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HIDDEN STRUCTURES ON SEMISTABLE CURVES

by

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Abstract. — Let V be the ring of integers of a finite extension of $\mathbb{Q}_p$ and let X be a proper curve over V with semistable special fiber and smooth generic fiber. In this article we explicitly describe the Frobenius and monodromy operators on the log crystalline cohomology of X with values in a regular log F-isocrystal in terms of p-adic integration. We have a version for open curves and as an application we prove that two differently defined $\mathscr{L}$-invariants, attached to a split multiplicative at p new elliptic eigenform, are equal.

Résumé **(Structures cachées sur les courbes semi-stables).** — Soit V l'anneau des entiers d'une extension finie de $\mathbb{Q}_p$ et soit X une courbe propre sur V à fibre spéciale semistable et à fibre générique lisse. Dans cet article nous décrivons explicitement les opérateurs de Frobenius et de monodromie sur la cohomologie log cristalline de X à valeurs dans un log F-isocristal régulier, en termes d'intégration p-adique. Nous proposons une version pour les courbes ouvertes et en guise d'application nous prouvons que deux $\mathscr{L}$-invariants définis de façon différente, attachés à une forme modulaire nouvelle multiplicative en p, sont égaux.

1. Introduction

Let K be a finite extension of $\mathbb{Q}_p$ and X an algebraic variety over K. As Illusie remarked in *Cohomologie de de Rham et cohomologie étale p-adique* [I], "le groupe $H^1_{dR}(X/K)$ se trouve muni d'une structure plus riche qu'il n'y paraît de prime abord." This "hidden structure" has been discussed by many people including Berthelot and Ogus [BO] when X is proper with good reduction and more generally by Hyodo and Kato [HK]. In this paper, we expose it in the relative situation over a curve with semi-stable reduction using residues and p-adic integration. More precisely we study de Rham cohomology of a semi-stable curve with coefficients in the relative cohomology of a smooth proper family over that curve. The information on crystalline and de

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Rham cohomology of a curve with semi-stable reduction supplied by this article is similar to that of the theory of vanishing cycles for ℓ-adic cohomology.

Suppose K has residue field k and ring of integers V. Let $W := W(k)$ denote the ring of Witt-vectors with coefficients in k, K_0 its fraction field and we denote by σ the Frobenius automorphism of K_0. Let C_K be a smooth projective curve over K with a semi-stable model C over V. By this we mean that locally C is smooth over $\mathrm{Spec}(V)$ or étale over $\mathrm{Spec}\,(V[X,Y]/(XY-\pi))$, where π is a uniformizer of V. Denote by $\overline{C} := C \times_{\mathrm{Spec(V)}} \mathrm{Spec}(k)$, its special fiber and by Sing, the singular sub-scheme of $\overline{C}$.

Then the vector space $H^1_{dR}(C_K)$ has enough hidden structure so that one can recover the corresponding representation of $G_K = \mathrm{Gal}(\overline{K}/K)$ on the étale cohomology of $C_{\overline{K}}$, à la Fontaine. I.e. besides the Hodge filtration it has a K_0-lattice (the log-crystalline cohomology of $\overline{C}$ with $\mathbb{Q}_p$-coefficients) with linear monodromy and σ-semi-linear Frobenius operators. One can use this to describe the representation. This is true much more generally (see for example **[18]** and **[39]**.)

Let $g\colon Z \longrightarrow C$ be a flat proper morphism. Suppose P is a sub-scheme of C, finite and étale over V whose reduction is disjoint from Sing. Let $C^\times$ be the log formal scheme over V associated to the pair (C,P) (i.e. the formal completion of C along its special fiber together with the log-structure associated to P). Denote $g^{-1}(P)$ by D_P and let $Z^\times$ be the log formal scheme over V associated to the pair (Z, D_P). We'll abuse notation and also let $g\colon Z^\times \longrightarrow C^\times$ denote the morphism of log formal schemes induced by g. Then D_P is a divisor of Z and we will suppose from now on that $D_P \cup \overline{Z}$ is a reduced divisor with normal crossings. Here $\overline{Z}$ is the special fiber of Z. Suppose that the restriction of g induces a smooth proper map $(Z\backslash D_P) \longrightarrow (C\backslash P)$. Then, under all of the assumptions above $g\colon Z^\times \to C^\times$ is log smooth.

For example, if $C = X(N,p) := X_1(N) \times_{X(1)} X_0(p)$ where $(N,p)=1$ and $N>4$, $Z = E(N,p)$, the universal generalized elliptic curve over C with level structure and $f : Z \longrightarrow C$ is the natural map, then if one takes P to be the divisor of cusps on C, the quadruple (C, Z, f, P) satisfies the above conditions.

If $h,i,j \geq 0$, $S^{h\,i\,j}(Z/C,P)$ will denote the h-th hypercohomology group of the complex of sheaves, $\mathrm{Sym}^j G^i(Z/C,P) \stackrel{\mathrm{Sym}^j D}{\longrightarrow} \mathrm{Sym}^j G^i(Z/C,P) \otimes \Omega^1_{C_K/K}(\log(P_K))$, where

$$G^i(Z/C,P) = K \otimes_V \mathbb{R}^i g_* \Omega^\bullet_{Z^\times/C^\times} = K \otimes_V H^i_{dR}(Z^\times/C^\times)$$

and D is the Gauss-Manin connection.

The group $S^{h\,i\,j}(Z/C,P)$ naturally has a Hodge filtration which we call $\mathscr{F}^{h\,i\,j,\bullet}(Z/C,P)$. After choosing a branch of the p-adic logarithm on $K^\times$, we will use the rigid geometry of Z/C and p-adic integration to produce a K_0-lattice $S^{h\,i\,j}_{int}(Z/C,P)$ in $S^{h\,i\,j}(Z/C,P)$, a linear operator N_h^{int} on this lattice and make a σ-semi-linear operator Φ_h^{int} on $S^{h\,i\,j}(Z/C,P)$ such that $N_h^{\mathrm{int}}\Phi_h^{\mathrm{int}} = p\Phi_h^{\mathrm{int}}N_h^{\mathrm{int}}$.

A four-tuple $(M, F, N, \mathscr{F}^\bullet)$ where M is a finite dimensional vector space over K_0, F and N are σ-semi-linear and respectively linear operators on M such that $NF = pFN$ and $\mathscr{F}^\bullet$ is a decreasing exhaustive filtration of $M_K := M \otimes_{K_0} K$ by K-vector subspaces is called a filtered, Frobenius, monodromy (FFM) module over K (see **[19]**). The category of FFM-modules is an additive, tensor category with kernels, cokernels and a notion of short exact sequences but it is not abelian. Its subcategory of weakly admissible modules (which are now known to be admissible by [**13**]) is abelian, see also [**19**]. To a $\mathbb{Q}_p$-representation of G_K, Fontaine associated an FFM-module and if this representation "comes from geometry" one can recover it from the FFM-module.

