Astérisque
328, 2009, p. 1–44

THE SIGNATURE OPERATOR ON MANIFOLDS WITH A CONICAL SINGULAR STRATUM

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — We consider a Riemannian manifold, M, which can be compactified by adjoining a smooth compact oriented Riemannian manifold such that a neighbourhood of the singular stratum B, of codimension at least two, is given by a family of metric cones. Under the assumption that the middle cohomology of the cross-section vanishes, we show that there is a natural self-adjoint extension for the Dirac operator on forms with discrete spectrum, and we determine the condition of essential self-adjointness. We describe the boundary conditions analytically and construct a good parametrix which leads to the asymptotic expansion of a suitable resolvent trace as in our previous work. We also give a new proof of the local formula for the L^2-signature.

***Résumé* (Opérateur de signature sur les variétés avec une strate singulière conique)**

Nous considèrons une variété riemannienne M, qui peut être compactifiée en lui adjoignant une variété riemannienne C^∞ compacte orientée, telle qu'un voisinage de la strate singulière B, de codimension au moins deux, est donné par une famille de cônes métriques. Sous une hypothèse d'annulation de la cohomologie de la section du cône en dimension moitié, nous montrons qu'il existe une extension auto-adjointe naturelle de l'opérateur de Dirac agissant sur les formes qui est de spectre discret, et nous déterminons la condition sous laquelle l'opérateur de Dirac est essentiellement auto-adjoint. Nous décrivons les conditions de bord, et nous construisons une parametrix qui donne le développement asymptotique de la trace de la résolvante, comme dans un travail antérieur. Nous donnons aussi une preuve nouvelle de la formule locale pour la signature L^2.

Introduction

In this article, we analyze the signature operator on an oriented Riemannian manifold (M, g), of dimension $m = 4k$, with one compact singular stratum B of dimension

2010 Mathematics Subject Classification. — 53C20.
Key words and phrases. — Signature operator, closed extension, parametrix, Witt space.

This work was partially supported by SFB 288 and SFB 647 (HU Berlin).

h (the "horizontal dimension"), such that $m-h \geq 2$. A neighbourhood of the singular set is given by

$$U := U_{\varepsilon_0} := (0, \varepsilon_0) \times N, \ \varepsilon_0 \in (0, 1/2), \tag{0.1}$$

with an oriented compact Riemannian manifold N of dimension $4k-1$ and metric g^{TN}, and M decomposes as

$$M =: U_{\varepsilon_0} \cup M_{\varepsilon_0} \tag{0.2}$$

into points of distance at most and at least ε_0 of the singular set, respectively. For $\varepsilon \in (0, \varepsilon_0]$, we use analogous notation and write $U_\varepsilon, M_\varepsilon$, with

$$M = U_\varepsilon \cup M_\varepsilon.$$

We assume that the orientation on M and N induce the boundary orientation on U, such that $\{-\frac{\partial}{\partial t}, e_1, \ldots, e_{m-1}\}$ is oriented on U if $t \in (0, \varepsilon_0)$ and $\{e_1, \ldots, e_{m-1}\}$ is oriented on N. We assume in addition that the singularity is of the following special type. There is a fibration of oriented compact Riemannian manifolds,

$$\pi : Y \hookrightarrow N \to B, \tag{0.3}$$

with fibers $Y_b = \pi^{-1}(b)$, $b \in B$, of dimension $v := 4k-1-h \geq 1$ (the "vertical dimension"); in particular, B carries a metric g^{TB} such that π becomes a Riemannian submersion. Then the tangent bundle TN of N splits under g^{TN} into the *vertical* and the *horizontal* tangent bundle, consisting of the tangent vectors to the fibers and their orthogonal complement,

$$TN_p \ =: \ T_H N_p \oplus T_V N_p, \tag{0.4}$$

with induced metrics $g^{T_H N}$ and $g^{T_V N}$; the corresponding orthogonal projections in TN will be denoted by P_H and P_V, respectively. Next we assume that the metric $g^{TU} := g^{TM}|U$ takes the form

$$g^{TU} := dt^2 \oplus g^{T_H N} \oplus t^2 g^{T_V N}, \tag{0.5}$$

which we will call a metric of *conic type*. Thus, $M \cup B$ is a Riemannian pseudomanifold with one singular stratum of conic type.

The boundary of U is the Riemannian manifold

$$N_{\varepsilon_0} := (N, g^{TN}_{\varepsilon_0} := g^{T_H N} \oplus \varepsilon_0^2 g^{T_V N}). \tag{0.6}$$

The splitting of TN induces a splitting of the cotangent bundle,

$$T^*N =: T^*_H N \oplus T^*_V N,$$

into cotangent vectors annihilating $T_V N$ and $T_H N$, respectively. This splitting induces a bigrading of the exterior algebra ΛT^*N which will be important for our analysis; we write

$$\begin{aligned} \Lambda T^*N &= \Lambda T^*_H N \otimes \Lambda T^*_V N \\ &= \oplus_{j=p+q} \Lambda^p T^*_H N \otimes \Lambda^q T^*_V N =: \oplus_{p,q} \Lambda^{p,q} T^*N. \end{aligned} \tag{0.7}$$

The smooth sections of ΛT^*N and $\Lambda T^*_{H/V}N$ will be denoted $\lambda(N)$ and $\lambda_{H/V}(N)$, respectively, with degree or bidegree noted with superscripts.

Our main object will be the canonical Dirac operator associated with ΛT^*M,

$$D^\Lambda := D^\Lambda_M := d_M + d^\dagger_M, \tag{0.8}$$

with $d_M =: d$ the exterior derivative on M and $d^\dagger$ its formal adjoint with respect to the metric g^{TM}.

D defined on forms with compact support, denoted by $\lambda_c(M)$, is symmetric in $L^2(M, \Lambda T^*M) =: \lambda_{(2)}(M)$ but may not be essentially self-adjoint; we refer to the closure of this operator as $D^\Lambda_{\min} =: D_{\min}$, and $d_{\min}, d^\dagger_{\min}$ are defined analogously.

A specific self-adjoint extension of this operator can be defined via the Hilbert complex given by the operator $d_{\max}$ which arises from $d^\dagger$ as

$$d_{\max} := (d^\dagger_{\min})^*, \tag{0.9}$$

cf. [**11**, §3]; with a slight abuse of notation we denote this extension again by $D = D^\Lambda = D^\Lambda_M$, with domain $\mathscr{D} = \operatorname{dom} D$. In general, there will be many more self-adjoint extensions but D is of interest since its kernel gives the L^2-cohomology of M. If D is a Fredholm operator we have to break its symmetry to obtain a nontrivial index, e. g. by an anticommuting *supersymmetry* i. e. a self-adjoint involution of ΛT^*M. We will use multiplication by the complex volume element, τ_M, which splits

$$\Lambda T^*M =: \Lambda^+T^*M \oplus \Lambda^-T^*M$$

into ± 1-eigenbundles and analogously

$$\lambda(M) =: \lambda^+(M) \oplus \lambda^-(M),$$

with associated splitting $\sigma = \sigma^+ + \sigma^-$ on the level of forms. If τ_M maps $\mathscr{D}$ to itself than we can define the *Signature Operator* of M, with domain $\mathscr{D}^{\mathrm{sign}} = \mathscr{D}^+ = \frac{1}{2}(I + \tau_M)\mathscr{D}$, by

$$D^{\mathrm{sign}}_M := D^{\mathrm{sign}} := D^\Lambda_M|\mathscr{D}^+ : \mathscr{D}^+ \to \mathscr{D}^-. \tag{0.10}$$

We say that the *case of uniqueness* or the L^2-Stokes Theorem holds on M if

$$d_{\max} = d_{\min}. \tag{0.11}$$

In this case we have $\tau(\mathscr{D}) \subset \mathscr{D}$, and if D is also Fredholm, then so is D^{sign} and its index equals the L^2-signature of M,

$$\operatorname{ind} D^{\mathrm{sign}} = \operatorname{sign}_{(2)} M. \tag{0.12}$$

The above metric data define the crucial object in the analysis of the signature operator: the splitting of T^*N (induced by (0.4)) defines the "vertical de Rham operator" d_V (see (2.5)) and the metric $g_1^{T_VN}$ defines the adjoint $d_V^\dagger$, such that we can form the operator (see (2.31))

$$A_V := (d_V + d_V^\dagger)\alpha + \nu. \tag{0.13}$$

Here α is another supersymmetry on ΛT^*N and ν is an endomorphism (which are defined in (2.19) and (2.13)), and A_V is a first order symmetric differential operator on $C_c^1(N, \Lambda T^*N)$ which is fiberwise elliptic. Now M is called a *Witt space* if

$$H^{v/2}(Y) = 0. \tag{0.14}$$

We will see below (cf. Theorem 3.1) that (0.14) is essentially equivalent to the analytic condition

$$A_V \text{ is invertible,} \tag{0.15}$$

in the sense that the invertibility of A_V implies the Witt condition, whereas the Witt condition does not exclude the existence of zero eigenvalues but only of such which may be called inessential; indeed, they disappear under suitable rescalings of the fiber metric. *We will assume that M is a Witt space.*

Our results can then be summarized in the following theorems. We describe the Signature Operator on M by explicitly constructing its Green kernel which relates it to the symmetric operator $\tilde{D}$ defined as the restriction of $D_{\max}$ to the domain

$$\begin{aligned}\{\sigma \in \operatorname{dom} D_{\max} : \|\sigma^+\|_{\lambda_{(2)}(N_t)} &= O(t^{1/2-\varepsilon}) \text{ for every } \varepsilon > 0,\\ \|\sigma^-\|_{\lambda_{(2)}(N_t)} &= O(t^{-1/2+\eta}) \text{ for some } \eta > 0, t \to 0\};\end{aligned} \tag{0.16}$$

note that $\tilde{D}$ anticommutes with τ_M by construction.

Theorem 0.1. — *Let the Riemannian manifold (M, g^{TM}), of dimension $m = 4k$, be the top stratum of a Riemannian pseudomanifold, X, which is a Witt space with only one singular stratum B of conic type.*
1. *The operator $\tilde{D}$ defined by* (0.16) *is self-adjoint and discrete and anticommutes with τ_M.*
2. *If $|A_V| \geq \frac{1}{2}$, then $D^{\Lambda}_{M,\min}$ is essentially self-adjoint.*
3. *The case of uniqueness holds for M.*
4. *$D^{\mathrm{sign}} = \tilde{D}^+$.*

This theorem is well known in the case $h = 0$, cf. [**15**], [**12**], and part 2 and part 3 could also be deduced from Cheeger's work [**15**].

It is clear from part 4 of Theorem 0.1 that under the above conditions

$$\operatorname{ind} \tilde{D}^+ = \operatorname{ind} \tilde{D}^{\mathrm{sign}} = \operatorname{sign}_{(2)} M, \tag{0.17}$$

so it is natural to ask for a local formula analogous to Hirzebruch's Signature Theorem in the smooth case. Bismut and Cheeger [**6**, Thm. 5.7] have indicated the adiabatic construction of the homology L-class on the compact singular space associated with M, together with the corresponding L^2-index formula. A crucial role is played by the η-invariant, $\eta(N, g^{TN})$, of the Riemannian manifold (N, g^{TN}), as introduced by Atiyah, Patodi, and Singer in [**1**, Thm. (4.14)], and its adiabatic limit,

$$\tilde{\eta}(N, g^{TN}) := \lim_{\epsilon \to 0} \eta(N, g^{TN}_{\varepsilon}).$$

The adiabatic limit was first introduced and computed by Witten [**25**], as a gravitational anomaly, in case of a one-dimensional base. Witten's formula was proved rigorously by Cheeger [**16**], and independently by Bismut and Freed [**9**], [**10**]. The computation of the adiabatic limit for arbitrary dimensions and invertible fiber operators was given by Bismut and Cheeger [**6, 7**], who introduced the form $\tilde{\eta} = \tilde{\eta}(\pi, g^{TM}) \in \lambda(B)$ generalizing the η-invariant; the case of the signature operator was treated by Dai [**18**, Thm.0.3] who further introduced the τ-invariant associated to the Leray spectral sequence of the fibration (0.3). There has been done considerable work recently on the computation of L^2-cohomology groups of spaces which can be compactified as pseudo-manifolds of the type we consider here, cf. [**19**], [**20**], [**21**], and [**17**]. These calculations lead to topological formulas for $\operatorname{sign}_{(2)} M$, see [**17**, Cor.1.2] for Witt spaces and its extension in [**21**]. Combining these topological formulas with Dai's result quoted above gives the following local signature formula which was stated for even dimensional base spaces in [**8**, Thm.5.7]; in its formulation, we denote by $D_B^{\Lambda\otimes\mathscr{H}(Y)}$ the Dirac operator D_B^{Λ} twisted by the bundle of fiber harmonic forms.

Theorem 0.2. — *We have*

$$\operatorname{ind} D^{\operatorname{sign}} = \lim_{\varepsilon\to 0} \int_M L(\nabla^{TM}) - \int_B L(TB, \nabla^{TB}) \wedge \tilde{\eta} - \frac{1}{2}\eta(D_B^{\Lambda\otimes\mathscr{H}(Y)}).$$

We give here an analytic proof of [**17**, Cor.1.2] in the general case which should be applicable to more general situations; in combination with the results of Atiyah, Patodi, and Singer and Dai's computation, it yields the theorem. The parametrix construction which we give in this paper should, in principle, also lead directly to the local index formula but, so far, we have been unable to overcome the technical difficulties involved.

We also have considered the resolvent trace expansion. We have a proof of the following result, but its presentation would lengthen the paper unduly; we hope to include it in a more general result at some future time.

Theorem 0.3. — 1. *For $\mu \in \mathbb{R}\setminus\{0\}$ and $p > m$, the resolvent $(D - i\mu)^{-1}$ is in the Schatten-von Neumann class of order p in $L^2(M, \Lambda T^*M)$.*
2. *For $z \in \mathbb{R}$ and $l > m/2$, we have the asymptotic expansion*

$$\operatorname{tr}[D^2 + z^2]^{-l} \sim_{z\to\infty} z^{m-2l} \sum_{j\geq 0} a_j z^{-j} + \sum_{j\geq 2l-h} b_j z^{-j} \log z.$$

The plan of the article is as follows. In Section 1, we deal with general Dirac operators and derive some decomposition theorems which are induced by a fibration of the form (0.3) and are needed later on. These results are known for spin Dirac operators, see [**5**, pp. 56, 59].

In Section 2, we represent the signature operator D^{sign} on U in the form

$$D_M^{\operatorname{sign}} \simeq \frac{\partial}{\partial t} + A_H(t) + t^{-1} A_V,$$

acting on $C_c^1\big((0,\varepsilon_0), H^1(N,\Lambda T^*N)\big)$ (see (2.38)). Here $A_H(t)$ and A_V are first order differential operators which can be written as a Dirac operator plus a potential and $A_H(t)$ is linear in t, with derivative a bounded endomorphism, while A_V is given by (0.13). We also show (in Theorem 2.5) that the anticommutator $A_H A_V + A_V A_H$ is a first order vertical differential operator, a crucial fact for our analysis. The guiding principles here are the structure of Dirac systems, as developped in **[3]**, and the decomposition results from Sec. 1.

In Section 3, we obtain explicitly the spectral decomposition of the operators $A_V(b) := A_V|Y_b$ (cf. Theorem 3.1). By ellipticity, the spectrum is discrete. It consists of the *harmonic* eigenvalues $\mu = j - v/2, 0 \le j \le v$, generated by the harmonic forms on Y_b, and two families $\mu^\pm$ generated by the nonzero eigenvalues of the Laplacian on Y_b, with $\mu^+ \subset (-\frac{1}{2}, \infty)$ and $\mu^- \subset (-\infty, \frac{1}{2})$. When the metric on Y_b is scaled down, these eigenvalues tend respectively to $+\infty$ and $-\infty$.

Section 4 introduces appropriate boundary conditions for D^{sign}, based on the spectral analysis of Section 2. For the choice of boundary conditions and hence of a self-adjoint extension, only the small eigenvalues of A_V matter. We treat them by explicitly constructing the resolvent kernel by means of matrix Bessel functions, as introduced in **[13]**, and then use this kernel in constructing a good pseudodifferential parametrix for D^{sign} with operator valued symbol, again following the strategy developed in **[13]**. At the end of this section, we give the proof of Theorem 0.1.

In Section 5 we prove Theorem 0.2 by reducing the problem to an APS-type problem on M_ε, for sufficciently small $\varepsilon > 0$. We also prove various related results: a Kato type perturbation result for the APS projection (Theorem 5.9), a vanishing result which is crucial for our approach (Theorem 5.2), and a new identity involving Dai's τ-invariant (Theorem 5.4).

This paper started as a joint project with Bob Seeley to whom it owes a lot. The construction of the Signature Operator was essentially finished several years ago using a less explicit parametrix construction. The publication of the results has been delayed by an attempt to deduce the local signature formula directly from the resolvent expansion in Theorem 0.3. However, this goal has proved elusive so far; we hope that, nevertheless, the results presented here will be of independent value.

We wish to thank Bob Seeley for many years of fruitful exchange and cooperation. We are indebted to Jean-Michel Bismut, Xiaonan Ma, and Henri Moscovici for useful discussions. We are grateful for the support of Deutsche Forschungsgemeinschaft under various grants, especially SFB 288 and SFB 647, and for the generous hospitality of the Ohio State University, the Mittag-Leffler Institute, the University of Bergen, Kyoto University, and MSRI Berkeley. Special thanks are due to an anonymous referee for very helpful remarks based on an unduly preliminary version of this article.

1. Dirac operators on fibrations

In this section, we consider a Riemannian manifold (M, g^{TM}) which we assume to be oriented. For $X, Y \in TM$ we write

$$g^{TM}(X,Y) =: \langle X, Y\rangle_{TM} =: \langle X, Y\rangle,$$

if no confusion may arise, and we use similar notation for vector bundles. Moreover, we consider a second oriented Riemannian manifold (B, g^{TB}) and a Riemannian fibration

$$\pi = \pi_B^M : M \to B \tag{1.1}$$

with generic fiber F; we write

$$F_b := \pi^{-1}(b),\ \ b \in B. \tag{1.2}$$

We denote the bundle of tangent vectors to the fibers by $T_V M$. Then the fibration induces an orthogonal splitting

$$TM =: T_H M \oplus T_V M,\ g := g^{TM} =: g^{T_H M} \oplus g^{T_V M} =: g_H \oplus g_V,$$

with orthogonal projections $P_{H/V} : TM \to T_{H/V}M$. Note that $T_V M$ and its annihilator $T^*_H M$ are defined independent of the metric.

The bundle (TM, g^{TM}) has a distinguished metric connection, the Levi-Civita connection ∇^{TM}; all bundles associated to the principal bundle of orthonormal frames in TM inherit a metric and a metric connection from (TM, g^{TM}). This holds in particular for the exterior algebra of the cotangent bundle, ΛT^*M, and for the bundle of Clifford algebras, $Cl(TM)$, and its complexification, $\mathbb{C}l(TM) = Cl(TM) \otimes_{\mathbb{R}} \mathbb{C}$.

We are interested in the class of *Dirac bundles* as defined in [**23**, p. 114], i.e. the smooth hermitian bundles (E, h^E) over M equipped with hermitian connections ∇^E such that the following conditions are satisfied: There is a smooth bundle map cl from the tangent bundle, TM, to the skew-hermitian endomorphisms, $\mathrm{End}_{\mathrm{as}}\, E$, of E such that

$$\mathrm{cl}(X) \circ \mathrm{cl}(X) = -g(X,X) I_E,\ \ X \in TM, \tag{1.3}$$

which implies that cl extends to an algebra homomorphism

$$\mathrm{cl} : \mathbb{C}l(TM) \to \mathrm{End}\, E, \tag{1.4}$$

turning E into a left Clifford module. Moreover, ∇^E is required to be compatible with the Levi-Civita connection in the sense that

$$\nabla^E_X \mathrm{cl}(Y)\sigma = \mathrm{cl}(\nabla^{TM}_X Y)\sigma + \mathrm{cl}(Y)\nabla^E_X \sigma, \tag{1.5}$$

for $X, Y \in TM, \sigma \in C^1(M, E)$. A prototypical Dirac bundle is, of course, $\mathbb{C}l(TM)$ itself with the metric structure induced from g^{TM}. This bundle is canonically isomorphic to the exterior algebra bundle ΛT^*M, with Clifford action

$$\mathrm{cl}(X)\omega = \mathrm{w}(X^\flat)\omega - \mathrm{i}(X)\omega,\ \ X \in TM,\ \omega \in \Lambda T^*M,$$

where "w" and "i" refer to wedge and interior multiplication, respectively, while $\flat : TM \to T^*M$ denotes the "musical" isomorphism induced by g^{TM} with inverse $\sharp$. Note that these definitions extend naturally to Hilbert bundles over M.

The notion of Dirac bundle was introduced to define the *Dirac operator* naturally associated with it, i.e. the operator

$$D := D_M^E := \sum_{i=1}^m \mathrm{cl}(e_i)\nabla_{e_i}^E, \tag{1.6}$$

which we will regard as an unbounded operator in $L^2(M,E)$ with domain $C_c^1(M,E)$ if not stated otherwise. Then D is symmetric in $L^2(M,E)$ and essentially self-adjoint e.g. if M is complete.

To obtain a nontrivial index, the symmetry of D must be broken. This is achieved by a *supersymmetry* or grading, α, on E, i.e. by a self-adjoint involution $\alpha \in \operatorname{End} E$ which is parallel with respect to ∇^E and anticommutes with Clifford multiplication, and hence with D. Then the bundle E splits as

$$E = E^+ \oplus E^-,\ E^\pm = \frac{1}{2}(I \pm \alpha)E.$$

$\mathbb{C}l(TM)$ has a natural grading obtained by lifting the map $X \mapsto -X$ from TM to $\mathbb{C}l(TM)$, with the property that

$$\mathbb{C}l(TM)^\pm E^+ \subset E^\pm,\ \mathbb{C}l(TM)^\pm E^- \subset E^\mp,$$

for any graded Dirac bundle E.

We are now interested in splitting the Dirac operator $D = D_M^E$ along the fibration $\pi : M \to B$ into a "horizontal" and a "vertical" part. The notion of horizontality we use will be introduced below, while we will call a differential operator Q on $C_c^1(M,E)$ *vertical* if Q commutes with multiplication by functions pulled back from the base, i.e. if Q differentiates only in fiber directions; if Q is of first order this is also equivalent to saying that

$$\hat{Q}(\xi) = 0,\ \xi \in T_H^*M, \tag{1.7}$$

with $\hat{Q}$ the principal symbol of Q. The desired splitting of D will reflect the geometry of the fibration π, through the second fundamental form, which is defined for $X, Y \in T_VM$ and $Z \in T_HM$ by

$$\begin{aligned} \langle II(X,Y), Z\rangle &= \langle \nabla_Z X - P_V[Z,X], Y\rangle \\ &= \langle \nabla_X Z, Y\rangle \\ &= -\langle \nabla_X Y, Z\rangle; \end{aligned} \tag{1.8}$$

and through the curvature of π, which is for $Z_1, Z_2 \in T_HM$ defined as

$$R_{Z_1,Z_2} := -P_V[Z_1, Z_2].$$

Before we state the results on the splitting of D we need to introduce some notation concerning local orthonormal frames. We will always denote by $(e_i)_{i=1}^h$ and $(f_j)_{j=1}^v$ an oriented local orthonormal frame for T_HM and T_VM, respectively, where $h = \dim B$

and $v := \dim F$ denote the "horizontal" and "vertical" dimensions, with $h + v = m := \dim M$, and we assume that $\{e_1, \dots, f_v\}$ is oriented in TM. More specifically, we may assume that $(e_i)_{i=1}^h$ consists of the horizontal lifts of an oriented local orthonormal frame $(\underline{e}_i)_{i=1}^h$ for TB; if this frame is defined in some open set U then $(e_i)_{i=1}^h$ is defined in $\pi^{-1}(U)$.

There are two operators generated by D which naturally belong to the horizontal and the vertical space, respectively, to wit

$$\tilde{D}_H := \sum_{i=1}^{h} \mathrm{cl}(e_i)\nabla^E_{e_i}, \tag{1.9}$$

$$\tilde{D}_V := \sum_{j=1}^{v} \mathrm{cl}(f_j)\nabla^E_{f_j}, \tag{1.10}$$

such that $D = \tilde{D}_H + \tilde{D}_V$. However, these operators are not easy to interpret and in spite of having a symmetric principal symbol, they are not symmetric in general. This defect is easily cured as follows. Since D is symmetric on $C^1_c(M, E)$, i. e. $D = D^\dagger$, its formal adjoint, we obtain

$$\begin{aligned} D &= \frac{1}{2}(\tilde{D}_H + \tilde{D}_H^\dagger) + \frac{1}{2}(\tilde{D}_V + \tilde{D}_V^\dagger) \\ &=: D_H + D_V, \end{aligned} \tag{1.11}$$

with $D_{H/V}$ symmetric. But since $\tilde{D}_V$ has symmetric principal symbol, we see that

$$\tilde{D}_V^\dagger = \tilde{D}_V + \beta_1, \tag{1.12}$$

with some endomorphism $\beta_1 \in C^\infty(M, \mathrm{End}\, E)$ such that

$$D_H = \tilde{D}_H - \frac{1}{2}\beta_1, \tag{1.13}$$

$$D_V = \tilde{D}_V + \frac{1}{2}\beta_1; \tag{1.14}$$

note that β_1 is necessarily skew-symmetric.

Lemma 1.1. — 1. *In* (1.12), *we have*

$$\beta_1 = -v\,\mathrm{cl}(H_F), \tag{1.15}$$

where

$$H_F := -\frac{1}{v}\sum_{j=1}^{v} P_H \nabla^{TM}_{f_j} f_j$$

is the mean curvature vector field of the fibers of π.
2. *For any horizontal vector field* Z *on* M *we have*

$$\mathrm{cl}(Z)D_V + D_V\,\mathrm{cl}(Z) = 0. \tag{1.16}$$

Proof. — 1. We compute $\tilde{D}_V^\dagger$ by calculating for $\sigma_k \in C_c^1(M,E)$, $k=1,2$, the expression

$$
\begin{aligned}
(\tilde{D}_V\sigma_1,\sigma_2)_{L^2(M,E)} &- (\sigma_1,\tilde{D}_V\sigma_2)_{L^2(M,E)}\\
&= \sum_{j=1}^{v}\int_M \big(\langle \mathrm{cl}(f_j)\nabla^E_{f_j}\sigma_1,\sigma_2\rangle_E - \langle\sigma_1,\mathrm{cl}(f_j)\nabla^E_{f_j}\sigma_2\rangle_E\big)\\
&= \sum_{j=1}^{v}\int_M \big(-f_j\langle\sigma_1,\mathrm{cl}(f_j)\sigma_2\rangle_E + \langle\sigma_1,\mathrm{cl}(\nabla^{TM}_{f_j}f_j)\sigma_2\rangle_E\big)\\
&= \sum_{j=1}^{v}\int_M \big(-f_j\langle\sigma_1,\mathrm{cl}(f_j)\sigma_2\rangle_E + \langle\sigma_1,\mathrm{cl}(\nabla^{T_VM}_{f_j}f_j)\sigma_2\rangle_E\big)\\
&\quad - (\sigma_1, v\,\mathrm{cl}(H_F)\sigma_2)_{L^2(M,E)},
\end{aligned}
\tag{1.17}
$$

where we have used the properties (1.3) through (1.5). Now we introduce a vertical vector field, X, by setting

$$\langle X,Y\rangle_{T_VM} := \langle\sigma_1,\mathrm{cl}(Y)\sigma_2\rangle_E,\ Y\in C(M,T_VM).$$

Then it is easy to see that the integrand in (1.17) equals the divergence of $X|F_b$ and hence vanishes upon integration over F_b, for any $b\in B$. It follows that

$$\tilde{D}_V^\dagger - \tilde{D}_V = -v\,\mathrm{cl}(H_F),$$

as claimed.

2. We compute, using again the basic relations (1.3) through (1.5),

$$
\begin{aligned}
\mathrm{cl}(X)D_V + D_V\,\mathrm{cl}(X) &= \mathrm{cl}(X)\tilde{D}_V + \tilde{D}_V\,\mathrm{cl}(X) + v\langle X,H_F\rangle_{TM}\\
&= \sum_j \big(\mathrm{cl}(X)\,\mathrm{cl}(f_j)\nabla^E_{f_j} + \mathrm{cl}(f_j)\nabla^E_{f_j}\,\mathrm{cl}(X)\big) + v\langle X,H_F\rangle_{TM}\\
&= \sum_j \mathrm{cl}(f_j)\,\mathrm{cl}(\nabla^{TM}_{f_j}X) + v\langle X,H_F\rangle_{TM}\\
&= \Big(\sum_{j,l}\mathrm{cl}(f_j)\,\mathrm{cl}(f_l)\langle\nabla^{TM}_{f_j}X,f_l\rangle_{TM} + v\langle X,H_F\rangle_{TM}\Big)\\
&= \sum_{j\neq l}\mathrm{cl}(f_j)\,\mathrm{cl}(f_l)\langle X,\nabla^{TM}_{f_j}f_l\rangle_{TM}\\
&= 0.
\end{aligned}
$$

□

We will use below a stronger property of this decomposition, namely that (in the case of D^Λ)

$$\tilde{D}_{HV} := D_HD_V + D_VD_H \tag{1.18}$$

is a first order vertical differential operator. Note that while $\tilde{D}_{HV}$ is always of first order, it need not be vertical in general. But this can be achieved if we further modify the decomposition (1.11) by bringing in the curvature of π.

Theorem 1.2. — *Define a symmetric endomorphism field of E by*

$$\beta_2 := \frac{1}{4}\sum_{j;i,k}\langle\nabla^{TM}_{e_i}e_k, f_j\rangle\,\mathrm{cl}(f_j)\,\mathrm{cl}(e_k)\,\mathrm{cl}(e_i). \tag{1.19}$$

Then the operator

$$\tilde{D}_{HV} := (D_H+\beta_2)(D_V-\beta_2)+(D_V-\beta_2)(D_H+\beta_2) \tag{1.20}$$

is first order vertical.

Proof. — β_2 is clearly well defined. We compute

$$\tilde{D}_{HV} = (D_HD_V+D_VD_H)-(D_H\beta_2+\beta_2D_H)+(D_V\beta_2+\beta_2D_V) \tag{1.21}$$

$$=: I+II+III. \tag{1.22}$$

Since III is first order vertical, we compute the coefficient, γ_m, of $\nabla^E_{e_m}$ from I and II:

$$\begin{aligned}\gamma_m &= -\frac{v}{2}\big(\mathrm{cl}(H_F)\,\mathrm{cl}(e_m)+\mathrm{cl}(e_m)\,\mathrm{cl}(H_F)\big)+\sum_j \mathrm{cl}(f_j)\,\mathrm{cl}(\nabla^{TM}_{f_j}e_m)\\ &\quad+\big(\beta_2\,\mathrm{cl}(e_m)+\mathrm{cl}(e_m)\beta_2\big)\\ &= v\langle H_F,e_m\rangle+\sum_{j,k}\mathrm{cl}(f_j)\,\mathrm{cl}(f_k)\langle\nabla^{TM}_{f_j}e_m,f_k\rangle\\ &\quad+\sum_{j;i}\mathrm{cl}(f_j)\,\mathrm{cl}(e_i)\langle\nabla^{TM}_{f_j}e_m,e_i\rangle+\big(\beta_2\,\mathrm{cl}(e_m)+\mathrm{cl}(e_m)\beta_2\big)\\ &= \sum_{j;i}\mathrm{cl}(f_j)\,\mathrm{cl}(e_i)\langle\nabla^{TM}_{f_j}e_m,e_i\rangle+\big(\beta_2\,\mathrm{cl}(e_m)+\mathrm{cl}(e_m)\beta_2\big)\\ &= 0,\end{aligned}$$

if we plug in the definition of β_2 in the penultimate line. □

Our next goal is to interpret the new operators D_H and D_V as Dirac operators in a natural way. This is more obvious for D_V since the fibers F_b, $b\in B$, inherit a lot of structure from M and E. Indeed, denoting by $j_b : F_b\to M$ the inclusion map, we obtain a hermitian bundle with hermitian connection over F_b by defining

$$E_b := j_b^*E,\ h^{E_b} := j_b^*h^E,\ \nabla^{E_b} := j_b^*\nabla^E.$$

Clearly, the relations (1.3) and (1.4) remain valid, so what remains to be checked is the compatibility condition (1.5) which now needs to involve ∇^{TF_b}. To achieve this we are going to modify ∇^{E_b} as follows. For $X,Y\in TF_b\subset T_VM, Z\in TF_b^{\perp}\subset T_HM$, we introduce the shape operator $S=S_b$ of F_b by

$$S_ZX := -P_V(\nabla^{TM}_XZ),$$

such that

$$\begin{aligned}\langle S_ZX,Y\rangle_{TF_b} &:= \langle\nabla^{TM}_XY,Z\rangle_{TM}\\ &= -\langle \mathrm{II}_{F_b}(X,Y),Z\rangle.\end{aligned}$$

Then we define a new connection on TF by

$$\nabla_X^{E,b} := \nabla_X^{E_b} - \frac{1}{2}\sum_i \operatorname{cl}(S_{e_i}X)\operatorname{cl}(e_i), \tag{1.23}$$

which is clearly invariantly defined.

Theorem 1.3. — *The data $(E_b, h^{E_b}, \nabla^{E,b})$ define a Dirac bundle over F_b, for all $b \in B$, with Dirac operator*

$$D_V(b) := D_V|F_b. \tag{1.24}$$

Proof. — We compute with the notation used above:

$$\begin{aligned}
&\nabla_X^{E,b}\operatorname{cl}(Y) - \operatorname{cl}(Y)\nabla_X^{E,b} \\
&\qquad = \operatorname{cl}(\nabla_X^{TM}Y) - \frac{1}{2}\sum_i \big(\operatorname{cl}(S_{e_i}X)\operatorname{cl}(e_i)\operatorname{cl}(Y) - \operatorname{cl}(Y)\operatorname{cl}(S_{e_i}X)\operatorname{cl}(e_i)\big) \\
&\qquad = \operatorname{cl}(\nabla_X^{TM}Y) + \frac{1}{2}\sum_i \big(\operatorname{cl}(S_{e_i}X)\operatorname{cl}(Y) + \operatorname{cl}(Y)\operatorname{cl}(S_{e_i}X)\big)\operatorname{cl}(e_i) \\
&\qquad = \operatorname{cl}(\nabla_X^{TM}Y) - \sum_i \langle S_{e_i}X, Y\rangle_{TM}\operatorname{cl}(e_i) \\
&\qquad = \operatorname{cl}(\nabla_X^{TM}Y) - \sum_i \langle \nabla_X^{TM}Y, e_i\rangle_{TM}\operatorname{cl}(e_i) \\
&\qquad = \operatorname{cl}(\nabla_X^{TF_b}Y).
\end{aligned}$$

Next we compute the Dirac operator, $\bar{D}_V$, associated to $(E_b, h^{E_b}, \nabla^{E,b})$:

$$\begin{aligned}
\bar{D}_V &= \sum_j \operatorname{cl}(f_j)\nabla_{f_j}^{E,b} \\
&= \sum_j \operatorname{cl}(f_j)\nabla_{f_j}^{E} - \frac{1}{2}\sum_{i,j,k}\langle S_{e_i}f_j, f_k\rangle_{TM}\operatorname{cl}(f_j)\operatorname{cl}(f_k)\operatorname{cl}(e_i) \\
&= \sum_j \operatorname{cl}(f_j)\nabla_{f_j}^{E} - \frac{1}{2}\sum_{i,j,k}\langle \nabla_{f_j}^{TM}f_k, e_i\rangle_{TM}\operatorname{cl}(f_j)\operatorname{cl}(f_k)\operatorname{cl}(e_i) \\
&= \sum_j \operatorname{cl}(f_j)\nabla_{f_j}^{E} + \frac{1}{2}\sum_{i,j}\langle \nabla_{f_j}^{TM}f_j, e_i\rangle_{TM}\operatorname{cl}(e_i) \\
&= \sum_j \operatorname{cl}(f_j)\nabla_{f_j}^{E} - \frac{v}{2}\operatorname{cl}(H_F) \\
&= \tilde{D}_V - \frac{v}{2}\operatorname{cl}(H_F) \\
&= D_V.
\end{aligned}$$

□

To exhibit D_H as a Dirac operator, too, we have to extend our setting to smooth Dirac-Hilbert bundles. This does not require new definitions but only natural extensions, as indicated above. If we introduce the family of Hilbert spaces over B,

$$\mathscr{E}_b := L^2(F_b, E_b),\ b \in B, \tag{1.25}$$

and put $\mathscr{E} := \bigcup_{b\in B} \mathscr{E}_b$ then the restriction map

$$R : C_c^1(M,E) \to \Gamma(B,\mathscr{E}),\ R\sigma(b) := R_b\sigma := \sigma|F_b,\ b \in B, \tag{1.26}$$

is an isometry by the Fubini Theorem,

$$||\sigma||^2_{L^2(M,E)} = \int_B ||R\sigma(b)||^2_{\mathscr{E}_b} \operatorname{vol}_B(b). \tag{1.27}$$

We define a metric on $\mathscr{E}$ by setting

$$h^{\mathscr{E}}(b)\big(R\sigma_1, R\sigma_2\big) := \int_{F_b} h^E(b)\big(\sigma_1,\sigma_2\big) \operatorname{vol}_{F_b},\ \sigma_j \in C_c(M,E), j = 1,2, \tag{1.28}$$

and the Clifford action by

$$\mathrm{cl}_B(\underline{X})R\sigma := R\,\mathrm{cl}(X)\sigma,\ \sigma \in C_c(M,E), \tag{1.29}$$

where $\underline{X} \in TB$ with horizontal lift $X \in T_HM$. The connection requires again some modification: we put

$$\nabla^{\mathscr{E}}_{\underline{X}} R\sigma := R\nabla^E_X\sigma - \frac{1}{2}\sum_j R\,\mathrm{cl}(\nabla^{TM}_{f_j}X)\,\mathrm{cl}(f_j)\sigma, \tag{1.30}$$

where again $\underline{X} \in TB$ with horizontal lift $X \in TM$, and $\sigma \in C_c^1(M,E)$. Then we have the following pleasant interpretation of D_H.

Theorem 1.4. — *The data $(\mathscr{E}, h^{\mathscr{E}}, \nabla^{\mathscr{E}})$ define a (Hilbert-) Dirac bundle over B such that its Dirac operator, $D^{\mathscr{E}}$, is given by*

$$D^{\mathscr{E}}R\sigma := D^{\mathscr{E}}_B R\sigma := RD_H\sigma,$$

for $\sigma \in C_c^1(M,E)$.

2. Representation of the signature operator near the singularity

We now restrict the general considerations of the previous section to a manageable and important special case, namely the Dirac operator on differential forms on a manifold with a conic singular stratum. Hence we will assume in the remainder of this work that we deal with the geometric situation explained in the Introduction. Thus, we consider a Riemannian manifold (M, g^{TM}), of dimension $m = 4k$, such that for $\varepsilon \in (0, \varepsilon_0]$ we have decompositions

$$M := U_\varepsilon \cup M_\varepsilon, \tag{2.1}$$

where $(M_{\varepsilon_0}, g^{TM_{\varepsilon_0}})$ is a compact Riemannian manifold with boundary $\partial M_{\varepsilon_0} = N_{\varepsilon_0}$. We further assume that the singular part, U_{ε_0}, is a bundle of metric cones over another compact Riemannian manifold, (B, g^{TB}), as explained above.

In order to construct a self-adjoint Fredholm extension of the operator

$$D^{\Lambda}_{M,\min} := D_{\min} := (d_M + d_M^{\dagger})_{\min}, \tag{2.2}$$

we need to construct a good representation of D on U_{ε_0}. To obtain a nontrivial index, we use the supersymmetry leading to the signature operator which is defined, on any oriented Riemannian manifold (M, g^{TM}) and for any local orthonormal and oriented frame $(\tilde{e}_i)_{i=1}^m$ of tangent vectors, by

$$\begin{aligned} \tau_M := \tau_{M,g^{TM}} &:= \sqrt{-1}^{[(m+1)/2]}\, \mathrm{cl}(\tilde{e}_1)\dots\mathrm{cl}(\tilde{e}_m) \\ &= (-1)^k\, \mathrm{cl}(\tilde{e}_1)\dots\mathrm{cl}(\tilde{e}_m); \end{aligned} \tag{2.3}$$

note that τ anticommutes with any Dirac operator on sections with compact support if m is even. If the signature operator can be defined then it is derived from the maximal de Rham complex. Thus, we state next the decomposition of d_N under the Riemannian fibration (0.3), as described somewhat more generally in [**4**, Prop. 10.1]. For this, a few further preparations are needed.

In the decomposition (0.7),

$$\Lambda T^*N = \oplus_{p,q}\Lambda^{p,q}T^*_N,$$

we count the degree of forms by operators hd and vd of horizontal and vertical degree, respectively, that is,

$$\mathrm{hd}|\Lambda^{p,q}T^*N = p, \;\; \mathrm{vd}|\Lambda^{p,q}T^*N = q.$$

Furthermore, we note the natural isometry of hermitian bundles

$$\psi : \pi^*\Lambda T^*B \otimes \Lambda T^*_V M \to \Lambda T^*_H N \otimes \Lambda T^*_V N, \tag{2.4}$$

such that the smooth sections of ΛT^*N are generated over $C^\infty(N)$ by sections of the form $\pi^*\omega_1 \otimes \omega_2$, with $\omega_1 \in \lambda(B)$ and $\omega_2 \in \lambda_V(N)$. Thus we can define the first order vertical operator d_V figuring in (0.13) by

$$d_V(\pi^*\omega_1 \otimes \omega_2) := \pi^*\varepsilon_H\omega_1 \otimes d_F\omega_2, \tag{2.5}$$

where

$$\varepsilon_H := (-1)^{\mathrm{hd}}. \tag{2.6}$$

Finally, we note the following decomposition of the Levi-Civita connection on N,

$$\begin{aligned} \nabla^{TN} &:= (P_H\nabla^{TN}P_H + P_V\nabla^{TN}P_V) + P_H\nabla^{TN}P_V + P_V\nabla^{TN}P_H \\ &=: \nabla^{TN,\delta} + \nabla^{HV} + \nabla^{VH}, \end{aligned} \tag{2.7}$$

where $\nabla^{TN,\delta}$ is a connection while the other two terms are endomorphisms; observe that all operators in (2.7) act as derivations on tensors.

The decomposition of d_N for fibrations $\pi : N \to B$ then reads as follows.

Lemma 2.1. — *In local oriented orthonormal frames* $(e_i)_{i=1}^h$ *and* $(f_j)_{j=1}^v$ *for* $T_H N$ *and* $T_V N$*, respectively, we have*

$$d_N = \Big(\sum_i^h \mathrm{w}(e_i^\flat)\nabla_{e_i}^{TN,\delta} - \sum_{i;j,l}\langle\nabla_{f_j}^{TN} f_l, e_i\rangle_{TN}\,\mathrm{w}(e_i^\flat)\otimes\mathrm{w}(f_j^\flat)\mathrm{i}(f_l)\Big) \tag{2.8}$$

$$+\frac{1}{2}\sum_{i,k;j}\langle[e_k,e_i],f_j\rangle_{TN}\,\mathrm{w}(e_i^\flat)\,\mathrm{w}(e_k^\flat)\otimes\mathrm{i}(f_j) \tag{2.9}$$

$$+ d_V$$

$$=: d_H^{(1,0)} + d_H^{(2,-1)} + d_V^{(0,1)} \tag{2.10}$$

$$=: d_H^1 + d_H^2 + d_V. \tag{2.11}$$

In (2.8) *and* (2.9)*, the indices* i,k *run from 1 to* h *and indices* j,l *from 1 to* v*, while the upper indices in* (2.10) *indicate the change in bidegree effected by the respective operators; and* $d_V = d_V^{(0,1)}$ *is defined in* (2.5)*.*

Proof. — The proof follows straightforwardly from the well known representation

$$d_M = \sum_i \mathrm{w}(e_i^\flat)\nabla_{e_i}^{TM} + \sum_j \mathrm{w}(f_j^\flat)\nabla_{f_j}^{TM},$$

and the decomposition (2.7). □

We will use this result to determine the decomposition (1.11) for the fibration

$$\pi_{(0,\infty)} : U_\infty \to (0,\infty),$$

where we now allow ε to be any number with $0 < \varepsilon \le \infty$, by an obvious extension. This gives the *boundary representation* needed in the approach of Atiyah, Patodi, and Singer (APS) which will be applied here to reduce the index problem to an APS-type problem, cf. [**1**]. The geometry is, however, not cylindrical near the boundary as assumed in loc. cit. which will cause additional difficulties later.

We will base our analysis on the unitary transformation

$$\Psi_1 : L^2(\mathbb{R}_+, \mathbb{C}^2\otimes\lambda(N)) \to \lambda_{(2)}(U_\infty),$$
$$\Psi_1(\sigma_1,\sigma_2)(t) := \pi_N^* t^\nu \sigma_1(t) + dt\wedge\pi_N^* t^\nu\sigma_2(t), \tag{2.12}$$

where π_N denotes the canonical projection $U_\infty \to N$ and

$$\nu := \mathrm{vd} - \frac{v}{2}. \tag{2.13}$$

Ψ_1 generalizes the unitary transformation used in [**12**] for simple cones; note that it arises as the parallel transport along normal geodesics with respect to the metric connection defined by the fibration $\pi_{(0,\infty)} : U_\infty \to (0,\infty)$ according to Theorem 1.4. Then a straightforward calculation gives

Lemma 2.2. — *We have*

$$\Psi_1^{-1} d_{U_\infty} \Psi_1 = \begin{pmatrix} d_H^1 + t d_H^2 + t^{-1} d_V & 0 \\ \frac{\partial}{\partial t} + t^{-1}\nu & -d_H^1 - t d_H^2 - t^{-1} d_V \end{pmatrix}.$$

Taking adjoints and adding we obtain the transformation of $D^\Lambda_{U_\infty}$.

Corollary 2.3. — *With the notation*

$$\tilde{A}_H(t) := (d_H^1 + t d_H^2) + (d_H^1 + t d_H^2)^\dagger, \tag{2.14}$$

$$\tilde{A}_{0V} := d_V + d_V^\dagger, \tag{2.15}$$

$$\tilde{A}_0(t) := \tilde{A}_H(t) + t^{-1} \tilde{A}_{0V}, \tag{2.16}$$

and

$$\gamma := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.17}$$

we have

$$\Psi_1^{-1} D^\Lambda_{U_\infty} \Psi_1 =: \tilde{D}^\Lambda_{U_\infty} = \gamma\Big(\frac{\partial}{\partial t} + \begin{pmatrix} 0 & -\tilde{A}_H(t) \\ -\tilde{A}_H(t) & 0 \end{pmatrix} + t^{-1} \begin{pmatrix} \nu & -\tilde{A}_{0V} \\ -\tilde{A}_{0V} & -\nu \end{pmatrix}\Big) \tag{2.18}$$

To transform the signature operator we need to incorporate the self-adjoint involution τ_{U_∞} which defines it. From (2.3) it is easy to derive its transformation law:

Lemma 2.4. — *We have*

$$\begin{aligned} \tilde{\tau} := \Psi_1^{-1} \tau_M \Psi_1 &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \otimes \varepsilon_H^v \tau_H \otimes \tau_V \\ &=: \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes (-\alpha), \end{aligned} \tag{2.19}$$

where with oriented frames $\{e_1, \dots, e_h\}$, $\{f_1, \dots, f_v\}$ for $T_H N$ and $T_V N$, respectively, we have

$$\tau_H := \sqrt{-1}^{[(h+1)/2]} \operatorname{cl}(e_1) \dots \operatorname{cl}(e_h), \tag{2.20}$$

$$\tau_V := \sqrt{-1}^{[(v+1)/2]} \operatorname{cl}(f_1) \dots \operatorname{cl}(f_v); \tag{2.21}$$

note that ε_H^v and τ_H commute.

The signature operator transforms to the positive part of $\tilde{D}_{U_\infty}$ with respect to $\tilde{\tau}$:

$$\tilde{D}^{\mathrm{sign}}_{U_\infty} := \frac{1}{2}(I + \tilde{\tau}) \tilde{D}_{U_\infty} \frac{1}{2}(I + \tilde{\tau}). \tag{2.22}$$

To further transform $\tilde{D}^{\mathrm{sign}}_{U_\infty}$, we observe that the orthogonal projection onto the $+1$-eigenspace of $\tilde{\tau}$,

$$P^+(\tilde{\tau}) := \frac{1}{2}\begin{pmatrix} I & -\alpha \\ -\alpha & I \end{pmatrix}, \tag{2.23}$$

is conjugate to the standard projection

$$P := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \tag{2.24}$$

under the unitary transformation

$$U := \frac{1}{\sqrt{2}}\begin{pmatrix} I & \alpha \\ -\alpha & I \end{pmatrix}, \tag{2.25}$$

i. e.

$$P = U^{-1}P^+(\tilde{\tau})U, \tag{2.26}$$

or equivalently,

$$U^{-1}\tilde{\tau}U = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{2.27}$$

Now we obtain the final representation of $D^{\mathrm{sign}}_{U_\infty}$ by transforming all terms in (2.18) under U, observing the commutation relations

$$\nu\alpha = -\alpha\nu, \tag{2.28}$$

$$\tilde{A}(t)\alpha = \alpha\tilde{A}(t), \tag{2.29}$$

and using the notation

$$A_H(t) := \tilde{A}_H(t)\alpha,\ A_{0V}(t) := \tilde{A}_{0V}\alpha, \tag{2.30}$$

$$A_V := A_{0V} + \nu, \tag{2.31}$$

$$A_{(0)}(t) := A_H(t) + t^{-1}A_{(0)V}, \tag{2.32}$$

$$\Psi := \Psi_1 U, \tag{2.33}$$

where all operators are acting on $\lambda(N)$. We will call A_V the *cone coefficient* and

$$D^{\Lambda}_{\mathrm{cone}} := \gamma\Big(\frac{\partial}{\partial t} + t^{-1}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes A_V\Big), \tag{2.34}$$

the *cone operator*. Then the final result reads as follows.

Theorem 2.5. — 1. *We have*

$$\Psi^{-1}D^{\Lambda}_{U_\infty}\Psi = \gamma\Big(\frac{\partial}{\partial t} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes A(t)\Big), \tag{2.35}$$

and

$$\Psi^{-1}\tau_{U_\infty}\Psi = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2.36}$$

$$\Psi^{-1}D^{\Lambda}_{N_t}\tau_{N_t}\Psi = \begin{pmatrix} A_0(t) & 0 \\ 0 & A_0(t) \end{pmatrix}, \tag{2.37}$$

such that

$$\Psi^{-1}D^{\mathrm{sign}}_{U_\infty}\Psi = \frac{\partial}{\partial t} + A(t). \tag{2.38}$$

2.

$$A_H(0)A_V + A_V A_H(0) =: A_{HV} \tag{2.39}$$

is a first order vertical operator.

3. *If A_V is invertible then for t sufficiently small we have the estimate*

$$A(t)^2 \geq Ct^{-2}A_V^2 \tag{2.40}$$

with a positive constant C.

Proof. — 1. The transformation formulas are again verified by straightforward computations.

2. To prove (2.39) we use Theorem 1.2 which, after the appropriate transformations, shows that we can modify $A_H(t)$ and $t^{-1}A_V$ by adding a bounded endomorphism multiplied by t to each term, such that their anticommutator becomes first order vertical. This, however, is an algebraic condition so that, after multiplication with t, all operator coefficients in the resulting polynomial have to be first order vertical, in particular the leading one which is A_{HV}.

3. The estimate (2.40) is an easy consequence of (2.39). □

3. Spectral decomposition of the cone coefficient

We want to deal with the existence of self-adjoint extensions of the cone operator, $D^{\Lambda}_{\mathrm{cone}}(b)$, defined in (2.34). According to [**12**, Thm. 3.1], this operator is *essentially self-adjoint* in $L^2\big(\mathbb{R}_+, \mathbb{C}^2 \otimes H^0\big)$ with domain $C^1_c\big((0,\infty), \mathbb{C}^2 \otimes H^1\big)$ if and only if

$$|A_V(b)| \geq \frac{1}{2}, \tag{3.1}$$

where $b \in B$. If the condition (3.1) is violated, then the self-adjoint extensions of $D^{\Lambda}_{\mathrm{cone}}$ are classified by the Lagrangian subspaces of

$$V := \sum_{|\lambda|<\frac{1}{2}} \ker(A(b)-\lambda) \oplus \sum_{|\lambda|<\frac{1}{2}} \ker(A(b)+\lambda) \tag{3.2}$$

with respect to the standard symplectic form

$$\omega_b\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = x_1y_2 - x_2y_1.$$

It is therefore necessary to determine the small eigenvalues of $A_V(b)$; in fact, we will describe the full spectral resolution in Theorem 3.1 below.

For its proof we recall some well known material from Hodge theory. In what follows, we fix $b \in B$ and write $Y := Y_b$, with metric $g := g^{TY} = g^{T_V N}$, the closed submanifold of N which is the fiber over b under the fibration $\pi : N \to B$; we will also suppress the index "Y" if no confusion is to be expected. Thus we consider the Hodge Laplacian

$$\Delta := \Delta_Y = d_Y(d_Y)^\dagger + (d_Y)^\dagger d_Y =: dd^\dagger + d^\dagger d,$$

which defines the harmonic forms,

$$\mathscr{H}^j := \mathscr{H}^j(Y) = \ker \Delta^j \subset \lambda^j(Y) =: \lambda^j,$$

and the Hodge decomposition

$$\lambda^j := \mathscr{H}^j \oplus \lambda^j_{\mathrm{cl}} \oplus \lambda^j_{\mathrm{ccl}}, \tag{3.3}$$

$$\Delta^j_{\mathrm{cl/ccl}} := \Delta|\lambda^j_{\mathrm{cl/ccl}}. \tag{3.4}$$

Here the subscripts "cl" and "ccl" refer to closed and coclosed forms, respectively; the eigenspaces of $\Delta^j_{\mathrm{cl/ccl}}$ with eigenvalue $\kappa > 0$ will be denoted by $E^j_{\mathrm{cl/ccl}}(\kappa)$.

We also recall the following definitions and relations, where $* := *_Y$ denotes the Hodge star operator on Y and $v(2)$ the remainder of v mod 2:

$$\varepsilon_V|\lambda^j =: \varepsilon|\lambda^j = (-1)^j, \tag{3.5}$$

$$\alpha_i|\lambda^j := (-1)^{[(j+i)/2]}, \tag{3.6}$$

$$\alpha_0\alpha_1 = \varepsilon, \tag{3.7}$$

$$d^\dagger = (-1)^{v+1} * d * \varepsilon^v, \tag{3.8}$$

$$\tau_V|Y =: \tau = \sqrt{-1}^{[(v+1)/2]} * (-1)^{[v/2]}\alpha_{v(2)}, \tag{3.9}$$

$$d\tau = (-1)^{v+1}\tau d^\dagger. \tag{3.10}$$

Then we have

$$A_V(b) =: A_V = -\varepsilon_H^{v+1}\tau_H \otimes \big(d\tau + (-1)^{v+1}\tau d\big). \tag{3.11}$$

Next we introduce some spaces which are invariant under A_V (here and below, $j \in \mathbb{N} \cap [1, (v+1)/2)$ if not stated otherwise):

$$\tilde{\lambda}^j_h := \mathscr{H}^j \oplus \mathscr{H}^{v-j}, \tag{3.12}$$

$$\tilde{\lambda}^j_{\mathrm{cl}} := \lambda^j_{\mathrm{cl}} \oplus \lambda^{v+1-j}_{\mathrm{cl}}, \tag{3.13}$$

$$\tilde{\lambda}^j_{\mathrm{ccl}} := \lambda^{j-1}_{\mathrm{ccl}} \oplus \lambda^{v-1-j}_{\mathrm{ccl}}, \tag{3.14}$$

$$F^j_h := \mathscr{H}^j \oplus \mathscr{H}^{v-j}, \tag{3.15}$$

$$F^j_{\mathrm{cl}}(\kappa) := E^j_{\mathrm{cl}}(\kappa) \oplus E^{v+1-j}_{\mathrm{cl}}(\kappa), \tag{3.16}$$

$$F^j_{\mathrm{ccl}}(\kappa) := E^{j-1}_{\mathrm{ccl}}(\kappa) \oplus E^{v-1-j}_{\mathrm{ccl}}(\kappa). \tag{3.17}$$

It is then convenient to put

$$A^j_{V,h} := A_V | \tilde{\lambda}^j_h, \tag{3.18}$$

$$\begin{aligned} A^j_{V,\mathrm{cl}} - \frac{1}{2} &:= (A_V - \frac{1}{2}) | \tilde{\lambda}^j_{\mathrm{cl}} \\ &= \begin{pmatrix} j - \frac{v+1}{2} & -\varepsilon_H^{v+1} \tau_H \otimes d\tau \\ -\varepsilon_H^{v+1} \tau_H \otimes d\tau & -(j - \frac{v+1}{2}) \end{pmatrix}, \end{aligned} \tag{3.19}$$

$$\begin{aligned} A^j_{V,\mathrm{ccl}} + \frac{1}{2} &:= (A_V + \frac{1}{2}) | \tilde{\lambda}^j_{\mathrm{ccl}} \\ &= \begin{pmatrix} j - \frac{v+1}{2} & (-1)^v \varepsilon_H^{v+1} \tau_H \otimes \tau d \\ (-1)^v \varepsilon_H^{v+1} \tau_H \otimes \tau d & -(j - \frac{v+1}{2}) \end{pmatrix}. \end{aligned} \tag{3.20}$$

Then the spectral resolution of A_V can be expressed as follows.

Theorem 3.1. — 1. *$A^j_{V,h}$ has the eigenspaces $\mathscr{H}^j$ and $\mathscr{H}^{v-j}$ with eigenvalues $\pm(j - \frac{v}{2})$.*

2. *For $\kappa \in \operatorname{spec} \Delta^j_{\mathrm{cl}} \setminus \{0\}$, $A^j_{V,\mathrm{cl}} - \frac{1}{2}$ has two eigenspaces in $F^j_{\mathrm{cl}}(\kappa)$, with eigenvalues*

$$\mu^j_{\mathrm{cl},\pm}(\kappa) := \pm\sqrt{\kappa + (j - \frac{v+1}{2})^2},$$

and multiplicities $m^j_{\mathrm{cl},\pm}(\kappa)$.

3. *For $\kappa \in \operatorname{spec} \Delta^j_{\mathrm{ccl}} \setminus \{0\}$, $A^j_{V,\mathrm{ccl}} + \frac{1}{2}$ has two eigenspaces in $F^j_{\mathrm{ccl}}(\kappa)$, with eigenvalues*

$$\mu^j_{\mathrm{ccl},\pm}(\kappa) := \pm\sqrt{\kappa + (j - \frac{v+1}{2})^2},$$

and multiplicities $m^j_{\mathrm{ccl},\pm}(\kappa)$.

4. *If v is odd, then, for $\kappa > 0$, there are two more eigenspaces of $A^{(v+1)/2}_{V,\mathrm{cl}} \oplus A^{(v-1)/2)}_{V,\mathrm{ccl}}$ in $E^{(v+1)/2}_{\mathrm{cl}}(\kappa) \oplus E^{(v-1)/2}_{\mathrm{ccl}}(\kappa)$ with eigenvalues $\pm\sqrt{\kappa}$.*

5. *For $\kappa > 0$, the four eigenvalues of A_V in $F^j_{\mathrm{cl}}(\kappa) \oplus F^j_{\mathrm{ccl}}(\kappa)$ have the common multiplicity $2 \dim E^j_{\mathrm{cl}}(\kappa)$.*

Proof. — The first statement is obvious from Poincaré duality.

We compute next, using (3.18)

$$(A^j_{\rm cl} - \frac{1}{2})^2 = \begin{pmatrix} \Delta^j_{\rm cl} + (j - \frac{v+1}{2})^2 & 0 \\ 0 & \Delta^{v+1-j}_{\rm cl} + (j - \frac{v+1}{2})^2 \end{pmatrix}, \tag{3.21}$$

$$(A^j_{\rm ccl} + \frac{1}{2})^2 = \begin{pmatrix} \Delta^{j-1}_{\rm ccl} + (j - \frac{v+1}{2})^2 & 0 \\ 0 & \Delta^{v-1-(j-1)}_{\rm cl} + (j - \frac{v+1}{2})^2 \end{pmatrix}. \tag{3.22}$$

It follows that $F^j_{\rm cl}(\kappa) \oplus F^j_{\rm ccl}(\kappa)$ is invariant under A_V, and that A_V has the indicated eigenvalues on $F^j_{\rm cl}(\kappa) \oplus F^j_{\rm ccl}(\kappa)$. Moreover, we have unitary equivalences

$$\Delta^j_{\rm cl} \simeq \Delta^{v+1-j}_{\rm cl} \simeq \Delta^{j-1}_{\rm ccl} \simeq \Delta^{v-1-j}_{\rm ccl},$$

induced by the mappings $d\tau, \tau$, and τd, respectively. If we employ the bijective maps

$$\begin{pmatrix} 0 & -d\tau/-\tau d \\ d\tau/\tau d & 0 \end{pmatrix} : \tilde{\lambda}^j_{\rm cl/ccl} \mapsto \tilde{\lambda}^j_{\rm cl/ccl}, \tag{3.23}$$

$$\begin{pmatrix} 0 & (-1)^v \tau \\ \tau & 0 \end{pmatrix} : \tilde{\lambda}^j_{\rm cl} \mapsto \tilde{\lambda}^j_{\rm ccl}, \tag{3.24}$$

we see that the respective restrictions of A_V are unitarily equivalent under these maps up to the factor -1, which easily implies that the four eigenvalues on $F^j_{\rm cl}(\kappa) \oplus F^j_{\rm ccl}(\kappa)$ have the same multiplicities, and this must be $2 \dim E^j_{\rm cl}(\kappa)$, as asserted. This proves the assertions 2), 3), and 4), while 5) follows immediately from (3.18). □

4. A self-adjoint extension

With $D := D^\Lambda_M$ we associate the operators $D_{\min}$, i. e. the closure in $\lambda_{(2)}(M)$ of $D|\lambda_c(M)$, and $D_{\max} := D^*_{\min}$. In this section, we construct a suitable self-adjoint extension of the operator $D_{\min}$. For this, we introduce an operator family $G(\mu, D)$ for sufficiently large real μ, with $\operatorname{im} G(\mu, D)$ contained in the maximal domain of D, and

$$(D - i\mu)G(\mu, D)\tau = \tau, \; \tau \in L^2(M, \mathscr{E}). \tag{4.1}$$

Moreover, all the operators $G(\mu, D)$ map into a common domain on which D is symmetric. Hence this domain defines a self-adjoint extension of D, with resolvent $G(\mu, D)$. By a certain abuse of notation, we will denote this extension also by D.

We can naturally extend the conic fibers at hand to the infinite cones $C_{(0,\infty)}Y_b$, so we may and will assume that we are dealing with a fibration of infinite cones over B. The results can then be applied to U_ε by a standard cut-off procedure.

We obtain $G(\mu, D)$ as a pseudo-differential operator on B with operator valued symbol. For given $b_0 \in B$, choose $W_{b_0} := B_\delta(b_0)$, a ball on which the Hilbert bundle $\mathscr{E}$ is trivial. We identify forms $\tau \in C_c(W_{b_0}, \mathscr{E}|W_{b_0})$ with their representation in

$C_c(B_\delta^{\mathbb{R}^h}(0), \mathscr{E}_{b_0})$, and define a local parametrix $G_1(\mu, D, b_0)$ in the form

$$G_1(\mu, D, b_0)\tau(b) := \int_{\mathbb{R}^h} \exp(i\langle b, \beta\rangle) G(\mu, b, \beta)\hat{\tau}(\beta) \bar{d}\beta. \tag{4.2}$$

Here (b, β) are coordinates for $T^*W_{b_0}$, and $\bar{d}\beta := (2\pi)^{-h} d\beta$. These local parametrices are patched together in the usual way to make a global parametrix, $G_1(\mu, D)$, such that $(D - i\mu)G_1(\mu, D) - I$ decays in norm like $|\mu|^{-1}$, so that $G_1(\mu, D)$ serves as the leading term in a Neumann series for the resolvent $G(\mu, D)$.

We recall from Section 1 the decomposition $D = D_H + D_V$ and construct our operator $G(\mu, b, \beta)$ with the property that $\operatorname{im} G(\mu, b, \beta) \subset \mathscr{D}_{V,\max}(b)$, the domain of $D_{V,\max}(b)$, and

$$\big(D_{V,\max}(b) + i\,\mathrm{cl}(\beta^\sharp) - i\mu\big) G(\mu, b, \beta)\tau = \tau,\ \ \tau \in \Lambda T_b^* B \otimes \lambda_{(2)}(Y_b). \tag{4.3}$$

Just as above, $G(\mu, b, \beta)$ will define a self-adjoint extension of $D_V(b)$, with domain $\mathscr{D}_V(b)$. Note that, in view of Lemma 1.1, part 2, we have for $\sigma(b) \in \mathscr{D}_V(b)$

$$\begin{aligned} &||\big((D_V(b) + i\,\mathrm{cl}(\beta^\sharp)) - i\mu\big)\sigma(b)||^2_{\mathscr{D}_V(b)} \\ &\qquad = ||D_V(b)\sigma(b)||^2_{\mathscr{D}_V(b)} + (|\mu|^2 + |\beta|_b^2)||\sigma(b)||^2_{\mathscr{D}_V(b)}, \end{aligned} \tag{4.4}$$

where $|\beta|_b^2 := g^{T^*B}(b)(\beta, \beta)$.

With (2.34) we now write $D_V(b)$ in the form

$$\begin{aligned} D_V(b) &:= \gamma\left(\frac{d}{dt} + t^{-1}\tilde{A}(b)\right) \\ &:= \varepsilon_H \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\frac{d}{dt} + \frac{1}{t}\begin{pmatrix} D_{Y_b}\alpha_{Y_b} + \nu & 0 \\ 0 & -(D_{Y_b}\alpha_{Y_b} + \nu) \end{pmatrix}\right). \end{aligned} \tag{4.5}$$

The trivialization of $\mathscr{E}$ identifies the fibers $\mathscr{E}_b, b \in W_{b_0}$, with

$$L^2\big((0,\infty), \Lambda T_{b_0}^* B \otimes \mathbb{C}^2 \otimes \lambda_{(2)}(Y_{b_0})\big) =: L^2((0,\infty), H).$$

We will need the following description of the singularities of elements in the maximal domain of $\mathscr{D}_{V,\max}(b)$ (see [**12**, Lem.3.2]).

Lemma 4.1. — 1. *Any σ in $\mathscr{D}_{V,\max}(b)$ has a representation of the form*

$$\sigma(t) = \sum_{\lambda \in \operatorname{spec} A, |\lambda| < 1/2} t^{-\lambda} C_\lambda(\sigma) + O_\sigma(t^{1/2}|\log t|),\ t \to 0, \tag{4.6}$$

with certain linear forms C_λ.

2. *Each closed extension of $D_{V,\max}(b)$ is determined by linear relations between the coefficients C_λ for $|\lambda| < \frac{1}{2}$.*

3. *$\sigma \in \mathscr{D}_{V,\min}$ if and only if*

$$||\sigma(t)||_H = O_\sigma(t^{1/2}|\log t|),\ t \to 0. \tag{4.7}$$

Now, to construct $G(\mu, b, \beta)$, we split the spectrum of the operator $\tilde{A}(b)$ from (4.5), and treat separately the high and low eigenvalues. Arguing as in [**12**, Lemma 1.1] and making U_{b_0} smaller if necessary, we may then assume that, for some $\Lambda \geq 1$ with the property that $\Lambda \notin \operatorname{spec} \tilde{A}(b)$ for all $b \in W_{b_0}$, the spectral projection

$$Q_{>} := Q_{|\lambda| \geq \Lambda}(\tilde{A}(b)) \tag{4.8}$$

does not depend on $b \in W_{b_0}$ (here and below we denote, for any Borel subset $I \subset \mathbb{R}$, the corresponding spectral projection of a self-adjoint operator, A, by $Q_I(A)$).

In constructing $G(\mu, b, \beta)$, consider first the high eigenvalues of $\tilde{A}(b)$. We reduce $D_{V,\min}(b)$ by the spectral projection $Q_{>}$, which is independent of $b \in W_{b_0}$, and denote the resulting objects by a subscript $'' >''$. Since $|\tilde{A}(b)_{>}| \geq 1$, Lemma 4.1 shows that

$$D_V(b)_{>} := D_{V,\min}(b)_{>}$$

is essentially self-adjoint on compactly supported sections. Moreover, from [**12**, Lem.3.1], by a proof as in Lemma 2.2 there, for $\sigma \in \mathscr{D}_V(b)_{>}$ we have

$$(1/t)\tilde{A}(b)_{>}\sigma(t) \in L^2\big((0,\infty), H\big) \text{ hence also } \sigma' \in L^2\big((0,\infty), H\big).$$

It follows from this and (4.4) that

$$G(\mu, b, \beta)_{>} := \big(D_V(b)_{>} + i\,\mathrm{cl}(\beta^\sharp) - i\mu\big)^{-1}$$

satisfies the estimates

$$||\frac{\partial^j}{\partial \mu^j}\frac{\partial^{|\kappa|}}{\partial b^\kappa}\frac{\partial^{|\lambda|}}{\partial \beta^\lambda} G(\mu, b, \beta)_{>}||_{\mathscr{L}(\mathscr{E})} \leq C_{l,\kappa,\lambda}|\mu|^{-1-j}, \tag{4.9}$$

while from Lemma 4.1 we see that

$$G(\mu, b, \beta)_{>}\sigma(t) = O(t^{1/2}|\log t|), t \to 0. \tag{4.10}$$

As usual, the low eigenvalue case needs more care. We note first that the reduction with $Q_{<} := I - Q_{>}$ leads to the matrix equation

$$D_{V<}(b) := \gamma\big(\frac{d}{dt} + t^{-1}\tilde{A}(b)_{<}\big) \tag{4.11}$$

in $L^2((0,\infty), H_{<})$, $H_{<} = Q_{<}(H)$. In view of Lemma 4.1 this operator is not essentially self-adjoint with domain $C_c^1((0,\infty), H_{<})$ if there are "small" eigenvalues with modulus less than $1/2$. Hence we will construct an operator function satisfying the conditions

$$\big(D_{V,<}(b) + i\,\mathrm{cl}(\beta^\sharp) - i\mu\big)G(\mu, b, \beta)_{<} = I; \tag{4.12}$$

$$D_{V,<}(b)_{\max} \text{ is symmetric on } \operatorname{im} G(\mu, b, \beta)_{<}; \tag{4.13}$$

$$||\frac{\partial^j}{\partial \mu^j}\frac{\partial^{|\kappa|}}{\partial b^\kappa}\frac{\partial^{|\lambda|}}{\partial \beta^\lambda} G(\mu, b, \beta)_{<}||_{\mathscr{L}(\mathscr{E})} \leq C_{l,\kappa,\lambda}|\mu|^{-1-j}. \tag{4.14}$$

From (4.12) and (4.13), $D_{V<}(b)$, on $\operatorname{im} G(\mu, b, \beta)_{<}$, is self-adjoint.

The estimates (4.14), together with the Calderón- Vaillancourt Theorem (cf.[**14**]), will provide the necessary norm estimates on our pseudo-differential operator.

In order to carry out this construction, we now consider the following model case to which we will reduce our situation. We are given a finite dimensional complex

Hilbert space $(H, \langle , \rangle)$ and a Hermitian operator $A \in \mathscr{L}(H)$. Moreover, there are two self-adjoint involutions α_1, α_2 with the following properties:

$$\alpha_1\alpha_2 + \alpha_2\alpha_1 = 0, \tag{4.15}$$

$$\alpha_1 A - A\alpha_1 = 0, \tag{4.16}$$

$$\alpha_2 A + A\alpha_2 = 0. \tag{4.17}$$

We want to solve the equation

$$L(A)\sigma(t) := (\frac{d}{dt} + t^{-1}A + \mu\alpha_2)\sigma(t) = \tau(t),\ t > 0, \tag{4.18}$$

in $L^2(\mathbb{R}_+, H)$, for $\mu \in \mathbb{R}^*$. We transform H by introducing the subspaces $H^\pm := \frac{1}{2}(I \pm \alpha_1)(H)$ and the isomorphism $\mathbb{C}^2 \otimes H^+ \to H$ which is induced by

$$H^+ \oplus H^+ \ni (x_+, x_-) \mapsto x_+ + \alpha_2 x_- \in H.$$

Then our equation takes the form, with $A^+ := A|H^+$,

$$\left(\frac{d}{dt} + t^{-1}\begin{pmatrix} A^+ & 0 \\ 0 & -A^+ \end{pmatrix} + \mu\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right)\begin{pmatrix} \sigma_+ \\ \sigma_- \end{pmatrix}(t) = \begin{pmatrix} \tau_+ \\ \tau_- \end{pmatrix}(t). \tag{4.19}$$

If we multiply the operator occuring in (4.19) with its formal adjoint from the left, then we obtain the Bessel type operator

$$-\frac{d^2}{dt^2} + t^{-2}\begin{pmatrix} A^2 + A & 0 \\ 0 & A^2 - A \end{pmatrix} + \mu^2 I, \tag{4.20}$$

where we have now replaced A^+ by A to ease the notation, which should not cause confusion. Now we introduce the modified matrix Bessel functions in H^+ as solutions of the homogeneous equation associated with (4.20), following [**12**, Sec. 2]. Thus, if N is hermitian in $\mathscr{L}(H^+)$ with eigenvalues ν_j then we define the modified matrix Bessel function with respect to an orthonormal eigenbasis of N by

$$I_N(t)_{ij} := \delta_{ij} I_{\nu_j}(t),$$

and require that for any unitary operator U in H^+ we have

$$U^{-1} I_N(t) U =: I_{U^{-1}NU}(t),\ t > 0.$$

Likewise, we introduce

$$\frac{2}{\pi}\sin(\pi N) K_N(t) := I_{-N}(t) - I_N(t).$$

We can then prove the following result.

Theorem 4.2. — *For $\mu > 0$, the equation* (4.18) *admits the solution*

$$\begin{aligned}
&G(\mu, A)\begin{pmatrix} \tau_+ \\ \tau_- \end{pmatrix}(t) = \\
&= \int_0^t \mu(ts)^{1/2}\begin{pmatrix} K_{A+1/2}(\mu t)I_{A-1/2}(\mu s) & K_{A+1/2}(\mu t)I_{A+1/2}(\mu s) \\ K_{A-1/2}(\mu t)I_{A-1/2}(\mu s) & K_{A-1/2}(\mu t)I_{A+1/2}(\mu s) \end{pmatrix}\begin{pmatrix} \tau_+ \\ \tau_- \end{pmatrix}(s)ds \\
&\quad - \int_t^\infty \mu(ts)^{1/2}\begin{pmatrix} I_{A+1/2}(\mu t)K_{A-1/2}(\mu s) & -I_{A+1/2}(\mu t)K_{A+1/2}(\mu s) \\ -I_{A-1/2}(\mu t)K_{A-1/2}(\mu s) & I_{A-1/2}(\mu t)K_{A+1/2}(\mu s) \end{pmatrix}\begin{pmatrix} \tau_+ \\ \tau_- \end{pmatrix}(s)ds \\
&=: G_0(\mu, A)\tau(t) + G_\infty(\mu, A)\tau(t).
\end{aligned}$$

The operators $G_{0/\infty}(\mu, A)$ are bounded in $L^2(\mathbb{R}_+, H)$ and smooth functions of the variables $\mu \in [1, \infty)$ and $A \in \mathscr{L}_s(H)$, the space of Hermitian matrices on H, such that for $p, q \in \mathbb{Z}_+$

$$||D_A^p(\frac{\partial}{\partial\mu})^q G_{0/\infty}(\mu, A)||_{L^2(\mathbb{R}_+,H)} \le C_{p,q,A}\ \mu^{-1}. \tag{4.21}$$

Moreover, for $\sigma \in \operatorname{im} G(\mu, A)$ and t sufficiently small we have the estimates

$$||\sigma_+(t)||_H \le C_\varepsilon t^{1/2-\varepsilon}||\tau||_{L^2(\mathbb{R}_+,H)} \text{ for every } \varepsilon > 0, \tag{4.22}$$

$$||\sigma_-(t)||_H \le Ct^{-1/2+\delta}||\tau||_{L^2(\mathbb{R}_+,H)} \text{ for some } \delta > 0. \tag{4.23}$$

If $|A| \ge \frac{1}{2}$, then we have the better estimate

$$||\sigma(t)||_H \le Ct^{1/2}||\tau||_{L^2(\mathbb{R}_+,H)}. \tag{4.24}$$

Proof. — We begin with verifying that $G(\mu, A)\tau(t)$ is indeed a solution of (4.19). The well known conic scaling

$$\sigma(t) =: t^{1/2}\rho(\mu t)$$

transforms the homogeneous equation associated with (4.19) to

$$\left(\frac{d}{dt} + t^{-1}\begin{pmatrix} A + 1/2 & 0 \\ 0 & -A + 1/2 \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right)\begin{pmatrix} \rho_+ \\ \rho_- \end{pmatrix}(t) = 0. \tag{4.25}$$

The Bessel recursion relations (cf. [**12**, (2.5a,b)]),

$$I_N'(t) \pm t^{-1}NI_N(t) = I_{N\mp 1}(t),$$
$$K_N'(t) \pm t^{-1}NK_N(t) = -K_{N\mp 1}(t),$$

show at once that two solutions are given by

$$\rho_{c_+}(t) = \begin{pmatrix} I_{A+1/2}(t)c_+ \\ -I_{A-1/2}(t)c_+ \end{pmatrix},\ \rho_{c_-}(t) = \begin{pmatrix} K_{A+1/2}(t)c_- \\ K_{A-1/2}(t)c_- \end{pmatrix},\ c_\pm \in H^+.$$

It remains to note that (cf. [**24**, p. 68])

$$I_NK_{N+1}(t) + I_{N+1}K_N(t) = t^{-1},$$

from which we deduce that

$$\begin{pmatrix} I_{A+1/2}(t) & K_{A+1/2}(t) \\ -I_{A-1/2}(t) & K_{A-1/2}(t) \end{pmatrix} \begin{pmatrix} K_{A-1/2}(t) & -K_{A+1/2}(t) \\ I_{A-1/2}(t) & I_{A+1/2}(t) \end{pmatrix} = t^{-1} I_H.$$

Thus, $G(\mu, A)\tau$ is indeed a solution of (4.18).

To deduce the estimate (4.21), we perform some reductions of the operator $L(A)$. First, we select a number $\Lambda \leq \frac{1}{2}$ such that $|A| \leq \Lambda$, and we choose a number $\Lambda_1 \in [-1/2, 0], \Lambda_1 \notin \operatorname{spec} A$. Then we split, with obvious notation,

$$A = A_{>\Lambda_1} \oplus A_{<\Lambda_1}.$$

This splits $L(A) = L(A_{>\Lambda_1}) \oplus L(A_{<\Lambda_1})$, and conjugating with α_2 in the second summand allows us to assume that

$$A > -\frac{1}{2} \tag{4.26}$$

in what follows. By the same token, we can select numbers $\Lambda_j, j = 1, \ldots, N$, such that

$$\Lambda_j \notin \operatorname{spec} A, \ \ \Lambda_N > \Lambda; \tag{4.27}$$

$$\Lambda_j < \Lambda_{j+1} < \Lambda_j + 1. \tag{4.28}$$

Splitting $L(A)$ accordingly as a direct sum, we may further assume that for some $\Lambda^* \in [-\frac{1}{2}, \Lambda)$ we have

$$\Lambda^* < A < \Lambda^* + 1. \tag{4.29}$$

Under the assumption (4.29) we will next prove the estimates (4.21) using [**12**, Lemma 2.3], which is perfectly adapted to the situation at hand, at least for the operator $G_\infty(\mu, A)$. However, it is easily seen that $G_0(\mu, A)$ is essentially the adjoint operator to $G_\infty(\mu, A)$, up to permutations and sign changes of the matrix elements. Since we will base our estimate on estimates of the matrix elements, it is hence enough to deal with $G_\infty(\mu, A)$. These estimates for the modified matrix Bessel functions and their derivatives have been derived in [**12**, Lemmas 2.1, 2.2] and are combined in the statement that follows. We recall from loc. cit. that l denotes a positive function, defined for positive real numbers, which equals $-\log t$ for $t \leq 1/2$ and 1 for $t \geq 1$.

Lemma 4.3. — *The modified matrix Bessel functions $I_N(t), K_N(t)$ are smooth in $\mathscr{L}_s(H) \times (0, \infty)$, and if $N \in \mathscr{L}_s(H)$ satisfies the inequality*

$$-\infty < a \leq N \leq b < \infty,$$

then the estimates

$$||D_N^p(\frac{\partial}{\partial t})^q I_N(t)|| \leq C_{a,b,p,q} t^{a-q}(1+t)^{q-a-1/2} e^t \, l(t)^p, \tag{4.30}$$

$$||D_N^p(\frac{\partial}{\partial t})^q K_N(t)|| \leq C_{a,b,p,q} t^{-b-q}(1+t)^{b+q-1/2} e^{-t} \, l(t)^p, \tag{4.31}$$

hold for $p, q \in \mathbb{Z}_+$ and $t > 0$.

Now we use Lemma 4.3 in [**12**, Lemma 2.3] to derive the norm estimate (4.21) for $G_\infty(\mu, A)$ where, by the above reduction, we may assume that

$$-\frac{1}{2} < a \le A \le b < a+1. \tag{4.32}$$

The desired estimate follows from the following block matrix estimate for the kernel:

$$\begin{aligned}(4.33)\quad ||D_A^p(\frac{\partial}{\partial\mu})^q\big(\mu(ts)^{1/2}I_{A\pm 1/2}(\mu t)K_{A\pm 1/2}(\mu s)\big)||_{\mathscr{L}(H)}&\\ \le C_{pq}(\mu t)^a(\mu s)^{-b}(1+\mu t)^{-a}(1+\mu s)^b e^{\mu(t-s)}&.\end{aligned}$$

As mentioned above, the same estimate gives the result for $G_0(\mu, A)$.

For the statement on the domain, we use again the estimates (4.30), (4.31), this time with $p = q = 0$. Moreover, since the operators $A \pm 1/2$ can be simultaneously diagonalized, we may assume that $A = \nu I_{H^+}$ where $\nu > -1/2$. We write for $\sigma \in \operatorname{im} G(\mu, A)$

$$\begin{aligned}\sigma(t) = G(\mu, A)\tau(t) &= G_0(\mu, A)\tau(t) + G_\infty(\mu, A)\tau(t)\\ &=: \sigma_0(t) + \sigma_\infty(t).\end{aligned}$$

Then we observe that for $\operatorname{supp}\tau \subset (1, \infty)$ Lemma 4.3 implies immediately that, with $\underline{\nu} := \inf\operatorname{spec} A > -1/2$,

$$\begin{aligned}||\sigma_+(t)||_H &= O(t^{\underline{\nu}+1}) = O(t^{1/2}),\ t \to 0,\\ ||\sigma_-(t)||_H &= O(t^{\underline{\nu}}),\ t \to 0,\end{aligned}$$

such that we may assume that $\operatorname{supp}\tau \subset (0, 1]$. Next we have the estimate

$$\begin{aligned}||\sigma_0(t)||_H &\le C_\sigma \int_0^t (s/t)^{\underline{\nu}}||\tau(s)||_H ds\\ &\le C_\sigma(1+2\underline{\nu})^{-1/2}t^{1/2}||\tau||_{L^2(\mathbb{R}_+,H)},\end{aligned}$$

which proves (4.22) and (4.23) for σ_0.

For $\sigma_\infty(t)$, we have again to distinguish the $\pm$-components. Arguing as before, we arrive at the estimates

$$\begin{aligned}||\sigma_{\infty,+}(t)||_{H^+} &\le C_\sigma t^{\underline{\nu}+1}\int_t^1 s^{-\underline{\nu}}||\tau(s)||_H ds\\ &\le C_\sigma t^{1/2},\\ ||\sigma_{\infty,-}(t)||_{H^+} &\le C_\sigma t^{\underline{\nu}}\int_t^1 s^{-\underline{\nu}}||\tau(s)||_H ds\\ &\le C_\sigma t^{\underline{\nu}}.\end{aligned}$$

The proof is complete. □

Now we apply Theorem 4.2 to construct the desired operator symbol, $G(\mu,b,\beta)_<$. Recall that we want

$$G(\mu,b,\beta)_< = \big(\gamma(\frac{d}{dt}+t^{-1}\tilde{A}(b)_<)+i\,\mathrm{cl}(\beta^\sharp)-i\mu\big)^{-1}, \tag{4.34}$$

$$= \big(D_V(b)_< + i\,\mathrm{cl}(\beta^\sharp)-i\mu)\big)^{-1}, \tag{4.35}$$

for a suitable self-adjoint extension, $D_V(b)_<$, of the conic operator. We will define this extension by solving the matrix equation on the right hand side of (4.34) using Theorem 4.2 appropriately. Let us recall from (4.5) that we now deal with the following data:

$$H := \Lambda T^*_{b_0}B\otimes\mathbb{C}^2\otimes Q_<(\lambda_{(2)}(Y_{b_0})), \tag{4.36}$$

$$\gamma = \varepsilon_H\otimes\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}, \tag{4.37}$$

$$\tilde{A}(b)_< = \begin{pmatrix}A(b)_< & 0\\ 0 & -A(b)_<\end{pmatrix}, \tag{4.38}$$

$$A(b)_< = Q_<(D_{Y_b}\alpha_{Y_b}+\nu), \tag{4.39}$$

where $Q_< = I - Q_>$ and $Q_>$ is given by (4.8). Now we put

$$\tilde{\gamma} := i\gamma,\ \zeta=\zeta(\mu,\beta) := \mu\tilde{\gamma}-\tilde{\gamma}\,\mathrm{cl}(\beta^\sharp), \tag{4.40}$$

and noting that for $\beta\in T^*B$, $\tilde{\gamma}$ and $\mathrm{cl}(\beta^\sharp)$ anticommute while $\mathrm{cl}(\beta^\sharp)$ commutes with α_1 and A, one easily computes that

$$\zeta^\dagger = \zeta, \tag{4.41}$$

$$\zeta^2 = (\mu^2+|\beta|_b^2)I =: \tilde{\mu}(b,\beta)^2 I, \tag{4.42}$$

$$\zeta\tilde{A}(b)_< + \tilde{A}(b)_<\zeta = 0 \tag{4.43}$$

This allows us to introduce two anticommuting self-adjoint involutions, α_1,α_2, by

$$\alpha_1 := I\otimes\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}, \tag{4.44}$$

$$\tilde{\mu}\alpha_2 := \zeta. \tag{4.45}$$

Then we can state

Lemma 4.4. — *With this notation we have in $L^2(\mathbb{R}_+,H)$*

$$D_V(b)_< + i\,\mathrm{cl}(\beta^\sharp)-i\mu = \gamma\big(\frac{d}{dt}+t^{-1}\tilde{A}(b)_<+\tilde{\mu}\alpha_2\big), \tag{4.46}$$

and the following relations hold:

$$\alpha_1\alpha_2+\alpha_2\alpha_1 = 0, \tag{4.47}$$

$$\alpha_1\tilde{A}(b)_< - \tilde{A}(b)_<\alpha_1 = 0, \tag{4.48}$$

$$\alpha_2\tilde{A}(b)_< + \tilde{A}(b)_<\alpha_2 = 0. \tag{4.49}$$

Thus we are in the position to prove Theorem 0.1.

Proof of Theorem 0.1. — 1. We construct an operator $\hat{D}$ by the method of Theorem 4.2. The proof of Theorem 4.2 has to be modified somewhat since we have to verify the conditions (4.12) through (4.14) for the operator symbol

$$G(\mu,\beta,b)_< := \big(\gamma(\frac{d}{dt} + t^{-1}\tilde{A}(b)_< + \tilde{\mu}\alpha_2)\big)^{-1},$$

where now $\tilde{\mu}$ and α_2 depend on μ,β, and b. First we use [**12**, Lemma 1.1] to the effect that the spectral projections $Q_{(\Lambda_j,\Lambda_{j+1})}(\tilde{A}(b)_<)$ are locally independent of b. Observing next that $\tilde{\mu}$ as well as its b- derivatives are homogeneous in (μ,β) of degree one and using Lemma 4.3, we reduce the estimates (4.14) to (4.33) where μ is replaced by $\tilde{\mu}$.

(4.12) holds by construction, while for (4.13) we use the boundary conditions (4.22), (4.23) to calculate with $\sigma_1,\sigma_2 \in \operatorname{im} G(\mu,\beta,b)_<$

$$\begin{aligned}(D_{V,\max}(b)_<\sigma_1,\sigma_2) - (\sigma_1, D_{V,\max}(b)_<\sigma_2)& \\ = \lim_{t\to 0}\big(\langle\sigma_1^-,\sigma_2^+\rangle(t) - \langle\sigma_1^+,\sigma_2^-\rangle(t)\big) &= 0.\end{aligned} \tag{4.50}$$

That the operator $\hat{D}$ anticommutes with τ_M is obviously built into our construction. Finally, the discreteness is equivalent to the compactness of $G(\mu,D)$ which follows in turn from the compactness of the parametrix $G_1(\mu,D)$, by the form of the Neumann series. Now we choose $\psi \in C_c(M)$ with $\psi = 1$ on M_ε. Then $\psi G_1(\mu,D)$ is compact by interior regularity, while the estimate

$$||(1-\psi)G_1(\mu,D)|| \leq C\varepsilon^{2\delta}$$

follows from (4.22) and (4.23) for the low eigenvalues; since the estimate (4.24) also holds for the large eigenvalues, by (4.10), $G_1(\mu,D)$ is a limit of compact operators and hence compact.

Finally, since $\tilde{D}$ is a symmetric extension of $\hat{D}$ the two operators coincide.

2. If $|A_V| \geq \frac{1}{2}$, then elements in the domain of D_M^Λ satisfy the estimate (4.24). Now the assertion follows as in [**12**, Lemma 5.1].

3. The assertion holds if $|A_V| \geq \frac{1}{2}$ since then $D_{\min}$ is essentially self-adjoint, by part 2, and the case of uniqueness holds by [**11**, Lemma 3.3].

In the general case, we construct a smooth family of metrics, $g(\alpha)^{TM}$, such that

$$g(\alpha)^{TM} := \begin{cases} dt^2 \oplus g^{T_HN} \oplus \alpha^2t^2g^{T_VN} & \text{on } U_{\varepsilon_0/2}, \\ g^{TM} & \text{on } M_{\varepsilon_0}. \end{cases} \tag{4.51}$$

We denote by $D^\Lambda(\alpha) = D(\alpha)$ the corresponding self-adjoint operator defined by the maximal de Rham complex and choose $\alpha_0 > 0$ such that $D(\alpha_0)$ is essentially self-adjoint. Since all metrics $g(\alpha)^{TM}$ are mutually quasi-isometric, the case of uniqueness holds for all of them since it is a quasi-isometry invariant.

4. We use the notation of part 3 and note that $D^{\rm sign}(\alpha)$ is well defined for all α. To prove the asserted equality we show first that

$$\operatorname{ind} D^{\rm sign}(\alpha) = \operatorname{ind} D(\alpha)^+. \tag{4.52}$$

Since $\operatorname{ind} D^{\rm sign}(\alpha)$ is constant in $[\alpha_0, 1]$, this identity will follow from [**22**, Thm.IV,5.17] if we prove an estimate of the form

$$\hat\delta\big(D^{\rm sign}(\alpha_1), D^{\rm sign}(\alpha_2)\big) \le C_{\alpha_0}|\alpha_1 - \alpha_2|,\ \alpha_1, \alpha_2 \in [\alpha_0, 1], \tag{4.53}$$

where $\hat\delta$ denotes the gap function defined in [**22**, p.197]. One checks that for $\mu \ge 1$

$$\hat\delta\big(D^{\rm sign}(\alpha_1), D^{\rm sign}(\alpha_2)\big) \le ||G\big(\mu, D(\alpha_1)\big) - G\big(\mu, D(\alpha_2)\big)||_{\lambda_{(2)}(M)},$$

such that (4.53) will follow if we show e. g. that the function

$$[\alpha_0, 1] \ni \alpha \to G(\mu, D(\alpha)) \in \mathscr{L}\big(\lambda_{(2)}(M)\big)$$

is continuously differentiable. We fix a large $\mu > 1$ and write with our parametrix $G_1(\alpha) := G_1(\mu, D)(\alpha)$

$$(D(\alpha) - i\mu)G_1(\alpha) =: I - R(\alpha),$$

where

$$||R(\alpha)|| \le C < 1,\ \alpha \in [\alpha_0, 1].$$

Hence it is enough to prove the differentiability of $G_1(\alpha)$ and $R(\alpha)$. This is clear for the interior part, by interior regularity. For the boundary part involving high eigenvalues this is also clear from the Calderón-Vaillancourt Theorem since the image of $G_1(\alpha)_>$ does not depend on α. For the low eigenvalue part, however, we have to go back to the proof of Theorem 4.2.

Since $\tilde A(b, \alpha)_<$ depends smoothly on α and $G(\alpha)_<$ depends smoothly on $\tilde A(b, \alpha)_<$, we have to insure that the spectral splittings (Λ_j) can be made locally independent of α. This can be done for the spectral projections in many ways but using the spectral analysis of Sec. 3 we can take into account the special role of the eigenvalues $\pm 1/2$, as needed in the next step. The Hodge decomposition on Y_b can also be made locally independent of b and α, by conjugating the equation with a transformation function (cf. [**22**, II,§4.2]). Then the operator function splits into the harmonic, the closed, and the coclosed parts which have uniform spectral gaps around 0, 1/2, and -1/2, respectively, independent of the parameter values. Conjugating appropriately as before, we may reduce to the case $\tilde A(b, \alpha)_< > -1/2$ locally in b and α; since the corresponding solution operator is smooth in α, this completes the proof (4.52). □

Next we want to show that $D^{\rm sign}(\alpha)$ extends $\tilde D^+(\alpha)$ which will give the assertion in view of (4.52).

We choose $\sigma = \sigma^+ \in \operatorname{dom} \tilde D^+$ and may assume that $\operatorname{supp} \sigma \subset U_{\varepsilon_0}$. We decompose σ into its harmonic, closed, and coclosed part which all satisfy the estimate (4.22). By part 3 of Lemma 4.1 we see that all components of σ are in the minimal domain of the corresponding conic operator. Moreover, by the spectral decomposition described in part 3 of this proof all cone coefficients will not have $-\frac{1}{2}$ in their spectrum such that we can apply Lemma 5.12 in Section 5; we find that

$$\sigma' \in L^2\big((0,\varepsilon_0), H^0\big),\ t^{-1}\sigma \in L^2\big((0,\varepsilon_0), H^1\big).$$

The pseudodifferential construction of the parametrix shows next that

$$d_H\sigma, d_H^\dagger\sigma \in \lambda_{(2)}(M),$$

and Lemma 2.2 finally shows that

$$d_M\sigma \in \lambda_{(2)}(M)$$

and completes the proof. □

5. The index calculation

In this section, we want to compute the index of the signature operator, as constructed in Theorem 0.1. As noted there, the index is stable under scaling of the fiber metric; this rules out, according to Theorem 3.1, that small eigenvalues occur on the closed and coclosed subspaces, while we need an extra condition on the space $\mathscr{H}^{v/2}(Y)$ known as the *Witt condition*:

$$\mathscr{H}^{v/2}(Y) = 0. \tag{5.1}$$

Thus we may and will assume in what follows that

$$|A_V| \geq \frac{1}{2}, \tag{5.2}$$

which ensures, by Theorem 0.1 again, that we do not have to impose boundary conditions near the singularity. However, the crucial vanishing results we need will require in addition that

$$-\frac{1}{2} \notin \operatorname{spec} A_{V,\mathrm{cl}} \cup \operatorname{spec} A_{V,\mathrm{ccl}}. \tag{5.3}$$

In view of Theorem 3.1, this can also be achieved by scaling $g^{T_V N}$; thus we will assume in what follows (5.1) and

$$\operatorname{spec}|A_{V,\mathrm{cl}}| \cup \operatorname{spec}|A_{V,\mathrm{ccl}}| \subset [1/2 + C, \infty), \tag{5.4}$$

for some positive constant C.

We will reduce the index calculation to a problem of APS-Type, by splitting the operator as a sum at ∂U_ε, for a sufficiently small $\varepsilon \in (0, \varepsilon_0)$, using [**3**, Thm. H]. At ∂M_ε, we will introduce the boundary condition

$$Q_{\geq 0}(A(\varepsilon))\sigma(\varepsilon) = 0, \tag{5.5}$$

where A is the operator family from (2.32) and $Q_{>0}$ denotes the spectral projection onto the positive eigenspaces. At ∂U_ε, we impose the complementary boundary condition (cf. [**3**, Thm. 4.17]),

$$Q_{<0}(A(\varepsilon))\sigma(\varepsilon) = 0; \tag{5.6}$$

note that these boundary conditions are invariant under τ_M. These boundary conditions generate the operators $D^{\Lambda}_{U_\varepsilon,Q_{<0}(A(\varepsilon))}$ and $D^{\Lambda}_{M_\varepsilon,Q_{\geq 0}(A(\varepsilon))}$ by imposing the boundary conditions on the maximal domain of $D^{\mathrm{sign}}_{U_\varepsilon}$ and $D^{\mathrm{sign}}_{M_\varepsilon}$, respectively (note that no boundary condition is necessary at 0 in view of (5.2)). The boundary conditions are such that the following holds.

Theorem 5.1. — *$D^{\Lambda}_{U_\varepsilon,Q_{<0}(A(\varepsilon))}$ and $D^{\Lambda}_{M_\varepsilon,Q_{\geq 0}(A(\varepsilon))}$ are Fredholm operators, and we have the index identity*

$$\operatorname{ind} D^{\mathrm{sign}}_M = \operatorname{ind} D^{\Lambda,+}_{U_\varepsilon,Q_{<0}(A(\varepsilon))} + \operatorname{ind} D^{\Lambda,+}_{M_\varepsilon,Q_{\geq 0}(A(\varepsilon))}. \tag{5.7}$$

Proof. — The proof of (5.7) follows immediately from [**3**, Thm.4.17] (cf. Remark 5.17) with the following data for $0 < u < \varepsilon < \varepsilon_0/2$:

$$D_1^+ := \gamma(\frac{\partial}{\partial u} + A(\varepsilon + u)), \tag{5.8}$$

$$D_2^+ := -\gamma(\frac{\partial}{\partial u} - A(\varepsilon - u)), \tag{5.9}$$

$$B_1 := Q_{<0}(A(\varepsilon))(\operatorname{dom}|A(\varepsilon)|^{1/2}), \tag{5.10}$$

$$B_2 := Q_{\geq 0}(A(\varepsilon))\big(\operatorname{dom}|A(\varepsilon)|^{1/2}\big). \tag{5.11}$$

□

We show next that the index contribution from U_ε vanishes.

Theorem 5.2. — *Assume that* (5.3) *holds. Then for $\varepsilon \in (0, \varepsilon_0]$ and sufficiently small we have*

$$\operatorname{ind} D^{\Lambda,+}_{U_\varepsilon,Q_{<0}(A(\varepsilon))} = 0. \tag{5.12}$$

This theorem will be proved in Subsection 5.2.

Thus it remains to compute the index of an APS-type problem on the smooth compact manifold with boundary, M_ε. However, to apply [**1**, Thm. 3.10] we need to modify the metric on U_{ε_0}, making it cylindrical near $t = \varepsilon$. To this end we choose a smooth positive function ψ on $(0,\infty)$ such that $\psi(t) = t$ if $t \in (0, 1] \cup [4,\infty)$ and $\psi(t) = 1$ if $t \in [2, 3]$. Then we put for $\varepsilon < \varepsilon_0/4$

$$g_\varepsilon^{TU_{\varepsilon_0}} := dt^2 \oplus g^{T_H N} \oplus \varepsilon^2 \psi(t/\varepsilon)^2 g^{T_V N}, \tag{5.13}$$

$$g_\varepsilon^{TM}|M_{\varepsilon_0} := g^{TM}|M_{\varepsilon_0}, \tag{5.14}$$

$$g_\varepsilon^{TM}|U_{\varepsilon_0} := g_\varepsilon^{TU_{\varepsilon_0}}. \tag{5.15}$$

Moreover, we are not yet dealing with the correct boundary condition in order to apply the APS-Theorem. In fact, we have from (2.32)

$$A(t) = A_H(t) + t^{-1}(A_{0,V} + \nu) \tag{5.16}$$

$$= A_0(t) + t^{-1}\nu, \tag{5.17}$$

and it follows from (2.37) and Theorem 2.5 that $A_0(t) \simeq D^{\Lambda}_{N_t}\tau_{N_t}$ is the tangential operator corresponding to $D^{\mathrm{sign}}_{U_{\varepsilon_0}}$, acting in H^0 with domain H^1. The correct boundary condition can be achieved by applying the Agranovich-Dynin Theorem respectively its equivariant version, as stated e. g. in [**3**, Thm. 4.14]. Noting that $A_0(\varepsilon) = D_{N_\varepsilon}\tau_{N_\varepsilon}$ has even dimensional kernel, we obtain

Theorem 5.3. — *The pair of subspaces* $\big(Q_{<0}(A(\varepsilon))(H^0), Q_{\geq 0}(A_0(\varepsilon))(H^0)\big)$ *is a Fredholm pair in* H^0. *If we denote its Kato index by* $i(\varepsilon)$ *then*

$$\begin{aligned}\operatorname{ind} D^{\text{sign}}_{M_\varepsilon, Q_{<0}(A(\varepsilon))(H^0)} &= \operatorname{ind} D^{\text{sign}}_{M_\varepsilon, Q_{<0}(A_0(\varepsilon))(H^0)} + i(\varepsilon)\\ &=: \operatorname{ind} D^{\text{sign}}_{(M_\varepsilon, g^{TM}_\varepsilon), Q_{<0}(A_0(\varepsilon))(H^0)} + \big(\tau(\varepsilon) + \frac{1}{2}\dim\ker A_0(\varepsilon)\big).\end{aligned}$$

This result will be proved in Subsection 5.1.

To obtain an explicit index formula, we need to identify the integer $\tau(\varepsilon)$. To do so, we use the generalized Thom space associated with the fibration (0.3), as introduced by Cheeger and Dai in [**17**] which we denote by T_π. Then we show using [**17**, Thm.1.1] (note our choice of orientation)

Theorem 5.4. — *For* ε *sufficiently small, we have*

$$\tau(\varepsilon) = \operatorname{sign}_{(2)} T_\pi =: \tau,$$

where τ *denotes the invariant introduced in* [**18**, Thm.0.3].

This theorem will be proved in Subsection 5.3.

Now we obtain our final local index formula by combining Theorem 5.3 and Theorem 5.4 with the APS-Theorem [**1**, Thm. 3.10] and the result of Dai [**18**, Thm. 0.3] which evaluates the adiabatic limit of the eta-invariant for the signature operator, to get

$$\operatorname{sign}_{(2)} M = \lim_{\varepsilon\to 0}\int_{M_\varepsilon} L(TM, g^{TM}) - \int_B L(TB, g^{TB}) \wedge \tilde{\eta} - \frac{1}{2}\eta(A_{0,\mathscr{H}}(0)), \tag{5.18}$$

where the operator $A_{0,\mathscr{H}}$, the Dirac operator on ΛT^*M twisted by the harmonic forms on the fibers, is defined in (5.53).

Remark 5.5. — 1. Using arguments as in [**7**, Sec.VI], it follows that the transgression term of the L-class from g^{TM}_ε to g^{TM} goes to zero with ε.

2. It is desirable to give a direct proof of the equality $\tau(\varepsilon) = \tau$, without using [**17**, Thm.1.1].

5.1. Perturbations of regular projections. — We use the terminology introduced in [**3**, Sec. 2.1]. Thus we consider a self-adjoint operator A with domain H_A in the (complex) Hilbert space H which we assume to be discrete i. e. to have a compact resolvent. For a Borel subset $J \subset \mathbb{R}$ we denote by $Q_J := Q_J(A)$ the associated spectral projection, and we write $Q_{>0} := Q_{(0,\infty)}$ etc.

With A we associate its Sobolev chain $(H^s := H^s(A))_{s\in\mathbb{R}}$ restricting attention to $\{|s| \leq 1\}$. Thus for $s \in [0,1], H^s$ is the closure of H_A under the norm

$$||x||^2_s := ||(I + A^2)^{s/2}x||^2_H \tag{5.19}$$

and H^{-s} is its strong dual space under the norm (5.19).

An operator $S \in \mathscr{L}(H)$ will be called 1/2-smooth if it restricts to $H^{1/2}$, with restriction $\hat{S}$, and extends to $H^{-1/2}$, with extension $\tilde{S}$. S will be called (1/2-)smoothing if $\operatorname{im} \tilde{S} \subset H^{1/2}$. With these preparations we can define *regular* and *elliptic* projections for A which are introduced to characterize elliptic boundary conditions for the evolution operator associated with A (cf. [**3**, Secns. 1.4, 2.3]). If A comes with an anticommuting skew-adjoint unitary operator $\gamma \in \mathscr{L}(H)$,

$$\gamma A + A\gamma = 0, \tag{5.20}$$

then we can also define the Dirac operator associated with A, cf. [**3**, Sec.2.1]. The following formulation derives from [**3**, Prop.1.99].

Definition 5.6. — A 1/2-smooth orthogonal projection P in H is called *regular (with respect to A)* if and only if

$$\begin{aligned} &x \in H^{-1/2},\ \tilde{P}x = 0,\ Q_{\leq 0}x \in H^{1/2} \\ &\Rightarrow x \in H^{1/2}. \end{aligned}$$

A regular projection P is called *elliptic (with respect to A)* if (5.20) holds and

$$P_\gamma := \gamma^*(I - P)\gamma \tag{5.21}$$

is also regular.

For example, the spectral projections $Q_{>\Lambda}(A)$ are regular with respect to A for any $\Lambda \in \mathbb{R}$ since A is discrete, and since $(Q_{>\Lambda}(A))_\gamma = Q_{\geq\Lambda}(A)$ they are also elliptic.

Now we want to study perturbations of A in the sense of Kato, i. e. operators of the form $\tilde{A} := A + B$ where B is a symmetric operator in H defined on H_A with estimate

$$||Bx||_H \leq a||x||_H + b||Ax||_H,\ x \in H_A \tag{5.22}$$

for some constants $a, b \in \mathbb{R}_+$ with $b < 1$. Then $\tilde{A}$ is self-adjoint and discrete in H with domain H_A, by the Kato-Rellich Theorem, and the projection $Q_{>0}(\tilde{A})$ is elliptic with respect to $\tilde{A}$. We want to know under what conditions it is also elliptic with respect to A. To answer this question we need two preliminary results; the first one parallels [**3**, Prop.1.93].

Lemma 5.7. — *Let P be a 1/2-smooth orthogonal projection in H such that*

$$P = Q_{>0}(A) + R_1 + R_2, \tag{5.23}$$

where R_1 is smoothing and

$$\max\{||\hat{R}_2||, ||\tilde{R}_2||\} < 1. \tag{5.24}$$

Then P is elliptic with respect to A, and $\big(\operatorname{im}(I-P), \operatorname{im} Q_{>0}(A)\big)$ is a Fredholm pair in H.

Proof. — Consider $x \in H^{-1/2}$ with $\tilde{P}x = 0$ and $Q_{\leq 0}(A)x \in H^{1/2}$. We write $x = Q_{>0}(A)x + Q_{\leq 0}(A)x =: x_> + x_\leq$ and obtain from (5.23)

$$(I + \tilde{R}_2)x_> =: y \in H^{1/2},$$

hence from (5.24)

$$x_> = (I + \hat{R}_2)^{-1} y \in H^{1/2},$$

such that P is regular. It follows from (5.20) that γ induces a unitary operator in each H^s; since $Q_0(A)$ is smoothing, we infer that the representation (5.22) also holds for P_γ such that P is elliptic.

We see next that

$$P : H_> = \operatorname{im} Q_{>0}(A) \to H, x_> \mapsto (I + R_2 + R_1)x_>,$$

is a left Fredholm operator, by [3, Lemma A.11] and the compactness of R_1, hence, from [3, Lemma A.12], $\big(\operatorname{im}(I - P), H_>\big)$ is a left Fredholm pair. By the same token, we see that $(\operatorname{im} P, H_\leq)$ is a left Fredholm pair, too, which completes the proof of the lemma. □

The second lemma addresses smoothing perturbations.

Lemma 5.8. — *Assume that A and $A + B$ are both invertible. If B is smoothing then so is*

$$R := Q_{>0}(A + B) - Q_{>0}(A).$$

Proof. — For any invertible and discrete self-adjoint operator $\tilde{A}$ in H we have from [**22**, p.359] the strongly convergent integral representation

$$\frac{1}{2}\big(I - 2\,Q_{>0}(\tilde{A})\big) = \frac{1}{2\pi i}\int_{\operatorname{Re} z = 0} (\tilde{A} - z)^{-1} dz. \tag{5.25}$$

This implies that

$$R = \frac{1}{2\pi i}\int_{\operatorname{Re} z = 0} (A + B - z)^{-1} B (A - z)^{-1} dz. \tag{5.26}$$

By construction, R is $1/2$-smooth; to show that R is smoothing, we need to show the boundedness in H of the operator

$$\begin{aligned}
\tilde{R} &:= (I + A^2)^{1/4} R (I + A^2)^{1/4} \\
&= \big((I + A^2)^{1/4} |A + B|^{-1/2}\big)\big(|A + B|^{1/2} R |A|^{1/2}\big)\big(|A|^{-1/2}(I + A^2)^{1/4}\big) \\
&=: V\big(|A|^{1/2}|A + B|^{-1/2}\big)\big(|A + B|^{1/2} R |A|^{1/2}\big) V \\
&=: VW\big(|A + B|^{1/2} R |A|^{1/2}\big) V.
\end{aligned}$$

In view of (5.22), A and $A+B$ generate the same Hilbert spaces, with equivalent norms, in their respective Sobolev chains implying the boundedness in H of the operators V and W. From (5.26) we obtain for the remaining part the representation

$$\begin{aligned}(5.27)\qquad |A{+}B|^{1/2}R|A|^{1/2} &= \frac{1}{2\pi i}\int_{\operatorname{Re} z=0}(A+B-z)^{-1}|A+B|^{1/2}B|A|^{1/2}(A-z)^{-1}dz\\ &=: \frac{1}{2\pi i}\int_{\operatorname{Re} z=0}(A+B-z)^{-1}\tilde{B}(A-z)^{-1}dz.\end{aligned}$$

Now if B is smoothing then $\tilde{B}=|A+B|^{1/2}B|A|^{1/2}$ is bounded in H. Thus we may apply [**2**, Lemma A.1] to complete the proof. □

Now we can deduce the desired perturbation result.

Theorem 5.9. — *Assume that $A+B$ is a Kato perturbation of A with*

$$b<\frac{2}{3}. \tag{5.28}$$

Then $Q_{>0}(A+B)$ is elliptic with respect to A and the subspaces $Q_{\le 0}(A)(H)$ and $Q_{>0}(A+B)(H)$ form a Fredholm pair.

If B is bounded and $|A|\ge\mu$ where

$$\mu>\sqrt{2}||B||_H, \tag{5.29}$$

then

$$\operatorname{ind}\big(Q_{\le 0}(A)(H),Q_{>0}(A+B)(H)\big)=0. \tag{5.30}$$

Proof. — We show first that we may assume that

$$|A|\ge\Lambda \tag{5.31}$$

for any $\Lambda>0$. Indeed, if we put

$$f_\Lambda(t):=\begin{cases}t & \text{if } |t|\ge\Lambda,\\ \Lambda & \text{if } 0<t<\Lambda,\\ -\Lambda & \text{if } -\Lambda<t\le 0,\end{cases} \tag{5.32}$$

then for the operator

$$A_\Lambda:=f_\Lambda(A)$$

the following properties are easily verified:

(5.33) A_Λ is discrete and commutes with A,

(5.34) $A-A_\Lambda$ is smoothing,

(5.35) $|A|\le|A_\Lambda|$,

(5.36) $|A_\Lambda|\ge\Lambda\Rightarrow||A_\Lambda^{-1}||\le\Lambda^{-1}$,

(5.37) $Q_{>0}(A_\Lambda)=Q_{>0}(A)$.

We have

$$A + B = A_\Lambda + (A - A_\Lambda + B) =: A_\lambda + B_\Lambda,$$

such that $A_\Lambda + B_\Lambda$ is a Kato perturbation of A_Λ with the same constant b in (5.21) as for A and B. Hence, by (5.36) it is enough to prove the theorem under the assumption (5.30).

Next we note that (5.21) implies that

$$b' := ||BA^{-1}|| \tag{5.38}$$

can be chosen arbitrarily close to b. Thus we may also assume that both A and $A+B$ are invertible.

Now we want to show that for Λ sufficiently large and b satisfying the condition (5.28), the $1/2$-smooth operator

$$R_2 := Q_{>0}(A + B) - Q_{>0}(A)$$

satisfies the estimate (5.24). To do so, we proceed as in the proof of Lemma 5.8. We observe first that from the symmetry of R_2 in H and the obvious identity

$$\tilde{S} = (\widehat{S^*})',$$

where S' denotes the dual in $H^{-1/2}$ and S^* the adjoint operator in H, for any $1/2$-smooth operator S, it is enough to estimate $||R_2||_{1/2}$ or equivalently, the norm in H of the operator

$$(I + A^2)^{1/4} R_2 (I + A^2)^{-1/4} = VW|A + B|^{1/2} R_2 |A|^{-1/2} V^{-1}, \tag{5.39}$$

where V and W are the operators introduced in the proof of Lemma 5.8.

From the Spectral Theorem we see that for any $\delta > 0$ we may choose Λ so large that

$$\sup\{||V||_H, ||V^{-1}||_H\} \le 1 + \delta. \tag{5.40}$$

Next we estimate the H-norm of W by the maximum principle applied to the holomorphic function

$$z \mapsto e^{-z(1-z)} \langle |A|^z |A + B|^{-z} x, y\rangle_H \in \mathbb{C},\ \ x, y \in H_A,$$

in the strip $\{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}$ which reduces us to an estimate for $\operatorname{Re} z = 1$. Clearly, for $b' < 1$ we have

$$||\,|A|\,|A + B|^{-1}||_H \le (1 - b')^{-1},$$

hence also

$$||W||_H \le (1 - b')^{-1}. \tag{5.41}$$

It remains to estimate the norm of $|A+B|^{1/2} R_2 |A|^{-1/2}$ for which we invoke again [**2**, Lemma A.1]. There we choose $A_1 := A + B, A_2 := A, \alpha_1 := \alpha_2 := 1/2$ and find with $B(z) = |A + B|^{1/2} B |A|^{-1/2}$

$$|||A + B|^{1/2} R_2 |A|^{-1/2}||_H \le \frac{1}{2} ||B|A|^{-1}||_H \le \frac{1}{2} b'. \tag{5.42}$$

Combining (5.40), (5.41), and (5.42) we arrive at

$$||R_2||_{H^{1/2}} \leq \frac{1}{2}b'(1-b')^{-1}(1+\delta). \tag{5.43}$$

This can be made smaller than 1 if $b < \frac{2}{3}$.

For the proof of (5.30) we observe that this will follow from the estimate

$$||Q_{>0}(A) - Q_{>0}(A+B)||_H < 1, \tag{5.44}$$

which again is an easy consequence of [**2**, Lemma A.1], this time applied with $\alpha_1 := \alpha_2 := \frac{1}{2}$. □

From the proof of the theorem we get

Corollary 5.10. — *Lemma 5.8 holds without the assumption that A and $A+B$ are invertible.*

Remark 5.11. — Theorem 5.9 is stronger than needed for our application but it is useful in other situations and does not seem to be known.

Proof of Theorem 5.3. — From (2.31) we have

$$A(\varepsilon) = A_0(\varepsilon) + \varepsilon^{-1}\nu,$$

and since ν is bounded it follows from Lemma 5.7 and Theorem 5.9 that $Q_{<0}(A(\varepsilon))$ is elliptic with respect to $A_0(\varepsilon)$ and that $\big(Q_{<0}(A(\varepsilon)(H^0), Q_{\geq 0}(A_0(\varepsilon))(H^0)\big)$ is a Fredholm pair in H.

The index formula follows from [**3**, Thm.4.14]. □

5.2. A vanishing theorem. — The purpose of this subsection is the proof of Theorem 5.2. We abbreviate

$$D_\varepsilon^{\mathrm{sign}} := D^{\mathrm{sign}}_{U_\varepsilon, Q_{<0}(A(\varepsilon))}$$

and note that, in view of (5.2) and Theorem 0.1, a core for $D_\varepsilon^{\mathrm{sign}}$ is given by

$$\mathscr{D}_\varepsilon^{\mathrm{sign}} := \{\sigma \in C_c^1((0,\varepsilon], H^1) : Q_{<0}(A(\varepsilon))\sigma(\varepsilon) = 0\}.$$

We prove the theorem first in a special case.

Lemma 5.12. — *Assume that A satisfies the further condition*

$$-\frac{1}{2} \notin \operatorname{spec} A_V. \tag{5.45}$$

Then for sufficiently small ε and $\sigma \in \mathscr{D}_\varepsilon^{\mathrm{sign}}$ we have the a priori estimate

$$||D_\varepsilon^{\mathrm{sign}}\sigma||_{H^0} \geq (2\varepsilon)^{-1}||(A_V + \frac{1}{2})\sigma||_{H^0}. \tag{5.46}$$

Proof. — We write $H := H^0, A_{HV}(t) := A_H(t)A_V + A_V A_H(t)$, and compute for $\sigma \in \mathscr{D}_\varepsilon^{\mathrm{sign}}$

$$(5.47)\quad ||D_\varepsilon^{\mathrm{sign}}\sigma(t)||_H^2 = ||\sigma'(t)||_H^2 + ||A_H\sigma(t)||_H^2 + t^{-2}||A_V\sigma(t)||_H^2 \\ + t^{-1}\langle A_{HV}\sigma(t), \sigma(t)\rangle_H + 2\,\mathrm{Re}\langle\sigma'(t), A\sigma(t)\rangle.$$

Next we verify that

$$(5.48)\quad 2\,\mathrm{Re}\langle\sigma'(t), A\sigma(t)\rangle = \\ \frac{d}{dt}\langle\sigma(t), A\sigma(t)\rangle_H + t^{-2}\langle\sigma(t), A_V\sigma(t)\rangle_H - \langle\sigma(t), A_H'(0)\sigma(t)\rangle_H.$$

The assumption (5.45) implies that $A_V + \frac{1}{2}$ is invertible while (2.39) and (2.30) imply that $A_{VH}(t)$ is a first order vertical operator on $\lambda(N)$. Hence there is a constant $C_1 > 0$ such that

$$(5.49)\qquad ||A_{HV}(t)(A_V + \frac{1}{2})^{-1}||_H \leq C_1,\ t \in (0, \varepsilon].$$

Combining (5.47), (5.48), and (5.49) and abbreviating $B := A_{HV}(t)(A_V + \frac{1}{2})^{-1}$ we arrive at the inequality

$$(5.50)\quad ||D_\varepsilon^{\mathrm{sign}}\sigma(t)||_H^2 = \big(||\sigma'(t)||_H^2 - \frac{1}{4}t^{-2}||\sigma(t)||_H^2\big) + \frac{d}{dt}\langle\sigma(t), A\sigma(t)\rangle_H \\ + t^{-2}||(A_V + \frac{1}{2})\sigma(t)||_H^2 \\ + t^{-1}\langle(A_V + \frac{1}{2})\sigma(t), B^*\sigma(t)\rangle_H - \langle\sigma(t), A_H'(0)\sigma(t)\rangle_H.$$

Hardy's inequality and the boundary condition at ε imply that the first two terms are nonnegative after integration over $(0, \varepsilon]$. Thus for sufficiently small ε we obtain

$$||D_\varepsilon^{\mathrm{sign}}\sigma||_{L^2((0,\varepsilon],H)}^2 \geq (4\varepsilon)^{-2}||(A_V + \frac{1}{2})\sigma||_{L^2(0,\varepsilon],H)}^2. \qquad \square$$

Now we can give the

Proof of Theorem 5.2. — The condition of Lemma 5.12 is satisfied, in view if of assumption (5.4), either v is even or

$$\mathscr{H}^{(v-1)/2}(Y) = 0,$$

in which case we can actually assert that

$$\ker D_\varepsilon^\Lambda = 0.$$

In the general case, we have to work differently since this assertion will no longer be true. If (5.12) does not hold then v must be odd hence h must be even. In this case, A_V is invertible and we can deform the operator $D_\varepsilon^{\mathrm{sign}}$ to an operator with vanishing index as follows.

As in Section 1, we view the Hilbert space $H^0 = \lambda_{(2)}(N_1)$ as a Hilbert bundle $\mathcal{E} \to B$ where

$$\begin{aligned}\mathcal{E} &= \Lambda T^*B \otimes \lambda_{(2)}(F),\\ \lambda_{(2)}(F)_b &= \lambda_{(2)}(Y_b).\end{aligned}$$

In $\lambda_{(2)}(Y_b)$, we define a smooth family of projections, $P_{\mathcal{H}}(b)$, by

$$P_{\mathcal{H}} := \frac{1}{2\pi i}\int_{|z|=\delta}(\Delta_b - z)^{-1}dz,\ b \in B, \tag{5.51}$$

which splits

$$\lambda_{(2)}(F) =: \mathcal{H}_F \oplus \mathcal{H}_F^{\perp}. \tag{5.52}$$

Here $\mathcal{H}_F$ is the finite dimensional vector bundle over B formed by the harmonic forms on the fibers. We note next that the projection $I \otimes P_{\mathcal{H}}$ commutes with A_V and with the principal symbol of $A_H(t)$. Hence the operator

$$A^{\delta}(t) := I \otimes P_{\mathcal{H}} A(t) I \otimes P_{\mathcal{H}} + I \otimes (I - P_{\mathcal{H}}) A(t) I \otimes (I - P_{\mathcal{H}}) \tag{5.53}$$

$$=: A_{\mathcal{H}}(t) + A_{\mathcal{H}^{\perp}}(t) \tag{5.54}$$

differs from $A(t)$ by an operator of uniformly bounded norm,

$$A^{\delta}(t) =: A(t) + C(t),\ ||C(t)||_H \le C, t \in (0, \varepsilon].$$

It follows that $A^{\delta}(t)$ satisfies the estimate (2.40), possibly with a different constant; in particular, $A^{\delta}(t)$ is invertible and $Q_{\le 0}(A^{\delta}(t)) = Q_{<0}(A^{\delta}(t))$. We now deform the operator $D_{\varepsilon}^{\text{sign}}$ to the operator $D_{\varepsilon,\delta}^{\text{sign}}$ which is given on the core

$$\mathcal{D}_{\varepsilon,\delta}^{\text{sign}} := \{\sigma \in C_c^1((0,\varepsilon], H^1) : Q_{<0}(A^{\delta}(\varepsilon))\sigma(\varepsilon) = 0\} \tag{5.55}$$

by

$$D_{\varepsilon,\delta}^{\text{sign}}\sigma(t) = (\frac{\partial}{\partial t} + A^{\delta}(t))\sigma(t). \tag{5.56}$$

Since $D_{\varepsilon}^{\text{sign}}$ and $D_{\varepsilon,\delta}^{\text{sign}}$ differ by a uniformly bounded operator we obtain from Theorem 5.9 and [**3**, Thm.4.14] the identity

$$\begin{aligned}\operatorname{ind} D_{\varepsilon}^{\text{sign}} &= \operatorname{ind} D_{\varepsilon,\delta}^{\text{sign}} + \operatorname{ind}\big(Q_{<0}(A(\varepsilon))(H^0), Q_{>0}(A^{\delta}(\varepsilon))(H^0)\big)\\ &= \operatorname{ind} D_{\varepsilon,\mathcal{H}}^{\text{sign}} + \operatorname{ind} D_{\varepsilon,\mathcal{H}^{\perp}}^{\text{sign}} + \operatorname{ind}\big(Q_{<0}(A(\varepsilon))(H^0), Q_{>0}(A^{\delta}(\varepsilon))(H^0)\big).\end{aligned} \tag{5.57}$$

Here the operators $D_{\varepsilon,\mathcal{H}}^{\text{sign}}$ and $D_{\varepsilon,\mathcal{H}^{\perp}}^{\text{sign}}$ are formed as $D_{\varepsilon}^{\text{sign}}$ above, by replacing A^{δ} in (5.55) and (5.56) by $A_{\mathcal{H}}$ and $A_{\mathcal{H}^{\perp}}$, respectively.

Now $D_{\varepsilon,\mathcal{H}^{\perp}}^{\text{sign}}$ satisfies the assumptions of Lemma 5.12 such that

$$\operatorname{ind} D_{\varepsilon,\mathcal{H}^{\perp}}^{\text{sign}} = 0. \tag{5.58}$$

Next we observe that $A_{\mathscr{H}}(t)$ anticommutes with τ_B up to a uniformly bounded operator since it has the same principal symbol as the canonical Dirac operator on B with coefficients in $\mathscr{H}_F$, that is

$$\tau_B A_{\mathscr{H}}(t)\tau_B =: -A_{\mathscr{H}}(t) + \tilde{C}(t), \tag{5.59}$$

where $||\tilde{C}(t)||_H \leq \tilde{C},\ t \in (0, \varepsilon_0]$. Thus we find that $\tau_B D^{\mathrm{sign}}_{\varepsilon,\mathscr{H}}\tau_B$ is given on the core

$$\mathscr{D}^{\mathrm{sign}}_{\varepsilon,\mathscr{H}^\perp} := \{\sigma \in C^1_c((0,\varepsilon], H^1) : Q_{<0}(\tau_B A_{\mathscr{H}}(\varepsilon)\tau_B)\sigma(\varepsilon) = 0\} \tag{5.60}$$

by the operator

$$(\frac{\partial}{\partial t} + \tau_B A_{\mathscr{H}}(t)\tau_B)\sigma(t). \tag{5.61}$$

We compare this with the adjoint operator $(D^{\mathrm{sign}}_{\varepsilon,\mathscr{H}})^*$ which is given on its core

$$(\mathscr{D}^{\mathrm{sign}}_{\varepsilon,\mathscr{H}})^* = \{\sigma \in C^1_c((0,\varepsilon], H^1) : Q_{>0}(A_{\mathscr{H}}(\varepsilon))\sigma(\varepsilon) = 0\} \tag{5.62}$$

by

$$(D^{\mathrm{sign}}_{\varepsilon,\mathscr{H}})^*\sigma(t) = (-\frac{\partial}{\partial t} + A_{\mathscr{H}}(t))\sigma(t). \tag{5.63}$$

Using (5.59) and the invertibility of $A(t)$, and applying Theorem 5.9 once more, we see that

$$\operatorname{ind} D^{\mathrm{sign}}_{\varepsilon,\mathscr{H}} = \operatorname{ind} \tau_B D^{\mathrm{sign}}_{\varepsilon,\mathscr{H}}\tau_B = \operatorname{ind}(D^{\mathrm{sign}}_{\varepsilon,\mathscr{H}})^* = -\operatorname{ind} D^{\mathrm{sign}}_{\varepsilon,\mathscr{H}} = 0. \tag{5.64}$$

A final application of (2.40) in Theorem 5.9 shows that

$$\operatorname{ind}\big(Q_{<0}(A(\varepsilon))(H^0), Q_{>0}(A^\delta(\varepsilon))(H^0)\big) = 0$$

and completes the proof of Theorem 5.2. □

5.3. Generalized Thom spaces. — In this subsection we compute the L^2-signature of a generalized Thom space, as introduced in [**17**], and identify it as a normalized spectral flow associated with the family $A(t)$ introduced in (2.32). We describe the generalized Thom space associated with the fibration (0.3) as the cylinder $T := T_\pi := (0,2) \times N$ with its product orientation and equipped with a family of metrics depending on a parameter $\varepsilon \in (0, 1/2)$ as follows. We write the metric on T_π in the form

$$g^{TT_\pi}_\varepsilon = dt^2 \oplus g^{TN}_\varepsilon(t), \tag{5.65}$$

where $g^{TN}_\varepsilon(t)$ is a smooth family of Riemannian metrics on N with the property

$$g^{TN}_\varepsilon(t) := \begin{cases} g^{T_HN} \oplus t^2 g^{T_VN} & \text{if } 0 < t \leq 1/2, \\ (2-t)^2(\varepsilon^{-2}g^{T_HN} \oplus g^{T_VN}) & \text{if } 3/2 < t < 2. \end{cases} \tag{5.66}$$

Here $g^{TN} = g^{T_HN} \oplus g^{T_VN}$ denotes again the metric introduced in (0.6) where we assume that g^{T_VN} is approriately scaled, as detailed below. Note that $g^{TN}_\varepsilon(\varepsilon) = g^{TN}_\varepsilon(2-\varepsilon) = g^{T_HN} \oplus \varepsilon^2 g^{T_VN}$; note also that we use the opposite orientation as in

[**17**]. Since any two metrics in the family $(g_\varepsilon^{TT_\pi})_{0<\varepsilon<1/2}$ are quasi-isometric, they all compute the same L^2-signature and we find

$$\operatorname{sign}_{(2)} T_\pi = \operatorname{ind} D^{\mathrm{sign}}_{T_\pi, g_\varepsilon^{TT_\pi}}. \tag{5.67}$$

The computation of $\operatorname{sign}_{(2)} T_\pi$ is now a special case of our general index computation with two singular strata of dimension h and 0, respectively. We split the computation at $t = \varepsilon$ and $t = 2 - \varepsilon$ and obtain three parts, the cone bundle U_ε over B, the metric cone $C_\varepsilon N := C_{(2,2-\varepsilon)}(N, \varepsilon^{-2} g_\varepsilon^{TN}(\varepsilon))$ over $(N, \varepsilon^{-2} g_\varepsilon^{TN}(\varepsilon))$, and the cylinder $Z_\varepsilon := (\varepsilon, 2-\varepsilon) \times N$ equipped with a nonsingular metric. We are ready for the

Proof of Theorem 5.4. — Arguing as before we see that on $U_{\varepsilon_0} \cup C_{(2,2-\varepsilon_0)}(N, \varepsilon^{-2} g_\varepsilon^{TN})$, D^{sign} is unitarily equivalent to $\frac{\partial}{\partial t} + A_\varepsilon(t)$ acting in $L^2\big((0,\varepsilon_0) \cup (2-\varepsilon_0, 2), \lambda_{(2)}(N_1)\big)$, where

$$A_\varepsilon(t) = \begin{cases} A(t), & t \in (0, \varepsilon_0), \\ (2-t)^{-1} \varepsilon \big(A_0(\varepsilon) + \varepsilon^{-1}(\mathrm{td} - \frac{n}{2})\big), & t \in (2-\varepsilon_0, 2). \end{cases} \tag{5.68}$$

To formulate our boundary conditions conveniently we introduce the spaces

$$H_{(0),I}(\varepsilon/2-\varepsilon) := Q_I\big(A_{(0)}(\varepsilon/2-\varepsilon)\big)(H^0). \tag{5.69}$$

Next we want to apply Theorem 5.2 to the operator $D^{\mathrm{sign}}_{l/r,\varepsilon}$ which is defined by $\frac{\partial}{\partial t} + A_\varepsilon(t)$ on its core

$$\mathscr{D}^{\mathrm{sign}}_{l,\varepsilon} := \{\sigma \in C_c^1((0,\varepsilon], H^1) : Q_{<0}(A(\varepsilon))\sigma(\varepsilon) = 0\},$$

and

$$\mathscr{D}^{\mathrm{sign}}_{r,\varepsilon} := \{\sigma \in C_c^1([2-\varepsilon, 2), H^1) : Q_{>0}(A(2-\varepsilon))\sigma(2-\varepsilon) = 0\},$$

respectively. Theorem 5.2 obviously applies to $D^{\mathrm{sign}}_{l,\varepsilon}$ if the condition (5.4) is satisfied. For $D^{\mathrm{sign}}_{r,\varepsilon}$, we note that the role of A_V is now taken by the operator $A_{r,V} := \varepsilon\big(A_0(\varepsilon) + \varepsilon^{-1}(\mathrm{td} - n/2)\big)$, such that the analogue of (5.4) can be verified by a straightforward estimate using (2.39), (2.19), (2.30), and (2.31), after the approriate scaling of $g^{T_V N}$.

Consequently, we obtain

$$\operatorname{sign}_{(2)} T_\pi = D^{\mathrm{sign}}_{\varepsilon, Z_\varepsilon}, \tag{5.70}$$

where $D^{\mathrm{sign}}_{\varepsilon, Z_\varepsilon}$ denotes the signature operator on the cylinder $(Z_\varepsilon, g_\varepsilon^{TT_\pi})$ with core

$$\{\sigma \in H^1(Z_\varepsilon, \Lambda T^* Z_\varepsilon) : \sigma | \partial U_\varepsilon \in H_{<0}(\varepsilon), \sigma | \partial C_\varepsilon N \in H_{>0}(2-\varepsilon)\}. \tag{5.71}$$

Clearly, if we replace the boundary conditions in (5.71) by $H_{0,<0}(\varepsilon)$ and $H_{0,\geq 0}(\varepsilon)$, respectively, then the resulting operator on Z_ε will have index 0. Thus we obtain from [**3**, Thm.4.14] again

$$\begin{aligned} \operatorname{sign}_{(2)} T_\pi = {} & \operatorname{ind}\big(H_{\leq 0}(\varepsilon), H_{0,>0}(\varepsilon)\big) \\ & - \operatorname{ind}\big(H_{0,<0}(\varepsilon), H_{0,>0}(\varepsilon)\big) + \operatorname{ind}(H_{\geq 0}(2-\varepsilon), H_{0,<0}(\varepsilon)). \end{aligned} \tag{5.72}$$

Now we recall that

$$\begin{aligned}(5.73)\quad & \operatorname{ind}(H_{>0}(2-\varepsilon), H_{0,<0}(\varepsilon)) \\ &= \operatorname{ind}\Big(Q_{>0}\big(A_0(\varepsilon)+\varepsilon^{-1}(\mathrm{td}+n/2)\big)(H^0), Q_{<0}(A_0(\varepsilon))(H^0)\Big).\end{aligned}$$

We recall also that $A_0(\varepsilon)$ is unitarily equivalent to $D^{\Lambda}_{N_\varepsilon}\tau_{N_\varepsilon}$ such that, by Theorem 3.1, the eigenspaces of $A_0(\varepsilon)$ coincide with those of $A_0(\varepsilon)+\varepsilon^{-1}(\mathrm{td}+n/2)$. Hence the explicit computations of loc.cit. give

$$(5.74)\qquad \operatorname{ind}(H_{\geq 0}(2-\varepsilon), H_{0,<0}(\varepsilon)) = -\frac{1}{2}\dim\ker A_0(\varepsilon).$$

Finally, we use [**3**, Prop.A.13] to see that

$$\operatorname{ind}\big(H_{\leq 0}(\varepsilon), H_{0,>0}(\varepsilon)\big) - \operatorname{ind}\big(H_{0,<0}(\varepsilon), H_{0,>0}(\varepsilon)\big) = \operatorname{ind}\big(H_{\leq 0}(\varepsilon), H_{0,\geq 0}(\varepsilon)\big),$$

which gives finally

$$(5.75)\qquad \operatorname{sign}_{(2)}T_\pi = \operatorname{ind}\big(H_{\leq 0}(\varepsilon), H_{0,\geq 0}(\varepsilon)\big) - \frac{1}{2}\dim\ker A_0(\varepsilon)$$

$$(5.76)\qquad =: \tau(\varepsilon).$$

This completes the proof of Theorem 5.4. □

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Astérisque
328, 2009, p. 45–135

SMOOTH K-THEORY

by

Ulrich Bunke & Thomas Schick

Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — In this paper we consider smooth extensions of cohomology theories. In particular we construct an analytic multiplicative model of smooth K-theory. We further introduce the notion of a smooth K-orientation of a proper submersion $p\colon W \to B$ and define the associated push-forward $\hat{p}_! : \hat{K}(W) \to \hat{K}(B)$. We show that the push-forward has the expected properties as functoriality, compatibility with pull-back diagrams, projection formula and a bordism formula.

We construct a multiplicative lift of the Chern character $\hat{\mathbf{ch}} : \hat{K}(B) \to \hat{H}(B,\mathbb{Q})$, where $\hat{H}(B,\mathbb{Q})$ denotes the smooth extension of rational cohomology, and we show that $\hat{\mathbf{ch}}$ induces a rational isomorphism.

If $p\colon W \to B$ is a proper submersion with a smooth K-orientation, then we define a class $A(p) \in \hat{H}^{\mathrm{ev}}(W,\mathbb{Q})$ (see Lemma 6.17) and the modified push-forward $\hat{p}_!^A := \hat{p}_!(A(p) \cup \dots) : \hat{H}(W,\mathbb{Q}) \to \hat{H}(B,\mathbb{Q})$. One of our main results lifts the cohomological version of the Atiyah-Singer index theorem to smooth cohomology. It states that $\hat{p}_!^A \circ \hat{\mathbf{ch}} = \hat{\mathbf{ch}} \circ \hat{p}_!$.

Résumé **(K-théorie différentiable).** — Dans cet article, nous considérons des extensions différentielles des théories cohomologiques. En particulier, nous construisons un modèle analytique multiplicatif de la K-théorie différentielle. Nous introduisons les K-orientations différentielles d'une submersion propre $p\colon W \to B$. Nous contruisons une application d'intégration associée: $\hat{p}_!\colon \hat{K}(W) \to \hat{K}(B)$; et nous démontrons les propriétés attendues, telles que la fonctorialité, la compatibilité aux pull-backs, des formules de projection et de bordisme.

Nous construisons un relèvement multiplicatif du caractère de Chern $\hat{\mathbf{ch}} : \hat{K}(B) \to \hat{H}(B,\mathbb{Q})$, où $\hat{H}(B,\mathbb{Q})$ est une extension différentielle de la cohomologie rationnelle, et nous démontrons que $\hat{\mathbf{ch}}$ induit un isomorphisme rationnel.

Si $p\colon W \to B$ est une submersion propre munie d'une K-orientation différentielle, nous définissons une classe $A(p) \in \hat{H}^{\mathrm{ev}}(W,\mathbb{Q})$ (compare Lemma 6.17) et une application d'intégration modifiée $\hat{p}_!^A := \hat{p}_!(A(p) \cup \dots) : \hat{H}(W,\mathbb{Q}) \to \hat{H}(B,\mathbb{Q})$. L'un de

2010 Mathematics Subject Classification. — 19L10, 58J28.
Key words and phrases. — Deligne cohomology, smooth K-theory, Chern character, families of elliptic operators, Atiyah-Singer index theorem.

Thomas Schick was supported by Courant Research Center "Higher order structures in mathematics" via the German Initiative of Excellence.

nos résultats principaux est une version en cohomologie différentielle du théorème d'indice d'Atiyah-Singer, pour laquelle $\hat{p}^A_! \circ \hat{\mathbf{ch}} = \hat{\mathbf{ch}} \circ \hat{p}_!$.

1. Introduction

1.1. The main results

1.1.1. — In this paper we construct a model of a smooth extension of the generalized cohomology theory K, complex K-theory. Historically, the concept of smooth extensions of a cohomology theory started with smooth integral cohomology [**24**], also called real Deligne cohomology, see [**16**]. A second, geometric model of smooth integral cohomology is given in [**24**], where the smooth integral cohomology classes were called differential characters. One important motivation of its definition was that one can associate natural differential characters to hermitean vector bundles with connection which refine the Chern classes. The differential character in degree two even classifies hermitean line bundles with connection up to isomorphism. The multiplicative structure of smooth integral cohomology also encodes cohomology operations, see [**29**].

The holomorphic counterpart of the theory became an important ingredient of arithmetic geometry.

1.1.2. — Motivated by the problem of setting up lagrangians for quantum field theories with differential form field strength it was argued in [**27**], [**26**] that one may need smooth extensions of other generalized cohomology theories. The choice of the generalized cohomology theory is here dictated by a charge quantization condition, which mathematically is reflected by a lattice in real cohomology. Let N be a graded real vector space such that the field strength lives in $\Omega_{d=0}(B)\otimes N$, the closed forms on the manifold B with coefficients in N. Let $L(B) \subset H(B,N)$ be the lattice given by the charge quantization condition on B. Then one looks for a generalized cohomology theory h and a natural transformation $c : h(B) \to H(B,N)$ such that $c(h(B)) = L(B)$. It was argued in [**27**], [**26**] that the fields of the theory should be considered as cycles for a smooth extension $\hat{h}$ of the pair (h,c). For example, if $N = \mathbb{R}$ and the charge quantization leads to $L(B) = \mathtt{im}(H(B,\mathbb{Z}) \to H(B,\mathbb{R}))$, then the relevant smooth extension could be the smooth integral cohomology theory of [**24**].

In Subsection 1.2 we will introduce the notion of a smooth extension in an axiomatic way.

1.1.3. — [**26**] proposes in particular to consider smooth extensions of complex and real versions of K-theory. In that paper it was furthermore indicated how cycle models of such smooth extensions could look like. The goal of the present paper is to carry through this program in the case of complex K-theory.

1.1.4. — In the remainder of the present subsection we describe, expanding the abstract, our main results. The main ingredient is a construction of an analytic model of smooth K-theory, also called differentiable K-theory by some authors, using cycles and relations.

1.1.5. — Our philosophy for the construction of smooth K-theory is that a vector bundle with connection or a family of Dirac operators with some additional geometry should represent a smooth K-theory class tautologically. In this way we follow the outline in [**26**]. Our class of cycles is quite big. This makes the construction of smooth K-theory classes or transformations to smooth K-theory easy, but it complicates the verification that certain cycle level constructions out of smooth K-theory are well-defined. The great advantage of our choice is that the constructions of the product and the push-forward on the level of cycles are of differential geometric nature.

More precisely we use the notion of a geometric family which was introduced in [**19**] in order to subsume all geometric data needed to define a Bismut super-connection in one notion. A cycle of the smooth K-theory $\hat{K}(B)$ of a compact manifold B is a pair $(\mathcal{E}, \rho)$ of a geometric family $\mathcal{E}$ and an element $\rho \in \Omega(B)/\mathtt{im}(d)$, see Section 2. Therefore, cycles are differential geometric objects. Secondary spectral invariants from local index theory, namely η-forms, enter the definition of the relations (see Definition 2.10). The first main result is that our construction really yields a smooth extension in the sense of Definition 1.1.

1.1.6. — Our smooth K-theory $\hat{K}(B)$ is a contravariant functor on the category of compact smooth manifolds (possibly with boundary) with values in the category of $\mathbb{Z}/2\mathbb{Z}$-graded rings. This multiplicative structure is expected since K-theory is a multiplicative generalized cohomology theory, and the Chern character is multiplicative, too. As said above, the construction of the product on the level of cycles (Definition 4.1) is of differential-geometric nature. Analysis enters the verification of well-definedness. The main result is here that our construction produces a multiplicative smooth extension in the sense of Definition 1.2.

1.1.7. — Let us consider a proper submersion $p\colon W \to B$ with closed fibres which has a topological K-orientation. Then we have a push-forward $p_!\colon K(W) \to K(B)$, and it is an important part of the theory to extend this push-forward to the smooth extension.

For this purpose one needs a smooth refinement of the notion of a K-orientation which we introduce in 3.5. We then define the associated push-forward $\hat{p}_!\colon \hat{K}(W) \to \hat{K}(B)$, again by a differential-geometric construction on the level of cycles (17). We show that the push-forward has the expected properties: functoriality, compatibility with pull-back diagrams, projection formula, bordism formula.

1.1.8. — Let $\mathbf{V} = (V, h^V, \nabla^V)$ be a hermitean vector bundle with connection. In [**24**] a smooth refinement $\hat{\mathbf{ch}}(\mathbf{V}) \in \hat{H}(B, \mathbb{Q})$ of the Chern character was constructed. In the present paper we construct a lift of the Chern character $\mathbf{ch}\colon K(B) \to H(B, \mathbb{Q})$ to a multiplicative natural transformation of smooth cohomology theories (see (30))

$$\hat{\mathbf{ch}}\colon \hat{K}(B) \to \hat{H}(B, \mathbb{Q})$$

such that $\hat{\mathbf{ch}}(\mathbf{V}) = \hat{\mathbf{ch}}([\mathcal{V}, 0])$, where $\mathcal{V}$ is the geometric family determined by $\mathbf{V}$. We prove in Proposition 6.12 that the Chern character induces a natural isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded rings $\hat{K}(B) \otimes \mathbb{Q} \xrightarrow{\sim} \hat{H}(B, \mathbb{Q})$.

1.1.9. — If $p\colon W \to B$ is a proper submersion with a smooth K-orientation, then we define a class (see Lemma 6.17) $A(p) \in \hat{H}^{\mathrm{ev}}(W, \mathbb{Q})$ and the modified push-forward

$$\hat{p}_!^A := \hat{p}_!(A(p) \cup \cdots)\colon \hat{H}(W, \mathbb{Q}) \to \hat{H}(B, \mathbb{Q}).$$

Our index theorem 6.19 lifts the characteristic class version of the Atiyah-Singer index theorem to smooth cohomology. It states that the following diagram commutes:

$$\begin{tikzcd}
\hat{K}(W) \arrow[r, "\hat{\mathbf{ch}}"] \arrow[d, "\hat{p}_!"] & \hat{H}(W, \mathbb{Q}) \arrow[d, "\hat{p}_!^A"] \\
\hat{K}(B) \arrow[r, "\hat{\mathbf{ch}}"] & \hat{H}(B, \mathbb{Q}).
\end{tikzcd}$$

1.1.10. — In Subsection 1.2 we present a short introduction to the theory of smooth extensions of generalized cohomology theories. In Subsection 1.3 we review in some detail the literature about variants of smooth K-theory and associated index theorems. In Section 2 we present the cycle model of smooth K-theory. The main result is the verification that our construction satisfies the axioms given below. Section 3 is devoted to the push-forward. We introduce the notion of a smooth K-orientation, and we construct the push-forward on the cycle level. The main results are that the push-forward descends to smooth K-theory, and the verification of its functorial properties. In Section 4 we discuss the ring structure in smooth K-theory and its compatibility with the push-forward. Section 5 presents a collection of natural constructions of smooth K-theory classes. In Section 6 we construct the Chern character and prove the smooth index theorem.

1.2. A short introduction to smooth cohomology theories

1.2.1. — The first example of a smooth cohomology theory appeared under the name Cheeger-Simons differential characters in [**24**]. Given a discrete subring $\mathtt{R} \subset \mathbb{R}$ we have a functor[(1)] $B \mapsto \hat{H}(B,\mathtt{R})$ from smooth manifolds to $\mathbb{Z}$-graded rings. It comes with natural transformations

1. $R\colon \hat{H}(B,\mathtt{R}) \to \Omega_{d=0}(B)$ (curvature)
2. $I\colon \hat{H}(B,\mathtt{R}) \to H(B,\mathtt{R})$ (forget smooth data)
3. $a\colon \Omega(B)/\mathtt{im}(d) \to \hat{H}(B,\mathtt{R})$ (action of forms).

Here $\Omega(B)$ and $\Omega_{d=0}(B)$ denote the space of smooth real differential forms and its subspace of closed forms. The map a is of degree 1. Furthermore, one has the following properties, all shown in **[24]**.

1. The following diagram commutes
$$\begin{tikzcd} \hat{H}(B,\mathtt{R}) \arrow[r, "I"] \arrow[d, "R"] & H(B,\mathtt{R}) \arrow[d, "\mathtt{R}\to\mathbb{R}"] \\ \Omega_{d=0}(B) \arrow[r, "dR"] & H(B,\mathbb{R}), \end{tikzcd}$$
where dR is the de Rham homomorphism.
2. R and I are ring homomorphisms.
3. $R \circ a = d$,
4. $a(\omega) \cup x = a(\omega \wedge R(x))$, $\forall x \in \hat{H}(B,\mathtt{R})$, $\forall \omega \in \Omega(B)/\mathtt{im}(d)$,
5. The following sequence is exact:

$$H(B,\mathtt{R}) \to \Omega(B)/\mathtt{im}(d) \xrightarrow{a} \hat{H}(B,\mathtt{R}) \xrightarrow{I} H(B,\mathtt{R}) \to 0. \tag{1}$$

1.2.2. — Cheeger-Simons differential characters are the first example of a more general structure which is described for instance in the first section of **[26]**. In view of our constructions of examples for this structure in the case of bordism theories and K-theory, and the presence of completely different pictures like **[31]** we think that an axiomatic description of smooth cohomology theories is useful.

Let N be a $\mathbb{Z}$-graded vector space over $\mathbb{R}$. We consider a generalized cohomology theory h with a natural transformation of cohomology theories $c\colon h(B) \to H(B,N)$. The natural universal example is given by $N := h^* \otimes \mathbb{R}$, where c is the canonical transformation. Let $\Omega(B,N) := \Omega(B) \otimes_{\mathbb{R}} N$. To a pair (h,c) we associate the notion of a smooth extension $\hat{h}$. Note that manifolds in the present paper may have boundaries.

Definition 1.1. — *A smooth extension of the pair (h,c) is a functor $B \to \hat{h}(B)$ from the category of compact smooth manifolds to $\mathbb{Z}$-graded groups together with natural transformations*

1. $R\colon \hat{h}(B) \to \Omega_{d=0}(B,N)$ *(curvature)*
2. $I\colon \hat{h}(B) \to h(B)$ *(forget smooth data)*

(1) In the literature, this group is sometimes denoted by $\hat{H}(B,\mathbb{R}/\mathtt{R})$, possibly with a degree-shift by one.

3. $a\colon \Omega(B,N)/\mathtt{im}(d) \to \hat{h}(B)$ *(action of forms).*

These transformations are required to satisfy the following axioms:

1. *The following diagram commutes*

$$\begin{tikzcd} \hat{h}(B) \arrow[r, "I"] \arrow[d, "R"] & h(B) \arrow[d, "c"] \\ \Omega_{d=0}(B,N) \arrow[r, "dR"] & H(B,N). \end{tikzcd}$$

2. *We have*

$$R \circ a = d. \tag{2}$$

3. *a is of degree* 1.
4. *The following sequence is exact:*

$$h(B) \xrightarrow{c} \Omega(B,N)/\mathtt{im}(d) \xrightarrow{a} \hat{h}(B) \xrightarrow{I} h(B) \to 0. \tag{3}$$

The Cheeger-Simons smooth cohomology $B \mapsto \hat{H}(B,\mathtt{R})$ considered in 1.2.1 is the smooth extension of the pair $(H(\dots,\mathtt{R}),i)$, where $i\colon H(B,\mathtt{R}) \to H(B,\mathbb{R})$ is induced by the inclusion $\mathtt{R} \to \mathbb{R}$. The main object of the present paper, smooth K-theory, is a smooth extension of the pair $(K,\mathbf{ch}_{\mathbb{R}})$, and we actually work with the obvious $\mathbb{Z}/2\mathbb{Z}$-graded version of these axioms.

1.2.3. — If h is a multiplicative cohomology theory, then one can consider a $\mathbb{Z}$-graded ring N over $\mathbb{R}$ and a multiplicative transformation $c\colon h(B) \to H(B,N)$. In this case is makes sense to talk about a multiplicative smooth extension $\hat{h}$ of (h,c).

Definition 1.2. — *A smooth extension $\hat{h}$ of (h,c) is called multiplicative, if $\hat{h}$ together with the transformations R,I,a is a smooth extension of (h,c), and in addition*

1. *$\hat{h}$ is a functor to $\mathbb{Z}$-graded rings,*
2. *R and I are multiplicative,*
3. *$a(\omega) \cup x = a(\omega \wedge R(x))$ for $x \in \hat{h}(B)$ and $\omega \in \Omega(B,N)/\mathtt{im}(d)$.*

The smooth extension $\hat{H}(\dots,\mathtt{R})$ of ordinary cohomology $H(\dots,\mathtt{R})$ with coefficients in a subring $\mathtt{R} \subset \mathbb{R}$ considered in 1.2.1 is multiplicative. The smooth extension $\hat{K}$ of K-theory which we construct in the present paper is multiplicative, too.

1.2.4. — Consider two pairs (h_i,c_i), $i = 0,1$ as in 1.2.2 and a transformation of generalized cohomology theories $u\colon h_0 \to h_1$ such that $c_1 \circ h = c_0$. Then we define the notion of a natural transformation of smooth cohomology theories which refines u.

Definition 1.3. — *A natural transformation of smooth extensions $\hat{u}\colon \hat{h}_0 \to \hat{h}_1$ which refines u is a natural transformation $\hat{u}\colon \hat{h}_0(B) \to \hat{h}_1(B)$ such that the following diagram commutes:*

$$\begin{tikzcd}
\Omega(B,N)/\mathtt{im}(d) \arrow[r, "a"] \arrow[d, equal] & \hat{h}_0(B) \arrow[r, "I"] \arrow[d, "\hat{u}"] \arrow[rr, bend left, "R"] & h_0(B) \arrow[d, "u"] & \Omega_{d=0}(B,N) \arrow[d, equal] \\
\Omega(B,N)/\mathtt{im}(d) \arrow[r, "a"] & \hat{h}_1(B) \arrow[r, "I"] \arrow[rr, bend right, "R"] & h_1(B) & \Omega_{d=0}(B,N).
\end{tikzcd}$$

Our main example is the Chern character $\hat{\mathbf{ch}}\colon \hat{K}(B) \to \hat{H}(B,\mathbb{Q})$ which refines the ordinary Chern character $\mathbf{ch}\colon K(B) \to H(B,\mathbb{Q})$. The Chern character and its smooth refinements are actually multiplicative.

1.2.5. — One can show that two smooth extensions of $(H(\ldots,R),i)$ are canonically isomorphic (see [**44**] and [**22**, Section 4]). There is no uniqueness result for arbitrary pairs (h,c). Appropriate examples in the case of K-theory are presented in [**22**, Section 6]. In order to fix the uniqueness problem one has to require more conditions, which are all quite natural.

The projection $\mathtt{pr}_2\colon S^1\times B \to B$ has a canonical smooth K-orientation (see 4.3.2 for details). Hence we have a push-forward $(\hat{\mathtt{pr}}_2)_!\colon \hat{K}(S^1\times B) \to \hat{K}(B)$ (see Definition 3.18). This map plays the role of the suspension for the smooth extension. It is natural in B, and the following diagram commutes (see Proposition 3.19)

$$\begin{tikzcd}
\Omega(S^1\times B)/\mathtt{im}(d) \arrow[r, "a"] \arrow[d, "\int_{S^1\times B/B}"] & \hat{K}(S^1\times B) \arrow[r, "I"] \arrow[d, "(\hat{\mathtt{pr}}_2)_!"] \arrow[rr, bend left, "R"] & K(B) \arrow[d, "(\mathtt{pr}_2)_!"] & \Omega(S^1\times B) \arrow[d, "\int_{S^1\times B/B}"] \\
\Omega(B)/\mathtt{im}(d) \arrow[r, "a"] & \hat{K}(B) \arrow[r, "I"] \arrow[rr, bend right, "R"] & K(B) & \Omega(B).
\end{tikzcd} \tag{4}$$

Furthermore, it satisfies (see 4.6)

$$(\hat{\mathtt{pr}}_2)_! \circ \mathtt{pr}_2^* = 0. \tag{5}$$

We have the following theorem, also discovered by Wiethaup.

Theorem 1.4 **([22, Section 3, Section 4]).** — *There is a unique (up to isomorphism) smooth extension of the pair $(K,\mathbf{ch}_{\mathbb{R}})$ for which in addition the push-forward along*

$\mathrm{pr}_2\colon S^1 \times B \to B$ is defined, is natural in B, satisfies (5), and is such that (4) commutes. If we require the isomorphism to preserve $(\hat{\mathrm{pr}}_2)_!$, then it is also unique.

1.2.6. — The theory of [**31**] gives the following general existence result.

Theorem 1.5 **([31])**. — *For every pair (h, c) of a generalized cohomology theory and a natural transformation $h \to HN$ there exists a smooth extension $\hat{h}$ in the sense of Definition 1.1.*

A similar general result about multiplicative extensions is not known. Besides smooth extensions of ordinary cohomology and K-theory we have a collection of multiplicative extensions of bordism theories, again by an an explicit construction in a cycle model. The details can be found in [**23**].

1.2.7. — Let us now assume that (h, c) is multiplicative, and that $\hat{h}$ is a multiplicative smooth extension of the pair (h, c). Let $p\colon W \to B$ be a proper submersion with closed fibres. An h-orientation of p is given by a collection of compatible choices of h-Thom classes on representatives of the stable normal bundle of p. Equivalently, we can fix a Thom class on the vertical tangent bundle, and we will adopt this point of view in the present paper. If p is h-oriented, then we have a push-forward $p_!\colon h(W) \to h(B)$. It is an inportant question for applications and calculations how one can lift the push-forward to the smooth extensions.

In the case of smooth ordinary cohomology with coefficients in R it turns out that an ordinary orientation of p suffices in order to define $\hat{p}_!\colon \hat{H}(W, R) \to \hat{H}(B, R)$. This push-forward has been considered e.g. in [**16**], [**25**], [**35**]. We refer to 6.1.1 for more details.

A push-forward for more general pairs (h, c) has been considered in [**31**] without a discussion of functorial properties.

1.2.8. — The philosophy in the present paper is that the push-forward in K-theory is realized analytically using families of fibre-wise Dirac operators. Therefore, in the present paper a smooth K-orientation is given by a collection of geometric data which allows to define the push-forward on the level of cycles, which are given by families of Dirac type operators. We add a differential form to the data in order to capture the behaviour under deformations.

1.2.9. — We have cycle models of multiplicative smooth extensions of bordism theories Ω^G, where G in particular can be $SO, Spin, U, Spin^c$, see [**23**]. In these examples the natural transformation c is the genus associated to a formal power series $\phi(x) = 1 + a_1 x + \dots$ with coefficients in some graded ring. These bordism theories admit a theory of orientations and push-forward which is very similar to the case of K-theory. Concerning the product and the integration bordism theories turn out to be much simpler than ordinary cohomology. Motivated by this fact, in a joint project with M. Kreck we develop a bordism like version of the smooth extension of integral cohomology based on the notion of orientifolds.

We also have an equivariant version of the theory of the present paper for finite groups which will be presented in a future publication.

1.3. Related constructions

1.3.1. — Recall that [**31**] provides a topological construction of smooth K-theory. In this subsection we review the literature about analytic variants of smooth K-theory and related index theorems. Note that we will completely ignore the development of holomorphic variants which are more related to arithmetic questions than to topology. This subsection will use the language which is set up later in the paper. It should be read in detail only after obtaining some familiarity with the main definitions (though we tried to give sufficiently many forward references).

1.3.2. — Let $p\colon W \to B$ be a proper submersion with closed fibres. To give a K-orientation of p is equivalent to give a $Spin^c$-structure on its vertical bundle $T^v p$. The K-orientation of p yields, by a stable homotopy construction, a push-forward $p_!\colon K(W) \to K(B)$. Let $\hat{\mathbf{A}}(T^v p)$ denote the $\hat{\mathbf{A}}$-class of the vertical bundle, and let $c_1(L^2) \in H^2(W, \mathbb{Z})$ be the cohomology class determined by the $Spin^c$-structure (see 3.1.6). The "index theorem for families" in the characteristic class version states that

$$\mathbf{ch}(p_!(x)) = \int_{W/B} \hat{\mathbf{A}}(T^v p) \cup e^{\frac{1}{2}c_1(L^2)} \cup \mathbf{ch}(x), \qquad \forall x \in K(W).$$

If one realizes the push-forward in an analytic model, then this statement is indeed an index theorem for families of Dirac operators.

1.3.3. — The cofibre of the map of spectra $K \to H\mathbb{R}$ induced by the Chern character represents a generalized cohomology theory $K\mathbb{R}/\mathbb{Z}$, called $\mathbb{R}/\mathbb{Z}$-K-theory. It is a module theory over K-theory and therefore also admits a push-forward for K-oriented proper submersions. This push-forward is again defined by constructions in stable homotopy theory. An analytic/geometric model of $\mathbb{R}/\mathbb{Z}$-K-theory was proposed in [**32**], [**33**]. This led to the natural question whether there is an analytic description of the push-forward in $\mathbb{R}/\mathbb{Z}$-K-theory. This question was solved in [**37**]. The solution gives a topological interpretation of ρ-invariants.

Furthermore, in [**37**] a Chern character from $\mathbb{R}/\mathbb{Z}$-K-theory to cohomology with $\mathbb{R}/\mathbb{Q}$-coefficients has been constructed, and an index theorem has been proved.

Let us now explain the relation of these constructions and results with the present paper. In the present paper we define the flat theory $\hat{K}_{\mathrm{flat}}(B)$ as the kernel of the curvature $R\colon \hat{K}(B) \to \Omega_{d=0}(B)$. It turns out that $\hat{K}_{\mathrm{flat}}(B)$ is isomorphic to $K\mathbb{R}/\mathbb{Z}(B)$ up to a degree-shift by one (Proposition 2.25). One can actually represent all classes of $K^0_{\mathrm{flat}}(B)$ by pairs $(\mathcal{E}, \rho)$, where $\mathcal{E}$ is a geometric family with zero-dimensional fibre (see 2.1.4). If one restricts to these special cycles, then our model of $K^0_{\mathrm{flat}}(B)$ and the model of $K\mathbb{R}/\mathbb{Z}^{-1}(B)$ of [**37**] coincide.

By an inspection of the constructions one can further check that the restriction of our cycle level push-forward (17) to these particular flat cycles is the same as the one in [**37**]. At a first glance our push-forward of flat classes seems to depend on a smooth refinement of the topological K-orientation of the map p, but it is in fact independent of these geometric choices as can be seen using the homotopy invariance of the flat theory. The comparison with [**37**] shows that the restriction of our push-forward to flat classes coincides with the homotopy theorists' one.

The restriction of our smooth lift of the Chern character $\hat{\mathbf{ch}}\colon \hat{K}(B) \to \hat{H}(B, \mathbb{Q})$ (see Theorem 6.2) to the flat theories exactly gives the Chern character of [**37**]

$$\hat{\mathbf{ch}}\colon \hat{K}_{\mathrm{flat}}(B) \to \hat{H}_{\mathrm{flat}}(B, \mathbb{Q})$$

(using our notation and the isomorphism of $\hat{H}^*_{\mathrm{flat}}(B) \cong H^{*-1}(B, \mathbb{R}/\mathbb{Q})$). If we restrict our index theorem 6.19 to flat classes, then it specializes to

$$\hat{\mathbf{ch}}(\hat{p}_!(x)) = \int_{W/B} \hat{\mathbf{A}}(T^v p) \cup e^{\frac{1}{2}c_1(L^2)} \cup \hat{\mathbf{ch}}(x), \qquad \forall x \in \hat{K}(W),$$

and this is exactly the index theorem of [**37**].

In this sense the present paper is a direct generalization of [**37**] from the flat to the general case.

1.3.4. — The analytic model of $\mathbb{R}/\mathbb{Z}$-K-theory and the analytic construction of the push-forward in [**37**] fits into a series of constructions of homotopy invariant functors with a push-forward which encodes secondary spectral invariants. Let us mention the two examples in [**38**] which are based on flat bundles or flat bundles with duality, respectively. The spectral geometric invariants in these examples are the analytic torsion forms of [**15**] and the η-forms introduced e.g. in [**12**]. The functoriality of the push-fowards under compositions is discussed in [**18**] and [**21**]. But these construction do not fit (at least at the moment) into the world of smooth cohomology theory, and it is still an open problem to interpret the push-forward in topological terms.

Let us also mention the paper [**43**] devoted to smooth lifts of Chern classes.

1.3.5. — In [**9**], [**8**] several variants of functors derived from K-theory are considered. In the following we recall the names of these groups used in that reference and explain, if possible, their relation with the present paper.

1. relative K-theory K_{rel}: the cycles are triples (V, ∇^V, f) of $\mathbb{Z}/2\mathbb{Z}$-graded flat vector bundles and an odd selfadjoint bundle automorphism f (which need not be parallel).

2. free multiplicative K-theory K_{ch} (also called transgressive in [**8**]): it is essentially[(2)] a model of $\hat{K}^0$ based on cycles of the form $(\mathcal{E},\rho)$, where $\mathcal{E}$ is a geometric family with zero-dimensional fibre coming from a geometric vector bundle (see 2.1.4).
3. multiplicative K-theory MK: it is the same model of K^0_{flat} as in [**37**], see 1.3.3.
4. flat K-theory K_{flat}: it is the Grothendieck group of flat vector bundles.

Besides the definition of these groups and the investigation of their interrelation the main topic of [**9**], [**8**] is the construction of push-forward operations. In the following we will only discuss multiplicative and transgressive K-theory since they are related to the present paper. The difference to the constructions of [**37**] and the present paper is that Berthomiau's analytic push-forward (which we denote here by $p^B_!$) does not use the $Spin^c$-Dirac operator but the fibre-wise de Rham complex. From the point of view of analysis the difference is essentially that the class $\hat{\mathbf{A}}(T^vp)\cup e^{\frac{1}{2}c_1(L^2)}$ or the corresponding differential form has to be replaced by the Euler class $E(T^vp)$ or the Euler form of the vertical bundle.

The advantage of working with the de Rham complex is that in order to define the push-forward $p^B_!$ one does not need a $Spin^c$-structure. If there is one, then one can actually express $p^B_!$ in terms of $\hat{p}_!$ as

$$p^B_!(x)=\hat{p}_!(x\cup s^*),$$

where $s^*\in K(W)$ is the class of the dual of the spinor bundle $S^c(T^vp)$, or the $\hat{K}(W)$-class represented by the geometric version of this bundle in the case of transgressive K-theory, respectively. The point here is that the Dirac operator induced by the de Rham complex is the $Spin^c$-Dirac operator twisted by $S^c(T^vp)^*$.

As said above, the homotopy theorists' $p_!$ is the push-forward associated to a K-orientation of p. In contrast, the homotopy theorists' version of $p^B_!$ is the Gottlieb-Becker transfer.

The motivation of [**9**] , [**8**] to define the push-forward with the de Rham complex is that it is compatible with the push-forward for flat K-theory. The push-forward of a flat vector bundle is expressed in terms of fibre-wise cohomology which forms again a flat vector bundle on the base. This additional structure also plays a crucial role in [**38**], [**15**], [**18**], and [**21**]. If one interprets the push-forward using the $Spin^c$-calculus, then the flat connection is lost. Let us mention that the first circulated version of the present paper predates the papers [**9**] , [**8**] which actually adapt some of our ideas.

1.3.6. — The topics of [**11**] are two index theorems involving $\hat{H}(B,\mathbb{Q})$-valued characteristic classes. Here we only review the first one, since the second is related to flat

(2) The connections are not assumed to be hermitean and the corresponding differential forms have complex coefficients.

vector bundles. (Compare also [**39**] for a "flat version"). Let us formulate the result of [**11**] in the language of the present paper.

