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FAMILIES OF AUTOMORPHIC FORMS ON DEFINITE QUATERNION ALGEBRAS AND TEITELBAUM'S CONJECTURE

by

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Abstract. — The main goal of this note is to describe a new proof of the "exceptional zero conjecture" of Mazur, Tate and Teitelbaum. This proof relies on Teitelbaum's approach to the $\mathscr{L}$-invariant based on the Cerednik-Drinfeld theory of p-adic uniformisation of Shimura curves.

Résumé **(Familles de formes automorphes sur les algèbres quaternioniques et conjecture de Teitelbaum)**

Cet article fournit une nouvelle démonstration de la conjecture de Mazur, Tate et Teitelbaum sur les « zéros exceptionnels » des fonctions L p-adiques. Cette démonstration repose sur une définition de l'invariant $\mathscr{L}$ proposée par Teitelbaum, qui repose sur la théorie de l'uniformisation p-adique des courbes de Shimura.

Introduction

Let $f = \sum a_n q^n$ be a newform of even weight $k_0+2 \geq 2$ on $\Gamma_0(Np)$, where $N \geq 4$ is a positive integer and p is a prime which does not divide N. We denote by $L(f,s)$ the complex L-function attached to f, and by $L(f,\chi,s)$ its twist by a Dirichlet character χ. A theorem of Shimura asserts the existence of a complex period Ω_f such that the special values

$$L(f,\chi,j)/\Omega_f \quad \text{with } 1 \leq j \leq k_0+1$$

belong to the subfield K_f of $\mathbb{C}$ generated by the Fourier coefficients of f, and even to its ring of integers. These special values (when χ ranges over the Dirichlet characters of p-power conductor) can be interpolated p-adically, yielding the Mazur-Swinnerton-Dyer p-adic L-function $L_p(f,s)$, a p-adic analytic function whose definition depends on the choice of Ω_f. Denote by

$$L^*(f,\chi,1+k_0/2) := L(f,\chi,1+k_0/2)/\Omega_f,$$

the *algebraic part* of $L(f,\chi,s)$ at the central critical point $s = 1+k_0/2$.

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The modular form f is said to be *split multiplicative* if

$$f|U_p = p^{k_0/2} f.$$

In that case, $L_p(f,s)$ has a so-called *exceptional zero* at $s = 1+k_0/2$ arising from the p-adic interpolation process. In fact, like its classical counterpart, the p-adic L-function $L_p(f,s)$ has a functional equation of the form

$$L_p(f, k_0+2-s) = \epsilon_p(f)\langle N\rangle^{s-1-k_0/2} L_p(f,s), \tag{1}$$

and the sign $\epsilon_p(f) = \pm 1$ that appears in this equation is related to the the sign $\epsilon_\infty(f)$ in the classical functional equation for $L(f,s)$ by the rule

$$\epsilon_p(f) = \begin{cases} -\epsilon_\infty(f) & \text{if } f \text{ is split multiplicative;} \\ \epsilon_\infty(f) & \text{otherwise.} \end{cases}$$

In the case where f is a split multiplicative newform, Mazur, Tate and Teitelbaum made the following conjecture in [**18**]:

Conjecture 1. — *There exists a constant $\mathscr{L}(f) \in \mathbb{C}_p$, which depends only on the restriction of the Galois representation attached to f to a decomposition group at p, and such that*

$$L_p'(f,\chi,1+k_0/2) = \mathscr{L}(f) L^*(f,\chi,1+k_0/2), \tag{2}$$

for all χ with $\chi(-1) = \chi(p) = 1$.

The constant $\mathscr{L}(f)$, which Mazur, Tate and Teitelbaum called the *L-invariant*, was only defined in [**18**] in the weight two case $k_0 = 0$. In the higher weight case $k_0 > 0$, several a priori inequivalent definitions of $\mathscr{L}(f)$ were subsequently proposed.

1. In [**23**], Teitelbaum offered the first definition for $\mathscr{L}(f)$. This invariant, denoted $\mathscr{L}_T(f)$, is based on the Cerednik-Drinfeld theory of p-adic uniformisation of Shimura curves and is only defined for modular forms which are the Jacquet-Langlands lift of a modular form on a Shimura curve uniformized by Drinfeld's p-adic upper half plane. This occurs, for example, when the conductor of f can be written as a product of three pairwise relatively prime integers of the form

$$pN = pN^+N^-,$$

where N^- is the square-free product of an *odd* number of prime factors. A modular form which satisfies this condition will be said to be *p-adically uniformisable.*
2. Coleman [**5**] then proposed an analogous but more general invariant $\mathscr{L}_C(f)$ by working directly with p-adic integration on the modular curve attached to the group $\Gamma_0(p) \cap \Gamma_1(N)$.
3. Fontaine and Mazur [**17**] gave a definition for the so-called *Fontaine-Mazur $\mathscr{L}$-invariant* $\mathscr{L}_{FM}(f)$ in terms of the filtered, Frobenius monodromy module of the p-adic Galois representation attached to f.

4. In **[19]**, Orton has introduced yet a further $\mathscr{L}$-invariant $\mathscr{L}_O(f)$, based on the group cohomology of arithmetic subgroups of $\mathbf{GL}_2(\mathbb{Z}[1/p])$, extending to forms of higher weight the approach taken in **[12]** for $k_0 = 0$.
5. Finally Breuil defined in **[2]** the $\mathscr{L}$-invariant $\mathscr{L}_{Br}(f)$ in terms of the p-adic representation of $\mathbf{GL}_2(\mathbb{Q}_p)$ attached by him to f.

We now know that all the above $\mathscr{L}$-invariants are equal (when they are defined) as result of work of many people, which we briefly list below (see **[9]** for a more detailed account of these various articles and preprints).

The equality of the $\mathscr{L}$-invariants $\mathscr{L}_C(f)$ and $\mathscr{L}_{FM}(f)$ was proved in **[7]** by making explicit the comparison isomorphism between the p-adic étale cohomology and log-crystalline cohomology of the modular curve $X_0(Np)$ with respective coefficients. The equality of $\mathscr{L}_T(f)$ and $\mathscr{L}_C(f)$ (when they are both defined) was proved in **[16]** by interpreting $\mathscr{L}_T(f)$ as the $\mathscr{L}$-invariant of a filtered, Frobenius monodromy module. Breuil proved in **[2]** the equality $\mathscr{L}_{Br}(f) = \mathscr{L}_O(f)$, which is a manifestation of the local-global compatibility for the p-adic Langlands correspondence.

It was first observed by Greenberg and Stevens for weight two (in **[15]**) and by Stevens in general (in **[22]**) that p-adic deformations of f, i.e. p-adic families of modular eigenforms are relevant for conjecture (1). To describe these objects precisely, let

$$\mathscr{W} := \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{Q}_p^\times)$$

denote the weight space, viewed as the $\mathbb{Q}_p$-points of a rigid analytic space. There is a natural inclusion $\mathbb{Z} \subset \mathscr{W}$ by sending k to the function $x \mapsto x^k$. Write $A(U)$ for the ring of rigid analytic functions on U, for any affinoid disk $U \subset \mathscr{W}$.

A *p-adic family of eigenforms* interpolating f is the data of a disk U with $k_0 \in U$, and of a formal q-expansion

$$f_\infty = \sum_{n=1}^{\infty} a_n q^n, \tag{3}$$

with coefficients in $A(U)$ satisfying:

1. For every $k \in U \cap \mathbb{Z}^{\geq 0}$,
$$f_k := \sum_{n=1}^{\infty} a_n(k) q^n$$
is the q-expansion of a normalized eigenform of weight $k+2$ on the congruence group $\Gamma_1(p) \cap \Gamma_0(N)$;
2. $f_{k_0} = f$.

The existence and essential uniqueness of the family f_∞ interpolating f is proved in **[6]**.

Greenberg and Stevens for weight two and Stevens in general first proved that $\mathscr{L}_C(f) = -2(\mathrm{dlog} a_p)_{\kappa=k_0}$. Colmez generalized the Galois cohomology calculations in **[15]** by working inside Fontaine's rings and proved the equality $\mathscr{L}_{FM}(f) = -2(\mathrm{dlog} a_p)_{\kappa=k_0}$ in **[11]**. He also proved the equality $\mathscr{L}_{Br}(f) = -2(\mathrm{dlog} a_p)_{\kappa=k_0}$ in

[**10**] by using the p-adic local Langlands correspondence for trianguline representations. Let us remark that in fact the quantity $\mathscr{L}(f)_{NoName} := -2(\mathrm{dlog} a_p)_{\kappa=k_0}$ behaves like an $\mathscr{L}$-invariant: it satisfies the equation (2) of conjecture 1 (see [**22**]) and it is a local invariant of f in the sense that it is invariant to twists of f by Dirichlet characters trivial at p (in fact it is invariant to all twists by Dirichlet characters.)

Conjecture (1) was first proved in [**15**] for weight two, and several different proofs have been announced in the higher weight case:

1. By Kato-Kurihara-Tsuji, working with the invariant $\mathscr{L}_{FM}(f)$;
2. By Glenn Stevens, working with $\mathscr{L}_C(f)$;
3. By Orton, working with $\mathscr{L}_O(f)$ in [**19**];
4. By Emerton working with $\mathscr{L}_{Br}(f)$ in [**14**].

The first two proofs are still unpublished but an account of the approach of Kato-Kurihara-Tsuji can be found in [**8**] while Stevens gave a series of lectures on his theory during the *Automorphic Forms* semester in Paris, 1998. Notes to these lectures, to which we will refer as [**22**], although not yet published circulated widely in the mathematical community and greatly influenced articles like [**3**], [**4**] and the present note. As these notes have not been published we will sketch proofs of all the results quoted from them.

The main goal of this note is to describe a new proof of Conjecture 1 which applies to forms which are p-adically uniformisable.

Theorem 2. — *Assume that f is p-adically uniformisable. Then*

$$L_p'(f,\chi,1+k_0/2) = \mathscr{L}_T(f)L^*(f,\chi,1+k_0/2), \tag{4}$$

for all Dirichlet characters χ satisfying $\chi(-1)=\chi(p)=1$.

Our proof of Theorem 2 is based on Teitelbaum's definition of the L-invariant: this is why it needs to be assumed that f is p-adically uniformisable. Thus the Ceradnik-Drinfeld theory of p-adic uniformisation of Shimura curves and the Jacquet-Langlands correspondence, which play no role in the earlier proofs of Stevens and Kato-Kurihara-Tsuji, are key ingredients in our approach. Section 1 supplies the necessary definitions concerning automorphic forms on definite quaternion algebras, and Section 2 recalls a few basic facts concerning p-adic integration on Shimura curves, including Teitelbaum's theory of the "p-adic Poisson kernel" and his definition of the invariant $\mathscr{L}_T(f)$.

Guided by the Jacquet-Langlands correspondence between classical modular forms and automorphic forms on quaternion algebras, Section 3 describes a theory of p-adic families of automorphic forms on definite quaternion algebras, based on ideas of Stevens, Buzzard and Chenevier. The resulting structures are used to prove the following theorem in Section 4, which relates Teitelbaum's L-invariant to the derivative of the Fourier coefficient $a_p(k)$ with respect to k.

Theorem 3. — *Suppose that f is p-adically uniformisable. Then*

$$\mathscr{L}_T(f) = -2\mathrm{dlog}(a_p)_{\kappa=k_0}. \tag{5}$$

The ideas of Orton in [**19**], which are recalled in Section 5, make it apparent that the definition of the invariants $\mathscr{L}_T(f)$ and $\mathscr{L}_O(f)$ are very similar in flavour. The calculations of Sections 1 to 4, when transposed to the context of a modular form on $\mathbf{GL}_2(\mathbb{Q})$, with the "integration on $\mathscr{H}_p \times \mathscr{H}$" defined in terms of modular symbols playing the role of the p-adic line integrals on Drinfeld's upper half-plane, leads to the proof of the following analogue of Theorem 3, which is described in Section 6:

Theorem 4. — *Let f be a modular form of weight k on $\Gamma_0(N)$ which is split multiplicative at p. Then*

$$\mathscr{L}_O(f) = -2\mathrm{dlog}(a_p)_{\kappa=k_0}. \tag{6}$$

Theorem 2 now follows directly from Theorems 3 and 4, in light of Orton's proof of Conjecture (1).

The remainder of the text will focus on explaining the proofs of Theorems 3 and 4, which are independent (both in their statement, and their formulation) of the existence and basic properties of either the p-adic L-function or the p-adic Galois representation attached to f and the Coleman family interpolating it.

We emphasize that the proof of Theorem 2 owes much to the ideas that are already present in the earlier (although still unpublished) approaches of Stevens and Kato-Kurihara-Tsuji. The main virtues (and drawbacks) of our method are inherently the same as those in Teitelbaum's approach to defining the $\mathscr{L}$-invariant: a gain in simplicity (because the method involves p-adic integration on a Mumford curve rather than a modular curve, and requires no information about Galois representations) offset by a certain loss of generality (since the method only applies to automorphic forms that can be obtained as the Jacquet-Langlands lift of a modular form on a p-adically uniformized Shimura curve). A second, less immediately apparent advantage of our approach lies in the insights arising from the connection that is drawn between the two-variable p-adic L-function $L_p(k,s)$ attached to f_∞ and the p-adic uniformisation of Shimura curves. In particular, the new ideas introduced in this article form the basis for the proof of the main result of [**1**], which, in the case where f corresponds to a modular elliptic curve E over $\mathbb{Q}$ and $\epsilon_\infty(f) = -\epsilon_p(f) = -1$, relates the leading term of $L_p(k,s)$ at the central critical point $(k,s) = (2,1)$ to the formal group logarithm of a global point on $E(\mathbb{Q})$.

1. Automorphic forms on quaternion algebras

Suppose from now on that f is p-adically uniformisable, so that its level pN can be factored as

$$pN = pN^+N^-, \text{ where } \gcd(N^+, N^-) = 1, \tag{7}$$

and where N^- is square-free and has an *odd* number of prime factors. Let B denote the quaternion algebra over $\mathbb{Q}$ ramified exactly at $N^-\infty$, and let $\mathscr{R}$ denote a maximal order in B. For each ℓ not dividing N^- we fix an isomorphism

$$\iota_\ell : B \otimes \mathbb{Q}_\ell \cong M_2(\mathbb{Q}_\ell), \text{ with } \iota_\ell(\mathscr{R} \otimes \mathbb{Z}_\ell) = M_2(\mathbb{Z}_\ell).$$

Let $\widehat{\mathbb{Z}}$ denote the profinite completion of $\mathbb{Z}$ and let $\hat{B} := B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$.

