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ERRATUM TO THE ARTICLE: GLOBAL APPLICATIONS TO RELATIVE (φ, Γ)-MODULES, I

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by

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1. The errors to be corrected

The present erratum is meant to correct two errors in the article [**2**]. The first is an error in the definition of Faltings's topology ([**2**], §4.2 following [**4**], page 214) and was pointed out to us by Ahmed Abbes. The correction follows suggestions of A. Abbes based on ideas behind the notion of oriented product of toposes introduced by P. Deligne and L. Illusie. The second error is the statement and proof of Proposition 4.4.2, 6) and 7). We thank A. Abbes for discussions regarding this issue as well.

Our basic setting is the following. Let $p > 0$ be a prime integer, k a perfect field of characteristic p, $W := \mathbb{W}(k)$ the ring of Witt vectors with coefficients in k and $K := \mathrm{Frac}(W)$ be the fraction field of W. We denote by $\overline{K}$ an algebraic closure of K and by $G_K := \mathrm{Gal}(\overline{K}/K)$. We fix a field M such that $K \subset M \subset \overline{K}$. Let X be a smooth scheme of finite type or a smooth formal scheme topologically of finite type defined over W.

2. Faltings's topology

The algebraic case. We first suppose that X/W is a smooth scheme of finite type. We define the category E_{X_M} as follows:

a) the objects are pairs of morphisms of schemes $(g\colon U \to X, f\colon W \to U_M)$, where g is an étale morphism and f is a finite and étale morphism. We will usually write (U, W) to denote this object in order to shorten the notation.

b) a morphism $(U', W') \to (U, W)$ in E_{X_M} is a pair of morphisms (α, β) where $\alpha\colon U' \to U$ is a morphism of schemes over X and $\beta\colon W' \to W$ is a morphism of schemes which makes the following diagram commute

$$\begin{array}{ccc} W' & \xrightarrow{\beta} & W \\ \downarrow & & \downarrow \\ U'_M & \xrightarrow{\alpha_M} & U_M \end{array}$$

c) Let (U,W) be an object of E_{X_M} and let $\{(U_i,W_i)\to(U,W)\}_{i\in I}$ be a family of morphisms in E_{X_M}. We say that this is a covering family of type (α) if

(α): $\{U_i\to U\}_{i\in I}$ is a covering in X^{et}, by which we denote étale site of X and $W_i\cong W\times_U U_i$ for every i.

and of type (β) if

(β): $U_i\cong U$ for every $i\in I$ and $\{W_i\to W\}_{i\in I}$ is a covering in $(X_M)^{\mathrm{et}}$, which denotes the étale site of X_M.

We endow the category E_{X_M} with the topology T_{X_M} generated by the covering families of type (α) and (β) and call it Faltings's topology associated to the data (X,M). We denote the associated site by $\mathfrak{X}_M$ and the topos of sheaves of sets on $\mathfrak{X}_M$ by $\mathrm{Sh}(\mathfrak{X}_M)$.

Remark 2.1. — Our definition of the site $\mathfrak{X}_M$ and its associated topology in [**2**] §4.2 is the original definition from [**4**], page 214. Such definition is wrong in the sense that the presheaf $\overline{\mathscr{O}}_{\mathfrak{X}_M}$ defined in definition 5.4.1 of [**2**] (and in [**4**] page 219-221) is not a sheaf in general and its sheafification does not have the required properties in order to relate it to relative Fontaine's theory (see [**3**] Example 2.2 for a simple counter example). With the new definition above the presheaf $\overline{\mathscr{O}}_{\mathfrak{X}_M}$ is a sheaf and has the required properties. For a detailed proof see [**3**, Proposition 2.11]. We remark, though, that the description of the associated topos in [**4**] corresponds to the definition of the topology given above and not to the topology given in loc. cit.

Remark 2.2. — One could define the category $E^{\mathrm{Z}}_{X_M}$ in an analogue way by replacing the étale topology X^{et} with the Zariski topology X^{Zar}, i.e. an object is a pair of morphisms $(g\colon U\to X, f\colon W\to U_M)$ such that g is an open immersion and f is a finite étale morphism. If one now endows the category $E^{\mathrm{Z}}_{X_M}$ with the covering families as defined in section 4.2 of [**2**], then the presheaf $\overline{\mathscr{O}}_{\mathfrak{X}^{\mathrm{Z}}_M}$ would be in fact a sheaf. So in this setting the definition of the topology given in section 4.2 of [**2**] is the right one. However to prove the results of [**2**], namely those of GAGA type, one needs to work with X^{et}.

Remark 2.3. — In the definition of the coverings of type (β) we allow $\{W_i\to W\}_{i\in I}$ to be a covering in $(X_M)^{\mathrm{et}}$. However for every $i\in I$, the composition $W_i\to W\to U_M\cong U_{i,M}$ is a finite étale morphism and so the morphism $W_i\to W$ is finite étale. As W has only a finite number of connected components, I contains a finite subset I' such that the family $\{W_i\to W\}_{i\in I'}$ is a covering in $(X_M)^{\mathrm{et}}$, i.e. it is a covering in $(X_M)^{\mathrm{fet}}$, by which we have denoted the finite étale topology on X_M.

We'll now give an alternative definition of the topology T_{X_M} and study some of its properties supplying details which are missing from the literature.

Definition 2.4. — Let $\{(U_{ij},W_{ij})\to(U,W)\}_{i\in I,j\in J}$ be a family of morphisms in E_{X_M}. We say this family is a *strict covering family* of (U,W) if

a) For every $i\in I$ there exists U_i object of X^{et} such that $U_i\cong U_{ij}$ for every $j\in J$;

b) The family $\{U_i\to U\}_{i\in I}$ is a covering family in X^{et}.

c) For every fixed $i \in I$ the family $\{W_{ij} \to W \times_U U_i\}_{j\in J}$ is a covering in $(X_M)^{\mathrm{et}}$.