Let $p\colon W \to B$ be a proper submersion with closed fibres with a fibre-wise *spin*-structure over a compact base B. The spin structure induces a $Spin^c$-structure, and we choose a representative of a smooth K-orientation $o := (g^{T^vp}, T^hp, \tilde{\nabla}, 0)$, where $\tilde{\nabla}$ is indced from the Levi-Civita connection on T^vp (see 3.1.9 for details). Let $\mathbf{V} = (V, h^V, \nabla^V)$ be a geometric vector bundle over W with associated geometric family $\mathcal{V}$ (compare 2.1.4). Then we can form the geometric family $\mathcal{E} := p_!\mathbf{V}$ (see 3.7) over B.

The family of Dirac operators $D(\mathcal{E})$ acts on sections of a bundle of Hilbert spaces $H(\mathcal{E}) \to B$. The geometric structures of the K-orientation o and $\mathbf{V}$ induce a connection $\nabla^{H(\mathcal{E})}$ (it is the connection part of the Bismut superconnection [**7**, Prop. 10.15] associated to this situation). We assume that the family of Dirac operators of $D(\mathcal{E})$ has a kernel bundle $K := \ker(D(\mathcal{E}))$. This bundle has an induced metric h^K. The projection of $\nabla^{H(\mathcal{E})}$ to K gives a hermitean connection ∇^K. We thus get a geometric bundle $\mathbf{K} := (K, h^K, \nabla^K)$, and an associated geometric family $\mathcal{K}$ (see 5.3.1). The index theorem in [**11**] calculates the smooth Chern character $\hat{\mathbf{ch}}(\mathbf{K}) \in \hat{H}(B, \mathbb{Q})$ of [**24**] and states:

$$\hat{\mathbf{ch}}(\mathbf{K}) = \hat{p}_!(\hat{\hat{\mathbf{A}}}(\mathbf{T^v p}) \cup \hat{\mathbf{ch}}(\mathbf{V})) + a(\eta^{BC}(\mathcal{E})),$$

where we refer to (33) and 5.3.3 for notation.

Note that this theorem could also be derived from our index Theorem 6.19. By Corollary 5.5, (17) , our special choice of o, and Theorem 6.19 (the marked step) we have

$$\begin{aligned}\hat{\mathbf{ch}}(\mathbf{K}) - a(\eta^{BC}(\mathcal{E})) = \hat{\mathbf{ch}}[\mathcal{K}, \eta^{BC}(\mathcal{E})] = \hat{\mathbf{ch}}[\mathcal{E}, 0] = \hat{\mathbf{ch}}([p_!\mathcal{V}, 0]) = \hat{\mathbf{ch}}(p_!([\mathcal{V}, 0]))\\ \stackrel{!}{=} \hat{p}_!^K(\hat{\mathbf{ch}}(\mathcal{V})) = p_!(\hat{\hat{\mathbf{A}}}(\mathbf{T^v p}) \cup \hat{\mathbf{ch}}(\mathbf{V})).\end{aligned}$$

Acknowledgement. — We thank Moritz Wiethaup for explaining to us his insights and result. We further thank Mike Hopkins and Dan Freed for their interest in this work and many helpful remarks. We thank the referee for many helpful comments which lead to considerable improvements of the exposition.

2. Definition of smooth K-theory via cycles and relations

2.1. Cycles

2.1.1. — One goal of the present paper is to construct a multiplicative smooth extension of the pair $(K, \mathbf{ch}_\mathbb{R})$ of the multiplicative generalized cohomology theory K, complex K-theory, and the composition $\mathbf{ch}_\mathbb{R}\colon K \stackrel{\mathbf{ch}}{\to} H\mathbb{Q} \to H\mathbb{R}$ of the Chern character with the natural map from ordinary cohomology with rational to real coefficients induced by the inclusion $\mathbb{Q} \to \mathbb{R}$. In this section we define the smooth K-theory group $\hat{K}(B)$ of a smooth compact manifold, possibly with boundary, and construct the natural transformations R, I, a. The main result of the present section is that our construction really yields a smooth extension in the sense of Definition 1.1. Wi discuss the multiplicative structure in Section 4.

Our restriction to compact manifolds with boundary is due to the fact that we work with absolute K-groups. One could in fact modify the constructions in order to produce compactly supported smooth K-theory or relative smooth K-theory. But in the present paper, for simplicity, we will not discuss relative smooth cohomology theories.

2.1.2. — We define the smooth K-theory $\hat{K}(B)$ as the group completion of a quotient of a semigroup of isomorphism classes of cycles by an equivalence relation. We start with the description of the cycles.

Definition 2.1. — *Let B be a compact manifold, possibly with boundary. A cycle for a smooth K-theory class over B is a pair $(\mathcal{E}, \rho)$, where $\mathcal{E}$ is a geometric family, and $\rho \in \Omega(B)/\mathtt{im}(d)$ is a class of differential forms.*

2.1.3. — The notion of a geometric family has been introduced in [**19**] in order to have a short name for the data needed to define a Bismut super-connection [**7**, Prop. 10.15]. For the convenience of the reader we are going to explain this notion in some detail.

Definition 2.2. — *A geometric family over B consists of the following data:*

1. *a proper submersion with closed fibres $\pi \colon E \to B$,*
2. *a vertical Riemannian metric $g^{T^v\pi}$, i.e. a metric on the vertical bundle $T^v\pi \subset TE$, defined as $T^v\pi := \ker(d\pi \colon TE \to \pi^*TB)$.*
3. *a horizontal distribution $T^h\pi$, i.e. a bundle $T^h\pi \subseteq TE$ such that $T^h\pi \oplus T^v\pi = TE$.*
4. *a family of Dirac bundles $V \to E$,*
5. *an orientation of $T^v\pi$.*

Here, a family of Dirac bundles consists of

1. a hermitean vector bundle with connection (V, ∇^V, h^V) on E,
2. a Clifford multiplication $c \colon T^v\pi \otimes V \to V$,
3. on the components where $\dim(T^v\pi)$ has even dimension a $\mathbb{Z}/2\mathbb{Z}$-grading z.

We require that the restrictions of the family Dirac bundles to the fibres $E_b := \pi^{-1}(b)$, $b \in B$, give Dirac bundles in the usual sense (see [**19**, Def. 3.1]):

1. The vertical metric induces the Riemannian structure on E_b,
2. The Clifford multiplication turns $V_{|E_b}$ into a Clifford module (see [**7**, Def.3.32]) which is graded if $\dim(E_b)$ is even.

3. The restriction of the connection ∇^V to E_b is a Clifford connection (see [**7**, Def.3.39]).

A geometric family is called even or odd, if $\dim(T^v\pi)$ is even-dimensional or odd-dimensional, respectively.

2.1.4. — Here is a simple example of a geometric family with zero-dimensional fibres. Let $V \to B$ be a complex $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle. Assume that V comes with a hermitean metric h^V and a hermitean connection ∇^V which are compatible with the $\mathbb{Z}/2\mathbb{Z}$-grading. The geometric bundle (V, h^V, ∇^V) will usually be denoted by $\mathbf{V}$.

We consider the submersion $\pi := \mathtt{id}_B \colon B \to B$. In this case the vertical bundle is the zero-dimensional bundle which has a canonical vertical Riemannian metric $g^{T^v\pi} := 0$, and for the horizontal bundle we must take $T^h\pi := TB$. Furthermore, there is a canonical orientation of p. The geometric bundle $\mathbf{V}$ can naturally be interpreted as a family of Dirac bundles on $B \to B$. In this way $\mathbf{V}$ gives rise to a geometric family over B which we will usually denote by $\mathcal{V}$.

2.1.5. — In order to define a representative of the negative of the smooth K-theory class represented by a cycle $(\mathcal{E}, \rho)$ we introduce the notion of the opposite geometric family.

Definition 2.3. — *The opposite $\mathcal{E}^{\mathrm{op}}$ of a geometric family $\mathcal{E}$ is obtained by reversing the signs of the Clifford multiplication and the grading (in the even case) of the underlying family of Clifford bundles, and of the orientation of the vertical bundle.*

2.1.6. — Our smooth K-theory groups will be $\mathbb{Z}/2\mathbb{Z}$-graded. On the level of cycles the grading is reflected by the notions of even and odd cycles.

Definition 2.4. — *A cycle $(\mathcal{E}, \rho)$ is called even (or odd, resp.), if $\mathcal{E}$ is even (or odd, resp.) and $\rho \in \Omega^{\mathrm{odd}}(B)/\mathtt{im}(d)$ (or $\rho \in \Omega^{\mathrm{ev}}(B)/\mathtt{im}(d)$, resp.).*

2.1.7. — Let $\mathcal{E}$ and $\mathcal{E}'$ be two geometric families over B. An isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{E}'$ consists of the following data:

$$\begin{tikzcd} V \arrow[rr, "F"] \arrow[d] & & V' \arrow[d] \\ E \arrow[rr, "f"] \arrow[dr, "\pi"] & & E' \arrow[dl, "\pi'"'] \\ & B & \end{tikzcd}$$

where

1. f is a diffeomorphism over B,
2. F is a bundle isomorphism over f,

3. f preserves the horizontal distribution, the vertical metric and the orientation.
4. F preserves the connection, Clifford multiplication and the grading.

Definition 2.5. — *Two cycles $(\mathcal{E},\rho)$ and $(\mathcal{E}',\rho')$ are called isomorphic if $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic and $\rho=\rho'$. We let $G^*(B)$ denote the set of isomorphism classes of cycles over B of parity $*\in\{ev,odd\}$.*

2.1.8. — Given two geometric families $\mathcal{E}$ and $\mathcal{E}'$ we can form their sum $\mathcal{E}\sqcup_B\mathcal{E}'$ over B. The underlying proper submersion with closed fibres of the sum is $\pi\sqcup\pi'\colon E\sqcup E'\to B$. The remaining structures of $\mathcal{E}\sqcup_B\mathcal{E}'$ are induced in the obvious way.

Definition 2.6. — *The sum of two cycles $(\mathcal{E},\rho)$ and $(\mathcal{E}',\rho')$ is defined by*

$$(\mathcal{E},\rho)+(\mathcal{E}',\rho'):=(\mathcal{E}\sqcup_B\mathcal{E}',\rho+\rho').$$

The sum of cycles induces on $G^*(B)$ the structure of a graded abelian semigroup. The identity element of $G^*(B)$ is the cycle $0:=(\varnothing,0)$, where $\varnothing$ is the empty geometric family.

2.2. Relations

2.2.1. — In this subsection we introduce an equivalence relation $\sim$ on $G^*(B)$. We show that it is compatible with the semigroup structure so that we get a semigroup $G^*(B)/\sim$. We then define the smooth K-theory $\hat{K}^*(B)$ as the group completion of this quotient.

In order to define $\sim$ we first introduce a simpler relation "paired" which has a nice local index-theoretic meaning. The relation $\sim$ will be the equivalence relation generated by "paired".

2.2.2. — The main ingredients of our definition of "paired" are the notions of a taming of a geometric family $\mathcal{E}$ introduced in [**19**, Def. 4.4], and the η-form of a tamed family [**19**, Def. 4.16].

In this paragraph we shortly review the notion of a taming. For the definition of eta-forms we refer to [**19**, Sec. 4.4]. In the present paper we will use η-forms as a black box with a few important properties which we explicitly state at the appropriate places below.

If $\mathcal{E}$ is a geometric family over B, then we can form a family of Hilbert spaces $(H_b)_{b\in B}$, where $H_b:=L^2(E_b,V_{|E_b})$. If $\mathcal{E}$ is even, then this family is in addition $\mathbb{Z}/2\mathbb{Z}$-graded. The geometric family $\mathcal{E}$ gives rise to a family of Dirac operators $(D(\mathcal{E}_b))_{b\in B}$, where $D(\mathcal{E}_b)$ is an unbounded selfadjoint operator on H_b, which is odd in the even case.

A pre-taming of $\mathcal{E}$ is a family $(Q_b)_{b\in B}$ of selfadjoint operators $Q_b\in B(H_b)$ given by a smooth fibrewise integral kernel $Q\in C^\infty(E\times_B E,V\boxtimes V^*)$. In the even case we assume in addition that Q_b is odd, i.e. that it anticommutes with the grading z. The pre-taming is called a taming if $D(\mathcal{E}_b) + Q_b$ is invertible for all $b \in B$.

The family of Dirac operators $(D(\mathcal{E}_b))_{b\in B}$ has a K-theoretic index which we denote by $\mathtt{index}(\mathcal{E}) \in K(B)$. If the geometric family $\mathcal{E}$ admits a taming, then the associated family of Dirac operators operators admits an invertible compact perturbation, and hence $\mathtt{index}(\mathcal{E}) = 0$. Vice versa, if $\mathtt{index}(\mathcal{E}) = 0$ and the even part is empty or has a component with $\dim(T^v\pi) > 0$, then by [**19**, Lemma. 4.6] the geometric family admits a taming.

If the even part of $\mathcal{E}$ has zero-dimensional fibres, then the existence of a taming may require some stabilization. This means that we must add a geometric family $\mathcal{V} \sqcup_B \mathcal{V}^{\mathrm{op}}$ (see 2.1.4 and Definition 2.3), where $\mathbf{V}$ is the bundle $B \times \mathbb{C}^n \to B$ for sufficiently large n.

2.2.3.

Definition 2.7. — *A geometric family $\mathcal{E}$ together with a taming will be denoted by $\mathcal{E}_t$ and called a tamed geometric family.*

Let $\mathcal{E}_t$ be a taming of the geometric family $\mathcal{E}$ by the family $(Q_b)_{b\in B}$.

Definition 2.8. — *The opposite tamed family $\mathcal{E}_t^{\mathrm{op}}$ is given by the taming $(-Q_b)_{b\in B}$ of $\mathcal{E}^{\mathrm{op}}$.*

2.2.4. — The local index form $\Omega(\mathcal{E}) \in \Omega(B)$ is a differential form canonically associated to a geometric family. For a detailed definition we refer to [**19**, Def..4.8], but we can briefly formulate its construction as follows. The vertical metric $T^v\pi$ and the horizontal distribution $T^h\pi$ together induce a connection $\nabla^{T^v\pi}$ on $T^v\pi$ (see 3.1.3 for more details). Locally on E we can assume that $T^v\pi$ has a spin structure. We let $S(T^v\pi)$ be the associated spinor bundle. Then we can write the family of Dirac bundles V as $V = S \otimes W$ for a twisting bundle (W, h^W, ∇^W, z^W) with metric, metric connection, and $\mathbb{Z}/2\mathbb{Z}$-grading which is determined uniquely up to isomorphism. The form $\hat{A}(\nabla^{T^v\pi}) \wedge \mathbf{ch}(\nabla^W) \in \Omega(E)$ is globally defined, and we get the local index form by applying the integration over the fibre $\int_{E/B} \colon \Omega(E) \to \Omega(B)$:

$$\Omega(\mathcal{E}) := \int_{E/B} \hat{A}(\nabla^{T^v\pi}) \wedge \mathbf{ch}(\nabla^W).$$

The local index form is closed and represents a cohomology class $[\Omega(\mathcal{E})] \in H_{dR}(B)$. We let $\mathbf{ch}_{dR} \colon K(B) \to H_{dR}(B)$ be the composition

$$\mathbf{ch}_{dR} \colon K(B) \stackrel{\mathbf{ch}}{\to} H(B;\mathbb{Q}) \stackrel{can}{\to} H_{dR}(B).$$

The characteristic class version of the index theorem for families is

Theorem 2.9 **([3])**. — $\mathbf{ch}_{dR}(\mathtt{index}(\mathcal{E})) = [\Omega(\mathcal{E})]$.

A proof using methods of local index theory has been given by [**10**]. For a presentation of the proof we refer to [**7**]. An alternative proof can be obtained from [**19**, Thm.4.18] by specializing to the case of a family of closed manifolds.

2.2.5. — If a geometric family $\mathcal{E}$ admits a taming $\mathcal{E}_t$ (see Definition 2.7), then we have $\mathtt{index}(\mathcal{E}) = 0$. In particular, the local index form $\Omega(\mathcal{E})$ is exact. The important feature of local index theory in this case is that it provides an explicit form whose boundary is $\Omega(\mathcal{E})$ (see equation (6) below).

Let $\mathcal{E}_t$ be a tamed geometric family over B. In [**19**, Def. 4.16] we have defined the η-form $\eta(\mathcal{E}_t) \in \Omega(B)$. By [**19**, Theorem 4.13]) it satisfies

$$d\eta(\mathcal{E}_t) = \Omega(\mathcal{E}). \tag{6}$$

The first construction of η-forms has been given in [**12**], [**13**], [**14**] under the assumption that $\ker(D(\mathcal{E}_b))$ vanishes or has constant dimension. The variant which we use here has also been considered in [**37**], [**41**], [**40**].

Since the analytic details of the definition of the η-form $\eta(\mathcal{E}_t)$ are quite complicated we will not repeat them here but refer to [**19**, Def. 4.16]. For most of the present paper we can use the construction of the η-form as a black box refering to [**19**] for details of the construction and the proofs of properties. Exceptions are arguments involving adiabatic limits for which we use [**21**] as the reference.

2.2.6. — Now we can introduce the relations "paired" and $\sim$.

Definition 2.10. — *We call two cycles $(\mathcal{E}, \rho)$ and $(\mathcal{E}', \rho')$ paired if there exists a taming $(\mathcal{E} \sqcup_B \mathcal{E}'^{\mathrm{op}})_t$ such that*

$$\rho - \rho' = \eta((\mathcal{E} \sqcup_B \mathcal{E}'^{\mathrm{op}})_t).$$

We let $\sim$ denote the equivalence relation generated by the relation "paired".

Lemma 2.11. — *The relation "paired" is symmetric and reflexive.*

Proof. — In order to show that "paired" is reflexive and symmetric we are going to employ the relation [**19**, Lemma 4.12]

$$\eta(\mathcal{E}_t^{\mathrm{op}}) = -\eta(\mathcal{E}_t). \tag{7}$$

Let $\mathcal{E}$ be a geometric family over B, and let H_b denote the Hilbert space of sections of the Dirac bundle along the fibre over $b \in B$. The family $\mathcal{E} \sqcup_B \mathcal{E}^{\mathrm{op}}$ has an involution τ which flips the components, the signs of the Clifford multiplications, the grading and the orientations. We use the same symbol τ in order to denote the action of τ on the Hilbert space of sections of the Dirac bundle of $\mathcal{E}_b \sqcup_B \mathcal{E}_b^{\mathrm{op}}$. The latter can be identified with $H_b \oplus H_b^{\mathrm{op}}$, and in this picture $\tau = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Note that τ anticommutes with

$$D_b := D(\mathcal{E}_b \sqcup_B \mathcal{E}_b^{\mathrm{op}}) = \begin{pmatrix} D(\mathcal{E}_b) & 0 \\ 0 & -D(\mathcal{E}_b) \end{pmatrix}.$$

We choose an even, compactly supported smooth function $\chi\colon \mathbb{R} \to [0,\infty)$ such that $\chi(0) = 1$ and form $Q_b := \tau\chi(D_b)$. This operator also anticommutes with D_b, and $(D_b + Q_b)^2 = D_b^2 + \chi^2(D_b)$ is positive and therefore invertible for all $b \in B$. The family $(Q_b)_{b\in B}$ thus defines a taming $(\mathcal{E} \sqcup_B \mathcal{E}^{\mathrm{op}})_t$.

The involution $\sigma := \left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$ on the Hilbert space $H_b \oplus H_b^{\mathrm{op}}$ is induced by an isomorphism

$$(\mathcal{E} \sqcup_B \mathcal{E}^{\mathrm{op}})_t \cong (\mathcal{E} \sqcup_B \mathcal{E}^{\mathrm{op}})_t^{\mathrm{op}}.$$

Because of the relation (7) we have $\eta\left((\mathcal{E} \sqcup_B \mathcal{E}^{\mathrm{op}})_t\right) = 0$. It follows that $(\mathcal{E}, \rho)$ is paired with $(\mathcal{E}, \rho)$.

Assume now that $(\mathcal{E}, \rho)$ is paired with $(\mathcal{E}', \rho')$ via the taming $(\mathcal{E} \sqcup_B \mathcal{E}'^{\mathrm{op}})_t$ so that $\rho - \rho' = \eta\left((\mathcal{E} \sqcup_B \mathcal{E}'^{\mathrm{op}})_t\right)$. Then $(\mathcal{E} \sqcup_B \mathcal{E}'^{\mathrm{op}})_t^{\mathrm{op}}$ is a taming of $\mathcal{E}' \sqcup_B \mathcal{E}^{\mathrm{op}}$ such that $\rho' - \rho = \eta\left((\mathcal{E} \sqcup_B \mathcal{E}'^{\mathrm{op}})_t^{\mathrm{op}}\right)$, again by (7). It follows that $(\mathcal{E}', \rho')$ is paired with $(\mathcal{E}, \rho)$. □

Lemma 2.12. — *The relations "paired" and $\sim$ are compatible with the semigroup structure on $G^*(B)$.*

Proof. — In fact, if $(\mathcal{E}_i, \rho_i)$ are paired with $(\mathcal{E}_i', \rho_i')$ via tamings $(\mathcal{E}_i \sqcup_B \mathcal{E}_i'^{\mathrm{op}})_t$ for $i = 0, 1$, then $(\mathcal{E}_0, \rho_0) + (\mathcal{E}_0', \rho_0')$ is paired with $(\mathcal{E}_1, \rho_1) + (\mathcal{E}_1', \rho_1')$ via the taming

$$\left(\mathcal{E}_0 \sqcup_B \mathcal{E}_1 \sqcup_B (\mathcal{E}_0' \sqcup_B \mathcal{E}_1')^{\mathrm{op}}\right)_t := (\mathcal{E}_0 \sqcup_B \mathcal{E}_0'^{op})_t \sqcup_B (\mathcal{E}_1 \sqcup_B \mathcal{E}_1'^{\mathrm{op}})_t.$$

In this calculation we use the additivity of the η-form [**19**, Lemma 4.12]

$$\eta(\mathcal{E}_t \sqcup_B \mathcal{F}_t) = \eta(\mathcal{E}_t) + \eta(\mathcal{F}_t).$$

The compatibilty of $\sim$ with the sum follows from the compatibility of "paired". □

We get an induced semigroup structure on $G^*(B)/\sim$.

Lemma 2.13. — *If $(\mathcal{E}_0, \rho_0) \sim (\mathcal{E}_2, \rho_2)$, then there exists a cycle $(\mathcal{E}', \rho')$ such that $(\mathcal{E}_0, \rho_0) + (\mathcal{E}', \rho')$ is paired with $(\mathcal{E}_2, \rho_2) + (\mathcal{E}', \rho')$.*

Proof. — Let $(\mathcal{E}_0, \rho_0)$ be paired with $(\mathcal{E}_1, \rho_1)$ via a taming $(\mathcal{E}_0 \sqcup_B \mathcal{E}_1^{\mathrm{op}})_t$, and $(\mathcal{E}_1, \rho_1)$ be paired with $(\mathcal{E}_2, \rho_2)$ via $(\mathcal{E}_1 \sqcup_B \mathcal{E}_2^{\mathrm{op}})_t$. Then $(\mathcal{E}_0, \rho_0) + (\mathcal{E}_1, \rho_1)$ is paired with $(\mathcal{E}_2, \rho_2) + (\mathcal{E}_1, \rho_1)$ via the taming

$$\left((\mathcal{E}_0 \sqcup_B \mathcal{E}_1) \sqcup_B (\mathcal{E}_2 \sqcup_B \mathcal{E}_1)^{\mathrm{op}}\right)_t := (\mathcal{E}_0 \sqcup_B \mathcal{E}_1^{\mathrm{op}})_t \sqcup_B (\mathcal{E}_1 \sqcup_B \mathcal{E}_2^{\mathrm{op}})_t.$$

If $(\mathcal{E}_0, \rho_0) \sim (\mathcal{E}_2, \rho_2)$, then there is a chain $(\mathcal{E}_{1,\alpha}, \rho_{1,\alpha})$, $\alpha = 1, \ldots, r$ with $(\mathcal{E}_{1,1}, \rho_{1,1}) = (\mathcal{E}_0, \rho_0)$, $(\mathcal{E}_{1,r}, \rho_{1,r}) = (\mathcal{E}_2, \rho_2)$, such that $(\mathcal{E}_{1,\alpha}, \rho_{1,\alpha})$ is paired with $(\mathcal{E}_{1,\alpha+1}, \rho_{1,\alpha+1})$.

The assertion of the lemma follows from an $(r-1)$-fold application of the argument above. □

2.3. Smooth K-theory

2.3.1. — In this subsection we define the contravariant functor $B \to \hat{K}(B)$ from compact smooth manifolds to $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups. Recall the definition 2.6 of the semigroup of isomorphism classes of cycles. By Lemma 2.12 we can form the semigroup $G^*(B)/\sim$.

Definition 2.14. — *We define the smooth K-theory $\hat{K}^*(B)$ of B to be the group completion of the abelian semigroup $G^*(B)/\sim$.*

If $(\mathcal{E}, \rho)$ is a cycle, then let $[\mathcal{E}, \rho] \in \hat{K}^*(B)$ denote the corresponding class in smooth K-theory.

We now collect some simple facts which are helpful for computations in $\hat{K}(B)$ on the level of cycles.

Lemma 2.15. — *We have $[\mathcal{E}, \rho] + [\mathcal{E}^{\mathrm{op}}, -\rho] = 0$.*

Proof. — We show that $(\mathcal{E}, \rho) + (\mathcal{E}^{\mathrm{op}}, -\rho) = (\mathcal{E} \sqcup_B \mathcal{E}^{\mathrm{op}}, 0)$ is paired with $0 = (\varnothing, 0)$. In fact, this relation is given by the taming $((\mathcal{E} \sqcup_B \mathcal{E}^{\mathrm{op}}) \sqcup_B \varnothing^{\mathrm{op}})_t = (\mathcal{E} \sqcup \mathcal{E}^{\mathrm{op}})_t$ introduced in the proof of Lemma 2.11 with $\eta((\mathcal{E} \sqcup_B \mathcal{E}^{\mathrm{op}})_t) = 0$. □

Lemma 2.16. — *Every element of $\hat{K}^*(B)$ can be represented in the form $[\mathcal{E}, \rho]$.*

Proof. — An element of $\hat{K}^*(B)$ can be represented by a difference $[\mathcal{E}_0, \rho_0] - [\mathcal{E}_1, \rho_1]$. Using Lemma 2.15 we get $[\mathcal{E}_0, \rho_0] - [\mathcal{E}_1, \rho_1] = [\mathcal{E}_0, \rho_0] + [\mathcal{E}_1^{\mathrm{op}}, -\rho_1] = [\mathcal{E}_0 \sqcup_B \mathcal{E}_1^{\mathrm{op}}, \rho_0 - \rho_1]$. □

Lemma 2.17. — *If $[\mathcal{E}_0, \rho_0] = [\mathcal{E}_1, \rho_1]$, then there exists a cycle $(\mathcal{E}', \rho')$ such that $(\mathcal{E}_0, \rho_0) + (\mathcal{E}', \rho')$ is paired with $(\mathcal{E}_1, \rho_1) + (\mathcal{E}', \rho')$.*

Proof. — The relation $[\mathcal{E}_0, \rho_0] = [\mathcal{E}_1, \rho_1]$ implies that there exists a cycle $(\tilde{\mathcal{E}}, \tilde{\rho})$ such that $(\mathcal{E}_0, \rho_0) + (\tilde{\mathcal{E}}, \rho) \sim (\mathcal{E}_1, \rho_1) + (\tilde{\mathcal{E}}, \tilde{\rho})$. The assertion now follows from Lemma 2.13. □

2.3.2. — In this paragraph we extend $B \mapsto \hat{K}^*(B)$ to a contravariant functor from smooth manifolds to $\mathbb{Z}/2\mathbb{Z}$-graded groups. Let $f\colon B_1 \to B_2$ be a smooth map. Then we have to define a map $f^*\colon \hat{K}^*(B_2) \to \hat{K}(B_1)$. We will first define a map of abelian semigroups $f^*\colon G^*(B_2) \to G^*(B_1)$, and then we show that it passes to $\hat{K}$.

If $\mathcal{E}$ is a geometric family over B_2, then we can define an induced geometric family $f^*\mathcal{E}$ over B_1. The underlying submersion and vector bundle of $f^*\mathcal{E}$ are given by the

Cartesian diagram

$$\begin{tikzcd} f^*V \arrow[r] \arrow[d] & V \arrow[d] \\ f^*E \arrow[r, "F"] \arrow[d, "f^*\pi"] & E \arrow[d, "\pi"] \\ B_1 \arrow[r, "f"] & B_2. \end{tikzcd}$$

The metric $g^{T^v f^*\pi}$ and the orientation of $T^v f^*\pi$ are defined such that $dF\colon T^v f^*\pi \to F^*T^v\pi$ is an isometry and orientation preserving. The horizontal distribution $T^h f^*\pi$ is given by the condition that $dF(T^h f^*\pi) \subseteq F^*T^h\pi$. Finally, the Dirac bundle structure of f^*V is induced from the Dirac bundle structure on V in the usual way. For $b_2 \in B_2$ let H_{b_2} be the Hilbert space of sections of V along the fibre E_{b_2}. If $b_1 \in B_1$ satisfies $f(b_1) = b_2$, then we can identify the Hilbert space of sections of f^*V along the fibre $f^*E_{b_1}$ canonically with H_{b_2}. If $(Q_{b_2})_{b_2 \in B_2}$ defines a taming $\mathcal{E}_t$ of $\mathcal{E}$, then the family $(Q_{f(b_1)})_{b_1 \in B}$ is a taming $f^*\mathcal{E}_t$ of $f^*\mathcal{E}$. We have the following relation of η-forms:

$$\eta(f^*\mathcal{E}_t) = f^*\eta(\mathcal{E}_t). \tag{8}$$

In order to see this note the following facts. The geometric family $\mathcal{E}$ gives rise to a bundle of Hilbert spaces $H(\mathcal{E}) \to B_2$ with fibres $H(\mathcal{E})_{b_2} = H_{b_2}$, using the notation introduced above. We have a natural isomorphism $H(f^*\mathcal{E}) \cong f^*H(\mathcal{E})$. The geometry of $\mathcal{E}$ together with the taming induces a family of super-connections $A_s(\mathcal{E}_t)$ on H parametrized by $s \in (0, \infty)$ (see [**19**, 4.4.4] for explicit formulas). By construction we have $f^*A_s(\mathcal{E}_t) = A_s(f^*\mathcal{E}_t)$. The η-form $\eta(\mathcal{E}_t)$ is defined as an integral of the trace of a family of operators on $H(\mathcal{E})$ (with differential form coefficients) built from $\partial_s A_s(\mathcal{E}_t)$ and $A_s(\mathcal{E})^2$ [**19**, Definition 4.16]. Equation (8) now follows from $f^*\partial_s A_s(\mathcal{E}_t) = \partial_s A_s(f^*\mathcal{E}_t)$ and $f^*A_s(\mathcal{E})^2 = A_s(f^*\mathcal{E}_t)^2$.

If $(\mathcal{E}, \rho) \in G(B_2)$ then we define $f^*(\mathcal{E}, \rho) := (f^*\mathcal{E}, f^*\rho) \in G(B_2)$. The pull-back preserves the disjoint union and opposites of geometric families. In particular, f^* is a semigroup homomorphism. Assume now that $(\mathcal{E}, \rho)$ is paired with $(\mathcal{E}', \rho')$ via the taming $(\mathcal{E} \sqcup_{B_2} \mathcal{E}'^{\mathrm{op}})_t$. Then we can pull back the taming as well and get a taming $f^*(\mathcal{E} \sqcup_{B_2} \mathcal{E}'^{\mathrm{op}})_t$ of $f^*\mathcal{E} \sqcup_{B_1} f^*\mathcal{E}'^{\mathrm{op}}$. Equation (8) now implies that $f^*(\mathcal{E}, \rho)$ is paired with $f^*(\mathcal{E}', \rho')$ via the taming $f^*(\mathcal{E} \sqcup_{B_2} \mathcal{E}'^{\mathrm{op}})_t$.

Hence, the pull-back f^* passes to $G^*(B)/\sim$, and being a semigroup homomorphism, it induces a map of group completions

$$f^*\colon \hat{K}^*(B_2) \to \hat{K}^*(B_1).$$

Evidently, $(\mathtt{id}_B)^* = \hat{\mathtt{id}}_{\hat{K}^*(B)}$. Let $f'\colon B_0 \to B_1$ be another smooth map. If $\mathcal{E}$ is a geometric family over B_2, then $(f \circ f')^*\mathcal{E}$ is isomorphic to $f'^*f^*\mathcal{E}$. This observation implies that

$$f'^* f^* = (f \circ f')^* \colon \hat{K}^*(B_2) \to \hat{K}(B_0).$$

This finishes the construction of the contravariant functor $\hat{K}^*$ on the level of morphisms.

2.4. Natural transformations and exact sequences

2.4.1. — In this subsection we introduce the transformations R, I, a, and we show that they turn the functor $\hat{K}$ into a smooth extension of $(K, \mathbf{ch}_{\mathbb{R}})$ in the sense of Definition 1.1.

2.4.2. — We first define the natural transformation

$$I \colon \hat{K}(B) \to K(B); \quad [\mathcal{E}, \rho] \mapsto \mathtt{index}(\mathcal{E}).$$

We must show that I is well-defined. Consider $\tilde{I} \colon G(B) \to K(B)$ defined by $\tilde{I}(\mathcal{E}, \rho) := \mathtt{index}(\mathcal{E})$. If $(\mathcal{E}, \rho)$ is paired with $(\mathcal{E}', \rho')$, then the existence of a taming $(\mathcal{E} \sqcup_B \mathcal{E}'^{\mathrm{op}})_t$ implies that $\mathtt{index}(\mathcal{E}) = \mathtt{index}(\mathcal{E}')$. The relation

$$\mathtt{index}(\mathcal{E} \sqcup_B \mathcal{E}') = \mathtt{index}(\mathcal{E}) + \mathtt{index}(\mathcal{E}') \tag{9}$$

together with Lemma 2.13 now implies that $\tilde{I}$ descends to $G(B)/\sim$. The additivity (9) and the definition of $\hat{K}(B)$ as the group completion of $G(B)/\sim$ implies that $\tilde{I}$ further descends to the homomorphism $I \colon \hat{K}(B) \to K(B)$.

The relation $\mathtt{index}(f^*\mathcal{E}) = f^*\mathtt{index}(\mathcal{E})$ shows that I is a natural transformation of functors from smooth manifolds to $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups.

2.4.3.

Lemma 2.18. — *For every compact manifold B, the transformation $I \colon \hat{K}(B) \to K(B)$ is surjective.*

Proof. — We discuss even and odd degrees seperately. In the even case, a K-theory class $\xi \in K(B)$ is represented by a $\mathbb{Z}/2Z$-graded vector bundle V on B. Simply choose a hermitean metric and a connection on V. We obtain a resulting geometric family $\mathbf{V}$ on B, with underlying submersion $\mathtt{id} \colon B \to B$ (i.e. 0-dimensional fibres) as in 2.1.4, and clearly $I(\mathbf{V}) = \mathtt{index}(\mathbf{V}) = [V] = \xi \in K^0(B)$.

For odd degrees, the statement is proved in [**19**, 3.1.6.7]. □

2.4.4. — We consider the functor $B \mapsto \Omega^*(B)/\mathtt{im}(d)$, $* \in \{ev, odd\}$ as a functor from manifolds to $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups. We construct a parity-reversing natural transformation

$$a \colon \Omega^*(B)/\mathtt{im}(d) \to \hat{K}^*(B); \qquad \rho \mapsto [\varnothing, -\rho].$$

2.4.5. — Let $\Omega^*_{d=0}(B)$ be the group of closed forms of parity $*$ on B. Again we consider $B \mapsto \Omega^*_{d=0}(B)$ as a functor from smooth manifolds to $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups. We define a natural transformation

$$R\colon \hat{K}(B) \to \Omega_{d=0}(B); \qquad [\mathcal{E}, \rho] \mapsto \Omega(\mathcal{E}) - d\rho.$$

Again we must show that R is well-defined. We will use the relation (6) of the η-form and the local index form, and the obvious properties of local index forms

$$\Omega(\mathcal{E} \sqcup_B \mathcal{E}') = \Omega(\mathcal{E}) + \Omega(\mathcal{E}'), \quad \Omega(\mathcal{E}^{\mathrm{op}}) = -\Omega(\mathcal{E}).$$

We start with

$$\tilde{R}\colon G(B) \to \Omega(B); \quad (\mathcal{E}, \rho) \mapsto \Omega(\mathcal{E}) - d\rho.$$

Since $\Omega(\mathcal{E})$ is closed, $\tilde{R}(\mathcal{E}, \rho)$ is closed. If $(\mathcal{E}, \rho)$ is paired with $(\mathcal{E}', \rho')$ via the taming $(\mathcal{E} \sqcup_B \mathcal{E}'^{\mathrm{op}})_t$, then $\rho - \rho' = \eta((\mathcal{E} \sqcup_B \mathcal{E}'^{\mathrm{op}})_t)$. It follows

$$\begin{aligned} R(\mathcal{E}, \rho) = \Omega(\mathcal{E}) - d\rho &= \Omega(\mathcal{E}) - d\rho' - d\eta((\mathcal{E} \sqcup_B \mathcal{E}'^{\mathrm{op}})_t) \\ &= \Omega(\mathcal{E}) - d\rho' - \Omega(\mathcal{E}) - \Omega(\mathcal{E}'^{\mathrm{op}}) = \Omega(\mathcal{E}') - d\rho' = R(\mathcal{E}', \rho'). \end{aligned}$$

Since $\tilde{R}$ is additive it descends to $G(B)/\sim$ and finally to the map $R\colon \hat{K}(B) \to \Omega_{d=0}(B)$. It follows from $\Omega(f^*\mathcal{E}) = f^*\Omega(\mathcal{E})$ that R is a natural transformation.

2.4.6. — The natural transformations satisfy the following relations:

Lemma 2.19. — $R \circ a = d$, $\mathbf{ch}_{dR} \circ I = [\dots] \circ R$.

Proof. — The first relation is an immediate consequence of the definition of R and a. The second relation is the local index theorem 2.9. □

2.4.7. — Via the embedding $H_{dR}(B) \subseteq \Omega(B)/\mathtt{im}(d)$, the Chern character $\mathbf{ch}_{dR}\colon K(B) \to H_{dR}(B)$ can be considered as a natural transformation

$$\mathbf{ch}_{dR}\colon K(B) \to \Omega(B)/\mathtt{im}(d).$$

Proposition 2.20. — *The following sequence is exact:*

$$K(B) \xrightarrow{\mathbf{ch}_{dR}} \Omega(B)/\mathtt{im}(d) \xrightarrow{a} \hat{K}(B) \xrightarrow{I} K(B) \to 0.$$

We give the proof in the following couple of subsection.

2.4.8. — We start with the surjectivity of $I\colon \hat{K}(B) \to K(B)$. The main point is the fact that every element $x \in K(B)$ can be realized as the index of a family of Dirac operators by Lemma 2.18. So let $x \in K(B)$ and $\mathcal{E}$ be a geometric family with $\mathtt{index}(\mathcal{E}) = x$. Then we have $I([\mathcal{E}, 0]) = x$.

2.4.9. — Next we show exactness at $\hat{K}(B)$. For $\rho \in \Omega(B)/\mathtt{im}(d)$ we have $I \circ a(\rho) = I([\emptyset, -\rho]) = \mathtt{index}(\emptyset) = 0$, hence $I \circ a = 0$. Consider a class $[\mathcal{E}, \rho] \in \hat{K}(B)$ which satisfies $I([\mathcal{E}, \rho]) = 0$. We can assume that the fibres of the underlying submersion are not zero-dimensional. Indeed, if necessary, we can replace $\mathcal{E}$ by $\mathcal{E} \sqcup_B (\tilde{\mathcal{E}} \sqcup_B \tilde{\mathcal{E}}^{\mathrm{op}})$ for some even family with nonzero-dimensional fibres without changing the smooth K-theory class by Lemma 2.15. Since $\mathtt{index}(\mathcal{E}) = 0$ this family admits a taming $\mathcal{E}_t$ (2.2.2). Therefore, $(\mathcal{E}, \rho)$ is paired with $(\emptyset, \rho - \eta(\mathcal{E}_t))$. It follows that $[\mathcal{E}, \rho] = a(\eta(\mathcal{E}_t) - \rho)$.

2.4.10. — In order to prepare the proof of exactness at $\Omega(B)/\mathtt{im}(d)$ in 2.4.11 we need some facts about the classification of tamings of a geometric family $\mathcal{E}$. The main idea is to measure the difference between tamings of $\mathcal{E}$ using a local index theorem for $\mathcal{E} \times [0,1]$ (compare [**19**, Cor. 2.2.19]). Let us assume that the underlying submersion $\pi \colon E \to B$ decomposes as $E = E^{\mathrm{ev}} \sqcup_B E^{\mathrm{odd}}$ such that the restriction of π to the even and odd parts is surjective with nonzero- and even-dimensional and odd-dimensional fibres, and which is such that the Clifford bundle is nowhere zero-dimensional. If $\mathtt{index}(\mathcal{E}) = 0$, then there exists a taming $\mathcal{E}_t$ (see 2.2.2). Assume that $\mathcal{E}_{t'}$ is a second taming. Both tamings together induce a boundary taming of the family with boundary $(\mathcal{E} \times [0,1])_{bt}$. In [**19**] we have discussed in detail geometric families with boundaries and the operation of taking a boundary of a geometric family with boundary. In the present case $\mathcal{E} \times [0,1]$ has two boundary faces labeled by the endpoints $\{0,1\}$ of the interval. We have $\partial_0(\mathcal{E} \times [0,1]) \cong \mathcal{E}$ and $\partial_1(\mathcal{E} \times [0,1]) \cong \mathcal{E}^{\mathrm{op}}$. A boundary taming $(\mathcal{E} \times [0,1])_{bt}$ is given by tamings of $\partial_i(\mathcal{E} \times [0,1])$ for $i = 0,1$ (see [**19**, Def. 2.1.48]). We use $\mathcal{E}_t$ at $\mathcal{E} \times \{0\}$ and $\mathcal{E}^{\mathrm{op}}_{t'}$ at $\mathcal{E} \times \{1\}$.

The boundary tamed family has an index $\mathtt{index}((\mathcal{E} \times [0,1])_{bt}) \in K(B)$ which is the obstruction against extending the boundary taming to a taming [**19**, Lemma 2.2.6]. The construction of the local index form extends to geometric families with boundaries. Because of the geometric product structure of $\mathcal{E} \times [0,1]$ we have $\Omega(\mathcal{E} \times [0,1]) = 0$. The index theorem for boundary tamed families [**19**, Theorem 2.2.18] gives

$$\mathbf{ch}_{dR} \circ \mathtt{index}((\mathcal{E} \times [0,1])_{bt}) = [\eta(\mathcal{E}_t) - \eta(\mathcal{E}_{t'})].$$

On the other hand, given $x \in K(B)$ and $\mathcal{E}_t$, since we have chosen our family $\mathcal{E}$ sufficiently big, there exists a taming $\mathcal{E}_{t'}$ such that $\mathtt{index}((\mathcal{E} \times [0,1])_{bt}) = x$.

To prove this, we argue as follows. Given tamings $\mathcal{E}_t$ and $\mathcal{E}_{t'}$ we obtain a family $D(\mathcal{E}_t, \mathcal{E}_{t'})$ of perturbed Dirac operators over $B \times \mathbb{R}$ which restricts to $D(\mathcal{E}_t)$ on $B \times \{\beta\}$ for $\beta < 0$, and to $D(\mathcal{E}_{t'})$ for $\beta \geq 1$, and which interpolates these families for $\beta \in [0,1]$. Since the restriction of $D(\mathcal{E}_t, \mathcal{E}_{t'})$ is invertible outside of a compact subset of $B \times \mathbb{R}$ (note that B is compact) it gives rise to a class $[\mathcal{E}_t, \mathcal{E}_{t'}] \in KK(\mathbb{C}, C(B) \otimes C_0(\mathbb{R}))$. The Dirac operator on $\mathbb{R}$ provides a class $[\partial] \in KK(C_0(\mathbb{R}), \mathbb{C})$, and one checks —using the method of connections as in [**17**, proof of Proposition 2.11] or directly working with the unbounded picture [**4**]— that $D(\mathcal{E}\times[0,1])_{bt}$ represents the Kasparov product

$$[\mathcal{E}_t,\mathcal{E}_{t'}]\otimes_{C_0(\mathbb{R})}[\partial]\in KK(\mathbb{C},C(B)).$$

The map

$$K_c(B\times\mathbb{R})\stackrel{\sim}{\to}KK(\mathbb{C},C(B)\otimes C_0(\mathbb{R}))\stackrel{\cdot\otimes_{C_0(\mathbb{R})}[\partial]}{\to}KK(\mathbb{C},C(B))\stackrel{\sim}{\to}K(B)$$

is by [**34**, Paragraph 5, Theorem 7] the inverse of the suspension isomorphism, so in particular surjective. It remains to see that one can exhaust $KK(\mathbb{C},C(B)\otimes C_0(\mathbb{R}))$ with classes of the form $[\mathcal{E}_t,\mathcal{E}_{t'}]$ by varying the taming $\mathcal{E}_{t'}$.

We sketch an argument in the even-dimensional case. The odd-dimensional case is similar. For a separable infinite-dimensional Hilbert space H let $GL_1(H)\subset GL(H)$ be the group of invertible operators of the form $1+K$ with $K\in K(H)$ compact. The space $GL_1(H)$ has the homotopy type of the classifying space for K^1. The bundle of Hilbert spaces $H(\mathcal{E})^+\to B$ gives rise to a (canonically trivial, up to homotopy) bundle of groups $GL_1(H(\mathcal{E})^+)\to B$ by taking $GL_1(\dots)$ fibrewise (it is here where we use that the family is sufficiently big so that $H(\mathcal{E})^+$ is infinite-dimensional). Let $\Gamma(GL_1(H(\mathcal{E})^+))$ be the topological group of sections. Then we have an isomorphism $\pi_0\Gamma(GL_1(H(\mathcal{E})^+))\cong K^1(B)$. Let $x\in K^1(B)$ be represented by a section $s\in\Gamma(GL_1(H(\mathcal{E})^+))$. We can approximate $s-1$ by a smooth family of smoothing operators. Therefore we can assume that $s-1$ is given by a smooth fibrewise integral kernel (a pretaming in the language of [**19**])[(3)].

There is a bijection between tamings $\mathcal{E}_{t'}$ and sections $s\in\Gamma(GL_1(H(\mathcal{E})^+))$ of this type which maps $\mathcal{E}_{t'}$ to $s:=D^+(\mathcal{E}_t)^{-1}D^+(\mathcal{E}_{t'})$. The map which associates the KK-class $[\mathcal{E}_t,\mathcal{E}_{t'}]$ to the section s is just one realization of the suspension isomorphism $K^1(B)\to K^0_c(B\times\mathbb{R})$ (using the Kasparov picture of the latter group). In particular we see that all classes in $K^0_c(B\times\mathbb{R})$ arise as $[\mathcal{E}_t,\mathcal{E}_{t'}]$ for various tamings $\mathcal{E}_{t'}$.

2.4.11. — We now show exactness at $\Omega(B)/\mathtt{im}(d)$. Let $x\in K(B)$. Then we have $a\circ\mathbf{ch}_{dR}(x)=[\varnothing,-\mathbf{ch}_{dR}(x)]$. We choose a geometric family $\mathcal{E}$ as in 2.4.10 and set $\tilde{\mathcal{E}}:=\mathcal{E}\sqcup_B\mathcal{E}^{\mathrm{op}}$. In the proof of Lemma 2.11 we have constructed a taming $\tilde{\mathcal{E}}_t$ such that $\eta(\tilde{\mathcal{E}}_t)=0$. Using the discussion 2.4.10 we choose a second taming $\tilde{\mathcal{E}}_{t'}$ such that $\mathtt{index}((\tilde{\mathcal{E}}\times[0,1])_{bt})=-x$, hence $\eta(\tilde{\mathcal{E}}_{t'})=\mathbf{ch}_{dR}(x)$. By the taming $\tilde{\mathcal{E}}_{t'}$ we see that the cycle $(\tilde{\mathcal{E}},0)$ pairs with $(\varnothing,-\mathbf{ch}_{dR}(x))$. On the other hand, via $\tilde{\mathcal{E}}_t$ the cycle $(\tilde{\mathcal{E}},0)$ pairs with 0. It follows that $(\varnothing,-\mathbf{ch}_{dR}(x))\sim 0$ and hence $a\circ\mathbf{ch}_{dR}=0$.

Let now $\rho\in\Omega(B)/\mathtt{im}(d)$ be such that $a(\rho)=[\varnothing,-\rho]=0$. Then by Lemma 2.17 there exists a cycle $(\hat{\mathcal{E}},\hat{\rho})$ such that $(\hat{\mathcal{E}},\hat{\rho}-\rho)$ pairs with $(\hat{\mathcal{E}},\hat{\rho})$. Therefore there exists a taming $\mathcal{E}_{t'}$ of $\mathcal{E}:=\hat{\mathcal{E}}\sqcup_B\hat{\mathcal{E}}^{\mathrm{op}}$ such that $\eta(\mathcal{E}_{t'})=-\rho$.

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(3) Alternatively one can directly produce such a section using the setup described in [**42**].

Let $\mathcal{E}_t$ be the taming with vanishing η-form constructed in the proof of Lemma 2.11. The two tamings induce a boundary taming $(\mathcal{E}\times[0,1])_{bt}$ such that $\mathbf{ch}_{dR}\circ\mathtt{index}((\mathcal{E}\times[0,1])_{bt}) = -\eta(\mathcal{E}_{t'}) = \rho$. This shows that ρ is in the image of $\mathbf{ch}_{dR}$. □

2.4.12. — We now improve Lemma 2.13. This result will be very helpful in verifying well-definedness of maps out of smooth K-theory, e.g. the smooth Chern character.

Lemma 2.21. — *If $[\mathcal{E}_0,\rho_0]=[\mathcal{E}_1,\rho_1]$ and at least one of these families has a higher-dimensional component, then $(\mathcal{E}_0,\rho_0)$ is paired with $(\mathcal{E}_1,\rho_1)$.*

Proof. — By Lemma 2.13 there exists $[\mathcal{E}',\rho']$ such that $(\mathcal{E}_0,\rho_0)+(\mathcal{E}',\rho')$ is paired with $(\mathcal{E}_1,\rho_1)+(\mathcal{E}',\rho')$ by a taming $\left(\mathcal{E}_0\sqcup_B\mathcal{E}'\sqcup_B(\mathcal{E}_1\sqcup_B\mathcal{E}')^{\mathrm{op}}\right)_t$. We have

$$\rho_1-\rho_0=\eta\left((\mathcal{E}_0\sqcup_B\mathcal{E}'\sqcup_B(\mathcal{E}_1\sqcup_B\mathcal{E}')^{\mathrm{op}})_t\right).$$

Since $\mathtt{index}(\mathcal{E}_0)=\mathtt{index}(\mathcal{E}_1)$ there exists a taming $(\mathcal{E}_0\sqcup_B\mathcal{E}_1^{\mathrm{op}})_t$. Furthermore, there exists a taming $(\mathcal{E}'\sqcup_B\mathcal{E}'^{\mathrm{op}})_t$ with vanishing η-invariant (see the proof of Lemma 2.11). These two tamings combine to a taming $\left(\mathcal{E}_0\sqcup_B\mathcal{E}'\sqcup_B(\mathcal{E}_1\sqcup_B\mathcal{E}')^{\mathrm{op}}\right)_{t'}$. There exists $\xi\in K(B)$ such that

$$\begin{aligned}\mathbf{ch}_{dR}(\xi) &= \eta\left((\mathcal{E}_0\sqcup_B\mathcal{E}'\sqcup_B(\mathcal{E}_1\sqcup_B\mathcal{E}')^{\mathrm{op}})_t\right)-\eta\left((\mathcal{E}_0\sqcup_B\mathcal{E}'\sqcup_B(\mathcal{E}_1\sqcup_B\mathcal{E}')^{\mathrm{op}})_{t'}\right)\\ &= \eta\left((\mathcal{E}_0\sqcup_B\mathcal{E}'\sqcup_B(\mathcal{E}_1\sqcup_B\mathcal{E}')^{\mathrm{op}})_t\right)-\eta\left((\mathcal{E}_0\sqcup_B\mathcal{E}_1^{\mathrm{op}})_t\right).\end{aligned}$$

We can now adjust (using 2.4.10) the taming $(\mathcal{E}_0\sqcup_B\mathcal{E}_1^{\mathrm{op}})_t$ such that we can choose $\xi=0$. It follows that $\rho_1-\rho_0=\eta\left((\mathcal{E}_0\sqcup_B\mathcal{E}_1^{\mathrm{op}})_t\right)$. □

2.5. Comparison with the Hopkins-Singer theory and the flat theory

2.5.1. — An important consequence of the axioms 1.1 for a smooth generalized cohomology theory is the homotopy formula. Let $\hat{h}$ be a smooth extension of a pair (h,c). Let $x\in\hat{h}([0,1]\times B)$, and let $i_k\colon B\to\{k\}\times B\subset[0,1]\times B$, $k=0,1$, be the inclusions.

Lemma 2.22. — *We have*

$$i_1^*(x)-i_0^*(x)=a\left(\int_{[0,1]\times B/B}R(x)\right).$$

Proof. — Let $p\colon[0,1]\times B\to B$ denote the projection. If $x=p^*y$, then on the one hand the left-hand side of the equation is zero. On the other hand, $R(x)=p^*R(y)$ so that $\int_{[0,1]\times B/B}R(x)=0$, too.

Since p is a homotopy equivalence there exists $\bar{y}\in h(B)$ such that $I(x)=p^*(\bar{y})$. Because of the surjectivity of I we can choose $y\in\hat{h}(B)$ such that $I(y)=\bar{y}$. It follows that $I(x-p^*y)=0$. By the exactness of (3) there exists a form $\omega\in\Omega(I\times B)/\mathtt{im}(d)$

such that $x - p^*y = a(\omega)$. By Stokes' theorem we have the equality $i_1^*\omega - i_0^*\omega = \int_{[0,1]\times B/B} d\omega$ in $\Omega(B)/\mathtt{im}(d)$. By (2) we have $d\omega = R(a(\omega))$. It follows that

$$\int_{[0,1]\times B/B} d\omega = \int_{[0,1]\times B/B} R(a(\omega)) = \int_{[0,1]\times B/B} R(x - p^*y) = \int_{[0,1]\times B/B} R(x).$$

This implies

$$i_1^*x - i_0^*x = i_1^*a(\omega) - i_0^*a(\omega) = a\left(i_1^*\omega - i_0^*\omega\right) = a(\int_{[0,1]\times B/B} R(x)\Big) . \qquad \square$$

2.5.2. — Let $\hat{h}$ be a smooth extension of a pair (h, c). We use the notation introduced in 1.2.2.

Definition 2.23. — *The associated flat functor is defined by*

$$B \mapsto \hat{h}_{\mathrm{flat}}(B) := \ker\{R\colon \hat{h}(B) \to \Omega_{d=0}(B, N)\}.$$

Recall that a functor F from smooth manifolds is homotopy invariant, if for the two embeddings $i_k\colon B \to \{k\} \times B \to [0,1] \times B$, $k = 0, 1$, we have $F(i_0) = F(i_1)$. As a consequence of the homotopy formula Lemma 2.22 the functor $\hat{h}_{\mathrm{flat}}$ is homotopy invariant.

In interesting cases it is part of a generalized cohomology theory. The map $c\colon h \to HN$ gives rise to a cofibre sequence in the stable homotopy category $h \xrightarrow{c} HN \to h_{N,\mathbb{R}/\mathbb{Z}}$ which defines a spectrum $h_{N,\mathbb{R}/\mathbb{Z}}$.

Proposition 2.24. — *If $\hat{h}$ is the Hopkins-Singer extension of (h, c), then we have a natural isomorphism*

$$\hat{h}_{\mathrm{flat}}(B) \cong h_{N,\mathbb{R}/\mathbb{Z}}(B)[-1].$$

In the special case that $N = h^* \otimes_{\mathbb{Z}} \mathbb{R}$ this is [**31**, (4.57)].

2.5.3. — In the case of K-theory and the Chern character $\mathbf{ch}_{\mathbb{R}}\colon K \to H(K^* \otimes_{\mathbb{Z}} \mathbb{R})$ one usually writes $K\mathbb{R}/\mathbb{Z} := h_{K^*\otimes_{\mathbb{Z}}\mathbb{R},\mathbb{R}/\mathbb{Z}}$. The functor $B \mapsto K\mathbb{R}/\mathbb{Z}(B)$ is called $\mathbb{R}/\mathbb{Z}$-K-theory. Since $\mathbb{R}/\mathbb{Z}$ is an injective abelian group we have a universal coefficient formula

$$K\mathbb{R}/\mathbb{Z}^*(B) \cong \mathtt{Hom}(K_*(B), \mathbb{R}/\mathbb{Z}), \tag{10}$$

where $K_*(B)$ denotes the K-homology of B. A geometric interpretation of $\mathbb{R}/\mathbb{Z}$-K-theory was first proposed in [**32**], [**33**]. In these references it was called multiplicative K-theory. The analytic construction of the push-forward has been given in [**37**].

2.5.4.

Proposition 2.25. — *There is a natural isomorphism of functors*

$$\hat{K}_{\mathrm{flat}}(B) \cong K\mathbb{R}/\mathbb{Z}(B)[-1].$$

Proof. — In the following (the paragraphs 2.5.5, 2.5.6) we sketch two conceptually very different arguments. For details we refer to [**22**, Section 5, Section 7].

2.5.5. — In the first step one extends $\hat{K}_{\mathrm{flat}}$ to a reduced cohomology theory on smooth manifolds. The reduced group of a pointed manifold is defined as the kernel of the restriction to the point. The missing structure is a suspension isomorphism. It is induced by the map $\hat{K}(B) \to \hat{K}(S^1 \times B)$ given by $x \mapsto \mathtt{pr}_1^* x_{S^1} \cup \mathtt{pr}_2^* x$, where $x_{S^1} \in \hat{K}^1(S^1)$ is defined in Definition 5.6, and the $\cup$-product is defined below in 4.1. The inverse is induced by the push-forward $(\hat{\mathtt{pr}}_2)_! \colon \hat{K}(S^1 \times B) \to \hat{K}(B)$ along $\mathtt{pr}_2 \colon S^1 \times B \to B$ introduced below in 3.18. Finally one verifies the exactness of mapping cone sequences.

In order to identify the resulting reduced cohomology theory with $\mathbb{R}/\mathbb{Z}$-K-theory one constructs a pairing between $\hat{K}_{\mathrm{flat}}$ and K-homology, using an analytic model as in [**37**]. This pairing, in view of the universal coefficient formula (10) gives a map of cohomology theories $\hat{K}_{\mathrm{flat}}(B) \to K\mathbb{R}/\mathbb{Z}(B)[-1]$ which is an isomorphism by comparison of coefficients.

2.5.6. — The second argument is based on the comparison with the Hopkins-Singer theory. We let $B \mapsto \hat{K}_{HS}(B)$ denote the version of the smooth K-theory functor defined by Hopkins-Singer [**31**]. In [**22**, Section 5] we show that there is a unique natural isomorphism $\hat{K}^{\mathrm{ev}} \xrightarrow{\sim} \hat{K}^{\mathrm{ev}}_{HS}$. In view of 2.24 we get the isomorphism

$$\hat{K}^{\mathrm{ev}}_{\mathrm{flat}}(B) \xrightarrow{\sim} \hat{K}^{\mathrm{ev}}_{HS,\mathrm{flat}}(B) \xrightarrow{\sim} K\mathbb{R}/\mathbb{Z}^{\mathrm{ev}}[-1](B).$$

In [**22**] we furthermore show that using the integration for $\hat{K}$ and the suspension isomorphism for $K\mathbb{R}/\mathbb{Z}$ this isomorphism extends to the odd parts. □

2.5.7. — Many of the interesting examples given in Section 5 can be understood (at least to a large extend) already at this stage. We recommend to look them up now, if one is less interested in structural questions. This should also serve as a motivation for the constructions in Sections 3 and 4.

3. Push-forward

3.1. K-orientation

3.1.1. — The groups $Spin(n)$ and $Spin^c(n)$ fit into exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d] & Spin(n) \arrow[r] \arrow[d] & SO(n) \arrow[r] \arrow[d, "\mathtt{id}"] & 1 \\
1 \arrow[r] & U(1) \arrow[r, "i"] & Spin^c(n) \arrow[r, "\pi"] & SO(n) \arrow[r] & 1
\end{tikzcd}$$

$$1 \to \mathbb{Z}/2\mathbb{Z} \to Spin^c(n) \stackrel{(\lambda,\pi)}{\to} U(1) \times SO(n) \to 1$$

such that $\lambda \circ i \colon U(1) \to U(1)$ is a double covering. Let $P \to B$ be an $SO(n)$-principal bundle. We let $Spin^c(n)$ act on P via the projection π.

Definition 3.1. — *A $Spin^c$-reduction of P is a diagram*

$$\begin{tikzcd}
Q \arrow[rr, "f"] \arrow[dr] & & P \arrow[dl] \\
& B &
\end{tikzcd},$$

where $Q \to B$ is a $Spin^c(n)$-principal bundle and f is $Spin^c(n)$-equivariant.

3.1.2. — Let $p\colon W \to B$ be a proper submersion with vertical bundle T^vp. We assume that T^vp is oriented. A choice of a vertical metric g^{T^vp} gives an SO-reduction $SO(T^vp)$ of the frame bundle $\mathtt{Fr}(T^vp)$, the bundle of oriented orthonormal frames.

Usually one calls a map between manifolds K-oriented if its stable normal bundle is equipped with a K-theory Thom class. It is a well-known fact [**1**] that this is equivalent to the choice of a $Spin^c$-structure on the stable normal bundle. Finally, isomorphism classes of choices of $Spin^c$-structures on T^vp and the stable normal bundle of p are in bijective correspondence. So for the purpose of the present paper we adopt the following definition.

Definition 3.2. — *A topological K-orientation of p is a $Spin^c$-reduction of $SO(T^vp)$.*

In the present paper we prefer to work with $Spin^c$-structures on the vertical bundle since it directly gives rise to a family of Dirac operators along the fibres. The goal of this section is to introduce the notion of smooth K-orientation which refines a given topological K-orientation.

3.1.3. — In order to define such a family of Dirac operators we must choose additional geometric data. If we choose a horizontal distribution T^hp, then we get a connection ∇^{T^vp} which restricts to the Levi-Civita connection along the fibres. Its construction goes as follows. First one chooses a metric g^{TB} on B. It induces a horizontal metric g^{T^hp} via the isomorphism $dp_{|T^hp}\colon T^hp \xrightarrow{\sim} p^*TB$. We get a metric $g^{T^vp}\oplus g^{T^hp}$ on $TW\cong T^vp\oplus T^hp$ which gives rise to a Levi-Civita connection. Its projection to T^vp is ∇^{T^vp}. Finally one checks that this connection is independent of the choice of g^{TB}.

3.1.4. — The connection ∇^{T^vp} can be considered as an $SO(n)$-principal bundle connection on the frame bundle $SO(T^vp)$. In order to define a family of Dirac operators, or better, the Bismut super-connection we must choose a $Spin^c$-reduction $\tilde{\nabla}$ of ∇^{T^vp}, i.e. a connection on the $Spin^c$-principal bundle Q which reduces to ∇^{T^vp}. If we think of the connections ∇^{T^vp} and $\tilde{\nabla}$ in terms of horizontal distributions $T^hSO(T^vp)$ and T^hQ, then we say that $\tilde{\nabla}$ reduces to ∇^{T^vp} if $d\pi(T^hQ)=\pi^*(T^hSO(T^vp))$.

3.1.5. — The $Spin^c$-reduction of $\mathtt{Fr}(T^vp)$ determines a spinor bundle $S^c(T^vp)$, and the choice of $\tilde{\nabla}$ turns $S^c(T^vp)$ into a family of Dirac bundles.

In this way the choices of the $Spin^c$-structure and $(g^{T^vp},T^hp,\tilde{\nabla})$ turn $p\colon W\to B$ into a geometric family $\mathcal{W}$.

3.1.6. — Locally on W we can choose a $Spin$-structure on T^vp with associated spinor bundle $S(T^vp)$. Then we can write $S^c(T^vp)=S(T^vp)\otimes L$ for a hermitean line bundle L with connection. The spin structure is given by a $Spin$-reduction $q\colon R\to SO(T^vp)$ (similar to 3.1) which can actually be considered as a subbundle of Q. Since q is a double covering and thus has discrete fibres, the connection ∇^{T^vp} (in contrast to the $Spin^c$-case) has a unique lift to a $Spin(n)$-connection on R. The spinor bundle $S(T^vp)$ is associated to R and has an induced connection. In view of the relations of the groups 3.1.1 the square of the locally defined line bundle L is the globally defined bundle $L^2\to W$ associated to the $Spin^c$-bundle Q via the representation $\lambda\colon Spin^c(n)\to U(1)$. The connection $\tilde{\nabla}$ thus induces a connection on ∇^{L^2}, and hence a connection on the locally defined square root L. Note that vice versa, ∇^{L^2} and ∇^{T^vp} determine $\tilde{\nabla}$ uniquely.

3.1.7. — We introduce the form

$$c_1(\tilde{\nabla}) := \frac{1}{4\pi i}R^{L^2} \tag{11}$$

which would be the Chern form of the bundle L in case of a global $Spin$-structure. Let $R^{\nabla^{T^vp}}\in\Omega^2(W,\mathtt{End}(T^vp))$ denote the curvature of ∇^{T^vp}. The closed form

$$\hat{\mathbf{A}}(\nabla^{T^vp}) := \mathtt{det}^{1/2}\left(\frac{\frac{R^{\nabla^{T^vp}}}{4\pi}}{\sinh\left(\frac{R^{\nabla^{T^vp}}}{4\pi}\right)}\right)$$

represents the $\hat{\mathbf{A}}$-class of $T^v p$.

Definition 3.3. — *The relevant differential form for local index theory in the $Spin^c$-case is*

$$\hat{\mathbf{A}}^c(\tilde{\nabla}) := \hat{\mathbf{A}}(\nabla^{T^v p}) \wedge e^{c_1(\tilde{\nabla})}.$$

If we consider $p\colon W \to B$ with the geometry $(g^{T^v p}, T^h p, \tilde{\nabla})$ and the Dirac bundle $S^c(T^v p)$ as a geometric family $\mathcal{W}$ over B, then by comparison with the description 2.2.4 of the local index form $\Omega(\mathcal{W})$ we see that

$$\int_{W/B} \hat{\mathbf{A}}^c(\tilde{\nabla}) = \Omega(\mathcal{W}).$$

3.1.8. — The dependence of the form $\hat{\mathbf{A}}^c(\tilde{\nabla})$ on the data is described in terms of the transgression form. Let $(g_i^{T^v p}, T_i^h p, \tilde{\nabla}_i)$, $i = 0, 1$, be two choices of geometric data. Then we can choose geometric data $(\overline{g}^{T^v p}, \overline{T}^h p, \overline{\tilde{\nabla}})$ on $\overline{p} = \mathtt{id}_{[0,1]} \times p\colon [0,1] \times W \to [0,1] \times B$ (with the induced $Spin^c$-structure on $T^v \overline{p}$) which restricts to $(g_i^{T^v p}, T_i^h p, \tilde{\nabla}_i)$ on $\{i\} \times B$. The class

$$\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_1, \tilde{\nabla}_0) := \int_{[0,1]\times W/W} \hat{\mathbf{A}}^c(\overline{\tilde{\nabla}}) \in \Omega(W)/\mathtt{im}(d)$$

is independent of the extension and satisfies

$$d\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_1, \tilde{\nabla}_0) = \hat{\mathbf{A}}^c(\tilde{\nabla}_1) - \hat{\mathbf{A}}^c(\tilde{\nabla}_0). \tag{12}$$

Definition 3.4. — *The form $\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_1, \tilde{\nabla}_0)$ is called the transgression form.*

Note that we have the identity

$$\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_2, \tilde{\nabla}_1) + \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_1, \tilde{\nabla}_0) = \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_2, \tilde{\nabla}_0). \tag{13}$$

As a consequence we get the identities

$$\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}, \tilde{\nabla}) = 0, \quad \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_1, \tilde{\nabla}_0) = -\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_0, \tilde{\nabla}_1). \tag{14}$$

3.1.9. — We can now introduce the notion of a smooth K-orientation of a proper submersion $p\colon W \to B$. We fix an underlying topological K-orientation of p (see Definition 3.2) which is given by a $Spin^c$-reduction of $SO(T^v p)$. In order to make this precise we must choose an orientation and a metric on $T^v p$.

We consider the set $\mathcal{O}$ of tuples $(g^{T^v p}, T^h p, \tilde{\nabla}, \sigma)$ where the first three entries have the same meaning as above (see 3.1.3), and $\sigma \in \Omega^{\mathrm{odd}}(W)/\mathtt{im}(d)$. We introduce a relation $o_0 \sim o_1$ on $\mathcal{O}$: Two tuples $(g_i^{T^v p}, T_i^h p, \tilde{\nabla}_i, \sigma_i)$, $i = 0, 1$ are related if and only if $\sigma_1 - \sigma_0 = \tilde{\hat{\mathbf{A}}}(\tilde{\nabla}_1, \tilde{\nabla}_0)$. We claim that $\sim$ is an equivalence relation. In fact, symmetry and reflexivity follow from (14), while transitivity is a consequence of (13).

Definition 3.5. — *The set of smooth K-orientations which refine a fixed underlying topological K-orientation of $p\colon W \to B$ is the set of equivalence classes $\mathcal{O}/\sim$.*

3.1.10. — Note that $\Omega^{\mathrm{odd}}(W)/\mathtt{im}(d)$ acts on the set of smooth K-orientations. If $\alpha \in \Omega^{\mathrm{odd}}(W)/\mathtt{im}(d)$ and $(g^{T^v p}, T^h p, \tilde{\nabla}, \sigma)$ represents a smooth K-orientation, then the translate of this orientation by α is represented by $(g^{T^v p}, T^h p, \tilde{\nabla}, \sigma + \alpha)$. As a consequence of (13) we get:

Corollary 3.6. — *The set of smooth K-orientations refining a fixed underlying topological K-orientation is a torsor over $\Omega^{\mathrm{odd}}(W)/\mathtt{im}(d)$.*

3.1.11. — If $o = (g^{T^v p}, T^h p, \tilde{\nabla}, \sigma) \in \mathcal{O}$ represents a smooth K-orientation, then we will write

$$\hat{\mathbf{A}}^c(o) := \hat{\mathbf{A}}^c(\tilde{\nabla}), \quad \sigma(o) := \sigma.$$

3.2. Definition of the Push-forward

3.2.1. — We consider a proper submersion $p\colon W \to B$ with a choice of a topological K-orientation. Assume that p has closed fibres. Let $o = (g^{T^v p}, T^h p, \tilde{\nabla}, \sigma)$ represent a smooth K-orientation which refines the given topological one. To every geometric family $\mathcal{E}$ over W we want to associate a geometric family $p_!\mathcal{E}$ over B.

Let $\pi\colon E \to W$ denote the underlying proper submersion with closed fibres of $\mathcal{E}$ which comes with the geometric data $g^{T^v\pi}$, $T^h\pi$ and the family of Dirac bundles (V, h^V, ∇^V).

The underlying proper submersion with closed fibres of $p_!\mathcal{E}$ is

$$q := p \circ \pi\colon E \to B.$$

The horizontal bundle of π admits a decomposition $T^h\pi \cong \pi^*T^v p \oplus \pi^*T^h p$, where the isomorphism is induced by $d\pi$. We define $T^h q \subseteq T^h\pi$ such that $d\pi\colon T^h q \cong \pi^*T^h p$. Furthermore we have an identification $T^v q = T^v\pi \oplus \pi^*T^v p$. Using this decomposition we define the vertical metric $g^{T^v q} := g^{T^v\pi} \oplus \pi^* g^{T^v p}$. The orientations of $T^v\pi$ and $T^v p$ induce an orientation of $T^v q$. Finally we must construct the Dirac bundle $p_!\mathcal{V} \to E$. Locally on W we choose a *Spin*-structure on $T^v p$ and let $S(T^v p)$ be the spinor bundle. Then we can write $S^c(T^v p) = S(T^v p) \otimes L$ for a hermitean line bundle with connection. Locally on E we can choose a *Spin*-structure on $T^v\pi$ with spinor bundle $S(T^v\pi)$. Then we can write $V = S(T^v\pi) \otimes Z$, where Z is the twisting bundle of V, a hermitean vector bundle with connection ($\mathbb{Z}/2\mathbb{Z}$-graded in the even case). The local spin structures on $T^v\pi$ and $\pi^*T^v p$ induce a local *Spin*-structure on $T^v q = T^v\pi \oplus \pi^*T^v p$. Therefore locally we can define the family of Dirac bundles $p_!V := S(T^v q) \otimes \pi^*L \otimes Z$. It is easy to see that this bundle is well-defined independent of the choices of local *Spin*-structures and therefore is a globally defined family of Dirac bundles.

Definition 3.7. — *Let $p_!\mathcal{E}$ denote the geometric family given by $q\colon E\to B$ and $p_!V\to E$ with the geometric structures defined above.*

It immediately follows from the definitions, that $p_!(\mathcal{E}^{\mathrm{op}})\cong(p_!\mathcal{E})^{\mathrm{op}}$.

3.2.2. — Let $p\colon W\to B$ be a proper submersion with a smooth K-orientation represented by o. In 3.2.1 we have constructed for each geometric family $\mathcal{E}$ over W a push-forward $p_!\mathcal{E}$. Now we introduce a parameter $\lambda\in(0,\infty)$ into this construction.

Definition 3.8. — *For $\lambda\in(0,\infty)$ we define the geometric family $p_!^\lambda\mathcal{E}$ as in* 3.2.1 *with the only difference that the metric on $T^vq=T^v\pi\oplus\pi^*T^vp$ is given by $g_\lambda^{T^vq}=\lambda^2g^{T^v\pi}\oplus\pi^*g^{T^vp}$.*

More specifically, we use scaling invariance of the spinor bundle to canonically identify the Dirac bundle for the metric g_λ locally with $p_!V:=S(T^vq)\otimes\pi^*L\otimes Z$ (for g_1). This uses the description of $S(T^vp)$ in terms of tensor products of $S(T^v\pi)$ and $\pi^*S(T^vp)$ (compare [**19**, Section 2.1.2]) and the scaling invariance of $S(T^v\pi)$. However, with this identification the Clifford multiplication by vectors in $T^vq=T^v\pi\oplus\pi^*T^vp$ is rescaled on the summand $T^v\pi$ by λ. The connection is slightly more complicated, but converges for $\lambda\to0$ to some kind of sum connection.

The family of geometric families $p_!^\lambda\mathcal{E}$ is called the adiabatic deformation of $p_!\mathcal{E}$. There is a natural way to define a geometric family $\mathcal{F}$ on $(0,\infty)\times B$ such that its restriction to $\{\lambda\}\times B$ is $p_!^\lambda\mathcal{E}$. In fact, we define $\mathcal{F}:=(\mathtt{id}_{(0,\infty)}\times p)_!((0,\infty)\times\mathcal{E})$ with the exception that we take the appropriate vertical metric. Note again that the underlying bundle can be canonically identified with $(0,\infty)\times p_!V$. In the following, we work with this identifications throughout.

Although the vertical metrics of $\mathcal{F}$ and $p_!^\lambda\mathcal{E}$ collapse as $\lambda\to0$ the induced connections and the curvature tensors on the vertical bundle T^vq converge and simplify in this limit. This fact is heavily used in local index theory, and we refer to [**7**, Sec 10.2] for details. In particular, the integral

$$\tilde\Omega(\lambda,\mathcal{E}):=\int_{(0,\lambda)\times B/B}\Omega(\mathcal{F}) \tag{15}$$

converges, and we have

(16)
$$\mathtt{lim}_{\lambda\to0}\Omega(p_!^\lambda\mathcal{E})=\int_{W/B}\hat{\mathbf{A}}^c(o)\wedge\Omega(\mathcal{E}),\quad \Omega(p_!^\lambda\mathcal{E})-\int_{W/B}\hat{\mathbf{A}}^c(o)\wedge\Omega(\mathcal{E})=d\tilde\Omega(\lambda,\mathcal{E}).$$

3.2.3. — Let $p\colon W\to B$ be a proper submersion with closed fibres with a smooth K-orientation represented by o. We now start with the construction of the push-forward $p_!\colon\hat K(W)\to\hat K(B)$. For $\lambda\in(0,\infty)$ and a cycle $(\mathcal{E},\rho)$ we define

$$\hat p_!^\lambda(\mathcal{E},\rho):=[p_!^\lambda\mathcal{E},\int_{W/B}\hat{\mathbf{A}}^c(o)\wedge\rho+\tilde\Omega(\lambda,\mathcal{E})+\int_{W/B}\sigma(o)\wedge R([\mathcal{E},\rho])]\in\hat K(B). \tag{17}$$

Since $\hat{\mathbf{A}}^c(o)$ and $R([\mathcal{E},\rho])$ are closed, the maps

$$\Omega(W)/\mathtt{im}(d) \ni \rho \mapsto \int_{W/B} \hat{\mathbf{A}}^c(o)\wedge\rho \in \Omega(B)/\mathtt{im}(d),$$

$$\Omega(W)/\mathtt{im}(d) \ni \sigma(o) \mapsto \int_{W/B} \sigma(o)\wedge R([\mathcal{E},\rho]) \in \Omega(B)/\mathtt{im}(d)$$

are well-defined. It immediately follows from the definition that $\hat{p}_!^\lambda\colon G(W)\to\hat{K}(B)$ is a homomorphism of semigroups.

3.2.4. — The homomorphism $\hat{p}_!^\lambda\colon G(W)\to\hat{K}(B)$ commutes with pull-back. More precisely, let $f\colon B'\to B$ be a smooth map. Then we define the submersion $p'\colon W'\to B'$ by the Cartesian diagram

$$\begin{tikzcd} W' \arrow[r, "F"] \arrow[d, "p'"] & W \arrow[d, "p"] \\ B' \arrow[r, "f"] & B. \end{tikzcd}$$

The differential $dF\colon TW'\to F^*TW$ induces an isomorphism $dF\colon T^vW' \xrightarrow{\sim} F^*T^vW$. Therefore the metric, the orientation, and the $Spin^c$-structure of T^vp induce by pull-back corresponding structures on T^vp'. We define the horizontal distribution T^hp' such that $dF(T^hp')\subseteq F^*T^hp$. Finally we set $\sigma' := F^*\sigma$. The representative of a smooth K-orientation given by these structures will be denoted by $o' := f^*o$. An inspection of the definitions shows:

Lemma 3.9. — *The pull-back of representatives of smooth K-orientations preserves equivalence and hence induces a pull-back of smooth K-orientations.*

Recall from 3.1.5 that the representatives o and o' of the smooth K-orientations enhance p and p' to geometric families $\mathcal{W}$ and $\mathcal{W}'$. We have $f^*\mathcal{W}\cong\mathcal{W}'$.

Note that we have $F^*\hat{\mathbf{A}}^c(o) = \hat{\mathbf{A}}^c(o')$. If $\mathcal{E}$ is a geometric family over W, then an inspection of the definitions shows that $f^*p_!(\mathcal{E})\cong p'_!(F^*\mathcal{E})$. The following lemma now follows immediately from the definitions

Lemma 3.10. — *We have $f^*\circ\hat{p}_!^\lambda = \hat{p'}_!^\lambda\circ F^*\colon G(W)\to\hat{K}(B')$.*

3.2.5.

Lemma 3.11. — *The class $\hat{p}_!^\lambda(\mathcal{E},\rho)$ does not depend on $\lambda\in(0,\infty)$.*

Proof. — Consider $\lambda_0<\lambda_1$. Note that

$$\hat{p}_!^{\lambda_1}(\mathcal{E},\rho) - \hat{p}_!^{\lambda_0}(\mathcal{E},\rho) = [p_!^{\lambda_1}\mathcal{E},\tilde{\Omega}(\lambda_1,\mathcal{E})] - [p_!^{\lambda_0}\mathcal{E},\tilde{\Omega}(\lambda_0,\mathcal{E})].$$

Consider the inclusion $i_\lambda\colon B \to \{\lambda\} \times B \subset [\lambda_0, \lambda_1] \times B$ and let $\mathcal{F}$ be the family over $[\lambda_0, \lambda_1] \times B$ as in 3.2.2 such that $p_!^\lambda \mathcal{E} = i_\lambda^* \mathcal{F}$. We apply the homotopy formula Lemma 2.22 to $x = [\mathcal{F}, 0]$:

$$
\begin{aligned}
i_{\lambda_1}^*(x) - i_{\lambda_0}^*(x) = a\left(\int_{[\lambda_0,\lambda_1]\times B/B} R(x)\right) &= a\left(\int_{[\lambda_0,\lambda_1]\times B/B} \Omega(\mathcal{F})\right) \\
&= a\left(\tilde{\Omega}(\lambda_1, \mathcal{E}) - \tilde{\Omega}(\lambda_0, \mathcal{E})\right),
\end{aligned}
$$

where the last equality follows directly from the definition of $\tilde{\Omega}$. This equality is equivalent to

$$
[p_!^{\lambda_1}\mathcal{E}, \tilde{\Omega}(\lambda_1, \mathcal{E})] = [p_!^{\lambda_0}\mathcal{E}, \tilde{\Omega}(\lambda_0, \mathcal{E})]. \qquad \square
$$

In view of this Lemma we can omit the superscript λ and write $\hat{p}_!(\mathcal{E}, \rho)$ for $\hat{p}_!^\lambda(\mathcal{E}, \rho)$.

3.2.6. — Let $\mathcal{E}$ be a geometric family over W which admits a taming $\mathcal{E}_t$. Recall that the taming is given by a family of smoothing operators $(Q_w)_{w\in W}$.

We have identified the Dirac bundle of $p_!^\lambda \mathcal{E}$ with the Dirac bundle of $p_!^1 \mathcal{E}$ in a natural way in 3.2.2. The λ-dependence of the Dirac operator takes the form

$$
D(p_!^\lambda \mathcal{E}) = \lambda^{-1} D(\mathcal{E}) + (D^H + R(\lambda)),
$$

where D^H is the horizontal Dirac operator, and $R(\lambda)$ is of zero order and remains bounded as $\lambda \to 0$. We now replace $D(\mathcal{E})$ by the invertible operator $D(\mathcal{E}) + Q$. Then for small $\lambda > 0$ the operator

$$
\lambda^{-1}(D(\mathcal{E}) + Q) + (D^H + R(\lambda))
$$

is invertible. To see this, we consider its square which has the structure

$$
\lambda^{-2}(D(\mathcal{E}) + Q)^2 + \lambda^{-1}\{D(\mathcal{E}) + Q, (D^H + R(\lambda))\} + (D^H + R(\lambda))^2.
$$

The anticommutator $\{D(\mathcal{E}), D^H + R(\lambda)\}$ is a first-order vertical operator which is thus dominated by a multiple of the positive second order $(D(\mathcal{E}) + Q)^2$. The remaining parts of the anticommutator are zero-order and therefore also dominated by multiples of $(D(\mathcal{E}) + Q)^2$. The last summand is a square of a selfadjoint operator and hence non-negative.

The family of operators along the fibres of $p_!\mathcal{E}$ induced by Q is not a taming since it is not given by a family of integral operators along the fibres of $p_!E \to B$. In order to understand its structure note the following. For $b \in B$ the fibre of $(p_!\mathcal{E})_b$ is the total space of the bundle $E_{|W_b} \to W_b$. The integral kernel Q induces a family of smoothing operators on the bundle of Hilbert spaces $H(\mathcal{E}_{|W_b}) \to W_b$. Using the natural identification

$$
H(p_!\mathcal{E})_b \cong L^2(W, S(T^v p) \otimes H(\mathcal{E}_{|W_b}))
$$

we get the induced operator on $H(p_!\mathcal{E})_b$. We will call a family of operators with this structure a generalized taming.

Now recall that the η-form $\eta(\mathcal{F}_t)$ of a tamed or generalized tamed family $\mathcal{F}_t$ is built from a family of superconnections $A_s(\mathcal{F}_t)$ parametrized by $s \in (0,\infty)$ (see [**19**, 2.2.4.3]). For $0 < s < 1$ the family coincides with the usual rescaled Bismut superconnection and is independent of the taming. Therefore the taming does not affect the analysis of $\partial_s A_s(\mathcal{F}_t)\mathrm{e}^{-A_s(\mathcal{F}_t)^2}$ for $s \to 0$. In the interval $s \in [1,2]$ the family $A_s(\mathcal{F}_t)$ smoothly connects with the family of superconnections given by

$$A_s(\mathcal{F}_t) = sD(\mathcal{F}_t) + \text{terms with higher form degree}$$

for $s \geq 2$. In order to define the η-form $\eta(\mathcal{F}_t)$ the main points are:

1. For small s the family $A_s(\mathcal{F}_t)$ behaves like the Bismut superconnection. The formula (6) $d\eta(\mathcal{F}_t) = \Omega(\mathcal{F})$ only depends on the behavior of $A_s(\mathcal{F}_t)$ for small s. Therefore this formula continues to hold for generalized tamings.
2. $\partial_s A_s(\mathcal{F}_t)\mathrm{e}^{-A_s(\mathcal{F}_t)^2}$ is given by a family of integral operators with smooth integral kernel. This holds true for tamed families as well as for familes which are tamed in the generalized sense explained above. A proof can be based on Duhamel's principle.
3. The integral kernel of $\partial_s A_s(\mathcal{F}_t)\mathrm{e}^{-A_s(\mathcal{F}_t)^2}$ together with all derivatives vanishes exponentially as $s \to \infty$. This follows by spectral estimates from the invertibility and selfadjointness of $D(\mathcal{F}_t)$. Now the invertibility of $D(\mathcal{F}_t)$ is exactly the desired effect of a taming or generalized taming.

Coming back to our iterated fibre bundle we see that we can use the generalized taming for sufficiently small $\lambda > 0$ like a taming in order to define an η-form which we will denote by $\eta(p_!^\lambda \mathcal{E}_t)$. To be precise, this eta form is associated to the family of operators

$$A_s(p_!^\lambda \mathcal{E}) + \chi(s\lambda^{-1})s\lambda^{-1}Q, \quad s \in (0,\infty),$$

where χ vanishes near zero and is equal to 1 on $[1,\infty)$. This means that we switch on the taming at time $s \sim \lambda$, and we rescale it in the same way as the vertical part of the Dirac operator.

We can control the behaviour of $\eta(p_!^\lambda \mathcal{E}_t)$ in the adiabatic limit $\lambda \to 0$.

Theorem 3.12. — *We have*

$$\lim_{\lambda\to 0}\eta(p_!^\lambda \mathcal{E}_t) = \int_{W/B} \hat{\mathbf{A}}^c(o) \wedge \eta(\mathcal{E}_t).$$

Proof. — To write out a formal proof of this theorem seems too long for the present paper, without giving fundamental new insights. Instead we point out the following references. Adiabatic limits of η-forms of twisted signature operators were studied in [**21**, Section 5]. The same methods apply in the present case. The L-form in [**21**, Section 5] is the local index form of the signature operator. In the present case it must be replaced by the form $\hat{\mathbf{A}}^c(o)$, the local index form of the $Spin^c$-Dirac operator. The absence of small eigenvalues simplifies matters considerably. $\square$

Since the geometric family $p_!^\lambda \mathcal{E}$ admits a generalized taming it follows that $\mathtt{index}(p_!^\lambda \mathcal{E}) = 0$. Hence we can also choose a taming $(p_!^\lambda \mathcal{E})_t$. The latter choice together with the generalized taming induce a generalized boundary taming of the family $p_!^\lambda \mathcal{E} \times [0,1]$ over B. The index theorem [**19**, Theorem 2.2.18] can be extended to generalized boundary tamed families (by copying the proof) and gives:

Lemma 3.13. — *The difference of η-forms $\eta((p_!^\lambda \mathcal{E})_t) - \eta(p_!^\lambda \mathcal{E}_t)$ is closed. Its de Rham cohomology class satisfies*

$$[\eta((p_!^\lambda \mathcal{E})_t) - \eta(p_!^\lambda \mathcal{E}_t)] \in \mathbf{ch}_{dR}(K(B)).$$

3.2.7. — We now show that $\hat{p}_! \colon G(W) \to \hat{K}(B)$ passes through the equivalence relation $\sim$. Since $\hat{p}_!$ is additive it suffices by Lemma 2.13 to show the following assertion.

Lemma 3.14. — *If $(\mathcal{E}, \rho)$ is paired with $(\tilde{\mathcal{E}}, \tilde{\rho})$, then $\hat{p}_!(\mathcal{E}, \rho) = \hat{p}_!(\tilde{\mathcal{E}}, \tilde{\rho})$.*

Proof. — Let $(\mathcal{E} \sqcup_W \tilde{\mathcal{E}}^{\mathrm{op}})_t$ be the taming which induces the relation between the two cycles, i.e. $\rho - \tilde{\rho} = \eta\left((\mathcal{E} \sqcup_W \tilde{\mathcal{E}}^{\mathrm{op}})_t\right)$. In view of the discussion in 3.2.6 we can choose a taming $p_!^\lambda (\mathcal{E} \sqcup \tilde{\mathcal{E}}^{\mathrm{op}})_t$.

$$[p_!^\lambda \mathcal{E}, 0] - [p_!^\lambda \tilde{\mathcal{E}}, 0] = [p_!^\lambda (\mathcal{E} \sqcup_W \tilde{\mathcal{E}}^{\mathrm{op}}), 0] = a\left(\eta\left(p_!^\lambda (\mathcal{E} \sqcup_W \tilde{\mathcal{E}}^{\mathrm{op}})_t\right)\right).$$

By Proposition 2.20 and Lemma 3.13 we can replace the taming by the generalized taming and still get

$$[p_!^\lambda \mathcal{E}, 0] - [p_!^\lambda \tilde{\mathcal{E}}, 0] = a\left(\eta\left(p_!^\lambda (\mathcal{E} \sqcup_W \tilde{\mathcal{E}}^{\mathrm{op}})_t\right)\right).$$

For sufficiently small $\lambda > 0$ we thus get

$$\begin{aligned} \hat{p}_!(\mathcal{E}, \rho) - \hat{p}_!(\tilde{\mathcal{E}}, \tilde{\rho}) &= a\left(\eta\left(p_!^\lambda (\mathcal{E} \sqcup_W \tilde{\mathcal{E}}^{\mathrm{op}})_t\right)\right) - \int_{W/B} \hat{\mathbf{A}}^c(o) \wedge (\rho - \tilde{\rho}) \\ &\quad + \tilde{\Omega}(\lambda, \mathcal{E}) - \tilde{\Omega}(\lambda, \tilde{\mathcal{E}})). \end{aligned}$$

We now go to the limit $\lambda \to 0$ and use Theorem 3.12 in order to get

$$\begin{aligned} \hat{p}_!(\mathcal{E}, \rho) - \hat{p}_!(\tilde{\mathcal{E}}, \tilde{\rho}) = a\left(\int_{W/B} \hat{\mathbf{A}}^c(o) \wedge \eta\left((\mathcal{E} \sqcup_W \tilde{\mathcal{E}}^{\mathrm{op}})_t\right)\right) \\ = -\int_{W/B} \hat{\mathbf{A}}^c(o) \wedge (\rho - \tilde{\rho}) = 0. \quad \square \end{aligned}$$

We let $\hat{p}_!\colon \hat{K}(W) \to \hat{K}(B)$ denote the map induced by the construction (17). Though not indicated in the notation until now this map may depend on the choice of the representative of the smooth K-orientation o (later in Lemma 3.17 we will see that it only depends on the smooth K-orientation).

3.2.8. — Let $p\colon W \to B$ be a proper submersion with closed fibres with a smooth K-orientation represented by o. We now have constructed a homomorphism

$$\hat{p}_!\colon \hat{K}(W) \to \hat{K}(B).$$

In the present paragraph we study the compatibilty of this construction with the curvature map $R\colon \hat{K} \to \Omega_{d=0}$.

Definition 3.15. — *We define the integration of forms $p^o_!\colon \Omega(W) \to \Omega(B)$ by*

$$p^o_!(\omega) = \int_{W/B} (\hat{\mathbf{A}}^c(o) - d\sigma(o)) \wedge \omega.$$

Since $\hat{\mathbf{A}}^c(o) - d\sigma(o)$ is closed we also have a factorization

$$p^o_!\colon \Omega(W)/\mathtt{im}(d) \to \Omega(B)/\mathtt{im}(d).$$

Lemma 3.16. — *For $x \in \hat{K}(W)$ we have $R(\hat{p}_!(x)) = p^o_!(R(x))$.*

Proof. — Let $x = (\mathcal{E}, \rho)$. We insert the definitions, $R(x) = \Omega(\mathcal{E}) - d\rho$, and (16) in the marked step.

$$\begin{aligned} R(\hat{p}_!(x)) &= \Omega(p^\lambda_!\mathcal{E}) - d\Big(\int_{W/B} \hat{\mathbf{A}}^c(o) \wedge \rho + \tilde{\Omega}(\lambda, \mathcal{E}) + \int_{W/B} \sigma(o) \wedge R(x)\Big) \\ &\overset{!}{=} \Omega(p^\lambda_!\mathcal{E}) - \int_{W/B} \hat{\mathbf{A}}^c(o) \wedge d\rho + \int_{W/B} \hat{\mathbf{A}}^c(o) \wedge \Omega(\mathcal{E}) - \Omega(p^\lambda_!\mathcal{E}) - \int_{W/B} d\sigma(o) \wedge R(x) \\ &= \int_{W/B} (\hat{\mathbf{A}}^c(o) - d\sigma(o)) \wedge R(x) = p^o_!(R(x)). \quad \square \end{aligned}$$

3.2.9. — Our constructions of the homomorphisms

$$\hat{p}_!\colon \hat{K}(W) \to \hat{K}(B), \quad p^o_!\colon \Omega(W) \to \Omega(B)$$

involve an explicit choice of a representative $o = (g^{T^vp}, T^hp, \tilde{\nabla}, \sigma)$ of the smooth K-orientation lifting the given topological K-orientation of p. In this paragraph we show:

Lemma 3.17. — *The homomorphisms $\hat{p}_!\colon \hat{K}(W) \to \hat{K}(B)$ and $p^o_!\colon \Omega(W) \to \Omega(B)$ only depend on the smooth K-orientation represented by o.*

Proof. — Let $o_k := (g_k^{T^vp}, T_k^h p, \tilde{\nabla}_k, \sigma_k)$, $k = 0, 1$ be two representatives of a smooth K-orientation. Then we have $\sigma_1 - \sigma_0 = \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_1, \tilde{\nabla}_0)$. For the moment we indicate by a superscript $\hat{p}_!^k$ which representative of the smooth K-orientation is used in the definition. Let $\omega \in \Omega(W)$. Then using (12) we get

$$\begin{aligned} p_!^{o_1}(\omega) - p_!^{o_0}(\omega) &= \int_{W/B} (\hat{\mathbf{A}}^c(o_1) - \hat{\mathbf{A}}^c(o_0) - d(\sigma_1 - \sigma_0)) \wedge \omega \\ &= \int_{W/B} (\hat{\mathbf{A}}^c(\tilde{\nabla}_1) - \hat{\mathbf{A}}^c(\tilde{\nabla}_0) - d\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_1, \tilde{\nabla}_0)) \wedge \omega = 0. \end{aligned}$$

We now consider the projection $\overline{p}\colon [0,1] \times W \to [0,1] \times B$ with the induced topological K-orientation. It can be refined to a smooth K-orientation $\overline{o}$ which restricts to o_k at $\{k\} \times B$. Let $q\colon [0,1] \times W \to W$ be the projection and $x \in \hat{K}(W)$. Furthermore let $i_k\colon B \to \{k\} \times B \to [0,1] \times B$ be the embeddings. The following chain of equalities follows from the homotopy formula Lemma 2.22, the curvature formula Lemma 3.16, Stokes' theorem and the definition of $\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_1, \tilde{\nabla}_0)$, and finally from the fact that $o_0 \sim o_1$.

$$\begin{aligned} \hat{p}_!^1(x) - \hat{p}_!^0(x) = i_1^*\hat{\overline{p}}_! q^*(x) - i_0^*\hat{\overline{p}}_! q^*(x) &= a\left(\int_{[0,1]\times B/B} R(\hat{\overline{p}}_! q^* x)\right) \\ &= a\left(\int_{[0,1]\times B/B} \overline{p}_!^{\overline{o}} R(q^*(x))\right) = a\left(\int_{[0,1]\times B/B} \overline{p}_!^{\overline{o}} q^*(R(x))\right) \\ &= a\left(\int_{[0,1]\times B/B} \int_{[0,1]\times W/[0,1]\times B} (\hat{\mathbf{A}}^c(\overline{o}) - d\sigma(\overline{o})) \wedge q^* R(x)\right) \\ &= a\left(\int_{W/B} [\int_{[0,1]\times W/W} (\hat{\mathbf{A}}^c(\overline{o}) - d\sigma(\overline{o}))] \wedge R(x)\right) \\ &= a\left(\int_{W/B} [\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_1, \tilde{\nabla}_0) - (\sigma(o_1) - \sigma(o_0))] \wedge R(x)\right) = 0. \end{aligned}$$ □

3.2.10. — Let $p\colon W \to B$ be a proper submersion with closed fibres with a topological K-orientation. We choose a smooth K-orientation which refines the topological K-orientation. In this case we say that p is smoothly K-oriented.

Definition 3.18. — *We define the push-forward $\hat{p}_!\colon \hat{K}(W) \to \hat{K}(B)$ to be the map induced by* (17) *for some choice of a representative of the smooth K-orientation*

We also have well-defined maps

$$p_!^o\colon \Omega(W) \to \Omega(B), \quad p_!^o\colon \Omega(W)/\mathtt{im}(d) \to \Omega(B)/\mathtt{im}(d)$$

given by integration of forms along the fibres. Let us state the result about the compatibility of $\hat{p}_!$ with the structure maps of smooth K-theory as follows.

Proposition 3.19. — *The following diagrams commute:*

$$\begin{tikzcd} K(W) \arrow[r, "\mathbf{ch}_{dR}"] \arrow[d, "p_!"] & \Omega(W)/\mathtt{im}(d) \arrow[r, "a"] \arrow[d, "p^o_!"] & \hat{K}(W) \arrow[r, "I"] \arrow[d, "\hat{p}_!"] & K(W) \arrow[d, "p_!"] \\ K(B) \arrow[r, "\mathbf{ch}_{dR}"] & \Omega(B)/\mathtt{im}(d) \arrow[r, "a"] & \hat{K}(B) \arrow[r, "I"] & K(B) \end{tikzcd} \tag{18}$$

$$\begin{tikzcd} \hat{K}(W) \arrow[r, "R"] \arrow[d, "\hat{p}_!"] & \Omega_{d=0}(W) \arrow[d, "p^o_!"] \\ \hat{K}(B) \arrow[r, "R"] & \Omega_{d=0}(B). \end{tikzcd} \tag{19}$$

Proof. — The maps between the topological K-groups are the usual push-forward maps defined by the K-orientation of p. The other two are defined above. The square (19) commutes by Lemma 3.16. The right square of (18) commutes because we have the well-known fact from index theory

$$\mathtt{index}(p_!(\mathcal{E})) = p_!(\mathtt{index}(\mathcal{E})).$$

Let $\omega \in \Omega(W)/\mathtt{im}(d)$. Then we have

$$\begin{aligned}\hat{p}_!(a(\omega)) = [\varnothing, \int_{W/B} \sigma(o) \wedge d\omega - \int_{W/B} \hat{\mathbf{A}}^c(o) \wedge \omega] \\ = [\varnothing, -\int_{W/B} (\hat{\mathbf{A}}^c(o) - d\sigma(o)) \wedge \omega] = a\left(p_!(\omega)\right).\end{aligned}$$

This shows that the middle square in (18) commutes. Finally, the commutativity of the left square in (18) is a consequence of the Chern character version of the family index theorem

$$\mathbf{ch}_{dR}(p_!(x)) = \int_{W/B} \hat{\mathbf{A}}^c(T^v p) \wedge \mathbf{ch}_{dR}(x), \quad x \in K(W). \qquad \square$$

If $f\colon B' \to B$ is a smooth map then we consider the Cartesian diagram

$$\begin{tikzcd} W' \arrow[r, "F"] \arrow[d, "p'"] & W \arrow[d, "p"] \\ B' \arrow[r, "f"] & B. \end{tikzcd}$$

We equip p' with the induced smooth K-orientation (see 3.2.4).

Lemma 3.20. — *The following diagram commutes:*

$$\begin{tikzcd} \hat{K}(W) \arrow[r, "F^*"] \arrow[d, "p_!"] & \hat{K}(W') \arrow[d, "p'_!"] \\ \hat{K}(B) \arrow[r, "f^*"] & \hat{K}(B'). \end{tikzcd}$$

Proof. — This follows from Lemma 3.10. □

3.3. Functoriality

3.3.1. — We now discuss the functoriality of the push-forward with respect to iterated fibre bundles. Let $p\colon W \to B$ be as before together with a representative of a smooth K-orientation $o_p = (g^{T^vp}, T^hp, \tilde{\nabla}_p, \sigma(o_p))$. Let $r\colon B \to A$ be another proper submersion with closed fibres with a topological K-orientation which is refined by a smooth K-orientation represented by $o_r := (g^{T^vr}, T^hr, \tilde{\nabla}_r, \sigma(o_r))$.

We can consider the geometric family $\mathcal{W} := (W \to B, g^{T^vp}, T^hp, S^c(T^vp))$ and apply the construction 3.2.2 in order to define the geometric family $r_!^\lambda(\mathcal{W})$ over A. The underlying submersion of the family is $q := r \circ p\colon W \to A$. Its vertical bundle has a metric $g_\lambda^{T^vq}$ and a horizontal distribution T^hq. The topological $Spin^c$-structures of T^vp and T^vr induce a topological $Spin^c$-structure on $T^vq = T^vp \oplus p^*T^vr$. The family of Clifford bundles of $p_!\mathcal{W}$ is the spinor bundle associated to this $Spin^c$-structure.

In order to understand how the connection $\tilde{\nabla}_q^\lambda$ behaves as $\lambda \to 0$ we choose local spin structures on T^vp and T^vr. Then we write $S^c(T^vp) \cong S(T^vp) \otimes L_p$ and $S^c(T^vr) \cong S(T^vr) \otimes L_r$ for one-dimensional twisting bundles with connection L_p, L_r. The two local spin structures induce a local spin structure on $T^vq \cong T^vp \oplus p^*T^vr$. We get $S^c(T^vq) \cong S(T^vq) \otimes L_q$ with $L_q := L_p \otimes p^*L_r$. The connection ∇_q^{λ, T^vq} converges as $\lambda \to 0$. Moreover, the twisting connection on L_q does not depend on λ at all. Since ∇_q^{λ,T^vq} and ∇_q^L determine $\tilde{\nabla}_q^\lambda$ (see 3.1.5) we conclude that the connection $\tilde{\nabla}_q^\lambda$ converges as $\lambda \to 0$. We introduce the following notation for this adiabatic limit:

$$\tilde{\nabla}^{\mathrm{adia}} := \mathtt{lim}_{\,\lambda \to 0} \tilde{\nabla}_q^\lambda .$$

3.3.2. — We keep the situation described in 3.3.1.

Definition 3.21. — *We define the composite $o_q^\lambda := o_r \circ_\lambda o_p$ of the representatives of smooth K-orientations of p and r by*

$$o_q^\lambda := (g_\lambda^{T^vq}, T^hq, \tilde{\nabla}_q^\lambda, \sigma(o_q^\lambda)),$$

where

$$\sigma(o_q^\lambda) := \sigma(o_p) \wedge p^*\hat{\mathbf{A}}^c(o_r) + \hat{\mathbf{A}}^c(o_p) \wedge p^*\sigma(o_r) - \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}^{\mathrm{adia}}, \tilde{\nabla}_q^\lambda) - d\sigma(o_p) \wedge p^*\sigma(o_r).$$

Lemma 3.22. — *This composition of representatives of smooth $\hat{K}$-orientations preserves equivalence and induces a well-defined composition of smooth K-orientations which is independent of λ.*

Proof. — We first show that o_q^λ is independent of λ. In view of 3.1.9 for $\lambda_0 < \lambda_1$ we must show that $\sigma(o_q^{\lambda_1}) - \sigma(o_q^{\lambda_0}) = \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_q^{\lambda_1}, \tilde{\nabla}_q^{\lambda_0})$. In fact, inserting the definitions and using (13) and (14) we have

$$\sigma(o_q^{\lambda_1}) - \sigma(o_q^{\lambda_0}) = -\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}^{\mathrm{adia}}, \tilde{\nabla}_q^{\lambda_1}) + \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}^{\mathrm{adia}}, \tilde{\nabla}_q^{\lambda_0}) = \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_q^{\lambda_1}, \tilde{\nabla}_q^{\lambda_0}).$$

Let us now take another representative o'_p. The following equalities hold in the limit $\lambda \to 0$.

$$\begin{aligned}
\sigma(o_q) - \sigma(o'_q) &= (\sigma(o_p) - \sigma(o'_p)) \wedge p^*\hat{\mathbf{A}}^c(o_r) \\
&\quad + (\hat{\mathbf{A}}^c(o_p) - \hat{\mathbf{A}}^c(o'_p)) \wedge p^*\sigma(o_r) - d(\sigma(o_p) - \sigma(o'_p)) \wedge p^*\sigma(o_r) \\
&= \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_p, \tilde{\nabla}'_p) \wedge p^*\hat{\mathbf{A}}^c(o_r) \\
&\quad + (\hat{\mathbf{A}}^c(\tilde{\nabla}_p) - \hat{\mathbf{A}}^c(\tilde{\nabla}'_p) - d\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_p, \tilde{\nabla}'_p)) \wedge p^*\sigma(o_r) \\
&= \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_q^{\mathrm{adia}}, \tilde{\nabla}_q^{\prime\mathrm{adia}}).
\end{aligned}$$

The last equality uses (12) and that in the adiabatic limit

$$\hat{\mathbf{A}}^c(\tilde{\nabla}_q^{\mathrm{adia}}) = \hat{\mathbf{A}}^c(\tilde{\nabla}_p) \wedge p^*\hat{\mathbf{A}}^c(\nabla_r), \tag{20}$$

which implies a corresponding formula for the adiabatic limit of transgressions,

$$\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_q^{\mathrm{adia}}, \tilde{\nabla}_q^{\prime\mathrm{adia}}) = \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_p, \tilde{\nabla}'_p) \wedge p^*\hat{\mathbf{A}}^c(\nabla_r).$$

Next we consider the effect of changing the representative o_r to the equivalent one o'_r. We compute in the adiabatic limit

$$\begin{aligned}
&\sigma(o_q) - \sigma(o'_q) \\
&\quad = \sigma(o_p) \wedge (p^*\hat{\mathbf{A}}^c(o_r) - p^*\hat{\mathbf{A}}^c(o'_r)) + (\hat{\mathbf{A}}^c(o_p) - d\sigma(o_p)) \wedge p^*(\sigma(o_r) - \sigma(o'_r)) \\
&\qquad = \sigma(o_p) \wedge dp^*\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_r, \tilde{\nabla}'_r) + (\hat{\mathbf{A}}^c(o_p) - d\sigma(o_p)) \wedge p^*\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_r, \tilde{\nabla}'_r) \\
&\qquad\qquad = \hat{\mathbf{A}}^c(o_p) \wedge p^*\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_r, \tilde{\nabla}'_r) = \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_q^{\mathrm{adia}}, \tilde{\nabla}_q^{\prime\mathrm{adia}}).
\end{aligned}$$

In the last equality we have used again (20) and the corresponding equality

$$\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_q^{\mathrm{adia}}, \tilde{\nabla}_q^{\prime\mathrm{adia}}) = \hat{\mathbf{A}}^c(o_p) \wedge p^*\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_r, \tilde{\nabla}'_r). \qquad \square$$

3.3.3. — We consider the composition of proper K-oriented submersions

$$\begin{tikzcd} W \arrow[r, "p"] \arrow[rr, bend right, "q"'] & B \arrow[r, "r"] & A \end{tikzcd}$$

with representatives of smooth K-orientations o_p of p and o_r of r. We let $o_q := o_r \circ o_p$ be the composition. These choices define push-forwards $\hat{p}_!$, $\hat{r}_!$ and $\hat{q}_!$ in smooth K-theory.

Theorem 3.23. — *We have the equality of homomorphisms $\hat{K}(W) \to \hat{K}(A)$*

$$\hat{q}_! = \hat{r}_! \circ \hat{p}_!.$$

Proof. — We calculate the push-forwards and the composition of the K-orientations using the parameter $\lambda = 1$ (though we do not indicate this in the notation). We take a class $[\mathcal{E}, \rho] \in \hat{K}(W)$. The following equality holds since $\lambda = 1$:

$$q_!\mathcal{E} = r_!(p_!\mathcal{E}).$$

So we must show that

$$\begin{aligned}(21)\int_{W/A} \hat{\mathbf{A}}^c(o_q) \wedge \rho + \tilde{\Omega}(q, 1, \mathcal{E}) + \int_{W/A} \sigma(o_q) \wedge R([\mathcal{E}, \rho]) \\ \equiv \int_{B/A} \hat{\mathbf{A}}^c(o_r) \wedge \left[\int_{W/B} \hat{\mathbf{A}}^c(o_p) \wedge \rho + \tilde{\Omega}(p, 1, \mathcal{E}) + \int_{W/B} \sigma(o_p) \wedge R([\mathcal{E}, \rho])\right] \\ + \tilde{\Omega}(r, 1, p_!\mathcal{E}) + \int_{B/A} \sigma(o_r) \wedge R(p_![\mathcal{E}, \rho]),\end{aligned}$$

where $\equiv$ means equality modulo $\mathtt{im}(d) + \mathbf{ch}_{dR}(K(A))$. The form $\Omega(q, 1, \mathcal{E})$ is given by (15). Since in the present paragraph we consider these transgression forms for various bundles we have included the projection q as an argument.

By Proposition 3.19 we have

$$R(\hat{p}_![\mathcal{E}, \rho]) = \int_{W/B} (\hat{\mathbf{A}}^c(o_p) - d\sigma(o_p)) \wedge R([\mathcal{E}, \rho]).$$

Next we observe that

$$(22)\ \ \tilde{\Omega}(q, 1, \mathcal{E}) \equiv \tilde{\Omega}(r, 1, p_!\mathcal{E}) + \int_{W/A} \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}^{\mathrm{adia}}, \tilde{\nabla}_q) \wedge \Omega(\mathcal{E}) + \int_{B/A} \hat{\mathbf{A}}^c(o_r) \wedge \tilde{\Omega}(p, 1, \mathcal{E}),$$

(where $\equiv$ means equality up to $\mathtt{im}(d)$). To see this we consider the two-parameter family $r_!^\lambda \circ p_!^\mu(\mathcal{E})$, $\lambda, \mu > 0$, of geometric families. There is a natural geometric family $\mathcal{F}$ over $(0,1]^2 \times A$ which restricts to $r_!^\lambda \circ p_!^\mu(\mathcal{E})$ on $\{(\lambda, \mu)\} \times A$ (see 3.2.2 for the one-parameter case). Note that the local index form $\Omega(\mathcal{F})$ extends by continuity to $[0,1]^2 \times A$. If $P\colon [0,1] \hookrightarrow [0,1]^2$ is a path, then one can form the integral $\int_{P \times A/A} \Omega(\mathcal{F}_{|P \times A})$, the transgression of the local index form of $r_!^\lambda \circ p_!^\mu(\mathcal{E})$ along the path P. The following square indicates four paths in the (λ, μ)-plane. The arrows are labeled by the evaluations of $\Omega(\mathcal{F})$ (which follow from the adiabatic limit formula 16), and their integrals, the corresponding transgression forms:

$$\begin{array}{ccc} (0,1) & \xrightarrow[\Omega(r_!^\lambda \circ p_!(\mathcal{E}))]{\tilde{\Omega}(r,1,p_!\mathcal{E})} & (1,1) \\ {\scriptstyle \int_{B/A} \hat{\mathbf{A}}^c(o_r) \wedge \Omega(p_!^\mu \mathcal{E})} \Big\uparrow {\scriptstyle \int_{B/A} \hat{\mathbf{A}}^c(o_r) \wedge \tilde{\Omega}(p,1,\mathcal{E})} & & {\scriptstyle \Omega(r_! \circ p_!^\mu(\mathcal{E}))} \Big\uparrow {\scriptstyle \tilde{\Omega}(q,1,\mathcal{E})} \\ (0,0) & \xrightarrow[\int_{W/A} \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_q, \tilde{\nabla}^{\mathrm{adia}}) \wedge \Omega(\mathcal{E})]{\int_{W/A} \hat{\mathbf{A}}^c(o_r \circ_\lambda o_p) \wedge \Omega(\mathcal{E})} & (1,0). \end{array}$$

Note the equality $r_! \circ p^\mu_!(\mathcal{E}) = q^\mu_!(\mathcal{E})$ which is relevant for the right vertical path. Also note that for the lower horizontal path that , as $\mu \to 0$, the fibres of $\mathcal{E}$ are scaled to zero, whereas the fibres of p are scaled by λ. The latter is exactly the effect of the scaled composition $o_r \circ_\lambda o_p$ of orientations defined in 3.3.1, explaining its appearence in the above formula. The equation (22) follows since the transgression is additive under composition of paths, and since the transgression along a closed contractible path gives an exact form.

We now insert Definition 3.21 of $\sigma(o_q)$ in order to get

$$\begin{aligned}
&\int_{W/A} \sigma(o_q) \wedge R([\mathcal{E},\rho]) \\
&\qquad = \int_{W/A} \Big[\sigma(o_p) \wedge p^* \hat{\mathbf{A}}^c(o_r) + \hat{\mathbf{A}}^c(o_p) \wedge p^*\sigma(o_r) \\
&\qquad\quad -d\sigma(o_p) \wedge p^*\sigma(o_r) - \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}^{\mathrm{adia}}, \tilde{\nabla}_q)\Big] \wedge R([\mathcal{E},\rho]) \\
&\qquad = \int_{W/A} \Big[\sigma(o_p) \wedge p^* \hat{\mathbf{A}}^c(o_r) + \hat{\mathbf{A}}^c(o_p) \wedge p^*\sigma(o_r) - d\sigma(o_r) \wedge p^*\sigma(o_r)\Big] \wedge R([\mathcal{E},\rho]) \\
&\qquad\quad - \int_{W/A} \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}^{\mathrm{adia}}, \tilde{\nabla}_q) \wedge \Omega(\mathcal{E}) + \int_{W/A} \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}^{\mathrm{adia}}, \tilde{\nabla}_q) \wedge d\rho \\
&\qquad = \int_{W/A} \Big[\sigma(o_p) \wedge p^* \hat{\mathbf{A}}^c(o_r) + \hat{\mathbf{A}}^c(o_p) \wedge p^*\sigma(o_r) - d\sigma(o_p) \wedge p^*\sigma(o_r)\Big] \wedge R([\mathcal{E},\rho]) \\
&\qquad\quad - \int_{W/A} \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}^{\mathrm{adia}}, \tilde{\nabla}_q) \wedge \Omega(\mathcal{E}) + \int_{W/A} \Big(\hat{\mathbf{A}}^c(o_p) \wedge p^* \hat{\mathbf{A}}^c(o_r) - \hat{\mathbf{A}}^c(o_q)\Big) \wedge \rho. \qquad (23)
\end{aligned}$$

We insert (23) and (22) into the left-hand side of (21).

$$\begin{aligned}
&\int_{W/A} \hat{\mathbf{A}}^c(o_q) \wedge \rho + \tilde{\Omega}(q,1,\mathcal{E}) + \int_{W/A} \sigma(o_q) \wedge R([\mathcal{E},\rho]) \\
&\qquad \equiv \int_{W/A} \hat{\mathbf{A}}^c(o_q) \wedge \rho \\
&\qquad\quad + \tilde{\Omega}(r,1,p_!\mathcal{E}) + \int_{W/A} \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}^{\mathrm{adia}}, \tilde{\nabla}_q) \wedge \Omega(\mathcal{E}) + \int_{B/A} \hat{\mathbf{A}}^c(o_r) \wedge \tilde{\Omega}(p,1,\mathcal{E}) \\
&\qquad\quad + \int_{W/A} \Big[\sigma(o_p) \wedge p^* \hat{\mathbf{A}}^c(o_r) + \hat{\mathbf{A}}^c(o_p) \wedge p^*\sigma(o_r) - d\sigma(o_p) \wedge p^*\sigma(o_r)\Big] \wedge R([\mathcal{E},\rho]) \\
&\qquad\quad - \int_{W/A} \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}^{\mathrm{adia}}, \tilde{\nabla}_q) \wedge \Omega(\mathcal{E}) + \int_{W/A} \Big(\hat{\mathbf{A}}^c(o_p) \wedge p^* \hat{\mathbf{A}}^c(o_r) - \hat{\mathbf{A}}^c(o_q)\Big) \wedge \rho \\
&\qquad = \tilde{\Omega}(r,1,p_!\mathcal{E}) + \int_{B/A} \hat{\mathbf{A}}^c(o_r) \wedge \tilde{\Omega}(p,1,\mathcal{E}) \\
&\qquad\quad + \int_{W/A} \Big[\sigma(o_p) \wedge p^* \hat{\mathbf{A}}^c(o_r) + \hat{\mathbf{A}}^c(o_p) \wedge p^*\sigma(o_r) - d\sigma(o_p) \wedge p^*\sigma(o_r)\Big] \wedge R([\mathcal{E},\rho])
\end{aligned}$$

$$+\int_{W/A} \hat{\mathbf{A}}^c(o_p)\wedge p^*\hat{\mathbf{A}}^c(o_r)\wedge\rho.$$

An inspection shows that this is exactly the right-hand side of (21). □

4. The cup product

4.1. Definition of the product

4.1.1. — In this section we define and study the cup product

$$\cup\colon \hat{K}(B)\otimes\hat{K}(B)\to\hat{K}(B).$$

It turns smooth K-theory into a functor on manifolds with values in $\mathbb{Z}/2\mathbb{Z}$-graded rings and into a multiplicative extension of the pair $(K,\mathbf{ch}_{\mathbb{R}})$ in the sense of Definition 1.2.

4.1.2. — Let $\mathcal{E}$ and $\mathcal{F}$ be geometric families over B. The formula for the product involves the product $\mathcal{E}\times_B\mathcal{F}$ of geometric families over B. The detailed description of the product is easy to guess, but let us employ the following trick in order to give an alternative definition.

Let $p\colon F\to B$ be the proper submersion with closed fibres underlying $\mathcal{F}$. Let us for the moment assume that the vertical metric, the horizontal distribution, and the orientation of p are complemented by a topological $Spin^c$-structure together with a $Spin^c$-connection $\tilde{\nabla}$ as in 3.2.1. The Dirac bundle $\mathcal{V}$ of $\mathcal{F}$ has the form $\mathcal{V}\cong W\otimes S^c(T^vp)$ for a twisting bundle W with a hermitean metric and unitary connection (and $\mathbb{Z}/2\mathbb{Z}$-grading in the even case), which is uniquely determined up to isomorphism. Let $p^*\mathcal{E}\otimes W$ denote the geometric family which is obtained from $p^*\mathcal{E}$ by twisting its Dirac bundle with δ^*W, where $\delta\colon E\times_B F\to F$ denotes the underlying proper submersion with closed fibres of $p^*\mathcal{E}$. Then we have

$$\mathcal{E}\times_B\mathcal{F}\cong p_!(p^*\mathcal{E}\otimes W).$$

This description may help to understand the meaning of the adiabatic deformation which blows up $\mathcal{F}$, which in this notation is given by $p_!^\lambda(p^*\mathcal{E}\otimes W)$.

In the description of the product of geometric families we could interchange the roles of $\mathcal{E}$ and $\mathcal{F}$.

If the vertical bundle of $\mathcal{E}$ does not have a global $Spin^c$-structure, then it has at least a local one. In this case the description above again gives a complete description of the local geometry of $\mathcal{E}\times_B\mathcal{F}$.

4.1.3. — We now proceed to the definition of the product in terms of cycles. In order to write down the formula we assume that the cycles $(\mathcal{E},\rho)$ and $(\mathcal{F},\theta)$ are homogeneous of degree e and f, respectively.

Definition 4.1. — *We define*

$$(\mathcal{E},\rho)\cup(\mathcal{F},\theta):=[\mathcal{E}\times_B\mathcal{F},(-1)^e\Omega(\mathcal{E})\wedge\theta+\rho\wedge\Omega(\mathcal{F})-(-1)^e d\rho\wedge\theta].$$

Proposition 4.2. — *The product is well-defined. It turns $B\mapsto\hat K(B)$ into a functor from smooth manifolds to unital graded-commutative rings.*

Proof. — We first show that this product is bilinear and compatible with the equivalence relation $\sim$ (2.10). The product is obviously biadditive and natural with respect to pull-backs along maps $B'\to B$. We now show that the product preserves the equivalence relation in the first argument. Assume that $\mathcal{E}$ admits a taming $\mathcal{E}_t$. Then we have $(\mathcal{E},\rho)\sim(\emptyset,\rho-\eta(\mathcal{E}_t))$. Using the latter representative we get

$$\begin{aligned}(\emptyset,\rho-\eta(\mathcal{E}_t))\cup(\mathcal{F},\theta)&=[\emptyset,(\rho-\eta(\mathcal{E}_t))\wedge\Omega(\mathcal{F})-(-1)^e d\rho\wedge\theta+(-1)^e d\eta(\mathcal{E}_t)\wedge\theta]\\&=[\emptyset,\rho\wedge\Omega(\mathcal{F})+(-1)^e\Omega(\mathcal{E})\wedge\theta-(-1)^e d\rho\wedge\theta-\eta(\mathcal{E}_t)\wedge\Omega(\mathcal{F})].\end{aligned}$$

On the other hand, similar to in 3.2.6, the taming $\mathcal{E}_t$ induces a generalized taming $(\mathcal{E}\times_B\mathcal{F})_t$. Using Lemma 3.13 and argueing as in the proof of Lemma 3.14 we get

$$\begin{aligned}&[\mathcal{E}\times_B\mathcal{F},(-1)^e\Omega(\mathcal{E})\wedge\theta+\rho\wedge\Omega(\mathcal{F})-(-1)^e d\rho\wedge\sigma]\\&\qquad=[\emptyset,(-1)^e\Omega(\mathcal{E})\wedge\theta+\rho\wedge\Omega(\mathcal{F})-(-1)^e d\rho\wedge\sigma-\eta((\mathcal{E}\times_B\mathcal{F})_t)].\end{aligned}$$

It suffices to show that

$$\eta(\mathcal{E}_t)\wedge\Omega(\mathcal{F})-\eta((\mathcal{E}\times_B\mathcal{F})_t)\in\mathtt{im}(\mathbf{ch}_{dR}). \tag{24}$$

We will actually show that this difference is exact.

We first consider the adiabatic limit in which we blow up the metric of $\mathcal{F}$. We get from Theorem 3.12

$$\mathtt{lim}_{\,\mathrm{adia}}\eta((\mathcal{E}\times_B\mathcal{F})_t)=\eta(\mathcal{E}_t)\wedge\Omega(\mathcal{F}). \tag{25}$$

In order to see this we use that $\mathcal{E}\times_B\mathcal{F}\cong p_!(p^*\mathcal{E}\otimes W)$ (see 4.1.2), where $p\colon F\to B$ and $W\to F$ is the twisting bundle of this family. The taming $\mathcal{E}_t$ induces a taming $p^*\mathcal{E}_t$, and hence a taming $(p^*\mathcal{E}\otimes W)_t$. It follows from standard properties of the induced superconnection on a tensor product bundle (alternatively one can use the special case of Theorem 3.12 where the second fibration has zero-dimensional fibres) that $\eta(p^*\mathcal{E}\otimes W)_t=p^*\eta(\mathcal{E}_t)\wedge\mathbf{ch}(\nabla^W)$. From Theorem 3.12 we get ($\tilde\nabla$ is associated to p)

$$\begin{aligned}&\mathtt{lim}_{\,\mathrm{adia}}\eta((\mathcal{E}\times_B\mathcal{F})_t)=\mathtt{lim}_{\,\lambda\to0}\eta(p_!^\lambda(p^*\mathcal{E}\otimes W)_t)\\&\qquad=\eta(\mathcal{E}_t)\wedge\left(\int_{F/B}\hat{\mathbf{A}}^c(\tilde\nabla)\wedge\mathbf{ch}(\nabla^W)\right)=\eta(\mathcal{E}_t)\wedge\Omega(\mathcal{F}).\end{aligned}$$

As in 3.2.2 we now let $\mathcal{G}_t$ be the tamed family over $(0,\infty)\times B$ with underlying projection $r\colon(0,\infty)\times E\times_B F\to(0,\infty)\times B$ which restricts to $p_!^\lambda(p^*\mathcal{E}\otimes W)_t$ on

$\{\lambda\} \times B$. Then we have $d\eta(\mathcal{G}_t) = \Omega(\mathcal{G})$. Using the formulas for $\nabla^{T^v r}$ given in [**7**, Prop. 10.2] we observe that $i_{\partial_\lambda^H} R^{\nabla^{T^v r}} = 0$, where ∂_λ^H is a horizontal lift of ∂_λ. This implies that $i_{\partial_\lambda} d\eta(\mathcal{G}_t) = i_{\partial_\lambda}\Omega(\mathcal{G}) = 0$. We get

$$\eta(p_!^\lambda(p^*\mathcal{E}\otimes W)_t) - \eta(p_!^1(p^*\mathcal{E}\otimes W)_t) = d\int_{[\lambda,1]\times B/B} \eta(\mathcal{G}^t).$$

The exactness of the difference (24) now follows by taking the limit $\lambda \to 0$ and the fact that the range of d is closed since $\mathtt{lim}_{\lambda\to 0}\eta(p_!^\lambda(p^*\mathcal{E}\otimes W)_t) = \eta(\mathcal{E}_t)\wedge\Omega(\mathcal{F})$ by (25) and $\eta(p_!^1(p^*\mathcal{E}\otimes W)_t) = \eta((\mathcal{E}\times_B\mathcal{F})_t)$ by construction.

In order to avoid repeating this argument for the second argument we show that the product is graded commutative. Note that $\mathcal{E}\times_B\mathcal{F} \cong \mathcal{F}\times_B\mathcal{E}$ except if both families are odd, in which case $\mathcal{E}\times_B\mathcal{F} \cong (\mathcal{F}\times_B\mathcal{E})^{\mathrm{op}}$

$$\begin{aligned}
[\mathcal{E},\rho]\cup[\mathcal{F},\theta] &= [\mathcal{E}\times_B\mathcal{F}, (-1)^e\Omega(\mathcal{E})\wedge\theta + \rho\wedge\Omega(\mathcal{F}) - (-1)^e d\rho\wedge\theta] \\
&= [(-1)^{ef}\mathcal{F}\times_B\mathcal{E}, (-1)^{e+e(f-1)}\theta\wedge\Omega(\mathcal{E}) + (-1)^{f(e-1)}\Omega(\mathcal{F})\wedge\rho - \rho\wedge d\theta] \\
&= [(-1)^{ef}\mathcal{F}\times_B\mathcal{E}, (-1)^{ef}\theta\wedge\Omega(\mathcal{E}) + (-1)^{ef}(-1)^f\Omega(\mathcal{F})\wedge\rho - (-1)^{ef}(-1)^f d\theta\wedge\rho] \\
&= (-1)^{ef}[\mathcal{F},\theta]\cup[\mathcal{E},\rho].
\end{aligned}$$

4.1.4. — We now have a well-defined $\mathbb{Z}/2\mathbb{Z}$-graded commutative product

$$\cup\colon \hat{K}(B)\otimes\hat{K}(B)\to\hat{K}(B).$$

We show next that it is associative. First of all observe that the fibre product of geometric families is associative. Let e, f, g be the parities of the homogeneous classes $[\mathcal{E},\rho]$, $[\mathcal{F},\theta]$, and $[\mathcal{G},\kappa]$.

$$\begin{aligned}
&([\mathcal{E},\rho]\cup[\mathcal{F},\theta])\cup[\mathcal{G},\kappa] \\
&\quad= [\mathcal{E}\times_B\mathcal{F}, (-1)^e\Omega(\mathcal{E})\wedge\theta + \rho\wedge\Omega(\mathcal{F}) - (-1)^e d\rho\wedge\theta]\cup[\mathcal{G},\kappa] \\
&\quad= [\mathcal{E}\times_B\mathcal{F}\times_B\mathcal{G}, ((-1)^e\Omega(\mathcal{E})\wedge\theta + \rho\wedge\Omega(\mathcal{F}) - (-1)^e d\rho\wedge\theta)\wedge\Omega(\mathcal{G}) \\
&\qquad +(-1)^{e+f}\Omega(\mathcal{E}\times_B\mathcal{F})\wedge\kappa \\
&\qquad -(-1)^{e+f}d((-1)^e\Omega(\mathcal{E})\wedge\theta + \rho\wedge\Omega(\mathcal{F}) - (-1)^e d\rho\wedge\theta)\wedge\kappa] \\
&\quad= [\mathcal{E}\times_B\mathcal{F}\times_B\mathcal{G}, (-1)^e\Omega(\mathcal{E})\wedge\theta\wedge\Omega(\mathcal{G}) + \rho\wedge\Omega(\mathcal{F})\wedge\Omega(\mathcal{G}) \\
&\qquad -(-1)^e d\rho\wedge\theta\wedge\Omega(\mathcal{G}) + (-1)^{e+f}\Omega(\mathcal{E})\wedge\Omega(\mathcal{F})\wedge\kappa - (-1)^{e+f}\Omega(\mathcal{E})\wedge d\theta\wedge\kappa \\
&\qquad -(-1)^{e+f}d\rho\wedge\Omega(\mathcal{F})\wedge\kappa + (-1)^{e+f}d\rho\wedge d\theta\wedge\kappa]
\end{aligned}$$

On the other hand

$$
\begin{aligned}
&[\mathcal{E},\rho]\times([\mathcal{F},\theta]\times[\mathcal{G},\kappa])\\
&\quad= [\mathcal{E},\rho]\times[\mathcal{F}\times_B\mathcal{G},(-1)^f\Omega(\mathcal{F})\wedge\kappa+\theta\wedge\Omega(\mathcal{G})-(-1)^f d\theta\wedge\kappa]\\
&\quad= [\mathcal{E}\times_B\wedge\mathcal{F}\times_B\mathcal{G},(-1)^e\Omega(\mathcal{E})\wedge((-1)^f\Omega(\mathcal{F})\wedge\kappa+\theta\wedge\Omega(\mathcal{G})-(-1)^f d\theta\wedge\kappa)\\
&\qquad+\rho\wedge\Omega(\mathcal{F}\times_B\mathcal{G})-(-1)^e d\rho\wedge((-1)^f\Omega(\mathcal{F})\wedge\kappa+\theta\wedge\Omega(\mathcal{G})-(-1)^f d\theta\wedge\kappa)]\\
&\quad= [\mathcal{E}\times_B\mathcal{F}\times_B\mathcal{G},(-1)^{e+f}\Omega(\mathcal{E})\wedge\Omega(\mathcal{F})\wedge\kappa+(-1)^e\Omega(\mathcal{E})\wedge\theta\wedge\Omega(\mathcal{G})\\
&\qquad-(-1)^{e+f}\Omega(\mathcal{E})\wedge d\theta\wedge\kappa+\rho\wedge\Omega(\mathcal{F})\wedge\Omega(\mathcal{G})-(-1)^{e+f}d\rho\wedge\Omega(\mathcal{F})\wedge\kappa\\
&\qquad-(-1)^e d\rho\wedge\theta\wedge\Omega(\mathcal{G})+(-1)^{e+f}d\rho\wedge d\theta\wedge\kappa]
\end{aligned}
$$

By an inspection we see that the two right-hand sides agree.

4.1.5. — Let us observe that the unit $1\in\hat{K}(B)$ is simply given by $(B\times\mathbb{C},0)$, i.e. the trivial 0-dimensional family with fibre the graded vector space $\mathbb{C}$ concentrated in even degree, and with curvature form 1. The definition shows that this is actually a unit on the level of cycles. This finishes the proof of Proposition 4.2. □

4.1.6. — In this paragraph we study the compatibility of the cup product in smooth K-theory with the cup product in topological K-theory and the wedge product of differential forms.

Lemma 4.3. — *For $x,y\in\hat{K}(B)$ we have*

$$R(x\cup y)=R(x)\wedge R(y),\qquad I(x\cup y)=I(x)\cup I(y).$$

Furthermore, for $\alpha\in\Omega(B)/\mathtt{im}(d)$ we have

$$a(\alpha)\cup x=a(\alpha\wedge R(x)).$$

Proof. — Straightforward calculation using the definitions. □

Corollary 4.4. — *With the $\cup$-product smooth K-theory $\hat{K}$ is a multiplicative extension of the pair $(K,\mathbf{ch}_{\mathbb{R}})$.*

4.2. Projection formula

4.2.1. — Let $p\colon W\to B$ be a proper submersion with closed fibres with a smooth K-orientation represented by o. In this case we have a well-defined push-forward $\hat{p}_!\colon\hat{K}(W)\to\hat{K}(B)$. The explicit formula in terms of cycles is (17). The projection formula states the compatibility of the push-forward with the $\cup$-product.

Proposition 4.5. — *Let $x\in\hat{K}(W)$ and $y\in\hat{K}(B)$. Then*

$$\hat{p}_!(p^*y\cup x)=y\cup\hat{p}_!(x).$$

Proof. — Let $x = [\mathcal{F}, \sigma]$ and $y = [\mathcal{E}, \rho]$. By an inspection of the constructions we observe that the projection formula holds true on the level of geometric families

$$p_!(p^*\mathcal{E} \times_W \mathcal{F}) \cong \mathcal{E} \times_B p_!\mathcal{F}.$$

This implies

$$\Omega(p_!^\lambda(p^*\mathcal{E} \times_W \mathcal{F})) = \Omega(\mathcal{E}) \wedge \Omega(p_!^\lambda(\mathcal{F})).$$

Consequently we have $\tilde{\Omega}(\lambda, p^*\mathcal{E} \times_W \mathcal{F}) = (-1)^e \Omega(\mathcal{E}) \wedge \tilde{\Omega}(\lambda, \mathcal{F})$. Inserting the definitions of the product and the push-forward we get up to exact forms

$$\begin{aligned}
&\hat{p}_!(p^*y \cup x) \\
&\quad= \hat{p}_!([p^*\mathcal{E} \times_W \mathcal{F}, (-1)^e p^*\Omega(\mathcal{E}) \wedge \sigma + p^*\rho \wedge \Omega(\mathcal{F}) - (-1)^e p^* d\rho \wedge \sigma]) \\
&\quad= [p_!(p^*\mathcal{E} \times_W \mathcal{F}), \int_{W/B} \hat{\mathbf{A}}^c(o) \wedge [(-1)^e p^*\Omega(\mathcal{E}) \wedge \sigma + p^*\rho \wedge \Omega(\mathcal{F}) - (-1)^e p^* d\rho \wedge \sigma] \\
&\qquad + \int_{W/B} \sigma(o) \wedge R(p^*y \cup x) + \tilde{\Omega}(1, p^*\mathcal{E} \times_W \mathcal{F})] \\
&\quad= [\mathcal{E} \times_B p_!\mathcal{F}, \rho \wedge \int_{W/B} \hat{\mathbf{A}}^c(o) \wedge \Omega(\mathcal{F}) + (-1)^e \Omega(\mathcal{E}) \wedge \int_{W/B} \hat{\mathbf{A}}^c(o) \wedge \sigma \\
&\qquad + (-1)^e \Omega(\mathcal{E}) \wedge \tilde{\Omega}(1, \mathcal{F}) \\
&\qquad - \rho \wedge \int_{W/B} \hat{\mathbf{A}}^c(o) \wedge d\sigma + (-1)^e R(y) \wedge \int_{W/B} \sigma(o) \wedge R(x)].
\end{aligned} \tag{26}$$

Up to exact forms we have

$$\begin{aligned}
&\rho \wedge \int_{W/B} \hat{\mathbf{A}}^c(o) \wedge \Omega(\mathcal{F}) + (-1)^e \Omega(\mathcal{E}) \wedge \int_{W/B} \hat{\mathbf{A}}^c(o) \wedge \sigma \\
&+ (-1)^e \Omega(\mathcal{E}) \wedge \tilde{\Omega}(1, \mathcal{F}) \\
&- \rho \wedge \int_{W/B} \hat{\mathbf{A}}^c(o) \wedge d\sigma + (-1)^e R(y) \wedge \int_{W/B} \sigma(o) \wedge R(x) \\
= \; &(-1)^e \Omega(\mathcal{E}) \wedge \left(\int_{W/B} \hat{\mathbf{A}}^c(o) \wedge \sigma + \tilde{\Omega}(1, \mathcal{F}) + \int_{W/B} \sigma(o) \wedge R(x) \right) \\
&+ \rho \wedge \int_{W/B} \hat{\mathbf{A}}^c(o) \wedge (\Omega(\mathcal{F}) - d\sigma)) - (-1)^e d\rho \wedge \int_{W/B} \sigma(o) \wedge R(x) \\
= \; &(-1)^e \Omega(\mathcal{E}) \wedge \left(\int_{W/B} \hat{\mathbf{A}}^c(o) \wedge \sigma + \tilde{\Omega}(1, \mathcal{F}) + \int_{W/B} \sigma(o) \wedge R(x) \right) \\
&+ \rho \wedge \int_{W/B} (\hat{\mathbf{A}}^c(o) - d\sigma(o)) \wedge R(x) \\
= \; &(-1)^e \Omega(\mathcal{E}) \wedge \left(\int_{W/B} \hat{\mathbf{A}}^c(o) \wedge \sigma + \tilde{\Omega}(1, \mathcal{F}) + \int_{W/B} \sigma(o) \wedge R(x) \right) \\
&+ \rho \wedge R(\hat{p}_! x).
\end{aligned}$$

Thus the form component of (26) is exactly the one needed for the product $y \cup p_!(x)$. $\square$

4.3. Suspension

4.3.1. — We consider the projection $\mathtt{pr}_2\colon S^1 \times B \to B$. The goal of this subsection is to verify the relation

$$(\hat{\mathtt{pr}}_2)_! \circ \mathtt{pr}_2^* = 0$$

which is an important ingredient in the uniqueness result Theorem 1.4.

4.3.2. — The projection $\mathtt{pr}_2$ fits into the Cartesian diagram

$$\begin{tikzcd} S^1 \times B \arrow[r, "\mathtt{pr}_1"] \arrow[d, "\mathtt{pr}_2"] & S^1 \arrow[d, "p"] \\ B \arrow[r, "r"] & *. \end{tikzcd}$$

We choose the metric g^{TS^1} of unit volume and the bounding spin structure on TS^1. This spin structure induces a $Spin^c$ structure on TS^1 together with the connection $\tilde{\nabla}$. In this way we get a representative o of a smooth K-orientation of p. By pull-back we get the representative r^*o of a smooth K-orientation of $\mathtt{pr}_2$ which is used to define $(\hat{\mathtt{pr}}_2)_!$.

4.3.3. — Using the projection formula Proposition 4.5 we get for $x \in \hat{K}(B)$

$$(\hat{\mathtt{pr}}_2)_!(\mathtt{pr}_2^*(x)) = (\hat{\mathtt{pr}}_2)_!(\mathtt{pr}_2^*(x) \cup 1) = x \cup (\hat{\mathtt{pr}}_2)_! 1.$$

Using the compatibility of the push-forward with Cartesian diagrams Lemma 3.20 we get

$$(\hat{\mathtt{pr}}_2)_! 1 = (\hat{\mathtt{pr}}_2)_!(\mathtt{pr}_1^*(1)) = r^* \hat{p}_!(1).$$

We let $\mathcal{S}^1$ denote the geometric family over $*$ given by $p\colon S^1 \to *$ with the geometry described above. Since S^1 has the bounding *Spin*-structure the Dirac operator is invertible and has a symmetric spectrum. The family $\mathcal{S}^1$ therefore has a canonical taming $\mathcal{S}^1_t$ by the zero smoothing operator, and we have $\eta(\mathcal{S}^1_t) = 0$. This implies

$$\hat{p}_!(1) = [\mathcal{S}^1, 0] = [\emptyset, \eta(\mathcal{S}^1_t)] = [\emptyset, 0] = 0.$$

Corollary 4.6. — *We have* $(\hat{\mathtt{pr}}_2)_! \circ \mathtt{pr}_2^* = 0$.

5. Constructions of natural smooth K-theory classes

5.1. Calculations

5.1.1.

Lemma 5.1. — *We have*

$$\hat{K}^*(*) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{R}/\mathbb{Z} & * = 1. \end{cases}$$

Proof. — We use the exact sequence given by Proposition 2.20. The assertion follows from the obvious identities

$$\hat{K}^0(*) \cong K^0(*) \cong \mathbb{Z}, \quad \hat{K}^1(*) \cong \Omega^{\mathrm{ev}}(*)/\mathbf{ch}_{dR}(K^0(*)) \cong \mathbb{R}/\mathbb{Z}. \qquad \square$$

5.1.2.

Lemma 5.2. — *There are exact sequences*

$$\begin{array}{rcl} 0 \to \mathbb{R}/\mathbb{Z} \to & \hat{K}^0(S^1) & \to \mathbb{Z} \to 0 \\ 0 \to C^\infty(S^1)/\mathbb{Z} \to & \hat{K}^1(S^1) & \to \mathbb{Z} \to 0. \end{array}$$

Proof. — These assertions again follow from Proposition 2.20 and the identifications

$$K^0(S^1) \cong \mathbb{Z}, \quad K^1(S^1) \cong \mathbb{Z}, \quad \Omega^{\mathrm{ev}}(S^1)/\mathbf{ch}_{dR}(K^0(S^1)) \cong C^\infty(S^1)/\mathbb{Z}. \qquad \square$$

5.1.3. — Let $\mathbf{V} := (V, h^V, \nabla^V, z)$ be a geometric $\mathbb{Z}/2\mathbb{Z}$-graded bundle over S^1 such that $\dim(V^+) = \dim(V^-)$. Let $\mathcal{V}$ denote the corresponding geometric family. By Lemma 5.2 the class $[\mathcal{V}, 0] \in \hat{K}^0(S^1)$ satisfies $I([\mathcal{V}, 0]) = 0$ and hence corresponds to an element of $\mathbb{R}/\mathbb{Z}$. This element is calculated in the following lemma. Let $\phi^\pm \in U(n)/conj$ denote the holonomies of $V^\pm$ (well defined modulo conjugation in the group $U(n)$).

Lemma 5.3. — *We have*

$$[\mathcal{V}, 0] = a\left(\frac{1}{2\pi i} \log \frac{\mathtt{det}(\phi^+)}{\mathtt{det}(\phi^-)}\right).$$

Proof. — We consider the map $q\colon S^1 \to *$ with the canonical K-orientation 4.3.2. By Proposition 3.19 we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{R}/\mathbb{Z} & \xrightarrow{\sim} & \Omega^1(S^1)/(\mathtt{im}(d) + \mathtt{im}(\mathbf{ch}_{dR})) & \xrightarrow{a} & \hat{K}^1(S^1) \\ \downarrow{=} & & \downarrow{q^o_!} & & \downarrow{\hat{q}_!} \\ \mathbb{R}/\mathbb{Z} & \xrightarrow{\sim} & \Omega^0(*)/\mathtt{im}(\mathbf{ch}_{dR}) & \xrightarrow{a} & \hat{K}^0(*). \end{array}$$

In order to determine $[\mathcal{V}, 0]$ it therefore suffices to calculate $\hat{q}_!([\mathcal{V}, 0])$. Now observe that $q\colon S^1 \to *$ is the boundary of $p\colon D^2 \to *$. Since the underlying topological K-orientation of q is given by the bounding $Spin$-structure we can choose a smooth K-orientation of p with product structure which restricts to the smooth K-orientation of q. The bundle $\mathbf{V}$ is topologically trivial. Therefore we can find a geometric bundle $\mathbf{W} = (W, h^W, \nabla^W, z)$, again with product structure, on D^2 which restricts to $\mathbf{V}$ on the boundary. Let $\mathcal{W}$ denote the corresponding geometric family over D^2. Later we prove the bordism formula Proposition 5.18. It gives

$$\hat{q}_!([\mathcal{V}, 0]) = [\varnothing, p_! R([\mathcal{W}, 0])] = -a\left(\int_{D^2/*} \Omega^2(\mathcal{W})\right).$$

Note that

$$\Omega^2(\mathcal{W}) = \mathbf{ch}_2(\nabla^W) = \mathbf{ch}_2(\nabla^{\mathtt{det}(W^+)}) - \mathbf{ch}_2(\nabla^{\mathtt{det}(W^-)}) = \frac{-1}{2\pi i}\left[R^{\nabla^{\mathtt{det} W^+}} - R^{\mathtt{det}\nabla^{W^-}}\right].$$

The holonomy $\mathtt{det}(\phi^\pm) \in U(1)$ of $\mathtt{det}(\mathbf{V}^\pm)$ is equal to the integral of the curvature of $\mathtt{det}\mathbf{W}^\pm$:

$$\log \mathtt{det}(\phi^\pm) = \int_{D^2} R^{\nabla^{\mathtt{det}(W^\pm)}}.$$

It follows that

$$\hat{q}_!([\mathcal{V}, 0]) = a\left(\frac{1}{2\pi i}\log\frac{\mathtt{det}(\phi^+)}{\mathtt{det}(\phi^-)}\right).$$ □

5.2. The smooth K-theory class of a mapping torus

5.2.1. — Let $\mathcal{E}$ be a geometric family over a point and consider an automorphism ϕ of $\mathcal{E}$. Then we can form the mapping torus $T(\mathcal{E}, \phi) := (\mathbb{R} \times \mathcal{E})/\mathbb{Z}$, where $n \in \mathbb{Z}$ acts on $\mathbb{R}$ by $x \mapsto x + n$, and by ϕ^n on $\mathcal{E}$. The product $\mathbb{R} \times \mathcal{E}$ is a $\mathbb{Z}$-equivariant geometric family over $\mathbb{R}$ (the pull-back of $\mathcal{E}$ by the projection $\mathbb{R} \to *$). The geometric structures descend to the quotient and turn the mapping torus $T(\mathcal{E}, \phi)$ into a geometric family over $S^1 = \mathbb{R}/\mathbb{Z}$. In the present subsection we study the class

$$[T(\mathcal{E}, \phi), 0] \in \hat{K}(S^1).$$

In the following we will assume that the parity of $\mathcal{E}$ is even, and that $\mathtt{index}(\mathcal{E}) = 0$.

5.2.2. — Let $\dim\colon K^0(S^1) \to \mathbb{Z}$ be the dimension homomorphism, which in this case is an isomorphism. Since $\dim I([T(\mathcal{E}, \phi), 0]) = \dim(\mathtt{index}(\mathcal{E})) = 0$ we have in fact $[T(\mathcal{E}, \phi), 0] \in \mathbb{R}/\mathbb{Z} \subset \hat{K}^0(S^1)$, where we consider $\mathbb{R}/\mathbb{Z}$ as a subgroup of $\hat{K}^0(S^1)$ according to Lemma 5.2.

Let $V := \ker(D(\mathcal{E}))$. This graded vector space is preserved by the action of ϕ. We use the same symbol in order to denote the induced action on V.

We form the zero-dimensional family $\mathcal{V} := (\mathbb{R} \times V)/\mathbb{Z}$ over S^1. This bundle is isomorphic to the kernel bundle of $T(\mathcal{E}, \phi)$. The bundle of Hilbert spaces of the family $T(\mathcal{E}, \phi) \sqcup_{S^1} \mathcal{V}^{\mathrm{op}}$ has a canonical subbundle of the form $\mathcal{V} \oplus \mathcal{V}^{\mathrm{op}}$. We choose the taming $(T(\mathcal{E}, \phi) \sqcup_{S^1} \mathcal{V}^{\mathrm{op}})_t$ which is induced by the isomorphism

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

on this subbundle. Note that $[T(\mathcal{E},\phi),0]=[\mathcal{V},\eta((T(\mathcal{E},\phi)\sqcup_{S^1}\mathcal{V}^{\mathrm{op}})_t)]$. Since the pull-back of $(T(\mathcal{E},\phi)\sqcup_{S^1}\mathcal{V}^{\mathrm{op}})_t$ under $\mathbb{R}\to\mathbb{R}/\mathbb{Z}$ is isomorphic to a tamed family pulled back under $\mathbb{R}\to *$ we see that the one-form $\eta((T(\mathcal{E},\phi)\sqcup_{S^1}\mathcal{V}^{\mathrm{op}})_t)=0$.

5.2.3. — Thus it remains to evaluate $[T(\mathcal{E},\phi),0]=[\mathcal{V},0]\in\mathbb{R}/\mathbb{Z}$. By Lemma 5.3 this number can be expressed in terms of the holonomy of the determinant bundle $\mathtt{det}(\mathcal{V})$. Let $\phi^\pm\in\mathtt{Aut}(V^\pm)$ be the induced transformations.

Proposition 5.4. — *We have* $[T(\mathcal{E},\phi),0]=[\frac{1}{2\pi i}\log(\frac{\mathtt{det}\phi^+}{\mathtt{det}\phi^-})]_{\mathbb{R}/\mathbb{Z}}$. *In particular, if* $D(\mathcal{E})$ *is invertible, then* $[T(\mathcal{E},\phi),0]=0$.

5.3. The smooth K-theory class of a geometric family with kernel bundle

5.3.1. — Let $\mathcal{E}$ be an even-dimensional geometric family over the base B. By $(D_b)_{b\in B}$ we denote the associated family of Dirac operators on the family of Hilbert spaces $(H_b)_{b\in B}$. The geometry of $\mathcal{E}$ induces a connection ∇^H on this family (the connection part of the Bismut superconnection [**7**, Prop. 10.15]). We assume that $\dim(\ker(D_b))$ is constant. In this case we can form a vector bundle $K:=\ker(D)$. The projection of ∇^H to K gives a connection ∇^K. Hence we get a geometric bundle $\mathbf{K}:=(K,h^K,\nabla^K)$ and an associated geometric family $\mathcal{K}$ (see 2.1.4).

5.3.2. — The sum $\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}}$ has a natural taming $(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})_t$ which is given by

$$\begin{pmatrix}0 & u\\ u^* & 0\end{pmatrix}\in\mathtt{End}(H_b\oplus K_b^{\mathrm{op}}),$$

where $u\colon K_b\to H_b$ is the embedding. We thus have the following equality in $\hat{K}(B)$:

$$[\mathcal{E},0]=[\mathcal{K},\eta((\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})_t)].$$

5.3.3. — Under the standing assumption that $\dim(\ker(D_b))$ is constant we also have the η-form of Bismut-Cheeger $\eta^{BC}(\mathcal{E})\in\Omega(B)$ (see [**14**], [**13**], [**12**]). Since other authors use $\eta^{BC}(\mathcal{E})$, in the following two paragraphs we shall analyse the relation between this and $\eta((\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})_t)$.

We form the geometric family $[0,1]\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})$ over B. The taming $(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})_t$ induces a boundary taming at $\{0\}\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})$. In index theory the boundary taming is used to construct a perturbation of the Dirac operator which is invertible at $-\infty$ of $(-\infty,1]\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})$ (see [**19**] for details). On the other side $\{1\}\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})$ we consider APS-boundary conditions. We thus get a family of perturbed Dirac operators on $(-\infty,1]\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})$. The L^2-boundary condition at $\{-\infty\}\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})$ and the APS-boundary condition at $\{1\}\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})$ together imply the Fredholm property (which can be checked locally for the various boundary components or ends). In this way the family of Dirac operators on $[0,1]\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})$ gives rise to a family of Fredholm operators. We will denote this structure by $([0,1]\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}}))_{bt,APS}$.

The Chern character of its index $\mathtt{index}(([0,1]\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}}))_{bt,APS})\in K(B)$ can be calculated using the methods of local index theory.

5.3.4. — Using 2.4.10 we can choose a possibly different taming $(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})_{t'}$ such that the corresponding index $\mathtt{index}(([0,1]\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}}))_{bt',APS})\in K(B)$ vanishes. In this case we can extend the boundary taming to a taming $\mathtt{index}(([0,1]\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}}))_{t',APS})$.

We set up the method of local index theory as usual by forming the family of rescaled Bismut superconnections $A_s:=A_s(([0,1]\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}}))_{t',APS})$ which take the tamings and boundary tamings into account as explained in [**19**, 2.2.4.3], see also 3.2.6. Invertibility of $D(([0,1]\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}}))_{t',APS})$ ensures exponential vanishing of the integral kernel of $\mathtt{e}^{-A_s^2}$ for $s\to\infty$. The usual transgression integral expresses the local index form $\Omega([0,1]\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}}))$ as a sum of contributions of the boundary components or ends (see [**19**, proof of Lemma 2.2.15]). These contributions can be calculated separately for each part.

Because of the product structure we have $\Omega([0,1]\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}}))=0$. The contribution of the boundary $\{1\}\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})$ is given by the proof of the APS-index theorem of [**14**], [**13**], [**12**], and it is equal to $\eta^{BC}(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})=\eta^{BC}(\mathcal{E})$. The second equality holds true, since the Dirac operator for $\mathcal{K}^{\mathrm{op}}$ is trivial. The contribution of the boundary $\{0\}\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})$ is calculated in the proof of [**19**, Lemma 2.2.15] and equal to $-\eta((\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})_{t'})$. Therefore we have $\eta^{BC}(\mathcal{E})=\eta((\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})_{t'})$ (note that we calculate modulo exact forms). We now use 2.4.10 and a relative index theorem (compare (28)) in order to see that

$$\eta((\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})_{t'})-\eta((\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}})_{t})=\mathbf{ch}_{dR}(\mathtt{index}(([0,1]\times(\mathcal{E}\sqcup_B\mathcal{K}^{\mathrm{op}}))_{bt,APS}))\in\mathbf{ch}_{dR}(K(B)).$$

Using Proposition 2.20 we get:

Corollary 5.5. — *We have* $[\mathcal{E},0]=[\mathcal{K},\eta^{BC}(\mathcal{E})]$.

5.3.5. — Let $p\colon W\to B$ be a proper submersion with closed fibres with a smooth K-orientation represented by o. Let $\mathbf{V}$ be a geometric vector bundle over W, and let $\mathcal{V}$ denote the associated geometric family. Then we can form the geometric family $\mathcal{E}:=p_!\mathcal{V}$ (see Definition 3.7). Assume that the kernel of the family of Dirac operators $(D(\mathcal{E}_b))_{b\in B}$ has constant dimension, forming thus the kernel bundle $\mathcal{K}$. Since $\mathcal{V}$ has zero-dimensional fibres we have $\tilde\Omega(1,\mathcal{V})=0$. From (17) we get

$$\begin{aligned}\hat p_![\mathcal{V},\rho]&=[p_!\mathcal{V},\int_{W/B}\hat{\mathbf{A}}^c(o)\wedge\rho+\int_{W/B}\sigma(o)\wedge(\Omega(\mathcal{V})-d\rho)]\\&=[\mathcal{E},\int_{W/B}\hat{\mathbf{A}}^c(o)\wedge\rho+\int_{W/B}\sigma(o)\wedge(\Omega(\mathcal{V})-d\rho)]\end{aligned}$$

$$= [\mathcal{K}, \eta^{BC}(\mathcal{E}) + \int_{W/B} \hat{\mathbf{A}}^c(o) \wedge \rho + \int_{W/B} \sigma(o) \wedge (\Omega(\mathcal{V}) - d\rho)].$$

5.4. A canonical $\hat{K}^1$-class on S^1

5.4.1. — We construct in a natural way an element $x_{S^1} \in \hat{K}^1(S^1)$ coming from the Poincaré bundle over $S^1 \times S^1$. Let us identify $S^1 \cong \mathbb{R}/\mathbb{Z}$. We consider the complex line bundle $L := (\mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \mathbb{C})/\mathbb{Z}$ over $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, where the $\mathbb{Z}$-action is given by $n(s,t,z) = (s+n, t, \exp(-2\pi i n t)z)$. On $\mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \mathbb{C} \to \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ we have the $\mathbb{Z}$-equivariant connection $\nabla := d + 2\pi i s dt$ with curvature $R^\nabla = 2\pi i ds \wedge dt$. This connection descends to a connection ∇^L on L. The unitary line bundle with connection $\mathbf{L} := (L, h^L, \nabla^L)$ gives a geometric family $\mathcal{L}$ over $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. It represents $v := [\mathcal{L}, 0] \in \hat{K}^0(\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z})$. Note that $R(v) = 1 + ds \wedge dt$. We now consider the projection $p\colon \mathbb{R}/\mathbb{Z} \times \mathbb{R}/Z \to \mathbb{R}/\mathbb{Z}$ on the second factor. This fibre bundle has a natural smooth $\hat{K}$-orientation $(g^{T^v p}, T^h p, \tilde{\nabla}, 0)$. The vertical metric and the horizontal distribution come from the metric of S^1 and the product structure. Moreover, $T^v p$ is trivialized by the S^1-action. Hence it has a preferred orientation. We take the bounding *Spin*-structure on the fibres which induces the $Spin^c$-structure and the connection $\tilde{\nabla}$.

Definition 5.6. — *We define* $x_{S^1} := \hat{p}_! v \in \hat{K}^1(S^1)$.

5.4.2. — We have $R(x_{S^1}) = dt$. Let $t \in S^1$. Then we compute $t^* x_{S^1} \in \hat{K}^1(*) \cong \mathbb{R}/\mathbb{Z}$ (identification again as in Lemma 5.2). Note that $0^* x_{S^1}$ is represented by the trivial line bundle over S^1. Since we choose the bounding spin structure, the corresponding Dirac operator is invertible. Its spectrum is symmetric and its η-invariant vanishes (compare 4.3.3). Therefore we have $0^* x_{S^1} = 0$. It now follows by the homotopy formula (or by an explicit computation of η-invariants), that

$$t^* x_{S^1} = -t. \tag{27}$$

5.4.3. — Let $f\colon B \to S^1$ be given. Then we define

Definition 5.7. — $<f> := f^* x_{S^1} \in \hat{K}^1(B)$.

Assume now that we have two such maps $f, g\colon B \to S^1$. As an interesting illustration we characterize

$$<f> \cup <g> \in \hat{K}^0(B).$$

It suffices to consider the universal example $B = T^2 = S^1 \times S^1$. We consider the projections $\mathtt{pr}_i\colon S^1 \times S^1 \to S^1$, $i = 1, 2$. Let $x := \hat{\mathtt{pr}}_1^* x_{S^1}$ and $y := \hat{\mathtt{pr}}_2^* x_{S^1}$. Then we must compute $x \cup y \in \hat{K}^0(T^2)$. We identify $T^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ with coordinates s, t. First note that $R(x \cup y) = R(x) \cup R(y) = ds \wedge dt$. Thus the class $x \cup y - v + 1$ is flat, i.e.

$$x \cup y - v + 1 \in K^0_{\mathrm{flat}}(T^2).$$

In fact, since $K^0(T^2)$ is torsion-free, we have

$$K^0_{\mathrm{flat}}(T^2) \cong H^{\mathrm{odd}}(T^2)/\mathtt{im}(\mathbf{ch}_{dR}) = \mathbb{R}^2/\mathbb{Z}^2.$$

In order to determine this element we must compute its holonomies along the circles $S^1 \times 0$ and $0 \times S^1$. The holonomy of v along these circles is trivial. Since $0^*x = 0$ and $0^*y = 0$ we see that $x \times y$ also has trivial holonomies along these circles. Therefore we conclude

Proposition 5.8. — *We have* $x \cup y = v - 1$.

Now we solve our original problem. The two maps f, g induce a map $f \times g \colon B \to T^2$.

Corollary 5.9. — *We have* $<f> \cup <g> = (f \times g)^* v - 1$.

5.5. The product of S^1-valued maps and line-bundles

5.5.1. — Let $f \colon B \to S^1$ be a smooth map and $\mathbf{L} := (L, \nabla^L, h^L)$ be a hermitean line bundle with connection over B. It gives rise to a geometric family $\mathcal{L}$ (see 2.1.4). We consider the smooth K-theory classes $<f>$ and $<\mathbf{L}> := [\mathcal{L}, 0] - 1$. It is again interesting to determine the class

$$<f> \cup <\mathbf{L}> \in \hat{K}^1(B).$$

An explicit answer is only known in special cases.

First we compute the curvature:

$$R(<f> \cup <\mathbf{L}>) = R(<f>) \wedge R(<\mathbf{L}>) = df \wedge (e^{c_1(\nabla^L)} - 1),$$

where $df := f^* dt$ and $c_1(\nabla^L) := -\frac{1}{2\pi i} R^{\nabla^L}$.

5.5.2. — Note that the degree-one component of the odd form $R(<f> \cup <\mathbf{L}>)$ vanishes. Let now $q \colon \Sigma \to B$ be a smooth map from an oriented closed surface. Then $R(q^*(<f> \cup <\mathbf{L}>)) = q^* R((<f> \cup <\mathbf{L}>)) = 0$. Therefore

$$q^*(<f> \cup <\mathbf{L}>) \in \hat{K}^1_{\mathrm{flat}}(\Sigma) \cong H^{\mathrm{ev}}(\Sigma, \mathbb{R})/\mathtt{im}(\mathbf{ch}) \cong \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z},$$

where the first component corresponds to $H^0(\Sigma, \mathbb{R})$ and the second to $H^2(\Sigma, \mathbb{R})$. In order to evaluate the first component we restrict to a point. Since the restriction of $<\mathbf{L}>$ to a point vanishes, the first component of $q^*(<f> \cup <\mathbf{L}>)$ vanishes. Therefore it remains to determine the second component.

5.5.3. — Let us assume that q^*L is trivial. We choose a trivialization. Then we can define the transgression Chern form $\tilde{c}_1(\nabla^{q^*L}, \nabla^{\mathrm{triv}}) \in \Omega^1(\Sigma)$ such that $d\tilde{c}_1(\nabla^{q^*L}, \nabla^{\mathrm{triv}}) = q^* c_1(\nabla^L)$. By the homotopy formula we have

$$q^* <\mathbf{L}> = [\varnothing, -\tilde{c}_1(\nabla^{q^*L}, \nabla^{\mathrm{triv}})].$$

In this special case we can compute

$$\begin{aligned} q^*(< f > \cup < \mathbf{L} >) &= q^* < f > \cup\ \ q^* < \mathbf{L} > \\ &= < q^* f > \cup\ \ q^* < \mathbf{L} > = [\emptyset, q^* df \wedge \tilde{c}_1(\nabla^{q^* L}, \nabla^{\mathrm{triv}})]. \end{aligned}$$

We see that the second component is

$$\left[\int_\Sigma q^* df \wedge \tilde{c}_1(\nabla^{q^* L}, \nabla^{\mathrm{triv}})\right]_{\mathbb{R}/\mathbb{Z}} .$$

We do not know a good answer in the general case where $q^* L$ is non-trivial.

5.6. A bi-invariant $\hat{K}^1$- class on $SU(2)$

5.6.1. — Let G be a group acting on the manifold M.

Definition 5.10. — *A class $x \in \hat{K}(M)$ is called invariant, if $g^* x = x$ for all $x \in G$.*

5.6.2. — For example, the class $x_{S^1} \in \hat{K}^1(S^1)$ defined in 5.6 is not invariant under the action L_t, $t \in S^1$, of S^1 on itself. Note that $R(x_{S^1}) = dt$ is invariant. Therefore $L_t^* x_{S^1} - x_{S^1} \in \mathbb{R}/\mathbb{Z}$. In fact by (27) we have

$$L_t^* x_{S^1} - x_{S^1} = -t.$$

Since dt is the only invariant form with integral one we see that the only way to produce an invariant smooth refinement of the generator of $H^1(S^1, \mathbb{Z}) \cong \mathbb{Z}$ would be to perturb x_{S^1} by a class $b \in H^0(S^1, \mathbb{R}/\mathbb{Z})$. But b is of course homotopy invariant, hence $L_t^* b = b$. We conclude that the generator of $H^1(S^1, \mathbb{Z})$ (and also every non-trivial multiple) does not admit any invariant lift.

5.6.3. — The situation is different for simply-connected groups. Let us consider the following example. The group $G := SU(2) \times SU(2)$ acts on $SU(2)$ by $(g_1, g_2)h := g_1 h g_2^{-1}$. Let $\mathtt{vol}_{SU(2)} \in \Omega^3(SU(2))$ denote the normalized volume form. Furthermore we let $i \colon * \to SU(2)$ denote the embedding of the identity.

Proposition 5.11. — *For $k \in \mathbb{Z}$ there exists a unique class $x_{SU(2)}(k) \in \hat{K}^1(SU(2))$ such that $R(x_{SU(2)}) = k\mathtt{vol}_{SU(2)}$ and $i^* x = 0$. This element is $SU(2) \times SU(2)$-invariant*

Proof. — Assume, that $x, y \in \hat{K}^1(SU(2))$ satisfy $R(x) = R(y)$. Then we have $x - y \in \hat{K}^1_{\mathrm{flat}}(SU(2)) \cong K^1_{\mathrm{flat}}(S^3) \cong \mathbb{R}/\mathbb{Z}$. Since $i^* x = i^* y = 0$ we have in fact that $x = y$. Therefore, if the class $x_{SU(2)}(k)$ exists, then it is unique.

We show the existence of an invariant class in an abstract manner. Note that $k\mathtt{vol}_{SU(2)}$ represents a class $\mathbf{ch}(Y)$ for some $Y \in K^1(S^3)$. In terms of classifying maps, Y for $k=1$ is given by the embedding $SU(2)\to U(2)\to U(\infty)\cong K^1$. We have the exact sequence

$$0\to\Omega^{\mathrm{ev}}(SU(2))/\mathtt{im}(\mathbf{ch}_{dR})\xrightarrow{a}\hat{K}^1(SU(2))\xrightarrow{I}K^1(SU(2))\to 0.$$

Therefore we can choose any class $y\in\hat{K}^1(SU(2))$ such that $I(y)=Y$. Then the continuous group cocycle $G\ni t\to c(t)=t^*y-y\in\Omega^{\mathrm{ev}}(SU(2))/\mathtt{im}(\mathbf{ch}_{dR})$ represents an element $[c]\in H^1_c(G,\Omega^{\mathrm{ev}}(SU(2))/\mathtt{im}(\mathbf{ch}_{dR}))$.