Let $\Sigma = \prod_\ell \Sigma_\ell$ be any compact open subgroup of $\hat{B}^\times$, and let V be any $\mathbb{Q}_p$-vector space equipped with a right action by Σ_p. The following definition is taken from Section 4 of [**3**].

Definition 1.1. — A V-valued automorphic form on B of level Σ is a function

$$\varphi : \hat{B}^\times \longrightarrow V \quad \text{satisfying} \quad \varphi(bs\sigma) = \varphi(s)\sigma_p, \tag{8}$$

for all $b \in B^\times$, $s \in \hat{B}^\times$, and $\sigma \in \Sigma$, where σ_p denotes the component of σ at p.

The space of all V-valued automorphic forms on B of level Σ will be denoted $S(\Sigma, V)$. It is equipped with the action of Hecke operators T_ℓ with $\ell \not| N$ as well as the operator U_p, defined as in [**3**], section 4.

Let

$$\tilde{\Gamma} = \iota_p \left(\mathscr{R}[1/p]^\times \cap \prod_{\ell \neq p} \Sigma_\ell \right),$$

and let Γ denote the subgroup of $\tilde{\Gamma}$ of elements of determinant 1. The strong approximation theorem for B asserts that

$$\hat{B}^\times = B^\times \mathbf{GL}_2(\mathbb{Q}_p)\Sigma,$$

so that we may write

$$S(\Sigma, V) = \{\varphi : \mathbf{GL}_2(\mathbb{Q}_p) \longrightarrow V \quad | \quad \varphi(\gamma g u) = \varphi(g)u\} \tag{9}$$

for all $\gamma \in \tilde{\Gamma}$, $g \in \mathbf{GL}_2(\mathbb{Q}_p)$, and $u \in \Sigma_p$.

We will be mostly interested in a specific choice of level structure Σ. Let $\Sigma(N,p) := \prod_\ell \Sigma_\ell \subset \hat{B}^\times$ be the compact open subgroup defined by

- $\Sigma_p = \iota_p^{-1}(\Gamma_0(p\mathbb{Z}_p))$;
- $\Sigma_\ell = (\mathscr{R} \otimes \mathbb{Z}_\ell)^\times$, if ℓ divides N^-;
- $\Sigma_\ell = \iota_\ell^{-1}(\Gamma_1(N\mathbb{Z}_\ell))$ if ℓ divides N^+;
- $\Sigma_\ell = (\mathscr{R} \otimes \mathbb{Z}_\ell)^\times$, otherwise.

The group $\Sigma(N,1)$ is defined in a similar way, with $\Gamma_0(p\mathbb{Z}_p)$ replaced by $\mathbf{GL}_2(\mathbb{Z}_p)$ in the definition of Σ_p.

Weights. If k is a positive integer, let $\mathscr{P}_k := \mathbb{Q}_p[z]^{deg \leq k}$ be the space of polynomials of degree $\leq k$, equipped with the right action of $\mathbf{GL}_2(\mathbb{Q}_p)$ given by

$$(P\beta)(z) = (cz+d)^k P\left(\frac{az+b}{cz+d}\right), \text{ for } \beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbb{Q}_p).$$

Let $V_k = \mathrm{Hom}_{\mathbb{Q}_p}(\mathscr{P}_k, \mathbb{Q}_p)$ denote its $\mathbb{Q}_p$-dual, equipped with the left action given by

$$(\beta h)(P) = h(P\beta) \text{ for } P \in \mathscr{P}_k \text{ and } h \in V_k.$$

We may also make V_k into a right $\mathbf{GL}_2(\mathbb{Q}_p)$-module by the rule

$$h\beta = \beta^{-1}h, \text{ for } h \in V_k \text{ and } \beta \in \mathbf{GL}_2(\mathbb{Q}_p).$$

The module V_k is isomorphic to $\mathscr{P}_k$ as a $\mathbf{GL}_2(\mathbb{Q}_p)$-module, and hence the following definition of the space of (classical) automorphic forms on B of weight $k+2$ and level $\Sigma(N,p)$ is equivalent to the one given in Section 4 of [**3**]:

$$S_{k+2}(N,p) := S(\Sigma(N,p), V_k).$$

Of crucial importance for our arguments is the Hecke operator U_p acting on the space $S_{k+2}(N,p)$, whose precise definition we now describe. Let α_1 be the matrix $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ and decompose the double coset space $\Sigma_p\alpha_1\Sigma_p$ as a disjoint union of left cosets:

$$\Sigma_p\alpha_1\Sigma_p = \cup_{j=1}^{p}\alpha_j\Sigma_p.$$

Then

$$(U_p\varphi)(g) = \det(\alpha_1)^{k/2}\sum_j \varphi(g\alpha_j)\alpha_j^{-1}.$$

It is useful to have a geometric interpretation of automorphic forms in terms of certain functions on the edges of the Bruhat-Tits tree $\mathscr{T}$ of $\mathbf{PGL}_2(\mathbb{Q}_p)$. Recall that $\mathscr{T}$ denotes the tree whose vertices are in bijection with the homothety classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$, two vertices being joined by an (unordered) edge if they admit representatives which are contained one in the other with index p. Let $\mathscr{T}_0$ and $\mathscr{T}_1$ denote the set of vertices and edges of $\mathscr{T}$ respectively, and let $\mathscr{E}(\mathscr{T})$ denote the set of *ordered edges* of $\mathscr{T}$, i.e., the set of ordered pairs of adjacent vertices. If $e = (v_s, v_t)$ is such an ordered edge, we will call the vertex $s(e) := v_s$ the *source* of e, and $t(e) := v_t$ its *target*. The edge $\bar{e} := (v_t, v_s)$ obtained from e by interchanging its source and target is called the edge *opposite* to e.

Let v_* be the vertex associated to the homothety class of the standard lattice $\mathbb{Z}_p^2$. The index p sublattices of $\mathbb{Z}_p^2$ are naturally in bijection with $\mathbb{P}^1(\mathbb{F}_p)$ by setting

$$L_j := \{(x,y) \in \mathbb{Z}_p^2 \text{ such that } [x:y] \equiv j \pmod{p}\}, \quad j = 0,1,\dots,p-1,\infty.$$

Let v_j be the vertex associated to the homothety class of L_j, and let

$$e_j = (v_*, v_j) \in \mathscr{E}(\mathscr{T}).$$

A vertex in $\mathscr{T}_0$ is said to be *even* or *odd* if its distance from v_* is even or odd. Likewise, an ordered edge in $\mathscr{E}(\mathscr{T})$ is even (resp. odd) if its source is even (resp. odd).

The groups $\mathbf{GL}_2(\mathbb{Q}_p)$ and $\mathbf{PGL}_2(\mathbb{Q}_p)$ act naturally on $\mathscr{T}$ via their left action on $\mathbb{Q}_p^2$, viewed as column vectors. The resulting actions of these groups on $\mathscr{T}_0$, $\mathscr{T}_1$, and $\mathscr{E}(\mathscr{T})$ are transitive, while the subgroup $\mathbf{P}\mathrm{SL}_2(\mathbb{Q}_p)$ preserves the even and odd elements in $\mathscr{T}_0$ and $\mathscr{E}(\mathscr{T})$. The stabilizer of v_* in $\mathbf{PGL}_2(\mathbb{Q}_p)$ is the group $\mathbf{PGL}_2(\mathbb{Z}_p)$, while the stabilizer of the ordered edge e_∞ is the projective image of the group $\Gamma_0(p\mathbb{Z}_p)$. Hence the assignment $g \mapsto ge_\infty$ identifies the quotient $\mathbf{PGL}_2(\mathbb{Q}_p)/\Gamma_0(p\mathbb{Z}_p)$ with $\mathscr{E}(\mathscr{T})$.

If $g \in \mathbf{GL}_2(\mathbb{Q}_p)$ we denote by $|g| := p^{\mathrm{ord}_p(\det(g))}$. To each $\eta \in S_{k+2}(N,p)$, viewed as a function on $\mathbf{GL}_2(\mathbb{Q}_p)$ via the description (9), is attached a V_k-valued, Γ-invariant function c_η on $\mathscr{E}(\mathscr{T})$ by setting, for all $e = ge_\infty$ with $g \in \mathbf{GL}_2(\mathbb{Q}_p)$, and for all $P \in \mathscr{P}_k$,

$$c_\eta(e)(P) := |g|^{-k/2}(g\eta(g))(P). \tag{10}$$

It is easy to see that the expression on the right of equation (10) depends only on the class of g in $\mathbf{PGL}_2(\mathbb{Q}_p)/\Gamma_0(p\mathbb{Z}_p)$, so that the value of c_η is well-defined. Moreover, if γ is any element of Γ, and $e = ge_\infty$ is any edge in $\mathscr{E}(\mathscr{T})$, we have

$$\begin{aligned} c_\eta(\gamma e)(P) &= |\gamma g|^{-k/2}(\gamma g\eta(\gamma g))(P) = \gamma(|g|^{-k/2}g\eta(g))(P) \\ &= (\gamma c_\eta(e))(P) = c_\eta(e)(P\gamma). \end{aligned}$$

Since η can be recovered from the datum of c_η, the assignment $\eta \mapsto c_\eta$ identifies $\eta \in S_{k+2}(N,p)$ with an element c_η in the space $\mathscr{C}(\mathscr{E}, V_k)^\Gamma$ of Γ-invariant V_k-valued functions on $\mathscr{E}(\mathscr{T})$. Let us spell out the action of the Hecke operator U_p which is deduced from this identification.

Lemma 1.2. — *For all $\eta \in S_{k+2}(N,p)$, we have*

$$(c_{U_p\eta})(e) = p^{k/2} \sum_{\substack{s(e')=t(e) \\ e' \neq \bar{e}}} c_\eta(e').$$

Proof. — This follows from a direct calculation. ☐

2. Teitelbaum's L-invariant

Let f be the normalized eigenform of weight k_0+2 on $\Gamma_0(N)$ that was discussed in the introduction. The definition of Teitelbaum's invariant $\mathscr{L}_T(f)$ rests crucially on the Jacquet-Langlands correspondence which associates to f an automorphic form on a definite quaternion algebra in the sense of the previous section.

Theorem 2.1. — *There exists an automorphic form $\phi \in S_{k_0+2}(N,p)$ which is an eigenform for the Hecke operators and satisfies*

$$\phi|T_\ell = a_\ell(k_0)\phi, \quad \text{for all } \ell \nmid Np, \qquad \phi|U_p = p^{k_0/2}\phi.$$

This ϕ is unique up to multiplication by a non-zero scalar in $\mathbb{C}_p^\times$.

Let $\phi \in S_{k_0+2}(N,p)$ be the modular form obtained from f via Theorem 2.1, and recall the Γ-equivariant V_k-valued function c_ϕ on $\mathscr{E}(\mathscr{T})$ that was associated to it in the previous section. A function c on $\mathscr{E}$ is called a *harmonic cocycle* if

$$c(\bar{e}) = -c(e), \quad \sum_{s(e)=v} c(e) = 0, \quad \text{for all } v \in \mathscr{T}_0.$$

Lemma 2.2. — *The function c_ϕ attached to ϕ is a V_{k_0}-valued harmonic cocycle on $\mathscr{T}$.*

Proof. — The fact that $c_\phi(\bar{e}) = -c_\phi(e)$ follows directly from the fact that the Atkin-Lehner involution W_p acts as multiplication by -1 on c_ϕ. Let v be any vertex of $\mathscr{E}$ and let e be an ordered edge of $\mathscr{T}$ satisfying $t(e) = v$. Since $c_\phi|U_p = p^{k_0/2}c_\phi$, it follows from the description of U_p given in Lemma 1.2 that

$$p^{k_0/2}c_\phi(e) = (c_\phi|U_p)(e) = p^{k_0/2} \sum_{\substack{s(e')=v \\ e'\neq\bar{e}}} c_\phi(e'),$$

so that

$$\sum_{s(e')=v} c_\phi(e') = 0$$

for all $e \in \mathscr{E}(\mathscr{T})$ and $v = t(e)$. □

We now explain how the cocycle c_ϕ gives rise to a *locally analytic distribution* on $\mathbb{P}^1(\mathbb{Q}_p)$, denoted μ_ϕ. To do this, let $W := \mathbb{Q}_p^2 - \{0\}$, equipped with its natural p-adic topology. There is a natural continuous projection

$$\pi : W \longrightarrow \mathbb{P}^1(\mathbb{Q}_p), \qquad \pi((x,y)) = x/y.$$

If L is any $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^2$, let $L' := L - pL$ be the compact open subset of W consisting of the primitive vectors in L. If $e = (s,t) \in \mathscr{E}(\mathscr{T})$ is an ordered edge of $\mathscr{T}$, let L_s and L_t denote $\mathbb{Z}_p$-lattices whose homothety classes correspond to the source and the target of e respectively, chosen in such a way that L_s contains L_t with index p. To the edge e are associated the subset $W_e \subset W$ and the compact open subset $U_e \subset \mathbb{P}^1(\mathbb{Q}_p)$ by the rules

$$W_e = L'_s \cap L'_t, \quad U_e = \pi(W_e).$$

Note that the set W_e depends on the choice of L_s and L_t, so that W_e is only well-defined (as a function of e) up to multiplication by elements of $\mathbb{Q}_p^\times$. The subset U_e, on the other hand, depends only on e and not on the choices of representative lattices L_s and L_t that were made to define it.

Let us now briefly recall some of the theory of locally analytic distributions. Let X be a compact open subset of $W \subset \mathbb{Q}_p^2$. For each integer $n \geq 0$, denote by $B[X, p^{-n}]$ the affinoid subdomain of $\mathbb{C}_p^2$ given by

$$B[X, p^{-n}] := \{z \in \mathbb{C}_p^2 \quad | \text{ there exists } x \in X \text{ with } |z - x| \leq p^{-n}\}.$$

The region $B[X, p^{-n}]$ is a finite disjoint union of closed polydisks of radius p^{-n} defined over $\mathbb{Q}_p$. Therefore $B[X, p^{-n}]$ is also defined over $\mathbb{Q}_p$. Let $A_n(X)$ denote the $\mathbb{Q}_p$-affinoid algebra of $B[X, p^{-n}]$. It is a Banach algebra over $\mathbb{Q}_p$ under the spectral norm,

$$\|h\|_{A_n(X)} := \sup_{z\in B[X,p^{-n}]} |h(z)|.$$

If $m \geq n \geq 0$, restriction defines a continuous map

$$A_n(X) \longrightarrow A_m(X).$$

The direct limit

$$\mathscr{A}(X,\mathbb{Q}_p) := \lim_{\rightarrow,n} A_n(X)$$

is called the space of *locally analytic functions on* X. It is endowed with the inductive limit of the Banach topologies on each of the $A_n(X)$'s. Let

$$D_n(X) := \mathrm{Hom}_{\mathrm{cont}}(A_n(X),\mathbb{Q}_p)$$

denote the $\mathbb{Q}_p$-Banach-dual to $A_n(X)$ and let

$$\mathscr{D}(X,\mathbb{Q}_p) := \lim_{\leftarrow,n} D_n(X) = \mathrm{Hom}_{\mathrm{cont}}(\mathscr{A}(X,\mathbb{Q}_p),\mathbb{Q}_p).$$

This space, endowed with the projective limit of the Banach topologies of the $D_n(X)$'s, is called the space of *locally analytic distributions on* X. It is a Fréchet space over $\mathbb{Q}_p$.

