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HODGE-TATE AND DE RHAM REPRESENTATIONS IN THE IMPERFECT RESIDUE FIELD CASE

BY KAZUMA MORITA

ABSTRACT. – Let K be a p-adic local field with residue field k such that $[k : k^p] = p^e < +\infty$ and V be a p-adic representation of $\mathrm{Gal}(\overline{K}/K)$. Then, by using the theory of p-adic differential modules, we show that V is a Hodge-Tate (resp. de Rham) representation of $\mathrm{Gal}(\overline{K}/K)$ if and only if V is a Hodge-Tate (resp. de Rham) representation of $\mathrm{Gal}(\overline{K^{\mathrm{pf}}}/K^{\mathrm{pf}})$ where K^{pf}/K is a certain p-adic local field with residue field the smallest perfect field k^{pf} containing k.

RÉSUMÉ. – Soit K un corps local p-adique de corps résiduel k tel que $[k : k^p] = p^e < +\infty$ et soit V une représentation p-adique de $\mathrm{Gal}(\overline{K}/K)$. Nous utilisons la théorie des modules différentiels p-adiques pour montrer que V est une représentation de Hodge-Tate (resp. de Rham) de $\mathrm{Gal}(\overline{K}/K)$ si et seulement si V est une représentation de Hodge-Tate (resp. de Rham) de $\mathrm{Gal}(\overline{K^{\mathrm{pf}}}/K^{\mathrm{pf}})$ où K^{pf}/K est un certain corps local p-adique de corps résiduel le plus petit corps parfait k^{pf} contenant k.

1. Introduction

Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$. Choose an algebraic closure $\overline{K}$ of K and put $G_K = \mathrm{Gal}(\overline{K}/K)$. By a p-adic representation of G_K, we mean a finite dimensional vector space V over $\mathbb{Q}_p$ endowed with a continuous action of G_K. In the case $e = 0$ (i.e. k is perfect), following Fontaine, we can classify p-adic representations of G_K by using the p-adic periods rings B_{HT}, B_{dR}, B_{st} and B_{cris} (Hodge-Tate, de Rham, semi-stable and crystalline representations). In the general case (i.e. k is not necessarily perfect), Hyodo constructed the imperfect residue field version of the ring B_{HT} and Tsuzuki and several authors constructed that of the ring B_{dR}. By using these rings, we can define the imperfect residue field version of Hodge-Tate and de Rham representations of G_K in the evident way ([3], [7], [8], [9], [12]).

Now, we shall state the main result of this article. Let us fix some notations. Fix a lifting $(b_i)_{1\leq i\leq e}$ of a p-basis of k in $\mathscr{O}_K$ (the ring of integers of K) and for each $m \geq 1$, fix a p^m -th root b_i^{1/p^m} of b_i in $\overline{K}$ satisfying $(b_i^{1/p^{m+1}})^p = b_i^{1/p^m}$. Put $K^{(\mathrm{pf})} = \cup_{m\geq 1} K(b_i^{1/p^m}, 1 \leq i \leq e)$

and K^{pf} = the p-adic completion of $K^{(\mathrm{pf})}$. These fields depend on the choice of a lifting of a p-basis of k in $\mathscr{O}_K$. Since K^{pf} becomes a complete discrete valuation field with perfect residue field, we can apply theories in the perfect residue field case to p-adic representations of $G_{K^{\mathrm{pf}}} = \mathrm{Gal}(\overline{K^{\mathrm{pf}}}/K^{\mathrm{pf}})$ where we choose an algebraic closure $\overline{K^{\mathrm{pf}}}$ of K^{pf} containing $\overline{K}$. Note that, if V is a p-adic representation of G_K, it can be also regarded as a p-adic representation of $G_{K^{\mathrm{pf}}}$ (see Section 2.2 for details). Our main result is the following.

THEOREM 1.1. – *Let K be a complete discrete valuation field of characteristic* 0 *with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$ and V be a p-adic representation of G_K. Let K^{pf} be the field extension of K defined as above. Then, we have the following equivalences*

1. *V is a Hodge-Tate representation of G_K if and only if V is a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$,*
2. *V is a de Rham representation of G_K if and only if V is a de Rham representation of $G_{K^{\mathrm{pf}}}$.*

In the case of Hodge-Tate representations, Tsuji [11] had proved a more refined theorem based on this article. This paper is organized as follows. In Section 2, we shall review the definitions and basic known facts on Hodge-Tate and de Rham representations, first in the perfect residue field case and then in the imperfect residue field case. In Section 3, we shall review the theory of p-adic differential modules which play a central role in this article. In Section 4, by using the theory of p-adic differential modules, we shall prove the main theorem, first for Hodge-Tate representations and then for de Rham representations.

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2. Preliminaries on Hodge-Tate and de Rham representations

2.1. Hodge-Tate and de Rham representations in the perfect residue field case

(See [4] and [5] for details.) Let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic $p > 0$. Choose an algebraic closure $\overline{K}$ of K and consider its p-adic completion $\mathbb{C}_p$. Put

$$\widetilde{\mathbb{E}} = \varprojlim_{x \mapsto x^p} \mathbb{C}_p = \{(x^{(0)}, x^{(1)}, \dots) \mid (x^{(i+1)})^p = x^{(i)}, x^{(i)} \in \mathbb{C}_p\}$$

and let $\widetilde{\mathbb{E}}^+$ denote the set of $x = (x^{(i)}) \in \widetilde{\mathbb{E}}$ such that $x^{(0)} \in \mathscr{O}_{\mathbb{C}_p}$ where $\mathscr{O}_{\mathbb{C}_p}$ denotes the ring of integers of $\mathbb{C}_p$. For two elements $x = (x^{(i)})$ and $y = (y^{(i)})$ of $\widetilde{\mathbb{E}}$, their sum and product are defined by $(x+y)^{(i)} = \lim_{j\to+\infty}(x^{(i+j)} + y^{(i+j)})^{p^j}$ and $(xy)^{(i)} = x^{(i)}y^{(i)}$. These sum and product make $\widetilde{\mathbb{E}}$ a perfect field of characteristic $p > 0$ ($\widetilde{\mathbb{E}}^+$ is a subring of $\widetilde{\mathbb{E}}$). Let $\epsilon = (\epsilon^{(n)})$ be an element of $\widetilde{\mathbb{E}}$ such that $\epsilon^{(0)} = 1$ and $\epsilon^{(1)} \neq 1$. Then, $\widetilde{\mathbb{E}}$ is the completion of an algebraic closure of $k((\epsilon - 1))$ for the valuation defined by $v_{\mathbb{E}}(x) = v_p(x^{(0)})$ where v_p denotes the p-adic valuation of $\mathbb{C}_p$ normalized by $v_p(p) = 1$. The field $\widetilde{\mathbb{E}}$ is equipped with a continuous action of the Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ with respect to the topology defined by the valuation $v_{\mathbb{E}}$. Put $\widetilde{\mathbb{A}}^+ = W(\widetilde{\mathbb{E}}^+)$ (the ring of Witt vectors with coefficients in $\widetilde{\mathbb{E}}^+$) and $\widetilde{\mathbb{B}}^+ = \widetilde{\mathbb{A}}^+[1/p] = \{\sum_{k\gg-\infty} p^k[x_k] \mid x_k \in \widetilde{\mathbb{E}}^+\}$ where $[*]$ denotes the Teichmüller lift of $* \in \widetilde{\mathbb{E}}^+$. This ring $\widetilde{\mathbb{B}}^+$ is equipped with a surjective homomorphism

$$\theta : \widetilde{\mathbb{B}}^+ \twoheadrightarrow \mathbb{C}_p : \sum p^k[x_k] \mapsto \sum p^k x_k^{(0)}.$$

If $\tilde{p} = (p^{(n)})$ denotes an element of $\widetilde{\mathbb{E}}^+$ such that $p^{(0)} = p$, we can show that $\mathrm{Ker}\,(\theta)$ is the principal ideal generated by $\omega = [\tilde{p}] - p$. The ring $B^+_{\mathrm{dR},K}$ is defined to be the $\mathrm{Ker}\,(\theta)$-adic completion of $\widetilde{\mathbb{B}}^+$

$$B^+_{\mathrm{dR},K} = \varprojlim_{n\geq 0} \widetilde{\mathbb{B}}^+/(\mathrm{Ker}\,(\theta)^n).$$

This is a discrete valuation ring and $t = \log([\epsilon])$ which converges in $B^+_{\mathrm{dR},K}$ is a generator of the maximal ideal. Put $B_{\mathrm{dR},K} = B^+_{\mathrm{dR},K}[1/t]$. This ring $B_{\mathrm{dR},K}$ becomes a field and is equipped with an action of the Galois group G_K and a filtration defined by $\mathrm{Fil}^i B_{\mathrm{dR},K} = t^i B^+_{\mathrm{dR},K}$ ($i \in \mathbb{Z}$). Then, $(B_{\mathrm{dR},K})^{G_K}$ is canonically isomorphic to K. Thus, for a p-adic representation V of G_K, $D_{\mathrm{dR},K}(V) = (B_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K-vector space. We say that a p-adic representation V of G_K is a de Rham representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{dR},K}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\mathrm{dR},K}(V)).$$

Furthermore, we say that a p-adic representation V of G_K is a potentially de Rham representation of G_K if there exists a finite field extension L/K in $\overline{K}$ such that V is a de Rham representation of G_L. It is known that a potentially de Rham representation V of G_K is a de Rham representation of G_K (see [5, 3.9]).

