cl}(W)^\times$ preserves the set of pure spinors, and the map $\phi \mapsto N_\phi$ is equivariant. That is, if $\widetilde{A} \in \mathrm{Pin}(W)$ lifts $A \in \mathrm{O}(W)$, then

$$N_{\varrho(\widetilde{A})\phi} = A(N_\phi).$$

Consider again the case $W = \mathbb{V}$. Then 2-forms $\omega \in \wedge^2 V^*$ and bivectors $\pi \in \wedge^2(V)$ define orthogonal transformations

$$A^{-\omega}(v \oplus \alpha) = v \oplus (\alpha + \iota_v\omega), \quad A^{-\pi}(v \oplus \alpha) = (v + \iota_\alpha\pi) \oplus \alpha.$$

Their lifts act in the spin representation as follows:

$$\varrho(\widetilde{A}^{-\omega})\phi = e^{-\omega}\,\phi,\quad \varrho(\widetilde{A}^{-\pi})\phi = e^{-\iota(\pi)}\phi. \tag{8}$$

1.6. Morphisms. — It is easy to see that the group of orthogonal transformations of $\mathbb{V}$ preserving the 'polarization'

$$0 \longrightarrow V^* \longrightarrow \mathbb{V} \longrightarrow V \longrightarrow 0 \tag{9}$$

(i.e., taking the subspace V^* to itself) is the semi-direct product $\wedge^2 V^* \rtimes \mathrm{GL}(V) \subset \mathrm{O}(\mathbb{V})$, where $\omega \in \wedge^2 V^*$ acts as $A^{-\omega}$ and $\mathrm{GL}(V)$ acts in the natural way on V and by the conjugate transpose on V^*.

More generally, for vector spaces V and V', we define the set of *morphisms from* $\mathbb{V}$ *to* $\mathbb{V}'$ **[33]** to be

$$\mathrm{Hom}(V, V') \times \wedge^2 V^*,$$

with the following composition law:

$$(\Phi_1, \omega_1) \circ (\Phi_2, \omega_2) = (\Phi_1 \circ \Phi_2,\ \omega_2 + \Phi_2^* \omega_1). \tag{10}$$

Given $w = v \oplus \alpha \in \mathbb{V}$ and $w' = v' \oplus \alpha' \in \mathbb{V}'$, we write

$$w \sim_{(\Phi,\omega)} w' \quad \Leftrightarrow \quad v' = \Phi(v),\ \Phi^* \alpha' = \alpha + \iota_v \omega.$$

In particular, taking $V' = V$ and $\Phi = \mathrm{id}$ we have $w \sim_{(\mathrm{id},\omega)} w'$ if and only $w' = A^{-\omega}(w)$. The *graph of a morphism* (Φ, ω) is the subspace

$$\Gamma_{(\Phi,\omega)} = \{(w', w) \in \mathbb{V}' \times \mathbb{V} \,|\, w \sim_{(\Phi,\omega)} w'\}. \tag{11}$$

We have $\Gamma_{(\Phi_1,\omega_1)\circ(\Phi_2,\omega_2)} = \Gamma_{(\Phi_1,\omega_1)} \circ \Gamma_{(\Phi_2,\omega_2)}$ under composition of relations. The morphisms (Φ, ω) are 'isometric', in the sense that

$$w_1 \sim_{(\Phi,\omega)} w_1',\quad w_2 \sim_{(\Phi,\omega)} w_2' \Rightarrow \langle w_1, w_2 \rangle = \langle w_1', w_2' \rangle. \tag{12}$$

Equivalently, $\Gamma_{(\Phi,\omega)}$ is Lagrangian in $\mathbb{V}' \oplus \overline{\mathbb{V}}$. We write

$$\begin{aligned} \ker(\Phi, \omega) &= \{w \in \mathbb{V} \,|\, w \sim_{(\Phi,\omega)} 0\}, \\ \mathrm{ran}(\Phi, \omega) &= \{w' \in \mathbb{V}' \,|\, \exists w \in \mathbb{V} \colon w \sim_{(\Phi,\omega)} w'\}. \end{aligned}$$

Thus $\ker(\Phi, \omega) = \{(v, -\iota_v \omega) \,|\, v \in \ker(\Phi)\}$ while $\mathrm{ran}(\Phi, \omega) = \mathrm{ran}(\Phi) \oplus (V')^*$.

Definition 1.6. — Let $(\Phi, \omega) \colon \mathbb{V} \to \mathbb{V}'$ be a morphism, and $E \in \mathrm{Lag}(\mathbb{V})$. We define the *forward image* $E' \in \mathrm{Lag}(\mathbb{V}')$ to be the Lagrangian subspace

$$E' := \Gamma_{(\Phi,\omega)} \circ E = \{w' \in \mathbb{V}' \,|\, \exists\, w \in E \colon w \sim_{(\Phi,\omega)} w'\}.$$

Similarly, for $F' \in \mathrm{Lag}(\mathbb{V}')$ the *backward image* is defined as the Lagrangian subspace

$$F := F' \circ \Gamma_{(\Phi,\omega)} = \{w \in \mathbb{V} \,|\, \exists\, w' \in F' \colon\ w \sim_{(\Phi,\omega)} w'\}.$$

The proof that forward and backward images of Lagrangian subspaces are Lagrangian is parallel to the similar statement in the symplectic category of Guillemin-Sternberg [**30**] (see also Weinstein [**54**]). It is simple to check that the composition $E' = \Gamma_{(\Phi,\omega)} \circ E$ is transverse if and only if $\ker(\Phi,\omega) \cap E = \{0\}$. Similarly, the composition $F = F' \circ \Gamma_{(\Phi,\omega)}$ is transverse if and only if $\mathrm{ran}(\Phi,\omega) + F' = \mathbb{V}'$ (equivalently, if and only if $\mathrm{ran}(\Phi) + \mathrm{ran}(F') = V'$).

Remark 1.7. — As in the symplectic category [**30, 54**], one could consider morphisms given by arbitrary *Lagrangian relations*, i.e. Lagrangian subspaces $\Gamma \subset \mathbb{V}' \oplus \overline{\mathbb{V}}$ (see e.g. [**16**]). The graphs (11) of morphisms (Φ,ω) are exactly those Lagrangian relations preserving the 'polarization' (9), in the sense that $\Gamma \circ V^* = (V')^*$ (where the composition is transverse), see [**33**].

The (Φ,ω)-relation may also be interpreted in terms of the spinor representations of $\mathrm{Cl}(\mathbb{V})$ and $\mathrm{Cl}(\mathbb{V}')$:

Lemma 1.8. — *Suppose* $(\Phi,\omega)\colon \mathbb{V} \to \mathbb{V}'$ *is a morphism, and* $w \in \mathbb{V},\ w' \in \mathbb{V}'$. *Then*

$$w \sim_{(\Phi,\omega)} w' \quad\Leftrightarrow\quad \varrho(w)(e^{\omega}\Phi^*\psi') = e^{\omega}\Phi^*(\varrho(w')\psi'), \qquad \psi' \in \wedge(V')^*. \tag{13}$$

Proof. — This follows from $(\epsilon(\alpha)+\iota_v)(e^{\omega}\Phi^*\psi') = e^{\omega}(\epsilon(\alpha+\iota_v\omega)+\iota_v)\Phi^*\psi'$, for $v\oplus\alpha \in \mathbb{V}$. □

Lemma 1.9. — *Suppose* $(\Phi,\omega)\colon \mathbb{V} \to \mathbb{V}'$ *is a morphism, and* ψ' *is a pure spinor defining a Lagrangian subspace* F'. *Then* $\psi = e^{\omega}\Phi^*\psi'$ *is non-zero if and only if the composition* $F = F' \circ \Gamma_{(\Phi,\omega)}$ *is transverse, and in that case it is a pure spinor defining* F.

Proof. — Suppose $w \in F$, i.e. $w \sim_{(\Phi,\omega)} w'$ with $w' \in F' = N_{\psi'}$. Then $w \in N_{\psi}$ by Equation (13). Thus $F \subset N_{\psi}$. For $\psi \neq 0$, this is an equality since F is Lagrangian. □

Example 1.10. — Suppose $E, F \subset \mathbb{V}$ are Lagrangian, with defining pure spinors ϕ, ψ. Let $E^{\top}$ be the image of E under the map $v\oplus\alpha \mapsto v\oplus(-\alpha)$. Then $\phi^{\top}$ is a pure spinor defining $E^{\top}$. Consider the diagonal inclusion $\mathrm{diag}\colon V \to V \times V$, so that $\mathrm{diag}^*(\phi^{\top}\otimes\psi) = \phi^{\top}\wedge\psi$ is just the wedge product. The wedge product is non-zero if and only if the composition $E^{\top}\wedge F := (E^{\top}\times F)\circ\Gamma_{\mathrm{diag}}$ is transverse. This is the case, for instance, if E and F are transverse (since the top degree part of $\phi^{\top}\wedge\psi$ is non-zero in this case). Explicitly,

$$E^{\top}\wedge F = \{v\oplus\alpha \mid \exists v\oplus\alpha_1 \in E,\ v\oplus\alpha_2 \in F\colon \alpha = \alpha_2 - \alpha_1\}.$$

Note that $\mathrm{ran}(E^{\top}\wedge F) = \mathrm{ran}(E)\cap\mathrm{ran}(F)$, with 2-form the difference of the restrictions of the 2-forms on $\mathrm{ran}(E)$ and $\mathrm{ran}(F)$. Note also that $(A^{-\omega}(E))^{\top}\wedge(A^{-\omega}(F)) = E^{\top}\wedge F$ for all $\omega \in \wedge^2 V^*$.

This "wedge product" operation of Lagrangian subspaces was noticed independently by Gualtieri, see [**29**].

1.7. Dirac spaces. — A *Dirac space* is a pair (V, E), where V is a vector space and $E \subset \mathbb{V}$ is a Lagrangian subspace. As remarked in Section 1.4, E determines a subspace $Q = \mathrm{ran}(E) = \mathrm{pr}_V(E) \subset V$ together with a 2-form $\omega_Q \in \wedge^2 Q^*$,

$$\omega_Q(v, v') = \langle \alpha, v' \rangle = -\langle \alpha', v \rangle \tag{14}$$

for arbitrary lifts $v \oplus \alpha, v' \oplus \alpha' \in E$ of $v, v' \in Q$. The kernel of ω_Q is the subspace $\ker(E) = E \cap V$. Conversely, any subspace Q equipped with a 2-form ω_Q determines a Lagrangian subspace $E = \{v \oplus \alpha \in \mathbb{V} |\ v \in Q,\ \alpha|_Q = \omega_Q(v, \cdot)\}$. The gauge transformation $A^{-\omega}$ by a 2-form $\omega \in \wedge^2 V^*$ preserves Q, while ω_Q changes by the pull-back of ω to Q.

Definition 1.11. — Let $(\mathbb{V}, E)$ and $(\mathbb{V}', E')$ be Dirac spaces. A *Dirac morphism* $(\Phi, \omega)\colon (V, E) \to (V', E')$ is a morphism (Φ, ω) with $E' = \Gamma_{(\Phi,\omega)} \circ E$. It is called a *strong Dirac morphism*[(2)] if this composition is transverse, i.e.,

$$\ker(\Phi, \omega) \cap E = \{0\}.$$

Clearly, the composition of strong Dirac morphisms is again a strong Dirac morphism. Note that the definition of a Dirac morphism $(\Phi, \omega)\colon (V, E) \to (V', E')$ amounts to the existence of a linear map $\widehat{\mathfrak{a}}\colon E' \to E$, assigning to each $w' \in E'$ an element of E to which it is (Φ, ω)-related:

$$\widehat{\mathfrak{a}}(w') \sim_{(\Phi,\omega)} w' \qquad \forall w' \in E'. \tag{15}$$

The map $\widehat{\mathfrak{a}}$ is completely determined by its V-component

$$\mathfrak{a} = \mathrm{pr}_V \circ \widehat{\mathfrak{a}}\colon E' \to V,$$

since $\widehat{\mathfrak{a}}(v' \oplus \alpha') = v \oplus (\Phi^*\alpha' + \iota_v \omega)$ where $v = \mathfrak{a}(v' \oplus \alpha')$. Hence (Φ, ω) is a Dirac morphism if and only if there exists a map $\mathfrak{a}\colon E' \to V$, such that the corresponding map $\widehat{\mathfrak{a}}$ takes values in E.

Lemma 1.12. — *For a strong Dirac morphism $(\Phi, \omega)\colon (V, E) \to (V', E')$, the map $\widehat{\mathfrak{a}}$ satisfying* (15) *is unique. Its range is given by*

$$\mathrm{ran}(\widehat{\mathfrak{a}}) = E \cap \ker(\Phi, \omega)^{\perp}. \tag{16}$$

Proof. — The map $\widehat{\mathfrak{a}}$ associated to a Dirac morphism is unique up to addition of elements in $E \cap \ker(\Phi, \omega)$. Hence, it is unique precisely if the Dirac morphism is strong. Its range consists of all $w \in E$ which are (Φ, ω)-related to some element of $w' \in E'$. By (12), the subspace $\{w \in \mathbb{V} \,|\, \exists w' \in \mathbb{V}'\colon w \sim_{(\Phi,\omega)} w'\}$ is orthogonal to $\ker(\Phi, \omega)$. Hence, by a dimension count it coincides with $\ker(\Phi, \omega)^{\perp}$. On the other hand, if $w \in E$ lies in this subspace, it is automatic that $w' \in E'$ since $E' = \Gamma_{(\Phi,\omega)} \circ E$. $\square$

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(2) In the particular case when $\omega = 0$, Dirac morphisms are also called *forward Dirac maps* [**15, 16**], and strong Dirac morphisms are called *Dirac realizations* [**14**].

Example 1.13. — Let $E \subset \mathbb{V}$ be a Lagrangian subspace, and let ω_Q be the corresponding 2-form on $Q = \mathrm{ran}(E)$. Let $\iota_Q : Q \to V$ be the inclusion. Then $(\iota_Q, \omega_Q) \colon (Q, Q) \to (V, E)$ is a strong Dirac morphism. Equivalently $(\iota_Q, 0) \colon (Q, \mathrm{Gr}_{\omega_Q}) \to (V, E)$ is a strong Dirac morphism. Here $\mathfrak{a}(v \oplus \alpha) = \iota_Q(v)$.

Example 1.14. — Suppose $\pi \in \wedge^2 V$ and $\pi' \in \wedge^2 V'$. Then $(\Phi, 0) \colon (V, \mathrm{Gr}_\pi) \to (V', \mathrm{Gr}_{\pi'})$ is a Dirac morphism if and only if $\Phi(\pi) = \pi'$. It is automatically strong (since $\ker(\mathrm{Gr}_\pi) = 0$), with $\mathfrak{a}(v' \oplus \alpha') = \pi^\sharp(\Phi^* \alpha')$.

Proposition 1.15. — *Suppose $(\Phi, \omega) \colon (V, E) \to (V', E')$ is a Dirac morphism, and that F' is a Lagrangian subspace transverse to E'. Let ϕ be a pure spinor defining E, and ψ' a pure spinor defining F'. Then $\psi := e^\omega \Phi^* \psi'$ is non-zero, and is a pure spinor defining the backward image $F = F' \circ \Gamma_{(\Phi,\omega)}$. Moreover, the following are equivalent:*

a. *(Φ, ω) is a* strong *Dirac morphism,*
b. *the backward image F is transverse to E,*
c. *The pairing $(\phi, \psi)_{\wedge(V^*)} \in \det(V^*)$ is non-zero, that is, it is a volume form on V.*

Proof. — By (6), we may write $\psi' = e^{-\omega_{Q'}} \theta'$, where $\omega_{Q'}$ is a 2-form on $Q' = \mathrm{ran}(F')$, and $\theta' \in \wedge^{\mathrm{top}}(V'/\mathrm{ran}(F'))^*$. Identifying $(V'/\mathrm{ran}(F'))^*$ with the annihilator of $\mathrm{ran}(F')$, this gives

$$\begin{aligned} \psi \neq 0 &\Leftrightarrow \Phi^* \theta' \neq 0 \\ &\Leftrightarrow \ker(\Phi^*) \cap \mathrm{ann}(\mathrm{ran}(F')) = 0 \\ &\Leftrightarrow \{w' \in F' |\ 0 \sim_{(\Phi,\omega)} w'\} = \{0\}. \end{aligned}$$

(Indeed, $0 \sim_{(\Phi,\omega)} w'$ if and only if $w' = 0 \oplus \alpha'$ with $\Phi^* \alpha' = \{0\}$. Moreover $w' \in F' = (F')^\perp$ if and only if $\alpha' \in \mathrm{ann}(\mathrm{ran}(F'))$.) But the condition $0 \sim_{(\Phi,\omega)} w'$ implies that $w' \in E'$. Since $E' \cap F' = 0$ it follows that $\{w' \in F' |\ 0 \sim_{(\Phi,\omega)} w'\} = \{0\}$, hence $\psi \neq 0$. Lemma 1.9 shows that it is a pure spinor defining the backward image F.

$(a) \Leftrightarrow (b)$. By definition, $E \cap F$ consists of all $w \in E$ such that $w \sim_{(\Phi,\omega)} w'$ for some $w' \in F'$. Since $E' = \Gamma_{(\Phi,\omega)} \circ E$, this element w' also lies in E', and hence $w' = 0$. Thus,

$$E \cap F = E \cap \ker(\Phi, \omega),$$

which is zero precisely if the Dirac morphism (Φ, ω) is strong. $(b) \Leftrightarrow (c)$ is immediate from Theorem 1.3. □

1.8. Lagrangian splittings. — Suppose W is a vector space with split bilinear form. By a *Lagrangian splitting* of W we mean a direct sum decomposition $W = E \oplus F$ into transverse Lagrangian subspaces.

Lemma 1.16. — *Let W be a vector space with split bilinear form $\langle \cdot, \cdot \rangle$. There is a 1-1 correspondence between projection operators $\mathsf{p} \in \mathrm{End}(W)$ with the property $\mathsf{p} + \mathsf{p}^t = 1$, and Lagrangian splittings $W = E \oplus F$. (Here p^t is the transpose with respect to the inner product on W.)*

Proof. — A Lagrangian splitting of W into transverse Lagrangian subspaces is equivalent to a projection operator whose kernel and range are isotropic. For any projection operator $\mathsf{p} = \mathsf{p}^2$, the range $\mathrm{ran}(\mathsf{p})$ is isotropic if and only if $\mathsf{p}^t\mathsf{p} = 0$, while $\ker(\mathsf{p}) = \mathrm{ran}(1-\mathsf{p})$ is isotropic if and only if $(1-\mathsf{p})^t(1-\mathsf{p}) = 0$. If both the kernel and the range of p are isotropic, then

$$1 - (\mathsf{p} + \mathsf{p}^t) = (1-\mathsf{p})^t(1-\mathsf{p}) - \mathsf{p}^t\mathsf{p} = 0.$$

Conversely, if p is a projection operator with $\mathsf{p} + \mathsf{p}^t = 1$, then $\mathsf{p}^t\mathsf{p} = (1-\mathsf{p})\mathsf{p} = 0$, and similarly $(1-\mathsf{p})^t(1-\mathsf{p}) = 0$. □

Again, we specialize to the case $W = \mathbb{V}$. Suppose $\mathbb{V} = E \oplus F$ is a Lagrangian splitting, with associated projection operator p. The property $\mathsf{p} + \mathsf{p}^t = 1$ implies that there is a bivector $\pi \in \wedge^2 V$ defined by

$$\pi^\sharp(\alpha) = -\,\mathrm{pr}_V(\mathsf{p}(\alpha)), \quad \alpha \in V^*, \tag{17}$$

that is, $\pi(\alpha, \beta) = -\langle \mathsf{p}(\alpha), \beta\rangle = \langle \alpha, \mathsf{p}(\beta)\rangle, \quad \alpha, \beta \in V^*$. If $\{e_i\}$ is a basis of E and $\{f^i\}$ is the dual basis of F, then

$$\pi = \tfrac{1}{2}\,\mathrm{pr}_V(e_i) \wedge \mathrm{pr}_V(f^i). \tag{18}$$

The graph of the bivector π was encountered in Example 1.10 above:

Proposition 1.17. — *The graph of the bivector π is given by*

$$\mathrm{Gr}_\pi = E^\top \wedge F. \tag{19}$$

In particular, $\mathrm{ran}(\pi^\sharp) = \mathrm{ran}(E) \cap \mathrm{ran}(F)$, *and the symplectic 2-form on* $\mathrm{ran}(\pi^\sharp)$ *is the difference of the restrictions of the 2-forms on* $\mathrm{ran}(E), \mathrm{ran}(F)$. *If ϕ, ψ are pure spinors defining E, F, then*

$$\phi^\top \wedge \psi = e^{-\iota(\pi)}(\phi^\top \wedge \psi)^{[\mathrm{top}]}.$$

Proof. — Since both sides of (19) are Lagrangian subspaces, it suffices to prove the inclusion $\supset$. Let $v \oplus \alpha \in E^\top \wedge F$. Hence, there exist α_1, α_2 with $\alpha = \alpha_2 - \alpha_1$ and $v \oplus \alpha_1 \in E$, $v \oplus \alpha_2 \in F$. Thus $v \oplus \alpha_1 = -\mathsf{p}(\alpha)$, which implies that $\pi^\sharp(\alpha) = -\,\mathrm{pr}_V\,\mathsf{p}(\alpha) = v$. The description of $\mathrm{ran}\,\pi^\sharp = \mathrm{ran}(\mathrm{Gr}_\pi)$ is immediate from (19), see the discussion in Example 1.10. The formula for $\phi^\top \wedge \psi$ follows since both sides are pure spinors defining the Lagrangian subspace Gr_π, with the same top degree part. □

Proposition 1.18. — *Suppose $\mathbb{V} = E \oplus F$ is a Lagrangian splitting, defining a bivector π. If $\varepsilon \in \wedge^2 E$, so that $F_\varepsilon = A^{-\varepsilon} F$ is a new Lagrangian complement to E, the bivector π_ε for the splitting $E \oplus F_\varepsilon$ is given by*

$$\pi_\varepsilon = \pi + \mathrm{pr}_V(\varepsilon),$$

where $\mathrm{pr}_V \colon \wedge E \to \wedge V$ *is the algebra homomorphism extending the projection to V.*

Proof. — Let ϕ, ψ be pure spinors defining E, F. Then F_ε is defined by the pure spinor $\psi_\varepsilon = \varrho(e^{-\varepsilon})\psi$. Using Remark 1.5(a), we obtain

$$\phi^\top \wedge \psi_\varepsilon = \phi^\top \wedge \varrho(e^{-\varepsilon})\psi = e^{-\iota(\mathrm{pr}_V(\varepsilon))}\phi^\top \wedge \psi.$$

The claim now follows from (1.17). □

Proposition 1.19. — *Let $(\Phi,\omega)\colon (V,E) \to (V',E')$ be a strong Dirac morphism. Suppose $F' \in \mathrm{Lag}(\mathbb{V}')$ is transverse to E', and F is its backward image under (Φ,ω). Then the bivectors for the Lagrangian splittings $\mathbb{V} = E \oplus F$ and $\mathbb{V}' = E' \oplus F'$ are Φ-related:*

$$\Phi(\pi) = \pi'.$$

Proof. — To prove $\Phi(\pi) = \pi'$, we have to show that $(\Phi, 0)\colon (V, \mathrm{Gr}_\pi) \to (V', \mathrm{Gr}_{\pi'})$ is a Dirac morphism:

$$\Gamma_{(\Phi,0)} \circ (E^\top \wedge F) = (E')^\top \wedge F'.$$

Since both sides are Lagrangian, it suffices to prove the inclusion $\supset$. If $v' \oplus \alpha' \in (E')^\top \wedge F'$, then $\alpha' = \alpha'_2 - \alpha'_1$, where $v' \oplus \alpha'_1 \in E'$ and $v' \oplus \alpha'_2 \in F'$. Since (Φ,ω) is a strong Dirac morphism for E, E', there is a unique element $v \oplus \alpha_1 \in E$ such that $v' = \Phi(v)$, $\Phi^*(\alpha'_1) = \alpha_1 + \iota_v\omega$. Let $\alpha_2 = \Phi^*(\alpha'_2) - \iota_v\omega$. Then $v \oplus \alpha_2 \in F$ since $v \oplus \alpha_2 \sim_{(\Phi,\omega)} v' \oplus \alpha_2$. Hence $v \oplus \Phi^*(\alpha') = v \oplus (\alpha_2 - \alpha_1) \in E^\top \wedge F$, proving that $v' \oplus \alpha' \in \Gamma_{(\Phi,0)} \circ (E^\top \wedge F)$. □

We next explain how a splitting $\mathbb{V}' = E' \oplus F'$ may be 'pulled back' under a linear map $\Phi\colon V \to V'$, given a bivector $\pi \in \wedge^2 V$ and a linear map $\mathfrak{a}\colon E' \to V$ satisfying suitable compatibility relations.

Theorem 1.20. — *Suppose that $\Phi\colon V \to V'$ is a linear map and $\omega \in \wedge^2 V^*$ a 2-form. Given a Lagrangian splitting $\mathbb{V}' = E' \oplus F'$, with associated projection $\mathsf{p}' \in \mathrm{End}(\mathbb{V}')$, there is a 1-1 correspondence between*

(i) *Lagrangian subspaces $E \subset \mathbb{V}$ such that $(\Phi,\omega)\colon (V,E) \to (V',E')$ is a strong Dirac morphism, and*
(ii) *Bivectors $\pi \in \wedge^2 V$ together with linear maps $\mathfrak{a}\colon E' \to V$, satisfying $\Phi \circ \mathfrak{a} = \mathrm{pr}_{V'}\big|_{E'}$ and*

$$\pi^\sharp \circ \Phi^* = -\mathfrak{a} \circ \mathsf{p}'\big|_{(V')^*}. \tag{20}$$

Under this correspondence, π is the bivector defined by the splitting $\mathbb{V} = E \oplus F$, where F is the backward image of F', and $\mathfrak{a}$ is the linear map defined by the strong Dirac morphism (Φ,ω) (see (15)).

Proof. — "$(i) \Rightarrow (ii)$". By Proposition 1.15, we know that the backward image F of F' is transverse to E. Let p and p' be the projections defined by the Lagrangian splittings $\mathbb{V} = E \oplus F$ and $\mathbb{V}' = E' \oplus F'$, and π, π' the corresponding bivectors. As in (15), the strong Dirac morphism (Φ,ω) defines a linear map $\widehat{\mathfrak{a}}\colon E' \to E$, taking

$w' \in E'$ to the unique element $w \in E$ such that $w \sim_{(\Phi,\omega)} w'$. We claim that for all $w \in \mathbb{V},\ w' \in V'$,

$$w \sim_{(\Phi,\omega)} w' \Rightarrow \mathsf{p}(w) = \widehat{\mathfrak{a}}(\mathsf{p}'(w')). \tag{21}$$

Indeed, let $w_1 = \mathsf{p}(w) \in E$, so that $w_2 = w - w_1 \in F$. There is a (unique) element $w_2' \in F'$ with $w_2 \sim_{(\Phi,\omega)} w_2'$, so let $w_1' = w' - w_2'$. Since $w_2 \sim_{(\Phi,\omega)} w_2'$, it follows that $w_1 \sim_{(\Phi,\omega)} w_1'$. Hence $w_1' \in E'$ by definition of E'. It follows that $\mathsf{p}(w) = w_1 = \widehat{\mathfrak{a}}(w_1') = \widehat{\mathfrak{a}}(\mathsf{p}'(w'))$, as claimed. In particular, since $\Phi^*\alpha' \sim_{(\Phi,\omega)} \alpha'$ for $\alpha' \in V'$, (21) implies that

$$\pi^\sharp(\Phi^*\alpha') = -\operatorname{pr}_V(\mathsf{p}(\Phi^*\alpha')) = -\operatorname{pr}_V(\widehat{\mathfrak{a}}(\mathsf{p}'(\alpha'))) = -\mathfrak{a}(\mathsf{p}'(\alpha')),\quad \alpha' \in (V')^*$$

where $\mathfrak{a} = \operatorname{pr}_V \circ \widehat{\mathfrak{a}}$.

"$(i) \Leftarrow (ii)$". Our aim is to construct the projection p with kernel $F := F' \circ \Gamma_{(\Phi,\omega)}$ and range E. We define p by the following equations, for $v, v_1, v_2 \in V$ and $\alpha, \alpha_1, \alpha_2 \in V^*$:

$$\begin{aligned}
\langle \mathsf{p}(v_1), v_2\rangle &= \langle \mathsf{p}'(\Phi(v_1)), \Phi(v_2)\rangle,\\
\langle \mathsf{p}(\alpha_1), \alpha_2\rangle &= -\pi(\alpha_1, \alpha_2),\\
\langle \mathsf{p}(v), \alpha\rangle &= \langle \mathfrak{a}^*\alpha,\ \Phi(v)\rangle + \pi(\iota_v\omega, \alpha),\\
\langle \mathsf{p}(\alpha), v\rangle &= \langle \alpha, v\rangle - \langle \mathfrak{a}^*\alpha,\ \Phi(v)\rangle - \pi(\iota_v\omega, \alpha),
\end{aligned}$$

where $\mathfrak{a}^* \colon V^* \to (E')^* = F'$ is the dual map to $\mathfrak{a}$. The linear map p defined in this way has the property $\mathsf{p} + \mathsf{p}^t = 1$. We claim that this linear map satisfies (21), where $\widehat{\mathfrak{a}} \colon E' \to \mathbb{V}$ is defined as follows,

$$\widehat{\mathfrak{a}}(w') = \mathfrak{a}(w') \oplus (\Phi^* \operatorname{pr}_{(V')^*}(w') - \iota_{\mathfrak{a}(w')}\omega).$$

For $w = v \oplus \iota_v\omega,\ w' = \Phi(v) \oplus 0$, (21) is easily checked using the definition of p. Hence it suffices to consider the case $w = \Phi^*\alpha',\ w' = \alpha'$ with $\alpha' \in (V')^*$. For all $v \in V$, using the definition of p and $\Phi \circ \mathfrak{a} = \operatorname{pr}_{V'}|_{E'}$, i.e., $\mathfrak{a}^* \circ \Phi^* = (\mathsf{p}')^t|_{(V')^*}$, we have:

$$\begin{aligned}
\langle \mathsf{p}(\Phi^*\alpha'), v\rangle &= \langle \alpha', \Phi(v)\rangle - \langle (\mathsf{p}')^t\alpha', \Phi(v)\rangle - \pi(\iota_v\omega, \Phi^*\alpha')\\
&= \langle \mathsf{p}'\alpha', \Phi(v)\rangle + \pi(\Phi^*\alpha', \iota_v\omega)\\
\langle \widehat{\mathfrak{a}}(\mathsf{p}'(\alpha')), v\rangle &= \langle \Phi^* \operatorname{pr}_{(V')^*} \mathsf{p}'(\alpha'), v\rangle - \omega(\mathfrak{a}(\mathsf{p}'(\alpha')), v)\\
&= \langle \mathsf{p}'\alpha', \Phi(v)\rangle + \omega(\pi^\sharp(\Phi^*\alpha'), v)
\end{aligned}$$

which shows $\langle \mathsf{p}(\Phi^*\alpha'), v\rangle = \langle \widehat{\mathfrak{a}}(\mathsf{p}'(\alpha')), v\rangle$. Similarly, for $\beta \in V^*$ we have, by (20),

$$\langle \mathsf{p}(\Phi^*\alpha'), \beta\rangle = -\langle \pi^\sharp(\Phi^*\alpha'), \beta\rangle = \langle \widehat{\mathfrak{a}}(\mathsf{p}'(\alpha')), \beta\rangle.$$

This proves (21). Equation (21) applies in particular to all elements $w \in F$, since these are by definition (Φ, ω)-related to elements $w' \in F'$. We hence see that $\mathsf{p}(w) = 0$ for all $w \in F$. This proves that $F \subset \ker(\mathsf{p})$. Taking orthogonals, $\operatorname{ran}(\mathsf{p}^t) \subset F$. In particular, the range of p^t is isotropic, i.e. $\mathsf{p}\mathsf{p}^t = 0$, and hence $\mathsf{p} - \mathsf{p}^2 = \mathsf{p}(1 - \mathsf{p}) = \mathsf{p}\mathsf{p}^t = 0$. Thus p is a projection. As before, we see that $\ker \mathsf{p} = \operatorname{ran}(1 - \mathsf{p})$ is isotropic as well, hence $F = \ker(\mathsf{p})$ since F is maximal isotropic. It remains to show that the Lagrangian subspace $E := \operatorname{ran}(\mathsf{p})$ satisfies $\Gamma_{(\Phi,\omega)} \circ E \subset E'$. Suppose $w \sim_{(\Phi,\omega)} w'$ for some $w \in E$. By (21), we also have $w = \mathsf{p}(w) \sim_{(\Phi,\omega)} \mathsf{p}'(w')$. Thus $0 \sim_{(\Phi,\omega)} (w' - \mathsf{p}'(w')) = (\mathsf{p}')^t(w')$.

Observe that $\mathrm{ran}(\Phi) \supset \Phi(\mathfrak{a}(E')) = \mathrm{ran}(E')$. Hence $\ker(\Phi^*) \subset \mathrm{ann}(\mathrm{ran}(E'))$. Since $E' \cap F' = 0$, it follows that

$$\ker(\Phi^*) \cap \mathrm{ann}(\mathrm{ran}(F')) = 0. \tag{22}$$

Using Equation (22), the relation $0 \sim_{(\Phi,\omega)} (\mathsf{p}')^t(w') \in F'$ implies that $(\mathsf{p}')^t(w') = 0$, i.e. $w' \in E'$. □

The proof shows that $\mathsf{p}|_V = \widehat{\mathfrak{a}} \circ \mathsf{p}' \circ \Phi$, whereas $h := \mathsf{p}|_{V^*} : V^* \to E$ is given by

$$h(\alpha) = (-\pi^\sharp(\alpha)) \oplus (\alpha - \Phi^* \operatorname{pr}_{(V')^*} \mathfrak{a}^*(\alpha) - \iota(\pi^\sharp(\alpha))\omega). \tag{23}$$

It follows that $E = \mathrm{ran}(\widehat{\mathfrak{a}}) + \mathrm{ran}(h)$. Projecting to V, it follows in particular that

$$\mathrm{ran}(E) = \mathrm{ran}(\mathfrak{a}) + \mathrm{ran}(\pi^\sharp). \tag{24}$$

2. Pure spinors on manifolds

A pure spinor on a manifold is simply a differential form whose restriction to any point is a pure spinor on the tangent space. The following discussion is carried out in the category of real manifolds and C^∞ vector bundles, but works equally well for complex manifolds with holomorphic vector bundles.

2.1. Dirac structures. — For any manifold M, we denote by $\mathbb{T}M = TM \oplus T^*M$ the direct sum of the tangent and cotangent bundles, with fiberwise inner product $\langle \cdot, \cdot \rangle$. The fiberwise Clifford action defines a bundle map

$$\varrho\colon \mathrm{Cl}(\mathbb{T}M) \to \mathrm{End}(\wedge T^*M). \tag{25}$$

The same symbol will denote the action of sections of $\mathrm{Cl}(\mathbb{T}M)$ on sections of $\wedge T^*M$, i.e. differential forms. The bilinear pairing will be denoted by

$$(\cdot, \cdot)_{\wedge T^*M} \colon \wedge T^*M \otimes \wedge T^*M \to \det(T^*M), \tag{26}$$

and the same notation will be used for sections. Thus $(\phi, \phi')_{\wedge T^*M} = (\phi^\top \wedge \phi')^{[\mathrm{top}]}$ for differential forms $\phi, \phi' \in \Gamma(\wedge T^*M) = \Omega(M)$. An *almost Dirac structure* on M is a smooth Lagrangian subbundle $E \subset \mathbb{T}M$. The pair (M, E) is called an almost Dirac manifold. A *pure spinor defining* E is a nonvanishing differential form $\phi \in \Omega(M)$ such that $\phi|_m$ is a pure spinor defining E_m, for all m. Equivalently, ϕ is a nonvanishing section of the line bundle $(\wedge T^*M)^E$. Thus E is globally represented by a pure spinor if and only if the line bundle $(\wedge T^*M)^E$ is orientable. (Otherwise, one may still use pure spinors to describe E locally.)

Let $\eta \in \Omega^3(M)$ be a closed 3-form. A direct computation shows that the spinor representation defines a bilinear bracket $[\![\cdot, \cdot]\!]_\eta \colon \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ by the condition:

$$\varrho([\![x_1, x_2]\!]_\eta)\psi = [[\mathrm{d} + \eta, \varrho(x_1)], \varrho(x_2)]\psi, \quad \psi \in \Omega(M),\ x_i \in \Gamma(\mathbb{T}M), \tag{27}$$

where the brackets on the right-hand side are graded commutators of operators on $\Omega(M)$. The bracket $[\![\cdot,\cdot]\!]_\eta$ is the *η-twisted Courant bracket* [**35, 50**]. [(3)] (For more on the definition of $[\![\cdot,\cdot]\!]_\eta$ as a 'derived bracket', see e.g. [**9, 36, 48**].) The operator on $\Omega(M)$ defined by

$$[\varrho(x_1),[\varrho(x_2),[\varrho(x_3),\mathrm{d}+\eta]]]$$

is multiplication by a function

$$\Upsilon(x_1,x_2,x_3)=-\langle[\![x_3,x_2]\!]_\eta,\,x_1\rangle\in C^\infty(M). \tag{28}$$

Given an almost Dirac structure $E\subset\mathbb{T}M$, let Υ^E denote the restriction of the trilinear form $(x_1,x_2,x_3)\mapsto\Upsilon(x_1,x_2,x_3)$ to the sections of E. In contrast to Υ, the trilinear form Υ^E is *tensorial* and *skew-symmetric*. The resulting element

$$\Upsilon^E\in\Gamma(\wedge^3E^*)$$

is called the *η-twisted Courant tensor* of E.

Definition 2.1. — A *Dirac structure* on a manifold M is an almost Dirac structure E together with a closed 3-form η such that its η-twisted Courant tensor vanishes: $\Upsilon^E=0$. The triple (M,E,η) is called a *Dirac manifold.*

For E an almost Dirac structure one can always choose a complementary almost Dirac structure F such that $E\oplus F=\mathbb{T}M$. (This is parallel to a well-known fact from symplectic geometry [**51**, Proposition 8.2], with a similar proof.) As a vector bundle, $F\cong E^*$ with pairing induced by the inner product on $\mathbb{T}M$. We have:

Proposition 2.2. — *Let E be an almost Dirac structure on M, and F be a complementary almost Dirac structure. Suppose E is represented by a pure spinor $\phi\in\Omega(M)$. Then there is a unique section $\sigma^E\in\Gamma(E^*)$ (depending on ϕ) such that*

$$(d+\eta)\phi=\varrho(-\Upsilon^E+\sigma^E)\phi.$$

Here we view Υ^E and σ^E as sections of $\wedge F\subset\mathrm{Cl}(\mathbb{T}M)$.

Proof. — Choose a Lagrangian subbundle F complementary to E. Since

$$\Gamma(\wedge F)\to\Omega(M),\ x\mapsto\varrho(x)\phi$$

is an isomorphism, there is a unique odd element $x\in\Gamma(\wedge F)\subset\Gamma(\mathbb{T}M)$ such that $(\mathrm{d}+\eta)\phi=\varrho(x)\phi$. To see that x has filtration degree 3, let x_1,x_2,x_3 be three sections of E. Since $\varrho(x_i)\phi=0$, it follows that

$$\begin{aligned}\varrho([x_1,[x_2,[x_3,x]]])\phi&=[[[\varrho(x_1),[\varrho(x_2),[\varrho(x_3),\varrho(x)]]]\phi=\varrho(x_1x_2x_3)\varrho(x)\phi\\&=\varrho(x_1x_2x_3)(\mathrm{d}+\eta)\phi=[[[\varrho(x_1),[\varrho(x_2),[\varrho(x_3),\mathrm{d}+\eta]]]\phi=\Upsilon^E(x_1,x_2,x_3)\phi,\end{aligned}$$

(3) This definition agrees with the non-skew symmetric version of the Courant bracket [**40, 50**], called the *Dorfman bracket* in [**28**]; the η-term in the bracket, however, differs from the one in [**50**] by a sign.

proving that the Clifford commutator $[x_1, [x_2, [x_3, x]]] = \iota(x_1)\iota(x_2)\iota(x_3)x$ (contraction of $x \in \Gamma(\wedge(E^*))$ with sections of E) is a scalar. This implies that x has filtration degree 3, and that the degree 3 part of x is $-\Upsilon^E$. □

We hence see that an almost Dirac structure $E \subset \mathbb{T}M$ is integrable if and only if

$$(\mathrm{d} + \eta)\phi \in \varrho(\mathbb{T}M)\phi,$$

for any pure spinor $\phi \in \Omega(M)$ (locally) representing E. The characterization of the integrability condition $\Upsilon^E = 0$ in terms of pure spinors was observed by Gualtieri **[28]**, see also **[9]**.

Examples of Dirac structures (for a given η) include graphs of 2-forms $\omega \in \Omega(M)$ with $\mathrm{d}\omega = \eta$, as well as graphs of bivector fields $\pi \in \mathfrak{X}^2(M)$ defining *η-twisted Poisson structures* **[35, 50]** in the sense that $\frac{1}{2}[\pi, \pi]_{\mathrm{Sch}} + \pi^\sharp(\eta) = 0$. One may also consider *complex* Dirac structures on M, given by complex Lagrangian subbundles $E \subset \mathbb{T}M^{\mathbb{C}}$ satisfying $\Upsilon^E = 0$. The defining pure spinors are complex-valued differential forms ϕ on M, given as nonvanishing sections of $(\wedge T^*M^{\mathbb{C}})^E$. If E is a Dirac structure, then its image E^c under the complex conjugation mapping is a Dirac structure defined by the complex conjugate spinor ϕ^c. E is called a *generalized complex structure* **[28, 32]** if $E \cap E^c = 0$.

Suppose $E \subset \mathbb{T}M$ is a Dirac structure. The vanishing of the Courant tensor implies that E is a *Lie algebroid*, with anchor given by the natural projection on TM, and Lie bracket $[\cdot, \cdot]_E$ on $\Gamma(E)$ given by the restriction of the Courant bracket $[\![\cdot, \cdot]\!]_\eta$. From the theory of Lie algebroids, it follows that the generalized distribution $\mathrm{ran}(E)$ is integrable (in the sense of Sussmann) **[24]**. The generalized foliation having $\mathrm{ran}(E)$ as its tangent distribution is called the *Dirac foliation*. For any leaf $Q \subset M$ of the Dirac foliation, the collection of 2-forms on T_mQ (defined as in (14)) defines a smooth 2-form $\omega_Q \in \Omega^2(Q)$ with

$$\mathrm{d}\omega_Q = i_Q^*\eta,$$

where $i_Q \colon Q \to M$ is the inclusion (for a proof, see e.g. **[47**, Proposition 6.10**]**). If E is the graph of a Poisson bivector π (with $\eta = 0$), this is the usual symplectic foliation.

2.2. Dirac morphisms. — Suppose $\Phi \colon M \to M'$ is a smooth map, and $\omega \in \Omega^2(M)$ is a 2-form. As in the linear case, we view the pair (Φ, ω) as a 'morphism', with composition rule (10). Given sections $x \in \Gamma(\mathbb{T}M)$ and $x' \in \Gamma(\mathbb{T}M')$, we will write

$$x \sim_{(\Phi,\omega)} x' \;\Leftrightarrow\; \forall m \in M \colon x_m \sim_{((d\Phi)_m, \omega_m)} x'_{\Phi(m)}.$$

In terms of the spinor representation, this is equivalent to the condition

$$e^\omega \Phi^*(\varrho(x')\psi') = \varrho(x)(e^\omega \Phi^*(\psi')), \quad \psi' \in \Omega(M').$$

Using the definition (27) of the Courant bracket as a derived bracket, one obtains:

***Lemma 2.3* (Stienon-Xu).** — **[53**, Lemma 2.2**]** *Let M, M' be manifolds with closed 3-forms η, η', $\Phi \colon M \to M'$ a smooth map, and $\omega \in \Omega^2(M)$ a 2-form such that $\Phi^*\eta' =$*

$\eta + d\omega$. *Then*

$$x_i \sim_{(\Phi,\omega)} x_i',\ i=1,2\ \Rightarrow [\![x_1,x_2]\!]_\eta \sim_{(\Phi,\omega)} [\![x_1',x_2']\!]_{\eta'}.$$

That is, the morphism $(\Phi,\omega)\colon M \to M'$ intertwines both the inner product and the (η- resp. η'-twisted) Courant brackets on $\mathbb{T}M$ and $\mathbb{T}M'$.

Definition 2.4. — a. Suppose (M,E) and (M',E') are almost Dirac manifolds. A morphism $(\Phi,\omega)\colon M \to M'$ is called a *(strong) almost Dirac morphism* $(\Phi,\omega)\colon (M,E) \to (M,E')$ if $((\mathrm{d}\Phi)_m,\omega_m)\colon (T_mM,E_m) \to (T_{\Phi(m)}M',E'_{\Phi(m)})$ is a linear (strong) Dirac morphism for all $m \in M$.

b. Suppose (M,E,η) and (M',E',η') are Dirac manifolds. A (strong) almost Dirac morphism $(\Phi,\omega)\colon M \to M'$ is called a *(strong) Dirac morphism* $(\Phi,\omega)\colon (M,E,\eta) \to (M',E',\eta')$ if $\eta + \mathrm{d}\omega = \Phi^*\eta'$.

For $\omega = 0$, strong Dirac morphisms coincide with the *Dirac realizations* of [**14**].

Example 2.5. — If (M,E,η) is a Dirac manifold, then so is $(M,A^{-\omega}(E),\eta+\mathrm{d}\omega)$, for any 2-form ω, and (id_M,ω) is a Dirac morphism between the two. The Dirac structures E and $A^{-\omega}(E)$ are isomorphic as Lie algebroids; in particular, they define the same Dirac foliation. However, the 2-forms on the leaves of this foliation change by the pull-back of ω.

Example 2.6. — Any manifold M can be trivially viewed as a Dirac manifold $M = (M,TM,0)$. A strong Dirac morphism from M to pt is then the same thing as a symplectic 2-form on M. More generally, strong Dirac morphisms $M \to N$ are (special types of) symplectic fibrations.

Example 2.7. — If (M,E,η) is a Dirac manifold, and $Q \subset M$ is a leaf of the associated foliation of M, then the inclusion map defines a strong Dirac morphism $(\iota_Q,\omega_Q)\colon (Q,TQ,0) \to (M,E,\eta)$.

From the linear case, it follows that a strong almost Dirac morphism gives rise to a bundle map

$$\widehat{\mathfrak{a}}\colon \Phi^*E' \to E.$$

This is indeed a smooth bundle map: the projection $\mathbb{T}M \oplus \Phi^*\mathbb{T}M' \to \Phi^*\mathbb{T}M'$ restricts to a bundle isomorphism $\Gamma_\Phi \cap (E \oplus \Phi^*\mathbb{T}M') \to \Phi^*E'$, and $\widehat{\mathfrak{a}}$ is the inverse of this bundle isomorphism followed by the projection to $\mathbb{T}M$. We let

$$\mathfrak{a} = \mathrm{pr}_{TM} \circ \widehat{\mathfrak{a}}\colon\ \Phi^*E' \to \mathrm{ran}(E) \subset TM \tag{29}$$

Proposition 2.8. — *Suppose $(\Phi,\omega)\colon (M,E,\eta) \to (M',E',\eta')$ is a strong Dirac morphism. Then the induced bundle map $\widehat{\mathfrak{a}}\colon \Phi^*E' \to E$ is a* comorphism of Lie algebroids [**43**]. *That is, it is compatible with the anchor maps in the sense that*

$$d\Phi \circ \mathfrak{a} = \mathrm{pr}_{\Phi^*TM'}|_{\Phi^*E'},$$

and the induced map on sections

$$\widehat{\mathfrak{a}}\colon \Gamma(E') \to \Gamma(E),\ (\widehat{\mathfrak{a}}(x'))_m = \widehat{\mathfrak{a}}(x'_{\Phi(m)})$$

preserves brackets.

Proof. — Compatibility with the anchor is obvious. If x'_1, x'_2 are section of E', then (using Lemma 2.3) both $\widehat{\mathfrak{a}}(\Phi^*[x'_1, x'_2]_{E'})$ and $[\widehat{\mathfrak{a}}(\Phi^* x'_1), \widehat{\mathfrak{a}}(\Phi^* x'_2)]_E$ are sections of E which are (Φ, ω)-related to $[x'_1, x'_2]_{E'}$. Hence their difference is (Φ, ω)-related to 0. Since (Φ, ω) is a strong Dirac morphism, it follows that the difference is in fact 0. □

The second part of Proposition 2.8 shows that (29) defines a Lie algebra homomorphism $\mathfrak{a}\colon \Gamma(E') \to \mathfrak{X}(M)$. That is, the strong Dirac morphism defines an '*action*' of the Lie algebroid E' on the manifold M.

2.3. Bivector fields. — From the linear theory, we see that any Lagrangian splitting $\mathbb{T}M = E \oplus F$ defines a bivector field π on M. Furthermore,

$$e^{-\iota(\pi)}(\phi^\top \wedge \psi)^{[\mathrm{top}]} = \phi^\top \wedge \psi$$

for any pure spinors ϕ, ψ defining E, F. Recall that $(\phi^\top \wedge \psi)^{[\mathrm{top}]}$ is a volume form on M.

For an arbitrary volume form μ on M, and any bivector field $\pi \in \mathfrak{X}^2(M)$, one has the formula [**26**]

$$\mathrm{d}(e^{-\iota(\pi)}\mu) = \iota\big(-\tfrac{1}{2}[\pi, \pi]_{\mathrm{Sch}} + X_\pi\big)(e^{-\iota(\pi)}\mu). \tag{30}$$

Here $[\cdot, \cdot]_{\mathrm{Sch}}$ is the Schouten bracket on multivector fields, and X_π is the vector field on M defined by $\mathrm{d}\iota(\pi)\mu = -\iota(X_\pi)\mu$. If π is a Poisson bivector field, then $X_\pi \in \mathfrak{X}(M)$ is called the *modular vector field* of π with respect to the volume form μ [**56**]. (See [**37**] for modular vector fields for *twisted* Poisson structures.)

Theorem 2.9. — *Let π be the bivector field defined by the Lagrangian splitting $\mathbb{T}M = E \oplus F$. Let $\Upsilon^E \in \Gamma(\wedge^3 F)$ and $\Upsilon^F \in \Gamma(\wedge^3 E)$ be the Courant tensor fields of E, F.*

a) *The Schouten bracket of π with itself is given by the formula*

$$\tfrac{1}{2}[\pi, \pi]_{\mathrm{Sch}} = \mathrm{pr}_{TM}(\Upsilon^E) + \mathrm{pr}_{TM}(\Upsilon^F),$$

where $\mathrm{pr}_{TM}\colon \wedge E \to \wedge TM$ is the algebra homomorphism extending the projection $E \to TM$, and similarly for $\mathrm{pr}_{TM}\colon \wedge F \to \wedge TM$.

b) *Given pure spinors $\phi, \psi \in \Omega(M)$ defining E, F, let $\sigma^E \in \Gamma(F)$ and $\sigma^F \in \Gamma(E)$ be the unique sections such that*

$$(d + \eta)\phi = \varrho(-\Upsilon^E + \sigma^E)\phi, \qquad (d + \eta)\psi = \varrho(-\Upsilon^F + \sigma^F)\psi.$$

Then the vector field X_π defined using the volume form $\mu = (\phi^\top \wedge \psi)^{[\mathrm{top}]}$ is given by

$$X_\pi = \mathrm{pr}_{TM}(\sigma^F) - \mathrm{pr}_{TM}(\sigma^E).$$

Proof. — We may assume that E, F are globally defined by pure spinors ϕ, ψ. Using Remark 1.5(a), we have

$$\begin{aligned}
\mathrm{d}(\phi^\top \wedge \psi) &= (-1)^{|\phi|}\big(\phi^\top \wedge \mathrm{d}\psi + (\mathrm{d}\phi)^\top \wedge \psi\big)\\
&= (-1)^{|\phi|}\big(\phi^\top \wedge (\mathrm{d}+\eta)\psi + ((\mathrm{d}+\eta)\phi)^\top \wedge \psi\big)\\
&= (-1)^{|\phi|}\big(\phi^\top \wedge (\varrho(-\Upsilon^F + \sigma^F)\psi) + (\varrho(-\Upsilon^E + \sigma^E)\phi)^\top \wedge \psi\big)\\
&= \iota(\mathrm{pr}_{TM}(-\Upsilon^F + \sigma^F) + \mathrm{pr}_{TM}(-\Upsilon^E - \sigma^E))(\phi^\top \wedge \psi).
\end{aligned}$$

On the other hand, $\phi^\top \wedge \psi = e^{-\iota(\pi)}\mu$ gives, by (30),

$$\mathrm{d}(\phi^\top \wedge \psi) = \iota(-\tfrac{1}{2}[\pi,\pi]_{\mathrm{Sch}} + X_\pi)(\phi^\top \wedge \psi).$$

Applying the star operator $\star$ for μ, and using that $\star(\phi^\top \wedge \psi)$ is invertible, it follows that

$$\mathrm{pr}_{TM}(-\Upsilon^F + \sigma^F) + \mathrm{pr}_{TM}(-\Upsilon^E - \sigma^E) = -\tfrac{1}{2}[\pi,\pi]_{\mathrm{Sch}} + X_\pi. \qquad \square$$

As a special case, if both E, F are Dirac structures (i.e. integrable), then the corresponding bivector field π satisfies $[\pi,\pi]_{\mathrm{Sch}} = 0$, i.e., it is a Poisson structure. The symplectic leaves of π are the intersections of the leaves of the Dirac structures E with those of F. The fact that transverse Dirac structures (or equivalently *Lie bialgebroids*) define Poisson structures goes back to Mackenzie-Xu [**44**].

Proposition 2.10. — *Suppose $(\Phi,\omega)\colon (M,E) \to (M',E')$ is an almost Dirac morphism, and let $F' \subset \mathbb{T}M'$ be a Lagrangian subbundle complementary to E'. Then there is a smooth Lagrangian subbundle $F \subset \mathbb{T}M$ complementary to E, with the property that for all $m \in M$, F_m is the backward image of $F'_{\Phi(m)}$ under $(d_m\Phi, \omega_m)$. Furthermore:*

a. *The bivector fields π, π' defined by the splittings $\mathbb{T}M = E \oplus F$ and $\mathbb{T}M' = E' \oplus F'$ satisfy*
$$\pi \sim_\Phi \pi',$$
i.e. $(d\Phi)_m \pi_m = \pi'_{\Phi(m)}$ for all $m \in M$.

b. *The Courant tensors $\Upsilon^F \in \Gamma(\wedge^3 E)$ and $\Upsilon^{F'} \in \Gamma(\wedge^3 E')$ are related by*
$$\Upsilon^F = \widehat{\mathfrak{a}}(\Phi^*\Upsilon^{F'}),$$
using the extension of $\widehat{\mathfrak{a}}\colon \Gamma(\Phi^*E') \to \Gamma(E)$ to the exterior algebras.

c. *The bivector field π satisfies*
$$\tfrac{1}{2}[\pi,\pi]_{\mathrm{Sch}} = \mathfrak{a}(\Phi^*\Upsilon^{F'}) + \mathrm{pr}_{TM}(\Upsilon^E),$$
using the extension of $\mathfrak{a}\colon \Gamma(\Phi^*E') \to \Gamma(TM)$ to the exterior algebras.

d.
$$\pi^\sharp \circ \Phi^* = -\mathfrak{a} \circ \mathsf{p}'\colon T^*M' \to TM,$$
where $\mathsf{p}'\colon \mathbb{T}M' \to E'$ is the projection along F'.

e. *If ψ' is a pure spinor defining F', and $\psi = e^{\omega}\Phi^*\psi'$ the corresponding pure spinor defining F, the sections σ^F, $\sigma^{F'}$ are related by $\sigma_F = \widehat{\mathfrak{a}}(\Phi^*\sigma_F)$, that is,*

$$\sigma^F \sim_{(\Phi,\omega)} \sigma^{F'}.$$

Proof. — Let $\psi' \in \Omega(M')$ be a pure spinor (locally) representing F'. From the linear case (Proposition 1.15), it follows that $\psi = e^{\omega}\Phi^*\psi'$ is non-zero everywhere, and is a pure spinor representing a Lagrangian subbundle $F \subset \mathbb{T}M$ transverse to E. Now (a) follows from the linear case, see Proposition 1.19. We next verify (b), at any given point $m \in M$. Let $m' = \Phi(m)$. Given $(x_i)_m \in F_m$ for $i = 1, 2, 3$, let $(x_i')_{m'} \in F'_{m'}$ with

$$(x_i)_m \sim_{((d\Phi)_m,\ \omega_m)} (x_i')_{m'}.$$

Choose sections $x_i \in \Gamma(F)$, $x_i' \in \Gamma(F')$ extending the given values at m, m'. We have to show $\Upsilon^F(x_1, x_2, x_3)|_m = \Upsilon^{F'}(x_1', x_2', x_3')|_{m'}$. We calculate:

$$\Upsilon^F(x_1, x_2, x_3)\ \psi = \varrho(x_1x_2x_3)\,(\mathrm{d} + \eta)(e^{\omega}\Phi^*\psi') = \varrho(x_1x_2x_3)\, e^{\omega}\Phi^*(\mathrm{d} + \eta')\psi'$$

On the other hand,

$$(\Phi^*\Upsilon^{F'}(x_1', x_2', x_3'))\ \psi = e^{\omega}\Phi^*\Upsilon^{F'}(x_1', x_2', x_3')\ \psi' = e^{\omega}\Phi^*\varrho(x_1'x_2'x_3')\,(\mathrm{d} + \eta')\psi'$$

These two expressions coincide at m, proving (b). Theorem 2.9 together with (b) implies the statement (c). Part (d) follows from Proposition 1.20. Part (e) follows from (b) together with the definition of σ^F, $\sigma^{F'}$. ☐

Part (b) shows in particular that if F' is a Dirac structure, transverse to E', then its backward image is again a Dirac structure.

2.4. Dirac cohomology. — In this Section, we will discuss certain cohomology groups associated with any pair of transverse Dirac structures $E, F \subset \mathbb{T}M$ and a given volume form μ on M. We assume that E, F are given by pure spinors ϕ, ψ, normalized by the condition $(\phi, \psi)_{\wedge T^*M} = \mu$. Let $\sigma^E \in \Gamma(F), \sigma^F \in \Gamma(E)$ be sections defined as in Theorem 2.9, and denote

$$\sigma = \sigma^F - \sigma^E \in \Gamma(\mathbb{T}M).$$

Replacing ϕ, ψ with $\tilde{\phi} = f\phi$, $\tilde{\psi} = f^{-1}\psi$, for f a nonvanishing function on M, this section changes by a closed 1-form:

$$\tilde{\sigma} = \sigma - f^{-1}\mathrm{d}f. \tag{31}$$

Indeed, letting let p be the projection from $\mathbb{T}M$ to E along F we have $\tilde{\sigma}^F = \sigma^F - \mathsf{p}(f^{-1}\mathrm{d}f)$, $\tilde{\sigma}^E = \sigma^E + (I - \mathsf{p})(f^{-1}\mathrm{d}f)$.

We define the *Dirac cohomology groups* associated to a triple (E, F, μ) as the cohomology of the operators

$$\not{\partial}_+ = d + \eta + \varrho(\sigma), \quad \not{\partial}_- = d + \eta - \varrho(\sigma)$$

on $\Omega(M)$, restricted to the subspace on which they square to zero:

$$H_{\pm}(E, F, \mu) := \ker(\not{\partial}_{\pm})/\ker(\not{\partial}_{\pm}) \cap \mathrm{im}(\not{\partial}_{\pm}) \equiv H(\ker \not{\partial}^2_{\pm}, \not{\partial}_{\pm}). \tag{32}$$

The pure spinors ϕ, ψ define classes in $H_+(E,F,\mu)$ and $H_-(E,F,\mu)$, respectively, since $\eth_+\phi = 0$ and $\eth_-\psi = 0$. The Dirac cohomology groups are independent of the choice of defining spinors ϕ, ψ: Changing the pure spinors by a function f as above, (31) shows that the operators $\eth_\pm$ change by conjugation, $\tilde{\eth}_+ = f\eth_+ f^{-1}$ and $\tilde{\eth}_- = f^{-1}\eth_- f$.

Example 2.11. — Let M be a manifold with volume form μ. Consider transverse Dirac structures $E = \mathrm{Gr}_\omega$ for some closed 2-form ω, and $F = T^*M$. In this case, one can choose $\phi = e^{-\omega}, \psi = \mu$. We obtain $\eta = 0$, $\sigma = 0$, $\eth_\pm = \mathrm{d}$, and the Dirac cohomology groups $H_\pm(TM, T^*M, \mu)$ coincide with the de Rham cohomology of M.

Example 2.12. — Let M be a manifold with volume form μ and with a Poisson bivector π. Let $E = TM, F = \mathrm{Gr}_\pi$. The choice $\phi = 1, \psi = e^{-\iota(\pi)}\mu$ gives $\eth_- = d - \iota(X_\pi)$, where X_π is the modular vector field. The operator $\eth_-^2 = -\mathscr{L}(X_\pi)$ vanishes on differential forms invariant under the flow generated by X_π. The Dirac cohomology $H_-(TM, \mathrm{Gr}_\pi, \mu) = H(\Omega(M)^{X_\pi}, d - \iota(X_\pi))$ resembles the Cartan model of equivariant cohomology for circle actions.

Let π be the Poisson structure defined by the splitting $\mathbb{T}M = E \oplus F$, and $X_\pi = \mathrm{pr}_{TM}\,\sigma$ the modular vector field. Let

$$H_\pi(M) = H(\Omega(M)^{X_\pi}, d - \iota(X_\pi)). \tag{33}$$

By Remark 1.5(a) there is a pairing

$$H_+(E,F,\mu) \otimes H_-(E,F,\mu) \to H_\pi(M)$$

given on representatives by the formula $u \otimes v \mapsto u^\top \wedge v$. The pure spinors ϕ, ψ define cohomology classes $[\phi] \in H_+(E,F,\mu), [\psi] \in H_-(E,F,\mu)$, and $[\phi^\top \wedge \psi] \in H_\pi(M)$. If M is compact, the integration map $\int_M : \Omega(M)^{X_\pi} \to \mathbb{R}$ descends to $H_\pi(M)$. Hence

$$\int_M \phi^\top \wedge \psi = \int_M \mu > 0$$

shows that the cohomology classes $[\phi] \in H_+(E,F,\mu), [\psi] \in H_-(E,F,\mu)$ are both nonzero.

There is the following version of functoriality with respect to strong Dirac morphisms for Dirac cohomology.

Proposition 2.13. — *Let $(\Phi, \omega) \colon (M, E, \eta) \to (M', E', \eta')$ be a strong Dirac morphism, and let $F' \subset \mathbb{T}M'$ be a Dirac structure transverse to E', with backward image F. Assume that E, E' are defined by pure spinors ϕ, ϕ' such that the corresponding sections σ^E and $\sigma^{E'}$ vanish. Let ψ' and $\psi = e^\omega \Phi^*\psi'$ be pure spinors defining F' and F, and let μ' and μ be the resulting volume forms. Then $e^\omega \circ \Phi^*$ intertwines $\eth_-$ and $\eth'_-$, and hence induces a map in Dirac cohomology $e^\omega\Phi^* \colon H_-(E', F', \mu') \to H_-(E, F, \mu)$ taking $[\psi']$ to $[\psi]$.*

Proof. — Since $\sigma^E, \sigma^{E'}$ vanish we have $\sigma = \sigma^F$ and $\sigma' = \sigma^{F'}$. By Proposition 2.10 (e), the map $e^\omega\Phi^*$ intertwines the Clifford actions of σ^F and $\sigma^{F'}$, while on the other hand this map also intertwines $\mathrm{d}+\eta$ with $\mathrm{d}+\eta'$. Hence it intertwines $\eth_-$ with $\eth'_-$. □

2.5. Classical dynamical Yang-Baxter equation. — The following result describes the Courant tensor of Lagrangian subbundles defined by elements in $\Gamma(\wedge^2 E)$.

***Proposition 2.14* (Liu-Weinstein-Xu [40]).** — *Let $\mathbb{T}M = E \oplus F$ be a splitting into Lagrangian subbundles, where both E, F are integrable relative to the closed 3-form η, and let us identify $F^* = E$. Given a section $\varepsilon \in \Gamma(\wedge^2 E)$, defining a section $A^{-\varepsilon} \in \Gamma(\mathrm{O}(\mathbb{T}M))$, let $F_\varepsilon = A^{-\varepsilon}(F)$ be the Lagrangian subbundle spanned by the sections $x + \iota_x \varepsilon$ for $x \in \Gamma(F) = \Gamma(E^*)$. Then the Courant tensor $\Upsilon_\varepsilon \in \Gamma(\wedge^3 E)$ of F_ε is given by the formula:*

$$\Upsilon_\varepsilon = d_F \varepsilon + \tfrac{1}{2}[\varepsilon, \varepsilon]_E.$$

Here $[\cdot,\cdot]_E$ is the Lie algebroid bracket of E, and $d_F \colon \Gamma(\wedge^\bullet F^*) \to \Gamma(\wedge^{\bullet+1} F^*)$ is the Lie algebroid differential of F.

Remark 2.15. — The result in **[40]** is stated only for $\eta = 0$. However, since the statement is local, one may use a gauge transformation by a local primitive of η to reduce to this case.

We are interested in the following special case: Let $M = \mathfrak{g}^*$, with its standard linear Poisson structure $\pi_{\mathfrak{g}^*} \in \Gamma(\wedge^2 T\mathfrak{g}^*) = C^\infty(\mathfrak{g}^*) \otimes \wedge^2 \mathfrak{g}^*$, and put $F = T\mathfrak{g}^*$ and $E = \mathrm{Gr}_{\pi_{\mathfrak{g}}^*}$. The bundle E is spanned by sections $\mathscr{A}_0(\xi) \oplus \langle \theta_0, \xi \rangle$ for $\xi \in \mathfrak{g}$, where $\mathscr{A}_0(\xi)$ is the generating vector fields for the co-adjoint action, and $\langle \theta_0, \xi \rangle \in \Omega^1(\mathfrak{g}^*)$ is the 'constant' 1-form defined by ξ. The trivialization $E = \mathfrak{g}^* \times \mathfrak{g}$ defined by these sections identifies E with the action algebroid for the co-adjoint action: The bracket on $\Gamma(E) = C^\infty(\mathfrak{g}^*, \mathfrak{g})$ is defined by the Lie bracket on $\mathfrak{g}$ via the Leibniz rule, and the anchor map is given by the action map $\mathscr{A}_0 \colon \mathfrak{g} \to T\mathfrak{g}^*$. For $\varepsilon \in \Gamma(\wedge^2 E)$, the bracket $[\varepsilon, \varepsilon]_E$ is given by the Schouten bracket on $\wedge \mathfrak{g}$. On the other hand we may view $\varepsilon \in C^\infty(\mathfrak{g}^*, \wedge^2 \mathfrak{g})$ as a 2-form on $\mathfrak{g}^*$, and then $\mathrm{d}\varepsilon = \mathrm{d}_F \varepsilon$ is just its exterior differential. The resulting equation reads

$$\mathrm{d}\varepsilon + \tfrac{1}{2}[\varepsilon, \varepsilon]_{\mathrm{Sch}} = \Upsilon_\varepsilon.$$

If Υ_ε is a multiple of the structure constants tensor, this is a special case of the *classical dynamical Yang-Baxter equation* (CDYBE) **[5, 25]**. We will see below how a solution arises from the Cartan-Dirac structure on G.

For more information on the relation between Dirac structures and the CDYBE, see the work of Liu-Xu **[41]** and Bangoura-Kosmann-Schwarzbach **[10]**.

3. Dirac structures on Lie groups

In this Section, we will study Dirac structures over Lie groups G with bi-invariant pseudo-Riemannian metrics. This will be based on the existence of a canonical isomorphism

$$\mathbb{T}G \cong G \times (\mathfrak{g} \oplus \overline{\mathfrak{g}})$$

preserving scalar products and Courant brackets. In the subsequent section, we will describe a corresponding isomorphism of spinor modules.

3.1. The isomorphism $\mathbb{T}G \cong G\times(\mathfrak{g}\oplus\overline{\mathfrak{g}})$. — Let G be a Lie group (not necessarily connected), and let $\mathfrak{g}$ be its Lie algebra. We denote by $\xi^L, \xi^R \in \mathfrak{X}(G)$ the left-, right-invariant vector fields on G which are equal to $\xi \in \mathfrak{g} = T_eG$ at the group unit. Let $\theta^L, \theta^R \in \Omega^1(G)\otimes\mathfrak{g}$ be the left-, right-Maurer-Cartan forms, i.e. $\iota(\xi^L)\theta^L = \iota(\xi^R)\theta^R = \xi$. They are related by $\theta^R_g = \mathrm{Ad}_g(\theta^L_g)$, for all $g \in G$. The adjoint action of G on itself will be denoted $\mathcal{A}_{\mathsf{ad}}$ (or simply $\mathcal{A}$, if there is no risk of confusion). The corresponding infinitesimal action is given by the vector fields

$$\mathcal{A}_{\mathsf{ad}}(\xi) = \xi^L - \xi^R.$$

Suppose that the Lie algebra $\mathfrak{g}$ of G carries an *invariant inner product*. By this we mean an Ad-invariant, non-degenerate symmetric bilinear form B, not necessarily positive definite. Equivalently, B defines a bi-invariant pseudo-Riemannian metric on G. Given B, we can define the bi-invariant 3-form $\eta \in \Omega^3(G)$,

$$\eta := \frac{1}{12}B(\theta^L, [\theta^L, \theta^L])$$

Since η is bi-invariant, it is closed, and so it defines an η-twisted Courant bracket $[\![\cdot,\cdot]\!]_\eta$ on G. The conjugation action $\mathcal{A}_{\mathsf{ad}}$ extends to an action of $D = G\times G$ on G, by

$$\mathcal{A}\colon D \to \mathrm{Diff}(G), \quad \mathcal{A}(a, a') = l_{a'} \circ r_{a^{-1}}, \tag{34}$$

where $l_a(g) = ag$ and $r_a(g) = ga$. The corresponding infinitesimal action

$$\mathcal{A}\colon \mathfrak{d} \to \mathfrak{X}(G), \quad \mathcal{A}(\xi, \xi') = \xi^L - (\xi')^R$$

lifts to a map

$$\mathsf{s}\colon \mathfrak{d} \to \Gamma(\mathbb{T}G), \qquad \mathsf{s}(\xi, \xi') = \mathsf{s}^L(\xi) + \mathsf{s}^R(\xi'), \tag{35}$$

where

$$\mathsf{s}^L(\xi) = \xi^L \oplus \tfrac{1}{2}B(\theta^L, \xi), \quad \mathsf{s}^R(\xi') = -(\xi')^R \oplus \tfrac{1}{2}B(\theta^R, \xi').$$

Let us equip $\mathfrak{d}$ with the bilinear form $B_\mathfrak{d}$ given by $+B$ on the first $\mathfrak{g}$-summand and $-B$ on the second $\mathfrak{g}$-summand. Thus $\mathfrak{d} = \mathfrak{g}\oplus\overline{\mathfrak{g}}$ is an example of a Lie algebra with invariant split bilinear form.

Proposition 3.1. — *The map $\mathsf{s}\colon \mathfrak{d} \to \Gamma(\mathbb{T}G)$ is D-equivariant, and satisfies*

$$\langle \mathsf{s}(\zeta_1), \mathsf{s}(\zeta_2)\rangle = B_\mathfrak{d}(\zeta_1, \zeta_2), \quad [\![\mathsf{s}(\zeta_1), \mathsf{s}(\zeta_2)]\!]_\eta = \mathsf{s}([\zeta_1, \zeta_2]) \tag{36}$$

for all $\zeta_1, \zeta_2 \in \mathfrak{d}$. Furthermore,

$$\Upsilon(\mathsf{s}(\zeta_1), \mathsf{s}(\zeta_2), \mathsf{s}(\zeta_3)) = B_\mathfrak{d}\big(\zeta_1, [\zeta_2, \zeta_3]\big) \tag{37}$$

for all $\zeta_i \in \mathfrak{d}$, where $\Upsilon\colon \Gamma(\mathbb{T}G)^{\otimes 3} \to C^\infty(G)$ was defined in (28).

Proof. — The D-equivariance of the map s is clear. Let ϱ be the Clifford action of $\mathbb{T}G$ on $\wedge T^*G$. We have $[\varrho(\mathsf{s}^L(\xi)), \mathrm{d}+\eta] = \mathcal{L}(\xi^L)$ and $[\varrho(\mathsf{s}^R(\xi)), \mathrm{d}+\eta] = -\mathcal{L}(\xi^R)$, thus

$$[\mathrm{d}+\eta,\ \varrho(\mathsf{s}(\zeta))] = \mathcal{L}(\mathcal{A}(\zeta))$$

for all $\zeta \in \mathfrak{d}$. This proves the second Equation in (36), while the first Equation is obvious. Finally, (37) follows from (36) and the definition of Υ. Hence,

$$\varrho([\![\mathsf{s}(\zeta_1), \mathsf{s}(\zeta_2)]\!]_\eta) = [[\mathrm{d}+\eta, \varrho(\mathsf{s}(\zeta_1)],\ \varrho(\mathsf{s}(\zeta_2))] = \varrho(\mathsf{s}([\zeta_1, \zeta_2])).$$

□

Put differently, the map s defines a D-equivariant isometric isomorphism

$$\mathbb{T}G \cong G \times \mathfrak{d}, \tag{38}$$

identifying the η-twisted Courant bracket on $\mathbb{T}G$ with the unique Courant bracket on $G \times \mathfrak{d}$ which agrees with the Lie bracket on $\mathfrak{d}$ on constant sections.

3.2. η-twisted Dirac structures on G. — Using (38), we see that any Lagrangian subspace $\mathfrak{s} \subset \mathfrak{d}$ defines a Lagrangian subbundle

$$E^{\mathfrak{s}} \cong G \times \mathfrak{s},$$

spanned by the sections $\mathsf{s}(\zeta)$ with $\zeta \in \mathfrak{s}$. The Lagrangian subbundle $E^{\mathfrak{s}}$ is invariant under the action of the subgroup of D preserving $\mathfrak{s}$. Let $\Upsilon^{\mathfrak{s}} \in \wedge^3 \mathfrak{s}^*$ be defined as

$$\Upsilon^{\mathfrak{s}}(\zeta_1, \zeta_2, \zeta_3) = B_{\mathfrak{d}}\big(\zeta_1, [\zeta_2, \zeta_3]\big),\ \ \zeta_i \in \mathfrak{s}. \tag{39}$$

By (37), the Courant tensor $\Upsilon^{E^{\mathfrak{s}}}$ is just $\Upsilon^{\mathfrak{s}}$, using the sections s to identify $(E^{\mathfrak{s}})^* \cong G \times \mathfrak{s}^*$. In particular, we see that $\mathfrak{s}$ defines a Dirac structure if and only if $\mathfrak{s}$ is a Lie subalgebra. To summarize:

Any Lagrangian subalgebra $\mathfrak{s} \subset \mathfrak{d}$ defines an η-twisted Dirac structure $E^{\mathfrak{s}}$.

The Dirac structure $E^{\mathfrak{s}}$ is invariant under the action of any Lie subgroup normalizing $\mathfrak{s}$, and in particular under the action of the subgroup $S \subset D$ integrating $\mathfrak{s}$. As a Lie algebroid, $E^{\mathfrak{s}}$ is just the action algebroid for this S-action. In particular, its leaves are just the components of the S-orbits on G. The 2-form on the orbit $\mathcal{O} = \mathcal{A}(S)g$ of an element $g \in G$ is the S-invariant form $\omega_{\mathcal{O}}$ given as follows: for $\zeta_i = (\xi_i, \xi_i') \in \mathfrak{s}$,

$$\begin{aligned} \omega_{\mathcal{O}}(\mathcal{A}(\zeta_1), \mathcal{A}(\zeta_2))|_g &= \tfrac{1}{2}\langle B(\theta^L, \xi_1) + B(\theta^R, \xi_1'),\ \xi_2^L - (\xi_2')^R\rangle \\ &= \tfrac{1}{2} B(\xi_2 - \mathrm{Ad}_{g^{-1}} \xi_2', \xi_1 + \mathrm{Ad}_{g^{-1}} \xi_1') \\ &= \tfrac{1}{2}(B(\mathrm{Ad}_g \xi_2, \xi_1') - B(\xi_2', \mathrm{Ad}_g \xi_1)), \end{aligned} \tag{40}$$

using $B(\xi_1, \xi_2) = B(\xi_1', \xi_2')$ since $\mathfrak{s}$ is Lagrangian. By the general theory from Section 2.1, these 2-forms satisfy $\mathrm{d}\omega_{\mathcal{O}} = \iota_{\mathcal{O}}^* \eta$, where $\iota_{\mathcal{O}}\colon \mathcal{O} \to G$ is the inclusion. The kernel of $\omega_{\mathcal{O}}$ equals $\ker(E^{\mathfrak{s}})$, i.e. it is spanned by all $\mathcal{A}(\zeta)$ such that the T^*G-component of $\mathsf{s}(\zeta)$ is zero:

$$\ker(\omega_{\mathcal{O}}|_g) = \{\mathcal{A}(\zeta)|_g \mid \zeta = (\xi, \xi') \in \mathfrak{s},\ \ \mathrm{Ad}_g \xi + \xi' = 0\}. \tag{41}$$

Remark 3.2. — For $\mathfrak{g}$ a complex semi-simple Lie algebra, a complete classification of Lagrangian subalgebras of $\mathfrak{d}$ was obtained by Karolinsky [**34**]. The Poisson geometry of the variety of Lagrangian subalgebras of $\mathfrak{d}$ was studied in detail by Evens–Lu [**27**].

Remark 3.3. — If $\mathfrak{d} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ is a splitting into two Lagrangian subalgebras (i.e., $(\mathfrak{d}, \mathfrak{s}_1, \mathfrak{s}_2)$ is a Manin triple), one obtains two transverse Dirac structures $E^{\mathfrak{s}_1}, E^{\mathfrak{s}_2}$. As discussed after Theorem 2.9, such a pair of transverse Dirac structures gives rise to a Poisson structure on G, with symplectic leaves the intersections of the orbits of S_1, S_2. For $\mathfrak{g}$ a complex semi-simple Lie algebra, the Manin triples were classified by Delorme [**22**]. See Evens–Lu [**27**] for a wealth of information regarding Poisson structures obtained from Lagrangian subalgebras. An example will be worked out in Section 3.6 below.

Remark 3.4. — We may also use this construction to obtain generalized complex (and Kähler) structures [**28**] on even-dimensional real Lie groups K, with complexification $G = K^{\mathbb{C}}$. Indeed, let $\mathfrak{s} \subset \mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$ be a Lagrangian subalgebra such that

$$\mathfrak{s} \cap \mathfrak{s}^c = \{0\}, \tag{42}$$

where $\mathfrak{s}^c$ denotes the complex conjugate of $\mathfrak{s}$. Then the associated Dirac structure $E^{\mathfrak{s}} \subset \mathbb{T}G$ satisfies $E^{\mathfrak{s}} \cap (E^{\mathfrak{s}})^c = \{0\}$ along K. Hence it defines a generalized complex structure on K. For a concrete example, suppose K is compact, and let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+$ be a triangular decomposition. (That is, $\mathfrak{t} = \mathfrak{t}_K^{\mathbb{C}}$ is the complexification of a maximal Abelian subalgebra, and $\mathfrak{n}_+, \mathfrak{n}_-$ are the sums of the positive, negative root spaces). Then

$$\mathfrak{s} = (\mathfrak{n}^+ \oplus 0) \oplus \mathfrak{l} \oplus (0 \oplus \mathfrak{n}^-) \subset \mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$$

has the desired property, for any Lagrangian subspace $\mathfrak{l} \subset \mathfrak{t} \oplus \bar{\mathfrak{t}}$ with $\mathfrak{l} \cap \mathfrak{l}^c = \{0\}$ (i.e., $\mathfrak{l}$ is a linear generalized complex structure on the vector space $\mathfrak{t}_K$). The generalized complex structures on Lie groups considered in Gualtieri [**28**, Example 6.39] are examples of this construction.

3.3. The Cartan-Dirac structure.

— The simplest example of a Lagrangian subalgebra is the diagonal $\mathfrak{s} = \mathfrak{g}_\Delta \hookrightarrow \mathfrak{d}$, with corresponding S the diagonal subgroup $G_\Delta \subset D$. The associated Dirac structure E_G is spanned by the sections $\mathsf{e}(\xi) := \mathsf{s}(\xi, \xi)$:

$$E_G = \operatorname{span}\{\mathsf{e}(\xi) \mid \xi \in \mathfrak{g}\} \subset \mathbb{T}G, \tag{43}$$

$$\mathsf{e}(\xi) = \big(\xi^L - \xi^R, B(\tfrac{\theta^L + \theta^R}{2}, \xi)\big).$$

We call E_G the *Cartan-Dirac structure*, see [**15, 39, 50**]. This Dirac structure was introduced independently by Alekseev, Ševera, and Strobl in the mid-1990's. The $G_\Delta \cong G$-action is just the action by conjugation on G, hence the Dirac foliation is given by the conjugacy classes $\mathscr{C} \subset G$. The formula (40) specializes to the 2-form on conjugacy classes introduced in [**31**]:

$$\omega_{\mathscr{C}}(\mathscr{A}_{\mathsf{ad}}(\xi_1), \mathscr{A}_{\mathsf{ad}}(\xi_2)) = -\tfrac{1}{2} B((\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}})\xi_1, \xi_2),$$

The kernel at $g \in \mathscr{C}$ is the span of vector fields $\mathscr{A}_{\mathsf{ad}}(\xi)|_g$ with $\mathrm{Ad}_g \xi + \xi = 0$. The anti-diagonal in $\mathfrak{g} \oplus \bar{\mathfrak{g}}$ is a G-invariant Lagrangian complement to the diagonal, and hence defines a G-invariant Lagrangian subbundle F_G complementary to E_G, spanned by $\mathsf{f}(\xi) = \mathsf{s}(\xi/2, -\xi/2)$:

$$F_G = \mathrm{span}\{\mathsf{f}(\xi)|\ \xi \in \mathfrak{g}\} \subset \mathbb{T}G, \tag{44}$$
$$\mathsf{f}(\xi) = \big(\tfrac{\xi^L+\xi^R}{2}, B(\tfrac{\theta^L-\theta^R}{4}, \xi)\big).$$

The 1/2 factors in the definition of $\mathsf{f}(\xi)$ are introduced so that $\langle \mathsf{e}(\xi), \mathsf{f}(\xi')\rangle = B(\xi, \xi')$.

Let $\Xi \in \wedge^3(\mathfrak{g})$ be the structure constants tensor of $\mathfrak{g}$, normalized as follows:

$$\iota(\xi_3)\iota(\xi_2)\iota(\xi_1)\Xi = \tfrac{1}{4} B(\xi_1, [\xi_2, \xi_3]_{\mathfrak{g}}). \tag{45}$$

Let $\mathsf{e}\colon \wedge\mathfrak{g} \to \Gamma(\wedge E_G)$ be the extension of $\mathsf{e}\colon \mathfrak{g} \to \Gamma(E_G)$ as an algebra homomorphism. Thus $\mathsf{e}(\Xi)$ is a section of $\wedge^3(E_G)$.

Lemma 3.5. — *The Courant tensor of F_G is given by :*

$$\Upsilon^{F_G} = \mathsf{e}(\Xi).$$

Proof. — This follows from (37) since $B_{\mathfrak{d}}(\zeta_1, [\zeta_2, \zeta_3]_{\mathfrak{d}}) = \frac{1}{4}B(\xi_1, [\xi_2, \xi_3]_{\mathfrak{g}})$ for $\zeta_i = (\xi_i/2,\ -\xi_i/2)$. □

The element Ξ also defines a trivector field, $\mathscr{A}_{\mathsf{ad}}(\Xi) \in \mathfrak{X}^3(G)$. Theorem 2.9 implies that the bivector field $\pi_G \in \mathfrak{X}^2(G)$ defined by the Lagrangian splitting $\mathbb{T}G = E \oplus F$ satisfies

$$\tfrac{1}{2}[\pi_G, \pi_G]_{\mathrm{Sch}} = \mathscr{A}_{\mathsf{ad}}(\Xi).$$

To give an explicit formula for π_G, let v_i, v^i be B-dual bases of $\mathfrak{g}$, i.e. $B(v_i, v^j) = \delta_i^j$.

Proposition 3.6. — *The bivector field π_G is given by*

$$\pi_G = \tfrac{1}{2}\sum_i v^{i,L} \wedge v_i^R. \tag{46}$$

Proof. — By (18), we have

$$\pi_G = \tfrac{1}{2}\sum_i \big((v_i)^L - (v_i)^R\big) \wedge \frac{(v^i)^L + (v^i)^R}{2}.$$

Since $\sum_i v^{i,L} \wedge v_i^L = \sum_i v^{i,R} \wedge v_i^R$, this simplifies to the expression in (46). □

The bivector field π_G was first considered in [**1, 2**].

3.4. Group multiplication. — In this Section, we will examine the behavior of the Cartan-Dirac structure under group multiplication,

$$\mathrm{Mult}\colon G \times G \to G, \quad (a, b) \mapsto ab.$$

For any differential form $\beta \in \Omega(G)$, we will denote by $\beta^i \in \Omega(G \times G)$ its pull-back to the i'th factor, for $i = 1, 2$. We will use similar notation for vector fields on $G \times G$, and for sections of the bundle $\mathbb{T}(G \times G)$. Let $\varsigma \in \Omega^2(G \times G)$ denote the 2-form

$$\varsigma = -\tfrac{1}{2}B(\theta^{L,1}, \theta^{R,2}). \tag{47}$$

A direct computation shows that

$$\mathrm{Mult}^* \eta \quad = \quad \eta^1 + \eta^2 + \mathrm{d}\varsigma, \tag{48}$$

hence we have a multiplication morphism

$$(\mathrm{Mult}, \varsigma)\colon (G, \eta) \times (G, \eta) = (G \times G, \eta^1 + \eta^2) \to (G, \eta).$$

Remark 3.7. — This is expressed more conceptually in terms of the simplicial model $B_pG = G^p$ of the classifying space BG. Let $\partial_i\colon G^p \to G^{p-1},\ 0 \le i \le p$ be the 'face maps' given as $\partial_i(g_1, \ldots, g_p) = (g_1, \ldots, g_i g_{i+1}, \ldots, g_p)$, while ∂_0 omits the first entry g_1, and ∂_p omits the last entry g_p. Let $\delta = \sum_{i=0}^{p} \partial_i^*\colon \Omega^\bullet(G^{p-1}) \to \Omega^\bullet(G^p)$. Then δ commutes with the de-Rham differential, turning $\bigoplus_{p,q} \Omega^q(G^p)$ into a double complex. The total differential on $\Omega^q(G^p)$ is $\mathrm{d} + (-1)^q \delta$. Then $\eta \in \Omega^3(G)$ and $\varsigma \in \Omega^2(G^2)$ define a cocycle of degree 4 (see [**55**]):

$$\mathrm{d}\eta = 0, \quad \partial\eta = -\mathrm{d}\varsigma, \quad \partial\varsigma = 0. \tag{49}$$

(If G is compact, simple, and simply connected, and B the basic inner product, this pair is the Bott-Shulman representative of the generator of $H^4(BG) \cong H^3(G)$.) The second condition is just the property (48) used above. Using the third property, one may verify that the multiplication morphism is associative, in the sense that

$$(\mathrm{Mult}, \varsigma) \circ \big((\mathrm{Mult}, \varsigma) \times (\mathrm{id}_G, 0)\big) = (\mathrm{Mult}, \varsigma) \circ \big((\mathrm{id}_G, 0) \times (\mathrm{Mult}, \varsigma)\big).$$

We will compare the morphism $(\mathrm{Mult}, \varsigma)$ with the groupoid multiplication of $\mathfrak{d}$, viewed as the pair groupoid over $\mathfrak{g}$: writing $\zeta = (\xi, \xi'),\ \zeta_i = (\xi_i, \xi_i'),\ i = 1, 2$, the groupoid multiplication is

$$\zeta = \zeta_2 \circ \zeta_1 \ \Leftrightarrow\ \xi = \xi_2,\ \xi' = \xi_1',\ \xi_2' = \xi_1.$$

Proposition 3.8. — *The isomorphism $G \times \mathfrak{d} \to \mathbb{T}G$ defined by s intertwines the groupoid multiplication of $\mathfrak{d}$ with the morphism $(\mathrm{Mult}, \varsigma)$, in the sense that*

$$\zeta_2 \circ \zeta_1 = \zeta \quad \Leftrightarrow \quad \mathsf{s}^1(\zeta_1) + \mathsf{s}^2(\zeta_2) \sim_{(\mathrm{Mult}, \varsigma)} \mathsf{s}(\zeta), \tag{50}$$

for $\zeta, \zeta_1, \zeta_2 \in \mathfrak{d}$.

Proof. — Spelling out the relations (50), we have to show that, for all $\xi \in \mathfrak{g}$,

$$\begin{aligned} &\mathsf{s}^{R,1}(\xi) \sim_{(\mathrm{Mult}, \varsigma)} \mathsf{s}^R(\xi), \qquad \mathsf{s}^{L,2}(\xi) \sim_{(\mathrm{Mult}, \varsigma)} \mathsf{s}^L(\xi), \\ &\mathsf{s}^{L,1}(\xi) + \mathsf{s}^{R,2}(\xi) \sim_{(\mathrm{Mult}, \varsigma)} 0. \end{aligned} \tag{51}$$

The equivariance properties

$$\begin{aligned} &\mathrm{Mult}(ga, b) = g\,\mathrm{Mult}(a, b), \quad \mathrm{Mult}(a, bg^{-1}) = \mathrm{Mult}(a, b)g^{-1}, \\ &\mathrm{Mult}(ag^{-1}, gb) = \mathrm{Mult}(a, b) \end{aligned}$$

of the multiplication map imply the following relations of generating vector fields:

$$-\xi^{R,1} \sim_{\mathrm{Mult}} -\xi^R, \quad \xi^{L,2} \sim_{\mathrm{Mult}} \xi^L, \quad \xi^{L,1} - \xi^{R,2} \sim_{\mathrm{Mult}} 0.$$

This proves the 'vector field part' of the relations (51). The 1-form part is equivalent to the following three identities, which are verified by a direct computation:

$$\begin{aligned}
&\tfrac{1}{2}B(\theta^{R,1},\xi)+\iota(-\xi^{R,1})\varsigma=\tfrac{1}{2}\operatorname{Mult}^* B(\theta^R,\xi),\\
&\tfrac{1}{2}B(\theta^{L,2},\xi)+\iota(\xi^{L,2})\varsigma=\tfrac{1}{2}\operatorname{Mult}^* B(\theta^L,\xi)\,,\\
&\tfrac{1}{2}B(\theta^{L,1}+\theta^{R,2},\xi)+\iota(\xi^{L,1}-\xi^{R,2})\varsigma=0.
\end{aligned}$$ □

Theorem 3.9. — *The multiplication map* Mult: $G\times G\to G$ *extends to a strong Dirac morphism*

$$(\operatorname{Mult},\varsigma)\colon (G,E_G,\eta)\times(G,E_G,\eta)\to(G,E_G,\eta),$$

with $\varsigma\in\Omega^2(G\times G)$ *as defined above. In terms of the trivialization* $E_G=G\times\mathfrak{g}$, *the map* $\widehat{\mathfrak{a}}\colon \operatorname{Mult}^* E_G\to E_G\times E_G$ *associated with the strong Dirac morphism is given by the diagonal embedding* $\mathfrak{g}\to\mathfrak{g}\times\mathfrak{g}$. *Similarly, the inversion map* Inv: $G\to G,\ g\mapsto g^{-1}$ *extends to a Dirac morphism*

$$(\operatorname{Inv},0)\colon (G,E_G,\eta)\to(G,E_G^\top,-\eta).$$

Proof. — By Proposition 3.8, the sections $\mathsf{e}(\xi)=\mathsf{s}(\xi,\xi)$ satisfy

$$\mathsf{e}^1(\xi)+\mathsf{e}^2(\xi)\sim_{(\operatorname{Mult},\varsigma)}\mathsf{e}(\xi).$$

This shows that $(\operatorname{Mult},\varsigma)$ is a Dirac morphism. For any given point $(a,b)\in G\times G$, no non-trivial linear combination of $\mathsf{e}^1(\xi)|_a,\ \mathsf{e}^2(\xi')|_b$ is $(\operatorname{Mult},\varsigma)$-related to 0. Hence, the Dirac morphism $(\operatorname{Mult},\varsigma)$ is strong.

We have $\operatorname{Inv}^* B(\theta^L+\theta^R,\xi)=-B(\theta^L+\theta^R,\xi)$ and $\xi^L-\xi^R\sim_{\operatorname{Inv}}(\xi^L-\xi^R)$. Hence

$$\mathsf{e}(\xi)\sim_{(\operatorname{Inv},0)}\mathsf{e}(\xi)^\top$$

where $\mathsf{e}(\xi)^\top$ is the image of $\mathsf{e}(\xi)$ under the map $(v,\alpha)\to(v,-\alpha)$. Since $\operatorname{Inv}^*\eta=-\eta$, this shows that $(\operatorname{Inv},0)\colon (G,E_G,\eta)\to(G,E_G^\top,-\eta)$ is a Dirac morphism. □

Remark 3.10. — More generally, suppose that $\mathfrak{s}\subset\mathfrak{d}$ is a Lagrangian subalgebra, defining a Dirac structure $E^{\mathfrak{s}}$. Since $\mathfrak{g}_\Delta\circ\mathfrak{s}=\mathfrak{s}$, the same argument as in the proof above shows that $(\operatorname{Mult},\varsigma)$ is a strong Dirac morphism from $(G,E_G,\eta)\times(G,E^{\mathfrak{s}},\eta)$ to $(G,E^{\mathfrak{s}},\eta)$.

Let $\widetilde{F}_{G\times G}\subset\mathbb{T}(G\times G)$ be the backward image of the Lagrangian subbundle F_G under $(\operatorname{Mult},\varsigma)$. Since F_G is spanned by the sections $\mathsf{f}(\xi)=\frac{1}{2}(\mathsf{s}^L(\xi)-\mathsf{s}^R(\xi))$, (51) shows that $\widetilde{F}_{G\times G}$ is spanned by the sections

$$\tfrac{1}{2}(\mathsf{s}^{L,2}(\xi)-\mathsf{s}^{R,1}(\xi)),\quad \tfrac{1}{2}(\mathsf{s}^{L,1}(\xi)+\mathsf{s}^{R,2}(\xi)). \tag{52}$$

Since F_G is a complement to E_G, its backward image $\widetilde{F}_{G\times G}$ is a complement to $E_G^1\oplus E_G^2$ (see Proposition 1.15). Let us describe the element of $\wedge^2(E_G^1\oplus E_G^2)$ relating $\widetilde{F}_{G\times G}$ to the standard complement $F_G^1\oplus F_G^2$. Let $v_i\in\mathfrak{g}$ and $v^i\in\mathfrak{g}$ be B-dual bases, and put

$$\gamma=\tfrac{1}{2}(v_i)^1\wedge(v^i)^2\in\wedge^2(\mathfrak{g}\oplus\mathfrak{g}). \tag{53}$$

Let

$$\mathsf{e}(\gamma) = \tfrac{1}{2}\sum_i \mathsf{e}^1(v_i)\wedge \mathsf{e}^2(v^i) \in \Gamma(\wedge^2(E_G^1\oplus E_G^2)) \tag{54}$$

be the corresponding section.

Proposition 3.11. — *The Lagrangian complement* $\widetilde{F}_{G\times G} = F_G\circ\Gamma_{(\mathrm{Mult},\varsigma)}$ *is obtained from* $F_G^1\oplus F_G^2$ *by the bivector* $\mathsf{e}(\gamma)$*:*

$$\widetilde{F}_{G\times G} = A^{-\mathsf{e}(\gamma)}(F_G^1\oplus F_G^2).$$

Proof. — We compute $\iota(\mathsf{f}^1(\xi))\mathsf{e}(\gamma) = \mathsf{e}(\iota^1(\xi)\gamma) = \frac{1}{2}\mathsf{e}^2(\xi) = \frac{1}{2}(\mathsf{s}^{L,2}(\xi)+\mathsf{s}^{R,2}(\xi))$. Thus

$$\mathsf{f}^1(\xi)+\iota(\mathsf{f}^1(\xi))\mathsf{e}(\gamma) = \tfrac{1}{2}(\mathsf{s}^{L,1}(\xi)-\mathsf{s}^{R,1}(\xi)+\mathsf{s}^{L,2}(\xi)+\mathsf{s}^{R,2}(\xi))$$

is the sum of the sections in (52). Similarly, we find that $\mathsf{f}^2(\xi)+\iota(\mathsf{f}^2(\xi))\mathsf{e}(\gamma)$ is the difference of the sections in (52). □

The bivector field on $G\times G$ corresponding to the splitting $(E_G^1\times E_G^2)\oplus A^{-\mathsf{e}(\gamma)}(F_G^1\times F_G^2)$ of $\mathbb{T}(G\times G)$ is given by (see Proposition 1.18(i)),

$$\widetilde{\pi} = \pi_G^1+\pi_G^2+\mathscr{A}_{\mathsf{ad}}^{12}(\gamma), \tag{55}$$

where π_G is the bivector field for the splitting $\mathbb{T}G = E_G\oplus F_G$, and $\mathscr{A}_{\mathsf{ad}}^{12} = \mathscr{A}_{\mathsf{ad}}^1\oplus\mathscr{A}_{\mathsf{ad}}^2\colon \mathfrak{g}\oplus\mathfrak{g}\to\mathfrak{X}(G\times G)$. By Proposition 2.10(c) we have $\tilde{\pi}\sim_{\mathrm{Mult}}\pi$. Furthermore, Proposition 2.10(c) and Lemma 3.5, imply that the Schouten bracket $\frac{1}{2}[\widetilde{\pi},\widetilde{\pi}]_{\mathrm{Sch}}$ equals the trivector field $\mathscr{A}_{\mathsf{ad}}^{\mathrm{diag}}(\Xi)$, where $\mathscr{A}_{\mathsf{ad}}^{\mathrm{diag}}$ is the diagonal action on $G\times G$.

3.5. Exponential map. — We will now discuss the behavior of the Cartan-Dirac structure under the exponential map,

$$\exp\colon\mathfrak{g}\to G.$$

Let $\mathfrak{g}_\natural\subset\mathfrak{g}$ denote the set of regular points of the exponential map, that is, all points where $\mathrm{d}\exp$ is an isomorphism. We begin with some preliminaries concerning $\mathbb{T}\mathfrak{g}^*$, not using the inner product on $\mathfrak{g}$ for the time being. Let $\mathscr{A}_0$ be the action of $D_0 := \mathfrak{g}^*\rtimes G$ on $\mathfrak{g}^*$ by

$$\mathscr{A}_0(\beta,g)\nu = (\mathrm{Ad}_{g^{-1}})^*\nu-\beta.$$

This action lifts to an action by automorphisms of $\mathbb{T}\mathfrak{g}^*$, preserving the inner product as well as the (untwisted) Courant bracket. Let $\mathfrak{d}_0 = \mathfrak{g}^*\rtimes\mathfrak{g}$ be the Lie algebra of D_0, equipped with the canonical inner product defined by the pairing, and let $\mathscr{A}_0\colon\mathfrak{d}_0\to\mathfrak{X}(\mathfrak{g}^*)$ be the infinitesimal action. To simplify notation, we denote the constant vector field defined by $\beta\in\mathfrak{g}^*$ by $\beta_0 = \mathscr{A}_0(\beta,0)$, and write $\mathscr{A}_0(\xi) = \mathscr{A}_0(0,\xi)$. Let $\theta_0\in\Omega^1(\mathfrak{g}^*)\otimes\mathfrak{g}^*$ be the tautological 1-form, defined by $\iota(\beta_0)\theta_0 = \beta$. Consider the D_0-equivariant map

$$\mathsf{s}_0\colon\mathfrak{d}_0\to\Gamma(\mathbb{T}\mathfrak{g}^*),\quad \mathsf{s}_0(\beta,\xi) = \mathscr{A}_0(\beta,\xi)\oplus\langle\theta_0,\xi\rangle. \tag{56}$$

Then $\langle \mathsf{s}_0(\zeta), \mathsf{s}_0(\zeta')\rangle = B_{\mathfrak{d}_0}(\zeta,\zeta')$, showing that s_0 defines a D_0-equivariant isometric isomorphism

$$\mathbb{T}\mathfrak{g}^* \cong \mathfrak{g}^* \times \mathfrak{d}_0.$$

A direct computation shows that this isomorphism is compatible with the Courant bracket $[\![\cdot,\cdot]\!]_0$ on $\mathbb{T}\mathfrak{g}^*$ and the Lie bracket on $\mathfrak{d}_0$.

Since $\mathfrak{g} \subset \mathfrak{d}_0$ is a Lagrangian Lie subalgebra, the sections $\mathsf{e}_0(\xi) := \mathsf{s}_0(0,\xi)$ span a Dirac structure $E_{\mathfrak{g}^*} \subset \mathbb{T}\mathfrak{g}^*$. Since $E_{\mathfrak{g}^*} \cap T\mathfrak{g}^* = 0$, this Dirac structure is of the form $E_{\mathfrak{g}^*} = \mathrm{Gr}_{\pi_{\mathfrak{g}^*}}$ for a Poisson bivector field $\pi_{\mathfrak{g}^*}$ satisfying

$$\iota(\langle\theta_0,\xi\rangle)\pi_{\mathfrak{g}^*} = \mathscr{E}_0(\xi), \quad \xi\in\mathfrak{g}. \tag{57}$$

The Poisson structure $\pi_{\mathfrak{g}^*}$ is just the standard linear Poisson structure on $\mathfrak{g}^*$. Similarly, the sections $\mathsf{f}_0(\beta) := \mathsf{s}_0(\beta,0)$ span the Lagrangian subspace $F_{\mathfrak{g}^*} = T\mathfrak{g}^*$, which is complementary to $E_{\mathfrak{g}^*}$.

Let us now use the invariant inner product B on $\mathfrak{g}$ to identify $\mathfrak{g}^* \cong \mathfrak{g}$. Let

$$\varpi \in \Omega^2(\mathfrak{g}), \quad \mathrm{d}\varpi = \exp^*\eta \tag{58}$$

be the primitive of $\exp^*\eta \in \Omega^3(\mathfrak{g})$ defined by the de Rham homotopy operator for the radial homotopy.

Proposition 3.12. — *The sections $\mathsf{e}_0(\xi)$ and $\mathsf{e}(\xi)$ are* $(\exp,\varpi)$*-related:*

$$\mathsf{e}_0(\xi) \sim_{(\exp,\varpi)} \mathsf{e}(\xi). \tag{59}$$

Similarly, over the subset $\mathfrak{g}_\natural \subset \mathfrak{g}$, one has

$$\mathsf{f}_0(\xi) + \mathsf{e}_0(C\,\xi) \sim_{(\exp,\varpi)} \mathsf{f}(\xi), \tag{60}$$

where $C\colon \mathfrak{g}_\natural \to \mathrm{End}(\mathfrak{g})$ is given by the formula:

$$C|_\nu = \big(1/2\coth(z/2) - 1/z\big)\big|_{z=\mathrm{ad}_\nu}, \quad \nu\in\mathfrak{g}_\natural. \tag{61}$$

Proof. — Recall that β_0 denotes the 'constant vector field' $\mathscr{E}_0(\beta,0)$. We extend the notation $(\cdot)_0$ to $\mathfrak{g}^* \cong \mathfrak{g}$-valued functions on $\mathfrak{g}^* \cong \mathfrak{g}$: For instance, the vector field corresponding to the function $\nu \mapsto -\mathrm{ad}_\xi\,\nu = \mathrm{ad}_\nu\,\xi$ is $(\mathrm{ad}_\nu\,\xi)_0 = \mathscr{E}_0(\xi)$.

The vector field part of the relation (59) says that $\mathscr{E}_0(\xi) \sim_{\exp} \xi^L - \xi^R = \mathscr{E}_{\mathsf{ad}}(\xi)$, which follows by the G-equivariance of exp. The 1-form part of (59) is equivalent to the following property **[3]** of ϖ:

$$\iota(\mathscr{E}_0(\xi))\varpi = \tfrac{1}{2}\exp^* B(\theta^L+\theta^R,\xi) - B(\theta_0,\xi).$$

Since exp is a local diffeomorphism over $\mathfrak{g}_\natural$, the section $\mathsf{f}(\xi)$ of $\mathbb{T}G$ is $(\exp,\varpi)$-related to a unique section $\widetilde{\mathsf{f}}(\xi)$ of $\mathbb{T}\mathfrak{g}|_{\mathfrak{g}_\natural}$. Since inner products are preserved under the $(\exp,\varpi)$-relation (see (12)) we have

$$\langle \mathsf{e}_0(\xi'), \widetilde{\mathsf{f}}_0(\xi)\rangle = \langle \mathsf{e}(\xi'),\mathsf{f}_0(\xi)\rangle = B(\xi',\xi) = \langle \mathsf{e}_0(\xi'),\mathsf{f}_0(\xi)\rangle$$

for all $\xi' \in \mathfrak{g}$, showing that the $F_{\mathfrak{g}}$-component of $\widetilde{\mathsf{f}}_0(\xi)$ is equal to $\mathsf{f}_0(\xi)$. It follows that $\widetilde{\mathsf{f}}_0(\xi) = \mathsf{f}_0(\xi) + \mathsf{e}_0(C(\xi))$, where C is defined by $B(\xi', C(\xi)) = \langle \mathsf{f}_0(\xi'), \widetilde{\mathsf{f}}_0(\xi)\rangle$. To compute C, we re-write (60) in the equivalent form (using (59)):

$$\mathsf{f}_0(\xi) \sim_{(\exp,\varpi)} \mathsf{f}(\xi) - \mathsf{e}(C(\xi))$$

Again, we write out the vector field and 1-form parts of this relation:

$$\begin{aligned} \xi_0 &= \tfrac{1}{2}\exp^*(\xi^L + \xi^R) - \mathscr{a}_0(C(\xi)), \\ \iota(\xi_0)\varpi &= \tfrac{1}{4}\exp^* B(\theta^L - \theta^R, \xi) - \tfrac{1}{2}\exp^* B(\theta^L + \theta^R, C(\xi)). \end{aligned} \tag{62}$$

We now verify that C given by (61) satisfies these two equations. Let $T, U^L, U^R\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ be the functions defined by

$$\iota(\xi_0)\varpi = B(\theta_0, T\,\xi), \quad \exp^*\theta^L = U^L\,\theta_0, \quad \exp^*\theta^R = U^R\,\theta_0.$$

It is known that (for the first identity, see e.g. [**45**])

$$T|_\nu = \left(\frac{\sinh(z) - z}{z^2}\right)\Big|_{z=\mathrm{ad}_\nu}, \quad U^L|_\nu = \left(\frac{1-e^{-z}}{z}\right)\Big|_{z=\mathrm{ad}_\nu}, \quad U^R|_\nu = \left(\frac{e^z - 1}{z}\right)\Big|_{z=\mathrm{ad}_\nu}.$$

Note that U^L and U^R are transposes relative to the inner product on $\mathfrak{g}$, and that they are invertible over $\mathfrak{g}_\natural$. Their definitions imply that

$$\exp^*\xi^L = ((U^L)^{-1}\,\xi)_0, \quad \exp^*\xi^R = ((U^R)^{-1}\,\xi)_0.$$

The first equation in (62) becomes

$$\xi_0 = \left(\left(\frac{(U^L)^{-1} + (U^R)^{-1}}{2} - \mathrm{ad}_\nu\, C\right)\xi\right)_0$$

which follows from the identity

$$1 = \frac{1}{2}\left(\frac{z}{1-e^{-z}} + \frac{z}{e^z - 1}\right) - z\left(\frac{1}{2}\coth\left(\frac{z}{2}\right) - \frac{1}{z}\right).$$

In a similar fashion, the second equation in (62) follows from the identity

$$\frac{\sinh(z) - z}{z^2} = \frac{1}{4}\left(\frac{e^z-1}{z} - \frac{1-e^{-z}}{z}\right) - \frac{1}{2}\left(\frac{e^z-1}{z} + \frac{1-e^{-z}}{z}\right)\left(\frac{1}{2}\coth(\frac{z}{2}) - \frac{1}{z}\right). \quad \square$$

As an immediate consequence of (59), we obtain

Theorem 3.13. — *The exponential map and the 2-form ϖ define a Dirac morphism*

$$(\exp, \varpi)\colon (\mathfrak{g}, E_{\mathfrak{g}}, 0) \to (G, E_G, \eta).$$

It is a strong Dirac morphism over the open subset $\mathfrak{g}_\natural \subset \mathfrak{g}$.

Let $\widetilde{F}_{\mathfrak{g}}$ be the backward image (defined over $\mathfrak{g}_\natural$) of F_G under $(\exp, \varpi)$, and let

$$\varepsilon \in C^\infty(\mathfrak{g}_\natural, \wedge^2\mathfrak{g})$$

be the unique map such that the associated orthogonal transformation $A^{-\mathsf{e}_0(\varepsilon)} \in \Gamma(\mathrm{O}(\mathbb{T}\mathfrak{g}_\natural))$ takes $F_{\mathfrak{g}}$ to $\widetilde{F}_{\mathfrak{g}}$. By (60), this section is given by $\iota_\xi\varepsilon = C(\xi)$, with C given by (61).

Let $[\varepsilon,\varepsilon]_{\mathrm{Sch}} \in C^\infty(\mathfrak{g}_\natural, \wedge^3\mathfrak{g})$ be defined using the Schouten bracket on $\wedge\mathfrak{g}$, and $\mathrm{d}\varepsilon \in C^\infty(\mathfrak{g}_\natural, \wedge^3\mathfrak{g})$ the exterior differential of ε, viewed as a 2-form on $\mathfrak{g}_\natural$.

Proposition 3.14. — *The map ε satisfies the classical dynamical Yang-Baxter equation:*

$$d\varepsilon + \tfrac{1}{2}[\varepsilon,\varepsilon]_{\mathrm{Sch}} = \Xi. \tag{63}$$

Proof. — Proposition 2.14 and the discussion following it show that $\mathrm{d}\varepsilon + \frac{1}{2}[\varepsilon,\varepsilon]_{\mathrm{Sch}}$ equals the Courant tensor of $\widetilde{F_{\mathfrak{g}}}$ (relative to the complementary subbundle $E_{\mathfrak{g}}$). By Lemma 3.5, together with Proposition 2.10, $\Upsilon^{\widetilde{F_{\mathfrak{g}}}} = \Xi$. □

This solution of the classical dynamical Yang-Baxter equation was obtained in [**5**], using a different argument. As a special case of Proposition 1.18, the map ε relates the linear Poisson bivector $\pi_{\mathfrak{g}}$ on $\mathfrak{g} \cong \mathfrak{g}^*$ with the pull-back $\exp^*\pi_G \in \mathfrak{X}^2(\mathfrak{g}_\natural)$ of the bivector field (46) on G:

$$\exp^*\pi_G = \pi_{\mathfrak{g}} + \mathscr{R}_0(\varepsilon).$$

3.6. The Gauss-Dirac structure. — In this Section we assume that $G = K^{\mathbb{C}}$ is a complex Lie group, given as the complexification of a compact, connected Lie group K of rank l. Thus the Cartan-Dirac structure E_G will be regarded as a holomorphic Dirac structure on the complex Lie group G. We will show that G carries another interesting Dirac structure besides the Cartan-Dirac structure. An important feature of this Dirac structure is that the corresponding Dirac foliation has an open dense leaf.

Take the bilinear form B on $\mathfrak{g}$ to be the complexification of a positive definite invariant inner product on $\mathfrak{k}$. Let T_K be a maximal torus in K, with complexification $T = T_K^{\mathbb{C}}$. Let

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+ \tag{64}$$

be the triangular decomposition relative to some choice of positive Weyl chamber, where $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) is the nilpotent subalgebra given as the sum of positive (resp. negative) root spaces. For every root α, let e_α be a corresponding root vector, with the normalization $B(\overline{e}_\alpha, e_\alpha) = 1$ and $e_{-\alpha} = \overline{e}_\alpha$. The unipotent subgroups corresponding to $\mathfrak{n}_\pm$ are denoted $N_\pm$. Recall that the multiplication map

$$j\colon N_- \times T \times N_+ \to G,\ (g_-, g_0, g_+) \mapsto g_- g_0 g_+ \tag{65}$$

is a diffeomorphism onto its image $\mathscr{O} \subset G$, called the *big Gauss cell*. The big Gauss cell is open and dense in G, and the inverse map $j^{-1}\colon \mathscr{O} \to N_- \times T \times N_+$ is known as the Gauss decomposition. Consider $\mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$ as Section 3.1. Then

$$\mathfrak{s} = \{(\xi_+ + \xi_0) \oplus (\xi_- - \xi_0) \in \mathfrak{d} \mid \ \xi_- \in \mathfrak{n}_-,\ \xi_0 \in \mathfrak{t},\ \xi_+ \in \mathfrak{n}_+\} \tag{66}$$

is a Lagrangian subalgebra of $\mathfrak{d}$, corresponding to the subgroup

$$S = \{(g_+ t,\ g_- t^{-1}) \in G \times G \,|\, g_- \in N_-,\ t \in T,\ g_+ \in N_+\}$$

of $D = G \times G$. Since $\mathfrak{s}$ is transverse to the diagonal $\mathfrak{g}_\Delta$, the corresponding Lagrangian subbundle $\widehat{F}_G := E^{\mathfrak{s}}$ is transverse to the Cartan-Dirac structure E_G:

$$\mathbb{T}G = E_G \oplus \widehat{F}_G.$$

We shall refer to it as to *Gauss-Cartan* splitting.

Unlike the complement F_G defined by the anti-diagonal, $\widehat{F}_G$ is integrable (since $\mathfrak{s}$ is a subalgebra), and it defines a Dirac manifold $(G, \widehat{F}_G, \eta)$. We refer to $\widehat{F}_G$ as the *Gauss-Dirac structure*. Its leaves are the orbits of S as a subgroup of D,

$$\mathcal{A}(g_+t, g_-t)(g) = g_-t^{-1}gt^{-1}g_+^{-1}. \tag{67}$$

The S-orbit of the group unit is exactly the big Gauss cell. Let $\omega_\mathcal{O}$ be the 2-form on $\mathcal{O}$, and $j^*\omega_\mathcal{O}$ its pull-back to $N_- \times T \times N_+$.

Proposition 3.15. — *The pull-back of the 2-form $\omega_\mathcal{O}$ on the big Gauss cell $N_- \times T \times N_+$ is given by:*

$$j^*\omega_\mathcal{O} = -\tfrac{1}{2} B(\theta_-^L, \mathrm{Ad}_{g_0}\, \theta_+^R). \tag{68}$$

Here $\theta_\pm^L$, $\theta_\pm^R$ are the Maurer-Cartan-forms on $N^\pm$, and g_0 is the T-component (i.e. projection of $N_- \times T \times N_+$ to the middle factor).

Proof. — Let $\omega \in \Omega^2(N_- \times T \times N_+)$ denote the 2-form on the right hand side of (68). Since both ω and $\omega_\mathcal{O}$ are S-invariant, it suffices to check that $j^*\omega_\mathcal{O} = \omega$ at the group unit $g = e$. At the group unit, the formula (40) for $\omega_\mathcal{O}$ simplifies to

$$\omega_\mathcal{O}(\mathcal{A}(\zeta_1), \mathcal{A}(\zeta_2))|_e = \tfrac{1}{2}(B(\xi_1', \xi_2) - B(\xi_2', \xi_1)), \tag{69}$$

for $\zeta_1 = (\xi_1, \xi_1'),\ \zeta_2 = (\xi_2, \xi_2') \in \mathfrak{s} \subset \mathfrak{d}$. Its kernel is

$$\ker(\omega_\mathcal{O}|_e) = \{\mathcal{A}(\zeta)\big|_e \mid \zeta = (\xi_0, -\xi_0), \xi_0 \in \mathfrak{t}\} = T_e(T)$$

which coincides with the kernel of $-\frac{1}{2}B(\theta_-^L, \theta_+^R)|_e$. Moreover, it is clear that $T_e(N_+)$ and $T_e(N_-)$ are isotropic subspaces for both 2-forms. Hence it is enough to compare on tangent vectors $\mathcal{A}(\zeta_1), \mathcal{A}(\zeta_2)$ for ζ_i of the form $\zeta_1 = (0, \xi_-)$ with $\xi_- \in \mathfrak{n}_-$, and $\zeta_2 = (\xi_+, 0)$ with $\xi_+ \in \mathfrak{n}_+$. (69) gives,

$$\omega_\mathcal{O}(\mathcal{A}(0, \xi_-), \mathcal{A}(\xi_+, 0))|_e = \tfrac{1}{2}B(\xi_+, \xi_-).$$

Since $j^*\mathcal{A}(\xi_+, 0)|_e = (0, 0, \xi_+) \in \mathfrak{n}_+ \subset \mathfrak{g} = T_eG$ and $j^*\mathcal{A}(0, \xi_-)|_e = (-\xi_-, 0, 0)$, the right hand side of (68) gives exactly the same answer. $\square$

Since F_G and $\widehat{F}_G$ are both complements to the Cartan-Dirac structure E_G, they are related by an element in $\Gamma(\wedge^2 E_G)$. To compute this element, let $\mathfrak{p}$ be the anti-diagonal in $\mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$, and let $\mathfrak{g}_\Delta \cong \mathfrak{g}$ be the diagonal. Let

$$\mathfrak{r} = \sum e_{-\alpha} \wedge e_\alpha \in \wedge^2 \mathfrak{g} \tag{70}$$

be the classical r-matrix.

Lemma 3.16. — *The bivector taking $\mathfrak{p}$ to $\mathfrak{s}$ is the image $\mathfrak{r}_\Delta \in \wedge^2 \mathfrak{g}_\Delta$ of the classical $\mathfrak{r}$-matrix under the diagonal embedding $\mathfrak{g} \to \mathfrak{g}_\Delta \subset \mathfrak{d}$.*

Proof. — Let $\mathfrak{g} \oplus \mathfrak{g}^*$ carry the bilinear form defined by the pairing, and consider the isometric isomorphism

$$\mathfrak{g} \oplus \mathfrak{g}^* \to \mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}, \quad \xi \oplus \mu \mapsto (\xi + \tfrac{B^\sharp(\mu)}{2}) \oplus (\xi - \tfrac{B^\sharp(\mu)}{2}).$$

This isomorphism takes $\mathfrak{g} = \mathfrak{g} \oplus 0$ to the diagonal $\mathfrak{g}_\Delta$, and $\mathfrak{g}^*$ to the anti-diagonal, $\mathfrak{p}$. The graph $\mathrm{Gr}_\mathfrak{r} \subset \mathfrak{g} \oplus \mathfrak{g}^*$ of the bivector $\mathfrak{r}$ is spanned by vectors of the form

$$0 \oplus B^\flat(\xi_0), \quad e_\alpha \oplus B^\flat(e_\alpha), \quad e_{-\alpha} \oplus (-B^\flat(e_{-\alpha})),$$

for $\xi_0 \in \mathfrak{t}$ and positive roots α. The isomorphism $\mathfrak{g} \oplus \mathfrak{g}^* \cong \mathfrak{d}$ takes these vectors to

$$\xi_0/2 \oplus (-\xi_0/2), \quad 0 \oplus e_{-\alpha}, \; e_\alpha \oplus 0.$$

Hence, it defines an isomorphism $\mathrm{Gr}_\mathfrak{r} \cong \mathfrak{s}$. □

Corollary 3.17. — *The orthogonal transformation $A^{-\mathsf{e}(\mathfrak{r})} \in \Gamma(\mathrm{O}(\mathbb{T}G))$ takes F_G to $\widehat{F}_G$.*

Proof. — This follows from Lemma 3.16 and the isomorphism $\mathbb{T}G \cong G \times \mathfrak{d}$. □

The Gauss-Cartan splitting $\mathbb{T}G = E_G \oplus \widehat{F_G}$ also defines a bivector field $\widehat{\pi}_G$, and Proposition1.18 implies that it is related to the bivector field π_G (46) by

$$\widehat{\pi}_G = \pi_G + \mathscr{A}_{\mathsf{ad}}(\mathfrak{r}).$$

Since $\widehat{F_G}$ is integrable, this bivector field is in fact a Poisson structure on G – see the remarks before Proposition 2.10. (On the other hand, unlike π_G, the Poisson structure is not invariant under the full adjoint action, but is only T-invariant.)

Proposition 3.18. — *The Poisson structure $\widehat{\pi}_G$ associated with the Gauss-Cartan splitting $\mathbb{T}G = E_G \oplus \widehat{F_G}$ is given by the formula:*

$$\widehat{\pi}_G = \tfrac{1}{2}\sum_i e_i^L \wedge (e^i)^R - \sum_{\alpha \succ 0} e_{-\alpha}^L \wedge e_\alpha^R + \tfrac{1}{2}\mathfrak{r}^L + \tfrac{1}{2}\mathfrak{r}^R.$$

Here e_i is a basis of $\mathfrak{t}$, with B-dual basis e^i, and $\mathfrak{r}^L$, $\mathfrak{r}^R$ are the left-, right-invariant bivector fields defined by $\mathfrak{r}$. The symplectic leaves of this Poisson structure are the connected components of the intersections of conjugacy classes in G with the orbits of the action (67).

This Poisson structure was first defined by Semenov-Tian-Shansky, see [**49**].

Proof. — The vectors

$$\tfrac{1}{2}(e_i \oplus (-e_i)), \; 0 \oplus (-e_{-\alpha}), \quad e_\alpha \oplus 0$$

form basis of $\mathfrak{s}$ that is dual (relative to the bilinear form on $\mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$) to the basis

$$e^i \oplus e^i, \quad e_\alpha \oplus e_\alpha, \quad e_{-\alpha} \oplus e_{-\alpha}$$

of the diagonal. Using the formula (18) for the bivector field, we obtain

$$\begin{aligned}\widehat{\pi}_G &= \tfrac{1}{2}\sum_i((e^i)^L - (e^i)^R)\wedge \tfrac{e_i^L+e_i^R}{2} + \tfrac{1}{2}\sum_{\alpha\succ 0}(e_\alpha^L - e_\alpha^R)\wedge(-e_{-\alpha}^L) + \tfrac{1}{2}\sum_{\alpha\succ 0}(e_{-\alpha}^L - e_{-\alpha}^R)\wedge(-e_\alpha)^R\\ &= \tfrac{1}{2}\sum_i e_i^L\wedge(e^i)^R - \sum_{\alpha\succ 0} e_{-\alpha}^L\wedge e_\alpha^R + \tfrac{1}{2}\mathfrak{r}^L + \tfrac{1}{2}\mathfrak{r}^R.\end{aligned}$$

Here we have used that the left- and right-invariant bivector fields generated by

$$\sum_i e_i\wedge e^i = \sum_i e_i\wedge e^i + \sum_{\alpha\succ 0} e_{-\alpha}\wedge e_\alpha + \sum_{\alpha\succ 0} e_\alpha\wedge e_{-\alpha}$$

coincide. □

Remark 3.19. — The Lagrangian subalgebra $\mathfrak{s}$ defines a Manin triple $(\mathfrak{d}=\mathfrak{g}\oplus\overline{\mathfrak{g}}, \mathfrak{g}_\Delta, \mathfrak{s})$, which induces a Poisson-Lie group structure on the double $D = G\times G$. The Poisson structure $\widehat{\pi}_G$ is the push-forward image of this Poisson-Lie structure under the natural projection $D\to D/G\cong G$, see e.g. [**1**, Sec. 3.6].

4. Pure spinors on Lie groups

In the previous section we identified $\mathbb{T}G\cong G\times\mathfrak{d}$ as Courant algebroids. In particular, we have an identification $\mathrm{Cl}(\mathbb{T}G)\cong G\times\mathrm{Cl}(\mathfrak{d})$ of Clifford algebra bundles. In this section, we will complement this isomorphism of Clifford bundles by an isomorphism of spinor modules,

$$\wedge T^*G\cong G\times\mathrm{Cl}(\mathfrak{g}),$$

where $\mathrm{Cl}(\mathfrak{g})$ is given the structure of a spinor module over $\mathrm{Cl}(\mathfrak{d})$. The differential $\mathrm{d}+\eta$ on $\Omega(G)$ intertwines with a certain differential d_{Cl} on $\mathrm{Cl}(\mathfrak{g})$. Hence, given a pure spinor $x\in\mathrm{Cl}(\mathfrak{g})$ defining a Lagrangian subspace $\mathfrak{s}\subset\mathfrak{d}$, one directly obtains a pure spinor $\phi^{\mathfrak{s}}\in\Omega(G)$ defining $E^{\mathfrak{s}}$. We will also obtain expressions for $(\mathrm{d}+\eta)\phi^{\mathfrak{s}}$ from the properties of x.

4.1. $\mathrm{Cl}(\mathfrak{g})$ as a spinor module over $\mathrm{Cl}(\mathfrak{g}\oplus\overline{\mathfrak{g}})$. — Recall from Examples 1.2 and 1.4 that for any vector space V with inner product B, the Clifford algebra $\mathrm{Cl}(V)$ may be viewed as a spinor module over $\mathrm{Cl}(V\oplus\overline{V})$. In the special case that $V=\mathfrak{g}$ is a Lie algebra, with B an invariant inner product, there is more structure that we now discuss.

Let $\lambda\colon\mathfrak{g}\to\wedge^2\mathfrak{g}$ be the map defined by the condition $-\iota(\xi_2)\lambda(\xi_1)=[\xi_1,\xi_2]_{\mathfrak{g}}$ (see Section 1.1), and let $\Xi\in\wedge^3\mathfrak{g}$ be the structure constants tensor (45). Then

$$\{\lambda(\xi_1),\lambda(\xi_2)\}=\lambda([\xi_1,\xi_2]_{\mathfrak{g}}),\quad \{\lambda(\xi_1),\xi_2\}=[\xi_1,\xi_2]_{\mathfrak{g}}$$
$$\{\Xi,\xi\}=-\frac{1}{4}\lambda(\xi),\qquad \{\Xi,\Xi\}=0$$

for all $\xi_1,\xi_2,\xi\in\mathfrak{g}$. The quantizations of these elements have similar properties: Let

$$\tau\colon\mathfrak{g}\to\mathrm{Cl}(\mathfrak{g}),\quad \tau(\xi)=q(\lambda(\xi)). \tag{71}$$

Then

$$[\tau(\xi_1), \tau(\xi_2)]_{\mathrm{Cl}} = \tau([\xi_1, \xi_2]_{\mathfrak{g}}), \quad [\tau(\xi_1), \xi_2]_{\mathrm{Cl}} = [\xi_1, \xi_2]_{\mathfrak{g}},$$
$$[q(\Xi), \xi]_{\mathrm{Cl}} = -\frac{1}{4}\tau(\xi), \qquad [q(\Xi), q(\Xi)]_{\mathrm{Cl}} \in \mathbb{K}.$$

(One can show (cf. [**4**]) that the constant $[q(\Xi), q(\Xi)]_{\mathrm{Cl}}$ is $\frac{1}{24}$ times the trace of the Casimir operator in the adjoint representation.) This last identity implies that the derivation

$$\mathrm{d}^{\mathrm{Cl}} = -4[q(\Xi), \cdot]_{\mathrm{Cl}} : \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g}) \tag{72}$$

squares to 0. We call d^{Cl} the *Clifford differential* [**4, 38**].

For the Lie algebra $\mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$, with bilinear form $B \oplus (-B)$, the corresponding elements $\Xi_{\mathfrak{d}}$ and $\lambda_{\mathfrak{d}}$ in $\wedge\mathfrak{d} = \wedge\mathfrak{g} \otimes \wedge\mathfrak{g}$ are given by

$$\Xi_{\mathfrak{d}} = \Xi \otimes 1 + 1 \otimes \Xi, \qquad \lambda_{\mathfrak{d}}(\xi, \xi') = \lambda(\xi) \otimes 1 - 1 \otimes \lambda(\xi'), \quad \text{for } (\xi, \xi') \in \mathfrak{d}.$$

Note also that $q(\Xi_{\mathfrak{d}})^2 = 0$. Consider the Clifford algebra $\mathrm{Cl}(\mathfrak{g})$ as a spinor module over $\mathrm{Cl}(\mathfrak{d})$, with Clifford action given on generators $\zeta = (\xi, \xi') \in \mathfrak{d}$ by

$$\varrho^{\mathrm{Cl}}(\xi, \xi') = l^{\mathrm{Cl}}(\xi) - r^{\mathrm{Cl}}(\xi').$$

Then the Clifford differential d^{Cl} is implemented as a Clifford action:

$$\mathrm{d}^{\mathrm{Cl}} = -4\varrho^{\mathrm{Cl}}(q(\Xi_{\mathfrak{d}})).$$

The elements $\tau_{\mathfrak{d}}(\zeta) = q(\lambda_{\mathfrak{d}}(\zeta))$ generate a $\mathfrak{d}$-action on $\mathrm{Cl}(\mathfrak{g})$, with generators

$$\mathscr{L}^{\mathrm{Cl}}(\zeta) = l^{\mathrm{Cl}}(\tau(\xi)) - r^{\mathrm{Cl}}(\tau(\xi')) = \varrho^{\mathrm{Cl}}(\tau(\zeta)).$$

Note that

$$\mathscr{L}^{\mathrm{Cl}}(\zeta) = [\varrho^{\mathrm{Cl}}(\zeta), \mathrm{d}^{\mathrm{Cl}}], \tag{73}$$

which implies that

$$[\varrho^{\mathrm{Cl}}(\zeta_1), [\varrho^{\mathrm{Cl}}(\zeta_2), \mathrm{d}^{\mathrm{Cl}}]] = \varrho^{\mathrm{Cl}}([\zeta_1, \zeta_2]).$$

Let $\mathfrak{s} \subset \mathfrak{d}$ be a Lagrangian subspace, and recall the definition of $\Upsilon^{\mathfrak{s}}$ given in (39). Given a Lagrangian complement $\mathfrak{p}$ to $\mathfrak{s}$, let $\mathrm{pr}_{\mathfrak{s}} : \mathfrak{d} \to \mathfrak{s}$ be the projection along $\mathfrak{p}$, and define a linear functional $\sigma^{\mathfrak{s}} \in \mathfrak{s}^*$ by

$$\langle \sigma^{\mathfrak{s}},\ \xi \rangle = \tfrac{1}{2}\,\mathrm{trace}(\mathrm{pr}_{\mathfrak{s}} \circ \mathrm{ad}_{\xi}\big|_{\mathfrak{s}}), \quad \xi \in \mathfrak{s}. \tag{74}$$

If $\mathfrak{s}$ is a Lagrangian subalgebra (i.e. $\Upsilon^{\mathfrak{s}} = 0$), we may omit $\mathrm{pr}_{\mathfrak{s}}$ in this formula; in this case $\sigma^{\mathfrak{s}}$ equals $-\frac{1}{2}$ times the modular character of the Lie algebra $\mathfrak{s}$.