If $\{(U_{ij}, W_{ij}) \to (U,W)\}_{i\in I, j\in J}$ is a strict covering family of (U,W) and for every $i \in I$, U_i is the object defined by a) of definition 2.4 then we will denote this family by $\{(U_i, W_{ij}) \to (U,W)\}_{i\in I, j\in J}$.

Remark 2.5. — Let us observe that if $\{(U_i, W_{ij}) \to (U,W)\}_{i\in I, j\in J}$ is a strict covering family of (U,W), then $\{(U_i, W_{ij}) \to (U,W)\}_{i\in I, j\in J}$ is the composite

$$\Big(\{(U_i, W_{ij}) \longrightarrow (U_i, W \times_U U_i)\}_{j\in J}\Big)_{i\in I} \circ \Big(\{(U_i, W\times_U U_i) \longrightarrow (U,W)\}_{i\in I}\Big)$$

and for every $i \in I$ $\{(U_i, W_{ij}) \to (U_i, W\times_U U_i)\}_{j\in J}$ is a covering of type (β) while $\{(U_i, W\times_U U_i) \to (U,W)\}_{i\in I}$ is a covering of type (α). Therefore the strict covering families are coverings in $\mathfrak{X}_M$.

On the other hand, clearly coverings of type (α) and (β) are strict coverings and therefore the strict covering families also generate the topology T_{X_M}.

Proposition 2.6. — *The finite projective limits are representable in E_{X_M}.*

Proof. — It suffices to show that given morphisms

$$(U', W') \longrightarrow (U,W) \longleftarrow (U'', W'')$$

the fiber product of (U', W') and (U'', W'') over (U,W) exists. We define it as follows: $(U', W') \times_{(U,W)} (U'', W'') := (U' \times_U U'', W' \times_W, W'')$, with the map $\gamma\colon W' \times_W W'' \to (U' \times_U U'')_M = U_M \times_{U_M} U''_M$ induced by the fiber product of the maps $W' \to U'_M$ and $W'' \to U''_M$.

We have to check that γ is finite and étale. For this let us remark that γ is the composition of the natural maps

$$W' \times_W W'' \longrightarrow W' \times_{U_M} W \times_{U_M} W'' \longrightarrow U'_M \times_{U_M} U''_M.$$

As $W' \to U'_M$ and $W'' \to U''_M$ are finite étale maps, the base changes $W' \times_{U_M} W \to U'_M \times_{U_M} W$ and $W'' \times_{U_M} W \to U''_M \times_{U_M} W$ are finite and étale. Therefore the natural map $W' \times_{U_M} W \times_{U_M} W'' \to U'_M \times_{U_M} U''_M \times_{U_M} W$ is finite and étale. Now as the map $W \to U_M$ is finite and étale it follows that the natural map $(U'_M \times_{U_M} U''_M) \times_{U_M} W \to U'_M \times_{U_M} U''_M$ is finite étale. Let us now examine the map $\rho\colon W' \times_W W'' \to W' \times_{U_M} W \times_{U_M} W''$. Let us consider the diagonal $\Delta_W\colon W \to W \times_{U_M} W$. As $W \to U_M$ is a finite and étale map, Δ_W is an open and closed morphism. Consider the diagram

$$\begin{array}{ccc} & & W \times_{U_M} W \\ & & \downarrow \\ W' & \longrightarrow & W \end{array}$$

where the vertical arrow is the projection on the first component.

Let us observe that the map $W' \to W' \times_{U_M} W$ defined by the identity and the map $W' \to W$ is the pull back of Δ_W via the map $W' \to W$ in the above diagram. It follows that $W' \to W' \times_{U_M} W$ is an open and closed morphism which implies that it is finite and étale. Similarly the morphism $W'' \to W'' \times_{U_M} W$ is finite and étale which implies that ρ is finite and étale. Finally, the object of E_{X_M} defined above $(U' \times_U U'', W' \times_W W'')$ obviously satisfies the universal property of the fiber product. □

Remark 2.7. — The category E_{X_M} with the strict covering families does not form a pretopology. Indeed due to proposition 2.6 the strict covering families satisfy PT0, PT1 and PT3 of [**1**, Def. II.1.3], but contrary to what was stated in [**2**] in the formal setting and as was pointed out to us by A. Abbes they do not satisfy PT2. However, the covering families of the pretopology PT_{X_M} generated by the strict covering families are composite of a finite number of strict covering families (or composite of a finite number of covering families of type (α) and (β)).

It follows from a direct check or from [**1**, Cor. II.2.3] that a presheaf on E_{X_M} is a sheaf if and only if it satisfies the exactness properties for the strict covering families. Moreover, the next lemma 2.8 and [**1**, Rmk. II.3.3] show that one can use strict coverings in order to compute the sheaf associated to a presheaf as done in [**2**].

Lemma 2.8. — *Let (U, W) be an object of E_{X_M}. Then the strict covering families of (U, W) are cofinal in the collection of all covering families of (U, W) in* PT_{X_M}.

Proof. — Consider a covering family $\mathscr{C}$ of (U, W) in PT_{X_M}. By Remark 2.7 $\mathscr{C}$ is a composite of n strict covering families $\mathscr{C} = \mathscr{C}_n \to \mathscr{C}_{n-1} \to \cdots \to \mathscr{C}_1$. We will prove by induction on n that we can find a covering family of every open of $\mathscr{C}$ such that the induced covering of (U, W) is a strict covering family.

For $n = 1$ there is nothing to prove so let us assume that $n = 2$. We write $\mathscr{C}_1 = \{(U_i, W_{ij})\}$ and $\mathscr{C}_2 = \{(U_{ij\alpha}, W_{ij\alpha\beta})\}$ such that $\{(U_{ij\alpha}, W_{ij\alpha\beta}) \to (U_i, W_{ij})\}_{\alpha\beta}$ are strict coverings for every i, j. For fixed i, j, α we denote by $I_{ij\alpha}$ the set over which the β's vary. For every i let us choose a finite set M_i of indices j such that the family $\{W_{ij} \to U_i \times_U W\}_{j \in M_i}$ is a covering in X_M^{et}.

