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SOME REGULARITIES AND SINGULARITIES APPEARING IN THE STUDY OF POLYNOMIALS AND OPERATORS

by

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Abstract. — We apply the viewpoint of singularity theory to the following problems: how does the decomposition of a polynomial P as the product of polynomials behave under perturbations of P? How do the eigenvalues, eigenspaces and more generally invariant subspaces of an operator A behave under perturbations of A? We give a characterization of the regular situations and describe completely the singular ones in some moderately degenerate situations.

***Résumé* (Quelques regularités et singularités apparaissant dans l'étude des polynômes et des opérateurs)**

Nous appliquons le point de vue de la théorie des singularités aux deux problèmes suivants : comment la décomposition d'un polynôme P comme produit de polynômes se comporte-t-elle quand on perturbe P ? Comment les valeurs propres, vecteurs propres et plus généralement sous-espaces invariants d'un opérateur A se comportent-ils quand on perturbe A ? Nous caractérisons les situations régulières et décrivons complètement celles qui sont singulières mais pas trop dégénérées.

Introduction

In the study of bifurcations of dynamical systems one has to deal frequently with the following situation: as a parameter varies one considers the variation of an eigenvalue or of the invariant line generated by the corresponding eigenvector of the linearization of the dynamical system at a certain point. It often happens that those elements vary smoothly with the parameter, which is known to be the case if the eigenvalue is simple. But nevertheless the system undergoes a bifurcation if the eigenvalue crosses a certain subset of the plane (the unit circle, the imaginary axis, etc.). A second, more complex, situation happens when the eigenvalue becomes multiple, since then its variation with the parameter ceases to be smooth. The same situations occur when instead of an

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invariant line one needs to consider an invariant subspace of dimension greater than one.

During the years we have meditated on these questions and have arrived at various forms of expressing the (essentially known) conditions for the smooth variation of those elements (see for example **[5, 4]** for recent versions). One of those forms seems especially suited for studying, in terms of singularities of mappings, the situations where that variation ceases to be smooth. In this article we describe the simplest of those singularities.

The results. — We begin by a study of the simplest singularities of the polynomial multiplication map:

$$\mathrm{Mult} : \mathrm{MP}(n) \times \mathrm{MP}(m) \to \mathrm{MP}(n+m)$$

where $\mathrm{MP}(n)$ will denote the space of monic polynomials of degree n over $\mathbf{K}$, which will be either the real or the complex field. The rank of this map at a point (f, g) can be expressed in terms of the degree of the greatest common divisor $\gcd(f, g)$ so that it is a local diffeomorphism precisely when this degree is 0, i.e. when the factors are relatively prime. And we can describe completely the singularities of Mult when this degree is 1 (Theorem 1). Then we proceed to study the higher corank singularities of Mult; here our results are not as sharp, but we have a complete geometric description of many cases and an algebraic description of the rest.

As a byproduct of Theorem 1 we give an interesting description of the classical resultant of two polynomials and we obtain the relation between the singularities of Mult we describe and the resultant set $\mathrm{Res}(f, g) = 0$.

Then we apply Theorem 1 (and its corollary, Theorem 3, which generalizes it to the multiplication of an arbitrary number of factors) to study the singularities of the (monic) characteristic polynomial map

$$\chi : \mathrm{M}(n \times n) \to \mathrm{MP}(n)$$

where $\mathrm{M}(n \times n)$ denotes the space of $n \times n$ matrices with entries in $\mathbf{K}$. We will view each $M \in \mathrm{M}(n \times n)$ as a linear mapping $\mathbb{C}^n \to \mathbb{C}^n$ and always take into account all its *complex* eigenvalues. We determine the matrices at which χ is a submersion and give a description of its simplest singularities (Theorem 5).

All the above is used to study the singularities of the eigenvalues of operators. For that, we introduce the set of all *proper elements* of a Banach space E over $\mathbf{K}$ to be the space of triples consisting of a linear operator on E, an invariant line and the corresponding eigenvalue:

$$\mathrm{Eig}(E) := \{(\lambda, L, A) \in \mathbf{K} \times \mathbf{P}(E) \times \mathrm{End}(E) \,:\, A(L) \subseteq L \textit{ and } A|_L = \lambda\}.$$

Here $\mathbf{P}(E)$ denotes the projectivization of E and $\mathrm{End}(E)$ the space of continuous linear endomorphisms of E. There is a natural projection $\Pi : \mathrm{Eig}(E) \mapsto \mathrm{End}(E)$ on the third factor.

The basic fact here (Theorem 6) is that $\mathrm{Eig}(E)$ is a *smooth object*, actually an analytic manifold modelled on $\mathrm{End}(E)$, provided with a projection Π onto $\mathrm{End}(E)$.

Therefore it is a kind a *resolution* of all the singularities associated to eigenvalue problems.

We show that, not surprisingly, Π is a local diffeomorphism precisely at those points where λ is a simple eigenvalue of A. And we can describe completely the singularities of Π when λ is a geometrically simple eigenvalue of A of finite multiplicity (Theorem 8). In the finite dimensional case this means simply that it has only one corresponding invariant line, while in infinite dimensions there are some technical additional conditions. We also show that the mapping that forgets the invariant line is regular (in this case an immersion) precisely when the eigenvalue is geometrically simple, a fact that is useful in the proof of the singularity part of Theorem 8.

In section D, we generalize this to invariant subspaces of dimension greater than one. In fact, this was the starting point of the whole story: in [**2**], we explained that the theory of formal normal forms for dynamical systems is an easy consequence of the Jordan decomposition of endomorphisms. Thinking about the generalization of this approach to *families*, we came to the conclusion that each characteristic space F_0 of an endomorphism A_0 of $\mathbf{C}^n$ must have the following stability property: every nearby endomorphism A has a unique invariant subspace $F(A)$ of the same dimension as $F_0 = F(A_0)$ and close to it, depending analytically on A. This is an easy result but it is not so well-known(1), and in section D we consider (and extend) it in the spirit of singularity theory.

The singularities. — The singularities found in Theorems 1, 8, 15, 16 are a certain type of Morin singularities which we will call *swallowtails*:

The *standard k–swallowtail* is the map

$$\mathrm{SW}_k : \mathbf{K}^{k-1} \to \mathbf{K}^{k-1}$$

defined by

$$\mathrm{SW}_k(a_1, \dots, a_{k-2}, u) := (a_1, \dots, a_{k-2}, u^k + a_{k-2}u^{k-2} + \cdots + a_1 u)$$

For us a *k–swallowtail* will be any map germ between two Banach spaces which is diffeomorphic to the germ at 0 of a map of the form

$$\mathrm{SW}_k \times \mathrm{Id} : \mathbf{K}^{k-1} \times E \to \mathbf{K}^{k-1} \times E$$

for some Banach space E. When $\mathbf{K} = \mathbf{C}$ but E is *real*—a situation occurring whenever a real polynomial or endomorphism has nonreal roots or eigenvalues—we call such a map a *complex swallowtail.*

Interesting examples of k-swallowtails are the evaluation map

$$\begin{aligned} \mathrm{ev} : \mathrm{MP}(k) \times \mathbf{K} &\to \mathrm{MP}(k) \times \mathbf{K} \\ (P, a) &\mapsto \big(P, P(a)\big) \end{aligned}$$

and the mapping

$$(a_1, \dots, a_{k-1}, a) \mapsto (aa_1, a_1 + aa_2, a_2 + aa_3, \dots, a_{k-2} + aa_{k-1}, a_{k-1} + a)$$

(1) The finite dimensional case led us to Theorem 1...

The second example shows that all swallowtails can be given by maps all of whose coordinate functions are polynomials of degree at most 2, a fact that we have not seen in the literature.

These examples, and some of their variants, will play an important role in the proofs of the theorems.

The singularities in Theorem 5 will be *k–swallowtail deformations*, by which we mean any map germ between two Banach spaces which is diffeomorphic to the germ at 0 of a map

$$G : E \times E' \to E$$

such that $G(x, 0)$ is a k–swallowtail, where E, E' are Banach spaces.

There are many k–swallowtail deformations between spaces of the same dimension, so this term does not describe a precise singularity type. And though it is possible, in principle, to describe them all, there remains to do so specifically for the singularities of χ.

We will show by examples that in all cases the singularities that are not swallowtails are more complicated than those one could expect from the classification results of singularities of mappings.

We hope to give soon some applications of these results to bifurcation problems of dynamical systems.

In the Appendices we recall the main properties of the singularities we will use in the text and describe completely the main examples. We also provide an introduction to the results on continuous linear maps between Banach spaces needed in the text, referring to Rudin's beautiful book **[10]** for more details about this magnificent theory.

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A. Singularities of Polynomial Multiplication

Polynomial multiplication defines a map

$$\text{Mult} : \text{MP}(n) \times \text{MP}(m) \to \text{MP}(n + m)$$

We are interested in describing the regular points and the singularities of the map Mult . We will denote by $\gcd(P, Q)$ the monic greatest common divisor of the monic polynomials P and Q.

Theorem 1. — *For (P_0, Q_0) in $\text{MP}(n) \times \text{MP}(m)$,*

(i) *The corank of the differential $D\,\text{Mult}(P_0, Q_0)$ is the degree of $\gcd(P_0, Q_0)$.*
(ii) *In particular,* Mult *is a local diffeomorphism at (P_0, Q_0) if and only if $\gcd(P_0, Q_0) = 1$.*
(iii) *The mapping* Mult *is a $(k + 1)$–swallowtail at (P_0, Q_0) for some positive integer k if, and only if, $\deg \gcd(P_0, Q_0) = 1$, the integer k being the maximum of the multiplicities in P_0 and Q_0 of their common root.*

(iv) *If* $\mathbf{K} = \mathbf{R}$*, the mapping* Mult *is a complex* $(k+1)$*–swallowtail at* (P_0, Q_0) *for some positive integer* k *if, and only if,* $\gcd(P_0, Q_0)$ *is an irreducible polynomial of degree* 2*, the integer* k *being the maximum of the multiplicities in* P_0 *and* Q_0 *of their complex conjugate common roots*(2).

Proof. — The tangent space of $\mathrm{MP}(n)$ at any point is the set of polynomials of degree less that n. The derivative of Mult at (P_0, Q_0) is then given by

$$(P, Q) \mapsto P_0 Q + P Q_0.$$

Therefore its image, being the set of multiples of $\gcd(P_0, Q_0)$ by polynomials of degree less than $n + m - \deg \gcd(P_0, Q_0)$, has this dimension. This proves (i) and therefore (ii).

If $\gcd(P_0, Q_0) = x - \alpha$ then $x - \alpha$ must divide one of P_0, Q_0 with multiplicity 1 and the other one with multiplicity k. By changing the variable in the polynomials (which induces a diffeomorphism of $\mathrm{MP}(n)$) we can assume $\alpha = 0$.

Consider first the case $P_0 = x$, $Q_0 = x^k$. Then Mult is given by

$$\mathrm{Mult}\left(x + a, x^k + \sum_{i=0}^{k-1} a_i x^i\right) = \sum_{i=0}^{k+1} (a_{i-1} + a a_i) x^i$$

(putting $a_k = 1$, $a_{k+1} = a_{-1} = 0$) or, in coordinates $(a, a_{k-1}, \ldots, a_0)$, by

$$\mathrm{Mult}(a_0, \ldots, a_{k-1}, a) = (a a_0, a_0 + a a_1, a_1 + a a_2, \ldots, a_{k-1} + a),$$

which is a $(k+1)$–swallowtail by example 2 in Appendix A.

In general, let $P_0 = x P_1$, $Q_0 = x^k Q_1$, where P_1, Q_1 are not divisible by x. Then, setting $m_1 := m - k$ and $n_1 := n - 1$, we have a commutative diagram:

$$\begin{array}{ccccc} \mathrm{MP}(n) & \times & \mathrm{MP}(m) & \to & \mathrm{MP}(m+n) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{MP}(1) \times \mathrm{MP}(n_1) & \times & \mathrm{MP}(k) \times \mathrm{MP}(m_1) & \to & \mathrm{MP}(k+1) \times \mathrm{MP}(m_1 + n_1) \end{array}$$

where all maps are given by multiplication. By Theorem 1, the vertical arrows are local diffeomorphisms at (x, P_1), (x^k, Q_1) and $(x^{k+1}, P_1 Q_1)$ respectively. The lower map is the product of the multiplication $\mathrm{MP}(1) \times \mathrm{MP}(k) \to \mathrm{MP}(k+1)$, which we have just seen to be a $(k+1)$–swallowtail at (x, x^k), and the multiplication $\mathrm{MP}(n_1) \times \mathrm{MP}(m_1) \to \mathrm{MP}(m_1 + n_1)$, a local diffeomorphism at (P_1, Q_1) b Theorem 1. Therefore the upper multiplication map is diffeomorphic to the lower one, which is a $(k+1)$–swallowtail. This proves the "if" in (i). As for the "only if", just notice that, when the degree of $\gcd(P_0, Q_0)$ is greater than 1, the corank of $D\,\mathrm{Mult}$ is greater than 1 and Mult cannot be a swallowtail at that point. This proves (iii).

(2) Or, in other words, the greatest integer k such that $\gcd(P_0, Q_0)^k$ divides P_0 or Q_0.

Let us prove (iv). In the "if", the same diagram as for (iii) reduces the problem to the case where $P_0 = (x-\alpha)(x-\bar\alpha)$ and $Q_0 = (x-\alpha)^k(x-\bar\alpha)^k$, $\alpha \in \mathbf{C}\setminus\mathbf{R}$. Applying (ii) with $m = n = 1$ (resp. $m = n = k$), we see that every complex polynomial P (resp. Q) of degree 2 (resp. $2k$) close enough to P_0 (resp. Q_0) writes in a (locally) unique fashion $P = P_1P_2$, $P_j \in \mathrm{MP}(1)$ (resp. $Q = Q_1Q_2$, $Q_j \in \mathrm{MP}(k)$), where (P_1, P_2) (resp. (Q_1, Q_2)) is the image of P (resp. Q) by the local inverse of Mult at $(x-\alpha, x-\bar\alpha)$ resp. $\big((x-\alpha)^k, (x-\bar\alpha)^k\big)$). This uniqueness property implies that, for real P and Q, we must have $P_2 = \bar P_1$ and $Q_2 = \bar Q_1$. Thus, in that case, the mappings $P \mapsto P_1$ and $Q \mapsto Q_1$ are real analytic local diffeomorphisms, identifying $(P, Q) \mapsto PQ$ to $(P_1, Q_1) \mapsto P_1Q_1$, which is a complex $(k+1)$–swallowtail by (iii).

For the "only if", we observe that any other corang 2 singularity (P_0, Q_0) has a common real root. In any neighborhood of (P_0, Q_0) we can find a pair (P_1, Q_1) with a single common *simple* real root and by (iii) the singular set of Mult is a codimension 1 smooth manifold near (P_1, Q_1). Therefore, at a complex swallowtail, as the singular set of Mult is locally of real codimension 2, it cannot be diffeomorphic to the singularity at (P_0, Q_0). □

Remarks. — Theorem 1 (ii), which can be found in **[3]** (Exercice 1, p. 234), extends the well-known result that a simple root varies smoothly with the coefficients of the polynomial (consider the local inverse of Mult).