We claim that this cohomology group is trivial. Note that $\Omega^{\mathrm{ev}}(SU(2))/\mathtt{im}(\mathbf{ch}_{dR})\cong\Omega^0(SU(2))/\mathbb{Z}\oplus\Omega^2(SU(2))/\mathtt{im}(d)$. Since $\Omega^2(SU(2))/\mathtt{im}(d)$ is a real topological vector space with a continuous action of the compact group G we immediately conclude that $H^1_c(G,\Omega^2(SU(2))/\mathtt{im}(d))=0$ by the usual averaging argument. We consider the exact sequence of G-spaces

$$0\to\mathbb{Z}\to\Omega^0(SU(2))\to\Omega^0(SU(2))/\mathbb{Z}\to 0.$$

Since G is simply-connected we see that taking continuous functions from $G\times\cdots\times G$ with values in these spaces, we obtain again exact sequences of $\mathbb{Z}$-modules. It follows that we have a long exact sequence in continuous cohomology. The relevant part reads

$$H^1_c(G,\mathbb{Z})\to H^1_c(G,\Omega^0(SU(2)))\to H^1_c(G,\Omega^0(SU(2))/\mathbb{Z})\to H^2_c(G,\mathbb{Z}).$$

Since $\mathbb{Z}$ is discrete and G is connected we see that $H^i_c(G,\mathbb{Z})=0$ for $i\geq 1$. Therefore,

$$H^1_c(G,\Omega^0(SU(2)))\cong H^1_c(G,\Omega^0(SU(2))/\mathbb{Z}).$$

But $\Omega^0(SU(2))$ is again a continuous representation of G on a real vector space so that $H^1_c(G,\Omega^0(SU(2)))=0$. The claim follows.

We now can choose $w\in\Omega^{\mathrm{ev}}(SU(2))/\mathtt{im}(\mathbf{ch}_{dR})$ such that $t^*w-w=t^*y-y$ for all $t\in G$. We can further assume that $i^*w=i^*y$ by adding a constant. Then we set $x_{SU(2)}(k)=y-w\in\hat{K}^1(SU(2))$. This element has the required properties. □

It is an interesting problem to write down an invariant cycle which represents the class $x_{SU(2)}$.

5.6.4. — Note that $x_{SU(2)}(k)=kx_{SU(2)}(1)$. Let $\Sigma\subset SU(2)$ be an embedded oriented hypersurface. Then $R(x_{SU(2)}(1))_{|\Sigma}=0$ so that $(x_{SU(2)})_{|\Sigma}\in\hat{K}^1_{\mathrm{flat}}(\Sigma)$. Since $x_{SU(2)}(1)$ evaluates trivially on points we have in fact

$$(x_{SU(2)}(1))_{|\Sigma}\in\ker\left(\hat{K}^1_{\mathrm{flat}}(\Sigma)\to\hat{K}^1_{\mathrm{flat}}(*)\right)\cong\mathbb{R}/\mathbb{Z}.$$

This number can be determined by integration over Σ. Formally, let $p\colon\Sigma\to\{*\}$ be the projection. If we choose some smooth K-orientation, then we can ask for $\hat{p}_!(x_{SU(2)}(1))_{|\Sigma}\in\hat{K}^1_{\mathrm{flat}}(*)\cong\mathbb{R}/\mathbb{Z}$. The hypersurface Σ decomposes $SU(2)$ in two parts $SU(2)^{\pm}_{\Sigma}$. Let $SU(2)^+_{\Sigma}$ be the part such that $\partial SU(2)^+_{\Sigma}$ has the orientation given by Σ. We choose a K-orientation o of the projection $q\colon SU(2)^+_{\Sigma}\to *$ which has a product structure such that $\sigma(o)=0$ and $\hat{\mathbf{A}}^c(o)=1$. In order to get the latter equality we choose a $Spin^c$-structure coming from a spin structure. The smooth K-orientation of q induces a smooth K-orientation of p. Then $q\colon SU(2)^+_\Sigma \to *$ provides a zero-bordism of Σ, and of $(x_{SU(2)}(1))_{|\Sigma}$. Therefore, we have by Proposition 5.18

$$\hat{p}_!(x_{SU(2)}(1))_{|\Sigma} = \left[\varnothing, \int_{SU(2)^+_\Sigma} R(x_{SU(2)}(1))\right] = -[\mathtt{vol}(SU(2)^+_\Sigma)]_{\mathbb{R}/\mathbb{Z}},$$

where $[\lambda]_{\mathbb{R}/\mathbb{Z}}$ denotes the class of $\lambda \in \mathbb{R}$. Note that the identification $\hat{K}^1_{\mathrm{flat}}(*) \cong \mathbb{R}/\mathbb{Z}$ is induced by $a\colon \mathbb{R} \cong \Omega^{\mathrm{odd}}(*)/\mathtt{im}(d) \to K^1_{\mathrm{flat}}(*)$ given by $\lambda \mapsto [\varnothing, -\lambda]$. This explains the minus sign in the second equality above.

5.7. Invariant classes on homogeneous spaces

5.7.1. — Some of the arguments from the $SU(2)$-case generalize. Let G be a compact connected and simply-connected Lie group and G/H be a homogenous space.

Given $Y \in K(G/H)$ we can find a lift $y \in \hat{K}(G/H)$. We form the cocycle $G \ni g \mapsto c(g) := g^*y - y \in \Omega(G/H)/\mathtt{im}(\mathbf{ch}_{dR})$. Since $\Omega(G/H)/\mathtt{im}(\mathbf{ch}_{dR})$ is the quotient of a vector space by a lattice and G is connected and simply-connected we can use the arguments as in the $SU(2)$-case in order to conclude that $H^1_c(G, \Omega(G/H)/\mathtt{im}(\mathbf{ch}_{dR})) = 0$. Therefore we can choose the lift y such that $g^*y = y$ for all $g \in G$. In particular, $R(y) \in \Omega(G/H)$ is now an invariant form representing $\mathbf{ch}(Y)$. Note that an invariant form is in general not determined by this condition.

5.7.2. — If we specialize to the case that G/H is symmetric, then invariant forms exactly represent the cohomology. In this case we see that two choices of invariant lifts y_0, y_1 of Y have the same curvature so that $y_1 - y_0 \in \hat{K}_{\mathrm{flat}}(G/H)$. Since the y_i also have the same index, we indeed have $y_1 - y_0 \in H(G/H, \mathbb{R})/\mathtt{im}(\mathbf{ch}_{dR})$. We have thus shown the following lemma.

Lemma 5.12. — *Assume that G/H is a symmetric space with G connected and simply connected. Then every $Y \in K(G/H)$ has an invariant lift $y \in \hat{K}(G/H)$ which is uniquely determined up to $H(G/H, \mathbb{R})/\mathtt{im}(\mathbf{ch}_{dR})$.*

5.7.3. — We can apply this in certain cases. First we write $S^{2n+1} \cong Spin(2n+2)/Spin(2n+1)$, $n \geq 1$. Note that $K^1(S^{2n+1}) \cong \mathbb{Z}$. Since $H^{\mathrm{ev}}(S^{2n+1}, \mathbb{R})/\mathtt{im}(\mathbf{ch}_{dR}) = \mathbb{R}/\mathbb{Z}$ is concentrated in degree zero we have the following result.

Corollary 5.13. — *Let $n \geq 1$. For each $k \in \mathbb{Z}$ there is a unique $x_{S^{2n+1}}(k) \in \hat{K}^1(S^{2n+1})$ which is invariant, has index $k \in \mathbb{Z} \cong K^1(S^{2n+1})$, and evaluates trivially on points.*

5.7.4. — In the even-dimensional case we write $S^{2n} \cong Spin(2n+1)/Spin(2n)$, $n \geq 1$. Note that $K^0(S^{2n}) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H^{\mathrm{odd}}(S^{2n}, \mathbb{R})/\mathtt{im}(\mathbf{ch}_{dR}) = 0$.

Corollary 5.14. — *For each $k \in \mathbb{Z}$ there is a unique $x_{S^{2n}}(k) \in \hat{K}^0(S^{2n})$ which is invariant and has index $k \in \mathbb{Z} \cong \tilde{K}^0(S^{2n})$, and evaluates trivially on points*

5.7.5. — We write $\mathbb{CP}^n := SU(n+1)/S(U(1) \times U(n))$. Then $H^{\mathrm{odd}}(\mathbb{CP}^n, \mathbb{R})/\mathtt{im}(\mathbf{ch}_{dR}) = 0$. Therefore we conclude:

Lemma 5.15. — *For each $Y \in K^0(\mathbb{CP}^n)$ there is a unique $SU(n+1)$-invariant class $y_{\mathbb{CP}^n}(Y) \in \hat{K}^0(\mathbb{CP}^n)$ such that $I(y_{\mathbb{CP}^n}(Y)) = Y$.*

5.7.6. — Let G be a connected and simply-connected Lie group. Let $T \subset G$ be a maximal torus. Then we have a G-map $P\colon G/T \times T \to G$, $P([g], t) := gtg^{-1}$, where G acts on the left-hand side by $g([h], t) := ([gh], t)$, and by conjugation on the right-hand side. Let $x \in \hat{K}^*(G)$ be an invariant element. It is an interesting question how P^*x looks like.

Let us consider the special case $G = SU(2)$ and $x_{SU(2)} = x_{SU(2)}(1) \in \hat{K}^1(SU(2))$. In this case we have $T = S^1$ and $G/T \cong \mathbb{CP}^1$. First we compute the curvature of $P^*x_{SU(2)}$. For this we must compute $P^*\mathtt{vol}_{SU(2)}$ which is given by Weyl's integration formula. We have

$$P^*\mathtt{vol}_{SU(2)} = \mathtt{vol}_{\mathbb{CP}^1} \wedge 4\sin^2(2\pi t)dt.$$

There is a unique class $z \in \hat{K}^1(S^1)$ with curvature $4\sin^2(2\pi t)dt$ such that $0^*z = 0$. Furthermore, there is a unique class $<\mathbf{L}> \in \hat{K}^0(\mathbb{CP}^1)$ with curvature $\mathtt{vol}_{\mathbb{CP}^1}$ which is in fact the class $<\mathbf{L}>$ considered in 5.5.1 associated to the canonical line bundle $\mathbf{L}$ on $\mathbb{CP}^1$.

The product $<\mathbf{L}> \cup z$ has now the same curvature as $P^*x_{SU(2)}$. We conclude that

$$P^*x_{SU(2)} - <\mathbf{L}> \cup z \in H^{\mathrm{ev}}(\mathbb{CP}^1 \times S^1, \mathbb{R})/\mathtt{im}(\mathbf{ch}_{dR}).$$

Now note that

$$\begin{aligned} & H^{\mathrm{ev}}(\mathbb{CP}^1 \times S^1, \mathbb{R})/\mathtt{im}(\mathbf{ch}_{dR}) \\ \cong \;& \big(H^0(\mathbb{CP}^1, \mathbb{R}) \otimes H^0(S^1, \mathbb{R}) \oplus H^2(\mathbb{CP}^1, \mathbb{R}) \otimes H^0(S^1, \mathbb{R})\big)/\mathtt{im}(\mathbf{ch}_{dR}) \\ \cong \;& \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z}. \end{aligned}$$

The first component can be determined by evaluating the difference $P^*x_{SU(2)} - <\mathbf{L}> \cup z$ at a point. Since $x_{SU(2)}$ is trivial on points, this first component vanishes. The second component can be determined by evaluating $P^*x_{SU(2)} - <\mathbf{L}> \cup z$ at $\mathbb{CP}^1 \times \{0\}$. Note that $P^*_{\mathbb{CP}^1 \times \{0\}} x_{SU(2)} = 0$, since $P_{|\mathbb{CP}^1 \times \{0\}}$ is constant. Furthermore, $0^*z = 0$ implies that $<\mathbf{L}> \cup z_{|\mathbb{CP}^1 \times \{0\}} = 0$. Thus we have shown (using $S^2 \cong \mathbb{CP}^1$):

Lemma 5.16. — *We have $P^*x_{SU(2)} = x_{S^2}(1) \cup z$.*

5.8. Bordism

5.8.1. — A zero bordism of a geometric family $\mathcal{E}$ over B is a geometric family $\mathcal{W}$ over B with boundary such that $\mathcal{E} = \partial\mathcal{W}$. The notion of a geometric family with boundary is explained in [**19**]. It is important to note that in our set-up a geometric family with boundary always has a product structure.

Proposition 5.17. — *If $\mathcal{E}$ admits a zero bordism $\mathcal{W}$, then in $\hat{K}^*(B)$ we have the identity*

$$[\mathcal{E}, 0] = [\varnothing, \Omega(\mathcal{W})].$$

Proof. — Since $\mathcal{E}$ admits a zero bordism we have $\mathtt{index}(\mathcal{E}) = 0$ so that $\mathcal{E}$ admits a taming $\mathcal{E}_t$. This taming induces a boundary taming $\mathcal{W}_{bt}$. The obstruction against extending the boundary taming to a taming of $\mathcal{W}$ is $\mathtt{index}(\mathcal{W}_{bt}) \in K(B)$ [**19**, Lemma 2.2.6].

Let us assume for simplicity that $\mathcal{E}$ is not zero-dimensional. Otherwise we may have to stabilize in the following assertion. Using 2.4.10 we can adjust the taming $\mathcal{E}_t$ such that $\mathtt{index}(\mathcal{W}_{bt}) = 0$. At this point we employ a version of the relative index theorem [**17**]

$$\mathtt{index}(\mathcal{W}_{bt'}) = \mathtt{index}(\mathcal{W}_{bt}) + \mathtt{index}((\mathcal{E} \times [0,1])_{bt}), \tag{28}$$

where $\mathcal{E}_t$ and $\mathcal{E}_{t'}$ define the boundary taming $(\mathcal{E} \times [0,1])_{bt}$.

If $\mathtt{index}(\mathcal{W}_{bt}) = 0$, then we can extend the boundary taming $\mathcal{W}_{bt}$ to a taming $\mathcal{W}_t$. We now apply the identity [**19**, Thm. 2.2.13]: $\Omega(\mathcal{W}) = d\eta(\mathcal{W}_t) - \eta(\mathcal{E}_t)$. Note that this equality is more precise than needed since it holds on the level of forms without factoring by $\mathtt{im}(d)$. We see that $(\mathcal{E}, 0)$ is paired with $(\varnothing, \Omega(\mathcal{W}))$. This implies the assertion. $\square$

5.8.2. — Let $p\colon W \to B$ be a proper submersion from a manifold with boundary W which restricts to a submersion $q := p_{|\partial W}\colon V := \partial W \to B$. We assume that p has a topological K-orientation and a smooth K-orientation represented by o_p which refines the topological K-orientation. We assume that the geometric data of o_p has a product structure near V (see [**19**, Section 2.1] for a detailed discussion of such product structures). Recall $o_p = (g^{T^v p}, T^h p, \tilde{\nabla}_p, \sigma_p)$. By the assumption of a product structure we have a quadruple $(g^{T^v q}, T^h q, \tilde{\nabla}_q, \sigma_q)$ and an isomorphism of a neighbourhood of $p_{|\partial W}\colon \partial W \to B$ with the bundle $\mathcal{E} \times [0,1) \xrightarrow{\mathtt{pr}_{\mathcal{E}}} \mathcal{E} \xrightarrow{p} B$ such that the geometric data are related as follows.

1. $T^v p_{|\mathcal{E}\times[0,1)} \cong \mathtt{pr}_{\mathcal{E}}^* T^v q \oplus \mathtt{pr}_{[0,1)}^* T[0,1)$ and $g^{T^v p}_{|\mathcal{E}\times[0,1)} = \mathtt{pr}_{\mathcal{E}}^* g^{T^v q} + \mathtt{pr}_{[0,1)}^* dr^2$, where $r \in [0,1)$ is the coordinate.
2. $T^h p_{|\mathcal{E}\times[0,1)} = \mathtt{pr}_{\mathcal{E}}^* T^h q$.
3. $(\sigma_p)_{|\mathcal{E}\times[0,1)} = \mathtt{pr}_{\mathcal{E}}^* \sigma_q$.

4. The $Spin^c$-structure on T^vq and the canonical $Spin^c$-structure on $T[0,1)$ induce a $Spin^c$-structure on the vertical bundle $T^v \cong \mathtt{pr}_{\mathcal{E}} T^v\mathcal{E} \oplus \mathtt{pr}^*_{[0,1)} T[0,1)$ of $\mathcal{E}\times[0,1)$ in a canonical way so that the associated spinor bundle is $S(T^v) = \mathtt{pr}^*_{\mathcal{E}} S^c(T^vq)$ or $\mathtt{pr}^*_{\mathcal{E}} S^c(T^vq)\otimes\mathbb{C}^2$ depending on the dimension of T^vq. In particular, the connection $\tilde{\nabla}_q$ gives rise to a connection $\tilde{\nabla}_{\mathrm{prod}}$. The product structure identifies the restricted $Spin^c$-structure of $T^vp_{|\mathcal{E}\times[0,1)}$ with this product $Spin^c$-structure such that $\tilde{\nabla}_{|\mathcal{E}\times[0,1)}$ becomes $\tilde{\nabla}_{\mathrm{prod}}$.

From this description we deduce that

$$\hat{\mathbf{A}}^c(\tilde{\nabla})_{|\mathcal{E}\times[0,1)} = \mathtt{pr}^*_{\mathcal{E}}\hat{\mathbf{A}}^c(\tilde{\nabla}_q),\quad \hat{\mathbf{A}}^c(o_p)_{|\mathcal{E}\times[0,1)} = \mathtt{pr}^*_{\mathcal{E}}\hat{\mathbf{A}}^c(o_q).$$

It is now easy to see that the restriction of representatives (with product structure) preserves equivalence and gives a well-defined restriction of smooth K-orientations. We have the following version of bordism invariance of the push-forward in smooth K-theory.

Proposition 5.18. — *For $y\in\hat{K}(W)$ we set $x := y_{|V}\in\hat{K}(V)$. Then we have*

$$\hat{q}_!(x) = [\emptyset, p^o_! R(y)].$$

Proof. — Let $y = [\mathcal{E},\rho]$. We compute using (17), Proposition 5.17, Stokes' theorem, Definition 3.15, and the adiabatic limit $\lambda\to 0$ at the marked equality

$$\begin{aligned}
\hat{q}_!(x) &= [q^\lambda_!\mathcal{E}_{|V}, \int_{V/B}\hat{\mathbf{A}}^c(o_q)\wedge\rho + \tilde{\Omega}(\lambda,\mathcal{E}_{|V}) + \int_{V/B}\sigma(o_q)\wedge R(x)]\\
&= [\emptyset, \Omega(p^\lambda_!\mathcal{E}) + \int_{V/B}\hat{\mathbf{A}}^c(o_q)\wedge\rho + \tilde{\Omega}(\lambda,\mathcal{E}_{|V}) + \int_{V/B}\sigma(o_q)\wedge R(x)]\\
&\overset{!}{=} [\emptyset, \int_{W/B}\Big(\hat{\mathbf{A}}^c(o_p)\wedge\Omega(\mathcal{E}) - \hat{\mathbf{A}}^c(o_p)\wedge d\rho - d\sigma(o_p)\wedge R(y)\Big)]\\
&= [\emptyset, \int_{W/B}(\hat{\mathbf{A}}^c(o_p) - d\sigma(o_p))\wedge R(y)] = [\emptyset, p^o_! R(y)].
\end{aligned}$$

□

5.9. $\mathbb{Z}/k\mathbb{Z}$-invariants

5.9.1. — Here we associate to a family of $\mathbb{Z}/k\mathbb{Z}$-manifolds over B a class in $\hat{K}_{\mathrm{flat}}(B)$.

Definition 5.19. — *A geometric family of $\mathbb{Z}/k\mathbb{Z}$-manifolds is a triple $(\mathcal{W},\mathcal{E},\phi)$, where $\mathcal{W}$ is a geometric family with boundary, $\mathcal{E}$ is a geometric family without boundary, and $\phi\colon \partial\mathcal{W}\xrightarrow{\sim} k\mathcal{E}$ is an isomorphism of the boundary of $\mathcal{W}$ with k copies of $\mathcal{E}$.*

We define $u(\mathcal{W},\mathcal{E},\phi) := [\mathcal{E}, -\frac{1}{k}\Omega(\mathcal{W})]\in\hat{K}(B)$.

Lemma 5.20. — *We have $u(\mathcal{W},\mathcal{E},\phi)\in\hat{K}_{\mathrm{flat}}(B)$. This class is a k-torsion class. It only depends on the underlying differential-topological data.*

Proof. — We first compute by 5.17

$$ku(\mathcal{W},\mathcal{E},\phi)=k[\mathcal{E},-\frac{1}{k}\Omega(\mathcal{W})]=[k\mathcal{E},-\Omega(\mathcal{W})]=[\varnothing,0]=0.$$

This implies that $R(u(\mathcal{W},\mathcal{E},\phi))=0$ so that $u(\mathcal{W},\mathcal{E},\phi)\in\hat{K}_{\mathrm{flat}}(B)$. Independence of the geometric data is now shown by a homotopy argument. □

5.9.2. — We now explain the relation of this construction to the $\mathbb{Z}/k\mathbb{Z}$-index of Freed-Melrose [**28**].

Lemma 5.21. — *Let $B=*$ and $\dim(\mathcal{W})$ be even. Then $u(\mathcal{W},\mathcal{E},\phi)\in\hat{K}^1_{\mathrm{flat}}(*)\cong\mathbb{R}/\mathbb{Z}$. Let $i_k\colon\mathbb{Z}/k\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ the embedding which sends $1+k\mathbb{Z}$ to $\frac{1}{k}$. Then*

$$i_k(\mathtt{index}_a(\bar{W}))=u(\mathcal{W},\mathcal{E},\phi),$$

where $i_k(\mathtt{index}_a(\bar{W}))\in\mathbb{Z}/k\mathbb{Z}$ is the index of the $\mathbb{Z}/k\mathbb{Z}$-manifold $\bar{W}$ (the notation of [**28**]*).*

Proof. — We recall the definition of $\mathtt{index}_a(\bar{W})$. In our language is can be stated as follows. Since $\mathtt{index}(\mathcal{E})=0$ we can choose a taming $\mathcal{E}_t$. We let k copies of $\mathcal{E}_t$ induce the boundary taming $\mathcal{W}_{bt}$. We have

$$\mathtt{index}_a(\bar{W})=\mathtt{index}(\mathcal{W}_{bt})+k\mathbb{Z}.$$

In fact it is easy to see that a change of the taming $\mathcal{E}_t$ leads to change of the index $\mathtt{index}(\mathcal{W}_{bt})$ by a multiple of k. We can now prove the Lemma using [**19**, Thm. 2.2.18].

$$\begin{aligned}u(\mathcal{W},\mathcal{E},\phi)&=[\mathcal{E},-\frac{1}{k}\Omega(\mathcal{W})]=[\varnothing,-\eta(\mathcal{E}_t)-\frac{1}{k}\Omega(\mathcal{W})]\\&=[\varnothing,-\frac{1}{k}\mathtt{index}(\mathcal{W}_{bt})]=a\left(\frac{1}{k}\mathtt{index}(\mathcal{W}_{bt})\right)=i_k(\mathtt{index}_a(\bar{W}))\in\mathbb{R}/\mathbb{Z}.\end{aligned}$$ □

5.10. *Spin^c*-bordism invariants

5.10.1. — Let π be a finite group. We construct a transformation

$$\phi\colon\Omega^{Spin^c}(BU(n)\times B\pi)\to\hat{K}_{\mathrm{flat}}(*).$$

Let $f\colon M\to BU(n)\times B\pi$ represent $[M,f]\in\Omega^{Spin^c}(BU(n)\times B\pi)$. This map determines a covering $p\colon\tilde{M}\to M$ and an n-dimensional complex vector bundle $V\to M$. We choose a Riemannian metric g^{TM} and a $Spin^c$-extension $\tilde{\nabla}$ of the Levi-Civita connection ∇^{TM}. These structures determine a smooth K-orientation of $t\colon M\to *$. We further fix a metric h^V and a connection ∇^V in order to define a geometric bundle $\mathbf{V}:=(V,h^V,\nabla^V)$ and the associated geometric family $\mathcal{V}$ (see 2.1.4). The pull-back of g^{TM} and $\tilde{\nabla}$ via $\tilde{M}\to M$ fixes a smooth K-orientation of $\tilde{t}\colon\tilde{M}\to *$.

We define the geometric families $\mathcal{M}:=t_!\mathcal{V}$ and $\tilde{\mathcal{M}}:=\tilde{t}_!(p^*\mathcal{V})$ over $*$. Then we set

$$\phi([M,f]):=[\tilde{\mathcal{M}}\sqcup_*|\pi|\mathcal{M}^{\mathrm{op}},0]\in\hat{K}_{\mathrm{flat}}(*).$$

By a homotopy argument we see that this class is independent of the choice of geometry. We now argue that it only depends on the bordism class of $[M, f]$.

The construction is additive. Let now $[M, f]$ be zero-bordant by $[W, F]$. Then we have a zero bordism $\tilde{W}$ of $\tilde{M}$ over W. Note that the bundles also extend over the bordism. The local index form of $\tilde{\mathcal{W}} \sqcup_B |\pi|\mathcal{W}$ vanishes. We conclude by 5.17, that $[\tilde{\mathcal{M}} \sqcup_B |\pi| \cdot \mathcal{M}^{\mathrm{op}}, 0] = 0$.

In this construction we can replace $E\pi \to B\pi$ by any finite covering.

5.10.2. — This construction allows the following modification. Let $\rho \in Rep(\pi)_0$ be a virtual zero-dimensional representation of π. It defines a flat vector bundle $F_\rho \to B\pi$. To $[M, f]$ we associate the geometric family $\mathcal{M}_\rho := t_!(\mathcal{L})$, where $\mathcal{L}$ is the geometric family associated to the geometric bundle $\mathbf{V} \otimes (\mathtt{pr}_2 \circ f)^* F_\rho$. We define

$$\phi_\rho \colon \Omega_*^{Spin^c}(BU(n) \times B\pi) \to \hat{K}_{\mathrm{flat}}(*)$$

such that $\phi_\rho[M, f] := [\mathcal{M}_\rho, 0]$. Here we need not to assume that π is finite. This is the construction of ρ-invariants in the smooth K-theory picture.

The first construction is a special case of the second with the representation $\rho = \mathbb{C}(\pi) \oplus (\mathbb{C}^{|\pi|})^{\mathrm{op}}$.

5.10.3. — We now discuss a parametrized version. Let B be some compact manifold and X be some topological space. Then we can define the parametrized bordism group $\Omega_*^{Spin^c}(X/B)$. Its cycles are pairs $(p \colon W \to B, f \colon W \to X)$ of a proper topologically K-oriented submersion p and a continuous map f. The bordism relation is defined correspondingly.

There is a natural transformation

$$\phi \colon \Omega_*^{Spin^c}((BU(n) \times B\pi)/B) \to \hat{K}^*_{\mathrm{flat}}(B).$$

It associates to $x = (p \colon W \to B, f \colon W \to BU(n) \times B\pi)$ the class $[\tilde{\mathcal{W}} \sqcup_B |\pi| \cdot \mathcal{W}^{\mathrm{op}}, 0]$. In this formula $p \colon \tilde{W} \to W$ is again the π-covering classified by $\mathtt{pr}_2 \circ f$. We define the geometric family $\mathcal{W}$ using some choice of geometric structures and the twisting bundle V, where V is classified by the first component of f. The family $\tilde{\mathcal{W}}$ is obtained from $\tilde{W}$ and $p^* V$ using the lifted geometric structures. Again, the class $\phi(x)$ is flat and independent of the choices of geometry. Using 5.17 one checks that ϕ passes through the bordism relation.

Again there is the following modification. For $\rho \in \mathtt{Rep}(\pi)_0$ we can define

$$\phi_\rho \colon \Omega_*^{Spin^c}((BU(n) \times B\pi)/B) \to \hat{K}^*_{\mathrm{flat}}(B).$$

It associates to $x = (p \colon W \to B, f \colon W \to BU(n) \times B\pi)$ the class $[\mathcal{W}_\rho]$ of the geometric manifold $\mathcal{W}$ with twisting bundle $V \otimes (\mathtt{pr}_2 \circ f)^* F_\rho$. These classes are K-theoretic higher ρ-invariants. It seems promising to use this picture to draw geometric consequences using these invariants.

5.11. The e-invariant

5.11.1. — A framed n-manifold M is a manifold with a trivialization $TM \cong M \times \mathbb{R}^n$. More general, a bundle of framed n-manifolds over B is a fibre bundle $\pi\colon E \to B$ with a trivialization $T^v\pi \cong E \times \mathbb{R}^n$.

Proposition 5.22. — *A bundle of framed n-manifolds $\pi\colon E \to B$ has a canonical smooth K-orientation which only depends on the homotopy class of the framing.*

Proof. — The framing $T^v\pi \cong E \times \mathbb{R}^n$ induces a vertical Riemannian metric $g^{T^v\pi}$ and an isomorphism $SO(T^v\pi) \cong E \times SO(n)$. Hence we get an induced vertical orientation and a $Spin$-structure which determines a $Spin^c$-structure, and thus a K-orientation of π. We choose a horizontal distribution $T^h\pi$ which gives rise to a connection $\nabla^{T^v\pi}$. Since our $Spin^c$-structure comes from a $Spin$-structure, this connection extends naturally to a $Spin^c$-connection $\tilde{\nabla}$ of trivial central curvature.

The trivial connection ∇^{triv} on $T^v\pi$ induced by the framing also lifts naturally to the trivial $Spin^c$-connection $\tilde{\nabla}^{\mathrm{triv}}$. The quadruple

$$o := (g^{T^v\pi}, T^h\pi, \tilde{\nabla}, \tilde{\hat{\mathbf{A}}}{}^c(\tilde{\nabla}, \tilde{\nabla}^{\mathrm{triv}}))$$

defines a smooth K-orientation of π which refines the given underlying topological K-orientation.

We claim that this orientation is independent of the choice of the vertical distribution $T^h\pi$. Indeed, if $T^h\pi$ is a second horizontal distribution with associated $Spin^c$-connection $\tilde{\nabla}'$, then we set

$$o' := (g^{T^v\pi}, T^h\pi', \tilde{\nabla}', \hat{\mathbf{A}}^c(\tilde{\nabla}', \tilde{\nabla}^{\mathrm{triv}})).$$

Since

$$\tilde{\hat{\mathbf{A}}}{}^c(\tilde{\nabla}', \tilde{\nabla}^{\mathrm{triv}}) - \tilde{\hat{\mathbf{A}}}{}^c(\tilde{\nabla}, \tilde{\nabla}^{\mathrm{triv}}) = \tilde{\hat{\mathbf{A}}}{}^c(\tilde{\nabla}', \tilde{\nabla})$$

we have $o \sim o'$ in view of the Definition 3.1.9.

Let us now consider a second framing of $T^v\pi$ which is homotopic to the first. In induces a second trivial connection $\tilde{\nabla}'^{triv}$ and a metric $g'^{T^v\pi}$. We therefore get a connection $\tilde{\nabla}'$ and and a second representative of a smooth K-orientation $o' := (g'^{T^v\pi}, T^h\pi, \tilde{\nabla}', \tilde{\hat{\mathbf{A}}}{}^c(\tilde{\nabla}', \tilde{\nabla}'^{triv}))$. In fact, the homotopy between the framings provides a connection $\tilde{\nabla}^{h,triv}$ on $I \times E$. Since this connection is flat we see that $\tilde{\hat{\mathbf{A}}}{}^c(\tilde{\nabla}'^{triv}, \tilde{\nabla}^{\mathrm{triv}}) = 0$. From

$$\tilde{\hat{\mathbf{A}}}{}^c(\tilde{\nabla}', \tilde{\nabla}'^{triv}) = \tilde{\hat{\mathbf{A}}}{}^c(\tilde{\nabla}', \tilde{\nabla}) + \tilde{\hat{\mathbf{A}}}{}^c(\tilde{\nabla}, \tilde{\nabla}^{\mathrm{triv}}) + \tilde{\hat{\mathbf{A}}}{}^c(\tilde{\nabla}^{\mathrm{triv}}, \tilde{\nabla}'^{triv})$$

we get

$$\tilde{\hat{\mathbf{A}}}{}^c(\tilde{\nabla}', \tilde{\nabla}'^{triv}) - \tilde{\hat{\mathbf{A}}}{}^c(\tilde{\nabla}, \tilde{\nabla}^{triv}) = \tilde{\hat{\mathbf{A}}}{}^c(\tilde{\nabla}', \tilde{\nabla})$$

and thus $o \sim o'$. □

Since $\tilde{\nabla}^{\mathrm{triv}}$ is flat we have

$$\hat{\mathbf{A}}^c(o) - d\sigma(o) = \hat{\mathbf{A}}(\tilde{\nabla}) - d\tilde{\hat{\mathbf{A}}}(\tilde{\nabla}, \tilde{\nabla}^{\mathrm{triv}}) = 1.$$

Assume that the fibre dimension n satisfies $n \geq 1$. According to Lemma 3.16 the curvature of $\hat{\pi}_!(1)$ is given by

$$R(\hat{\pi}_!(1)) = \int_{E/B} (\hat{\mathbf{A}}^c(o) - d\sigma(o)) \wedge 1 = \int_{E/B} 1 \wedge 1 = 0$$

Definition 5.23. — *If $\pi\colon E \to B$ is a bundle of framed manifolds of fibre dimension $n \geq 1$, then we define a differential topological invariant*

$$e(E \to B) := -\hat{\pi}_!(1) \in \hat{K}^{-n}_{\mathrm{flat}}(B).$$

In the following we will explain in some detail that this is a higher generalization of the Adams e-invariant. The stable homotopy groups of the sphere $\pi_n := \pi^s_n(S^0)$ have a decreasing filtration

$$\cdots \subseteq \pi^2_n \subseteq \pi^1_n \subseteq \pi^0_n = \pi_n$$

related to the MSpin-based Adams-Novikov spectral sequence. The e-invariant is a homomorphism

$$e\colon \pi^1_{4n-1}/\pi^2_{4n-1} \to \mathbb{R}/\mathbb{Z}.$$

A closed framed $4n-1$-dimensional manifold M represents a class $[M] \in \pi_{4n-1}$ under the Pontrjagin-Thom identification of framed bordism with stable homotopy. In the indicated dimension $\pi_{4n-1} = \pi^1_{4n-1}$ so that $[M]$ is actually a boundary of a compact $4n$-dimensional *Spin*-manifold N. As explained in **[2]** (see also **[36]**) the e-invariant $e[M]$ can be calculated as follows. One chooses a connection ∇^{TN} on TN which restricts to the trivial connection ∇^{triv} on TM given by the framing. Then

$$e([M]) = \left[\int_N \hat{\mathbf{A}}(\nabla)\right]_{\mathbb{R}/\mathbb{Z}}.$$

We now consider $q\colon M \to *$ as a bundle of framed manifolds over the point and identify $\mathbb{R}/\mathbb{Z} \xrightarrow{\sim} \hat{K}^{-4n+1}_{\mathrm{flat}}(*)$ by $[u] \mapsto a(u) = [\varnothing, -u]$, $u \in \mathbb{R}$.

Lemma 5.24. — *Under these identifications we have $e(M \to *) = e([M])$.*

Proof. — We choose a metric g^{TM} on M which induces the representative

$$o := (g^{TM}, 0, \tilde{\nabla}, \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}, \nabla^{\mathrm{triv}}))$$

of the smooth K-orientation on q. The *Spin*-structure of N induces a *Spin*^{c}-structure. We choose a Riemannian metric g^{TN} on N with a product structure near the boundary which extends g^{TM} and induces the *Spin*- and *Spin*^{c}-connections ∇^N and $\tilde{\nabla}^N$. Note that $\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}^N, \tilde{\nabla}^{TN})$ extends $\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}, \tilde{\nabla}^{\mathrm{triv}})$. Therefore $o^N := (g^{TN}, 0, \tilde{\nabla}^N, \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}^N, \tilde{\nabla}^{TN}))$ represents a smooth K-orientation of $p\colon N \to *$

which extends the orientation o of $q\colon M \to *$. We can now apply the bordism formula Proposition 5.18 in the marked step and get

$$\begin{aligned}e(M \to *) = -\hat{q}_!(1) \overset{!}{=} a(p_!(R(1))) = \left[\int_{N/*} (\hat{\mathbf{A}}^c(o^N) - d\sigma(o^N)) \wedge 1\right]_{\mathbb{R}/\mathbb{Z}} \\ = \left[\int_{N/*} \hat{\mathbf{A}}^c(\tilde{\nabla}^N) - d\tilde{\hat{\mathbf{A}}}(\tilde{\nabla}^N, \tilde{\nabla}^{TN})\right]_{\mathbb{R}/\mathbb{Z}} = \left[\int_{N/*} \hat{\mathbf{A}}^c(\tilde{\nabla}^{TN})\right]_{\mathbb{R}/\mathbb{Z}} \\ = \left[\int_{N/*} \hat{\mathbf{A}}(\nabla^{TN})\right]_{\mathbb{R}/\mathbb{Z}} = e([M]). \quad \square\end{aligned}$$

Using the method of Subsection 5.3 or the APS index theorem it is now easy to reproduce the result of [**2**]

$$e([M]) = \left[\eta^0(M) - \int_M \hat{\mathbf{A}}(\tilde{\nabla}, \tilde{\nabla}^{\mathrm{triv}})\right]_{\mathbb{R}/\mathbb{Z}}.$$

6. The Chern character and a smooth Grothendieck-Riemann-Roch theorem

6.1. Smooth rational cohomology

6.1.1. — Let $Z_{k-1}(B)$ be the group of smooth singular cycles on B. The picture of $\hat{H}(B, \mathbb{Q})$ as Cheeger-Simons differential characters

$$\hat{H}^k(B, \mathbb{Q}) \subset \mathtt{Hom}(Z_{k-1}(B), \mathbb{R}/\mathbb{Q})$$

is most appropriate to define the integration map. By definition (see [**24**]) a homomorphism $\phi \in \mathtt{Hom}(Z_{k-1}(B), \mathbb{R}/\mathbb{Q})$ is a differential character if and only if there exists a form $R(\phi) \in \Omega^k_{d=0}(B)$ such that

$$\phi(\partial c) = \left[\int_c R(\phi)\right]_{\mathbb{R}/\mathbb{Q}} \tag{29}$$

for all smooth k-chains $c \in C_k(B)$. It is shown in [**24**] that $R(\phi)$ is uniquely determined by ϕ. In fact, the map $R\colon \hat{H}^k(B, \mathbb{Q}) \to \Omega^k_{d=0}(B)$ is the curvature transformation in the sense of Definition 1.1.

Assume that T is a closed oriented manifold of dimension n with a triangulation. Then we have a map $\tau\colon Z^{k-1}(B) \to Z^{k-1+n}(T \times B)$. If $\sigma\colon \Delta^{k-1} \to B$ is a smooth singular simplex, then the triangulation of $T \times \Delta^{k-1}$ gives rise to a $k-1+n$ chain $\tau(\sigma)\colon = \mathtt{id} \times \sigma\colon T \times \Delta \to T \times B$. The integration

$$(\hat{\mathtt{pr}}_2)_!\colon \hat{H}(T \times B, \mathbb{Q}) \to \hat{H}(B, \mathbb{Q})$$

is now induced by $\tau^*\colon \mathtt{Hom}(Z^{k-1+n}(T \times B), \mathbb{R}/\mathbb{Q}) \to \mathtt{Hom}(Z^{k-1}(B), \mathbb{R}/\mathbb{Q})$. Alternative definitions of the integration (for proper oriented submersions) are given in [**31**], [**30**].

Another construction of the integration has been given in **[25]**, where also a projection formula (the analog of 4.5 for smooth cohomology) is proved. This picture is used in **[35]** in particular to establish functoriality.

We will also need the following bordism formula which we prove using yet another characterization of the push-forward. We consider a proper oriented submersion $q\colon W \to B$ such that $\dim(T^v q) = n$. Let $x \in \hat{H}^r(W,\mathbb{Q})$ and $f\colon \Sigma \to B$ be a smooth map from a closed oriented manifold of dimension $r-n-1$. We get a pull-back diagram

$$\begin{array}{ccc} U & \xrightarrow{F} & W \\ \downarrow & & \downarrow{\scriptstyle q} \\ \Sigma & \xrightarrow{f} & B \end{array}$$

The orientations of Σ and $T^v q$ induce an orientation of U. Note that $f^*\hat{q}_!(x)$ and F^*x are flat classes for dimension reasons. Therefore $F^*x \in H^{r-1}(U,\mathbb{R}/\mathbb{Q})$ and $f^*\hat{q}_!(x) \in H^{r-n-1}(\Sigma,\mathbb{R}/\mathbb{Q})$. The compatibility of the push-forward with Cartesian diagrams implies the following relation in $\mathbb{R}/\mathbb{Q}$:

$$<f^*\hat{q}_!(x),[\Sigma]> = <F^*x,[U]>.$$

If we let $f\colon \Sigma \to B$ vary, then these numbers completely characterize the push-forward $\hat{p}_!(x) \in \hat{H}^{r-n}(B,\mathbb{Q})$. We will use this fact in the argument below.

6.1.2. — Let now $p\colon V \to B$ be a proper oriented submersion from a manifold with boundary such that $\partial V \cong W$ and $p_{|W} = q$. Assume that $x \in \hat{H}(V,\mathbb{Q})$.

Lemma 6.1. — *In $\hat{H}(B,\mathbb{Q})$ we have the equality*

$$\hat{q}_!(x_{|W}) = -a\left(\int_{V/B} R(x)\right).$$

Proof. — Assume that $x \in \hat{H}^r(V,\mathbb{Q})$. Let $f\colon \Sigma \to B$ be as above and form the Cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{z} & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Sigma & \xrightarrow{f} & B. \end{array}$$

The oriented manifold Z has the boundary $\partial Z \cong U$. Using (29) at the marked equality we calculate

$$<f^*\hat{q}_!(x_{|W}),[\Sigma]> = <F^*x_{|W},[U]> = <(z^*x)_{|U},[U]> \overset{!}{=} \left[\int_Z R(z^*x)\right]_{\mathbb{R}/\mathbb{Q}}$$
$$= \left[\int_\Sigma \int_{Z/\Sigma} R(z^*x)\right]_{\mathbb{R}/\mathbb{Q}} = \left[\int_\Sigma f^* \int_{V/B} R(x)\right]_{\mathbb{R}/\mathbb{Q}} = -<f^*a\left(\int_{V/B} R(x)\right),[\Sigma]>.$$

This implies the assertion. □

6.2. Construction of the Chern character

6.2.1. — We start by recalling the classical smooth characteristic classes of Cheeger-Simons. A complex vector bundle $V \to B$ has Chern classes $c_i \in H^{2i}(B,\mathbb{Z})$, $i \geq 1$. If we add the geometric data of a hermitean metric and a metric connection, then we get the geometric bundle $\mathbf{V} = (V, h^V, \nabla^V)$. In [**24**] the Chern classes have been refined to smooth integral cohomology-valued Chern classes

$$\hat{c}_i(\mathbf{V}) \in \hat{H}^{2i}(B,\mathbb{Z})$$

(see 1.2.1 for an introduction to smooth ordinary cohomology). In particular, the class $\hat{c}_1(\mathbf{V}) \in \hat{H}^2(B,\mathbb{Z})$ classifies isomorphism classes of hermitean line bundles with connection.

The embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}$ induces a natural map $\hat{H}(B,\mathbb{Z}) \to \hat{H}(B,\mathbb{Q})$, and we let $\hat{c}_{\mathbb{Q}}(\mathbf{V}) \in \hat{H}^2(B,\mathbb{Q})$ denote the image of $\hat{c}_1(\mathbf{V}) \in \hat{H}^2(B,\mathbb{Z})$ under this map.

6.2.2. — The smooth Chern character $\hat{\mathbf{ch}}$ which we will construct is a natural transformation

$$\hat{\mathbf{ch}} \colon \hat{K}(B) \to \hat{H}(B,\mathbb{Q})$$

of smooth cohomology theories. In particular, this means that the following diagrams commute (compare Definition 1.3)

$$\begin{tikzcd} \Omega(B)/\mathtt{im}(d) \arrow[r, "a"] \arrow[d, equal] & \hat{K}(B) \arrow[r, "I"] \arrow[d, "\hat{\mathbf{ch}}"] & K(B) \arrow[d, "\mathbf{ch}"] \\ \Omega(B)/\mathtt{im}(d) \arrow[r, "a"] & \hat{H}(B,\mathbb{Q}) \arrow[r, "I"] & H(B,\mathbb{Q}) \end{tikzcd} \quad , \quad \begin{tikzcd} \hat{K}(B) \arrow[r, "R"] \arrow[d, "\hat{\mathbf{ch}}"] & \Omega_{d=0}(B) \arrow[d, equal] \\ \hat{H}(B,\mathbb{Q}) \arrow[r, "R"] & \Omega_{d=0}(B) \end{tikzcd} . \tag{30}$$

In addition we require that the even and odd Chern characters are related by suspension, which in the smooth case amounts to the commutativity of the following diagram

$$\begin{tikzcd} \hat{K}^0(S^1 \times B) \arrow[r, "\hat{\mathbf{ch}}"] \arrow[d, "(\hat{\mathtt{pr}}_2)_!"] & \hat{H}^{\mathrm{ev}}(S^1 \times B,\mathbb{Q}) \arrow[d, "(\hat{\mathtt{pr}}_2)_!"] \\ \hat{K}^1(B) \arrow[r, "\hat{\mathbf{ch}}"] & \hat{H}^{\mathrm{odd}}(B,\mathbb{Q}) \end{tikzcd} . \tag{31}$$

The smooth K-orientation of $\mathtt{pr}_2 \colon S^1 \times B \to B$ is as in 4.3.2.

Theorem 6.2. — *There exists a unique natural transformation* $\hat{\mathbf{ch}} \colon \hat{K}(B) \to \hat{H}(B,\mathbb{Q})$ *such that* (30) *and* (31) *commute.*

Note that naturality means that $\hat{\mathbf{ch}} \circ f^* = f^* \circ \hat{\mathbf{ch}}$ for every smooth map $f \colon B' \to B$. The proof of this theorem occupies the remainder of the present subsection.

6.2.3.

Proposition 6.3. — *If the smooth Chern character* $\hat{\mathbf{ch}}$ *exists, then it is unique.*

Proof. — Assume that $\hat{\mathbf{ch}}$ and $\hat{\mathbf{ch}}'$ are two smooth Chern characters. Consider the difference $\Delta := \hat{\mathbf{ch}} - \hat{\mathbf{ch}}'$. It follows from the diagrams above that Δ factors through an odd natural transformation

$$\bar{\Delta} \colon K(B) \to H(B, \mathbb{R}/\mathbb{Q}).$$

Indeed, the left diagram of (30) gives a factorization

$$K(B) \to (\mathtt{im} \colon \Omega(B)/\mathtt{im}(d) \to \hat{H}(B, \mathbb{Q})),$$

and the right square in (30) refines it to $\bar{\Delta}$.

6.2.4. — We now use the following topological fact. Let P be a space of the homotopy type of a countable CW-complex. It represents a contravariant set-valued functor $W \mapsto P(W) := [W, P]$ on the category of compact manifolds. We further consider some abelian group V.

Lemma 6.4. — *A natural transformation of functors* $N \colon P(B) \to H^j(B, V)$ *on the category of compact manifolds is necessarily induced by a class* $N \in H^j(P, V)$.

Proof. — There exists a countable directed diagram $\mathcal{M}$ of compact manifolds such that $\mathtt{hocolim}\, \mathcal{M} \cong P$ in the homotopy category. Hence we have a short exact sequence

$$0 \to \mathtt{lim}^1 H(\mathcal{M}, V) \to H(P, V) \to \mathtt{lim}\, H(\mathcal{M}, V) \to 0.$$

If $x \in P(P)$ is the tautological class, then the pull-back of $N(x)$ to the system $\mathcal{M}$ gives an element in $\mathtt{lim}\, H(\mathcal{M}, V)$. A preimage in $H(P, V)$ induces the natural transformation. □

In our application, $P = \mathbb{Z} \times BU$, and the relevant cohomology $H^{\mathrm{odd}}(\mathbb{Z} \times BU, \mathbb{R}/\mathbb{Q})$ is trivial. Therefore $\bar{\Delta} \colon K^0(B) \to H^{\mathrm{odd}}(B, \mathbb{R}/\mathbb{Q})$ vanishes

6.2.5. — Next we observe that $(\hat{\mathtt{pr}}_2)_! \colon \hat{K}(S^1 \times B) \to \hat{K}(B)$ is surjective. In fact, we have

$$(\hat{\mathtt{pr}}_2)_!(\mathtt{pr}_1^* x_{S^1} \cup \mathtt{pr}_2^*(x)) = x \tag{32}$$

by the projection formula 4.5 and $\hat{p}_!(x_{S^1}) = 1$ for $p \colon S^1 \to *$, where $x_S^1 \in \hat{K}(S^1)$ was defined in 5.6. Hence (31) implies that $\bar{\Delta} \colon K^1(B) \to H^{\mathrm{ev}}(B, \mathbb{R}/\mathbb{Q})$ vanishes, too. □

6.2.6. — In view of Proposition 6.3 it remains to show the existence of the smooth Chern character. We first construct the even part

$$\hat{\mathbf{ch}}\colon \hat{K}^0(B) \to \hat{H}^{\mathrm{ev}}(B,\mathbb{Q})$$

using the splitting principle. We will define $\hat{\mathbf{ch}}$ as a natural transformation of functors such that the following conditions hold.

1. $\hat{\mathbf{ch}}[\mathcal{L},0] = e^{\hat{c}_{\mathbb{Q}}(\mathbf{L})} \in \hat{H}^{\mathrm{ev}}(B,\mathbb{Q})$, where $\mathcal{L}$ is the geometric family given by a hermitean line bundle with connection $\mathbf{L}$, and $\hat{c}_{\mathbb{Q}}(\mathbf{L}) \in \hat{H}^2(B,\mathbb{Q})$ is derived from the Cheeger-Simons Chern class which classifies the isomorphism class of $\mathbf{L}$ (6.2.1).
2. $R \circ \hat{\mathbf{ch}} = R$
3. $\hat{\mathbf{ch}} \circ a = a$

Once this is done, the resulting $\hat{\mathbf{ch}}$ automatically satisfies (30). For this it suffices to show that $\mathbf{ch} \circ I = I \circ \hat{\mathbf{ch}}$. We consider the following diagram

$$\begin{tikzcd}
\hat{K}(B) \arrow[r, "\hat{\mathbf{ch}}"] \arrow[rr, bend left, "R"] \arrow[d, "I"] & \hat{H}(B,\mathbb{Q}) \arrow[r, "R"] \arrow[d, "I"] & \Omega_{d=0}(B) \arrow[d] \\
K(B) \arrow[r, "\mathbf{ch}"] & H(B,\mathbb{Q}) \arrow[r, "i"] & H(B,\mathbb{R})
\end{tikzcd}$$

The outer square and the right square commute. It follows from 2. that the upper triange commutes. Since i is injective we conclude that the left square commutes, too.

6.2.7. — In the construction of the Chern character $\hat{\mathbf{ch}}$ we will use the splitting principle. If $x \in \hat{K}^0(B)$, then there exists a $\mathbb{Z}/2\mathbb{Z}$-graded hermitean vector bundle with connection $\mathbf{V} = (V, h^V, \nabla^V)$ such that $x = [\mathcal{V}, \rho]$ for some $\rho \in \Omega^{\mathrm{odd}}(B)/\mathtt{im}(d)$, where $\mathcal{V}$ is the zero-dimensional geometric family with underlying Dirac bundle $\mathbf{V}$. We will call V the splitting bundle for x. Let $F(V^{\pm}) \to B$ be the bundle of full flags on $V^{\pm}$ and $p\colon F(V) := F(V^+) \times_B F(V^-) \to B$. Then we have a decomposition $p^*V^{\pm} \cong \oplus_{L\in I^{\pm}} L$ for some ordered finite sets $I^{\pm}$ of line bundles over $F(V)$. For $L \in I^{\pm}$ let $\mathbf{L}$ denote the bundle with the induced metric and connection, and let $\mathcal{L}$ be the corresponding zero-dimensional geometric family. Then we have $p^*x = \sum_{L\in I^+}[\mathcal{L},0] - \sum_{L\in I^-}[\mathcal{L},0] + a(\sigma)$ for some $\sigma \in \Omega^{\mathrm{odd}}(F(V))/\mathtt{im}(d)$. The properties above thus uniquely determine $p^*\hat{\mathbf{ch}}(x)$.

Lemma 6.5. — *The following pull-back operations are injective:*

1. $p^*\colon H^*(B,\mathbb{Q}) \to H^*(F(V),\mathbb{Q})$,
2. $p^*\colon H^*(B,\mathbb{R}) \to H^*(F(V),\mathbb{R})$
3. $p^*\colon H^*(B,\mathbb{R}/\mathbb{Q}) \to H^*(F(V),\mathbb{R}/\mathbb{Q})$
4. $p^*\colon \hat{H}^*(B,\mathbb{Q}) \to \hat{H}^*(F(V),\mathbb{Q})$

5. $p^*: \Omega(B) \to \Omega(F(V))$.

Proof. — The assertion is a classical consequence of the Leray-Hirsch theorem in the cases 1., 2., and 3. In case 5., it follows from the fact that p is surjective and a submersion. It remains to discuss the case 4. Let $x \in \hat{H}^*(B, \mathbb{Q})$. Assume that $p^*x = 0$. Then in particular $p^*R(x) = R(p^*x) = 0$ so that from 5. also $R(x) = 0$. Thus $x \in H(B, \mathbb{R}/\mathbb{Q})$. We now apply 3. and see that $p^*x = 0$ implies $x = 0$. □

In view of Proposition 6.3 we see that a natural transformation $\hat{\mathbf{ch}}: \hat{K}^0(B) \to \hat{H}^{\mathrm{ev}}(B, \mathbb{Q})$ is uniquely determined by the conditions 1., 2., and 3. formulated in 6.2.6.

6.2.8.

Proposition 6.6. — *There exists a natural transformation* $\hat{\mathbf{ch}}: \hat{K}^0(B) \to \hat{H}^{\mathrm{ev}}(B, \mathbb{Q})$ *which satisfies the conditions 1. to 3. formulated in* 6.2.6.

We give the proof of this Proposition in the next couple of subsections. Let $x := [\mathcal{E}, \rho] \in \hat{K}^0(B)$, and $V \to B$ be a splitting bundle for x with bundle of flags $p: F(V) \to B$. We choose a geometry $\mathbf{V} := (V, h^V, \nabla^V)$ and let $\mathcal{V}$ denote the associated geometric family[(4)]. In order to avoid stabilizations we can and will always assume that $\mathcal{E}$ has a non-zero dimensional component. Then we have

$$p^*I(x) = \sum_{\epsilon \in \{\pm 1\}, L \in I^\epsilon} \epsilon I([\mathcal{L}, 0]).$$

We define $\mathcal{F} := \bigsqcup_{B, \epsilon \in \{\pm 1\}, L \in I^\epsilon} \mathcal{L}^\epsilon$. Then we can find a taming $(p^*\mathcal{E} \sqcup_{F(V)} \mathcal{F}^{\mathrm{op}})_t$, and

$$p^*x = \sum_{\epsilon \in \{\pm 1\}, L \in I^\epsilon} \epsilon([\mathcal{L}, 0]) - a(p^*\rho - \eta((p^*\mathcal{E} \sqcup_{F(V)} \mathcal{F}^{\mathrm{op}})_t)).$$

We now set

$$p^*\hat{\mathbf{ch}}(x) = \hat{\mathbf{ch}}(p^*x) := \sum_{\epsilon \in \{\pm 1\}, L \in I^\epsilon} \epsilon \exp(\hat{c}_{\mathbb{Q}}(\mathbf{L})) + a(\eta((p^*\mathcal{E} \sqcup_{F(V)} \mathcal{F}^{\mathrm{op}})_t)) - a(p^*\rho).$$

This construction a priori depends on the choices of the representative of x, the splitting bundle $V \to B$, and the taming $(\mathcal{E} \sqcup_{F(V)} \mathcal{F}^{\mathrm{op}})_t$.

(4) It was suggested by the referee that one should use the Chern character $\hat{\mathbf{ch}}(V) \in \hat{H}^{\mathrm{ev}}(B, \mathbb{Q})$ constructed in [**24**]. The Ansatz would be

$$\hat{\mathbf{ch}}(x) := \hat{\mathbf{ch}}(\mathbf{V}) + \eta((\mathcal{E} \sqcup_B \mathcal{V}^{\mathrm{op}})_t).$$

In order to show that this is independent of the choice of $\mathbf{V}$ one would need to show an equation like

$$\hat{\mathbf{ch}}(\mathbf{V}) - \hat{\mathbf{ch}}(\mathbf{V}') = a(\eta((\mathcal{V}^{\mathrm{op}} \sqcup \mathcal{V}')_t)).$$

Since after all we know that the Chern character exists this equation is true, but we do not know a *simple* direct proof. Therefore we opted for the variant to give a complete and independent proof.

6.2.9. — In this paragraph we show that this construction is independent of the choices.

Proposition 6.7. — *Assume that there exists a class $z \in \hat{H}^{\mathrm{ev}}(B, \mathbb{Q})$ such that*

$$p^* z = \sum_{\epsilon \in \{\pm 1\}, L \in I^\epsilon} \epsilon \exp(\hat{c}_{\mathbb{Q}}(\mathbf{L})) + a(\eta((p^* \mathcal{E} \sqcup_{F(V)} \mathcal{F}^{\mathrm{op}})_t)) - a(p^* \rho)$$

for one set of choices. Then z is determined by $x \in \hat{K}^0(B)$.

Proof. — If $(\mathcal{E}', \rho')$ is another representative of x, then we have $\mathtt{index}(\mathcal{E}) = \mathtt{index}(\mathcal{E}')$. Therefore we can take the same splitting bundle for $\mathcal{E}'$. The following Lemma (together with Lemma 6.5) shows that z does not depend on the choice of the representative of x.

Lemma 6.8. — *We have*

$$a(\eta((p^* \mathcal{E} \sqcup_{F(V)} \mathcal{F}^{\mathrm{op}})_t) - p^* \rho) = a(\eta((p^* \mathcal{E}' \sqcup_{F(V)} \mathcal{F}^{\mathrm{op}})_t) - p^* \rho')$$

Proof. — In fact, by Lemma 2.21 there is a taming $(\mathcal{E}' \cup \mathcal{E}^{\mathrm{op}})_t$ such that $\rho' - \rho = \eta\left((\mathcal{E}' \cup \mathcal{E}^{\mathrm{op}})_t\right)$. Therefore the assertion is equivalent to

$$a\left[\eta\left((p^* \mathcal{E} \sqcup_{F(V)} \mathcal{F}^{\mathrm{op}})_t\right) - \eta\left((p^* \mathcal{E}' \sqcup_{F(V)} \mathcal{F}^{\mathrm{op}})_t\right) + p^* \eta\left((\mathcal{E}' \sqcup_{F(V)} \mathcal{E}^{\mathrm{op}})_t\right)\right] = 0.$$

But this is true since this sum of η-forms represents a rational cohomology class of the form $\mathbf{ch}_{dR}(\xi)$. This follows from 2.4.10 and the fact

$$p^* \mathcal{E} \sqcup_{F(V)} \mathcal{F}^{\mathrm{op}} \sqcup_{F(V)} p^* \mathcal{E}'^{\mathrm{op}} \sqcup_{F(V)} \mathcal{F} \sqcup_{F(V)} p^* \mathcal{E}' \sqcup_{F(V)} p^* \mathcal{E}^{\mathrm{op}}$$

admits another taming with vanishing η-form (as in the proof of Lemma 2.11). □

6.2.10. — Next we discuss what happens if we vary the splitting bundle. Thus let $V' \to B$ be another $\mathbb{Z}/2\mathbb{Z}$-graded bundle which represents $\mathtt{index}(\mathcal{E})$. Let $p' \colon F(V') \to B$ be the associated splitting bundle.

Lemma 6.9. — *Assume that we have classes $c, c' \in \hat{H}(B, \mathbb{Q})$ such that*

$$p^* c = \sum_{\epsilon \in \{\pm 1\}, L \in I^\epsilon} \epsilon \exp(\hat{c}_{\mathbb{Q}}(\mathbf{L})) + a\left(\eta\left((p^* \mathcal{E} \sqcup_{F(V)} \mathcal{F}^{\mathrm{op}})_t\right) - p^* \rho\right)$$

and

$$p'^* c' = \sum_{\epsilon \in \{\pm 1\}, L \in I'^\epsilon} \epsilon \exp(\hat{c}_{\mathbb{Q}}(\mathbf{L}')) + a\left(\eta\left((p'^* \mathcal{E} \sqcup_{F(V')} \mathcal{F}'^{\mathrm{op}})_t\right) - p'^* \rho\right).$$

Then we have $c = c'$.

Proof. — Note that the right-hand sides depend on the geometric bundles $\mathbf{V}, \mathbf{V}'$ since they depend on the induced connections on the line bundle summands. We first discuss a special case, namely that $\mathbf{V}'$ is obtained from $\mathbf{V}$ by stabilization, i.e. $\mathbf{V}' = \mathbf{V} \oplus B \times (\mathbb{C}^m \oplus (\mathbb{C}^m)^{\mathrm{op}})$. In this case there is a natural embedding $i\colon F(\mathbf{V}) \hookrightarrow F(\mathbf{V}')$ which is induced by extension of the flags in V by the standard flag in $\mathbb{C}^m$. We can factor $p = p' \circ i$. Furthermore, there exists subsets $S^\epsilon \subset I'^\epsilon$ of line bundles (the last m line bundles in the natural order) and a natural bijection $I'^\epsilon \cong I^\epsilon \sqcup S^\epsilon$. If $L \in S^\epsilon$, then i^*L is trivial with the trivial connection. We thus have

$$p^*(c' - c) = a\left[i^*\eta\left((p'^*\mathcal{E} \cup \mathcal{F}'^{\mathrm{op}})_t\right) - \eta\left((p^*\mathcal{E} \cup \mathcal{F}^{\mathrm{op}})_t\right)\right]$$

It is again easy to see that this difference of η-forms represents a rational cohomology class in the image of $\mathbf{ch}_{dR}$. Therefore, $p^*(c' - c) = 0$ and hence $c = c'$ by Lemma 6.5.

Since the bundle V represents the index of $\mathcal{E}$, two choices are always stably isomorphic as hermitean bundles. Using the special case above we can reduce to the case where $\mathbf{V}$ and $\mathbf{V}'$ only differ by the connection.

We argue as follows. We have $p^*R(c' - c) = R(p^*(c' - c)) = 0$ by an explicit computation. Therefore $c' - c \in H^{\mathrm{odd}}(B, \mathbb{R}/\mathbb{Q})$. Since any two connections on V can be connected by a family we conclude that $p^*(c' - c) = 0$ by a homotopy argument. The assertion now follows. □

This finishes the proof of Proposition 6.7. □

6.2.11. — In order to finish the construction of the Chern character in the even case it remains to verify the existence clause in Proposition 6.7. Let $x := [\mathcal{E}, \rho] \in \hat{K}(B)$ be such that $\mathcal{E}$ has a non-zero dimensional component. Let $V \to B$ be a splitting bundle and $p\colon F(V) \to B$ be as above.

Lemma 6.10. — *We have*

$$z := \sum_{\epsilon \in \{\pm 1\}, L \in I^\epsilon} \epsilon \exp(\hat{c}_{\mathbb{Q}}(\mathbf{L})) + a\left[\eta\left((p^*\mathcal{E} \cup \mathcal{F}^{\mathrm{op}})_t\right) - p^*\rho\right] \in \mathtt{im}(p^*).$$

Proof. — We use a Mayer-Vietoris sequence argument. Let us first recall the Mayer-Vietoris sequence for smooth rational cohomology. Let $B = U \cup V$ be an open covering of B. Then we have the exact sequence

$$\cdots \to H(U \cap V, \mathbb{R}/\mathbb{Q}) \to \hat{H}(B, \mathbb{Q}) \to \hat{H}(U, \mathbb{Q}) \oplus \hat{H}(V, \mathbb{Q}) \to \hat{H}(U \cap V, \mathbb{Q}) \to H(B, \mathbb{Q}) \to \cdots$$

which continues to the left and right by the Mayer-Vietoris sequences of $H(\dots, \mathbb{R}/\mathbb{Q})$ and $H(\dots, \mathbb{Q})$.

We choose a finite covering of B by contractible subsets. Let U be one of these. Note that $\mathtt{index}(\mathcal{E})_{|U} \in \mathbb{Z}$. Thus $x_{|U} = [U \times W, \theta]$ for some form θ and $\mathbb{Z}/2\mathbb{Z}$-graded vector space W. Then we have by 1. and 3. that $c_U\colon = \hat{\mathbf{ch}}(x_{|U}) = \dim(W) - a(\theta)$. This can be seen using the splitting bundle $F(B\times\mathbb{C}^n)$. Moreover, $p^*c_U = p^*[\dim(W)-a(\theta)] = z_{|p^{-1}U}$ by Proposition 6.7.

Assume now that we have already constructed $c_V \in \hat{H}(V,\mathbb{Q})$ such that $p^*c_V = z_{|p^{-1}V}$, where V is a union V of these subsets. Let U be the next one in the list.

We show that we can extend c_V to $c_{V\cup U}$. We have $(c_U)_{|U\cap V} = (c_V)_{|U\cap V}$ by the injectivity of the pull-back $p^*\colon \hat{H}(U\cap V,\mathbb{Q}) \to \hat{H}(p^{-1}(U\cap V),\mathbb{Q})$, Lemma 6.5. The Mayer-Vietoris sequence implies that we can extend c_V by c_U to $U\cup V$. □

6.2.12. — We now construct the odd part of the Chern character. In fact, by (31) and (32) we are forced to define

$$\hat{\mathbf{ch}}\colon \hat{K}^1(B) \to \hat{H}^{\mathrm{odd}}(B,\mathbb{Q})$$

by

$$\hat{\mathbf{ch}}(x) := (\hat{\mathrm{pr}}_2)_!(\hat{\mathbf{ch}}(x_{S^1}\cup x)).$$

Lemma 6.11. — *The diagrams* (30) *and* (31) *commute.*

Proof. — The even case of (30) has been checked already. The diagram (31) commutes by construction. The odd case of (30) follows from the Projection formula 4.5 and the even case. □

This finishes the proof of Theorem 6.2

6.3. The Chern character is a rational isomorphism and multiplicative

6.3.1. — Note that $\hat{H}(B,\mathbb{Q})$ is a $\mathbb{Q}$-vector space, and that the sequence (1) is an exact sequence of $\mathbb{Q}$-vector spaces. The Chern character extends to a rational version

$$\hat{\mathbf{ch}}_{\mathbb{Q}}\colon \hat{K}_{\mathbb{Q}}(B) \to \hat{H}(B,\mathbb{Q}),$$

where $\hat{K}_{\mathbb{Q}}(B) := \hat{K}(B)\otimes_{\mathbb{Z}}\mathbb{Q}$.

Proposition 6.12. — $\hat{\mathbf{ch}}_{\mathbb{Q}}\colon \hat{K}_{\mathbb{Q}}(B) \to \hat{H}(B,\mathbb{Q})$ *is an isomorphism.*

Proof. — By (30) we have the commutative diagram

$$\begin{tikzcd}
K_{\mathbb{Q}}(B) \arrow[r, "\mathbf{ch}_{dR}"] \arrow[d, "\mathbf{ch}_{\mathbb{Q}}"] & \Omega(B)/\mathtt{im}(d) \arrow[r, "a"] \arrow[d, equal] & \hat{K}_{\mathbb{Q}}(B) \arrow[r, "I"] \arrow[d, "\hat{\mathbf{ch}}_{\mathbb{Q}}"] & K_{\mathbb{Q}}(B) \arrow[r] \arrow[d, "\mathbf{ch}_{\mathbb{Q}}"] & 0\ , \\
H(B,\mathbb{Q}) \arrow[r] & \Omega(B)/\mathtt{im}(d) \arrow[r] & \hat{H}(B,\mathbb{Q}) \arrow[r, "I"] & H(B,\mathbb{Q}) \arrow[r] & 0
\end{tikzcd}$$

whose horizontal sequences are exact. Since $\mathbf{ch}_{\mathbb{Q}}\colon K_{\mathbb{Q}}(B) \to H(B,\mathbb{Q})$ is an isomorphism we conclude that $\hat{\mathbf{ch}}_{\mathbb{Q}}$ is an isomorphism by the Five Lemma. □

6.3.2. — We can extend $\hat{K}_{\mathbb{Q}}$ to a smooth cohomology theory if we define the structure maps as follows:

1. $R\colon \hat{K}_{\mathbb{Q}}(B) \to \Omega_{d=0}(B)$ is the rational extension of $R\colon \hat{K}(B) \to \Omega_{d=0}(B)$.
2. $I\colon \hat{K}_{\mathbb{Q}}(B) \stackrel{I\otimes \mathtt{id}_{\mathbb{Q}}}{\to} K(B)_{\mathbb{Q}} \stackrel{\mathbf{ch}_{\mathbb{Q}}}{\to} H(B,\mathbb{Q})$,
3. $a\colon \Omega(B)/\mathtt{im}(d) \stackrel{a}{\to} \hat{K}(B) \stackrel{\dots\otimes 1}{\to} \hat{K}_{\mathbb{Q}}(B)$.

The commutative diagrams (30) now imply:

Corollary 6.13. — *The rational Chern character induces an isomorphism of smooth cohomology theories refining the isomorphism $\mathbf{ch}_{\mathbb{Q}}\colon K_{\mathbb{Q}} \to H\mathbb{Q}$ (in the sense of Definition 1.3).*

6.3.3.

Proposition 6.14. — *The smooth Chern character*

$$\hat{\mathbf{ch}}\colon \hat{K}(B) \to \hat{H}(B,\mathbb{Q})$$

is a ring homomorphism.

Proof. — Since the target of $\hat{\mathbf{ch}}$ is a $\mathbb{Q}$-vector space it suffices to show that $\hat{\mathbf{ch}}_{\mathbb{Q}}\colon \hat{K}_{\mathbb{Q}}(B) \to \hat{H}(B,\mathbb{Q})$ is a ring homomorphism. Using that $\hat{\mathbf{ch}}_{\mathbb{Q}}$ is an isomorphism of smooth extensions of rational cohomology we can use the rational Chern character in order to transport the product on $\hat{K}_{\mathbb{Q}}(B)$ to a second product $\cup_K$ on $\hat{H}(B,\mathbb{Q})$. It remains to show that $\cup$ and $\cup_K$ coincide. Hence the following Lemma finishes the proof of Proposition 6.14.

6.3.4.

Lemma 6.15. — *There is a unique product on smooth rational cohomology.*

Proof. — Assume that we have two products $\cup_k$, $k = 0,1$. We consider the bilinear transformation $\mathbf{B}\colon \hat{H}(B,\mathbb{Q}) \times \hat{H}(B,\mathbb{Q}) \to \hat{H}(B,\mathbb{Q})$ given by

$$(x,y) \mapsto \mathbf{B}(x,y) := x \cup_1 y - x \cup_0 y.$$

We first consider the curvature. Since a product is compatible with the curvature (1.2, 2.) we get

$$R(\mathbf{B}(x,y)) = R(x \cup_1 y) - R(x \cup_0 y) = R(x) \wedge R(y) - R(x) \wedge R(y) = 0.$$

Therefore, by (1) the bilinear form factors over an odd transformation

$$\mathbf{B}\colon \hat{H}(B,\mathbb{Q}) \times \hat{H}(B,\mathbb{Q}) \to H(B,\mathbb{R}/\mathbb{Q}).$$

Furthermore, for $\omega \in \Omega(B)/\mathtt{im}(d)$ we have by 1.2, 2.

$$\mathbf{B}(a(\omega),y) = a(\omega) \cup_1 y - a(\omega) \cup_0 y = a(\omega \wedge R(y)) - a(\omega \wedge R(y)) = 0.$$

Similarly, $\mathbf{B}(x, a(\omega)) = 0$. Again by (1) $\mathbf{B}$ has a factorization over a natural bilinear transformation

$$\bar{\mathbf{B}}\colon H(B, \mathbb{Q}) \times H(B, \mathbb{Q}) \to H(B, \mathbb{R}/\mathbb{Q}).$$

We consider the restriction $\bar{\mathbf{B}}^{p,q}$ of $\bar{\mathbf{B}}$ to $H^p(B, \mathbb{Q}) \times H^q(B, \mathbb{Q})$.

The functor from finite CW-complexes to sets

$$W \to H^p(W, \mathbb{Q}) \times H^q(W, \mathbb{Q})$$

is represented by a product of Eilenberg MacLane spaces

$$P^{p,q} := H\mathbb{Q}^p \times H\mathbb{Q}^q.$$

The spaces $H\mathbb{Q}^p$, and hence P has the homotopy type of countable CW-complexes. Therefore we can apply Lemma 6.4 and conclude that $\bar{\mathbf{B}}^{p,q}$ is induced by a cohomology class $b \in H(P^{p,q}, \mathbb{R}/\mathbb{Q})$. We finish the proof of Lemma 6.15 by showing that $b = 0$. To this end we analyse the candidates for b and show that they vanish either for degree reasons, or using the fact that $\bar{\mathbf{B}}^{p,q}$ is bilinear.

Consider a homomorphism of $\mathbb{Q}$-vector spaces $w\colon \mathbb{R}/\mathbb{Q} \to \mathbb{Q}$. It induces a transformation $w_*\colon H(B, \mathbb{R}/\mathbb{Q}) \to H(B, \mathbb{Q})$. In particular we can consider $w_* b \in H(P^{p,q}, \mathbb{Q})$.

1. First of all if p, q are both even, then $w_* b \in H^{\mathrm{odd}}(P^{p,q,}, \mathbb{Q})$ vanishes since $P^{p,q}$ does not have odd-degree rational cohomology at all.
2. Assume now that p, q are both odd. The odd rational cohomology of $P^{p,q}$ is additively generated by the classes $1 \times x_q$ and $x_p \times 1$, where $x_p \in H^p(H\mathbb{Q}^p, \mathbb{Q})$ and $x_q \in H^q(H\mathbb{Q}^q, \mathbb{Q})$. It follows that
$$w_* b = c \cdot x_p \times 1 + d \cdot 1 \times x_q$$
for some rational constants c, d. Consider odd classes $u_p \in H^p(B, \mathbb{Q})$ and $v_q \in H^q(B, \mathbb{Q})$. The form of b implies that
$$w_* \circ \bar{\mathbf{B}}^{p,q}(u_p, v_q) = c \cdot u_p \times 1 + d \cdot 1 \times v_q.$$
This can only be bilinear if all c and d vanish. Hence $b = 0$.
3. Finally we consider the case that p is even and q is odd (or vice versa, q is even and p is odd). In this case b is an even class. The even cohomology of $P^{p,q}$ is additively generated by the classes $x_p^n \times 1$, $n \geq 0$. Therefore $w_* b = \sum_{n \geq 0} c_n x_p^n \times 1$ for some rational constants c_n, $n \geq 0$. Let $u_p \in H^p(B, \mathbb{Q})$ and $v_q \in H^q(B, \mathbb{Q})$. Then we have
$$w_* \circ \bar{\mathbf{B}}^{p,q}(u_p, v_q) = \sum_{n \geq 0} c_n\ u_p^n.$$
This is only bilinear if $c_n = 0$ for all $n \geq 0$, hence $w_* b = 0$.

Since we can choose $w_*\colon \mathbb{R}/\mathbb{Q} \to \mathbb{Q}$ arbitrary we conclude that $b = 0$. □

This also finishes the proof of the Proposition 6.14. □

6.4. Riemann Roch theorem

6.4.1. — Let $p\colon W \to B$ be a proper submersion with a smooth K-orientation o. The Riemann Roch theorem asserts the commutativity of a diagram

$$\begin{array}{ccc} \hat{K}(W) & \xrightarrow{\hat{\mathbf{ch}}} & \hat{H}(W,\mathbb{Q}) \\ \downarrow{\scriptstyle p_!} & & \downarrow{\scriptstyle \hat{p}^A_!} \\ \hat{K}(B) & \xrightarrow{\hat{\mathbf{ch}}} & \hat{H}(B,\mathbb{Q}). \end{array}$$

Here $\hat{p}^A_!$ is the composition of the cup product with a smooth rational cohomology class $\hat{\hat{\mathbf{A}}}^c(o)$ and the push-forward in smooth rational cohomology. The Riemann Roch theorem refines the characteristic class version of the ordinary index theorem for families.

We will first give the details of the definition of the push-forward $\hat{p}^A_!$. In order to show the Riemann Roch theorem we then show that the difference

$$\Delta := \hat{\mathbf{ch}} \circ \hat{p}_! - \hat{p}^A_! \circ \hat{\mathbf{ch}}$$

vanishes.

This is proved in several steps. First we use the compatibilites of the push-forward with the transformations a, I, R in order to show that Δ factors over a map

$$\bar{\Delta}\colon K(W) \to H(B,\mathbb{R}/\mathbb{Q}).$$

In the next step we show that Δ is natural with respect to the pull-back of fibre bundles, and that it does neither depend on the smooth nor on the topological K-orientations of p.

We then show that Δ vanishes in the special case that $B = *$. The argument is based on the bordism invariance Proposition 5.18 and some calculation of rational $Spin^c$-bordism groups.

Finally we use the functoriality of the push-forward Proposition 3.23 in order to reduce the case of a general B to the special case of a point.

6.4.2. — We consider a proper submersion $p\colon W \to B$ with closed fibres with a smooth K-orientation represented by $o = (g^{T^vp}, T^hp, \tilde{\nabla}, \sigma)$. In the following we define a refinement $\hat{\hat{\mathbf{A}}}(o) \in \hat{H}^{\mathrm{ev}}(W,\mathbb{Q})$ of the form $\hat{\mathbf{A}}^c(o) \in \Omega^{\mathrm{ev}}(W)$. The geometric data of o determines a connection ∇^{T^vp} (see 2.2.4, 3.1.3) and hence a geometric bundle $\mathbf{T^vp} := (T^vp, g^{T^vp}, \nabla^{T^vp})$. According to [**24**] we can define Pontrjagin classes

$$\hat{p}_i(\mathbf{T^vp}) \in \hat{H}^{4i}(W,\mathbb{Z}), \quad i \geq 1.$$

The $Spin^c$-structure gives rise to a hermitean line bundle $L^2 \to W$ with connection ∇^{L^2} (see 3.1.6). A choice of a local spin structure amounts to a choice of a local square root L of L^2 (this bundle was considered already in 3.1.3) such that $S^c(T^vp) \cong$

$S(T^v p)\otimes L$ as hermitean bundles with connections. We set $\mathbf{L}^2 := (L^2, h^{L^2}, \nabla^{L^2})$. In particular, we have

$$\frac{1}{2\pi i} R^{\tilde{\nabla}^{L^2}} = 2c_1(\tilde{\nabla}).$$

Again using **[24]** we get a class

$$\hat{c}_1(\mathbf{L}^2) \in \hat{H}^2(W, \mathbb{Z})$$

with curvature $R(\hat{c}_1(\mathbf{L}^2)) = 2c_1(\tilde{\nabla})$.

6.4.3. — Inserting the classes $\hat{p}_i(\mathbf{T^v p})$ into that $\hat{\mathbf{A}}$-series $\hat{\mathbf{A}}(p_1, p_2, \dots) \in \mathbb{Q}[[p_1, p_2, \dots]]$ we can define

$$\hat{\hat{\mathbf{A}}}(\mathbf{T^v p}) := \hat{\mathbf{A}}(\hat{p}_1(\mathbf{T^v p}), \hat{p}_2(\mathbf{T^v p}), \dots) \in \hat{H}^{\mathrm{ev}}(W, \mathbb{Q}). \tag{33}$$

Let $\hat{c}_{\mathbb{Q}}(\mathbf{L}^2) \in \hat{H}^2(W, \mathbb{Q})$ denote the image of $\hat{c}_1(\mathbf{L}^2)$ under the natural map $\hat{H}^2(W, \mathbb{Z}) \to \hat{H}^2(W, \mathbb{Q})$.

Definition 6.16. — *We define*

$$\hat{\hat{\mathbf{A}}}^c(o) := \hat{\hat{\mathbf{A}}}(\mathbf{T^v p}) \wedge e^{\frac{1}{2}\hat{c}_{\mathbb{Q}}(\mathbf{L}^2)} \in \hat{H}^{\mathrm{ev}}(W, \mathbb{Q}).$$

Note that $R(\hat{\hat{\mathbf{A}}}^c(o)) = \hat{\mathbf{A}}^c(o)$.

Lemma 6.17. — *The class*[5]

$$\hat{\hat{\mathbf{A}}}^c(o) - a(\sigma(o)) \in \hat{H}^{\mathrm{ev}}(W, \mathbb{Q})$$

only depends on the smooth K-orientation represented by o.

Proof. — This is a consequence of the homotopy formula Lemma 2.22. Given two representatives o_0, o_1 of a smooth K-orientation we can choose a representative $\tilde{o}$ of a smooth K-orientation on $\mathtt{id}_{\mathbb{R}} \times p\colon \mathbb{R}\times W \to \mathbb{R}\times B$ which restricts to o_k on $\{k\}\times B$, $k = 0, 1$. The construction of the class $\hat{\hat{\mathbf{A}}}^c(o)$ is compatible with pull-back. Therefore by the definition of the transgression form 3.4 we have

$$\hat{\hat{\mathbf{A}}}^c(o_1) - \hat{\hat{\mathbf{A}}}^c(o_0) = i_1^*\hat{\hat{\mathbf{A}}}^c(\tilde{o}) - i_0^*\hat{\hat{\mathbf{A}}}^c(\tilde{o}) = a\left[\int_{[0,1]\times W/W} R(\hat{\hat{\mathbf{A}}}^c(\tilde{o}))\right] = a\left[\tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_1, \tilde{\nabla}_0)\right].$$

By the definition of equivalence of representatives of smooth K-orientations we have

$$\sigma(o_1) - \sigma(o_0) = \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}_1, \tilde{\nabla}_0).$$

Therefore

$$\hat{\hat{\mathbf{A}}}^c(o_1) - a(\sigma(o_1)) = \hat{\hat{\mathbf{A}}}^c(o_0) - a(\sigma(o_0)). \qquad \square$$

[5] This class is denoted by $A(p)$ in the abstract and in 1.1.9.

6.4.4. — We use the class $\hat{\hat{\mathbf{A}}}^c(o) \in \hat{H}^{\mathrm{ev}}(W,\mathbb{Q})$ in order to define the push-forward

$$\hat{p}_!^A := \hat{p}_!([\hat{\hat{\mathbf{A}}}^c(o) - a(\sigma(o))] \cup \dots) \colon \hat{H}(W,\mathbb{Q}) \to \hat{H}(B,\mathbb{Q}), \tag{34}$$

where $\hat{p}_! \colon \hat{H}(W,\mathbb{Q}) \to \hat{H}(B,\mathbb{Q})$ is the push-forward in smooth rational cohomology (see 6.1.1) fixed by the underlying ordinary orientation of p. By Lemma 6.17 also $\hat{p}_!^A$ only depends to the smooth K-orientation of p and not on the choice of the representative.

If $f \colon B' \to B$ is a smooth map then we consider the pull-back diagram

$$\begin{tikzcd} W' \arrow[r, "F"] \arrow[d, "p'"] & W \arrow[d, "p"] \\ B' \arrow[r, "f"] & B. \end{tikzcd}$$

The smooth K-orientation o of p induces (see 3.2.4) a smooth K-orientation o' of p'. We have $\hat{\hat{\mathbf{A}}}(o') = F^*\hat{\hat{\mathbf{A}}}(o)$ and $\hat{p}_!'^A \circ F^* = f^* \circ \hat{p}_!^A$.

6.4.5. — As in 3.3.3 we consider the composition of proper smoothly K-oriented submersions

$$\begin{tikzcd} W \arrow[r, "p"] \arrow[rr, bend right, "q"'] & B \arrow[r, "r"] & A . \end{tikzcd}$$

The composition $q := r \circ p$ has an induced smooth K-orientation (Definition 3.21 and Lemma 3.22). In this situation we have push-forwards $\hat{p}_!^A$, $\hat{r}_!^A$ and $\hat{q}_!^A$ in smooth rational cohomology given by (34).

Lemma 6.18. — *We have the equality*

$$\hat{r}_!^A \circ \hat{p}_!^A = \hat{q}_!^A$$

of maps $\hat{H}(W,\mathbb{Q}) \to \hat{H}(B,\mathbb{Q})$.

Proof. — We choose representatives of smooth K-orientations o_p of p and o_r of r, and we let $o_q^\lambda := o_p \circ_\lambda o_r$ be the composition. We consider the class (see Definition 3.21)

$$\begin{aligned} &\hat{\hat{\mathbf{A}}}^c(o_q^\lambda) - a(\sigma(o_q^\lambda)) = \hat{\hat{\mathbf{A}}}^c(o_q^\lambda) \\ &- a\left(\sigma(o_p) \wedge p^*\hat{\mathbf{A}}^c(o_r) + \hat{\mathbf{A}}^c(o_p) \wedge p^*\sigma(o_r) - \tilde{\hat{\mathbf{A}}}^c(\tilde{\nabla}^{\mathrm{adia}}, \tilde{\nabla}_q^\lambda) - d\sigma(o_p) \wedge p^*\sigma(o_r)\right). \end{aligned}$$

By Lemma 6.17 and Lemma 3.22 this class is independent of λ. If we let $\lambda \to 0$, then the connection ∇^{T^vq} tends to the direct sum connection $\nabla^{T^vp} \oplus p^*\nabla^{T^vr}$. Furthermore, the transgression $\tilde{\hat{\mathbf{A}}}^c(\tilde\nabla^{\mathrm{adia}},\tilde\nabla^\lambda_q)$ tends to zero. Therefore

$$\begin{aligned}
&\mathtt{lim}_{\lambda\to 0}[\hat{\hat{\mathbf{A}}}^c(o_q^\lambda)-a(\sigma(o_q^\lambda))]\\
&=\quad \hat{\hat{\mathbf{A}}}^c(o_p)\cup p^*\hat{\hat{\mathbf{A}}}^c(o_r)-a\left(\sigma(o_p)\wedge p^*\hat{\mathbf{A}}^c(o_r)+\hat{\mathbf{A}}^c(o_p)\wedge p^*\sigma(o_r)-d\sigma(o_p)\wedge p^*\sigma(o_r)\right)\\
&=\quad (\hat{\hat{\mathbf{A}}}^c(o_p)-a(\sigma(o_p)))\cup p^*(\hat{\hat{\mathbf{A}}}^c(o_r)-a(\sigma(o_r))).
\end{aligned}$$

For $x\in\hat H(W,\mathbb{Q})$ we get, using the projection formula and the functorialty $\hat q_!=\hat r_!\circ\hat p_!$, for the push-forward in smooth rational cohomology

$$\begin{aligned}
\hat r_!^A\circ\hat p_!^A(x)&=\hat r_!\left(\left[\hat{\hat{\mathbf{A}}}^c(o_r)-a(\sigma(o_r))\right]\cup\hat p_!\left(\left[\hat{\hat{\mathbf{A}}}^c(o_p)-a(\sigma(o_p))\right]\cup x\right)\right)\\
&=\hat q_!\left(p^*\left[\hat{\hat{\mathbf{A}}}^c(o_r)-a(\sigma(o_r))\right]\cup\left[\hat{\hat{\mathbf{A}}}^c(o_p)-a(\sigma(o_p))\right]\cup x\right)\\
&=\hat q_!\left((\hat{\hat{\mathbf{A}}}^c(o_q^a)-a(\sigma(o_q^a)))\cup x\right)=\hat q_!^A(x).
\end{aligned}$$

□

6.4.6. — Recall Definition 3.18 that the smooth K-orientation determines a push-down

$$\hat p_!\colon\hat K(W)\to\hat K(B).$$

We can now formulate the index theorem.

Theorem 6.19. — *The following square commutes*

$$\begin{array}{ccc}
\hat K(W) & \xrightarrow{\hat{\mathbf{ch}}} & \hat H(W,\mathbb{Q})\\
\Big\downarrow{\scriptstyle \hat p_!} & & \Big\downarrow{\scriptstyle \hat p_!^A}\\
\hat K(B) & \xrightarrow{\hat{\mathbf{ch}}} & \hat H(B,\mathbb{Q}).
\end{array}$$

Proof. — We consider the difference

$$\Delta:=\hat{\mathbf{ch}}\circ\hat p_!-\hat p_!^A\circ\hat{\mathbf{ch}}.$$

It suffices to show that $\Delta=0$.

6.4.7. — Let $x\in\hat K(W)$.

Lemma 6.20. — *We have* $R(\Delta(x))=0$.

Proof. — This Lemma is essentially equivalent to the local index theorem. We have by Definition 3.15 and Lemma 3.16

$$R(\hat{\mathbf{ch}}\circ\hat p_!(x))=R(\hat p_!(x))=p_!(R(x))=\int_{W/B}\left(\hat{\mathbf{A}}^c(o)-d\sigma(o)\right)\wedge R(x).$$

On the other hand, since $R\left(\hat{\hat{\mathbf{A}}}^c(o)-a(\sigma(o))\right)=\hat{\mathbf{A}}^c(o)-d\sigma(o)$ we get

$$R\left(\hat p_!^A\circ\hat{\mathbf{ch}}(x)\right)=\int_{W/B}\left(\hat{\mathbf{A}}^c(o)-d\sigma(o)\right)\wedge R(\hat{\mathbf{ch}}(x))=\int_{W/B}\left(\hat{\mathbf{A}}^c(o)-d\sigma(o)\right)\wedge R(x).$$

Therefore $R(\Delta(x))=0$. $\square$

6.4.8.

Lemma 6.21. — *We have* $I(\Delta(x))=0$

Proof. — This is the usual index theorem. Indeed,

$$I(\hat{\mathbf{ch}}\circ\hat{p}_!(x))=\mathbf{ch}\circ I(\hat{p}_!(x))=\int_{W/B}\hat{\mathbf{A}}^c(T^vp)\cup\mathbf{ch}(I(x))$$

and

$$I\left(\hat{p}_!^A\circ\hat{\mathbf{ch}}(x)\right)=\int_{W/B}\hat{\mathbf{A}}^c(T^vp)\cup I(\hat{\mathbf{ch}}(x))=\int_{W/B}\hat{\mathbf{A}}^c(T^vp)\cup\mathbf{ch}(I(x)).$$

The equality of the right-hand sides proves the Lemma. Alternatively one could observe that the Lemma is a consequence of Lemma 6.20. $\square$

6.4.9. — Let $\omega\in\Omega(W)/\mathtt{im}(d)$.

Lemma 6.22. — *We have* $\Delta(a(\omega))=0$.

Proof. — We have by Proposition 3.19

$$\hat{\mathbf{ch}}\circ\hat{p}_!(a(\omega))=\hat{\mathbf{ch}}\circ a(p_!(\omega))=a\left(\int_{W/B}\left(\hat{\mathbf{A}}^c(o)-d\sigma(o)\right)\wedge\omega\right).$$

On the other hand, by (30) and

$$\left[\hat{\hat{\mathbf{A}}}^c(o)-a(\sigma(o))\right]\cup a(\omega)=a\left(R\left(\hat{\hat{\mathbf{A}}}(o)-a(\sigma(o))\right)\wedge\omega\right)=a\left(\left(\hat{\mathbf{A}}^c(o)-d\sigma(o)\right)\wedge\omega\right),$$

we have $\hat{p}_!^A\circ\hat{\mathbf{ch}}(a(\omega))=\hat{p}_!^A(a(\omega))=a\left(\int_{W/B}\left(\hat{\mathbf{A}}^c(o)-d\sigma(o)\right)\wedge\omega\right)$. $\square$

6.4.10. — Let o_0,o_1 represents two smooth refinements of the same topological K-orientation of p. Assume that Δ_k is defined with the choice o_k, $k=0,1$.

Lemma 6.23. — *We have* $\Delta_0=\Delta_1$.

Proof. — We can assume that $o_k=(g^{T^vp},T^hp,\tilde{\nabla},\sigma_k)$ for $\sigma_k\in\Omega^{\mathrm{odd}}(W)/\mathtt{im}(d)$.

Then we have for $x\in\hat{K}(W)$

$$\begin{aligned}\Delta_1(x)-\Delta_0(x) &= -a\left(\int_{W/B}(\sigma_1-\sigma_0)\wedge R(x)\right)+\int_{W/B}a(\sigma_1-\sigma_0)\cup\hat{\mathbf{ch}}(x)\\ &= -a\left(\int_{W/B}(\sigma_1-\sigma_0)\wedge R(x)\right)+\int_{W/B}a\left[(\sigma_1-\sigma_0)\wedge R\circ\hat{\mathbf{ch}}(x)\right]\\ &= 0\end{aligned}$$

since $R\circ\hat{\mathbf{ch}}(x)=R(x)$ and $a\circ\int_{W/B}=\int_{W/B}\circ a$. $\square$

6.4.11. — It follows from Lemma 6.20 and (1) that Δ factorizes through a transformation

$$\Delta\colon \hat{K}(W) \to H(B, \mathbb{R}/\mathbb{Q}).$$

By Lemma 6.22 and 2.20 the map Δ factors over a map

$$\bar{\Delta}\colon K(W) \to H(B, \mathbb{R}/\mathbb{Q}).$$

This map only depends on the topological K-orientation of p. It is our goal to show that $\bar{\Delta} = 0$.

6.4.12. — Next we want to show that the transformation $\bar{\Delta}$ is natural. For the moment we write $\Delta_p := \bar{\Delta}$. Let $f\colon B' \to B$ be a smooth map and form the Cartesian diagram

$$\begin{tikzcd} W' \arrow[r, "F"] \arrow[d, "p'"] & W \arrow[d, "p"] \\ B' \arrow[r, "f"] & B. \end{tikzcd}$$

The map p' is a proper submersion with closed fibres which has an induced topological K-orientation.

Lemma 6.24. — *We have the equality of maps $K(W) \to H(B', \mathbb{R}/\mathbb{Q})$*

$$\Delta_{p'} \circ F^* = f^* \circ \Delta_p.$$

Proof. — This follows from the naturality of $\hat{\mathbf{ch}}$, $\hat{p}_!$, and $\hat{p}_!^A$ with respect to the base B. □

6.4.13.

Lemma 6.25. — *If $\mathtt{pr}_2\colon S^1 \times B \to B$ is the trivial bundle with the topological K-orientation given by the bounding spin structure, then $\Delta_{\mathtt{pr}_2}\colon K^0(S^1 \times B) \to H^{\mathrm{odd}}(B, \mathbb{R}/\mathbb{Q})$ vanishes.*

Proof. — The odd Chern character is defined such that for $x \in K^0(S^1 \times B)$ we have $\hat{\mathbf{ch}}_1((\hat{\mathtt{pr}}_2)_! x) = (\hat{\mathtt{pr}}_2)_! \hat{\mathbf{ch}}_0(x)$ (see (31)). With the choice of the smooth K-orientation of $\mathtt{pr}_2$ given in 4.3.2 we have $\hat{\hat{\mathbf{A}}}(o) - a(\sigma(o)) = 1$ so that $\hat{p}_!^A = \hat{p}_!$. This implies the lemma. □

6.4.14. — The group $H^2(W, \mathbb{Z})$ acts simply transitive on the set of $Spin^c$-structures of $T^v p$. Let $Q \to W$ be a unitary line bundle classified by $c_1(Q) \in H^2(W, \mathbb{Z})$. We choose a hermitean connection ∇^Q and form the geometric line bundle $\mathbf{Q} := (Q, h^Q, \nabla^Q)$. Let $o := (T^v p, T^h p, \tilde{\nabla}, \rho)$ represent a smooth K-orientation refining the given topological K-orientation of p. Note that $\tilde{\nabla}$ is completely determined by the Clifford connection on the Spinor bundle $S^c(T^v p)$. The spinor bundle of the shift of the topological K-orientation by $c_1(Q)$ is given by $S^c(T^v p)' = S^c(T^v p) \otimes Q$. We construct a corresponding smooth K-orientation $o' = (T^v p, T^h p, \tilde{\nabla} \otimes \nabla^Q, \rho)$. We let $\hat{p}_!$ and $\hat{p}'_!$ denote the corresponding push-forwards in smooth K-theory. Let $\mathcal{Q}$ be the geometric family over W with zero-dimensional fibre given by the bundle $\mathbf{Q}$ (see 2.1.4). The push-forwards $\hat{p}_!$ and $\hat{p}'_!$ are now related as follows:

Lemma 6.26. — *We have*

$$\hat{p}'_!(x) = \hat{p}_!([\mathcal{Q}, 0] \cup x), \qquad \forall x \in \hat{K}(W).$$

Proof. — Let $x = [\mathcal{E}, \rho]$. By an inspection of the constructions leading to Definition 3.7 we see that

$$p'^{\lambda}_! \mathcal{E} = p^{\lambda}_! (\mathcal{Q} \times_W \mathcal{E}).$$

Furthermore we have $c_1(\tilde{\nabla} \otimes \nabla^Q) = c_1(\tilde{\nabla}) + c_1(\nabla^Q)$ so that

$$\hat{\mathbf{A}}^c(o') = \hat{\mathbf{A}}^c(o) \wedge e^{c_1(\nabla^Q)}.$$

On the other hand, since $\Omega(\mathcal{Q}) = e^{c_1(\nabla^Q)}$ we have

$$[\mathcal{Q}, 0] \cup [\mathcal{E}, \rho] = [\mathcal{Q} \times_W \mathcal{E}, e^{c_1(\nabla^Q)} \wedge \rho]$$

Using the explicit formula (17) we get

$$\hat{p}'_!([\mathcal{E}, \rho]) - \hat{p}_!([\mathcal{Q}, 0] \cup [\mathcal{E}, \rho]) = [\emptyset, \tilde{\Omega}'(\lambda, \mathcal{E}) - \tilde{\Omega}(\lambda, \mathcal{E})]$$

for all small $\lambda > 0$. Since both transgression forms vanish in the limit $\lambda = 0$ we get the desired result. □

In the notation of 6.4.2 we have $\mathbf{L}' = \mathbf{L} \otimes \mathbf{Q}$. Therefore

$$\hat{c}_{\mathbb{Q}}(\mathbf{L}'^2) = \hat{c}_{\mathbb{Q}}(\mathbf{L}^2) + 2\hat{c}_{\mathbb{Q}}(\mathbf{Q})$$

and hence we can express $\hat{p}'^{,A}_!$ according to (34) as

$$\hat{p}'^{A}_!(x) = \hat{p}_! \left[\left(\hat{\hat{\mathbf{A}}}^c(o) \cup e^{\hat{c}_{\mathbb{Q}}(\mathbf{Q})} - a(\sigma(o)) \right) \cup x \right].$$

6.4.15. — As before, let $p \colon W \to B$ be a proper oriented submersion which admits topological K-orientations.

Lemma 6.27. — *If $\Delta_p = 0$ for some topological K-orientation of p, then it vanishes for every topological K-orientation of p.*

Proof. — We fix the K-orientation of p such that $\Delta_p = 0$ and let p' denote the same map with the topological K-orientation shifted by $c_1(Q) \in H^2(W, \mathbb{Z})$. We continue to use the notation of 6.4.14. We choose a representative o of a smooth K-orientation of p refining the topological K-orientation. For simplicity we take $\sigma(o) = 0$. Furthermore, we take o' as above. Using $\hat{\mathbf{ch}}([Q,0]) = e^{\hat{c}_{\mathbb{Q}}(\mathbf{Q})}$ and the multiplicativity of the Chern character we get

$$\begin{aligned}\hat{p}_!'^A \circ \hat{\mathbf{ch}}(x) - \hat{\mathbf{ch}} \circ \hat{p}_!'(x) &= \hat{p}_! \left[\hat{\hat{\mathbf{A}}}^c(o) \cup e^{\hat{c}_{\mathbb{Q}}(\mathbf{Q})} \cup \hat{\mathbf{ch}}(x)\right] - \hat{\mathbf{ch}} \circ \hat{p}_! \left([Q,0] \cup x\right) \\ &= \hat{p}_! \left[\hat{\hat{\mathbf{A}}}^c(o) \cup \hat{\mathbf{ch}}([Q,0]) \cup \hat{\mathbf{ch}}(x)\right] - \hat{p}_!^A \circ \hat{\mathbf{ch}}\left([Q,0] \cup x\right) \\ &= \hat{p}_!^A \circ \hat{\mathbf{ch}}([Q,0] \cup x) - \hat{p}_!^A \circ \hat{\mathbf{ch}}([Q,0] \cup x) = 0. \end{aligned}$$

□

6.4.16. — We now consider the special case that $B = *$ and W is an odd-dimensional $Spin^c$-manifold. Since $H(*, \mathbb{R}/\mathbb{Q}) \cong \mathbb{R}/\mathbb{Q}$ we get a homomorphism

$$\Delta_p \colon K(W) \to \mathbb{R}/\mathbb{Q}.$$

Proposition 6.28. — *If $B \cong *$, then $\Delta_p = 0$.*

Proof. — First note that Δ_p is trivial on $K^1(W)$ for degree reasons. It therefore suffices to study $\Delta_p \colon K^0(W) \to \mathbb{R}/\mathbb{Q}$. Let $x \in K^0(W)$ be classified by $\xi \colon W \to \mathbb{Z} \times BU$. It gives rise to an element $[\xi] \in \Omega^{Spin^c}_{\dim(W)}(\mathbb{Z} \times BU)$ of the $Spin^c$-bordism group of $\mathbb{Z} \times BU$.

Lemma 6.29. — *If $[\xi] = 0$, then $\Delta_p = 0$.*

Proof. — Assume that $[\xi] = 0$. In this case there exists a compact $Spin^c$-manifold V with boundary $\partial V \cong W$ (as $Spin^c$-manifolds), and a map $\nu \colon V \to \mathbb{Z} \times BU$ such that $\nu_{|\partial V} = \xi$.

We can choose a $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $E \to V$ which represents the class of ν in $K^0(V)$. We refine E to a geometric bundle $\mathbf{E} := (E, h^E, \nabla^E)$ and form the associated geometric family $\mathcal{E}$ with zero-dimensional fibre.

We choose a representative $\tilde{o}$ of a smooth K-orientation of the map $q \colon V \to *$ which refines the topological K-orientation given by the $Spin^c$-structure and which has a product structure near the boundary. For simplicity we assume that $\sigma(\tilde{o}) = 0$. The restriction of $\tilde{o}$ to the boundary ∂V defines a smooth K-orientation of p.

We let $\hat{y} := [\mathcal{E}, 0] \in \hat{K}(V)$, and we define $\hat{x} := \hat{y}_{|\partial V}$ such that $I(\hat{x}) = x$. By Proposition 5.18 we have

$$\hat{\mathbf{ch}} \circ \hat{p}_!(\hat{x}) = \hat{\mathbf{ch}} \circ \hat{p}_!(\hat{y}_{|W}) = \hat{\mathbf{ch}}([\varnothing, q_!(R(\hat{y}))]) = -a\left(\int_V \hat{\mathbf{A}}^c(\tilde{o}) \wedge R(\hat{y})\right).$$

On the other hand, the bordism formula for the push-forward in smooth rational cohomology, Lemma 6.1, gives

$$\hat{p}_!^A \circ \hat{\mathbf{ch}}(\hat{x}) = \hat{p}_! \left(\hat{\hat{\mathbf{A}}}^c(o) \cup \hat{\mathbf{ch}}(\hat{x})\right) = \hat{p}_! \left(\hat{\hat{\mathbf{A}}}^c(\tilde{o})_{|W} \cup \hat{\mathbf{ch}}(\hat{y})_{|W}\right) = -a\left(\int_V \hat{\mathbf{A}}^c(\tilde{o}) \wedge R(\hat{y})\right).$$

These two formulas imply that $\Delta_p = 0$. □

6.4.17. — We now finish the proof of Proposition 6.28. We claim that there exists $c \in \mathbb{N}$ such that $c[\xi] = 0$. In view of Lemma 6.29 we then have

$$0 = \Delta_{cp} = c\Delta_p,$$

and this implies the Proposition since the target $\mathbb{R}/\mathbb{Q}$ of Δ_p is a $\mathbb{Q}$-vector space.

Note that the graded ring $\Omega_*^{Spin^c} \otimes \mathbb{Q}$ is concentrated in even degrees. Using that $\Omega_*^{SO} \otimes \mathbb{Q}$ is concentrated in even degrees, one can see this as follows. In [**45**, p. 352] it is shown that the homomorphism $Spin^c \to U(1) \times SO$ induces an injection $\Omega_*^{Spin^c} \to \Omega_*^{SO}(BU(1))$. Since $H_*(BU(1), \mathbb{Z}) \cong \mathbb{Z}[z]$ with $\deg(z) = 2$ lives in even degrees, we see using the Atiyah-Hirzebruch spectral sequence that $\Omega^{SO}(BU(1)) \otimes \mathbb{Q}$ lives in even degrees, too. This implies that $\Omega_*^{Spin^c} \otimes \mathbb{Q}$ is concentrated in even degrees.

Since $H_*(\mathbb{Z} \times BU, \mathbb{Z})$ is also concentrated in even degrees it follows again from the Atiyah-Hirzebruch spectral sequence that $\Omega_*^{Spin^c}(\mathbb{Z} \times BU) \otimes \mathbb{Q}$ is concentrated in even degrees.

Since $[\xi]$ is of odd degree we conclude the claim that $c[\xi] = 0$ for an appropriate $c \in \mathbb{N}$. □

This finishes the proof of Proposition 6.28. □

6.4.18. — We now consider the general case. Let $p\colon W \to B$ be a proper submersion with closed fibres with a topological K-orientation.

Proposition 6.30. — *We have $\Delta_p = 0$.*

We give the proof in the next couple of subsections.

6.4.19. — For a closed oriented manifold Z let $\mathtt{PD}\colon H^*(Z, \mathbb{Q}) \xrightarrow{\sim} H_*(Z, \mathbb{Q})$ denote the Poincaré duality isomorphism.

Lemma 6.31. — *The group $H_*(B, \mathbb{Q})$ is generated by classes of the form $f_*\left(\mathtt{PD}(\hat{\mathbf{A}}^c(TZ))\right)$, where Z is a closed $Spin^c$-manifold and $f\colon Z \to B$.*

Proof. — We consider the sequence of transformations of homology theories

$$\Omega_*^{Spin^c}(B) \xrightarrow{\alpha} K_*(B) \xrightarrow{\beta} H_*(B, \mathbb{Q}).$$

The transformation α is the K-orientation of the $Spin^c$-cobordism theory, and β is the homological Chern character. We consider all groups as $\mathbb{Z}/2\mathbb{Z}$-graded. The homological Chern character is a rational isomorphism. Furthermore one knows by [**5**], [**6**] that $\Omega_*^{Spin^c}(B) \xrightarrow{\alpha} K_*(B)$ is surjective. It follows that the composition

$$\beta \circ \alpha\colon \Omega^{Spin^c}(B) \otimes \mathbb{Q} \to H^*(B, \mathbb{Q})$$

is surjective. An explicit description of $\beta \circ \alpha$ is given as follows. Let $x \in \Omega^{Spin^c}(B)$ be represented by a map $f\colon Z \to B$ from a closed $Spin^c$-manifold Z to B. Let

$\mathtt{PD}\colon H^*(Z,\mathbb{Q}) \xrightarrow{\sim} H_*(Z,\mathbb{Q})$ denote the Poincaré duality isomorphism. Then we have

$$\beta\circ\alpha(x) = f_*\left(\mathtt{PD}(\hat{\mathbf{A}}^c(TZ))\right). \qquad \square$$

6.4.20. — For the proof of Proposition 6.30 we first consider the case that p has even-dimensional fibres, and that $x \in K^0(W)$. By Lemma 6.31, in order to show that $\Delta_p(x) = 0$, it suffices to show that all evaluations $\Delta_p(x)\left(f_*(\mathtt{PD}(\hat{\mathbf{A}}^c(TZ)))\right)$ vanish. In the following, if x denotes a K-theory class, then $\hat{x}$ denotes a smooth K-theory class such that $I(\hat{x}) = x$.

We choose a representative o_q of a smooth K-orientation which refines the topological K-orientation of the map $q\colon Z \to *$ induced by the $Spin^c$-structure on TZ. Furthermore, we consider the diagram with a Cartesian square

$$\begin{tikzcd}
V \arrow[r, "F"] \arrow[d, "r"] \arrow[dd, bend right=60, "s"'] & W \arrow[d, "p"] \\
Z \arrow[r, "f"] \arrow[d, "q"] & B. \\
*
\end{tikzcd}$$

In the present case $\Delta_p(x) \in H^{\mathrm{odd}}(B,\mathbb{R}/\mathbb{Q})$, and we can assume that Z is odd-dimensional. We calculate

$$\begin{aligned}
\Delta_p(x)\left(f_*(\mathtt{PD}(\hat{\mathbf{A}}^c(TZ)))\right) &= f^*\Delta_p(x)\left(\mathtt{PD}(\hat{\mathbf{A}}^c(TZ))\right) \\
&\stackrel{\text{Lemma 6.24}}{=} \Delta_r(F^*x)\left(\mathtt{PD}(\hat{\mathbf{A}}^c(TZ))\right) \\
&= (\hat{\mathbf{A}}^c(\nabla^{TZ}) \cup \Delta_r(F^*x))[Z] \\
&= \int_Z \hat{\mathbf{A}}^c(o) \wedge \Delta_r(F^*x) \\
&= \hat{q}_!\left(\hat{\hat{\mathbf{A}}}^c(o_q) \cup \Delta_r(F^*x)\right) \\
&= \hat{q}_!^A\left(\Delta_r(F^*\hat{x})\right) \\
&= \hat{q}_!^A\left[\hat{\mathbf{ch}}\circ\hat{r}_!(F^*\hat{x}) - \hat{r}_!^A\circ\hat{\mathbf{ch}}(F^*\hat{x})\right] \\
&= \hat{q}_!^A\circ\hat{\mathbf{ch}}\circ\hat{r}_!(F^*\hat{x}) - \hat{s}_!^A\circ\hat{\mathbf{ch}}(F^*\hat{x}) \\
&\stackrel{\text{Proposition 6.28}}{=} \hat{\mathbf{ch}}\circ\hat{q}_!\circ\hat{r}_!(F^*\hat{x}) - \hat{s}_!^A\circ\hat{\mathbf{ch}}(F^*\hat{x}) \\
&= \hat{\mathbf{ch}}\circ\hat{s}_!(F^*\hat{x}) - \hat{s}_!^A\circ\hat{\mathbf{ch}}(F^*\hat{x}) \\
&= \Delta_s(F^*x) \stackrel{\text{Proposition 6.28}}{=} 0.
\end{aligned}$$

We thus have shown that

$$0 = \Delta_p\colon K^0(W) \to H^{\mathrm{odd}}(B,\mathbb{R}/\mathbb{Q})$$

if p has even-dimensional fibres.

6.4.21. — If p has odd-dimensional fibres and $x \in K^1(W)$, then we can choose $y \in K^0(S^1 \times W)$ such that $(\hat{\mathtt{pr}}_2)_!(y) = x$. Since $p \circ \mathtt{pr}_2$ has even-dimensional fibres we get using the Lemmas 6.18 and 3.23

$$\begin{aligned}\Delta_p(x) &= \hat{\mathbf{ch}} \circ \hat{p}_! \circ (\hat{\mathtt{pr}}_2)_!(\hat{y}) - \hat{p}_!^A \circ \hat{\mathbf{ch}} \circ (\hat{\mathtt{pr}}_2)_!(\hat{y}) \\ &\overset{\text{Lemma 6.25}}{=} \hat{\mathbf{ch}} \circ (\widehat{p \circ \mathtt{pr}_2})_!(\hat{y}) - \hat{p}_!^A \circ (\hat{\mathtt{pr}}_2)_!^A \circ \hat{\mathbf{ch}}(\hat{y}) \\ &= \hat{\mathbf{ch}} \circ (\widehat{p \circ \mathtt{pr}_2})_!(\hat{y}) - (\widehat{p \circ \mathtt{pr}_2})_!^A \circ \hat{\mathbf{ch}}(\hat{y}) = \Delta_{p\circ\mathtt{pr}_2}(y) = 0.\end{aligned}$$

Therefore, if p has odd-dimensional fibres,

$$0 = \Delta_p \colon K^1(W) \to H^{\mathrm{odd}}(B, \mathbb{R}/\mathbb{Q}).$$

6.4.22. — Let us now consider the case that p has even-dimensional fibres, and that $x \in K^1(W)$. In this case we consider the diagram

$$\begin{tikzcd} S^1 \times W \arrow[r, "\mathtt{Pr}_2"] \arrow[d, "t:=\mathtt{id}_{S^1}\times p"] & W \arrow[d, "p"] \\ S^1 \times B \arrow[r, "\mathtt{pr}_2"] & B. \end{tikzcd}$$

We choose a class $y \in K^0(S^1 \times W)$ such that $(\mathtt{Pr}_2)_!(y) = x$. We further choose a smooth refinement $\hat{y} \in \hat{K}^0(S^1 \times W)$ of y and set $\hat{x} := (\hat{\mathtt{Pr}}_2)_!(\hat{y})$. Then we calculate using the Lemmas 6.18 and 3.23

$$\begin{aligned}\Delta_p(x) &= \hat{\mathbf{ch}} \circ \hat{p}_!(\hat{x}) - \hat{p}_!^A \circ \hat{\mathbf{ch}}(\hat{x}) \\ &= \hat{\mathbf{ch}} \circ \hat{p}_! \circ (\hat{\mathtt{Pr}}_2)_!(\hat{y}) - \hat{p}_!^A \circ \hat{\mathbf{ch}} \circ (\hat{\mathtt{Pr}}_2)_!(\hat{y}) \\ &\overset{\text{Lemma 6.25}}{=} \hat{\mathbf{ch}} \circ \hat{p}_! \circ (\hat{\mathtt{Pr}}_2)_!(\hat{y}) - \hat{p}_!^A \circ (\hat{\mathtt{Pr}}_2)_!^A \circ \hat{\mathbf{ch}} \circ (\hat{y}) \\ &= \hat{\mathbf{ch}} \circ (\widehat{p \circ \mathtt{Pr}_2})_!(\hat{y}) - (\widehat{p \circ \mathtt{Pr}_2})_!^A \circ \hat{\mathbf{ch}}(\hat{y}) \\ &= \hat{\mathbf{ch}} \circ (\widehat{\mathtt{pr}_2 \circ t})_!(\hat{y}) - (\widehat{\mathtt{pr}_2 \circ t})_!^A \circ \hat{\mathbf{ch}}(\hat{y}) \\ &= \hat{\mathbf{ch}} \circ \hat{\mathtt{pr}}_{2!} \circ \hat{t}_!(\hat{y}) - \hat{\mathtt{pr}}_{2!}^A \circ \hat{t}_!^A \circ \hat{\mathbf{ch}}(\hat{y}) \\ &\overset{\text{Lemma 6.25}}{=} (\hat{\mathtt{pr}}_2)_!^A \left[\hat{\mathbf{ch}} \circ \hat{t}_!(\hat{y}) - \hat{t}_!^A \circ \hat{\mathbf{ch}}(\hat{y})\right] \\ &= (\hat{\mathtt{pr}}_2)_!^A \circ \Delta_t(y) = 0.\end{aligned}$$

Therefore, if p has even-dimensional fibres,

$$0 = \Delta_p \colon K^1(W) \to H^{\mathrm{ev}}(B, \mathbb{R}/\mathbb{Q}).$$

6.4.23. — In the final case p has odd-dimensional fibres and $x \in K^0(W)$. In this case we consider the sequence of projections

$$S^1 \times S^1 \times W \overset{\mathtt{pr}_{23}}{\to} S^1 \times W \overset{\mathtt{pr}_2}{\to} W.$$

We choose a class $y \in K^0(S^1 \times S^1 \times W)$ such that $(\mathtt{pr}_2 \circ \mathtt{pr}_{23})_!(y) = x$. We further choose a smooth refinement $\hat{y} \in \hat{K}^0(S^1 \times S^1 \times W)$ of y and set $\hat{x} := (\widehat{\mathtt{pr}_2 \circ \mathtt{pr}_{23}})_!(\hat{y})$. Then we calculate using the already known cases and the Lemmas 6.18 and 3.23,

$$\begin{aligned}
\Delta_p(x) &= \hat{\mathbf{ch}} \circ \hat{p}_!(\hat{x}) - \hat{p}_!^A \circ \hat{\mathbf{ch}}(\hat{x}) \\
&= \hat{\mathbf{ch}} \circ \hat{p}_! \circ (\hat{\mathtt{pr}}_2)_! \circ (\hat{\mathtt{pr}}_{23})_!(\hat{y}) - \hat{p}_!^A \circ \hat{\mathbf{ch}} \circ (\hat{\mathtt{pr}}_2)_! \circ (\hat{\mathtt{pr}}_{23})_!(\hat{y}) \\
&= \hat{\mathbf{ch}} \circ (\widehat{p \circ \mathtt{pr}_2})_! \circ (\hat{\mathtt{pr}}_{23})_!(\hat{y}) - \hat{p}_!^A \circ \hat{\mathbf{ch}} \circ (\widehat{\mathtt{pr}_2 \circ \mathtt{pr}_{23}})_!(\hat{y}) \\
&= (\widehat{p \circ \mathtt{pr}_2})_!^A \circ \hat{\mathbf{ch}} \circ (\hat{\mathtt{pr}}_{23})_!(\hat{y}) - \hat{p}_!^A \circ (\widehat{\mathtt{pr}_2 \circ \mathtt{pr}_{23}})_!^A \circ \hat{\mathbf{ch}}(\hat{y}) \\
&= (\widehat{p \circ \mathtt{pr}_2})_!^A \circ \Delta_{\mathtt{pr}_{23}}(\hat{y}) \stackrel{\text{Lemma 6.25}}{=} 0.
\end{aligned}$$

This finishes the proof of Theorem 6.19. □

7. Conclusion

We have now constructed a geometric model for smooth K-theory, built out of geometric families of Dirac-type operators. We equipped it with a compatible multiplicative structure, and we have given an explicit construction of a push-down map for fibre bundles with all the expected properties. For the verification of these properties we heavily used local index theory.

We presented a collection of natural examples of smooth K-theory classes and showed in particular that several known secondary analytic-geometric invariants can be understood in this framework very naturally. This involved also the consideration of bordisms in this framework.

Finally, we constructed a smooth lift of the Chern character and proved a smooth version of the Grothendieck-Riemann-Roch theorem. This also involved certain considerations from homotopy theory which are special to K-theory.

Important open questions concern the construction of equivariant versions of this theory, or even better versions which work for orbifolds or similar singular spaces.

In a different direction, we have addressed the construction of geometric models of smooth bordism theories along similar lines in **[23]**; using singular bordism this has also been achieved for smooth ordinary cohomology in **[20]**.

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Astérisque
328, 2009, p. 137–160

AN EXPLICIT PROOF OF THE GENERALIZED GAUSS-BONNET FORMULA

by

Henri Gillet & Fatih M. Ünlü

To Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — In this paper we construct an explicit representative for the Grothendieck fundamental class $[Z] \in \mathrm{Ext}^r(\mathcal{O}_Z, \Omega^r_X)$ of a complex submanifold Z of a complex manifold X when Z is the zero locus of a real analytic section of a holomorphic vector bundle E of rank r on X. To this data we associate a super-connection A on $\bigwedge^* E^\vee$, which gives a "twisted resolution" T^* of $\mathcal{O}_Z$ such that the "generalized super-trace" of $\frac{1}{r!}A^{2r}$, which is a map of complexes from T^* to the Dolbeault complex $\mathcal{A}^{r,*}_X$, represents $[Z]$. One may then read off the Gauss-Bonnet formula from this map of complexes.

***Résumé* (Une démonstration explicite de la formule de Gauss-Bonnet généralisée)**
Dans cet article nous construisons un représentant explicite de la classe fondamentale de Grothendieck $[Z] \in \mathrm{Ext}^r(\mathcal{O}_Z, \Omega^r_X)$ d'une sous-variété Z dans une variété lisse complexe X quand Z est le lieu des zéros d'une section réelle analytique d'un fibré vectoriel holomorphe E de rang r sur X. Nous associons à cette donnée une super-connection A sur $\bigwedge^* E^\vee$, qui fournit une « résolution tordue » T^* de $\mathcal{O}_Z$ telle que la « super-trace généralisée » de $\frac{1}{r!}A^{2r}$, qui est un morphisme de complexes de T^* vers le complexe de Dolbeault $\mathcal{A}^{r,*}_X$, représente $[Z]$. On peut alors lire la formule de Gauss-Bonnet à partir de cette application entre complexes.

Introduction

If X is a complex manifold, and τ is a holomorphic section, transverse to the zero section, of the dual $E^\vee$ of a rank r holomorphic vector bundle, it is well known that

2010 Mathematics Subject Classification. — 57R20, 32C35.
Key words and phrases. — Differential geometry, algebraic geometry, characteristic classes, Gauss-Bonnet formula.

The first author was supported in part by NSF Grants DMS 0100587 and DMS 0500762.

the fundamental class of the locus Z of zeros of τ is equal to the top Chern class of the bundle $E^\vee$:

$$[Z] = c_r(E^\vee) = (-1)^r c_r(E)$$

For Hodge cohomology, this is the fact that the image of the Grothendieck fundamental class

$$[Z] \in \mathrm{Ext}^r(\mathcal{O}_Z, \Omega^r_X)$$

under the map

$$\mathrm{Ext}^r(\mathcal{O}_Z, \Omega^r_X) \to \mathrm{Ext}^r(\mathcal{O}_X, \Omega^r_X) = \mathrm{H}^r(X, \Omega^r_X)$$

coincides with the top Chern class of $E^\vee$. Proofs of this result tend to be indirect, i.e. they depend on the axioms for cycle classes and Chern classes, and comparison with "standard" cases.

However, one may observe that the section τ gives rise to an explicit global Koszul resolution

$$K^*(\tau) = (\bigwedge{}^* E^\vee, \iota_\tau) \to \mathcal{O}_Z\ ,$$

and so the theorem can be rephrased as saying that image of $[Z]$ under the map:

$$\mathrm{Ext}^r(K^*(\tau), \Omega^r_X) \to \mathrm{Ext}^r(\mathcal{O}_X, \Omega^r_X)$$

induced by the isomorphism $\mathcal{O}_X \simeq K^0(\tau)$, is the top Chern class of $E^\vee$. Our first result is to show that a choice of connection $\widetilde{\nabla}$ on E, determines, via Chern-Weil theory applied to superconnections, an *explicit* map of complexes from the Koszul complex $K^*(\tau)$ to the Dolbeault complex of Ω^r_X, which represents the Grothendieck fundamental class and the restriction of which to the degree zero component $\mathcal{O}_X$ of the Koszul complex is *precisely* multiplication by the r-th Chern form of $E^\vee$.

One motivation for the current paper was to obtain a better understanding of the proof by Toledo and Tong of the Hirzebruch-Riemann-Roch theorem in [**12**]. In that paper the authors used local Koszul resolutions of the structure sheaf of the diagonal $\Delta_X \subset X \times X$ to construct the Grothendieck fundamental class $[\Delta_X]$, and then to compute $\chi(X, \mathcal{O}_X)$ as the degree of the restriction of the appropriate Kunneth-component of $[\Delta_X]$ to the diagonal. For such a computation one needs only the existence of a "nice" representative of the Grothendieck fundamental class in some neighborhood of the diagonal. However the diagonal Δ_X is *not* in general the zero set of a holomorphic section of a vector bundle. Instead one can use the "holomorphic exponential map" (see the article [**10**] for an exposition) to construct, in a neighborhood of the diagonal, a *real analytic* section of $p^*(T_X)$, which vanishes exactly on the diagonal. (Here $p : X \times X \to X$ is the projection onto the first factor.) Thus we are led to consider what happens if we ask only that τ be real analytic rather than holomorphic. In our second main result, we use the theory of superconnections and twisted complexes in the style of Brown [**5**], and of Toledo and Tong (*op. cit.*) to construct a map from the Dolbeault resolution of $K^*(\tau)$ to that of Ω^r_X representing the Grothendieck fundamental class and which restricts to the r-th Chern form of $E^\vee$. An important tool in this construction is a non-commutative version of the supertrace for endomorphisms of Grassman algebras.

We should also remark that instead of working in the real analytic category, one can make a very similar argument in the algebraic category, using formal schemes.

Let us now give a more detailed outline of the paper. Recall that the section τ gives rise to a natural Koszul resolution $K(\tau)^* \to \mathcal{O}_Z$, in which $K(\tau)^{-j} = \bigwedge^j \mathcal{E}$. Here $\mathcal{E}$ is the sheaf of holomorphic sections of E. Choose a connection $\nabla : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}_X^{1,0} \otimes \mathcal{E}$ of type $(1,0)$ ($\mathcal{A}_X^{1,0}$ being the sheaf of real analytic $(1,0)$-forms on X) on $\mathcal{E}$, such that $\nabla^2 = 0$. Let $\widetilde{\nabla} = \nabla + \overline{\partial}$ be the associated connection. We view ∇ as acting not only on $\mathcal{E}$, but on all tensor constructions on $\mathcal{E}$. Then our first result is:

***Theorem* (A).** — *The connection ∇ and the section τ determine a map of complexes, from the Koszul resolution $K(\tau)^*$ of $\mathcal{O}_Z$, to the Dolbeault resolution $\mathcal{A}_X^{r,*}[r]$ of $\Omega_X^r[r]$*

$$\psi : K(\tau)^* \to \mathcal{A}_X^{r,*}[r]$$

the degree $-r$ component of which is $\frac{1}{r!}(\imath_{\nabla(\tau)})^r$, and the degree 0 component $K(\tau)^0 = \mathcal{O}_X \to \mathcal{A}_X^{r,*}[r]^0 = \mathcal{A}_X^{r,r}$ of which is represented by the r-th Chern form of $(E^\vee, \widetilde{\nabla})$. In general ψ is given by a linear algebra construction involving ∇ and the curvature $R = [\nabla, \overline{\partial}]_s$ of $\widetilde{\nabla}$, and we have:

- *The class in $\mathrm{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega_X^r)$ represented by ψ is the Grothendieck fundamental class $[Z]$.*
- *The image of $[Z]$ in $\mathrm{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_X, \Omega_X^r) \simeq H^{r,r}(X, \mathbb{C})$, is represented by the degree zero component of ψ, which is equal to the r-th Chern form $c_r(E^\vee, \widetilde{\nabla})$*

It follows immediately that the image of $[Z]$ in $H^{r,r}(X, \mathbb{C})$ is equal to $c_r(E^\vee)$.

The proof of Theorem A is contained in Section 5. (cf. Theorem 5.5 and Corollary 5.6).

In the second half of the paper, we extend Theorem A to the case where Z is the zero locus of a real analytic section of $E^\vee$. It is no longer the case that τ determines a Koszul resolution of $\mathcal{O}_Z$, but instead we get a resolution of $\mathcal{A}_X^{0,*} \otimes \mathcal{O}_Z$. In order to get a complex that is quasi-isomorphic to $\mathcal{O}_Z$, we construct a resolution of the Dolbeault resolution $\mathcal{A}_X^{0,*} \otimes \mathcal{O}_Z$ of $\mathcal{O}_Z$, by constructing a *twisted differential*, δ, in the sense of Toledo and Tong [**13**], on $\mathcal{A}_X^{0,*} \otimes \bigwedge^* \mathcal{E}$.

A key tool in extending Theorem A to this situation is the notion of the "generalized supertrace" of an endomorphism of the exterior algebra of a finitely generated projective module. Suppose that V is a (locally) free module of finite rank r over a commutative ring k. Then the generalized supertrace is a map

$$\mathrm{Tr}_\Lambda : \mathrm{End}_k(\textstyle\bigwedge^* V) \to \bigwedge^* V^\vee$$

(c.f. Definition 6.1). If A is a graded-commutative algebra over k, we can extend this to a map

$$\mathrm{Tr}_\Lambda : \mathrm{End}_A(A \widehat{\otimes} \textstyle\bigwedge^* V) \to A \widehat{\otimes} \bigwedge^* V^\vee$$

Here $\widehat{\otimes}$ denotes the "super" or graded tensor product. If $\varphi \in \mathrm{End}_A(\bigwedge^* V)$, then the degree 0 component of $\mathrm{Tr}_\Lambda(\varphi)$ is the usual super-trace of φ. The key property of Tr_Λ (which is proved in Section 3.) is:

Proposition. — *Assume that $\varphi \in \mathrm{End}_A(\bigwedge^* V)$, and let $\delta \in \mathrm{End}_A(\bigwedge^* V)$ be an A-linear superderivation. Then*

$$\mathrm{Tr}_\Lambda[\delta, \varphi]_s = [\delta, \mathrm{Tr}_\Lambda(\varphi)]_s$$

***Theorem* (B)**. — *Let Z be a complex submanifold of X such that there exists a holomorphic vector bundle $\pi : E \to X$ and $\tau \in \Gamma(X, \mathcal{A}_X \otimes \mathcal{E}^\vee)$ such that $\imath_\tau : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}_X \otimes \mathcal{I}_Z$ is surjective. Then*

- *There is a superconnection δ of type (0,1), on the super-bundle $\bigwedge^* E$, such that:*
  1. *$\delta^2 = 0$, so δ defines a differential on $\mathcal{A}_X^{0,*} \otimes \bigwedge^* \mathcal{E}$,*
  2. *the component of δ of degree -1 with respect to the grading on $\bigwedge^* E$ is the Koszul differential $\imath_\tau$,*
  3. *If we write δ for the induced differential on $\mathcal{A}_X^{0,*} \otimes \bigwedge^* \mathcal{E}$, then the map $\bigwedge^0 \mathcal{E} = \mathcal{O}_X \to \mathcal{O}_Z$ induces a quasi-isomorphism of complexes:*

$$(\mathcal{A}_X^{0,*} \otimes {\textstyle\bigwedge^*} \mathcal{E}, \delta) \xrightarrow{\sim} (\mathcal{A}_X^{0,*} \otimes \mathcal{O}_Z, \overline{\partial}) \xleftarrow{\sim} \mathcal{O}_Z$$

- *Let R_A be the curvature of the superconnection $A = \nabla + \delta$ on $\bigwedge^* E$. Then the generalized supertrace of $\frac{1}{r!} R_A^r$ defines a map of complexes*

$$\mathcal{A}_X^{0,*} \otimes {\textstyle\bigwedge^*} \mathcal{E} \to \mathcal{A}_X^{r,*}[r],$$

  which, via the quasi-isomorphisms in part 1), represents the Grothendieck fundamental class $[Z]$,
- *The image of $[Z]$ in $H^{r,r}(X, \mathbb{C})$ is represented by the degree 0 component of the generalized supertrace of $\frac{1}{r!} R_A^r$, i.e., by the super-trace of $\frac{1}{r!} R_A^r$, which by Quillen* **[11]** *is an (r,r)-form representing the Chern character $\mathrm{ch}_r(\bigwedge^* E)$.*

The proof of Theorem B is contained in Proposition 8.4, Theorem 10.3, and Corollary 10.5. We would like to thank the referee for comments which let to a substantial improvement in the organization of the paper.

1. Superobjects

Throughout this paper we will use the language of *super-objects*. We include here basic definitions and properties for the convenience of the reader and to fix notation. We omit the details and proofs, which may be found in **[11]** and **[4]**.

Let k be a commutative ring with unity .

Definition 1.1. — *A k-module V with a $\mathbb{Z}/2\mathbb{Z}$-grading is called a k-supermodule.*

Remark 1.2. — In the same spirit, a $\mathbb{Z}/2\mathbb{Z}$-graded object in an additive category is called a *superobject*. As realizations of this general definition, we will be dealing with super algebras, super vector bundles on a smooth manifold, and sheaves of superalgebras on a topological space, etc.

We will write V^+ and V^- for the degree 0 (mod 2) and degree 1 (mod 2) parts of V and we will call them the even and the odd parts of V respectively. Let $\nu \in V$ be a homogeneous element. We say $|\nu| = 0$ if $\nu \in V^+$ and $|\nu| = 1$ if $\nu \in V^-$.

$\mathrm{End}_k(V)$ is also a k-supermodule with the grading

$$\begin{aligned}\mathrm{End}_k(V)^+ &= \mathrm{Hom}_k(V^+, V^+) \oplus \mathrm{Hom}_k(V^-, V^-)\\ \mathrm{End}_k(V)^- &= \mathrm{Hom}_k(V^+, V^-) \oplus \mathrm{Hom}_k(V^-, V^+)\end{aligned}$$

Moreover, the algebra of endomorphisms $\mathrm{End}_k(V)$ is a k-superalgebra with this grading. If no confusion is likely to arise, we will suppress the mention of the ring k from now on.

Definition 1.3. — *Let A be a superalgebra. The supercommutator of two elements of A is*

$$[a, b]_s = ab - (-1)^{|a||b|}ba$$

where a and b are homogeneous. The supercommutator is extended bilinearly to non-homogeneous a and b.

If the supercommutator $[\,,]_s : A \otimes A \to A$ is the zero map, then A is called a commutative superalgebra. The exterior algebra of a free module M with the $\mathbb{Z}/2\mathbb{Z}$-grading $\bigwedge^+ M = \bigoplus_{p\ even} \bigwedge^p M$ and $\bigwedge^- M = \bigoplus_{p\ odd} \bigwedge^p M$ is a commutative superalgebra.

Let V be finitely generated and projective. Assume that $\frac{1}{2} \in k$. Giving a $\mathbb{Z}/2\mathbb{Z}$-grading on V is equivalent to giving an involution $\epsilon \in \mathrm{End}_k(V)$, that is $\epsilon^2 = I$. The even and the odd parts are the eigenspaces corresponding to the eigenvalues $+1$ and -1 respectively. In the same fashion, the $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{End}_k(V)$ can be given by the involution

$$\rho(\varphi) = \epsilon \circ \varphi \circ \epsilon$$

where $\varphi \in \mathrm{End}_k(V)$.

Definition 1.4. — *Let $\varphi \in \mathrm{End}_k(V)$. The supertrace of φ, denoted by $\mathrm{tr}_s(\varphi)$, is defined to be*

$$\mathrm{tr}_s(\varphi) = \mathrm{tr}(\epsilon \circ \varphi)$$

where 'tr' is the usual trace map.

Lemma 1.5. — *The supertrace vanishes on supercommutators.*

Proof. — Cf. [**11**]. □

Let A and B be superalgebras. We define the *super tensor product* of A and B, denoted by $A \widehat{\otimes} B$, to be the k-module $A \otimes B$ with the $\mathbb{Z}/2\mathbb{Z}$-grading

$$\begin{aligned}(A \widehat{\otimes} B)^+ &= (A^+ \otimes B^+) \oplus (A^- \otimes B^-)\\ (A \widehat{\otimes} B)^- &= (A^+ \otimes B^-) \oplus (A^- \otimes B^+)\end{aligned}$$

and the algebra structure

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|} a_1 a_2 \otimes b_1 b_2$$

for homogeneous elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$. As usual the product is extended bilinearly.

Definition 1.6. — *Let A be a superalgebra and $\delta \in \operatorname{End}_k(A)$ be homogeneous. We will call δ a superderivation if it satisfies the super-Leibniz formula*

$$\delta(a_1 a_2) = \delta(a_1)a_2 + (-1)^{|\delta||a_1|}a_1\delta(a_2)$$

for homogeneous $a_1, a_2 \in A$. We will call a non-homogeneous element of $\operatorname{End}_k(A)$ *a superderivation, if its even and odd components are superderivations.*

2. Sheaves on Real Analytic Manifolds

While we could use the C^∞ Dolbeault complex in the proof of the first main theorem, for consistency we will work with real-analytic forms throughout this paper. In this section, we shall recall the results that we need.

Theorem 2.1. — *Let M be a real analytic manifold which is countable at infinity and let $\mathcal{F}$ be a coherent analytic sheaf on M. Then*

$$H^p(M, \mathcal{F}) = 0 \qquad \textit{for} \quad p > 0$$

Proof. — Cf. Proposition 2.3 of [**3**]. □

We denote the sheaf of real analytic functions on M by $\mathcal{A}_M$, while if X is a complex manifold, we shall write $\mathcal{A}_X^{p,q}$ for the sheaf of (p,q)-forms with real analytic coefficients. It is a classical result (see [**6**]) that the real-analytic Dolbeault complex is a resolution of the sheaf Ω_X^p of holomorphic p-forms. It follows from the theorem, therefore, that if $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_X$-modules, then the cohomology groups $H^q(X, \mathcal{E})$ may be computed as the cohomology of the real analytic Dolbeault complex $\mathcal{A}_X^{0,*} \otimes_{\mathcal{O}_X} \mathcal{E}(X)$.

Corollary 2.2. — *Let $\mathcal{F}$ be a locally free sheaf of $\mathcal{A}_M$-modules of finite rank. Then $\mathcal{F}$ is a projective object in the category of coherent sheaves of $\mathcal{A}_M$-modules.*

Proof. — Cf. Lemma 2.7 of [**3**]. □

It follows immediately that any vector bundle on a complex manifold admits a real analytic connection, since to give such a connection is the same as splitting the Atiyah sequence.

Proposition 2.3. — *Let X_1 and X_2 be complex spaces. The canonical projection*

$$\pi_1 : X_1 \times X_2 \to X_1$$

is flat.

Proof. — Cf. [**7**] (Proposition 3.17 on page 155). □

Corollary 2.4. — *Let X be a complex manifold. The sheaf $\mathcal{A}_X$ is a flat sheaf of $\mathcal{O}_X$-algebras.*

Proof. — Let $\overline{X}$ denote the complex manifold with the opposite complex structure and $\triangle : X \to X \times \overline{X}$ be the diagonal embedding. Let $\pi_1 : X \times \overline{X} \to X$ be the projection onto the first component. Let $x \in X$ be any point. The stalks $\mathcal{C}_{X,x}$ and $\mathcal{O}_{X\times\overline{X},\triangle(x)}$ are canonically isomorphic. Hence the result follows from the proposition applied to the map $\pi_1 : X \times \overline{X} \to X$. □

It follows immediately that if $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules, then the cohomology groups $H^q(X, \mathcal{F})$ may be computed as the cohomology of the real analytic Dolbeault complex $\mathcal{C}_X^{0,*} \otimes_{\mathcal{O}_X} \mathcal{F}(X)$.

3. Superconnections and the Chern Character

Let us recall the definition and basic properties of superconnections from **[11]**.

We assume that X is a real analytic manifold of dimension n. However, everything in this section applies verbatim to the smooth case. We denote the exterior algebra of the sheaf of real analytic differential forms on X, which is a sheaf of commutative superalgebras, by $\mathcal{C}_X^*$. Let $E = E^+ \oplus E^-$ be a real analytic super vector bundle on X. We will write $\mathcal{E}$ for the sheaf of real analytic sections of E, and $\mathcal{C}_X^*(\mathcal{E})$ for $\mathcal{C}_X^* \widehat{\otimes}_{\mathcal{C}_X} \mathcal{E}$.

Definition 3.1. — *A $\mathbb{C}$-linear endomorphism A of $\mathcal{C}_X^*(\mathcal{E})$ of odd degree is called a superconnection on E if it satisfies the super-Leibniz rule*

$$A(\omega \otimes s) = d\omega \otimes s + (-1)^{|\omega|}\omega \wedge A(s)$$

for local sections ω, s of $\mathcal{C}_X^*$ and $\mathcal{E}$ respectively.

If X is an almost complex manifold and if A satisfies the following version of the super-Leibniz rule

$$A(\omega \otimes s) = \overline{\partial}\omega \otimes s + (-1)^{|\omega|}\omega \wedge A(s)$$

then, it is called a superconnection of type $(0,1)$ (or simply a $(0,1)$-superconnection).

A^2 is called the curvature of the superconnection and is denoted by R_A. The curvature of A satisfies the identity

$$R_A(\omega \otimes s) = \omega \wedge R_A(s)$$

for local sections ω and s of $\mathcal{C}_X^*$ and $\mathcal{E}$ respectively. Thus R_A can be thought as a section of the sheaf of superalgebras $\mathcal{C}_X^* \widehat{\otimes}\, \mathcal{E}nd_{\mathcal{C}_X}(E)$ where $\mathcal{E}nd_{\mathcal{C}_X}(E)$ denotes the sheaf of endomorphisms of the bundle E.

We extend the supertrace to a map $\operatorname{tr}_s : \mathcal{C}_X^* \widehat{\otimes}\, \mathcal{E}nd_{\mathcal{C}_X}(E) \to \mathcal{C}_X^*$ by the formula

$$\operatorname{tr}_s(\omega \otimes \varphi) = \omega \operatorname{tr}_s(\varphi)$$

for local sections ω and φ of $\mathcal{C}_X^*$ and $\mathcal{E}nd_{\mathcal{C}_X}(E)$.

Proposition 3.2. — *Let n be a non-negative integer. The differential form $\operatorname{tr}_s(R_A^n)$ is closed and its cohomology class does not depend on the choice of the superconnection A.*

Proof. — Cf. **[11]**. □

Theorem 3.3. — *The differential form*

$$\operatorname{tr}_s(\exp(R_A)) \tag{3.1}$$

represents the class $\operatorname{ch}(E^+) - \operatorname{ch}(E^-)$ *in cohomology.*

Proof. — Cf. [**11**]. □

Remark 3.4. — The reader is warned that we omit the usual factor of $(\frac{i}{2\pi})$ from (3.1), following the convention in algebraic geometry.

4. The Grothendieck Fundamental Class

General references for this section are [**9**] and [**1**].

Let X be a compact complex manifold of dimension n. We denote the sheaf of holomorphic functions and the sheaf of holomorphic k-forms on X by $\mathcal{O}_X$ and Ω_X^k respectively. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are sheaves of $\mathcal{O}_X-$modules. We write $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ for the sheaf of $\mathcal{O}_X$-morphisms from $\mathcal{F}$ to $\mathcal{G}$ and $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ for $\Gamma(X,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}))$. The derived functors of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ (resp. $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$) will be denoted by $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ (resp. $\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$). We simply write $\mathcal{F}^\vee$ for the dual of $\mathcal{F}$.

The abelian groups $\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ and sheaves $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ are related by the following spectral sequence

$$E_2^{i,j} = H^i(X, \mathcal{E}xt^j_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})) \Rightarrow \operatorname{Ext}^{i+j}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}))$$

Let Y be a complex submanifold of X of codimension p. We denote the sheaf of ideals defining Y by $\mathcal{I}$. In this situation, one has that

$$\begin{aligned}
\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_Y,\Omega_X^p) &= 0 \qquad \text{for } i<p \qquad \text{and}\\
\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_Y,\Omega_X^p) &= \textstyle\bigwedge^p(\mathcal{I}/\mathcal{I}^2)^\vee \otimes \mathcal{O}_Y \otimes \Omega_X^p\\
&= \mathcal{H}om_{\mathcal{O}_X}(\textstyle\bigwedge^p(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_Y \otimes \Omega_X^p).
\end{aligned}$$

All tensor products are taken over $\mathcal{O}_X$ unless stated otherwise. It follows that the edge homomorphism

$$\operatorname{Ext}^p_{\mathcal{O}_X}(\mathcal{O}_Y,\Omega_X^p) \to H^0(\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_Y,\Omega_X^p))$$

is an isomophism, and so

$$\operatorname{Ext}^p_{\mathcal{O}_X}(\mathcal{O}_Y,\Omega_X^p) = \operatorname{Hom}_{\mathcal{O}_X}(\textstyle\bigwedge^p(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_Y \otimes \Omega_X^p).$$

Therefore there is a class $[Y]$ in $\operatorname{Ext}^p_{\mathcal{O}_X}(\mathcal{O}_Y,\Omega_X^p)$ which corresponds to the homomorphism of sheaves

$$\begin{gathered}
\textstyle\bigwedge^p(\mathcal{I}/\mathcal{I}^2) \to \mathcal{O}_Y \otimes \Omega_X^p\\
f_1\wedge\cdots\wedge f_p \mapsto df_1\wedge\cdots\wedge df_p.
\end{gathered}$$

The class $[Y]$ is called the Grothendieck fundamental class of Y in X.

5. Koszul Factorizations

In this section, we prove Theorem A of the Introduction. The proof is contained in Proposition 5.4 and Theorem 5.5.

Let $\pi : E \to X$ be a holomorphic vector bundle of rank r and let ∇ be a flat real analytic connection of type $(1,0)$ on E. For instance, ∇ can be taken as the $(1,0)$ part of the canonical connection associated to a real analytic hermitian structure on E. We write $\widetilde{\nabla}$ for $\overline{\partial} + \nabla$. We will denote the induced connection, and the $(1,0)$-connection on the dual bundle $E^\vee$ using the same symbols. However R will be used exclusively to denote the curvature of the induced connection on $E^\vee$. Throughout this section we assume that $\tau \in \Gamma(X, \mathscr{E}^\vee)$. Let $\imath_\tau : \mathscr{A}_X \otimes \bigwedge^p \mathscr{E} \to \mathscr{A}_X \otimes \bigwedge^{p-1} \mathscr{E}$ be contraction by τ as usual. We extend $\imath_\tau$ to an odd superderivation of the sheaf of commutative superalgebras $\mathscr{A}^*_X \widehat{\otimes} \bigwedge \mathscr{E}$. Note that $\nabla(\tau) = \imath_\tau \circ \nabla + \partial \circ \imath_\tau : \mathscr{A}_X \otimes \mathscr{E} \to \mathscr{A}^{1,0}_X$ and therefore $\nabla(\tau)$ can be considered as an element of $\Gamma(X, \mathscr{A}^{1,0}_X \otimes \mathscr{E}^\vee)$. We write $\imath_{\nabla(\tau)} : \mathscr{A}_X \otimes \bigwedge^p \mathscr{E} \to \mathscr{A}^{1,0}_X \otimes \bigwedge^{p-1} \mathscr{E}$ and $\imath_{R(\tau)} : \mathscr{A}_X \otimes \bigwedge^p \mathscr{E} \to \mathscr{A}^{1,1}_X \otimes \bigwedge^{p-1} \mathscr{E}$ for the contractions with $\nabla(\tau) \in \Gamma(X, \mathscr{A}^{1,0}_X \otimes \mathscr{E}^\vee)$ and $R(\tau) \in \Gamma(X, \mathscr{A}^{1,1}_X \otimes \mathscr{E}^\vee)$ respectively. We extend $\imath_{\nabla(\tau)}$ (resp. $\imath_{R(\tau)}$) to an even (resp. odd) superderivation of $\mathscr{A}^*_X \widehat{\otimes} \bigwedge \mathscr{E}$.

We state two facts without proof

$$\begin{aligned} [\overline{\partial}, \imath_{\nabla(\tau)}]_s &= \imath_{R(\tau)} \\ [\imath_{\nabla(\tau)}, \imath_{R(\tau)}]_s &= 0. \end{aligned}$$

Lemma 5.1. — *For $1 \leqslant p \leqslant r$ the following diagram is commutative*

$$\begin{tikzcd} \mathscr{Z}^{r-p,r-p}_X \otimes \bigwedge^p \mathscr{E} \arrow[r, "\frac{1}{p!}(\imath_{\nabla(\tau)})^p"] \arrow[d, "\imath_{R(\tau)}"'] & \mathscr{A}^{r,r-p}_X \arrow[d, "\overline{\partial}"'] \\ \mathscr{Z}^{r-p+1,r-p+1}_X \otimes \bigwedge^{p-1} \mathscr{E} \arrow[r, "\frac{1}{(p-1)!}(\imath_{\nabla(\tau)})^{p-1}"] & \mathscr{A}^{r,r-p+1}_X \end{tikzcd}$$

where $\mathscr{Z}^{p,p}_X$ denote the sheaf of $\overline{\partial}$-closed (not necessarily ∂-closed) forms of type (p,p).

Proof. — We have

$$\begin{aligned}
\overline{\partial} \circ \frac{1}{p!}\,(\imath_{\nabla(\tau)})^p &= \frac{1}{p!}\,[\overline{\partial}, (\imath_{\nabla(\tau)})^p]_s \\
&= \frac{1}{p!}\,\sum_{j=0}^{p-1} (\imath_{\nabla(\tau)})^j \circ [\overline{\partial}, \imath_{\nabla(\tau)}]_s \circ (\imath_{\nabla(\tau)})^{p-j-1} \\
&= \frac{1}{p!}\,\sum_{j=0}^{p-1} (\imath_{\nabla(\tau)})^j \circ \imath_{R(\tau)} \circ (\imath_{\nabla(\tau)})^{p-j-1} \\
&= \frac{1}{p!}\,\sum_{j=0}^{p-1} (\imath_{\nabla(\tau)})^{p-1} \circ \imath_{R(\tau)} \\
&= \frac{1}{p!}\,p\,(\imath_{\nabla(\tau)})^{p-1} \circ \imath_{R(\tau)} = \frac{1}{(p-1)!}\,(\imath_{\nabla(\tau)})^{p-1}. \qquad \square
\end{aligned}$$

Let $\phi_p : \bigwedge^p \mathscr{E} \to \mathscr{H}om_{\mathscr{O}_X}(\bigwedge^{r-p} \mathscr{E}, \bigwedge^r \mathscr{E})$ be the isomorphism given by

$$\phi_p : \alpha \mapsto (\beta \mapsto \beta \wedge \alpha) \qquad \text{for} \qquad \alpha \in \textstyle\bigwedge^p \mathscr{E}, \quad \beta \in \bigwedge^{r-p} \mathscr{E}, \text{ and } 0 \leqslant p \leqslant r$$

We will identify the sheaves $\mathscr{H}om_{\mathscr{O}_X}(\bigwedge^{r-p} \mathscr{E}, \bigwedge^r \mathscr{E})$ and $\bigwedge^{r-p} \mathscr{E}^\vee \otimes \bigwedge^r \mathscr{E}$ via the canonical isomorphism between them.

Lemma 5.2. — *The following diagram is commutative for* $1 \leqslant p \leqslant r$

$$\begin{array}{ccc}
\bigwedge^p \mathscr{E} & \xrightarrow{(\phi_p^{-1}\otimes 1)\circ(\bigwedge^{r-p} R\otimes 1)\circ\phi_p} & \bigwedge^p \mathscr{E} \otimes \mathscr{Z}_X^{r-p,r-p} \\
\imath_\tau \downarrow & & \downarrow \imath_{R(\tau)} \\
\bigwedge^{p-1} \mathscr{E} & \xrightarrow{(\phi_{p-1}^{-1}\otimes 1)\circ(\bigwedge^{r-p+1} R\otimes 1)\circ\phi_{p-1}} & \bigwedge^{p-1} \mathscr{E} \otimes \mathscr{Z}_X^{r-p+1,r-p+1}
\end{array}$$

where $\bigwedge^p R : \bigwedge^p \mathscr{E}^\vee \to \bigwedge^p \mathscr{E}^\vee \otimes \mathscr{Z}_X^{p,p}$ *is defined by* $\bigwedge^p R\,(e^1 \wedge \cdots \wedge e^p) = R(e^1) \wedge \cdots \wedge R(e^p)$ *for local sections* $e^1, \ldots, e^p$ *of* $\mathscr{E}^\vee$*.*

Proof. — The lemma is an immediate consequence of the following commutative diagrams. (Note that $\wedge\tau$ and $\wedge R(\tau)$ denote right multiplication by τ and $R(\tau)$ respectively).

$$\begin{array}{ccccc}
\bigwedge^p \mathscr{E} & \xrightarrow{\phi_p} & \bigwedge^{r-p} \mathscr{E}^\vee \otimes \bigwedge^r \mathscr{E} & \xrightarrow{\bigwedge^{r-p} R\otimes 1} & \bigwedge^{r-p} \mathscr{E}^\vee \otimes \bigwedge^r \mathscr{E} \otimes \mathscr{Z}_X^{r-p,r-p} \\
\imath_\tau \downarrow & & \wedge\tau \downarrow & & \wedge R(\tau) \downarrow \\
\bigwedge^{p-1} \mathscr{E} & \xrightarrow{\phi_{p-1}} & \bigwedge^{r-p+1} \mathscr{E}^\vee \otimes \bigwedge^r \mathscr{E} & \xrightarrow{\bigwedge^{r-p+1} R\otimes 1} & \bigwedge^{r-p+1} \mathscr{E}^\vee \otimes \bigwedge^r \mathscr{E} \otimes \mathscr{Z}_X^{r-p+1,r-p+1}
\end{array}$$

and

$$\begin{array}{ccc}
\bigwedge^{r-p}\mathscr{E}^\vee\otimes\bigwedge^r\mathscr{E}\otimes\mathscr{Z}_X^{r-p,r-p} & \xrightarrow{\phi_p^{-1}\otimes 1} & \bigwedge^p\mathscr{E}\otimes\mathscr{Z}_X^{r-p,r-p}\\
{\scriptstyle\wedge R(\tau)}\big\downarrow & & {\scriptstyle\imath_{R(\tau)}}\big\downarrow\\
\bigwedge^{r-p+1}\mathscr{E}^\vee\otimes\bigwedge^r\mathscr{E}\otimes\mathscr{Z}_X^{r-p+1,r-p+1} & \xrightarrow{\phi_{p-1}^{-1}\otimes 1} & \bigwedge^{p-1}\mathscr{E}\otimes\mathscr{Z}_X^{r-p+1,r-p+1}.
\end{array}$$

It is worth mentioning that R can be written as a matrix of $\overline{\partial}$-closed forms of type $(1,1)$ with respect to any given local holomorphic framing. Consequently the image of the mapping $\bigwedge^p R:\bigwedge^p\mathscr{E}^\vee\to\bigwedge^p\mathscr{E}^\vee\otimes\mathscr{A}_X^{p,p}$ lies in $\bigwedge^p\mathscr{E}^\vee\otimes\mathscr{Z}_X^{p,p}$. Since τ is a holomorphic section, a similar remark applies to the mappings $\imath_{R(\tau)}$ and $\wedge R(\tau)$. □

Proposition 5.3. — *The following diagram is commutative*

$$\begin{array}{ccccccccc}
\bigwedge^r\mathscr{E} & \xrightarrow{\imath_\tau} & \bigwedge^{r-1}\mathscr{E} & \xrightarrow{\imath_\tau} & \cdots & \xrightarrow{\imath_\tau} & \mathscr{E} & \xrightarrow{\imath_\tau} & \mathscr{O}_X\\
{\scriptstyle\psi_r=\frac{1}{r!}(\imath_{\nabla(\tau)})^r}\big\downarrow & & {\scriptstyle\psi_{r-1}}\big\downarrow & & & & {\scriptstyle\psi_1}\big\downarrow & & {\scriptstyle\psi_0=\det(R)}\big\downarrow\\
\mathscr{A}_X^{r,0} & \xrightarrow{\overline{\partial}} & \mathscr{A}_X^{r,1} & \xrightarrow{\overline{\partial}} & \cdots & \xrightarrow{\overline{\partial}} & \mathscr{A}_X^{r,r-1} & \xrightarrow{\overline{\partial}} & \mathscr{A}_X^{r,r}.
\end{array}$$

where $\psi_p=\frac{1}{p!}(\imath_{\nabla(\tau)})^p\circ(\phi_p^{-1}\otimes 1)\circ(\bigwedge^{r-p}R\otimes 1)\circ\phi_p$.

Proof. — This is an immediate result of the previous two lemmas. □

The symbol $f:C^*\xrightarrow{\sim}D^*$ is used to denote that f is a quasi-isomorphism.

Proposition 5.4. — *The morphism $\psi\in\mathscr{H}om_{\mathscr{O}_X}(\bigwedge^*\mathscr{E},\mathscr{A}_X^{r,*}[r])$ represents the Grothendieck fundamental class of Z in $\mathscr{E}xt^r_{\mathscr{O}_X}(\mathscr{O}_Z,\Omega^r_X)$.*

Proof. — The morphism (which is a map of complexes) $\psi\in\mathscr{H}om_{\mathscr{O}_X}(K(\tau)^*,\mathscr{A}_X^{r,*}[r])$ gives us an r-cocycle, denoted by $[\psi]$, in the double complex $\mathscr{H}om^*_{\mathscr{O}_X}(K(\tau)^\cdot,\mathscr{A}_X^{r,\cdot})$. We have the quasi-isomorphisms

$$\mathscr{H}om^*_{\mathscr{O}_X}(K(\tau)^\cdot,\mathscr{A}_X^{r,\cdot})\xrightarrow{\sim}K(\tau)^*\otimes\det(\mathscr{E}^\vee)\otimes\mathscr{A}_X^{r,*}\xrightarrow{\sim}\mathscr{O}_Z\otimes\det(\mathscr{E}^\vee)\otimes\mathscr{A}_X^{r,*}.$$

Under these maps, $[\psi]$ is mapped to $\psi_r:\bigwedge^r\mathscr{E}\to\mathscr{A}_X^{r,0}\pmod{\mathscr{I}}$.

We have $\tau=\sum_i\alpha_i e^i$ with respect to some local holomorphic framing $\{e^1,\dots,e^r\}$ of $E^\vee$. Then

$$\begin{aligned}
\psi_r(e_1\wedge\cdots\wedge e_r) &= \frac{1}{r!}(\imath_{\nabla(\tau)})^r(e_1\wedge\cdots\wedge e_r)\\
&= \imath_{\nabla(\tau)}(e_1)\wedge\cdots\wedge\imath_{\nabla(\tau)}(e_r)\\
&= \partial\alpha_1\wedge\cdots\wedge\partial\alpha_r \pmod{\mathscr{I}}\\
&= d\alpha_1\wedge\cdots\wedge d\alpha_r \pmod{\mathscr{I}}.
\end{aligned}$$

Then the result follows from the fact that the morphism defining Grothendieck fundamental class of Z in $\mathrm{Hom}_{\mathscr{O}_X}(\bigwedge^r(\mathscr{I}/\mathscr{I}^2),\mathscr{O}_Z\otimes\Omega^r_X)$ is mapped to $\psi_r:\bigwedge^r\mathscr{E}\to\mathscr{A}_X^{r,0}$ (mod $\mathscr{I}$) under the following sequence of quasi-isomorphisms

$$\bigwedge^r(\mathscr{I}/\mathscr{I}^2)^\vee\otimes\Omega^r_X\xrightarrow{\sim}\mathscr{O}_Z\otimes\det(\mathscr{E}^\vee)\otimes\Omega^r_X\xrightarrow{\sim}\mathscr{O}_Z\otimes\det(\mathscr{E}^\vee)\otimes\mathscr{A}_X^{r,*}.$$ □

Theorem 5.5. — *The map of complexes $\psi \in \operatorname{Hom}_{\mathcal{O}_X}(\bigwedge^* \mathcal{E}, \mathcal{A}_X^{r,*}[r])$ represents the Grothendieck fundamental class of Z in $\operatorname{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^r_X)$. Moreover the image of ψ in $H^r(X, \Omega^r_X)$ is the r-th Chern form $c_r(E^\vee, \widetilde{\nabla})$.*

Proof. — Since ψ represents the Grothendieck fundamental class locally, it does so globally. The second result follows from the fact that $\psi_0 = \det(R)$ by Proposition 5.3 and that $\det(R)$ is the r-th Chern form of the pair $(E^\vee, \widetilde{\nabla})$. □

We obtain immediately

Corollary 5.6. — *Let $\pi : E \to X$ be a holomorphic vector bundle of rank r and $\tau : X \to E^\vee$ be a holomorphic section which is transverse to the zero section. If Z is the complex submanifold where τ vanishes, then the fundamental class of Z in Dolbeault cohomology is represented by the r-th Chern form $c_r(E^\vee, \widetilde{\nabla})$.*

Notice that the standard proofs of this result (for example in [**8**]) implicitly use the axioms defining Chern classes.

6. Generalized Supertraces

The heart of this section is Proposition 6.4 which will be used in Section 9 to construct a map of complexes that represents the Grothendieck fundamental class.

Let V be a finitely generated projective module over a commutative ring with unity k, and let $V^\vee$ be its dual. Let $\langle\, ,\rangle : V^\vee \otimes_k V \to k$ be the pairing defined by $\langle s, t\rangle = s(t)$ for $s \in V^\vee$ and $t \in V$. We extend $\langle\, ,\rangle$ to a pairing between $\bigwedge^m V^\vee$ and $\bigwedge^m V$ by

$$\langle u, v\rangle = \det\langle u^i, v_j\rangle$$

where $u = u^1 \wedge \cdots \wedge u^m \in \bigwedge^m V^\vee$ and $v = v_1 \wedge \cdots \wedge v_m \in \bigwedge^m V$. It is easy to check that

$$\langle u, v\rangle = (\imath_{u^m} \circ \cdots \circ \imath_{u^1})(v_1 \wedge \cdots \wedge v_m)$$

where $\imath_{u^j}$ denotes contraction by u^j. We denote the exterior algebra of V by $\bigwedge V$ with the usual grading for which $\bigwedge^n V$ has degree n. Then

$$\operatorname{Hom}_k(\textstyle\bigwedge V, \bigwedge V)$$

is naturally graded with $\operatorname{Hom}_k(\bigwedge^m V, \bigwedge^n V)$ having degree $(n-m)$.

Definition 6.1. — *Let $\varphi \in \operatorname{Hom}_k(\bigwedge V, \bigwedge V)$. If φ has degree $(-n)$ with $n \geqslant 0$, we define the generalized supertrace of φ, denoted by $\operatorname{Tr}_\Lambda(\varphi) \in \bigwedge^n V^\vee$ (as opposed to the supertrace tr_s), as follows:*

$$(\operatorname{Tr}_\Lambda(\varphi))(\eta) = (-1)^{|\eta|}\operatorname{tr}_s(l_\eta \circ \varphi)$$

where $l_\eta \in \operatorname{End}_k(\bigwedge V)$ is left multiplication by η for some $\eta \in \bigwedge^n V$. If φ has positive degree, then $\operatorname{Tr}_\Lambda(\varphi)$ is defined to be 0.

Clearly when $n = 0$, we have $\mathrm{Tr}_\Lambda = \mathrm{tr}_s$.

Let $i : \bigwedge V^\vee \to \mathrm{End}_k(\bigwedge V)$ be the inclusion defined by $i(\alpha)(\beta) = \langle \alpha, \beta \rangle$ for $\alpha \in \bigwedge^m V^\vee$ and $\beta \in \bigwedge^m V$, and $i(\alpha)(\beta) = 0$ if $\beta \notin \bigwedge^m V$. We will often identify $\bigwedge V^\vee$ with its image under i, and think of $\mathrm{Tr}_\Lambda(\varphi)$ as belonging to $\mathrm{End}_k(\bigwedge V)$.

Remark 6.2. — We have the identity

$$\mathrm{Tr}_\Lambda \circ i = \mathrm{id}_{\bigwedge V^\vee}.$$

Remark 6.3. — Let $\tau \in V^\vee$ and let $\imath_\tau$ denote contraction by τ, which is a superderivation of $\bigwedge V$ of degree -1. It is straightforward to check that

$$\mathrm{Tr}_\Lambda(\imath_\tau) = \begin{cases} 0 & \text{if} \quad \operatorname{rank} V \geqslant 2 \\ \tau & \text{if} \quad \operatorname{rank} V = 1. \end{cases}$$

An important feature of the supertrace is that it vanishes on supercommutators. However, this is *not* true of the generalized trace Tr_Λ. Instead we have:

Proposition 6.4. — *Assume that $\varphi \in \mathrm{End}_k(\bigwedge V)$, and let $\delta \in \mathrm{End}_k(\bigwedge V)$ be a k-linear superderivation. Then*

$$\mathrm{Tr}_\Lambda[\delta, \varphi]_s = [\delta, \mathrm{Tr}_\Lambda(\varphi)]_s.$$

In order to prove the proposition we need a lemma.

Lemma 6.5. — *Assume that $\varphi \in \mathrm{Hom}_k(\bigwedge V, \bigwedge V)$ is of degree $-n \leqslant 0$, and let δ be a k-linear superderivation of degree j with $-n + j \leqslant 0$. Then*

$$\mathrm{Tr}_\Lambda[\delta, \varphi]_s = [\delta, \mathrm{Tr}_\Lambda(\varphi)]_s.$$

Proof. — Let $\eta \in \bigwedge^{n-j} V$ be any element. Then

$$\begin{aligned} \mathrm{Tr}_\Lambda[\delta, \varphi]_s(\eta) &= \mathrm{Tr}_\Lambda(\delta \circ \varphi - (-1)^{-nj} \varphi \circ \delta)(\eta) \\ &= (-1)^{|\eta|} \mathrm{tr}_s(l_\eta \circ \delta \circ \varphi) - (-1)^{|\eta| - nj} \mathrm{tr}_s(l_\eta \circ \varphi \circ \delta) \\ &\quad \text{by definition of } \mathrm{Tr}_\Lambda \\ &= (-1)^{|\eta|} \mathrm{tr}_s(l_\eta \circ \delta \circ \varphi) - (-1)^{n-nj} \mathrm{tr}_s(\delta \circ l_\eta \circ \varphi) \\ &\quad \text{since } \mathrm{tr}_s([\delta, l_\eta \circ \varphi]_s) = 0 \\ &= (-1)^{n-nj+1} \mathrm{tr}_s\left([\delta, l_\eta]_s \circ \varphi\right) \\ &= (-1)^{n-nj+1} \mathrm{tr}_s(l_{\delta(\eta)} \circ \varphi) \\ &= -(-1)^{-nj} \mathrm{Tr}_\Lambda(\varphi)(\delta(\eta)) \quad \text{by definition of } \mathrm{Tr}_\Lambda \\ &= [\delta, \mathrm{Tr}_\Lambda(\varphi)]_s(\eta) \quad \text{since } \delta \circ \mathrm{Tr}_\Lambda(\varphi) = 0. \end{aligned}$$

□

Proof of Proposition 6.4. — We can write $\varphi = \sum_n \varphi_n$ and $\delta = \sum_j \delta_j$ where φ_n is the degree n component of φ, and δ_j is the degree j component of δ. Note that a superderivation of $\bigwedge V$ is necessarily of degree greater than or equal to -1. Moreover, $\mathrm{Tr}_\Lambda[\delta_j,\varphi_n]_s = 0$ unless $n+j \leqslant 0$. Then

$$\begin{aligned}
\mathrm{Tr}_\Lambda[\delta,\varphi]_s &= \sum_{j\geqslant -1} \mathrm{Tr}_\Lambda[\delta_j,\varphi]_s \\
&= \sum_{j\geqslant -1}\sum_{n\leqslant -j} \mathrm{Tr}_\Lambda[\delta_j,\varphi_n]_s \\
&= \sum_{j\geqslant -1}\sum_{n\leqslant -j} [\delta_j,\mathrm{Tr}_\Lambda(\varphi_n)]_s \quad \text{by Lemma 6.5} \\
&= [\delta,\mathrm{Tr}_\Lambda(\varphi)]_s.
\end{aligned}$$

□

7. Twisted Complexes

In this section, we give a brief exposition of twisted complexes, which were introduced by E. Brown in [**5**]. These shall be used in Section 8 to construct “global resolutions" of $\mathcal{O}_Z$ by locally free $\mathcal{O}_X$-modules. (cf. Proposition 8.4). The reader is referred to [**14**] for an extensive study of the use of twisted complexes in the duality theory of complex manifolds.

Let **A** be an abelian category.