In particular, if $g\colon Z \to C$ is as above then

$$M_{int}^{h\,i\,j}(Z/C,P) := (S_{\rm int}^{h\,i\,j}(Z/C,P), \Phi_h^{\rm int}, N_h^{\rm int}, \mathscr{F}^{h\,i\,j,\bullet}(C,P))$$

is an FFM-module over K.

We will prove,

Theorem 1.1. — *The FFM-module $M_{int}^{h\,i\,j}(Z/C,P)$ is the one associated to*

$$\mathscr{V}^{h\,i\,j}(Z/C,P) := H^h_{\acute{e}t}((C-P)_{\overline{K}}, \mathrm{Sym}^j(R^i g_{*,\acute{e}t}\mathbb{Q}_p))$$

via Fontaine theory. In particular,

$$\mathscr{V}^{h\,i\,j}(Z/C,P) \cong \big(B_{\rm st}\otimes (M_{\rm int}^{h\,i\,j}(Z/C,P))\big)^{\Phi=Id,N=0} \cap \mathrm{Fil}^0\big(B_{\rm dR}\otimes_K M_{\rm int}^{h,i,j}(Z/C,P)_K\big).$$

We obtain our theorem from results of Faltings [**17**], which we now describe.

Let us denote by $\overline{C}^{\times}$ the scheme $\overline{C}$ with the inverse image log structure from $C^{\times}$. Suppose $\mathscr{E}$ is a filtered logarithmic F-isocrystal on $\overline{C}^{\times}$. Such an object associates to the "enlargements" (thickenings) of $\overline{C}^{\times}$ (see [**32**] for the non-logarithmic case and [**16**], [**34**],[**35**] in general) coherent sheaves in a compatible way. We will recall the precise definitions in Sections 3.3 and 6. The notion of an F-isocrystal and it's initial development is due to Berthelot and Ogus [**2**], [**32**]. The notion of a filtered logarithmic F-isocrystal was defined by Faltings in [**16**] and developed by Shiho in [**34**] and [**35**]. In particular, one gets from $\mathscr{E}$ a coherent sheaf $\mathscr{E}_{C^{\times}}$ on C_K with an integrable connection D with logarithmic singularities at P. Therefore, if g, Z, C and P are as above, there is a filtered log-F isocrystal $\mathscr{E}^{i\,j}_{Z/C}$ on $\overline{C}^{\times}$ which associates to the enlargement $C^{\times}$, $\mathrm{Sym}^j G^i(Z/C,P)$.

In [**17**], Faltings associated étale local systems on C, $\mathbb{L}(\mathscr{E})$ to certain (very special) filtered log-F isocrystals, $\mathscr{E}$, and made families of FFM-modules, $(H^h_{\rm deg}(\mathscr{E}), \Phi_h^{\rm deg}, N_h^{\rm deg}, \mathscr{F}^{h,\bullet}_{\rm deg})$ (see Section 2.1 for more details). Let us very briefly describe $H^h_{\rm deg}(\mathscr{E})$. It is the log crystalline cohomology on $\overline{C}$, with a certain log structure $\overline{C}^{\times\times}$, with values in $\mathscr{E}$. As $\overline{C}$ is a reduced divisor with normal crossings in C, let $C^{\times\times}$ be C with the log-structure induced by $\overline{C}\cup P$. Let $\overline{C}^{\times\times}$ be $\overline{C}$ with the pull back log structure. Similarly, let $\mathrm{Spec}(V)^{\times}$ be $\mathrm{Spec}(V)$ with the log structure given by the closed point, let $\mathrm{Spec}(k)^{\times}$ be $\mathrm{Spec}(k)$ with the pull-back log structure and let $\mathrm{Spec}(W)^{\times}$ be $\mathrm{Spec}(W)$ with the Teichmüller lift of the log structure on $\mathrm{Spec}(k)^{\times}$. Then $\mathscr{E}$ is a filtered log F-isocrystal on $\overline{C}^{\times\times}$ over $\mathrm{Spec}(W)^{\times}$ and we set $H^h_{\rm deg}(\mathscr{E}) := H^h_{\rm cris}(\overline{C}^{\times\times}/\mathrm{Spec}(W)^{\times}, \mathscr{E})$ for $h\geq 0$. It is proved in [**17**] that the étale cohomology $H^h_{\rm et}((C-P)_{\overline{K}}, \mathbb{L}(\mathscr{E}))$ and these FFM-modules are associated to each other via Fontaine's theory. In the case, $\mathscr{E} = \mathscr{E}^{i\,j}_{Z/C}$, $H^h_{\rm deg}(\mathscr{E})\otimes_{K_0} K = S^{h\,i\,j}(Z/C,P)$,

$\mathscr{F}^{h,\bullet}_{\deg}$ is the Hodge filtration and $H^h_{\text{et}}((C-P)_{\overline{K}}, \mathbb{L}(\mathscr{E})) = \mathscr{V}^{hij}(Z/C,P)$. In this paper, we will extend the definitions in [C1] of FFM-modules $H^h_{int}(\mathscr{E})$ to regular (see Section 6) logarithmic F-isocrystals $\mathscr{E}$ on $\overline{C}^{\times}$ over Spec(W) and prove

$$H^h_{\deg}(\mathscr{E}) = H^h_{\text{int}}(\mathscr{E})$$

for all $h \geq 0$, when all the irreducible components of $\overline{C}$ are absolutely irreducible.

We have several applications of our theorem. We first point out that our descriptions of the operators Φ^{int}_h, N^{int}_h are more explicit than those of the corresponding operators defined by Hyodo-Kato in ([**23**]) and Faltings in ([**17**]). If $C = X(N,p)$, with $(N,p)=1$ and $N>4$ (see the notations above) and $\mathscr{E} = \text{Sym}^j G^1(E/C,P)$ then we prove that the rank of $N^{\deg}_1$ on $H^1_{\text{cris}}(\overline{C}^{\times,\times}/\text{Spec}(W)^{\times}, \mathscr{E})^{p-new}$ is exactly half the dimension over K_0 of this vector space (see Corollary 7.4.) As a consequence we derive that if f is a p-new cuspidal eigenform of weight $k=j+2$ on $X(N,p)$ and V_f denotes the p-adic G_K-representation attached to f, then V_f is semi-stable but *not* crystalline (Corollary 7.5). This was proved in [**33**] in a very indirect way, using the local Langlands correspondence and results of Carayol on the rank of the monodromy operator on the ℓ-adic ($\ell \neq p$) Weil-Deligne representation attached to f.