Now we fix i, j, α and denote $M_{ij} := M_i \cup \{j\}$. Let x denote a geometric point of $U_{ij\alpha}$ and let x_i denote the image of x in U_i. For every $j' \in M_{ij}$, because $\{U_{ij'\alpha'} \to U_i\}_{\alpha'}$ is a covering in X^{et} there is an α' and a geometric point x' of $U_{ij'\alpha'}$ mapping to x_i. We denote

$$U_{ij\alpha x} := \times_{U_i} U_{ij'\alpha'} \text{ where the product is over } j' \in M_{ij}.$$

Then keeping in mind that $j \in M_{ij}$, we have a natural projection map $U_{ij\alpha x} \to U_{ij\alpha}$ such that there is a geometric point of $U_{ij\alpha x}$ mapping under it to x. Therefore the collection $\{U_{ij\alpha x} \to U_{ij\alpha}\}_x$ is a covering in X^{et}.

For every i, j, α as above, for every geometric point x of $U_{ij\alpha}$, $j' \in M_{ij}$ and $\beta \in I_{ij\alpha}$ we denote by

$$W_{ijj'\alpha\beta x} := W_{ij'\alpha'\beta} \times_{U_{ij'\alpha'}} U_{ij\alpha x}.$$

In particular the collection $\{(U_{ij\alpha x}, W_{ijj'\alpha\beta x}) \to (U_{ij'\alpha'}, W_{ij'\alpha'\beta})\}_x$ is a covering family of type α), i.e. it is a strict covering family. Putting together all these covering families for varying i, j, j', α, β and x we obtain a refinement $\mathscr{D} \to \mathscr{C}_2$.

We observe that (1) the family $\{W_{ij'\alpha'\beta} \to W_{ij'} \times_{U_i} U_{ij'\alpha'}\}_{\beta \in I_{ij\alpha}}$ is a covering in X_M^{et} and (2) The family $\{(W_{ij'} \times_{U_i} U_{ij\alpha x} \to W \times_U U_{ij\alpha x}\}_{j' \in M_{ij}}$ is also a covering in X_M^{et}. It follows that for all i, j, α, x the family

$$\{W_{ijj'\alpha\beta x} = W_{ij'\alpha\beta} \times_{U_{ij'\alpha'}} U_{ij\alpha x} \longrightarrow W \times_U U_{ij\alpha x}\}_{j' \in M_{ij}, \beta \in I_{ij\alpha}}$$

is a covering family and hence the family $\{(U_{ij\alpha x}, W_{ijj'\alpha\beta x}) \to (U, W)\}_{ijj'\alpha\beta x}$ is a strict covering family as claimed. This ends the case $n = 2$.

Suppose now that the statement of the lemma is true for a chain of N strict covering families and let us prove it for $n = N + 1$. By induction we can refine $\mathscr{C}_N$ by a strict covering family $\mathscr{C}'_N \to \mathscr{C}_N$ such that the induced covering of (U, W) is a strict covering family. But strict covering families are stable by fiber product therefore by replacing $\mathscr{C}_N$ by $\mathscr{C}'_N$ and $\mathscr{C}_{N+1}$ by its base change $\mathscr{C}'_{N+1}$ via $\mathscr{C}'_N \to \mathscr{C}_N$ we are reduced again to the case $n = 2$. I.e. there is a refinement $\mathscr{C}''$ of $\mathscr{C}'_{N+1}$, which is strict such that the covering $\mathscr{C}'' \to (U, W)$ is strict. Therefore the covering $\mathscr{C}'' \to \mathscr{C}_{N+1}$ is a refinement (it is not necessarily strict) such that the family $\mathscr{C}'' \to (U, W)$ is a strict covering family. This proves the claim. □

The formal case. — The definition of the topology is treated in detail in §2 of [**3**]. We notice that in the formal setting the definition given in loc. cit. is the correct one. Contrary to 2.2 in the formal setting even if we work with the Zariski site of X instead of the étale site, the correct topology is not the one defined originally by Faltings. The analogues of Lemma 2.8 and of the fact that $\overline{\mathscr{O}}_{\mathfrak{X}_M}$ is a sheaf in the formal case, not proved in loc. cit., are similar to the ones in the algebraic case and are left to the reader.

3. Geometric points of $\mathfrak{X}_M$

Let us first point out that Proposition 4.4.2, 6) and 7) of [**2**] (both statements and proofs) are true if $M_0 = K$ and if we use the pointed site $\mathfrak{X}_M^{\bullet}$. Let us recall that M_0 is the completion of the maximal unramified extension of K in M. However, in general, (using notations as in the Proposition 4.4.2) the scheme $\mathrm{Spec}(\mathscr{O}^{\text{sh}}_{X,\hat{x}} \otimes_{\mathscr{O}_K} M)$ has $[M_0 : K]$ components which have to be accounted for. It is possible to refine the argument in [**2**] and repare that proof. Here we prefer to give a new and conceptually clearer proof of Proposition 4.4.2 for the site $\mathfrak{X}_M$ based on results in [**1**]. We will first refine the notion of "geometric point" of $\mathfrak{X}_M$.

Geometric points of $\mathfrak{X}_M$. — According to [**1**] a point of $\mathfrak{X}_M$ is simply a morphism of toposes Sets $\to \mathrm{Sh}(\mathfrak{X}_M)$. In this section we will give an explicit description of a particular class of points of $\mathfrak{X}_M$ arising from morphisms of sites $\mathfrak{X}_M \to$ Sets, which we call *geometric points*. We show that they are enough to separate sheaves (this will correct the proof in [**2**] in the algebraic setting.)

Definition 3.1. — We define a geometric point of $\mathfrak{X}_M$ to be a pair (x, y) where

a) x is a geometric point of X. We denote by $X_x := \mathrm{Spec}(\mathscr{O}^{\mathrm{sh}}_{X,x})$ i.e. the spectrum of the strict henselization of the local ring of X at x and by $X_{x,M} := \mathrm{Spec}(\mathscr{O}^{\mathrm{sh}}_{X,x} \otimes_{\mathscr{O}_K} M)$.

b) y is a geometric point of $X_{x,M}$. We may think of y as a geometric point of X_M which specializes to x in other words $y\colon \mathrm{Spec}(\Omega) \to X_x$ where Ω is an algebraically closed field containing M.