Proposition 4.1. — *Let $\mathfrak{s} \subset \mathfrak{d}$ be a Lagrangian subspace, with defining pure spinor $x \in \mathrm{Cl}(\mathfrak{g})$. Choose a Lagrangian complement $\mathfrak{p} \cong \mathfrak{s}^*$ to $\mathfrak{s}$ to view $\Upsilon^{\mathfrak{s}}$ as an element of the Clifford algebra $\mathrm{Cl}(\mathfrak{d})$. Then*

$$d^{\mathrm{Cl}}x = \varrho^{\mathrm{Cl}}(-\Upsilon^{\mathfrak{s}} + \sigma^{\mathfrak{s}})x.$$

In particular, $\mathfrak{s}$ is a Lie subalgebra if and only if the defining pure spinor x is 'integrable', in the sense that

$$d^{\mathrm{Cl}}\, x \in \varrho^{\mathrm{Cl}}(\mathfrak{d})x.$$

Proof. — The choice of a Lagrangian complement identifies $\mathfrak{d} = \mathfrak{s} \oplus \mathfrak{s}^*$, with bilinear form given by the pairing. Using a basis e_i of $\mathfrak{s}$ and a dual basis f^i of $\mathfrak{s}^*$, we have

$$\begin{aligned}4\Xi_{\mathfrak{d}} =&\tfrac{1}{6}\textstyle\sum_{ijk} B_{\mathfrak{d}}([e_i,e_j],e_k) f^i\wedge f^j\wedge f^k + \tfrac{1}{2}\textstyle\sum_{ijk} B_{\mathfrak{d}}([e_j,e_k],f^i)\ e_i\wedge f^j\wedge f^k\\ &+\tfrac{1}{2}\sum_{ijk} B_{\mathfrak{d}}([f^j,f^k],e_i)\ e_j\wedge e_k\wedge f^i + \tfrac{1}{6}\textstyle\sum_{ijk} B_{\mathfrak{d}}([f^j,f^k],f^i)\ e_j\wedge e_k\wedge e_i.\end{aligned}$$

The quantization map takes the last two terms into the left ideal $\mathrm{Cl}(\mathfrak{d})\mathfrak{s}$, and it takes the second term to

$$\tfrac{1}{2}\sum_{ik} B_{\mathfrak{d}}([e_i,e_k],f^i)\ f^k + \tfrac{1}{2}\sum_{ijk} B_{\mathfrak{d}}([e_j,e_k],f^i)\ f^j f^k e_i = -\sigma^{\mathfrak{s}} \mod \mathrm{Cl}(\mathfrak{d})\mathfrak{s}.$$

This gives

$$-4q(\Xi_{\mathfrak{d}}) = -\Upsilon^{\mathfrak{s}} + \sigma^{\mathfrak{s}} \mod \mathrm{Cl}(\mathfrak{d})\mathfrak{s},$$

from which the result is immediate. □

Let us now assume that the adjoint action $\mathrm{Ad}\colon G\to \mathrm{O}(\mathfrak{g})$ lifts to a group homomorphism

$$\tau\colon G\to \mathrm{Pin}(\mathfrak{g})\subset \mathrm{Cl}(\mathfrak{g}) \tag{75}$$

to the double cover $\mathrm{Pin}(\mathfrak{g})\to \mathrm{O}(\mathfrak{g})$. If G is connected, this is automatic if $\pi_1(G)$ is torsion free. Note that (75) is consistent with our previous notation $\tau(\xi) = q(\lambda(\xi))$, since [**4**]

$$\tau(\xi) = \frac{d}{dt}\Big|_{t=0}\tau(\exp t\xi).$$

We will write $\mathsf{N}(g) = \mathsf{N}(\tau(g)) = \pm 1$ for the image under the norm homomorphism, and $|g| = |\tau(g)|$ for the parity of $\tau(g)$. Since $\tau(g)$ lifts Ad_g, one has $(-1)^{|g|} = \det(\mathrm{Ad}_g)$. The definition of the Pin group implies that conjugation by $\tau(g)$ is the *twisted adjoint action*,

$$\tau(g)x\tau(g^{-1}) = \widetilde{\mathrm{Ad}}_g(x) := (-1)^{|g||x|}\,\mathrm{Ad}_g(x) \tag{76}$$

(using the extension of $\mathrm{Ad}_g\in \mathrm{O}(\mathfrak{g})$ to an automorphism of the Clifford algebra). This twisted adjoint action extends to an action of the group D on $\mathrm{Cl}(\mathfrak{g})$,

$$\mathscr{A}^{\mathrm{Cl}}(a,a')(x) = \tau(a)x\tau((a')^{-1}). \tag{77}$$

4.2. The isomorphism $\wedge T^*G\cong G\times \mathrm{Cl}(\mathfrak{g})$. — Let us now fix a generator $\mu\in \det(\mathfrak{g})$, and consider the corresponding star operator $\star\colon \wedge\mathfrak{g}^*\to\wedge\mathfrak{g}$, see Remark 1.5(b). The star operator satisfies

$$\mathrm{Ad}_g\circ\star = (-1)^{|g|}\star\circ\mathrm{Ad}^*_{g^{-1}}\,. \tag{78}$$

We use the trivialization by left-invariant forms to identify $\wedge T^*G\cong G\times\wedge\mathfrak{g}^*$. Applying $\star$ pointwise, we obtain an isomorphism $q\circ\star\colon \wedge T^*_gG \xrightarrow{\sim} \mathrm{Cl}(\mathfrak{g})$ for each $g\in G$. Let us define the linear map

$$\mathscr{R}\colon \mathrm{Cl}(\mathfrak{g})\to\Omega(G),\quad \mathscr{R}(x)|_g = (q\circ\star)^{-1}(x\tau(g)). \tag{79}$$

We denote by $\mu^* \in \det(\mathfrak{g}^*)$ the dual generator, defined by $\iota((\mu^*)^\top)\mu = 1$, and let μ_G be the left-invariant volume form on G defined by μ^*.

Proposition 4.2. — *The map (79) has the following properties:*

a. *$\mathcal{R}$ intertwines the Clifford actions, in the sense that*

$$\mathcal{R}(\varrho^{\mathrm{Cl}}(\zeta)x) = \varrho(\mathsf{s}(\zeta))\mathcal{R}(x), \quad \forall x \in \mathrm{Cl}(\mathfrak{g}),\ \zeta \in \mathfrak{d}.$$

Up to a scalar function, $\mathcal{R}$ is uniquely characterized by this property.

b. *$\mathcal{R}$ intertwines differentials:*

$$\mathcal{R}(d^{\mathrm{Cl}}(x)) = (d+\eta)\mathcal{R}(x), \quad \forall x \in \mathrm{Cl}(\mathfrak{g}).$$

c. *$\mathcal{R}$ satisfies has the following D-equivariance property: For any $h = (a, a') \in D$, and at any given point $g \in G$,*

$$\mathcal{A}(h^{-1})^* \mathcal{R}(x) = (-1)^{|a|(|g|+|x|)} \mathcal{R}(\mathcal{A}^{\mathrm{Cl}}(h)x).$$

d. *$\mathcal{R}$ relates the bilinear pairings on the Clifford modules $\mathrm{Cl}(\mathfrak{g})$ and $\Omega(G)$ as follows: At any given point $g \in G$, and for all $x, x' \in \mathrm{Cl}(\mathfrak{g})$,*

$$(\mathcal{R}(x), \mathcal{R}(x'))_{\wedge T^*G} = (-1)^{|g|(\dim G+1)}\ \mathsf{N}(g)\ (x, x')_{\mathrm{Cl}(\mathfrak{g})}\ \ \mu_G. \tag{80}$$

Here the pairing $(\cdot,\cdot)_{\mathrm{Cl}(\mathfrak{g})}$ is viewed as scalar-valued, using the trivialization of $\det(\mathfrak{g})$ *defined by μ. (Cf. Remark 1.5.)*

Notice that the signs in part (c), (d) disappear if G is connected.

Proof. — (a) Given $\xi \in \mathfrak{g}$, let $\epsilon(\xi) : \wedge\mathfrak{g} \to \wedge\mathfrak{g}$ be defined by $\epsilon(\xi)\xi' = \xi \wedge \xi'$. Then

$$l^{\mathrm{Cl}}(\xi) \circ q = q \circ (\epsilon(\xi) + \tfrac{1}{2}\iota(B^\flat(\xi))), \quad r^{\mathrm{Cl}}(\xi) \circ q = q \circ (\epsilon(\xi) - \tfrac{1}{2}\iota(B^\flat(\xi))).$$

Since the star operator exchanges exterior multiplication and contraction, we have

$$\star^{-1} \circ q^{-1} \circ \varrho^{\mathrm{Cl}}(\xi, \xi') = \Big(\iota(\xi - \xi') + \epsilon\left(B^\flat\left(\tfrac{\xi+\xi'}{2}\right)\right)\Big) \circ \star^{-1} \circ q^{-1}.$$

On the other hand,

$$(\varrho^{\mathrm{Cl}}(\xi, \xi')x)\tau(g) = (\xi x - (-1)^{|x|}x\xi')\tau(g) = \varrho^{\mathrm{Cl}}(\xi, \mathrm{Ad}_{g^{-1}}\xi')(x\tau(g)).$$

This implies that, at $g \in G$,

$$\mathcal{R}(\varrho^{\mathrm{Cl}}(\xi, \xi')x) = \Big(\iota(\xi - \mathrm{Ad}_{g^{-1}}\xi') + \epsilon\left(B^\flat\left(\tfrac{\xi+\mathrm{Ad}_{g^{-1}}\xi'}{2}\right)\right)\Big)\mathcal{R}(x),$$

which is precisely the Clifford action of $\mathsf{s}(\xi, \xi')$ since

$$\mathsf{s}(\xi, \xi') = (\xi - \mathrm{Ad}_{g^{-1}}\xi') \oplus B^\flat\left(\tfrac{\xi+\mathrm{Ad}_{g^{-1}}\xi'}{2}\right)$$

under left-trivialization $\mathbb{T}G \cong G \times (\mathfrak{g} \oplus \mathfrak{g}^*)$. This shows that $\mathcal{R}$ intertwines the Clifford actions of $\mathrm{Cl}(\mathfrak{d}) \cong \mathrm{Cl}(\mathbb{T}_g G)$. By the uniqueness properties of spinor modules, $\mathcal{R}$ is uniquely characterized by this property up to a scalar.

(b) From the global equivariance property in (c), verified below, we obtain the infinitesimal equivariance: $\mathcal{L}(\mathcal{A}(\zeta))\mathcal{R}(x) = \mathcal{R}(\mathcal{L}^{\mathrm{Cl}}(\zeta)x)$. Since $[\varrho(\mathsf{s}(\zeta)), \mathrm{d}+\eta] = \mathcal{L}(\mathcal{A}(\zeta))$ and $[\varrho^{\mathrm{Cl}}(\zeta), \mathrm{d}^{\mathrm{Cl}}] = \mathcal{L}^{\mathrm{Cl}}(\zeta)$, this gives

$$\begin{aligned}\varrho(\mathsf{s}(\zeta))\big((\mathrm{d}+\eta)\mathcal{R}(x) - \mathcal{R}(\mathrm{d}^{\mathrm{Cl}}x)\big) &= \mathcal{L}(\mathcal{A}(\zeta))\mathcal{R}(x) - \mathcal{R}(\varrho^{\mathrm{Cl}}(\zeta)\mathrm{d}^{\mathrm{Cl}}x)\\ &= \mathcal{L}(\mathcal{A}(\zeta))\mathcal{R}(x) - \mathcal{R}(\mathcal{L}^{\mathrm{Cl}}(\zeta)x).\end{aligned}$$

That is, the map $(\mathrm{d}+\eta)\circ\mathcal{R} - \mathcal{R}\circ\mathrm{d}^{\mathrm{Cl}}\colon \mathrm{Cl}(\mathfrak{g}) \to \Gamma(\mathbb{T}G)$ intertwines the Clifford actions, and hence agrees with $\mathcal{R}$ up to a scalar function. Since its parity is opposite to that of $\mathcal{R}$, that function is zero.

(c) We have to show that for all $a \in G$,

$$l_a^*\mathcal{R}(x) = \mathcal{R}(x\tau(a)), \quad r_a^*\mathcal{R}(x) = (-1)^{|a|(|g|+|x|)}\mathcal{R}(\tau(a)x). \tag{81}$$

In terms of the left-trivialization $\wedge T^*G = G \times \wedge\mathfrak{g}^*$,

$$(l_a^*\mathcal{R}(x))|_g = \mathcal{R}(x)\big|_{ag}, \quad (r_a^*\mathcal{R}(x))\big|_g = \mathrm{Ad}^*_{a^{-1}}(\mathcal{R}(x)\big|_{ga}).$$

(Here $\mathrm{Ad}^*_{a^{-1}}$ stands for the contragredient action on $\wedge\mathfrak{g}^*$, not for a pull-back on $\Omega(G)$.) We compute, using (76) and (78):

$$\begin{aligned}\mathrm{Ad}^*_{a^{-1}}\left(\mathcal{R}(x)\big|_{ga}\right) &= (-1)^{|a|}\star^{-1}\, q^{-1}\,\mathrm{Ad}_a(x\tau(ga))\\ &= (-1)^{|a|}\,(-1)^{|a|(|x|+|g|+|a|)}\star^{-1}\, q^{-1}(\tau(a)x\tau(g))\\ &= (-1)^{|a|(|x|+|g|)}\mathcal{R}(\tau(a)x)\big|_g\end{aligned}$$

The equivariance property with respect to left translations is immediate from the definition.

(d) Use the generator $\mu \in \det(\mathfrak{g})$ and μ_G to trivialize both $\det(\mathfrak{g})$ and $\det(\wedge T^*G)$. By Remark 1.5(b) and Example 1.4 we have, at $g \in G$,

$$(\mathcal{R}(x), \mathcal{R}(x'))_{\wedge T^*G} = (x\tau(g), x'\tau(g))_{\mathrm{Cl}(\mathfrak{g})}.$$

This is computed as follows:

$$\begin{aligned}\mathrm{str}(\tau(g)^\top x^\top x'\tau(g)) &= (-1)^{|g|(|g|+|x|+|x'|)}\,\mathrm{str}(\tau(g)\tau(g)^\top x^\top x')\\ &= \mathsf{N}(g)\,(-1)^{|g|(1+|x|+|x'|)}\,\mathrm{str}(x^\top x')\end{aligned}$$

Finally, replace $|x|+|x'|$ with $\dim G$, using that $(x,x')_{\mathrm{Cl}(\mathfrak{g})}$ vanishes unless $|x|+|x'| = \dim G \mod 2$. □

As an immediate consequence of Propositions 4.1 and 4.2, we have

Corollary 4.3. — *If $x \in \mathrm{Cl}(\mathfrak{g})$ is a pure spinor defining a Lagrangian subspace $\mathfrak{s} \subset \mathfrak{d}$, then the differential form $\phi^{\mathfrak{s}} := \mathcal{R}(x) \in \Omega(G)$ is a pure spinor defining the Lagrangian subbundle $E^{\mathfrak{s}}$. It satisfies the differential equation*

$$(d+\eta)\phi^{\mathfrak{s}} = \varrho(\mathsf{s}(-\Upsilon^{\mathfrak{s}} + \sigma^{\mathfrak{s}}))\phi^{\mathfrak{s}}, \tag{82}$$

where $\sigma^{\mathfrak{s}} \in \mathfrak{s}^*$ is defined as in (74) *(using a complementary Lagrangian subspace $\mathfrak{p} \cong \mathfrak{s}^* \subset \mathfrak{d}$). Let $H \subset D$ be a subgroup preserving $\mathfrak{s}$, and define the character $u^{\mathfrak{s}}\colon H \to \mathbb{K}^\times$ by $\mathscr{A}^{\mathrm{Cl}}(h)x = u^{\mathfrak{s}}(h)x$. Then*

$$\mathscr{A}(h^{-1})^*\phi^{\mathfrak{s}} = (-1)^{|a|(|g|+|x|)}\, u^{\mathfrak{s}}(h)\phi^{\mathfrak{s}} \tag{83}$$

for all $h = (a, a') \in H$, and at any given point $g \in G$.

We are mainly interested in pure spinors defining the Cartan-Dirac structure E_G and its Lagrangian complement F_G. These are obtained by taking $x = 1$ and $x = q(\mu)$ in the above:

Proposition 4.4. — *Let $\phi_G, \psi_G \in \Omega(G)$ be the differential forms*

$$\phi_G = \mathscr{R}(1), \quad \psi_G = \mathscr{R}(q(\mu)). \tag{84}$$

Then ϕ_G, ψ_G are pure spinors defining the Lagrangian subbundles E_G, F_G. They satisfy the differential equations,

$$(d + \eta)\phi_G = 0, \quad (d + \eta)\psi_G = -\varrho(\mathsf{e}(\Xi))\psi_G. \tag{85}$$

The equivariance properties under the adjoint action of G read

$$\mathscr{A}_{\mathsf{ad}}(a^{-1})^*\phi_G = (-1)^{|a||g|}\phi_G, \quad \mathscr{A}_{\mathsf{ad}}(a^{-1})^*\psi_G = (-1)^{|a|(|g|+1)}\psi_G.$$

We will refer to ϕ_G as the *Cartan-Dirac spinor.*

Proof. — It is clear that the diagonal $\mathfrak{g}_\Delta \subset \mathfrak{d}$ is defined by the pure spinor $x = 1$. Similarly, the anti-diagonal $\mathfrak{p} \subset \mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$ is defined by the pure spinor $q(\mu) \in \mathrm{Cl}(\mathfrak{g})$:

$$\varrho^{\mathrm{Cl}}(\xi, -\xi)q(\mu) = \xi q(\mu) + (-1)^{\dim G} q(\mu)\xi = 0.$$

Hence ϕ_G, ψ_G are pure spinors defining E_G, F_G. The equivariance properties are special cases of (83), since both $\mathfrak{g}_\Delta$ and $\mathfrak{p}$ are preserved under G_Δ. Here we are using $|1| = 0$, $|q(\mu)| = \dim G \mod 2$, while $u^{\mathfrak{p}}(a) = (-1)^{|a|(1+\dim G)}$ by the calculation:

$$\tau(a)q(\mu)\tau(a^{-1}) = (-1)^{|a|\dim G} q(\mathrm{Ad}_a(\mu)) = (-1)^{|a|(1+\dim G)} q(\mu).$$

The differential equation for ϕ_G follows since $\mathrm{d}^{\mathrm{Cl}}(1) = 0$. It remains to check the differential equation for ψ_G. Since the anti-diagonal satisfies $[\mathfrak{p}, \mathfrak{p}]_{\mathfrak{d}} \subset \mathfrak{g}_\Delta$, the element $\sigma^{\mathfrak{p}} \in \mathfrak{p}^*$ is just zero. On the other hand, the element $\Upsilon^{\mathfrak{p}}$ is given by Ξ_Δ, the image of Ξ under the the map $\wedge\mathfrak{g} \xrightarrow{\sim} \wedge\mathfrak{g}_\Delta$. Hence $\mathsf{s}(\Xi_\Delta) = \mathsf{e}(\Xi)$, confirming that ψ_G satisfies (85). □

Remarks 4.5. —
a. The map $\mathscr{R}$ depends on the choice of generator $\mu \in \det(\mathfrak{g})$, via the star operator: Replacing μ with $t\mu$ changes $\mathscr{R}$ to $t^{-1}\mathscr{R}$. Hence, the definition of $\psi_G = \mathscr{R}(q(\mu))$ is independent of the choice of μ.
b. Since $(1, q(\mu))_{\mathrm{Cl}(\mathfrak{g})} = \mu$, the bilinear pairing between ϕ_G, ψ_G equals the volume form, up to a sign:

$$\big(\phi_G, \psi_G\big)_{\wedge T^*G} = \mathsf{N}(g)(-1)^{|g|(\dim G+1)}\, \mu_G.$$

Proposition 4.6. — *Over the open subset $\mathcal{U}$ of G where $1+\mathrm{Ad}_g$ is invertible, the pure spinor ψ_G is given by the formula:*

$$\psi_G = \det{}^{1/2}\big(\tfrac{1+\mathrm{Ad}_g}{2}\big) \exp\Big(\tfrac{1}{4}\, B\Big(\tfrac{1-\mathrm{Ad}_g}{1+\mathrm{Ad}_g}\ \theta^L, \theta^L\Big)\Big),$$

at any given point $g \in \mathcal{U}$. (The square root depends on the choice of lift $\tau\colon G \to \mathrm{Pin}(\mathfrak{g})$.)

Note that the exponent in this formula becomes singular where $1 + \mathrm{Ad}_g$ fails to be invertible, but these singularities are compensated by the zeroes of the factor $\det^{1/2}\big(\frac{1+\mathrm{Ad}_g}{2}\big)$. One proof of this formula is given in [**47**]; here is an outline of an alternative approach.

Sketch of proof. — One easily checks that over $\mathcal{U}$, F_G coincides with the graph of the 2-form $\omega_F := -\frac{1}{4}\, B\Big(\frac{1-\mathrm{Ad}_g}{1+\mathrm{Ad}_g}\ \theta^L, \theta^L\Big)$. Hence $\psi_G|_{\mathcal{U}} = f\exp(-\omega_F)$ for some nonvanishing function $f \in C^\infty(\mathcal{U})$, with $f(e) = 1$. Equation (85) reads, after dividing by $f\exp(-\omega_F)$,

$$\mathrm{d}\log(f) + \eta + \exp(\omega_F)\varrho(\mathsf{e}(\Xi))\big(\exp(-\omega_F)\big) = 0.$$

Taking the form degree 1 parts of both sides of this equation, one obtains the following condition on f:

$$\mathrm{d}\log(f) + \Big(\exp(\omega_F)\varrho(\mathsf{e}(\Xi))\big(\exp(-\omega_F)\big)\Big)_{[1]} = 0.$$

f is uniquely determined by this Equation with the initial condition $f(e) = 1$. It is straightforward (though slightly cumbersome) to verify that $f(g) = \det^{1/2}\big(\frac{1+\mathrm{Ad}_g}{2}\big)$ solves this equation. □

If G is connected, one has $\det(1+\mathrm{Ad}_g) \neq 0$ on a dense open subset of G. However, for a disconnected group G it vanishes on the components with $\det(\mathrm{Ad}_g) = -1$.

Example 4.7. — Let $G = \mathrm{O}(2)$. Here $\mathrm{O}(\mathfrak{g}) = \mathbb{Z}_2$ and $\mathrm{Pin}(\mathfrak{g}) = \mathbb{Z}_4$. There are two possible lifts $\mathrm{O}(\mathfrak{g}) \to \mathrm{Pin}(\mathfrak{g})$. Let $\theta \in \Omega^1(G)$ be the left-invariant Maurer-Cartan-form (using the isomorphism $\mathfrak{g} = \mathbb{R}$ defined by a generator $\mu \in \det(\mathfrak{g}) = \mathfrak{g}$). One finds that on $\mathrm{SO}(2) \subset \mathrm{O}(2)$, $\phi_G = \theta$, while $\psi_G = 1$. On the non-identity component $\mathrm{O}(2)\backslash \mathrm{SO}(2)$, the roles are reversed: $\psi_G = \pm\theta$ and $\phi_G = \pm 1$. (The signs depend on the choice of lift.) Observe that ϕ_G, ψ_G given by these formulas have the correct equivariance properties.

4.3. Group multiplication. — In this section, we will examine the composition of the map $\mathcal{R}\colon \mathrm{Cl}(\mathfrak{g}) \to \Omega(G)$ with the pull-back under group multiplication. It will be convenient to work with the element $\Lambda \in \mathrm{Cl}(\mathfrak{g}) \otimes \Omega(G)$, defined by the property

$$\mathcal{R}(x) = \mathrm{str}(x\Lambda)$$

where we have extended $\mathrm{str}\colon \mathrm{Cl}(\mathfrak{g}) \to \wedge^{[\mathrm{top}]}(\mathfrak{g}) = \mathbb{K}$ to the tensor product with $\Omega(G)$. The properties of $\mathcal{R}$ under the Clifford action translate into

$$(l^{\mathrm{Cl}}(\xi) + \varrho(s^R(\xi)))\Lambda = 0, \quad (-r^{\mathrm{Cl}}(\xi) + \varrho(s^L(\xi)))\Lambda = 0.$$

Thus Λ is itself a pure spinor for the action of $\mathrm{Cl}(\mathfrak{d}) \times \mathrm{Cl}(\mathbb{T}G)$, defining a Lagrangian subbundle of $\mathfrak{d} \times \mathbb{T}G$. The equivariance properties (81) of $\mathscr{R}$ translate into

$$l_a^*\Lambda = \tau(a^{-1})\Lambda, \qquad r_{a^{-1}}^*\Lambda = \Lambda\tau(a)$$

The first identity is immediate, while for the second identity is obtained by the calculation:

$$\begin{aligned}\mathrm{str}(x\, r_{a^{-1}}^*\Lambda) &= r_{a^{-1}}^*\mathscr{R}(x) = (-1)^{|a|(|x|+|g|)}\mathscr{R}(\tau(a)x)\\ &= (-1)^{|a|(|x|+|g|)}\,\mathrm{str}(\tau(a)x\Lambda)\\ &= \mathrm{str}(x\Lambda\tau(a)).\end{aligned}$$

(Note that $|\Lambda| = |g|$ at $g \in G$.) We finally observe that the pull-back of Λ to the group unit is simply

$$i_e^*\Lambda = 1 \in \mathrm{Cl}(\mathfrak{g}). \tag{86}$$

Let $\Lambda^1, \Lambda^2 \in \mathrm{Cl}(\mathfrak{g}) \otimes \Omega(G \times G)$ be the pull-back to the first, second G-factor, and recall the 2-form $\varsigma \in \Omega^2(G \times G)$ from (47).

Proposition 4.8. — *The pull-back of Λ under group multiplication satisfies*

$$e^{\varsigma}\,\mathrm{Mult}^*\,\Lambda = \Lambda^1\Lambda^2, \tag{87}$$

using the product in the algebra $\mathrm{Cl}(\mathfrak{g}) \otimes \Omega(G \times G)$.

Proof. — Using (51), we find that both sides of (87) are annihilated by the following operators:

$$l^{\mathrm{Cl}}(\xi) + \varrho(s^{R,1}(\xi)), \quad -r^{\mathrm{Cl}}(\xi) + \varrho(s^{L,2}(\xi)), \qquad \varrho(s^{L,1}(\xi) + s^{R,2}(\xi)).$$

Hence the two sides of (87) are pure spinors, defining the same Lagrangian subbundle of $\mathfrak{d} \times \mathbb{T}(G \times G)$. So the two sides agree up to a scalar function.

The 2-form ς is invariant under $l_{a,1}$ (left multiplication by a on the first factor) and $r_{a^{-1},2}$ (right multiplication by a^{-1} on the second factor). From the equivariance of Λ, and since $\mathrm{Mult} \circ l_{a,1} = l_a \circ \mathrm{Mult}$ and $\mathrm{Mult} \circ r_{a^{-1},2} = r_{a^{-1}} \circ \mathrm{Mult}$, we obtain the following equivariance property of $e^{\varsigma}\,\mathrm{Mult}^*\,\Lambda$:

$$\begin{aligned}(l_{a,1})^*(e^{\varsigma}\,\mathrm{Mult}^*\,\Lambda) &= \tau(a^{-1})\ (e^{\varsigma}\,\mathrm{Mult}^*\,\Lambda),\\ (r_{a^{-1},2})^*(e^{\varsigma}\,\mathrm{Mult}^*\,\Lambda) &= (e^{\varsigma}\,\mathrm{Mult}^*\,\Lambda)\tau(a).\end{aligned}$$

The product $\Lambda^1\Lambda^2$ has a similar equivariance property. Hence, to verify (87) it suffices to compare the two sides at $(e,e) \in G \times G$. But by (86), both sides of (87) pull back to 1 at (e,e). □

We will use Proposition 4.8 to obtain a formula for the pull-back of $\psi_G = \mathscr{R}(q(\mu))$, the pure spinor defining the Lagrangian subbundle $F_G \subset \mathbb{T}G$. Recall the element $\gamma \in \wedge^2(\mathfrak{g} \oplus \mathfrak{g})$ from (53).

Theorem 4.9. — *The pull-back of ψ_G under group multiplication is given by the formula*

$$e^{\varsigma}\,\mathrm{Mult}^*\,\psi_G = \varrho(\exp(-\mathsf{e}(\gamma)))\,(\psi_G^1\otimes\psi_G^2)$$

Note that *up to a scalar function*, this identity follows from Proposition 3.11.

Proof. — The element γ enters the following formula (cf. [**4**, Lemma 3.1]) , relating the product $\mathrm{Mult}^{\mathrm{Cl}}$ in $\mathrm{Cl}(\mathfrak{g})$ with the wedge product $\mathrm{Mult}^{\wedge}$ in $\wedge(\mathfrak{g})$:

$$q^{-1}\circ\mathrm{Mult}^{\mathrm{Cl}} = \mathrm{Mult}^{\wedge}\circ\exp(-\iota^{\wedge}(\gamma))\circ q^{-1}\colon \mathrm{Cl}(\mathfrak{g}\oplus\mathfrak{g})\to\wedge(\mathfrak{g}).$$

Since $\mathrm{str}\circ l^{\mathrm{Cl}}(q(\mu))\circ q\colon\ \wedge\mathfrak{g}\to\mathbb{K}$ is simply the augmentation map, we have

$$\psi_G = \mathscr{R}(q(\mu)) = \mathrm{str}(q(\mu)\Lambda) = q^{-1}(\Lambda)_{[0]},$$

where the subscript indicates the degree 0 part with respect to $\wedge\mathfrak{g}$. Using (87), we calculate:

$$\begin{aligned}
e^{\varsigma}\,\mathrm{Mult}^*\,\psi_G &= q^{-1}(\Lambda^1\Lambda^2)_{[0]}\\
&= q^{-1}\circ\big(\mathrm{Mult}^{\mathrm{Cl}}(\Lambda^1\otimes\Lambda^2)\big)_{[0]}\\
&= \big(\mathrm{Mult}^{\wedge}\circ\exp(-\iota^{\wedge}(\gamma))\circ q^{-1}(\Lambda^1\otimes\Lambda^2)\big)_{[0]}\\
&= \exp(-\mathsf{e}(\gamma))\circ\big(\mathrm{Mult}^{\wedge}\circ q^{-1}(\Lambda^1\otimes\Lambda^2)\big)_{[0]}\\
&= \exp(-\mathsf{e}(\gamma))\circ(\psi_G^1\otimes\psi_G^2).
\end{aligned}$$

Here we used that $(\iota^{\mathrm{Cl}}(\xi)+\varrho(\mathsf{e}(\xi)))\Lambda=0$, hence $(\iota^{\wedge}(\gamma)-\varrho(\mathsf{e}(\gamma)))q^{-1}(\Lambda^1\otimes\Lambda^2)=0$. $\square$

4.4. Exponential map. — Let us return to our description (Section 3.5) $\mathbb{T}\mathfrak{g}^*=\mathfrak{g}^*\times\mathfrak{d}_0$ of the Courant algebroid over $\mathfrak{g}^*$, where $\mathfrak{d}_0=\mathfrak{g}^*\rtimes\mathfrak{g}$.

Let $\wedge\mathfrak{g}^*$ be the contravariant spinor module over $\mathrm{Cl}(\mathfrak{d}_0)$ (cf. Section 1.4), with Clifford action denoted $\varrho^{\wedge}$. Let $\mathrm{d}^{\wedge}$ be the exterior algebra differential. For all $w=(\beta,\xi)\in\mathfrak{d}_0$ one has

$$L^{\wedge}(w) := [\mathrm{d}^{\wedge},\varrho^{\wedge}(w)] = \mathrm{d}^{\wedge}\beta-(\mathrm{ad}_\xi)^*.$$

One easily checks that $L^{\wedge}(w)$ defines an action of the Lie algebra $\mathfrak{d}_0$. This action exponentiates to an action of the group D_0, given as

$$\mathscr{A}^{\wedge}(\beta,g)y = \exp(\mathrm{d}^{\wedge}\beta)\wedge(\mathrm{Ad}_{g^{-1}})^*y,$$

The function

$$\tau_0\colon\mathfrak{g}^*\to\wedge\mathfrak{g}^*,\quad \tau_0(\beta)=\exp(\mathrm{d}^{\wedge}\beta)\in\wedge\mathfrak{g}^*$$

is the counterpart to the function $\tau\colon G\to\mathrm{Cl}(\mathfrak{g})$. The D_0-action commutes with the differential, and it is straightforward to check that the Clifford action is D_0-equivariant:

$$\mathscr{A}^{\wedge}(\beta,g)\big(\varrho^{\wedge}(w)y\big) = \varrho^{\wedge}(\mathrm{Ad}_{(\beta,g)}\,w)\big(\mathscr{A}^{\wedge}(\beta,g)y\big),$$

for $w\in\mathfrak{d}_0,\ (\beta,g)\in D_0,\ y\in\wedge\mathfrak{g}^*$.

Choose a generator $\mu \in \det(\mathfrak{g}^*)$, and let $\star\colon \wedge\mathfrak{g} \to \wedge\mathfrak{g}^*$ be the associated star operator [4]. Let X_π denote the modular vector field of the Kirillov-Poisson structure $\pi_{\mathfrak{g}^*}$, relative to the translation-invariant volume form $\mu_{\mathfrak{g}^*} \in \Gamma(\det(T^*\mathfrak{g}^*))$ defined by the dual generator $\mu^* \in \det(\mathfrak{g})$. (Recall that $X_\pi = 0$ if $\mathfrak{g}$ is unimodular.) Define a linear map

$$\mathscr{R}_0\colon \wedge\mathfrak{g}^* \to \Omega(\mathfrak{g}^*),$$

given at any point $\nu \in \mathfrak{g}^*$ by

$$\mathscr{R}_0(y) = \star^{-1}(y \wedge \tau_0(\nu)) \in \wedge\mathfrak{g} = \wedge T^*_\nu\mathfrak{g}^*.$$

Parallel to Proposition 4.2, we have,

Proposition 4.10. — a. *The map $\mathscr{R}_0$ intertwines the Clifford actions of $\mathfrak{d}_0$:*

$$\mathscr{R}_0 \circ \varrho^\wedge(w) = \varrho(\mathsf{s}_0(w)) \circ \mathscr{R}_0, \quad w \in \mathfrak{d}_0.$$

It is uniquely determined by this property, up to a scalar function.

b. *The map $\mathscr{R}_0$ intertwines the differentials, up to contraction by the modular vector field:*

$$\mathscr{R}_0 \circ d^\wedge = (d - \iota(X_\pi)) \circ \mathscr{R}_0.$$

c. *$\mathscr{R}_0$ has the equivariance property, for all $h = (\beta, a) \in D_0 = \mathfrak{g}^* \rtimes G$,*

$$(\mathscr{A}_0(h^{-1}))^*\mathscr{R}_0(y) = \det(\mathrm{Ad}_a)\ \mathscr{R}_0(\mathscr{A}^\wedge(h)y).$$

d. *$\mathscr{R}_0$ preserves the bilinear pairings on the spinor modules $\wedge\mathfrak{g}^*$, $\Omega(\mathfrak{g}^*)$, in the sense that*

$$(\mathscr{R}_0(y), \mathscr{R}_0(y'))_{\wedge T^*\mathfrak{g}^*} = (y, y')_{\wedge\mathfrak{g}^*}\ \mu_{\mathfrak{g}^*}$$

for all $y, y' \in \wedge\mathfrak{g}^*$.

Proof. — Each of the statements (a),(c),(d) is proved by a direct computation, parallel to those in Proposition 4.2. To prove (b), we first note that (c) implies the infinitesimal equivariance, for $(\beta, \xi) \in \mathfrak{d}_0$,

$$\big(\mathscr{L}(\mathscr{A}_0(\beta,\xi)) - \mathrm{tr}(\mathrm{ad}_\xi)\big)\mathscr{R}_0(y) = \mathscr{R}_0(\mathscr{L}_0^\wedge(\beta,\xi)y). \tag{88}$$

Since $\iota(X_\pi)\langle\theta_0, \xi\rangle = \mathrm{tr}(\mathrm{ad}_\xi)$, we have

$$\mathscr{L}(\mathscr{A}_0(\beta,\xi)) - \mathrm{tr}(\mathrm{ad}_\xi) = [(\mathrm{d} - \iota(X_\pi)), \varrho(\mathsf{s}_0(\beta,\xi))].$$

Hence we can re-write (88) as

$$[(\mathrm{d} - \iota(X_\pi)), \varrho(\mathsf{s}_0(\beta,\xi))]\ \mathscr{R}_0(y) = \mathscr{R}_0\Big([\mathrm{d}^\wedge, \varrho^\wedge(\beta,\xi)]\Big).$$

Together with (a), this implies that the linear map

$$(\mathrm{d} - \iota(X_\pi)) \circ \mathscr{R}_0 - \mathscr{R}_0 \circ \mathrm{d}^\wedge\colon \wedge\mathfrak{g}^* \to \Omega(\mathfrak{g}^*) \tag{89}$$

[4] Note that in the previous Section, μ denoted a generator of $\det(\mathfrak{g})$, and hence the star operator went from $\wedge\mathfrak{g}^* \to \wedge\mathfrak{g}$. This change in notation is intended, since our aim is to compare the Poisson manifold $\mathfrak{g}^*$ with the Dirac manifold G.

intertwines the Clifford actions of $\mathfrak{d}_0$. Since the parity of this map is opposite to that of $\mathcal{R}_0$, the uniqueness assertion in (a) implies that (89) is zero. $\square$

As before, we may use this map to construct pure spinors $\mathcal{R}_0(y) \in \Omega(\mathfrak{g}^*)$ from pure spinors $y \in \wedge\mathfrak{g}^*$.

The element $y = 1$ is the pure spinor defining the Lagrangian subspace $\mathfrak{g} \subset \mathfrak{d}_0$, and its image $\phi_{\mathfrak{g}^*} = \mathcal{R}_0(1)$ defines the Lagrangian subbundle $E_{\mathfrak{g}^*}$ (spanned by the sections $\mathsf{e}_0(\xi)$). The pure spinor $y = \mu \in \wedge\mathfrak{g}^*$ defines a Lagrangian complement $\mathfrak{g}^* \subset \mathfrak{d}_0$, and its image $\psi_{\mathfrak{g}^*} = \mathcal{R}_0(\mu) = 1$ defines the Lagrangian subbundle $F_{\mathfrak{g}^*} = T\mathfrak{g}^*$ (spanned by the sections $\mathsf{f}_0(\beta)$). For the bilinear pairing between these pure spinors, we obtain

$$(\phi_{\mathfrak{g}^*}, \psi_{\mathfrak{g}^*})_{\wedge T^*\mathfrak{g}^*} = \mu_{\mathfrak{g}^*}.$$

since $(1,\mu)_{\wedge\mathfrak{g}^*} = \mu$.

Lemma 4.11. — *The pure spinor $\phi_{\mathfrak{g}^*}$ is given by the formula*

$$\phi_{\mathfrak{g}^*} = (-1)^{n(n-1)/2} e^{-\iota(\pi_{\mathfrak{g}^*})} \mu_{\mathfrak{g}^*}$$

where $n = \dim G$.

Proof. — The Kirillov-Poisson bivector on $\mathfrak{g}^*$ is given by $\pi_{\mathfrak{g}^*}|_\nu = -\mathrm{d}^\wedge\nu \in \wedge^2\mathfrak{g}^* = \wedge^2 T_\nu\mathfrak{g}^*$. That is, $\tau_0 = \exp(-\pi_{\mathfrak{g}^*})$. The Lemma follows since $\star$ intertwines exterior product with contractions, and since $\star^{-1}(1) = (\mu^*)^\top = (-1)^{n(n-1)/2}\mu^*$. $\square$

Let us now return to our original setting where $\mathfrak{g}$ carries an invariant inner product B, used to identify $\mathfrak{g} \cong \mathfrak{g}^*$. We take the generators $\mu \in \det(\mathfrak{g})$ (from the last section) and $\mu \in \det(\mathfrak{g}^*)$ (from the present section) to be equal under this identification.

Let $\mu_{\mathfrak{g}}$ be the translation invariant volume form on $\mathfrak{g} \cong \mathfrak{g}^*$, and μ_G the corresponding left-invariant volume form on G. Let $J \in C^\infty(\mathfrak{g})$ be the Jacobian of the exponential map, defined by $\exp^*\mu_G = J\,\mu_{\mathfrak{g}}$. Recall that $\mathfrak{g}_\natural \subset \mathfrak{g}$ is the dense open subset where exp is a local diffeomorphism, i.e where $J \neq 0$. With $\varpi \in \Omega^2(\mathfrak{g})$ as in Section 3.5, we have:

Proposition 4.12. — *Over the subset $\mathfrak{g}_\natural$, the maps $\mathcal{R}_0\colon \wedge\mathfrak{g} \to \Omega(\mathfrak{g})$ and $\mathcal{R}\colon \mathrm{Cl}(\mathfrak{g}) \to \Omega(G)$ are related as follows:*

$$\exp^*(\mathcal{R}(x)) = J^{1/2} e^{-\varpi} \varrho(\widetilde{A}^{-\mathsf{e}_0(\varepsilon)})(\mathcal{R}_0(y)), \tag{90}$$

for $x = q(y)$. Here $\varepsilon \in C^\infty(\mathfrak{g}_\natural, \wedge^2\mathfrak{g})$ is the solution of the classical dynamical Yang-Baxter equation, cf. Proposition 3.14, and $J^{1/2} \in C^\infty(\mathfrak{g})$ is a smooth square root of J, equal to 1 at the origin.

Proof. — The map $\widetilde{\mathcal{R}}_0\colon \wedge\mathfrak{g} \to \Omega(\mathfrak{g}_\natural)$ given as

$$\widetilde{\mathcal{R}}_0(y) = e^{-\varpi} \varrho(\widetilde{A}^{-\mathsf{e}_0(\varepsilon)}) \exp^* \mathcal{R}(q(y))$$

intertwines the $\mathrm{Cl}(\mathfrak{d}_0)$-actions, hence it coincides with $\widetilde{\mathscr{R}}_0 = f\,\mathscr{R}_0$ for a scalar function. To find f, we consider bilinear pairings. Note that

$$\begin{aligned}(\widetilde{\mathscr{R}}_0(y), \tilde{\mathscr{R}}_0(y'))_{\wedge T^*\mathfrak{g}} &= (\exp^* \mathscr{R}(q(y)), \exp^* \mathscr{R}(q(y')))_{\wedge T^*\mathfrak{g}} \\ &= \exp^* \big(\mathscr{R}(q(y)),\ \mathscr{R}(q(y'))\big)_{\wedge T^*G}.\end{aligned}$$

Taking $y' = 1,\ y = \mu$ we obtain

$$f^2\mu_{\mathfrak{g}} = f^2\,(\mathscr{R}_0(\mu),\ \mathscr{R}_0(1))_{\wedge T^*\mathfrak{g}} = (\widetilde{\mathscr{R}}_0(\mu), \widetilde{\mathscr{R}}_0(1))_{\wedge T^*\mathfrak{g}} = \exp^*\mu_G = J\,\mu_{\mathfrak{g}}.$$

This shows that $f^2 = J$. □

Remark 4.13. — Of course, $\exp^*(R(x))$ is defined globally on all of $\mathfrak{g}$, not only on $\mathfrak{g}_\natural$. It follows from the Proposition that $J^{1/2}\exp(\mathsf{e}_0(\varepsilon))$ extends smoothly to all of $\mathfrak{g}$. Hence, the expression

$$J^{1/2}\exp(\varepsilon)$$

extends smoothly to a global function $\mathfrak{g} \to \wedge\mathfrak{g}$. For a direct proof, see **[5]**.

Applying the proposition to $y = 1$ and $y = \mu$, we find in particular that

$$\begin{aligned}\exp^*\phi_G &= J^{1/2}\,e^{-\varpi}\,\phi_{\mathfrak{g}}, \\ \exp^*\psi_G &= J^{1/2}\,e^{-\varpi}\,\varrho(\widetilde{A}^{-\mathsf{e}_0(\varepsilon)})(1).\end{aligned} \tag{91}$$

4.5. The Gauss-Dirac spinor. — We return to the set-up of Section 3.6, with $G = K^{\mathbb{C}}$ denoting the complexification of a compact Lie group, with Cartan subgroup $T = T_K^{\mathbb{C}}$. Recall that the Gauss-Dirac structure $\widehat{F}_G$ is defined by the Lagrangian subspace $\mathfrak{s} \subset \mathfrak{d}$, with basis the collection of all $e_\alpha \oplus 0$, $0 \oplus e_{-\alpha}$, $e_i \oplus (-e_i)$ where $\alpha \succ 0$ are positive roots and $i = 1, \ldots, l = \mathrm{rank}(G)$. The element

$$x = \prod_{\alpha\succ 0} e_\alpha e_{-\alpha}\ \prod_i e_i \in \mathrm{Cl}(\mathfrak{g}) \tag{92}$$

is non-zero and is annihilated by the Clifford action of $\mathfrak{s}$; hence it is a pure spinor defining $\mathfrak{s}$. Note that x satisfies

$$\tau(h_+)x = x,\quad x\tau(h_-^{-1}) = x,\quad \tau(h_0)x\tau(h_0) = h_0^{2\rho}\,x$$

for all $h_+ \in N_+$, $h_- \in N_-$, $h_0 \in T$. Here $\rho = \frac{1}{2}\sum_{\alpha\succ 0}\alpha$, and $t \mapsto t^{2\rho} \in \mathbb{C}^\times$ is the character of T defined by the weight 2ρ. Hence,

$$\widehat{\psi}_G = \mathscr{R}(x) \in \Omega(G)$$

is a pure spinor defining $\widehat{F}_G$. We refer to $\widehat{\psi}_G$ as the *Gauss-Dirac spinor*. Its equivariance properties are:

$$l_{h_+}^*\widehat{\psi}_G = \widehat{\psi}_G,\quad r_{h_-^{-1}}^*\widehat{\psi}_G = \widehat{\psi}_G,\quad l_{h_0}^* r_{h_0}^*\widehat{\psi}_G = h_0^{2\rho}\widehat{\psi}_G.$$

That is, $\widehat{\psi}_G$ is invariant up to the character, given by the group homomorphism $S \to T$ followed by the 2ρ-character.

Since the big Gauss cell $\mathcal{O} = N_-TN_+ \subset G$ is dense in G, the equivariance property, together with the fact that the pull-back of ψ_G to the group unit is equal to $\mathrm{str}(x) = 1$, completely characterizes the pure spinor ψ_G, and allows us to give an explicit formula. Recall the 2-form $\omega_{\mathcal{O}}$ on the big Gauss cell, given by (68):

Proposition 4.14. — *The restriction of the pure spinor $\widehat{\psi}_G$ to the big Gauss cell $\mathcal{O} = j(N_- \times T \times N_+)$ is given by the formula,*

$$\widehat{\psi}_G\big|_{\mathcal{O}} = g_0^\rho \exp(-\omega_{\mathcal{O}}).$$

Here $g_0\colon \mathcal{O} \to T$ is the composition of the Gauss decomposition $j^{-1}\colon \mathcal{O} \to N_- \times T \times N_+$ with projection to the middle factor.

Proof. — Both sides are pure spinors defining the Gauss-Dirac structure over $\mathcal{O}$, with the same equivariance property under S, and both sides pull back to 1 at the group unit e. □

We now compare the Gauss-Dirac spinor $\widehat{\psi}_G$ with the pure spinor ψ_G from Proposition 4.4.

Proposition 4.15. — *The pure spinors $\psi_G, \widehat{\psi}_G$ are related by a twist by the r-matrix $\mathfrak{r}$:*

$$\widehat{\psi}_G = \varrho(\exp(-\mathsf{e}(\mathfrak{r}))\psi_G.$$

Proof. — Let $\mathfrak{r}_\Delta \in \wedge^2\mathfrak{d}$ be the image of $\mathfrak{r}$ under the diagonal inclusion $\mathfrak{g} \hookrightarrow \mathfrak{d}$. We will show that

$$x = \varrho^{\mathrm{Cl}}(\exp(-\mathfrak{r}_\Delta))q(\mu). \tag{93}$$

The proposition follows from this identity by applying the map $\mathcal{R}$. Up to a scalar, (93) holds since both sides are pure spinors defining the same Lagrangian subspace. To determine the scalar, we apply the super-trace to both sides. Recall that the spinor action of elements $\xi_\Delta \in \mathfrak{g}_\Delta \subset \mathfrak{g}$ is given by Clifford commutator with the corresponding element $\xi \in \mathfrak{g}$. Since the super-trace vanishes on Clifford commutators, it follows that

$$\mathrm{str}(\varrho^{\mathrm{Cl}}(\exp(-\mathfrak{r}))q(\mu)) = \mathrm{str}(q(\mu)) = 1 = \mathrm{str}(x). \qquad \square$$

Let us next compute the Clifford differential $\mathrm{d}^{\mathrm{Cl}} = -4[q(\Xi), \cdot]$ of the element (92). Let $\rho = \frac{1}{2}\sum_{\alpha\succ 0}\alpha \in \mathfrak{t}^*$ be the half-sum of positive (real) roots.

Lemma 4.16. — *The quantization of the structure constant tensor satisfies,*

$$-4q(\Xi) = 2\pi\sqrt{-1}\rho \mod \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+.$$

Here B is used to identify $\mathfrak{g}^* \cong \mathfrak{g}$.

Proof. — By definition,

$$-4q(\Xi) = \frac{1}{6}\sum B([e^a, e^b], e^c)e_a e_b e_c,$$

using a basis e_a of $\mathfrak{g}$, with B-dual basis e^a. Take this basis to be the Cartan-Weil basis, and use the Clifford relations to write factors $e_{-\alpha}$ to the left and factors e_α to the right. Then

$$-4q(\Xi) \in \mathrm{Cl}(\mathfrak{g})^T \subset \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+ \oplus \mathrm{Cl}(\mathfrak{t}).$$

(For a T-equivariant element in $\mathrm{Cl}(\mathfrak{g})$, the T-weight of the $\mathfrak{n}_-$-factors must be compensated by the T weights of the $\mathfrak{n}_+$-factors.) Since $-4q(\Xi)$ is an odd element of filtration degree 3, and since Ξ has no component in $\wedge^3\mathfrak{t}$, it follows that

$$-4q(\Xi) \in \mathfrak{t} \oplus \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+.$$

To compute the $\mathfrak{t}$-component, we calculate the constant component of

$$[\xi, -4q(\Xi)]_{\mathrm{Cl}} = \mathrm{d}^{\mathrm{Cl}}\xi = q(\lambda(\xi))$$

for any $\xi \in \mathfrak{t}$. We have

$$\lambda(\xi) = -\sum_{\alpha\succ 0}[\xi, e_{-\alpha}] \wedge e_\alpha = 2\pi\sqrt{-1}\sum_{\alpha\succ 0}\langle\alpha,\xi\rangle e_{-\alpha}\wedge e_\alpha,$$

hence (see Sternberg [**52**, Equation (9.25)])

$$q(\lambda(\xi)) = 2\pi\sqrt{-1}\ \sum_{\alpha\succ 0}\langle\alpha,\xi\rangle e_{-\alpha}e_\alpha + 2\pi\sqrt{-1}\ \langle\rho,\xi\rangle.$$ □

As a consequence, we obtain,

Proposition 4.17. — *The element $x = \prod_{\alpha\succ 0} e_\alpha e_{-\alpha}\ \prod_i e_i$ satisfies,*

$$\left(d^{\mathrm{Cl}} - 2\pi\sqrt{-1}\iota^{\mathrm{Cl}}(\rho)\right)x = 0.$$

Proof. — d^{Cl} is given as the Clifford commutator with $-4q(\Xi)$. Since x is annihilated under both left and right multiplication by elements of $\mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$, it follows that

$$\mathrm{d}^{\mathrm{Cl}}(x) = 2\pi\sqrt{-1}[\rho, x]_{\mathrm{Cl}}.$$ □

As a consequence, the Gauss-Dirac spinor satisfies the differential equation:

$$\left(\mathrm{d} + \eta - 2\pi\sqrt{-1}\varrho(\mathsf{e}(\rho))\right)\widehat{\psi}_G = 0. \tag{94}$$

In fact, there is a more general version of this Equation, stated in the following Proposition. For any (real) dominant weight λ of G (not to be confused with the map λ above), let $\Delta_\lambda \in C^\infty(G)$ be the function

$$\Delta_\lambda(g) = \frac{\langle v_\lambda, g\cdot v_\lambda\rangle}{\langle v_\lambda, v_\lambda\rangle},$$

where v_λ is a highest weight vector in the irreducible unitary representation $(V_\lambda, \langle\cdot,\cdot\rangle)$ of highest weight λ. The function Δ_λ is invariant under the left-action of N_-, under the right-action of N_+, and under the T-action it satisfies

$$\Delta_\lambda(tg) = \Delta_\lambda(gt) = t^\lambda\Delta_\lambda(g). \tag{95}$$

Since $\Delta_\lambda(e) = 1$, it follows that $\Delta_\lambda \neq 0$ on the big Gauss cell. We are interested in the product $\Delta_\lambda\widehat{\psi}_G$. Away from the zeroes of Δ_λ, this is a pure spinor defining the

Gauss-Dirac structure. Similar to $\widehat{\psi}_G$, it is invariant under the left-action of N_- and the right-action of N_+, and satisfies

$$l_t^*(\Delta_\lambda\widehat{\psi}_G) = r_t^*(\Delta_\lambda\widehat{\psi}_G) = t^{\lambda+\rho}(\Delta_\lambda\widehat{\psi}_G) \tag{96}$$

for all $t \in T$.

Proposition 4.18. — *For any dominant weight λ, the product $\Delta_\lambda\widehat{\psi}_G$ satisfies the differential equation:*

$$\big(d + \eta - 2\pi\sqrt{-1}\varrho(\mathsf{e}(\lambda+\rho))\big)\Delta_\lambda\widehat{\psi}_G = 0, \tag{97}$$

where B is used to identify $\mathfrak{g}^* \cong \mathfrak{g}$.

Proof. — Let $\mathfrak{s} \subset \mathfrak{d}$ be the Lagrangian subalgebra (66) defining the Gauss-Dirac structure. We have, for all $\zeta = (\xi, \xi') \in \mathfrak{s}$,

$$\begin{aligned}
&\varrho(\mathsf{s}(\zeta))\big(\mathrm{d} + \eta - 2\pi\sqrt{-1}\varrho(\mathsf{e}(\lambda+\rho))\big)\,\Delta_\lambda\widehat{\psi}_G\\
&= \Big[\varrho(\mathsf{s}(\zeta)),\ \mathrm{d} + \eta - 2\pi\sqrt{-1}\varrho(\mathsf{e}(\lambda+\rho))\Big]\Delta_\lambda\widehat{\psi}_G\\
&= \big(\mathscr{L}(\xi^L - (\xi')^R)) - 2\pi\sqrt{-1}B(\xi-\xi', \lambda+\rho)\big)\Delta_\lambda\widehat{\psi}_G = 0,
\end{aligned}$$

where the last equality follows from the equivariance properties (96) of $\Delta_\lambda\widehat{\psi}_G$. (Note that for the elements of the form $\zeta = (\xi, 0)$ with $\xi \in \mathfrak{n}_+$ or $\zeta = (0, \xi)$ with $\xi \in \mathfrak{n}_-$, the inner product with $\lambda + \rho \in \mathfrak{t}$ vanishes.) Hence, the left hand side of (97) is annihilated by all $\mathsf{s}(\zeta)$, for $\zeta \in \mathfrak{s}$. Hence it is a function times $\widehat{\psi}_G$, and thus vanishes since it has parity opposite to that of $\widehat{\psi}_G$. □

Remark 4.19. — The holomorphic Dirac structure $\widehat{F}_G$ on $G = K^{\mathbb{C}}$ restricts to a complex Dirac structure $\widehat{F}_G|_K = \widehat{F}_K$ on the real Lie group K, with defining pure spinor the pull-back (restriction) $\widehat{\psi}_K$ of $\widehat{\psi}_G$. On the other hand, $E_G|_K = (E_K)^{\mathbb{C}}$. In the notation of Section 2.4, applied to the Gauss-Cartan-splitting $(\mathbb{T}K)^{\mathbb{C}} = E_K^{\mathbb{C}} \oplus \widehat{F}_K$, we have $\sigma = 2\pi\sqrt{-1}e(\rho) \in \Gamma((\mathbb{T}K)^{\mathbb{C}})$, thus $\eth_\pm = \mathrm{d} + \eta \pm 2\pi\sqrt{-1}\varrho(\mathsf{e}(\rho))$. As usual, $\eth_+\phi_K = 0$, $\eth_-\widehat{\psi}_K = 0$ (the second equation is the pull-back of (94) to K). Let μ be the bi-invariant (real) volume form on K defined by $\phi_K, \widehat{\psi}_K$. Since $\eth_\pm^2 = \pm 2\pi\sqrt{-1}\,\mathscr{L}(\mathscr{C}_{\mathrm{ad}}(\rho))$, the Dirac cohomology groups $H_\pm(E_K^{\mathbb{C}}, \widehat{F}_K, \mu)$ are the cohomology groups of $\eth_\pm$ on the space of $\mathscr{C}_{\mathrm{ad}}(\rho)$-invariant complex-valued differential forms on K. These may be computed by the standard localization argument ([**12**], see also [**33**]): The set of zeroes of the vector field $\mathscr{C}_{\mathrm{ad}}(\rho)$ on K is just the maximal torus T_K, and the pull-back to T_K intertwines $\eth_\pm$ with $d \pm 2\pi\sqrt{-1}B(\theta_T, \rho))$, with θ_T the Maurer-Cartan form on T_K. Hence, by localization the pull-back to T_K induces an isomorphism,

$$H_\pm(E_K^{\mathbb{C}}, \widehat{F}_K, \mu) \cong H(\Omega(T_K)^{\mathbb{C}}, \mathrm{d} \pm 2\pi\sqrt{-1}B(\theta_T, \rho))$$

Since ρ is a weight, it defines a T_K-character t^ρ, and the operators $\mathrm{d} \pm 2\pi\sqrt{-1}B(\theta_T, \rho)$ are obtained from d by conjugation by $t^{\pm\rho}$. Hence $H_\pm(E_K^{\mathbb{C}}, \widehat{F}_K, \mu) \cong H(T_K)^{\mathbb{C}}$.

5. q-Hamiltonian G-manifolds

In this section, we use the techniques developed in this paper to extend the theory of group-valued moment maps, as developed in [**3, 8**] for the case of compact Lie groups, to more general settings.

5.1. Dirac morphisms and group-valued moment maps.

— We briefly recall the definitions.

Definition 5.1. — A *quasi-Hamiltonian* $\mathfrak{g}$*-manifold* (or simply *q-Hamiltonian* $\mathfrak{g}$*-manifold*) is a manifold M with a Lie algebra action $\mathcal{A}_M\colon \mathfrak{g} \to \mathfrak{X}(M)$, a 2-form ω, and a $\mathfrak{g}$-equivariant *moment map* $\Phi\colon M \to G$ such that

$$\begin{aligned} &\mathrm{d}\omega = \Phi^*\eta \\ (98)\qquad &\iota(\mathcal{A}_M(\xi))\omega = \Phi^* B(\xi, \tfrac{\theta^L+\theta^R}{2}) \qquad \text{(moment map condition)} \\ &\ker(\omega_m) = \{\mathcal{A}_M(\xi)_m \mid \operatorname{Ad}_{\Phi(m)} \xi = -\xi\} \qquad \text{(minimal degeneracy condition).} \end{aligned}$$

If the action of $\mathfrak{g}$ extends to an action of the Lie group G, and if ω and Φ are equivariant for the action of G, we speak of a *q-Hamiltonian G-manifold.*

The first two conditions in (98) imply that ω is $\mathfrak{g}$-invariant (see [**3**]). As shown by Bursztyn-Crainic [**14**], the definition of a q-Hamiltonian space may be restated in Dirac geometric terms (see also Xu [**57**] for another interpretation).

Theorem 5.2. — *There is a 1-1 correspondence between q-Hamiltonian $\mathfrak{g}$-manifolds, and manifolds M together with a strong Dirac morphism*

$$(99)\qquad (\Phi, \omega)\colon (M, TM, 0) \to (G, E_G, \eta).$$

More precisely, $(M, \mathcal{A}_M, \omega, \Phi)$ satisfies the first two conditions if and only if (Φ, ω) is a Dirac morphism, and in this case the third condition is equivalent to this Dirac morphism being strong.

Proof. — Let $(M, \mathcal{A}_M, \omega, \Phi)$ be a q-Hamiltonian $\mathfrak{g}$-space. Given $m \in M$, let $E'_{\Phi(m)}$ be the forward image of T_mM under $((\mathrm{d}\Phi)_m, \omega_m)$:

$$E'_{\Phi(m)} = \{(\mathrm{d}\Phi(v), \alpha) \mid v \in T_mM,\ (\mathrm{d}\Phi)^*_m\alpha = \iota(v)\omega_m\}.$$

Taking v of the form $\mathcal{A}_M(\xi)_m$ for $\xi \in \mathfrak{g}$, and using the moment map condition, we see $E'_{\Phi(m)} \supset (E_G)_{\Phi(m)}$. In fact, one has equality since both are Lagrangian subspaces. This shows that (Φ, ω) is a Dirac morphism. In particular,

$$(\mathrm{d}\Phi)_m(\ker(\omega_m)) = \ker((E_G)_{\Phi(m)}) = \{\mathcal{A}_{\mathsf{ad}}(\xi)_{\Phi(m)} \mid \operatorname{Ad}_{\Phi(m)} \xi = -\xi\}.$$

Hence, the minimal degeneracy condition holds if and only $(\mathrm{d}\Phi)_m$ restricts to an isomorphism on $\ker(\omega_m)$, i.e. if and only if (Φ, ω) is a strong Dirac morphism. Conversely, given a strong Dirac morphism (99), the associated map $\mathfrak{a}$ defines a $\mathfrak{g}$-action $\mathcal{A}_M(\xi) = \mathfrak{a}(\Phi^*\mathsf{e}(\xi))$ on M, for which the map Φ is $\mathfrak{g}$-equivariant. The above argument then shows that $(M, \mathcal{A}_M, \omega, \Phi)$ is a q-Hamiltonian $\mathfrak{g}$-space. $\square$

Remark 5.3. — As a consequence of this result (or rather its proof), we see that if $(M, \mathcal{A}_M, \omega, \Phi)$ satisfies the first two conditions in (98), then the third condition (minimal degeneracy) is equivalent to the transversality property [**15, 57**]

$$\ker(\omega) \cap \ker(\mathrm{d}\Phi) = \{0\}.$$

Remark 5.4. — There is a similar result for q-Hamiltonian G-manifolds. Here, it is necessary to assume the existence of a G-action on M for which the Dirac morphism (Φ, ω) is equivariant, and such that the infinitesimal action coincides with that defined by $\mathfrak{a}$.

Example 5.5. — By Example 2.7, the inclusion of the conjugacy classes $\mathcal{C}$ in G, with 2-forms defined by the Cartan-Dirac structure, defines a strong Dirac morphism $(\iota_{\mathcal{C}}, \omega_{\mathcal{C}})$. Thus, conjugacy classes are q-Hamiltonian G-manifolds.

Using our results on the Cartan-Dirac structure, it is now straightforward to deduce the basic properties of q-Hamiltonian spaces $(M, \mathcal{A}_M, \omega, \Phi)$. In contrast with the original treatment in [**3**], the discussion works equally well for non-compact Lie groups, and also in the holomorphic category.

***Theorem 5.6* (Fusion)**. — *Let $(M, \mathcal{A}_M, \Phi, \omega)$ be a q-Hamiltonian $G \times G$-manifold. Let $\mathcal{A}_{\mathrm{fus}}$ be the diagonal G-action, $\Phi_{\mathrm{fus}} = \mathrm{Mult} \circ \Phi$, and $\omega_{\mathrm{fus}} = \omega + \Phi^* \varsigma$, with $\varsigma \in \Omega^2(G^2)$ the 2-form defined in* (47). *Then $(M, \mathcal{A}_{\mathrm{fus}}, \Phi_{\mathrm{fus}}, \omega_{\mathrm{fus}})$ is a q-Hamiltonian G-manifold. (An analogous statement holds for q-Hamiltonian $\mathfrak{g} \times \mathfrak{g}$-manifolds.)*

Proof. — Since

$$(\Phi_{\mathrm{fus}}, \omega_{\mathrm{fus}}) = (\mathrm{Mult}, \varsigma) \circ (\Phi, \omega)$$

is a composition of two strong Dirac morphism, it is itself a strong Dirac morphism from $(M, TM, 0)$ to (G, E_G, η). The induced map $M \times \mathfrak{g} = \Phi_{\mathrm{fus}}^* E_G \to TM$ is a composition of the map $\mathrm{Mult}^* E_G \to E_{G \times G}$ defined by the strong Dirac morphism $(\mathrm{Mult}, \varsigma)$, with the map $\Phi^* E_G \times E_G \to TM$ given by the strong Dirac morphism (Φ, ω). If we use the sections $\mathsf{e}(\xi)$ to identify $E_G \cong G \times \mathfrak{g}$, the latter map is the $\mathfrak{g} \times \mathfrak{g}$-action on M, while the former is the diagonal inclusion $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$. This confirms that the resulting action is just the diagonal action. $\square$

If $M = M_1 \times M_2$ is a direct product of two q-Hamiltonian manifolds, the quadruple $(M, \mathcal{A}_{\mathrm{fus}}, \Phi_{\mathrm{fus}}, \omega_{\mathrm{fus}})$ is called the *fusion product* of M_1, M_2. In particular we obtain products of conjugacy classes as new examples of q-Hamiltonian G-spaces.

Suppose $(M, \mathcal{A}_M, \omega_0, \Phi_0)$ is a Hamiltonian $\mathfrak{g}$-manifold: That is, ω_0 is symplectic, and $\Phi_0 \colon M \to \mathfrak{g}^*$ is the moment map for a Hamiltonian $\mathfrak{g}$-action on M. As is well-known, this is equivalent to Φ_0 being a Poisson map from the symplectic manifold (M, ω_0) to the Poisson manifold $(\mathfrak{g}^*, \pi_{\mathfrak{g}^*})$. But this is also equivalent to

$$(\Phi_0, \omega_0) \colon (M, TM, 0) \to (\mathfrak{g}^*, \mathrm{Gr}_\pi, 0)$$

being a strong Dirac morphism. A *Hamiltonian G-manifold* comes with a G-action on M integrating the $\mathfrak{g}$-action, and such that the Dirac morphism (Φ_0, ω_0) is equivariant.

Given an invariant inner product B on $\mathfrak{g}$, used to identify $\mathfrak{g}^* \cong \mathfrak{g}$, we may compose the Dirac morphism (Φ_0, ω_0) with the Dirac morphism $(\exp, \varpi)$ from Theorem 3.13, and obtain:

***Theorem 5.7* (Exponentials).** — *Suppose $(M, \mathcal{A}_M, \omega_0, \Phi_0)$ is a Hamiltonian G-manifold, and let $\omega = \omega_0 + \Phi_0^* \varpi$, $\Phi = \exp \circ \Phi_0$. Then $(M, \mathcal{A}_M, \omega, \Phi)$ satisfies the first two conditions in* (98). *On $M_\natural = \Phi_0^{-1}(\mathfrak{g}_\natural)$, the third condition (minimal degeneracy) holds as well, thus $(M_\natural, \mathcal{A}_M, \omega, \Phi)$ is a q-Hamiltonian G-manifold. (Similar statements hold for q-Hamiltonian $\mathfrak{g}$-manifolds.)*

5.2. Volume forms. — Any symplectic manifold (M, ω) carries a distinguished volume form, given as the top degree component $\exp(\omega)^{[\dim M]} = \frac{1}{n!}\omega^n$. For a q-Hamiltonian G-manifold $(M, \mathcal{A}_M, \omega, \Phi)$, the 2-form ω is usually degenerate, hence $\exp(\omega)^{[\mathrm{top}]}$ will have zeroes. Nevertheless, any q-Hamiltonian G-manifold carries a distinguished volume form, provided the adjoint action $\mathrm{Ad}\colon G \to \mathrm{O}(\mathfrak{g})$ lifts to $\mathrm{Pin}(\mathfrak{g})$:

***Theorem 5.8* (Volume forms).** — *Suppose the adjoint action $\mathrm{Ad}\colon G \to \mathrm{O}(\mathfrak{g})$ lifts to $\mathrm{Pin}(\mathfrak{g})$, and let $\psi_G \in \Omega(G)$ be the pure spinor defined by this lift. For any q-Hamiltonian G-manifold $(M, \mathcal{A}_M, \omega, \Phi)$, the differential form*

$$\mu_M = (\exp(\omega) \wedge \Phi^* \psi_G)^{[\dim M]} \tag{100}$$

is a volume form. It has the equivariance property $\mathcal{A}_M(g)^* \mu_M = \det(\mathrm{Ad}_g)\, \mu_M$. More generally, if $(M, \mathcal{A}_M, \omega, \Phi)$ satisfies the first two conditions in (98), *the form μ_M is non-zero exactly at those points where ω satisfies the minimal degeneracy condition.*

Of course, the factor $\det(\mathrm{Ad}_g) = \pm 1$ is trivial if G is connected.

Proof. — Since ψ_G is a pure spinor defining the complementary Lagrangian subbundle F_G, and since (Φ, ω) is a strong Dirac morphism, the pull-back $\Phi^* \psi_G$ is non-zero everywhere. Furthermore, $\exp(\omega)\Phi^*\psi_G$ is a pure spinor defining the backward image F of F_G under the Dirac morphism (Φ, ω). Since F is transverse to TM (see Proposition 1.15), the top degree part of $\exp(\omega)\Phi^*\psi_G$ is nonvanishing. More generally, if $(M, \mathcal{A}_M, \omega, \Phi)$ only satisfies the first two conditions in (98), then the above argument applies at all points of M where (Φ, ω) is a strong Dirac morphism. But these are exactly the points where $\Phi^*\psi_G$ is non-zero.

The equivariance property of μ_M is a direct consequence of the equivariance properties of ϕ_G and ψ_G described in Proposition 4.4. □

The volume form μ_M is called the *Liouville volume form* of the q-Hamiltonian G-manifold $(M, \mathcal{A}_M, \omega, \Phi)$. Let $|\mu_M|$ be the associated measure. If the moment Φ is proper, the push-forward $\Phi_* |\mu_M|$ is a well-defined measure on G, called the *Duistermaat-Heckman measure.*

***Remark 5.9*.** — For the case of compact Lie groups, the q-Hamiltonian Liouville forms and Duistermaat-Heckman measures were introduced in [**8**]. The fact that μ_M is a volume form was verified by 'direct computation'. However, the argument in [**8**] does not extend to non-compact Lie groups.

Remark 5.10. — The expression $\exp(\omega)\Phi^*\psi_G$ entering the definition of the volume form μ_M satisfies the differential equation

$$(101) \qquad (\mathrm{d} + \iota(\mathscr{A}_M(\Xi)))\Big(\exp(\omega)\Phi^*\psi_G\Big) = 0.$$

This follows from the differential equation (85) for ψ_G together with Remark 1.5(a).

Proposition 5.11. — *Suppose $(M, \mathscr{A}_M, \omega, \Phi)$ is a q-Hamiltonian G-manifold, and that* Ad *lifts to the Pin group. Then M is even-dimensional if* $\det(\mathrm{Ad}_\Phi) = +1$*, and odd-dimensional if* $\det(\mathrm{Ad}_\Phi) = -1$*. In particular, it is even-dimensional when G is connected, and in this case M carries a canonical orientation.*

Proof. — The construction of ψ_G in terms of the map $\mathscr{R}$ (see Proposition 4.4) shows that the form ψ_G has even degree at points $g \in G$ with $\det(\mathrm{Ad}_g) = 1$, and odd degree at points with $\det(\mathrm{Ad}_g) = -1$. Hence, the parity of the volume form μ_M is determined by the parity of $\det(\mathrm{Ad}_\Phi)$. If G is connected, the lift of Ad (which exists by assumption) is unique, and $\det(\mathrm{Ad}_g) \equiv 1$. □

Without the existence of a lift to $\mathrm{Pin}(\mathfrak{g})$, the form ψ_G is only defined locally, up to sign. That is, we still obtain a G-invariant *measure* on M, given locally as $(e^\omega \Phi^*\psi_G)^{[\mathrm{top}]}$. It is interesting to specialize these results to conjugacy classes:

Theorem 5.12. — *Suppose G is a connected Lie group, whose Lie algebra carries an invariant inner product B. Then:*

a. *Every conjugacy class $\mathscr{C} \subset G$ carries a distinguished invariant measure (depending only on B).*
b. *The conjugacy class $\mathscr{C}$ of $g \in G$ is even-dimensional if and only if* $\det(\mathrm{Ad}_g) = +1$.
c. *If the adjoint action $G \to \mathrm{O}(\mathfrak{g})$ lifts to* $\mathrm{Pin}(\mathfrak{g})$*, then every conjugacy class carries a distinguished orientation.*

Example 5.13. — Consider the conjugacy classes of $G = \mathrm{O}(2)$: If $g \in \mathrm{SO}(2)$, the conjugacy class of g is zero-dimensional, consisting of either one or two points. On the other hand, the circle $\mathrm{O}(2)\setminus\mathrm{SO}(2) \cong S^1$ forms a single conjugacy class. Similarly, for $G = \mathrm{O}(3)$, the elements $g \in G$ with $\det(g) = -1$ have $\det(\mathrm{Ad}_g) = 1$. Each of these form a single 2-dimensional conjugacy class. The group $\mathrm{SO}(3)$ is the simplest example where the adjoint action $G \to \mathrm{SO}(\mathfrak{g})$ (which in this case is just the identity map) does not lift to the spin group. Indeed the conjugacy class of rotations by 180^o is isomorphic to $\mathbb{R}P(2)$, hence non-orientable.

Example 5.14. — Suppose G carries an involution σ, such that the corresponding involution of $\mathfrak{g}$ preserves B. Form the semi-direct product $G \rtimes \mathbb{Z}_2$, where the action of $\mathbb{Z}_2$ is generated by the involution σ. The $G \rtimes \mathbb{Z}_2$-conjugacy class of the element (e, σ) is isomorphic to the homogeneous space $M = G/G^\sigma$, which therefore is an example of a q-Hamiltonian $G \rtimes \mathbb{Z}_2$-space. The 2-form on M is just zero. Let us compute the

Liouville measure on M, for the case that the restriction of B to $\mathfrak{g}^\sigma = \ker(\sigma - 1)$ is still non-degenerate. Let $e_1, \ldots, e_n$ be a basis of $\mathfrak{g}$, with $B(e_i, e_j) = \pm\delta_{ij}$, such that $e_1 \ldots, e_k$ are a basis of $\mathfrak{g}^\sigma$. Then

$$\widetilde{\sigma} = 2^{(n-k)/2} e_{k+1} \cdots e_n \in \mathrm{Pin}(\mathfrak{g})$$

is a lift of σ. Note that $\widetilde{\sigma}^2 = \pm 1$, with sign depending on $n-k$. Taking $\mu = e_1 \wedge \cdots \wedge e_n$ as the Riemannian volume form on $\mathfrak{g}$, we have

$$\widetilde{\sigma} q(\mu) = \pm 2^{(n-k)/2} e_1 \ldots e_k$$

so $\star q^{-1}(\widetilde{\sigma} q(\mu)) = \pm 2^{(n-k)/2} e_{k+1} \wedge \cdots \wedge e_n$. We conclude that the Liouville measure on $M = G/G^\sigma$ coincides with the G-invariant measure defined by the metric on $(\mathfrak{g}^\sigma)^\perp \subset \mathfrak{g}$.

***Proposition 5.15* (Volume form for 'fusions').** — *The volume form of a q-Hamiltonian $G \times G$-manifold $(M, \mathcal{A}_M, \omega, \Phi)$ (as in Theorem 5.6) coincides with the volume form of its fusion $(M, \mathcal{A}_{\mathrm{fus}}, \omega_{\mathrm{fus}}, \Phi_{\mathrm{fus}})$:*

$$(\exp(\omega)\, \Phi^* \psi_{G\times G})^{[\dim M]} = (\exp(\omega_{\mathrm{fus}})\, \Phi_{\mathrm{fus}}^* \psi_G)^{[\dim M]}.$$

Proof. — Using Theorem 4.9, we have

$$\begin{aligned}
\exp(\omega_{\mathrm{fus}})\, \Phi_{\mathrm{fus}}^* \psi_G &= \exp(\omega + \Phi^* \varsigma)\, \Phi^*\, \mathrm{Mult}^*\, \psi_G \\
&= \exp(\omega)\, \Phi^* \big(\varrho(\exp(-\mathsf{e}(\gamma)))\psi_G^1 \otimes \psi_G^2\big) \\
&= \exp(-\iota(\mathcal{A}_M(\gamma)))(\exp(\omega)\Phi^* \psi_{G\times G}),
\end{aligned}$$

where we used Remark 1.5(a) for the last equality. Since the operator $\exp(-\iota(\mathcal{A}_M(\gamma)))$ does not affect the top degree part, the proof is complete. □

Example 5.16. — An important example of a q-Hamiltonian G-space is the double $D(G) = G \times G$, with moment map the commutator $\Phi(a, b) = aba^{-1}b^{-1}$. As explained in [**8**] the double is obtained by fusion, as follows: Start by viewing the Lie group G as a homogeneous space $G = G \times G/G_\Delta$, where G_Δ is the diagonal subgroup. Since G_Δ is the fixed point set for the involution σ of $G \times G$ switching the two factors, we see as in Example 5.14 that G is a q-Hamiltonian $(G \times G) \rtimes \mathbb{Z}_2$-space, with moment map $a \mapsto (a, a^{-1}, \sigma)$. The Liouville measure is simply the Haar measure on G. Fusing two copies, the direct product $G \times G$ becomes a q-Hamiltonian $G \times G$-space. Finally, passing to the diagonal action one arrives at the double $D(G)$. By Proposition 5.15, the resulting Liouville measure on $D(G)$ is just the Haar measure.

***Proposition 5.17* (Volume form for 'exponentials').** — *Let $(M, \mathcal{A}_M, \Phi_0, \omega_0)$ be a Hamiltonian G-space, and $(M, \mathcal{A}_M, \Phi, \omega)$ its 'exponential', as in Theorem 5.7. Then*

$$(\exp(\omega)\Phi^* \psi_G)^{[\dim M]} = \Phi_0^* J^{1/2}\ \exp(\omega_0)^{[\dim M]}.$$

Proof. — Using the relation (91) between $\exp^*\psi_G$ and $\psi_{\mathfrak{g}} = 1$, we find

$$\begin{aligned}\exp(\omega)\,\Phi^*\psi_G &= \exp(\omega_0 + \Phi_0^*\varpi)\Phi_0^*\exp^*\psi_G\\ &= \exp(\omega_0)\Phi_0^* J^{1/2}\varrho(\widetilde{A}^{-\mathsf{e}_0(\varepsilon)})(1)\\ &= \Phi_0^* J^{1/2}\exp(-\iota(\mathcal{E}_M(\varepsilon)))\exp(\omega_0).\end{aligned}$$

Since $\exp(-\iota(\mathcal{E}_M(\varepsilon)))$ does not affect the top degree part, the proof is complete. □

5.3. The volume form in terms of the Gauss-Dirac spinor. — Suppose now that K is a *compact* Lie group, with complexification $G = K^{\mathbb{C}}$, and let $B\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ be the complexification of a positive definite inner product on $\mathfrak{k}$. In this case, as discussed in Section 3.6, E_G has a second Lagrangian complement $\widehat{F}_G$, defined by the Gauss-Dirac spinor $\widehat{\psi}_G$. Its pull-back to $K\subset G$, denoted by $\widehat{\psi}_K$, is thus a complex-valued pure spinor defining a (complex) Lagrangian complement $\widehat{F}_K\subset(\mathbb{T}K)^{\mathbb{C}}$.

Given a q-Hamiltonian K-space $(M,\mathcal{E}_M,\Phi,\omega)$, the complex differential form $\exp(\omega)\Phi^*\widehat{\psi}_K$ is related to $\exp(\omega)\Phi^*\psi_K$ by the r-matrix,

$$\exp(\omega)\Phi^*\widehat{\psi}_K = \exp(-\iota(\mathcal{E}_M(\mathfrak{r})))\Big(\exp(\omega)\Phi^*\psi_K\Big).$$

Since $\exp(-\iota(\mathcal{E}_M(\mathfrak{r})))$ does not affect the top degree part, it follows that we can write our volume form also in terms of $\widehat{\psi}_K$:

$$\mu_M = \Big(\exp(\omega)\Phi^*\widehat{\psi}_K\Big)^{[\dim M]}.$$

Remark 5.18. — Let $\tilde{F}_M$ be the backward image of $\widehat{F}_K$ under the strong Dirac morphism $(\Phi,\omega)\colon(M,TM,0)\to(K,E_K,\eta)$. Since $\tilde{F}_M$ is transverse to $TM^{\mathbb{C}}$, it is given by a graph of a (complex-valued) bivector π, and $H_-(TM^{\mathbb{C}},\tilde{F}_M,\mu_M)\cong H_\pi(M) = H(\Omega(M)^{X_\pi}, d-\iota(X_\pi))$. A simple calculation shows that $X_\pi = 2\pi\sqrt{-1}\,\mathcal{E}_M(\rho)$ (where B is used to identify $\mathfrak{k}^*\cong\mathfrak{k}$).

The pure spinors $\phi_M = 1$ and ϕ_K satisfy $d\phi_M = 0$ and $(d+\eta)\phi_K = 0$. Hence, by Proposition 2.13 the map $e^\omega\Phi^*$ descends to Dirac cohomology, $H_-(E_K^{\mathbb{C}},\widehat{F}_K,\mu_K)\to H_\pi(M)$. In particular, $\eth_-\widehat{\psi}_K = 0$ implies that $\exp(\omega)\Phi^*\widehat{\psi}_K$ is closed under $\mathrm{d} - 2\pi\sqrt{-1}\iota(\mathcal{E}_M(\rho))$. For M is compact, the class $[e^\omega\Phi^*\widehat{\psi}_K]$ in $H_\pi(M)$ is nonvanishing because its integral is $\int_M\mu_M>0$.

Let $\Delta_\lambda\colon G\to\mathbb{C}$ be the holomorphic functions introduced in Section 4.5.

Proposition 5.19. — *For any dominant weight λ, the complex differential form $\exp(\omega)\Phi^*(\Delta_\lambda\widehat{\psi}_K)$ satisfies the differential equation*

$$(d - 2\pi\sqrt{-1}\iota(\mathcal{E}_M(\lambda+\rho)))\Big(\exp(\omega)\Phi^*(\Delta_\lambda\widehat{\psi}_K)\Big) = 0. \tag{102}$$

Her B_K is used to identify $\mathfrak{k}^*\cong\mathfrak{k}$.

Proof. — This follows from the differential equation for the Gauss-Dirac spinor, Proposition 4.18, together with Remark 1.5(a). □

As remarked in [**6**], the orthogonal projection of $\dim V_\lambda \Delta_\lambda|_K$ to the K-invariant functions on K coincides with the irreducible character χ_λ of highest weight λ. Thus,

$$\begin{aligned}\int_M \exp(\omega)\Phi^*(\Delta_\lambda \widehat{\psi}_K) &= \int_M |\mu_M| \Phi^* \Delta_\lambda \\ &= \int_K \Phi_* |\mu_M|\, \Delta_\lambda = (\dim V_\lambda)^{-1} \int_K \chi_\lambda\, \Phi_* |\mu_M|.\end{aligned}$$

On the other hand, by (102) the integral may be computed by localization [**12**] to the zeroes of the vector field $\mathscr{A}_M(\lambda+\rho)$. As shown in [**6**], the 2-form ω pulls back to symplectic forms $\omega_Z = \iota_Z^* \omega$ on the components Z of the zero set, and the restriction $\Phi_Z = \iota_Z^* \Phi$ takes values in T. Since $\iota_T^*(\Delta_\lambda \widehat{\psi}_K)(t) = t^{\lambda+\rho}$ for $t \in T$, one obtains the following formula for the Fourier coefficients of the q-Hamiltonian Duistermaat-Heckman measure:

$$\int_K \chi_\lambda \Phi_* |\mu_M| = \dim V_\lambda \sum_{Z \subset \mathscr{A}_M(\lambda+\rho)^{-1}(0)} \int_Z \frac{\exp(\omega_Z)(\Phi_Z)^{\lambda+\rho}}{\mathrm{Eul}(\nu_Z, 2\pi\sqrt{-1}(\lambda+\rho))}.$$

Here $\mathrm{Eul}(\nu_Z,\cdot)$ is the T-equivariant Euler form of the normal bundle. This formula was proved in [**6**], using a more elaborate argument. Taking $\lambda = 0$, one obtains a formula for the volume $\int_M |\mu_M|$ of M.

5.4. q-Hamiltonian q-Poisson $\mathfrak{g}$-manifolds. — Just as any symplectic 2-form determines a Poisson bivector π, any q-Hamiltonian G-manifold carries a distinguished bivector field π. However, since ω is not non-degenerate π is not simply obtained as an inverse, and also π is not generally a Poisson structure.

Suppose $(M, \mathscr{A}_M, \omega, \Phi)$ is a q-Hamiltonian $\mathfrak{g}$-manifold, or equivalently that (Φ, ω) is a strong Dirac morphism $(M, TM, 0) \to (G, E_G, \eta)$. Let $\widetilde{F} \subset \mathbb{T}M$ be the backward image of F_G under this Dirac morphism. It is a complement to TM, hence it is of the form $\widetilde{F} = \mathrm{Gr}_\pi$ for some $\mathfrak{g}$-invariant bivector field $\pi \in \mathfrak{X}^2(M)$. By Proposition 2.10(c), the Schouten bracket of this bivector field with itself satisfies

$$\tfrac{1}{2}[\pi,\pi]_{\mathrm{Sch}} = \mathscr{A}_M(\Xi). \tag{103}$$

Let $\mathsf{p}' \colon \mathbb{T}G \to E_G$ be the projection along F_G. Let $\{v_a\}$ and $\{v^a\}$ be bases of $\mathfrak{g}$ with $B(v_a, v^b) = \delta_a^b$. Then $\mathsf{p}'(x') = \sum_a \langle x, \mathsf{f}(v_a)\rangle \mathsf{e}(v^a)$ for all $x' \in \Gamma(\mathbb{T}G)$. For $\alpha' \in \Omega^1(G) \subset \Gamma(\mathbb{T}G)$, we have $\langle \alpha', \mathsf{f}(v^a)\rangle = \frac{1}{2}\langle \alpha', v_a^L + v_a^R\rangle\, \mathsf{e}(v^a)$. Hence, (20) shows that

$$\pi^\sharp \Phi^* \alpha' = -\sum_a \Phi^* \langle \alpha', \tfrac{v_a^L + v_a^R}{2}\rangle\, \mathscr{A}_M(v^a), \quad \alpha' \in \Omega^1(G), \tag{104}$$

and, by (24), we have:

$$\mathrm{ran}(\mathscr{A}_M) + \mathrm{ran}(\pi^\sharp) = TM. \tag{105}$$

This last condition can be viewed as a counterpart to the invertibility of a Poisson bivector defined by a symplectic form. Dropping this condition, one arrives at the following definition:

Definition 5.20. — [**1, 2**] A *q-Hamiltonian q-Poisson* $\mathfrak{g}$*-manifold* is a manifold M, together with a Lie algebra action $\mathscr{A}_M : \mathfrak{g} \to \mathfrak{X}(M)$, a $\mathfrak{g}$-invariant bivector field π, and a $\mathfrak{g}$-equivariant *moment map* $\Phi : M \to G$, such that conditions (103) and (104) are satisfied. If the $\mathfrak{g}$-action on M integrates to a G-action, such that π, Φ are G-equivariant, we speak of a *q-Hamiltonian q-Poisson G-manifold.*

Example 5.21. — The basic example of a Hamiltonian Poisson G-manifold is provided by the coadjoint action on $M = \mathfrak{g}^*$, with $\pi = \pi_{\mathfrak{g}^*}$ the Kirillov bivector and moment map the identity map. Similarly, the quadruple $(G, \mathscr{A}_{\mathsf{ad}}, \pi_G, \mathrm{id})$, with π_G the bivector field (46), is a q-Hamiltonian q-Poisson G-manifold.

The techniques in this paper allow us to give a much simpler proof to the following theorem from [**14**]:

Theorem 5.22. — *There is a 1-1 correspondence between q-Hamiltonian q-Poisson* $\mathfrak{g}$*-manifolds* $(M, \mathscr{A}_M, \pi, \Phi)$*, and Dirac manifolds* (M, E_M, η_M) *equipped with a strong Dirac morphism*

$$(\Phi, 0) : (M, E_M, \eta_M) \to (G, E_G, \eta). \tag{106}$$

Under this correspondence, $\mathrm{ran}(E_M) = \mathrm{ran}(\mathscr{A}_M) + \mathrm{ran}(\pi^\sharp)$.

Proof. — Suppose $(\Phi, 0) : (M, E_M, \eta_M) \to (G, E_G, \eta)$ is a strong Dirac morphism. Consider the bundle map $\mathfrak{a} : \Phi^* E_G \to TM$ defined by Φ (see Section 2.2). By Proposition 2.10(c), the vector fields $\mathscr{A}_M(\xi) = \mathfrak{a}(\mathsf{e}(\xi)) \in \mathfrak{X}(M)$ define a Lie algebra action of $\mathfrak{g}$ on M for which Φ is equivariant. Note also that since $\mathrm{ran}(\mathfrak{a}) \subset \mathrm{ran}(E_M)$, this action preserves the leaves $Q \subset M$ of E_M. In fact, the bundle E_M is $\mathfrak{g}$-invariant: If $E_M = \mathrm{Gr}_\omega$ this follows from the $\mathfrak{g}$-invariance of ω (see comment after Def. 5.1), and in the general case it follows since $E_M|_Q$ is invariant, for any leaf Q. Let F_M be the backward image of F_G under $(\Phi, 0)$, and $\pi \in \mathfrak{X}^2(M)$ be the bivector field defined by the splitting $\mathbb{T}M = E_M \oplus F_M$. Then π is $\mathfrak{g}$-invariant (since E_M, F_M are). Equation (103) follows from Proposition 2.10(d), while Equation (104) is a consequence of Theorem 1.20, Equation (20).

Conversely, given a quasi-Poisson $\mathfrak{g}$-manifold $(M, \mathscr{A}_M, \pi, \Phi)$, let $\mathfrak{a} : \Phi^* E_G \to TM$ be the bundle map given on sections by $\Phi^* \mathsf{e}(\xi) \mapsto \mathscr{A}_M(\xi)$. The $\mathfrak{g}$-equivariance of Φ implies that $\Phi \circ \mathfrak{a} = \mathrm{pr}_{\Phi^* TG}|_{\Phi^* E_G}$. Theorem 1.20 provides a Lagrangian splitting $\mathbb{T}M = E_M \oplus F_M$ such that F_M is the backward image of F_G and E_G is the forward image of E_M. It remains to check the integrability condition of E_M relative to the 3-form $\eta_M = \Phi^* \eta$. Let $\Upsilon^E \in \Gamma(\wedge^3 F_M)$ be the Courant tensor of E_M. We have to show that $\Upsilon^E = 0$, or equivalently that $\Gamma(E_M)$ is closed under the η_M-twisted Courant bracket. Recall that E_M is spanned by the sections of two types:

$$\widehat{\mathscr{A}}_M(\xi) := \widehat{\mathfrak{a}}(\Phi^* \mathsf{e}(\xi)) = \mathscr{A}_M(\xi) \oplus \Phi^* B(\tfrac{\theta^L + \theta^R}{2}, \xi)$$

for $\xi \in \mathfrak{g}$, and sections $h(\alpha)$, for $\alpha \in \Omega^1(M)$, where the map h is defined as in (23), with $\mathbb{V}$ replaced with $\mathbb{T}M$, and with $\omega = 0$. Since $\widehat{\mathfrak{a}}$ is a comorphism of Lie algebroids (cf. Proposition 2.8), we have

$$\llbracket\widehat{\mathcal{C}}_M(\xi_1),\widehat{\mathcal{C}}_M(\xi_2)\rrbracket_{\eta_M}=\widehat{\mathcal{C}}_M([\xi_1,\xi_2]). \tag{107}$$

Furthermore, since π is $\mathfrak{g}$-invariant, it follows from (23) that the map h is $\mathfrak{g}$-equivariant, and therefore

$$[\varrho(h(\alpha)),[\varrho(\widehat{\mathcal{C}}_M(\xi)),\mathrm{d}+\Phi^*\eta]]=[\varrho(h(\alpha)),\mathcal{L}(\mathcal{C}_M(\xi))]=-\varrho(h(\mathcal{L}(\mathcal{C}_M(\xi)\alpha)).$$

Thus

$$\llbracket\widehat{\mathcal{C}}_M(\xi),h(\alpha)\rrbracket_{\eta_M}=h(\mathcal{L}(\mathcal{C}_M(\xi))\alpha) \tag{108}$$

by definition of the Courant bracket. Equations (107) and (108) show that $\llbracket\widehat{\mathcal{C}}_M(\xi),\cdot\rrbracket_{\eta_M}$ preserves $\Gamma(E_M)$. Thus $\Upsilon^E(x_1,x_2,x_3)$ vanishes if one of the three sections $x_i\in\Gamma(E_M)$ lies in the range of $\widehat{\mathcal{C}}_M$. It remains to show that $\Upsilon^E(h(\alpha_1),h(\alpha_2),h(\alpha_3))=0$ for all 1-forms α_i, or equivalently that $h^*\Upsilon^E=0$, where $h^*\colon F_M\to TM$ is the dual map to $h\colon T^*M\to E_M=F_M^*$. Since $h=\mathsf{p}|_{T^*M}$, where $\mathsf{p}:\mathbb{T}M\to E_M$ is the projection along F_M (see (23)), we have $h^*=\mathrm{pr}_{TM}|_{F_M}$. Thus, we must show that $\mathrm{pr}_{TM}\Upsilon^E=0$. By Proposition 2.10(b), and the defining property of q-Hamiltonian q-Poisson spaces, we have

$$\mathrm{pr}_{TM}(\Upsilon^F)=\mathfrak{a}(\Phi^*\Upsilon^{F_G})=\mathcal{C}_M(\Xi)=\tfrac{1}{2}[\pi,\pi]_{\mathrm{Sch}}.$$

On the other hand, Theorem 2.9(a) gives $\mathrm{pr}_{TM}(\Upsilon^E)+\mathrm{pr}_{TM}(\Upsilon^F)-\frac{1}{2}[\pi,\pi]_{\mathrm{Sch}}=0$. Taking the two results together, we obtain $\mathrm{pr}_{TM}(\Upsilon^E)=0$ as desired. □

As an immediate consequence, the data $(M,\mathcal{C}_M,\pi,\Phi)$ defining a q-Hamiltonian q-Poisson G-manifold are equivalent to the data of a G-equivariant Dirac manifold (M,E_M,η_M), equipped with a G-equivariant Dirac morphism $(\Phi,0)$, for which the G-action on M integrates the $\mathfrak{g}$-action defined by the Dirac morphism.

***Proposition 5.23* (Fusion).** — *Suppose $(M,\mathcal{C}_M,\pi,\Phi)$ is a q-Hamiltonian q-Poisson $\mathfrak{g}\times\mathfrak{g}$-manifold. Let $\mathcal{C}_{\mathrm{fus}}$ be the diagonal $\mathfrak{g}$-action, $\Phi_{\mathrm{fus}}=\mathrm{Mult}\circ\Phi$, and $\pi_{\mathrm{fus}}=\pi+\mathcal{C}_M(\gamma)$. Then $(M,\mathcal{C}_{\mathrm{fus}},\pi_{\mathrm{fus}},\Phi_{\mathrm{fus}})$ is a q-Hamiltonian q-Poisson $\mathfrak{g}$-manifold.*

Proof. — By Theorem 5.22, the given q-Poisson $\mathfrak{g}\times\mathfrak{g}$-manifold corresponds to a Dirac manifold (M,E_M,η_M) such that $(\Phi,0)$ is a Dirac morphism into $(G,E_G,\eta)\times(G,E_G,\eta)$. Thus, $\eta_M=\Phi^*(\eta_G^1+\eta_G^2)$. The bivector field π is defined by the Lagrangian splitting $\mathbb{T}M=E_M\oplus F_M$, where F_M is the backward image of $F_G^1\oplus F_G^2$ under $(\Phi,0)$. Composing with $(\mathrm{Mult},\varsigma)$ (cf. Thm. 3.9), we obtain a strong Dirac morphism,

$$(\Phi_{\mathrm{fus}},\Phi^*\varsigma)\colon(M,E_M,\eta_M)\to(G,E_G,\eta),$$

which in turn defines a q-Hamiltonian q-Poisson $\mathfrak{g}$-manifold. Let $\widetilde{F}_M$ be the backward image of F_G under this Dirac morphism. By Proposition 3.11, $\widetilde{F}$ is related to F by the section $\widehat{\mathcal{C}}_M(\gamma)\in\Gamma(\wedge^2E_M)$, where $\widehat{\mathcal{C}}_M\colon\mathfrak{g}\times\mathfrak{g}\to E_M$ is the map defined by the Dirac morphism $(\Phi,0)$. Hence, by Proposition 1.18, the bivector for the new splitting $\mathbb{T}M=E_M\oplus\widetilde{F}_M$ is $\pi_{\mathrm{fus}}=\pi+\mathcal{C}_M(\gamma)$. □

***Proposition 5.24* (Exponentials).** — *Suppose $(M, \mathcal{A}_M, \pi_0, \Phi_0)$ is a Hamiltonian Poisson $\mathfrak{g}$-manifold. That is, $\mathcal{A}_M$ is a $\mathfrak{g}$-action on M, π_0 is a $\mathfrak{g}$-invariant Poisson structure, and $\Phi_0 \colon M \to \mathfrak{g}$ is a $\mathfrak{g}$-equivariant moment map generating the given action on M. Assume that $\Phi_0(M) \subset \mathfrak{g}_\natural$, and let*

$$\Phi = \exp \circ \Phi_0, \quad \pi = \pi_0 + \mathcal{A}_M(\Phi_0^* \varepsilon)$$

where $\varepsilon \in C^\infty(\mathfrak{g}_\natural, \wedge^2 \mathfrak{g})$ is the solution of the CDYBE defined in Section 3.5. Then $(M, \mathcal{A}_M, \pi, \Phi)$ is a q-Hamiltonian q-Poisson $\mathfrak{g}$-manifold.

Proof. — It is well-known that $(M, \mathcal{A}_M, \pi_0, \Phi_0)$ is a Hamiltonian $\mathfrak{g}$-manifold if and only if $\Phi_0 \colon M \to \mathfrak{g}^*$ is a Poisson map, i.e., if and only if

$$(\Phi_0, 0) \colon (M, E_M, 0) \to (\mathfrak{g}^*, E_{\pi_{\mathfrak{g}^*}}, 0)$$

is a strong Dirac morphism, with $E_M = \mathrm{Gr}_{\pi_0}$ and $E_{\mathfrak{g}^*} = \mathrm{Gr}_{\pi_{\mathfrak{g}^*}}$. Using B to identify $\mathfrak{g}^* \cong \mathfrak{g}$, and composing with the strong Dirac morphism $(\exp, \varpi)$, one obtains the strong Dirac morphism

$$(\Phi, \Phi_0^* \varpi) \colon (M, E_M, 0) \to (G, E_G, \eta),$$

which in turn gives rise to a q-Hamiltonian q-Poisson $\mathfrak{g}$-manifold $(M, \mathcal{A}_M, \pi, \Phi)$. The backward image $\widetilde{F}_M \subset \mathbb{T}M$ of F_G under the Dirac morphism $(\Phi, \Phi_0^* \varpi)$ is a Lagrangian complement to $E_M = \mathrm{Gr}_\pi$. Let $\widehat{\mathfrak{a}} \colon \Phi_0^* E_{\mathfrak{g}} \to E_M$ be defined by the Dirac morphism $(\Phi_0, 0)$, and put $\widehat{\mathcal{A}}_M(\xi) = \widehat{\mathfrak{a}} \circ \Phi_0^* \mathsf{e}_0(\xi)$. As explained in Section 3.5, $\widetilde{F}_M$ is related the Lagrangian complement $F_M = TM$ by the section $\widehat{\mathcal{A}}_M(\Phi_0^* \varepsilon)$. Hence, $\pi = \pi_0 + \mathcal{A}_M(\Phi_0^* \varepsilon)$. □

5.5. $\mathfrak{k}^*$-valued moment maps. — Let K be any Lie group. An ordinary Hamiltonian Poisson K-manifold is a triple (M, π, Φ) where M is a K-manifold, $\pi \in \mathfrak{X}^2(M)$ is an invariant Poisson structure, and $\Phi \colon M \to \mathfrak{k}^*$ is a K-equivariant map satisfying the moment map condition,

$$\pi^\sharp(\mathrm{d}\langle \Phi, \xi \rangle) = \mathcal{A}_M(\xi).$$

The moment map condition is equivalent to Φ being a Poisson map. The following result implies that $\mathfrak{k}^*$-valued moment maps can be viewed as special cases of $G = \mathfrak{k}^* \rtimes K$-valued moment maps. Let $\mathfrak{g} = \mathfrak{k}^* \rtimes \mathfrak{k}$ carry the invariant inner product given by the pairing.

Proposition 5.25. — *The inclusion map $j \colon \mathfrak{k}^* \hookrightarrow \mathfrak{k}^* \rtimes K = G$ is a strong Dirac morphism $(j, 0)$, as well as a backward Dirac morphism, relative to the Kirillov-Poisson structure on $\mathfrak{k}^*$ and the Cartan-Dirac structure on G. The backward image of F_G under this Dirac morphism is $F_{\mathfrak{k}^*} = T\mathfrak{k}^*$. The pure spinor ψ_G on $G = \mathfrak{k}^* \rtimes K$ satisfies*

$$j^* \psi_G = 1.$$

Proof. — The Cartan-Dirac structure E_G is spanned by the sections $\mathsf{e}(w)$ for $w = (\beta, \xi) \in \mathfrak{g}$, while $E_{\mathfrak{k}^*}$ is spanned by the sections $\mathsf{e}_0(\xi)$ for $\xi \in \mathfrak{k}$. The first part of the Proposition will follow once we show that $\mathsf{s}_0(\beta, \xi) \sim_{(j,0)} \mathsf{s}(\beta, \xi)$, i.e.

$$\mathsf{e}_0(\xi) \sim_{(j,0)} \mathsf{e}(\beta, \xi), \quad \mathsf{f}_0(\beta) \sim_{(j,0)} \mathsf{f}(\beta, \xi). \tag{109}$$

The vector field part of the first relation follows since the inclusion $j\colon \mathfrak{k}^* \hookrightarrow \mathfrak{k}^* \rtimes K$ is equivariant for the conjugation action of $G = \mathfrak{k}^* \rtimes K$. (Here, the $\mathfrak{k}^*$-component of G acts trivially on $\mathfrak{k}^*$, while the K-component acts by the co-adjoint action.) For the 1-form part, we note that the pull-back of the Maurer-Cartan forms $\theta^L, \theta^R \in \Omega^1(G) \otimes \mathfrak{g}$ to the subgroup $\mathfrak{k}^* \subset G$ is the Maurer-Cartan form for additive group $\mathfrak{k}^*$, i.e.

$$j^*\theta^L = j^*\theta^R = \theta_0$$

where the 'tautological 1-form' $\theta_0 \in \Omega^1(\mathfrak{k}^*) \otimes \mathfrak{k}^*$ is defined as in Section 3.5. Thus

$$j^* B\big(\tfrac{\theta_G^L + \theta_G^R}{2}, (\beta, \xi)\big) = B(\theta_0, (\beta, \xi)) = \langle \theta_0, \xi \rangle.$$

This verifies the first relation in (109); the second one is checked similarly.

Since the adjoint action $\mathrm{Ad}\colon G \to \mathrm{O}(\mathfrak{g})$ is trivial over $\mathfrak{k}^*$, the lift $\tau\colon G \to \mathrm{Pin}(\mathfrak{g}) \subset \mathrm{Cl}(\mathfrak{g})$ satisfies $\tau|_{\mathfrak{k}^*} = 1$. It follows that the pure spinor $\psi_G = \mathscr{R}(q(\mu))$ satisfies $j^*\psi_G = 1$. $\square$

Corollary 5.26. — *Let (M, π) be a Poisson manifold. Then $\Phi\colon M \to \mathfrak{k}^*$ is a Poisson map if and only if the composition $j \circ \Phi\colon M \to G$ is a strong Dirac morphism*

$$(j \circ \Phi, 0)\colon (M, \mathrm{Gr}_\pi, 0) \to (G, E_G, \eta).$$

Put differently, Hamiltonian Poisson K-manifolds are q-Hamiltonian q-Poisson $\mathfrak{k}^* \rtimes K$-manifolds for which the moment map happens to take values in $\mathfrak{k}^*$.

As a special case, a Hamiltonian K-manifold (M, ω, Φ) (with ω a symplectic 2-form, and Φ satisfying the moment map condition $\iota(\mathscr{C}_M(\xi))\omega = \mathrm{d}\langle \Phi, \xi \rangle$) is equivalent to a q-Hamiltonian $G = \mathfrak{k}^* \rtimes K$-space for which the moment map takes values in $\mathfrak{k}^*$. Since $j^*\psi_G = 1$, its q-Hamiltonian volume form coincides with the usual Liouville form $(\exp \omega)^{[\mathrm{top}]}$.

6. K^*-valued moment maps

For a Poisson Lie group K, J.-H. Lu **[42]** introduced another type of group-valued moment map, taking values in the dual Poisson Lie group K^*. For a compact Lie group K, with its standard Poisson structure, this moment map theory turns out to be equivalent to the usual $\mathfrak{k}^*$-valued one. In this Section, we will re-examine this equivalence using the techniques developed in this paper.

6.1. Review of K^*-valued moment maps. — The theory of Poisson-Lie groups were introduced by Drinfeld in [**23**], see e.g. [**18**] for an overview and bibliography. Suppose K is a connected Poisson Lie group, with Poisson structure defined by a Manin triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{k}')$. (That is, $\mathfrak{g}$ is a Lie algebra with an invariant split inner product, and $\mathfrak{k}, \mathfrak{k}'$ are complementary Lagrangian subalgebras.) Use the paring to identify $\mathfrak{k}' = \mathfrak{k}^*$, and let K^* be the associated dual Poisson Lie group. We assume that $\mathfrak{g}$ integrates to a Lie group G (the double) such that K, K^* are subgroups and the product map $K \times K^* \to G$ is a diffeomorphism. The left action of K on G descends to a *dressing action* $\mathscr{A}_{K^*}$ on K^* (viewed as a homogeneous space G/K). The Poisson structure on K^*, or equivalently its graph $E_{K^*} = \mathrm{Gr}_{\pi_{K^*}} \subset \mathbb{T}K^*$, may be expressed in terms of the infinitesimal dressing action, as the span of sections

$$e_{K^*}(\xi) = \mathscr{A}_{K^*}(\xi) \oplus \langle \theta^R_{K^*}, \xi \rangle$$

for $\xi \in \mathfrak{k}$. Here $\theta^R_{K^*} \in \Omega^1(K^*) \otimes \mathfrak{k}^*$ is the right-invariant Maurer-Cartan form for K^*. Note that as a Lie algebroid, E_{K^*} is just the action algebroid.

For the remainder of this Section 6, we will assume that K is a compact real Lie group. The *standard Poisson structure* on K is described as follows. Let $G = K^{\mathbb{C}}$ be the complexification, with Lie algebra $\mathfrak{g}$, and let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \quad G = KAN$$

be the Iwasawa decompositions. Here $\mathfrak{a} = \sqrt{-1}\mathfrak{t}_K$, $A = \exp \mathfrak{a}$ and $N = N_+$ (using the notation from Section 3.6). We denote by B_K an invariant inner product on $\mathfrak{k}$, and let $\langle \cdot, \cdot \rangle$ be the imaginary part of $2B_K^{\mathbb{C}}$. Then $(\mathfrak{g}, \mathfrak{k}, \mathfrak{a} \oplus \mathfrak{n})$ (where $\mathfrak{g}$ is viewed as a real Lie algebra) is a Manin triple. Thus K becomes a Poisson Lie group, with dual Poisson Lie group $K^* = AN$.

A *K^*-valued Hamiltonian $\mathfrak{k}$-manifold*, as defined by Lu [**42**], is a symplectic manifold (M, ω) together with a Poisson map $\Phi \colon M \to K^*$. Equivalently, $(\Phi, \omega) \colon (M, TM, 0) \to (K^*, E_{K^*}, 0)$ is a strong Dirac morphism. The Poisson map Φ induces a $\mathfrak{k}$-action on M, and if this action integrates to an action of K we speak of a *K^*-valued Hamiltonian K-manifold*. An interesting feature is that ω is not K-invariant, in general: Instead, the action map $K \times M \to M$ is a Poisson map. Accordingly, the volume form $(\exp \omega)^{[\mathrm{top}]}$ is not K-invariant. However, let $\Phi^A \colon M \to A$ be the composition of Φ with projection $K^* = AN \to A$, and $(\Phi^A)^{2\rho} \colon M \to \mathbb{R}_{>0}$ its image under the homomorphism $T \to \mathbb{C}^\times$, $t \mapsto t^{2\rho}$ defined by the sum of positive roots. By [**7**, Theorem 5.1], the product

$$(\Phi^A)^{2\rho} \ (\exp \omega)^{[\mathrm{top}]} \tag{110}$$

is a K-invariant volume form. The proof in [**7**] uses a tricky argument; one of the goals of this Section is to give a more conceptual explanation.

6.2. P-valued moment maps.— To explain the origin of the volume form (110), we will use the notion of a P-valued moment map introduced in [**3**, Section 10]. Let

$g \mapsto g^c$ denote the complex conjugation map on G, and let

$$I(g) \equiv g^\dagger = (g^{-1})^c.$$

On the Lie algebra level, let $\xi \mapsto \xi^c$ denote conjugation, and $\xi^\dagger = -\xi^c$. We have $K = \{g \in G |\ g^\dagger = g^{-1}\}$. Let

$$P = \{g^\dagger g |\ g \in G\}$$

denote the subset of 'positive definite' elements in G. Then P is a submanifold fixed under I, and the product map defines the Cartan decomposition $G = KP$. Let E_G be the (holomorphic) Dirac structure on G defined by the inner product

$$B := \tfrac{1}{\sqrt{-1}} B_K^{\mathbb{C}}.$$

Since $(\theta^L)^\dagger = I^*\theta^R,\ (\theta^R)^\dagger = I^*\theta^L$, the Cartan 3-form on G satisfies satisfies $\eta^c = I^*\eta$, thus $\eta_P := \iota_P^*\eta$ is real-valued. Similarly, the pull-backs of the 1-forms $B(\frac{\theta^L+\theta^R}{2}, \xi)$ for $\xi \in \mathfrak{k}$ are real-valued. It follows that the sections

$$\mathsf{e}_P(\xi) := \mathsf{e}(\xi)|_P$$

are real-valued. Letting $E_P \subset \mathbb{T}P$ be the subbundle spanned by these sections, it follows that (P, E_P, η_P) is a real Dirac manifold, with $(E_P)^{\mathbb{C}} = E_G|_P$. As a Lie algebroid, E_P is just the action algebroid for the K-action on P. Similarly, the sections $\mathsf{f}_P(\xi) := \mathsf{f}(\xi)|_P$ are real-valued, defining a complement F_P to E_P. The bundle F_P is defined by the (real-valued) pure spinor, $\psi_P := \iota_P^*\psi_G \in \Omega(P)$.

Remark 6.1. — Since $\det(\mathrm{Ad}_g + 1) > 0$ for $g \in P$ (all eigenvalues of Ad_g are strictly positive), one finds that $\ker(E_P) = \{0\}$. Hence E_P is the graph of a bivector π_P with $\frac{1}{2}[\pi_P, \pi_P] = \pi_P^\sharp(\eta_P)$.

A *P-valued Hamiltonian $\mathfrak{k}$-manifold* [**3**, Section 10] is a manifold M together with a strong Dirac morphism $(\Phi_1, \omega_1)\colon (M, TM, 0) \to (P, E_P, \eta_P)$. For any such space we obtain, as for the q-Hamiltonian setting, an invariant volume form

$$(\exp(\omega_1) \wedge \Phi_1^*\psi_P)^{[\mathrm{top}]}. \tag{111}$$

Here ψ_P may be replaced by $\widehat{\psi}_P$, the pull-back of the Gauss-Dirac spinor.(5) By Proposition 5.19, the expression $\exp\omega_1 \wedge \Phi_1^*(\Delta_\lambda\ \widehat{\psi}_P)$ is closed under the differential $\mathrm{d} - 2\pi\iota(\mathscr{C}_M(\lambda+\rho)))$, for any dominant weight λ.

6.3. Equivalence between K^*-valued and P-valued moment maps. — To relate the K^*-valued theory with the P-valued theory, we use the K-equivariant diffeomorphism

$$\kappa\colon K^* \to P,\ g \mapsto g^\dagger g.$$

(5) In Section 5.3, B was taken as the complexification of B_K, while here we have an extra factor $\sqrt{-1}$. This amounts to a simple rescaling of the bilinear form $B_K^{\mathbb{C}}$, not affecting any of the results.

Note that this map takes values in the big Gauss cell, $\mathcal{O} = N_-KN \subset G$. Let $\varpi_{\mathcal{O}}$ denote the (complex) 2-form on the big Gauss cell, and $\varpi_{K^*} = \kappa^*\omega_{\mathcal{O}}$. It is easy to check that ϖ_{K^*} is real-valued. One can check that

$$e_{K^*}(\xi) \sim_{(\kappa,\varpi_{K^*})} e_P(\xi)$$

for all $\xi \in \mathfrak{k}$: The vector field part of this relation is equivalent to the $\mathfrak{k}$-equivariance, while the 1-form part is verified in [**3**, Section 10]. It follows that (κ, ϖ_{K^*}) is a Dirac isomorphism from $(K^*, E_{K^*}, 0)$ onto (P, E_P, η_P).

Thus, if (M, ω, Φ) is a K^*-valued Hamiltonian $\mathfrak{k}$-manifold, then (M, ω_1, Φ_1) with $\omega_1 = \omega + \Phi^*\varpi_{K^*}$ and $\Phi_1 = \kappa \circ \Phi$ is a P-valued Hamiltonian $\mathfrak{k}$-manifold. In particular, we obtain an invariant volume form on M,

$$\big(\exp(\omega + \Phi^*\varpi_{K^*}) \wedge \Phi^*\kappa^*\widehat{\psi}_P\big)^{[\mathrm{top}]}.$$

Using the explicit formula (Proposition 4.14) for the Gauss-Dirac spinor, we obtain

$$\kappa^*\widehat{\psi}_P = a^{2\rho}\exp(-\varpi_{K^*}),$$

where $a\colon K^* \to A$ is projection to the A-factor. Hence,

$$\exp(\omega + \Phi^*\varpi_{K^*}) \wedge \Phi^*\kappa^*\psi_P = (\Phi^A)^{2\rho}\exp(\omega),$$

identifying the volume form for the associated P-valued space with the volume form (110).

Proposition 6.2. — *For any K^*-valued Hamiltonian $\mathfrak{k}$-space (M, ω, Φ), the volume form $(\Phi^A)^{2\rho}(\exp\omega)^{[\mathrm{top}]}$ is $\mathfrak{k}$-invariant. Moreover, for all dominant weights λ the differential form*

$$(\Phi^A)^{2(\lambda+\rho)}\exp(\omega)$$

is closed under the differential $d - 2\pi\mathcal{A}_M(B_K^\sharp(\lambda+\rho))$.

Proof. — Invariance follows from the identification with the volume form for the associated P-valued space. The second claim follows from Proposition 5.19, since the function Δ_λ from Section 4.5 satisfies $\kappa^*\Delta_\lambda = a^{2\lambda}$. □

The differential equation permits a computation of the integrals $\int_M (\Phi^A)^{2(\lambda+\rho)}(\exp(\omega))^{[\mathrm{top}]}$ by localization [**12**] to the zeroes of the vector field $\mathcal{A}_M(B_K^\sharp(\lambda+\rho))$, similar to the formula in 5.3.

6.4. Equivalence between P-valued and $\mathfrak{k}^*$-valued moment maps. — Finally, let us express the correspondence [**3**, Section 10] between P-valued moment maps and $\mathfrak{k}^*$-valued moment maps in terms of Dirac morphisms. The exponential map for $G = K^{\mathbb{C}}$ restricts to a diffeomorphism

$$\exp_{\mathfrak{p}}\colon \mathfrak{p} := \sqrt{-1}\mathfrak{k} \to P := \exp(\sqrt{-1}\mathfrak{k}).$$

Let $\varpi \in \Omega^2(\mathfrak{g})$ be the primitive of $\exp^*\eta$ defined in (58), and $\varpi_{\mathfrak{p}}$ its pull-back to $\mathfrak{p}$. Since η_P is real-valued, so is $\varpi_{\mathfrak{p}}$, and $\mathrm{d}\varpi_{\mathfrak{p}} = (\exp|_{\mathfrak{p}})^*\eta_P$. Similarly, $J_{\mathfrak{p}} := J|_{\mathfrak{p}} > 0$. The formulas for $\varpi_{\mathfrak{p}}$ and $J_{\mathfrak{p}}$ are similar to those for the Lie algebra $\mathfrak{k}$, but with sinh functions replaced by sin functions. Use $B^\sharp = \sqrt{-1}B^\sharp_K$ to identify $\mathfrak{k}^* \cong \mathfrak{p}$. By Proposition 3.12,

$$\mathsf{e}_0(\xi) \sim_{(\exp_\mathfrak{p}, \varpi_\mathfrak{p})} \mathsf{e}_P(\xi), \quad \xi \in \mathfrak{k}.$$

Hence $(\exp_\mathfrak{p}, \varpi_\mathfrak{p})$ is a Dirac (iso)morphism from $(\mathfrak{k}^*, E_{\mathfrak{k}^*}, 0)$ to (P, E_P, η_P). This sets up a 1-1 correspondence between P-valued and $\mathfrak{k}^*$-valued Hamiltonian $\mathfrak{k}$-spaces. Thinking of the latter as given by strong Dirac morphisms (Φ_0, ω_0) to $(\mathfrak{k}^*, E_{\mathfrak{k}^*}, 0)$, the correspondence reads

$$(\Phi_1, \omega_1) = (\exp_\mathfrak{p}, \varpi_\mathfrak{p}) \circ (\Phi_0, \omega_0).$$

The volume forms are related by $(\exp(\omega_1) \wedge \Phi_1^* \psi_P)^{[\mathrm{top}]} = J_\mathfrak{p}^{1/2} \exp(\omega_0)^{[\mathrm{top}]}$.

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Astérisque
327, 2009, p. 201–287

INDEX, ETA AND RHO INVARIANTS ON FOLIATED BUNDLES

by

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Dedicated to Jean-Michel Bismut on the occasion of his sixtieth birthday

Abstract. — We study primary and secondary invariants of leafwise Dirac operators on foliated bundles. Given such an operator, we begin by considering the associated regular self-adjoint operator $\mathcal{D}_m$ on the maximal Connes-Skandalis Hilbert module and explain how the functional calculus of $\mathcal{D}_m$ encodes both the leafwise calculus and the monodromy calculus in the corresponding von Neumann algebras. When the foliation is endowed with a holonomy invariant transverse measure, we explain the compatibility of various traces and determinants. We extend Atiyah's index theorem on Galois coverings to foliations. We define a foliated rho-invariant and investigate its stability properties for the signature operator. Finally, we establish the foliated homotopy invariance of such a signature rho-invariant under a Baum-Connes assumption, thus extending to the foliated context results proved by Neumann, Mathai, Weinberger and Keswani on Galois coverings.

***Résumé* (Indices, invariants êta et rho de fibrés feuilletés).** — Nous étudions certains invariants primaires et secondaires associés aux opérateurs de Dirac le long des feuilles de fibrés feuilletés. Etant donné un tel opérateur, nous considérons d'abord l'opérateur auto-adjoint régulier $\mathcal{D}_m$ qui lui est associé sur le module de Hilbert maximal de Connes-Skandalis, puis nous expliquons comment le calcul fonctionnel de $\mathcal{D}_m$ permet de coder le calcul longitudinal ainsi que le calcul sur les fibres de monodromie dans les algèbres de von Neumann correspondantes. Lorsque le feuilletage admet une mesure transverse invariante par holonomie, nous expliquons la compatibilité de diverses traces et déterminants. Nous étendons le théorème de l'indice pour les revêtements d'Atiyah aux feuilletages. Nous définissons l'invariant rho feuilleté et étudions ses propriétés de stabilité lorsque l'opérateur en question est l'opérateur de signature. Finalement, nous établissons l'invariance par homotopie feuilletée de l'invariant rho de l'opérateur de signature le long des feuilles sous une hypothèse de Baum-Connes, prolongeant ainsi au contexte feuilleté des résultats prouvés par Neumann, Mathai, Weinberger et Keswani dans le cadre des revêtements galoisiens.

2010 Mathematics Subject Classification. — 52J20, 58J28, 58J42, 19k56.
Key words and phrases. — Measured foliations, foliated bundles, groupoids, eta-invariants, rho-invariants, maximal foliation-C^*-algebra, Baum-Connes map, foliated homotopy invariance.

Introduction and main results

The Atiyah-Singer index theorem on closed compact manifolds is regarded nowadays as a classic result in mathematics. The original result has branched into several directions, producing new ideas and new results. One of these directions consists in considering elliptic differential operators on the following hierarchy of geometric structures:

- fibrations and operators that are elliptic in the fiber directions; for example, a product fibration $M \times T \to T$ and a family $(D_\theta)_{\theta \in T}$ of elliptic operators on M continuously parametrized by T;
- Galois Γ-coverings and Γ-equivariant elliptic operators;
- measured foliations and operators that are elliptic along the leaves;
- general foliations and, again, operators that are elliptic along the leaves.

One pivotal example, going through all these situations, is the one of foliated bundles. Let $\Gamma \to \tilde{M} \to M$ be a Galois Γ-cover of a smooth compact manifold M, let T be a compact manifold on which Γ acts by diffeomorphism. We can consider the diagonal action of Γ on $\tilde{M} \times T$ and the quotient space $V := \tilde{M} \times_\Gamma T$, which is a compact manifold, a bundle over M and carries a foliation $\mathcal{F}$. This foliation is obtained by considering the images of the fibers of the trivial fibration $\tilde{M} \times T \to T$ under the quotient map $\tilde{M} \times T \to \tilde{M} \times_\Gamma T$ and is known as a *foliated bundle*. More generally, we could allow T to be a compact topological space with an action of Γ by homeomorphisms, obtaining what is usually called a *foliated space* or a *lamination*. We then consider a family of elliptic differential operators $(\tilde{D}_\theta)_{\theta \in T}$ on the product fibration $\tilde{M} \times T \to T$ and we assume that it is Γ-equivariant; it therefore yields a leafwise differential operator $D = (D_L)_{L \in V/\mathcal{F}}$ on V, which is elliptic along the leaves of $\mathcal{F}$. Notice that, if $\dim T > 0$ and $\Gamma = \{1\}$ then we are in the family situation; if $\dim T = 0$ and $\Gamma \neq \{1\}$, then we are in the covering situation; if $\dim T > 0$, $\Gamma \neq \{1\}$ and T admits a Γ-invariant Borel measure ν, then we are in the measured foliation situation and if $\dim T > 0$, $\Gamma \neq \{1\}$ then we are dealing with a more general foliation.

In the first three cases, there is first of all a *numeric* index: for families this is quite trivially the integral over T of the locally constant function that associates to θ the index of D_θ; for Γ-coverings we have the Γ-index of Atiyah and for measured foliations we have the measured index introduced by Connes. These last two examples involve the definition of a von Neumann algebra endowed with a suitable trace. More generally, and this applies also to general foliations, one can define *higher indices*, obtained by pairing the index class defined by an elliptic operator with suitable (higher) cyclic cocycles. In the case of foliated bundles there is a formula for these higher indices, due to Connes [**18**], and recently revisited by Gorokhovsky and Lott [**23**] using a generalization of the Bismut superconnection [**13**]. See also [**39**]. Since our main focus here are numeric (versus higher) invariants, we go back to the case of measured foliated bundles, thus assuming that T admits a Γ-invariant measure ν.

The index is of course a global object, defined in terms of the kernel and cokernel of operators. However, one of its essential features is the possibility of localizing it near the diagonal using the remainders produced by a parametrix for D. On a closed manifold this crucial property is encoded in the so-called Atiyah-Bott formula:

$$\operatorname{ind}(D) = \operatorname{Tr}(R_0^N) - \operatorname{Tr}(R_1^N)\,, \quad \forall N \geq 1 \tag{1}$$

if $R_1 = \mathrm{Id} - DQ$ and $R_0 = \mathrm{Id} - QD$ are the remainders of a parametrix Q. Similar results hold in the other two contexts: Γ-coverings and measured foliations. One important consequence of formula (1) and of the analogous one on Γ-coverings is Atiyah's index theorem on a Γ-covering $\tilde{M} \to M$, stating the equality of the index on M and the von Neumann Γ-index on $\tilde{M}$. Informally, the index upstairs is equal to the index downstairs. On a measured foliation, for example on a foliated bundle $(\tilde{M} \times_\Gamma T, \mathcal{F})$ associated to a Γ-space T endowed with a Γ-invariant measure ν, we also have an index upstairs and an index downstairs, depending on whether we consider the Γ-equivariant family $(\tilde{D}_\theta)_{\theta \in T}$ or the longitudinal operator $D = (D_L)_{L \in V/\mathcal{F}}$; the analogue of formula (1) allows to prove the equality of these two indices. (This phenomenon is well known to experts; we explain it in detail in Section 4.)

Now, despite its many geometric applications, the index remains a very coarse spectral invariant of the elliptic differential operator D, depending only on the spectrum near zero. Especially when considering geometric operators, such as Dirac-type operators, and related geometric questions involving, for example, the diffeomorphism type of manifolds or the moduli space of metrics of positive scalar curvature, one is led to consider more involved spectral invariants. The eta invariant, introduced by Atiyah, Patodi and Singer on odd dimensional manifolds, is such an invariant. This invariant is highly non-local (in contrast to the index) and involves the whole spectrum of the operator. It is, however, too sophisticated: indeed, a small perturbation of the operator produces a variation of the corresponding eta invariant. In geometric questions one considers rather a more stable invariant, the rho invariant, typically a difference of eta invariants having the same local variation. The Cheeger-Gromov rho invariant on a Galois covering $\tilde{M} \to M$ of an odd dimensional manifold M is the most famous example; it is precisely defined as the difference of the Γ-eta invariant on $\tilde{M}$, defined using the Γ-trace of Atiyah, and of the Atiyah-Patodi-Singer eta invariant of the base M. Notice that the analogous difference for the indices (in the even dimensional case) would be equal to zero because of Atiyah's index theorem on coverings; the Cheeger-Gromov rho invariant is thus a genuine *secondary invariant.* The Cheeger-Gromov rho invariant is usually defined for a Dirac-type operator $\tilde{D}$ and we bound ourselves to this case from now on; we denote it by $\rho_{(2)}(\tilde{D})$. Here are some of the stability properties of rho:

- let (M,g) be an oriented riemannian manifold and let $\tilde{D}^{\text{sign}}$ be the signature operator on $\tilde{M}$ associated to the Γ-invariant lift of g to $\tilde{M}$: then $\rho_{(2)}(\tilde{D}^{\text{sign}})$ is metric independent and a diffeomorphism invariant of M;
- let M be a spin manifold and assume that the space $\mathscr{R}^+(M)$ of metrics with positive scalar curvature is non-empty. Let $g \in \mathscr{R}^+(M)$ and let $\tilde{D}_g^{\text{spin}}$ be the spin Dirac operator associated to the Γ-invariant lift of g. Then the function $\mathscr{R}^+(M) \ni g \to \rho_{(2)}(\tilde{D}_g^{\text{spin}})$ is constant on the connected components of $\mathscr{R}^+(M)$

There are easy examples, involving lens spaces, showing that $\rho_{(2)}(\tilde{D}^{\text{sign}})$ is *not* a homotopy invariant and that $\mathscr{R}^+(M) \ni g \to \rho_{(2)}(\tilde{D}_g^{\text{spin}})$ is *not* the constant function equal to zero. For purely geometric applications of these two results see, for example, **[15]** and **[46]**. These two properties can be proved in general, regardless of the nature of the group Γ. However, when Γ is *torsion-free*, then the Cheeger-Gromov rho invariant enjoys particularly strong stability properties. Let $\Gamma = \pi_1(M)$ and let $\tilde{M} \to M$ be the universal cover. Then in a series of papers **[29]**, **[30]**, **[31]**, Keswani, extending work of Neumann **[41]**, Mathai **[36]** and Weinberger **[57]**, establishes the following fascinating theorem:

- if M is orientable, Γ is torsion free and the Baum-Connes map $K_*(B\Gamma) \to K_*(C^*_{\max}\Gamma)$ is an isomorphism, then $\rho_{(2)}(\tilde{D}^{\text{sign}})$ is a *homotopy invariant* of M;
- if M is in addition spin and $\mathscr{R}^+(M) \neq \varnothing$ then $\rho_{(2)}(\tilde{D}_g^{\text{spin}}) = 0$ for any $g \in \mathscr{R}^+(M)$.

(The second statement is not explicitly given in the work of Keswani but it follows from what he proves; for a different proof of Keswani's result see the recent paper **[45]**.) Informally: *when Γ is torsion free and the maximal Baum-Connes map is an isomorphism, the Cheeger-Gromov rho invariant behaves like an index, i.e. like a primary invariant: more precisely, it is a homotopy invariant for the signature operator and it is equal to zero for the spin Dirac operator associated to a metric of positive scalar curvature.*

Let us now move on in the hierarchy of geometric structures and consider a foliated bundle $(V := \tilde{M} \times_\Gamma T, \mathscr{F})$, with $\tilde{M} \to M$ the universal cover of an odd dimensional compact manifold and T a compact Γ-space endowed with a Γ-invariant Borel (probability) measure ν. We are also given a Γ-equivariant family of Dirac-type operators $\tilde{D} := (\tilde{D}_\theta)_{\theta \in T}$ on the product fibration $\tilde{M} \times T \to T$ and let $D = (D_L)_{L \in V/\mathscr{F}}$ be the induced longitudinally elliptic operator on V. One is then led to the following natural questions:

1. Can one define a foliated rho invariant $\rho_\nu(D; V, \mathscr{F})$?
2. What are its stability properties if $\tilde{D} = \tilde{D}^{\text{sign}}$ and $\tilde{D} = \tilde{D}^{\text{spin}}$?

3. If the isotropy groups of the action of Γ on T are torsion free and the maximal Baum-Connes map with coefficients
$$K_*^\Gamma(E\Gamma; C(T)) \to K_*(C(T) \rtimes_{\max} \Gamma)$$
is an isomorphism, is $\rho_\nu(V, \mathcal{F}) := \rho_\nu(D^{\text{sign}}; V, \mathcal{F})$ a foliated homotopy invariant?

The goal of this paper is to give an answer to these three questions. Along the way we shall present in a largely self-contained manner the main results in index theory and in the theory of eta invariants on foliated bundles.

This work is organized as follows. In Section 1 we introduce the maximal C^*-algebra $\mathcal{A}_m$ associated to the Γ-space T or, more precisely, to the groupoid $\mathcal{G} := T \rtimes \Gamma$. We endow this C^*-algebra with two traces τ^ν_{reg} and τ^ν_{av}, ν denoting as before the Γ-invariant Borel measure on T. We then define two von Neumann algebras $W^*_{\text{reg}}(\mathcal{G})$, $W^*_{\text{av}}(\mathcal{G})$ with their respective traces; we define representations $\mathcal{A}_m \to W^*_{\text{reg}}(\mathcal{G})$, $\mathcal{A}_m \to W^*_{\text{av}}(\mathcal{G})$ and show compatibility of the traces involved.

In Section 2 we move to foliated bundles, giving the definition, studying the structure of the leaves, introducing the monodromy groupoid G and the associated maximal C^*-algebra $\mathcal{B}_m$. We then introduce two von Neumann algebras, $W^*_\nu(G)$ and $W^*_\nu(V, \mathcal{F})$, to be thought of as the one upstairs and the one downstairs respectively, with their respective traces τ^ν, $\tau^\nu_{\mathcal{F}}$. We introduce representations $\mathcal{B}_m \to W^*_\nu(G)$, $\mathcal{B}_m \to W^*_\nu(V, \mathcal{F})$ and define two compatible traces, also denoted τ^ν_{reg} and τ^ν_{av}, on the C^*-algebra $\mathcal{B}_m$. We then prove an explicit formula for these two traces on $\mathcal{B}_m$. We end Section 2 with a proof of the Morita isomorphism $K_0(\mathcal{A}_m) \simeq K_0(\mathcal{B}_m)$ and its compatibility with the morphisms
$$\tau^\nu_{\text{reg},*}, \tau^\nu_{\text{av},*} : K_0(\mathcal{A}_m) \to \mathbb{C}, \quad \tau^\nu_{\text{reg},*}, \tau^\nu_{\text{av},*} : K_0(\mathcal{B}_m) \to \mathbb{C}$$
induced by the two pairs of traces on $\mathcal{A}_m$ and $\mathcal{B}_m$ respectively.

In Section 3 we move to more analytic questions. We define a natural $\mathcal{A}_m$-Hilbert module $\mathcal{E}_m$ with associated C^*-algebra of compact operators $\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m)$ isomorphic to $\mathcal{B}_m$; we show how $\mathcal{E}_m$ encodes both the L^2-spaces of the fibers of the product fibration $\tilde{M} \times T \to T$ and the L^2-spaces of the leaves of $\mathcal{F}$. We then introduce a Γ-equivariant pseudodifferential calculus, showing in particular how 0-th order operators extend to bounded $\mathcal{A}_m$-linear operators on $\mathcal{E}_m$ and how negative order operators extend to compact operators. We then move to *unbounded regular* operators, for example operators defined by a Γ-equivariant Dirac family $\tilde{D} := (\tilde{D}_\theta)_{\theta \in T}$ and study quite carefully the functional calculus associated to such an operator. We then treat Hilbert-Schmidt operators and trace class operators in our two von Neumann contexts and give sufficient conditions for an operator to be trace class. We study once again various compatibility issues (this material will be crucial later on).

In Section 4 we introduce, in the even dimensional case, the two indices $\operatorname{ind}^{\mathrm{up}}_\nu(\tilde{D})$, $\operatorname{ind}^{\mathrm{down}}_\nu(D)$ with $\tilde{D}=(\tilde{D}_\theta)_{\theta\in T}$ and $D:=(D_L)_{L\in V/\mathcal{F}}$, and show the equality

$$\operatorname{ind}^{\mathrm{up}}_\nu(\tilde{D})=\operatorname{ind}^{\mathrm{down}}_\nu(D).$$

This is the analogue of Atiyah's index theorem on Galois coverings. We also introduce the relevant index class, in $K_0(\mathcal{B}_m)$, and show how the von Neumann indices can be recovered from it and the two morphisms,

$$\tau^\nu_{\mathrm{reg},*}:K_0(\mathcal{B}_m)\to\mathbb{C}\,,\quad \tau^\nu_{\mathrm{av},*}:K_0(\mathcal{B}_m)\to\mathbb{C}$$

defined by the traces $\tau^\nu_{\mathrm{reg}}:\mathcal{B}_m\to\mathbb{C}$, $\tau^\nu_{\mathrm{av}}:\mathcal{B}_m\to\mathbb{C}$.

In Section 5 we introduce the two eta invariants $\eta^\nu_{\mathrm{up}}(\tilde{D})$, $\eta^\nu_{\mathrm{down}}(D)$ and, finally, the foliated rho-invariant $\rho_\nu(D;V,\mathcal{F})$ as the difference of the two. This answers the first question raised above. We end this section establishing an important link between the rho invariant and the determinant of certain paths.