If one writes down the Jacobian matrix of Mult at a point (f,g) with respect to the standard bases of the vector spaces of (non-necessarily monic) polynomials involved, one obtains the transpose of the usual Sylvester matrix, whose determinant is one of the definitions of the resultant of the polynomials (g, f). Therefore we have an interesting equality:

$$\mathrm{Res}(f,g) = J\,\mathrm{Mult}(g,f) \quad \text{(Jacobian determinant).}$$

which is natural since both sides of the equality vanish precisely when f, g have a common complex root. One can take this as a definition of the resultant and use it to prove its basic properties. The change from (f, g) to (g, f) in the right-hand side is only a sign convention, as in fact the definitions of the resultant by different authors only coincide up to sign: see for example **[11, 6]**.

The regular points of the variety $\mathrm{Res}(f, g) = 0$ are precisely the pairs with gcd of degree 1, since the points of higher corank have to be singularities of $J\,\mathrm{Mult}$ (see Appendix 1). By (iii), this regular set can be still stratified according to the singularity type of Mult, i.e. according to the order of the swallowtails. Thus, the singularity type of Mult gives more information than the singularities of the resultant variety. This is a first answer to a question by Bernard Teissier about the relation between those two singularities.

We now turn to the singularities of corank ≥ 2 involving only simple common roots:

Proposition 1. — *Given* $P_0 \in \mathrm{MP}(n)$ *and* $Q_0 \in \mathrm{MP}(m)$, *assume that all the (complex) roots of* $\gcd(P_0, Q_0)$ *are simple.*

(i) *If* $\mathbf{K} = \mathbf{C}$, *then, denoting by* $\alpha_1, \ldots, \alpha_d$ *the roots of* $\gcd(P_0, Q_0)$, *the map* Mult *is the product of* d *swallowtails of respective orders* $k_1 + 1, \ldots, k_d + 1$, *where* k_j *denotes the maximum of the multiplicities of the root* α_j *in* P_0 *and* Q_0.

(ii) *If* $\mathbf{K} = \mathbf{R}$, *then, denoting by* $\alpha_1, \ldots, \alpha_r$ *the real roots of* $\gcd(P_0, Q_0)$ *and by* $\alpha_{r+1}, \bar{\alpha}_{r+1}, \ldots, \alpha_d, \bar{\alpha}_d$ *its other roots, the map* Mult *is the product of* r *real swallowtails of respective orders* $k_1 + 1, \ldots, k_r + 1$ *and* c *complex swallowtails of respective orders* $k_{r+1} + 1, \ldots, k_d + 1$, *where* k_j *denotes the maximum of the multiplicities of the root* α_j *in* P_0 *and* Q_0.

Proof. — We establish (i) by induction on d. Theorem 1 (ii) tells us that (i) is true if $d = 1$. Given $d > 1$, assume (i) true for $d - 1$. Then, in the situation of (i), exchanging P_0 and Q_0 if necessary, we have $P_0 = (x - \alpha_d)P_1$, $Q_0 = (x - \alpha_d)^{k_d} Q_1$ and $\gcd(P_0, Q_0) = (x - \alpha_d) \gcd(P_1, Q_1)$. By the induction hypothesis, Mult is at (P_1, Q_1) the product of $d-1$ swallowtails of respective orders $k_1+1, \ldots, k_{d-1}+1$. Now, setting $m_1 := m - k_d$ and $n_1 := n - 1$, we have a commutative diagram:

$$\begin{array}{ccccc} \mathrm{MP}(n) & \times & \mathrm{MP}(m) & \rightarrow & \mathrm{MP}(m+n) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{MP}(1) \times \mathrm{MP}(n_1) & \times & \mathrm{MP}(k_d) \times \mathrm{MP}(m_1) & \rightarrow & \mathrm{MP}(k_d + 1) \times \mathrm{MP}(m_1 + n_1) \end{array}$$

where all maps are given by multiplication, and we conclude as in the proof of Theorem 1 (iii).

This also proves (ii) if $c = 0$. Otherwise, exchanging P_0 and Q_0 if necessary, we have that $P_0 = (x - \alpha_{k_d})(x - \bar{\alpha}_{k_d})P_1$, $Q_0 = (x - \alpha_{k_d})^{k_d}(x - \bar{\alpha}_{k_d})^{k_d} Q_1$ and $\gcd(P_0, Q_0) = (x - \alpha_{k_d})(x - \bar{\alpha}_{k_d}) \gcd(P_1, Q_1)$. Using Theorem 1 (iv), we conclude as for (i). □

Remark. — In the situation of (ii) with $r = 2$, $c = 0$ and $k_1 = k_2 = 1$, we get (a suspension of) the "twice folded handkerchief", product of two one-dimensional folds. More generally in the situation of (ii) with $r = 2$, $c = 0$ and $k_1 + k_2 = k$, the critical set of Mult is locally the (singular) union of two smooth hypersurfaces intersecting at (P_0, Q_0). In particular, the critical set of Mult has codimension 1, making more precise the end of the proof of Theorem 1 (iv) in this case.

Let us now consider all the singularities of corank $\nu \geq 2$ with a single common root:

Proposition 2. — *Given* $P_0 \in \mathrm{MP}(n)$ *and* $Q_0 \in \mathrm{MP}(m)$,

(i) *Assume* $\gcd(P_0, Q_0) = (x - \alpha)^\nu$, $\alpha \in \mathbf{K}$, $\nu \geq 2$. *Then, denoting by* $k \geq \nu$ *the maximum of the multiplicities of the root* α *in* P_0 *and* Q_0, *the map* Mult *has a singularity at* (P_0, Q_0) *which is diffeomorphic to the singularity at* 0 *of the map*

$$\begin{array}{ccc} (a, b, u) & \longmapsto & \big(a, b, f_{\nu,k}(b, u),\big) \\ \mathbf{K}^{m+n-k-\nu} \times \mathbf{K}^k \times \mathbf{K}^\nu & \longrightarrow & \mathbf{K}^{m+n-k-\nu} \times \mathbf{K}^k \times \mathbf{K}^\nu \end{array}$$

given by $f_{\nu,k} = (f_{\nu,k,1}, \dots, f_{\nu,k,\nu})$ *and, if* $b = (b_1, \dots, b_k)$,

$$f_{\nu,k,\ell}(b,u) \quad := \quad \sum_{\|m\|=k+\ell} \frac{|m|!}{m!} u^m + \sum_{j=1}^{k} b_j \sum_{\|m\|=k+\ell-j} \frac{|m|!}{m!} u^m \,,\ 1 \le \ell \le \nu,$$

where $m = (m_1, \dots, m_\nu) \in \mathbf{N}^\nu$, $m! = m_1! \cdots m_\nu!$, $|m| := m_1 + \cdots + m_\nu$, $\|m\| := m_1 + 2m_2 + \cdots + \nu m_\nu$ *and* $u^m := u_1^{m_1} \cdots u_\nu^{m_\nu}$.

(ii) *If* $\mathbf{K} = \mathbf{R}$ *and* $\gcd(P_0, Q_0) = (x-\alpha)^\nu (x-\bar{\alpha})^\nu$, $\alpha \in \mathbf{C} \setminus \mathbf{R}$, $\nu \ge 2$, *then, denoting by* $k \ge \nu$ *the maximum of the multiplicities of the root* α *in* P_0 *and* Q_0, *the map* Mult *has a singularity at* (P_0, Q_0) *which is diffeomorphic to the singularity at* 0 *of the map*

$$\begin{array}{ccc} (a,b,u) & \longmapsto & (a, b, f_{\nu,k}(b,u),) \\ \mathbf{R}^{m+n-2k-2\nu} \times \mathbf{C}^k \times \mathbf{C}^\nu & \longrightarrow & \mathbf{R}^{m+n-2k-2\nu} \times \mathbf{C}^k \times \mathbf{C}^\nu, \end{array}$$

where $f_{\nu,k}$ *is as in (i).*

Proof. — (i) If $\gcd(P_0, Q_0) = (x-\alpha)^\nu$ then $x - \alpha$ must divide one of the two polynomials P_0, Q_0 with multiplicity ν and the other one with multiplicity $k \ge \nu$. As in the proof of Proposition 1, we can assume $\alpha = 0$ and reduce the general case to the case $P_0 = x^\nu$, $Q_0 = x^k$. Then, Mult is given by

$$\mathrm{Mult}\Big(x^\nu - \sum_{i=1}^{\nu} u_i x^{\nu-i}, x^k + \sum_{j=1}^{k} v_j x^{k-j}\Big) = \sum_{j=0}^{k+\nu} \Big(v_j - \sum_{i=1}^{\nu} v_{j-i} u_i\Big) x^{k+\nu-j}$$

where

$$\text{(1)} \qquad \begin{cases} v_0 = 1 \\ v_j = 0 \quad \text{for } j < 0 \text{ and for } j > k \end{cases}$$

or, taking $u_1, \dots, u_\nu, v_1, \dots, v_k$ as coordinates, by

$$\mathrm{Mult}(v_1, \dots, v_k, u_1, \dots, u_\nu) = \Big(v_j - \sum_{i=1}^{\nu} v_{j-i} u_i\Big)_{1 \le j \le k+\nu}.$$

Denoting by $b_1, \dots, b_{k+\nu}$ the components of the right-hand side, we shall express the variables $v_1, \dots, v_k$ as functions of $b_1, \dots, b_k$ and $u_1, \dots, u_\nu$ by solving the equations

$$\text{(2)} \qquad b_j = v_j - \sum_{i=1}^{\nu} v_{j-i} u_i$$

for $1 \le j \le k$, clearly an invertible linear system with respect to $v_1, \dots, v_k$. Then, the equations (2) with $k+1 \le j \le k+\mu$ will yield the required expression

$$\text{(3)} \qquad b_j = -f_{\nu,k,j-k}(b_1, \dots, b_k, u_1, \dots, u_\nu)\,,\ k+1 \le j \le k+\mu.$$

To do all this at once, we consider (2) for all $j \in \mathbf{Z}$ and, using (1), rewrite it as

$$(4) \qquad \begin{cases} b_j = 0 & \text{for } j < 0 \\ b_0 = 1 & \\ v_j = b_j + \sum_{i=1}^{\nu} v_{j-i}\, u_i & \text{for } j > 0 \end{cases}$$

We claim that these conditions imply that

$$(5) \qquad v_j \quad = \quad \sum_{\|m\| \leq j} \frac{|m|!}{m!}\, b_{j-\|m\|}\, u^m \ , \ j \in \mathbf{Z},$$

hence (3) because of (1).

Indeed, by (4), we know that (5) is true for all $j < 0$. Given $j \geq 0$, we can therefore make the induction hypothesis that (5) is true for all $j - i$, $1 \leq i \leq \mu$, hence, by (4),

$$v_j \quad = \quad b_j + \sum_{i=1}^{\nu} \sum_{\|n\| \leq j-i} \frac{|n|!}{n!}\, b_{j-i-\|n\|}\, u^{n+\delta_i}$$

where n lies in $\mathbf{N}^\nu$ and $(\delta_1, \ldots, \delta_\nu)$ denotes the canonical basis of $\mathbf{K}^\nu$. Now, for each $m \in \mathbf{N}^\nu$, we have $m = n + \delta_i$ with $n \in \mathbf{N}^\nu$ if and only if m_i is positive, in which case $\|m\| = \|n\| + i$, hence

$$\begin{aligned} v_j \quad &= \quad b_j + \sum_{1 \leq \|m\| \leq j} \sum_{m_i \neq 0} \frac{|m - \delta_i|!}{(m - \delta_i)!}\, b_{j-\|m\|}\, u^m \\ &= \quad b_j + \sum_{1 \leq \|m\| \leq j} \frac{(|m| - 1)!}{m!} \Big(\sum_{m_i \neq 0} m_i \Big)\, b_{j-\|m\|}\, u^m \\ &= \quad b_j + \sum_{1 \leq \|m\| \leq j} \frac{|m|!}{m!}\, b_{j-\|m\|}\, u^m \ , \end{aligned}$$

proving (5). From this particular case, we deduce (i) in general as in the proof of Theorem 1 (iii). The proof of (ii) is that of Theorem 1 (iv). □

Remarks. — Of course, for $\nu = 1$, the proof of Proposition 2 applies and is nothing but the proof of Theorem 1 (iii), the function $f_{1,k}$ being essentially the evaluation map of $\mathrm{MP}(k+1)$.

In the situation of (i) with $\nu = 2$ and $\mathbf{K} = \mathbf{R}$, the germ of Mult at (P_0, Q_0) is analytically diffeomorphic to the germ at 0 of $(a, b, u) \mapsto \big(a, b, f_{2,k}(b, u)\big)$, $a \in \mathbf{R}^{m+n-k-2}$, $b = (b_1, \ldots, b_k) \in \mathbf{R}^k$, whose critical set is the set of zeros of the determinant $J_u f_{2,k}(b, u) := \big(\partial_{u_1} f_{2,k,1} \partial_{u_2} f_{2,k,2} - \partial_{u_1} f_{2,k,2} \partial_{u_2} f_{2,k,1}\big)(b, u)$. Since the latter is a quadratic form in the variable b satisfying $J_u f_{2,k}(b, 0) = b_k^2 - b_{k-1}^2$, this gives a more precise idea of the shape of the critical set of Mult, which is a singular hypersurface, as shown at the end of the proof of Theorem 1 (iv).

We can now glue together Theorem 1 (iv) and Proposition 2 as in the proof of Proposition 1 to obtain an algebraic description of all the singularities of Mult:

Theorem 2. — *Given* $P_0 \in \mathrm{MP}(n)$ *and* $Q_0 \in \mathrm{MP}(m)$,

(i) *If* $\mathbf{K} = \mathbf{C}$, *then, denoting by* $\alpha_1, \ldots, \alpha_d$ *the roots of* $\gcd(P_0, Q_0)$ *and by* $\nu_1, \ldots, \nu_d$ *their respective multiplicities, the germ of* Mult *at* (P_0, Q_0) *is diffeomorphic to the germ at* 0 *of the map*

$$\begin{array}{ccc} \mathbf{K}^p \times \prod_1^d \left(\mathbf{K}^{k_i} \times \mathbf{K}^{\nu_i}\right) & \longrightarrow & \mathbf{K}^p \times \prod_1^d \left(\mathbf{K}^{k_i} \times \mathbf{K}^{\nu_i}\right) \\ \left(a, (b_1, x_1), \ldots, (b_d, x_d)\right) & \longmapsto & \left(a, f_{\nu_1,k_1}(b_1, x_1), \ldots, f_{\nu_d,k_d}(b_d, x_d)\right), \end{array}$$

where $k_i \geq \nu_i$ *denotes the maximum of the multiplicities of the root* α_i *in* P_0 *and* Q_0, *and* $p = m + n - |k| - |\nu|$.

(ii) *If* $\mathbf{K} = \mathbf{R}$, *then, denoting by* $\alpha_1, \ldots, \alpha_r$ *the real roots of* $\gcd(P_0, Q_0)$, *by* $\alpha_{r+1}, \bar{\alpha}_{r+1}, \ldots, \alpha_d, \bar{\alpha}_d$ *its other roots, by* ν_j *the multiplicity of the root* α_j *and setting* $d := r + c$, *the germ of* Mult *at* (P_0, Q_0) *is diffeomorphic to the germ at* 0 *of the map*

$$\left(a, (b_1, x_1), \ldots, (b_d, x_d)\right) \longmapsto \left(a, f_{\nu_1,k_1}(b_1, x_1), \ldots, f_{\nu_d,k_d}(b_d, x_d)\right),$$

of $\mathbf{R}^p \times \prod_1^r \left(\mathbf{R}^{k_i} \times \mathbf{R}^{\nu_i}\right) \times \prod_{r+1}^d \left(\mathbf{C}^{k_i} \times \mathbf{C}^{\nu_i}\right)$ *into itself, where* $k_i \geq \nu_i$ *denotes the maximum of the multiplicities of the root* α_i *in* P_0 *and* Q_0, *and* $p = m + n - \sum_1^r (k_i + \nu_i) - 2\sum_{r+1}^d (k_i + \nu_i)$.