Definition 7.1. — *Let $M = \{M^{p,q}\}_{p,q\in\mathbb{Z}}$ be a bigraded object in* **A**, *and let δ be a differential of total degree +1 on the associated graded object $T(M)^i = \bigoplus_{p+q=i} M^{p,q}$. The pair (M,δ) is called a twisted complex if δ preserves the filtration with respect to the grading by the first degree on M. In this case, δ is called the twisting differential of the pair (M,δ).*

We can write $\delta = \sum_{k\geqslant 0} a_k$ where $a_k \in \prod_{p,q\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{O}}(M^{p,q}, M^{p+k,q-k+1})$. The fact that $\delta^2 = 0$ entails the following, which is called the *twisting cocycle condition*,

$$\sum_{i=0}^{n} a_i a_{n-i} = 0 \qquad \text{for} \quad n \geqslant 0. \tag{7.1}$$

Consider the cases where n=0, 1, 2

$$a_0^2 = 0 \tag{7.2}$$

$$a_0a_1 + a_1a_0 = 0 \tag{7.3}$$

$$a_0a_2 + a_1^2 + a_2a_0 = 0 \tag{7.4}$$

$(M^{p,*},a_0)$ is a cochain complex for each $p \in \mathbb{Z}$ by (7.2). The totality of maps $(-1)^q a_1^{p,q} : M^{p,q} \to M^{p+1,q}$ gives us a map of complexes from $(M^{p,*},a_0)$ to $(M^{p+1,*},a_0)$ by (7.3). Equation (7.4) entails that $-a_1^2 : (M^{p,*},a_0) \to (M^{p+2,*},a_0)$ is chain homotopic to the zero map, and the chain homotopy is given by $a_2 \in \prod_{p,q\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{O}}(M^{p,q}, M^{p+2,q-1})$.

By definition $(T(M)^*, \delta)$ is a filtered differential object with the filtration given by $F^k = \bigoplus_{i \geqslant k} M^{i,j}$. Hence there exists a spectral sequence with

$$E_1^{p,q} = H^q(M^{p,*}, a_0) \Rightarrow H^{p+q}(T(M)^*, \delta).$$

We note that $(H^*(M^{p,*}, a_0), a_1)$ is a cochain complex because of (7.4). As a result the E_2 terms of the spectral sequence are

$$E_2^{p,q} = H^p(H^q(M^{**}, a_0), a_1).$$

***Remark** 7.2*. — The reader may observe that there is a formal similarity between twisting cocyles and flat superconnections.

***Example** 7.3*. — Every double complex (with anticommuting differentials) is a twisted complex with $a_k = 0$ for $k \geqslant 2$.

***Example** 7.4*. — Let (C^*, d) be a bounded cochain complex in $\mathbf{A}$, and for simplicity assume that $C^k \neq 0$ only for $0 \leqslant k \leqslant n$ for some $n \in \mathbb{N}$. Suppose we are given projective resolutions $(P^{p,*}, \alpha_p)$ of C^p for each p with augmentation maps $\epsilon_p : (P^{p,*}, \alpha_p) \to C^p$. Then one has cochain maps $\beta_p : (P^{p,*}, \alpha_p) \to (P^{p+1,*}, \alpha_{p+1})$ lifting $d : C^p \to C^{p+1}$.

$$\begin{tikzcd}
 & \vdots \arrow[d, "\alpha_0"'] & \vdots \arrow[d, "\alpha_1"'] & & \vdots \arrow[d, "\alpha_n"'] & \\
0 \arrow[r] & P^{0,-1} \arrow[r, "\beta_{-1}"] \arrow[d, "\alpha_0"'] & P^{1,-1} \arrow[r, "\beta_{-1}"] \arrow[d, "\alpha_1"'] & \cdots \arrow[r, "\beta_{-1}"] & P^{n,-1} \arrow[r] \arrow[d, "\alpha_n"'] & 0 \\
0 \arrow[r] & P^{0,0} \arrow[r, "\beta_0"] \arrow[d, "\epsilon_0"'] & P^{1,0} \arrow[r, "\beta_0"] \arrow[d, "\epsilon_1"'] & \cdots \arrow[r, "\beta_0"] & P^{n,0} \arrow[r] \arrow[d, "\epsilon_n"'] & 0 \\
0 \arrow[r] & C^0 \arrow[r, "d"] \arrow[d] & C^1 \arrow[r, "d"] \arrow[d] & \cdots \arrow[r, "d"] & C^n \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & & 0 &
\end{tikzcd}$$

Now we will construct maps a_k for $k \geqslant 0$ in order to make $(P^{**}, \delta = \sum_{k \geqslant 0} a_k)$ a twisted complex. (Note that $a_k = 0$ for $k \geqslant n + 1$).

We set $a_0 = \{(-1)^p \alpha_p\}_{p \in \mathbb{Z}}$ and $a_1 = \{\beta_q\}_{q \in \mathbb{Z}}$. Then the twisting cocycle condition is satisfied for $n = 0$ and $n = 1$. Now assume that a_k is constructed. It is easy to check that $-\sum_{i=1}^{k} a_i a_{k-i+1} : (P^{p,*}, a_0) \to (P^{p+k+1,*}[-k+1], a_0)$ is a cochain map. If we consider $-\sum_{i=1}^{k} a_i a_{k-i+1}$ as a map from the complex $(P^{p,*}, a_0)$ to the augmented complex $(P^{p+k+1,*}[-k+1] \to C^{p+k+1}[-k+1] \to 0)$, then it is homotopic to the zero map being a chain map from a complex of projectives to an acyclic complex. Let a_{k+1}

be a chain homotopy between $-\sum_{i=1}^{k} a_1 a_{k-i+1}$ and the zero map, then one has

$$a_{k+1}a_0 + a_0 a_{k+1} + \sum_{i=1}^{k} a_i a_{k-i+1} = 0$$

which is equivalent to the twisting cocycle condition for $n = k+1$. This completes the induction.

Moreover, if we endow C^* with the *filtration bête*, i.e. the filtration defined by

$$\sigma_{\geqslant p}(C)^i = \begin{cases} 0 & \text{if} \quad i < p \\ C^i & \text{if} \quad i \geqslant p \end{cases}$$

then, the augmentation map $\epsilon : T(P^{**}) \to C^*$ respects the filtrations and is a quasi-isomorphism of the associated graded objects. Therefore, the cochain complex $(T(P^{**}), \delta)$ is quasi-isomorphic to (C^*, d).

Definition 7.5. — *Let C be an object in $\mathbf{A}$. A twisted complex (M, δ) is called a twisted resolution of C if*

$$H^i(T(M)^*, \delta) = \begin{cases} C & \textit{if} \quad i = 0 \\ 0 & \textit{if} \quad i \neq 0. \end{cases}$$

8. Koszul-Dolbeault twisted resolutions

The main result of this section is Proposition 8.4 wherein we use twisted complexes to construct a global resolution of $\mathcal{O}_Z$ by locally free $\mathcal{A}_X$-modules.

Let X be a compact complex manifold of dimension n. We write $\mathcal{A}_X$ and $\mathcal{A}_X^{p,q}$ for the sheaf of real analytic complex valued functions and for the sheaf of real analytic differential forms of type (p,q) respectively. Let $\pi : E \to X$ be a holomorphic vector bundle of rank r. We will denote the sheaf of holomorphic sections of E by $\mathcal{E}$. The dual bundle and its sheaf of sections will be denoted by $E^\vee$ and $\mathcal{E}^\vee$ respectively.

Definition 8.1. — *A connection of type (0,1) (or simply a (0,1)-connection) on E is a $\mathbb{C}$-linear map*

$$\overline{D} : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}_X^{0,1} \otimes \mathcal{E}$$

satisfying the Leibniz rule

$$\overline{D}(fs) = \overline{\partial} f \otimes s + f\overline{D}(s)$$

for local sections f of $\mathcal{A}_X$ and s of $\mathcal{E}$.

Let $\overline{D}$ be a $(0,1)$-connection on E, and let $\langle\, ,\rangle$ denote the pairing between E and $E^\vee$. One defines a (0,1)-connection on $E^\vee$ (which will also be denoted by $\overline{D}$) by the formula

$$\overline{\partial}\langle s, t\rangle = \langle \overline{D}s, t\rangle + \langle s, \overline{D}t\rangle$$

for local sections s and t of $\mathcal{A}_X \otimes \mathcal{E}$ and $\mathcal{A}_X \otimes \mathcal{E}^\vee$ respectively. Finally we extend $\overline{D}$ to a $\mathbb{C}$-linear superderivation of odd degree of the sheaf of superalgebras $\mathcal{A}_X^* \widehat{\otimes} \bigwedge \mathcal{E}$. As a result, $\overline{D}$ is a $(0,1)$-superconnection on the bundle $\bigwedge E$. Let τ be a global section of $\mathcal{A}_X \otimes \mathcal{E}^\vee$. We extend $\imath_\tau$, the contraction by τ, to an odd degree superderivation of $\mathcal{A}_X^* \widehat{\otimes} \bigwedge \mathcal{E}$ which acts trivially on $\mathcal{A}(X)$.

Lemma 8.2. — *Let $\tau \in \Gamma(X, \mathcal{A}_X \otimes \mathcal{E}^\vee)$ be any section and let $\overline{D}$ be a $(0,1)$-connection on E such that $\overline{D}(\tau) = 0$. The following diagram is anti-commutative for $p, q \geqslant 0$*

$$\begin{array}{ccc} \mathcal{A}_X^{0,q} \otimes \bigwedge^p \mathcal{E} & \xrightarrow{\overline{D}} & \mathcal{A}_X^{0,q+1} \otimes \bigwedge^p \mathcal{E} \\ \imath_\tau \big\downarrow & & \imath_\tau \big\downarrow \\ \mathcal{A}_X^{0,q} \otimes \bigwedge^{p-1} \mathcal{E} & \xrightarrow{\overline{D}} & \mathcal{A}_X^{0,q+1} \otimes \bigwedge^{p-1} \mathcal{E}. \end{array}$$

Proof. — We have $\overline{D} \circ \imath_\tau + \imath_\tau \circ \overline{D} = [\overline{D}, \imath_\tau]_s = \imath_{\overline{D}(\tau)}$. □

Lemma 8.3. — *Let Z be a complex submanifold of X such that there exists a holomorphic vector bundle $\pi : E \to X$ and a section$\tau \in \Gamma(X, \mathcal{A}_X \otimes \mathcal{E}^\vee)$, vanishing along Z, such that the induced map $\imath_\tau : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}_X \otimes \mathcal{I}_Z$ is surjective. Then there exists a $(0,1)$-connection $\overline{D}$ on the bundle E such that $\overline{D}(\tau) = 0$.*

Proof. — We have the following diagram

$$\begin{array}{ccccccc} & & \mathcal{A}_X \otimes \mathcal{E} & & & & \\ & & \imath_{\overline{\partial}(\tau)} \big\downarrow & & & & \\ \mathcal{A}_X^{0,1} \otimes \mathcal{E} & \xrightarrow{\imath_\tau} & \mathcal{A}_X^{0,1} & \xrightarrow{p} & \mathcal{A}_X^{0,1} \otimes \mathcal{O}_Z & \longrightarrow & 0. \end{array}$$

There exists an $\mathcal{A}_X$-linear map $\theta : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}_X^{0,1} \otimes \mathcal{E}$ such that $\imath_\tau \circ \theta = \imath_{\overline{\partial}(\tau)}$ since $p \circ \imath_{\overline{\partial}(\tau)} = 0$ and $\mathcal{A}_X \otimes \mathcal{E}$ is projective by Corollary 2.2. Then $\overline{D} = \overline{\partial} - \theta$ is the desired $(0,1)$-connection since

$$[\overline{D}, \imath_\tau]_s = [\overline{\partial}, \imath_\tau]_s - [\theta, \imath_\tau]_s = 0. \qquad \square$$

Proposition 8.4. — *Let Z be a complex submanifold of X such that there exists a holomorphic vector bundle $\pi : E \to X$ and $\tau \in \Gamma(X, \mathcal{A}_X \otimes \mathcal{E}^\vee)$ such that $\imath_\tau : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}_X \otimes \mathcal{I}_Z$ is surjective. There is a twisted resolution (cf. Definition 7.5) $(M^{l,m}, \delta = \sum_{k\geqslant 0} a_k)$ of $\mathcal{O}_Z$ with $M^{l,m} = \mathcal{A}_X^{0,l} \otimes \bigwedge^m \mathcal{E}$, $\quad a_0 = \imath_\tau$, $\quad a_1 = \overline{D}$, and where the a_k are $\mathcal{A}_X^*$-linear superderivations for $k \geqslant 2$.*

The quasi-isomorphisms are the augmentation map $\epsilon : (T(M)^*, \delta) \to (\mathcal{A}_X^{0,*} \otimes \mathcal{O}_Z, \overline{\partial})$ and the inclusion $\mathcal{O}_Z \to (\mathcal{A}_X^{0,*} \otimes \mathcal{O}_Z, \overline{\partial})$.

Proof. — The fact that $(\imath_\tau)^2 = 0$ and that $\overline{D}$ and $\imath_\tau$ anticommute implies that the twisting cocycle condition (defined in Section 7) is satisfied for $n = 0$ and $n = 1$. We will construct superderivations $a_k, \quad k \geqslant 2$ satisfying the twisting cocycle condition by induction.

$$
\begin{array}{ccccccc}
\mathcal{A}_X & \xrightarrow{\overline{\partial}} & \mathcal{A}_X^{0,1} & \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} & \mathcal{A}_X^{0,n} \\
{\scriptstyle \imath_\tau}\uparrow & & {\scriptstyle \imath_\tau}\uparrow & & {\scriptstyle \imath_\tau}\uparrow \\
\mathcal{A}_X \otimes \mathcal{E} & \xrightarrow{\overline{D}} & \mathcal{A}_X^{0,1} \otimes \mathcal{E} & \xrightarrow{\overline{D}} \cdots \xrightarrow{\overline{D}} & \mathcal{A}_X^{0,n} \otimes \mathcal{E} \\
{\scriptstyle \imath_\tau}\uparrow & & {\scriptstyle \imath_\tau}\uparrow & & {\scriptstyle \imath_\tau}\uparrow \\
\vdots & & \vdots & & \vdots \\
{\scriptstyle \imath_\tau}\uparrow & & {\scriptstyle \imath_\tau}\uparrow & & {\scriptstyle \imath_\tau}\uparrow \\
\mathcal{A}_X \otimes \bigwedge^{r-1} \mathcal{E} & \xrightarrow{\overline{D}} & \mathcal{A}_X^{0,1} \otimes \bigwedge^{r-1} \mathcal{E} & \xrightarrow{\overline{D}} \cdots \xrightarrow{\overline{D}} & \mathcal{A}_X^{0,n} \otimes \bigwedge^{r-1} \mathcal{E} \\
{\scriptstyle \imath_\tau}\uparrow & & {\scriptstyle \imath_\tau}\uparrow & & {\scriptstyle \imath_\tau}\uparrow \\
\mathcal{A}_X \otimes \bigwedge^{r} \mathcal{E} & \xrightarrow{\overline{D}} & \mathcal{A}_X^{0,1} \otimes \bigwedge^{r} \mathcal{E} & \xrightarrow{\overline{D}} \cdots \xrightarrow{\overline{D}} & \mathcal{A}_X^{0,n} \otimes \bigwedge^{r} \mathcal{E}.
\end{array}
$$

We have the diagram

$$
\begin{array}{ccccc}
 & & \mathcal{A}_X \otimes \mathcal{E} & \xrightarrow{\imath_\tau} & \mathcal{A}_X \\
 & & {\scriptstyle -a_1^2}\downarrow & & {\scriptstyle 0}\downarrow \\
\mathcal{A}_X^{0,2} \otimes \bigwedge^2 \mathcal{E} & \xrightarrow{\imath_\tau} & \mathcal{A}_X^{0,2} \otimes \mathcal{E} & \xrightarrow{\imath_\tau} & \mathcal{A}_X^{0,2}.
\end{array}
$$

There exists an $\mathcal{A}_X$-linear map $a_2 : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}_X^{0,2} \otimes \bigwedge^2 \mathcal{E}$ such that $-a_1^2 = \imath_\tau \circ a_2$ since $\mathcal{A}_X^{0,q} \otimes \mathcal{E}$ is projective by Lemma 2.2. We extend a_2 to an odd superderivation of $\mathcal{A}_X^* \widehat{\otimes} \bigwedge \mathcal{E}$ which acts trivially on $\mathcal{A}_X^*$. The twisting cocycle condition for $n = 2$ is satisfied since both $-a_1^2$ and $a_2 \circ \imath_\tau + \imath_\tau \circ a_2$ are superderivations that act trivially on $\mathcal{A}_X^*$ and agree on $\mathcal{A}_X \otimes \mathcal{E}$.

Now assume that a_k is constructed. Thus we have the diagram

$$
\begin{array}{ccccc}
 & & \mathcal{A}_X \otimes \mathcal{E} & \xrightarrow{\imath_\tau} & \mathcal{A}_X \\
 & & {\scriptstyle \mu}\downarrow & & {\scriptstyle 0}\downarrow \\
\mathcal{A}_X^{0,k+1} \otimes \bigwedge^{k+1} \mathcal{E} & \xrightarrow{\imath_\tau} & \mathcal{A}_X^{0,k+1} \otimes \bigwedge^{k} \mathcal{E} & \xrightarrow{\imath_\tau} & \mathcal{A}_X^{0,k+1} \otimes \bigwedge^{k-1} \mathcal{E}
\end{array}
$$

where $\mu = -\sum_{i=1}^{k} a_i a_{k-i+1}$. It is straightforward to check that μ is $\mathcal{A}_X$-linear. Hence there exists a map $a_{k+1} : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}_X^{0,k+1} \otimes \bigwedge^{k+1} \mathcal{E}$ such that $\mu = \imath_\tau \circ a_{k+1}$. The extension of a_{k+1} to an odd superderivation of $\mathcal{A}_X^* \widehat{\otimes} \bigwedge \mathcal{E}$ which acts trivially on $\mathcal{A}_X^*$ satisfies the twisting cocycle condition by applying the argument used in the previous paragraph. Note that $a_k = 0$ for $k \geqslant n+1$, so the induction ends after a finite number of steps.

Let ϵ denote the augmentation map from the twisted complex $(T^* = T(M)^*, \delta)$ to the complex $(\mathcal{A}_X^{0,*} \otimes \mathcal{O}_Z, \overline{\partial})$. If we endow the latter complex with *filtration bête*, then ϵ becomes a map of filtered complexes which is a quasi-isomorphism of the associated graded objects. Hence, (T^*, δ) is quasi-isomorphic to $(\mathscr{A}_X^{0,*} \otimes \mathscr{O}_Z, \overline{\partial})$, which in turn is quasi-isomorphic to $\mathscr{O}_Z$. □

Note that δ is a flat superconnection of type $(0,1)$ on the superbundle $\bigwedge E$.

9. Koszul Factorizations II

In this section, we will construct a map from the twisted complex $T^* = T(M)^*$ of Proposition 8.4 to the Dolbeault complex $\mathscr{A}_X^{r,*}[r]$ by using the generalized supertraces of Section 6. The precise argument is contained in Corollary 9.3. In the next section, we will prove that this map represents the Grothendieck fundamental class of Z in X.

We extend the generalized trace (cf. Definition 6.1) to a map

$$\mathrm{Tr}_\Lambda : \mathscr{A}_X^* \widehat{\otimes}\ \mathscr{E}nd_{\mathscr{O}_X}(\bigwedge E) \to \mathscr{A}_X^* \widehat{\otimes} \bigwedge \mathscr{E}^\vee$$

by the formula

$$\mathrm{Tr}_\Lambda(\omega \otimes \varphi) = \omega\, \mathrm{Tr}_\Lambda(\varphi)$$

for local sections ω and φ of $\mathscr{A}_X^*$ and $\mathscr{E}nd_{\mathscr{O}_X}(\bigwedge E)$ respectively.

Proposition 9.1. — *Let φ be a section of the sheaf of superalgebras $\mathscr{A}_X^* \widehat{\otimes}\ \mathscr{E}nd_{\mathscr{O}_X}(\bigwedge E)$ and δ be the twisting differential of Proposition 8.4. Then*

$$\mathrm{Tr}_\Lambda[\delta, \varphi]_s = [\delta, \mathrm{Tr}_\Lambda(\varphi)]_s$$

Proof. — We observe that, for local sections $\omega_1 \otimes \varphi_1, \omega_2 \otimes \varphi_2$ of $\mathscr{A}_X^* \widehat{\otimes}\ \mathscr{E}nd_{\mathscr{O}_X}(\bigwedge E)$, one has that

$$[\omega_1 \otimes \varphi_1, \omega_2 \otimes \varphi_2]_s = (-1)^{|\varphi_1||\omega_1|} \omega_1 \wedge \omega_2 \otimes [\varphi_1, \varphi_2]_s.$$

This follows from a straightforward computation and the fact that $\mathscr{A}_X^*$ is supercommutative. Therefore

$$\mathrm{Tr}_\Lambda[\omega_1 \otimes \varphi_1, \omega_2 \otimes \varphi_2]_s = (-1)^{|\varphi_1||\omega_1|} \omega_1 \wedge \omega_2 \otimes \mathrm{Tr}_\Lambda[\varphi_1, \varphi_2]_s.$$

Assume that $\tilde{\delta}$ is a section of $\mathscr{A}_X^* \widehat{\otimes}\ \mathscr{E}nd_{\mathscr{O}_X}(\bigwedge E)$ which is a superderivation. Since the supercommutator and Tr_Λ are additive, we may assume (without any loss of generality) that $\tilde{\delta} = \omega_1 \otimes \varphi_1$ and $\varphi = \omega_2 \otimes \varphi_2$. Then

$$\begin{aligned}
\mathrm{Tr}_\Lambda[\tilde{\delta}, \varphi]_s &= (-1)^{|\varphi_1||\omega_1|} \omega_1 \wedge \omega_2 \otimes \mathrm{Tr}_\Lambda[\varphi_1, \varphi_2]_s \\
&= (-1)^{|\varphi_1||\omega_1|} \omega_1 \wedge \omega_2 \otimes [\varphi_1, \mathrm{Tr}_\Lambda(\varphi_2)]_s \quad \text{by Proposition 6.4} \\
&= [\omega_1 \otimes \varphi_1, \omega_2 \otimes \mathrm{Tr}_\Lambda(\varphi_2)]_s \\
&= [\tilde{\delta}, \mathrm{Tr}_\Lambda(\varphi)]_s.
\end{aligned}$$

We further observe that $\delta - \overline{\partial}$ is a section of $\mathscr{A}_X^* \widehat{\otimes}\ \mathscr{E}nd_{\mathscr{O}_X}(\bigwedge E)$ which is a sum of superderivations. As a result we have

$$\mathrm{Tr}_\Lambda[\delta - \overline{\partial}, \varphi]_s = [\delta - \overline{\partial}, \mathrm{Tr}_\Lambda(\varphi)]_s.$$

We finally observe that

$$[\overline{\partial}, \varphi]_s = [\overline{\partial}, \omega_2 \otimes \varphi_2]_s = \overline{\partial}\omega_2 \otimes \varphi_2$$

and

$$[\overline{\partial}, \mathrm{Tr}_\Lambda(\varphi)]_s = [\overline{\partial}, \mathrm{Tr}_\Lambda(\omega_2 \otimes \varphi_2)]_s = [\overline{\partial}, \omega_2 \otimes \mathrm{Tr}_\Lambda(\varphi_2)]_s = \overline{\partial}\omega_2 \otimes \mathrm{Tr}_\Lambda(\varphi_2).$$

Then the assertion follows from the equation

$$\mathrm{Tr}_\Lambda[\overline{\partial}, \varphi]_s = [\overline{\partial}, \mathrm{Tr}_\Lambda(\varphi)]_s. \qquad \square$$

Recall that the differential δ is a flat superconnection of type $(0,1)$ on the super vector bundle $\bigwedge E$. If we let $A = \nabla + \delta$ (recall that ∇ is a flat $(1,0)$-connection on E), then A is a superconnection on $\bigwedge E$. The curvature of A, denoted by R_A, is given by the formula

$$R_A = A^2 = (\nabla + \delta)^2 = \nabla^2 + \nabla \circ \delta + \delta \circ \nabla + \delta^2 = [\nabla, \delta]_s$$

since $\nabla^2 = \delta^2 = 0$.

Corollary 9.2. — *Let $\psi = \frac{1}{r!} R_A^r$. Then*

$$[\delta, \mathrm{Tr}_\Lambda(\psi)]_s = 0.$$

Proof. — This follows from the fact that $[\delta, R_A]_s = 0$ and the proposition. $\square$

Corollary 9.3. — *Let $\psi = \frac{1}{r!} R_A^r$. Then*

$$\overline{\partial} \circ \mathrm{Tr}_\Lambda(\psi) = \mathrm{Tr}_\Lambda(\psi) \circ \delta.$$

In other words, $\mathrm{Tr}_\Lambda(\psi)$ is a cochain map from the twisted complex $(T^* = T(M)^*, \delta)$ of Proposition 8.4 (which is quasi-isomorphic to $\mathcal{O}_Z$) to the Dolbeault complex $(\mathcal{A}_X^{r,*}[r], \overline{\partial})$ (which is quasi-isomorphic to $\Omega_X^r[r]$).

Proof. — We have

$$\begin{aligned}
[\delta, \mathrm{Tr}_\Lambda(\psi)]_s &= \sum_{j \geqslant 0} \sum_{m-n \geqslant j-1} [a_j, \mathrm{Tr}_\Lambda(\psi_{m,n})]_s \\
&= \sum_{j \geqslant 0} \sum_{m-n \geqslant j-1} (a_j \circ \mathrm{Tr}_\Lambda(\psi_{m,n}) - \mathrm{Tr}_\Lambda(\psi_{m,n}) \circ a_j) \\
&\quad \text{since } \mathrm{Tr}_\Lambda(\psi_{m,n}) \text{ are of even degree} \\
&= \overline{\partial} \circ \mathrm{Tr}_\Lambda(\psi) - \sum_{j \geqslant 0} \sum_{m-n \geqslant j-1} \mathrm{Tr}_\Lambda(\psi_{m,n}) \circ a_j \\
&= \overline{\partial} \circ \mathrm{Tr}_\Lambda(\psi) - \mathrm{Tr}_\Lambda(\psi) \circ \delta.
\end{aligned}$$

Then the assertion follows from Corollary 9.2. $\square$

Corollary 9.3 (combined with the Lemmas 10.1 and 10.4) can be seen as a generalization of Proposition 5.3 to the real-analytic case.

10. Comparison with the Grothendieck Class

As a result of Corollary 9.3, $\mathrm{Tr}_\Lambda(\psi)$ gives us an element in $\mathrm{Hom}_{\mathcal{O}_X}(T^*, \mathcal{A}_X^{r,*}[r])$, and therefore a class in $\mathrm{Ext}^r_{\mathcal{O}_X}(T^*, \mathcal{A}_X^{r,*})$. We can identify $\mathrm{Ext}^r_{\mathcal{O}_X}(T^*, \mathcal{A}_X^{r,*})$ with the group $\mathrm{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^r_X)$ since T^* and $\mathcal{A}_X^{r,*}$ are quasi-isomorphic to $\mathcal{O}_Z$ and Ω^r_X respectively. Now we shall prove that the class of $\mathrm{Tr}_\Lambda(\psi)$ in $\mathrm{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^r_X)$, denoted by $[\mathrm{Tr}_\Lambda(\psi)]$, is the Grothendieck fundamental class. But we need two preliminary lemmas first.

Lemma 10.1. — *Let $\psi = \frac{1}{r!}R_A^r$. We write $\psi = \sum \psi_{m,n}$ where $\psi_{m,n} \in \mathcal{A}(X) \otimes \mathrm{Hom}_{\mathcal{A}_X}(\bigwedge^m \mathcal{E}, \bigwedge^n \mathcal{E})$. Then*

$$\psi_{r,0} = \frac{1}{r!}(\imath_{\nabla(\tau)})^r.$$

Proof. — We have $R_A = [\nabla, \delta]_s = \imath_{\nabla(\tau)} + \sum_{k\geqslant 1} \nabla(a_k)$. In this sum, the only summand that lowers the Koszul degree is the term $\imath_{\nabla(\tau)}$. Therefore, the only term that lowers the Koszul degree by r in $\frac{1}{r!}R_A^r$ is $\frac{1}{r!}(\imath_{\nabla(\tau)})^r$. □

Therefore, we have the following equality for the restriction of $\mathrm{Tr}_\Lambda(\psi)$ to $\mathcal{A}_X \otimes \bigwedge^r \mathcal{E}$

$$\mathrm{Tr}_\Lambda(\psi)|_{\mathcal{A}_X \otimes \bigwedge^r \mathcal{E}} = \frac{1}{r!}(\imath_{\nabla(\tau)})^r.$$

Lemma 10.2. — *Let A^*, B^*, and C^* be cochain complexes; and $f : A^* \to B^*$ and $g : B^* \to C^*$ be maps of complexes such that the composition $g \circ f$ is homotopic to the zero map. There exists a map $l(f) : A^* \to \mathrm{Cone}(g)^*[-1]$ such that the following triangle is commutative*

$$\begin{tikzcd} A^* \arrow[r, "l(f)"] \arrow[dr, "f"'] & \mathrm{Cone}(g)^*[-1] \arrow[d, "\mathrm{Pr}"] \\ & B^* \end{tikzcd}$$

where $\mathrm{Pr} : \mathrm{Cone}(g)^*[-1] \to B^*$ is the projection map.

Proof. — Exercise. □

Theorem 10.3. — *$[\mathrm{Tr}_\Lambda(\psi)] \in \mathrm{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^r_X)$ is the Grothendieck fundamental class.*

Proof. — Since $\mathrm{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^r_X) \cong H^0(X, \mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^r_X))$, we need only prove that the classes agree locally.

Let $x \in X$ be a point and U be a neighborhood of x such that the restriction of E to U is trivial, with $\{f_1, \dots, f_r\}$ a local holomorphic framing of E over U and $\{f^1, \dots, f^r\}$ the dual framing for $E^\vee$; we identify E and $E^\vee$ with the trivial bundle via these framings. Assume that Z has holomorphic equations $\{z_1, \dots, z_r\}$ in U, and is hence the zero set of the $\nu = z_1 f^1 + \cdots + z_r f^r$ of $E^\vee$. Then the Koszul complex $K(\nu)^*$ over U is quasi-isomorphic to $\mathcal{O}_Z|_U$ where $K(\nu)^{-i} = \bigwedge^i \mathcal{O}_U^r$ and the differentials are contractions by ν .

We will construct a quasi-isomorphism $\tilde{u} : K(\nu)^* \to T^*|_U$ such that the class of the composition $\mathrm{Tr}_\Lambda(\psi)|_U \circ \tilde{u}$ in

$$\mathcal{E}xt^r_{\mathcal{O}_X}(K(\nu)^*, \mathcal{A}_U^{r,*}[r]) \cong \mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_Z|_U, \Omega^r_U)$$

is the restriction of the Grothendieck fundamental class of Z.

Step 1: We first define a map u from $K(\nu)^*$ to $K(\tau)^*|_U$. By the assumptions on Z and τ, there exist $u_{ji} \in \Gamma(U, \mathcal{A}_X)$ such that $z_i = \sum_j u_{ji}\alpha_j$ where $\tau = \alpha_1 e^1 + \cdots + \alpha_r e^r$. We let $u_0 : \mathcal{O}_U \to \mathcal{A}_U$ be the inclusion and $u_{-1} : \mathcal{O}^r_U \to \mathcal{A}_U \otimes \mathcal{E}|_U$ be the map which sends f_i to $\sum_j u_{ji}e_j$. Then we extend u_{-1} to a map of Koszul complexes by setting $u_{-k} = \bigwedge^k u_{-1}$. Therefore u_{-r} is given by multiplication by $\det(u_{ij})$.

Step 2: Next, we shall extend $u : K(\nu)^* \to K(\tau)^*|_U$ to a map $\tilde{u} : K(\nu)^* \to T^*|_U$. The twisted complex T^* is a filtered complex with respect to Dolbeault degree. Let us denote this (decreasing) filtration by F^i, i.e. $F^i = F^i(T^*) = \bigoplus_{j \geqslant i} \mathcal{A}_X^{0,j} \otimes \bigwedge \mathcal{E}$. Then one has $Gr^i_F = \mathcal{A}_X^{0,i} \otimes K(\tau)^*$.

The map $(-\delta + \imath_\tau) : K(\tau)^* \to F^1[1]$ is a map of complexes, since $(-\delta + \imath_\tau) \circ \imath_\tau = -\delta \circ (-\delta + \imath_\tau)$. Hence

$$(-\delta + \imath_\tau) \circ u : K(\nu)^* \to F^1[1]|_U$$

is a cochain map. Moreover, $(-\delta + \imath_\tau) \circ u$ is homotopic to the zero map since $K(\nu)^*$ is a complex of free $\mathcal{O}_X$-modules; $F^1[1]$ is acyclic in negative degrees; and $((-\delta + \imath_\tau) \circ u) : \mathcal{O}_U \to F^1[1]^0|_U$ is the zero map. (Note that u_0 is the inclusion of $\mathcal{O}_U$ into $\mathcal{A}_U$, and $(-\delta + \imath_\tau)_0 = \bar{\partial}$). Consequently, there exists an extension

$$\tilde{u} = l(u) : K(\nu)^* \to \mathrm{Cone}(-\delta + \imath_\tau)^*[-1]$$

by Lemma 10.2. This is the desired extension since $T^* = \mathrm{Cone}(-\delta + \imath_\tau)^*[-1]$.

Step 3: We now prove that $\tilde{u} : K(\nu)^* \to T^*|_U$ is a quasi-isomorphism. Let $i : \mathcal{O}_Z|_U \to \mathcal{A}_U^{0,*} \otimes \mathcal{O}_Z|_U$ be the inclusion and $p : K(\nu)^* \to \mathcal{O}_Z|_U$ be the augmentation map. Then we have a commutative diagram

$$\begin{tikzcd} K(\nu)^* \arrow[r, "\tilde{u}"] \arrow[d, "p"'] & T^*|_U \arrow[d, "\epsilon"'] \\ \mathcal{O}_Z|_U \arrow[r, "i"] & \mathcal{A}_U^{0,*} \otimes \mathcal{O}_Z|_U \end{tikzcd}$$

in which the vertical arrows and the bottom horizontal arrow are quasi-isomorphisms. As a result, $\tilde{u}$ is a quasi-isomorphism.

Step 4: Let $\eta = \mathrm{Tr}_\Lambda(\psi) \circ \tilde{u}$. Thus the degree $(-r)$ component of η is given by the composition $\frac{1}{r!}(\imath_{\nabla(\tau)})^r \circ \det(u_{ij})$. Hence

$$\begin{aligned}
\eta_{-r}(f_1 \wedge \cdots \wedge f_r) &= \frac{1}{r!}(\imath_{\nabla(\tau)})^r(\det(u_{ij})e_1 \wedge \cdots \wedge e_r) \\
&= \det(u_{ij})\frac{1}{r!}(\imath_{\nabla(\tau)})^r(e_1 \wedge \cdots \wedge e_r) \\
&= \det(u_{ij})\partial\alpha_1 \wedge \cdots \wedge \partial\alpha_r \pmod{\mathscr{I}} \\
&= \partial z_1 \wedge \cdots \wedge \partial z_r \pmod{\mathscr{I}} \\
&= dz_1 \wedge \cdots \wedge dz_r \pmod{\mathscr{I}}.
\end{aligned}$$

By Proposition 5.4 η represents the Grothendieck fundamental class of $Z \cap U$ in U. Since $\tilde{u}$ is a quasi-isomorphism, so is $\mathrm{Tr}_\Lambda(\psi)|_U$. Since $\mathrm{Tr}_\Lambda(\psi)$ represents the Grothendieck fundamental class locally, it does so globally. □

Lemma 10.4. — *Let $\psi = \frac{1}{r!}R_A^r$. We have*

$$\mathrm{Tr}_\Lambda(\psi)|_{\mathscr{O}_X} = \frac{1}{r!}\mathrm{tr}_s(\psi).$$

Proof. — Omitted. □

Corollary 10.5. — *The image of the Grothendieck fundamental class in $H^r(X, \Omega_X^r)$ is represented by the (r, r) degree part of the Chern character form of the superbundle $\bigwedge E$ equipped with the superconnection $A = \nabla + \delta$.*

Proof. — This follows from the theorem, Lemma 10.4, and Theorem 3.3. □

This completes the proof of Theorem B. □

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Astérisque
328, 2009, p. 161–206

TORSION INVARIANTS FOR FAMILIES

by

Sebastian Goette

Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — We give an overview over the higher torsion invariants of Bismut-Lott, Igusa-Klein and Dwyer-Weiss-Williams, including some more or less recent developments.

***Résumé* (Invariants de torsion en familles).** — On expose la théorie des invariants de torsion supérieures de Bismut-Lott, Igusa-Klein et Dwyer-Weiss-Williams, ainsi que ses développements récents.

The classical Franz-Reidemeister torsion τ_{FR} is an invariant of manifolds with acyclic unitarily flat vector bundles [**62**], [**33**]. In contrast to most other algebraic-topological invariants known at that time, it is invariant under homeomorphisms and simple-homotopy equivalences, but not under general homotopy equivalences. In particular, it can distinguish homeomorphism types of homotopy-equivalent lens spaces. Hatcher and Wagoner suggested in [**39**] to extend τ_{FR} to families of manifolds $p\colon E \to B$ using pseudoisotopies and Morse theory. A construction of such a higher Franz-Reidemeister torsion τ was first proposed by John Klein in [**48**] using a variation of Waldhausen's A-theory. Other descriptions of τ were later given by Igusa and Klein in [**45**], [**46**].

In this overview, we will refer to the construction in [**42**]. Let $p\colon E \to B$ be a family of smooth manifolds, and let $F \to E$ be a unitarily flat complex vector bundle of rank r such that the fibrewise cohomology with coefficients in F forms a unipotent bundle over B. Using a function $h\colon E \to \mathbb{R}$ that has only Morse and

2010 Mathematics Subject Classification. — 58J52 (57R22 55R40).
Key words and phrases. — Bismut-Lott torsion, Igusa-Klein torsion, Dwyer-Weiss-Williams torsion, higher analytic torsion, higher Franz-Reidemeister torsion.

Supported in part by DFG special programme "Global Differential Geometry".

birth-death singularities along each fibre of p, and with trivialised fibrewise unstable tangent bundle, one constructs a homotopy class of maps $\xi_h(M/B;F)$ from B to a classifying space $Wh^h(M_r(\mathbb{C}),U(r))$. Now, the higher torsion $\tau(E/B;F)\in H^{4\bullet}(B;\mathbb{R})$ is defined as the pull-back of a certain universal cohomology class $\tau\in H^{4\bullet}\big(Wh^h(M_r(\mathbb{C}),U(r));\mathbb{R}\big)$.

On the other hand, Ray and Singer defined an analytic torsion $\mathcal{T}_{\mathrm{RS}}$ of unitarily flat complex vector bundles on compact manifolds in [**61**] and conjectured that $\mathcal{T}_{\mathrm{RS}}=\tau_{\mathrm{FR}}$. This conjecture was established independently by Cheeger [**26**] and Müller [**59**]. The most general comparison result was given by Bismut and Zhang in [**17**] and [**18**]. In [**64**], Wagoner predicted the existence of a "higher analytic torsion" that detects homotopy classes in the diffeomorphism groups of smooth closed manifolds. Such an invariant was defined later by Bismut and Lott in [**16**].

Kamber and Tondeur constructed characteristic classes $\mathrm{ch}^{\mathrm{o}}(F)\in H^{\mathrm{odd}}(M;\mathbb{R})$ of flat vector bundles $F\to M$ in [**47**] that provide obstructions towards finding a parallel metric. If $p\colon E\to B$ is a smooth bundle of compact manifolds and $F\to E$ is flat, Bismut and Lott proved a Grothendieck-Riemann-Roch theorem relating the characteristic classes of F to those of the fibrewise cohomology $H(E/B;F)\to B$. The higher analytic torsion form $\mathcal{T}(T^HE,g^{TX},g^F)$ appears in a refinement of this theorem to the level of differential forms. Its component in degree 0 equals the Ray-Singer analytic torsion of the fibres, and the refined Grothendieck-Riemann-Roch theorem implies a variation formula for the Ray-Singer torsion that was already discovered in [**17**].

In [**30**], Dwyer, Weiss and Williams gave yet another approach to higher torsion. They defined three generalised Euler characteristics for bundles $p\colon E\to B$ of homotopy finitely dominated spaces, topological manifolds, and smooth manifolds, respectively, with values in certain bundles over B. A flat complex vector bundle $F\to E$ defines a homotopy class of maps from E to the algebraic K-theory space $K(\mathbb{C})$. The Euler characteristics above give analogous maps $B\to K(\mathbb{C})$ for the fibrewise cohomology $H(E/B;F)\to B$. If F is fibrewise acyclic, these maps lift to three different generalisations of Reidemeister torsion, given again as sections in certain bundles over B. By comparing the three characteristics for smooth manifold bundles, Dwyer, Weiss and Williams also showed that the Grothendieck-Riemann-Roch theorem in [**16**] holds already on the level of classifying maps to $K(\mathbb{C})$.

Bismut-Lott torsion $\mathcal{T}(E/B;F)$ and Igusa-Klein torsion $\tau(E/B;F)$ are very closely related. For particularly nice bundles, this was proved by Bismut and the author in [**12**] and [**36**], [**37**]. We will establish the general case in [**38**]. Igusa also gave a set of axioms in [**44**] that characterise $\tau(E/B;F)$ and hopefully also $\mathcal{T}(E/B;F)$ when F is trivial. Badzioch, Dorabiała and Williams recently gave a cohomological version of the smooth Dwyer-Weiss-Williams torsion in [**3**]. Together with Klein, they proved in [**2**] that it satisfies Igusa's axioms as well. On the other hand, the other two torsions in **[30]** are definitely coarser than Bismut-Lott and Igusa-Klein torsion, because they do not depend on the differentiable structure. They might however be related to the Bismut-Lott or Igusa-Klein torsion of a virtual flat vector bundle F of rank zero, see Remark 7.5 below.

Let us now recall one of the most import applications of higher torsion invariants. It is possible to construct two smooth manifold bundles $p_i\colon E_i \to B$ for $i = 0$, 1 with diffeomorphic fibres, such that there exists a homeomorphism $\varphi\colon E_0 \to E_1$ with $p_0 = p_1 \circ \varphi$ and with a lift to an isomorphism of vertical tangent bundles, but no such diffeomorphism. The first example of such bundles p_i was constructed by Hatcher, and it was later proved by Bökstedt that p_0 and p_1 are not diffeomorphic in the sense above **[19]**. Igusa showed in **[42]** that the higher torsion invariants $\tau(E_i/B;\mathbb{C})$ differ, and by **[36]**, the Bismut-Lott torsions $\mathcal{T}(E_i/B;\mathbb{C})$ differ as well. Hatcher's example can be generalised to construct many different smooth structures on bundles $p\colon E \to B$. We expect that higher torsion invariants distinguish many of these different structures, but not all of them.

One may wonder why one wants to consider so many different higher torsion invariants, in particular, if some of them are conjectured to provide the same information. We will see that different constructions of these invariants give rise to different applications. Since Hatcher's example and its generalisations come with natural fibrewise Morse functions, the difference of the Igusa-Klein torsions of different smooth structures is sometimes easy to compute. Due to Igusa's axiomatic approach, one can also understand the topological meaning of Igusa-Klein torsion. On the other hand, one can classify smooth structures on a topological manifold bundle $p\colon E \to B$ in a more abstract way as classes of sections in a certain bundle of classifying spaces over B. These section spaces fit well into the framework of generalised Euler characteristics and Dwyer-Weiss-Williams torsion. But some extra work is necessary to recover cohomological information from this approach.

Finally, Bismut-Lott torsion is defined using the language of local index theory. The proofs of some interesting properties of Bismut-Lott torsion were inspired by parallel results in the setting of the classical Atiyah-Singer family index theorem or the Grothendieck-Riemann-Roch theorem in Arakelov geometry. Bismut-Lott torsion is defined for any flat vector bundle $F \to E$, whereas Igusa-Klein torsion and Dwyer-Weiss-Williams torsion can only be defined if the fibrewise cohomology is of a special type. This makes Bismut-Lott torsion useful for other applications, for example in the definition of a secondary K-theory by Lott **[52]**. Heitsch and Lazarov generalised Bismut-Lott torsion to foliations **[40]**, so one may try to use it to detect different smooth structures on a given foliation, which induce the same structures on the space of leaves. Finally, Bismut and Lebeau recently defined higher torsion invariants using a hypoelliptic Laplacian on the cotangent bundle [**8**], [**15**]. Conjecturally, this torsion can give some information about the fibrewise geodesic flow.

This overview is organised as follows. We start by discussing the index theorem for flat vector bundles by Bismut and Lott in Section 1. In Sections 2 and 3, we introduce Bismut-Lott torsion and state some properties and applications that are inspired by local index theory. In Section 4 and 5, we introduce Igusa-Klein torsion and relate it to Bismut-Lott torsion using two different approaches. Section 6 is devoted to generalised Euler characteristics and Dwyer-Weiss-Williams torsion. In Section 7, we discuss smooth structures on fibre bundles and a possible generalisation to foliations. Finally, we sketch the hypoelliptic operator on the cotangent bundle and its torsion due to Bismut and Lebeau in Section 8.

We have tried to keep the notation and the normalisation of the invariants consistent throughout this paper; as a result, both will disagree with most of the references. In particular, we use the Chern normalisation of [**12**], which is the only normalisation for which Theorem 3.7 and a few other results hold. To keep this paper reasonably short, only the most basic versions of some of the theorems on higher torsion will be explained. Thus we will not discuss some non-trivial generalisations of the theorems below to fibre bundles with group actions. We will also only give hints towards the relation with the classical Atiyah-Singer family index theorem or the Grothendieck-Riemann-Roch theorem in algebraic geometry. Finally, we will not discuss the interesting refinements and generalisations of classical Franz-Reidemeister torsion and Ray-Singer torsion for single manifolds that have been invented in the last few years.

Acknowledgements. — This paper is a somewhat extended version of a series of lectures at the Chern Institute at Tianjin in 2007, whose support and hospitality we highly appreciated. The author was supported in part by the DFG special programme "Global Differential Geometry".

We are grateful to J.-M. Bismut for introducing us to higher torsion, and also to U. Bunke, W. Dorabiała, K. Igusa, K. Köhler, X. Ma, B. Williams and W. Zhang, from whom we learned many different aspects of this intriguing subject. We also thank the anonymous referee for her or his helpful comments.

1. An Index Theorem for Flat Vector Bundles

There exists a theory of characteristic classes of flat vector bundles that is parallel to the theory of Chern classes and Chern-Weil differential forms. These classes have been constructed by Kamber and Tondeur [**47**], and are closely related to the classes used by Borel [**20**] to study the algebraic K-theory of number fields.

Analytic torsion forms made their first appearance in a local index theorem for these Kamber-Tondeur classes by Bismut and Lott [**16**]. Refinements of this theorem have later been given by Dwyer, Weiss and Williams [**30**] and by Bismut [**7**] and Ma and Zhang [**56**].

1.1. Characteristic classes for flat vector bundles. — Before we introduce Kamber-Tondeur forms, let us first recall classical Chern-Weil theory. Let $V \to M$ be a complex vector bundle, and let ∇^V be a connection on V with curvature $(\nabla^V)^2 \in \Omega^2(M; \operatorname{End} V)$. Then one defines the Chern character form

$$\operatorname{ch}\big(V, \nabla^V\big) = \operatorname{tr}_V\Big(e^{-\frac{(\nabla^V)^2}{2\pi i}}\Big) \in \Omega^{\text{even}}(M; \mathbb{C}) \,. \tag{1.1}$$

This form is closed because the covariant derivative $[\nabla^V, (\nabla^V)^2]$ of the curvature vanishes by the Bianchi identity, so

$$d \operatorname{ch}\big(V, \nabla^V\big) = \operatorname{tr}_V\Big(\Big[\nabla^V, e^{-\frac{(\nabla^V)^2}{2\pi i}}\Big]\Big) = 0 \,. \tag{1.2}$$

If $\nabla^{V,0}$ and $\nabla^{V,1}$ are two connections on V, one can choose a connection $\nabla^{\tilde V}$ on the natural extension $\tilde V$ of V to $M \times [0,1]$ with $\nabla^{\tilde V}|_{M\times\{i\}} = \nabla^{V,i}$ for $i = 0, 1$. Stokes' theorem then implies

$$\begin{aligned} &\operatorname{ch}\big(V, \nabla^{V,1}\big) - \operatorname{ch}\big(V, \nabla^{V,0}\big) = d\, \widetilde{\operatorname{ch}}\big(V, \nabla^{V,0}, \nabla^{V,1}\big), \\ \text{with} \qquad &\widetilde{\operatorname{ch}}\big(V, \nabla^{V,0}, \nabla^{V,1}\big) = \int_0^1 \iota_{\frac{\partial}{\partial t}} \operatorname{ch}\Big(\tilde V, \nabla^{\tilde V}\Big)\, dt. \end{aligned} \tag{1.3}$$

Thus, the class $\operatorname{ch}(V)$ of $\operatorname{ch}(V, \nabla^V)$ in de Rham cohomology is independent of ∇^V. Moreover, $\widetilde{\operatorname{ch}}(V, \nabla^{V,0}, \nabla^{V,1})$ is independent of the choice of $\nabla^{\tilde V}$ up to an exact form.

Now let $F \to M$ be a flat vector bundle, so F comes with a fixed connection ∇^F such that $(\nabla^F)^2 = 0$. We choose a metric g^F on F and define the adjoint connection $\nabla^{F,*}$ with respect to g^F such that

$$dg(v, w) = g\big(\nabla^F v, w\big) + g\big(v, \nabla^{F,*} w\big) \tag{1.4}$$

for all sections v, w of F. Then the form

$$\operatorname{ch}^{\circ}\big(F, g^F\big) = \pi i\, \widetilde{\operatorname{ch}}\big(F, \nabla^F, \nabla^{F,*}\big) \in \Omega^{\text{odd}}(M; \mathbb{R}) \tag{1.5}$$

is real, odd and also closed, because

$$d \operatorname{ch}^{\circ}\big(F, g^F\big) = \pi i \operatorname{ch}\big(F, \nabla^{F,*}\big) - \pi i \operatorname{ch}\big(F, \nabla^F\big) = 0 \,. \tag{1.6}$$

Clearly, if g^F is parallel with respect to ∇^F, then $\operatorname{ch}^{\circ}(F, g^F) = 0$.

Let $g^{F,0}$, $g^{F,1}$ be two metrics on F. Proceeding as in (1.3), one constructs a form $\widetilde{\operatorname{ch}}^{\circ}(F, g^{F,0}, g^{F,1}) \in \Omega^{\text{even}}(M)$ such that

$$\operatorname{ch}^{\circ}\big(F, g^{F,1}\big) - \operatorname{ch}^{\circ}\big(F, g^{F,0}\big) = d\, \widetilde{\operatorname{ch}}^{\circ}\big(F, g^{F,0}, g^{F,1}\big) \,. \tag{1.7}$$

So again, the de Rham cohomology class $\mathrm{ch}^{\mathrm{o}}(F)$ of $\mathrm{ch}^{\mathrm{o}}(F,g^F)$ does not depend on the choice of metric g^F — but of course, it depends on the flat connection ∇^F. Note that the form $\widetilde{\mathrm{ch}}^{\mathrm{o}}(F,g^{F,0},g^{F,1})$ is again naturally well-defined up to an exact form.

1.1. Definition. — The forms $\mathrm{ch}^{\mathrm{o}}_k(F,g^F) = \mathrm{ch}^{\mathrm{o}}(F,g^F) \in \Omega^{2k-1}(M)$ are called *Kamber-Tondeur forms*, and their classes $\mathrm{ch}^{\mathrm{o}}_k(F) \in H^{2k-1}(M;\mathbb{R})$ are called *Kamber-Tondeur classes* or *Borel classes.*

Note that in the literature, there are at least three different normalisations of these classes. There are however good reasons to stick to the normalisation here, see Section 3.4.

For later reference, we give a more explicit construction of the Kamber-Tondeur forms. If we define a connection $\nabla^{\tilde F}$ over $p\colon M\times[0,1]\to M$ that interpolates between ∇^F and $\nabla^{F,*}$ by

$$\nabla^{\tilde F} = (1-t)\,p^*\nabla^F + tp^*\nabla^{F,*}\ , \tag{1.8}$$

then by flatness of ∇^F and $\nabla^{F,*}$,

$$\big(\nabla^{\tilde F}\big)^2 = -t(1-t)\,p^*\big(\nabla^{F,*}-\nabla^F\big)^2 - p^*\big(\nabla^{F,*}-\nabla^F\big)\,dt\ . \tag{1.9}$$

From this formula and (1.3), (1.5) one deduces that there exist rational multiples c_k of $(2\pi i)^k$ such that

$$\begin{aligned} \mathrm{ch}^{\mathrm{o}}\big(F,g^F\big) &= \sum_{k=0}^{\infty} c_k\,\mathrm{tr}_F\big(\omega(F,g^F)^{2k+1}\big)\ ,\\ \text{with}\qquad \omega\big(F,g^F\big) &= \nabla^{F,*}-\nabla^F = (g^F)^{-1}\,[\nabla^F,g^F] \in \Omega^1(M;\mathrm{End}\,V)\ . \end{aligned} \tag{1.10}$$

Bismut and Lott use the real, odd and closed differential forms

$$\mathrm{tr}_F\Big(\omega\big(F,g^F\big)e^{\frac{\omega(F,g^F)^2}{2\pi i}}\Big) \tag{1.11}$$

and their cohomology classes instead of ch^{o}, which is more convenient for some of the following constructions. It is not hard to see that these forms are given by a similar formula as (1.10), but with different constants $c_k \in (2\pi i)^k\mathbb{Q}$. We prefer the Chern normalisation given by (1.5) for reasons explained in Remark 3.8.

The Chern-Weil classes like $\mathrm{ch}(V)$ vanish whenever V admits a flat connection. Similarly, the classes $\mathrm{ch}^{\mathrm{o}}(F)$ vanish whenever F admits a ∇^F-parallel metric. We will see that there are more analogies between these constructions. A good overview can be found in the introduction to [**52**].

1.2. The cohomological index theorem. — The central theme in [**16**] is a family index theorem for flat vector bundles in terms of their Kamber-Tondeur classes. The analytic index in question is given by fibrewise cohomology. More precisely, let $p\colon E \to B$ be a smooth proper submersion, in other words, a smooth fibre bundle with n-dimensional compact fibres, to be denoted M. Let (F, ∇^F) be a flat vector bundle then we consider the vector bundles $H^k(E/B; F) \to B$, whose fibres over $x \in B$ are given as the twisted de Rham cohomology

$$H^k(E/B; F)_x = H^k\big(\Omega^\bullet(E_x; F|_{E_x}), \nabla^F\big) . \tag{1.12}$$

The bundles $H^k(E/B; F)$ naturally carry the Gauß-Manin connection ∇^H, which is again flat. The analytic index is thus given by the virtual flat vector bundle

$$H(E/B; F) = \bigoplus_{k=0}^{\dim M} (-1)^k H^k(E/B; F) . \tag{1.13}$$

The topological index is given by the Becker-Gottlieb transfer of [**4**]. Recall that the Becker-Gottlieb transfer is given as a stable homotopy class of maps $tr_{E/B}\colon S^\bullet B_+ \to S^\bullet E_+$. It acts on de Rham cohomology by

$$tr_{E/B}^*\, \alpha = \int_{E/B} e(TM)\, \alpha \in H^k(B; \mathbb{R}) \tag{1.14}$$

for all $\alpha \in H^k(E; \mathbb{R})$, where $e(TM) \in H^n(E; o(TM) \otimes \mathbb{R})$ denotes the Chern-Weil theoretic Euler class of the vertical tangent bundle $TM = \ker dp \subset TE$, and $\int_{E/B}$ denotes integration over the fibre. Here is a cohomological version of the family index theorem.

***1.2. Theorem* (Bismut and Lott [16]).** — *For all smooth proper submersions $p\colon E \to B$ and all flat vector bundles $F \to E$,*

$$\mathrm{ch}^{\mathrm{o}}(H(E/B; F)) = tr_{E/B}^*\, \mathrm{ch}^{\mathrm{o}}(F) \in H^{\mathrm{odd}}(B, \mathbb{R}) . \tag{1.15}$$

One notes that $tr_{E/B}^*$ preserves the degree of differential forms and cohomology classes. For this reason, an analogous result holds for the classes constructed in (1.11), and in fact for all classes of the form (1.10), independent of the choice of the constants c_k.

The cohomological index theorem can be refined as follows, see also Section 3.5. Following [**27**], to a vector bundle $V \to M$ with connection ∇^V, one associates a Cheeger-Simons differential character $\widehat{\mathrm{ch}}(V, \nabla^V)$, from which both the rational Chern character $\mathrm{ch}(V) \in H^{\mathrm{even}}(M; \mathbb{Q})$ and the Chern-Weil form $\mathrm{ch}(V, \nabla^V) \in \Omega^{\mathrm{even}}(M)$ can be read off. If ∇^V is a flat connection, then $\widehat{\mathrm{ch}}(V, \nabla^V)$ becomes a cohomology class in $H^{\mathrm{odd}}(M; \mathbb{C}/\mathbb{Q})$. It has already been observed in [**16**] that its imaginary part is given by

$$\operatorname{Im}\widehat{\mathrm{ch}}\big(V,\nabla^V\big)=\mathrm{ch}^{\mathrm{o}}(V)\in H^{\mathrm{odd}}(M;\mathbb{R})\,. \tag{1.16}$$

***1.3. Theorem* (Bismut [7], Ma and Zhang [56]).** — *For all smooth proper submersions and all flat vector bundles $F\to E$,*

$$\widehat{\mathrm{ch}}\big(H(E/B;F),\nabla^H\big)=tr^*_{E/B}\,\widehat{\mathrm{ch}}\big(F,\nabla^F\big)\in H^{\mathrm{odd}}(B;\mathbb{C}/\mathbb{Q})\,. \tag{1.17}$$

It is natural to ask if the same theorem holds on the level of flat vector bundles on B. A flat vector bundle $F\to E$, or more generally, a bundle of finitely generated projective R-modules for some ring R, is classified by a map from E to the classifying space $BGL(R)\times K_0(R)$. Following Quillen, there is a natural map from $BGL(R)$ to the algebraic K-theory space $K(R)$. Thus, we may associate to F the corresponding homotopy class $[F]$ of maps from E to $K(R)$, which is slightly coarser than the class of F in the K-theory of finitely generated projective R-module bundles on E.

***1.4. Theorem* (Dwyer, Weiss and Williams [30]).** — *If $p\colon E\to B$ is a bundle of smooth closed manifolds, then*

$$[H(E/B;F)]=tr^*_{E/B}\,[F] \tag{1.18}$$

in the homotopy classes of maps $B\to K(R)$.

Although both sides of (1.18) exist in a much more general situation, the smooth bundle structure is needed in the proof of the theorem, see Section 6.1 below, in particular Theorem 6.3. Theorem 1.2 can be deduced from Theorem 1.4 because the class ch^{o} can already be defined on $K(R)$.

1.3. A refined index theorem. — There is another possible refinement of Theorem 1.2, where one replaces de Rham cohomology classes by differential forms. For this, one first chooses metrics g^{TM} and g^F on the bundles $TM\to E$ and $F\to E$, and a horizontal complement T^HE of $TM\subset TE$. These data give rise to a natural connection ∇^{TM} on TM by [**6**]. Thus, one can consider the Chern-Weil theoretic Euler form $e(TM,\nabla^{TM})$.

We also have a natural decomposition

$$\Omega^\bullet(E;F)=\Omega^\bullet(B;\Omega^\bullet(E/B;F)) \tag{1.19}$$

using $TE=T^HE\oplus TM$, and an L^2-metric on the infinite dimensional bundle $\Omega^\bullet(E/B;F)\to B$ of vertical forms twisted by F. Regarding $H^\bullet(E/B;F)$ as the subbundle of fibrewise harmonic forms, we get a metric $g^H_{L^2}$ on $H^\bullet(E/B;F)$. Bismut and Lott now construct a form $\mathscr{T}(T^HE,g^{TM},g^F)$ on B that depends natural on the data, the *analytic torsion form,* see Section 2.2 below.

***1.5. Theorem* (Bismut and Lott [16]).** — *In the situation above,*

$$d\,\mathcal{T}\big(T^HE, g^{TM}, g^F\big) = \int_{E/B} e\big(TM, \nabla^{TM}\big)\,\mathrm{ch}^\circ\big(F, g^F\big) - \mathrm{ch}^\circ\big(H, g^H_{L^2}\big)\,. \tag{1.20}$$

In the theory of flat vector bundles, this result plays the same role as the η-forms in the heat kernel proof of the classical family index theorem [**6**], [**5**], see also [**9**] and [**28**]. The holomorphic torsion forms similarly arise in a double transgression formula [**14**] in the Riemann-Roch-Grothendieck theorem for proper holomorphic submersions in Kähler geometry. This analogy with η-forms and holomorphic torsion forms has inspired most of the constructions and results of the following two sections.

2. Construction of the Bismut-Lott torsion

In this section, we recall the construction of the torsion forms occurring in Theorem 1.5. As in [**16**], we start with a finite-dimensional toy model that will be of independent interest. We then present the original construction of $\mathcal{T}(T^HM, g^{TM}, g^F)$ by Bismut and Lott, and also a construction using η-forms by Ma and Zhang.

2.1. A finite-dimensional model. — Consider flat vector bundles $V^k \to M$ and parallel vector bundle homomorphisms $a_k\colon V^k \to V^{k+1}$, such that

$$0 \longrightarrow V^0 \xrightarrow{\ a_0\ } V^1 \xrightarrow{\ a_1\ } \cdots \xrightarrow{\ a_{n-1}\ } V^n \longrightarrow 0 \tag{2.1}$$

forms a cochain complex over each point in M. Then

$$A' = \nabla^V + a \tag{2.2}$$

is a superconnection, which is flat because

$$(A')^2 = a^2 + \big[\nabla^V, a\big] + \big(\nabla^V\big)^2\,, \tag{2.3}$$

and each term on the right hand side vanishes by assumption. We will call the pair $(V, \nabla^V + a)$ a *parallel family of* (finite-dimensional) *cochain complexes.*

If we fix a metric g^{V^k} on each V^k, we can consider the adjoint connection $\nabla^{V,*}$ as in (1.4), and let $a_k^*\colon V^{k+1} \to V^k$ be the adjoint of a_k with respect to g^{V^k} and $g^{V^{k+1}}$. Then we obtain another flat superconnection

$$A'' = \nabla^{V,*} + a^*. \tag{2.4}$$

As in Hodge theory, the fibrewise cohomology of (V, a) is represented by $H = \ker(a + a^*) \subset V$. Projection of ∇^V onto H defines a connection ∇^H on H. One checks that ∇^H is independent of g^V, and in fact, ∇^H is the natural Gauß-Manin connection. Let g^H_V denote the restriction of g^V to H.

Bismut and Lott then define a differential form $T(\nabla^V + a, g^V) \in \Omega^{\mathrm{even}}(M)$ and obtain a finite-dimensional analogue of Theorem 1.5.

***2.1. Theorem* (Bismut and Lott, [16]).** — *In the situation above,*

$$dT\big(\nabla^V + a, g^V\big) = \mathrm{ch}^{\mathrm{o}}\big(V, g^V\big) - \mathrm{ch}^{\mathrm{o}}\big(H, g^H_V\big)\ . \tag{2.5}$$

The core of the proof is the construction of $T(\nabla^V + a, g^V)$ that we now describe. On the pullback $\tilde V$ of V to $\tilde M = M \times (0,\infty)$, we introduce two flat superconnections

$$\begin{aligned}\tilde A' &= \nabla^V + \sqrt{t}a - \frac{N^V}{2t}\,dt\ ,\\ \tilde A'' &= \nabla^{V,*} + \sqrt{t}a^* + \frac{N^V}{2t}\,dt\ ,\end{aligned} \tag{2.6}$$

where $N^V \in \operatorname{End} V$ acts on V^k as multiplication by k. The difference of the two superconnections above is an endomorphism

$$\tilde X = \tilde A'' - \tilde A' = \omega\big(V, g^V\big) + \sqrt{t}(a^* - a) + \frac{N^V}{t}\,dt \in \Omega^\bullet\big(\tilde M, \operatorname{End}\tilde V\big)\ . \tag{2.7}$$

We also define the supertrace by

$$\mathrm{str}_V = \mathrm{tr}_V \circ (-1)^{N^V} : \Omega^\bullet(\,\cdot\,, \operatorname{End} V) \to \Omega^\bullet(\,\cdot\,)\ . \tag{2.8}$$

For convenience, we stick to the conventions of [**16**]. In analogy with (1.11), the form

$$(2\pi i)^{\frac{1-N^M}{2}}\,\mathrm{str}\,V\Big(\tilde X e^{\tilde X^2}\Big) \in \Omega^{\mathrm{odd}}(\tilde M) \tag{2.9}$$

is real, odd and closed. By (2.7), we have

$$\lim_{t\to 0} \mathrm{str}\,V\left(\tilde X e^{\tilde X^2}\right)\Big|_{M\times\{t\}} = \mathrm{str}_V\left(\omega\big(V, g^V\big) e^{\omega(V,g^V)^2}\right)\ . \tag{2.10}$$

To understand the limit for $t \to \infty$, note that $a^* - a$ is a skew-adjoint operator. In particular, the "finite dimensional Laplacian" $-(a^* - a)^2$ has nonnegative eigenvalues, and its kernel is given by the "harmonic elements" H. In particular, the "heat operator" $e^{t(a^*-a)^2}$ converges to the orthogonal projection onto H as t tends to infinity. More generally, it is proved in [**16**] that

$$\lim_{t\to\infty} \mathrm{str}_V\left(\tilde X e^{\tilde X^2}\right)\Big|_{M\times\{t\}} = \mathrm{str}_H\left(\omega\big(H, g^H_V\big) e^{\omega(H,g^H_V)^2}\right). \tag{2.11}$$

Because the form in (2.9) is closed, the forms in (2.10) and (2.11) belong to the same cohomology class. Thus we have already proved a finite-dimensional version of Theorem1.2. To define the torsion form, we have to integrate the form in (2.9) over $(0,\infty)$. We note that

$$\iota_{\frac{\partial}{\partial t}}\,\mathrm{str}_V\Big(\tilde X e^{\tilde X^2}\Big)\Big|_{M\times\{t\}} = \mathrm{str}_V\left(\frac{N^V}{t}(1 + 2\tilde X^2)\,e^{\tilde X^2}\right)\Big|_{M\times\{t\}}. \tag{2.12}$$

Unfortunately, the integral over (2.12) diverges both for $t \to 0$ and for $t \to \infty$. However, the divergence can be compensated easily. For any $\mathbb{Z}$-graded vector bundle V, we define

$$\chi(V)=\sum_k(-1)^k\operatorname{rk}V^k\qquad\text{and}\qquad\chi'(V)=\sum_k(-1)^k k\operatorname{rk}V^k. \tag{2.13}$$

Then it is proved in **[16]** that the integral

$$\begin{aligned}\int_0^\infty\Big((2\pi i)^{-\frac{N^M}{2}}\operatorname{str}_V\Big(N^V(1+2\tilde X^2)e^{\tilde X^2}\Big)-\chi'(H)&\\ -(\chi'(V)-\chi'(H))(1-2t)e^{-t}\Big)\frac{dt}{t}&\in\Omega^{\mathrm{even}}(M)\end{aligned} \tag{2.14}$$

converges and gives a torsion form for the characteristic classes considered in (1.11). Adjusting the coefficients c_k in (1.10), we obtain the form $T(\nabla^V+a,g^V)$ needed for Theorem 2.1.

2.2. Definition. — The *Bismut-Lott torsion* of the parallel family of cochain complexes (V,∇^V+a) is defined as

$$\begin{aligned}T\big(\nabla^V+a,g^V\big)=-\int_0^1\Big(\frac{s(1-s)}{2\pi i}\Big)^{\frac{N^M}{2}}\int_0^\infty\Big(\operatorname{str}_V\Big(N^V(1+2\tilde X^2)e^{\tilde X^2}\Big)&\\ -\chi'^H-(\chi'(V)-\chi'(H))(1-2t^2)e^{-t^2}\Big)\frac{dt}{2t}\,ds\quad&\in\Omega^{\mathrm{even}}(M).\end{aligned} \tag{2.15}$$

Proof of Theorem 2.1. — Let N^M act on $\Omega^k(M)$ as multiplication by k. Because the form (2.9) is closed, it follows from (2.10) and (2.11) that

$$\begin{aligned}dT\big(\nabla^V+a,g^V\big)&=\frac12\int_0^1\Big(\frac{s(1-s)}{2\pi i}\Big)^{\frac{N^M-1}{2}}\Big(\lim_{t\to0}\operatorname{str}_V\Big(\tilde Xe^{\tilde X^2}\Big)\Big|_{M\times\{t\}}\\ &\qquad\qquad-\lim_{t\to\infty}\operatorname{str}_V\Big(\tilde Xe^{\tilde X^2}\Big)\Big|_{M\times\{t\}}\Big)\\ &=\operatorname{ch}^{\mathrm o}\big(V,g^V\big)-\operatorname{ch}^{\mathrm o}\big(H,g^H_V\big).\quad\square\end{aligned} \tag{2.16}$$

2.3. Remark. — The correction terms in Definition 2.2 are constant and only affect the Bismut-Lott torsion in degree 0. They are chosen such that

$$\begin{aligned}T\big(\nabla^V+a,g^V\big)^{[0]}_x&=\frac12\sum_k(-1)^k k\log\det\big(-(a^*-a)^2\big|_{V^k\cap H^{k\perp}}\big)\\ &=\frac12\sum_k(-1)^k\log\det\big(aa^*\big|_{V^k\cap\operatorname{im}a}\big)\,.\end{aligned} \tag{2.17}$$

But this is just one way to represent the Franz-Reidemeister torsion of the cochain complex (V_x,a) with metric g^V for $x\in M$. Hence $T(\nabla^V+a,g^V)$ is called a “higher torsion form”.

2.2. The Bismut-Lott torsion form. — As in [**16**], Section 3, we now translate the construction of $T(\nabla^V + a, g^V)$ to the infinite-dimensional family of fibrewise de Rham complexes.

Let $p\colon E \to B$ be a smooth proper submersion with typical fibre M, and let $TM = \ker dp \subset TE$. As in Section 1.3, we fix $T^HE \subset TE$ such that $TE = TM \oplus T^HE$. Because $T^HE \cong p^*TB$, we can identify vector fields on B with their pullback to E, which we call basic vector fields.

Let $F \to E$ be a flat vector bundle, then we may regard the flat connection ∇^F as a differential on the total complex $\Omega^\bullet(E;F)$. Using the splitting (1.19), we may also regard ∇^F as a superconnection on the infinite-dimensional bundle $\Omega^\bullet(E/B;F) \to B$ with

$$\mathbb{A}' = \nabla^F = d^M + \nabla^{\Omega^\bullet(E/B;F)} + \iota_\Omega \tag{2.18}$$

by [**5**]. Here, d^M denotes the fibrewise differential on $\Omega^\bullet(E/B;F)$, $\nabla^{\Omega^\bullet(E/B;F)}$ is the connection induced by the Lie derivative by basic vector fields, and Ω is the vertical component of the Lie bracket of two basic vector fields on E.

In analogy with (2.4), we also define an adjoint superconnection

$$\mathbb{A}'' = d^{M,*} + \nabla^{\Omega^\bullet(E/B;F),*} + \varepsilon_\Omega \tag{2.19}$$

with respect to the fibrewise L^2-metric g_{L^2} on $\Omega^\bullet(E/B;F)$. Let $\tilde B = B \times (0,\infty)$, $\tilde E = E \times (0,\infty)$ and $\tilde F = F \times (0,\infty)$, and let t be the coordinate of $(0,\infty)$. Then we define superconnections

$$\begin{aligned} \tilde{\mathbb{A}}' &= \sqrt{t}\, d^M + \nabla^{\Omega^\bullet(\tilde E/\tilde B;\tilde F)} + \frac{1}{\sqrt{t}}\,\iota_\Omega - \frac{N^{\tilde E/\tilde B}}{2t}\,dt\ , \\ \tilde{\mathbb{A}}'' &= \sqrt{t}\, d^{M,*} + \nabla^{\Omega^\bullet(\tilde E/\tilde B;\tilde F),*} + \frac{1}{\sqrt{t}}\,\varepsilon_\Omega + \frac{N^{\tilde E/\tilde B}}{2t}\,dt\ , \end{aligned} \tag{2.20}$$

where now $N^{\tilde E/\tilde B}$ acts on $\Omega^k(\tilde E/\tilde B;\tilde F)$ as multiplication by k. Then

$$\begin{aligned} \tilde{\mathbb{X}} = \tilde{\mathbb{A}}'' - \tilde{\mathbb{A}}' = \sqrt{t}\big(d^{M,*} - d^M\big) &+ \omega\big(\Omega^\bullet(\tilde E/\tilde B;\tilde F), g_{L^2}\big) \\ &+ \frac{1}{\sqrt{t}}(\varepsilon_\Omega - \iota_\Omega) + \frac{N^{\tilde E/\tilde B}}{t}\,dt \quad \in \Omega^\bullet\big(\tilde B; \operatorname{End}\Omega^\bullet(\tilde E/\tilde B;\tilde F)\big). \end{aligned} \tag{2.21}$$

Note that $d^{M,*} - d^M$ is a skew-adjoint fibrewise elliptic differential operator, whereas the other terms on the right hand side involve no differentiation at all. The operator $-\tilde{\mathbb{X}}^2$ can be regarded as a generalised Laplacian along the fibres of p. If the metric g^F is parallel along the fibres, then $-\tilde{\mathbb{X}}^2$ is precisely the curvature of the Bismut superconnection, which already appeared in the heat equation proof of the Atiyah-Singer families index theorem [**6**]. In particular, the fibrewise odd heat operator $\tilde{\mathbb{X}}e^{\tilde{\mathbb{X}}^2}$

is well-defined and of trace class. Using Getzler rescaling, one proves

$$\lim_{t\to 0} \operatorname{str}_{\Omega^\bullet(\tilde E/\tilde B;\tilde F)}\left(\tilde{\mathbb X} e^{\tilde{\mathbb X}^2}\right)\Big|_{B\times\{t\}} = \int_{E/B} e\big(TM,\nabla^{TM}\big)\ \operatorname{str}\left(\omega\big(F,g^F\big)e^{\omega(F,g^F)^2}\right) \tag{2.22}$$

in analogy with (2.10). Similarly, if we identify $H = H^\bullet(E/B;F)$ with the fibre-wise harmonic differential forms, equipped with the restriction $g^H_{L^2}$ of the L^2-metric on $\Omega^\bullet(E/B;F)$, then

$$\lim_{t\to\infty} \operatorname{str}_{\Omega^\bullet(\tilde E/\tilde B;\tilde F)}\left(\tilde{\mathbb X} e^{\tilde{\mathbb X}^2}\right)\big|_{B\times\{t\}} = \operatorname{str}_H\left(\omega\big(H,g^H_{L^2}\big)e^{\omega(H,g^H_{L^2})^2}\right) \tag{2.23}$$

as in (2.11). To obtain the torsion form, we have to take care of some divergent terms and of the coefficients in (1.10) as before.

2.4. Definition. — The Bismut-Lott torsion is defined as

$$\begin{aligned}\mathcal T\big(T^HE,g^{TM},g^F\big) &= -\int_0^1\left(\frac{s(1-s)}{2\pi i}\right)^{\frac{N^B}{2}}\int_0^\infty\Bigg(\operatorname{str}_{\Omega^\bullet(\tilde E/\tilde B;\tilde F)}\left(N^{\tilde E/\tilde B}\big(1+2\tilde{\mathbb X}^2\big)e^{\tilde{\mathbb X}^2}\right)-\chi'(H)\\ &\qquad -\Big(\frac{\chi(M)\dim M\operatorname{rk}F}{2}-\chi'(H)\Big)(1-2t)e^{-t}\Bigg)\frac{dt}{2t}\,ds \quad\in\Omega^{\mathrm{even}}(B)\,.\end{aligned} \tag{2.24}$$

Proof of Theorem 1.5. — As in the proof of Theorem 2.1, this follows from (2.22) and (2.23), because the form

$$\operatorname{str}_{\Omega^\bullet(\tilde E/\tilde B;\tilde F)}\left(\tilde{\mathbb X} e^{\tilde{\mathbb X}^2}\right)\in\Omega^\bullet(B\times(0,\infty);\mathbb C) \tag{2.25}$$

is closed. □

2.5. Remark. — Again, the correction terms in (2.24) are constant scalars. They are chosen such that

$$\mathcal T\big(T^HE,g^{TM},g^F\big)^{[0]}_x = \frac12\sum_{k=0}^{\dim M}(-1)^k k\log\operatorname{Det}\left(-\big(d^{M,*}-d^M\big)^2\big|_{\Omega^k(M_x;F)\cap H^{k\perp}}\right)\,, \tag{2.26}$$

where "Det" denotes a zeta-regularised determinant. The right hand side is precisely the Ray-Singer analytic torsion of the fibre M_x. Hence $\mathcal T(T^HE,g^{TM},g^F)$ is called a Bismut-Lott torsion form.

2.3. Elementary Properties. — From Theorem 1.5, one can derive a variation formula for Bismut-Lott torsion. If we choose $T^H_j E$, g^{TM}_j, g^F_j for $j = 0, 1$, let $\nabla^{TM,j}$ denote the corresponding connections on TM, and let $g^{H,j}_{L^2}$ denote the corresponding L^2-metrics on H. As in (1.3), there exists a Chern-Simons Euler class $\tilde{e}(TM, \nabla^{TM,0}, \nabla^{TM,1})$ such that

$$d\tilde{e}\big(TM, \nabla^{TM,0}, \nabla^{TM,1}\big) = e\big(TM, \nabla^{TM,1}\big) - e\big(TM, \nabla^{TM,0}\big) . \tag{2.27}$$

***2.6. Theorem* (Bismut and Lott [16]).** — *Modulo exact forms on B,*

$$\begin{aligned}\mathcal{T}\big(T^H_1 E, g^{TM}_1, g^F_1\big) &- \mathcal{T}\big(T^H_0 E, g^{TM}_0, g^F_0\big)\\ = \int_{E/B} &\Big(\tilde{e}\big(TM, \nabla^{TM,0}, \nabla^{TM,1}\big)\, \mathrm{ch}^\circ\big(F, g^F_0\big) + e\big(TM, \nabla^{TM,1}\big)\, \widetilde{\mathrm{ch}}^\circ\big(F, g^F_0, g^F_1\big)\Big)\\ &- \widetilde{\mathrm{ch}}^\circ\left(H^\bullet(E/B;F), g^{H,0}_{L^2}, g^{H,1}_{L^2}\right) .\end{aligned} \tag{2.28}$$

A variation formula like this has already been proved for the Ray-Singer torsion in [**17**]. Theorem 2.6 is a direct consequence of Theorem 1.5. Similar variation formulas exist for η-forms [**10**] and holomorphic torsion forms [**14**].

***2.7. Corollary* (Bismut and Lott [16]).** — *If the fibres of $p\colon E \to B$ are odd-dimensional and $F \to E$ is fibrewise acyclic, then $\mathcal{T}(T^H E, g^{TM}, g^F)$ defines an even cohomology class on B that is independent of the choices of $T^H E$, g^{TM} and g^F.*

There is another situation where $\mathcal{T}(T^H E, g^{TM}, g^F)$ defines a cohomology class, at least its higher degree components. Assume that g^F_0 and g^F_1 are both parallel with respect to ∇^F. Then

$$g^F_t = (1-t) g^F_0 + t g^F_1 \tag{2.29}$$

is a parallel metric on F for all $t \in [0,1]$. Put the metric $g^{\tilde{F}}|_{F\times\{t\}} = g^F_t$ on the pullback $\tilde{F}$ to $\tilde{E} = E \times [0,1]$, then

$$\omega\big(\tilde{F}, g^{\tilde{F}}\big) = (g^F_t)^{-1} \frac{\partial}{\partial t} g^F_t \, dt \in \Omega^1(E \times [0,1]; \operatorname{End} \tilde{F}) \tag{2.30}$$

because $\omega(\tilde{F}, g^{\tilde{F}})|_{E\times\{t\}} = 0$ by (1.10). In particular

$$\widetilde{\mathrm{ch}}^\circ\big(F, g^F_0, g^F_1\big) = c_0 \int_0^1 \mathrm{tr}_F \left((g^F_t)^{-1} \frac{\partial}{\partial t} g^F_t\right) dt \in \Omega^0(E) \tag{2.31}$$

is in fact just a constant function on E.

2.8. Definition. — If the bundles $F \to E$ and $H(E/B;F) \to B$ admit parallel metrics g^F and g^H, one defines the higher analytic torsion or *Bismut-Lott torsion* as

$$\mathcal{T}(E/B;F) = \mathcal{T}\big(T^H E, g^{TM}, g^F\big)^{[\geq 2]} + \widetilde{\mathrm{ch}}^\circ\big(H, g^H, g^H_{L^2}\big)^{[\geq 2]} \quad \in \Omega^{\geq 2}(B) . \tag{2.32}$$

It follows from Theorems 1.5 and 2.6 that $\mathcal{T}(E/B;F)$ defines a cohomology class in $H^{\geq 2}(B;\mathbb{R})$ that is independent of T^HE, g^{TM}, g^F and g^H, as long as g^F and g^H are parallel metrics.

3. Properties of Bismut-Lott torsion

Since η-forms, analytic torsion forms and holomorphic torsion forms are parallel objects in three somewhat similar theories, one can try to translate any result concerning one of those three objects into theorems on the other two. In this section, we present a few results on higher torsion that where at least partially motivated by results on η-forms or on holomorphic torsion forms. In particular, we recall results by Ma and Bunke on torsion forms of iterated fibrations, and of Bunke, Bismut and the author about the relation with equivariant Ray-Singer torsion. Most of these theorems have not yet been proved for Igusa-Klein or Dwyer-Weiss-Williams torsion. We also discuss Ma and Zhang's construction using η-invariants of subsignature operators.

One should mention at this point that in the theory of flat vector bundles, we are only considering proper submersions. The reason is that the direct image of a flat vector bundle under other maps like open or closed embeddings is in general not given by a flat vector bundle. Another reason is that there is no suitable analogue of the Becker-Gottlieb transfer for general maps. For this reason, many beautiful results for η-invariants and holomorphic torsion have no counterpart for Bismut-Lott torsion.

3.1. A transfer formula. — Consider a smooth proper submersion $p_1\colon E \to B$ with typical fibre M as before, and assume that $p_2\colon D \to E$ is another smooth proper submersion with fibre N. Then $p_3 = p_1 \circ p_2$ is again a smooth proper submersion, and its fibre L maps to M with fibre N. Let $F \to D$ be a flat vector bundle, then we have higher direct images

$$K = \bigoplus_{k=0}^{\dim N} (-1)^k H^\bullet(D/E;F) \to E \tag{3.1}$$
$$\text{and}\quad H = \bigoplus_{k=0}^{\dim L} (-1)^k H^\bullet(D/B;F) \to B.$$

Note that H is not the higher direct image of K under p_1. Instead, there is a fibrewise Leray-Serre spectral sequence over B with E_2-term $H^\bullet(E/B;K)$ that converges to H. Beginning with E_2, the higher terms in this spectral sequence are given by parallel families of finite-dimensional cochain complexes $\left(E_k, \nabla^{E_k} + d_k\right)$ over B. Of course, $E_n = E_\infty$ and $d_n = 0$ for all sufficiently large n.

We now choose compatible complements of the vertical tangent bundles for all three fibrations, fibrewise Riemannian metrics, and a metric on the bundle F. Again, these data induce connections on the three vertical tangent bundles TM, TN and $TL \cong TN \oplus p_2^*TM$. They also induce L^2-metrics on the flat vector bundles H and E_k over B for $k \geq 2$ and on $K \to E$. We need the Chern-Simons Euler form $\tilde{e}$, which is constructed in analogy with $\widetilde{\mathrm{ch}}$ in (1.3), and we also need another finite-dimensional torsion form $T(H, E_\infty, g^H, g^{E_\infty})$ relating the filtered flat vector bundle H to its graded version $E_\infty = E_n$ for n sufficiently large.

***3.1. Theorem* (Transfer formula, Ma [55]).** — *Modulo exact forms on B, we have*

$$\begin{aligned}(3.2)\quad \mathscr{T}\big(T^H D, g^{TL}, g^F\big) &= \int_{E/B} e\big(TM, \nabla^{TM}\big)\, \mathscr{T}\big(H^H D \oplus T^H L, g^{TN}, g^F\big) \\ &\quad + \mathscr{T}\big(T^H E, g^{TM}, g^K\big) + \sum_{k=2}^{\infty} T\left(\nabla^{E_k} + d_k, g^{E_k}\right) + T\big(H, E_\infty, g^H, g^{E_\infty}\big) \\ &\quad + \int_{D/B} \tilde{e}\left(TL, \nabla^{TL}, \nabla^{TN} \oplus p_2^* \nabla^{TM}\right) \mathrm{ch}^{\mathrm{o}}\big(F, g^F\big)\,.\end{aligned}$$

The first two terms on the right hand side should be regarded as torsion forms of the terms E_0 and E_1 of the Leray-Serre spectral sequence. The sum of the torsions of the remaining terms is of course finite. The theorem says in other words that the analytic torsion form of the total fibration is the sum of the torsion forms of all terms in the Leray-Serre spectral sequence and two natural correction terms. A similar formula for holomorphic torsion forms has been proved by Ma [**53**], [**54**]. For η-invariants of signature operators, an analogous result is due to Bunke and Ma [**25**].

3.2. Lott's Secondary K-theory of flat bundles. — In Arakelov geometry, one studies arithmetic Chow groups, which constitute a simultaneous refinement of classical Chow groups and of de Rham forms, see [**63**] for an introduction. The central objects in this theory are algebraic vector bundles over arithmetic schemes, together with Hermitian metrics on the corresponding holomorphic vector bundles over the complex points of those schemes, which form classical complex algebraic varieties. To construct the "complex algebraic" part of the direct image of such vector bundles, one needs the holomorphic torsion forms of Bismut and Köhler [**14**]. To establish elementary properties of this direct image construction, one needs deep results on holomorphic torsion forms. Thus, Arakelov geometry has been one of the main motivations for the many results on holomorphic torsion by Bismut and others. For this reason, it is tempting to have a similar theory for flat vector bundles over smooth manifolds, where Bismut-Lott torsion plays the role of holomorphic torsion forms.

Lott's K-theory of flat vector bundles with vanishing Kamber-Tondeur classes is a first step in this direction. But note that there are no objects corresponding to Chow cycles, and that we can take direct images only for submersions, for reasons explained at the beginning of this section. Thus we cannot expect a theory that is as rich as arithmetic Chow theory. Nevertheless, some nice results are motivated by Lott's construction.

We consider triples (F, g^F, α), where $F \to M$ is a flat vector bundle, equipped with a metric g^F, and $\alpha \in \Omega^{\mathrm{even}}(M)/d\Omega^{\mathrm{odd}}(M)$ satisfies

$$\mathrm{ch}^{\mathrm{o}}\big(F, g^F\big) - d\alpha = 0 \in \Omega^{\mathrm{odd}}(M). \tag{3.3}$$

A short exact sequence

$$0 \longrightarrow F_1 \xrightarrow{a_1} F_2 \xrightarrow{a_2} F_3 \longrightarrow 0 \tag{3.4}$$

of flat vector bundles and parallel linear maps can be interpreted as a parallel family of acyclic chain complexes $(F, \nabla^F + a)$. Let g^{F_1}, g^{F_2}, g^{F_3} be metrics on these bundles. By Theorem 2.1, the higher torsion form of this family satisfies

$$dT\big(\nabla^F + a, g^F\big) = \mathrm{ch}^{\mathrm{o}}\big(F_2, g^{F_2}\big) - \mathrm{ch}^{\mathrm{o}}\big(F_1, g^{F_1}\big) - \mathrm{ch}^{\mathrm{o}}\big(F_3, g^{F_3}\big). \tag{3.5}$$

3.2. Definition. — Lott's secondary K-group $\overline{K}^0(M)$ is the abelian group generated by triples (F, g^F, α) subject to

1. the condition (3.3), and
2. the relation

$$T\big(\nabla^F + a, g^F\big) = \alpha_2 - \alpha_1 - \alpha_3 \in \Omega^{\mathrm{even}}(M)/d\Omega^{\mathrm{odd}}(M)$$

for each short exact sequence (3.4).

In fact, Lott considers groups $\overline{K}^0_R(M)$ in **[52]**. Here, R is a ring satisfying a few technical assumptions with a representation $\rho : R \to \operatorname{End} \mathbb{C}^n$, and all flat vector bundles arise from local systems of R-modules by tensoring with $\mathbb{C}^n$. Similarly, relations come from short exact sequences of such local systems.

Let now $p\colon E \to B$ be a proper submersion with fibre M. We choose T^HE and g^{TM} as before.

3.3. Definition. — Let (F, g^F, α) be a generator of $\overline{K}^0_R(M)$ and let $g^H_{L^2}$ denote the L^2-metric on the virtual vector bundle

$$H = \bigoplus_{k=0}^{\dim M} (-1)^k H^\bullet(E/B; F) \to B.$$

Then the push-forward of (F, g^F, α) is defined as

$$p_!(F, g^F, \alpha) = \bigg(H,\ g^H_{L^2},\ \int_{E/B} e\big(TM, \nabla^{TM}\big)\, \alpha - \mathscr{T}\big(T^HE, g^{TM}, g^F\big)\bigg).$$

Lott then verifies that $p_!$ defines a push-forward map

$$p_! \colon \overline{K}^0_R(E) \to \overline{K}^0_R(B). \tag{3.6}$$

Moreover, on the level of K-theory, the push-forward is independent of the choices of $T^H E$ and g^{TM}.

***3.4. Theorem* (Bunke [24]).** — *Lott's secondary K-groups together with the pushforward define a functor from the category of smooth proper submersions to the category of abelian groups.*

The proof is based on Ma's Theorem 3.1. Bunke shows that if $p_1 \colon E \to B$ and $p_2 \colon D \to E$ are smooth proper submersions, then

$$(p_1 \circ p_2)_! = p_{1!} \circ p_{2!} \colon \overline{K}^0_R(D) \to \overline{K}^0_R(B). \tag{3.7}$$

A similar push-forward in secondary L-theory has been defined by Bunke and Ma [**25**], correcting an older definition by Lott [**52**].

3.3. Rigidity of Kamber-Tondeur classes. — In this section, we discuss the dependence of the Kamber-Tondeur forms and the torsion forms on the flat structure on the bundle F. Let $V \to M$ be a vector bundle and assume that $(\nabla^{V,t})_{t\in[0,1]}$ is a family of flat connections on V. If we define a connection $\nabla^{\tilde V}$ on the pull-back $\tilde V$ of V to $M \times [0,1]$ such that $\nabla^{\tilde V}|_{M\times\{t\}} = \nabla^{V,t}$, then the connection $\nabla^{\tilde V}$ will in general not be flat. In particular, the arguments in (1.3) and (1.7) are not applicable here. If we fix a family of metrics (g^V_t) on V, we have a family $(\nabla^{V,t,*})_{t\in[0,1]}$ of adjoint connections that are again flat. Let now $\tilde V$ denote the pullback of V to $M \times [0,1]^2$ and construct $\nabla^{\tilde V}$ such that

$$\nabla^{\tilde V}\big|_{M\times\{s\}\times[0,1]} = (1-s)\nabla^{V,t} + s\nabla^{V,t,*}. \tag{3.8}$$

We define forms $L\left((\nabla^{V,t}, g^V_t)_t\right) \in \Omega^{\mathrm{even}}(M)$ by

$$L\left((\nabla^{V,t}, g^V_t)_t\right) = \pi i \int_0^1 \int_0^1 \iota_{\frac{\partial}{\partial s}} \iota_{\frac{\partial}{\partial t}} \operatorname{ch}\big(\tilde V, \nabla^{\tilde V}\big)\, dt\, ds. \tag{3.9}$$

Because $\nabla^{V,t}$ and $\nabla^{V,t,*}$ are flat, for $s \in \{0,1\}$, we have

$$\operatorname{ch}\big(\tilde V, \nabla^{\tilde V}\big)\big|_{M\times\{0,1\}\times[0,1]} = \begin{cases} \frac12 \operatorname{tr}_V\left(\frac{\partial}{\partial t}\nabla^{V,t}\right)\, dt & s=0,\\ \frac12 \operatorname{tr}_V\left(\frac{\partial}{\partial t}\nabla^{V,t,*}\right)\, dt & s=1. \end{cases} \tag{3.10}$$

Hence it follows from Stokes' theorem that

$$dL\left((\nabla^{V,t})_t, g^V\right)^{[\geq 2]} = \operatorname{ch}^\circ\left(V_1, g^V_1\right)^{[\geq 3]} - \operatorname{ch}^\circ\left(V_0, g^V_0\right)^{[\geq 3]}, \tag{3.11}$$

where V_t denotes the flat vector bundle $(V, \nabla^{V,t})$.

One can show that $L((\nabla^{V,t}, g^V_t)_t)$ changes by exact forms if one replaces $(\nabla^{V,t})_t$ by a homotopic path of flat connections. On the other hand, if $\nabla^{V,1} = \nabla^{V,0}$, then the cohomology class of $L((\nabla^{V,t},g^V_t)_t)$ depends on the homotopy class of the loop $(\nabla^{V,t})_t$ in the space of flat connections.

Now assume that V is $\mathbb{Z}$-graded and that $(V^\bullet,\nabla^{V,t}+a_t)$ is a parallel family of cochain complexes on M such that the fibrewise cohomology $H^\bullet(V,a_t)$ has the same rank for all t. Then we obtain a family of flat Gauß-Manin connections $(\nabla^{H,t})_t$ and a family of metrics $g^H_{V,t}$ on a fixed vector bundle $H\to M$.

***3.5. Theorem* (Rigidity, Bismut and G. [12]).** — *Under these assumptions,*

$$T\left(\nabla^{V,1}+a_1,g^V\right)^{[\geq 2]}-T\left(\nabla^{V,0}+a_0,g^V\right)^{[\geq 2]} = L\left((\nabla^{V,t},g^V_t)_t\right)^{[\geq 2]}-L\big((\nabla^{H,t},g^H_{V,t})_t\big)^{[\geq 2]}. \tag{3.12}$$

Similarly, let $(\nabla^{F,t})_t$ be a family of flat connections on $F\to E$ such that the fibrewise cohomology $H^\bullet(E/B;F_t)$ has the same rank for all t. Then we again have a family $(H_t,g^H_{L^2,t})$ of flat vector bundles over B.

***3.6. Theorem* (Rigidity, Bismut and G. [12]).** — *Under these assumptions,*

$$\mathscr{T}\big(T^HE,g^{TM},g^{F_1}\big)^{[\geq 2]}-\mathscr{T}\big(T^HE,g^{TM},g^{F_0}\big)^{[\geq 2]} = \int_{M/B} e\big(TM,\nabla^{TM}\big)\,L\left((\nabla^{F,t},g^F_t)_t\right)^{[\geq 2]}-L\big((\nabla^{H,t},g^H_{L^2,t})_t\big)^{[\geq 2]}. \tag{3.13}$$

Because $T(T^HE,g^{TM},g^F)^{[0]}$ equals the Ray-Singer analytic torsion, we cannot expect Theorems 3.5 and 3.6 to hold for the scalar part of the Bismut-Lott torsion, too. In fact these theorems as well as the construction of $\mathscr{T}(E/B;F)$ indicate that the "higher" Bismut-Lott torsion has a different topological meaning than the Ray-Singer torsion.

3.4. Equivariant analytic torsions. — Let us assume that $p\colon E\to B$ is associated to a G-principal bundle $P\to B$ for some compact, connected Lie group G. In particular, G acts by isometries on the fibre (M,g^{TM}). Let $F\to M$ be a G-equivariant flat vector bundle such that elements X of the Lie algebra g of G act by $\nabla^F_{X^M}$, where X^M is the corresponding Killing field on M. Then the induced vector bundle

$$P\times_G F\longrightarrow E=P\times_G M, \tag{3.14}$$

which we will again call F, is also flat. A G-equivariant fibre bundle connection T^HP defines T^HE, and we also fix a G-invariant metric on F. Let $\Omega\in\Omega^2(B;\mathfrak{g})$ denote the curvature of T^HP.

It was already observed in **[16]** and **[50]** that in this situation, the Bismut-Lott torsion is given by an Ad-invariant formal power series $\mathscr{T}_{\mathfrak{g}}(g^{TM},g^F)\in\mathbb{C}[\![\mathfrak{g}^*]\!]$ on $\mathfrak{g}$, such that

$$\mathcal{T}\big(T^HE, g^{TM}, g^F\big) = \mathcal{T}_{\frac{\Omega}{2\pi i}}\big(g^{TM}, g^F\big) \in \Omega^{\mathrm{even}}(B). \tag{3.15}$$

On the other hand, there is a G-equivariant generalisation of the Ray-Singer analytic torsion. If $g \in G$ acts by isometries on M and preserves ∇^F, put

$$\begin{aligned} \vartheta_g\big(g^{TM}, g^F\big)(s) &= -\operatorname{str}\big(N^M g(d_M + d_M^*)^{-2s}\big) \\ \text{and} \quad \mathcal{T}_g\big(g^{TM}, g^F\big) &= \frac{\partial}{\partial s}\vartheta_g\big(g^{TM}, g^F\big)(s). \end{aligned} \tag{3.16}$$

Inspired by results of Bismut, Berline and Vergne about the equality of two notions of the equivariant index [**5**], one can ask if the infinitesimal equivariant Bismut-Lott torsion $\mathcal{T}_{\mathfrak{g}}(g^{TM}, g^F)$ is related to the equivariant torsion $\mathcal{T}_G(g^{TM}, g^F)$. Bunke proved in [**22**] and [**23**] that both equivariant torsions can be computed from the G-equivariant Euler characteristic of M up to a constant when G is connected and F satisfies some technical assumptions. From Bunke's results, one can deduce a relation between both equivariant torsions in some interesting special cases.

To state a more general relation between both equivariant torsions, we need the infinitesimal Euler form $e_{\mathfrak{g}}(TM, \nabla^{TM}) \in \Omega^\bullet(M)[\![\mathfrak{g}^*]\!]$ and an equivariant Mathai-Quillen current $\psi_X(TM, \nabla^{TM})$ on M such that

$$d\psi_X\big(TM, \nabla^{TM}\big) = e_X\big(TM, \nabla^{TM}\big) - e\big(TM_X, \nabla^{TM_X}\big)\delta_{M_X}, \tag{3.17}$$

where δ_{M_X} is the Dirac current of integration over the fixpoint set M_X of the Killing field X^M, for $X \in \mathfrak{g}$. Finally, for a proper submersion $p\colon E \to B$ with typical fibre M and a fibrewise G-action, there exists an even closed form $V_X(E/S, T^HE, g^{TM})$ that is locally computable on E, vanishes for even-dimensional fibres, and satisfies

$$V_{rX}\big(E/S, T^HE, g^{TM}\big) = \frac{1}{|r|} r^{-\frac{N^B}{2}} V_X\big(E/S, T^HE, g^{TM}\big) \tag{3.18}$$

for all $r \in \mathbb{R}\backslash\{0\}$. In particular, the class $V_X(E/S) \in H^{\mathrm{even}}(B;\mathbb{R})$ is independent of T^HE and g^{TM}. Let $V_X(M) = V_X(E/S)^{[0]}$ denote the scalar part.

***3.7. Theorem* (Bismut and G. [13]).** — *For $X \in \mathfrak{g}$, the equivariant torsions are related by*

$$\begin{aligned} \mathcal{T}_X\big(g^{TM}, g^F\big) &- \mathcal{T}_{e^X}\big(g^{TM}, g^F\big) \\ &= \int_M \psi_X\big(TM, \nabla^{TM}\big)\operatorname{ch}^\circ\big(F, g^F\big) + V_X(M)\operatorname{rk} F\,. \end{aligned} \tag{3.19}$$

Similar results for equivariant η-invariants have been proved in [**35**], and for equivariant holomorphic torsion by Bismut and the author in [**11**].

3.8. Remark. — In $\mathcal{T}_{e^X}(g^{TM}, g^F)$, all powers of X occur simultaneously. Thus, Theorem 3.7 can only hold for one choice of constants c_k in (1.10), and this is precisely the so-called Chern normalisation introduced in [**12**] and also used in this overview. The Chern normalisation is also needed for Lott's noncommutative higher torsion classes in [**51**], see Remark 7.6 below.

The theorem above is of course compatible with Bunke's computations. As a simple application, we can use Köhler's computation of the equivariant analytic torsion on compact symmetric spaces [**49**] to compute the Bismut-Lott torsion of bundles with compact symmetric fibres and compact structure groups. The case of sphere bundles will be important later. Let ζ denote the Riemann ζ-function. We define an additive characteristic class ${}^0J(W)$ for a vector bundle $W \to M$ by

$$ {}^0J(W) = \frac{1}{2}\sum_{k=0}^{\infty} \zeta'(-2k)\,\mathrm{ch}(W)^{[4k]} \in H^\bullet(M;\mathbb{R}). \tag{3.20} $$

***3.9. Corollary* (Sphere bundles, Bunke [23], Bismut and G. [12])**

Let $E \to B$ be the unit n-sphere bundle of an oriented real vector bundle $W \to B$. Then

$$ \mathcal{T}(E/B;\mathbb{C}) = \chi(S^n)\ {}^0J(W). $$

The meaning of the class $V_X(M)$ is not quite clear from Theorem 3.7. As Bismut explains in [**8**], the Bismut-Lott torsion of a smooth proper submersion $p\colon E \to B$ is formally given by evaluating V on the generator of the natural S^1 action on the fibrewise free loop space $L_B E$, viewed as a bundle over B. Although the flat vector bundle F and its cohomology are not visible in this approach, many properties of $V_X(E/S)$ proved in [**13**] mirror well-known properties of Bismut-Lott torsion, including the behaviour under iterated fibrations in Section 3.1 and under Witten deformation in Section 5.1.

3.5. The Ma-Zhang subsignature operator. — In Section 1.1, we have constructed Kamber-Tondeur forms by lifting the Chern character to flat vector bundles. In Section 2.2, we have constructed the torsion form as a correction term in a family index theorem. Thus, Bismut-Lott torsion is a double transgression of the Chern character. Ma and Zhang first produce an η-invariant, which can be regarded as a transgression of the Chern character. Then they derive Bismut-Lott torsion from a transgression of η-forms in [**56**]. In other words, they get torsion forms by a different double transgression. On the way, they give a new analytic proof of Theorems 1.2 and 1.3. Dai and Zhang have recently given a related construction in [**29**], where Bismut-Lott torsion appears in the adiabatic limit of a Bismut-Freed connection form that is related to Ma and Zhang's η-invariant.

Let $p\colon E \to B$ be a proper submersion of closed manifolds, where B is oriented, and let $F \to E$ be a flat vector bundle, then F is rationally trivial in the topological K-theory of E. Thus there exists an isomorphism $qF \cong E \times \mathbb{C}^{q \operatorname{rk} F}$ for some positive integer q. Let ∇^0 denote the trivial flat connection on $E \times \mathbb{C}^{q \operatorname{rk} F}$, then

$$\widehat{\mathrm{ch}}\big(F, \nabla^F\big) = \frac{1}{q} \widetilde{\mathrm{ch}}\big(\nabla^0, \nabla^{qF}\big) \in H^\bullet(E; \mathbb{C}/\mathbb{Q}). \tag{3.21}$$

Choose $T^H E$, g^{TM}, g^F as before. We also choose a metric g^{TB} on B and put $g^{TE} = g^{TM} \oplus p^* g^{TB}$ using the splitting $TE = TM \oplus T^H E$. Let $W \to B$ be a Hermitian vector bundle with metric g^W and connection ∇^W. Ma and Zhang consider two operators $D^{W,F}_{\mathrm{sig}}$ and $\hat{D}^{W,F}_{\mathrm{sig}}$ on $\Omega^\bullet(E; p^*W \oplus F)$. Whereas $D^{W,F}_{\mathrm{sig}}$ is an honest Dirac-operator if g^F is parallel, the operator $\hat{D}^{W,F}_{\mathrm{sig}}$ differentiates only in the directions of the fibres. These operators should be viewed as "quantisations" in the sense of [**5**], applied to the Bismut type superconnection $\tilde{\mathbb{A}} = \frac{1}{2}(\tilde{\mathbb{A}}' + \tilde{\mathbb{A}}'')$ and the operator $\tilde{\mathbb{X}}$ of (2.21).

If B is odd-dimensional, then

$$D^{W,F}_{\mathrm{sig}}(r) = D^{W,F}_{\mathrm{sig}} + ir\hat{D}^{W,F}_{\mathrm{sig}} \tag{3.22}$$

is a selfadjoint operator on $\Omega^{\mathrm{even}}(B; \Omega^\bullet(E/B; p^*W \oplus F))$ for all $r \in \mathbb{R}$. The *reduced η-invariant* of $D^{W,F}_{\mathrm{sig}}(r)$ is as usual defined as

$$\overline{\eta}\left(D^{W,F}_{\mathrm{sig}}(r)\right) = \frac{1}{2}\left(\eta\big(D^{W,F}_{\mathrm{sig}}(r)\big) + \dim\ker\big(D^{W,F}_{\mathrm{sig}}(r)\big)\right) \in \mathbb{R}/\mathbb{Z}. \tag{3.23}$$

For the virtual bundle $H(E/B; F) \to B$, one defines similarly

$$\overline{\eta}\left(D^{W,H}_{\mathrm{sig}}(r)\right) = \sum_k (-1)^k\, \overline{\eta}\left(D^{W,H^k(E/B;F)}_{\mathrm{sig}}(r)\right) \in \mathbb{R}/\mathbb{Z}. \tag{3.24}$$

For $\varepsilon > 0$, let $D^{W,F}_{\mathrm{sig},\varepsilon}(r)$ denote the analogous operator, where the metric g^{TB} has been replaced by $\frac{1}{\varepsilon} g^{TB}$. The reduced η-invariants are related in the adiabatic limit $\varepsilon \to 0$.

***3.10. Theorem* (Ma and Zhang [67], [56]).** — *One has*

$$\lim_{\varepsilon \to 0} \overline{\eta}\left(D^{W,F}_{\mathrm{sig},\varepsilon}(r)\right) = \overline{\eta}\left(D^{W,H}_{\mathrm{sig}}(r)\right) \in \mathbb{R}/\mathbb{Z}\,. \tag{3.25}$$

Proof of Theorem 1.3. — The proof for the imaginary part of $\widehat{\mathrm{ch}}$ uses the identities

$$\begin{aligned}
&\frac{\partial}{\partial r}\bigg|_{r=0} \overline{\eta}\left(D^{W,F}_{\mathrm{sig},\varepsilon}(r)\right) = \int_B L(TB)\,\mathrm{ch}(W)\, tr^*_{E/B} \sum_{k=0}^\infty c'_k \operatorname{Im} \widehat{\mathrm{ch}}(F)^{[2k+1]} \\
\text{and } &\frac{\partial}{\partial r}\bigg|_{r=0} \overline{\eta}\left(D^{W,H}_{\mathrm{sig}}(r)\right) = \int_B L(TB)\,\mathrm{ch}(W) \sum_{k=0}^\infty c'_k \operatorname{Im} \widehat{\mathrm{ch}}(H)^{[2k+1]}
\end{aligned} \tag{3.26}$$

for some constants $c'_k \neq 0$, where $L(TB)$ denotes the Hirzebruch L-class. Because $H^{\mathrm{even}}(B; \mathbb{R})$ is spanned by the values of $L(TB)\,\mathrm{ch}(W)$ for all complex vector bundles W, one gets the imaginary part of (1.17) in Theorem 1.3 from Theorem 3.10 by comparison of coefficients in (3.26).

The real part also follows from Theorem 3.10 because

$$\overline{\eta}\big(D^{W,F}_{\mathrm{sig},\varepsilon}\big) - \operatorname{rk} F\,\overline{\eta}\,\big(D^{W}_{\mathrm{sig},\varepsilon}\big) = \int_B L(TB)\operatorname{ch}(W)\,tr^*_{E/B}\operatorname{Re}\widehat{\operatorname{ch}}(F) \in \mathbb{R}/\mathbb{Q} \tag{3.27}$$

and a similar equation holds for the two virtual bundles $H(E/B;F) \to B$ and $\operatorname{rk} F \cdot H(E/B;\mathbb{C}) \to B$. To complete the proof, one needs that

$$\widehat{\operatorname{ch}}(H(E/B;\mathbb{C})) = 0 \in H^\bullet(B;\mathbb{C}/\mathbb{Q})\ , \tag{3.28}$$

which was already proved for evendimensional M by Bismut in [**7**]. □

To recover Bismut-Lott torsion and Theorem 1.5 from this approach, one considers a generalised η-form

$$\hat{\eta}_r = (2\pi i)^{-\frac{N^B+1}{2}} \int_0^\infty \left(\operatorname{tr}_s\left(\left(\frac{\partial}{\partial t}\left(\tilde{\mathbb{A}}_t + \frac{ir}{2}\tilde{\mathbb{X}}_t\right)\right) e^{-\left(\tilde{\mathbb{A}}_t + \frac{ir}{2}\tilde{\mathbb{X}}_t\right)^2}\right) - \frac{ir\cdot a}{\sqrt{1+r^2}}t^{-\frac{3}{2}}\right) dt \tag{3.29}$$

for some locally computable function $a\colon B \to \mathbb{R}$.

***3.11. Theorem* (Ma and Zhang [56]).** — *For certain constants $c''_k \neq 0$, one has*

$$\left.\frac{\partial \hat{\eta}_r}{\partial r}\right|_{r=0} = \sum_{k=0}^{\infty} c''_k\, d\mathscr{T}(T^H E, g^{TM}, g^F)^{[2k+1]}. \tag{3.30}$$

Dai and Zhang will give a more explicit construction in [**29**]. These last results seem to indicate a strong relation between Bismut-Lott torsion and η-forms that still has to be explored. A similar relation has been established by Braverman and Kappeler in a definition of complex-valued Ray-Singer torsion in [**21**] for single manifolds.

4. Igusa-Klein torsion

We have seen in Sections 1–3 how to establish an index theorem for flat vector bundles using methods from local index theory for families, and how to discover Bismut-Lott torsion in a natural refinement of this index theorem. It is somewhat surprising that homotopy theoretical methods from differential topology lead to an invariant that is very closely related to Bismut-Lott torsion. There are several slightly different approaches to this topological higher torsion by Igusa and Klein [**48**], [**42**], [**45**], [**46**]. In this section, we focus on Igusa-Klein torsion as described in [**42**]. In Section 6, we discuss the approach by Dwyer, Weiss and Williams [**30**].

4.1. Generalised Morse functions and filtered complexes. — It is well-known that smooth manifolds admit Morse functions. If $p\colon E \to B$ is a smooth proper submersion, then in general, there is no function $h\colon E \to \mathbb{R}$ that is a Morse function on every fibre of p. However, by results of Igusa [**41**] and Eliashberg and Mishachev [**31**], there always exist generalised Morse functions.

By a birth-death singularity of $h\colon E \to \mathbb{R}$, we mean a fibrewise critical point of type A2 that is unfolded over B. In other words, there exist k, a function h_0 on B and coordinates $u_1, \dots$ on B and $x_1, \dots, x_n$ along the fibres such that locally,

$$h(x,u) = h_0(u) + \frac{x_n^3}{3} - u_1 x_n - \frac{x_1^2 + \cdots + x_k^2}{2} + \frac{x_{k+1}^2 + \cdots + x_{n-1}^2}{2} .$$

Birth-death singularities occur over a two-sided immersed submanifold $B_0 \subset B$ given by $u_1 = 0$ in the coordinates above. Two fibrewise Morse critical points of adjacent indices over the "positive" side of B come together in a fibrewise cubical singularity. In a neighbourhood over the "negative" side, the function is regular.

Let $C = C_{\mathrm{M}} \cup C_{\mathrm{bd}} \subset E$ denote the submanifold of fibrewise critical points of h. Note that the submanifold C_{M} of Morse fibrewise critical points of h locally covers B, and that the submanifold C_{bd} of birth-death critical points locally bounds two components of C_{M}. After fixing a fibrewise metric g^{TM}, the negative eigenspaces of the Hessian of h form a vector bundle $T^uM \subset TM|_{C_{\mathrm{M}}}$ over C_{M}, whose rank is given by the Morse index $\operatorname{ind} h$. At the birth-death singularities C_{bd}, the natural extension of T^uM of the two adjacent components of C_{M} differ by an oriented trivial line bundle, the "cubical direction".

4.1. Definition. — A *generalised fibrewise Morse function* on $p\colon E \to B$ is a function $h\colon E \to \mathbb{R}$ that has only Morse and birth-death type fibrewise singularities. A *framed function* is a generalised fibrewise Morse function together with trivialisations of T^uM over each connected component of C_{M} that extend up to the boundary, such that the two frames at each point of C_{bd} differ only by the preferred generator of the cubical direction.

***4.2. Theorem* (Igusa [41])**. — *Let $p\colon E \to B$ be a smooth fibre bundle with typical fibre M. If $\dim M \geq \dim B$, there exists a framed function, and if $\dim M > \dim B$, it is unique up to homotopy.*

Here, uniqueness up to homotopy means that if $h_0, h_1\colon E \to \mathbb{R}$ are two framed functions, then there exists a framed function $h\colon E \times [0,1] \to \mathbb{R}$ that restricts to h_j at $E \times \{j\}$ for $j = 0, 1$.

If the dimension of the fibres is too small to apply Theorem 4.2, one can take cross products with manifolds of Euler number 1, for example $\mathbb{R}P^{2n}$. One can check that this will not alter the torsion classes of Igusa and Klein that we are going to introduce, so that the following constructions are valid for fibre bundles of arbitrary dimensions.

4.2. Filtered chain complexes and the Whitehead space. — We assume that we are given a smooth fibre bundle $p\colon E \to B$ and a proper framed function $h\colon E \to \mathbb{R}$ with finitely many fibrewise critical points over small subsets of B. Let $F \to E$ be a flat vector bundle.

Over a small open subset $U \subset B$, one can use h to filter the singular chain complexes of the fibres over U. The filtered chain complexes are quasiisomorphic to a filtered chain complex on the vector space

$$V'_x = \bigoplus_{C \in C'_M|_x} F_c. \tag{4.1}$$

Here $C'_M|_x$ is a subset of $C_{\mathrm{M}}|_x$, where some pairs of components of C_{M} near birth-death singularities are omitted. Both the filtration and the quasiisomorphism are natural and unique up to contractible choice.

Moreover, the two leaves of C_{M} near a birth-death singularity generate a direct summand isomorphic to

$$0 \longrightarrow F \xrightarrow{\ \mathrm{id}\ } F \longrightarrow 0 \tag{4.2}$$

after applying another quasiisomorphism that is again unique up to contractible choice. Adding or deleting a subcomplex of the form (4.2) is called an *elementary expansion* or *elementary collapse.*

Suppose now that the flat bundle F is fibrewise acyclic and comes with an R-structure for a suitable ring R as in Section 3.2 above. Also assume that the holonomy of F if contained in some group $G \subset GL_r(R)$, with $r = \operatorname{rk} F$. A typical choice would be $R = M_r(\mathbb{C})$ and $G = U(r)$ with $r \in \mathbb{N}$. In [**42**], Igusa constructs a classifying space for acyclic locally filtered finite dimensional chain complexes over R with holonomy G, up to filtered quasiisomorphisms and elementary expansions and collapses. This space is called the *acyclic Whitehead space* $Wh^h(R,G)$. We give a slightly more explicit description in Section 5.2.

***4.3. Theorem* (Igusa [42]).** — *Each generalised fibrewise Morse function $h\colon E \to \mathbb{R}$ gives rise to a classifying map*

$$\xi_h(E/B;F)\colon B \longrightarrow Wh^h(R,G) \tag{4.3}$$

that is unique up to homotopy.

Together with Theorem 4.2, one can associate to a smooth fibre bundle $p\colon E \to B$ as above and a flat, fibrewise acyclic vector bundle $F \to E$ with R-structure and holonomy group G a unique homotopy class of maps

$$\xi(E/B;F)=\xi_h(E/B;F)\colon B\longrightarrow Wh^h(R,G), \tag{4.4}$$

by choosing h to be a framed function.

Assume that G preserves a Hermitian metric, in other words, that F carries a parallel metric. Then Igusa constructs cohomology classes

$$\tau=\sum_{k=1}^{\infty}\tau_{2k}\qquad\text{with}\qquad\tau_{2k}\in H^{4k}(Wh^h(R,G);\mathbb{R}) \tag{4.5}$$

that are related to the Kamber-Tondeur classes of Section 1.1. These classes are natural under pairs of compatible ring and group homomorphisms $(R,G)\to(S,H)$. In particular, it is enough to construct them for $R=M_r(\mathbb{C})$ and $G=U(n)$.

4.4. Definition. — The *Igusa-Klein torsion* of a smooth fibre bundle $p\colon E\to B$ as above and a flat fibrewise acyclic vector bundle $F\to E$ with a parallel Hermitian metric is defined as

$$\tau(E/B;F)=\xi(E/B;F)^*\,\tau\in H^{4\bullet}(B;\mathbb{R})\ . \tag{4.6}$$

Igusa also explains how to define $\xi(E/B;F)$ and $\tau(E/B;F)$ if $H^\bullet(E/B;F)\to B$ is a trivial bundle, or more generally, a globally filtered flat vector bundle such that the associated graded vector bundle is trivial. In other words, the flat cohomology bundle $H^\bullet(E/B;F)\to B$ is then given by a unipotent representation of $\pi_1 B$.

4.5. Remark. — The map $\xi(E/B;F)$ of (4.4) is a higher torsion invariant in its own right. In fact, most of the properties of $\tau(E/B;F)$ in the next subsection already hold at the level of $\xi(E/B;F)$. Moreover, $\xi(E/B;F)$ is well-defined even if F carries no parallel metric. However, the cohomology class $\tau(E/B;F)$ makes it possible to compare Igusa-Klein torsion with Bismut-Lott torsion.

4.3. Properties of the Igusa-Klein torsion. — Assume that the fibre bundle $p\colon E\to B$ arises by gluing two families $p_i\colon E_i\to B$ for $i=1$, 2 along their fibrewise boundary $\partial_B E_1=\partial_B E_2$. Then there exist a framed function $h\colon E\to\mathbb{R}$ such that $h|_{E_1}\geq 0$ and $h|_{E_2}\leq 0$. Igusa proves in **[42]** that the corresponding classifying map $\xi(E/B;F)\colon B\to Wh^h(R,G)$ "splits" in an appropriate sense, at least if R is a field and the cohomology bundles $H^\bullet(E_i/B;F|_{E_i})\to B$ are unipotent as above.

Let $DE_i:=E_i\cup E_i\to B$ denote the fibrewise double of E_i, and let $F_i\to DE_i$ denote the flat vector bundle induced by $F|_{E_i}$. Then the splitting above has the following consequence, in a wording suggested by Bunke.

***4.6. Theorem* (Additivity, Igusa [44]).** — *If $E = E_1 \cup E_2 \to B$ is as above and the bundles $H^\bullet(E_i/B; F|_{E_i}) \to B$ are unipotent, then*

$$2\tau(E/B;F) = \tau(DE_1/B;F_1) + \tau(DE_2/B;F_2). \tag{4.7}$$

Suppose that $p\colon E \to B$ is a $(2n-1)$-sphere bundle with structure group $U(1)^n \subset O(2n)$. Then E is the fibrewise join of n circle bundles over B. The Igusa-Klein torsion of circle bundles has been computed explicitly in [**42**], and Theorem 4.6 gives the Igusa-Klein torsion of p. By the splitting principle for vector bundles and naturality of the Igusa-Klein torsion, one can now compute the higher torsion of all unit sphere bundles in Euclidean vector bundles. We use the same normalisation as for Bismut-Lott torsion. Let ${}^0J$ denote the characteristic class defined in Equation (3.20).

***4.7. Theorem* (Sphere bundles, Igusa [42]).** — *Let $V \to B$ be an oriented Euclidean vector bundle with unit sphere bundle $p\colon E \to B$. Then*

$$\tau(E/B;F) = 2\,{}^0J(V)\;. \tag{4.8}$$

Note that this agrees with the computations of the Bismut-Lott torsion in Corollary 3.9 if the fibres are odd-dimensional.

Assume that $h\colon E \to \mathbb{R}$ is a generalised fibrewise Morse function for $p\colon E \to B$ that is not framed. Because at the birth-death singularities C_{bd}, the natural extensions of T^uM at the two adjacent components of C_{M} are stably isomorphic, we have a class ${}^0J(T^uM) \in H^\bullet(C;\mathbb{R})$. Let $\hat p = p|_C$ and let C^j_{M} denote the fibrewise Morse critical points of Morse index j, then there exists a well-defined push-down map

$$\hat p_*\alpha = \sum_{j=0}^{\dim M} (-1)^j (p|_{C^j_{\mathrm{M}}})_* \, (\alpha|_{C^j_{\mathrm{M}}}) \in H^\bullet(B) \tag{4.9}$$

for all $\alpha \in H^\bullet(C)$. One can compute the Igusa-Klein torsion of $p\colon E \to B$ using the classifying map $\xi_h(E/B;F)$ even though h is not framed.

***4.8. Theorem* (Framing principle, Igusa [42], [43]).** — *In the situation above,*

$$\tau(E/B;F) = \xi_h(E/B;F)^*\tau - 2\,\hat p_*{}^0J(T^uM)\ \mathrm{rk}\,F. \tag{4.10}$$

As an example, suppose that $p\colon E \to B$ is the fibrewise suspension of the unit sphere bundle in a vector bundle $V \to B$. Then there exists a fibrewise Morse function with only two fibrewise critical points, and the unstable tangent bundle at the fibrewise maximums is isomorphic to the pullback of V. In this case, Theorems 4.7 and 4.8 give the same Igusa-Klein torsion for $E \to B$.

5. Bismut-Lott = Igusa-Klein

The higher analytic torsion of Bismut and Lott and the higher Franz-Reidemeister torsion of Igusa and Klein are defined using rather different methods. Nevertheless, it was noticed that both torsions assign special values of the Bloch-Wigner dilogarithm to acyclic flat line bundles over circle bundles over S^2 [**45**], [**16**]. In this section, we describe two approaches to prove that both torsions agree. The first, due to Bismut and the author, is inspired both by the proof of a general Cheeger-Müller theorem in [**17**], [**18**] and by the constructions of Igusa-Klein torsion using Morse theory [**42**], [**48**]. The second approach classifies all invariants of smooth fibre bundles satisfying two simple axioms [**44**]. It is also suitable to compare Igusa-Klein torsion with the Dwyer-Weiss-Williams construction in [**30**], see [**2**] and Theorem 6.5 below. We also give some consequences of the equality of both torsions.

5.1. The Witten deformation. — Let $p\colon E \to B$ be a smooth proper submersion, and let $F \to E$ be a flat vector bundle. We assume that there exists a fibrewise Morse function $h\colon E \to \mathbb{R}$ such that the fibrewise gradient field ∇h satisfies the Thom-Smale transversality condition on every fibre of p. A nontrivial example is given by the fibrewise suspension of a unit sphere bundle at the end of Section 4.3.

Let $o(T^uM) \to C$ denote the orientation bundle of $T^uM \to C_{\mathrm{M}}$, which extends naturally to the birth-death singularities. Recall that $C_{\mathrm{M}} = \bigcup_j C^j_{\mathrm{M}}$, where C^j_{M} is the set of fibrewise critical points of Morse index j. We define a finite-dimensional $\mathbb{Z}$-graded vector bundle

$$V = \bigoplus_j V^j \longrightarrow B,$$

with

$$V^j = \bigl(p|_{C^j_{\mathrm{M}}}\bigr)_*(F \otimes o(T^uM)). \tag{5.1}$$

This bundle carries a flat connection ∇^V induced by ∇^F, and a fibrewise Thom-Smale differential a. Then a is parallel, so by [**16**], there exists a torsion form

$$T\bigl(\nabla^V + a, g^V\bigr) \in H^\bullet(B;\mathbb{R}) \tag{5.2}$$

as in Theorem 2.1 for all metrics g^V induced by metrics g^F on F.

We choose a horizontal subbundle T^HE and a fibrewise Riemannian metric g^{TM} as in Sections 1.3 and 2.2. Then there exists a Mathai-Quillen current $\psi(\nabla^{TM}, g^{TM})$ on the total space of TM, such that

$$d\bigl((\nabla h)^*\psi(\nabla^{TM}, g^{TM})\bigr) = e\bigl(TM, \nabla^{TM}\bigr) - \delta_C, \tag{5.3}$$

where δ_C denotes the alternating sum of the currents of integration over C^j_{M}.

Recall that we have defined two metrics g_V^H and $g_{L^2}^H$ on the flat vector bundle

$$H = H^\bullet(M/B;F) \cong H^\bullet(V,a) \to B. \tag{5.4}$$

***5.1. Theorem* (Bismut and G. [12]).** — *Modulo exact forms on B,*

$$\begin{aligned}\mathcal{T}\big(T^HM, g^{TM}, g^F\big) &= T\big(\nabla^V + a, g^V\big) + \widetilde{\mathrm{ch}}^\circ\big(H, g_{L^2}^H, g_V^H\big)\\ &\quad + \int_{M/B} (\nabla h)^*\psi\big(\nabla^{TM}, g^{TM}\big)\cdot \mathrm{ch}^\circ\big(F, g^F\big) + \hat{p}_*\,{}^0J(T^sM - T^uM)\,\mathrm{rk}\,F.\end{aligned} \tag{5.5}$$

This theorem is proved using the Witten deformation of the fibrewise de Rham complex by h as in [**17**], [**18**]. By Theorems 1.5 and 2.1 and (1.7) and (5.3), taking the exterior derivative in Theorem 5.1 gives a trivial identity. The first three terms on the right hand side can be guessed that way. On the other hand, the last term contains topological information related to Igusa's framing principle.

In fact, if F carries a parallel metric, then $T(\nabla^V + a, g^V)^{[\geq 2]} = 0$ by the axiomatic description of T in [**16**], and the metric g_V^H is parallel, too. Recall the Becker-Gottlieb transfer $tr_{E/B}^*\colon H^\bullet(E) \to H^\bullet(B)$ of (1.14). In this case, Theorem 5.1 reduces to

$$\begin{aligned}\mathcal{T}(E/B;F) &= \hat{p}_*\,{}^0J(T^sM - T^uM)\,\mathrm{rk}\,F\\ &= \hat{p}_*\,{}^0J(TM|_C)\,\mathrm{rk}\,F - 2\hat{p}_*\,{}^0J(T^uM)\,\mathrm{rk}\,F\\ &= \tau(E/B;F) + tr_{E/B}^*\,{}^0J(TM)\,\mathrm{rk}\,F,\end{aligned} \tag{5.6}$$

where we have used a families version of the Poincaré-Hopf theorem, the framing principle of Theorem 4.8, and the triviality of the classifying map $\xi_h(E/B;F)\colon B \to Wh^h(R,G)$. This already explains the similarity of Corollary 3.9 and Theorem 4.7 for suspended unit sphere bundles.

5.2. Analytic Igusa-Klein torsion. — Let us assume again that $h\colon E \to \mathbb{R}$ is a fibrewise Morse function. We still consider the $\mathbb{Z}$-graded flat vector bundle $V \to B$ of 5.1 with connection ∇^V and metric g^V induced from F. The function h acts by multiplication on $F|_C$, giving rise to a selfadjoint endomorphism h of V. An endomorphism of V is called *h-upper triangular* if it maps each λ-eigenvector of h to the sum of the μ-eigenspaces with $\mu > \lambda$.

For a generic fibrewise Riemannian metric g^{TM}, the fibrewise gradient ∇h will satisfy the Smale transversality condition over an open dense subset of B. Over this subset, the Thom-Smale cochain differential is a parallel, h-upper triangular endomorphism a of V. The various differentials a over different points along a path in B are conjugated by endomorphisms of V of the type $\mathrm{id} + b$, where b is again h-upper triangular. As one moves around in a small circle on B, these endomorphisms compose to an automorphism of (V,a) that is homotopic to the identity by an h-upper triangular homotopy. These various homotopies are again related by h-upper triangular higher homotopies, and so on. If the cohomology bundle H is unipotent, then all these structures are encoded in Igusa's map $\xi_h(E/B;F)\colon B\to Wh^h(R,G)$ of Theorem 4.3.

One may also consider these algebraic structures as a singular superconnection on V. If $R=M_r(\mathbb{R})$ or $R=M_r(\mathbb{C})$, there exists a smooth flat superconnection

$$A'=\nabla^V+a_0+a_1+\cdots \tag{5.7}$$

of total degree one with h-upper triangular

$$a_j\in\Omega^j\big(B;\operatorname{End}^{1-j}(V)\big)=\Omega^j\Big(B;\bigoplus_k\operatorname{Hom}\big(V^k,V^{k+1-j}\big)\Big) \tag{5.8}$$

and an $\Omega^\bullet(B)$-linear quasiisomorphism

$$I\colon\big(\Omega^\bullet(E;F),\nabla^F\big)\to\big(\Omega^\bullet(B;V),A'\big) \tag{5.9}$$

by **[36]**. The map I arises as a modification of the classical "integration over the unstable cells", and it maps forms supported on $h^{-1}(\lambda,\infty)$ to the sum of the μ-eigenspaces of $h\in\operatorname{End}V$ with $\mu\geq\lambda$. Moreover, the pair (A',I) is uniquely determined up to contractible choice by h and g^{TM}. It is shown in **[37]** that for acyclic F, Igusa's map $\xi_h(E/B;F)\colon B\to Wh^h(R,G)$ also classifies (A',I) up to a natural notion of homotopy.

The finite-dimensional torsion form of Definition 2.2 is only well-defined for flat superconnections of the form ∇^V+a_0. In **[36]**, **[37]** a torsion form $T(A',\nabla^V,g^V)$ is constructed using the fact that $A'-\nabla^V$ is a form on B with values in a nilpotent subalgebra of End V, which may vary over B. We can still construct a metric g^H_V on

$$H=H^\bullet(E/B;F)=H^\bullet(V,a_0)\to B \tag{5.10}$$

as in Section 2.1. Then we still have

$$dT\big(A',\nabla^V,g^V\big)=\operatorname{ch}^\circ\big(V,g^V\big)-\operatorname{ch}^\circ\big(H,g^H_V\big). \tag{5.11}$$

5.2. Theorem **([36], [37])**. — *Modulo exact forms on B,*

$$\begin{aligned}\mathcal{T}\big(T^HE,g^{TM},g^F\big)&=T\big(A',\nabla^V,g^V\big)+\operatorname{ch}^\circ\big(H,g^H_{L^2},g^H_V\big)\\&\quad+\int_{E/B}(\nabla h)^*\psi\big(\nabla^{TM},g^{TM}\big)\operatorname{ch}^\circ\big(F,g^F\big)+\hat p_*\,{}^0J(T^uM-T^sM)\operatorname{rk}F.\end{aligned} \tag{5.12}$$

If both F and H carry parallel metrics, we can construct a cohomology class as in Definition 2.8. Let g^V be the induced parallel metric on V.

5.3. Definition. — The analytic Igusa-Klein torsion is defined as

$$T(E/B;F)=T\big(A',\nabla^V,g^V\big)^{[\geq2]}+\widetilde{\operatorname{ch}}^\circ\big(H,g^H,g^H_V\big)\in H^{\text{even},\geq2}(B;\mathbb{R}). \tag{5.13}$$

To justify the name, assume that g^F is parallel and the bundle $H\to B$ is a trivial flat bundle. Then both $\tau(E/B;F)$ and $T(E/B;F)$ are defined.

***5.4. Theorem* ([37]).** — *Under these assumptions,*

$$T(E/B;F) = \xi_h(E/B;F)^*\,\tau \in H^{4\bullet,\geq 4}(B;\mathbb{R}). \tag{5.14}$$

In contrast to the situation in the previous Section 5.1, these cohomology classes will be nontrivial in general. Also note that $T(A', \nabla^V, g^V)$ can still be constructed for generalised fibrewise Morse functions h as in Definition 4.1. In this context, Theorem 5.4 still holds. A generalisation of Theorem 5.2 will be proved in [**38**]. As in (5.6), one can now compare Bismut-Lott torsion and Igusa-Klein torsion.

***5.5. Theorem* ([37], [38]).** — *If F carries a parallel metric and $H \to B$ is a trivial flat bundle, then*

$$\mathcal{T}(E/B;F) = \tau(E/B;F) + tr^*_{E/B}\,{}^0J(TM)\ \mathrm{rk}\,F. \tag{5.15}$$

5.3. Axioms for higher torsions. — In this section, we consider all smooth proper submersions $p\colon E \to B$ with oriented fibres, such that the flat cohomology bundle $H^\bullet(E/B;\mathbb{C}) \to B$ is unipotent in the sense of Sections 4.2, 4.3. We will consider characteristic classes $\tau(E/B) \in H^\bullet(B;\mathbb{R})$ of such fibre bundles that are natural under pullback. Such a class is called *additive* if it satisfies a gluing formula as in Theorem 4.6.

Let $W \to E$ be an oriented real vector bundle of rank $n+1$, and let $S \to E$ be its unit n sphere bundle. Then $H^\bullet(S/B;\mathbb{C}) \to B$ is still unipotent. A characteristic class τ as above is said to satisfy the *transfer relation* if

$$\tau(S/B) = \chi(S^n)\,\tau(E/B) + tr^*_{E/B}\tau(S/E) \in H^\bullet(B;\mathbb{R}). \tag{5.16}$$

For the analytic torsion, the analogous result is a special case of Ma's transfer Theorem 3.1.

***5.6. Definition*.** — A *higher torsion invariant* in degree k is a characteristic class $\tau_k(E/B) \in H^k(B;\mathbb{R})$ for all $p\colon E \to B$ as above that is natural under pullback, additive, and satisfies the transfer relation (5.16).

***5.7. Theorem* (Igusa [44]).** — *Higher torsion invariants exist in degree $4k$ for all $k > 0$, and every higher torsion invariant is a linear combination of*

$$tr^*_{E/B}\,{}^0J(TM)^{[4k]}, \tag{even}$$

$$\textit{and}\quad \tau_{2k}(E/B;\mathbb{C}) + tr^*_{E/B}\,{}^0J(TM)^{[4k]}. \tag{odd}$$

Note that even higher torsion invariants vanish for $p\colon E \to B$ if the fibres are odd-dimensional, and vice versa. The even classes $tr^*_{E/B}\,{}^0J(TM)^{[4k]}$ are called *Miller-Morita-Mumford classes* in [**44**], because they generalise the classes for surface bundles introduced in [**57**], [**58**], [**60**]. The odd higher torsion classes are multiples of the Bismut-Lott torsion $\mathcal{T}(E/B;\mathbb{C})$ under the assumptions of Theorem 5.5.

The following result follows from the proof of uniqueness in Theorem 5.7.

***5.8. Theorem* (Igusa [44]).** — *For fibre bundles $p\colon E \to B$ as above,*

$$\tau_{2k}(E/B;\mathbb{C}) \in H^{4k}(B;\zeta'(-2k)\mathbb{Q}) \,. \tag{5.17}$$

Theorem 5.7 could in principle also be used to prove Theorem 5.5. Unfortunately, additivity of the Bismut-Lott torsion is only known as a consequence of Theorem 5.5. Another consequence of this result is a more general transfer formula for Igusa-Klein torsion as in Ma's Theorem 3.1, including the case of fibre products. By Theorems 3.7 and 5.5, Igusa-Klein torsion is also related to equivariant torsion in the case of fibre bundles with compact structure groups. Finally, Theorems 3.6 and 5.5 describe the variation of Igusa-Klein torsion under changes of the flat bundle $F \to E$.

We already mentioned the smooth Dwyer-Weiss-Williams torsion. Its definition is given in [**30**], see Section 6.2 below. In [**3**], corresponding cohomology classes in $H^{4k}(B;\mathbb{R})$ are constructed. Additivity and the transfer relation have recently been proved in [**2**]. This implies that cohomological smooth Dwyer-Weiss-Williams torsion shares all the other properties mentioned above. It also implies a more general transfer formula for Igusa-Klein torsion.

6. Dwyer-Weiss-Williams torsion

In this section, we present the homotopy theoretical approach to generalised Euler characteristics and higher torsion invariants in [**30**] and [**3**], and we sketch the proof of Theorem 1.4. Dwyer, Weiss and Williams construct three generalised Euler characteristics for fibrations $p\colon E \to B$, which contain information about the existence of a topological or even smooth bundle of manifolds that is fibre homotopy equivalent to p. If $F \to E$ is a fibrewise acyclic bundle of R-modules, then these Euler characteristics can be lifted to three different higher torsion invariants.

6.1. The topological index theorem. — The Waldhausen K-theory $A(E)$ of a space E is the K-theory of a certain category of retractive spaces over E [**65**]. It is a homotopy invariant functor, but not excisive, so it does not define a generalised homology theory. One can however define an excisive functor $A^{\%}$ by putting

$$A^{\%}(E) = \Omega^\infty(E_+ \wedge A(*)) \,. \tag{6.1}$$

Here, E_+ is the disjoint union of E and a basepoint $*$, and Ω^∞ is the infinite loop space construction. Weiss and Williams construct a natural assembly map

$$\alpha : A^{\%}(E) \longrightarrow A(E) \tag{6.2}$$

in [**66**]. We will also need the spectrum

$$Q(E_+) = \Omega^\infty \Sigma^\infty (E_+) = \lim_k \Omega^k \Sigma^k (E_+) , \tag{6.3}$$

where Σ denotes the reduced suspension. For a fibration $p\colon E \to B$, one has relative functors $A_B(E) \to B$, $A_B^{\%}(E) \to B$ and $Q_B(E_B) \to B$, which behave almost as fibrations over B where the functors above have been applied fibrewise to $p\colon E \to B$.

The *homotopy Euler characteristic*

$$\chi^h(E/B) \colon B \longrightarrow A_B(E) \tag{6.4}$$

is a section of $A_B(E) \to B$. It is defined as the class of $E \times S^0$ over E in $A_B(E)$ if the fibres of p are *homotopy finitely dominated*, that is homotopy equivalent to retracts of finite CW complexes. If B is a point, then $\chi^h(E)$ encodes precisely the Euler number and the Wall finiteness obstruction of the fibre. A flat vector bundle $F \to E$, or more generally, a bundle of finitely generated projective R-modules for some ring R, induces a map $\lambda_F \colon A(E) \to K(R)$ induced by taking homology relative to E with coefficients in F. For the proof of Theorem 1.4, one uses that the composition of maps

$$B \xrightarrow{\chi^h(E/B)} A_B(E) \longrightarrow A(E) \xrightarrow{\lambda_F} K(R) \tag{6.5}$$

classifies the fibrewise cohomology $H(E/B; F) \to B$ as a virtual bundle and thus gives the left hand side of (1.18).

If $p\colon E \to B$ is a bundle of topological manifolds, there exists a vertical tangent microbundle $TM \to E$. It has an Euler class $e(TM)$ with coefficients in $A_B^{\%}(E)$. Let $\wp$ denote the generalised fibrewise Poincaré duality [**30**]. Then one can define a *topological Euler characteristic* χ^t of p with the property that

$$\chi^t(E/B) = \wp\, e(TM) \colon B \longrightarrow A_B^{\%}(E). \tag{6.6}$$

The fibrewise assembly of (6.2) maps it to $A_B(E)$. One has a Poincaré-Hopf type index theorem.

***6.1. Theorem* (Dwyer, Weiss and Williams [30]).** — *For a bundle $p\colon E \to B$ of compact topological manifolds, the sections $\chi^h(E/B)$ and $\alpha \circ \chi^t(E/B)$ of $A_B(E) \to B$ are homotopic by a preferred path of sections.*

Conversely, if $\chi^h(E/B)$ lifts to $A_B^{\%}(E)$, then p is fibre homotopy equivalent to a bundle of compact topological manifolds.

If the vertical tangent bundle $TM \to E$ is a topological disc bundle, then $p\colon E \to B$ is called a *regular manifold bundle*, which includes the important special case of a proper submersion. In this case, one can define the Becker Euler class $b(TM)$ with coefficients in the sphere spectrum. Its fibrewise Poincaré dual gives the Becker-Gottlieb transfer, regarded as a section

$$\chi^d(E/B) = tr_{E/B} = \wp\, b(TM) \colon B \longrightarrow Q_B(E_B). \tag{6.7}$$

Even though Becker-Gottlieb transfer is already defined for fibrations with homotopy finitely dominated fibres, we can regard it as a third generalised Euler characteristic χ^d for regular manifold bundles by (6.7). There is a natural unit map $\eta\colon Q_B(E_B) \to A^{\%}_B(E)$, and we have another Poincaré-Hopf type index theorem.

***6.2. Theorem* (Dwyer, Weiss and Williams [30]).** — *For a bundle $p\colon E \to B$ of closed regular topological manifolds, the sections $\chi^t(E/B)$ and $\eta \circ tr_{E/B}$ of $A^{\%}_B(E) \to B$ are homotopic by a preferred path of sections.*

Proof of Theorem 1.4. — We regard the homotopy class of maps $E \to K(R)$ induced by the finitely generated projective R-module bundle $F \to E$. As in (6.5), this map can be written as a composition

$$E \longrightarrow Q(E) \xrightarrow{\alpha \circ \eta} A(E) \xrightarrow{\lambda_F} K(R) . \tag{6.8}$$

Thus, the right hand side of (1.18) in Theorem 1.4 is classified by the composition

$$B \xrightarrow{tr_{E/B}} Q_B(E) \xrightarrow{\alpha \circ \eta} A_B(E) \longrightarrow A(E) \xrightarrow{\lambda_F} K(R) . \tag{6.9}$$

By Theorems 6.1 and 6.2, this map is homotopic to (6.5), which classifies the left hand side of (1.18). This completes the proof. □

One notes that both sides of (1.18) in Theorem 1.4 are defined for a fibration $p\colon E \to B$ with homotopy finitely dominated fibres. However, for Theorem 6.2 one needs the regular structure coming from the smooth bundle structure. It is somewhat surprising that the existence of a smooth fibre bundle structure is necessary to compare the various Euler characteristics above.

***6.3. Theorem* (Dwyer, Weiss and Williams [30]).** — *Let $p\colon E \to B$ be a fibration with homotopy finitely dominated fibres. If $\chi^h(E/B)$ lifts to $Q_B(E_B) \to B$, then p is fibre homotopy equivalent to a bundle of smooth manifolds.*

6.2. Topological higher Reidemeister torsion. — Suppose that $F \to E$ is a bundle of finitely generated projective R-modules that is fibrewise acyclic. Then the three Euler characteristics $\chi^h(E/B)$, $\chi^t(E/B)$ and $tr_{E/B}$ of the previous subsection can be lifted to higher Reidemeister torsions.

Assume first that $p\colon E \to B$ is a fibration with homotopy finitely dominated fibres. If F is fibrewise acyclic, then the composition in (6.5) is canonically homotopic to the trivial map $B \to K(R)$. For a single space M, this gives an element $\tau^h(M;F)$ in the homotopy fibre

$$\Phi^h(M;F) = \operatorname{hofib}(\lambda_F) \tag{6.10}$$

of $\lambda_F\colon A(M) \to K(R)$ over $\chi^h(M) \in A(M)$. For the fibration p, we get a lift

$$\tau^h(E/B;F)\colon B \longrightarrow \Phi^h(E/B;F) = \operatorname{hofib}_B(\lambda_F) \tag{6.11}$$

of $\chi^h(E/B)$, where the fibres of $\Phi^h(E/B;F) \to B$ are the homotopy fibres of λ_F.

If $p\colon E \to B$ is a bundle of topological manifolds, we similarly get a lift of $\chi^t(E/B) : B \to A^{\%}_B(E)$ to

$$\tau^t(E/B;F)\colon B \longrightarrow \Phi^t(E/B;F) = \mathrm{hofib}_B(\lambda_F \circ \alpha). \tag{6.12}$$

If $p\colon E \to B$ is a bundle of smooth or regular manifolds, one gets a lift of $\chi^d(E/B) = tr_{E/B}\colon B \to Q_B(E_B)$ to

$$\tau^d(E/B;F)\colon B \longrightarrow \Phi^d(E/B;F) = \mathrm{hofib}_B(\lambda_F \circ \alpha \circ \eta). \tag{6.13}$$

6.4. Definition. — If $F \to E$ is fibrewise acyclic, then $\tau^h(E/B;F)$, $\tau^t(E/B;F)$ and $\tau^d(E/B;F)$ are called the *homotopy, topological* and *smooth Dwyer-Weiss-Williams torsion*, respectively, whenever they are defined.

The natural maps α and η induce maps

$$\begin{aligned} \alpha\colon \Phi^t(E/B;F) &\longrightarrow \Phi^h(E/B;F) \\ \text{and}\quad \eta\colon \Phi^d(E/B;F) &\longrightarrow \Phi^t(E/B;F). \end{aligned} \tag{6.14}$$

By Theorems 6.1 and 6.2, the Dwyer-Weiss-Williams torsions are related up to a preferred fibrewise homotopy by

$$\tau^h(E/B;F) \sim \alpha\tau^t(E/B;F) \quad \text{and} \quad \tau^t(E/B;F) \sim \eta\tau^d(E/B;F) \tag{6.15}$$

if they are defined.

We will see in the next Section 7 that Bismut-Lott torsion and Igusa-Klein torsion can detect different smooth bundle structures on a given topological manifold bundle $p\colon E \to B$. Thus, $\mathcal{T}(E/B;F)$ and $\tau(E/B;F)$ cannot be recovered from $\tau^h(E/B;F)$ or $\tau^t(E/B;F)$. On the other hand, we do not know any example yet where the difference $\tau(E/B;F_1) - \tau(E/B;F_0)$ depends on the smooth fibre bundle structure if F_0, $F_1 \to E$ are two flat vector bundles of the same rank with unipotent fibrewise cohomology bundles. It is thus natural to ask if one can recover $\tau(E/B;F_1) - \tau(E/B;F_0)$ or $\mathcal{T}(E/B;F_1) - \mathcal{T}(E/B;F_0)$ from $\tau^t(E/B;F_1) - \tau^t(E/B;F_0)$ or even from $\tau^h(E/B;F_1) - \tau^h(E/B;F_0)$. Let us note at this point that additivity of the topological Dwyer-Weiss-Williams torsion $\tau^t(E/B;F)$ and of the underlying Euler characteristic $\chi^t(E/B)$ of (6.6) has been established in [**1**].

In [**3**], a cohomological version of $\tau^d(E/B;F)$ is constructed. It is still defined if $H^\bullet(E/B;F) \to B$ is a unipotent bundle. The following result has recently been proved using Igusa's axioms.

***6.5. Theorem* (Badzioch, Dorabiała, Klein and Williams [2]).** — *For any $k > 0$, the cohomological smooth Dwyer-Weiss-Williams torsion of* [**3**] *in degree $4k$ is nontrivial and proportional to the Igusa-Klein torsion in the same degree.*

In addition, it would be nice to have a natural map from $\Phi^d(E/B;F)$ to Igusa's Whitehead space $Wh^h(R,G)$ that sends $\tau^d(E/B;F)$ to the map $\xi(E/B;F)$ of (4.4).

7. Exotic smooth bundles

Consider two smooth proper submersions $p_i\colon E_i \to B$ for $i = 0, 1$. It is possible that the fibres of p_0 and p_1 are diffeomorphic, and that there exists a homeomorphism $\varphi\colon E_0 \to E_1$ such that $p_0 = p_1 \circ \varphi$, but no such diffeomorphism. If this is the case, then p_0 and p_1 are isomorphic as topological, but not as smooth fibre bundles over B. In this case, we will say that p_1 gives an *exotic smooth bundle structure* on the bundle p_0. Of course, in many cases there is no distinguished standard smooth bundle structure, so the term "exotic" may be misleading. Higher torsion invariants detect some exotic smooth bundle structures, as we will explain in this section. We also recall Heitsch-Lazarov torsion, which might be useful to detect exotic smooth structures on foliations.

7.1. Hatcher's example. — It is well known that the higher stable homotopy groups of spheres are finite, whereas some higher homotopy groups of the orthogonal group are not. More precisely, if m is sufficiently large with respect to k, then the kernel of the J-homomorphism

$$J_{4k-1}\colon \pi_{4k-1}(O(m)) \longrightarrow \pi_{n+4k-1}(S^m) \tag{7.1}$$

contains an infinite cyclic subgroup. An element $\gamma \in \ker J_{4k-1}$ can be used to construct a family of embeddings $\tilde\gamma_q\colon S^m \times D^{n-1} \to S^m \times D^{n-1}$ for $q \in D^{4k}$, if n is sufficiently large, which are given by a pair of linear maps $S^m \to S^m$ and $D^{n-1} \to D^{n-1}$ for $q \in S^{4k-1} = \partial D^{4k}$. Glueing $D^{m+1} \times D^{n-1}$ to $S^m \times D^m$ along $S^m \times D^{n-1} \subset \partial(D^{m+1} \times D^{n-1})$ for all $q \in D^{4k}$, one obtains an $(m+n)$-disc bundle over D^{4k} together with a canonical trivialisation over S^{4k-1}. Thus, this bundle can be extended to a smooth disc bundle

$$p_\gamma\colon E_\gamma \longrightarrow S^{4k} = D^{4k} \cup_{S^{4k-1}} D^{4k}, \tag{7.2}$$

as described in [**42**] and [**36**].

This disc bundle was first constructed by Hatcher. Bökstedt proved that for $\gamma \neq 0$, the bundle p_γ is homeomorphic, but not diffeomorphic to a trivial disc bundle in the sense above [**19**]. Note that p_γ carries a fibrewise Morse function $h\colon E_\gamma \to \mathbb{R}$ with two critical points of index 0 and m in the part $S^m \times D^n$ of the fibre, and another one of index $m+1$ on $D^{m+1} \times D^{n-1}$. The corresponding family of Thom-Smale complexes is trivial, but h is not framed. If $W_\gamma \to S^{4k}$ denotes the $\mathbb{R}^n$-bundle with clutching function $\gamma|_{S^{4k-1}}$, Igusa's framing principle gives

$$\tau(E_\gamma/S^{4k};\mathbb{C}) = 2(-1)^m\, {}^0J(W_\gamma) \neq 0 \in H^{4k}(S^{4k},\mathbb{R}), \tag{7.3}$$

see Theorem 4.8 and [**42**].

To construct a smooth proper submersion, we take the fibrewise double $DE_\gamma \to S^{4k}$. Its Igusa-Klein torsion of Definition 4.4 is given by

$$\tau(DE_\gamma/S^{4k};\mathbb{C}) = 2\big((-1)^m - (-1)^n\big)\,{}^0J(W_\gamma), \tag{7.4}$$

which vanishes precisely if the fibres are even-dimensional. By Theorem 5.5, this agrees with the Bismut-Lott torsion $\mathcal{T}(DE_\gamma/S^{4k};\mathbb{C})$ of Definition 2.8, see [**36**].

If $p\colon E \to B$ is a smooth proper submersion with $\dim B = 4k$ and $\dim M$ odd and sufficiently large, then one can take out r copies of $D^{4k} \times D^{\dim M}$ from E and glue in r copies of $E_\gamma|_{D^{4k}}$ instead. This gives an exotic smooth bundle $p_r\colon E_r \to B$. If either Bismut-Lott torsion or Igusa-Klein torsion are defined for some flat bundle $F \to E$, then this torsion will change by $\pm 2r\,{}^0J(W_\gamma)\,\mathrm{rk}\,F \in H^{4k}(B;\mathbb{R})$ if B is oriented. Igusa also constructs a *difference torsion* satisfying

$$\tau(E_r/B, E/B; F) = \pm 2r\,{}^0J(W_\gamma)\,\mathrm{rk}\,F \tag{7.5}$$

even if $H^\bullet(E/B;F) \cong H^\bullet(E_r/B;F) \to B$ is not a unipotent bundle.

We still assume that B is oriented and that $\dim M$ is odd and sufficiently large. The gluing construction above can be generalised to construct a discrete family of exotic smooth bundles $p_\nu\colon E_\nu \to B$ such that the values of their difference torsions $\tau(E_r/B, E/B; F)$ form a lattice in the space

$$\bigoplus_{k=1}^{\infty} \wp\,\mathrm{im}\Big(p_*\colon H_{\dim B-4k}(E) \longrightarrow H_{\dim B-4k}(B)\Big) \subset \bigoplus_{k=1}^{\infty} H^{4k}(B) \tag{7.6}$$

of classes that are Poincaré dual to classes pushed down from E. This is an ongoing project with Igusa.

7.2. The space of stable exotic smooth structures.

— There are two natural questions: can higher torsion detect all exotic smooth bundle structures, and can all these structures be constructed? To answer these questions, one wants to understand the space of all such exotic smooth bundle structures. As Williams pointed out, a certain stable version of this space can be analysed using the methods of the paper [**30**].

We start with a bundle $p\colon E \to B$ of compact topological n-manifolds, equipped with a vector bundle $V \to E$ of rank n. A smooth manifold bundle $p'\colon E' \to B$ is called a *fibrewise tangential smoothing* of $(E/B, V)$ if there exists a homeomorphism $\varphi\colon E' \to E$ with $p' = p \circ \varphi$ and a vector bundle isomorphism $\ker(dp') \to V$ over φ. Let $\mathcal{S}_B(E, V)$ denote the space of all fibrewise tangential smoothings. By considering total spaces of closed, even-dimensional linear disk bundles $\pi : D(\xi) \subset \xi \to E$ after rounding off the corners, we construct the space of *stable fibrewise tangential smoothings*

$$\mathcal{S}^s_B(E,V) = \varinjlim \mathcal{S}_B(D(\xi), \pi^*(V \oplus \xi))\ , \tag{7.7}$$

where the limit is taken over all vector bundles.

Let $\mathscr{H}(*)$ be the stable h-cobordism space, and construct a fibration $\mathscr{H}^{\%}_B(E)$ with fibres $\Omega^\infty(M_+ \wedge \mathscr{H}(*))$ as in (6.1).

7.1. Theorem **(Dwyer, Weiss and Williams [30])**. — *If $(E/B, V)$ admits stable fibrewise tangential smoothings, then $\mathscr{S}^s_B(E,V)$ is homotopy equivalent to the space of sections of $\mathscr{H}^{\%}_B(E) \to B$.*

In other words, the group $\pi_0\Gamma_B\mathscr{H}^{\%}_B(E)$ of homotopy classes of sections acts simply transitively on the isomorphism classes of stable fibrewise tangential smoothings.

7.2. Theorem **(Igusa and G.)**. — *If the fibres and base of $p\colon E \to B$ are closed oriented manifolds, then*

$$\pi_0\big(\mathscr{S}^s_B(E,V)\big) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{k=1}^{\infty} H_{\dim B-4k}(E;\mathbb{Q}). \tag{7.8}$$

In special cases, this was already known, see [**32**]. Thus, if $p_i\colon D(\xi_i) \to B$ are stable fibrewise tangential smoothings for $i = 0$, 1, we can define the *relative Dwyer-Weiss-Williams torsion* $\tau^{d/t}_{2k}(p_0,p_1) \in H^{4k}(B;\mathbb{Q})$ for $k \geq 1$ as the Poincaré dual of the image of the corresponding difference class in $H_{\dim B-4k}(B;\mathbb{Q})$.

7.3. Theorem **(Igusa and G.)**. — *In the situation above, the Igusa-Klein difference torsion is a scalar multiple of the relative Dwyer-Weiss-Williams torsion.*

Details will appear elsewhere.

7.4. Remark. — In general, the space in (7.8) has higher rank than the space in (7.6). This implies that higher torsion cannot detect all rational stable fibrewise tangential smoothings. It does not help to chose different flat vector bundles $F \to E$ either. One reason is that E could be simply connected. Another reason is the fact that in (7.5) and its analogue in the more general setting of (7.6), the flat vector bundle F only contributes by its rank.

7.5. Remark. — Thus the difference of the Igusa-Klein or Bismut-Lott torsions of $E \to B$ with two different flat vector bundles of the same rank seems to be independent of the smooth structure in the examples known so far. This observation leads to the question if this difference can be computed already from the topological or the homotopy Dwyer-Weiss-Williams torsion. Theorem 3.6 shows that under special assumptions, it can even be computed using the Becker-Gottlieb transfer only.

7.6. Remark. — In the special case of aspherical fibres M, Lott defines a noncommutative higher analytic torsion form with coefficients in a certain subalgebra of $C^*_{\mathrm{r}}\pi_1(M)$

in [51]. Lott asks if this invariant detects all rational exotic structures. To the author's knowledge, this question is still open. More generally, one would like to have a similar invariant for arbitrary fibres that can detect all rational stable exotic smooth structures.

7.3. Heitsch-Lazarov torsion for foliations. — Let E be a smooth closed manifold with a smooth foliation $\mathcal{F}$. Since in general, the space of leaves $E/\mathcal{F}$ is ill-behaved, we consider a foliation groupoid whose elements are classes of paths on the leaves of $\mathcal{F}$. We will assume that this groupoid $\mathcal{G}$ lies between the homotopy and the holonomy groupoid, and that it is Hausdorff and thus given by a smooth manifold and two submersions r, $s\colon \mathcal{G} \to E$. We will assume that the strong Novikov-Shubin invariants of the leafwise Hodge-Laplacians are positive.

Heitsch and Lazarov give a generalisation of Bismut-Lott torsion in a setting that essentially avoids noncommutative methods [40]. Thus, let $\Omega_c^\bullet(E/\mathcal{F})$ denote the Häfliger Forms, that is, the coinvariants under $\mathcal{F}$ in the space of compactly supported de Rham forms on a complete transversal to $\mathcal{F}$. The cohomology $H_c^\bullet(E/\mathcal{F})$ of $(\Omega_c^\bullet(E/\mathcal{F}), d)$ resembles the compactly supported de Rham cohomology of a manifold.

Let $F \to E$ be a flat vector bundle with metric g^F. If one fixes a complement $T^H E$ to $T\mathcal{F} \subset TE$ and a leafwise metric $g^{T\mathcal{F}}$, there exists a natural connection $\nabla^{T\mathcal{F}}$ on $T\mathcal{F} \to E$. Using integration along the leaves, one defines

$$\int_{\mathcal{F}} e\big(T\mathcal{F}, \nabla^{T\mathcal{F}}\big)\, \mathrm{ch}^\circ\big(F, g^F\big) \in \Omega_c^\bullet(E/\mathcal{F}). \tag{7.9}$$

Let $P\colon \Omega^\bullet(\mathcal{F}; F) \to H^\bullet = H^\bullet(\mathcal{F}; F)$ denote the projection of the leafwise forms with values in F onto the harmonic forms. Using P, one defines

$$\mathrm{ch}^\circ\big(H, g^H_{L^2}\big) \in \Omega_c^\bullet(E/\mathcal{F}) \tag{7.10}$$

in analogy with (1.5). As in Definition 2.4, Heitsch and Lazarov then construct a higher analytic torsion form $\mathcal{T}(T^H E, g^{T\mathcal{F}}, g^F) \in \Omega_c^\bullet(E/\mathcal{F})$.

***7.7. Theorem* (Heitsch and Lazarov [40]).** — *In the situation above,*

$$d\mathcal{T}\big(T^H E, g^{T\mathcal{F}}, g^F\big) = \int_{\mathcal{F}} e\big(T\mathcal{F}, \nabla^{T\mathcal{F}}\big)\mathrm{ch}^\circ\big(F, g^F\big) - \mathrm{ch}^\circ\big(H, g^H_{L^2}\big) \in \Omega_c^\bullet(E/\mathcal{F}). \tag{7.11}$$

Heitsch and Lazarov need a large positive lower bound for the strong leafwise Novikov-Shubin invariants. Thus they only regard examples with compact leaves. It seems however, that uniform positivity of the Novikov-Shubin invariants is sufficient to prove Theorem 7.7. This is an ongoing joint project with Azzali.

Given $\mathcal{F}$ as above with $\dim \mathcal{F}$ odd and $\dim E - \dim \mathcal{F} = 4k$, one can remove r disjoint foliated regions $D^{4k} \times D^{\dim \mathcal{F}}$ and glue in r copies of the disc bundle $E_\gamma|_{D^{4k}}$ of Section 7.1. It would be interesting to know if the Heitsch-Lazarov torsion $\mathcal{T}(T^HE, g^{T\mathcal{F}}, g^F)$ then changes by $\pm 2r\,{}^0J(W_\gamma)\operatorname{rk} F$ as in (7.5). In this case, we would have a new foliation $\mathcal{F}_r$ on E that is homeomorphic, but not diffeomorphic to the original foliation $\mathcal{F}$, and thus "exotic". Note that $\mathcal{F}$ and $\mathcal{F}_r$ have the same dynamics, since on a complete transversal that does not meet the modified regions, nothing changes. More generally, one would like to classify these exotic smooth structures, construct as many as possible explicitly, and see which of them can be distinguished by Heitsch-Lazarov torsion, or a noncommutative generalisation of it.

8. The hypoelliptic Laplacian and Bismut-Lebeau torsion

In **[8]** and **[15]**, Bismut and Lebeau consider an analytic torsion form that is defined using a hypoelliptic operator $\mathfrak{A}^2_{b,\pm}$ on differential forms on the total space T^*M of the vertical cotangent bundle of the family $p\colon E \to B$. While the fibrewise Hodge Laplacian generates a Brownian motion on the fibres M of p, the operator $\mathfrak{A}^2_{b,\pm}$ generates a stochastic version of the geodesic flow on T^*M, where the velocities are perturbed by a Brownian motion for $b \in (0,\infty)$. As $b \to 0$, this process converges in an appropriate sense to the classical Brownian motion on M. On the other hand, as $b \to \infty$, one recovers the unperturbed geodesic flow. One motivation to study the family of operators $(\mathfrak{A}_b)$ is Fried's conjecture, which relates the torsion of a single manifold M to the closed orbits of a certain class of flows on M, see **[34]** for an overview.

8.1. The hypoelliptic Laplacian on the cotangent bundle. — Let $p\colon E \to B$ be a smooth proper submersion, and let $F \to E$ be a flat vector bundle as before. Let $\pi\colon T^*M \to E$ denote the vertical cotangent bundle. If one fixes $T^HE \subset TE$ and g^{TM} as before, one obtains a splitting

$$TT^*M \cong \pi^*(T^HE \oplus TM \oplus T^*M) \tag{8.1}$$

and a corresponding splitting of the bundle $\Omega^\bullet(T^*M/B; \pi^*F)$. We regard the bundle $\tilde{E} = E \times (0,\infty)^2 \to \tilde{B} = B \times (0,\infty)^2$, and let (b,t) denote the coordinates of $(0,\infty)^2$.

On the vertical part $TM \oplus T^*M$ of TT^*M, one defines a metric $\mathfrak{g}$ by

$$\mathfrak{g} = \begin{pmatrix} \frac{1}{t}\, g^{TM} & \mathrm{id}_{T^*M} \\ \mathrm{id}_{TM} & 2t\, g^{T^*M} \end{pmatrix} : TM \oplus T^*M \longrightarrow \big(TM \oplus T^*M\big)^* . \tag{8.2}$$

Together with a metric g^F on $F \to E$ and the symplectic volume form on $TM \oplus T^*M$, one obtains an L^2-metric $\mathfrak{g}$ on the bundle $\Omega^\bullet_0(T^*M/B; \pi^*F) \to B$ of compactly supported forms. On this bundle, there exists a $\mathfrak{g}$-isometric involution u with

$$(u\alpha)_{(q,v)} = \begin{pmatrix} \mathrm{id}_{TM} & 2t\, g^{T^*M} \\ 0 & -\mathrm{id}_{T^*M} \end{pmatrix} \alpha_{(q,-v)} \tag{8.3}$$

for all $q \in E$, $v \in T_q^*M$, and thus, one can define a nondegenerate Hermitian form $\mathfrak{h}$ of signature (∞, ∞) by

$$\mathfrak{h}(\alpha, \beta) = \mathfrak{g}(u\alpha, \beta). \tag{8.4}$$

As before, let $\mathbb{A}' = d_E$ denote the total exterior derivative on $\Omega^\bullet(T^*M; \pi^*F)$, regarded as a superconnection on the bundle $\Omega_0^\bullet(T^*M/B; \pi^*F)$. Define $\bar{\mathbb{A}}'$ as the $\mathfrak{h}$-adjoint of $\mathbb{A}'$. Again, $\mathbb{A}'$ and $\bar{\mathbb{A}}'$ are flat superconnections.

One has the canonical one-form $\vartheta \in \Omega^1(T^*M)$, with

$$d\vartheta = \omega^H + \omega^V \in \Gamma\big(\pi^*\big(\Lambda^2(T^H E)^* \oplus T^*M \otimes TM\big)\big) \subset \Omega^2(T^*M), \tag{8.5}$$

where ω^V is the standard symplectic form on the cotangent bundle of each fibre of p. Consider the Hamiltonians

$$\mathscr{H}_\pm(q, v) = \pm \frac{t^2}{2b^2} \, \|v\|^2_{T^*M}. \tag{8.6}$$

Then for $b = t = 1$, the ω^V-gradient of $\mathscr{H}_+$ is the generator

$$\mathrm{sgrad}\mathscr{H}_+|_{(q,v,b,t)} = g^{T^*M}(v) \in T_q M \tag{8.7}$$

of the geodesic flow on T^*M over the fibres M of p.

Regard the flat superconnections

$$\mathfrak{A}'_\pm = e^{-(\mathscr{H}_\pm - \omega^H)} \mathbb{A}' e^{\mathscr{H}_\pm - \omega^H} \quad \text{and} \quad \bar{\mathfrak{A}}'_\pm = e^{\mathscr{H}_\pm - \omega^H} \bar{\mathbb{A}}' e^{-(\mathscr{H}_\pm - \omega^H)}. \tag{8.8}$$

Then there exists an $\mathfrak{h}$-selfadjoint superconnection $\mathfrak{A}_\pm$ and an $\mathfrak{h}$-skew adjoint endomorphism $\mathfrak{X}$ with

$$\mathfrak{A}_\pm = \frac{1}{2}(\mathfrak{A}'_\pm + \bar{\mathfrak{A}}'_\pm) \quad \text{and} \quad \mathfrak{X}_\pm = \bar{\mathfrak{A}}'_\pm - \mathfrak{A}'_\pm. \tag{8.9}$$

One finally defines

$$\mathfrak{A}_{b,t,\pm} = \mathfrak{A}_\pm|_{B\times\{(b,t)\}} \quad \text{and} \quad \mathfrak{X}_{b,t,\pm} = \mathfrak{X}_\pm|_{B\times\{(b,t)\}} \,. \tag{8.10}$$

The operator $\mathfrak{A}^2_{b,t,\pm} = -\mathfrak{X}^2_{b,t,\pm}$ is the sum of a harmonic oscillator along the fibres of $\pi\colon T^*M \to E$, the Lie derivative by $\mathrm{sgrad}\,\mathscr{H}_\pm$, and some terms of lower order or smaller growth at infinity. In particular, $\frac{\partial}{\partial u} - \mathfrak{A}^2_{b,t,\pm}$ is hypoelliptic in the sense of Hörmander, for an extra variable $u \in \mathbb{R}$. By **[15]**, the restriction $\mathfrak{A}^{[0],2}_{b,t,\pm}$ of the operator $\mathfrak{A}^2_{b,t,\pm}$ to the fibres of $(p\circ\pi)\colon T^*M \to B$ has discrete spectrum and compact resolvent. Recall that $n = \dim M$.