These definitions can be extended without difficulty to the case where X is a compact open subset of the projective space $\mathbb{P}^1(\mathbb{Q}_p)$. (see **[22]**.)

Following the approach described in **[23]**, the harmonic cocycle c_ϕ can be used to define a locally analytic distribution μ_ϕ on $\mathbb{P}^1(\mathbb{Q}_p)$, determined by the property:

$$\int_{U_e} P(t)\mu_\phi(t) = c_\phi(e)(P), \tag{11}$$

for all $e \in \mathscr{E}(\mathscr{T})$ and $P \in \mathscr{P}_{k_0}$.

Let $\mathscr{H}_p := \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p)$ denote the p-adic upper half-plane. In **[23]**, the distribution μ_ϕ is used to define a rigid analytic function

$$\psi = \psi_f : \mathscr{H}_p \longrightarrow \mathbb{C}_p$$

by the rule

$$\psi(z) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \left(\frac{1}{t-z}\right) d\mu_\phi(t). \tag{12}$$

By Theorem 3 of **[23]**, the function ψ is a rigid analytic modular form on $\Gamma\backslash\mathscr{H}_p$ of weight k_0+2, i.e., it satisfies the relation

$$\psi(\gamma z) = (cz+d)^{k_0+2}\psi(z), \qquad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

The p-adic Coleman line integral attached to ψ, a polynomial $P \in V_{k_0}$, and two endpoints τ_1 and $\tau_2 \in \mathscr{H}_p$ is defined in terms of the distribution μ_ϕ by the rule

$$\int_{\tau_1}^{\tau_2} \psi(z)P(z)dz := \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log\left(\frac{t-\tau_2}{t-\tau_1}\right) P(t)\mu_\phi(t). \tag{13}$$

This formula can be used as a definition for the Coleman line integral in this setting, in light of Teitelbaum's theory of the "p-adic Poisson kernel". (See **[23]** for a more complete discussion.) In particular, it satisfies the additivity properties suggested by the line integral notation.

Let us now fix base points $v_0 \in \mathscr{T}_0$ and $z_0 \in \mathscr{H}_p$. (For example, one could take $v_0 = v_*$, but this is not necessary.) The harmonic cocycle c_ϕ gives rise (after extending scalars from V_{k_0} to $V_{k_0} \otimes \mathbb{C}_p$) to a $V_{k_0} \otimes \mathbb{C}_p$-valued one-cocycle on Γ (where V_{k_0} is viewed as a left Γ-module) defined by the rule:

$$\kappa_\phi^{\mathrm{ord}}(\gamma)(P) = \sum_{e: v_0 \to \gamma v_0} c_\phi(e)(P), \tag{14}$$

where the sum is taken over the ordered edges in the path joining v_0 to γv_0. Likewise, the associated rigid analytic modular form ψ gives rise to the $V_{k_0} \otimes \mathbb{C}_p$-valued one-cocycle on Γ defined by

$$\kappa_\phi^{\log}(\gamma)(P) = \int_{z_0}^{\gamma z_0} \psi(z) P(z) dz. \tag{15}$$

The images $[\kappa_\phi^{\mathrm{ord}}]$ and $[\kappa_\phi^{\log}]$ of $\kappa_\phi^{\mathrm{ord}}$ and $\kappa_\phi^{\log}$ in $H^1(\Gamma, V_{k_0} \otimes \mathbb{C}_p)$ are independent of the choices of v_0 and z_0 that were made to define them. These classes lie in the one-dimensional f-isotypic component of $H^1(\Gamma, V_{k_0} \otimes \mathbb{C}_p)$ for the action of the Hecke operators. Furthermore, Theorem 1 of **[23]** shows that the class of $\kappa_\phi^{\mathrm{ord}}$ is non-zero. We are now in a position to recall the definition of $\mathscr{L}_T(f)$ given in **[23]**.

Definition 2.3. — The Teitelbaum L-invariant attached to f is the unique scalar $\mathscr{L}_T(f) \in \mathbb{C}_p$ such that

$$[\kappa_\phi^{\log}] = \mathscr{L}_T(f)[\kappa_\phi^{\mathrm{ord}}].$$

Note that multiplying ϕ, and the resulting cocycle and locally analytic distribution, by a non-zero scalar multiplies both κ_{ord} and $\kappa_{\log}$ by that same scalar and hence does not affect the value of $\mathscr{L}_T(f)$, which is therefore a genuine invariant of f (once the factorisation (7) has been fixed) in light of the uniqueness of ϕ described in Theorem 2.1.

3. Families of automorphic forms on B

The group $\mathbf{GL}_2(\mathbb{Q}_p)$ acts naturally on $W := \mathbb{Q}_p^2 - \{0\}$ on the left, by viewing elements of W as non-zero column vectors. Of considerable importance is the resulting action of the scalar matrices in $\mathbb{Z}_p^\times$, which commutes with the $\mathbf{GL}_2(\mathbb{Q}_p)$ action, and preserves L' for any $\mathbb{Z}_p$-lattice $L \subset \mathbb{Q}_p^2$. This latter action is denoted by

$$\lambda \cdot (x, y) := (\lambda x, \lambda y).$$

Recall the standard lattice $L_* = \mathbb{Z}_p^2$, and let $\mathscr{A}(L'_*, \mathbb{Q}_p)$ denote as above the space of locally analytic $\mathbb{Q}_p$-valued functions on L'_*. It is equipped with a right action by $\mathbf{GL}_2(\mathbb{Z}_p)$ given by:

$$(f|u)(x, y) = f(ax + by, cx + dy) \quad \text{for } u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbb{Z}_p).$$

Let

$$\mathbb{D} := \mathscr{D}(L'_*, \mathbb{Q}_p)$$

be the space of locally analytic distributions on L'_*. The natural, continuous left action of $\mathbf{GL}_2(\mathbb{Z}_p)$ can be turned into a right action by the rule:

$$\mu \cdot u := u^{-1}\mu, \qquad \text{for } u \in \mathbf{GL}_2(\mathbb{Z}_p), \quad \mu \in \mathbb{D}.$$

Let $R := \mathscr{D}(\mathbb{Z}_p^\times, \mathbb{Q}_p)$ be the $\mathbb{Q}_p$-algebra of locally analytic distributions on $\mathbb{Z}_p^\times$.

The $\mathbb{Z}_p^\times$-action on L'_* equips $\mathbb{D}$ with a natural R-module structure

$$R \times \mathbb{D} \longrightarrow \mathbb{D} \text{ sending } (\alpha, \mu) \text{ to } \alpha \cdot \mu,$$

where $\alpha \cdot \mu$ is defined by the rule:

$$\int_{L'_*} F(x,y)(\alpha\cdot\mu)(x,y) := \int_{\mathbb{Z}_p^\times \times L'_*} F(tx,ty)\alpha(t)\mu(x,y),$$

where $F(x,y)$ belongs to $\mathscr{A}(L'_*, \mathbb{Q}_p)$ and the variables of integration t and (x,y) range over $\mathbb{Z}_p^\times$ and L'_* respectively.

Let us now fix an integer $k_0 \geq 0$ and let U be an affinoid disk defined over the finite extension K of $\mathbb{Q}_p$ such that $k_0 \in U \subset \mathscr{W}$. Let $A(U)$ denote the K-affinoid algebra of U. Then we have a natural $\mathbb{Q}_p$-algebra homomorphism $R \longrightarrow A(U)$ defined by rule

$$\alpha \to (\kappa \to \int_{\mathbb{Z}_p^\times} \kappa(t)\alpha(t)), \qquad \text{for all } \alpha \in R, \kappa \in U. \tag{16}$$

Remark 3.1. — Let $\kappa \in U(K)$, then κ can be uniquely written $\kappa = \epsilon(t)\chi(t)\langle t\rangle^c$ for $\epsilon : \mathbb{Z}_p^\times \longrightarrow K^\times$ a character of order dividing $p-1$, $\chi : \mathbb{Z}_p^\times \longrightarrow K^\times$ a character of order a power of p and $c \in \mathscr{O}_K$. So we may think of κ as determined by the pair $(\epsilon\chi, c)$. Let us remark that if K is fixed and the radius r of U is small enough the associated pair $(\epsilon\chi, c)$ is characterized by: $\epsilon(t) = (\frac{t}{\langle t\rangle})^{k_0}$, $\chi(t) = 1$ and $|c - k_0| \leq r$. In other words κ is entirely determined by c.

Denote by $\mathbb{D}_U := A(U)\hat{\otimes}_R\mathbb{D}$ and let $\mathbf{GL}_2(\mathbb{Q}_p)$ act on the right on $\mathbb{D}_U$ via its action on $\mathbb{D}$.

A natural R-module structure on $\mathbb{S}(N,p)$ is obtained by setting

$$(\alpha\cdot\Phi)(g) := \alpha\cdot\Phi(g), \text{ for } \alpha \in R, \ \Phi \in \mathbb{S}(N,p), \text{ and } g \in \mathbf{GL}_2(\mathbb{Q}_p).$$

Definition 3.2. — Fix k_0 and U as above. The space

$$\mathbb{S}_U(N,p) := S(\Sigma(N,1), \mathbb{D}_U)$$

is called the space of *p-adic families of automorphic forms on B* of level $\Sigma(N,p)$ parametrized by weights in U.

Remark 3.3. — Note that the space $\mathbb{S}_U(N,p)$ is defined using a level structure $\Sigma(N,1)$ in which the prime p has been removed. In other words, these functions satisfy an equivariance property, on the right, by the full group $\mathbf{GL}_2(\mathbb{Z}_p)$ and not just $\Gamma_0(p\mathbb{Z}_p)$.

The terminology introduced in Definition 3.2 is justified by the fact that $\mathbb{S}_U(N,p)$ is equipped with natural Hecke-equivariant *specialization maps* ρ_k to $S_{k+2}(N,p)$ for every even integer $k \geq 0$ in U. In order to define ρ_k, it is convenient to introduce

$$W_\infty := L'_* \cap L'_\infty = \mathbb{Z}_p^\times \oplus p\mathbb{Z}_p \subset W,$$

where $L_\infty = \mathbb{Z}_p \oplus p\mathbb{Z}_p$, as before. If $P \in \mathscr{P}_k$ is a polynomial (of degree $\leq k$), let

$$\tilde{P}(x,y) = y^k P(x/y)$$

denote the corresponding homogeneous polynomial in x and y of degree k. More generally let $\kappa \in U$ and let $k \geq 0$ be an integer and define for $X = L'_*$ or W:

$$\begin{aligned}\mathscr{A}^{(\kappa)}(X) := \quad & \{f : L'_* \longrightarrow K \quad \text{locally analytic} \quad | \quad f(tx,ty) = \kappa(t)f(x,y) \\ & \qquad\qquad\qquad\qquad \text{for all } t \in \mathbb{Z}_p^\times, (x,y) \in X\}\end{aligned}$$

and

$$\begin{aligned}\mathscr{A}_k^{(\kappa)}(W) := \quad & \{f : W \longrightarrow K \text{ locally analytic} \mid \quad f(tx,ty) = \kappa(t)f(x,y) \\ & \text{and } f(px,py) = p^k f(x,y) \text{ for all } t \in \mathbb{Z}_p^\times, (x,y) \in W\}\end{aligned}$$

Let us fix $\kappa \in U(K)$ and define

$$B_\kappa : A(U) \times \mathbb{D} \longrightarrow \mathrm{Hom}_{cont,\mathbb{Q}_p}(\mathscr{A}^\kappa(L'_*), K)$$

by

$$B_\kappa(\alpha,\mu)(f) := \alpha(\kappa) \int_{L'_*} f(x,y)\mu(x,y),$$

where $f \in \mathscr{A}^{(\kappa)}(L'_*)$, $\alpha \in A(U)$, and $\mu \in \mathbb{D}$. Moreover we have

$$\begin{aligned}|B_\kappa(\alpha,\mu)(f)|_K \quad &= \quad |\alpha(\kappa)|_K \cdot |\int_{L'_*} f\mu|_{\mathbb{Q}_p} \leq ||\kappa|| \cdot ||\alpha||_{A(U)} \cdot ||f|| \cdot ||\mu||_{\mathbb{D}} \\ &= \quad ||\kappa|| \cdot ||f|| \cdot ||(\alpha,\mu)||.\end{aligned}$$

Therefore, for every $(\alpha,\mu) \in A(U) \times \mathbb{D}$, $B_\kappa(\alpha,\mu)$ is continuous and $\mathbb{Q}_p$-linear, therefore an element of $\mathrm{Hom}_{cont,\mathbb{Q}_p}(\mathscr{A}^{(\kappa)}(L'_*), K)$, and B_κ is a continuous, $\mathbb{Q}_p$-bilinear map. Moreover if $r \in R$, $\alpha \in A(U)$, $\mu \in \mathbb{D}$ we have

$$\begin{aligned}B_\kappa(\alpha, r\mu)(f) \quad &= \quad \alpha(\kappa \int_{L'_*} f(r\mu) = \int_{\mathbb{Z}_p^\times \times L'_*} f(tx,ty)r(t)\mu(x,y) \\ &= \quad \alpha(\kappa)r(\kappa) \int_{L'_*} f(x,y)\mu(x,y) = B_\kappa(r\alpha,\mu)(f).\end{aligned}$$

By the universal property of completed tensor product, there is a unique continuous, $\mathbb{Q}_p$-linear map $L_\kappa : A(U)\hat{\otimes}_R\mathbb{D} \longrightarrow \mathrm{Hom}_{cont,\mathbb{Q}_p}(\mathscr{A}^\kappa(L'_*), K)$ such that the following diagram is commutative

$$\begin{array}{ccc} A(U)\times\mathbb{D} & \xrightarrow{B_\kappa} & \mathrm{Hom}_{cont,\mathbb{Q}_p}(\mathscr{A}^{(\kappa)}(L'_*),K) \\ \downarrow & & || \\ A(U)\hat{\otimes}_R\mathbb{D} & \xrightarrow{L_\kappa} & \mathrm{Hom}_{cont,\mathbb{Q}_p}(\mathscr{A}^{(\kappa)}(L'_*),K) \end{array}$$

Finally, if $\mu\in\mathbb{D}_U=A(U)\hat{\otimes}_R\mathbb{D}$ and $f\in\mathscr{A}^{(\kappa)}(L'_*)$ we denote

$$L_\kappa(\mu)(f)=\int_{L'_*} f\mu.$$

Let now $k\geq 0$ be an integer such that $k\in U$ and $P\in\mathscr{P}_k$. Let us remark that as the $\mathbb{Z}_p^\times$-action on L'_* preserves W_∞, the function $\tilde{P}\chi_{W_\infty}\in\mathscr{A}^{(k)}(L'_*)$ where χ_{W_∞} is the characteristic function of W_∞ in L'_*. Let $\Phi\in\mathbb{S}_U(N,p)$.