Products of p monic polynomials. — Theorem 1 has the following obvious generalisation:

Theorem 3. — *Given integers* $m_1, \ldots, m_p$, $p > 1$, *denote again the multiplication map by* $\mathrm{Mult} : \mathrm{MP}(m_1) \times \cdots \times \mathrm{MP}(m_p) \to \mathrm{MP}(m_1 + \cdots + m_p)$. *Then, for each* $(P_1, \ldots, P_p) \in \mathrm{MP}(m_1) \times \cdots \times \mathrm{MP}(m_p)$:

(i) *The corank of* $D\,\mathrm{Mult}(P_1, \ldots, P_p)$ *is the degree of* $\gcd(P_1 \cdots P_p / P_j)_{1 \leq j \leq p}$.

(ii) *In particular,* Mult *is a local diffeomorphism at* $(P_1, \ldots, P_p)$ *if, and only if* $\gcd(P_i, P_j) = 1$ *for* $1 \leq i < j \leq p$.

(iii) *The map* Mult *is a* $(k+1)$*–swallowtail at* $(P_1, \ldots, P_p)$ *for some positive integer* k *if, and only if, it has corank one. If this is the case, there exist* $i, j \in \{1, \ldots, p\}$ *and* $\alpha \in \mathbf{K}$ *such that*

$$\gcd\left(P_\ell, P_m\right) = \begin{cases} x - \alpha & \text{if } \{\ell, m\} = \{i, j\} \\ 1 & \text{otherwise,} \end{cases}$$

and k *is the maximum of the multiplicities of the root* α *in* P_i *and* P_j.

(iv) *If* $\mathbf{K} = \mathbf{R}$, *the map* Mult *is a complex* $(k+1)$*–swallowtail at* $(P_1, \ldots, P_p)$ *for some positive integer* k *if, and only if, there exist* $i, j \in \{1, \ldots, p\}$ *and* $\alpha \in \mathbf{C} \setminus \mathbf{R}$ *such that*

$$\gcd\left(P_\ell, P_m\right) = \begin{cases} (x - \alpha)(x - \bar{\alpha}) & \text{if } \{\ell, m\} = \{i, j\} \\ 1 & \text{otherwise,} \end{cases}$$

and k *is the maximum of the multiplicities of the root* α *in* P_i *and* P_j.

The following result, whose proof is that of Proposition 2, describes the simplest singularities of higher corank:

Theorem 4. — *Given* $(P_1, \dots, P_p) \in \mathrm{MP}(m_1) \times \cdots \times \mathrm{MP}(m_p)$*, assume that all the (complex) roots of* $\gcd(P_1 \cdots P_p/P_j)_{1\le j\le p}$ *are simple.*

(i) *If* $\mathbf{K} = \mathbf{C}$*, then, denoting the roots by* $\alpha_1, \dots, \alpha_d$*, the map* Mult *is the product of* d *swallowtails of respective orders* $k_1 + 1, \dots, k_d + 1$*, where* k_j *denotes the maximum of the multiplicities of the root* α_j *in* $P_1, \dots, P_p$*.*

(ii) *If* $\mathbf{K} = \mathbf{R}$*, then, denoting the real roots of* $\gcd(P_1 \cdots P_p/P_j)_{1\le j\le p}$ *by* $\alpha_1, \dots, \alpha_r$ *and its other roots by* $\alpha_{r+1}, \bar\alpha_{r+1}, \dots, \alpha_d, \bar\alpha_d$*, the map* Mult *is the product of* r *real swallowtails of respective orders* $k_1+1, \dots, k_r+1$ *and* c *complex swallowtails of respective orders* $k_{r+1} + 1, \dots, k_d + 1$*, where* k_j *denotes the maximum of the multiplicities of the root* α_j *in* $P_1, \dots, P_p$*.*

Remark. — For $p > 2$, when $\gcd(P_1 \cdots P_p/P_j)_{1\le j\le p}$ has multiple roots, they can be common to three or more of the P_j's, yielding other singularities which deserve a better study. For example, when $p = 3$, $\gcd(P_1P_2P_3/P_j)_{1\le j\le 3} = x - \alpha$ and α is a simple root of all three polynomials, the singularity we get is a suspension of the germ at $0 \in \mathbf{C}$ of $z \mapsto \big(|z|^2, \Im(z^3)\big)$.

B. Singularities of the characteristic polynomial function

Let $\mathrm{M}(n \times n)$ be the space of $n \times n$ matrices with entries in $\mathbf{K}$. We will view each $M \in \mathrm{M}(n \times n)$ as a linear mapping $\mathbb{C}^n \to \mathbb{C}^n$ and always take into account all its *complex* eigenvalues. We will denote by

$$\chi : \mathrm{M}(n \times n) \to \mathrm{MP}(n)$$

the mapping sending M to its monic characteristic polynomial:

$$\chi(M) := \det(xI - M).$$

We are interested in the regular points and the simplest singularities of the mapping χ. Recall that an eigenvalue of $M \in \mathrm{M}(n \times n)$ is called *simple* if it is a simple root of $\chi(M)$.

We will call the eigenvalue λ of $M \in \mathrm{M}(n \times n)$ *geometrically simple* if the corresponding eigenspace is a line.

Theorem 5. — *Let* $M_0 \in \mathrm{M}(n \times n)$*. Then*

(i) *The map* χ *is regular at* M_0 *if, and only if, all the eigenvalues of* M_0 *are geometrically simple.*

(ii) *The rank of* $D\chi(M_0)$ *is the degree* $m(M_0)$ *of the minimal polynomial of* M_0*. In other words, the corank of* $D\chi(M_0)$ *equals*

$$\sum_{\lambda\in\sigma(M_0)} (d_\lambda - m_\lambda),$$

where d_λ *is the dimension of the characteristic space* E_λ *of* M_0 *associated to* λ*, and* m_λ *is the smallest integer* m *such that* $E_\lambda = \operatorname{Ker}(\lambda I - M_0)^m$*.*[3]

(iii) *The map* χ *is a* $(k+1)$*–swallowtail deformation at* M_0 *if, and only if, it has corank* 1*, the integer* k *being as follows: all eigenvalues of* M_0 *are geometrically simple except one, for which* $m_\lambda = k$ *and* $d_\lambda = k+1$*.*

Proof. — We can assume $\mathbf{K} = \mathbf{C}$ since even in the real case all definitions involve the complex numbers and the regularity of χ does not depend on the field. We will denote the Jordan block of order n and eigenvalue λ by $J_n(\lambda)$:

$$J_n(\lambda) = \begin{cases} \lambda & \text{if } n = 1 \\ \lambda I_n + \begin{pmatrix} 0 & I_{n-1} \\ 0 & 0 \end{pmatrix} & \text{for } n > 1. \end{cases}$$

Let us prove the "if" part of (i), first in the case where $M_0 = J_n(\lambda)$. To see that χ is a submersion at M_0 we can also assume $\lambda = 0$, since we can compose with a translation in the space of matrices and with a change of variable in the space of polynomials. In this case χ admits a section, sending a polynomial to its *companion matrix*, defined as follows: we let

$$\operatorname{Comp} : \mathrm{MP}(n) \to \mathrm{M}(n \times n),$$

be given by

$$\operatorname{Comp}\Big(x^n + \sum_{i=0}^{n-1} a_i x^i\Big) := J_n(0) + \begin{pmatrix} 0 & \cdots & 0 \\ -a_0 & \cdots & -a_{n-1} \end{pmatrix}.$$

Then $\chi(\operatorname{Comp}(P)) = P$ and in particular χ is a submersion.

If M_0 has only geometrically simple eigenvalues then, changing coordinates, we may assume that it is block-diagonal, of the form (as a map)

$$M_0 = J_{m_1}(\lambda_1) \times \cdots \times J_{m_p}(\lambda_p),$$

where the λ_j's are all different. Now, the restriction of χ to the vector subspace of $\mathrm{M}(n \times n)$ consisting of block-diagonal matrices (linear maps) $M_1 \times \cdots \times M_p$ with $M_j \in \mathrm{M}(m_j \times m_j)$ is already a submersion at M_0: indeed, it is the composed map of

— the map $M_1 \times \cdots \times M_p \overset{\chi^p}{\longmapsto} \big(\chi(M_1), \dots, \chi(M_p)\big)$, which is a submersion at M_0 by what we have just done, and

— the product map

$$\mathrm{MP}(m_1) \times \cdots \times \mathrm{MP}(m_p) \ni (P_1, \dots, P_p) \mapsto P_1 \cdots P_p \in \mathrm{MP}(n),$$

which is a local diffeomorphism at $\chi^p(M_0) = \big((x-\lambda_1)^{m_1}, \dots, (x-\lambda_p)^{m_p}\big)$ by Theorem 3 (i) since the λ_j's are all different.

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[3] This is the size of the largest Jordan block with eigenvalue λ or, equivalently, the multiplicity of λ as a root of the minimal polynomial of M_0.

To prove (ii) and therefore the "only if" part of (i), we can again assume that $M_0 = J_{m_1}(\lambda_1) \times \cdots \times J_{m_p}(\lambda_p)$, where the λ_j's may not be all different. Then, writing each $A \in \mathrm{M}(n \times n)$ in block form

$$A = \begin{pmatrix} a_1^1 & \cdots & a_p^1 \\ \vdots & \ddots & \vdots \\ a_1^p & \cdots & a_p^p \end{pmatrix}, \quad a_i^j : \mathbf{K}^{m_i} \to \mathbf{K}^{m_j},$$

we notice[(4)] that $D\chi(M_0)A = D\chi(M_0)(a_1^1 \times \cdots \times a_p^p)$. Therefore, the corank of $D\chi(M_0)$ is the corank of the differential at M_0 of the the restriction of χ to the space of all $M_1 \times \cdots \times M_p$ with $M_j \in \mathrm{M}(m_j \times m_j)$. Now, we have seen that this restriction is the composed map of the submersion χ^p and the map $\mathrm{Mult} : \mathrm{MP}(m_1) \times \cdots \times \mathrm{MP}(m_p) \to \mathrm{MP}(m_1 + \cdots + m_p)$. Thus, the corank of $D\chi(M_0)$ is that of $D\,\mathrm{Mult}\left(\chi^p(M_0)\right) = D\,\mathrm{Mult}\left((x-\lambda_1)^{m_1}, \ldots, (x-\lambda_p)^{m_p}\right)$. By Theorem 3 (ii), this is indeed the degree of $\gcd\left(\prod_{i \neq j}(x-\lambda_i)^{m_i}\right)_{1 \leq j \leq p} = \prod_{\lambda \in \sigma(M_0)}(x-\lambda)^{d_\lambda - m_\lambda}$.

To prove (iii), still assuming that $M_0 = J_{m_1}(\lambda_1) \times \cdots \times J_{m_p}(\lambda_p)$, just notice the following two facts:

— If the corank of $D\chi(M_0)$ is greater than 1, then χ is not a $(k+1)$–swallowtail deformation at M_0.
— If $D\chi(M_0)$ has corank 1, then, by (ii) and Theorem 3 (iii), the map Mult is a $(k+1)$–swallowtail at $\chi^p(M_0)$ with just the right k. Therefore χ, being the composed map of Mult with a local submersion, is a $(k+1)$–swallowtail deformation at M_0. □

Remarks on the real case. — If $\mathbf{K} = \mathbf{R}$, it follows from Theorem 3 (iv) that χ is a $(k+1)$–*complex* swallowtail deformation at M_0 when all eigenvalues of M_0 are geometrically simple except one pair $\{\lambda, \bar{\lambda}\}$, for which $\lambda \in \mathbf{C} \setminus \mathbf{R}$, $m_\lambda = k$ and $d_\lambda = k+1$.

This is coherent with the following (maybe not so well-known) Jordan normal form theorem in the real case: every endomorphism A of a real vector space E of finite dimension n is conjugate to a block-diagonal endomorphism of $\prod_1^r \mathbf{R}^{m_j} \times \prod_{r+1}^p \mathbf{C}^{m_j}$ of the form $J_{m_1}(\lambda_1) \times \cdots \times J_{m_p}(\lambda_p)$, where the λ_j's and the $\bar{\lambda}_j$'s are the eigenvalues of A, real for $j \leq r$ and nonreal for $j > r$.

Higher singularites. — It follows from the above arguments that all the singularities of χ are deformations of singularities of Mult for 2 or more factors. We shall not dwell on this fact for the time being.

(4) Using the fact that $\det A$ is an n–linear function of the columns of A, implying that $D\chi(M_0)A$ is the sum of the determinants of the n matrices obtained each by replacing one column of $xI - M_0$ by the corresponding column of $-A$.

C. Singularities of eigenvalues of linear operators

First we will describe several linear spaces, manifolds and maps related to the Banach space E.

All closed hyperplanes $H \ni 0$ are isomorphic as Banach spaces. We will use the notation E_0 for their common type(5).

Let $\mathbf{P}(E)$ be the projective space associated to E, that is, the space of all one-dimensional linear subspaces of E. Then $\mathbf{P}(E)$ has a natural analytic (algebraic) Banach manifold structure modelled on E_0, defined in the usual way—it is a connected component of the Grassmannian $\mathbf{G}(E)$ described in Section D.

Let $\mathrm{End}(E)$ be the space of bounded linear operators from E to E. We will denote the identity operator by 1 and its multiple by a scalar k also by k. If $A \in \mathrm{End}(E)$ we denote by $\sigma(A)$ its spectrum.

More generally, if we have two Banach spaces E_1, E_2 over $\mathbf{K}$ we will denote by $\mathscr{B}(E_1, E_2)$ the space of continuous linear maps from E_1 to E_2.

C1. The manifold of proper elements. — The *manifold of proper elements* of E is the space

$$\mathrm{Eig}(E) := \{(\lambda, L, A) \in \mathbf{K} \times \mathbf{P}(E) \times \mathrm{End}(E) \,:\, A(L) \subseteq L \text{ and } A|_L = \lambda\}.$$

That is, the space of triples consisting of a linear operator, an invariant line and the corresponding eigenvalue. The specification of the eigenvalue λ is redundant but useful, as we shall see.