Given a geometric point (x, y) as above we denote by $G_{(x,y)} := \pi_1(X_{x,M}; y)$. The site of the finite and étale coverings of the connected component of $X_{x,M}$ containing y is equivalent via a functor which we denote $\rho_{(x,y)}$ to the site of finite $G_{(x,y)}$-sets, which we denote by $\mathrm{FSets}_{G_{(x,y)}}$.

We consider the functor $\iota_{(x,y)}\colon E_{X_M} \longrightarrow \mathrm{FSets}_{G_{(x,y)}}$ defined by

$$\iota_{(x,y)}(U, W) := \rho_{(x,y)}\big((X_{x,M} \times_{X_M} W)_y\big),$$

where the index y denotes the inverse image in $X_{x,M} \times_{X_M} W$ of the connected component of $X_{x,M}$ containing y. This functor (or rather its composition with the forgetful functor $\mathrm{FSets}_{G_{(x,y)}} \longrightarrow \mathrm{Sets}$) sends covering families to covering families, final objects to final objects and commutes with finite projective limits therefore defines a morphism of sites and so it induces a morphism of toposes from Sets to the topos associated to $\mathfrak{X}_M$ i. e., it defines a point.

Definition 3.2. — Let $\mathscr{F}$ be a presheaf on E_{X_M}, we denote by $\mathscr{F}_{(x,y)} := \iota^*_{(x,y)}(\mathscr{F})$ and call it the stalk of $\mathscr{F}$ at (x, y).

We'd like now to explicitly describe $\mathscr{F}_{(x,y)}$. Let I_x denote the category of pointed étale neighbourhoods of x in X i. e., the category of pairs (U, x') where $U \to X$ is an étale morphism and x' is a geometric point of U over x. We define $J_{(x,y)}$ to be the category of pairs $\big((U, x'), (W, y')\big)$ where

a) $i := (U, x')$ is an object of I_x,

b) (U, W) is an object of E_{X_M},

c) y' is a geometric point of W lifting the point $\mathrm{Spec}(\Omega) \xrightarrow{y} X_{x,M} \longrightarrow U_M$, where the last map above is the base change to M of the map $X_x \longrightarrow U$ induced by x'.

For every $i = (U, x') \in I_x$ we let $J_{i,y} \subset J_{(x,y)}$ denote the full subcategory of pairs of the form $\big((U, x'), (W, y')\big)$, i.e. with constant (U, x') first component. Given a morphism $i'' = (U'', x'') \to (U, x') = i$ we have a natural functor $J_i \to J_{i''}$ given by

$$\big((U, x'), (W, y')\big) \longrightarrow \big((U'', x''), (U'' \times_U W, (x'', y'))\big).$$

In this way $J_{(x,y)} \to I_x$ becomes a fibred category. For a presheaf $\mathscr{F}$ on E_{X_M} and $i = (U, x') \in I_x$ we denote by $\mathscr{F}_{i,y}$ the limit of $\mathscr{F}(U, W)$ for $\big((U, x'), (W, y')\big) \in J_{i,y}$ and let us observe ([**1**, IV.6.8.3]), that as a set $\mathscr{F}_{(x,y)}$ is the limit of $\mathscr{F}_{i,y}$ for $i \in I_x$. Then, $\mathscr{F}_{i,y}$ considered as a set with the discrete topology is endowed with a continuous action of $\pi_1(U_M; y)$ (where we think of y as a geometric point of U_M via the composition $\mathrm{Spec}(\Omega) \xrightarrow{y} X_{x,M} \xrightarrow{x'_M} U_M$). Given a morphism $i'' = (U'', x'') \to (U, x') = i$ in I_x we have a morphism $\mathscr{F}_{i,y} \to \mathscr{F}_{i'',y}$ compatible with the continuous group homomorphism

$\pi_1(U''_M; y) \to \pi_1(U_M; y)$ and therefore $\mathscr{F}_{(x,y)}$ is canonically endowed with a continuous action of $G_{(x,y)}$. We have used here that $G_{(x,y)} \cong \varprojlim_{i} \pi_1(U_M; y)$, where $i = (U, x') \in I_x$. In fact more is true namely if G is any finite group we have

$$(*) \quad \mathrm{Hom}(G_{(x,y)}, G) \cong \varinjlim_{i} \mathrm{Hom}(\pi_1(U_M; y), G).$$

Let $\mathscr{F}$ be a presheaf of abelian groups on E_{X_M} and (x, y) a geometric point of $\mathfrak{X}_M$.

Lemma 3.3. — *The group* $\mathrm{H}^n_{\mathrm{cont}}(G_{(x,y)}, \mathscr{F}_{(x,y)})$ *coincides with the direct limit over* $i = (U, x') \in I_x$ *of* $\mathrm{H}^n(\pi_1(U_M; y), \mathscr{F}_{i,y})$.