Our main result is also used in [**24**] in order to give an explicit description of the image of the p-adic Abel-Jacobi map applied to Heegner cycles on certain Shimura curves in terms of extension classes in the category of FFM-modules. In particular a p-adic Gross-Zagier formula for higher weight modular forms is proved in that paper.

Finally, another application of our results is to get an explicit description of the Mazur-Tate-Teitelbaum $\mathscr{L}$-invariants which we now describe.

Suppose now that $k \geq 0$ is an integer and $(M, F, N, \mathscr{F}^{\bullet})$ is a FFM-module over K such that $\mathscr{F}^i M$ is M_K for $i \leq k$ and it is 0 for $i \geq k+2$. Suppose $\mathscr{H}$ is a commutative $\mathbb{Z}_p$-algebra free of finite rank which acts on M such that $\mathscr{F}^{k+1}M$ is a rank 1 $\mathscr{H}_{\mathbb{Q}_p} := \mathscr{H} \otimes \mathbb{Q}_p$-submodule,

$$M_K = \mathscr{F}^{k+1}M \oplus (N \otimes 1_K)M_K$$

and $N \otimes 1_K \colon \mathscr{F}^{k+1}M \longrightarrow (N \otimes 1_K)M_K$ is a non-zero $\mathscr{H}_{\mathbb{Q}_p}$-isomorphism. Then, if $v \in M$ is an eigenvector for F such that $(N \otimes 1_K)M_K = \mathscr{H}_{\mathbb{Q}_p} \cdot Nv$, the $\mathscr{L}$-invariant $\mathscr{L}(M)$ of $(M, F, N, \mathscr{F}(D)^{\bullet})$ is the unique element in $\mathscr{H}_{\mathbb{Q}_p}$ such that

$$v - \mathscr{L}(M)Nv \in \mathscr{F}^{k+1}M.$$

The general definition of an $\mathscr{L}$-invariant becomes arithmetically significant when we attach it to a cuspidal newform on $X(N,p)$ of weight $k+2$ (as above), with $k \geq 0$ even, which is split multiplicative at p. This means precisely that $a_p = p^{k/2}$ (see [**29**].) The quest for an $\mathscr{L}$-invariant which is intimately connected to the relationship between complex and p-adic L-functions was initiated by Mazur-Tate-Teitelbaum (86) in [**30**]. There, a definition in the weight 2 case was offered. Its relationship with values of L-functions was established by Greenberg and Stevens using Hida theory (91) in [**20**]. Teitelbaum proposed the first definition in the higher weight case under some restrictions on the level using the uniformization of Shimura curves by the p-adic upper half plane (90) in [38] (his definition does not involve a FFM-module but see [24]), the first author of the present paper offered a definition using the FFM-module $M_{int}^{1\,i\,j}(E(N,p)/X(N,p),\mathrm{Cusps})$ and $\mathscr{H}$ is the Hecke-algebra acting on $X(N,p)$, in [8]. Finally, Fontaine-Mazur defined an $\mathscr{L}$-invariant associated to a cusp form as above using the FFM-module $D_{st}(V)$, where V is the local Galois representation attached to the cusp form and D_{st} is Fontaine's functor (see [19]) in [29]. The algebra $\mathscr{H}$ is again the Hecke algebra acting on $X(N,p)$. K. Kato, M. Kurihara and T. Tsuji established the connection between the $\mathscr{L}$-invariant of Fontaine and Mazur and special values of the complex and p-adic L-functions while G. Stevens has established the connection between the $\mathscr{L}$-invariant defined in [8] and special values of the complex and p-adic L-functions using p-adic families of modular forms, see [37]. The result of Kato, Kurihara and Tsuji has not yet been published. The present paper together with the results in [24] establishes the equality of all the $\mathscr{L}$-invariants (whenever they are defined). Of course, the results of Kato-Kurihara-Tsuji and Stevens togeher also imply (indirectly) the equality of the $\mathscr{L}$-invariants defined in [8] and the corresponding Fontaine-Mazur $\mathscr{L}$-invariants.

We mention that P. Colmez also proved (in [12]) a formula giving the $\mathscr{L}$-invariant of Fontaine-Mazur as derivative of a family of eigenvalues of Frobenius. Together with the result of Stevens mentioned above involving the $\mathscr{L}$-invariant defined in [8], this gives another local proof of the equality of the two $\mathscr{L}$-invariants we consider.

In [21] Grosse-Klönne extended the Hyodo-Kato theory and showed that there are natural Frobenius and monodromy operators on the de Rham cohomology of a quite general rigid space. He has been able to explicitly compute these when the space is a quotient of a p-adic symmetric domain.

Writing this paper we had two options, namely to present the definitions, statements and proofs in the most general case (the logarithmic case), which would have made the notations very complicated and would have obscured the ideas of the proofs or, to first present some of the definitions, statements and proofs in the non-logarithmic case, then to give the definitions and make the precise statements in general and leave it to the reader to check that the same proofs go through with the obvious adjustments. We choose to do the latter.

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Thus, it should be understood that the paper owes much to this report and we are very grateful to its author for his/her help.

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2. Definitions of the operators

Let $K, V, k, W, K_0, C_K, C, P, \overline{C}, \overline{P}$ be as in Section 1. Let us recall that we suppose that the reduction of P, $\overline{P}$ does not meet the singular divisor of $\overline{C}$. We endow the formal completion of C along its special fiber with the natural log structure defined by the divisor P and denote the resulting formal log scheme by $C^\times$. We let $\overline{C}^\times$ denote the log scheme $\overline{C}$ with the inverse image log structure. We also denote by $C^{\times\times}$ the formal completion of C along its special fiber with log structure given by the divisor with normal crossings $P \cup \overline{C}$. We denote $\overline{C}^{\times\times}$ the scheme $\overline{C}$ with the inverse image log structure. Let $\mathscr{E}$ be a filtered log F-isocrystal on $\overline{C}^\times$. We fix a uniformizer π of K and fix the branch, log, of the p-adic logarithm in $K^\times$ such that $\log(\pi) = 0$. Then, if $\mathscr{E}$ is regular (see below) there are two ways to attach a family of FFM-modules to $\mathscr{E}$, as we shall explain below.