Definition 3.4. — The *specialization map in weight* $k+2$ is the map

$$\rho_k:\mathbb{S}_U(N,p)\longrightarrow S_{k+2}(N,p)$$

defined by

$$\rho_k(\Phi)(g)(P)=\int_{W_\infty}\tilde{P}(x,y)\Phi(g)(x,y),$$

for $g\in\mathbf{GL}_2(\mathbb{Q}_p)$ and $P\in\mathscr{P}_k$.

Remark 3.5. — Note that the stabilizer of the ordered edge $e_\infty:=([L_*],[L_\infty])$ in $\mathbf{PGL}_2(\mathbb{Z}_p)$, and therefore of W_∞, is the image of $\Gamma_0(p\mathbb{Z}_p)$. This is why the prime p arises in the level of the specialization $\rho_k(\Phi)$, even though Φ was taken to be equivariant under the larger group $\Sigma(N,1)$.

The group $\hat{B}^\times$ can be written as a finite disjoint union of double cosets

$$\hat{B}^\times=\cup_{i=1}^q B^\times d_i\Sigma(N,p),$$

for elements d_i, $i=1,...,q$ in $\hat{B}^\times$. The condition $N\geq 4$ insures that the groups $d_i^{-1}B^\times d_i\cap\Sigma(N,p)$ are trivial, so that there is a natural identification

$$\mathbb{S}(N,p)\longrightarrow\mathbb{D}^q,\ \text{given by}\ \varphi\longrightarrow(\varphi(d_i))_{1\leq i\leq q}.$$

The spaces R and $\mathbb{D}$ with their natural topologies are Fréchet spaces. Thus $\mathbb{S}(N,p)$ inherits from $\mathbb{D}$ a topology under which it becomes a Fréchet space (just like R and $\mathbb{D}$). Moreover $\mathbb{S}_U(N,p)=S(\Sigma(N,1),\mathbb{D}_U)\cong\mathbb{S}(N,p)\hat{\otimes}_R A(U)$. See Section 4 of [**3**] for more details.

Definition 3.6. — Let M be a Fréchet space which is an R-module. We'll say that M is an orthonormalizable R-module if, for each $n\geq 0$ there are orthonormalizable $R_n=D_n(\mathbb{Z}_p^\times)$-Banach modules M_n such that $M\cong\lim\limits_{\leftarrow,n}M_n$ as R-modules.

Theorem 3.7. — *The Fréchet spaces* $\mathbb{D}$ *and* $\mathbb{S}(N,p)$ *are orthonormalizable* R*-modules.*

Proof. — We have a natural projection $\pi : W_\infty \longrightarrow \mathbb{Z}_p$ given by $(x, y) \longrightarrow y/x$, whose fibers are isomorphic to $\mathbb{Z}_p^\times$. Moreover π is equipped with a natural continuous section s defined as follows. For each $i = 0, 1, ..., p-1, \infty \in \mathbb{P}^1(\mathbb{F}_p)$ let $C_i \subset \mathbb{P}^1(\mathbb{Q}_p)$ denote the residue class of i. The we can write L'_* as the disjoint union $(\mathbb{Z}_p^\times \oplus p\mathbb{Z}_p) \cup (\mathbb{Z}_p \oplus \mathbb{Z}_p^\times)$ such that $\pi(\mathbb{Z}_p^\times \oplus p\mathbb{Z}_p) = C_\infty$ and $\pi(\mathbb{Z}_p \oplus \mathbb{Z}_p^\times) = \mathbb{P}^1(\mathbb{Q}_p) - C_\infty = \cup_{i=0}^{p-1} C_i$. Define $s : \mathbb{P}^1(\mathbb{Q}_p) \longrightarrow L'_*$ by $s(z) = (1, 1/z)$ if $z \in C_\infty$ and $s(z) = (z, 1)$ else. Then both π, s are locally analytic functions and they induce locally analytic isomorphisms:

$$u : L'_* \longrightarrow \mathbb{Z}_p^\times \oplus \mathbb{P}^1(\mathbb{Q}_p) \text{ and } v : \mathbb{Z}_p^\times \oplus \mathbb{P}^1(\mathbb{Q}_p) \longrightarrow L'_*$$

by: $(u(x, y) = (x, \pi(x, y))$ and $v(a, z) = as(z)$.

Moreover we have actions of L'_* and $\mathbb{Z}_p^\times \oplus \mathbb{P}^1(\mathbb{Q}_p)$ as follows: if $\alpha \in \mathbb{Z}_p^\times, (x, y) \in L'_*, (a, z) \in \mathbb{Z}_p^\times \oplus \mathbb{P}^1(\mathbb{Q}_p)$ then $\alpha(x, y) = (\alpha x, \alpha y)$ and $\alpha(a, z) = (\alpha a, z)$. Then both u, v are equivariant with respect to these actions and they induce, for each $n \geq 1$ natural isomorphisms as Banach spaces

$$A_n(L'_*) \cong A_n(\mathbb{Z}_p^\times) \hat{\otimes} A_n(\mathbb{P}^1(\mathbb{Q}_p)).$$

By duality they induce $D_n(\mathbb{Z}_p^\times)$-linear isomorphisms

$$\begin{aligned} D_n(L'_*) &= \mathrm{Hom}_{\mathrm{cont}, \mathbb{Q}_p}(A_n(L'_*), \mathbb{Q}_p) \\ &\cong \mathrm{Hom}_{\mathrm{cont}}(A_n(\mathbb{Z}_p^\times) \hat{\otimes} A_n(\mathbb{P}^1(\mathbb{Q}_p)), \mathbb{Q}_p) \\ &\cong \mathrm{Hom}_{\mathrm{cont}}(A_n(\mathbb{P}^1(\mathbb{Q}_p)), D_n(\mathbb{Z}_p^\times)). \end{aligned}$$

The last term in the sequence naturally contains $D_n(\mathbb{P}^1(\mathbb{Q}_p)) \hat{\otimes} D_n(\mathbb{Z}_p^\times)$ as the subspace of *completely continuous* (or compact) $\mathbb{Q}_p$-linear maps from $A_n(\mathbb{P}^1(\mathbb{Q}_p))$ to $D_n(\mathbb{Z}_p^\times)$. (See **[20]** section 4.) Since $D_n(\mathbb{P}^1(\mathbb{Q}_p))$ is a Banach space over $\mathbb{Q}_p$, it is orthonormalizable and therefore $D_n(\mathbb{P}^1(\mathbb{Q}_p)) \hat{\otimes} D_n(\mathbb{Z}_p^\times)$ is an orthonormalizable Banach module over $D_n(\mathbb{Z}_p^\times)$. Now we claim that the natural inclusions above induce isomorphisms

$$\lim_{\leftarrow, n} D_n(\mathbb{P}^1(\mathbb{Q}_p)) \hat{\otimes} D_n(\mathbb{Z}_p^\times) \longrightarrow \lim_{\leftarrow, n} \mathrm{Hom}_{\mathrm{cont}, \mathbb{Q}_p}(A_n(\mathbb{P}^1(\mathbb{Q}_p)), D_n(\mathbb{Z}_p^\times)).$$

The map above is clearly injective. Let us show that it is surjective. Let

$$(f_n)_n \in \lim_{\leftarrow, n} \mathrm{Hom}_{\mathrm{cont}}(A_n(\mathbb{P}^1(\mathbb{Q}_p)), D_n(\mathbb{Z}_p^\times)).$$

We have the following commutative diagram:

$$\begin{array}{ccc} A_n(\mathbb{P}^1(\mathbb{Q}_p)) & \xrightarrow{f_n} & D_n(\mathbb{Z}_p^\times) \\ \downarrow r_n & & \uparrow s_n \\ A_{n+1}(\mathbb{P}^1(\mathbb{Q}_p)) & \xrightarrow{f_{n+1}} & D_{n+1}(\mathbb{Z}_p^\times), \end{array}$$

where r_n is the restriction and s_n is dual to restriction. Therefore,

$$f_n = s_n f_{n+1} r_n,$$

and because r_n is the restriction induced by the inclusion

$$B[\mathbb{P}^1(\mathbb{Q}_p), p^{-n-1}] \subset B[\mathbb{P}^1(\mathbb{Q}_p), p^{-n}],$$

it is completely continuous. (See [**20**], Section 8.) Therefore f_n is completely continuous for all $n \geq 0$. So we have an isomorphism as R-modules $\mathbb{D} \cong \lim_{\to,n} D_n(\mathbb{P}^1(\mathbb{Q}_p)\hat{\otimes} D_n(\mathbb{Z}_p^\times)$ which implies that $\mathbb{D}$ is an orthonormalizable R-module. As $\mathbb{S}(N,p) \cong \mathbb{D}^q$ it is an orthonormalizable R-module as well. □

Corollary 3.8. — *Let U be an affinoid disk contained in the weight space $\mathscr{W}$. Then $\mathbb{D}_U$ and $\mathbb{S}_U(N,p)$ are orthonormalizable $A(U)$-modules.*

Theorem 3.7 can be used to define actions of Hecke operators T_ℓ for ℓ not dividing Np and U_p, as in Sections 6 and 8 of [**3**]. The following theorem now follows from a standard argument.

Theorem 3.9. — *Let U be an affinoid disk contained in the weight space $\mathscr{W}$. The operator $U_p : \mathbb{S}_U(N,p) \longrightarrow \mathbb{S}_U(N,p)$ is a compact $A(U)$-linear operator.*

Proof. — See [**22**] and [**3**]. □

Recall the Coleman family f_∞ of eigenforms on $\Gamma_1(N) \cap \Gamma_0(p)$ interpolating f that is given in equation (3) of the introduction. The Fourier coefficients $a_n(k)$ of f_∞ correspond to elements of $A(U)$ for some rigid analytic disk U containing k_0 and contained in the weight space $\mathscr{W}$. We will be making crucial use of the following "Jacquet-Langlands correspondence" applied to the family f_∞.

***Theorem 3.10* (G.Chenevier, [4])**. — *To the expense of possibly shrinking U, there exists an eigenfamily $\Phi \in \mathbb{S}_U(N,p)$ such that*

$$\Phi|T_\ell = a_\ell \Phi \quad \text{for } (\ell, Np) = 1 \qquad \text{and} \qquad \Phi|U_p = a_p \Phi.$$

4. A geometric interpretation of p-adic families of automorphic forms

In this section, we attach to any family $\Phi \in \mathbb{S}_U(N,p)$ a collection of locally analytic distributions $(\mu_L)_{L \subset \mathbb{Q}_p^2}$ on W, indexed by the $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$.

Definition 4.1. — Let us first fix a weight $\kappa \in U$. Let $L = gL_*$ be a $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^2$, for some $g \in \mathbf{GL}_2(\mathbb{Q}_p)$. The distribution μ_L on $\mathscr{A}^{(\kappa)}(W)$ is defined by

$$\int_W F(z)\mu_L(z) = \int_{L'} F(z)\mu_L(z) := \int_{L'_*} (F|g)(z)\Phi(g) = \int_{L'_*} F(gz)\Phi(g),$$

where $F : W \longrightarrow \mathbb{Q}_p$ is any function in $\mathscr{A}^{(\kappa)}(W)$.

Note that if $F \in \mathscr{A}^{(\kappa)}(W)$ then $(F|g) \in \mathscr{A}^{(\kappa)}(W)$ for any $g \in \mathbf{GL}_2(\mathbb{Q}_p)$ and that μ_L is supported, by definition, on the compact subset L' of W.

Here are some elementary properties of the collection $\{\mu_L\}$.

1. The distribution μ_L is well defined, i.e. it does not depend on the choice of g. Indeed, let $g_1, g_2 \in \mathbf{GL}_2(\mathbb{Q}_p)$ be such that $g_1 L_* = g_2 L_* = L$. Then $g_1 = g_2 u$ with $u \in \mathbf{GL}_2(\mathbb{Z}_p)$ and we have

$$\begin{aligned}\int_{L'_*} (F|g_1)(z)\Phi(g_1) &= \int_{L'_*} (F|g_2u)(z)\Phi(g_2u) = \int_{L'_*} (F|g_2u)(z)(u^{-1}\Phi(g_2)) \\ &= \int_{L'_*} (F|g_2uu^{-1})(z)\Phi(g_2) = \int_{L'_*} (F|g_2)(z)\Phi(g_2),\end{aligned}$$

for all functions $F \in \mathscr{A}^{(\kappa)}(W)$.

2. Let γ be any element of $\tilde{\Gamma}$. Then

$$\int_{(\gamma L)'} F(z)\mu_{\gamma L}(z) = \int_{L'} (F|\gamma)(z)\mu_L(z)$$

for all locally analytic functions F in the space $\mathscr{A}^{(\kappa)}(W)$. In particular for $\gamma = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \in \tilde{\Gamma}$ we have

$$\int_{(pL)'} F(z)\mu_{pL}(z) = \int_{L'} F(pz)\mu_L(z).$$

3. For any $\alpha \in A(U)$ and any lattice $L \subset \mathbb{Q}_p^2$, there is a natural multiplication $\alpha \cdot \mu_L$, such that $\alpha \cdot \mu_L$ is a locally analytic distribution on L', and the family $(\alpha \cdot \mu_L)_{L \subset \mathbb{Q}_p^2}$ is associated to $\alpha\Phi \in \mathbb{S}(N,p)$ by the procedure described above.

The specialization map

$$\rho_k : \mathbb{S}_U(N,p) \longrightarrow S_{k+2}(N,p)$$

can be reinterpreted geometrically as a map assigning a V_k-valued cocycle on $\mathscr{T}$ to a family of distributions $(\mu_L)_L$ indexed by lattices in $\mathbb{Q}_p^2$ and satisfying properties 1 to 3 above. More precisely, for all $P \in \mathscr{P}_k$, let

$$\tilde{P}(x,y) := y^k P(x/y)$$

denote the homogeneous polynomial in x and y, satisfying $\tilde{P}(z,1) = P(z)$. Let us also denote $|L| := p^{\mathrm{ord}_p(\det(B))}$, for B any $\mathbb{Z}_p$-basis of L.