8.1. Theorem **(Bismut and Lebeau [15])**. — *The operator* $\mathfrak{A}'^{[0]}_{b,t,\pm}$ *acts on the generalised 0-eigenspace* $\ker(\mathfrak{A}^{[0],2N}_{b,t,\pm})$ *of the operator* $\mathfrak{A}^{[0],2}_{b,t,\pm}$, *where* $N \gg 0$, *and for all* $k \in \mathbb{Z}$ *and all* b, $t > 0$,

$$\begin{aligned} H^k(\ker(\mathfrak{A}^{[0],2N}_{b,+}), \mathfrak{A}'^{[0]}_{b,+}) &\cong H^k_+(E/B;F) = H^k(E/B;F), \\ H^{k+n}(\ker(\mathfrak{A}^{[0],2N}_{b,-}), \mathfrak{A}'^{[0]}_{b,-}) &\cong H^{k+n}_-(E/B;F) = H^k(E/B;F \otimes o(TM)) \,. \end{aligned} \tag{8.11}$$

Note that $H^{n+k}_-(E/B;F^*) \cong H^{n-k}_+(E/B;F)$ by fibrewise Poincaré duality.

8.2. Bismut-Lebeau torsion. — We can now explain the higher torsion $\mathcal{T}_{b,\pm}$ of the cotangent bundle defined by Bismut and Lebeau in [**15**]. We will see how it fits into Igusa's axiomatic framework of [**44**], see Section 5.3 above. At the moment, Bismut-Lebeau torsion is only defined for small positive values of b. A definition for all $b > 0$ would be nicer because as the hypoelliptic Laplacian converges to the generator of the geodesic flow as $b \to \infty$, one hopes to recover some information about the fibrewise geodesic flow from the higher torsion.

The Hermitian form $\mathfrak{h}$ of (8.4) restricts to a nondegenerate Hermitian form $\mathfrak{h}^{H_\pm}_b$ on $H^\bullet_\pm(E/B;F)$, so one still has characteristic forms $\mathrm{ch}^\circ(H^\bullet_\pm(E/B;F), \mathfrak{h}^{H_\pm}_b) \in \Omega^{\mathrm{odd}}(B)$.

Bismut and Lebeau also show that the heat operator $e^{-\mathfrak{A}^2_{b,t,\pm}}$ is a smoothing operator and of trace class. Analytic torsion forms $\mathcal{T}_{b,\pm}(T^HE, g^{TM}, g^F) \in \Omega^{\mathrm{even}}B$ can thus be defined as in Section 2.2. They satisfy the following analogue of Theorem 1.5.

8.2. Theorem **(Bismut and Lebeau [15])**. — *For* $b > 0$ *sufficiently small,*

$$d\mathcal{T}_{b,\pm}\big(T^HE, g^{TM}, g^F\big) = \int_{E/B} e\big(TM, \nabla^{TM}\big)\,\mathrm{ch}^\circ\big(F, g^F\big) - \mathrm{ch}^\circ\big(H^\bullet_\pm(E/B;F), \mathfrak{h}^{H_\pm}_b\big) \,. \tag{8.12}$$

Note that $\mathrm{ch}^\circ(H^\bullet_-(E/B;F)) = (-1)^n \,\mathrm{ch}^\circ(H^\bullet_+(E/B;F))$ by (8.11). This gives no contradiction in (8.12) because for odd n, the first term on the right hand side vanishes.

It is now natural to compare $\mathcal{T}_{b,\pm}$ with the Bismut-Lott torsion $\mathcal{T}$ of Section 2.2. Recall that we have defined a metric $g^H_{L^2}$ on $H(E/B;F)$ in Section 1.3. Let $g^{H_\pm}_{L^2}$ denote the induced metric on $H_\pm(E/B;F)$.

8.3. Theorem **(Bismut and Lebeau [15])**. — *For* $b > 0$ *sufficiently small, the Hermitian form* $(\pm 1)^n\,\mathfrak{h}^{H_\pm}_b$ *is positive definite, and modulo exact forms on* B *one has*

$$\mathcal{T}_{b,\pm}\big(T^HE, g^{TM}, g^F\big) = (\pm 1)^n\,\mathcal{T}\big(T^HE, g^{TM}, g^F\big) - \widetilde{\mathrm{ch}}^\circ\big(H_\pm(E/B;F), g^H_{L^2}, \mathfrak{h}^{H_\pm}_b\big) \pm \mathrm{tr}^*_{E/B}\,{}^0J(TM)\,\mathrm{rk}\,F \,. \tag{8.13}$$

8.4. Remark. — If one applies the exterior derivative d on B to (8.13), the result is compatible with Theorems 1.5 and 8.2, which explains the first two terms on the right hand side of (8.13).

The last term is a Miller-Morita-Mumford class in Igusa's sense. It cannot be guessed from Theorems 1.5 and 8.2. But if we believe that $\mathscr{T}_{b,\pm}\big(T^HE, g^{TM}, g^F\big)$ is a higher torsion invariant in the sense of Definition 5.6, then it is not surprising that such a class appears here. On the other hand, it is surprising that $\mathscr{T}_{b,\pm}\big(T^HE, g^{TM}, g^F\big)$ is given by the same linear combination of the classes in Theorem 5.7 as Igusa-Klein torsion in the following special case. If F is acyclic and $E \to B$ admits a fibrewise Morse function, then

$$\mathscr{T}_{b,-}\big(T^HE, g^{TM}, g^F\big) = (-1)^n\,\tau(E/B;F) \tag{8.14}$$

by comparison with Theorem 5.5. If h has trivial stable tangent bundle T^sM in an analogous sense to Definition 4.1, the class $\mathscr{T}_{b,+}\big(T^HE, g^{TM}, g^F\big)$ equals $\xi_h(E/B;F)^*\,\tau$ up to sign. Conjecturally, these equations hold even if there is no fibrewise Morse function. This coincidence indicates a relation between the Bismut-Lebeau analytic torsion and Igusa-Klein torsion that is even deeper than Theorems 5.5 or 5.7.

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Astérisque
328, 2009, p. 207–221

BOUNDARIES OF POSITIVE HOLOMORPHIC CHAINS AND THE RELATIVE HODGE QUESTION

by

F. Reese Harvey & H. Blaine Lawson, Jr

Dedicated to Jean-Michel Bismut on the occasion of his 60^{th} birthday

Abstract. — We characterize the boundaries of positive holomorphic chains in an arbitrary complex manifold.

We then consider a compact oriented real submanifold M of dimension $2p-1$ in a compact Kähler manifold X and address the question of which relative homology classes in $H_{2p}(X, M; \mathbf{Z})$ are represented by positive holomorphic chains. Specifically, we define what it means for a class $\tau \in H_{2p}(X, M; \mathbf{Z})$ to be of type (p, p) and positive. It is then shown that τ has these properties if and only if $\tau = [T + S]$ where T is a positive holomorphic chain with $dT = \partial\tau$ and S is a positive (p, p)-current with $dS = 0$.

Résumé **(Bords de chaînes holomorphes positives et la question de Hodge relative)**

On donne une caractérisation des chaînes holomorphes positives dans une variété complexe générale.

On considère une sous-variété compacte orientée réelle M de dimension $2p-1$ dans une variété X compacte kählerienne, et on étudie les classes d'homologie relative $H_{2p}(X, M; \mathbf{Z})$ qui sont représentables par une chaîne holomorphe positive. On décrit les classes $\tau \in H_{2p}(X, M; \mathbf{Z})$ de type (p, p) positives. On montre que τ possède cette propriété si et seulement si $\tau = [T + S]$ où T est une chaîne holomorphe telle que $dT = \partial\tau$ et S est un courant (p, p) positif tel que $dS = 0$.

1. Introduction

In the first part of this note we establish a general result concerning boundaries of positive holomorphic chains in a complex manifold X. In the second part we address the "Relative Hodge Question": *When is a homology class $\tau \in H_{2p}(X, M; \mathbf{Z})$ represented by a positive holomorphic chain?* Assuming M is a real $(2p-1)$-dimensional submanifold we are able to give a surprisingly full answer.

2010 Mathematics Subject Classification. — 32C25, 32V99, 32Q99.
Key words and phrases. — Positive holomorphic chain, boundary, relative homology.

Partially supported by the N.S.F.

We begin our discussion of the first part by presenting some interesting special cases which are quite different in nature. The first main theorem is then formulated and proved in Section 2.

To start, suppose X compact and let Γ be a current of dimension $2p-1$ in X. By a positive holomorphic p-chain with boundary Γ we mean a finite sum $V = \sum_k m_k V_k$ with $m_k \in \mathbf{Z}^+$ and V_k an irreducible complex analytic variety of dimension p and finite volume in $X - \mathrm{supp}\Gamma$, such that $dV = \Gamma$ as currents on X.

Equip X with a hermitian metric and let ω denote its associated $(1,1)$-form. A real $(2p-1)$-form α will be called a (p,p)*-positive linking form* if

$$d^{p,p}\alpha + \frac{1}{p!}\omega^p \geq 0 \qquad \text{(strongly positive)}$$

where $d^{p,p}\alpha$ denotes the (p,p)-component of $d\alpha$. (See **[11]** or **[10]** for the definition of strongly and weakly positive currents.) The numbers $\int_\Gamma \alpha$ with α as above, will be called the (p,p)*-linking numbers* of Γ.

Theorem 1.1. — *Let $\Gamma = \sum_{k=1}^N n_k\Gamma_k$ be an integer linear combination of compact, mutually disjoint, C^1-submanifolds of dimension $2p-1$ in X, each of which has a real analytic point. Then $\Gamma = dV$ where V is a positive holomorphic p-chain if and only if the (p,p)-linking numbers of Γ are bounded below.*

Note 1.2. — The condition that the linking numbers of Γ are bounded below is easily seen to be independent of the choice of hermitian metric on X. However, for any given metric we have the precise statement that Γ bounds a positive holomorphic chain of mass $\leq \Lambda$ if and only if

$$\int_\Gamma \alpha \geq -\Lambda \qquad \text{for all } (p,p)\text{-positive linking forms } \alpha \tag{1.1}$$

Note 1.3. — We shall actually prove the theorem in the more general situation where Γ is allowed to have a “scar" set and the real analyticity assumption is replaced by a weaker “push-out” hypothesis (see Section 2). When $p > 1$, this hypothesis is satisfied at any point where the boundary is smooth and its Levi form has at least one negative eigenvalue. In all these cases, one has regularity at almost all points of Γ. This boundary regularity is discussed in **[12]** and **[10]**.

Remark 1.4. — When X is a projective surface and $p=1$, a much stronger result is conjectured: namely, Γ bounds a positive holomorphic 1-chain if and only if

$$\int_\Gamma d^c u \geq -\Lambda \qquad \text{for all } u \in C^\infty(X) \text{ with } dd^c u + \omega \geq 0. \tag{1.2}$$

Functions u with $dd^c u + \omega \geq 0$ are called *quasi-plurisubharmonic*. They were introduced by Demailly and play an important role in complex analysis **[2]**, **[7]**. Condition (1.2) is equivalent to the condition that

$$\frac{1}{\ell}\mathrm{Link}_{\mathbf{P}}(\Gamma, Z) \geq -\Lambda \qquad \text{for all positive divisors } Z \text{ in } X - \Gamma$$

of sections $\sigma \in H^0(X, \mathcal{O}(\ell))$, $\ell > 0$, where $\mathrm{Link}_{\mathbf{P}}$ denotes the *projective linking number* introduced in [**17**]. In this form the conjecture extends to all dimensions and codimensions (for X projective) and is a consequence of the above case: $p = 1$ in surfaces. All this is established in [**16, 17**] where the conjectures are also related to the projective hull introduced in [**15**].

Although the hypothesis of Theorem 1.1 is conjecturally too strong for projective manifolds, it does give the "correct" result in the general case. For example, if X is a non-algebraic K3-surface, there appears to be no simpler condition characterizing the boundaries of positive holomorphic 1-chains.

Quite different characterizations of the boundaries of (not necessarily positive) holomorphic chains in projective manifolds appear in [**3**], [**4, 5**] and [**14**].

Remark 1.5. — The Linking Condition (1.1) forces the components of Γ to be maximally complex CR-manifolds. Maximal complexity is equivalent to the assertion that $\Gamma = \Gamma_{p-1,p} + \Gamma_{p,p-1}$ where $\Gamma_{r,s}$ denotes the Dolbeault component of Γ in bidimension (r, s). To see that this must hold, note that any $\alpha \in \mathcal{E}^{r,2p-1-r}(X)$ with $r \neq p-1, p$ satisfies $d^{p,p}\alpha + \omega \geq 0$ since $d^{p,p}\alpha = 0$.

Theorem 1.1 extends to characterize boundaries of compactly supported holomorphic chains in certain non-compact spaces. A complex n-manifold X is called *q-convex* if there exists a proper exhaustion function $f : X \to \mathbf{R}^+$ such that $dd^c f$ has at least $n - q + 1$ strictly positive eigenvalues outside some compact subset of X.

Theorem 1.6. — *Theorem 1.1 remains valid (for compactly supported holomorphic chains V) in any q-convex hermitian manifold with $q \leq p$.*

Remark 1.7. — If X is 1-convex (i.e., strongly pseudoconvex), then Theorem 1.1 is valid for all p. If, further, X admits a proper exhaustion which is strictly plurisubharmonic everywhere (i.e., X is Stein), much stronger results are known. Condition (1.1) implies maximal complexity, and for $p > 1$ this condition alone implies that Γ bounds a holomorphic p-chain [**12**]. Condition (1.1) also implies the *moment condition*: $\Gamma(\alpha) = 0$ for all $(p, p-1)$-forms α with $\overline{\partial}\alpha = 0$. When $p = 1$ this implies that Γ bounds a holomorphic 1-chain [**12**]. Results of this kind go back to Wermer [**21**].

Analogous remarks apply to results of [**13**] in the q-convex spaces $\mathbf{P}^n - \mathbf{P}^{n-q}$.

Remark 1.8. — Condition (1.1) implies that $\int_\Gamma \alpha \geq 0$ for all α with $d^{p,p}\alpha \geq 0$. If X is a Stein manifold embedded in some $\mathbf{C}^n$, this in turn implies that the linking number $\mathrm{Link}(\Gamma, Z) \geq 0$ for all algebraic subvarieties Z of codimension p in $\mathbf{C}^n - \Gamma$. By Alexander-Wermer [**1**], [**22**] this last condition alone implies that Γ bounds a positive holomorphic p-chain in X.

Theorem 1.1 also holds "locally", that is, it extends to any non-compact hermitian manifold X where neither Γ nor V are assumed to have compact support.

Theorem 1.9. — *Suppose X is a non-compact hermitian manifold, and let $\Gamma = \sum_j n_j \Gamma_j$ be a locally finite integral combination of disjointly embedded C^1-submanifolds of dimension $2p-1$, each of which has a real analytic point. Then Γ is the boundary of a holomorphic p-chain V of mass $M(V) \leq \Lambda$ (whose support is a closed but not necessarily compact analytic subvariety of $X - \mathrm{supp}\Gamma$) if and only if $\int_\Gamma \alpha \geq -\Lambda$ for all (p,p)-positive linking forms α with compact support on X.*

In the last section of this paper we further weaken our hypotheses on Γ to an assumption that each component Γ_k be *residual* at some point. (See § 3 for the definition.) The concept of residual submanifolds leads to questions of some independent interest.

In Section 3 we address a question related to the Characterization Theorems above. Let $j : M \subset X$ be a compact oriented real submanifold of dimension $2p-1$ in a compact Kähler manifold X. Represent the relative homology group $H_{2p}(X, M; \mathbf{R})$ by $2p$-currents T on X with $dT = j_* S$ for some $(2p-1)$-current S on M. One can ask: *When does a given class $\tau \in H_{2p}(X, M; \mathbf{R})$ contain a positive holomorphic chain?*

As a first step we show that for every T as above and every d-closed form φ on X the pairing $T(\varphi)$ depends only on the relative class $\tau = [T]$. This allows us to introduce a real Hodge filtration on $H_{2p}(X, M; \mathbf{Z})_{\mathrm{mod\ tor}}$ which extends the standard one on the subgroup $H_{2p}(X; \mathbf{Z})_{\mathrm{mod\ tor}}$. It also allows us to formulate the following.

Definition 1.10. — A class $\tau \in H_{2p}(X, M; \mathbf{R})$ is a *positive (p,p)-class* if $\tau(\varphi) \geq 0$ for all $2p$-forms φ with $d\varphi = 0$ and $\varphi^{p,p} \geq 0$.

Theorem 1.11. — *Let $M \subset X$ be as above and suppose each component of M has a real analytic point. Let $\tau \in H_{2p}(X, M; \mathbf{Z})_{\mathrm{mod\ tor}}$ be a positive (p,p)-class. Then there exists a positive holomorphic p-chain V on X with $dV = \partial\tau$ and a positive (p,p)-current S with $dS = 0$ such that $\tau = [V + S]$.*

In particular, if the positive classes in $H_{p,p}(X; \mathbf{Q})$ are represented by positive holomorphic chains with rational coefficients, then so are all the positive classes in $H_{p,p}(X, M; \mathbf{Q})$.

Remark 1.12. — This last result is a strengthening of the previous ones (in the Kähler case). Let τ be as in Theorem 1.11 and note that $\Gamma = \partial\tau = \sum_k n_k [M_k]$ where $M_1, \ldots, M_\ell$ are the connected components of M and the n_k's are integers. If τ is a positive (p,p)-class, then $\tau(d\alpha + \frac{1}{p!}\omega^p) \geq 0$ whenever $d^{p,p}\alpha + \frac{1}{p!}\omega^p \geq 0$. Therefore for any (p,p)-positive linking form α we have $\Gamma(\alpha) = (\partial\tau)(\alpha) = \tau(d\alpha) = \tau(d^{p,p}\alpha) = \tau(d^{p,p}\alpha + \frac{1}{p!}\omega^p) - \tau(\frac{1}{p!}\omega^p) \geq -\tau(\frac{1}{p!}\omega^p)$, and we conclude from Theorem 1.1 that Γ bounds a positive holomorphic p-chain V. Theorem 1.11 asserts that, moreover, the absolute class $\tau - [V]$ is represented by a positive (p,p)-current.

2. The Characterization Theorem

In this section we prove a general theorem which implies all of the results discussed in § 1 except Theorem 1.11. We shall assume throughout that X is a hermitian manifold which is not necessarily compact.

Definition 2.1. — Suppose there exists a closed subset Σ_Γ of Hausdorff $(2p-1)$-measure zero and an oriented, properly embedded, $(2p-1)$-dimensional C^1 submanifold of $X - \Sigma_\Gamma$ with connected components $\Gamma_1, \Gamma_2, \ldots$. If for given integers $n_1, n_2, \ldots$,

$$\Gamma = \sum_{k=1}^{\infty} n_k \Gamma_k$$

defines a current of locally finite mass in X which is d-closed, then Γ will be called a *scarred* $(2p-1)$*–cycle of class* C^1 in X. By a unique choice of orientation on Γ_k we assume each $n_k > 0$.

Example 2.2. — Any real analytic $(2p-1)$-cycle is automatically a scarred $(2p-1)$-cycle (see Federer [**6**, p. 433]).

Definition 2.3. — By a *positive holomorphic p-chain with boundary* Γ in X we mean a sum $V = \sum_k m_k V_k$ with $m_k \in \mathbf{Z}^+$ and V_k an irreducible p-dimensional complex analytic subvariety of $X - \mathrm{supp}\Gamma$ such that V has locally finite mass in X and $dV = \Gamma$ as currents.

Definition 2.4. — Suppose Γ is an embedded $(2p-1)$-dimensional oriented submanifold of a complex manifold. We say that Γ can be *pushed out at* $p \in \Gamma$ if there exists a complex p-dimensional submanifold-with-boundary $(V, -\Gamma)$ containing the point p (i.e., $\partial V = -\Gamma$ as oriented manifolds).

Our main result is the following.

Theorem 2.5. — *Let Γ be a scarred $(2p-1)$-cycle of class C^1 in X such that each component Γ_k can be pushed out at some point. Then $\Gamma = dV$ where V is a positive holomorphic p-chain with mass $M(V) \le \Lambda$ if and only if the (p,p)-linking numbers of Γ are bounded below by $-\Lambda$.*

Remark 2.6. — We say Γ is *two sided at p* if there exists a complex p-dimensional submanifold V near p with $\Gamma \subset V$ near p. Note that if Γ is real analytic and maximally complex at p, then Γ is two-sided at p. Note also that if Γ is two-sided at p, then Γ can be pushed out at p.

The proof of Theorem 2.5 has two parts. First the linking condition is shown to be equivalent to the existence of a weakly positive current T of bidimension p, p satisfying $dT = \Gamma$. In the second part it is shown that the existence of a positive T with $dT = \Gamma$ together with the pushout hypothesis on Γ implies the existence of a positive holomorphic chain V with boundary Γ.

2.1. Solving $dT = \Gamma$ for T positive

Theorem 2.7. — *Let $\Gamma \in \mathscr{D}'_{2p-1}(X)$ be an arbitrary current of dimension $2p-1$ on X. Then $dT = \Gamma$ for some weakly positive (p,p)-current with mass $M(T) \le \Lambda$ if and only if the linking condition*

$$\int_\Gamma a \ge -\Lambda \tag{2.1}$$

is satisfied for all compactly supported, strongly positive (p,p)-linking forms α on X.

Proof. — Let

$$S \equiv \{\alpha \in \mathscr{D}^{2p-1}(X) : d^{p,p}\alpha + \tfrac{1}{p!}\omega^p \ge 0 \quad \text{(strongly positive)}\}$$

and let

$$C \equiv \{\Gamma \in \mathscr{D}'_{2p-1}(X) : \Gamma = dT \text{ for some } T \ge 0 \text{ (weakly positive) with } M(T) \le 1\}.$$

It suffices to prove the theorem for $\Lambda = 1$. In this case the theorem states that $\Gamma \in C$ if and only if $\Gamma \in S^0$, where $S^0 \equiv \{\Gamma \in \mathscr{D}'_{2p-1}(X) : \Gamma(\alpha) \ge -1 \text{ for all } \alpha \in S\}$ is the *polar* of S. So we must prove that

$$C = S^0.$$

Note that C is a closed convex set in $\mathscr{D}'_{2p-1}(X)$ since the set of weakly positive (p,p)-currents T with $M(T) \le 1$ is compact in $\mathscr{D}'_{p,p}(X)$. Hence by the Bipolar Theorem **[19]** $C = (C^0)^0$, and it will suffice to prove that $C^0 = S$.

To see this first note that

$$T\left(d^{p,p}\alpha + \frac{1}{p!}\omega^p\right) = (dT)(\alpha) + M(T) \tag{2.2}$$

for all weakly positive (p,p)-currents T and all $\alpha \in \mathscr{D}^{2p-1}(X)$. If, in addition, $\alpha \in S$ and $\Gamma \in C$, then $0 \le \Gamma(\alpha) + M(T) \le \Gamma(\alpha) + 1$, so that $S \subseteq C^0$.

It remains to show that $C^0 \subseteq S$. Choose $\Gamma = dT$ with $T = \delta_x\xi$ where ξ is a weakly positive (p,p)-vector of mass norm one at $x \in X$. Note that $\Gamma \in C$. By (2.2) we have $(d^{p,p}\alpha + \frac{1}{p!}\omega^p)_x(\xi) = \Gamma(\alpha) + M(T) = \Gamma(\alpha) + 1$. If $\alpha \in C^0$, then $\Gamma(\alpha) \ge -1$ which proves that $\alpha \in S$. □

2.2. Replacing the positive solution by a holomorphic chain

Theorem 2.8. — *Suppose Γ is a scarred $(2p-1)$-cycle (of class C^1) in an arbitrary complex manifold X. Assume each component Γ_k of Γ can be pushed out at some point. If $\Gamma = dT$ for some weakly positive (p,p)-current T, then there exists a positive holomorphic p-chain V with $\Gamma = dV$ and $T - V \ge 0$, so in particular, $M(V) \le M(T)$ and $\operatorname{supp}(V) \subset \operatorname{supp}(T)$.*

The proof depends on the following local result.

Lemma 2.9. — *Suppose Γ is an oriented connected $(2p-1)$-dimensional submanifold near $0 \in \Gamma$ in $\mathbf{C}^n$.*

(1) *If Γ can be pushed out at $0 \in \Gamma$ and $r\Gamma = dT$ for some $T \geq 0$ and $r > 0$, then Γ is two-sided near 0. That is, near 0 there exists a (unique) complex p-dimensional subvariety V containing Γ, so that $V = V^+ \cup \Gamma \cup V^-$ and $dV^{\pm} = \pm\Gamma$.*

(2) *If Γ is two-sided near 0 and $r\Gamma = dT$ for some $T \geq 0$ and $r > 0$, then*

$$T = rV^+ + S \qquad \text{with } S \geq 0 \text{ and } dS = 0.$$

Proof. — By the push-out hypothesis we have that $-\Gamma = dZ$ for some irreducible subvariety Z of $B(0,R) - \Gamma$. By taking a small piece V^- of Z we may assume that the positive current $T^+ \equiv T + rV^- \geq 0$ has boundary Γ^+ which does *not* contain the origin. Consider the subset

$$E_r(T^+) = \{z : \Theta(T^+, z) \geq r\} \subset B(0,R) - \Gamma^+$$

where $\Theta(T^+, z)$ denotes the standard density, or Lelong number, of T^+ at z. Since $dT^+ = 0$ in $B(0,R) - \Gamma^+$ we know by a fundamental theorem of Siu [**20**] that

$$E_r(T^+) \text{ is a complex subvariety of complex dimension } \leq p \text{ and}$$
$$T^+ - rW \geq 0 \text{ where } W \text{ is the } p\text{-dimensional part of } E_r(T^+).$$

Since $E_r(T^+)$ contains V^-, it must have an irreducible p-dimensional component $V \supset V^-$, defined in a neighborhood of the origin. This proves (1). Since $V \subset W$, we have $T^+ - rV \geq 0$. Note also that $d(T^+ - rV) = 0$ near the origin. This proves (2) since $S \equiv T - rV^+ = T^+ - rV$. □

Corollary 2.10. — Under the hypotheses of Lemma 2.9 (1), $-\Gamma$ can also be pushed out at 0.

Proof of Theorem 2.8. — As an easy consequence of Siu's Theorem (See, for example, Theorem 2.4, p. 638 in [**9**]), there exist irreducible p-dimensional subvarieties V_j of $X - \operatorname{supp}\Gamma$ and positive constants c_j so that

$$T = \sum_{j=1}^{\infty} c_j V_j + R \tag{2.3}$$

where $R \geq 0$ and, for each $c > 0$, the complex subvariety $E_c(R)$ is of dimension $\leq p - 1$. This representation (2.3) is unique. (Note that $R \geq 0$ implies that the mass of T dominates the mass of $\sum_j c_j V_j$ on any set.)

Near the point where Γ_1 can be pushed out, Lemma 2.9 (with $r = n_1$) implies that

$$T = n_1 V^+ + S \qquad \text{with } S \geq 0 \text{ and } dS = 0. \tag{2.4}$$

By uniqueness V^+ must be contained in one of the V_j, say V_1. Moreover, since $S \geq 0$ we have $c_1 \geq n_1$. This implies $\widetilde{T} \equiv T - n_1 V_1 \geq 0$.

Near the point where Γ_1 can be pushed out we have $dV_1 = \Gamma_1$. Hence, on X we have $dV_1 = \Gamma_1 + \sum_{k=2}^{\infty} m_k \Gamma_k$ with $m_k \in \mathbf{Z}$. Consequently,

$$d\widetilde{T} = \sum_{k=2}^{\infty} (n_k - n_1 m_k)\Gamma_k,$$

and so we have eliminated one of components of the boundary. Now the coefficients in this sum may not all be positive, and to make them all positive we may have to reverse the orientation of some of the Γ_k. However, by Corollary 2.10 these orientation-reversed components can also be pushed out at some point. Hence, $\widetilde{\Gamma} = d\widetilde{T} = \sum_{k=2}^{\infty} \widetilde{n}_k \widetilde{\Gamma_k}$ satisfies all the hypotheses of Theorem 2.8.

If Γ has only a finite number of components, then we are done by induction on the number of components. If not, then by continuing this process we obtain a sequence of positive currents $\widetilde{T}_k = T - (n_1 V_1 + n_2' V_2 + \cdots + n_k' V_k)$ where the n_j' are positive integers and

$$d\widetilde{T}_k = \sum_{j=k+1}^{\infty} n_{kj} \Gamma_j.$$

Since $T - \widetilde{T}_k \geq 0$, we may assume, by passing to a subsequence, that $\{\widetilde{T}_k\}_{k=1}^{\infty}$ converges in mass norm to a positive current $\widetilde{T}_\infty$, which must be flat since each $\widetilde{T}_k$ is a normal current. Note that $\operatorname{supp}(d\widetilde{T}_\infty) \subset \Sigma_\Gamma$ and recall that, by assumption, the scar set Σ_Γ has Hausdorff $(2p-1)$-measure zero. Hence, by [**6**, 4.1.20], we have $d\widetilde{T}_\infty = 0$. We conclude that $V = \sum n_j' V_j = T - \widetilde{T}_\infty$ is a positive holomorphic chain with $dV = \Gamma$. □

Proof of Theorems 1.1 and 1.9. — Remarks 1.5 and 2.6 show that if $\Gamma = \sum_k m_k \Gamma_k$ satisfies the linking hypothesis, then Γ_k is two-sided at any real analytic point. □

Proof of Theorem 1.6. — It suffices to show that when X is q-convex for $q \leq p$, then Theorem 2.7 also holds with Γ and T having compact support. For this we change the definitions of S and C in the proof of Theorem 2.7 by permitting the α's in S to have arbitrary support and restricting the T's in C to have compact support. The argument will carry through as before once it is established that the cone C is closed in the weak topology. This follows from standard compactness theorems and the following fact. Suppose $f : X \to \mathbf{R}^+$ is the proper exhaustion with $n-q+1$ positive eigenvalues on $\{x : f(x) \geq 1\}$. If $T \in \mathscr{P}_{p,p}(X)$ for $p \geq q$, then

$$\operatorname{supp} T \subset \Big\{ x \in X : f(x) \leq \max\Big\{1, \sup_{\operatorname{supp} dT} f\Big\}\Big\}. \qquad \square$$

3. Relative Hodge Classes and Representability

In this chapter we address the question of when a relative homology class can be represented by a positive holomorphic chain. More specifically, let X be a compact Kähler manifold and $M \subset X$ a smooth orientable compact submanifold of real dimension $2p-1$. Then we have the two closely related questions:

Relative Hodge Question. — *Which classes in $H_{2p}(X, M; \mathbf{Z})$ can be represented by holomorphic chains?*

3.1. Relative Hodge Question (Positive Version). — *Which classes in $H_{2p}(X, M; \mathbf{Z})$ can be represented by positive holomorphic chains?*

We shall work in the space $\overline{H}_{2p}(X, M; \mathbf{Z}) = H_{2p}(X, M; \mathbf{Z})/\mathrm{Tor}$ where Tor is the torsion subgroup and use the following Relative de Rham Theorem. Consider the short exact sequence of chain complexes of Fréchet spaces

$$0 \longrightarrow \mathcal{E}^*(X, M) \longrightarrow \mathcal{E}^*(X) \xrightarrow{j^*} \mathcal{E}^*(M) \longrightarrow 0, \tag{3.1}$$

where $j : M \to X$ denotes the inclusion, and the dual sequence of topological dual spaces

$$0 \longleftarrow \frac{\mathcal{E}'_*(X)}{j_*\mathcal{E}'_*(M)} \longleftarrow \mathcal{E}'_*(X) \xleftarrow{j_*} \mathcal{E}'_*(M) \longleftarrow 0. \tag{3.2}$$

The complex $\mathcal{E}^*(X, M)$, consisting of forms which vanish when restricted to M, computes the relative cohomology $H^*(X, M; \mathbf{R})$, and the complex $\mathcal{E}'_*(X, M) \equiv \mathcal{E}'_*(X)/j_*\mathcal{E}'_*(M)$ computes the relative homology $H_*(X, M; \mathbf{R})$.

The Relative de Rham Theorem states that:

$$H^k(X, M; \mathbf{R}) \text{ and } H_k(X, M; \mathbf{R}) \text{ are dual to each other.}$$

This can be proven as follows. Consider the dual triples

$$\mathcal{E}^{k-1}(X, M) \xrightarrow{d} \mathcal{E}^k(X, M) \xrightarrow{d} \mathcal{E}^{k+1}(X, M)$$
$$\mathcal{E}'_{k-1}(X, M) \xleftarrow{d} \mathcal{E}'_k(X, M) \xleftarrow{d} \mathcal{E}'_{k+1}(X, M)$$

where $H^k(X, M; \mathbf{R}) = Z/B$ using the cycles Z and boundaries B in the first sequence, and $H_k(X, M; \mathbf{R}) = \widetilde{Z}/\widetilde{B}$ using the cycles $\widetilde{Z}$ and boundaries $\widetilde{B}$ in the second sequence. By the Hahn-Banach Theorem it suffices to show that B and $\widetilde{B}$ are closed. These spaces are images of continuous linear maps. If they are of finite codimension in Z and $\widetilde{Z}$ respectively, then they are closed by a standard result in functional analysis. Thus is remains to show that $H^k(X, M; \mathbf{R})$ and $H_k(X, M; \mathbf{R})$ are finite dimensional. That $H^k(X, M; \mathbf{R})$ is finite dimensional follows from the long exact sequence

$$\cdots \longrightarrow H^{k-1}(M; \mathbf{R}) \longrightarrow H^k(X, M; \mathbf{R}) \longrightarrow H^k(X; \mathbf{R}) \longrightarrow H^k(M; \mathbf{R}) \longrightarrow \cdots$$

derived from (3.1) and the fact that $H^{k-1}(M; \mathbf{R})$ and $H^k(X; \mathbf{R})$ are finite dimensional by the standard de Rham Theorem. That $H_k(X, M; \mathbf{R})$ is finite dimensional follows similarly from the long exact sequence derived from (3.2).

In the special case $k = 2p = \dim M + 1$ we have:

$$H^{2p}(X, M; \mathbf{R}) = Z/B \qquad \text{where } \begin{cases} Z = \{\varphi \in \mathcal{E}^{2p}(X) : d\varphi = 0\} \\ \text{and } B = d\mathcal{E}^{2p-1}(X, M) \end{cases} \tag{3.3}$$

and

$$H_{2p}(X, M; \mathbf{R}) = \widetilde{Z}/\widetilde{B} \qquad \text{where } \begin{cases} \widetilde{Z} = \{T \in \mathcal{E}'_{2p}(X) : dT \in j_*\mathcal{E}'_{2p-1}(M)\} \\ \text{and } \widetilde{B} = d\mathcal{E}'_{2p+1}(X). \end{cases} \tag{3.4}$$

It is an interesting fact, established in the next section, that the group $\overline{H}_{2p}(X, M; \mathbf{Z}) \subset H_{2p}(X, M; \mathbf{R})$ carries a "real Hodge filtration". A key point is the following lemma.

Lemma 3.1. — *Fix $\tau \in H_{2p}(X, M; \mathbf{R})$. If $T, T' \in \mathscr{E}_{2p}(X, M)$ are relatively closed currents representing τ, then*

$$T(\varphi) = T'(\varphi) \quad \textit{for all } \varphi \in \mathscr{E}^{2p}(X) \textit{ with } d\varphi = 0. \tag{3.5}$$

Hence, the notion of $\tau(\varphi)$ is well defined for such φ. Furthermore,

$$dT = dT' = \sum_{j=1}^{L} r_j [M_j] \tag{3.6}$$

where $M = M_1 \cup \cdots \cup M_L$ is the decomposition into connected components and the r_j are real numbers. Thus, $\partial\tau = \sum_{j=1}^{L} r_j [M_j]$ is well defined.

Proof. — Since T and T' both represent $\tau \in \widetilde{Z}/\widetilde{B}$ and $\widetilde{B} = d\mathscr{E}'_{2p+1}(X)$ by (3.4), we have $T - T' = dR$ for $R \in \mathscr{E}'_{2p+1}(X)$. This proves (3.5) and that $dT = dT'$. Since $T \in \widetilde{Z}$, (3.4) says that $dT = j_* u$ where $u \in \mathscr{E}_{2p-1}(M)$. This implies that $du = 0$. Hence, u is a locally constant function on M. □

Definition 3.2. — A class $\tau \in H_{2p}(X, M; \mathbf{R})$ is called *positive* if $\tau(\varphi) \geq 0$ for all closed, real $2p$-forms φ such that the component

$$\varphi^{p,p} \geq 0 \quad \text{(is weakly positive) on } X.$$

If τ is positive, then it is of type (p, p) as defined in 4.1 below.

Proposition 3.3. — *A class $\tau \in H_{2p}(X, M; \mathbf{R})$ is positive if and only if it is represented (in the complex $\mathscr{E}_*(X, M)$) by a strongly positive current of type (p, p).*

This proposition will be proved below. We first observe that it leads to the following main result.

Theorem 3.4. — *Suppose $\tau \in \overline{H}_{2p}(X, M; \mathbf{Z})$ is positive. Suppose each component of M has a real analytic point (or, more generally, is two-sided at some point). Then there exists a positive holomorphic p-chain V on X with $dV = \partial\tau$. Furthermore, there exists a positive d-closed (p, p)-current S with $\tau = [V + S]$.*

In particular, if the positive classes in $H_{2p}(X; \mathbf{Q})$ are all represented by positive holomorphic chains with rational coefficients, then so are all the positive classes in $H_{2p}(X, M; \mathbf{Q})$.

Thus for example, given any real analytic M in a Grassmann manifold X, we conclude that every positive class in $H_{2p}(X, M; \mathbf{Z})$ carries a positive holomorphic chain. However there are projective manifolds X with positive (p, p)-classes in $\overline{H}_{2p}(X; \mathbf{Z})$ which do not carry positive holomorphic cycles. In fact, for every integer $k \geq 2$ there exists an abelian variety X of complex dimension $2k$ and a class $\tau \in H_{2k}(X; \mathbf{Z})$ which is represented by a positive (k,k)-current and also by an algebraic k-cycle, but τ is not represented by a positive algebraic k-cycle (see [**18**]).

Proof. — By Proposition 3.3 and (3.6) in Lemma 3.1, the class τ is represented by a positive (p,p)-current T with $dT = \partial\tau = \sum_i n_i[M_i]$ for integers n_i (cf. the argument for (3.2) above.) Applying Theorem 2.8 with $\Gamma = dT$, we deduce the existence of a positive holomorphic chain V with

$$dV = dT \text{ and } T - V \geq 0. \qquad \square$$

Proof of Proposition 3.3. — Consider the closed convex cones

$$P \equiv \{\varphi \in \mathscr{E}^{2p}(X) : \varphi^{p,p} \text{ is weakly positive}\} \subset \mathscr{E}^{2p}(X)$$
$$\widetilde{P} \equiv \{T \in \mathscr{E}'_{2p}(X) : T = T_{p,p} \text{ is strongly positive}\} \subset \mathscr{E}'_{2p}(X)$$

These are polars of each other in the dual pair $\mathscr{E}^{2p}(X)$, $\mathscr{E}'_{2p}(X)$. Moreover, by the Relative de Rham Theorem and (3.3) and (3.4) we have:

(i) $\widetilde{B} \subset \mathscr{E}'_{2p}(X)$ is closed (in the weak topology).
(ii) Z and $\widetilde{B}$ are polars of each other in the dual pair $\mathscr{E}^{2p}(X)$, $\mathscr{E}'_{2p}(X)$.
(iii) $B \subset \mathscr{E}^{2p}(X)$ is closed.
(iv) B and $\widetilde{Z}$ are polars of each other in the dual pair $\mathscr{E}^{2p}(X)$, $\mathscr{E}'_{2p}(X)$.

Lemma 3.5. — *The subset $\widetilde{P} + \widetilde{B}$ is closed in the standard topology on $\mathscr{E}'_{2p}(X)$.*

Proof. — Let $\{T_i\} \subset \mathscr{P}_{p,p}$ and $\{dS_i\} \subset \widetilde{B}$ be sequences such that

$$T_i + dS_i \longrightarrow R \qquad \text{weakly in } \mathscr{E}'_{2p}(X)$$

Let ω denote the Kähler form on X. Then

$$p!M(T_i) = T_i\left(\omega^p\right) = \left(T_i + dS_i\right)\left(\omega^p\right) \longrightarrow R\left(\omega^p\right),$$

and so the masses $M(T_i)$ are uniformly bounded. By the compactness theorem for positive currents there is a subsequence, again denoted by T_i, converging to a positive current T. Hence, $dS_i \longrightarrow R - T$ weakly, and since d has closed range, there exists $S \in \mathscr{E}'_{2p+1}(X)$ with $dS = R - T$. $\square$

Proposition 3.6. — *We have*

$$[(P \cap Z) + B]^0 = (\widetilde{P} \cap \widetilde{Z}) + \widetilde{B}.$$

Proof. — By standard principles we have $[(P\cap Z)+B]^0 = (P \cap Z)^0 \cap B^0 = \overline{(P^0 + Z^0)} \cap B^0$. By (ii), (iv) and Lemma 3.5 we have $\overline{(P^0 + Z^0)} \cap B^0 = \overline{(\widetilde{P} + \widetilde{B})} \cap \widetilde{Z} = (\widetilde{P} + \widetilde{B}) \cap \widetilde{Z}$. Finally it is easy to see that $(\widetilde{P} + \widetilde{B}) \cap \widetilde{Z} = (\widetilde{P} \cap \widetilde{Z}) + \widetilde{B}$ since $\widetilde{B} \subset \widetilde{Z}$. $\square$

To complete the proof of Proposition 3.3 choose a current $T \in \widetilde{Z}$ which represents the class τ. By hypothesis T is in the polar of $(P \cap Z) + B$. Therefore, by Proposition 3.6 and (3.4), $T = T_0 + dS$ with $T_0 \in \widetilde{P}$. $\square$

4. A Real Hodge Filtration on $H_{2p}(X, M; \mathbf{R})$

Definition 4.1. — A homology class $\tau \in H_{2p}(X, M; \mathbf{R})$ is of *filtration level* k if $\tau(\varphi) = 0$ for all closed complex valued forms φ of type (r, s) with $r > p+k$. Classes of filtration level 0 are called *type* (p, p).

Note 4.2. — This induces a real Hodge filtration $F^k H_{2p}(X, M; \mathbf{R})$ on $H_{2p}(X, M; \mathbf{R})$ which extends the basic one $F^k H_{2p}(X; \mathbf{R}) = \bigoplus_{r=0}^{k} \{H_{p-r,p+r}(X) \oplus H_{p+r,p-r}(X)\}_{\mathbf{R}}$ on $H_{2p}(X; \mathbf{R})$.

Proposition 4.3. — *Suppose $\tau \in H_{2p}(X, M; \mathbf{R})$ has filtration level k. Then τ is represented by a current*

$$T \in \left\{ \mathscr{E}'_{p-k,p+k}(X) \oplus \cdots \oplus \mathscr{E}'_{p+k,p-k}(X) \right\}_{\mathbf{R}}, \tag{4.1}$$

and therefore,

$$dT \in \left\{ \mathscr{E}'_{p-k-1,p+k}(X) \oplus \cdots \oplus \mathscr{E}'_{p+k,p-k-1}(X) \right\}_{\mathbf{R}}. \tag{4.2}$$

In particular, if τ is of type (p, p), then $\tau = [T]$ for a some (p, p)-current T, and each non-zero boundary component of $\partial\tau$ is maximally complex (cf. **[12]***).*

Proof. — We start by establishing (4.2). Write $\partial\tau = \sum_j r_j [M_j]$ as in Lemma 3.1. Choose any smooth form $\psi \in \mathscr{E}^{r,s}(X)$ with $r + s = 2p - 1$ and either $r > p + k$ or $s > p+k$. Then $0 = \tau(d\psi) = (\partial\tau)(\psi) = \sum_j r_j \int_{M_j} \psi$. Since ψ is arbitrary, we conclude that for each M_j with $r_j \neq 0$, the Dolbeault components

$$[M_j]_{r,s} = 0 \quad \text{if either } s > p + k \text{ or } r > p + k.$$

This gives (4.2). When $k = 0$ this means M_j is maximally complex.

Consider the case where τ is of type (p, p) with $2p \leq n$. Choose a current T representing τ. Then by standard harmonic theory $T_{2p,0} = h_{2p,0} - \overline{\partial}\beta$ where h is harmonic (in particular, smooth) and $\beta \in \mathscr{E}'_{2p,1}(X)$. Then $[T - d\beta]_{2p,0} = T_{2p,0} - \overline{\partial}\beta = h_{2p,0} \equiv h$ and because $T(\overline{*}h) = \|h\|^2 = 0$ (since $\tau = [T]$ is type (p, p)), we have $h = 0$. Thus replacing T by $T - \overline{\partial}\beta - \partial\overline{\beta}$ we can assume $T_{2p,0} = T_{0,2p} = 0$.

If $p = 1$, we are done. If $p > 1$, we note that $\overline{\partial}T_{2p-1,1} = M_{2p-1,0} = 0$, and so $T_{2p-1,1} = h_{2p-1,1} + \overline{\partial}\beta$ where $h_{2p-1,1}$ is harmonic and $\beta \in \mathscr{E}'_{2p-1,2}(X)$. We conclude as above that $h_{2p-1,1} = 0$, and then replace T by $T - \overline{\partial}\beta - \partial\overline{\beta}$ so that $T_{2p-1,1} = T_{1,2p-1} = 0$. Continuing in this fashion gives the result. All other cases are entirely analogous and details are left to the reader. $\square$

5. Residual Currents

Definition 5.1. — Let R be a weakly positive, d-closed (p, p)-current. Then R is *residual* if for each $c > 0$ the complex dimension of the subvariety $E_c(R) = \{z : \Theta(R, z) \geq c\}$ is $\leq p - 1$.

Suppose T is a a weakly positive, d-closed (p,p)-current defined in the complement of supp(Γ) where Γ is a scarred $2p-1$ cycle (of class C^1). By the main result of [**8**] (see Theorem 6, p. 71 and the note added in proof) T has locally finite mass across supp(Γ). That is, T has a unique "extension by zero" across supp(Γ). Let T also denote this extension. It follows from the two support theorems of Federer [**6**, 4.1.15, 4.1.20] that $dT = \sum_{k=1}^{\infty} r_k\Gamma_k$ with constants $r_k \in \mathbf{R}$.

Definition 5.2. — The set supp(Γ) is *residual* if each residual current R on $X - \text{supp}(\Gamma)$ satisfies $dR = 0$ on X.

Proposition 5.3. — *If each component of* Γ *has a two-sided point, then* supp(Γ) *is residual.*

Proof. — Suppose R is a residual current on $X - \text{supp}(\Gamma)$ with $dR = \sum_k r_k\Gamma_k$, $r_k \in \mathbf{R}$. Near a two-sided point of one of the components, say Γ_1, we have $dR = r_1\Gamma_1$. By Lemma 2.9 we can write R locally as $R = r_1V^+ + S$ with $S \geq 0$ and $dS = 0$ across Γ_1. This contradicts the hypothesis that R is residual unless $r_1 = 0$. □

Remark 5.4. — This Proposition combined with the first half of Lemma 2.9 and the next result provides a second proof of Theorem 2.5.

Theorem 5.5. — *Suppose* Γ *is a scarred* $2p-1$ *cycle (of class* C^1*) in an arbitrary complex manifold* X*. Assume each component* Γ_k *of* Γ *is residual at some point. If* $\Gamma = dT$ *for some weakly positive* (p,p)*-current* T *on* X*, then there exists a positive holomorphic* p*-chain* V *with* $\Gamma = dV$ *and* $T - V \geq 0$*.*

Proof. — Suppose $\Gamma = dT$ as in the theorem and consider the decomposition $T = S + R$ into a positive *real-coefficient holomorphic chain* $S = \sum_{j=1}^{\infty} c_jV_j$ plus a residual current R (on $X - \Gamma$), Now $dR = \sum_{k=1}^{\infty} r_k\Gamma_k$ for some $r_k \in \mathbf{R}$, but by the hypothesis each r_k must be zero. Hence, $\Gamma = dS$ bounds a positive real-coefficient holomorphic chain. □

Proposition 5.6. — *Let* Γ *be a scarred* $2p-1$ *cycle in an arbitrary complex manifold* X*. If* $\Gamma = dS$ *bounds a positive real-coefficient holomorphic chain* $S = \sum_{j=1}^{\infty} c_jV_j$*, then* $\Gamma = dV$ *bounds a positive (integer-coefficient) holomorphic chain* V *with* $S - V \geq 0$ *(and therefore also* supp$V \subseteq$ supp$S$*).*

Proof. — By hypothesis $d\left(\sum_{j=1}^{\infty} c_jV_j\right) = \sum_{k=1}^{\infty} n_k\Gamma_k$. Near a regular point x in Γ_1 each V_j satisfies $dV_j = \epsilon_j\Gamma_1$ with $\epsilon_j \in \{-1, 0, 1\}$. By uniqueness there is at most one of the subvarieties V_j with boundary Γ_1. Relabel so that $dV_1 = \Gamma_1$. Now there are two cases.

Case 1. — $dV_j = 0$ for all $j \geq 2$. In this case we must have $c_1 = n_1$, and we can eliminate the component Γ_1 from Γ.

Case 2. — $-\Gamma_1$ bounds exactly one of the subvarieties V_j, $j \geq 2$. Relabel so that $-\Gamma_1 = dV_2$. In this case $c_1 - c_2 = n_1$. Note that $V_1 + V_2$ is a subvariety without boundary near the point x on Γ_1. Set $\widetilde{S} = S - n_1 V_1 = c_2(V_1 + V_2) + \sum_{j=3}^{\infty} c_j V_j$. Then $\widetilde{S}$ is positive and $d\widetilde{S} = 0$ near the point x. Consequently, $d\widetilde{S} = \sum_{j=2}^{\infty} b_j V_j$. Finally, the b_j's must be integers. In fact $b_j = n_j - \epsilon_{1j} n_1$ where $dV_1 = \epsilon_{1j}\Gamma_j$ defines $\epsilon_{1j} \in \{-1, 0, 1\}$. Hence, we can eliminate the component Γ_1 from Γ in this case as well.

The proof can now be completed exactly as in the last paragraph of the proof of Theorem 2.8. □

Question 5.7. — Which (maximally complex) $(2p-1)$-dimensional submanifolds are residual? Note that if Γ is two-sided, then Γ is residual. Moreover, if Γ is one-sided, then Γ has a natural orientation so that $\Gamma = dW$ where W is complex, and in this case the residual property is equivalent to the following uniqueness property:

$$\text{If } T \geq 0 \text{ satisfies } dT = \Gamma, \text{ then } T = W + S \text{ with } S \geq 0 \text{ and } dS = 0.$$

If Γ is zero-sided, then Γ is residual if and only if

$$T \geq 0 \text{ and } \operatorname{supp}\{dT\} \subset \Gamma \qquad \Rightarrow \qquad dT = 0.$$

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Astérisque
328, 2009, p. 223–235

MIRZAKHARNI'S RECURSION FORMULA IS EQUIVALENT TO THE WITTEN-KONTSEVICH THEOREM

by

Kefeng Liu & Hao Xu

Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — In this paper, we give a proof of Mirzakhani's recursion formula of Weil-Petersson volumes of moduli spaces of curves using the Witten-Kontsevich theorem. We also describe properties of intersections numbers involving higher degree κ classes.

Résumé **(La formule de récurrence de Mirzakhani est équivalente au théorème de Witten-Kontsevich)**

Dans cet article, nous démontrons la formule de récurrence de Mirzakhani sur les volumes de Weil-Petersson des espaces de module de courbes en utilisant le théorème de Witten-Kontsevich. Nous donnons aussi des propriétés des nombres d'intersection associées aux classes κ de degré supérieur.

1. Introduction

Following the notation of Mulase and Safnuk [**21**], let $\mathscr{M}_{g,n}(\mathbf{L})$ denote the moduli space of bordered Riemann surfaces with n geodesic boundary components of specified lengths $\mathbf{L} = (L_1, \ldots, L_n)$ and let $\mathrm{Vol}_{g,n}(\mathbf{L})$ denote its Weil-Petersson volume $\mathrm{Vol}(\mathscr{M}_{g,n}(\mathbf{L}))$. Using her remarkable generalization of the McShane identity, Mirzakhani [**19**] proved a beautiful recursion formula for these Weil-Petersson volumes

$$\begin{aligned}\mathrm{Vol}_{g,n}(\mathbf{L}) = \frac{1}{2L_1} \sum_{\substack{g_1+g_2=g \\ \underline{n}=I\coprod J}} \int_0^{L_1} \int_0^{\infty} \int_0^{\infty} & xyH(t,x+y) \\ & \times \mathrm{Vol}_{g_1,n_1}(x,\mathbf{L}_I)\mathrm{Vol}_{g_2,n_2}(y,\mathbf{L}_J)dxdydt \\ + \frac{1}{2L_1}\int_0^{L_1} \int_0^{\infty} \int_0^{\infty} & xyH(t,x+y)\mathrm{Vol}_{g-1,n+1}(x,y,L_2,\ldots,L_n)dxdydt\end{aligned}$$

2010 Mathematics Subject Classification. — 14H10, 14H81.
Key words and phrases. — Weil-Petersson volume, Mirzakhani recursion formula, Witten-Kontsevich theorem.

$$+\frac{1}{2L_1}\sum_{j=2}^{n}\int_0^{L_1}\int_0^{\infty} x\big(H(x,L_1+L_j)+H(x,L_1-L_j)\big)$$
$$\times \operatorname{Vol}_{g,n-1}(x,L_2,\dots,\hat{L_j},\dots,L_n)dxdt,$$

where the kernel function

$$H(x,y)=\frac{1}{1+e^{(x+y)/2}}+\frac{1}{1+e^{(x-y)/2}}.$$

Using symplectic reduction, Mirzakhani [**20**] showed the following relation

$$\frac{\operatorname{Vol}_{g,n}(2\pi\mathbf{L})}{(2\pi^2)^{3g+n-3}}=\frac{1}{(3g+n-3)!}\int_{\mathcal{M}_{g,n}}(\kappa_1+\sum_{i=1}^{n}L_i^2\psi_i)^{3g+n-3}$$
$$=\sum_{\substack{d_0+\cdots+d_n\\=3g+n-3}}\prod_{i=0}^{n}\frac{1}{d_i!}\langle\kappa_1^{d_0}\prod\tau_{d_i}\rangle_{g,n}\prod_{i=1}^{\infty}L_i^{2d_i}.$$

Combining with her recursion formula of Weil-Petersson volumes, Mirzakhani [**20**] found a new proof of the celebrated Witten-Kontsevich theorem.

By taking derivatives with respect to $\mathbf{L}=(L_1,\dots,L_n)$ in Mirzakhani's recursion, Mulase and Safnuk [**21**] obtained the following enlightening recursion formula of intersection numbers which is equivalent to Mirzakhani's recursion.

$$(2d_1+1)!!\langle\prod_{j=1}^{n}\tau_{d_j}\kappa_1^a\rangle_g$$
$$=\sum_{j=2}^{n}\sum_{b=0}^{a}\frac{a!}{(a-b)!}\frac{(2(b+d_1+d_j)-1)!!}{(2d_j-1)!!}\beta_b\langle\kappa_1^{a-b}\tau_{b+d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g$$
$$+\frac{1}{2}\sum_{b=0}^{a}\sum_{r+s=b+d_1-2}\frac{a!}{(a-b)!}(2r+1)!!(2s+1)!!\beta_b\langle\kappa_1^{a-b}\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1}$$
$$+\frac{1}{2}\sum_{b=0}^{a}\sum_{\substack{c+c'=a-b\\ I\coprod J=\{2,\dots,n\}}}\sum_{r+s=b+d_1-2}\frac{a!}{c!c'!}(2r+1)!!(2s+1)!!\beta_b$$
$$\times\langle\kappa_1^{c}\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa_1^{c'}\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},$$

where

$$\beta_b=(2^{2b+1}-4)\frac{\zeta(2b)}{(2\pi^2)^b}=(-1)^{b-1}2^b(2^{2b}-2)\frac{B_{2b}}{(2b)!}.$$

Safnuk [**23**] gave a proof of the above differential form of Mirzakhani's recurson formula using localization techniques, but he also used the Mirzakhani-McShane formula. The relationship between Mirzakhani's recurson and matrix integrals has been studied by Eynard-Orantin [**7**] and Eynard [**6**].

Indeed, when $a = 0$, Mulase-Safnuk differential form of Mirzakhani's recursion is just the Witten-Kontsevich theorem [**14, 24**] in the form of DVV recursion relation [**4**]. There are several other new proofs of Witten-Kontsevich theorem [**3, 12, 13, 22**] besides Mirzakhani's proof [**20**].

More discussions about Weil-Petersson volumes from the point of view of intersection numbers can be found in the papers [**5, 10, 18, 26**].

In Section 2, we show that Mirzakhani's recursion formula is essentially equivalent to the Witten-Kontsevich theorem via a formula from [**11**] expressing κ classes in terms of ψ classes. In Section 3, we present certain results of intersection numbers involving higher degree κ classes.

Acknowledgements. — We would like to thank Chiu-Chu Melissa Liu for helpful discussions. We also thank the referees for helpful suggestions.

2. Proof of Mirzakhani's recursion formula

We first give three lemmas. The following lemma can be found in [**21**].

Lemma 2.1. — *The constants β_b in Mirzakhani's recursion satisfy the following:*

$$\sum_{k=0}^{\infty}\beta_k x^k = \frac{\sqrt{2x}}{\sin\sqrt{2x}}.$$

And its inverse:

$$\Big(\sum_{k=0}^{\infty}\beta_k x^k\Big)^{-1} = \frac{\sin\sqrt{2x}}{\sqrt{2x}} = \sum_{k=0}^{\infty}\frac{(-1)^k 2^k}{(2k+1)!}x^k.$$

Proof. — Since

$$\sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}x^{2n} = \frac{x}{2}\frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}} = \frac{x}{2i}\cot\frac{x}{2i},$$

we have

$$\sum_{k=0}^{\infty}\beta_k x^k = \sqrt{2x}\Big(\cot\sqrt{\frac{x}{2}} - \cot\sqrt{2x}\Big) = \frac{\sqrt{2x}}{\sin\sqrt{2x}}. \qquad \square$$

The following elementary result is crucial to our proof.

Lemma 2.2. — *Let $F(m,n)$ and $G(m,n)$ be two functions defined on $\mathbb{N}\times\mathbb{N}$, where $\mathbb{N} = \{0,1,2,\ldots\}$ is the set of nonnegative integers. Let α_k and β_k be real numbers that satisfy*

$$\sum_{k=0}^{\infty}\alpha_k x^k = \Big(\sum_{k=0}^{\infty}\beta_k x^k\Big)^{-1}.$$

Then the following two identities are equivalent:

$$G(m,n) = \sum_{k=0}^{m}\alpha_k F(m-k, n+k), \quad \forall\ (m,n)\in\mathbb{N}\times\mathbb{N},$$

$$F(m,n) = \sum_{k=0}^{m} \beta_k G(m-k, n+k), \quad \forall\ (m,n) \in \mathbb{N} \times \mathbb{N}.$$

Proof. — Assume the first identity holds, then we have

$$\begin{aligned}
\sum_{i=0}^{m} \beta_i G(m-i, n+i) &= \sum_{i=0}^{m} \beta_i \sum_{j=0}^{m-i} \alpha_j F(m-i-j, n+i+j) \\
&= \sum_{k=0}^{m} \sum_{i+j=k} (\beta_i \alpha_j) F(m-k, n+k) \\
&= \sum_{k=0}^{m} \delta_{k0} F(m-k, n+k) \\
&= F(m,n).
\end{aligned}$$

So we proved the second identity. The proof of the other direction is the same. □

The fact that intersection numbers involving both κ classes and ψ classes can be reduced to intersection numbers involving only ψ classes was already known to Witten [9], and has been developed by Arbarello-Cornalba [**2**], Faber [**8**] and Kaufmann-Manin-Zagier [**11**] into a nice combinatorial formalism.

Lemma 2.3 **([11])**. — *For $m > 0$,*

$$\langle \prod_{j=1}^{n} \tau_{d_j} \kappa_1^m \rangle_g = \sum_{k=1}^{m} \frac{(-1)^{m-k}}{k!} \sum_{\substack{m_1+\cdots+m_k=m \\ m_i>0}} \binom{m}{m_1, \ldots, m_k} \langle \prod_{j=1}^{n} \tau_{d_j} \prod_{j=1}^{k} \tau_{m_j+1} \rangle_g.$$

Proof. — (sketch) Let $\pi_{n+p,n} : \overline{\mathcal{M}}_{g,n+p} \longrightarrow \overline{\mathcal{M}}_{g,n}$ be the morphism which forgets the last p marked points and denote $\pi_{n+p,n*}(\psi_{n+1}^{a_1+1} \cdots \psi_{n+p}^{a_p+1})$ by $R(a_1, \ldots, a_p)$, then we have the formula from [**2**]

$$R(a_1, \ldots, a_p) = \sum_{\sigma \in \mathbb{S}_p} \prod_{\substack{\text{each cycle } c \\ \text{of } \sigma}} \kappa_{\sum_{j \in c} a_j},$$

where we write any permutation σ in the symmetric group $\mathbb{S}_p$ as a product of disjoint cycles.

A formal combinatorial argument [**11**] leads to the following inversion equation

$$\kappa_{a_1} \cdots \kappa_{a_p} = \sum_{k=1}^{p} \frac{(-1)^{p-k}}{k!} \sum_{\substack{\{1,\ldots,p\} = S_1 \coprod \cdots \coprod S_k \\ S_k \neq \varnothing}} R(\sum_{j \in S_1} a_j, \ldots, \sum_{j \in S_k} a_j),$$

from which the result follows easily. □

Proposition 2.4. — *We have*

$$\sum_{b=0}^{a}(-1)^b\binom{a}{b}\frac{(2(d_1+b)+1)!!}{(2b+1)!!}\langle\tau_{d_1+b}\prod_{i=2}^{n}\tau_{d_i}\kappa_1^{a-b}\rangle_g$$
$$=\sum_{j=2}^{n}\frac{(2d_1+2d_j-1)!!}{(2d_j-1)!!}\langle\kappa_1^a\tau_{d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g$$
$$+\frac{1}{2}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\langle\kappa_1^a\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1}$$
$$+\frac{1}{2}\sum_{\substack{c+c'=a\\ I\coprod J=\{2,\dots,n\}}}\binom{a}{c}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\langle\kappa_1^c\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa_1^{c'}\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.$$

Proof. — Let LHS and RHS denote the left and right hand side of the equation respectively. By Lemma 2.3 and the Witten-Kontsevich theorem, we have

$$(2d_1+1)!!\langle\prod_{j=1}^{n}\tau_{d_j}\kappa_1^a\rangle_g$$
$$=(2d_1+1)!!\sum_{k=0}^{a}\frac{(-1)^{a-k}}{k!}\sum_{\substack{m_1+\cdots+m_k=a\\ m_i>0}}\binom{a}{m_1,\dots,m_k}\langle\prod_{j=1}^{n}\tau_{d_j}\prod_{j=1}^{k}\tau_{m_j+1}\rangle_g$$
$$=\sum_{k=0}^{a}\frac{(-1)^{a-k}}{k!}\sum_{\substack{m_1+\cdots+m_k=a\\ m_i>0}}\binom{a}{m_1,\dots,m_k}$$
$$\times\Bigg(\sum_{j=2}^{n}\frac{(2(d_1+d_j)-1)!!}{(2d_j-1)!!}\langle\tau_{d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\prod_{i=1}^{k}\tau_{m_i+1}\rangle_g$$
$$+\sum_{j=1}^{k}\frac{(2(d_1+m_j)+1)!!}{(2m_j+1)!!}\langle\tau_{d_1+m_j}\prod_{i=2}^{n}\tau_{d_i}\prod_{i\neq j}\tau_{m_i+1}\rangle_g$$
$$+\frac{1}{2}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\prod_{i=2}^{n}\tau_{d_i}\prod_{i=1}^{k}\tau_{m_i+1}\rangle_{g-1}$$
$$+\frac{1}{2}\sum_{\substack{I\coprod J=\{2,\dots,n\}\\ I'\coprod J'=\{1,\dots,k\}}}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!$$
$$\times\langle\tau_r\prod_{i\in I}\tau_{d_i}\prod_{i\in I'}\tau_{m_i+1}\rangle_{g'}\langle\tau_s\prod_{i\in J}\tau_{d_i}\prod_{i\in J'}\tau_{m_i+1}\rangle_{g-g'}\Bigg)$$

$$
\begin{aligned}
&= \sum_{j=2}^{n} \frac{(2d_1+2d_j-1)!!}{(2d_j-1)!!} \langle \kappa_1^a \tau_{d_1+d_j-1} \prod_{i\neq 1,j} \tau_{d_i}\rangle_g \\
&+ \frac{1}{2} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \langle \kappa_1^a \tau_r \tau_s \prod_{i\neq 1} \tau_{d_i}\rangle_{g-1} \\
&+ \frac{1}{2} \sum_{\substack{c+c'=a \\ I\coprod J=\{2,\ldots,n\}}} \binom{a}{c} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \langle \kappa_1^c \tau_r \prod_{i\in I} \tau_{d_i}\rangle_{g'} \langle \kappa_1^{c'} \tau_s \prod_{i\in J} \tau_{d_i}\rangle_{g-g'} \\
&+ \sum_{k=0}^{a} \frac{(-1)^{a-k}}{k!} \sum_{\substack{m_1+\cdots+m_k=a \\ m_i>0}} \binom{a}{m_1,\ldots,m_k} \\
&\times \sum_{j=1}^{k} \frac{(2(d_1+m_j)+1)!!}{(2m_j+1)!!} \langle \tau_{d_1+m_j} \prod_{i=2}^{n} \tau_{d_i} \prod_{i\neq j} \tau_{m_i+1}\rangle_g \\
&= RHS + \sum_{k\geq 0} \frac{(-1)^{a-k-1}}{(k+1)!} \sum_{b=1}^{a} \sum_{\substack{m_1+\cdots+m_k=a-b \\ m_i>0}} \binom{a}{b} \binom{a-b}{m_1,\ldots,m_k} \\
&\times (k+1) \frac{(2(d_1+b)+1)!!}{(2b+1)!!} \langle \tau_{d_1+b} \prod_{i=2}^{n} \tau_{d_i} \prod_{i=1}^{k} \tau_{m_i+1}\rangle_g \\
&= RHS - \sum_{b=1}^{a} (-1)^b \binom{a}{b} \frac{(2(d_1+b)+1)!!}{(2b+1)!!} \langle \tau_{d_1+b} \prod_{i=2}^{n} \tau_{d_i} \kappa_1^{a-b}\rangle_g \\
&= RHS - LHS + (2d_1+1)!! \langle \prod_{j=1}^{n} \tau_{d_j} \kappa_1^a \rangle_g.
\end{aligned}
$$

So we have proved $RHS = LHS$. □

Proposition 2.4 is also implicitly contained in the arguments of Mulase and Safnuk [**21**].

Theorem 2.5. — *We have*

$$
\begin{aligned}
&\frac{(2d_1+1)!!}{a!} \langle \prod_{j=1}^{n} \tau_{d_j} \kappa_1^a \rangle_g \\
&\quad = \sum_{b=0}^{a} \sum_{j=2}^{n} \frac{(2(b+d_1+d_j)-1)!!}{(a-b)!(2d_j-1)!!} \beta_b \langle \kappa_1^{a-b} \tau_{b+d_1+d_j-1} \prod_{i\neq 1,j} \tau_{d_i}\rangle_g \\
&\quad + \frac{1}{2} \sum_{b=0}^{a} \sum_{r+s=b+d_1-2} \frac{(2r+1)!!(2s+1)!!}{(a-b)!} \beta_b \langle \kappa_1^{a-b} \tau_r \tau_s \prod_{i\neq 1} \tau_{d_i}\rangle_{g-1}
\end{aligned}
$$

$$+\frac{1}{2}\sum_{b=0}^{a}\sum_{\substack{c+c'=a-b\\ I\coprod J=\{2,\ldots,n\}}}\sum_{r+s=b+d_1-2}\frac{(2r+1)!!(2s+1)!!}{c!c'!}\beta_b$$
$$\times\langle\kappa_1^c\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa_1^{c'}\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},$$

where the constants β_k are given by

$$(\sum_{k=0}^{\infty}\beta_k x^k)^{-1}=\frac{\sin\sqrt{2x}}{\sqrt{2x}}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(2k+1)!!}x^k.$$

Proof. — Denote the LHS by $F(a,d_1)$. Let

$$G(a,d_1)=\sum_{j=2}^{n}\frac{(2(d_1+d_j)-1)!!}{a!(2d_j-1)!!}\langle\kappa_1^a\tau_{d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g$$
$$+\frac{1}{2}\sum_{r+s=d_1-2}\frac{(2r+1)!!(2s+1)!!}{a!}\langle\kappa_1^a\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1}$$
$$+\frac{1}{2}\sum_{\substack{c+c'=a\\ I\coprod J=\{2,\ldots,n\}}}\sum_{r+s=d_1-2}\frac{(2r+1)!!(2s+1)!!}{c!c'!}\times\langle\kappa_1^c\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa_1^{c'}\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},$$

Note that Proposition 2.4 is just

$$\sum_{b=0}^{a}\frac{(-1)^b}{b!(2b+1)!!}F(a-b,d_1+b)=G(a,d_1).$$

By Lemmas 2.1 and 2.2, we have

$$F(a,d_1)=\sum_{b=0}^{a}\beta_b G(a-b,d_1+b)=RHS.$$

So we conclude the proof. □

3. Higher Weil-Petersson volumes

Mirzakhani's formula provides a recursive way of computing the following Weil-Petersson volumes of moduli spaces of curves

$$WP(g):=\int_{\overline{\mathcal{M}}_{g,n}}\kappa_1^{3g-3+n}.$$

Mirzakhani's formula resorts to intersection numbers of mixed ψ and κ classes.

A natural question is whether there exist an explicit formula expressing $WP(g)$ in terms of those $WP(g')$ with $g'<g$. Recall the following beautiful formula due to Itzykson-Zuber [**9**].

Proposition 3.1 **(Itzykson-Zuber).** — *Let* $g \geq 0$. *Then*

$$\phi_{g+1} = \frac{25g^2-1}{24}\phi_g + \frac{1}{2}\sum_{m=1}^{g}\phi_{g+1-m}\phi_m,$$

where $\phi_0 = -1, \phi_1 = \frac{1}{24}$ *and*

$$\phi_g = \frac{(5g-5)(5g-3)}{2^g(3g-3)!}\langle\tau_2^{3g-3}\rangle_g, \quad g \geq 2.$$

By projection formula, we have

$$\langle\tau_2^{3g-3}\rangle_g = \langle\kappa_1^{3g-3}\rangle_g + \cdots,$$

where $\cdots$ denote terms involving higher degree kappa classes. Also note that $\langle\kappa_1^{3g-3}\rangle_g$ is conjecturally [**16**] the largest term in the right hand side.

To our disappointment, so far, all recursion formulae for $WP(g)$ stemming from the Witten-Kontsevich theorem involve either ψ class or higher degree κ classes inevitably.

Mirzakhani, Mulase and Safnuk's arguments use Wolpert's formula [**25**]

$$\kappa_1 = \frac{1}{2\pi^2}\omega_{WP},$$

where ω_{WP} is the Weil-Petersson Kähler form. We have no similar formulae for higher degree κ classes. So a priori κ_1 may be rather special in the intersection theory. However, as we will see, this is not the case.

First we fix notations as in [**11**]. Consider the semigroup N^∞ of sequences $\mathbf{m} = (m(1), m(2), \ldots)$ where $m(i)$ are nonnegative integers and $m(i) = 0$ for sufficiently large i.

Let $\mathbf{m}, \mathbf{t}, \mathbf{a_1}, \ldots, \mathbf{a_n} \in N^\infty$, $\mathbf{m} = \sum_{i=1}^n \mathbf{a_i}$, $\mathbf{m} \geq \mathbf{t}$ and $\mathbf{s} := (s_1, s_2, \ldots)$ be a family of independent formal variables.

$$|\mathbf{m}| := \sum_{i\geq 1} im(i), \quad ||\mathbf{m}|| := \sum_{i\geq 1} m(i), \quad \mathbf{s}^{\mathbf{m}} := \prod_{i\geq 1} s_i^{m(i)}, \quad \mathbf{m}! := \prod_{i\geq 1} m(i)!,$$

$$\binom{\mathbf{m}}{\mathbf{t}} := \prod_{i\geq 1}\binom{m(i)}{t(i)}, \quad \binom{\mathbf{m}}{\mathbf{a_1}, \ldots, \mathbf{a_n}} := \prod_{i\geq 1}\binom{m(i)}{a_1(i), \ldots, a_n(i)}.$$

Let $\mathbf{b} \in N^\infty$, we denote a formal monomial of κ classes by

$$\kappa(\mathbf{b}) := \prod_{i\geq 1}\kappa_i^{b(i)}.$$

We are interested in the following intersection numbers

$$\langle\kappa(\mathbf{b})\tau_{d_1}\cdots\tau_{d_n}\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \kappa(\mathbf{b})\psi_1^{d_1}\cdots\psi_n^{d_n}.$$

When $d_1 = \cdots = d_n = 0$, these intersection numbers are called higher Weil-Petersson volumes of moduli spaces of curves. The details of the following discussions are contained in [**17**].

The following lemma is a direct generalization of Lemma 2.2.

Lemma 3.2. — *Let $F(\mathbf{L},n)$ and $G(\mathbf{L},n)$ be two functions defined on $N^\infty\times\mathbb{N}$, where $\mathbb{N}=\{0,1,2,\ldots\}$ is the set of nonnegative integers. Let $\alpha_{\mathbf{L}}$ and $\beta_{\mathbf{L}}$ be real numbers depending only on $\mathbf{L}\in N^\infty$ that satisfy $\alpha_{\mathbf{0}}\beta_{\mathbf{0}}=1$ and*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\alpha_{\mathbf{L}}\beta_{\mathbf{L}'}=0,\qquad \mathbf{b}\neq 0.$$

Then the following two identities are equivalent:

$$G(\mathbf{b},n)=\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\alpha_{\mathbf{L}}F(\mathbf{L}',n+|\mathbf{L}|),\quad \forall\ (\mathbf{b},n)\in N^\infty\times\mathbb{N},$$
$$F(\mathbf{b},n)=\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\beta_{\mathbf{L}}G(\mathbf{L}',n+|\mathbf{L}|),\quad \forall\ (\mathbf{b},n)\in N^\infty\times\mathbb{N}.$$

We may generalize Mirzakhani's recursion formula to include higher degree κ classes.

Theorem 3.3. — *There exist (uniquely determined) rational numbers $\alpha_{\mathbf{L}}$ depending only on $\mathbf{L}\in N^\infty$, such that for any $\mathbf{b}\in N^\infty$ and $d_j\geq 0$, the following recursion relation of mixed ψ and κ intersection numbers holds.*

$$\begin{aligned}
&(2d_1+1)!!\langle\kappa(\mathbf{b})\prod_{j=1}^{n}\tau_{d_j}\rangle_g\\
&=\sum_{j=2}^{n}\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\alpha_{\mathbf{L}}\binom{\mathbf{b}}{\mathbf{L}}\frac{(2(|\mathbf{L}|+d_1+d_j)-1)!!}{(2d_j-1)!!}\langle\kappa(\mathbf{L}')\tau_{|\mathbf{L}|+d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g\\
&+\frac{1}{2}\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\ \sum_{r+s=|\mathbf{L}|+d_1-2}\alpha_{\mathbf{L}}\binom{\mathbf{b}}{\mathbf{L}}(2r+1)!!(2s+1)!!\langle\kappa(\mathbf{L}')\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1}\\
&\quad+\frac{1}{2}\sum_{\substack{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b}\\ I\coprod J=\{2,\ldots,n\}}}\ \sum_{r+s=|\mathbf{L}|+d_1-2}\alpha_{\mathbf{L}}\binom{\mathbf{b}}{\mathbf{L},\mathbf{e},\mathbf{f}}(2r+1)!!(2s+1)!!\\
&\qquad\qquad\times\langle\kappa(\mathbf{e})\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa(\mathbf{f})\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.
\end{aligned}$$

These tautological constants $\alpha_{\mathbf{L}}$ can be determined recursively from the following formula

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\frac{(-1)^{||\mathbf{L}||}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!}=0,\qquad \mathbf{b}\neq 0,$$

namely

$$\alpha_{\mathbf{b}}=\mathbf{b}!\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ \mathbf{L}'\neq\mathbf{0}}}\frac{(-1)^{||\mathbf{L}'||-1}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!},\qquad \mathbf{b}\neq 0,$$

with the initial value $\alpha_{\mathbf{0}}=1$.

Theorem 3.4. — *We have*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}(-1)^{||\mathbf{L}||}\binom{\mathbf{b}}{\mathbf{L}}\frac{(2d_1+2|\mathbf{L}|+1)!!}{(2|\mathbf{L}|+1)!!}\langle\kappa(\mathbf{L}')\tau_{d_1+|\mathbf{L}|}\prod_{j=2}^{n}\tau_{d_j}\rangle_g$$
$$=\sum_{j=2}^{n}\frac{(2(d_1+d_j)-1)!!}{(2d_j-1)!!}\langle\kappa(\mathbf{b})\tau_{d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g$$
$$+\frac{1}{2}\sum_{r+s=|d_1|-2}(2r+1)!!(2s+1)!!\langle\kappa(\mathbf{b})\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1}$$
$$+\frac{1}{2}\sum_{\substack{\mathbf{e}+\mathbf{f}=\mathbf{b}\\ I\coprod J=\{2,\ldots,n\}}}\sum_{r+s=d_1-2}\binom{\mathbf{b}}{\mathbf{e}}(2r+1)!!(2s+1)!!$$
$$\times\langle\kappa(\mathbf{e})\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa(\mathbf{f})\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.$$

Theorem 3.3 and Theorem 3.4 implies each other through Lemma 3.2.

Both Theorems 3.3 and 3.4 are effective recursion formulae for computing higher Weil-Petersson volumes with the three initial values

$$\langle\tau_0\kappa_1\rangle_1=\frac{1}{24},\qquad\langle\tau_0^3\rangle_0=1,\qquad\langle\tau_1\rangle_1=\frac{1}{24}.$$

From the following Proposition 3.4, we have

$$\langle\kappa(\mathbf{b})\rangle_g=\frac{1}{2g-2}\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}(-1)^{||\mathbf{L}||}\binom{\mathbf{b}}{\mathbf{L}}\langle\tau_{|\mathbf{L}|+1}\kappa(\mathbf{L}')\rangle_g.$$

We have computed a table of $\alpha_{\mathbf{L}}$ for all $|\mathbf{L}|\leq 15$ and have written a Maple program [**1**] implementing Theorems 3.3 and 3.4.

In fact, we find that ψ and κ classes are compatible in the sense that recursions of pure ψ classes can be neatly generalized to recursions including both ψ and κ classes by the same proof as Proposition 2.4. In view of Theorem 3.8 below, this can be rephrased as differential equations governing generating functions of ψ classes also govern generating functions of mixed ψ and κ classes.

We present some examples below.

Proposition 3.5. — *Let* $\mathbf{b}\in N^\infty$ *and* $d_j\geq 0$. *Then*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}(-1)^{||\mathbf{L}||}\binom{\mathbf{b}}{\mathbf{L}}\langle\tau_{|\mathbf{L}|+1}\prod_{j=1}^{n}\tau_{d_j}\kappa(\mathbf{L}')\rangle_g=(2g-2+n)\langle\prod_{j=1}^{n}\tau_{d_j}\kappa(\mathbf{b})\rangle_g.$$

The above proposition is a generalization of the dilaton equation. In the special case $\mathbf{b}=(m,0,0,\ldots)$, it has been proved by Norman Do and Norbury [**5**].

Proposition 3.6. — *Let* $\mathbf{b}\in N^\infty$. *Then*

$$
\begin{aligned}
\langle\tau_0\tau_1\prod_{j=1}^n \tau_{d_j}\kappa(\mathbf{b})\rangle_g &= \frac{1}{12}\langle\tau_0^4\prod_{j=1}^n \tau_{d_j}\kappa(\mathbf{b})\rangle_g \\
&\quad + \frac{1}{2}\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ \underline{n}=I\coprod J}} \binom{\mathbf{b}}{\mathbf{L}}\langle\tau_0^2\prod_{i\in I}\tau_{d_i}\kappa(\mathbf{L})\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\kappa(\mathbf{L}')\rangle_{g-g'}.
\end{aligned}
$$

The above proposition, together with the projection formula, can be used to derive an effective recursion formula for higher Weil-Petersson volumes [**17**] (without ψ classes).

Let $\mathbf{s} := (s_1, s_2, \dots)$ and $\mathbf{t} := (t_0, t_1, t_2, \dots)$, we introduce the following generating function

$$
G(\mathbf{s},\mathbf{t}) := \sum_g \sum_{\mathbf{m},\mathbf{n}} \langle\kappa_1^{m_1}\kappa_2^{m_2}\cdots\tau_0^{n_0}\tau_1^{n_1}\cdots\rangle_g \frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!}\prod_{i=0}^\infty \frac{t_i^{n_i}}{n_i!},
$$

where $\mathbf{s}^{\mathbf{m}} = \prod_{i\geq 1} s_i^{m_i}$.

Following Mulase and Safnuk [**21**], we introduce the following family of differential operators for $k \geq -1$,

$$
\begin{aligned}
V_k = &-\frac{1}{2}\sum_{\mathbf{L}}(2(|\mathbf{L}|+k)+3)!!\frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!(2|\mathbf{L}|+1)!!}\mathbf{s}^{\mathbf{L}}\frac{\partial}{\partial t_{|\mathbf{L}|+k+1}} \\
&+\frac{1}{2}\sum_{j=0}^\infty \frac{(2(j+k)+1)!!}{(2j-1)!!}t_j\frac{\partial}{\partial t_{j+k}} + \frac{1}{4}\sum_{d_1+d_2=k-1}(2d_1+1)!!(2d_2+1)!!\frac{\partial^2}{\partial t_{d_1}\partial t_{d_2}} \\
&+\frac{\delta_{k,-1}t_0^2}{4}+\frac{\delta_{k,0}}{48}.
\end{aligned}
$$

Theorem 3.7 **([17, 21])**. — *The recursion of Theorem 3.4 implies*

$$
V_k \exp(G) = 0.
$$

Moreover, we can check directly that the operators V_k, $k \geq -1$ satisfy the Virasoro relations

$$
[V_n, V_m] = (n-m)V_{n+m}.
$$

The Witten-Kontsevich theorem states that the generating function for ψ class intersections

$$
F(t_0, t_1, \dots) = \sum_g \sum_{\mathbf{n}} \langle\prod_{i=0}^\infty \tau_i^{n_i}\rangle_g \prod_{i=0}^\infty \frac{t_i^{n_i}}{n_i!}
$$

is a τ-function for the KdV hierarchy.

Theorem 3.8 **([17, 21])**. — *We have*

$$
G(\mathbf{s}, t_0, t_1, \dots) = F(t_0, t_1, t_2 + p_2, t_3 + p_3, \dots),
$$

where p_k are polynomials in $\mathbf{s}$ *given by*

$$p_k = \sum_{|\mathbf{L}|=k-1} \frac{(-1)^{||\mathbf{L}||-1}}{\mathbf{L}!}\mathbf{s}^{\mathbf{L}}.$$

In particular, for any fixed values of $\mathbf{s}$, $G(\mathbf{s},\mathbf{t})$ *is a* τ*-function for the KdV hierarchy.*

At a final remark, it would be interesting to prove that $\alpha_{\mathbf{L}}$ in Theorem 3.3 are positive for all $\mathbf{L} \in N^{\infty}$. This problem is kindly pointed out to us by a referee.

More generally the question can be formulated as following: two sequences $\alpha_{\mathbf{L}}$ and $\beta_{\mathbf{L}}$ with $\alpha_{\mathbf{0}} = \beta_{\mathbf{0}} = 1$ are said to be inverse to each other if they satisfy

$$\left(\sum_{\mathbf{L}} \alpha_{\mathbf{L}}\mathbf{s}^{\mathbf{L}}\right) \cdot \left(\sum_{\mathbf{L}} \beta_{\mathbf{L}}\mathbf{s}^{\mathbf{L}}\right) = 1.$$

Find sufficient conditions on $\beta_{\mathbf{L}}$ such that $\alpha_{\mathbf{L}} > 0$ for all $\mathbf{L}$.

We conjecture that $\alpha_{\mathbf{L}}$ are positive when $\sum_{\mathbf{L}} \beta_{\mathbf{L}}\mathbf{s}^{\mathbf{L}}$ equals any of the following.

$$\sum_{\mathbf{L}} \frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!(2|\mathbf{L}|+1)!!}\mathbf{s}^{\mathbf{L}}, \quad \sum_{\mathbf{L}} \frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!(2|\mathbf{L}|-1)!!}\mathbf{s}^{\mathbf{L}}, \quad \sum_{\mathbf{L}} \frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!|\mathbf{L}|!}\mathbf{s}^{\mathbf{L}}.$$

The latter two arise when we consider Hodge integrals involving λ classes [**17**].

For works on the positivity criteria of coefficients of reciprocal power series of a single variable, see for example [**15**]. However it seems there is no literature dealing with the coefficients of reciprocal series of several variables.

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Astérisque
328, 2009, p. 237–253

FORMES AUTOMORPHES ET THÉORÈMES DE RIEMANN-ROCH ARITHMÉTIQUES

par

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À Jean-Michel Bismut, avec admiration

Résumé. — Nous construisons trois familles de formes automorphes au moyen du théorème de Riemann-Roch arithmétique et de la formule de Lefschetz arithmétique. Deux de ces familles ont déjà été construites par Yoshikawa et notre construction met en lumière leur origine arithmétique.

Abstract **(Automorphic forms and arithmetic Riemann-Roch theorems).** — We construct three families of automorphic forms using the arithmetic Riemann-Roch theorem and the arithmetic Lefschetz formula. Two of these families have already been constructed by Yoshikawa and our construction displays their arithmetic origin.

1. Introduction

Le but de ce texte est de donner une interprétation arithmétique et géométrique à trois familles de formes automorphes d'expression analytique. Plus précisément, on démontre que ces formes automorphes sont algébriques et entières, lorsque les espaces sous-jacents ont des modèles entiers.

La première est la famille de formes modulaires de Siegel construite par Yoshikawa dans [**26**] (voir aussi [**14**] pour une autre construction). Notre calcul démontre une version légèrement affaiblie d'une conjecture de Yoshikawa sur les coefficients de Fourier de ces formes modulaires.

La deuxième est la famille de formes modulaires d'Igusa « produit des thêta constantes paires », souvent notées χ_g. Les formes modulaires χ_g dégénèrent au voisinage des variétés abéliennes munies d'un diviseur thêta singulier et notre calcul

***Classification mathématique par sujets* (2010).** — 11F55, 14G40.
Mots clefs. — Formes automorphes, théorème de Riemann-Roch arithmétique.

fournit une expression géométrique pour cette dégénérescence, dans le cas où le diviseur thêta singulier est défini sur un corps fini.

La troisième est la famille de formes automorphes à coefficients sur certains espaces de modules de surfaces K3 ; lorsque l'involution est sans point fixe, elles coïncident avec certaines fonctions Φ de Borcherds (cf. [**27**, Sec. 8]). Cette famille de formes est construite par Yoshikawa dans [**27**, Th. 5.2]. Notre calcul démontre en particulier que les fonctions Φ ci-dessus sont d'origine arithmétique.

Dans l'appendice, nous formulons une extension conjecturale de la formule de Lefschetz arithmétique, où des irrégularités sont autorisées sur les fibres finies. Cette formule n'est pas appliquée dans le présent texte mais elle représente un moyen théorique d'étudier la dégénérescence de la deuxième forme modulaire de Yoshikawa lorsqu'on considère une surface K3 avec involution définie sur un corps de nombres et ayant mauvaise réduction en certaines places finies.

Dans ce texte, nous utiliserons librement la terminologie et les résultats énoncés dans la section 4 de [**13**] (article dans lequel la formule de Lefschetz arithmétique mentionnée plus haut est démontrée). Par ailleurs, nous utiliserons la terminologie et les résultats de [**11**] (article dans lequel le théorème de Riemann-Roch arithmétique en degré 1 est démontré).

L'objet du présent texte est de présenter des calculs. Pour une introduction au théorème de Riemann-Roch arithmétique et à la formule de Lefschetz arithmétique, nous suggérons de consulter les articles originaux cités dans le dernier paragraphe, ainsi que [**8**] ou encore les notes [**22**].

Les résultats de la partie 4 ont fait l'objet d'une communication par les auteurs lors de la conférence « Arithmetic Algebraic Geometry » organisée au R.I.M.S. (Université de Kyoto, Japon) en septembre 2006.

Remerciements. — Une partie de ce travail a été réalisée alors que le premier auteur était professeur invité au R.I.M.S. ; il lui est très agréable de remercier cette institution pour son hospitalité et les conditions de travail exceptionnelles dont il a pu bénéficier. Nos remerciements vont également à K.-I. Yoshikawa, pour toutes les explications qu'il nous a fournies sur ses travaux, ainsi qu'au rapporteur pour sa lecture très attentive du manuscrit. Enfin, les auteurs sont reconnaissants à S. Tang de leur avoir signalé une erreur dans la formulation initiale de la Conjecture 5.1.

2. Les formes modulaires de Yoshikawa de premier type

Soit S le spectre d'un anneau arithmétique. Soit B une variété arithmétique sur S. Dans ce texte, on appellera *variété arithmétique* sur S un schéma intègre et régulier, qui est quasi-projectif sur S. Soit $\pi : \mathcal{A} \to B$ un schéma abélien de dimension relative

g. Soit $h : \Theta \to B$ un morphisme lisse et propre de dimension relative $g-1$ et $\theta : \Theta \hookrightarrow \mathcal{A}$ une B-immersion fermée. On suppose que $\mathcal{O}(\Theta)$ est un fibré relativement ample et que le degré de $\mathcal{O}(\Theta)$ est $g!$ sur chaque fibre géométrique de $\mathcal{A}/B$. Une hypothèse équivalente est que la caractéristique d'Euler de $\mathcal{O}(\Theta)$ vaut 1 sur chaque fibre géométrique de $\mathcal{A}/B$.

Nous noterons $T\Theta := \Omega^{\vee}_{\Theta/B}$ et $\mathrm{T}\mathcal{A} := \Omega^{\vee}_{\mathcal{A}/B}$. Nous écrirons $u : B \to \mathcal{A}$ pour la section unité et $\mathrm{T}\mathcal{A}_0$ pour $u^*\mathrm{T}\mathcal{A}$. Nous écrirons aussi $\omega := u^* \det(\Omega_{\mathcal{A}/B})$. On note $\overline{\mathcal{O}}(\Theta)$ le fibré $\mathcal{O}(\Theta)$ muni de sa métrique de Moret-Bailly (voir [**16**, Par. 3.2]) et on pose $\overline{L} := \overline{\mathcal{O}}(\Theta) \otimes \pi^* u^* \overline{\mathcal{O}}(\Theta)^{\vee}$. La forme $2\pi \cdot \mathrm{c}_1(\overline{L})$ définit une structure de fibration Kählerienne sur $\mathcal{A}$ au sens de [**4**, Par. 1]. Soit N le fibré conormal de l'immersion θ.

Le *morphisme de Gauss* est défini de la manière suivante. Le morphisme naturel $\mathrm{T}\Theta \hookrightarrow \theta^*\mathrm{T}\mathcal{A}$ induit un morphisme $\Theta \to \mathrm{Gr}(g-1, \theta^*\mathrm{T}\mathcal{A})$. Utilisant les isomorphismes naturels $\mathrm{Gr}(g-1, \theta^*\mathrm{T}\mathcal{A}) \simeq \theta^* \mathrm{Gr}(g-1, \mathrm{T}\mathcal{A})$, $\mathrm{Gr}(g-1, \pi^*\mathrm{T}\mathcal{A}_0) \simeq \pi^* \mathrm{Gr}(g-1, \mathrm{T}\mathcal{A}_0)$ et $\mathrm{T}\mathcal{A} \simeq \pi^*\mathrm{T}\mathcal{A}_0$, on obtient un morphisme naturel $\Theta \to h^* \mathrm{Gr}(g-1, \mathrm{T}\mathcal{A}_0)$. Si l'on compose ce dernier avec la projection naturelle de $h^* \mathrm{Gr}(g-1, \mathrm{T}\mathcal{A}_0) = \mathrm{Gr}(g-1, \mathrm{T}\mathcal{A}_0) \times_B \Theta$ sur le premier facteur, on obtient le morphisme de Gauss $\gamma : \Theta \to \mathrm{Gr}(g-1, \mathrm{T}\mathcal{A}_0)$. On note $p : P := \mathrm{Gr}(g-1, \mathrm{T}\mathcal{A}_0) \to B$ l'application structurale. Pour la définition de $\mathrm{Gr}(\cdot, \cdot)$ voir [**7**, App. B.5.7]. On notera

$$\mathcal{E} : 0 \to E \to p^*T\mathcal{A}_0 \to Q \to 0$$

la suite exacte universelle sur P. Si l'on munit $p^*T\mathcal{A}_0$ de la métrique image réciproque de celle de $T\mathcal{A}_0$ et les fibrés E et Q des métriques induites, on obtient à partir de $\mathcal{E}$ une suite exacte métrisée que nous noterons $\overline{\mathcal{E}}$.

Lemme 2.1. — *Le morphisme de Gauss est génériquement fini de degré $g!$.*

Démonstration. — Le fait que le morphisme de Gauss est génériquement fini (ou autrement dit, qu'il induit une extension finie de corps de fonctions $\kappa(\Theta)|\kappa(\mathrm{Gr}(g-1, \mathrm{T}\mathcal{A}_0))$ est démontré dans [**2**, Th. 4]. Pour calculer son degré, nous considérons le calcul suivant dans la théorie de Chow de $\mathcal{A}$:

$$\begin{aligned} g! &\overset{(1)}{=} \pi_*(\mathrm{c}_1(\mathcal{O}(\Theta))^g) = \pi_*(\theta_*(1)\,\mathrm{c}_1(\mathcal{O}(\Theta))^{g-1}) \overset{(2)}{=} h_*(\mathrm{c}_1(\theta^*(\mathcal{O}(\Theta)))^{g-1}) \\ &\overset{(3)}{=} h_*(\mathrm{c}_1(N^{\vee})^{g-1}) \overset{(4)}{=} p_*\gamma_*(\gamma^*(\mathrm{c}_1(Q)^{g-1})) = \deg(\gamma) p_*(\mathrm{c}_1(Q)^{g-1}) \\ &\overset{(5)}{=} \deg(\gamma). \end{aligned}$$

L'égalité (1) est justifiée par le théorème de Riemann-Roch (appliqué au morphisme π et au fibré $\mathcal{O}(\Theta)$), l'égalité (2) est justifiée par la formule de projection, l'égalité (3) est justifiée par la formule d'adjonction, l'égalité (4) est une conséquence la définition de P et pour finir (5) est une conséquence du fait que le degré de $\mathcal{O}(1)$ sur un espace projectif au-dessus d'un corps vaut 1. □

Nous appliquons maintenant le théorème de Riemann-Roch arithmétique à Θ.

Lemme 2.2. — *Les égalités suivantes*

$$\begin{aligned}\widehat{\mathrm{c}}_1(R^\bullet h_*(\overline{\mathcal{O}}_\Theta)) &= (-1)^g \widehat{\mathrm{c}}_1(R^0\pi_*(\overline{\Omega}^g_{\mathcal{A}})) + \log(\frac{g!!}{(g-1)!!}) \\ &= (-1)^{g+1}\widehat{\mathrm{c}}_1(\overline{\mathrm{T}\mathcal{A}}_0) + \log(\frac{g!!}{(g-1)!!})\end{aligned}$$

sont vérifiées.

On rappelle que par définition de la double factorielle :

$$\frac{g!!}{(g-1)!!} = \frac{g(g-2)\cdots(1+(1+(-1)^g)/2)}{(g-1)(g-3)\cdots(1+(1-(-1)^g)/2)}\ .$$

Démonstration. — Soit M une variété Kählerienne de dimension g et de forme de Kähler $\underline{\omega}$. Soit $k \geqslant 0$ et soit $\nu \in \mathrm{H}^k(M, \mathcal{O}_M)$. On dispose de la formule suivante :

$$|\nu|^2_{L_2} := \frac{i^k(-1)^{k(k+1)/2}}{(2\pi)^g(g-k)!}\int_M \nu\wedge\overline{\nu}\wedge\underline{\omega}^{g-k}. \tag{1}$$

Voir [**15**, Par. 2.3]. Par ailleurs, considérons la suite exacte longue de cohomologie

$$\begin{array}{rl} 0 & \to\ R^0\pi_*\mathcal{O} \to R^0h_*\mathcal{O}_\Theta \to 0 \\ & \to\ R^1\pi_*\mathcal{O} \to R^1h_*\mathcal{O}_\Theta \to 0 \\ & \to\ \cdots \to \\ 0 & \to\ R^{g-1}\pi_*\mathcal{O} \to R^{g-1}h_*\mathcal{O}_\Theta \to R^g\pi_*\mathcal{O}(-\Theta) \xrightarrow{\star} R^g\pi_*\mathcal{O} \to 0 \end{array}$$

née de la suite exacte

$$0 \to \mathcal{O}(-\Theta) \to \mathcal{O} \to \mathcal{O}_\Theta \to 0.$$

On remarque aussi que la flèche $\star$ est un isomorphisme car $R^g\pi_*\mathcal{O}$ est localement libre de rang 1. Toutes les flèches reliant deux objets non-nuls dans la suite exacte longue sont donc des isomorphismes. En particulier les faisceaux de cohomologie $R^kh_*\mathcal{O}_\Theta$ sont localement libres. De plus, en comparant la formule (1) sur les fibres de $\mathcal{A}(\mathbb{C})$ et sur les fibres de $\Theta(\mathbb{C})$, on conclut que pour tout entier k tel que $0 \leqslant k \leqslant g-1$, on a

$$\widehat{\mathrm{c}}_1(R^k\pi_*(\overline{\mathcal{O}}_{\mathcal{A}})) = \widehat{\mathrm{c}}_1(R^kh_*(\overline{\mathcal{O}}_\Theta)) - \log(g-k). \tag{2}$$

On utilisé ici le fait que la forme de Kähler sur les fibres est donnée par $2\pi\cdot\mathrm{c}_1(\overline{\mathcal{O}}(\Theta))$. Si l'on combine cette dernière égalité avec l'égalité

$$\widehat{\mathrm{c}}_1(R^\bullet\pi_*(\overline{\mathcal{O}}_{\mathcal{A}})) = 0 \tag{3}$$

on obtient la première égalité du lemme. L'égalité (3) est une conséquence immédiate du théorème de Riemann-Roch arithmétique appliqué à π et $\overline{\mathcal{O}}_{\mathcal{A}}$ et de l'annulation de la torsion analytique du fibré trivial d'une variété abélienne de dimension $\geqslant 2$ (cf.

[**20**, Par. 5, p. 173]). La deuxième égalité du lemme est une conséquence de la formule de projection. □

Le théorème de Riemann-Roch arithmétique appliqué à h et au fibré hermitien trivial sur Θ donne l'égalité suivante dans $\widehat{\mathrm{CH}}^{\leqslant 1}(B)_{\mathbb{Q}}$:

$$\begin{aligned}
\widehat{\mathrm{ch}}(R^{\bullet}h_*\overline{\mathscr{O}}_\Theta) - T(\Theta, \overline{\mathscr{O}}_\Theta) &= h_*(\widehat{\mathrm{Td}}(\overline{\mathrm{T}h})) - \int_{\Theta/B} R(\mathrm{T}h)\,\mathrm{Td}(\mathrm{T}h) \\
&= g!\; p_*(\widehat{\mathrm{Td}}(\overline{E})) - g!\; \int_{P/B} R(E)\,\mathrm{Td}(E) \\
&= g! \int_{P/B} \mathrm{Td}^{-1}(\overline{Q})\widetilde{\mathrm{Td}}(\overline{\mathscr{E}}) + g!\; p_*(\widehat{\mathrm{Td}}^{-1}(\overline{Q})\widehat{\mathrm{Td}}(p^*\overline{\mathrm{T}\mathscr{A}_0})) \\
&- g! \int_{P/B} R(E)\,\mathrm{Td}(E).
\end{aligned}$$

Remarquons à présent que pour tout $k \geqslant 0$, on a

$$\widehat{\mathrm{Td}}^{-1}(\overline{Q})^{[k]} = -\frac{(-1)^{k+1}}{(k+1)!}\widehat{\mathrm{c}}_1(\overline{Q})^k$$

(on rappelle que par définition $\mathrm{Td}^{-1}(x) = (1 - \exp(-x))/x$). Par ailleurs, selon [**17**, Par. 8.2]

$$p_*(\widehat{\mathrm{c}}_1(\overline{Q})^g)) = [\sum_{l=1}^{g-1} \mathscr{H}_l] + \widehat{\mathrm{c}}_1(\overline{\mathrm{T}\mathscr{A}_0});$$

ici $\mathscr{H}_l := \sum_{k=1}^{l} \frac{1}{k}$ est le l-ème nombre harmonique. On peut maintenant calculer

$$\begin{aligned}
&g!\; p_*(\widehat{\mathrm{Td}}^{-1}(\overline{Q})\widehat{\mathrm{Td}}(p^*\overline{\mathrm{T}\mathscr{A}_0}))^{[\leqslant 1]} \\
&\quad = -g!\; (1 + \frac{1}{2}\widehat{\mathrm{c}}_1(\overline{\mathrm{T}\mathscr{A}_0})) \cdot \Big(\frac{(-1)^g}{g!}\deg(Q) + \frac{(-1)^{g+1}}{(g+1)!}(\widehat{\mathrm{c}}_1(\overline{\mathrm{T}\mathscr{A}_0}) + [\sum_{l=1}^{g-1}\mathscr{H}_l])\Big) \\
&\quad = -g!\; (1 + \frac{1}{2}\widehat{\mathrm{c}}_1(\overline{\mathrm{T}\mathscr{A}_0})) \cdot \Big(\frac{(-1)^g}{g!} + \frac{(-1)^{g+1}}{(g+1)!}(\widehat{\mathrm{c}}_1(\overline{\mathrm{T}\mathscr{A}_0}) + [\sum_{l=1}^{g-1}\mathscr{H}_l])\Big) \\
&\quad = -(1 + \frac{1}{2}\widehat{\mathrm{c}}_1(\overline{\mathrm{T}\mathscr{A}_0})) \cdot \Big((-1)^g + \frac{(-1)^{g+1}}{(g+1)}(\widehat{\mathrm{c}}_1(\overline{\mathrm{T}\mathscr{A}_0}) + [\sum_{l=1}^{g-1}\mathscr{H}_l])\Big)
\end{aligned}$$

Si l'on combine ce dernier calcul avec le Lemme 2.2, il vient

$$\begin{aligned} T(\Theta, \overline{\mathscr{O}}_\Theta) &= -g! \int_{P/B} \mathrm{Td}^{-1}(\overline{Q})\widetilde{\mathrm{Td}}(\overline{\mathscr{E}}) \\ &- (-1)^g \left(\frac{g+3}{2g+2}\right) \widehat{\mathrm{c}}_1(\overline{\mathrm{T}\mathscr{C}_0}) - \frac{(-1)^g}{(g+1)} \sum_{l=1}^{g-1} \mathscr{H}_l \\ &+ g! \int_{P/B} R(E)\,\mathrm{Td}(E) + \log\left(\frac{g!!}{(g-1)!!}\right). \end{aligned}$$

Enfin, on a le

Lemme 2.3. — *L'égalité*

$$\int_{P/B} \mathrm{Td}^{-1}(\overline{Q})\widetilde{\mathrm{Td}}(\overline{\mathscr{E}}) = (-1)^g \sum_{p=0}^{[\frac{g}{2}-1]} \frac{\zeta_{\mathbb{Q}}(-1-2p)}{(2p+1)!\,(g-2p-1)!} \mathscr{H}_{2p+1}$$

est vérifiée.

Ici $\zeta_{\mathbb{Q}}(\cdot)$ désigne la fonction zêta de Riemann.

Démonstration. — Il suffit de démontrer l'égalité dans le cas où $S = \operatorname{Spec}\mathbb{C}$ et B est un point. Nous le supposerons donc pendant la preuve du lemme.

À toute série formelle symétrique ϕ on peut associer une classe caractéristique encore notée ϕ et une classe secondaire de Bott-Chern $\tilde{\phi}$ comme dans [**9**, §1].

D'après le théorème fondamental des fonctions symétriques, pour tout fibré vectoriel F la classe $\phi(F)$ s'exprime comme combinaison linéaire finie des classes de Chern de F :

$$\phi(F) = \sum \phi_{i_1,\ldots,i_k} c_{i_1}(F) \cdots c_{i_k}(F).$$

Soit $\overline{\mathscr{F}}$ une suite exacte courte de fibrés vectoriels hermitiens sur une variété complexe X :

$$\overline{\mathscr{F}} : 0 \to \overline{F}' \to \overline{F} \to \overline{F}'' \to 0$$

telle que les métriques sur F' et F'' sont induites par celle sur F et telle que le fibré hermitien $\overline{F}$ est plat, ou même simplement projectivement plat (voir [**23**, §6 et Theorem 4] pour la définition de cette notion et ses conséquences). On déduit de cette dernière hypothèse que $c(\overline{F}' \oplus \overline{F}'') = c(\overline{F})$ en tant que formes, ce qui en appliquant [**9**, Proposition 1.3.1] à chacun des termes monomiaux de :

$$\tilde{\phi}(\overline{\mathscr{F}}) = \sum \phi_{i_1,\ldots,i_k} \widetilde{c_{i_1} \cdots c_{i_k}}(\overline{\mathscr{F}}),$$

montre dans $\widetilde{A}(X) := A(X)/(\operatorname{Im}\partial + \operatorname{Im}\overline{\partial})$ l'égalité :

$$\tilde{\phi}(\overline{\mathscr{F}}) = \sum \phi_{i_1,\ldots,i_k} \sum_{q=1}^{k} c_{i_1}(\overline{F}) \cdots c_{i_{q-1}}(\overline{F}) \cdot \tilde{c}_q(\overline{\mathscr{F}}) \cdot c_{i_{q+1}}(\overline{F}) \cdots c_{i_k}(\overline{F}).$$

Appliquons ce qui précède pour la classe de Todd et la suite exacte universelle $\overline{\mathscr{E}}$. Il vient, en remarquant de plus que $c\,(p^*\overline{\mathrm{T}\mathscr{A}_0}) = 1$, l'égalité :

$$\widetilde{\mathrm{Td}}(\overline{\mathscr{E}}) = \sum_{k=1}^{g} \mathrm{Td}_k\, \tilde{c}_k(\overline{\mathscr{E}}).$$

On sait par ailleurs [**12**, p. 14 Remark 2] que $\mathrm{Td}_k = \mathrm{Td}_{1,\dots,1}$ (k fois). On peut donc écrire :

$$\widetilde{\mathrm{Td}}(\overline{\mathscr{E}}) = \frac{1}{2}\,\tilde{c}_1(\overline{\mathscr{E}}) + \sum_{k=2}^{g} \frac{B_k}{k!}\,\tilde{c}_k(\overline{\mathscr{E}}),$$

où $B_k = -k\,\zeta_{\mathbb{Q}}(1-k)$ pour $k \geqslant 2$ est le k-ième nombre de Bernoulli.

On tire de [**10**, Proposition 5.3] que $\tilde{c}_1(\overline{\mathscr{E}}) = 0$ et que pour $2 \leqslant k \leqslant g$ on a :

$$\tilde{c}_k(\overline{\mathscr{E}}) = (-1)^{k-1}\mathscr{H}_{k-1}\, c_1^{k-1}(\overline{Q}),$$

où $\mathscr{H}_{k-1}$ est le $(k-1)$-ième nombre harmonique.

Mettant ce qui précède bout-à-bout en tenant compte de la nullité des nombres B_{2p+1} pour $p > 0$, on peut finalement écrire :

$$\begin{aligned}
&\int_{\mathbb{P}^{g-1}(\mathbb{C})} \mathrm{Td}^{-1}(\overline{Q})\widetilde{\mathrm{Td}}(\overline{\mathscr{E}}) \\
&\quad = \int_{\mathbb{P}^{g-1}(\mathbb{C})} \left(\sum_{q\geqslant 0} \frac{\left(-c_1(\overline{Q})\right)^q}{(q+1)!}\right) \sum_{p=0}^{[\frac{g}{2}-1]} \frac{\zeta_{\mathbb{Q}}(-1-2p)}{(2p+1)!}\mathscr{H}_{2p+1}\, c_1^{2p+1}(\overline{Q}) \\
&\quad = (-1)^g \sum_{p=0}^{[\frac{g}{2}-1]} \frac{\zeta_{\mathbb{Q}}(-1-2p)}{(2p+1)!\,(g-2p-1)!}\mathscr{H}_{2p+1}.
\end{aligned}$$

□

Enfin, en utilisant la suite exacte universelle $\mathscr{E}$, on calcule que

$$\begin{aligned}
&\int_{P/B} R(E)\,\mathrm{Td}(E) = -\int_{P/B} R(Q)\,\mathrm{Td}^{-1}(Q) \\
&= \int_{P/B} \Big(\big(\sum_{l\geqslant 0}(-1)^{l+1}\frac{x^l}{(l+1)!}\big)\cdot\big(\sum_{m\geqslant 1,\ m \text{ impair}} (2\zeta'_{\mathbb{Q}}(-m) + \mathscr{H}_m\zeta_{\mathbb{Q}}(-m))\cdot\frac{x^m}{m!}\big)\Big)
\end{aligned}$$

où l'on a posé $x = c_1(Q)$. Ainsi

$$\int_{P/B} R(E)\,\mathrm{Td}(E) = -\sum_{k=0}^{[\frac{g}{2}-1]} (-1)^{g-2k}\,\frac{2\zeta'_{\mathbb{Q}}(-1-2k) + \zeta_{\mathbb{Q}}(-1-2k)\mathscr{H}_{2k+1}}{(2k+1)!(g-2k-1)!}.$$

Le théorème suivant résume maintenant nos calculs :

Théorème 2.4. — *L'égalité*

$$\begin{aligned} T(\Theta, \overline{\mathcal{O}}_\Theta) &= (-1)^g \left(\frac{g+3}{2g+2}\right) \widehat{\mathrm{c}}_1(\overline{\omega}) \\ &- (-1)^g \sum_{k=0}^{[\frac{g}{2}-1]} \binom{g}{2k+1} \left(2\zeta'_{\mathbb{Q}}(-1-2k) + 2\zeta_{\mathbb{Q}}(-1-2k)\mathscr{H}_{2k+1}\right) \\ &- \frac{(-1)^g}{(g+1)} \sum_{l=1}^{g-1} \mathscr{H}_l + \log\left(\frac{g!!}{(g-1)!!}\right) \end{aligned}$$

est vérifiée.

Cette dernière égalité implique notamment qu'il existe un nombre entier $l \in \mathbb{N}^*$, un nombre réel C et une forme modulaire μ pour le groupe d'Igusa $\Gamma(1,2)$, tels que

$$\exp(T(\Theta, \overline{\mathcal{O}}_\Theta))^l = \|C \cdot \mu\|_{\mathrm{Pet}}^{2l(-1)^{g+1}(g+3)/(2g+2)} \tag{4}$$

et que μ est définie sur $\mathbb{Q}$. L'égalité (4) est une forme affaiblie de la première partie de la Conjecture 6.1 de Yoshikawa dans [**26**]. Une autre conséquence de l'égalité (4) est une forme affaiblie de la première assertion du « Main Theorem » dans l'introduction de [**26**]. Dans les deux cas, il s'agit d'une forme affaiblie parce que le nombre l n'est pas effectif et que le groupe d'Igusa $\Gamma(1,2)$ est plus petit que le groupe de Siegel.

Remarque. — Il est probable que le nombre l peut être déterminé en faisant usage dans les calculs ci-dessus du théorème d'Adams-Riemann-Roch arithmétique démontré dans [**21**] (qui tient compte des phénomènes de torsion) plutôt que du théorème de Riemann-Roch arithmétique. Nous n'avons cependant pas effectué ce calcul.

3. Les formes modulaires d'Igusa « produit des thêta constantes paires »

Les formes modulaires décrites via la torsion analytique de $\mathcal{O}_\Theta$ dans la dernière section coïncident avec la forme modulaire d'Igusa « produits des fonctions thêta paires » lorsque $g = 2, 3$ (cf. [**26**, après le « Main Theorem »]). On peut se demander si ces dernières formes modulaires peuvent aussi être interprétées via le théorème de Riemann-Roch arithmétique. Nous allons montrer dans cette section qu'une pareille interprétation est possible. Le théorème de Riemann-Roch arithmétique est ici appliqué à $\mathcal{O}(\Theta)$; il s'agit alors d'un cas particulier de la « formule clé » de Moret-Bailly.

On continue avec les mêmes hypothèses que dans la section 2. On suppose de plus que Θ est symétrique, i.e. invariant par l'action de l'inversion $[-1]$ du schéma en groupes $\mathcal{A}/B$; on notera $\mathcal{A}_{[-1]}$ (resp. $\Theta_{[-1]}$) le schéma des points fixes de $[-1]$ dans $\mathcal{A}$ (resp. dans Θ). Par ailleurs, on suppose que 2 est inversible sur B.

Le théorème du cube (cf. par ex. [**6**, 4.1.23]) implique que

$$8 \cdot \widehat{\mathrm{c}}_1(\overline{\mathscr{O}}(\Theta)|_{\mathscr{A}_{[-1]}}) = 8 \cdot \widehat{\mathrm{c}}_1(u^*\overline{\mathscr{O}}(\Theta)).$$

Par ailleurs, la formule clé de Moret-Bailly (cf. [**16**, Th. 3.3]) affirme que

$$8 \cdot \widehat{\mathrm{c}}_1(u^*\overline{\mathscr{O}}(\Theta)) = 4 \cdot \widehat{\mathrm{c}}_1(\overline{\omega}) + 2g\log(4\pi).$$

On en déduit que $8 \cdot \widehat{\mathrm{c}}_1(\overline{\mathscr{O}}(\Theta)|_{\mathscr{A}_{[-1]}}) = 4 \cdot \widehat{\mathrm{c}}_1(\overline{\omega}) + 2g\log(4\pi)$. Comme Θ est lisse sur B, le schéma $\Theta_{[-1]}$ est régulier et donc ouvert dans $\mathscr{A}_{[-1]}$. Soit $g : \mathscr{A}_{[-1]}\backslash\Theta_{[-1]} \to B$ le morphisme de structure. On cherche à calculer

$$g_*(\widehat{\mathrm{c}}_1(\overline{\mathscr{O}}(\Theta)|_{\mathscr{A}_{[-1]}\backslash\Theta_{[-1]}})).$$

Vu que le fibré en droites $\mathscr{O}(\Theta)$ est canoniquement trivialisé sur $\mathscr{A}_{[-1]}\backslash\Theta_{[-1]}$, on est ramené à un calcul analytique sur une fibre complexe arbitraire de $\mathscr{A} \to B$. Nous en fixons une et la nommons A. Nous rappelons l'expression explicite de la métrique de Moret-Bailly de $\mathscr{O}(\Theta)$ donnée dans [**16**, Par. 3, (3.2.2)]. Si Ω est une matrice $g \times g$ complexe du demi-plan de Siegel représentant A, on dispose de la formule

$$\|s_\Theta(z)\| = \det(\Im(\Omega))^{1/4}\exp(-\pi{}^t y(\Im(\Omega))^{-1}y)|\theta(z,\Omega)|$$

où $\theta(z,\Omega)$ est la fonction θ de Riemann associée à Ω, $z = x + iy$ et s_Θ est la section canonique de $\mathscr{O}(\Theta)$, restreinte à A. Si l'on utilise la formule de changement de coordonnées

$$x + iy = x_1 + \Omega(y_1) = (x - \Re(\Omega)(\Im(\Omega))^{-1}y) + \Omega((\Im(\Omega))^{-1}y)$$

on peut réécrire, en utilisant la symétrie de $\Im(\Omega)$,

$$\|s_\Theta(z)\| = \det(\Im(\Omega))^{1/4}\exp(-\pi{}^t y_1\Im(\Omega)y_1)|\theta(z,\Omega)|.$$

On calcule maintenant

$$\begin{aligned} &g_*(\widehat{\mathrm{c}}_1(\overline{\mathscr{O}}(\Theta)|_{\mathscr{A}_{[-1]}\backslash\Theta_{[-1]}})) \\ &\qquad = -2\log|\prod_{a,b}\det(\Im(\Omega))^{1/4}\ \exp(-\pi{}^t a\Im(\Omega)a)\theta(\Omega(a)+b,\Omega)| \\ &\qquad = -\log|\det(\Im(\Omega))|^{2^{2g-2}+2^{g-2}} - \log|\prod\theta\begin{bmatrix}a\\b\end{bmatrix}(0,\Omega)|^2 \\ &\qquad =: -\log\|\chi_g(\Omega)\|^2_{\mathrm{Pet}} \end{aligned}$$

où $\chi_g(\cdot)$ est la forme modulaire d'Igusa « produit des thêta constantes paires » (cf. [**18**, chap. II]) et $\|\cdot\|_{\mathrm{Pet}}$ est la norme associée à la métrique de Petersson. La somme porte sur les couples $(a,b) \in \mathbb{Q}^g$ tels que $a, b \in \{0, 1/2\}$ et tels que $\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](0,\Omega) \neq 0$. Il y a $2^{2g} - 2^{g-1}(2^g - 1) = 2^{2g-1} + 2^{g-1}$ tels couples. Ceci résulte par exemple de la formule de Lefschetz habituelle appliquée à $[-1]$ agissant sur $\Theta_{\mathbb{C}}|_A$. Par ailleurs,

$$8 \cdot g_*(\widehat{\mathrm{c}}_1(\overline{\mathscr{O}}(\Theta)|_{\mathscr{A}_{[-1]}\backslash\Theta_{[-1]}})) = 4 \cdot (2^{2g-1} + 2^{g-1})\widehat{\mathrm{c}}_1(\overline{\omega}) + 2g \cdot (2^{2g-1} + 2^{g-1})\log(4\pi)$$

ce qui implique la

Proposition 3.1. — *L'égalité*

$$(2^{2g-2}+2^{g-2})\widehat{\mathrm{c}}_1(\overline{\omega})+\frac{g}{4}(2^{2g-1}+2^{g-1})\log(4\pi)=-\log\|\chi_g(\Omega)\|^2_{\mathrm{Pet}}$$

est vérifiée.

Supposons à présent que S est le spectre d'un anneau de Dedekind et affaiblissons les hypothèses précédentes en supposant seulement que Θ est lisse au-dessus d'un ouvert non-vide V de B. Supposons de plus que $B=S$ et que le schéma $\mathcal{A}_{[-1]}$ est la réunion disjointe de 2^{2g} sections $B\to\mathcal{A}$.

Soit maintenant

$$Z:=\mathrm{Zar}(\mathcal{A}_{[-1],V}\backslash\Theta_{[-1],V})$$

l'adhérence schématique de $\mathcal{A}_{[-1],V}\backslash\Theta_{[-1],V}$. On cherche alors à calculer

$$g_*(\widehat{\mathrm{c}}_1(\overline{\mathcal{O}}(\Theta)|_Z)).$$

Soit $u_1,\dots,u_{(2^{2g-1}+2^{g-1})}$ une énumération des sections formant Z. Une variante du calcul fait plus haut donne alors la

Proposition 3.2. — *L'égalité*

$$\begin{aligned}(2^{2g-2}+2^{g-2})\widehat{\mathrm{c}}_1(\overline{\omega})&+\frac{g}{4}(2^{2g-1}+2^{g-1})\log(4\pi)\\&=\sum_{j=1}^{2^{2g-1}+2^{g-1}}\ \sum_{P\in u_j\cap\Theta}\mathrm{long}_{\mathcal{O}_{\mathcal{A},P}}(\mathcal{O}_{u_j\cap\Theta})\cdot\pi_*(P)-\log\|\chi_g(\Omega)\|^2_{\mathrm{Pet}}\end{aligned}$$

est vérifiée.

4. Les formes modulaires de Yoshikawa de deuxième type

On se donne un schéma régulier et intègre $\mathcal{B}$, quasi-projectif sur un anneau arithmétique D de corps de fractions K et tel que 2 est inversible sur D. On se donne également un schéma intègre et régulier $\mathcal{T}$ et un morphisme projectif, plat $f:\mathcal{T}\to\mathcal{B}$ tel que $f_K:\mathcal{T}_K\to\mathcal{B}_K$ est lisse et dont on note ω le faisceau canonique relatif associé. On notera $B:=\mathcal{B}_K$ (resp. $T:=\mathcal{T}_K$) la fibre générique de B (resp. $\mathcal{T}$) sur D.

On suppose que T est un schéma en surfaces K3 sur B. Par définition, cela signifie que les fibres géométriques de f_K sont des surfaces K3. On munit $T(\mathbb{C})$ d'une structure de fibration Kählerienne ν. On suppose aussi que le morphisme d'adjonction

$$f^*f_*\omega\to\omega$$

est un isomorphisme. Munissons ω de la métrique induite par la structure de fibration Kählerienne et $f^*f_*\omega$ de la métrique image réciproque par f de la métrique L^2. On écrira η pour la classe secondaire $\widetilde{\mathrm{ch}}(\overline{\mathcal{F}})$ de la suite exacte de fibrés

$$\mathcal{F}:0\to f^*f_*\omega\to\omega\to 0\to 0$$

munie de ces métriques.

On suppose de plus qu'il existe un automorphisme d'ordre 2 de $\mathcal{T}$ sur $\mathcal{B}$; on dispose ainsi d'une action de μ_2 sur $\mathcal{T}$; on note $\mathcal{T}_{\mu_2}$ son schéma des points fixes et l'on suppose que la restriction $f_{\mu_2} : \mathcal{T}_{\mu_2} \to \mathcal{B}$ du morphisme f est lisse. On notera g l'action de l'automorphisme sur $T(\mathbb{C})$ et η_g la restriction de la classe η à $T_g := \mathcal{T}_{\mu_2}(\mathbb{C})$. On suppose également que la fibration Kählerienne est équivariante pour l'action de g ; on note ν_g la restriction de ν à T_g. Enfin on notera d (resp. d_g) la dimension relative de T (resp. T_g) sur B.

On applique la formule de Lefschetz arithmétique au morphisme f, à l'action de μ_2 sur $\mathcal{T}$ et au fibré trivial $\mathcal{O} := \mathcal{O}_{\mathcal{T}}$ muni de sa métrique triviale. On note N le fibré conormal de $\mathcal{T}_{\mu_2}$ dans $\mathcal{T}$ et on le munit de la métrique induite par la structure de fibration Kählerienne. On obtient

$$\begin{aligned}\widehat{\mathrm{c}}_{1,\mu_2}(R^0 f_* \overline{\mathcal{O}}) - \widehat{\mathrm{c}}_{1,\mu_2}(R^1 f_* \overline{\mathcal{O}}) + \widehat{\mathrm{c}}_{1,\mu_2}(R^2 f_* \overline{\mathcal{O}}) \\ = f_{\mu_2 *}(\widehat{\mathrm{Td}}_{\mu_2}(\overline{Tf}))^{[1]} - \int_{T_g/B} \mathrm{Td}_g(Tf_{\mathbb{C}}) R_g(Tf_{\mathbb{C}}) + T_g(\overline{\mathcal{O}}).\end{aligned}$$

On calcule dans $\widehat{\mathrm{CH}}^{\leqslant 2}(\mathcal{T}_{\mu_2})_{\mathbb{Q}}$:

$$\begin{aligned}\widehat{\mathrm{Td}}_{\mu_2}(\overline{Tf}) &= \widehat{\mathrm{ch}}_{\mu_2}(1-\overline{N})^{-1}\widehat{\mathrm{Td}}(\overline{Tf}_{\mu_2}) \\ &= \frac{1}{2}(1+\frac{1}{2}\widehat{\mathrm{c}}_1(\overline{N}) + \frac{1}{4}\widehat{\mathrm{c}}_1(\overline{N})^2)^{-1}\widehat{\mathrm{Td}}(\overline{Tf}_{\mu_2}) \\ &= \Big(\frac{1}{2} - \frac{1}{4}\widehat{\mathrm{c}}_1(\overline{N})\Big)\widehat{\mathrm{Td}}(\overline{Tf}_{\mu_2}).\end{aligned}$$

Par ailleurs, dans $\widehat{\mathrm{CH}}^{\leqslant 2}(\mathcal{T}_{\mu_2})_{\mathbb{Q}}$, on a $\widehat{\mathrm{Td}}(\overline{Tf}_{\mu_2}) = 1 - \frac{1}{2}\widehat{\mathrm{c}}_1(\overline{\omega}_{\mu_2}) + \frac{1}{12}\widehat{\mathrm{c}}_1(\overline{\omega}_{\mu_2})^2$, où $\overline{\omega}_{\mu_2}$ est le fibré des différentielles relatives de $\mathcal{T}_{\mu_2}$ sur $\mathcal{B}$, muni de la métrique induite. La partie de degré 2 de $\widehat{\mathrm{Td}}_{\mu_2}(\overline{Tf})$ est donc la partie de degré 2 de l'expression

$$\Big(\frac{1}{2} - \frac{1}{4}\widehat{\mathrm{c}}_1(\overline{N})\Big)\Big(1 - \frac{1}{2}\widehat{\mathrm{c}}_1(\overline{\omega}_{\mu_2}) + \frac{1}{12}\widehat{\mathrm{c}}_1(\overline{\omega}_{\mu_2})^2\Big)$$

qui est

$$\frac{1}{8}\widehat{\mathrm{c}}_1(\overline{N})\widehat{\mathrm{c}}_1(\overline{\omega}_{\mu_2}) + \frac{1}{24}\widehat{\mathrm{c}}_1(\overline{\omega}_{\mu_2})^2. \tag{5}$$

On rappelle qu'on dispose d'une suite exacte équivariante

$$0 \to N \to \Omega \to \omega_{\mu_2} \to 0$$

sur $\mathcal{T}_{\mu_2}$. Pour des raisons de rang, cette suite est isométriquement scindée. On a donc

$$\widehat{\mathrm{c}}_1(N) = f_{\mu_2}^* \widehat{\mathrm{c}}_1(f_* \overline{\omega}) - \eta_g - \widehat{\mathrm{c}}_1(\overline{\omega}_{\mu_2})$$

dans $\widehat{\mathrm{CH}}^1(\mathscr{T}_{\mu_2})$. On peut donc évaluer l'expression (5) comme

$$\frac{1}{8}\Big(f_{\mu_2}^*\widehat{\mathrm{c}}_1(f_*\overline{\omega}) - \eta_g - \widehat{\mathrm{c}}_1(\overline{\omega}_{\mu_2})\Big)\widehat{\mathrm{c}}_1(\overline{\omega}_{\mu_2}) + \frac{1}{24}\widehat{\mathrm{c}}_1(\overline{\omega}_{\mu_2})^2$$
$$= \frac{1}{8}f_{\mu_2}^*\widehat{\mathrm{c}}_1(f_*\overline{\omega})\widehat{\mathrm{c}}_1(\overline{\omega}_{\mu_2}) - \frac{1}{12}\widehat{\mathrm{c}}_1(\overline{\omega}_{\mu_2})^2 - \frac{1}{8}c_1(\overline{\omega}_{\mu_2})\eta_g.$$

Par ailleurs, on calcule :

$$\begin{aligned}
&\int_{T_g/B}\mathrm{Td}_g(Tf_{\mathbb{C}})R_g(Tf_{\mathbb{C}})\\
&= -2\int_{T_g/B}\Big((2\zeta'_{\mathbb{Q}}(-1,-1)+\zeta_{\mathbb{Q}}(-1,-1))c_1(N) + (2\zeta'_{\mathbb{Q}}(-1)+\zeta_{\mathbb{Q}}(-1))c_1(\omega_{\mu_2})\Big)\\
&= -2\int_{T_g/B}\Big((6\zeta'_{\mathbb{Q}}(-1)+(3-\log(16))\zeta_{\mathbb{Q}}(-1))c_1(N)\\
&\qquad\qquad + (2\zeta'_{\mathbb{Q}}(-1)+\zeta_{\mathbb{Q}}(-1))c_1(\omega_{\mu_2})\Big)\\
&= -2\int_{T_g/B}\Big(-(6\zeta'_{\mathbb{Q}}(-1)+(3-\log(16))\zeta_{\mathbb{Q}}(-1)) + (2\zeta'_{\mathbb{Q}}(-1)+\zeta_{\mathbb{Q}}(-1))\Big)c_1(\omega_{\mu_2})\\
&= -2\int_{T_g/B}\Big(-4\zeta'_{\mathbb{Q}}(-1)+(\log(16)-2)\zeta_{\mathbb{Q}}(-1)\Big)c_1(\omega_{\mu_2})\\
&= -2G\Big(-4\zeta'_{\mathbb{Q}}(-1)+(\log(16)-2)\zeta_{\mathbb{Q}}(-1)\Big)
\end{aligned}$$

où G est une fonction localement constante sur $B(\mathbb{C})$. En un point $P\in B(\mathbb{C})$, G vaut

$$\sum_{C\subseteq T_{g,P}}(2\cdot\mathrm{genre}(C)-2)$$

où la somme porte sur les composantes connexes C de la fibre $T_{g,P}$ de T_g au-dessus de P. Pour résumer, on obtient

$$\begin{aligned}
&\widehat{\mathrm{c}}_{1,\mu_2}(R^0f_*\overline{\mathcal{O}}) - \widehat{\mathrm{c}}_{1,\mu_2}(R^1f_*\overline{\mathcal{O}}) + \widehat{\mathrm{c}}_{1,\mu_2}(R^2f_*\overline{\mathcal{O}})\\
&\qquad = \frac{G}{8}\widehat{\mathrm{c}}_1(f_*\overline{\omega}) - \frac{1}{12}f_{\mu_2*}\widehat{\mathrm{c}}_1(\overline{\omega}_{\mu_2})^2 + T_g(\overline{\mathcal{O}}) - \frac{1}{8}\int_{T_g/B}c_1(\overline{\omega}_{\mu_2})\eta_g\\
&\qquad\qquad + 2G\Big(-4\zeta'_{\mathbb{Q}}(-1)+(\log(16)-2)\zeta_{\mathbb{Q}}(-1)\Big).
\end{aligned}$$

Ceci implique en particulier le

Théorème 4.1. — *Supposons que toutes les fibres géométriques de f sont des surfaces K3, alors on a :*

$$\begin{aligned}
-\log\left|\frac{1}{d!(2\pi)^d}\int_{T/B}\nu^d\right| \\
&= \frac{G-8}{8}\widehat{c}_1(f_*\overline{\omega}) - \frac{1}{12}f_{\mu_2 *}\widehat{c}_1(\overline{\omega}_{\mu_2})^2 + T_g(\overline{\mathscr{O}}) - \frac{1}{8}\int_{T_g/B} c_1(\overline{\omega}_{\mu_2})\eta_g \\
&\qquad + 2G\Big(-4\zeta'_{\mathbb{Q}}(-1) + (\log(16)-2)\zeta_{\mathbb{Q}}(-1)\Big).
\end{aligned}$$

Remarque. — Lorsque $\mathscr{T}_{\mu_2}$ est vide, on trouve :

$$-\log\left|\frac{1}{d!(2\pi)^d}\int_{T/B}\nu^d\right| + \widehat{c}_1(f_*\overline{\omega}) = T_g(\overline{\mathscr{O}}) \tag{6}$$

sous les hypothèses du théorème 4.1. On aurait pu montrer directement ce résultat en appliquant le théorème de Riemann-Roch arithmétique au quotient $\mathscr{T}/\mu_2$. Borcherds **[5]** avait montré la trivialité (modulo torsion) de $f_*\omega$ en construisant explicitement une section non-nulle Φ de $(f_*\omega)^{\otimes 4}$. Pappas **[19]** redémontre (indépendamment de ce qui précède) le fait que $f_*\omega$ est de torsion sur $B(\mathbb{C})$ en appliquant le théorème de Grothendieck-Riemann-Roch au schéma $\mathscr{T}/\mu_2$. L'identité (6) montre que ces deux démonstrations, *a priori* totalement différentes, ne sont que les deux versants d'une même application du théorème de Riemann-Roch arithmétique. On notera que l'identité (6) implique même que l'ordre de $f_*\omega$ est une puissance de 2.

On peut exprimer la quantité $\frac{1}{12}f_{\mu_2 *}\widehat{c}_1(\overline{\omega}_{\mu_2})^2$ du Théorème 4.1 au moyen de la torsion analytique des fibres de T_g sur $B(\mathbb{C})$, via le théorème de Riemann-Roch arithmétique. On obtient

$$\begin{aligned}
\frac{1}{12}f_{\mu_2 *}\widehat{c}_1(\overline{\omega}_{\mu_2})^2 &= f_*(\widehat{\mathrm{Td}}(\mathscr{T}/\mathscr{B}))^{[1]} \\
&= -T(\overline{\mathscr{O}}_g) - \log\left|\frac{1}{d_g!(2\pi)^{d_g}}\int_{T_g/B}\nu_g^{d_g}\right| + \widehat{c}_1(f_{\mu_2 *}\overline{\omega}_{\mu_2}) \\
&\qquad - \int_{T_g/B}(2\zeta'_{\mathbb{Q}}(-1) + \zeta_{\mathbb{Q}}(-1))c_1(\omega_{\mu_2})
\end{aligned}$$

dans $\widehat{\mathrm{CH}}^1(\mathscr{B})_{\mathbb{Q}}$. Si l'on juxtapose cette dernière expression à celle du Théorème 4.1, on obtient

$$\begin{aligned}
\widehat{c}_1(f_{\mu_2 *}\overline{\omega}_{\mu_2}) + \frac{8-G}{8}\widehat{c}_1(f_*\overline{\omega}) \\
&= T_g(\overline{\mathscr{O}}) + T(\overline{\mathscr{O}}_g) - \frac{1}{8}\int_{T_g/B} c_1(\overline{\omega}_{\mu_2})\eta_g + \int_{T_g/B}(2\zeta'_{\mathbb{Q}}(-1) + \zeta_{\mathbb{Q}}(-1))c_1(\omega_{\mu_2})
\end{aligned}$$

$$
\begin{aligned}
&+ 2G\Big(- 4\zeta'_{\mathbb{Q}}(-1) + (\log(16) - 2)\zeta_{\mathbb{Q}}(-1)\Big) \\
&+ \log\left|\frac{1}{d!(2\pi)^d}\int_{T/B}\nu^d\right| + \log\left|\frac{1}{d_g!(2\pi)^{d_g}}\int_{T_g/B}\nu_g^{d_g}\right| \\
&= T_g(\overline{\mathscr{O}}) + T(\overline{\mathscr{O}}_g) - \frac{1}{8}\int_{T_g/B} c_1(\overline{\omega}_{\mu_2})\eta_g - 6G\zeta'_{\mathbb{Q}}(-1) - \frac{2G}{3}\log(2) + \frac{G}{4} \\
&+ \log\left|\frac{1}{d!(2\pi)^d}\int_{T/B}\nu^d\right| + \log\left|\frac{1}{d_g!(2\pi)^{d_g}}\int_{T_g/B}\nu_g^{d_g}\right|
\end{aligned}
$$

sous les hypothèses du Théorème 4.1. On suppose maintenant que $D = \mathbb{C}$; les hypothèses du Théorème 4.1 sont alors automatiquement satisfaites. Supposons par ailleurs que $f_*\omega$ a une section analytique trivialisante de norme L^2 constante. Ceci est le cas par exemple si la famille $\mathscr{T}$ est munie d'un marquage (cf. [**27**, Par. 1.2 (b)] pour cette notion). Soit κ un entier tel que le fibré $f_*\omega^{\otimes(8-G)\kappa} \otimes (\det f_{\mu_2 *}\omega_{\mu_2})^{\otimes 8\kappa}$ est trivial. Le fibré $(\det f_{\mu_2 *}\omega_{\mu_2})^{\otimes(-8\kappa)}$ est alors analytiquement trivial. Il existe donc t une section analytique trivialisante de $(\det f_{\mu_2 *}\omega_{\mu_2})^{\otimes 8\kappa}$ satisfaisant l'égalité

$$
\begin{aligned}
|t|_{L^2}^{-\frac{1}{4\kappa}} &= e^{T_g(\overline{\mathscr{O}})} \cdot e^{T(\overline{O}_g)} \\
&\cdot \left|\frac{1}{d!(2\pi)^d}\int_{T/B}\nu^d\right| \cdot \left|\frac{1}{d_g!(2\pi)^{d_g}}\int_{T_g/B}\nu_g^{d_g}\right| \cdot \exp(-\frac{1}{8}\int_{T_g/B} c_1(\overline{\omega}_{\mu_2})\eta_g).
\end{aligned}
$$

Écrivons $\mathrm{Vol}(T_g) := |\frac{1}{d_g!(2\pi)^{d_g}}\int_{T_g/B}\nu_g^{d_g}|$ et $\mathrm{Vol}(T) := |\frac{1}{d!(2\pi)^d}\int_{T/B}\nu^d|$. Soit r_+ (resp. r_-) la dimension du sous-espace de $H^2(T(\mathbb{C})_b, \mathbb{C})$ invariant par g (resp. celui où g agit par -1) ; b étant un élément générique de $B(\mathbb{C})$. Remarquons que par la formule du point fixe holomorphe et la formule de Gauss-Bonnet généralisée (cf. [**25**, Example 3.8, chap. III, sec. 3, p.96] pour cette dernière), on a l'égalité

$$1 - 0 + r_+ - r_- + 1 - 0 = -G$$

et par ailleurs, le formulaire [**3**, VIII, 3.] nous assure que $r_+ + r_- = 22$. On en déduit que

$$G = 20 - 2r_+ .$$

On reprend maintenant l'expression pour $|t|_{L^2}^{-\frac{1}{4\kappa}}$ et on calcule

$$
\begin{aligned}
&e^{T_g(\overline{\mathscr{O}})}.e^{T(\overline{O}_g)} \cdot \mathrm{Vol}(T) \cdot \mathrm{Vol}(T_g) \cdot \exp(-\frac{1}{8}\int_{T_g/B} c_1(\overline{\omega}_{\mu_2})\eta_g) \\
&\quad = \; e^{T_g(\overline{\mathscr{O}})} \cdot e^{T(\overline{O}_g)} \cdot \mathrm{Vol}(T) \cdot \mathrm{Vol}(T_g) \\
&\qquad\qquad \cdot \exp\left(-\frac{1}{8}\int_{T_g/B} c_1(\overline{\omega}_{\mu_2})(\eta_g + \log|\mathrm{Vol}(T)|)\right) \cdot \mathrm{Vol}(T)^{\frac{G}{8}}
\end{aligned}
$$

$$= e^{T_g(\overline{\mathcal{O}})} \cdot e^{T(\overline{O}_g)} \cdot \mathrm{Vol}(T)^{G/8+1} \cdot \mathrm{Vol}(T_g) \\ \cdot \exp\left(-\frac{1}{8}\int_{T_g/B} c_1(\overline{\omega}_{\mu_2})(\eta_g + \log|\mathrm{Vol}(T)|)\right).$$

Par ailleurs, on a

$$G/8+1 = \frac{G+8}{8} = \frac{20-2r_+ + 8}{8} = \frac{14-r_+}{4}$$

et on conclut que

$$|t|_{L^2}^{-\frac{1}{4\kappa}} = e^{T_g(\overline{\mathcal{O}})} \cdot e^{T(\overline{O}_g)} \\ \cdot \mathrm{Vol}(T)^{\frac{14-r_+}{4}} \cdot \mathrm{Vol}(T_g) \cdot \exp\left(-\frac{1}{8}\int_{T_g/B} c_1(\overline{\omega}_{\mu_2})(\eta_g + \log|\mathrm{Vol}(T)|)\right).$$

Il s'agit de l'égalité du théorème principal [**27**, Main Th., Introduction] de Yoshikawa.

5. Appendice : une formule du point fixe singulière conjecturale en théorie d'Arakelov

Soit D un anneau arithmétique d'anneau de fractions K et supposons que D est régulier. Soit $f : X \to \operatorname{Spec} D$ un schéma intègre, projectif sur D, dont la fibre sur K est lisse. Soit $h : Z \to \operatorname{Spec} D$ un schéma intègre et régulier, projectif sur D, dont la fibre sur K est lisse. Soit j une D-immersion fermée $X \hookrightarrow Z$. On munit Z d'une métrique Kählerienne ω_Z et on munit X de la structure ω_X induite. On se donne un nombre entier $n \geqslant 1$ et des structures μ_n-équivariantes sur X et Z telle que f, h, j soient μ_n-équivariants et que la structure ω_Z soit $\mu_n(\mathbb{C})$-invariante (D est supposé muni de la structure équivariante triviale). Soit enfin $R(\mu_n) = \mathbb{Z}/(1-T^n)$ le groupe de Grothendieck des μ_n-comodules de type fini sur $\mathbb{Z}$. On choisit un racine primitive n-ième de l'unité ζ_n et une $R(\mu_n)$-algèbre $\mathscr{R}$ telle que les éléments $1-T^k$ ($k = 1, \dots, n-1$) sont inversibles dans $\mathscr{R}$.

Soit N le fibré conormal de l'immersion $Z_{\mu_n} \hookrightarrow Z$. Soit enfin $\sum_i r_i \overline{E}_i$ une $\mathscr{R}$ combinaison linéaire finie de fibrés hermitiens sur Z_{μ_n} tels que $\sum_i r_i \overline{E}_i = (\Lambda_{-1}(\overline{N}))^{-1}$ dans $\widehat{\mathrm{K}}_0^{\mu_n}(Z_{\mu_n}) \otimes_{R(\mu_n)} \mathscr{R}$.

Soit $\overline{E}$ un fibré hermitien μ_n-équivariant sur X.

On remarque que l'immersion $X_{\mathbb{C}} \hookrightarrow Z_{\mathbb{C}}$ est régulière et on a donc

$$\underline{\mathrm{Tor}}^k_{\mathcal{O}_Z}(j_*E, \mathcal{O}_{Z_{\mu_n}})_{\mathbb{C}} \simeq j_{\mathbb{C}*}(\Lambda^k(F) \otimes E_{\mathbb{C}}),$$

où F est un fibré localement libre défini sur $X_{\mu_n,\mathbb{C}}$ par la suite exacte

$$\mathscr{F} : 0 \to F \to N_{Z_{\mu_n,\mathbb{C}}/Z_{\mathbb{C}}} \to N_{X_{\mu_n,\mathbb{C}}/X_{\mathbb{C}}} \to 0$$

(voir [**1**, Exp. VII, Prop. 2.5]). Nous munissons le fibré F de la métrique induite par $N_{Z_{\mu_n}/Z}$.

Pour tout $l \geqslant 0$, les fibrés cohérents $R^l h_*(E_i \otimes \underline{\mathrm{Tor}}^k_{\mathcal{O}_Z}(j_*E, \mathcal{O}_{Z_{\mu_n}}))$ (qui sont localement libres sur la fibre générique) peuvent être munis de métriques hermitiennes via l'isomorphisme naturel

$$R^l h_*(E_i \otimes \underline{\mathrm{Tor}}^k_{\mathcal{O}_Z}(j_*E, \mathcal{O}_{Z_{\mu_n}}))_{\mathbb{C}} \simeq R^l f_{\mathbb{C}*}(j^*(E_{i,\mathbb{C}}) \otimes \Lambda^k(F) \otimes E_{\mathbb{C}}).$$

Par abus de notation, on notera $R^l h_*(\overline{E}_i \otimes \underline{\mathrm{Tor}}^k_{\mathcal{O}_Z}(j_*\overline{E}, \overline{\mathcal{O}}_{Z_{\mu_n}}))$ le fibré cohérent hermitien sur D (« hermitian coherent sheaf » en anglais) correspondant.

Conjecture 5.1. — *L'égalité*

$$\begin{aligned}\sum_{l\geqslant 0}(-1)^l R^l f_*(\overline{E}) - T_g(\overline{E}_{\mathbb{C}}) \quad &= \quad \sum_i r_i \sum_{l,k\geqslant 0}(-1)^{l+k} R^l h_*(\overline{E}_i \otimes \underline{\mathrm{Tor}}^k_{\mathcal{O}_Z}(j_*\overline{E}, \overline{\mathcal{O}}_{Z_{\mu_n}}))\\ &- \quad \sum_i r_i \sum_{k\geqslant 0}(-1)^k T_g(j^*_{\mu_n}(\overline{E}_{i,\mathbb{C}}) \otimes \Lambda^k(\overline{F}) \otimes \overline{E}_{\mathbb{C}}|_{X_{\mu_n}})\\ &+ \quad \int_{X_{\mu_n}} \mathrm{Td}(\overline{\mathrm{T}X}_{\mathbb{C}})\,\mathrm{ch}_g(\overline{E}_{\mathbb{C}})\,\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}})\,\mathrm{Td}_g^{-1}(\overline{F})\\ &- \quad \int_{X_{\mu_n}} \mathrm{Td}_g(\mathrm{T}X_{\mathbb{C}})\,\mathrm{ch}_g(E_{\mathbb{C}}) R_g(N_{X_{\mu_n,\mathbb{C}}/X_{\mathbb{C}}})\end{aligned}$$

est vérifiée dans $\widehat{\mathrm{K}}_0^{\mu_n\prime}(D) \otimes_{R(\mu_n)} \mathcal{R}$.

Cette conjecture est inspirée par la formule [**24**, Th. 3.5].

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Astérisque
328, 2009, p. 255–296

THE INDEX OF PROJECTIVE FAMILIES OF ELLIPTIC OPERATORS: THE DECOMPOSABLE CASE

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — An index theory for projective families of elliptic pseudodifferential operators is developed under two conditions. First, that the twisting, i.e. Dixmier-Douady, class is in $\mathrm{H}^2(X;\mathbb{Z})\cup\mathrm{H}^1(X;\mathbb{Z})\subset\mathrm{H}^3(X;\mathbb{Z})$ and secondly that the 2-class part is trivialized on the total space of the fibration. One of the features of this special case is that the corresponding Azumaya bundle can be refined to a bundle of smoothing operators. The topological and the analytic index of a projective family of elliptic operators associated with the smooth Azumaya bundle both take values in twisted K-theory of the parameterizing space and the main result is the equality of these two notions of index. The twisted Chern character of the index class is then computed by a variant of Chern-Weil theory.

Résumé **(L'indice des familles projectives d'opérateurs elliptiques: le cas décomposable)**
Une théorie de l'indice pour des familles projectives d'opérateurs pseudodifférentiels elliptiques est développée sous les deux conditions suivantes: la classe de Dixmier-Douady est dans $\mathrm{H}^2(X;\mathbb{Z})\cup\mathrm{H}^1(X;\mathbb{Z})\subset\mathrm{H}^3(X;\mathbb{Z})$, et la partie de degré deux est trivialisée sur l'espace total de la fibration. Le fibré d'Azumaya correspondant peut alors être raffiné en un fibré d'opérateurs régularisants. Les indices topologiques et analytiques d'une famille projective d'opérateurs elliptiques associée au fibré d'Azumaya lisse sont à valeurs dans la K-théorie tordue de la base de la famille et le résultat principal est l'égalité de ces deux indices. Le caractère de Chern tordu de la famille est calculé par une variante de la théorie de Chern-Weil.

2010 Mathematics Subject Classification. — 19K56, 58G10, 58G12, 58J20, 58J22.
Key words and phrases. — Twisted K-theory, index theorem, decomposable Dixmier-Douady invariant, smooth Azumaya bundle, Chern Character, twisted cohomology.

We would like to thank the referees for their helpful suggestions. The first author thanks the Australian Research Council for support. The second author acknowledges the support of the National Science Foundation under grant DMS0408993.

Introduction

The basic object leading to twisted K-theory for a space, X, can be taken to be a principal PU-bundle $\mathscr{P} \longrightarrow X$, where $\mathrm{PU} = \mathrm{U}(\mathscr{H})/\mathrm{U}(1)$ is the group of projective unitary operators on some separable infinite-dimensional Hilbert space $\mathscr{H}$. Circle bundles over X are classified up to isomorphism by their Chern classes in $\mathrm{H}^2(X;\mathbb{Z})$ and analogously principal PU bundles are classified by $\mathrm{H}^3(X;\mathbb{Z})$ with the element $\delta(\mathscr{P})$ being the Dixmier-Douady invariant of $\mathscr{P}$. Just as $\mathrm{K}^0(X)$, the ordinary K-theory group of X, may be identified with the group of homotopy classes of maps $X \longrightarrow \mathscr{F}(\mathscr{H})$ into the Fredholm operators on $\mathscr{H}$, the twisted K-theory group $\mathrm{K}^0(X;\mathscr{P})$ may be identified with the homotopy classes of sections of the bundle $\mathscr{P} \times_{\mathrm{PU}} \mathscr{F}$ arising from the conjugation action of PU on $\mathscr{F}$. The action of PU on the compact operators, $\mathscr{K}$, induces the *Azumaya* bundle, $\mathscr{A}$. The K-theory, in the sense of C^* algebras, of the space of continuous sections of this bundle, written $\mathrm{K}^0(X;\mathscr{A})$, is naturally identified with $\mathrm{K}^0(X;\mathscr{P})$. From an analytic viewpoint $\mathscr{A}$ is more convenient to deal with than $\mathscr{P}$ itself.

In the case of circle bundles isomorphisms are classified up to homotopy by an element of $\mathrm{H}^1(X;\mathbb{Z})$, corresponding to the homotopy class of a smooth map $X \longrightarrow \mathrm{U}(1)$. Similarly, $\delta \in \mathrm{H}^3(X;\mathbb{Z})$ determines $\mathscr{P}$ up to isomorphism with the isomorphism class determined up to homotopy by an element of $\mathrm{H}^2(X;\mathbb{Z})$, corresponding to the fact that PU is a $K(\mathbb{Z},2)$. The result is that $\mathrm{K}^0(X;\mathscr{A})$ depends as a group on the choice of Azumaya bundle with DD invariant δ up to an action of $\mathrm{H}^2(X;\mathbb{Z})$.

In [**20**] we extended the index theorem for a family of elliptic operators, giving the equality of the analytic and the topological index maps in K-theory, to the case of twisted K-theory where the twisting class is a torsion element of $\mathrm{H}^3(X;\mathbb{Z})$. In this paper we prove a similar index equality in the case of twisted K-theory when the index class is decomposable

$$\delta = \alpha \cup \beta,\ \alpha \in \mathrm{H}^1(X;\mathbb{Z}),\ \beta \in \mathrm{H}^2(X;\mathbb{Z}), \tag{1}$$

and the fibration $\phi : Y \longrightarrow X$ is such that $\phi^*\beta = 0$ in $\mathrm{H}^2(Y;\mathbb{Z})$.

Under the assumption (1), that the class δ is *decomposed,* we show below that there is a choice of principal PU bundle with class δ such that the classifying map above, $c_P : X \longrightarrow K(\mathbb{Z};3)$ factors through $\mathrm{U}(1) \times \mathrm{PU}$. Twisting by a homotopically nontrivial map $\kappa : X \longrightarrow \mathrm{PU}$ does not preserve this property, so in this decomposed case there is indeed a natural choice of smooth Azumaya bundle, $\mathscr{J}$, up to homotopically trivial isomorphism and this induces a choice of twisted K-group determined by the decomposition of δ; we denote this well-defined twisted K-group by

$$\mathrm{K}^0(X;\alpha,\beta) = \mathrm{K}^0(X;\mathscr{A}), \quad \mathscr{A} = \overline{\mathscr{J}}. \tag{2}$$

The effect on smoothness of the assumption of decomposability on the Dixmier-Douady class can be appreciated by comparison with the simpler case of degree 2. Thus, if $\alpha_1 \cup \alpha_2 \in \mathrm{H}^2(X;\mathbb{Z})$ is a decomposed class, $\alpha_i \in \mathrm{H}^1(X;\mathbb{Z})$ for $i = 1, 2$, then the associated line bundle is the pull-back of the Poincaré line bundle associated to a polarization on the 2-torus under the map $u_1 \times u_2$, where the $u_i \in \mathscr{C}^\infty(X;\mathrm{U}(1))$

represent the α_i. This is to be contrasted with the general case in with the line bundle is the pull-back from a classifying space such as PU, and is only unique up to twisting by a smooth map $\kappa' : X \longrightarrow \mathrm{U}(1)$.

The data we use to define a smooth Azumaya bundle is:

– A smooth function

$$u \in \mathscr{C}^\infty(X; \mathrm{U}(1)) \tag{3}$$

the homotopy class of which represents $\alpha \in \mathrm{H}^1(X, \mathbb{Z})$.

– A Hermitian line bundle (later with unitary connection)

$$\begin{array}{c} L \\ \downarrow \\ X \end{array} \tag{4}$$

with Chern class $\beta \in \mathrm{H}^2(X; \mathbb{Z})$.

– A smooth fiber bundle of compact manifolds

$$\begin{tikzcd} Z \arrow[r, dash] & Y \arrow[d, "\phi"] \\ & X \end{tikzcd} \tag{5}$$

such that $\phi^*\beta = 0$ in $\mathrm{H}^2(Y; \mathbb{Z})$.

– An explicit global unitary trivialization

$$\gamma : \phi^*(L) \xrightarrow{\simeq} Y \times \mathbb{C}. \tag{6}$$

These hypotheses are satisfied by taking $Y = \tilde{L}$, the circle bundle of L, and then there is a natural choice of γ in (6). This corresponds to the 'natural' smooth Azumaya bundle associated to the given decomposition of $\delta = \alpha \cup \beta$ and we take $\mathrm{K}^0(X; \alpha, \beta)$ in (2) to be defined by this Azumaya bundle, discussed as a warm-up exercise in Section 1. In Appendix C it is observed that any fibration for which β is a multiple of a degree 2 characteristic class of $\phi : Y \longrightarrow X$ satisfies the hypothesis in (5).

In general, the data (3) – (6) are shown below to determine an infinite rank 'smooth Azumaya bundle', which we denote $\mathscr{S}(\gamma)$. This has fibres isomorphic to the algebra of smoothing operators on the fibre, Z, of Y with Schwartz kernels consisting of the smooth sections of a line bundle $J(\gamma)$ over Z^2. The completion of this algebra of 'smoothing operators' to a bundle with fibres modelled on the compact operators has Dixmier-Douady invariant $\alpha \cup \beta$.

In outline the construction of $\mathscr{S}(\gamma)$ proceeds as follows; details may be found in Section 3. The trivialization (6) induces a groupoid character $Y^{[2]} \longrightarrow \mathrm{U}(1)$, where $Y^{[2]}$ is the fiber product of two copies of fibration. Combined with the choice (3) this gives a map from $Y^{[2]}$ into the torus and hence by pull-back the line bundle $J = J(\gamma)$. This line bundle is *primitive* in the sense that under lifting by the three projection maps

$$\tilde{L}^{[3]} \overset{\pi_S}{\underset{\pi_F}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{\pi_C}}}}} \tilde{L}^{[2]} \tag{7}$$

(corresponding respectively to the left two, the outer two and the right two factors) there is a natural isomorphism

$$\pi_S^* J \otimes \pi_F^* J = \pi_C^* J. \tag{8}$$

This is enough to give the space of global sections, $\mathcal{C}^\infty(Y^{[2]}; J \otimes \Omega_R)$, where Ω_R is the fiber-density bundle on the right factor, a fibrewise product isomorphic to the smoothing operators on Z. Indeed, if z represents a fiber variable then

$$A \circ B(x, z, z') = \int_Z A(x, z, z'') \cdot B(x, z'', z') \tag{9}$$

where $\cdot$ denotes the isomorphism (8) which gives the identification

$$J_{(z,z'')} \otimes J_{(z'',z')} \simeq J_{(z,z')} \tag{10}$$

needed to interpret the integral in (9). The naturality of the isomorphism corresponds to the associativity of this product.

Then the smooth Azumaya bundle is defined in terms of its space of global sections

$$\mathcal{C}^\infty(X; \mathcal{S}(\gamma)) = \mathcal{C}^\infty(Y^{[2]}; J(\gamma)). \tag{11}$$

As remarked above, $J(\gamma)$, and hence also the Azumaya bundle, depends on the particular global trivialization (6). Two trivializations, γ_i, $i = 1, 2$ as in (6) determine

$$\gamma_{12} : Y \longrightarrow \mathrm{U}(1), \ \gamma_{12}(y)\gamma_2(y) = \gamma_1(y) \tag{12}$$

which fixes an element $[\gamma_{12}] \in \mathrm{H}^1(Y; \mathbb{Z})$ and hence a line bundle K_{12} over Y with Chern class $[\gamma_{12}] \cup [\phi^*\alpha]$. Then

$$J(\gamma_2) \simeq (K_{12}^{-1} \boxtimes K_{12}) \otimes J(\gamma_1) \tag{13}$$

with the isomorphism consistent with primitivity.

Pulling back to Y, $\phi^* \mathcal{A}(\gamma)$ is trivialized as an Azumaya bundle and this trivialization induces an isomorphism of twisted and untwisted K-theory

$$\mathrm{K}^0(Y; \phi^* \mathcal{A}(\gamma)) \xrightarrow{\simeq} \mathrm{K}^0(Y). \tag{14}$$

In fact there are stable isomorphisms between the different smooth Azumaya bundles and these induce natural and consistent isomorphisms

$$\mathrm{K}^0(X; \mathcal{A}(\gamma)) \xrightarrow{\simeq} \mathrm{K}^0(X; \alpha, \beta). \tag{15}$$

The proof may be found in Section 4.

The transition maps for the local presentation of the smooth Azumaya bundle, $\mathcal{S}(\gamma)$, determined by the data (3) – (6), are given by multiplication by smooth functions. Thus they also preserve the corresponding spaces of differential, or pseudodifferential, operators on the fibres; the corresponding algebras of twisted fibrewise pseudodifferential operators are therefore well defined. Moreover, since the principal symbol of a pseudodifferential operator is invariant under conjugation by (non-vanishing) functions there is a well-defined symbol map from the pseudodifferential extension of the Azumaya bundle, with values in the usual symbol space on $T^*(Y/X)$ (so with no additional twisting). The trivialization of the Azumaya bundle over Y, and hence over $T^*(Y/X)$, means that the class of an elliptic element can also be interpreted as an element of $\mathrm{K}^0_{\mathrm{c}}(T^*(Y/X);\rho^*\phi^*\mathcal{A}(\gamma))$ where $\rho : T(Y/X) \longrightarrow Y$ is the bundle projection. This leads to the analytic index map,

$$\mathrm{ind}_{\mathrm{a}} : \mathrm{K}^0_{\mathrm{c}}(T^*(Y/X);\rho^*\phi^*\mathcal{A}(\gamma)) \longrightarrow \mathrm{K}^0(X;\mathcal{A}(\gamma)). \tag{16}$$

The topological index can be defined using the standard argument by embedding of the fibration Y into the product fibration $\pi : \mathbb{R}^N \times X \longrightarrow X$ for large N. Namely, the Azumaya bundle is trivialized over Y and this trivialization extends naturally to a fibred collar neighborhood Ω of Y embedded in $\mathbb{R}^N \times X$. Thus, the usual Thom map $\mathrm{K}^0_{\mathrm{c}}(T^*(Y/X)) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/X)$ is trivially lifted to a map for the twisted K-theory, which then extends by excision to a map giving the topological index as the composite with Bott periodicity:

$$\begin{aligned}\mathrm{ind}_{\mathrm{t}} : \mathrm{K}^0_{\mathrm{c}}(T^*(Y/X);\rho^*\phi^*\mathcal{A}(\gamma)) &\longrightarrow \mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/X);\rho^*\tilde{\pi}^*\mathcal{A}(\gamma)) \\ &\longrightarrow \mathrm{K}^0_{\mathrm{c}}(T^*(\mathbb{R}^N/X);\rho^*\pi^*\mathcal{A}(\gamma)) \longrightarrow \mathrm{K}^0(X;\mathcal{A}(\gamma)).\end{aligned} \tag{17}$$

In the proof of the equality of these two index maps we pass through an intermediate step using an index map given by semiclassical quantization of smoothing operators, rather than standard pseudodifferential quantiztion. This has the virtue of circumventing the usual problems with multiplicativity of the analytic index even though it is somewhat less familiar. A fuller treatment of this semiclassical approach can be found in **[21]** so only the novelties, such as they are, in the twisted case are discussed here. The more conventional route, as used in **[20]**, is still available but is technically more demanding. In particular it is worth noting that the semiclassical index map, as defined below, is well-defined even for a general fibration – without assuming that $\phi^*\beta = 0$. Indeed, this is essential in the proof, since the product fibration $\mathbb{R}^N \times X$ does not have this property.

For a fixed fibration the index maps induced by two different trivializations γ may be compared and induce a commutative diagram

$$\begin{tikzcd}
\mathrm{K}^0_{\mathrm{c}}(T^*(Y/X)) \arrow[r, "\simeq"] \arrow[dd, "{[K_{12}]\times}"] & \mathrm{K}^0_{\mathrm{c}}(T^*(Y/X);\rho^*\phi^*\mathcal{A}(\gamma_1)) \arrow[r, "\mathrm{ind}(\gamma_1)"] & \mathrm{K}^0(X;\mathcal{A}(\gamma_1)) \arrow[d, "\simeq"] \\
& & \mathrm{K}^0(X;\alpha,\beta) \\
\mathrm{K}^0_{\mathrm{c}}(T^*(Y/X)) \arrow[r, "\simeq"] & \mathrm{K}^0_{\mathrm{c}}(T^*(Y/X);\rho^*\phi^*\mathcal{A}(\gamma_2)) \arrow[r, "\mathrm{ind}(\gamma_2)"] & \mathrm{K}^0(X;\mathcal{A}(\gamma_2)). \arrow[u, "\simeq"']
\end{tikzcd} \tag{18}$$

This follows from the proof of the index theorem.

The smoothness of the Azumaya bundle here allows us to give an explicit Chern-Weil formulation for the index in twisted cohomology. We recall that the twisted deRham cohomology $\mathrm{H}^*(X;\delta)$ is obtained by deforming the deRham differential to $d+\delta\wedge$, where

$$\delta = \bar{\alpha} \wedge \frac{\bar{\beta}}{2\pi i}.$$

Here $\bar{\alpha} = u^*(\theta)$ is the closed 1-form on X with integral periods, where θ is the Cartan-Maurer 1-form on $\mathrm{U}(1)$ and $\bar{\beta}$ is the closed 2-form with integral periods which is the curvature of the hermitian connection γ on L.

We remark that our results easily extend to the case when the Dixmier-Douady class is the sum of decomposable classes, ie when it is in the $\mathbb{Z}$-span of $\mathrm{H}^2(X;\mathbb{Z}) \cup \mathrm{H}^1(X;\mathbb{Z})$. The Azumya bundle in this case is the tensor product of the decomposable Azumaya bundles as defined in this paper. The case of an arbitrary, not necessarily decomposable, Dixmier-Douady invariant is postponed to a subsequent paper where the twisted index theorem is treated in full. The general case uses pseudodifferential operators valued in the Azumaya bundle, rather than the pseudodifferential operator extension of the smooth Azumaya bundle as discussed here. Again, the semiclassical index map extends without difficulty to this general case.

In outline the paper proceeds as follows. The special case of the circle bundle is discussed in §1 and §2 contains the geometry of the general decomposable case. The smooth Azumaya bundle corresponding to a decomposable Dixmier-Douady class is constructed in §3 and some examples are also given. In §4, (15) is proved. The analytic index maps is defined in §5 using spaces of projective elliptic operators but including the case of twisted families of Dirac operators. The topological index is defined in §6. The Chern-Weil representative of the twisted Chern Character is studied in §7. Semiclassical versions of the index maps are introduced in §8 and §9 contains the proof of the equality of these two indices. In §10, the Chern character of the index is computed. In Appendix A the formulation of the Dixmier-Douady invariant in terms of differential characters is explored and in Appendix B it is computed using Čech cohomology (following a similar computation by Brylinski). Appendix C contains a discussion of the conditions on a fibration under which a line bundle from the base is trivial when lifted to the total space. It also contains the description of a canonical projective family of Dirac operators on a Riemann surface.

1. Trivialization by the circle bundle

An element $\beta \in \mathrm{H}^2(X;\mathbb{Z})$ for a compact manifold X, represents an isomorphism class of line bundles over X. Let L be such a line bundle with Hermitian inner product h and unitary connection ∇^L. We proceed to outline the construction of the smooth Azumaya bundle in the special case, alluded to above, where $Y = \tilde{L}$ is the circle bundle of L. This is carried out separately since this case gives a *natural* choice of the smooth Azumaya bundle, and hence the twisted K-group. The corresponding twisted cohomology is also identified with the cohomology of a subcomplex of the deRham complex over $\tilde{L}$.

From $u \in \mathscr{C}^\infty(X; \mathrm{U}(1))$ construct the principal $\mathbb{Z}$-bundle

$$\begin{tikzcd} \mathbb{Z} \arrow[r, dash] & \hat{X} \arrow[d, "\pi"] \\ & X \end{tikzcd} \tag{1.1}$$

with total space the possible values of $\frac{1}{2\pi i}\log u$ over points of X and with $\mathscr{C}^\infty$ structure determined by the smoothness of a local branch of this function. Thus $f = \frac{1}{2\pi i}\log u \in \mathscr{C}^\infty(\hat{X}; \mathbb{R})$ is a well defined smooth function and under deck transformations

$$f(\hat{x}+n) = f(\hat{x}) + n \ \forall\ \hat{x} \in \hat{X},\ n \in \mathbb{Z}. \tag{1.2}$$

Let $p : \tilde{L} \longrightarrow X$ be the circle bundle of L; pulled back to $\tilde{L}$, L is canonically trivial. If ∇^L is an Hermitian connection on L then pulled back to $\tilde{L}$ it is of the form $d+\gamma$ on the trivialization of L, with $\gamma \in \mathscr{C}^\infty(\tilde{L}, \Lambda^1)$ a principal $\mathrm{U}(1)$-bundle connection form in the usual sense. That is, under the action

$$m : \mathrm{U}(1) \times \tilde{L} \longrightarrow \tilde{L}, \tag{1.3}$$

$m^*\gamma = id\theta + \gamma$. This corresponds to the 'fiber shift map' on the fiber product

$$s : \tilde{L}^{[2]} = \tilde{L} \times_X \tilde{L} \longrightarrow \mathrm{U}(1),\ \tilde{l}_1 = s(\tilde{l}_1, \tilde{l}_2)\tilde{l}_2 \text{ in } \tilde{L}_x \tag{1.4}$$

in that $d\log s = p_1^*\gamma - p_2^*\gamma$ is the difference of the pull-back of the connection form from the two factors. From the character s a bundle, J, can be constructed from the trivial bundle over the fiber product $Q = \tilde{L} \times_X \tilde{L} \times_X \hat{X}$ corresponding to the identification

$$(\tilde{l}_1, \tilde{l}_2, \hat{x}+n, z) \simeq (\tilde{l}_1, \tilde{l}_2, \hat{x}, s(\tilde{l}_1, \tilde{l}_2)^n z). \tag{1.5}$$

Thus, J is associated to Q as a principal $\mathbb{Z}$-bundle over $\tilde{L}^{[2]}$. The primitivity property (8) follows from the multiplicativity property of s, that $s(l_1, l_2)s(l_2, l_3) = s(l_1, l_2)$ for any three points in a fixed fiber, which in turn follows from (1.4). The connection $d + f d\log s$ on the trivial bundle over Q descends to a connection on J which has curvature equal to a difference

$$\omega_J = \frac{1}{2\pi i}\overline{\alpha} \wedge d\log s = \overline{\alpha} \wedge \frac{1}{2\pi i}(p_1^*\gamma - p_2^*\gamma),\ \overline{\alpha} = df. \tag{1.6}$$

By definition, the space of global sections of the smooth Azumaya bundle is

$$\mathscr{C}^\infty(X; \mathscr{A}) = \mathscr{C}^\infty(\tilde{L}^{[2]}; J), \tag{1.7}$$

where the product on the right hand side is given by composition of Schwartz kernels.