2.1. The definition via degeneration. — We first briefly review the definition given by G. Faltings in [**17**]. We give more details in later sections. By deformation theory, the pair (C, P) can be regarded as the fiber at the point π of $\mathscr{S} := \mathrm{Spf}(W[[t]])$ over W, of a pair $(\mathfrak{X}, \mathscr{P})$ consisting of a family of curves $\mathfrak{X}$ defined over $\mathscr{S}$ and a smooth divisor $\mathscr{P}$ of $\mathfrak{X}$ over $\mathscr{S}$. Let $\mathfrak{X}^\times$ denote the log formal scheme $\mathfrak{X}$ with the log structure given by the divisor $\mathscr{P}$. Let $f : \mathfrak{X} \longrightarrow \mathscr{S}$ denote the structure morphism. Let $\mathscr{Y}$ denote the fiber of this morphism at $t = 0$. Then $\mathscr{P}$ and $\mathscr{Y}$ are disjoint and $\mathscr{Y}$ is a divisor of $\mathfrak{X}$ with normal crossings. We denote by $\mathfrak{X}^{\times\times}$ the formal scheme $\mathfrak{X}$ with the log structure associated to the divisor $\mathscr{P} \cup \mathscr{Y}$. If we let $X = \mathfrak{X}^{\mathrm{rig}}$, $S = \mathscr{S}^{\mathrm{rig}}$ and $\mathscr{P}^{\mathrm{rig}} := P_X$ denote the rigid analytic spaces over K_0 associated to $\mathfrak{X}$, $\mathscr{S}$ and $\mathscr{P}$ respectively and if $f \colon X \longrightarrow S$ is the induced morphism then we have

i) $X \longrightarrow \mathrm{Spec}(K_0)$ is smooth

ii) $Y := f^{-1}(0) = \mathscr{Y}^{\mathrm{rig}}$ is a semi-stable curve over K_0

iii) $P_0 := P_X \cap Y$ is disjoint from the singular divisor of Y

iv) $f|_{X^*} \colon X^* = (X - Y) \longrightarrow S^* = (S - \{0\})$ is smooth.

The evaluation of $\mathscr{E}$ on $\mathfrak{X}^\times$ is a coherent $\mathscr{O}_X$-module $\mathscr{E}_{\mathfrak{X}^\times}$, with a relative, logarithmic, integrable connection $D_{\mathrm{X/S}}$. Let us denote by $K^\bullet_{\mathrm{X/S}}$ the complex of sheaves on X

$$\mathscr{E}_{\mathfrak{X}^\times} \xrightarrow{\mathrm{D_{X/S}}} \mathscr{E}_{\mathfrak{X}^\times} \otimes_{\mathscr{O}_X} \Omega^1_{X/S}(\log(Y \cup P_X)).$$

The relative connection $D_{\mathrm{X/S}}$ is induced from the absolute connection:

$$\mathscr{E}_{\mathfrak{X}^\times} \xrightarrow{\mathrm{D_{X/K_0}}} \mathscr{E}_{\mathfrak{X}^\times} \otimes_{\mathscr{O}_X} \Omega^1_{X/K_0}(\log(P_X))$$

by composing with the natural map: $\Omega^1_{X/K_0}(\log(P_X)) \longrightarrow \Omega^1_{X/S}(\log(Y \cup P_X))$.

See Section 3.3 and Section 6. We denote by $\mathbb{H}^i$ the i-th logarithmic relative de Rham cohomology group of X/S with coefficients in $\mathscr{E}_{\mathfrak{X}^\times}$, i.e. the sheaf $\mathbb{R}^i f_*(K^\bullet_{\mathrm{X/S}})$ for $i = 0, 1, 2$. For every i, $\mathbb{H}^i$ is a free $\mathscr{O}_S$-module with an integrable, regular-singular connection

$$\nabla_i \colon \mathbb{H}^i \longrightarrow \mathbb{H}^i \otimes_{\mathscr{O}_S} \Omega^1_{S/K_0}(\log 0).$$

Fix a parameter t on S, with $t(0) = 0$. The Frobenius on $\mathscr{E}$ together with the Frobenius φ on S which sends t to t^p and acts on the coefficients as the absolute Frobenius on K_0, endow $\mathbb{H}^i$ with a φ-semi-linear, horizontal (with respect to ∇_i) Frobenius operator

$$\Phi_i \colon \varphi^* \mathbb{H}^i \longrightarrow \mathbb{H}^i.$$

If s is a point of S, let $\mathbb{H}^i_s$ denote the fiber of $\mathbb{H}^i$ at s. The i-th logarithmic de Rham cohomology of C_K, with coefficients in $\mathscr{E}_{C^\times}$, $H^i_{dR}\big(C_K, \mathscr{E}_{C^\times}\big)$ is canonically isomorphic to $\mathbb{H}^i_\pi$. (Recall, P is the fiber of P_X at $s = \pi$.) We denote these groups by $H^i(C, P, \mathscr{E})$. On the other hand, $\mathbb{H}^i_0$ is canonically isomorphic to the logarithmic de Rham cohomology of Y with coefficients in $\mathscr{E}_{\mathscr{Y}^\times}$, i.e. the i-th hypercohomology on Y of the complex of sheaves

$$\mathscr{E}_{\mathscr{Y}^\times} \overset{D_{\mathscr{Y}/W}}{\longrightarrow} \mathscr{E}_{\mathscr{Y}^\times} \otimes_{\mathscr{O}_{\mathscr{Y}}} \Omega^1_{\mathscr{Y}^{\times\times}/\mathrm{Spf}(W)^\times},$$

where $\mathscr{Y}^{\times\times}$ is the formal scheme $\mathscr{Y}$ with the inverse image log structure from $\mathfrak{X}^{\times\times}$. We denote this group by $H^i(Y, P_0, \mathscr{E})$.

Now let $H^i_{\deg}(\mathscr{E})$ denote the FFM-module $(H^i(Y, P_0, \mathscr{E}), \Phi^{\deg}_i, N^{\deg}_i, \mathscr{F}^\bullet_{\deg})$, where the operators are defined as follows

the monodromy operator: $N^{\deg}_i := \mathrm{Res}_0(\nabla_i) \colon H^i(Y, P_0, \mathscr{E}) \longrightarrow H^i(Y, P_0, \mathscr{E})$,
and

the Frobenius operator: $\Phi^{\deg}_i := \Phi_i|_{H^i(Y,P_0,\mathscr{E})} \colon H^i(Y, P_0, \mathscr{E}) \longrightarrow H^i(Y, P_0, \mathscr{E})$.

These operators satisfy $N^{\deg}_i \Phi^{\deg}_i = p \Phi^{\deg}_i N^{\deg}_i$.