Lemma 4.2. — *For each even integer $k \geq 0$, the Γ-invariant cocycle on $\mathscr{T}$ attached to the specialisation $\rho_k(\Phi)$,*

$$c_{\Phi,k} : \mathscr{E}(\mathscr{T}) \longrightarrow V_k$$

is expressed in terms of the system of distributions $(\mu_L)_L$ associated to Φ by the rule:

$$c_{\Phi,k}(e)(P) = |L|^{-k/2} \int_{W_e} \tilde{P}(x,y)\mu_L(x,y).$$

where the lattice L above is any representative of the origin of e.

Proof. — The proof is a direct consequence of the definitions. □

Let $k \geq 0$ be an even integer and let $\kappa \in U$. Let us recall that we have defined the space $\mathscr{A}_k^{(\kappa)}(W)$ of locally analytic functions on W, homogeneous of degree κ for the action of $\mathbb{Z}_p^\times$ and homogeneous of degree k for the action of p on W. Let us remark that if P is a locally meromorphic function on $\mathbb{P}^1(\mathbb{Q}_p)$ with at worst a pole of order k at ∞ then $\tilde{P}(x,y) := y^k P(x/y) \in \mathscr{A}_k^{(k)}(W)$. In particular if $P \in \mathscr{P}_k$ then $\tilde{P} \in \mathscr{A}_k^{(k)}(W)$.

Suppose now that $\Phi \in \mathbb{S}_U(N,p)$ is an eigenvector for the operator U_p, so that

$$\Phi | U_p = a_p \Phi, \text{ with } a_p \in A(U).$$

Lemma 4.3. — *Suppose that $L_2 \subset L_1$ are $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$ with $[L_1 : L_2] = p$. Let $\epsilon = ([L_1],[L_2]) \in \mathscr{E}(\mathscr{T})$ be the corresponding edge. Then*

$$\int_{W_\epsilon} F(x,y)\mu_{L_2}(x,y) = \int_{W_\epsilon} F(x,y)(a_p\mu_{L_1})(x,y),$$

for every locally analytic function F in $\mathscr{A}_k^{(\kappa)}(W)$, where $k \geq 0$ is an even integer and $\kappa \in U$.

Proof. — Let $D_k^{(\kappa)}(W)$ be the continuous dual of $\mathscr{A}_k^{(\kappa)}(W)$. We will extend the definition in 4.2 and will attach to Φ a $D_k^{(\kappa)}(W)$-valued cocycle on $\mathscr{T}$ as follows: let $C_{\Phi,\kappa,k} : \mathscr{E}(T) \longrightarrow D_k^{(\kappa)}(W)$ be defined by

$$C_{\Phi,\kappa,k}(e)(F) := |L|^{-k/2} \int_{W_e} F(x,y)\mu_L(x,y),$$

where $e = [L, L']$ with L, L' lattices in $\mathbb{Q}_p^2$ such that $L' \subset L$ has index p and $F \in \mathscr{A}_k^{(\kappa)}(W)$. Let us remark that due to the homogeneity of F with respect to the action of p, the definition is independent of the choice of L, L'. Then $C_{\Phi,\kappa,k}$ enjoys the same formal properties as $c_{\Phi,k}$, in particular we have

$$C_{U_p\Phi,\kappa,k}(e) = p^k \sum_{s(e')=t(e), e' \neq \bar{e}} C_{\Phi,\kappa,k}(e'),$$

for all $e \in \mathscr{E}(\mathscr{T})$.

Let us now prove the lemma. We have

$$\begin{aligned} |L_1|^{-k/2} \int_{W_\epsilon} F(x,y)(a_p\mu_L)(x,y) &= C_{U_p\Phi,\kappa,k}(\epsilon)(F) \\ &= p^k \sum_{s(\epsilon')=t(\epsilon), \epsilon' \neq \bar{\epsilon}} C_{\Phi,\kappa,k}(\epsilon')(F). \end{aligned}$$

For every ϵ' in the above sum let us choose lattices $\epsilon' = ([L_2],[L_{\epsilon'}])$, then we have

$$\begin{aligned} (17)\qquad |L_1|^{-k/2} \int_{W_\epsilon} F(x,y)(a_p\mu_{L_1})(x,y) &= |L_1|^{-k/2} p^k \sum_{\epsilon'} \int_{W_{\epsilon'}} F(x,y)\mu_{L_2}(x,y) = \\ &= |L_1|^{-k/2} \int_{W_\epsilon} F(x,y)\mu_{L_2}(x,y). \end{aligned}$$

□

For every $\mathbb{Z}_p$-lattice $L \subset \mathbb{Q}_p^2$ we define a distribution $\pi_*(\mu_L)$ on $\mathbb{P}^1(\mathbb{Q}_p)$ by the formula

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} P(t)\pi_*(\mu_L)(t) := |L|^{-k_0/2} \int_W \tilde{P}(x,y)\mu_L(x,y),$$

where P is any locally meromorphic function on $\mathbb{P}^1(\mathbb{Q}_p)$, with at worst a pole of order k_0 at ∞.

Assume now that $\rho_{k_0}(\Phi) = \phi$, where $\phi \in S_{k_0+2}(N,p)$ is the automorphic form on B attached to f via Theorem 2.1. In particular, Φ is an eigenvector for U_p whose associated eigenvalue $a_p(k)$ satisfies

$$a_p(k_0) = p^{k_0/2}.$$

Recall the distribution μ_ϕ attached to ϕ that was defined in Section 2.

Proposition 4.4. — *For all $\mathbb{Z}_p$-lattices L in $\mathbb{Q}_p^2$,*

$$\pi_*(\mu_L) = \mu_\phi.$$

Proof. — First note that the function $(x,y) \longrightarrow \tilde{P}(x,y)$ is a locally analytic homogeneous function of degree k_0 with respect to the action of $\mathbb{Q}_p^\times$ on W, in particular $\tilde{P} \in \mathscr{A}_{k_0}^{(k_0)}(W)$. The relationship between μ_{pL} and μ_L described after Definition 4.1 implies that $\pi_*(\mu_L)$ only depends on the homothety class of L. Moreover, let L_1 and L_2 be any two $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$. Suppose without loss of generality that L_2 is contained in L_1 with index p, and that $|L_1| = 1$, and $|L_2| = p$. Let $e = ([L_1],[L_2])$ be the corresponding edge. Using Lemma 4.3 we have

$$\begin{aligned}
\int_{U_e} P(t)\pi_*(\mu_{L_2})(t) &= |L_2|^{-k_0/2} \int_{W_e} \tilde{P}(x,y)\mu_{L_2}(x,y) \\
&= p^{-k_0/2} \int_{W_e} \tilde{P}(x,y)(a_p\mu_{L_1})(x,y) \\
&= p^{-k_0/2} a_p(k_0) \int_{W_e} \tilde{P}(x,y)\mu_{L_1}(x,y) \\
&= \int_{U_e} P(t)\pi_*(\mu_{L_1})(t).
\end{aligned}$$

Arguing in the same way for $\overline{e} = ([(1/p)L_2],[L_1])$, one finds that

$$\int_{U_{\overline{e}}} P(t)\pi_*(\mu_{L_2})(t) = \int_{U_{\overline{e}}} P(t)\pi_*(\mu_{L_1})(t),$$

for all locally meromorphic functions P on $\mathbb{P}^1(\mathbb{Q}_p)$ with at worst a pole of order k_0 at ∞. Because $\mathbb{P}^1(\mathbb{Q}_p) = U_e \cup U_{\overline{e}}$, we conclude that the distribution $\pi_*(\mu_L)$ is independent of the lattice L.

On the other hand, for all P as above we have

$$\begin{aligned}\int_{U_e} P(t)\mu_\phi(t) &= C_{\Phi,k_0,k_0}(e)(\tilde{P}) = |L_1|^{-k_0/2}\int_{W_e}\tilde{P}(x,y)\mu_{L_1}(x,y)\\ &= \int_{U_e} P(t)\pi_*(\mu_{L_1})(t).\end{aligned}$$

Similarly, it follows that

$$\int_{U_{\overline{e}}} P(t)\mu_\phi(t) = \int_{U_{\overline{e}}} P(t)\pi_*(\mu_{L_2})(t),$$

which allows us to conclude. □

Given $\tau \in \mathbb{P}^1(\mathbb{C}_p)$, let $\bar{\tau}$ denote the natural image of τ in $\mathbb{P}^1(\bar{\mathbb{F}}_p)$ obtained by reducing τ modulo the maximal ideal of the ring of integers of $\mathbb{C}_p$. Let $\mathscr{H}_p(\mathbb{Q}_p^{ur})$ denote the *unramified p-adic upper half-plane*, consisting of elements in $\mathbb{Q}_p^{ur} - \mathbb{Q}_p$. Finally, let

$$r : \mathscr{H}_p(\mathbb{Q}_p^{ur}) \longrightarrow \mathscr{T}_0$$

denote the so-called *reduction map* which is determined by the rules

1. $r(\tau) = v_*$ if and only if $\bar{\tau} \notin \mathbb{P}^1(\mathbb{F}_p)$;
2. $r(\gamma\tau) = \gamma r(\tau)$ for all $\gamma \in \mathbf{PGL}_2(\mathbb{Q}_p)$.

(See Chapter 5 of **[13]**, for example, for more details.)

We will now extend the definition to a more general class of functions. Let us fix $\tau \in \mathscr{H}_p(\mathbb{Q}_p^{ur})$, $k_0 \geq 0$ an integer and let U be an affinoid disk containing k_0 and contained in the weight space $\mathscr{W}$. Let $P \in \mathscr{A}^{(k_0)}(L'_*)$ and $\mu \in \mathbb{D}_U$. We'd like to define

$$\int_{L'_*} \log(x-\tau y)P(x,y)\mu,$$

where the branch of log in the above formula and to the end of this article is such that $\log(p) = 0$.

Let $F : U(K) \times L'_* \longrightarrow \mathbb{C}_p$ be defined by:

$$F(\kappa,(x,y)) = P(x,y)(x-\tau y)^{\kappa-k_0}.$$

By the above expression we mean the following. Suppose first that the radius r of U is small enough and let κ be determined by the pair (ϵ, c) as in remark 3.1. Here ϵ is the character $t \longrightarrow (\frac{t}{\langle t\rangle})^{k_0}$ and $c \in \mathscr{O}_K$ such that $|c-k_0| \leq r$. Then by $(x-\tau y)^{\kappa-k_0}$ we mean $(x-\tau y)^{c-k_0} = \exp((c-k_0)\log(x-\tau y))$.

Let us remark that if $t \in \mathbb{Z}_p^\times$, we have $F(\kappa,(tx,ty)) = \kappa(t)F(\kappa,(x,y))$, i.e. $F(\kappa,-) \in \mathscr{A}^{(\kappa)}(L'_*)$.

Lemma 4.5. — *Let $\mu \in \mathbb{D}_U$. The function $U(K) \longrightarrow \mathbb{C}_p$ defined by*

$$\kappa \longrightarrow \int_{L'_*} F(\kappa,(x,y))\mu(x,y),$$

is analytic near k_0.

Proof. — Let us remark that we have the following expansion

$$F(\kappa,(x,y)) = P(x,y)\sum_{n=0}^{\infty}\frac{(\kappa-k_0)^n}{n!}\log^n(x-\tau y),$$

which converges for all $(x,y)\in L'_*$ as $\log(x-\tau y)\in p\mathscr{O}_{\mathbb{Q}_p^{ur}}$. Moreover, for all $n\geq 0$ the function $(x,y)\to P(x,y)\log^n(x-\tau y)$ is locally analytic, more precisely $P(x,y)\log^n(x-\tau y)\in A_m(L'_*)$ with m depending only on τ and P. Let us fix an orthonormal basis $\{\mu_i\}_{i=0}^{\infty}$ of $D_m(L'_*)$, so that the m-th component of μ in $A(U)\hat{\otimes}_{R_m}D_m(L'_*)$, $\mu^{(m)}$ can be uniquely written

$$\mu^{(m)} = \sum_{i=0}^{\infty}\alpha_i\otimes\mu_i, \text{ where } \alpha_i\in A(U) \text{ with } ||\alpha_i||\longrightarrow 0.$$

We have, according to our definition

$$\begin{aligned}\int_{L'_*}F(\kappa,(x,y))\mu &= \int_{L'_*}F(\kappa,(x,y))\mu^{(m)} = \sum_{i=0}^{\infty}\alpha_i(\kappa)\int_{L'_*}F(\kappa,(x,y))\mu_i\\ &= \sum_{i=0}^{\infty}\alpha_i(\kappa)\sum_{n=0}^{\infty}\frac{(\kappa-k_0)^n}{n!}\int_{L'_*}P(x,y)\log^n(x-\tau y)\mu_i(x,y).\end{aligned}$$

The lemma now follows from the fact that $\alpha_i(\kappa)$ is analytic around k_0 for all $i\geq 0$. □

Let notations be as above, i.e. let $\mu\in\mathbb{D}_U$ and $P\in\mathscr{A}^{(k_0)}(L'_*)$.

Definition 4.6. — We define $\int_{L'_*}P(x,y)\log(x-\tau y)\mu(x,y)$ to be

$$\left(\frac{d}{d\kappa}\int_{L'_*}F(\kappa,(x,y))\mu\right)_{\kappa=k_0}.$$

Remark 4.7. — Let us give an explicit formula for $\int_{L'_*}P(x,y)\log(x-\tau y)\mu(x,y)$. Let us suppose that $P(x,y)\log^n(x-\tau y)\in A_m(L'_*)$ for some m independent of n and let us fix an orthonormal basis $\{\mu_i\}_{i=0}^{\infty}$ as in the proof of lemma 4.5. We write $\mu=\sum_{i=0}^{\infty}\alpha_i\otimes\mu_i$, with $\alpha_i\in A(U)$ such that $||\alpha_i||\longrightarrow 0$. Then we have

$$\begin{aligned}\int_{L'_*}P(x,y)\log(x-\tau y)\mu(x,y) &= \sum_{i=0}^{\infty}(\frac{d}{d\kappa}\alpha_i)_{\kappa=k_0}\int_{L'_*}P(x,y)\mu_i(x,y)+\\ &+ \sum_{i=0}^{\infty}\alpha_i(k_0)\int_{L'_*}P(x,y)\log(x-\tau y)\mu_i(x,y).\end{aligned}$$

Let now $\Phi \in \mathbb{S}_U(N,p)$ and let $\{\mu_L\}_{L\subset\mathbb{Q}_p^2}$ be the family of distributions attached to it. Let as above $\tau \in \mathscr{H}_p(\mathbb{Q}_p^{ur})$, $P \in \mathscr{A}^{(k_0)}(W)$ and define $f(x,y) := P(x,y)\log(x-\tau y)$. Let $L \subset \mathbb{Q}_p^2$ be a lattice and let $g \in \mathbf{GL}_2(\mathbb{Q}_p)$ be such that $L = gL_*$. For $z=(x,y)\in W$, $(f|g)(z)$ can be written

$$(f|g)(z) = f(gz) = C(g,\tau)(P|g)(x,y) + (P|g)(x,y)\log(x-\tau' y),$$

where $C(g,\tau)$ is independent of (x,y) and $\tau' \in \mathscr{H}_p(\mathbb{Q}_p^{ur})$. Therefore it makes sense to define

$$\begin{aligned}\int_W P(x,y)\log(x-\tau y)\mu_L \quad &:= \quad C(g,\tau)\int_{L'_*}(P|g)(x,y)\mu_{L_*}(x,y) + \\ &+ \quad \int_{L'_*}(P|g)(x,y)\log(x-\tau' y)\mu_{L_*}(x,y).\end{aligned}$$

We are now ready to define the main object of this section. Given $\tau \in \mathscr{H}_p(\mathbb{Q}_p^{ur})$, let $v_\tau = r(\tau) \in \mathscr{T}_0$ and let L_τ be any $\mathbb{Z}_p$-lattice in the homothety class of v_τ. Recall the rigid analytic modular form ψ defined in equation (12) of Section 2.