Define $B_{\mathrm{HT},K}$ to be the associated graded algebra to the filtration $\mathrm{Fil}^i B_{\mathrm{dR},K}$. The quotient $\mathrm{gr}^i B_{\mathrm{HT},K} = \mathrm{Fil}^i B_{\mathrm{dR},K}/\mathrm{Fil}^{i+1} B_{\mathrm{dR},K}$ ($i \in \mathbb{Z}$) is a one-dimensional $\mathbb{C}_p$-vector space spanned by the image of t^i. Thus, we obtain the presentation

$$B_{\mathrm{HT},K} = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}_p(i)$$

where $\mathbb{C}_p(i) = \mathbb{C}_p \otimes \mathbb{Z}_p(i)$ is the Tate twist. Then, $(B_{\mathrm{HT},K})^{G_K}$ is canonically isomorphic to K. Thus, for a p-adic representation V of G_K, $D_{\mathrm{HT},K}(V) = (B_{\mathrm{HT},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K-vector space. We say that a p-adic representation V of G_K is a Hodge-Tate representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{HT},K}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\mathrm{HT},K}(V)).$$

Furthermore, we say that a p-adic representation V of G_K is a potentially Hodge-Tate representation of G_K if there exists a finite field extension L/K in $\overline{K}$ such that V is a Hodge-Tate representation of G_L. It is known that a potentially Hodge-Tate representation V of G_K is a Hodge-Tate representation of G_K (see [5, 3.9]). Since we have $\mathrm{gr} B_{\mathrm{dR},K} \simeq \bigoplus_{i\in\mathbb{Z}} \mathbb{C}_p(i)$, if V is a de Rham representation of G_K, there exists a G_K-equivariant isomorphism $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \simeq \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} \mathbb{C}_p(n_j)$ ($n_j \in \mathbb{Z}$). Thus, it follows that a de Rham representation V of G_K is a Hodge-Tate representation of G_K.

2.2. Hodge-Tate and de Rham representations in the imperfect residue field case

Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$. Choose an algebraic closure $\overline{K}$ of K and put $G_K = \mathrm{Gal}(\overline{K}/K)$. As in the introduction, fix a lifting $(b_i)_{1\leq i\leq e}$ of a p-basis of k in $\mathscr{O}_K$ (the ring of integers of K) and for each $m \geq 1$, fix a p^m -th root b_i^{1/p^m} of b_i in $\overline{K}$ satisfying $(b_i^{1/p^{m+1}})^p = b_i^{1/p^m}$. Put

$$K^{(\mathrm{pf})} = \cup_{m\geq 0} K(b_i^{1/p^m}, 1 \leq i \leq e) \quad \text{and} \quad K^{\mathrm{pf}} = \text{the } p\text{-adic completion of } K^{(\mathrm{pf})}.$$

These fields depend on the choice of a lifting of a p-basis of k in $\mathscr{O}_K$. Since $K^{(\mathrm{pf})}$ is a Henselian discrete valuation field, we have an isomorphism $G_{K^{\mathrm{pf}}} = \mathrm{Gal}(\overline{K^{\mathrm{pf}}}/K^{\mathrm{pf}}) \simeq G_{K^{(\mathrm{pf})}} = \mathrm{Gal}(\overline{K}/K^{(\mathrm{pf})})$ ($\subset G_K$) where we choose an algebraic closure $\overline{K^{\mathrm{pf}}}$ of K^{pf} containing $\overline{K}$. With this isomorphism, we identify $G_{K^{\mathrm{pf}}}$ with a subgroup of G_K. We have a bijective map from the set of finite extensions of $K^{(\mathrm{pf})}$ contained in $\overline{K}$ to the set of finite extensions of K^{pf} contained in $\overline{K^{\mathrm{pf}}}$ defined by $L \to LK^{\mathrm{pf}}$. Furthermore, LK^{pf} is the p-adic completion of L. Hence, we have an isomorphism of rings

$$\mathscr{O}_{\overline{K}}/p^n\mathscr{O}_{\overline{K}} \simeq \mathscr{O}_{\overline{K^{\mathrm{pf}}}}/p^n\mathscr{O}_{\overline{K^{\mathrm{pf}}}}$$

where $\mathscr{O}_{\overline{K}}$ and $\mathscr{O}_{\overline{K^{\mathrm{pf}}}}$ denote the rings of integers of $\overline{K}$ and $\overline{K^{\mathrm{pf}}}$. Thus, the p-adic completion of $\overline{K}$ is isomorphic to the p-adic completion of $\overline{K^{\mathrm{pf}}}$, which we will write $\mathbb{C}_p$. As in Subsection 2.1, construct the rings $\widetilde{\mathbb{E}}^+$ and $\widetilde{\mathbb{A}}^+ = W(\widetilde{\mathbb{E}}^+)$ from this $\mathbb{C}_p$. Let k^{pf} denote the perfect residue field of K^{pf} and put $\mathscr{O}_{K_0} = \mathscr{O}_K \cap W(k^{\mathrm{pf}})$. Let $\alpha : \mathscr{O}_K \otimes_{\mathscr{O}_{K_0}} \widetilde{\mathbb{A}}^+ \twoheadrightarrow \mathscr{O}_{\overline{K}}/p\mathscr{O}_{\overline{K}}$ be the natural surjection and define $\widetilde{\mathbb{A}}^+_{(K)}$ to be $\widetilde{\mathbb{A}}^+_{(K)} = \varprojlim_{n\geq 0}(\mathscr{O}_K \otimes_{\mathscr{O}_{K_0}} \widetilde{\mathbb{A}}^+)/(\mathrm{Ker}\,(\alpha))^n$. Let $\theta_K : \widetilde{\mathbb{A}}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \twoheadrightarrow \mathbb{C}_p$ be the natural extension of $\theta : \widetilde{\mathbb{A}}^+[1/p] \twoheadrightarrow \mathbb{C}_p$. Define $B^+_{\mathrm{dR},K}$ to be the $\mathrm{Ker}\,(\theta_K)$-adic completion of $\widetilde{\mathbb{A}}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$

$$B^+_{\mathrm{dR},K} = \varprojlim_{n\geq 0}(\widetilde{\mathbb{A}}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)/(\mathrm{Ker}\,(\theta_K)^n).$$