Proof. — For every $i = (U, x') \in I_x$ we have natural maps $\mathrm{H}^n(\pi_1(U_M; y), \mathscr{F}_{i,y}) \to \mathrm{H}^n_{\mathrm{cont}}(G_{(x,y)}, \mathscr{F}_{(x,y)})$ compatible with the transition maps, therefore we have a natural map

$$\varphi\colon \varinjlim_{i \in I_x} \mathrm{H}^n(\pi_1(U_M; y), \mathscr{F}_{i,y}) \longrightarrow \mathrm{H}^n_{\mathrm{cont}}(G_{(x,y)}, \mathscr{F}_{(x,y)}).$$

a) injectivity of φ. Let $i = (U, x') \in I_x$ and $g \in \mathrm{H}^n(\pi_1(U_M; y), \mathscr{F}_{i,y})$ such that $\varphi(g) = 0$. Then g is represented by a continuous cocycle $G\colon \pi_1(U_M; y)^n \to \mathscr{F}_{i,y}$ and $\varphi(g)$ is represented by the composition G':

$$G^n_{(x,y)} \longrightarrow \pi_1(U_M; y)^n \xrightarrow{G} \mathscr{F}_{i,y} \longrightarrow \mathscr{F}_{(x,y)}.$$

As $\varphi(g) = 0$, there is a continuous cochain $F\colon G^{n-1}_{(x,y)} \to \mathscr{F}_{(x,y)}$ such that $dF = G'$. As F is continuous, it factors via H^{n-1}, for H a finite quotient of $G_{(x,y)}$ and as the image $F(H^{n-1})$ is finite it ends up in the image of $\mathscr{F}_{i',y}$ for a certain $i'' = (U'', x'') \in I_x$. Shrinking U'' if necessary we may suppose that H is also a quotient of $\pi_1(U''_M; y)$. We obtain a continuous cochain $F'\colon \pi_1(U''_M; y)^{n-1} \to \mathscr{F}_{i'',y}$ such that $dF' = G|_{i'',y}$ and therefore the image of g in the inductive limit is 0.

b) surjectivity of φ. Let $f \in \mathrm{H}^n_{\mathrm{cont}}(G_{(x,y)}, \mathscr{F}_{(x,y)})$ and let $F\colon G^n_{(x,y)} \to \mathscr{F}_{(x,y)}$ denote a continuous cocycle defining it. As F is continuous, it factors via a finite quotient of $G_{(x,y)}$ and so it factors via a continuous cocycle $F'\colon \pi_1(U''_M; y) \to \mathscr{F}_{i'',y}$, where $i'' = (U'', x'') \in I_x$. The image f' of the cohomology class of F' in $\varinjlim_{i \in I_x} \mathrm{H}^n(\pi_1(U_M; y), \mathscr{F}_{i,y})$ has the property $\varphi(f') = f$. □

Proposition 3.4. — *Let* $\mathscr{F}, \mathscr{G}$ *be sheaves on* $\mathfrak{X}_M$ *and let* α, $\beta\colon \mathscr{F} \longrightarrow \mathscr{G}$ *be two morphisms of sheaves. Suppose that for all geometric points* (x, y) *of* $\mathfrak{X}_M$ *we have* $\alpha_{(x,y)} = \beta_{(x,y)}$. *Then* $\alpha = \beta$.

In other words the proposition states that the geometric points are "enough" points of $\mathfrak{X}_M$.

Proof. — Let (U, W) be an object of E_{X_M} and $s \in \mathscr{F}(U, W)$. We'd like to show that there is a covering $\{(U_i, W_i) \to (U, W)\}_i$ such that $\alpha(s)|_{(U_i, W_i)} = \beta(s)|_{(U_i, W_i)}$ for every i.

First if W is not irreducible and $\{W_a\}_a$ are its irreducible components the family $\{(U, W_a) \to (U, W)\}_a$ is a covering and we have $\mathscr{F}(U, W) \cong \prod_a \mathscr{F}(U, W_a)$ as $\mathscr{F}$

is a sheaf. So it is enough to prove the above property for every component s_a of s, in other words we may suppose that W is irreducible. Similarly we may assume without loss of generality that U is irreducible. Choose a geometric point x of U and a geometric point y' of W whose image y in U_M specializes to x (i.e. such that (x,y) is a geometric point of $\mathfrak{X}_M$). Then $\big((U,x),(W,y')\big) \in J_{(x,y)}$ and this induces a morphism $\mathscr{F}(U,W) \to \mathscr{F}_{(x,y)}$. By hypothesis the images of $\alpha(s)$ and $\beta(s)$ via this morphism coincide, thus there exists $\big((U_{(x,y')},x),(W_{(x,y')},y')\big) \in J_{(x,y)}$ such that the restrictions of $\alpha(s),\beta(s)$ to $(U_{(x,y')},W_{(x,y')})$ are equal. In fact using the finite family $\{(U_{(x,y')},W_{(x,y')}\}_{y'}$ one can manufacture a pair $\big((U^{(x)},x'),(W^{(x)},y')\big)$ such that

i) $(U^{(x)},W^{(x)}) \in E_{X_M}$ and we have a morphism $(U^{(x)},W^{(x)}) \to (U,W)$;

ii) $U^{(x)}$ and $W^{(x)}$ are connected, x' maps to x and y' maps to y.

iii) the images of $\alpha(s)$ and $\beta(s)$ in $\mathscr{G}(U^{(x)},W^{(x)})$ are equal.

It follows that the family $\{U^{(x)} \to U\}_x$ is a covering in X^{et}. Let us denote by A' (respectively B') the integral closure of $\mathscr{O}_K$ in U (respectively in $U^{(x)}$). In particular, $U \otimes_{A'} \overline{K}$ (resp. $U^{(x)} \otimes_{B'} \overline{K}$) is irreducible. Moreover, $A' \subset B'$ is an extension of finite extensions of $\mathscr{O}_K$. Possibly replacing $U^{(x)}$ with a connected component of the base change of $U^{(x)}$ to the normalization of $\mathscr{O}_K$ in a Galois closure of $A'_K \subset B'_K$, we may assume that B'_K/A'_K is finite and Galois with group denoted by G'. Let A (resp. B) be the subfield of the residue field of y (resp. y') defined as the composite of A' (resp. B') and M. Note that G' is the Galois group of $B' \otimes_{\mathscr{O}_K} M$ over $A' \otimes_{\mathscr{O}_K} M$ and A and B are direct factors of $A' \otimes_{\mathscr{O}_K} M$ (resp. $B' \otimes_{\mathscr{O}_K} M$). Then, $A \subset B$ is still Galois with group G which is the subgroup of G' fixing the direct factor $B \subset B' \otimes_{\mathscr{O}_K} M$. Choose $H := \{1,\sigma_1,\ldots,\sigma_n\} \subset G'$ so that $\big(B' \otimes_{\mathscr{O}_K} M\big) \otimes_{A' \otimes_{\mathscr{O}_K} M} A \cong \prod_{h\in H} h(B)$. Let $U_{1,M}$ (resp. $U^{(x)}_{1,M}$) be the connected component of U_M (resp. $U^{(x)}_M$) containing y (resp. y'). Then, $U_{1,M} = U_K \otimes_K M \otimes_{A'\otimes M} A$ and $U^{(x)}_{1,M} = U^{(x)}_K \otimes_K M \otimes_{B'\otimes M} B$. In particular, the connected components of $U^{(x)}_M$ over $U_{1,M}$ are

$$U^{(x)}_M \times_{U_M} U_{1,M} \cong \amalg_{h\in H} U^{(x)}_M \otimes_{B'\otimes_K M} h(B).$$