We still have to define the filtration on $(H^i_{\deg}(\mathscr{E}))_K := H^i(Y, P_0, \mathscr{E}) \otimes_{K_0} K$. For this let us recall from [**4**] (this was also proved in [**17**]) that the triple $(\mathbb{H}^i, \nabla_i, \Phi_i)$ is determined by the triple $(H^i(Y, P_0, \mathscr{E}), N^{\deg}_i, \Phi^{\deg}_i)$. More precisely we have a natural, horizontal, Frobenius-equivariant isomorphism of $\mathscr{O}_S$-modules

$$(\mathbb{H}^i, \nabla_i, \Phi_i) \cong (H^i(Y, P_0, \mathscr{E}) \otimes_{K_0} \mathscr{O}_S, (\nabla_i)', \Phi^{\deg}_i \otimes \varphi),$$

where the connection $(\nabla_i)'$ is defined by,

$$(\nabla_i)'(h \otimes x) = N^{\deg}_i(h) \otimes x \frac{dt}{t} + h dx, \quad \text{for all} \quad h \in H^i(Y, P_0, \mathscr{E}), x \text{ section of } \mathscr{O}_S.$$

Here a few comments are in order. For $i = 0, 2$ the pair $(\mathbb{H}^i, \nabla_i)$ is very simple. Namely, let $i = 0$. Then $\mathbb{H}^0 = (\mathscr{E}_{\mathfrak{X}^\times}(X))^{\mathrm{D}_{X/S}} =: E_{X/S}$ and the connection ∇_0 is the composition

$$E_{X/S} \stackrel{\mathrm{D}_{X/K_0}}{\longrightarrow} E_{X/S} \otimes_{\mathscr{O}_S} \Omega^1_S \longrightarrow E_{X/S} \otimes_{\mathscr{O}_S} \Omega^1_S(\log(0)),$$

where D_{X/K_0} is the absolute connection mentioned at the beginning of this section. Therefore $N_0^{deg} = \mathrm{Res}_0(\nabla_0) = 0$ and so applying the above we get that $\mathbb{H}^0 \cong H^0(Y, P_0, \mathscr{E}) \otimes_{K_0} \mathscr{O}_S$ and $(\nabla_0)'$ (therefore also ∇_0) is the trivial connection. The same happens for $i = 2$ by Poincaré duality (see [**17**]).

∇_1 is not trivial in general so let us define $\mathbb{H}^1_{\log} = \mathbb{H}^1 \otimes_{\mathscr{O}_S} \mathscr{O}_S[\ell(t)]$, where $\ell(t)$ is a variable. We endow $\mathbb{H}^1_{\log}$ with the connection $\nabla_1(\log) := \nabla_1 \otimes 1 + 1 \otimes d$ where $d : \mathscr{O}_S[\ell(t)] \longrightarrow \mathscr{O}_S[\ell(t)] \otimes_{\mathscr{O}_S} \Omega^1_{S/K_0}(\log(0))$ is defined by $d(\ell(t)) = 1 \otimes \dfrac{dt}{t}$.

For all $h \in H^1(Y, P_0, \mathscr{E})$ the sections of $\mathbb{H}^1_{\log}$

$$h \otimes 1 - N_1^{\deg}(h) \otimes \ell(t)$$

are horizontal for $\nabla_1(\log)$ hence the connection $\nabla_1(\log)$ is trivial.

Therefore, letting $\mathbb{H}^i_{\log} = \mathbb{H}^i$ if $i = 0, 2$ we have for $i = 0, 1, 2$ and every K-point $s \neq 0$ of S natural identifications (by parallel transport, see [**14**])

$$\big(H^i_{\deg}(\mathscr{E})\big)_K = H^i(Y, P_0, \mathscr{E}) \otimes_{K_0} K \cong (\mathbb{H}^i_{\log})_s$$

where by $(\mathbb{H}^i_{\log})_s$ we denote the pull back of $\mathbb{H}^i_{\log}$ by the map $\mathscr{O}_S[\ell(t)] \longrightarrow K$ sending $t \to s$ and $\ell(t) \to \log(s)$, where let us recall that the branch of the logarithm chosen at the beginning of this section is such that $\log(\pi) = 0$. In particular, for $s = \pi$ we have $(\mathbb{H}^i_{\log})_\pi = \mathbb{H}^i_\pi = H^i_{dR}(C_K, \mathscr{E}_{C^\times}(\log(P))$ and we define the filtration on $\big(H^i_{\deg}(\mathscr{E})\big)_K$ to be the inverse image under this isomorphism of the Hodge filtration on $H^i_{dR}(C_K, \mathscr{E}_{C^\times}(\log(P))$.

Remark 2.1. — *Actually Faltings does not mention the basis of horizontal sections defined above in* [**17**] *and it seems to us that he does not identify fibers of* $\mathbb{H}^i_{\log}$ *(see also the remark before Lemma 2.1 in* [**17**]*).*

2.2. The definition via p-adic integration. — We generalize the definition given in [**8**] when $\mathscr{E}$ is regular. As pointed out above, the evaluation of $\mathscr{E}$ on $C^\times$ is a coherent $\mathscr{O}_{C_K}$-module with a regular singular (at P) integrable connection $D\colon \mathscr{E}_{C^\times} \longrightarrow \mathscr{E}_{C^\times} \otimes_{\mathscr{O}_{C_K}} \Omega^1_{C_K/K}(\log(P))$. Recall that we have denoted by $H^i(C, P, \mathscr{E})$ the K-vector spaces $H^i_{dR}(C_K, \mathscr{E}_{C^\times}(\log(P))$, for $i = 0, 1, 2$.. The following lemma will be proved in Section 3.3

Lemma 2.2. — *The connection D has a basis of horizontal sections on every residue class of C_K.*

We'll assume that the components of $\overline{C}$ are smooth, absolutely irreducible and there are at least two of them. Also suppose that the singular points of the reduction are defined over k.

For $i = 0, 2$ we have the K_0-lattices in $H^i(C, P, \mathscr{E})$, $H^i_{\mathrm{int}}(\mathscr{E}) := H^i_{\mathrm{cris}}(\overline{C}^{\times\times}, \mathscr{E})$ with the respective Frobenii and zero monodromies. The filtrations on $H^i(C, P, \mathscr{E})$ are the respective Hodge filtrations.

For $i = 1$ the situation is more complicated. For an admissible covering $\mathscr{D}$ of a rigid space let $G := G(\mathscr{D})$ be the graph whose vertices $v(G)$ are the elements of $\mathscr{D}$ and whose oriented edges $\epsilon(G)$ correspond to ordered triples $e := (U, V, W)$ where $U \neq V \in \mathscr{D}$ and $A_e := W$ is a connected component of $U \cap V$. Also, if e is such an edge then its *origin* $a(e)$ is U and its *end* $b(e)$ is V. We set $\tau(e) = (V, U, W)$. If $v \in v(G(\mathscr{D}))$ we will denote by U_v the element of $\mathscr{D}$ corresponding to it. We choose and fix a system of representatives $e(G)$ of the quotient set $\epsilon(G)/\tau$.