This is a K-algebra equipped with an action of the Galois group G_K. Let $\widetilde{b_i}$ denote $(b_i^{(n)}) \in \widetilde{\mathbb{E}}^+$ such that $b_i^{(0)} = b_i$ and then the series which defines $\log([\widetilde{b_i}]/b_i)$ converges to an element t_i in $B^+_{\mathrm{dR},K}$. Then, the ring $B^+_{\mathrm{dR},K}$ becomes a local ring with the maximal ideal $m_{\mathrm{dR}} = (t, t_1, \ldots, t_e)$. Define a filtration on $B^+_{\mathrm{dR},K}$ by $\mathrm{fil}^i B^+_{\mathrm{dR},K} = m^i_{\mathrm{dR}}$. Then, the homomorphism

$$f : B^+_{\mathrm{dR},K^{\mathrm{pf}}}[[t_1, \ldots, t_e]] \to B^+_{\mathrm{dR},K}$$

is an isomorphism of filtered algebras (see [3, Proposition 2.9]). From this isomorphism, it follows easily that

$$i : B^+_{\mathrm{dR},K^{\mathrm{pf}}} \hookrightarrow B^+_{\mathrm{dR},K} \quad \text{and} \quad p : B^+_{\mathrm{dR},K} \twoheadrightarrow B^+_{\mathrm{dR},K^{\mathrm{pf}}} : \ t_i \mapsto 0$$

are $G_{K^{\mathrm{pf}}}$-equivariant homomorphisms and the composition

$$p \circ i : B^+_{\mathrm{dR},K^{\mathrm{pf}}} \hookrightarrow B^+_{\mathrm{dR},K} \twoheadrightarrow B^+_{\mathrm{dR},K^{\mathrm{pf}}}$$

is an identity. Put $B_{\mathrm{dR},K} = B^+_{\mathrm{dR},K}[1/t]$. Then, K is canonically embedded in $B_{\mathrm{dR},K}$ and we have a canonical isomorphism $(B_{\mathrm{dR},K})^{G_K} = K$. Thus, for a p-adic representation V of G_K, $D_{\mathrm{dR},K}(V) = (B_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K-vector space. We say that a p-adic representation V of G_K is a de Rham representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{dR},K}(V) \quad \text{(we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\mathrm{dR},K}(V)\text{)}.$$

Furthermore, we say that a p-adic representation V of G_K is a potentially de Rham representation of G_K if there exists a finite field extension L/K in $\overline{K}$ such that V is a de Rham representation of G_L. We can show that a potentially de Rham representation V of G_K is a de Rham representation of G_K in the same way as in the perfect residue field case.

Define a filtration on $B_{\mathrm{dR},K}$ to be

$$\begin{aligned}\mathrm{Fil}^0 B_{\mathrm{dR},K} &= \sum_{n=0}^{\infty} t^{-n}\mathrm{fil}^n B^+_{\mathrm{dR},K} = B^+_{\mathrm{dR},K}[\frac{t_1}{t}, \ldots, \frac{t_e}{t}],\\ \mathrm{Fil}^i B_{\mathrm{dR},K} &= t^i \mathrm{Fil}^0 B_{\mathrm{dR},K} \ (i \in \mathbb{Z}).\end{aligned}$$

Define $B_{\mathrm{HT},K}$ to be the associated graded algebra to this filtration. Since the quotient $\mathrm{gr}^i B_{\mathrm{HT},K} = \mathrm{Fil}^i B_{\mathrm{dR},K}/\mathrm{Fil}^{i+1} B_{\mathrm{dR},K}$ $(i \in \mathbb{Z})$ is given by $\mathrm{gr}^i B_{\mathrm{HT},K} = t^i \mathbb{C}_p[\frac{t_1}{t}, \ldots, \frac{t_e}{t}]$, we obtain the presentation

$$B_{\mathrm{HT},K} = \mathbb{C}_p[t, t^{-1}, \frac{t_1}{t}, \ldots, \frac{t_e}{t}] = B_{\mathrm{HT},K^{\mathrm{pf}}}[\frac{t_1}{t}, \ldots, \frac{t_e}{t}].$$

From this presentation, it follows easily that

$$i : B_{\mathrm{HT},K^{\mathrm{pf}}} \hookrightarrow B_{\mathrm{HT},K} \quad \text{and} \quad p : B_{\mathrm{HT},K} \twoheadrightarrow B_{\mathrm{HT},K^{\mathrm{pf}}} : \ t_i/t \mapsto 0$$

are $G_{K^{\mathrm{pf}}}$-equivariant homomorphisms and the composition

$$p \circ i : B_{\mathrm{HT},K^{\mathrm{pf}}} \hookrightarrow B_{\mathrm{HT},K} \twoheadrightarrow B_{\mathrm{HT},K^{\mathrm{pf}}}$$

is an identity. The field K is canonically embedded in $B_{\mathrm{HT},K}$ and we have $(B_{\mathrm{HT},K})^{G_K} = K$. Thus, for a p-adic representation V of G_K, $D_{\mathrm{HT},K}(V) = (B_{\mathrm{HT},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K-vector space. We say that a p-adic representation V of G_K is a Hodge-Tate representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{HT},K}(V) \quad \text{(we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\mathrm{HT},K}(V)\text{)}.$$

Furthermore, we say that a p-adic representation V of G_K is a potentially Hodge-Tate representation of G_K if there exists a finite field extension L/K in $\overline{K}$ such that V is a Hodge-Tate representation of G_L. We can show that a potentially Hodge-Tate representation V of G_K is a Hodge-Tate representation of G_K in the same way as in the perfect residue field case.

3. Preliminaries on p-adic differential modules

In this section, we shall review the theory of p-adic differential modules which plays an important role in this article. First, let us fix the notations. Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < \infty$ and V be a p-adic representation of G_K. Define $K^{(\mathrm{pf})}$ and K^{pf} as in the introduction and in Subsection 2.2. Put $K^{(\mathrm{pf})}_{\infty} = \cup_{m\geq 0} K^{(\mathrm{pf})}(\zeta_{p^m})$ (resp. $K^{\mathrm{pf}}_{\infty} = \cup_{m\geq 0} K^{\mathrm{pf}}(\zeta_{p^m})$) where ζ_{p^m} denotes a primitive p^m-th root of unity in $\overline{K}$ (resp. $\overline{K^{\mathrm{pf}}}$) such that $(\zeta_{p^{m+1}})^p = \zeta_{p^m}$. Let $\hat{K}^{\mathrm{pf}}_{\infty}$ denote the p-adic completion of K^{pf}_{∞}. These fields $K^{(\mathrm{pf})}_{\infty}$, K^{pf}_{∞} and $\hat{K}^{\mathrm{pf}}_{\infty}$ depend on the choice of a lifting of a p-basis of k in $\mathscr{O}_K$. Then, we have the following inclusions

$$K^{(\mathrm{pf})}_{\infty} \subset K^{\mathrm{pf}}_{\infty} \subset \hat{K}^{\mathrm{pf}}_{\infty}.$$

Let H denote the kernel of the cyclotomic character $\chi : G_{K^{\mathrm{pf}}} \to \mathbb{Z}_p^*$. Then, the Galois group H is isomorphic to the subgroup $\mathrm{Gal}(\overline{K}/K^{(\mathrm{pf})}_{\infty})$ of G_K. Define $\Gamma_K = G_K/H$. Let Γ_0 denote the subgroup $\mathrm{Gal}(K^{(\mathrm{pf})}_{\infty}/K^{(\mathrm{pf})})$ ($\simeq G_{K^{\mathrm{pf}}}/H$) of Γ_K. Let Γ_i ($1 \leq i \leq e$) be the subgroup of Γ_K such that actions of $\beta_i \in \Gamma_i$ ($1 \leq i \leq e$) satisfy $\beta_i(\zeta_{p^m}) = \zeta_{p^m}$ and $\beta_i(b_j^{1/p^m}) = b_j^{1/p^m}$ ($i \neq j$) and define the homomorphism $c_i : \Gamma_i \to \mathbb{Z}_p$ such that we have $\beta_i(b_i^{1/p^m}) = b_i^{1/p^m}\zeta_{p^m}^{c_i(\beta_i)}$. Then, the homomorphism c_i defines an isomorphism $\Gamma_i \simeq \mathbb{Z}_p$ of profinite groups. With this, we can see that there exist isomorphisms of profinite groups

$$\Gamma_K \simeq \Gamma_0 \ltimes (\oplus_{i=1}^{e}\Gamma_i) \simeq \Gamma_0 \ltimes \mathbb{Z}_p^{\oplus e}.$$

3.1. Definitions of p-adic differential modules

We shall review the definitions of p-adic differential modules and have the following diagram, for a p-adic representation V of G_K,

$$\begin{array}{ccc}
(B^+_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)^H & \overset{\theta_K}{\twoheadrightarrow} & (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H \\
\cup & & \cup \\
D^+_{\mathrm{dif}}(V) & \twoheadrightarrow & D_{\mathrm{Sen}}(V) \\
\cup & & \cup \\
D^+_{e\text{-}\mathrm{dif}}(V) & \twoheadrightarrow & D_{\mathrm{Bri}}(V).
\end{array}$$

3.1.1. *The module* $D_{\mathrm{Sen}}(V)$. – In the article [10], Sen shows that, for a p-adic representation V of $G_{K^{\mathrm{pf}}}$, the $\hat{K}^{\mathrm{pf}}_{\infty}$-vector space $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ has dimension $d = \dim_{\mathbb{Q}_p} V$ and the union of the finite dimensional K^{pf}_{∞}-subspaces of $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ stable under Γ_0 ($\simeq G_{K^{\mathrm{pf}}}/H$) is a K^{pf}_{∞}-vector space of dimension d stable under Γ_0 (called $D_{\mathrm{Sen}}(V)$). We have $\mathbb{C}_p \otimes_{K^{\mathrm{pf}}_{\infty}} D_{\mathrm{Sen}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ and the natural map $\hat{K}^{\mathrm{pf}}_{\infty} \otimes_{K^{\mathrm{pf}}_{\infty}} D_{\mathrm{Sen}}(V) \to (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ is an isomorphism. Furthermore, if $\gamma \in \Gamma_0$ is close enough to 1, then the series of operators on $D_{\mathrm{Sen}}(V)$

$$\frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))}\sum_{k\geq 1}\frac{(1-\gamma)^k}{k}$$

converges to a K^{pf}_{∞}-linear derivation $\nabla^{(0)} : D_{\mathrm{Sen}}(V) \to D_{\mathrm{Sen}}(V)$ and does not depend on the choice of γ.