Theorem 6. — *The set* $\mathrm{Eig}(E)$ *is an analytic (algebraic) Banach submanifold of the manifold* $\mathbf{K} \times \mathbf{P}(E) \times \mathrm{End}(E)$*, modelled on* $\mathrm{End}(E)$*.*

Proof. — Given $(\lambda_0, L_0, A_0) \in \mathrm{Eig}(E)$, choose $x \in L_0 \setminus \{0\}$ and a complementary subspace H of L_0. Identifying $E = \mathbf{K}x \oplus H$ to $\mathbf{K} \times H$ we can identify each line $L \in \mathbf{P}(E)$ transversal to H to the unique $h \in H$ satisfying $(1, h) \in L$ (6) and write every operator $A \in \mathrm{End}(E)$ in matrix form

$$A \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a \in \mathbb{K}, b \in H^*, c \in H, d \in \mathrm{End}(H)$$

Hence in particular

$$A_0 = \begin{pmatrix} \lambda_0 & b_0 \\ 0 & d_0 \end{pmatrix}.$$

(5) For E the Hilbert space l^2 (or any of the well known Banach spaces from Functional Analysis) E_0 is isomorphic to E. However, there are examples of infinite dimensional Banach spaces where this is not the case [**7**].
(6) Therefore, L_0 corresponds to $h = 0$.

In these identifications, the relation $(\lambda, L, A) \in \mathrm{Eig}(E)$ reads

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ h \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ h \end{pmatrix},$$

that is

$$\begin{aligned} a &= \lambda - bh \\ c &= (\lambda - d)h. \end{aligned} \tag{6}$$

In other words, the open subset of $\mathrm{Eig}(E)$ consisting of those (λ, L, A) such that L is not contained in H admits the parametrisation

$$\mathrm{End}(E) \ni \begin{pmatrix} \lambda & b \\ h & d \end{pmatrix} \mapsto \left(\lambda, h, \begin{pmatrix} \lambda - bh & b \\ (\lambda - d)h & d \end{pmatrix}\right)$$

as the graph of the polynomial map defined by (6). □

Corollary. — *For $(\lambda_0, L_0, A_0) \in \mathrm{Eig}(E)$, there is an analytic function $A(\lambda, L)$ defined in a neighborhood of (λ_0, L_0) such that $A(\lambda_0, L_0) = A_0$ and that the nonzero elements of L are eigenvectors of $A(\lambda, L)$ with eigenvalue λ.*

Proof. — Just take $A(\lambda, h) = \begin{pmatrix} \lambda - b_0 h & b_0 \\ (\lambda - d_0)h & d_0 \end{pmatrix}$ modulo the identifications of the previous proof. □

This function is clearly not unique, and although it is possible from the proof of the theorem to describe all of them, much more interesting is the question whether λ, L are analytic functions of the operator A. To present our version of the (classical) answer, we consider the following geometric reformulation:

Question. — Let Π be the projection from $\mathrm{Eig}(E)$ to $\mathrm{End}(E)$ which forgets the first two components. When is it a local diffeomorphism? What are its simplest singularities?

C2. The Immersion Theorem

Definition. — If λ is an eigenvalue of $A \in \mathrm{End}(E)$ with eigenvector x and L is the line generated by x, then A induces an operator $\dot{A}$ from the quotient E/L to itself. We will say that λ is a *simple* eigenvalue of A if we have $\lambda \notin \sigma(\dot{A})$.
We call λ a *geometrically simple* eigenvalue of A if
(i) the corresponding eigenspace is a line (i.e. $\dim \mathrm{Ker}(A - \lambda) = 1$) and,
(ii) the image of the operator $\lambda - A$ is a direct factor[7].
We call λ a geometrically simple eigenvalue of A *of (finite) multiplicity* $k \geq 1$ if

[7] Meaning that it is closed and admits a closed complementary subspace.

$\dim \operatorname{Ker}(A-\lambda)^k = k$, and λ is not in the spectrum of the endomorphism of the quotient $E/\operatorname{Ker}(A-\lambda)^k$ induced by A.
Let

$$\operatorname{Val}(E) := \{(\lambda, A) \in \mathbf{K} \times \operatorname{End}(E) \,:\, \lambda \text{ is an eigenvalue of } A\}.$$

Theorem 7. — *Let $j : \operatorname{Eig}(E) \to \mathbf{K} \times \operatorname{End}(E)$ be defined by $j(\lambda, L, A) := (\lambda, A)$, hence $j\big(\operatorname{Eig}(E)\big) = \operatorname{Val}(E)$.*

(i) *The map j is an immersion in the neighbourhood of $(\lambda_0, L_0, A_0) \in \operatorname{Eig}(E)$ if, and only if, λ_0 is a geometrically simple eigenvalue of A_0.*

(ii) *The set of those $(\lambda, A) \in \operatorname{Val}(E)$ where λ is a geometrically simple eigenvalue of A with finite multiplicity is a manifold modelled on* $\operatorname{End}(E)$.

Proof. — (i) In terms of the parametrization of $\operatorname{Eig}(E)$ introduced in the proof of Theorem 6, j is the map

$$j : \begin{pmatrix} \lambda & b \\ h & d \end{pmatrix} \longmapsto \left(\lambda, \begin{pmatrix} \lambda - bh & b \\ (\lambda - d)h & d \end{pmatrix}\right),$$

(λ_0, L_0, A_0) and A_0 being identified to the same matrix

$$A_0 = \begin{pmatrix} \lambda_0 & b_0 \\ 0 & d_0 \end{pmatrix}. \tag{7}$$

Therefore, the derivative of j at (λ_0, L_0, A_0) is the map

$$Dj(\lambda_0, L_0, A_0) : \begin{pmatrix} \lambda & b \\ h & d \end{pmatrix} \longmapsto \left(\lambda, \begin{pmatrix} \lambda - b_0 h & b \\ (\lambda_0 - d_0)h & d \end{pmatrix}\right), \tag{8}$$

which vanishes if and only if $\lambda = 0$, $b = 0$, $d = 0$ and

$$b_0 h = 0\,, \quad (\lambda_0 - d_0)h = 0. \tag{9}$$

As (9) writes

$$\begin{pmatrix} \lambda_0 & b_0 \\ 0 & d_0 \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} = \lambda_0 \begin{pmatrix} 0 \\ h \end{pmatrix},$$

we have[8] $\operatorname{Ker} Dj(\lambda_0, L_0, A_0) \neq \{0\}$ if and only if $A_0 v = \lambda_0 v$ for some $v \notin L_0$. From this (i) follows for $\dim E < \infty$. In infinite dimensions, we have to check that, moreover, the image of $Dj(\lambda_0, L_0, A_0)$ is a direct factor if and only if so is the image of $\lambda_0 - A_0$. Now, by (7)–(8), this amounts to proving that the image of

$$\begin{pmatrix} \lambda \\ h \end{pmatrix} \longmapsto \left(\lambda, \begin{pmatrix} \lambda - b_0 h \\ (\lambda_0 - d_0)h \end{pmatrix}\right)$$

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(8) Recall that, in our parametrization, L_0 is generated by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

is a direct factor in $\mathbf{K} \times E$ if and only if the image of

$$\begin{pmatrix} \lambda \\ h \end{pmatrix} \longmapsto \begin{pmatrix} -b_0 h \\ (\lambda_0 - d_0)h \end{pmatrix}$$

is a direct factor in E. In other words, setting $e := \left(1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$, we should prove that $\mathbf{K}e \oplus \big(\{0\} \times \operatorname{Im}(\lambda_0 - A_0)\big)$ has a closed complement S in $\mathbf{K} \times E$ if and only if $\operatorname{Im}(\lambda_0 - A_0)$ has a closed complement S_0 in E, which is obvious: take $S = \{0\} \times S_0$ to get the "if" part and $\{0\} \times S_0 = (\{0\} \times E) \cap (\mathbf{K}e \oplus S)$ to obtain the converse.

(ii) Given $(\lambda_0, L_0, A_0) \in \operatorname{Eig}(E)$, we should prove that if λ_0 is a simple eigenvalue of A_0 of finite multiplicity k then, near (λ_0, A_0), the subset $\operatorname{Val}(E)$ consists solely of the image by j of a neighbourhood of (λ_0, L_0, A_0), a consequence of the following

Lemma. — *For each sequence $(\lambda_n, L_n, A_n)_{n\geq 1}$ in $\operatorname{Eig}(E)$ such that (λ_n, A_n) converges to (λ_0, A_0), the line L_n tends to L_0 when $n \to \infty$.*

Proof. — For $\dim E < \infty$, the projective space $\mathbf{P}(E)$ is compact and every convergent subsequence of (L_n) must tend to a line L invariant by $A_0 = \lim A_n$ and such that $A_0|_L = \lim \lambda_n = \lambda_0$, hence $L = L_0$, proving the lemma.

When E is infinite dimensional, the subspace $K_0 := \operatorname{Ker}(\lambda_0 - A_0)^k$ admits an A_0–invariant closed complement S by the Hahn-Banach theorem. Identifying $E = \mathbf{K}_0 \oplus S$ to $K_0 \times S$, we can write $A_n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}$ with $\alpha_n \in \operatorname{End}(K_0)$, $\beta_n \in \mathscr{B}(S, K_0)$, $\gamma_n \in \mathscr{B}(K_0, S)$, $\delta_n \in \operatorname{End}(S)$ for all $n \in \mathbf{N}$ and, as K_0 is A_0–invariant, $\gamma_0 = 0$ and $\lambda_0 \notin \sigma(\delta_0)$. For $n \geq 1$, choose a generator $u_n = (v_n, w_n)$ of L_n with $|u_n| = 1$. The S–component of the relation $(\lambda_n - A_n)u_n = 0$ reads

$$-\gamma_n v_n + (\lambda_n - \delta_n)w_n = 0. \tag{10}$$

Now, as $\lambda_0 - \delta_0$ is invertible, so is $\lambda_n - \delta_n$ for large enough n. Therefore, (10) can be written

$$w_n = (\lambda_n - \delta_n)^{-1}\gamma_n v_n, \tag{11}$$

hence in particular

$$\lim_{n\to\infty} w_n = 0 \tag{12}$$

since v_n is bounded, $\lim(\lambda_n - \delta_n)^{-1} = (\lambda_0 - \delta_0)^{-1}$ and $\lim \gamma_n = \gamma_0 = 0$. By (11), for large enough n, the K_0–component of the relation $(\lambda_n - A_n)u_n = 0$ becomes

$$\big((\lambda_n - \alpha_n) + \beta_n(\lambda_n - \delta_n)^{-1}\gamma_n\big)v_n = 0. \tag{13}$$

By (12), $|v_n|$ tends to 1 when $n \to \infty$. Therefore, for large enough n, the vector v_n generates a line $L'_n \subset K_0$ and, by (13), the hypotheses of the finite dimensional case are satisfied by the sequence $(0, L'_n, A'_n) \in \operatorname{Eig}(K_0)$ defined by the formula $A'_n := (\lambda_n - \alpha_n) + \beta_n(\lambda_n - \delta_n)^{-1}\gamma_n$, as $\lim A'_n = \lambda_0 - \alpha_0$. It follows that the line L'_n tends to the one-dimensional kernel of $\lambda_0 - \alpha_0$, namely L_0. Therefore, by (12), we do have $\lim L_n = \lim L'_n = L_0$. □

Remark. — Let $\pi : \mathrm{Val}(E) \to \mathrm{End}(E)$ be the natural map, induced by the projection $\mathbf{K} \times \mathrm{End}(E) \to \mathrm{End}(E)$. When λ is geometrically simple this map is diffeomorphic to the natural map $\Pi : \mathrm{Eig}(E) \to \mathrm{End}(E)$ via the local diffeomorphism $\mathrm{Eig}(E) \approx \mathrm{Val}(E)$ defined by j.

The singularities of $\mathrm{Val}(E)$ might be of some interest. In the finite dimensional case, $\mathrm{Val}(E)$ is an algebraic subset, given by the equation $\chi(A)(\lambda) = 0$, whose regular part contains the points with geometrically simple eigenvalue. Near each such regular point, it can be proven that $\tilde{\chi} : (\lambda, A) \mapsto \big(\lambda, \chi(A)\big)$ is a submersion of $\mathrm{Val}(E)$ into $\mathrm{Root}(n)$ (see Example 1c).

C3. Singularities of eigenvalues

Theorem 8. — *Let* $\Pi : \mathrm{Eig}(E) \to \mathrm{End}(E)$ *be the natural map* $(\lambda, L, A) \mapsto A$.

(i) *The map* Π *is a local diffeomorphism near* $(\lambda_0, L_0, A_0) \in \mathrm{Eig}(E)$ *if, and only if,* λ_0 *is a simple eigenvalue of* A_0.

(ii) *The dimension of the kernel of* $D\Pi(\lambda_0, L_0, A_0)$ *equals the dimension of the kernel of* $\lambda_0 - \dot{A}_0$.

(iii) *The map* Π *is a* k*–swallowtail at* (λ_0, L_0, A_0) *if (and, for* $\dim E < \infty$*, only if)* λ_0 *is a geometrically simple eigenvalue of* A_0 *with mutiplicity* k.

Proof. — In terms of the parametrization of $\mathrm{Eig}(E)$ introduced in the proof of Theorem 6, Π is the map

$$\Pi : \begin{pmatrix} \lambda & b \\ h & d \end{pmatrix} \longmapsto \begin{pmatrix} \lambda - bh & b \\ (\lambda - d)h & d \end{pmatrix}$$

and (λ_0, L_0, A_0) identifies to the matrix

$$A_0 = \begin{pmatrix} \lambda_0 & b_0 \\ 0 & d_0 \end{pmatrix}$$

Therefore, the derivative of Π at (λ_0, L_0, A_0) is

$$D\Pi(\lambda_0, L_0, A_0) : \begin{pmatrix} \lambda & b \\ h & d \end{pmatrix} \longmapsto \begin{pmatrix} \lambda - b_0 h & b \\ (\lambda_0 - d_0)h & d \end{pmatrix},$$

which is an isomorphism if, and only if, $\lambda_0 - d_0$ is an automorphism of H, i.e. $\lambda_0 \notin \sigma(d_0)$ or, equivalently, $\lambda_0 \notin \sigma(\dot{A}_0)$. By the Inverse Function Theorem in Banach spaces this condition is equivalent to Π being a local diffeomorphism in the neighborhood of (λ_0, L_0, A_0).

The same argument proves (ii).

Proof of the "if" part of (iii) *when* $\dim E = k$. — We can assume that $\lambda_0 = 0$ and $A_0 = J_k(0)$. Then, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Val}(\mathbf{K}^k) & \overset{\tilde{\chi}}{\to} & \mathrm{Root}(k) \\ \downarrow \pi & & \downarrow \bar{\pi} \\ \mathrm{End}(\mathbf{K}^k) & \overset{\chi}{\to} & \mathrm{MP}(k) \end{array}$$

where $\pi : (\lambda, A) \mapsto A$ equals Π up to the local diffeomorphism j near $\big(0, J_k(0)\big)$ and $\bar{\pi} : (\lambda, P) \mapsto P$ is a k–swallowtail by example 1c of section A. The map χ is a local submersion since it admits the local section Comp (see the proof of Theorem 3), and so is $\tilde{\chi} : (\lambda, A) \mapsto \big(\lambda, \chi(A)\big)$ since it admits the local section $(\lambda, P) \mapsto \big(\lambda, \mathrm{Comp}(P)\big)$. In particular, the (algebraic) fiber

$$F := \chi^{-1}(x^k) \subset \mathrm{End}(\mathbf{K}^k)$$

is a submanifold near $J_k(0)$ and so is

$$\tilde{\chi}^{-1}(0, x^k) = \{0\} \times F \approx F$$

near $\big(0, J_k(0)\big) \in \mathrm{Val}(\mathbf{K}^k)$. As χ is a submersion, there exists a local diffeomorphism defined near $J_k(0)$ and of the form

$$\begin{array}{ccc} \mathrm{End}(\mathbf{K}^k) & \xrightarrow{g} & \mathrm{MP}(k) \times F \\ A & \longmapsto & \big(\chi(A), f(A)\big) \end{array}$$

such that

$$f(A) = A \quad \text{for} \quad A \in F.$$

It follows at once that, near $\big(0, J_k(0)\big)$, the map

$$\begin{array}{ccc} \mathrm{Val}(\mathbf{K}^k) & \xrightarrow{\tilde{g}} & \mathrm{Root}(k) \times F \\ (\lambda, A) & \longmapsto & \big(\tilde{\chi}(A), f(A)\big) \end{array}$$

is a local diffeomorphism. As the diagram

$$\begin{array}{ccc} \mathrm{Val}(\mathbf{K}^k) & \overset{\tilde{g}}{\to} & \mathrm{Root}(k) \times F \\ \downarrow \pi & & \downarrow \bar{\pi} \times \mathrm{Id} \\ \mathrm{End}(\mathbf{K}^k) & \overset{g}{\to} & \mathrm{MP}(k) \times F \end{array}$$

is commutative and $\bar{\pi}$ is a k–swallowtail, so is π.