By construction $U^{(x)}_{1,M}$ is the image of $W^{(x)}$. For every $h \in H$ let $\rho_h \colon U^{(x)}_{h,M} \cong U^{(x)}_{1,M}$ be the isomorphism as $U_{1,M}$-schemes defined by h. Then $\rho_h^*(W^{(x)}) \to W \times_{U_M} U^{(x)}_{h,M}$ is surjective, so if we define $W_{(x)} := \amalg_{h\in H} \rho_h^*(W^{(x)})$, then $W_{(x)}$ covers $W \times_U U^{(x)}$, i.e. the pair $(U^{(x)},W_{(x)})$ is an object of E_{X_M} for every x and the family $\{(U^{(x)},W_{(x)}) \to (U,W)\}_x$ is a strict covering. Moreover, the images of $\alpha(s)$ and $\beta(s)$ in $\mathscr{G}(U^{(x)},W^{(x)})$ are equal. It follows that the images of $\alpha(s)$ and $\beta(s)$ in $\mathscr{G}(U^{(x)},W_{(x)})$ are equal for every x, and therefore we have $\alpha(s)=\beta(s)$. This ends the proof of the proposition. □

Morphisms of topoi. — Let us recall the functors defined in section §4.3 of **[2]** and add a few new details. We work in the algebraic setting, i.e. assume that X is a smooth scheme of finite type over $\mathscr{O}_K$.

a) $v = v_{X,M} \colon X^{\mathrm{et}} \longrightarrow E_{X_M}$ defined by $v(U) := (U, U_{\overline{K}})$. It sends covering families to covering families, final objects to final objects and commutes with fiber projects (hence with finite projective limits). In particular it defines a morphism of sites $v\colon X^{\mathrm{et}} \longrightarrow \mathfrak{X}_M$ and we denote by v_*, v^* the direct and inverse image functors on the respective topoi.

b) $\rho = \rho_{X,M}\colon X_M^{\mathrm{fet}} \longrightarrow E_{X_M}$ defined by $w(W) := (X, W)$. This functor sends covering families to covering families, final objects to final objects and commutes with finite projective limits. It induces therefore a morphism of sites $\rho\colon X_M^{\mathrm{fet}} \longrightarrow \mathfrak{X}_M$.

c) Let U be an open in X^{et} and let us denote by $\mathfrak{U}_M$ Faltings's site associated to the pair (U, M). Then $\mathfrak{U}_M = \mathfrak{X}_M|_{(U,U_M)}$. Let $\rho_U\colon U_M^{\mathrm{fet}} \longrightarrow \mathfrak{U}_M$ be the morphism of sites as above. Denote buy ρ_U^* and $\rho_{U,*}$ the induced morphism of toposes.

d) Using the same notations as at d) above we also have a natural morphism of sites

$$j_U\colon \mathfrak{X}_M \longrightarrow \mathfrak{U}_M, \text{ defined by } j_{U.M}(V, W) := (V \times_X U, W \times_X U).$$

Then $j_U^*\colon \mathrm{Sh}(\mathfrak{X}_M) \longrightarrow \mathrm{Sh}(\mathfrak{U}_M)$ admits an exact left adjoint given by the functor $j_{U,!}\colon \mathrm{Sh}(\mathfrak{U}_M) \longrightarrow \mathrm{Sh}(\mathfrak{X}_M)$ defined by extension by zero, i.e. more precisely

$$j_{U!}(\mathscr{F})(V, W) := \prod_\xi F(\xi(V, W)), \text{ where } \xi \in \mathrm{Hom}\big((V, W), (U, U_M)\big).$$

It follows that j_U^* sends injective sheaves to injective sheaves.

3.1. Explicit description of $\mathrm{R}^i v_*$. — Let $\mathscr{F}$ be a sheaf of abelian groups on $\mathfrak{X}_M$.

Lemma 3.5. — *For every U object of X^{et} and $n \geq 0$ there is a functorial homomorphism $f_{n,U}\colon \mathrm{H}^n(U_M^{\mathrm{fet}}, \rho_{U,*}(\mathscr{F})) \longrightarrow \mathrm{H}^0(U, \mathrm{R}^n v_*(\mathscr{F}))$. Moreover, if we denote by $\mathscr{H}^n_{\mathrm{Gal}}(\mathscr{F})$ the sheaf on X^{et} associated to the presheaf $U \mapsto \mathrm{H}^n(U_M^{\mathrm{fet}}, \rho_{U,*}(\mathscr{F}))$, then the family of morphisms $\{f_{n,U}\}_U$ defines a morphism of sheaves $f_n\colon \mathscr{H}^n_{\mathrm{Gal}}(\mathscr{F}) \longrightarrow \mathrm{R}^n v_*(\mathscr{F})$.*

Proof. — We have the following obvious equalities:

$$\mathrm{H}^0(U_M^{\mathrm{fet}}, \rho_{U,*}(\mathscr{F})) = \mathrm{H}^0(U, v_*(\mathscr{F})) = \mathrm{H}^0(\mathfrak{U}_M, j_U^*(\mathscr{F})).$$

Using the fact that j_U^* is an exact functor we obtain a Leray spectral sequence

$$\mathrm{H}^i(U_M^{\mathrm{fet}}, \mathrm{R}^j \rho_{U,*}(\mathscr{F})) \Longrightarrow \mathrm{H}^{i+j}(\mathfrak{U}_M, j_U^*(\mathscr{F}))$$

which induces for every $n \geq 0$ a morphism $g_{n,U}\colon \mathrm{H}^n(U_M^{\mathrm{fet}}, \rho_{U,*}(\mathscr{F})) \longrightarrow \mathrm{H}^n(\mathfrak{U}_M, j_U^*(\mathscr{F}))$.