3.1.2. *The module* $D_{\mathrm{Bri}}(V)$. – In the article [2], Brinon generalizes Sen's work above. For a p-adic representation V of G_K, he shows that the union of the finite dimensional $K_\infty^{(\mathrm{pf})}$-subspaces of $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ stable under Γ_K is a $K_\infty^{(\mathrm{pf})}$-vector space of dimension d stable under Γ_K (we call it $D_{\mathrm{Bri}}(V)$). We have $\mathbb{C}_p \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ and the natural map $\hat{K}_\infty^{\mathrm{pf}} \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V) \to (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ is an isomorphism. As in the case of $D_{\mathrm{Sen}}(V)$, the $K_\infty^{(\mathrm{pf})}$-vector space $D_{\mathrm{Bri}}(V)$ is endowed with the action of the $K_\infty^{(\mathrm{pf})}$-linear derivation $\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))}$ if $\gamma \in \Gamma_0$ is close enough to 1. In addition to this operator $\nabla^{(0)}$, if $\beta_i \in \Gamma_i$ is close enough to 1, then the series of operators on $D_{\mathrm{Bri}}(V)$

$$\frac{\log(\beta_i)}{c_i(\beta_i)} = -\frac{1}{c_i(\beta_i)} \sum_{k \geq 1} \frac{(1-\beta_i)^k}{k}$$

converges to a $K_\infty^{(\mathrm{pf})}$-linear derivation $\nabla^{(i)} : D_{\mathrm{Bri}}(V) \to D_{\mathrm{Bri}}(V)$ and does not depend on the choice of β_i.

3.1.3. *The module* $D^+_{e\text{-dif}}(V)$. – In the article [1], Andreatta and Brinon generalize Fontaine's work [6]. For a p-adic representation V of G_K, they show that the union of $K_\infty^{(\mathrm{pf})}[[t, t_1, \ldots, t_e]]$-submodules of finite type of $(B^+_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)^H$ stable under Γ_K is a free $K_\infty^{(\mathrm{pf})}[[t, t_1, \ldots, t_e]]$-module of rank d stable under Γ_K (we call it $D^+_{e\text{-dif}}(V)$). We have $B^+_{\mathrm{dR},K} \otimes_{K_\infty^{(\mathrm{pf})}[[t,t_1,\ldots,t_e]]} D^+_{e\text{-dif}}(V) = B^+_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V$ and the natural map

$$(B^+_{\mathrm{dR},K})^H \otimes_{K_\infty^{(\mathrm{pf})}[[t,t_1,\ldots,t_e]]} D^+_{e\text{-dif}}(V) \to (B^+_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)^H$$

is an isomorphism. The $K_\infty^{(\mathrm{pf})}[[t, t_1, \ldots, t_e]]$-module $D^+_{e\text{-dif}}(V)$ is endowed with the action of the $K_\infty^{(\mathrm{pf})}$-linear derivations $\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))}$ if $\gamma \in \Gamma_0$ is close enough to 1 and $\nabla^{(i)} = \frac{\log(\beta_i)}{c_i(\beta_i)}$ $(1 \leq i \leq e)$ if $\beta_i \in \Gamma_i$ is close enough to 1.

3.1.4. *The module* $D^+_{\mathrm{dif}}(V)$. – For a p-adic representation V of G_K, define $D^+_{\mathrm{dif}}(V)$ to be $\varprojlim_r (K_\infty^{\mathrm{pf}}[[t, t_1, \ldots, t_e]] \otimes_{K_\infty^{(\mathrm{pf})}[[t,t_1,\ldots,t_e]]} D^{+,(r)}_{e\text{-dif}}(V))$ where we put $D^{+,(r)}_{e\text{-dif}}(V) = D^+_{e\text{-dif}}(V)/(t, t_1, \ldots, t_e)^r D^+_{e\text{-dif}}(V)$. One can verify that $D^+_{\mathrm{dif}}(V)$ is the union of $K_\infty^{\mathrm{pf}}[[t, t_1, \ldots, t_e]]$-submodules of finite type of $(B^+_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)^H$ stable under Γ_0 $(\simeq G_{K^{\mathrm{pf}}}/H)$ and is a free $K_\infty^{\mathrm{pf}}[[t, t_1, \ldots, t_e]]$-module of rank d stable under Γ_0. Furthermore, we have $B^+_{\mathrm{dR},K} \otimes_{K_\infty^{\mathrm{pf}}[[t,t_1,\ldots,t_e]]} D^+_{\mathrm{dif}}(V) = B^+_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V$ and the natural map $(B^+_{\mathrm{dR},K})^H \otimes_{K_\infty^{\mathrm{pf}}[[t,t_1,\ldots,t_e]]} D^+_{\mathrm{dif}}(V) \to (B^+_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)^H$ is an isomorphism. As in the case of $D^+_{e\text{-dif}}(V)$, the $K_\infty^{\mathrm{pf}}[[t, t_1, \ldots, t_e]]$-module $D^+_{\mathrm{dif}}(V)$ is endowed with the action of the K_∞^{pf}-linear derivation $\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))}$ if $\gamma \in \Gamma_0$ is close enough to 1.

Remark 3.1. –

1. The preceding results in Subsection 3.1.1 are obtained when V is a p-adic representation of $G_L = \mathrm{Gal}(\overline{L}/L)$ where L is a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p > 0$ and we choose an algebraic closure $\overline{L}$ of L. However, in Subsection 3.1.1, for simplicity, we stated the results in the case $L = K^{\mathrm{pf}}$.
2. Note that, though many people denote the p-adic differential module constructed by Fontaine in [6] by $D^+_{\mathrm{dif}}(V)$, the module $D^+_{\mathrm{dif}}(V)$ in Subsection 3.1.4 is a little different from this module.

3.2. Some properties of differential operators

We shall describe the action of derivations $\{\nabla^{(i)}\}_{i=0}^{e}$ on $D_{\mathrm{Bri}}(V)$ and $D^+_{e\text{-dif}}(V)$. First, by a standard argument, we can show that, if $x \in D_{\mathrm{Bri}}(V)$ (resp. $D^+_{e\text{-dif}}(V)$), we have

$$\nabla^{(0)}(x) = \lim_{\gamma\to 1}\frac{\gamma(x)-x}{\chi(\gamma)-1} \quad \text{and} \quad \nabla^{(i)}(x) = \lim_{\beta_i\to 1}\frac{\beta_i(x)-x}{c_i(\beta_i)}.$$

With this, we can easily describe the actions of $K_\infty^{(\mathrm{pf})}$-linear derivations $\{\nabla^{(i)}\}_{i=0}^{e}$ on $K_\infty^{(\mathrm{pf})}[[t,t_1,\dots,t_e]] = D^+_{e\text{-dif}}(\mathbb{Q}_p)$ where $\mathbb{Q}_p$ is equipped with the structure of p-adic representations of G_K induced by the trivial action of G_K.