Proof of the "if" part of (iii) *in general.* — If λ_0 is a geometrically simple eigenvalue of A_0 with multiplicity k, we can choose a closed complement F of $\mathrm{Ker}(\lambda_0 - A_0)^k$ and identify $\mathrm{Ker}(\lambda_0 - A_0)^k$ to $\mathbf{K}^k$ so that $E = \mathrm{Ker}(\lambda_0 - A_0)^k \oplus F$ identifies to $\mathbf{K}^k \times F$ and

$$A_0 = \begin{pmatrix} J_k(\lambda_0) & b_0 \\ 0 & d_0 \end{pmatrix} \quad \text{with} \quad b_0 \in \mathscr{B}(F, \mathbf{K}^k),\ d_0 \in \mathrm{End}(F),\ \lambda_0 \notin \sigma(d_0).$$

More generally, every $A \in \operatorname{End}(E)$ writes

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ a \in \operatorname{End}(\mathbf{K}^k),\ b \in \mathscr{B}(F, \mathbf{K}^k),\ c \in \mathscr{B}(\mathbf{K}^k, F),\ d \in \operatorname{End}(F).$$

Note that the graph of $h \in \mathscr{B}(\mathbf{K}^k, F)$ is invariant by A if and only if

$$c + dh = h(a + bh)$$

or, equivalently, if and only if

$$\begin{aligned} a &= \alpha - bh \\ c &= h\alpha - dh \end{aligned}$$

for some $\alpha \in \operatorname{End}(\mathbf{K}^k)$. The following crucial observation is a particular case of the proof of Theorem 10 hereafter:

Lemma 1. — *The polynomial map*

$$\begin{pmatrix} \alpha & b \\ h & d \end{pmatrix} \longmapsto \begin{pmatrix} \alpha - bh & b \\ h\alpha - dh & d \end{pmatrix}$$

is a local diffeomorphism $\big(\operatorname{End}(E), A_0\big) \to \big(\operatorname{End}(E), A_0\big)$.

Proof of Lemma 1 As its derivative at A_0 is

$$\begin{pmatrix} \alpha & b \\ h & d \end{pmatrix} \longmapsto \begin{pmatrix} \alpha - b_0 h & b \\ hJ_k(\lambda_0) - dh & d \end{pmatrix},$$

we should show that the continuous linear map $h \mapsto hJ_k(\lambda_0) - d_0 h$ of $\mathscr{B}(\mathbf{K}^k, F)$ into itself is an isomorphism, i.e. that, for each $c \in \mathscr{B}(\mathbf{K}^k, F)$, the equation $hJ_k(\lambda_0) - d_0 h = c$ or, equivalently, $hJ_k(0) + (\lambda_0 - d_0)h = c$, which (as we have $\lambda_0 \notin \sigma(d_0)$) can be written

$$h = (\lambda_0 - d_0)^{-1}\big(c - hJ_k(0)\big), \tag{14}$$

has a unique solution. Now, this is obvious since, denoting the canonical basis of $\mathbf{K}^k$ by $(e_1, \ldots, e_k)$, (14) is equivalent to the triangular system

$$he_j = \begin{cases} (\lambda_0 - d_0)^{-1} ce_j & \text{if } j = k \\ (\lambda_0 - d_0)^{-1}(ce_j - he_{j+1}) & \text{for } 1 \leq j < k. \end{cases}$$

The following result concludes the proof of the "if" part of Theorem 8 (iii):

Lemma 2. — *We have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Val}(\mathbf{K}^k)\times\mathscr{B}(F,\mathbf{K}^k)\times\mathscr{B}(\mathbf{K}^k,F)\times\mathrm{End}(F) & \to & \mathrm{Val}(E)\\ \downarrow \pi\times\mathrm{Id}\times\mathrm{Id}\times\mathrm{Id} & & \downarrow\pi\\ \mathrm{End}(\mathbf{K}^k)\times\mathscr{B}(F,\mathbf{K}^k)\times\mathscr{B}(\mathbf{K}^k,F)\times\mathrm{End}(F) & \to & \mathrm{End}(E)\end{array}$$

$$\begin{array}{ccc} \big((\lambda,\alpha),b,h,d\big) & \mapsto & \left(\lambda,\begin{pmatrix}\alpha-bh & b\\ h\alpha-dh & d\end{pmatrix}\right)\\ \downarrow & & \downarrow\\ (\alpha,b,h,d)) & \mapsto & \begin{pmatrix}\alpha-bh & b\\ h\alpha-dh & d\end{pmatrix},\end{array}$$

where the horizontal arrows are local diffeomorphisms at $\big((\lambda_0,J_k(\lambda_0)),b_0,0,d_0\big)$ *and* $\big(J_k(\lambda_0),b_0,0,d_0\big)$ *respectively, the right vertical one is diffeomorphic to* Π *(since* λ *is geometrically simple) and the left one is a swallowtail by the particular case* $E=\mathbf{K}^k$ *already treated.*

Proof of Lemma 2 The two things we do not already know are the following:

(a) the upper arrow does send $\mathrm{Val}(\mathbf{K}^k)\times\mathscr{B}(F,\mathbf{K}^k)\times\mathscr{B}(\mathbf{K}^k,F)\times\mathrm{End}(F)$ into $\mathrm{Val}(E)$

(b) it is a local diffeomorphism.

To obtain (a), just notice that $\begin{pmatrix}\alpha-bh & b\\ h\alpha-dh & d\end{pmatrix}\begin{pmatrix}x\\ hx\end{pmatrix}=\lambda\begin{pmatrix}x\\ hx\end{pmatrix}$ if and only if $\alpha x=\lambda x$.

By Lemma 1, to get (b), we should show that, for (λ,α,b,h,d) close enough to $\big(\lambda_0,J_k(\lambda_0),b_0,0,d_0\big)$, we have $\begin{pmatrix}\alpha-bh & b\\ h\alpha-dh & d\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\lambda\begin{pmatrix}x\\ y\end{pmatrix}$ if and only if $\alpha x=\lambda x$ and $y=hx$. Now, setting $z:=y-hx$, the first relation reads

$$\begin{cases}(\alpha-\lambda)x=bz\\ h(\alpha-\lambda)x=(\lambda-d)z\end{cases}$$

or, equivalently,

$$\begin{cases}(\alpha-\lambda)x=bz\\ (\lambda-d-hb)z=0.\end{cases}$$

As λ_0-d_0 is invertible, so is $\lambda-d-hb$ for (λ,b,h,d) close enough to $(\lambda_0,b_0,0,d_0)$, in which case our system is equivalent to $z=0$ and $(\alpha-\lambda)x=0$, proving Lemma 2.

Proof of the "only if" part of (iii). When the eigenspace associated to λ is not a line there is (at least) a circle of invariant lines with the same eigenvalue mapping to the same operator. This cannot happen in a swallowtail, proving our result since we assume[9] $\dim E<\infty$. □

[9] Though the result might well be true in general.

C4. Simple eigenvalues. — Theorem 8 (i) is a version of the following well-known result:

Corollary. — *For* $(\lambda_0, L_0, A_0) \in \mathrm{Eig}(E)$ *the following assertions are equivalent:*

(i) There are (necessarily unique) analytic functions $\lambda(A), \mathcal{B}(A)$ *defined near* A_0 *such that* $\lambda(A_0) = \lambda_0, \mathcal{B}(A_0) = L_0$ *and* $\big(\lambda(A), \mathcal{B}(A), A\big) \in \mathrm{Eig}(E)$.

(ii) λ_0 *is a simple eigenvalue of* A_0.

Proof. — Just observe that such functions provide a local inverse of Π. □

Remarks. — Actually, under the conditions of the corollary there is also an analytic function $v(A)$ (which is not unique) defined in a neighborhood of A such that $v(A)$ is an eigenvector of A with eigenvalue $\lambda(A)$. As we said above, this result is classical and many proofs have been given of it. For a proof that uses, as we do here, the Implicit Function Theorem in Banach spaces see for example [**3**], exercice 14 p. 268. Proofs using this method have been known for a long time, see for example [**9**].

Part (iii) of Theorem 8 is related to the usual expansion of the eigenvalue as a power series on roots of the parameters [**8**].

In the finite dimensional case, for $(\lambda, L, M) \in \mathrm{Eig}(\mathbf{K}^n)$ and $P = \chi(M)$ we always have $P(\lambda) = 0$, and λ is simple if and only if the derivative $P'(\lambda)$ is nonzero. It can be checked that $\mathrm{Simp}(\lambda, L, M) := P'(\lambda)$ equals the Jacobian of Π at (λ, L, M) in the coordinates introduced in the proof of Theorem 6.

Since at the singularities with higher corank of Π the jacobian determinant is a singular function (see Appendix A1), it follows that the regular points of the variety $\mathrm{Simp} = 0$ are precisely the points with a geometrically simple eigenvalue. This regular set can be still stratified according to the singularity type of Π, i.e. according to the order of the swallowtails.

In the case of geometrically multiple eigenvalues, we have observed that there is (at least) a circle of invariant lines with the same eigenvalue mapping to the same operator. This means that the singularity type is not finite, and is therefore very degenerate from the viewpoint of singularity theory. Nevertheless, it is a kind of blow-down map that could be described combining the blow-down singularities of the map j and some simpler singularities.

The group $\mathrm{GL}(E)$ of invertible endomorphisms acts naturally by conjugation on both $\mathrm{Eig}(E)$ and $\mathrm{End}(E)$ and the mapping Π is equivariant. Therefore Π maps the stratification by orbits of the first space into the second one and is some kind of (partial) resolution of the latter's singularities.

In the finite dimensional case, Arnold [**1**] has given a complete description of the stratification by orbits of $M(n \times n)$ which is possible to lift to those of $\mathrm{Eig}(\mathbf{K}^n)$. The images of our swallowtail singularities can be observed in the slice around the corresponding orbit. Although no explicit statement appears it is probable that some version of our results was known to Arnold.

D. Singularities of linear operators and invariant subspaces

D1. Grassmannians. — We will denote by $\mathbf{G}(E)$ the set of closed linear subspaces S of E with a closed complement. For each such S and each pair (V, W) of complementary closed subspaces of E such that $E = S \oplus W$, the subspace S is (in the identification of $E = V \oplus W$ to $V \times W$) the graph of a unique continuous linear map $h = h_{V,W}(S)$ of V into W.

Proposition. — *The charts $h_{V,W}$ make $\mathbf{G}(E)$ into an analytic Banach manifold, which is impure*(10) *but Hausdorff.*

Proof. — The intersection of the domains of two such maps $h_{V,W}$ and h_{V_1,W_1} can be nonempty only if V_1 is isomorphic to V and W_1 to W. When this is the case, each $v \in E$ writes $(x, y) \in V \times W$ in one identification and $(x_1, y_1) \in V_1 \times W_1$ in the other, and there exists a unique invertible transition matrix

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ a \in \mathscr{B}(V_1, V)\,,\ b \in \mathscr{B}(W_1, V)\,,\ c \in \mathscr{B}(V_1, W)\,,\ d \in \mathscr{B}(W_1, W)$$

such that the first expression of v is obtained from the second by the formula

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

If $S \in \mathbf{G}(E)$ lies in the domain of both $h_{V,W}$ and h_{V_1,W_1}, then $h := h_{V,W}(S)$ and $h_1 := h_{V_1,W_1}(S)$ satisfy

$$c + dh_1 = h\,(a + bh_1)\,.$$

Now, $a + bh_1$ is an isomorphism of V_1 onto V, as it is obtained by composing the isomorphism $x_1 \mapsto (x_1, h_1 x_1)$ of V_1 onto S and the isomorphism $(x, y) \mapsto x$ of S onto V in the identification of E to $V \times W$. In other words, the domain of the transition map $h_{V,W} \circ h_{V_1,W_1}^{-1}$ is the open subset of $\mathscr{B}(V_1, W_1)$ consisting of those h_1 such that $a + bh_1$ is an isomorphism, and

$$h_{V,W} \circ h_{V_1,W_1}^{-1}(h_1) = (c + dh_1)\,(a + bh_1)^{-1}\,.$$

To see that $\mathbf{G}(E)$ is Hausdorff (which we do not care much about), one can proceeed as follows:

- If E is a Hilbert space, each $S \in \mathbf{G}(E)$ can be identified to the orthogonal projector onto S, yielding a natural embedding of $\mathbf{G}(E)$ into $\mathrm{End}(E)$. The image is the smooth *real*(11) algebraic subset of $\mathrm{End}(E)$ defined by the equations $P^2 =$

(10) Meaning the following: each $h_{V,W}$ applies only to subspaces S isomorphic to V and such that E/S is isomorphic to W. It follows for example that two subspaces S which do not have the same dimension or the same codimension lie in different connected components—which have various dimensions when E is finite dimensional. In particular, the projective space $\mathbf{P}(E)$ is a connected component of $\mathbf{G}(E)$.
(11) If $\mathbf{K} = \mathbf{C}$, the holomorphic manifold $\mathbf{G}(E)$ is compact for $\dim E < \infty$ and therefore cannot be embedded as a holomorphic submanifold of the complex linear space $\mathrm{End}(E)$.

P and $(1-P^*)P + P^*(1-P) = 0$, where P^* denotes the adjoint of P. See [**3**], exercice 31 p. 271.

– Otherwise, for each $p \in E$, it is quite easy to prove (using the charts $h_{V,W}$) that the function $\mathrm{dist}_p : \mathbf{G}(E) \to \mathbf{R}$ defined by

$$\mathrm{dist}_p(S) = \mathrm{dist}(p, S) := \inf_{q \in S} |q - p|$$

is continuous. Given distinct elements S, S_1 of $\mathbf{G}(E)$, exchanging them if necessary, there exists $p \in S_1 \setminus S$ and we have $\mathrm{dist}(p, S_1) = 0 < D := \mathrm{dist}(p, S)$, hence two disjoint open subsets $\mathrm{dist}_p^{-1}(-\infty, D/2) \supset S_1$ and $\mathrm{dist}_p^{-1}(D/2, +\infty) \supset S$. □

D2. The manifold of invariant subspaces. — The manifold of invariant subspaces of E is the space

$$\mathrm{Inv}(E) := \{(S, A) \in \mathbf{G}(E) \times \mathrm{End}(E) \,:\, A(S) \subseteq S\}.$$

That is, the space of pairs consisting of a linear operator and an invariant subspace with a closed complement. Compare with the definition of $\mathrm{Eig}(E)$. This manifold resembles more the manifold $\mathrm{Eig}(E)$ (which it contains), than the more complicated Grassmannian:

Theorem 9. — *The subset* $\mathrm{Inv}(E)$ *is an analytic submanifold modelled on* $\mathrm{End}(E)$.