The 'Dixmier-Douady twisting', given the decomposed form, corresponds to two different trivializations of J. Over any open set $U \subset X$ where u has a smooth logarithm, J is trivial using the section of $\hat{X}$ this gives. On the other hand, over any open set $U \subset X$ over which $\tilde{L}$ has a smooth section τ, the character in (1.5) is decomposed as the product $s(\tilde{l}_1, \tilde{l}_2) = s_\tau(\tilde{l}_1)s_\tau(\tilde{l}_2)^{-1}$ where $s_\tau(\tilde{l}) = s(\tilde{l}, \tau(p(\tilde{l})))$. This allows a line bundle K to be defined over the preimage of U in $\tilde{L}$ by the identification of the trivial bundle

$$(\tilde{l}, \hat{x}+n, z) \simeq (\tilde{l}, \hat{x}, s_\tau(\tilde{l})^n z). \tag{1.8}$$

Clearly then J may be identified with $K \boxtimes K'$, where K' is the dual, over U. In terms of a local trivialization in both senses over a small open set $U \subset X$, in which $L_U = U \times \mathbb{C}$, $\tilde{L}_U = U \times \mathbb{S}$, $\hat{X}_U = U \times \mathbb{Z}$, $a(x, \theta, \theta') \in \mathscr{C}^\infty(U \times \mathbb{Z} \times \mathbb{S} \times \mathbb{S})$ satisfies

$$a(x, n, \theta, \theta') = e^{in\theta} a(x, 0, \theta, \theta') e^{-in\theta'}. \tag{1.9}$$

This twisted conjugation means that $\mathscr{A}$ is a bundle of algebras, modelled on the smoothing operators on the circle with (1.9) giving local algebra trivializations. In this case the Azumaya bundle is associated with the principal $\mathrm{U}(1) \times \mathbb{Z}$ bundle $\tilde{L} \times_X \hat{X}$ and to the projective representation of this structure group through its central extension to the Heisenberg group.

The corresponding construction in the general case is quite similar and is described in §3.

The 3-twisted cohomology on X, with twisting form $\overline{\delta} = \overline{\alpha} \wedge \overline{\beta}$, is the target for the twisted Chern character discussed below. Here $\overline{\alpha}$ is a closed 1-form and $\overline{\beta}$ is the curvature 2-form on X for the Hermitian connection on L. Thus, on $\tilde{L}$, $d\gamma = (2\pi i) p^* \overline{\beta}$. In fact the $\overline{\delta}$-twisted deRham cohomology on X can be expressed as the cohomology of a subcomplex of the ordinary (total) deRham complex on $\tilde{L}$.

Proposition 1. — *The even and odd degree subspaces of $\mathscr{C}^\infty(\tilde{L}, \Lambda^*)$ fixed by the conditions with respect to the infinitesmal generator of the* U(1) *action on $\tilde{L}$*

$$\mathscr{L}_{\partial/\partial\theta} \tilde{v} = 0, \ \iota_{\partial/\partial\theta} \tilde{v} = \frac{p^* \overline{\alpha}}{2\pi} \wedge \tilde{v}, \ \tilde{v} \in \mathscr{C}^\infty(\tilde{L}, \Lambda^*) \tag{1.10}$$

are mapped into each other by the standard deRham differential which has cohomology groups canonically isomorphic to the $\overline{\delta}$-twisted deRham cohomology on X.

Proof. — The conditions in (1.10) are preserved by d since it commutes with the Lie derivative and given the first condition

$$\iota_{\partial/\partial\theta} d\tilde{v} = \mathscr{L}_{\partial/\partial\theta} \tilde{v} - d(\frac{p^* \overline{\alpha}}{2\pi} \wedge \tilde{v}) = \frac{p^* \overline{\alpha}}{2\pi} \wedge d\tilde{v}. \tag{1.11}$$

If $\tilde{v}$ satisfies (1.10) then $v' = \tilde{v} - \frac{\gamma}{2\pi i} \wedge p^* \overline{\alpha} \wedge \tilde{v}$ satisfies

$$\mathscr{L}_{\partial/\partial\theta} v' = 0, \ \iota_{\partial/\partial\theta} v' = 0 \Longrightarrow v' = p^* v, \ v \in \mathscr{C}^\infty(X; \Lambda^*). \tag{1.12}$$

Conversely if $v \in \mathscr{C}^\infty(X; \Lambda^*)$ then $\tilde{v} = p^* v + \frac{\gamma}{2\pi i} \wedge \overline{\alpha} \wedge p^* v$ satisfies (1.10). Thus every form satisfying (1.10) can be written uniquely

$$\tilde{v} = \exp\left(\frac{\gamma \wedge p^* \overline{\alpha}}{2\pi i}\right) p^* v = p^* v + \frac{\gamma \wedge p^* \overline{\alpha}}{2\pi i} \wedge p^* v. \tag{1.13}$$

Under this isomorphism d is clearly conjugated to $d + \overline{\delta} \wedge$ proving the Proposition. □

2. Geometry of the decomposed class

For a given line bundle L over X consider a fiber bundle (5) such that L is trivial when lifted to the total space. As discussed above, the circle bundle $\tilde{L}$ is an example. A more general discussion of this condition can be found in Appendix C. An explicit trivialization of the lift, γ, as in (6) is equivalent to a global section which is the preimage of 1 :

$$s' : Y \longrightarrow \phi^*(\tilde{L}). \tag{2.1}$$

Over each fiber of Y, the image is fixed so this determines a map

$$s(z_1, z_2) = s'(z_1)(s'(z_2))^{-1}$$

which is well-defined on the fiber product and is a groupoid character:

$$\begin{gathered} s : Y^{[2]} \longrightarrow \mathrm{U}(1), \\ s(z_1, z_2)s(z_2, z_3) = s(z_1, z_3) \ \forall\ z_i \in Y \text{ with } \phi(z_i) = x,\ i = 1, 2, 3,\ \forall\ x \in X. \end{gathered} \tag{2.2}$$

Conversely one can start with a unitary character s of $Y^{[2]}$ and recover L as the associated Hermitian line bundle

$$\begin{gathered} L = Y \times \mathbb{C}/\simeq_s, \\ (z_1, t) \simeq_s (z_2, s(z_2, z_1)t) \ \forall\ t \in \mathbb{C},\ \phi(z_1) = \phi(z_2). \end{gathered} \tag{2.3}$$

The connection on L lifts to a connection

$$\phi^*\nabla^L = d + \gamma, \quad \gamma \in \mathscr{C}^\infty(Y; \Lambda^1), \quad \pi_1^*\gamma - \pi_2^*\gamma = d\log s \text{ on } Y^{[2]} \tag{2.4}$$

on the trivial bundle $\phi^*(L)$. Conversely any 1-form on Y with this property defines a connection on L.

Now, let $Q = Y^{[2]} \times_X \hat{X}$ be the fiber product of $Y^{[2]}$ and $\hat{X}$, so as a bundle over X it has typical fiber $Z^2 \times \mathbb{Z}$; it is also a principal $\mathbb{Z}$-bundle over $Y^{[2]}$. The data above determines an action of $\mathbb{Z}$ on the trivial bundle $Q \times \mathbb{C}$ over Q, namely

$$T_n : (z_1, z_2, \hat{x}; w) \longrightarrow (z_1, z_2, \hat{x} + n, s(z_1, z_2)^{-n}w) \ \forall\ n \in \mathbb{Z}. \tag{2.5}$$

Let J be the associated line bundle over $Y^{[2]}$

$$J = (Q \times \mathbb{C})/\simeq, \quad (z_1, z_2, \hat{x}; w) \simeq T_n(z_1, z_2, \hat{x}; w) \ \forall\ n \in \mathbb{Z}. \tag{2.6}$$

The fiber of J at $(z_1, z_2) \in Y^{[2]}$ such that $\phi(z_1) = \phi(z_2) = x$ is

$$J_{z_1,z_2} = \hat{X}_x \times \mathbb{C}/\simeq, \quad (\hat{x} + n, w) \simeq (\hat{x}, s(z_1, z_2)^n w). \tag{2.7}$$

Lemma 1. — *The connection $d + f d\log s$ on $Q \times \mathbb{C}$ descends to a connection ∇^J on J which has curvature*

$$F_{\nabla^J} = \pi_1^*\mu - \pi_2^*\mu,\ \ \mu = df \wedge \frac{\gamma}{2\pi i} \in \mathscr{C}^\infty(Y; \Lambda^2),\ \ Y^{[2]} \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} Y. \tag{2.8}$$

Moreover $d\mu = \phi^*(\overline{\delta})$, for the uniquely determined 3-form on X, $\overline{\delta} = \overline{\alpha} \wedge \overline{\beta} \in \mathscr{C}^\infty(X; \Lambda^3)$, where $df = \phi^*(\overline{\alpha})$ and $d\gamma = 2\pi i\phi^*(\overline{\beta})$ represent the characteristic class of $\hat{X}$ and the first Chern class of L respectively.

Proof. — Clearly the 1-form $f d\log s$ has the correct transformation law under the $\mathbb{Z}$ action on $Y^{[2]} \times_X \hat{X}$ to give a connection on J. Its curvature is $\overline{\alpha} \wedge \frac{d\log s}{2\pi i}$ where $\overline{\alpha} = \frac{1}{2\pi i} d\log u$. If γ is the connection form for the trivialization of L on Y then

$$d\log s = \pi_1^*\gamma - \pi_2^*\gamma \text{ in } Y^{[2]} \tag{2.9}$$

from which (2.8), together with the remainder of the Lemma, follows. □

3. Smooth Azumaya bundle

We proceed to show how to associate to the data (3) – (6) discussed above a smooth Azumaya bundle over X. That is, we construct a locally trivial bundle with fibres modelled on the smoothing operators on the sections of a line bundle over the fibres of Y and having completion with Dixmier-Douady invariant $\alpha \cup \beta$. Note that this Azumaya bundle *does depend* on the trivialization data in (6); we will therefore denote it $J(\gamma)$. The effect of changing this trivialization is discussed in Lemma 4 below.

First consider local trivializations of the data.

Proposition 2. — *A section of ϕ, over on open set $U \subset X$, $\tau : U \longrightarrow \phi^{-1}(U)$, induces a trivialization of L over U and an isomorphism of $J(\gamma)$ over the open subset $V = \phi^{-1}(U) \times_U \phi^{-1}(U)$ of $Y^{[2]}$, with*

$$J\big|_V \cong_\tau \operatorname{Hom}(K_\tau) = K_\tau \boxtimes K'_\tau \tag{3.1}$$

for a line bundle K_τ over $\phi^{-1}(U) \subset Y$, where K'_τ denotes the line bundle dual to K_τ. Another choice of section $\tau' : U \longrightarrow \phi^{-1}(U)$, determines another line bundle $K_{\tau'}$ over $\phi^{-1}(U) \subset Y$, satisfying

$$K_\tau = K_{\tau'} \otimes \phi^*(L_{\tau,\tau'}), \tag{3.2}$$

where $L_{\tau,\tau'} = (\tau, \tau')^* J$ is the fixed local line bundle over U.

Proof. — A local section of ϕ induces a local trivialization of the character s,

$$s(z_1, z_2) = \chi_\tau(z_1)\chi_\tau^{-1}(z_2), \quad \chi_\tau(z) = s(z, \tau(\phi(z))) \text{ on } \phi^{-1}(U) \subset Y. \tag{3.3}$$

This trivializes L over U, identifying it with $\tau^*\mathbb{C}$ with connection $d + \tau^*\gamma$.

The line bundle K_τ over $\phi^{-1}(U)$ associated to the $\mathbb{Z}$ bundle $\phi^{-1}(U) \times_U \hat{X}_U$ by the identification $(z, \hat{x} + n, w) \simeq (z, \hat{x}, \chi_\tau(z)^n w)$ then satisfies (3.1). The line bundle $K_{\tau'}$ is similarly defined over $\phi^{-1}(U)$, satisfying (3.1) with τ' substituted for τ. The relation (3.2) follows from (3.1) and its modification with τ' substituted for τ. □

Such a section of Y will induce a local trivialization of the smooth Azumaya bundle in which it becomes the smoothing operators on the fibres of Y acting on sections of K_τ :

$$\mathscr{J}_\tau = \Psi^{-\infty}(\phi^{-1}(U)/U; K_\tau). \tag{3.4}$$

Using Proposition 2, we get the local patching,

$$\mathscr{A}_\tau = \mathscr{A}_{\tau'} \otimes \Psi^{-\infty}(\phi^{-1}(U)/U; \phi^*(L_{\tau,\tau'})). \tag{3.5}$$

Rather than use this as a definition we adopt an *a priori* global definition by trivializing over Y.

Definition 1. — For any $x \in X$, the fiber of the smooth Azumaya bundle associated to the geometric data in §2 is

$$\mathscr{A}_x = \mathscr{C}^\infty(Y_x^2; J\big|_{Y_x^2} \otimes \Omega_R) \tag{3.6}$$

where Ω_R is the fiber density bundle on the right factor of Y_x^2. Globally, we have a natural identification,

$$\mathscr{C}^\infty(X, \mathscr{A}) \cong \mathscr{C}^\infty(Y^{[2]}, J \otimes \Omega_R). \tag{3.7}$$

Thus a smooth section of $\mathscr{A}$ over any open set $U \subset X$ is just a smooth section of $J \otimes \Omega_R$, where $\Omega_R = \pi_R^*\Omega$, over the preimage of U in $Y^{[2]}$.

Of course, we need to show that $\mathscr{A}$ is a bundle of algebras over X with local trivializations as indicated in (3.4). To see this globally, observe that J has the same 'primitivity' property as for $\tilde{L}$ in §1 with respect to the groupoid structure.

Lemma 2. — *If*

$$Y^{[3]} \overset{\pi_S}{\underset{\pi_F}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{\pi_C}}}}} Y^{[2]} \tag{3.8}$$

are the three projections (respectively onto the two left-most, the outer two and the two right-most factors – the notation stands for 'second', 'composite' and 'first' for operator composition) then there is a natural isomorphism

$$\pi_S^* J \otimes \pi_F^* J \xrightarrow{\simeq} \pi_C^* J \tag{3.9}$$

and moreover J carries a connection ∇^J which respects this primitivity property.

Proof. — The identity (3.9) is evident from the definition of J and Proposition 2. The naturality property for (3.9) corresponds to an identity on $Y^{[4]}$. Namely if J' is the dual of J then the tensor product of the pull-backs under the four projections $Y^{[4]} \longrightarrow Y^{[3]}$ of the combination $\pi_S^* J \otimes \pi_F^* J \otimes \pi_C^* J'$ over $Y^{[3]}$ is naturally trivial. That this trivialization is equal to the tensor product of the four trivializations from (3.9) follows again from the definition of J.

By Proposition 2, a section $\tau : U \longrightarrow Y$ of ϕ over the open subset U of X defines an isomorphism $J\big|_V \cong_\tau \operatorname{Hom}(K_\tau) = K_\tau \boxtimes K'_\tau$ where $V = \phi^{-1}(U) \times_U \phi^{-1}(U)$ is the open subset of $Y^{[2]}$. A choice of connection ∇^τ on K_τ induces a connection ∇^V on $J\big|_V$ which clearly respects the primitivity property. A global connection preserving the primitivity property can then be constructed using a partition of unity on X. $\square$

As a consequence of Lemma 2 there is a lifting map

$$\mathscr{C}^\infty(Y^{[2]}; J \otimes \Omega_R) \xrightarrow{\pi_F^*} \mathscr{C}^\infty(Y^{[3]}; \pi_S J' \otimes \pi_C J \otimes \pi_F^* \Omega_R) \xrightarrow{\simeq} \Psi^{-\infty}(Y^{[2]}/Y; J') \tag{3.10}$$

which embeds into an algebra, namely the smoothing operators on sections of J' on the fibres of $Y^{[2]}$ as a fibration over Y (projecting onto the first factor).

Proposition 3. — *Lifting $\mathscr{C}^\infty(Y^{[2]}; J \otimes \Omega_R)$ to $Y^{[3]}$ under the projection off the left-most factor (the 'first' projection in terms of composition) embeds it as a subalgebra of the smoothing operators on sections of J' as a bundle over $Y^{[2]}$ on the fibres of the projection onto the right factor such that the lift of the bundle of algebras over X is equal to the bundle of algebras over Y.*

This justifies (3.4). As discussed below it also shows that, as an Azumaya bundle, the completion of $\mathscr{J}$ is $\mathscr{A} = \mathscr{A}(\gamma)$.

Proof. — It only remains to show that composition of two local sections of $\mathscr{J}$ in the algebra of fiber smoothing operators gives another section of the Azumaya bundle. However, this follows from (3.4), which in turn is a consequence of Lemma 2 applied to the local decomposition of J in (3.1). $\square$

An infinite rank Azumaya bundle $\mathscr{A}$, over a topological space X, is a bundle of star algebras with local isomorphisms with the trivial bundle of compact operators, $\mathscr{K}(\mathscr{H})$, on a fixed separable but infinite-dimensional Hilbert space $\mathscr{H}$. The Dixmier-Douady invariant of $\mathscr{A}$ is an element of $\mathrm{H}^3(X;\mathbb{Z})$. It classifies the bundle up to stable isomorphism (i.e. after tensoring with $\mathscr{K}$) and can be realized in terms of Čech cohomology or alternatively in terms of classifying spaces as follows. The group of $*$-automorphism of $\mathscr{K}$ is $\mathrm{PU}(\mathscr{H}) = \mathrm{U}(\mathscr{H})/\mathrm{U}(1)$, the projective unitary group of the Hilbert space acting by conjugation. Thus the fiber trivializations of $\mathscr{A}$ form a principal $\mathrm{PU}(\mathscr{H})$-bundle over X. Since $\mathrm{PU}(\mathscr{H}) = K(\mathbb{Z}, 2)$ is an Eilenberg-Maclane space, this bundle, and hence $\mathscr{A}$, is classified up to isomorphism by an homotopy class of maps $X \longrightarrow B\,\mathrm{PU}(\mathscr{H}) = K(\mathbb{Z}, 3)$ which represents, and is equivalent to, the Dixmier-Douady invariant.

The Chern class of a line bundle L over a space X has a similar representation. Taking an Hermitian structure and passing to the associated circle bundle $\tilde{L}$ over X one can consider the Hilbert bundle $L^2(\tilde{L}/X)$ of Lebesgue square integrable functions on the fibres of the circle bundle. Each point $l \in \tilde{L}$ defines a unitary operator on the fiber through that point, namely multiplication by $U(\hat{l}) = \exp(i\theta_{\hat{l}})\times$ where the normalization is such that $\exp(i\theta_{\hat{l}})(\hat{l}) = 1$. Changing $\hat{l}$ within the fiber changes $U(\hat{l})$ to $\exp(i\theta')U(\hat{l}')$ so this defines a map

$$X \longrightarrow \mathrm{PU}(L^2(\tilde{L}/X)) \tag{3.11}$$

into the bundle of projective unitary operators on the fibres of the Hilbert bundle. By Kuiper's theorem any Hilbert bundle is trivial (in the uniform topology) and the trivialization is natural up to homotopy. Thus the map (3.11) lifts to a unique homotopy class of maps

$$X \longrightarrow \mathrm{PU}(\mathscr{H}) = K(\mathbb{Z}, 2) \tag{3.12}$$

and this represents, and is equivalent to, the first Chern class. This follows from the evident fact that $\tilde{L}$ is isomorphic to the pull-back of the canonical circle bunde, $\mathrm{U}(\mathscr{H})/\mathrm{PU}(\mathscr{H})$ over $\mathrm{PU}(\mathscr{H})$.

Now consider the decomposed case under consideration here. Over the given space X we have both a map $u \in \mathscr{C}^\infty(X; \mathrm{U}(1))$ and a line bundle L. Passing to the classifying map (3.11) this gives a unique homotopy class of maps

$$X \longrightarrow \mathrm{U}(1) \times \mathrm{PU}(\mathscr{H}). \tag{3.13}$$

Proposition 4. — *The completion of the smooth Azumaya bundle $\mathscr{J}$ associated above to* (3) – (6) *to an Azumaya bundle $\mathscr{A} = \mathscr{A}(\gamma)$, has Dixmier-Douady invariant $\alpha \cup \beta \in \mathrm{H}^3(X; \mathbb{Z})$ which is represented by the composite of* (3.13) *with the classifying map $\mathrm{U}(1) \times \mathrm{PU}(\mathscr{H}) \longrightarrow K(\mathbb{Z}, 3)$ induced by the projectivisation of the basic representation of the Heisenberg group $\mathbb{Z} \times \mathrm{U}(1) \longrightarrow \mathrm{PU}(\mathscr{H})$.*

Proof. — The classifying space BG of a topological group G is defined up to homotopy as the quotient $*/G$ of a contractible space on which G acts freely. In particular it follows that (always up to homotopy)

$$B(G_1 \times G_2) \simeq BG_1 \times BG_2 \tag{3.14}$$

and if $H \subset G$ is a closed subgroup then there is a well defined homotopy class of maps

$$BH \longrightarrow BG. \tag{3.15}$$

Recall that the basic representation of the Heisenberg group H arises from the actions of $\mathrm{U}(1)$ and $\mathbb{Z}$ on $L^2(\mathbb{S})$ (or $\mathscr{C}^\infty(\mathbb{S})$) respectively by translation and multiplication by $e^{in\theta}$. These commute up to scalars, which is the action of the center of H as a central extension

$$\mathrm{U}(1) \longrightarrow H \longrightarrow \mathbb{Z} \times \mathrm{U}(1) \tag{3.16}$$

and so embeds

$$\mathbb{Z} \times \mathrm{U}(1) \hookrightarrow \mathrm{PU}(\mathscr{H}) \tag{3.17}$$

as a subgroup of the projective unitary group on $L^2(\mathbb{S})$. By (3.14) and (3.15) this induces an homotopy class of continuous maps

$$\Delta : \mathrm{U}(1) \times \mathrm{PU} \simeq B(\mathbb{Z} \times \mathrm{U}(1)) \longrightarrow K(\mathbb{Z}, 3). \tag{3.18}$$

So the claim in the Proposition is that under this map the pull-back of the degree 3 generator of the cohomology of $K(\mathbb{Z}, 3)$ is the Dixmier-Douady invariant of $\mathscr{A}$ and is equal to $\alpha \cup \beta$ in $\mathrm{H}^3(X, \mathbb{Z})$.

The first statement follows from the fact that the PU bundle to which $\mathscr{A}$ is associated is obtained from the $\mathbb{Z} \times \mathrm{U}(1)$ bundle $\hat{X} \times_X \tilde{L}$ by extending the structure group using (3.17). The second statement follows from the fact that under the map (3.18) the generating 3-class $\delta_{\mathrm{DD}} \in \mathrm{H}^3(K(\mathbb{Z},3),\mathbb{Z})$ pulls back to $\alpha' \cup \beta'$ where $\alpha' \in \mathrm{H}^1(\mathbb{S},\mathbb{Z})$ and $\beta' \in \mathrm{H}^2(\mathrm{PU},\mathbb{Z})$ are the generators, that is,

$$\Delta^*\delta = \alpha' \cup \beta'. \tag{3.19}$$

Indeed, the degree 3-cohomology of $\mathrm{U}(1) \times \mathrm{PU}$ has a single generator, so (3.19) must be correct up to a multiple on the right side. Thus it is enough to check one example, to determine that the multiple is equal to one. Take $X = \mathbb{S} \times \mathbb{S}^2$ with u the identity on $\mathbb{S}$ and L the standard line bundle over the sphere. We know that the induced map (3.12) for the sphere generates the second homotopy group of PU and pulls back to the fundamental class on $\mathbb{S}^2$. Thus it suffices to note that the PU bundle over $\mathbb{S} \times \mathbb{S}^2$ with which our smooth Azumaya bundle is associated in this case is just obtained by the clutching construction from the trivial bundle over $[0, 2\pi] \times \mathbb{S}^2$ using this map. □

An interesting special case of this construction, close to the lifting to the circle bundle described in §1, arises when $\beta \in \mathrm{H}^2(X;\mathbb{Z})$ is thought of as the first Chern class of a complex vector bundle rather than a line bundle. Then Y can be taken to be the associated principal bundle

$$\begin{array}{ccc} \mathrm{U}(n) & \text{———} & P \\ & & \big\downarrow \\ & & X. \end{array}$$

Since the abelianization of $\mathrm{U}(n)$ is canonically isomorphic to $\mathrm{U}(1)$, any character (i.e. 1-dimensional unitary representation) of $\mathrm{U}(n)$ factorizes through $\mathrm{U}(1)$, and conversely, any character of $\mathrm{U}(1)$ lifts to a character of $\mathrm{U}(n)$. A $\mathrm{U}(1)$-central extension of the group $\mathrm{U}(n) \times \mathbb{Z}$ arises in the form of a generalized Heisenberg group. Namely the group product on $H_n = \mathrm{U}(n) \times \mathbb{Z} \times \mathrm{U}(1)$ can be taken to be

$$(g_1, n_1, z_1)(g_2, n_2, z_2) = (g_1 g_2, n_1 + n_2, \det(g_1)^{n_2} z_1 z_2).$$

Then

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow H_n \longrightarrow \mathrm{U}(n) \times \mathbb{Z} \longrightarrow 1$$

is a central extension.

4. Stable Azumaya isomorphism

We proceed to show that the twisted K-groups, $\mathrm{K}^0(X; \mathscr{A}(\gamma))$, defined through the possible data (3) – (6) corresponding to a fixed decomposition (1) are all naturally isomorphic, as indicated in (15). This is a consequence of the Morita invariance of the C^* K-groups and the existence of stabilized isomorphisms between the various Azumaya bundles.

For a smooth 1-parameter family of trivializations, as in (6), so depending smoothly on $t \in [0, 1]$, the K-groups $\mathrm{K}^0(X; \mathscr{A}(\gamma(t)))$ are all naturally isomorphic. Since two such trivializations differ by a smooth map $Y \longrightarrow \mathrm{U}(1)$, the K-group can only depend on the homotopy class of this map, when the other data is fixed. It is also the case that K-theory of C^* algebras admits Morita equivalence. That is, the K-group of $\mathcal{A}$ is naturally isomorphic to the K-group of $\mathcal{A} \otimes \mathcal{K}$. Kuiper's theorem shows that the completion of the smoothing operators on any fiber bundle over a space, and acting on sections of any vector bundle, V, over the fiber bundle, is naturally isomorphic up to homotopy to the trivial Azumaya bundle $\mathcal{K}$. It follows that the 'twisted' K-theory of a space, computed with respect to such a bundle is naturally isomorphic to the untwisted K-theory. More generally, taking the smooth Azumaya bundle $\mathcal{S}(\gamma)$ and tensoring with the bundle of smoothing operators, $\Psi^{-\infty}(\psi; V)$, on any other fiber bundle $\psi : Y' \longrightarrow X$, over the same base, gives an Azumaya bundle with the same twisted K-theory, $\mathrm{K}^0(X; \mathcal{A}(\gamma))$. This proves:

Lemma 3. — *If $\psi : Y' \longrightarrow X$ is a fibration of compact manifolds and $\mathcal{S}(\gamma)$ is the Azumaya bundle associated to data* (3) – (6) *then there is a natural isomorphism of twisted K-theory*

$$\mathrm{K}^0(X; \mathcal{A}(\gamma)) \xrightarrow{\simeq} \mathrm{K}^0(X; \mathcal{A}(\gamma')) \tag{4.1}$$

where γ' is the trivialization obtained by pulling back γ to the product bundle $Y \times_X Y' \longrightarrow X$.

Applying this result to the initial Azumaya bundle in § 1 and the general case, shows that $\mathrm{K}^0(X; \alpha, \beta)$ and $\mathrm{K}^0(X; \mathcal{A}(\gamma))$ are each naturally isomorphic to some (possibly different) $\mathrm{K}^0(X; \mathcal{A}(\gamma'))$ where γ' is a trivialization of the lift of $\tilde{L}$ to $\tilde{L} \times_X Y$, obtained in the two cases by lifting the trivialization from $\tilde{L}$ or Y to the fibre product. Thus it remains to consider two different trivializations over the same fibration.

Proposition 5. — *If γ_i are two trivializations of $\phi^* L$ over Y as in* (6) *then there is an embedding of algebras, unique up to homotopy,*

$$\mathcal{S}(\gamma_2) \longrightarrow \mathcal{S}(\gamma_1) \boxtimes \Psi^{-\infty}(\mathbb{T}^2; K) \tag{4.2}$$

for a line bundle over the 2-torus which induces natural isomorphisms

$$\mathrm{K}^0(X; \mathcal{A}(\gamma_2)) \xrightarrow{\simeq} \mathrm{K}^0(X; \mathcal{A}_{12})) \xrightarrow{\simeq} \mathrm{K}^0(X; \mathcal{A}(\gamma_1)) \tag{4.3}$$

where $\mathcal{A}_{12}$ is the completion of $\mathcal{S}(\gamma_1) \boxtimes \Psi^{-\infty}(\mathbb{T}^2; K)$.

Proof. — This is really an adaptation of the proof of the index theorem via embedding. First, we recall the discussion above, which shows that the primitive line bundle $J(\gamma_2)$ is isomorphic to $J(\gamma_1) \otimes (K_{12} \boxtimes K'_{12})$ for a line bundle K_{12} over Y pulled back from a line bundle K over $\mathbb{T}$ by a smooth map $\kappa_{12} : Y \longrightarrow \mathbb{R}^2$. This map embeds Y as a subfibration of $\phi \circ \pi_1 : Y \times \mathbb{T}^2 \longrightarrow X$. Let $N \longrightarrow Y$ be the normal bundle to this embedding. Given a metric this carries a field of harmonic oscillators on the fibres, the ground states of which give the desired embedding.

Let $v(z, \zeta)$ be the L^2-orthonormalized ground state on the fiber over $z \in Y$. Then

$$\mathcal{C}^\infty(Y^{[2]}; J(\gamma)) \ni a(z_1, z_2) \longmapsto \tilde{a} = v(z_1, \zeta_1) a(z_1, z_2) v(z_2, \zeta_2) \in \mathcal{S}(V^{[2]}; J(\gamma_2)) \tag{4.4}$$

is an embedding. Moreover, this is an embedding of algebras with the algebra structure on the right given by Schwartz-smoothing operators. Now consider the bundle $J(\gamma_1) \boxtimes$

$K \boxtimes K'$ over $Y^{[2]} \times \mathbb{T}^2 \times \mathbb{T}^2$. Restricted to the image of the embedding of $Y^{[2]}$ given by κ_{12} acting in both fibres, this is isomorphic to $J(\gamma_2)$ since by construction K pulls back to K_{12} over Y. Now, consider an embedding of V, using the collar neighborhood theorem, as a neigborhood, $\Omega \subset Y \times \mathbb{T}^2$, of the image of $Y^{[2]}$ under this embedding. The bundle K, pulled back to $Y \times \mathbb{T}^2$ by the projection onto $\mathbb{T}^2$ can be deformed to a bundle $\tilde{K}$, which is equal over Ω to the pull back under the normal retraction of its restriction, K_{12}, to the image of Y. Then the embeddign (4.4) embeds $\mathscr{C}^\infty(Y^{[2]}; J(\gamma_2)$ as a subalgebra of $\mathscr{C}^\infty((Y')^{[2]}; J(\gamma_1) \otimes \tilde{K} \boxtimes \tilde{K}')$, $Y' = Y \times \mathbb{T}^2$. Moreover, using the full spectral expansion of the harmonic oscillator, the completion of the image is Morita equivalent to the whole subalgebra with support in the compact manifold with boundary which is the closure of $\Omega \subset Y'$. This in turn is Morita equivalent to the whole algebra and hence, after another deformation of $\tilde{K}$ back to K over $\mathbb{T}^2$ to $\mathscr{A}_{12}$ in (4.3). This gives the first isomorphism in (4.3). The second follows from stabilization by the compact operators on K over $\mathbb{T}^2$ as discussed above, completing the proof. □

Proof of (15). — As noted above this is a corollary of Proposition 5 and the preceeding discussion. Namely this provides a stabilized isomorphism, unique up to homotopy, of the Azumaya bundle in §1 with that constructed over $\tilde{L} \times_X Y$ by lifting the trivialization over $\tilde{L}$ to the fiber product. The same is true by lifting the trivialization over Y to the fiber product. Then the Proposition constructs a stable isomorphism, again unique up to homotopy, of the two lifts to $\tilde{L} \times_X \times Y \times \mathbb{T}^2$. These stable isomorphisms project to a unique isomorphism of the twisted K-groups, as in (15), consistent under composition. □

Lemma 4. — *The Azumaya bundle $\mathscr{J}(\gamma)$, lifted to Y, is completion isomorphic to the trivial bundle $\mathscr{K}$, with the isomorphism fixed up to homotopy, and this induces the natural isomorphisms* (14).

Proof. — The primitivity condition on J shows that when lifted to the second two factors of $Y^{[3]}$ it is isomorphic to the bundle over $Y^{[3]}$ of which the elements of $\Psi(Y^{[2]}/Y; J')$, the smoothing operators on the fibers of $Y^{[2]}$ as a bundle over Y, are (density-valued) sections. As noted above, Kuiper's theorem shows that the completion of $\Psi(Y^{[2]}/Y; J')$ is naturally, up to homotopy, isomorphic to the trivial Azumaya bundle $\mathscr{K}$, from which (14) follows. □

5. Analytic index

We now proceed to define the analytic index map (16) using the constructions in §2, §3 and §4. The first step is to define the projective bundle of pseudodifferential operators. We do this by direct generalization of Definition 1. So, for any $\mathbb{Z}_2$-graded bundle $\mathbb{E} = (E_+, E_-)$ over Y set

$$\Psi^\ell(Y/X; \mathscr{A} \otimes \mathbb{E}) = I^\ell(Y^{[2]}, \mathrm{Diag}; J \otimes \mathrm{Hom}(\mathbb{E}) \otimes \Omega_R) \tag{5.1}$$

where $\mathrm{Hom}(\mathbb{E}) = E_- \boxtimes E'_+$ over $Y^{[2]}$ and I^ℓ is the space of (classical) conormal distributions. As is typical in projective index theory, the Schwartz kernel of the projective family of elliptic operators is globally defined, even though one only has local families of elliptic operators with a compatibility condition on triple overlaps given by a phase factor. More precisely, definition (5.1) means that on any open set in $Y^{[2]}$ over which J is trivialized as $\mathrm{Hom}(K_\tau)$ as in Proposition 2, the kernel is that of a family of pseudodifferential operators on the fibres of Y acting from sections of E_+ to sections of E_-. It follows from the standard case that (3.4) also extends immediately to show that if $\tau : U \longrightarrow Y$ is a section over an open set, then

$$(5.2) \quad \Psi^\ell(\phi^{-1}(U)/U; \mathcal{A} \otimes \mathbb{E}) \cong_\tau \Psi^\ell(\phi^{-1}(U)/U; K_\tau \otimes \mathbb{E}) \xrightarrow{\sigma_\ell} \mathcal{C}^\infty(S^*(\phi^{-1}(U)/U); \mathrm{hom}(\mathbb{E}) \otimes N_\ell),$$

where we have used the fact that $\mathrm{hom}(K_\tau)$ is canonically trivial. The principal symbol map here is invariant under conjugation by functions and hence well-defined independent of the trivialization; N_ℓ is the trivial line bundle corresponding to functions of homogeneity ℓ on $T^*(\phi^{-1}(U)/U)$ and $\mathrm{hom}(\mathbb{E})$ is the bundle (over $S^*(\phi^{-1}(U)/U)$) of homomorphisms from E_+ to E_-. Thus the usual composition properties of pseudodifferential operators extend without any difficulty as do the symbolic properties. More precisely,

Lemma 5. — *The spaces of smooth sections of $\Psi^\ell(Y/X; \mathcal{A} \otimes \mathbb{E})$ form graded modules under composition and the principal symbol defined through* (5.2) *is independent of τ and gives a multiplicative short exact sequence for any ℓ :*

$$(5.3) \quad \Psi^{\ell-1}(Y/X; \mathcal{A} \otimes \mathbb{E}) \hookrightarrow \Psi^\ell(Y/X; \mathcal{A} \otimes \mathbb{E}) \xrightarrow{\sigma_\ell} \mathcal{C}^\infty(S^*(Y/X; p^* \mathrm{hom}(\mathbb{E}) \otimes N_\ell)).$$

Proof. — The theory of conormal distributional sections of a complex vector bundle with respect to a submanifold, implicit already in Hörmander's paper [**18**], shows that these have well-defined principal symbols which are homogeneous sections over the conormal bundle of the submanifold, in this case the fibre diagonal, of the pull-back of the bundle tensored with a density bundle. In this case, as for pseudodifferential operators, the density bundles cancel. Moreover the bundle J is canonically trivial over the (fiber) diagonal in $Y^{[2]}$ by the primitivity property of J. The symbol in (5.3) therefore does not involve any twisting – it takes values in the same space as in the untwisted case, and is a well-defined homogeneous section of the homomorphism bundle of E (hence section of that bundle tensored with the homogeneity bundle N_l) on the fibre cotangent bundle – which is canonically the conormal bundle of the fibre diagonal, as claimed. □

With the trivialization κ fixed, the symbol of a projective family of elliptic pseudodifferential operators determines an element in $\mathrm{K}^0_\mathrm{c}(T^*(Y/X))$ We now show that the index of such a projective elliptic family is an element in twisted K-theory of the base, $\mathrm{K}^0(X, \mathcal{A})$. More precisely, let $P \in \Psi^m(Y/X; \mathcal{A} \otimes \mathbb{E})$ be a projective family of elliptic operators. This means that the symbol is invertible in the usual sense, so from the standard ellipticity construction (using iteration over ℓ in the sequence (5.3)) P has a parametrix $Q \in \Psi^{-m}(Y/X; \mathscr{A}\otimes\mathbb{E}_-)$, where $\mathbb{E}_- = (E_-, E_+)$, such that $S_0 = 1 - QP \in \Psi^{-\infty}(Y/X; \mathscr{A}\otimes E_+)$ and $S_1 = 1 - PQ \in \Psi^{-\infty}(Y/X; \mathscr{A}\otimes E_-)$. Then the index is realized using the idempotent

$$E_1 = \begin{pmatrix} 1 - S_0^2 & Q(S_1 + S_1^2) \\ S_1 P & S_1^2 \end{pmatrix} \in M_2(\Psi^{-\infty}(Y/X; \mathscr{A}\otimes\mathbb{E})^\dagger).$$

Here, $\dagger$ denotes the unital extension of the algebra. It is standard to verify that E_1 is an idempotent.

Then, as in the usual case, the analytic index of P expressed in terms of idempotents is

$$\begin{gathered}\operatorname{ind}_a(P) = [E_1 - E_0] \in \mathrm{K}_0(\Psi^{-\infty}(Y/X; \mathscr{A}\otimes\mathbb{E})) \text{ where} \\ E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\Psi^{-\infty}(Y/X; \mathscr{A}\otimes\mathbb{E})^\dagger).\end{gathered} \tag{5.4}$$

That $\operatorname{ind}_a(P)$ is a well-defined element in the K-theory follows from invariance of K-theory under Morita equivalence of algebras. Thus, the inclusion

$$\mathscr{C}^\infty(X, \mathscr{A}) = \Psi^{-\infty}(Y/X; \mathscr{A}) \hookrightarrow \Psi^{-\infty}(Y/X; \mathscr{A}\otimes\mathbb{E}),$$

induces a natural isomorphism of $\mathrm{K}_0(\Psi^{-\infty}(Y/X; \mathscr{A}\otimes\mathbb{E}))$ and $\mathrm{K}^0(X; \mathscr{A})$. Therefore we have defined the *analytic index* of any projective family of elliptic pseudodifferential operators.

To see that this fixes the map,

$$\operatorname{ind}_a : \mathrm{K}^0_{\mathrm{c}}(T^*(Y/X); \rho^*\phi^*\mathscr{A}) \longrightarrow \mathrm{K}^0(X, \mathscr{A}) \tag{5.5}$$

we need, as usual, to check homotopy invariance, invariance under bundle isomorphisms and stability. However, this all follows as in the standard case.

Of particular geometric interest are examples arising from projective families of (twisted) Dirac operators. If the fibres of Y are even-dimensional and consistently oriented, let $\mathrm{Cl}(Y/X)$ denote the bundle of Clifford algebras associated to some family of fiber metrics and let $\mathbb{E}$ be a $\mathbb{Z}_2$-graded hermitian Clifford module over Y with unitary Clifford connection $\nabla^{\mathbb{E}}$.

This data determines a family of (twisted) Dirac operators $\eth_{\mathbb{E}}$ acting fibrewise on the sections of $\mathbb{E}$. We can further twist $\eth_{\mathbb{E}}$ by a connection ∇^τ of the line bundle K_τ over $\phi^{-1}(U) \subset Y$ for contractible open subsets $U \subset X$. In this way, we get a projective family of (twisted) Dirac operators $\eth_{\mathscr{A}\otimes\mathbb{E}} \in \Psi^1(Y/X; \mathbb{E}\otimes\mathscr{A})$ which can be viewed as a family of twisted Dirac operators acting on a projective Hilbert bundle $\mathbb{P}(\phi_*(\mathbb{E}\otimes K_\tau))$ over X. Here the local bundle $\phi_*(\mathbb{E}\otimes K_\tau)$ is given by $U \times L^2(\phi^{-1}(U)/U; \mathbb{E}\otimes K_\tau)$ for contractible open subsets $U \subset X$.

The above projective Dirac family can be globally defined as follows. Consider the delta distributional section $\delta_Z^{\mathbb{E},J} \in I^\bullet(Y^{[2]}, J\otimes \mathrm{Hom}(\mathbb{E})\otimes\Omega_R)$, which is supported on the fibrewise diagonal in $Y^{[2]}$. Let ${}^L\nabla^{\mathbb{E}}$ denote the unitary Clifford connection acting on the left variables, and ∇^J a connection on J which is compatible with the primitive property of J. Then

$$(1 \otimes {}^L\nabla^{\mathbb{E}} + \nabla^J \otimes 1)\delta_Z^{\mathbb{E},J} \in I^{\bullet-1}(Y^{[2]}, J \otimes \mathrm{Hom}(\mathbb{E}) \otimes T^*(Y/X) \otimes \Omega_R),$$

and composition with the contraction given by Clifford multiplication gives

$$c \circ (1 \otimes {}^L\nabla^{\mathbb{E}} + \nabla^J \otimes 1)\delta_Z^{\mathbb{E},J} \in I^{\bullet-1}(Y^{[2]}, J \otimes \mathrm{Hom}(\mathbb{E}) \otimes \Omega_R),$$

which represents the Schwartz kernels of the projective family of (twisted) Dirac operators denoted above by $\eth_{\mathcal{A}\otimes\mathbb{E}}$.

6. The topological index

In this section we define the *topological index* map for the setup in the previous section,

$$\mathrm{ind}_t : \mathrm{K}_{\mathrm{c}}^0(T(Y/X); \rho^*\phi^*\mathcal{A}) \longrightarrow \mathrm{K}^0(X; \mathcal{A}). \tag{6.1}$$

It is defined in terms of Gysin maps in twisted K-theory, which have been studied in the case of torsion twists in [**20**], which extends routinely to the general case as in [**10, 8**]. In the particular case that we consider here, there are several simplifications that we shall highlight.

We first recall some functorial properties of twisted K-theory. Let $F : Z \longrightarrow X$ be a smooth map between compact manifolds. Then the pullback map,

$$F^! : \mathrm{K}^0(X, \mathcal{A}) \longrightarrow \mathrm{K}^0(Z, F^*\mathcal{A}),$$

is well defined.

Lemma 6. — *There is a canonical isomorphism,*

$$j_! : \mathrm{K}^0(X, \mathcal{A}) \cong \mathrm{K}_c^0(X \times \mathbb{R}^{2N}, \pi_1^*\mathcal{A}),$$

determined by Bott periodicity, where the inclusion $j : X \to X \times \mathbb{R}^{2N}$ is onto the zero section. Here $\pi_1 : X \times \mathbb{R}^{2N} \to X$ is the projection onto the first factor.

Proof. — First notice that $\mathrm{K}^\bullet(X, \mathcal{A}) = \mathrm{K}_\bullet(C^\infty(X, \mathcal{A}))$ and $\mathrm{K}_c^\bullet(X \times \mathbb{R}^{2N}, \pi_1^*\mathcal{A}) = \mathrm{K}_\bullet(C_c^\infty(X \times \mathbb{R}^{2N}; \pi_1^*\mathcal{A}))$. But $C_c^\infty(X \times \mathbb{R}^{2N}; \pi_1^*\mathcal{A}) = C^\infty(X, \mathcal{A}) \otimes C_c^\infty(\mathbb{R}^{2N})$. So the lemma follows from Bott periodicity for the K-theory of (smooth) operators algebras. $\square$

For the fiber bundle $\phi : Y \longrightarrow X$ of compact manifolds, we know that there is an embedding $i : Y \longrightarrow X \times \mathbb{R}^N$, cf. [**5**] §3. The fibrewise differential is an embedding $Di : T(Y/X) \longrightarrow X \times \mathbb{R}^{2N}$ with complex normal bundle $\mathcal{N}$.

Let $\mathcal{A}$ be the smooth Azumaya algebra over X as defined earlier in §3; there is a fixed trivialization of $\phi^*\mathcal{A}$. Let $\mathcal{A}_{\mathcal{N}}$ be the lift of $\phi^*\mathcal{A}$ to $\mathcal{N}$. Let $\rho : T(Y/X) \longrightarrow Y$ be the projection map. Then since $\phi^*\mathcal{A}$ is trivialized, we have the commutative diagram

$$
\begin{array}{ccc}
\mathrm{K}_c^0(T(Y/X),\rho^*\phi^*\mathcal{A}) & \xrightarrow{Di_!:} & \mathrm{K}_c^0(\mathcal{N},\mathcal{A}_{\mathcal{N}}) \\
\cong_{\kappa^{-1}}\Big\downarrow & & \Big\downarrow\cong \\
\mathrm{K}_c^0(T(Y/X)) & \xrightarrow{Di_!:} & \mathrm{K}_c^0(\mathcal{N}),
\end{array}
\tag{6.2}
$$

where $Di_!$ in the lower horizontal arrow is given by $\xi = (\xi^+,\xi^-,G) \longmapsto \pi^*\xi\otimes(\pi^*\mathcal{S}^+,\pi^*\mathcal{S}^-,c(v))$. Here $\xi=(\xi^+,\xi^-)$ is pair of vector bundles over $T(Y/X)$, $G:\xi^+\longrightarrow\xi^-$ a bundle map between them which is an isomorphism outside a compact subset and $(\pi^*\mathcal{S}^+,\pi^*\mathcal{S}^-,c(v))$ is the usual Thom class of the complex vector bundle $\mathcal{N}$, where π is the projection map of $\mathcal{N}$ and $\mathcal{S}^\pm$ denotes the bundle of half spinors on $\mathcal{N}$. On the the right hand side the the graded pair of vector bundle data is

$$(\pi^*\xi^+\otimes\pi^*\mathcal{S}^+\oplus\pi^*\xi^-\otimes\pi^*\mathcal{S}^-,\pi^*\xi^+\otimes\pi^*\mathcal{S}^-\oplus\pi^*\xi^-\otimes\pi^*\mathcal{S}^+)$$

with map between them being

$$\begin{bmatrix} G & c(v) \\ c(v) & G \end{bmatrix},\ v\in\mathcal{N}.$$

This is an isomorphism outside a compact subset of $\mathcal{N}$ and defines a class in $\mathrm{K}_{\mathrm{c}}^0(\mathcal{N})$ which is independent of choices, provided the trivialization of $\phi^*(\mathcal{A})$ is kept fixed. Then the usual Thom isomorphism theorem asserts that $Di_!$ is an isomorphism. The upper horizontal arrow is defined in the same way by tensoring with the same Thom class.

Now, $\mathcal{N}$ is diffeomorphic to a tubular neighborhood $\mathcal{U}$ of the image of Y; let $\Phi:\mathcal{U}\longrightarrow\mathcal{N}$ denote this diffeomorphism. Then the induced map in K-theory gives isomorphisms,

$$\Phi^*:\mathrm{K}_c^0(\mathcal{N})\cong\mathrm{K}_c^0(\mathcal{U}),\qquad \Phi^*:\mathrm{K}_c^0(\mathcal{N},\mathcal{A}_{\mathcal{N}})\cong\mathrm{K}_c^0(\mathcal{U},\Phi^*(\mathcal{A}_{\mathcal{N}})).$$

We will next show that the inclusion $i_{\mathcal{U}}:\mathcal{U}\to X\times R^{2N}$ of the open set $\mathcal{U}$ in $X\times R^{2N}$ induces a natural extension map

$$(i_{\mathcal{U}})_!:\mathrm{K}_{\mathrm{c}}^0(\mathcal{U},\Phi^*(\mathcal{A}_{\mathcal{N}}))\longrightarrow\mathrm{K}_{\mathrm{c}}^0(X\times R^{2N},\pi_1^*\mathcal{A})$$

To see this, we need to show that the restriction $i_{\mathcal{U}}^*\pi_1^*\mathcal{A}$ is trivialized. Note that $\phi\circ\tau=\pi_1\circ i_{\mathcal{U}}$, where $\tau:\mathcal{U}\longrightarrow Y$ is equal to the composition, $\lambda\circ\Phi$, and $\lambda:\mathcal{N}\longrightarrow Y$ the projection map. Since $(\phi\circ\tau)^*\mathcal{A}=\tau^*\phi^*\mathcal{A}$ is trivializable because $\phi^*\mathcal{A}$ is trivialized, it follows that $i_{\mathcal{U}}^*\pi_1^*\mathcal{A}$ is trivialized.

We have the following commutative diagram.

$$
\begin{tikzcd}
\mathrm{K}^0_c(T(Y/X)) \arrow[r, "Di_!", "\cong"'] \arrow[d, "\rho^*\kappa_*"', "\cong"] & \mathrm{K}^0_c(\mathcal{N}) \arrow[r, "\Phi^*", "\cong"'] \arrow[d, "\cong"] & \mathrm{K}^0_c(\mathcal{U}) \arrow[d, "\cong"] \\
\mathrm{K}^0_c(T(Y/X), \rho^*\phi^*\mathcal{A}) \arrow[r, "Di_!", "\cong"'] \arrow[rrdd, "\widetilde{\mathrm{ind}}_t"'] & \mathrm{K}^0_c(\mathcal{N}, \mathcal{A}_{\mathcal{N}}) \arrow[r, "\Phi^*", "\cong"'] & \mathrm{K}^0_c(\mathcal{U}, \Phi^*(\mathcal{A}_{\mathcal{N}})) \arrow[d, "(i_{\mathcal{U}})_!"] \\
& & \mathrm{K}^0_{\mathrm{c}}(X \times R^{2N}, \pi_1^*\mathcal{A}) \arrow[d, "\cong"', "j_!^{-1}"] \\
& & \mathrm{K}^0_{\mathrm{c}}(X, \mathcal{A}).
\end{tikzcd}
\tag{6.3}
$$

The composition of the maps in the diagram above defines the Gysin map in twisted K-theory,

$$Di_! : \mathrm{K}^0_{\mathrm{c}}(T(Y/X)) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X \times \mathbb{R}^{2N}, \pi_1^*\mathcal{A}).$$

Here we have used the fact that since $\pi = \pi_1 \circ i$ it follows that $Di^*\pi_1^*\mathcal{A} = \rho^*\phi^*\mathcal{A}$ is trivialized. Now define the *topological index*, as the map

$$\mathrm{ind}_t = j_!^{-1} \circ Di_! : \mathrm{K}^0_{\mathrm{c}}(T(Y/X)) \longrightarrow \mathrm{K}^0(X; \mathcal{A}), \tag{6.4}$$

where we apply the Thom isomorphism in Lemma 6 to see that the inverse $j_!^{-1}$ exists. We also note that $\widetilde{\mathrm{ind}}_t \circ \rho^*\kappa_* = \mathrm{ind}_t$, consistent with the corresponding analytic indices.

The source is untwisted since $\mathcal{A}$ is trivialized by κ, as an Azumaya bundle, when pulled back to Y. The identification of twisted and untwisted K-theory in (16) depends on the choice of trivialization (6) but then so does the Azumaya bundle and these choices do not change the index map $\widetilde{\mathrm{ind}}_{\mathrm{t}}$.

7. Twisted Chern character

First we recall an explicit formula for the odd Chern character in the untwisted case. For any compact manifold (of positive dimension), Z, the group of invertible, smoothing, perturbations of the identity operator

$$G^{-\infty}(Z) = \{a \in \Psi^{-\infty}(Z); \exists\ (\mathrm{Id} + a)^{-1} = \mathrm{Id} + b,\ b \in \Psi^{-\infty}(Z)\} \tag{7.1}$$

is classifying for odd K-theory. So there is a canonical identification of the odd K-theory of a compact manifold X with the (smooth) homotopy classes of (smooth) maps

$$\mathrm{K}^1(X) = [X; G^{-\infty}(Z)]. \tag{7.2}$$

The odd Chern character is then represented in deRham cohomology by the pull-back of the universal Chern character on $G^{-\infty}(Z)$:

$$\begin{aligned}\mathrm{Ch} &= \sum_{k=0}^{\infty} c_k \operatorname{tr}\left((A^{-1}dA)^{2k+1}\right) \\ &= -\frac{1}{2\pi i}\int_0^1 \operatorname{tr}\left((A^{-1}dA)\exp\Big(\frac{t(1-t)}{2\pi i}(A^{-1}dA)^2\Big)\right)dt,\ \ A = \mathrm{Id}+a.\end{aligned} \tag{7.3}$$

Here $dA = da$, as for finite dimensional Lie groups, is the natural identification of $T_aG^{-\infty}$ with $\Psi^{-\infty}(Z)$ coming from the fact that $G^{-\infty}(Z)$ is an open (and dense) set in $\Psi^{-\infty}(Z)$. Thus for an odd K-class

$$\begin{gathered} a : X \longrightarrow G^{-\infty}(Z),\ \ \mathrm{Ch}([a]) = [a^* \mathrm{Ch}] \in \mathrm{H}^{\mathrm{odd}}(X), \\ a^* \mathrm{Ch} = -\frac{1}{2\pi i}\int_0^1 \operatorname{tr}\left((\mathrm{Id}+a)^{-1}da\exp\Big(\frac{t(1-t)}{2\pi i}((\mathrm{Id}+a)^{-1}da)^2\Big)\right)dt \end{gathered} \tag{7.4}$$

where now the differential can be interpreted in the usual way for functions valued in the fixed vector space $\Psi^{-\infty}(Z)$.

For any fiber bundle $\phi : Y \longrightarrow X$, with typical fiber Z, $\mathrm{K}^1(X)$ is also naturally identified with the abelian group of homotopy classes of sections of the bundle of groups over X with fiber $G^{-\infty}(Z_x)$ at $x \in X$. That is, the twisting by the diffeomorphism group does not affect this property. The formula (7.4) can be extended to this geometric setting by choosing a connection on ϕ, i.e. a lift of vector fields from X to Y. Indeed, such a connection can be identified as a connection on the bundle $\mathscr{C}^\infty(Y/X)$, with fibres $\mathscr{C}^\infty(Z_x)$ (and space of sections $\mathscr{C}^\infty(Y)$), as a differential operator

$$\begin{gathered} \nabla : \mathscr{C}^\infty(Y) \longrightarrow \mathscr{C}^\infty(Y;\phi^*T^*X), \\ \nabla(hg) = (dh)g + h\nabla g,\ \ h \in \mathscr{C}^\infty(X),\ \ g \in \mathscr{C}^\infty(Y). \end{gathered} \tag{7.5}$$

The curvature of such a connection (extended to a superconnection), is a first order differential operator on the fibres $w = \nabla^2/2\pi i \in \mathrm{Diff}^1(Y/X;\mathbb{C},\phi^*\Lambda^2X)$ from the trivial bundle to the 2-form bundle lifted from the base. The connection on Y induces a connection on $\Psi^{-\infty}(Y/X)$, as a bundle of operators on $\mathscr{C}^\infty(Y/X)$, acting by conjugation and then (7.4) is replaced by

$$\begin{gathered} \mathrm{Ch}(A) = \\ -\frac{1}{2\pi i}\int_0^1 \operatorname{tr}\left((A^{-1}\nabla A)\exp\Big((1-t)w + tA^{-1}wA + \frac{t(1-t)}{2\pi i}(A^{-1}\nabla A)^2\Big)\right)dt, \\ A : X \longmapsto G^{-\infty}(Y/X),\ \ \pi\circ A = \mathrm{Id}. \end{gathered} \tag{7.6}$$

Note that any such section is homotopic to a section which is a finite rank perturbation of the identity, in which case (7.6) becomes the more familiar formula. The same conclusions, and formula hold, if the bundle of groups of smoothing perturbations of the identity acting on a vector bundle over Y, $G^{-\infty}(Y/X;E)$, is considered, provided the connection (and curvature) are lifted to a connection on E.

Note that (7.6) can also be considered as the pull-back of a universal form on the total space of the fibration $G^{-\infty}(Y/X)$. It then has the property that restricted to a fiber, so that the curvature vanishes, one recovers the original form in (7.3).

The case of immediate interest arises from a circle bundle $p : \tilde{L} \longrightarrow X$. As explained in §3 we consider the fiber product $\tilde{L}^{[2]}$ fibred over $\tilde{L}$ with the fibres taken to be in the second factor, with the smoothing operators acting on sections of J. Of course these operators are acting on the restriction of this line bundle to each fiber, which is a circle, so they can always be identified on each fiber with ordinary smoothing operators. On the other hand J has the primitivity property of Lemma 2 which allows us to identify the smoothing operators on sections of J on the fibres of $\tilde{L}^{[2]}$ with $\mathscr{C}^\infty(\tilde{L}^{[3]}; \pi_F^* J)$ as in Proposition 3. An explicit fiber density factor is not needed since this is supplied naturally by the Hermitian structure.

Proposition 6. — *Suppose $a \in \mathscr{C}^\infty(\tilde{L}^{[2]}; J)$ is such that $A = \mathrm{Id} + a$ is everywhere invertible over X. Then the odd Chern character of* $\mathrm{Id} + a$, *as a form on $\tilde{L}$ computed with respect to a unitary connection on L and the primitive connection of Lemma 1 on J, with combined curvature Ω,*

$$
\begin{aligned}
(7.7)\quad & \mathrm{Ch}_{\mathscr{A}}(A) = \\
& -\frac{1}{2\pi i}\int_0^1 \mathrm{tr}\left((A^{-1}\nabla A)\exp\left((1-t)\Omega + tA^{-1}\Omega A + t(1-t)(A^{-1}\nabla A)^2\right)\right) dt \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \in \mathscr{C}^\infty(\tilde{L}; \Lambda^{\mathrm{odd}})
\end{aligned}
$$

is closed and satisfies the conditions in (1.10).

Proof. — That the Chern form (7.7) is closed follows from the standard properties. To see the other stated properties, we choose a section of $\tilde{L}$ over an open set $U \subset X$ over which u has a smooth logarithm and set $f = \frac{1}{2\pi i}\log u$. In terms of the induced trivializations

$$\tilde{L}_U = U \times \mathbb{S},\ \tilde{L}_U^{[2]} = U \times \mathbb{S} \times \mathbb{S}, \tag{7.8}$$

let the fiber variables be θ_1 and θ_2. The operators are acting in the θ_2 variable and the lifted connection on $\tilde{L}$ as a fibration over $\tilde{L}$ is therefore

$$\nabla = d_x + d_{\theta_1} + \overline{\gamma}\partial_{\theta_2} \tag{7.9}$$

where $\overline{\gamma} \in \mathscr{C}^\infty(U; \Lambda)$ is the local connection form for L. The corresponding connection on $\mathscr{C}^\infty(\tilde{L}^{[2]}/\tilde{L}; J)$ in terms of this trivialization and of the connection on J from Lemma 1

$$\nabla_J = d_x + d_{\theta_1} + \overline{\gamma}\partial_{\theta_2} + f d\theta_1 - \overline{\gamma} f. \tag{7.10}$$

The curvature is

$$\Omega = \nabla_J^2/2\pi i = \overline{\beta}(\partial_{\theta_2} - f) + \frac{1}{2\pi i}\overline{\alpha} \wedge \gamma,\ \gamma = d\theta_1 + \overline{\gamma}. \tag{7.11}$$

In terms of this local trivialization $a = a(x, \theta_2, \theta_3)$ is independent of the first (parameter) fiber variable. Inserting (7.11) into (7.7) observe that the two terms in (7.11) commute so

$$\mathrm{Ch}_{\mathscr{A}}(A) = e^{\frac{\overline{\alpha}\wedge\gamma}{2\pi i}} v,\ \ v \in \mathscr{C}^\infty(U; \Lambda^{\mathrm{odd}}) \tag{7.12}$$

satisfies the conditions of (1.10). □

Note that under a deck transformation of $\hat{X}$, i.e. integral shift of f by $n \in \mathbb{N}$, each term undergoes conjugation by $\exp(in\theta)$ and the Chern form itself is therefore unchanged.

It follows from Proposition 1 that $\mathrm{Ch}_{\mathscr{A}}(A)$ defines an element in the twisted cohomology of X, given explicitly by the form v in (7.12). Although the proof above is written out for sections of J over $\tilde{L}^{[2]}$ the passage to matrix-valued sections is merely notational, so it applies essentially unchanged to elements of

$$\begin{aligned} G(X; \mathscr{A} \otimes M(N, \mathbb{C})) = \\ \{a \in \mathscr{C}^\infty(\tilde{L}^{[2]}; J \otimes M(N, \mathbb{C}); \mathrm{Id}_{N\times N} + a(x) \text{ is invertible for all } x \in X\}. \end{aligned} \tag{7.13}$$

***Lemma** 7.* — *The Chern form* (7.7) *descends to represent the twisted Chern character*

$$G(X; \mathscr{A} \otimes M(N, \mathbb{C}))/\!\sim\ = \mathrm{K}^1(X; \mathscr{A}) \longrightarrow \mathrm{H}^{\mathrm{odd}}(X; \delta) \tag{7.14}$$

where the equivalence relation on invertible matrix-valued sections of the Azumaya bundle is homotopy and stability.

Proof. — The invariance of the twisted cohomology class under stabilization follows directly from the definition. Invariance under homotopy follows as usual from the fact that the construction is universal and the form is closed, so is closed for a homotopy when interpreted as a family over $X \times [0,1]$ and this proves the invariance of the cohomology class.

It also follows directly from the definition that the twisted Chern character behaves appropriately under the Thom isomorphism for a complex (or symplectic) vector bundle $w : W \longrightarrow X$. That is, there is a commutative diagram with horizontal isomorphisms

$$\begin{tikzcd} \mathrm{K}^1(X; \mathscr{A}) \arrow[r, "\boxtimes b"] \arrow[d, "\mathrm{Ch}_{\mathscr{A}}"'] & \mathrm{K}^1(W; w^*\mathscr{A}) \arrow[d, "\mathrm{Ch}_{\mathscr{A}}"] \\ \mathrm{H}^{\mathrm{odd}}(X; \delta) \arrow[r, "\wedge \mathrm{Td}"] & \mathrm{H}^{\mathrm{odd}}_{\mathrm{c}}(W; w^*\delta) \end{tikzcd} \tag{7.15}$$

As in the case of the bundle of groups $G^{-\infty}(Y/X)$ the form (7.7) is again the pull-back from the total space of the bundle of groups $G^{-\infty}(X; \mathscr{A})$ of invertible sections of the unital extension of the Azumaya bundle and then restricting this universal form to a fiber one again recovers the standard odd Chern character in (7.3). This is enough to show that the Chern form here does represent the twisted Chern character as widely discussed in the literature, for instance recently by Atiyah and Segal in

[4]. Namely they remark that the Chern character as they describe it (in the even case), which is determined by universality under pull-back from the twisted PU bundle over $K(\mathbb{Z},3)$, is actually determined by its pull-back to the 3-sphere. The PU bundle over $\mathbb{S}^3$ with generating DD class is trivial over points and so can be transferred to $(0,\pi)\times\mathbb{S}^2$ to be trivial outside a compact set and thence to $\mathbb{S}\times\mathbb{S}^2$ where it reduces to the twisted bundle again with generating DD class. As shown in [4], the universal twisted Chern character over the 3-sphere is determined by multiplicativity and the fact that it restricts to the standard Chern character on the fibres over points. The odd case follows by suspension so the deRham version of the Chern character above does correspond with more topological definitions. $\square$

We need some extensions of this discussion of the odd twisted Chern character. In particular we need to discuss the even case. However, the context needs to be broadened to cover operators on the fibres of a trivializing bundle Y as in (3) – (6). Finally the relative case is needed for the discussion of the Chern character of the symbol and the index formula in twisted cohomology. Fortunately these are all straightforward generalizations of the untwisted case.

We start with the extension of the odd twisted Chern character to the more general geometric case under discussion here. Thus, instead of being over $\tilde{L}^{[2]}$, the bundle J is defined over $Y^{[2]}$. Still, when lifted to the fiber product

$$\tilde{Y}^{[2]}=\tilde{L}\times_X Y^{[2]}, \tag{7.16}$$

J is reduced to a the exterior tensor product

$$p^*J=\tilde{J}\boxtimes\tilde{J}' \text{ over } \tilde{Y}^{[2]} \tag{7.17}$$

where $\tilde{J}$ is a line bundle over $\tilde{Y}=\tilde{L}\times_X Y$. Namely, there is a character property for $s: Y^{[2]}\longrightarrow \mathrm{U}(1)$, which is determined by the trivialization of L over Y, when lifted to $\tilde{Y}^{[2]}$:

$$s(z_1,z_2)=\tilde{s}(z_1,\tilde{l})\tilde{s}(z_2,\tilde{l})^{-1},\ \ \tilde{s}:\tilde{Y}\longrightarrow \mathrm{U}(1). \tag{7.18}$$

Here $\tilde{s}$ is fixed by the demand that it intertwine the trivializations over $\tilde{L}$ and Y. Thus, using $\tilde{s}$ to define $\tilde{J}$ by the same procedure as previously used to define J, (7.17) follows.

From this point the discussion proceeds as before. That is, the Azumaya bundle $\mathscr{A}_Y$ acting on the fibres of Y over X lifts to $\tilde{Y}$, acting on the same fibres but now over $\tilde{L}$, into a subalgebra of $\Psi^{-\infty}(\tilde{Y}/\tilde{L};\tilde{J})$. Then, as above, the odd Chern character for invertible sections of the unital extension of the Azumaya bundle is a differential form on $\tilde{L}$ satisfying (1.10) and so defines the twisted odd Chern character in this more general geometric setting.

Next, consider the even twisted Chern character. To do so, recall that for a complex vector bundle E embedded as a subbundle of some trivial $\mathbb{C}^N$ over a manifold X the curvature, and Chern character, can be written in terms of an idempotent e projecting onto the range as the 2-form valued homomorphism $\omega_E=e(de)(1-e)(de)e/2\pi i$. There is a similar formula if E is embedded in a possibly non-trivial bundle F with connection ∇_F which is projected onto E using e. The same formula applies in the case of a subbundle of $\mathscr{C}^\infty(Y/X;E)$ given by a family of idempotents e. In the untwisted case, the K-theory of X can be represented by formal differences of finite rank idempotents in the fibres of $\mathscr{C}^\infty(Y/X;E)$, giving finite dimensional bundles. In general, in the twisted case, the K-theory is interpreted as the C^* K-theory of a non-unital algebra (the completion of $\mathscr{A}$ in the compacts), it is necessary to take pairs of infinite rank idempotents in $\mathbb{C}^N\otimes\mathscr{A}^\dagger$ with differences valued in $\mathbb{C}^N\otimes\mathscr{A}$. In fact it is enough to take single idempotents in $\mathbb{C}^N\otimes\mathscr{A}^\dagger$ with constant unital part $e_0\in M(N,\mathbb{C})$ and consider the formal difference $e\ominus e_0$ to generate the K-theory. For the untwisted case, the usual Chern character is given by

$$\operatorname{tr}(\exp(\frac{\omega_E^2}{2\pi i})-e_0) \tag{7.19}$$

as can be shown by suspension from the odd case if desired. Here all terms in $\Lambda^{>0}$ involve a derivative of e and hence are smoothing, as is the normalized term of form degree zero, so the trace functional can be applied.

To carry this discussion of the even Chern character to the twisted case, we can proceed precisely as above. Namely, given an idempotent section, e, of $\mathbb{C}^N\otimes\mathscr{A}_Y^\dagger$ as a bundle over X with constant unital term e_0 one can compute the Chern form (7.19) after lifting the idempotent to $\mathbb{C}^N\otimes\Psi^{-\infty}(\tilde{Y}/\tilde{L};\tilde{J}\otimes\mathbb{C}^N)$ as discussed above. Then, for the same reason, the form satisfies (1.10) and defines the even Chern character as a twisted deRham form on X.

The final extension is to the relative case to handle the Chern character of the symbol of a pseudodifferential operator. As discussed in **[1]** for any real vector bundle $W\longrightarrow Y$ (here applied to $T^*(Y/X)$) the compactly supported cohomology of W can be obtained directly as from the relative deRham complex of SW, the sphere bundle of W, and Y. This involves the same odd Chern class on SW (which is no longer closed) and the even Chern class on Y which 'corrects' the failure of the odd form to be closed. The extension to the twisted case just combines the two cases discussed above; this is briefly considered in §10.

8. Semiclassical quantization

To avoid the usual complications which arise in the proof of the index theorem, especially concerning the multiplicativity of the analytic index – although they are no worse in the present twisted setting than the standard one – we introduce another definition of the index map using semiclassical pseudodifferential operators. This approach is discussed in more detail in **[21]** but the underlying notion of a semiclassical family of pseudodifferential operators is well established in the literature **[15]**. The method of 'asymptotic morphism' of Connes and Higson is closely related to the notion of semiclassical limit.

Proposition 7. — *Let $\psi : M \longrightarrow B$ be a fibre bundle of possibly non-compact manifolds then the modules*

$$\Psi^{\ell}_{\mathrm{c,scl}}(M/B;\mathbb{E}) \subset \mathscr{C}^{\infty}((0,1)_{\epsilon};\Psi^{\ell}_{\mathrm{c}}(M/B;\mathbb{E}))$$

of semiclassical families of classical, uniformly compactly-supported, pseudodifferential operators on the fibres of ψ are well defined for any $\mathbb{Z}_2$-graded bundle $\mathbb{E}$, have a global multiplicative exact symbol sequence

$$0 \longrightarrow \epsilon\Psi^{\ell}_{\mathrm{c,scl}}(M/B;\mathbb{E}) \hookrightarrow \Psi^{\ell}_{\mathrm{c,scl}}(M/B;\mathbb{E}) \xrightarrow{\sigma_{\mathrm{scl}}} S^{\ell}_{\mathrm{c}}(T^*(M/B);\hom(\mathbb{E})) \longrightarrow 0 \tag{8.1}$$

and completeness property

$$\bigcap_j \epsilon^j\Psi^{\ell}_{\mathrm{c,scl}}(M/B;\mathbb{E}) = \dot{\mathscr{C}}^{\infty}([0,1);\Psi^{\ell}_{\mathrm{c}}(M/B;\mathbb{E})). \tag{8.2}$$

Note that the space of functions on the right in (8.1) consists of the *global* classical symbols on $T^*(M/B)$, with compact support in the base M, not the quotient by the symbols of order $\ell - 1$. The space on the right in (8.2) consists of the smooth families of pseudodifferential operators with uniformly compact support in the usual sense, depending smoothly on the additional parameter $\epsilon \in [0,1)$ down to $\epsilon = 0$ where they vanish with all derivatives. Thus, by iteration, the semiclassical symbol in (8.1) captures the complete behaviour of these operators as $\epsilon \downarrow 0$.

To define the semiclassical index maps, one for each parity, we only need the smoothing operators of this type, for $\ell = -\infty$; indeed this is the key to their utility. In this special case the Schwartz kernels of the operators are easily described explicitly. Namely they correspond to the subspace of $\mathscr{C}^{\infty}((0,1) \times M^{[2]}_{\psi};\mathrm{Hom}(\mathbb{E}) \otimes \Omega_R))$ consisting of those functions which have support in some set $(0,1) \times K$ with $K \subset M^{[2]}_{\psi}$ compact, which tend to 0 rapidly with all derivatives as $\epsilon \downarrow 0$ in any closed set in $M^{[2]}_{\psi}$ disjoint from the diagonal and which near each point of the diagonal take the from

$$\epsilon^{-d}K(\epsilon, b, z, z', \frac{z - z'}{\epsilon})|dz'| \tag{8.3}$$

where K is a smooth bundle homomorphism which is uniformly Schwartz in the last variable and d is the fiber dimension.

As with usual pseudodifferential operators, there is no obstruction to defining $\Psi^{\ell}_{\mathrm{scl}}(Y/X;\mathscr{A}\otimes\mathbb{E})$ either by transferring the kernels directly to sections of $J \otimes \mathrm{Hom}(\mathbb{E})$ over $Y^{[2]}$ or by using the local form (3.4).

Proposition 8. — *The space of invertible elements in the unital extension of the semiclassical twisted smoothing operators defines an odd index map via the diagram*

(8.4)

$$\begin{tikzcd}[column sep=small] & \bigcup_N \left\{(A,B) \in \Psi^{-\infty}_{\mathrm{scl}}(M/B;\mathscr{A}\otimes\mathbb{C}^N);(\mathrm{Id}+A)^{-1} = \mathrm{Id}+B\right\} \arrow[dl, "{[\mathrm{Id}+\sigma_{\mathrm{scl}}(A)]}"'] \arrow[dr, "{[(\mathrm{Id}+A)\big|_{\epsilon=\frac12}]}"] & \\ \mathrm{K}^1_{\mathrm{c}}(T^*(Y/X)) \arrow[rr, "\mathrm{ind}^1_{\mathrm{scl}}"] & & \mathrm{K}^1_{\mathrm{c}}(X;\mathscr{A}). \end{tikzcd}$$

Proof. — The space on the top line in (8.4) consists of the invertible perturbations of the identity by semiclassical smoothing operators, with the inverse of the same form. Thus it follows that $\mathrm{Id} + \sigma_{\mathrm{scl}}(A)$ is invertible as a smooth family of $N \times N$ matrices over $T^*(Y/X)$, reducing to the identity at infinity. It therefore defines an element of odd K-theory giving the map on the left. Conversely, the invertibility of $\mathrm{Id} + a$ for a symbol a implies, using the exactness of the symbol sequence, that $\mathrm{Id} + A$, where $\sigma_{\mathrm{scl}}(A) = a$, is invertible at least for small ϵ. Modifying the semiclassical family to remain invertible for $\epsilon \in (0, 1)$ shows that this map is surjective. The map on the right, defined by restriction to $\epsilon = \frac{1}{2}$ (or any other positive value) immediately gives an element of the odd twisted K-theory of the base.

To see that the 'odd semiclassical index', or push-forward map, is defined from this diagram it suffices to note that the 'quantized' class on the right only depends on the class on the left up to homotopy and stability, which as usual follows directly from the properties of the algebra. □

For this odd index there is a companion even index map. Recall that a compactly supported K-class can be defined by a smooth map into $N \times N$ matrices which takes values in the idempotents and is constant outside a compact set, where the class can be identified with the difference of the projection and the limiting constant projection.

Proposition 9. — *If $a \in \mathcal{C}^\infty_c(T^*(Y/X); \mathbb{C}^N)$ is such that $\Pi_\infty + a$ takes values in the idempotents, where $\Pi_\infty \in M(N, \mathbb{C})$, then a has a semiclassical quantization*

$$A \in \Psi^{-\infty}_{\mathrm{scl}}(Y/X; \mathcal{A} \otimes \mathbb{C}^N), \ \sigma_{\mathrm{scl}}(A) = a, \tag{8.5}$$

such that $(\Pi_\infty + A)^2 = \Pi_\infty + A$ and this leads to a well-defined even semiclassical index map

$$\mathrm{K}^0_{\mathrm{c}}(T^*(Y/X)) \underset{\mathrm{ind}^0_a}{\overset{\mathrm{ind}^0_{\mathrm{scl}}}{\underset{=}{\rightrightarrows}}} \mathrm{K}^0_{\mathrm{c}}(X; \mathcal{A}) \tag{8.6}$$

analogous to (8.4) *and as indicated, equal to the analytic index as defined in §5.*

Proof. — Certainly a quantization of a exists by the surjectivity of the symbol map. Moreover the idempotent $\Pi_\infty + a$ can be extended to a 'formal' idempotent, meaning that, using the symbol calculus, the quantization can be arranged to be idempotent up to infinite order error at $\epsilon = 0$. The error terms of order $-\infty$ in the semiclassical smoothing algebra are simply smoothing operators vanishing to infinite order with ϵ. Use of the functional calculus then allows one to perturb the quantization by such a term to give a true idempotent for small $\epsilon > 0$. Then stretching the parameter arranges this for $\epsilon \in (0, 1)$. The pair of this projection, for any $\epsilon > 0$, and the limiting constant projection, Π_∞ defines a K-class. The existence of the map (8.6) then follows in view of the homotopy invariance and stability of this construction.

To see the equality with the analytic index as previously defined is the major step in the proof of the index theorem. This amounts to a construction giving both this semiclassical index map and the usual analytic index map at the same time. The two index maps, semiclassical and analytic are based on two different models for the compactly supported K-theory of $T^*(Y/X)$ – or more generally of a vector bundle W. The first reduces to the set of projection-valued smooth maps $W \longrightarrow M(N, \mathbb{C})$ into matrices which are constant outside a compact set. The second is defined in terms of triples, consisting of a pair of vector bundles over the base together with an isomorphism between their lifts to S^*W.

These two models can be combined into a larger one, in which the set of objects are triples (E, F, a) where E and F are vector bundles over $\overline{W}$, the radial compactification of W, given directly as smooth projection-valued matrices into some $\mathbb{C}^N$ and where a intertwines these two smooth families of projections over S^*W, the boundary of the radial compactification. The equivalence relation $(E_1, F_1, a_1) \simeq (E_2, F_2, a_2)$ is generated by isomorphisms, meaning smooth intertwinings of E_1, and E_2 and of F_1 and F_2 over $\overline{W}$ which also intertwine the isomorphisms over S^*W, the boundary, plus stability. This again gives $\mathrm{K_c}(W)$.

Standard arguments show that any such class in this general sense is equivalent to an 'analytic class' in which the bundles are lifted from the base, or a 'semiclassical class' in which the projections are constant outside a compact set and the isomorphism between them is the identity – in fact the second projection can be taken to be globally constant. Moreover equivalence is preserved under these reductions.

Using these more general triples a combined analytic-semiclassical quantization procedure may be defined by first taking semiclassical quantizations of the projections E, F to actual semiclassical families P, Q which are projections; the classical symbols of these projections can be chosen to be independent of ϵ. This is again the standard argument for quantizations of idempotents which is outlined above. Then the isomorphism a can be quantized to a pseudodifferential operator A in the ordinary sense but this can be chosen to satisfy $AP(\frac{1}{2}) = A = Q(\frac{1}{2})A$ so it 'acts between' the images of $P(\frac{1}{2})$ and $Q(\frac{1}{2})$. This is accomplished by choosing some A' with symbol a and replacing it by the 'Toeplitz operator' $A = Q(\frac{1}{2})A'P(\frac{1}{2})$ which necessarily has the same symbol.

Then A is relatively elliptic, in the sense that it has a parametrix B satisfying $BQ(\frac{1}{2}) = B = P(\frac{1}{2})B$ and with $AB - Q(\frac{1}{2})$ and $BA - P(\frac{1}{2})$ smoothing operators. The analytic-semiclassical index can now be defined using the using the same formula as the analytic index above. That it is well-defined involves the standard homotopy and stability arguments.

Finally then this map clearly reduces to the analytic and semiclassical index maps on the corresponding subsets of data and hence these two maps must be equal. The introduction of the Dixmier-Douady twisting makes essentially no difference to these constructions so the equality in (8.6) follows. $\square$

Proposition 10. — *The appropriate form of Bott periodicity can be proved directly giving commutative diagrams*

$$\begin{tikzcd} \mathrm{K}^1_{\mathrm{c}}(T^*(Y/X)) \arrow[r] \arrow[d, "\mathrm{ind}^1_{\mathrm{scl}}"'] & \mathrm{K}^0_{\mathrm{c}}(T^*(Y\times\mathbb{R})/(X\times\mathbb{R})) \arrow[d, "\mathrm{ind}^0_{\mathrm{scl}}"] \\ \mathrm{K}^1(X;\mathcal{C}) \arrow[r] & \mathrm{K}^0(X\times\mathbb{R};\mathcal{C}) \end{tikzcd} \tag{8.7}$$

where the horizontal maps are the clutching construction and

$$\begin{tikzcd} \mathrm{K}^0_{\mathrm{c}}(T^*(Y/X)) \arrow[r] \arrow[d, "\mathrm{ind}_{\mathrm{scl}}\, 1^0"'] & \mathrm{K}^1_{\mathrm{c}}(T^*(Y\times\mathbb{R})/(X\times\mathbb{R})) \arrow[d, "\mathrm{ind}^1_{\mathrm{scl}}"] \\ \mathrm{K}^0(X;\mathcal{C}) \arrow[r] & \mathrm{K}^1(X\times\mathbb{R};\mathcal{C}) \end{tikzcd} \tag{8.8}$$

where the inverses of the horizontal isomorphism are the Toeplitz index maps.

Corollary 1. — *To prove the equality of the analytic and topological index maps it suffices to prove the equality of the odd semiclassical and odd topological index maps.*

Proof. — Suppose we have proved the equality of odd semiclassical and odd topological index maps

$$\begin{tikzcd} \mathrm{K}^1_{\mathrm{c}}(T^*(Y/X)) \arrow[r, shift left, "\mathrm{ind}^1_{\mathrm{scl}}"] \arrow[r, shift right, "\mathrm{ind}^1_{\mathrm{t}}"'] \arrow[r, phantom, "="] & \mathrm{K}^1_{\mathrm{c}}(X;\mathcal{C}). \end{tikzcd} \tag{8.9}$$

Both the topological and the semiclassical index maps give commutative diagrams as in Proposition 10, so it follows that the more standard, even, versions of these maps are also equal. □

Lemma 8. — *For an iterated fibration of manifolds*

$$\begin{tikzcd} Z' \arrow[r, dash] & M' \arrow[d, "\psi"] \\ Z \arrow[r, dash] & M \arrow[d, "\phi"] \\ & B \end{tikzcd} \tag{8.10}$$

the semiclassical index gives a commutative diagram

$$\begin{tikzcd} \mathrm{K}_c^1(T^*(M'/Y)) \arrow[dd, "\mathrm{ind}^1_{\mathrm{scl}}"] \arrow[dr, "\mathrm{ind}^1_{\mathrm{scl}}"] & \\ & \mathrm{K}_c^1(T^*(M/Y)) \arrow[dl, "\mathrm{ind}^1_{\mathrm{scl}}"] \\ \mathrm{K}_c^1(Y) & \end{tikzcd} \tag{8.11}$$

where the map on the top right is the semiclassical index map for the fibration of M' over M pulled back to $T^*(M/Y)$.

Proof. — The commutativity of (8.11) follows from the use of a double semiclassical quantization, with different parameters in the two fibres (see the extensive discussion in [**21**]). □

Lemma 9. — *For any complex, or real-symplectic, vector bundle W over a manifold X the semiclassical index implements the Thom isomorphism*

$$\mathrm{K}_c^1(W) \underset{\mathrm{Thom}}{\overset{\mathrm{ind}^1_{\mathrm{scl}}}{\underset{\longrightarrow}{\overset{\longrightarrow}{=}}}} \mathrm{K}_c^1(X). \tag{8.12}$$

Proof. — This again follows from the use of semiclassical quantization in the 'isotropic' of (pseudodifferential) Weyl algebra of operators on a symplectic vector space. The resulting symbol map is shown, in [**21**], to be an isomorphism using the argument of Atiyah. Since the Thom map constructed this way is homotopy invariant it applies to to case of a complex vector bundle where the 'positive' sympectic structure on the underlying real bundle is fixed up to homotopy. □

9. The index theorem

The odd topological index is defined as the composite map arising from an embedding so we wish to prove the commutativity of the diagram

$$\begin{tikzcd} \mathrm{K}_{\mathrm{c}}^1(T^*(Y/X)) \arrow[r] \arrow[drr, "\mathrm{ind}_{\mathrm{scl}}"'] & \mathrm{K}_{\mathrm{c}}^1(T^*(\Omega/X); \mathscr{A}) \arrow[r, "\iota^*"] \arrow[dr, "\mathrm{ind}_{\mathrm{scl}}"] & \mathrm{K}_{\mathrm{c}}^1(T^*(\mathbb{R}^M) \times X); \mathscr{A}) \arrow[d, shift right, "\mathrm{ind}_{\mathrm{scl}}"'] \arrow[d, shift left, "\mathrm{Thom}"] \arrow[d, phantom, "="] \\ & & \mathrm{K}_{\mathrm{c}}^1(X; \mathscr{A}). \end{tikzcd} \tag{9.1}$$

Here Ω is a collar neighbourhood of Y embedded in $\mathbb{R}^M$, so is isomorphic to the normal bundle to Y. Thus, it suffices to prove commutativity in three places. The last of these is equality of the two maps on the right, that the semiclassical index map implements the Thom isomorphism (or in this trivial case, Bott periodicity). The second is 'excision' which is immediate from the definition of the semiclassical index.

The first commutativity, for the triangle on the left corresponds to multiplicativity of the semiclassical index which in this case reduces to (a special case of) Lemma 8.

This leads to the main theorem. Here we tacitly identify the tangent and cotangent bundles via a Riemannian metric.

***Theorem 1* (The index theorem in K-theory).** — *Let $\phi : Y \longrightarrow X$ be a fiber bundle of compact manifolds, together with the other data in (3) – (6). Let $\mathscr{A}$ be the smooth Azumaya bundle over X as defined in §3 and $P \in \Psi^{\bullet}(Y/X, \mathscr{A} \otimes \mathbb{E})$ be a projective family of elliptic pseudodifferential operators acting on the projective Hilbert bundle $\mathbb{P}(\phi_*(\mathbb{E} \otimes K_\tau))$ over X, with symbol $p \in \mathrm{K}_c(T(Y/X))$, then*

$$\mathrm{ind}_a(P) = \mathrm{ind}_t(p) \in K^0(X, \mathscr{A}). \tag{9.2}$$

10. The Chern character of the index

As discussed above, the index map in K-theory can be considered as acting on the untwisted K-theory, with compact supports, of $T^*(Y/X)$, via the identification with the (trivially) twisted K-theory coming from the original choice of data (3) – (6). The Chern character for the symbol class in the standard setting,

$$\mathrm{K}^0_{\mathrm{c}}(T^*(Y/X)) \longrightarrow \mathrm{H}^{\mathrm{even}}_{\mathrm{c}}(T^*(Y/X)) \tag{10.1}$$

can be represented explicitly in terms of symbol data and connections in a relative version of the formulæ in §7 following Fedosov [**13**]. A K-class is represented by bundles (E_+, E_-) over Y and an elliptic symbol a identifying them over $S^*(Y/X)$. It is convenient to use the relative interpretation of the cohomology from [**1**]. Thus one can take the explicit representative

$$\begin{aligned} \mathrm{Ch}([(E_+, E_-, a)] &= (\widetilde{\mathrm{Ch}}(a), \mathrm{Ch}(E_+) - \mathrm{Ch}(E_-)), \\ \widetilde{\mathrm{Ch}}(a) &= -\frac{1}{2\pi i}\int_0^1 \mathrm{tr}\left(a^{-1}(\nabla a)e^{w(t)}\right) dt \text{ where} \\ w(t) &= (1-t)F_+ + ta^{-1}F_-a + \frac{1}{2\pi i}t(1-t)(a^{-1}\nabla a)^2 \text{ and} \\ \mathrm{Ch}(E_\pm) &= \mathrm{tr}\exp(F_\pm/2\pi i), \quad F_\pm = \nabla^2_\pm. \end{aligned} \tag{10.2}$$

Here $\nabla_\pm$ are connections on $E_\pm$ over Y and ∇ is the induced connection on $\hom(E_+, E_-)$ lifted to $S^*(Y/X)$. Note that the underlying relative complex is the direct sum of the deRham complexes with differential

$$\mathscr{C}^\infty(S^*(Y/X); \Lambda^*) \oplus \mathscr{C}^\infty(Y; \Lambda^*), \quad D = \begin{pmatrix} d & \pi^* \\ 0 & -d \end{pmatrix}. \tag{10.3}$$

In our twisted case, as shown in §7 the line bundle J over $Y^{[2]}$ decomposes as $\tilde{J} \boxtimes \tilde{J}'$ when lifted to $\tilde{Y}^{[2]}$ which has an additional fiber factor of $\tilde{L}$. The discussion of the Chern character therefore carries over directly to this relative setting.

Proposition 11. — *For any element of* $\mathrm{K}^0_c(T^*(Y/X))$ *represented by (untwisted) data* (E_+, E_-, a) *the twisted Chern character of the image in* $\mathrm{K}_c(T^*(Y/X); \rho^*\phi^*\mathscr{A})$ *is represented by the pair of forms after lifting to* $\tilde{L}$ *and trivializing* J *as in* (7.17)

$$\mathrm{Ch}_{\rho^*\phi^*\mathscr{A}}([(E_+, E_-, a)]) = (\widetilde{\mathrm{Ch}}_{\mathscr{A}}(a), \mathrm{Ch}_{\mathscr{A}}(E_+) - \mathrm{Ch}_{\mathscr{A}}(E_-)) \tag{10.4}$$

in the subcomplex of the relative deRham complex fixed by (1.10), *and* $\rho\colon T^*(Y/X) \longrightarrow Y$ *is the projection.*

Of course the point of this discussion is that these forms do give the analogue of the index formula in (twisted) cohomology.

Theorem 2. — *For the twisted index map* (5.5) *the twisted Chern character is given by the push-forward of the differential form in* (10.4)

$$\begin{aligned} &\mathrm{Ch}_{\mathscr{A}} \circ \mathrm{ind} : \mathrm{K}^0_c(T^*(Y/X); \rho^*\phi^*\mathscr{A}) \simeq \mathrm{K}^0_c(T^*(Y/X)) \longrightarrow \mathrm{H}^{\mathrm{even}}(X, \delta), \\ &\mathrm{Ch}_{\mathscr{A}} \circ \mathrm{ind}(p) = (-1)^n \phi_*\rho_* \left\{\rho^*\mathrm{Todd}(T^*(Y/X) \otimes \mathbb{C}) \wedge \mathrm{Ch}_{\rho^*\phi^*\mathscr{A}}(p)\right\}, \end{aligned} \tag{10.5}$$

where $\mathrm{Todd}(T^*(Y/X) \otimes \mathbb{C})$ *denotes the Todd class of the complexified vertical cotangent bundle and* $p = [(E_+, E_-, a)]$ *as identified in Proposition 11.*

Proof. — By the Index Theorem in K-theory, Theorem 1 of the previous section, it suffices to compute the twisted Chern character of the topological index of the projective family of elliptic pseudodifferential operators. We begin with by recalling the basic properties of the twisted Chern character. As before, we assume that the primitive line bundle J defining the smooth Azumaya bundle $\mathscr{A}$ is endowed with a fixed connection respecting the primitive property. It gives a homomorphism,

$$\mathrm{Ch}_{\mathscr{A}} : \mathrm{K}^0(X, \mathscr{A}) \longrightarrow \mathrm{H}^{\mathrm{even}}(X, \delta), \tag{10.6}$$

satisfying the following properties.

1. The Chern character is *functorial* under smooth maps in the sense that if $f\colon W \longrightarrow X$ is a smooth map between compact manifolds, then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{K}^0(X, \mathscr{A}) & \xrightarrow{f^!} & \mathrm{K}^0(W, f^*\mathscr{A}) \\ \big\downarrow \mathrm{Ch}_{\mathscr{A}} & & \big\downarrow \mathrm{Ch}_{f^*\mathscr{A}} \\ \mathrm{H}^{\mathrm{even}}(X, \delta) & \xrightarrow{f^*} & \mathrm{H}^{\mathrm{even}}(W, f^*\delta). \end{array} \tag{10.7}$$

   Here the pullback primitive line bundle ${f^{[2]}}^*J$ defining the pullback smooth Azumaya bundle $f^*\mathscr{A}$ is endowed with the pullback of the fixed connection respecting the primitive property.

2. The Chern character respects the structure of $\mathrm{K}^0(X,\mathscr{A})$ as a module over $\mathrm{K}^0(X)$, in the sense that the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{K}^0(X)\times\mathrm{K}^0(X,\mathscr{A}) & \longrightarrow & \mathrm{K}^0(X,\mathscr{A}) \\
\Big\downarrow{\scriptstyle \mathrm{Ch}\times\mathrm{Ch}_{\mathscr{A}}} & & \Big\downarrow{\scriptstyle \mathrm{Ch}_{\mathscr{A}}} \\
\mathrm{H}^{\mathrm{even}}(X,\mathbb{Q})\times\mathrm{H}^{\mathrm{even}}(X,\delta) & \longrightarrow & \mathrm{H}^{\mathrm{even}}(X,\delta)
\end{array}
\tag{10.8}
$$

where the top horizontal arrow is the action of $\mathrm{K}^0(X)$ on $\mathrm{K}^0(X,\mathscr{A})$ given by tensor product and the bottom horizontal arrow is given by the cup product.

The theorem now follows rather routinely from the index theorem in K-theory, Theorem 1. The key step to getting the formula is the analog of the Riemann-Roch formula in the context of twisted K-theory, which we now give details.

Let $\pi : E \longrightarrow X$ be a spin$\mathbb{C}$ vector bundle over X, $i : X \longrightarrow E$ the zero section embedding, and $F\in\mathrm{K}^0(X,\mathscr{A})$. Then using the properties of the twisted Chern character as above, we compute,

$$
\begin{aligned}
\mathrm{Ch}_{\pi^*\mathscr{A}}(i_!F) &= \mathrm{Ch}_{\pi^*\mathscr{A}}(i_!1\otimes\pi^*F)\\
&= \mathrm{Ch}(i_!1)\wedge\mathrm{Ch}_{\pi^*\mathscr{A}}(\pi^*F).
\end{aligned}
$$

The standard Riemann-Roch formula asserts that

$$\mathrm{Ch}(i_!1) = i_*\mathrm{Todd}(E)^{-1}.$$

Therefore we deduce the following Riemann-Roch formula for twisted K-theory,

$$\mathrm{Ch}_{\pi^*\mathscr{A}}(i_!F) = i_*\left\{\mathrm{Todd}(E)^{-1}\wedge\mathrm{Ch}_{\mathscr{A}}(F)\right\}. \tag{10.9}$$

We need to compute $\mathrm{Ch}_{\mathscr{A}}(\mathrm{ind}_t\, p)$ where

$$p = [E_+,E_-,a]\in\mathrm{K}^0_{\mathrm{c}}(T(Y/X))\cong\mathrm{K}^0_{\mathrm{c}}(T(Y/X),\rho^*\phi^*\mathscr{A}).$$

We will henceforth identify $T(Y/X)\cong T^*(Y/X)$ via a Riemannian metric. Recall from §6 that the topological index, $\mathrm{ind}_t = j_!^{-1}\circ(Di)_!$ where $i : Y\hookrightarrow X\times\mathbb{R}^{2N}$ is an embedding that commutes with the projections $\phi : Y\longrightarrow X$ and $\pi_1 : X\times\mathbb{R}^{2N}\longrightarrow X$, and $j : X\hookrightarrow X\times\mathbb{R}^{2N}$ is the zero section embedding. Therefore

$$\mathrm{Ch}_{\mathscr{A}}(\mathrm{ind}_t\, p) = \mathrm{Ch}_{\mathscr{A}}(j_!^{-1}\circ(Di)_!p)$$

By the Riemann-Roch formula for twisted K-theory (10.9),

$$\mathrm{Ch}_{\pi_1^*\mathscr{A}}(j_!F) = j_*\,\mathrm{Ch}_{\mathscr{A}}(F)$$

since $\pi_1 : X\times\mathbb{R}^{2N}\longrightarrow X$ is a trivial bundle. Since $\pi_{1*}j_*1 = (-1)^n$, it follows that for $\xi\in\mathrm{K}^0_{\mathrm{c}}(X\times\mathbb{R}^{2N},\pi_1^*\mathscr{A})$, one has

$$\mathrm{Ch}_{\mathscr{A}}(j_!^{-1}\xi) = (-1)^n\pi_{1*}\,\mathrm{Ch}_{\pi_1^*\mathscr{A}}(\xi)$$

Therefore

$$\mathrm{Ch}_{\mathscr{A}}(j_!^{-1}\circ(Di)_!p) = (-1)^n\pi_{1*}\,\mathrm{Ch}_{\pi_1^*\mathscr{A}}((Di)_!p) \tag{10.10}$$

By the Riemann-Roch formula for twisted K-theory (10.9),

$$\mathrm{Ch}_{\pi_1^*\mathscr{B}}((Di)_!p) = (Di)_*\left\{\rho^*\mathrm{Todd}(\mathscr{N})^{-1} \wedge \mathrm{Ch}_{\rho^*\phi^*\mathscr{B}}(p)\right\} \tag{10.11}$$

where $\mathscr{N}$ is the complexified normal bundle to the embedding $Di : T(Y/X) \longrightarrow X \times T\mathbb{R}^{2N}$, that is, $\mathscr{N} = X \times T\mathbb{R}^{2N}/Di(T(Y/X)) \otimes \mathbb{C}$. Therefore $\mathrm{Todd}(\mathscr{N})^{-1} = \mathrm{Todd}(T(Y/X) \otimes \mathbb{C})$ and (10.11) becomes

$$\mathrm{Ch}_{\pi_1^*\mathscr{B}}((Di)_!p) = (Di)_*\left\{\rho^*\mathrm{Todd}(T(Y/X) \otimes \mathbb{C}) \wedge \mathrm{Ch}_{\rho^*\phi^*\mathscr{B}}(p)\right\}.$$

Therefore (10.10) becomes

(10.12)
$$\begin{aligned}\mathrm{Ch}_{\mathscr{B}}(j_!^{-1} \circ (Di)_!p) &= (-1)^n\pi_{1*}(Di)_*\left\{\rho^*\mathrm{Todd}(T(Y/X) \otimes \mathbb{C}) \wedge \mathrm{Ch}_{\rho^*\phi^*\mathscr{B}}(p)\right\}\\ &= (-1)^n\phi_*\rho_*\left\{\rho^*\mathrm{Todd}(T(Y/X) \otimes \mathbb{C}) \wedge \mathrm{Ch}_{\rho^*\phi^*\mathscr{B}}(p)\right\}\end{aligned}$$

since $\phi_*\rho_* = \pi_{1*}(Di)_*$. Therefore

$$\mathrm{Ch}_{\mathscr{B}}(\mathrm{ind}_t\, p) = (-1)^n\phi_*\rho_*\left\{\rho^*\mathrm{Todd}(T(Y/X) \otimes \mathbb{C}) \wedge \mathrm{Ch}_{\rho^*\phi^*\mathscr{B}}(p)\right\}, \tag{10.13}$$

proving Theorem 2. □

Appendix A
Differential characters

We will refine Lemma 1 to an equality of differential characters. For an account of differential characters, see [**17, 12**].

We first relate a connection $\tilde{\gamma}$ on $\tilde{L}$ to the 1-form γ on Y. Consider the commutative diagram

$$\begin{tikzcd} \phi^*(\tilde{L}) \arrow[r, "\tilde{\phi}"'] \arrow[d, "pr_1"'] & \tilde{L} \arrow[d, "\pi"] \\ Y \arrow[r, "\phi"'] & X \end{tikzcd} \tag{A.1}$$

where $\phi^*(\tilde{L}) = Y \times \mathbb{S}$ as observed earlier, and $pr_1 : Y \times \mathbb{S} \longrightarrow Y$ denotes projection to the first factor. Then the connection 1-form $\tilde{\gamma}$ on $\tilde{L}$ with curvature equal to β is related to the 1-form γ on Y by $\tilde{\phi}^*(\tilde{\gamma}) = \gamma + \theta$ where θ is the Cartan-Maurer 1-form on $\mathbb{S}$. If $\iota : Y \to Y \times \mathbb{S}$ denotes the inclusion map into the first factor, then $\iota^*\tilde{\phi}^*(\tilde{\gamma}) = \gamma$.

Now the circle bundle $\tilde{L}$ has a section $\tilde{\tau} : X \setminus M_1 \to \tilde{L}$, where M_1 is a codimension 2 submanifold of X. We define a section $\tau : X \setminus M_1 \to Y$ such that $\tilde{\phi} \circ \iota \circ \tau = \tilde{\tau}$. Then we have a well defined singular 1-form $\varphi_1 := \tilde{\tau}^*(\tilde{\gamma}) = \tau^*(\gamma)$ on X with the property that $d\varphi_1 = \beta$. The differential character associated to φ_1 is (cf. [**11**])

$$S(\varphi_1)(z) = \varphi_1(z') + \beta(c)$$

where $z, z' \in Z_1(X, \mathbb{Z})$ and $c \in C_2(X, \mathbb{Z})$ is such that $\partial c = z - z'$, where $z' \cap M_1 = \varnothing$.

The smooth map $u : X \to \mathbb{R}/\mathbb{Z}$ gives rise to a singular function φ_0 on X as follows. If $t \in \mathbb{R}/\mathbb{Z}$ is a regular value for u, then $M_0 := u^{-1}(t)$ is a codimension 1 submanifold of X, and the Cartan-Maurer 1-form θ on $\mathbb{R}/\mathbb{Z}$ is exact on $\mathbb{R}/\mathbb{Z} \setminus \{t\}$, say dg, where g is a smooth function on $\mathbb{R}/\mathbb{Z} \setminus \{t\}$. Then the pullback function $\varphi_0 = u^*(g)$ is a smooth function on $X \setminus M_0$, ie it is a singular function on X such that $d\varphi_0 = u^*(\theta) = \alpha$ is the associated global smooth 1-form on X with integer periods.

With μ as in Lemma 1, $\varphi_2 := \tau^*(\mu) = d\varphi_0 \wedge \varphi_1$ is a singular 2-form on X, whose associated differential character is

$$S(\varphi_2)(z) = d\varphi_0 \wedge \varphi_1(z') + \alpha \wedge \beta(c)$$

where $z, z' \in Z_2(X, \mathbb{Z})$ and $c \in C_3(X, \mathbb{Z})$ is such that $\partial c = z - z'$, where $z' \cap M_1 = \varnothing$. By the argument given above, it is also the differential character associated to the Azumaya bundle $\mathscr{A}$ with connection.

Lemma 10. — *In the notation above, $S(\varphi_2) = S(\varphi_0) \star S(\varphi_1)$, where $\star$ denotes the Cheeger-Simons product of differential characters.*

Proof. — First note that by **[12]**, the field strength of $S(\varphi_0) \star S(\varphi_1)$ is $\bar{\alpha} \wedge \bar{\beta}$, which by Lemma 1 is equal to $\bar{\delta}$ which is the field strength of $S(\varphi_2)$. That is,

$$S(\varphi_0) \star S(\varphi_1)(\partial c) = S(\varphi_2)(\partial c)$$

for every degree 3 integral cochain c.

By **[12]**, we see that the characteristic class of $S(\varphi_0) \star S(\varphi_1)$ is equal to the cup product $\alpha \cup \beta$, and by Appendix B, also equal to δ, which is the characteristic class of $S(\varphi_2)$. Note that the image in real cohomology of α and β is equal to $[\bar{\alpha}]$ and $[\bar{\beta}]$ respectively.

According to **[11]**, if $z \in Z_2(X, \mathbb{Z})$ is transverse to M_1, then

$$S(\varphi_0) \star S(\varphi_1)(z) = -d\varphi_0 \wedge \varphi_1(z) + \sum_{p \in z \cap M_1} \varphi_0(p)$$

In particular, if $z \cap M_1 = \varnothing$, then

$$S(\varphi_0) \star S(\varphi_1)(z) = S(\varphi_2)(z),$$

proving the lemma. □

Appendix B
Čech class of the Azumaya bundle

Suppose that there is a line bundle K over Y such that $J \cong K \boxtimes K'$. That is, in this case, $\mathscr{C}^\infty(X, \mathscr{A}) \cong \Psi^{-\infty}(Y/X, K)$ is the algebra of smoothing operators along the fibres of $\phi : Y \longrightarrow X$ acting on sections of K, and therefore has trivial Dixmier-Douady invariant. Here $\mathscr{A}_x = \Psi^{-\infty}(\phi^{-1}(x), K\big|_{\phi^{-1}(x)})$ for all $x \in X$.

Conversely, suppose that the Dixmier-Douady invariant of $\mathscr{A}$ is trivial, where $\mathscr{C}^\infty(X, \mathscr{A}) = \mathscr{C}^\infty(Y^{[2]}, J)$. Then there is a line bundle K over Y such that $J \cong K \boxtimes K'$. To see this, we use the connection ∇^J preserving the primitive property of

J, and Lemma 1 to see that $d\mu = \phi^* dB$, for some global 2-form $B \in \Omega^2(X)$. Then $d(\mu - \phi^*(B)) = 0$, and $\pi_1^*(\mu - \phi^*(B)) - \pi_2^*(\mu - \phi^*(B)) = F_{\nabla^J}$. So $\mu - \phi^*(B)$ is a closed 2-form on Y and can be chosen to have integral periods, since F_{∇^J} has integral periods (this is clear from the Čech description below). Therefore there is a line bundle K on Y with connection, whose curvature is equal to $\mu - \phi^*(B)$ such that $J \cong K \boxtimes K'$.

Suppose that J_1 and J_2 are two primitive line bundles over $Y^{[2]}$ and let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the corresponding Azumaya bundles, that is, $\mathcal{C}^\infty(X, \mathcal{A}_j) = \mathcal{C}^\infty(Y^{[2]}, J_j)$, $j = 1, 2$. Then we conclude by the argument above that $\mathcal{A}_1 \cong \mathcal{A}_2$ if and only if there is a line bundle K over Y such that $J_1 \cong J_2 \otimes (K \boxtimes K')$.

The main result that we want to show here is the following.

Lemma 11. — *Suppose $\phi : Y \longrightarrow X$, $L \longrightarrow X$ and $u : X \longrightarrow \mathrm{U}(1)$ are as in the introduction. Then the Dixmier-Douady class of the Azumaya bundle $\mathcal{A}$ constructed from this data as in §3, is equal to $\alpha \cup \beta$, where $\alpha \in \mathrm{H}^1(X, \mathbb{Z})$ is the cohomology defined by u and $\beta \in \mathrm{H}^2(X, \mathbb{Z})$ is the Chern class of L.*

Proof. — As noted above, the Dixmier-Douady invariant of $\mathcal{A}$ is the degree 3 cohomology class on X associated to the primitive line bundle J over $Y^{[2]}$.

As argued in §2, the line bundle L gives rise to a character $s : Y^{[2]} \to \mathrm{U}(1)$. Suppose that $\tau_i : U_i \to Y$ are local sections of Y. Then it is clear from §2 that $c_{ij} := s(\tau_i, \tau_j)$ defines a U(1)-valued Cech 1-cocycle representing the first Chern class of L.

Using the same local sections of Y, we see that $J_{ij} := (\tau_i \times \tau_j)^* J = L_j^{-n_{ij}}$, where $n_{jk} : U_j \cap U_k \to \mathbb{Z}$ denotes the transition functions of $\hat{X}$. If s_j is a local nowhere zero section of L_j, then $\sigma_{ij} := s_j^{-n_{ij}}$ is a local nowhere zero section of J_{ij}. We compute,

$$\sigma_{ij}\sigma_{jk} = s_j^{-n_{ij}} s_k^{-n_{jk}} \tag{B.1}$$

$$= c_{jk}^{-n_{ij}} s_k^{-n_{ij}} s_k^{-n_{jk}} \tag{B.2}$$

$$= c_{kj}^{-n_{ji}} s_k^{-n_{ik}} = c_{kj}^{-n_{ji}} \sigma_{ik}. \tag{B.3}$$

Therefore the U(1)-valued Cech 2-cocycle associated to J is $d_{ijk} := c_{kj}^{-n_{ji}}$ But it is well known (cf. equation (1-18), page 29, **[9]**) that the right hand side represents the cup product of the Cech cocycles $[c]$ and $[-n]$, that is, $[d] = [c] \cup [-n] = -\beta \cup \alpha = \alpha \cup \beta \in \mathrm{H}^3(X, \mathbb{Z})$, proving the lemma. □

Appendix C

The universal case

Let $\phi : Y \to X$ be a fibre bundle of compact manifolds, $L \to X$ a line bundle over X with the property that the pullback $\phi^*(\beta) = 0$ in $\mathrm{H}^2(Y, \mathbb{Z})$, where $\beta \in \mathrm{H}^2(X, \mathbb{Z})$ is the first Chern class of L.

Lemma 12. — *In the notation above, $\phi^*(\beta) = 0$ in $\mathrm{H}^2(Y, \mathbb{Z})$ if and only if there is a $\tilde{\beta} \in \mathrm{H}^2(B\mathrm{Diff}(Z), \mathbb{Z})$ such that $\beta = f^*(\tilde{\beta})$ in $\mathrm{H}^2(X, \mathbb{Z})$, where $f\colon\ X \to B\mathrm{Diff}(Z)$ is the classifying map for $\phi : Y \to X$, and Z is the typical fiber of $\phi : Y \to X$.*

This follows in a straightforward way from standard algebraic topology. The direction that we will mainly use is trivial to prove, viz. if there is a class $\tilde{\beta} \in \mathrm{H}^2(B\mathrm{Diff}(Z), \mathbb{Z})$ such that $\beta = f^*(\tilde{\beta})$ in $\mathrm{H}^2(X, \mathbb{Z})$, then $\phi^*(\beta) = 0$ in $\mathrm{H}^2(Y, \mathbb{Z})$.

Therefore we see that given any fibre bundle of compact manifolds $\phi : Y \to X$ with typical fiber Z, and $\beta \in f^*(\mathrm{H}^2(B\mathrm{Diff}(Z), \mathbb{Z})) \subset \mathrm{H}^2(X, \mathbb{Z})$ (that is, if β is a characteristic class of the fiber bundle $\phi : Y \to X$), then $\phi^*(\beta) = 0$ in $\mathrm{H}^2(Y, \mathbb{Z})$, satisfying the hypotheses of our main index theorem.

But what are line bundles on $B\mathrm{Diff}(Z)$? Since roughly speaking, $B\mathrm{Diff}(Z) = \mathrm{Metrics}(Z)/\mathrm{Diff}(Z)$, where $\mathrm{Metrics}(Z)$ denotes the contractible space of all Riemanian metrics on Z, the theory of anomalies in gravity constructs line bundles on $B\mathrm{Diff}(Z)$ via determinant line bundles of index bundles of families of twisted Dirac operators obtained by varying the Riemannian metric on Z, cf. [**2**].

In particular, let $\phi : Y \to X$ be a fibre bundle of compact manifolds, with typical fiber a compact Riemann surface Σ_g of genus $g \geq 2$. Then $T(Y/X)$ is an oriented rank 2 bundle over Y. Define $\beta = \phi_*(e \cup e) \in \mathrm{H}^2(X, \mathbb{Z})$, where $e := e(T(Y/X)) \in \mathrm{H}^2(Y, \mathbb{Z})$ is the Euler class of $T(Y/X)$. By naturality of this construction, $\beta = f^*(e_1)$, where $e_1 \in \mathrm{H}^2(B\mathrm{Diff}(\Sigma_g), \mathbb{Z})$ and $f\colon X \to B\mathrm{Diff}(\Sigma_g)$ is the classifying map for $\phi : Y \to X$. e_1 is known as the universal first Mumford-Morita-Miller class, and β is the first Mumford-Morita-Miller class of $\phi : Y \to X$, cf. Chapter 4 in [**22**]. Therefore by Lemma 12, we have the following.

Lemma 13. — *In the notation above, let $\phi : Y \to X$ be a fibre bundle of compact manifolds, with typical fiber a compact Riemann surface Σ_g of genus $g \geq 2$, and let $\beta \in \mathrm{H}^2(X, \mathbb{Z})$ be a multiple of the first Mumford-Morita-Miller class of $\phi : Y \to X$. Then $\phi^*(\beta) = 0$ in $\mathrm{H}^2(Y, \mathbb{Z})$.*

Such choices satisfy the hypotheses of our main index theorem. In fact, if $\phi : Y \longrightarrow X$ be as above, and in addition let X be a closed Riemann surface. Then Proposition 4.11 in [**22**] asserts that $\langle e_1, [X] \rangle = \mathrm{Sign}(Y)$, where $\mathrm{Sign}(Y)$ is the signature of the 4-dimensional manifold Y, which is originally a result of Atiyah. As a consequence, Morita is able to produce infinitely many surface bundles Y over X that have nontrivial first Mumford-Morita-Miller class.

On the other hand, given any $\beta \in \mathrm{H}^2(X, \mathbb{Z})$, we know that there is a fibre bundle of compact manifolds $\phi : Y \to X$ such that $\phi^*(\beta) = 0$ in $\mathrm{H}^2(Y, \mathbb{Z})$. In fact we can choose Y to be the total space of a principal $\mathrm{U}(n)$ bundle over X with first Chern class β. Here we can also replace $\mathrm{U}(n)$ by any compact Lie group G such that $\mathrm{H}^1(G, \mathbb{Z})$ is nontrivial and torsion-free, such as the torus $\mathbb{T}^n$.

Lemma 14. — *Let $\phi : Y \to X$ be a fibre bundle of compact manifolds with typical fiber Z and $\beta \in \mathrm{H}^2(X, \mathbb{Z})$. Let $\pi : P \to X$ be a principal $\mathrm{U}(n)$-bundle whose first Chern* *class is β. Then the fibred product $\phi \times \pi : Y \times_X P \to X$ is a fiber bundle with typical fiber $Z \times \mathrm{U}(n)$, and has the property that $(\phi \times \pi)^*(\beta) = 0$ in $\mathrm{H}^2(Y \times_X P, \mathbb{Z})$.*

This follows from the obvious commutativity of the following diagram,

$$\begin{array}{ccc} Y \times_X P & \xrightarrow{pr_1} & Y \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle \phi} \\ P & \xrightarrow{\pi} & X. \end{array} \tag{C.1}$$

Hence this data also satisfy the hypotheses of our main index theorem.

The construction of the universal fibre bundle of Riemann surfaces which we will describe next, is well known, cf. [**6, 14, 2**]. Let Σ be a compact Riemann surface of genus g greater than 1, $\mathfrak{M}_{(-1)}$ the space of all hyperbolic metrics on Σ of curvature equal to -1, and $\mathrm{Diff}_+(\Sigma)$ the group of all orientation preserving diffeomorphisms of Σ. Then the quotient

$$\mathfrak{M}_{(-1)}/\mathrm{Diff}_+(\Sigma) = \mathcal{M}_g$$

is a noncompact orbifold, namely the moduli space of Riemann surfaces of genus equal to g. The fact that $\mathcal{M}_g$ has singularities can be dealt with in several ways, for instance by going to a finite smooth cover, and the noncompactness of $\mathcal{M}_g$ can be dealt with for instance by considering compact submanifolds. We will however not deal with these delicate issues in the discussion below. The group $\mathrm{Diff}_+(\Sigma)$ also acts on $\Sigma \times \mathfrak{M}_{(-1)}$ via $g(z, h) = (g(z), g^*h)$ and the resulting smooth fibre bundle,

$$\pi : Y = (\Sigma \times \mathfrak{M}_{(-1)})/\mathrm{Diff}_+(\Sigma) \longrightarrow \mathfrak{M}_{(-1)}/\mathrm{Diff}_+(\Sigma) = \mathcal{M}_g \tag{C.2}$$

is the *universal bundle* of genus g Riemann surfaces. The classifying map for (C.2) is the identity map on $\mathcal{M}_g$ so π is maximally nontrivial in a sense made precise below.

As defined above, let

$$e_1 = e_1(Y/\mathcal{M}_g) = \pi_*(e \cup e) \in \mathrm{H}^2(\mathcal{M}_g; \mathbb{Z})$$

be the first Mumford-Morita-Miller class of $\pi\colon Y \to \mathcal{M}_g$.

A theorem of Harer [**22, 16**] asserts that:

$$\mathrm{H}^2(\mathcal{M}_g; \mathbb{Q}) = \mathbb{Q}(e_1);$$
$$\mathrm{H}^1(\mathcal{M}_g; \mathbb{Q}) = \{0\}.$$

Our next goal is to define a line bundle $\mathcal{L}$ over $\mathcal{M}_g$ such that $c_1(\mathcal{L}) = ke_1$ for some $k \in \mathbb{Z}$. This line bundle then automatically has the property that $\pi^*(\mathcal{L})$ is trivializable since e_1 is a characteristic class of the fibre bundle $\pi : Y \longrightarrow \mathcal{M}_g$. This is exactly the data that is needed to define a projective family of Dirac operators. The line bundle $\mathcal{L}$ turns out to be a power of the determinant line bundle of the virtual vector bundle Λ known as the Hodge bundle, which is defined using the Gysin map in K-theory.

$$\Lambda = \pi_!(T(Y/\mathcal{M}_g)) \in \mathrm{K}^0(\mathcal{M}_g).$$

Then $\det(\Lambda)$ is actually a line bundle over $\mathcal{M}_g$. Next we need the following special Grothendieck-Riemann-Roch (GRR) calculation.

Lemma 15. — *In the notation above, one has the following identity of first Chern classes,*

$$c_1(\pi_!(T(Y/\mathcal{M}_g)) = \frac{13}{12}\pi_*(c_1(T(Y/\mathcal{M}_g))^2).$$

Proof. — By the usual GRR calculation [**3**], we have

$$\mathrm{ch}(\pi_!(T(Y/\mathcal{M}_g)) = \pi_* \left(\mathrm{Todd}(T(Y/\mathcal{M}_g) \cup \mathrm{Ch}(T(Y/\mathcal{M}_g))\right).$$

Now

$$\mathrm{Todd}(x) = 1 + \frac{x}{2} + \frac{x^2}{12} + \dots$$

and

$$\mathrm{Ch}(x) = 1 + x + \frac{x^2}{2} + \dots$$

where $x = c_1(T(Y/\mathcal{M}_g))$. Therefore the degree 4 component is

$$[\mathrm{Todd}(x)\mathrm{Ch}(x)]_{(4)} = \frac{13}{12}x^2.$$

That is, the degree 2 component of the GRR formula in our case is

$$c_1(\pi_!(T(Y/\mathcal{M}_g)) = \frac{13}{12}\pi_*(x^2). \qquad \square$$

Observing that $c_1(T(Y/\mathcal{M}_g) = e$ and

$$c_1(\pi_!(T(Y/\mathcal{M}_g)) = c_1(\Lambda) = c_1(\det(\Lambda)),$$

the lemma above shows that $c_1(\det(\Lambda)) = \frac{13}{12}e_1$. Setting $\mathcal{L} = \det(\Lambda)^{\otimes 12}$, we obtain

Corollary 2. — *In the notation above, $\mathcal{L}$ is a line bundle over $\mathcal{M}_g$ and one has the following identity:*

$$c_1(\mathcal{L}) = 13e_1.$$

We next construct a canonical projective family of Dirac operators on the Riemann surface Σ. We enlarge the parametrizing space by taking the product with the circle $\mathbb{T}$. Applying the main construction in the paper, we get a primitive line bundle $J \longrightarrow Y^{[2]}$, where we denote the pullback of Y over $\mathbb{T} \times \mathcal{M}_g$ by the same symbol. By the construction at the end of §5, we obtain a projective family of Dirac operators $\eth_J$ on the Riemann surface Σ, parametrized by $\mathbb{T} \times \mathcal{M}_g$, having analytic index,

$$\mathrm{Index}_a(\eth_J) \in \mathrm{K}^0(\mathbb{T} \times \mathcal{M}_g; a \cup e_1),$$

where $a \in H^1(\mathbb{T}; \mathbb{Z})$ is the generator.

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Astérisque
328, 2009, p. 297–338

INDEX OF TRANSVERSALLY ELLIPTIC OPERATORS

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60^{th} birthday

Abstract. — In the 70's, the notion of analytic index has been extended by Atiyah and Singer to the class of transversally elliptic operators. They did not, however, give a general cohomological formula for the index. This was accomplished many years later by Berline and Vergne. The Berline-Vergne formula is an integral of a non-compactly supported equivariant form on the cotangent bundle, and depends on rather subtle growth conditions for these forms.

This paper gives an alternative expression for the index, where the non-compactly supported form is replaced with a compactly supported one, but with generalized coefficients.

***Résumé* (Indice d'opérateurs transversalement elliptiques).** — Dans les années 70, Atiyah et Singer ont étendu la notion d'indice analytique au cadre des opérateurs transversalement elliptiques. Néanmoins, ils ne donnaient pas de formule cohomologique générale pour cet indice. Ce problème a été résolu bien des années après par Berline et Vergne. La formule de Berline-Vergne exprime l'indice comme l'intégrale sur un fibré cotangent d'une forme équivariante à support non-compact: ici une propriété de croissance très particulière de cette forme est requise pour assurer l'existence de l'intégrale.

Le but de ce travail est de donner une autre formulation de cet indice, où la forme équivariante à support non-compact est remplacée par une forme équivariante à support compact, mais avec des coefficients généralisés.

1. Introduction

Let M be a compact manifold. The Atiyah-Singer formula for the index of an *elliptic* pseudo-differential operator P on M with *elliptic* symbol σ on $\mathbf{T}^*M$ involves

2010 Mathematics Subject Classification. — 19L47, 19L10, 55N25.
Key words and phrases. — index, transversally elliptic, equivariant cohomology, generalized coefficients.

integration over the non compact manifold $\mathbf{T}^*M$ of the Chern character $\mathrm{Ch}_c(\sigma)$ of σ multiplied by the square of the $\widehat{\mathrm{A}}$-genus of M:

$$\mathrm{index}(P) = (2i\pi)^{-\dim M} \int_{\mathbf{T}^*M} \widehat{\mathrm{A}}(M)^2 \, \mathrm{Ch}_c(\sigma).$$

Here σ, the principal symbol of P, is a morphism of vector bundles on $\mathbf{T}^*M$ invertible outside the zero section of $\mathbf{T}^*M$ and the Chern character $\mathrm{Ch}_c(\sigma)$ is supported on a compact neighborhood of M embedded in $\mathbf{T}^*M$ as the zero section. It is important that the representative of the Chern character $\mathrm{Ch}_c(\sigma)$ is compactly supported to perform integration.

Assume that a compact Lie group K (with Lie algebra $\mathfrak{k}$) acts on M. If the elliptic operator P is K-invariant, then $\mathrm{index}(P)$ is a smooth function on K. The equivariant index of P can be expressed similarly as the integral of the equivariant Chern character of σ multiplied by the square of the equivariant $\widehat{\mathrm{A}}$-genus of M: for $X \in \mathfrak{k}$ small enough,

$$\mathrm{index}(P)(\mathrm{e}^X) = (2i\pi)^{-\dim M} \int_{\mathbf{T}^*M} \widehat{\mathrm{A}}(M)^2(X) \, \mathrm{Ch}_c(\sigma)(X). \tag{1}$$

Here $\mathrm{Ch}_c(\sigma)(X)$ is a compactly supported closed equivariant differential form, that is a differential form on $\mathbf{T}^*M$ depending smoothly of $X \in \mathfrak{k}$, and closed for the equivariant differential D. The result of the integration determines a smooth function on a neighborhood of e in K and similar formulae can be given near any point of K. Formula (1) is a "delocalization" of the Atiyah-Bott-Segal-Singer formula, in the sense of Bismut [**9**].

The delocalized index formula (1) can be adapted to new cases such as:

- Index of transversally elliptic operators.
- L^2-index of some elliptic operators on some non-compact manifolds (Rossmann formula for discrete series [**20**]).

Indeed, in these two contexts, the index exists in the sense of generalized functions but cannot be always computed in terms of fixed point formulae. A "delocalized" formula will however continue to have a meaning, as we explain now for transversally elliptic operators.

The invariant operator P with symbol $\sigma(x, \xi)$ on $\mathbf{T}^*M$ is called transversally elliptic, if it is elliptic in the directions transverse to K-orbits. In this case, the operator P has again an index which is a generalized function on K [**1**]. A very simple example of transversally elliptic operator is the operator 0 on $L^2(K)$: its index is the trace of the action of K in $L^2(K)$, that is the δ-function on K. At the opposite side, K-invariant elliptic operators are of course transversally elliptic, and index of such operators are smooth functions on K given by Formula (1). Thus a cohomological formula must incorporate these two extreme cases. Such a cohomological formula was given in Berline-Vergne [**7, 8**]. We present here a new point of view, where the equivariant Chern character $\mathrm{Ch}_c(\sigma)(X)$ entering in Formula (1) is replaced by a Chern character with generalized coefficients, but still *compactly supported.* Let us briefly explain the construction.

Let $\mathbf{T}^*_K M \subset \mathbf{T}^*M$ be the cone formed by the covectors $\xi \in \mathbf{T}^*_x M$ which vanish on tangent vectors to the orbit $K \cdot x$. Let $\mathrm{supp}(\sigma)$ be the support of the symbol σ of a transversally elliptic operator P. By definition, the intersection $\mathrm{supp}(\sigma) \cap \mathbf{T}^*_K(M)$ is compact. By the Quillen super-connection construction, the Chern character $\mathrm{Ch}(\sigma)(X)$ is a closed equivariant differential form supported near the closed set $\mathrm{supp}(\sigma)$. Using the Liouville 1-form ω of $\mathbf{T}^*M$, we construct a closed equivariant form $\mathrm{One}(\omega)$ supported near $\mathbf{T}^*_K M$. Outside $\mathbf{T}^*_K M$, one has indeed the equation $1 = D(\frac{\omega}{D\omega})$, where the inverse of the form $D\omega$ is defined by $-i\int_0^\infty e^{itD\omega}dt$, an integral which is well defined in the generalized sense, that is tested against a smooth compactly supported density on $\mathfrak{k}$. Thus using a function χ equal to 1 on a small neighborhood of $\mathbf{T}^*_K M$, the closed equivariant form

$$\mathrm{One}(\omega)(X) = \chi + d\chi \frac{\omega}{D\omega(X)}, \quad X \in \mathfrak{k},$$

is well defined, supported near $\mathbf{T}^*_K M$, and represents 1 in cohomology. Remark that

$$\mathrm{Ch}_c(\sigma, \omega) := \mathrm{Ch}(\sigma)(X)\mathrm{One}(\omega)(X)$$

is *compactly supported.* We prove that, for $X \in \mathfrak{k}$ small enough, we have

$$\text{(2)} \qquad \mathrm{index}(P)(\mathrm{e}^X) = (2i\pi)^{-\dim M} \int_{\mathbf{T}^*M} \widehat{\mathrm{A}}(M)^2(X)\,\mathrm{Ch}(\sigma)(X)\,\mathrm{One}(\omega)(X).$$

This formula is thus similar to the delocalized version of the Atiyah-Bott-Segal-Singer equivariant index theorem. We have just localized the formula for the index near $\mathbf{T}^*_K M$ with the help of the form $\mathrm{One}(\omega)$, equal to 1 in cohomology, but supported near $\mathbf{T}^*_K M$.

When P is elliptic we can furthermore localize on the zeros of VX (the vector field on M produced by the action of X) and we obtain the Atiyah-Bott-Segal-Singer fixed point formulae for the equivariant index of P. However the main difference is that usually we cannot obtain a fixed point formula for the index. For example, the index of a transversally elliptic operator P where K acts freely is a generalized function on K supported at the origin. Thus in this case the use of the form $\mathrm{One}(\omega)$ is essential. Its role is clearly explained in the example of the 0 operator on S^1 given at the end of this introduction.

We need also to define the formula for the index at any point $s \in K$, in terms of integrals over $\mathbf{T}^*M(s)$, where $M(s)$ is the fixed point submanifold of M under the action of s. The compatibility properties (descent method) between the formulae at different points s are easy to prove, thanks to a localization formula adapted to this generalized setting.

In the Berline-Vergne cohomological formula for the index of P, the Chern character $\mathrm{Ch}_c(\sigma)(X)$ in Formula (1) was replaced by a Chern character $\mathrm{Ch}_{BV}(\sigma,\omega)(X)$ depending also of the Liouville 1-form ω. This Chern character $\mathrm{Ch}_{BV}(\sigma,\omega)$ is constructed for "good symbols" σ. It looks like a Gaussian in the transverse directions, and is oscillatory in the directions of the orbits. Our new point of view defines the *compactly supported* product class $\mathrm{Ch}(\sigma)\mathrm{One}(\omega)$ in a straightforward way. We proved in [**17**] that the classes $\mathrm{Ch}_{BV}(\sigma,\omega)$ and $\mathrm{Ch}(\sigma)\mathrm{One}(\omega)$ are equivalent in an appropriate cohomology space, so that our new cohomological formula gives the analytic index. Nevertheless, we will see in this paper that it is technically simpler to work with the compactly supported equivariant form $\mathrm{Ch}(\sigma)\mathrm{One}(\omega)$ rather than with the equivariant form $\mathrm{Ch}_{BV}(\sigma,\omega)$ which has subtle growth on the cotangent bundle. So we choose to prove directly the equality between the analytic index and the cohomological index, and we show that our formula in terms of the product class $\mathrm{Ch}(\sigma)\mathrm{One}(\omega)$ is natural. We follow the same line as Atiyah-Singer: functoriality with respect to products and free actions. The compatibility with the free action reduces basically to the case of the zero operator on K, and the calculation is straightforward. The typical calculation is shown below. The multiplicativity property is more delicate, but is based on a general principle on multiplicativity of relative Chern characters that we proved in a preceding article [**17**]. Thus, following Atiyah-Singer [**1**], we are reduced to the case of S^1 acting on a vector space. The basic examples are then the pushed symbol with index $-\sum_{n=1}^{\infty} e^{in\theta}$ and the index of the tangential $\overline{\partial}$ operators on odd dimensional spheres. We include at the end a general formula due to Fitzpatrick [**11**] for contact manifolds.

Let us finally point out that there are many examples of transversally elliptic operators of great interest. The index of elliptic operators on orbifolds are best understood as indices of transversally elliptic operators on manifolds where a group K acts with finite stabilizers. The restriction to the maximal compact subgroup K of a representation of the discrete series of a real reductive group are indices of transversally elliptic operators [**16**]. More generally, there is a canonical transversally elliptic operator on any prequantized Hamiltonian manifold with proper moment map (under some mild assumptions) [**16**], [**22**]. Furthermore, as already noticed in Atiyah-Singer, and systematically used in [**15**], transversally elliptic operators associated to symplectic vector spaces with proper moment maps and to cotangent manifolds $\mathbf{T}^*K$ are the local building pieces of any K-invariant elliptic operator.

Example 1.1. — *Let us check the validity of* (2) *in the example of the zero operator* 0_{S^1} *from* $S^1\times\mathbb{C}$ *to* $S^1\times\{0\}$. *This operator is* S^1*-transversally elliptic and its index* *is equal to*

$$\delta_1(\mathrm{e}^{iX}) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{ikX}, \quad X \in \mathrm{Lie}(S^1) \simeq \mathbb{R}.$$

The principal symbol σ *of* 0_{S^1} *is the zero morphism* $\mathbf{T}^*S^1 \times \mathbb{C} \to \mathbf{T}^*S^1 \times \{0\}$. *Hence* $\mathrm{Ch}(\sigma)(X) = 1$. *The equivariant class* $\widehat{\mathrm{A}}(S^1)^2(X)$ *is also equal to* 1. *Thus the right hand side of* (2) *becomes*

$$(2i\pi)^{-1} \int_{\mathbf{T}^*S^1} \mathrm{One}(\omega)(X).$$

The cotangent bundle $\mathbf{T}^*S^1$ *is parametrized by* $(\mathrm{e}^{i\theta}, \xi) \in S^1 \times \mathbb{R}$. *The Liouville* 1*-form is* $\omega = -\xi d\theta$ *: the symplectic form* $d\omega = d\theta \wedge d\xi$ *gives the orientation of* $\mathbf{T}^*S^1$. *Since* $VX = -X\frac{\partial}{\partial\theta}$, *we have* $D\omega(X) = d\theta \wedge d\xi - X\xi$.

Let $g \in \mathscr{C}^\infty(\mathbb{R})$ *with compact support and equal to* 1 *in a neighborhood of* 0. *Then* $\chi = g(\xi^2)$ *is a function on* $\mathbf{T}^*S^1$ *which is supported in a neighborhood of* $\mathbf{T}^*_{S^1}S^1 = \text{zero section}$. *We look now at the equivariant form* $\mathrm{One}(\omega)(X) = \chi + d\chi \wedge (-i\omega)\int_0^\infty \mathrm{e}^{itD\omega(X)}\, dt$. *We have*

$$\begin{aligned} \mathrm{One}(\omega)(\mathrm{e}^{i\theta}, \xi, X) &= g(\xi^2) + g'(\xi^2)2\xi d\xi \wedge (i\xi d\theta) \int_0^\infty \mathrm{e}^{it(d\theta\wedge d\xi - X\xi)}\, dt \\ &= g(\xi^2) - id\theta \wedge d(g(\xi^2)) \left(\int_0^\infty \mathrm{e}^{-itX\xi}\, \xi dt\right). \end{aligned}$$

If we make the change of variable $t\xi \to t$ *in the integral* $\int_0^\infty \mathrm{e}^{-itX\xi}\,\xi dt$ *we get*

$$\mathrm{One}(\omega)(\mathrm{e}^{i\theta}, \xi, X) = \begin{cases} g(\xi^2) - id\theta \wedge d(g(\xi^2)) \left(\int_0^\infty \mathrm{e}^{-itX}\, dt\right), & \text{if} \quad \xi \geq 0; \\ g(\xi^2) + id\theta \wedge d(g(\xi^2)) \left(\int_{-\infty}^0 \mathrm{e}^{-itX}\, dt\right), & \text{if} \quad \xi \leq 0. \end{cases}$$

Finally, since $-\int_{\xi\geq 0} d(g(\xi^2)) = \int_{\xi\leq 0} d(g(\xi^2)) = 1$, *we have*

$$(2i\pi)^{-1} \int_{\mathbf{T}^*S^1} \mathrm{One}(\omega)(X) = \int_{-\infty}^{\infty} \mathrm{e}^{-itX}\, dt.$$

The generalized function $\delta_0(X) = \int_{-\infty}^\infty \mathrm{e}^{-itX}\, dt$ *satisfies*

$$\int_{\mathrm{Lie}(S^1)} \delta_0(X)\varphi(X) dX = \mathrm{vol}(S^1, dX)\varphi(0)$$

for any function $\varphi \in \mathscr{C}^\infty(\mathrm{Lie}(S^1))$ *with compact support. Here* $\mathrm{vol}(S^1, dX) = \int_0^{2\pi} dX$ *is also the volume of* S^1 *with the Haar measure compatible with* dX.

Finally, we see that (2) *corresponds to the following equality of generalized functions*

$$\delta_1(\mathrm{e}^{iX}) = \delta_0(X),$$

which holds for $X \in \mathrm{Lie}(S^1)$ *small enough.*

2. The analytic index

2.1. Generalized functions. — Let K be a compact Lie group. We denote by $\hat{K}$ the set of equivalence classes of finite dimensional irreducible (complex) representations of K. If $\tau \in \hat{K}$, we denote by V_τ the representation space of τ and by $k \mapsto \mathrm{Tr}(k,\tau)$ its character. Let τ^* be the dual representation of K in V_τ^*.

We denote by $C^{-\infty}(K)$ the space of generalized functions on K and by $C^{-\infty}(K)^K$ the subspace of generalized functions invariant by conjugaison. The space $C^\infty(K)$ of smooth functions on K is naturally a subspace of $C^{-\infty}(K)$. We will often use the notation $\Theta(k)$ to denote a generalized function Θ on K, although (in general) the value of Θ on a particular point k of K does not have a meaning. By definition, Θ is a linear form on the space of smooth densities on K. If dk is a Haar measure on K and $\Phi \in C^\infty(K)$, we denote by $\int_K \Theta(k)\Phi(k)dk$ the value of Θ on the density Φdk.

Any invariant generalized function on K is expressed as $\Theta(k) = \sum_{\tau\in\hat{K}} n_\tau \,\mathrm{Tr}(k,\tau)$ where the Fourier coefficients n_τ have at most a polynomial growth [**21**].

2.2. Symbols and pseudo-differential operators. — Let M be a compact manifold with a smooth action of a compact Lie group K. We consider the closed subset $\mathbf{T}^*_K M$ of the cotangent bundle $\mathbf{T}^*M$, union of the spaces $(\mathbf{T}^*_K M)_x, x \in M$, where $(\mathbf{T}^*_K M)_x \subset \mathbf{T}^*_x M$ is the orthogonal of the tangent space at x to the orbit $K\cdot x$. Let $\mathcal{E}^\pm$ be two K-equivariant complex vector bundles over M. We denote by $\Gamma(M,\mathcal{E}^\pm)$ the space of smooth sections of $\mathcal{E}^\pm$. Let $P : \Gamma(M,\mathcal{E}^+) \to \Gamma(M,\mathcal{E}^-)$ be a K-invariant pseudo-differential operator of order m. Let $p : \mathbf{T}^*M \to M$ be the natural projection. The principal symbol $\sigma(P)$ of P is a bundle map $p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ which is homogeneous of degree m, defined over $\mathbf{T}^*M \setminus M$.

The operator P is elliptic if its principal symbol $\sigma(P)(x,\xi)$ is invertible for all $(x,\xi) \in \mathbf{T}^*M$ such that $\xi \neq 0$. The operator P is said to be K-transversally elliptic if its principal symbol $\sigma(P)(x,\xi)$ is invertible for all $(x,\xi) \in \mathbf{T}^*_K M$ such that $\xi \neq 0$.

Using a K- invariant function χ on $\mathbf{T}^*M$ identically equal to 1 in a neighborhood of M and compactly supported, then $\sigma_P(x,\xi) := (1-\chi(x,\xi))\sigma(P)(x,\xi)$ is a morphism from $p^*\mathcal{E}^+$ to $p^*\mathcal{E}^-$ defined on the whole space $\mathbf{T}^*M$ and which is *almost homogeneous*: $\sigma_P(x,t\xi) = t^m\sigma_P(x,\xi)$ for $t>1$ and ξ large enough. We consider the support of the morphism σ_P,

$$\mathrm{supp}(\sigma_P) := \{(x,\xi) \in \mathbf{T}^*M \mid \sigma_P(x,\xi) \text{ is not invertible}\}$$

which is a closed K-invariant subset of $\mathbf{T}^*M$.

When P is elliptic, then $\mathrm{supp}(\sigma_P)$ is compact, and the morphism σ_P gives rise to a $\mathbf{K}^0$-theory class $[\sigma_P] \in \mathbf{K}^0_K(\mathbf{T}^*M)$ which does not depend on the choice of χ. Similarly, when P is K-transversally elliptic, then $\mathrm{supp}(\sigma_P) \cap \mathbf{T}^*_K M$ is compact and the morphism σ_P gives rise to a $\mathbf{K}^0$-theory class $[\sigma_P|_{\mathbf{T}^*_K M}] \in \mathbf{K}^0_K(\mathbf{T}^*_K M)$ which does not depend on the choice of χ.

Recall the definition of the K-equivariant index of a pseudo-differential operator P which is K-transversally elliptic. Let us choose a K-invariant metric on M and K-invariant Hermitian structures on $\mathscr{E}^{\pm}$. Then the adjoint P^* of P is also a K-transversally elliptic pseudo-differential operator.

If P is elliptic, its kernel $\ker P := \{s \in \Gamma(M, \mathscr{E}^+) |\ Ps = 0\}$ is finite dimensional, and the K-equivariant index of P is the invariant function $\operatorname{index}^K(P)(k) = \operatorname{Tr}(k, \ker P) - \operatorname{Tr}(k, \ker P^*)$.

If P is K-transversally elliptic, its kernel $\ker P$ is not finite dimensional, but it has finite multiplicities: for any irreducible representation $\tau \in \hat{K}$, the multiplicity $m_\tau(P) := \dim(\hom_K(V_\tau, \ker P))$ is finite, and $\tau \mapsto m_\tau(P)$ has at most a polynomial growth [**1**]. We define then an invariant generalized function on K by setting

$$\operatorname{Tr}(k, \ker P) := \sum_{\tau \in \hat{K}} m_\tau(P) \operatorname{Tr}(k, \tau).$$

Definition 2.1. — *The K-equivariant index of a K-transversally elliptic pseudo-differential operator P is the generalized function*

$$\operatorname{index}^K(P)(k) = \operatorname{Tr}(k, \ker P) - \operatorname{Tr}(k, \ker P^*).$$

We recall

Theorem 2.2 **(Atiyah-Singer)**. — *The K-equivariant index of a K-invariant elliptic pseudo-differential operator P depends only of $[\sigma_P] \in \mathbf{K}^0_K(\mathbf{T}^* M)$.*

— *The K-equivariant index of a K-transversally elliptic pseudo-differential operator P depends only of $[\sigma_P|_{\mathbf{T}^*_K M}] \in \mathbf{K}^0_K(\mathbf{T}^*_K M)$.*

— *Each element in $\mathbf{K}^0_K(\mathbf{T}^*_K M)$ is represented by the class $[\sigma_P|_{\mathbf{T}^*_K M}]$ of a K-transversally elliptic pseudo differential operator P of order m. Similarly, each element in $\mathbf{K}^0_K(\mathbf{T}^* M)$ is represented by the class $[\sigma_P]$ of a K-invariant elliptic pseudo differential operator P of order m.*

Thus we can define

$$\operatorname{index}^{K,M}_a : \mathbf{K}^0_K(\mathbf{T}^*_K M) \to C^{-\infty}(K)^K \tag{3}$$

by setting, for P a K-transversally elliptic pseudo-differential operator of order m, $\operatorname{index}^{K,M}_a([\sigma_P|_{\mathbf{T}^*_K M}]) = \operatorname{index}^K(P)$. Similarly, we can define in the elliptic setting

$$\operatorname{index}^{K,M}_a : \mathbf{K}^0_K(\mathbf{T}^* M) \to \mathscr{C}^{\infty}(K)^K. \tag{4}$$

Note that we have a natural restriction map $\mathbf{K}^0_K(\mathbf{T}^*M) \to \mathbf{K}^0_K(\mathbf{T}^*_K M)$ which makes the following diagram

$$
\begin{tikzcd}
\mathbf{K}^0_K(\mathbf{T}^*M) \arrow[r] \arrow[d, "\mathrm{index}_a^{K,M}"'] & \mathbf{K}^0_K(\mathbf{T}^*_K M) \arrow[d, "\mathrm{index}_a^{K,M}"] \\
\mathscr{C}^{\infty}(K)^K \arrow[r] & \mathscr{C}^{-\infty}(K)^K
\end{tikzcd} \tag{5}
$$

commutative.

Let $R(K)$ be the representation ring of K. Using the trace, we will consider $R(K)$ as a sub-ring of $\mathscr{C}^{\infty}(K)^K$. The map (3) and (4) are homomorphisms of $R(K)$-modules and will be called the analytic indices.

Remark 2.3. — *In order to simplify the notations we will make no distinction between an element $V \in R(K)$, and its trace function $k \to \mathrm{Tr}(k, V)$ which belongs to $\mathscr{C}^{\infty}(K)^K$. For example, the constant function* 1 *on K is identified with the trivial representation of K.*

Let H be a compact Lie group acting on M and commuting with the action of K. Then the space $\mathbf{T}^*_K M$ is provided with an action of $K \times H$. If $[\sigma] \in \mathbf{K}^0_{K\times H}(\mathbf{T}^*_K M)$, we can associate to $[\sigma]$ a virtual trace class representation of $K \times H$. Indeed, we can choose as representative of $[\sigma]$ the symbol of a H-invariant and K-transversally elliptic operator P. Then $\ker P - \ker P^*$ is a trace class representation of $K \times H$. Thus we can define a $R(K \times H)$-homomorphism:

$$\mathrm{index}_a^{K,H,M} : \mathbf{K}^0_{K\times H}(\mathbf{T}^*_K M) \to C^{-\infty}(K \times H)^{K\times H}.$$

Obviously $\mathbf{T}^*_{K\times H}(M)$ is contained in $\mathbf{T}^*_K(M)$ so we have a restriction morphism $r : \mathbf{K}^0_{K\times H}(\mathbf{T}^*_K M) \to \mathbf{K}^0_{K\times H}(\mathbf{T}^*_{K\times H} M)$. We see that

$$\mathrm{index}_a^{K,H,M} = \mathrm{index}_a^{K\times H,M} \circ r.$$

However, it is easy to see that for a H-invariant and K-transversally elliptic symbol $[\sigma]$, $\mathrm{index}_a^{K,H,M}([\sigma])(k, h)$ is a generalized function on $K \times H$ which is smooth relative to the variable $h \in H$ (see [**1**], remark p. 17). In particular we can restrict $\mathrm{index}_a^{K,H,M}([\sigma])$ to $K \times H'$, for a subgroup H' of H. We can also multiply $\mathrm{index}_a^{K,H,M}([\sigma])(k, h)$ by generalized functions $\Psi(h)$ on H.

2.3. Functoriality properties of the analytic index. — We have defined for any compact $K \times H$-manifold M a $R(K \times H)$-morphism

$$\mathrm{index}_a^{K,H,M} : \mathbf{K}^0_{K\times H}(\mathbf{T}^*_K M) \to \mathscr{C}^{\infty}(H, C^{-\infty}(K))^{K\times H},$$

where $\mathscr{C}^{\infty}(H, C^{-\infty}(K))^{K\times H}$ denote the subspace of invariant generalized functions $\Theta(k, h)$ on $K \times H$ which are smooth relative to the variable $h \in H$.

Let us recall some basic properties of the analytic index map:

– **[N1]** If $M = \{\text{point}\}$, then $\mathrm{index}_a^{K,M}$ is the trace map $R(K) \hookrightarrow \mathscr{C}^\infty(K)^K$.
– **[Diff]** Compatibility with diffeomorphisms: if $f : M_1 \to M_2$ is a $K \times H$-diffeomorphism then $\mathrm{index}_a^{K,H,M_1} \circ f^*$ is equal to $\mathrm{index}_a^{K,H,M_2}$.
– **[Morph]** If $\phi : H' \to H$ is a Lie group morphism, we have $\phi^* \circ \ \mathrm{index}_a^{K,H,M} = \mathrm{index}_a^{K,H',M}$.

2.3.1. *Excision.* — Let U be a non-compact K-manifold. Lemma 3.6 of [**1**] tell us that, for any open K-embedding $j : U \hookrightarrow M$ into a compact manifold, we have a pushforward map $j_* : \mathbf{K}^0_K(\mathbf{T}^*_K U) \to \mathbf{K}^0_K(\mathbf{T}^*_K M)$.

Let us rephrase Theorem 3.7 of [**1**].

***Theorem 2.4* (Excision property).** — *The composition*

$$\mathbf{K}^0_K(\mathbf{T}^*_K U) \xrightarrow{j_*} \mathbf{K}^0_K(\mathbf{T}^*_K M) \overset{\mathrm{index}_a^{K,M}}{\longrightarrow} \mathscr{C}^{-\infty}(K)^K$$

is independent of the choice of $j : U \hookrightarrow M$: we denote this map $\mathrm{index}_a^{K,U}$.

Note that a *relatively compact* K-invariant open subset U of a K-manifold admits an open K-embedding $j : U \hookrightarrow M$ into a compact K-manifold. So the index map $\mathrm{index}_a^{K,U}$ is defined in this case. An important example is when $U \to N$ is a K-equivariant vector bundle over a compact manifold N : we can imbed U as an open subset of the real projective bundle $\mathbb{P}(U \oplus \mathbb{R})$.

2.3.2. *Exterior product.* — Let us recall the multiplicative property of the analytic index for the product of manifolds that was proved by Atiyah-Singer in [**1**]. Consider a compact Lie group K_2 acting on two manifolds M_1 and M_2, and assume that another compact Lie group K_1 acts on M_1 commuting with the action of K_2.

The external product of complexes on $\mathbf{T}^*M_1$ and $\mathbf{T}^*M_2$ induces a multiplication (see [**1**]):

$$\odot_{\mathrm{ext}} : \mathbf{K}^0_{K_1\times K_2}(\mathbf{T}^*_{K_1}M_1) \times \mathbf{K}^0_{K_2}(\mathbf{T}^*_{K_2}M_2) \longrightarrow \mathbf{K}^0_{K_1\times K_2}(\mathbf{T}^*_{K_1\times K_2}(M_1\times M_2)).$$

Let us recall the definition of this external product. For $k = 1, 2$, we consider equivariant morphisms[(1)] $\sigma_k : \mathscr{E}_k^+ \to \mathscr{E}_k^-$ on $\mathbf{T}^*M_k$. We consider the equivariant morphism on $\mathbf{T}^*(M_1 \times M_2)$

$$\sigma_1 \odot_{\mathrm{ext}} \sigma_2 : \mathscr{E}_1^+ \otimes \mathscr{E}_2^+ \oplus \mathscr{E}_1^- \otimes \mathscr{E}_2^- \longrightarrow \mathscr{E}_1^- \otimes \mathscr{E}_2^+ \oplus \mathscr{E}_1^+ \otimes \mathscr{E}_2^-$$

defined by

$$\sigma_1 \odot_{\mathrm{ext}} \sigma_2 = \begin{pmatrix} \sigma_1 \otimes \mathrm{Id} & -\mathrm{Id} \otimes \sigma_2^* \\ \mathrm{Id} \otimes \sigma_2 & \sigma_1^* \otimes \mathrm{Id} \end{pmatrix}. \tag{6}$$

(1) In order to simplify the notation, we do not make the distinctions between vector bundles on $\mathbf{T}^*M$ and on M.

We see that the set $\operatorname{supp}(\sigma_1 \odot \sigma_2) \subset \mathbf{T}^*M_1 \times \mathbf{T}^*M_2$ is equal to $\operatorname{supp}(\sigma_1) \times \operatorname{supp}(\sigma_2)$.

We suppose now that the morphisms σ_k are respectively K_k-transversally elliptic. Since $\mathbf{T}^*_{K_1\times K_2}(M_1 \times M_2) \neq \mathbf{T}^*_{K_1}M_1 \times \mathbf{T}^*_{K_2}M_2$, the morphism $\sigma_1 \odot_{\mathrm{ext}} \sigma_2$ is not necessarily $K_1 \times K_2$-transversally elliptic. Nevertheless, if σ_2 is taken *almost homogeneous* of order $m = 0$, then the morphism $\sigma_1 \odot_{\mathrm{ext}} \sigma_2$ is $K_1 \times K_2$-transversally elliptic (see Lemma 4.9 in [**17**]). So the exterior product $a_1 \odot_{\mathrm{ext}} a_2$ is the $\mathbf{K}^0$-theory class defined by $\sigma_1 \odot_{\mathrm{ext}} \sigma_2$, where $a_k = [\sigma_k]$ and σ_2 is taken almost homogeneous of order $m = 0$.

***Theorem 2.5* (Multiplicative property).** — *For any $[\sigma_1] \in \mathbf{K}^0_{K_1\times K_2}(\mathbf{T}^*_{K_1}M_1)$ and any $[\sigma_2] \in \mathbf{K}^0_{K_2}(\mathbf{T}^*_{K_2}M_2)$ we have*

$$\operatorname{index}_a^{K_1\times K_2, M_1\times M_2}([\sigma_1] \odot_{\mathrm{ext}} [\sigma_2]) = \operatorname{index}_a^{K_1,K_2,M_1}([\sigma_1]) \operatorname{index}_a^{K_2,M_2}([\sigma_2]).$$

2.3.3. *Free action.* — Let K and G be two compact Lie groups. Let P be a compact manifold provided with an action of $K \times G$. We assume that the action of K is free. Then the manifold $M := P/K$ is provided with an action of G and the quotient map $q : P \to M$ is G-equivariant. Note that we have the natural identification of $\mathbf{T}^*_K P$ with $q^*\mathbf{T}^*M$, hence $(\mathbf{T}^*_K P)/K \simeq \mathbf{T}^*M$ and more generally

$$(\mathbf{T}^*_{K\times G}P)/K \simeq \mathbf{T}^*_G M.$$

This isomorphism induces an isomorphism

$$Q^* : \mathbf{K}^0_G(\mathbf{T}^*_G M) \to \mathbf{K}^0_{K\times G}(\mathbf{T}^*_{K\times G}P).$$

Let $\mathcal{E}^{\pm}$ be two G-equivariant complex vector bundles on M and $\sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ be a G-transversally elliptic symbol. For any finite dimensional irreducible representation (τ, V_τ) of K, we form the G-equivariant complex vector bundle $\mathcal{V}_\tau := P \times_K V_\tau$ on M. We consider the morphism

$$\sigma_\tau := \sigma \otimes \mathrm{Id}_{V_\tau} : p^*(\mathcal{E}^+ \otimes \mathcal{V}_\tau) \to p^*(\mathcal{E}^- \otimes \mathcal{V}_\tau)$$

which is G-transversally elliptic.

The following theorem was obtained by Atiyah-Singer in [**1**].

***Theorem 2.6* (Free action property).** — *We have the following equality in $\mathcal{C}^{-\infty}(K \times G)^{K\times G}$: for $(k, g) \in K \times G$*

$$\operatorname{index}_a^{K\times G,P}(Q^*[\sigma])(k, g) = \sum_{\tau\in\hat{K}} \operatorname{Tr}(k, \tau) \operatorname{index}_a^{G,M}([\sigma_{\tau^*}])(g).$$

2.4. Basic examples

2.4.1. *Bott symbols.* — Let W be a Hermitian vector space. For any $v \in W$, we consider on the $\mathbb{Z}_2$-graded vector space $\wedge W$ the following odd operators: the exterior multiplication $\mathrm{m}(v)$ and the contraction $\iota(v)$. The contraction $\iota(v)$ is an odd derivation of $\wedge W$ such that $\iota(v)w = (w, v)$ for $w \in \wedge^1 W = W$.

The Clifford action of W on $\wedge W$ is defined by the formula

$$\mathbf{c}(v) = \mathrm{m}(v) - \iota(v) \tag{7}$$

Then $\mathbf{c}(v)$ is an odd operator on $\wedge W$ such that $\mathbf{c}(v)^2 = -\|v\|^2 Id$. So $\mathbf{c}(v)$ is invertible when $v \neq 0$.

Consider the trivial vector bundles $\mathcal{E}^{\pm} := W \times \wedge^{\pm} W$ over W with fiber $\wedge^{\pm} W$. The Bott morphism $\mathrm{Bott}(W) : \mathcal{E}^{+} \to \mathcal{E}^{-}$ is defined by

$$\mathrm{Bott}(W)(v, w) = (v, \mathbf{c}(v)w). \tag{8}$$

Consider now a Euclidean vector space V. Then its complexification $V_{\mathbb{C}}$ is an Hermitian vector space. The cotangent bundle $\mathbf{T}^*V$ is identified with $V_{\mathbb{C}}$: we associate to the covector $\xi \in \mathbf{T}^*_v V$ the element $v + i\hat{\xi} \in V_{\mathbb{C}}$, where $\xi \in V^* \to \hat{\xi} \in V$ is the identification given by the Euclidean structure.

Then $\mathrm{Bott}(V_{\mathbb{C}})$ defines an elliptic symbol on V which is equivariant relative to the action of the orthogonal group $O(V)$. Its analytic index is computed in [**3**].

Proposition 2.7. — *We have* [**N2**]*:* $\qquad \mathrm{index}_a^{O(V),V}(\mathrm{Bott}(V_{\mathbb{C}})) = 1.$

Remark 2.8. — *If V and W are two Euclidean vector spaces we see that the symbol* $\mathrm{Bott}((V \times W)_{\mathbb{C}})$ *is equal to the product* $\mathrm{Bott}(V_{\mathbb{C}}) \odot \mathrm{Bott}(W_{\mathbb{C}})$*. Then for $(g, h) \in O(V) \times O(W)$, the multiplicative property tells us that*

$$\mathrm{index}_a^{O(V\times W),V\times W}(\mathrm{Bott}((V \times W)_{\mathbb{C}}))(g, h)$$

is equal to the product $\mathrm{index}_a^{O(V),V}(\mathrm{Bott}(V_{\mathbb{C}}))(g)\, \mathrm{index}_a^{O(W),W}(\mathrm{Bott}(W_{\mathbb{C}}))(h)$.

For any fixed $g \in O(V)$, the vector space V decomposes as an orthogonal sum $\oplus_i V_i$ of g-stable subspaces, where either $\dim V_i = 1$ *and g acts on V_i as ± 1, or* $\dim V_i = 2$ *and g acts on V_i as a rotation.*

Hence [**N2**] *is satisfied for any Euclidean vector space if one checks it for the cases:*

- *$V = \mathbb{R}$ with the action of the group $O(V) = \mathbb{Z}_2$,*
- *$V = \mathbb{R}^2$ with the action of the group $SO(V) = S^1$.*

2.4.2. *Atiyah symbol.* — In the following example, we denote, for any integer k, by $\mathbb{C}_{[k]}$ the vector space $\mathbb{C}$ with the action of the circle group S^1 given by : $t \cdot z = t^k z$.

The Atiyah symbol is the S^1-equivariant morphism on $N = \mathbf{T}^*\mathbb{C}_{[1]} \simeq \mathbb{C}_{[1]} \times \mathbb{C}_{[1]}$

$$\begin{aligned} \sigma_{\mathrm{At}} : N \times \mathbb{C}_{[0]} &\longrightarrow N \times \mathbb{C}_{[1]} \\ (\xi, v) &\longmapsto (\xi, \sigma_{\mathrm{At}}(\xi)v) \end{aligned}$$

defined by $\sigma_{\mathrm{At}}(\xi) = \xi_2 + i\xi_1$ for $\xi = (\xi_1, \xi_2) \in \mathbf{T}^*\mathbb{C}_{[1]}$.

The symbol σ_{At} is not elliptic since $\mathrm{supp}(\sigma_{\mathrm{At}}) = \{\xi_1 = i\xi_2\} \subset \mathbb{C}^2$ is not compact. But $\mathbf{T}^*_{S^1}\mathbb{C}_{[1]} = \{(\xi_1, \xi_2) \mid \mathrm{Im}\,(\xi_1\overline{\xi_2}) = 0\}$ and $\mathrm{supp}(\sigma_{\mathrm{At}}) \cap \mathbf{T}^*_{S^1}\mathbb{C}_{[1]} = \{(0,0)\}$: the symbol σ_{At} is S^1-transversally elliptic. Atiyah-Singer compute its analytic index in [**1**]. Another computation is done in the Appendix of [**8**].

Proposition 2.9. — *We have* [**N3**]*:* $\quad \operatorname{index}_a^{S^1,\mathbb{C}}(\sigma_{\mathrm{At}})(t) = -\sum_{k=1}^{\infty} t^k$.

2.5. Unicity of the index. — The next theorem is the crucial point that we use in the next sections to give a cohomological formula for the index of a transversally elliptic operator (see [**7, 8**]). Note that, in the elliptic case, Atiyah-Segal-Singer used similar strategy to prove that the analytical index coincides with the topological one [**2, 3, 4, 5**].

Suppose that for any compact Lie groups K and H, and any compact $K \times H$-manifold M, we have a map of $R(K \times H)$-modules:

$$\mathbb{I}^{K,H,M} : \mathbf{K}^0_{K\times H}(\mathbf{T}^*_K M) \to \mathscr{C}^\infty(H, C^{-\infty}(K))^{K\times H}.$$

Theorem 2.10. — *Suppose that the maps* $\mathbb{I}^-$ *satisfy*

— *the normalization conditions* [**N1**], [**N2**] *and* [**N3**],

— *the functorial properties* **Diff** *and* **Morph**,

— *the "excision property", the "multiplicative property" and the "free action property".*

Then $\mathbb{I}^-$ *coincides with the analytic index map* index_a^-.

3. The cohomological index

Let N be a manifold, and let $\mathscr{A}(N)$ be the algebra of differential forms on N. We denote by $\mathscr{A}_c(N)$ the subalgebra of compactly supported differential forms. We will consider on $\mathscr{A}(N)$ and $\mathscr{A}_c(N)$ the $\mathbb{Z}_2$-grading in even or odd differential forms.

Let K be a compact Lie group with Lie algebra $\mathfrak{k}$. We suppose that the manifold N is provided with an action of K. We denote $X \mapsto VX$ the corresponding morphism from $\mathfrak{k}$ into the Lie algebra of vectors fields on N: for $n \in N$,

$$V_n X := \frac{d}{d\epsilon} \exp(-\epsilon X) \cdot n|_{\epsilon=0}.$$

Let $\mathscr{A}^\infty(\mathfrak{k}, N)$ be the $\mathbb{Z}_2$-graded algebra of equivariant smooth functions $\alpha : \mathfrak{k} \to \mathscr{A}(N)$. Its $\mathbb{Z}_2$-grading is the grading induced by the exterior degree. Let $D = d - \iota(VX)$ be the equivariant differential: $(D\alpha)(X) = d(\alpha(X)) - \iota(VX)\alpha(X)$. Here the operator $\iota(VX)$ is the contraction of a differential form by the vector field VX. Let $\mathscr{H}^\infty(\mathfrak{k}, N) := \operatorname{Ker} D / \operatorname{Im} D$ be the equivariant cohomology algebra with C^∞-coefficients. It is a module over the algebra $\mathscr{C}^\infty(\mathfrak{k})^K$ of K-invariant C^∞-functions on $\mathfrak{k}$.

The sub-algebra $\mathscr{A}_c^\infty(\mathfrak{k}, N) \subset \mathscr{A}^\infty(\mathfrak{k}, N)$ of equivariant differential forms with compact support is defined as follows : $\alpha \in \mathscr{A}_c^\infty(\mathfrak{k}, N)$ if there exists a compact subset $\mathscr{K}_\alpha \subset N$ such that the differential form $\alpha(X) \in \mathscr{A}(N)$ is supported on $\mathscr{K}_\alpha$ for any

$X \in \mathfrak{k}$. We denote $\mathcal{H}_c^\infty(\mathfrak{k}, N)$ the corresponding cohomology algebra: it is a $\mathbb{Z}_2$-graded algebra.

Let $\mathcal{A}^{-\infty}(\mathfrak{k}, N)$ be the space of *generalized* equivariant differential forms. An element $\alpha \in \mathcal{A}^{-\infty}(\mathfrak{k}, N)$ is, by definition, a $\mathcal{C}^{-\infty}$ map $\alpha : \mathfrak{k} \to \mathcal{A}(N)$ which is equivariant relative to the actions of K on $\mathfrak{k}$ and $\mathcal{A}(N)$ (see [**12**]). The value taken by α on a smooth compactly supported density $Q(X)dX$ on $\mathfrak{k}$ is denoted by $\int_{\mathfrak{k}} \alpha(X)Q(X)dX \in \mathcal{A}(N)$. We have $\mathcal{A}^\infty(\mathfrak{k}, N) \subset \mathcal{A}^{-\infty}(\mathfrak{k}, N)$ and we can extend the differential D to $\mathcal{A}^{-\infty}(\mathfrak{k}, N)$ [**12**]. We denote by $\mathcal{H}^{-\infty}(\mathfrak{k}, N)$ the corresponding cohomology space. Note that $\mathcal{A}^{-\infty}(\mathfrak{k}, N)$ is a module over $\mathcal{A}^\infty(\mathfrak{k}, N)$ under the wedge product, hence the cohomology space $\mathcal{H}^{-\infty}(\mathfrak{k}, N)$ is a module over $\mathcal{H}^\infty(\mathfrak{k}, N)$.

The sub-space $\mathcal{A}_c^{-\infty}(\mathfrak{k}, N) \subset \mathcal{A}^{-\infty}(\mathfrak{k}, N)$ of generalized equivariant differential forms with compact support is defined as follows : $\alpha \in \mathcal{A}_c^{-\infty}(\mathfrak{k}, N)$ if there exist a compact subset $\mathcal{K}_\alpha \subset N$ such that the differential form $\int_{\mathfrak{k}} \alpha(X)Q(X)dX \in \mathcal{A}(N)$ is supported on $\mathcal{K}_\alpha$ for any compactly supported density $Q(X)dX$. We denote $\mathcal{H}_c^{-\infty}(\mathfrak{k}, N)$ the corresponding space of cohomology. The $\mathbb{Z}_2$-grading on $\mathcal{A}(N)$ induces a $\mathbb{Z}_2$-grading on the cohomology spaces $\mathcal{H}^{-\infty}(\mathfrak{k}, N)$ and $\mathcal{H}_c^{-\infty}(\mathfrak{k}, N)$.

If $\mathcal{U}$ is a K-invariant open subset of $\mathfrak{k}$, ones defines also $\mathcal{H}^{-\infty}(\mathcal{U}, N)$ and $\mathcal{H}_c^{-\infty}(\mathcal{U}, N)$. If N is equipped with a K-invariant orientation, the integration over N defines a morphism

$$\int_N : \mathcal{H}_c^{-\infty}(\mathcal{U}, N) \longrightarrow \mathcal{C}^{-\infty}(\mathcal{U})^K.$$

3.1. Restrictions of generalized functions. — Let K be a compact Lie group with Lie algebra $\mathfrak{k}$. In this section, we recall the notions of restriction of invariant generalized functions defined on the Lie group K or on the Lie algebra $\mathfrak{k}$. For more details, see [**10**].

For any $s \in K$ (resp. $S \in \mathfrak{k}$), we denote $K(s)$ (resp. $K(S)$) the stabilizer subgroup: the corresponding Lie algebra is denoted $\mathfrak{k}(s)$ (resp. $\mathfrak{k}(S)$).

For any $s \in K$, we consider a (small) open $K(s)$-invariant neighborhood $\mathcal{U}_s$ of 0 in $\mathfrak{k}(s)$ such that the map $[k, Y] \mapsto ks\,\mathrm{e}^Y\,k^{-1}$ is an open embedding of $K \times_{K(s)} \mathcal{U}_s$ on an open neighborhood of the conjugacy class $K \cdot s := \{ksk^{-1},\ k \in K\} \simeq K/K(s)$.