Lemma 3.2. – *The actions of $K_\infty^{(\mathrm{pf})}$-linear derivations $\{\nabla^{(i)}\}_{i=0}^{e}$ on $K_\infty^{(\mathrm{pf})}[[t,t_1,\dots,t_e]]$ are given by $\nabla^{(0)} = t\frac{d}{dt}$ and $\nabla^{(i)} = t\frac{d}{dt_i}$ $(1 \le i \le e)$.*

Proof. – Since $\{\nabla^{(j)}\}_{j=0}^{e}$ are $K_\infty^{(\mathrm{pf})}$-linear derivations and we can see that we have $\nabla^{(j)}(t_k) = 0$ $(j \neq k)$ and $\nabla^{(i)}(t) = 0$ $(i \neq 0)$, it suffices to show that we have $\nabla^{(0)}(t) = t$ and $\nabla^{(i)}(t_i) = t$. These follow from

$$\nabla^{(0)}(t) = \lim_{\gamma\to 1}\frac{\gamma(t)-t}{\chi(\gamma)-1} = \lim_{\gamma\to 1}\frac{\chi(\gamma)t-t}{\chi(\gamma)-1} = t$$
$$\nabla^{(i)}(t_i) = \lim_{\beta_i\to 1}\frac{\beta_i(t_i)-t_i}{c_i(\beta_i)} = \lim_{\beta_i\to 1}\frac{(t_i+c_i(\beta_i)t)-t_i}{c_i(\beta_i)} = t. \qquad \square$$

We extend naturally actions of $K_\infty^{(\mathrm{pf})}$-linear derivations $\{\nabla^{(i)}\}_{i=0}^{e}$ on $K_\infty^{(\mathrm{pf})}[[t,t_1,\dots,t_e]]$ to $K_\infty^{(\mathrm{pf})}[[t,t_1,\dots,t_e]][t^{-1}]$ $(\subset B_{\mathrm{dR},K})$ by putting $\nabla^{(0)}(t^{-1}) = -t^{-1}$ and $\nabla^{(i)}(t^{-1}) = 0$ $(1 \le i \le e)$. Now, we compute the bracket $[\ ,\ ]$ of derivations $\{\nabla^{(i)}\}_{i=0}^{e}$ on $D_{\mathrm{Bri}}(V)$ (resp. $D^+_{e\text{-dif}}(V)$).

Proposition 3.3. – *On the p-adic differential module $D_{\mathrm{Bri}}(V)$ (resp. $D^+_{e\text{-dif}}(V)$), we have $[\nabla^{(0)},\nabla^{(i)}] = \nabla^{(i)}$ $(i \neq 0)$ and $[\nabla^{(i)},\nabla^{(j)}] = 0$ $(i,j \neq 0)$.*

Proof. – The second equality follows from the commutativity of β_i and β_j. For the first equality, we have the relation $\gamma\beta_i = \beta_i^{\chi(\gamma)}\gamma$. Then, since we have

$$\lim_{h\to 0}\frac{a^{h+1}-a}{(h+1)-1} = a\log(a),$$

we obtain

$$\begin{aligned}
[\nabla^{(0)},\nabla^{(i)}](*) &= \lim_{\gamma\to 1}\frac{\gamma-1}{\chi(\gamma)-1}\lim_{\beta_i\to 1}\frac{\beta_i-1}{c_i(\beta_i)}(*) - \lim_{\beta_i\to 1}\frac{\beta_i-1}{c_i(\beta_i)}\lim_{\gamma\to 1}\frac{\gamma-1}{\chi(\gamma)-1}(*)\\
&= \lim_{\beta_i\to 1}\lim_{\gamma\to 1}\frac{\gamma\beta_i-\gamma-\beta_i+1}{(\chi(\gamma)-1)c_i(\beta_i)}(*) - \lim_{\beta_i\to 1}\lim_{\gamma\to 1}\frac{\beta_i\gamma-\gamma-\beta_i+1}{(\chi(\gamma)-1)c_i(\beta_i)}(*)\\
&= \lim_{\beta_i\to 1}\lim_{\gamma\to 1}\frac{\beta_i^{\chi(\gamma)}\gamma-\beta_i\gamma}{(\chi(\gamma)-1)c_i(\beta_i)}(*)\\
&= \lim_{\beta_i\to 1}\frac{\beta_i\log(\beta_i)}{c_i(\beta_i)}(*)\\
&= \nabla^{(i)}(*). \qquad \square
\end{aligned}$$

PROPOSITION 3.4. – *The action of the $K_\infty^{(\mathrm{pf})}$-linear derivation $\nabla^{(i)}$ $(i \neq 0)$ on $D_{\mathrm{Bri}}(V)$ is nilpotent.*

Proof. – From the equality $\nabla^{(0)}\nabla^{(i)} - \nabla^{(i)}\nabla^{(0)} = \nabla^{(i)}$, we get $\nabla^{(0)}(\nabla^{(i)})^r - (\nabla^{(i)})^r\nabla^{(0)} = r(\nabla^{(i)})^r$ and $\mathrm{tr}(r(\nabla^{(i)})^r) = 0$ for all $r \in \mathbb{N}$. Since the characteristic of $K_\infty^{(\mathrm{pf})}$ is 0, we obtain $\mathrm{tr}((\nabla^{(i)})^r) = 0$ for all $r \in \mathbb{N}$. As is well known in linear algebra, this shows that the action of the $K_\infty^{(\mathrm{pf})}$-linear derivation $\nabla^{(i)}$ $(i \neq 0)$ on $D_{\mathrm{Bri}}(V)$ is nilpotent. □

NOTATION . – For simplicity, put

$$R = K_\infty^{(\mathrm{pf})}[t, \frac{t_1}{t}, \dots, \frac{t_e}{t}] \quad \text{or} \quad K_\infty^{(\mathrm{pf})}[[t, t_1, \dots, t_e]].$$

PROPOSITION 3.5. – *Let M be a finitely generated free $R[1/t]$-module endowed with $K_\infty^{(\mathrm{pf})}$-linear derivations $\{\nabla^{(i)}\}_{i=0}^{e}$ which satisfy the same properties in Lemma 3.2 and Proposition 3.3. Assume that we can choose a basis $\{g_j\}_{j=1}^{d}$ of M over $R[1/t]$ such that $\nabla^{(0)}(g_j) = 0$. Then, the action of $\nabla^{(i)}$ $(i \neq 0)$ on this basis is given by $\nabla^{(i)}(g_j) = t\sum_{k=1}^{d} c_k g_k$ where c_k is an element of R such that $\nabla^{(0)}(c_k) = 0$.*

Proof. – Since $\{g_j\}_{j=1}^{d}$ forms a basis of M over $R[1/t]$, we can write, for $i \neq 0$,

$$\nabla^{(i)}(g_j) = \sum_{k=1}^{d} a_k g_k \quad (a_k \in R[1/t]). \tag{3.1}$$

Then, the relation $[\nabla^{(0)}, \nabla^{(i)}] = \nabla^{(i)}$ $(i \neq 0)$ of Proposition 3.3 says that we have $\sum_{k=1}^{d} \nabla^{(0)}(a_k)g_k = \sum_{k=1}^{d} a_k g_k$. Note that we have $\nabla^{(0)}(g_j) = 0$ by hypothesis. Hence, we obtain the differential equation $\nabla^{(0)}(a_k) = a_k$. Define an element c_k of $R[1/t]$ to be a_k/t. Then, we can see that c_k satisfies $\nabla^{(0)}(c_k) = a_k/t - a_k/t = 0$ and that c_k is contained in R. Thus, the solution of the differential equation $\nabla^{(0)}(a_k) = a_k$ in R[1/t] has the following form

$$a_k = c_k t \tag{3.2}$$

where c_k is an element of R such that $\nabla^{(0)}(c_k) = 0$. Hence, from (3.1) and (3.2), we obtain, for $i \neq 0$, $\nabla^{(i)}(g_j) = t\sum_{k=1}^{d} c_k g_k$ where c_k is an element of R such that $\nabla^{(0)}(c_k) = 0$. □

COROLLARY 3.6. – *With notations as in Proposition 3.5 above, we have the following presentation*

$$(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(g_j) = t^{k_1+\cdots+k_e} \sum_{k=1}^{d} c_k g_k$$

where c_k is an element of R such that $\nabla^{(0)}(c_k) = 0$.

4. Proof of the main theorem

In this section, we keep the notation and the assumption of Section 3.