In Section 6 we study the stability properties of the foliated rho invariant, showing in particular that for the signature operator it is metric independent and a foliated diffeomorphism invariant. This answers the second question raised above.

Finally, in Sections 7, 8 and 9 we prove the foliated homotopy invariance of the signature rho-invariant under a Baum-Connes assumption, following ideas of Keswani. In order to keep this paper in a reasonable size, we establish this result under the additional assumption that the foliated homotopy equivalence is induced by an equivariant fiber homotopy equivalence of the fibration defining the foliated bundle (we call this foliated homotopy equivalences *special*). Thus, Section 7 contains preparatory material on determinants and Bott-periodicity; Section 8 gives a sketch of the proof of the homotopy invariance and Section 9 contains the details. With these three sections we give an answer, at least partially, to the third question raised above. Most of the material explained in the previous part of the paper goes into the rather complicated proof. Some of our results are also meant to clarify statements in the work of Keswani.

Acknowledgements. — We thank Paul Baum, Heath Emerson, James Heitsch, Steve Hurder, Yuri Kordyukov, Eric Leichtnam, Hervé Oyono-Oyono, Ken Richardson, George Skandalis and Stéphane Vassout for interesting discussions. We also thank the two anonymous referees for a careful and critical reading of the original manuscript and for some valuable suggestions. Most of this reasearch was carried out while the first author was visiting Università di Roma La Sapienza and while the second author was visiting Université Paul Verlaine-Metz. Financial support for these visits was provided by Université Paul Verlaine-Metz, under the program *professeurs invités*, the CNR-CNRS bilateral project GENCO (Noncommutative Geometry) and the *Ministero Istruzione Università Ricerca*, Italy, under the project *Spazi di Moduli e Teorie di Lie*.

Notations

M	closed manifold	
Γ	fundamental group of M	
$\tilde{M}$	universal cover of M	
F	fundamental domain for the deck transformations	
T	a compact Γ-space	
ν	a Γ-invariant Borel measure on T	
$\mathcal{G}$	the groupoid $T \rtimes \Gamma$	1.1
$\Gamma(\theta)$	isotropy group of Γ at $\theta \in T$	1.1
$\mathcal{A}_c$	$= C_c(T \times \Gamma)$: algebraic crossed product algebra	1.1
$\mathcal{A}_r$	$= C(T) \rtimes_r \Gamma$: reduced crossed product algebra	1.2
$\mathcal{A}_m$	$= C(T) \rtimes_{\max} \Gamma$: maximal crossed product algebra	1.2
V	$= \tilde{M} \times_\Gamma T$: the foliated space	2.1
G	the groupoid $(\tilde{M} \times \tilde{M} \times T)/\Gamma$	2.2
$\mathcal{B}_c$	the compactly supported convolution algebra of G	2.2
$\mathcal{B}_r$	the regular completion of $\mathcal{B}_c$	2.2
$\mathcal{B}_m$	the maximal completion of $\mathcal{B}_c$	2.2
$\mathcal{B}_c^E$	the modified algebra when $E \to V$ is a vector bundle	2.2
τ^ν_{reg}	regular trace of $\mathcal{A}_r$ or $\mathcal{A}_m$	1.4
τ^ν_{av}	trivial or averaged trace on $\mathcal{A}_m$	1.4
also τ^ν_{reg}	regular trace of $\mathcal{B}_r$ or $\mathcal{B}_m$	2.4
also τ^ν_{av}	trivial or averaged trace on $\mathcal{B}_m$	2.4
$\hat{E} \to \tilde{M} \times T$	the Γ-equivariant lift of E	
$W^*_{\mathrm{av}}(\mathcal{G})$	average von Neumann algebra of $\mathcal{G}$	1.3
$W^*_{\mathrm{reg}}(\mathcal{G})$	regular von Neumann algebra of $\mathcal{G}$	1.3
π^{reg}	regular representation of $\mathcal{A}_r$ in $W^*_{\mathrm{reg}}(\mathcal{G})$	1.3
π^{av}	average representation of $\mathcal{A}_m$ in $W^*_{\mathrm{av}}(\mathcal{G})$	1.3
τ^ν	trace on $W^*_{\mathrm{reg}}(\mathcal{G})$	1.4
also τ^ν	trace on $W^*_{\mathrm{av}}(\mathcal{G})$	1.4
$W^*_\nu(G; E)$	regular von Neumann algebra of G with coefficients in E	2.3
$W^*_\nu(V, \mathcal{F}; E)$	leafwise von Neumann algebra with coefficients in E	2.3
τ^ν	trace on $W^*_\nu(G; E)$	2.4

$\tau^{\nu}_{\mathcal{F}}$	trace on $W^*_{\nu}(V,\mathcal{F};E)$	2.4
$\mathcal{E}_c$	the prehilbertian $\mathcal{A}_c$-module $C_c^{\infty,0}(\tilde{M}\times T,\hat{E})$	3.1
$\mathcal{E}_r$	the completion of $\mathcal{E}_c$ into a Hilbert $\mathcal{A}_r$-module	3.1
$\mathcal{E}_m$	the completion of $\mathcal{E}_c$ into a Hilbert $\mathcal{A}_m$-module	3.1
$D=(D_L)_{L\in V/\mathcal{F}}$	leafwise geometric operator	3.3
$\tilde{D}=(\tilde{D}_\theta)_{\theta\in T}$	Γ-invariant fiberwise geometric operator	3.3
$\mathcal{D}_m$	the induced regular operator on $\mathcal{E}_m$	3.3
$\mathcal{D}_r$	the induced regular operator on $\mathcal{E}_r$	3.3
$\mathrm{Ind}(\mathcal{D}_m)$	the index class of $\mathcal{D}_m$ in $K_*(\mathcal{A}_m)$	4.2
$\mathrm{IND}(\mathcal{D}_m)$	the index class of $\mathcal{D}_m$ in $K_*(\mathcal{B}_m)$	4.2
$\mathrm{ind}^{\nu}_{\mathrm{up}}(\tilde{D})$	measured index upstairs	4.2
$\mathrm{ind}^{\nu}_{\mathrm{down}}(\tilde{D})$	measured index downstairs	4.2
$\eta^{\nu}_{\mathrm{up}}(\tilde{D})$	eta invariant upstairs	5.1
$\eta^{\nu}_{\mathrm{down}}(D)$	eta invariant downstairs	5.1
$\rho^{\nu}(D;V,\mathcal{F})$	foliated rho invariant associated to D	5.2
$\rho^{\nu}(V,\mathcal{F})$	foliated rho invariant for the signature operator	6.2

1. Group actions

1.1. The discrete groupoid $\mathcal{G}$. — Let Γ be a discrete group. Let T be a compact topological space on which the group Γ acts by homeomorphisms on the left. We shall assume that T is endowed with a Γ-invariant Borel measure ν; this is a non-trivial hypothesis. Thus (T,ν) is a compact Borel measured space on which Γ acts by measure preserving homeomorphisms. We shall assume that ν is a probability measure. We consider the crossed product groupoid $\mathcal{G}:=T\rtimes\Gamma$; thus the set of arrows is $T\times\Gamma$, the set of units is T,

$$s(\theta,\gamma)=\gamma^{-1}\theta \quad \text{and} \quad r(\theta,\gamma)=\theta.$$

The composition law is given by

$$(\gamma'\theta,\gamma')\circ(\theta,\gamma)=(\gamma'\theta,\gamma'\gamma)\,.$$

We denote by $\mathcal{A}_c$ the convolution $\star$-algebra of compactly supported continuous functions on $\mathcal{G}$ and by $L^1(\mathcal{G})$ the Banach $\star$-algebra which is the completion of $\mathcal{A}_c$ with respect to the Banach norm $\|\cdot\|_1$ defined by

$$\|f\|_1:=\max\{\sup_{\theta\in T}\sum_{\gamma\in\Gamma}|f(\theta,\gamma)|;\sup_{\theta\in T}\sum_{\gamma\in\Gamma}|f(\gamma^{-1}\theta,\gamma^{-1})|\}.$$

The convolution operation and the adjunction are fixed by the following formulae

$$(f*g)(\theta,\gamma)=\sum_{\gamma_1\in\Gamma}f(\theta,\gamma_1)g(\gamma_1^{-1}\theta,\gamma_1^{-1}\gamma) \text{ and } f^*(\theta,\gamma)=\overline{f(\gamma^{-1}\theta,\gamma^{-1})}.$$

For $\theta \in T$ we shall denote by $\Gamma(\theta)$ the isotropy group of the point θ: $\Gamma(\theta) := \{\gamma \in \Gamma \mid \gamma\theta = \theta\}$. So, $\Gamma(\theta)$ is a subgroup of Γ and the orbit of θ under the action of Γ, denoted $\Gamma\theta$, can be identified with $\Gamma/\Gamma(\theta)$. Finally, we recall that $\mathcal{G}^\theta := r^{-1}(\theta)$ and that $\mathcal{G}_\theta := s^{-1}(\theta)$.

1.2. C^*-algebras associated to the discrete groupoid $\mathcal{G}$. — For any $\theta \in T$, we define the regular $*$-representation $\pi_\theta^{\mathrm{reg}}$ of $\mathcal{A}_c$ in the Hilbert space $\ell^2(\Gamma)$, viewed as $\ell^2(\mathcal{G}_\theta)$, by the following formula

$$\pi_\theta^{\mathrm{reg}}(f)(\xi)(\gamma) := \sum_{\gamma' \in \Gamma} f(\gamma\theta, \gamma{\gamma'}^{-1})\xi(\gamma').$$

It is easy to check that this formula defines a $*$-representation $\pi_\theta^{\mathrm{reg}}$ which is L^1 continuous. Moreover, we complete $L^1(\mathcal{G})$ with respect to the norm $\sup_{\theta\in T} \|\pi_\theta^{\mathrm{reg}}(\cdot)\|$ and obtain a C^*-algebra $\mathcal{A}_r$. The C^*-algebra $\mathcal{A}_r$ is usually called the regular C^*-algebra of the groupoid $\mathcal{G}$, it will also be denoted with the symbols $C_r^*(\mathcal{G})$ or $C(T) \rtimes_r \Gamma$.

If we complete the Banach $*$-algebra $L^1(\mathcal{G})$ with respect to all continuous $*$-representations, then we get the C^*-algebra $\mathcal{A}_m$, usually called the maximal C^*-algebra of the groupoid $\mathcal{G}$. See [**49**] for more details on these constructions. Other notations for $\mathcal{A}_m$ are $C_m^*(\mathcal{G})$ and $C(T) \rtimes_m \Gamma$.

By construction, any continuous $*$-homomorphism from $L^1(\mathcal{G})$ to a C^*-algebra B yields a C^*-algebra morphism from $\mathcal{A}_m$ to B. In particular, the homomorphism π^{reg} yields a C^*-algebra morphism

$$\pi^{\mathrm{reg}} : \mathcal{A}_m \longrightarrow \mathcal{A}_r.$$

1.3. von Neumann algebras associated to the discrete groupoid $\mathcal{G}$. — At the level of measure theory, recall that we have fixed once for all a Γ-invariant borelian probability measure ν on T. We associate with $\mathcal{G}$ two von Neumann algebras that will be important for our purpose.

The first one is the regular von Neumann algebra $W^*_{\mathrm{reg}}(\mathcal{G})$. It is the algebra $L^\infty(T, B(\ell^2\Gamma); \nu)^\Gamma$ of Γ-equivariant essentially bounded families of bounded operators on $\ell^2\Gamma$, so it acts on the Hilbert space $L^2(T\times\Gamma, \nu)$. An element T of $W^*_{\mathrm{reg}}(\mathcal{G})$ is thus (a class of) a familly $(T_\theta)_{\theta\in T}$ of operators in $\ell^2(\Gamma)$, which satisfies the following properties:

- For any $\xi \in L^2(T\times\Gamma)$ the map $\theta \mapsto <T_\theta\xi_\theta, \xi_\theta>$ is Borel measurable where $\xi_\theta(\gamma) := \xi(\theta, \gamma)$;
- $\theta \mapsto \|T_\theta\|$ is ν-essentially bounded on T;
- For any $\gamma \in \Gamma$, we have $T_{\gamma\theta} = \gamma T_\theta$.

Notice that if we denote by $R_\gamma^* : \ell^2\Gamma \to \ell^2\Gamma$ the operator

$$(R_\gamma^*\xi)(\alpha) := \xi(\alpha\gamma),$$

then $\gamma T := R^*_\gamma \circ T \circ R^*_{\gamma^{-1}}$ for any $T \in B(\ell^2\Gamma)$. That $W^*_{\rm reg}(\mathcal{G})$ is a von Neumann algebra is clear since it is the commutant of a unitary group associated with the action of Γ. The $*$-representation $\pi^{\rm reg}$ is then valued in $W^*_{\rm reg}(\mathcal{G})$ as can be checked easily, and we have the $*$-representation

$$\pi^{\rm reg} : \mathcal{C}_{\rm r} \longrightarrow W^*_{\rm reg}(\mathcal{G}).$$

This $*$-representation then extends to the maximal C^*-algebra $\mathcal{C}_m$.

The second von Neumann algebra that will be important for us will be called the average von Neumann algebra $W^*_{\rm av}(\mathcal{G})$ and we proceed now to define it. We set $\mathcal{G}_0 := (T \times \Gamma)/\sim$ where we identify (θ, γ) with $(\theta, \gamma\alpha)$ whenever $\alpha\theta = \theta$. Then $\mathcal{G}_0$ is Borel and an element T of $W^*_{\rm av}(\mathcal{G})$ is (a class of) a family $(T_\theta)_{\theta \in T}$ of operators in $\ell^2(\Gamma\theta)$, which satisfies the properties:

- For any measurable (as a function on $\mathcal{G}_0$) ν-square integrable section ξ of the Borel field $\ell^2(\Gamma/\Gamma(\theta))$ over T, the map $\theta \mapsto <T_\theta\xi_\theta, \xi_\theta>$ is Borel measurable where $\xi_\theta[\gamma] := \xi[\theta, \gamma]$
- $\theta \mapsto \|T_\theta\|$ is ν-essentially bounded on T;
- For any $\gamma \in \Gamma$, we have $T_{\gamma\theta} = \gamma T_\theta$;

Here we denote by $R^*_\gamma : \ell^2(\Gamma/\Gamma(\theta)) \to \ell^2(\Gamma/\Gamma(\gamma\theta))$ the isomorphism given by $(R^*_\gamma\xi)[\alpha] := \xi[\alpha\gamma]$, and $\gamma T := R^*_\gamma \circ T \circ R^*_{\gamma^{-1}}$. Again $W^*_{\rm av}(\mathcal{G})$ is a von Neumann algebra; for more details on this constructions see for instance **[21]**, **[20]**

There is an interesting representation π^{av} of $L^1(\mathcal{G})$ in $W^*_{\rm av}(\mathcal{G})$ defined as follows. Let $f \in C_c(\mathcal{G})$; for any $\theta \in T$, we set

$$\pi^{av}_\theta(f)(\xi)(x = [\alpha]) := \sum_{y \in \Gamma/\Gamma(\theta)} \sum_{[\beta]=y} f(\alpha\theta, \alpha\beta^{-1})\xi(y), \quad \xi \in \ell^2(\Gamma/\Gamma(\theta)).$$

Remark 1.1. — *If we identify $\Gamma/\Gamma(\theta)$ with the orbit $\Gamma\theta$ then π^{av} becomes*

$$\pi^{av}_\theta(f)(\xi)(\theta') = \sum_{\theta'' \in \Gamma\theta} \sum_{\alpha\theta''=\theta'} f(\theta', \alpha)\xi(\theta'') = \sum_{\alpha \in \Gamma} f(\theta', \alpha)\xi(\alpha^{-1}\theta').$$

Proposition 1.2. — *For any $f \in L^1(\mathcal{G})$ and any $\theta \in T$, the operator $\pi^{av}_\theta(f)$ is bounded and the family $\pi^{av}(f) = (\pi^{av}_\theta(f))_{\theta \in T}$ defines a continuous $*$-representation of $L^1(\mathcal{G})$ with values in $W^*_{\rm av}(\mathcal{G})$. Hence, π^{av} yields a $*$-representations of the maximal C^*-algebra $\mathcal{C}_m$ in $W^*_{\rm av}(\mathcal{G})$.*

Proof. — If we set for any $f \in C_c(\mathscr{G})$, $f_0(\theta,\theta') := \sum_{\gamma\theta=\theta'} f(\theta',\gamma)$, then for $g \in C_c(\mathscr{G})$ we have:

$$\begin{aligned}
(f*g)_0(\theta,\theta') &= \sum_{\gamma.\theta=\theta'} (f*g)(\theta',\gamma)\\
&= \sum_{\gamma\theta=\theta'}\sum_{\gamma_1\in\Gamma} f(\theta',\gamma_1)g(\gamma_1^{-1}\theta',\gamma_1^{-1}\gamma)\\
&= \sum_{\theta''\in\Gamma.\theta}\quad \sum_{\gamma_1^{-1}.\theta'=\theta''\ ,\ \gamma_2^{-1}.\theta''=\theta} f(\theta',\gamma_1)g(\theta'',\gamma_2)\\
&= \sum_{\theta''\in\Gamma.\theta} f_0(\theta'',\theta')g_0(\theta,\theta'')\\
&= (f_0*g_0)(\theta,\theta').
\end{aligned}$$

Since $\pi^{av}(f)$ is simply convolution by the kernel f_0, we deduce that π is a representation of the convolution algebra $\mathscr{A}_c$. Now, the kernel $(f^*)_0$ is given by

$$(f^*)_0(\theta,\theta') = \sum_{\gamma\theta=\theta'} \overline{f}(\gamma^{-1}\theta',\gamma^{-1}) = \sum_{\alpha\theta'=\theta} \overline{f}(\theta,\alpha) = \overline{f_0(\theta',\theta)}.$$

It remains to prove that π^{av} is L^1-continuous. But, we have:

$$\begin{aligned}
\|\pi_\theta^{\mathrm{av}}(f)\xi\|_2^2 &= \sum_{\theta'\in\Gamma\theta} |\sum_{\gamma\in\Gamma} f(\theta',\gamma)\xi(\gamma^{-1}\theta')|^2\\
&\le \sum_{\theta'\in\Gamma\theta} (\sum_{\gamma\in\Gamma}|f(\theta',\gamma)|)\times(\sum_{\gamma\in\Gamma}|f(\theta',\gamma)|.|\xi(\gamma^{-1}\theta')|^2)\\
&\le \|f\|_1\sum_{\theta'\in\Gamma\theta}\sum_{\gamma\in\Gamma}|f(\theta',\gamma)|.|\xi(\gamma^{-1}\theta')|^2\\
&\le \|f\|_1\sum_{\gamma\in\Gamma}\sum_{\theta''\in\Gamma\theta}|\xi(\theta'')|^2|f(\gamma\theta'',\gamma)|\\
&\le \|f\|_1^2\|\xi\|_2^2.
\end{aligned}$$

So, $\|\pi^{\mathrm{av}}(f)\| = \sup_{\theta\in T}\|\pi_\theta^{\mathrm{av}}(f)\| \le \|f\|_1$. □

We therefore deduce the existence of a $*$-homomorphism of C^*-algebras:

$$\pi^{\mathrm{av}} : \mathscr{A}_m \longrightarrow W^*_{\mathrm{av}}(\mathscr{G}).$$

1.4. Traces. — For any non negative element $T = (T_\theta)_{\theta\in T}$ of the von Neumann algebra $W^*_{\mathrm{reg}}(\mathscr{G})$ (resp. $W^*_{\mathrm{av}}(\mathscr{G})$), we set

$$\tau^\nu(T) := \int_T <T_\theta(\delta_e),\delta_e> d\nu(\theta),$$

where in the regular case, δ_e stands for the δ function at the unit e of Γ, while in the second case it is the δ function of the class $[e]$ in $\Gamma/\Gamma(\theta)$.

Proposition 1.3. — *The functional τ^ν induces a faithful normal positive finite trace.*

Proof. — Positivity is clear since T is non negative in the von Neumann algebra if and only if for ν-almost every θ the operator T_θ is non negative. If the non negative element $T=(T_\theta)_{\theta\in T}$ satisfies $\tau^\nu(T)=0$ then $<T_\theta(\delta_e),\delta_e>=0$ for ν-almost every θ. But, the Γ-equivariance of T implies that

$$<T_\theta(\delta_\gamma),\delta_\gamma>=0,\quad \forall\gamma\in\Gamma \text{ and } \nu \;\; a.e.$$

Therefore, $T_\theta=0$ for ν-almost every θ and hence $T=0$ in $W^*_{\text{reg}}(\mathscr{G})$. In the second case, the proof is similar again by Γ-equivariance and by replacing δ_γ by $\delta_{[\gamma]}$.

If $T(n)\uparrow T$ is an increasing sequence of non negative operators which converges in the von Neumann algebra to T, then for ν-almost every θ, the sequence $T(n)_\theta$ increases to T_θ. But then since the state $<\cdot(\delta_e),\delta_e>$ is normal, the conclusion follows by Beppo-Levi's property for ν.

If now T is in the von Neumann algebra $W^*_{\text{reg}}(\mathscr{G})$ then writing T_θ as an infinite matrix in $\ell^2\Gamma$ and using the Γ equivariance we deduce that

$$T_{\gamma\theta}^{\alpha,\beta}=T_\theta^{\alpha\gamma,\beta\gamma}.$$

If we now consider a second operator S in $W^*_{\text{reg}}(\mathscr{G})$, then we have

$$(T_\theta S_\theta)^{e,e}=\sum_{\gamma\in\Gamma}T_\theta^{e,\gamma}\,S_\theta^{\gamma,e}=\sum_{\gamma\in\Gamma}\,S_{\gamma\theta}^{e,\gamma^{-1}}T_{\gamma\theta}^{\gamma^{-1},e},$$

by the Γ-equivariance property. The Γ-invariance of measure ν can now be applied to yield that $\tau^\nu(TS)=\tau^\nu(ST)$. A similar proof works for the von Neumann algebra $W^*_{\text{av}}(\mathscr{G})$. $\square$

We define the functionals τ^ν_{reg} and τ^ν_{av} on $\mathscr{A}_c$ by setting for $f\in\mathscr{A}_c$

(2) $$\tau^\nu_{\text{reg}}(f):=\int_T f(\theta,e)d\nu(\theta),$$

(3) $$\tau^\nu_{\text{av}}(f):=\int_T\left[\sum_{g\in\Gamma(\theta)}f(\theta,g)\right]d\nu(\theta).$$

Lemma 1.4. — *1. We have $\tau^\nu\circ\pi^{\text{reg}}=\tau^\nu_{\text{reg}}$ and $\tau^\nu\circ\pi^{\text{av}}=\tau^\nu_{\text{av}}$.*

2. Hence, τ^ν_{reg} and τ^ν_{av} extend to finite traces on $\mathscr{A}_r$ and $\mathscr{A}_m$.

Proof. — The statement for the regular trace is classical and we thus omit the (easy) proof. We consider for any $f\in L^1(\mathscr{G})$ the Borel family of operators $(\pi^{\text{av}}_\theta(f))_{\theta\in T}$ defined in the previous paragraph. For any $f\in\mathscr{A}_c$, denote as before by f_0 the function

$$f_0(\theta,\theta'):=\sum_{\gamma\theta=\theta'}f(\theta',\gamma).$$

Then we know that $\pi^{\mathrm{av}}(f)$ is given as convolution with f_0. If $f \in \mathscr{A}_c$, then we have, using the identification $\Gamma/\Gamma(\theta) \equiv \Gamma\theta$:

$$\begin{aligned}\int_T <\pi_\theta^{\mathrm{av}}(f)\delta_\theta, \delta_\theta> d\nu(\theta) &= \int_T f_0(\theta,\theta) d\nu(\theta)\\ &= \int_T \sum_{\gamma\in\Gamma(\theta)} f(\theta,\gamma) d\nu(\theta)\\ &= \tau_{\mathrm{av}}^{\nu}(f).\end{aligned}$$

□

As a Corollary of the above Lemma notice that the traces $\tau^\nu_{\mathrm{reg}} : \mathscr{A}_r \to \mathbb{C}$ and $\tau^\nu_{\mathrm{av}} : \mathscr{A}_m \to \mathbb{C}$ induce group homomorphisms

$$\tau^\nu_{\mathrm{reg},*} : K_0(\mathscr{A}_r) \to \mathbb{R}\,, \quad \tau^\nu_{\mathrm{av},*} : K_0(\mathscr{A}_m) \to \mathbb{R}. \tag{4}$$

2. Foliated spaces

2.1. Foliated spaces. — Let M be a compact manifold without boundary and let Γ denote its fundamental group and $\tilde{M}$ its universal cover. The group Γ acts by homemorphisms on the compact topological space T and hence acts on the right, freely and properly, on the space $\tilde{M} \times T$ by the formula

$$(\tilde{m},\theta)\gamma := (\tilde{m}\gamma, \gamma^{-1}\theta), \quad (\tilde{m},\theta) \in \tilde{M}\times T \text{ and } \gamma \in \Gamma.$$

The quotient space of $\tilde{M}\times T$ under this action is denoted by V. We assume as before the existence of a Γ-invariant probability measure ν. If we want to be specific about the action of Γ on T we shall consider it as a homomorphism $\Psi : \Gamma \to \mathrm{Homeo}(T)$. We do not assume the action to be locally free [(1)] .

If $p : \tilde{M}\times T \to V$ is the natural projection then the leaves of a lamination on V are given by the projections $L_\theta = p(\tilde{M}_\theta)$, where θ runs through the compact space T, and

$$\tilde{M}_\theta := \tilde{M} \times \{\theta\}\,. \tag{5}$$

It is easy to check that this is a lamination of V with smooth leaves and possibly complicated transverse structure according to the topology of T, see for instance [**11**]. By definition, it is easy to check that the leaf L_θ coincides with the leaf $L_{\theta'}$ if and only if θ' belongs to the orbit $\Gamma\theta$ of θ under the action of Γ in T. We shall refer to this lamination by $(V,\mathscr{F})$ and sometimes shall call it a foliated space or, more briefly, a foliation. If $\Gamma(\theta)$ is the isotropy group of $\theta \in T$ then we see from the definition of L_θ that L_θ is diffeomorphic to the quotient manifold $\tilde{M}/\Gamma(\theta)$ through the map $L_\theta \to \tilde{M}/\Gamma(\theta)$ given by $[\tilde{m}',\theta'] \to [\tilde{m}'\gamma]$, if $\theta' = \gamma\theta$. Note however that L_θ is also

(1) Recall that an action is locally free if given $\gamma \in \Gamma$ and open set U in T such that $\gamma(\theta) = \theta$ for any $\theta \in U$ then $\gamma = 1$.

diffeomorphic to $\tilde{M}/\Gamma(\theta')$ for any $\theta' \in \Gamma\theta$. Moreover the monodromy cover of a leaf L is obtained by choosing $\theta \in T$ such that $L = L_\theta$ and by using the composite map

$$\tilde{M} \to \tilde{M}_\theta \to (\tilde{M}_\theta)/\Gamma(\theta) \simeq L_\theta = L.$$

which is a monodromy cover of L corresponding to θ.

Notice that the set of $\theta \in T$ for which $\Gamma(\theta)$ is non-trivial has in general positive measure. This is the case, for instance, when there exists a subgroup Γ_1 of Γ whose action on T has the property that $\nu(T^{\Gamma_1}) > 0$, where T^{Γ_1} is the fixed-point subspace defined by Γ_1. In fact, one can construct simple examples where the measure of the set of $\theta \in T$ for which $\Gamma(\theta)$ is non-trivial is any value in $(0,1)$. See Example 2.2 for a specific situation.

Example 2.1. — *As an easy example where this situation occurs naturally, consider any Galois covering $\tilde{M}'$ of M with structure group Γ' such that $\pi_1(\tilde{M}') \neq 1$. Assume the existence of a locally free Γ'-action $\Psi' : \Gamma' \to \mathrm{Homeo}(T)$ on T and let V be the resulting foliated space. Assume the existence of an invariant measure ν on T. Since Γ' is a quotient of $\Gamma := \pi_1(M)$ we have a natural group homomorphism $\pi : \Gamma \to \Gamma'$ and thus an action $\Psi := \Psi' \circ \pi$ of Γ on T. By definition ν is also Γ-invariant. The isotropy group of this action at $\theta \in T$ is at least as big as the fundamental group of $\tilde{M}'$. Notice that one can show that*

$$(\tilde{M} \times T)/\Gamma = (\tilde{M}' \times T)/\Gamma' \equiv V.$$

Summarizing: V is a lamination where the set of leaves with non-trivial monodromy has measure equal to $\nu(T) = 1$.

Example 2.2. — *Take M to be any manifold whose fundamental group is a free product of copies of $\mathbb{Z}$, for example a connected sum of $\mathbb{S}^1 \times \mathbb{S}^2$'s, so that now Γ is the free group of rank k. Let $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$ be the generators. Let T be $\mathbb{S}^2$. Let $C \subset \mathbb{S}^2$ be a parallel and let $U \subset \mathbb{S}^2$ one of the two hemispheres bounded by C. Let $\Psi(\gamma_1)$ be any measure-preserving diffeomorphism of $\mathbb{S}^2$ that fixes U. We then define Ψ on the other generators in an arbitrary measure-preserving way. Then any point θ in U would have nontrivial isotropy group $\Gamma(\theta)$. Clearly, one can jazz up this example by selecting any T and finding a single homeomorphism whose fixed point set is a set of nonzero measure.*

Example 2.3. — *Following* **[38]** *we now give an example of a lamination with the set of leaves with non-trivial monodromy of positive measure and, in addition, of a rather complicated sort. Take a (generalized) Cantor set K of positive Lebesgue measure in the unit circle. Choose now a homeomorphism ϕ of the circle admitting K as the fixed point set. Let M be any closed odd dimensional manifold with $\pi_1(M) = \mathbb{Z}$. Consider the foliated space V obtained by suspension of ϕ: thus $V = \tilde{M} \times_{\mathbb{Z}} \mathbb{S}^1$ with $\mathbb{Z} = \pi_1(\mathbb{S}^1)$*

acting on $\mathbb{S}^1$ via ϕ and acting by deck transformations on $\tilde{M}$. The set of $\theta \in S^1$ such that $\{\gamma \in \mathbb{Z} | \gamma\theta = \theta\}$ is non-trivial is equal to K, hence it has positive measure. Using **[38]** *page 105/106, we can find a Radon ϕ-invariant measure ν on $\mathbb{S}^1$ and $\nu(K) > 0$. Notice that in this class of examples, although the measure is diffuse, one can even ensure that the set of leaves with non-trivial holonomy has positive transverse measure. These laminations show up in the study of aperiodic tillings and especially of quasi-crystals. In* **[11]** *for instance, the measured foliated index for such laminations, a primary invariant, is used to solve the gap-labelling conjecture. The authors expect potential applications of the foliated rho invariant to aperiodic solid physics.*

2.2. The monodromy groupoid and the C^*-algebra of the foliation. — Let $\tilde{M}$, Γ and T be as before. We define the monodromy groupoid G as the quotient space $(\tilde{M} \times \tilde{M} \times T)/\Gamma$ of $\tilde{M} \times \tilde{M} \times T$ by the right diagonal action

$$(\tilde{m}, \tilde{m}', \theta)\gamma := (\tilde{m}\gamma, \tilde{m}'\gamma, \gamma^{-1}\theta).$$

The groupoid structure is clear: the space of units $G^{(0)}$ is the space $V = \tilde{M} \times_\Gamma T$, the source and range maps are given by

$$s[\tilde{m}, \tilde{m}', \theta] = [\tilde{m}', \theta] \text{ and } r[\tilde{m}, \tilde{m}', \theta] = [\tilde{m}, \theta],$$

where the brackets denote equivalence classes modulo the action of the group Γ

It is not difficult to show that G can be identified in a natural way with the usual monodromy groupoid associated to the foliated space $(V, \mathscr{F})$, as defined, for example, in **[44]**. More precisely given a smooth path $\alpha : [0,1] \to L$, with L a leaf, choose any lift $\tilde{\beta} : [0,1] \to \tilde{M}$ of the projection of the path α in M through the natural projection $V \to M$. Then there exists a unique $\theta \in T$ with $\alpha(0) = [\tilde{\beta}(0), \theta]$ and we obtain in this way a well defined element $[\tilde{\beta}(0), \tilde{\beta}(1), \theta]$ of G which only depends on the leafwise homotopy class of α with fixed end-points. This furnishes the desired isomorphism.

We fix now a Lebesgue class measure dm on M and the corresponding Γ-invariant measure $d\tilde{m}$ on $\tilde{M}$. We denote by $\mathscr{B}_c$ the convolution $*$-algebra of continuous compactly supported functions on G. For $f, g \in \mathscr{B}_c$ we have:

$$(f * g)[\tilde{m}, \tilde{m}', \theta] = \int_{\tilde{M}} f[\tilde{m}, \tilde{m}'', \theta] g[\tilde{m}'', \tilde{m}', \theta] d\tilde{m}'' \quad \text{and} \quad f^*[\tilde{m}, \tilde{m}', \theta] = \overline{f[\tilde{m}', \tilde{m}, \theta]}.$$

More generally, let E be a hermitian continuous longitudinally smooth vector bundle over V; thus E is a continuous bundle over V such that its restriction to each leaf is smooth **[38]**. Consider $\mathrm{END}(E) := (s^*E)^* \otimes (r^*E) = \mathrm{Hom}(s^*E, r^*E)$, a bundle of endomorphisms over G. We consider $\mathscr{B}_c^E := C_c^{\infty,0}(G, \mathrm{END}(E))$ the space of continuous longitudinally smooth sections of END(E); this is also a $*$-algebra with product and adjoint given by

$$(f_1 * f_2)[\tilde{m}, \tilde{m}', \theta] = \int_{\tilde{M}} f_1[\tilde{m}, \tilde{m}'', \theta] \circ f_2[\tilde{m}'', \tilde{m}', \theta]\, d\tilde{m}'',$$

$$f^*[\tilde{m}, \tilde{m}', \theta] = (f[\tilde{m}', \tilde{m}, \theta])^*.$$

Let $\widehat{E}$ be its lift to $\widetilde{M} \times T$; denote by H_θ the Hilbert space $H_\theta = L^2(\widetilde{M} \times \{\theta\}; \widehat{E}_{|\widetilde{M} \times \{\theta\}})$. Any $f \in \mathscr{B}_c^E$ can be viewed as a smooth kernel acting on H_θ by the formula

$$\pi_\theta^{\mathrm{reg}}(f)(\xi)(\tilde{m}) := \int_{\tilde{M}} f[\tilde{m}, \tilde{m}', \theta](\xi(\tilde{m}'))d\tilde{m}', \quad \text{for any } \xi \in H_\theta\,,$$

and this defines a $*$-representation π_θ^{reg} in H_θ . We point out that the representation π_θ^{reg} is continuous for the L^1 norm defined by:

$$\|f\|_1 := \max\{ \sup_{(\tilde{m},\theta)\in \tilde{M}\times T} \int_{\tilde{M}} \|f[\tilde{m}, \tilde{m}', \theta]\|_E d\tilde{m}'\,; \sup_{(\tilde{m},\theta)\in \tilde{M}\times T} \int_{\tilde{M}} \|f[\tilde{m}', \tilde{m}, \theta]\|_E d\tilde{m}'\}.$$

If we complete $\mathscr{B}_c$ with respect to the C^* norm

$$\|f\|_{\mathrm{reg}} := \sup_{\theta\in T} \|\pi_\theta^{reg}(f)\|,$$

then we get $\mathscr{B}_r^E$, the regular C^*-algebra of the groupoid G with coefficients in E. When $E = V \times \mathbb{C}$ then we denote this C^*-algebra simply by $\mathscr{B}_r$ In the same way, if we complete $\mathscr{B}_c$ with respect to all L^1 continuous $*$-representations, then we get the maximal C^*-algebra of the groupoid G, that will be denoted by $\mathscr{B}_m^E$ and simply by $\mathscr{B}_m$ when $E = V \times \mathbb{C}$.

2.3. von Neumann Algebras of foliations. — The material in this paragraph is classical; for more details see for instance **[18]**, **[25]** **[8]**, **[17]**, **[35]**.

The representation π^{reg} defined above takes value in the regular von Neumann algebra of the groupoid G. More precisely, the regular von Neumann algebra $W_\nu^*(G; E)$ of G with coefficients in E, acts on the Hilbert space $H = L^2(T \times \tilde{M}, \widehat{E}; \nu \otimes d\tilde{m})$, and is by definition the space of families $(S_\theta)_{\theta\in T}$ of bounded operators on $L^2(\tilde{M}, \widehat{E})$ such that

- For any $\gamma \in \Gamma$, $S_{\gamma\theta} = \gamma S_\theta$ where γS_θ is defined using the action of Γ on the equivariant vector bundle $\widehat{E}$;
- The map $\theta \mapsto \|S_\theta\|$ is ν-essentially bounded on T;
- For any $(\xi, \eta) \in H^2$, the map $\theta \mapsto {<}S_\theta(\xi_\theta), \eta_\theta{>}$ is Borel measurable.

The von Neumann algebra $W_\nu^*(G; E)$ is a type II_∞ von Neumann algebra as we shall see later. It is easy to see that for any $S \in \mathscr{B}_r^E$, the operator $\pi^{\mathrm{reg}}(S)$ belongs to $W_\nu^*(G; E)$.

In the same way we define a leafwise von Neumann algebra that we shall denote by $W^*_\nu(V,\mathcal{F};E)$; this algebra acts on the Hilbert space [21] $H=\int^\oplus L^2(L_\theta,E|_{L_\theta})d\nu(\theta)$ where L_θ is, as before, the leaf in V corresponding to θ. Equivalently, and using the identification of the leaves with quotient of $\tilde M$ under isotropy, $W^*_\nu(V,\mathcal{F};E)$ can be described as the set of families $(S_\theta)_{\theta\in T}$ of bounded operators on $(L^2(\tilde M_\theta/\Gamma(\theta),E|_\theta)_{\theta\in T}$ such that

- The map $\theta\mapsto\|S_\theta\|$ is ν-essentially bounded on T.
- For any square integrable sections ξ,η of the Borel field $(L^2(\tilde M_\theta/\Gamma(\theta),E_\theta))_{\theta\in T}$, the map $\theta\mapsto <S_\theta(\xi_\theta),\eta_\theta>$ is Borel measurable.
- $S_{\gamma\theta}=\gamma S_\theta$, for any $(\theta,\gamma)\in T\times\Gamma$.

Notice that $\Gamma(\gamma\theta)=\gamma\Gamma(\theta)\gamma^{-1}$ and hence the definition of γS_θ is clear.

Proposition 2.4. — *There is a well defined representation $\pi_{\rm av}$ from the maximal C^*algebra $\mathcal{B}^E_m$ to the leafwise von Neumann algebra $W^*_\nu(V,\mathcal{F};E)$ such that for $f\in C_c(G,\mathrm{END}(E))$ the operator $(\pi_{\rm av}(f))_\theta$ is given by the kernel*

$$f_0(x,y)=0 \textit{ if } L_x\neq L_y \textit{ and } f_0([\tilde m,\theta],[\tilde m',\theta]):=\sum_{\gamma\in\Gamma(\theta)} f[\tilde m,\tilde m'\gamma,\theta].$$

Proof. — For simplicity we take E the product line bundle. For $f\in C_c(G)$ the formula

$$((\pi_{\rm av}(f))_\theta\,\xi)\,(x):=\int_{L_\theta} f_0(x,y)\xi(y)dy,\quad \xi\in L^2(L_\theta),x\in L_\theta\subset V.$$

defines a bounded operator on $L^2(L_\theta)$. Indeed the sum on the RHS in the definition of f_0 is finite since f is compactly supported. Moreover, when restricted to the leaf L_x the kernel f_0 is supported within a uniform neighborhood of the diagonal of L_x. We have:

$$\begin{aligned}
\|\pi_{\rm av}(f))_\theta\,(\xi)\|_2^2 &= \int_{L_\theta}|\int_{L_\theta} f_0(x,x')\xi(x')dx'|^2dx\\
&\leq \int_{L_\theta}\left(\int_{L_\theta}|f_0(x,x')|dx'\right)\left(\int_{L_\theta}|f_0(x,x')||\xi(x')|^2dx'\right)dx\\
&\leq \|f_0\|_1\int_{L_\theta}|\xi(x')|^2\int_{L_\theta}|f_0(x,x')|dxdx'\\
&\leq \|f_0\|_1^2\|\xi\|_2^2.
\end{aligned}$$

Here $\|f_0\|_1$ stands for the L^1-norm

$$\mathrm{Max}(\sup_{x'\in V}\int_{L_x}|f_0(x,x')|dx,\sup_{x\in V}\int_{L_x}|f_0(x,x')|dx').$$

Therefore, we have

$$\sup_{\theta\in T}\|\pi_{\rm av}(f)\|\leq\|f_0\|_1.$$

But now it is easy to check that $\|f_0\|_1 \leq \|f\|_1$. On the other hand $\pi_{\rm av}$ is a $*$-representation; since for $f, g \in C_c(G)$ one has, with proof similar to the one given for $\mathcal{G}$,

$$(f * g)_0 = f_0 * g_0 \text{ and } (f^*)_0 = (f_0)^*.$$

To sum up, these arguments prove that π_{av} on $\mathcal{B}_c$ extends to a continuous $*$-representation of the $\mathcal{B}_m$ in the von Neumann algebra $W^*_\nu(V, \mathcal{F})$. This completes the proof. □

2.4. Traces. — We fix once and for all a fundamental domain F for the free and proper action of Γ on $\tilde{M}$. Let χ be the characteristic function of F. Then we set for any non-negative element $S \in W^*_\nu(G; E)$,

$$\tau^\nu(S) := \int_T \operatorname{tr}(M_\chi \circ S_\theta \circ M_\chi) d\nu(\theta),$$

where tr is the usual trace of a non-negative operator on a Hilbert space.

We shall also denote by χ the induced function $\chi \otimes 1_T$, i.e. the characteristic function of $F \times T$ in $\tilde{M} \times T$. Since $F \times T$ is a fundamental domain for the free and proper action of Γ on $\tilde{M} \times T$, we shall also denote by χ_θ the same function χ but viewed as the characteristic function of F inside a given leaf L_θ, which is the image under the projection $\tilde{M} \times T \to V$ of $\tilde{M} \times \{\theta\}$. We define a functional τ^ν on the leafwise von Neumann algebra $W^*_\nu(V, \mathcal{F}; E)$, by setting for any non-negative element $S \in W^*_\nu(V, \mathcal{F}; E)$

$$\tau^\nu_{\mathcal{F}}(S) := \int_T \operatorname{tr}(M_{\chi_\theta} \circ S_\theta \circ M_{\chi_\theta}) d\nu(\theta),$$

where the M_{χ_θ} appearing in the integrand is the multiplication operator in the L^2 space of sections over $\tilde{M}_\theta/\Gamma(\theta)$, by the characteristic function χ_θ of F viewed in $\tilde{M}_\theta/\Gamma(\theta)$.

Proposition 2.5. — *With the above notations we have:*

- *the functional τ^ν yields a positive semifinite normal faithful trace on $W^*_\nu(G, E)$;*
- *the functional $\tau^\nu_{\mathcal{F}}$ yields a positive semifinite normal faithful trace on $W^*_\nu(V, \mathcal{F}; E)$.*

Proof. — If $R = S^*S \in W^*_\nu(G; E)$, then for any $\theta \in T$,

$$M_\chi \circ R_\theta \circ M_\chi = (S_\theta M_\chi)^*(S_\theta M_\chi) \geq 0.$$

Therefore, $\operatorname{tr}(M_\chi \circ R_\theta \circ M_\chi) \geq 0$ and hence $\tau^\nu(R) \geq 0$. Moreover, $\tau^\nu(R) = 0$ if and only if $M_\chi R_\theta M_\chi = 0$ for ν-almost every θ. The Γ equivariance of R implies the relations

$$M_{\gamma_1\chi} R_{\gamma\theta} M_{\gamma_2\chi} = U_\gamma \left[M_{\gamma^{-1}\gamma_1\chi} R_\theta M_{\gamma^{-1}\gamma_2\chi}\right] U_{\gamma^{-1}}, \quad \gamma, \gamma_1, \gamma_2 \in \Gamma.$$

The same relations hold for S. In particular,

$$M_{\gamma\chi} R_\theta M_{\gamma\chi} = U_\gamma \left[M_\chi R_{\gamma^{-1}\theta} M_\chi \right] U_{\gamma^{-1}} = 0.$$

Since ν is Γ-invariant, we deduce that $M_{\gamma\chi} R_\theta M_{\gamma\chi} = 0$ ν almost everywhere. Thus

$$\sum_{\gamma' \in \Gamma} (M_{\gamma'\chi} S_\theta M_{\gamma\chi})^* (M_{\gamma'\chi} S_\theta M_{\gamma\chi}) = M_{\gamma\chi} R_\theta M_{\gamma\chi} = 0, \quad \nu - \text{a.e. } \theta \in T.$$

As a consequence, we get that for ν almost every $\theta \in T$ and for any $\gamma, \gamma' \in \Gamma$,

$$M_{\gamma'\chi} S_\theta M_{\gamma\chi} = 0,$$

which proves that $S = 0$ in $W^*_\nu(G, E)$ and whence $R = 0$ in $W^*_\nu(G, E)$. On the other hand for any non negative $A, B \in W^*_\nu(G; E)$, we have

$$\begin{aligned} M_\chi A_\theta B_\theta M_\chi &= \sum_{\gamma \in \Gamma} M_\chi A_\theta M_{\gamma\chi} B_\theta M_\chi \\ &= \sum_{\gamma \in \Gamma} M_\chi A_\theta (U_\gamma M_\chi U_{\gamma^{-1}}) B_\theta M_\chi \\ &= \sum_{\gamma \in \Gamma} M_\chi U_\gamma A_{\gamma^{-1}\theta} M_\chi B_{\gamma^{-1}\theta} U_{\gamma^{-1}} M_\chi \\ &= \sum_{\gamma \in \Gamma} U_\gamma \left[M_{\gamma^{-1}\chi} A_{\gamma^{-1}\theta} M_\chi B_{\gamma^{-1}\theta} M_{\gamma^{-1}\chi} \right] U_{\gamma^{-1}} \end{aligned}$$

and so,

$$\begin{aligned} \operatorname{tr}(M_\chi A_\theta B_\theta M_\chi) &= \sum_{\gamma \in \Gamma} \operatorname{tr} \left[M_{\gamma^{-1}\chi} A_{\gamma^{-1}\theta} M_\chi B_{\gamma^{-1}\theta} M_{\gamma^{-1}\chi} \right] \\ &= \sum_{\gamma \in \Gamma} \operatorname{tr} \left[M_\chi B_{\gamma^{-1}\theta} M_{\gamma^{-1}\chi} A_{\gamma^{-1}\theta} M_\chi \right]. \end{aligned}$$

Now the Γ-invariance of ν yields again

$$\begin{aligned} \tau^\nu(AB) &= \int_T \operatorname{tr}(M_\chi A_\theta B_\theta M_\chi) d\nu(\theta) = \int_T \sum_{\gamma \in \Gamma} \operatorname{tr} \left[M_\chi B_\theta M_{\gamma^{-1}\chi} A_\theta M_\chi \right] d\nu(\theta) \\ &= \int_T \operatorname{tr}(M_\chi B_\theta A_\theta M_\chi) d\nu(\theta) = \tau^\nu(BA). \end{aligned}$$

The normality is a consequence of normality of tr and of the Beppo-Levi property. That τ^ν is semi-finite is straightforward.

Finally, according to our description of the leafwise von Neumann algebra $W^*_\nu(V, \mathcal{F}; E)$, its elements are also equivariant Borel families. So, the proof of the first item is readily adapted to take care of the quotients by the isotropy groups. □

Recall the two $*$-representations

$$\pi_{\text{reg}} : \mathcal{B}^E_r \to W^*_\nu(G, E), \quad \pi_{\text{av}} : \mathcal{B}^E_m \to W^*_\nu(V, \mathcal{F}; E).$$

Corollary 2.6. — *The two functionals $\tau^\nu_{\mathrm{reg}} := \tau^\nu \circ \pi_{\mathrm{reg}}$ and $\tau^\nu_{\mathrm{av}} := \tau^\nu_{\mathcal{G}} \circ \pi_{\mathrm{av}}$ are traces on the C^*-algebras $\mathcal{B}^E_r$ and $\mathcal{B}^E_m$ respectively*[2]*. Moreover they are explicitly given, for $f \in \mathcal{B}^E_c$ longitudinally smooth by the formulas*

$$\tau^\nu_{\mathrm{reg}}(f) := \int_{F\times T} \mathrm{tr}_{E_{[\tilde m,\theta]}}(f[\tilde m, \tilde m, \theta])d\tilde m d\nu(\theta) \tag{6}$$

$$\tau^\nu_{\mathrm{av}}(f) := \int_{F\times T} \sum_{\gamma\in\Gamma(\theta)} \mathrm{tr}_{E_{[\tilde m,\theta]}}(f[\tilde m, \tilde m\gamma, \theta])d\tilde m d\nu(\theta). \tag{7}$$

Proof. — We only need to show the two formulas (6) and (7). The first one is tautological, so we only sketch the proof of the second one. Let then $f \in \mathcal{B}_c$ longitudinally smooth be fixed. The operator $[\pi_{\mathrm{av}}(f)]_\theta$ acts on $L^2(L_\theta, E)$ with Schwartz kernel f_0 given by

$$f_0([\tilde m, \theta], [\tilde m', \theta]) = \sum_{\gamma\in\Gamma(\theta)} f[\tilde m, \tilde m'\gamma, \theta].$$

Therefore, the operator $M_\chi[\pi_{\mathrm{av}}(f)]_\theta M_\chi$ has Schwartz kernel supported in $F \times F$ viewed in $L_\theta \times L_\theta$. Recall that L_θ is identified with $\tilde M/\Gamma(\theta)$. We deduce

$$\tau^\nu[\pi_{\mathrm{av}}(f)] = \int_{F\times T} f_0([\tilde m, \theta], [\tilde m, \theta])d\mu_\theta(\tilde m)d\nu(\theta),$$

with $d\mu_\theta(\tilde m)$ being the measure induced by $d\tilde m$ on the leaf through θ. Whence, the formula is readily deduced. □

In the sequel we shall also denote by τ^ν_{reg} the resulting trace on the maximal C^*-algebra $\mathcal{B}^E_m$, obtained via the natural epimorphism $\mathcal{B}^E_m \to \mathcal{B}^E_r$.

Remark 2.7. — *The proof of the tracial property of τ^ν_{reg} and τ^ν_{av} can also be carried out directly. Here are the details (we only treat the averaged trace τ^ν_{av} and for simplicity we take E equal to the product line bundle). Let f, f' be two elements of $C_c(G)$. We have:*

$$(f * f')[\tilde m, \tilde m', \theta] = \int_F \sum_{\alpha\in\Gamma} f[\tilde m, \tilde m''\alpha, \theta]f'[\tilde m''\alpha, \tilde m', \theta]d\tilde m''.$$

[2] These traces will not be finite in general.

Hence we deduce

$$\begin{aligned}
\tau^\nu_{\rm av}(f*f') &= \int_{F\times F\times T}\sum_{\gamma\in\Gamma(\theta)}\sum_{\alpha\in\Gamma} f[\tilde m,\tilde m'\alpha,\theta]f'[\tilde m'\alpha,\tilde m\gamma,\theta]d\tilde m' d\tilde m d\nu(\theta)\\
&= \int_{F\times F\times T}\sum_{\gamma\in\Gamma(\theta)}\sum_{\alpha\in\Gamma} f'[\tilde m',\tilde m\gamma\alpha^{-1},\alpha\theta]f[\tilde m\alpha^{-1},\tilde m',\alpha\theta]d\tilde m' d\tilde m d\nu(\theta)\\
&= \int_{F\times F}\sum_{\alpha\in\Gamma}\int_T\sum_{\gamma'\in\Gamma(\theta')} f'[\tilde m'\gamma'^{-1},\tilde m\alpha^{-1},\theta']f[\tilde m\alpha^{-1},\tilde m',\theta']d\tilde m' d\tilde m d\nu(\theta')\\
&= \int_{F\times T}\sum_{\gamma'\in\Gamma(\theta')}(f'*f)[\tilde m\gamma'^{-1},\tilde m',\theta']d\tilde m' d\tilde m d\nu(\theta').
\end{aligned}$$

Now note that since $\gamma'\in\Gamma(\theta')$, we have

$$(f'*f)[\tilde m\gamma'^{-1},\tilde m',\theta'] = (f'*f)[\tilde m,\tilde m'\gamma',\theta'].$$

Therefore, we get

$$\tau^\nu_{\rm av}(f*f') = \tau^\nu_{\rm av}(f'*f).$$

Proposition 2.8. — *1. The trace $\tau^\nu_{\rm reg}$ induces a group homomorphism $\tau^\nu_{{\rm reg},*}$: $K_0(\mathcal{B}^E_r)\longrightarrow\mathbb{R}$.*

2. The trace $\tau^\nu_{\rm av}$ induces a group homomorphism $\tau^\nu_{{\rm av},*}: K_0(\mathcal{B}^E_m)\longrightarrow\mathbb{R}$.

Proof. — We only sketch the proof of this classical result: one shows, for instance, that $L^1(W^*_\nu(G;E))\cap\mathcal{B}^E_r$, with $L^1(W^*_\nu(G;E))$ the Schatten-ideal of τ^ν-trace class operators, is dense holomorphically closed in $\mathcal{B}^E_r$. Similarly $L^1(W^*_\nu(V,\mathcal{F};E))\cap\pi_{\rm av}(\mathcal{B}^E_m)$ is dense and holomorphically closed in $\pi_{\rm av}(\mathcal{B}^E_m)$; this finishes the proof by using the definition of τ^ν_{av}. □

2.5. Compatibility with Morita isomorphisms. — The goal of this subsection is to prove the compatibility between the different traces defined so far and the isomorphisms induced in K-theory by Morita equivalence.

Recall the C^*-algebras $\mathcal{A}_r$ and $\mathcal{A}_m$ associated to the groupoid $\mathcal{G}:=T\rtimes\Gamma$. Let $\mathcal{K}$ denote as usual the C^*-algebra of compact operators on a Hilbert space.

Proposition 2.9. — *There are isomorphisms of C^*-algebras:*

$$\mathcal{B}_r\simeq\mathcal{A}_r\otimes\mathcal{K}\ ,\qquad \mathcal{B}_m\simeq\mathcal{A}_m\otimes\mathcal{K}. \tag{8}$$

Proof. — We fix $\tilde m_0\in\tilde M$ and consider the subgroupoid $G(\tilde m_0)$ consisting of the elements which start and end in the image of $\{\tilde m_0\}\times T$ in V:

$$G(\tilde m_0)=\{[\tilde m_0,\tilde m_0\alpha,\theta],\theta\in T \text{ and } \alpha\in\Gamma\}.$$

Notice that the composition in $G(\tilde m_0)$ can be expressed in the following way:

$$[\tilde m_0,\tilde m_o\alpha',\theta']\circ[\tilde m_0,\tilde m_o\alpha,\alpha'\theta'] = [\tilde m_0,\tilde m_0\alpha\alpha',\theta']\,.$$

Then there is a *groupoid isomorphism* between $G(\tilde{m}_0)$ and the groupoid $\mathcal{G}$ given by

$$[\tilde{m}_0, \tilde{m}_0\alpha, \theta] \longmapsto (\theta, \alpha^{-1}).$$

In particular the reduced (respectively maximal) C^*-algebras associated to $G(\tilde{m}_0)$ and $\mathcal{G}$ are isomorphic: $C^*_r(G(\tilde{m}_0)) \simeq \mathcal{A}_r$ (respectively $C^*_m(G(\tilde{m}_0)) \simeq \mathcal{A}_m$). Now the main result in **[27]**, see also **[7]**, together with the fact that the image of $\{\tilde{m}_0\} \times T$ in V intersects every leaf of the foliation, we deduce that the stable C^*-algebra $\mathcal{B}_r$ is isomorphic to the tensor product C^*-algebra $\mathcal{A}_r \otimes \mathcal{K}$. In the same way, the C^*-algebra $\mathcal{B}_m$ is isomorphic to the tensor product C^*-algebra $\mathcal{A}_m \otimes \mathcal{K}$, using the maximal version of the stability theorem which is valid as pointed out in **[27]**. □

Denote by $\mathcal{M}_{\mathrm{r}} : K_0(\mathcal{A}_r) \to K_0(\mathcal{B}_r)$ and $\mathcal{M}_{\mathrm{m}} : K_0(\mathcal{A}_m) \to K_0(\mathcal{B}_m)$ the isomorphisms induced in K-theory by the isomorphisms (8).

Proposition 2.10. — *The following diagrams are commutative*

$$\begin{tikzcd} & K_0(\mathcal{B}_r) \arrow[d, "\tau^{\nu}_{\mathrm{reg},*}"] \\ K_0(\mathcal{A}_r) \arrow[ur, "\mathcal{M}_{\mathrm{r}}"] \arrow[r, "\tau^{\nu}_{\mathrm{reg},*}"'] & \mathbb{R} \end{tikzcd} \qquad \begin{tikzcd} & K_0(\mathcal{B}_m) \arrow[d, "\tau^{\nu}_{\mathrm{av},*}"] \\ K_0(\mathcal{A}_m) \arrow[ur, "\mathcal{M}_{\mathrm{m}}"] \arrow[r, "\tau^{\nu}_{\mathrm{av},*}"'] & \mathbb{R}. \end{tikzcd}$$

Proof. — Let us identify T with a fiber of the flat bundle $V = \tilde{M} \times_\Gamma T \to M$. Let Ω be an open connected submanifold of $\tilde{M}$ contained in a fundamental domain F of the action of Γ. Let U be the projection in V of $\Omega \times T$. Then $U \to T$ is an open neighborhood of T in V such that the induced foliation on U is given by the fibres of $U \to T$. The subgroupoid G_U^U of G consisting of homotopy classes of paths drawn in leaves, starting and ending in U, can be describe as

$$G_U^U = \{[\tilde{m}, \tilde{m}'\gamma, \theta] \in \frac{\Omega \times \tilde{M} \times T}{\Gamma}, [\tilde{m}, \theta] \in U \text{ and } [\tilde{m}'\gamma, \theta] \in U\}.$$

An easy inspection of the groupoid laws in G_U^U shows that the bijection

$$[\tilde{m}, \tilde{m}'\gamma, \theta] \longmapsto (\tilde{m}, \tilde{m}', \theta, \gamma^{-1}) \in \Omega \times \Omega \times (T \rtimes \Gamma),$$

is an isomorphism of groupoids, so that the reduced (resp. maximal) C^*-algebra of G_U^U is isomorphic to $\mathcal{K}(L^2\Omega) \otimes [C(T) \rtimes_r \Gamma]$ (resp. $\mathcal{K}(L^2\Omega) \otimes [C(T) \rtimes_m \Gamma]$). Recall that $\mathcal{K}(L^2\Omega)$ denotes the nuclear C^*-algebra of compact operators in the Hilbert space $L^2\Omega$.

If we now fix a continuous compactly supported function φ on Ω with L^2 norm equal to 1 then for any continuous compactly supported function $\xi \in \mathscr{A}_c$, we set:

$$T(\xi)[\tilde{m},\tilde{m}',\theta] := \sum_{\gamma,\gamma'\in\Gamma} \varphi(\tilde{m}\gamma)\overline{\varphi(\tilde{m}'\gamma')}\xi(\gamma^{-1}\theta,\gamma^{-1}\gamma').$$

Since φ is supported in a fundamental domain, it is clear that only one couple (γ,γ') gives a non trivial contribution. Moreover, the function $T(\xi)$ is well defined on G and is supported inside G_U^U. The map T is a $*$-homomorphism from the algebra $\mathscr{A}_c$ to the algebra $\mathscr{B}_c$ which implements the Morita isomorphisms $\mathscr{M}_{\mathrm{r}}$ and $\mathscr{M}_{\mathrm{m}}$ in K-theory. Indeed, we have:

$$\begin{aligned}
T(\xi)*T(\xi')[\tilde{m},\tilde{m}',\theta] &= \int_{\tilde{M}} T(\xi)[\tilde{m},\tilde{m}'',\theta]T(\xi')[\tilde{m}'',\tilde{m}',\theta]d\tilde{m}'' \\
&= \sum_{\alpha,\alpha',\beta,\beta'\in\Gamma} \varphi(\tilde{m}\alpha)\overline{\varphi(\tilde{m}'\beta')}\int_{\tilde{M}}\overline{\varphi(\tilde{m}''\beta)}\varphi(\tilde{m}''\alpha')d\tilde{m}''\times \\
&\quad\ \xi(\alpha^{-1}\theta,\alpha^{-1}\beta)\xi'(\alpha'^{-1}\theta,\alpha'^{-1}\beta') \\
&= \sum_{\alpha,\alpha'\in\Gamma} \varphi(\tilde{m}\alpha)\overline{\varphi(\tilde{m}'\alpha')}\sum_{\beta\in\Gamma}\xi(\alpha^{-1}\theta,\alpha^{-1}\beta)\xi'(\beta^{-1}\theta,\beta^{-1}\alpha') \\
&= \sum_{\alpha,\alpha'\in\Gamma} \varphi(\tilde{m}\alpha)\overline{\varphi(\tilde{m}'\alpha')}(\xi*\xi')(\alpha^{-1}\theta,\alpha^{-1}\alpha') \\
&= T(\xi*\xi')[\tilde{m},\tilde{m}',\theta].
\end{aligned}$$

Hence, we conclude that

$$T(\xi)*T(\xi') = T(\xi*\xi').$$

In a similar way one checks that $(T(\xi))^* = T(\xi^*)$.
T extends to a morphism between the corresponding reduced C^*-algebras. More precisely, let $f \in L^2(\tilde{M})$, then the regular representation π^{reg} is given for any $\tilde{m}\in\tilde{M}$ by:

$$(\pi^{\mathrm{reg}}T(\xi))_\theta(f)(\tilde{m}) = \int_{\tilde{M}}\sum_{\gamma',\gamma\in\Gamma}\varphi(\tilde{m}\gamma)\overline{\varphi(\tilde{m}'\gamma')}\xi(\gamma^{-1}\theta,\gamma^{-1}\gamma')f(\tilde{m}')d\tilde{m}'.$$

Denote by $g:\Gamma\to\mathbb{C}$ the function given by

$$g(\gamma') := \int_{\tilde{M}}\overline{\varphi(\tilde{m}'\gamma'^{-1})}f(\tilde{m}')d\tilde{m}',$$

then, one easily shows that the function g belongs to $\ell^2(\Gamma)$ and that its ℓ^2-norm can be estimated as follows:

$$\begin{aligned}
\|g\|_2^2 &= \sum_{\gamma'} |g(\gamma')|^2 = \sum_{\gamma'} \left| \int_{\tilde M} \overline{\phi(\tilde m' \gamma'^{-1})} f(\tilde m')\, d\tilde m \right|^2 \\
&= \sum_{\gamma'} \left| \int_{F\gamma'} \overline{\phi(\tilde m' \gamma'^{-1})} f(\tilde m')\, d\tilde m' \right|^2 \leq \sum_{\gamma'} \int_{F\gamma'} |f(\tilde m')|^2 d\tilde m' = \|f\|_2^2.
\end{aligned}$$

If we recall the regular representation of the algebra $\mathscr{A}_c$, denoted also by π^{reg}, then, using g we can write:

$$(\pi_{\mathrm{reg}} T(\xi))_\theta(f)(\tilde m) = \sum_{\gamma\in\Gamma} \phi(\tilde m\gamma) \sum_{\gamma'\in\Gamma} \xi(\gamma^{-1}\theta, \gamma^{-1}\gamma') g(\gamma'^{-1}) = \sum_{\gamma\in\Gamma} \phi(\tilde m\gamma)(\pi_\theta^{\mathrm{reg}}(\xi))(g)(\gamma^{-1})$$

Therefore, if we compute the L^2-norm of the function $(\pi^{\mathrm{reg}} T(\xi))_\theta(f)$ we get:

$$\begin{aligned}
\|(\pi_{\mathrm{reg}} T(\xi))_\theta(f)\|_2^2 &= \int_{\tilde M} \left| \sum_{\gamma\in\Gamma} \phi(\tilde m\gamma) \pi_\theta^{\mathrm{reg}}(\xi)(g)(\gamma^{-1}) \right|^2 d\tilde m \\
&= \sum_{\alpha\in\Gamma} \int_{F\alpha^{-1}} \left| \phi(\tilde m\alpha) \pi_\theta^{\mathrm{reg}}(\xi)(g)(\alpha^{-1}) \right|^2 d\tilde m \\
&= \sum_{\alpha\in\Gamma} \left| \pi_\theta^{\mathrm{reg}}(\xi)(g)(\alpha^{-1}) \right|^2 \int_{F\alpha^{-1}} |\phi(\tilde m\alpha)|^2 d\tilde m \\
&= \|\pi_\theta^{\mathrm{reg}}(\xi)(g)\|_2^2 \\
&\leq \|\xi\|_{\mathscr{A}_r}^2 \|g\|_2^2 \leq \|\xi\|_{\mathscr{A}_r}^2 \|f\|_2^2.
\end{aligned}$$

Summarizing: $\sup_{\theta\in T} \|(\pi^{\mathrm{reg}}(T\xi))_\theta\| \leq \|\xi\|_{\mathscr{A}_r}$ so that $\|T(\xi)\|_{\mathscr{B}_r} \leq \|\xi\|_{\mathscr{A}_r}$ as required. It thus remains to show compatibility of the traces with respect to the homomorphism T, and only on the compactly supported functions. Let us start with the regular trace. We have:

$$\begin{aligned}
\tau^\nu_{\mathrm{reg}}(T(\xi)) &= \int_{F\times T} T(\xi)[\tilde m, \tilde m, \theta] d\tilde m d\nu(\theta) \\
&= \int_T \xi(\theta, 1) \int_{\tilde M} |\varphi(\tilde m)|^2 d\tilde m d\nu(\theta) \\
&= \int_T \xi(\theta, 1) d\nu(\theta) \\
&= \tau^\nu_{\mathrm{reg}}(\xi).
\end{aligned}$$

Note that when $\tilde m \in \Omega$, only $\gamma = 1$ contributes to the sum defining $T(\xi)$.

Let us now check, briefly, that T induces a morphism between the maximal C^*-algebras. It suffices to show that T is continuous with respect to the L^1-norms on the groupoids $\mathscr{G}$ and G. But for $\xi \in \mathscr{A}_c$ and for any $\tilde{m} \in \Omega$ we have

$$\begin{aligned} \int_{\tilde{M}} |(T\xi)[\tilde{m}, \tilde{m}', \theta]| \, d\tilde{m}' &\leq |\phi(\tilde{m})| \int_{\tilde{M}} |\phi(\tilde{m}')| \, d\tilde{m}' \left(\sum_{\gamma' \in \Gamma} |\xi(\theta, \gamma')| \right) \\ &\leq \|\phi\|_1 \|\phi\|_\infty \|\xi\|_1. \end{aligned}$$

Hence,

$$\|T(\xi)\|_1 \leq \|\phi\|_1 \|\phi\|_\infty \|\xi\|_1 \, .$$

Now let us check the compatibility with the average trace τ^{ν}_{av}. We have, for $\xi \in \mathscr{A}_c$:

$$\begin{aligned} \tau^{\nu}_{\mathrm{av}}(T(\xi)) &= \int_{F \times T} \sum_{\gamma \in \Gamma(\theta)} T(\xi)[\tilde{m}, \tilde{m}\gamma, \theta] d\tilde{m} \; d\nu(\theta) \\ &= \int_{\Omega \times T} \sum_{\gamma \in \Gamma(\theta)} T(\xi)[\tilde{m}, \tilde{m}\gamma, \theta] d\tilde{m} \; d\nu(\theta) \end{aligned}$$

$$= \int_T \sum_{\gamma \in \Gamma(\theta)} \xi(\theta, \gamma) \int_\Omega |\varphi(\tilde{m})|^2 d\tilde{m}\, d\nu(\theta)$$
$$= \int_T \sum_{\gamma \in \Gamma(\theta)} \xi(\theta, \gamma) d\nu(\theta)$$
$$= \tau_{\rm av}^\nu(\xi).$$

Note that in the expression $T(\xi)[\tilde{m}, \tilde{m}\gamma, \theta]$ for $\tilde{m} \in \Omega$, only the couple $(1, \gamma)$ contributes non trivially to the sum. $\square$

3. Hilbert modules and Dirac operators

3.1. Connes-Skandalis Hilbert module. — Recall that $V = \tilde{M} \times_\Gamma T$ where $\tilde{M} \to M$ is the universal Γ-covering of the closed manifold M and where Γ acts by homeomorphisms on the compact space T. We fix a hermitian vector bundle E over V and we denote by $\widehat{E}$ its pull-back to $\tilde{M} \times T$. We define a right action of the convolution algebra $\mathcal{A}_c = C_c(T \rtimes \Gamma) \equiv C_c(\mathcal{G})$ on the space $\mathcal{E}_c = C_c^{\infty,0}(\widetilde{M} \times T; \widehat{E})$, of compactly supported sections of the vector bundle $\widehat{E}$ which are smooth with respect to the $\tilde{M}$ variable and continuous with respect to the T variable, as follows.

$$(\xi f)(\tilde{m}, \theta) = \sum_{\gamma \in \Gamma} \xi(\tilde{m}\gamma^{-1}, \gamma\theta) f(\gamma\theta, \gamma), \quad \xi \in \mathcal{E}_c, \quad f \in \mathcal{A}_c.$$

A $\mathcal{A}_c$-valued inner product $<.;.>$ on $\mathcal{E}_c$ is also defined by **[27]**

$$<\xi_1; \xi_2>(\theta, \gamma) := \int_{\widetilde{M}} <\xi_1(\tilde{m}, \gamma^{-1}\theta); \xi_2(\tilde{m}\gamma^{-1}, \theta)>_{E_{[\tilde{m},\theta]}} d\tilde{m},$$

where $<.;.>_E$ is the hermitian scalar product that we have fixed of the vector bundle E. A classical computation shows that these operations endow the space $\mathcal{E}_c$ with the structure of a pre-Hilbert module over the algebra $\mathcal{A}_c$.

As in the previous sections, we denote by $\mathcal{A}_r$ and $\mathcal{A}_m$ the reduced and maximal C^*-algebras of the groupoid $\mathcal{G}$. Recall that there is a natural C^*-algebra morphism

$$\lambda : \mathcal{A}_m \longrightarrow \mathcal{A}_r.$$

The pre-Hilbert $\mathcal{A}_c$-module $\mathcal{E}_c$ can be completed with respect to the reduced C^*-norm to yield a right Hilbert C^*-module over $\mathcal{A}_r$ that we shall denote by $\mathcal{E}_r$. In the same way, we can complete $\mathcal{E}_c$ with respect to the maximal C^*-norm and define the Hilbert C^*-module $\mathcal{E}_m$ over the C^*-algebra $\mathcal{A}_m$. It is then clear that the natural map $\mathcal{E}_c \longrightarrow \mathcal{E}_r$, extends to a morphism of Hilbert modules $\mathcal{E}_m \to \mathcal{E}_r$. More precisely, we have a well defined linear map

$$\varrho : \mathcal{E}_m \longrightarrow \mathcal{E}_r \text{ such that } \varrho(\xi f) = \varrho(\xi)\lambda(f) \quad f \in \mathcal{A}_m \text{ and } \xi \in \mathcal{E}_m.$$

We denote as in the previous sections by G the monodromy groupoid

$$G := \frac{\tilde{M} \times \tilde{M} \times T}{\Gamma}.$$

The algebra $\mathscr{B}_c^E$ of smooth compactly supported sections of the bundle $\mathrm{END}(E)$ over G is faithfully represented in $\mathscr{E}_c$ by the formula [**19**]

$$\chi(\varphi)(\xi)(\tilde{m}, \theta) := \int_{\tilde{M}} \varphi[\tilde{m}, \tilde{m}', \theta]\xi(\tilde{m}', \theta)d\tilde{m}', \quad \varphi \in C_c^\infty(G, \mathrm{END}(E)), \xi \in \mathscr{E}_c.$$

Recall that $\mathscr{B}_r^E$ and $\mathscr{B}_m^E$ are respectively the reduced and maximal C^*-algebras associated with G and with coefficients in E. Given a C^*-algebra A and a Hilbert A-module $\mathscr{E}$, the algebra $B_A(\mathscr{E})$ consists of bounded adjointable A-linear morphisms of $\mathscr{E}$. Recall also that the C^*-algebra $\mathscr{K}_A(\mathscr{E})$ of A-compact operators is the completion in $B_A(\mathscr{E})$ of the subalgebra of A-finite rank operators. The following proposition is proved in [**27**], see also [**39**] and [**7**].

Proposition 3.1. — *For any $\varphi \in \mathscr{B}_c^E$, the map $\chi(\varphi) : \mathscr{E}_c \to \mathscr{E}_c$ is $\mathscr{A}_c$-linear and the morphism χ extends to continuous $*$-representations*

$$\chi_r : \mathscr{B}_r^E \longrightarrow \mathscr{K}_{\mathscr{A}_r}(\mathscr{E}_r) \text{ and } \chi_m : \mathscr{B}_m^E \longrightarrow \mathscr{K}_{\mathscr{A}_m}(\mathscr{E}_m),$$

which are C^*-algebra isomorphisms.

Notice that the proof of this proposition is usually given for the holonomy groupoid of the foliation; however the same argument applies to the monodromy groupoid. Note also that the proof is usually given for the reduced C^*-algebra but it remains valid for the maximal C^*algebra too [**27**] [Remarque 5].

For any $\theta \in T$, we have defined in Subsection 1.3 a representation π_θ^{av} of the maximal C^*-algebra $\mathscr{A}_m$ in the Hilbert space $\ell^2(\Gamma/\Gamma(\theta))$. By using Remark 1.1 we can write

$$\pi_\theta^{av}(f)(\xi)(\theta') := \sum_{\theta'' \in \Gamma.\theta} \sum_{\gamma\theta'' = \theta'} f(\theta', \gamma)\xi(\theta''), \quad f \in \mathscr{A}_c, \xi \in \ell^2(\Gamma\theta) \text{ and } \theta' \in \Gamma\theta.$$

Using the $\mathscr{A}_m$-Hilbert module $\mathscr{E}_m$ together with the representation π_θ^{av}, one defines the Hilbert space

$$\mathscr{H}_\theta^{av} := \mathscr{E}_m \otimes_{\pi_\theta^{\mathrm{av}}} \ell^2(\Gamma\theta).$$

Similarly

$$\mathscr{H}_\theta^{reg} := \mathscr{E}_m \otimes_{\pi_\theta^{reg}} \ell^2(\Gamma).$$

Lemma 3.2. — *There exists an isomorphism of Hilbert spaces, Φ_θ, between $\mathscr{H}_\theta^{av}$ and the Hilbert space $L^2(L_\theta, E)$ of square integrable sections of the vector bundle E over* *the leaf L_θ through θ, induced by the formula*

$$\Phi_\theta(\xi\otimes f)(\tilde m,\theta) := \sum_{\gamma\in\Gamma} f(\gamma\theta)\ \left[\xi(\tilde m\gamma^{-1},\gamma\theta)\right], \quad \xi\in\mathscr{E}_c \text{ and } f\in C_c(\Gamma\theta).$$

Similarly there exists an isomorphism Ψ_θ of Hilbert spaces between $\mathscr{H}_\theta^{reg}$ and $L^2(\tilde M_\theta,\widehat{E})$ induced by the formula

$$\Psi_\theta(\xi\otimes\delta_\gamma)(\tilde m) := \xi(\tilde m\gamma^{-1},\gamma\theta).$$

where δ_γ denotes the delta function at Γ.

Proof. — If $\alpha\in\Gamma(\theta)$ then we can write for $\xi\in\mathscr{E}_c$:

$$\begin{aligned}\Phi_\theta(\xi\otimes f)(\tilde m\alpha^{-1},\theta) &= \sum_{\gamma\in\Gamma} f(\gamma\theta)\ \left[\xi(\tilde m\alpha^{-1}\gamma^{-1},\gamma\theta)\right]\\ &= \sum_{\beta\in\Gamma} f(\beta\theta)\ \xi(\tilde m\beta^{-1},\beta\theta)\\ &= \Phi_\theta(\xi\otimes f)(\tilde m,\theta).\end{aligned}$$

Hence, $\Phi_\theta(\xi\otimes f)$ is a smooth section of $\hat E$ over $\tilde M_\theta$ which is $\Gamma(\theta)$-invariant. Moreover, if $f=\delta_{\gamma\theta}$ and if we denote by $K_{\gamma\theta}$ the (compact) support of ξ in $\tilde M\times\{\gamma\theta\}$ then the support of $\Phi_\theta(\xi\otimes\delta_{\gamma\theta})$ is contained in

$$[K_{\gamma\theta}\cdot\gamma]\cdot\Gamma(\theta),$$

and hence is $\Gamma(\theta)$-compact.

Let now $g\in\mathscr{A}_c$ be given. Then we have

$$\begin{aligned}\Phi_\theta(\xi g\otimes f)(\tilde m,\theta) &= \sum_{\gamma\in\Gamma} f(\gamma\theta)(\xi g)(\tilde m\gamma^{-1},\gamma\theta)\\ &= \sum_{\gamma\in\Gamma} f(\gamma\theta)\sum_{\alpha\in\Gamma} g(\alpha\gamma\theta,\alpha)\xi(\tilde m\gamma^{-1}\alpha^{-1},\alpha\gamma\theta)\\ &= \sum_{\theta',\theta''\in\Gamma\theta}\ \sum_{\gamma\theta=\theta',\alpha\theta'=\theta''} f(\theta')g(\theta'',\alpha)\xi(\tilde m(\alpha\gamma)^{-1},\theta'')\\ &= \sum_{\theta',\theta''\in\Gamma\theta}\ \sum_{\beta\theta=\theta'',\alpha\theta'=\theta''} f(\theta')g(\theta'',\alpha)\xi(\tilde m\beta^{-1},\theta'').\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}\Phi_\theta(\xi\otimes\pi_\theta^{\mathrm{av}}(g)(f))(\tilde m,\theta) &= \sum_{\theta''\in\Gamma\theta}\pi_\theta^{\mathrm{av}}(g)(f)(\theta'')\sum_{\gamma_1\theta=\theta''}\xi(\tilde m\gamma_1^{-1},\theta'')\\ &= \sum_{\theta'',\theta'\in\Gamma\theta}\ \sum_{\delta\theta'=\theta'',\delta_1\theta=\theta''} f(\theta')g(\theta'',\delta)\xi(\tilde m\delta_1^{-1},\theta'').\end{aligned}$$

Hence, we obtain the equality $\Phi_\theta(\xi g\otimes f)=\Phi_\theta(\xi\otimes\pi_\theta^{\mathrm{av}}(g)(f))$.

In order to finish the proof, we need to identify the scalar product on the Hilbert space $\mathcal{H}_\theta^{av}$. We have

$$\begin{aligned}
<\xi\otimes f,\xi\otimes f> &= <\pi_\theta^{av}(<\xi,\xi>)(f), f> \\
&= \sum_{\theta'\in\Gamma\theta} \pi_\theta^{av}(<\xi,\xi>)(f)(\theta')\overline{f(\theta')} \\
&= \sum_{\theta'\in\Gamma\theta} \overline{f(\theta')}\sum_{\beta\in\Gamma}<\xi,\xi>(\theta',\beta)f(\beta^{-1}\theta') \\
&= \sum_{\theta',\theta''\in\Gamma\theta} \overline{f(\theta')}f(\theta'')\sum_{\beta\theta''=\theta'}\int_{\tilde M}<\xi(\tilde m,\beta^{-1}\theta'),\xi(\tilde m\beta^{-1},\theta')>d\tilde m \\
&= \sum_{\theta',\theta''\in\Gamma\theta} \overline{f(\theta')}f(\theta'')\sum_{\alpha\theta'=\theta''}\int_{\tilde M}<\xi(\tilde m,\alpha\theta'),\xi(\tilde m\alpha,\theta')>d\tilde m
\end{aligned}$$

On the other hand, if we view $\Phi_\theta(\xi\otimes f)$ as a section over the leaf L_θ through θ, then we can use a fundamental domain F_θ for the free and proper action of the isotropy group $\Gamma(\theta)$ on $\tilde M$ and write

$$<\Phi_\theta(\xi\otimes f),\Phi_\theta(\xi\otimes f)> = \int_{F_\theta} <\Phi_\theta(\xi\otimes f)(\tilde m,\theta),\Phi_\theta(\xi\otimes f)(\tilde m,\theta)> d\tilde m$$

$$= \sum_{\theta_1,\theta_2\in\Gamma\theta} f(\theta_1)\overline{f(\theta_2)}\sum_{\gamma_1\theta=\theta_1,\gamma_2\theta=\theta_2}\int_{F_\theta}<\xi(\tilde m\gamma_1^{-1},\theta_1),\xi(\tilde m\gamma_2^{-1},\theta_2)>d\tilde m$$

We fix a section $\varphi:\Gamma\theta\to\Gamma$ of the map $\gamma\mapsto\gamma\theta$. Then $\beta=\varphi(\theta_1)^{-1}\gamma_1$ is an element of the isotropy group $\Gamma(\theta)$ and we have

$$\begin{aligned}
&<\Phi_\theta(\xi\otimes f),\Phi_\theta(\xi\otimes f)> \\
&= \sum_{\theta'',\theta'\in\Gamma\theta} f(\theta'')\overline{f(\theta')}\sum_{\gamma_2\theta=\theta'}\sum_{\beta\in\Gamma(\theta)}\int_{F_\theta}<\xi(\tilde m\beta^{-1}\varphi(\theta'')^{-1},\theta''),\xi(\tilde m\gamma_2^{-1},\theta')>d\tilde m \\
&= \sum_{\theta'',\theta'\in\Gamma\theta} f(\theta'')\overline{f(\theta')}\sum_{\gamma_2\theta=\theta'}\sum_{\beta\in\Gamma(\theta)}\int_{F_\theta\beta^{-1}\varphi(\theta'')^{-1}}\times \\
&\quad <\xi(\tilde m_1,\theta''),\xi(\tilde m_1\varphi(\theta'')\beta\gamma_2^{-1},\theta')>d\tilde m_1 \\
&= \sum_{\theta'',\theta'\in\Gamma\theta} f(\theta'')\overline{f(\theta')}\sum_{\alpha\theta'=\theta''}\sum_{\beta\in\Gamma(\theta)}\int_{F_\theta\beta^{-1}\varphi(\theta'')^{-1}}<\xi(\tilde m_1,\theta''),\xi(\tilde m_1\alpha,\theta')>d\tilde m
\end{aligned}$$

Setting $\delta=\varphi(\theta_1)\beta^{-1}\varphi(\theta_1)^{-1}$ and noticing that a fundamental domain $F_{\theta''}$ is equal to $F_\theta\varphi(\theta'')^{-1}$ we get

$$\begin{aligned}
&<\Phi_\theta(\xi\otimes f),\Phi_\theta(\xi\otimes f)> \\
&\quad = \sum_{\theta',\theta''\in\Gamma\theta} f(\theta'')\overline{f(\theta')}\sum_{\alpha\theta'=\theta''}\sum_{\delta\in\Gamma(\theta'')}\int_{(F_\theta\varphi(\theta'')^{-1})\delta}<\xi(\tilde m_1,\theta''),\xi(\tilde m_1\alpha,\alpha^{-1}\theta'')>d\tilde m
\end{aligned}$$

$$= \sum_{\theta',\theta''\in\Gamma\theta} f(\theta'')\overline{f(\theta')} \sum_{\alpha\theta'=\theta''} \sum_{\delta\in\Gamma(\theta'')} \int_{F_{\theta''}\delta} <\xi(\tilde m_1,\theta''),\xi(\tilde m_1\alpha,\alpha^{-1}\theta'')>d\tilde m$$

$$= \sum_{\theta',\theta''\in\Gamma\theta} f(\theta'')\overline{f(\theta')} \sum_{\alpha\theta'=\theta''} \int_{\tilde M} <\xi(\tilde m_1,\alpha\theta'),\xi(\tilde m_1\alpha,\theta')>d\tilde m.$$

Hence $<\xi\otimes f,\xi\otimes f> = <\Phi_\theta(\xi\otimes f),\Phi_\theta(\xi\otimes f)>$. It now remains to show that Φ_θ is surjective. Let η be a smooth compactly supported section over the leaf L_θ and denote by $\tilde\eta$ its lift into a $\Gamma(\theta)$-invariant section over $\tilde M\times\theta$ and by ξ_0 any extension of $\tilde\eta$ into a leafwise smooth continuous section over $\tilde M\times T$. Let φ be a smooth function on $\tilde M$ such that $\sum_{\alpha\in\Gamma(\theta)}\alpha\varphi=1$ and such that for any compact set K in $L_\theta\simeq\tilde M/\Gamma(\theta)$, the intersection of the support of φ with the inverse image of K, under the projection $\tilde M\to L_\theta$, is compact in $\tilde M$. We view φ as a function on $\tilde M\times T$ independent of the T variable and set

$$\xi := \varphi\xi_0.$$

Then $\xi\in C_c^{\infty,0}(\tilde M\times T,\tilde E)$ and one checks immediately that $\Phi_\theta(\xi\otimes\delta_\theta)=\eta$. The proof of the second isomorphism is simpler and is left as an exercise. □

Recall that we have defined two representations, that we have both denoted π^{av}, respectively of the C^*-algebras $\mathcal{A}_m$ and $\mathcal{B}_m^E$ in the corresponding von Neumann algebras of the discrete groupoid $\mathcal{G}$ and of the monodromy groupoid G with coefficients in the vector bundle E:

$$\pi^{\mathrm{av}} : \mathcal{A}_m \to W^*_{\mathrm{av}}(\mathcal{G})\,, \quad \pi^{\mathrm{av}} : \mathcal{B}_m^E \to W^*_\nu(V,\mathcal{F};E)\,.$$

Recall also that we have defined a $*$-representation χ_m of $\mathcal{B}_m^E$ in the compact operators of the Hilbert module $\mathcal{E}_m$:

$$\chi_m : \mathcal{B}_m^E \to \mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m)\,.$$

Proposition 3.3. — *Let S be a given element of $\mathcal{B}_m^E$. Then we have*

$$\pi_\theta^{av}(S) = \Phi_\theta\circ\left[\chi_m(S)\otimes_{\pi_\theta^{av}} I_{\ell^2(\Gamma\theta)}\right]\circ\Phi_\theta^{-1}.$$

with $\Phi_\theta:\mathcal{E}_m\otimes_{\pi_\theta^{\mathrm{av}}}\ell^2(\Gamma\theta)\to L^2(L_\theta,E)$ the isomorphism given in Lemma 3.2. In the same way, we have

$$\pi_\theta^{reg}(S) = \Psi_\theta\circ\left[\chi_r(S)\otimes_{\pi_\theta^{reg}} I_{\ell^2(\Gamma)}\right]\circ\Psi_\theta^{-1}.$$

with $\Psi_\theta:\mathcal{E}_m\otimes_{\pi_\theta^{reg}}\ell^2(\Gamma)\to L^2(\tilde M_\theta,\widehat{E})$ the second isomorphism given in Lemma 3.2.

Proof. — Let us fix an element $k \in C_c^{\infty,0}(G; \mathrm{END}(E))$ and give the proof for $S = k$. We compute for $\xi \in \mathcal{E}_c$ and $f \in C_c[\Gamma\theta]$:

$$\begin{aligned}
\Phi_\theta(\chi(k)(\xi) \otimes f)(\tilde m, \theta) &= \sum_{\gamma\in\Gamma} f(\gamma\theta)\chi(k)(\xi)(\tilde m\gamma^{-1}, \gamma\theta) \\
&= \sum_{\gamma\in\Gamma} f(\gamma\theta) \int_{\tilde M} k[\tilde m\gamma^{-1}, \tilde m', \gamma\theta]\xi(\tilde m', \gamma\theta) d\tilde m' \\
&= \sum_{\gamma\in\Gamma} f(\gamma\theta) \int_{\tilde M} k[\tilde m, \tilde m'\gamma, \theta]\xi(\tilde m', \gamma\theta) d\tilde m' \\
&= \sum_{\gamma\in\Gamma} f(\gamma\theta) \int_{\tilde M} k[\tilde m, \tilde m_1, \theta]\xi(\tilde m_1\gamma^{-1}, \gamma\theta) d\tilde m_1.
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
\pi_\theta^{av}(k)(\Phi_\theta(\xi \otimes f))(\tilde m, \theta) &= \int_{F_\theta} \sum_{\alpha\in\Gamma(\theta)} k[\tilde m, \tilde m'\alpha, \theta] \sum_{\gamma\in\Gamma} f(\gamma\theta)\xi(\tilde m'\gamma^{-1}, \gamma\theta) d\tilde m' \\
&= \sum_{\gamma\in\Gamma} f(\gamma\theta) \sum_{\alpha\in\Gamma(\theta)} \int_{F_\theta\alpha} k[\tilde m, \tilde m'', \theta]\xi(\tilde m''\alpha^{-1}\gamma^{-1}, \gamma\theta) d\tilde m'' \\
&= \sum_{\gamma'\in\Gamma} f(\gamma'\theta) \sum_{\alpha\in\Gamma(\theta)} \int_{F_\theta\alpha} k[\tilde m, \tilde m'', \theta]\xi(\tilde m''{\gamma'}^{-1}, \gamma'\theta) d\tilde m'' \\
&= \sum_{\gamma'\in\Gamma} f(\gamma'\theta) \int_{\tilde M} k[\tilde m, \tilde m'', \theta]\xi(\tilde m''{\gamma'}^{-1}, \gamma'\theta) d\tilde m''.
\end{aligned}$$

So we get

$$\Phi_\theta(\chi(k)(\xi) \otimes f) = \pi_\theta^{av}(k)(\Phi_\theta(\xi \otimes f))$$

which proves the first statement by continuity. We omit the proof of the second statement as it is similar and in fact easier. □

3.2. Γ-equivariant pseudodifferential operators. — This subsection is devoted to a brief overview of the pseudodifferential calculus relevant to our study. All stated results are known and we therefore only sketch the proofs.

Let $\mathcal{E}_c$ be as before $C_c^{\infty,0}(\tilde M \times T, \widehat{E})$ endowed with its structure of pre-Hilbert $\mathcal{A}_c$-module. Recall that if we complete the prehilbertian module $\mathcal{E}_c$ with respect to the regular norm on $\mathcal{A}_c$ then we get a Hilbert C^*-module $\mathcal{E}_r$ over the regular C^*-algebra $\mathcal{A}_r$. In the same way, completing $\mathcal{A}_c$ with respect to the maximal C^*-norm yields a Hilbert C^*-module $\mathcal{E}_m$ over the maximal C^*-algebra $\mathcal{A}_m$. We fix two vector bundles E and F over V and we denote by $\widehat{E}$ and $\widehat{F}$ their pullbacks to $\tilde M \times T$ into Γ-equivariant vector bundles; we let $\widehat{E}_\theta$ be the restriction of $\widehat{E}$ to $\tilde M_\theta$. We set, as before, $\tilde M_\theta := \tilde M_\theta$.

Definition 3.4. — *Let $P : C_c^{\infty,0}(\tilde{M} \times T, \widehat{E}) \to C^{\infty,0}(\tilde{M} \times T, \widehat{F})$ be a linear map. We shall say that P defines a pseudodifferential operator of order m on the monodromy groupoid G if there is a continuous family of order m pseudodifferential operators $(P_\theta)_{\theta \in T}$,*

$$P_\theta : C_c^\infty(\tilde{M}_\theta, \widehat{E}_\theta) \to C^\infty(\tilde{M}_\theta, \widehat{F}_\theta)$$

satisfying:
(1) $(P\xi)(\tilde{m}, \theta) = (P_\theta \xi(\cdot, \theta))(\tilde{m} \times \{\theta\})$
(2) *P is Γ-equivariant: $R_\gamma^* P R_{\gamma^{-1}}^* = P$;*
(3) *the Schwartz kernel of P, K_P, which can be thought of as a Γ-invariant distributional section on $\tilde{M} \times \tilde{M} \times T$, is of Γ-compact support, i.e. the image of the support in $(\tilde{M} \times \tilde{M} \times T)/\Gamma =: G$ is a compact set.*

Notice that (2) can be then restated as: $P_{\gamma\theta} = \gamma P_\theta \ \forall \theta \in T$, $\forall \gamma \in \Gamma$, exactly as in the definition of the regular von Neumann algebra. The notion of continuity for families of pseudodifferential operators is classical and will not be recalled here, see, for example, [**10**], [**42**], [**32**], [**55**], [**56**]. Finally, because of the third condition P maps $C_c^{\infty,0}(\tilde{M} \times T, \widehat{E})$ into $C_c^{\infty,0}(\tilde{M} \times T, \widehat{F})$.

Notice that a Γ-equivariant continuous family of differential operators acting between the sections of two equivariant vector bundles is an example of a pseudodifferential operator on G.

If $m \in \mathbb{Z}$, we shall denote by $\Psi_c^m(G; \widehat{E}, \widehat{F})$ the space of pseudodifferential operators of order $\leq m$ from $\widehat{E}$ to $\widehat{F}$[(3)] . We set

$$\Psi_c^\infty(G; \widehat{E}, \widehat{F}) := \bigcup_{m \in \mathbb{Z}} \Psi_c^m(G; \widehat{E}, \widehat{F}) \text{ and } \Psi_c^{-\infty}(G; \widehat{E}, \widehat{F}) := \bigcap_{m \in \mathbb{Z}} \Psi_c^m(G; \widehat{E}, \widehat{F}).$$

Using condition (3) , it is not difficult to check that the space $\Psi_c^\infty(G; \widehat{E}, \widehat{E})$ is a filtered algebra. Moreover, assigning to P its formal adjoint $P^* = (P_\theta^*)_{\theta \in T}$ gives $\Psi_c^\infty(G; \widehat{E}, \widehat{E})$ the structure of an involutive algebra; the formal adjoint is defined also for $P \in \Psi_c^m(G; \widehat{E}, \widehat{F})$ and it is then an alement in $\Psi_c^m(G; \widehat{F}, \widehat{E})$.

Remark 3.5. — *Notice that Definition 3.4 fits into the general framework of pseudodifferential calculus on groupoids, as developed by Connes and many others. More precisely, let $P = (P_\theta)_{\theta \in T}$ be a pseudodifferential operator on G as in Definition 3.4. For any $\theta \in T$ and any $x = [\tilde{m}, \theta] \in L_\theta$ the diffeomorphism*

$$\rho_{x,\theta} : \tilde{M} \to G^x = r^{-1}(x) \quad \text{given by} \quad \rho_{x,\theta}(\tilde{m}') = [\tilde{m}, \tilde{m}', \theta],$$

(3) The notation for this space of operators is not unique: in [**34**] it is denoted $\Psi^*_{\rtimes,c}(\tilde{M} \times T; \widehat{E}, \widehat{F})$ with $\rtimes$ denoting equivariance and c denoting again *of* Γ-*compact support*; in [**39**] it is simply denoted as $\Psi^*_\Gamma(\widehat{E}, \widehat{F})$.

allows to define a pseudodifferential operator P_x on G^x with coefficients in s^*E, viz. $P_x := (\rho_{x,\theta}^{-1})^* \circ P_\theta \circ (\rho_{x,\theta})^*$. It is easy to check that P_x only depends on x and that the family $(P_x)_{x\in V}$ is a pseudodifferential operator on G in the sense of Connes. Conversely if we are given now a pseudodifferential operator $(P_x)_{x\in V}$ in the sense of Connes, then a choice of a base point m_0 in M allows to construct $P = (P_\theta)_{\theta\in T}$ satisfying the assumptions of Definition 3.4, viz. $P_\theta := \rho^*_{x(\theta),\theta} \circ P_{x(\theta)} \circ (\rho^{-1}_{x(\theta),\theta})^*$ with $x(\theta) = [\tilde{m}_0, \theta]$ and $[\tilde{m}_0] = m_0$.

Remark 3.6. — *According to* **[18]** *a pseudodifferential operator as in Connes, admits a well defined distributional Schwartz kernel over G. It is easy to check that this Schwartz kernel coincides with our K_P when the two families correspond as in the previous remark.*

Remark 3.7. — *The construction explained in remark 3.5. also allows to establish an identification between Connes' von Neumann algebra* **[18]** *for the groupoid G and our von Neumann algebra $W^*_\nu(G, E)$. It is easy to check that Connes' trace* **[18]** *corresponds to our trace τ^ν through this identification.*

Lemma 3.8. — *A pseudodifferential operator P of order m yields an $\mathcal{A}_c$-linear map $\mathcal{P} : \mathcal{E}_c \to \mathcal{F}_c$. Moreover, the following identity holds in $\mathcal{A}_c$: $<\mathcal{P}u, v> = <u, \mathcal{P}^*v>$ $\forall u \in \mathcal{E}_c$, $\forall v \in \mathcal{F}_c$.*

Proof. — Let $\xi \in \mathcal{E}_c$ and let $f \in \mathcal{A}_c$. By definition $(\xi f)(\cdot) = \sum_\gamma (R^*_{\gamma^{-1}}\xi)(\cdot) f(\gamma\pi(\cdot), \gamma)$ with $\pi : \tilde{M} \times T \to T$ the projection. Hence:

$$\begin{aligned}
\mathcal{P}(\xi f) &= \mathcal{P}\left(\sum_\gamma (R^*_{\gamma^{-1}}\xi)(\cdot) f(\gamma\pi(\cdot), \gamma)\right) \\
&= \sum_\gamma \left(\mathcal{P}\left(R^*_{\gamma^{-1}}\xi\right)(\cdot)\right) f(\gamma\pi(\cdot), \gamma) \\
&= \sum_\gamma \left(R^*_{\gamma^{-1}}\mathcal{P}\xi\right)(\cdot) f(\gamma\pi(\cdot), \gamma) = (\mathcal{P}\xi) f
\end{aligned}$$

where in the second equality we have used the fact that $\mathcal{P}$ commutes with multiplication by functions in $C(T)$ (indeed, $\mathcal{P}$ is given by a continuous family) and in the third equality we have used the Γ-equivariance. The equality $<\mathcal{P}u, v> = <u, \mathcal{P}^*v>$ is established in a straightforward way. □

Proposition 3.9. — *Let $\mathcal{P}$ be a pseudodifferential operator of order m between $\mathcal{E}_c$ and $\mathcal{F}_c$. Then we have:*

1. *If $m \leq 0$ then $\mathcal{P}$ extends to a bounded adjointable $\mathcal{A}_m$-linear operator $\overline{\mathcal{P}}_m$ from $\mathcal{E}_m$ to $\mathcal{F}_m$ and to a bounded adjointable $\mathcal{A}_r$-linear operator $\overline{\mathcal{P}}_r$ from $\mathcal{E}_r$ to $\mathcal{F}_r$.*

2. *If $m < 0$, then $\overline{\mathcal{P}}_m$ is an $\mathcal{A}_m$-linear compact operator from $\mathcal{E}_m$ to $\mathcal{F}_m$ and $\overline{\mathcal{P}}_r$ is an $\mathcal{A}_r$-linear compact operator from $\mathcal{E}_r$ to $\mathcal{F}_r$.*

Proof. — We only sketch the arguments, following [**55**]. For simplicity we take E and F to be the trivial line bundles, so that $\mathcal{E}_c = \mathcal{F}_c$. We give the proof for the maximal completion, the proof for the regular completion being the same.

For the first item, one applies the classical argument of Hörmander, see for example [**53**], reducing the continuity of order zero pseudodifferential operators to that of the smoothing operators. We omit the details.

For the second item, one starts with $\mathcal{P}$ of order $< -n$, with n equal to the dimension of $\tilde{M}$. Then $\mathcal{P}$ is given by integration against a continuous compactly supported element in G; in other words $\mathcal{P} = \chi(K)$, with $K \in C_c(G)$. We already know that such an element extends to a compact operator $\overline{\mathcal{P}}$ on $\mathcal{E}_m$, see 3.1. If $\mathcal{P}$ is of order less than $-n/2$ then we consider $\mathcal{Q} := \mathcal{P}^*\mathcal{P}$ which is of order less then $-n$ and symmetric. We know that $\mathcal{Q}$ extends to a (compact) bounded operator on $\mathcal{E}_m$; thus if $f \in \mathcal{E}_c$ then, in particular, $\|\mathcal{P} f\|^2 \leq C\|f\|^2$ which means that $\mathcal{P}$ extends to a bounded operator $\overline{\mathcal{P}}$ on $\mathcal{E}_m$. Similarly $\mathcal{P}^*$ extends to a bounded operator $\overline{\mathcal{P}^*}$ and by density we obtain that $\overline{\mathcal{P}}$ is adjointable with adjoint equal to $\overline{\mathcal{P}^*}$. Now, again by density, we have $\overline{\mathcal{Q}} = \overline{\mathcal{P}}^*\overline{\mathcal{P}}$; thus we can take the square root of $\overline{\mathcal{Q}}$ which will be again compact since $\overline{\mathcal{Q}}$ is. Using the polar decomposition for $\overline{\mathcal{P}}$ we can finally conclude that $\overline{\mathcal{P}}$ is compact which is what we need to prove.

If the order of $\mathcal{P}$ is $m < 0$ then we fix $\ell \in \mathbb{N}$ such that $m2^\ell < -n$; then we proceed inductively, considering $(\mathcal{P}^*\mathcal{P})^{2^\ell}$ and applying the above argument. □

Let $P = (P_\theta)_{\theta \in T}$ be an element in $\Psi_c^\ell(G)$; its principal symbol $\sigma_\ell(P)$ defines a Γ-equivariant function on the vertical cotangent bundle $T_V^*(\tilde{M} \times T)$ to the trivial fibration $\tilde{M} \times T \to T$; equivalently, $\sigma_\ell(P)$ is a function on the longitudinal cotangent bundle $T^*\mathcal{F}$ to the foliation $(V, \mathcal{F})$. If, more generally, $P \in \Psi_c^\ell(G; \widehat{E}, \widehat{F})$, then its principal symbol will be a Γ-equivariant section of the bundle $\operatorname{Hom}(\pi_V^*(\widehat{E}), \pi_V^*(\widehat{F})) := \pi_V^*(\widehat{E^*}) \otimes \pi_V^*(\widehat{F})$ with $\pi_V : T_V^*(\tilde{M} \times T) \to (\tilde{M} \times T)$ the natural projection; equivalently, $\sigma_\ell(P)$ is a section of the bundle $\operatorname{Hom}(\pi_{\mathcal{F}}^* E, \pi_{\mathcal{F}}^* F)$ over the longitudinal cotangent bundle $\pi_{\mathcal{F}} : T^*\mathcal{F} \to V$. We shall say that P is elliptic if its principal symbol $\sigma_\ell(P)$ is invertible on non-zero cotangent vectors. We end this subsection by stating the following fundamental and classical result whose proof can be found, for example in the work of Connes [**17**], see also [**38**]. (Notice that in this particular case the proof can be easily done directly, mimicking the classic one on a closed compact manifold.)

Theorem 3.10. — *Let $P \in \Psi_c^\ell(G; \widehat{E}, \widehat{F})$ be elliptic; then there exists $Q \in \Psi_c^{-\ell}(G; \widehat{F}, \widehat{E})$ such that*

$$\operatorname{Id} - PQ \in \Psi_c^{-\infty}(G; \widehat{F}, \widehat{F}), \qquad \operatorname{Id} - QP \in \Psi_c^{-\infty}(G; \widehat{E}, \widehat{E}). \tag{9}$$

Notice that in our definition elements in $\Psi_c^{-\infty}$ are of Γ-compact support: this applies in particular to both $S := \mathrm{Id} - PQ$ and $R := \mathrm{Id} - QP$.

We end this subsection by observing that it is also possible to introduce Sobolev modules $\mathscr{E}^{(k)}$ and prove the usual properties of pseudodifferential operators, see [**56**]. For simplicity we consider the case $k \in \mathbb{N}$. In order to give the definition, we fix an elliptic differential operator of order k, P; for example $P = D^k$, with D a Dirac type operator. This is a regular unbounded operator (see the next subsection). We consider the domain of its extension $\mathrm{Dom}\mathscr{P}_m$ and we endow it with the $\mathscr{A}_m$-valued scalar product

$$<s,t>_k := <s,t> + <\mathscr{P}_m s, \mathscr{P}_m t>.$$

This defines the Sobolev module of order k, $\mathscr{E}^{(k)}$. One can prove for these modules the usual properties:

- different choices of P yield compatible Hilbert module structures;
- if $k > \ell$ we have $\mathscr{E}^{(k)} \hookrightarrow \mathscr{E}^{(\ell)}$ and the inclusion in $\mathscr{A}_m$-compact
- if $R \in \Psi_c^m(G, E)$ then R extends to a bounded operator $\mathscr{E}^{(k)} \to \mathscr{E}^{(k-m)}$.

Since we shall make little use of these properties, we omit the proofs.

3.3. Functional calculus for Dirac operators. — Let $\tilde{D} = (\tilde{D}_\theta)_{\theta\in T}$ be a Γ-equivariant family of Dirac-type operators acting on the sections of a Γ-equivariant vertical hermitian Clifford module $\widehat{E}$ endowed with a Γ-equivariant connection. We shall make the usual assumptions on the connection and on the Clifford action ensuring that each $\tilde{D}_\theta$ is formally self-adjoint. Recall that $\tilde{D} = (\tilde{D}_\theta)_{\theta\in T} \in \Psi_c^1(G; \widehat{E})$ and that $\tilde{D}$ induces a $\mathscr{A}_c$-linear operator on $\mathscr{E}_c$ that we have denoted by $\mathscr{D}$.

Proposition 3.11. — *The operator $\mathscr{D}$ is closable in $\mathscr{E}_r$ and in $\mathscr{E}_m$. Moreover, the closures $\mathscr{D}_r$ and $\mathscr{D}_m$ on the Hilbert modules $\mathscr{E}_r$ and $\mathscr{E}_m$ respectively, are regular and self-adjoint operators.*

Proof. — We give a classical proof based on general results described for instance in [**55**]. Since the densely defined operator $\mathscr{D}$ is formally self-adjoint, it is closable with symmetric closures in $\mathscr{E}_r$ and $\mathscr{E}_m$ respectively. Let $\tilde{Q} \in \Psi_c^{-1}(G, \widehat{E})$ be a formally self-adjoint parametrix for $\tilde{D}$:

$$\mathrm{Id} - \tilde{D}\tilde{Q} = \tilde{S}\,, \qquad \mathrm{Id} - \tilde{Q}\tilde{D} = \tilde{R}\,.$$

For simplicity, we denote by π the regular or the maximal representation, by $\mathscr{E}_\pi$ the corresponding Hilbert module and by $\mathscr{D}_\pi$ the closure of $\mathscr{D}$. Since $\tilde{Q}$ has negative order, it extends into a bounded operator on $\mathscr{E}_\pi$, denoted by $\overline{\mathscr{Q}}_\pi$, or simply by $\mathscr{Q}_\pi$. On the other hand, we know that the zero-th order pseudodifferential operator $\tilde{D}\tilde{Q}$ extends to a bounded $\mathscr{A}_\pi$-linear operator on $\mathscr{E}_\pi$. If ξ belongs to the domain of this closure (which is $\mathscr{E}_\pi$) then there exists ξ_n in $C_c^{\infty,0}(\tilde{M} \times T, \widehat{E})$ converging in the π-norm to

ξ and such that $(\tilde{D}\tilde{Q})\xi_n$ is convergent in the π norm. We deduce that $\mathcal{Q}_\pi(\xi)$ is well defined and is the limit of $\tilde{Q}\xi_n$. Hence we deduce that $\mathcal{Q}_\pi\xi$ belongs to the domain of $\mathcal{D}_\pi$ and that $\mathrm{Im}\mathcal{Q}_\pi \subset \mathrm{Dom}\,\mathcal{D}_\pi$. Hence, $\mathcal{D}_\pi\mathcal{Q}_\pi$ is a bounded operator which coincides with the extension of $\tilde{D}\tilde{Q}$ and we have with obvious notation,

$$\mathcal{D}_\pi\mathcal{Q}_\pi = I - \mathcal{S}_\pi\,,$$

so $\mathcal{Q}_\pi^*\mathcal{D}_\pi^* \subset (\mathcal{D}_\pi\mathcal{Q}_\pi)^* = I - \mathcal{S}_\pi^*$ and hence $\mathrm{Dom}(\mathcal{D}_\pi^*) \subset \mathrm{Im}(\mathcal{Q}_\pi^*) + \mathrm{Im}(\mathcal{S}_\pi^*)$. Since $\mathcal{Q}_\pi$ is self-adjoint we deduce that

$$\mathrm{Dom}(\mathcal{D}_\pi^*) \subset \mathrm{Im}(\mathcal{Q}_\pi) + \mathrm{Im}(\mathcal{S}_\pi^*) \subset \mathrm{Dom}(\mathcal{D}_\pi).$$

The last inclusion is a consequence of the fact that $\mathcal{S}_\pi^*$ is induced by a smoothing Γ-compactly supported operator. So $\mathcal{D}_\pi$ is self-adjoint. Now, the graph of $\mathcal{D}_\pi$, $G(\mathcal{D}_\pi)$, is given by

$$G(\mathcal{D}_\pi) = \{(\mathcal{Q}_\pi(\eta) + \mathcal{S}_\pi(\eta'), \mathcal{D}_\pi(\mathcal{Q}_\pi(\eta)) + \mathcal{D}_\pi(\mathcal{S}_\pi(\eta')), \eta, \eta' \in \mathcal{E}_\pi\}.$$

Hence $G(\mathcal{D}_\pi)$, which is closed in $\mathcal{E}_\pi \times \mathcal{E}_\pi$, coincides with the image of a bounded morphism U of $\mathcal{C}_\pi$-modules given by

$$U = \begin{pmatrix} \mathcal{Q}_\pi & \mathcal{S}_\pi \\ \mathcal{D}_\pi\mathcal{Q}_\pi & \mathcal{D}_\pi\mathcal{S}_\pi \end{pmatrix}.$$

Now, as a general fact, the image of such morphism, when closed, is always orthocomplemented. Thus $\mathcal{D}_\pi$ is regular. □

Recall that we established in Lemma 3.2 isomorphisms of Hilbert spaces

$$\Phi_\theta : \mathcal{E}_m \otimes_{\pi_\theta^{\mathrm{av}}} \ell^2(\Gamma\theta) \to L^2(L_\theta, E)\,, \quad \Psi_\theta : \mathcal{E}_m \otimes_{\pi_\theta^{reg}} \ell^2(\Gamma) \to L^2(\tilde{M}_\theta, \widehat{E}).$$

Proposition 3.12. — *Let $\psi : \mathbb{R} \to \mathbb{C}$ be a continuous bounded function. Then for any $\theta \in T$, the bounded operator, acting on $L^2(L_\theta, E)$, given by*

$$\Phi_\theta \circ \left[\psi(\mathcal{D}_m) \otimes_{\pi_\theta^{av}} I_{\ell^2(\Gamma\theta)}\right] \circ \Phi_\theta^{-1},$$

coincides with the operator $\psi(D_{L_\theta})$ where D_{L_θ} is our Dirac type operator acting on the leaf L_θ.
In the same way the operator, acting on $L^2(\tilde{M}_\theta, \widehat{E}_\theta)$, given by

$$\Psi_\theta \circ \left[\psi(\mathcal{D}_m) \otimes_{\pi_\theta^{reg}} I_{\ell^2(\Gamma)}\right] \circ \Psi_\theta^{-1},$$

coincides with the operator $\psi(\tilde{D}_\theta)$.

Proof. — We prove only the first result, the proof of the second is similar. Since the operator $\mathcal{D}_m$ is a regular self-adjoint operator, its continuous functional calculus is well defined. See [**55**]. Let $\xi \in \mathscr{E}_c$ and let $f \in C_c[\Gamma\theta]$, then we have

$$\begin{aligned}\Phi_\theta(\mathscr{D}_m(\xi)\otimes f)(\tilde m,\theta) &= \sum_{\gamma\in\Gamma} f(\gamma\theta)(\mathscr{D}\xi)(\tilde m\gamma^{-1},\gamma\theta)\\ &= \sum_{\gamma\in\Gamma} f(\gamma\theta)[R^*_{\gamma^{-1}}(\mathscr{D}\xi)](\tilde m,\theta)\\ &= \sum_{\gamma\in\Gamma} f(\gamma\theta)\mathscr{D}(R^*_{\gamma^{-1}}\xi)(\tilde m,\theta).\end{aligned}$$

On the other hand, the action of the operator D_{L_θ} on the image of Φ_θ is given by

$$(D_{L_\theta}\circ\Phi_\theta)(\xi\circ f)(\tilde m,\theta) = \sum_{\gamma\in\Gamma} f(\gamma\theta)\tilde D_\theta([\gamma^{-1}\xi]_\theta)(\tilde m).$$

Since by definition of $\mathscr{D}$ we have $\mathscr{D}(\gamma^{-1}\xi)(\tilde m,\theta) = \tilde D_\theta([\gamma^{-1}\xi]_\theta)(\tilde m)$ we obtain that

$$\Phi_\theta\circ(\mathscr{D}_m\otimes I)\circ\Phi_\theta^{-1} = D_{L_\theta}.$$

If ψ is as above then we get as a consequence of the definition of functional calculus,

$$\begin{aligned}\psi(D_{L_\theta}) &= \psi\left(\Phi_\theta\circ(\mathscr{D}_m\otimes I)\circ\Phi_\theta^{-1}\right)\\ &= \Phi_\theta\circ\psi(\mathscr{D}_m\otimes I)\circ\Phi_\theta^{-1}.\end{aligned}$$

By uniqueness of the functional calculus we also deduce that $\psi(\mathscr{D}_m\otimes I) = \psi(\mathscr{D}_m)\otimes I$, and hence the proof is complete. □

Before proving the main result of this Subection, we recall two technical results about trace class operators. First we establish two useful Lemmas. The first one is classical and generalizes [**53**] Proposition A.3.2 while the second one is an easy extension of similar results for coverings established in [**1**].

Lemma 3.13. — *Let $S \in W^*_\nu(G,E)$; then the following statements are equivalent:*

- *S is τ^ν Hilbert-Schmidt (i.e. $\tau^\nu(S^*S) < +\infty$);*
- *there exists a measurable section K_S of $\mathrm{END}(E)$ over G such that for ν-almost every θ the operator S_θ is given on $L^2(\tilde M_\theta, \widehat{E}_\theta)$ by*

$$(S_\theta\xi)(\tilde m) = \int_{\tilde M} K_S(\tilde m,\tilde m',\theta)\xi(\tilde m')d\tilde m',$$

with

$$\int_{\tilde M\times F\times T} \mathrm{tr}\left(K_S(\tilde m,\tilde m',\theta)^* K_S(\tilde m,\tilde m',\theta)\right)\, d\tilde m\, d\tilde m'\, d\nu(\theta) \;<\; +\infty$$

where we interpret K_S as a Γ-equivariant section on $\tilde M\times\tilde M\times T$.

Moreover in this case the τ^ν Hilbert-Schmidt norm of S, $\|S\|^2_{\nu-\mathrm{HS}} := \tau^\nu(S^*S)$, is given by

$$\|S\|^2_{\nu-\mathrm{HS}} = \int_{\tilde M\times F\times T} \mathrm{tr}\left(K_S(\tilde m,\tilde m',\theta)^* K_S(\tilde m,\tilde m',\theta)\right)\, d\tilde m\, d\tilde m'\, d\nu(\theta)\,.$$

Proof. — We have, by definition,

$$\|S\|^2_{\nu-\mathrm{HS}} = \tau^\nu(S^*S) = \int_T \|S_\theta M_\chi\|^2_{\mathrm{HS}}\, d\nu(\theta)$$

where the integrand involves the usual Hilbert-Schmidt norm in $L^2(\tilde{M}_\theta, \widehat{E}_\theta)$. Therefore the proof is easily deduced using [**53**][page 251] $\square$

Lemma 3.14. — *Let S be a positive selfadjoint operator in $W^*_\nu(G,E)$; then the following statements are equivalent:*

- $\tau^\nu(S) < +\infty$*;*
- *for any smooth compactly supported function ϕ on $\tilde{M}$, the measurable function*

$$T \ni \theta \longrightarrow \mathrm{Tr}(M_{\overline{\phi}} \circ S_\theta \circ M_\phi)$$

  is ν-integrable on T, where the trace is the usual trace for bounded operators on the Hilbert space $L^2(\tilde{M}, \widehat{E})$;
- *for any smooth compactly supported function ϕ on $\tilde{M}$, the function*

$$T \ni \theta \longrightarrow \|S_\theta^{1/2} \circ M_\phi\|^2_{\mathrm{HS}}$$

  is ν-integrable on T.

Proof. — We follow the techniques in [**1**] and use Lemma 3.13. The second and third items are clearly equivalent. Assume that $\tau^\nu(S) < +\infty$ and let ϕ be a smooth compactly supported function on $\tilde{M}$ with uniform norm $\|\phi\|_\infty$. We let Γ_ϕ be a finite subset of Γ such that the support of ϕ lies in the union $\cup_{\gamma\in\Gamma_\phi} F\gamma$. Here F is a fundamenal domain as before. Then $S^{1/2}$ is τ^ν Hilbert-Schmidt and if $K_{S^{1/2}}$ is its Schwartz kernel, then we easily deduce

$$\begin{aligned}
&\int_{\tilde{M}\times\tilde{M}\times T} |\phi(\tilde{m}')|^2 \ \mathrm{tr}\left(K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta)^* K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta)\right)\ d\tilde{m}\, d\tilde{m}'\, d\nu(\theta)\\
=\ & \sum_{\gamma\in\Gamma_\phi} \int_{\tilde{M}\times F\gamma\times T} |\phi(\tilde{m}')|^2 \ \mathrm{tr}\left(K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta)^* K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta)\right) d\tilde{m}\, d\tilde{m}'\, d\nu(\theta)\\
\leq\ & \|\phi\|^2_\infty \times \sum_{\gamma\in\Gamma_\phi} \int_{\tilde{M}\times F\gamma\times T} \mathrm{tr}\left(K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta)^* K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta)\right) d\tilde{m}\, d\tilde{m}'\, d\nu(\theta)\\
=\ & \|\phi\|^2_\infty \times \sum_{\gamma\in\Gamma_\phi} \int_{\tilde{M}\times F\times T} \mathrm{tr}\left(K_{S^{1/2}}(\tilde{m}\gamma^{-1},\tilde{m}',\gamma\theta)^* K_{S^{1/2}}(\tilde{m}\gamma^{-1},\tilde{m}',\gamma\theta)\right) d\tilde{m}\, d\tilde{m}'\, d\nu(\theta)\\
=\ & \|\phi\|^2_\infty \times \mathrm{Card}(\Gamma_\phi) \times \tau^\nu(S) \quad < +\infty.
\end{aligned}$$

Conversely, let ϕ be any nonnegative smooth compactly supported function on $\tilde{M}$ such that $\phi\chi = \chi$, where χ is the characteristic function of F. Then we have

$$\begin{aligned}\tau^\nu(S) &= \int_T \operatorname{Tr}(M_\chi \circ S_\theta \circ M_\chi)\, d\nu(\theta)\\ &= \int_T \operatorname{Tr}(M_\chi \circ M_\phi \circ S_\theta \circ M_\phi \circ M_\chi)\, d\nu(\theta)\\ &\leq \int_T \operatorname{Tr}(M_\phi \circ S_\theta \circ M_\phi)\, d\nu(\theta) < +\infty. \qquad \square\end{aligned}$$

Proposition 3.15. — *Let $S = (S_\theta)_{\theta\in T}$ be an element of the von Neumann algebra $W^*_\nu(G;E)$. We assume that S_θ is an integral operator with smooth kernel for any θ in T and that the resulting Schwartz kernel K_S is a Borel bounded section over G.*

– *If S is positive and self-adjoint, then S is τ^ν trace class and we have*

$$\tau^\nu(S) = \int_{F\times T} \operatorname{tr}(K_S(\tilde m, \tilde m, \theta)) d\tilde m d\nu(\theta), \tag{10}$$

where F is a fundamental domain in $\tilde{M}$ and where in the right hand side we interpret $K(S)$ as a Γ-equivariant section on $\tilde M \times \tilde M \times T$.

– *If S is assumed to be τ^ν trace class, then formula* (10) *holds.*

Proof. — Let us prove the first item. Let ϕ be a smooth compactly supported function on $\tilde M$. The operator $M_\phi \circ S_\theta \circ M_{\overline{\phi}}$ acting on $L^2(\tilde M_\theta, \widehat{E})$ has a smooth compactly supported Schwartz kernel and is therefore trace class with

$$\operatorname{Tr}(M_\phi \circ S_\theta \circ M_{\overline{\phi}}) = \int_{\tilde M_\theta} |\phi(\tilde m)|^2 K_S(\tilde m, \tilde m, \theta)\, d\tilde m\,.$$

Since K_S is bounded as a section over G and since ν is a borelian measure, we have

$$\int_T \operatorname{Tr}(M_\phi \circ S_\theta \circ M_{\overline{\phi}}) d\nu(\theta) \,<\, +\infty\,.$$

This shows, using Lemma 3.14, that S is τ^ν trace class and also that $S^{1/2}$ is τ^ν Hilbert-Schmidt. By Lemma 3.13 we deduce that the $S^{1/2}$ is an integral operator with measurable Schwartz kernel $K_{S^{1/2}}$ satisfying

$$\|S^{1/2}\|^2_{\mathrm{HS}} := \int_{\tilde M\times F\times T} \operatorname{tr}\left(K_{S^{1/2}}(\tilde m, \tilde m', \theta)^* K_{S^{1/2}}(\tilde m, \tilde m', \theta)\right) d\tilde m\, d\tilde m' d\nu(\theta) \,<\, +\infty\,.$$

On the other hand we also have

$$\begin{aligned}K_S(\tilde m, \tilde m, \theta) &= \int_{\tilde M} K_{S^{1/2}}(\tilde m, \tilde m', \theta) K_{S^{1/2}}(\tilde m', \tilde m, \theta)\, d\tilde m\\ &= \int_{\tilde M} K_{S^{1/2}}(\tilde m, \tilde m', \theta) K_{S^{1/2}}(\tilde m, \tilde m', \theta)^*\, d\tilde m\end{aligned}$$

The last equality employs the fact that $S^{1/2}$ is selfadjoint. Taking pointwise traces we get:

$$\begin{aligned}\operatorname{tr} K_S(\tilde m,\tilde m,\theta) &= \int_{\tilde M} \operatorname{tr}\left(K_{S^{1/2}}(\tilde m,\tilde m',\theta)K_{S^{1/2}}(\tilde m,\tilde m',\theta)^*\right)\, d\tilde m\\ &= \int_{\tilde M} \operatorname{tr}\left(K_{S^{1/2}}(\tilde m,\tilde m',\theta)^*K_{S^{1/2}}(\tilde m,\tilde m',\theta)\right)\, d\tilde m\,.\end{aligned}$$

Therefore

$$\tau^\nu(S)=\|S^{1/2}\|^2_{\mathrm{HS}}=\int_{F\times T}\operatorname{tr} K_S(\tilde m,\tilde m,\theta)d\tilde m d\nu(\theta)\,.$$

This finishes the proof of the first item.

Regarding the second item, assume now that S is τ^ν trace class i.e. $\tau^\nu(|S|)$ is finite. Write $S=U|S|$ for the polar decomposition of S in $W^*_\nu(G,E)$. Then the operators $|S|^{1/2}$ and $U|S|^{1/2}$ are τ^ν Hilbert-Schmidt and thus have L^2 Schwartz kernels $K_{|S|^{1/2}}$ and $K_{U|S|^{1/2}}$. Using Lemma 3.13 and the polarization identity we deduce:

$$\begin{aligned}&<U|S|^{1/2},|S|^{1/2}>_{\mathrm{HS}}\\ =&\int_{\tilde M\times F\times T}\operatorname{tr} K_{U|S|^{1/2}}(\tilde m,\tilde m',\theta)K_{|S|^{1/2}}(\tilde m,\tilde m',\theta)^*\, d\tilde m\, d\tilde m'\, d\nu(\theta)\\ =&\int_{\tilde M\times F\times T}\operatorname{tr} K_{U|S|^{1/2}}(\tilde m,\tilde m',\theta)K_{|S|^{1/2}}(\tilde m',\tilde m,\theta)\, d\tilde m\, d\tilde m'\, d\nu(\theta)\\ =&\int_{F\times T}\operatorname{tr} K_S(\tilde m,\tilde m,\theta)\, d\tilde m\, d\nu(\theta)\,.\end{aligned}$$

Hence

$$\tau^\nu(S)=<U|S|^{1/2},|S|^{1/2}>_{\mathrm{HS}}=\int_{F\times T}\operatorname{tr} K_S(\tilde m,\tilde m,\theta)\, d\tilde m\, d\nu(\theta)\,.$$

The proof is complete. □

Remark 3.16. — *A proof similar to the one given above shows, as in* **[1]** *(Proposition 4.16), that if $R=(R_\theta)_{\theta\in T}$ has a continuous (or even Borel bounded) leafwise smooth Schwartz kernel with Γ-compact support, then R is τ^ν trace class with $\tau^\nu(R)=\int_{F\times T}\operatorname{tr} K_R(\tilde m,\tilde m,\theta)d\tilde m\, d\nu(\theta)$.*
A similar statement holds for a leafwise operator in $W^*_\nu(V,\mathcal{F};E)$ with a Borel bounded leafwise smooth Schwartz kernel which is supported within a uniform C-neighbourhood, $C\in\mathbb{R}, C>0$, of the diagonal of every leaf.

Proposition 3.17. — *Let $\psi:\mathbb{R}\to\mathbb{C}$ be a measurable rapidly decreasing function. The the operator $\psi(\tilde D):=(\psi(\tilde D_\theta))_{\theta\in T}$ satisfies the assumption of Proposition 3.15 (second item). In particular $\psi(\tilde D)$ has a bounded fiberwise-smooth Schwartz kernel K_ψ and we have*

$$\tau^\nu(\psi(\tilde D))=\int_{F\times T}\operatorname{tr}(K_\psi[\tilde m,\tilde m,\theta])d\tilde m\ d\nu(\theta).$$

Proof. — Using [**38**], Theorem 7.36 (which is in fact valid for any measurable rapidly decreasing function) we know that K_ψ is bounded and fiberwise smooth and that $\psi(\tilde{D}) \in W^*_\nu(G,E)$. Therefore it remains to show that $\psi(\tilde{D})$ is τ^ν trace class since then we can simply apply Proposition 3.15 (second item). But $|\psi(\tilde{D})| = |\psi|(\tilde{D})|$ and $|\psi|$ is a measurable rapidly decreasing function; therefore $|\psi(\tilde{D})|$ has a bounded fiberwise smooth Schwartz and thus satisfies the assumptions of Proposition 3.15 (first item). We conclude that $\psi(\tilde{D})$ is τ^ν trace class. □

A statement similar to the one just proved holds for the leafwise Dirac-type operator $D := (D_L)_{L\in V/\mathcal{G}}$. In order to keep this paper to a reasonable size we state the corresponding proposition without proof. See [**52**].

Proposition 3.18. — *Let $\psi : \mathbb{R} \to \mathbb{C}$ be a measurable rapidly decreasing function. Then the operator $\psi(D) := (\psi(D_L))_{L\in V/\mathcal{G}}$ is $\tau^\nu_{\mathcal{G}}$ trace class, has a leafwise smooth Schwartz kernel K_ψ which is bounded as a measurable section over the equivalence relation $\cup_{L\in V/\mathcal{G}} L\times L$,and we have*

$$\tau^\nu_{\mathcal{G}}(\psi(D)) = \int_{F\times T} \operatorname{tr}(K_\psi([\tilde{m},\theta],[\tilde{m},\theta]))d\tilde{m}\; d\nu(\theta)$$

where now $F\times T$ is viewed as a subset in V.

We are now in position to prove the main results of this section.

Theorem 3.19. — *Let for simplicity $\psi : \mathbb{R} \to \mathbb{C}$ be a Schwartz class function. Then $\psi(\mathcal{D}_m) \in \mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m)$ and the element $\chi_m^{-1}(\psi(\mathcal{D}_m)) \in \mathcal{B}^E_m$ admits a finite τ^ν_{av} trace and also a finite τ^ν_{reg} trace. Moreover*

- *$\tau^\nu_{av}(\chi_m^{-1}(\psi(\mathcal{D}_m)) = \tau^\nu_{\mathcal{G}}\left[(\psi(D_L))_{L\in V/F}\right]$ where $(\psi(D_L))_{L\in V/F}$ is the corresponding element in the leafwise von Neumann algebra $W^*_\nu(V,\mathcal{G};E)$ and $\tau^\nu_{\mathcal{G}}$ is the trace on this von Neumann algebra as defined in Subsection 2.4.*
- *$\tau^\nu_{reg}(\chi_m^{-1}(\psi(\mathcal{D}_m)) = \tau^\nu\left[(\psi(\tilde{D}_\theta))_{\theta\in T}\right]$ where $(\psi(\tilde{D}_\theta))_{\theta\in T}$ is the corresponding element in the regular von Neumann algebra $W^*_\nu(G,E)$ and τ^ν is the trace on this von Neumann algebra as defined in Subsection 2.4.*

Proof. — We know from Corollary 2.6 that $\tau^\nu_{av} = \tau^\nu_{\mathcal{G}}\circ\pi^{av}$. Therefore

$$\begin{aligned}\tau^\nu_{av}(\chi_m^{-1}(|\psi(\mathcal{D}_m)|)) &= \tau^\nu_{\mathcal{G}}\left[(\pi^{av}\circ\chi_m^{-1})(|\psi(\mathcal{D}_m)|)\right]\\ &= \tau^\nu_{\mathcal{G}}\left((\Phi_\theta\circ\left[|\psi|(\mathcal{D}_m)\otimes_{\pi^{av}_\theta} I\right]\circ\Phi_\theta^{-1})_{\theta\in T}\right).\end{aligned}$$

The last equality is a consequence of Proposition 3.3. Now, using Proposition 3.12, we finally deduce

$$\tau^\nu_{av}(\chi_m^{-1}(|\psi(\mathcal{D}_m)|)) = \tau^\nu_{\mathcal{G}}\left(((|\psi|(D_L))_{L\in V/F}\right) < +\infty.$$

Hence we see from Proposition 3.1 that $\chi_m^{-1}(|\psi(\mathscr{D}_m)|)$ is trace class and the same computation with ψ instead of $|\psi|$ finishes the proof of the first item. The second item is proved repeating the same argument. $\square$

4. Index theory

Let $\tilde{M}$, Γ, and T be as in the previous sections and let $(V, \mathscr{F})$, with $V = \tilde{M} \times_\Gamma T$, the associated foliated bundle. We assume in this section only that the manifold M is even dimensional and hence that the leaves of our foliation are even dimensional. Let E be a continuous longitudinally smooth hermitian vector bundle on V and let $\widehat{E}$ be its lift to $\tilde{M} \times T$. Let $\tilde{D} = (\tilde{D}_\theta)_{\theta \in T}$ be as in the previous section a Γ-equivariant continuous family of Dirac-type operators. The bundle E is $\mathbb{Z}_2$-graded, $E = E^+ \oplus E^-$, and the operator $\tilde{D}$ is odd and essentially self-adjoint, i.e.

$$\tilde{D}_\theta = \begin{pmatrix} 0 & \tilde{D}_\theta^- \\ \tilde{D}_\theta^+ & 0 \end{pmatrix} \quad \forall \theta \in T$$

and $(\tilde{D}_\theta^-)^* = \tilde{D}_\theta^+$. Let $D := (D_L)_{L \in V/F}$ be the longitudinal operator induced by $\tilde{D}$ on the leaves of the foliation $(V, \mathscr{F})$.

4.1. The numeric index. — We consider for each θ the orthogonal projection $\tilde{\Pi}_\theta^\pm$ onto the L^2-null space of the operator $\tilde{D}_\theta^\pm$. Similarly, on each leaf L, we consider the orthogonal projections $\Pi_L^\pm$ onto the L^2-null space of the operator D_L. It is well known that these orthogonal projections are smoothing operators, but of course are not localized in a compact neighborhood of the unit space V, viewed as a subspace of the graph of the foliation equivalence relation.

Proposition 4.1

- *The family $\tilde{\Pi}^\pm := (\tilde{\Pi}_\theta^\pm)_{\theta \in T}$ belongs to the regular von Neumann algebra $W_\nu^*(G, E^\pm)$. Moreover it is a τ^ν trace class operator.*
- *The family $\Pi^\pm := (\Pi_L^\pm)_{L \in V/F}$ belongs to the leafwise von Neumann algebra $W_\nu^*(V, \mathscr{F}; E^\pm)$. Moreover it is a $\tau_{\mathscr{F}}^\nu$ trace class operator.*

Proof. — As we have already mentioned, for any Borel bounded function $f : \mathbb{R} \to \mathbb{C}$, the operator $f(\tilde{D})$ (respectively $f(D)$) belongs to the von Neumann algebra $W_\nu^*(G, E)$ (to the von Neumann algebra $W_\nu^*(V, \mathscr{F}; E)$). Hence, $\tilde{\Pi}^\pm$ belongs to $W_\nu^*(G, E^\pm)$ and $\Pi^\pm$ belongs to $W_\nu^*(V, \mathscr{F}; E^\pm)$.