Consider

$$\mathscr{C} = \{\mathrm{red}^{-1} Z \colon Z \text{ is a component of } \overline{C}\},$$

where $\mathrm{red}\colon C_K = C^{\mathrm{rig}} \longrightarrow \overline{C}$ is the reduction map. Then $\mathscr{C}$ is an admissible open cover of C_K by wide opens (see [**7**]). Let $G = G(\mathscr{C})$, $v(G)$ be the vertices of G and $\epsilon(G)$, the edges of G. If $v \in v(G)$, C_v will denote the corresponding component of $\overline{C}$. We also set $C_v^0 = C_v - \bigcup_{w \neq v} C_w$. In this situation, for each $e \in e(G)$, A_e is an oriented wide open annulus. Given Lemma 2.2, there is a natural residue map

$$\mathrm{Res}_e \colon H^1_{dR}(A_e, \mathscr{E}_{C^\times}) \cong H^0_{dR}(A_e, \mathscr{E}_{C^\times}) = (\mathscr{E}_{C^\times}|_{A_e})^D.$$

We will sometimes abuse notation and allow Res_e to denote the composition of Res_e with the natural map from $H^1(C, P, \mathscr{E})$ to $H^1_{dR}(A_e, \mathscr{E}_{C^\times})$.

Elements of $H^1(C, P, \mathscr{E})$ are represented by pairs of collections

$$(\{\omega_v\}_{v \in v(G)}, \{f_e\}_{e \in e(G)})$$

where $\omega_v \in (\mathscr{E}_C \otimes \Omega^1_{U_v})(\log P_v))(U_v)$ and $f_e \in \mathscr{E}(A_e)$ are such that

$$\omega_{a(e)}|_{A_e} - \omega_{b(e)}|_{A_e} = Df_e$$

for all $e \in e(G)$. We denote $P \cap U_v$ by P_v. From the Mayer-Vietoris exact sequence corresponding to the covering $\mathscr{C}$ we get a short exact sequence

(1)

$$\begin{aligned} 0 \to (\oplus_{e \in e(G)} H^0_{dR}(A_e, \mathscr{E}_{C^\times}))/(\oplus_{v \in v(G)} H^0_{dR}(U_v, \mathscr{E}_{C^\times}(\log(P_v)))) &\xrightarrow{\iota} H^1(C, P, \mathscr{E}) \\ \xrightarrow{\gamma} \mathrm{Ker}(\oplus_{v \in v(G)} H^1_{dR}(U_v, \mathscr{E}_{C^\times}(\log(P_v))) &\to \oplus_{e \in e(G)} H^1_{dR}(A_e, \mathscr{E}_{C^\times})) \to 0. \end{aligned}$$

First, let us observe that the left and right terms in the exact sequence (1) have natural K_0-lattices, with Frobenii. To see this, note that $H^0_{dR}(A_e, \mathscr{E}_{C^\times})$ contains a natural K_0-lattice, namely $H^0_{\mathrm{cris}}(x_e, \mathscr{E})$, where x_e is the point of $\overline{C}$ corresponding to the edge e, and it has a natural Frobenius. Therefore we get a natural K_0-lattice with a Frobenius on the left module of the exact sequence (1) which will be denoted $H^{0,1}(C)$ and $F_{0,\mathrm{cris}}$ respectively. Moreover, for $v \in v(G)$, $H^1_{dR}(U_v, \mathscr{E}_{C^\times}(\log(P_v)))$ contains a natural K_0-lattice with a Frobenius, namely the first log crystalline cohomology with coefficients in $\mathscr{E}$ of the component corresponding to the vertex v, $C_v^{\times\times}$ where the log structure is the one induced by the log structure on $\overline{C}^{\times\times}$. See [**16**]. Therefore, the right module of the exact sequence (1) has a natural K_0-lattice, denoted $H^{1,0}(C)$, with a Frobenius denoted $F_{1,\mathrm{cris}}$. To define a K_0-lattice, $H^1_{\mathrm{int}}(\mathscr{E})$ of $H^1(C,P,\mathscr{E})$, together with a Frobenius operator Φ_1^{int} and a monodromy operator N_1^{int} we'll first split the exact sequence (1) by defining a section s of ι. This can be done if the log F-isocrystal $\mathscr{E}$ is regular.

Definition 2.3. — *We say that the log F-isocrystal $\mathscr{E}$ on $\overline{C}^{\times}$ is* regular *if for every $v \in v(G)$ and x closed point of $C_v - \overline{P}$ the characteristic polynomials of Frobenius on $H^0_{\mathrm{cris}}(x,\mathscr{E})$ and $H^1_{\mathrm{cris}}(C_v^{\times\times},\mathscr{E}))$ are relatively prime.*

Remark 2.4. — *It will be proved in Section 6 that the definition (2.3) is satisfied by all log F-isocrystals on $\overline{C}^{\times}$ coming from a family of schemes $Z \longrightarrow C$ as in the Section 1.*

For the rest of the section we'll assume that $\mathscr{E}$ is regular. Let $\omega \in H^1(C,P,\mathscr{E}_{C^\times})$ be represented by the hypercocycle $(\{\omega_v\}_v,\{f_e\}_e)$ as above. If $v \in v(G)$ one can define a p-adic integral of ω_v, λ_v, on $U_v - P_v$, which depends on our choice of the logarithm and is well defined up to a rigid horizontal section of $\mathscr{E}_{C^\times}|_{U_v}$ (see Section 5.2). Then $s(\omega)$ will be represented by the cocycle $(\{g_e\}_e)$, where

$$g_e = f_e - (\lambda_{a(e)}|_{A_e} - \lambda_{b(e)}|_{A_e}).$$

Let u be the corresponding section of γ. Then define $H^1_{\mathrm{int}}(\mathscr{E})$ to be the FFM-module, where the underlying K_0-vector space is $\iota(H^{0,1}(C)) + u(H^{1,0}(C))$ and the Frobenius operator, $\Phi_1^{\mathrm{int}}(\omega)$, is

$$\iota(F_{0,\mathrm{cris}}(s(\omega)) + u(F_{1,\mathrm{cris}}(\gamma(\omega)).$$

Moreover, the monodromy operator, N_1^{int}, is defined to be the composition

$$\iota \circ \oplus_{e\in e(G)} \mathrm{Res}_e.$$

The operators satisfy the relation,

$$N_1^{\mathrm{int}}\Phi_1^{\mathrm{int}} = p\Phi_1^{\mathrm{int}}N_1^{\mathrm{int}}.$$

Finally the filtration on $(\iota(H^{0,1}(C)) + u(H^{1,0}(C))) \otimes_{K_0} K = H^1(C,P,\mathscr{E})$ is the Hodge filtration.