4.1. Main theorem for Hodge-Tate representations

PROPOSITION 4.1 ([10, Section (2.3)]). – *If V is a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$, there exists a Γ_0-equivariant isomorphism of K_∞^{pf}-vector spaces*

$$D_{\mathrm{Sen}}(V) \simeq \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} K_\infty^{\mathrm{pf}}(n_j) \quad (n_j \in \mathbb{Z}).$$

REMARK 4.2. – In general, if L denotes a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p > 0$ and V is a Hodge-Tate representation of $G_L = \mathrm{Gal}(\overline{L}/L)$ where we choose an algebraic closure $\overline{L}$ of L, Sen shows that there exists a G_L/H-equivariant isomorphism of $L_\infty(= \cup_{m\geq 1} L(\zeta_{p^m}))$-vector spaces ([10, Section (2.3)])

$$D_{\mathrm{Sen}}(V) \simeq \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} L_\infty(n_j) \quad (n_j \in \mathbb{Z}).$$

COROLLARY 4.3. – *For a p-adic representation V of G_K, assume that V is a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$. Then, there exists a $\nabla^{(0)}$- equivariant isomorphism of $K_\infty^{(\mathrm{pf})}$-vector spaces*

$$D_{\mathrm{Bri}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} K_\infty^{(\mathrm{pf})}(n_j) \quad (n_j \in \mathbb{Z}).$$

Here, $\simeq_{\nabla^{(0)}}$ denotes a $\nabla^{(0)}$-equivariant isomorphism. Furthermore, the multiplicity of $\{n_j\}_{j=1}^d$ is the same as that of $\{n_j\}_{j=1}^d$ in Proposition 4.1.

Proof. – From the presentation of Proposition 4.1, the action of the K_∞^{pf}-linear derivation $\nabla^{(0)}$ on $D_{\mathrm{Sen}}(V)$ is semi-simple and its eigenvalues are integers. Thus, the action of the $K_\infty^{(\mathrm{pf})}$-linear derivation $\nabla^{(0)}$ on the subspace $D_{\mathrm{Bri}}(V)$ of $D_{\mathrm{Sen}}(V)$ is also semi-simple and its eigenvalues are the same. Therefore, we obtain a $\nabla^{(0)}$-equivariant isomorphism $D_{\mathrm{Bri}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^d K_\infty^{(\mathrm{pf})}(n_j)$ $(n_j \in \mathbb{Z})$. By tensoring $K_\infty^{\mathrm{pf}} \otimes_{K_\infty^{(\mathrm{pf})}}$ over both sides, we obtain $K_\infty^{\mathrm{pf}} \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^d K_\infty^{\mathrm{pf}}(n_j)$ $(n_j \in \mathbb{Z})$. Furthermore, since we have $K_\infty^{\mathrm{pf}} \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V) \hookrightarrow D_{\mathrm{Sen}}(V)$ by definition and both sides have the same dimension d over K_∞^{pf}, we obtain $K_\infty^{\mathrm{pf}} \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V) = D_{\mathrm{Sen}}(V)$ and can see that the multiplicity of $\{n_j\}_{j=1}^d$ is the same as that of $\{n_j\}_{j=1}^d$ in Proposition 4.1. $\square$

THEOREM 4.4. – *Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$ and V be a p-adic representation of G_K. Let K^{pf} be the field extension of K defined as before. Then, V is a Hodge-Tate representation of G_K if and only if V is a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$.*

Proof. – We shall prove the main theorem in two parts.

(1) V: HT rep. of $G_K \Rightarrow V$: HT rep. of $G_{K^{\mathrm{pf}}}$. – Since V is a Hodge-Tate representation of G_K, there exists a G_K-equivariant isomorphism of $B_{\mathrm{HT},K}$-modules

$$B_{\mathrm{HT},K} \otimes_{\mathbb{Q}_p} V \simeq (B_{\mathrm{HT},K})^{d=\dim_{\mathbb{Q}_p} V}. \tag{4.1}$$

Now, by tensoring $B_{\mathrm{HT},K^{\mathrm{pf}}} \otimes_{B_{\mathrm{HT},K}}$ (which is induced by the $G_{K^{\mathrm{pf}}}$-equivariant surjection $p : B_{\mathrm{HT},K} \twoheadrightarrow B_{\mathrm{HT},K^{\mathrm{pf}}} : t_i/t \mapsto 0$) over (4.1), we obtain a $G_{K^{\mathrm{pf}}}$-equivariant isomorphism of $B_{\mathrm{HT},K^{\mathrm{pf}}}$-modules

$$B_{\mathrm{HT},K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_p} V \simeq (B_{\mathrm{HT},K^{\mathrm{pf}}})^d.$$

This means that V is a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$.

(2) V: HT rep. of $G_{K^{\mathrm{pf}}} \Rightarrow V$: HT rep. of G_K. – For simplicity, put $R = K_\infty^{(\mathrm{pf})}[t, \frac{t_1}{t}, \ldots, \frac{t_e}{t}]$. We shall construct the $K_\infty^{(\mathrm{pf})}$-linearly independent elements $\{f_j^{(*)}\}_{j=1}^{d=\dim_{\mathbb{Q}_p} V}$ of $R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$ ($\subset B_{\mathrm{HT},K} \otimes_{\mathbb{Q}_p} V$) such that $\nabla^{(i)}(f_j^{(*)}) = 0$ for all $0 \leq i \leq e$ and $1 \leq j \leq d$.

(A) Construction of $\{f_j^{(*)}\}_{j=1}^d \in R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$. – From the presentation of Corollary 4.3 above, if we twist by some powers of t, we obtain a basis $\{f_j\}_{j=1}^d$ of $R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$ over $R[1/t]$ such that $\nabla^{(0)}(f_j) = 0$ for all $1 \leq j \leq d$. Thus, by applying Corollary 3.6 to the $R[1/t]$-module $R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$ generated by $\{f_j\}_{j=1}^d$, we can deduce

$$(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(f_j) = t^{k_1+\cdots+k_e} \sum_{k=1}^{d} c_k f_k \tag{4.2}$$

where c_k is an element of R such that $\nabla^{(0)}(c_k) = 0$. Furthermore, since the action of $K_\infty^{(\mathrm{pf})}$-linear derivation $\nabla^{(i)}$ ($i \neq 0$) on $D_{\mathrm{Bri}}(V)$ is nilpotent by Proposition 3.4, if we take $n \in \mathbb{N}$ large enough, we obtain

$$(\nabla^{(i)})^n(f_j) = 0 \quad \text{for all } 1 \leq j \leq d \text{ and } 1 \leq i \leq e. \tag{4.3}$$

Define an element $f_j^{(*)}$ of $R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$ by

$$f_j^{(*)} = \sum_{0 \leq k_1, \ldots, k_e} (-1)^{k_1+\cdots+k_e} \frac{t_1^{k_1} \cdots t_e^{k_e}}{k_1! \cdots k_e! t^{k_1+\cdots+k_e}} (\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(f_j).$$

Note that this series is a finite sum by (4.3) and thus $f_j^{(*)}$ actually defines an element of $R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$. Then, it follows easily that we have $\nabla^{(i)}(f_j^{(*)}) = 0$ for all $1 \leq i \leq e$ and $1 \leq j \leq d$ by using the Leibniz rule. Furthermore, by using (4.2) and the fact $\nabla^{(0)}(f_j) = 0$, we can deduce that we have $\nabla^{(0)}(f_j^{(*)}) = 0$ for all $1 \leq j \leq d$.

(B) $\{f_j^{(*)}\}_{j=1}^d \in R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$ is linearly independent over $K_\infty^{(\mathrm{pf})}$. – By the presentation of $f_j^{(*)}$, we have

$$f_j^{(*)} = f_j + g_j \qquad (g_j \in (\frac{t_1}{t}, \ldots, \frac{t_e}{t})(B_{\mathrm{HT},K} \otimes_{\mathbb{Q}_p} V)).$$

Since $\{f_j\}_{j=1}^d$ forms a basis of $R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$ over $R[1/t]$, it is, in particular, linearly independent over $K_\infty^{(\mathrm{pf})}$ $(\subset R[1/t])$. Thus, $\{\overline{f_j} = \overline{f_j}^{(*)}\}_{j=1}^d$ ($-$ denotes the reduction modulo $(t_1, \dots, t_e)$) is linearly independent over $K_\infty^{(\mathrm{pf})}$ and we can see that $\{f_j^{(*)}\}_{j=1}^d$ is linearly independent over $K_\infty^{(\mathrm{pf})}$ in $R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$.