Recall on the other hand from Propositions 3.17, 3.18 that $e^{-\tilde{D}^2}$ is τ^ν trace class and that e^{-D^2} is $\tau_{\mathscr{F}}^\nu$ trace class. Hence the proof is complete since

$$\tilde{\Pi} = \tilde{\Pi} e^{-\tilde{D}^2} \quad \text{and} \quad \Pi = \Pi e^{-D^2}. \qquad \square$$

Definition 4.2. — *We define the monodromy index of $\tilde{D}$ as*

$$\mathrm{ind}^{\nu}_{\mathrm{up}}(\tilde{D}) := \tau^{\nu}(\tilde{\Pi}^{+}) - \tau^{\nu}(\tilde{\Pi}^{-}) \tag{11}$$

We define the leafwise index of D as

$$\mathrm{ind}^{\nu}_{\mathrm{down}}(D) := \tau^{\nu}_{\mathcal{G}}(\Pi^{+}) - \tau^{\nu}_{\mathcal{G}}(\Pi^{-}). \tag{12}$$

As $\tilde{D}^{+}$ is elliptic, we can find a Γ-equivariant family of parametrices $\tilde{Q} := (\tilde{Q}_{\theta})_{\theta\in T}$ of Γ-compact support with remainders $\tilde{R}_{+}$ and $\tilde{R}_{-}$; the remainder families are Γ-equivariant, smoothing and of Γ-compact support, i.e.

$$\tilde{R}_{+} = I - \tilde{Q}\tilde{D}^{+} \quad \text{and} \quad \tilde{R}_{-} = I - \tilde{D}^{+}\tilde{Q}\,; \quad \tilde{R}_{\pm} \in \Psi^{-\infty}_{c}(G, E^{\pm})\,.$$

We know that $\tilde{R}_{\pm}$ are both τ^{ν} trace class. Let Q, R_{+}, R_{-} be the longitudinal operators induced on $(V,\mathcal{F})$; thus $Q, R_{+} \in W^{*}_{\nu}(V,\mathcal{F};E^{+})$ and $R_{-} \in W^{*}_{\nu}(V,\mathcal{F};E^{-})$ with $R_{\pm}$ $\tau^{\nu}_{\mathcal{G}}$ trace class, see Remark 3.16.

Proposition 4.3. — *For any $N\in\mathbb{N}, N\geq 1$, the following formulas hold:*

$$\mathrm{ind}^{\nu}_{\mathrm{up}}(\tilde{D}) = \tau^{\nu}(\tilde{R}_{+})^{N} - \tau^{\nu}(\tilde{R}_{-})^{N}\,, \qquad \mathrm{ind}^{\nu}_{\mathrm{down}}(D) = \tau^{\nu}_{\mathcal{G}}(R_{+})^{N} - \tau^{\nu}_{\mathcal{G}}(R_{-})^{N}. \tag{13}$$

Proof. — Let $N=1$; then the proof given by Atiyah in [**1**] extends easily to the present context. Replacing the parametrix $\tilde{Q}$ by $\tilde{Q}_{N} := \tilde{Q}(1+\tilde{R}_{-}+\tilde{R}^{2}_{-}+\cdots+\tilde{R}^{N-1}_{-}$, which is again a parametrix, reduces the general case to the one treated by Atiyah. □

Using these formulas we shall now sketch the proof of the precise analogue of Atiyah's index theorem on coverings.

Proposition 4.4. — *The monodromy index and the leafwise index coincide:*

$$\mathrm{ind}^{\nu}_{\mathrm{up}}(\tilde{D}) = \mathrm{ind}^{\nu}_{\mathrm{down}}(D). \tag{14}$$

Proof. — Given $\epsilon>0$ we can choose a parametrix $\tilde{Q}\in\Psi^{-1}_{c}(G;\widehat{E}^{-},\widehat{E}^{+})$ with the property that the two remainders $\tilde{R}_{\pm} = (\tilde{R}_{\pm})_{\theta}$, $\theta\in T$ are such that each $(\tilde{R}_{\pm})_{\theta}$ is supported within an ϵ-neighbourhood of the diagonal in $\tilde{M}_{\theta}\times\tilde{M}_{\theta}$. Let $\mathcal{R}_{\pm,m} : \mathcal{E}^{\pm}_{m} \to \mathcal{E}^{\mp}_{m}$ be the induced operators on the $\mathcal{A}_{m}$-Hilbert modules $\mathcal{E}^{\pm}_{m}$; since $\tilde{R}_{\pm}$ are smoothing and of Γ-compact support we certainly know that $\mathcal{R}_{\pm,m}$ are $\mathcal{A}_{m}$-compact operators. Let $K^{\pm} := \chi^{-1}_{m}(\mathcal{R}_{\pm,m}) \in \mathcal{B}^{E^{\pm}}_{m}$; $K^{\pm}$ is simply given by the Schwartz kernel of $\tilde{R}_{\pm}$ and is in fact an element in $\mathcal{B}^{E^{\pm}}_{c}$. In particular $K^{\pm}$ has finite $\tau^{\nu}_{\mathrm{reg}}$ trace and τ^{ν}_{av} trace. By arguments very similar (in fact easier) to those establishing Theorem 3.19 we know that

$$\tau^{\nu}(\tilde{R}_{\pm}) = \tau^{\nu}_{\mathrm{reg}}(\chi^{-1}_{m}\mathcal{R}_{\pm,m}) \equiv \tau^{\nu}_{\mathrm{reg}}(K^{\pm})\,, \quad \tau^{\nu}_{\mathcal{G}}(R_{\pm}) = \tau^{\nu}_{\mathrm{av}}(\chi^{-1}_{m}\mathcal{R}_{\pm,m}) \equiv \tau^{\nu}_{\mathrm{av}}(K^{\pm})\,. \tag{15}$$

Thus, from (13), it suffices to show that

$$\tau^{\nu}_{\mathrm{reg}}(K^{\pm}) = \tau^{\nu}_{\mathrm{av}}(K^{\pm})\,.$$

We can write

$$\begin{aligned}\tau^\nu_{\mathrm{av}}(K^\pm) &= \int_{F\times T}\sum_{\gamma\in\Gamma(\theta)} K^\pm[\tilde m,\tilde m\gamma,\theta]\,d\tilde m\,d\nu(\theta)\\ &= \int_{F\times T} K^\pm[\tilde m,\tilde m,\theta]\,d\tilde m\,d\nu(\theta)+\int_{F\times T}\sum_{\gamma\in\Gamma(\theta);\gamma\neq e} K^\pm[\tilde m,\tilde m\gamma,\theta]\,d\tilde m\,d\nu(\theta)\\ &= \tau^\nu_{\mathrm{reg}}(K^\pm)+\int_{F\times T}\sum_{\gamma\in\Gamma(\theta);\gamma\neq e} K^\pm[\tilde m,\tilde m\gamma,\theta]\,d\tilde m\,d\nu(\theta).\end{aligned}$$

Choosing ϵ small enough we can ensure that $K^\pm[\tilde m,\tilde m\gamma,\theta]=0\ \forall\gamma\in\Gamma(\theta)$, $\gamma\neq e$. The proof is complete. □

Remark 4.5. — *The possibility of localizing a parametrix in an arbitrary small neighbourhood of the diagonal plays a crucial role in the proof of the above proposition. There are more general situations, for example foliated flat bundles $\tilde M\times_\Gamma T$ with $\tilde M$ a manifold with boundary, where it is not possible to localize the parametrix. In these cases the analogue of Atiyah's index theorem does* **not** *hold.*

4.2. The index class in the maximal C^*-algebra. — Let $\tilde D^+$ be as in the previous subsection. As before we consider a parametrix $\tilde Q:=(\tilde Q_\theta)_{\theta\in T}\in\Psi_c^{-1}(G;\widehat{E}^-,\widehat{E}^+)$ with remainders $\tilde R_+$ and $\tilde R_-$. The family $\tilde Q$ defines a bounded $\mathcal{A}_m$-linear operator $\mathcal{Q}_m$ from $\mathcal{E}_m^-$ to $\mathcal{E}_m^+$. The families $\tilde R_+$ and $\tilde R_-$ define $\mathcal{A}_m$-linear compact operators $\mathcal{R}_{\pm,m}$ on the Hilbert modules $\mathcal{E}_m^\pm$ respectively.

We now define idempotents p,p_0 in $M_{2\times 2}(\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m)\oplus\mathbb{C})$ by setting

$$p:=\begin{pmatrix}\mathcal{R}^2_{+,m} & \mathcal{R}_{+,m}(I+\mathcal{R}_{+,m})\mathcal{Q}_m\\ \mathcal{R}_{-,m}\mathcal{D}^+_m & I-\mathcal{R}^2_{-,m}\end{pmatrix},\quad p_0:=\begin{pmatrix}0&0\\0&I\end{pmatrix}. \tag{16}$$

We thus get a K_0-class $[p-p_0]\in K_0(\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m))$.

Definition 4.6. — *The (maximal) index class* $\mathrm{IND}(\mathcal{D}_m)\in K_0(\mathcal{B}_m)$ *associated to the family $\tilde D$ is, by definition, the image under the composite isomorphism*

$$K_0(\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m))\to K_0(\mathcal{B}^E_m)\to K_0(\mathcal{B}_m)$$

of the class $[p-p_0]$.

One also considers the index class in $K_0(\mathcal{A}_m)$:

$$\mathrm{Ind}(\mathcal{D}_m):=\mathcal{M}^{-1}_{\max}(\mathrm{IND}(\mathcal{D}_m))\in K_0(\mathcal{A}_m) \tag{17}$$

with $\mathcal{M}_{\max}:K_0(\mathcal{A}_m)\to K_0(\mathcal{B}_m)$ the Morita isomorphism considered in Proposition 2.10.

Recall now the morphisms $\tau^{\nu}_{\mathrm{av},*} : K_0(\mathcal{B}_m) \to \mathbb{C}$ and $\tau^{\nu}_{\mathrm{reg},*} : K_0(\mathcal{B}_r) \to \mathbb{C}$. Using the natural morphism $K_0(\mathcal{B}_m) \to K_0(\mathcal{B}_r)$ we view both morphisms with domain $K_0(\mathcal{B}_m)$:

$$\tau^{\nu}_{\mathrm{av},*} : K_0(\mathcal{B}_m) \to \mathbb{C}\,, \qquad \tau^{\nu}_{\mathrm{reg},*} : K_0(\mathcal{B}_m) \to \mathbb{C}. \tag{18}$$

Recall also that using the natural morphism $K_0(\mathcal{A}_m) \to K_0(\mathcal{A}_r)$ we have induced morphisms

$$\tau^{\nu}_{\mathrm{av},*} : K_0(\mathcal{A}_m) \to \mathbb{C}\,, \qquad \tau^{\nu}_{\mathrm{reg},*} : K_0(\mathcal{A}_m) \to \mathbb{C}. \tag{19}$$

Proposition 4.7. — *Let* $\mathrm{IND}(\mathcal{D}_m) \in K_0(\mathcal{B}_m)$ *and* $\mathrm{Ind}(\mathcal{D}_m) \in K_0(\mathcal{A}_m)$ *be the two index classes introduced above. Then the following formulas hold:*

$$\mathrm{ind}^{\nu}_{\mathrm{up}}(\tilde{D}) = \tau^{\nu}_{\mathrm{reg},*}(\mathrm{IND}(\mathcal{D}_m)) = \tau^{\nu}_{\mathrm{reg},*}(\mathrm{Ind}(\mathcal{D}_m)), \tag{20}$$

$$\mathrm{ind}^{\nu}_{\mathrm{down}}(D) = \tau^{\nu}_{\mathrm{av},*}(\mathrm{IND}(\mathcal{D}_m)) = \tau^{\nu}_{\mathrm{av},*}(\mathrm{Ind}(\mathcal{D}_m)). \tag{21}$$

Consequently, from (14), *we have the following fundamental equality:*

$$\tau^{\nu}_{\mathrm{reg},*}(\mathrm{Ind}(\mathcal{D}_m)) = \tau^{\nu}_{\mathrm{av},*}(\mathrm{Ind}(\mathcal{D}_m)). \tag{22}$$

Proof. — We only need to prove the first equality in each equation, for the second one is a consequence of the definition of $\mathrm{Ind}(\mathcal{D}_m) \in K_0(\mathcal{A}_m)$ and the compatibilty result explained in Proposition 2.10. For the first equality we apply (13) with $N = 2$ and the parametrix $\tilde{Q}$. Using now (15), (16) we get

$$\begin{aligned} \mathrm{ind}^{\nu}_{\mathrm{up}}(\tilde{D}) &= \tau^{\nu}((\tilde{R}_+)^2) - \tau^{\nu}((\tilde{R}_-)^2) \\ &= \tau^{\nu}_{\mathrm{reg}}((\mathcal{R}_{+,m})^2) - \tau^{\nu}_{\mathrm{reg}}((\mathcal{R}_{-,m})^2) \\ &= \tau^{\nu}_{\mathrm{reg},*}(\mathrm{IND}(\mathcal{D}_m)). \end{aligned}$$

The proof of the other one is similar. □

Remark 4.8. — *The equalities in Proposition 4.7 can be rephrased as the equality between the numeric C^*-algebraic index and the von Neumann index. Notice once again that there are more general situations where this proposition does not hold, in the sense that there exists a well defined von Neumann index but there does not exist a well-defined C^*-algebraic index we can equate it to. The simplest example is given by a fibration of compact manifolds $L \to V \to T$ with V and L manifolds with boundary. The von Neumann index defined by the family of Atiyah-Patodi-Singer boundary conditions is certainly well defined (this is the integral over T of the function that assigns to $\theta \in T$ the APS index of D^+_θ). On the other hand, unless the boundary family associated to $(D^+_\theta)_{\theta\in T}$ is invertible, there is not a well defined Atiyah-Patodi-Singer index class in $K_0(C(T)) = K^0(T)$. For more on higher Atiyah-Patodi-Singer index theory on foliated bundles see* [**34**], [**33**].

4.3. The signature operator for odd foliations. — We briefly review the definition of the leafwise signature operator in the odd case. Recall that when $\dim(M) = 2m-1$, the leafwise signature operator is defined as the operator D^{sign} acting on leafwise differential forms on V, defined on even forms of degree $2k$ by

$$D^{\text{sign}}_{ev} = i^m(-1)^{k+1}(* \circ d - d \circ *),$$

and on odd forms of degree $2k-1$ by

$$D^{\text{sign}}_{od} = i^m(-1)^{m-k}(d \circ * + * \circ d),$$

where d is the leafwise de Rham differential and $*$ is the usual Hodge operator along the leaves associated with the Riemannian metric on the foliation [**38**]. An easy computation shows that the two operators D^{sign}_{odd} and D^{sign}_{ev} are conjugate so that their invariants coincide and it is sufficient to work with one of them. In contrast with [**3**], D^{sign} will be in the sequel the operator D^{sign}_{od}. Using the lifted structure to the fibers of the monodromy covers $\tilde{M} \times \{\theta\}$ of the leaves, we consider in the same way the Γ-equivariant family of signature operators $\tilde{D}^{\text{sign}} = (\tilde{D}^{\text{sign}}_\theta)_{\theta\in T}$ which actually coincides with the lift of D^{sign} as can be easily checked. The following is well known, see [**2**], [**3**] for the first part and [**28**] for the second:

Recall that the K_1 index of D^{sign} is the class of the Cayley transform of D^{sign}, see for instance [**28**].

Proposition 4.9. — *The operator D^{sign} is a leafwise elliptic essentially self-adjoint operator whose K_1 index class is a leafwise homotopy invariant of the foliation.*

The square of D^{sign} is proportional to the Laplace operator along the leaves and hence it is leafwise elliptic. The proof that D^{sign}_{ev} is formally self-adjoint is straightforward, see [**3**], and classical elliptic theory on foliations of compact spaces allows to deduce that it is essentially self-adjoint. Now D^{sign} is unitarily equivalent to D^{sign}_{ev} and hence is also formally self-adjoint. We shall get back to the index class later on. The homotopy invariance means that if $f : (V, \mathcal{F}) \to (V', \mathcal{F}')$ is a leafwise oriented leafwise homotopy equivalence between odd dimensional foliations, then with obvious notations we have

$$f_* \operatorname{Ind}(D^{\text{sign}}) = \operatorname{Ind}(D^{\text{sign}'})$$

where f_* is the isomorphism induced by the Morita equivalence implemented by f [**28**].

5. Foliated rho invariants

Recall that T is a compact Hausdorff space on which the discrete countable group Γ acts by homeomorphisms, M is a compact closed manifold with fundamental group Γ and universal cover $\tilde{M}$ and that $V = \tilde{M} \times_\Gamma T$ is the induced foliated space. We are also given a Borel measure ν on T which is Γ-invariant. We assume in the present section that M is odd dimensional and whence that the leaves of the induced foliation $\mathcal{F}$ of V are odd dimensional. We fix as in the previous section a Dirac-type operator along the leaves of the foliation $(V, \mathcal{F})$ acting on the vector bundle E. We denote by D this operator acting leafwise, so $D = (D_L)_{L\in V/F}$ where each D_L is an elliptic Dirac-type operator on the leaf L acting on the restriction of E to L. We also consider the lifted operator $\tilde{D}$ to the monodromy groupoid G of the foliation (V, F) as defined in Section 3.2. More precisely, $\tilde{D} = (\tilde{D}_\theta)_{\theta\in T}$ is a Γ-equivariant continuous family of Dirac type operators on $\tilde{M}$.

5.1. Foliated eta and rho invariants. — The construction of foliated eta invariants was first given independently in the two references [**47**] [**43**] and the two definitions work in fact for general measured foliations. Notice that [**47**] works with the measurable groupoid defined by foliation, whereas [**43**] works with the holonomy groupoid. As we shall clarify in a moment, the choice of the groupoid does make a difference for these non-local invariants. We give in this paragraph a self-contained treatment of these two definitions following our set-up, but using the monodromy groupoid instead of the holonomy groupoid considered in [**43**].

We denote by k_t and $\tilde{k}_t$ the longitudinally smooth uniformly bounded Schwartz kernels of the operators $\varphi_t(D)$ and $\varphi_t(\tilde{D})$ obtained using the function $\varphi_t(x) := xe^{-t^2x^2}$ for $t > 0$. See Lemma 3.17.

Lemma 5.1. — *(Bismut-Freed estimate) There exists a constant $C \geq 0$ such that for any $(\tilde{m}, \theta) \in \tilde{M} \times T$, we have:*

$$|\operatorname{tr}(k_t([\tilde{m}, \theta], [\tilde{m}, \theta]))| \leq C \quad \textit{and} \quad |\operatorname{tr}(\tilde{k}_t([\tilde{m}, \tilde{m}, \theta]))| \leq C, \ \textit{for } t \leq 1.$$

Proof. — A proof of these estimates appear already in [**47**]. We give nevertheless a sketch of the argument.
The Bismut-Freed estimate on a closed odd dimensional compact manifold M is a pointwise estimate on the vector-bundle trace of the Schwartz kernel of $D\exp(-t^2D^2)$ restricted to the diagonal. See the original article [**14**] but also [**37**]. As explained for example in the latter reference the Bismut-Freed estimate is ultimately a consequence of Getzler rescaling for the heat kernel of a Dirac laplacian on the even dimensional manifold obtained by crossing M with S^1. Since these arguments are purely local, they easily extend to our foliated case, using the compactness of $V := \tilde{M} \times_\Gamma T$ in order to control uniformly the constants appearing in the poinwise estimate. □

The operators D^2 and $\tilde{D}^2$ (as well as the operators $|D|$ and $|\tilde{D}|$) are non negative operators which are affiliated respectively with the von Neumann algebra $W^*_\nu(V, \mathcal{F}; E)$ and the von Neumann algebra $W^*_\nu(G; E)$. (This means that their sign operators as well as their spectral projections belong to the von Neumann algebra.) Moreover, according to the usual pseudodifferential estimates along the leaves (see for instance **[55]**, **[8]**), the resolvents of these operators belong respectively to the C^*-algebras $\mathcal{K}(W^*_\nu(V,\mathcal{F};E),\tau^\nu_{\mathcal{F}})$ of $\tau^\nu_{\mathcal{F}}$-compact elements in $W^*_\nu(V,\mathcal{F};E)$ and $\mathcal{K}(W^*_\nu(G;E),\tau^\nu)$ of τ^ν-compact elements of $W^*_\nu(G;E)$. We recall that these compact operators are roughly defined using for instance the vanishing at infinity of the singular numbers, and we refer, for example, to **[8]** for the precise definition of these ideals. Set

$$D^2 = \int_0^{+\infty} \lambda dE_\lambda \quad \text{and} \quad \tilde{D}^2 = \int_0^{+\infty} \lambda d\tilde{E}_\lambda,$$

for the spectral decompositions in their respective von Neumann algebras. So E_λ and $\tilde{E}_\lambda$ are the spectral projections corresponding to $(-\infty,\lambda)$. Since the traces are normal on both von Neumann algebras,

$$N(\lambda) = \tau^\nu_{\mathrm{av}}(E_\lambda) \quad \text{and} \quad \tilde{N}(\lambda) = \tau^\nu_{\mathrm{reg}}(\tilde{E}_\lambda),$$

are well defined finite (Proposition 5.6 in the next subsection) non-decreasing and non-negative functions, which are right continuous. Hence there are Borel-Stieljes measures ϑ and $\tilde{\vartheta}$ on $\mathbb{R}$, such that:

$$\tau^\nu_{\mathcal{F}}(f(D)) = \int_{\mathbb{R}} f(x)d\vartheta(x) \text{ and } \tau^\nu(f(\tilde{D})) = \int_{\mathbb{R}} f(x)d\tilde{\vartheta}(x),$$

for any Borel function $f:\mathbb{R}\to[0,+\infty]$. Since N and $\tilde{N}$ are finite, the measures ϑ and $\tilde{\vartheta}$ are easily proved to be σ-finite.

Proposition 5.2. — *The functions* $t\mapsto \tau^\nu_{\mathcal{F}}(De^{-t^2D^2})$ *and* $t\mapsto\tau^\nu(\tilde{D}e^{-t^2\tilde{D}^2})$ *are Lebesgue integrable on* $(0,+\infty)$.

Proof. — We have

$$|\tau^\nu_{\mathcal{F}}(De^{-t^2D^2})| \le \tau^\nu_{\mathcal{F}}(|D|e^{-t^2D^2}) \quad \text{and} \quad |\tau^\nu(\tilde{D}e^{-t^2\tilde{D}^2})| \le \tau^\nu(|\tilde{D}|e^{-t^2\tilde{D}^2}).$$

Therefore and since the function $x\mapsto |x|e^{-t^2x^2}$ is rapidly decreasing, we know from Propositions 3.17 and 3.18 that for any $t>0$

$$\tau^\nu_{\mathcal{F}}(|D|e^{-t^2D^2}) < +\infty \text{ and } \tau^\nu(|\tilde{D}|e^{-t^2\tilde{D}^2}) < +\infty.$$

We also have the formulae

$$\tau^\nu_{\mathcal{F}}(|D|e^{-t^2D^2}) = \int_{\mathbb{R}_+} \sqrt{x}e^{-t^2x}d\vartheta(x) \quad \text{and} \quad \tau^\nu(|\tilde{D}|e^{-t^2\tilde{D}^2}) = \int_{\mathbb{R}_+} \sqrt{x}e^{-t^2x}d\tilde{\vartheta}(x).$$

Therefore, by Tonelli's theorem

$$\begin{aligned}\int_1^{+\infty} \tau_{\mathcal{G}}^{\nu}(|D|e^{-t^2D^2})dt &= \int_0^{\infty}\sqrt{x}\int_1^{\infty} e^{-t^2x}\,dt\,d\vartheta(x)\\ &= \int_0^{\infty}\sqrt{x}e^{-x}\int_1^{\infty} e^{-(t^2-1)x}\,dt\,d\vartheta(x)\\ &= \frac{1}{2}\int_0^{\infty}\sqrt{x}e^{-x}\int_0^{\infty} x^{-1/2}(u+x)^{-1/2}e^{-u}\,du\,d\vartheta(x)\\ &\leq \frac{1}{2}\Big(\int_0^{\infty} e^{-x}d\vartheta(x)\Big)\Big(\int_0^{\infty} u^{-1/2}e^{-u}du\Big)\\ &= \frac{\sqrt{\pi}}{2}\tau_{\mathcal{G}}^{\nu}(e^{-D^2}).\end{aligned}$$

The same proof show that

$$\int_1^{+\infty}\tau^{\nu}(|\tilde{D}|e^{-t^2\tilde{D}^2})dt < +\infty.$$

On the other hand, we have

$$\begin{aligned}\int_0^1 |\tau_{\mathcal{G}}^{\nu}(De^{-t^2D^2})|dt &\leq \int_0^1\int_{F\times T} |\operatorname{tr}(k_t([\tilde{m},\theta],[\tilde{m},\theta])|\,d\tilde{m}\,d\nu(\theta)\,dt\\ &\leq \int_0^1\int_{F\times T} C\,d\tilde{m}\,d\nu(\theta)\,dt\\ &= C\times \operatorname{vol}(V, d\tilde{m}\otimes\nu) < +\infty.\end{aligned}$$

Again, the same proof works as well for the regular trace and the regular von Neumann algebra. □

We are now in position to define the foliated eta invariants.

Definition 5.3. — *We define the up and down eta invariants of our longitudinal Dirac type operator by the formulae*

$$\eta_{\mathrm{up}}^{\nu}(\tilde{D}) := \frac{2}{\sqrt{\pi}}\int_0^{+\infty}\tau^{\nu}(\tilde{D}e^{-t^2\tilde{D}^2})dt \quad \textit{and} \quad \eta_{\mathrm{down}}^{\nu}(D) := \frac{2}{\sqrt{\pi}}\int_0^{+\infty}\tau_{\mathcal{G}}^{\nu}(De^{-t^2D^2})dt.$$

Since the traces on both von Neumann algebras are positive, the two eta invariants are real numbers.

Definition 5.4. — *The foliated rho invariant associated to the longitudinal Dirac type operator D on the foliated flat bundle $(V,\mathcal{F})$ is defined as*

$$\rho^{\nu}(D;V,\mathcal{F}) := \eta_{\mathrm{up}}^{\nu}(\tilde{D}) - \eta_{\mathrm{down}}^{\nu}(D)$$

with $\tilde{D}$ the lift of D to the monodromy cover.

We are mainly interested in the present paper in the leafwise signature operator D^{sign} and its leafwise lift to the monodromy groupoid $\tilde{D}^{\mathrm{sign}}$. In this case, we can state the following convenient result.

Lemma 5.5. — *Denote by Δ_j the Laplace operator on leafwise j-forms. Then the foliated eta invariant of the operator D^{sign} on $(V,\mathcal{F})$ is given by*

$$\eta^\nu(D^{\mathrm{sign}};V,\mathcal{F})=\frac{1}{\sqrt{\pi}}\int_0^{+\infty}\tau^\nu_{\mathcal{G}}(*de^{-t^2\Delta_{m-1}})dt=\frac{1}{\sqrt{\pi}}\int_0^{+\infty}\tau^\nu_{\mathcal{G}}(d*e^{-t^2\Delta_m})dt.$$

Similar statements hold for the lifted family $\tilde{D}^{\mathrm{sign}}$.

Proof. — This is an immediate consequence of a straightforward leafwise version of the computation made in [**2**, p. 67-68]. □

5.2. Eta invariants and determinants of paths. — We review the notion of (log-)determinants of paths, adapting the work of de La Harpe-Skandalis [**24**] to our context. Recall that M is odd dimensional. For any von Neumann algebra $\mathcal{M}$ endowed with a positive semifinite faithful normal trace τ, we denote by $L^1(\mathcal{M},\tau)$ the Schatten space of summable τ-measurable operators in the sense of [**22**]. Recall that $L^1(\mathcal{M},\tau)\cap\mathcal{M}$ is a two sided $*$-ideal in $\mathcal{M}$. By Propositions 3.17, 3.18 we have for any rapidly decreasing Borel function ψ

$$\psi(\tilde{D}):=(\psi(\tilde{D}_\theta))_{\theta\in T}\in L^1(W^*_\nu(G;E),\tau^\nu)\cap W^*_\nu(G;E)$$

$$\psi(D):=(\psi(D_{L_\theta}))_{\theta\in T}\in L^1(W^*_\nu(V,\mathcal{F};E),\tau^\nu_{\mathcal{G}})\cap W^*_\nu(V,\mathcal{F};E).$$

We set $\tilde{D}=\tilde{U}|\tilde{D}|$ and $D=U|D|$ for the polar decompositions in the corresponding von Neumann algebras. Then, this decomposition obviously coincides with the leafwise decompositions

$$\tilde{D}_\theta=\tilde{U}_\theta|\tilde{D}_\theta|\text{ and }D_L=U_L|D_L|.$$

For any $\theta\in T$ with $L=L_\theta$, we write the spectral decompositions:

$$|\tilde{D}_\theta|=\int_0^{+\infty}\lambda d\tilde{E}^\theta_\lambda\text{ and }|D_L|=\int_0^{+\infty}\lambda dE^L_\lambda.$$

As we have already remarked, the collection of partial isometries $\tilde{U}=(\tilde{U}_\theta)_{\theta\in T}$ (resp. $U=(U_{L_\theta})_{\theta\in T}$) as well as that of spectral projections $\tilde{E}_\lambda=(\tilde{E}^\theta_\lambda)_{\theta\in T}$ (resp. $E_\lambda=(E^{L_\theta}_\lambda)_{\theta\in T}$), all belong to $W^*_\nu(G;E)$ (resp. $W^*_\nu(V,\mathcal{F};E)$).

Proposition 5.6. — *We have $\tau^\nu(\tilde{E}_\lambda)<+\infty$ and $\tau^\nu_{\mathcal{G}}(E_\lambda)<+\infty$ for any $\lambda\in\mathbb{R}_+$.*

Proof. — We know that for any $\lambda<0$ the operator $(|\tilde{D}|-\lambda)^{-1}$ is τ^ν-compact in $W^*_\nu(G;E)$. In the same way, the operator $(|D|-\lambda)^{-1}$ is $\tau^\nu_{\mathcal{G}}$-compact in $W^*_\nu(V,\mathcal{F};E)$ [**17**]. Hence the spectral projections of $(|\tilde{D}|-\lambda)^{-1}$ are τ^ν-finite and the spectral projections of $(|D|-\lambda)^{-1}$ are $\tau^\nu_{\mathcal{G}}$-finite. This completes the proof. □

For any $t > 0$, the t-th singular number of the operator $\tilde{D}$ with respect to the probability measure ν is defined by **[22]**

$$\mu_t(\tilde{D}) = \inf\{\||\tilde{D}|\tilde{p}\|, \tilde{p} = \tilde{p}^2 = \tilde{p}^* \in W^*_\nu(G;E) \text{ and } \int_T \mathrm{tr}(M_\chi(I - \tilde{p}_\theta)M_\chi)d\nu(\theta) \le t\}.$$

In the same way, we define

$$\mu_t(D) = \inf\{\||D|p\|, p = p^2 = p^* \in W^*_\nu(V, \mathcal{F}; E) \text{ and } \int_T \mathrm{tr}(M_\chi(I - p_{L_\theta})M_\chi)d\nu(\theta) \le t\}.$$

From Proposition 5.6, we deduce that $0 \le \mu_t(\tilde{D}) = \mu_t(|\tilde{D}|) < +\infty$ and $0 \le \mu_t(D) = \mu_t(|D|) < +\infty$. The spectral measure of $|\tilde{D}|$ with respect to the probability measure ν is denoted $\tilde{\mu}$, while the spectral measure of $|D|$ is denoted μ. So for $\tilde{D}$ for instance we have

$$\mu(B) = \int_T \mathrm{tr}(M_\chi 1_B(|D_{L_\theta}|)M_\chi)d\nu(\theta)$$

for any Borel subset B of the spectrum of $|D|$ and

$$\tilde{\mu}(\tilde{B}) = \int_T \mathrm{tr}(M_\chi 1_{\tilde{B}}(|\tilde{D}_\theta|)M_\chi)d\nu(\theta)$$

for any Borel subset $\tilde{B}$ of the spectrum of $|\tilde{D}|$.

We denote by $\mathcal{IK}_{E,\mathrm{reg}}$ (resp. $\mathcal{IK}_{E,\mathrm{triv}}$) the subgroup of invertible operators in $W^*_\nu(G;E)$ (resp. in $W^*_\nu(V,\mathcal{F};E)$) which differ from the identity by an element of the ideal $\mathcal{K}(W^*_\nu(G;E),\tau^\nu)$ (resp. $\mathcal{K}(W^*_\nu(V,\mathcal{F};E),\tau^\nu_{\mathcal{F}})$). The subgroup of bounded operators which differ from the identity by a τ^ν-summable (resp. $\tau^\nu_{\mathcal{F}}$-summable) operator will be denoted $\mathcal{I}L^1_{E,\mathrm{reg}}$ (resp. $\mathcal{I}L^1_{E,\mathrm{triv}}$).

Whenever possible we shall refer to both von Neumann algebras $W^*_\nu(G;E)$ and $W^*_\nu(V,\mathcal{F};E)$ as $\mathcal{M}$. We shall then use the notation $\mathcal{IK}$ (resp. $\mathcal{I}L^1$) and denote by τ the corresponding trace.

Lemma 5.7. — *The space $\mathcal{I}L^1$ (resp. $\mathcal{IK}$) is a subgroup of the group of invertibles* $\mathrm{GL}(\mathcal{M})$ *of the von Neumann algebra $\mathcal{M}$.*

Proof. — We only need to check the stability for taking inverses. Let then $I + T$ be an invertible element in $\mathcal{M}$ such that $T \in L^1(\mathcal{M},\tau)$ (resp. $\mathcal{K}(\mathcal{M},\tau)$). Then we can write

$$(I+T)^{-1} - I = (I+T)^{-1}(I-(I+T)) = -(I+T)^{-1}T \in L^1(\mathcal{M},\tau) \text{ (resp. } \mathcal{K}(\mathcal{M},\tau)).$$

□

Proposition 5.8. — *Let $\gamma : [0,1] \to \mathcal{IK}$ be a continuous path for the uniform norm. For any $\epsilon > 0$, there exists a continuous piecewise affine path $\gamma_\epsilon : [0,1] \to \mathcal{I}L^1$ such* *that for any $t \in [0,1]$ we have $\|\gamma(t)-\gamma_\epsilon(t)\| \leq \epsilon$. Moreover, if $\gamma(0)$ and $\gamma(1)$ belong to $\mathscr{I}L^1$, then we can insure that $\gamma_\epsilon(i)=\gamma(i)$ for $i=0,1$.*

Proof. — Since γ is continous for the operator norm, we can find $\delta>0$ such that

$$|t-s| \leq \delta \Rightarrow \|\gamma(t)-\gamma(s)\| \leq \epsilon/3.$$

We subdivide $[0,1]$ into $0=x_0<x_1<\cdots<x_n=1$ so that $|x_{j+1}-x_j| \leq \delta$ for any j. On the other hand, the ideal $L^1(\mathcal{M},\tau)\cap\mathcal{M}$ is dense in $\mathcal{K}(\mathcal{M},\tau)$ for the uniform norm. Therefore, for any $j=0,\cdots,n$, we can find $\gamma_\epsilon(x_j)\in B(\gamma(x_j),\epsilon/9)$, the ball centered at $\gamma(x_j)$ with radius $\epsilon/9$, such that $\gamma_\epsilon(x_j)\in\mathscr{I}L^1$. We then define a path $\gamma_\epsilon:[0,1]\to\mathscr{I}L^1$ which is affine on every interval $[x_j,x_{j+1}]$ and prescribed by the values $\gamma_\epsilon(x_j)$ for $j=0,\cdots,n$. The path γ_ϵ is then continuous and differentiable outside the finite set $\{x_j, j=0,\cdots,n\}$. Moreover, for $t\in[x_j,x_{j+1}]$ we have

$$\begin{aligned}\|\gamma_\epsilon(t)-\gamma_\epsilon(x_j)\| &= t\times\|\gamma_\epsilon(x_{j+1})-\gamma_\epsilon(x_j)\| \leq \|\gamma_\epsilon(x_{j+1})-\gamma_\epsilon(x_j)\|\\ &\leq \|\gamma(x_{j+1})-\gamma(x_j)\|+2\epsilon/9 \leq 5\epsilon/9.\end{aligned}$$

Therefore,

$$\|\gamma(t)-\gamma_\epsilon(t)\| \leq \|\gamma(t)-\gamma(x_j)\|+\|\gamma(x_j)-\gamma_\epsilon(x_j)\|+\|\gamma_\epsilon(x_j)-\gamma_\epsilon(t)\| \leq \epsilon/3+\epsilon/9+5\epsilon/9=\epsilon.$$

□

Definition 5.9. — *Given a continuous piecewise C^1 path $\gamma:[0,1]\to\mathscr{I}L^1$ for the L^1-norm in $\mathcal{M}$, we define the determinant $w^\tau(\gamma)$ by the formula*

$$w^\tau(\gamma) := \frac{1}{2\pi\sqrt{-1}}\int_0^1 \tau(\gamma(t)^{-1}\gamma'(t))dt.$$

When $\mathcal{M}$ is $W^*_\nu(G,E)$ this determinant will be denoted by $w^\nu(\gamma)$ while when $\mathcal{M}$ is equal to $W^*_\nu(V,\mathcal{F};E)$ this determinant will be denoted $w^\nu_{\mathcal{F}}(\gamma)$.

We summarize the properties of the determinant in the following

Proposition 5.10. — *Let $\gamma:[0,1]\to\mathscr{I}L^1$ be a continuous piecewise C^1 path for the L^1-norm.*

1. *Assume that*

$$\|\gamma(t)-I\|_1<1, \qquad \text{for any } t\in[0,1].$$

Then, for any $t\in[0,1]$ the operator $\mathrm{Log}(\gamma(t))$ is well defined in the von Neumann algebra and we have

$$w^\tau(\gamma)=\frac{1}{2\pi\sqrt{-1}}\left[\tau(\mathrm{Log}(\gamma(1)))-\tau(\mathrm{Log}(\gamma(0)))\right].$$

2. *There exists $\delta_\gamma > 0$ such that for any continuous piecewise C^1 path $\alpha : [0,1] \to \mathscr{I}L^1$ for the L^1 norm, with*
$$\|\alpha(t) - \gamma(t)\|_1 \leq \delta_\gamma \text{ and } \alpha(i) = \gamma(i), i = 0, 1,$$
we have $w^\tau(\alpha) = w^\tau(\gamma)$.
3. *If γ is a continuous piecewise C^1 path for the uniform norm, then the determinant $w^\tau(\gamma)$ is well defined. Moreover, $w^\tau(\gamma)$ only depends on the homotopy class of γ with fixed endpoints and with respect to the uniform norm.*

Proof. — This proposition is a straightforward extension of the corresponding results in [**24**]. We give a brief outline of the proof here for the benefit of the reader. It is clear in the first item, since τ is a positive trace, that the function $t \mapsto \mathrm{Log}(\gamma(t))$ is well defined (using for instance the series) and is a piecewise smooth path. Moreover, we have
$$\frac{d}{dt}\tau(\mathrm{Log}(\gamma(t)) = \tau(\gamma^{-1}(t)\frac{d\gamma}{dt}(t).$$
This completes the proof of the first item.

Let α be a continuous piecewise C^1 path satisfying the assumptions of the second item. We consider the continuous piecewise C^1 loop $\beta : [0,1] \to \mathscr{I}L^1$ given by $\beta(t) = \gamma(t)^{-1}\alpha(t)$ which satisfies $\beta(0) = \beta(1) = I$. We have
$$\|\beta(t) - I\|_1 \leq \|\gamma(t)^{-1}\| \times \|\gamma(t) - \beta(t)\|_1.$$
Therefore, with $\delta_\gamma = \frac{1}{\inf_{t\in[0,1]} \|\gamma(t)^{-1}\|}$, we are done using the first item.

The rest of the proof is similar and is omitted. □

Definition 5.11. — *Let $\gamma : [0,1] \to \mathscr{I}\mathscr{K}$ be a continous path for the uniform norm such that $\gamma(0)$ and $\gamma(1)$ are in $\mathscr{I}L^1$. We define the determinant $w^\tau(\gamma)$ by $w^\tau(\gamma) := w^\tau(\alpha)$, for any continuous piecewise C^1 path $\alpha : [0,1] \to \mathscr{I}L^1$ such that*
$$\|\alpha(t) - \gamma(t)\|_1 \leq \delta_\gamma \text{ and } \alpha(i) = \gamma(i), i = 0, 1.$$

Remark 5.12. — *It is clear from the previous proposition that the above definition is well posed.*

We now set
$$\varphi(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds, \quad \psi_t(x) := -e^{i\pi\varphi(tx)} \quad \text{and}$$
$$f_t(x) := xe^{-t^2x^2} \quad \text{for } x \in \mathbb{R}, \text{ and any } t \geq 0.$$

Then the function $1 - \psi_t$, the derivative ψ_t' and the function f_t are Schwartz class functions for any $t > 0$. Using the results of the previous sections, we deduce that the operators $I - \psi_t(\mathscr{D}_m)$, $\psi_t{}'(\mathscr{D}_m)$ and $f_t(\mathscr{D}_m)$ are $\mathscr{A}_m$-compact operators on the Hilbert module $\mathscr{E}_m$. Moreover, their images under the representations in the von Neumann algebras $W^*_\nu(G;E)$ and $W^*_\nu(V,\mathcal{F};E)$ are trace class operators. Note also that the operator $\psi_t(\mathcal{D}_m)$ is invertible with inverse given by $-e^{-i\pi\varphi(t\mathcal{D}_m)}$, so $\psi_t(\mathcal{D}_m)$ is a smooth path of invertibles in $\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m)$ whose image under $\pi^{reg}\circ\chi_m^{-1}$ in $W^*_{\rm reg}(G;E)$ is also a smooth path of invertibles in $\mathcal{I}L^1_{E,\rm reg}$. The same result holds for the image under $\pi^{av}\circ\chi_m^{-1}$ in $W^*_\nu(V,\mathcal{F};E)$. We denote by

$$\gamma^{\rm reg}(\mathcal{D}_m)\equiv(\gamma_t^{\rm reg}(\mathcal{D}_m))_{t\geq 0}:=\big((\pi^{reg}\circ\chi_m^{-1})(\psi_t(\mathcal{D}_m))\big)_{t\geq 0}$$

and

$$\gamma^{\rm av}(\mathcal{D}_m)\equiv(\gamma_t^{\rm av}(\mathcal{D}_m))_{t\geq 0}:=\big((\pi^{av}\circ\chi_m^{-1})(\psi_t(\mathcal{D}_m))\big)_{t\geq 0}$$

the resulting smooth paths in the two von Neumann algebras. Using the traces τ^ν and $\tau^\nu_{\mathcal{F}}$, we define

$$w^\nu_{{\rm reg},\epsilon}(\mathcal{D}_m):=w^\nu(\gamma^{{\rm reg},\epsilon}(\mathcal{D}_m))\text{ and }w^\nu_{{\rm av},\epsilon}(\mathcal{D}_m):=w^\nu_{\mathcal{F}}(\gamma^{{\rm av},\epsilon}(\mathcal{D}_m)).$$

with $\gamma^{{\rm reg},\epsilon}(\mathcal{D}_m)$ the path $\big((\pi^{reg}\circ\chi_m^{-1})(\psi_t(\mathcal{D}_m))\big)_{t\geq\epsilon}^{t\leq 1/\epsilon}$ and similarly for $\gamma^{{\rm av},\epsilon}(\mathcal{D}_m)$

Theorem 5.13. — *The following relations hold:*

$$\lim_{\epsilon\to 0}w^\nu_{{\rm reg},\epsilon}(\mathcal{D}_m)=\frac{1}{2}\eta^\nu_{\rm up}(\tilde{D})\quad\textit{and}\quad\lim_{\epsilon\to 0}w^\nu_{{\rm av},\epsilon}(\mathcal{D}_m)=\frac{1}{2}\eta^\nu_{\rm down}(D)$$

and hence

$$2\rho^\nu(D;V,\mathcal{F})=\lim_{\epsilon\to 0}[w^\nu_{{\rm reg},\epsilon}(\mathcal{D}_m)-w^\nu_{{\rm av},\epsilon}(\mathcal{D}_m)].$$

Proof. — We have by definition and by straightforward computation

$$\begin{aligned}\gamma_t^{\rm reg}(\mathcal{D}_m)^{-1}\frac{d}{dt}\gamma_t^{\rm reg}(\mathcal{D}_m)&=(\pi^{\rm reg}\circ\chi_m^{-1})\left(i\pi\mathcal{D}_m\frac{2}{\sqrt{\pi}}e^{-t^2\mathcal{D}_m^2}\right)\\&=2i\sqrt{\pi}(\pi^{\rm reg}\circ\chi_m^{-1})\left(f_t(\mathcal{D}_m)\right).\end{aligned}$$

But we know by Proposition 3.12 that

$$(\pi^{\rm reg}\circ\chi_m^{-1})\left(f_t(\mathcal{D}_m)\right)=(f_t(\tilde{D}_\theta))_{\theta\in T},$$

where $(\tilde{D}_\theta)_{\theta\in T}$ is the Γ-invariant Dirac type family. Hence we get

$$\gamma_t^{\rm reg}(\mathcal{D}_m)^{-1}\frac{d}{dt}\gamma_t^{\rm reg}(\mathcal{D}_m)=2i\sqrt{\pi}(f_t(\tilde{D}_\theta))_{\theta\in T},$$

where this equality holds in the von Neumann algebra $W^*_\nu(G;E)$. Applying the trace τ^ν, integrating over $(0,+\infty)$ and dividing by $2i\pi$, we obtain

$$\begin{aligned}\lim_{\epsilon\to 0}w^\nu_{{\rm reg},\epsilon}(\mathcal{D}_m)&=\lim_{\epsilon\to 0}\frac{1}{\sqrt{\pi}}\int_\epsilon^{1/\epsilon}\tau^\nu((f_t(\tilde{D}_\theta))_{\theta\in T})dt\\&=\frac{1}{\sqrt{\pi}}\int_0^{+\infty}\tau^\nu((f_t(\tilde{D}_\theta))_{\theta\in T})dt=\frac{1}{2}\eta^\nu_{\rm up}(\tilde{D}).\end{aligned}$$

The proof of the second equality is similar and one uses the equality

$$(\pi^{\rm av}\circ\chi_m^{-1})\left(f_t(\mathcal{D}_m)\right)=(f_t(D_L))_{L\in V/F},$$

which is proved in Proposition 3.12 . □

6. Stability properties of ρ^ν for the signature operator

6.1. Leafwise homotopies. — Let Γ, T and $\tilde{M}$ be as in the previous sections. Let $V := \tilde{M} \times_\Gamma T$ be the associated foliated flat bundle. Assume that $\tilde{M}'$ is another Γ-coverings and let T' be a compact space endowed with a continuous action of Γ by homeomorphisms. We consider $\tilde{M}' \times T$ and the foliated flat bundle $V' := \tilde{M}' \times_\Gamma T$.

Definition 6.1. — *Let $(V, \mathcal{F})$ and $(V', \mathcal{F}')$ be two foliated spaces. A leafwise map $f : (V, \mathcal{F}) \to (V', \mathcal{F}')$ is a continuous map such that*

- *The image under f of any leaf of $(V, \mathcal{F})$ is contained in a leaf of $(V', \mathcal{F}')$.*
- *The restriction of f to any leaf of $(V, \mathcal{F})$ is a smooth map between smooth leaves.*

Remark 6.2. — *1. We do not assume, that the leafwise derivatives to all orders of f are also continuous.*

2. If V and V' are smooth manifolds and $f : V \to V'$ is a differentiable map, then f is a leafwise map if and only if $f_* : T(V) \to T(V')$ sends $T\mathcal{F}$ to $T\mathcal{F}'$.

Roughly speaking, a leafwise map induces a "continuous map" between the quotient spaces of leaves. When the foliations are trivial, a leafwise map $f : M \times T \to M' \times T'$ is given by

$$f(m, \theta) = (h(m, \theta), k(\theta)), \quad (m, \theta) \in M \times T,$$

where k and h are continous and h is smooth with respect to the first variable.

An easy example of a leafwise map occurs when f is the quotient of a leafwise map $\tilde{f} : \tilde{M} \times T \to \tilde{M}' \times T'$ between the two trivial foliations, which is (Γ, Γ')-equivariant with respect to a group homomorphism $\alpha : \Gamma \to \Gamma'$. We shall get back to this example more explicitely later on. It is easy to construct a leafwise map between V and V' which is not the quotient of a (Γ, Γ') equivariant leafwise map $\tilde{f}$. Moreover, if $\tilde{f}$ exists then it is not unique: indeed, for example, if $\delta \in Z(\Gamma) \subset \Gamma$ is an element in the center of Γ, then the leafwise map $\tilde{f}_\delta := \tilde{f} \circ \delta^*$ (where $\delta^* : \tilde{M} \times T \to \tilde{M} \times T$ is the diffeomorphism induced by the action of δ on the right), is equivariant with respect to the same homomorphism $\alpha : \Gamma \to \Gamma'$ (because $\delta \in Z(\Gamma)$) and also induces f.

Given a foliated space $(V, \mathcal{F})$ in the sense of **[38]**, a subspace W of $(V, \mathcal{F})$ will be called a transversal to the foliation if for any $w \in W$ there exists a distinguished neighborhood U_w of w in V which is homeomorphic to $\mathbb{R}^p \times (U_w \cap W)$. Then one can show that the intersection of W with any leaf L of $(V, \mathcal{F})$ is a discrete subspace of L, that is a zero dimensional submanifold of L. Such a transversal is complete if it intersects all the leaves. In our example of foliated bundle $V = \tilde{M} \times_\Gamma T$, any fiber of

$V \to M$ is a complete transversal which is in addition compact, and any open subset of such fiber is a transversal.

Definition 6.3. — *1. Let $(V, \mathcal{F})$ be a foliated space. Two leafwise maps $f, g : (V, \mathcal{F}) \to (V', \mathcal{F}')$ are leafwise homotopic if there exists a leafwise map $H : (V \times [0,1], \mathcal{F} \times [0,1]) \to (V', \mathcal{F}')$ such that $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$.*

2. Let $(V, \mathcal{F})$ and $(V', \mathcal{F}')$ be two foliated spaces. A leafwise map $f : (V, \mathcal{F}) \to (V', \mathcal{F}')$ is a leafwise homotopy equivalence, if there exists a leafwise map $g : (V', \mathcal{F}') \to (V, \mathcal{F})$ such that

- *$g \circ f$ is leafwise homotopic to the identity of $(V, \mathcal{F})$.*
- *$f \circ g$ is leafwise homotopic to the identity of $(V', \mathcal{F}')$.*

3. We shall say that the foliations $(V, \mathcal{F})$ and $(V', \mathcal{F}')$ are (strongly) leafwise homotopy equivalent if there exists a leafwise homotopy equivalence from $(V, \mathcal{F})$ to $(V', \mathcal{F}')$.

Note that according to the above definition, the homotopies in (2) are supposed to preserve the leaves.

It is a classical fact that two leafwise homotopy equivalent compact foliated spaces $(V, \mathcal{F})$ and $(V', \mathcal{F}')$ have necessarily the same leaves dimension [**9**]. Note also that each leafwise homotopy equivalence sends a transversal to a transversal.

Lemma 6.4. — *A leafwise homotopy equivalence induces a local homeomorphism between transversals to the foliations.*

Proof. — See also [**9**]. Let f be the leafwise homotopy equivalence with homotopy inverse g, and denote by $h : [0,1] \times V \to V$ the $C^{\infty,0}$ homotopy between gf and the identity. Let $w \in V$. Let W be an open transversal of $(V, \mathcal{F})$ through $w \in W$. Take a distinguished chart U' in $(V', \mathcal{F}')$ which is an open neighborhood of $f(w)$ and which is homeomorphic to $D' \times W'$ for some transversal W' at $f(w)$. Then one finds an open distinguished chart U in $(V, \mathcal{F})$ such that $f(U) \subset U'$. Reducing W if necessary we can assume that U is homeomorphic to $D \times W$ for some disc D. Now, it is clear that since f is leafwise, it induces a map $\hat{f} : W \to W'$. By the same reasonning, we can assume furthermore that $g(U')$ is contained in a distinguished chart U_1 in $(V, \mathcal{F})$, homeomorphic to $D_1 \times W_1$.

The homotopy h induces a continous map $\hat{h} : W \to W_1$ and this map (or its reduction to a smaller domain) is simply the holonomy of the path $t \mapsto h(t, w)$. Hence $\hat{h}$ is locally invertible and it s clear that $\hat{h}^{-1}\hat{g}$ is a continuous inverse for $\hat{f}$. □

When $V = \tilde{M} \times_\Gamma T$ and $V' = \tilde{M}' \times_{\Gamma'} T'$, a particular case of leafwise homotopy equivalence is given by the quotient of an equivariant leafwise homotopy equivalence between $\tilde{M} \times T$ and $\tilde{M}' \times T'$. Recall that a fiberwise smooth map $\tilde{f} : \tilde{M} \times T \to \tilde{M}' \times T'$

is a continous map which can be written in the form

$$f(\tilde{m},\theta) = (h(\tilde{m},\theta), k(\theta)), \quad (\tilde{m},\theta) \in \tilde{M} \times T,$$

with h smooth with respect to the first variable. If $\alpha : \Gamma \to \Gamma'$ is a group homomorphism, then the fiberwise map $\tilde{f} : \tilde{M} \times T \to \tilde{M}' \times T'$ is α-equivariant if $\tilde{f}((\tilde{m},\theta)\gamma) = (\tilde{f}(\tilde{m},\theta))\alpha(\gamma)$.

In the following definition we extend the action of Γ and Γ' on $\tilde{M} \times T$ and $\tilde{M}' \times T'$ to $\tilde{M} \times [0,1] \times T$ and $\tilde{M}' \times [0,1] \times T'$ respectively, by declaring the action trivial on the $[0,1]$ factor.

Definition 6.5. — *We shall say that $f : (V,\mathcal{F}) \to (V',\mathcal{F}')$ is a special homotopy equivalence if there exist continuous maps $\tilde{f} : \tilde{M} \times T \to \tilde{M}' \times T'$, $\tilde{g} : \tilde{M}' \times T' \to \tilde{M} \times T$, $H : \tilde{M} \times [0,1] \times T \to \tilde{M} \times T$, $H' : \tilde{M}' \times [0,1] \times T' \to \tilde{M}' \times T'$, and group homomorphisms $\alpha : \Gamma \to \Gamma'$, $\beta : \Gamma' \to \Gamma$ such that:*

- *$\tilde{f}$, $\tilde{g}$, H and H' are fiberwise smooth;*
- *$\tilde{f}$ is α-equivariant; $\tilde{g}$ is β-equivariant; H is Γ-equivariant, H' is Γ'-equivariant;*
- *the restriction of H to $\tilde{M} \times \{0\} \times T$ (resp. of H' to $\tilde{M}' \times \{0\} \times T'$) is the identity map and the restriction of H to $\tilde{M} \times \{1\} \times T$ (resp. of H' to $\tilde{M}' \times \{1\} \times T'$) is $\tilde{g} \circ \tilde{f}$ (resp. $\tilde{f} \circ \tilde{g}$);*
- *$f : (V,\mathcal{F}) \to (V',\mathcal{F}')$ is induced by $\tilde{f} : \tilde{M} \times T \to \tilde{M}' \times T'$.*

If there exists such a special homotopy equivalence, we say that $(V,\mathcal{F})$ and $(V',\mathcal{F}')$ are special homotopy equivalent.

Lemma 6.6. — *If the pairs $(V,\mathcal{F})$ and $(V',\mathcal{F}')$ are special homotopy equivalent, then they are leafwise homotopic equivalent.*

Proof. — The equivariance of $\tilde{H}$ and $\tilde{H}'$ with respect to α and β, and the trivial action on the $[0,1]$ factor, allows to induce leafwise maps $H : V \times [0,1] \to V$ and $H' : V' \times [0,1] \to V'$ by setting,

$$H([\tilde{m},\theta];t) := [H(\tilde{m},t,\theta)] \text{ and } H'([\tilde{m}',\theta'];t) := [H'(\tilde{m}',t,\theta')].$$

In the same way the maps $\tilde{f}$ and $\tilde{g}$ induce leafwise maps f and g which are leafwise homotopy equivalences through the homotopies H and H'. □

Lemma 6.7. — *If $f : (V,\mathcal{F}) \to (V',\mathcal{F}')$ is a special homotopy equivalence induced by $\tilde{f}(\tilde{m},\theta) = (h(\tilde{m},\theta), k(\theta))$ as in the previous definition, then $\alpha : \Gamma \to \Gamma'$ is an isomorphism and $k : T \to T'$ is an equivariant homeomorphism.*

Proof. — Let $\tilde{f}$ and $\tilde{g}$ be equivariant leafwise smooth maps which give a special homotopy equivalence as in the above definition. We denote by k and k' the continuous equivariant maps induced by $\tilde{f}$ and $\tilde{g}$ on T and T' respectively. So,

$$k : T \to T' \text{ and } k' : T' \to T.$$

Since our homotopies send leaves to leaves, the composite maps $k' \circ k$ and $k \circ k'$ are identity maps. Moreover, if α and β are the group homomorphisms corresponding to the equivariance property of $\tilde{f}$ and $\tilde{g}$ respectively, then the homotopy $\tilde{H}$ satisfies

$$\tilde{H}((\tilde{m}, t, \theta)\gamma) = \tilde{H}(\tilde{m}, t, \theta)(\beta \circ \alpha)(\gamma), \quad \forall t \in [0, 1].$$

Therefore, applying this relation to $t = 0$, we get $\beta \circ \alpha = id_\Gamma$. The same argument gives the relation $\alpha \circ \beta = id_{\Gamma'}$. □

Remark 6.8. — *As already remaked, easy examples show that the foliations $(V, \mathcal{F})$ and $(V', \mathcal{F}')$ can be leafwise homotopy equivalent with non isomorphic groups Γ and Γ' and non homeomorphic spaces T and T'.*

6.2. $\rho^\nu(V, \mathcal{F})$ is metric independent. — We fix a continuous leafwise smooth Riemannian metric g on (V, F). g is lifted to a Γ-equivariant leafwise metric $\tilde{g}$ on $\tilde{M} \times T$, see **[38]**. So $\tilde{g} = (\tilde{g}(\theta))_{\theta \in T}$, where $\tilde{g}(\theta)$ is a metric on $\tilde{M} \times \{\theta\}$ and we assume that this structure is transversely continuous and equivariant with respect to the action of Γ. In what follows we shall refer to the bundle of exterior powers of the cotangent bundle as the Grassmann bundle. Consider the Γ-equivariant vector bundle $\widehat{E}$ over $\tilde{M} \times T$, obtained by pulling back from V the longitudinal Grassmann bundle E of the foliation $(V, \mathcal{F})$. Assume for the sake of simplicity of signs that the dimension of M is $4\ell - 1$ that is in the notations of Section 4, $m = 2\ell$. Consider the associated Γ-equivariant family of signature operators $(\tilde{D}_\theta^{\text{sign}})_{\theta \in T}$ associated with $\tilde{g}$, as defined in Section 4. We denote by D^{sign} the longitudinal signature operator on $(V, \mathcal{F})$ associated with the leafwise metric g acting on leafwise $2\ell - 1$ forms.

Recall that ν is a Γ-invariant Borel measure on T. We have defined in Subsection 5.1 a foliated rho-invariant $\rho^\nu(D^{\text{sign}}; V, \mathcal{F})$. We want to investigate the behavior of $\rho^\nu(D^{\text{sign}}; V, \mathcal{F})$ under a change of metric and under a leafwise diffeomorphism. First, we deal with the invariance of ρ^ν with respect to a change of metric. Up to constant, we can replace $\rho^\nu(D^{\text{sign}}; V, \mathcal{F})$, as it is usual, see **[3]** **[16]**, by the ρ invariant of the foliation $(V, \mathcal{F})$ defined as:

$$\rho^\nu(V, \mathcal{F}; g) := \frac{1}{\sqrt{\pi}} \int_0^\infty \left[\tau^\nu(\tilde{*}\tilde{d}e^{-t\tilde{\Delta}}) - \tau^\nu_{\mathcal{F}}(*de^{-t\Delta})\right] \frac{dt}{\sqrt{t}},$$

where $\tilde{\Delta}$ and Δ are the Laplace operators on leafwise $2\ell - 1$ forms, associated with the metrics $\tilde{g}$ and g respectively.

Proposition 6.9. — *Let Γ, $\tilde{M}$, T, ν and $(V, \mathcal{F})$ be as above. Let $(g_u)_{u \in [0,1]}$ be a continuous leafwise smooth one-parameter family of continuous leafwise smooth metrics* *on $(V, \mathcal{F})$. Then*

$$\rho^\nu(V, \mathcal{F}; g_0) = \rho^\nu(V, \mathcal{F}; g_1). \tag{23}$$

Proof. — The proof of this proposition in the case where T is reduced to a point was first given by J. Cheeger and M. Gromov in [**16**]. The Cheeger-Gromov proof extends to the general case of measured foliations and in particular to the case of foliated bundles and we proceed to explain the easy modifications needed for foliated bundles. Let D_u, for $0 \leq u \leq 1$, be the leafwise operator on $2\ell - 1$ leafwise differential forms of $(V, \mathcal{F})$, given by $D_u = *_u \circ d$, where d is the leafwise de Rham operator and $*_u$ is the leafwise Hodge operator associated with the metric g_u. It is easy to see that $u \mapsto \tau^\nu_\mathcal{F}(D_u e^{-t\Delta_u})$ is smooth. Since V is compact, the elliptic estimates along the leaves are uniform and we have for instance

$$\mathcal{R}(e^{-r\Delta_0}) \subset \mathrm{Dom}(\Delta_u), \quad \forall r > 0 \text{ and } u \in [0, 1].$$

Here $\mathcal{R}$ denotes the range of an operator and Dom the domain. Therefore, we can follow the steps of the proof in [**16**] and deduce the fundamental relation

$$\frac{d}{du}|_{u=0} \tau^\nu_\mathcal{F}(D_u e^{-t\Delta_u}) = \tau^\nu_\mathcal{F}(\frac{d*}{du}(0) d e^{-t\Delta_0}) + 2t \frac{d}{dt} \tau^\nu_\mathcal{F}(\frac{d*}{du}(0) d e^{-t\Delta_0})).$$

Using integration by parts, we then deduce

$$\sqrt{\pi} \frac{d}{du}|_{u=0} \int_\epsilon^A \tau^\nu_\mathcal{F}(D_u e^{-t\Delta_u}) \frac{dt}{\sqrt{t}} = 2\sqrt{A} \tau^\nu_\mathcal{F}(\frac{d*}{du}(0) d e^{-A\Delta_0}) - 2\sqrt{\epsilon} \tau^\nu_\mathcal{F}(\frac{d*}{du}(0) d e^{-\epsilon\Delta_0}).$$

Using the normality of the trace $\tau^\nu_\mathcal{F}$ and the spectral decomposition in the type II_∞ von Neumann algebra $W^*_\nu(V, \mathcal{F}; E)$, we deduce that

$$\lim_{A \to +\infty} 2\sqrt{A} \tau^\nu_\mathcal{F}(\frac{d*}{du}(0) d e^{-A\Delta_0}) = 0.$$

Now, the same estimates are as well valid in the type II_∞ von Neumann algebra $W^*_\nu(G; E)$ with the normal trace τ^ν. Hence, we are reduced to comparing the limits as $\epsilon \to 0$ of the difference

$$2\sqrt{\epsilon} \tau^\nu_\mathcal{F}(\frac{d*}{du}(0) d e^{-\epsilon\Delta_0}) - 2\sqrt{\epsilon} \tau^\nu(\frac{d\tilde{*}}{du}(0) d e^{-\epsilon\tilde{\Delta}_0}).$$

Replacing the heat operators by corresponding parametrices which are localized near the units V, in the two groupoids involved, see for instance [**17**], the limit of the two terms in the above difference is proved to be the same by classical arguments, which finishes the proof. □

According to the previous proposition we can now denote by $\rho^\nu(V, \mathcal{F})$ the signature rho invariant associated to any metric as before. All the leafwise maps considered in the rest of the paper are assumed to respect the orientations.

If we are now given a leafwise smooth homeomorphism $f : V \longrightarrow V'$, then we can transport the leafwise metric g from V to f_*g on V' and form the corresponding signature operator $D^{\mathrm{sign}\,\prime}$ along the leaves of $(V', \mathcal{F}')$ and also the Γ-equivariant signature operator $\tilde{D}^{\mathrm{sign}\,\prime} = (\tilde{D}^{\mathrm{sign}\,\prime}_{\theta'})_{\theta' \in T'}$ corresponding to the lifted Γ-invariant metric. Finally, the Γ-invariant measure ν on T, yields a holonomy invariant transverse measure $\Lambda(\nu)$ on the foliation $(V, \mathcal{F})$. The leafwise smooth homeomorphism f sends transversals to transversals and allows to transport the measure $\Lambda(\nu)$ into a holonomy invariant transverse measure $f_*\Lambda(\nu)$ on $(V', \mathcal{F}')$. Such a measure yields by restriction to a fiber a Γ'-invariant measure ν' on T' so that $f_*\Lambda(\nu) = \Lambda(\nu')$. More precisely, a fiber $V'_{m'_0}$ of the fibration $V' \to M'$ is a transversal to the foliation $\mathcal{F}'$ and hence the holonomy invariant transverse measure $f_*\Lambda(\nu)$ restricts to a measure on $V'_{m'_0}$. On the other hand, by fixing $\tilde{m}'_0$ with $[\tilde{m}'_0] = m'_0$ we get an identification of $V'_{m'_0}$ with the space T'. It is an easy exercise to check that the corresponding mesure on T' through this identification is Γ'-invariant and that the associated holonomy invariant transverse measure on the foliation $(V', \mathcal{F}')$ is precisely $f_*\Lambda(\nu)$.

Proposition 6.10. — *With the above notations, we have the following equalities for the eta invariants associated with the two signature operators D^{sign} and $D^{\mathrm{sign}\,\prime}$:*

$$\eta^{\nu}_{\mathrm{down}}(D^{\mathrm{sign}}) = \eta^{\nu'}_{\mathrm{down}}(D^{\mathrm{sign}\,\prime}) \quad \textit{and} \quad \eta^{\nu}_{\mathrm{up}}(\tilde{D}^{\mathrm{sign}}) = \eta^{\nu'}_{\mathrm{up}}(\tilde{D}^{\mathrm{sign}\,\prime}).$$

Proof. — Let us prove, for example, the second equality (the first one will be obtained in a similar way). Let W be the regular von Neumann algebra associated to $(V, \mathcal{F})$, the vertical Grassmann bundle $\widehat{E}$ and g. Let τ be the trace defined by g and ν and let W' and τ' be the corresponding objects, associated to $(V', \mathcal{F}')$, f_*g and the transported measure ν' under the leafwise smooth homeomorphism f. The leafwise smooth homeomorphism f lifts to a leafwise smooth homeomorphism $\tilde{f}$ between the monodromy groupoids G and G'. More precisely, for any $x \in V$ f lifts to a diffeomorphism $\tilde{f}_x : G_x \to G'_{f(x)}$ which induces, by the pull-back of forms, a unitary U_x between the spaces of L^2-forms. Recall that the metric on $(V', \mathcal{F}')$ is f_*g. The signature operator on $G'_{f(x)}$ associated with the metric f_*g is easily identified with the push-forward operator under $\tilde{f}$, that is the conjugation of the signature operator on G_x by the unitary U_x. Hence the functional calculus of $\tilde{D}^{\mathrm{sign}\,\prime}_{f(x)}$ is also the conjugation of the functional calculus of $\tilde{D}^{\mathrm{sign}}_x$ by U_x. So, in particular, for any $x \in V$ we have

$$\tilde{D}^{\mathrm{sign}\,\prime}_{f(x)} \exp(-t(\tilde{D}^{\mathrm{sign}\,\prime}_{f(x)})^2) = U_x \tilde{D}^{\mathrm{sign}}_x \exp(-t(\tilde{D}^{\mathrm{sign}}_x)^2) U_x^{-1}.$$

Now, by definition of the trace τ' associated with the image measure ν', one easily shows that

$$\tau'(U_x \tilde{D}^{\mathrm{sign}}_x \exp(-t(\tilde{D}^{\mathrm{sign}}_x)^2) U_x^{-1}) = \tau(\tilde{D}^{\mathrm{sign}}_x \exp(-t(\tilde{D}^{\mathrm{sign}}_x)^2)).$$

Therefore, the $f_*\Lambda(\nu)$ measured eta invariant of the G'-invariant family $(\tilde{D}^{\rm sign}{}'_{x'})_{x'\in V'}$ as defined by Peric in [**43**] coincides with the $\Lambda(\nu)$ measured eta invariant of the G-invariant family $(\tilde{D}^{\rm sign}_x)_{x\in V}$. On the other hand and as we already observed, these Peric measured eta invariants coincide with ours for the Γ'-invariant and Γ-invariant families of signature operators on $\tilde{M}'\times T'$ and $\tilde{M}\times T$ respectively. Hence the proof is complete. □

Corollary 6.11. — *Let $(V,\mathcal{F},\nu)$ and $(V',\mathcal{F}',\nu')$ be two foliated bundles as above and assume that there exists a leafwise smooth homeomorphism between $(V,\mathcal{F})$ and $(V',\mathcal{F}')$ such that $f_*\nu=\nu'$. Then*

$$\rho^\nu(V,\mathcal{F})=\rho^{\nu'}(V',\mathcal{F}')\,.$$

Proof. — We use the two previous propositions. The first one allows to compute $\rho^\nu(V,\mathcal{F})$ using any metric g. Then we apply the same proposition to $\rho^{f_*\nu}(V',\mathcal{F}')$ and compute it using the image metric f_*g. Finally, the second proposition allows to finish the proof. □

7. Loops, determinants and Bott periodicity

As before, let $\mathcal{A}_m$ be the maximal C^*-algebra of the groupoid $T\rtimes\Gamma$; let $\mathcal{E}_m$ be the $\mathcal{A}_m$-Hilbert module considered in the previous sections. Thus $\mathcal{E}_m$ is obtained by completion of the $\mathcal{A}_c$-Module $C^\infty_c(\widetilde{M}\times T,\widehat{E})$. Let $\mathcal{D}_m$ be the regular unbounded $\mathcal{A}_m$-linear operator induced by a Γ-equivariant family of Dirac operators. Let

$$\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m):=\{A\in\mathcal{B}_{\mathcal{A}_m}(\mathcal{E}_m)\text{ such that }A-\mathrm{Id}\in\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m)\text{ and }A\text{ is invertible}\}.$$

Let $\Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m))$ be the space of homotopy classes of loops in $\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m)$ which contain the identity operator. Then, using the inverse of the Bott isomorphism $\beta^{-1}:\Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m))\to K_0(\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m))$, the isomorphism $(\chi_m^{-1})_*:K_0(\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m))\to K_0(\mathcal{B}^E_m)$ induced by $\chi_m:\mathcal{B}^E_m\to K_{\mathcal{A}_m}(\mathcal{E}_m)$, and the inverse of the Morita isomorphism $\mathcal{M}_m:K_0(\mathcal{A}_m)\to K_0(\mathcal{B}^E_m)$ of Proposition 2.10, we obtain an isomorphism

$$\Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m))\xrightarrow{\beta^{-1}}K_0(\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m))\xrightarrow{(\chi_m^{-1})_*}K_0(\mathcal{B}^E_m)\xrightarrow{\mathcal{M}_m^{-1}}K_0(\mathcal{A}_m).$$

We denote by $\Theta:\Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m))\to K_0(\mathcal{A}_m)$ the composition of these isomorphisms. Recall the representations

$$\pi^{\rm reg}:\mathcal{B}^E_m\to W^*_\nu(G;E)\,;\quad \pi^{\rm av}:\mathcal{B}^E_m\to W^*_\nu(V,\mathcal{F};E).$$

Given a morphism α between two C^*-algebras, we denote, with obvious abuse of notation, by $\Omega\alpha$ the induced map on homotopy classes of loops. We thus obtain maps $\Omega\pi^{\rm reg}$, $\Omega\chi_m^{-1}$, $\Omega\pi^{\rm av}$; we define

$$\sigma^{\rm reg}:\Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m))\to\Omega(\mathcal{IK}(W^*_\nu(G;E)))\,;$$

$$\sigma^{\mathrm{av}} : \Omega(\mathscr{I}\mathcal{K}_{\mathscr{A}_m}(\mathscr{E}_m)) \to \Omega(\mathscr{I}\mathcal{K}(W^*_\nu(V,\mathscr{F};E)))$$

with

$$\sigma^{\mathrm{reg}} := \Omega\pi^{\mathrm{reg}} \circ \Omega\chi_m^{-1}\,, \quad \sigma^{\mathrm{av}} := \Omega\pi^{\mathrm{av}} \circ \Omega\chi_m^{-1}\,.$$

Recall, finally, that if ℓ is a loop in $\mathscr{I}L^1(W^*_\nu(V,\mathscr{F};E))$, or more generally in $\mathscr{I}\mathcal{K}(W^*_\nu(V,\mathscr{F};E))$, then ℓ has a well defined (log-)determinant $w^\nu_{\mathscr{F}}(\ell) \in \mathbb{C}$. Similarly, if ℓ is a loop in $\mathscr{I}L^1(W^*_\nu(G;E))$, or more generally in $\mathscr{I}\mathcal{K}(W^*_\nu(G;E))$, then ℓ has a well defined (log-)determinant $w^\nu(\ell) \in \mathbb{C}$.

Proposition 7.1. — *The following diagram commutes:*

$$\begin{tikzcd}
\Omega(\mathscr{I}\mathcal{K}_{\mathscr{A}_m}(\mathscr{E}_m)) \arrow[r, "\Theta"] \arrow[d, "\sigma^{\mathrm{av}}"] & K_0(\mathscr{A}_m) \arrow[d, "\tau^\nu_{\mathrm{av},*}"] \\
\Omega(\mathscr{I}\mathcal{K}(W^*_\nu(V,\mathscr{F};E))) \arrow[r, "w^\nu_{\mathscr{F}}"] & \mathbb{C}.
\end{tikzcd}$$

Similarly, the following diagram commutes:

$$\begin{tikzcd}
\Omega(\mathscr{I}\mathcal{K}_{\mathscr{A}_m}(\mathscr{E}_m)) \arrow[r, "\Theta"] \arrow[d, "\sigma^{\mathrm{reg}}"] & K_0(\mathscr{A}_m) \arrow[d, "\tau^\nu_{\mathrm{av},*}"] \\
\Omega(\mathscr{I}\mathcal{K}(W^*_\nu(G;E))) \arrow[r, "w^\nu"] & \mathbb{C}.
\end{tikzcd}$$

Proof. — Recall that for a C^*-algebra A the (inverse of the) Bott isomorphism β : $K_0(A) \to K_1(SA)$ is given by the map $[p] \to [(\exp(2\pi i t p)]$; as there will be several C^*-algebras involved, we denote this map by β_A. We observe that

$$\beta_{\mathscr{B}^E_m} \circ (\chi_m^{-1})_* = \Omega(\chi_m^{-1})_* \circ \beta_{\mathcal{K}_{\mathscr{A}_m}(\mathscr{E}_m)}\,.$$

Therefore,

$$\begin{aligned}
\Omega\pi^{\mathrm{av}} \circ \beta_{\mathscr{B}^E_m} \circ (\chi_m^{-1})_* \circ \beta^{-1}_{\mathcal{K}_{\mathscr{A}_m}(\mathscr{E}_m)} &= \Omega\pi^{\mathrm{av}} \circ \Omega(\chi_m^{-1})_* \circ (\beta_{\mathcal{K}_{\mathscr{A}_m}(\mathscr{E}_m)} \circ \beta^{-1}_{\mathcal{K}_{\mathscr{A}_m}(\mathscr{E}_m)}) \\
&= \Omega\pi^{\mathrm{av}} \circ \Omega(\chi_m^{-1})_* \\
&= \sigma^{\mathrm{av}}.
\end{aligned}$$

On the other hand, by definition of $\Omega\pi^{\mathrm{av}}$,

$$\Omega\pi^{\mathrm{av}} \circ \beta_{\mathscr{B}^E_m} = \beta_{\mathcal{K}(W^*_\nu(V,\mathscr{F};E))} \circ \pi^{\mathrm{av}}_*;$$

therefore

$$\begin{aligned}
w^\nu_{\mathscr{F}} \circ \Omega\pi^{\mathrm{av}} \circ \beta_{\mathscr{B}^E_m} &= w^\nu_{\mathscr{F}} \circ \beta_{\mathcal{K}(W^*_\nu(V,\mathscr{F};E))} \circ \pi^{\mathrm{av}}_* \\
&= \tau^\nu_{\mathscr{F}} \circ \pi^{\mathrm{av}}_* \\
&= \tau^\nu_{\mathrm{av},*}
\end{aligned}$$

where τ^{ν}_{av} is the trace on $\mathscr{B}^E_m$ as defined in Subsection 2.4 and with the equality $w^{\nu}_{\mathscr{G}} \circ \beta_{\mathscr{K}(W^*_{\nu}(V,\mathscr{F};E))} = \tau^{\nu}_{\mathscr{G}}$ proved by direct computation. To finish the proof we simply apply Proposition 2.10. □

Definition 7.2. — *We shall denote by*

$$w^{\nu}_{\mathrm{av}} : \Omega(\mathscr{I}\mathscr{K}_{\mathscr{A}_m}(\mathscr{E}_m)) \to \mathbb{C} \quad \textit{and} \quad w^{\nu}_{\mathrm{reg}} : \Omega(\mathscr{I}\mathscr{K}_{\mathscr{A}_m}(\mathscr{E}_m)) \to \mathbb{C}$$

the compositions $w^{\nu}_{\mathscr{G}} \circ \sigma^{\mathrm{av}}$ *and* $w^{\nu} \circ \sigma^{\mathrm{reg}}$ *respectively.*

We can summarize the previous proposition by the following two equations

$$w^{\nu}_{\mathrm{av}} = \tau^{\nu}_{\mathrm{av},*} \circ \Theta\,, \qquad w^{\nu}_{\mathrm{reg}} = \tau^{\nu}_{\mathrm{reg},*} \circ \Theta. \tag{24}$$

Remark 7.3. — *Definition 7.2 can be extended to a path in* $\mathscr{I}\mathscr{K}_{\mathscr{A}_m}(\mathscr{E}_m)$ *provided the two extreme points are mapped by* $\pi^{\mathrm{reg}} \circ \chi_m^{-1}$ *and* $\pi^{\mathrm{av}} \circ \chi_m^{-1}$ *into* τ^{ν} *trace class and* $\tau^{\nu}_{\mathscr{G}}$ *trace class perturbations of the identity respectively.*

8. On the homotopy invariance of rho on foliated bundles

Before plunging into foliated bundles and the foliated homotopy invariance of the signature rho invariant defined in Section 5, we digress briefly and treat a general orientable measured foliation $(V,\mathscr{F})$. We denote by Λ the holonomy invariant transverse measure. We fix a longitudinal riemannian metric on $(V,\mathscr{F})$ and we denote by D^{sign} the associated longitudinal signature operator. Let G be the monodromy groupoid associated to $(V,\mathscr{F})$. Then, as already remarked, Peric has defined in **[43]** a foliated eta invariant $\eta^{\Lambda}(\tilde{D}^{\mathrm{sign}})$, with $\tilde{D}^{\mathrm{sign}}$ the lift of D^{sign} to the monodromy covers, a G-equivariant operator on G. The work of Peric employs the holonomy groupoid, but is is not difficult to see that his arguments apply to the monodromy groupoid as well. Ramachandran, on the other hand, has defined in **[47]** an eta invariant $\eta^{\Lambda}(D^{\mathrm{sign}})$ using the measurable groupoid defined by the foliation, as we have already observed. We infer that the definition of foliated rho invariant is basically present in the literature. It suffices to define $\rho^{\Lambda}(D^{\mathrm{sign}}) := \eta^{\Lambda}(\tilde{D}^{\mathrm{sign}}) - \eta^{\Lambda}(D^{\mathrm{sign}})$. Assume now that G^x_x is torsion-free for any $x \in V$, then Connes has defined in **[18]** a Baum-Connes map $K_*(BG) \to K_*(C^*_{\mathrm{reg}}(V,\mathscr{F}))$ which factors through a maximal Baum-Connes map with values in the K-theory of the maximal C^*-algebra $C^*_{\max}(V,\mathscr{F})$. Here BG is the classifying space of the monodromy groupoid, see **[18]**, page 126. If $(V,\mathscr{F})$ is equal to the foliated bundle $V = \tilde{M} \times_{\Gamma} T$, then BG is given by the homotopy quotient $E\Gamma \times_{\Gamma} T$, with $E\Gamma$ equal to the universal space for Γ principal bundles. The Baum-Connes conjecture states that the Baum-Connes map is an isomorphism. We shall make a stronger assumption here, namely that the *maximal* Baum-Connes map is an isomorphism. This is a non trivial assumption and even if it is known to be satisfied for instance for amenable actions, there are examples where it fails to be true. The general conjecture one would then like to make goes as follows.

Let $(V', \mathcal{F}')$ be another foliation, endowed with a holonomy invariant transverse measure Λ' and let $f : (V, \mathcal{F}) \to (V', \mathcal{F}')$ be a leafwise measure preserving homotopy equivalence.

Conjecture. — *If G_x^x is torsion-free for any $x \in V$ and $K_*(BG) \to K_*(C^*_{\max}(V, \mathcal{F}))$ is an isomorphism, then $\rho^{\Lambda}(D^{\mathrm{sign}}) = \rho^{\Lambda'}(D^{\mathrm{sign}\,\prime})$.*

We shall now specialize to foliated bundles. Let Γ, T and $\tilde{M}$ be as in the previous sections. Let $V := \tilde{M} \times_{\Gamma} T$ and let $(V, \mathcal{F})$ be the associated foliated bundle. We assume the existence of a Γ-invariant measure on T; let $\Lambda(\nu)$ be the associated holonomy invariant transverse measure on $(V, \mathcal{F})$. Let $D = (D_L)_{L \in V/\mathcal{F}}$ be a longitudinal Dirac-type operator. Let $\tilde{D} = (\tilde{D}_\theta)_{\theta \in T}$ be the associated Γ-equivariant family of Dirac operators. As already remarked the rho invariant $\rho^{\Lambda(\nu)}(D)$ defined above, is indeed equal to our rho invariant $\rho^{\nu}(D; V, \mathcal{F})$. Assume now that $\tilde{M}'$ is the Γ' universal covering of a compact manifold M' and let T' be a compact space endowed with a continuous action of Γ' by homeomorphisms. We consider $\tilde{M}' \times T'$ and the foliated bundle $V' := \tilde{M}' \times_{\Gamma'} T'$. Let $(V', \mathcal{F}')$ be the associated foliated space. We assume the existence of a Γ'-invariant measure ν' on T' and we let $\Lambda(\nu')$ be the associated transverse measure on $(V', \mathcal{F}')$. Given a measure preserving foliated homotopy equivalence $f : V \to V'$, we can apply the general conjecture stated above to the invariants $\rho^{\Lambda(\nu)}(D)$, $\rho^{\Lambda(\nu')}(D')$ with D and D' denoting now the signature operators. We obtain in this way a conjecture about the homotopy invariance of the signature rho invariant $\rho^{\nu}(V, \mathcal{F})$ defined and studied in this paper; we shall deal with the general conjecture on foliated spaces in a different paper. In the rest of this section, we shall tackle the homotopy invariance of rho for the special homotopy equivalences descending from equivariant homotopies $\tilde{f} : \tilde{M} \times T \to \tilde{M}' \times T'$ as described in the previous section.

8.1. The Baum-Connes map for the discrete groupoid $T \rtimes \Gamma$. — In order to tackle the homotopy invariance of our $\rho^{\nu}(V, \mathcal{F})$ we first need to describe in the most geometric way the Baum-Connes map relevant to foliated bundles. This subsection is thus devoted to recall the definition of the Baum-Connes map with coefficients in the $\Gamma - C^*$-algebra $C(T)$ and, more importantly, to give a very geometric description of it. There are indeed several definitions available in the literature, with proofs of their compatibility sometime missing. The differences are all concentrated in the domain and, consequently, in the definition of the application; the target is always the same, namely $K_*(C(T) \rtimes_r \Gamma)$ (which is nothing but $K_*(\mathcal{A}_r)$ in our notation). Notice that if T is a point, we also have two different possibilities for the classical Baum-Connes map, depending on whether we consider, on the left hand side, the Baum-Douglas definition of K-homology or, instead, Kasparov's definition; although the compatibility of the two pictures has been assumed for many years, a complete proof only appeared recently, see the paper [**6**]. Going back to our more general situation, we begin with the Baum-Connes-Higson definition [**5**], which is given is terms of Kasparov KK-theory and the intersection product:

$$\mu_{BCH} : K_j^{\Gamma}(\underline{E}\Gamma; C(T)) \to K_j(C(T) \rtimes_r \Gamma). \tag{25}$$

The group on the left is, by definition,

$$\lim_{X \subset \underline{E}\Gamma} KK_{\Gamma}^j(C_0(X), C(T))$$

with the direct limit taken over the directed system of all Γ-compact subset of $\underline{E}\Gamma$. Similarly, there is a maximal Baum-Connes-Higson map:

$$\mu_{BCH} : K_j^{\Gamma}(\underline{E}\Gamma; C(T)) \to K_j(C(T) \rtimes_m \Gamma). \tag{26}$$

Next, we have the original definition of Baum and Connes [**4**], with the left hand side defined in terms of Gysin maps:

$$\mu_{BC} : K^j(T, \Gamma) \to K_j(C(T) \rtimes_r \Gamma). \tag{27}$$

We are not aware of a published proof of the compatibility of these two maps.
There is a third description of the Baum-Connes map with coefficients in $C(T)$: consider as set of cycles the (isomorphism classes of) pairs $(X, E \to X \times T)$ where X is a spin$_c$ proper Γ-manifold and E is a Γ-equivariant vector bundle on $X \times T$; define the usual Baum-Douglas equivalence relation on these cycles, bordism, direct sum and bundle modification; we obtain a group that we denote by $K_j^{\mathrm{geo}}(T \rtimes \Gamma)$ with $j = \dim M \bmod 2$. The Baum-Connes map in this case is denoted

$$\mu_{\rtimes} : K_j^{\mathrm{geo}}(T \rtimes \Gamma) \to K_j(C(T) \rtimes_r \Gamma) \tag{28}$$

and is very simply described as the map that associates to $[X, E \to X \times T]$ the index class of the Γ-equivariant family $(D_\theta)_{\theta \in T}$, with D_θ the spin$_c$ Dirac operator on X twisted by $E\big|_{X \times \{\theta\}}$. Also in this case we have a maximal version of the map:

$$\mu_{\rtimes} : K_j^{\mathrm{geo}}(T \rtimes \Gamma) \to K_j(C(T) \rtimes_m \Gamma). \tag{29}$$

Thanks to the Ph.D. thesis of Jeff Raven [**48**] it is now established that the two groups $K_j^{\Gamma}(\underline{E}\Gamma; C(T))$ and $K_j^{\mathrm{geo}}(T \rtimes \Gamma)$ are isomorphic and the two pairs of maps (25), (28) and (26), (29) are compatible; the proof of Raven's isomorphism is far from being trivial. Notice that, as in [**29**], we can consider orientable manifolds instead of spin$_c$ manifolds; thus the set of cycles for this version of Raven's group is given by pairs $(X, E \to X \times T)$ with X an orientable proper riemannian Γ-manifold and E a Γ-equivariant vector bundle on $X \times T$ endowed with an equivariant Clifford-module structure with respect to the Clifford algebra bundle of T^*X. Introduce on these cycles the equivalence relation given by bordism, direct sum and bundle modification as in **[29]** (Subsection 2.2, pages 59 and 60). The resulting group will be isomorphic to $K_j^{\mathrm{geo}}(T \rtimes \Gamma)$ and the resulting Baum-Connes map will be compatible. In the rest of this work we look at the stability properties of our foliated rho-invariant for the signature operator under a bijectivity hypothesis on the map (29). However, in order to exhibit examples we do need to use the compatibility between (26) and (29); indeed almost all examples where the Baum-Connes assumption is satisfied are proved using the Baum-Connes-Higson description.

8.2. Homotopy invariance of $\rho^\nu(V, \mathcal{F})$ for special homotopy equivalences. — We can state the main result of this section as follows:

Theorem 8.1. — *Let $V := \tilde{M} \times_\Gamma T$ and $V' := \tilde{M}' \times_{\Gamma'} T'$ be two foliated flat bundles, with Γ and Γ' discrete* **torsion-free** *groups*[(4)]*. Assume that there exists a* **special** *leafwise homotopy equivalence $f : (V, \mathcal{F}) \to (V', \mathcal{F}')$ and let $k : T \to T'$ be the induced equivariant homeomorphism . Let ν' be a Γ'-invariant measure on T'; let $\nu := k^*\nu'$ be the corresponding Γ-invariant measure on T. Assume that the Baum-Connes map* (28) *for the maximal C^*-algebra*

$$\mu_\rtimes : K_j^{\mathrm{geo}}(T \rtimes \Gamma) \longrightarrow K_*(C(T) \rtimes_{\max} \Gamma)$$

is **bijective**. *Then*

$$\rho^\nu(V, \mathcal{F}) = \rho^{\nu'}(V', \mathcal{F}') . \tag{30}$$

Remark 8.2. — *This theorem has been extended by Benameur and Roy to non special leafwise homotopy equivalences in* **[12]**. *The main ingredient is the use of an appropriate Hilbert bimodule associated with the equivalence.*

Sketch of the proof. — We follow the method of Keswani, see **[30]**, **[31]** and **[29]**. We simply denote the relevant signature operators by $D' = (D'_{L'})_{L' \in V'/\mathcal{F}'}$, $\tilde{D}' = (\tilde{D}'_\theta)_{\theta \in T}$, $\mathcal{D}'_m$ and $D = (D_L)_{L \in V/\mathcal{F}}$, $\tilde{D} = (\tilde{D}_\theta)_{\theta \in T}$, $\mathcal{D}_m$. We shall first assume that $T = T'$ and $\Gamma = \Gamma'$. Consider, with obvious notation, the trivial Γ-equivariant fibration $(\tilde{M}' \sqcup -\tilde{M}) \times T \to T$ as well as the foliated space $(X, \mathcal{F}^\sqcup)$, with $X := V' \sqcup (-V)$ and $\mathcal{F}^\sqcup$ induced by $\mathcal{F}$ and $\mathcal{F}'$. The longitudinal Grassmann bundles on V' and $-V$ define a longitudinally smooth bundle H over the foliated space X. Let $\widehat{H}$ be the equivariant vector bundle on $(\tilde{M}' \sqcup -\tilde{M}) \times T \to T$ obtained by pulling back the bundle H. All the constructions explained in the previous sections extend to $(\tilde{M}' \sqcup -\tilde{M}) \times T \to T$ and $\widehat{H}$ as well as to $(X, \mathcal{F}^\sqcup)$ and H. More precisely, we treat $(\tilde{M}' \sqcup -\tilde{M}) \times \{\theta\}$ as the leaf of the product foliation even if it is not connected and we consider the induced lamination $\mathcal{F}^\sqcup$. So the leaves are not connected for us. Clearly, we can define the

[(4)] The assumption on Γ and Γ' can be replaced by the weaker assumption that the isotropy groups are torsion-free, as can be checked in the proof.

C^*-algebra $\mathscr{B}_m^H$ as the completion of the convolution algebra of compactly supported continuous sections over the corresponding monodromy groupoid $G^{\sqcup}$, with respect to the direct sum of the regular representations in $L^2(\tilde{M}', \tilde{E}') \oplus L^2(\tilde{M}, \tilde{E})$. Note that $G^{\sqcup}$ can be identified with the space

$$G^{\sqcup} = [(\tilde{M}' \sqcup -\tilde{M}) \times (\tilde{M}' \sqcup -\tilde{M}) \times T]/\Gamma.$$

The reader should note that $\mathscr{B}_m^H$ is different from the C^*-algebra of the monodromy groupoid of the disjoint union of the two foliations $(V', \mathscr{F}')$ and $(V, \mathscr{F})$, and that $\mathscr{B}_m^H$ is Morita equivalent to the C^*-algebra $\mathscr{A}_m$. Indeed, we then have a well defined $\mathscr{A}_m$-Hilbert module $\mathscr{H}_m$ (this is nothing but $\mathscr{E}'_m \oplus \mathscr{E}_m$) as well as an isomorphism $\chi_m : \mathscr{B}_m^H \to \mathscr{K}_{\mathscr{A}_m}(\mathscr{H}_m)$ constructed in the same way as in the previous sections. Now, there are again representations

$$\pi^{\mathrm{reg}} = (\pi_\theta^{\mathrm{reg}})_{\theta \in T} : \mathscr{B}_m^H \to W_\nu^*(G^{\sqcup}; H), \quad \pi^{\mathrm{av}} = (\pi_\theta^{\mathrm{av}})_{\theta \in T} : \mathscr{B}_m^H \to W_\nu^*(X, \mathscr{F}^{\sqcup}; H).$$

Here, the von Neumann algebras $W_\nu^*(G^{\sqcup}; H)$ and $W_\nu^*(X, \mathscr{F}^{\sqcup}; H)$ are defined using ν-essentially bounded families over T as in the previous sections, except that the operators act on the direct sums of the Hilbert spaces. Said differently, we are again simply allowing disconnected leaves. Finally, the previous constructions of traces and determinants on foliations, work as well for these two von Neumann algebras. So, extending obviously the constructions of Section 7, using the composition operation of Hilbert modules, we can consider determinants

$$w_{\mathrm{reg}}^{\sqcup} : \Omega(\mathscr{I}\mathscr{K}_{\mathscr{A}_m}(\mathscr{H}_m)) \to \mathbb{C}, \quad w_{\mathrm{av}}^{\sqcup} : \Omega(\mathscr{I}\mathscr{K}_{\mathscr{A}_m}(\mathscr{H}_m)) \to \mathbb{C}.$$

Following the notation of Subsection 5.2, consider the path in $\mathscr{I}\mathscr{K}_{\mathscr{A}_m}(\mathscr{H}_m)$

$$\mathscr{W}_\epsilon := \left(\psi_t(\mathscr{D}'_m) \oplus (\psi_t(\mathscr{D}_m))^{-1}\right)_{t=\epsilon}^{t=1/\epsilon}.$$

Consider $w_{\mathrm{reg}}^{\sqcup}(\mathscr{W}_\epsilon)$ and $w_{\mathrm{av}}^{\sqcup}(\mathscr{W}_\epsilon)$ (one can easily show that the determinants of these paths are indeed well defined, see Remark 7.3). The proof proceeds along the following steps:

- we connect $\psi_\epsilon(\mathscr{D}'_m) \oplus (\psi_\epsilon(\mathscr{D}_m))^{-1}$ to the identity using the small time path ST_ϵ. This step is based on the injectivity of the Baum-Connes map and on the homotopy invariance of the signature index class;
- we connect $\psi_{1/\epsilon}(\mathscr{D}'_m) \oplus (\psi_{1/\epsilon}(\mathscr{D}_m))^{-1}$ to the identity via the large time path $LT_{1/\epsilon}$. This step is based on the surjectivity of the Baum-Connes map, on the foliated homotopy invariance of the space of leafwise harmonic forms and on the homotopy invariance of the signature index class;
- we obtain in this way a loop ℓ in $\mathscr{I}\mathscr{K}_{\mathscr{A}_m}(\mathscr{H}_m)$, i.e. an element of $\Omega(\mathscr{I}\mathscr{K}_{\mathscr{A}_m}(\mathscr{H}_m))$;
- we prove that $w_{\mathrm{reg}}^{\sqcup}(LT_{1/\epsilon})$ and $w_{\mathrm{av}}^{\sqcup}(LT_{1/\epsilon})$ are well defined and that

$$w_{\mathrm{reg}}^{\sqcup}(LT_{1/\epsilon}) \to 0 \quad \text{and} \quad w_{\mathrm{av}}^{\sqcup}(LT_{1/\epsilon}) \to 0 \quad \text{as} \quad \epsilon \downarrow 0 \tag{31}$$

– we prove that $w^{\sqcup}_{\mathrm{reg}}(ST_\epsilon)$ and $w^{\sqcup}_{\mathrm{av}}(ST_\epsilon)$ are well defined and

$$(w^{\sqcup}_{\mathrm{reg}}(ST_\epsilon) - w^{\sqcup}_{\mathrm{av}}(ST_\epsilon)) \to 0 \quad \text{as} \quad \epsilon \downarrow 0 \tag{32}$$

Now consider the map $\Theta : \Omega(\mathscr{I}\mathscr{K}_{\mathscr{C}_m}(\mathscr{H}_m)) \to K_0(\mathscr{C}_m)$. By the surjectivity of the Baum-Connes map one proves, using Θ, the following fundamental equality:

$$w^{\sqcup}_{\mathrm{reg}}(\ell) - w^{\sqcup}_{\mathrm{av}}(\ell) = 0 \tag{33}$$

which means that

$$(w^{\sqcup}_{\mathrm{reg}}(\mathscr{W}_\epsilon) - w^{\sqcup}_{\mathrm{av}}(\mathscr{W}_\epsilon)) + (w^{\sqcup}_{\mathrm{reg}}(LT_{1/\epsilon}) - w^{\sqcup}_{\mathrm{av}}(LT_{1/\epsilon}) + (w^{\sqcup}_{\mathrm{reg}}(ST_\epsilon) - w^{\sqcup}_{\mathrm{av}}(ST_\epsilon)) = 0.$$

Taking the limit as $\epsilon \downarrow 0$, using (31), (32) and recalling that

$$\lim_{\epsilon \downarrow 0}(w^{\sqcup}_{\mathrm{reg}}(\mathscr{W}_\epsilon) - w^{\sqcup}_{\mathrm{av}}(\mathscr{W}_\epsilon)) = \rho^{\nu'}(V', \mathscr{F}') - \rho^{\nu}(V, \mathscr{F})$$

we end the proof in the particular case $T = T'$ and $\Gamma = \Gamma'$. In the general case we know that, since we have assumed the special homotopy equivalence, T and T' are homeomorphic and that the two groups are isomorphic. Therefore, the above proof can be adapted easily. □

9. Proof of the homotopy invariance for special homotopy equivalences: details

We shall now provide more details for the proof of Theorem 8.1; most of our work in the previous sections will go into the proof. We shall work under the additional assumption that $T = T'$ and $\Gamma = \Gamma'$.

9.1. Consequences of surjectivity I: equality of determinants. — The following proposition will play a crucial role in our analysis. Recall that we have defined traces $\tau^{\nu}_{\mathrm{reg},*} : K_0(\mathscr{C}_m) \to \mathbb{C}$ and $\tau^{\nu}_{\mathrm{av},*} : K_0(\mathscr{C}_m) \to \mathbb{C}$; where in our notation $\mathscr{C}_m := C(T) \rtimes_m \Gamma$.

Proposition 9.1. — *Assume the Baum-Connes map*

$$\mu_{\rtimes} : K_0^{\mathrm{geo}}(T \rtimes \Gamma) \to K_0(C(T) \rtimes_m \Gamma)$$

surjective; then

$$\tau^{\nu}_{\mathrm{reg},*} = \tau^{\nu}_{\mathrm{av},*}.$$

Proof. — According to the definition of $K_0^{\mathrm{geo}}(T \rtimes \Gamma)$, we know that each K-theory class $\alpha \in K_0(C(T) \rtimes_m \Gamma)$ is, by the surjectivity of $\mu_{\rtimes}$, the index class associated to a Γ-equivariant family of Dirac-type operators on manifolds without boundary. Using formula (22) (which is a consequence of the analogue of Atiyah's index theorem on coverings and the Atiyah-Bott formula), we end the proof. □

Proposition 9.2. — *If the Baum-Connes map $\mu_{\rtimes} : K_0^{\mathrm{geo}}(T \rtimes \Gamma) \to K_0(C(T) \rtimes_m \Gamma)$ is surjective, then w^ν_{av} and w^ν_{reg} coincide on $\Omega(\mathcal{I}\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m))$.*

Proof. — Recall that $w^\nu_{\mathrm{av}} : \Omega(\mathcal{I}\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m)) \to \mathbb{C}$ and $w^\nu_{\mathrm{reg}} : \Omega(\mathcal{I}\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m)) \to \mathbb{C}$ are defined by passing to the respective von Neumann algebras and then taking the de La Harpe–Skandalis (log-)determinant there (see Definition 7.2): in formulae

$$w^\nu_{\mathrm{av}} := w^\nu \circ \sigma^{\mathrm{av}}\,, \quad w^\nu_{\mathrm{reg}} := w^\nu \circ \sigma^{\mathrm{reg}}.$$

Using the commutative diagram of Proposition 7.1, as summarized in formula (24), and the equality of traces on K_0 given by Proposition 9.1, we immediately conclude the proof. $\square$

Corollary 9.3. — *Let $V = \tilde{M} \times_\Gamma T$ and $V' = \tilde{M}' \times_\Gamma T$ be two homotopy equivalent foliated bundles as in the previous subsection, i.e. through a special homotopy equivalence. Let $\mathcal{H}_m = \mathcal{E}'_m \oplus \mathcal{E}_m$ be the $\mathcal{A}_m$-Hilbert module associated to the disjoint union of $\tilde{M}' \times T$ and $-(\tilde{M} \times T)$. Let ℓ be a loop in $\Omega(\mathcal{I}\mathcal{K}_{\mathcal{A}_m}(\mathcal{H}_m))$.*
If the Baum-Connes map $\mu_{\rtimes}$ is surjective, then

$$w^{\sqcup}_{\mathrm{av}}(\ell) = w^{\sqcup}_{\mathrm{reg}}(\ell). \tag{34}$$

If we consider, in particular, the loop $\ell \in \Omega(\mathcal{I}\mathcal{K}_{\mathcal{A}_m}(\mathcal{H}_m))$ defined in the sketch of the proof of Theorem 8.1, then we have justified formula (33).

9.2. Consequences of surjectivity II: the large time path. — Let $V = \tilde{M} \times_\Gamma T$ and $V' = \tilde{M}' \times_\Gamma T$ be two homotopy equivalent foliated bundles as in the previous subsection, i.e. through a special homotopy equivalence with $\Gamma = \Gamma'$ and $T = T'$. We consider the Cayley transforms of the regular operators $\mathcal{D}_m : \mathcal{E}_m \to \mathcal{E}_m$ and $\mathcal{D}'_m : \mathcal{E}'_m \to \mathcal{E}'_m$:

$$\mathcal{U} := (\mathcal{D}_m - i\mathrm{Id})(\mathcal{D}_m + i\mathrm{Id})^{-1}\,, \quad \mathcal{U}' := (\mathcal{D}'_m - i\mathrm{Id})(\mathcal{D}'_m + i\mathrm{Id})^{-1}.$$

Let $\tilde{f} : \tilde{M} \times T \to \tilde{M}' \times T$ be a fiberwise smooth equivariant map inducing the special homotopy equivalence between $(V, \mathcal{F})$ and $(V', \mathcal{F}')$; let g and $\tilde{g}$ be choices for the homotopy inverses of f and $\tilde{f}$, with $\tilde{g} : \tilde{M}' \times T \to \tilde{M} \times T$ inducing g. This notation should not cause any trouble even if the metrics are denoted by the same letters. Following **[30]** (Section 3) one can construct a path of unitaries in $\mathcal{H}_m = \mathcal{E}'_m \oplus \mathcal{E}_m$, $\mathcal{V}(t), t \in [0, 2]$, connecting $\mathcal{U}' \oplus \mathcal{U}^{-1} = \mathcal{V}(0)$ to the identity $\mathrm{Id}_{\mathcal{H}_m} = \mathcal{V}(2)$. The path $\mathcal{V}(t), t \in [0, 2]$ (which is denoted $\mathbf{W}(t)$ in **[30]**) is obtained by defining a perturbation $\sigma(t)$ of the grading operator defining the signature operator; the definition of $\sigma(t)$, which is due to Higson and Roe, employs the pull back operator defined by the homotopy equivalence $\tilde{g}$ (precomposed and composed respectively with an extension to $\mathcal{E}_m$ and $\mathcal{E}'_m$ of the smoothing operators $(\phi(\tilde{D}_\theta))_{\theta \in T}$, $(\phi(\tilde{D}'_\theta))_{\theta \in T}$, ϕ being a rapidly decreasing smooth function with compactly supported Fourier transform). We omit the actual definition of $\mathcal{V}(t)$ since it is somewhat lengthy and refer instead to [**30**], pages 968-969.

Recall that our goal is to construct a path connecting $\psi_{1/\epsilon}(\mathcal{D}'_m) \oplus (\psi_{1/\epsilon}(\mathcal{D}_m))^{-1}$, (where $\psi_\alpha(x) = -\exp(i\pi \frac{2}{\sqrt{\pi}} \int_0^{\alpha x} e^{-u^2} du)$, to the identity on $\mathcal{H}_m$.

To this end, notice that the Cayley transform of the operator $\mathcal{D}_m$ can be expressed as $-\exp(i\pi\chi(\mathcal{D}_m))$, with $\pi\chi(x) = 2\arctan(x)$.

Definition 9.4. — [**30**] *A chopping function is an odd continuous function $\mu : \mathbb{R} \to \mathbb{C}$ such that $|\mu(x)| \leq 1$ and $\lim_{x\to\pm\infty} \mu(x) = \pm 1$.*

Both $\chi(x) := \frac{2}{\pi}\arctan(x)$ and $\phi(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$ are chopping functions. Two chopping functions μ_1 and μ_2 can be homotoped one to the other via the straight line homotopy $k_s = (1-s)\mu_1 + s\mu_2$. Thus $\mathcal{U}' \oplus \mathcal{U}^{-1}$, which is equal to $-\exp(i\pi\chi(\mathcal{D}'_m)) \oplus -\exp(-i\pi\chi(\mathcal{D}_m))$, can be joined to

$$-\exp(i\pi\phi(\mathcal{D}'_m)) \oplus -\exp(-i\pi\phi(\mathcal{D}_m)), \quad \phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$

via the path $\mathcal{K}(s) := -\exp(i\pi k_s(\mathcal{D}'_m)) \oplus -\exp(-i\pi k_s(\mathcal{D}_m))$. We denote by LT the concatenation of $\mathcal{K}(s)$ and $\mathcal{V}(t)$. So LT is a path joining $-\exp(i\pi\phi(\mathcal{D}'_m)) \oplus -\exp(-i\pi\phi(\mathcal{D}_m))$, with $\phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$, to the identity.

Definition 9.5. — *Let $\epsilon > 0$ be fixed. The large time path $LT_{1/\epsilon}$ is the path obtained from the above construction but with the operators $\mathcal{D}_m$ and $\mathcal{D}'_m$ replaced by $\frac{1}{\epsilon}\mathcal{D}_m$ and $\frac{1}{\epsilon}\mathcal{D}'_m$ respectively. The large time path connects*

$$\psi_{1/\epsilon}(\mathcal{D}'_m) \oplus (\psi_{1/\epsilon}(\mathcal{D}_m))^{-1}, \quad \textit{with} \quad \psi_{1/\epsilon}(x) = -\exp\left(i\pi(\frac{2}{\sqrt{\pi}} \int_0^{x/\epsilon} e^{-u^2} du)\right),$$

to the identity.

For later use, we notice that

$$\psi_{1/\epsilon} = -\exp(i\pi\phi_{1/\epsilon}), \quad \text{with} \quad \phi_{1/\epsilon}(x) = \frac{2}{\sqrt{\pi}} \int_0^{x/\epsilon} e^{-u^2} du. \tag{35}$$

For each fixed $\epsilon > 0$ $LT_{1/\epsilon}$ is a path in $\mathcal{I}\mathcal{K}_{\mathcal{A}_m}(\mathcal{H}_m)$ (we recall that this is the group consisting of the operators $A \in \mathcal{B}_{\mathcal{A}_m}(\mathcal{H}_m)$ such that $A - \mathrm{Id} \in \mathcal{K}_{\mathcal{A}_m}(\mathcal{H}_m)$ and A is invertible). In order to show this property we first recall that at the end of Subsection 3.2, Sobolev modules $\mathcal{E}_m^{(\ell)}$ were introduced and the compactness of the inclusion $\mathcal{E}_m^{(\ell)} \hookrightarrow \mathcal{E}_m^{(k)}$, $\ell > k$ was stated. Observe then that if χ is any chopping function with the property that $\chi' \sim 1/x^2$ as $|x| \to \infty$, then, using the compactness of the inclusion of the Sobolev module $\mathcal{E}_m^{(1)}$ into $\mathcal{E}_m$, one proves easily that $-\exp(i\pi\chi(\mathcal{D}_m) \in \mathcal{I}\mathcal{K}_{\mathcal{A}_m}(\mathcal{H}_m)$. Notice now that both $\frac{2}{\pi}\arctan(x)$ and $\frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$ satisfy this condition; thus $LT_{1/\epsilon} \in \mathcal{I}\mathcal{K}_{\mathcal{A}_m}$.

9.3. The determinants of the large time path. — Recall the isomorphism $\chi_m : \mathcal{B}^H_m \to K_{\mathcal{A}_m}(\mathcal{H}_m)$, and the representations

$$\pi^{\mathrm{reg}} : \mathcal{B}^H_m \to W^*_\nu(G^{\sqcup}; H)\,; \quad \pi^{\mathrm{av}} : \mathcal{B}^E_m \to W^*_\nu(X, \mathcal{F}^{\sqcup}; H)\,, \text{ with } X = [\tilde{M} \sqcup (-\tilde{M}')] \times_\Gamma T.$$

Proceeding as in Section 7, we can use χ_m^{-1} and π^{reg} in order to define a path $\sigma^{\mathrm{reg}}(LT_{1/\epsilon})$ in $\mathcal{IK}(W^*_\nu(G^{\sqcup}; H))$. The end-points of this path are τ^ν trace class perturbations of the identity; thus, see Remark 7.3, the determinant $w^\nu(\sigma^{\mathrm{reg}}(LT_{1/\epsilon}))$ is well defined and we can set

$$w^\nu_{\mathrm{reg}}(LT_{1/\epsilon}) := w^\nu(\sigma^{\mathrm{reg}}(LT_{1/\epsilon}))\,.$$

Similarly,

$$w^\nu_{\mathrm{av}}(LT_{1/\epsilon}) := w^\nu_{\mathcal{F}^{\sqcup}}(\sigma^{\mathrm{av}}(LT_{1/\epsilon}))$$

is well defined (and we recall that $\mathcal{F}^{\sqcup}$ is the foliation induced on X by the foliations $\mathcal{F}$ and $\mathcal{F}'$ on V and V' respectively).

Proposition 9.6. — *As $\epsilon \downarrow 0$ we have*

$$w^\nu_{\mathrm{reg}}(LT_{1/\epsilon}) \longrightarrow 0\,, \quad w^\nu_{\mathrm{av}}(LT_{1/\epsilon}) \longrightarrow 0. \tag{36}$$

Proof. — Fix $\epsilon > 0$ and recall that $LT_{1/\epsilon}$ is the composition of two paths: the path $\mathcal{V}_{1/\epsilon}$, connecting

$$-\exp(i\pi\chi(\frac{1}{\epsilon}\mathcal{D}'_m)) \oplus -\exp(-i\pi\chi(\frac{1}{\epsilon}\mathcal{D}_m)) \text{ (with } \chi(x) = \frac{2}{\pi}\arctan(x)) \quad \text{to} \quad \mathrm{Id}_{\mathcal{H}_m}\,,$$

and the straight line path $\mathcal{K}_{1/\epsilon}$, connecting $\psi_{1/\epsilon}(\mathcal{D}'_m)) \oplus (\psi_{1/\epsilon}(\mathcal{D}_m)))^{-1}$ to

$$-\exp(i\pi\chi(\frac{1}{\epsilon}\mathcal{D}'_m)) \oplus -\exp(-i\pi\chi(\frac{1}{\epsilon}\mathcal{D}_m)).$$

Consider $\sigma^{\mathrm{reg}}(LT_{1/\epsilon})$ in $\mathcal{IK}(W^*_\nu(G^{\sqcup}; H))$; for the signature family $\tilde{P}$ associated to $\tilde{M}' \sqcup (-\tilde{M}) \times T \to T$ denote by $\tilde{\Pi} := (\tilde{\Pi}_\theta)_{\theta \in T}$ the element in $W^*_\nu(G^{\sqcup}; H)$ defined by the family of orthogonal projections onto the null space. Then, proceeding as in Keswani [**30**], one can show that $\sigma^{\mathrm{reg}}(LT_{1/\epsilon})$ converges strongly to the path

$$\tilde{\mathcal{V}}_\infty(t) = \begin{cases} -\tilde{\Pi} + \tilde{\Pi}^\perp\,, & t \in [-1, 3/2] \\ -e(t)\tilde{\Pi} + \tilde{\Pi}^\perp\,, & t \in [3/2, 2] \end{cases} \tag{37}$$

with

$$e(t) = -\begin{pmatrix} \exp(2\pi i t) & 0 \\ 0 & \exp(-2\pi i t) \end{pmatrix}.$$

More precisely: $\sigma^{\mathrm{reg}}(\mathcal{K}_{1/\epsilon})$ converges strongly to the constant path $\tilde{\Pi} + \tilde{\Pi}^\perp$, whereas $\sigma^{\mathrm{reg}}(\mathcal{V}_{1/\epsilon})$ (is homotopic, with fixed end-points, to a path that) converges strongly to $\tilde{\mathcal{V}}_\infty(t)$. Similarly, if we denote by $\Pi \in W^*_\nu(X, \mathcal{F}^{\sqcup}; H)$ the projection onto the null space of the longitudinal signature operator on X, then $\sigma^{\mathrm{av}}(LT_{1/\epsilon})$ converges strongly to the path

$$\mathcal{V}_\infty(t) = \begin{cases} -\Pi + \Pi^\perp\,, & t \in [-1, 3/2] \\ -e(t)\Pi + \Pi^\perp\,, & t \in [3/2, 2]. \end{cases} \tag{38}$$

We can now end the proof[5]. Recall the function $\phi_{1/\epsilon}(x) := \frac{2}{\sqrt{\pi}}\int_0^{x/\epsilon} e^{-u^2}du$, see formula (35); consider the function $\alpha(x)$ equal to zero for $x = 0$, equal to 1 for $x > 0$ and equal to -1 for $x < 0$; let $\varpi_{1/\epsilon}(t) = (1-t)\phi_{1/\epsilon} + t\alpha$ be the straight-line path joining $\phi_{1/\epsilon}$ to α; consider the path

$$X_{1/\epsilon}(t) := -\exp(i\pi\varpi_{1/\epsilon}(t)(\tilde{D}') \oplus -\exp(-i\pi\varpi_{1/\epsilon}(t)(\tilde{D}))\,.$$

We notice that as $\epsilon \to 0$, $\phi_{1/\epsilon}$ converges pointwise to α. Using once again the spectral theorem for unbounded operators this means that, in the strong topology,

$$\phi_{1/\epsilon}(\tilde{P}) \longrightarrow \alpha(\tilde{P}) \quad \text{as} \quad \epsilon \downarrow 0 \tag{39}$$

where we recall that $\tilde{P}$ denotes the signature family on $(\tilde{M}' \sqcup (-\tilde{M})) \times T \to T$. We go back to the path $X_{1/\epsilon}(t)$, which is a path in $W^*_\nu(G^\sqcup; H)$ joining $\sigma^{\mathrm{reg}}(\psi_{1/\epsilon}(\mathcal{D}'_m)) \oplus (\psi_{1/\epsilon}(\mathcal{D}_m)))^{-1})$, i.e. $\psi_{1/\epsilon}(\tilde{D}')) \oplus (\psi_{1/\epsilon}(\tilde{D})))^{-1}$, to the constant path $-\tilde{\Pi}+\tilde{\Pi}^\perp$. Consider the loop $\gamma_{1/\epsilon}$ in $W^*_\nu(G^\sqcup; H)$ obtained by the concatenation of $X_{1/\epsilon}(t)$, $\tilde{\mathcal{V}}_\infty(t)$ and the reverse of $\sigma^{\mathrm{reg}}(LT_{1/\epsilon})$. By the above results the loop $\gamma_{1/\epsilon}$ is strongly null homotopic, thus its determinant is equal to zero. Summarizing:

$$w^\nu(\sigma^{\mathrm{reg}}(LT_{1/\epsilon})) = w^\nu(\tilde{\mathcal{V}}_\infty) + w^\nu(X_{1/\epsilon})$$

which can be rewritten as

$$w^\nu_{\mathrm{reg}}(LT_{1/\epsilon}) = w^\nu(\tilde{\mathcal{V}}_\infty) + w^\nu(X_{1/\epsilon}).$$

Computing

$$\tilde{\mathcal{V}}_\infty(t)^{-1}\frac{d\tilde{\mathcal{V}}_\infty(t)}{dt} = \begin{cases} 0\,, & t \in [-1, 3/2] \\ (2\pi i)\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}\tilde{\Pi}\,, & t \in [3/2, 2] \end{cases}$$

and recalling that the von Neumann dimension of the null space of the signature operator is a foliated homotopy invariant, see **[26]**, we deduce that $w^\nu(\tilde{\mathcal{V}}_\infty(t)) = 0$. Thus the first part of the proposition will follow from the following result:

$$w^\nu(X_{1/\epsilon}) \longrightarrow 0 \quad \text{as} \quad \epsilon \downarrow 0\,.$$

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[5] Notice that the proof given by Keswani for coverings contains a few imprecisions; the argument given here corrects them.

However, this is clear from (39) and the normality of the trace, given that, by direct computation,

$$w^\nu(X_{1/\epsilon}) = \frac{1}{2\pi i}\tau^\nu\left(\phi_{1/\epsilon}(\tilde P) - \alpha(\tilde P)\right).$$

Essentially the same argument, using the strong convergence of $\sigma^{\mathrm{av}}(LT_{1/\epsilon})$ to $\mathscr{V}_\infty$ (see (38)), shows that $w^\nu_{\mathrm{av}}(LT_{1/\epsilon}) \longrightarrow 0$. □

9.4. Consequences of injectivity: the small time path. — So far, we have connected the $t = 1/\epsilon$ endpoint of the path

$$\mathscr{W}_\epsilon := \left(\psi_t(\mathscr{D}'_m) \oplus (\psi_t(\mathscr{D}_m))^{-1}\right)_{t=\epsilon}^{t=1/\epsilon} \quad \text{in} \quad \mathscr{I}\mathscr{K}_{\mathscr{A}_m}(\mathscr{H}_m)$$

to the identity using the large time path $LT_{1/\epsilon}$. We also showed that

$$\lim_{\epsilon\to 0}(w^\nu_{\mathrm{reg}}(LT_{1/\epsilon}) - w^\nu_{\mathrm{av}}(LT_{1/\epsilon})) = 0\,.$$

We now wish to close up the concatenation of $\mathscr{W}_\epsilon$ and $LT_{1/\epsilon}$ to a loop based at the identity. This step will be achieved through the small time path ST_ϵ, a path in $\mathscr{I}\mathscr{K}_{\mathscr{A}_m}(\mathscr{H}_m)$ connecting the $t = \epsilon$ end point of $\mathscr{W}_\epsilon$ to the identity. We shall want to ensure that

$$\lim_{\epsilon\to 0}(w^\nu_{\mathrm{reg}}(ST_\epsilon) - w^\nu_{\mathrm{av}}(ST_\epsilon)) = 0\,. \tag{40}$$

The existence of *a path* connecting $\left(\psi_\epsilon(\mathscr{D}'_m) \oplus (\psi_\epsilon(\mathscr{D}_m))^{-1}\right)$ to the identity is in fact not difficult and follows from the proof of the Hilsum-Skandalis theorem; what is more delicate is the construction of a path satisfying the crucial property (40). It is here that the injectivity of the Baum-Connes map is used, as we proceed now to explain in more details.

Let $V = \tilde M \times_\Gamma T$ and $V' = \tilde M' \times_\Gamma T$ be two homotopy equivalent foliated bundles as in the previous subsections, with $\tilde M$ and $\tilde M'$ orientable. We fix leafwise Γ-equivariant metrics on $\tilde M \times T$ and $\tilde M' \times T$. We denote by $\tilde D = (\tilde D_\theta)$, $D = (D_L)_{L\in V/\mathscr{F}}$ and $\mathscr{D}_m$ respectively the Γ-equivariant signature family, the longitudinal signature operator on $(V, \mathscr{F})$ and the $\mathscr{A}_m$-linear signature operator on the $\mathscr{A}_m$-Hilbert module $\mathscr{E}_m$. We fix similar notations for $V' = \tilde M' \times_\Gamma T$ and we let as usual $\mathscr{H}_m = \mathscr{E}'_m \oplus \mathscr{E}_m$. We denote only in the rest of this paragraph by Λ and Λ' the vertical Grassmann bundles on $\tilde M \times T$ and $\tilde M' \times T$ respectively. Consider the index classes $\mathrm{Ind}(\mathscr{D}_m)$, $\mathrm{Ind}(\mathscr{D}'_m)$, two elements in $K_1(\mathscr{A}_m)$. By the foliated homotopy invariance of the signature index class we know that $\mathrm{Ind}(\mathscr{D}_m) = \mathrm{Ind}(\mathscr{D}'_m)$. On the other hand, using the very definition of the Baum-Connes map $\mu_\rtimes$, we have $\mathrm{Ind}(\mathscr{D}_m) = \mu_\rtimes[\tilde M, \Lambda \to \tilde M \times T]$ and $\mathrm{Ind}(\mathscr{D}'_m) = \mu_\rtimes[\tilde M', \Lambda' \to \tilde M' \times T]$, so that, by the assumed injectivity of $\mu_\rtimes$ we infer that

$$[\tilde M, \Lambda \to \tilde M \times T] = [\tilde M', \Lambda' \to \tilde M' \times T] \quad \text{in} \quad K_1^{\mathrm{geo}}(T \rtimes \Gamma)\,. \tag{41}$$

This is the information we want to use. Before stating the main result of this subsection we give a convenient definition.

Definition 9.7. — *We shall say that a chopping function χ is controlled if*

- *the derivative of χ is a Schwartz function;*
- *the Fourier transform of χ is supported in $[-1,1]$*[(6)]*;*
- *the functions χ^2-1 and $\chi(\chi^2-1)$ are Schwartz and their Fourier transforms are supported in $[-1,1]$.*

For the existence of such a function, see **[40]**.

Theorem 9.8. — *If $[\tilde{M},\Lambda \to \tilde{M}\times T] = [\tilde{M}',\Lambda' \to \tilde{M}'\times T]$ in $K_1^{\mathrm{geo}}(T\rtimes\Gamma)$ then there exist a Γ-proper manifold Y, a longitudinally smooth Γ-equivariant vector bundle $\widehat{L}\to Y\times T$ and a continuous s-path of Γ-equivariant families on Y*

$$\tilde{B}_s := (\tilde{B}_{s,\theta})_{\theta\in T} \quad s\in(0,1)$$

such that

1. *for each $s\in(0,1)$ and $\theta\in T$, $(\tilde{B}_{s,\theta})$ is a first order elliptic differential operator on Y acting on the sections of $\widehat{L}|_{\tilde{Y}\times\{\theta\}}$*
2. *the $\mathcal{A}_m$-Hilbert bundle $\mathcal{L}_m$ obtained by completing $C_c^\infty(Y\times T,\widehat{L})$ contains $\mathcal{E}'_m\oplus\mathcal{E}_m$ as an orthocomplemented submodule; thus there is an orthogonal decomposition $\mathcal{L}_m = (\mathcal{E}'_m\oplus\mathcal{E}_m)\oplus(\mathcal{E}'_m\oplus\mathcal{E}_m)^\perp$*
3. *for any controlled chopping function χ the path $-\exp(i\pi\chi(\mathcal{B}_s))$ is norm continuous in the space of bounded operators in $\mathcal{L}_m$ (here, for $s\in(0,1)$, $\mathcal{B}_s$ denotes the regular $\mathcal{A}_m$-linear operator defined by the family $(\tilde{B}_{s,\theta})_{\theta\in T}$);*
4. *we have, in norm topology,*

$$\begin{aligned}\lim_{s\to1}(-\exp(i\pi\chi(\mathcal{B}_s))) &= \mathrm{Id}_{\mathcal{L}_m}\\ \lim_{s\to0}(-\exp(i\pi\chi(\mathcal{B}_s))) &= \big(-\exp(i\pi\chi(\mathcal{D}'_m))\oplus-\exp(-i\pi\chi(\mathcal{D}_m)\big)\oplus\mathrm{Id}_\perp\end{aligned}$$

   with $\mathrm{Id}_\perp$ denoting the identity on $(\mathcal{E}'_m\oplus\mathcal{E}_m)^\perp$.
5. *$-\exp(i\pi\chi(\mathcal{B}_s))\in\mathcal{IK}_{\mathcal{A}_m}$.*

Proof. — If $[\tilde{M},\Lambda\to\tilde{M}\times T] = [\tilde{M}',\Lambda'\to\tilde{M}'\times T]$ in $K_1^{\mathrm{geo}}(T\rtimes\Gamma)$, then we know that we can pass from $(\tilde{M},\Lambda\to\tilde{M}\times T)$ to $(\tilde{M}',\Lambda'\to\tilde{M}'\times T)$ through a finite number of equivalences. The most delicate one is bordism, so we assume directly that we have a manifold X endowed with a proper action of Γ, a Γ-equivariant bundle $\widehat{H}$ on $X\times T$, a proper Γ-manifold with boundary Z' and an equivariant vector bundle $\widehat{F'}$ on $Z'\times T$

(6) Notice that it is impossible to have, as required in **[30]**, that $\hat{\chi}$ is smooth and compactly supported (since, otherwise, χ itself , which is the Fourier transform of $\hat{\chi}$, would be rapidly decreasing and thus **not** a chopping function).

such that the boundary of Z' is equal to X and $\widehat{F}'$ restricted to $\partial Z' \times T$ is equal to $\widehat{H}$. Consider the manifold with cylindrical ends, Z, obtained by attaching to Z' a cylinder $[0,\infty) \times X$; consider the cylinder $W = X \times \mathbb{R}$; these are proper Γ manifolds if we extend the action to be trivial in the cylindrical direction; extend bundles to the cylindrical parts in the obvious way. The Γ manifold Y appearing in the statement of the theorem is the disjoint union of Z, $-Z$ and W, as in [**29**]. The bundle $\widehat{L}$ is given in terms of $\widehat{H}$ and its extension to the cylindrical parts. The equivariant families $\tilde{B}_s$, $s \in (0,1)$, appearing in the statement of the theorem are *explicitly defined* (in [**29**] see: the last displayed formula page 70; the last displayed formula page 72; the second displayed formula page 76 and the first displayed formula page 77). We shall see an example in a moment. The common feature of these operators is that they are Dirac-type on all of Y but look like an harmonic oscillator along the cylindrical ends. Since we have extended the action in a trivial way to the $\mathbb{R}$-direction of the cylindrical ends we can decompose the Hilbert module defined on the cylinder $(X \times \mathbb{R}) \times T$ as $\mathscr{E}_m(X) \otimes_{\mathbb{C}} L^2(\mathbb{R})$. Using the spectral decomposition of the harmonic oscillator we see, as in [**29**], that there is an orthogonal decomposition of Hilbert modules $\mathscr{E}_m(X \times \mathbb{R}) = (\mathscr{E}_m(X \times \mathbb{R}))' \oplus^{\perp} (\mathscr{E}_m(X \times \mathbb{R}))''$ with $(\mathscr{E}_m(X \times \mathbb{R}))'$ equal to the tensor product of $\mathscr{E}_m(X)$ with the 1-dimensional space generated by the kernel of the harmonic oscillator and $(\mathscr{E}_m(X \times \mathbb{R}))''$ equal to the tensor product of $\mathscr{E}_m(X)$ with the orthogonal space to this kernel in $L^2(\mathbb{R})$. In particular, $(\mathscr{E}_m(X \times \mathbb{R}))' \simeq \mathscr{E}_m(X)$, so that the Hilbert module $\mathscr{L}_m$ obtained by completing $C^{\infty,0}_c(Y \times T, \widehat{H})$ does contain $\mathscr{E}_m(X)$ as an orthocomplemented submodule. Regarding the statements involving the continuity and limiting properties of $-\exp(i\pi\mathscr{B}_s)$, we shall treat only the first of the four steps proving Theorem 5.1.10 in [**29**]. Thus Y is the cylinder $X \times \mathbb{R}$ and

$$\mathscr{B}_t = \begin{pmatrix} 0 & \mathscr{D}_X \\ \mathscr{D}_X & 0 \end{pmatrix} + \frac{1}{t}\begin{pmatrix} x & \partial_x \\ -\partial_x & -x \end{pmatrix} \quad \text{with} \quad t \in (0,1].$$

The operator $\mathscr{B}_t$ restricted to $(\mathscr{E}_m(X \times \mathbb{R}))'$ is precisely $\begin{pmatrix} 0 & \mathscr{D}_X \\ \mathscr{D}_X & 0 \end{pmatrix}$. Let us consider $\mathscr{B}_t$ restricted to the orthocomplement $(\mathscr{E}_m(X \times \mathbb{R}))''$ and denote it $\mathscr{C}_t$, so that

$$\mathscr{B}_t = \begin{pmatrix} 0 & \mathscr{D}_X \\ \mathscr{D}_X & 0 \end{pmatrix} \oplus \mathscr{C}_t\,.$$

We can prove the norm-resolvent continuity of $\mathscr{C}_t$ (this notion extends to the C^*-algebraic framework) exactly as in [**29**]; we also obtain that $f(\mathscr{C}_t)$ goes to 0 in norm as $t \to 0$ for any rapidly decreasing function f. Using the fact that $\chi^2 - \mathrm{Id}$ is indeed rapidly decreasing we see that $\chi^2(\mathscr{C}_t) - \mathrm{Id}$ goes to zero in norm as $t \to 0$. A similar statement holds for $\chi(\mathscr{C}_t)(\chi^2(\mathscr{C}_t)-\mathrm{Id})$. Then, writing as in [**30**]

$$-\exp(i\pi z)=h(\pi^2(1-z^2))+(i\pi z)g(\pi^2(1-z^2))$$

with h and g entire, we prove that $-\exp(i\pi\chi(\mathscr{C}_t))$ converges in norm to the identity on $(\mathscr{E}_m(X\times\mathbb{R}))''$, so that $-\exp(i\pi\chi(\mathscr{B}_t))$ converges to (two copies of) $-\exp(i\pi\mathscr{D}_X)\oplus\mathrm{Id}_\perp$ as $t\to 0$. Of course, it is not true in this case that $-\exp(i\pi\chi(\mathscr{B}_t))$ converges to the identity as $t\to 1$ but the idea is that there will be further paths of operators in $\mathscr{I}\mathscr{K}_{\mathscr{A}_m}$ with the property that their concatenation will produce the desired path, joining $-\exp(i\pi\mathscr{D}_X)\oplus\mathrm{Id}_\perp$ to the identity up to stabilization. For the bordism relation these paths are obtained by adapting to our context, as we have done above, the remaining three paths appearing in the treatment of the bordism relation in [**29**]; see in particular the Subsections 5.1.2, 5.1.3, 5.1.4 there.

Finally, let us comment about cycles that are equivalent through a bundle modification. We are thus considering, in general,

$$(X,E\to X\times T)\sim(X',E'\to X'\times T)\equiv(\widehat{X},\widehat{E}\to\widehat{X}\times T)$$

where, as explained for example in [**29**], $\widehat{X}$ is a sphere bundle $S^{2n}\to\widehat{X}\xrightarrow{\pi}X$ and $\widehat{E}$ is the tensor product of $(\pi\times\mathrm{Id}_T)^*(E)$ and a certain bundle V built out of the Grassmann bundle of $\widehat{X}$; V is defined originally on $\widehat{X}$ and then extended trivially on all of $\widehat{X}\times T$. Consider the two T-families of Dirac-type operators defined by the equivariant Clifford modules $E\to X\times T$ and $E'\to X'\times T$ respectively and denote them briefly by $P=(P_\theta)_{\theta\in T}$ and $P'=(P'_\theta)_{\theta\in T}$ (for this argument we thus forget about the tilde). Let $\mathscr{E}_m$ and $\mathscr{E}'_m$ be the two Hilbert modules associated to these data and let $\mathscr{P}$ and $\mathscr{P}'$ be the regular operators defined by the two families above. Then we want to show that there exist

(i) an orthogonal decomposition of Hilbert modules $\mathscr{E}'_m=\mathscr{E}_m\oplus\mathscr{E}_m^\perp$;

(ii) a continuous s-path of Γ-equivariant first order differential operators $R_s:=(R_{s,\theta})_{\theta\in T}$, $s\in[0,2)$, on

$\widehat{X}$ with $R_0=P'$ and with regular extensions $\mathscr{R}_s$, $s\in[0,2)$;

(iii) for any controlled chopping function χ the path $-\exp(i\pi\chi(\mathscr{R}_s))$ is norm continuous in the space of

bounded operators in $\mathscr{E}'_m$;

(iv) $(-\exp(i\pi\chi(\mathscr{R}_s)))\longrightarrow(-\exp(i\pi\chi(\mathscr{P})))\oplus\mathrm{Id}_\perp$ as $s\to 2$.