Definition 4.8. — For all $P \in \mathscr{P}_{k_0}$, the *indefinite integral* attached to τ and ψ is defined by the formula

$$\int^\tau \psi(z)P(z)dz := |L_\tau|^{-k_0/2}\int_W \log(x-\tau y)\tilde{P}(x,y)\mu_{L_\tau}(x,y), \tag{18}$$

where the branch of the p-adic log used above is the one satisfying $\log(p)=0$.

Note that because

$$\log(px-p\tau y)\tilde{P}(px,py) = p^{k_0}\log(x-\tau y)\tilde{P}(x,y),$$

formula (18) only depends on the homothety class of L_τ, so that the indefinite integral is well-defined.

The main properties of the indefinite integral of Definition 4.8 are summarized in the following two propositions.

Proposition 4.9. — *For all $\gamma \in \Gamma$ and $P \in \mathscr{P}_{k_0}$,*

$$\int^{\gamma\tau}\psi(z)P(z)dz = \int^\tau \psi(z)(P\gamma)(z)dz.$$

Proof. — Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Then

$$\int^{\gamma\tau}\psi(z)P(z)dz = |\gamma L_\tau|^{-k_0/2}\int_W \log(x-(\gamma\tau)y)\tilde{P}(x,y)\mu_{\gamma L_\tau}(x,y).$$

Performing the change of variables

$$\begin{pmatrix} u \\ v \end{pmatrix} = \gamma^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} dx-by \\ -cx+ay \end{pmatrix},$$

we obtain

$$\begin{aligned}\int^{\gamma\tau}\psi(z)P(z)dz &= |L_\tau|^{-k_0/2}\int_W \log\left(\frac{u-\tau v}{c\tau+d}\right)(\widetilde{P\gamma})(u,v)\mu_{L_\tau}(u,v)\\ &= \int^{\tau}\psi(z)(P\gamma)(z)dz\\ &\quad -\log(c\tau+d)|L_\tau|^{-k_0/2}\int_W(\widetilde{P\gamma})(u,v)\mu_{L_\tau}(u,v).\end{aligned}$$

On the other hand by proposition 4.4 we have

$$|L_\tau|^{-k_0/2}\int_W(\widetilde{P\gamma})(u,v)\mu_{L_\tau}(u,v) = \int_{\mathbb{P}^1(\mathbb{Q}_p)}(P\gamma)(t)\mu_\phi(t) = 0.$$

Proposition 4.9 follows. □

The next proposition relates the indefinite integral to the p-adic line integral of equation (13).

Proposition 4.10. — *Let $\tau_1,\tau_2 \in \mathscr{H}_p(\mathbb{Q}_p^{ur})$ and let $v_i = r(\tau_i) = [L_i] \in \mathscr{T}_0$ be the corresponding vertices. For all $P \in \mathscr{P}_{k_0}$,*

$$\begin{aligned}&\int^{\tau_2}\psi(z)P(z)dz - \int^{\tau_1}\psi(z)P(z)dz\\ &\quad = \int_{\tau_1}^{\tau_2}\psi(z)P(z)dz + 2p^{-k_0/2}a_p'(k_0)\sum_{e:v_1\to v_2}c_\phi(e)(P).\end{aligned}$$

Proof. — Suppose without loss of generality that $L_2 \subset L_1$ and $[L_1 : L_2] = p$. Let $e = ([L_1],[L_2]) \in \mathscr{E}(\mathscr{T})$. Then

$$\int^{\tau_2}\psi(z)P(z)dz - \int^{\tau_1}\psi(z)P(z)dz \tag{19}$$

$$= |L_2|^{-k_0/2}\int_W \log(x-\tau_2 y)\tilde{P}(x,y)\mu_{L_2}(x,y) \tag{20}$$

$$- |L_1|^{-k_0/2}\int_W \log(x-\tau_1 y)\tilde{P}(x,y)\mu_{L_1}(x,y) \tag{21}$$

$$= |L_2|^{-k_0/2}\int_W \log\left(\frac{x-\tau_2 y}{x-\tau_1 y}\right)\tilde{P}(x,y)\mu_{L_2}(x,y) \tag{22}$$

$$+ \int_W \log(x-\tau_1 y)\tilde{P}(x,y)\left(|L_2|^{-k_0/2}\mu_{L_2} - |L_1|^{-k_0/2}\mu_{L_1}\right). \tag{23}$$

By proposition 4.4, the first integral (22) appearing in the last expression is equal to

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} \log\left(\frac{t-\tau_2}{t-\tau_1}\right) P(t)\mu_\phi(t) = \int_{\tau_1}^{\tau_2} \psi(z)P(z)dz.$$

In order to calculate the second integral (23), we will need the following describing the distribution $\alpha \cdot \mu_L$ for $\alpha \in A(U)$.

Lemma 4.11. — *Let α be an element of $A(U)$. For all $e \in \mathscr{E}(\mathscr{T})$, $\tau \in \mathscr{H}_p$, and $P \in \mathscr{P}_{k_0}$,*

$$\int_{W_e} \log(x-\tau y)\tilde{P}(x,y)(\alpha\mu_L)(x,y) = \alpha'(k_0)|L|^{k_0/2}c_\phi(e)(P)$$
$$+\alpha(k_0)\int_{W_e} \log(x-\tau y)\tilde{P}(x,y)\mu_L(x,y).$$

Proof of Lemma 4.11: This proof is a consequence of the following calculation:

$$\begin{aligned}
\int_{W_e} & \log(x-\tau y)\tilde{P}(x,y)(\alpha\mu_L)(x,y) \\
&= \frac{d}{d\kappa}\left((\alpha(\kappa)(\int_{W_e} \tilde{P}(x,y)(x-\tau y)^{\kappa-k_0}\mu_L(x,y)\right)_{\kappa=k_0} \\
&= \alpha'(k_0)\int_{W_e} \tilde{P}(x,y)\mu_L(x,y) \\
&\quad +\alpha(k_0)\int_{W_e} \log(x-\tau y)\tilde{P}(x,y)\mu_L(x,y) \\
&= \alpha'(k_0)|L|^{k_0/2}\int_{U_e} P(z)\mu_\phi(z) + \alpha(k_0)\int_{W_e} \log(x-\tau y)\tilde{P}(x,y)\mu_L(x,y).
\end{aligned}$$

This proves the lemma.

End of proof of Proposition 4.10: We return to the evaluation of the integral

$$J := \int_W \log(x-\tau_1 y)\tilde{P}(x,y)\left(|L_2|^{-k_0/2}\mu_{L_2} - |L_1|^{-k_0/2}\mu_{L_1}\right)$$

appearing in (23). It is useful to express J as a sum of two contributions J_e and $J_{\bar{e}}$ obtained by integrating over the disjoint subsets W_e and $W_{\bar{e}}$ of W associated to the ordered edge $e = ([L_1],[L_2])$ of $\mathscr{T}$. By Lemma 4.3,

$$\begin{aligned}
J_e &= \int_{W_e} \log(x-\tau_1 y)\tilde{P}(x,y)\left(|L_2|^{-k_0/2}a_p - |L_1|^{-k_0/2}\right)\mu_{L_1}(x,y) \\
&= |L_1|^{-k_0/2}\int_{W_e} \log(x-\tau_1 y)\tilde{P}(x,y)\left((p^{-k_0/2}a_p-1)\mu_{L_1}\right)(x,y).
\end{aligned}$$

Now applying Lemma 4.11 with $\alpha = p^{-k_0/2}a_p - 1$, and noting that $\alpha(k_0) = 0$, we find

$$J_e = p^{-k_0/2}a_p'(k_0)c_\phi(e)(P). \tag{24}$$

On the other hand, $\bar{e} = ([(1/p)L_2], [L_1])$ and $[(1/p)L_2 : L_1] = p$ so we have $\mu_{L_1}|_{W_{\bar{e}}} = a_p \mu_{(1/p)L_2}|_{W_{\bar{e}}}$ and the same computation gives:

$$\begin{aligned} J_{\bar{e}} &:= \int_{W_{\bar{e}}} \log(x - \tau_1 y) \tilde{P}(x,y) \left(|L_2|^{-k_0/2} \mu_{L_2} - |L_1|^{-k_0/2} \mu_{L_1} \right) \\ &= -p^{-k_0/2} a_p'(k_0) c_\phi(\bar{e})(P) = p^{-k_0/2} a_p'(k_0) c_\phi(e)(P). \end{aligned}$$

Therefore

$$J = J_e + J_{\bar{e}} = 2p^{-k_0/2} a_p'(k_0) c_\phi(e)(P).$$

This concludes the proof of Proposition 4.10. □

We are now able to prove Theorem 3 of the introduction:

Theorem 4.12. — *Let $\mathscr{L}_T(f)$ denote Teitelbaum's L-invariant attached to f. Then*

$$-2p^{-k_0/2} a_p'(k_0) = \mathscr{L}_T(f).$$

Proof. — Let Φ be the family of automorphic forms associated to f_∞ by Theorem 3.10. Fix $\tau \in \mathscr{H}_p(\mathbb{Q}_p^{ur})$ and let $v_\tau = [L_\tau] \in \mathscr{T}_0$ be the corresponding vertex. Let $h_\tau \in V_{k_0} \otimes \mathbb{C}_p$ be the map sending $P \in \mathscr{P}_{k_0}$ to

$$h_\tau(P) := \int^\tau \psi(z) P(z) dz.$$

For all $\gamma \in \Gamma$ and $P \in \mathscr{P}_{k_0}$, Proposition 4.10 gives

$$h_{\gamma\tau}(P) - h_\tau(P) = \int_\tau^{\gamma\tau} \psi(z) P(z) dz + 2p^{-k_0/2} a_p'(k_0) \sum_{e: v_\tau \to \gamma(v_\tau)} c_\phi(e)(P).$$

In the notations of Section 2 this formula can be rewritten as

$$h_{\gamma\tau} - h_\tau = \kappa_\phi^{\log}(\gamma) + 2p^{-k_0/2} a_p'(k_0) \kappa_\phi^{\mathrm{ord}}(\gamma).$$

On the other hand, Proposition 4.9 implies that

$$h_{\gamma\tau} - h_\tau = \gamma h_\tau - h_\tau$$

is a $V_{k_0} \otimes \mathbb{C}_p$-valued coboundary for Γ. It follows that

$$[\kappa_\phi^{\log}] = -2p^{-k_0/2} a_p'(k_0) [\kappa_\phi^{\mathrm{ord}}].$$

Theorem 4.12 now follows from Definition 2.3 of $\mathscr{L}_T(f)$. □

5. Orton's $\mathscr{L}$-invariant

This section recalls the definition of Orton's $\mathscr{L}$-invariant, which involves the theory of modular symbols. The reader is referred to **[19]** for more details.

Write Δ for the group $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ of degree zero divisors supported on the rational cusps of the Poincaré upper half plane. For any unitary commutative ring A of $\mathbb{C}$, let $\mathscr{P}_k(A)$ denote the A-algebra of polynomials of degree $\leq k$ with coefficients in A, and let $V_k(A)$ be the A-dual of $\mathscr{P}_k(A)$. When A is a subfield of $\mathbb{C}$, the group $\mathbf{GL}_2(\mathbb{Q})$ acts on the right on $\mathscr{P}_k(A)$ by the rule

$$(P\gamma)(z) = (cz+d)^k P(\gamma z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This induces a *right* action of $\mathbf{GL}_2(\mathbb{Q})$ on $V_k(A)$ by setting

$$(\phi\gamma)(P) = \phi(P\gamma^{-1}).$$

A modular symbol with values in a $\mathbf{GL}_2(\mathbb{Q})$-module $\mathscr{V}$ is a homomorphism from Δ to $\mathscr{V}$. The space of all such modular symbols is denoted

$$MS(\mathscr{V}) := \hom(\Delta, \mathscr{V}).$$

It is equipped with a right $\mathbf{GL}_2(\mathbb{Q})$-action by the rule

$$(m\gamma)(\delta) = m(\gamma\delta)\gamma,$$

for $m \in MS(\mathscr{V})$, $\delta \in \Delta$, and $\gamma \in \mathbf{GL}_2(\mathbb{Q})$. If the divisor δ is of the form $(s)-(r)$, write $m\{r \to s\}$ for $m(\delta)$.

A modular eigenform g of weight $k+2$ on $\Gamma_0(Np)$ gives rise to a $\Gamma_0(Np)$-invariant $V_k(\mathbb{C})$-valued modular symbol

$$\Psi_g : \Delta \longrightarrow V_k(\mathbb{C})$$

by the rule

$$\Psi_g(\delta)(P) = 2\pi i \int_\delta g(z)P(z)dz,$$

with $\delta \in \Delta$ and $P \in \mathscr{P}_k(\mathbb{C})$. Write $\Psi_g^{\pm}$ for the projection of Ψ_g to the $\pm$-eigenspace of $\mathrm{Hom}(\Delta, V_k(\mathbb{C}))$ for the action of the involution

$$c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let K_g be the extension of $\mathbb{Q}$ generated by the Hecke eigenvalues of g. By a result of Shimura, there exist complex periods $\Omega_g^{\pm}$ such that

$$\Phi_g^{\pm} = \Psi_g^{\pm}/\Omega_g^{\pm}$$

takes values in $V_k(K_g)$. Note that the modular symbols Ψ_g and $\Phi_g^{\pm}$ are all $\Gamma_0(Np)$-invariant.