Proof. — For each chart $h_{V,W}$, every $S \in \mathbf{G}(E)$ such that $E = S \oplus W$ identifies to the linear map $h = h_{V,W}(S)$ of which it is the graph in the identification of $E = V \oplus W$ to $V \times W$. In this identification, every $A \in \mathrm{End}(E)$ reads as usual

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a \in \mathrm{End}(V), b \in \mathscr{B}(W, V), c \in \mathscr{B}(V, W), d \in \mathrm{End}(W)$$

and we have $(S, A) \in \mathrm{Inv}(E)$ if and only if

$$c + dh = h(a + bh) \tag{15}$$

or, equivalently, if and only if

$$\begin{aligned} a &= \alpha - bh \\ c &= h\alpha - dh \end{aligned}$$

for some $\alpha \in \mathrm{End}(\mathbf{K}^k)$. In other words:

– The image of the restriction of $h_{V,W} \times \mathrm{Id}_{\mathrm{End}(E)}$ to $\mathrm{Inv}(E)$ is the smooth algebraic submanifold of $\mathscr{B}(V, W) \times \mathrm{End}(E)$ defined by (15), which is the graph $c = h(a + bh) - dh$.

– This graph admits the *global* parametrization

$$\Phi_{V,W} : \begin{pmatrix} \alpha & b \\ h & d \end{pmatrix} \longmapsto \left(h, \begin{pmatrix} \alpha - bh & b \\ h\alpha - dh & d \end{pmatrix}\right) \tag{16}$$

by $\mathrm{End}(E)$. □

Corollary. — *For all* $(S_0, A_0) \in \mathrm{Inv}(E)$*, there is an analytic function* $A(S)$ *defined in a neighborhood of* (S_0) *such that* $A(S_0) = A_0$ *and* $A(S)$ *is an operator with invariant subspace* S*.*

Proof. — Given any complementary subspace W of $V := S_0$, with the notation of the above proof, we have

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ 0 & d_0 \end{pmatrix}$$

and we can take $A\big(h_{V,W}^{-1}(h)\big) := \varphi_{V,W}\begin{pmatrix} a_0 & b_0 \\ h & d_0 \end{pmatrix}$, where $\varphi_{V,W}$ is the second component of the parametrization $\Phi_{V,W}$ defined by (16). □

Again, a more interesting question is whether S is an analytic function of the operator A. Or, in our terms, whether the map Π of $\mathrm{Inv}(E)$ into $\mathrm{End}(E)$ which forgets the first component is a local diffeomorphism.

D3. Simple invariant subspaces. — For $(S, A) \in \mathrm{Inv}(E)$, let a denote the restriction of A to S and $\dot{A}$ the induced endomorphism of the quotient E/S.

Proposition and definition. — *The following three conditions are equivalent:*

(a) *We have that* $\sigma(a) \cap \sigma(\dot{A}) = \varnothing$*.*
(b) *The mapping* $h \mapsto ha - \dot{A}h$ *is an automorphism of* $\mathcal{B}(S, E/S)$*.*
(c) *The mapping* $h \mapsto h\dot{A} - ah$ *is an automorphism of* $\mathcal{B}(E/S, S)$*.*

When they are satisfied, we call S *a* simple *invariant subspace of* A *(compare with the definition of a simple eigenvalue).*

Equivalence between (a), (b), (c) follows from Proposition B.6 in Appendix B.

Theorem 10. — *The restriction* $\Pi : \mathrm{Inv}(E) \to \mathrm{End}(E)$ *of the canonical projection* $\mathbf{G}(E) \times \mathrm{End}(E) \to \mathrm{End}(E)$ *is a local diffeomorphism in the neighborhood of* $(S_0, A_0) \in \mathrm{Inv}(E)$ *if, and only if,* S_0 *is a simple invariant subspace of* A_0*.*

Proof. — Given any complementary subspace W of $V := S_0$, we can read everything in the chart $(S, A) \mapsto \Phi_{V,W}^{-1}\big(h_{V,W}(S), A\big)$. Then, Π is the map

$$\begin{pmatrix} \alpha & b \\ h & d \end{pmatrix} \longmapsto \begin{pmatrix} \alpha - bh & b \\ h\alpha - dh & d \end{pmatrix}$$

of $\mathrm{End}(E)$ into itself and (S_0, A_0) equals

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ 0 & d_0 \end{pmatrix}$$

It follows that

$$d\Pi(S_0, A_0) : \begin{pmatrix} \alpha & b \\ h & d \end{pmatrix} \longmapsto \begin{pmatrix} \alpha - b_0 h & b \\ h a_0 - d_0 h & d \end{pmatrix}$$

is an isomorphism if, and only if, the mapping $h \to h a_0 - d_0 h$ is an automorphism of $\mathscr{B}(V, W)$. As d_0 identifies to $\dot{A}_0$, the theorem follows from the Inverse Function Theorem in Banach spaces and the characterization (b) of simple invariant subspaces. □

Theorem 11. — *Given $(S_0, A_0) \in \mathrm{Inv}(E)$, if S_0 is a simple invariant subspace of A_0, then:*

(i) *The subspace S_0 admits a unique A_0–invariant closed complement S_1, which is a simple invariant subspace of A_0.*
(ii) *Such a pair of complementary invariant subspaces exists for all $A \in \mathrm{End}(E)$ close enough to A_0. More precisely, there exists a unique analytic map germ $(V_0, V_1) : \big(\mathrm{End}(E), A_0\big) \to \big(\mathbf{G}(E)^2, (S_0, S_1)\big)$ such that $V_0(A)$ and $V_1(A)$ are complementary A–invariant subspaces.*

Proof. — Assertion (ii) clearly follows from (i) and Theorem 10, as V_0 and V_1 are the first components of the maps obtained by inverting the local diffeomorphisms $\big(\mathrm{Inv}(E), (S_0, A_0)\big) \to \mathrm{End}(E)$ and $\big(\mathrm{Inv}(E), (S_1, A_0)\big) \to \mathrm{End}(E)$ induced by Π.

To prove (i), denote by W any closed complementary subspace of S_0. In the identification of $E = S_0 \oplus W$ to $S_0 \times W$, we can as usual write

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ 0 & d_0 \end{pmatrix}$$

and notice that S_1, if it exists, must be the "graph" $\{(x, y) \in S_0 \times W \,:\, x = hy\}$ of a map $h \in \mathscr{B}(W, S_0)$, whose invariance is expressed by the equation

$$a_0 h + b_0 = h d_0.$$

Now, as d_0 identifies to $\dot{A}_0$, the characterization (c) of simple invariant subspaces implies that this equation has a unique solution h, namely the inverse image of b_0 by the automorphism $h \mapsto h d_0 - a_0 h$ of $\mathscr{B}(W, S_0)$. □

D4. Existence of simple invariant subsets. — Under the hypothesis of Theorem 11, in the identification of $E = S_0 \oplus S_1$ to $S_0 \times S_1$, the operator A_0 writes

$$A_0 = \begin{pmatrix} a_0 & 0 \\ 0 & a_1 \end{pmatrix}$$

and therefore $\sigma(A_0)$ is the disjoint union of $\sigma(a_0)$ and $\sigma(a_1) = \sigma(\dot{A}_0)$. In fact, each decomposition of $\sigma(A)$ as the disjoint union of two compact subsets yields a unique decomposition of E as the direct sum of two closed invariant subspaces:

Theorem 12. — *Assume that* $\mathbf{K} = \mathbf{C}$*. If the spectrum of* $A_0 \in \operatorname{End}(E)$ *is the union of two disjoint nonempty compact subsets* σ_0 *and* σ_1*, then:*

(i) *For* $j = 0, 1$*, there exists a unique invariant subspace* S_j *of* A_0 *such that the spectra of the maps* $a_j \in \operatorname{End}(S_j)$ *and* $\dot{A}_j \in \operatorname{End}(E/S_j)$ *induced by* A_0 *are* σ_j *and* $\sigma_{j\pm 1}$ *respectively (in particular,* S_j *is simple).*

(ii) *The* A_0*–invariant subspaces* S_0 *and* S_1 *are complementary. Thus, in the identification of* $E = S_0 \oplus S_1$ *to* $S_0 \times S_1$*,*

$$A_0 = \begin{pmatrix} a_0 & 0 \\ 0 & a_1 \end{pmatrix}.$$

(iii) *Therefore, by Theorem 11, there exist uniquely determined analytic germ* V_j : $\big(\operatorname{End}(E), A_0\big) \to \big(\mathbf{G}(E), S_j\big)$*,* $j = 0, 1$*, such that* $V_0(A)$ *and* $V_1(A)$ *are complementary* A*–invariant subspaces for each* A *in their domains.*

Definition Under the hypotheses of Theorem 12, we call S_j the *invariant subspace of* A_0 *associated to* σ_j*.*

Proof of Theorem 12 By Theorem 11, the subspace S_j exists and is unique if and only if the pair (S_0, S_1) exists and is unique. Therefore, our problem is the following:

- find a projector[12] $P \in \operatorname{End}(E)$ such that $S_0 := \operatorname{Im} P$ and $S_1 := \operatorname{Ker} P$ have the required properties
- prove that P is unique.

Lemma. — *Given a bounded open subset* U *of* $\mathbf{C}$ *with smooth boundary, containing* σ_0 *and such that* σ_1 *lies in its exterior, we have the following:*

(a) *The map* $P \in \operatorname{End}(E)$ *defined by*

$$P := \frac{1}{2\pi i} \int_{\partial U} (z - A_0)^{-1} dz$$

is a projector.

(b) *As* P *commutes with* A_0*, the complementary subspaces* $S_0 := \operatorname{Im} P$ *and* $S_1 :=$ $\operatorname{Ker} P$ *are invariant by* A_0*.*

(c) *Moreover, the maps* $a_0 \in \operatorname{End}(S_0)$ *and* $a_1 \in \operatorname{End}(S_1)$ *induced by* A_0 *satisfy* $\sigma(a_0) = \sigma_0$ *and* $\sigma(a_1) = \sigma_1$*.*

Proof of the lemma. — We can enlarge U into a bounded open subset U_1 of $\mathbf{C}$ containing ∂U and σ_0 and such that σ_1 lies in its exterior. Then[13] $P = f(A_0)$, where the holomorphic function $f : \mathbf{C} \setminus \partial U_1 \to \mathbf{C}$ is given by

$$f(z) := \begin{cases} 1 & \text{for } z \in U_1 \\ 0 & \text{otherwise.} \end{cases}$$

(12) Endomorphism P such that $P^2 = P$.

(13) See **[10]**, chapter 10 from 10.21 to 10.29, for an account of the beautiful theory sometimes called *holomorphic functional calculus.*

Since $f(z)^2 = f(z)$ for all $z \in \mathbf{C} \setminus \partial U_1$, we have $f(A_0)^2 = f(A_0)$, hence (a).

As (b) is obvious, let us prove (c). Given $\lambda \in \mathbf{C} \setminus \sigma(A_0)$ observe that $(A_0 - \lambda)P = P(A_0 - \lambda)$ equals $g(A_0)$, where the holomorphic function $g : \mathbf{C} \setminus \partial U_1 \to \mathbf{C}$ is given by $g(z) = (z - \lambda)f(z)$. It follows that

(A) the spectrum of $(A_0 - \lambda)P$ is $g\big(\sigma(A_0)\big) = (\sigma_0 - \lambda) \cup \{0\}$

(B) similarly, the spectrum of $(A_0 - \lambda)(1 - P)$ is $(\sigma_1 - \lambda) \cup \{0\}$.

Now, in the identification $v \mapsto (Pv, v - Pv)$ of E to $S_0 \times S_1$, we have

$$(A_0 - \lambda)P = \begin{pmatrix} a_0 - \lambda & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_0 - \lambda = \begin{pmatrix} a_0 - \lambda & 0 \\ 0 & a_1 - \lambda \end{pmatrix},$$

hence, by (A)–(B),

$$\begin{aligned} (\sigma(a_0) - \lambda) \cup (\sigma(a_1) - \lambda) &= \sigma(A_0 - \lambda) = (\sigma_0 - \lambda) \cup (\sigma_1 - \lambda) \\ (\sigma(a_0) - \lambda) \cup \{0\} &= \sigma\big((A_0 - \lambda)P\big) = (\sigma_0 - \lambda) \cup \{0\} \\ (\sigma(a_1) - \lambda) \cup \{0\} &= \sigma\big((A_0 - \lambda)(1 - P)\big) = (\sigma_1 - \lambda) \cup \{0\}, \end{aligned}$$

implying (c) since λ belongs neither to σ_0, nor to σ_1. □

Proof that P is unique. — Let $Q \in \mathrm{End}(E)$ be a projector with the required properties. As $\mathrm{Im}\, Q$ and $\mathrm{Ker}\, Q = \mathrm{Im}(1 - Q)$ are complementary subspaces invariant by A_0, we have, for all $v \in E$,

$$QA_0v + (1 - Q)A_0v = A_0v = A_0\big(Qv + (1 - Q)v\big) = A_0Qv + A_0(1 - Q)v\,,$$

hence $A_0Qv = QA_0v$ and therefore

$$A_0Q = QA_0.$$

Identifying $E = S_0 \oplus S_1$ to $S_0 \times S_1$, we can write

$$A_0 = \begin{pmatrix} a_0 & 0 \\ 0 & a_1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} q_0 & b \\ c & q_1 \end{pmatrix}.$$

The commutation relation implies that $a_0b = ba_1$ and $ca_0 = a_1c$, hence $b = 0$ and $c = 0$ by the characterizations (b)–(c) of a simple invariant subspace. It follows that

$$Q = \begin{pmatrix} q_0 & 0 \\ 0 & q_1 \end{pmatrix},$$

where $q_j \in \mathrm{End}(S_j)$ is a projector which commutes with a_j for $j = 0, 1$. For every $\lambda \in \mathbf{C}$, we have

$$\sigma\big((A_0 - \lambda)Q\big) = (\sigma_0 - \lambda) \cup \{0\}$$

since the spectrum of $A_0 - \lambda$ restricted to $\mathrm{Im}\, Q$ is $\sigma_0 - \lambda$ and Q is not the identity (otherwise σ_1 would be empty). Now, we also have

$$\sigma\big((A_0 - \lambda)Q\big) = \sigma\big((a_0 - \lambda)q_0\big) \cup \sigma\big((a_1 - \lambda)q_1\big)$$

and[14], as $a_j - \lambda$ commutes with q_j,

$$\sigma\big((a_j - \lambda)q_j\big) \subset \sigma(a_j - \lambda)\sigma(q_j) = (\sigma_j - \lambda)\sigma(q_j) \subset (\sigma_j - \lambda) \cup \{0\}.$$

It follows that we must have $\sigma\big((a_1 - \lambda)q_1\big) = \{0\}$ for every λ and therefore $\sigma(q_1) = 0$, hence $q_1 = 0$ since q_1 is a projector, yielding

$$\operatorname{Im} Q \subset S_0.$$

Replacing Q by $1 - Q$ in this argument, we obtain the inclusion

$$\operatorname{Im}(1 - Q) \subset S_1.$$

As $E = S_0 \oplus S_1 = \operatorname{Im} Q \oplus \operatorname{Im}(1 - Q)$, it follows that $\operatorname{Im} Q = S_0$ and $\operatorname{Im}(1 - Q) = S_1$, hence $Q = P$. □

Remarks. — Instead of deducing part (iii) of the theorem from the inverse function theorem, one can use directly the observation that, for A close enough to A_0, the formula

$$P(A) := \frac{1}{2\pi i} \int_{\partial U} (z - A)^{-1} dz$$

defines a projector. As it depends analytically on A, so do its image $V_0(A)$ and its kernel $V_1(A)$, which are invariant by A since $P(A)A = AP(A)$. This type of proof (and the result) are well-known, although it must be said that in the standard reference on the subject ([**8**], Chapter IV, Section 4, Theorem 3.16) the result is obscured by unnecessary "additional" hypotheses.