(C) Conclusion. – Therefore, on the K-vector space generated by $\{f_j^{(*)}\}_{j=1}^d$, $\log(\gamma)$ and $\{\log(\beta_i)\}_{i=1}^e$ act trivially ($\Leftrightarrow \nabla^{(0)}(f_j^{(*)}) = 0$ and $\nabla^{(i)}(f_j^{(*)}) = 0$ for all $1 \le i \le e$ and $1 \le j \le d$). Thus, this means that Γ_K acts on this K-vector space via finite quotient and there exists a finite field extension L/K in $K_\infty^{(\mathrm{pf})}$ such that $\{f^{(*)}\}_{j=1}^d$ forms a basis of $D_{\mathrm{HT},L}(V)$ over L. Since a potentially Hodge-Tate representation of G_K is a Hodge-Tate representation of G_K, this completes the proof. $\square$

4.2. Main theorem for de Rham representations

LEMMA 4.5. – *For a p-adic representation V of G_K, assume that V is a de Rham representation of $G_{K^{\mathrm{pf}}}$. Then, we can choose a basis $\{h_j\}_{j=1}^{d=\dim_{\mathbb{Q}_p} V}$ of $D_{\mathrm{dif}}^+(V)[1/t]$ over $K_\infty^{\mathrm{pf}}[[t, t_1, \dots, t_e]][1/t]$ such that the action of Γ_0 on $\{h_j\}_{j=1}^d$ is trivial.*

Proof. – Since V is a de Rham representation of $G_{K^{\mathrm{pf}}}$, there exists a basis $\{h_j\}_{j=1}^d$ of $B_{\mathrm{dR},K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_p} V$ over $B_{\mathrm{dR},K^{\mathrm{pf}}}$ such that the action of $G_{K^{\mathrm{pf}}}$ on $\{h_j\}_{j=1}^d$ is trivial. We can see that these elements $\{h_j\}_{j=1}^d$ are contained in $D_{\mathrm{dif}}^+(V)[1/t]$ by definition. For each j, if we twist h_j by some power of t, we obtain an element g_j of $B_{\mathrm{dR},K^{\mathrm{pf}}}^+ \otimes_{\mathbb{Q}_p} V$ such that $g_j \notin tB_{\mathrm{dR},K^{\mathrm{pf}}}^+ \otimes_{\mathbb{Q}_p} V$. Then, it follows that g_j is contained in $D_{\mathrm{dif}}^+(V)$ and satisfies $\overline{g_j} \neq 0$ ($-$ denotes the reduction modulo $(t, t_1, \dots, t_e) D_{\mathrm{dif}}^+(V)$). Since $D_{\mathrm{dif}}^+(V)$ is a free module of rank d over the local ring $K_\infty^{\mathrm{pf}}[[t, t_1, \dots, t_e]]$ and $\{\overline{g_j}\}_{j=1}^d$ forms a basis of $D_{\mathrm{Sen}}(V)$ over K_∞^{pf}, the lifting $\{g_j\}_{j=1}^d$ of $\{\overline{g_j}\}_{j=1}^d$ in $D_{\mathrm{dif}}^+(V)$ forms a basis of $D_{\mathrm{dif}}^+(V)$ over $K_\infty^{\mathrm{pf}}[[t, t_1, \dots, t_e]]$. Thus, it follows that $\{h_j\}_{j=1}^d$ forms a basis of $D_{\mathrm{dif}}^+(V)[1/t]$ over $K_\infty^{\mathrm{pf}}[[t, t_1, \dots, t_e]][1/t]$. $\square$

With notations as above, note that, since we have the inclusion $D_{e\text{-dif}}^+(V) \hookrightarrow D_{\mathrm{dif}}^+(V)[1/t]$ by definition, any element g of $D_{e\text{-dif}}^+(V)$ can be written as $g = \sum_{k=l}^{+\infty} (\sum_{j=1}^d a_{jk} h_j) t^k$ $(a_{jk} \in K_\infty^{\mathrm{pf}}[[t_1, \dots, t_e]])$.

REMARK 4.6. – Keep the notation as in Lemma 4.5. Since we assume that V is a de Rham representation of $G_{K^{\mathrm{pf}}}$, by Corollary 4.3, there exists a basis $\{v_j\}_{j=1}^d$ of $D_{\mathrm{Bri}}(V)$ over $K_\infty^{(\mathrm{pf})}$ such that $\nabla^{(0)}(v_j) = n_j v_j$. Put $M = \mathrm{Max}(n_j)_{j=1}^d$. Then, for an element $g \in D_{e\text{-dif}}^+(V)$, there exists an element $\sum_{k=n}^{+\infty} (\sum_{j=1}^d c_{jk} h_j) t^k$ of $(t, t_1, \dots, t_e) D_{e\text{-dif}}^+(V)$ such that we can write

$$g = \sum_{k=m}^{M} (\sum_{j=1}^{d} b_{jk} h_j) t^k + \sum_{k=n}^{+\infty} (\sum_{j=1}^{d} c_{jk} h_j) t^k \quad (b_{jk}, c_{jk} \in K_\infty^{\mathrm{pf}}[[t_1, \dots, t_e]]).$$

Thus, $g' = \sum_{k=m}^{M} (\sum_{j=1}^d b_{jk} h_j) t^k$ defines an element of $D_{e\text{-dif}}^+(V)$.

LEMMA 4.7. – *With notations as above, for an element $g' = \sum_{k=m}^{M} (\sum_{j=1}^d b_{jk} h_j) t^k$ of $D_{e\text{-dif}}^+(V)$, each $(\sum_{j=1}^d b_{jk} h_j) t^k$ is contained in $D_{e\text{-dif}}^+(V)$.*

Proof. – We shall prove this lemma by induction on the smallest degree of g' with respect to t. Since we have $g' - (\sum_{j=1}^{d} b_{jm}h_j)t^m \in D^+_{e\text{-dif}}(V)$ if $(\sum_{j=1}^{d} b_{jm}h_j)t^m$ is contained in $D^+_{e\text{-dif}}(V)$, it suffices to show that $(\sum_{j=1}^{d} b_{jm}h_j)t^m$ is contained in $D^+_{e\text{-dif}}(V)$. Since the $K^{\mathrm{pf}}_\infty[[t_1,\dots,t_e]]$-linear derivation $\nabla^{(0)}$ acts trivially on $\{h_j\}_{j=1}^{d}$, we have

$$\prod_{k=m+1}^{M}(\nabla^{(0)}-k)(g') = (\prod_{k=m+1}^{M}(m-k))(\sum_{j=1}^{d} b_{jm}h_j)t^m.$$

It follows that $(\sum_{j=1}^{d} b_{jm}h_j)t^m$ is contained in $D^+_{e\text{-dif}}(V)$ since the action of $\nabla^{(0)}$ on $D^+_{e\text{-dif}}(V)$ is stable. Thus, this completes the proof. □

PROPOSITION 4.8. – *For a p-adic representation V of G_K, assume that V is a de Rham representation of $G_{K^{\mathrm{pf}}}$. Then, there exists a $\nabla^{(0)}$-equivariant isomorphism of $K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]$-modules*

$$D^+_{e\text{-dif}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]](n_j) \quad (n_j \in \mathbb{Z}).$$

Proof. – Since V is also a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$, by Corollary 4.3, there exists a basis $\{v_j\}_{j=1}^{d}$ of $D^+_{e\text{-dif}}(V)/(t,t_1,\dots,t_e)D^+_{e\text{-dif}}(V) \simeq D_{\mathrm{Bri}}(V)$ over $K^{(\mathrm{pf})}_\infty$ such that it gives a $\nabla^{(0)}$-equivariant isomorphism of $K^{(\mathrm{pf})}_\infty$-vector spaces

$$D^+_{e\text{-dif}}(V)/(t,t_1,\dots,t_e)D^+_{e\text{-dif}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d} K^{(\mathrm{pf})}_\infty(n_j) : v_j \mapsto t^{n_j}.$$