Similarly, for any $S \in \mathfrak{k}$, we consider a (small) open $K(S)$-invariant neighborhood $\mathcal{U}_S$ of 0 in $\mathfrak{k}(S)$ such that the map $[k, Y] \mapsto \mathrm{Ad}(k)(S + Y)$ is a open embedding of $K \times_{K(S)} \mathcal{U}_S$ on an open neighborhood of the adjoint orbit $K \cdot S \simeq K/K(S)$.

Note that the map $Y \mapsto [\mathrm{e}, Y]$ realizes $\mathcal{U}_s$ (resp. $\mathcal{U}_S$) as a $K(s)$-invariant submanifold of $K \times_{K(s)} \mathcal{U}_s$ (resp. $K \times_{K(S)} \mathcal{U}_S$).

Let Θ be a generalized function on K invariant by conjugation. For any $s \in K$, Θ defines a K-invariant generalized function on $K \times_{K(s)} \mathcal{U}_s \hookrightarrow K$ which admits a restriction to the submanifold $\mathcal{U}_s$ that we denote

$$\Theta\|_s \in \mathcal{C}^{-\infty}(\mathcal{U}_s)^{K(s)}.$$

This notation means that we restrict Θ to the slice $\mathcal{U}_s$. If Θ is smooth, we have $\Theta\|_s(Y) = \Theta(s\,\mathrm{e}^Y)$ for any $Y \in \mathcal{U}_s$.

Similarly, let θ be a K-invariant generalized function on $\mathfrak{k}$. For any $S \in \mathfrak{k}$, θ defines a K-invariant generalized function on $K \times_{K(S)} \mathcal{U}_S \hookrightarrow \mathfrak{k}$ which admits a restriction to the submanifold $\mathcal{U}_S$ that we denote

$$\theta\|_S \in \mathcal{C}^{-\infty}(\mathcal{U}_S)^{K(S)}.$$

If θ is smooth, we have $\theta\|_S(Y) = \theta(S+Y)$ for any $Y \in \mathcal{U}_S$.

We have $K(se^S) = K(s) \cap K(S)$ for any $S \in \mathcal{U}_s$. Let $\Theta\|_s \in \mathcal{C}^{-\infty}(\mathcal{U}_s)^{K(s)}$ be the restriction of a generalized function $\Theta \in \mathcal{C}^{-\infty}(K)^K$. For any $S \in \mathcal{U}_s$, the generalized function $\Theta\|_s$ admits a restriction $(\Theta\|_s)\|_S$ which is a $K(s\,\mathrm{e}^S)$-invariant generalized function defined in a neighborhood of 0 in $\mathfrak{k}(s) \cap \mathfrak{k}(S) = \mathfrak{k}(s\,\mathrm{e}^S)$.

Lemma 3.1 **([10])**. — *Let* $\Theta \in \mathcal{C}^{-\infty}(K)^K$.

— *For* $s \in K$, *and* $S \in \mathcal{U}_s$, *we have the following equality of generalized functions defined in a neighborhood of* 0 *in* $\mathfrak{k}(s\,\mathrm{e}^S)$

$$(\Theta\|_s)\|_S = \Theta\|_{s\,\mathrm{e}^S}. \tag{9}$$

— *Let* $s, k \in K$. *We have the following equality of generalized functions defined in a neighborhood of* 0 *in* $\mathfrak{k}(s)$

$$\Theta\|_s = \Theta\|_{ksk^{-1}} \circ \mathrm{Ad}(k). \tag{10}$$

When $\Theta \in \mathcal{C}^{\infty}(K)^K$ is smooth, condition (9) is easy to check: for $Y \in \mathfrak{k}(s\,\mathrm{e}^S)$, we have

$$(\Theta\|_s)\|_S(Y) = \Theta\|_s(S+Y) = \Theta(s\,\mathrm{e}^{S+Y}) = \Theta(s\,\mathrm{e}^S\,\mathrm{e}^Y) = \Theta\|_{s\,\mathrm{e}^S}(Y).$$

We have the following

Theorem 3.2 **([10])**. — *Let* K *be a compact Lie group. Consider a family of generalized function* $\theta_s \in \mathcal{C}^{-\infty}(\mathcal{U}_s)^{K(s)}$. *We assume that the following conditions are verified.*

— Invariance*: for any* k *and* $s \in K$, *we have the following equality of generalized functions defined in a neighborhood of* 0 *in* $\mathfrak{k}(s)$

$$\theta_s = \theta_{ksk^{-1}} \circ \mathrm{Ad}(k).$$

— Compatibility*: for every* $s \in K$ *and* $S \in U_s$, *we have the following equality of generalized functions defined in a neighborhood of* 0 *in* $\mathfrak{k}(s\,\mathrm{e}^S)$

$$\theta_s\|_S = \theta_{s\,\mathrm{e}^S}.$$

Then there exists a unique generalized function $\Theta \in C^{-\infty}(K)^K$ such that, for any $s \in K$, the equality $\Theta\|_s = \theta_s$ holds in $\mathscr{C}^{-\infty}(\mathscr{U}_s)^{K(s)}$.

3.2. Integration of bouquet of equivariant forms. — Let K be a compact Lie group acting on a compact manifold M. We are interested in the invariant functions on K that can be defined by integrating equivariant forms on $\mathbf{T}^*M$.

Let ω be the Liouville 1-form on $\mathbf{T}^*M$. For any $s \in K$, we denote $M(s)$ the fixed points set $\{x \in M \mid sx = x\}$. Similarly, for any $S \in \mathfrak{k}$, we denote $M(S) \subset M$ the subset fixed by the 1-parameter subgroup $\exp(\mathbb{R}S)$. As K is compact, $M(s)$ and $M(S)$ are submanifolds of M, and $\mathbf{T}^*(M(s)) = (\mathbf{T}^*M)(s)$. The cotangent bundle $\mathbf{T}^*M(s)$ is a symplectic submanifold of $\mathbf{T}^*M$ and the restriction $\omega|_{\mathbf{T}^*M(s)}$ is equal to the Liouville 1-form ω_s on $\mathbf{T}^*M(s)$. The manifolds $\mathbf{T}^*M(s)$ are oriented by their symplectic form $d\omega_s$.

For any $s \in K$, the tangent bundle $\mathbf{T}M$, when restricted to $M(s)$, decomposes as

$$\mathbf{T}M|_{M(s)} = \mathbf{T}M(s) \oplus \mathscr{N}.$$

Let $\underline{s}$ be the linear action induced by s on the bundle $\mathbf{T}M|_{M(s)}$: here $\mathbf{T}M(s)$ is the kernel of $\underline{s} - \mathrm{Id}$, and the normal bundle $\mathscr{N}$ is equal to the image of $\underline{s} - \mathrm{Id}$.

Let ∇ be a K-equivariant connection on the the tangent bundle $\mathbf{T}M$. It induces $K(s)$-equivariant connections : $\nabla^{0,s}$ on the bundle $\mathbf{T}M(s)$ and $\nabla^{1,s}$ on the bundle $\mathscr{N}$. For $i = 0, 1$, we consider the equivariant curvature $R_i(Y), Y \in \mathfrak{k}(s)$ of the connections $\nabla^{i,s}$. We will use the following equivariant forms

Definition 3.3. — *We consider the following smooth closed $K(s)$-equivariant forms on $M(s)$: the equivariant $\widehat{\mathrm{A}}$-genus of the manifold $M(s)$*

$$\widehat{\mathrm{A}}(M(s))(Y) = \det{}^{1/2}\left(\frac{R_0(Y)}{\mathrm{e}^{R_0(Y)/2} - \mathrm{e}^{-R_0(Y)/2}}\right)$$

which is defined for Y in a (small) neighborhood $\mathscr{U}_s$ of $0 \in \mathfrak{k}(s)$, and

$$\mathrm{D}_s(\mathscr{N})(Y) = \det\left(1 - \underline{s}\,\mathrm{e}^{R_1(Y)}\right).$$

In the previous definition, the equivariant form $\widehat{\mathrm{A}}(M(s))$ may be understood as the exponential of the characteristic form associated to the power series $\frac{1}{2}\log(\frac{z}{\mathrm{e}^{z/2} - \mathrm{e}^{-z/2}})$ (see [**6**], Section 1).

The manifold $M(s)$ may have several connected components C_i. We denote by $\dim M(s)$ the locally constant function on $M(s)$ equal to $\dim C_i$ on C_i. In the formulas of the cohomological index, we will use the following closed equivariant form on $M(s)$.

Definition 3.4. — *We consider the smooth closed equivariant form on $M(s)$*

$$\Lambda_s(Y) := (2i\pi)^{-\dim M(s)}\,\frac{\widehat{\mathrm{A}}(M(s))^2(Y)}{\mathrm{D}_s(\mathscr{N})(Y)}$$

which is defined for Y in a (small) neighborhood $\mathcal{U}_s$ of $0 \in \mathfrak{k}(s)$.

Here $\mathcal{U}_s$ is a small $K(s)$-invariant neighborhood of 0 in $\mathfrak{k}(s)$. It is chosen so that : $\mathrm{ad}(k)\mathcal{U}_s = \mathcal{U}_{ksk^{-1}}$, and $M(s) \cap M(S) = M(s\,\mathrm{e}^S)$ for any $s \in K$ and any $S \in \mathcal{U}_s$. For any $s \in K$ and any $S \in \mathcal{U}_s$, let $\mathcal{N}_{(s,S)}$ be the normal bundle of $M(s\,\mathrm{e}^S) = M(s) \cap M(S)$ in $M(s)$. Let $Z \in \mathfrak{k}(se^S)$. Let $R(Z)$ be the $K(s\,\mathrm{e}^S)$-equivariant curvature of an invariant Euclidean connection on $\mathcal{N}_{(s,S)}$. Let

$$\mathrm{Eul}\left(\mathcal{N}_{(s,S)}\right)(Z) := (-2\pi)^{-\mathrm{rank}\,\mathcal{N}_{(s,S)}/2}\ \det{}_o^{1/2}(R(Z))$$

be its $K(s\,\mathrm{e}^S)$-equivariant Euler form. Recall that S induces a complex structure J_S on the bundle $\mathcal{N}_{(s,S)}$: the action of S is linear on the fibers of $\mathcal{N}_{(s,S)}$ and we take $J_S = S(-S^2)^{-1/2}$. The square root $\det{}_o^{1/2}$ is computed using the orientation o defined by this complex structure. Remark that $\mathrm{Eul}\left(\mathcal{N}_{(s,S)}\right)(Z)$ is invertible near $Z = S$, as S acts by an invertible map on the bundle $\mathcal{N}_{(s,S)}$.

Note that the diffeomorphism $k : \mathbf{T}^*M(s) \to \mathbf{T}^*M(ksk^{-1})$ induces a map $k : \mathcal{A}_c^\infty(\mathcal{U}_s, \mathbf{T}^*M(s)) \to \mathcal{A}_c^\infty(\mathcal{U}_{ksk^{-1}}, \mathbf{T}^*M(ksk^{-1}))$. It is easy to check that the family $\Lambda_s \in \mathcal{A}^\infty(\mathcal{U}_s, M(s))$ satisfies :

$$k \cdot \Lambda_s = \Lambda_{ksk^{-1}} \quad \text{in} \quad \mathcal{H}^\infty(\mathcal{U}_s, M(s)), \tag{11}$$

$$\Lambda_{s\,\mathrm{e}^S}(Z) = (-1)^r \frac{\Lambda_s|_{M(s\,\mathrm{e}^S)}}{\mathrm{Eul}(\mathcal{N}_{(s,S)})^2}(S+Z) \quad \text{in} \quad \mathcal{H}^\infty(\mathcal{U}', M(s\,\mathrm{e}^S)), \tag{12}$$

where $\mathcal{U}' \subset \mathfrak{k}(s\,\mathrm{e}^S)$ is a small invariant neighborhood of 0, and $r = \frac{1}{2}\mathrm{rank}_{\mathbb{R}}\,\mathcal{N}_{(s,S)}$.

Let $\gamma_s \in \mathcal{A}_c^\infty(\mathcal{U}(s), \mathbf{T}^*M(s))$ be a family of closed equivariant forms with compact support. We look now at the family of *smooth* invariant functions

$$\theta(\gamma)_s(Y) = \int_{\mathbf{T}^*M(s)} \Lambda_s(Y)\gamma_s(Y), \quad Y \in \mathcal{U}_s.$$

Lemma 3.5. — *The family $\theta(\gamma)_s$ defines an invariant function $\Theta(\gamma) \in C^\infty(K)^K$ if*

$$k \cdot \gamma_s = \gamma_{ksk^{-1}} \quad \text{in} \quad \mathcal{H}_c^\infty(\mathcal{U}_s, \mathbf{T}^*M(s)),$$

and

$$\gamma_{s\,\mathrm{e}^S}(Z) = \gamma_s|_{\mathbf{T}^*M(s\,\mathrm{e}^S)}(S+Z) \quad \text{in} \quad \mathcal{H}_c^\infty(\mathcal{U}', \mathbf{T}^*M(s\,\mathrm{e}^S)),$$

where $\mathcal{U}' \subset \mathfrak{k}(s\,\mathrm{e}^S)$ is a small invariant neighborhood of 0.

Proof. — The proof, that can be found in **[10]** and **[7, 8]**, follows directly from the localization formula in equivariant cohomology. Note that the square $\mathrm{Eul}(\mathcal{N}_{(s,S)})^2$ is equal to the equivariant Euler form of the normal bundle of $\mathbf{T}^*M(s\,\mathrm{e}^S)$ in $\mathbf{T}^*M(s)$. □

In this article, the equivariant forms γ_s that we use are the Chern forms attached to a transversally elliptic symbol. Since they have *generalized coefficients*, we give an extension of Lemma 3.5 to this setting in Section 3.5 (see Theorem 3.18).

3.3. The Chern character with support. — In this subsection, we recall some constructions and some results of [**18**].

Let M be a K-manifold. Let $p : \mathbf{T}^*M \to M$ be the projection.

Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be a Hermitian K-equivariant super-vector bundle over M. Let $\sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ be a K-equivariant symbol. Recall that $\mathrm{supp}(\sigma) \subset \mathbf{T}^*M$ is the set where σ is not invertible. In this section, we do not assume that $\mathrm{supp}(\sigma)$ is compact.

Choose a K-invariant super-connection $\mathbb{A}$ on $p^*\mathcal{E}$, without 0 exterior degree term. As in [**18, 19**], we deform $\mathbb{A}$ with the help of σ : we consider the family of super-connections

$$\mathbb{A}^\sigma(t) = \mathbb{A} + it\, v_\sigma,\ t \in \mathbb{R},$$

on $\mathcal{E}$ where $v_\sigma = \begin{pmatrix} 0 & \sigma^* \\ \sigma & 0 \end{pmatrix}$ is an odd endomorphism of $\mathcal{E}$ defined with the help of the Hermitian structure. Let $\mathbf{F}(\sigma, \mathbb{A}, t)(X), X \in \mathfrak{k}$, be the equivariant curvature of $\mathbb{A}^\sigma(t)$.

We denote by $\mathbf{F}(X), X \in \mathfrak{k}$, the equivariant curvature of $\mathbb{A}$: we have $\mathbf{F}(X) = \mathbb{A}^2 + \mu^{\mathbb{A}}(X)$ where $\mu^{\mathbb{A}}(X) \in \mathcal{A}(\mathbf{T}^*M, \mathrm{End}(p^*\mathcal{E}))$ is the moment of $\mathbb{A}$ [**6**]. Then $\mathbf{F}(\sigma, \mathbb{A}, t)(X) \in \mathcal{A}(\mathbf{T}^*M, \mathrm{End}(p^*\mathcal{E}))^+$ is given by:

$$\mathbf{F}(\sigma, \mathbb{A}, t)(X) = -t^2 v_\sigma + it[\mathbb{A}, v_\sigma] + \mathbf{F}(X). \tag{13}$$

Let $\mathrm{Str} : \mathcal{A}(\mathbf{T}^*M, \mathrm{End}(p^*\mathcal{E})) \to \mathcal{A}(\mathbf{T}^*M)$ be the super-trace. Let $t \in \mathbb{R}$. Consider the K-equivariant forms on $\mathbf{T}^*M$:

$$\begin{aligned}
\mathrm{Ch}(\mathbb{A})(X) &= \mathrm{Str}(\mathrm{e}^{\mathbf{F}(X)}),\\
\mathrm{Ch}(\mathbb{A}, t)(X) &= \mathrm{Str}\left(\mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)(X)}\right),\\
\eta(\sigma, \mathbb{A}, t)(X) &= -i\,\mathrm{Str}\left(v_\sigma\, \mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)(X)}\right),\\
\beta(\sigma, \mathbb{A}, t)(X) &= \int_0^t \eta(\sigma, \mathbb{A}, r)(X)dr.
\end{aligned}$$

The forms $\mathrm{Ch}(\mathbb{A})$, $\mathrm{Ch}(\mathbb{A}, t)$ and $\beta(\sigma, \mathbb{A}, t)$ are equivariant forms on $\mathbf{T}^*M$ with C^∞-coefficients. We have on $\mathbf{T}^*M$ the relation $D(\beta(\sigma, \mathbb{A}, t)) = \mathrm{Ch}(\mathbb{A}) - \mathrm{Ch}(\mathbb{A}, t)$.

We show in [**18**] that the equivariant forms $\mathrm{Ch}(\mathbb{A}, t)$ and $\eta(\sigma, \mathbb{A}, t)$ tends to zero exponentially fast on the open subset $\mathbf{T}^*M \setminus \mathrm{supp}(\sigma)$, when t goes to infinity. Hence the integral

$$\beta(\sigma, \mathbb{A})(X) = \int_0^\infty \eta(\sigma, \mathbb{A}, t)(X)dt$$

defines an equivariant form with $\mathcal{C}^\infty$-coefficients on $\mathbf{T}^*M \setminus \operatorname{supp}(\sigma)$, and we have $D(\beta(\sigma, \mathbb{A})) = \operatorname{Ch}(\mathbb{A})$ on $\mathbf{T}^*M \setminus \operatorname{supp}(\sigma)$.

We will now define the Chern character with support of σ. For any invariant open neighborhood U of $\operatorname{supp}(\sigma)$, we consider the algebra $\mathcal{A}_U(\mathbf{T}^*M)$ of differential forms on $\mathbf{T}^*M$ which are supported in U. Let $\mathcal{A}_U^\infty(\mathfrak{k}, \mathbf{T}^*M)$ be the vector space of equivariant differential forms $\alpha : \mathfrak{k} \to \mathcal{A}_U(\mathbf{T}^*M)$ which are supported in U : $\mathcal{A}_U^\infty(\mathfrak{k}, \mathbf{T}^*M)$ is a subspace of $\mathcal{A}^\infty(\mathfrak{k}, \mathbf{T}^*M)$ which is stable under the derivative D. Let $\mathcal{H}_U^\infty(\mathfrak{k}, \mathbf{T}^*M)$ be the corresponding cohomology space.

The following proposition follows easily:

Proposition 3.6 **([18])**. — *Let U be a K-invariant open neighborhood of* $\operatorname{supp}(\sigma)$*. Let $\chi \in \mathcal{C}^\infty(\mathbf{T}^*M)$ be a K-invariant function, with support contained in U and equal to* 1 *in a neighborhood of* $\operatorname{supp}(\sigma)$*. The equivariant differential form on $\mathbf{T}^*M$*

$$c(\sigma, \mathbb{A}, \chi) = \chi \operatorname{Ch}(\mathbb{A}) + d\chi\, \beta(\sigma, \mathbb{A})$$

is equivariantly closed and supported in U. Its cohomology class $\operatorname{Ch}_U(\sigma)$ in $\mathcal{H}_U^\infty(\mathfrak{k}, \mathbf{T}^*M)$ does not depend on the choice of $(\mathbb{A}, \chi)$, nor on the Hermitian structure on $\mathcal{E}$.

Definition 3.7. — *We define the "Chern character with support"* $\operatorname{Ch}_{\sup}(\sigma)$ *as the collection $(\operatorname{Ch}_U(\sigma))_U$, where U runs over K-invariant open neighborhood of* $\operatorname{supp}(\sigma)$*.*

In practice, the Chern character with support $\operatorname{Ch}_{\sup}(\sigma)$ will be identified with a class $\operatorname{Ch}_U(\sigma) \in \mathcal{H}_U^\infty(\mathfrak{k}, \mathbf{T}^*M)$, where U is a "sufficiently" small neighborhood of $\operatorname{supp}(\sigma)$.

When σ is *elliptic*, that is when $\operatorname{supp}(\sigma)$ is compact, we can choose $\chi \in \mathcal{C}^\infty(\mathbf{T}^*M)^K$ with compact support, and we denote

$$\operatorname{Ch}_c(\sigma) \in \mathcal{H}_c^\infty(\mathfrak{k}, \mathbf{T}^*M) \tag{14}$$

the class defined by the equivariant form with *compact support* $c(\sigma, \mathbb{A}, \chi)$.

We introduce now the bouquet of Chern characters with support.

Let $s \in K$. Then the action of s on $\mathcal{E}|_{M(s)}$ is given by $s^{\mathcal{E}}$, an even endomorphism of $\mathcal{E}|_{M(s)}$. The restriction of ω to $\mathbf{T}^*M(s)$ is the canonical 1-form ω_s of $\mathbf{T}^*M(s)$.

The super-connection $\mathbb{A} + itv_\sigma$ restricts to a super-connection on $p^*\mathcal{E}|\mathbf{T}^*M(s)$. Its curvature $\mathbf{F}(\sigma, \mathbb{A}, t)$ restricted to $\mathbf{T}^*M(s) = N(s)$ gives an element of $\mathcal{A}(N(s), \operatorname{End}(p^*\mathcal{E}|N(s)))$. To avoid further notations, if χ is a function on M, we still denote by χ its restriction to $M(s)$, by σ the restriction of σ to $\mathbf{T}^*M(s)$, by $\mathbf{F}(\sigma, \mathbb{A}, t)$ the restriction of $\mathbf{F}(\sigma, \mathbb{A}, t)$ to $\mathbf{T}^*M(s)$, etc.

For $Y \in \mathfrak{k}(s)$, we introduce the following $K(s)$-equivariant forms on $\mathbf{T}^*M(s)$:

$$\begin{aligned}
\mathrm{Ch}_s(\mathbb{A})(Y) &= \mathrm{Str}(s^{\mathcal{E}}\, \mathrm{e}^{\mathbf{F}(Y)}),\\
\eta_s(\sigma,\mathbb{A},t)(Y) &= -i\,\mathrm{Str}\left(v_\sigma\, s^{\mathcal{E}}\, \mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)(Y)}\right),\\
\beta_s(\sigma,\mathbb{A})(Y) &= \int_0^\infty \eta_s(\sigma,\mathbb{A},t)(Y)dt.
\end{aligned}$$

Then $\beta_s(\sigma,\mathbb{A})$ is a well defined $K(s)$-equivariant form with $\mathscr{C}^\infty$-coefficients on $\mathbf{T}^*M(s) \setminus \mathrm{supp}(\sigma) \cap \mathbf{T}^*M(s)$. We have similarly $d\beta_s(\sigma,\mathbb{A}) = \mathrm{Ch}_s(\mathbb{A})$ outside $\mathrm{supp}(\sigma)\cap\mathbf{T}^*M(s)$.

The bouquet of Chern characters $(\mathrm{Ch}_{\sup}(\sigma,s))_{s\in K}$ can be constructed as follows.

Proposition 3.8. — *Let U be a $K(s)$-invariant open neighborhood of $\mathrm{supp}(\sigma)\cap\mathbf{T}^*M(s)$ in $\mathbf{T}^*M(s)$. Let $\chi \in \mathscr{C}^\infty(\mathbf{T}^*M(s))$ be a $K(s)$-invariant function, with support contained in U and equal to 1 in a neighborhood of $\mathrm{supp}(\sigma)\cap\mathbf{T}^*M(s)$. The equivariant differential form on $\mathbf{T}^*M(s)$*

$$c_s(\sigma,\mathbb{A},\chi)(Y) = \chi\,\mathrm{Ch}_s(\mathbb{A})(Y) + d\chi\,\beta_s(\sigma,\mathbb{A})(Y),\quad Y\in\mathfrak{k}(s),$$

is equivariantly closed and supported in U. Its cohomology class $\mathrm{Ch}_U(\sigma,s)$ in $\mathscr{H}^\infty_U(\mathfrak{k},\mathbf{T}^*M(s))$ does not depend on the choice of $(\mathbb{A},\chi)$, nor on the Hermitian structure on $\mathcal{E}$.

The proof of this proposition is entirely similar to the proof of Proposition 3.6.

Definition 3.9. — *We define the "Chern character with support" $\mathrm{Ch}_{\sup}(\sigma,s)$ as the collection $(\mathrm{Ch}_U(\sigma,s))_U$, where U runs over the $K(s)$-invariant open neighborhood of $\mathrm{supp}(\sigma)\cap\mathbf{T}^*M(s)$ in $\mathbf{T}^*M(s)$.*

Lemma 3.10. — *Let $s\in K$ and $S\in K(s)$. Then for all $Y\in\mathfrak{k}(s)\cap\mathfrak{k}(S)$, one has*

$$c_{s\,\mathrm{e}^S}(\sigma,\mathbb{A},\chi)(Y) = c_s(\sigma,\mathbb{A},\chi)(S+Y)|_{N(s)\cap N(S)}.$$

Proof. — Let $N=\mathbf{T}^*M$. We have to compare the following forms on $N(s)\cap N(S)$

$$\begin{aligned}
\mathrm{Ch}_{s\,\mathrm{e}^S}(\mathbb{A})(Y) &= \mathrm{Str}(s^{\mathcal{E}}\,\mathrm{e}^{S^{\mathcal{E}}}\,\mathrm{e}^{\mathbf{F}(Y)}),\\
\mathrm{Ch}_s(\mathbb{A})(S+Y) &= \mathrm{Str}(s^{\mathcal{E}}\,\mathrm{e}^{\mathbf{F}(S+Y)}),
\end{aligned}$$

as well as the following forms

$$\begin{aligned}
\eta_{s\,\mathrm{e}^S}(\sigma,\omega,\mathbb{A},t)(Y) &= -i\,\mathrm{Str}\left(v_\sigma\, s^{\mathcal{E}}\,\mathrm{e}^{S^{\mathcal{E}}}\,\mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)(Y)}\right),\\
\eta_s(\sigma,\mathbb{A},t)(S+Y) &= -i\,\mathrm{Str}\left(v_\sigma\, s^{\mathcal{E}}\,\mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)(S+Y)}\right).
\end{aligned}$$

For $S\in\mathfrak{k}$, the equivariant curvature $\mathbf{F}(\sigma,\mathbb{A},t)(S+Y)$ on $N(S)$ is equal to $S^{\mathcal{E}} + \mathbf{F}(\sigma,\mathbb{A},t)(Y)$ as the vector field VS vanishes on $N(S)$. Furthermore, above $N(s)\cap N(S)$, the endomorphism $\mathbf{F}(\sigma,\mathbb{A},t)(Y)$ commutes with $S^{\mathcal{E}}$, for $Y\in\mathfrak{k}(S)\cap\mathfrak{k}(s)$. Thus the result follows. □

Then, for any open neighborhood U of $\operatorname{supp}(\sigma)$, the family $(\mathrm{Ch}_U(\sigma, s))_{s\in K}$ forms a bouquet of cohomology classes in the sense of [**10**].

3.4. The Chern character of a transversally elliptic symbol. — We keep the same notations than in the previous sections. We will use definition and results of [**17**] specialized to the case where N is $\mathbf{T}^*M$.

We denote by ω the Liouville form on $\mathbf{T}^*M$. In local coordinates (q,p) then $\omega = -\sum_a p_a dq_a$. The two-form $\Omega = d\omega = \sum_a dq_a \wedge dp_a$ gives a symplectic structure to $\mathbf{T}^*M$. The orientation of $\mathbf{T}^*M$ is the orientation determined by the symplectic structure (our convention for the canonical 1-form ω differs from [**7**], but the symplectic form Ω is the same).

The moment map for the action of K on $(\mathbf{T}^*M, \Omega)$ is the map $f_\omega : \mathbf{T}^*M \to \mathfrak{k}^*$ defined by $\langle f_\omega(x,\xi), X\rangle = \langle \xi, V_x X\rangle$: we have $D\omega(X) = \Omega + \langle f_\omega, X\rangle$.

Remark that $\mathbf{T}^*_K M$ is the set of zeroes of f_ω. Recall [**14, 17**] how to associate to the 1-form ω a K-equivariant form $\mathrm{One}(\omega)$ with generalized coefficients supported near $\mathbf{T}^*_K M$.

On the complement of $\mathbf{T}^*_K M$, the K-equivariant form

$$\beta(\omega) = -i\omega \int_0^\infty \mathrm{e}^{itD\omega}\, dt \tag{15}$$

is well defined as a K-equivariant form with generalized coefficients, and it is obvious to check that $D\beta(\omega) = 1$ outside $\mathbf{T}^*_K M$.

Proposition 3.11 ([**14, 17**]). — *Let U' be a K-invariant open neighborhood of $\mathbf{T}^*_K M$. Let $\chi' \in \mathscr{C}^\infty(\mathbf{T}^*M)$ be a K-invariant function, with support contained in U' and equal to 1 in a neighborhood of $\mathbf{T}^*_K M$. The equivariant differential form on $\mathbf{T}^*M$*

$$\mathrm{One}(\omega, \chi') = \chi' + d\chi'\, \beta(\omega)$$

is closed, with generalized coefficients, and supported in U'. Its cohomology class $\mathrm{One}_{U'}(\omega)$ in $\mathscr{H}^{-\infty}_{U'}(\mathfrak{k}, \mathbf{T}^*M)$ does not depend on the choice of χ'.

This proposition allows us to make the following definition.

Definition 3.12. — *We will denote $\mathrm{One}(\omega)$ the collection $(\mathrm{One}_{U'}(\omega))_{U'}$.*

It is immediate to verify that

$$\mathrm{One}(\omega, \chi') = 1 + D\Big((\chi' - 1)\beta(\omega)\Big). \tag{16}$$

Thus, if we do not impose support conditions, the K-equivariant form $\mathrm{One}(\omega, \chi')$ represents 1 in $\mathscr{H}^{-\infty}(\mathfrak{k}, \mathbf{T}^*M)$. The notation One is suggestive of this fact.

We consider now a K-transversally elliptic symbol σ on M. We have the Chern character $\mathrm{Ch}_{\mathrm{sup}}(\sigma)$ which is an equivariant form with $\mathscr{C}^\infty$-coefficients supported near

$\operatorname{supp}(\sigma)$, and the equivariant form $\operatorname{One}(\omega)$ with $\mathscr{C}^{-\infty}$-coefficients supported near $\mathbf{T}^*_K M$. Since $\operatorname{supp}(\sigma) \cap \mathbf{T}^*_K M$ is compact, the product

$$\operatorname{Ch}_{\sup}(\sigma) \wedge \operatorname{One}(\omega)$$

defines an equivariant form *with compact support* with $\mathscr{C}^{-\infty}$-coefficients. We summarize the preceding discussion by the

Theorem 3.13 **([17])**. — *Let σ be a K-transversally elliptic symbol. Let U, U' be respectively K-invariant open neighborhoods of* $\operatorname{supp}(\sigma)$ *and $\mathbf{T}^*_K M$ such that $\overline{U \cap U'}$ is compact. The product*

$$\operatorname{Ch}_U(\sigma) \wedge \operatorname{One}_{U'}(\omega)$$

defines a compactly supported class in $\mathscr{H}_c^{-\infty}(\mathfrak{k}, \mathbf{T}^* M)$ which depends uniquely of $[\sigma|_{\mathbf{T}^*_K M}] \in \mathbf{K}^0_K(\mathbf{T}^*_K M)$.

Definition 3.14. — *We define $\operatorname{Ch}_c(\sigma, \omega) \in \mathscr{H}_c^{-\infty}(\mathfrak{k}, \mathbf{T}^* M)$ to be the equivariant class of* $\operatorname{Ch}_U(\sigma) \wedge \operatorname{One}_{U'}(\omega)$

We will use the notation

$$\operatorname{Ch}_c(\sigma, \omega) = \operatorname{Ch}_{\sup}(\sigma) \wedge \operatorname{One}(\omega)$$

which summarizes the fact that the class with compact support $\operatorname{Ch}_c(\sigma, \omega)$ is represented by the product

$$c(\sigma, \mathbb{A}, \chi) \wedge \operatorname{One}(\omega, \chi') \tag{17}$$

where $\chi, \chi' \in \mathscr{C}^\infty(\mathbf{T}^* M)^K$ are equal to 1 respectively in a neighborhood of $\operatorname{supp}(\sigma)$ and $\mathbf{T}^*_K M$, and furthermore the product $\chi\chi'$ is compactly supported.

Remark 3.15. — *If σ is elliptic, one can take χ with compact support, and $\chi' = 1$ on $\mathbf{T}^* M$ in Equation* (17)*. We see then that*

$$\operatorname{Ch}_c(\sigma, \omega) = \operatorname{Ch}_c(\sigma) \quad \text{in} \quad \mathscr{H}_c^{-\infty}(\mathfrak{k}, \mathbf{T}^* M).$$

Let $s \in K$. Similarly, we denote by $\operatorname{One}(\omega_s, \chi')$ the closed $K(s)$-equivariant form on $\mathbf{T}^* M(s)$ associated to the canonical 1-form $\omega_s = \omega | \mathbf{T}^* M(s)$ and a function $\chi' \in \mathscr{C}^\infty(\mathbf{T}^* M(s))^{K(s)}$ equal to 1 in a neighborhood of $\mathbf{T}^*_{K(s)} M(s)$. For any $K(s)$-invariant neighborhood $U' \subset \mathbf{T}^* M(s)$ of $\mathbf{T}^*_{K(s)} M(s)$, we denote

$$\operatorname{One}_{U'}(\omega_s) \in \mathscr{H}_{U'}^{-\infty}(\mathfrak{k}(s), \mathbf{T}^* M(s))$$

the class defined by $\operatorname{One}(\omega_s, \chi')$ when χ' is supported in U'. We denote $\operatorname{One}(\omega_s)$ the collection $(\operatorname{One}_{U'}(\omega, s))_{U'}$.

We defined, in Section 3.3, the family of Chern classes $(\operatorname{Ch}_{\sup}(\sigma, s))_{s \in K}$ for any K-invariant symbol. We now define a family $(\operatorname{Ch}_c(\sigma, \omega, s))_{s \in K}$ with compact support and $\mathcal{C}^{-\infty}$-coefficients when σ is a K-transversally elliptic symbol on M. Note that the restriction $\sigma|\mathbf{T}^*M(s)$ of a K-transversally elliptic symbol on M is a $K(s)$-transversally elliptic symbol on the submanifold $M(s)$.

The following proposition is proved in an entirely similar way than Theorem 3.13.

Proposition 3.16. — *Let σ be a K-transversally elliptic symbol. Let $s \in K$.*

Let $U, U' \subset \mathbf{T}^*M(s)$ be respectively $K(s)$-invariant open neighborhoods of $\mathrm{supp}(\sigma|\mathbf{T}^*M(s))$ *and $\mathbf{T}^*_{K(s)}M$ such that $\overline{U \cap U'}$ is compact. The product*

$$\mathrm{Ch}_U(\sigma, s) \wedge \mathrm{One}_{U'}(\omega_s)$$

defines a compactly supported class in $\mathcal{H}_c^{-\infty}(\mathfrak{k}(s), \mathbf{T}^*M(s))$ which depends uniquely of $[\sigma|_{\mathbf{T}^*_{K(s)}M(s)}]$.

Definition 3.17. — *We define* $\mathrm{Ch_c}(\sigma, \omega, s) \in \mathcal{H}_c^{-\infty}(\mathfrak{k}(s), \mathbf{T}^*M(s))$ *to be the equivariant class of* $\mathrm{Ch}_U(\sigma) \wedge \mathrm{One}_{U'}(\omega)$

The notation

$$\mathrm{Ch}_c(\sigma, \omega, s) = \mathrm{Ch}_{\sup}(\sigma, s) \wedge \mathrm{One}(\omega_s)$$

summarizes the fact that the class with compact support $\mathrm{Ch_c}(\sigma, \omega, s)$ is represented by $c_s(\sigma, \mathbb{A}, \chi) \wedge \mathrm{One}(\omega_s, \chi')$ where $\chi, \chi' \in \mathcal{C}^{\infty}(\mathbf{T}^*M(s))^{K(s)}$ are chosen so that $\chi\chi'$ is compactly supported.

3.5. Definition of the cohomological index. — Let K be a compact Lie group and let M be a compact K-manifold. The aim of this section is to define the cohomological index

$$\mathrm{index}_c^{K,M} : \mathbf{K}^0_K(\mathbf{T}^*_K M) \to \mathcal{C}^{-\infty}(K)^K.$$

For any $[\sigma] \in \mathbf{K}^0_K(\mathbf{T}^*_K M)$, the generalized function $\mathrm{index}_c^{K,M}([\sigma])$ will be described through their restrictions $\mathrm{index}_c^{K,M}([\sigma])\|_s$, $s \in K$ (see Section 3.1).

In Subsection 3.2, we have introduced for any $s \in K$, the closed equivariant form on $M(s)$

$$\Lambda_s(Y) := (2i\pi)^{-\dim M(s)} \frac{\widehat{\mathrm{A}}(M(s))^2(Y)}{\mathrm{D}_s(\mathcal{N})(Y)}.$$

We wish to prove first the following theorem.

Theorem 3.18. — *Let σ be a K-transversally elliptic symbol. There exists a unique invariant generalized function* $\mathrm{index}_c^{K,M}([\sigma])$ *on K satisfying the following equations. Let $s \in K$. For every $Y \in \mathfrak{k}(s)$ sufficiently small,*

$$\mathrm{index}_c^{K,M}([\sigma])\|_s(Y) = \int_{\mathbf{T}^*M(s)} \Lambda_s(Y)\ \mathrm{Ch}_{\sup}(\sigma, s)(Y)\, \mathrm{One}(\omega_s)(Y). \tag{18}$$

As $\mathrm{Ch_c}(\sigma,\omega,s) = \mathrm{Ch_{sup}}(\sigma,s)(Y)\,\mathrm{One}(\omega_s)(Y)$ is *compactly supported*, the integral (18) of equivariant differential forms with generalized coefficients defines a generalized function on a neighborhood of zero in $\mathfrak{k}(s)$. However, we need to prove that the different local formulae patch together (see Theorem 3.2). The proof of this theorem occupies the rest of this subsection. Once this theorem is proved, we can make the following definition.

Definition 3.19. — *Let σ be a K-transversally elliptic symbol. The cohomological index of σ is the invariant generalized function* $\mathrm{index}_c^{K,M}([\sigma])$ *on K satisfying Equation* (18)*. We also rewrite the formula for the cohomological index as*

$$\mathrm{index}_c^{K,M}([\sigma])\|_s(Y) = \int_{\mathbf{T}^*M(s)} \Lambda_s(Y)\ \mathrm{Ch_c}(\sigma,\omega,s)(Y). \tag{19}$$

In particular, when $s = e$ is the identity of the group K, Equation (18) *becomes*

$$\mathrm{index}_c^{K,M}([\sigma])(\mathrm{e}^X) = (2i\pi)^{-\dim M}\int_{\mathbf{T}^*M} \widehat{\mathrm{A}}(M)^2(X)\,\mathrm{Ch_{sup}}(\sigma)(X)\,\mathrm{One}(\omega)(X). \tag{20}$$

Remark 3.20. — *In* (18)*,* (20) *and* (19) *we take for the integration the symplectic orientation on the cotangent bundles.*

Let us now prove Theorem 3.18.

Proof. — The right hand side of (18) defines a $K(s)$-invariant generalized function $\theta_s(Y)$ on a neighborhood $\mathcal{U}_s$ of 0 in $\mathfrak{k}_s$. Following Theorem 3.2, the family $(\theta_s)_{s\in K}$ defines an invariant generalized function on K, if the *invariance condition* and the *compatibility condition* are satisfied. The invariance condition is easy to check. We will now prove the compatibility condition.

Let $s \in K$, and $S \in \mathcal{U}_s$. We have to check that the restriction $\theta_s\|_S$ coincides with $\theta_{s\,\mathrm{e}^S}$ in a neighborhood of 0 in $\mathfrak{k}(s)\cap\mathfrak{k}(S) = \mathfrak{k}(s\,\mathrm{e}^S)$. We conduct the proof only for s equal to the identity e, as the proof for s general is entirely similar.

For a differential form α on a manifold N, we denote by *support*$(\alpha) \subset N$ its support. For a smooth equivariant form $X \to \eta(X)$, its support *support*(η) is defined as the smallest closed subset containing *support*$(\eta(X))$ for all $X \in \mathfrak{k}$.

We have to compute the restriction at $\theta_e\|_S$ of the generalized invariant function

$$\theta_e(X) := \int_{\mathbf{T}^*M} \Lambda_e(X)\ \mathrm{Ch_{sup}}(\sigma)(X)\,\mathrm{One}(\omega)(X), \quad X \in \mathcal{U}_e.$$

For this purpose, we choose a particular representative of the class $\mathrm{Ch_{sup}}(\sigma)\mathrm{One}(\omega)$ in $\mathcal{H}_c^{-\infty}(\mathfrak{k},\mathbf{T}^*M)$. Since this class only depends of $[\sigma|_{\mathbf{T}^*_K M}] \in \mathbf{K}^0_K(\mathbf{T}^*_K M)$, we choose a transversally elliptic symbol σ_h which is almost homogeneous of degree 0 and such that $[\sigma_h|_{\mathbf{T}^*_K M}] = [\sigma|_{\mathbf{T}^*_K M}]$.

One can show **[17]** that the moment map $f_\omega : \mathbf{T}^*M \to \mathfrak{k}^*$ is *proper* when restricted to the support $\mathrm{supp}(\sigma_h)$. We represent $\mathrm{Ch_{sup}}(\sigma_h)$ by the form $c(\sigma_h,\mathbb{A},\chi_\sigma)$ where χ_σ

is a function on $\mathbf{T}^*M$ such that $support(\chi_\sigma) \cap \{\|f_\omega\|^2 \leq 1\}$ is compact. For this choice of χ_σ, the equivariant form $\alpha(X) = \Lambda_e(X)\, c(\sigma_h, \mathbb{A}, \chi_\sigma)(X)$ is thus such that $support(\alpha) \cap \{\|f_\omega\|^2 \leq 1\}$ is compact. It is defined for X small enough. Multiplying by a smooth invariant function of X with small compact support and equal to 1 in a neighborhood of 0, we may find $\alpha(X)$ defined for all $X \in \mathfrak{k}$ and which coincides with $\Lambda_e(X)\, c(\sigma_h, \mathbb{A}, \chi_\sigma)(X)$ for X small enough.

We choose χ supported in $\{\|f_\omega\|^2 < 1\}$ and equal to 1 on $\{\|f_\omega\|^2 \leq \epsilon\}$, and define $\mathrm{One}(\omega)$ with this choice of χ. Then $\alpha(X)\mathrm{One}(\omega)(X)$ is compactly supported.

We will now prove the following result:

Proposition 3.21. — *Let $\alpha(X)$ be a closed equivariant form with C^∞-coefficients on $N := \mathbf{T}^*M$ such that $\{\|f_\omega\|^2 \leq 1\} \cap support(\alpha)$ is compact. Define the generalized function $\theta \in C^{-\infty}(\mathfrak{k})^K$ by*

$$\theta(X) := \int_N \alpha(X)\mathrm{One}(\omega)(X). \tag{21}$$

Then, the restriction $\theta\|_S$ is given, for $Y = Z - S$ sufficiently close to 0 by

$$\theta\|_S(Y) = (-1)^r \int_{N(S)} \frac{\alpha|_{N(S)}(Z)}{\mathrm{Eul}(\mathscr{N}_S)^2(Z)} \mathrm{One}(\omega_S)(Z). \tag{22}$$

Here $\mathscr{N}_S$ denotes the normal bundle of $M(S)$ in M, and $r = \frac{1}{2}(\dim M - \dim M(S))$.

Remark 3.22. — *The integral (22) is defined using the symplectic orientation $o(\omega_S)$ on $N(S) = \mathbf{T}^*M(S)$. The linear action of S on the normal bundle $\mathscr{N}'_S$ of $N(S)$ in N induces a complex structure J_S : let $o(J_S)$ be corresponding orientation of the fibers of $\mathscr{N}'_S$. We have then on $N(S)$ the orientation $o(S)$ such that $o(\omega) = o(S)o(J_S)$. One can check that $(-1)^r$ is the quotient between $o(S)$ and $o(\omega_S)$.*

Let us apply the last proposition to the form $\alpha(X) = \Lambda_e(X)\,\mathrm{Ch}_{\sup}(\sigma_h)(X)$. If we use (12) and Lemma 3.10, we see that $(-1)^r \frac{\alpha|_{N(S)}}{\mathrm{Eul}(\mathscr{N}_S)^2}(S+Y)$ is equal to $\Lambda_{\mathrm{e}^S}(Y)\,\mathrm{Ch}_{\sup}(\sigma_h, \mathrm{e}^S)(Y)$. Hence Proposition 3.21 tells us that the the restriction of $\theta_e\|_S$ is equal to θ_{e^S} : Theorem 3.18 is proved.

Proof. — We now concentrate on the proof of Proposition 3.21.

Remark that if α is compactly supported, we can get rid of the forms $\mathrm{One}(\omega)$ and $\mathrm{One}(\omega_S)$ in the integrals (21) and (22), since they are equal to 1 in cohomology. In this case, the proposition is just the localization formula, as $\mathrm{Eul}(\mathscr{N}_S)^2$ is the Euler class of the normal bundle of $\mathbf{T}^*M(S)$ in $\mathbf{T}^*M$.

The proof will follow the same scheme as the usual localization formula (see [**6**]) and will use the fact that $\alpha|_{\mathfrak{k}(S)}$ is exact outside the set of zeroes of S. To extend the proof of the localization formula in our setting, we have to bypass the fact that the restriction of $\mathrm{One}(\omega)$ to $\mathfrak{k}(S)$ has no meaning, since $\mathrm{One}(\omega)$ is an equivariant form with generalized coefficients. However, we will use in a crucial way the fact that the closed equivariant form $\mathrm{One}(\omega)$ is the limit of smooth equivariant forms

$$\mathrm{One}^T(\omega)(X) = \chi + d\chi \int_0^T (-i\omega\, \mathrm{e}^{itD\omega(X)})dt.$$

Here $D\left(\mathrm{One}^T(\omega)\right) = d\chi\, \mathrm{e}^{iTD\omega}$ tends to zero as T goes to infinity.

Let $\mathrm{One}^T(\omega)(Z)$ be the restriction of $\mathrm{One}^T(\omega)(X)$ to $\mathfrak{k}(S)$. We write $f_\omega = f_\omega^S + f_\omega^{\mathfrak{q}}$ relative to the $K(S)$-invariant decomposition $\mathfrak{k}^* = \mathfrak{k}(S)^* \oplus \mathfrak{q}$. Then the family of $K(S)$-equivariant forms

$$\mathrm{e}^{itD\omega(Z)} = \mathrm{e}^{itd\omega}\, \mathrm{e}^{it\langle f_\omega^S, Z\rangle}$$

tends to 0 outside $\{f_\omega^S = 0\}$, as t goes to ∞. Since $d\chi$ can be non-zero on the subset $\{f_\omega^S = 0\}$, the family of $\mathfrak{k}(S)$-equivariant forms $\mathrm{One}^T(\omega)(Z)$ does not have a limit when $T \to \infty$ in general.

Consider the sub-manifold $N(S) := \mathbf{T}^*M(S)$ of $N := \mathbf{T}^*M$. Note that $f_\omega^{\mathfrak{q}}$ vanishes on $N(S)$. Let $\mathcal{V}$ be an invariant tubular neighborhood of $N(S)$ which is contained in $\{\|f_\omega^{\mathfrak{q}}\|^2 \le \frac{\epsilon}{2}\}$. We are interested in the restriction $\mathrm{One}^T(\omega)|_{\mathcal{V}}(Z)$ to $\mathcal{V}$. Since the function χ is equal to 1 on $\{\|f_\omega\|^2 \le \epsilon\}$, we see that $d\chi|_{\mathcal{V}}$ is equal to zero in the neighborhood $\mathcal{V} \cap \{\|f_\omega^S\|^2 \le \frac{\epsilon}{2}\}$ of $\mathcal{V} \cap \{f_\omega^S = 0\}$. Hence the limit

$$\mathrm{One}(\omega)|_{\mathcal{V}}(Z) = \lim_{T\to\infty} \mathrm{One}^T(\omega)|_{\mathcal{V}}(Z), \quad Z \in \mathfrak{k}(S), \tag{23}$$

defines a $K(S)$-equivariant form with generalized coefficients on $\mathcal{V}$. Note that the restriction of $\mathrm{One}(\omega)|_{\mathcal{V}}$ to $N(S) \subset \mathcal{V}$ is the $K(S)$-equivariant form $\mathrm{One}(\omega_S)$ associated to the Liouville 1-form ω_S on $\mathbf{T}^*M(S)$.

The generalized function $\theta \in \mathcal{C}^{-\infty}(\mathfrak{k})^K$ is the limit, as T goes to infinity, of the family of smooth functions

$$\theta^T(X) := \int_N \alpha(X)\mathrm{One}^T(\omega)(X).$$

Here the equivariant forms $\alpha^T = \alpha\, \mathrm{One}^T(\omega)$ stay supported in the fixed compact set $\mathcal{K} := \{\|f_\omega\|^2 \le 1\} \cap \mathit{support}(\alpha)$.

The proof will be completed if we show that the family of smooth functions $\theta^T(Z), Z \in \mathfrak{k}(S)$, converge to the generalized function

$$\theta'(Z) := (-1)^r \int_{N(S)} \frac{\alpha(Z)}{\mathrm{Eul}(\mathcal{N}_S)^2(Z)} \mathrm{One}(\omega_S)(Z),$$

as T goes to infinity, and when Z varies in a small neighborhood of S in $\mathfrak{k}(S)$.

Let U be a relatively compact invariant neighborhood of $\mathcal{K}$ in N. Let $\chi' \in \mathcal{C}^\infty(U)^{K(S)}$ be such that χ' is supported in $\mathcal{V} \cap U$, and $\chi' = 1$ in a neighborhood of $U(S) = N(S) \cap U$. Here $\mathcal{V}$ is a tubular neighborhood of $N(S)$ satisfying the conditions for the existence of the limit (23).

Choose a K-invariant metric $\langle -,-\rangle$ on $\mathbf{T}N$. Let λ be the $K(S)$-invariant 1-form on N defined by $\lambda = \langle VS, -\rangle$. Note that $D(\lambda)(S) = d\lambda - \|VS\|^2$ is invertible outside $N(S)$. One sees that

$$\mathrm{P}_{\chi'}(Z) = \chi' + d\chi' \frac{\lambda}{D\lambda(Z)}$$

is a $K(S)$-equivariant form on U for Z in a small neighborhood of S. The following equation of $K(S)$-equivariant forms on U is immediate to verify:

$$1 = \mathrm{P}_{\chi'} + D\left((1-\chi')\frac{\lambda}{D\lambda}\right). \tag{24}$$

Since the $K(S)$-equivariant forms

$$\alpha^T(Z) := \alpha(Z)\mathrm{One}^T(\omega)(Z)$$

are supported in U, one can multiply (24) by α^T. We have then the following relations between *compactly supported* $K(S)$-equivariant forms on N:

$$\begin{aligned} \alpha^T &= \mathrm{P}_{\chi'}\alpha^T + D\left((1-\chi')\frac{\lambda}{D\lambda}\right)\alpha^T \\ &= \mathrm{P}_{\chi'}\alpha^T + D\left((1-\chi')\frac{\lambda}{D\lambda}\alpha^T\right) + (1-\chi')\frac{\lambda}{D\lambda}D(\alpha^T). \end{aligned}$$

According to this equation, we divide the function $\theta^T(Z)$ in two parts

$$\theta^T(Z) = A^T(Z) + B^T(Z), \quad \text{for} \quad Z - S \quad \text{small}$$

with

$$A^T(Z) = \int_N \mathrm{P}_{\chi'}(Z)\alpha^T(Z)$$

and

$$B^T(Z) = \int_N (1-\chi')\frac{\lambda}{D\lambda(Z)}D\alpha^T(Z) = \int_N (1-\chi')\frac{\lambda}{D\lambda(Z)}\alpha(Z)d\chi\, \mathrm{e}^{iTD\omega(Z)}.$$

Let $p : \mathcal{V} \to N(S)$ be the projection, and let $i : N(S) \to \mathcal{V}$ be the inclusion. Since the form $\mathrm{P}_{\chi'}(Z)$ is supported in $\mathcal{V}$, the family of smooth equivariant forms $\mathrm{P}_{\chi'}(Z)\alpha(Z)\mathrm{One}^T(\omega)(Z)$ converges to

$$\mathrm{P}_{\chi'}(Z)\alpha(Z)\mathrm{One}(\omega)|_{\mathcal{V}}(Z)$$

as T goes to ∞, by our previous computation of the limit (23). Hence the functions $A^T(Z)$ converge to

$$\begin{aligned}\int_{\mathcal{V}} \mathrm{P}_{\chi'}(Z)\alpha(Z)\mathrm{One}(\omega)|_{\mathcal{V}}(Z) &= \int_{\mathcal{V}} \mathrm{P}_{\chi'}(Z)\; p^*\circ i^*\left(\alpha\, \mathrm{One}(\omega)|_{\mathcal{V}}\right)(Z) \quad [1]\\ &= \int_{N(S),o(S)} p_*(\mathrm{P}_{\chi'})(Z)\; \alpha|_{N(S)}(Z)\, \mathrm{One}(\omega_S)(Z) \quad [2]\\ &= \int_{N(S),o(S)} \frac{\alpha|_{N(S)}(Z)}{\mathrm{Eul}(\mathcal{N}_S)^2(Z)}\mathrm{One}(\omega_S)(Z) \quad [3]\\ &= (-1)^r\int_{N(S)} \frac{\alpha|_{N(S)}(Z)}{\mathrm{Eul}(\mathcal{N}_S)^2(Z)}\mathrm{One}(\omega_S)(Z). \quad [4]\end{aligned}$$

Points [1] and [2] are due to the fact that $\alpha(Z)\mathrm{One}(\omega)|_{\mathcal{V}}(Z)$ is equal to $p^*\circ i^*\left(\alpha\, \mathrm{One}(\omega)|_{\mathcal{V}}\right)(Z)$ in $\mathcal{H}^{-\infty}(\mathfrak{k}(S),\mathcal{V})$ and that $\mathrm{P}_{\chi'}$ has a compact support relative to the fibers of p (here p_* denotes the integration along the fibers). For point [3], we use then that $p_*(\mathrm{P}_{\chi'})$ multiplied by the Euler class[(2)] of $\mathcal{V}$ is equal to the restriction of $\mathrm{P}_{\chi'}$ to $N(S)$, which is identically equal to 1. In [4], we use the symplectic orientation for the integration.

Let us show that the integral $\int_{\mathfrak{k}(S)} B^T(Z)\varphi(Z)dZ$ tends to 0, as T goes to infinity, for any $\varphi\in\mathcal{C}^\infty(\mathfrak{k}(S))^{K(S)}$ supported in a small neighborhood of S. As $\det_{\mathfrak{k}/\mathfrak{k}(S)}(Z)$ does not vanish when $Z-S$ remains small enough, it is enough to show that

$$I(T) := \int_{N\times\mathfrak{k}(S)} (1-\chi')\frac{\lambda}{D\lambda(Z)}D\alpha^T(Z)\varphi(Z)\det{}_{\mathfrak{k}/\mathfrak{k}(S)}(Z)dZ$$

tends to 0, as T goes to infinity. We have

$$I(T) := \int_{N\times\mathfrak{k}(S)} \mathrm{e}^{iTD\omega(Z)}\,\eta(Z)\det{}_{\mathfrak{k}/\mathfrak{k}(S)}(Z)dZ$$

where $\eta(Z)=(\chi'-1)\frac{\lambda}{D\lambda(Z)}\alpha(Z)d\chi\varphi(Z)$ is a compactly supported $K(S)$-equivariant form on N with $\mathcal{C}^\infty$-coefficients, which is defined for all $Z\in\mathfrak{k}(S)$. Furthermore we have $\eta(Z)=0$ for Z outside a small neighborhood of S and

$$\mathit{support}(\eta)\cap\{f_\omega=0\}=\varnothing.$$

There exists a K-equivariant form $\Gamma:\mathfrak{k}\to\mathcal{A}(N)$ such that $\Gamma(Z)=\eta(Z)$ for any $Z-S$ small in $\mathfrak{k}(S)$. Indeed we define $\Gamma(X)=k\cdot\eta(Z)$ for any choice of k,Z such that $k\cdot Z=X$. Here X varies in a (small) neighborhood of $K\cdot S$. As $\eta(Z)$ is zero when Z is not near S, the map $X\mapsto\Gamma(X)$ is supported on a compact neighborhood of $K\cdot S$ in $\mathfrak{k}$. We see also that

$$\mathit{support}(\Gamma)\cap\{f_\omega=0\}=\varnothing. \tag{25}$$

---

(2) The Euler form of the vector bundle $\mathcal{V}\to N(S)$ is equal to the square of the Euler form of the normal bundle $\mathcal{N}_S$ of $M(S)$ in M.

Condition (25) implies that the integral $J(T) := \int_{\mathfrak{k}\times N} \mathrm{e}^{iTD\omega(X)}\,\Gamma(X)dX$ goes to 0, as T goes to infinity. But $I(T) = J(T)$. Indeed, write $X = k\cdot Z$ and apply the Weyl integration formula. We obtain

$$\begin{aligned} J(T) &= \int_{\mathfrak{k}(S)} \Big(\int_{K\times N} \mathrm{e}^{iTD\omega(k\cdot Z)}\,\Gamma(k\cdot Z)dk\Big) \det{}_{\mathfrak{k}/\mathfrak{k}(S)}(Z)dZ \\ &= \int_{\mathfrak{k}(S)} \int_{K\times N} k\cdot \Big(\mathrm{e}^{iTD\omega(Z)}\,\eta(Z)\Big)dk\ \det{}_{\mathfrak{k}/\mathfrak{k}(S)}(Z)dZ. \end{aligned}$$

Integration on the K-manifold N is invariant under diffeomorphisms, thus

$$J(T) = \int_{\mathfrak{k}(S)} \int_N \mathrm{e}^{iTD\omega(Z)}\,\eta(Z) \det{}_{\mathfrak{k}/\mathfrak{k}(S)}(Z)dZ = I(T).$$

We have shown that the family of smooth function $B^T(Z)$ goes to 0, as T goes to infinity. The proof of Proposition 3.21 is then completed. □

Let H be a compact Lie group acting on M and commuting with the action of K. Then the space $\mathbf{T}^*_K M$ is provided with an action of $K\times H$.

Lemma 3.23. — *If* $[\sigma] \in \mathbf{K}^0_{K\times H}(\mathbf{T}^*_K M)$, *then the cohomological index* $\mathrm{index}_c^{K,H,M}([\sigma])(k,h) \in C^{-\infty}(K\times H)^{K\times H}$ *is smooth relative to the variable* $h\in H$.

Proof. — We have to prove that for any $s=(s_1,s_2)\in K\times H$, the generalized function

$$\mathrm{index}_c^{K,H,M}([\sigma])\|_s(Y_1,Y_2)$$

which is defined for (Y_1,Y_2) in a neighborhood of 0 in $\mathfrak{k}(s_1)\times\mathfrak{h}(s_2)$, is smooth relative to the variable $Y_2\in\mathfrak{h}(s_2)$. We check it for $s=e$.

We have

$$\mathrm{index}_c^{K,H,M}([\sigma])\|_e(X,Y) = \int_{\mathbf{T}^*M} \Lambda_e(X,Y)\,\mathrm{Ch}_{\sup}(\sigma)(X,Y)\mathrm{One}(\omega)(X,Y) \tag{26}$$

for $(X,Y)\in\mathfrak{k}\times\mathfrak{h}$ in a neighborhood of 0. The equivariant class with compact support $\mathrm{Ch}_{\sup}(\sigma)\mathrm{One}(\omega)$ is represented by the product $c(\sigma,\mathbb{A},\chi)\mathrm{One}(\omega,\chi')$ where (χ,χ') is chosen so that $\chi=1$ in a neighborhood of $\mathrm{supp}(\sigma)$, $\chi'=1$ in a neighborhood of $\mathbf{T}^*_{K\times H}M$, and $\chi\chi'$ is compactly supported.

Since σ is K-transversally elliptic, the set $\mathrm{supp}(\sigma)\cap\mathbf{T}^*_K M$ is compact. Hence we can choose (χ,χ') so that $\chi'=1$ in a neighborhood of $\mathbf{T}^*_K M$ and $\chi\chi'$ is compactly supported. It easy to check that the equivariant form $\mathrm{One}(\omega,\chi')(X,Y)$ is then smooth relative to the variable $Y\in\mathfrak{h}$. This show that the right hand side of (26) is smooth relative to the variable $Y\in\mathfrak{h}$. □

Remark 3.24. — *We will denote* $\mathrm{Ch}^1_{\mathrm{c}}(\sigma,\omega)(X,Y)$ *the* $K\times H$*-equivariant form defined by the product* $c(\sigma,\mathbb{A},\chi)\mathrm{One}(\omega,\chi')$ *where* (χ,χ') *is chosen so that* $\chi=1$ *in a neighborhood of* $\mathrm{supp}(\sigma)$, $\chi'=1$ *in a neighborhood of* $\mathbf{T}^*_K M$, *and* $\chi\chi'$ *is compactly supported.*

The equivariant form $\mathrm{Ch}^1_{\mathrm{c}}(\sigma,\omega)(X,Y)$ *is compactly supported and is smooth relative to the variable* $Y \in \mathfrak{h}$.

4. The cohomological index coincides with the analytic one

In this section, we now prove that the cohomological index is equal to the analytical index. The main difficulty in the proof of this result in Berline-Vergne [**7, 8**] was to prove that their formulae were defining generalized functions which, moreover, were compatible with each other. The heart of this new proof is the fact the Chern character with compact support is multiplicative. Thus we rely heavily here on the results of [**17**], so that the proof is now easy.

Theorem 4.1. — *The analytic index of a transversally elliptic operator* P *on a* K*-manifold* M *is equal to* $\mathrm{index}_c^{K,M}([\sigma_P])$.

To prove that the cohomological index is equal to the analytic index, following the Atiyah-Singer algorithm, we need only to verify that the cohomological index satisfies the properties that we listed of the analytic index:

- Invariance by diffeomorphism : **Diff**,
- Functorial with respect to subgroups : **Morph**,
- Excision property,
- Free action properties,
- Multiplicative properties,
- Normalization conditions [**N1**], [**N2**] and [**N3**].

The invariance by diffeomorphism, the functoriality with respect to subgroups and the excision property are obviously satisfied by $\mathrm{index}_c^{K,M}$.

4.1. Free action. — We now prove that the cohomological index satisfies the free action property. We consider the setting of Subsection 2.3.3. The action of K on the bundle $\mathbf{T}^*_K P$ is free and the quotient $\mathbf{T}^*_K P/K$ admit a canonical identification with $\mathbf{T}^* M$. Then we still denote by

$$q : \mathbf{T}^*_K P \to \mathbf{T}^* M$$

the quotient map by K: it is a G-equivariant map such that $q^{-1}(\mathbf{T}^*_G M) = \mathbf{T}^*_{K\times G} P$.

We choose a G-invariant connection θ for the principal fibration $q : P \to M$ of group K. With the help of this connection, we have a direct sum decomposition

$$\mathbf{T}^* P = \mathbf{T}^*_K P \oplus P \times \mathfrak{k}^*.$$

Let $\pi_1 : \mathbf{T}^* P \to \mathbf{T}^*_K P$ and $\pi_2 : \mathbf{T}^* P \to P \times \mathfrak{k}^*$ be the projections on each factors. Let

$$Q : \mathbf{T}^* P \to \mathbf{T}^* M$$

be the map $q \circ \pi_1$.

Let σ be a G-transversally elliptic morphism on $\mathbf{T}^*M$. Its pull-back $Q^*\sigma$ is then a $K\times G$-transversally elliptic morphism on $\mathbf{T}^*P$: we have $\mathrm{supp}(Q^*\sigma) = Q^{-1}(\mathrm{supp}(\sigma))$ and then $\mathrm{supp}(Q^*\sigma)\cap \mathbf{T}^*_{K\times G}P = q^{-1}(\mathrm{supp}(\sigma)\cap \mathbf{T}^*_G M)$ is compact.

Theorem 4.2. — *Let $P \to M$ be a principal fibration with a free right action of K, provided with a left action of G. Consider a class $[\sigma] \in \mathbf{K}^0_G(\mathbf{T}^*_G M)$ and its pull-back by Q : $[Q^*\sigma] \in \mathbf{K}^0_{K\times G}(\mathbf{T}^*_{K\times G}P)$. Then we have the equality of generalized functions: for $(k,g)\in K\times G$*

$$\mathrm{index}_c^{K\times G,P}([Q^*\sigma])(k,g) = \sum_{\tau\in\hat{K}} \mathrm{Tr}(k,\tau)\,\mathrm{index}_c^{G,M}([\sigma_{\tau^*}])(g).$$

The rest of this section is devoted to the proof. We have to check that for any $(s,s')\in K\times G$ we have the following equality of generalized functions defined in a neighborhood of $\mathfrak{k}(s)\times\mathfrak{g}(s')$:

$$\mathrm{index}_c^{K\times G,P}([Q^*\sigma])\|_{(s,s')}(X,Y) = \sum_{\tau\in\hat{K}} \mathrm{Tr}(s\,\mathrm{e}^X,\tau)\,\mathrm{index}_c^{G,M}([\sigma_{\tau^*}])\|_{s'}(Y). \tag{27}$$

We conduct the proof of (27) only for $(s,s') = (e,e)$ the identity of $K\times G$. This proof can be adapted to the general case by using the same arguments as Berline-Vergne [**8**].

First, we analyze the left hand side of (27) at $(s,s')=(e,e)$.

We consider the $K\times G$-invariant 1-form $\nu = \langle\xi,\theta\rangle$ on $P\times\mathfrak{k}^*$: here $\theta\in\mathcal{A}^1(P)\otimes\mathfrak{k}$ is our connection form, and ξ is the variable in $\mathfrak{k}^*$. We have

$$D\nu(X,Y) = d\nu + \langle\xi,\mu(Y)-X\rangle,\quad X\in\mathfrak{k},\quad Y\in\mathfrak{g}. \tag{28}$$

where $\mu(Y) = -\theta(VY)\in C^\infty(P)\otimes\mathfrak{k}$.

We associate to ν the $K\times G$-equivariant form with *generalized coefficients* $\beta(-\nu)(X,Y) = i\nu\int_0^\infty \mathrm{e}^{-itD\nu(X,Y)}\,dt$, $(X,Y)\in\mathfrak{k}\times\mathfrak{g}$, which is defined on the open subset $P\times\mathfrak{k}^*\setminus\{0\}$. One checks that $\beta(-\nu)(X,Y)$ is smooth relative to the variable $Y\in\mathfrak{g}$. Let $\chi_{\mathfrak{k}^*}\in\mathcal{C}^\infty(\mathfrak{k}^*)^K$ be a function with compact support and equal to 1 near 0. Then

$$\mathrm{One}(-\nu)(X,Y) := \chi_{\mathfrak{k}^*} + d\chi_{\mathfrak{k}^*}\beta(-\nu)(X,Y) \tag{29}$$

is a closed equivariant form on $P\times\mathfrak{k}^*$, with compact support, and which is smooth relative to the variable $Y\in\mathfrak{g}$.

Let σ be a G-transversally elliptic morphism on $\mathbf{T}^*M$. Its pull-back $Q^*\sigma$ is then a $K\times G$-transversally elliptic morphism on $\mathbf{T}^*P$. Let ω_P and ω_M be the Liouville 1-forms on $\mathbf{T}^*P$ and $\mathbf{T}^*M$ respectively. We have defined the equivariant Chern classes with compact support $\mathrm{Ch}_c(\sigma,\omega_M)\in\mathcal{H}_c^{-\infty}(\mathfrak{g},\mathbf{T}^*M)$ and $\mathrm{Ch}_c(Q^*\sigma,\omega_P)\in\mathcal{H}_c^{-\infty}(\mathfrak{k}\times\mathfrak{g},\mathbf{T}^*P)$.

Proposition 4.3. — *We have the following equality*

$$\mathrm{Ch}_c(Q^*\sigma,\omega_P)(X,Y)=Q^*\Big(\mathrm{Ch}_c(\sigma,\omega_M)\Big)(Y)\wedge\pi_2^*\Big(\mathrm{One}(-\nu)\Big)(X,Y)$$

in $\mathcal{H}_c^{-\infty}(\mathfrak{k}\times\mathfrak{g},\mathbf{T}^*P)$. *Note that the product on the right hand side is well defined since* $\mathrm{One}(-\nu)(X,Y)$ *is smooth relative to the variable* $Y\in\mathfrak{g}$.

Proof. — The proof which is done in **[17]** follows from the relation

$$\omega_P=Q^*(\omega_M)-\pi_2^*(\nu). \tag{30}$$

□

We now analyze the term

$$\mathrm{index}_c^{K\times G,P}([Q^*\sigma])\|_{(e,e)}(X,Y)=(2i\pi)^{-\dim P}\int_{\mathbf{T}^*P}\widehat{\mathrm{A}}(P)^2\,\mathrm{Ch}_c(Q^*\sigma,\omega_P)(X,Y).$$

An easy computation gives that $\widehat{\mathrm{A}}(P)^2(X,Y)=j_{\mathfrak{k}}(X)^{-1}q^*\widehat{\mathrm{A}}(M)^2(Y)$, with $j_{\mathfrak{k}}(X)=\det_{\mathfrak{k}}\left(\frac{e^{\mathrm{ad}(X)/2}-e^{-\mathrm{ad}(X)/2}}{\mathrm{ad}(X)}\right)$. If we use Proposition 4.3, we see that

$$\begin{aligned}
&\mathrm{index}_c^{K\times G,P}([Q^*\sigma])\|_{(e,e)}(X,Y)\\
&=\frac{(2i\pi)^{-\dim P}}{j_{\mathfrak{k}}(X)}\int_{\mathbf{T}^*P}\pi_1^*\circ q^*\Big(\widehat{\mathrm{A}}(M)^2\,\mathrm{Ch}_c(\sigma,\omega_M)\Big)(Y)\wedge\pi_2^*\mathrm{One}(-\nu)(X,Y)\\
&=\frac{(2i\pi)^{-\dim P}}{j_{\mathfrak{k}}(X)}\int_{\mathbf{T}^*_KP}q^*\Big(\widehat{\mathrm{A}}(M)^2\,\mathrm{Ch}_c(\sigma,\omega_M)\Big)(Y)\wedge\int_{\mathfrak{k}^*}\mathrm{One}(-\nu)(X,Y). \qquad (31)
\end{aligned}$$

Let us compute the integral $\int_{\mathfrak{k}^*}\mathrm{One}(-\nu)(X,Y)$.

We choose a K-invariant scalar product on $\mathfrak{k}$ and an orthonormal basis $E^1,\dots,E^r$ of $\mathfrak{k}$, with dual basis $E_1,\dots,E_r$: we write $X=\sum_k X_kE^k$ for $X\in\mathfrak{k}$, and $\xi=\sum_k\xi_kE_k$ for $\xi\in\mathfrak{k}^*$. Let $\theta_k=\langle E_k,\theta\rangle$ be the 1-forms on P associated to the connection 1-form. Let $\mathrm{vol}(K,dX^o)$ be the volume of K computed with the Haar measure compatible with the volume form $dX^o=dX_1\dots dX_r$.

We have $d\nu=\sum_k\xi_kd\theta_k+d\xi_k\theta_k$, and (30) gives that

$$(d\omega_P)^{\dim P}=q^*\left(d\omega_M\right)^{\dim M}\wedge\theta_r\cdots\theta_1\wedge\pi_2^*\left(d\xi_1\cdots d\xi_r\right).$$

So, in the integral (31), the vector space $\mathfrak{k}^*$ is oriented by the volume form $d\xi^o=d\xi_1\cdots d\xi_r$, and $\mathbf{T}^*_KP$ is oriented by $q^*\left(d\omega_M\right)^{\dim M}\wedge\theta_r\cdots\theta_1$.

Let $\Theta=d\theta+\frac{1}{2}[\theta,\theta]\in\mathcal{A}^2(P)\otimes\mathfrak{k}$ be the curvature of θ. The equivariant curvature of θ is

$$\Theta(Y)=\mu(Y)+\Theta.$$

Then $\Theta(Y)\in\mathcal{A}(P)\otimes\mathfrak{k}$ is horizontal, and the element $\Theta\in\mathcal{A}^2(P)\otimes\mathfrak{k}$ is nilpotent. If φ is a $\mathcal{C}^\infty$ function on $\mathfrak{k}$, then $\varphi(\Theta(Y))$ is computed via the Taylor series expansion at $\mu(Y)(p)$ and $\varphi(\Theta(Y))$ is a horizontal form on P which depends smoothly and G-equivariantly of $Y \in \mathfrak{g}$. When $\varphi \in \mathscr{C}^\infty(\mathfrak{k})$ is K-invariant, the form $\varphi(\Theta(Y))$ is basic, hence we can look at it as a differential form on M which depends smoothly and G-equivariantly of $Y \in \mathfrak{g}$.

Definition 4.4. — *Let $\delta(X - \Theta(Y))$ be the $K \times G$-equivariant form on P defined by the relation*

$$\int_{\mathfrak{k}\times\mathfrak{g}} \delta(X - \Theta(Y))\varphi(X, Y)dXdY := \mathrm{vol}(K, dX) \int_{\mathfrak{g}} \varphi(\Theta(Y), Y)dY,$$

for any $\varphi \in \mathscr{C}^\infty(\mathfrak{k} \times \mathfrak{g})$ with compact support. Here $\mathrm{vol}(K, dX)$ is the volume of K computed with the Haar measure compatible with dX.

One sees that $\delta(X - \Theta(Y))$ is a $K \times G$-equivariant form on P which depends smoothly of the variable $Y \in \mathfrak{g}$.

Lemma 4.5. — *Let $\mathfrak{k}^*$ be oriented by the volume form $d\xi^o = d\xi_1 \cdots d\xi_r$. Then*

$$\int_{\mathfrak{k}^*} \mathrm{One}(-\nu)(X, Y) = (2i\pi)^{\dim K} \delta(X - \Theta(Y)) \frac{\theta_r \cdots \theta_1}{\mathrm{vol}(K, dX^o)}.$$

Proof. — Take $\chi_{\mathfrak{k}^*}(\xi) = g(\|\xi\|^2)$ where $g \in \mathscr{C}_c^\infty(\mathbb{R})$ is equal to 1 in a neighborhood of 0. Let $\varphi \in \mathscr{C}_c^\infty(\mathfrak{k})$ and let $\widehat{\varphi}(\xi) = \int_{\mathfrak{k}} \mathrm{e}^{i\langle \xi, X\rangle} \varphi(X)dX^o$ be its Fourier transform relative to dX^o.

To compute the integral over the fiber $\mathfrak{k}^*$ of $\mathrm{One}(-\nu)(X, Y)$, only the highest exterior degree term in $d\xi$ will contribute to the integral. This term comes only from the term $d\chi_{\mathfrak{k}^*}\beta(-\nu)(X, Y)$ in $\mathrm{One}(-\nu)(X, Y) := \chi_{\mathfrak{k}^*} + d\chi_{\mathfrak{k}^*}\beta(-\nu)(X, Y)$. We compute

$$\begin{aligned}
\int_{\mathfrak{k}} \Big(\int_{\mathfrak{k}^*} \mathrm{One}(-\nu)(X, Y)\Big) \varphi(X)dX^o &= \int_{\mathfrak{k}^*} \Big(\int_{\mathfrak{k}} \mathrm{One}(-\nu)(X, Y)\varphi(X)dX^o\Big) \\
&= \int_{\mathfrak{k}^*} d\chi_{\mathfrak{k}^*}(i\nu) \Big(\int_0^\infty \mathrm{e}^{-it(d\nu + \langle \xi, \mu(Y)\rangle)}\, \widehat{\varphi}(t\xi)dt\Big) \\
&= \int_0^\infty \underbrace{\Big(\int_{\mathfrak{k}^*} d\chi_{\mathfrak{k}^*}(i\nu)\, \mathrm{e}^{-it(d\nu + \langle \xi, \mu(Y)\rangle)}\, \widehat{\varphi}(t\xi)\Big)}_{I(t)} dt.
\end{aligned}$$

Since $d\nu = \sum_k \xi_k d\theta_k + d\xi_k \theta_k$, the differential form $d\chi_{\mathfrak{k}^*}(i\nu)\, \mathrm{e}^{-itd\nu}$ is equal to

$$2i\, g'(\|\xi\|^2)(\sum_j \xi_j d\xi_j)(\sum_k \xi_k \theta_k) \prod_l (1 - itd\xi_l \theta_l)\, \mathrm{e}^{-it\langle \xi, d\theta\rangle},$$

and its component $[d\chi_{\mathfrak{k}^*}(i\nu)\, \mathrm{e}^{-itd\nu}]_{\max}$ of highest exterior degree in $d\xi$ is

$$\begin{aligned}
[d\chi_{\mathfrak{k}^*}(i\nu)\, \mathrm{e}^{-itd\nu}]_{\max} &= -2(-i)^r t^{r-1} g'(\|\xi\|^2)\|\xi\|^2 \prod_j (d\xi_j \wedge \theta_j)\, \mathrm{e}^{-it\langle \xi, d\theta\rangle} \\
&= -2(i)^r t^{r-1}\, \theta_r \cdots \theta_1\, g'(\|\xi\|^2)\|\xi\|^2\, \mathrm{e}^{-it\langle \xi, d\theta\rangle}\, d\xi^o.
\end{aligned}$$

So for $t > 0$ we have

$$
\begin{aligned}
I(t) &= -2(i)^r t^{r-1}\theta_r\cdots\theta_1\left(\int_{\mathfrak{k}^*} g'(\|\xi\|^2)\|\xi\|^2\, \mathrm{e}^{-it\langle\xi,d\theta+\mu(Y)\rangle}\,\widehat{\varphi}(t\xi)d\xi^o\right)\\
&= (i)^r\theta_r\cdots\theta_1\left(\int_{\mathfrak{k}^*}\left[(-2g'(\tfrac{\|\xi\|^2}{t^2})\tfrac{\|\xi\|^2}{t^3}\right]\mathrm{e}^{-i\langle\xi,d\theta+\mu(Y)\rangle}\,\widehat{\varphi}(\xi)d\xi^o\right)\\
&= (i)^r\theta_r\cdots\theta_1\frac{d}{dt}\left(\int_{\mathfrak{k}^*} g(\tfrac{\|\xi\|^2}{t^2})\,\mathrm{e}^{-i\langle\xi,d\theta+\mu(Y)\rangle}\,\widehat{\varphi}(\xi)d\xi^o\right).
\end{aligned}
$$

Finally $\int_{\mathfrak{k}}\left(\int_{\mathfrak{k}^*}\mathrm{One}(-\nu)(X,Y)\right)\varphi(X)dX^o$ is equal to

$$
\begin{aligned}
\int_0^\infty I(t)dt &= (i)^r\theta_r\cdots\theta_1\left(\int_{\mathfrak{k}^*}\mathrm{e}^{-i\langle\xi,d\theta+\mu(Y)\rangle}\,\widehat{\varphi}(\xi)d\xi^o\right)\\
&= (2i\pi)^r\ \theta_r\cdots\theta_1\ \varphi(d\theta+\mu(Y))\\
&= (2i\pi)^r\ \theta_r\cdots\theta_1\ \varphi(\Theta+\mu(Y))\\
&= (2i\pi)^r\left(\int_{\mathfrak{k}}\delta(X-\Theta(Y))\varphi(X)dX^o\right)\frac{\theta_r\cdots\theta_1}{\mathrm{vol}(K,dX^o)}.
\end{aligned}
$$

□

The last lemma shows that $\mathrm{index}_c^{K\times G,P}([Q^*\sigma])\|_{(e,e)}(X,Y)$ is equal to

$$
\frac{(2i\pi)^{-\dim M}}{j_{\mathfrak{k}}(X)}\int_{\mathbf{T}^*_K P} q^*\left(\widehat{\mathrm{A}}(M)^2\,\mathrm{Ch}_c(\sigma,\omega_M)\right)(Y)\,\delta(X-\Theta(Y))\frac{\theta_r\cdots\theta_1}{\mathrm{vol}(K,dX^o)}
$$

$$
= \frac{(2i\pi)^{-\dim M}}{j_{\mathfrak{k}}(X)}\int_{\mathbf{T}^*M}\widehat{\mathrm{A}}(M)^2(Y)\,\mathrm{Ch}_c(\sigma,\omega_M)(Y)\,\delta_o(X-\Theta(Y)). \tag{32}
$$

Here $\delta_o(X-\Theta(Y))$ denotes the closed $K\times G$-equivariant form M defined by the relation

$$
\int_{\mathfrak{k}}\delta_o(X-\Theta(Y))\varphi(X)dX = \mathrm{vol}(K,dX)\overline{\varphi}(\Theta(Y))
$$

for any $\varphi\in\mathscr{C}_c^\infty(\mathfrak{k})$. Here $\overline{\varphi}(X) := \mathrm{vol}(K,dk)^{-1}\int_K\varphi(kX)dk$ is the K-invariant function obtained by averaging φ.

Now we analyze the right hand side of (27) at $(s,s')=(e,e)$. Here the Chern class $\mathrm{Ch}_{\mathrm{sup}}(\sigma_{\tau^*})(Y)$ is equal to $\mathrm{Ch}_{\mathrm{sup}}(\sigma)(Y)\,\mathrm{Ch}(\mathcal{V}_{\tau^*})(Y)$ where the equivariant Chern character $\mathrm{Ch}(\mathcal{V}_{\tau^*})(Y)$ is represented by $\mathrm{Tr}(\mathrm{e}^{\Theta(Y)},\tau^*)$. Hence $\mathrm{Ch}_{\mathrm{c}}(\sigma_{\tau^*},\omega_M)(Y) = \mathrm{Ch}_{\mathrm{c}}(\sigma,\omega_M)(Y)\,\mathrm{Tr}(\mathrm{e}^{\Theta(Y)},\tau^*)$. So the generalized function $\sum_{\tau\in\hat{K}}\mathrm{Tr}(\mathrm{e}^X,\tau)\,\mathrm{index}_c^{G,M}([\sigma_{\tau^*}])\|_e(Y)$ is equal to

$$
(2i\pi)^{-\dim M}\int_{\mathbf{T}^*M}\widehat{\mathrm{A}}(M)^2(Y)\,\mathrm{Ch}_c(\sigma,\omega_M)(Y)\,\Xi(X,\Theta(Y)) \tag{33}
$$

where $\Xi(X,X')$ is a generalized function on a neighborhood of 0 in $\mathfrak{k}\times\mathfrak{k}$ defined by the relation $\Xi(X,X') = \sum_{\tau\in\hat{K}}\mathrm{Tr}(\mathrm{e}^X,\tau)\,\mathrm{Tr}(\mathrm{e}^{X'},\tau^*)$.

The Schur orthogonality relation shows that

$$
\Xi(X,X') = j_{\mathfrak{k}}(X)^{-1}\delta_o(X-X').
$$

In other words, $\Xi(X,X')$ is smooth relative to the variable X' and for any $\varphi \in \mathscr{C}^\infty(\mathfrak{k})^K$ which is supported in a small neighborhood of 0, we have $\operatorname{vol}(K,dX)\varphi(X') = \int_{\mathfrak{k}} \Xi(X,X') j_{\mathfrak{k}}(X)\varphi(X)dX$.

Finally, we have proved that the generalized functions (32) and (33) coincide: the proof of (27) is then completed for $(s,s') = (e,e)$.

4.2. Multiplicative property. — We consider the setting of Subsection 2.3.2. We will check that the cohomological index satisfies the *Multiplicative property* (see Theorem 2.5).

Let M_1 be a compact $K_1 \times K_2$-manifold, and let M_2 be a K_2-manifold. We consider the product $M := M_1 \times M_2$ with the action of $K := K_1 \times K_2$.

***Theorem 4.6* (Multiplicative property).** — *For any* $[\sigma_1] \in \mathbf{K}^0_{K_1\times K_2}(\mathbf{T}^*_{K_1} M_1)$ *and any* $[\sigma_2] \in \mathbf{K}^0_{K_2}(\mathbf{T}^*_{K_2} M_2)$ *we have*

$$\operatorname{index}_c^{K,M}([\sigma_1] \odot_{\mathrm{ext}} [\sigma_2]) = \operatorname{index}_c^{K_1,K_2,M_1}([\sigma_1]) \operatorname{index}_c^{K_2,M_2}([\sigma_2]). \tag{34}$$

The product on the right hand side of (34) is well defined since $\operatorname{index}_c^{K_1,K_2,M_1}([\sigma_1])(k_1,k_2)$ *is a generalized function on* $K_1 \times K_2$ *which is smooth relative to the variable* $k_2 \in K_2$ *(see Lemma 3.23).*

Proof. — Let σ_1 be a morphism on $\mathbf{T}^*M_1$, which is $K_1 \times K_2$-equivariant and K_1-transversally elliptic. Let σ_2 be a morphism on $\mathbf{T}^*M_2$, which is K_2-transversally elliptic. The morphism σ_2 can be chosen so that it is almost homogeneous of degree 0. Then the product $\sigma := \sigma_1 \odot_{\mathrm{ext}} \sigma_2$ is a K-transversally elliptic morphism on $\mathbf{T}^*M$, and $[\sigma] = [\sigma_1] \odot_{\mathrm{ext}} [\sigma_2]$ in $\mathbf{K}^0_K(\mathbf{T}^*_K M)$.

We have to show that for any $s = (s_1,s_2) \in K_1 \times K_2$, we have

$$\begin{aligned}\operatorname{index}_c^{K,M}([\sigma])\|_s(Y_1,Y_2) = \\ \operatorname{index}_c^{K_1,K_2,M_1}([\sigma_1])\|_{s_1}(Y_1,Y_2) \operatorname{index}_c^{K_2,M_2}([\sigma_2])\|_{s_2}(Y_2)\end{aligned} \tag{35}$$

for (Y_1,Y_2) in a neighborhood of 0 in $\mathfrak{k}_1(s_1) \times \mathfrak{k}_2(s_2)$. We conduct the proof only for s equal to the identity e, as the proof for s general is entirely similar.

For $k = 1,2$, let $\pi_k : \mathbf{T}^*M \to \mathbf{T}^*M_k$ be the projection. The Liouville 1-form ω on $\mathbf{T}^*(M_1 \times M_2)$ is equal to $\pi_1^*\omega_1 + \pi_2^*\omega_2$, where ω_k is the Liouville 1-form on $\mathbf{T}^*M_k$.

We have three index formulas:

$$\begin{aligned}
\operatorname{index}_c^{K,M}([\sigma])\|_e(X_1,X_2) &:= (2i\pi)^{-\dim M} \int_{\mathbf{T}^*M} \widehat{\mathrm{A}}(M)^2 \operatorname{Ch_c}(\sigma,\omega)(X_1,X_2),\\
\operatorname{index}_c^{K,M_1}([\sigma_1])\|_e(X_1,X_2) &:= (2i\pi)^{-\dim M_1} \int_{\mathbf{T}^*M_1} \widehat{\mathrm{A}}(M_1)^2 \operatorname{Ch}^1_{\mathrm{c}}(\sigma_1,\omega_1)(X_1,X_2),\\
\operatorname{index}_c^{K_2,M_2}([\sigma_2])\|_e(X_2) &:= (2i\pi)^{-\dim M_2} \int_{\mathbf{T}^*M_2} \widehat{\mathrm{A}}(M_2)^2 \operatorname{Ch_c}(\sigma_2,\omega_2)(X_2).
\end{aligned}$$

Following Remark 3.24, $\mathrm{Ch}^1_{\mathrm{c}}(\sigma_1,\omega_1)(X_1,X_2)$ denotes a closed equivariant form with compact support which represents the class $\mathrm{Ch}_{\mathrm{c}}(\sigma_1,\omega_1)$, and which is *smooth* relative to $X_2\in\mathfrak{k}_2$.

It is immediate to check that $\widehat{\mathrm{A}}(M)^2(X_1,X_2) = \widehat{\mathrm{A}}(M_1)^2(X_1,X_2)\widehat{\mathrm{A}}(M_2)^2(X_2)$. Hence Equality (35) follows from the following identity in $\mathscr{H}_c^{-\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,\mathbf{T}^*M)$ that we proved in [**17**]:

$$\pi_1^*\,\mathrm{Ch}^1_{\mathrm{c}}(\sigma_1,\omega_1)(X_1,X_2)\wedge\pi_2^*\,\mathrm{Ch}_{\mathrm{c}}(\sigma_2,\omega_2)(X_2)=\mathrm{Ch}_{\mathrm{c}}(\sigma,\omega)(X_1,X_2). \qquad\square$$

4.3. Normalization conditions

4.3.1. *Atiyah symbol.* — Let $V:=\mathbb{C}_{[1]}$ be equipped with the canonical action of S^1. The Atiyah symbol σ_{At} was introduced in Subsection 2.4.2 : it is a S^1-transversally elliptic symbol on V. It is the first basic example of a "pushed" symbol (see Subsection 5.1).

We consider on V the Euclidean metric $(v,w)=\Re(v\overline{w})$: it gives at any $v\in V$ identifications $\mathbf{T}_vV\simeq\mathbf{T}^*_vV\simeq\mathbb{C}_{[1]}$. So in this example we will make no distinction between vectors fields and 1-forms on V. Let $\kappa(\xi_1)=i\xi_1$ be the vector field on V associated to the action of S^1 : $\kappa=-VX$ where $X=i\in\mathrm{Lie}(S^1)$.

Let σ_V be the symbol on the complex vector space V: at any $(\xi_1,\xi_2)\in\mathbf{T}^*V$, $\sigma_V(\xi_1,\xi_2):\wedge^0V\to\wedge^1V$ acts by multiplication by ξ_2. We see then that

$$\sigma_{\mathrm{At}}(\xi_1,\xi_2)=\sigma_V(\xi_1,\xi_2+\kappa(\xi_1)).$$

The symbol σ_{At} is obtained by "pushing" the symbol σ by the vector field κ.

We can attached to the 1-form κ, the equivariant form $\mathrm{One}(\kappa)$ which is defined on V, and localized near $\{\kappa=0\}=\{0\}\subset V$. Since the support of σ_V is the zero section, the equivariant Chern character $\mathrm{Ch}_{\sup}(\sigma_V)$ is an equivariant form on $\mathbf{T}^*V$ which is compactly supported in the fibers of $p:\mathbf{T}^*V\to V$. Then the product $\mathrm{Ch}_{\sup}(\sigma_V)p^*\mathrm{One}(\kappa)$ defines an equivariant form with compact support on $\mathbf{T}^*V$.

Here we will use the relation (see Proposition 5.5)

$$\mathrm{Ch}_{\sup}(\sigma_{\mathrm{At}})\mathrm{One}(\omega)=\mathrm{Ch}_{\sup}(\sigma_V)\,p^*\mathrm{One}(\kappa)\quad\text{in}\quad\mathscr{H}_c^{-\infty}(\mathfrak{k},\mathbf{T}^*V). \tag{36}$$

Using (36), we now compute the cohomological index of the Atiyah symbol.

Proposition 4.7. — *We have*

$$[\mathbf{N3}]\qquad \mathrm{index}_c^{S^1,V}([\sigma_{\mathrm{At}}])(\mathrm{e}^{i\theta})=-\sum_{n=1}^{\infty}\mathrm{e}^{in\theta}.$$

Proof. — We first prove the equality above when $s=\mathrm{e}^{i\theta}$ is not equal to 1. Then, near s, the generalized function $-\sum_{n=1}^{\infty}\mathrm{e}^{in\theta}$ is analytic and given by $-\frac{s}{1-s}$.

Now, at a point $s \in S^1$ different from 1, the fixed point set $V(s)$ is $\{0\}$. The character $\mathrm{Ch}_s(\mathscr{E})$ is $(1-s)$, and the form $D_s(\mathscr{N})$ is $(1-s)(1-s^{-1})$. Thus

$$\mathrm{index}_c^{S^1,V}(s) = \frac{(1-s)}{(1-s)(1-s^{-1})} = -\frac{s}{1-s}.$$

This shows the equality of both members in Proposition 4.7 on the open set $s \neq 1$ of S^1.

We now compute near $s = 1$. Thanks to Formula (36) we have

$$\mathrm{index}_c^{S^1,V}(\sigma_{\mathrm{At}})\|_1(\theta) = (2i\pi)^{-2} \int_{\mathbf{T}^*V} \widehat{\mathrm{A}}(V)^2(\theta)\ \mathrm{Ch}_{\mathrm{sup}}(\sigma_V)(\theta)\, p^*\mathrm{One}(\kappa)(\theta).$$

The Chern character with support $\mathrm{Ch}_{\mathrm{sup}}(\sigma_V)(\theta)$ is proportional to the S^1-equivariant Thom form of the real vector bundle $\mathbf{T}^*V \to V$. More precisely, calculation already done in **[18]** shows that

$$\mathrm{Ch}_{\mathrm{sup}}(\sigma_V)(\theta) = (2i\pi)\frac{\mathrm{e}^{i\theta} - 1}{i\theta}\mathrm{Thom}(\mathbf{T}^*V)(\theta).$$

However the symplectic orientation on $\mathbf{T}^*V \simeq \mathbb{C}^2$ is the opposite of the orientation given by its complex structure. Now

$$\widehat{\mathrm{A}}(V)^2(\theta) = \frac{(i\theta)(-i\theta)}{(1-\mathrm{e}^{i\theta})(1-\mathrm{e}^{-i\theta})}.$$

Thus we obtain

$$\mathrm{index}_c^{S^1,V}(\sigma_{\mathrm{At}})\|_1(\theta) = \frac{-i\theta}{(1-\mathrm{e}^{-i\theta})}\frac{1}{2i\pi}\int_V \mathrm{One}(\kappa)(\theta).$$

As $\frac{(1-\mathrm{e}^{-i\theta})}{-i\theta} = -\int_{-1}^0 \mathrm{e}^{ix\theta}\,dx$, we see that $\frac{(1-\mathrm{e}^{-i\theta})}{-i\theta}(-\sum_{n=1}^\infty \mathrm{e}^{in\theta}) = \int_0^\infty \mathrm{e}^{ix\theta}\,dx$. It remains to show

$$\frac{1}{2i\pi}\int_V \mathrm{One}(\kappa)(\theta) = \int_0^\infty \mathrm{e}^{ir\theta}\,dr. \tag{37}$$

We have $D\kappa(\theta) = \theta(x^2+y^2) + 2dx \wedge dy$. Take a function g on $\mathbb{R}$ with compact support and equal to 1 on a neighborhood of 0. Let $\chi = g(x^2+y^2)$. Then

$$\begin{aligned}\mathrm{One}(\kappa)(\theta) &= \chi - id\chi \wedge \kappa \int_0^\infty \mathrm{e}^{itD\kappa(\theta)}\,dt \\ &= g(x^2+y^2) - 2ig'(x^2+y^2)dx\wedge dy \int_0^\infty (x^2+y^2)\,\mathrm{e}^{i\theta t(x^2+y^2)}\,dt\end{aligned}$$

$$= g(x^2+y^2) - 2ig'(x^2+y^2)dx\wedge dy \int_0^\infty \mathrm{e}^{i\theta t}\, dt.$$

Finally we obtain (37) since $\int_V -2ig'(x^2+y^2)dx\wedge dy = 2i\pi$. This completes the proof. □

4.3.2. *Bott symbols.* — We will check here that the cohomological index satisfies the condition [**N2**]: $\quad \mathrm{index}_c^{O(V),V}(\mathrm{Bott}(V_{\mathbb{C}})) = 1$, for any Euclidean vector space V.

We have explain in Remark 2.8 that it is sufficient to prove [**N2**] for the cases:

- $V=\mathbb{R}$ with the action of the group $O(V)=\mathbb{Z}/2\mathbb{Z}$,
- $V=\mathbb{R}^2$ with the action of the group $SO(V)=S^1$.

Let $V=\mathbb{R}$ with the multiplicative action of $\mathbb{Z}_2$. We have to check that $\mathrm{index}_c^{\mathbb{R},\mathbb{Z}_2}(\mathrm{Bott}(\mathbb{C}))(\epsilon)=1$ for $\epsilon\in\mathbb{Z}_2$. When $\epsilon=1$, we have

$$\mathrm{index}_c^{\mathbb{R},\mathbb{Z}_2}(\mathrm{Bott}(\mathbb{C}))(1) = (2i\pi)^{-1}\int_{\mathbf{T}^*\mathbb{R}} \widehat{\mathrm{A}}(\mathbb{R})^2\,\mathrm{Ch_c}(\mathrm{Bott}(\mathbb{C})).$$

Here $\widehat{\mathrm{A}}(\mathbb{R})^2=1$. We have proved in [**18**] that the class $\mathrm{Ch_c}(\mathrm{Bott}(\mathbb{C}))\in\mathscr{H}_c^2(\mathbf{T}^*\mathbb{R})$ is equal to $2i\pi$ times the Thom form of the oriented vector space of $\mathbb{R}^2\simeq\mathbf{T}^*\mathbb{R}$. Hence $\mathrm{index}_c^{\mathbb{R},\mathbb{Z}_2}(\mathrm{Bott}(\mathbb{C}))(1)=1$. When $\epsilon=-1$, the space $\mathbf{T}^*\mathbb{R}(\epsilon)$ is reduced to a point. We see that $\mathrm{Ch_c}(\mathrm{Bott}(\mathbb{C}),\epsilon)=2$, $D_\epsilon(\mathscr{N})=\det(1-\epsilon)=2$. Then $\mathrm{index}_c^{\mathbb{R},\mathbb{Z}_2}(\mathrm{Bott}(\mathbb{C}))(1) = \frac{\mathrm{Ch_c}(\mathrm{Bott}(\mathbb{C}),\epsilon)}{D_\epsilon(\mathscr{N})}=1$.

Let $V=\mathbb{R}^2$ with the rotation action of S^1. Like before $\mathrm{index}_c^{\mathbb{R}^2,S^1}(\mathrm{Bott}(\mathbb{C}^2))(1)$ is equal to 1 since the Chern class $\mathrm{Ch_c}(\mathrm{Bott}(\mathbb{C}^2))$ is equal to $(2i\pi)^2$ times the Thom form of the oriented vector space of $\mathbb{R}^4\simeq\mathbf{T}^*\mathbb{R}^2$. When $\mathrm{e}^{i\theta}\neq 1$, the space $\mathbf{T}^*\mathbb{R}^2(\mathrm{e}^{i\theta})$ is reduced to a point. We see that $\mathrm{Ch_c}(\mathrm{Bott}(\mathbb{C}),\mathrm{e}^{i\theta}) = D_{\mathrm{e}^{i\theta}}(\mathscr{N}) = 2(1-\cos(\theta))$. Then $\mathrm{index}_c^{\mathbb{R}^2,S^1}(\mathrm{Bott}(\mathbb{C}^2))(\mathrm{e}^{i\theta})=1$.

5. Examples

5.1. Pushed symbols. — Let M be a K-manifold and $N=\mathbf{T}^*M$. Let $\mathcal{E}^\pm\to M$ be two K-equivariant complex vector bundles on M and $\sigma: p^*\mathcal{E}^+\to p^*\mathcal{E}^-$ be a K-equivariant symbol which is supposed to be **invertible exactly outside the zero section** : the set $\mathrm{supp}(\sigma)$ coincides with the zero section of $\mathbf{T}^*M$.

If M is compact, σ defines an elliptic symbol on $\mathbf{T}^*M$, thus a fortiori a transversally elliptic symbol.

Here we assume M non compact. Following Atiyah's strategy [**1**], we can "push" the symbol σ outside the zero section, by means of a K-invariant real 1-form κ on M. This construction provides new transversally elliptic symbols. We recall some definitions of [**17**]:

Definition 5.1. — *Let κ be a real K-invariant 1-form on M. Define $f_\kappa : M \to \mathfrak{k}^*$ by $\langle f_\kappa(x), X\rangle = \langle \kappa(x), V_x X\rangle$. We define the subset C_κ of M by $C_\kappa = f_\kappa^{-1}(0)$. We call C_κ the critical set of κ.*

We define the symbol $\sigma(\kappa)$ on M by

$$\sigma(\kappa)(x,\xi) = \sigma(x, \xi + \kappa(x)), \quad \text{for} \quad (x,\xi) \in \mathbf{T}^*M.$$

Thus $\sigma(\kappa)$ is not invertible at (x,ξ) if and only if $\xi = -\kappa(x)$, and then $(x,\xi) \in \operatorname{supp}(\sigma(\kappa)) \cap \mathbf{T}^*_K M$ if $\xi = -\kappa(x)$ and $\langle \kappa(x), V_x X\rangle = 0$ for all $X \in \mathfrak{k}$. Thus

$$\operatorname{supp}(\sigma(\kappa)) \cap \mathbf{T}^*_K M = \{(x, -\kappa(x)) \mid x \in C_\kappa\}.$$

If C_κ is compact, then the morphism $\sigma(\kappa)$ is transversally elliptic.

Using a K-invariant metric on $\mathbf{T}M$, we can associate to a K-invariant vector field $\mathcal{K}$ on M a K-invariant real 1-form.

Example 5.2. — *Let $S \in \mathfrak{k}$ be a central element of $\mathfrak{k}$ such that the set of zeroes of VS is compact. Then the associated form $\kappa_S(\bullet) = \langle VS, \bullet\rangle$ is a K-invariant real 1-form such that C_{κ_S} is compact. Indeed the value of f_{κ_S} on S is $\|VS\|^2$, so that the set C_{κ_S} coincides with the fixed point set $M(S)$.*

Definition 5.3. — *If κ is a K-invariant real 1-form on M such that C_κ is compact, the transversally elliptic symbol*

$$\sigma(\kappa)(x,\xi) = \sigma(x, \xi + \kappa(x))$$

is called the pushed symbol of σ by κ.

Example 5.4. — *The Atiyah symbol is a pushed symbol defined on $M = \mathbb{R}^2$ (see Subsection 4.3.1).*

We construct as in (15) the K-equivariant differential form

$$\beta(\kappa)(X) = -i\kappa \wedge \int_0^\infty \mathrm{e}^{itD\kappa(X)}\, dt$$

which is defined on $M \setminus C_\kappa$. We choose a compactly supported function χ_κ on M identically 1 near C_κ. Then the K-equivariant form

$$\operatorname{One}(\kappa)(X) = \chi_\kappa + d\chi_\kappa \beta(\kappa)(X)$$

defined a class in $\mathcal{H}_c^{-\infty}(\mathfrak{k}, M)$.

The K-equivariant form $\operatorname{One}(\kappa)$ is congruent to 1 in cohomology without support conditions. Indeed one verify that $\operatorname{One}(\kappa) = 1 + D((\chi_\kappa - 1)\beta(\kappa))$.

Let $p : \mathbf{T}^*M \to M$ be the projection. We can multiply the K-equivariant form $p^*\operatorname{One}(\kappa)(X)$ with $C^{-\infty}$-coefficients by the K-equivariant form $\operatorname{Ch}_{\sup}(\sigma)(X)$. In this way, we obtain a K-equivariant form with compact support on $\mathbf{T}^*M$.

Proposition 5.5. — *The K-equivariant form* $\mathrm{Ch}_{\mathrm{sup}}(\sigma)p^*\mathrm{One}(\kappa)$ *represents the class* $\mathrm{Ch}_{\mathrm{c}}(\sigma(\kappa),\omega)$ *in* $\mathcal{H}_c^{-\infty}(\mathfrak{k},\mathbf{T}^*M)$.

Proof. — By definition the class $\mathrm{Ch}_{\mathrm{c}}(\sigma(\kappa),\omega)$ is represented by the product $\mathrm{Ch}_{\mathrm{sup}}(\sigma(\kappa))\mathrm{One}(\omega)$. We first prove $\mathrm{Ch}_{\mathrm{sup}}(\sigma(\kappa))\mathrm{One}(\omega) = \mathrm{Ch}_{\mathrm{sup}}(\sigma(\kappa))\mathrm{One}(p^*\kappa)$ in $\mathcal{H}_c^{-\infty}(\mathfrak{k},\mathbf{T}^*M)$.

Indeed if $(x,\xi)\in\mathrm{supp}(\sigma(\kappa))$, then $\xi=-\kappa(x)$. Thus $\langle\omega(x,\xi),v\rangle = -\langle\xi,p_*v\rangle = \langle\kappa(x),p_*v\rangle$ where v is any tangent vector at $(x,\xi)\in\mathbf{T}^*M$. So the 1-forms ω and $p^*\kappa$ coincides on the support of $\sigma(\kappa)$. Thus $\mathrm{Ch}_{\mathrm{sup}}(\sigma(\kappa))\mathrm{One}(\omega) = \mathrm{Ch}_{\mathrm{sup}}(\sigma(\kappa))p^*\mathrm{One}(\kappa)$ as consequence of ([**17**], Corollary 3.12).

Let us prove now that $\mathrm{Ch}_{\mathrm{sup}}(\sigma(\kappa))p^*\mathrm{One}(\kappa) = \mathrm{Ch}_{\mathrm{sup}}(\sigma)p^*\mathrm{One}(\kappa)$. Consider the family of symbols on M defined by $\sigma_t(x,\xi)=\sigma(x,\xi+t\kappa(x))$ for $t\in[0,1]$: we have $\sigma_0=\sigma$ and $\sigma_1=\sigma(\kappa)$.

On a compact neighborhood $\mathcal{U}$ of C_κ, the support of σ_t stays in the compact set $\{(x,\xi)\,:\,x\in\mathcal{U},\xi=-t\kappa(x)\}$ when t varies between 0 and 1. It follows from ([**17**], Theorem 3.11) that all the classes $\mathrm{Ch}_{\mathrm{sup}}(\sigma_t)p^*\mathrm{One}(\kappa)$, $t\in[0,1]$ coincides in $\mathcal{H}_c^{-\infty}(\mathfrak{k},\mathbf{T}^*M)$. □

Similarly for any $s\in K$, we consider the restriction κ_s of the form κ to $M(s)$. We finally obtain the following formula:

Theorem 5.6. — *For any $s\in K$ and $X\in\mathfrak{k}(s)$ small, the cohomological index* $\mathrm{index}_c^{K,M}([\sigma(\kappa)])\|_s(Y)$ *is given on $\mathfrak{k}(s)$ by the integral formula:*

$$\int_{\mathbf{T}^*M(s)} \Lambda_s(Y)\ \mathrm{Ch}_{\mathrm{sup}}(\sigma,s)(Y)\,\mathrm{One}(\kappa_s)(Y).$$

In particular, when $s=e$ we get

$$\mathrm{index}_c^{K,M}([\sigma(\kappa)])\|_e(X) = (2i\pi)^{-\dim M}\int_{\mathbf{T}^*M}\widehat{\mathrm{A}}(M)^2(X)\ \mathrm{Ch}_{\mathrm{sup}}(\sigma)(X)\,\mathrm{One}(\kappa)(X).$$

An interesting situation is when the manifold M is oriented, and is equipped with a K-invariant Spin structure. Let $\mathcal{S}_M\to M$ be the corresponding spinor bundle. We associate to any K-equivariant complex vector bundle $E\to M$ the K-invariant symbol $\sigma_{spin}^E : p^*(\mathcal{S}_M^+\otimes E)\to p^*(\mathcal{S}_M^-\otimes E)$: its support is exactly the zero section of the cotangent bundle. For any invariant 1-form κ on M such that C_κ is compact we consider the transversally elliptic symbol $\sigma_{spin}^E(\kappa)$.

We have proved in [**18**] that

$$\mathrm{Ch}_{\mathrm{sup}}(\sigma_{spin}^E)(X) = (2i\pi)^{\dim M}\widehat{\mathrm{A}}(M)^{-1}(X)\ \mathrm{Ch}(E)(X)\,\mathrm{Thom}(\mathbf{T}^*M)(X).$$

Hence Theorem 5.6 tells us that

$$\mathrm{index}_c^{K,M}([\sigma_{spin}^E(\kappa)])\|_e(X) = \int_M \widehat{\mathrm{A}}(M)(X)\ \mathrm{Ch}(E)(X)\,\mathrm{One}(\kappa)(X).$$

5.2. Contact manifolds. — The following geometric example is taken from **[11]**.

Let M be a compact manifold of dimension $2n+1$. Suppose that M carries a *contact* 1-form α ; that is, $E = \ker(\alpha)$ is a hyperplane distribution of $\mathbf{T}M$, and the restriction of the 2-form $d\alpha$ to E is symplectic. In this context, the Reeb vector field $\mathbf{Y}$ is uniquely determined by the conditions $\alpha(\mathbf{Y}) = 1$ and $\mathscr{L}(\mathbf{Y})\alpha = 0$. We have then canonical decompositions $\mathbf{T}M = E \oplus \mathbb{R}\mathbf{Y}$ and $\mathbf{T}^*M = E^* \oplus E^0$ with $E^0 = \mathbb{R}\alpha$.

Let J be a K-invariant complex structure on the bundle E which is compatible with the symplectic structure $d\alpha$. We equipped the bundle E^* with the complex structure J^* defined by $J^*(\xi) := \xi \circ J$ for any cotangent vector ξ. We note that the complex bundle (E^*, J^*) is the complex dual of the vector bundle (E, J).

We consider the $\mathbb{Z}_2$-graded complex vector bundle $\mathscr{E} := \wedge_{J^*} E^*$. The Clifford action defines a bundle map $\mathbf{c} : E^* \to \mathrm{End}_{\mathbb{C}}(\mathscr{E})$. We consider now the symbol on M

$$\sigma_b : p^*(\mathscr{E}^+) \to p^*(\mathscr{E}^-)$$

defined by $\sigma_b(x,\xi) = \mathbf{c}(\xi')$ where ξ' is the projection of $\xi \in \mathbf{T}^*M$ on E^*.

We see that the support of σ_b is equal to $E^0 \subset \mathbf{T}^*M$: σ_b is not an elliptic symbol.

Let K be a compact Lie group acting on M, which leaves α invariant. Then E, E^* are K-equivariant complex vector bundles, and the complex struture J can be chosen K-invariant. The morphism σ_b is then K-equivariant.

We suppose for the rest of this section that

$$E^0 \cap \mathbf{T}^*_K M = \text{ zero section of } \mathbf{T}^*M. \tag{38}$$

It means that for any $x \in M$, the map $f_\alpha(x) : X \mapsto \alpha_x(V_x X)$ is not the zero map. Under this hypothesis the symbol σ_b is transversally elliptic.

Under the hypothesis (38), we can define the following closed equivariant form on M with $\mathscr{C}^{-\infty}$-coefficients

$$\mathscr{J}_\alpha(X) := \alpha \int_{\mathbb{R}} \mathrm{e}^{itD\alpha(X)}\, dt.$$

For any $\varphi \in \mathscr{C}^\infty_c(\mathfrak{k})$, the expression $\int_{\mathfrak{k}} \mathscr{J}_\alpha(X)\varphi(X)dX := \alpha \int_{\mathbb{R}} \mathrm{e}^{itd\alpha}\,\widehat{\varphi}(tf_\alpha)dt$ is a well defined differential form on M since the map $f_\alpha : M \to \mathfrak{k}^*$ as an empty 0-level set.

Let $\mathrm{Todd}(E)(X)$ be the equivariant Todd class of the complex vector bundle (E, J). We have the following

Theorem 5.7 **([11])**. — *For any $X \in \mathfrak{k}$ sufficiently small,*

$$\mathrm{index}_c^{K,M}([\sigma_b])\|_e(X) = (2i\pi)^{-n} \int_M \mathrm{Todd}(E)(X)\mathscr{J}_\alpha(X).$$

Proof. — Consider the equivariant form with compact support $\mathrm{Ch}_{\sup}(\sigma_b)\mathrm{One}(\omega)$. The Chern form $\mathrm{Ch}_{\sup}(\sigma_b)$ attached to the complex vector bundle E^* is computed in **[17]** as follows. Let $\mathrm{Thom}(E^*)(X)$ be the equivariant Thom form, and let $\mathrm{Todd}(E^*)(X)$ be the equivariant Todd form. We have proved in **[17]**, that

$$\mathrm{Ch}_{\sup}(\sigma_b) = (2i\pi)^n \,\mathrm{Todd}(E^*)(X)^{-1}\mathrm{Thom}(E^*)(X). \tag{39}$$

Let $[\mathbb{R}]$ be the trivial vector bundle over M. We work with the isomorphism $E^* \oplus [\mathbb{R}] \simeq \mathbf{T}^*M$ who sends (x, ξ, t) to $(x, \xi + t\alpha(x))$. We consider the invariant 1-form λ on $E^* \oplus [\mathbb{R}] \simeq \mathbf{T}^*M$ defined by

$$\lambda = -\underline{t}\; p^*(\alpha)$$

Here $p : E^* \oplus [\mathbb{R}] \to M$ is the projection, and $\underline{t}$ denotes the function that sends (x, ξ, t) to t.

It is easy to check that the form λ and the Liouville 1-form ω are equal on the support of σ_b. Thus

$$\mathrm{Ch}_{\sup}(\sigma_b)\mathrm{One}(\omega) = \mathrm{Ch}_{\sup}(\sigma_b)\mathrm{One}(\lambda) \quad \text{in} \quad \mathscr{H}_c^{-\infty}(\mathfrak{k}, \mathbf{T}^*M),$$

as consequence of **[17]**, Corollary 3.12. We have then

$$\mathrm{index}_c^{K,M}([\sigma_b])\|_e(X) = (2i\pi)^{-\dim M} \int_{\mathbf{T}^*M} \widehat{\mathrm{A}}(M)^2(X)\, \mathrm{Ch}_{\sup}(\sigma_b)(X)\mathrm{One}(\lambda)(X).$$

The integral of $\mathrm{Ch}_{\sup}(\sigma_b)(X)\mathrm{One}(\lambda)(X)$ on the fibers of $\mathbf{T}^*M$ is then equal to the product

$$\left(\int_{E^*\, fiber} \mathrm{Ch}_{\sup}(\sigma_b)(X)\right)\left(\int_{\mathbb{R}} \mathrm{One}(\lambda)(X)\right)$$

If we uses (39), we see that the integral $\int_{E^*\, fiber} \mathrm{Ch}_{\sup}(\sigma_b)(X)$ is equal to $(2i\pi)^n \mathrm{Todd}(E^*)(X)^{-1}$. A small computation gives that $\int_{\mathbb{R}} \mathrm{One}(\lambda)(X)$ is equal to $(2i\pi)\mathscr{J}_\alpha(X)$. The proof is now completed since $\widehat{\mathrm{A}}(M)^2(X)\mathrm{Todd}(E^*)(X)^{-1} = \mathrm{Todd}(E)(X)$. □

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Astérisque
328, 2009, p. 339–354

CM STABILITY AND THE GENERALIZED FUTAKI INVARIANT II

by

Sean Timothy Paul & Gang Tian

Dedicated to Jean-Michel Bismut on the occasion of his 60^{th} birthday

Abstract. — The Mabuchi K-energy map is exhibited as a singular metric on the refined CM polarization of any equivariant family $\mathbf{X} \xrightarrow{p} S$. Consequently we show that the generalized Futaki invariant is the leading term in the asymptotics of the reduced K-energy of the generic fiber of the map p. Properness of the K-energy implies that the generalized Futaki invariant is strictly negative.

***Résumé* (CM-stabilité et invariant de Futaki généralisé II).** — On interpréte la K-énergie de Mabuchi comme une métrique singulière sur la CM-polarisation raffinée d'une famille équivariante $\mathbf{X} \xrightarrow{p} S$. Nous montrons que l'invariant de Futaki généralisé est le terme principal de l'asymptotique de la K-énergie réduite de la fibre générique de l'application p. Si la K-énergie est propre, alors l'invariant de Futaki généralisé est strictement négatif.

1. Introduction

1.1. Statement of results. — Throughout this paper $\mathbf{X}$ and S denote smooth, proper complex projective varieties satisfying the following conditions.

1. $\mathbf{X} \subset S \times \mathbb{P}^N$; $\mathbb{P}^N$ denotes the complex projective space of *lines* in $\mathbb{C}^{N+1}$.
2. $p := p_1 : \mathbf{X} \to S$ is flat of relative dimension n, degree d with Hilbert polynomial P.
3. $L|_{\mathbf{X}_z}$ is very ample and the embedding $\mathbf{X}_z := p_1^{-1}(z) \overset{L}{\hookrightarrow} \mathbb{P}^N$ is given by a complete linear system for $z \in S$.

2010 Mathematics Subject Classification. — Primary: 53C55; Secondary: 14D06.
Key words and phrases. — Futaki Invariant, Stability, Mabuchi Energy, CM polarization.

Support in part by a National Science foundation Grant Support in part by a National Science foundation Grant.

4. There is an action of $G := SL(N+1, \mathbb{C})$ on the data compatible with the projection and the standard action on $\mathbb{P}^N$.

It is well known that (1) and (3) imply that

$$\mathbb{P}(p_{1*}L) \cong S \times \mathbb{P}^N. \tag{1.1}$$

Which in turn is equivalent to the existence of a line bundle $\mathscr{C}$ on S such that

$$p_{1*}L \cong \underbrace{\bigoplus \mathscr{C}}_{N+1}. \tag{1.2}$$

Below Chow$(\mathbf{X}/S)$ denotes the Chow form of the family $\mathbf{X}/S$, μ is the coefficient of k^{n-1} in $P(k)$, and $\mathscr{M}_n$ is the coefficient of $\binom{m}{n}$ in the CGKM expansion of $\det(p_{1*}L^{\otimes m})$ for $m >> 0$. A complete discussion of these notions is given in "*CM Stability and the Generalized Futaki Invariant I*". We refer the reader to that paper for the basic definitions and constructions that are used in the present article.

We define an invertible sheaf on S as follows.

***Definition 1* (The Refined CM polarization[(1)])**. — *We have*

$$\mathbb{L}_1(\mathbf{X}/S) := \{\mathrm{Chow}(\mathbf{X}/S) \otimes \mathscr{C}^{d(n+1)}\}^{n(n+1)+\mu} \otimes \mathscr{M}_n^{-2(n+1)} \tag{1.3}$$

With the family $p_1 : \mathbf{X} \to S$ fixed throughout, we will denote $\mathbb{L}_1(\mathbf{X}/S)$ by $\mathbb{L}_1$ in the remainder of the paper.

Our first result exhibits the Mabuchi energy as a *singular* Hermitian metric on $\mathbb{L}_1$.

Theorem 1. — *Let $|| \ ||$ be any smooth Hermitian metric on $\mathbb{L}_1^{-1}$.[(2)] Then there is a continuous function $\Psi_S : S \setminus \Delta \to (-\infty, c)$ such that for all $z \in S/\Delta$*

$$d(n+1)\nu_{\omega|_{\mathbf{X}_z}}(\varphi_\sigma) = \log\left(e^{(n+1)\Psi_S(\sigma z)}\frac{|| \ ||^2(\sigma z)}{|| \ ||^2(z)}\right). \tag{1.4}$$

Here c denotes a constant which depends only on the choice of background Kähler metrics on S and $\mathbf{X}$, Δ denotes the discriminant locus of the map p_1, and $\omega|_{\mathbf{X}_z}$ denotes the restriction of the Fubini Study form of $\mathbb{P}^N$ to the fiber $\mathbf{X}_z$.

Remark 1. — *This should be compared with the main result in Section 8 of* **[17]**. *The principal contribution of our present work is the observation that the whole theory in Section 8 of* **[17]** *should be recast from the beginning with the sheaf $\mathbb{L}_1$.*

Let $X \hookrightarrow \mathbb{P}^N$ be an n dimensional projective variety with Hilbert polynomial P. Let $Hilb_m(X)$ denote the mth Hilbert point of X (see **[12]** for further information). If λ is a one parameter subgroup of G then it is known (see **[12]**) that the weight,

(1) We use this terminology in order to distinguish this sheaf from one introduced by the second author in (**[17]**).
(2) $\mathbb{L}_1^{-1}$ denotes the dual of $\mathbb{L}_1$.

$w_\lambda(m)$, of $Hilb_m(X)$ with respect to λ is a *polynomial* in m of degree at most $n+1$. That is,

$$w_\lambda(m) = a_{n+1}(\lambda)m^{n+1} + a_n(\lambda)m^n + \cdots$$

Then the ratio may be expanded as follows.

$$\frac{w_\lambda(m)}{mP(m)} = F_0(\lambda) + F_1(\lambda)\frac{1}{m} + \cdots + F_l(\lambda)\frac{1}{m^l} + \ldots$$

***Definition 2* (Donaldson ([5]))**. — *$F_1(\lambda)$ is the generalized Futaki invariant of X with respect to λ.*

In our previous paper we have shown the following.

***Theorem* (The weight of the Refined CM polarization)**. — i) *There is a natural G linearization on the line bundle $\mathbb{L}_1$.*

ii) *Let λ be a one parameter subgroup of G. Let $z \in \mathfrak{Hilb}^P_{\mathbb{P}^N}(\mathbb{C})$. Let $w_\lambda(z)$ denote the weight of the restricted $\mathbb{C}^*$ action (whose existence is asserted in i)) on $\mathbb{L}_1^{-1}|_{z_0}$ where $z_0 = \lambda(0)z$. Then*

$$w_\lambda(z) = F_1(\lambda). \tag{1.5}$$

The main result of the paper is the following corollary of (1.4) and (1.5).

***Corollary 1* (Algebraic asymptotics of the Mabuchi energy)**. — *Let $\varphi_{\lambda(t)}$ be the Bergman potential associated to an algebraic 1psg λ of G, and let $z \in S \setminus \Delta$. Then there is an asymptotic expansion*

$$d(n+1)\nu_{\omega|_{\mathbf{X}_z}}(\varphi_{\lambda(t)}) - \Psi_S(\lambda(t)) = F_1(\lambda)\log(|t|^2) + O(1) \ \text{as} \ |t| \to 0. \tag{1.6}$$

Moreover $\Psi_S(\lambda(t)) = \psi(\lambda)\log(|t|^2) + O(1)$ where $\psi(\lambda) \in \mathbb{Q}_{\geq 0}$. Moreover, $\psi(\lambda) \in \mathbb{Q}_+$ if and only if $\lambda(0)\mathbf{X}_z = \mathbf{X}_{\lambda(0)z}$ (the limit cycle[(3)] of $\mathbf{X}_z$ under λ) has a component of multiplicity greater than one. Here $O(1)$ denotes any quantity which is bounded as $|t| \to 0$.

Moser iteration and a refined Sobolev inequality (see **[11]** and **[7]**) yield the following.

Corollary 2. — *If $\nu_{\omega|_{\mathbf{X}_z}}$ is proper (bounded from below) then the generalized Futaki invariant of $\mathbf{X}_z$ is strictly negative (nonnegative) for all $\lambda \in G$.*

Remark 2. — *We call the left hand side of (1.6) the* reduced *K-Energy along λ. We also point out that while it is certainly the case that $F_1(\lambda)$ may be defined for any subscheme of $\mathbb{P}^N$ it evidently only controls the behavior of the K-Energy when $\lambda(0)\mathbf{X}_z$ is reduced.*

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(3) See **[12]** pg. 61.

Remark 3. — *The precise constant $d(n+1)$ in front of ν_ω is not really crucial, since what really matters is the sign* of $F_1(\lambda)+\psi(\lambda)$. *That $\Psi_S(\lambda(t))$ has logarithmic singularities can be deduced from* **[13]**.

Remark 4. — *We emphasize that we do not assume the limit cycle is smooth.*

2. Background and Motivation

Let (X,ω) be a compact Kähler manifold (ω not necessarily a Hodge class) and $P(X,\omega) := \{\varphi \in C^\infty(X) : \omega_\varphi := \omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi > 0\}$ the space of Kähler potentials. This is the usual description of all Kähler metrics in the same class as ω (up to translations by constants). It is not an overstatement to say that the most basic problem in Kähler geometry is the following

$$\textit{Does there exist } \varphi \in P(X,\omega) \textit{ such that } \operatorname{Scal}(\omega_\varphi) \equiv \mu? \ (*)$$

This is a fully nonlinear *fourth order* elliptic partial differential equation for φ. μ is a constant, the average of the scalar curvature, it depends only on $c_1(X)$ and $[\omega]$. When $c_1(X) > 0$ and ω represents the *anticanonical* class a simple application of the Hodge Theory shows that $(*)$ is equivalent to the *Monge-Ampere equation.*

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{F-\kappa\varphi} \quad (\kappa = 1) \qquad (**)$$

where F denotes the Ricci potential. When $\kappa = 0$ this is the celebrated Calabi problem solved by S.T.Yau and when $\kappa < 0$ this was solved by Aubin and Yau independently in the 70's. It is well known that $(*)$ is actually a *variational* problem. There is a natural energy on the space $P(X,\omega)$ whose critical points are those φ such that ω_φ has constant scalar curvature (csc). This energy was introduced by T. Mabuchi (**[10]**) in the 1980's. It is called the *K-Energy map* (denoted by ν_ω) and is given by the following formula

$$\nu_\omega(\varphi) := -\frac{1}{V}\int_0^1\int_X \dot{\varphi}_t(\operatorname{Scal}(\varphi_t) - \mu)\omega_t^n\, dt.$$

Above, φ_t is a smooth path in $P(X,\omega)$ joining 0 with φ. The K-Energy does not depend on the path chosen. In fact there is the following well known formula for ν_ω where $O(1)$ denotes a quantity which is bounded on $P(X,\omega)$.

$$\nu_\omega(\varphi) = \int_X \log\left(\frac{{\omega_\varphi}^n}{\omega^n}\right)\frac{{\omega_\varphi}^n}{V} - \mu(I_\omega(\varphi) - J_\omega(\varphi)) + O(1)$$

$$J_\omega(\varphi) := \frac{1}{V}\int_X \sum_{i=0}^{n-1}\frac{\sqrt{-1}}{2\pi}\frac{i+1}{n+1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i\wedge{\omega_\varphi}^{n-i-1}$$

$$I_\omega(\varphi) := \frac{1}{V}\int_X \varphi(\omega^n - {\omega_\varphi}^n).$$

We have written down the K-energy in the case when $\omega = c_1(X)$. Observe that ν_ω is essentially the *difference* of two positive terms. What is of interest for us is that the problem $(*)$ is not only a variational problem but a *minimization* problem. With this said we have the following fundamental result.

***Theorem* (S. Bando and T. Mabuchi [1]).** — *If $\omega = c_1(X)$ admits a Kähler Einstein metric then $\nu_\omega \geq 0$. The absolute minimum is taken on the solution to $(**)$ (which is unique up to automorphisms of X).*

Therefore a *necessary* condition for the existence of a Kähler Einstien metric is a bound from below on ν_ω. In order to get a *sufficient* condition one requires that the K-energy *grow* at a certain rate. Precisely, it is required that the K-Energy be *proper.* This concept was introduced by the second author in [**17**].

Definition 3. — *ν_ω is **proper** if there exists a strictly increasing function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ (where $\lim_{T\longrightarrow\infty} f(T) = \infty$) such that $\nu_\omega(\varphi) \geq f(J_\omega(\varphi))$ for all $\varphi \in P(M, \omega)$.*

***Theorem* ([17])**. — *Assume that $Aut(X)$ is discrete. Then $\omega = c_1(X)$ admits a Kähler Einstein metric if and only if ν_ω is proper.*

The next result was established by the second author and Xiuxiong Chen. It holds in an *arbitrary* Kähler class ω. An alternative proof of this was given by Donaldson for polarized projective manifolds.

***Theorem* ([3])**. — *If ω admits a metric of csc then $\nu_\omega \geq 0$.*

In this paper our interest is to test for a lower bound of ν_ω along the large but finite dimensional group G of *matrices* in the polarized case. When we restrict our attention to G we make the connection with Mumfords' Geometric Invariant Theory. The past couple of years have witnessed quite a bit of activity on this problem due to this connection.

To put things in historical perspective consider the various formulations of the Futaki invariant.

i) 1983 Futaki ([**6**]) introduces his invariant as a lie algebra character on a Fano manifold X

$$F_\omega : \eta(X) \longrightarrow \mathbb{C}.$$

ii) 1986 Mabuchi (see [**10**]) integrates the Futaki invariant with the introduction of the K-energy map. The linearization of the K-energy along orbits of holomorphic vector fields is the real part of the Futaki invariant.

iii) 1992 Ding and Tian ([**4**]) introduced the *generalized* Futaki invariant. Here the *jumping of complex structures* is introduced. The limit of the derivative of the K-Energy map is identified with the generalized Futaki invariant of $X^{\lambda(0)}$ provided this limit has at most *normal* singularities.

iv) 1997 The CM polarization is defined (see [**17**]) for *smooth* families, as the relative canonical bundle is explicitly involved in the definition. K-Stability is defined in terms of special degenerations and the generalized Futaki invariant.

v) 1999 Yotov formulated the generalized Futaki Invariant in terms of equivariant Chow groups of a *normal* variety.

vi) 2002 For an *arbitrary scheme* Donaldson ([**5**]) defined the weight $F_1(\lambda)$. This is identified with the limit of the derivative of K-energy (by [**4**]) when the limit cycle is a *smooth* (or normal) scheme.

Remark 5. — *We hope that we have clarified the role of the CM polarization. The main point is that once the CM polarization is extended to the Hilbert scheme (*[**14**]*) the polarization computes the precise asymptotics of the K-energy of any generic fiber of the map* $\mathbf{X} \to S$*. This extension was made possible by an application of the Knudsen Mumford expansion of the determinant of direct images of perfect complexes of sheaves (see* [**8**]*). In fact,* $\psi(\lambda)$ *already appeared in work of the second author (see* [**17**]*). Despite this, the role of* $\psi(\lambda)$ *becomes more precise in the present work.*

3. Algebraic potentials

In order to connect these notions to the K-Energy map we now give an account of how to associate an admissible potential $\varphi_{\lambda(t)}$ to a one parameter subgroup of G. In order to detect properness (conjecturally) one restricts attention to the subspace of *Bergman metrics* inside $P(M,\omega)$ since these metrics are *dense* in $P(M,\omega)$ (see [**16**], [**15**], [**19**], [**2**]). By definition these metrics are induced by the Kodaira embeddings furnished by the polarization L. The construction is as follows. We have an embedding

$$X \xrightarrow[L]{} \mathbb{P}(H^0(X,L)^*) = \mathbb{P}^N$$

furnished by some basis $\{S_0,\ldots,S_N\}$ of $H^0(X,L)$. Observe that with the natural Hermitian metric on $H^0(X,L)$, the induced Fubini-Study metric on $\mathbb{P}^N$ is related to the curvature of the initial metric on L by the formula

$$\omega_{FS}|_X = \omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(\sum_{i=0}^{N}||S_i||^2\right).$$

We conclude that

$$\log\left(\sum_{i=0}^{N}||S_i||^2\right) \in P(X,\omega).$$

Let $\sigma \in SL(N+1,\mathbb{C})$, then

$$\sigma^*(\omega_{FS}) = \omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_\sigma.$$

Where φ_σ is given by the formula

$$\varphi_\sigma = \log\left(\frac{||\sigma z||^2}{||z||^2}\right).$$

We let $\{T_0, \dots T_N\}$ denote the corresponding change of basis

$$\begin{pmatrix} \sigma_{00} & \dots & \sigma_{0N} \\ \sigma_{10} & \dots & \sigma_{1N} \\ \dots & \dots & \dots \\ \sigma_{N0} & \dots & \sigma_{NN} \end{pmatrix} \begin{pmatrix} S_0 \\ \dots \\ \dots \\ S_N \end{pmatrix} = \begin{pmatrix} T_0 \\ \dots \\ \dots \\ T_N \end{pmatrix}.$$

Then we have

$$\varphi_\sigma|_X = \log\left(\frac{\sum_{i=0}^N ||T_i||^2}{\sum_{i=0}^N ||S_i||^2}\right).$$

Putting these ingredients together gives

$$\sigma^*\omega_{FS}|_X = \omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(\sum_{i=0}^N ||T_i||^2\right). \tag{3.1}$$

Therefore, if we *fix* a basis of $H^0(X, L)$ we get a natural map

$$SL(N+1, \mathbb{C}) \to P(X, \omega).$$

A *one parameter subgroup* of $SL(N+1, \mathbb{C})$ is an algebraic(4) homomorphism

$$\lambda : \mathbb{C}^* \to SL(N+1, \mathbb{C}).$$

Any such $\lambda(t)$ can be diagonalised. That is, we may assume that $\lambda(t)$ takes values in the standard maximal torus $H \cong (\mathbb{C}^*)^N$ of $SL(N+1, \mathbb{C})$.

$$\lambda(t) = \begin{pmatrix} t^{m_0} & \dots & \dots & 0 \\ 0 & t^{m_1} & \dots & 0 \\ 0 & \dots & \dots & t^{m_N} \end{pmatrix}.$$

The exponents m_i satisfy

$$\sum_{0 \le i \le N} m_i = 0.$$

We arrive at the following formula.

$$\varphi_{\lambda(t)}(z) := \log\left(\sum_{0\le j\le N} |t|^{2m_j}||S_j||^2(z)\right).$$

Now we may consider the K-energy map as a function on $SL(N+1, \mathbb{C})$.

(4) "Algebraic" means that the matrix coefficients $\lambda(t)_{i,j} \in \mathbb{C}[t, t^{-1}]$.

4. Singular Hermitian metrics

Proof of Theorem 1. — In part I of this work the authors provided the following formula for the first Chern class of $\mathbb{L}_1$.

(4.1)
$$c_1(\mathbb{L}_1) = p_{1*}\left((n+1)c_1(K_{\mathbf{X}/S})c_1(L)^n + \mu c_1(L)^{n+1}\right) \quad K_{\mathbf{X}/S} := K_{\mathbf{X}} \otimes p_1^*(K_S^\vee).$$

(4.1) allows us to exhibit the K-energy map as a *singular* metric on the CM polarization (see [**17**]). Recall that $p^{-1}(z) = \mathbf{X}_z \subset \mathbb{P}^N$, where $z \in S_\infty := S \setminus \Delta$. We define

$$G\mathbf{X}_z := \{(\sigma, y) \in G \times \mathbb{P}^N : y \in \sigma\mathbf{X}_z\}.$$

Observe that $G\mathbf{X}_z$ is biholomorphic to $G \times \mathbf{X}_z$. Then we have the following diagram, where p_z denotes the evaluation map, i.e. $p_z(\sigma) := \sigma z$.

$$\begin{tikzcd}
p_z^*(\mathbf{X}) \cong G\mathbf{X}_z \arrow[r, "p_{z,2}"] \arrow[d, "p_{z,1}"'] & \mathbf{X} \arrow[r, hook, "\iota"] \arrow[d, "p"'] & S \times \mathbb{P}^N \arrow[r, "p_2"] \arrow[dl, "\pi"] & \mathbb{P}^N \\
G \arrow[r, "p_z"] & S & &
\end{tikzcd}$$

Given $z \in B \setminus \Delta$ we can consider $K_{\mathbf{X}_z}$, the canonical bundle of the fiber $\mathbf{X}_z$. These fit together holomorphically into a line bundle K_∞ on $\mathbf{X} \setminus p^{-1}(\Delta)$. On the other hand, the relative canonical bundle K_p of the map p exists and lives on *all* of $\mathbf{X}$.

$$K_p := K_{\mathbf{X}} \otimes p^* K_S^{-1}$$

When we restrict this sheaf to $\mathbf{X} \setminus p^{-1}(\Delta)$ we have an isomorphism

$$K_p \cong K_\infty.$$

$\iota^* p_2^* \omega_{FS}$ restricts to a Kähler metric on $p^{-1}(z)$ ($z \in S_\infty$) and hence induces a Hermitian metric on the bundle K_∞. We denote its curvature by $R(\iota^* p_2^*(\omega_{FS}))$. Let $g_{\mathbf{X}}$ and g_S denote two Kähler metrics on $\mathbf{X}$ and S respectively. In this way we obtain a metric on the relative canonical bundle K_p. We let R_f denote its curvature

$$R_p := R(g_{\mathbf{X}}) - p^* R(g_S).$$

In this way we obtain *two* metrics on the relative canonical bundle over the smooth locus. The crucial point is the following fact.

The curvatures of these metrics are not the same.

The relation between them is given in the following proposition (see [**17**] Lemma 8.5 pg. 31).

***Proposition 1* ("$\partial\overline{\partial}$ lemma along the fibers").** — *There is a smooth function $\Psi : \mathbf{X} \setminus p^{-1}(\Delta) \to \mathbb{R}$ such that*

1) $R(g_{\mathbf{X}}) - p^*R(g_S) + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\Psi = R(\iota^* p_2^*(\omega_{FS}))$;

2) $\Psi \leq C$, for some constant C.

Example 1. — *(The universal family of hypersurfaces of degree d in $\mathbb{C}P^{n+1}$)*

$$\begin{aligned} S &:= \mathbb{P}(H^0(\mathbb{C}P^{n+1}, \mathcal{O}(d))) \\ \mathbf{X} &:= \{([f],[z]) \in S \times \mathbb{C}P^{n+1} \ | f(z) = 0\} \\ p &:= p_1 \quad (\textit{projection onto the first factor}). \end{aligned}$$

Let $|||.|||$ denote any norm on $H^0(\mathbb{C}P^{n+1}, \mathcal{O}(d))$, with associated Fubini-Study metric ω_S. Then a computation shows that

$$\Psi(([f],[z])) = \log\left(\frac{\sum_{i=0}^{n+1} |\frac{\partial f}{\partial z_i}(z)|^2}{|||f|||^2 ||z||^{2(d-1)}}\right).$$

The next result is a *pointwise* version of (4.1).

Proposition 2. — *There is a continuous Hermitian metric $||\ ||$ on $\mathbb{L}_1^{-1}$ such that, in the sense of currents we have*

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(||\ ||^2) = (n+1)p_*(R(g_{\mathbf{X}}) - p^*R(g_S))p_2^*(\omega_{FS})^n + \mu p_* p_2^*(\omega_{FS})^{n+1}.$$

Proof. — See Proposition 4.3 pg. 2576 of **[13]**. □

Now we pull back the curvature form of K_∞ to $G\mathbf{X}_z$

$$R_{G|\mathbf{X}_z} := p_{z,2}^*(R(\pi_2^*(\omega_{FS}))).$$

Recall that for $\sigma \in G$ we define φ_σ by the relation

$$\sigma^*\omega_{FS} = \omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_\sigma.$$

Let $\nu_{\omega,z}(\sigma)$ denote the K energy of $(\mathbf{X}_z, \omega_{FS})$ applied to the potential φ_σ. With these notations in place we have the following result.

***Proposition 3* (The complex Hessian of the K-Energy map on G)**

For every smooth compactly supported (N^2+2N-1, N^2+2N-1) form η on G we have

$$d(n+1)\int_G \nu_{\omega,z}(\varphi_\sigma)\partial\overline{\partial}\eta = \int_{G\mathbf{X}_z} ((n+1)R_{G|\mathbf{X}_z} + \mu p_2^*(\omega_{FS})) \wedge p_2^*(\omega_{FS})^n \wedge p_{z,1}^*\eta.$$

The proof of Proposition 3 appears in the next section after some standard preliminaries on Bott Chern classes.

4.1. Bott Chern secondary classes. — Let ϕ be a $GL_N(\mathbb{C})$ invariant polynomial on $M_{N\times N}(\mathbb{C})$ homogeneous of degree d. ϕ_1 denotes the *complete polarization* of ϕ. Let E be a holomorphic vector bundle of rank N over a base X. Let h_1 and h_0 be two Hermitian metrics on E and $\frac{\sqrt{-1}}{2\pi}R(h_i)$ the curvatures. Then we define the *Bott-Chern class* $BC(\phi, E; h_0, h_1)$ by the expression

$$BC(\phi, E; h_0, h_1) := \int_0^1 \phi_1(h_t^{-1}\dot{h}_t, \overbrace{\frac{\sqrt{-1}}{2\pi}R_t, \ldots, \frac{\sqrt{-1}}{2\pi}R_t}^{d-1})dt \in \Omega_X^{(d-1,d-1)} \tag{4.2}$$

where h_t is any piecewise C^1 path of Hermitian metrics joining h_0 and h_1. The point of the construction is the following identity:

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}BC(\phi, E; h_0, h_1) = \left(\frac{\sqrt{-1}}{2\pi}\right)^d (\phi(R_{h_0}) - \phi(R_{h_1})).$$

Let $d = n + 1$ where, $n = dim(X)$ in this case $BC(\phi, E; h_0, h_1)$ has top dimension and we may introduce the *Donaldson Functional associated to* ϕ.

$$D_E(h_0, h_1) := \int_X BC(\phi, E; h_0, h_1). \tag{4.3}$$

When h_0 is fixed, we consider it to be a functional on $\mathcal{M}_E$ (the space of hermitian metrics on E). In what follows we take $\phi = Ch_{n+1}$, the $n+1^{st}$ component of the chern character. We can extend the Donaldson functional to "virtual bundles" $\mathcal{E} = E - F$ by observing that a Hermitian metric h on $\mathcal{E}$ is just a pair of metrics, one on E and one on F:

$$h = (h^E, h^F).$$

We set

$$BC(\phi, \mathcal{E}; h_0, h_1) := BC(\phi, E; h_0^E, h_1^E) - BC(\phi, F; h_0^F, h_1^F). \tag{4.4}$$

Let $h : Y \to \mathcal{M}_{\mathcal{E}}$ be a smooth map, where Y is a complex manifold of dimension m.

Lemma 4.1. — *Let ϕ be homogeneous of degree $n + 1$ and h_0 a fixed metric on $\mathcal{E}$. Then for all smooth compactly supported forms ψ of type $(m - 1, m - 1)$ we have the identity*

$$\frac{\sqrt{-1}}{2\pi}\int_Y D_{\mathcal{E}}(\phi; h_0, h(y))\partial_Y\overline{\partial}_Y\psi = \int_{Y\times X} \phi(R(\frac{\sqrt{-1}}{2\pi}h(y))) \wedge \pi_1^*(\psi). \tag{4.5}$$

Next we want to realize the Mabuchi K-energy as the Donaldson functional, with respect to the polynomial $\phi = Ch_{n+1}$, of a certain virtual bundle to be defined below. Then proposition (3) follows at once from the preceding lemma.

Let X be a complex projective manifold (in our present application X is a smooth fiber of $\mathbf{X} \xrightarrow{p} S$), and let L be the restriction of $\underline{O}(1)$ to X. Let φ be a kahler potential. The two metrics h_{FS} and $e^{-\varphi}h_{FS}$ induce metrics on the canonical bundle $\mathscr{K}$. We consider the virtual bundle

$$2^{n+1}\mathscr{E} := (n+1)(\mathscr{K}^{-1}-\mathscr{K})(L-L^{-1})^n - \mu(L-L^{-1})^{n+1}. \tag{4.6}$$

Here μ is the average of the scalar curvature. We need to calculate the following terms.

$$\begin{aligned} &BC(\phi;\mathscr{K}^{-1}\otimes L^{n-2j},h_0,h_1)\\ &BC(\phi;\mathscr{K}\otimes L^{n-2j},h_0,h_1)\\ &BC(\phi;L^{n+1-2j},h_0,h_1). \end{aligned} \tag{4.7}$$

The path of metrics for the first two expression are given as follows.

$$\begin{aligned} h_{\mathscr{K}^{-1}\otimes L^{n-2j},t} &:= \det(g_{\alpha\overline{\beta}} + t\frac{\partial^2}{\partial z_\alpha\partial\overline{z_\beta}}\varphi)e^{-t(n-2j)\varphi}h_{FS}^{n-2j}\\ h_{\mathscr{K}\otimes L^{n-2j},t} &:= \det(g_{\alpha\overline{\beta}} + t\frac{\partial^2}{\partial z_\alpha\partial\overline{z_\beta}}\varphi)^{-1}e^{-t(n-2j)\varphi}h_{FS}^{n-2j}. \end{aligned} \tag{4.8}$$

The complete polarization of ϕ is given by

$$\phi_1(B,A\ldots A) = tr(BA^n) \quad A,B\in M_k(\mathbb{C}). \tag{4.9}$$

Therefore,

(4.10)

$$\begin{aligned} BC(\mathscr{K}^{-1}\otimes L^{n-2j},h_0,h_1) &= \int_0^1(\Delta_{t\varphi}\varphi-(n-2j)\varphi)((n-2j)\omega_{t\varphi}+Ric_{\omega_t})^n dt\\ BC(\mathscr{K}\otimes L^{n-2j},h_0,h_1) &= -\int_0^1(\Delta_{t\varphi}\varphi+(n-2j)\varphi)((n-2j)\omega_{t\varphi}-Ric_{\omega_t})^n dt. \end{aligned}$$

Similarly we have

$$BC(L^{n+1-2j},h_0,h_1) = -(n+1-2j)^{n+1}\int_0^1\varphi\omega_t^n dt \quad \omega_t := \omega + t\partial\overline{\partial}\varphi. \tag{4.11}$$

We see that

(4.12)

$$BC((L-L^{-1})^{n+1},h_{FS},e^{-\varphi}h_{FS}) = -\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}(n+1-2j)^{n+1}\int_0^1\varphi\omega_t^n dt.$$

Now we need the following numerical identity.

$$\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}(n+1-2j)^i = \begin{cases}0 & i<n+1 \text{ or } i=n+2\\ (n+1)!2^{n+1} & i=n+1.\end{cases} \tag{4.13}$$

It follows at once that

$$\int_X BC((L-L^{-1})^{n+1},h_{FS},e^{-\varphi}h_{FS}) = -(n+1)!2^{n+1}\int_0^1\int_X\varphi\omega_t^n dt. \tag{4.14}$$

It follows from (3.9) that

$$BC(\mathcal{K}^{-1}\otimes L^{n-2j}) = \int_0^1 \Delta_{t\varphi}\varphi\sum_{i=0}^{n}\binom{n}{i}(n-2j)^i Ric_t^{n-i}\omega_t^i - \int_0^1\sum_{i=0}^{n}\binom{n}{i}(n-2j)^{i+1}\varphi Ric_t^{n-i}\omega_t^i.$$

We use the identity (4.13) to see that

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}BC(\mathcal{K}^{-1}\otimes L^{n-2j}) = n!2^n\int_0^1\left(\Delta_t\varphi\omega_t^n - \varphi nRic_t\omega_t^{n-1}\right)dt. \tag{4.15}$$

Similarly we have the second term

$$\sum_{j=0}^{n}(-1)^{j+1}\binom{n}{j}BC(\mathcal{K}\otimes L^{n-2j}) = n!2^n\int_0^1\left(\Delta_t\varphi\omega_t^n - \varphi nRic_t\omega_t^{n-1}\right)dt. \tag{4.16}$$

The next lemma follows at once from summing up (4.15), (4.16), and (4.14).

Lemma 4.2. — *Let $D(\mathcal{E}, h_{FS}, e^{-\varphi}h_{FS})$ denote the Donaldson functional of Ch_{n+1} with respect to $\mathcal{E}$. Then the following identity holds.*

$$D(\mathcal{E}, h_{FS}, e^{-\varphi}h_{FS}) = \nu_\omega(\varphi) \tag{4.17}$$

Let $\varphi = \varphi_\sigma$ and apply 4.5 to Lemma 4.2 to conclude the proof of Proposition 3. □

Next we observe that the identity

$$R_{G|\mathbf{X}_z} = p_{2,z}^*\left(R(g_{\mathbf{X}}) - p^*R(g_S) + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\Psi\right) \tag{4.18}$$

together with the previous lemmas yields the following corollary.

Corollary 3. — *The function*

$$\sigma\in G\to D(\sigma) := d(n+1)\nu_{\omega,z}(\sigma) - \log\left(e^{(n+1)\Psi_S(\sigma z)}\frac{||\ ||^2(\sigma z)}{||\ ||^2(z)}\right)$$

is pluriharmonic. Where we have defined $\Psi_S(z) := \int_{\{y\in f^{-1}(z)\}}\Psi(y)p_2^*(\omega_{FS})^n$.

Moreover $\Psi_S(z)\le C$ on $S\setminus\Delta$, extends continuously to the locus of reduced fibers, and $\lim_{z\to z_\infty}\Psi_S(z) = -\infty$ whenever $\mathbf{X}_{z_\infty}$ is non-reduced.

Remark 6. — *The construction of Ψ and Ψ_S as well as their behavior on the locus of singular fibers can be seen directly in Example 1. The general case is treated in Lemma 8.5 pg. 31 in* [**17**].

Since $\pi_1(G) = 1$ there is a (nonvanishing) entire function ξ on G such that

$$D(\sigma) = \log(|\xi(\sigma)|^2).$$

An analysis of the growth of this function on the standard compactification $\overline{G}$

$$\overline{G} := \{[(w_{ij}, z)]\in\mathbb{P}^{(N+1)^2} : \det(w_{ij}) = z^{N+1}\}$$

reveals that it must reduce to a *constant*.

Tying everything together establishes our main result.

***Theorem 1* (The K-Energy as a singular metric on $\mathbb{L}_1^{-1}$).** — *We have*

$$d(n+1)\nu_{\omega,z}(\sigma) = \log\left(e^{(n+1)\Psi_S(\sigma z)}\frac{||\ ||^2(\sigma z)}{||\ ||^2(z)}\right). \tag{4.19}$$

We proceed to the proof of Corollary 1. First substitute $\sigma = \lambda(t)$ in (4.19). Then we have the string of identities.

$$\begin{aligned}
d(n+1)\nu_{\omega,z}(\lambda(t)) &= \log\left(e^{(n+1)\Psi_S(\lambda(t)z)}\frac{||\ ||^2(\lambda(t)z)}{||\ ||^2(z)}\right)\\
&= (n+1)\Psi_S(\lambda(t)z) + \log\left(\frac{||\ ||^2(\lambda(t)z)}{||\ ||^2(z)}\right)\\
&= (n+1)\Psi_S(\lambda(t)z) + \log\left(\frac{||\ ||^2(t^{w_\lambda(z)-w_\lambda(z)}\lambda(t)z)}{||\ ||^2(z)}\right)\\
&= (n+1)\Psi_S(\lambda(t)z) + w_\lambda(z)\log(|t|^2) + O(1)\\
&= F_1(\lambda)\log(|t|^2) + (n+1)\Psi_S(\lambda(t)z) + O(1).
\end{aligned}$$

The passage from line 3 to 4 follows from the defining property of the weight (see the introduction to [**14**]). The passage from line 4 to 5 is the statement of (1.5).

Rationality of the contribution from $\Psi_S(\lambda(t)z)$ follows easily from [**13**] Theorem 3.5 pg. 2564 and Zhiqin Lu's explicit computation of the asymptotics of the K-Energy on hypersurfaces (see [**9**]). This completes the proof of Corollary 1. □

4.2. Properness Implies that $F_1(\lambda) < 0$.

— Let $X := \mathbf{X}_z$ a smooth fiber of p. Recall that the algebraic potential associated to a one parameter subgroup λ is given by

$$\varphi_t := \varphi_{\lambda(t)} = \log(\sum_{i=0}^{N} t^{2q_i}||S_i||^2).$$

Then, as we have seen, $\varphi_t \in P(X,\omega)$. Following Yau [**18**], our plan is to use the standard Moser iteration to control $Osc(\varphi_t)$ by $I_\omega(\varphi_t)$. Define

$$\varphi_- := \mathrm{Max}\{-\varphi_t, 1\} \geq 1.$$

Let $p \in \mathbb{Z}_+$. Then we have the (obvious) inequality

$$\varphi_-^p\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi\wedge\omega_\varphi^{n-1} \leq \varphi_-^p\omega_\varphi^n.$$

Trivially this implies

$$\int_X \varphi_-^p\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi\wedge\omega_\varphi^{n-1} \leq \int_X \varphi_-^p\omega_\varphi^n \leq \int_X \varphi_-^{p+1}\omega_\varphi^n.$$

Next integrate by parts on the leftmost side of this inequality

$$\begin{aligned}\int_X \varphi_-^p \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi\wedge\omega_\varphi^{n-1} &= -\int_X \frac{\sqrt{-1}}{2\pi}\partial\varphi_-^p\wedge\overline{\partial}\varphi\,\omega_\varphi^{n-1}\\ &= \int_X \frac{\sqrt{-1}}{2\pi}\partial\varphi_-^p\wedge\overline{\partial}\varphi_-\omega_\varphi^{n-1}\\ &= \frac{4p}{(p+1)^2}\frac{\sqrt{-1}}{2\pi}\int_X \partial\varphi_-^{\frac{p+1}{2}}\wedge\overline{\partial}\varphi_-^{\frac{p+1}{2}}\wedge\omega_\varphi^{n-1}.\end{aligned}$$

Since $\varphi_-\geq 1$ we deduce the gradient estimate

$$\frac{4p}{(p+1)^2}\frac{\sqrt{-1}}{2\pi}\int_X \partial\varphi_-^{\frac{p+1}{2}}\wedge\overline{\partial}\varphi_-^{\frac{p+1}{2}}\wedge\omega_\varphi^{n-1}\leq \int_X\varphi_-^p\omega_\varphi^n\leq\int_X\varphi_-^{p+1}\omega_\varphi^n.$$

We concentrate on the outermost inequality

$$\frac{4p}{n(p+1)^2}\int_X ||\nabla_{\varphi_t}\varphi_-^{\frac{p+1}{2}}||_{\varphi_t}^2\omega_{\varphi_t}^n\leq\int_X\varphi_-^{p+1}\omega_\varphi^n.$$

Now we invoke the Sobolev inequality

$$\left(\int_X \varphi_-^{\frac{(p+1)n}{n-1}}\frac{\omega_\varphi^n}{V}\right)^{\frac{n-1}{n}}\leq \mathscr{C}(\varphi_t)\left(\int_X ||\nabla_{\varphi_t}\varphi_-^{\frac{p+1}{2}}||_{\varphi_t}^2\frac{\omega_{\varphi_t}^n}{V}+\int_X\varphi_-^{p+1}\frac{\omega_\varphi^n}{V}\right).$$

$\mathscr{C}(\varphi_t)$ is the Sobolev constant of the metric $\omega+\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_t$. Concerning this constant we have the crucial

Proposition 4 **([11], [7])**. — *There is a positive constant $\delta=\delta(n)$ such that for all $\sigma\in SL(N+1,\mathbb{C})$ we have*

$$\mathscr{C}(\varphi_\sigma)<\delta.$$

This follows from the fact the complex projective subvarieties are *minimal* as Riemannian submanifolds of $\mathbb{P}^N$ and hence have vanishing mean curvature.

Therefore,

$$\left(\int_X \varphi_-^{\frac{(p+1)n}{n-1}}\frac{\omega_\varphi^n}{V}\right)^{\frac{n-1}{n}}\leq n(p+1)\delta\int_X\varphi_-^{p+1}\frac{\omega_\varphi^n}{V}.$$

Now extract the $p+1$st root of both sides to get

$$\left(\int_X \varphi_-^{\frac{(p+1)n}{n-1}}\frac{\omega_\varphi^n}{V}\right)^{\frac{n-1}{n(p+1)}}\leq (n(p+1)\delta)^{\frac{1}{p+1}}\left(\int_X\varphi_-^{p+1}\frac{\omega_\varphi^n}{V}\right)^{\frac{1}{p+1}}.$$

Now we start the standard iteration: Let $p_0:=1$ and $p_{j+1}+1:=\frac{n}{n-1}(p_j+1)$. Then we have that

$$\begin{aligned}||\varphi_-||_{p_{j+1}+1}&\leq C^{\frac{1}{p_j+1}}(p_j+1)^{\frac{1}{p_j+1}}||\varphi_-||_{p_j+1}\leq\cdots\\ &\leq C^{\sum_{i=0}^{j}\frac{1}{p_i+1}}\prod_{i=0}^{j}(p_i+1)^{\frac{1}{p_j+1}}||\varphi_-||_2.\end{aligned}$$

That is to say

$$||\varphi_-||_{p_{j+1}+1} \leq C^{\sum_{i=0}^{\frac{1}{p_i+1}}} \prod_{i=0}^{j}(p_i+1)^{\frac{1}{p_j+1}}||\varphi_-||_2.$$

It is not hard to check that the infinite product converges. Taking limits as $j \longrightarrow \infty$ gives

$$||\varphi_-||_\infty \leq C\left(\int_X \varphi_-^2 \frac{\omega_\varphi^n}{V}\right)^{\frac{1}{2}} \leq ||\varphi_-||_\infty^{\frac{1}{2}}\, C\left(\int_X \varphi_- \frac{\omega_\varphi^n}{V}\right)^{\frac{1}{2}}.$$

Which implies

$$||\varphi_-||_\infty \leq C^2\left(\int_X \varphi_- \frac{\omega_\varphi^n}{V}\right).$$

Since $\varphi_t \leq C$ as $t \to 0$ we have

$$-\mathrm{inf}_X \varphi_t \leq C_1 \int_X (-\varphi_t)\frac{\omega_\varphi^n}{V} + C_2.$$

Now, by the Green identity we deduce

$$\mathrm{Osc}_X(\varphi_t) := \mathrm{Sup}_X(\varphi_t) - \mathrm{Inf}_X(\varphi_t) \leq C_1\left(\int_X \varphi_t\omega^n - \int_X \varphi_t\omega_\varphi^n\right) + C_2.$$

Using the properness assumption gives:

$$f(\mathrm{Osc}_X(\varphi_t)) \leq \nu_\omega(\varphi_t).$$

Now we are prepared to complete the proof of the corollary.

Case 1: Assume that $X^{\lambda(0)} \neq X$ and moreover that $X^{\lambda(0)}$ is reduced, then by the same argument as in [**17**] we have

$$\mathrm{lim}_{t\to 0}\mathrm{Osc}_X(\varphi_t) \to \infty.$$

Consequently we deduce that

$$\mathrm{lim}_{t\to 0}\nu_\omega(\varphi_t) \to \infty.$$

Corollary 1 yields the precise asymptotics [(5)]

$$\nu_\omega(\varphi_{\lambda(t)}) = F_1(\lambda)\log(t^2) + O(1).$$

This forces the desired sign $F_1(\lambda) < 0$.

Case 2: If $X^{\lambda(0)}$ is nonreduced, then $\Psi(\lambda(t)) \to -\infty$, however, under the properness assumption the K-Energy is bounded from below, and we again have that $F_1(\lambda) < 0$. This completes the proof of Corollary 2. □

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(5) Recall that when $X^{\lambda(0)}$ is multiplicity free $\Psi(\lambda(t)) = O(1)$.

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Astérisque
328, 2009, p. 355–393

CALABI–YAU THREEFOLDS OF BORCEA–VOISIN, ANALYTIC TORSION, AND BORCHERDS PRODUCTS

by

Ken-ichi Yoshikawa

Dedicated to Professor Jean-Michel Bismut on his sixtieth birthday

Abstract. — For a class of Borcea–Voisin threefolds, we give an explicit formula for the BCOV invariant [**3**], [**14**] as a function on the moduli space. For those Calabi–Yau threefolds, the BCOV invariant is expressed as the Petersson norm of the tensor product of a certain Borcherds lift on the Kähler moduli of a Del Pezzo surface and the Dedekind η-function. As a by-product, we construct an automorphic form on the orthogonal modular variety associated to the odd unimodular lattice of signature $(2, m)$, $m \leq 10$, which vanishes exactly on the Heegner divisor of norm (-1)-vectors.

Résumé **(Variétés de Calabi-Yau de dimension trois de type Borcea–Voisin, torsion analytique, et produits de Borcherds)**

Pour une classe de variétés de Borcea–Voisin, nous donnons une formule explicite de l'invariant de BCOV [**3**], [**14**] comme une fonction sur l'espace de modules. Pour ces variétés de Calabi–Yau de dimension trois, l'invariant de BCOV s'exprime comme la norme du produit tensoriel d'un relèvement de Borcherds à l'espace des modules kählériens d'une surface de Del Pezzo et de la fonction η de Dedekind. Nous construisons une forme automorphe sur la variété modulaire orthogonale associée au réseau unimodulaire impair de signature $(2, m)$, $m \leq 10$, qui s'annule exactement sur le diviseur de Heegner des vecteurs de norme -1.

1. Introduction

In [**33**], Ray–Singer introduced the notion of analytic torsion for compact Kähler manifolds. Their definition was extended to arbitrary holomorphic Hermitian vector bundles over a compact Kähler manifold by Quillen [**32**] and Bismut–Gillet–Soulé [**7**]. Let $\xi \to X$ be a holomorphic Hermitian vector bundle over a compact Kähler

2010 Mathematics Subject Classification. — 58J52, 14J32, 14J28, 11F22, 32N10, 32N15.
Key words and phrases. — Analytic torsion, Calabi-Yau threefold, Borcherds product.

The author is partially supported by the Grants-in-Aid for Scientific Research (B) 19340016 and (S) 17104001, JSPS.

manifold and let $\zeta_q(s)$ be the spectral zeta function of the Hodge–Kodaira Laplacian acting on the space of $(0,q)$-forms on X with values in ξ. Then the real number

$$\tau(X,\xi) = \exp[-\sum_{q\geq 0}(-1)^q q\, \zeta'_q(0)]$$

is called the analytic torsion of ξ. The most fundamental results in the theory of analytic torsion such as the first variational formula, the second variational formula and the comparison formula for complex immersions were obtained by Bismut–Gillet–Soulé and Bismut–Lebeau as the corresponding results in the theory of Quillen metrics, i.e., the anomaly formula, the curvature formula and the immersion formula for Quillen metrics [**7**], [**8**],...

In [**3**], Bershadsky–Cecotti–Ooguri–Vafa introduced the following combination of analytic torsions for a compact Kähler manifold X

$$\prod_{p\geq 0}\tau(X,\Omega^p_X)^{(-1)^p p},$$

which we call the BCOV torsion of X. They studied the BCOV torsion as a function on the moduli space of Calabi–Yau threefolds and used it to extend the mirror symmetry conjecture to higher-genus Gromov–Witten invariants [**2**], [**3**].

In [**14**], the notion of BCOV invariant was introduced for Calabi–Yau threefolds by Fang–Lu–Yoshikawa, which they obtained using the BCOV torsion and a certain Bott–Chern secondary class. (See Sect. 5.1 for the definition.) The BCOV invariant of a Calabi–Yau threefold X is denoted by $\tau_{\mathrm{BCOV}}(X)$. Then $\tau_{\mathrm{BCOV}}(X)$ depends only on the isomorphism class of X, while the BCOV torsion does depend on the choice of a Kähler metric on X. Because of this invariance property, the BCOV invariant τ_{BCOV} gives rise to a function on the moduli space of Calabi–Yau threefolds and is identified with the partition function F_1 in [**3**]. In this paper, we give an explicit formula for the BCOV invariant for a class of Calabi–Yau threefolds studied by Borcea [**9**] and Voisin [**36**]. (See [**14**] for some other examples including the quintic mirror threefolds and the FHSV models.) Let us explain our results.

Let S be a $K3$ surface and let $\theta\colon S\to S$ be an anti-symplectic holomorphic involution. Let T be an elliptic curve and let $-1_T\colon T\to T$ be the involution defined as $-1_T(x)=-x$. Let $X_{(S,\theta,T)}$ be the blow-up of the orbifold $(S\times T)/\theta\times(-1)_T$ along the singular locus. Then $X_{(S,\theta,T)}$ is a smooth Calabi–Yau threefold equipped with the following two fibrations. Let $\pi_1\colon X_{(S,\theta,T)}\to S/\theta$ be the elliptic fibration with constant fiber T induced from the projection $(S\times T)/\theta\times(-1)_T\to S/\theta$ and let $\pi_2\colon X_{(S,\theta,T)}\to T/(-1_T)$ be the $K3$-fibration with constant fiber S induced from the projection $(S\times T)/\theta\times(-1)_T\to T/(-1_T)$. The triplet $(X_{(S,\theta,T)},\pi_1,\pi_2)$ is called the Borcea–Voisin threefold associated with (S,θ,T). The moduli space of the triplet $(X_{(S,\theta,T)},\pi_1,\pi_2)$ is determined by the lattice $H^2_-(S,\mathbf{Z})$, the anti-invariant part of the θ-action on $H^2(S,\mathbf{Z})$. By [**28**], $H^2_-(S,\mathbf{Z})$ is isometric to a primitive 2-elementary sublattice of the $K3$-lattice $\mathbb{L}_{K3}$. Let $\Lambda\subset\mathbb{L}_{K3}$ be a sublattice of rank $r(\Lambda)$. A Borcea–Voisin threefold $(X_{(S,\theta,T)},\pi_1,\pi_2)$ is of type Λ if $H^2_-(S,\mathbf{Z})$ is isometric to Λ. Since θ is anti-symplectic, there exist Borcea–Voisin threefolds of type Λ if and only if $\Lambda\subset\mathbb{L}_{K3}$ is a primitive 2-elementary sublattice of signature $(2,r(\Lambda)-2)$.

Some Borcea–Voisin threefolds are related to Del Pezzo surfaces. Let V be a Del Pezzo surface and set $\deg V=c_1(V)^2\in\mathbf{Z}_{>0}$. Let $H(V,\mathbf{Z})$ be the total cohomology group of V, which is equipped with the cup-product $\langle\cdot,\cdot\rangle_V$. Then the sublattice $H^2(V,\mathbf{Z})\subset H(V,\mathbf{Z})$ is Lorentzian. Let $H(V,\mathbf{Z})(2)$ be the lattice $(H(V,\mathbf{Z}),2\langle\cdot,\cdot\rangle_V)$. By the classification of primitive 2-elementary Lorentzian sublattices of $\mathbb{L}_{K3}$ [**29**], there exist Borcea–Voisin threefolds of type $H(V,\mathbf{Z})(2)$. Let us explain their moduli space briefly.

Let $\mathscr{K}_V\subset H^2(V,\mathbf{R})$ be the Kähler cone of V, let $\mathscr{C}^+_V\subset H^2(V,\mathbf{R})$ be the component of the positive cone of $H^2(V,\mathbf{R})$ with $\mathscr{K}_V\subset\mathscr{C}^+_V$ and let $\mathrm{Eff}(V)\subset H^2(V,\mathbf{Z})$ be the set of effective classes on V. The tube domain $H^2(V,\mathbf{R})+i\,\mathscr{C}^+_V$ is isomorphic to a bounded symmetric domain of type IV and its subdomain $H^2(V,\mathbf{R})+i\,\mathscr{K}_V$ is called the complexified Kähler cone of V. Let $\mathfrak{H}$ be the complex upper half-plane. By assigning $(X_{(S,\theta,T)},\pi_1,\pi_2)$ the periods of (S,θ) and T, the coarse moduli space of Borcea–Voisin threefolds of type $H(V,\mathbf{Z})(2)$ is isomorphic to the quotient of the tube domain $(H^2(V,\mathbf{R})+i\,\mathscr{C}^+_V)\times\mathfrak{H}$ by the group $O^+(H(V,\mathbf{Z}))\times SL_2(\mathbf{Z})$ with some divisor removed (cf. Theorem 3.7), where $O^+(H(V,\mathbf{Z}))$ is the group of isometries of $H(V,\mathbf{Z})$ preserving $H^2(V,\mathbf{R})+i\,\mathscr{C}^+_V$. Hence τ_{BCOV} is regarded as an $O^+(H(V,\mathbf{Z}))\times SL_2(\mathbf{Z})$-invariant function on a certain Zariski open subset of $(H^2(V,\mathbf{R})+i\,\mathscr{C}^+_V)\times\mathfrak{H}$. The goal of this paper is to give an explicit formula for τ_{BCOV} as a function on $(H^2(V,\mathbf{R})+i\,\mathscr{C}^+_V)\times\mathfrak{H}$ for Borcea–Voisin threefolds of type $H^2(V,\mathbf{Z})(2)$. Let us explain the infinite product appearing in the formula.