The existence of the s-path $R_s:=(R_{s,\theta})_{\theta\in T}$, $s\in[0,2)$, is proved following the arguments in [**29**], Subsection 5.2: thus we write $P'=P^0+P^1+Z^0$ where for each $\theta\in T$, P^1_θ is a vertical operator on the fiber bundle $S^{2n}\to\widehat{X}\xrightarrow{\pi}X$, P^0_θ is a horizontal operator defined in terms of P_θ and Z^0_θ is a 0-th order operator. Define R_s, for $s\in[0,1]$ as $R_s:=P^0+P^1+(1-s)Z^0$ so that $R_0=P'$ as required. Next observe, as in [**29**], that for each $\theta\in T$ the vertical operator P^1_θ has a one-dimensional kernel, when

restricted to each sphere of the sphere bundle $S^{2n} \to \widehat{X} \xrightarrow{\pi} X$; using the orthogonal projection onto the null space of these operators on spheres we obtain an orthogonal decomposition $\mathscr{E}'_m = \mathscr{U} \oplus \mathscr{U}^\perp$ with $\mathscr{U}$ isomorphic to $\mathscr{E}_m$. We can now define R_s for $s \in [1,2)$; consider R_1 and its extension to $\mathscr{E}'_m$ which is diagonal with respect to the orthogonal decomposition. The restriction of $\mathscr{R}_1$ to $\mathscr{U}$ is, by definition, $\mathscr{P}^0$, given that $\mathscr{P}^1$ is zero on $\mathscr{U}$; using the isomorphism between $\mathscr{U}$ and $\mathscr{E}_m$, $\mathscr{P}^0$ can be connected to $\mathscr{P}$, since they differ by the extension of a 0-th order operator Z_1 (it will suffice to consider $P^0 + (s-1)Z_1$, $s \in [1,2]$). For the restriction of $\mathscr{R}_1 = \mathscr{P}^1 + \mathscr{P}^0$ to $\mathscr{U}^\perp$ we consider instead the open path $\mathscr{P}^0 + \frac{1}{2-s}\mathscr{P}^1$, $s \in [1,2)$. Summarizing, we have defined a continuous s-path of regular operators $\mathscr{R}_s$, $s \in [0,2)$. Using the fact that $(\mathscr{P}^1)^2$ is strictly positive on $\mathscr{U}^\perp$ one can prove the stated continuity properties, as well as the crucial fact that $(-\exp(i\pi\chi(\mathscr{R}_s))) \longrightarrow (-\exp(i\pi\chi(\mathscr{P}))) \oplus \mathrm{Id}_\perp$ as $s \to 2$.
Putting together the above two constructions, the one for the bordism relation and the one for the bundle modification relation, one can end the proof of the first four items in the statement of the theorem. We finally tackle the property that $-\exp(i\pi\chi(\mathscr{B}_s)) \in \mathscr{I}\mathscr{K}_{\mathscr{A}_m}$. From the fact that χ is controlled, it suffices to show that $f(\mathscr{B}_s)$ is in $\mathscr{K}_{\mathscr{A}_m}$ if f is rapidly decreasing; let us see this property for the case of the cylinder considered above. With respect to the above decomposition,

$$f(\mathscr{B}_s) = f(\begin{pmatrix} 0 & \mathscr{D}_X \\ \mathscr{D}_X & 0 \end{pmatrix}) \oplus f(\mathscr{C}_t)\,.$$

and it suffices to see that $f(\mathscr{C}_t)$ is compact. Write $f(\mathscr{C}_t) = (f(\mathscr{C}_t)(\mathscr{C}_t^2)^N) \circ (\mathscr{C}_t^2)^{-N}$, where we recall that $\mathscr{C}_t^2$ is positive. Since f is rapidly decreasing the first operator is bounded; thus we are left with the task of proving that $(\mathscr{C}_t^2)^{-N}$ is compact. Recall that $\mathscr{C}_t^2$ is the restriction to $(\mathscr{E}_m(X \times \mathbb{R}))''$ of $(\mathscr{D}^2 \otimes \mathrm{Id}_{2\times 2} + t^{-2}X^2)$, with $X = \begin{pmatrix} x & \partial_x \\ -\partial_x & -x \end{pmatrix}$.
Write $(\mathscr{C}_t^2)^{-N}$ in terms of the heat kernel, using the inverse Mellin transform:

$$(\mathscr{C}_t^2)^{-N} = \frac{1}{(N-1)!}\int_0^\infty \exp(-t\mathscr{C}_t^2)t^{N-1}dt\,.$$

Observe that the heat kernel of $(\mathscr{D}^2 \otimes \mathrm{Id}_{2\times 2} + t^{-2}X^2)$ decouples. Using again the invertibility of $\mathscr{C}_t^2$, the properties of the heat kernel of $\mathscr{D}^2$ and, more importantly, of the heat kernel of the harmonic oscillator, it is not difficult to end the proof. □

Let $\chi_\epsilon(x) := \chi(\epsilon x)$. Then, up to a harmless stabilization, the above theorem allows us to connect $\big(-\exp(i\pi\chi_\epsilon(\mathscr{D}'_m)) \oplus -\exp(-i\pi\chi_\epsilon(\mathscr{D}_m)\big)$ to the identity; we denote by $\gamma_1^\epsilon \in \mathscr{I}\mathscr{K}_{\mathscr{A}_m}$, $\gamma_1^\epsilon \equiv (\gamma_1^\epsilon(s))_{s\in[0,1]}$ the resulting path. Recall, however, that our goal is rather to connect $\big(-\exp(i\pi\phi_\epsilon(\mathscr{D}'_m)) \oplus -\exp(-i\pi\phi_\epsilon(\mathscr{D}_m)\big)$ to the identity, with $\phi_\epsilon(x) = \frac{2}{\sqrt{\pi}}\int_0^{\epsilon x} e^{-u^2}du$. Take the linear homotopy between the two chopping functions χ and ϕ and set

$$\mathcal{M}(t) := t\left(\chi(\epsilon\mathcal{D}'_m) \oplus -\chi(\epsilon\mathcal{D}_m)\right) + (1-t)\left(\phi(\epsilon\mathcal{D}'_m) \oplus -\phi(\epsilon\mathcal{D}_m)\right).$$

Consider then the path

$$\gamma_2^\epsilon(t) = -\exp(i\pi\mathcal{M}(t)).$$

Definition 9.9. — *The small time path ST_ϵ is the path obtained by the concatenation of γ_1^ϵ and γ_2^ϵ.*

So ST_ϵ is a path in $\mathcal{IK}_{\mathcal{A}_m}$ and connects $\psi_\epsilon(\mathcal{D}'_m)\oplus(\psi_\epsilon(\mathcal{D}_m))^{-1} \equiv \big(-\exp(i\pi\phi_\epsilon(\mathcal{D}'_m))$ $\oplus - \exp(-i\pi\phi_\epsilon(\mathcal{D}_m))$ to the identity.

9.5. The determinants of the small time path. — Let $(X, \mathcal{F})$, $X = Z\times_\Gamma T$, be a foliated bundle as in the proof of Theorem 9.8. Let L be a continuous longitudinally smooth vector bundle on X as in Theorem 9.8 and let $\mathcal{L}_m$ be the associated Hilbert $\mathcal{A}_m$-module. Let $\mathcal{B}_m^L$ be the maximal C^*-algebra associated to the groupoid $G^Z := (Z\times Z\times T)/\Gamma$. Recall the isomorphism $\chi_m : \mathcal{B}_m^L \to K_{\mathcal{A}_m}(\mathcal{L}_m)$, and the representations

$$\pi^{\rm reg} : \mathcal{B}_m^L \to W^*_\nu(G^Z; L)\,; \quad \pi^{\rm av} : \mathcal{B}_m^L \to W^*_\nu(X, \mathcal{F}; L)\,.$$

Proceeding as in Section 7, we can use χ_m^{-1} and $\pi^{\rm reg}$ in order to define a path $\sigma^{\rm reg}(ST_\epsilon)$ in $\mathcal{IK}(W^*_\nu(G^Z; L))$. The end-points of this path are τ^ν trace class perturbations of the identity; thus, see Remark 7.3, the determinant $w^\nu(\sigma^{\rm reg}(ST_\epsilon))$ is well defined and we can set

$$w^\nu_{\rm reg}(ST_\epsilon) := w^\nu(\sigma^{\rm reg}(ST_\epsilon)).$$

Similarly,

$$w^\nu_{\rm av}(ST_\epsilon) := w^\nu_{\mathcal{F}}(\sigma^{\rm av}(ST_\epsilon))$$

is well defined. The goal of this subsection is to indicate a proof of the following

Theorem 9.10. — *As $\epsilon \downarrow 0$ we have*

$$w^\nu_{\rm reg}(ST_\epsilon) - w^\nu_{\rm av}(ST_\epsilon) \longrightarrow 0\,. \tag{42}$$

Proof. — To simplify the notation we shall assume that the injectivity radius of $(\tilde M_\theta, \tilde g_\theta)$ is greater or equal to 1 for each $\theta \in T$; we also assume that for each $\theta \in T$ the distance between $\tilde m$ and $\tilde m\gamma$ is greater than 1 for each $\tilde m \in \tilde M_\theta$ and for each $\gamma \in \Gamma(\theta)$, $\gamma \neq e$. We begin by a few preliminary remarks. Recall that ST_ϵ is the concatenation of two paths: γ_1^ϵ and γ_2^ϵ. Using the fundamental Proposition 3.12 we observe that

$$\sigma^{\rm reg}(\gamma_1^\epsilon(t)) \equiv \sigma^{\rm reg}(-\exp(i\pi\chi(\mathcal{B}_t)) = -\exp(i\pi\chi(\tilde B_t))$$

and

$$\sigma^{\rm av}(\gamma_1^\epsilon(t)) \equiv \sigma^{\rm av}(-\exp(i\pi\chi(\mathcal{B}_t)) = -\exp(i\pi\chi(B_t))$$

with $B_t = ((B_t)_L)_{L\in X/\mathcal{G}}$ the longitudinal differential operator induced by the Γ-equivariant family $\tilde{B}_t$. (Once again, here and before the statement of Theorem 9.10 we are using, is a slight extension of the results proved in Section 3, allowing for manifolds with cylindrical ends and operators that are modeled like harmonic oscillators along the ends). Similarly, up to a harmless stabilization by $\mathrm{Id}_\perp$ (that will in any case disappear after taking determinants), we can write

$$\sigma^{\mathrm{reg}}(\gamma_2^\epsilon(t)) = -\exp i\pi\left(t\chi(\epsilon\tilde{P}) + (1-t)\phi(\epsilon\tilde{P})\right),$$

$$\sigma^{\mathrm{av}}(\gamma_2^\epsilon(t)) = -\exp i\pi\left(t\chi(\epsilon P) + (1-t)\phi(\epsilon P)\right)$$

where $\tilde{P}$ and P are the signature operators on $(\tilde{M}'\sqcup(-\tilde{M}))\times T\to T$ and on $(X,\mathcal{G}^\sqcup)$ respectively (this is the notation we had introduced in the subsection on the large time path). One can prove that for $j=1,2$ the paths $\sigma^{\mathrm{reg}}(\gamma_j^\epsilon)$ and $\sigma^{\mathrm{av}}(\gamma_j^\epsilon)$ are all made of *trace class* perturbations of the identity. Moreover, the determinants of these two paths are well defined *individually* and without the regularizing procedure explained in Proposition 5.8. We shall justify this claim in a moment. This property granted, we can break the proof of (42) into two distinct statements:

$$w^\nu_{\mathrm{reg}}(\gamma_1^\epsilon) - w^\nu_{\mathrm{av}}(\gamma_1^\epsilon) \longrightarrow 0\,. \tag{43}$$

$$w^\nu_{\mathrm{reg}}(\gamma_2^\epsilon) - w^\nu_{\mathrm{av}}(\gamma_2^\epsilon) \longrightarrow 0\,. \tag{44}$$

We now tackle (44) which is slightly easier since it involves exclusively operators on manifolds without boundary.

First we observe that to each operator $\tilde{P}_\theta$ and P_L we can apply the results of [**54**], [**51**]. In particular, using the properties of χ, which is of controlled type, and ϕ we have:

1. $\chi(\tilde{P}_\theta)$ and $\chi(P_L)$, are given by 0-th order pseudodifferential operators with Schwartz kernel localized in an uniform R-neighbourhood of the diagonal (remember that the Fourier transform of χ is compactly supported); we shall assume without loss of generality that $R=1$;
2. $\phi(\tilde{P}_\theta)$ and $\phi(P_L)$ are each one the sum of a 0-th order pseudodifferential operators with Schwartz kernel localized in an uniform $R=1$-neighbourhood of the diagonal and of an integral operator with smooth kernel;
3. if $\tilde{\chi}$ denotes the linear chopping function equal to $sign(x)$ for $|x|>1$ and equal to x for $|x|\leq 1$ then $(\chi(\tilde{P}_\theta)-\tilde{\chi}(\tilde{P}_\theta))_{\theta\in T}$ and $(\phi(\tilde{P}_\theta)-\tilde{\chi}(\tilde{P}_\theta))_{\theta\in T}$ are τ^ν trace class elements given by longitudinally smooth kernels (indeed, the differences $\chi-\tilde{\chi}$ and $\phi-\tilde{\chi}$ are rapidly decreasing);
4. similarly, $(\chi(P_L)-\tilde{\chi}(P_L))_{L\in X/\mathcal{G}^\sqcup}$ and $(\phi(P_L)-\tilde{\chi}(P_L))_{L\in X/\mathcal{G}^\sqcup}$ are $\tau^\nu_{\mathcal{G}^\sqcup}$ trace class elements given by uniformly bounded longitudinally smooth kernels;

5. consequently, $(\chi(\tilde{P}_\theta) - \phi(\tilde{P}_\theta))_{\theta\in T}$ and $(\chi(P_L) - \phi(P_L))_{L\in X/\mathscr{G}^\sqcup}$ are both trace class elements given by longitudinally smooth kernels; indeed it suffices to write $(\chi(\tilde{P}_\theta) - \phi(\tilde{P}_\theta))_{\theta\in T} = ((\chi(\tilde{P}_\theta) - \tilde{\chi}(\tilde{P}_\theta))_{\theta\in T} + (\tilde{\chi}(\tilde{P}_\theta) - \phi(\tilde{P}_\theta))_{\theta\in T}$.

Notice that these properties imply easily the claim we have made about the determinants of $\sigma^{\mathrm{reg}}(\gamma_2^\epsilon)$ and $\sigma^{\mathrm{av}}(\gamma_2^\epsilon)$. We go back to our goal, i.e. proving (44). We observe that since γ_2^ϵ is defined in terms of a linear homotopy, we have, by direct computation,

$$w^\nu_{\mathrm{reg}}(\gamma_2^\epsilon) = -\frac{1}{2}\tau^\nu\left(\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})\right) \quad w^\nu_{\mathrm{av}}(\gamma_2^\epsilon) = -\frac{1}{2}\tau^\nu_{\mathscr{G}^\sqcup}\left(\chi(\epsilon P) - \phi(\epsilon P)\right).$$

Write

$$\tau^\nu\left(\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})\right) = \tau^\nu\left((\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon) - (\phi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon)\right)$$

with $\phi(\epsilon\tilde{P})_\epsilon$ a compression of $\phi(\epsilon\tilde{P})$ to a Γ-equivariant ϵ-neighbourhood of $\{(\tilde{m},\tilde{m},\theta), \tilde{m}\in\tilde{M}, \theta\in T\}$ in $\tilde{M}\times\tilde{M}\times T$. Both $\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon)$ and $\phi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon)$ are individually τ^ν trace class: indeed the first term is the ϵ-compression of a longitudinally smooth kernel (since $\chi(\epsilon\tilde{P})$ is already ϵ-local) and it is therefore τ^ν trace class; the second term can be written as the sum $(\phi(\epsilon\tilde{P}) - \chi(\epsilon\tilde{P})) + (\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon)$ and both terms are trace class; thus

$$\tau^\nu\left(\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})\right) = \tau^\nu\left((\chi(\epsilon\tilde{P}) - (\phi(\epsilon\tilde{P})_\epsilon\right) - \tau^\nu\left(\phi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon\right).$$

A similar expression can be written for $\tau^\nu_{\mathscr{G}^\sqcup}(\chi(\epsilon P) - \phi(\epsilon P))$. Consider now the difference $w^\nu_{\mathrm{reg}}(\gamma_2^\epsilon) - w^\nu_{\mathrm{av}}(\gamma_2^\epsilon)$ which is the sum

$$\left(\tau^\nu(\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon) - \tau^\nu_{\mathscr{G}^\sqcup}(\chi(\epsilon P) - \phi(\epsilon P)_\epsilon)\right) + \tag{45}$$

$$\left(\tau^\nu(\phi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon) - \tau^\nu_{\mathscr{G}^\sqcup}(\phi(\epsilon P) - \phi(\epsilon P)_\epsilon)\right). \tag{46}$$

As already remarked the two differences $\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon$ and $\chi(\epsilon P) - (\phi(\epsilon P))_\epsilon$ are given by longitudinally smooth kernel which are supported in an ϵ-neighbourhood of the diagonal. Proceeding as in the proof of Proposition 4.4 we shall now prove that $\tau^\nu(\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon) - \tau^\nu_{\mathscr{G}^\sqcup}(\chi(\epsilon P) - \phi(\epsilon P)_\epsilon)$ is in fact equal to zero for ϵ small enough. Indeed, consider the Γ-equivariant family $\chi(\epsilon\tilde{P})$; we know that $\chi(\epsilon\tilde{P}) \in \Psi^0_c(G,E)$. Similarly, consider $\phi(\epsilon\tilde{P})_\epsilon \in \Psi^0_c(G,E)$. We know that $\chi(\epsilon\tilde{P}) - (\phi(\epsilon\tilde{P}))_\epsilon \in \Psi^{-\infty}_c(G,E)$ and that this operator extends to an element $\mathscr{P}^\epsilon_{\chi,\phi} \in \mathscr{K}_{\mathscr{A}_m}(\mathscr{H}_m)$. Observe now that

$$(\mathscr{P}^\epsilon_{\chi,\phi})\otimes_{\pi^{\mathrm{reg}}_\theta}\mathrm{Id} = \chi(\epsilon\tilde{P}_\theta) - \phi(\epsilon\tilde{P}_\theta)_\epsilon, \quad (\mathscr{P}^\epsilon_{\chi,\phi})\otimes_{\pi^{\mathrm{av}}_\theta}\mathrm{Id} = \chi(\epsilon P_L) - \phi(\epsilon P_L)_\epsilon \text{ with } L = L_\theta.$$

Using Theorem 3.19 we thus can write

$$\tau^\nu(\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P}))_\epsilon) - \tau^\nu_{\mathscr{G}^\sqcup}(\chi(\epsilon P) - \phi(\epsilon P))_\epsilon) = \tau^\nu_{\mathrm{reg}}(\mathscr{P}^\epsilon_{\chi,\phi}) - \tau^\nu_{\mathrm{av}}(\mathscr{P}^\epsilon_{\chi,\phi})$$

where we have omitted the isomorphism $\chi_m^{-1} : \mathscr{K}_{\mathscr{A}_m}(\mathscr{H}_m) \to \mathscr{B}^H_m$. Taking ϵ small enough and proceeding precisely as in the proof of Proposition 4.4 we see that the right hand side is equal to zero for ϵ small enough (it is in this last step that we use the fact that $\mathscr{P}^\epsilon_{\chi,\phi}$ is given by an ϵ-localized smoothing kernel). Finally, the terms in the second summand of (45) are individually zero since they are trace class elements given by longitudinally smooth kernels which restrict to zero on the diagonal. Summarizing: $w^{\nu}_{\rm reg}(\gamma^{\epsilon}_2) - w^{\nu}_{\rm av}(\gamma^{\epsilon}_2) = 0$ for ϵ small enough.

We are left with the task of proving that γ^{ϵ}_1 has well defined determinants and that

$$\lim_{\epsilon\to 0} w^{\nu}_{\rm reg}(\gamma^{\epsilon}_1) - w^{\nu}_{\rm av}(\gamma^{\epsilon}_1) = 0\,. \tag{47}$$

To this end we begin by writing explicitly the left hand side:

$$w^{\nu}_{\rm reg}(\gamma^{\epsilon}_1) = \frac{1}{2\pi i}\int_0^1 \tau^{\nu}\left((-\exp(-i\pi\chi(\epsilon\tilde{B}_t)))\frac{d}{dt}(-\exp(i\pi\chi(\epsilon\tilde{B}_t)))\right)dt \tag{48}$$

$$w^{\nu}_{\rm av}(\gamma^{\epsilon}_1) = \frac{1}{2\pi i}\int_0^1 \tau^{\nu}_{\mathcal{G}}\left((-\exp(-i\pi\chi(\epsilon B_t)))\frac{d}{dt}(-\exp(i\pi\chi(\epsilon B_t)))\right)dt \tag{49}$$

provided the right hand sides make sense. To see why the last statement is true, we begin by making a general comment on the traces we are using. Remember that the two paths of operators $\tilde{B}_s$ and B_s, $s\in(0,1)$, are defined on foliated bundles that might have as leaves manifolds with cylindrical ends. We define the two relevant von Neumann algebras in the obvious way and we define the two traces τ^{ν} and $\tau^{\nu}_{\mathcal{G}}$ as we did in Subsection 2.4. Needless to say, an arbitrary smoothing operator will not be trace class on such a foliation, since its Schwartz kernel might not be integrable in the cylindrical direction. (This is the typical situation for the heat kernel associated to a Dirac operator which restrict to a $\mathbb{R}^+$-invariant operator $\frac{d}{dt}+D_{\partial}$ along the cylindrical ends.) We now write

$$\exp(i\pi z) = h(\pi^2(1-z^2)) + (i\pi z)g(\pi^2(1-z^2))$$

with h and g entire. Recall that χ is of controlled type; we shall now see that this implies that $1-\chi^2(\tilde{B}_t)$ is τ^{ν} trace class and $1-\chi^2(B_t)$ is $\tau^{\nu}_{\mathcal{G}}$ trace class. Moreover these operators are given by longitudinally smooth kernels that are supported within a uniform $(R=1)$-neighbourhood of the diagonal. These statements are clear when $(\tilde{M},\Lambda\to\tilde{M}\times T)\sim(\tilde{M}',\Lambda'\to\tilde{M}'\times T)$ through a bundle modification or a direct sum of vector bundles (indeed, from our discussion of the bundle modification relation in the proof of Theorem 9.8, it is clear that in this case we remain within the category of foliations of compact manifolds without boundary and it suffices to apply **[50]** for the latter property and **[22]** for the first). If $(\tilde{M},\Lambda\to\tilde{M}\times T)\sim(\tilde{M}',\Lambda'\to\tilde{M}'\times T)$ through a bordism, then we use the fact that $\tilde{B}_{\theta,t}$ and $(B_t)_L$ are again of bounded propagation speed and restrict to harmonic oscillators along the cylinders of the relevant manifolds with cylindrical ends (this is needed in order to make claims about the trace class property). For the trace class property we also make use of the results in **[22]**, proceeding as in **[29]** but using singular numbers instead of eigenvalues.

Using $\exp(i\pi z) = h(\pi^2(1-z^2)) + (i\pi z)g(\pi^2(1-z^2))$ we can then conclude, as in **[30]** Lemma 4.1.7, that

$$\sigma^{\mathrm{reg}}(\gamma_1^\epsilon(t)) \equiv -\exp(i\pi\chi(\epsilon\tilde{B}_t)) \quad \text{and} \quad \sigma^{\mathrm{av}}(\gamma_1^\epsilon(t)) \equiv -\exp(i\pi\chi(\epsilon B_t)), \quad t \in [0,1]$$

are piecewise continuosly differentiable in the L^1 norm and that they both have a well defined (log-)determinant, as we had claimed (notice that in the proof of Lemma 4.1.7 in **[30]** only the controlled property of χ is used).

Having justified (48) and (49), we next make the following:

Claim. — *There exists polynomials p_1, p_2 such that, uniformly in $s \in [0,1]$,*

$$||\chi(\tilde{B}_s) - \chi(\epsilon\tilde{B}_s)||_1 < p_1(\frac{1}{\epsilon}), \quad ||\chi(B_s) - \chi(\epsilon B_s)||_1 < p_2(\frac{1}{\epsilon}). \tag{50}$$

Assume the claim; then using the inequality

$$||AB||_1 \le ||A||_1 ||B||_\infty, \quad A \in L^1(\mathcal{M}, \tau) \cap \mathcal{M}, \quad B \in \mathcal{M}$$

which is valid in any Von Neumann algebra $\mathcal{M}$ endowed with a faithful normal trace τ, one can show, proceeding exactly as in Lemma 4.2.8 of **[30]**, that there exist polynomials q_1 and q_2 such that, uniformly in $s \in [0,1]$,

$$||\chi^2(\epsilon\tilde{B}_s) - \mathrm{Id}||_1 < q_1(\frac{1}{\epsilon}), \quad ||\chi^2(\epsilon B_s) - \mathrm{Id}||_1 < q_2(\frac{1}{\epsilon}). \tag{51}$$

We first end the proof of (47) using (51).
For any entire function $f(z) = \sum_{n=0}^\infty a_n z^n$ we define $[f(z)]_N := \sum_{n=0}^N a_n z^n$. Consider the entire function h in the decomposition $\exp(i\pi z) = h(\pi^2(1-z^2)) + (i\pi z)g(\pi^2(1-z^2))$. Proceeding as in Lemma 4.2.6 in **[30]** we show using the first inequality in (51) that for each $\alpha > 0$ there exists an $\epsilon > 0$ and an integer N_ϵ such that

- $||h(\pi^2(\mathrm{Id} - \chi^2(\epsilon\tilde{B}_s))) - [h(\pi^2(\mathrm{Id} - \chi^2(\epsilon\tilde{B}_s)))]_{N_\epsilon}||_1 < \alpha$
- $[h(\pi^2(\mathrm{Id} - \chi^2(\epsilon\tilde{B}_s)))]_{N_\epsilon}$ is of propagation less than 1.

Remark here that N_ϵ is in fact fixed by ϵ and, with our conventions, can be set to be equal to the integral part of $1/\epsilon$. Thus the left hand side of the above inequality can be thought of as a positive function of ϵ, converging to 0 when $\epsilon \downarrow 0$. A similar statement can be made for the derivative of $h(\epsilon\tilde{B}_s)$ with respect to s. Applying the same reasoning to the second summand in the decomposition $\exp(i\pi z) = h(\pi^2(1-z^2)) + (i\pi z)g(\pi^2(1-z^2))$ we conclude as in **[30]** Lemma 4.2.10, that for each $\alpha > 0$ there exists an $\epsilon > 0$ and an integer N_ϵ such that

$$\begin{aligned} \Big| &\int_0^1 \tau^\nu \left((-\exp(-i\pi\chi(\epsilon\tilde{B}_t))) \frac{d}{dt} (-\exp(i\pi\chi(\epsilon\tilde{B}_t))) \right) dt \\ &- \int_0^1 \tau^\nu \left(([-\exp(-i\pi\chi(\epsilon\tilde{B}_t))]_{N_\epsilon}) \frac{d}{dt} ([-\exp(i\pi\chi(\epsilon\tilde{B}_t))]_{N_\epsilon}) \right) dt \Big| < \alpha. \end{aligned} \tag{52}$$

Similarly, using the second inequality in the claim and the second inequality in (51), we can prove that for each $\alpha > 0$ there exists a $\delta > 0$ and an integer N_δ such that

$$\begin{aligned}(53)\qquad \Big| \int_0^1 \tau^\nu_{\mathscr{G}} \Big((-\exp(-i\pi\chi(\delta B_t))) \frac{d}{dt}(-\exp(i\pi\chi(\delta B_t))) \Big)\, dt \\ - \int_0^1 \tau^\nu_{\mathscr{G}} \Big(([-\exp(-i\pi\chi(\delta B_t))]_{N_\delta}) \frac{d}{dt}([-\exp(i\pi\chi(\delta B_t))]_{N_\delta}) \Big)\, dt \Big| < \alpha.\end{aligned}$$

Since the left hand sides of the inequalities (52), (53) can be thought of as positive functions of ϵ and δ converging to 0 as $\epsilon \downarrow 0$ and $\delta \downarrow 0$, it is clear that we can ensure the existence of a common value, say η and N_η, for which both inequalities are satisfied. Consider again the difference $|w^\nu_{\rm reg}(\gamma^\epsilon_1) - w^\nu_{\rm av}(\gamma^\epsilon_1)|$ that we rewrite as $|A_\epsilon + B_\epsilon + C_\epsilon|$ with

$$\begin{aligned} A_\epsilon &:= w^\nu_{\rm reg}\gamma^\epsilon_1 - \int_0^1 \tau^\nu \Big(([-\exp(-i\pi\chi(\epsilon\tilde{B}_t))]_{N_\epsilon}) \frac{d}{dt}([-\exp(i\pi\chi(\epsilon\tilde{B}_t))]_{N_\epsilon}) \Big)\, dt \\ B_\epsilon &:= \int_0^1 \tau^\nu \Big(([-\exp(-i\pi\chi(\epsilon\tilde{B}_t))]_{N_\epsilon}) \frac{d}{dt}([-\exp(i\pi\chi(\epsilon\tilde{B}_t))]_{N_\epsilon}) \Big)\, dt \\ &\quad - \int_0^1 \tau^\nu_{\mathscr{G}} \Big(([-\exp(-i\pi\chi(\epsilon B_t))]_{N_\epsilon}) \frac{d}{dt}([-\exp(i\pi\chi(\epsilon B_t))]_{N_\epsilon}) \Big)\, dt \\ C_\epsilon &:= \int_0^1 \tau^\nu_{\mathscr{G}} \Big(([-\exp(-i\pi\chi(\epsilon B_t))]_{N_\epsilon}) \frac{d}{dt}([-\exp(i\pi\chi(\epsilon B_t))]_{N_\epsilon}) \Big)\, dt - w^\nu_{\rm av}\gamma^\epsilon_1. \end{aligned}$$

We know that for each $\alpha > 0$ there exists a common ϵ such that $|A_\epsilon| < \alpha$ and $|C_\epsilon| < \alpha$. On the other hand, using the fact that $[-\exp(i\pi\chi(\epsilon B_t))]_{N_\epsilon}$ is of propagation equal to 1, we can prove, proceeding as in Proposition 4.4, that there exists ϵ such that $B_\epsilon = 0$. Thus we have proved (47) modulo the claim.

We shall prove the claim for the particular case of the cylinder; let us prove, for example, the first inequality. Consider

$$\tilde{B}_t = \begin{pmatrix} 0 & \tilde{D} \\ \tilde{D} & 0 \end{pmatrix} + \frac{1}{t}\begin{pmatrix} x & \partial_x \\ -\partial_x & -x \end{pmatrix} \quad \text{with} \quad t \in (0,1].$$

Observe that the left hand side of the first inequality in the claim is nothing but the last term in inequality (4.3) in **[29]**. Proceed now exactly as in the part of the proof of Lemma 4.7 in **[29]** that begins with the inequality (4.3). It is not difficult to realize that the proof given there, i.e. the proof of the first inequality in the claim, can be easily adapted to our von Neumann context using singular numbers and the results of Fack and Kosaki. More precisely, the operator $\tilde{B}^2_t$ can be diagonalized with respect to the eigenfunctions of the operator X^2, with

$$X = \begin{pmatrix} x & \partial_x \\ -\partial_x & -x \end{pmatrix}.$$

The functional calculus of $\tilde{B}_t^2$ is then reduced to the functional calculus of the operator

$$\tilde{D}' + \frac{1}{t}\lambda_k, \quad \text{with} \quad \tilde{D}' = \begin{pmatrix} \tilde{D}^2 & 0 \\ 0 & \tilde{D}^2 \end{pmatrix}$$

and where λ_k is an eigenvalue of X^2 as in **[29]** . Now the L^1-norm $\|\chi(\tilde{B}_s) - \chi(\epsilon\tilde{B}_s)\|_1$ is given by the sum over k of L^1-norms in corresponding von Neumann algebras of the operator $(\chi - \chi_\epsilon)(\tilde{D}' + \lambda_k)$. By **[22]**, this L^1-norm is expressed in terms of the singular numbers $\mu_s^\nu(\tilde{D}' + \lambda_k) = \mu_s^\nu(\tilde{D}') + \lambda_k$. This reduces the estimate to the similar estimate of the singular numbers of $\tilde{D}'$ exactly as in **[29]**. This latter being a leafwise elliptic second order differential operator, we can use the estimate $\mu_s(\tilde{D}') \sim s^{2/p}$ where p is the dimension of the leaves, see for instance **[8]**. Hence the proof of the first inequality of the claim is completed following the steps of **[29]**. The proof of the second inequality in the claim is similar. Thus we have proved the claim and thus (47) in the case of cylinders. For manifolds with cylindrical ends we split the relevant statements into purely cylindrical ones and statements on compact foliated bundles, as in **[29]**. We end here our explanation of the proof of (47). The proof of Theorem 8.1 is now complete. □

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Astérisque
327, 2009, p. 289–360

DIRECT IMAGE FOR SOME SECONDARY K-THEORIES

by

Alain Berthomieu

This article is dedicated to J.-M. Bismut, for his sixtieth birthday

Abstract. — The real counterpart of relative K-theory (considered in the complex setting in [**4**]) is considered here, some direct image under proper submersion is constructed, and a Grothendieck-Riemann-Roch theorem for Johnson-Nadel-Chern-Simons classes is proved. Metric properties are also studied.

This needs to revisit the construction of η-forms in the case where the direct image is provided by the vertical Euler (de Rham) operator. A direct image under proper submersions of some "non hermitian smooth" or "free multiplicative" K-theory is deduced (in the same context).

Double submersions are also studied to establish some functoriality properties of these direct images.

Résumé **(Image directe pour certaines K-théories secondaires).** — On construit un morphisme d'image directe par submersion propre pour la version réelle de la K-théorie relative (considérée dans [**4**] dans un contexte holomorphe), et un théorème de type Grothendieck-Riemann-Roch est établi pour les classes de Johnson-Nadel-Chern-Simons. On étudie aussi des propriétés métriques.

Ceci nécessite de construire des formes η (de transgression du théorème d'indice des familles) dans le cas où l'image directe est définie par l'operateur d'Euler (de Rham) des fibres. On en déduit également un morphisme d'image directe pour une K-théorie « lisse non hermitienne » ou « multiplicative libre ».

La question de la fonctorialité de ces images directes pour des doubles submersions est également abordée.

1. Introduction

In [**35**], Nadel proposed characteristic classes (also considered by Johnson [**23**], see infra) for triples (E, F, f) where E and F are holomorphic vector bundles on

2010 Mathematics Subject Classification. — primary: 14F05, 19E20, 57R20, secondary: 14F40, 19D55, 53C05, 55R50.

Key words and phrases. — Relative K-theory, multiplicative K-theory, smooth K-theory, local families index, Chern-Simons, direct image, pushforward.

some Kähler manifold X, and $f\colon E \xrightarrow{\sim} F$ is a C^∞ vector bundle isomorphism. He conjectured that if X is projective, his classes, which take their values in $H^{(0,\mathrm{odd})}(X)$, were projections of the image by the Abel-Jacobi map of the difference of the Chow group valued Chern classes of E and F. Inspired by **[26]** §6, I developped in **[4]** a notion of relative K-theory which appeared as suitably adapted to describe such triples considered by Nadel. This theory measures the kernel of the forgetful map from the K^0-theory of holomorphic vector bundles on X to the usual topological K^0-theory. As such, if X is projective, any pointed fine moduli space of vector bundles on X naturally maps to this relative K-theory. Moreover, it is rationally isomorphic to the Chow subgroup of homologically trivial cycles.

In [**4**], Johnson-Nadel classes were extended by considering a suitable projection of the Chern-Simons transgression form associated to compatible connections on E and F. The obtained characteristic class was proved to solve a generalised form of Nadel's conjecture.

I realised very recently that D. Johnson already obtained partial results in this direction: in **[24]** it seems that the same classes as considered by Nadel were defined, and in **[23]** some weaker version (than in [**4**]) of the classes were constructed and a weaker version of the "generalized Nadel conjecture" was proved.

[**4**] also contains direct images and Grothendieck-Riemann-Roch type results for relative K-theory and its characteristic class, for submersions and immersions of smooth projective varieties.

One of the goals of this article is to study the counterpart of this theory in the context of complex flat vector bundles over some real smooth manifold M. The corresponding relative K-theory was defined by Karoubi **[26]** §6 and studied by Karoubi and Dupont **[17]**. It is here described from objects of the form $(E, \nabla_E, F, \nabla_F, f)$ where f is a smooth vector bundle isomorphism between complex vector bundles E and F endowed with flat connections ∇_E and ∇_F (see Definition 4). If M is compact, the pointed algebraic variety $\mathscr{V}_F$ of flat vector bundle structures on some fixed topological vector bundle on M naturally maps to this relative K-theory.

If $\pi\colon M \longrightarrow B$ is a proper submersion, I construct here (see Definition 26 and Theorem 27) a direct image morphism $\pi_*\colon K^0_{\mathrm{rel}}(M) \longrightarrow K^0_{\mathrm{rel}}(B)$. The main technical problem consists in finding a vector bundle isomorphism (or something equivalent) between representatives of $\pi_!E$ and $\pi_!F$ as virtual flat vector bundles on B in such a way that the direct image becomes natural and functorial.

The counterpart here of Johnson-Nadel classes is simply given by Chern-Simons transgression forms in odd degree de Rham cohomology:

$$\mathscr{N}_{\mathrm{ch}}(E, \nabla_E, F, \nabla_F, f) = \big[\widetilde{\mathrm{ch}}(\nabla_E, f^*\nabla_F)\big] \in H^{\mathrm{odd}}_{dR}(X). \tag{1}$$

Because of its rigidity properties, this Chern-Simons class may essentially detect different connected components of the above algebraic variety $\mathscr{V}_F$, and the class of the determinant line bundle (see §2.3).

A Grothendieck-Riemann-Roch type theorem for $\mathscr{N}_{\mathrm{ch}}$ (Theorem 29) is obtained as a by-product of the constructions performed in pursuing the second goal of the article, namely the study of "free multiplicative" or "non hermitian smooth" K-theory. This K-theory, denoted by $\widehat{K}_{\mathrm{ch}}$ is generated by triples of the form (E, ∇, α) where ∇ is a connection on the complex vector bundle E over M and α is an odd degree differential form defined modulo exact forms. Relations are direct sum and if $f\colon E \longrightarrow F$ is any smooth vector bundle isomorphism:

$$(E, \nabla_E, \alpha) = \big(F, \nabla_F, \alpha + \widetilde{\mathrm{ch}}(\nabla_E, f^*\nabla_F)\big) \tag{2}$$

(Here $\widetilde{\mathrm{ch}}$ is again a Chern-Simons transgression form). K^0_{rel} and $\widehat{K}_{\mathrm{ch}}$ are related by a commutative diagram whose lines are exact sequences (see Proposition 10):

$$\begin{tikzcd}
K^1_{\mathrm{top}}(M) \arrow[r] \arrow[d, "\|"] & K^0_{\mathrm{rel}}(M) \arrow[r] \arrow[d, "\mathscr{N}_{\mathrm{ch}}"] & K^0_{\mathrm{flat}}(M) \arrow[r] \arrow[d] & K^0_{\mathrm{top}}(M) \arrow[d, "\|"] \\
K^1_{\mathrm{top}}(M) \arrow[r, "\mathrm{ch}"] & \Omega^{\mathrm{odd}}(M)/d\Omega^{\mathrm{even}}(M) \arrow[r] & \widehat{K}_{\mathrm{ch}}(M) \arrow[r] & K^0_{\mathrm{top}}(M)
\end{tikzcd} \tag{3}$$

In this diagram, $\Omega^\bullet(M)$ denotes differential forms, K_{top} denotes ordinary K-theory, and K^0_{flat} denotes the K^0 theory of the category of flat bundles modulo exact sequences. For any vector bundle E on M endowed with a flat connection ∇_E, the image in $\widehat{K}_{\mathrm{ch}}(M)$ of $(E, \nabla_E) \in K^0_{\mathrm{flat}}(M)$ is the triple $(E, \nabla_E, 0)$.

On one hand, Karoubi's multiplicative K-theory **[26]** **[27]** **[28]** consists of quotients (the form α being defined modulo greater subgroups than only exact forms) of subgroups (defined by restrictions on the Chern-Weil character form $\mathrm{ch}(\nabla)$) of this theory. These subgroups and constraints stemm from natural filtrations of the de Rham complex of M suitably adapted to the geometry studied. In **[28]**, Karoubi studies foliations for which he constructs generalisations of the Godbillon-Vey invariant, and holomorphic and algebraic varieties for which known characteristic classes for holomorphic or algebraic vector bundles are shown to factor through the suitable multiplicative K-theory. Poutriquet **[36]** studies the context of conical singularities. The corresponding multiplicative K-theory he constructs shows interesting similarities with intersection cohomology. Felisatti and Neumann **[18]** generalise the concept of multiplicative K-theory to simplicial manifolds with applications to classifying spaces of Lie groups and Lie groupoids.

As an example, the multiplicative K-theory adapted to the study of flat bundles is the subgroup of $\widehat{K}_{\mathrm{ch}}$ generated by triples (E, ∇, α) such that

$$\mathrm{ch}(\nabla) - d\alpha \in \mathbb{Z} \subset \Omega^{\mathrm{even}}(M) \tag{4}$$

Removing this constraint would justify the name "free multiplicative" K-theory. Direct image results for $\widehat{K}_{\mathrm{ch}}$ should have corollaries for "nonfree" multiplicative K-theories under mild compatibility conditions on the filtrations of the de Rham complex used to define them.

On the other hand, Bunke and Schick [**14**] defined a smooth (hermitian) K-theory, which coincides with the subgroup of $\widehat{K}_{\mathrm{ch}}$ generated by triples (E, ∇, α) where α is a real form and ∇ respects some hermitian metric on E. Bunke and Schick's smooth K-theory is motivated by quantum field theory considerations [**19**] and it fits in the general framework of smooth extensions of generalized cohomology theories [**20**] [**21**]. Among other examples, Bunke and Schick construct interesting smooth K-theory canonical classes on homogeneous spaces and generalisations of parametrized ρ-invariants [**14**] §5.

Allowing nonunitary connections (and nonreal forms) would justify the name "non hermitian smooth K-theory". Anyway, the hermitian restriction would prevent from obtaining a natural morphism $K^0_{\mathrm{flat}}(M) \longrightarrow \widehat{K}_{\mathrm{ch}}(M)$ because of the existence of nonunitary flat vector bundles.

The obstruction for a flat bundle (E, ∇_E) to be unitary can be detected by characteristic classes similar to $\mathscr{N}_{\mathrm{ch}}(E, \nabla_E, E, \nabla_E^*, \mathrm{Id}_E)$ where ∇_E^* is the adjoint connection of ∇_E with respect to any hermitian metric on E (22). Such classes were first considered by Kamber and Tondeur [**25**], they correspond to the imaginary part of Chern-Cheeger-Simons classes [**15**], (see [**11**] Proposition 1.14). Karoubi proved in [**26**] §6.31 that they could detect some Borel generators of algebraic K-theory of integer rings in number fields [**12**]. See also [**11**] §I(g) for an interpretation as stable characteristic classes arising from stable continuous cohomology of $GL(\mathbb{C})$.

Here this Borel-Kamber-Tondeur class is extended to $\widehat{K}_{\mathrm{ch}}$. It is not always a cohomology class, but rather a purely imaginary differential form defined modulo exact forms (see Definition 16).

Moreover, a direct image morphism for $\widehat{K}_{\mathrm{ch}}$ under proper submersions is constructed (Theorem 31), which is compatible with the usual (sheaf theoretic) direct image of flat vector bundles (using fiberwise twisted de Rham cohomology, see Definition 22). This is performed from the families analytic index of the fiberwise twisted Euler operator together with a suitable η-form which is a non hermitian generalisation of that of Bunke [**13**] (Theorem 28). Functoriality is established only for the "nonfree" multiplicative subgroup of $\widehat{K}_{\mathrm{ch}}$ subject to the constraint (4), using some universal characterisation of the η-form.

Finally the symmetries induced by the fiberwise Hodge star operator are studied. Reality (resp. vanishing) properties of the pushforwards are established in the even (resp. odd) dimensional fibre case (Theorems 32 and 33).

The paper is organized as follows: the definitions of K-theories and characteristic classes, and their mutual relations are given in §2, the pushforward morphisms are defined and all the theorems are stated in §3, the construction of the direct image for relative K-theory is performed in §4, the construction of the η-form and all its consequences are detailed in §5, and §6 is devoted to results about symmetries induced by the fiberwise Hodge star operator. Finally, double fibrations are studied in §7. This paper is a reformulation of previously diffused preprints. I apologize for some changes of title, names and notations between earlier versions and this one.

I am very grateful to Thomas Schick, Sebastian Goette, Kiyoshi Igusa, Xiaonan Ma, Weiping Zhang and Xianzhe Dai for their kind invitations to Oberwolfach and Tianjin conferences and to Ulrich Bunke for giving me the idea to free multiplicative K-theory of the constraint (4).

2. Various K-theories

After recalling some facts about Chern-Simons transgression in §2.1, the definitions of all the K-theory groups considered here are given in §2.2. §2.3 is devoted to the counterpart of Johnson-Nadel's classes defined in [**4**], §2.4 to the diagrams and exact sequences in which these K-groups enter, §2.5 and §2.6 to hermitian metrics and the extended Borel-Kamber-Tondeur class on $\widehat{K}_{\mathrm{ch}}$.

2.1. Preliminaries

2.1.1. *Connections and vector bundle morphisms.* — Let M be a smooth manifold. Let E and F be two vector bundles on M. Two vector bundles isomorphisms f and $g\colon E \xrightarrow{\sim} F$ are called isotopic if there exists a smooth family $(f_t)_{t\in[0,1]}$ of isomorphisms $f_t\colon E \xrightarrow{\sim} F$ such that $f_0 = f$ and $f_1 = g$. Suppose that E and F are endowed with connections ∇_E and ∇_F respectively (which need not be flat). A vector bundle morphism (which does not need to be an isomorphism) $f\colon E \longrightarrow F$ is parallel if $\nabla_F \circ f = f \circ \nabla_E$. For three vector bundles E', E and E'' endowed with connections $\nabla_{E'}$, ∇_E and $\nabla_{E''}$, the short exact sequence

$$0 \longrightarrow E' \xrightarrow{i} E \xrightarrow{p} E'' \longrightarrow 0 \tag{5}$$

is parallel if the morphisms i and p are parallel with respect to $\nabla_{E'}$, ∇_E and $\nabla_{E''}$. Parallel longer exact sequences or complexes of vector bundles are defined in a similar obvious way. In such parallel long exact sequences (or complexes), the kernel or image subbundles are respected by the connections of their ambient bundles (which are not supposed to be flat), so that cokernel or coimage bundles inherit natural connections (which need not be flat). Thus, longer parallel exact sequences (or complexes) can be decomposed, in the classical way, in several short exact sequences (see (56) and (57)) which turn out to be themselve parallel.

2.1.2. *Chern-Simons transgression forms.* — For any vector bundle G on M the vector space of smooth differential forms on M with values in G will be denoted by $\Omega^\bullet(M,G)$. A connection ∇_E on the smooth vector bundle E on M gives rise to an exterior differential operator d^{∇_E} on $\Omega^\bullet(M,E)$. Its square is the exterior product with an element of $\Omega^2(M,\mathrm{End}E)$ (in particular, it does not differentiate). This element of $\Omega^2(M,\mathrm{End}E)$ is the curvature of ∇_E and will be denoted by ∇_E^2. Chern-Weil theory associates to E and ∇_E the following complex differential form on M

$$\mathrm{ch}(\nabla_E) = \mathrm{Tr}\,\exp\left(-\frac{1}{2\pi i}\nabla_E^2\right) = \phi\mathrm{Tr}\exp(-\nabla_E^2) \tag{6}$$

where ϕ is the operator on even degree differential forms which divides $2k$-degree forms by $(2\pi i)^k$. This form is closed, its de Rham cohomology class is independent of ∇_E and equals the image of the Chern character of E in $H^{\mathrm{even}}(M,\mathbb{C})$.

Consider $p_1\colon M\times[0,1]\to M$ (the projection on the first factor) and the bundle $\widetilde{E}=p_1^*E$ on $M\times[0,1]$, choose any connection $\widetilde{\nabla}_E$ on $\widetilde{E}$, denote for all $t\in[0,1]$ by $\nabla_{E,t}$ the restriction $\widetilde{\nabla}_E|_{M\times\{t\}}$. Extend ϕ to odd forms by deciding that ϕ divides $(2k-1)$-degree forms by $(2\pi i)^k$, and define

$$\begin{aligned}\widetilde{\mathrm{ch}}(\nabla_{E,0},\nabla_{E,1}) &= \int_{[0,1]}\mathrm{ch}(\widetilde{\nabla}_E) = -\int_0^1 \phi\mathrm{Tr}\left(\frac{\partial\nabla_{E,t}}{\partial t}\exp\left(-\nabla_{E,t}^2\right)\right)dt\\ &= -\frac{1}{2\pi i}\int_0^1\mathrm{Tr}\left(\frac{\partial\nabla_{E,t}}{\partial t}\exp\left(-\frac{1}{2\pi i}\nabla_{E,t}^2\right)\right)dt.\end{aligned}\tag{7}$$

Modifying $\widetilde{\nabla}_E$ (without changing $\nabla_{E,0}$ nor $\nabla_{E,1}$) changes $\widetilde{\mathrm{ch}}(\nabla_{E,0},\nabla_{E,1})$ by addition of an exact form. This form is a “transgression” form in the sense that:

$$d\widetilde{\mathrm{ch}}(\nabla_{E,0},\nabla_{E,1}) = \mathrm{ch}(\nabla_{E,1})-\mathrm{ch}(\nabla_{E,0}) \tag{8}$$

Its class in $\Omega^{\mathrm{odd}}(M,\mathbb{C})/d\Omega^{\mathrm{even}}(M,\mathbb{C})$ is functorial by pull-backs, and locally gauge invariant, which means that $\widetilde{\mathrm{ch}}(\nabla,g^*\nabla)$ is an exact form if g is a global smooth automorphism of E isotopic to the identity.

If $\nabla_{E,2}$ is a third connection on E, Chern-Simons forms verify the following cocycle equality (modulo exact forms):

$$\widetilde{\mathrm{ch}}(\nabla_{E,0},\nabla_{E,2}) = \widetilde{\mathrm{ch}}(\nabla_{E,0},\nabla_{E,1})+\widetilde{\mathrm{ch}}(\nabla_{E,1},\nabla_{E,2}). \tag{9}$$

In particular $\widetilde{\mathrm{ch}}(\nabla_{E,0},\nabla_{E,1}) = -\widetilde{\mathrm{ch}}(\nabla_{E,1},\nabla_{E,0})$.

Let $\nabla_{E,i}\oplus\nabla_{F,i}$ be the canonical direct sum connections on $E\oplus F$ associated to $\nabla_{E,i}$ and $\nabla_{F,i}$, the additivity of the Chern character form (6) for such direct sum connections yields the following equality (modulo exact forms):

$$\widetilde{\mathrm{ch}}(\nabla_{E,0}\oplus\nabla_{F,0},\nabla_{E,1}\oplus\nabla_{F,1}) = \widetilde{\mathrm{ch}}(\nabla_{E,0},\nabla_{E,1})+\widetilde{\mathrm{ch}}(\nabla_{F,0},\nabla_{F,1}). \tag{10}$$

Consider a short exact sequence as in (5), and a bundle morphism $s\colon E \to E'$ such that $s \circ i$ is the identity of E'. Then $s \oplus p\colon E \xrightarrow{\sim} E' \oplus E''$ is an isomorphism.

Lemma 1. — *$\widetilde{\mathrm{ch}}\big(\nabla_E, (s\oplus p)^*(\nabla_{E'} \oplus \nabla_{E''})\big)$ vanishes if the exact sequence is parallel with respect to $\nabla_{E'}$, ∇_E and $\nabla_{E''}$.*

Proof. — The fact that i and p are parallel means that with respect to the decomposition $E \cong E' \oplus E''$ (provided by the isomorphism $s \oplus p$), the connections ∇_E and $(s\oplus p)^*(\nabla_{E'} \oplus \nabla_{E''})$ differ from a one-form ω with values in $\mathrm{Hom}(E'', E')$.

Consider the path of connections $\nabla_t = \nabla_E - t\omega$. Then, ω is upper triangular with respect to the decomposition $E \cong E' \oplus E''$, and thus ∇_t^2 too. But ω has vanishing diagonal terms. Consequently, the trace vanishes in the Formula (7) applied to this situation, and this proves the lemma. □

2.2. Definitions of the considered K-groups

2.2.1. *Topological K-theory*

Definition 2. — *The topological K^0-group $K^0_{\mathrm{top}}(M)$ is the free abelian group generated by isomorphism classes of smooth complex vector bundles on M modulo direct sum.*

Let $p_1\colon M \times S^1 \to M$ be the projection on the first factor, the topological K^1-group $K^1_{\mathrm{top}}(M)$ is the quotient group $K^0_{\mathrm{top}}(M \times S^1)/p_1^* K^0_{\mathrm{top}}(M)$.

$K^1_{\mathrm{top}}(M)$ is isomorphic to the kernel of the restriction map $\iota^*\colon K^0_{\mathrm{top}}(M \times S^1) \to K^0_{\mathrm{top}}(M \times \{pt\})$ where pt is some point in S^1 and $\iota\colon pt \to S^1$ the inclusion map. One can also describe $K^1_{\mathrm{top}}(M)$ as generated by global smooth automorphisms g_E of any vector bundle E on M; the corresponding element of $K^0_{\mathrm{top}}(M \times S^1)$ is the formal difference of the vector bundle obtained by gluing using g_E the restrictions to $M \times \{1\}$ and $M \times \{0\}$ of the pull-back of E on $M \times [0,1]$, minus the pull-back of E on $M \times S^1$. Any element of $K^1_{\mathrm{top}}(M)$ can be represented in this way with some trivial vector bundle as E.

2.2.2. *K^0-theory of the category of flat bundles*. — The connection ∇_E on the vector bundle E on M is said to be flat if its curvature $\nabla_E^2 \in \Omega^2(M, \mathrm{End}E)$ vanishes. The couple (E, ∇_E) is then called a flat vector bundle. Two flat vector bundles (E, ∇_E) and (F, ∇_F) are isomorphic if there exists some vector bundle isomorphism $f\colon E \xrightarrow{\sim} F$ which is parallel with respect to ∇_E and ∇_F.

Definition 3. — *The group $K^0_{\mathrm{flat}}(M)$ is the quotient of the free abelian group generated by isomorphism classes of flat vector bundles, by the following relation:*

$$(E, \nabla_E) = (E', \nabla_{E'}) + (E'', \nabla_{E''}) \qquad \textit{if} \qquad 0 \to E' \xrightarrow{i} E \xrightarrow{p} E'' \to 0 \tag{11}$$

is a parallel exact sequence.

If a flat vector bundle (E,∇_E) admits some subbundle which is respected by ∇_E, then the subbundle and the quotient bundle inherit connections, which are both flat (a similar result is proved in [**11**] Proposition 2.5). Following the comment of the end of §2.1.1, longer parallel exact sequences (or complexes) of flat vector bundles can be decomposed in short parallel exact sequences of flat vector bundles (see (56) and (57)).

2.2.3. *Relative K-theory.* — Consider now on M quintuples $(E,\nabla_E,F,\nabla_F,f)$ where (E,∇_E) and (F,∇_F) are flat vector bundles on M, and $f\colon E\xrightarrow{\sim}F$ is a smooth isomorphism. Two objects $(E,\nabla_E,F,\nabla_F,f)$ and $(G,\nabla_G,H,\nabla_H,h)$ are isomorphic if there are parallel isomorphisms $\varphi_E\colon E\xrightarrow{\sim}G$ and $\varphi_F\colon F\xrightarrow{\sim}H$ which verify that $h=\varphi_F\circ f\circ\varphi_E^{-1}$.

Definition 4. — *$K^0_{\mathrm{rel}}(M)$ is the quotient of the free abelian group generated by such isomorphism classes of quintuples modulo the following relations:*

(i) $(E,\nabla_E,F,\nabla_F,f)=0$ *if f is isotopic to some parallel isomorphism.*

(ii) $$(E,\nabla_E,F,\nabla_F,f)+(G,\nabla_G,H,\nabla_H,h)=\\ =(E\oplus G,\nabla_E\oplus\nabla_G,F\oplus H,\nabla_F\oplus\nabla_H,f\oplus h)$$

(iii) $(E,\nabla_E,E'\oplus E'',\nabla_{E'}\oplus\nabla_{E''},s\oplus p)$ *vanishes in $K^0_{\mathrm{rel}}(M)$ if there is a short exact sequence of flat bundles as in* (11) *above and if $s\colon E\to E'$ is a smooth bundle map such that $s\circ i$ is the identity of E'.*

Remark 5. — Note that $(E,\nabla_E,F,\nabla_F,f)=(E,\nabla_E,F,\nabla_F,g)$ if f and g are isotopic, that $(E,\nabla_E,F,\nabla_F,f)+(F,\nabla_F,G,\nabla_G,g)=(E,\nabla_E,G,\nabla_G,g\circ f)$, and that $(E,\nabla_E,F,\nabla_F,f)=(E',\nabla_{E'},F',\nabla_{F'},f')+(E'',\nabla_{E''},F'',\nabla_{F''},f'')$ if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0\\ & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 \end{array}\tag{12}$$

is a commutative diagram whose lines are short exact sequences in the category of flat vector bundles (on M).

In fact the first one and the third one of these three relations are together equivalent to (i), (ii) and (iii) so that they can be used to provide an alternative definition of $K^0_{\mathrm{rel}}(M)$ (see [**4**] §2.1 for details).

Independently, relation (iii) above is equivalent to the following

(iii)′ $(E'\oplus E'',\nabla_{E'}\oplus\nabla_{E''},E,\nabla_E,i+j)$ vanishes in $K^0_{\mathrm{rel}}(M)$ if there is a short exact sequence of flat bundles as in (11) above and if $j\colon E''\to E$ is a smooth bundle map such that $p\circ j$ is the identity of E''.

In fact, $i+j$ is isotopic to $(s\oplus p)^{-1}$.

2.2.4. *"Free multiplicative" or "non hermitian smooth" K-theory.* — Consider some triple (E,∇_E,α) where E is a smooth complex vector bundle on M, ∇_E a connection on E and α an odd degree differential form defined modulo exact forms. Two such objects $(E_1,\nabla_{E_1},\alpha_1)$ and $(E_2,\nabla_{E_2},\alpha_2)$ will be equivalent if there is some smooth vector bundle isomorphism $f\colon E_1 \xrightarrow{\sim} E_2$ such that

$$\alpha_2 = \alpha_1 + \widetilde{\mathrm{ch}}(\nabla_{E_1}, f^*\nabla_{E_2}). \tag{13}$$

This is compatible with iterated changes of connections (see (9)).

Definition 6. — *The group $\widehat{K}_{\mathrm{ch}}(M)$ is the quotient of the free abelian group generated by such equivalence classes of triples modulo direct sum (of the vector bundles, with direct sum connection and sum of the differential forms).*

The Chern character on $\widehat{K}_{\mathrm{ch}}(M)$ is the map

$$\dddot{\mathrm{ch}}\colon (E,\nabla_E,\alpha)\in \widehat{K}_{\mathrm{ch}}(M) \longmapsto \mathrm{ch}(\nabla_E) - d\alpha \in \Omega^{\mathrm{even}}(M,\mathbb{C}). \tag{14}$$

Equations (8) and (10) ensure that $\dddot{\mathrm{ch}}$ is well defined.

The kernel of $\dddot{\mathrm{ch}}$ will be denoted $K^{-1}_{\mathbb{C}/\mathbb{Z}}(M)$ following [**30**] Definition 3. The preimage $MK^0(M)$ of $\mathbb{Z}$ by $\dddot{\mathrm{ch}}$ was considered by Karoubi in [**26**] §7.5 and [**28**] EXEMPLE 3. Of course, $MK^0(M)\cong \mathbb{Z}\oplus K^{-1}_{\mathbb{C}/\mathbb{Z}}(M)$ is the subgroup of $\widehat{K}_{\mathrm{ch}}(M)$ generated by the triples (E,∇_E,α) as above, but subjected to the extra condition:

$$d\alpha = \mathrm{ch}(\nabla_E) - \mathrm{rk}E. \tag{15}$$

This is why $\widehat{K}_{\mathrm{ch}}$ is considered as "unrestricted" with respect to $MK^0(M)$, and called "free" multiplicative K-theory. The relation with the smooth K-theory considered by Bunke and Schick in [**14**] will be explained in §2.6.

2.3. Chern-Simons class on relative K-theory

Definition 7. — *The Chern-Simons class on $K^0_{\mathrm{rel}}(M)$ is defined as*

$$\mathscr{N}_{\mathrm{ch}}(E,\nabla_E,F,\nabla_F,f) = \left[\widetilde{\mathrm{ch}}(\nabla_E, f^*\nabla_F)\right] \in H^{\mathrm{odd}}(M,\mathbb{C}) \tag{16}$$

(of course $\widetilde{\mathrm{ch}}(\nabla_E, f^*\nabla_F)$ is closed since $\mathrm{ch}(\nabla_E)$ and $\mathrm{ch}(f^*\nabla_F)$ both equal $\mathrm{rk}E$).

Arguments as in [**4**] Theorem 3.5 and its corollary allow to prove the

Proposition 8. — *$\mathscr{N}_{\mathrm{ch}}$ induces a group morphism from $K_0^{\mathrm{rel}}(M)$ to $H^{\mathrm{odd}}(M,\mathbb{C})$.*

Arguments as in [**4**] §5.1 and 5.2 or [**35**] allow to prove the following facts:

- Let Φ multiply $2k$ and $(2k-1)$-degree forms by $k!$, then $\Phi\mathrm{ch}$ is the Chern character without denominators. The nonintegrality of $\Phi\mathscr{N}_{\mathrm{ch}}(E,\nabla_E,F,\nabla_F,f)$ detects the fact that $(E,\nabla_E)\neq(F,\nabla_F)\in K_0^{\mathrm{flat}}(M)$.

- The nonintegrality of the degree≥ 3 components of $\Phi\mathscr{N}_{\mathrm{ch}}(E,\nabla_E,F,\nabla_F,f)$ detects the fact that (F,∇_F) cannot be obtained from (E,∇_E) through a deformation of flat bundles, where a deformation of flat bundles on M is a smooth vector bundle $\widetilde{E}$ on $M\times[0,1]$ with a connection $\widetilde{\nabla}$ whose restriction to $E_t=\widetilde{E}|_{M\times\{t\}}$ is flat for any point $t\in[0,1]$ and such that $(E_0,\widetilde{\nabla}|_{M\times\{0\}})\cong(E,\nabla_E)$ and $(E_1,\widetilde{\nabla}|_{M\times\{1\}})\cong(F,\nabla_F)$
- The nonnullity of the degree≥ 3 components of $\mathscr{N}_{\mathrm{ch}}(E,\nabla_E,F,\nabla_F,f)$ detects the fact that (F,∇_F) cannot be obtained from (E,∇_E) through a deformation of flat bundles, for which the parallel transport along $[0,1]$ would be isotopic to f.
- If (F,∇_F) can be obtained from (E,∇_E) by deformation of flat bundles, then the degree 1 component of $\mathscr{N}_{\mathrm{ch}}$ modulo integral cohomology detects the variation of the determinant line.

The third statement is known as the rigidity of higher classes of flat bundles.

Remark 9. — Let $(E,\nabla_E,F,\nabla_F,f)\in K^0_{\mathrm{rel}}(M)$. Define $\omega=f^*\nabla_F-\nabla_E$ (then of course $\omega\in\Omega^1(M,\mathrm{End}E)$). It can be proved as in [**4**] Lemma 4.3 that in fact

$$\widetilde{\mathrm{ch}}(\nabla_E,f^*\nabla_F)=-\sum_{r=1}^{\left[\frac{\dim M}{2}\right]}\left(\frac{1}{2\pi i}\right)^r\frac{(r-1)!}{(2r-1)!}\mathrm{Tr}(\omega^{2r-1}).\tag{17}$$

(This is of course a particular property of flat connections and cannot be generalised to any connections). Thus $\mathscr{N}_{\mathrm{ch}}(E,\nabla_E,F,\nabla_F,f)$ can be computed in the same way as the classes studied in [**35**] and [**11**].

2.4. Relations between the preceding K-groups. — The Chern character $\mathrm{ch}\colon K^0_{\mathrm{top}}(M)\longrightarrow H^{\mathrm{even}}(M,\mathbb{C})$ is obtained by considering the de Rham cohomology class of the form of (6).

Consider some element β of $K^1_{\mathrm{top}}(M)$. Represent it by some vector bundle over $M\times S^1$. Integrate along S^1 the Chern character of this bundle, the obtained class in $H^{\mathrm{odd}}(M,\mathbb{C})$ is the Chern character of β. If β is represented by some global automorphism g_E of some vector bundle E on M as in the construction after Definition 2, then it follows from (7) that for any connection ∇ on E

$$\mathrm{ch}(\beta)=\widetilde{\mathrm{ch}}(\nabla,g_E^*\nabla).\tag{18}$$

If $E=\mathbb{C}^N$ is trivial (then denote g_E by $g_{\mathbb{C}^N}$), let $d_{\mathbb{C}^N}$ be its canonical trivial flat connection, the formula

$$\beta\in K^1_{\mathrm{top}}(M)\longmapsto(\mathbb{C}^N,d_{\mathbb{C}^N},\mathbb{C}^N,d_{\mathbb{C}^N},g_{\mathbb{C}^N})\tag{19}$$

defines a group morphism φ (see [**4**] Proposition 2.2 for a proof).

$K^0_{\rm rel}(M)$, $K^0_{\rm flat}(M)$ and $\widehat{K}_{\rm ch}(M)$ are related by the following morphisms:

$$\begin{array}{ccc} K^0_{\rm rel}(M) \xrightarrow{\partial} K^0_{\rm flat}(M) & & K^0_{\rm flat}(M) \xrightarrow{\kappa} \widehat{K}_{\rm ch}(M) \\ (E,\nabla_E,F,\nabla_F,f) \longmapsto (F,\nabla_F)-(E,\nabla_E) & \text{and} & (E,\nabla_E)\longmapsto (E,\nabla_E,0) \end{array} \tag{20}$$

κ is well defined thanks to Lemma 1, and takes its values in $MK^0(M)$.

Let $\gamma \in \Omega^{\rm odd}(M,\mathbb{C})/d\Omega^{\rm even}(M,\mathbb{C})$. It is easily checked that the following element $a(\gamma) = (E,\nabla_E,\alpha+\gamma)-(E,\nabla_E,\alpha)$ of $\widehat{K}_{\rm ch}(M)$ is independent on the choice of (E,∇_E,α) of $\widehat{K}_{\rm ch}(M)$ used to compute it. $a(\gamma)\in MK^0(M)$ if and only if γ is closed. Consider the obvious forgetful maps from $K^0_{\rm flat}$ or $\widehat{K}_{\rm ch}$ to $K^0_{\rm top}$:

Proposition 10. — *This diagram commutes. Its lines are exact sequences:*

$$\begin{array}{ccccccc} K^1_{\rm top}(M) & \xrightarrow{\varphi} & K^0_{\rm rel}(M) & \xrightarrow{\partial} & K^0_{\rm flat}(M) & \longrightarrow & K^0_{\rm top}(M) \\ \downarrow\| & & \downarrow \mathscr{N}_{\rm ch} & & \downarrow\kappa & & \downarrow\| \\ K^1_{\rm top}(M) & \xrightarrow{\rm ch} & H^{\rm odd}(M,\mathbb{C}) & \xrightarrow{a} & MK^0(M) & \longrightarrow & K^0_{\rm top}(M). \end{array} \tag{21}$$

In this diagram, the part "$H^{\rm odd}(M,\mathbb{C}) \xrightarrow{a} MK^0(M)$" can be replaced by "$\Omega^{\rm odd}(M,\mathbb{C})/d\Omega^{\rm even}(M,\mathbb{C}) \xrightarrow{a} \widehat{K}_{\rm ch}(M)$" without losing the commutativity nor the exactness of the second line.

Proof. — A proof of the exactness of the first line can be found (in the holomorphic setting) in [**4**]. A proof of the exactness of the second line can be found in [**27**] Théorème 5.3. This proof generalises easily to the proposed modified second line. The commutativity of the right square is tautological. The commutativity of the left square follows from (16), (18) and (19). The commutativity of the central square is a consequence of the compatibility of (20) and (13) with the definitions of a and of $\mathscr{N}_{\rm ch}$. The proposed replacement in the middle of the second line has no influence on the commutativity of the squares. □

2.5. Symmetries associated to hermitian metrics. — For any complex vector bundle E on M endowed with a hermitian metric h^E and a connection ∇_E, the adjoint connection ∇^*_E of ∇_E is defined as follows:

$$h^E(\nabla^*_{E\mathrm{v}}\sigma,\theta) = \mathrm{v}.h^E(\sigma,\theta) - h^E(\sigma,\nabla_{E\mathrm{v}}\theta) \tag{22}$$

where σ and θ are local sections of E, v is a tangent vector, $\mathrm{v}.f$ is the derivative of the function f along v, $\nabla^*_{E\mathrm{v}}\sigma$ is the derivative of σ along v with respect to the connection ∇^*_E and accordingly for $\nabla_{E\mathrm{v}}\theta$. Of course $(\nabla^*_E)^* = \nabla_E$, (and $\nabla_E = \nabla^*_E$ if and only if ∇_E respects the hermitian metric h^E).

Adjoint connections allow to define conjugation involutions on the above considered K-groups (on the model of complex conjugation):

Definition 11. — *The conjugate elements of* $(E,\nabla_E)\in K^0_{\mathrm{flat}}(M)$*, or* $(E,\nabla_E,F,\nabla_F,f)\in K^0_{\mathrm{rel}}(M)$*, or* $(E,\nabla_E,\alpha)\in\widehat{K}_{\mathrm{ch}}(M)$ *are defined by:*

$$\begin{aligned}(E,\nabla_E)^c&=(E,\nabla^*_E)\in K^0_{\mathrm{flat}}(M)\\(E,\nabla_E,F,\nabla_F,f)^c&=(E,\nabla^*_E,F,\nabla^*_F,f)\in K^0_{\mathrm{rel}}(M)\\(E,\nabla_E,\alpha)^c&=(E,\nabla^*_E,\overline{\alpha})\in\widehat{K}_{\mathrm{ch}}(M).\end{aligned}\tag{23}$$

Lemma 12. — *The above formulae define involutive group automorphisms. Moreover*

$$\begin{aligned}\mathscr{N}_{\mathrm{ch}}(E,\nabla^*_E,F,\nabla^*_F,f)&=\overline{\mathscr{N}_{\mathrm{ch}}(E,\nabla_E,F,\nabla_F,f)}\\\dddot{\mathrm{ch}}(E,\nabla^*_E,\overline{\alpha})&=\overline{\dddot{\mathrm{ch}}(E,\nabla_E,\alpha)}.\end{aligned}\tag{24}$$

Proof. — The curvatures ∇^2_E and ∇^{*2}_E are mutually skew adjoint, so that ∇^*_E is flat if and only if ∇_E is. Thus (E,∇^*_E) and $(E,\nabla^*_E,F,\nabla^*_F,f)$ really define classes in $K^0_{\mathrm{flat}}(M)$ and $K^0_{\mathrm{rel}}(M)$ respectively.

If h^E_1 and h^E_2 are two different hermitian metrics on E, define the global automorphism g_E of E by the following formula, valid for any local sections σ and θ of E:

$$h^E_2(\sigma,\theta)=h^E_1(g_E(\sigma),\theta).\tag{25}$$

Call $\nabla^*_{E,1}$ and $\nabla^*_{E,2}$ the adjoint of ∇_E relatively to h^E_1 and h^E_2 respectively, then $\nabla^*_{E,1}$ and $\nabla^*_{E,2}=g_E^{-1}\nabla^*_{E,1}g_E$ are gauge equivalent. This proves that in the first and third cases, the K-theory classes of the proposed elements are independent of the hermitian metrics used to define them.

Note that g_E is isotopic to the identity of E. If one chooses two hermitian metrics on F and compute in the same way the corresponding automorphism g_F, then f and $g_F\circ f\circ g_E^{-1}$ are isotopic. This proves the independence on the hermitian metrics on E and on F of the class of $(E,\nabla^*_E,F,\nabla^*_F,f)$ in $K^0_{\mathrm{rel}}(M)$.

Consider a parallel exact sequence of the form (11) where E', E and E'' are endowed with hermitian metrics. Its transpose

$$0\to E''\xrightarrow{p^*}E\xrightarrow{i^*}E'\to 0\tag{26}$$

turns out to be a parallel exact sequence with respect to the adjoint connections on E'', E' and E. This proves the first statement (on K^0_{flat}) of the lemma. The second formula of the lemma associates to any quintuple of the same form as in relation (iii) in Definition 4 a quintuple of the form appearing in relation $(iii)'$ in Remark 5. This proves the second statement (on K^0_{rel}) of the lemma.

The fact that the curvatures of ∇_E and ∇^*_E are mutually skew adjoint has the following consequence (which proves the last statement (on $\dddot{\mathrm{ch}}$) of the lemma):

$$\mathrm{ch}(\nabla^*_E)=\overline{\mathrm{ch}(\nabla_E)}.\tag{27}$$

Finally, considering the adjoint connection of $\widetilde{\nabla}_E$ in Formula (7) yields (using (27)) the following relation modulo exact forms:

$$\widetilde{\mathrm{ch}}(\nabla^*_{E,0}, \nabla^*_{E,1}) = \overline{\widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1})} \tag{28}$$

where $\nabla^*_{E,0}$ and $\nabla^*_{E,1}$ are adjoint of $\nabla_{E,0}$ and $\nabla_{E,1}$ with respect to possibly different hermitian metrics on E. The compatibility of the third line of (23) with relation (13) follows. This proves the third statement (on $\widehat{K}_{\mathrm{ch}}$) of the lemma. The statement on $\mathscr{N}_{\mathrm{ch}}$ is a direct consequence of (28). □

(27) and (28) imply that $\mathrm{ch}(\nabla_{E,0})$ and $\widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1})$ are real forms if $\nabla_{E,0}$ and $\nabla_{E,1}$ respect (possibly different) hermitian metrics on E.

Elements of $K^0_{\mathrm{rel}}(M)$ of the form $(E, \nabla_E, E, \nabla^*_E, \mathrm{Id}_E)$ are purely imaginary with respect to this conjugation; conversely, the subgroup of $K^0_{\mathrm{rel}}(M)$ generated by such elements is equal to, or of index 2 in, the purely imaginary part of $K^0_{\mathrm{rel}}(M)$. This is because (see the beginning of Remark 5)

$$\begin{aligned}(E, \nabla_E, F, \nabla_F, f) - {}&(E, \nabla^*_E, F, \nabla^*_F, f) = \\ &= (F, \nabla_F, F, \nabla^*_F, \mathrm{Id}_F) - (E, \nabla_E, E, \nabla^*_E, \mathrm{Id}_E).\end{aligned} \tag{29}$$

2.6. Borel-Kamber-Tondeur class on $\widehat{K}_{\mathrm{ch}}$. — In the notation of (25), the fact that g_E is isotopic to the identity proves that ∇^{*0}_E and ∇^{*1}_E are locally gauge invariant. Thus

Lemma 13. — *Let E be a vector bundle with connection ∇_E. Let ∇^*_E be the adjoint of ∇_E with respect to any hermitian metric on E. The class of $\widetilde{\mathrm{ch}}(\nabla^*_E, \nabla_E)$ in $\Omega^{\mathrm{odd}}(M,\mathbb{C})/d\Omega^{\mathrm{even}}(M,\mathbb{C})$ is independent of the metric h^E.*

Moreover, it is a purely imaginary form, since:

$$\widetilde{\mathrm{ch}}(\nabla^*_E, \nabla_E) = -\widetilde{\mathrm{ch}}(\nabla_E, \nabla^*_E) = -\overline{\widetilde{\mathrm{ch}}(\nabla^*_E, \nabla_E)}. \tag{30}$$

Consider the connection $\nabla^u_E = \frac{1}{2}(\nabla_E + \nabla^*_E)$; it respects h^E, and

$$\widetilde{\mathrm{ch}}(\nabla^*_E, \nabla_E) = \widetilde{\mathrm{ch}}(\nabla^*_E, \nabla^u_E) + \widetilde{\mathrm{ch}}(\nabla^u_E, \nabla_E) = 2i\mathfrak{Im}\big(\widetilde{\mathrm{ch}}(\nabla^u_E, \nabla_E)\big). \tag{31}$$

Finally, if $\nabla_{E,0}$ and $\nabla_{E,1}$ are connections on E, then the cocycle condition (9) produces the following relation modulo exact forms

$$\begin{aligned}\widetilde{\mathrm{ch}}(\nabla^*_{E,1}, \nabla_{E,1}) - \widetilde{\mathrm{ch}}(\nabla^*_{E,0}, \nabla_{E,0}) &= \widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1}) - \widetilde{\mathrm{ch}}(\nabla^*_{E,0}, \nabla^*_{E,1}) \\ &= 2i\,\mathfrak{Im}\,\widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1}).\end{aligned} \tag{32}$$

Remark 14. — It is proved in [**11**], proof of Proposition 1.14, that $\widetilde{\mathrm{ch}}(\nabla_E, \nabla_E^u)$ is purely imaginary if ∇_E is flat. In this case, (31) holds without $i\,\mathfrak{Im}$. Moreover

$$\widetilde{\mathrm{ch}}(\nabla_E^*, \nabla_E) = \frac{1}{\pi}\sum_{j=0}^{\infty} \frac{2^{2j} j!}{(2j+1)!} c_{2j+1}(E) \tag{33}$$

is the degree decomposition of $\widetilde{\mathrm{ch}}(\nabla_E^*, \nabla_E)$, where the $c_k(E)$ are the classes considered by Bismut and Lott (see [**11**] formulae (0.2) and (1.34)). These are exactly the imaginary part of the Chern-Cheeger-Simons classes of flat complex vector bundles ([**15**] and [**11**] Proposition 1.14).

Lemma 15. — *If $E_{\mathbb{R}}$, is a real vector bundle on M with connections $\nabla_{E_{\mathbb{R}},0}$ and $\nabla_{E_{\mathbb{R}},1}$, and if E is its complexification with associated connections ∇_0 and ∇_1, then, up to exact forms, $\widetilde{\mathrm{ch}}(\nabla_0, \nabla_1)$ is real in degrees $4k+3$ and purely imaginary in degrees $4k+1$. In particular, $\widetilde{\mathrm{ch}}(\nabla_0^*, \nabla_0)$ vanishes in degrees $4k+3$.*

Proof. — Suppose that E is endowed with a hermitian form which is the complexification of a real scalar product on $E_{\mathbb{R}}$, and use the path of connections $\nabla_t = (1-t)\nabla^* + t\nabla$, then the lemma follows from formulae (7) and (30) by counting the i, and from Lemma 13. □

Definition 16. — *For any $(E, \nabla_E, \alpha) \in \widehat{K}_{\mathrm{ch}}(M)$, its Borel-Kamber-Tondeur class $\mathfrak{B}(E, \nabla_E, \alpha)$ is the class in $\Omega^{\mathrm{odd}}(M, \mathbb{C})/d\Omega^{\mathrm{even}}(M, \mathbb{C})$ of the differential form:*

$$\mathfrak{B}(E, \nabla_E, \alpha) = \widetilde{\mathrm{ch}}(\nabla_E^*, \nabla_E) - \alpha + \overline{\alpha} \tag{34}$$

where ∇_E^* is the adjoint of ∇_E for any hermitian metric on E.

Relations (28) and (13) imply that $\mathfrak{B}$ is a morphism from $\widehat{K}_{\mathrm{ch}}(M)$ to the subgroup of purely imaginary forms in $\Omega^{\mathrm{odd}}(M)/d\Omega^{\mathrm{even}}(M)$. Moreover, from (32):

$$d\mathfrak{B}(E, \nabla_E, \alpha) = 2i\,\mathfrak{Im}\big(\dddot{\mathrm{ch}}(E, \nabla_E, \alpha)\big).$$

It follows from Lemma 15 that if E is the complexification of a real bundle $E_{\mathbb{R}}$ on M with connection ∇_E coming from a connection on $E_{\mathbb{R}}$, then $\mathfrak{B}(E, \nabla, 0)$ vanishes in degrees $4k+3$ for any integer k.

Any vector bundle admits some hermitian metric and some connection which respects it, so that using relation (13), one checks that $\mathfrak{B}$ is twice the operation of taking the imaginary part with respect to the conjugation defined in Lemma 12, i.e. twice the projection on the second factor of

$$\widehat{K}_{\mathrm{ch}}(M) = \mathrm{Ker}\mathfrak{B} \oplus i\Omega^{\mathrm{odd}}(M, \mathbb{R})/d\Omega^{\mathrm{even}}(M, \mathbb{R}). \tag{35}$$

$\mathrm{Ker}\mathfrak{B}$ coincides with the smooth K-theory $\widehat{K}^0(M)$ considered by Bunke and Schick in [**14**]. In fact any vector bundle V on M endowed with some hermitian metric h^V and unitary connection ∇_V defines some geometric family with zero-dimensional fibre

$\mathscr{V} = (V, h^V, \nabla_V)$ (see **[14]** §2.1.4), then $(V, \nabla_V, \alpha) \longmapsto (\mathscr{V}, \alpha)$ defines a map from $\mathrm{Ker}\mathfrak{B}$ to $\widehat{K}(B)$ which can be proved to be an isomorphism by the five lemma.

$\mathfrak{B}$ sends $MK^0(M)$ into $iH^{\mathrm{odd}}(M, \mathbb{R})$. From Remark 14, one sees that the imaginary part of Cheeger-Chern-Simons classes **[15]** studied by Bismut and Lott in **[11]** factor through $\mathfrak{B}$ and the second morphism defined in (20). This justifies the interest of adding the part $i\Omega^{\mathrm{odd}}(M, \mathbb{R})/d\Omega^{\mathrm{even}}(M, \mathbb{R})$ to Bunke and Schick's smooth K^0-theory, in order to take into account all flat connections.

Finally, the K-theory with coefficients in $\mathbb{R}/\mathbb{Z}$ considered by Lott in **[30]** Definition 7 is $K^{-1}_{\mathbb{R}/\mathbb{Z}}(M) = \mathrm{Ker}\mathfrak{B} \cap K^{-1}_{\mathbb{C}/\mathbb{Z}}(M) = \mathrm{Ker}\mathfrak{B} \cap \mathrm{Ker}\,\dddot{\mathrm{ch}}$.

3. Direct images for K-groups

Let M and B be smooth real manifolds possibly with boundary and $\pi\colon M \to B$ a smooth proper submersion. The goal of this part is to define direct images morphisms from K-theories on M to K-theories on B in each case precedingly reviewed, and to state all the theorems proved in this paper.

The direct image $\pi_!$ for K^0_{flat} is constructed from fiberwise twisted de Rham cohomology (see Definition 22). This is compatible with the forgetful map $K^0_{\mathrm{flat}} \longrightarrow K^0_{\mathrm{top}}$ and the pushforward π^{Eu}_* on K^0_{top} associated to the fiberwise twisted Euler operator (Definitions 17 and 20 and Lemma 23). The notion of "link", which is a generalisation of the concept of vector bundle isomorphism (see Definition 24) is used to solve the problem of defining a pushforward $\pi_*\colon K^0_{\mathrm{rel}}(M) \longrightarrow K^0_{\mathrm{rel}}(B)$. (As stated in the introduction, it consists for any $(E, \nabla_E, F, \nabla_F, f) \in K^0_{\mathrm{rel}}(M)$ in finding some link naturally associated to f between $\pi_!(E, \nabla_E)$ and $\pi_!(F, \nabla_F)$ on B). Finally for $\widehat{K}_{\mathrm{ch}}$, the ingredient is the Chern-Simons analog for transgressing the families index theorem, known as η-form (see Theorem 28). As in the case of topological K-theory, the pushforward $\pi^{\mathrm{Eu}}_!$ is here associated to the fiberwise Euler operator.

In some cases, some more preliminary is needed to be able to state the entire definitions. The proofs are delayed to the subsequent sections.

The fibres of π are supposed to be compact without boundary, orientable, and modelled on the closed manifold Z. For $y \in B$, $\pi^{-1}(y)$ will be denoted Z_y.

3.1. The case of topological K-theory

3.1.1. *Preliminary: construction of family index bundles.* — Let ξ be a smooth complex vector bundle on M. Let TZ^* be the dual of TZ. For any $y \in B$, the infinite dimensional spaces

$$\mathscr{E}^{\pm}_y = \mathscr{C}^{\infty}\left(Z_y, \wedge^{\mathrm{even}}_{\mathrm{odd}} T^*Z \otimes \xi\right) = \Omega^{\mathrm{even}}_{\mathrm{odd}}(Z_y, \xi) \tag{36}$$

are fibres over y of infinite rank vector bundles $\mathscr{E}^+$ and $\mathscr{E}^-$ on B such that

$$\mathscr{C}^\infty(B, \mathscr{E}^\pm) \cong \mathscr{C}^\infty \left(M, \wedge^{\text{even}}_{\text{odd}} T^*Z \otimes \xi\right) \tag{37}$$

(see [**11**] (3.1) to (3.6)). Choose some connection ∇_ξ on ξ, the vertical exterior differential operator $d^{\nabla_\xi} : \Omega^\bullet(Z_y, \xi) \longrightarrow \Omega^{\bullet+1}(Z_y, \xi)$ will be considered as an odd endomorphism of the $\mathbb{Z}_2$-graded vector bundle $\mathscr{E} = \mathscr{E}^+ \oplus \mathscr{E}^-$ (d^{∇_ξ} depends only on the restriction of ∇_ξ to the fibres of π). Choose some smooth hermitian metric h^ξ on ξ and euclidean metric g^Z on TZ, from which a volume form $d\mathrm{Vol}_Z$ along the fibres of π, and an inner product $(\ |\ )_Z$ on $\wedge^\bullet T^*Z \otimes \xi$ are deduced. One obtains on $\mathscr{E}$ the L^2 scalar product (where α and $\beta \in \mathscr{E}^+_y \oplus \mathscr{E}^-_y$):

$$\langle \alpha, \beta \rangle_{L^2} = \int_{Z_y} (\alpha|\beta)_Z d\mathrm{Vol}_Z. \tag{38}$$

Let $(d^{\nabla_\xi})^*$ be the formal adjoint of d^{∇_ξ} for this metric.

Let μ^+ and μ^- be complex vector bundles on B with hermitian metrics h^+ and h^-. For any bundle map $\psi \colon (\mathscr{E}^+ \oplus \mu^+) \to (\mathscr{E}^- \oplus \mu^-)$ of everywhere finite rank, call ψ^* the adjoint of ψ with respect to $h^\pm$ and $\langle\ ,\ \rangle_{L^2}$, and set

$$\begin{aligned} \mathscr{D}^{\nabla_\xi +}_\psi &= \left(d^{\nabla_\xi} + (d^{\nabla_\xi})^*\right) + \psi \colon \mathscr{E}^+ \oplus \mu^+ \longrightarrow \mathscr{E}^- \oplus \mu^-, \\ \mathscr{D}^{\nabla_\xi -}_\psi &= \left(d^{\nabla_\xi} + (d^{\nabla_\xi})^*\right) + \psi^* \colon \mathscr{E}^- \oplus \mu^- \longrightarrow \mathscr{E}^+ \oplus \mu^+. \end{aligned} \tag{39}$$

These are elliptic operators on Z_y so that their kernels are finite dimensional.

Definition 17. — *A triple (μ^+, μ^-, ψ) as above such that* $\dim \mathrm{Ker} \mathscr{D}^{\nabla_\xi \pm}_\psi$ *are constant (independent of $y \in B$) will be called "suitable" in the sequel.*

In that case, the kernels of $\mathscr{D}^{\nabla_\xi \pm}_\psi$ are vector bundles $\mathscr{H}^\pm$ on B, they will be called kernel bundles. The couple $(\mathscr{H}^+ \oplus \mu^-, \mathscr{H}^- \oplus \mu^+)$ is called a couple of family index bundles for ξ.

If ζ is another vector bundle on M with hermitian metric and connection ∇_ζ, and if (ν^+, ν^-, φ) is a suitable triple for ζ, call $\mathscr{K}^\pm$ the kernel bundles $\mathrm{Ker} \mathscr{D}^{\nabla_\zeta \pm}_\varphi$ *then the couple $(\mathscr{H}^+ \oplus \mu^- \oplus \mathscr{K}^- \oplus \nu^+, \mathscr{H}^- \oplus \mu^+ \oplus \mathscr{K}^+ \oplus \nu^-)$ will be called a couple of family index bundles for $\xi - \zeta$.*

3.1.2. *Definition of the direct image morphism for K^0_{top} and K^1_{top}*

Proposition 18. — *If B is compact, then for any ξ on M endowed with any connection ∇_ξ and any hermitean metric h^ξ, there exists suitable data (μ^+, μ^-, ψ).*

This is proved in [**2**] Proposition (2.2) (see also [**3**] Lemma 9.30). The following classical result will be precised in Theorem 25 below.

Lemma 19. — *If $(\mathscr{G}^+_1, \mathscr{G}^-_1)$ and $(\mathscr{G}^+_2, \mathscr{G}^-_2)$ are couples of family index bundles for the same vector bundle ξ on M (for different metrics or connections or suitable data),* *then*

$$[\mathscr{G}_1^+]-[\mathscr{G}_1^-] = [\mathscr{G}_2^+]-[\mathscr{G}_2^-] \in K^0_{\mathrm{top}}(B). \tag{40}$$

The same holds if $(\mathscr{G}_1^+,\mathscr{G}_1^-)$ *and* $(\mathscr{G}_2^+,\mathscr{G}_2^-)$ *are couples of family index bundles for the couples of vector bundle* $\xi_1-\zeta_1$ *and* $\xi_2-\zeta_2$ *such that*

$$[\xi_1]-[\zeta_1] = [\xi_2]-[\zeta_2] \in K^0_{\mathrm{top}}(M). \tag{41}$$

Definition 20. — *If B is compact, then for any vector bundle ξ on M, take any couple $(\mathscr{G}^+,\mathscr{G}^-)$ of family index bundles for ξ and put*

$$\pi_*^{\mathrm{Eu}}([\xi]) = [\mathscr{G}^+]-[\mathscr{G}^-] \ \in K^0_{\mathrm{top}}(B). \tag{42}$$

If B is not compact, $\pi_*^{\mathrm{Eu}}([\xi])$ is defined in the same way on compact subsets of B and by inductive limit (or using the stability properties of vector bundles **[22]** *§8 Theorems 1.2 and 1.5) on the whole B.*

The above lemma proves that π_*^{Eu} is a morphism from $K_0^{\mathrm{top}}(M)$ to $K_0^{\mathrm{top}}(B)$. It is the one associated to the fiberwise Euler operator (see **[2]** Definition 2.3: if d^Z is the as above constructed d^{∇_ξ} in the case where ξ is the trivial rank one complex vector bundle with canonical connection and metric, then the fiberwise Euler operator is d^Z+d^{Z*} acting on vertical differential forms $\mathbb{Z}_2$-graded by the parity of their degree). This is in contrast with the case of **[8]**, **[14]**, **[30]** and **[7]** §1, where the direct image is associated to the fiberwise *Spin* or *Spin*^{c} Dirac operator, but compatible with **[11]**, **[31]**, **[32]** and **[7]** §§2 and 3.

Fiberwise twisted Euler operators of the form $\mathscr{D}^{\nabla_\xi}$ can be pulled back on fibered products (here $\widetilde{B}\longrightarrow B$ is any differentiable map):

$$\begin{array}{ccc} \widetilde{B}\times_B M & \longrightarrow & M \\ \downarrow & & \downarrow \\ \widetilde{B} & \longrightarrow & B \end{array} \tag{43}$$

(the model of the fibre may not change). The additional data (μ^+,μ^-,ψ) used to construct the direct image can also be pulled back in such situations, and this makes the construction of families index bundles functorial. Thus π_*^{Eu} is also functorial by pullbacks on fibered products. This justifies the following

Definition 21. — *The direct image morphism $\pi_*^{\mathrm{Eu}}\colon K^1_{\mathrm{top}}(M)\longrightarrow K^1_{\mathrm{top}}(B)$ is the morphism induced (on quotients) by $\pi_*^{\mathrm{Eu}}\colon K^0_{\mathrm{top}}(M\times S^1)\longrightarrow K^0_{\mathrm{top}}(B\times S^1)$.*

3.2. The case of the K^0-theory of flat bundles. — Consider some flat vector bundle (E, ∇_E) on M. The de Rham cohomology $H^\bullet(Z, E)$ of the fibres of π with coefficients in E provides ($\mathbb{Z}$-graded) vector bundles on B, which are endowed with flat connections in a canonical way, see **[11]** §III (f). Put $\pi_!^+ E = H^{\mathrm{even}}(Z, E)$ and $\pi_!^- E = H^{\mathrm{odd}}(Z, E)$ (they are smooth vector bundles on B, whose definition depends on ∇_E). and call $\nabla_{\pi_!^+ E}$ and $\nabla_{\pi_!^- E}$ their canonical flat connections.

Definition 22. — *$(\pi_!^+ E, \nabla_{\pi_!^+ E})$ and $(\pi_!^- E, \nabla_{\pi_!^- E})$ will be called the sheaf theoretic direct images of (E, ∇_E). The direct image morphism $\pi_! \colon K^0_{\mathrm{flat}}(M) \to K^0_{\mathrm{flat}}(B)$ is given by:*

$$(E, \nabla_E) \longmapsto (\pi_!^+ E, \nabla_{\pi_!^+ E}) - (\pi_!^- E, \nabla_{\pi_!^- E}).$$

The definition of $\pi_!$ is justified by the following fact: for a parallel short exact sequence of flat bundles as in (11)

$$0 \longrightarrow (E', \nabla_{E'}) \xrightarrow{\ i\ } (E, \nabla_E) \xrightarrow{\ p\ } (E'', \nabla_{E''}) \longrightarrow 0 \tag{44}$$

the long exact sequence in cohomology reads

$$\begin{array}{ccccc} \pi_!^+ E' & \xrightarrow{[i]} & \pi_!^+ E & \xrightarrow{[p]} & \pi_!^+ E'' \\ \uparrow & & & & \downarrow \\ \pi_!^- E'' & \xleftarrow{[p]} & \pi_!^- E & \xleftarrow{[i]} & \pi_!^- E' \end{array} \tag{45}$$

and all the morphisms in (45) are parallel. This diagram decomposes in several short parallel exact sequences of flat vector bundles as was remarked at the ends of §2.1.1 and §2.2.2. Thus $\pi_!(E, \nabla_E) = \pi_!(E', \nabla_{E'}) + \pi_!(E'', \nabla_{E''}) \in K^0_{\mathrm{flat}}(B)$. This proves that the above definition of $\pi_!$ fits with relation (11).

The following result is needed to define the direct image for K^0_{rel}:

Lemma 23. — *The following diagram commutes:*

$$\begin{array}{ccc} K^0_{\mathrm{flat}}(M) & \longrightarrow & K^0_{\mathrm{top}}(M) \\ {\scriptstyle \pi_!}\big\downarrow & & \big\downarrow{\scriptstyle \pi_*^{\mathrm{Eu}}} \\ K^0_{\mathrm{flat}}(B) & \longrightarrow & K^0_{\mathrm{top}}(B). \end{array} \tag{46}$$

Proof. — Let (E, ∇_E) be any flat vector bundle over M. By the Hodge theory of the fibres of π, the $H^\pm(Z_y, E)$ are isomorphic to $\mathrm{Ker}\big(d^{\nabla_E} + (d^{\nabla_E})^*\big)^\pm$ on Z_y. (They are of constant dimension, whatever the riemannian metric on M and the hermitian metric on E may be). Thus $(\{0\}, \{0\}, 0)$ is a suitable triple in this situation. The couple $(\pi_!^+ E, \pi_!^- E)$ is thus isomorphic to a couple of family index bundles for E, so that $[\pi_!^+ E] - [\pi_!^- E] = \pi_*^{\mathrm{Eu}}[E] \in K^0_{\mathrm{top}}(B)$. □

3.3. The case of relative K-theory

3.3.1. *The notion of "link".* — For four smooth vector bundles E, F, G, and H on M such that

$$[E]-[F]=[G]-[H]\in K^0_{\mathrm{top}}(M)$$

there exists some vector bundle K on M and some $\mathscr{C}^\infty$ isomorphism

$$\ell\colon E\oplus H\oplus K\xrightarrow{\sim}F\oplus G\oplus K. \tag{47}$$

Definition 24. — *These (K,ℓ) will be called a "link between $E-F$ and $G-H$".*

Two such links (K_1,ℓ_1) and (K_2,ℓ_2) are equivalent if there exists some vector bundle L on M such that the two following isomorphisms are isotopic

$$E\oplus H\oplus K_1\oplus K_2\oplus L\xrightarrow[\ell_2\oplus\mathrm{Id}_{K_1}\oplus\mathrm{Id}_L]{\ell_1\oplus\mathrm{Id}_{K_2}\oplus\mathrm{Id}_L}F\oplus G\oplus K_1\oplus K_2\oplus L. \tag{48}$$

The equivalence class of a link (K,ℓ) will be denoted by $[\ell]$. The set of equivalence classes of links between $E-F$ and $G-H$ will be denoted by $\mathscr{L}^{G-H}_{E-F}$.

Of course a link between $E-F$ and $G-H$ is also a link between $E-G$ and $F-H$ or $H-G$ and $F-E$ or $H-F$ and $G-E$. Any link is equivalent to some other one with a trivial vector bundle as K. Moreover, if (K,ℓ) is a link between $E-F$ and $G-H$, then (K,ℓ^{-1}) will be a link between $G-H$ and $E-F$ (or $F-H$ and $E-G$ and so on). Its equivalence class will be denoted either by $[\ell^{-1}]$ or $[\ell]^{-1}$.

The identity of $K\oplus K$ is isotopic to the switch of the two copies of K, thus (K,ℓ) as in (47) is equivalent to itself. It is also obviously equivalent to $(K\oplus L,\ell\oplus\mathrm{Id}_L)$ (for any vector bundle L). It follows that any link is equivalent to a link of the form (47) where K is a trivial vector bundle.

Links can be pulled back, and added (for direct sum of data). Moreover, two links (L,ℓ) between $E-F$ and $G-H$, and (M,ℓ') between $G-H$ and $J-K$ can be composed as $(L\oplus M\oplus G\oplus H,\ell\oplus\ell')$ between $E-F$ and $J-K$; this composition is easily checked to be associative. The equivalence class of the composed link will be denoted by $[\ell'\circ\ell]$ or $[\ell']\circ[\ell]$.

$K^1_{\mathrm{top}}(M)$ acts freely transitively on $\mathscr{L}^{G-H}_{E-F}$. The element β of $K^1_{\mathrm{top}}(M)$ represented by the global smooth automorphism g_N of the vector bundle N maps the equivalence class α of (K,ℓ) to the equivalence class $\beta\alpha$ of $(K\oplus N,\ell\oplus g_N)$.

3.3.2. *Definition of the direct image for K^0_{rel}*

Theorem 25. — *Let ξ be any vector bundle on M, let $(\mathscr{F}^+,\mathscr{F}^-)$ and $(\mathscr{G}^+,\mathscr{G}^-)$ be two couples of family index bundles for (the same) ξ, then there exists a canonical element $[\ell^{\mathscr{G}}_{\mathscr{F}}]\in\mathscr{L}^{\mathscr{G}^+-\mathscr{G}^-}_{\mathscr{F}^+-\mathscr{F}^-}$. It is canonical in the sense of the following global compatibility property: if $(\mathscr{H}^+,\mathscr{H}^-)$ is another couple of family index bundles for ξ, then one has $[\ell^{\mathscr{H}}_{\mathscr{F}}]=[\ell^{\mathscr{H}}_{\mathscr{G}}]\circ[\ell^{\mathscr{G}}_{\mathscr{F}}]$.*

This extends trivially to couples of family index bundles for $\xi-\zeta$ (for any vector bundles ξ and ζ on M).

Moreover, if $[\xi_1]-[\zeta_1]=[\xi_2]-[\zeta_2]\in K^0_{\mathrm{top}}(M)$ and if $(\mathcal{F}_1^+,\mathcal{F}_1^-)$ and $(\mathcal{F}_2^+,\mathcal{F}_2^-)$ are couples of family index bundles for $\xi_1-\zeta_1$ and $\xi_2-\zeta_2$ respectively, then there exists a canonical map $\pi_\ell\colon \mathcal{L}^{\xi_2-\zeta_2}_{\xi_1-\zeta_1}\longrightarrow \mathcal{L}^{\mathcal{F}_2^+-\mathcal{F}_2^-}_{\mathcal{F}_1^+-\mathcal{F}_1^-}$. It is canonical in the sense of the following global compatibility property: if $[\xi_3]-[\zeta_3]=[\xi_1]-[\zeta_1]\in K^0_{\mathrm{top}}(M)$ and if $(\mathcal{F}_3^+,\mathcal{F}_3^-)$ is a couple of family index bundles for $\xi_3-\zeta_3$, then for any $\alpha\in\mathcal{L}^{\xi_2-\zeta_2}_{\xi_1-\zeta_1}$ and $\beta\in\mathcal{L}^{\xi_3-\zeta_3}_{\xi_2-\zeta_2}$, one has $\pi_\ell(\beta\circ\alpha)=\pi_\ell(\beta)\circ\pi_\ell(\alpha)\in\mathcal{L}^{\mathcal{F}_3^+-\mathcal{F}_3^-}_{\mathcal{F}_1^+-\mathcal{F}_1^-}$.

If $\xi_1=\xi_2$ and $\zeta_1=\zeta_2$, then $[\ell^{\mathcal{F}_2}_{\mathcal{F}_1}]=\pi_\ell(\mathrm{Id}_{\xi\oplus\zeta})$.

π_ℓ is compatible with the actions by K^1_{top} in the following sense: if $\alpha\in\mathcal{L}^{\xi_2-\zeta_2}_{\xi_1-\zeta_1}$ and $\beta\in K^1_{\mathrm{top}}$, then $\pi_\ell(\beta\alpha)=\pi_*^{\mathrm{Eu}}(\beta)\pi_\ell(\alpha)$.

If (E,∇_E), (F,∇_F), (G,∇_G) and (H,∇_H) are flat vector bundles on M, and if $\ell\colon E\oplus H\oplus K\xrightarrow{\sim}F\oplus G\oplus K$ is a link between $E-F$ and $G-H$, it is possible to find a link $\ell'\colon E\oplus H\oplus\mathbb{C}^n\xrightarrow{\sim}F\oplus G\oplus\mathbb{C}^n$ equivalent to ℓ (by adding $\mathrm{Id}_{K'}\colon K'\xrightarrow{\sim}K'$ for some K' such that $K\oplus K'\cong\mathbb{C}^n$). The obtained element

$$(E\oplus H\oplus\mathbb{C}^n,\nabla_E\oplus\nabla_H\oplus d_{\mathbb{C}^n},F\oplus G\oplus\mathbb{C}^n,\nabla_F\oplus\nabla_G\oplus d_{\mathbb{C}^n},\ell')\in K^0_{\mathrm{rel}}(M)$$

does not depend on the choice of ℓ' and depends on ℓ only through its equivalence class (this can be checked using (48) with L replaced in the same way by some trivial bundle).

For some element $(E,\nabla_E,F,\nabla_F,f)$ of $K_0^{\mathrm{rel}}(M)$, Consider the sheaf theoretic direct images $(\pi_!^+E,\nabla_{\pi_!^+E})$ and $(\pi_!^-E,\nabla_{\pi_!^-E})$ of (E,∇_E), and $(\pi_!^+F,\nabla_{\pi_!^+F})$ and $(\pi_!^-F,\nabla_{\pi_!^-F})$ of (F,∇_F). Following the proof of Lemma 23, $(\pi_!^+E,\pi_!^-E)$ and $(\pi_!^+F,\pi_!^-F)$ are couples of family index bundles for E and F respectively. Using the above Theorem 25 (especially π_ℓ), one obtains an equivalence class of links between $\pi_!^+E-\pi_!^-E$ and $\pi_!^+F-\pi_!^-F$ as image by π_ℓ of the equivalence class of $f\colon E\to F$ (which is a link between $E-\{0\}$ and $F-\{0\}$).

Definition 26. — *We define*

$$\pi_*(E,\nabla_E,F,\nabla_F,f)=\big(\pi_!^+E\oplus\pi_!^-F,\nabla_{\pi_!^+E}\oplus\nabla_{\pi_!^-F},\pi_!^-E\oplus\pi_!^+F,\nabla_{\pi_!^-E}\oplus\nabla_{\pi_!^+F},\pi_\ell([f])\big).$$

Theorem 27. — *This defines a morphism $K_0^{\mathrm{rel}}(M)\xrightarrow{\pi_*}K_0^{\mathrm{rel}}(B)$ which enters in the following commutative diagram (with lines modeled on the first line of (21)):*

$$\begin{array}{ccccccc}
K^1_{\mathrm{top}}(M) & \xrightarrow{\varphi} & K^0_{\mathrm{rel}}(M) & \xrightarrow{\partial} & K^0_{\mathrm{flat}}(M) & \longrightarrow & K^0_{\mathrm{top}}(M)\\
{\scriptstyle\pi_*^{\mathrm{Eu}}}\downarrow & & {\scriptstyle\pi_*}\downarrow & & {\scriptstyle\pi_!}\downarrow & & \downarrow{\scriptstyle\pi_*^{\mathrm{Eu}}}\\
K^1_{\mathrm{top}}(B) & \xrightarrow{\varphi} & K^0_{\mathrm{rel}}(B) & \xrightarrow{\partial} & K^0_{\mathrm{flat}}(B) & \longrightarrow & K^0_{\mathrm{top}}(B).
\end{array}\tag{49}$$

3.4. The case of multiplicative, or smooth, K^0-theory

3.4.1. *Transgression of the family index theorem.* — Let $F_{\mathbb{R}}$ be a real vector bundle over M endowed with a euclidean metric and a unitary connection $\nabla_{F_{\mathbb{R}}}$. The curvature $\nabla^2_{F_{\mathbb{R}}}$ is a two-form with values in antisymmetric endomorphisms of $F_{\mathbb{R}}$. Define $e(\nabla_{F_{\mathbb{R}}})$ to be zero if $F_{\mathbb{R}}$ is of odd rank (as real vector bundle) and to be the Pfaffian of $\frac{1}{2\pi}\nabla^2_{F_{\mathbb{R}}}$ if $F_{\mathbb{R}}$ is of even rank. One obtains a closed real differential form whose degree equals the rank of $F_{\mathbb{R}}$, whose de Rham cohomology class $e(F_{\mathbb{R}})$ is independent on $\nabla_{F_{\mathbb{R}}}$ (and on the euclidean metric on $F_{\mathbb{R}}$) and coincides with the image of the Euler class of $F_{\mathbb{R}}$ in $H^{\bullet}(M,\mathbb{C})$. (This is the Chern-Weil version of the Euler class).

The vertical tangent bundle TZ of the submersion π, which is the subbundle of TM consisting of vectors tangent to the fibres of π, will be supposed to be globally orientable along M. If ξ is a vector bundle on M and F^+ and F^- are vector bundles on B such that $[F^+]-[F^-]=\pi_*^{\mathrm{Eu}}[\xi]\in K^0_{\mathrm{top}}(B)$, the cohomological counterpart of the families index theorem asserts that

$$\mathrm{ch}(F^+)-\mathrm{ch}(F^-)=\int_Z e(TZ)\mathrm{ch}(\xi)\ \in H^{\mathrm{even}}(B,\mathbb{C}) \tag{50}$$

where $\int_Z$ stands for integration along the fibres of π.

Choose any smooth complementary subbundle T^HM of TZ in TM. Of course $T^HM\cong\pi^*TB$. Let P^{TZ} be the projection of TM onto TZ with kernel T^HM. Endow TZ with some riemannian metric g^Z. All riemannian metrics on M which coincide with g^Z on TZ and make TZ and T^HM orthogonal give rise to Levi-Civita connections ∇_{LC} on TM which all project to the same connection $\nabla_{TZ}=P^{TZ}\nabla_{LC}$ on TZ.

Let ∇_ξ, ∇_{F^+} and ∇_{F^-} be connections on ξ, F^+ and F^- respectively. It follows from (50) that $\mathrm{ch}(\nabla_{F^+})-\mathrm{ch}(\nabla_{F^-})$ and $\int_Z e(\nabla_{TZ})\wedge\mathrm{ch}(\nabla_\xi)$ are cohomologous differential forms on B. The following theorem is a non hermitian analogue of results of Bunke [**13**]:

Theorem 28. — *Let $[\ell]=([\ell_K])_{K\ \mathit{compact}\ \subset B}$ be any collection of mutually compatible equivalence classes of links between restrictions of F^+-F^- and couples of family index bundles for ξ on compact subsets of B. There exists a way to associate to such data (ξ, ∇_ξ, g^Z, T^HM, F^+, ∇_{F^+}, F^-, ∇_{F^-} and $[\ell]$) an element $\eta(\nabla_\xi,\nabla_{TZ},\nabla_{F^+},\nabla_{F^-},[\ell])$ of $\Omega^{\mathrm{odd}}(M,\mathbb{C})/d\Omega^{\mathrm{even}}(M,\mathbb{C})$ with properties*

(a) $d\eta(\nabla_\xi,\nabla_{TZ},\nabla_{F^+},\nabla_{F^-},[\ell])=\int_Z e(\nabla_{TZ})\mathrm{ch}(\nabla_\xi)-\mathrm{ch}(\nabla_{F^+})+\mathrm{ch}(\nabla_{F^-})$
(b) *η is natural by pullbacks on fibered products as in* (43).
(c) *η is additive for direct sums of vector bundles ξ and $F^\pm$ with direct sum connections (and direct sum of links).*
(d) $\eta(\nabla_E,\nabla_{TZ},\nabla_{\pi_!^+E},\nabla_{\pi_!^-E},[\mathrm{Id}])=0$ *if (E,∇_E) is a flat bundle on M with sheaf theoretic direct images $(\pi_!^+E,\nabla_{\pi_!^+E})$ and $(\pi_!^-E,\nabla_{\pi_!^-E})$.*

Moreover η with these properties is unique for vector bundles ξ with vanishing rational Chern classes on M.

In statement (d), [Id] stands for the trivial link between $\pi_!^+E - \pi_!^-E$ and itself, when using $(\pi_!^+E, \pi_!^-E)$ as couple of family index bundles for E (see the proof of Lemma 23). In statement (b), the vertical tangent bundle $\widetilde{TZ}$ of $\widetilde{B}\times_B M$ is naturally isomorphic to the pullback of the vertical tangent bundle TZ of M, the connection on $\widetilde{TZ}$ is supposed to be the pullback connection of ∇_{TZ}. The statement (a) is seen as a Chern-Simons like transgression of the family index Theorem (50). As a first consequence of this:

Theorem 29. — *For any $(E,\nabla_E,F,\nabla_F,f)\in K^0_{\mathrm{rel}}(M)$, one has*

$$\mathscr{N}_{\mathrm{ch}}\big(\pi_*(E,\nabla_E,F,\nabla_F,f)\big) = \int_Z e(TZ)\mathscr{N}_{\mathrm{ch}}(E,\nabla_E,F,\nabla_F,f).$$

This "Riemann-Roch-Grothendieck" theorem for K^0_{rel} is a cohomological formula, it does not need the Chern-Weil version of the Euler class in its expression.

3.4.2. *Direct image for multiplicative/smooth K^0-theory*

Definition 30. — *Let $(\xi,\nabla_\xi,\alpha)\in\widehat{K}_{\mathrm{ch}}(M)$, take any vector bundles F^+ and F^- such that $[F^+]-[F^-]=\pi_*^{\mathrm{Eu}}[\xi]\in K^0_{\mathrm{top}}(B)$, choose any connections ∇_{F^+} on F^+ and ∇_{F^-} on F^-, take any collection of equivalence classes of links $[\ell]$ between F^+-F^- and any families index bundles for ξ on compact subsets of B, and define the direct image of (ξ,∇_ξ,α) by*

(51)
$$\pi_!^{\mathrm{Eu}}(\xi,\nabla_\xi,\alpha) = \Big(F^+,\nabla_{F^+},\int_Z e(\nabla_{TZ})\alpha\Big) - \big(F^-,\nabla_{F^-},\eta(\nabla_\xi,\nabla_{TZ},\nabla_{F^+},\nabla_{F^-},[\ell])\big).$$

This definition is intended to obtain the following property:

$$\dddot{\mathrm{ch}}\big(\pi_!(\xi,\nabla_\xi,\alpha)\big) = \int_Z e(\nabla_{TZ})\wedge\dddot{\mathrm{ch}}(\xi,\nabla_\xi,\alpha) \tag{52}$$

which implies that $\pi_!^{\mathrm{Eu}}$ sends $MK_0(M)$ to $MK_0(B)$, and $K^{-1}_{\mathbb{C}/\mathbb{Z}}(M)$ to $K^{-1}_{\mathbb{C}/\mathbb{Z}}(B)$.

Theorem 31. — *$\pi_!^{\mathrm{Eu}}(\xi,\nabla_\xi,\alpha)$ as defined above does not depend on the choices of F^+, F^-, ∇_{F^+}, ∇_{F^-} nor $[\ell]$.*

(51) *defines a morphism $\pi_!^{\mathrm{Eu}}\colon\widehat{K}_{\mathrm{ch}}(M)\longrightarrow\widehat{K}_{\mathrm{ch}}(B)$. The following diagrams commute (the lines of* (54) *are modeled on the modified second line of* (21)*):*

$$\begin{array}{ccc} K^0_{\mathrm{flat}}(M) & \longrightarrow & \widehat{K}_{\mathrm{ch}}(M) \\ {\scriptstyle\pi_!}\big\downarrow & & \big\downarrow{\scriptstyle\pi_!^{\mathrm{Eu}}} \\ K^0_{\mathrm{flat}}(B) & \longrightarrow & \widehat{K}_{\mathrm{ch}}(B), \end{array} \tag{53}$$

(54)
$$\begin{array}{ccccccc}
K^1_{\mathrm{top}}(M) & \xrightarrow{\mathrm{ch}} & \Omega^{\mathrm{odd}}(M,\mathbb{C})/d\Omega^{\mathrm{even}}(M,\mathbb{C}) & \xrightarrow{a} & \widehat{K}_{\mathrm{ch}}(M) & \longrightarrow & K^0_{\mathrm{top}}(M) \\
{\scriptstyle \pi^{\mathrm{Eu}}_*}\big\downarrow & & {\scriptstyle \int_Z e(\nabla_{TZ})\wedge\bullet}\big\downarrow & & \big\downarrow{\scriptstyle \pi^{\mathrm{Eu}}_!} & & \big\downarrow{\scriptstyle \pi^{\mathrm{Eu}}_*} \\
K^1_{\mathrm{top}}(B) & \xrightarrow{\mathrm{ch}} & \Omega^{\mathrm{odd}}(B,\mathbb{C})/d\Omega^{\mathrm{even}}(B,\mathbb{C}) & \xrightarrow{a} & \widehat{K}_{\mathrm{ch}}(B) & \longrightarrow & K^0_{\mathrm{top}}(B).
\end{array}$$
Moreover $\mathfrak{B}\big(\pi_!(\xi,\nabla_\xi,\alpha)\big) = \int_Z e(\nabla_{TZ})\wedge\mathfrak{B}(\xi,\nabla_\xi,\alpha) \in \Omega^{\mathrm{odd}}(B)/d\Omega^{\mathrm{even}}(B)$.

Here the morphism denoted by $\int_Z e(\nabla_{TZ})\wedge\bullet$ is integration along the fibre after product with $e(\nabla_{TZ})$ (i.e. $\alpha\mapsto\int_Z e(\nabla_{TZ})\wedge\alpha$). It vanishes if $\dim Z$ is odd.

The relation concerning $\mathfrak{B}$ implies that $\pi^{\mathrm{Eu}}_!$ sends $\widehat{K}^0(M)$ to $\widehat{K}^0(B)$ (Bunke and Schick's K-theory, see the end of §2.6 after (35)) and $K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ to $K^{-1}_{\mathbb{R}/\mathbb{Z}}(B)$.

3.5. Hermitian symmetry and functoriality results

3.5.1. *Direct images and symmetries.* — The conjugations on K^0_{flat}, K^0_{rel} and $\widehat{K}_{\mathrm{ch}}$ were defined in Definition 11.

Theorem 32. — *If* $\dim Z$ *is even, then* $\pi_!$ *on* K^0_{flat}, π_* *on* K^0_{rel} *and* $\pi^{\mathrm{Eu}}_!$ *on* $\widehat{K}_{\mathrm{ch}}$ *are all real in the sense that:*
$$\pi_!\big((E,\nabla_E)^c\big) = \big(\pi_!(E,\nabla_E)\big)^c \in K^0_{\mathrm{flat}}(B),$$
$$\pi_*\big((E,\nabla_E,F,\nabla_F,f)^c\big) = \big(\pi_!(E,\nabla_E,F,\nabla_F,f)\big)^c \in K^0_{\mathrm{rel}}(B),$$
$$\pi^{\mathrm{Eu}}_!\big((\xi,\nabla_\xi,\alpha)^c\big) = \big(\pi^{\mathrm{Eu}}_!(\xi,\nabla_\xi,\alpha)\big)^c \in \widehat{K}_{\mathrm{ch}}(B).$$

In fact the last statement of this theorem is a consequence of the last statement (about $\mathfrak{B}$) of the preceding one, and of the facts stated just before (35).

Theorem 33. — *If* $\dim Z$ *is odd, then* π^{Eu}_* *on* K_{top} *and* $\pi^{\mathrm{Eu}}_!$ *on* $\widehat{K}_{\mathrm{ch}}$ *both vanish.*

If $\dim Z$ *is odd, then there exists a map* $\pi_\leftarrow\colon K^0_{\mathrm{flat}}(M)\longrightarrow K^0_{\mathrm{rel}}(B)$ *such that* $\pi_! = \partial\circ\pi_\leftarrow$ *(on* K^0_{flat}*) and* $\pi_* = \pi_\leftarrow\circ\partial$ *(on* K^0_{rel}*).*

$\pi_\leftarrow$ *is purely imaginary in the sense that if* $(E,\nabla_E)\in K^0_{\mathrm{flat}}$*:*
$$\pi_\leftarrow\big((E,\nabla_E)^c\big) = -\big(\pi_\leftarrow(E,\nabla_E)\big)^c \in K^0_{\mathrm{rel}}(B).$$

Moreover, $\mathscr{N}_{\mathrm{ch}}\circ\pi_\leftarrow$ *vanishes.*

3.5.2. *Double fibrations.* — Consider two submersions $\pi_1\colon M\longrightarrow B$ and $\pi_2\colon B\longrightarrow S$ and the composed submersion $\pi_2\circ\pi_1\colon M\longrightarrow S$. The following classical results

$$(55)\qquad \begin{aligned}(\pi_2\circ\pi_1)^{\mathrm{Eu}}_* &= \pi^{\mathrm{Eu}}_{2*}\circ\pi^{\mathrm{Eu}}_{1*}\colon K^\bullet_{\mathrm{top}}(M)\longrightarrow K^\bullet_{\mathrm{top}}(S)\\ (\pi_2\circ\pi_1)_! &= \pi_{2!}\circ\pi_{1!}\colon K^0_{\mathrm{flat}}(M)\longrightarrow K^0_{\mathrm{flat}}(S)\end{aligned}$$

will be reproved or explained during the proof of the following

Theorem 34. — $(\pi_2\circ\pi_1)_* = \pi_{2*}\circ\pi_{1*}\colon K^0_{\mathrm{rel}}(M)\longrightarrow K^0_{\mathrm{rel}}(S)$.

Only a partial result is obtained for multiplicative K-theory:

Theorem 35. — *The restriction to* $MK^0(M)$ *of* $\pi_{2!}^{\mathrm{Eu}} \circ \pi_{1!}^{\mathrm{Eu}}$ *and* $(\pi_2 \circ \pi_1)_!^{\mathrm{Eu}}$ *coincide.*

4. Proof of Theorems 25 and 27

4.1. Proof of Theorem 25. — The link between any two couples of family index bundles for the same vector bundle ξ is obtained by an intermediary link with some special couple of ("positive kernel") family index bundles (see Definition 37 in §4.1.2). It is proved in §4.1.2 that any couple can be linked with some special one, and that all these links are mutually compatible, the general link is then obtained in two steps by a homotopy technique in §4.1.3 and §4.1.4. B is supposed to be compact in §4.1.2 and §4.1.3.

4.1.1. *Links and exact sequences of vector bundles.* — Consider a short exact sequence of complex vector bundles on M:

$$0 \longrightarrow E' \xrightarrow{\ i\ } E \xrightarrow{\ p\ } E'' \longrightarrow 0.$$

Take any morphisms $s\colon E \longrightarrow E'$ and $j\colon E'' \longrightarrow E$ such that $s \circ i = \mathrm{Id}_{E'}$ and $p \circ j = \mathrm{Id}_{E''}$, then, as was remarked just after Remark 5, $i + j$ and $(s \oplus p)^{-1}$ are isotopic isomorphisms from $E' \oplus E''$ to E. They thus provide the same equivalence class of link between $(E' \oplus E'') - \{0\}$ and $E - \{0\}$, or any equivalent combination. Take any hermitian metrics on E', E and E'', and consider the adjoints i^* and p^* with respect to these metrics. Then $s \oplus p$ and $i^* \oplus p$ are isotopic, and so are $i + j$ and $i + p^*$. This is because autoadjoint automorphisms (here $i^* \circ i$ and $p \circ p^*$) are always isotopic to the identity.

Consider now a longer complex of vector bundles on M:

$$0 \longrightarrow E^0 \xrightarrow{\ v_0\ } E^1 \xrightarrow{\ v_1\ } \cdots \xrightarrow{\ v_{k-1}\ } E^k \longrightarrow 0. \tag{56}$$

It may not be an exact sequence, but the v_i are supposed to be of everywhere constant rank. Call H^k the cohomology of this complex in degree k. The H^k are vector bundles on M. Choose some hermitian metrics h^i on the E^i, and consider the associated adjoints v_i^* of the v_i. By finite dimensional Hodge theory one has canonical isomorphisms $H^i \cong \mathrm{Ker}(v_i + v_{i+1}^*)$. Let $\iota_i\colon H^i \hookrightarrow E^i$ be induced by the inclusion of $\mathrm{Ker}(v_i + v_i^*)$ and $p_i\colon E^i \longrightarrow H^i$ by the orthogonal projection on $\mathrm{Ker}(v_i + v_i^*)$. Denote by E^+ and E^- the direct sums $\oplus_{i\mathrm{even}} E^i$ and $\oplus_{i\mathrm{odd}} E^i$ respectively, and accordingly for H^+, H^-, v_+, v_-, v_+^*, v_-^*, ι_+, ι_-, p_+ and p_-. The isomorphism $v_+ + v_-^* + p_+ + \iota_-\colon E^+ \oplus H^- \xrightarrow{\ \sim\ } E^- \oplus H^+$ is isotopic to $\big(v_- + v_+^* + p_- + \iota_+\big)^{-1}$. This is because $p_\pm$ and $\iota_\pm$ are mutually adjoint.

Definition 36. — *The equivalence class of links between $E^+ - E^-$ and $H^+ - H^-$ (or any equivalent combinations) associated to the complex* (56) *is the common class defined by anyone of these two isomorphisms.*

This definition is justified by the independence on the hermitian metrics. This class of link is not modified by isotopy of the complex, i.e. smooth homotopy of the morphisms such that any of them stays of same constant rank. This class of links can be described in the same terms from the following exact sequence

$$0 \longrightarrow H^+ \xrightarrow{\iota_+} E^+ \xrightarrow{v_+ + v_-^*} E^- \xrightarrow{p_-} H^- \longrightarrow 0.$$

It is left as an exercise to check that it is the same class as the one obtained from the composition of links associated to the following short exact sequences

(57)
$$0 \longrightarrow \mathrm{Ker} v_i \longrightarrow E^i \xrightarrow{v_i} \mathrm{Im} v_i \longrightarrow 0, \qquad 0 \longrightarrow \mathrm{Im} v_{i-1} \longrightarrow \mathrm{Ker} v_i \longrightarrow H^i \longrightarrow 0$$

which enter in the canonical decomposition of (56) in short exact sequences.

4.1.2. *Link with "positive kernel" family index bundles.* — Consider as in §3.1.1 some vector bundle ξ on M with hermitian metric h^ξ and connection ∇_ξ. Take some vertical riemannian metric g^Z on TZ and consider some triple (μ^+, μ^-, ψ) as in §3.1.1, with which a vertical modified de Rham operator $\mathscr{D}_\psi^{\nabla_\xi \pm}$ is computed. The triple (μ^+, μ^-, ψ) may be not suitable.

If B is compact, there exists some vector bundle λ on B and some bundle morphism $\varphi\colon \lambda \to \mathscr{E}^- \oplus \mu^-$ such that $\mathscr{D}_{\psi+\varphi}^{\nabla_\xi +}$ is surjective, as can be proved in exactly the same way as in **[2]** Proposition 2.2, or **[3]** Lemma 9.30 or **[29]** Lemma 8.4 of chapter III. This proves the existence of suitable triples in general.

Definition 37. — *A (suitable) triple which has the same surjectivity property as $(\mu^+ \oplus \lambda, \mu^-, \psi + \varphi)$ will hereafter be called a "positive kernel" triple; the obtained couple of family index bundles $((\mathrm{Ker}\mathscr{D}_{\psi+\varphi}^{\nabla_\xi +} \oplus \mu^-, \mu^+ \oplus \lambda)$ in the above example) will be called "couple of positive kernel family index bundles".*

Suppose now that (μ^+, μ^-, ψ) were suitable and gave rise to kernel bundles $\mathscr{H}^\pm$. Choose λ and φ as above. Let $P^{\mathscr{H}^-}$ be the projector from $\mathscr{E}^- \oplus \mu^-$ onto $\mathscr{H}^-$ with kernel $\mathrm{Im}\mathscr{D}_\psi^{\nabla_\xi +}$. The following sequence of vector bundles on B is exact:

$$\begin{array}{c} 0 \longrightarrow \mathscr{H}^+ \longrightarrow \mathrm{Ker}\mathscr{D}_{\psi+\varphi}^{\nabla_\xi +} \longrightarrow \lambda \xrightarrow{P^{\mathscr{H}^-} \circ \varphi} \mathscr{H}^- \longrightarrow 0 \\ (\sigma, v, w) \longmapsto w \end{array} \tag{58}$$

($\mathrm{Ker}\mathscr{D}_{\psi+\varphi}^{\nabla_\xi +}$ is a subbundle of $\mathscr{E}^+ \oplus \mu^+ \oplus \lambda$ on which its elements are decomposed). This provides an equivalence class of links between $\mathscr{H}^+ - \mathscr{H}^-$ and $\mathrm{Ker}\mathscr{D}_{\psi+\varphi}^{\nabla_\xi +} - \lambda$ as in Definition 36. An equivalence class of links between $(\mathscr{H}^+ \oplus \mu^-) - (\mathscr{H}^- \oplus \mu^+)$ and $(\mathrm{Ker}\mathscr{D}^{\nabla_\xi +}_{\psi+\varphi} \oplus \mu^-) - (\lambda \oplus \mu^+)$ is trivially deduced.

Lemma 38. — *Classes of links obtained in this way are mutually compatible.*

Proof. — Suppose that λ' and φ' satisfy the same surjectivity hypothesis as λ and φ with respect to $\mu^\pm$ and ψ. Then $\lambda \oplus \lambda'$ and $\varphi \oplus \varphi'$ also do. On the other hand, the same construction can be performed starting from the triple $(\mu^+ \oplus \lambda, \mu^-, \psi + \varphi)$ and using λ' and φ', or starting from the triple $(\mu^+ \oplus \lambda', \mu^-, \psi + \varphi')$ and using λ and φ. One obtains in each case some equivalence class of links between two of the couples $(\mathscr{H}^+ \oplus \mu^-) - (\mathscr{H}^- \oplus \mu^+)$, $(\mathrm{Ker}\mathscr{D}^{\nabla_\xi +}_{\psi+\varphi} \oplus \mu^-) - (\lambda \oplus \mu^+)$, $(\mathrm{Ker}\mathscr{D}^{\nabla_\xi +}_{\psi+\varphi'} \oplus \mu^-) - (\lambda' \oplus \mu^+)$ or $(\mathrm{Ker}\mathscr{D}^{\nabla_\xi +}_{\psi+\varphi+\varphi'} \oplus \mu^-) - (\lambda \oplus \lambda' \oplus \mu^+)$.

These links are all compatible (in the sense of composition of links) as can be checked by considering the exact sequence (58) associated either to $\lambda \oplus \lambda'$ and $\varphi + t\varphi'$ with t varying along $[0,1]$ or to $\lambda \oplus \lambda'$ and $s\varphi + \varphi'$ with $s \in [0,1]$. □

4.1.3. *Deformation of ψ, h^ξ and ∇_ξ.* — Consider two triples (μ^+, μ^-, ψ_0) and (μ^+, μ^-, ψ_1) with same μ^+ and μ^-. Take the product with the interval $[0,1]$ and consider some everywhere finite rank $\widetilde{\psi}\colon \mathscr{E}^+ \oplus \mu^+ \longrightarrow \mathscr{E}^- \oplus \mu^-$ over $B \times [0,1]$ with restrictions $\widetilde{\psi}|_{B\times\{0\}} = \psi_0$ and $\widetilde{\psi}|_{B\times\{1\}} = \psi_1$. The pullback of ξ on $B \times [0,1]$ is endowed with any (not necessarily pullback) hermitian metric and connection.

If B is compact, one can perform the above construction over $B \times [0,1]$, finding some positive kernel triple $\big(\mu^+ \oplus \lambda, \mu^-, \widetilde{\psi} + \widetilde{\varphi}\big)$ over $B \times [0,1]$. An isotopy class of bundle isomorphism $\mathrm{Ker}\mathscr{D}^{\nabla_\xi +}_{\psi_0+\widetilde{\varphi}|_{B\times\{0\}}} \cong \mathrm{Ker}\mathscr{D}^{\nabla_\xi +}_{\psi_1+\widetilde{\varphi}|_{B\times\{1\}}}$ is obtained by parallel transport along $[0,1]$. This produces an equivalence class of links between the couples $\big(\mathrm{Ker}\mathscr{D}^{\nabla_\xi +}_{\psi_0+\widetilde{\varphi}|_{B\times\{0\}}} \oplus \mu^-\big) - (\mu^+ \oplus \lambda)$ and $\big(\mathrm{Ker}\mathscr{D}^{\nabla_\xi +}_{\psi_1+\widetilde{\varphi}|_{B\times\{1\}}} \oplus \mu^-\big) - (\mu^+ \oplus \lambda)$.

Suppose (μ^+, μ^-, ψ_0) and (μ^+, μ^-, ψ_1) are both suitable triples with associated kernel bundles $\mathscr{H}_0^\pm$ and $\mathscr{H}_1^\pm$ (and with respect to not necessarily same metric and connection on ξ). On $B \times \{0\}$, the construction of the preceding paragraph produces an equivalence class of links between $(\mathscr{H}_0^+ \oplus \mu^-) - (\mathscr{H}_0^- \oplus \mu^+)$ and $\mathrm{Ker}\mathscr{D}^{\nabla_\xi +}_{\psi_0+\widetilde{\varphi}|_{B\times\{0\}}} - \lambda$, and similarly on $B \times \{1\}$. This three links compose to produce an equivalence class of links between $(\mathscr{H}_0^+ \oplus \mu^-) - (\mathscr{H}_0^- \oplus \mu^+)$ and $(\mathscr{H}_1^+ \oplus \mu^-) - (\mathscr{H}_1^- \oplus \mu^+)$.

Lemma 39. — *This equivalence class of links does not depend on λ, $\widetilde{\varphi}$ and $\widetilde{\psi}$.*

Proof. — The independence on λ and $\widetilde{\varphi}$ follows from Lemma 38 (and the fonctoriality of links by pullbacks). The independence on the choice of $\widetilde{\psi}$ can be proved by deforming it to any other choice (with fixed boundary values), and make the above construction on $B \times [0,1] \times [0,1]$. □

4.1.4. *General construction (and proof of Theorem 25).* — If two suitable triples (μ_0^+,μ_0^-,ψ_0) and (μ_1^+,μ_1^-,ψ_1) give rise to couples of family index bundles $(\mathscr{F}^+,\mathscr{F}^-)$ and $(\mathscr{G}^+,\mathscr{G}^-)$, one performs the preceding construction starting from the bundles $\mu_0^+\oplus\mu_1^+$ and $\mu_0^-\oplus\mu_1^-$ and the two morphisms ψ_0 extended by 0 on μ_1^+ and ψ_1 extended by 0 on μ_1^-. One obtains an equivalence class of links between the couples $(\mathscr{F}^+\oplus\mu_1^+\oplus\mu_1^-)-(\mathscr{F}^-\oplus\mu_1^-\oplus\mu_1^+)$ and $(\mathscr{G}^+\oplus\mu_0^+\oplus\mu_0^-)-(\mathscr{G}^-\oplus\mu_0^-\oplus\mu_0^+)$. On compact subsets of B, one defines $[\ell_{\mathscr{F}}^{\mathscr{G}}]$ as the composition of this class of link with the trivial links between $\mathscr{F}^+-\mathscr{F}^-$ and $(\mathscr{F}^+\oplus\mu_1^+\oplus\mu_1^-)-(\mathscr{F}^-\oplus\mu_1^-\oplus\mu_1^+)$ and between $(\mathscr{G}^+\oplus\mu_0^+\oplus\mu_0^-)-(\mathscr{G}^-\oplus\mu_0^-\oplus\mu_0^+)$ and $\mathscr{G}^+-\mathscr{G}^-$. One obtains a projective family of equivalence classes of links on compact subsets of B. Stability properties of vector bundles **[22]** §8 Proposition 1.4 can be used to prove that these links can be described with isomorphisms of the form $\mathscr{F}^+\oplus\mathscr{G}^-\oplus\mathbb{C}^N\xrightarrow{\sim}\mathscr{F}^-\oplus\mathscr{G}^+\oplus\mathbb{C}^N$ with some fixed N, and such that two such isomorphisms are always isotopic. It is then possible to obtain a global link by inductive limit on an exhaustion by compact subsets with an iterative deformation procedure to fix the isomorphism at finite distance.