Of course, Theorem 12 enables us to associate to every decomposition of $\sigma(A_0)$ as the union of finitely many mutually disjoint nonempty compact subsets $\sigma_1, \ldots, \sigma_p$ the decomposition $E = S_1 \oplus \cdots \oplus S_p$, where S_j denotes the A_0–invariant subspace associated to σ_j.

In finite dimensions, we can consider the maximal decomposition of $\sigma(A_0)$ defined by $\sigma_j = \{\lambda_j\}$, where the λ_j's are the eigenvalues of A_0. The subspace S_j is then called the *characteristic subspace* of A_0 associated to λ_j. In that case, as mentioned in the introduction, Theorem 12 (iii) (or, rather, Theorem 10) tells us something which deserves being better known: every A close enough to A_0 admits an invariant subspace $V_j(A)$ of the same dimension as the characteristic subspace S_j, unique in a suitable neighborhood of S_j and depending analytically on A even though the eigenvalues do not and the eigenspaces may explode—for generic A, the subspace $V_j(A)$ is the direct sum of one-dimensional eigenspaces corresponding to mutually distinct eigenvalues of A close to λ_j.

Theorem 13. — *Theorem 12 holds if* $\mathbf{K} = \mathbf{R}$, *provided* σ_0 *and (therefore)* σ_1 *are invariant by complex conjugation.*

[14] See [**10**], Theorem 11.23.

Proof. — On the complex Banach space $E_{\mathbf{C}} := E \oplus iE$ obtained from E by complexification, there is a conjugation

$$v + iw \mapsto \overline{v + iw} := v - iw, \ v, w \in E.$$

Denoting again by A_0 the complexified endomorphism $v + iw \mapsto A_0 v + iA_0 w$, the identity

$$A_0 \overline{z} = \overline{A_0 z}$$

implies that the complementary A_0–invariant subspaces $S_0, S_1 \subset E_{\mathbf{C}}$ obtained from Theorem 12 satisfy

$$\overline{S}_j = S_j. \tag{17}$$

Indeed, $\overline{S}_j$ is an A_0–invariant subspace of which A_0 induces an endomorphism with spectrum $\overline{\sigma}_j = \sigma_j$, hence (17) since S_j is unique.

It follows that S_j is the complexification of the real A_0–invariant subspace $S_j \cap E$, that $E = (S_0 \cap E) \oplus (S_1 \cap E)$ and, of course, that the spectrum of the endomorphism of $S_j \cap E$ induced by A_0 is σ_j. □

Theorem 14. — *If $\mathbf{K} = \mathbf{R}$, then, given $A_0 \in \operatorname{End}(E)$:*

(i) *If $\sigma(A_0) \cap \mathbf{R} = \varnothing$, there exist a* complex *Banach space F and an analytic local map $I : \big(\operatorname{End}(E), A_0\big) \to \operatorname{Iso}(E, F)$ (space of continuous isomorphisms of E onto F) such that every $I(A)_* A := I(A) \circ A \circ I(A)^{-1}$ is a $\mathbf{C}$–linear operator whose spectrum is the intersection $\sigma^+(A)$ of $\sigma(A)$ with the upper half-plane $\Im z > 0$.*

(ii) *More generally, if $\sigma(A_0) \cap \mathbf{C} \setminus \mathbf{R}$ is compact*[15]*, there exist Banach spaces S, F with F* complex *and an analytic $I : \big(\operatorname{End}(E), A_0\big) \to \operatorname{Iso}(E, S \times F)$ such that every $I(A)_* A$ is block diagonal $a(A) \times d(A)$, the endomorphism $d(A)$ of F is $\mathbf{C}$–linear, $\sigma\big(a(A)\big) = \sigma(A) \cap \mathbf{R}$ and $\sigma\big(d(A)\big) = \sigma^+(A)$.*

Proof. — (i) By Theorem 12, applied with $\sigma_0 := \sigma^+(A_0)$ to the complexified map $A_{0\mathbf{C}}$ of A_0, there is a unique analytic local map $P : \big(\operatorname{End}(E), A_0\big) \to \operatorname{End}(E_{\mathbf{C}})$ such that

- each $P(A)$ is a projector whose kernel and image are invariant by $A_{\mathbf{C}}$
- the map $A_{\mathbf{C}}$ induces endomorphisms of $\ker P(A)$ and $\operatorname{Im} P(A)$ whose spectra are $\overline{\sigma^+(A)}$ and $\sigma^+(A)$ respectively.

The projection $v \mapsto \Re v$ of $E_{\mathbf{C}}$ onto E, restricted to $\operatorname{Im} P(A)$, is an isomorphism and, denoting the inverse map by $r(A)$, the map $r(A)_* A$ clearly is the $\mathbf{C}$–linear endomorphism of $\operatorname{Im} P(A)$ induced by $A_{\mathbf{C}}$. The image of $r(A)$ depends on A but we can make it constant by composing $r(A)$ with $P(A_0)$, as the latter induces an isomorphism of $\operatorname{Im} P(A)$ onto $\operatorname{Im} P(A_0)$ for A close enough to A_0. This proves (i) with $F := \operatorname{Im} P(A_0)$ and $P(A) := P(A_0) \circ r(A)$.

(ii) By Theorem 13, applied to A_0 with $\sigma_0 := \sigma(A_0) \cap \mathbf{R}$, the same argument as in the proof of (i) shows that there exist real Banach spaces S, V and an analytic $J : \big(\operatorname{End}(E), A_0\big) \to \operatorname{Iso}(E, S \times V)$ such that every $J(A)_* A$ has the form $a(A) \times D(A)$

[15] Which is automatically the case for $\dim E < \infty$.

with $\sigma\big(a(A)\big) = \sigma(A) \cap \mathbf{R}$ and $\sigma\big(D(A)\big) = \sigma(A) \cap \mathbf{C} \setminus \mathbf{R}$. Applying (i) to $D(A)$, we get what we want. □

Remark. — This easy complexification result is extremely useful in the theory of normal forms of dynamical systems.

D5. Singularities of invariant subspaces. — In infinite dimensions, the map $\Pi : \mathrm{Inv}(E) \to \mathrm{End}(E)$ can have a *very* wild singularity at (S_0, A_0) when S_0 is not a simple invariant subspace of A_0. We shall only consider the simplest cases, which do occur naturally, at least for compact or Fredholm operators and in particular in finite dimensions.

Theorem 15. — *Assume that $(S_0, A_0) \in \mathrm{Inv}(E)$ satisfies $\sigma(a_0) \cap \sigma(\dot{A}_0) = \{\lambda_0\}$, where λ_0 is a geometrically simple eigenvalue of A_0 of multiplicity k* and *a simple eigenvalue of a_0 or $\dot{A}_0$. Then Π is a k–swallowtail at (S_0, A_0).*

Proof. — Our hypothesis implies that $\sigma(a_0) = \tau_0 \cup \{\lambda_0\}$, $\sigma(\dot{A}_0) = \tau_1 \cup \{\lambda_0\}$ and $\sigma(A_0) = \tau_0 \cup \{\lambda_0\} \cup \tau_1$, where τ_0, τ_1 are disjoint compact subsets, which may be empty and do not contain λ_0.

First assume that λ_0 is a simple eigenvalue of a_0. If $\tau_0 = \varnothing$, our theorem is the “if” part of Theorem 8 (iii). Otherwise, we proceed as in the proof of Theorem 8 (iii) to show that the contribution of τ_0 (and τ_1) to the singularity is trivial.

If λ_0 is a simple eigenvalue of $\dot{A}_0$, then the contribution of τ_0 and τ_1 to the singularity is trivial, which reduces the problem to the case where $E = \mathbf{K}^k$. Then, the hypotheses of Theorem 8 (iii) are satisfied by the transposed map $A_0^* \in \mathrm{End}(E^*)$ and the line $S_0^{\perp}$, hence our theorem since $(A, S) \mapsto (A^*, S^{\perp})$ is a diffeomorphism of the open subset of $\mathrm{Inv}(\mathbf{K}^k)$ associated to hyperplanes S onto $\mathrm{Eig}(\mathbf{K}^{k*})$, fibered over the isomorphism $A \mapsto A^*$. □

Remark. — For $k > 1$, the hypotheses of Theorem 15 imply that S_0 *admits no A_0–invariant complement.*

Theorem 16. — *Assume that $\mathbf{K} = \mathbf{R}$ and that $(S_0, A_0) \in \mathrm{Inv}(E)$ satisfies $\sigma(a_0) \cap \sigma(\dot{A}_0) = \{\lambda_0, \bar{\lambda}_0\}$, where $\lambda_0 \in \mathbf{C} \setminus \mathbf{R}$ is a geometrically simple eigenvalue of A_0 of multiplicity k* and *a simple eigenvalue of a_0 or $\dot{A}_0$. Then Π is a* complex *k–swallowtail at (S_0, A_0).*

Proof. — As in the proof of Theorem 15, we have that $\sigma(a_0) = \tau_0 \cup \{\lambda_0, \bar{\lambda}_0\}$, $\sigma(\dot{A}_0) = \tau_1 \cup \{\lambda_0, \bar{\lambda}_0\}$ and $\sigma(A_0) = \tau_0 \cup \{\lambda_0, \bar{\lambda}_0\} \cup \tau_1$, where τ_0, τ_1 are disjoint compact subsets, invariant by complex conjugation, which may be empty and do not contain λ_0. The contributions of τ_0 and τ_1 to the singularity are trivial, reducing us to the case where $E = \mathbf{R}^{2k}$ and $\lambda_0, \bar{\lambda}_0$ are the only (geometrically simple) eigenvalues of A_0. Therefore, our problem is to prove a real version of Theorem 8 (iii) in its simplest case, assuming that S_0 has dimension 2 (if it has codimension 2, the same duality argument as in the proof of Theorem 15 applies).

By Theorem 14, there exists an isomorphism $I(A)$ of $\mathbf{R}^{2k}$ onto $\mathbf{C}^k$, depending analytically on A such that $I(A)_*A := I(A)\circ A\circ I(A)^{-1}$ is a *complex* endomorphism of $\mathbf{C}^k$, that λ_0 is the sole, geometrically simple, eigenvalue of $P(A_0)_*A_0$ and that $P(A_0)S_0$ is the corresponding *complex* one-dimensional eigenspace. Applying Theorem 8 (iii) to $I(A)_*A$, we do get what we want for A. □

APPENDICES

Appendix A
Some Useful Singularities

We will recall in this appendix some simple properties of singularities, especially of the swallowtail type, and we describe some examples of them that are used in the main text.

Two function germs $f, g : E_1, 0 \to E_2, 0$ are called *diffeomorphic* (16) if there are local diffeomorphisms φ of $E_1, 0$ and ψ of $E_2, 0$ such that $g\circ\varphi = \psi\circ f$.

A singular function germ $f : \mathbf{K}^n, 0 \to \mathbf{K}^n, 0$ is called *good* (in the sense of Whitney) if its jacobian determinant $Jf : \mathbf{K}^n, 0 \to \mathbf{K}, 0$ is regular at 0.

It is clear that a good map is of corank 1, because if two rows of the jacobian matrix of f vanish at 0, then the jacobian determinant is at least of order 2 and so is singular at the origin. In fact, the good function germs are exactly those such that $j^1 f$ is transversal to the stratum Σ^1 of mappings of corank 1.

Swallowtails. — The *standard k–swallowtail* is the mapping

$$\begin{aligned} SW_k : \mathbf{K}^{k-1} &\to \mathbf{K}^{k-1} \\ SW_k(a_1, \dots, a_{k-2}, u) &:= (a_1, \dots, a_{k-2}, u^k + a_{k-2}u^{k-2} + \cdots + a_1 u). \end{aligned}$$

In other words, it is the universal unfolding of the map $u \to u^k$.

For us a *k–swallowtail* will be any map germ between two Banach spaces which is diffeomorphic to the germ at 0 of a suspension of the standard one, that is, a map of the form

$$SW_k \times \mathrm{Id} : \mathbf{K}^{k-1} \times E \to \mathbf{K}^{k-1} \times E$$

for some Banach space E. In other words, it is a versal unfolding of the map $u \to u^k$.

A k–swallowtail has the following properties:

(i) It is a stable map germ of corank 1.

(ii) It is a Morin singularity of type Σ^{1_k}.

(iii) It is a *good* function in the sense of Whitney.

(16) Usually called *left-right equivalent.*

Example 1. — The evaluation map

$$\begin{array}{rcl} \mathrm{ev} : \mathrm{MP}(k) \times \mathbf{K} & \to & \mathrm{MP}(k) \times \mathbf{K} \\ (P, a) & \mapsto & \big(P, P(a)\big) \end{array}$$

is a k–swallowtail. Indeed, if we restrict (in the souce and target) to the subspace of polynomials without terms of degree $k-1$ and 0 we get the standard $k-$swallowtail. For the whole space of monic polynomials we need only put aside those coefficients by the standard translation procedures: Let

$$\begin{array}{rcl} P(x) & = & x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0 \\ Q(x) & := & P(x - a_{k-1}/k) \\ Q_0(x) & := & Q(x) - Q(0). \end{array}$$

Then, the map ev factors as:

$$\begin{array}{ccc} (P, a) & \to & (Q_0, a + a_{k-1}/k, a_{k-1}, Q(0)) \\ \downarrow & & \downarrow \\ (P, P(a)) & \leftarrow & (Q_0, Q_0(a + a_{k-1}/k), a_{k-1}, Q(0)) \end{array}$$

and the second vertical arrow is of the form $\mathrm{SW}_k \times \mathrm{Id}$ while the two horizontal ones are diffeomorphisms.

Example 1a. — The evaluation map restricted to polynomials P with $a_{k-1} = 0$ is also a k–swallowtail.

This is because in the above factorization of ev one can eliminate the third component from the right-hand spaces.

Example 1b. — The evaluation map restricted to polynomials P with $a_0 = 0$ is also a k–swallowtail.

This is because in the above factorization of ev one can eliminate the fourth component $Q(0)$ from the right-hand spaces since it is determined from the other ones by the relation $0 = Q_0(a + a_{k-1}/k) + Q(0)$ (the last expression equals $P(0) = a_0$).

Example 1c. — Let $\mathrm{Root}(n) = \{(a, P) \in \mathbf{K} \times \mathrm{MP}(n)$ such that $P(a) = 0\}$, which is diffeomorphic to $\mathbf{K}^n$ since the defining equation can be solved for a_0. The map $\mathrm{Root}(n) \to \mathrm{MP}(n)$ which sends (a, P) to P is also a k–swallowtail.