Remark 2.5. — *The same construction can be performed for every fiber X_s where $s \in S^* = S - \{0\}$, i.e., we have residue maps $\mathrm{Res}^{(s)}$, monodromy operators $N^{\mathrm{int}}_{(i,s)}$ and Frobenii $\Phi^{\mathrm{int}}_{(i,s)}$, for $i = 0,1,2$.*

The main result of this paper is

Theorem 2.6. — *Suppose that $\mathscr{E}$ is a regular filtered log F-isocrystal on $\overline{C}^{\times}$. Then the isomorphism $H^i(Y,P_0,\mathscr{E}) \otimes_{K_0} K \cong (\mathbb{H}^i_{\log})_\pi$ obtained by parallel transport yields an isomorphism of FFM-modules $H^i_{\deg}(\mathscr{E}) \cong H^i_{\mathrm{int}}(\mathscr{E})$.*

Remark 2.7. — *Actually regularity is only needed in order to compare the K_0-lattices and the Frobenii. We shall prove the equality of the monodromy operators (tensored with the identity of K) without any restriction.*

Theorem 2.6 is an easy consequence of the definitions for $i = 0, 2$. The next sections of the paper will be devoted to the proof of this theorem for $i = 1$. We'll first prove the theorem (2.6) in the non-logarithmic case (i.e. $\overline{P}$ is the void set) and then we'll provide all the necessary definitions and results so that the reader should be able to fill in the details of the proof in the logarithmic case.

3. F-Isocrystals

3.1. Formal schemes, rigid analytic spaces and weak completions. — In this section we review some constructions and results on formal schemes, rigid analytic spaces and weak completions which will be used later in the paper.

3.1.1. *The functor* rig. — We recall a standard construction in rigid analytic geometry, the functor "rig" (for more details see Section 02 of [**1**] or [**25**]). This is a functor from the category of locally noetherian formal V-schemes (or formal W-schemes) to the category of rigid analytic spaces over K (respectively K_0).

Let $\mathfrak{X}$ be a locally noetherian formal scheme over $\mathrm{Spf}(V)$ (the case where V is replaced by W is treated in the same way) having the property that the scheme $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I})_{\mathrm{red}}$ is locally of finite type, where $\mathscr{I}$ is an ideal of definition of $\mathfrak{X}$. To the formal scheme $\mathfrak{X}$ we attach a rigid analytic space $X := \mathfrak{X}^{\mathrm{rig}}$ over K as follows.

We first suppose that $\mathfrak{X}$ is affine, $\mathfrak{X} = \mathrm{Spf}(A)$, let $I = H^0(\mathfrak{X}, \mathscr{I})$ and fix $f_1, f_2, \ldots, f_r$ a set of generators of the ideal I. For every $n \geq 1$ define the V-algebra

$$B_n := A\langle T_1, T_2, \ldots, T_r\rangle/(f_1^n - \pi T_1, f_2^n - \pi T_2, \ldots, f_r^n - \pi T_r),$$

where π is a uniformizer of V, and as usual, $A\langle T_1, T_2, \ldots, T_r\rangle$ denotes the p-adic (or π-adic) completion of the polynomial ring $A[T_1, T_2, \ldots, T_r]$. The conditions on $\mathfrak{X}$ imply that the k-algebra

$$B_n/\pi B_n \cong A/(\pi, f_1^n, f_2^n, \ldots, f_r^n)[T_1, T_2, \ldots, T_r]$$

is of finite type which implies that B_n itself is topologically of finite type. Therefore $B_n \otimes_V K$ is a Tate-algebra over K. For $m > n \geq 1$ we have canonical V-algebra homomorphisms $B_m \longrightarrow B_n$ sending $T_i \to f_i^{m-n} T_i$ for all $1 \leq i \leq r$. The induced morphism of affinoids $\mathrm{Spm}(B_n \otimes K) \longrightarrow \mathrm{Spm}(B_m \otimes K)$ identifies the source with the affinoid sub-domain of the target given by $|f_i| \leq |\pi|^{1/n}$, $1 \leq i \leq r$. We define $X := \mathfrak{X}^{\mathrm{rig}}$ to be the inductive limit of $\mathrm{Spm}(B_n \otimes K)$, where these affinoids form, by definition, an admissible covering of X. In fact one can prove that $\mathfrak{X}^{\mathrm{rig}}$ is independent of the ideal of definition $\mathscr{I}$ and of the choice of generators $f_1, f_2, \ldots, f_r$ and that it is functorial in $\mathfrak{X}$.

If the ideal of definition of $\mathfrak{X}$ is $\pi\mathscr{O}_{\mathfrak{X}}$, i.e. $\mathfrak{X}$ is a p-adic formal V-scheme topologically of finite type, then $\mathfrak{X}^{\mathrm{rig}}$ is the usual "generic fiber of $\mathfrak{X}$" à la Raynaud.

Let $\mathfrak{X}, \mathfrak{X}^{\mathrm{rig}}$ be as above. Then one can define a reduction (or specialization) map $\mathrm{red} : \mathfrak{X}^{\mathrm{rig}} \longrightarrow \mathfrak{X}$ as follows. For $m > n \geq 1$ the natural V-algebra homomorphisms

$A \longrightarrow B_m \longrightarrow B_n$ induce the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Spm}(B_m \otimes K) & \stackrel{\mathrm{red}}{\longrightarrow} & \mathrm{Spf}(B_m) & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow & & \| \\ \mathrm{Spm}(B_n \otimes K) & \stackrel{\mathrm{red}}{\longrightarrow} & \mathrm{Spf}(B_n) & \longrightarrow & \mathfrak{X} \end{array}$$

Here the morphisms red : $\mathrm{Spm}(B_n \otimes K) \longrightarrow \mathrm{Spf}(B_n)$ are the usual reduction maps for p-adic formal schemes and their generic fibers, i.e. defined as follows. Let $x \in \mathrm{Spm}(B_n \otimes K)$ be a point and let m_x be the respective maximal ideal. Then $K(x) := (B_n \otimes K)/m_x$ is a finite extension of K and we have V-algebra morphisms: $B_n \longrightarrow B_n \otimes K \longrightarrow K(x)$. We define $\mathrm{red}(x)$ to be the point of $\mathrm{Spf}(B_n)$ corresponding to the unique closed point of the finite, local V-algebra which is the image of B_n in $K(x)$.

The morphism $\mathrm{red} : \mathfrak{X}^{\mathrm{rig}} \longrightarrow \mathfrak{X}$ is obtained by gluing the morphisms $\mathrm{Spm}(B_n \otimes K) \longrightarrow \mathfrak{X}$ in the above diagram.

For a general $\mathfrak{X}$, we obtain $\mathfrak{X}^{\mathrm{rig}}$ and the morphism red : $\mathfrak{X}^{\mathrm{rig}} \longrightarrow \mathfrak{X}$ by taking an affine cover $\{\mathscr{U}_i\}_i$ of $\mathfrak{X}$ and gluing $\mathscr{U}_i^{\mathrm{rig}}$ and $\mathrm{red}_{\mathscr{U}_i^{\mathrm{rig}}}$.