Let f be the newform on $\Gamma_0(Np)$ considered in the introduction. Fix a choice of sign $w_\infty \in \{-1, 1\}$ and let

$$\Phi_f = \begin{cases} \Phi_f^+ & \text{if } w_\infty = 1; \\ \Phi_f^- & \text{if } w_\infty = -1. \end{cases}$$

be the modular symbol in $MS(V_{k_0}(K_f))$ attached to f. Define

$$\tilde{\Gamma} = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}[1/p]) : N \mid c \;\text{ and }\; \det(\gamma) = p^{2h}, \;\text{for } h \in \mathbb{Z} \right\}.$$

Write Γ for the group of elements in $\tilde{\Gamma}$ having determinant one. For $(s)-(r) \in \Delta$, define a harmonic cocycle

$$c_f\{r \to s\} : \mathscr{E}(\mathscr{T}) \longrightarrow V_{k_0}(\bar{\mathbb{Q}}_p)$$

by the rule

$$c_f\{r \to s\}(e)(P) = \Phi_f((\gamma s) - (\gamma r))(P\gamma^{-1}),$$

where $\gamma \in \Gamma$ is such that $\gamma e = e_\infty$. This definition is independent of the choice of γ such that $\gamma e = e_\infty$: for if γ' is another such element, the element $\gamma'\gamma^{-1}$ belongs to $\Gamma_0(Np)$, the stabiliser of e_∞. The claim then follows from the $\Gamma_0(Np)$-invariance of Φ_f.

The cocycle $c_f\{r \to s\}$ gives rise to a locally analytic distribution on $\mathbb{P}^1(\mathbb{Q}_p)$, denoted $\mu_f\{r \to s\}$, and determined by setting

$$\int_{U_e} P(t)\mu_f\{r \to s\}(t) = c_f\{r \to s\}(e)(P)$$

for all $e \in \mathscr{E}(\mathscr{T})$ and $P \in \mathscr{P}_{k_0}(\bar{\mathbb{Q}}_p)$, and extending to functions on $\mathbb{P}^1(\mathbb{Q}_p)$ which are locally analytic on $\mathbb{Q}_p$ and have a pole of order at most k_0 at infinity. Note the analogy between this definition and the definition of the locally analytic distribution μ_ϕ in equation (11) of Section 2.

The following definition is modelled on the description of the Coleman line integral given in equation (13) of Section 2.

Definition 5.1. — For $\tau_1, \tau_2 \in \mathscr{H}_p$ and $r, s \in \mathbb{P}^1(\mathbb{Q})$, the *definite double integral* is defined by

$$\int_{\tau_1}^{\tau_2}\int_r^s \omega_f P = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log\left(\frac{t-\tau_2}{t-\tau_1}\right) P(t)\mu_f\{r \to s\}(t).$$

The notation ω_f in Definition 5.1 is meant to suggest that the definite double integral should be thought of as the integration of a form of parallel weight (k_0+2, k_0+2) on $\mathscr{H}_p \times \mathscr{H}$ associated to f. This point of view is explained in detail in [**13**], Chapter 9 and [**19**], Chapter 2.

Set $\mathscr{P}_k = \mathscr{P}_k(\bar{\mathbb{Q}}_p)$, $V_k = V_k(\bar{\mathbb{Q}}_p)$ and write

$$\mathscr{M}_k := MS(V_k) = \mathrm{Hom}(\Delta, V_k).$$

The following definitions are motivated by the definition of the 1-cocycles $\kappa_\phi^{\mathrm{ord}}$ and $\kappa_\phi^{\log}$ given in equations (14) and (15) respectively.

Definition 5.2. —

1. The 1-cocycle $\kappa_f^{\mathrm{ord}} \in Z^1(\Gamma, \mathscr{M}_{k_0})$ is defined by choosing $v \in \mathscr{T}_0$ and setting

$$\kappa_f^{\mathrm{ord}}(\gamma)\{r \to s\}(P) = \sum_{e : v \to \gamma v} c_f\{r \to s\}(P)(e).$$

2. The 1-cocycle $\kappa_f^{\log} \in Z^1(\Gamma, \mathscr{M}_{k_0})$ is defined by choosing $\tau \in \mathscr{H}_p$ and setting

$$\kappa_f^{\log}(\gamma)\{r \to s\}(P) = \int_\tau^{\gamma\tau} \int_r^s \omega_f P.$$

Lemma 5.3. —

1. *The image* $[\kappa_f^{\mathrm{ord}}]$ *of* κ_f^{ord} *in* $H^1(\Gamma, \mathscr{M}_{k_0})$ *is independent of the choice of base vertex* v.
2. *The image* $[\kappa_f^{\log}]$ *of* $\kappa_f^{\log}$ *in* $H^1(\Gamma, \mathscr{M}_{k_0})$ *is independent of the choice of base point* $\tau \in \mathscr{H}_p$.

Proof. — See [**19**], Lemma 5.1 and 5.2. Note that the one-cocycles κ_f^{ord} and $\kappa_f^{\log}$ are denoted $\widetilde{oc}_{f,v}$ and $\widetilde{lc}_{f,\tau}$ respectively in [**19**]. □

Proposition 5.4. — *The class* $[\kappa_f^{\mathrm{ord}}]$ *is non-zero.*

Proof. — Proposition 13 of Section II.2.8 of [**21**], applied to the case $M = \mathscr{M}_{k_0}$ and $G = \Gamma$ acting on $\mathscr{T}$, yields a linear transformation

$$\theta : H^0(\Gamma_0(Np), \mathscr{M}_{k_0}) \xrightarrow{\theta} H^1(\Gamma, \mathscr{M}_{k_0})$$

whose kernel is identified with the p-old subspace of the space of modular symbols on $\Gamma_0(Np)$. The map θ is described explicitly in Section 3.1 of [**12**], where it is shown that $\theta(\Phi_f) = [\kappa_f^{\mathrm{ord}}]$. (Although [**12**] assumes $k_0 = 0$, the treatment of the general case is no different.) Proposition 5.4 follows from the fact that the form f is new at p. □

Let

$$H^1(\Gamma, \mathscr{M}_{k_0})^f, \quad H^1(\Gamma, \mathscr{M}_{k_0})^{f,w_\infty} \subset H^1(\Gamma, \mathscr{M}_{k_0})$$

denote, respectively, the f-isotypic subspace and its w_∞-eigenspace for the action of the involution c defined at the beginning of this section. The classes $[\kappa_f^{\mathrm{ord}}]$ and $[\kappa_f^{\log}]$ both belong to $H^1(\Gamma, \mathscr{M}_{k_0})^{f,w_\infty}$. In [**19**], Proposition 7.1, it is shown that this space is one-dimensional over $\mathbb{C}_p$. This makes it possible to define Orton's $\mathscr{L}$-invariant $\mathscr{L}_O(f)$ in a way which parallels closely Definition 2.3 of Teitelbaum's $\mathscr{L}$-invariant.

Definition 5.5. — The Orton L-invariant attached to f is the unique scalar $\mathscr{L}_O(f) \in \mathbb{C}_p$ such that

$$[\kappa_f^{\log}] = \mathscr{L}_O(f)[\kappa_f^{\mathrm{ord}}].$$

Remark 5.6. — Note that $\mathscr{L}_O(f)$ depends a priori on the choice of sign w_∞ which determines whether Φ_f is taken to be the even or odd modular symbol attached to f. Hence there are two a priori distinct Orton $\mathscr{L}$-invariants attached to f, which could be denoted $\mathscr{L}_O^+(f)$ and $\mathscr{L}_O^-(f)$. A by-product of our study of $\mathscr{L}_O(f)$ is a direct proof that these two invariants are in fact equal. (Cf. Theorem 6.8.)

Recall the Shimura period Ω_f (depending on the choice of sign w_∞) that was used to define the modular symbol attached to f. Let $L_p(f,\chi,s)$ denote the Mazur-Tate-Teitelbaum p-adic L-function attached to f and χ, which is constructed in terms of the modular symbol of f and hence depends on the choice of Ω_f. Set $w = 1$ if f is split multiplicative at p, and $w = -1$ if f is non-split multiplicative at p. The following theorem of Orton is crucial for the proof of Theorem 2 of the Introduction.

***Theorem 5.7* (Orton).** — *For all Dirichlet characters χ satisfying $\chi(p) = w$ and $\chi(-1) = w_\infty$,*

$$L'_p(f,\chi,1+k_0/2) = \mathscr{L}_O(f)L^*(f,\chi,1+k_0/2).$$

It is worth noting that it is at this stage, and this stage only, that a connection is made between the cohomologically-defined $\mathscr{L}$-invariants and special values of L-series.

Let us briefly recall some of the ideas that go in Orton's proof of Theorem 5.7. Fix a positive integer c prime to Np. For any positive integer ν prime to c, define an embedding $\Psi_\nu : \mathbb{Q}\times\mathbb{Q} \to M_2(\mathbb{Q})$ by setting

$$\Psi_\nu(a,a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad \Psi_\nu(c,0) = \begin{pmatrix} c & \nu \\ 0 & 0 \end{pmatrix}.$$

When ν varies in a full set of representatives for $(\mathbb{Z}/c\mathbb{Z})^\times$, Ψ_ν describes the set of all Γ-conjugacy classes of oriented optimal embeddings of conductor c: see **[12]**, Section 2. Set

$$r_{\Psi_\nu} = \infty, \quad s_{\Psi_\nu} = -\nu/c, \quad \gamma_{\Psi_\nu} = \Psi_\nu(p^u,p^{-u}) = \begin{pmatrix} p^u & (p^u - p^{-u})\nu/c \\ 0 & p^{-u} \end{pmatrix},$$

where u denotes the order of p^2 in $(\mathbb{Z}/c\mathbb{Z})^\times$. The element γ_{Ψ_ν} is a generator for the image of $\Psi(\mathbb{Q}^\times\times\mathbb{Q}^\times)\cap\Gamma$ in $\mathbf{PGL}_2(\mathbb{Q})$. Moreover, r_{Ψ_ν}, resp., s_{Ψ_ν} is the repulsive, resp., attractive fixed point for the action of $\Psi(\gamma_{\Psi_\nu})$ on $\mathbb{P}^1(\mathbb{Q})$. Define the polynomial

$$P_{\Psi_\nu}(z) = (cz+\nu)^{k_0/2} \in \mathscr{P}_{k_0}.$$

Note that P_{Ψ_ν} is invariant under the weight k_0+2 action of γ_{Ψ_ν}.

The one-cocycles κ_f^{ord} and $\kappa_f^{\log}$ can be used to associate to the embedding Ψ_ν the following numerical invariants:

$$(25)\qquad J_{\Psi_\nu}^{\mathrm{ord}} = \kappa_f^{\mathrm{ord}}(\gamma_{\Psi_\nu})\{r_{\Psi_\nu}\to s_{\Psi_\nu}\}(P_{\Psi_\nu}) = \sum_{e:v\to\gamma_{\Psi_\nu}v} c_f\{r_{\Psi_\nu}\to s_{\Psi_\nu}\}(P_{\Psi_\nu})(e);$$

$$(26)\qquad J_{\Psi_\nu}^{\log} = \kappa_f^{\log}(\gamma_{\Psi_\nu})\{r_{\Psi_\nu}\to s_{\Psi_\nu}\}(P_{\Psi_\nu}) = \int_{\tau_{\Psi_\nu}}^{\gamma\tau_{\Psi_\nu}}\int_{r_{\Psi_\nu}}^{s_{\Psi_\nu}} \omega_f P_{\Psi_\nu}.$$

Remark 5.8. — Note that

$$b(\gamma_{\Psi_\nu})\{r_{\Psi_\nu}\to s_{\Psi_\nu}\}(P_{\Psi_\nu}) = 0$$

for any coboundary $b \in B^1(\Gamma, \mathscr{M}_{k_0})$. This implies that the quantities $J_{\Psi_\nu}^{\mathrm{ord}}$ and $J_{\Psi_\nu}^{\log}$ do not depend on the choice of cocycles representing the cohomology classes $[\kappa_f^{\mathrm{ord}}]$ and $[\kappa_f^{\log}]$ respectively, and hence, by the definitiono f $\mathscr{L}_O(f)$, that

$$J_{\Psi_\nu}^{\log} = \mathscr{L}_O(f) J_{\Psi_\nu}^{\mathrm{ord}}. \tag{27}$$

Let χ be a Dirichlet character of conductor c, such that $\chi(p) = w$ and $\chi(-1) = w_\infty$. The following formula of Orton relates the numerical invariants $J_{\Psi_\nu}^{\mathrm{ord}}$ and $J_{\Psi_\nu}^{\log}$ defined in (25) and (26) to special values of L-series, and derivatives of the corresponding p-adic L-functions, respectively:

$$\sum_{\nu \in (\mathbb{Z}/c\mathbb{Z})^\times} \chi(\nu) J_{\Psi_\nu}^{\mathrm{ord}} = (2u) L^*(f, \chi, 1 + k_0/2); \tag{28}$$

$$\sum_{\nu \in (\mathbb{Z}/c\mathbb{Z})^\times} \chi(\nu) J_{\Psi_\nu}^{\log} = (2u) L_p'(f, \chi, 1 + k_0/2). \tag{29}$$

The first formula is Corollary 6.1 of **[19]**, while the second formula is Corollary 6.2 of **[19]**.

Theorem 5.7 now follows directly from these formulae and equation (27).

6. Distribution-valued modular symbols

Recall from the introduction the p-adic family of eigenforms

$$f_\infty = \sum_{n=1}^{\infty} a_n q^n, \qquad \text{with } a_n \in A(U)$$

interpolating the given newform $f = \sum_{n=1}^{\infty} a_n q^n$ of weight $k_0 + 2$ on $\Gamma_0(Np)$. This means that

$$f_k = \sum_{n=1}^{\infty} a(k) q^n$$

is a normalised eigenform of weight $k+2$ on $\Gamma_1(N) \cap \Gamma_0(p)$, for all $k \in U \cap \mathbb{Z}_{\geq 0}$, and that $f_{k_0} = f$. Let Φ_{f_k} be the modular symbol in $MS(V_k)$ defined in section 5, associated to the choice of sign w_∞. Note that Φ_{f_k} also depends on a choice of complex period Ω_{f_k}, and thus is only really well-defined up to multiplication by a non-zero scalar. Two modular symbols m_1 and m_2 in $MS(V_k)$ are said to be equivalent if there exists a non-zero scalar $\lambda \in \mathbb{C}_p^\times$ such that $m_1 = \lambda m_2$; one then writes $m_1 \sim m_2$.