After Borcherds [**12**] and Gritsenko–Nikulin [**16**], we introduce the following infinite product $\Phi_V(z)$ on the complexified Kähler cone $H^2(V,\mathbf{R})+i\,\mathscr{K}_V$:

$$\Phi_V(z)=e^{\pi i\langle c_1(V),z\rangle_V}\prod_{\alpha\in\mathrm{Eff}(V)}\left(1-e^{2\pi i\langle\alpha,z\rangle_V}\right)^{c^{(0)}_{\deg V}(\alpha^2)}$$
$$\times\prod_{\beta\in\mathrm{Eff}(V),\ \beta/2\equiv c_1(V)/2\mod H^2(V,\mathbf{Z})}\left(1-e^{\pi i\langle\beta,z\rangle_V}\right)^{c^{(1)}_{\deg V}(\beta^2/4)},$$

where $c^{(0)}_k(m)$ and $c^{(1)}_k(m)$ are the m-th Fourier coefficients of the modular forms

$$f^{(0)}_k(\tau)=\eta(\tau)^{-8}\eta(2\tau)^{8}\eta(4\tau)^{-8}\,\theta_{\mathbb{A}_1^+}(\tau)^k,\quad f^{(1)}_k(\tau)=-8\,\eta(4\tau)^{8}\eta(2\tau)^{-16}\,\theta_{\mathbb{A}_1^++\frac12}(\tau)^k,$$

respectively. Here $\eta(\tau)$ is the Dedekind η-function and $\theta_{\mathbb{A}_1^+}(\tau)$, $\theta_{\mathbb{A}_1^++1/2}(\tau)$ are the theta series of the A_1-lattice. Let $A_{H(V,\mathbf{Z})(2)}$ be the discriminant group of the lattice $H(V,\mathbf{Z})(2)$ and let $\{\mathbf{e}_\gamma\}_{\gamma\in A_{H(V,\mathbf{Z})(2)}}$ be the standard basis of $\mathbf{C}[A_{H(V,\mathbf{Z})(2)}]$, the group ring of $A_{H(V,\mathbf{Z})(2)}$. In Sects. 4.3, 4.4 and 6.2, we shall prove that $\Phi_V(2z)^2$ is the Borcherds lift [**12**] of the $\mathbf{C}[A_{H(V,\mathbf{Z})(2)}]$-valued elliptic modular form

$$f^{(0)}_{\deg V}(\tau)\,\mathbf{e}_0 + \sum_{\gamma\in A_{H(V,\mathbf{Z})(2)}}\ \sum_{m\equiv 2\gamma^2 \mod 4} c^{(0)}_{\deg V}(m)\,q^{m/4}\,\mathbf{e}_\gamma + f^{(1)}_{\deg V}(\tau)\,\mathbf{e}_{\mathbf{1}_{H(V,\mathbf{Z})(2)}}$$

with respect to the lattice $H(V,\mathbf{Z})(2)$. Here $\mathbf{1}_{H(V,\mathbf{Z})(2)} \in A_{H(V,\mathbf{Z})(2)}$ is the characteristic element and $q=\exp(2\pi i\tau)$. As a result, $\Phi_V(z)$ converges when $(\operatorname{Im} z)^2 \gg 0$ and extends to an automorphic form on $H^2(V,\mathbf{R}) + i\,\mathscr{C}^+_V$ for $O^+(H(V,\mathbf{Z}))$ of weight $\deg V+4$ vanishing exactly on the Heegner divisor of norm (-1)-vectors of $H(V,\mathbf{Z})$. If $\operatorname{Exc}(V) \subset H^2(V,\mathbf{Z})$ denotes the exceptional classes on V, the following functional equations hold by the automorphic property of $\Phi_V(z)$ (cf. Sect. 6.3):

(a) $\Phi_V(z+l) = \Phi_V(z)$ for all $l \in H^2(V,\mathbf{Z})$ with $\langle l, c_1(V)\rangle_V \equiv 0 \mod 2$.
(b) $\Phi_V(g(z)) = \pm\Phi_V(z)$ for all $g \in O^+(H^2(V,\mathbf{Z}))$.
(c) $\Phi_V(-\frac{z}{\langle z,z\rangle_V} + \delta) = -(-\langle z,z\rangle_V)^{\deg V+4}\,\Phi_V(z+\delta)$ for all $\delta \in \operatorname{Exc}(V)$.
(d) $\Phi_V(-\frac{2z}{\langle z,z\rangle_V}) = (-\frac{\langle z,z\rangle_V}{2})^{\deg V+4}\,\Phi_V(z)$.

Since $c_1(V)/2$ is a Weyl vector of $H^2(V,\mathbf{Z})(2)$, the Fourier expansion of $\Phi_V(2z)$ is of Lie type in the sense of [**18**] by (a), (b). Hence there exists a Borcherds superalgebra whose denominator function is $\Phi_V(2z)$. This Borcherds superalgebra is obtained as an automorphic correction [**17**] of the Kac-Moody algebra defined by the generalized Cartan matrix $(2\langle c_1(E), c_1(E')\rangle_V)_{E,E'\in\operatorname{Exc}(V)}$. (See Question 4.4.)

Let $\|\Phi_V\|$ and $\|\eta\|$ be the Petersson norms of $\Phi_V(z)$ and $\eta(\tau)$, respectively. Then $\|\Phi_V\|^2\cdot\|\eta^{24}\|^2$ is a function on $(H^2(V,\mathbf{R}) + i\,\mathscr{C}^+_V)\times\mathfrak{H}$ invariant under the action of $O^+(H(V,\mathbf{Z}))\times SL_2(\mathbf{Z})$. The following (cf. Theorems 5.7 and 6.4) is the main result of this paper.

Theorem 1.1. — *If V is a Del Pezzo surface with $1 \le \deg V \le 6$, then there exists a constant $C_{\deg V}$ depending only on $\deg V$ such that the following equation of functions on the moduli space of Borcea–Voisin threefolds of type $H(V,\mathbf{Z})(2)$ holds:*

$$\tau_{\mathrm{BCOV}} = C_{\deg V}\,\|\Phi_V\|^2\cdot\|\eta^{24}\|^2.$$

Under the identification of τ_{BCOV} with F_1 in B-model [**2**], [**3**], it follows from Theorem 1.1 that the conjecture of Harvey–Moore [**19**, Sect. 7] holds for Borcea–Voisin threefolds of type $H(V,\mathbf{Z})(2)$ when $1 \le \deg V \le 6$, since Φ_V is the denominator function of a Borcherds superalgebra.

After Theorem 1.1, the conjecture of Bershadsky–Cecotti–Ooguri–Vafa [**2**], [**3**] seems to predict that the elliptic Gromov–Witten invariants of the mirror of Borcea–Voisin threefolds of type $H(V,\mathbf{Z})(2)$ are expressed as certain linear combinations of the Fourier coefficients $c^{(0)}_{\deg V}(m)$, $c^{(1)}_{\deg V}(m)$. If this is the case, the invariant of $K3$

surfaces with involution constructed in [**37**] would be the Borcherds lift of an elliptic modular form whose Fourier coefficients are elliptic Gromov–Witten invariants of some Calabi–Yau threefolds by the structure theorem [**38**, Th. 0.1]. However, since the Borcea–Voisin construction of mirrors [**9**], [**36**] does not apply to Borcea–Voisin threefolds of type $H(V,\mathbf{Z})(2)$, we do not know the existence of mirrors for those Borcea–Voisin threefolds as well as their elliptic Gromov–Witten invariants.

This paper is organized as follows. In Sect. 2, we recall some definitions and results about lattices. In Sect. 3, we recall Borcea–Voisin threefolds and study their moduli space. In Sect. 4, we introduce the automorphic form Φ_m, which will be identified with $\varPhi_V$ in Sect. 6. In Sect. 5, we recall the BCOV invariant of a Calabi–Yau threefold and we prove the main theorem. In Sect. 6, we rewrite the automorphic form Φ_m as an automorphic form on the complexified Kähler cone of a Del Pezzo surface to give an identification between Φ_m and $\varPhi_V$.

Acknowledgements. — The author thanks the referee for helpful comments, which inspired Question 5.18.

2. Lattices and orthogonal modular varieties

A free $\mathbf{Z}$-module of finite rank endowed with a non-degenerate, integral, symmetric bilinear form is called a lattice. We often identify a non-degenerate, integral, symmetric matrix with the corresponding lattice. The rank of a lattice L is denoted by $r(L)$. The signature of L is denoted by $\mathrm{sign}(L) = (b^+(L), b^-(L))$. A lattice L is *Lorentzian* if $\mathrm{sign}(L) = (1, r(L) - 1)$. For a lattice L with bilinear form $\langle\cdot,\cdot\rangle$, we denote by $L(k)$ the lattice with bilinear form $k\langle\cdot,\cdot\rangle$. The set of roots of L is defined by $\Delta_L := \{d \in L;\ \langle d,d\rangle = -2\}$. The isometry group of L is denoted by $O(L)$. For $r \in L\otimes\mathbf{R}$, the reflection $s_r \in O(L\otimes\mathbf{R})$ is defined by $s_r(x) = x - 2\frac{\langle x,r\rangle}{\langle r,r\rangle} r$ for $x \in L\otimes\mathbf{R}$. If $\delta \in L$ and $\delta^2 = -1$ or $\delta^2 = -2$, then $s_\delta \in O(L)$. The subgroup of $O(L)$ generated by the reflections $\{s_\delta\}_{\delta\in\Delta_L}$ is called the *Weyl group* of L and is denoted by $W(L)$. The dual lattice of L is defined by $L^\vee := \mathrm{Hom}_{\mathbf{Z}}(L,\mathbf{Z}) \subset L\otimes\mathbf{Q}$. We set $A_L := L^\vee/L$. A lattice L is *unimodular* if $A_L = 0$. A lattice L is *even* if $\langle x,x\rangle \in 2\mathbf{Z}$ for all $x \in L$. A lattice is *odd* if it is not even. A sublattice $M \subset L$ is *primitive* if L/M has no torsion elements.

2.1. 2-elementary lattices. — Set $\mathbf{Z}_2 := \mathbf{Z}/2\mathbf{Z}$. An even lattice L is 2-*elementary* if there is an integer $l \geq 0$ with $A_L \cong \mathbf{Z}_2^l$. For a 2-elementary lattice L, we set $l(L) := \dim_{\mathbf{Z}_2} A_L$.

Let $\mathbb{U} = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and let $\mathbb{A}_1$, $\mathbb{E}_8$ be the *negative-definite* Cartan matrix of type A_1, E_8 respectively, which are identified with the corresponding even lattices. Then $\mathbb{U}$ and

$\mathbb{E}_8$ are unimodular, and $\mathbb{A}_1$ is 2-elementary. The lattice

$$\mathbb{L}_{K3} := \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8 \oplus \mathbb{E}_8$$

is called the $K3$ *lattice*. For a sublattice $\Lambda \subset \mathbb{L}_{K3}$, set $\Lambda^\perp := \{l \in \mathbb{L}_{K3};\ \langle l, \Lambda\rangle = 0\}$.

For a primitive 2-elementary Lorentzian sublattice $M \subset \mathbb{L}_{K3}$, let I_M be the involution on $M \oplus M^\perp$ defined as $I_M(x,y) = (x,-y)$ for $(x,y) \in M \oplus M^\perp$. Then I_M extends uniquely to an involution on $\mathbb{L}_{K3}$ by [**28**, Cor. 1.5.2].

Let L be an even 2-elementary lattice. Since A_L is a vector space over $\mathbf{Z}_2$, the mapping $A_L \ni \gamma \to \gamma^2 \in \frac{1}{2}\mathbf{Z}/\mathbf{Z} \cong \mathbf{Z}_2$ is $\mathbf{Z}_2$-linear. Since the discriminant bilinear form on A_L is non-degenerate, there is a unique element $\mathbf{1}_L \in A_L$ such that $\langle \gamma, \mathbf{1}_L\rangle \equiv \gamma^2 \mod \mathbf{Z}$ for all $\gamma \in A_L$. If $L = L' \oplus L''$, then $\mathbf{1}_L = \mathbf{1}_{L'} \oplus \mathbf{1}_{L''}$.

2.2. Lorentzian lattices. — Let L be a Lorentzian lattice. The set $\mathscr{C}_L := \{v \in L \otimes \mathbf{R};\ v^2 > 0\}$ is called the positive cone of L, which consists of two connected components. Let $\mathscr{C}_L^+$ be one of the connected components of $\mathscr{C}_L$. For $\lambda \in L \otimes \mathbf{R}$, we set $h_\lambda := \{v \in \mathscr{C}_L^+;\ \langle v, \lambda\rangle = 0\}$. Define $(\mathscr{C}_L^+)^o := \mathscr{C}_L^+ \setminus \bigcup_{\delta \in \Delta_L} h_\delta$. The Weyl group $W(L)$ acts simply transitively on the set of connected components of $(\mathscr{C}_L^+)^o$. Each connected component of $(\mathscr{C}_L^+)^o$ is called a *Weyl chamber* of L. Let $\mathscr{W}$ be a Weyl chamber of L. A hyperplane $h_d \subset L \otimes \mathbf{R}$, $d \in \Delta_L^+$ is called a *wall* of $\mathscr{W}$ if $\dim(h_d \cap \overline{\mathscr{W}}) = r(L) - 1$, where $\overline{\mathscr{W}}$ is the closure of $\mathscr{W}$ in $L \otimes \mathbf{R}$. We set $\Pi(L, \mathscr{W}) := \{d \in \Delta_L;\ d \cdot \mathscr{W} > 0,\ h_d \text{ is a wall of } \mathscr{W}\}$, which is the minimal set of roots defining $\mathscr{W}$, i.e.,

$$\mathscr{W} = \{v \in \mathscr{C}_L^+;\ \langle v, d\rangle > 0,\ \forall\, d \in \Pi(L, \mathscr{W})\}. \tag{2.1}$$

In (2.1), each inequality $\langle v, d\rangle > 0$, $d \in \Pi(L, \mathscr{W})$ is essential. A vector $\varrho \in L \otimes \mathbf{Q}$ is called a *Weyl vector* of $(L, \mathscr{W})$ if $\langle \varrho, d\rangle = 1$ for all $d \in \Pi(L, \mathscr{W})$.

2.3. Lattices of signature $(2,n)$. — Let Λ be a lattice with $\operatorname{sign}(\Lambda) = (2, r(\Lambda)-2)$. Define

$$\Omega_\Lambda := \{[\eta] \in \mathbf{P}(\Lambda \otimes \mathbf{C});\ \langle \eta, \eta\rangle = 0,\ \langle \eta, \bar\eta\rangle > 0\}.$$

Then Ω_Λ consists of two connected components $\Omega_\Lambda^\pm$, each of which is isomorphic to a bounded symmetric domain of type IV of dimension $r(\Lambda) - 2$. The group $O(\Lambda)$ acts on Ω_Λ projectively. We set $O^+(\Lambda) := \{g \in O(\Lambda);\ g(\Omega_\Lambda^\pm) = \Omega_\Lambda^\pm\}$. Then $O^+(\Lambda)$ acts on Ω_Λ^+ properly discontinuously, and the quotient

$$\mathscr{M}_\Lambda := \Omega_\Lambda / O(\Lambda) = \Omega_\Lambda^+ / O^+(\Lambda)$$

is an analytic space. The Baily–Borel–Satake compactification of $\mathscr{M}_\Lambda$ is denoted by $\mathscr{M}_\Lambda^*$. Then $\mathscr{M}_\Lambda^*$ is an irreducible normal projective variety with $\dim(\mathscr{M}_\Lambda^* \setminus \mathscr{M}_\Lambda) \leq 1$. For $\lambda \in \Lambda \otimes \mathbf{R}$, set

$$H_\lambda := \{[\eta] \in \Omega_\Lambda;\ \langle \eta, \lambda\rangle = 0\}.$$

Then $H_\lambda \neq \varnothing$ if and only if $\langle\lambda,\lambda\rangle < 0$. We define

$$\mathscr{D}_\Lambda := \bigcup_{d\in\Delta_\Lambda} H_d, \qquad\qquad \Omega_\Lambda^o := \Omega_\Lambda \setminus \mathscr{D}_\Lambda.$$

The reduced divisor $\mathscr{D}_\Lambda$ is called the *discriminant locus* of Ω_Λ. We define the subsets $H_d^o \subset H_d$ ($d \in \Delta_\Lambda$) and $\mathscr{D}_\Lambda^o \subset \mathscr{D}_\Lambda$ by

$$H_d^o := \{[\eta] \in \Omega_\Lambda^+;\ O^+(\Lambda)_{[\eta]} = \{\pm 1,\ \pm s_d\}\}, \qquad \mathscr{D}_\Lambda^o := \sum_{d\in\Delta_\Lambda/\pm 1} H_d^o.$$

Since $O(\Lambda)$ preserves $\mathscr{D}_\Lambda$ and $\mathscr{D}_\Lambda^o$, we define

$$\overline{\mathscr{D}}_\Lambda := \mathscr{D}_\Lambda/O(\Lambda), \qquad\qquad \overline{\mathscr{D}}_\Lambda^o := \mathscr{D}_\Lambda^o/O(\Lambda) \subset \overline{\mathscr{D}}_\Lambda.$$

Then $\overline{\mathscr{D}}_\Lambda^o \cap \operatorname{Sing}\mathscr{M}_\Lambda = \varnothing$ by [**38**, Prop. 1.9 (5)] and $\Omega_\Lambda^o \cup \mathscr{D}_\Lambda^o$ is a Zariski open subset of Ω_Λ such that $\Omega_\Lambda \setminus (\Omega_\Lambda^o \cup \mathscr{D}_\Lambda^o)$ has codimension at least 2 by [**37**, Prop. 1.9 (2)].

When $\Lambda = \mathbb{U}(N) \oplus L$, a vector of $\Lambda \otimes \mathbf{C}$ is denoted by (m, n, v), where $m, n \in \mathbf{C}$ and $v \in L \otimes \mathbf{C}$. The tube domain $L \otimes \mathbf{R} + i\,\mathscr{C}_L$ is identified with Ω_Λ via the map

$$L \otimes \mathbf{R} + i\,\mathscr{C}_L \ni z \to [(-z^2/2, 1/N, z)] \in \Omega_\Lambda \subset \mathbf{P}(\Lambda \otimes \mathbf{C}), \qquad z \in L \otimes \mathbf{C}. \tag{2.2}$$

The component of Ω_Λ corresponding to $L \otimes \mathbf{R} + i\,\mathscr{C}_L^+$ via (2.2) is written as Ω_Λ^+.

3. Calabi–Yau threefolds of Borcea–Voisin

An irreducible, smooth, compact Kähler n-fold X with canonical line bundle K_X is *Calabi–Yau* if

$$(1) \quad K_X \cong \mathscr{O}_X, \qquad\qquad (2) \quad H^q(X, \mathscr{O}_X) = 0 \quad (0 < q < n).$$

A two-dimensional Calabi–Yau manifold is called a $K3$ surface. In this section, we recall a class of Calabi–Yau threefolds studied by Borcea [**9**] and Voisin [**36**].

3.1. $K3$ surfaces with involution and their moduli space. — Let S be a $K3$ surface. Then $H^2(S, \mathbf{Z})$ endowed with the cup-product pairing is isometric to the $K3$ lattice $\mathbb{L}_{K3}$. An isometry of lattices $\alpha\colon H^2(S, \mathbf{Z}) \cong \mathbb{L}_{K3}$ is called a *marking* of S, and the pair (S, α) is called a *marked $K3$ surface*. The period of a marked $K3$ surface (S, α) is defined by

$$\pi(S, \alpha) := [\alpha(\eta)] \in \mathbf{P}(\mathbb{L}_{K3} \otimes \mathbf{C}), \qquad \eta \in H^0(S, K_S) \setminus \{0\}.$$

Let $M \subset \mathbb{L}_{K3}$ be a sublattice. A $K3$ surface equipped with a holomorphic involution $\theta\colon S \to S$ is called a 2-*elementary $K3$ surface of type M* if

$$\theta^* = \alpha^{-1} \circ I_M \circ \alpha, \qquad \theta^*|_{H^0(S,K_S)} = -1.$$

By the global Torelli theorem [**31**], [**13**] and by [**28**, Cor. 1.5.2], there exists a 2-elementary $K3$ surface of type M if and only if $M \subset \mathbb{L}_{K3}$ is a primitive 2-elementary Lorentzian sublattice.

Let (S,θ) be a 2-elementary $K3$ surface of type M and let α be a marking with $\theta^* = \alpha^{-1} \circ I_M \circ \alpha$. Let $\eta \in H^0(S, K_S) \setminus \{0\}$. Then $\pi(S,\alpha) \in \Omega^o_{M^\perp}$. By [**37**, Th. 1.8] and [**38**, Prop. 11.2], the $O(M^\perp)$-orbit of $\pi(S,\alpha)$ is independent of the choice of a marking α with $\theta^* = \alpha^{-1} \circ I_M \circ \alpha$. The period of (S,θ) is defined as the $O(M^\perp)$-orbit

$$\varpi_M(S,\theta) := O(M^\perp) \cdot \pi(S,\alpha) \in \Omega_{M^\perp}/O(M^\perp) = \mathcal{M}_{M^\perp}.$$

By [**37**, Th. 1.8], the period map induces an isomorphism from the coarse moduli space of 2-elementary $K3$ surfaces of type M to the analytic space

$$\mathcal{M}^o_{M^\perp} := \Omega^o_{M^\perp}/O(M^\perp) = (\Omega^+_{M^\perp} \setminus \mathcal{D}_{M^\perp})/O^+(M^\perp).$$

Theorem 3.1. — *Let $x \in \overline{\mathcal{D}}^o_{M^\perp}$ and let $C \subset \mathcal{M}^*_{M^\perp}$ be an irreducible projective curve passing through x. Assume that $x \in C \setminus \operatorname{Sing} C$ and that C intersects $\overline{\mathcal{D}}^o_{M^\perp}$ transversally at x. Then there exist a pointed smooth projective curve (B,y), a neighborhood U of y, a holomorphic map $f\colon (B,y) \to (C,x)$, a smooth projective threefold $\mathcal{W}$ with an involution $\theta\colon \mathcal{W} \to \mathcal{W}$, and a surjective holomorphic map $p\colon \mathcal{W} \to B$ satisfying the following properties:*

(1) *$f(B) = C$ and the map $f|_U\colon (U,y) \to (f(U),x)$ is an isomorphism.*
(2) *The projection $p\colon \mathcal{W} \to B$ is $\mathbf{Z}_2$-equivariant with respect to the $\mathbf{Z}_2$-action on $\mathcal{W}$ induced by θ and with respect to the trivial $\mathbf{Z}_2$-action on B.*
(3) *For every $b \in U \setminus \{y\}$, $(\mathcal{W},\theta)|_{p^{-1}(b)}$ is a 2-elementary $K3$ surface of type M such that $\varpi_M((\mathcal{W},\theta)|_{p^{-1}(b)}) = f(b)$.*

Proof. — See [**37**, Th. 2.8]. □

For a 2-elementary $K3$ surface (S,θ), we define $S^\theta := \{x \in S;\ \theta(x) = x\}$.

Proposition 3.2. — *Let (S,θ) be a 2-elementary $K3$ surface of type M and set*

$$g(M) := (22 - r(M) - l(M))/2, \qquad k(M) := (r(M) - l(M))/2.$$

If $M \not\cong \mathbb{U}(2) \oplus \mathbb{E}_8(2), \mathbb{U} \oplus \mathbb{E}_8(2)$, then there exist a smooth irreducible curve C of genus $g(M)$ and (-2)-curves $E_1, \dots, E_{k(M)}$ such that $S^\theta = C \amalg E_1 \amalg \cdots \amalg E_{k(M)}$.

Proof. — See [**29**, Th. 4.2.2]. □

3.2. Elliptic curves and elliptic fibrations. — Let $\mathfrak{H} = \{\tau \in \mathbf{C};\ \operatorname{Im}\tau > 0\}$ be the complex upper half-plane and let $\mathfrak{M}$ be the modular curve

$$\mathfrak{M} := SL_2(\mathbf{Z})\backslash\mathfrak{H}.$$

For an elliptic curve T, let $\Omega(T) \in \mathfrak{M}$ denote the period of T. Let $-1_T\colon T \to T$ be the holomorphic involution that assigns $x \in T$ the inverse $-x \in T$. Let $j(T) \in \mathbf{C}$ denote the value of the j-invariant of T. If T is isomorphic to the cubic curve of $\mathbf{P}^2$ defined by the inhomogeneous equation $y^2 = 4x^3 - g_2\,x - g_3$, then

$$j(T) = \frac{g_2^3}{g_2^3 - 27\,g_3^2}.$$

The j-invariant induces an identification between $\mathfrak{M}$ with the complex plane $\mathbf{C}$.

Let S be a compact complex surface, let B be a compact Riemann surface, and let $f\colon S \to B$ be a surjective holomorphic map. We set $S_b := f^{-1}(b)$ for $b \in B$. Let $\Delta_{S/B} \subset B$ be the set of critical values of f. Then $f\colon S \to B$ is an *elliptic fibration* if S_b is an elliptic curve for every $b \in B \setminus \Delta_{S/B}$. The analytic invariant of an elliptic fibration $f\colon S \to B$ is the meromorphic function on B defined as $j_{S/B}(b) := j(S_b)$ for $b \in B \setminus \Delta_{S/B}$. For an elliptic fibration $f\colon S \to B$, we set $B^o := B \setminus \Delta_{S/B}$, $S^o := f^{-1}(B^o)$ and $f^o := f|_{S^o}$.

Let $f\colon S \to B$ be an elliptic fibration with a holomorphic section $\sigma\colon B \to S$. By [**1**, Chap. V Prop. 9.1], the elliptic fibration $f^o\colon S^o \to B^o$ is canonically isomorphic to the Jacobian fibration $(R^1 f_* \mathcal{O}_S / R^1 f_* \mathbf{Z})|_{B^o} \to B^o$ such that $\sigma(b)$ is identified with the identity element of the Jacobian $H^1(S_b, \mathcal{O}_{S_b})/H^1(S_b, \mathbf{Z})$. Hence there exists a holomorphic involution -1_{S^o} on S^o such that $-1_{S^o}|_{S_b} = -1_{S_b}$ for all $b \in B^o$. When -1_{S^o} extends to a holomorphic involution on S, we call the elliptic fibration $f\colon S \to B$ with a holomorphic section *admissible*.

3.3. Borcea–Voisin threefolds and their moduli space. — Let (S, θ) be a 2-elementary $K3$ surface. Let T be an elliptic curve. Let $T[2]$ denote the 2-torsion points of T, which is the set of fixed points of -1_T.

Define a holomorphic involution on $S \times T$ by $\iota := \theta \times (-1_T)$, which acts trivially on $H^0(S \times T, K_{S\times T})$. By identifying the generator of $\mathbf{Z}_2$ with the involutions θ, -1_T and ι, the group $\mathbf{Z}_2$ acts holomorphically on S, T, $S \times T$, respectively. The set of fixed points of ι, $(S \times T)^\iota = S^\theta \times T[2]$, is the disjoint union of four copies of the curve S^θ. After Borcea [**9**] and Voisin [**36**], we make the following

Definition 3.3. — For a 2-elementary $K3$ surface (S, θ) and an elliptic curve T, let $X_{(S,\theta,T)}$ be the resolution of $S \times T/\mathbf{Z}_2$ defined as the blow-up of $S \times T/\mathbf{Z}_2$ along $\mathrm{Sing}\,(S \times T/\mathbf{Z}_2) \cong (S \times T)^\iota$. Let $\pi_1\colon X_{(S,\theta,T)} \to S/\mathbf{Z}_2$ and $\pi_2\colon X_{(S,\theta,T)} \to T/\mathbf{Z}_2$ be the projections induced from the projections $\mathrm{pr}_1\colon S \times T \to S$ and $\mathrm{pr}_2\colon S \times T \to T$, respectively. The triplet $(X_{(S,\theta,T)}, \pi_1, \pi_2)$ is called the *Borcea–Voisin threefold associated with* (S, θ, T). Two Borcea–Voisin threefolds $(X_{(S,\theta,T)}, \pi_1, \pi_2)$ and $(X_{(S',\theta',T')}, \pi_1', \pi_2')$ are isomorphic if there exist isomorphisms of complex manifolds

$$f\colon X_{(S,\theta,T)} \to X_{(S',\theta',T')}, \qquad g\colon S/\mathbf{Z}_2 \to S'/\mathbf{Z}_2, \qquad h\colon T/\mathbf{Z}_2 \to T'/\mathbf{Z}_2$$

such that $\pi'_1 \circ f = g \circ \pi_1$ and $\pi'_2 \circ f = h \circ \pi_2$.

By Borcea [**9**] and Voisin [**36**], $X_{(S,\theta,T)}$ is a Calabi–Yau threefold, which is equipped with the elliptic fibration $\pi_1 \colon X_{(S,\theta,T)} \to S/\mathbf{Z}_2$ with constant fiber T and with the $K3$-fibration $\pi_2 \colon X_{(S,\theta,T)} \to T/\mathbf{Z}_2$ with constant fiber S.

We recall another construction of $X_{(S,\theta,T)}$. Let $q \colon \widetilde{S \times T} \to S \times T$ be the blow-up of $S \times T$ along the curve $\Sigma_{(S,\theta,T)} := S^\theta \times T[2] = (S \times T)^{\theta \times (-1_T)}$. Let $\widetilde{\theta \times (-1_T)}$ be the involution on $\widetilde{S \times T}$ induced from $\theta \times (-1_T)$. We consider the $\mathbf{Z}_2$-action on $\widetilde{S \times T}$ induced from $\widetilde{\theta \times (-1_T)}$, so that $q \colon \widetilde{S \times T} \to S \times T$ is $\mathbf{Z}_2$-equivariant. Since $\theta \times (-1_T)$ acts as -1 on the normal bundle $N_{\Sigma_{(S,\theta,T)}/(S \times T)}$, $\widetilde{\theta \times (-1_T)}$ acts trivially on the exceptional divisor $q^{-1}(\Sigma_{(S,\theta,T)})$. Hence

$$(\widetilde{S \times T})^{\widetilde{\theta \times (-1_T)}} = q^{-1}(\Sigma_{(S,\theta,T)}).$$

Since $\widetilde{\theta \times (-1_T)}$ acts as the reflection with respect to the hypersurface $q^{-1}(\Sigma_{(S,\theta,T)})$, we have $K_{\widetilde{S \times T}} \cong \mathcal{O}_{\widetilde{S \times T}}(q^{-1}(\Sigma_{(S,\theta,T)}))$ and $K_{\widetilde{S \times T}/\mathbf{Z}_2} \cong \mathcal{O}_{\widetilde{S \times T}/\mathbf{Z}_2}$. Hence $\widetilde{S \times T}/\mathbf{Z}_2$ is a Calabi–Yau threefold. The natural projection $(\widetilde{S \times T})/\mathbf{Z}_2 \to (S \times T)/\mathbf{Z}_2$ induces an isomorphism

$$X_{(S,\theta,T)} \cong (\widetilde{S \times T})/\mathbf{Z}_2 = (\widetilde{S \times T})/\widetilde{\theta \times (-1_T)}. \tag{3.1}$$

By (3.1), the projections $\pi_1 \colon X_{(S,\theta,T)} \to S/\mathbf{Z}_2$ and $\pi_2 \colon X_{(S,\theta,T)} \to T/\mathbf{Z}_2$ are induced from the projections $\mathrm{pr}_1 \colon \widetilde{S \times T} \to S$ and $\mathrm{pr}_2 \colon \widetilde{S \times T} \to T$.

Definition 3.4. — Let $\Lambda \subset \mathbb{L}_{K3}$ be a primitive 2-elementary sublattice with signature $(2, r(\Lambda) - 2)$. A Borcea–Voisin threefold $(X_{(S,\theta,T)}, \pi_1, \pi_2)$ is of *type* Λ if there exists an isometry of lattices $H^2_-(S, \mathbf{Z}) := \{l \in H^2(S, \mathbf{Z});\ \theta^* l = -l\} \cong \Lambda$.

Notice that when $X_{(S,\theta,T)}$ is a Borcea–Voisin threefold of type Λ, (S, θ) is a 2-elementary $K3$ surface of type $\Lambda^\perp$.

Lemma 3.5. — *Let (S, θ) and (S', θ') be 2-elementary $K3$ surfaces of type $\Lambda^\perp$, and let T and T' be elliptic curves. Then the Borcea–Voisin threefolds $(X_{(S,\theta,T)}, \pi_1, \pi_2)$ and $(X_{(S',\theta',T')}, \pi'_1, \pi'_2)$ are isomorphic if and only if $(S, \theta) \cong (S', \theta')$ and $T \cong T'$.*

Proof. — Let $f \colon X_{(S,\theta,T)} \to X_{(S',\theta',T')}$, $g \colon S/\mathbf{Z}_2 \to S'/\mathbf{Z}_2$ and $h \colon T/\mathbf{Z}_2 \to T'/\mathbf{Z}_2$ be isomorphisms as in Definition 3.3. Let $\bar{t} = \{\pm t\} \in T/\mathbf{Z}_2$ be a regular value of π_2 and set $\overline{t'} := h(\bar{t}) \in T'/\mathbf{Z}_2$. Since $t \neq -t$, we have $\pi_2^{-1}(\bar{t}) = (S \times \{t\} \amalg S \times \{-t\})/\mathbf{Z}_2 \cong S$. Similarly, we have $(\pi'_2)^{-1}(\overline{t'}) \cong S'$. We obtain the involutions $\theta \colon S \to S$ and $\theta' \colon S' \to S'$ as the non-trivial covering transformations of the projections $\pi_1 \colon S = \pi_2^{-1}(\bar{t}) \to S/\mathbf{Z}_2$ and $\pi'_1 \colon S' = (\pi'_2)^{-1}(\overline{t'}) \to S'/\mathbf{Z}_2$, respectively. The isomorphism of fibers $f|_{\pi_2^{-1}(\bar{t})} \colon \pi_2^{-1}(\bar{t}) \to (\pi_2')^{-1}(\overline{t'})$ is an isomorphism from S to S' such that $\theta = (f|_{\pi_2^{-1}(\bar{t})})^{-1} \circ \theta' \circ f|_{\pi_2^{-1}(\bar{t})}$. This proves that $(S,\theta) \cong (S',\theta')$.

Let $x \in (S \setminus S^{\theta})/\mathbf{Z}_2$ be a regular value of π_1 and set $x' := g(x) \in S'/\mathbf{Z}_2$. Since $T = \pi_1^{-1}(x)$ and $T' = \pi_1^{-1}(x')$, the map $f|_{\pi_1^{-1}(x)}$ is an isomorphism from T to T'.

Conversely, if $(S,\theta) \cong (S',\theta')$ and $T \cong T'$, then it is obvious by construction that $(X_{(S,\theta,T)}, \pi_1, \pi_2) \cong (X_{(S',\theta',T')}, \pi_1', \pi_2')$. This proves the lemma. □

By Lemma 3.5, the following definition makes sense.

Definition 3.6. — Let $(X_{(S,\theta,T)}, \pi_1, \pi_2)$ be a Borcea–Voisin threefold of type Λ. The point $\widetilde{\varpi}_{\Lambda}(X_{(S,\theta,T)}, \pi_1, \pi_2)$ is defined as the pair of the periods of (S,θ) and T, i.e.,

$$\widetilde{\varpi}_{\Lambda}(X_{(S,\theta,T)}, \pi_1, \pi_2) := (\varpi_{\Lambda^{\perp}}(S,\theta), \Omega(T)) \in \mathcal{M}_{\Lambda}^{o} \times \mathfrak{M}.$$

Let $p \colon \mathcal{X} \to B$ be a proper, surjective holomorphic submersion between smooth complex spaces. Let $p_1 \colon (\mathcal{S}, \vartheta) \to B$ be a family of 2-elementary $K3$ surfaces of type Λ and let $p_2 \colon \mathcal{T} \to B$ be a family of elliptic curves with a holomorphic section. Then $\mathcal{T}$ is equipped with an involution $-1_{\mathcal{T}}$ which induces $-1_{p_2^{-1}(b)}$ for every $b \in B$. With respect to the trivial $\mathbf{Z}_2$-action on B, $p_2 \colon \mathcal{T} \to B$ is $\mathbf{Z}_2$-equivariant. Let $\pi_1 \colon \mathcal{X} \to \mathcal{S}/\mathbf{Z}_2$ and $\pi_2 \colon \mathcal{X} \to \mathcal{T}/\mathbf{Z}_2$ be surjective holomorphic maps such that $p = p_1 \circ \pi_1 = p_2 \circ \pi_2$. Then the quintet $(p \colon \mathcal{X} \to B, p_1 \colon (\mathcal{S}, \vartheta) \to B, p_2 \colon \mathcal{T} \to B, \pi_1, \pi_2)$ is called a *family of Borcea–Voisin threefold of type* Λ if $(p^{-1}(b), \pi_1|_{p^{-1}(b)}, \pi_2|_{p^{-1}(b)})$ is a Borcea–Voisin threefold of type Λ for all $b \in B$.

Theorem 3.7. — *The coarse moduli space of Borcea–Voisin threefolds of type Λ is isomorphic to $\mathcal{M}_{\Lambda}^{o} \times \mathfrak{M}$ via the map $\widetilde{\varpi}_{\Lambda}$.*

Proof. — By Lemma 3.5, the set of isomorphism classes of Borcea–Voisin threefold of type Λ is identified with $\mathcal{M}_{\Lambda}^{o} \times \mathfrak{M}$ via the map $\widetilde{\varpi}_{\Lambda}$. Since the period map $\varpi_{\Lambda^{\perp}}$ (resp. Ω) is holomorphic for every family of 2-elementary $K3$ surfaces of type $\Lambda^{\perp}$ (resp. elliptic curves), $\widetilde{\varpi}_{\Lambda}$ is also holomorphic for every family of Borcea–Voisin threefold of type Λ by Definition 3.6. □

By Theorem 3.7 and [**38**, Cor. 8.3], the coarse moduli space of Borcea–Voisin threefolds of type Λ is quasi-affine if $r(\Lambda) \leq 12$.

3.4. Degenerations of Borcea–Voisin threefolds

Theorem 3.8. — *Let $(\mathfrak{p}, \mathfrak{q}) \in \overline{\mathcal{D}}_{\Lambda}^{o} \times \mathfrak{M}$ and let $C \subset \mathcal{M}_{\Lambda}^{*}$ be an irreducible projective curve passing through $\mathfrak{p}$. Assume that $\mathfrak{p} \in C \setminus \operatorname{Sing} C$ and that C intersects $\overline{\mathcal{D}}_{\Lambda}^{o}$ transversally at $\mathfrak{p}$. Then there exist an irreducible projective fourfold $\mathcal{X}$, a pointed compact Riemann surface $(B, \mathfrak{b})$, a neighborhood U of $\mathfrak{b}$, a surjective flat holomorphic map $\pi \colon \mathcal{X} \to B$, and a holomorphic map $f \colon (B, \mathfrak{b}) \to (C, \mathfrak{p})$ satisfying*

(1) *$f(B) = C$ and the map $f|_U : (U, \mathfrak{b}) \to (f(U), \mathfrak{p})$ is an isomorphism;*
(2) *for all $b \in U \setminus \{\mathfrak{b}\}$, $\pi^{-1}(b)$ is the Calabi–Yau threefold underlying a Borcea–Voisin threefold $(\pi^{-1}(b), \pi_{1,b}, \pi_{2,b})$ of type Λ such that*

$$\widetilde{\varpi}_\Lambda(\pi^{-1}(b), \pi_{1,b}, \pi_{2,b}) = (f(b), \mathfrak{q}).$$

Proof. — By Theorem 3.1, there exist a pointed smooth projective curve $(B, \mathfrak{b})$, a neighborhood U of $\mathfrak{b}$, a holomorphic map $f : (B, \mathfrak{b}) \to (C, \mathfrak{p})$, a smooth projective threefold $\mathscr{W}$ with an involution $\theta : \mathscr{W} \to \mathscr{W}$, and a surjective holomorphic map $p : \mathscr{W} \to B$ satisfying Theorem 3.1 (1), (2), (3).

Let T be an elliptic curve with $\Omega(T) = \mathfrak{q} \in \mathfrak{M}$. Let Σ be the union of all 2-dimensional components of $(\mathscr{W} \times T)^{\theta \times (-1_T)} = \mathscr{W}^\theta \times T[2]$. Let $q : \widetilde{\mathscr{W} \times T} \to \mathscr{W} \times T$ be the blow-up of $\mathscr{W} \times T$ along Σ. Since $\theta \times (-1_T)$ acts as -1 on the normal bundle $N_{\Sigma/(\mathscr{W} \times T)}$ and since $q^{-1}(\Sigma) = \mathbf{P}(N_{\Sigma/(\mathscr{W} \times T)})$, $\theta \times (-1_T)$ lifts to an involution $\mathscr{I}$ on $\widetilde{\mathscr{W} \times T}$, which acts trivially on the exceptional divisor $q^{-1}(\Sigma)$.

We consider the $\mathbf{Z}_2$-action on $\widetilde{\mathscr{W} \times T}$ induced from $\mathscr{I}$, so that $q : \widetilde{\mathscr{W} \times T} \to \mathscr{W} \times T$ is $\mathbf{Z}_2$-equivariant. Set $\mathscr{X} := (\widetilde{\mathscr{W} \times T})/\mathbf{Z}_2 = (\widetilde{\mathscr{W} \times T})/\mathscr{I}$. Then $\mathscr{X}$ is an irreducible projective fourfold. Since the projections $p : \mathscr{W} \to B$, $\mathrm{pr}_1 : \mathscr{W} \times T \to \mathscr{W}$, and $q : \widetilde{\mathscr{W} \times T} \to \mathscr{W} \times T$ are $\mathbf{Z}_2$-equivariant, the composite $p \circ \mathrm{pr}_1 \circ q : \widetilde{\mathscr{W} \times T} \to B$ is $\mathbf{Z}_2$-equivariant and induces a surjective holomorphic map $\pi : \mathscr{X} \to B$. Since $\mathscr{X}$ is irreducible and $\dim B = 1$, $\pi : \mathscr{X} \to B$ is a flat holomorphic map.

For $b \in U \setminus \{\mathfrak{b}\}$, set $W_b := p^{-1}(b)$, $\theta_b := \theta|_{W_b}$ and $\Sigma_b := \Sigma \cap (W_b \times T)$. Then (W_b, θ_b) is a 2-elementary $K3$ surface of type $\Lambda^\perp$ and $\Sigma_b = W_b^{\theta_b} \times T[2]$ by Theorem 3.1 (3). Let $q_b : \widetilde{W_b \times T} \to W_b \times T$ be the blow-up along Σ_b. Since $W_b \times T$ intersects Σ transversely, we get $q^{-1}(W_b \times T) = \widetilde{W_b \times T}$ and $q_b = q|_{q^{-1}(W_b \times T)}$. Thus

$$(p \circ \mathrm{pr}_1 \circ q)^{-1}(b) = q^{-1} \circ (\mathrm{pr}_1)^{-1} \circ p^{-1}(b) = q^{-1}(W_b \times T) = \widetilde{W_b \times T}. \tag{3.2}$$

Since $p \circ \mathrm{pr}_1 \circ q$ is $\mathbf{Z}_2$-equivariant, $\mathscr{I}$ preserves the fibers of $p \circ \mathrm{pr}_1 \circ q$. Set $\mathscr{I}_b := \mathscr{I}|_{\widetilde{W_b \times T}}$. Since $q \circ \mathscr{I} \circ q^{-1}|_{(\mathscr{W} \times T) \setminus \Sigma} = \theta \times (-1_T)|_{(\mathscr{W} \times T) \setminus \Sigma}$ by the definition of $\mathscr{I}$, we get

$$q \circ \mathscr{I}_b \circ q^{-1}|_{(W_b \times T) \setminus \Sigma_b} = \theta_b \times (-1_T)|_{(W_b \times T) \setminus \Sigma_b}.$$

Since $q_b|_{\widetilde{(W_b \times T)} \setminus q_b^{-1}(\Sigma_b)} : \widetilde{(W_b \times T)} \setminus q_b^{-1}(\Sigma_b) \to (W_b \times T) \setminus \Sigma_b$ is an isomorphism,

$$\mathscr{I}_b|_{\widetilde{(W_b \times T)} \setminus q^{-1}(\Sigma_b)} = q_b^{-1} \circ (\theta_b \times (-1_T)) \circ q_b|_{\widetilde{(W_b \times T)} \setminus q^{-1}(\Sigma_b)} = \widetilde{\theta_b \times (-1_T)}|_{\widetilde{(W_b \times T)} \setminus q^{-1}(\Sigma_b)}$$

for all $b \in U \setminus \{\mathfrak{b}\}$. Since both of $\mathscr{I}_b$ and $\widetilde{\theta_b \times (-1_T)}$ are defined on $\widetilde{W_b \times T}$, this implies that

$$\mathscr{I}_b = \widetilde{\theta_b \times (-1_T)}. \tag{3.3}$$

By (3.1), (3.2), (3.3), we get

$$\pi^{-1}(b) = (p \circ \mathrm{pr}_1 \circ q)^{-1}(b)/\mathbf{Z}_2 = \widetilde{(W_b \times T)}/\mathscr{I}_b = X_{(W_b,\theta_b,T)}. \tag{3.4}$$

Consider the projections $\pi_{1,b}\colon X_{(W_b,\theta_b,T)} \to W_b/\mathbf{Z}_2$ and $\pi_{2,b}\colon X_{(W_b,\theta_b,T)} \to T/\mathbf{Z}_2$. Then the triplet $(\pi^{-1}(b), \pi_{1,b}, \pi_{2,b})$ is a Borcea–Voisin threefold of type Λ. Since $\varpi_{\Lambda^\perp}(W_b, \theta_b) = f(b)$ by Theorem 3.1 (3) and since $\Omega(T) = \mathfrak{q}$, we get $\widetilde{\varpi}_\Lambda(\pi^{-1}(b), \pi_{1,b}, \pi_{2,b}) = (f(b), \mathfrak{q})$ by (3.4). This completes the proof. □

Theorem 3.9. — *Let $\mathfrak{p} \in \mathscr{M}^o_\Lambda$. Let $p\colon \mathscr{E} \to B$ be an admissible elliptic fibration over a compact Riemann surface with a holomorphic section such that $\mathscr{E}$ is projective. Then there exist an irreducible projective fourfold $\mathscr{X}$ and a surjective flat holomorphic map $\pi\colon \mathscr{X} \to B$ such that $\pi^{-1}(b)$ is the Calabi–Yau threefold underlying a Borcea–Voisin threefold $(\pi^{-1}(b), \pi_{1,b}, \pi_{2,b})$ of type Λ such that*

$$\widetilde{\varpi}_\Lambda(\pi^{-1}(b), \pi_{1,b}, \pi_{2,b}) = (\mathfrak{p}, \Omega(p^{-1}(b))), \qquad b \in B^o.$$

Proof. — Set $E_b := p^{-1}(b)$ for $b \in B^o$. Let $-1_{\mathscr{E}}$ be the holomorphic involution on $\mathscr{E}$ preserving the fibers of p such that $-1_{\mathscr{E}}|_{E_b} = -1_{E_b}$ for all $b \in B^o$. Let (S, θ) be a 2-elementary $K3$ surface of type $\Lambda^\perp$ with $\mathfrak{p} = \varpi_{\Lambda^\perp}(S, \theta)$. Then $S \times \mathscr{E}$ is equipped with the $\mathbf{Z}_2$-action induced from the involution $\theta \times (-1_{\mathscr{E}})$. Let $\mathscr{E}[2]$ denote the set of fixed points of $-1_{\mathscr{E}}$. The fixed point set of $\theta \times (-1_{\mathscr{E}})$ is given by $S^\theta \times \mathscr{E}[2]$. Since $\dim \mathscr{E}[2] = 1$, we get $\dim(S^\theta \times \mathscr{E}[2]) = 2$, where $S^\theta \times \mathscr{E}[2]$ may not be pure dimensional. Let Σ be the union of all 2-dimensional components of $S^\theta \times \mathscr{E}[2]$. Then Σ is the disjoint union of smooth complex surfaces. Let $q\colon \widetilde{S \times \mathscr{E}} \to S \times \mathscr{E}$ be the blow-up along Σ. As in the proof of Theorem 3.8, $\theta \times (-1_{\mathscr{E}})$ lifts to an involution $\mathscr{I}$ on $\widetilde{S \times \mathscr{E}}$, which induces a $\mathbf{Z}_2$-action on $\widetilde{S \times \mathscr{E}}$. Then $q\colon \widetilde{S \times \mathscr{E}} \to S \times \mathscr{E}$ is $\mathbf{Z}_2$-equivariant.

Set $\mathscr{X} := (\widetilde{S \times \mathscr{E}})/\mathbf{Z}_2$, which is an irreducible projective fourfold. Since the projections $q\colon \widetilde{S \times \mathscr{E}} \to S \times \mathscr{E}$, $\mathrm{pr}_2\colon S \times \mathscr{E} \to \mathscr{E}$, and $p\colon \mathscr{E} \to B$ are $\mathbf{Z}_2$-equivariant, the composite map $p \circ \mathrm{pr}_2 \circ q\colon \widetilde{S \times \mathscr{E}} \to B$ is $\mathbf{Z}_2$-equivariant and induces a holomorphic surjection $\pi\colon \mathscr{X} \to B$.

Let $b \in B^o$. Let $\widetilde{S \times E_b}$ be the blow-up of $S \times E_b$ along $S^\theta \times E_b[2] = (S \times E_b) \cap \Sigma$. Since $S \times E_b$ intersects Σ transversally, we get $q^{-1}(S \times E_b) = \widetilde{S \times E_b}$. Thus

$$(p \circ \mathrm{pr}_2 \circ q)^{-1}(b) = q^{-1} \circ (\mathrm{pr}_2)^{-1} \circ p^{-1}(b) = q^{-1}(S \times E_b) = \widetilde{S \times E_b}. \tag{3.5}$$

Since $p \circ \mathrm{pr}_2 \circ q$ is $\mathbf{Z}_2$-equivariant, $\mathscr{I}$ preserves the fibers of $p \circ \mathrm{pr}_2 \circ q$. Set $\mathscr{I}_b := \mathscr{I}|_{\widetilde{S \times E_b}}$. Since $-1_{\mathscr{E}}|_{E_b} = -1_{E_b}$, we get

$$I_b = \widetilde{\theta \times (-1_{E_b})} \tag{3.6}$$

as before in the proof of Theorem 3.8. By (3.5), (3.6), we get

$$\pi^{-1}(b) = (p \circ \mathrm{pr}_2 \circ q)^{-1}(b)/\mathbf{Z}_2 = \widetilde{(S \times E_b)}/\mathscr{I}_b = X_{(S,\theta,E_b)}. \tag{3.7}$$

Consider the projections $\pi_{1,b}\colon X_{(S,\theta,E_b)} \to S/\mathbf{Z}_2$ and $\pi_{2,b}\colon X_{(S,\theta,E_b)} \to E_b/\mathbf{Z}_2$. Then $(\pi^{-1}(b), \pi_{1,b}, \pi_{2,b})$ is a Borcea–Voisin threefold of type Λ. Since $\varpi_{\Lambda^\perp}(S,\theta) = \mathfrak{p}$, we get $\widetilde{\varpi}_\Lambda(\pi^{-1}(b), \pi_{1,b}, \pi_{2,b}) = (\mathfrak{p}, \Omega(E_b))$. □

Example 3.10. — We consider the pencil of plane cubics

$$S := \{((x:y:z),(t_0:t_1)) \in \mathbf{P}^2 \times \mathbf{P}^1;\ t_0 y^2 z = 4t_0 x^3 - 3t_1 x z^2 - t_0 z^3\},$$

$B := \mathbf{P}^1$, $p := \mathrm{pr}_2\colon S \to \mathbf{P}^1$. Then $p\colon S \to \mathbf{P}^1$ is an elliptic fibration equipped with a section $\sigma\colon \mathbf{P}^1 \ni t = (t_0:t_1) \to ((0:1:0),t) \in S$. When t is a regular value of p, $\sigma(t)$ is the identity element of $p^{-1}(t)$. The involution

$$-1_S\colon S \ni ((x:y:z),(t_0:t_1)) \to ((x:-y:z),(t_0:t_1)) \in S$$

induces the map $-1_{p^{-1}(t)}$ when t is a regular value of p. Let $(\mathscr{E}, -1_{\mathscr{E}}) \to (S, -1_S)$ be an equivariant resolution of the singularity of S and set $\widetilde{p} := q \circ p$. Then $\widetilde{p}\colon \mathscr{E} \to \mathbf{P}^1$ is an admissible elliptic fibration with section. Since $j_{\mathscr{E}/\mathbf{P}^1}(t) = \frac{27t^3}{27(t^3-1)}$, $1/j_{\mathscr{E}/\mathbf{P}^1}(t)$ is a local coordinate of $\mathbf{P}^1$ near the set $\{(t_0:t_1) \in \mathbf{P}^1;\ t_0^3 = t_1^3\} \subset \Delta_{\mathscr{E}/\mathbf{P}^1}$.

3.5. Borcea–Voisin threefolds of exceptional type. — Let 1_k denote the $k \times k$-identity matrix. For $\ell, m \in \mathbf{Z}$, we set

$$\mathbb{I}_{\ell,m} := \begin{pmatrix} 1_\ell & 0 \\ 0 & -1_m \end{pmatrix}, \qquad \mathbb{I}_{\ell,m}(2) = 2\begin{pmatrix} 1_\ell & 0 \\ 0 & -1_m \end{pmatrix},$$

which are identified with the corresponding lattices. Then $\mathbb{I}_{1,m}$ is an odd unimodular lattice and $\mathbb{I}_{1,m}(2)$ is a 2-elementary lattice. For $m \geq 0$, we define

$$\Lambda_m := \mathbb{U}(2) \oplus \mathbb{I}_{1,m-1}(2) \quad (m \geq 1), \qquad \Lambda_0 := \mathbb{I}_{2,0}(2).$$

By the classification of primitive 2-elementary Lorentzian sublattices of $\mathbb{L}_{K3}$ [**29**, p. 1434 Table 1], there exists a Borcea–Voisin threefold of type Λ_m if $0 \leq m \leq 9$.

Remark 3.11. — Let X be the Calabi–Yau threefold underlying a Borcea–Voisin threefold of type Λ and let $\pi\colon (\mathfrak{X}, X) \to (\mathrm{Def}(X), [X])$ be the Kuranishi family of X. We define the Borcea–Voisin locus $\mathrm{Def}(X)_{\mathrm{BV}} \subset \mathrm{Def}(X)$ as follows: $u \in \mathrm{Def}(X)_{\mathrm{BV}}$ if there exist a 2-elementary $K3$ surface (S_u, θ_u) of type $\Lambda^\perp$ and an elliptic curve T_u such that $\pi^{-1}(u) = X_{(S_u,\theta_u,T_u)}$. Comparing $\dim \mathrm{Def}(X)$ (cf. [**9**], [**36**]) and $\dim(\mathscr{M}^o_\Lambda \times \mathfrak{M})$, we have $\mathrm{Def}(X) = \mathrm{Def}(X)_{\mathrm{BV}}$ if and only if Λ is isometric to one of Λ_m $(0 \leq m \leq 9)$, $\mathbb{U}(2) \oplus \mathbb{U}(2)$, $\mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(2)$. When Λ is isometric to one of these lattices, then the Weil–Petersson metric on $\mathrm{Def}(X)$ coincides with the Bergman metric on $\Omega_\Lambda \times \mathfrak{H}$ (cf. Proof of Lemma 5.8). Notice that even if the moduli space is covered by a bounded symmetric domain, the Weil–Petersson metric does not necessarily coincide with the Bergman metric. For example, the moduli space of quintic mirror threefolds is covered by $\mathfrak{H}$, but the curvature of the Weil–Petersson metric is positive on some domain of the moduli space.

Lemma 3.12. — *Let $(X_{(S,\theta,T)}, \pi_1, \pi_2)$ be a Borcea–Voisin threefold of type Λ. If Λ is isometric to one of Λ_m $(0 \le m \le 9)$, $\mathbb{U}(2) \oplus \mathbb{U}(2)$, $\mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(2)$, then*

$$h^{1,2}(X_{(S,\theta,T)}) + \frac{\chi(X_{(S,\theta,T)})}{12} + 3 = 14. \tag{3.8}$$

Proof. — Set $N := \dim H^0(S^\theta, \mathbf{C})$ and $N' := \frac{1}{2} \dim H^1(S^\theta, \mathbf{C})$. By **[9]**, **[36]**, we get

$$h^{1,1}(X_{(S,\theta,T)}) = 11 - 5N - N', \qquad h^{1,2}(X_{(S,\theta,T)}) = 11 + 5N' - N. \tag{3.9}$$

Assume $\Lambda \cong \Lambda_m$ $(0 \le m \le 9)$ or $\Lambda \cong \mathbb{U}(2) \oplus \mathbb{U}(2)$. Set $r := r(\Lambda)$, $r^\perp := r(\Lambda^\perp)$ and $l^\perp := l(\Lambda^\perp) = l(\Lambda)$. Then $r^\perp = 22 - r$ and $l^\perp = r$. By Proposition 3.2,

$$N = 1 + \frac{r^\perp - l^\perp}{2}, \qquad N' = 11 - \frac{r^\perp + l^\perp}{2}. \tag{3.10}$$

By (3.9) and (3.10), we get

$$h^{1,1}(X_{(S,\theta,T)}) = 5r^\perp - 39, \qquad h^{1,2}(X_{(S,\theta,T)}) = 21 - r^\perp. \tag{3.11}$$

Since $\chi(X_{(S,\theta,T)}) = 2(h^{1,1}(X_{(S,\theta,T)}) - h^{1,2}(X_{(S,\theta,T)}))$, we get

$$\chi(X_{(S,\theta,T)}) = 12(r^\perp - 10). \tag{3.12}$$

The result follows from (3.11) and (3.12) in this case.

Assume $\Lambda \cong \mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(2)$. Then $\Lambda^\perp \cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$ and a 2-elementary $K3$ surface of type $\Lambda^\perp$ is the universal covering of an Enriques surface. Hence $N = N' = 0$ in this case. Since $h^{1,1}(X_{(S,\theta,T)}) = h^{1,2}(X_{(S,\theta,T)}) = 11$ and $\chi(X_{(S,\theta,T)}) = 0$ in this case, we get the result. □

4. Odd unimodular lattices and Borcherds products

In this section, we assume that Λ is a lattice of signature $(2, r(\Lambda) - 2)$.

4.1. Automorphic forms. — We fix a vector $l_\Lambda \in \Lambda \otimes \mathbf{R}$ with $\langle l_\Lambda, l_\Lambda \rangle \ge 0$. Hence $H_{l_\Lambda} = \varnothing$. We define

$$j_\Lambda(\gamma, [z]) := \frac{\langle \gamma(z), l_\Lambda \rangle}{\langle z, l_\Lambda \rangle}, \qquad [z] \in \Omega_\Lambda^+, \quad \gamma \in O^+(\Lambda).$$

Then $j_\Lambda(\gamma, \cdot)$ is a nowhere vanishing holomorphic function on Ω_Λ^+. A holomorphic function $f \in \mathscr{O}(\Omega_\Lambda^+)$ is called an *automorphic form on Ω_Λ^+ for $O^+(\Lambda)$ of weight p* if

$$f(\gamma \cdot [z]) = \chi(\gamma)\, j_\Lambda(\gamma, [z])^p\, f([z]), \qquad [z] \in \Omega_\Lambda^+, \quad \gamma \in O^+(\Lambda),$$

where $\chi \in \mathrm{Hom}(O^+(\Lambda), \mathbf{C}^*)$ is a character. For an automorphic form f on Ω_Λ^+ for $O^+(\Lambda)$ of weight p, the Petersson norm $\|f\|$ is the C^∞ function on Ω_Λ^+ defined as

$$\|f([z])\|^2 := K_\Lambda([z])^p\,|f([z])|^2, \qquad K_\Lambda([z]) := \frac{\langle z, \bar{z}\rangle}{|\langle z, l_\Lambda\rangle|^2}.$$

Since $O^+(\Lambda)/[O^+(\Lambda), O^+(\Lambda)]$ is finite when $r(\Lambda) \geq 5$, $\|f\|^2$ is $O^+(\Lambda)$-invariant.

Let ω_Λ be the Kähler form of the Bergman metric on Ω_Λ^+:

$$\omega_\Lambda := -dd^c \log K_\Lambda = \frac{1}{2\pi i}\partial\bar{\partial}\log K_\Lambda.$$

For a divisor D on Ω_Λ^+, δ_D denotes the Dirac δ-current on Ω_Λ^+ with support D.

4.2. Borcherds product associated with 2-elementary lattices. — For $\tau \in \mathfrak{H}$, set $q = e^{2\pi i\tau}$. The Dedekind η-function is defined by

$$\eta(\tau) := q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1 - q^n).$$

The theta series of the positive-definite A_1-lattice $\mathbb{A}_1^+ = \langle 2\rangle$ are defined by

$$\theta_{\mathbb{A}_1^+}(\tau) := \sum_{n\in\mathbf{Z}} q^{n^2}, \qquad \theta_{\mathbb{A}_1^+ + 1/2}(\tau) := \sum_{n\in\mathbf{Z}} q^{(n+\frac{1}{2})^2}.$$

Define $f_k^{(0)}(\tau), f_k^{(1)}(\tau) \in \mathcal{O}(\mathfrak{H})$ and the series $\{c_k^{(0)}(\ell)\}_{\ell\in\mathbf{Z}}$, $\{c_k^{(1)}(\ell)\}_{\ell\in\mathbf{Z}+k/4}$ by

$$\begin{array}{lll} f_k^{(0)}(\tau) & = \sum_{l\in\mathbf{Z}} c_k^{(0)}(\ell)\, q^\ell & := \eta(\tau)^{-8}\eta(2\tau)^8\eta(4\tau)^{-8}\,\theta_{\mathbb{A}_1^+}(\tau)^k, \\ f_k^{(1)}(\tau) & = \sum_{l\in k/4+\mathbf{Z}} 2c_k^{(1)}(\ell)\, q^\ell & := -16\,\eta(4\tau)^8\eta(2\tau)^{-16}\,\theta_{\mathbb{A}_1^+ + 1/2}(\tau)^k. \end{array}$$

We define holomorphic functions $g_k^{(i)}(\tau) \in \mathcal{O}(\mathfrak{H})$, $i \in \mathbf{Z}/4\mathbf{Z}$ by

$$g_k^{(i)}(\tau) := \sum_{\ell\equiv i \bmod 4} c_k^{(0)}(\ell)\, q^{\ell/4}.$$

Let $\mathbf{C}[A_\Lambda]$ be the group ring of the discriminant group A_Λ and let $\{\mathbf{e}_\gamma\}_{\gamma\in A_\Lambda}$ be its standard basis. Recall that the element $\mathbf{1}_\Lambda \in A_\Lambda$ was defined in Sect. 2.1. If Λ is 2-elementary and $r(\Lambda) \leq 12$, the $\mathbf{C}[A_\Lambda]$-valued holomorphic function on $\mathfrak{H}$

$$F_\Lambda(\tau) := f_{12-r(\Lambda)}^{(0)}(\tau)\,\mathbf{e}_0 + 2^{\frac{r(\Lambda)-l(\Lambda)}{2}} \sum_{\gamma\in A_\Lambda} g_{12-r(\Lambda)}^{(2\gamma^2)}(\tau)\,\mathbf{e}_\gamma + f_{12-r(\Lambda)}^{(1)}(\tau)\,\mathbf{e}_{\mathbf{1}_\Lambda}$$

is a modular form for $Mp_2(\mathbf{Z})$ of type ρ_Λ in the sense of [**12**, Sect. 2] by [**38**, Th. 7.7].

Let $N \in \{1, 2\}$ and let L be a 2-elementary Lorentzian lattice. Let $\mathcal{W}$ be a Weyl chamber of L. We set $\Lambda := \mathbb{U}(N) \oplus L$ and $l_\Lambda = (1, 0, 0)$ in Sect. 4.1. By [**12**, Th. 13.3], the following infinite product on $L\otimes\mathbf{R}+i\,\mathscr{W}$ converges absolutely when $(\mathrm{Im}\,z)^2\gg 0$ and it extends to an automorphic form on Ω^+_Λ for $O^+(\Lambda)$:

$$
\begin{aligned}
\Psi_\Lambda(z,F_\Lambda) := e^{2\pi i\langle\varrho(L,F_L,\mathscr{W}),z\rangle} &\prod_{\lambda\in L,\,\lambda\cdot\mathscr{W}>0,\,\lambda^2\geq -2}\left(1-e^{2\pi i\langle\lambda,z\rangle}\right)^{c^{(0)}_{12-r(\Lambda)}(\lambda^2/2)} \\
\times &\prod_{\lambda\in 2L^\vee,\,\lambda\cdot\mathscr{W}>0,\,\lambda^2\geq -2}\left(1-e^{\pi i N\langle\lambda,z\rangle}\right)^{2^{\frac{r(\Lambda)-l(\Lambda)}{2}}c^{(0)}_{12-r(\Lambda)}(\lambda^2/2)} \\
\times &\prod_{\lambda\in(\mathbf{1}_L+L),\,\lambda\cdot\mathscr{W}>0,\,\lambda^2\geq 0}\left(1-e^{2\pi i\langle\lambda,z\rangle}\right)^{2c^{(1)}_{12-r(\Lambda)}(\lambda^2/2)},
\end{aligned}
\tag{4.1}
$$

where $\varrho(L,F_L,\mathscr{W})\in L\otimes\mathbf{Q}$ is the Weyl vector of $(L,F_L,\mathscr{W})$. See [**12**, Th. 10.4] for an explicit formula for $\varrho(L,F_L,\mathscr{W})$. We refer to [**38**] for more about $\Psi_\Lambda(\cdot,F_\Lambda)$.

4.3. A Borcherds product associated with Λ_m. — Let $m\geq 1$. We fix a basis $\{h,d_1,\cdots,d_{m-1}\}$ of $\mathbb{I}_{1,m-1}(2)$ over $\mathbf{Z}$ such that

$$\langle h,h\rangle=2,\qquad\langle h,d_i\rangle=0,\qquad\langle d_i,d_j\rangle=-2\delta_{ij}\quad(1\leq i,j\leq m-1).$$

We define

$$\varrho_m:=\frac{1}{2}(3h-d_1-\cdots-d_{m-1})\in\mathbb{I}_{1,m-1}(2)^\vee=\mathbb{I}_{1,m-1}(1/2)$$

and

$$\mathit{\Pi}_m:=\{d\in\Delta_{\mathbb{I}_{1,m-1}(2)};\,\langle\varrho_m,d\rangle=1\}.$$

When $m\leq 9$, $\varrho_m^2>0$ and $\mathit{\Pi}_m$ is finite. See [**27**, Th. 26.2] for an explicit formula for $\mathit{\Pi}_m$. Let $\mathscr{W}_m$ be the Weyl chamber of $\mathscr{C}_{\mathbb{I}_{1,m-1}(2)}$ containing ϱ_m. Set

$$\mathrm{Aut}(\mathscr{W}_m):=\{g\in O(\mathbb{I}_{1,m-1}(2));\,g(\mathscr{W}_m)=\mathscr{W}_m\}.$$

Proposition 4.1. — *If $1\leq m\leq 9$, then the following hold:*

(1) *ϱ_m is a Weyl vector of $(\mathbb{I}_{1,m-1}(2),\mathscr{W}_m)$.*
(2) *$\mathit{\Pi}_m$ is the set of simple roots of $(\mathbb{I}_{1,m-1}(2),\mathscr{W}_m)$.*
(3) *$\mathscr{W}_m=\{v\in\mathbb{I}_{1,m-1}(2)\otimes\mathbf{R};\,v^2>0,\,\langle v,d\rangle>0\ \forall\,d\in\mathit{\Pi}_m\}$.*
(4) *$\{v\in\mathbb{I}_{1,m-1}(2)\otimes\mathbf{R};\,\langle v,d\rangle\geq 0,\,\forall\,d\in\mathit{\Pi}_m\}\subset\overline{\mathscr{C}^+_{\mathbb{I}_{1,m-1}(2)}}\subset\sum_{d\in\mathit{\Pi}_m}\mathbf{R}_{\geq 0}d$.*

Proof. — Since $\varrho(\mathbb{I}_{1,m-1}(2),F_{\mathbb{I}_{1,m-1}(2)},\mathscr{W}_m)=2\varrho_m$ by [**12**, Th. 10.4], we get (1) by [**38**, Th. 7.11 (2)]. We get the inclusion $\mathit{\Pi}(\mathbb{I}_{1,m-1}(2),\mathscr{W}_m)\subset\mathit{\Pi}_m$ by the definition of a Weyl vector of $(\mathbb{I}_{1,m-1}(2),\mathscr{W}_m)$. We prove the converse inclusion. Let $\delta\in\mathit{\Pi}(\mathbb{I}_{1,m-1}(2),\mathscr{W}_m)$. Since $\mathrm{Aut}(\mathscr{W}_m)$ acts transitively on $\mathit{\Pi}_m$ by [**27**, Cor. 26.7 (ii)],

$$\mathit{\Pi}_m=\mathrm{Aut}(\mathscr{W}_m)\cdot\delta\subset\mathrm{Aut}(\mathscr{W}_m)\cdot\mathit{\Pi}(\mathbb{I}_{1,m-1}(2),\mathscr{W}_m).$$

Since $\mathrm{Aut}(\mathscr{W}_m)$ preserves $\mathit{\Pi}(\mathbb{I}_{1,m-1}(2),\mathscr{W}_m)$, we get $\mathit{\Pi}_m\subset\mathit{\Pi}(\mathbb{I}_{1,m-1}(2),\mathscr{W}_m)$. This proves (2). We get (3) by (2.1) and (2).

Since $O(\mathbb{I}_{1,m-1}(2))/W(\mathbb{I}_{1,m-1}(2))$ is finite by [**29**, Cor. 4.2.3], the first inclusion of (4) follows from [**30**, Th. 1.4.3 and (1.4.5)]. Since $\overline{\mathscr{C}^{+}_{\mathbb{I}_{1,m-1}(2)}}$ is a self-dual cone and since $\sum_{d\in\Pi_m}\mathbf{R}_{\geq 0}d$ is the dual cone of $\{v\in\mathbb{I}_{1,m-1}(2)\otimes\mathbf{R};\ \langle v,d\rangle\geq 0,\ \forall\, d\in\Pi_m\}$, the second inclusion of (4) is a consequence of the first inclusion of (4). □

Theorem 4.2. — *If $1\leq m\leq 10$, then the following hold:*

(1) *There exists an automorphic form Φ_m on $\Omega^{+}_{\Lambda_m}$ for $O^{+}(\Lambda_m)$ of weight $14-m$ with zero divisor $\mathscr{D}_{\Lambda_m}$ such that*

$$\Phi_m(z)^2=\Psi_{\Lambda_m}(z,F_{\Lambda_m}).$$

(2) *The following identity holds for $z\in\mathbb{I}_{1,m-1}(2)\otimes\mathbf{R}+i\,\mathscr{W}_m$ with $(\operatorname{Im} z)^2\gg 0$:*

$$\Phi_m(z)=e^{2\pi i\langle\varrho_m,z\rangle}\prod_{\delta\in\{0,1\}}\prod_{\lambda\in\Pi_m^{+(\delta)}}\left(1-e^{2\pi i\langle\lambda,z\rangle}\right)^{c^{(\delta)}_{10-m}(\lambda^2/2)},$$

where $\Pi_m^{+(\delta)}:=\{\lambda\in\delta\varrho_m+\mathbb{I}_{1,m-1}(2);\ \lambda\cdot\mathscr{W}_m>0,\ \lambda^2\geq 2(\delta-1)\}$.

Proof. — Since $r(\Lambda_m)=l(\Lambda_m)$, we deduce from [**38**, Th. 8.1] that the weight of $\Psi_{\Lambda_m}(z,F_{\Lambda_m})$ is $2(14-m)$ and that $\operatorname{div}(\Psi_{\Lambda_m}(z,F_{\Lambda_m}))=2\,\mathscr{D}_{\Lambda_m}$. We set $\varphi=\|\Psi_{\Lambda_m}(z,F_{\Lambda_m})\|$ in [**37**, Th. 3.17]. Since we may choose $\nu(\Lambda_m^{\perp})=1$ in [**37**, Th. 3.17], we get the existence of an automorphic form Φ_m on $\Omega^{+}_{\Lambda_m}$ for $O^{+}(\Lambda_m)$ of weight $14-m$ with zero divisor $\mathscr{D}_{\Lambda_m}$. Comparing the weights and zeros, we get $\Phi_m^2=\Psi_{\Lambda_m}(\cdot,F_{\Lambda_m})$. This proves (1).

By [**12**, Th. 10.4], we get $\varrho(L,F_L,\mathscr{W})=2\varrho_m$ when $L=\mathbb{I}_{1,m-1}(2)$ and $\mathscr{W}=\mathscr{W}_m$. Since $\mathbb{I}_{1,m-1}(2)=\mathbb{A}_1^{+}\oplus\mathbb{A}_1\oplus\cdots\oplus\mathbb{A}_1$ and since $\mathbf{1}_{\mathbf{Z}h}=h/2$, $\mathbf{1}_{\mathbf{Z}d_i}=d_i/2$, we get $\mathbf{1}_{\mathbb{I}_{1,m-1}(2)}=(h+d_1+\cdots+d_{m-1})/2\equiv\varrho_m \mod \mathbb{I}_{1,m-1}(2)$. Since $L=\mathbb{I}_{1,m-1}(2)=2L^{\vee}$, $N=2$ and $r(\mathbb{I}_{2,m}(2))=l(\mathbb{I}_{2,m}(2))$ in (4.1), we get

$$\begin{aligned}\Phi_m(z)^2&=\Psi_{\Lambda_m}(z,F_{\Lambda_m})\\&=e^{2\pi i\langle 2\varrho_m,z\rangle}\prod_{\lambda\in\Pi_m^{+(0)}}\left(1-e^{2\pi i\langle\lambda,z\rangle}\right)^{2c^{(0)}_{10-m}(\frac{\lambda^2}{2})}\prod_{\lambda\in\Pi_m^{+(1)}}\left(1-e^{2\pi i\langle\lambda,z\rangle}\right)^{2c^{(1)}_{10-m}(\frac{\lambda^2}{2})}\\&=\left[e^{2\pi i\langle\varrho_m,z\rangle}\prod_{\delta\in\{0,1\}}\prod_{\lambda\in\Pi_m^{+(\delta)}}\left(1-e^{2\pi i\langle\lambda,z\rangle}\right)^{c^{(\delta)}_{10-m}(\lambda^2/2)}\right]^2.\end{aligned}$$

This proves (2). □

We study the invariance property of Φ_m. Recall that $W(\mathbb{I}_{1,m-1}(2))$ is the Weyl group of $\mathbb{I}_{1,m-1}(2)$. By Proposition 4.1 (3) and the definition of Π_m, we have

$$\operatorname{Aut}(\mathscr{W}_m)=\{g\in O^{+}(\mathbb{I}_{1,m-1}(2));\ g(\varrho_m)=\varrho_m\}.$$

By [**27**, Th. 23.9], $\mathrm{Aut}(\mathscr{W}_m)$ $(4 \leq m \leq 9)$ is isomorphic to the Weyl group of the root system of type $A_1 \times A_2$, A_4, D_5, E_6, E_7, E_8, respectively. Since the Weyl group $W(\mathbb{I}_{1,m-1}(2))$ acts transitively on the set of Weyl chambers of $\mathbb{I}_{1,m-1}(2)$, $O^+(\mathbb{I}_{1,m-1}(2))$ is generated by the reflection groups $W(\mathbb{I}_{1,m-1}(2))$ and $\mathrm{Aut}(\mathscr{W}_m)$.

Proposition 4.3. — *If* $1 \leq m \leq 9$, *then the following hold:*

(1) *For all* $r \in \mathbb{I}_{1,m-1}(2)^\vee$ *with* $\langle r, \varrho_m \rangle \equiv 0 \mod 2$,
$$\Phi_m(z+r) = \Phi_m(z).$$

(2) *For all* $w \in W(\mathbb{I}_{1,m-1}(2))$,
$$\Phi_m(w(z)) = \det(w)\, \Phi_m(z).$$

(3) *For all* $g \in \mathrm{Aut}(\mathscr{W}_m)$,
$$\Phi_m(g(z)) = \Phi_m(z).$$

Proof. — We get (1) by the infinite product expansion of Φ_m in Theorem 4.2 (2). Since $\mathrm{Aut}(\mathscr{W}_m)$ preserves ϱ_m and $\mathscr{W}_m$, $\Pi_m^{+\,(0)}$ and $\Pi_m^{+\,(1)}$ are $\mathrm{Aut}(\mathscr{W}_m)$-invariant. We get (3) by the infinite product expansion of Φ_m in Theorem 4.2.(2).

Since $O^+(\mathbb{I}_{1,m-1}(2)) \subset O^+(\mathbb{U}(2) \oplus \mathbb{I}_{1,m-1}(2))$ and since Φ_m is an automorphic form for $O^+(\mathbb{U}(2) \oplus \mathbb{I}_{1,m-1}(2))$, there is a character $\epsilon \in \mathrm{Hom}(O^+(\mathbb{I}_{1,m-1}(2)), \mathbf{C}^*)$ such that $\Phi_m(g(z)) = \epsilon(g)\, \Phi_m(z)$ for all $g \in O^+(\mathbb{I}_{1,m-1}(2))$. Since $W(\mathbb{I}_{1,m-1}(2))$ is generated by the reflections $\{s_\delta;\, \delta \in \Delta_{\mathbb{I}_{1,m-1}(2)}\}$, it suffices to prove $\epsilon(s_\delta) = -1$ for all $\delta \in \Delta_{\mathbb{I}_{1,m-1}(2)}$. Since $s_\delta^2 = 1$, we get $\epsilon(s_\delta) \in \{\pm 1\}$. If $\epsilon(s_\delta) = 1$, the vanishing order of Φ_m along the divisor H_δ would be an even integer, which contradicts Theorem 4.2 (1), i.e., $\mathrm{div}(\Phi_m) = \mathscr{D}_{\mathbb{I}_{1,m-1}(2)}$. Hence we get $\epsilon(s_\delta) = -1$. □

Question 4.4. — By Proposition 4.3 (1) and the infinite product expansion in Theorem 4.2, $\Phi_m(z)$ has a Fourier expansion with integral Fourier coefficients. By the same argument as in [**17**, Proof of Th. 2.3 (a)] (cf. [**21**]), we see that $\Phi_m(z)$ has a Fourier expansion of Lie type in the sense of [**18**, Def. 2.5.1]. Namely, the Fourier expansion of $\Phi_m(z)$ with respect to the cusp defined by a primitive isotropic vector of $\mathbb{U}(2)$ is of the form:
$$\sum_{w \in W(\mathbb{I}_{1,m-1}(2))} \det(w) \{e^{2\pi i \langle w(\varrho_m), z\rangle} - \sum_{r \in (\mathbb{I}_{1,m-1}(2) + \mathbf{Z}\varrho_m) \cap \overline{\mathscr{W}}_m \setminus \{0\}} m(r)\, e^{2\pi i \langle w(\varrho_m + r), z\rangle}\},$$
where $m(r) \in \mathbf{Z}$ for all $r \in (\mathbb{I}_{1,m-1}(2) + \mathbf{Z}\varrho_m) \cap \overline{\mathscr{W}}_m \setminus \{0\}$. To get this Fourier expansion, we used Propositions 4.1 and 4.3 (1), (2) instead of [**17**, Prop. 2.2, Eqs (2.2), (2.3)]. Since $\Phi_m(z)$ has a Fourier expansion of Lie type, there exists by [**17**, Sect. 3 and p. 222 Statement 6.8'], [**18**, Sect. 2.5] a Borcherds superalgebra $\mathfrak{g}_m$ such that $\mathfrak{g}_m$ is an automorphic correction of the Kac–Moody algebra defined by the generalized Cartan matrix $\langle d, \delta \rangle_{d,\delta \in \Pi_m}$ and such that $\Phi_m(z)$ is the denominator function of $\mathfrak{g}_m$.

Since $\Phi_m(z)$ has the $\mathrm{Aut}(\mathscr{W}_m)$-invariance by Proposition 4.3 (3), it is very likely that there is an $\mathrm{Aut}(\mathscr{W}_m)$-action on $\mathfrak{g}_m$ inducing the $\mathrm{Aut}(\mathscr{W}_m)$-invariance of $\Phi_m(z)$.

In Theorem 6.4 below, we shall see that $\Phi_m(z)$ is regarded as an automorphic form on the Kähler moduli of a Del Pezzo surface of degree $10-m$. A more interesting question is the construction of $\Phi_m(z)$ from the geometry of Del Pezzo surface. Is $\Phi_m(z)$ (or equivalently $\varPhi_V(z)$ in Sect. 6) related to the Borcherds superalgebra constructed in [**20**] for a Del Pezzo surface of degree $10-m$?

4.4. Borcherds products associated with the odd unimodular lattices. —

We identify $\mathbb{I}_{1,m-1}\otimes\mathbf{R}+i\,\mathscr{C}^{+}_{\mathbb{I}_{1,m-1}}$ with $\Omega^{+}_{\mathbb{U}\oplus\mathbb{I}_{1,m-1}}$ by the isomorphism (2.2).

Theorem 4.5. — *For $1\le m\le 10$, $\Phi_m(z/2)$ is an automorphic form on $\Omega^{+}_{\mathbb{U}\oplus\mathbb{I}_{1,m-1}}$ for $O^{+}(\mathbb{U}\oplus\mathbb{I}_{1,m-1})$ of weight $14-m$ with zero divisor $\sum_{d\in\mathbb{U}\oplus\mathbb{I}_{1,m-1},\,d^2=-1}H_d$.*

Proof. — Set $L=\mathbb{I}_{1,m-1}(2)$. Hence $L(\frac{1}{2})=\mathbb{I}_{1,m-1}$. By (2.2), $\Omega_{\mathbb{U}(2)\oplus L}=\Omega_{\Lambda_m}$ is isomorphic to $L\otimes\mathbf{R}+i\,\mathscr{C}_L$ via the map

$$\iota\colon L\otimes\mathbf{R}+i\,\mathscr{C}_L\ni z\to\left[\left(-\frac{1}{2}\langle z,z\rangle_L,\frac{1}{2},z\right)\right]\in\Omega_{\mathbb{U}(2)\oplus L}. \tag{4.2}$$

Identify $\mathbb{U}$ with $\mathbb{U}(2)$ via the identity map of the Abelian groups underlying them. The lattice $\mathbb{U}\oplus L(1/2)$ is an *odd* unimodular lattice. The map (2.2) gives the following identification between $L(1/2)\otimes\mathbf{R}+i\,\mathscr{C}_{L(1/2)}$ and $\Omega_{\mathbb{U}\oplus L(1/2)}$:

$$\iota'\colon L(1/2)\otimes\mathbf{R}+i\,\mathscr{C}_{L(1/2)}\ni z\to\left[\left(-\frac{1}{2}\langle z,z\rangle_{L(1/2)},1,z\right)\right]\in\Omega_{\mathbb{U}\oplus L(1/2)}. \tag{4.3}$$

The identity map of the free $\mathbf{Z}$-modules underlying $\Lambda_m=\mathbb{U}(2)\oplus L$ and $\mathbb{U}\oplus L(1/2)$ induces an isomorphism from $\Omega_{\mathbb{U}(2)\oplus L}$ to $\Omega_{\mathbb{U}\oplus L(1/2)}$. This isomorphism is denoted by $I\colon\Omega_{\mathbb{U}(2)\oplus L}\ni[z]\to[z]\in\Omega_{\mathbb{U}\oplus L(1/2)}$. By (4.2) and (4.3), we get

$$(\iota')^{-1}\circ I\circ\iota(z)=2z. \tag{4.4}$$

By (4.2), (4.3), (4.4), an automorphic form $\Psi(z)$ on $L(1/2)\otimes\mathbf{R}+i\,\mathscr{C}_{L(1/2)}$ for $O^{+}(\mathbb{U}\oplus L(1/2))$ is identified with the automorphic form $\Psi((\iota')^{-1}\circ I\circ\iota(z))=\Psi(2z)$ on $L\otimes\mathbf{R}+i\,\mathscr{C}_L$ for $O^{+}(\mathbb{U}(2)\oplus L)$ via the identity map $I\colon\Omega_{\mathbb{U}(2)\oplus L}\to\Omega_{\mathbb{U}\oplus L(1/2)}$. In particular, $\Phi_m(z/2)$ is an automorphic form on $\Omega^{+}_{\mathbb{U}\oplus\mathbb{I}_{1,m-1}}$ for $O^{+}(\mathbb{U}\oplus\mathbb{I}_{1,m-1})$ of weight $14-m$. Since the zero divisor of $\Phi_m(z/2)$ on $\Omega^{+}_{\mathbb{U}\oplus\mathbb{I}_{1,m-1}}$ coincides with the zero divisor of $\Phi_m(z)$ on $\Omega^{+}_{\mathbb{U}(2)\oplus\mathbb{I}_{1,m-1}(2)}$, we get

$$\mathrm{div}(\Phi_m(z/2))=\sum_{d\in\Delta_{\Lambda_m}}H_d=\sum_{d\in\mathbb{U}\oplus\mathbb{I}_{1,m-1},\,d^2=-1}H_d.$$

This proves the theorem. $\square$

Remark 4.6. — Let $\mathfrak{e}$, $\mathfrak{e}'$ be primitive isotropic vectors of Λ_m. By [**28**, Prop. 1.17.1], there exists $g \in O(\Lambda_m)$ with $g(\mathfrak{e}) = \mathfrak{e}'$ if and only if $\mathfrak{e}^\perp/\mathfrak{e} \cong (\mathfrak{e}')^\perp/\mathfrak{e}'$. Since $\mathfrak{e}^\perp/\mathfrak{e}$ is a unimodular Lorentzian lattice of signature $(1, m-1)$, Λ_m has a unique $O(\Lambda_m)$-orbit of primitive isotropic vectors if $m \neq 2, 10$. If $m = 2, 10$, there exist two $O(\Lambda_m)$-orbits of primitive isotropic vectors: If we set $\mathbb{V} := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, then $\Lambda_2 = \mathbb{U} \oplus \mathbb{V}$ and $\Lambda_{10} = \mathbb{U} \oplus \mathbb{V} \oplus \mathbb{E}_8$. Let $\mathfrak{e}$ (resp. $\mathfrak{e}'$) be a primitive isotropic vector of $\mathbb{U}$ (resp. $\mathbb{V}$). Then $\mathfrak{e}^\perp/\mathfrak{e}$ is an odd unimodular lattice, while $(\mathfrak{e}')^\perp/\mathfrak{e}'$ is an even unimodular lattice. Hence $\mathfrak{e}$ and $\mathfrak{e}'$ do not lie on the same $O(\Lambda_m)$-orbit. Since the choice of an $O(\Lambda_m)$-orbit of an isotropic vector of Λ_m corresponds to the choice of a zero-dimensional cusp of $\mathscr{M}^*_{\Lambda_m}$, $\mathscr{M}^*_{\Lambda_m}$ has a unique zero-dimensional cusp if $3 \leq m \leq 9$.

4.5. The Borcherds Φ-function and Φ_{10}. — By [**12**, Th. 13.3], [**38**, Th. 8.1], $\Psi_{\mathbb{U}(2)\oplus\mathbb{U}(2)\oplus\mathbb{E}_8(2)}(\cdot, F_{\mathbb{U}(2)\oplus\mathbb{U}(2)\oplus\mathbb{E}_8(2)})$ is a meromorphic function on $\mathscr{M}_{\mathbb{U}(2)\oplus\mathbb{U}(2)\oplus\mathbb{E}_8(2)}$ without zeros and poles and hence is a constant function. By comparing the exponents of the infinite product (4.1), this implies that the Fourier coefficients of $f_0^{(0)}(\tau)$ and $f_0^{(1)}(\tau)$ satisfy the following relation:

$$c_0^{(0)}(2m) + c_0^{(1)}(2m) = 0, \qquad m \in \mathbf{Z}. \tag{4.5}$$

Since $\eta(2\tau)^{-16}\eta(4\tau)^8 = \prod_{n=1}^\infty (1-q^{2n})(1+q^{2n})^{-1}$, we get by the definition of $f_0^{(1)}(\tau)$

$$c_0^{(0)}(2m-1) = 0, \qquad m \in \mathbf{Z}. \tag{4.6}$$

Let $\Lambda = \mathbb{U}(2) \oplus \mathbb{U} \oplus \mathbb{E}_8(2)$. The weight of $\Psi_\Lambda(\cdot, F_\Lambda)$ is 4 by [**12**, Th. 13.3], [**38**, Th. 8.1]. The automorphic form $\Psi_\Lambda(\cdot, F_\Lambda)$ is the Borcherds Φ-function of dimension 10 (cf. [**11**]). We set $N = 2$, $L = \mathbb{U} \oplus \mathbb{E}_8(2)$ and $\rho = ((0,1), 0_{E_8(2)})$ in (4.1). Then $\varrho(L, , F_L \mathscr{W}) = \rho$ by [**12**, Th. 10.4]. Substituting this into (4.1) and using (4.5), (4.6), we get the expression in [**11**]:

$$\Psi_\Lambda(z, F_\Lambda) = e^{2\pi i\langle\rho, z\rangle} \prod_{\lambda \in \Delta_L^+ \cup (L \cap \overline{\mathscr{C}}_L^+)} (1 - e^{2\pi i \langle\lambda, z\rangle})^{\epsilon(\lambda) c_0^{(0)}(\lambda^2/2)},$$

which is the denominator function of the fake monster algebra [**10**, Sect. 14 Example 3]. Here $\epsilon(\lambda) = 1$ when $\lambda \in 2L^\vee$. When $\lambda \in L \setminus (2L^\vee)$, we set $\epsilon(\lambda) = 1$ if $\lambda^2/2 \notin 2\mathbf{Z}$ and $\epsilon(\lambda) = -1$ if $\lambda^2/2 \in 2\mathbf{Z}$. Then $\Psi_\Lambda(\cdot, F_\Lambda)$ is identified with Φ_{10} as follows.

Using the basis $\{h, d_1, \ldots, d_9\}$ of $\mathbb{I}_{1,9}(2)$ with Gram matrix $\mathbb{I}_{1,9}(2)$, we define

$$K := \{k \in \mathbb{I}_{1,9}(2);\ \langle k, d_9\rangle = \langle k, 3h - \sum_{i=1}^8 d_i\rangle = 0\} \cong \mathbb{E}_8(2),$$

where the last isometry follows from e.g. [**27**, Th. 25.4]. We set

$$\mathfrak{f} := (3h - \sum_{i=1}^9 d_i)/2 = \varrho_9, \qquad \mathfrak{f}' := (3h - \sum_{i=1}^8 d_i + d_9)/2.$$

Then $\mathfrak{f}^2 = (\mathfrak{f}')^2 = 0$ and $\langle \mathfrak{f}, \mathfrak{f}' \rangle = 1$. We define $L := \mathbf{Z}\mathfrak{f} + \mathbf{Z}\mathfrak{f}' + \mathbf{Z}h + \sum_{i=1}^{9} \mathbf{Z}d_i$, which is equipped with the bilinear form induced from $\mathbb{I}_{1,9}(2)$. Since

$$(4.7)\quad \mathbf{Z}h \oplus \mathbf{Z}d_1 \oplus \cdots \oplus \mathbf{Z}d_9 = \mathbf{Z}(3h - \sum_{i=1}^{8} d_i) \oplus \mathbf{Z}d_9 \oplus K = \mathbf{Z}(\mathfrak{f}' + \mathfrak{f}) \oplus \mathbf{Z}(\mathfrak{f}' - \mathfrak{f}) \oplus K$$

and hence $L = \mathbf{Z}\mathfrak{f} \oplus \mathbf{Z}\mathfrak{f}' \oplus K$, we get $L \cong \mathbb{U} \oplus \mathbb{E}_8(2)$. Since $\mathbb{I}_{1,9}(2) \subset L$, we have the inclusion of lattices $\Lambda_{10} = \mathbb{U}(2) \oplus \mathbb{I}_{1,9}(2) \subset \mathbb{U}(2) \oplus L = \Lambda$, which yields the identification $\Omega_{\Lambda_{10}} = \Omega_\Lambda$. Since $O(\Lambda_{10}) = \{g \in O(\Lambda);\, g(\Lambda_{10}) = \Lambda_{10}\} \subset O(\Lambda)$, an automorphic form on $\Omega^+_{\Lambda_{10}}$ for $O^+(\Lambda_{10})$ is identified with an automorphic form on Ω^+_Λ for the cofinite subgroup $O^+(\Lambda_{10}) \subset O^+(\Lambda)$.

Theorem 4.7. — *Under the identification $\Omega^+_{\Lambda_{10}} = \Omega^+_\Lambda$ and the inclusion of groups $O^+(\Lambda_{10}) \subset O^+(\Lambda)$ induced from the inclusion of lattices $\Lambda_{10} \subset \Lambda$ as above,*

$$\Phi_{10} = \Psi_\Lambda(\cdot, F_\Lambda).$$

Proof. — We prove $\Delta_{\Lambda_{10}} = \Delta_\Lambda$. Since $\Lambda_{10} \subset \Lambda$ and hence $\Delta_{\Lambda_{10}} \subset \Delta_\Lambda$, it suffices to prove $\Delta_{\Lambda_{10}} \supset \Delta_\Lambda$. Let $d = (a, b, m, n, \lambda) \in \Delta_\Lambda$, where $(a, b) \in \mathbb{U}(2)$, $(m, n) \in \mathbb{U}$, and $\lambda \in \mathbb{E}_8(2)$. Since $d^2 = 4ab + 2mn + \lambda^2 = -2$ and $\lambda^2 \equiv 0 \mod 4$, we get $mn \equiv 1 \mod 2$ and hence $m \equiv n \equiv 1 \mod 2$. By (4.7), we get

$$m\,\mathfrak{f} + n\,\mathfrak{f}' + \lambda = \frac{m+n}{2}(\mathfrak{f} + \mathfrak{f}') + \frac{n-m}{2}(\mathfrak{f}' - \mathfrak{f}) + \lambda \in \mathbb{I}_{1,9}(2).$$

This proves $d \in \Lambda_{10} = \mathbb{U}(2) \oplus \mathbb{I}_{1,9}(2)$. Since $\Delta_{\Lambda_{10}} = \Delta_\Lambda$ via the inclusion $\Lambda_{10} \subset \Lambda$, both of Φ_{10} and $\Psi_\Lambda(\cdot, F_\Lambda)$ are automorphic forms on Ω^+_Λ for $O^+(\Lambda_{10})$ of weight 4 with zero divisor $\mathscr{D}_\Lambda$. Hence $\Phi_{10} = \mathrm{Const.}\, \Psi_\Lambda(\cdot, F_\Lambda)$ by the Koecher principle. Comparing $\lim_{z \to +i\infty} \Phi_{10}(z)$ and $\lim_{z \to +i\infty} \Psi_\Lambda(z, F_\Lambda)$, we get the result. □

5. The BCOV invariant of Borcea–Voisin threefolds

5.1. The BCOV invariant of Calabi–Yau threefolds. — Let X be a compact Kähler manifold with Kähler form γ. Let $D := \sqrt{2}(\bar\partial + \bar\partial^*)$ be the Dirac operator of (X, γ) and let $\square_{p,q} := D^2$ be the Laplacian of (X, γ) acting on (p, q)-forms on X. Let $\zeta_{p,q}(s)$ be the spectral zeta function of $\square_{p,q}$. After Ray-Singer **[33]**, Bismut-Gillet-Soulé **[7]**, and Bershadsky-Cecotti-Ooguri-Vafa **[3]**, we make the following:

Definition 5.1. — The *BCOV torsion* of (X, γ) is the real number defined by

$$\mathscr{T}_{\mathrm{BCOV}}(X, \gamma) := \exp[-\sum_{p,q \geq 0} (-1)^{p+q} pq\, \zeta'_{p,q}(0)].$$

Assume that X is a Calabi–Yau n-fold. Let $\mathrm{Vol}(X, \gamma) = (2\pi)^{-n} \int_X \gamma^n / n!$ be the volume of (X, γ) and let $c_i(X, \gamma)$ denote the i-th Chern form of (TX, γ). Let η be a nowhere vanishing holomorphic n-form on X, whose L^2-norm is defined as $\|\eta\|_{L^2}^2 = (2\pi)^{-n}(\sqrt{-1})^{n^2}\int_X \eta\wedge\bar{\eta}$. Define

$$\mathscr{A}(X,\gamma) := \mathrm{Vol}(X,\gamma)^{\frac{\chi(X)}{12}} \exp\left[-\int_X \log\left(\frac{(\sqrt{-1})^{n^2}\eta\wedge\bar{\eta}}{\gamma^n/n!}\cdot\frac{\mathrm{Vol}(X,\gamma)}{\|\eta\|_{L^2}^2}\right)\frac{c_n(X,\gamma)}{12}\right].$$

Set $b_2(X) := \dim H^2(X,\mathbf{R})$. Let $\{\mathbf{e}_1,\dots,\mathbf{e}_{b_2(X)}\}$ be an integral basis of the free $\mathbf{Z}$-module $H^2(X,\mathbf{Z})_{\mathrm{fr}} := H^2(X,\mathbf{Z})/\mathrm{Torsion}$. Let κ be a Kähler class on X, and let $\mathrm{Vol}_{L^2}(H^2(X,\mathbf{Z}),\kappa)$ be the covolume of $H^2(X,\mathbf{Z})$ with respect to κ, i.e.,

$$\mathrm{Vol}_{L^2}(H^2(X,\mathbf{Z}),\kappa) := \det\left(\langle\mathbf{e}_i,\mathbf{e}_j\rangle_{L^2,\kappa}\right) = \mathrm{Vol}(H^2(X,\mathbf{R})/H^2(X,\mathbf{Z})_{\mathrm{fr}},\langle\cdot,\cdot\rangle_{L^2,\kappa}).$$

Definition 5.2. — When X is a Calabi–Yau threefold, define

$$\tau_{\mathrm{BCOV}}(X) := \frac{\mathscr{A}(X,\gamma)\,\mathscr{T}_{\mathrm{BCOV}}(X,\gamma)}{\mathrm{Vol}(X,\gamma)^3\,\mathrm{Vol}_{L^2}(H^2(X,\mathbf{Z}),[\gamma])}.$$

We call $\tau_{\mathrm{BCOV}}(X)$ the *BCOV invariant* of X.

The following result is a consequence of the curvature formula for Quillen metrics [**7**, Th. 0.1].

Theorem 5.3. — *When X is a Calabi–Yau threefold, $\tau_{\mathrm{BCOV}}(X)$ is independent of the choice of a Kähler metric on X. In particular, $\tau_{\mathrm{BCOV}}(X)$ is an invariant of X.*

Proof. — See [**14**, Th. 4.16]. □

5.2. The singularity of the BCOV invariant. — The following result is an application of the immersion formula for Quillen metrics [**8**], [**5**] (cf. [**39**]).

Theorem 5.4. — *Let $\mathscr{X}$ be an irreducible projective algebraic fourfold and let S be a compact Riemann surface. Let $\pi\colon\mathscr{X}\to S$ be a surjective, flat holomorphic map. Let $\mathscr{D}\subset S$ be a reduced divisor and set $\mathscr{X}^o := \mathscr{X}\setminus\pi^{-1}(\mathscr{D})$, $S^o := S\setminus\mathscr{D}$, $\pi^o := \pi|_{\mathscr{X}^o}$. Let $0\in\mathscr{D}$, and let (U,t) be a coordinate neighborhood of S centered at 0 such that $U\setminus\{0\}$ is isomorphic to the unit punctured disc in $\mathbf{C}$. If $\pi^o\colon\mathscr{X}^o\to S^o$ is a smooth morphism whose fibers are Calabi–Yau threefolds, then there exists $\alpha\in\mathbf{R}$ such that*

$$\log\tau_{\mathrm{BCOV}}(X_t) = \alpha\log|t|^2 + O(\log(-\log|t|^2)) \qquad (t\to 0).$$

Proof. — See [**14**, Th. 9.1]. □

For a Borcea–Voisin threefold $(X_{(S,\theta,T)},\pi_1,\pi_2)$ of type Λ, set

$$\tau^{\Lambda}_{\mathrm{BCOV}}(\varpi_{\Lambda^\perp}(S,\theta),\varOmega(T)) := \tau_{\mathrm{BCOV}}(X_{(S,\theta,T)}).$$

By Theorem 3.7, $\tau^{\Lambda}_{\mathrm{BCOV}}$ is a function on $\mathscr{M}^o_\Lambda\times\mathfrak{M}$.

Proposition 5.5. — *Let $(\mathfrak{p},\mathfrak{q}) \in \overline{\mathscr{D}}^{o}_{\Lambda} \times \mathfrak{M}$ and let $C \subset \mathscr{M}^{*}_{\Lambda}$ be an irreducible projective curve passing through $\mathfrak{p}$. Assume that $\mathfrak{p} \in C \setminus \operatorname{Sing} C$ and that C intersects $\overline{\mathscr{D}}^{o}_{\Lambda}$ transversally at $\mathfrak{p}$. Let (V,s) be a coordinate neighborhood of $\mathfrak{p}$ in C centered at $\mathfrak{p}$ satisfying $(\mathscr{M}^{*}_{\Lambda} \setminus \mathscr{M}^{o}_{\Lambda}) \cap \overline{V} = \{\mathfrak{p}\}$ and $\sup_{z\in V}|s(z)| < 1$. Then there exist constants $\alpha \in \mathbf{R}$ and $K \in \mathbf{R}_{>0}$ such that for all $z \in V \setminus \{\mathfrak{p}\}$,*

$$\left|\overline{\tau}^{\Lambda}_{\mathrm{BCOV}}|_{V}(z,\mathfrak{q}) + \alpha \log|s(z)|^{2}\right| \leq K \log(-\log|s(z)|^{2}). \tag{5.1}$$

Proof. — Let $\pi\colon \mathscr{X} \to B$, $f\colon (B,\mathfrak{b}) \to (C,\mathfrak{p})$, and $(U,\mathfrak{b}) \subset (B,\mathfrak{b})$ be the same as in Theorem 3.8. Choosing U sufficiently small, $f^{*}s$ is a coordinate on U centered at $\mathfrak{b}$. It suffices to prove (5.1) when $V = f(U)$. By Theorem 5.4 applied to the family $\pi\colon \mathscr{X} \to B$, there exist constants $\alpha \in \mathbf{R}$ and $K \in \mathbf{R}_{>0}$ such that for all $b \in U \setminus \{\mathfrak{b}\}$,

$$\left|\overline{\tau}^{\Lambda}_{\mathrm{BCOV}}|_{f(U)}(f(b),\mathfrak{q}) + \alpha \log|s(f(b))|^{2}\right| \leq K \log(-\log|s(f(b))|^{2}), \tag{5.2}$$

because $\overline{\tau}^{\Lambda}_{\mathrm{BCOV}}|_{f(U)}(f(b),\mathfrak{q}) = \tau_{\mathrm{BCOV}}(X_{(W_b,\theta_b,T)})$ by Theorem 3.8 (2). By setting $z = f(b)$, Estimate (5.1) follows from (5.2). □

Proposition 5.6. — *Let $\mathfrak{p} \in \mathscr{M}^{o}_{\Lambda}$. Let $p\colon \mathscr{E} \to B$ be an admissible elliptic fibration over a compact Riemann surface with a holomorphic section such that $\mathscr{E}$ is projective. For $\mathfrak{b} \in j^{-1}_{\mathscr{E}/B}(\{\infty\})$, let (V,s) be a coordinate neighborhood of $\mathfrak{b}$ in B centered at $\mathfrak{b}$ satisfying $\sup_{z\in V}|s(z)| < 1$ and $V \cap j^{-1}_{\mathscr{E}/B}(\{\infty\}) = \{\mathfrak{b}\}$. Then there exist constants $\beta \in \mathbf{R}$ and $K \in \mathbf{R}_{>0}$ such that for all $z \in V \setminus \{\mathfrak{b}\}$,*

$$\left|\overline{\tau}^{\Lambda}_{\mathrm{BCOV}}(\mathfrak{p}, j(E_b)) + \beta \log|s(z)|^{2}\right| \leq K \log(-\log|s(z)|^{2}). \tag{5.3}$$

Proof. — Let $\pi\colon \mathscr{X} \to B$ be the same as in Theorem 3.9. The result follows from Theorem 5.4 applied to the family $\pi\colon \mathscr{X} \to B$. □

5.3. The BCOV invariant of Borcea–Voisin threefolds of type Λ_m. — Let $\Delta(\tau) := \eta(\tau)^{24} = q\prod_{n=1}^{\infty}(1-q^{n})^{24}$ be the Jacobi Δ-function. Then $\Delta(\tau)$ is a cusp form on $\mathfrak{H}$ for $SL_2(\mathbf{Z})$ of weight 12. Let $\|\Delta(\tau)\|^{2} := (\operatorname{Im}\tau)^{12}|\Delta(\tau)|^{2}$ be the Petersson norm of $\Delta(\tau)$, which is a $SL_2(\mathbf{Z})$-invariant C^{∞} function on $\mathfrak{H}$. We often regard $\|\Delta(\tau)\|^{2}$ as a function on $\mathfrak{M} = SL_2(\mathbf{Z})\backslash\mathfrak{H}$.

Theorem 5.7. — *Assume that $m = 0$ or $4 \leq m \leq 9$ and set $\|\Phi_m\| := 1$ when $m = 0$. Then there exists a constant C_m depending only on m such that for every Borcea–Voisin threefold $(X_{(S,T)},\pi_1,\pi_2)$ of type Λ_m,*

$$\tau_{\mathrm{BCOV}}(X_{(S,\theta,T)}) = C_m\,\|\Phi_m(\varpi_{\Lambda_m^{\perp}}(S,\theta))\|^{2} \cdot \|\Delta(\Omega(T))\|^{2}. \tag{5.4}$$

Since Φ_m is the denominator function of a Borcherds superalgebra (cf. Question 4.4), Theorem 5.7 implies that the conjecture of Harvey–Moore [**19**, Sect. 7 Conjecture] holds for Borcea–Voisin threefolds of type Λ_m, $4 \leq m \leq 9$.

For the proof of Theorem 5.7, we need some intermediate results. Let

$$\varPi_m \colon \Omega_{\Lambda_m} \times \mathfrak{H} \to \mathscr{M}_{\Lambda_m} \times \mathfrak{M}$$

be the natural projection and set

$$\tau_{\mathrm{BCOV}}^{\Lambda_m} := \varPi_m^* \overline{\tau}_{\mathrm{BCOV}}^{\Lambda_m}.$$

By Theorems 3.7 and 5.3, $\tau_{\mathrm{BCOV}}^{\Lambda_m}$ is an $O^+(\Lambda_m) \times SL_2(\mathbf{Z})$-invariant C^∞ function on $\Omega_{\Lambda_m}^o \times \mathfrak{H}$. Set

$$\overline{F}_m := \log \left[\frac{\overline{\tau}_{\mathrm{BCOV}}^{\Lambda_m}}{\|\Phi_m\|^2 \|\Delta\|^2} \right].$$

Then $\overline{F}_m$ is a function on $\mathscr{M}_{\Lambda_m}^o \times \mathfrak{M}$. Set

$$F_m := \varPi_m^* \overline{F}_m,$$

which is an $O(\Lambda_m)^+ \times SL_2(\mathbf{Z})$-invariant C^∞ function on $\Omega_{\Lambda_m}^o \times \mathfrak{H}$.

Lemma 5.8. — *If $0 \leq m \leq 9$, then F_m is pluri-harmonic on $\Omega_{\Lambda_m}^o \times \mathfrak{H}$.*

Proof. — Let $X = X_{(S,\theta,T)}$ be the Calabi–Yau threefold underlying a Borcea–Voisin threefold of type Λ_m and let $\pi \colon (\mathfrak{X}, X) \to (\mathrm{Def}(X), [X])$ be the Kuranishi family of X. Similarly, let $\pi' \colon ((\mathfrak{S}, \Theta), (S, \theta)) \to (\mathrm{Def}(S,\theta), [(S,\theta)])$ and $\pi'' \colon (\mathfrak{T}, T) \to (\mathrm{Def}(T), [T])$ be the Kuranishi family of (S,θ) and T, respectively. Comparing the dimensions of the Kuranishi spaces (cf. Remark 3.11 and (3.11)), we have an isomorphism of germs $(\mathrm{Def}(S,\theta), [(S,\theta)]) \times (\mathrm{Def}(T), [T]) \cong (\mathrm{Def}(X), [X])$, which is induced by the map

$$(\mathrm{Def}(S,\theta), [(S,\theta)]) \times (\mathrm{Def}(T), [T]) \ni (s,t) \to [X_{(S_s,\theta_s,T_t)}] \in (\mathrm{Def}(X), [X]).$$

We regard $\mathrm{Def}(X)$ as a small open subset of $\Omega_{\Lambda_m}^o \times \mathfrak{H}$. Similarly, we regard $\mathrm{Def}(S,\theta)$ and $\mathrm{Def}(T)$ as small open subsets of $\Omega_{\Lambda_m}^o$ and $\mathfrak{H}$, respectively.

Let $\xi' \in H^0(\mathrm{Def}(S,\theta), \pi_* K_{\mathfrak{S}/\mathrm{Def}(S,\theta)})$, $\xi'' \in H^0(\mathrm{Def}(T), \pi_* K_{\mathfrak{T}/\mathrm{Def}(T)})$ and $\xi \in H^0(\mathrm{Def}(X), \pi_* K_{\mathfrak{X}/\mathrm{Def}(X)})$ be nowhere vanishing relative canonical forms, respectively. Then $\xi|_{(s,t)}$ is a non-zero holomorphic 3-form on $X_{(S_s,\theta_s,T_t)}$. Let $\|\xi\|_{L^2}^2$ be the C^∞ function on $\mathrm{Def}(X) \subset \Omega_{\Lambda_m}^o \times \mathfrak{H}$ defined as

$$\|\xi\|_{L^2}^2(s,t) = \left| \int_{X_{(S_s,\theta_s,T_t)}} \xi|_{(s,t)} \wedge \overline{\xi|_{(s,t)}} \right|, \qquad (s,t) \in \mathrm{Def}(X).$$

We define the functions $\|\xi'\|_{L^2}^2 \in C^\infty(\mathrm{Def}(S,\theta))$ and $\|\xi''\|_{L^2}^2 \in C^\infty(\mathrm{Def}(T))$ in the same manner. Since the holomorphic 3-form $\xi'|_s \wedge \xi''|_t$ on $(S_s \times T_t)/\theta_s \times (-1)_{T_t}$ lifts to a holomorphic 3-form on $X_{(S_s,\theta_s,T_t)}$, there is a nowhere vanishing holomorphic function $\psi \in \mathscr{O}(\mathrm{Def}(X))$ such that

$$\|\xi\|_{L^2}^2 = |\psi|^2 \, \|\xi'\|_{L^2}^2 \|\xi''\|_{L^2}^2.$$

Let ω_{WP} be the Weil-Petersson form on $\Omega_{\Lambda_m}^o \times \mathfrak{H}$. Then $\log \|\xi\|_{L^2}^2$ is a local potential function of ω_{WP} (cf. [**14**, Sect. 4.2]). Similarly, $\log \|\xi'\|_{L^2}^2$ is a local potential function of ω_{Λ_m} (cf. [**38**, Eq. (5.4)]). Let $\omega_{\mathfrak{H}}$ be the Kähler form of the Poincaré metric on $\mathfrak{H}$, i.e.,

$$\omega_{\mathfrak{H}} = -dd^c \log \operatorname{Im} \tau.$$

Then $\log \|\xi''\|^2_{L^2}$ is a local potential function of $\omega_{\mathfrak{H}}$.

Since ω_{WP}, ω_{Λ_m}, $\omega_{\mathfrak{H}}$ have potentials $\|\xi\|^2_{L^2}$, $\|\xi'\|^2_{L^2}$, $\|\xi''\|^2_{L^2}$ respectively, we have

$$\omega_{\mathrm{WP}}|_{\mathrm{Def}(X)} = -dd^c \log \|\xi\|^2_{L^2} = -dd^c \log(\|\xi'\|^2_{L^2}\|\xi''\|^2_{L^2}) = \omega_{\Lambda_m}|_{\mathrm{Def}(S,\theta)} + \omega_{\mathfrak{H}}|_{\mathrm{Def}(T)},$$

which implies the following equation of $(1,1)$-forms on $\Omega^o_{\Lambda_m} \times \mathfrak{H}$:

$$\omega_{\mathrm{WP}} = \omega_{\Lambda_m} + \omega_{\mathfrak{H}}. \tag{5.5}$$

Let $\mathrm{Ric}(\omega_{\mathrm{WP}})$, $\mathrm{Ric}(\omega_{\Lambda_m})$, $\mathrm{Ric}(\omega_{\mathfrak{H}})$ be the Ricci-forms of ω_{WP}, ω_{Λ_m}, $\omega_{\mathfrak{H}}$, respectively. By (5.5), we get

$$\mathrm{Ric}(\omega_{\mathrm{WP}}) = \mathrm{Ric}(\omega_{\Lambda_m}) + \mathrm{Ric}(\omega_{\mathfrak{H}}) = -m\,\omega_{\Lambda_m} - 2\,\omega_{\mathfrak{H}}, \tag{5.6}$$

where we used [**22**, Th. 4.1] and the explicit formula for the Bergman kernel [**23**, p. 34] to get the second equality. Notice that $K_\Lambda([z])^{-(r(\Lambda)-2)}$ is the Bergman kernel of Ω_Λ up to a constant by [**23**, p. 34].

Let $h^{1,2}$ and χ denote the Hodge number and the Euler characteristic of a Borcea–Voisin threefold of type Λ_m (cf. (3.11), (3.12)). By [**14**, Th. 4.14], Lemma 3.12, (5.5), (5.6), we get the following equation of C^∞ $(1,1)$-forms on $\Omega^o_{\Lambda_m} \times \mathfrak{H}$:

$$dd^c \log \tau^{\Lambda_m}_{\mathrm{BCOV}} = -\left(h^{1,2} + \frac{\chi}{12} + 3\right)\omega_{\mathrm{WP}} - \mathrm{Ric}(\omega_{\mathrm{WP}}) = -(14-m)\,\omega_{\Lambda_m} - 12\,\omega_{\mathfrak{H}}. \tag{5.7}$$

Since Φ_m is an automorphic form on $\Omega^+_{\Lambda_m}$ for $O^+(\Lambda_m)$ of weight $14-m$ with zero divisor $\mathscr{D}_{\Lambda_m}$ by Theorem 4.2 and since $\Delta(\tau)$ is an elliptic modular form for $SL_2(\mathbf{Z})$ without zeros on $\mathfrak{H}$, we get the following equation on $\Omega^o_{\Lambda_m} \times \mathfrak{H}$

$$-dd^c \log(\|\Phi_m\|^2 \|\Delta\|^2) = (14-m)\,\omega_{\Lambda_m} + 12\,\omega_{\mathfrak{H}},$$

which, together with (5.7), yields the desired equation $dd^c F_m = 0$ on $\Omega^o_{\Lambda_m} \times \mathfrak{H}$. This proves the lemma. □

Lemma 5.9. — *Let $\Delta \subset \mathbf{C}$ be the unit disc and set $\Delta^* := \Delta\setminus\{0\}$. Let f be a real-valued pluri-harmonic function on $\Delta^* \times \Delta^n$. Assume the existence of real-valued functions $\alpha(z)$ and $C(z)$ on Δ^n such that for all $|t| < \frac{1}{2}$ and $z \in \Delta^n$,*

$$|f(t,z) - \alpha(z)\,\log|t|^2| \le C(z)\,\log(-\log|t|).$$

Then $\alpha(z)$ is a constant function on Δ^n and there exists a real-valued pluri-harmonic function $\varphi(t,z)$ on Δ^{n+1} such that the following equation holds on $\Delta^* \times \Delta^n$:

$$f(t,z) = \alpha\,\log|t|^2 + \varphi(t,z), \qquad \alpha = \alpha(0).$$

In particular, the following identity of currents on Δ^{n+1} holds

$$dd^c f = \alpha\,\delta_{\{0\}\times\Delta^n}.$$

Proof. — Fix $z \in \Delta^n$. Since $dd^c f = 0$ on $\Delta^* \times \Delta^n$, we can put $P = f(\cdot, z)$, $\alpha = \alpha(z)$, $q = 0$ in [**37**, Prop. 3.11]. For each $z \in \Delta^n$, there exists by [**37**, Prop. 3.11] a harmonic function $\varphi(\cdot, z)$ on Δ satisfying the following the equation on $\Delta^* \times \{z\}$:

$$f(t, z) = \alpha(z) \log |t|^2 + \varphi(t, z).$$

By the same argument as in [**6**, pp. 54-75, Proof of Prop. 10.2 (ii)], $\alpha(z)$ is a constant function on Δ^n and $\varphi(t, z)$ is a pluri-harmonic function on Δ^{n+1}. □

Lemma 5.10. — *Let $0 \leq m \leq 9$. For every $d \in \Delta_{\Lambda_m}$, there exists $\alpha(d) \in \mathbf{R}$ such that the following equation of currents on $\Omega_{\Lambda_m} \times \mathfrak{H}$ holds:*

$$dd^c F_m = \sum_{d \in \Delta_{\Lambda_m}/\{\pm 1\}} \alpha(d)\, \delta_{H_d \times \mathfrak{H}}. \tag{5.8}$$

Proof. — Since the result is obvious when $m = 0$, we assume $1 \leq m \leq 9$. By [**37**, Prop. 1.9 (2)], there is a Zariski closed subset $Z_m \subset \Omega_{\Lambda_m}$ of codimension ≥ 2 such that $\Omega^o_{\Lambda_m} \cup \mathscr{D}^o_{\Lambda_m} = \Omega_{\Lambda_m} \setminus Z_m$. Let $P \subset \Omega^o_{\Lambda_m} \cup \mathscr{D}^o_{\Lambda_m}$ be a small polydisc and set $H := P \cap \mathscr{D}^o_{\Lambda_m}$. Choosing P smaller if necessary, we may assume that H is a smooth hypersurfaces of P. By the same argument as in [**37**, Sect. 7 Step 1], there is a system of coordinates $(f_1, \ldots, f_m)$ on P such that $f_1, \ldots, f_m$ extend to meromorphic functions on $\mathscr{M}^*_{\Lambda_m}$ and such that H is defined by the equation $f_1 = 0$. By Proposition 5.5 and Lemma 5.8, there exist real-valued functions $\alpha(f_2, \ldots, f_m, \tau)$ and $C(f_2, \ldots, f_m, \tau)$ defined on $H \times \mathfrak{H}$ such that the following estimate holds on $(P \setminus H) \times \mathfrak{H}$:

$$|F_m(f_1, f_2, \ldots, f_m, \tau) - \alpha(f_2, \ldots, f_m, \tau) \log |f_1|^2| \leq C(f_2, \ldots, f_m, \tau) \log(-\log |f_1|^2).$$

By Lemma 5.9 applied to $F_m|_{(P\setminus H)\times\mathfrak{H}}$, α is a constant function on $H \times \mathfrak{H}$ and the equation of currents $dd^c F_m|_{P\times\mathfrak{H}} = \alpha\, \delta_{H\times\mathfrak{H}}$ holds on $P \times \mathfrak{H}$. This implies (5.8) on $\Omega^o_{\Lambda_m} \cup \mathscr{D}^o_{\Lambda_m} = \Omega_{\Lambda_m} \setminus Z_m$. By [**35**, p. 53 Th. 1], Eq. (5.8) holds on Ω_{Λ_m}. □

Lemma 5.11. — *Let $m = 0$ or $4 \leq m \leq 9$. Then F_m is pluri-harmonic on $\Omega_{\Lambda_m} \times \mathfrak{H}$. In particular, $\overline{F}_m$ extends to a pluri-harmonic function on $\mathscr{M}_{\Lambda_m} \times \mathfrak{M}$.*

Proof. — When $m = 0$, Ω_{Λ_m} is a point and $\Delta_{\Lambda_m} = \emptyset$. The result follows from (5.8) in this case. We assume $4 \leq m \leq 9$. By Lemma 5.9, it suffices to prove $\alpha(d) = 0$ for all $d \in \Delta_{\Lambda_m}/\{\pm 1\}$. Let $\gamma \in O^+(\Lambda_m)$. Since F_m is $O^+(\Lambda_m)$-invariant and hence $\gamma^* dd^c F_m = dd^c F_m$, we get by (5.8)

$$\sum_{d \in \Delta_{\Lambda_m}/\{\pm 1\}} \alpha(d)\, \delta_{H_d \times \mathfrak{H}} = \gamma^*\Big(\sum_{d \in \Delta_{\Lambda_m}/\{\pm 1\}} \alpha(d)\, \delta_{H_d \times \mathfrak{H}}\Big) = \sum_{d \in \Delta_{\Lambda_m}/\{\pm 1\}} \alpha(d)\, \delta_{H_{\gamma(d)} \times \mathfrak{H}}.$$

Hence $\alpha(\gamma(d)) = \alpha(d)$ for all $d \in \Delta_{\Lambda_m}/\{\pm 1\}$ and $\gamma \in O^+(\Lambda_m)$. Since $m \geq 4$, $\Delta_{\Lambda_m}/\{\pm 1\}$ consists of a unique $O^+(\Lambda_m)$-orbit by [**38**, Prop. 11.8]. There exists $\alpha \in \mathbf{R}$ such that $\alpha(d) = \alpha$ for all $d \in \Delta_{\Lambda_m}/\{\pm 1\}$. Replacing F_m by $-F_m$ if necessary, we may assume that $\alpha \geq 0$.

Let $\mathfrak{q} \in \mathfrak{H}$ be an arbitrary point. Set $f_m := F_m|_{\Omega_{\Lambda_m} \times \{\mathfrak{q}\}}$. Equation (5.8) restricted to $\Omega_{\Lambda_m} \times \{\mathfrak{q}\}$ yields that

$$dd^c f_m = dd^c(F_m|_{\Omega_{\Lambda_m}\times\{\mathfrak{q}\}}) = \alpha \sum_{d\in\Delta_{\Lambda_m}/\{\pm1\}} \delta_{H_d\times\{\mathfrak{q}\}}. \tag{5.9}$$

Assume $\alpha \neq 0$. By (5.9) and the $O^+(\Lambda_m)$-invariance of $F_m|_{\Omega_{\Lambda_m}\times\{\mathfrak{q}\}}$, we may set $\varphi = f_m$, $p = q = 0$ in [**37**, Th. 3.17]. Then there would exist by [**37**, Th. 3.17] an integer $\nu \geq 1$ and an $O^+(\Lambda_m)$-invariant meromorphic function ψ on Ω_{Λ_m} such that

$$f_m = \alpha \log|\psi|^{2/\nu}, \qquad \operatorname{div}(\psi) = \nu\mathscr{D}_{\Lambda_m}.$$

Since $\dim(\mathscr{M}^*_{\Lambda_m} \setminus \mathscr{M}_{\Lambda_m}) \leq \dim \mathscr{M}^*_{\Lambda_m} - 2$ when $m \geq 3$, we deduce from the Levi extension theorem [**1**, Th. I.8.7] that ψ descends to a meromorphic function $\widetilde{\psi}$ on $\mathscr{M}^*_{\Lambda_m}$. Since $\operatorname{div}(\widetilde{\psi}) = \nu\overline{\mathscr{D}}_{\Lambda_m}$ by the relation $\operatorname{div}(\psi) = \nu\mathscr{D}_{\Lambda_m}$, we get a contradiction that the divisor of the meromorphic function $\widetilde{\psi}$ on the compact complex space $\mathscr{M}^*_{\Lambda_m}$ is non-zero and effective. Hence $\alpha(d) = \alpha = 0$ for all $d \in \Delta_{\Lambda_m}$. □

Lemma 5.12. — *Let* $\mathrm{pr}_2 \colon \mathscr{M}_{\Lambda_m} \times \mathfrak{M} \to \mathfrak{M}$ *be the projection. If* $m = 0$ *or* $4 \leq m \leq 9$, *then there exists a harmonic function* ϕ_m *on* $\mathfrak{M}$ *such that* $\overline{F}_m = (\mathrm{pr}_2)^*\phi_m$.

Proof. — Since $\mathscr{M}_{\Lambda_m}$ is a point when $m = 0$, the result is obvious in this case. We assume $4 \leq m \leq 9$. By Lemma 5.11, $\overline{F}_m$ extends to a pluri-harmonic function on $\mathscr{M}_{\Lambda_m} \times \mathfrak{M}$ when $4 \leq m \leq 9$. Since $\dim(\mathscr{M}^*_{\Lambda_m} \setminus \mathscr{M}_{\Lambda_m}) \leq \dim \mathscr{M}^*_{\Lambda_m} - 2$ when $m \geq 3$ and since $\mathscr{M}^*_{\Lambda_m}$ is normal, $\overline{F}_m$ extends to a pluri-harmonic function on $\mathscr{M}^*_{\Lambda_m} \times \mathfrak{M}$ by [**15**, Satz 4]. Since $\mathscr{M}^*_{\Lambda_m}$ is compact, $\overline{F}_m$ is constant on every slice $\mathscr{M}^*_{\Lambda_m} \times \{\mathfrak{q}\}$, $\mathfrak{q} \in \mathfrak{M}$, by the maximum principle. This proves the lemma. □

5.4. Proof of Theorem 5.7. — Let $\mathfrak{M}^*$ be the compactification of the modular curve $\mathfrak{M} = SL_2(\mathbf{Z})\backslash\mathfrak{H}$ and set $\infty := \mathfrak{M}^* \setminus \mathfrak{M}$. The j-function induces an isomorphism $j \colon \mathfrak{M}^* \cong \mathbf{P}^1$ with $j(\infty) = \infty$ and $j(\mathfrak{M}) = \mathbf{C}$, such that $1/j$ is a local coordinate of $\mathfrak{M}^*$ centered at ∞. Since $j(\tau) = q^{-1} + O(1)$ and $\Delta(\tau) = q + O(q^2)$ near $\tau = +i\infty$, the following estimate holds near $j = \infty$:

$$\log\|\Delta\|^2 = \log|j|^2 + O(\log\log|j|). \tag{5.10}$$

Let (S, θ) be a 2-elementary $K3$ surface of type Λ_m with period $\mathfrak{p} \in \mathscr{M}^o_{\Lambda_m}$. Let $p \colon \mathscr{E} \to B$ be an admissible elliptic surface with a holomorphic section such that $\mathscr{E}$ is projective and such that there exists a singular fiber of type I_1, i.e., a nodal rational curve with a unique node. Such an elliptic fibration exists by Example 3.10. Set $E_b := p^{-1}(b)$ for $b \in B$. Let $\mathfrak{b} \in \Delta_{\mathscr{E}/B}$ be such that $E_{\mathfrak{b}}$ is a nodal rational curve with a unique node. Then $1/j_{\mathscr{E}/B}$ is a local coordinate of B near $\mathfrak{b}$. By (5.3), (5.10) and the definition of $\overline{F}_m$, there exists $\gamma \in \mathbf{R}$ such that as $b \to \mathfrak{b}$,

$$\begin{aligned}\overline{F}_m(\mathfrak{p}, j_{\mathscr{E}/B}(b)) &= \log \overline{\tau}^{\Lambda_m}_{\mathrm{BCOV}}(\mathfrak{p}, j_{\mathscr{E}/B}(b)) - \log \|\Phi_m(\mathfrak{p})\|^2 - \log \|\Delta(\Omega(E_b))\|^2 \\ &= \gamma \log |j_{\mathscr{E}/B}(b)|^2 + O(\log\log |j_{\mathscr{E}/B}(b)|^2).\end{aligned} \tag{5.11}$$

Since $\overline{F}_m = (\mathrm{pr}_2)^*\phi_m$ and since ϕ_m is a harmonic function on $\mathfrak{M} = \mathbf{P}^1 \setminus \{\infty\}$, we deduce from (5.11) the following estimate near $j = \infty$:

$$\phi_m(j) = \gamma \log |j|^2 + O(\log\log |j|). \tag{5.12}$$

Assume that $\gamma \neq 0$. Since ϕ_m is a harmonic function on $\mathfrak{M} = \mathbf{P}^1 \setminus \{\infty\}$, $\partial\phi_m$ must be a meromorphic 1-form on $\mathbf{P}^1$ with divisor $\mathrm{div}(\partial\phi_m) = -\{\infty\}$ by (5.12). Namely, $\partial\phi_m$ is a logarithmic 1-form on $\mathbf{P}^1$ with a unique pole at ∞. This contradicts the residue theorem. Hence $\gamma = 0$ and ϕ_m extends to a harmonic function on $\mathbf{P}^1$. By the maximum principle, ϕ_m is a constant. This proves that $\overline{F}_m = \mathrm{pr}_2^*\phi_m$ is also a constant. This completes the proof of Theorem 5.7. □

The proof contains technical difficulties when $1 \leq m \leq 3$; when $m = 3$, $\overline{\mathscr{D}}_{\Lambda_m}$ is not irreducible by [**38**, Prop. 11.8] and we can not get Lemma 5.11 by the same argument; when $m = 1, 2$, the boundary locus $\mathscr{M}^*_{\Lambda_m} \setminus \mathscr{M}_{\Lambda_m}$ is a divisor of $\mathscr{M}^*_{\Lambda_m}$ and the Hartogs extension theorem does not apply in Lemmas 5.11 and 5.12.

Conjecture 5.13. — Equation (5.4) holds when $1 \leq m \leq 3$.

5.5. Factorization of the BCOV invariant for Borcea–Voisin threefolds. — Let (X, γ) be a compact Kähler manifold. Let G be a compact Lie group acting holomorphically on X and preserving γ. Recall that $\square_{0,q}$ is the Laplacian acting on C^∞ $(0,q)$-forms on X. Let $\sigma(\square_{0,q})$ be the spectrum of $\square_{0,q}$. For $\lambda \in \sigma(\square_{0,q})$, let $E_{0,q}(\lambda)$ be the eigenspace of $\square_{0,q}$ with respect to the eigenvalue λ. Since G preserves γ, $E_{0,q}(\lambda)$ is a finite-dimensional unitary representation of G. For $g \in G$ and $s \in \mathbf{C}$, set

$$\zeta_{0,q}(g)(s) := \sum_{\lambda \in \sigma(\square_{0,q}) \setminus \{0\}} \mathrm{Tr}\,(g|_{E_{0,q}(\lambda)})\, \lambda^{-s}.$$

Then $\zeta_{0,q}(g)(s)$ converges absolutely when $\mathrm{Re}\, s > \dim X$, admits a meromorphic continuation to the complex plane $\mathbf{C}$, and is holomorphic at $s = 0$. The *equivariant analytic torsion* of (X, γ) is the class function on G defined by

$$\tau_G(X, \gamma)(g) := \exp[-\sum_{q \geq 0} (-1)^q q\, \zeta'_{0,q}(g)(0)].$$

When $g = 1$, $\tau_G(X, \gamma)(1)$ is denoted by $\tau(X, \gamma)$. We refer to [**4**], [**25**] for more about equivariant analytic torsion.

Let (S, θ) be a 2-elementary $K3$ surface of type M. Identify $\mathbf{Z}_2$ with the subgroup of $\mathrm{Aut}(S)$ generated by θ. Let γ be a $\mathbf{Z}_2$-invariant Kähler form on S and let η be a nonzero holomorphic 2-form on S. Let $S^\theta = \sum_i C_i$ be the decomposition into the connected components. In [**37**], we introduced the number

$$\tau_M(S,\theta) := \mathrm{vol}(S,\gamma)^{\frac{14-r(M)}{4}} \tau_{\mathbf{Z}_2}(S,\gamma)(\theta) \prod_i \mathrm{Vol}(C_i,\gamma|_{C_i})\tau(C_i,\gamma|_{C_i})$$
$$\times \exp\left[\frac{1}{8}\int_{S^\theta} \log\left(\frac{\eta\wedge\bar\eta}{\gamma^2/2!}\cdot\frac{\mathrm{Vol}(S,\gamma)}{\|\eta\|_{L^2}^2}\right)\Bigg|_{S^\theta} c_1(S^\theta,\gamma|_{S^\theta})\right].$$

By [**37**], $\tau_M(S,\theta)$ is an invariant of the pair (S,θ), so that τ_M descends to a function on $\mathscr{M}^o_{M^\perp}$, the coarse moduli space of 2-elementary $K3$ surfaces of type M.

Theorem 5.14. — *If $m=0$ or $3\leq m\leq 9$, there exists a constant C_{Λ_m} depending only on Λ_m such that for every 2-elementary $K3$ surface (S,θ) of type $\Lambda_m^\perp$,*

$$\tau_{\Lambda_m^\perp}(S,\theta) = C_{\Lambda_m}\,\|\Phi_m(\varpi_{\Lambda_m^\perp}(S,\theta))\|^{-\frac{1}{2}}.$$

Proof. — Since $\mathscr{M}^o_{\Lambda_m}$ is a point when $m=0$, the result is obvious in this case. When $3\leq m\leq 9$, the result follows from [**38**, Th. 9.1] and Theorem 4.2 (1). □

Let E be an elliptic curve and let γ be a Kähler form on E. Let ξ be a nonzero holomorphic 1-form on E. We set

$$\tau_{\mathrm{elliptic}}(E) := \mathrm{Vol}(E,\gamma)\,\tau(E,\gamma)\,\exp\left[\frac{1}{12}\int_E \log\left(\frac{\xi\wedge\bar\xi}{\gamma}\right)c_1(E,\gamma)\right].$$

Since $\chi(E)=\int_E c_1(E,\gamma)=0$, $\tau(E)_{\mathrm{elliptic}}$ is independent of the choice of ξ.

Lemma 5.15. — *The following identity holds:*

$$\tau_{\mathrm{elliptic}}(E) = \|\Delta(\Omega(E))\|^{-\frac{1}{6}}.$$

Proof. — The result follows from [**7**, Th. 0.2] and the Kronecker limit formula. □

Theorem 5.16. — *Assume $m=0$ or $4\leq m\leq 9$. The following identity holds for every Borcea–Voisin threefold $(X_{(S,\theta,T)},\pi_1,\pi_2)$ of type Λ_m:*

$$\tau_{\mathrm{BCOV}}(X_{(S,\theta,T)}) = C_m C_{\Lambda_m}^4\,\tau_{\Lambda_m^\perp}(S,\theta)^{-4}\,\tau_{\mathrm{elliptic}}(T)^{-12}.$$

Proof. — The result follows from Theorems 5.7 and 5.14 and Lemma 5.15. □

Conjecture 5.17. — If $\Lambda\subset\mathbb{L}_{K3}$ is a primitive 2-elementary sublattice with $\mathrm{sign}(\Lambda)=(2,r(\Lambda)-2)$, then there exist constants $a(\Lambda)$, $b(\Lambda)$, $C(\Lambda)$ depending only on Λ such that for every Borcea–Voisin threefold $(X_{(S,\theta,T)},\pi_1,\pi_2)$ of type Λ,

$$\tau_{\mathrm{BCOV}}(X_{(S,\theta,T)}) = C(\Lambda)\,\tau_{\Lambda^\perp}(S,\theta)^{a(\Lambda)}\,\tau_{\mathrm{elliptic}}(T)^{b(\Lambda)}.$$

If this conjecture holds, then an explicit formula for the BCOV invariant of the Borcea–Voisin threefolds of type Λ will be obtained from [**38**, Th. 0.1] when $r(\Lambda)\leq 11$ or $(r(\Lambda),\delta(\Lambda))=(12,1)$.

Question 5.18. — Let $X_{(S,\theta,T)}$ be a Borcea–Voisin threefold and let $\pi\colon X_{(S,\theta,T)} \to (S\times T)/\mathbf{Z}_2$ be the projection with exceptional divisor $E := \pi^{-1}(\mathrm{Sing}\,(S\times T)/\mathbf{Z}_2)$. Then E has the structure of a $\mathbf{P}^1$-bundle over $\mathrm{Sing}\,(S\times T)/\mathbf{Z}_2$, whose fiber has negative intersection number with E.

Let γ be a Kähler metric on $(S\times T)/\mathbf{Z}_2$ in the sense of orbifolds and let γ_ϵ be a family of Kähler metrics on $X_{(S,\theta,T)}$ converging to γ as $\epsilon\to 0$ such that

$$[\gamma_\epsilon] = \pi^*[\gamma] - \epsilon\, c_1([E]), \qquad 0<\epsilon\ll 1.$$

It is very likely that $\mathscr{T}_{\mathrm{BCOV}}(X_{(S,\theta,T)},\gamma_\epsilon)$, $\mathscr{A}(X_{(S,\theta,T)},\gamma_\epsilon)$, $\mathrm{Vol}_{L^2}(H^2(X_{(S,\theta,T)},\mathbf{Z}),[\gamma_\epsilon])$ admit the following asymptotic expansions as $\epsilon\to 0$:

$$\begin{aligned}
&\log \mathscr{T}_{\mathrm{BCOV}}(X_{(S,\theta,T)},\gamma_\epsilon) = \alpha_1 \log\epsilon + \beta_1 + o(1),\\
&\log \mathscr{A}(X_{(S,\theta,T)},\gamma_\epsilon) = \alpha_2 \log\epsilon + \beta_2 + o(1),\\
&\log \mathrm{Vol}_{L^2}(H^2(X_{(S,\theta,T)},\mathbf{Z}),[\gamma_\epsilon]) = \alpha_3 \log\epsilon + \beta_3 + o(1).
\end{aligned}$$

It is worth asking explicit formulae for β_1, β_2, β_3, which will lead to direct proofs of Theorems 5.7 and 5.16 and Conjecture 5.13 (and possibly Conjecture 5.17).

Question 5.19. — As an application of the arithmetic Lefschetz formula [**24**], the arithmetic counterpart of the invariant τ_M and hence Φ_m was studied by Maillot–Rössler [**26**]. After [**26**] and Theorem 5.16, it is worth asking the arithmetic counterpart of the BCOV invariant for general Calabi–Yau threefolds.

6. Automorphic forms on the Kähler moduli of a Del Pezzo surface

6.1. Del Pezzo surfaces. — A compact connected smooth complex surface V is a *Del Pezzo surface* if its anti-canonical line bundle K_V^{-1} is ample. The integer $\deg V := c_1(V)^2$ is called the degree of V. Then $1\leq \deg V\leq 9$. Throughout this section, V *is a Del Pezzo surface.* A Del Pezzo surface of degree $d\neq 8$ is isomorphic to the blow-up of $\mathbf{P}^2$ at $9-d$ points in general position. A Del Pezzo surface of degree 8 is isomorphic to the blow-up of $\mathbf{P}^2$ at one point or to $\mathbf{P}^1\times\mathbf{P}^1$. If $\deg V = d$, then $H^2(V,\mathbf{Z})$ equipped with the cup-product is isometric to $\mathbb{I}_{1,9-d}$ or to $\mathbb{U}$. Let $\langle\cdot,\cdot\rangle_V$ denote the cup-product pairing on the total integral cohomology lattice of V

$$H(V,\mathbf{Z}) := H^0(V,\mathbf{Z})\oplus H^2(V,\mathbf{Z})\oplus H^4(V,\mathbf{Z}).$$

We have an isometry of lattices $(H(V,\mathbf{Z}),\langle\cdot,\cdot\rangle_V)\cong \mathbb{U}\oplus\mathbb{I}_{1,9-\deg V}$ if $V\not\cong\mathbf{P}^1\times\mathbf{P}^1$ and $(H(V,\mathbf{Z}),\langle\cdot,\cdot\rangle_V)\cong\mathbb{U}\oplus\mathbb{U}$ if $V\cong\mathbf{P}^1\times\mathbf{P}^1$. The $\mathbf{Z}$-module $H^0(V,\mathbf{Z})$ (resp. $H^4(V,\mathbf{Z})$) has natural generators $[1]$ (resp. $[V]^\vee$) such that $\langle[1],[V]^\vee\rangle_V = 1$.

Let $1\leq m\leq 9$ and let $P_1,\dots,P_{m-1}$ be $m-1$ points of $\mathbf{P}^2$ in general position. Let $\pi\colon V\to\mathbf{P}^2$ be the blow-up of $\mathbf{P}^2$ at $P_1,\dots,P_{m-1}$. Then V is a Del Pezzo surface of degree $10-m$. Set $E_i = \pi^{-1}(P_i)$. Then $E_1,\dots,E_{m-1}$ are (-1)-curves of V. Set

$H := \pi^* c_1(\mathcal{O}_{\mathbf{P}^2}(1)) \in H^2(V, \mathbf{Z})$ and $D_i := c_1([E_i])$, where $[E_i]$ is the line bundle on V defined by the divisor E_i. Then $\{H, D_1, \ldots, D_{m-1}\}$ is a basis of $H^2(V, \mathbf{Z})$ over $\mathbf{Z}$ with Gram matrix $\mathbb{I}_{1,m-1}$. By the adjunction formula, we have

$$c_1(V) = c_1(K_V^{-1}) = 3H - (D_1 + \cdots + D_{m-1}).$$

Recall that the basis $\{h, d_1, \ldots, d_{m-1}\}$ of $\mathbb{I}_{1,m-1}(2)$ and the Weyl vector $\varrho_m \in \mathbb{I}_{1,m-1}(2)^\vee$ were defined in Sect. 4.3. Let $\mathfrak{i}\colon H^2(V, \mathbf{Z}) \to \mathbb{I}_{1,m-1}(2)$ be the isomorphism of $\mathbf{Z}$-modules defined by

$$\mathfrak{i}(H) = h, \qquad \mathfrak{i}(D_i) = d_i \qquad (1 \le i \le m-1).$$

The following identities hold:

$$\langle \mathfrak{i}(v), \mathfrak{i}(w) \rangle_{\mathbb{I}_{1,m-1}(2)} = 2\langle v, w \rangle_V, \qquad \forall\, v, w \in H^2(V, \mathbf{Z}), \tag{6.1}$$

$$\mathfrak{i}(c_1(V)) = 2\varrho_m. \tag{6.2}$$

Set

$$\mathrm{Exc}(V) := \{c_1([C]) \in H^2(V, \mathbf{Z});\ C \text{ is a } (-1)\text{-curve on } V\}.$$

By [**27**, Th. 26.2 (i)],

$$\mathfrak{i}(\mathrm{Exc}(V)) = \Pi_m. \tag{6.3}$$

The set of effective classes on V is the subset of $H^2(V, \mathbf{Z})$ defined by

$$\mathrm{Eff}(V) := \{c_1(L) \in H^2(V, \mathbf{Z});\ L \in H^1(V, \mathcal{O}_V^*),\ h^0(L) > 0\}.$$

We set $\mathrm{Eff}(V)_{\ge m} := \{\alpha \in \mathrm{Eff}(V);\ \alpha^2 \ge m\}$ for $m \in \mathbf{Z}$. Let $\mathcal{K}_V \subset H^2(V, \mathbf{R})$ be the set of Kähler classes on V. By Nakai's criterion [**1**, Chap. IV Cor. 5.4], $\mathcal{K}_V$ is the cone of $H^2(V, \mathbf{R})$ given by $\mathcal{K}_V = \{x \in H^2(V, \mathbf{R});\ x^2 > 0,\ \langle x\ , \alpha \rangle_V > 0,\ \forall\, \alpha \in \mathrm{Eff}(V)\}$. If D is an irreducible projective curve on V with arithmetic genus $a(D)$, we get $c_1([D])^2 = 2a(D) - 2 + \deg(K_V^{-1}|_D) \ge 2a(D) - 1 \ge -1$ by the adjunction formula and the ampleness of K_V^{-1}. If $c_1([D])^2 = -1$ for an irreducible curve $D \subset V$, then $a(D) = 0$ and D must be a (-1)-curve by [**27**, Th. 26.2 (i)]. Hence $c_1([D])^2 \ge 0$ if $c_1([D]) \notin \mathrm{Exc}(V)$. Since $H^2(V, \mathbf{R})$ is a Lorentzian vector space, this implies that

$$\mathcal{K}_V = \{x \in H^2(V, \mathbf{R});\ x^2 > 0,\ \langle x\ , \delta \rangle_V > 0,\ \forall\, \delta \in \mathrm{Exc}(V)\}. \tag{6.4}$$

By Proposition 4.1 (3) and (6.3), (6.4), we get

$$\mathcal{W}_m = \mathfrak{i}(\mathcal{K}_V). \tag{6.5}$$

Lemma 6.1. — *Let L be a holomorphic line bundle on V with $c_1(L)^2 \ge -1$. Then $c_1(L) \cdot \mathcal{K}_V > 0$ if and only if L is effective, i.e., $h^0(L) > 0$.*

Proof. — Assume that $c_1(L)\cdot\mathscr{K}_V>0$ and $c_1(L)^2\geq -1$. By the Riemann-Roch theorem, $h^0(L)-h^1(L)+h^0(K_V\otimes L^{-1})=1+\{\langle c_1(V),c_1(L)\rangle_V+c_1(L)^2\}/2$. Since $c_1(V)\in\mathscr{K}_V$ and $c_1(L)^2\geq -1$, we get $h^0(L)+h^0(K_V\otimes L^{-1})\geq 1$. Since K_V^{-1} is ample, we get $\langle c_1(K_V^{-1}),c_1(K_V\otimes L^{-1})\rangle_V=-c_1(K_V^{-1})^2-\langle c_1(K_V^{-1}),c_1(L)\rangle_V<0$ by the condition $c_1(L)\cdot\mathscr{K}_V>0$. It follows from Nakai's criterion [**1**, Chap. 4 Cor. 5.4] that $K_V\otimes L^{-1}$ is not effective, i.e., $h^0(K_V\otimes L^{-1})=0$. Thus we get $h^0(L)>0$.

If $h^0(L)>0$ and $c_1(L)^2\geq -1$, then L is effective and hence $\langle c_1(L),\kappa\rangle_V>0$ for every Kähler class $\kappa\in H^2(V,\mathbf{R})$ on V. This proves the converse. □

Recall that the subset $\Pi_m^{+\,(\delta)}$ was defined in Theorem 4.2 (2).

Lemma 6.2. — *The following identities hold:*

(1) $\mathfrak{i}^{-1}(\Pi_m^{+(0)})=\mathrm{Eff}(V)_{\geq -1}$.
(2) $\mathfrak{i}^{-1}(\Pi_m^{+(1)})=\{\alpha\in H^2(V,\mathbf{Q});\ 2\alpha\in\mathrm{Eff}(V)_{\geq 0},\ \alpha\equiv c_1(V)/2 \mod H^2(V,\mathbf{Z})\}$.

Proof. — By (6.1), (6.5), the result is a consequence of Lemma 6.1 and the definition of $\Pi_m^{+\,(\delta)}$. □

6.2. An automorphic form on the Kähler moduli of V. — The *complexified Kähler cone* of V is the tube domain of $H^2(V,\mathbf{C})$ defined as $H^2(V,\mathbf{R})+i\,\mathscr{K}_V$. Recall that $\mathscr{C}_{H^2(V,\mathbf{Z})}$ is the positive cone of the Lorentzian vector space $H^2(V,\mathbf{Z})$. Let $\mathscr{C}_V^+$ be the component of $\mathscr{C}_{H^2(V,\mathbf{Z})}$ containing $\mathscr{K}_V$. The complexified Kähler cone of V is regarded as an open subset of $\Omega^+_{H(V,\mathbf{Z})}$ via (2.2):

$$H^2(V,\mathbf{R})+i\,\mathscr{K}_V\ni\eta\to\left[[1]+\eta-\frac{\eta^2}{2}[V]^\vee\right]\in\Omega^+_{H(V,\mathbf{Z})}.$$

Definition 6.3. — Define a formal infinite product $\Phi_V(w)$ on $H^2(V,\mathbf{R})+i\,\mathscr{K}_V$ by

$$\begin{aligned}\Phi_V(w):=e^{\pi i\langle c_1(V),w\rangle_V}&\prod_{\alpha\in\mathrm{Eff}(V)}\left(1-e^{2\pi i\langle\alpha,w\rangle_V}\right)^{c^{(0)}_{\deg V}(\alpha^2)}\\ &\times\prod_{\beta\in\mathrm{Eff}(V),\ \beta/2\equiv c_1(V)/2 \mod H^2(V,\mathbf{Z})}\left(1-e^{\pi i\langle\beta,w\rangle_V}\right)^{c^{(1)}_{\deg V}(\beta^2/4)}.\end{aligned}$$

This is an analogue of similar infinite products for algebraic $K3$ surfaces [**16**].

Theorem 6.4. — *The following identity holds:*

$$\Phi_V(w)=\Phi_{10-\deg V}(\mathfrak{i}(w)/2).$$

In particular, $\Phi_V(w)$ converges absolutely for $w\in H^2(V,\mathbf{R})+i\,\mathscr{K}_V$ with $(\mathrm{Im}\,w)^2\gg 0$. Under the identification $H^2(V,\mathbf{R})+i\,\mathscr{C}^+_{H^2(V,\mathbf{Z})}\cong\Omega^+_{H(V,\mathbf{Z})}$ given by (2.2), Φ_V extends to an automorphic form on $\Omega_{H(V,\mathbf{Z})}$ for $O^+(H(V,\mathbf{Z}))$ of weight $\deg V+4$ with zero divisor $\sum_{\delta\in H(V,\mathbf{Z}),\,\delta^2=-1}H_\delta$.

Proof. — Set $m = 10 - \deg V$. By Theorem 4.2 (2), we get

$$\begin{aligned}\Phi_m(\mathfrak{i}(w)/2) &= e^{2\pi i\langle \varrho_m, \mathfrak{i}(w)/2\rangle} \prod_{\delta\in\mathbf{Z}_2} \prod_{\lambda\in\Pi_m^{+(\delta)}} \left(1 - e^{2\pi i\langle\lambda,\mathfrak{i}(w)/2\rangle}\right)^{c^{(\delta)}_{10-m}(\lambda^2/2)} \\ &= e^{\pi i\langle c_1(V),w\rangle_V} \prod_{\delta\in\mathbf{Z}_2} \prod_{\alpha\in\mathfrak{i}^{-1}(\Pi_m^{+(\delta)})} \left(1 - e^{2\pi i\langle\alpha,w\rangle_V}\right)^{c^{(\delta)}_{10-m}(\alpha^2)} \\ &= e^{\pi i\langle c_1(V),w\rangle_V} \prod_{\alpha\in\mathrm{Eff}(V)} \left(1 - e^{2\pi i\langle\alpha,w\rangle_V}\right)^{c^{(0)}_{\deg V}(\alpha^2)} \\ &\quad\times \prod_{2\alpha\in\mathrm{Eff}(V),\ \alpha\equiv c_1(V)/2 \mod H^2(V,\mathbf{Z})} \left(1 - e^{2\pi i\langle\alpha,w\rangle_V}\right)^{c^{(1)}_{\deg V}(\alpha^2)} \\ &= \Phi_V(w),\end{aligned}$$

where the second equality follows from (6.1), (6.2) and the third equality follows from Lemma 6.2 and the vanishings $c_m^{(0)}(\ell) = 0$ for $\ell < -1$ and $c_m^{(1)}(\ell) = 0$ for $\ell < 0$. The rest of the theorem follows from Theorems 4.2 (1) and 4.5. $\square$

Remark 6.5. — Let Λ be the total cohomology lattice of a $K3$ surface. In [**12**, Example 15.2], Borcherds constructed an $O^+(\Lambda)$-invariant real analytic function on the Grassmannian $G^+(\Lambda)$ with singularities along the subgrassmannians orthogonal to vectors of Λ of norm -2. The automorphic form Φ_V may be regarded as an analogue of this Borcherds' function for Del Pezzo surfaces.

Let (S,θ) be a 2-elementary $K3$ surface of type M with $M^\perp \cong H(V,\mathbf{Z})(2)$. By definition, there is an isometry $\mathfrak{j}\colon H^2_-(S,\mathbf{Z}) \to H(V,\mathbf{Z})(2)$. By (2.2), there is a vector $\widehat{\varpi}_M(S,\theta,\mathfrak{j}) \in H^2(V,\mathbf{R}) + i\,\mathscr{C}_V^+$ with

$$\mathfrak{j}(H^{2,0}(S,\mathbf{C})) = \left[[1] + \widehat{\varpi}_M(S,\theta,\mathfrak{j}) - \frac{1}{2}\widehat{\varpi}_M(S,\theta,\mathfrak{j})^2\,[V]^\vee\right] \in \Omega^+_{H(V,\mathbf{Z})}.$$

Theorem 6.6. — *If* $\deg V \leq 7$*, there is a constant* $C_{\deg V}$ *depending only on* $\deg V$ *such that for every* 2*-elementary* $K3$ *surface* (S,θ) *of type* M *with* $M^\perp \cong \Lambda_{10-\deg V}$*,*

$$\tau_M(S,\theta) = C_{\deg V}\ \|\Phi_V(\widehat{\varpi}_M(S,\theta,\mathfrak{j}))\|^{-\frac{1}{2}}.$$

Notice that the left hand side is a function on the moduli space of 2-elementary $K3$ surfaces of type M, while the right hand side is a function on the Kähler moduli of the Del Pezzo surface V.

Proof. — Since $\mathfrak{i}\circ\mathfrak{j}\colon H^2_-(S,\mathbf{Z}) \to \mathbb{I}_{1,m-1}(2)$ is an isometry of lattices, the point $(-\frac{1}{4}\widehat{\varpi}_M(S,\theta,\mathfrak{j})^2, 1, \frac{1}{2}\mathfrak{i}(\widehat{\varpi}_M(S,\theta,\mathfrak{j}))) \in \Omega^+_{\Lambda_m}$ is the period of (S,θ). By Theorems 4.2, 5.13 and 6.4, we get

$$\tau_M(S,\theta) = C_M \left\| \Phi_m \left(\frac{1}{2} \mathfrak{i}(\widehat{\varpi}_M(S,\theta,\mathfrak{j})) \right) \right\|^{-\frac{1}{2}} = C_M \left\| \varPhi_V(\widehat{\varpi}_M(S,\theta,\mathfrak{j})) \right\|^{-\frac{1}{2}}.$$

Since the isometry class of M is determined by $\deg V$, we get the result. □

6.3. The functional equations of $\varPhi_V$. — Let $H^2(V,\mathbf{Z})_0$ be the maximal *even* sublattice of $H^2(V,\mathbf{Z})$:

$$H^2(V,\mathbf{Z})_0 := \{\alpha \in H^2(V,\mathbf{Z});\ \langle \alpha, c_1(V)\rangle_V \equiv 0 \mod 2\}.$$

Set $W(V) := \{g \in O^+(H^2(V,\mathbf{Z}));\ g(c_1(V)) = c_1(V)\}$. By [**27**, Th. 23.9], $W(V)$ is the Weyl group of the root system with root lattice $c_1(V)^\perp \subset H^2(V,\mathbf{Z})_0$. Set

$$\Gamma_V := H^2(V,\mathbf{Z})_0 \rtimes O^+(H^2(V,\mathbf{Z})), \qquad \widetilde{W}(V) := c_1(V)^\perp \rtimes W(V) \subset \Gamma_V.$$

Then $\widetilde{W}(V)$ is the affine Weyl group of the root system with root lattice $c_1(V)^\perp$. The group Γ_V preserves both of $H^2(V,\mathbf{R}) + i\,\mathscr{K}_V$ and $H^2(V,\mathbf{R}) + i\,\mathscr{C}_V^+$ and is regarded as a subgroup of $O^+(H(V,\mathbf{Z}))$ by the following injective homomorphism $\varphi\colon \Gamma_V \to O^+(H(V,\mathbf{Z}))$: For $(a,x,b) = a[1] + x + b[V]^\vee$,

$$\varphi_\lambda(a,x,b) := \begin{cases} a[1] + (x + a\lambda) + \left(b - \frac{\lambda^2}{2}a - \langle \lambda, x\rangle_V\right)[V]^\vee & (\lambda \in H^2(V,\mathbf{Z})_0), \\ a[1] + \lambda(x) + b[V]^\vee & (\lambda \in O^+(H^2(V,\mathbf{Z}))). \end{cases}$$

Then $\varphi(\Gamma_V)$ is the stabilizer of the isotropic vector $[1] \in H^0(V,\mathbf{Z})$ in $O^+(H(V,\mathbf{Z}))$.

Let G_V be the subgroup of $O^+(H(V,\mathbf{Z}))$ generated by the set

$$\varphi(\Gamma_V), \qquad \{s_{[1]+\delta}\}_{\delta\in \mathrm{Exc}(V)}, \qquad s_{[1]-[V]^\vee}, \qquad -1.$$

Following [**11**, Sect. 2], one can verify that G_V is a cofinite subgroup of $O^+(H(V,\mathbf{Z}))$ when $1 \le \deg V \le 7$. We give explicit functional equations of $\varPhi_V$ for the above system of generators of G_V. We set $\Lambda = H(V,\mathbf{Z})$ and $l_{H(V,\mathbf{Z})} = [V]^\vee$ in Sect. 4.1.

Let $\mathfrak{W}^{(1)}(V)$ be the subgroup of $O^+(H^2(V,\mathbf{Z}))$ generated by the reflections $\{s_\delta\}_{\delta\in\mathrm{Exc}(V)}$. Since $\mathscr{K}_V$ is a fundamental domain for the $\mathfrak{W}^{(1)}(V)$-action on $\mathscr{C}_V^+$ and since $W(V)$ is the stabilizer of $\mathscr{K}_V$ in $O^+(H^2(V,\mathbf{Z}))$, $O^+(H^2(V,\mathbf{Z}))$ is generated by $\mathfrak{W}^{(1)}(V)$ and $W(V)$. Let $\epsilon\colon O^+(H^2(V,\mathbf{Z})) \to \{\pm 1\}$ be the character such that $\epsilon(g) = 1$ for $g \in W(V)$ and $\epsilon(g) = \det(g)$ for $g \in \mathfrak{W}^{(1)}(V)$.

By Proposition 4.3 (1), (2), (3), we get the following equations for $\varphi(\Gamma_V)$:

$$\begin{aligned} &\text{(a)} \quad \varPhi_V(w + l) = \varPhi_V(w), && \forall\, l \in H^2(V,\mathbf{Z})_0, \\ &\text{(b)} \quad \varPhi_V(g(w)) = \epsilon(g)\,\varPhi_V(w), && \forall\, g \in O^+(H^2(V,\mathbf{Z})). \end{aligned}$$

In particular, $\varPhi_V(w)$ is invariant under the action of the affine Weyl group $\widetilde{W}(V)$.

Let $\delta \in \mathrm{Exc}(V)$. Since

$$\begin{aligned} & s_{[1]+\delta}\left([1]+(w+\delta)-\frac{(w+\delta)^2}{2}[V]^\vee\right) \\ & = -\langle w,w\rangle_V \left\{[1]+\left(-\frac{w}{\langle w,w\rangle_V}+\delta\right)-\frac{1}{2}\left(-\frac{w}{\langle w,w\rangle_V}+\delta\right)^2[V]^\vee\right\} \end{aligned}$$

and since Φ_V vanishes of order 1 on $H_{[1]+\delta}$, the automorphic property of Φ_V with respect to $s_{[1]+\delta}$ (cf. Sect. 4.1) implies that

$$\text{(c)} \qquad \Phi_V\left(-\frac{w}{\langle w,w\rangle_V}+\delta\right) = -(-\langle w,w\rangle_V)^{\deg V+4}\,\Phi_V(w+\delta), \qquad \forall\,\delta \in \mathrm{Exc}(V).$$

Since $s_{[1]-[V]^\vee}([1]+w-\frac{w^2}{2}[V]^\vee) = -\frac{w^2}{2}[1]+w+[V]^\vee$, the automorphic property of Φ_V with respect to $s_{[1]-[V]^\vee}$ implies that $\Phi_V(-\frac{2w}{\langle w,w\rangle_V}) = \epsilon(-\frac{\langle w,w\rangle_V}{2})^{\deg V+4}\,\Phi_V(w)$, $\epsilon \in \{\pm 1\}$. Since $[1]-[V]^\vee \in H(V,\mathbf{Z})$ is a vector of norm -2 and since Φ_V does not vanish on the divisor $H_{[1]-[V]^\vee} \subset \Omega^+_{H(V,\mathbf{Z})}$, we get $\epsilon = 1$, i.e.,

$$\text{(d)} \qquad \Phi_V\left(-\frac{2w}{\langle w,w\rangle_V}\right) = \left(-\frac{\langle w,w\rangle_V}{2}\right)^{\deg V+4}\Phi_V(w).$$

Remark 6.7. — When $1 \leq \deg V \leq 7$, the conditions $\mathrm{div}(\Phi_V) = \sum_{\delta\in H(V,\mathbf{Z}),\,\delta^2=-1} H_\delta$ and (a), (b), (c), (d) are sufficient to characterize Φ_V up to a constant, since $|O^+(H(V,\mathbf{Z}))/G_V| < \infty$.

6.4. Borcherds Φ-function as an analogue of Φ_V for Enriques surfaces. — Consider the case $N = 1$ and $L = \mathbb{U}(2) \oplus \mathbb{E}_8(2)$ in (4.1). Then $L^\vee = \frac{1}{2}L$, $\mathbf{1}_L = 0$, $\Delta_L = \varnothing$. By [**12**, Th. 10.4], we get $\varrho(L, F_L, \mathscr{W}) = 0$. Substituting these into (4.1), we get another expression of the Borcherds Φ-function [**12**, Example 13.7]

$$\text{(6.6)} \qquad \Psi_{\mathbb{U}\oplus L}(z, F_{\mathbb{U}\oplus L}) = \prod_{\lambda\in L\cap\overline{\mathscr{C}}^+_L}\left(\frac{1-e^{\pi i\langle\lambda,z\rangle_L}}{1+e^{\pi i\langle\lambda,z\rangle_L}}\right)^{c_0^{(0)}(\lambda^2/2)},$$

which is the Fourier expansion of the Borcherds Φ-function at the *level* 1 *cusp* and is the denominator function of the fake monster superalgebra [**34**]. We see that (6.6) is regarded as an analogue of Theorem 6.4 in the case of Enriques surfaces.

Let S be an Enriques surface [**1**, Chap. VIII] and let $p\colon \widetilde{S} \to S$ be the universal covering. Let $\theta \in \pi_1(\widetilde{S})$ be the generator. Hence $S = \widetilde{S}/\theta$. Assume that S contains no rational curves. Let $\mathscr{K}_S \subset H^2(S,\mathbf{R})$ be the Kähler cone of S. We define the infinite product Φ_S on the complexified Kähler cone $H^2(S,\mathbf{R}) + i\,\mathscr{K}_S$ by

$$\text{(6.7)} \qquad \Phi_S(w) := \prod_{\alpha\in H^2(S,\mathbf{Z})\cap\overline{\mathscr{K}}_S}\left(\frac{1-e^{2\pi i\langle\alpha,w\rangle_S}}{1+e^{2\pi i\langle\alpha,w\rangle_S}}\right)^{c_0^{(0)}(\alpha^2)}.$$

We set $H_+(\widetilde{S},\mathbf{Z}) := \{v \in H(\widetilde{S},\mathbf{Z}) = H^0(\widetilde{S},\mathbf{Z})\oplus H^2(\widetilde{S},\mathbf{Z})\oplus H^4(\widetilde{S},\mathbf{Z});\ \theta^* v = v\}$. Then $H_+(\widetilde{S},\mathbf{Z}) \cong \mathbb{U}\oplus H^2(\widetilde{S},\mathbf{Z}) \cong \mathbb{U}\oplus\mathbb{U}(2)\oplus\mathbb{E}_8(2) = \mathbb{U}\oplus L$. The pull-back $p^*\colon H^2(S,\mathbf{Z}) \to H^2(\widetilde{S},\mathbf{Z})$ induces the following embedding:

$$p^*\colon H^2(S,\mathbf{R}) + i\,\mathscr{K}_S \hookrightarrow H^2_+(\widetilde{S},\mathbf{R}) + i\,\mathscr{C}^+_{H^2_+(\widetilde{S},\mathbf{Z})} \cong \Omega^+_{H_+(\widetilde{S},\mathbf{Z})},$$

where $H^2_+(\widetilde{S},\mathbf{Z}) = H^2(\widetilde{S},\mathbf{Z}) \cap H_+(\widetilde{S},\mathbf{Z})$ and the last isomorphism is given by (2.2). By (6.6), Φ_S is an automorphic form on $\Omega^+_{H(\widetilde{S},\mathbf{Z})}$ for $O^+(H_+(\widetilde{S},\mathbf{Z}))$ of weight 4.

There is a formula for the analytic torsion of a Ricci-flat Enriques surface [**37**, Th. 8.3] analogous to Theorem 6.6: For every Ricci-flat Enriques surface (S,ω),

$$\mathrm{Vol}(S,\omega)^{\frac{1}{2}}\tau(S,\omega) = \mathrm{Const.}\,\|\Phi_S(\widehat{\varpi}(\widetilde{S},\theta))\|^{-\frac{1}{2}}.$$

Question 6.8. — After Theorem 4.7, it is worth asking the limiting situation in Theorem 6.4. Let W be the blow-up of $\mathbf{P}^2$ at 9 points. Is Φ_{10} regarded as an automorphic form on $H^2(W,\mathbf{R}) + i\,\mathscr{C}^+_W$? If this is the case, the Fourier expansion of the Borcherds Φ-function at the *level* 2 *cusp* would be regarded as an automorphic form on the complexified Kähler cone of W by Theorem 4.7. The case when these 9 points are given by the intersection of two generic cubics in $\mathbf{P}^2$ will be the most interesting, in which case W is a rational elliptic surface.

Question 6.9. — Let X be a smooth projective surface with $h^1(\mathscr{O}_X) = h^2(\mathscr{O}_X) = 0$. As before, the tube domain $H^2(X,\mathbf{R}) + i\,\mathscr{C}^+_X$ is isomorphic to a bounded symmetric domain of type IV of dimension $b_2(X)$. As we have seen, there is a nice automorphic form on $H^2(X,\mathbf{R}) + i\,\mathscr{C}^+_X$ when X is a Del Pezzo surface or an Enriques surface. Is there a canonical way of constructing a nice Borcherds product on $H^2(X,\mathbf{R}) + i\,\mathscr{C}^+_X$? For example, when X is of general type with $h^1(\mathscr{O}_X) = h^2(\mathscr{O}_X) = 0$ or when X is rational, is there an analogue of the Borcherds Φ-function for X?

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