Since $D^+_{e\text{-dif}}(V)$ is a free module of rank d over the local ring $K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]$, any lifting $\{g_j\}_{j=1}^{d}$ of $\{v_j\}_{j=1}^{d}$ in $D^+_{e\text{-dif}}(V)$ forms a basis of $D^+_{e\text{-dif}}(V)$ over $K^{(\mathrm{pf})}_\infty[[t,\, t_1,\dots,t_e]]$. Let $\{h_j\}_{j=1}^{d}$ denote a basis of $D^+_{\mathrm{dif}}(V)[1/t]$ over $K^{\mathrm{pf}}_\infty[[t,\, t_1,\dots,t_e]][1/t]$ such that $\nabla^{(0)}(h_j) = 0$ obtained in Lemma 4.5. Then, we may assume that each g_j is written as $g_j = \sum_{k=m}^{M}(\sum_{l=1}^{d} b_{kl}h_l)\, t^k$ ($b_{kl} \in K^{\mathrm{pf}}_\infty[[t_1,\dots,t_e]]$) where we take $M \in \mathbb{N}$ as in Remark 4.6. Now, define an element f_j of $D^+_{e\text{-dif}}(V)$ (Lemma 4.7 above) by

$$f_j = (\sum_{l=1}^{d} b_{n_j l}h_l)t^{n_j}.$$

It is easy to see $\nabla^{(0)}(f_j) = n_j f_j$. Therefore, the rest is to show that $\{f_j\}_{j=1}^{d}$ forms a basis of $D^+_{e\text{-dif}}(V)$ over $K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]$. To prove that $\{f_j\}_{j=1}^{d}$ is a lifting of $\{v_j\}_{j=1}^{d}$, it suffices to show $g_j - f_j \in (t,t_1,\dots,t_e)D^+_{e\text{-dif}}(V)$. For each g_j, put $s_k = (\sum_{l=1}^{d} b_{kl}h_l)t^k \in D^+_{e\text{-dif}}(V)$ (Lemma 4.7 above). Since we have $\nabla^{(0)}(\overline{s_k}) = k\overline{s_k}$ ($\overline{}$ denotes the reduction modulo $(t,t_1,\dots,t_e)$) and this means that $\overline{s_k}$ is an eigenvector of $\nabla^{(0)}$, it follows that the elements $\{v_j, \overline{s_k} \neq 0\}_{k\neq n_j}$ are linearly independent over $K^{(\mathrm{pf})}_\infty$ in $D_{\mathrm{Bri}}(V)$. Since we have $v_j = \sum_{k=m}^{M}\overline{s_k}$ by definition, it follows that we obtain $\overline{s_k} = 0$ for $k \neq n_j$. This means that we have $s_k \in (t,t_1,\dots,t_e)D^+_{e\text{-dif}}(V)$ ($k \neq n_j$) and $g_j - f_j \in (t,t_1,\dots,t_e)D^+_{e\text{-dif}}(V)$. Thus, this completes the proof. □

REMARK 4.9. – In general, it is evident from the proof that, if L denotes a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p > 0$ and V is a de Rham representation of $G_L = \mathrm{Gal}(\overline{L}/L)$ where we choose an algebraic closure $\overline{L}$ of L, we have a $\nabla^{(0)}$-equivariant isomorphism of $L_\infty[[t]]$-modules

$$D^+_{\mathrm{dif}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} L_\infty[[t]](n_j) \quad (n_j \in \mathbb{Z}).$$

THEOREM 4.10. – *Let K be a complete discrete valuation field of characteristic* 0 *with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$ and V be a p-adic representation of G_K. Let K^{pf} be the field extension of K defined as before. Then, V is a de Rham representation of G_K if and only if V is a de Rham representation of $G_{K^{\mathrm{pf}}}$.*

Proof. – We shall prove the main theorem in two parts.

(1) V: dR rep. of $G_K \Rightarrow V$: dR rep. of $G_{K^{\mathrm{pf}}}$. – Since V is a de Rham representation of G_K, there exists a G_K-equivariant isomorphism of $B_{\mathrm{dR},K}$-modules

$$B_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V \simeq (B_{\mathrm{dR},K})^{d=\dim_{\mathbb{Q}_p} V}. \tag{4.4}$$

Now, by tensoring $B_{\mathrm{dR},K^{\mathrm{pf}}} \otimes_{B_{\mathrm{dR},K}}$ (which is induced by the $G_{K^{\mathrm{pf}}}$-equivariant surjection $p : B_{\mathrm{dR},K} \twoheadrightarrow B_{\mathrm{dR},K^{\mathrm{pf}}} : t_i \mapsto 0$) over (4.4), we obtain a $G_{K^{\mathrm{pf}}}$-equivariant isomorphism of $B_{\mathrm{dR},K^{\mathrm{pf}}}$-modules

$$B_{\mathrm{dR},K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_p} V \simeq (B_{\mathrm{dR},K^{\mathrm{pf}}})^d.$$

This means that V is a de Rham representation of $G_{K^{\mathrm{pf}}}$.

(2) V: dR rep. of $G_{K^{\mathrm{pf}}} \Rightarrow V$: dR rep. of G_K. – For simplicity, put $R = K_\infty^{(\mathrm{pf})}[[t, t_1, \ldots, t_e]]$. We shall construct the $K_\infty^{(\mathrm{pf})}$-linearly independent elements $\{f_j^{(*)}\}_{j=1}^{d=\dim_{\mathbb{Q}_p} V}$ of $R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$ $(\subset B_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)$ such that $\nabla^{(i)}(f_j^{(*)}) = 0$ for all $0 \le i \le e$ and $1 \le j \le d$.

(A) Construction of $\{f_j^{(*)}\}_{j=1}^d \in R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$. – From the presentation of Proposition 4.8 above, if we twist by some powers of t, we obtain a basis $\{f_j\}_{j=1}^d$ of $R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$ over $R[1/t]$ such that $\nabla^{(0)}(f_j) = 0$ for all $1 \le j \le d$. Thus, by applying Corollary 3.6 to the $R[1/t]$-module $R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$ generated by $\{f_j\}_{j=1}^d$, we can deduce

$$(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(f_j) = t^{k_1+\cdots+k_e} \sum_{k=1}^d c_k f_k \tag{4.5}$$

where c_k is an element of R such that $\nabla^{(0)}(c_k) = 0$. Define an element $f_j^{(*)}$ of $R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$ by

$$f_j^{(*)} = \sum_{0 \le k_1, \ldots, k_e} (-1)^{k_1+\cdots+k_e} \frac{t_1^{k_1} \cdots t_e^{k_e}}{k_1! \cdots k_e! t^{k_1+\cdots+k_e}} (\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(f_j).$$

Note that this series converges in $R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$ for $(t_1, \ldots, t_e)$-adic topology by (4.5) and thus $f_j^{(*)}$ actually defines an element of $R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$. Then, it follows easily that we have $\nabla^{(i)}(f_j^{(*)}) = 0$ for all $1 \leq i \leq e$ and $1 \leq j \leq d$ by using the Leibniz rule. Furthermore, by using (4.5) and the fact $\nabla^{(0)}(f_j) = 0$, we can deduce that we have $\nabla^{(0)}(f_j^{(*)}) = 0$ for all $1 \leq j \leq d$.

(B) $\{f_j^{(*)}\}_{j=1}^d \in R[1/t]\otimes_R D^+_{e\text{-dif}}(V)$ *is linearly independent over* $K_\infty^{(\mathrm{pf})}$. – By the presentation of $f_j^{(*)}$, we have

$$f_j^{(*)} = f_j + g_j \qquad (g_j \in (t_1, \dots, t_e)(B_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)).$$

Since $\{f_j\}_{j=1}^d$ forms a basis of $R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$ over $R[1/t]$, it is, in particular, linearly independent over $K_\infty^{(\mathrm{pf})}$ ($\subset R[1/t]$). Thus, $\{\overline{f_j} = \overline{f_j}^{(*)}\}_{j=1}^d$ ($-$ denotes the reduction modulo $(t_1, \dots, t_e)$) is linearly independent over $K_\infty^{(\mathrm{pf})}$ and we can see that $\{f_j^{(*)}\}_{j=1}^d$ is linearly independent over $K_\infty^{(\mathrm{pf})}$ in $R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$.

(C) Conclusion. – Therefore, on the K-vector space generated by $\{f_j^{(*)}\}_{j=1}^d$, $\log(\gamma)$ and $\{\log(\beta_i)\}_{i=1}^e$ act trivially ($\Leftrightarrow \nabla^{(0)}(f_j^{(*)}) = 0$ and $\nabla^{(i)}(f_j^{(*)}) = 0$ for all $1 \leq i \leq e$ and $1 \leq j \leq d$). Thus, this means that Γ_K acts on this K-vector space via finite quotient and there exists a finite field extension L/K in $K_\infty^{(\mathrm{pf})}$ such that $\{f^{(*)}\}_{j=1}^d$ forms a basis of $D_{\mathrm{dR},L}(V)$ over L. Since a potentially de Rham representation of G_K is a de Rham representation of G_K, this completes the proof. $\square$

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