Let us denote by $T^n(\mathscr{F})$ the sheaf associated to the presheaf on X^{et}, $U \mapsto \mathrm{H}^n(\mathfrak{U}_M, j_U^*(\mathscr{F}))$. Then the family of functors $\{T^n\}_n$ is a family of delta-functors which is universal. The property of the family being delta-functors follows from the fact that j_U^* is exact. The universality of the family follows from the fact that if $\mathscr{G}$ is an injective sheaf on $\mathfrak{X}_M$ then $j_U^*(\mathscr{G})$ is an injective sheaf on $\mathfrak{U}_M$ and therefore $T^n(\mathscr{G}) = 0$ for all $n \geq 1$.

For every sheaf $\mathscr{F}$ of abelian groups on $\mathfrak{X}_M$ we also have a natural isomorphism $v_*(\mathscr{F}) \cong T^0(\mathscr{F})$ and because the family of functors $\{\mathrm{R}^n v_*\}_n$ is a universal family of delta-functors, we have canonical isomorphisms $\mathrm{R}^n v_*(\mathscr{F}) \cong T^n(\mathscr{F})$ for every $n \geq 0$.

So finally we define $f_{n,U}$ by the composition

$$\mathrm{H}^n(U_M^{\mathrm{fet}}, \rho_{U,*}(\mathscr{F})) \xrightarrow{g_{n,U}} \mathrm{H}^n(\mathfrak{U}_M, j_U^*(\mathscr{F})) \longrightarrow \mathrm{H}^0(U, T^n(\mathscr{F})) \cong \mathrm{H}^0(U, \mathrm{R}^n v_*(\mathscr{F})). \quad \square$$

Theorem 3.6. — *For every sheaf of abelian groups $\mathscr{F}$ on $\mathfrak{X}_M$ the morphism of sheaves*

$$f_n \colon \mathscr{H}^n_{\mathrm{Gal}}(\mathscr{F}) \longrightarrow \mathrm{R}^n v_*(\mathscr{F})$$

defined in lemma 3.5 is an isomorphism of abelian sheaves on X^{et} for every $n \geq 0$.

Proof. — Let x be a geometric point of X. We will prove that f_n induces an isomorphism of stalks: $\big(\mathscr{H}^n_{\mathrm{Gal}}(\mathscr{F})\big)_x \cong \big(\mathrm{R}^n v_*(\mathscr{F})\big)_x$. For this let I_x denote the filtered category of pointed étale neighbourhoods (U, x') of x in X^{et} which as schemes are affine. Let us remark that I_x is a co-filtering direct system. We will define three coherent, filtered sites and toposes over I_x and consider their projective limit (see chapter VI of [**1**] for the definitions and main properties of these notions).

• The fibred site Et $\longrightarrow I_x$ (see §VI.7.2.1 of [**1**]). For every $(U, x') \in I_x$ we define $\mathrm{ET}|_{(U,x')} := U^{\mathrm{et}}$ to be the étale site of U. If $h \colon (U, x') \to (V, x'')$ is a morphism in I_x then define the functor $h^* \colon \mathrm{ET}|_{(V,x'')} \longrightarrow \mathrm{ET}_{(U,x')}$ to be the base change functor, i.e. $h^*(Z \to V) := (Z \times_V U \to U)$. This functor sends covering families to covering families, commutes with finite projective limits and sends final objects to final objects. In particular it defines a morphism of sites [**1**, IV.4.9.2]. One checks that it induces a fibred topos which we denote by Sh(ET).

• The fibred site $\mathrm{FET}_M \longrightarrow I_x$. For every (U, x') in I_x we define $\mathrm{FET}_M|_{(U,x')} := U_M^{\mathrm{fet}}$ finite étale site of U_M. If $h \colon (U, x') \to (V, x'')$ is a morphism in I_x we define the functor $h^* \colon \mathrm{FET}_M|_{(V,x'')} \longrightarrow \mathrm{FET}_M|_{(U,x')}$ to be the base change induced by $h_M \colon U_M \to V_M$, i.e. $h^*(W \to V_M) := (W \times_V U \to U_M)$. This functor sends covering families to covering families, commutes with finite projective limits and sends final objects to final objects. In particular it defines a morphism of sites. It defines a fibred topos denoted $\mathrm{Sh}(\mathrm{FET}_M)$.

• The fibred site $\overline{\mathfrak{X}}_M \longrightarrow I_x$. For every (U, x') is I_x we define $\overline{\mathfrak{X}}_M|_{(U,x')} := \mathfrak{X}_M|_{(U,U_M)} = \mathfrak{U}_M$, i.e. Faltings's site associated to U. If $h \colon (U, x') \to (V, x'')$ is a morphism in I_x we define the functor $h^* \colon \mathfrak{X}_M|_{(V,x'')} \longrightarrow \overline{\mathfrak{X}}_M|_{(U,x')}$ by base change, i.e. $h^*\big((Z, W) \to (V, V_M)\big) := \big(Z \times_V U, W \times_V U) \to (U.U_M)\big)$. As before h^* defines a morphism of sites and we denote by $\mathrm{Sh}(\overline{\mathfrak{X}}_M)$ the associated fibred topos.

The main properties of these fibred sites and toposes are summarized in the next lemma.