Assume throughout this section that $f = f_{k_0}$ is *split* multiplicative at p, so that

$$w = +1, \qquad a_p(k_0) = p^{k_0/2}.$$

As in section 3, let $\mathbb{D}$ be the space of locally analytic distributions on L'_*, with $L_* = \mathbb{Z}_p^2$. Recall that the $\mathbb{Q}_p$-algebra R of locally analytic distributions on $\mathbb{Z}_p^\times$ acts on $\mathbb{D}$. The space $MS_{\Gamma_0(N)}(\mathbb{D})$ of $\Gamma_0(N)$-invariant $\mathbb{D}$-valued modular symbols is equipped with a natural action of the Hecke operators T_n with $p \nmid n$, as well as an action of R arising from the R-module structure on $\mathbb{D}$. Let us fix as in the previous sections an affinoid disk U, defined over K, containing k_0 and contained in the weight space $\mathscr{W}$ and let $\mathbb{D}_U := \mathbb{D}\hat{\otimes}_R A(U)$.

Proposition 6.1. — *There exists a distribution-valued modular symbol $\Phi_{f_\infty} \in MS(\mathbb{D}_U)$ satisfying the following properties:*

1. *($\Gamma_1(N)$-invariance) Φ_{f_∞} is $\Gamma_0(N)$-invariant, that is,*
$$\Phi_{f_\infty}(\gamma\delta)\cdot\gamma = \Phi_{f_\infty}(\delta)$$
for all $\gamma \in \Gamma_1(N)$.
2. *(Weight specialisation) Following the notations of Definition 3.4, for $k \in U(K)\cap \mathbb{Z}_{\geq 0}$ and $P \in \mathscr{P}_k$, define a V_k-valued modular symbol*
$$\rho_k(\Phi_{f_\infty}) : \Delta \longrightarrow V_k$$
by the rule
$$\rho_k(\Phi_{f_\infty})(\delta)(P) = \int_{W_\infty} \tilde{P}(x,y)\Phi_{f_\infty}(\delta)(x,y).$$
Then,
$$\rho_k(\Phi_{f_\infty}) \sim \Phi_{f_k}, \qquad \textit{and } \rho_{k_0}(\Phi_{f_\infty}) = \Phi_f.$$

Proof. — When f has weight 2 (i.e., $k_0 = 0$), the existence of a modular symbol with values in the module of bounded distributions on L_* is proved in [**15**]. In general, it follows from results of Stevens in [**22**]. □

For a divisor $\delta = (s)-(r)$ in Δ, write $\mu_{L_*}\{r \to s\}$ for the locally analytic distribution $\Phi_{f_\infty}(\delta)$. It will be viewed as a distribution on W, supported on L'_*.

Definition 6.2. — For any lattice L in $\mathbb{Q}_p^2$, the locally analytic distribution $\mu_L\{r \to s\}$ on W is defined by the rule
$$\int_W F(x,y)\mu_L\{r\to s\}(x,y) = \int_{L'_*} F(g^{-1}(x,y))\mu_{L_*}\{gr \to gs\}(x,y),$$
where $F : W \longrightarrow \mathbb{Q}_p$ is any locally analytic function, and $g \in \tilde{\Gamma}$ is any element such that $gL = L_*$.

Note that the above definition does not depend on the choice of $g \in \tilde{\Gamma}$ such that $gL = L_*$: if g' is another element of $\tilde{\Gamma}$ such that $g'L = L_*$, it follows that $g'g^{-1}$ belongs to the stabiliser of L_* in $\tilde{\Gamma}$, which is the group $\Gamma_0(N)$. The claim then follows from the $\Gamma_0(N)$-invariance of Φ_{f_∞}, stated in part 1 of Proposition 6.1.

The system of distributions $\mu_L\{r \to s\}$ satisfies similar properties to those of the system μ_L introduced in section 4. Since the proofs of these new properties are analogous to those presented in section 4, details are usually omitted.

Lemma 6.3. — *Let $\kappa \in U(K)$, $L_2 \subset L_1$ be $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$ with $[L_1 : L_2] = p$, and let $e = ([L_1],[L_2]) \in \mathscr{E}(\mathscr{T})$ be the corresponding edge. Then*
$$\mu_{L_2}\{r\to s\}|_{W_e} = a_p\mu_{L_1}\{r \to s\}|_{W_e},$$

so that

$$\int_{W_e} F(x,y)\mu_{L_2}\{r\to s\}(x,y) = \int_{W_e} F(x,y)(a_p\mu_{L_1}\{r\to s\})(x,y),$$

for every locally analytic function $F\in\mathscr{A}^{(\kappa)}(W)$.

Proof. — The proof is identical to that of lemma 4.3. □

Let π be as before the projection of W onto $\mathbb{P}^1(\mathbb{Q}_p)$. For every $\mathbb{Z}_p$-lattice $L\subset\mathbb{Q}_p^2$, define a locally analytic distribution $\pi_*(\mu_L\{r\to s\})$ on $\mathbb{P}^1(\mathbb{Q}_p)$ by the formula

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} P(t)\pi_*(\mu_L\{r\to s\})(t) = |L|^{-k_0/2}\int_W \eta^{k_0}P(x/y)\mu_L\{r\to s\}(x,y),$$

where P is any locally meromorphic function on $\mathbb{P}^1(\mathbb{Q}_p)$ with a pole of order at most k_0 at ∞.

Proposition 6.4. — *For all* $\mathbb{Z}_p$*-lattices* L *in* $\mathbb{Q}_p^2$,

$$\pi_*(\mu_L\{r\to s\}) = \mu_f\{r\to s\},$$

where μ_f *is the locally analytic distribution on* $\mathbb{P}^1(\mathbb{Q}_p)$ *defined in section 5.*

Proof. — The proof is similar to that of Proposition 4.4. It uses lemma 6.3 instead of lemma 4.3, and part 3 of Proposition 6.1, which guarantees that the specialisation at k_0 of Φ_{f_∞} is the modular symbol attached to f. □

Let $\tau\in\mathscr{H}_p(\mathbb{Q}_p^{ur})$ and let $v_\tau=[L_\tau]\in\mathscr{T}_0$ be the vertex corresponding to τ under the reduction map. The following definition is modelled on that of the indefinite integral of Definition 4.8:

Definition 6.5. — For all $P\in\mathscr{P}_{k_0}$, the *indefinite integral* attached to $\tau\in\mathscr{H}_p$, to $r,s\in\mathbb{P}^1(\mathbb{Q})$, and to f is defined by the formula

$$\int^{\tau}\!\!\int_r^s \omega_f P = |L_\tau|^{-k_0/2}\int_W \log(x-\tau y)\tilde{P}(x,y)\mu_{L_\tau}\{r\to s\}(x,y). \tag{30}$$

Since

$$\log(px-p\tau y)\tilde{P}(px,py) = p^{k_0}\log(x-\tau y)\tilde{P}(x,y),$$

formula (30) depends only on the homothety class of L_τ, and hence only on τ. The main properties of the indefinite double integral of Definition 6.5 are summarized in the following two propositions.

Proposition 6.6. — *For all* $\gamma\in\Gamma$ *and* $P\in\mathscr{P}_{k_0}$,

$$\int^{\gamma\tau}\!\!\int_{\gamma r}^{\gamma s}\omega_f P = \int^{\tau}\!\!\int_r^s \omega_f P\ \gamma.$$

Proof. — It is identical to the proof of Proposition 4.9. □

Proposition 6.7. — *Let $\tau_1, \tau_2 \in \mathscr{H}_p(\mathbb{Q}_p^{ur})$, and let $v_1 = [L_1], v_2 = [L_2] \in \mathscr{T}_0$ be the corresponding vertices under the reduction map. For all $P \in \mathscr{P}_{k_0}$,*

$$\int^{\tau_2}\int_r^s \omega_f P - \int^{\tau_1}\int_r^s \omega_f P$$
$$= \int_{\tau_1}^{\tau_2}\int_r^s \omega_f P + 2p^{-k_0/2} a_p'(k_0) \sum_{e: v_1 \to v_2} c_f\{r \to s\}(e)(P).$$

Proof. — Assume without loss of generality that $L_2 \subset L_1$ and $[L_1 : L_2] = p$. Set $e = ([L_1], [L_2]) \in \mathscr{E}(\mathscr{T})$. Then

$$\int^{\tau_2}\int_r^s \omega_f P - \int^{\tau_1}\int_r^s \omega_f P$$
$$= |L_2|^{-k_0/2} \int_W \log(x - \tau_2 y)\tilde{P}(x,y)\mu_{L_2}\{r \to s\}(x,y)$$
$$- |L_1|^{-k_0/2} \int_W \log(x - \tau_1 y)\tilde{P}(x,y)\mu_{L_1}\{r \to s\}(x,y)$$
$$= I_{\log} + I_{\text{ord}},$$

where

$$I_{\log} = |L_2|^{-k_0/2} \int_W \log\left(\frac{x - \tau_2 y}{x - \tau_1 y}\right) \tilde{P}(x,y)\mu_{L_2}\{r \to s\}(x,y),$$

and

$$I_{\text{ord}} = \int_W \log(x - \tau_1 y)\tilde{P}(x,y) \left(|L_2|^{-k_0/2}\mu_{L_2}\{r \to s\} - |L_1|^{-k_0/2}\mu_{L_1}\{r \to s\}\right).$$

Using Proposition 6.4, and the fact that the function involved in the integral defining $I_{\log}$ is constant along the fibers of π, one finds that

$$I_{\log} = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log\left(\frac{t - \tau_2}{t - \tau_1}\right) P(t)\mu_f\{r \to s\}(t) = \int_{\tau_1}^{\tau_2}\int_r^s \omega_f P.$$

Now, write the integral defining I_{ord} as the sum of two contributions J_e and $J_{\bar{e}}$, obtained by integrating over the disjoint subsets W_e and $W_{\bar{e}}$. By Lemma 6.3,

$$\begin{aligned} J_e &= \int_{W_e} \log(x - \tau_1 y)\tilde{P}(x,y) \left(|L_2|^{-k_0/2}\mu_{L_2} - |L_1|^{-k_0/2}\mu_{L_1}\right)\{r \to s\}(x,y) \\ &= \int_{W_e} \log(x - \tau_1 y)\tilde{P}(x,y) \left(|L_2|^{-k_0/2}a_p - |L_1|^{-k_0/2}\right)\mu_{L_1}\{r \to s\}(x,y) \\ &= |L_1|^{-k_0/2} \int_{W_e} \log(x - \tau_1 y)\tilde{P}(x,y) \left((p^{-k_0/2}a_p - 1)\mu_{L_1}\{r \to s\}\right)(x,y). \end{aligned}$$

The formula

$$\int_{W_e} \log(x - \tau y)\tilde{P}(x,y)(\alpha\mu_L\{r \to s\})(x,y) = \tag{31}$$
$$\alpha'(k_0)|L|^{k_0/2}c_f\{r \to s\}(e)(P) + \alpha(k_0)\int_{W_e} \log(x - \tau y)\tilde{P}(x,y)\mu_L\{r \to s\}(x,y)$$

($e \in \mathscr{E}(\mathscr{T})$, $\alpha \in A(U)$, $\tau \in \mathscr{H}_p$, and $P \in \mathscr{P}_{k_0}$) is obtained by adapting the approach that is followed in the proof of lemma 4.11. By applying (31) with $\alpha = p^{-k_0/2}a_p - 1$, the above expression for J_e becomes

$$J_e = p^{-k_0/2} a_p'(k_0) c_f\{r \to s\}(e)(P).$$

Moreover, a similar argument proves that

$$J_{\bar{e}} = -p^{-k_0/2} a_p'(k_0) c_f\{r \to s\}(\bar{e})(P) = p^{-k_0/2} a_p'(k_0) c_f\{r \to s\}(e)(P).$$

Hence,

$$J_{\text{ord}} = J_e + J_{\bar{e}} = 2p^{-k_0/2} a_p'(k_0) c_f\{r \to s\}(e)(P),$$

as was to be shown. □

We are now ready to prove the main result of this section.

Theorem 6.8. — *The equality*

$$-2p^{-k_0/2} a_p'(k_0) = \mathscr{L}_O(f)$$

holds. In particular, Orton's $\mathscr{L}$-invariant $\mathscr{L}_O(f)$ is independent of the choice of sign w_∞ that was made in defining it.

Proof. — Let τ be a point in $\mathscr{H}_p(\mathbb{Q}_p^{ur})$, and let $v_\tau = [L_\tau] \in \mathscr{T}_0$ be the corresponding vertex. Fix a divisor $(s)-(r)$ in Δ. Given $\tau \in \mathscr{H}_p$, one defines an $\mathscr{P}_{k_0}^\vee$-valued modular symbol h_τ by the rule

$$h_\tau\{r \to s\}(P) = \int^\tau \int_r^s \omega_f P.$$

Proposition 6.7 gives

$$h_\tau\{\gamma^{-1}r \to \gamma^{-1}s\}(\gamma^{-1}P) - h_\tau\{r \to s\}(P) = h_{\gamma\tau}\{r \to s\}(P) - h_\tau\{r \to s\}(P)$$

$$= \int_\tau^{\gamma\tau} \int_r^s \omega_f P + 2p^{-k_0/2} a_p'(k_0) \sum_{e: v_\tau \to \gamma(v_\tau)} c_f\{r \to s\}(e)(P), \tag{32}$$

for all $\gamma \in \Gamma$. In the notations of Definition 5.2 of Section 5, this relation can be rewritten as

$$\begin{aligned} & h_\tau\{\gamma^{-1}r \to \gamma^{-1}s\}(\gamma^{-1}P) - h_\tau\{r \to s\}(P) \\ &= \kappa_f^{\log}(\gamma)\{r \to s\} + 2p^{-k_0/2} a_p'(k_0) \kappa_f^{\text{ord}}(\gamma)\{r \to s\}. \end{aligned} \tag{33}$$

Since the expression on the left of (33) is a $\mathscr{M}_{k_0}$-valued one-coboundary, it follows upon projecting this equation to $H^1(\Gamma, \mathscr{M}_{k_0})$ that

$$[\kappa_f^{\log}] = -2p^{-k_0/2} a_p'(k_0) [\kappa_f^{\text{ord}}].$$

Theorem 6.8 is now a direct consequence of Definition 5.5 of $\mathscr{L}_O(f)$. □

Corollary 6.9. —

1. *The equality $\mathscr{L}_T(f) = \mathscr{L}_0(f)$ holds.*

2. *For all Dirichlet characters χ satisfying $\chi(p) = 1$,*

$$L'_p(f, \chi, 1 + k_0/2) = \mathscr{L}_T(f) L^*(f, \chi, 1 + k_0/2).$$

Proof. — Part 1 follows by combining theorem 6.8 with theorem 4.12. Part 2 follows by combining part 1 of this corollary with theorem 5.7. □

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