Definition 40. — *$[\ell_{\mathscr{F}}^{\mathscr{G}}]$ is the equivalence class of links between $\mathscr{F}^+-\mathscr{F}^-$ and $\mathscr{G}^+-\mathscr{G}^-$ obtained in this way.*

The independence on the various choices follows from Lemma 39.

For three suitable triples, the construction (for compact B) can be adapted so that the restriction to $B\times\{\frac12\}$ corresponds to the third data, this proves the compatibility of these links with respect to mutual composition. Now the equivalence of links on compacts propagates in the inductive limit along an exhaustion by compact subsets.

If ξ_1, ζ_1, ξ_2 and ζ_2 are vector bundles on M such that $[\xi_1]-[\zeta_1]=[\xi_2]-[\zeta_2]$ in $K^0_{\mathrm{top}}(M)$, consider some vector bundle isomorphism $\ell\colon\xi^+\oplus\zeta^-\oplus L\xrightarrow{\sim}\xi^-\oplus\zeta^+\oplus L$ as in (47). Let $(\mathscr{F}_i^+,\mathscr{F}_i^-)$, $(\mathscr{G}_i^+,\mathscr{G}_i^-)$ and $(\mathscr{L}^+,\mathscr{L}^-)$ be couples of family index bundles for ξ_i, ζ_i and L respectively for $i=1$ and 2. (It is always possible to choose L such that it admits family index bundles on the whole B: it suffices to take L trivial with canonical metric and connection). Thus $(\mathscr{F}_1^+\oplus\mathscr{G}_2^+\oplus\mathscr{L}^+,\mathscr{F}_1^-\oplus\mathscr{G}_2^-\oplus\mathscr{L}^-)$ and $(\mathscr{F}_2^+\oplus\mathscr{G}_1^+\oplus\mathscr{L}^+,\mathscr{F}_2^-\oplus\mathscr{G}_1^-\oplus\mathscr{L}^-)$ are couples of family index bundles for the same vector bundle modulo the isomorphism ℓ.

Definition 41. — *$\pi_\ell([\ell])$ is the equivalence class of links obtained between these couples using Definition 40, and interpreted as an equivalence class of links between $(\mathscr{F}_1^+\oplus\mathscr{G}_1^-)-(\mathscr{F}_1^-\oplus\mathscr{G}_1^+)$ and $(\mathscr{F}_2^+\oplus\mathscr{G}_2^-)-(\mathscr{F}_2^-\oplus\mathscr{G}_2^+)$.*

The fact that $[\ell_{\mathscr{F}}^{\mathscr{G}}]=\pi_\ell([\mathrm{Id}])$ is tautological.

But if one takes a different link from the identity, and the same couples of family index bundles at both boundaries, one obtains a realisation of the direct image π_*^{Eu} on K^1_{top} by gluing the ends and applying Definition 21. The last statement of the theorem is a consequence of this fact and the obvious compatibility of the whole construction with direct sums.

The independence of $\pi_\ell([\ell])$ on the choice of L and ℓ (in some same equivalence class of links see (48)) is due to the above facts and to the invariance of $\pi_\ell([\ell])$ under isotopy of ℓ. The canonicity of $\pi_\ell([\ell])$ is a direct consequence of the corresponding property of $\ell_{\mathcal{G}}^{\mathcal{G}}$.

4.2. Proof of Theorem 27. — This result is a consequence of a compatibility result (Proposition 43) of some canonical link obtained from Theorem 25 and another one obtained from Definition 36 applied to some long exact sequence in cohomology. This second link is computed in §4.2.2 as a composition of two pieces. The compatibility proof then uses a geometric deformation, in which the canonical link is proved to decompose in two pieces too. The fit of each piece of one link with its counterpart in the other one is proved in §4.2.3 and §4.2.4.

As remarked just before Definition 40, an equivalence of links on compact subsets propagates in the inductive limit along an exhaustion by compacts. So, in this whole section, B can be supposed to be compact without restriction.

4.2.1. *Reduction of the problem*

Lemma 42. — *Suppose that* (E^i, ∇_{E^i}) *are flat vector bundles on* M *entering in the following parallel complex:*

$$0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^k \longrightarrow 0. \tag{59}$$

Take the same notations E^+, E^-, H^+ *and* H^- *as in Definition 36 and define the connections* $\nabla_{E^+} = \bigoplus_{i\ \mathrm{even}} \nabla_{E^i}$ *and* $\nabla_{E^-} = \bigoplus_{i\ odd} \nabla_{E^i}$ *and accordingly for* ∇_{H^+} *and* ∇_{H^-}. *Let* $[\ell]$ *be the equivalence class of links between* $E^+ - E^-$ *and* $H^+ - H^-$ *associated to* (59) *from Definition 36. Then*

$$(E^+ \oplus H^-, \nabla_{E^+} \oplus \nabla_{H^-}, E^- \oplus H^+, \nabla_{E^-} \oplus \nabla_{H^+}, [\ell]) = 0 \in K^0_{\mathrm{rel}}(M).$$

Proof. — The decomposition of the complex (59) in several short exact sequences as in (57) gives rise to short exact sequences of flat vector bundles as was remarked at the ends of §2.1.1 and §2.2.2. Through this decomposition, $[\ell]$ is reduced to canonical links associated to short exact sequences as in relation (*iii*) of Definition 4 (as was remarked after Definition 36) and the lemma follows. □

Consider now the exact sequence (44). Denote by $\pi_!^i E$ the i^{th} degree de Rham cohomology of the fibres of π with coefficients in the restriction of (E, ∇_E) (to the fibres) and similarly for E' and E''. The associated long exact sequence in cohomology (45) also reads:

$$0 \longrightarrow \pi_!^0 E' \xrightarrow{[i]} \pi_!^0 E \xrightarrow{[p]} \pi_!^0 E'' \longrightarrow \pi_!^1 E' \xrightarrow{[i]} \cdots \xrightarrow{[p]} \pi_!^{\dim Z} E'' \longrightarrow 0. \tag{60}$$

Let $[i+j]$ be the equivalence class of links corresponding to (44) constructed at the beginning of §4.1.1.

Proposition 43. — *$\pi_\ell([i+j])$ coincides with the equivalence class of links between $(\pi_!^+E' \oplus \pi_!^+E'') - (\pi_!^-E' - \pi_!^-E'')$ and $\pi_!^+E - \pi_!^-E$ associated to* (60).

The proof of this proposition is delayed in the following paragraphs.

We are now in position to prove Theorem 27 using Proposition 43.

The definition of π_* on K^0_{rel} is clearly compatible with the isotopy of f. If f is parallel, then $\pi_\ell([f])$ is itself a parallel isomorphism between $\pi_!^+E \oplus \pi_!^-F$ and $\pi_!^-E \oplus \pi_!^+F$. This proves the compatibility of π_* with relation (i) of Definition 4. π_* is also obviously compatible with direct sums as in relation (ii) of Definition 4. The compatibility of π_* with relation (iii) is a direct consequence of the above proposition and Lemma 42.

The commutativity of the right square of diagram (49) was proved in Lemma 23. The commutativity of the central square of diagram (49) is tautological. The commutativity of the left square of diagram (49) is a consequence of the last statement of Theorem 25.

4.2.2. *Sheaf theoretic direct images and short exact sequences.* — Back to the model exact sequence (44), consider E' as a subbundle of E. The vertical exterior differential operator d^{∇_E} respects the subbundle (over B) $\Omega^\bullet(Z,E')$ of the vertical de Rham complex $(\Omega^\bullet(Z,E), d^{\nabla_E})$. This filtration $0 \subset \Omega(M,E') \subset \Omega(M,E)$ gives rise to some spectral sequence, and to some filtration $0 \subset FH^\bullet(Z,E) \subset H^\bullet(Z,E)$ of the fiberwise cohomology of E. The (E_0,d_0)-term of this spectral sequence is the direct sum of the fiberwise de Rham complexes of E' and of E''; consequently, the E_1-term is the direct sum $\pi_!E' \oplus \pi_!E''$ of the fiberwise cohomology of E' and of E''.

Let $s\colon E \to E'$ be a smooth vector bundle morphism such that $s\circ i$ is the identity of E', then E'' will be identified with the subbundle $\mathrm{Ker} s$ of E so that E will be identified with $E' \oplus E''$. Thus E inherits two flat connections ∇_E and $\nabla_{E'} \oplus \nabla_{E''}$, whose difference is (as was used in Lemma 1 in a nonflat context) a one form ω with values in $\mathrm{Hom}(E'',E')$. On any closed E''-valued form, d^{∇_E} applies as $\omega\wedge$ so that the operator d_1 of the spectral sequence is given by

$$d_1 = [\omega\wedge]\colon H^\bullet(Z,E'') \longrightarrow H^{\bullet+1}(Z,E'). \tag{61}$$

This is exactly the linking maps of the exact diagram (45), and the spectral sequence converges at E_2 which is the filtrated fiberwise cohomology of E.

Thus the exact diagram (45) decomposes in two exact sequences:

$$0 \longrightarrow FH^\pm(Z,E) \longrightarrow \pi_!^\pm E' \xrightarrow{[\omega]} \pi_!^\mp E'' \longrightarrow H^\mp(Z,E)/FH^\mp(Z,E) \longrightarrow 0. \tag{62}$$

The canonical link associated to (45)-(60) is the direct sum of the two canonical links of these two exact sequences modulo the canonical isotopy class of isomorphism between (graded) cohomology and (graded) filtrated cohomology.

$$\pi_!^\bullet E = H^\bullet(Z,E) \cong FH^\bullet(Z,E) \oplus \big(H^\bullet(Z,E)/FH^\bullet(Z,E)\big). \tag{63}$$

4.2.3. *"Adiabatic" limit of harmonic forms.* — Put $\nabla_\theta = (\nabla_{E'} \oplus \nabla_{E''}) + \theta\omega$ for any $\theta \in [0,1]$, then $\nabla_E = \nabla_1$, and ∇_θ is flat for any $\theta \in [0,1]$. Moreover, the flat bundles (E,∇_θ) and (E,∇_E) are isomorphic for any $\theta > 0$ through the automorphism $\mathrm{Id}_{E'} \oplus \theta \mathrm{Id}_{E''}$ of E. For any $\theta > 0$, d^{∇_θ} (as d^{∇_E}) also respects the subbundle $\Omega(Z,E')$ of $\Omega(Z,E)$, and the associated spectral sequence is isomorphic to the preceding one if θ is positive, so that the considerations of the preceding paragraph apply verbatim for $\theta \in (0,1]$.

Put any riemanian metric on M, and endow $E \cong E' \oplus E''$ with a direct sum hermitian metric. The Hodge theory of the fibres of π provides for any $\theta \in (0,1]$ an isomorphism between the (graded) kernel $\mathscr{H}_\theta^\bullet$ of the fiberwise Euler-de Rham operator $\mathscr{D}_\theta = d^{\nabla_\theta} + (d^{\nabla_\theta})^*$ and the cohomology of the de Rham complex associated with d^{∇_θ}. In particular, the dimension of $\mathscr{H}_\theta^i$ is constant for any i when θ goes over $(0,1]$. The isomorphism class provided by parallel transport along $(0,1]$ of $\mathscr{H}_\theta$ is isotopic to the twist of the de Rham cohomology by $\mathrm{Id}_{E'} \oplus \theta\mathrm{Id}_{E''}$.

Let $d^{\nabla_{E'}}$ and $d^{\nabla_{E''}}$ be the fiberwise exterior differential operators on $\Omega(Z,E')$ and $\Omega(Z,E'')$ respectively obtained from $\nabla_{E'}$ and $\nabla_{E''}$, and define $\mathscr{D}' = d^{\nabla_{E'}} + (d^{\nabla_{E'}})^*$ and $\mathscr{D}'' = d^{\nabla_{E''}} + (d^{\nabla_{E''}})^*$. Then $\mathscr{D}_\theta = \mathscr{D}' + \mathscr{D}'' + \theta(\omega + \omega^*)$ so that one has a continuous family of elliptic operators on $B \times [0,1]$. Suppose that B is compact, this ensures the positivity of the minimum positive eigenvalue of $\mathscr{D}' + \mathscr{D}''$ along all B, which will be denoted by $\lambda_{\min}$. There exists $\varepsilon > 0$ such that $\theta\omega$ is bounded by $\frac{1}{5}\lambda_{\min}$ in L^2 norm for all $\theta \le \varepsilon$. Then for any $y \in B$ and any $\theta \le \varepsilon$, $\mathscr{D}_\theta$ has no eigenvalue equal to $\pm\frac{\lambda_{\min}}{2}$. Thus the (graded) direct sum $\mathscr{F}_\theta^\bullet$ of eigenspaces of $\mathscr{D}_\theta$ corresponding to eigenvalues belonging to $[-\frac{\lambda_{\min}}{2}, \frac{\lambda_{\min}}{2}]$ is a finite rank vector bundle on $B \times [0,\varepsilon]$ whose restriction $\mathscr{F}_0$ to $B \times \{0\}$ equals $\mathrm{Ker}\mathscr{D}' \oplus \mathrm{Ker}\mathscr{D}''$.

As θ converges to 0, $\frac{1}{\theta}P^{\mathscr{F}_\theta} d^{\nabla_\theta} P^{\mathscr{F}_\theta}$ converges to $P^{\mathscr{F}_0}(\omega\wedge)P^{\mathscr{F}_0}$ and this is the image of $[\omega\wedge]$ through the Hodge isomorphism $\mathscr{F}_0 \cong \pi_! E' \oplus \pi_! E''$.

This proves that $\mathscr{H}_\theta$ converges to the kernel $\mathscr{H}_0$ of $P^{\mathscr{F}_0}(\omega\wedge)P^{\mathscr{F}_0}$ as θ converges to 0, because the dilation factor $\frac{1}{\theta}$ does not modify the kernels. This limit subspace $\mathscr{H}_0$ is identified by Hodge isomorphism $\mathscr{F}_0 \cong H(Z,E') \oplus H(Z,E'')$ with the filtrated fiberwise cohomology of E as seen around Equation (61). Consequently, the parallel transport along $[0,1]$ for $\mathscr{H}$ is, modulo the Hodge isomorphisms, in the same isotopy class as the isomorphism (63) between fiberwise cohomology of E and its filtrated counterpart.

4.2.4. *End of proof of Proposition 43.* — Clearly

$$[\mathcal{H}_\theta^+] - [\mathcal{H}_\theta^+] = [\mathcal{F}_0^+] - [\mathcal{F}_0^-] = \pi_*^{\mathrm{Eu}}([E]) \in K^0_{\mathrm{top}}(B)$$

for any positive θ. Following the construction of canonical links, the equivalence class of links $\pi_\ell([i+j])$ is isomorphic (modulo Hodge isomorphisms at the boundaries) to the class of links between $[\mathcal{F}_0^+] - [\mathcal{F}_0^-]$ and $[\mathcal{H}_1^+] - [\mathcal{H}_1^-]$ obtained by parallel transport along $[0,1]$ of some kernel bundle on $B \times [0,1]$ associated to the above model deformation of d^{∇_E} and canonical links at the boundaries.

However we will cut at some $\theta \in (0,\varepsilon]$ to perform the construction. In fact over $B \times (0,1]$, the triple $(\{0\},\{0\},0)$ is suitable (because of fiberwise Hodge theory). Over $B \times [0,\varepsilon]$, one has $\mathbb{Z}$-graded vector subbundles $\mathcal{F}_\theta$ and $\mathcal{H}_\theta$ of $\Omega(Z,E)$ which are all respected by d^{∇_θ} and $\mathcal{D}_\theta$. Let $\mathcal{P}^{\mathcal{F}_\theta}$ be the orthogonal projection onto $\mathcal{F}_\theta$, the triple $(\{0\},\{0\},-\mathcal{P}^{\mathcal{F}_\theta}\mathcal{D}_\theta)$ is suitable, with associated kernel bundles $\mathcal{F}_\theta^\pm$.

To describe the canonical link between $\mathcal{F}_\theta^+ - \mathcal{F}_\theta^-$ and $\mathcal{H}_\theta^+ - \mathcal{H}_\theta^-$ of Definition 40 over $B \times (0,\varepsilon]$, one observes that we are in the special case studied in §4.1.3. On $B \times (0,\varepsilon] \times [0,1]$, one puts (following the notations of §4.1.3) $\widetilde{\psi} = -(1-t)\mathcal{P}^{\mathcal{F}_\theta\mathcal{D}_\theta}$, $\lambda = \mathcal{F}_\theta^-$ and $\widetilde{\varphi} = \mathrm{Id}_{\mathcal{F}_\theta^-}$. The obtained kernel bundle is the kernel of

$$t\mathcal{D}_\theta + \mathrm{Id}_{\mathcal{F}_\theta^-} : \mathcal{F}_\theta^+ \oplus \mathcal{F}_\theta^- \longrightarrow \mathcal{F}_\theta^-$$

i.e. $\mathcal{K}_t = \{(\sigma, -t\mathcal{D}_\theta\sigma)/\sigma \in \mathcal{F}_\theta^+\}$. For $t=0$ the link between $\mathcal{F}_\theta^+ - \mathcal{F}_\theta^-$ and itself is tautological. For $t=1$ the link is associated to the exact sequence (58)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}_\theta^+ & \longrightarrow & \mathcal{K}_1 & \longrightarrow & \mathcal{F}_\theta^- & \longrightarrow \mathcal{H}_\theta^- & \longrightarrow 0 \\ & & & & (\sigma, -\mathcal{D}_\theta\sigma) & \longmapsto & -\mathcal{D}_\theta\sigma & & \end{array}$$

which is isotopic through the obvious isomorphism $\mathcal{K}_t^+ \cong \mathcal{F}_\theta^+$ (obtained for any t by parallel transport along $[0,t]$) to the exact sequence

$$0 \longrightarrow \mathcal{H}_\theta^+ \longrightarrow \mathcal{F}_\theta^+ \xrightarrow{-\mathcal{D}_\theta} \mathcal{F}_\theta^- \longrightarrow \mathcal{H}_\theta^- \longrightarrow 0. \tag{64}$$

It follows that $\pi_\ell([i+j])$ is the composition of the Hodge isomorphism $\pi_!E \cong \mathcal{H}_1$, the parallel transport $\mathcal{H}_1 \cong \mathcal{H}_\theta$, the canonical equivalence class of links between $\mathcal{H}_\theta^+ - \mathcal{H}_\theta^-$ and $\mathcal{F}_\theta^+ - \mathcal{F}_\theta^-$ associated to (64) as in Definition 36, the parallel transport again $\mathcal{F}_\theta \cong \mathcal{F}_0$ and the Hodge isomorphism again $\mathcal{F}_0 \cong \pi_!E' \oplus \pi_!E''$.

The convergence of $\frac{1}{\theta}P^{\mathcal{F}_\theta}d^{\nabla_\theta}P^{\mathcal{F}_\theta}$ to $P^{\mathcal{F}_0}(\omega\wedge)P^{\mathcal{F}_0}$ as θ converges to 0 proves that the equivalence class of links between $\mathcal{H}_\theta^+ - \mathcal{H}_\theta^-$ and $\mathcal{F}_\theta^+ - \mathcal{F}_\theta^-$ converges to the equivalence class of links between $\mathcal{H}_0^+ - \mathcal{H}_0^-$ and $\mathcal{F}_0^+ - \mathcal{F}_0^-$ provided via the Hodge isomorphisms $\mathcal{F}_0 \cong \pi_!E' \oplus \pi_!E''$ and $\mathcal{H}_0 \cong FH(Z,E) \oplus H(Z,E)/FH(Z,E)$ by Definition 36 and (62).

In the case of compact B, Proposition 43 follows from this convergence and the compatibility of the adiabatic limit of harmonic forms with (63) checked in the preceding paragraph. In the case of noncompact B one concludes using the fact that K^1_{top} is stable by inductive limit along an exhaustion of compact sets, so that two equivalence classes of links whose restrictions to any compact subset agree are equal.

5. η-forms

The goal of this section is to prove Theorems 28, 29 and 31. The construction of η-forms occupies three paragraphs: preliminaries of algebraic nature are given in §5.1, the adaptation to suitable triples of the construction of family index transgression forms is performed in §5.2. In §5.3, the construction is completed, and the existence part of Theorem 28 is proved. The anomaly formulae obtained in (91) and (92) allow to complete the proof of Theorem 28 and to prove Theorems 29 and 31 in §5.4.

5.1. $\mathbb{Z}_2$-graded theory

5.1.1. *$\mathbb{Z}_2$-graded bundles and superconnections.* — Consider a complex vector space V which decomposes as $V = V^+ \oplus V^-$, with a $\mathbb{Z}_2$-graduation operator $\tau|_{V^\pm} = \pm \mathrm{Id}|_{V^\pm}$. The supertrace of $a \in \mathrm{End} V$ is defined by $\mathrm{Tr}_s a = \mathrm{Tr}(\tau \circ a)$, (this is the trace on V^+ minus the trace on V^-). $\mathrm{End}V$ is also $\mathbb{Z}_2$-graded (even endomorphisms respect both parts V^+ and V^- and odd ones exchange them). The supercommutator in $\mathrm{End}V$ is defined for pure degree objects as

$$[a,b] = ab - (-1)^{\deg a \deg b} ba$$

and bilinearly extended to $\mathrm{End}V$. This is such that the supertrace vanishes on supercommutators.

Suppose now that E is a $\mathbb{Z}_2$ graded vector bundle on M, that is $E = E^+ \oplus E^-$ where E^+ and E^- are complex vector bundles themselve. The supertrace is defined as above and extends naturally on $\mathrm{End}E$-valued differential forms, with values in ordinary differential forms. $\mathrm{End}E$-valued differential forms inherit a global $\mathbb{Z}_2$-graduation, ordinary differential forms being $\mathbb{Z}_2$-graduated by the parity of their degree. They act on E-valued differential forms and multiply in the following way

$$(\alpha \mathbin{\widehat{\otimes}} a)(\beta \mathbin{\widehat{\otimes}} b) = (-1)^{\deg a \deg \beta} (\alpha \wedge \beta) \widehat{\otimes} (ab) \tag{65}$$

where $\alpha \widehat{\otimes} a$ and $\beta \widehat{\otimes} b$ are decomposed tensors in the graded tensor product of differential forms with either $\mathrm{End}E$ or E. The supercommutator of $\mathrm{End}E$-valued differential forms is defined in the same way as above but by considering the global graduation. With this convention, the supertrace allways vanishes on supercommutators.

A superconnection A on E is the sum of a connection ∇ which respects the decomposition of E and of a globally odd $\mathrm{End}E$-valued differential form ω. Its curvature is its square $A^2 = (\nabla + \omega)^2 = \nabla^2 + [\nabla, \omega] + \omega^2$, a global even End$E$-valued differential form (A^2 is not a differential operator).

Following (6), denote by $\mathrm{ch}(A) = \phi \mathrm{Tr}_s \exp -A^2$ the Chern-Weil form representing the Chern character of any superconnection A. It is an even degree differential form on M. The space of superconnections on E is convex (and of course contains ordinary connections) so that the preceding Chern-Weil and Chern-Simons theory also works for superconnections (especially Formula (7)). Thus $\mathrm{ch}(A)$ is closed and its cohomology class is the same as the Chern character of E in complex cohomology (this means $\mathrm{ch}(E^+) - \mathrm{ch}(E^-)$ because of the $\mathbb{Z}_2$-graduation).

5.1.2. *Special adjunction.* — A hermitian metric on $E = E^+ \oplus E^-$ will be supposed to make this decomposition orthogonal. Let β be a differential form and $a \in \mathrm{End}E$, the adjoint of a will be denoted by a^*. For End$E$-valued differential forms, there are two notions of adjunction: the ordinary adjoint of $\beta \,\widehat{\otimes}\, a$ is $\overline{\beta} \,\widehat{\otimes}\, a^*$, while its special adjoint is

$$(\beta \,\widehat{\otimes}\, a)^S = (-1)^{\frac{\deg\beta(\deg\beta - 1)}{2} + \deg\beta \deg a} \overline{\beta} \,\widehat{\otimes}\, a^* \tag{66}$$

following the convention implicitely used in **[11]** §I(c) and (d). If ω_1 and ω_2 are any (multidegree) End$E$-valued differential forms, denote by ω_1^S and ω_2^S their special adjoints, then for the product (65):

$$(\omega_1 \omega_2)^S = \omega_2^S \omega_1^S. \tag{67}$$

Denote by ω^* the usual adjoint of ω, the relations between usual and special adjunctions and the supertrace is as follows:

$$\mathrm{Tr}_s(\omega^*) = \overline{\mathrm{Tr}_s(\omega)} \qquad \text{and} \qquad \phi\mathrm{Tr}_s(\omega^S) = \overline{\phi\mathrm{Tr}_s(\omega)} \tag{68}$$

in particular, $\phi\mathrm{Tr}_s(\omega)$ is real if ω is a special autoadjoint (multidegree) End$E$-valued differential form.

Let ω be some globally odd End$E$-valued differential form, and $A = \nabla + \omega$ a superconnection on E. Then the adjoint of A is defined by

$$A^S = \nabla^* + \omega^S. \tag{69}$$

Thus $\frac{1}{2}(A + A^S)$ is the sum of $\nabla^u = \frac{1}{2}(\nabla + \nabla^*)$ (which respects the hermitian metric of E) and of some special autoadjoint End$E$-valued differential form of globally odd degree. The following adjunction and commutation rules

$$[\nabla^*, \omega^S] = [\nabla, \omega]^S \qquad \text{and} \qquad (\nabla^*)^2 = -(\nabla^2)^* = (\nabla^2)^S \tag{70}$$

have the following consequences

$$(A^S)^2 = (A^2)^S \qquad \text{and} \qquad \mathrm{ch}(A^S) = \overline{\mathrm{ch}(A)}. \tag{71}$$

In particular, $\mathrm{ch}\big(\frac{1}{2}(A+A^S)\big)$ is a real form. Finally, Lemma 13 and formulae (28) and (31) are also valid in the context of superconnections.

5.2. Adaptation of Bismut's superconnection

5.2.1. *Definition of Bismut and Lott's Levi-Civita superconnection.* — Remember the definitions of P^{TZ} and T^HM from §3.4.1. Let $y \in B$. For any vector $\mathrm{u} \in T_yB$, its horizontal lift u^H is a global section of the restriction of T^HM to $Z_y = \pi^{-1}(y)$ such that at any point of Z_y one has $\pi_*\mathrm{u}^H = \mathrm{u}$.

Consider some vector bundle ξ on M with a connection ∇_ξ and hermitian metric h^ξ. ∇_ξ is not supposed to be flat nor to respect h^ξ. Remember the definition of $\mathscr{E}$ from (37). The flow associated to vector fields of the form u^H send fibres of π to fibres of π diffeomorphically, so that there is some fiberwise Lie differentiation operator $\mathscr{L}^{\nabla_\xi}_{\mathrm{u}^H}$ which acts on ξ-valued vertical differential forms $\mathscr{E}$ (it is defined using the connection ∇_ξ). Put then for any local section σ of $\mathscr{E}$ (see **[11]** Definition 3.2)

$$\overline{\nabla}_{\mathrm{u}}\sigma = \mathscr{L}^{\nabla_\xi}_{\mathrm{u}^H}\sigma. \tag{72}$$

$\overline{\nabla}$ is a connection on $\mathscr{E}$ as can be proved following **[11]** (3.8) to (3.10).

If u and v are vector fields defined on a neighbourhood of $y \in B$, then the vector field $P^{TZ}[\mathrm{u}^H, \mathrm{v}^H]$ on $Z_y = \pi^{-1}(y)$ depends on the values of u and v at y only. Let $\iota_T\colon \wedge^2 TB \longrightarrow \mathrm{End}^{\mathrm{odd}}(\mathscr{E})$ be the operator which to u and $\mathrm{v} \in T_yB$ associates the interior product by $-P^{TZ}[\mathrm{u}^H, \mathrm{v}^H]$ in $\wedge^\bullet T^*Z \otimes \xi$. ι_T can be extended to a globally odd $\mathrm{End}\mathscr{E}$-valued differential form (of differential form degree 2) on B.

$\overline{\nabla} + d^{\nabla_\xi} + \iota_T$ is a superconnection on $\mathscr{E}$ in the sense of §5.1.1 and also of **[37]**, **[3]** Definitions 1.37 and 9.12 and **[6]**. It can be proved to coincide with the total exterior differential operator d^M on ξ-valued differential forms (defined using ∇_ξ) on M through the identification (37) as in **[3]** Proposition 10.1 (the proof of **[11]** §III (b) cannot be adapted here because $(d^M)^2 \neq 0$ if ∇_ξ is not flat).

Remember the definition of metric data g^Z, h^ξ, $(\ |\ )_Z$ and $\langle\ ,\ \rangle_{L^2}$ from §3.1.1 and (38). Define the adjoint connection $\overline{\nabla}^S$ of $\overline{\nabla}$ as in (22) by the following formula, valid for any element u of the tangent bundle of B and any local sections σ and θ of $\mathscr{E}$:

$$\langle \overline{\nabla}^S_{\mathrm{u}}\sigma, \theta\rangle_{L^2} = \mathrm{u}.\langle\sigma, \theta\rangle_{L^2} - \langle\sigma, \overline{\nabla}_{\mathrm{u}}\theta\rangle_{L^2}. \tag{73}$$

Let $T\wedge\colon \wedge^2 TB \longrightarrow \mathrm{End}^{\mathrm{odd}}(\mathscr{E})$ be the operator which associates to u and $\mathrm{v} \in T_yB$ the exterior product in $\mathscr{E}_y$ by the one form $(-P^{TZ}[\mathrm{u}^H, \mathrm{v}^H])^\flat$ (the dual through g^Z to the vector field $-P^{TZ}[\mathrm{u}^H, \mathrm{v}^H]$) on $\pi^{-1}(y)$. Before and after being extended to a globally odd $\mathrm{End}\mathscr{E}$-valued differential form on B (of differential form degree 2), $T\wedge$ is the adjoint of ι_T, so that $\iota_T - T\wedge$ is a special autoadjoint $\mathrm{End}\mathscr{E}$-valued differential form in the sense of paragraph 5.1.

d^{∇_ξ} and its adjoint $(d^{\nabla_\xi})^*$ as defined in §3.1.1 are also mutually special adjoint as $\mathrm{End}\mathscr{E}$-valued differential forms (with differential form degree 0). The superconnection $\overline{\nabla}^S + (d^{\nabla_\xi})^* - T\wedge$ is the adjoint of the superconnection $\overline{\nabla} + d^{\nabla_\xi} + \iota_T$ in the sense of [**11**] §I(b) and Proposition 3.7, (and (69) above).

The relevant Bismut-Levi-Civita superconnection in this context is defined for any $t > 0$ as in [**11**] (3.50) (and also (3.49), (3.30) and Proposition 3.4) by:

$$C_t = \frac{1}{2}(\overline{\nabla} + \overline{\nabla}^S) + \frac{\sqrt{t}}{2}\left(d^{\nabla_\xi} + (d^{\nabla_\xi})^*\right) + \frac{1}{2\sqrt{t}}(\iota_T - T\wedge). \tag{74}$$

In the case of a fibered product of the form (43), the construction of C_t is functorial if the horizontal subspace $T^H(\widetilde{B} \times_B M)$ is taken to be the subspace of $T(\widetilde{B} \times_B M)$ consisting of vectors which are sent to $T^H M$ by the tangent map of $\widetilde{B} \times_B M \longrightarrow M$. (It is not always isomorphic to the pullback of $T^H M$).

5.2.2. *Properties and asymptotics of the Chern character of C_t.* — C_t^2 is a fiberwise positive second order elliptic differential operator so that its heat kernel $\exp -C_t^2$ is trace class. The Chern character of C_t is defined to be

$$\mathrm{ch}(C_t) = \phi \mathrm{Tr}_s \exp -C_t^2.$$

Lemma 44. — *$\mathrm{ch}(C_t)$ is a real form. It is a constant integer if ∇_ξ is flat.*

Proof. — The superconnection C_t is for any t the half sum of $\overline{\nabla} + \sqrt{t} d^{\nabla_\xi} + \frac{1}{\sqrt{t}}\iota_T$ and its adjoint $\overline{\nabla}^S + \sqrt{t}(d^{\nabla_\xi})^* - \frac{1}{\sqrt{t}}T\wedge$. The reality of $\mathrm{ch}(C_t)$ follows from (71) and the comment after it. The case of flat connection ∇_ξ is treated in [**11**] Theorem 3.15. □

Remember the definition of the Euler form e and the connection ∇_{TZ} from paragraph 3.4.1, and put $\nabla_\xi^u = \frac{1}{2}(\nabla_\xi + \nabla_\xi^*)$ as in (31).

Proposition 45. — *As t tends to 0, $\mathrm{ch}(C_t)$ has for any $k \geq 1$ an asymptotic of the form*

$$\begin{cases} \mathrm{ch}(C_t) = \sum_{j=0}^{k-1} t^{j+\frac{1}{2}} A_j + \mathscr{O}(t^{k+\frac{1}{2}}) & \textit{if } \dim Z \textit{ is odd} \\ \mathrm{ch}(C_t) = \sum_{j=0}^{k-1} t^{j} B_j + \mathscr{O}(t^{k}) & \textit{if } \dim Z \textit{ is even} \end{cases}$$

in either case:

$$\lim_{t\to 0} \mathrm{ch}(C_t) = \int_Z e(\nabla_{TZ}) \wedge \mathrm{ch}(\nabla_\xi^u). \tag{75}$$

Proof. — The asymptotics with $\sum_{j=-\frac{1}{2}\dim Z}^{k-1}$ are classical results on heat kernels (see [**3**] §§2.5 and 2.6 and appendix after §9.7).

The limit formula (and thus the vanishing of the terms A_j and/or B_j for negative j) is a consequence of [**11**] (3.76). The connection ∇_ξ is supposed to be flat in [**11**], which is not the case here: thus formula [**11**] (3.52) does not hold true here. However, consider $\mathscr{R}$ defined as in [**11**] (3.56) without taking [**11**] (3.52) into account, then the $z = 0$

case of the Lichnerowicz-type formula of [**11**] Theorem 3.11 holds true here. Thus the rescaling formula [**11**] (3.75) and its consequence [**11**] (3.76) remain true here. (This is only a matter of Clifford degrees which has nothing to do with the fact that ∇_ξ be flat or not).

In particular, if $\dim Z$ is odd, then the same argument as in [**11**] (3.79) applies, and both sides of the equality (75) vanish. □

5.2.3. *Calculating C_t for the product with the real line.* — Consider now the product manifold $\widetilde{M} = \mathbb{R} \times M$ and its obvious submersion $\widetilde{\pi} = \mathrm{Id}_{\mathbb{R}} \times \pi$ onto $\widetilde{B} = \mathbb{R} \times B$. Extend ξ tautologically to $\widetilde{M}$ with constant (with respect to s) hermitian metric and connection $d_{\mathbb{R}} + \nabla_\xi$ (where $d_{\mathbb{R}} = ds \frac{\partial}{\partial s}$ is the trivial canonical differential along $\mathbb{R}$). Consider any smooth real positive function f on $\mathbb{R}$ such that $f(1) = 1$ and endow the vertical tangent bundle of $\widetilde{\pi}$ with the metric $\frac{1}{f(s)} g^Z$. Choose $T^H\widetilde{M} = T\mathbb{R} \oplus T^H M$ as horizontal bundle of $\widetilde{\pi}$. Let's calculate the Bismut-Lott Levi-Civita superconnection $\widetilde{C}_t$ in this context.

The equivalent here of the connection $\overline{\nabla}$ defined in (72) is simply equal to $d_{\mathbb{R}} + \overline{\nabla}$. The vertical exterior differential operator d^{∇_ξ} is unchanged, and so is the operator ι_T (defined at the beginning of §5.2).

The volume form of the fibres of $\widetilde{\pi}$ on $\{s\} \times B$ is equal to $f(s)^{-\frac{\dim Z}{2}}$ times the corresponding volume form of the fibres on $\{1\} \times B$. The punctual scalar product between vertical differential forms of degree k on $\{s\} \times B$ is equal to the one on $\{1\} \times B$ multiplied by $f(s)^k$. Call $\widetilde{\mathscr{E}}$ the infinite rank vector bundle on $\widetilde{B}$ of ξ-valued vertical differential forms, and define $N_V \in \mathrm{End}\mathscr{E}$ or $\mathrm{End}\widetilde{\mathscr{E}}$ to be the operator which multiplies vertical differential forms by their degree. The global L^2 scalar product on the restriction of $\widetilde{\mathscr{E}}$ to $\{s\} \times B$ is thus equal to $f(s)^{N_V - \frac{\dim Z}{2}} \langle\ ,\ \rangle_{L^2}$ (where $\langle\ ,\ \rangle_{L^2}$ defined in (38) is the one on $\{1\} \times B$).

It follows that the adjoint of d^{∇_ξ} is $f(s)(d^{\nabla_\xi})^*$ (if $(d^{\nabla_\xi})^*$ is its adjoint on $\{1\} \times B$) and the adjoint of ι_T is $\frac{1}{f(s)} T\wedge$ (if $T\wedge$ is its adjoint on $\{1\} \times B$). In the same way, following (22), one has $(d_{\mathbb{R}} + \overline{\nabla})^S = d_{\mathbb{R}} + ds \frac{f'(s)}{f(s)} \left(N_V - \frac{\dim Z}{2}\right) + \overline{\nabla}^S$.

Thus if $C_{t,s}$ denotes the Bismut-Lott Levi-Civita superconnection on $\{s\} \times B$:

$$\begin{aligned} C_{t,s} &= \tfrac{1}{2}\left(\overline{\nabla} + \overline{\nabla}^S\right) + \tfrac{\sqrt{t}}{2}\left(d^{\nabla_\xi} + f(s)(d^{\nabla_\xi})^*\right) + \tfrac{1}{2\sqrt{t}}\left(\iota_T - \tfrac{1}{f(s)} T\wedge\right), \\ \widetilde{C}_t &= C_{t,s} + d_{\mathbb{R}} + \tfrac{1}{2} ds \tfrac{f'(s)}{f(s)}\left(N_V - \tfrac{\dim Z}{2}\right). \end{aligned}$$

One then computes:

$$
\begin{aligned}
[d_{\mathbb{R}}, C_{t,s}] &= \frac{\sqrt{t}}{2} f'(s) ds (d^{\nabla_\xi})^* + \frac{f'(s)}{2\sqrt{t} f(s)^2} ds\, T\wedge \\
[N_V, C_{t,s}] &= \frac{\sqrt{t}}{2}\left(d^{\nabla_\xi} - f(s)(d^{\nabla_\xi})^*\right) + \frac{1}{2\sqrt{t}}\left(-\iota_T - \frac{1}{f(s)} T\wedge\right) \\
\left[d_{\mathbb{R}} + \frac{1}{2} ds \frac{f'(s)}{f(s)} N_V\, ,\, C_{t,s}\right] &= \frac{\sqrt{t}}{4} ds \frac{f'(s)}{f(s)}\left(d^{\nabla_\xi} + f(s)(d^{\nabla_\xi})^*\right) + \\
&\quad + ds \frac{f'(s)}{4\sqrt{t} f(s)}\left(-\iota_T + \frac{1}{f(s)} T\wedge\right) \\
&= t\, ds \frac{f'(s)}{f(s)} \frac{\partial C_{t,s}}{\partial t} \\
\widetilde{C}_t^2 &= C_{t,s}^2 + \left[d_{\mathbb{R}} + \frac{1}{2} ds \frac{f'(s)}{f(s)}\left(N_V - \frac{\dim Z}{2}\right)\, ,\, C_{t,s}\right] \\
&= C_{t,s}^2 + t\, ds \frac{f'(s)}{f(s)} \frac{\partial C_{t,s}}{\partial t}
\end{aligned}
$$

$$
\operatorname{Tr}_s \exp(-\widetilde{C}_t^2) = \operatorname{Tr}_s \exp(-C_{t,s}^2) - t\, ds \frac{f'(s)}{f(s)} \operatorname{Tr}_s\left(\frac{\partial C_{t,s}}{\partial t} \exp(-C_{t,s}^2)\right). \tag{76}
$$

5.2.4. $t \longrightarrow 0$ *asymptotics of the infinitesimal transgression form.* — The transgression Formula (7) yields here

$$
\frac{d}{dt}\operatorname{ch}(C_t) = -d\left[\phi \operatorname{Tr}_s\left(\frac{\partial C_t}{\partial t} \exp -C_t^2\right)\right]
$$

so that for any $0 < S < T < +\infty$

$$
\operatorname{ch}(C_S) - \operatorname{ch}(C_T) = d\left[\int_S^T \phi \operatorname{Tr}_s\left(\frac{\partial C_t}{\partial t} \exp -C_t^2\right) dt\right]. \tag{77}
$$

Proposition 46. — *One has the following estimate*

$$
\text{as} \quad t \to 0: \qquad \phi \operatorname{Tr}_s\left(\frac{\partial C_t}{\partial t} \exp -C_t^2\right) = \begin{cases} \mathcal{O}(1) & \textit{if } \dim Z \textit{ is even,} \\ \mathcal{O}(t^{-\frac{1}{2}}) & \textit{if } \dim Z \textit{ is odd.} \end{cases} \tag{78}
$$

Proof. — This will be proved with the technique proposed in **[3]** Theorem 10.32: apply Proposition 45 on $\widetilde{M}$, one obtains because of the factor t appearing in (76) an asymptotic of the form

$$
\phi \operatorname{Tr}_s\left(\frac{\partial C_t}{\partial t} \exp -C_t^2\right) = \begin{cases} \displaystyle\sum_{j=-1}^{k-1} E_j t^j + \mathcal{O}(t^k) & \text{if } \dim Z \text{ is even,} \\ \displaystyle\sum_{j=0}^{k-1} E_j t^{j-\frac{1}{2}} + \mathcal{O}(t^{k-\frac{1}{2}}) & \text{if } \dim Z \text{ is odd.} \end{cases}
$$

This proves the assertion for odd dimensional fibres.

Let $\widetilde{\nabla}_{TZ}$ be the Levi-Civita connection on the vertical tangent bundle of the submersion $\widetilde{\pi}$ over $\widetilde{M}$ (as defined at §3.4.1). If $\dim Z$ is even, let $\int_Z$ denote the integral along the fibres of $\widetilde{\pi}$, then E_{-1} is the factor of ds in the decomposition of the form $\int_Z e(\widetilde{\nabla}_{TZ})\mathrm{ch}(\nabla^u_\xi)$ with respect to $\Omega(\widetilde{B},\mathbb{C}) = C^\infty\big(\mathbb{R},\Omega(B,\mathbb{C})\big)\oplus ds\wedge C^\infty\big(\mathbb{R},\Omega(B,\mathbb{C})\big)$. This is because the Chern character is functorial by pullbacks. However, $\widetilde{\nabla}_{TZ}$ is not the pullback of ∇_{TZ}. A direct calculation from the classical formula for Levi-Civita connections (see **[3]** formula (1.18)) yields

$$\widetilde{\nabla}_{TZ} = d_{\mathbb{R}} + \nabla_{TZ} + \frac{f'(s)}{2f(s)}ds$$

so that $\widetilde{\nabla}^2_{TZ} = \nabla^2_{TZ}$ because $d_{\mathbb{R}}$ and ds both commute with ∇_{TZ}. Thus the curvature of $\widetilde{\nabla}_{TZ}$ is the pullback of the one of ∇_{TZ} and neither $e(\widetilde{\nabla}_{TZ})$ nor $\mathrm{ch}(\nabla^u_\xi)$ have a ds component. This proves the vanishing of E_{-1}. □

5.2.5. *Adapting C_t to some suitable triple.* — Let χ be a smooth real increasing function on $\mathbb{R}_+$ which vanishes on $[0,\frac{1}{2}]$ and equals 1 on $[1,+\infty)$. Consider some suitable triple (μ^+,μ^-,ψ) with respect to ξ, h^ξ, ∇_ξ and g^Z in the sense of Definition 17. Put some hermitian metrics $h^\pm$ on $\mu^\pm$ and some connection ∇_μ on $\mu^+\oplus\mu^-$ which respects the decomposition. Denote $\frac{1}{2}(\overline{\nabla}+\overline{\nabla}^*)$ by $\overline{\nabla}^u$. Consider the following t-depending superconnection on $(\mathscr{E}^+\oplus\mu^+)\oplus(\mathscr{E}^-\oplus\mu^-)$:

$$B_t = \overline{\nabla}^u \oplus \nabla_\mu + \frac{\sqrt{t}}{2}\mathscr{D}^{\nabla_\xi}_{\chi(t)\psi} + \frac{1}{2\sqrt{t}}(\iota_T - T\wedge) = C_t \oplus \nabla_\mu + \frac{\sqrt{t}}{2}\chi(t)\,(\psi+\psi^*)\,. \tag{79}$$

B_t^2 is as C_t^2 a fiberwise positive second order elliptic operator, so that its heat kernel is trace class. Its Chern character is defined as is $\mathrm{ch}(C_t)$, the supertrace being the trace on $\mathrm{End}(\mathscr{E}^+\oplus\mu^+)$ minus the trace on $\mathrm{End}(\mathscr{E}^-\oplus\mu^-)$.

Lemma 47. — $\mathrm{ch}(B_t)$ *is real if ∇_μ respects h^+ and h^-. For $t\le\frac{1}{2}$, one has*

$$\mathrm{ch}(B_t) = \mathrm{ch}(C_t) + \mathrm{ch}(\nabla_\mu). \tag{80}$$

Proof. — The equality is obvious. ψ is of differential form degree 0 so that ψ^* is the special adjoint of ψ. The reality follows from (71) (as does Lemma 44). □

Call $\mathscr{H}^\pm = \mathrm{Ker}\mathscr{D}^{\nabla_\xi\pm}_\psi$ and $P^{\mathscr{H}^\pm}$ the orthogonal projection $\mathscr{E}^\pm\oplus\mu^\pm\longrightarrow\mathscr{H}^\pm$, (and $P^{\mathscr{H}} = P^{\mathscr{H}^+}\oplus P^{\mathscr{H}^-}$). The associated connection on $\mathscr{H}=\mathscr{H}^+\oplus\mathscr{H}^-$ is

$$\nabla_{\mathscr{H}} = P^{\mathscr{H}}\left(\overline{\nabla}^u\oplus\nabla_\mu\right)P^{\mathscr{H}}. \tag{81}$$

This connection respects the decomposition $\mathscr{H}^+\oplus\mathscr{H}^-$, and it also respects the hermitian metric on $\mathscr{H}$ obtained by restriction of $\langle\ ,\ \rangle_{L^2}\oplus h^\pm$ provided ∇_μ respects $h^\pm$ (this can be proved by a direct elementary computation).

It is proved in [**3**] Theorem 9.26 that:

$$\lim_{t\to+\infty} \mathrm{ch}(B_t) = \mathrm{ch}(\nabla_{\mathscr{H}}) \tag{82}$$

in the sense of any $\mathscr{C}^\ell$ norm on any compact subset of B.

Both B_t and its Chern character are functorial by pullbacks on fibered products as in (43) (if the horizontal subspace of the source manifold is taken as described at the end of §5.2). Note also that the construction can be performed with any smooth function χ on $B \times \mathbb{R}_+$ which vanishes on $B \times [0, \varepsilon]$ and equals 1 on $B \times [A, +\infty)$ for any $0 < \varepsilon < A$, and which is increasing with respect to the variable in $\mathbb{R}_+$. This is of course not essential, but will be useful to prove some independence of the constructed forms on the choice of the function χ.

5.2.6. *$t \longrightarrow +\infty$ asymptotics of the infinitesimal transgression form.* — For any $0 < S < T < +\infty$, the counterpart of (77) for B_t is here

$$\mathrm{ch}(B_S) - \mathrm{ch}(B_T) = d\left[\int_S^T \phi\mathrm{Tr}_s\left(\frac{\partial B_t}{\partial t}\exp -B_t^2\right)dt\right]. \tag{83}$$

Lemma 48. — *$\phi\mathrm{Tr}_s\left(\frac{\partial B_t}{\partial t}\exp -B_t^2\right)$ is a real form if ∇_μ respects $h^\pm$ (the hermitian metrics on $\mu^\pm$). If not, this form is changed into its complex conjugate if ∇_μ is changed into its adjoint connection with respect to $h^\pm$.*

If ∇_ξ is flat and if the suitable triple used in the construction of B_t is the trivial one $(\{0\},\{0\},0)$, then:

$$\phi\mathrm{Tr}_s\left(\frac{\partial B_t}{\partial t}\exp -B_t^2\right) = \phi\mathrm{Tr}_s\left(\frac{\partial C_t}{\partial t}\exp -C_t^2\right) = 0.$$

Proof. — The second assertion is proved in [**32**]. It is reproved here as a direct consequence of (76), of the last assertion of Lemma 44 (and the fact that if ∇_ξ is flat on ξ over M, then $d_{\mathbb{R}} + \nabla_\xi$ is also flat on the pullback of ξ over $\widetilde{M}$).

In general, $\exp -B_t^2$ is a globally even $\mathrm{End}\mathscr{E}$-valued differential form, so that its supercommutator with $\frac{\partial B_t}{\partial t}$ is their usual commutator; and it is special autoadjoint if ∇_μ respects $h^\pm$ on $\mu^\pm$ (if not, the two forms obtained from mutually adjoint connections on μ are mutually special adjoint).

On the other hand, $\frac{\partial B_t}{\partial t}$ is for any t a special autoadjoint $\mathrm{End}\mathscr{E}$-valued differential form, so that the product $\frac{\partial B_t}{\partial t}\exp -B_t^2$ is the special adjoint of $(\exp -B_t^2)\frac{\partial B_t}{\partial t}$ (if ∇_μ respect $h^\pm$). Thus

$$\phi\mathrm{Tr}_s\left(\frac{\partial B_t}{\partial t}\exp -B_t^2\right) = \phi\mathrm{Tr}_s\left((\exp -B_t^2)\frac{\partial B_t}{\partial t}\right) = \overline{\phi\mathrm{Tr}_s\left(\frac{\partial B_t}{\partial t}\exp -B_t^2\right)}$$

and the reality follows (the case when ∇_μ does not respect $h^\pm$ is similar). □

Proposition 49. — *One has the following estimate:*

$$\text{as} \quad t \to +\infty : \qquad \phi \mathrm{Tr}_s \left(\frac{\partial B_t}{\partial t} \exp -B_t^2 \right) = \mathcal{O}(t^{-\frac{3}{2}}).$$

Proof. — The $t \to +\infty$ asymptotic is proved by the adaptation of [**3**] Theorem 9.23 which is proposed (though not detailed) at the end of §9.3 of [**3**]. (Here $\chi(t)$ is constant on a neighbourhood of $+\infty$, so that the arguments of the proof of Theorems 9.7 and 9.23 of [**3**] apply). □

This estimate together with formulae (80) and (78) prove the convergence of the integral $\int_0^{+\infty} \phi \mathrm{Tr}_s \left(\frac{\partial B_t}{\partial t} \exp -B_t^2 \right) dt$. It follows from (82), (83), (80), and Proposition 45 that this integral is a transgression form in the following sense:

$$(84) \quad d \left[\int_0^{+\infty} \phi \mathrm{Tr}_s \left(\frac{\partial B_t}{\partial t} \exp -B_t^2 \right) dt \right] = \int_Z e(\nabla_{TZ}) \wedge \mathrm{ch}(\nabla_\xi^u) + \mathrm{ch}(\nabla_\mu) - \mathrm{ch}(\nabla_{\mathcal{H}})$$

(where $\mathrm{ch}(\nabla_\mu) = \mathrm{ch}(\nabla_{\mu^+}) - \mathrm{ch}(\nabla_{\mu^-})$ and accordingly for $\mathrm{ch}(\nabla_{\mathcal{H}})$). The preceding considerations about functoriality apply here, so that this transgression form is functorial by pullbacks on fibered products as in (43) (if the horizontal subspace of the source manifold is taken as described at the end of §5.2).

5.3. Proof of the first part of Theorem 28

5.3.1. *Chern-Simons transgression and links*. — Let E, F, G and H be vector bundles on M with connections ∇_E, ∇_F, ∇_G and ∇_H. Suppose there exists some link (K, ℓ) between $E - F$ and $G - H$ as in (47). One associates to (K, ℓ) the differential form (defined modulo exact forms)

$$(85) \qquad \widetilde{\mathrm{ch}}([\ell]) = \widetilde{\mathrm{ch}}\big(\nabla_E \oplus \nabla_H \oplus \nabla_K, \ell^*[\nabla_F \oplus \nabla_G \oplus \nabla_K]\big)$$

for some connection ∇_K on K. It is easily checked from (9) and (10) that the class of this form modulo exact forms does not depend on the choice of ∇_K and is not modified by changing (K, ℓ) by an equivalent link. It is possible to choose a unitary ∇_K, so that $\widetilde{\mathrm{ch}}([\ell])$ is a real form (modulo exact forms) if it happens that ∇_E, ∇_F, ∇_G and ∇_H are all unitary connections. And of course

$$(86) \qquad d\widetilde{\mathrm{ch}}([\ell]) = \mathrm{ch}(\nabla_E) + \mathrm{ch}(\nabla_H) - \mathrm{ch}(\nabla_F) - \mathrm{ch}(\nabla_G).$$

For the composition of two links ℓ and ℓ', and any connections on the considered bundles one obtains (modulo exact forms and always from (9) and (10)):

$$(87) \qquad \widetilde{\mathrm{ch}}([\ell' \circ \ell]) = \widetilde{\mathrm{ch}}([\ell]) + \widetilde{\mathrm{ch}}([\ell']).$$

5.3.2. *Definition of the η-form and check of its properties.* — Consider now some vector bundle ξ with connection ∇_ξ and hermitian metric h^ξ on M, some horizontal tangent vector space T^HM and vertical metric g^Z for the submersion $\pi\colon M \to B$, and vector bundles F^+ and F^- on B such that

$$[F^+] - [F^-] = \pi_*^{\mathrm{Eu}}[\xi] \in K^0_{\mathrm{top}}(B).$$

Put any connections ∇_{F^+} on F^+ and ∇_{F^-} on F^- and choose some equivalence class of links $[\ell]$ between $F^+ - F^-$ and some family index bundles $(\mathscr{H}^+ \oplus \mu^-) - (\mathscr{H}^- \oplus \mu^+)$ provided by any suitable triple (μ^+, μ^-, ψ) (with connections $\nabla_{\mathscr{H}}$ and ∇_μ, $\mathscr{H}^\pm$ being the kernel bundles).

Definition 50. — *The families Chern-Simons transgression form is the (inductive limit of the) class modulo exact forms of the following differential form on (compact subsets of) B:*

$$\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell]) = \int_0^{+\infty} \phi \mathrm{Tr}_s \left(\frac{\partial B_t}{\partial t} \exp -B_t^2 \right) dt + \\ + \int_Z e(\nabla_{TZ}) \wedge \widetilde{\mathrm{ch}}(\nabla_\xi^u, \nabla_\xi) + \widetilde{\mathrm{ch}}([\ell])$$

where $\widetilde{\mathrm{ch}}([\ell])$ is computed with the connections ∇_μ, $\nabla_{\mathscr{H}}$ and $\nabla_{F^\pm}$.

If B is noncompact, the above construction produces some projective collection of elements of $\Omega^{\mathrm{odd}}(K, \mathbb{C})/d\Omega^{\mathrm{even}}(K, \mathbb{C})$ on compact submanifolds (with boundary and of the same dimension as B. In fact, this will be fully established in Proposition 51 below. The properties checked just hereafter are local and will also be valid for a noncompact B). This gives rise to an unambiguous object in $\Omega^{\mathrm{odd}}(B, \mathbb{C})/d\Omega^{\mathrm{even}}(B, \mathbb{C})$ (which can be constructed by an analogue procedure to the one which was sketched just before Definition 40).

It follows from (84), (8) and (86) that the form $\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell])$ verifies the transgression formula stated as property (a) in Theorem 28.

The $\int_0^{+\infty} \phi \mathrm{Tr}_s \left(\frac{\partial B_t}{\partial t} \exp -B_t^2 \right) dt$ part of η is functorial by pullback on fibered products as in (43) as was remarked at the end of subSection 5.2.6 just after the proof of Proposition 49. The $\widetilde{\mathrm{ch}}$ are both functorial, as was remarked just before Equation (9), and $e(\nabla_{TZ})$ too, under the assumption on horizontal subspaces of the end of §5.2. This proves the naturality property (b) for η.

$\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell])$ is additive in the following sense: let ξ_1 and ξ_2 be bundles on M with connections ∇_{ξ_1} and ∇_{ξ_2}, let F_1^+, F_1^-, F_2^+ and F_2^- be bundles with connections on B such that $[F_1^+] - [F_1^-] = \pi_*^{\mathrm{Eu}}[\xi_1]$ and $[F_2^+] - [F_2^-] = \pi_*^{\mathrm{Eu}}[\xi_2]$ in $K^0_{\mathrm{top}}(B)$. Let $[\ell_1]$ be some link between $F_1^+ - F_1^-$ and some (couple of) family index bundles for ξ_1 on B, and correspondingly for $[\ell_2]$. The additivity (for direct sums) of the topological direct image construction ensures that $\ell_1 \oplus \ell_2$ provides an equivalence class of link between $(F_1^+ \oplus F_2^+) - (F_1^- \oplus F_2^-)$ and bundles on B which form a couple of family index bundles for $\xi_1 \oplus \xi_2$. Then

$$\begin{aligned}\eta(\nabla_{\xi_1} \oplus \nabla_{\xi_2}, &\nabla_{TZ}, \nabla_{F_1^+} \oplus \nabla_{F_2^+}, \nabla_{F_1^-} \oplus \nabla_{F_2^-}, [\ell_1 \oplus \ell_2]) = \\ &= \eta(\nabla_{\xi_1}, \nabla_{TZ}, \nabla_{F_1^+}, \nabla_{F_1^-}, [\ell_1]) + \eta(\nabla_{\xi_2}, \nabla_{TZ}, \nabla_{F_2^+}, \nabla_{F_2^-}, [\ell_2]).\end{aligned}$$

This additivity is a direct consequence of the fact that the Chern character and the supertrace entering the construction of $\int_0^{+\infty} \phi \mathrm{Tr}_s \left(\frac{\partial B_t}{\partial t} \exp -B_t^2\right) dt$ are additive for direct sums, and accordingly for Chern-Simons transgressions (10). Property (c) of Theorem 28 is thus established for η.

The vanishing of $\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{\pi_!^+\xi}, \nabla_{\pi_!^-\xi}, [\mathrm{Id}])$ for any flat bundle (ξ, ∇_ξ) is a consequence of the first statement of Lemma 48 and of [**11**] Proposition 3.14 and Theorem 3.17: Lemma 48 proves that the integrand of the first term in the definition of η vanishes for all $t > 0$ (if it is computed using the trivial suitable triple $(\{0\}, \{0\}, 0)$). In particular, the link [Id] in the third term $\widetilde{\mathrm{ch}}([\mathrm{Id}])$ is trivial as link, but it links $\pi_!^+\xi - \pi_!^-\xi$ endowed with their sheaf theoretic direct image flat connections $\nabla_{\pi_!^+\xi}$ and $\nabla_{\pi_!^-\xi}$, with $\pi_!^+\xi - \pi_!^-\xi$ endowed with their metric connections $\nabla_{\mathscr{H}^+}$ and $\nabla_{\mathscr{H}^-}$ obtained by the projection on the kernel of the fiberwise Dirac operator (82).

It is proved in [**11**] Proposition 3.14 that $\nabla_{\mathscr{H}^+} = \nabla^u_{\pi_!^+\xi}$ and accordingly on F^-, and in [**11**] Theorem 3.17 (see also Remark 14 above) that, up to exact forms

$$\widetilde{\mathrm{ch}}(\nabla_{\pi_!^+\xi}, \nabla^u_{\pi_!^+\xi}) - \widetilde{\mathrm{ch}}(\nabla_{\pi_!^-\xi}, \nabla^u_{\pi_!^+\xi}) = \int_Z e(\nabla_{TZ}) \widetilde{\mathrm{ch}}(\nabla_\xi, \nabla^u_\xi).$$

Thus the two last terms in the definition of $\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{\pi_!^+\xi}, \nabla_{\pi_!^-\xi}, [\mathrm{Id}])$ mutually compensate, and the property (d) of Theorem 28 is established for η.

5.3.3. *Invariance properties of* η. — The proof of the first part of Theorem 28 is thus reduced to the following

Proposition 51. — *$\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell])$ does not depend on h^ξ, nor on the function χ nor on the construction of topological direct image and the choice of data used in it, provided the class of link $[\ell]$ is modified by composition with the canonical link between the obtained representatives of the topological direct image when they are changed.*

$\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell])$ of course depends on the other data in a way which will be precised later in §5.4.1.

Proof. — This will be proved in two steps.

First step: independence on h^ξ, χ, and on deformation of ψ. — Consider the submersion $\pi \times \mathrm{Id}_{[0,1]} : M \times [0,1] \longrightarrow B \times [0,1]$. The vertical tangent space of $\pi \times \mathrm{Id}_{[0,1]}$ is simply the pullback to $M \times [0,1]$ of the one of π, and it will be supposed to be endowed with a pullback metric. Choose some horizontal subspace T^HM for π and pull it back on $M \times [0,1]$, where it is a suitable horizontal subspace with respect to $\pi \times \mathrm{Id}_{[0,1]}$. These choices of horizontal subspaces verify the conditions of the end of §5.2 with respect to the maps $B \times \{0\} \hookrightarrow B \times [0,1]$ and $B \times \{1\} \hookrightarrow B \times [0,1]$. Call $\widetilde{\nabla}_{TZ}$ the associated pullback connection on the vertical tangent bundle of $\pi \times \mathrm{Id}_{[0,1]}$.

Consider some vector bundle ξ on M, with connection ∇_ξ, and any pair of bundles F^+ and F^- on B with connections ∇_{F^+} and ∇_{F^-} such that $[F^+] - [F^-] = \pi_*^{\mathrm{Eu}}[\xi]$ in $K^0_{\mathrm{top}}(B)$, and some equivalence class of link $[\ell]$ between $F^+ - F^-$ and some couple of family index bundles for ξ. Pull back ξ on $M \times [0,1]$ and F^+ and F^- on $B \times [0,1]$ and call $\widetilde{\xi}$, $\widetilde{F}^+$ and $\widetilde{F}^-$ the pullbacks. Call $\widetilde{\nabla}_\xi$, $\widetilde{\nabla}_{F^+}$ and $\widetilde{\nabla}_{F^-}$ the pullback connections on them. Endow $\widetilde{\xi}$ with some not necessarily pullback hermitian metric $\widetilde{h}^\xi$ and choose any suitable data $(\widetilde{\mu}^+, \widetilde{\mu}^-, \widetilde{\psi})$ with respect to $\pi \times \mathrm{Id}_{[0,1]}$ providing kernel bundles $\widetilde{\mathscr{H}}^\pm = \mathrm{Ker}\mathscr{D}_{\widetilde{\psi}}^{\nabla_\xi \pm}$ on $B \times [0,1]$. Of course one has

$$[\widetilde{F}^+] - [\widetilde{F}^-] = [\widetilde{\mathscr{H}}^+ \oplus \widetilde{\mu}^-] - [\widetilde{\mathscr{H}}^- \oplus \widetilde{\mu}^+] = (\pi \times \mathrm{Id}_{[0,1]})_*^{\mathrm{Eu}}[\widetilde{\xi}] \in K^0_{\mathrm{top}}(B \times [0,1]).$$

$[\ell]$ naturally provides an equivalence class of link between $F^+ - F^-$ and the restrictions to $B \times \{0\}$ of $(\widetilde{\mathscr{H}}^+ \oplus \widetilde{\mu}^-) - (\widetilde{\mathscr{H}}^- \oplus \widetilde{\mu}^+)$, which can be extended (by parallel transport along $[0,1]$) to an equivalence class of link $[\widetilde{\ell}]$ on the whole $B \times [0,1]$ between $\widetilde{F}^+ - \widetilde{F}^-$ and $(\widetilde{\mathscr{H}}^+ \oplus \widetilde{\mu}^-) - (\widetilde{\mathscr{H}}^- \oplus \widetilde{\mu}^+)$.

Construct the differential form $\widetilde{\eta} = \eta(\widetilde{\nabla}_\xi, \widetilde{\nabla}_{TZ}, \widetilde{\nabla}_{F^+}, \widetilde{\nabla}_{F^-}, [\widetilde{\ell}])$ in the same way as in Definition 50 with respect to all these data on $M \times [0,1]$. This must be made using a smooth function $\widetilde{\chi}$ on $B \times [0,1] \times \mathbb{R}_+$ vanishing on $B \times [0,1] \times [0,\varepsilon]$, equal to 1 on $B \times [0,1] \times [A, +\infty)$ and increasing with respect to the variable in $\mathbb{R}^+$ as was sketched at the end of §5.2.5. The obtained form $\widetilde{\eta}$ verifies (a):

$$d\widetilde{\eta} = \int_Z e(\widetilde{\nabla}_{TZ})\mathrm{ch}(\widetilde{\nabla}_\xi) - \mathrm{ch}(\widetilde{\nabla}_{F^+}) + \mathrm{ch}(\widetilde{\nabla}_{F^-})$$

where $\int_Z$ stands for integration along the fibres of $\pi \times \mathrm{Id}_{[0,1]}$. Call η_0 and η_1 the restrictions of $\widetilde{\eta}$ to $B \times \{0\}$ and $B \times \{1\}$ respectively. Integrating this formula along $[0,1]$ provides that the following differential form on B is exact:

$$(88)\quad d\Big(\int_{[0,1]} \widetilde{\eta}\Big) = \eta_1 - \eta_0 + \int_{[0,1]} \int_Z e(\widetilde{\nabla}_{TZ})\mathrm{ch}(\widetilde{\nabla}_\xi) - \int_{[0,1]} \mathrm{ch}(\widetilde{\nabla}_{F^+}) + \int_{[0,1]} \mathrm{ch}(\widetilde{\nabla}_{F^-})$$

but $\widetilde{\nabla}_{TZ}$ and $\widetilde{\nabla}_\xi$ are pullback connections on $M \times [0,1]$ for the projection on the second factor $M \times [0,1] \longrightarrow M$ and accordingly for $\widetilde{\nabla}_{F^+}$ and $\widetilde{\nabla}_{F^-}$ on $B \times [0,1]$, so that their Chern characters or Euler form are pullback forms, and their integral along $[0,1]$ vanish. It follows that η_0 and η_1 are equal modulo exact forms.

Now η_0 and η_1 are both regular definitions of $\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell])$ as in Definition 50, because the class of link between $F^+ - F^-$ and the restrictions to $B \times \{1\}$

of $(\widetilde{\mathscr{H}}^{+} \oplus \widetilde{\mu}^{-}) - (\widetilde{\mathscr{H}}^{-} \oplus \widetilde{\mu}^{+})$ is in the equivalence class of $[\ell]$ (it can be deformed along $[0,1]$ to the one between the restrictions on $B \times \{0\}$) (and because of the functoriality property of η). This proves the independence of the class of η modulo exact forms on h^{ξ} and χ, and also that a deformation of the suitable triple does not modify the class of η modulo exact forms.

Second step: general independence on the suitable triple used. — First remark that if (μ^+, μ^-, ψ) is a suitable triple, then $(\mu^+ \oplus \zeta^+, \mu^- \oplus \zeta^-, \psi)$ is also suitable (ζ^+ and ζ^- are inert excess vector bundles) and gives rise to the same η. The same is true for $(\mu^+ \oplus \zeta, \mu^- \oplus \zeta, \psi \oplus \mathrm{Id}_\zeta)$ because the extra term due to Id_ζ appearing in $\phi\mathrm{Tr}_s\left(\frac{\partial B_t}{\partial t} \exp -B_t^2\right)$ is the supertrace on $\zeta \oplus \zeta$ of some $\mathrm{End}(\zeta \oplus \zeta)$-valued differential form whose diagonal terms are equal.

For some suitable triple (μ^+, μ^-, ψ), giving rise to kernel bundles $\mathscr{H}^{\pm}$, one associates to it some positive kernel triple $(\mu^+ \oplus \lambda, \mu^-, \psi + \varphi)$ as just before Definition 37. One puts on $B \times [0,1]$ the bundles $\widetilde{\mu}^+ = \mu^+ \oplus \lambda \oplus \mathscr{H}^-$, $\widetilde{\mu}^- = \mu^-$ and $\widetilde{\psi} = \psi + \cos(\frac{\pi}{2}t)\varphi + \sin(\frac{\pi}{2}t)\iota_{\mathscr{H}^-}$ where $\iota_{\mathscr{H}^-}$ is the obvious embedding of $\mathscr{H}^-$ into $\mathscr{E}^- \oplus \mu^-$. The obtained triple $(\widetilde{\mu}^+, \widetilde{\mu}^-, \widetilde{\psi})$ is a positive kernel triple with respect to $\pi \times \mathrm{Id}_{[0,1]}$. Its kernel bundle restricts to $(\mathrm{Ker}\mathscr{D}^{\nabla_\xi +}_{\psi+\varphi}) \oplus \mathscr{H}^-$ on $M \times \{0\}$ and $(\mathrm{Ker}\mathscr{D}^{\nabla_\xi +}_{\psi}) \oplus \lambda$ on $M \times \{1\}$. Thus applying the above considerations to this case, proves that η_1 constructed using $(\mu^+ \oplus \mathscr{H}^-, \mu^-, \psi + \iota^-_{\mathscr{H}})$ (corresponding to $M \times \{1\}$ with an inert copy of λ added to $\mu^+ \oplus \mathscr{H}^-$) and η_0 constructed using $(\mu^+ \oplus \lambda, \mu^-, \psi + \varphi)$ (corresponding to $M \times \{0\}$ with an inert copy of $\mathscr{H}^-$ added to $\mu^+ \oplus \lambda$) differ from an exact form; the parallel transport along $[0,1]$ from $(\mathrm{Ker}\mathscr{D}^{\nabla_\xi +}_{\psi+\varphi}) \oplus \mathscr{H}^-$ to $(\mathrm{Ker}\mathscr{D}^{\nabla_\xi +}_{\psi}) \oplus \lambda$ (following $\mathrm{Ker}(\mathscr{D}^{\nabla_\xi +}_{\widetilde{\psi}}|_{M \times \{t\}})$) is easily checked to lie in the equivalence class of the link between $(\mathrm{Ker}\mathscr{D}^{\nabla_\xi +}_{\psi}) - \mathscr{H}^-$ and $(\mathrm{Ker}\mathscr{D}^{\nabla_\xi +}_{\psi+\varphi}) - \lambda$ obtained from (58) and Definition 36. The lemma is thus proved in full generality. □

5.4. Anomaly formulae and their consequences

5.4.1. *Anomaly formulae.* — The Chern-Simons theory (7) also applies for the Euler class: for any real vector bundle $F_{\mathbb{R}}$ on M, consider $p_1 \colon M \times [0,1] \to M$ (the projection on the first factor) and the bundle $\widetilde{F_{\mathbb{R}}} = p_1^* F_{\mathbb{R}}$ on $M \times [0,1]$, choose any euclidean metric and unitary connection $\widetilde{\nabla}_{F_{\mathbb{R}}}$ on $\widetilde{F_{\mathbb{R}}}$, denote by $\nabla_{F_{\mathbb{R}},t} = \widetilde{\nabla}_{F_{\mathbb{R}}}|_{M \times \{t\}}$ the restrictions of $\widetilde{\nabla}_{F_{\mathbb{R}}}$ to $M \times \{t\}$ for all $t \in [0,1]$, and define

$$\widetilde{e}(\nabla_{F_{\mathbb{R}},0}, \nabla_{F_{\mathbb{R}},1}) = \int_{[0,1]} e(\widetilde{\nabla}_{F_{\mathbb{R}}}). \tag{89}$$

The class of $\widetilde{e}(\nabla_{F_{\mathbb{R}},0}, \nabla_{F_{\mathbb{R}},1})$ in $\Omega(M,\mathbb{C})/d\Omega(M,\mathbb{C})$ only depends on the limiting connections $\nabla_{F_{\mathbb{R}},0}$ and $\nabla_{F_{\mathbb{R}},1}$ and $\widetilde{e}(\nabla_{F_{\mathbb{R}},0}, \nabla_{F_{\mathbb{R}},1})$ verifies the following transgression formula

$$d\widetilde{e}(\nabla_{F_{\mathbb{R}},0}, \nabla_{F_{\mathbb{R}},1}) = e(\nabla_{F_{\mathbb{R}},1}) - e(\nabla_{F_{\mathbb{R}},0}).$$

It is also functorial by pull-backs, and locally gauge invariant, and verifies a similar cocycle property (9) as does $\widetilde{\mathrm{ch}}$. Moreover, making the product of $e(\widetilde{\nabla}_{F_{\mathbb{R}}})$ and $\mathrm{ch}(\widetilde{\nabla}_E)$ yields the following equality modulo exact forms:

(90)
$$\begin{aligned}\int_{[0,1]} e(\widetilde{\nabla}_{F_{\mathbb{R}}}) \wedge \mathrm{ch}(\widetilde{\nabla}_E) &= \widetilde{e}(\nabla_{F_{\mathbb{R}},0}, \nabla_{F_{\mathbb{R}},1}) \wedge \mathrm{ch}(\nabla_{E,0}) + e(\nabla_{F_{\mathbb{R}},1}) \wedge \widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1}) \\ &= e(\nabla_{F_{\mathbb{R}},0}) \wedge \widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1}) + \widetilde{e}(\nabla_{F_{\mathbb{R}},0}, \nabla_{F_{\mathbb{R}},1}) \wedge \mathrm{ch}(\nabla_{E,1}).\end{aligned}$$

Take now the same model as in the first step of the proof of Proposition 51, but with not necessarily pullback connections $\widetilde{\nabla}_\xi$ nor fiberwise riemannian metric $\widetilde{g}^Z$ nor horizontal space $\widetilde{T^HM}$. The obtained connection $\widetilde{\nabla}_{TZ}$ is of course not a pullback connection. Denote by ∇_ξ^0 and ∇_{TZ}^0 the connections on ξ and on TZ corresponding to data on $M \times \{0\}$ and by ∇_ξ^1 and ∇_{TZ}^1 their counterpart on $M \times \{1\}$. Consider pullbacks on $B \times [0,1]$ of some couple (F^+, F^-) of bundles on B with connections ∇_{F^+} and ∇_{F^-} such that $[F^+] - [F^-] = \pi_*^{\mathrm{Eu}}[\xi] \in K^0_{\mathrm{top}}(B)$ with pullback connections, the counterpart of (88) in this setting is

$$\begin{aligned}&\eta(\nabla_\xi^1, \nabla_{TZ}^1, \nabla_{F^+}, \nabla_{F^-}, [\ell]) - \eta(\nabla_\xi^0, \nabla_{TZ}^0, \nabla_{F^+}, \nabla_{F^-}, [\ell]) = \\ &\quad = \int_Z \Big[e(\nabla_{TZ}^0) \wedge \widetilde{\mathrm{ch}}(\nabla_\xi^0, \nabla_\xi^1) + \widetilde{e}(\nabla_{TZ}^0, \nabla_{TZ}^1) \wedge \mathrm{ch}(\nabla_\xi^1) \Big]\end{aligned} \tag{91}$$

where the integrand can be modified as in (90).

Now one also can change the bundles on B in the following way: take suitable (μ^+, μ^-, ψ) and call $\mathscr{H}^\pm = \mathrm{Ker}\mathscr{D}_\psi^{\nabla_\xi \pm}$, endow $\mathscr{H}^+ \oplus \mu^-$ and $\mathscr{H}^- \oplus \mu^+$ with any connections $\nabla^\uparrow$ and $\nabla^\downarrow$. Consider vector bundles F^+, F^-, G^+ and G^- on B such that $[F^+] - [F^-] = [G^+] - [G^-] = \pi_*^{\mathrm{Eu}}[\xi] \in K^0_{\mathrm{top}}(B)$, choose some connections ∇_{F^+}, ∇_{F^-}, ∇_{G^+} and ∇_{G^-} on them, and some links $[\ell_F]$ and $[\ell_G]$ between $F^+ - F^-$ or $G^+ - G^-$ respectively and $(\mathscr{H}^+ \oplus \mu^-) - (\mathscr{H}^- \oplus \mu^+)$. Then from the construction of η it follows that

$$\begin{aligned}\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell_F]) &= \eta(\nabla_\xi, \nabla_{TZ}, \nabla^\uparrow, \nabla^\downarrow, [\mathrm{Id}]) + \widetilde{\mathrm{ch}}([\ell_F]) \\ &= \eta(\nabla_\xi, \nabla_{TZ}, \nabla_{G^+}, \nabla_{G^-}, [\ell_G]) - \widetilde{\mathrm{ch}}([\ell_G]) + \widetilde{\mathrm{ch}}([\ell_F]) \\ &= \eta(\nabla_\xi, \nabla_{TZ}, \nabla_{G^+}, \nabla_{G^-}, [\ell_G]) + \widetilde{\mathrm{ch}}([\ell_F \circ \ell_G^{-1}])\end{aligned} \tag{92}$$

where of course $\widetilde{\mathrm{ch}}([\ell_F])$ and $\widetilde{\mathrm{ch}}([\ell_G])$ are computed with $\nabla_{F^\pm}$ or $\nabla_{G^\pm}$ respectively, and $\nabla^\uparrow$ and $\nabla^\downarrow$.

Formulae (91) and (92) give all the dependence of η on its data.

5.4.2. *End of proof of Theorem 28.* — If ξ has vanishing rational Chern classes, then some finite direct sum $\xi \oplus \xi \oplus \cdots \oplus \xi$ is topologically trivial on X. The anomaly formulae (which are consequences of properties (a) and (b)) then relate η for $\nabla_\xi \oplus \nabla_\xi \oplus \cdots \oplus \nabla_\xi$ (and any direct sum of copies of direct image representatives) and η for the canonical flat connection on the trivial bundle with corresponding flat direct image, which vanishes because of (d). Property (c) allows to simply divide by the number of copies of ξ to obtain the desired η, which is thus obtained using only (a), (b), (c) and (d).

Remark. — One could generalise to bundles ξ whose restrictions to the fibers of π have vanishing rational Chern classes by adding some property linking η for ξ and η for $\xi \otimes \pi^*\zeta$ where ζ is any bundle on B. Some more axioms are needed to obtain a general caracterisation.

One could hope to obtain a caracterisation of η modulo the image of $K^1_{\mathrm{top}}(B)$ by the Chern character, with no care of links of bundles on B with someones obtained by analytic families index construction. However, the fact that one must consider a not controlled finite number of copies of ξ would prevent to obtain more than a caracterisation modulo rational cohomology.

5.4.3. *Proof of Theorem 29.* — The anomaly formulae (91) and (92) yield in the situation of Theorem 29 that

$$\eta(\nabla_E, \nabla_{TZ}, \nabla_{\pi_!^+ E}, \nabla_{\pi_!^- E}, [\mathrm{Id}]) - \eta(\nabla_F, \nabla_{TZ}, \nabla_{\pi_!^+ F}, \nabla_{\pi_!^- F}, [\mathrm{Id}]) =$$
$$= \int_Z e(\nabla_{TZ})\widetilde{\mathrm{ch}}(\nabla_E, f^*\nabla_F) - \widetilde{\mathrm{ch}}\big(\pi_\ell([f])\big).$$

Both η vanish (this is property (d)), and that the right hand side vanishes is exactly the desired result in view of Definitions 7 and 26.

5.4.4. *Proof of Theorem 31.* — Let $(\xi, \nabla_\xi, \alpha) \in \widehat{K}_{\mathrm{ch}}(M)$. If F^+, F^-, G^+ and G^- are vector bundles on B such that $[G^+] - [G^-] = [F^+] - [F^-] = \pi_*^{\mathrm{Eu}}[\xi] \in K_0^{\mathrm{top}}(B)$. Consider any connections ∇_{F^+}, ∇_{F^-}, ∇_{G^+} and ∇_{G^-} on them. It follows from (92) that

$$\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell_F]) - \eta(\nabla_\xi, \nabla_{TZ}, \nabla_{G^+}, \nabla_{G^-}, [\ell_G]) = \widetilde{\mathrm{ch}}([\ell_F \circ \ell_G^{-1}]).$$

Formula (51) written with G^+ and G^- (with their connections) instead of F^+ and F^- thus provides the same class in $\widehat{K}_{\mathrm{ch}}(B)$ (see (10), (13) and (85)). $\pi_!^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha)$ is thus a well defined element in $\widehat{K}_{\mathrm{ch}}(B)$.

Suppose that $(\xi, \nabla_\xi, \alpha) = (\xi', \nabla_{\xi'}, \alpha') \in \widehat{K}_{\mathrm{ch}}(M)$, and that $f\colon \xi \to \xi'$ is some smooth vector bundle isomorphism, then

$$\alpha' = \alpha + \widetilde{\mathrm{ch}}(\nabla_\xi, f^*\nabla_{\xi'}) + \beta$$

where β is a closed form lying in the image of $K^1_{\mathrm{top}}(M)$ by the Chern character. Thus if $[F^+]-[F^-]=\pi^{\mathrm{Eu}}_*[\xi]\in K^0_{\mathrm{top}}(B)$ with connections ∇_{F^+} on F^+ and ∇_{F^-} on F^-, one has from (91) and (92) (for any suitable links $[\ell_\xi]$ and $[\ell_{\xi'}]$):

$$\begin{aligned}\eta(\nabla_\xi,\nabla_{TZ},\nabla_{F^+},\nabla_{F^-},[\ell_\xi])-\eta(\nabla_{\xi'},\nabla_{TZ},\nabla_{F^+},\nabla_{F^-},[\ell_{\xi'}])=\\ =\int_Z e(\nabla_{TZ})\wedge\widetilde{\mathrm{ch}}(\nabla_\xi,f^*\nabla_{\xi'})+\widetilde{\mathrm{ch}}(\ell_\xi\circ\ell_{\xi'}^{-1}).\end{aligned}$$

Remember the definition of $a\colon\Omega^{\mathrm{odd}}(M,\mathbb{C})/d\Omega^{\mathrm{even}}(M,\mathbb{C})\longrightarrow\widehat{K}_{\mathrm{ch}}(M)$ given just before Proposition 10. One obtains from the preceding equation:

$$\begin{aligned}&\pi^{\mathrm{Eu}}_!(\xi,\nabla_\xi,\alpha)-\pi^{\mathrm{Eu}}_!(\xi',\nabla_{\xi'},\alpha')=\\ &=a\Big(\int_Z e(\nabla_{TZ})\wedge\Big(\widetilde{\mathrm{ch}}(\nabla_\xi,f^*\nabla_{\xi'})+\alpha-\alpha'\Big)+\widetilde{\mathrm{ch}}(\ell_\xi\circ\ell_{\xi'}^{-1})\Big)\\ &=a\Big(\int_Z e(\nabla_{TZ})\wedge\beta\Big)+a\left(\widetilde{\mathrm{ch}}(\ell_\xi\circ\ell_{\xi'}^{-1})\right)\end{aligned}$$

which vanishes in $\widehat{K}_{\mathrm{ch}}(B)$, because $\widetilde{\mathrm{ch}}(\ell_\xi\circ\ell_{\xi'}^{-1})\in\mathrm{ch}\big(K^1_{\mathrm{top}}(B)\big)\subset H^{\mathrm{odd}}(B,\mathbb{C})$ and so does $\int_Z e(\nabla_{TZ})\wedge\beta$ by virtue of the cohomological version of Atiyah-Singer families index theorem for K^1_{top}.

Moreover the additivity of η for direct sums (property (c)) yields

$$\pi^{\mathrm{Eu}}_!(\xi_1\oplus\xi_2,\nabla_{\xi_1}\oplus\nabla_{\xi_2},\alpha_1+\alpha_2)=\pi^{\mathrm{Eu}}_!(\xi_1,\nabla_{\xi_1},\alpha_1)+\pi^{\mathrm{Eu}}_!(\xi_2,\nabla_{\xi_2},\alpha_2).$$

$\pi^{\mathrm{Eu}}_!$ is thus well defined as a morphism from $\widehat{K}_{\mathrm{ch}}(M)$ to $\widehat{K}_{\mathrm{ch}}(B)$.

The commutativity of diagram (53) is a consequence of property (d) of η.

The commutativity of the right and the central squares of diagram (54) are tautological. The commutativity of the left square of (54) is a consequence of the cohomological version of Atiyah-Singer families index theorem for K^1_{top}.

In the same way one has the following equality modulo exact forms:

$$\begin{aligned}\mathfrak{B}\big(\pi^{\mathrm{Eu}}_*(\xi,\nabla_\xi,\alpha)\big)=\widetilde{\mathrm{ch}}(\nabla^*_{F^+},\nabla_{F^+})-2i\mathfrak{Im}\Big(\int_Z e(\nabla_{TZ})\wedge\alpha\Big)\\ -\widetilde{\mathrm{ch}}(\nabla^*_{F^-},\nabla_{F^-})+2i\mathfrak{Im}\big(\eta(\nabla_\xi,\nabla_{TZ},\nabla_{F^+},\nabla_{F^-},[\ell])\big).\end{aligned}$$

Of course the connections on F^+ and on F^- can be supposed to respect some hermitian metrics on F^+ and F^- without changing the formula, and this makes and vanish the terms $\widetilde{\mathrm{ch}}(\nabla^*_{F^+},\nabla_{F^+})$ and $\widetilde{\mathrm{ch}}(\nabla^*_{F^-},\nabla_{F^-})$.

The reality considerations for $\widetilde{\mathrm{ch}}([\ell])$ between (85) and (86) and the last statement of Lemma 48 imply that $\eta(\nabla_\xi,\nabla_{TZ},\nabla_{F^+},\nabla_{F^-},[\ell])$ is a real form (modulo exact forms) if it happens that ∇_ξ, ∇_{F^+} and ∇_{F^-} respect some hermitian metrics on their bundles. Consider any connection ∇^u_ξ which respects some hermitian metrics on ξ. The reality of $\eta(\nabla_\xi^u, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell])$ and Formula (91) yield

$$\begin{aligned}\eta(\nabla_\xi,&\nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell]) = \\ &= \eta(\nabla_\xi^u, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell]) + \int_Z e(\nabla_{TZ}) \wedge \widetilde{\mathrm{ch}}(\nabla_\xi^u, \nabla_\xi)\end{aligned}$$

$$2i\mathfrak{Im}\big(\eta(\nabla_\xi,\nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell])\big) = \int_Z e(\nabla_{TZ}) \wedge 2i\mathfrak{Im}\big(\widetilde{\mathrm{ch}}(\nabla_\xi^u, \nabla_\xi)\big).$$

Now using (31)) one gets:

$$\begin{aligned}\mathfrak{B}\big(\pi_!^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha)\big) &= \int_Z e(\nabla_{TZ}) \wedge \big(\widetilde{\mathrm{ch}}(\nabla_\xi^*, \nabla_\xi) - 2i\mathfrak{Im}\alpha\big) \\ &= \int_Z e(\nabla_{TZ}) \wedge \mathfrak{B}(\xi, \nabla_\xi, \alpha)\end{aligned}$$

and the last statement of Theorem 31 is proved.

5.4.5. *Influence of the vertical metric and the horizontal distribution.* — If geometric data are changed on M, namely the vertical riemannian metric g^Z and/or the horizontal subspace T^HM, this changes the connection ∇_{TZ}, and this also changes the morphism $\pi_!^{\mathrm{Eu}}$.

Lemma 52. — *Let ∇_{TZ} and $\pi_!^{\mathrm{Eu}}$ be associated to data g^Z and T^HM, let $g^{Z'}$ and $T^{H'}M$ be other data and call ∇'_{TZ} and $\pi_!^{\mathrm{Eu}'}$ the associated connection on TZ and morphism from $\widehat{K}_{\mathrm{ch}}(M)$ to $\widehat{K}_{\mathrm{ch}}(B)$. Then, for any $(\xi, \nabla_\xi, \alpha)$ one has*

$$\pi_!^{\mathrm{Eu}'}(\xi, \nabla_\xi, \alpha) - \pi_!^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha) = -a\left(\int_Z \widetilde{e}(\nabla_{TZ}, \nabla'_{TZ})\dddot{\mathrm{ch}}(\xi, \nabla_\xi, \alpha)\right).$$

Proof. — If dim$Z$ is odd, $\pi_!^{\mathrm{Eu}}$ and $\pi_!^{\mathrm{Eu}'}$ will be proved to vanish in §6.3. $\widetilde{e}$ also vanishes. If dim$Z$ is even, it successively follows from (91) that

$$\eta(\nabla_\xi, \nabla'_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell]) - \eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell]) = \int_Z \widetilde{e}(\nabla_{TZ}, \nabla'_{TZ}) \wedge \mathrm{ch}(\nabla_\xi)$$

$$\begin{aligned}\pi_!^{\mathrm{Eu}'}&(\xi, \nabla_\xi, \alpha) - \pi_!^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha) = \\ &= a\left(\int_Z \big(e(\nabla'_{TZ}) - e(\nabla_{TZ})\big)\alpha - \int_Z \widetilde{e}(\nabla_{TZ}, \nabla'_{TZ}) \wedge \mathrm{ch}(\nabla_\xi)\right) = \\ &= a\left(\int_Z \widetilde{e}(\nabla_{TZ}, \nabla'_{TZ}) \wedge \big(-\mathrm{ch}(\nabla_\xi) + d\alpha\big)\right)\end{aligned}$$

this last equality is valid modulo exact forms because

$$d\big(\widetilde{e}(\nabla_{TZ}, \nabla'_{TZ})\alpha\big) = e(\nabla'_{TZ})\alpha - e(\nabla_{TZ})\alpha + (-1)^{\deg \widetilde{e}(\nabla_{TZ}, \nabla'_{TZ})}\widetilde{e}(\nabla_{TZ}, \nabla'_{TZ})d\alpha$$

and $\widetilde{e}(\nabla_{TZ}, \nabla'_{TZ})$ is of degree dim$Z - 1$, with dim$Z$ even. $\square$

If $\dim Z$ is even, and since $\widetilde{e}(\nabla_{TZ}, \nabla'_{TZ})$ is of degree $\dim Z - 1$, it follows that MK_0 is the biggest subgroup of $\widehat{K}_{\mathrm{ch}}$ on which there is no variation of $\pi_!^{\mathrm{Eu}}$ when geometric data g^Z and T^HM are changed. This gives a topological significance to the direct image morphism $\pi_!^{\mathrm{Eu}}$ on MK_0.

In the language of [**14**], the geometry of the fibration should be encoded into some smooth refinement of the used K-orientation, (here it is the one associated to the fiberwise Euler operator) and the restriction of $\pi_!^{\mathrm{Eu}}$ to $MK_0(M)$ would be independent of the choice of this smooth K-orientation.

6. Fiberwise Hodge symmetry

The goal of this part is to prove Theorems 32 and 33. All these results are consequences of symmetries induced by the fiberwise Hodge star operator. Paragraph §6.1 is essentially devoted to technical computations dealing with relations of this star operator with various geometrical features of the theory.

6.1. Symmetries induced on family index bundles

6.1.1. *The fiberwise Hodge $*$ operator.* — Here we will make constant use of the notations introduced in §3.1.1, §3.4.1 and §5.2.1. For any vertical tangent vector $\mathtt{w} \in TZ$, consider its dual one-form $\mathtt{w}^\flat$ (through the fiberwise riemannian metric g^Z), and its Clifford action on $\wedge^\bullet T^*Z \otimes \xi$

$$c(\mathtt{w}) = (\mathtt{w}^\flat \wedge) - \iota_{\mathtt{w}} \tag{93}$$

($\iota_{\mathtt{w}}$ denotes the interior product by $\mathtt{w}$); $c(\mathtt{w})$ is skewadjoint with respect to $(\ |\ )_Z$ and verifies $c(\mathtt{w})^2 = -g^Z(\mathtt{w}, \mathtt{w})$, it is an isometry if $g^Z(\mathtt{w}, \mathtt{w}) = 1$.

Consider the vertical Hodge operator $*_Z = c(\mathtt{e}_1)c(\mathtt{e}_2)\cdots c(\mathtt{e}_{\dim Z})$ for any orthonormal direct base $\mathtt{e}_1, \mathtt{e}_2, \ldots, \mathtt{e}_{\dim Z}$ of TZ. It is an isometry of $\mathscr{E}$ (endowed with $\langle\ ,\ \rangle_{L^2}$), and it has the same parity as $\dim Z$ (with respect to the $\mathbb{Z}_2$ grading of $\mathscr{E}$). Its inverse $*_Z^{-1} = (-1)^{\frac{1}{2}\dim Z(\dim Z+1)} *_Z$ is also its adjoint with respect to both $(\ |\ )_Z$ and $\langle\ ,\ \rangle_{L^2}$. Define the metrized exterior product of ξ-valued vertical differential forms by the following formula on decomposed tensors:

$$(\alpha \mathbin{\widehat{\otimes}} a) \underset{h^\xi}{\wedge} (\beta \mathbin{\widehat{\otimes}} b) = (\alpha \wedge \overline{\beta}) h^\xi(a, b)$$

(a sign $(-1)^{\deg\alpha\deg\beta}$ should be put on the right side if ξ would be $\mathbb{Z}_2$-graded, but this case will not be considered in the sequel, note also that this operation is independent of the riemannian vertical metric g^Z). Then for any $\gamma \in \mathscr{E}$ whose differential form degree is $\leq \deg\alpha$:

$$(\alpha \mathbin{\widehat{\otimes}} a) \underset{h^\xi}{\wedge} (*_Z\gamma) = (-1)^{\frac{1}{2}\deg\alpha(\deg\alpha-1)+\dim Z\deg\alpha} \big((\alpha \mathbin{\widehat{\otimes}} a)\big|\gamma\big)_Z d\mathrm{Vol}_Z. \tag{94}$$

6.1.2. *Symmetry induced by $*_Z$ on fiberwise twisted Euler operators.* — For any vector $\mathtt{w} \in TZ$, $c(\mathtt{w})$ commutes with $*_Z$ if $\dim Z$ is odd and it anticommutes with $*_Z$ if $\dim Z$ is even. It follows from the two preceding formulae that if ∇_ξ^* is associated to ∇_ξ and h^ξ as in (22), then for any γ and γ' in $\mathscr{E}$:

$$
\begin{aligned}
&d^Z(\gamma \underset{h^\xi}{\wedge} \gamma') = (d^{\nabla_\xi}\gamma) \underset{h^\xi}{\wedge} \gamma' + (-1)^{\deg\gamma}\gamma \underset{h^\xi}{\wedge} (d^{\nabla_\xi^*}\gamma') \\
&\text{so that} \qquad (d^{\nabla_\xi})^* = (-1)^{1+\frac{1}{2}\dim Z(\dim Z-1)} *_Z d^{\nabla_\xi^*} *_Z
\end{aligned}
\tag{95}
$$

from which one deduces that

$$
d^{\nabla_\xi} + (d^{\nabla_\xi})^* = -(-1)^{\dim Z} *_Z^{-1} \left(d^{\nabla_\xi^*} + (d^{\nabla_\xi^*})^*\right) *_Z . \tag{96}
$$

This formula can also be checked from [**11**] formulae (3.36), (1.30), (1.31) and the last sentence at the end of the first alinea of §III(d).

Suppose that (μ^+, μ^-, ψ) is a suitable triple with respect to ξ endowed with h^ξ and ∇_ξ, and produce kernel bundles $\mathscr{H}^+$ and $\mathscr{H}^-$. If $\dim Z$ is even, $*_Z$ respects the parity of vertical forms while $*_Z$ exchanges this parity if $\dim Z$ is odd. It then follows from (96) that:

Proposition 53. — *If* $\dim Z$ *is odd,* $\left(\mu^-, \mu^+, (*_Z \oplus \mathrm{Id}_{\mu^+}) \circ \psi^* \circ (*_Z^{-1} \oplus \mathrm{Id}_{\mu^-})\right)$ *is a suitable triple for* ξ *endowed with* h^ξ *and* ∇_ξ^*. *It produces kernel bundles* $(*_Z \oplus \mathrm{Id}_{\mu^-})\mathscr{H}^-$ *and* $(*_Z \oplus \mathrm{Id}_{\mu^+})\mathscr{H}^+$.

If $\dim Z$ *is even. The triple* $\left(\mu^+, \mu^-, -(*_Z \oplus \mathrm{Id}_{\mu^-}) \circ \psi \circ (*_Z^{-1} \oplus \mathrm{Id}_{\mu^+})\right)$ *is suitable with respect to* ξ *endowed with* h^ξ *and* ∇_ξ^*. *It produces kernel bundles* $(*_Z \oplus \mathrm{Id}_{\mu^+})\mathscr{H}^+$ *and* $(*_Z \oplus \mathrm{Id}_{\mu^-})\mathscr{H}^-$.

Indeed denote in both cases by Ψ the third element of the proposed triples, then the triple (μ^+, μ^-, Ψ) or (μ^-, μ^+, Ψ) for ∇_ξ^* is chosen so that (96) reads

$$
\begin{cases}
\mathscr{D}_\psi^{\nabla_\xi \pm} = -(*_Z \oplus \mathrm{Id}_{\mu^\mp})^{-1} \mathscr{D}_\Psi^{\nabla_\xi^* \pm} (*_Z \oplus \mathrm{Id}_{\mu^\pm}) & \text{if } \dim Z \text{ is even,} \\
\mathscr{D}_\psi^{\nabla_\xi \pm} = (*_Z \oplus \mathrm{Id}_{\mu^\mp})^{-1} \mathscr{D}_\Psi^{\nabla_\xi^* \mp} (*_Z \oplus \mathrm{Id}_{\mu^\pm}) & \text{if } \dim Z \text{ is odd.}
\end{cases}
\tag{97}
$$

6.1.3. *Odd dimensional fibre case.* — Suppose B is compact and the fibres of π are odd dimensional. Consider some positive kernel triple $(\lambda, \{0\}, \varphi)$ for ξ, which is supposed to be endowed with a connection ∇_ξ which respects the hermitian metric h^ξ. (There exists some as was mentioned just before Definition 37). It is here needed that φ vanishes on $\mathscr{E}^+$ (which is in fact the case in the above cited references [**2**] Proposition 2.2, or [**3**] Lemma 9.30 or [**29**] Lemma 8.4 of chapter III). Call $\mathscr{K}^+$ the associated kernel bundle. It follows from Proposition 53 that $\left(\{0\}, \lambda, (*_Z \oplus \mathrm{Id}_\lambda) \circ \varphi^* \circ *_Z^{-1}\right)$ is suitable and gives rise to kernel bundles $\{0\}$ and $(*_Z \oplus \mathrm{Id}_\lambda)\mathscr{K}^+ \subset \mathscr{E}^- \oplus \lambda$.

Lemma 54. — *The triple* $\left(\lambda, \lambda, \varphi + (*_Z \oplus \mathrm{Id}_\lambda) \circ \varphi^* \circ *_Z^{-1} + i^{1+\frac{1}{2}\dim Z(\dim Z+1)}\mathrm{Id}_\lambda\right)$ *is suitable with kernel bundles* $\{0\}$ *and* $\{0\}$.

The point about the factor of Id_λ is that it should be nonvanishing and purely imaginary if $*_Z^2 = \mathrm{Id}$ but real (and nonvanishing) if $*_Z^2 = -\mathrm{Id}$.

The vanishing of π_*^{Eu} on K^0_{top} in the case of odd dimensional fibres and compact B follows. If B is noncompact, one concludes using the fact that any element of $K^0_{\mathrm{top}}(B)$ whose restriction to any compact subset vanishes, is itself trivial. The vanishing of π_*^{Eu} on K^1_{top} is a consequence of its vanishing on K^0_{top}.

Proof. — Consider any element $(v,\sigma) \in \lambda \oplus \mathscr{E}^+$ belonging to the kernel bundle. The corresponding condition reads

$$\begin{cases} \big(d^{\nabla_\xi} + (d^{\nabla_\xi})^*\big)\sigma = -\varphi v \\ \qquad -\varphi^*(*_Z^{-1}\sigma) = i^{1+\frac{1}{2}\dim Z(\dim Z+1)} v. \end{cases}$$

Writing $\sigma = *_Z\sigma'$, one obtains the following

$$\big(d^{\nabla_\xi} + (d^{\nabla_\xi})^*\big) *_Z \sigma' = -i^{-1-\frac{1}{2}\dim Z(\dim Z+1)}\varphi\varphi^*\sigma'$$

$$\big\langle \big(d^{\nabla_\xi} + (d^{\nabla_\xi})^*\big) *_Z \sigma', \sigma'\big\rangle_{L^2} = -i^{-1-\frac{1}{2}\dim Z(\dim Z+1)}\langle\varphi^*\sigma', \varphi^*\sigma'\rangle_\lambda$$

where $\langle\ ,\ \rangle_\lambda$ is the scalar product on λ. It follows from (96), and the fact that ∇_ξ respects the hermitian metric, that $\big(d^{\nabla_\xi} + (d^{\nabla_\xi})^*\big)*_Z$ is selfadjoint if $*_Z^2 = \mathrm{Id}$ and antiselfadjoint if $*_Z^2 = -\mathrm{Id}$. Thus the right hand side of this equality is real whenever the left hand side is purely imaginary and conversely. In any case this proves that $\varphi^*\sigma'$ vanishes. Thus v vanishes, thus $\big(d^{\nabla_\xi} + (d^{\nabla_\xi})^*\big)\sigma$ vanishes. It follows that σ belongs to the (positive) kernel bundle associated to the triple $\big(\{0\}, \lambda, (*_Z \oplus \mathrm{Id}_\lambda)\circ\varphi^*\circ *_Z^{-1}\big)$. But this kernel bundle vanishes, and so does σ.

The proof of the vanishing of the cokernel is similar. □

Let $(\mathscr{F}^+, \mathscr{F}^-)$ be any couple of family index bundles for ξ. It follows from the preceding lemma and Theorem 25 that there exists a canonical link $\ell_{\mathscr{F}}^{\{0\}}$ between $\mathscr{F}^+ - \mathscr{F}^-$ and $\{0\} - \{0\}$. These canonical links are all compatible, this means that if $(\mathscr{G}^+, \mathscr{G}^-)$, and $(\mathscr{K}^+, \mathscr{K}^-)$ are couples of family index bundles for $\xi_1 - \zeta_1$ and $\xi_2 - \zeta_2$ which are linked through some link ℓ, then

$$\pi_\ell([\ell]) = [\ell_{\mathscr{F}}^{\{0\}}] \circ [\ell_{\mathscr{G}}^{\{0\}}]^{-1}. \tag{98}$$

This is because the same construction as in the proof of Lemma 54 can be performed on $M \times [0,1]$ compatibly with a deformation as was used in §4.1.4.

That $\pi_\ell([\ell])$ is constant, (i.e. $\pi_\ell([\ell])$ does not depend on $[\ell]$) is compatible with the action of K^1_{top} on links and the vanishing of π_*^{Eu} on K^1_{top}.

6.1.4. *Symmetry on canonical links.* — B is no longer supposed compact. Let ξ_1, ξ_2, ζ_1 and ζ_2 be bundles on M with connections ∇_{ξ_1}, ∇_{ξ_2}, ∇_{ζ_1} and ∇_{ζ_2} such that $[\xi_1] - [\zeta_1] = [\xi_2] - [\zeta_2] \in K^0_{\mathrm{top}}(M)$. Let $[\ell]$ be some equivalence class of link between

$\xi_1 - \zeta_1$ and $\xi_2 - \zeta_2$. One supposes that $\xi_1 - \zeta_1$ and $\xi_2 - \zeta_2$ admit respective couples of family index bundles $(\mathscr{F}^+, \mathscr{F}^-)$ and $(\mathscr{G}^+, \mathscr{G}^-)$.

Call $(\mathscr{F}^{*+}, \mathscr{F}^{*-})$ and $(\mathscr{G}^{*+}, \mathscr{G}^{*-})$ the respectively associated family index bundles for $\xi_1 - \zeta_1$ endowed with $\nabla^*_{\xi_1}$ and $\nabla^*_{\zeta_1}$ or for $\xi_2 - \zeta_2$ endowed with $\nabla^*_{\xi_2}$ and $\nabla^*_{\zeta_2}$ obtained using the symmetric triples of Proposition 53. There are isomorphisms (of the form $(*_Z \oplus \mathrm{Id}_{\mu^\pm})$)

$$\begin{array}{llll} \mathscr{F}^\pm \cong \mathscr{F}^{*\pm} & \text{and} & \mathscr{G}^\pm \cong \mathscr{G}^{*\pm} & \text{if } \dim Z \text{ is even,} \\ \mathscr{F}^\pm \cong \mathscr{F}^{*\mp} & \text{and} & \mathscr{G}^\pm \cong \mathscr{G}^{*\mp} & \text{if } \dim Z \text{ is odd.} \end{array}$$

This provides a link $\ell^*_{\mathscr{F}}$ between $\mathscr{F}^+ - \mathscr{F}^-$ and $\mathscr{F}^{*+} - \mathscr{F}^{*-}$ if $\dim Z$ is even or between $\mathscr{F}^+ - \mathscr{F}^-$ and $\mathscr{F}^{*-} - \mathscr{F}^{*+}$ if $\dim Z$ is odd. And a link $\ell^*_{\mathscr{G}}$ accordingly.

Remember the definition of $\pi_\ell([\ell])$ as an equivalence class of links between $\mathscr{F}^+ - \mathscr{F}^-$ and $\mathscr{G}^+ - \mathscr{G}^-$ from Definition 41. Denote by $\pi_\ell([\ell]\check{})$ the corresponding equivalence class of links between $\mathscr{F}^{*+} - \mathscr{F}^{*-}$ and $\mathscr{G}^{*+} - \mathscr{G}^{*-}$.

Proposition 55. — *These classes of links are compatible in the sense that*

$$\begin{cases} \pi_\ell([\ell]\check{}) = [\ell^*_{\mathscr{F}}]^{-1} \circ \pi_\ell([\ell]) \circ [\ell^*_{\mathscr{G}}] & \textit{if } \dim Z \textit{ is even,} \\ [\ell^{\{0\}}_{\mathscr{F}^*}] = [\ell^*_{\mathscr{F}}] \circ [\ell^{\{0\}}_{\mathscr{F}}]^{-1} & \textit{if } \dim Z \textit{ is odd.} \end{cases}$$

Proof. — The symmetry of family index bundles of Proposition 53 is valid on a deformation on $B \times [0,1]$ as was performed in §4.1.3 and used in §4.1.4 for the general construction of $\ell^{\mathscr{G}}_{\mathscr{F}}$. In the even dimensional fibre case, one obtains two constructions of $\ell^{\mathscr{G}}_{\mathscr{F}}$ and $\ell^{\mathscr{G}^*}_{\mathscr{F}^*}$ in exactly the same terms as in §4.1.3 and §4.1.4 which are mutually isomorphic through $*_Z$. Thus $[\ell^{\mathscr{G}^*}_{\mathscr{F}^*}] = [\ell^*_{\mathscr{F}}]^{-1} \circ [\ell^{\mathscr{G}}_{\mathscr{F}}] \circ [\ell^*_{\mathscr{G}}]$ and the first statement of the proposition follows from the fact that $\pi_\ell([\ell])$ is constructed as a particular case of some (inductive limit of) $[\ell^{\mathscr{G}}_{\mathscr{F}}]$.

In the odd dimensional fibre case, first remark that the links of type $[\ell^{\{0\}}_{\mathscr{F}}]$, though constructed under a compacity hypothesis, are globally valid for globally defined couple of family index bundles (if there exists some. This is because locally defined links between global objects yield global links, as was sketched just before Definition 40). One may then suppose that $\mathscr{G}^+ = \mathscr{G}^- = \{0\}$ (see (98)). The point is now that $*_Z$ exchanges the parity, so that a link obtained through some couple of positive kernel family index bundles (see Definition 37) is mapped by $*_Z$ to a link obtained through a couple of "negative kernel" family index bundles. The counterpart of (58) in this situation reads

$$0 \longrightarrow \mathscr{H}^+ \xrightarrow{\varphi^*} \lambda \longrightarrow \mathrm{Ker}\,\mathscr{D}^{\nabla_{\xi^-}}_{\psi \oplus \varphi^*} \longrightarrow \mathscr{H}^- \longrightarrow 0$$

where the two last maps are orthogonal projections (after inclusion of λ in $\lambda \oplus \mathscr{E}^-$). The proposition is reduced to prove that the equivalence class of links associated to this exact sequence (from Definition 36) equals $[\ell_{\mathscr{H}^+-\mathscr{H}^-}^{\lambda-\operatorname{Ker}\mathscr{D}_{\psi\oplus\varphi^*}^{\nabla_\xi-}}]$. This is a consequence of the fact that this link can be realised as a deformation, by an analogue construction to what was made in the second step of the proof of Proposition 51. □

6.1.5. *Symmetry on connections on the infinite rank bundle $\mathscr{E}$.* — Remember the definitions of the infinite rank bundle $\mathscr{E}$ from (36) and (37), and of the connections $\overline{\nabla}$ and $\overline{\nabla}^S$ on $\mathscr{E}$ from (72) and (73)

Consider the adjoint ∇_ξ^* of ∇_ξ, and the connection $\overline{\nabla}^{\check{}}$ on $\mathscr{E}$ which is associated to ∇_ξ^* in the same way as $\overline{\nabla}$ is associated to ∇_ξ through (72). Call $\overline{\nabla}^{\check{}S}$ the adjoint connection of $\overline{\nabla}^{\check{}}$ defined in the same way as was $\overline{\nabla}^S$ with respect to $\overline{\nabla}$ in (73). The reader is warned that the connection denoted here by $\overline{\nabla}^S$ corresponds to the connection denoted by $(\nabla^W)^*$ in **[11]** Proposition 3.7, and that $\overline{\nabla}^{\check{}}$ and $\overline{\nabla}^{\check{}S}$ here have no counterpart in **[11]**.

Lemma 56. — *For any vector* u *tangent to B, and any local section σ of $\mathscr{E}$*

$$\overline{\nabla}^{\check{}}_{\mathrm{u}}\sigma = *_Z^{-1}\big(\overline{\nabla}^S_{\mathrm{u}}(*_Z\sigma)\big) \qquad \textit{and} \qquad \overline{\nabla}^{\check{}S}_{\mathrm{u}}\sigma = *_Z^{-1}\big(\overline{\nabla}_{\mathrm{u}}(*_Z\sigma)\big).$$

Proof. — Remember the definition of ∇_{TZ} from §3.4.1. Denote allways by ∇_{TZ} the associated connection on $\wedge^\bullet T^*Z$, it is compatible with the Clifford action (93), so that its associated covariant derivative commutes with $*_Z$. Let $\nabla_{TZ\otimes\xi}$ be the connection on $\wedge^\bullet T^*Z\otimes\xi$ associated to ∇_{TZ} and ∇_ξ, its adjoint $\nabla^*_{TZ\otimes\xi}$ with respect to $(\ |\ )_Z$ is nothing but the connection on $\wedge^\bullet T^*Z\otimes\xi$ associated to ∇_{TZ} and ∇_ξ^*. Then the covariant derivatives associated to both $\nabla_{TZ\otimes\xi}$ and $\nabla^*_{TZ\otimes\xi}$ commute with $*_Z$.

Let u be some vector tangent to B, and u^H its horizontal lift. For any vector y tangent to the fibre, the vertical projection $P^{TZ}\nabla_{LC_{\mathrm{y}}}\mathrm{u}^H$ of the covariant derivative of u^H along y for the connection ∇_{LC} is independent of the global riemannian metric defining ∇_{LC}. Moreover, if v is another vertical tangent vector at the same point as y, then the scalar product $g^Z(P^{TZ}\nabla_{LC_{\mathrm{y}}}\mathrm{u}^H,\mathrm{v})$ is symmetric in y and v. As proved in **[11]** (3.27) and (3.32), if $(\mathsf{e}_1,\mathsf{e}_2,\dots,\mathsf{e}_{\dim Z})$ is an orthonormal base of TZ, then for any local section σ of $\mathscr{E}$, the connections $\overline{\nabla}$ and $\overline{\nabla}^S$ express locally on M as:

$$\begin{aligned}\overline{\nabla}_{\mathrm{u}}\sigma &= \nabla_{TZ\otimes\xi_{\mathrm{u}^H}}\sigma + \sum_{i \text{ and } k} g^Z(P^{TZ}\nabla_{LC_{\mathsf{e}_i}}\mathrm{u}^H,\mathsf{e}_k)\mathsf{e}_i^\flat\wedge(\iota_{\mathsf{e}_k}\sigma),\\ \overline{\nabla}^S_{\mathrm{u}}\sigma &= \nabla^*_{TZ\otimes\xi_{\mathrm{u}^H}}\sigma - \sum_{i \text{ and } k} g^Z(P^{TZ}\nabla_{LC_{\mathsf{e}_i}}\mathrm{u}^H,\mathsf{e}_k)\mathsf{e}_i^\flat\wedge(\iota_{\mathsf{e}_k}\sigma).\end{aligned} \tag{99}$$

The lemma follows from the obvious corresponding formulae for $\overline{\nabla}^{\check{}}$ and $\overline{\nabla}^{\check{}S}$, the fact that $\nabla_{TZ\otimes\xi}$ and $\nabla^*_{TZ\otimes\xi}$ commute with $*_Z$, the fact that

$$(\mathsf{e}_i^\flat\wedge)\iota_{\mathsf{e}_k}*_Z = -*_Z(\mathsf{e}_k^\flat\wedge)\iota_{\mathsf{e}_i}$$

for any i and k and the symmetry in e_i and e_k of $g^Z(P^{TZ}\nabla_{LC_{\mathsf{e}_i}}\mathrm{u}^H,\mathsf{e}_k)$. □

6.2. Proof of results about K^0_{flat} and K^0_{rel}

6.2.1. *End of proof of Theorem 32.* — Suppose that E is a vector bundle on M with a flat connection ∇_E (and hermitian metric h^E), and construct the associated objects $\mathscr{E}$, $\overline{\nabla}$ and $\overline{\nabla}^S$ as above. Let $P\colon \mathscr{E} \longrightarrow \mathrm{Ker}\big(d^{\nabla_E} + (d^{\nabla_E})^*\big)$ be the orthogonal projection, then it is proved in [**11**] Proposition 3.14 that $\nabla_{\pi_! E} \cong P\overline{\nabla}P$ and $\nabla^*_{\pi_! E} \cong P\overline{\nabla}^S P$ through the fiberwise Hodge isomorphism $\pi_! E \cong \mathrm{Ker}\big(d^{\nabla_E} + (d^{\nabla_E})^*\big)$.

Consider on E the adjoint connection ∇^*_E (which is flat). The direct image of the flat bundle (E, ∇^*_E) will be denoted by $\pi_!\check{} E$ and the flat connection on it by $\nabla_{\pi_!\check{} E}$ so that $\pi_!(E, \nabla^*_E) = (\pi_!\check{} E, \nabla_{\pi_!\check{} E})$.

As precedingly, call $\overline{\nabla}\check{}$ and $\overline{\nabla}\check{}^S$ the connections on $\mathscr{E}$ constructed from ∇^*_E as in (72) and (73), let $P\check{}\colon \mathscr{E} \longrightarrow \mathrm{Ker}\big(d^{\nabla^*_E} + (d^{\nabla^*_E})^*\big)$ be the orthogonal projection, then from [**11**] Proposition 3.14 again, $\nabla_{\pi_!\check{} E} \cong P\check{}\,\overline{\nabla}\check{}\,P\check{}$ and $\nabla^*_{\pi_!\check{} E} \cong P\check{}\,\overline{\nabla}\check{}^S P\check{}$ through the fiberwise Hodge isomorphism $\pi_!\check{} E \cong \mathrm{Ker}\big(d^{\nabla^*_E} + (d^{\nabla^*_E})^*\big)$.

It follows from (96) that $P\check{} = *_Z P *_Z^{-1}$ so that $*_Z$ directly provides a smooth isomorphism $\pi_! E \cong \pi_!\check{} E$. It then follows from the preceding Lemma 56 that through this isomorphism $\nabla^*_{\pi_! E} \cong \nabla_{\pi_!\check{} E}$ and $\nabla_{\pi_! E} \cong \nabla^*_{\pi_!\check{} E}$. Now $*_Z$ respects the $^+$ and $^-$ parts of $\mathscr{E}$ if $\dim Z$ is even, and exchanges them if $\dim Z$ is odd so that

$$\pi_!(E, \nabla^*_E) = (-1)^{\dim Z}(\pi_! E, \nabla^*_{\pi_! E}). \tag{100}$$

In particular, the first equation of Theorem 32 is proved.

Suppose now that $\dim Z$ is even. Then if $(E, \nabla_E, F, \nabla_F, f) \in K^0_{\mathrm{rel}}(M)$

$$\begin{aligned}
&\pi_*(E, \nabla^*_E, F, \nabla^*_F, f) = \\
&\quad = \big(\pi_!^{+\check{}} E \oplus \pi_!^{-\check{}} F, \nabla_{\pi_!^{+\check{}} E} \oplus \nabla_{\pi_!^{-\check{}} F}, \pi_!^{-\check{}} E \oplus \pi_!^{+\check{}} F, \nabla_{\pi_!^{-\check{}} E} \oplus \nabla_{\pi_!^{+\check{}} F}, \pi_\ell([f])\big) \\
&\quad = \big(\pi_!^{+} E \oplus \pi_!^{-} F, \nabla^*_{\pi_!^{+} E} \oplus \nabla^*_{\pi_!^{-} F}, \pi_!^{-} E \oplus \pi_!^{+} F, \nabla^*_{\pi_!^{-} E} \oplus \nabla^*_{\pi_!^{+} F}, \\
&\qquad\qquad [\ell^*_{\pi_! E} \oplus (\ell^*_{\pi_! F})^{-1}] \circ \pi_\ell([f]) \circ [(\ell^*_{\pi_! E})^{-1} \oplus \ell^*_{\pi_! F}]\big).
\end{aligned}$$

The reality of π_* in the case of even dimensional fibres (second statement of Theorem 32) follows from this, the first statement of Proposition 55 and the obvious compatibility of links of the form $\ell^*_{\mathcal{F}}$ with direct sums.

The last equation of Theorem 32 was proved just after its statement.

The proof of Theorem 32 is thus completed.

6.2.2. *Results on $\pi_{\leftarrow}$.* — If $\dim Z$ is odd, consider some $(E, \nabla_E) \in K^0_{\mathrm{flat}}(M)$, there is a link $[\ell^{\{0\}}_{\pi_! E}]$ between $\pi_!^+ E - \pi_!^- E$ and $\{0\} - \{0\}$ as defined at the end of §6.1.2 (see also the proof of Proposition 55),

Definition 57. — *For $(E, \nabla_E) \in K^0_{\mathrm{flat}}(M)$, one defines*

$$\pi_{\leftarrow}(E, \nabla_E) = \big(\pi_!^- E, \nabla_{\pi_!^- E}, \pi_!^+ E, \nabla_{\pi_!^+ E}, [\ell^{\{0\}}_{\pi_! E}]^{-1}\big) \in K^0_{\mathrm{rel}}(B). \tag{101}$$

Proposition 58. — *$\pi_{\leftarrow}$ defines a morphism from $K^0_{\mathrm{flat}}(M)$ to $K^0_{\mathrm{rel}}(B)$.*

The relation $\pi_! = \partial\circ\pi_{\leftarrow}$ is then tautological.

The last but one statement of Theorem 33 (that $\pi_{\leftarrow}$ is purely imaginary) is a direct consequence of the second statement of Proposition 55, since through the isomorphism induced by $*_Z$ one has

$$\begin{aligned}\pi_{\leftarrow}(E,\nabla_E^*) &= \big(\pi_!^{-\,\check{}}E,\nabla_{\pi_!^{-\,\check{}}E},\pi_!^{+\,\check{}}E,\nabla_{\pi_!^{+\,\check{}}E},[\ell^{\{0\}}_{\pi_!\check{}E}]^{-1}\big)\\ &= \big(\pi_!^{+}E,\nabla^*_{\pi_!^{+}E},\pi_!^{-}E,\nabla^*_{\pi_!^{-}E},[\ell^*_{\pi_!E}]\circ[\ell^{\{0\}}_{\pi_!E}]^{-1}\big)\end{aligned}$$

(and in general $\ell^*_{\mathscr{G}}=\ell^*_{\mathscr{G}^*}$ in the odd dimensional fibre case). In particular, if ∇_E respects some hermitian metric on E then

$$\begin{aligned}\pi_{\leftarrow}(E,\nabla_E) &= \big(\pi_!^{+}E,\nabla^*_{\pi_!^{+}E},\pi_!^{+}E,\nabla_{\pi_!^{+}E},[\mathrm{Id}]\big)\\ &= -\big(\pi_!^{-}E,\nabla^*_{\pi_!^{-}E},\pi_!^{-}E,\nabla_{\pi_!^{-}E},[\mathrm{Id}]\big)\end{aligned}$$

from which one deduces using (33), (34), Remark 14, [**11**] Theorem 0.1 and the vanishing of the Euler class of odd rank real vector bundles that:

$$\begin{aligned}\mathscr{N}_{\mathrm{ch}}\circ\pi_{\leftarrow}(E,\nabla_E) &= \frac{1}{2}\left(\mathfrak{B}(E,\nabla_E,0)-\mathfrak{B}(E,\nabla_E^*,0)\right)\\ &= \frac{1}{2}\int_Z e(TZ)\wedge\mathfrak{B}(E,\nabla_E,0)=0.\end{aligned}\tag{102}$$

The relation $\pi_* = \pi_{\leftarrow}\circ\partial$ on K^0_{rel} is proved by the following computation, which uses (98) and the compatibility of links of the form $\ell^{\{0\}}_{\mathscr{G}}$ with direct sums:

$$\begin{aligned}&\pi_*(E,\nabla_E,F,\nabla_F,f)=\\ &=\big(\pi_!^+E\oplus\pi_!^-F,\nabla_{\pi_!^+E}\oplus\nabla_{\pi_!^-F},\pi_!^-E\oplus\pi_!^+F,\nabla_{\pi_!^-E}\oplus\nabla_{\pi_!^+F},\pi_\ell([f])\big)\\ &=\big(\pi_!^+E,\nabla_{\pi_!^+E},\pi_!^-E,\nabla_{\pi_!^-E},[\ell^{\{0\}}_{\pi_!E}]\big)+\big(\pi_!^-F,\nabla_{\pi_!^-F},\pi_!^+F,\nabla_{\pi_!^+F},[\ell^{\{0\}}_{\pi_!F}]^{-1}\big).\end{aligned}$$

One deduces from this, Theorem 29 and the vanishing of the Euler class of odd rank real vector bundles that $\mathscr{N}_{\mathrm{ch}}\circ\pi_{\leftarrow}(E,\nabla_E)$ depends only on the topological K-theory class of E. Its vanishing for general $(E,\nabla_E)\in K^0_{\mathrm{flat}}(M)$ follows this, (102), the additivity of $\pi_{\leftarrow}$ and of $\mathscr{N}_{\mathrm{ch}}$ and the fact that there is some integer k such that the direct sum of k copies of E is topologically trivial on M.

The last statement remaining unproved in Theorem 33 is the vanishing of $\pi_!^{\mathrm{Eu}}$ on $\widehat{K}_{\mathrm{ch}}$. It is delayed to §6.3. Let us now prove the above proposition:

Proof. — The point to check is that $\pi_{\leftarrow}\big((E',\nabla_{E'})+(E'',\nabla_{E''})-(E,\nabla_E)\big)$ vanishes in $K^0_{\mathrm{rel}}(B)$ if E', E'' and E come from a parallel exact sequence like (11).

$(\pi_!^+E'\oplus\pi_!^+E'',\pi_!^-E'\oplus\pi_!^-E'')$ and $(\pi_!^+E,\pi_!^-E)$ are both couples of family index bundles for E (as topological vector bundle). They are thus canonically linked by

$[\ell^{\pi_!E}_{\pi_!E'\oplus\pi_!E''}]$. It follows from Lemma 42 that

(103)
$$\Big(\pi_!^+E'\oplus\pi_!^+E''\oplus\pi_!^-E,\nabla_{\pi_!^+E'}\oplus\nabla_{\pi_!^+E''}\oplus\nabla_{\pi_!^-E},$$
$$\pi_!^-E'\oplus\pi_!^-E''\oplus\pi_!^+E,\nabla_{\pi_!^-E'}\oplus\nabla_{\pi_!^-E''}\oplus\nabla_{\pi_!^+E},[\ell^{\pi_!E}_{\pi_!E'\oplus\pi_!E''}]\Big)=0\in K^0_{\mathrm{rel}}(B).$$

But it follows from (98) (which is also valid on noncompact B for globally defined couples of family index bundles as in the proof of Proposition 55) that

$$[\ell^{\pi_!E}_{\pi_!E'\oplus\pi_!E''}]=\big([\ell^{\{0\}}_{\pi_!E'}]\oplus[\ell^{\{0\}}_{\pi_!E''}]\big)\circ[\ell^{\{0\}}_{\pi_!E}]^{-1}=[\ell^{\{0\}}_{\pi_!E'}]\oplus[\ell^{\{0\}}_{\pi_!E''}]\oplus[\ell^{\{0\}}_{\pi_!E}]^{-1}$$

so that the right hand side of (103) is easily recognized (using relation (ii) of Definition 4) to be equal to $\pi_{\leftarrow}\big((E',\nabla_{E'})+(E'',\nabla_{E''})-(E,\nabla_E)\big)$. □

6.3. End of proof of Theorem 33. — The Formulas (99) and their obvious counterpart for $\overline{\nabla}^{\check{}}$ and $\overline{\nabla}^{\check{}S}$ prove that $\frac{1}{2}(\overline{\nabla}^{\check{}}+\overline{\nabla}^{\check{}S})=\frac{1}{2}(\overline{\nabla}+\overline{\nabla}^{S})=\overline{\nabla}^{u}$. Denote by $\check{C_t}$ the superconnection on $\mathscr{E}$ constructed from ∇^*_ξ as C_t is from ∇_ξ:

$$\check{C_t}=\overline{\nabla}^{u}+\frac{\sqrt{t}}{2}\big(d^{\nabla^*_\xi}+(d^{\nabla^*_\xi})^*\big)+\frac{1}{2\sqrt{t}}(\iota_T-T\wedge).$$

Remember the definition of B_t from (79), and let $\check{B_t}$ be the modified superconnection constructed as in (79) from $\check{C_t}$ (or ∇^*_ξ) and the suitable triple of Proposition 53 then

Lemma 59. — *We have*
$$\phi\mathrm{Tr}_s\left(\frac{\partial\check{B_t}}{\partial t}\exp-\check{B_t}^2\right)=(-1)^{\dim Z}\phi\mathrm{Tr}_s\left(\frac{\partial B_t}{\partial t}\exp-B_t^2\right).$$
In particular, $\phi\mathrm{Tr}_s\left(\frac{\partial C_t}{\partial t}\exp-C_t^2\right)$ vanishes if ∇_ξ respects h^ξ and $\dim Z$ *is odd.*

Proof. — For any $\mathtt{w}\in TZ$, $c(\mathtt{w})$ commutes with $*_Z$ if $\dim Z$ is odd and it anticommutes with $*_Z$ if $\dim Z$ is even. Thus $\iota_T-T\wedge=-c(T)$ also does. Then it follows from (96) that:
$$\frac{\sqrt{t}}{2}\big(d^{\nabla_\xi}+(d^{\nabla_\xi})^*\big)+\frac{1}{2\sqrt{t}}(\iota_T-T\wedge)=$$
$$=-(-1)^{\dim Z}*_Z^{-1}\left(\frac{\sqrt{t}}{2}\big(d^{\nabla^*_\xi}+(d^{\nabla^*_\xi})^*\big)+\frac{1}{2\sqrt{t}}(\iota_T-T\wedge)\right)*_Z.$$

Lemma 56 has the consequence that the covariant derivative with respect to $\overline{\nabla}^{u}$ commutes with $*_Z$. For $\mathbb{Z}_2$-graduation reasons, this proves that the exterior derivative associated with $\overline{\nabla}^{u}$ on $\mathrm{End}E$-valued differential forms on B supercommutes with $*_Z$.

Let N_H be the graduation operator on $\wedge T^*B$ which multiplies k-degree differential forms by k, the properties above give the following formula:

(104)
$$C_t=-(-1)^{\dim Z}(-1)^{N_H}*_Z^{-1}\check{C_t}*_Z(-1)^{N_H}.$$

Put $\mathrm{Id}_\mu = \mathrm{Id}_{\mu^+} \oplus \mathrm{Id}_{\mu^-}$. Then, using (97) instead of (96) one obtains

$$B_t = -(-1)^{\dim Z}(-1)^{N_H}(*_Z \oplus \mathrm{Id}_\mu)^{-1} B_{\check{t}}(*_Z \oplus \mathrm{Id}_\mu)(-1)^{N_H}. \tag{105}$$

Now it successively follows that

$$\begin{aligned}
B_t^2 &= (-1)^{N_H}(*_Z \oplus \mathrm{Id}_\mu)^{-1} B_{\check{t}}^2(*_Z \oplus \mathrm{Id}_\mu)(-1)^{N_H}\\
\exp(-B_t^2) &= (-1)^{N_H}(*_Z \oplus \mathrm{Id}_\mu)^{-1} \exp(-B_{\check{t}}^2)(*_Z \oplus \mathrm{Id}_\mu)(-1)^{N_H}\\
\frac{\partial B_t}{\partial t}\exp(-B_t^2) &=\\
&= -(-1)^{\dim Z}(-1)^{N_H}(*_Z \oplus \mathrm{Id}_\mu)^{-1}\frac{\partial B_{\check{t}}}{\partial t}\exp(-B_{\check{t}}^2)(*_Z \oplus \mathrm{Id}_\mu)(-1)^{N_H}.
\end{aligned}$$

In this context of infinite rank vector bundles, it remains true that the supertrace of the supercommutator of two L^2-bounded $\mathrm{End}\mathscr{E}$-valued differential forms, one of which is trace class, vanishes. Using the fact that Id_μ has the same parity as $*_Z$, one can apply this to $[(*_Z \oplus \mathrm{Id}_\mu)^{-1}, \omega(*_Z \oplus \mathrm{Id}_\mu)]$ to obtain

$$\mathrm{Tr}_s(\omega) = \mathrm{Tr}_s\big((*_Z \oplus \mathrm{Id}_\mu)^{-1}\omega(*_Z \oplus \mathrm{Id}_\mu)\big)$$

which is valid for any globally odd $\mathrm{End}(\mathscr{E} \oplus \mu)$-valued trace-class differential form ω, in particular for $\frac{\partial B_{\check{t}}}{\partial t}\exp(-B_{\check{t}}^2)$. Thus

$$\mathrm{Tr}_s\left(\frac{\partial B_t}{\partial t}\exp(-B_t^2)\right) = -(-1)^{\dim Z}(-1)^{N_H}\mathrm{Tr}_s\left(\frac{\partial B_{\check{t}}}{\partial t}\exp(-B_{\check{t}}^2)\right)(-1)^{N_H}.$$

The fact that this form and $\frac{\partial B_t}{\partial t}\exp(-B_t^2)$ and $\frac{\partial B_{\check{t}}}{\partial t}\exp(-B_{\check{t}}^2)$ are globally odd, implies that only their odd differential form degree parts contribute to their supertrace. The equation of the lemma follows. □

Suppose now that $\dim Z$ is odd. Denote by 0 the connection on the null rank vector bundle $\{0\}$ on B. Consider any element $(\xi, \nabla_\xi, \alpha) \in \widehat{K}_{\mathrm{ch}}(M)$. Choose any suitable data (μ^+, μ^-, ψ) giving rise to family index bundles $\mathscr{K}^+$ and $\mathscr{K}^-$. Let $[\ell_{\mathscr{K}}^{\{0\}}]$ be the canonical class of links between $\mathscr{K}^+ - \mathscr{K}^-$ and $\{0\} - \{0\}$ obtained just before (98). Then

$$\pi_!^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha) = \left(\{0\}, 0, \int_Z e(\nabla_{TZ}) \wedge \alpha\right) - \big(\{0\}, 0, \eta(\nabla_\xi, \nabla_{TZ}, 0, 0, [\ell_{\mathscr{K}}^{\{0\}}]^{-1})\big). \tag{106}$$

$e(\nabla_{TZ})$ vanishes. Theorem 33 is then reduced to the following lemma. Note that all arguments were local, but (106) and the following lemma have global meaning, so that the arguments also work for noncompact B.

Lemma 60. — *We have* $\eta(\nabla_\xi, \nabla_{TZ}, 0, 0, [\ell_{\mathscr{K}}^{\{0\}}]^{-1}) = 0$.