Lemma 3.7. — a) *The toposes Sh*(ET)*, Sh*(FET_M) *and Sh*$(\overline{\mathfrak{X}}_M)$ *are coherent.*

b) *There are natural coherent morphisms of fibred sites $v^* \colon \mathrm{ET} \longrightarrow \overline{\mathfrak{X}}_M$ and $\rho^* \colon \mathrm{FET}_M \longrightarrow \overline{\mathfrak{X}}_M$.*

Proof. — a) Let us observe that for every (U, x') in I_x given an object of either $\mathrm{ET}|_{(U,x')}$, $\mathrm{FET}_M|_{(U,x')}$ or $\overline{\mathfrak{X}}_M|_{(U,x')}$ we can extract for a covering family of that object a finite subfamily which is still a covering. Moreover all sites admit final objects, therefore following [**1**, VI.2.4.1] the associated toposes are coherent.

b) We define the morphism $v^*\colon \mathrm{ET} \longrightarrow \overline{\mathfrak{X}}_M$ as follows (see [**1**, VI.7.2.2]). For every (U,x') in I_x define $v^*|_{(U,x')}\colon \mathrm{ET}_{(U,x')} \longrightarrow \overline{\mathfrak{X}}_M|_{(U,x')}$ by $v^*|_{(U,x')}(V\to U) := ((V,V_M)\to(U,U_M))$. This is a morphism of sites and in fact a morphism of fibred sites as if h is a morphism in I_x then $v^*\circ h^* = h^*$. In particular, v^* induces a morphism of fibred toposes which is coherent.

We also define a morphism $\rho^*\colon \mathrm{FET}_M \longrightarrow \overline{\mathfrak{X}}_M$ of fibred sites as follows. Let (U,x') be an object of I_x, then $\rho^*|_{(U,x')}\colon \mathrm{FET}_M|_{(U,x')} \longrightarrow \overline{\mathfrak{X}}_M|_{(U,x')}$ is defined by $\rho^*|_{(U,x')}(W\to U_M) := ((U,W)\to(U,U_M))$. It induces a morphism of sites and so ρ^* is a morphism of fibred sites which induces a coherent morphism of fibred toposes. □

Now, by [**1**, Thm. VI.8.2.3], since I_x is a co-filtering direct system, the projective limits of toposes $\varprojlim \mathrm{Sh}(\overline{\mathfrak{X}}_M), \varprojlim \mathrm{Sh}(\mathrm{ET}), \varprojlim \mathrm{Sh}(\mathrm{FET}_M)$ exist. Moreover, as I_x consists of affine schemes, by [**1**, Thm. VII.5.2] it follows that $\varprojlim \mathrm{Sh}(\mathrm{ET})$ is the étale topos associated to the scheme $\lim\limits_{U\in I_x} U = X_x$, (the spectrum of the strict henselization of X at x). In particular the associated topos is the topos of the point x. Let

$$\varprojlim v\colon \varprojlim \mathrm{Sh}(\overline{\mathfrak{X}}_M) \longrightarrow \varprojlim \mathrm{Sh}(\mathrm{ET}) \text{ and } \varprojlim \rho\colon \varprojlim \mathrm{Sh}(\overline{\mathfrak{X}}_M) \longrightarrow \varprojlim \mathrm{Sh}(\mathrm{FET}_M)$$

be the induced morphisms on the projective limits of toposes [**1**, VI.8.1.4]. Let $\mathscr{F} \in \mathrm{Sh}(\mathfrak{X}_M)$ be our abelian sheaf. Then for every U in I_x we denote $j_U^*(\mathscr{F})$ its restriction to $\mathfrak{U}_M$. The family of these define a sheaf $\mathscr{F}' \in \mathrm{Sh}(\overline{\mathfrak{X}}_M)$ [**1**, VI.7.4.4]. From the lemma 3.7, [**1**, VI.8.7.1, VI.8.7.3] we deduce that we have

$$\mathrm{R}^n(\varprojlim v)(\mathscr{F}') \cong \varprojlim \mathrm{R}^n(v|_U)_*(j_U^*(\mathscr{F})) = \big(\mathrm{R}^n v_*(\mathscr{F})\big)_x.$$

We deduce from the Leray spectral sequence [**1**, Cor. VI.8.7.7] that $\mathrm{R}^n(\varprojlim v)(\mathscr{F}') = \mathrm{H}^n(\varprojlim \mathrm{Sh}(\overline{\mathfrak{X}}_M), \mathscr{F}')$. It follows from [**1**, VII.5.4] arguing as in [**1**, Lemma VII.5.6] that both $\varprojlim \mathrm{Sh}(\overline{\mathfrak{X}}_M)$ and $\varprojlim \mathrm{Sh}(\mathrm{FET}_M)$ coincide with the topos associated to the finite étale site of $X_{x,M}$ so that $\varprojlim \rho$ is an equivalence. In particular, $\mathrm{R}^n(\varprojlim \rho) = 0$ if $n \geq 1$. Therefore,

$$\big(\mathrm{R}^n v_*(\mathscr{F})\big)_x = \mathrm{R}^n(\varprojlim) v(\mathscr{F}') = \mathrm{H}^n(\varprojlim \mathrm{Sh}(\overline{\mathfrak{X}}_M), \mathscr{F}') = \varprojlim \mathrm{H}^n\big((U_M)^{\mathrm{fet}}, j_U^*(\mathscr{F})\big) = \big(\mathscr{H}^n_{\mathrm{Gal}}(\mathscr{F})\big)_x. \quad \square$$

This implies the following description of $\big(\mathrm{R}^n v_*(\mathscr{F})\big)_x$:

Corollary 3.8. — *We have a natural isomorphism* $\big(\mathrm{R}^n v_*(\mathscr{F})\big)_x \cong \mathrm{H}^n\big((X_{x,M})^{\mathrm{fet}}, \mathscr{F}|_{X_{x,M}}\big)$.

Assume that K is totally ramified in M so that $M_0 = K$. Let (x,y) be a geometric point of $\mathfrak{X}_M$. Thanks to 3.3 we have $\big(\mathrm{R}^n v_*(\mathscr{F})\big)_x \cong \mathrm{H}^n_{\mathrm{cont}}(G_{(x,y)}, \mathscr{F}_{(x,y)})$ which is what was claimed in [**2**].

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