This is because, in terms of the natural parametrizations of $\mathrm{Root}(n)$ and $\mathrm{MP}(n)$ the above map is given by

$$(a, Q) \mapsto \big(a, Q - Q(a)\big) \mapsto Q - Q(a) \cong \big(Q, -Q(a)\big),$$

where $Q(0) = 0$, which is diffeomorphic to the evaluation map for polynomials with $a_0 = 0$ (Example 1b).

Example 2. — The map

$$(a_1, \ldots, a_{k-1}, a) \to (aa_1, a_1 + aa_2, a_2 + aa_3, \ldots, a_{k-2} + aa_{k-1}, a_{k-1} + a)$$

is a k–swallowtail.

To see this, take as new coordinates $u = -a$ and the last $k-1$ components of the map:

$$b_i := a_i - ua_{i+1})$$

for $i = 1, \dots k-1$, where we take $a_k = 1$ (so $b_{k-1} := a_{k-1} - u)$). This yields inductively

$$a_{k-i} = u^i + \sum_{j=1}^{i} b_{k-j} u^{i-j}$$

and therefore

$$aa_1 = -u\Big(u^{k-1} + \sum_{j=1}^{k-1} b_{k-j} u^{k-1-j}\Big) = -\Big(u^k + \sum_{j=1}^{k-1} b_{k-j} u^{k-j}\Big)$$

so the mapping is equivalent to

$$(u, b_1, \dots, b_{k-1}) \to (u^k + b_{k-1}u^{k-1} + \cdots + b_1 u, b_1, \dots, b_{k-1})$$

which is essentially the evaluation map for polynomials with null constant term. As we have seen in example 1b above, it is a k–swallowtail.

If we put $a_{k-1} = -a$ on example 2 we get essentially the map:

$$(a_1, \dots, a_{k-2}, a) \to (aa_1, a_1 + aa_2, a_2 + aa_3, \dots, a_{k-2} - a^2).$$

As this corresponds to making $b_{k-1} = 0$ in the new coordinates, it is diffeomorphic to the standard k–swallowtail. This shows that every swallowtail can be given by polynomial functions of degree 2.

Complex Swallowtails. — The complex swallowtail

$$\mathrm{SW}_k : \mathbf{C}^{k-1} \to \mathbf{C}^{k-1}$$

can be considered as a real mapping

$$\mathrm{SW}_k : \mathbf{R}^{2k-2} \to \mathbf{R}^{2k-2}$$

which we will call the *standard complex swallowtail* and by a *complex swallowtail* we will mean any map diffeomorphic to one of its *real* suspensions. For example, the standard 2-swallowtail is the real map $(x, y) \mapsto (x^2 - y^2, 2xy)$.

The complex swallowtails are not stable as real maps. In fact they are very degenerate since their singular set is of codimension 2 and can explode into a subset of codimension 1 (for $k > 2$, in uncountably many ways).

Swallowtail deformations. — A *k–swallowtail deformation* is any map germ between two Banach spaces which is diffeomorphic to the germ at 0 of a map

$$G : E \times E' \to E$$

such that $G(x, 0)$ is a k–swallowtail, where E, E' are Banach spaces. The stability of the swallowtail implies that any k–swallowtail deformation is diffeomorphic to a simple form, essentially a k–swallowtail with coefficients that depend on the parameters.

A k-swallowtail has always corank 1 at its singular points. However, observe that, in a k–swallowtail deformation, the derivative with respect to the parameters may add the missing direction in the image of the derivative, in which case it is a submersion (this is due to the fact that we are thinking of it as a mapping where variables and parameters are not to be distinguished). Thus a k–swallowtail deformation can have corank 0 or 1.

The first case of a k–swallowtail deformation in our work corresponds to a double eigenvalue. In this case the map χ is

$$(a, b, c, d) \to (a + d, ad - bc)$$

which is easily seen to be diffeomorphic to

$$(a, b, c, d) \to (a, d^2 + bc)$$

This is a stable map, being the suspension of a Morse funcion.

Appendix B
Some useful facts about Banach spaces

We denote by E, F two Banach spaces over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ and by $\mathscr{B}(E, F)$ the space of continuous linear maps of E into F.

Proposition B.1. — *For $A \in \mathscr{B}(E, F)$, the following properties are equivalent:*

(i) *The map A is injective and its image* $\operatorname{Im} A$ *is closed, in which case we call A an* embedding.
(ii) *There exists $c > 0$ such that $c|Ax| \geq |x|$ for all $x \in E$.*
(iii) *There does not exist any sequence (x_n) in E such that $|x_n| = 1$ for all n and* $\lim_{n\to\infty} Ax_n = 0$.

Proof. — By the open mapping theorem, if (i) holds, then A induces an isomorphism A_1 of E onto $\operatorname{Im} A$ and $A_1^{-1} \circ A = \mathrm{Id}_E$, hence (iii) since $\lim_{n\to\infty} Ax_n = 0$ implies $\lim_{n\to\infty} x_n = A_1^{-1} \lim_{n\to\infty} Ax_n = 0$.

If (ii) does not hold, there is a sequence (y_n) in E satisfying $|y_n| > n|Ay_n|$ for all n and therefore the sequence $x_n := y_n/|y_n|$ is such that $|x_n| = 1$ for all n and $\lim_{n\to\infty} Ax_n = 0$, proving that (iii) implies (ii).

Finally, assuming (ii), the linear map A is clearly injective. Moreover, any sequence (x_n) in E such that Ax_n converges to some y satisfies $|x_n - x_p| \leq c|A(x_n - x_p)| = c|Ax_n - Ax_p|$ and therefore is Cauchy. It follows that (x_n) converges to some $x \in E$, which must satisfy $Ax = A \lim x_n = \lim Ax_n = y$, proving that $\operatorname{Im} A$ is closed and therefore that (ii) implies (i). □

Corollary B.2. — *The set of all embeddings $A \in \mathscr{B}(E, F)$ is open.*

Proof. — By Proposition B.1 (ii), the embeddings are exactly those A such that $|Ax|/|x|$ is bounded below by a positive constant on $E \setminus \{0\}$. □

Proposition B.3. — *Given $A \in \mathscr{B}(E, F)$, let $A^* \in \mathscr{B}(F^*, E^*)$ denote the* adjoint map $q \mapsto q \circ A$ *(also called* transposed map*).*

(i) *The map* A^* *is injective if and only if* $\operatorname{Im} A$ *is dense.*
(ii) *The subspace* A^*F^* *is closed if and only if* $\operatorname{Im} A$ *is closed.*
(iii) *In particular,* A^* *is an embedding if and only if* A *is onto.*
(iv) *The map* A^* *is onto if and only if* A *is an embedding.*

Proof. — As $A^*q(x) = q(Ax)$ for all $x \in E$, the kernel of A^* is the set $(\operatorname{Im} A)^\perp$ of those q which vanish on the image of A, hence (i). Assertion (ii) is Theorem 4.14 of [**10**], and (iii) follows at once from (i)–(ii).

Let us prove (iv). Clearly, for each $q \in F^*$, the function $A^*q : v \mapsto q(Av)$ vanishes on $\operatorname{Ker} A$, hence the inclusion $A^*F^* \subset (\operatorname{Ker} A)^\perp$. Therefore, if A^* is onto, then $(\operatorname{Ker} A)^\perp = E^*$, hence (Hahn-Banach) $\operatorname{Ker} A = \{0\}$, proving that A—which has closed image by (ii)—is an embedding. Conversely, if A is an embedding, then it induces an isomorphism A_1 of E onto $\operatorname{Im} A$. For each $p \in E^*$, the map $q := p \circ A_1^{-1} \in (\operatorname{Im} A)^*$ can be extended (Hahn-Banach) to a map $Q \in F^*$, which satisfies $A^*Q = Q \circ A = q \circ A = p$, proving that A^* is onto. □

Corollary B.4. — *The set of all surjective* $A \in \mathscr{B}(E,F)$ *is open.*

Proof. — As the linear map $A \mapsto A^*$ of $\mathscr{B}(E,F)$ into $\mathscr{B}(F^*,E^*)$ is continuous (isometric), this follows at once from Proposition B.3 and Corollary B.2. □

Proposition B.5. — *The subset of* $\mathscr{B}(E,F)$ *consisting of those* A *which are onto but not embeddings is open, and so is the subset consisting of those embeddings which are not onto.*

Proof. — By Proposition B.3 (iii)–(iv) and the fact that $A \mapsto A^*$ is continuous, we just have to prove the second assertion. Given an embedding A which is not onto and $y \in F \setminus \operatorname{Im} A$, the distance $2D$ from y to the closed subset $\operatorname{Im} A$ is positive, that is

$$|Ax - y| \geq 2D > 0 \text{ for all } x \in E \tag{18}$$

and, by Proposition B.1, there exists $C > 0$ such that

$$|Ax| > C|x| \text{ for all } x \in E. \tag{19}$$

We claim that every $B = A + u \in \mathscr{B}(E,F)$ close enough to A is an embedding and satisfies

$$|Bx - y| \geq D > 0 \text{ for all } x \in E, \tag{20}$$

proving our result. Indeed, (19) implies the inequality

$$|Bx| = |Ax + ux| \geq |Ax| - |ux| \geq (C - |u|)|x|,$$

proving that B is an embedding for $|u| < C$ and implying

$$|Bx - y| \geq |Bx| - |y| \geq (C - |u|)|x| - |y|,$$

which shows that (20) holds for

$$|u| < C \text{ and } |x| \geq \frac{D + |y|}{C - |u|}.$$

Therefore, all we have to prove is that it holds for small enough $|u| < C$ when x satisfies $|x| \leq \frac{D+|y|}{C-|u|}$. Now, this is clear as we then have

$$|Bx - y| = |Ax + ux - y| \geq |Ax - y| - |ux| \geq 2D - |u|\frac{D + |y|}{C - |u|}$$

by (18). □

Proposition B.6. — *Given two Banach spaces S, F and endomorphisms a, d of S and F respectively, the spectrum of the endomorphism $a^* - d_* : h \mapsto ha - dh$ of $\mathscr{B}(S, F)$ is $\sigma(a) - \sigma(d) := \{\lambda - \mu \,:\, \lambda \in \sigma(a), \mu \in \sigma(d)\}$.*

Proof. — Our endomorphism is the sum of the two commuting endomorphisms $a^* : u \mapsto ua$ and $-d_* : u \mapsto -du$. Now, a classical application of the Gel'fand transform ([**10**], Theorem 11.23) is that if two elements of a Banach algebra with unit[17] commute, the spectrum of their sum is included in the sum of their spectra. As, clearly, $\sigma(a^*) = \sigma(a)$ and $\sigma(d_*) = \sigma(d)$, we get the inclusion

$$\sigma(a^* - d_*) \subset \sigma(a) - \sigma(d).$$

Alternatively, one can use the holomorphic functional calculus to verify that, if $\sigma(a) \cap \sigma(d) = \varnothing$ then the mapping

$$k \longmapsto \frac{1}{2\pi i}\int_{\partial U}(\zeta I - d)^{-1}k(\zeta I - a)^{-1}d\zeta$$

(where U is a bounded open subset of $\mathbf{C}$ with smooth boundary, containing $\sigma(a)$ only, cf. subsection D4) is the inverse of $a^* - d_*$ and then use this particular case to prove the above inclusion.

There remains to prove that we have $\lambda - \mu \in \sigma(a^* - d_*)$ for all $\lambda \in \sigma(a)$ and $\mu \in \sigma(d)$. Replacing a by $a - \lambda$ and d by $d - \mu$, this amounts to proving the following

Lemma. — *If neither a, nor d is invertible, then $a^* - d_*$ is not invertible.*

Indeed[18], there are four possible situations:

- If a is not onto and d is not an embedding, then by Propositions B.1 and B.3 (iii), there exist a sequence y_n in F and a sequence p_n in S^* with $|y_n| = |p_n| = 1$ such that $p_n a$ and dy_n converge to 0. The sequence u_n in $\mathscr{B}(S, F)$ defined by $u_n(v) = p_n(v)y_n$ satisfies $|u_n| = 1$ and $\lim(a^* - d_*)u_n = 0$, proving that $a^* - d_*$ is not an embedding.
- Similarly, if a is not an embedding and d is not onto, there exist a sequence x_n in S and a sequence q_n in F^* with $|x_n| = |q_n| = 1$ such that $q_n d$ and ax_n converge to 0. It follows that the sequence φ_n in $\mathscr{B}(S, F)^*$ defined by $\varphi_n(u) := q_n u x_n$ satisfies $|\varphi_n| = 1$ and $\lim \varphi_n(a^* - d_*) = 0$, proving that $a^* - d_*$ is not onto.

(17) In our case, End $(\mathscr{B}(S, F))$.
(18) We are indebted to Georges Skandalis for what follows.

– If both a and d are non-surjective embeddings, it follows from Proposition B.5 that the set of those $\lambda \in \mathbf{C}$ such that both $a-\lambda$ and $b-\lambda$ are non-surjective embeddings is open. Going to the boundary, one of the maps $a-\lambda$ and $b-\lambda$ is not an embedding, and the other one is not onto by Corollary B.4. Thus, replacing a, d by $a-\lambda, d-\lambda$, we are in one of the previous two cases.
– Similarly, if both a and d are non-injective but onto, the set of those $\lambda \in \mathbf{C}$ such that both $a-\lambda$ and $b-\lambda$ are non-injective but onto is open by Proposition B.5. Going to the boundary, we are again in one of the previous first two cases: indeed, one of the maps $a-\lambda$ and $b-\lambda$ is not onto and the other one is not an embedding by Corollary B.2. □

References

[1] V. I. Arnol′d – "Matrices depending on parameters", *Uspehi Mat. Nauk* **26** (1971), p. 101–114.
[2] M. Chaperon – "Géométrie différentielle et singularités de systèmes dynamiques", *Astérisque* **138-139** (1986).
[3] ———, *Calcul différentiel et calcul intégral, 3e année*, Dunod, corrected and expanded second printing, 2008.
[4] M. Chaperon & S. López de Medrano – "On the Hopf bifurcation for flows", *C. R. Math. Acad. Sci. Paris* **340** (2005), p. 833–838.
[5] M. Chaperon, S. López de Medrano & J. L. Samaniego – "On sub-harmonic bifurcations", *C. R. Math. Acad. Sci. Paris* **340** (2005), p. 827–832.
[6] I. M. Gel′fand, M. M. Kapranov & A. V. Zelevinsky – *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications, Birkhäuser, 1994.
[7] W. T. Gowers – "A solution to Banach's hyperplane problem", *Bull. London Math. Soc.* **26** (1994), p. 523–530.
[8] T. Kato – *Perturbation theory for linear operators*, Corrected printing of the second edition, Springer, 1976, Grundlehren der Mathematischen Wissenschaften, Band 132.
[9] P. Rosenbloom – "Perturbation of linear operators in Banach spaces", *Arch. Math. (Basel)* **6** (1955), p. 89–101.
[10] W. Rudin – *Functional analysis*, McGraw-Hill Book Co., 1973.
[11] van der Waerden – *Modern algebra, in part a development from lectures by E. Artin and E. Noether*, vol. 1, F. Ungar, 1949.

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