Proof. — Compare $\eta(\nabla_\xi, \nabla_{TZ}, 0, 0, [\ell_{\mathscr{K}}^{\{0\}}]^{-1})$ with the form η computed from the "adjoint" triple $\big(\mu^-, \mu^+, (*_Z \oplus \mathrm{Id}_{\mu^+}) \circ \psi^* \circ (*_Z^{-1} \oplus \mathrm{Id}_{\mu^-})\big)$. They are of course equal because $\pi_!^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha)$ can be written with the same formula by simply replacing η by the other one. The terms of the form $\int_Z e(\nabla_{TZ})\widetilde{\mathrm{ch}}(\nabla_\xi^u, \nabla_\xi)$ vanish in both cases. The terms of the form $\int_0^1 \frac{\partial B_t}{\partial t}\exp(-B_t^2)$ are mutually opposite from the preceding lemma. Denote by $\mathcal{K}^{*\pm}$ the "adjoint" family index bundles, it follows from (87) and the second equation of Proposition 55 that

$$\widetilde{\mathrm{ch}}([\ell_{\mathcal{K}}^{\{0\}}]^{-1}) + \widetilde{\mathrm{ch}}([\ell_{\mathcal{K}}^{\{0\}}]^{-1}) = -\widetilde{\mathrm{ch}}([\ell_{\mathcal{K}}^*]).$$

But $\widetilde{\mathrm{ch}}([\ell_{\mathcal{K}}^*])$ vanishes because the $(*_Z + \mathrm{Id}_\mu)$ isomorphism respect the connections (81) on kernel bundles. Thus both η forms are mutually opposite. □

7. Double fibrations

Consider two proper submersions $\pi_1 \colon M \longrightarrow B$ and $\pi_2 \colon B \longrightarrow S$ and the composed submersion $\pi_2 \circ \pi_1 \colon M \longrightarrow S$. The goal of this section is to compare direct image with respect to $\pi_2 \circ \pi_1$ and the composition of the two direct images relative to π_1 and π_2 for topological, relative and multiplicative/smooth K-theories, and then to prove Theorems 34 and 35. Unless otherwise stated, S is supposed to be compact.

7.1. Topological K-theory. — Consider some vector bundle ξ on M, some couples of family index bundles $(\mathcal{H}^+, \mathcal{H}^-)$ and $(\mathcal{F}^+, \mathcal{F}^-)$ for ξ relatively to π_1 and to $\pi_2 \circ \pi_1$ respectively and some couple $(\mathcal{G}^+, \mathcal{G}^-)$ of family index bundles for $\mathcal{H}^+ - \mathcal{H}^-$ (with respect to π_2).

Theorem 61. — *There exists some canonical equivalence class of links $[\ell_{\mathcal{G}}^{\mathcal{F}}]$ between $\mathcal{G}^+ - \mathcal{G}^-$ and $\mathcal{F}^+ - \mathcal{F}^-$.*

This implies the functoriality of π^{Eu} for double submersions of compact manifolds, and hence in full generality (see Definition 20).

The canonicity is to understand in the same sense that in Theorem 25. The construction of $[\ell_{\mathcal{G}}^{\mathcal{F}}]$ uses the convergence of Euler operators under adiabatic limits. The point is to obtain some spectral convergence which allows to understand the behaviour of the kernel and of eigenvalues converging to 0 in this limit. We closely follow the analysis performed in **[5]** §5 with some analogue of **[5]** Theorem 5.28 and formula (5.118) as goal. In fact we want to connect family index bundles on M for ξ and on B for $\mathcal{H}^\pm$. We will combine spectral convergence with the fact that if $a > 0$ is such that the Euler operator along the fibres of $\pi_2 \circ \pi_1$ has no eigenvalue equal to a nor $-a$ along S, then the eigenspaces corresponding to eigenvalues lying in $[-a, +a]$ form vector bundles on S which are themselves family index bundles. (This was already used in §4.2.3).

7.1.1. *Fiberwise exterior differentials:*— We precise (in (108)) the decomposition of the exterior differential along horizontal and vertical differential form degrees corresponding to **[5]** Theorem 5.1.

The respective vertical tangent vector bundles associated with π_1, π_2 and $\pi_2 \circ \pi_1$ will be denoted by TM/B, TB/S and TM/S. Choose some connection ∇_ξ on ξ along M. Call $\mathscr{E}^{\pm}_{M/S}$ (resp. $\mathscr{E}^{\pm}_{M/B}$) the infinite rank vector bundles on S (resp. B) of even/odd degree differential forms along the fibres of $\pi_2 \circ \pi_1$ (resp. π_1) with values in ξ. Choose any smooth supplementary subbundle T^HM of TM/B in TM/S. Of course $T^HM \cong \pi_1^*TB/S$. On the fibre of $\pi_2 \circ \pi_1$ over any point s of S, one obtains for ξ-valued differential forms an isomorphism analogous to (37):

$$\mathscr{E}_{M/S} \cong \Omega\big(\pi_2^{-1}(\{s\}), \mathscr{E}_{M/B}\big). \tag{107}$$

For any $b \in B$ and any tangent vector $U \in T_bB/S$, call U^H its horizontal lift as a section of T^HM over $\pi_1^{-1}(b)$. For any $s \in S$, the construction of (72) produces a connection on the restriction of $\mathscr{E}_{M/B}$ over $\pi^{-1}(\{s\})$ which will be denoted $\overline{\nabla}$. We will denote by d^H the exterior differential operator on $\Omega\big(\pi_2^{-1}(\{s\}), \mathscr{E}_{M/B}\big) \cong \mathscr{E}_{M/S}$ associated to $\overline{\nabla}$.

The "vertical" differential operator d^{∇_ξ} will be denoted by $d^{M/B}$ on $\mathscr{E}_{M/B}$ and $d^{M/S}$ on $\mathscr{E}_{M/S}$. As was remarked at the beginning of §5.2,

$$d^{M/S} = d^{M/B} + d^H + \iota_T \qquad \text{(through the identification (107))} \tag{108}$$

where ι_T here stands for the restrictions to the fibres of $\pi_2 \circ \pi_1$ of the operator $\iota_T \in \wedge^2(TB/S)^* \otimes \mathrm{End}^{\mathrm{odd}}(\mathscr{E}_{M/B})$ of §5.2. We will consider this ι_T as an element of $\mathrm{End}\mathscr{E}_{M/S}$ (through the identification (107)).

7.1.2. *Fiberwise Euler operators.* — Here we precise (in (109) and 110) the dependence of the Euler operator on the parameter θ of the adiabatic limit. This corresponds to **[5]** Definition 5.5.

Endow TM/B with some (riemannian) metric $g^{M/B}$ and ξ with some hermitian metric h^ξ. Take some riemannian metric $g^{B/S}$ on TB/S. Put on TM/S the riemannian metric for which the decomposition $TM/S = TM/B \oplus T^HM$ is orthogonal and which coincides with $g^{M/B}$ and $\pi_1^*g^{B/S}$ on either parts. The adjoints in $\mathrm{End}\mathscr{E}_{M/S}$ will be considered with respect to the L^2 scalar product on $\mathscr{E}_{M/S}$ obtained from h^ξ and this riemannian metric (as in (38)). These are not the adjoints (neither usual nor special) considered on $\Omega\big(\pi_2^{-1}(\{s\}), \mathscr{E}_{M/B}\big)$ in §5.1. For instance, the adjoint ι_T^* of ι_T here is not $T\wedge$ as it was in §5.2.

Let $d^{M/B*}$ be the adjoint of $d^{M/B}$ with respect to $g^{M/B}$ and h^ξ as endomorphisms of $\mathscr{E}_{M/B}$, then $d^{M/B}$ and $d^{M/B*}$ are also adjoint as endomorphisms of $\mathscr{E}_{M/S}$ because of the choice of a riemannian submersion metric on M. Call d^{H*} the adjoint of d^H as endomorphism of $\mathcal{E}_{M/S}$ and put:

$$
\begin{aligned}
d^\theta &= d^H + \frac{1}{\theta} d^{M/B} + \theta \iota_T \qquad \text{and} \qquad d^{\theta *} = d^{H*} + \frac{1}{\theta} d^{M/B*} + \theta \iota_T^* \\
\mathcal{D}^H &= d^H + d^{H*} \qquad \text{and} \qquad \mathcal{D}^V = d^{M/B} + d^{M/B*} \qquad (109)\\
&\mathcal{D}^\theta = d^\theta + d^{\theta *} = \mathcal{D}^H + \frac{1}{\theta}\mathcal{D}^V + \theta(\iota_T + \iota_T^*).
\end{aligned}
$$

Let N_V be the endomorphism of $\mathcal{E}_{M/B}$ defined as in §5.2.3. N_V multiplies k-degree vertical forms by k, "vertical" meaning forms along the fibres of π_1. Using T^HM, N_V extends to an operator on $\mathcal{E}_{M/S}$ through the identification (107). Let g_θ be the riemannian metric on M such that TM/B and T^HM are orthogonal and which restricts to $g^{M/B}$ and $\frac{1}{\theta^2}\pi_1^* g^B$ on either part, the observation here is that

$$
\mathcal{D}^\theta = \theta^{N_V}(d^{M/S} + d_\theta^{M/S*})\theta^{-N_V} \qquad (110)
$$

where $d_\theta^{M/S*}$ is the adjoint of $d^{M/S}$ with respect to g_θ and h^ξ. The riemanian submersion metric chosen here simplifies considerably the form of $\mathcal{D}^\theta$ with respect to the case of **[5]** where such a choice is not allowed and forces more complicated conjugations than by θ^{N_V} (see **[5]** §5(a)).

7.1.3. *Introducing some intermediate suitable triple.* — In the adiabatic limit, the ξ-twisted Euler operator on M should converge to the Euler operator on B twisted by the kernel bundles on B for ξ with respect to π_1. In the general setting considered here, this forces to introduce some suitable triple with respect to π_1 in the global Euler operator. This is performed here, the induced 2×2-matrix decomposition of the modified $\mathcal{D}_\theta$ is presented and the first estimates on the matrix elements are obtained by analogy with **[5]** §5.

Consider some suitable triple (μ^+, μ^-, ψ) with respect to π_1 (and ξ with ∇_ξ and h^ξ and $g^{M/B}$). $\mu^\pm$ are endowed with some hermitian metrics. Choose some connection ∇_μ on $\mu^\pm$ (which respects either part) and consider the associated Euler operator $\mathcal{D}^{\nabla_\mu} = d^{\nabla_\mu} + (d^{\nabla_\mu})^*$ on $\Omega(B/S, \mu^\pm)$. Take some function χ as in §5.2.5. For $\theta \in (0,1]$, one puts

$$
\begin{aligned}
\mathcal{D}_\psi^\theta &= \mathcal{D}^\theta + \mathcal{D}^\mu + \frac{1-\chi(\theta)}{\theta}(\psi + \psi^*) \\
&= \mathcal{D}^H + \mathcal{D}^\mu + \frac{1}{\theta}\mathcal{D}^V_{(1-\chi(\theta))\psi} + \theta(\iota_T + \iota_T^*) \in \operatorname{End}^{\mathrm{odd}}\left(\mathcal{E}_{M/S} \oplus \Omega(B/S,\mu)\right)
\end{aligned} \qquad (111)
$$

where $\mathcal{D}^V_{(1-\chi(\theta))\psi}$ is obtained from $\mathcal{D}^V$ and $(1-\chi(\theta))\psi$ as $\mathcal{D}_\psi^{\nabla_\xi}$ is from $\mathcal{D}^{\nabla_\xi}$ and ψ in (39). Here ψ is extended to forms on B/S through the isomorphism (107). The choice of a riemannian submersion metric on TM ensures the compatibility of the adjunctions of ψ before and after extending it to forms on B/S. This result (111) corresponds to **[5]** Proposition 5.9 with $\theta = \frac{1}{T}$ and with the extra term $\theta(\iota_T + \iota_T^*)$.

There is a double decomposition associated to $\mathscr{D}_\psi^V$.

$$\mathscr{E}_{M/B} \oplus \mu^\pm = \operatorname{Ker}\mathscr{D}_\psi^V \oplus (\operatorname{Ker}\mathscr{D}_\psi^V)^\perp \tag{112}$$

which gives a double decomposition of infinite rank vector bundles on S:

$$\mathscr{E}_{M/S} \oplus \Omega(B/S, \mu^\pm) = \Omega\big(B/S, \operatorname{Ker}\mathscr{D}_\psi^V\big) \oplus \Omega\big(B/S, (\operatorname{Ker}\mathscr{D}_\psi^V)^\perp\big).$$

The choice of a riemannian submersion metric induces that the second one is orthogonal: let $p\colon \mathscr{E}_{M/S} \oplus \Omega(B/S, \mu^\pm) \longrightarrow \Omega\big(B/S, \operatorname{Ker}\mathscr{D}_\psi^V\big)$ be the orthogonal projection, it is the tensor product of the identity in $\Omega(B/S)$ and the orthogonal projection on the first factor of (112). Put $p^\perp = \operatorname{Id} - p$. For any positive θ one decomposes the operator $\mathscr{D}_\psi^\theta$ as a 2×2 matrix:

$$\mathscr{D}_\psi^\theta = \begin{pmatrix} p\mathscr{D}_\psi^\theta p & p\mathscr{D}_\psi^\theta p^\perp \\ p^\perp \mathscr{D}_\psi^\theta p & p^\perp \mathscr{D}_\psi^\theta p^\perp \end{pmatrix} =: \begin{pmatrix} A_1^\theta & A_2^\theta \\ A_3^\theta & A_4^\theta \end{pmatrix}.$$

As in §5.2.5, the vector bundle $\operatorname{Ker}\mathscr{D}_\psi^V$ is endowed with the restriction of the metric on $\mathscr{E}_{M/B} \oplus \mu^\pm$, and with the connection $\nabla_{\mathscr{H}}$ obtained by projecting the connection on $\mathscr{E}_{M/B} \oplus \mu$ onto it (in fact $p(\overline{\nabla} \oplus \nabla_\mu)p$, see [**5**] Theorem 5.1 and formula (5.34)). Because of the compatibility of orthogonal projections, the exterior differential operator on $\Omega\big(B/S, \operatorname{Ker}\mathscr{D}_\psi^V\big)$ associated to this connection is $d^{\nabla_{\mathscr{H}}} = p(d^H \oplus d^{\nabla_\mu})p$. Clearly $(d^{\nabla_{\mathscr{H}}})^* = p\big(d^{H*} \oplus (d^{\nabla_\mu})^*\big)p$. Define then $\mathscr{D}^{\nabla_{\mathscr{H}}} = d^{\nabla_{\mathscr{H}}} + (d^{\nabla_{\mathscr{H}}})^*$. For any $\theta \leq \frac{1}{2}$ (to ensure that $\chi(\theta) = 0$), one has

$$A_1^\theta = \mathscr{D}^{\nabla_{\mathscr{H}}} + \theta p(\iota_T + \iota_T^*)p. \tag{113}$$

Of course $p(\iota_T + \iota_T^*)p$ is a bounded operator in the L^2-topology, and this remark with the above equation replaces here equation (5.35) of Theorem 5.1 in [**5**].

In the same way, for $\theta \in [0, \frac{1}{2}]$

$$\begin{aligned} &A_2^\theta = p\big((d^H + d^{H*}) \oplus (d^{\nabla_\mu} + (d^{\nabla_\mu})^*)\big)p^\perp + \theta p(\iota_T + \iota_T^*)p^\perp \\ \text{and}\qquad &A_3^\theta = p^\perp\big((d^H + d^{H*}) \oplus (d^{\nabla_\mu} + (d^{\nabla_\mu})^*)\big)p + \theta p^\perp(\iota_T + \iota_T^*)p \end{aligned} \tag{114}$$

are uniformly bounded operators in the L^2-topology. This is because of the choice of the riemannian submersion metric and is a simplification with respect to the corresponding result Proposition 5.18 of [**5**].

7.1.4. *Estimates on the operator A_4^θ.* — First one wants to obtain results analogous to [**5**] Theorems 5.19 and 5.20. There are three differences between the situation here and [**5**]. The absence of conjugation (by C_T in [**5**] Definition 5.4 and (5.10)) due to the choice of a riemannian submersion metric is a simplification and does not create any obstacle; the presence here of the term $\theta(\iota_T + \iota_T^*)$ does not change these results because of the fact that $\iota_T + \iota_T^*$ is a bounded operator in the L^2-topology and because of its factor θ; more seriously, the commutator $[D^Z_\infty, D^H_\infty]$ in **[5]** which corresponds to $[\mathcal{D}^V, \mathcal{D}^H]$ in the notations here, is to be replaced by

$$[\mathcal{D}^V_\psi, \mathcal{D}^H + \mathcal{D}^\mu] = [\mathcal{D}^V, \mathcal{D}^H] + [\psi + \psi^*, \mathcal{D}^H + \mathcal{D}^\mu].$$

Of course the first term has the required majoration property **[5]** (5.67). The operator $\psi + \psi^*$ is a fiberwise kernel operator (along the fibres of π_1), and its kernel is smooth along the fibered double $M \times_B M$. Thus, if v is a smooth vector field on B, the commutator $\big[\psi+\psi^*, \nabla^{\wedge^\bullet T^*M/B\otimes\xi}_{\mathrm{v}^H} \oplus \nabla^\mu_{\mathrm{v}}\big]$ (where v^H is the horizontal lift of v, a section of T^HM), is a fiberwise kernel operator with globally smooth kernel. In particular, it is bounded in L^2-topology, and so is the (super)commutator $[\psi + \psi^*, \mathcal{D}^H + \mathcal{D}^\mu]$. The estimate **[5]** (5.67) then follows from **[5]** (5.61) (whose equivalent here holds true).

The conclusions of Theorems 5.19 and 5.20 of **[5]** remain thus valid here, namely the existence of some constant C such that for any $\theta \leq \frac{1}{2}$ and any section s of $\mathcal{E}_{M/S} \oplus \mu^\pm$

$$\|A^\theta_4(p^\perp s)\|_{L^2} \geq C\left(\|p^\perp s\|_{H^1} + \frac{1}{\theta}\|p^\perp s\|_{L^2}\right). \tag{115}$$

where $\|\ \|_{H^1}$ stands for the usual Sobolev H^1-norm.

Secondly, one needs some equivalent of **[5]** Proposition 5.22, particularly the estimate (5.71) contained in it. But the proof here is in fact easier than in **[5]** because Equation (114) provides a simplification of the corresponding Proposition 5.18 in **[5]**, the extra term $\theta(\iota_T + \iota_T^*)$ is a uniformly bounded operator, $(\psi+\psi^*)$ too, and $\frac{1}{\theta}(\psi+\psi^*)$ is part of A^θ_4, it does not disable the ellipticity of A^θ and it is taken into account in the obtained estimates: there exist constants c, C and θ_0 such that for any $\theta \leq \theta_0$, $\lambda \in \mathbb{C}$ such that $|\lambda| \leq \frac{c}{2\theta}$ and any $s \in \mathcal{E}_{M/S} \oplus \mu^\pm$,

$$\begin{aligned} &\|(\lambda - A^\theta_4)^{-1} p^\perp s\|_{L^2} \leq C\theta \|p^\perp s\|_{L^2} \\ &\|(\lambda - A^\theta_4)^{-1} p^\perp s\|_{H^1} \leq C \|p^\perp s\|_{L^2}. \end{aligned} \tag{116}$$

7.1.5. *Spectral convergence of Euler operators.* — Our goal is to follow kernel bundles. This makes us now introduce some suitable triple $(\zeta^+, \zeta^-, \varphi)$ for $\mathcal{D}^{\nabla_{\mathcal{H}}}$. We extend p and $p^\perp$ to $\mathcal{E}_{M/S} \oplus \zeta^\pm$ in the following way: p induces the identity on $\zeta^\pm$ and $p^\perp$ induces the null map on $\zeta^\pm$. Consider then

$$\begin{aligned} \mathcal{D}^\theta_{\psi,\varphi} &= \begin{pmatrix} p\big(\mathcal{D}^\theta_\psi + (1-\chi(\theta))(\varphi+\varphi^*)\big)p & p\mathcal{D}^\theta_\psi p^\perp \\ p^\perp \mathcal{D}^\theta_\psi p & p^\perp \mathcal{D}^\theta_\psi p^\perp \end{pmatrix} \\ &= \begin{pmatrix} A^\theta_1 + (1-\chi(\theta))(\varphi+\varphi^*) & A^\theta_2 \\ A^\theta_3 & A^\theta_4 \end{pmatrix}. \end{aligned}$$

(It is not essential that the same function χ appears here and in (111)). Equation (113) obviously leads to the following equality for $\theta \in [0, \frac{1}{2}]$:

$$A^\theta_1 + (1-\chi(\theta))(\varphi+\varphi^*) = \mathcal{D}^{\nabla_{\mathcal{H}}} + \varphi + \varphi^* + \theta p(\iota_T + \iota_T^*)p = \mathcal{D}^{\nabla_{\mathcal{H}}}_\varphi + \theta p(\iota_T + \iota_T^*)p \tag{117}$$

with the same remark (as after (113)) that $p(\iota_T + \iota_T^*)p$ is bounded.

Using this, the remark after (114) above and (116) instead of **[5]**(5.35), (5.49) and (5.71) respectively, the analysis performed in **[5]** §§5(d) and (g) applies here. The only difference is that the following equivalent here of the first line of **[5]** (5.89) (for the usual norm of bounded operators in L^2-topology) is NOT true

$$\Big\| \big(A_1^\theta + (1-\chi(\theta))(\varphi+\varphi^*) - \mathscr{D}_\varphi^{\nabla_{\mathscr{H}}}\big)(\lambda - \mathscr{D}_\varphi^{\nabla_{\mathscr{H}}})^{-1} \Big\| \leq C\theta^2(1+|\lambda|). \tag{118}$$

The set U_T (or $U_{\frac{1}{\theta}}$) where λ is supposed to lie, defined in **[5]** (5.76), is such that $|\lambda| \leq c_1 T$ (or $\frac{c_1}{\theta}$) and $\|(\lambda - \mathscr{D}_\varphi^{\nabla_{\mathscr{H}}})^{-1}\| \leq \frac{c_2}{4}$ for some constants c_1 and c_2. But only the following consequence of (118)

$$\Big\| \big(A_1^\theta + (1-\chi(\theta))(\varphi+\varphi^*) - \mathscr{D}_\varphi^{\nabla_{\mathscr{H}}}\big)(\lambda - \mathscr{D}_\varphi^{\nabla_{\mathscr{H}}})^{-1} \Big\| \leq C\theta$$

is needed for establishing the equivalent of **[5]** (5.90). This last estimate can be obtained here directly from the remark after (117) and the properties of $U_{\frac{1}{\theta}}$.

One obtains firstly the convergence of the resolvent of $\mathscr{D}_{\psi,\varphi}^\theta$ to any great enough positive integral power $(\lambda - \mathscr{D}_{\psi,\varphi}^\theta)^{-k}$ to $p(\lambda - \mathscr{D}_\varphi^{\nabla_{\mathscr{H}}})^{-k}p$ in the sense of the norm $\|A\|_1 = \operatorname{tr}(A^*A)^{\frac{1}{2}}$ (**[5]** Theorem 5.28), and secondly the convergence of the spectral projector of $\mathscr{D}_{\psi,\varphi}^\theta$ with respect to eigenvalues of absolute value bounded by some suitable positive constant a to the orthogonal projector onto the kernel of $\mathscr{D}_\varphi^{\nabla_{\mathscr{H}}}$ (**[5]** equation (5.118)). Thus

Theorem 62. — *There exists some $\varepsilon_2 > 0$ and $a > 0$, and a vector bundle $\mathscr{K}$ on $S \times [0, \varepsilon_2]$ such that $\mathscr{K}|_{S\times\{0\}} \cong \operatorname{Ker}\mathscr{D}_\varphi^{\nabla_{\mathscr{H}}}$ and $\mathscr{K}|_{S\times\{t\}}$ identifies with the direct sum of eigenspaces of $\mathscr{D}_{\psi,\varphi}^\theta$ corresponding to (all) eigenvalues lying in $[-a, +a]$.*

This is because all the used estimates are uniform along S, which is compact.

7.1.6. *Construction of the canonical link (proof of Theorem 61).* — The above eigenspaces are also eigenspaces for the squared operator $\big(\mathscr{D}_{\psi,\varphi}^\theta\big)^2$, they are thus naturally $\mathbb{Z}_2$-graded, and for any nonzero eigenvalue, $\mathscr{D}_{\psi,\varphi}^\theta$ exchanges bijectively the positive and negative degree parts. (In particular, there is no nonzero eigenvalue in $[-a, a]$ if $(\zeta^+, \zeta^-, \varphi)$ is a positive kernel triple).

In any case, on $S \times \{0\}$, $(\mathscr{K}^+, \mathscr{K}^-)$ are kernel bundles so that in $K^0_{\mathrm{top}}(S)$:

$$\begin{aligned}
[\mathscr{K}^+|_{S\times\{0\}} \oplus \zeta^-] - [\mathscr{K}^-|_{S\times\{0\}} \oplus \zeta^+] &= \pi_{2*}^{\mathrm{Eu}}\big([\operatorname{Ker}\mathscr{D}_\psi^{V+}] - [\operatorname{Ker}\mathscr{D}_\psi^{V-}]\big)\\
&= \pi_{2*}^{\mathrm{Eu}}\big(\pi_{1*}^{\mathrm{Eu}}([\xi]) + [\mu^+] - [\mu^-]\big).
\end{aligned}$$

The constructions of §4.1.2, §4.1.3 and §4.1.4 can be applied to $\mathscr{D}_\psi^\theta$ on any compact subset of $S \times (0, 1]$. This is because $\mathscr{D}_\psi^\theta$ is the sum of the fiberwise elliptic operator $\mathscr{D}^\theta \oplus \mathscr{D}^\mu$ and an order 0 pseudo-differential operator, which does not destroy the ellipticity. (Only ellitpic regularity is needed to construct suitable triples). This does not work on $[0,1]$ because of the explosion of A_4^θ (115).

Choose $\varepsilon_1 \in (0,\varepsilon_2)$ and some suitable triple $(\lambda^+,\lambda^-,\phi)$ for $\mathscr{D}_\psi^\theta$ with respect to the submersion $(\pi_2\circ\pi_1)\times \mathrm{Id}_{[\varepsilon_1,1]}$. One then obtains kernel bundles $\mathscr{L}^\pm$ on $S\times[\varepsilon_1,1]$ which verify the following equality in $K^0_{\mathrm{top}}(S)$ for any $\theta\in[\varepsilon_1,1]$

$$[(\mathscr{L}^+\oplus\lambda^-)|_{S\times\{\theta\}}]-[(\mathscr{L}^-\oplus\lambda^+)|_{S\times\{\theta\}}]=(\pi_2\circ\pi_1)^{\mathrm{Eu}}_*[\xi]+\pi^{\mathrm{Eu}}_{2*}([\mu^+]-[\mu^-]).$$

This is clear on $S\times\{1\}$ and spreads by parallel transport along $[\varepsilon_1,1]$.

One obtains a class of links between $\big(\mathscr{K}^+|_{S\times\{0\}}\oplus\zeta^-\big)-\big(\mathscr{K}^-|_{S\times\{0\}}\oplus\zeta^+\big)$ and $\big((\mathscr{L}^+\oplus\lambda^-)|_{S\times\{1\}}\big)-\big((\mathscr{L}^-\oplus\lambda^+)|_{S\times\{1\}}\big)$ by composing the parallel transport along $[0,\varepsilon_2]$ for $\mathscr{K}$, the canonical link between $\big(\mathscr{K}^+|_{S\times\{t\}}\oplus\zeta^-\big)-\big(\mathscr{K}^-|_{S\times\{t\}}\oplus\zeta^+\big)$ and $\big((\mathscr{L}^+\oplus\lambda^-)|_{S\times\{t\}}\big)-\big((\mathscr{L}^-\oplus\lambda^+)|_{S\times\{t\}}\big)$ of Theorem 25 (which may be applied to $\mathscr{D}^\theta_\psi$) for any $t\in[\varepsilon_1,\varepsilon_2]$ and parallel transport again along $[\varepsilon_1,1]$ for $\mathscr{L}$.

Choose any couple of family index bundles (ν^+,ν^-) for $\mu^+-\mu^-$.

Definition 63. — *The canonical equivalence class of links $[\ell^{\mathscr{F}}_{\mathscr{G}}]$ of Theorem 61 is obtained by composing the above link with the canonical links of Theorem 25 between $(\mathscr{G}^+\oplus\nu^+)-(\mathscr{G}^-\oplus\nu^-)$ and $\big(\mathscr{K}^+|_{S\times\{0\}}\oplus\zeta^-\big)-\big(\mathscr{K}^-|_{S\times\{0\}}\oplus\zeta^+\big)$ on one hand, and $\big((\mathscr{L}^+\oplus\lambda^-)|_{S\times\{1\}}\big)-\big((\mathscr{L}^-\oplus\lambda^+)|_{S\times\{1\}}\big)$ and $(\mathscr{F}^+\oplus\nu^+)-(\mathscr{F}^-\oplus\nu^-)$ on the other hand.*

This class of links is clearly independent of the choice of ν^+ or ν^-, or of the triples $(\lambda^+,\lambda^-,\phi)$ or $(\zeta^+,\zeta^-,\varphi)$ because of the global compatibility of links obtained from Theorem 25.

Now take two systems of suitable data (μ_1^+,μ_1^-,ψ_1) and (μ_2^+,μ_2^-,ψ_2) with respect to π_1 (and ξ with ∇_ξ and h^ξ and $g^{M/B}$). There is a link (as constructed in §4.1.3) between $\big(\mathrm{Ker}\,\mathscr{D}^{\psi_1+}_{M/B}\oplus\mu_1^-\big)-\big(\mu_1^+\oplus\mathrm{Ker}\,\mathscr{D}^{\psi_1-}_{M/B}\big)$ and $\big(\mathrm{Ker}\,\mathscr{D}^{\psi_2+}_{M/B}\oplus\mu_2^-\big)-\big(\mu_2^+\oplus\mathrm{Ker}\,\mathscr{D}^{\psi_2-}_{M/B}\big)$. This link is obtained by constructing a families index map for a submersion of the form $\pi_1\times\mathrm{Id}_{[0,1]}\colon M\times[0,1]\longrightarrow B\times[0,1]$. This construction can be extended to the case of a double submersion in the following form $M\times[0,1]\overset{\pi_1\times\mathrm{Id}_{[0,1]}}{\longrightarrow}B\times[0,1]\overset{\pi_2\times\mathrm{Id}_{[0,1]}}{\longrightarrow}S\times[0,1]$, and the compatibility of canonical links either for linked data or for one and for two submersions follows.

7.2. Flat and relative K-theory. — The first goal of this section is to explain why $(\pi_2\circ\pi_1)_!=\pi_{2!}\circ\pi_{1!}$ on K^0_{flat}: this is a by-product of the Leray spectral sequence (see §7.2.1). It is well known that the Leray spectral sequence fits with the adiabatic limit of the preceding section, the goal of §7.2.2 is to explain how this traduces in the language of links. This is needed in to prove Theorem 34 in §7.2.3.

7.2.1. *Leray spectral sequence.* — Consider some flat vector bundle (F, ∇_F) on M, let $G^\bullet = \pi_{1!}^\bullet F$ with flat connections $\nabla_{G^\bullet}$, and $H^\bullet = (\pi_2 \circ \pi_1)_!^\bullet F$ with flat connections $\nabla_{H^\bullet}$. Note that here the full $\mathbb{Z}$-graduation is needed and not only the parity $\mathbb{Z}_2$-graduation.

The vertical F-valued de Rham complex $\Omega^\bullet(M/S, F)$ along M/S is filtrated by the horizontal degree: for any p, $F^p\Omega^\bullet(M/S, F)$ consists of F-valued differential forms whose interior product with more than p elements of TM/B vanishes. Thus $H^\bullet$ is also filtrated from this filtration: $F^pH^\bullet$ consists of classes which can be represented by some element in $F^p\Omega^\bullet(M/S, F)$. This filtration is compatible with the flat connections of $H^\bullet$, so that for any p and k,

$$0 \longrightarrow F^{p+1}H^k \longrightarrow F^pH^k \longrightarrow F^pH^k/F^{p+1}H^k \longrightarrow 0 \tag{119}$$

is a parallel exact sequence of flat bundles. The corresponding flat connections will be respectively denoted by $\nabla_{F^{p+1}H^k}$, $\nabla_{F^pH^k}$ and $\nabla_{p/k}$.

It is proved in [**31**] Proposition 3.1 that the associated spectral sequence gives rise to flat vector bundles $(E_r^{p,q}, \nabla_r^{p,q})$ on S with flat (parallel) spectral sequence morphisms $d_r\colon E_r^{p,q} \longrightarrow E_r^{p-r,q+r+1}$ (and $E_{r+1}^{p,q} = \mathrm{Ker} d_r|_{E_r^{p,q}} / (\mathrm{Im} d_r \cap E_r^{p,q})$).

It is a classical fact (see [**31**] Theorem 2.1) that this spectral sequence is isomorphic to the Leray spectral sequence, and thus $E_2^{p,q} \cong H^p(B/S, G^q)$ while for all sufficiently great r one has $E_r^{p,q} \cong F^pH^{p+q}/F^{p+1}H^{p+q}$.

Put $E_r^+ = \bigoplus_{p+q \text{ even}} E_r^{p,q}$ and $E_r^- = \bigoplus_{p+q \text{ odd}} E_r^{p,q}$, and denote their direct sum (flat) connections by ∇_r^+ and ∇_r^-. Applying Lemma 42 to the complexes

$$\cdots \xrightarrow{d_r} E_r^{p+r,q-r-1} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p-r,q+r+1} \xrightarrow{d_r} \cdots \tag{120}$$

proves in particular that $[E_r^+, \nabla_r^+] - [E_r^-, \nabla_r^-] \in K^0_{\mathrm{flat}}(S)$ is independent of r.

For $r = 2$, this is nothing but $\pi_{2!}([G^+, \nabla_{G^+}] - [G^-, \nabla_{G^-}]) = \pi_{2!}\big(\pi_{1!}[F, \nabla_F]\big)$.

On the other hand, it follows from (119) that the element

$$[F^pH^\bullet, \nabla_{F^pH^\bullet}] + \sum_{i=0}^{p-1} [F^iH^\bullet/F^{i+1}H^\bullet, \nabla_{i/\bullet}] \in K^0_{\mathrm{flat}}(S)$$

is independent of p. For $p = 0$, it equals $[H^\bullet, \nabla_H] = (\pi_2 \circ \pi_1)_![F, \nabla_F]$, while for sufficienly great p and r, it equals $[E_r^+, \nabla_r^+] - [E_r^-, \nabla_r^-]$. Thus

Proposition. — *We have $\pi_{2!} \circ \pi_{1!} = (\pi_2 \circ \pi_1)_! \; : \; K^0_{\mathrm{flat}}(M) \longrightarrow K^0_{\mathrm{flat}}(S)$.*

7.2.2. *Compatibility of topological and sheaf theoretic links*

One has now two classes of links between $E_2^+ - E_2^- \cong \pi_{2!}\big(\pi_{1!}[F, \nabla_F]\big)$ and $H^+ - H^- \cong (\pi_2 \circ \pi_1)_![F, \nabla_F]$: the link $[\ell_{\mathrm{top}}]$ constructed in subSection 7.1.6 and the sheaf theoretic one $[\ell_{\mathrm{flat}}]$ obtained by combining links obtained using Definition 36 from (120) and (119). The geometric setting of adiabatic limit is here the same as in §7.1. The three triples (μ^+, μ^-, ψ), $(\zeta^+, \zeta^-, \varphi)$ and $(\lambda^+, \lambda^-, \phi)$ are taken trivial.

Proposition 64. — *We have* $[\ell_{\mathrm{top}}]=[\ell_{\mathrm{flat}}]$.

Proof. — *First step: Hodge theoretic version of the Leray spectral sequence.* — Such a theory was studied by various authors in various contexts [**33**] [**16**] [**5**] [**31**], the version corresponding to the situation here in explained in [**31**] §2 and §3. It can be summarized as follows: E_0 is nothing but $\mathscr{E}_{M/S}$ (see (107)) as global infinite rank vector bundle over S. Then there exists a nested sequence of vector subbundles $\widetilde{E}_r$ of $E_0=\widetilde{E}_0$ which are for all $r\geq 2$ of finite rank and endowed with canonical flat connections $\widetilde{\nabla}_r$. This sequence stabilizes for sufficiently great r. For any r, there is some canonical isomorphism $E_r\cong\widetilde{E}_r$ with the corresponding term of the Leray spectral sequence, for $r\geq 2$ it makes ∇_r and $\widetilde{\nabla}_r$ correspond to each other. All the $\widetilde{E}_r$ are naturally endowed with the restriction of the L^2 hermitian inner product on $\mathscr{E}_{M/S}$ (which needs here to be obtained from some riemannian submersion metric). Finally for any r, let $\widetilde{d}_r^*$ be the adjoint of the bundle endomorphism $\widetilde{d}_r$ corresponding to the operator d_r of the spectral sequence, and define $\widetilde{\mathscr{D}}_r=\widetilde{d}_r+\widetilde{d}_r^*$, then $\widetilde{E}_{r+1}:=\mathrm{Ker}\widetilde{\mathscr{D}}_r$.

For $r=0$, $d_0\cong d^{M/B}$, so that E_1 identifies through the Hodge theory of the fibres of π_1 with $\widetilde{E}_1=\Omega(B/S,\mathrm{Ker}\mathscr{D}^V)\cong\Omega(B/S,G^\bullet)$ in the notations of the preceding paragraph. Thus $\widetilde{E}_1$ identifies with vertical differential forms with values in $\pi_{1!}F$, where "vertical" is to understand with respect to the fibration π_2. Let $\widetilde{p}_1$ be the orthogonal projection of E_0 onto $\widetilde{E}_1$, then $\widetilde{d}_1=\widetilde{p}_1d^H$ acting on $\widetilde{E}_1$. It follows that $\widetilde{E}_2=\mathrm{Ker}(p_1\mathscr{D}^H|_{\widetilde{E}_1})$ identifies with vertical harmonic $G^\bullet$-valued differential forms, hence with $\pi_{2!}(\pi_{1!}F)$.

For any $r\geq 2$, $\widetilde{E}_r$ can be described as follows ([**31**] Proposition 2.1):

$$\begin{aligned}\widetilde{E}_r\ =\ \big\{s_0\in\mathscr{E}_{M/S}\ &\text{such that there exists } s_1,s_2,\dots,s_{r-1}\in\mathscr{E}_{M/S}\text{ verifying}\\ &\mathscr{D}^Vs_0=0,\ \mathscr{D}^Hs_0+\mathscr{D}^Vs_1=0\ \text{ and}\\ &(\iota_T+\iota_T^*)s_{i-2}+\mathscr{D}^Hs_{i-1}+\mathscr{D}^Vs_i=0\ \text{ for any } 2\leq i\leq r-1\big\}.\end{aligned}\tag{121}$$

Then in this description $\widetilde{\mathscr{D}}_rs_0=\widetilde{p}_r((\iota_T+\iota_T^*)s_{r-2}+\mathscr{D}^Hs_{r-1})$, where $\widetilde{p}_r$ is the orthogonal projection of $\mathscr{E}_{M/S}$ onto $\widetilde{E}_r$. One can then prove along the same lines as in [**5**] §VI (a) (especially formulae (6.13) and (6.15)) that

$$\begin{aligned}&d^{M/B}s_0=0,\ d^Hs_0+d^{M/B}s_1=0,\\ &\qquad\iota_Ts_{i-2}+d^Hs_{i-1}+d^{M/B}s_i=0\ \text{ for any } 2\leq i\leq r-1\\ &\qquad\quad\text{and}\ \ \widetilde{d}_rs_0=\widetilde{p}_r(\iota_Ts_{r-2}+d^Hs_{r-1}).\end{aligned}$$

Second step: convergence of harmonic forms. — Use now the convergence of the resolvent $\big(\lambda-\big(\frac{1}{\theta}\big)^{r-1}\mathscr{D}^\theta\big)^{-1}$ (here both μ and ψ vanish) to $\widetilde{p}_r(\lambda-\widetilde{\mathscr{D}}_r)^{-1}\widetilde{p}_r$ ([**31**] Theorem 2.2) for sufficiently large r. One can deduce that the orthogonal projection p_θ of $\mathscr{E}_{M/S}$ onto $\mathrm{Ker}\mathscr{D}^\theta$ converges at $\theta=0$ to $\widetilde{p}_r$. In other words $\mathrm{Ker}\mathscr{D}^\theta$ is the restriction to $S\times(0,1]$ of some vector bundle on $S\times[0,1]$ whose restriction to $S\times\{0\}$ is $\widetilde{E}_\infty$. There is a bigrading on $\mathscr{E}_{M/S}$, from (107) according to horizontal (i.e. corresponding to $\Omega^\bullet$) and vertical (corresponding to the grading of $\mathscr{E}_{M/B}$) degrees. $\widetilde{E}_\infty$ decomposes with respect to this bigrading [**5**] Theorem 6.1. Consider some $s_0\in\widetilde{E}_\infty^{p,q}$ and call $s_i^{p+i,q-i}$ for any i the corresponding component of the s_i introduced in (121). The above description of $\widetilde{d}_r$ proves that for any sufficiently large r the differential form $s_0+s_1^{p+1,q-1}+\dots s_r^{p+r,q-r}$ is closed. According to the scaling appearing in (109) the section $p_\theta(s_0+\theta s_1^{p+1,q-1}+\theta^2 s_2^{p+2,q-2}\dots\theta^r s_r^{p+r,q-r})$ is the rescaled harmonic form corresponding to some fixed cohomology class. Its convergence to s_0 at $\theta=0$ proves that the isomorphism between $\operatorname{Ker}\mathscr{D}^1$ ad $\widetilde{E}_\infty$ provided by the parallel transport along $[0,1]$ exactly corresponds to the isomorphism $[H^\bullet,\nabla_H]\cong[E_r^\bullet,\nabla_r]$ obtained at the end of §7.2.1 from (119).

Third step: eigenvalues converging to 0. — The convergence of the resolvent $\big(\lambda-(\frac{1}{\theta})^{r-1}\mathscr{D}^\theta\big)^{-1}$ to $\widetilde{p}_r(\lambda-\widetilde{\mathscr{D}}_r)^{-1}\widetilde{p}_r$ ([**31**] Theorem 2.2) for any r gives the following description of the vector bundle $\mathscr{K}$ of Theorem 62 over $S\times[0,\varepsilon_2]$: its restriction to $S\times\{\theta\}$ is the direct sum of eigenspaces of $\mathscr{D}_\theta$ corresponding to "little" modulus eigenvalues while its restriction to $S\times\{0\}$ is the direct sum of the $\widetilde{E}_r$, each $\widetilde{E}_r$ corresponding to eigenspaces associated to eigenvalues of order less than or equal to θ^{r-1}. For any positive θ, $(\mathscr{K},d^\theta)$ form a complex whose cohomology is $\mathscr{L}$. The convergence of the resolvents also prove that the operator $(\frac{1}{\theta})^{r-1}\mathscr{D}^\theta$ on the suitable eigensubspace converges to $\widetilde{\mathscr{D}}_r$, and accordingly for $(\frac{1}{\theta})^{r-1}d^\theta$ and $\widetilde{d}_r$.

By proceeding exactly as in §4.2.4, one obtains that the canonical class of links between $\mathscr{K}^\pm$ and $\mathscr{L}^\pm$ equals the canonical class of links associated by Definition 36 to the Leray spectral sequence ((120) for all r).

One may use the limit $t\to 0$ or $\varepsilon_1\to 0$ in the construction of ℓ_{top}. The two remaining components of the construction of ℓ_{top} (parallel transport along $[0,1]$ and $\ell_{\mathscr{K}}^{\mathscr{L}}$) were shown to be equal to the two components of $[\ell_{\mathrm{flat}}]$ (the links coming from filtration of cohomology and from the spectral sequence respectively). □

7.2.3. *Proof of Theorem 34.* — Consider some $(E,\nabla_E,F,\nabla_F,f)\in K^0_{\mathrm{rel}}(M)$, then $\pi_{2*}\circ\pi_{1*}(E,\nabla_E,F,\nabla_F,f)$ equals $\big(\pi_{2!}\circ\pi_{1!}(E,\nabla_E),\pi_{2!}\circ\pi_{1!}(F,\nabla_F),\pi_{2\ell}(\pi_{1\ell}([f]))\big)$ while $(\pi_2\circ\pi_1)_*(E,\nabla_E,F,\nabla_F,f)$ equals $\big((\pi_2\circ\pi_1)_!(E,\nabla_E),(\pi_2\circ\pi_1)_!(F,\nabla_F),(\pi_2\circ\pi_1)_\ell([f])\big)$.

Consider the pull-back $\widetilde{E}$ of E to $M\times[0,1]$ with some connection $\widetilde{\nabla}$ whose restrictions on $M\times\{0\}$ and $M\times\{1\}$ respectively equal ∇_E and $f^*\nabla_F$. There is a canonical (topological) class of link $[\widetilde{\ell}]$ between one-step and two-step direct images of $\widetilde{E}$ whose restrictions to $M\times\{0\}$ and $M\times\{1\}$ coincide with $[\ell^E_{\mathrm{top}}]$ and $[\ell^F_{\mathrm{top}}]$ (with obvious notations from the preceding subsection, this is because of the naturality of $[\ell_{\mathrm{top}}]$). Now $\pi_{2\ell}(\pi_{1\ell}([f]))$ and $(\pi_2\circ\pi_1)_\ell([f])$ both correspond to the parallel transport along

$[0,1]$. Thus

$$\begin{aligned}(122)\qquad \pi_{2*}\circ\pi_{1*}(E,\nabla_E,F,\nabla_F,f)-(\pi_2\circ\pi_1)_*(E,\nabla_E,F,\nabla_F,f)=\\ =\big(\pi_{2!}\circ\pi_{1!}(E,\nabla_E),(\pi_2\circ\pi_1)_!(E,\nabla_E),[\ell^E_{\mathrm{top}}]\big)\\ -\big(\pi_{2!}\circ\pi_{1!}(F,\nabla_F),(\pi_2\circ\pi_1)_!(F,\nabla_F),[\ell^F_{\mathrm{top}}]\big).\end{aligned}$$

But in both cases $[\ell_{\mathrm{top}}]=[\ell_{\mathrm{flat}}]$ and ℓ_{flat} is only obtained from parallel complexes of flat vector bundles (from either (120) or (119)). It follows from Lemma 42 that both terms in the right hand side of (122) vanish and this proves the theorem. The case of noncompact S follows directly from the fact that links of the form $[\ell_{\mathrm{flat}}]$ are globally defined.

7.3. Multiplicative and smooth K-theory

7.3.1. *Calculation of* $\pi^{\mathrm{Eu}}_{2!}\circ\pi^{\mathrm{Eu}}_{1!}-(\pi_2\circ\pi_1)^{\mathrm{Eu}}_!$. — Consider the vector bundles ξ on M, F^+ and F^- on B and G^+ and G^- on S (with connections ∇_ξ, ∇_{F^+}, ∇_{F^-}, ∇_{G^+} and ∇_{G^-}) such that

$$[F^+]-[F^-]=\pi^{\mathrm{Eu}}_{1*}[\xi]\in K^0_{\mathrm{top}}(B)\quad\text{ and }\quad [G^+]-[G^-]=(\pi_2\circ\pi_1)^{\mathrm{Eu}}_*[\xi]\in K^0_{\mathrm{top}}(S).$$

Choose some smooth supplementary subbundle T^HM/S of TM/S in TM, such that $T^HM/S\cap TM/B=T^HM$; then $\pi_{1*}T^HM/S$ is a smooth supplementary subbundle of TB/S in TB. One can define connections $\nabla_{TM/B}$, $\nabla_{TM/S}$ and $\nabla_{TB/S}$ on TM/B, TM/S and TB/S as at the beginning of the proof Lemma 56 from the choices of horizontal subspaces T^HM, T^HM/S and $\pi_{1*}T^HM/S$ respectively. Let $[\ell_F]$ and $[\ell_G]$ be equivalence classes of links between either F^+-F^- or G^+-G^- and couples of family index bundles (as in Definition 50), and denote $\eta_1=\eta(\nabla_\xi,\nabla_{TM/B},\nabla_{F^+},\nabla_{F^-},[\ell_F])$ and $\eta_{12}=\eta(\nabla_\xi,\nabla_{TM/S},\nabla_{G^+},\nabla_{G^-},[\ell_G])$:

$$\begin{aligned}\pi^{\mathrm{Eu}}_{1!}(\xi,\nabla_\xi,\alpha)=\left(F^+,\nabla_{F^+},\int_{M/B}e(\nabla_{TM/B})\alpha\right)-(F^-,\nabla_{F^-},\eta_1)\in\widehat{K}_{\mathrm{ch}}(B)\\ \text{and}\quad(\pi_2\circ\pi_1)^{\mathrm{Eu}}_!(\xi,\nabla_\xi,\alpha)=\\ =\left(G^+,\nabla_{G^+},\int_{M/S}e(\nabla_{TM/S})\alpha\right)-(G^-,\nabla_{G^-},\eta_{12})\in\widehat{K}_{\mathrm{ch}}(S).\end{aligned}$$

Take vector bundles H^{++}, H^{+-}, H^{-+} and H^{--} on S (with connections ∇_{++}, ∇_{+-}, ∇_{-+} and ∇_{--}) such that $\pi^{\mathrm{Eu}}_{2*}[F^\pm]=[H^{\pm+}]-[H^{\pm-}]\in K^0_{\mathrm{top}}(S)$. Consider some classes of links $[\ell_+]$ and $[\ell_-]$ between $H^{\pm+}-H^{\pm-}$ and couples of families index bundles and denote by $\eta_\pm$ the forms $\eta(\nabla_{F^\pm}, \nabla_{TB/S}, \nabla_{\pm+}, \nabla_{\pm-}, [\ell_\pm])$:

$$\pi_{2!}^{\mathrm{Eu}}\big(\pi_{1!}^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha)\big) = \left(H^{++}, \nabla_{++}, \int_{B/S} e(\nabla_{TB/S}) \int_{M/B} e(\nabla_{TM/B})\alpha\right)$$
$$-(H^{+-}, \nabla_{+-}, \eta_+) - \left(H^{-+}, \nabla_{-+}, \int_{B/S} e(\nabla_{TB/S})\eta_1\right) + (H^{--}, \nabla_{--}, \eta_-).$$

Now $G^+ - G^-$ and $(H^{++} \oplus H^{--}) - (H^{+-} \oplus H^{-+})$ are linked through $[\ell_G]$, $[\ell_+]$, $[\ell_-]$ and the construction of §7.1.6. Call $[\ell_{\mathrm{top}}]$ the resulting link and $\widetilde{\mathrm{ch}}([\ell_{\mathrm{top}}])$ the associated Chern-Simons form as in §5.3.1, then

$$\pi_{2!}^{\mathrm{Eu}}\big(\pi_{1!}^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha)\big) = \left(G^+, \nabla_{G^+}, \int_{M/S} \pi_1^*\big(e(\nabla_{TB/S})\big)e(\nabla_{TM/B})\alpha\right)$$
$$-\left(G^-, \nabla_{G^-}, \eta_+ - \eta_- - \widetilde{\mathrm{ch}}([\ell_{\mathrm{top}}]) + \int_{B/S} e(\nabla_{TB/S})\eta_1\right).$$

Choose any supplementary subbundle of T^HM in T^HM/S, it then identifies with π_1^*TB/S and is endowed with the connection $\pi_1^*\nabla_{TB/S}$. Denote by $\widetilde{e}_{M/B/S}$ the form $\widetilde{e}(\nabla_{TM/S}, \nabla_{TM/B} \oplus \pi_1^*\nabla_{!TB/S})$ defined in §89, then the following form

$$\widetilde{e}_{M/B/S}d\alpha + \big(e(\nabla_{TM/S}) - \pi_1^*\big(e(\nabla_{TB/S})\big)e(\nabla_{TM/B})\big)\alpha$$

is exact so that in $\widehat{K}_{\mathrm{ch}}(S)$:

$$\pi_{2!}^{\mathrm{Eu}}\big(\pi_{1!}^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha)\big) = \left(G^+, \nabla_{G^+}, \int_{M/S} e(\nabla_{TM/S})\alpha\right) - (G^-, \nabla_{G^-}, \widetilde{\eta}_{12})$$

with $$\widetilde{\eta}_{12} = \eta_+ - \eta_- - \widetilde{\mathrm{ch}}([\ell_{\mathrm{top}}]) + \int_{B/S} e(\nabla_{TB/S})\eta_1 - \int_{M/S} \widetilde{e}_{M/B/S}d\alpha.$$

For any $(\xi, \nabla_\xi, \alpha) \in MK_0(M)$, one has $d\alpha = \mathrm{ch}(\nabla_\xi) - \mathrm{rk}\xi$ but for degree reasons $\int_{M/S} \widetilde{e}_{M/B/S}$ vanishes (the degree of this form equals $\dim M - \dim S - 1$). Thus

Proposition 65. — $\big(\pi_{2!}^{\mathrm{Eu}} \circ \pi_{1!}^{\mathrm{Eu}} - (\pi_2 \circ \pi_1)_!^{\mathrm{Eu}}\big)(\xi, \nabla_\xi, \alpha) = a(\eta_{12} - \widetilde{\widetilde{\eta}}_{12})$ *where*

$$\widetilde{\widetilde{\eta}}_{12} = \eta_+ - \eta_- - \widetilde{\mathrm{ch}}([\ell_{\mathrm{top}}]) + \int_{B/S} e(\nabla_{TB/S})\eta_1 + \int_{M/S} \widetilde{e}_{M/B/S}\mathrm{ch}(\nabla_\xi).$$

An argument similar to just before Lemma 60 yields that this equality also holds true in the case of noncompact S.

7.3.2. *Proof of Theorem 35.* — It is easily verified that $\widetilde{\widetilde{\eta}}_{12}$ is additive in the sense of property (c) of Theorem 28, is functorial by pullbacks over fibered products (with double fibration structure!); a direct calculation proves that it verifies the same transgression formula (property (a) of Theorem 28) as η_{12}. In the case of a flat bundle (ξ, ∇_ξ), $F^\pm$ here correspond to $G^\pm$ in §7.2.1, $G^\pm$ here correspond to $H^\pm$ of §7.2.1, and $H^{\pm\pm}$ here correspond to $E_2^{\pm\pm}$ of §7.2.1: in any case, the suitable data are taken trivial because all bundles are flat, and thus all the forms η_+, η_- and η_1 vanish (property (d) of Theorem 28). Finally $\mathrm{ch}(\nabla_\xi) = \mathrm{rk}\xi$ so that the integral involving $\widetilde{e}_{M/B/S}$ vanishes, and $\widetilde{\mathrm{ch}}([\ell_{\mathrm{top}}])$ also vanishes, because of Proposition 64 and Lemma 1 ($[\ell_{\mathrm{flat}}]$ of Proposition 64 is obtained by using Definition 36 from parallel exact sequences of flat bundles).

The coincidence of η_{12} and $\widetilde{\widetilde{\eta}}_{12}$ for elements of $MK_0(M)$ is then obtained from the second statement of Theorem 28.

Remark. — It is likely that $\widetilde{\widetilde{\eta}}_{12} = \eta_{12}$ in any case, so that one would have

$$(\pi_2 \circ \pi_1)_!^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha) - \pi_{2!}^{\mathrm{Eu}}\big(\pi_{1!}^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha)\big) = a\left(\int_{M/S} \widetilde{e}_{M/B/S}\, \dddot{\mathrm{ch}}(\xi, \nabla_\xi, \alpha)\right)$$

for any $(\xi, \nabla_\xi, \alpha) \in \widehat{K}_{\mathrm{ch}}(M)$. This formula would be compatible with Theorem 28 and with anomaly formulae (91) and (92). A corresponding result is proved in [**14**], where the above discrepancy is compensated by a suitable composition of smooth K-orientations in the double fibration.

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Astérisque
327, 2009, p. 361–424

HERMITIAN VECTOR BUNDLES AND EXTENSION GROUPS ON ARITHMETIC SCHEMES II. THE ARITHMETIC ATIYAH EXTENSION

by

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Pour Jean-Michel Bismut

Abstract. — In a previous paper, we have defined arithmetic extension groups in the context of Arakelov geometry. In the present one, we introduce an arithmetic analogue of the Atiyah extension that defines an element — the arithmetic Atiyah class — in a suitable arithmetic extension group. Namely, if $\overline{E}$ is a hermitian vector bundle on an arithmetic scheme X, its arithmetic Atiyah class $\widehat{\mathrm{at}}_{X/\mathbb{Z}}(\overline{E})$ lies in the group $\widehat{\mathrm{Ext}}^1_X(E, E\otimes\Omega^1_{X/\mathbb{Z}})$, and is an obstruction to the algebraicity over X of the unitary connection on the vector bundle $E_{\mathbb{C}}$ over the complex manifold $X(\mathbb{C})$ that is compatible with its holomorphic structure.

In the first sections of this article, we develop the basic properties of the arithmetic Atiyah class which can be used to define characteristic classes in arithmetic Hodge cohomology.

Then we study the vanishing of the first Chern class $\hat{c}_1^H(\overline{L})$ of a hermitian line bundle $\overline{L}$ in the arithmetic Hodge cohomology group $\widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X, \Omega^1_{X/\mathbb{Z}})$. This may be translated into a concrete problem of diophantine geometry, concerning rational points of the universal vector extension of the Picard variety of X. We investigate this problem, which was already considered and solved in some cases by Bertrand, by using a classical transcendence result of Schneider-Lang, and we derive a finiteness result for the kernel of $\hat{c}_1^H$.

In the final section, we consider a geometric analog of our arithmetic situation, namely a smooth, projective variety X which is fibered on a curve C defined over some field k of characteristic zero. To any line bundle L over X is attached its relative Atiyah class $\mathrm{at}_{X/C}L$ in $H^1(X, \Omega^1_{X/C})$. We describe precisely when $\mathrm{at}_{X/C}L$ vanishes. In particular, when the fixed part of the relative Picard variety of X over C is trivial, this holds iff some positive power of L descends to a line bundle over C.

2010 Mathematics Subject Classification. — 14G40; 11J95, 14F05, 32L10.
Key words and phrases. — Arakelov geometry, hermitian vector bundles, extension groups, Atiyah class, transcendence and algebraic groups.

Résumé **(Fibrés vectoriels hermitiens et groupes d'extensions sur les schémas arithmétiques II. La classe d'Atiyah arithmétique)**

Dans un précédent article, nous avons défini des groupes d'extensions arithmétiques dans le contexte de la géométrie d'Arakelov. Dans le présent travail, nous introduisons un analogue arithmétique de l'extension d'Atiyah; sa classe dans un groupe d'extensions arithmétiques convenable définit la classe d'Atiyah arithmétique. Plus précisément, pour tout fibré vectoriel hermitien $\overline{E}$ sur un schéma arithmétique X, sa classe d'Atiyah arithmétique $\widehat{\mathrm{at}}_{X/\mathbb{Z}}(\overline{E})$ appartient au groupe $\widehat{\mathrm{Ext}}^1_X(E, E \otimes \Omega^1_{X/\mathbb{Z}})$ et constitue une obstruction à l'algébricité sur X de l'unique connection unitaire sur la fibré vectoriel $E_{\mathbb{C}}$ sur la variété complexe $X(\mathbb{C})$ qui soit compatible avec sa structure holomorphe.

Dans les premières sections de cet article, nous présentons la construction et les propriétés de base de la classe d'Atiyah, qui permettent notamment de définir des classes caractéristiques en cohomologie de Hodge arithmétique.

Nous étudions ensuite l'annulation de la première classe de Chern $\hat{c}_1^H(\overline{L})$ d'un fibré en droites hermitien $\overline{L}$ dans le groupe de cohomologie de Hodge arithmétique $\widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X, \Omega^1_{X/\mathbb{Z}})$. La détermination de tels fibrés en droites hermitiens se traduit en une question de géométrie diophantienne, concernant les points rationnels de l'extension vectorielle universelle de la variété de Picard de X. Nous étudions ce problème — qui a déjà été considéré, et résolu dans certains cas, par Bertrand — au moyen d'un classique résultat de transcendance dû à Schneider et Lang, et nous en déduisons un théorème de finitude sur le noyau de $\hat{c}_1^H$.

Dans la dernière section, nous étudions un analogue géométrique de la situation arithmétique précédente. A savoir, nous considérons une variété projective lisse X fibrée sur une courbe C, au dessus d'un corps de base k de caractéristique nulle et nous attachons à tout fibré en droites L sur X sa classe d'Atiyah relative $\mathrm{at}_{X/C}L$ dans $H^1(X, \Omega^1_{X/C})$. Nous déterminons quand cette classe $\mathrm{at}_{X/C}L$ s'annule. Notamment, lorsque la variété de Picard relative de X sur C n'a pas de partie fixe, cela se produit précisément lorsque une puissance non-nulle de L descend en un fibré en droites sur C.

0. Introduction

0.1. — This paper is a sequel to [**7**], where we have defined and investigated arithmetic extensions on arithmetic schemes, and the groups they define.

Recall that if X is a scheme over $\operatorname{Spec}\mathbb{Z}$, separated of finite type, whose generic fiber $X_{\mathbb{Q}}$ is smooth, then an arithmetic extension of vector bundles over X is the data $(\mathcal{E}, s)$ of a short exact sequence of vector bundles (that is, of locally free coherent sheaves of $\mathcal{O}_X$-modules) on the scheme X,

$$\mathcal{E} : 0 \longrightarrow G \xrightarrow{i} E \xrightarrow{p} F \longrightarrow 0, \tag{0.1}$$

and of a $\mathcal{C}^\infty$-splitting

$$s : F_{\mathbb{C}} \longrightarrow E_{\mathbb{C}},$$

invariant under complex conjugation, of the extension of $\mathcal{C}^\infty$-complex vector bundles on the complex manifold $X(\mathbb{C})$

$$\mathcal{E}_{\mathbb{C}} : 0 \longrightarrow G_{\mathbb{C}} \xrightarrow{i_{\mathbb{C}}} E_{\mathbb{C}} \xrightarrow{p_{\mathbb{C}}} F_{\mathbb{C}} \longrightarrow 0$$

that is deduced from $\mathcal{E}$ by the base change from $\mathbb{Z}$ to $\mathbb{C}$ and analytification.

For any two given vector bundles F and G over X, the isomorphism classes of the so-defined arithmetic extensions of F by G constitute a set $\widehat{\operatorname{Ext}}^1_X(F,G)$ that becomes an abelian group when equipped with the addition law defined by a variant of the classical construction of the Baer sum of 1-extensions of (sheaves of) modules[(1)].

Recall that a hermitian vector bundle $\overline{E}$ over X is a pair $(E, \|.\|)$ consisting of a vector bundle E over X and of a $\mathcal{C}^\infty$-hermitian metric, invariant under complex conjugation, on the holomorphic vector bundle $E_\mathbb{C}$ over $X(\mathbb{C})$. Examples of arithmetic extensions in the above sense are provided by admissible extensions

$$\overline{\mathcal{E}} : 0 \longrightarrow \overline{G} \xrightarrow{i} \overline{E} \xrightarrow{p} \overline{F} \longrightarrow 0 \tag{0.2}$$

of hermitian vector bundles over X, namely from the data of an extension

$$\mathcal{E} : 0 \longrightarrow G \xrightarrow{i} E \xrightarrow{p} F \longrightarrow 0$$

of the underlying $\mathcal{O}_X$-modules such that the hermitian metrics $\|.\|_{\overline{G}}$ and $\|.\|_{\overline{F}}$ on $G_\mathbb{C}$ and $F_\mathbb{C}$ are induced (by restriction and quotients) by the metric $\|.\|_{\overline{E}}$ on $E_\mathbb{C}$ (by means of the morphisms $i_\mathbb{C}$ and $p_\mathbb{C}$). Indeed, to any such admissible extension is naturally attached its orthogonal splitting, namely the $\mathcal{C}^\infty$-splitting

$$s_{\overline{\mathcal{E}}} : F_\mathbb{C} \longrightarrow E_\mathbb{C}$$

that maps $F_\mathbb{C}$ isomorphically onto the orthogonal complement $i_\mathbb{C}(G_\mathbb{C})^\perp$ of the image of $i_\mathbb{C}$ in $E_\mathbb{C}$. This splitting is invariant under complex conjugation, and $(\mathcal{E}, s_{\overline{\mathcal{E}}})$ is an arithmetic extension of F by G. For any two hermitian vector bundles $\overline{F}$ and $\overline{G}$ over X, this construction establishes a bijection from the set of isomorphism classes of admissible extension of the form (0.2) to the set $\widehat{\operatorname{Ext}}^1_X(F,G)$.

In **[7]** we studied basic properties of the so-defined arithmetic extension groups. In particular, we introduced the following natural morphisms of abelian groups:

– the “forgetful" morphism

$$\nu : \widehat{\operatorname{Ext}}^1_X(F,G) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_X}(F,G),$$

which maps the class of an arithmetic extension $(\mathcal{E}, s)$ to the one of the underlying extension $\mathcal{E}$ of $\mathcal{O}_X$-modules;

(1) Consider indeed two arithmetic extensions of F by G, say $\overline{\mathcal{E}}_\alpha := (\mathcal{E}_\alpha, s_\alpha)$, $\alpha = 1, 2$, defined by extensions of vector bundles $\mathcal{E}_\alpha : 0 \to G \xrightarrow{i_\alpha} E_\alpha \xrightarrow{p_\alpha} F \to 0$ and $\mathcal{C}^\infty$-splittings $s_\alpha : F_\mathbb{C} \to E_{\alpha,\mathbb{C}}$. We may define a vector bundle $E := \frac{\operatorname{Ker}(p_1 - p_2 : E_1 \oplus E_2 \to F)}{\operatorname{Im}((i_1, -i_2) : G \to E_1 \oplus E_2)}$ over X. The Baer sum of $\overline{\mathcal{E}}_1$ and $\overline{\mathcal{E}}_2$ is the arithmetic extension $\overline{\mathcal{E}}$ defined by the usual Baer sum of $\mathcal{E}_1$ and $\mathcal{E}_2$ — namely $\mathcal{E} : 0 \to G \xrightarrow{i} E \xrightarrow{p} F \to 0$ where the morphisms $i : G \to E$ and $p : E \to F$ are defined by $p([(g_1, g_2)]) := p_1(f_1) = p_2(f_2)$ and $i(g) := [(i_1(g), 0)] = [(0, i_2(g))]$ — equipped with the $\mathcal{C}^\infty$-splitting $s : F_\mathbb{C} \to E_\mathbb{C}$ defined by $s(e) := [(s_1(e), s_2(e))]$.

– the morphism

$$b : \mathrm{Hom}_{\mathscr{C}^\infty_{X(\mathbb{C})}}(F_\mathbb{C}, G_\mathbb{C})^{F_\infty} \longrightarrow \widehat{\mathrm{Ext}}^1_X(F, G),$$

defined on the real vector space $\mathrm{Hom}_{\mathscr{C}^\infty_{X(\mathbb{C})}}(F_\mathbb{C}, G_\mathbb{C})^{F_\infty}$ of $\mathscr{C}^\infty$-morphisms of vector bundles over $X(\mathbb{C})$ from $F_\mathbb{C}$ to $G_\mathbb{C}$, invariant under complex conjugation; it sends an element T in this space to the class of the arithmetic extension $(\mathscr{E}, s)$ where $\mathscr{E}$ is the trivial algebraic extension, defined by (0.1) with $E := G \oplus F$ and i and p the obvious injection and projection morphisms, and where s is given by $s(f) = (T(f), f)$;

– the morphism

$$\iota : \mathrm{Hom}_{\mathscr{O}_X}(F, G) \longrightarrow \mathrm{Hom}_{\mathscr{C}^\infty_{X(\mathbb{C})}}(F_\mathbb{C}, G_\mathbb{C})^{F_\infty}$$

which sends a morphism $\varphi : F \to G$ of vector bundles over X to the morphism of $\mathscr{C}^\infty$-complex vector bundles $\varphi_\mathbb{C} : F_\mathbb{C} \to G_\mathbb{C}$ deduced from φ by base change from $\mathbb{Z}$ to $\mathbb{C}$ and analytification;

– the morphism

$$\Psi : \widehat{\mathrm{Ext}}^1_X(F, G) \longrightarrow Z^{0,1}_{\overline{\partial}}(X_\mathbb{R}, F^\vee \otimes G),$$

that takes values in the real vector space

$$Z^{0,1}_{\overline{\partial}}(X_\mathbb{R}, F^\vee \otimes G) := Z^{0,1}_{\overline{\partial}}(X(\mathbb{C}), F^\vee_\mathbb{C} \otimes G_\mathbb{C})^{F_\infty}$$

of $\overline{\partial}$-closed forms of type $(0, 1)$ on $X(\mathbb{C})$ with coefficients in $F^\vee_\mathbb{C} \otimes G_\mathbb{C}$, invariant under complex conjugation. It maps the class of an arithmetic extension $(\mathscr{E}, s)$ to its "second fundamental form" $\Psi(\mathscr{E}, s)$ defined by

$$i_\mathbb{C} \circ \Psi(\mathscr{E}, s) = \overline{\partial}_{F^\vee_\mathbb{C} \otimes G_\mathbb{C}}(s).$$

We also established the following basic exact sequence:

$$\mathrm{Hom}_{\mathscr{O}_X}(F, G) \xrightarrow{\iota} \mathrm{Hom}_{\mathscr{C}^\infty_{X(\mathbb{C})}}(F_\mathbb{C}, G_\mathbb{C})^{F_\infty} \xrightarrow{b} \widehat{\mathrm{Ext}}^1_X(F, G) \xrightarrow{\nu} \mathrm{Ext}^1_{\mathscr{O}_X}(F, G) \to 0, \tag{0.3}$$

which displays the arithmetic extension group $\widehat{\mathrm{Ext}}^1_X(F, G)$ as an extension of the "classical" extension group $\mathrm{Ext}^1_{\mathscr{O}_X}(F, G)$ by a group of analytic type.

The sequel of [**7**] was devoted to the study of the groups $\widehat{\mathrm{Ext}}^1_X(F, G)$ when the base scheme is an arithmetic curve, that is, the spectrum $\mathrm{Spec}\, \mathscr{O}_K$ of the ring of integers of some number field K. In this special case, these extension groups appear as natural tools in geometry of numbers and reduction theory in their modern guise, namely the study of hermitian vector bundles over arithmetic curves and their admissible extensions.

In the present paper, we focus on a natural construction of arithmetic extensions attached to hermitian vector bundles over an arithmetic scheme X as above, their *arithmetic Atiyah extensions.* In contrast with the arithmetic extensions over arithmetic curves investigated in **[7]**, for which the interpretation as admissible extensions was crucial, the arithmetic Atiyah extensions are genuine examples of arithmetic extensions constructed as pairs $(\mathcal{E}, s)$ — where s is a $\mathcal{C}^\infty$-splitting of an extension $\mathcal{E}$ of vector bundles over X — and not derived from an admissible extension.

0.2. — Atiyah extensions of vector bundles were initially introduced by Atiyah **[2]** in the context of complex analytic geometry.

Namely, for any holomorphic vector bundle E over a complex manifold X, Atiyah introduces the holomorphic vector bundle $P^1_X(E)$ of jets of order one of sections of E, whose fiber $P^1_X(E)_x$ at a point x of X is by definition the space of sections of E over the first order thickening $x_1 := \operatorname{Spec} \mathcal{O}_{X,x}/\mathfrak{m}_x^2$ of x in X. Here, as usual, $\mathcal{O}_X$ denotes the sheaf of holomorphic functions over X, and $\mathfrak{m}_x$ the maximal ideal of its stalk $\mathcal{O}_{X,x}$ at x.

The vector bundle $P^1_X(E)$ fits into a short exact sequence of holomorphic vector bundles

$$\mathcal{A}t_X E : 0 \longrightarrow E \otimes \Omega^1_X \xrightarrow{i} P^1_X(E) \xrightarrow{p} E \longrightarrow 0, \tag{0.4}$$

where the morphisms i and p are defined as follows: for any point x in X, the map $i_x : E_x \otimes \Omega^1_{X,x} \to P^1_X(E)_x$ maps an element v in $E_x \otimes \Omega^1_{X,x} \simeq \operatorname{Hom}_{\mathbb{C}}(T_{X,x}, E_x)$ to the section of E over x_1 that vanishes at x and admits v as differential, while the map $p_x : P^1_X(E)_x \to E_x$ is simply the evaluation at x.

The Atiyah extension of E is precisely the extension $\mathcal{A}t_X E$ of E by $E \otimes \Omega^1_X$ so-defined. According to its very definition, its class $\mathrm{at}_X E$ in the group $\operatorname{Ext}^1_{\mathcal{O}_X}(E, E \otimes \Omega^1_X)$ which classifies extensions of holomorphic vector bundles of E by $E \otimes \Omega^1_X$ is the obstruction to the existence of a holomorphic connection

$$\nabla : E \longrightarrow E \otimes \Omega^1_X$$

on the vector bundle E.

The point of Atiyah's article **[2]** is that the class $\mathrm{at}_X E$ also leads to a straightforward construction of characteristic classes of E with values in the so-called Hodge cohomology groups of X

$$H^{p,p}(X) := H^p(X, \Omega^p_X). \tag{0.5}$$

For instance, Atiyah defines a first Chern class $c_1^H(E)$ in $H^{1,1}(X) = H^1(X, \Omega^1_X)$ as the image of $\mathrm{at}_X E$ by the morphism

$$\begin{array}{ccccc}
\operatorname{Ext}^1_{\mathcal{O}_X}(E, E \otimes \Omega^1_X) & \simeq & \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{E}nd\, E \otimes \Omega^1_X) & & \\
 & & \downarrow {\scriptstyle (\mathrm{Tr}_E \otimes \mathrm{id}_{\Omega^1_X}) \circ _} & & \\
 & & \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \Omega^1_X) & \simeq & H^1(X, \Omega^1_X)
\end{array}$$

deduced from the canonical trace morphism

$$\begin{array}{rrcl} \mathrm{Tr}_E : & \mathscr{E}nd\, E \simeq E^\vee \otimes E & \longrightarrow & \mathscr{O}_X, \\ & \lambda \otimes v & \mapsto & \lambda(v). \end{array}$$

Higher degree characteristic classes are constructed by means of the successive powers $(\mathrm{at}_X E)^p$ in $\mathrm{Ext}^p_{\mathscr{O}_X}(\mathscr{O}_X, (\mathscr{E}nd\, E)^{\otimes p} \otimes \Omega^p_X)$, where p denotes a positive integer. For instance, the p-th Segre class, associated to the p-th Newton polynomial $X_1^p + \cdots + X^p_{\mathrm{rk}\, E}$, may be constructed in the Hodge cohomology group $H^p(X, \Omega^p_X)$ as

$$s_p^H(E) := (\mathrm{Tr}^p_E \otimes \mathrm{id}_{\Omega^p_X}) \circ (\mathrm{at}_X E)^p,$$

where

$$\begin{array}{rrcl} \mathrm{Tr}^p_E : & (\mathscr{E}nd\, E)^{\otimes p} & \longrightarrow & \mathscr{O}_X, \\ & T_1 \otimes \cdots \otimes T_p & \mapsto & \mathrm{Tr}_E(T_1 \dots T_p). \end{array}$$

When the manifold X is compact and Kähler (*e.g.*, projective), the Hodge cohomology group $H^p(X, \Omega^p_X)$ may be identified with a subspace of the complex de Rham cohomology group $H^{2p}_{\mathrm{dR}}(X, \mathbb{C})$ of X, and Atiyah's construction of characteristic classes is compatible (up to normalization) to classical topological constructions.

The definition of the Atiyah class and the construction of the associated characteristic classes obviously make sense in a purely algebraic context, say over a base field k of characteristic zero. If X is a smooth algebraic scheme over k, for any vector bundle E over X, its Atiyah class $\mathrm{at}_{X/k}E$ is constructed as above, *mutatis mutandis*, as an element of the k-vector space $\mathrm{Ext}^1_{\mathscr{O}_X}(E, E \otimes \Omega^1_{X/k})$, and the associated characteristic classes are elements of the Hodge cohomology groups of X defined similarly to (0.5), but now using the Zariski topology of X instead of the analytic one, and the sheaf of Kähler differentials $\Omega^p_{X/k}$ instead of the holomorphic differential forms Ω^p_X.

These constructions are especially suited to smooth algebraic schemes X that are proper over k. In this case, the "Hodge to de Rham" spectral sequence degenerates, and the Hodge group $H^{p,p}(X)$ gets identified to a subquotient of the Hodge filtration of the algebraic de Rham cohomology group $H^{2p}_{\mathrm{dR}}(X/k) := H^{2p}(X, \Omega^{\cdot}_{X/k})$. Moreover, when X is proper over $k = \mathbb{C}$, this algebraic construction is compatible with the previous analytic one, as a consequence of the GAGA principle.

This algebraic version of Atiyah's constructions has been considerably extended by Illusie [**25**]. Instead of a smooth algebraic scheme over a field k, he considers a suitable morphism of ringed topoi $f : X \to S$, and associates Atiyah classes and characteristic classes to perfect complexes of sheaves of $\mathscr{O}_X$-modules; his definitions involve the cotangent complex $\mathbb{L}^{\cdot}_{X/S}$ of X over S, which in this general setting plays the role of the sheaf $\Omega^1_{X/k}$ attached to a smooth scheme X over the field k. Let us also mention the presentation of this "algebraic" theory and of some of its developments in the monograph of Angéniol and Lejeune-Jalabert [**1**], and the analytic construction of Buchweitz and Flenner [**8**], [**9**][(2)].

0.3. — Let us briefly describe our construction of arithmetic Atiyah classes.

Let $\overline{E} := (E, \|.\|_E)$ be a hermitian vector bundle over a scheme X which is separated and of finite type over $\mathbb{Z}$, and which for simplicity will be assumed smooth over $\mathbb{Z}$ in this introduction. The relative version of the exact sequence (0.4) defines the Atiyah extension of E over $\mathbb{Z}$:

$$\mathcal{A}t_{X/\mathbb{Z}}E : 0 \longrightarrow E \otimes \Omega^1_{X/\mathbb{Z}} \xrightarrow{i} P^1_{X/\mathbb{Z}}(E) \xrightarrow{p} E \longrightarrow 0. \tag{0.6}$$

Besides, according to a classical result of Chern and Nakano ([**10, 36**]), the holomorphic vector bundle $E^{\mathrm{hol}}_{\mathbb{C}}$ over the complex manifold $X(\mathbb{C})$, seen as $\mathcal{C}^\infty$-vector bundle, admits a unique connection $\nabla_{\overline{E}}$ that is unitary with respect to the hermitian metric $\|.\|_E$, and moreover is compatible with its holomorphic structure in the sense that its component $\nabla^{0,1}_{\overline{E}}$ of type $(0,1)$ coincides with the $\overline{\partial}$-operator $\overline{\partial}_{E_{\mathbb{C}}}$ with coefficients in the holomorphic vector bundle $E^{\mathrm{hol}}_{\mathbb{C}}$. The component $\nabla^{1,0}_{\overline{E}}$ of type $(1,0)$ of $\nabla_{\overline{E}}$ defines a $\mathcal{C}^\infty$-splitting $s_{\overline{E}}$ of the Atiyah extension of the holomorphic vector bundle $E^{\mathrm{hol}}_{\mathbb{C}}$:

$$\mathcal{A}t_{X(\mathbb{C})}E_{\mathbb{C}} : 0 \longrightarrow \Omega^1_{X(\mathbb{C})} \otimes E_{\mathbb{C}} \xrightarrow{i_{\mathbb{C}}} P^1_{X(\mathbb{C})}(E_{\mathbb{C}}) \xrightarrow{p_{\mathbb{C}}} E_{\mathbb{C}} \longrightarrow 0.$$

Namely, for any point x in $\mathcal{X}(\mathbb{C})$ and any e in E_x, $s_{\overline{E}}(e)$ is the section of E over x_1 that takes the value e at x and is killed by $\nabla^{1,0}_{\overline{E}}$.

Since the above analytic Atiyah extension $\mathcal{A}t_{X(\mathbb{C})}E_{\mathbb{C}}$ is precisely the extension deduced from $\mathcal{A}t_{X/\mathbb{Z}}E$ by the base change from $\mathbb{Z}$ to $\mathbb{C}$ and analytification, the pair $(\mathcal{A}t_{X/\mathbb{Z}}E, s_{\overline{E}})$ defines an arithmetic extension, the *arithmetic Atiyah extension* $\widehat{\mathcal{A}t}_{X/\mathbb{Z}}\overline{E}$ of the hermitian vector bundle $\overline{E}$. Its class $\widehat{\mathrm{at}}_{X/\mathbb{Z}}\overline{E}$ in $\widehat{\mathrm{Ext}}^1_X(E, E \otimes \Omega^1_{X/\mathbb{Z}})$ — the *arithmetic Atiyah class* of $\overline{E}$ — is mapped by the forgetful morphism ν to the “algebraic" Atiyah class $\mathrm{at}_{X/\mathbb{Z}}E$ of E in $\mathrm{Ext}^1_{\mathcal{O}_X}(E, E \otimes \Omega^1_{X/\mathbb{Z}})$ (defined by the extension $\mathcal{A}t_{X/\mathbb{Z}}E$) and by the “second fundamental form" morphism Ψ to the curvature form of the Chern-Nakano connection $\nabla_{\overline{E}}$ (up to a sign).

0.4. — In the first section of this article, we begin by reviewing the constructions of the Atiyah extension in the classical $\mathbb{C}$-analytic and algebraic frameworks. For the sake of simplicity, we deal with locally free coherent sheaves only, and follow a naive approach — we work with relative differentials, and not with their “correct" derived version defined by the cotangent complex. This naive approach is sufficient when one considers — as we shall in the sequel — relative situations $f : X \to S$ where X is

(2) These authors work in an analytic context as the original article [**2**], but extend the construction of Atiyah classes to complex of coherent analytic sheaves over possibly singular complex spaces. Like Illusie's construction, this requires to deal with the cotangent complex, now in an analytic context.

integral, and f is l.c.i. and generically smooth, in which case $\mathbb{L}^{\cdot}_{X/S}$ is quasi-isomorphic to $\Omega^1_{X/S}$.

Then, in Section 2, we construct the arithmetic Atiyah class in the following relative situation, which extends the one considered in the previous paragraphs. Consider arithmetic schemes X and S, flat over an arithmetic ring (R, Σ, F_∞) (in the sense of [**17**, 3.1.1]; see also [**7**, 1.1]), and a morphism of R-schemes $\pi : X \to S$, smooth over the fraction field K of R. Then, to any hermitian vector bundle $\overline{E}$ over X, we attach a class $\widehat{\mathrm{at}}_{X/S}\overline{E}$ in $\widehat{\mathrm{Ext}}^1_X(E, E \otimes \Omega^1_{X/S})$. Applying a trace morphism to this class, we define the *first Chern class* $\hat{c}_1^H(\overline{E})$ *of* $\overline{E}$ *in arithmetic Hodge cohomology*, that lies in the group

$$\widehat{H}^{1,1}(X/S) := \widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X, \Omega^1_{X/S}).$$

The class $\widehat{\mathrm{at}}_{X/S}\overline{E}$ and its trace $\hat{c}_1^H(\overline{E})$ satisfy compatibility properties with pull-back and tensor operations on hermitian vector bundles that extend well-known properties of the classical Atiyah and first Chern classes. In particular we construct a functorial homomorphism

$$\hat{c}_1^H : \widehat{\mathrm{Pic}}(X) \longrightarrow \widehat{H}^{1,1}(X/S)$$

from the group of isomorphism classes of hermitian line bundles over X to the arithmetic Hodge cohomology group.

In the last sections of this article, we investigate the kernel of this morphism. It trivially vanishes on the image of

$$\pi^* : \widehat{\mathrm{Pic}}(S) \longrightarrow \widehat{\mathrm{Pic}}(X),$$

and we may wonder "how large" this image $\pi^*(\widehat{\mathrm{Pic}}(S))$ is in $\ker \hat{c}_1^H$.

This question becomes a concrete problem of Diophantine geometry when the base arithmetic ring is a number field K equipped with a non-empty set Σ of embeddings $\sigma : K \hookrightarrow \mathbb{C}$ stable under complex conjugation, and when S is $\operatorname{Spec} K$ and X is projective over K. Indeed, in this case, the class of a hermitian line bundle $\overline{L} = (L, \|.\|_L)$ over X lies in the kernel of $\hat{c}_1^H$ precisely when L admits an algebraic connection $\nabla : L \to L \otimes \Omega^1_{X/K}$, defined over K, such that the induced holomorphic connection $\nabla_{\mathbb{C}} : L_{\mathbb{C}} \to L_{\mathbb{C}} \otimes \Omega^1_{X_\Sigma(\mathbb{C})}$ on the holomorphic line bundle $L_{\mathbb{C}}$ over

$$X_\Sigma(\mathbb{C}) := \coprod_{\sigma \in \Sigma} X_\sigma(\mathbb{C})$$

is unitary with respect to the hermitian metric $\|.\|_L$.

One easily checks that, if L has a torsion class in $\mathrm{Pic}(X)$ and if the metric $\|.\|_L$ has vanishing curvature on $X_\Sigma(\mathbb{C})$, then their exists such a connection. Moreover the converse implication, namely

I1$_{X,\Sigma}$: *if a hermitian line bundle* $\overline{L} = (L, \|.\|_L)$ *over* X *admits an algebraic connection* ∇ *defined over* K *such that the connection* $\nabla_{\mathbb{C}}$ *on* $L_{\mathbb{C}}$ *over* $X_\Sigma(\mathbb{C})$ *is unitary with* *respect to* $\|.\|_L$, *then* L *has a torsion class in* $\mathrm{Pic}(X)$ *and the metric* $\|.\|_L$ *has vanishing curvature,*

turns out to be equivalent with the following condition, where π denotes the structural morphism from X to $\operatorname{Spec} K$:

$\mathbf{I2}_{X,\Sigma}$: *the image of* $\pi^* : \widehat{\mathrm{Pic}}(\operatorname{Spec} K) \to \widehat{\mathrm{Pic}}(X)$ *has finite index in the kernel of*

$$\hat{c}_1^H : \widehat{\mathrm{Pic}}(X) \longrightarrow \widehat{H}^{1,1}(X/K).$$

The equivalent assertions $\mathbf{I1}_{X,\Sigma}$ and $\mathbf{I2}_{X,\Sigma}$ may be translated in terms of K-rational points of the universal vector extension of the Picard variety of X. In this formulation, their validity has been established by Bertrand [**4, 5**] when Σ has a unique element (necessarily a real embedding of K) and when this Picard variety admits "real multiplication"[(3)] as a consequence of the analytic subgroup theorem of Wüstholz ([**44**]).

Inspired by [**4, 5**] — which we tried to understand in more geometric terms, avoiding the explicit use of differential forms and their periods, but working with algebraic groups and their exponential maps— we establish in Section 3 the validity of $\mathbf{I1}_{X,\Sigma}$ and $\mathbf{I2}_{X,\Sigma}$ when Σ is arbitrary without any assumption on the Picard variety of X. The proof proceeds by reducing to the case where X is an abelian variety, and Σ has a unique or two conjugate elements. To handle this case, we use a classical transcendence theorem of Schneider-Lang characterizing Lie algebras of algebraic subgroups of commutative algebraic groups over number fields. Our argument is presented in the first part of Section 3, and may be read independently of the rest of the article.

The validity of $\mathbf{I1}_{X,\Sigma}$ and $\mathbf{I2}_{X,\Sigma}$ demonstrates that the first Chern class $\hat{c}_1^H(\overline{L})$ in the group $\widehat{H}^{1,1}(X/K)$ encodes quite non-trivial Diophantine informations. In a later part of this work, we plan to study characteristic classes of higher degree, with values in the arithmetic Hodge cohomology groups

$$\widehat{H}^{p,p}(X/S) := \widehat{\mathrm{Ext}}^p_X(\mathcal{O}_X, \Omega^p_{X/S})$$

defined as special instances of the higher arithmetic extension groups introduced in [**7**, 0.1], that are deduced from the powers of the arithmetic Atiyah class $\widehat{\mathrm{at}}_{X/S}\overline{E}$ using suitably defined products on the higher arithmetic extension groups.

Let us also indicate that, starting from the results in Section 3, one may derive finiteness results on $\ker \hat{c}_1^H/\pi^*(\widehat{\mathrm{Pic}}(S))$ for more general smooth projective morphisms $\pi : X \to S$ of arithmetic schemes over arithmetic rings, by considering the restriction of π over points of S rational over some number field. We leave this to the interested reader.

(3) Namely, if this Picard variety A has dimension g, the $\mathbb{Q}$-algebra $\mathrm{End}(A/K) \otimes_{\mathbb{Z}} \mathbb{Q}$ is assumed to be a totally real field of degree g over $\mathbb{Q}$. Actually, Bertrand establishes a more precise result, concerning g independent extensions of A by the additive group $\mathbb{G}_a$; see [**5**, Theorem 3, pages 13-14].

In the final section of the article, we establish a geometric analogue of condition $\mathbf{I1}_{X,\Sigma}$. We consider a smooth, projective, geometrically connected curve C over some field k of characteristic zero, its function field $K := k(C)$, and a smooth projective variety X over k equipped with a dominant k morphism $f : X \to C$, with geometrically connected fibers. To any line bundle L over X is attached its relative Atiyah class $\mathrm{at}_{X/C}L$ in $H^1(X, \Omega^1_{X/C})$. We show that, when the fixed part of the abelian variety $\mathrm{Pic}^0_{X_K/K}$ is trivial, the class $\mathrm{at}_{X/C}L$ vanishes iff some positive power of L is isomorphic to a line bundle of the form f^*M, where M is a line bundle over C. The proof relies on the Hodge Index Theorem expressed in the Hodge cohomology groups of X.

Considering the classical analogy between number fields and function fields, it may be interesting to observe that, when investigating the kernel of the relative Atiyah class of line bundles, a transcendence result — in the guise of a criterion for a subspace of the Lie algebra of a commutative algebraic group to define an algebraic subgroup — plays a key role in the "number field case", while our main tool in the "function field case" is intersection theory in Hodge cohomology.

In Appendix A, we describe arithmetic extension groups in terms of Čech cocycles. Based on this description, in the main part of the paper we calculate explicit Čech cocycles which represent the arithmetic Atiyah class and the first Chern class in arithmetic Hodge cohomology. Finally Appendix B summarizes basic facts concerning universal vector extensions of Picard varieties that are used in Sections 3 and 4.

It is a pleasure to thank A. Chambert-Loir and D. Bertrand for helpful discussions, and S. Kudla, M. Rapoport and J. Schwermer for invitations to the ESI in Vienna where part of the work on this paper was done. We are grateful to the TMR network 'Arithmetic geometry' and the DFG-Forschergruppe 'Algebraische Zykel und L-Funktionen' for their support and to the universities of Regensburg and Paris-Sud (Orsay) for their hospitality. Finally we wish to thank the referee for his careful reading and his helpful suggestions for improving the exposition.

1. Atiyah extensions in algebraic and analytic geometry

1.1. Definition and basic properties. — We consider simultaneously the algebraic and the analytic situation where $\pi : X \to S$ is a morphism of locally ringed spaces which is either

a) a separated morphism of finite presentation between schemes, or
b) an analytic morphism between complex analytic spaces.

We denote in both cases by $\mathcal{O}_X$ the structure sheaf of regular resp. holomorphic functions on X. Let I denote the ideal sheaf and

$$\Delta^{(1)} : X^{(1)} \longrightarrow X \times_S X$$

the first infinitesimal neighborhood of the diagonal $\Delta\colon\ X \to X \times_S X$. For $i = 1, 2$, let $q_i : X^{(1)} \to X$ denote the composition of $\Delta^{(1)}$ with the i-th projection. We identify $(\Omega^1_{X/S}, d)$ with the $\mathcal{O}_X$-module I/I^2 and the universal derivation

$$d : \mathcal{O}_X \longrightarrow I/I^2\,,\ d(\lambda) = q_2^*(\lambda) - q_1^*(\lambda). \tag{1.1}$$

The $\mathcal{O}_X$-modules $q_{1*}\mathcal{O}_{X^{(1)}}$ and $q_{2*}\mathcal{O}_{X^{(1)}}$ are canonically isomorphic as sheaves of $\mathcal{O}_S$-modules. We denote this $\mathcal{O}_S$-module by $P^1_{X/S}$ and observe that $P^1_{X/S}$ carries two natural $\mathcal{O}_X$-module structures via the left and right projection q_1 and q_2. The canonical extension

$$0 \longrightarrow I/I^2 \longrightarrow \mathcal{O}_{X\times_S X}/I^2 \longrightarrow \mathcal{O}_{X\times_S X}/I \longrightarrow 0$$

yields an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \Omega^1_{X/S} \longrightarrow P^1_{X/S} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{1.2}$$

for both $\mathcal{O}_X$-module structures. The left and right $\mathcal{O}_X$-module structures yield canonical but different $\mathcal{O}_X$-linear splittings of (1.2) which map $1 \bmod I$ to $1 \bmod I^2$.

1.1.1. — Let F denote a vector bundle (that is, a locally free coherent sheaf) on X. We obtain from (1.2) an exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{J}et^1_{X/S}(F)\colon 0 \longrightarrow F \otimes \Omega^1_{X/S} \xrightarrow{i_F} P^1_{X/S}(F) \xrightarrow{p_F} F \longrightarrow 0$$

where

$$P^1_{X/S}(F) = q_{1*}q_2^*F. \tag{1.3}$$

Indeed we have

$$P^1_{X/S}(F) = P^1_{X/S} \otimes F$$

where the tensor product is taken using the right $\mathcal{O}_X$-module structure on $P^1_{X/S}$, and then the sequence is viewed as sequence of $\mathcal{O}_X$-modules via the left $\mathcal{O}_X$-module structure. The canonical splitting of (1.2) for the right $\mathcal{O}_X$-module structure induces a canonical $\mathcal{O}_S$-linear splitting of $\mathcal{J}et^1_{X/S}(F)$. We obtain a canonical direct sum decomposition

$$P^1_{X/S}(F) = F \oplus (F \otimes \Omega^1_{X/S}) \tag{1.4}$$

of $\mathcal{O}_S$-modules. We use squared brackets $[\,,\,]$ when we refer to this decomposition. A straightforward calculation shows that, in terms of this decomposition, the left $\mathcal{O}_X$-module structure of $P^1_{X/S}(F)$ is given by

$$\lambda \cdot [f, \omega] = [\lambda \cdot f, \lambda \cdot \omega - f \otimes d\lambda] \tag{1.5}$$

for local sections λ of $\mathcal{O}_X$, f of F, and ω of $F \otimes \Omega^1_{X/S}$. It follows that there is a one-to-one correspondence

$$\left\{\begin{matrix} \mathcal{O}_X\text{-linear splittings} \\ s\colon\ F \to P^1_{X/S}(F) \text{ of } \mathcal{J}et^1_{X/S}(F) \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \text{algebraic resp. holomorphic} \\ \text{connections } \nabla\colon\ F \to F \otimes \Omega^1_{X/S} \end{matrix}\right\}.$$

Under this correspondence, a connection ∇ corresponds to the splitting s_∇ of $\mathcal{J}et^1_{X/S}(F)$ given by the formula

$$s_\nabla : F \longrightarrow P^1_{X/S}(F) = F \oplus (F \otimes \Omega^1_{X/S}) \ , \ f \longmapsto \big[f, -\nabla(f)\big]. \tag{1.6}$$

For instance, the "trivial" connection $\nabla := d$ on $E = \mathcal{O}_X$ is associated to the canonical left $\mathcal{O}_X$-linear splitting of (1.2).

1.1.2. — The extension $\mathcal{J}et^1_{X/S}(F)$ is called the *extension given by the* 1*-jets* or *principal parts of first order associated with* F. We denote the class of $\mathcal{J}et^1_{X/S}(F)$ in $\mathrm{Ext}^1_{\mathcal{O}_X}(F, F \otimes \Omega^1_{X/S})$ by $\mathrm{jet}^1_{X/S}(F)$ and abbreviate $\mathrm{jet}(F) = \mathrm{jet}^1_{X/S}(F)$ if X/S is clear from the context. We have followed in (1.1), (1.3), and (1.6) the conventions fixed in [**23**, 16.7], [**25**, III. (1.2.6.2)], and [**13**, (2.3.4)].

1.1.3. — We recall from [**2**, Propositions 6, 7 and 8] that the assignment

$$\begin{array}{ccc} \{\text{vector bundles on } X\} & \longrightarrow & \{\text{short exact sequences of } \mathcal{O}_X\text{-modules}\} \\ F & \longmapsto & \mathcal{J}et^1_{X/S}(F) \end{array}$$

defines an additive, exact functor. Furthermore $\mathcal{J}et^1_{X/S}(F)$ is a short exact sequence of vector bundles if π is smooth.

The following Lemma is a slight refinement of [**2**, Proposition 10].

***Lemma* 1.1.4**. — *Let E and F denote vector bundles on X.*

i) *Let*

$$B = \frac{\mathrm{Ker}\big(p_E \otimes \mathrm{id}_F - \mathrm{id}_E \otimes p_F : P^1_{X/S}(E) \otimes F \oplus E \otimes P^1_{X/S}(F) \to E \otimes F\big)}{\mathrm{Im}\big((i_E \otimes \mathrm{id}_F, -\mathrm{id}_E \otimes i_F) : E \otimes F \otimes \Omega^1_{X/S} \to P^1_{X/S}(E) \otimes F \oplus E \otimes P^1_{X/S}(F)\big)}.$$

denote the Baer sum of the extensions $\mathcal{J}et^1_{X/S}(E) \otimes F$ and $E \otimes \mathcal{J}et^1_{X/S}(F)$. There exists a canonical isomorphism

$$\varphi : P^1_{X/S}(E \otimes F) \longrightarrow B \tag{1.7}$$

which fits into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E \otimes F \otimes \Omega^1_{X/S} & \longrightarrow & P^1_{X/S}(E \otimes F) & \longrightarrow & E \otimes F & \longrightarrow & 0 \\ & & \| & & \downarrow \varphi & & \| & & \\ 0 & \longrightarrow & E \otimes F \otimes \Omega^1_{X/S} & \longrightarrow & B & \longrightarrow & E \otimes F & \longrightarrow & 0. \end{array}$$

Consequently we have

$$\mathrm{jet}^1_{X/S}(E \otimes F) = \mathrm{jet}^1_{X/S}(E) \otimes F + E \otimes \mathrm{jet}^1_{X/S}(F)$$

in $\mathrm{Ext}^1_{\mathcal{O}_X}(E \otimes F, E \otimes F \otimes \Omega^1_{X/S})$.

ii) *Let* ∇_E *and* ∇_F *denote connections on* E *and* F. *We equip the tensor product* $E \otimes F$ *with the product connection*

$$\nabla_{E\otimes F} = \nabla_E \otimes \mathrm{id}_F + \mathrm{id}_E \otimes \nabla_F. \tag{1.8}$$

The connections ∇_E, ∇_F, *and* $\nabla_{E\otimes F}$ *induce sections* s_E,s_F, *and* $s_{E\otimes F}$ *of* $\mathscr{J}et^1_{X/S}(E)$, $\mathscr{J}et^1_{X/S}(F)$, *and* $\mathscr{J}et^1_{X/S}(E\otimes F)$ *respectively. We have*

$$\varphi \circ s_{E\otimes F} = (s_E \otimes \mathrm{id}_F, \mathrm{id}_E \otimes s_F)$$

where the notation on the right hand side refers to the description of the Baer sum given above.

Proof. — i) Let $IM = \mathrm{Im}(i_E \otimes \mathrm{id}_F, -\mathrm{id}_E \otimes i_F)$. Recall that

$$P^1_{X/S}(E\otimes F) = (E\otimes F) \oplus (E \otimes F \otimes \Omega^1_{X/S}).$$

There exists a unique $\mathscr{O}_S$-linear map (1.7) which satisfies

$$\begin{aligned}
\varphi\big([e_0\otimes f_0, e_1 \otimes f_1 \otimes \alpha]\big) &= \big([e_0,0]\otimes f_0 + [0, e_1\otimes\alpha]\otimes f_1\big) \oplus \big(e_0 \otimes [f_0,0]\big) \bmod IM \\
&= \big([e_0,0]\otimes f_0\big) \oplus \big(e_0\otimes[f_0,0] + e_1 \otimes [0, f_1\otimes \alpha]\big) \bmod IM
\end{aligned}$$

for local sections e_0, e_1 of E, f_0, f_1 of F and α of $\Omega^1_{X/S}$. It is straightforward to check that φ is well defined and makes our diagram commutative. It remains to show that φ is also $\mathscr{O}_X$-linear. This follows from

$$\begin{aligned}
\varphi\big(\lambda\cdot[e_0\otimes f_0, 0]\big) &= \varphi\big([\lambda\cdot e_0 \otimes f_0, -e_0\otimes f_0 \otimes d\lambda]\big) \\
&= \big([\lambda\cdot e_0, 0]\otimes f_0 - [0, e_0\otimes d\lambda]\otimes f_0\big) \oplus \big(\lambda\cdot e_0 \otimes [f_0,0]\big) \bmod IM \\
&= \lambda\cdot\varphi\big([e_0\otimes f_0, 0]\big)
\end{aligned}$$

as φ induces the identity on $\Omega^1_{X/S}\otimes E \otimes F$.

ii) For local sections e of E and f of F, we get

$$\begin{aligned}
\varphi\circ s_{E\otimes F}(e\otimes f) &= \big([e, -\nabla e]\otimes f\big) \oplus \big(e \otimes [f, -\nabla f]\big) \quad \bmod\ IM \\
&= (s_E \otimes \mathrm{id}_F, \mathrm{id}_E \otimes s_F)(e\otimes f)
\end{aligned}$$

which proves ii). □

***Corollary* 1.1.5.** — *Let* E *be a vector bundle on* X *and denote*

$$j_E : \mathscr{O}_X \to E \otimes E^\vee \simeq \mathrm{End}(E)$$

the canonical morphism of vector bundles which maps 1 *to* id_E. *The Baer sum of* $\mathscr{J}et^1_{X/S}(E)\otimes E^\vee$ *and* $E \otimes \mathscr{J}et^1_{X/S}(E^\vee)$ *is canonically isomorphic to* $\mathscr{J}et^1_{X/S}(E\otimes E^\vee)$.

The pullback $\mathcal{J}et^1_{X/S}(E\otimes E^\vee)\circ j_E$ *of* $\mathcal{J}et^1_{X/S}(E\otimes E^\vee)$ *along* j_E *— defined as the upper extension in the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & E\otimes E^\vee\otimes\Omega^1_{X/S} & \to & Q & \to & \mathcal{O}_X & \to & 0 \\ & & \| & & \downarrow & & \downarrow j_E & & \\ 0 & \to & E\otimes E^\vee\otimes\Omega^1_{X/S} & \to & P^1_{X/S}(E\otimes E^\vee) & \to & E\otimes E^\vee & \to & 0 \end{array} \tag{1.9}$$

whose righthand square is cartesian (compare [**7**, App. A.4.2]) *— admits a canonical splitting.*

Proof. — The first statement follows from Lemma 1.1.4. The map j_E induces by functoriality a morphism from $\mathcal{J}et^1_{X/S}(\mathcal{O}_X)$ to $\mathcal{J}et^1_{X/S}(E\otimes E^\vee)$. Since the righthand side in (1.9) is cartesian, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1_{X/S} & \longrightarrow & P^1_{X/S}(\mathcal{O}_X) & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \downarrow j_E\otimes\mathrm{id}_{\Omega^1_{X/S}} & & \downarrow \varphi & & \| & & \\ 0 & \longrightarrow & E\otimes E^\vee\otimes\Omega^1_{X/S} & \longrightarrow & Q & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0. \end{array} \tag{1.10}$$

The canonical splitting s_d of $\mathcal{J}et^1_{X/S}(\mathcal{O}_X)$ (that correspond to the connection d on $\mathcal{O}_X$) induces via (1.10) the requested canonical splitting $\varphi\circ s_d$ of $\mathcal{J}et^1_{X/S}(E\otimes E^\vee)\circ j_E$. $\square$

***Lemma* 1.1.6.** — *Consider a commutative diagram*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f} & X \\ \downarrow \tilde{\pi} & & \downarrow \pi \\ \tilde{S} & \xrightarrow{g} & S \end{array}$$

in the category of locally ringed spaces where $\tilde{\pi}$ *and* π *are morphisms as in situation 1.1, a) or b). Let* E *be a vector bundle on* X *and denote by* f^* *the canonical map* $f^*\Omega^1_{X/S}\to\Omega^1_{\tilde{X}/\tilde{S}}$.

i) *There exists a canonical* $\mathcal{O}_{\tilde{X}}$*-linear map*

$$\phi : f^*P^1_{X/S}(E)\longrightarrow P^1_{\tilde{X}/\tilde{S}}(f^*E)$$

which makes the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^*E\otimes_{\mathcal{O}_{\tilde{X}}} f^*\Omega^1_{X/S} & \longrightarrow & f^*P^1_{X/S}(E) & \longrightarrow & f^*E & \longrightarrow & 0 \\ & & \downarrow \mathrm{id}_{f^*E}\otimes f^* & & \downarrow \phi & & \| & & \\ 0 & \longrightarrow & f^*E\otimes_{\mathcal{O}_{\tilde{X}}}\Omega^1_{\tilde{X}/\tilde{S}} & \longrightarrow & P^1_{\tilde{X}/\tilde{S}}(f^*E) & \longrightarrow & f^*E & \longrightarrow & 0. \end{array} \tag{1.11}$$

commutative. Consequently we have

$$(\mathrm{id}_{f^*E}\otimes f^*)\circ\mathrm{jet}^1_{X/S}(E)=\mathrm{jet}^1_{\tilde{X}/\tilde{S}}(f^*E)$$

in $\mathrm{Ext}^1_{\mathcal{O}_{\tilde{X}}}(f^*E, f^*E\otimes_{\mathcal{O}_{\tilde{X}}}\Omega^1_{\tilde{X}/\tilde{S}})$.

ii) *A connection ∇_E on E induces a splitting s_E of $\mathcal{J}et^1_{X/S}(E)$. The splitting*

$$s_{f^*E} := \phi \circ f^*(s_E)$$

induces a connection $f^*\nabla_E$ on f^*E which is uniquely determined by

$$(f^*\nabla_E)(f^*s) = f^*(\nabla_E\, s) := (\mathrm{id}_{f^*E} \otimes f^*)(f^{-1}(\nabla_E s)) \tag{1.12}$$

for local sections s of E.

Notice that the case where $\tilde{\pi}$ is as in situation 1.1, b) and π as in situation 1.1, a) is allowed.

Proof. — i) Observe that the upper sequence in (1.11) is exact as E is locally free. Recall that

$$f^*P^1_{X/S}(E) = \left[f^{-1}E \oplus f^{-1}(E \otimes_{\mathcal{O}_X} \Omega^1_{X/S})\right] \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_{\tilde{X}} \tag{1.13}$$

and

$$P^1_{\tilde{X}/\tilde{S}}(f^*E) = f^*E \oplus f^*E \otimes_{\mathcal{O}_{\tilde{X}}} \Omega^1_{\tilde{X}/\tilde{S}}. \tag{1.14}$$

By the very definitions of f^*E and $f^*(E \otimes_{\mathcal{O}_X} \Omega^1_{X/S})$, we have $f^{-1}\mathcal{O}_X$-linear canonical maps

$$f^{-1}E \longrightarrow f^*E$$

and

$$f^{-1}(E \otimes_{\mathcal{O}_X} \Omega^1_{X/S}) \to f^*(E \otimes_{\mathcal{O}_X} \Omega^1_{X/S}) \xrightarrow{\sim} f^*E \otimes_{\mathcal{O}_{\tilde{X}}} f^*\Omega^1_{X/S} \xrightarrow{\mathrm{id}_{f^*E} \otimes f^*} f^*E \otimes_{\mathcal{O}_{\tilde{X}}} \Omega^1_{\tilde{X}/\tilde{S}}.$$

The direct sum of these maps induces a $g^{-1}\mathcal{O}_S$-linear morphism

$$\left[f^{-1}E \oplus f^{-1}(E \otimes_{\mathcal{O}_X} \Omega^1_{X/S})\right] \longrightarrow f^*E \oplus f^*E \otimes_{\mathcal{O}_{\tilde{X}}} \Omega^1_{\tilde{X}/\tilde{S}}.$$

It is straightforward to check that this morphism is $f^{-1}\mathcal{O}_X$-linear for the module structure given by formula (1.5). Via (1.13) and (1.14), we obtain the desired morphism ϕ which fits by construction in the diagram (1.11).

ii) is a straightforward consequence of the construction of ϕ in the proof of i). □

1.2. Cotangent complex and Atiyah class. — In situation 1.1, a) resp. b), the cotangent complex $\mathbb{L}^{\cdot}_{X/S}$ is constructed in [**25**, II.1.2] resp. [**9**, 2.38] as an object in the derived category $D(\mathcal{O}_X-\mathrm{mod})$ of $\mathcal{O}_X$-modules. Consider $\Omega^1_{X/S}$ as a complex concentrated in degree zero. The cotangent complex $\mathbb{L}^{\cdot}_{X/S}$ comes with a natural morphism

$$\mathbb{L}^{\cdot}_{X/S} \longrightarrow \Omega^1_{X/S} \tag{1.15}$$

in $D(\mathcal{O}_X-\mathrm{mod})$ which is a quasi-isomorphism if X is smooth over S. Given a vector bundle E over X, the *Atiyah class* of E is defined in [**25**, IV.2.3] resp. [**9**, §3] as an element

$$\mathrm{at}_{X/S}(E) \in \mathrm{Ext}^1_{\mathcal{O}_X}(E, E \otimes^{\mathbb{L}} \mathbb{L}^{\cdot}_{X/S}) = \mathrm{Hom}_{D(\mathcal{O}_X-\mathrm{mod})}(E, E \otimes^{\mathbb{L}} \mathbb{L}^{\cdot}_{X/S}[1]).$$

If $X \xrightarrow{\pi} S$ is a morphism of schemes, the Atiyah class of Illusie maps under the morphism induced by (1.15) to the class (compare [**25**, Cor. IV.2.3.7.4])

$$\mathrm{jet}^1_{X/S}(E) \in \mathrm{Ext}^1_{\mathcal{O}_X}(E, E \otimes \Omega^1_{X/S}).$$

Furthermore, according to [**25**, Prop. II.1.2.4.2], (1.15) induces an isomorphism

$$H_0(\mathbb{L}^{\cdot}_{X/S}) \xrightarrow{\sim} \Omega^1_{X/S}. \tag{1.16}$$

If $X \xrightarrow{\pi} S$ is a smooth morphism of complex analytic spaces, the Atiyah class of Buchweitz and Flenner maps under the morphism induced by (1.15) to the *opposite* class of $\mathrm{jet}^1_{X/S}(E)$ ([**9**, 3.27]).

If the canonical morphism (1.15) is a quasi-isomorphism, we call $\mathcal{J}et^1_{X/S}(E)$ the *Atiyah extension associated with* E and denote it by $\mathcal{A}t_{X/S}(E)$.

The associated extension class $\mathrm{at}_{X/S}(F)$ equals the opposite of the Atiyah classes $\mathrm{At}(F)$ in [**9**] and $b(F)$ in [**2**, Section 4]. It coincides with the Atiyah class defined in [**1**]. Compare also [**9**, 2.4 and Rem. 3.17] for a discussion of signs related to the Atiyah class.

The following Lemma implies in particular that (1.15) is a quasi-isomorphism in the situations considered in the next sections.

***Lemma* 1.2.1**. — *Let $\pi : X \to S$ be a locally complete intersection (l.c.i.) morphism of schemes such that X is integral and π is generically smooth, in the sense that the smooth locus of π is dense in X. Then the morphism* (1.15) *is a quasi-isomorphism.*

Proof. — It is sufficient to show our claim locally on X as the formation of (1.15) is compatible with restrictions to open subsets. Hence we may assume that π admits a factorization

$$X \xrightarrow{j} Q \xrightarrow{q} S$$

where j is a regular immersion defined by some regular ideal sheaf J and q is smooth. We obtain an exact sequence

$$0 \longrightarrow J/J^2 \xrightarrow{\phi} j^*\Omega^1_{Q/S} \xrightarrow{\psi} \Omega^1_{X/S} \longrightarrow 0. \tag{1.17}$$

This is well known up to the injectivity of ϕ which holds as ϕ is a morphism of locally free sheaves which is injective over the smooth locus of π. The complex

$$J/J^2 \xrightarrow{\phi} j^*\Omega^1_{Q/S}$$

concentrated in degrees minus one and zero is a cotangent complex for f by [**25**, Cor. III.3.2.7]. Therefore it follows from the exactness of (1.17) on the left and the isomorphism (1.16) that (1.15) is in fact a quasi-isomorphism. □

1.3. $\mathscr{C}^\infty$-connections compatible with the holomorphic structure. — Let E denote a holomorphic vector bundle on a complex manifold X. Recall that a $\mathscr{C}^\infty$-connection

$$\nabla\colon\ A^0(X,E) \longrightarrow A^1(X,E)$$

on E is called *compatible with the holomorphic structure* if its $(0,1)$-part coincides with the Dolbeault operator, i.e. $\nabla^{0,1}=\overline{\partial}_E$. Consider the Atiyah extension associated with E

$$\mathscr{A}t_X(E)\colon 0 \longrightarrow E\otimes\Omega^1_X \xrightarrow{i_E} P^1_{X/\mathbb{C}}(E) \xrightarrow{p_E} E \longrightarrow 0.$$

In the same way as before, we obtain a one-to-one correspondence

$$\nabla \longleftrightarrow s_{\nabla^{1,0}}$$

between $\mathscr{C}^\infty$-connections on the vector bundle E which are compatible with the holomorphic structure and $\mathscr{C}^\infty$-splittings

$$s_{\nabla^{1,0}}: E \longrightarrow P^1_{X/\mathbb{C}}(E)\,,\ f \longmapsto [f,-\nabla^{1,0}(f)] \tag{1.18}$$

of the extension $\mathscr{A}t_X(E)$.

It is straightforward to check that this correspondence satisfies compatibility properties with tensor operations and pull back similar to the ones established in Lemma 1.1.4, Corollary 1.1.5, and Lemma 1.1.6 above.

The one-to-one correspondence described above extends in a straightforward way to the relative situation where X/S is a holomorphic family of complex manifolds. We leave the details of this construction to the interested reader.

Assume that E carries a hermitian metric h. A $\mathscr{C}^\infty$-connection ∇ on $\overline{E}=(E,h)$ is called *unitary* if and only if it satisfies

$$dh(s,t)=h(\nabla s,t)+h(s,\nabla t)\quad\text{for all } s,t\in A^0(X,E).$$

Recall that a hermitian holomorphic vector bundle $\overline{E}=(E,h)$ carries a unique unitary connection $\nabla_{\overline{E}}$ which is compatible with the holomorphic structure ([**10**], [**36**]; see also [**20**, Ch. 0.5] or [**43**, Sect. II.2]; this connection is sometimes called the *Chern connection* of (E,h)). Moreover the assignement $\overline{E}\mapsto\nabla_{\overline{E}}$ is compatible with direct sums, tensor products, duals and pull-backs.

***Lemma* 1.3.1**. — *Let $\overline{E}=(E,h)$ be a hermitian holomorphic vector bundle on X. Let $\nabla=\nabla_{\overline{E}}$ denote the unitary $\mathscr{C}^\infty$-connection on E which is compatible with the complex structure. The curvature form*

$$\nabla^2\in A^{1,1}\big(X,\mathscr{E}nd(E)\big)$$

and the second fundamental form

$$\alpha\in A^{0,1}\big(X,\mathscr{E}nd(E)\otimes\Omega^1_X\big)$$

associated with $\mathcal{A}t_X(E)$ *and its* $\mathcal{C}^\infty$*-splitting* $s_{\nabla^{1,0}}$ *as in* [**7**, A.5.2] *satisfy*

$$\alpha = -\nabla^2 \tag{1.19}$$

where we read the canonical isomorphism

$$A^{1,1}\big(X, \mathcal{E}nd(E)\big) \xrightarrow{\sim} A^{0,1}\big(X, \mathcal{E}nd(E) \otimes \Omega^1_X\big)\,,\ f \otimes (\alpha \wedge \beta) \mapsto (f \otimes \alpha) \wedge \beta.$$

(compare [**7**, 1.1.5]*) as an identification.*

Proof. — Recall from [**7**, A.5.2] that α is determined by

$$\overline{\partial}_{P^1_{X/\mathbb{C}}(E)\otimes E^\vee}\big(s_{\nabla^{1,0}}\big) = (i_E \otimes \mathrm{id}_{A^{0,1}_X})(\alpha).$$

It is sufficient to verify (1.19) locally on X. Hence we may assume that E admits a holomorphic frame. We describe ∇ and ∇^2 with respect to this frame by the connection matrix θ and the curvature matrix Θ. Following the conventions in [**43**, Ch. III], we have

$$\Theta_{ik} = d\theta_{ik} + \sum_j \theta_{ij} \wedge \theta_{jk}.$$

The connection matrix θ has type $(1,0)$ and the curvature matrix Θ has type $(1,1)$ by *loc. cit.* Hence the equality above becomes

$$\Theta = \overline{\partial}\theta. \tag{1.20}$$

Let $\tilde{\nabla}$ denote the connection on E whose connection matrix is zero. The associated splitting $s_{\tilde{\nabla}^{1,0}}$ of $\mathcal{A}t_X(E)$ is holomorphic. Hence (1.6) and (1.20) give

$$\overline{\partial}_{P^1_{X/\mathbb{C}}(E)\otimes E^\vee}\big(s_{\nabla^{1,0}}\big) = \overline{\partial}_{P^1_{X/\mathbb{C}}(E)\otimes E^\vee}\big(s_{\nabla^{1,0}} - s_{\tilde{\nabla}^{1,0}}\big) = -\overline{\partial}(\theta) = -\Theta = -\nabla^2. \qquad \square$$

2. The arithmetic Atiyah class of a vector bundle with connection

In this section we fix an arithmetic ring $R = (R, \Sigma, F_\infty)$ in the sense of [**17**, 3.1.1]. We denote K the fraction field of R, and we let $S := \operatorname{Spec} R$.

2.1. Definition and basic properties. — Let X be an integral arithmetic scheme over R (in the sense of [**17**], or [**7**, 1.1]) with a flat, l.c.i. structural morphism $\pi : X \to S$. Recall that the generic fiber X_K of X is smooth (by the very definition of an arithmetic scheme in *loc. cit.*), and observe that π satisfies the assumptions in Lemma 1.2.1.

Let E be a vector bundle on X. We consider the commutative square

$$\begin{array}{ccc} (X_\Sigma(\mathbb{C}), \mathcal{O}^{\mathrm{hol}}_{X_\Sigma}) & \xrightarrow{j} & (X, \mathcal{O}_X) \\ \downarrow \pi_{\mathbb{C}} & & \downarrow \pi \\ (S_\Sigma(\mathbb{C}), \mathcal{O}^{\mathrm{hol}}_{S_\Sigma}) & \xrightarrow{j_0} & (S, \mathcal{O}_S). \end{array}$$

Lemma 1.1.6 implies that the formation of the Atiyah extension of E is compatible with base change with respect to this diagram. More precisely, we have a canonical analytification isomorphism

$$P^1_{X/S}(E)^{\mathrm{hol}}_{\mathbb{C}} \xrightarrow{\sim} P^1_{X_\Sigma(\mathbb{C})/S_\Sigma(\mathbb{C})}(E^{\mathrm{hol}}_{\mathbb{C}})$$

where we put $F^{\mathrm{hol}}_{\mathbb{C}} = j^*F$ for every $\mathcal{O}_X$-module F.

2.1.1. — We have seen in 1.3 that there is a one-to-one correspondence between $\mathscr{C}^\infty$-connections

$$\nabla\colon\ A^0\big(X_\Sigma(\mathbb{C}), E_{\mathbb{C}}\big) \to A^1\big(X_\Sigma(\mathbb{C}), E_{\mathbb{C}}\big)$$

which are compatible with the holomorphic structure and commute with the action of F_∞, and sections

$$s_\nabla\colon\ E_{\mathbb{C}} \to P^1_{X/S}(E)_{\mathbb{C}}$$

such that $(\mathscr{A}t_{X/S}E, s_\nabla)$ is an arithmetic extension. This correspondence allows us to associate its *arithmetic Atiyah extension* $(\mathscr{A}t_{X/S}E, s_\nabla)$ and its *arithmetic Atiyah class*

$$\widehat{\mathrm{at}}_{X/S}(E,\nabla) \in \widehat{\mathrm{Ext}}^1_X(E, E\otimes\Omega^1_{X/S})$$

to any vector bundle E on X equipped with an F_∞-invariant $\mathscr{C}^\infty$-connection ∇ on $E_{\mathbb{C}}$ that is compatible with the holomorphic structure.

If $\overline{E}$ is a hermitian vector bundle over X, we obtain the *arithmetic Atiyah extension* $(\mathscr{A}t_{X/S}E, s_{\nabla_{\overline{E}}})$ *of* $\overline{E}$ and its *arithmetic Atiyah class*

$$\widehat{\mathrm{at}}_{X/S}(\overline{E}) := \widehat{\mathrm{at}}_{X/S}(E,\nabla_{\overline{E}}) \in \widehat{\mathrm{Ext}}^1_X(E, E\otimes\Omega^1_{X/S}),$$

where $\nabla_{\overline{E}}$ denotes the unitary connection on $E^{\mathrm{hol}}_{\mathbb{C}}$ over $X_\Sigma(\mathbb{C})$ that is compatible with the complex structure. As a direct consequence of this definition and Lemma 1.3.1, we get a formula for the “second fundamental form" (compare the introduction and [**7**, 2.3.1])

$$\Psi\big(\widehat{\mathrm{at}}_{X/S}(\overline{E})\big) \in A^{0,1}\big(X_{\mathbb{R}},\ \mathscr{E}nd(E)\otimes\Omega^1_{X/S}\big).$$

Namely

$$\Psi\big(\widehat{\mathrm{at}}_{X/S}(\overline{E})\big) = -R_{\overline{E}}, \tag{2.1}$$

under the canonical identification

$$A^{1,1}\big(X_{\mathbb{R}},\ \mathscr{E}nd(E)\big) = A^{0,1}\big(X_{\mathbb{R}},\ \mathscr{E}nd(E)\otimes\Omega^1_{X/S}\big),$$

where $R_{\overline{E}} := \nabla^2_{\overline{E}}$ denotes the curvature of $\overline{E}$.

In particular, when $\overline{E}$ is a hermitian line bundle over X,

$$\frac{1}{2\pi i}\,\Psi\big(\widehat{\mathrm{at}}_{X/S}(\overline{E})\big) = -\frac{1}{2\pi i}\,R_{\overline{E}} =: c_1(\overline{E}) \tag{2.2}$$

is the first Chern form of $\overline{E}$.

We collect basic properties of the arithmetic Atiyah class.

***Proposition* 2.1.2.** — i) *Let (E,∇_E) and (F,∇_F) be vector bundles on X equipped with F_∞-invariant $\mathscr{C}^\infty$-connections compatible with the holomorphic structure. We equip the tensor product $E\otimes F$ with the product connection. Then the equality*

$$\widehat{\mathrm{at}}_{X/S}(E\otimes F,\nabla_{E\otimes F})=\widehat{\mathrm{at}}_{X/S}(E,\nabla_E)\otimes F+E\otimes\widehat{\mathrm{at}}_{X/S}(F,\nabla_F)$$

holds in $\widehat{\mathrm{Ext}}^1_X(E\otimes F,E\otimes F\otimes\Omega^1_{X/S})$.

ii) *Let $\overline{E}$ and $\overline{F}$ be hermitian vector bundles on X, and $\overline{E}\otimes\overline{F}$ their tensor product equipped with the product hermitian metric. Then the equality*

$$\widehat{\mathrm{at}}_{X/S}(\overline{E}\otimes\overline{F})=\widehat{\mathrm{at}}_{X/S}(\overline{E})\otimes F+E\otimes\widehat{\mathrm{at}}_{X/S}(\overline{F})$$

holds in $\widehat{\mathrm{Ext}}^1_X(E\otimes F,E\otimes F\otimes\Omega^1_{X/S})$.

iii) *Let $\overline{E}$ be a hermitian vector bundle on X, and $\overline{E}^\vee$ the dual hermitian vector bundle. Then the equality*

$$\widehat{\mathrm{at}}_{X/S}(\overline{E})=-\widehat{\mathrm{at}}_{X/S}(\overline{E}^\vee)\tag{2.3}$$

holds in

$$\begin{aligned}\widehat{\mathrm{Ext}}^1_X(E,E\otimes\Omega^1_{X/S})&\simeq\widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X,E^\vee\otimes E\otimes\Omega^1_{X/S})\\&\simeq\widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X,(E^\vee)^\vee\otimes E^\vee\otimes\Omega^1_{X/S})\simeq\widehat{\mathrm{Ext}}^1_X(E^\vee,E^\vee\otimes\Omega^1_{X/S}),\end{aligned}\tag{2.4}$$

where the first and last isomorphisms in (2.4) *are the canonical isomorphisms in* [**7**, 2.4.6], *and the second one is deduced from the isomorphism $E^\vee\otimes E\simeq E\otimes E^\vee$ exchanging the two factors and the canonical biduality isomorphism $E\simeq(E^\vee)^\vee$.*

iv) *Let $f:X\to Y$ be a morphism of integral arithmetic schemes which are generically smooth l.c.i. over S. Let (E,∇_E) be a vector bundle on Y with F_∞-invariant $\mathscr{C}^\infty$-connection which is compatible with the holomorphic structure. The canonical map $f^*:f^*\Omega^1_{Y/S}\to\Omega^1_{X/S}$ induces a homomorphism*

$$\widehat{\mathrm{Ext}}^1_X(f^*E,f^*E\otimes f^*\Omega^1_{Y/S})\longrightarrow\widehat{\mathrm{Ext}}^1_X(f^*E,f^*E\otimes\Omega^1_{X/S})$$

by pushout along $\mathrm{id}_{f^*E}\otimes f^*$. We still denote the image of $f^*\widehat{\mathrm{at}}_{Y/S}(E,\nabla_E)$ under this map by $f^*\widehat{\mathrm{at}}_{Y/S}(E,\nabla_E)$ and equip $f^*E^{\mathrm{hol}}_{\mathbb{C}}$ with the connection $f^*\nabla_E$ described in (1.12). *Then we have the equality*

$$f^*\widehat{\mathrm{at}}_{Y/S}(E,\nabla_E)=\widehat{\mathrm{at}}_{X/S}(f^*E,f^*\nabla_E).$$

in $\widehat{\mathrm{Ext}}^1_X(f^*E,f^*E\otimes\Omega^1_{X/S})$.

v) *Let $f:X\to Y$ be a morphism of integral arithmetic schemes which are generically smooth l.c.i. over S. Let $\overline{E}$ denote a hermitian vector bundle on Y,* *and $f^*\overline{E}$ its pull-back on X. Then the inverse image $f^*\widehat{\mathrm{at}}_{Y/S}(\overline{E})$ may be defined in $\widehat{\mathrm{Ext}}^1_X(f^*E, f^*E \otimes \Omega^1_{X/S})$ as in* iv) *and satisfies*

$$f^*\widehat{\mathrm{at}}_{Y/S}(\overline{E}) = \widehat{\mathrm{at}}_{X/S}(f^*\overline{E}).$$

Proof. — Assertion i) follows from Lemma 1.1.4 and its variant for $\mathscr{C}^\infty$-connections compatible with the holomorphic structure, and assertion ii) is a direct consequence of i) and of the fact that the Chern connection of a tensor product of hermitian vector bundles coincides with the tensor product of their Chern connections. To establish iii), observe that Corollary 1.1.5 and the compatibility of the canonical splitting given there with holomorphic and hermitian structures leads to the equality

$$(E^\vee \otimes \widehat{\mathrm{at}}_{X/S}(\overline{E})) \circ j_E = -(\widehat{\mathrm{at}}_{X/S}(\overline{E}^\vee) \otimes E) \circ j_E$$

in $\widehat{\mathrm{Ext}}^1_X(\mathscr{O}_X, \mathscr{E}nd(E) \otimes \Omega^1_{X/S})$ where $. \circ j_E$ denotes the pushout along j_E. Equality (2.3) then follows from the very definitions of the isomorphisms in (2.4) in [**7**, Prop. 2.4.6]. Assertions iv) and v) follow from 1.1.6. □

Let $\overline{E}$ be a hermitian line bundle on X. We give a cocycle description of $\widehat{\mathrm{at}}(\overline{E})$ based on the description of arithmetic extension groups by Čech cocycles given in Appendix A.

***Proposition* 2.1.3.** — *Let $\overline{E} = (E, h)$ be a hermitian vector bundle of rank n on X. Choose an affine, open cover $\mathscr{U} = (U_i)_{i\in I}$ of X such that E admits a frame*

$$f_i : \mathscr{O}^n_{U_i} \xrightarrow{\sim} E|_{U_i}$$

over each U_i. For $i \in I$, we define

$$h_i := h(f_{i,\mathbb{C}}, f_{i,\mathbb{C}}) = \big(h(f_{i,\mathbb{C}}(e_l), f_{i,\mathbb{C}}(e_k))\big)_{1\le k,l\le n} \in \mathrm{Mat}_n\big(\mathscr{C}^\infty(U_{i,\Sigma}(\mathbb{C}), \mathbb{C})^{F_\infty}\big),$$

where $e_l := (\delta_{\alpha l})_{1\le\alpha\le n}$, and

$$\partial \log\, h_i := f_i \circ h_i^{-1} \circ (\partial h_i) \circ f_i^{-1} \in A^0\big(U_{i,\mathbb{R}}, End(E) \otimes \Omega^1_{X/S}\big).$$

For $i, j \in I$, we define

$$\begin{aligned} f_{ij} &:= f_j^{-1} \circ f_i &\in\ & \mathrm{Mat}_n\big(\mathscr{O}_X(U_{ij})\big) \\ \mathrm{dlog}\, f_{ij} &:= f_j \circ (df_{ij}) \circ f_i^{-1} &\in\ & \Gamma\big(U_{ij}, End(E) \otimes \Omega^1_{X/S}\big). \end{aligned}$$

Then the isomorphism

$$\hat\rho_{\mathscr{U},E,E\otimes\Omega^1_{X/S}} : \widehat{\mathrm{Ext}}^1_X(E, E \otimes \Omega^1_{X/S}) \to \check{H}^0\big(\mathscr{U}, C(\mathrm{ad}_{End(E)\otimes\Omega^1_{X/S}})\big)$$

constructed in Lemma A.0.1 *maps $\widehat{\mathrm{at}}_{X/S}(\overline{E})$ to the class of*

$$\big((-\mathrm{dlog}\, f_{ij})_{i,j\in I}, (-\partial\log h_i)_{i\in I}\big).$$

Proof. — Let ∇ denote the unitary connection on $E_{\mathbb{C}}$ which is compatible with the holomorphic structure. We compute a cocycle $\big((\alpha_{ij})_{i,j},(\beta_i)_i\big)$ which represents the image of the arithmetic extension $(\mathcal{A}t(E),s_\nabla)$ under $\hat{\rho}_{\mathcal{U},E,E\otimes\Omega^1_{X/S}}$. We follow the construction of $\hat{\rho}_{\mathcal{U},E,E\otimes\Omega^1_{X/S}}$ given in Appendix A. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}nd(E)\otimes\Omega^1_{X/S} & \longrightarrow & W & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0\\
 & & || & & \downarrow & & \downarrow j_E & & \\
0 & \longrightarrow & E\otimes\Omega^1_{X/S}\otimes E^\vee & \longrightarrow & P^1_{X/S}(E)\otimes E^\vee & \longrightarrow & E\otimes E^\vee & \longrightarrow & 0.
\end{array}$$

where the lower exact sequence is the extension $\mathcal{A}t(E)\otimes E^\vee$ and the upper exact sequence is the pullback $(\mathcal{A}t(E)\otimes E^\vee)\circ j_E$ of the lower exact sequence by j_E. There is a unique connection $\nabla_i\colon E|_{U_i}\to E|_{U_i}\otimes\Omega^1_{U_i/S}$ such that $\nabla_i(f_i)=0$. It satisfies

$$\nabla_j(f_i)=\nabla_j(f_j\cdot f_{ij})=f_j\cdot df_{ij},$$

where the frames f_i and f_j are seen as "line vectors" with entries sections of E. The connection ∇_i determines an $\mathcal{O}_{U_i}$-linear splitting s_{∇_i} of $\mathcal{A}t(E)$ over U_i as in (1.6). We write $j_E(1_X)=f_i\otimes f_i^\vee$, where $f_i^\vee$ denotes the dual frame of $E^\vee$ — which we may see as a "column vector" with entries sections of $E^\vee$ — and get

$$\begin{aligned}
\alpha_{ij} &= (s_{\nabla_j}\otimes\mathrm{id}_{E^\vee}-s_{\nabla_i}\otimes\mathrm{id}_{E^\vee})\circ j_E(1_X)\\
&= (-\nabla_j+\nabla_i)f_i\otimes f_i^\vee\\
&= \big(-f_j\cdot(df_{ij})\big)\otimes f_i^\vee\\
&= -\mathrm{dlog}\, f_{ij}.
\end{aligned}$$

We observe that we have

$$\nabla^{1,0}(f_i)=f_i\cdot h_i^{-1}\cdot(\partial h_i)$$

by [**43**, III.2, eq. (2.1)]. Hence

$$\begin{aligned}
\beta_i &= (s_{\nabla^{1,0}}\otimes\mathrm{id}_{E^\vee}-s_{\nabla_i}\otimes\mathrm{id}_{E^\vee})\circ j_E(1_X)\\
&= -f_i\circ h_i^{-1}\circ(\partial h_i)\circ f_i^{-1}\\
&= -\partial\log\, h_i.
\end{aligned}$$

Our claim follows. □

The properties of the arithmetic Atiyah class in Proposition 2.1.2 may be recovered by straightforward cocycle computations using Proposition 2.1.3.

2.1.4. — Let us indicate that there is a straightforward generalization of the construction of the arithmetic extension class $\mathrm{at}_{X/S}(E,\nabla)$ in $\widehat{\mathrm{Ext}}^1_X(E,E\otimes\Omega^1_{X/S})$ given above when S is a flat arithmetic scheme over $\operatorname{Spec} R$, X an integral arithmetic scheme equipped with a l.c.i. morphism $\pi: X\to S$, smooth over K, and ∇ is a relative $\mathcal{C}^\infty$-connection for $X_\Sigma(\mathbb{C})/S_\Sigma(\mathbb{C})$.

If the relative connection ∇ is induced by an absolute connection ∇_X via the canonical map

$$\Omega^1_{X/\operatorname{Spec} R} \to \Omega^1_{X/S}, \tag{2.5}$$

the relative and the absolute Atiyah class are related as follows. The commutative square

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ \downarrow \pi & & \downarrow \\ S & \longrightarrow & \operatorname{Spec} R \end{array}$$

induces by Lemma 1.1.6 a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E\otimes\Omega^1_{X/\operatorname{Spec} R} & \longrightarrow & P^1_{X/\operatorname{Spec} R}(E) & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \phi & & \| & & \\ 0 & \longrightarrow & E\otimes\Omega^1_{X/S} & \longrightarrow & P^1_{X/S}(E) & \longrightarrow & E & \longrightarrow & 0. \end{array} \tag{2.6}$$

which identifies $\mathscr{At}_{X/S}(E)$ with the pushout of $\mathscr{Jet}^1_{X/\operatorname{Spec} R}(E)$ along the canonical map (2.5). We have $s_\nabla = \phi_{\mathbb{C}} \circ s_{\nabla_X}$. Hence the pushout of the arithmetic extension $(\mathscr{Jet}^1_{X/\operatorname{Spec} R}E, s_\nabla)$ along the canonical map (2.5) is by its very definition in [**7**, 2.4.1] the arithmetic extension $(\mathscr{At}_{X/S}(E), s_\nabla)$.

2.2. The first Chern class in arithmetic Hodge cohomology

2.2.1. — For a hermitian vector bundle $\overline{E}$ on an arithmetic scheme X, flat and l.c.i. over $S = \operatorname{Spec} R$, we put

$$\hat{c}_1^H(\overline{E}) := \hat{c}_1^H(X/S, \overline{E}) := \operatorname{tr}_E \circ (\widehat{\operatorname{at}}_{X/S}(\overline{E}) \otimes E^\vee) \circ j_E \in \widehat{\operatorname{Ext}}^1(\mathcal{O}_X, \Omega^1_{X/S})$$

where $\operatorname{tr}_E : E\otimes E^\vee \to \mathcal{O}_X$ and $j_E : \mathcal{O}_X \to End(E) \simeq E\otimes E^\vee$ denote the canonical morphisms. We call $\hat{c}_1^H(\overline{E})$ the *first Chern class of $\overline{E}$ in arithmetic Hodge cohomology.*

When $\overline{E}$ is a hermitian line bundle, tr_E and j_E are the "obvious" isomorphisms, and $\hat{c}_1^H(\overline{E})$ is nothing else than $\widehat{\operatorname{at}}_{X/S}(\overline{E})$ in

$$\widehat{\operatorname{Ext}}^1_X(E, E\otimes\Omega^1_{X/S}) \simeq \widehat{\operatorname{Ext}}^1_X(\mathcal{O}_X, E^\vee\otimes E\otimes\Omega^1_{X/S}) \simeq \widehat{\operatorname{Ext}}^1_X(\mathcal{O}_X, \Omega^1_{X/S}).$$

Using the the description of the arithmetic Atiyah class in terms of Čech cocycles in Proposition 2.1.3, and the expression of the differential of the determinant in terms of the trace, we obtain, after a straightforward computation:

$$\hat{c}_1^H(\overline{E}) = \hat{c}_1^H(\det \overline{E}). \tag{2.7}$$

Proposition 2.1.3 also leads immediately to the following description of the first Chern class in arithmetic Hodge cohomology for hermitian line bundles:

***Lemma* 2.2.2.** — *Let $\overline{L}$ be a hermitian line bundle on an arithmetic scheme X. Choose an affine, open cover $\mathcal{U} = (U_i)_{i\in I}$ of X such that L admits a trivialization $l_i \in \Gamma(U_i, L)$ over U_i. Put*

$$f_{ij} := l_j^{-1} \cdot l_i \in \Gamma(U_{ij}, \mathcal{O}^*).$$

Then

$$\hat{\rho}_{\mathcal{U},\Omega^1_{X/S}}\big(\hat{c}_1^H(\overline{L})\big) = \big[(-\mathrm{dlog}\, f_{ij})_{i,j\in I}, (-\partial\mathrm{log}\,\|l_i\|^2)_{i\in I}\big].$$

2.2.3. — Let $\widehat{\mathrm{Pic}}(X)$ denote the group of isometry classes of hermitian line bundles on X. It follows immediately from Proposition 2.1.2 that the map

$$\hat{c}_1^H : \widehat{\mathrm{Pic}}(X) \longrightarrow \widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X, \Omega^1_{X/S})$$

is a group homomorphism which satisfies

$$\hat{c}_1^H(X/S,\,.\,) \circ f^* = f^* \circ \hat{c}_1^H(Y/S,\,.\,)$$

for every morphism $f\colon X \to Y$ of integral, flat, l.c.i, arithmetic S-schemes.

2.2.4. — We consider the diagrams

(2.8)

$$\begin{array}{ccccccc}
\mathcal{O}(X)^* & \overset{\log|.|^2}{\to} & A^{0,0}(X_{\mathbb{R}}) & \overset{a}{\to} & \widehat{\mathrm{Pic}}(X) & \to & \mathrm{Pic}(X) \\
\downarrow -\mathrm{dlog} & & \downarrow -\partial & & \downarrow \hat{c}_1^H & & \downarrow c_1^H \\
\Gamma(X, \Omega^1_{X/S}) & \to & A^0(X_{\mathbb{R}}, \Omega^1_{X/S}) & \overset{b}{\to} & \widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X, \Omega^1_{X/S}) & \overset{\nu}{\to} & \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \Omega^1_{X/S}).
\end{array}$$

and

$$(2.9)\qquad \begin{array}{ccc}
\widehat{\mathrm{Pic}}(X) & \xrightarrow{c_1} & A^{1,1}(X_{\mathbb{R}}) \\
\downarrow \hat{c}_1^H & & \downarrow \iota \\
\widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X, \Omega^1_{X/S}) & \xrightarrow{\Psi} & A^{0,1}(X_{\mathbb{R}}, \Omega^1_{X/S}).
\end{array}$$

Here $A^{p,p}(X_{\mathbb{R}})$ is by definition the space of *real* (p,p)-forms α on the complex manifold $X_\Sigma(\mathbb{C})$ which satisfy $F_\infty(\alpha) = (-1)^p\alpha$ (compare [**17**, 3.2.1]). The monomorphism ι is defined by

$$A^{p,p}(X_{\mathbb{R}}) \hookrightarrow A^{0,p}(X_{\mathbb{R}}, \Omega^p_{X/S}), \;\; \alpha \mapsto (2\pi i)^p\alpha$$

(compare [**7**, 1.1.5]). Furthermore we have used the following morphisms:

$$\begin{array}{rcl}
\log|.|^2 : \mathcal{O}(X)^* & \longrightarrow & A^{0,0}(X_{\mathbb{R}}), \;\; f \longmapsto \log|f|^2, \\
\mathrm{dlog}\colon \mathcal{O}(X)^* & \longrightarrow & \Gamma(X, \Omega^1_{X/S}), \;\; f \longmapsto f^{-1}df, \\
\Gamma(X, \Omega^1_{X/S}) & \longrightarrow & A^0(X_{\mathbb{R}}, \Omega^1_{X/S}), \;\; \alpha \longmapsto \alpha_{\mathbb{C}}, \\
\partial\colon A^{0,0}(X_{\mathbb{R}}) & \longrightarrow & A^0(X_{\mathbb{R}}, \Omega^1_{X/S}), \;\; f \longmapsto \partial f, \\
a\colon A^{0,0}(X_{\mathbb{R}}) & \longrightarrow & \widehat{\mathrm{Pic}}(X), \;\; f \longmapsto [(\mathcal{O}_X, \|.\|_f)] \text{ with } \|1_X\|_f^2 = \exp f, \\
b\colon A^0(X_{\mathbb{R}}, \Omega^1_{X/S}) & \longrightarrow & \widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X, \Omega^1_{X/S}), \, T \longmapsto \Big[0 \to \Omega^1_{X/S} \overset{\binom{\mathrm{id}}{0}}{\to} \Omega^1_{X/S} \oplus \mathcal{O}_X
\end{array}$$

$$\overset{(0,\mathrm{id})}{\to} \mathcal{O}_X \to 0, s := \binom{T}{\mathrm{id}}\Big]$$

(compare the introduction and [**7**, 2.2]),

$$\begin{aligned}
\widehat{\mathrm{Pic}}(X) &\longrightarrow \mathrm{Pic}(X)\,,\ [(L,\|\,.\,\|)] \longmapsto [L],\\
\nu\colon \widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X,\Omega^1_{X/S}) &\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X,\Omega^1_{X/S})\,,\ [(\mathcal{E},s)] \longmapsto [\mathcal{E}],\\
c_1^H\colon \mathrm{Pic}(X) &\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X,\Omega^1_{X/S})\,,\ [L] \longmapsto [\mathrm{tr}_L \circ \mathrm{at}_{X/S}(L) \circ i_L],\\
c_1\colon \widehat{\mathrm{Pic}}(X) &\longrightarrow A^{1,1}(X_{\mathbb{R}})\,,\ [\overline{L} = (L,\|\,.\,\|)] \longmapsto -(2\pi i)^{-1}\nabla^2_{\overline{L}},\\
\Psi\colon \widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X,\Omega^1_{X/S}) &\longrightarrow A^{0,1}(X_{\mathbb{R}},\Omega^1_{X/S}),\ \text{defined in [7, 2.3.1]}.
\end{aligned}$$

The horizontal lines in (2.8) are exact by [**18**, (2.5.2)] and [**7**, 2.2.1]. Observe the analogy between (2.8) and [**18**, (2.5.2)].

***Proposition* 2.2.5**. — *The diagrams* (2.8) *and* (2.9) *are commutative.*

Proof. — For f in $\mathcal{O}(X)^*$, we have

$$\partial \log |f|^2 = \frac{\partial(f\overline{f})}{f\overline{f}} = \frac{\partial f}{f} = \frac{df}{f} = \mathrm{dlog}\, f \tag{2.10}$$

which shows the commutativity of the left square in (2.8). The unitary connection ∇_f on $(\mathcal{O}_X, \|.\|_f)$ that is compatible with the holomorphic structure is given according to [**43**, III.2 formula (2.1)] by the formula

$$\nabla_f^{1,0}(1) = \partial f \in A^0(X_{\mathbb{R}}, \Omega^1_{X/S}).$$

Taking into account the correspondence between connections and splittings in 1.3 above (and notably the sign in (1.18)), it follows that the middle square commutes. The commutativity of the right square holds by definition. The square (2.9) is commutative by formula (2.2). □

3. Hermitian line bundles with vanishing arithmetic Atiyah class

This section is devoted to the proof of assertions $\mathbf{I1}_{X,\Sigma}$ and $\mathbf{I2}_{X,\Sigma}$ in the Introduction (see 0.4 *supra*).

In the first part of the section, we establish the special case of $\mathbf{I1}_{X,\Sigma}$ where X is an abelian variety and Σ has a unique or two conjugate elements. As mentioned in the Introduction, the validity of $\mathbf{I1}_{X,\Sigma}$ in this case has been established by Bertrand ([**4, 5**]) under suitable hypotheses of "real multiplication".

In the second part of the section, we use some classical properties of Picard varieties to extend $\mathbf{I1}_{X,\Sigma}$ to arbitrary smooth projective varieties X over number fields. Finally we establish $\mathbf{I2}_{X,\Sigma}$, which describes the kernel of the first class Chern in arithmetic Hodge cohomology $\hat{c}_1^H$ "up to a finite group".

3.1. Transcendence and line bundles with connections on abelian varieties. — The next paragraphs are devoted to the proof of the following theorem:

***Theorem* 3.1.1.** — *Let A be an abelian variety over a number field K, and (L, ∇) a line bundle over A equipped with a connection (defined over K).*

If there exists a field embedding $\sigma : K \hookrightarrow \mathbb{C}$ and a hermitian metric $\|.\|$ on the complex line bundle L_σ on $A_\sigma(\mathbb{C})$ such that the connection ∇_σ is unitary with respect to $\|.\|$, then L has a torsion class in $\mathrm{Pic}(A)$*, and the metric $\|.\|$ has vanishing curvature.*

Actually this implies that the connection ∇ is the unique one on L such that (L, ∇) has a torsion class in the group of isomorphism classes of line bundles with connections over A (see 3.2 *infra*).

Let us indicate that this result admits an alternative formulation in terms of universal vector extensions of abelian varieties and their maximal compact subgroups, in the spirit of Bertrand's articles [**4, 5**]:

***Theorem* 3.1.2.** — *Let B be an abelian variety over a number field K, $B^\#$ the universal vector extension of B, and P a point in $B^\#(K)$.*

If there exists a field embedding $\sigma : K \hookrightarrow \mathbb{C}$ such that the point P_σ belongs to the maximal compact subgroup of $B^\#_\sigma(\mathbb{C})$, then P is a torsion point in $B^\#(K)$.

Actually, for any given K and σ, the implications in the statement of Theorems 3.1.1 and 3.1.2 are equivalent when the abelian varieties A and B are dual to each other. This follows from the description of the universal vector extension $B^\#$ and of the maximal compact subgroup of $B^\#_\sigma(\mathbb{C})$ recalled in Appendix B (see notably B.6 applied to $k = K$ and $X = A$, in which case $E_{X/k} = B^\#$, and B.7 applied to $X = A_\sigma$, in which case $E_{X/\mathbb{C}}(\mathbb{C}) = B^\#_\sigma(\mathbb{C})$).

The formulation in Theorem 3.1.1 turns out to be more convenient for the proof, which will proceed along the following lines.

Firstly, the data (L, ∇) in Theorem 3.1.1 may be "translated" in terms of algebraic groups: the total space of the $\mathbb{G}_m$-torsor associated to L defines a commutative algebraic group $L^\times$, and the connection ∇ an hyperplane in its Lie algebra $\mathrm{Lie}\, L^\times$. Then an application of the theorem of Schneider-Lang to this situation will show that, *if there exists a family $(\gamma_1, \ldots, \gamma_g)$ of points in the lattice of periods Γ_{A_σ} of A_σ which constitutes a $\mathbb{C}$-basis of* $\mathrm{Lie}\, A_\sigma$ *such that the monodromy of the complex line bundle with connection $(L_\sigma, \nabla_\sigma)$ along each γ_i lies in $\overline{\mathbb{Q}}^*$, then L is torsion.*(4)

(4) *Added in proof.* After the acceptation of this article, we realized that related results had been obtained by Simpson, [**40**], Section 3. Namely Theorem 1 of *loc. cit.* establishes the validity of the previous criterion under the stronger assumption that the monodromy of $(L_\sigma, \nabla_\sigma)$ along any γ in Γ belongs to $\overline{\mathbb{Q}}^*$. Simpson's proof relies on transcendence results of Waldschmidt [**42**] concerning

This criterion easily leads to a derivation of Theorem 3.1.1 when the image of the embedding σ lies in $\mathbb{R}$. Indeed a simple "reality" argument then shows that the monodromy of $(L_\sigma, \nabla_\sigma)$ along the "real periods" of A_σ lies in $\{1, -1\}$.

When the image of σ does not lie in $\mathbb{R}$, we may assume that K is Galois over $\mathbb{Q}$, and consider the involution τ of K such that $\sigma \circ \tau = \overline{\sigma}$. It will turn out that we may apply the above criterion to the line bundle with connection on $A \times_K A_\tau$ defined as the external tensor product of (L, ∇) and (L_τ, ∇_τ) to establish that $L \boxtimes L_\tau$, hence L, is torsion.

3.1.3. *Line bundles with connections on abelian varieties.* — Let A be an abelian variety over a field k of characteristic zero, and L a line bundle over A. We may choose a rigidification of L, namely a trivialization $\phi : k \simeq L_e$ of its fiber at the zero element e of $A(k)$, or equivalently the vector $\ell := \phi(1)$ in $L_e \setminus \{0\}$.

In the sequel, we shall assume that the following equivalent[(5)] conditions are satisfied:

(i) *the line bundle L is algebraically equivalent to the trivial line bundle;*

(ii) *the Atiyah class* $\mathrm{at}_{A/k} L (= \mathrm{jet}^1_{A/k} L)$ *of L vanishes;*

(iii) *the line bundle L may be equipped with an algebraic connection ∇.*

Observe that the connection ∇ is necessarily flat[(6)] and that the set of connections on A is a torsor under the k-vector space $\Gamma(A, \Omega^1_{A/k}) \simeq (\mathrm{Lie}\, A)^\vee$ of regular 1-forms on A, which acts additively on this set.

Beside, the $\mathbb{G}_m$-torsor $L^\times$ defined by deleting the zero section from the total space[(7)] $\mathbb{V}(L^\vee)$ of L admits a unique structure of commutative algebraic group over k such that the diagram

$$0 \longrightarrow \mathbb{G}_{m,k} \xrightarrow{\phi} L^\times \xrightarrow{\pi} A \longrightarrow 0, \tag{3.1}$$

— where ϕ denotes the composite morphism $\mathbb{G}_{m,k} \overset{\phi}{\simeq} L^\times_e \hookrightarrow L^\times$ and π the restriction of the "structural morphism" from $\mathbb{V}(L^\vee)$ to A — becomes a short exact sequence of

exponentials of abelian integrals, which themselves are deduced from the Theorem of Schneider-Lang. The derivation of the previous criterion in 3.1.5 *infra* may be seen as a refined geometric variant of the arguments of Waldschmidt and Simpson.

(5) Indeed (ii) and (iii) are equivalent by the very definition of $\mathrm{at}_{A/k} L$, (ii) is equivalent to the rational vanishing of the first Chern class of L (hence (i) implies (ii)), and if the first Chern class of L vanishes rationally, one gets (i) from [**28**, II.2 Cor. 1 to Th. 2], as $\mathrm{Pic}^0_{A/k} = \mathrm{Pic}^\tau_{A/k}$ by [**35**, Cor. 6.8].

(6) To establish this, we may reduce to the case $k = \mathbb{C}$ and use transcendental arguments. We may also assume that k is algebraically closed, and observe that the curvature ∇^2 of an algebraic connection on L depends only on the isomorphism class of L and defines a morphism of algebraic groups over k from the dual abelian variety $A^\vee$ to the additive group $\Gamma(A, \Omega^2_{A/k}) (\simeq \wedge^2 (\mathrm{Lie}\, A)^\vee)$. Since A is proper and connected, any such morphism is zero.

(7) Namely, the spectrum of the quasi-coherent $\mathcal{O}_A$-algebra $\bigoplus_{n \in \mathbb{N}} L^{\vee \otimes n}$.

commutative algebraic groups. Its zero element is the k-point $\epsilon \in L^\times(k)$ defined by ℓ. (See for instance [**39**], VII.3.16.)

From (3.1), we derive a short exact sequence of k-vector spaces:

$$0 \longrightarrow \operatorname{Lie} \mathbb{G}_{m,k} \xrightarrow{\operatorname{Lie}\phi} \operatorname{Lie} L^\times \xrightarrow{\operatorname{Lie}\pi} \operatorname{Lie} A \longrightarrow 0. \tag{3.2}$$

Recall that a connection over a vector bundle on a smooth algebraic variety may be described *à la* Ehresmann as an equivariant splitting of the differential of the structural morphism of its frame bundle (see for instance [**30**], Chapter II, or [**41**], Chapter 8; the constructions of *loc. cit.* in a differentiable setting can be immediately transposed in the algebraic framework of smooth algebraic varieties). In the present situation, a connection ∇ on L may thus be seen as a $\mathbb{G}_{m,k}$-equivariant splitting of the surjective morphism of vector bundles over $L^\times$ defined by the differential of π:

$$D\pi : T_{L^\times} \longrightarrow \pi^* T_A.$$

In particular, its value at the unit element ϵ of $L^\times$ defines a k-linear splitting

$$\Sigma : \operatorname{Lie} A \longrightarrow \operatorname{Lie} L^\times$$

of (3.2).

Conversely, from any k-linear right inverse Σ of $\operatorname{Lie}\pi$, we deduce a $\mathbb{G}_m$-equivariant splitting of $D\pi$ by constructing its $L^\times$-equivariant extension to $L^\times$.

Through these constructions, connections on L and k-linear splittings of (3.2) correspond bijectively. Indeed, by means of the identification

$$\begin{array}{ccc} \operatorname{Lie} \mathbb{G}_{m,k} & \xrightarrow{\sim} & k \\ \lambda \cdot X \frac{\partial}{\partial X} & \longmapsto & \lambda, \end{array}$$

the set of k-linear splittings of (3.2) becomes naturally a torsor under $(\operatorname{Lie} A)^\vee$, and the above constructions are compatible with the (additive) actions of $(\operatorname{Lie} A)^\vee$ on the set of these splittings and on the set of connections on L.

This correspondence may also be described as follows. A linear splitting Σ as above may also be seen as a morphism $\tilde{\ell} : A_{e,1} \to L^\times_{\epsilon,1}$ from the first infinitesimal neighbourhood $A_{e,1}$ of e in A to the first infinitesimal neighbourhood $L^\times_{\epsilon,1}$ of ϵ in $L^\times$ which is a right inverse of the map $\pi_{\epsilon,1} : L^\times_{\epsilon,1} \to A_{e,1}$ deduced from π. In other words, $\tilde{\ell}$ is a section of L over $A_{e,1}$ such that $\tilde{\ell}(e) = l$. The connection ∇ associated to Σ is the unique one such that $\nabla\tilde{\ell}(e) = 0$.

3.1.4. *The complex case.* — If G is a commutative algebraic group over $\mathbb{C}$, its exponential map will be denoted $\exp_G$. It is the unique morphism of $\mathbb{C}$-analytic Lie groups

$$\exp_G : \operatorname{Lie} G \longrightarrow G(\mathbb{C})$$

whose differential at $0 \in \operatorname{Lie} G$ is $\mathrm{Id}_{\operatorname{Lie} G}$. Its kernel

$$\Gamma_G := \ker \exp_G$$

is a discrete additive subgroup of $\operatorname{Lie} G$. When G is connected, $\exp_G$ is a universal covering of $G(\mathbb{C})$, and Γ_G may be identified with the fundamental group $\pi_1(G(\mathbb{C}), 0_G)$, or with the homology group $H_1(G(\mathbb{C}), \mathbb{Z})$.

Let us go back to the situation considered in paragraph 3.1.3, in the case where the base field k is $\mathbb{C}$, and fix the algebraic connection ∇ on L.

Then the diagram

$$\begin{array}{ccc} \operatorname{Lie} L^{\times} & \xrightarrow{D\pi} & \operatorname{Lie} A \\ \downarrow \exp_{L^{\times}} & & \downarrow \exp_A \\ L^{\times}(\mathbb{C}) & \xrightarrow{\pi} & A(\mathbb{C}). \end{array}$$

is commutative. Consequently the morphism of groups

$$\exp_{L^{\times}} \circ \Sigma : \Gamma_A \longrightarrow L^{\times}(\mathbb{C})$$

takes its value in $\ker \pi \simeq \mathbb{C}^*$. It coincides with the monodromy representation

$$\rho : \Gamma_A = H_1(A(\mathbb{C}), \mathbb{Z}) \longrightarrow \mathbb{C}^*$$

of the line bundle with flat connection (L, ∇) — or more properly of the corresponding objects in the analytic category — over $A(\mathbb{C})$. Indeed, the horizontal $\mathbb{G}_{m,\mathbb{C}}$-equivariant foliation on $L^{\times}(\mathbb{C})$ defined by ∇ is translation invariant, and its leaves are precisely the translates in $L^{\times}(\mathbb{C})$ of the image of $\exp_{L^{\times}} \circ \Sigma$.

3.1.5. *An application of the Theorem of Schneider-Lang.* — To establish Theorem 3.1.1, we shall use the following classical transcendence result on commutative algebraic groups:

***Theorem* 3.1.6**. — *Let K be a number field and $\sigma : K \hookrightarrow \mathbb{C}$ a field embedding, and let G be a commutative algebraic group over K, and V a K-vector subspace of* $\operatorname{Lie} G$.

If there exists a basis $(\gamma_1, \ldots, \gamma_v)$ of the complex vector space V_σ such that, for every $i \in \{1, \ldots, v\}$, $\exp_{G_\sigma}(\gamma_i)$ belongs to $G(\overline{\mathbb{Q}})$, then V is the Lie algebra of some algebraic subgroup H of G.

We have denoted $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. By means of the embedding σ, it may be seen as an algebraic closure of K, and the group $G(\overline{\mathbb{Q}})$ of $\overline{\mathbb{Q}}$-rational points of G becomes a subgroup of the group $G_\sigma(\mathbb{C})$ of its complex points.

Observe also that the subgroup H whose existence is asserted in Theorem 3.1.6 may clearly be chosen connected, and then H is clearly unique, defined over K, and the group $H_\sigma(\mathbb{C})$ of its complex points coincides with $\exp_{G_\sigma}(V_\sigma)$.

Theorem 3.1.6 has been established by Lang ([**31**], IV.4, Theorem 2), who elaborated on some earlier work of Schneider on abelian functions and the transcendence of their values [**38**]. We refer the reader to [**42**] (where it appears as Théorème 5.2.1) for more details on Theorem 3.1.6 and its classical applications.

Let us point out that Theorem 3.1.6 is now subsumed by various renowned more recent results — namely, the transcendence criterion of Bombieri and the analytic subgroup theorem of Wüstholz. The reader may find a recent survey of these and related transcendence results on commutative algebraic groups in the monograph [**3**].

We now return to the situation considered in paragraph 3.1.3, where we assume that the base field k is a number field K.

Taking into account the relation in the complex case between the monodromy of connections on L and the exponential map of the algebraic group $L^{\times}$ described in 3.1.4, we may derive from the theorem of Schneider-Lang (Theorem 3.1.6 above) applied to the algebraic group $G = L^{\times}$:

***Corollary* 3.1.7.** — *Let A be an abelian variety of dimension g over a number field K, and (L, ∇) a line bundle over L equipped with a flat connection (defined over K).*

Let $\sigma : K \hookrightarrow \mathbb{C}$ be a field embedding, and let $\rho_\sigma : \Gamma_{A_\sigma} \longrightarrow \mathbb{C}^*$ denote the monodromy representation attached to the flat complex line bundle $(L_\sigma, \nabla_\sigma)$ over $A_\sigma(\mathbb{C})$.

If there exists $\gamma_1, \dots, \gamma_g$ in Γ_{A_σ} such that $(\gamma_1, \dots, \gamma_g)$ is a basis of the $\mathbb{C}$-vector space $\operatorname{Lie} A_\sigma$ *and such that, for every $i \in \{1, \dots, g\}$, $\rho_\sigma(\gamma_i)$ belongs to $\overline{\mathbb{Q}}^*$, then L has a torsion class in* $\operatorname{Pic}(A)$.

Observe that conversely, if n is a positive integer such that $L^{\otimes n} \simeq \mathcal{O}_A$, the unique connection ∇_L^{tor} on L such the n-th tensor power of the line bundle with connection $(L, \nabla_L^{\mathrm{tor}})$ is isomorphic to $(\mathcal{O}_A, d)$ is such that, for any $\sigma : K \hookrightarrow \mathbb{C}$, the image of the monodromy ρ_σ of $(L_\sigma, \nabla_{L,\sigma}^{\mathrm{tor}})$ lies in the n-th roots of unity, hence in $\overline{\mathbb{Q}}^*$. By elaborating slightly on the proof below, one may show that, with the notation of Corollary 3.1.7, the connection ∇ necessarily coincides with the connection ∇_L^{tor} so defined. We leave this to the interested reader.

Proof. — We consider the K-linear map $\Sigma : \operatorname{Lie} A \longrightarrow \operatorname{Lie} L^{\times}$ associated to the connection ∇ as in 3.1.3, and its image $V := \Sigma(\operatorname{Lie} A)$. The vectors $\tilde{\gamma}_i := \Sigma_\sigma(\gamma_i)$, $1 \leq i \leq g$, constitute a basis of the $\mathbb{C}$-vector space V_σ. Moreover the image $\exp_{L_\sigma^{\times}}(\tilde{\gamma}_i)$ of $\tilde{\gamma}_i$ by the exponential map of $L_\sigma^{\times}$ is the point of $L_{\sigma,e}^{\times} \simeq \mathbb{C}^*$ defined by the monodromy $\rho_\sigma(\gamma_i)$ of γ_i. According to our assumption, these images belong to $L^{\times}(\overline{\mathbb{Q}})$.

The theorem of Schneider-Lang now shows that V is the Lie algebra of a connected algebraic subgroup H of $L^{\times}$, defined over K. Since $\operatorname{Lie} \pi_{|H} : \operatorname{Lie} H = V \to \operatorname{Lie} A$ is an isomorphism of K-vector spaces, the morphism of algebraic groups $\pi_{|H} : H \to A$ is étale, and consequently H is an abelian variety over K and $\pi_{|H}$ an isogeny.

By the very construction of H as a subscheme of $L^\times$, the inverse image $\pi_{|H}^* L$ of L on H is trivial. If N denotes the degree of $\pi_{|H}$, it follows that $L^{\otimes N}$ — which is isomorphic to the norm, relative to $\pi_{|H}$, of $\pi_{|H}^* L$ — is a trivial line bundle. □

3.1.8. *Reality I.* — Let us keep the framework of paragraph 3.1.3, and suppose now that the base field k is $\mathbb{R}$.

The line bundle with connection (L, ∇) defines a real analytic line bundle with flat connection $(L^{\mathbb{R}}, \nabla^{\mathbb{R}})$ over the compact real analytic Lie group $A(\mathbb{R})$. Its monodromy defines a representation $\rho_{\mathbb{R}}$ of the fundamental group $\pi_1(A(\mathbb{R}), 0_A)$, or equivalently of the homology group $H_1(A(\mathbb{R})^\circ, \mathbb{Z})$ of the connected component of 0_A, with values in $\mathbb{R}^*$.

Actually the inclusion $\iota : A(\mathbb{R})^\circ \hookrightarrow A(\mathbb{C})$ defines an injective map of free abelian groups, of respective ranks g and $2g$,

$$\iota_* : H_1(A(\mathbb{R})^\circ, \mathbb{Z}) \longrightarrow H_1(A(\mathbb{C}), \mathbb{Z}),$$

and the monodromy representation $\rho_{\mathbb{R}}$ coincides with the restriction $\rho_{\mathbb{C}} \circ \iota_*$ of the monodromy representation

$$\rho_{\mathbb{C}} : H_1(A(\mathbb{C}), \mathbb{Z}) \longrightarrow \mathbb{C}^*$$

defined by the $\mathbb{C}$-analytic line bundle with flat connection $(L_{\mathbb{C}}, \nabla_{\mathbb{C}})$ over the compact $\mathbb{C}$-analytic Lie group $A(\mathbb{C})$. Moreover any $\mathbb{Z}$-basis of $\iota_*(H_1(A(\mathbb{R})^\circ, \mathbb{Z}))$ is a $\mathbb{C}$-basis of $H_1(A(\mathbb{C}), \mathbb{C}) \simeq \operatorname{Lie} A_{\mathbb{C}}$.

***Lemma* 3.1.9.** — *The following conditions are equivalent:*

(i) *There exists a hermitian metric $\|.\|$ on the complex line bundle $L_{\mathbb{C}}$ on $A(\mathbb{C})$ such that the connection $\nabla_{\mathbb{C}}$ is unitary with respect to $\|.\|$*[8]*.*

(ii) *The monodromy representation $\rho_{\mathbb{R}}$ takes its values in $\{1, -1\}$.*

Clearly Condition (i) is equivalent to:

(i′) *The monodromy representation $\rho_{\mathbb{C}}$ takes its values in $U(1) := \{z \in \mathbb{C} \mid |z| = 1\}$.*

In the sequel, we shall only use the implications (i) $\Rightarrow$ (i′) $\Rightarrow$ (ii), which are straightforward. To show (ii) $\Rightarrow$ (i′), let $\Gamma^+ := \iota_*(H_1(A(\mathbb{R})^\circ, \mathbb{Z}))$, and observe that the elements of $\Gamma_{A_{\mathbb{C}}}$ which are "purely imaginary" in $\operatorname{Lie} A_{\mathbb{C}} \simeq (\operatorname{Lie} A) \otimes_{\mathbb{R}} \mathbb{C}$ constitute a subgroup Γ^- of rank g such that $\Gamma^+ \cap \Gamma^- = \{0\}$, that $\Gamma / \Gamma^+ \oplus \Gamma^-$ is a 2-torsion group, and that the image $\rho_{\mathbb{C}}(\Gamma^-)$ of Γ^- by the monodromy representation lies in $U(1)$. We leave the details to the reader.

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[8] Or, equivalently, such that $\nabla_{\mathbb{C}}$ is the Chern connection associated to $\|.\|$.

3.1.10. *Reality II.* — In this paragraph, we still keep the framework of the paragraph 3.1.3, and we now assume that the base field k is $\mathbb{C}$. We may apply the considerations of the last paragraph to the abelian variety over $\mathbb{R}$ deduced from A by Weil restriction of scalar from $\mathbb{C}$ to $\mathbb{R}$. This leads to the following results, that we formulate without explicit reference to Weil restriction.

Let A_-, L_-, ∇_- be respectively the complex abelian variety, the line bundle over A_-, and the connection over L_- deduced from A, L, and ∇ by the base change $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}$ defined by complex conjugation.

Let us consider the complex abelian variety

$$B := A \times A_-,$$

the two projections

$$\mathrm{pr} : B \longrightarrow A \quad \text{ and } \mathrm{pr}_- : B \longrightarrow A_-,$$

and $(\tilde{L}, \tilde{\nabla})$ the line bundle with connection over B defined as the tensor product of $\mathrm{pr}^*(L, \nabla)$ and $\mathrm{pr}_-^*(L_-, \nabla_-)$.

Let $j : \operatorname{Lie} A \to \operatorname{Lie} A_-$ denote the canonical $\mathbb{C}$-antilinear isomorphism. It maps bijectively Γ_A onto Γ_{A_-}, and we may introduce the diagonal embedding

$$\begin{array}{rccc} \Delta : & \Gamma_A & \longrightarrow & \Gamma_A \oplus \Gamma_{A_-} \simeq \Gamma_B \\ & \gamma & \longmapsto & (\gamma, j(\gamma)). \end{array}$$

Observe that any $\mathbb{Z}$-basis $(\gamma_1, \ldots, \gamma_{2g})$ of Γ_A is a $\mathbb{R}$-basis of $\operatorname{Lie} A$, and consequently its image $(\Delta(\gamma_1), \ldots, \Delta(\gamma_{2g}))$ by Δ is a $\mathbb{C}$-basis of $\operatorname{Lie} B$.

Let ρ (resp. ρ_-, $\tilde{\rho}$) be the monodromy representation of Γ_A (resp. Γ_{A_-}, Γ_B) defined by the line bundle with connection (L, ∇) (resp. (L_-, ∇_-), $(\tilde{L}, \tilde{\nabla})$).

It is straightforward that, for any γ in Γ_A, the following relations hold:

$$\rho_-(j(\gamma)) = \overline{\rho(\gamma)},$$

and

$$\tilde{\rho}(\Delta(\gamma)) = \rho(\gamma).\rho_-(j(\gamma)) = |\rho(\gamma)|^2.$$

These observations establish:

***Lemma* 3.1.11**. — *If there exists a hermitian metric $\|.\|$ on the complex line bundle L on $A(\mathbb{C})$ such that the connection ∇ is unitary with respect to $\|.\|$, then the image $\Delta(\Gamma)$ of the diagonal embedding Δ contains a $\mathbb{C}$-basis of* $\operatorname{Lie} B$, *and is included in the kernel of the monodromy representation $\tilde{\rho}$ of $(\tilde{L}, \tilde{\nabla})$.*

3.1.12. *Conclusion of the proof of Theorem* 3.1.1. — The following statement is a straightforward consequence of Corollary 3.1.7 to the Theorem of Schneider-Lang, combined with Lemma 3.1.9 above:

***Corollary* 3.1.13.** — *Let A be an abelian variety over a number field K, and (L, ∇) a line bundle over A equipped with a flat connection defined over K, and let $\sigma : K \hookrightarrow \mathbb{C}$ be a field embedding that is* real, *namely such that its image $\sigma(K)$ lies in $\mathbb{R}$.*

If there exists a hermitian metric $\|.\|$ on the complex line bundle L_σ on $A_\sigma(\mathbb{C})$ such that the connection ∇_σ is unitary with respect to $\|.\|$, then L has a torsion class in $\operatorname{Pic}(A)$.

If we use Lemma 3.1.11 instead of Lemma 3.1.9, we may prove:

***Corollary* 3.1.14.** — *Let A be an abelian variety over a number field K, and (L, ∇) a line bundle over A equipped with a flat connection defined over K.*

Let $\sigma : K \hookrightarrow \mathbb{C}$ be a field embedding, and let τ be a (necessarily involutive) automorphism of the field K such that $\sigma \circ \tau = \overline{\sigma}$.

If there exists a hermitian metric $\|.\|$ on the complex line bundle L_σ on $A_\sigma(\mathbb{C})$ such that the connection ∇_σ is unitary with respect to $\|.\|$, then L has a torsion class in $\operatorname{Pic}(A)$.

Observe that when $\tau = \mathrm{Id}_K$ Corollary 3.1.14 reduces to Corollary 3.1.13 above. We have however chosen to present explicitly the statement of Corollary 3.1.13 and its proof above, since the basic idea behind the proofs of Corollaries 3.1.13 and 3.1.14 appears more clearly in the first one, which indeed has been inspired by Bertrand's proof in [**4**] and [**5**].

Proof of Corollary 3.1.14. As usual we denote A_τ, L_τ, and ∇_τ respectively the abelian variety over K, the line bundle over A_τ, and the connection over L_τ deduced from A, L, and ∇ by the base change $\operatorname{Spec} K \to \operatorname{Spec} K$ defined by τ. We may also introduce the abelian variety over K

$$B := A \times A_\tau,$$

the two projections

$$\mathrm{pr} : B \longrightarrow A \quad \text{ and } \mathrm{pr}_\tau : B \longrightarrow A_\tau,$$

and $(\tilde{L}, \tilde{\nabla})$ the line bundle with connection over B defined as the tensor product of $\mathrm{pr}^*(L, \nabla)$ and $\mathrm{pr}_\tau^*(L_\tau, \nabla_\tau)$.

Lemma 3.1.11 applied to $(A_\sigma, L_\sigma, \nabla_\sigma)$ shows that the hypotheses of Corollary 3.1.7 are satisfied by the abelian variety B over K, and the line bundle with connection $(\tilde{L}, \tilde{\nabla})$ over B. Consequently $\tilde{L}$ has a torsion class in $\operatorname{Pic}(B)$, and so L itself — which is isomorphic to the restriction of $\tilde{L}$ to $A \times \{e\} \simeq A$ — has a torsion class in $\operatorname{Pic}(A)$. $\square$

Finally consider K, A, (L, ∇), σ and $\|.\|$ as in the statement of Theorem 3.1.1.

Let us first show that L has a torsion class in K. To achieve this, let us choose a finite field extension K' of K admitting an automorphism τ and an embedding σ' in $\mathbb{C}$ that extends σ and satisfies $\sigma' \circ \tau = \overline{\sigma'}$ — for instance the subfield K' of $\mathbb{C}$ generated by $\sigma(K)$ and its image by complex conjugation. We may apply Corollary

3.1.14 to the number field K' equipped with the complex embedding σ', and to the abelian variety $A_{K'}$ and the line bundle with connection $(L_{K'}, \nabla_{K'})$ deduced from A and (L, ∇) by the base change $\operatorname{Spec} K' \to \operatorname{Spec} K$. Therefore $L_{K'}$ has a torsion class in $\operatorname{Pic}(A_{K'})$. Since the base change morphism

$$\operatorname{Pic}(A) \longrightarrow \operatorname{Pic}(A_{K'})$$

is injective, this indeed implies that L has a torsion class in $\operatorname{Pic}(A)$.

To complete the proof of Theorem 3.1.1, it is sufficient to observe that the curvature of $\|.\|$ — or equivalently, of the $\mathscr{C}^\infty$-connection $\nabla_{\mathscr{C}^\infty} = \nabla_\sigma + \overline{\partial}_{L_\sigma}$ on L_σ — vanishes for reason of type[9]: it is a 2-form on $A_\sigma(\mathbb{C})$ of type $(2,0)$, since ∇_σ is holomorphic, and purely imaginary, since $\nabla_{\mathscr{C}^\infty}$ is unitary.

3.2. Hermitian line bundles with vanishing arithmetic Atiyah class on smooth projective varieties over number fields. — Let K be a number field, and Σ a non-empty set of field embeddings of K in $\mathbb{C}$, stable under complex conjugation.

To these data is naturally attached the arithmetic ring in the sense of Gillet-Soulé ([**17**], 3.1.1) defined as the triple (K, Σ, F_∞) where F_∞ denotes the conjugate linear involution of $\mathbb{C}^\Sigma$ defined by $F_\infty(a_\sigma)_{\sigma\in\Sigma} := (\overline{a_{\overline{\sigma}}})_{\sigma\in\Sigma}$.

3.2.1. — Recall that, for any line bundle M over a smooth projective connected variety V over $\mathbb{C}$, the following conditions are equivalent, as a consequence of the GAGA principle and Hodge theory:

(a1) *the Atiyah class* $\mathrm{at}_{V/\mathbb{C}} M$ *of* M *vanishes in* $H^{1,1}(V/\mathbb{C}) := \operatorname{Ext}^1_{\mathscr{O}_V}(\mathscr{O}_V, \Omega^1_{V/\mathbb{C}})$;

(a2) *the first Chern class* $c_1(M^{\mathrm{hol}})$ *of the holomorphic line bundle* M^{hol} *over* $V(\mathbb{C})$ *deduced from* M *vanishes rationally* (that is, in $H^2(V(\mathbb{C}), \mathbb{Q})$, or equivalently in $H^2(V(\mathbb{C}), \mathbb{C})$);

(a3) *there exists a* $\mathscr{C}^\infty$*-hermitian metric* $\|.\|$ *with vanishing curvature on* M^{hol}.

Moreover, when they are satisfied, the metric $\|.\|$ is unique up to a constant factor in $\mathbb{R}_+^*$, and the $(1,0)$-part $\nabla^{1,0}$ of the $\mathscr{C}^\infty$-connection ∇ on M^{hol} that is unitary (for $\|.\|$) and compatible with the holomorphic structure is the unique integrable holomorphic connection ∇^{u}_M whose monodromy lies in $U(1) := \{z \in \mathbb{C} \mid |z| = 1\}$. Observe also that ∇^{u}_M algebraizes, and may be seen as as an "algebraic" connection on the line bundle M on the algebraic variety V over $\mathbb{C}$.

3.2.2. — Let X be a smooth, projective, geometrically connected scheme over K, and $E_{X/K}$ the universal vector extension of $\operatorname{Pic}^0_{X/K}$ (see Appendix B for basic facts on Picard varieties and their universal vector extensions).

(9) One could also argue that this curvature coincides with the one of the holomorphic connection ∇_σ, which vanishes, as recalled above.

In the sequel, we shall consider X and $\operatorname{Spec} K$ as arithmetic schemes over the arithmetic ring (K, Σ, F_∞).

In particular, a hermitian line bundle $\overline{L}$ over X is the data of a line bundle L over X and of a $\mathscr{C}^\infty$-hermitian metric $\|.\|_{\overline{L}}$, invariant under complex conjugation, on the holomorphic line bundle $L_{\mathbb{C}}^{\mathrm{hol}}$ over $X_\Sigma(\mathbb{C}) = \coprod_{\sigma\in\Sigma} X_\sigma(\mathbb{C})$.

According to the observations in 3.2.1, for any line bundle L over X, the following conditions are equivalent:

(b1) *the Atiyah class* $\mathrm{at}_{X/K}L$ *of* L *in* $H^{1,1}(X/K) := \operatorname{Ext}^1_{\mathscr{O}_X}(\mathscr{O}_X, \Omega^1_{X/K})$ *vanishes;*

(b2) *there exists a* $\mathscr{C}^\infty$*-hermitian metric* $\|.\|$ *with vanishing curvature, invariant under complex conjugation, on the holomorphic line bundle* $L_{\mathbb{C}}^{\mathrm{hol}}$ *over* $X_\Sigma(\mathbb{C})$.

When (b1) and (b2) are realized, the metric $\|.\|$ is unique, up to some multiplicative constant, on every component $X_\sigma(\mathbb{C})$ of $X_\Sigma(\mathbb{C})$.

Observe also that these conditions hold precisely when some positive power of the line bundle L is algebraically equivalent to zero(10) (see for instance [**28**, II.2 Cor. 1 to Th. 2]).

3.2.3. — Consider now a line bundle L on X satisfying Conditions (b1) and (b2) above, and let us choose a $\mathscr{C}^\infty$ hermitian metric $\|.\|$ on $L_{\mathbb{C}}$, as in Condition (b1) above.

We shall denote $\overline{L}$ the hermitian line bundle $(L, \|.\|)$ over X, and $\nabla_{\overline{L}}$ the unitary connection on $L_{\mathbb{C}}$ which is compatible with the holomorphic structure. It does not depend on the actual choice of $\|.\|$. Indeed, for any σ in Σ, the $(1,0)$-part $\nabla^{1,0}_{\overline{L}}$ of $\nabla_{\overline{L}}$ coincides with $\nabla^{\mathrm{u}}_{L_\sigma}$ over $X_\sigma(\mathbb{C})$.

It is a straightforward consequence of our definitions that the following conditions are equivalent:

(1) *the line bundle* L *admits a connection* $\nabla\colon L \to L\otimes\Omega^1_{X/K}$ *(over* K*) such that the induced holomorphic connection* $\nabla_{\mathbb{C}}$ *on* $L_{\mathbb{C}}$ *over* $X_\Sigma(\mathbb{C})$ *equals* $\nabla^{1,0}_{\overline{L}}$*, or equivalently such that for any* σ *in* Σ *the induced holomorphic connection* ∇_σ *on* L_σ *over* X_σ *equals* $\nabla^{\mathrm{u}}_{L_\sigma}$*;*

(2) *the class* $\hat{c}_1^H(\overline{L}) := \hat{c}_1^H(X/\operatorname{Spec} K, \overline{L})$*, or in other words the arithmetic Atiyah class* $\widehat{\mathrm{at}}_{X/K}(\overline{L})$*, vanishes in* $\hat{H}^{1,1}(X/K) := \widehat{\operatorname{Ext}}^1_X(\mathscr{O}_X, \Omega^1_{X/K})$*;*

Observe also that, when L is algebraically equivalent to zero, the pair $(L_{\mathbb{C}}, \nabla^{1,0}_{\overline{L}})$ — or equivalently the family $(L_\sigma, \nabla^{\mathrm{u}}_{L_\sigma})_{\sigma\in\Sigma}$ — determines a point $P = P_{\overline{L}}$ in the maximal compact subgroup of

$$E_{X/K}(\mathbb{R}) := \Big[\coprod_{\sigma\in\Sigma} E_{X/K}(\mathbb{C})\Big]^{F_\infty}.$$

(10) By definition a line bundle on X is algebraically equivalent to zero if and only if its restriction to the geometric fiber $X_{\overline{K}}$ is algebraically equivalent to zero.

(details of this construction may be found in the Appendix in B.7 and B.8), and Conditions (1) and (2) are also equivalent to:

(3) *the point $P_{\overline{L}}$ in the maximal compact subgroup of $E_{X/K}(\mathbb{R})$ is the image of a K-rational point of $E_{X/K}$.*

We claim that, *if a line bundle L over X defines a torsion point in* $\mathrm{Pic}(X)$, *then Conditions* (1) *and* (2) *are satisfied.*

Indeed, if n is a positive integer and $\alpha : \mathcal{O}_X \rightarrow L^{\otimes n}$ is an isomorphism of line bundles over X, we may introduce the connection ∇_L^{tor} on L, defined over K, such that the connection $\nabla_{L^{\otimes n}}^{\mathrm{tor}}$ on $L^{\otimes n}$ deduced from ∇_L^{tor} by taking its n-th tensor power makes α an isomorphism of line bundles with connections from $(\mathcal{O}_X, d)$ to $(L^{\otimes n}, \nabla_{L^{\otimes n}})$[11]. For any σ in Σ, the two connections $\nabla_{L,\sigma}^{\mathrm{tor}}$ and $\nabla_{L_\sigma}^{\mathrm{u}}$ on L_σ coincide, since the monodromy of $\nabla_{L,\sigma}^{\mathrm{tor}}$ lies in the n-th roots of unity. Consequently *Condition (1) is satisfied by $\nabla := \nabla_L^{\mathrm{tor}}$.*

3.2.4. — It turns out that, *conversely, if Conditions (1) and (2) hold, then L has a torsion class in* $\mathrm{Pic}(X)$ *and the connection ∇, uniquely defined by (1), necessarily coincides with ∇_L^{tor}.* This is basically the content of Theorems 3.1.1 and 3.1.2 when X is an abelian variety and Σ has one or two conjugate elements. It holds more generally for any X as above:

***Theorem* 3.2.5.** — *Let X be a smooth, projective, geometrically connected variety over K, and let $\pi : X \rightarrow \mathrm{Spec}\, K$ its structural morphism, that we consider as a morphism of arithmetic schemes over the arithmetic ring (K, Σ, F_∞).*

(i) Let $\overline{L} = (L, \|.\|_L)$ be a hermitian line bundle over X. If L admits an algebraic connection $\nabla : L \rightarrow L \otimes \Omega^1_{X/K}$ such that $\nabla_{\mathbb{C}}$ is unitary with respect to $\|.\|_L$, then L has a torsion class in $\mathrm{Pic}(X)$, *the metric $\|.\|_L$ has vanishing curvature, and ∇ coincides with ∇_L^{tor}.*

(ii) For any hermitian line bundle $\overline{L}$ on X, if the first Chern class $\hat{c}_1^H(\overline{L})$ in $\hat{H}^{1,1}(X/K) := \widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X, \Omega^1_{X/K})$ vanishes, then there exists a positive integer n such that $\overline{L}^{\otimes n}$ is isometric to the trivial bundle $\mathcal{O}_X$ equipped with a metric constant on every component $X_\sigma(\mathbb{C})$ of $X_\Sigma(\mathbb{C})$ — or equivalently, such that the class of $\overline{L}^{\otimes n}$ in $\widehat{\mathrm{Pic}}(X)$ belongs to the image of $\pi^* : \widehat{\mathrm{Pic}}(\mathrm{Spec}\, K) \rightarrow \widehat{\mathrm{Pic}}(X)$.

[11] More generally, for any two line bundles L and M over X, any connection ∇_M on M and any isomorphism $\alpha : M \rightarrow L^{\otimes n}$, there exists a unique connection ∇_L on L such that the connection $\nabla_{L^{\otimes n}}$ on $L^{\otimes n}$ deduced from ∇_L by taking its n-th tensor power makes α an isomorphism of line bundles with connections from (M, ∇_M) to $(L^{\otimes n}, \nabla_{L^{\otimes n}})$. It may be defined by the following identity, valid for any local regular section l of L: $n.l^{\otimes n-1} \otimes \nabla_L l = (\alpha \otimes \mathrm{Id}_{\Omega^1_{\mathrm{X/K}}})\nabla_M(\alpha^{-1}(l^{\otimes n}))$.

(iii) Let $P \in E_{X/K}(K)$ be a K-rational point of the universal vector extension $E_{X/K}$ that belongs to the maximal compact subgroup of $E_{X/K}(\mathbb{R})$. Then P is a torsion point in $E_{X/K}(K)$.

Proof. — We prove below that the assertions (i)–(iii) are equivalent for any given variety X as above. The isomorphism (B.9) will then show that it is sufficient to show (iii), hence any of the assertions (i)–(iii), for abelian varieties. In order to prove (i), we may choose σ in Σ and replace the set of embeddings Σ by $\{\sigma\}$ (resp. $\{\sigma, \overline{\sigma}\}$) if σ is a real (resp. complex) embedding. In this situation, (i) has been proved for abelian varieties as Theorem 3.1.1 in Section 3.1 *supra*.

For any given hermitian line bundle $\overline{L}$, the equivalence of the implications in (i) and (ii) is a straightforward consequence of the observations in 3.2.3 and of the implication

$$\hat{c}_1^H(\overline{L}) = 0 \Rightarrow c_1(\overline{L}) = 0,$$

which follows from the commutativity of (2.9).

To establish the implication (ii) $\Rightarrow$ (iii), consider P in $E_{X/K}(K)$ a K-rational point of the universal vector extension that belongs to the maximal compact subgroup of $E_{X/K}(\mathbb{R})$. Replacing K by a finite extension, we may assume that P is represented by a line bundle L algebraically equivalent to zero with an integrable connection ∇. If P belongs to the maximal compact subgroup of $E_{X/K}(\mathbb{R})$, we have $\nabla_{\mathbb{C}} = \nabla^{1,0}_{\overline{L}}$ where $\overline{L}$ carries a hermitian metric with curvature zero. As observed in 3.2.3 above, this implies that $\hat{c}_1^H(\overline{L}) = 0$. According to (ii), there exists some integer $m > 0$ such that $\overline{L}^{\otimes m}$ is isometric to the trivial bundle $\mathcal{O}_X$ with a constant metric. It follows that $(L, \nabla)^{\otimes m}$ is isomorphic to the trivial bundle $\mathcal{O}_X$ with the trivial connection, and consequently that P belongs to the m-torsion of $E_{X/K}(K)$.

Finally, we show the implication (iii) $\Rightarrow$ (ii). Let $\overline{L} = (L, \|.\|_L)$ be a hermitian line bundle over X such that the class $\hat{c}_1^H(\overline{L}) := \widehat{\mathrm{at}}_{X/K}(\overline{L})$ vanishes. Then $\mathrm{at}_{X/K}(L)$ vanishes too, and there exists a positive integer m such that $L^{\otimes m}$ is algebraically equivalent to zero. By replacing $\overline{L}$ by $\overline{L}^{\otimes m}$, we may therefore assume that L is algebraically equivalent to zero. As observed in 3.2.3, the point $P_{\overline{L}}$ associated to $(L_{\mathbb{C}}, \|.\|_L)$ lies in the maximal compact group of $E_{X/K}(\mathbb{R})$, and is the image of a K-rational point of $E_{X/K}$. According to (iii), it is a torsion point. This implies that L has a torsion class in $\mathrm{Pic}(X)$, and that $\nabla_{\overline{L}}$ coincides with the connection $\nabla^{\mathrm{tor}}_{L,\mathbb{C}}$. This establishes that $\overline{L}$ satisfies the conclusion of (i), and consequently, as observed above, of (ii). □

3.3. Finiteness results on the kernel of $\hat{c}_1^H$. — We may use Theorem 3.2.5 to investigate the kernel of the first Chern class in arithmetic Hodge cohomology. Indeed this Theorem easily leads to a derivation of the assertion $\mathbf{I2}_{X,\Sigma}$ in the Introduction (which conversely contains Part (ii) of Theorem 3.2.5):

***Corollary* 3.3.1.** — *The image of*

$$\pi^* : \widehat{\mathrm{Pic}}(\mathrm{Spec}\, K) \longrightarrow \widehat{\mathrm{Pic}}(X)$$

has finite index in the kernel of

$$\hat{c}_1^H : \widehat{\mathrm{Pic}}(X) \longrightarrow \widehat{H}^{1,1}(X/K).$$

Proof. — A hermitian metric with curvature zero on the trivial line bundle on X is constant on every component $X_\sigma(\mathbb{C})$ of $X_\Sigma(\mathbb{C})$. Therefore, if we introduce the canonical map

$$w : \widehat{\mathrm{Pic}}(X) \to \mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}_{X/K}(K),$$

then we have:

$$\mathrm{Ker}(\hat{c}_1^H) \cap \mathrm{Ker}(w) = \mathrm{Im}\left(\pi^* : \widehat{\mathrm{Pic}}(S) \to \widehat{\mathrm{Pic}}(X)\right).$$

Hence the map w induces an injection of

$$\frac{\mathrm{Ker}\left(\hat{c}_1^H : \widehat{\mathrm{Pic}}(X) \longrightarrow \widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X, \Omega^1_{X/K})\right)}{\mathrm{Im}\left(\pi^* : \widehat{\mathrm{Pic}}(\mathrm{Spec}\ K) \longrightarrow \widehat{\mathrm{Pic}}(X)\right)} \tag{3.3}$$

into $\mathrm{Pic}_{X/K}(K)$. Theorem 3.2.5 (iii) implies that the image of (3.3) is contained in the torsion subgroup of $\mathrm{Pic}_{X/K}(K)$[12]. This is a finite group as the Néron-Severi group

$$NS_{X/K}(\overline{K}) = \mathrm{Pic}_{X/K}(\overline{K}) / \mathrm{Pic}^0_{X/K}(\overline{K})$$

and $\mathrm{Pic}^0_{X/K}(K)$ are finitely generated abelian groups by [**29**, Th. 5.1] and the theorem of Mordell-Weil. □

We may also establish a similar finiteness result where the base scheme $\mathrm{Spec}\, K$ is replaced by an "arithmetic curve":

***Corollary* 3.3.2.** — *Let $\mathcal{O}_K$ denote the ring of integers in a number field K, and let us work over the arithmetic ring $(\mathcal{O}_K, \Sigma, F_\infty)$. Let S denote a non-empty open subset of* $\mathrm{Spec}\, \mathcal{O}_K$*, and let X be a smooth projective S-scheme with geometrically connected fibers. Then*

$$\frac{\mathrm{Ker}\left(\hat{c}_1^H : \widehat{\mathrm{Pic}}(X) \longrightarrow \widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X, \Omega^1_{X/S})\right)}{\mathrm{Im}\left(\pi^* : \widehat{\mathrm{Pic}}(S) \longrightarrow \widehat{\mathrm{Pic}}(X)\right)} \tag{3.4}$$

is a finite group.

Proof. — Let X_K denote the fiber of X over $\mathrm{Spec}\, K$. We consider X_K as an arithmetic scheme over the arithmetic field $K = (K, \Sigma, F_\infty)$. There is a canonical restriction map

$$\nu : \widehat{\mathrm{Pic}}(X) \longrightarrow \widehat{\mathrm{Pic}}(X_K).$$

[12] Actually this morphism factorizes through the torsion subgroup $\mathrm{Pic}(X)_{\mathrm{tor}}$ of $\mathrm{Pic}(X)$, and one may easily show that the so defined injection $\mathrm{Ker}(\hat{c}_1^H) / \mathrm{Im}\,(\pi^*) \to \mathrm{Pic}(X)_{\mathrm{tor}}$ is an isomorphism.

Any element in $\operatorname{Ker}\nu \cap \operatorname{Ker}\hat{c}_1^H(X/S, .)$ is generically trivial and carries a constant metric. The sequence

$$\operatorname{Pic}(S) \longrightarrow \operatorname{Pic}(X) \longrightarrow \operatorname{Pic}(X_K)$$

is exact as the fibers of X/S are integral. Hence

$$\operatorname{Ker}(\nu) \cap \operatorname{Ker}\hat{c}_1^H(X/S, .) \subseteq \operatorname{Im}\left(\pi^* : \widehat{\operatorname{Pic}}(S) \longrightarrow \widehat{\operatorname{Pic}}(X)\right).$$

Moreover ν maps $\operatorname{Im}\left(\pi^* : \widehat{\operatorname{Pic}}(S) \longrightarrow \widehat{\operatorname{Pic}}(X)\right)$ onto $\operatorname{Im}\left(\pi^* : \widehat{\operatorname{Pic}}(\operatorname{Spec} K) \longrightarrow \widehat{\operatorname{Pic}}(X_K)\right)$. Consequently it induces an embedding of (3.4) into (3.3). The latter group is finite by Theorem 3.2.5. Our claim follows. □

4. A geometric analogue

4.1. Line bundles with vanishing relative Atiyah class on fibered projective varieties

4.1.1. *Notation.* — In this section, we consider a smooth projective geometrically connected curve C over a field k of characteristic 0, and a smooth projective variety V over k equipped with a dominant k-morphism $\pi : V \to C$, with geometrically connected fibers.

Observe that the morphism π is flat, and smooth over an open dense subscheme of C, namely over the complement of the finite set Δ of closed points P in C such that the (scheme theoretic) fiber $\pi^*(P)$ is not smooth over k.

Let $K := k(C)$ denote the function field of C. The generic fiber V_K of π is a smooth projective geometrically connected variety over K. Conversely, according to Hironaka's resolution of singularities, any such variety over K may be constructed from the data of a k-variety V and of a k-morphism $\pi : V \to C$ as above.

Recall also that a divisor E in V is called *vertical* if it belongs to the group of divisors generated by components of closed fibers of π, or equivalently, if its restriction E_K to the generic fiber V_K of V vanishes.

In the sequel, we assume that the dimension n of V is at least 2. Moreover we choose an ample line bundle $\mathcal{O}(1)$ over V, we denote H its first Chern class in the Chow group $CH^1(X)$, and for any integral subscheme D of positive dimension in V and any line bundle L over V, we let:

$$\deg_{H,D} L := \deg_k(c_1(L).H^{\dim D-1}.[D]).$$

Actually, we shall use this definition only when D is a vertical divisor in V. Consequently, we could require $\mathcal{O}(1)$ to be ample relatively to π only. Besides, when $\dim D = 1$ the choice of $\mathcal{O}(1)$ is immaterial.

Observe that, if $\mathcal{O}(1)$ is very ample and defines a projective embedding $\iota : V \hookrightarrow \mathbb{P}_k^N$, then, for any general enough $(\dim D - 1)$-tuple $(H_1, \ldots, H_{\dim D-1})$ of projective hyperplanes in $\mathbb{P}_k^N$, the subscheme

$$C := D \cap \iota^{-1}(H_1) \cap \cdots \cap \iota^{-1}(H_{\dim D-1})$$

in $\mathbb{P}_k^N$ is integral, one-dimensional, and projective over k, and its class $[C]$ in $CH_1(X)$ coincides with $H^{\dim D-1}.[D]$. Consequently $\deg_{H,D} L$ is nothing but the degree $\deg_k c_1(L).[C]$ of the restriction of L to the "general linear section" C of D.

Let us recall that, if M is a smooth projective geometrically connected scheme over some field k_0 of characteristic zero, then the Picard functor Pic_{M/k_0} is representable by a separated group scheme over k_0, and that its identity component Pic^0_{M/k_0} is an abelian variety over k_0. A line bundle L over M is algebraically equivalent to zero[(13)] when the point in $\mathrm{Pic}_{M/k_0}(k_0)$ it defines belongs to $\mathrm{Pic}^0_{M/k_0}(k_0)$, or equivalently, if its class in the Néron-Severi group of M over k_0 — defined as $\mathrm{Pic}_{M/k_0}(k_0)/\mathrm{Pic}^0_{M/k_0}(k_0)$ — vanishes.

In particular, we may consider the identity component $\mathrm{Pic}^0_{V_K/K}$ of the Picard variety of the generic fiber V_K of π; it is an abelian variety over K, and we shall denote (B,τ) its K/k-trace. By definition, B is an abelian variety over k, and τ is a morphism of abelian varieties over K:

$$\tau : B_K \longrightarrow \mathrm{Pic}^0_{V_K/K}.$$

Since the base field k is assumed to be of characteristic zero, this morphism is actually a closed immersion. We refer the reader to Section 4.6 *infra* for a discussion and references concerning the definition of $\mathrm{Pic}^0_{V_K/K}$ and (B,τ).

4.1.2. — The following theorem may be seen as a geometric counterpart, valid over the function field $K := k(C)$, of the characterization of hermitian line bundles with **v**anishing arithmetic **A**tiyah class in Theorem 3.2.5 ii).

***Theorem* 4.1.3**. — *With the above notation, for any line bundle L over V, the following three conditions are equivalent:*

VA1 *The relative Atiyah class $\mathrm{at}_{V/C}(L)$ vanishes in*

$$\mathrm{Ext}^1_{\mathcal{O}_X}(L, L\otimes\Omega^1_{V/C}) \simeq H^1(V,\Omega^1_{V/C}).$$

VA2 *There exist a positive integer N and a line bundle M over C such that the line bundle $L^{\otimes N}\otimes\pi^*M$ is algebraically equivalent to zero.*

(13) The reader should beware that, here as in the previous section, we use a "geometric" definition of "algebraically equivalent to zero", related as follows to the one occuring in [**16**], 10.3: for any divisor D in M and any algebraic closure $\overline{k}_0$ of k_0, the line bundle $\mathcal{O}(D)$ is algebraically equivalent to zero in our "geometric" sense iff the divisor $D_{\overline{k}_0}$ on $M_{\overline{k}_0}$ is algebraically equivalent to zero in Fulton's sense. Also observe that (the first Chern class of) a line bundle on M algebraically equivalent to zero in the above sense is numerically equivalent to zero in the sense of Fulton [**16**], 19.1. In particular, with the notation of the previous paragraphs, for any line bundle L algebraically equivalent to zero over V, $\deg_{H,D} L$ vanishes.

VA3 *There exists a positive integer N such that the line bundle $L_K^{\otimes N}$ on V_K is algebraically equivalent to zero, and the attached K-rational point of the Picard variety* $\mathrm{Pic}^0_{V_K/K}$ *is defined by a k-rational point of the K/k-trace of* $\mathrm{Pic}^0_{V_K/K}$*. Moreover, for any component D of a closed fiber of π, the degree $\deg_{H,D} L$ vanishes.*

Observe that, for any closed point P of $C \setminus \Delta$, its fiber $D := \pi^*(P)$ is a divisor in V, smooth and geometrically connected over $k(P)$, and that, according to the projection formula,

$$\begin{aligned}\deg_{H,D} L &= \deg_k(c_1(L).H^{n-2}.[\pi^*(P)]) \\ &= \deg_k(\pi_*(c_1(L).H^{n-2}).[P]) \\ &= [k(P):k].\deg_K(c_1(L_K).c_1(\mathcal{O}(1)_K)^{\dim V_K - 1}.[V_K]).\end{aligned}$$

In particular, if some positive power $L_K^{\otimes N}$ of L_K is algebraically equivalent to zero, then $\deg_{H,D} L$ vanishes. Consequently, in condition **VA3**, we may require the vanishing of $\deg_{H,D} L$ only for components D of the supports of the singular fibers $\pi^*(P)$, where P varies in Δ.

The proof of the equivalence of conditions **VA1** and **VA2**, which uses the Hodge index theorem and basic properties of Hodge cohomology groups, will be presented in Sections 4.4 and 4.5 below. Then in Section 4.6 and 4.7 we shall recall some classical facts concerning the Picard variety $\mathrm{Pic}^0_{V_K/K}$ and its K/k-trace, and establish the equivalence of conditions **VA2** and **VA3**.

4.2. Variants and complements. — Before we enter into the proof of Theorem 4.1.3, we discuss some variants and related statements. Observe that the variants in 4.2.1 make Theorem 4.1.3 more similar to its "arithmetic counterpart" in Theorem 3.2.5 ii), whereas Proposition 4.2.4 would rather make less convincing the analogy between the arithmetic framework in Section 3 and the geometric framework of the present section.

4.2.1. — Recall that the following conditions are equivalent — when they hold, the Picard variety $\mathrm{Pic}^0_{V_K/K}$ will be said to have *no fixed part*:

NFP1 *The K/k-trace of* $\mathrm{Pic}^0_{V_K/K}$ *vanishes,* or in other terms, *for any abelian variety A over k, there is no non-zero morphism of abelian varieties over K from A_K to* $\mathrm{Pic}^0_{V_K/K}$.

NFP2 *The morphism of k-abelian varieties naturally deduced from $\pi : V \longrightarrow C$*

$$\pi^* : \mathrm{Pic}^0_{C/k} \longrightarrow \mathrm{Pic}^0_{V/k}$$

— which has a finite kernel — *is an isogeny.*

NFP3 *The injective morphism of k-vector spaces*

$$\pi^* : H^1(C, \mathcal{O}_C) \longrightarrow H^1(V, \mathcal{O}_V)$$

is an isomorphism.

NFP4 *The injective morphism of k-vector spaces*

$$\pi^* : H^0(C, \Omega^1_{C/k}) \longrightarrow H^0(V, \Omega^1_{V/k})$$

is an isomorphism.

A few comments on these conditions may be appropriate.

The finiteness of the kernel of π^* in **NFP2** may be derived by considering a smooth projective geometrically connected curve C' in V such that the morphism $\pi_{|C'} : C' \to C$ is finite. Let $i : C' \hookrightarrow V$ denote the inclusion morphism. The norm with respect to $\pi_{|C'}$ defines a morphism $\pi_{|C'*} : \mathrm{Pic}^0_{C'/k} \to \mathrm{Pic}^0_{C/k}$ of abelian varieties over k, and the morphisms of abelian varieties π^*, $\pi_{|C'*}$, $\pi^*_{|C'} : \mathrm{Pic}^0_{C/k} \to \mathrm{Pic}^0_{C'/k}$, and $i^* : \mathrm{Pic}^0_{V/k} \to \mathrm{Pic}^0_{C/k}$ satisfy the relations

$$\pi^*_{|C'} = i^* \circ \pi^*$$

and

$$\pi_{|C'*} \circ \pi^*_{|C'} = [\delta],$$

where $[\delta]$ denotes the morphism of multiplication by the degree δ of $\pi_{|C'}$ in $\mathrm{Pic}^0_{C/k}$. This immediately implies that the kernel of π^* is a subgroup of the δ-torsion in $\mathrm{Pic}^0_{C/k}$.

The injectivity of π^* in **NFP4** is a consequence of the generic smoothness of the dominant morphism π (recall that the base field k is assumed to have characteristic zero). The injectivity of π^* in **NFP3** and the equivalence of **NFP3** and **NFP4** follows from Hodge theory when $k = \mathbb{C}$, and therefore, by a standard base change argument, for any base field k of characteristic zero.

The equivalence of **NFP1** and **NFP2** follows from the description of the K/k-trace of $\mathrm{Pic}^0_{V_K/K}$ recalled in Proposition 4.6.1 below. Finally, the equivalence of **NFP2** and **NFP3** follows from the identification of $H^1(C, \mathcal{O}_C)$ (resp. $H^1(V, \mathcal{O}_V)$) with Lie $\mathrm{Pic}^0_{C/k}$ (resp. Lie $\mathrm{Pic}^0_{V/k}$).

As demonstrated by the theorem of Mordell-Weil-Lang-Néron, it is natural to require a no fixed part condition when searching for statements valid over function fields that are as close as possible to their arithmetic counterparts. This is indeed the case with Theorem 4.1.3. Namely, when $\mathrm{Pic}^0_{V_K/K}$ has no fixed part, Conditions **VA1-3** are also equivalent to the following ones, which look more closely like the conditions appearing in i) and ii) of the "arithmetic" Theorem 3.2.5:

VA2' *There exists a positive integer N and a line bundle M over C such that the line bundle $L^{\otimes N}$ is isomorphic to $\pi^* M$.*

VA3' *The class of L_K in the abelian group $\mathrm{Pic}_{V_K/K}(K)$ is torsion. Moreover, for any component D of a closed fiber of π, the degree $\deg_{H,D} L$ vanishes.*

Indeed, the equivalence of **VA3** and **VA3'** when **NFP1** holds is straightforward, and the equivalence of **VA2** and **VA2'** easily follows from **NFP2**.

4.2.2. — Generalizations of Theorem 4.1.3 concerning a smooth projective variety V over k fibered over a projective variety C of dimension > 1 may be deduced from its original version with C a curve by means of standard techniques, as in the proof of the Mordell-Weil-Lang-Néron theorem (*cf.* [**32**]). We leave this to the interested reader.

4.2.3. — Finally observe that when the base C is assumed to be affine instead of projective, the determination of line bundles with vanishing relative Atiyah class becomes a rather straightforward issue. For instance, we have:

***Proposition* 4.2.4.** — *Let C be an affine integral scheme of finite type over a field k of characteristic zero, and let $K := k(C)$ denote its function field. Let $\pi : V \to C$ be a smooth projective morphism, L a line bundle over V, and L_K the restriction of L to the generic fibre V_K of π. The following conditions are equivalent:*

(i) *the relative Atiyah class* $\mathrm{at}_{V/C}(L)$ *vanishes in*

$$\mathrm{Ext}^1_{\mathcal{O}_V}(L, L \otimes \Omega^1_{V/C}) \simeq H^1(V, \Omega^1_{V/C});$$

(ii) *the Atiyah class* $\mathrm{at}_{V_K/K}(L_K)$ *vanishes in* $H^1(V_K, \Omega_{V_K/K})$*;*

(iii) *some positive power of L_K is algebraically equivalent to zero over V_K.*

Proof. — The equivalence (i)⇔ (ii) follows from the identification

$$H^1(V, \Omega^1_{V/C}) \simeq H^0(C, R^1\pi_*\Omega^1_{V/C})$$

and from the fact that, since the base field has characteristic zero, by Hodge theory the coherent sheaf $R^1\pi_*\Omega^1_{V/C}$ is a locally free sheaf over C, the formation of which is actually compatible with any base change.

The equivalence (ii)⇔ (iii) holds since the base field K has characteristic zero (see for instance 4.3.2 below). □

4.3. Hodge cohomology and first Chern class. — In this section, we review some basic properties of the Hodge cohomology of smooth projective varieties over fields of characteristic zero. These properties are consequence of the duality theory for coherent sheaves on projective varieties, as explained in [**21**], exposé 149.

4.3.1. *Hodge cohomology groups.* — Let k be a field of characteristic zero, and $\mathbf{SmPr}_k$ the full subcategory of the category of k-schemes whose objects are smooth projective schemes V over k.

To any object V in $\mathbf{SmPr}_k$ are attached his *Hodge cohomology groups*:

$$H^{p,q}(V/k) := H^q(V, \Omega^p_{V/k}).$$

These are finite dimensional k-vector spaces, and vanish if $\max(p,q) > d := \dim V$. Moreover, the cup products

$$\begin{array}{rcl} H^{p,q}(V/k) \times H^{p',q'}(V/k) & \longrightarrow & H^{p+p',q+q'}(V/k) \\ (\alpha, \alpha') & \longmapsto & \alpha.\alpha', \end{array}$$

— defined as the compositions of the products

$$H^q(V, \Omega^p_{V/k}) \times H^{q'}(V, \Omega^{p'}_{V/k}) \longrightarrow H^{q+q'}(V, \Omega^p_{V/k} \otimes \Omega^{p'}_{V/k})$$

and of the mappings

$$H^{q+q'}(V, \Omega^p_{V/k} \otimes \Omega^{p'}_{V/k}) \longrightarrow H^{q+q'}(V, \Omega^{p+p'}_{V/k})$$

deduced from the exterior product $\wedge : \Omega^p_{V/k} \otimes \Omega^{p'}_{V/k} \longrightarrow \Omega^{p+p'}_{V/k}$ — make the direct sum $H^{*,*}(V/k) := \bigoplus_{(p,q)\in\mathbb{N}^2} H^{p,q}(V/k)$ a bigraded commutative[14] k-algebra.

Moreover, the "top-dimensional" Hodge cohomology group $H^{d,d}(V/k)$ is equipped with a canonical k-linear form:

$$\int_{V/k} . : H^{d,d}(V/k) \longrightarrow k,$$

and the attached k-bilinear map

$$\begin{array}{rrcl} <.,.>: & H^{*,*}(V/k) \times H^{*,*}(V/k) & \longrightarrow & k \\ & (\alpha, \beta) & \longmapsto & \int_{V/k} \alpha.\beta \end{array}$$

is a perfect pairing.

In particular, when V is a geometrically connected k-scheme, or equivalently when the linear map

$$\begin{array}{rcl} k & \longrightarrow & \Gamma(V, \mathcal{O}_V) = H^{0,0}(V/k) \\ \lambda & \longmapsto & \lambda.1_V \end{array}$$

is an isomorphism, then the "residue map" also is:

$$\int_{V/k} . : H^{d,d}(V/k) \xrightarrow{\sim} k.$$

Then we denote μ_V the unique element in $H^{d,d}(V/k)$ such that

$$\int_{V/k} \mu_V = 1.$$

These constructions are compatible in an obvious sense with extensions of the base field k. Let us also indicate that, when $k = \mathbb{C}$, the trace map

$$\int_{V/\mathbb{C}} . : H^{d,d}(V/\mathbb{C}) \longrightarrow \mathbb{C}$$

(14) Namely, for any α (resp. α') in $H^q(V, \Omega^p_{V/k})$ (resp. in $H^{q'}(V, \Omega^{p'}_{V/k})$), we have $\alpha.\alpha' = (-1)^{pp'+qq'}\alpha'.\alpha$.

satisfies the following compatibility relation with the Dolbeault isomorphism

$$\mathrm{Dolb}_{\Omega^d_{V/\mathbb{C}}} : H^d(V, \Omega^d_{V/\mathbb{C}}) \longrightarrow H^d_{\mathrm{Dolb}}(V, \Omega^d_{V/\mathbb{C}})$$

(we follow the notation of [**7**], A.5.1) and the integration of top degree forms:

$$\int_{V(\mathbb{C})} . : A^{d,d}(V(\mathbb{C})) \longrightarrow \mathbb{C}.$$

For any α in $A^{d,d}(V(\mathbb{C}))$, of class $[\alpha]$ in $H^d_{\mathrm{Dolb}}(V, \Omega^d_{V/\mathbb{C}})$, we have:

$$\int_{V/\mathbb{C}} \mathrm{Dolb}^{-1}_{\Omega^d_{V/\mathbb{C}}}([\alpha]) = \varepsilon_d \frac{1}{(2\pi i)^d} \int_{V(\mathbb{C})} \alpha,$$

where ε_d denotes a sign, function of d only, depending on the sign conventions followed in duality theory (we refer the reader to [**15**], Appendice, and [**37**] for discussions of this delicate issue).

4.3.2. *The first Chern class in Hodge cohomology.* — Any line bundle L over some V in $\mathbf{SmPr}_k$ admits a first Chern class $c_1(L)$ in $H^{1,1}(V/k)$. It may be defined as the class

$$\mathrm{at}_{X/k} L = \mathrm{jet}^1_{X/k} L$$

in

$$\mathrm{Ext}^1_{\mathcal{O}_V}(L, \Omega^1_{V/k} \otimes L) \simeq \mathrm{Ext}^1_{\mathcal{O}_V}(\mathcal{O}_V, \Omega^1_{V/k}) \tag{4.1}$$

$$\simeq H^1(V, \Omega^1_{V/k}). \tag{4.2}$$

of the extension given by the principal parts of first order associated with L

$$\mathcal{J}et^1_{X/k} L : 0 \longrightarrow \Omega^1_{X/k} \otimes L \longrightarrow P^1_{X/k}(L) \longrightarrow L \longrightarrow 0$$

(see Section 1.2 above). The isomorphism (4.1) is the (inverse of the) one defined by applying the functor $. \otimes L$ to complexes of $\mathcal{O}_V$-modules, without intervention of signs. The isomorphism (4.2) is the one discussed in [**7**], A.2 and A.4.

The so-defined first Chern class defines a morphism of abelian groups:

$$\begin{array}{rcl} \mathrm{Pic}(V) & \longrightarrow & H^1(V, \Omega^1_{V/k}) =: H^{1,1}(V/k) \\ {[L]} & \longmapsto & c_1(L). \end{array}$$

Moreover, this morphism factorizes through the Néron-Severi group

$$\mathrm{NS}_{V/k}(k) = \mathrm{Pic}_{V/k}(k)/\mathrm{Pic}^0_{V/k}(k);$$

the induced morphism on $\mathrm{NS}_{V/k}(k)$ vanishes precisely on its torsion subgroup $\mathrm{NS}_{V/k}(k)_{\mathrm{tor}}$ (compare for example [**28**, II.2 Cor. 1 to Th. 2]), and consequently defines an injective morphism of groups

$$c_1 : \mathrm{NS}_{V/k}(k)/\mathrm{NS}_{V/k}(k)_{\mathrm{tor}} \longrightarrow H^{1,1}(V/k).$$

In other words, for any line bundle L on V, the following two conditions are equivalent:

(i) the first Chern class $c_1(L)$ in $H^{1,1}(V/k)$ vanishes;

(ii) for some positive integer N, the line bundle $L^{\otimes N}$ over V is algebraically equivalent to zero.

Let us also recall that the construction of the first Chern class in Hodge cohomology is compatible with pull-back by k-morphisms. It is also compatible with intersection theory. In particular, we have:

***Proposition* 4.3.3.** — *For any d-tuple $D_1, \dots, D_d$ of divisors in some d-dimensional variety V in $\mathbf{SmPr}_k$, the following formula holds:*

$$\int_{V/k} c_1(\mathcal{O}(D_1)).\cdots.c_1(\mathcal{O}(D_d)) = \deg_k([D_1].\cdots.[D_d]), \tag{4.3}$$

where $[D_i]$ denotes the class of D_i in the Chow group $CH^1(V)$, $[D_1].\cdots.[D_d]$ their product in $CH^d(V) = CH_0(V)$ and

$$\deg_k : CH_0(V) \xrightarrow{\pi_*} CH_0(\operatorname{Spec} k) \simeq \mathbb{Z}$$

the degree map, attached to the structural morphism $\pi : V \to \operatorname{Spec} k$ of V.

In particular, if $d = 1$ and V is geometrically irreducible, then

$$c_1(\mathcal{O}(D)) = \deg_k D.\mu_V.$$

To establish the equality (4.3), one easily reduces to the case where k is algebraically closed and V is connected. Then it follows from **[21]**, exposé 149 (Théorème 1, Théorème 2, and its proof) when moreover the divisors $D_1, \dots, D_n$ and their successive intersections $D_1 \cap D_2$, $D_1 \cap D_2 \cap D_3, \dots, D_1 \cap D_2 \cap \cdots \cap D_n$ are smooth. Together with the invariance of both sides of (4.3) by linear equivalence of $D_1, \dots, D_n$ and Bertini theorem, this shows that (4.3) holds when $D_1, \dots, D_n$ are very ample. The general case of (4.3) follows by multilinearity.

4.4. An application of the Hodge Index Theorem. — Our proof of Theorem 4.1.3 will rely on an application of Hodge Index Theorem to projective varieties fibered over curves that we discuss in the present Section.

4.4.1. *The Hodge Index Theorem in Hodge cohomology.* — Let V be a smooth, projective, geometrically connected scheme over k, and let h be the first Chern class $c_1(\mathcal{O}(1))$ in $H^{1,1}(V/k)$ of some ample line bundle $\mathcal{O}(1)$ on V.

We shall use the following straightforward consequence of the Hodge Index Theorem (as formulated in **[29]**, Appendix 7) and of the compatibility of intersection theory and products in Hodge cohomology stated in Proposition 4.3.3:

***Proposition* 4.4.2.** — *When $d := \dim V \geq 2$, for any class α of $H^{1,1}(V/k)$ in the image of $c_1 : \operatorname{Pic}(V) \to H^{1,1}(V/k)$, the following conditions are equivalent:*

(i) $\alpha = 0$;

(ii) $\alpha^2.h^{d-2} = \alpha.h^{d-1} = 0$ *in* $H^{d,d}(V/k) \simeq k$.

4.4.3. *An application to projective varieties fibered over curves.* — We keep the notation of the previous paragraph, and assume that $d := \dim V$ is at least 2. Moreover, we consider a smooth geometrically connected projective curve C over k, and a dominant k-morphism $\pi : V \to C$. We shall denote K the function field $k(C)$ of C, $V_K := V \times_C \operatorname{Spec} K$ the generic fiber of π, and $\mathcal{O}(1)_K$ the pull-back of $\mathcal{O}(1)$ to V_K.

Let us introduce the following class in $H^{1,1}(V/k)$:

$$F := \pi^* \mu_C.$$

Observe that $\mu_C^2 = 0$ for dimension reasons, and that consequently $F^2 = 0$. Moreover Proposition 4.3.3 and the naturality of c_1 show that, for any divisor E on C,

$$c_1(\mathcal{O}(E)) = \deg_k E \cdot \mu_C$$

and

$$c_1(\mathcal{O}(\pi^*(E))) = \deg_k E \cdot F. \tag{4.4}$$

***Lemma* 4.4.4.** — *1) For any divisor D on V, $\int_{V/k} c_1(\mathcal{O}(D)).h^{d-1}$ coincides with the intersection number $\deg_k([D].[H]^{d-1})$, where H denotes the divisor of some non-zero rational section of $\mathcal{O}(1)$. In particular, it is an integer.*

2) We have:

$$\int_{V/k} F.h^{d-1} = \deg_{\mathcal{O}(1)_K} V_K.$$

In particular, the class F is not zero, and the image of $\pi^* : H^{1,1}(C/k) \to H^{1,1}(V/k)$ is precisely the k-line $k.F$.

Proof. — Assertion 1) is a special case of Proposition 4.3.3.

To establish 2), let us choose a divisor E with positive degree on C. We have

$$\deg_k([\pi^*(E)].[H]^{d-1}) = \deg_k([E].\pi_*([H]^{d-1})) = \deg_k E. \deg_{\mathcal{O}(1)_K} V_K, \tag{4.5}$$

by basic intersection theory. Besides, according to Proposition 4.3.3 and (4.4), the left-hand side of (4.5) is also equal to

$$\int_{V/k} c_1(\mathcal{O}(\pi^*(E))).c_1(\mathcal{O}(1))^{d-1} = \deg_k E. \int_{V/k} F.h^{d-1}.$$

Together with (4.5), this establishes the announced relation. □

***Proposition* 4.4.5.** — *With the above notation, for any class β of $H^{1,1}(V/k)$ in the image of c_1, the following conditions are equivalent:*

(i) *β belongs to $\mathbb{Q}.F$;*

(ii) *β belongs to $k.F$;*

(iii) *$\beta.\beta = \beta.F = 0$ in $H^{2,2}(V/k)$;*

(iv) $\beta^2.h^{d-2} = \beta.F.h^{d-2} = 0$ *in* $H^{d,d}(V/k) \simeq k$.

Proof. — The implications (i)⇒(ii)⇒(iii)⇒(iv) are straightforward. To establish the converse implications, observe that $\int_{V/k} \beta.h^{d-1} / \int_{V/k} F.h^{d-1}$ is a well defined rational number by Lemma 4.4.4, and consider the class

$$\alpha := \beta - \frac{\int_{V/k} \beta.h^{d-1}}{\int_{V/k} F.h^{d-1}}.F$$

in $H^{1,1}(V/k)$. It satisfies $\alpha.h^{d-1} = 0$ by its very definition (recall that $\int_{V/k}$ maps isomorphically $H^{d,d}(V/k)$ onto k). Moreover (4.4) shows that some positive multiple of α lies in the image of c_1. Finally, when condition (iv) holds, then α also satisfies $\alpha^2.h^{d-2} = 0$. Then, according to Proposition 4.4, α vanishes, or equivalently:

$$\beta = \frac{\int_{V/k} \beta.h^{d-1}}{\int_{V/k} F.h^{d-1}}.F.$$

This establishes (i). □

4.5. The equivalence of VA1 and VA2. — We keep the notation of the previous paragraph 4.4.3. In other words, the same hypotheses as in Theorem 4.1.3 are supposed to hold, except the connectedness of the geometric fibers of π.

The following result contains the equivalence of Conditions **VA1** and **VA2** in Theorem 4.1.3:

***Theorem* 4.5.1.** — *For any line bundle L over V, the following conditions are equivalent:*

(i) *The relative Atiyah class* $\mathrm{at}_{V/C}L$ *vanishes in* $H^{1,1}(V, \Omega^1_{V/C})$.

(ii)′ $c_1(L)$ *belongs to* $\mathbb{Q}.F$.

(ii)″ *There exists a positive integer N and a line bundle M over C such that* $c_1(L^{\otimes N} \otimes \pi^* M)$ *vanishes.*

Proof. — The equivalence (ii)′ ⇔ (ii)″ is straightforward.

To establish the implication (ii)′ ⇒ (i), consider the canonical exact sequence of sheaves of Kähler differentials on V,

$$0 \longrightarrow \pi^*\Omega^1_{C/k} \xrightarrow{i} \Omega^1_{V/k} \xrightarrow{p} \Omega^1_{V/C} \longrightarrow 0,$$

and the associated exact sequence of cohomology groups

$$H^1(V, \pi^*\Omega^1_{C/k}) \xrightarrow{H^1(i)} H^1(V, \Omega^1_{V/k}) \xrightarrow{H^1(p)} H^1(V, \Omega^1_{V/C}).$$

As a special case of Lemma 1.1.6, i), we have

$$\mathrm{at}_{V/C}L = H^1(p)(\mathrm{at}_{V/k}L). \tag{4.6}$$

Since F belongs to the image of $H^1(i)$, hence to the kernel of $H^1(p)$, this establishes the implication (ii)$'$ $\Rightarrow$ (i).

The implication (i)$\Rightarrow$(ii)$'$ will follow from the implication (iii)$\Rightarrow$(i) in Proposition 4.4.5 (applied to $\beta := c_1(L)$) combined with the following:

***Lemma* 4.5.2.** — *For any line bundle L over V, if the relative Atiyah class $\mathrm{at}_{V/C}L$ vanishes in $H^1(V, \Omega^1_{V/C})$, then $c_1(L).F$ and $c_1(L)^2$ vanish in $H^2(V, \Omega^2_{V/k})$.*

To establish this lemma, observe that the cup product

$$H^{1,1}(V/k) \otimes H^{1,1}(V/k) \longrightarrow H^{2,2}(V/k) \tag{4.7}$$

vanishes on $\operatorname{im} H^1(i) \otimes \operatorname{im} H^1(i)$. Indeed the map of sheaves of $\mathcal{O}_V$-modules defined as the composition

$$\pi^*\Omega^1_{C/k} \otimes \pi^*\Omega^1_{C/k} \xrightarrow{i\otimes i} \Omega^1_{V/k} \otimes \Omega^1_{V/k} \xrightarrow{.\wedge.} \Omega^2_{V/k}$$

vanishes by functoriality of the exterior product, since $\Omega^2_{C/k} = 0$. This entails the vanishing of the cup product (4.7) on $\ker H^1(p) \otimes \ker H^1(p)$ and on $\ker H^1(p) \otimes \operatorname{im} \pi^*$, where π^* denotes the pull-back map in Hodge cohomology $\pi^* : H^{1,1}(C/k) \to H^{1,1}(V/k)$.

According to (4.6), $\mathrm{at}_{V/C}L$ vanishes precisely when $c_1(L) = \mathrm{at}_{V/k}L$ belongs to $\ker H^1(p)$, in which case $c_1(L)^2$ and $c_1(L).F$ vanish in $H^2(V, \Omega^2_{V/k})$ by the observation above. This completes the proof of Lemma 4.5.2, hence of Theorem 4.5.1. □

4.6. The Picard variety of a variety over a function field. — In this paragraph, we recall some classical facts concerning the relations between the Picard varieties of C and V, and the K/k-trace of the Picard variety of the generic fiber V_K of V. (For modern presentations of Chow's classical theory of the K/k-trace of abelian varieties over K, we refer to **[11]** and Hindry's Appendix A in **[26]**.)

Let (B, τ) be the K/k-trace of $\mathrm{Pic}^0_{V_K/K}$. By construction, B is an abelian variety over k, and τ is a morphism of abelian varieties over K

$$\tau : B_K \longrightarrow \mathrm{Pic}^0_{V_K/K}.$$

The pair (B, τ) is characterized by the following universal property: for any abelian variety $\tilde{B}$ over k and any morphism of abelian varieties over K

$$\psi : \tilde{B}_K \longrightarrow \mathrm{Pic}^0_{V_K/K},$$

there exists a unique morphism

$$\beta : \tilde{B} \longrightarrow B$$

such that

$$\psi = \tau \circ \beta_K.$$

Actually, since our base field k has characteristic zero, τ is an embedding.

The inclusion $V_K \hookrightarrow V$ induces a morphism of abelian varieties over K

$$\phi : \mathrm{Pic}^0_{V/k,K} \longrightarrow \mathrm{Pic}^0_{V_K/K}.$$

According to the universal property above, there exists a unique morphism of abelian varieties over k

$$\alpha : \mathrm{Pic}^0_{V/k} \longrightarrow B$$

such that

$$\phi = \tau \circ \alpha_K.$$

Besides, we may consider the morphism

$$\pi^* : \mathrm{Pic}^0_{C/k} \longrightarrow \mathrm{Pic}^0_{V/k}$$

defined by functoriality from $\pi : V \to C$.

The following Proposition is established as Proposition 3.3 in [**24**], where references are made to similar earlier results due to Tate, Shioda, and Raynaud.

***Proposition* 4.6.1**. — *The morphism α is surjective, and the morphism π^* is an isogeny from $\mathrm{Pic}^0_{C/k}$ onto the abelian variety $(\ker\alpha)^\circ$ defined as the identity component of the k-group scheme $\ker\alpha$.*

In brief, the following diagram of abelian varieties over k

$$0 \longrightarrow \mathrm{Pic}^0_{C/k} \xrightarrow{\pi^*} \mathrm{Pic}^0_{V/k} \xrightarrow{\alpha} B \longrightarrow 0$$

is “exact up to some finite group schemes". Together with Poincaré’s reducibility theorem, this implies that the diagram of abelian groups

$$0 \longrightarrow \mathrm{Pic}^0_{C/k}(k) \xrightarrow{\pi^*} \mathrm{Pic}^0_{V/k}(k) \xrightarrow{\alpha} B(k) \longrightarrow 0 \tag{4.8}$$

is “exact up to some finite groups."

***Corollary* 4.6.2**. — *For any line bundle L over V, the following conditions are equivalent:*

(i) *There exists a positive integer N such that the class of $L_K^{\otimes N}$ in $\mathrm{Pic}_{V_K/K}(K)$ belongs to $\tau(B(k))$.*

(ii) *There exist a positive integer N and a line bundle L' over V, algebraically equivalent to zero, such that, over V_K,*

$$L_K^{\otimes N} \simeq L'_K.$$

(iii) *There exist a positive integer N, a line bundle L' over V, algebraically equivalent to zero, and a vertical divisor E over V such that, over V,*

$$L^{\otimes N} \simeq L' \otimes \mathcal{O}(E).$$

Proof. — The equivalence of (ii) and (iii) is straightforward. The one of (i) and (ii) follows from the "almost exactness" of (4.8) and the fact that any element of the group $\mathrm{Pic}^0_{V/k}(k)$ has a positive multiple that may be represented by an actual line bundle[15] over V, algebraically equivalent to zero. □

4.7. The equivalence of VA2 and VA3. — In this section, we complete the proof of Theorem 4.1.3 by establishing the equivalence of conditions **VA2** and **VA3**.

The implication **VA2⇒VA3** follows from the implication (ii)⇒(i) in Corollary 4.6.2 and from the invariance of $\deg_{H,D} L$ under algebraic equivalence of line bundles.

Conversely let us consider a line bundle L over V that satisfies **VA3**.

According to the implication (i)⇒(iii) in Corollary 4.6.2, we may find a positive integer N, a line bundle L' over V, algebraically equivalent to zero, and a vertical divisor E in V such that $L^{\otimes N} \simeq L' \otimes \mathcal{O}(E)$.

Moreover, for every vertical integral divisor D in V, we have

$$\deg_{H,D} L^{\otimes N} = N. \deg_{H,D} L = 0$$

by **VA3**, and

$$\deg_{H,D} L' = 0$$

since L' is algebraically equivalent to zero. Therefore,

$$\deg_{H,D} \mathcal{O}(E) = 0.$$

Lemma 4.7.1 below shows that, after possibly replacing L and L' by some positive power, the divisor E is of the form $\pi^*(E')$ for some divisor E' on C. Consequently,

$$L^{\otimes N} \otimes \pi^* \mathcal{O}(-E') \simeq L'$$

is algebraically equivalent to zero, and L satisfies **VA2**.

***Lemma* 4.7.1**. — *For any vertical divisor E on V, the following conditions are equivalent:*

(i) *For every vertical divisor D on V,*

$$\deg_{H,D} \mathcal{O}(E) = 0.$$

(ii) *There exist a divisor E' on C and a positive integer N such that*

$$N \cdot E = \pi^* E'.$$

[15] Indeed the functor $\mathrm{Pic}^0_{V/k}$ may be introduced via sheafification for the étale topology, hence given any α in $\mathrm{Pic}^0_{V/k}(k)$, we can find a finite (separable) extension k' of k and a line bundle M' on $V' := V \otimes_k k'$ that represents the image of α in $\mathrm{Pic}^0_{V/k}(k')$. Then $[k' : k].\alpha$ is represented by the line bundle $M := N_{V'/V}(M')$ on V defined as the norm of M'.

This is well known, at least when $n = 2$ and k is algebraically closed, in which case it is traditionally attributed to Zariski. We refer to **[14]** for a discussion of related results concerning intersection theory on surfaces, and to **[24]**, Lemme 2.1 for a similar result. We sketch a proof below for the sake of completeness.

Proof. — To establish the implication (ii)⇒(i), observe that, for any integral vertical divisor D on V, the following equality holds in the Chow group $CH^0(C)$

$$\pi_*(H^{n-2}.D) = 0. \tag{4.9}$$

(Indeed the class in $CH_1(V)$ of $H^{n-2}.D$ may be represented by a cycle in $Z_1(D)$, and consequently the left-hand side of (4.9) may be represented by a cycle in $Z_1(C)$ supported by $\pi(D)$. Since the latter is zero-dimensional, any such cycle vanishes.) Consequently, by the projection formula, for any divisor E' in C, we have

$$\begin{aligned}\deg_{H,D} \mathcal{O}(\pi^* E') &= \deg_k(H^{n-2}.D.\pi^* E') \\ &= \deg_k(\pi_*(H^{n-2}.D).E') \\ &= 0.\end{aligned}$$

To establish the implication (i)⇒(ii), we may assume that E is supported by the fiber $\pi^*(P)$ of some closed point P of C. Let $D_1, \dots, D_r$ be the components of $|\pi^*(P)|$, and let $n_1, \dots, n_r$ be the positive integers defined by the equality of divisors in V:

$$\pi^* P = \sum_{i=1}^{r} n_i.D_i.$$

We want to prove that if some divisor supported by $\pi^*(P)$, $E := \sum_{i=1}^{r} m_i.D_i$, satisfies

$$\deg_{H,D_j} \mathcal{O}(E) = 0,$$

for every $j \in \{1, \dots, r\}$, then E is a rational multiple of $\pi^*(P)$, that is, there exists m in $\mathbb{Q}$ such that

$$(m_1, \dots, m_r) = m(n_1, \dots, n_r).$$

In other words, we want to establish that the kernel of the symmetric quadratic form attached to the matrix $(q_{ij})_{1 \leq i,j \leq r}$ defined by

$$q_{ij} := \deg_k(H^{n-2}.D_i.D_j)$$

is included in the line $\mathbb{Q}.(n_1, \dots, n_r)$.

To establish this inclusion, observe that the converse implication (ii)⇒(i), applied to $D = D_i$ and $E = \pi^* P$, shows that

$$\sum_{j=1}^{r} q_{ij} n_j = 0$$

for every $i \in \{1, \ldots, r\}$. This yields the following expression for the quadratic form defined by the q_{ij}'s:

$$\sum_{i,j=1}^{r} q_{ij} m_i m_j = - \sum_{1 \leq i < j \leq r} q_{ij} n_i n_j \left(\frac{m_i}{n_i} - \frac{m_j}{n_j} \right)^2 .$$

The required property now follows from the following two observations:

1) For any two distinct elements i and j in $\{1, \ldots, r\}$, the cycle theoretic intersection $D_i.D_j$ of the Cartier divisors D_i and D_j is the cycle attached to the intersection scheme $D_i \cap D_j$, which is either empty or purely $(n-2)$-dimensional, and consequently, by the ampleness of H, the degree $q_{ij} := \deg_k(H^{n-2}.[D_i \cap D_j])$ *is non-negative, and positive if* $D_i \cap D_j$ *is not empty.*

2) The scheme $\pi^*(P)$ is connected, and consequently *there is no partition of* $\{1, \ldots, r\}$ *in two non-empty subsets* I *and* J *such that* $(i, j) \in I \times J \Rightarrow q_{ij} = 0$. □

Appendix A
Arithmetic extensions and Čech cohomology

Let X be an arithmetic scheme over an arithmetic ring $R = (R, \Sigma, F_\infty)$, E a quasi-coherent $\mathcal{O}_X$-module on X, and $\mathcal{U} = (U_i)_{i \in I}$ an affine, open covering of X. We fix a well ordering on I and consider the (alternating) Čech complex $\left(\mathcal{C}^{\cdot}(\mathcal{U}, E), \delta\right)$ where

$$\mathcal{C}^p(\mathcal{U}, E) := \prod_{i_0 < \cdots < i_p} E(U_{i_0 \ldots i_p}),$$

with the usual notation

$$U_{i_0 \ldots i_p} = U_{i_0} \cap \cdots \cap U_{i_p},$$

and where the differential $\delta : \mathcal{C}^p(\mathcal{U}, E) \to \mathcal{C}^{p+1}(\mathcal{U}, E)$ is given by the formula

$$(\delta\alpha)_{i_0, \ldots, i_{p+1}} := \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0, \ldots, \widehat{i_k}, \ldots, i_{p+1}} |_{U_{i_0} \cap \cdots \cap U_{i_{p+1}}} .$$

Recall from [**7**, 2.5] that we have a natural morphism of locally ringed spaces

$$\rho : (X_\Sigma(\mathbb{C}), \mathcal{C}^\infty_{X_\Sigma}) \longrightarrow (X_\Sigma(\mathbb{C}), \mathcal{O}^{\mathrm{hol}}_{X_\Sigma}) \longrightarrow (X, \mathcal{O}_X),$$

and that, if

$$E_{\mathbb{C}} := \rho^* E$$

denotes the $\mathcal{C}^\infty$-module over $X_\Sigma(\mathbb{C})$ deduced from E[16], there is a natural morphism of $\mathcal{O}_X$-modules, given by adjunction,

$$\mathrm{ad}_E \colon E \longrightarrow (\rho_* E_{\mathbb{C}})^{F_\infty}.$$

[16] Namely, when E is coherent and locally free, the sheaf of $\mathcal{C}^\infty$-sections over $X_\Sigma(\mathbb{C})$ of the holomorphic vector bundle $E^{\mathrm{hol}}_{\mathbb{C}}$ deduced from E.

It induces a morphism of Čech complexes

$$\mathscr{C}^{\cdot}(\mathscr{U}, \mathrm{ad}_E) \colon \mathscr{C}^{\cdot}(\mathscr{U}, E) \longrightarrow \mathscr{C}^{\cdot}(\mathscr{U}, (\rho_* E_{\mathbb{C}})^{F_\infty}).$$

Concerning cone constructions, in the sequel we use the sign conventions discussed in [**7**, A.1].

We consider the Čech hypercohomology $\check{H}^0\big(\mathscr{U}, C(\mathrm{ad}_E)\big)\big)$ of the cone $C(\mathrm{ad}_E)$ of ad_E with respect to the covering $\mathscr{U}$, namely the cohomology in degree zero of the cone $C\big(\mathscr{C}^{\cdot}(\mathscr{U}, \mathrm{ad}_E)\big)$. This cone is a complex of R-modules which starts as

$$0 \longrightarrow \mathscr{C}^0(\mathscr{U}, E) \xrightarrow{\binom{-\delta}{\mathrm{ad}_E}} \mathscr{C}^1(\mathscr{U}, E) \oplus \mathscr{C}^0(\mathscr{U}, (\rho_* E_{\mathbb{C}})^{F_\infty}) \xrightarrow{\left(\begin{smallmatrix} -\delta & 0 \\ \mathrm{ad}_E & \delta \end{smallmatrix}\right)} \mathscr{C}^2(\mathscr{U}, E) \oplus \mathscr{C}^1(\mathscr{U}, (\rho_* E_{\mathbb{C}})^{F_\infty})$$

where $\mathscr{C}^0(\mathscr{U}, E)$ sits in degree -1. Hence $\check{H}^0\big(\mathscr{U}, C(\mathrm{ad}_E)\big)$ is the quotient

$$\text{(A.1)} \qquad \frac{\Big\{(\alpha, \beta) \in \mathscr{C}^1(\mathscr{U}, E) \oplus \mathscr{C}^0\big(\mathscr{U}, (\rho_* E_{\mathbb{C}})^{F_\infty}\big) \,\big|\, \delta\alpha = 0 \wedge \mathrm{ad}_E(\alpha) = -\delta(\beta)\Big\}}{\Big\{\big(-\delta(\gamma), \mathrm{ad}_E(\gamma)\big) \,\big|\, \gamma \in \mathscr{C}^0(\mathscr{U}, E)\Big\}}.$$

According to the standard properties of the cone construction (in the category of R-modules) and the very definition of Čech cohomology as cohomology of the Čech complex, this group fits into a natural exact sequence:

$$\text{(A.2)} \quad \check{H}^0\big(\mathscr{U}, E)\big) \longrightarrow \check{H}^0\big(\mathscr{U}, (\rho_* E_{\mathbb{C}})^{F_\infty}\big) \longrightarrow \check{H}^0\big(\mathscr{U}, C(\mathrm{ad}_E)\big) \longrightarrow \check{H}^1\big(\mathscr{U}, E)\big) \longrightarrow \check{H}^1\big(\mathscr{U}, (\rho_* E_{\mathbb{C}})^{F_\infty})\big).$$

***Lemma* A.0.1**. — *Let E be quasi-coherent $\mathscr{O}_X$-module. There exists a canonical commutative diagram*

$$\begin{array}{ccccccccc}
\Gamma(X, E) & \rightarrow & A^0(X_{\mathbb{R}}, E) & \rightarrow & \widehat{\mathrm{Ext}}^1_X(\mathscr{O}_X, E) & \rightarrow & \mathrm{Ext}^1(\mathscr{O}_X, E) & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \hat{\rho}_{\mathscr{U},E} & & \downarrow \rho_{\mathscr{U},E} & \\
\check{H}^0(\mathscr{U}, E)) & \rightarrow & \check{H}^0(\mathscr{U}, (\rho_* E_{\mathbb{C}})^{F_\infty}) & \rightarrow & \check{H}^0(\mathscr{U}, C(\mathrm{ad}_E)) & \rightarrow & \check{H}^1(\mathscr{U}, E)) & \rightarrow 0
\end{array}$$

with exact horizontal lines where all vertical maps are isomorphisms.

Proof. — The upper exact sequence is established in [**7**, 2.2].

We have

$$\check{H}^1\big(\mathscr{U}, (\rho_* E_{\mathbb{C}})^{F_\infty})\big) = \check{H}^1\big(\rho^{-1}\mathscr{U}, (E_{\mathbb{C}})^{F_\infty})\big),$$

and the latter group is zero as Čech cohomology of a fine sheaf with respect to an open covering vanishes (see for instance [**19**, II.3.7 and II.5.2.3 (b)]). Consequently we obtain the lower exact sequence from (A.2).

The two left vertical maps are given by the natural isomorphisms induced by the restriction maps of the sheaves E and $(\rho_* E_{\mathbb{C}})^{F_\infty}$.

We now define $\rho_{\mathscr{U},E}$. Let

$$\mathscr{E} : 0 \longrightarrow E \longrightarrow F \xrightarrow{\pi} \mathscr{O}_X \longrightarrow 0$$

be an extension of $\mathcal{O}_X$-modules. The map π admits a section φ_i over each affine scheme U_i. The difference $\alpha_{ij} = \varphi_j|_{U_{ij}} - \varphi_i|_{U_{ij}}$ determines an element in $\Gamma(U_{ij}, E)$. The family $(\alpha_{ij})_{ij}$ defines a 1-cocycle in $\mathscr{C}^1(\mathcal{U}, E)$ whose class in $\check{H}^1(\mathcal{U}, E)$ does not depend on the choices of the φ_i. One obtains a canonical isomorphism (compare for example [**2**, Prop. 2])

$$\rho_{\mathcal{U},E} : \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, E) \longrightarrow \check{H}^1(\mathcal{U}, E)\,,\ [\mathcal{E}] \longmapsto \big[(\alpha_{ij})_{ij}\big].$$

Finally we define $\hat{\rho}_{\mathcal{U},E}$. Let $(\mathcal{E}, s)$ be an arithmetic extension with $\mathcal{E}$ as above. Choose the φ_i as before and define

$$\beta_i = s|_{U_i} - \mathrm{ad}_E(\varphi_i) \in A^{0,0}(U_{i,\mathbb{R}}, E).$$

We have $\mathrm{ad}_E(\alpha_{ij}) = \beta_i|_{U_{ij}} - \beta_j|_{U_{ij}}$. Hence the pair $\big((\alpha_{ij})_{ij}, (\beta_i)_i\big)$ determines an element $\hat{\rho}_{\mathcal{U},E}(\mathcal{E}, s)$ in (A.1), i.e. in $\check{H}^0\big(\mathcal{U}, C(\mathrm{ad}_E)\big)$. This class does not depend on the choices of the φ_i. Given different sections $\tilde{\varphi}_i$ which lead to cocycles $\big((\tilde{\alpha}_{ij})_{ij}, (\tilde{\beta}_i)_i\big)$ as above, we consider

$$\gamma \in \mathscr{C}^0(\mathcal{U}, E)\,,\ \gamma_i = \varphi_i - \tilde{\varphi}_i$$

and get

$$\begin{pmatrix} -\delta \\ \mathrm{ad}_E \end{pmatrix}(\gamma) = (\tilde{\alpha}, \tilde{\beta}) - (\alpha, \beta).$$

It is straightforward to check that

$$\hat{\rho}_{\mathcal{U},E} : \widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X, E) \longrightarrow \check{H}^0\big(\mathcal{U}, C(\mathrm{ad}_E)\big)\,,\ \big[(\mathcal{E}, s)\big] \longmapsto \big[(\alpha_{ij}), (\beta_i)\big]$$

is a group homomorphism which fits into the above commutative diagram. The five lemma implies that the map $\hat{\rho}_{\mathcal{U},E}$ is an isomorphism. □

***Corollary* A.0.2.** — *Let F, G be quasi-coherent $\mathcal{O}_X$-modules such that F is a vector bundle on X. There exists a canonical isomorphism*

$$\hat{\rho}_{\mathcal{U},F,G} : \widehat{\mathrm{Ext}}^1_X(F, G) \longrightarrow \check{H}^0\big(\mathcal{U}, C(\mathrm{ad}_{\mathcal{H}om(F,G)})\big)$$

which identifies $\widehat{\mathrm{Ext}}^1(F, G)$ with the quotient (A.1) *for $E = \mathcal{H}om(F, G)$.*

Proof. — It is proved in [**7**, 2.4.6] that there is a canonical isomorphism

$$\widehat{\mathrm{Ext}}^1_X(F, G) \xrightarrow{\sim} \widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X, \mathcal{H}om(F, G)) \tag{A.3}$$

which maps the class of an arithmetic extension $(\mathcal{E}, s)$ to the pushout of $(\mathcal{E}, s) \otimes F^\vee$ along the canonical map $j_F : \mathcal{O}_X \to F \otimes F^\vee$. Let $E = \mathcal{H}om(F, G)$. We define $\hat{\rho}_{\mathcal{U},F,G}$ as the composition of the isomorphisms (A.3) and $\hat{\rho}_{\mathcal{U},E}$ in Lemma A.0.1. □

Appendix B
The universal vector extension of a Picard variety

In this Appendix, we recall some basic facts concerning universal vector extensions of Picard varieties, which are essentially due to Messing and Mazur ([**34**], [**33**]). We show in particular that the universal vector extension of the Picard variety $\mathrm{Pic}^0_{X/k}$ of a smooth projective variety X over a field k of characteristic zero classifies line bundles with integrable connections (see (B.12) *infra*; this is certainly well-known but, to our knowledge, only the case where X is an abelian variety is treated in the literature). We also describe the maximal compact subgroups of the Lie groups defined by real and complex points of universal vector extensions.

B.1. Let S be a locally noetherian scheme. In the sequel, we consider a morphism $f : X \to S$ of schemes which satisfies the following assumptions:

i) The morphism f is projective, smooth with geometrically connected fibers.
ii) The Hodge to de Rham spectral sequence

$$E_1^{p,q} = R^q f_* \Omega^p_{X/S} \Rightarrow R^{p+q} f_* \Omega^{\cdot}_{X/S}$$

degenerates at E_1 and the sheaves $R^q f_* \Omega^p_{X/S}$ are locally free.
iii) The identity component $\mathrm{Pic}^0_{X/S}$ of the Picard scheme $\mathrm{Pic}_{X/S}$ is an abelian scheme.

We observe that i) implies that $\mathrm{Pic}_{X/S}$ is representable by a S-group scheme [**21**, n.232, Thm. 3.1] and that $f_*\mathcal{O}_X = \mathcal{O}_S$ holds universally [**22**, 7.8.6]. Furthermore i) implies ii) if S is of characteristic zero [**12**, Th. 5.5] and i) implies iii) if S is the spectrum of a field of characteristic zero [**6**, 8.4]. It is shown in [**27**, 8.3] that the formation of the coherent sheaves $R^q f_* \Omega^p_{X/S}$ and $R^n f_* \Omega^{\cdot}_{X/S}$ commutes with arbitrary base change if they are locally free for all $p, q \geq 0$ and all $n \geq 0$.

B.2. We consider the complex

$$\Omega^{\times}_{X/S} : 0 \longrightarrow \mathcal{O}_X^* \xrightarrow{\mathrm{dlog}} \Omega^1_{X/S} \xrightarrow{\mathrm{d}} \Omega^2_{X/S} \xrightarrow{\mathrm{d}} \cdots$$

where $\mathcal{O}_X^*$ sits in degree zero. The group

$$\mathrm{Pic}^{\#}(X/S) := H^1(X_{\mathrm{fppf}}, \Omega^{\times}_{X/S})$$

classifies isomorphism classes of pairs (L, ∇) where L is a line bundle on X and ∇ is an integrable connection

$$\nabla : L \longrightarrow L \otimes \Omega^1_{X/S}$$

relative to S [**34**, (2.5.3)]. We denote by

$$\mathrm{Pic}^{\#}_{X/S} := R^1 f_{\mathrm{fppf}*} \Omega^{\times}_{X/S}$$

the fppf-sheaf on the category of S-schemes associated to the presheaf

$$T \mapsto \mathrm{Pic}^{\#}(X \times_S T/T)$$

(see for instance [**6**, 8.1]). If $X_T = X \times_S T$ admits a section over T, we have [**34**, (2.6.4)]

$$\mathrm{Pic}^{\#}_{X/S}(T) = \mathrm{Coker}\big(\mathrm{Pic}(T) = \mathrm{Pic}^{\#}(T/T) \xrightarrow{f^*} \mathrm{Pic}^{\#}(X \times_S T/T)\big). \tag{B.1}$$

B.3. If T/S is a fpqc-morphism, we have

$$\mathrm{Pic}^{\#}_{X/S} \times_S T = \mathrm{Pic}^{\#}_{X_T/T}. \tag{B.2}$$

Indeed, this is obvious if T/S is fppf. Hence we may assume without loss of generality that X/S admits a section ε. This allows us to describe elements in $\mathrm{Pic}^{\#}_{X/S}(T)$ as isomorphism classes of triples (L, ∇, r) where L is a line bundle on X_T, ∇ is an integrable connection relative to T, and

$$r : \varepsilon^* L \xrightarrow{\sim} \mathcal{O}_T$$

is a rigidification. It follows from fpqc-descent that $\mathrm{Pic}^{\#}_{X/S}$ is in fact an fpqc-sheaf on S, which implies (B.2).

We will apply (B.2) in the situation where S is the spectrum of an arithmetic ring and T is the spectrum of $\mathbb{R}$ or $\mathbb{C}$.

B.4. The exact sequence of complexes

$$0 \longrightarrow \tau_{\geq 1}\Omega^{\cdot}_{X/S} \longrightarrow \Omega^{\times}_{X/S} \longrightarrow \mathcal{O}^*_X \to 0 \tag{B.3}$$

induces an exact sequence

(B.4)

$$H^1(X_{\mathrm{fppf}}, \tau_{\geq 1}\Omega^{\cdot}_{X/S}) \longrightarrow \mathrm{Pic}^{\#}(X/S) \longrightarrow H^1(X_{\mathrm{fppf}}, \mathcal{O}^*_X) \longrightarrow H^2(X_{\mathrm{fppf}}, \tau_{\geq 1}\Omega^{\cdot}_{X/S}).$$

Observe also that the first map in (B.4) is injective: this follows from the long exact sequence of H^0 's and H^1's associated with (B.3), from the vanishing of the map

$$\mathrm{dlog}\colon \Gamma(X, \mathcal{O}^*_X) \longrightarrow \Gamma(X, \Omega^1_{X/S})$$

(implied by Assumption B.1 i)), and the fppf-descent isomorphisms $\Gamma(X, \mathcal{O}^*_X) \simeq \Gamma(X_{\mathrm{fppf}}, \mathcal{O}^*_X)$ and $\Gamma(X, \Omega^1_{X/S}) \simeq \Gamma(X_{\mathrm{fppf}}, \Omega^1_{X/S})$.

Using fppf-descent and Assumption B.1 ii), one also gets:

$$H^1(X_{\mathrm{fppf}}, \mathcal{O}^*_X) = \mathrm{Pic}(X),$$

$$H^2(X_{\mathrm{fppf}}, \tau_{\geq 1}\Omega^{\cdot}_{X/S}) = H^2(X_{\mathrm{Zar}}, \tau_{\geq 1}\Omega^{\cdot}_{X/S}),$$

and

$$H^1(X_{\mathrm{fppf}}, \tau_{\geq 1}\Omega^{\cdot}_{X/S}) = \ker\big(H^0(X_{\mathrm{fppf}}, \Omega^1_{X/S}) \longrightarrow H^0(X_{\mathrm{fppf}}, \Omega^2_{X/S})\big) = \Gamma(S, f_*\Omega^1_{X/S}).$$

Sheafification of the exact sequence (B.4) and the injectivity of its first map yields an exact sequence of fppf-sheaves of abelian groups over S:

$$0 \longrightarrow f_*\Omega^1_{X/S} \longrightarrow \mathrm{Pic}^{\#}_{X/S} \longrightarrow \mathrm{Pic}_{X/S} \xrightarrow{c} R^2 f_* \tau_{\geq 1}\Omega^{\cdot}_{X/S}.$$

As there are no non-trivial homomorphisms from the abelian scheme $\mathrm{Pic}^0_{X/S}$ to the coherent sheaf $R^2 f_* \tau_{\geq 1}\Omega^{\cdot}_{X/S}$ by [**33**, Lemma p.9], we have $\mathrm{Pic}^0_{X/S} \subseteq \ker(c)$. Finally we obtain an extension of fppf-sheaves of abelian groups over S

$$0 \longrightarrow f_*\Omega^1_{X/S} \longrightarrow \mathrm{Pic}^{\#,0}_{X/S} \longrightarrow \mathrm{Pic}^0_{X/S} \longrightarrow 0 \tag{B.5}$$

where

$$\mathrm{Pic}^{\#,0}_{X/S} := \mathrm{Pic}^{\#}_{X/S} \times_{\mathrm{Pic}_{X/S}} \mathrm{Pic}^0_{X/S}.$$

B.5. The *universal vector extension* of the abelian scheme $\mathrm{Pic}^0_{X/S}$ is a group scheme $E_{X/S}$ which fits into an exact sequence of fppf-sheaves

$$0 \longrightarrow \mathbb{E}_{A/S} \longrightarrow E_{X/S} \longrightarrow \mathrm{Pic}^0_{X/S} \longrightarrow 0 \tag{B.6}$$

where $\mathbb{E}_{A/S}$ denotes the Hodge bundle of the dual abelian scheme

$$A := (\mathrm{Pic}^0_{X/S})^{\vee} \xrightarrow{\pi_A} S,$$

namely

$$\mathbb{E}_{A/S} := \pi_{A*}\Omega^1_{A/S}.$$

The universal vector extension may be characterized by its universal property: given an abelian fppf-sheaf E' and a vector group scheme M which fit into an extension of fppf-sheaves of abelian groups

$$0 \longrightarrow M \longrightarrow E' \longrightarrow \mathrm{Pic}^0_{X/S} \longrightarrow 0, \tag{B.7}$$

there exists a unique $\mathcal{O}_S$-linear morphism $\phi : \mathbb{E}_{A/S} \to M$ such that (B.7) is isomorphic to the pushout of (B.6) along ϕ.

By the universal property there exist unique morphisms α and β (of $\mathcal{O}_S$-modules and S-group schemes respectively) such that

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{E}_{A/S} & \to & E_{X/S} & \to & \mathrm{Pic}^0_{X/S} & \to & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \| & & \\ 0 & \to & f_*\Omega^1_{X/S} & \to & \mathrm{Pic}^{\#,0}_{X/S} & \to & \mathrm{Pic}^0_{X/S} & \to & 0 \end{array} \tag{B.8}$$

is a pushout diagram. The biduality of abelian schemes

$$\mathrm{Pic}^0_{X/S} \simeq \left(\mathrm{Pic}^0_{X/S}\right)^{\vee\vee} = A^{\vee} := \mathrm{Pic}^0_{A/S}$$

(see for instance [**6**, 8.1, Theorem 5]) yields a canonical isomorphism

$$E_{X/S} \xrightarrow{\sim} E_{A/S}. \tag{B.9}$$

It is furthermore shown in [**33**] and [**34**] that (B.8) with X replaced by A induces a canonical isomorphism

$$E_{A/S} \xrightarrow{\sim} \mathrm{Pic}^{\#,0}_{A/S}.$$

Assume that X/S admits a section ϵ. There exists a canonical morphism of S-schemes, the Albanese morphism of X over S relative to the “base point" ϵ,

$$\varphi : X \longrightarrow A,$$

that is characterized by the fact that the pullback of a Poincaré bundle for A over S (rigidified along 0) is isomorphic to a Poincaré bundle for X (rigidified along ϵ). The pullback along φ induces morphisms

$$\varphi^* : \mathbb{E}_{A/S} \longrightarrow f_*\Omega^1_{X/S}\,, \;\; \gamma \longmapsto \varphi^*\gamma$$

and (using description (B.1))

$$\varphi^* : \mathrm{Pic}^{\#,0}_{A/S} \longrightarrow \mathrm{Pic}^{\#,0}_{X/S}\,, \;\; [L,\nabla] \longmapsto [\varphi^*L, \varphi^*\nabla]$$

such that the diagram

$$\text{(B.10)}\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{E}_{A/S} & \longrightarrow & \mathrm{Pic}^{\#,0}_{A/S} & \longrightarrow & \mathrm{Pic}^0_{X/S} & \longrightarrow & 0 \\ & & \downarrow \varphi^* & & \downarrow \varphi^* & & \| & & \\ 0 & \longrightarrow & f_*\Omega^1_{X/S} & \longrightarrow & \mathrm{Pic}^{\#,0}_{X/S} & \longrightarrow & \mathrm{Pic}^0_{X/S} & \longrightarrow & 0 \end{array}$$

is commutative. The uniqueness assertion in the universal property implies that the maps α and β in (B.8) are given under the canonical identifications

$$E_{X/S} \xrightarrow{\sim} E_{A/S} \xrightarrow{\sim} \mathrm{Pic}^{\#,0}_{A/S}$$

by pullback along φ.

B.6. Let S be the spectrum of a field k of characteristic zero. For a projective, smooth, geometrically connected S-scheme X, our assumptions i)-iii) are satisfied, as explained in B.1.

Furthermore the morphism α becomes an isomorphism

$$\text{(B.11)}\qquad \alpha : \mathbb{E}_{A/k} := \Gamma(A, \Omega^1_{A/k}) \xrightarrow{\sim} \Gamma(X, \Omega^1_{X/k})$$

of k-vector spaces. Indeed, to establish that α is an isomorphism, we may replace k by a finite field extension, and therefore assume that $X(k)$ is not empty. If $\varphi : X \to A$ denotes the Albanese morphism associated to some base point ϵ in $X(k)$, α is given by pull back along φ, and is injective as X generates A as an abelian variety, and bijective for dimension reasons (compare for example [**6**, 8.4 Th. 1 b)]).

It follows that β is an isomorphism of k-group schemes

$$\text{(B.12)}\qquad \beta : E_{X/k} \xrightarrow{\sim} \mathrm{Pic}^{\#,0}_{X/k}.$$

In other words, $\mathrm{Pic}^{\#,0}_{X/k}$ becomes canonically isomorphic to the universal vector extension $E_{X/k}$ of $\mathrm{Pic}^0_{X/k}$.

When $X(k)$ is not empty, this isomorphism may be described as above, by means of the pull back along the Albanese map φ associated to any base point ϵ in $X(k)$, and using (B.1) we get a canonical isomorphism of abelian groups:

(B.13)
$$E_{X/k}(k) \simeq \left\{ (L,\nabla) \,\middle|\, \begin{array}{c} L \text{ line bundle algebraically equivalent to zero on } X \\ \nabla \text{ integrable connection on } L \end{array} \right\} \Big/ \sim ,$$

where $\sim$ denotes the obvious isomorphism relation between pairs (L,∇).

In general, when $X(k)$ is possibly empty, we may choose a Galois extension k'/k with Galois group Γ such that $X(k') \neq \varnothing$ and use the obvious identification

$$E_{X/k}(k) = E_{X_{k'}/k'}(k')^{\Gamma} \tag{B.14}$$

to reduce to the previous case.

B.7. If $k = \mathbb{C}$, the extension of commutative complex Lie groups

$$0 \to \Gamma(X, \Omega^1_{X/\mathbb{C}}) \longrightarrow E_{X/\mathbb{C}}(\mathbb{C}) \longrightarrow \mathrm{Pic}^0_{X/\mathbb{C}}(\mathbb{C}) \longrightarrow 0, \tag{B.15}$$

deduced from (B.6) by considering the complex points, admits the following description in the complex analytic category (compare [**34**, ex.(1.4)]).

The Lie algebra of $\mathrm{Pic}^0_{X/\mathbb{C}}$, hence of the complex Lie group $\mathrm{Pic}^0_{X/\mathbb{C}}(\mathbb{C})$, may be be identified with $H^1(X, \mathcal{O}_X)$, that is, by GAGA, with $H^1(X(\mathbb{C}), \mathcal{O}^{\mathrm{hol}}_{X(\mathbb{C})})$. By considering the exact sequence of sheaves over $X(\mathbb{C})$

$$0 \longrightarrow 2\pi i\mathbb{Z} \longrightarrow \mathcal{O}^{\mathrm{hol}}_{X(\mathbb{C})} \xrightarrow{\exp} \mathcal{O}^{\mathrm{hol},*}_{X(\mathbb{C})} \longrightarrow 0$$

and using GAGA, one obtains that the exponential map of $\mathrm{Pic}^0_{X/\mathbb{C}}$ defines an isomorphism of commutative complex Lie groups:

$$\frac{H^1(X(\mathbb{C}), \mathcal{O}^{\mathrm{hol}}_X)}{H^1(X(\mathbb{C}), 2\pi i\mathbb{Z})} \simeq \mathrm{Pic}^0_{X/\mathbb{C}}(\mathbb{C}). \tag{B.16}$$

The group of isomorphism classes of pairs (L,∇) where L is an algebraic line bundle over X and ∇ an integrable algebraic connection on L — or equivalently by GAGA, of pairs $(L^{\mathrm{hol}}, \nabla^{\mathrm{hol}})$ where L^{hol} is a holomorphic line bundle on the complex manifold $X(\mathbb{C})$ and ∇^{hol} an integrable, complex analytic connection on L^{hol} — may be identified with $H^1(X(\mathbb{C}), \mathbb{C}^*)$, by sending $[(L^{\mathrm{hol}}, \nabla^{\mathrm{hol}})]$ to the class of the rank one local system $\mathrm{Ker}(\nabla^{\mathrm{hol}})$. By considering the exponential sequence

$$0 \longrightarrow 2\pi i\mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow 0,$$

one sees that the group of classes of such pairs (L,∇) with L algebraically equivalent to zero may be identified with the subgroup of $H^1(X(\mathbb{C}), \mathbb{C}^*)$ that is the isomorphic image under the exponential map of

$$\frac{H^1(X(\mathbb{C}),\mathbb{C})}{H^1(X(\mathbb{C}),2\pi i\mathbb{Z})}.$$

Using the identification (B.13), we finally obtain an isomorphism

$$\frac{H^1(X(\mathbb{C}),\mathbb{C})}{H^1(X(\mathbb{C}),2\pi i\mathbb{Z})} \simeq E_{X/\mathbb{C}}(\mathbb{C}). \tag{B.17}$$

The analytic de Rham isomorphism

$$H^1(X(\mathbb{C}),\mathbb{C}) \simeq H^1(X(\mathbb{C}),\Omega^{\cdot\,\mathrm{hol}}_{X/\mathbb{C}})$$

and the Hodge filtration give rise to a short exact sequence of finite dimensional $\mathbb{C}$-vector spaces

$$0 \longrightarrow \Gamma(X(\mathbb{C}),\Omega^{1\,\mathrm{hol}}_{X/\mathbb{C}}) \longrightarrow H^1(X(\mathbb{C}),\mathbb{C}) \longrightarrow H^1(X(\mathbb{C}),\mathcal{O}^{\mathrm{hol}}_{X(\mathbb{C})}) \longrightarrow 0,$$

and then, by quotienting its second and third terms by $H^1(X(\mathbb{C}),2\pi i\mathbb{Z})$, to a short exact sequence of commutative complex Lie groups:

$$0 \longrightarrow \Gamma(X(\mathbb{C}),\Omega^{1\,\mathrm{hol}}_{X/\mathbb{C}}) \longrightarrow \frac{H^1(X(\mathbb{C}),\mathbb{C})}{H^1(X(\mathbb{C}),2\pi i\mathbb{Z})} \longrightarrow \frac{H^1(X(\mathbb{C}),\mathcal{O}^{\mathrm{hol}}_{X(\mathbb{C})})}{H^1(X(\mathbb{C}),2\pi i\mathbb{Z})} \longrightarrow 0, \tag{B.18}$$

It turns out that it coincides with the short exact sequence (B.15) when we take the GAGA isomorphism $\Gamma(X,\Omega^1_{X/\mathbb{C}}) \simeq \Gamma(X(\mathbb{C}),\Omega^{1\,\mathrm{hol}}_{X/\mathbb{C}})$ and the "exponential" isomorphisms (B.16) and (B.17) into account.

Observe that the maximal compact subgroup of the Lie group $E_{X/\mathbb{C}}(\mathbb{C})$ is precisely

$$\frac{H^1(X(\mathbb{C}),2\pi i\mathbb{R})}{H^1(X(\mathbb{C}),2\pi i\mathbb{Z})} \hookrightarrow \frac{H^1(X(\mathbb{C}),\mathbb{C})}{H^1(X(\mathbb{C}),2\pi i\mathbb{Z})} \simeq E_{X/\mathbb{C}}(\mathbb{C}). \tag{B.19}$$

It is a "real torus", of dimension the first Betti number of $X(\mathbb{C})$. Moreover, as a consequence of Hodge theory, the canonical morphism $E_{X/\mathbb{C}}(\mathbb{C}) \to \mathrm{Pic}^0_{X/\mathbb{C}}(\mathbb{C})$ in (B.15) maps this subgroup isomorphically (in the category of real Lie groups) onto $\mathrm{Pic}^0_{X/\mathbb{C}}(\mathbb{C})$.

In this way, we define a canonical splitting

$$\varsigma\colon \mathrm{Pic}^0_{X/\mathbb{C}}(\mathbb{C}) \longrightarrow E_{X/\mathbb{C}}(\mathbb{C}). \tag{B.20}$$

of (B.15) in the category of commutative real Lie groups, characterized by the fact that its image lies in — or equivalently, is — the maximal compact subgroup of $E_{X/\mathbb{C}}(\mathbb{C})$.

The injection $U(1) \hookrightarrow \mathbb{C}^*$ determines an injective morphism $H^1(X(\mathbb{C}),U(1)) \hookrightarrow H^1(X(\mathbb{C}),\mathbb{C}^*)$, and the maximal compact group (B.19) coincides with the preimage of $H^1(X(\mathbb{C}),U(1))$ under the exponential map. Consequently this group classifies the pairs (L,∇) as above, with L algebraically equivalent to zero, such that the monodromy of ∇^{hol} lies in $U(1)$. This shows that the real analytic splitting ς may also be described as follows: for any line bundle L over X that is algebraically equivalent to zero, we may equip $L_{\mathbb{C}}^{\mathrm{hol}}$ with its unique integrable, holomorphic connection ∇_L^u with unitary monodromy (*cf.* 3.2.1 *supra*); it algebraizes uniquely by GAGA, and the assignment

$$[L] \longmapsto \big[(L, \nabla_L^u)\big]$$

defines the group homomorphism (B.20).

B.8. If $k = \mathbb{R}$, the extension

$$0 \longrightarrow \Gamma(X, \Omega^1_{X/\mathbb{R}}) \longrightarrow E_{X/\mathbb{R}}(\mathbb{R}) \longrightarrow \mathrm{Pic}^0_{X/\mathbb{R}}(\mathbb{R}) \longrightarrow 0 \tag{B.21}$$

is obtained from the extension (B.15) by taking invariants under complex conjugation. We obtain again a canonical splitting

$$\varsigma_{\mathbb{R}} : \mathrm{Pic}^0_{X/\mathbb{R}}(\mathbb{R}) \longrightarrow E_{X/\mathbb{R}}(\mathbb{R})$$

since the splitting (B.20) is invariant under complex conjugation. The image of $\varsigma_{\mathbb{R}}$ is the unique maximal compact subgroup of $E_{X/\mathbb{R}}(\mathbb{R})$.

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