Under the notations and hypothesis at the beginning of the section, let Z be a closed sub-scheme of $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I})$. We denote by $\mathfrak{X}_{/Z}$ the formal completion of $\mathfrak{X}$ along Z. We have canonical morphisms $\mathfrak{X}_{/Z} \longrightarrow \mathfrak{X}$ and $(\mathfrak{X}_{/Z})^{\mathrm{rig}} \longrightarrow \mathfrak{X}^{\mathrm{rig}}$. The image of the latter morphism is an admissible open subset of $\mathfrak{X}^{\mathrm{rig}}$ which may be canonically identified with $\mathrm{red}^{-1}(Z) :=]Z[_{\mathfrak{X}}$ (see Proposition 0.2.7 of [**1**]).

3.1.2. *Formal models.* — Let $\mathfrak{X}$ be a p-adic formal V-scheme (or W-scheme), separated and topologically of finite type and let $X := \mathfrak{X}^{\mathrm{rig}}$. Assume that X is reduced and let U be an admissible affinoid open of X.

Lemma 3.1. — *There is a canonical p-adic formal scheme $\mathfrak{U}$ over V (respectively over W), depending on $\mathfrak{X}$, with a morphism $\mathfrak{U} \longrightarrow \mathfrak{X}$ whose generic fiber is the inclusion $U \subset X$.*

Proof. — Let, as usual $\mathfrak{X}_1$ denote the special fiber of $\mathfrak{X}$ and consider an affine open covering of $\mathfrak{X}_1$, $\{V_i\}_i$. Let $U_i := \mathrm{red}^{-1}(V_i) \cap U \subset U$, the family $\{U_i\}_i$ is an admissible covering of U and let us denote by $\mathfrak{U}_i := \mathrm{Spf}(A_i)$ where A_i is the sub-ring of functions of $\mathscr{O}_U(U_i)$ bounded by 1 (we say that $\mathfrak{U}_i$ is "the canonical formal model" of U_i). Let V_{ij} be the inverse image of $V_i \cap V_j$ under the map of special fibers $(\mathfrak{U}_i)_1 \longrightarrow \mathfrak{X}_1$. Then $U_i \cap U_j = \mathrm{red}_i^{-1}(V_{ij})$, where $\mathrm{red}_i : U_i \longrightarrow \mathfrak{U}_i$ is the reduction map and the canonical model of $U_i \cap U_j$ is the formal open sub-scheme of $\mathfrak{U}_i$ whose support is V_{ij} Therefore, one can glue the formal schemes $\mathfrak{U}_i$ along the canonical formal models of $U_i \cap U_j$ and obtain the required formal model of U. This is independent of the covering $\{V_i\}_i$, as one may take the covering of $\mathfrak{X}_1$ consisting of all the affine open sub-schemes. □

These formal models of affinoid opens of X have the following functorial property.

Let $\mathfrak{X}, \mathfrak{X}'$ be p-adic formal schemes, separated, topologically of finite type over V (or W) and let $X = \mathfrak{X}^{\mathrm{rig}}, X' = \mathfrak{X}'^{\mathrm{rig}}$ and assume that X, X' are reduced. Let

U, U' be admissible affinoid opens of X respectively X' and assume that we are given morphisms $f : U' \longrightarrow U$ and $g : (\mathfrak{X}')_1 \longrightarrow (\mathfrak{X})_1$ such that the following diagram commutes.

$$\begin{array}{ccccc} U' & \subset & X' & \stackrel{\text{red}}{\longrightarrow} & (\mathfrak{X}')_1 \\ f\downarrow & & & & g\downarrow \\ U & \subset & X & \stackrel{\text{red}}{\longrightarrow} & (\mathfrak{X})_1 \end{array}$$

Then there exists a canonical morphism $h : \mathfrak{U}' \longrightarrow \mathfrak{U}$ inducing f on generic fibers and such that $h_1 : (\mathfrak{U}')_1 \longrightarrow (\mathfrak{U})_1$ is compatible with g.

3.1.2.1. *Logarithmic structures.* — In this section we'd like to recall some basic notions in the theory of log schemes from [**26**], [**23**], Sections 2.8, 2.9 and [**34**].

Suppose A is a scheme (or a formal scheme or a rigid space). A morphism of sheaves of monoids on the Zariski site of A, $\alpha\colon M \to \mathcal{O}_A$, will be called a pre log structure on A. Call the pair (A, α) a pre log scheme (or formal pre log scheme) and denote it $A^\times$ and denote M, $M_{A^\times}$. A pre log scheme (A, α) is called a log scheme if α induces an isomorphism $\alpha^{-1}(\mathcal{O}_A^*) \cong \mathcal{O}_A^*$. The sheaf of log one forms $\omega_{A^\times}$ on A associated to α is the quasi-coherent sheaf $\Omega^1_A \oplus \mathcal{O}_A \otimes_{\mathcal{O}_A^*} M_{A^\times}$ subject to the relations $\alpha(m) \otimes m = \mathrm{d}\alpha(m)$, for $m \in M_{A^\times}$. One has a natural derivation on the exterior algebra of $\omega_{A^\times}$ over $\mathcal{O}_A$ such that $\mathrm{d}(1 \otimes m) = 0$, for $m \in M_{A^\times}$.

If P is a divisor on A, M_P is the sheaf $M_P(U) = \mathcal{O}_A(U) \cap \mathcal{O}_A^*(U - P)$ and $\alpha_P\colon M_P \to \mathcal{O}_A$ is the inclusion, then $A_P^\times =: (A, \alpha_P)$ is a log-scheme which is fine ("coherent" and "integral"). If A is noetherian and reduced and if A is a variety $\omega_{A_P^\times}$ is naturally isomorphic to $\Omega^1_A(\log P)$. If $P = \varnothing$, α_P is called the a trivial log structure on A.

G. Faltings defines and uses a more restricted notion of log-structures in [**16**] and [**17**] (see the appendix of [**26**] for the precise relationship between the two notions.)

Henceforth, all log structures will be fine.

Let $T^\times$ be a formal log scheme. Let us denote by T_0 the reduced sub-scheme of the closed sub-scheme of T corresponding to the ideal sheaf $p\mathcal{O}_T$. We have a closed immersion

$$\iota : T_0 \longrightarrow T$$

and we'll let $T_0^\times$ be the log scheme corresponding to the log structure on T_0

$$\iota^{-1}(M_{T^\times}) \longrightarrow \iota^{-1}(\mathcal{O}_T) \longrightarrow \mathcal{O}_{T_0}.$$

We use, as in [**26**] the notation ι^{-1} for the inverse image of a sheaf and ι^* for the inverse image of a log structure.

Let now $g\colon U^\times \to T^\times$ be a morphism of formal log schemes, $g = (f, h) : (U, M_{U^\times}) \to (T, M_{T^\times})$. Here $f : U \to T$ is a morphism of formal W-schemes and we have a commutative diagram

$$\begin{array}{ccc} f^{-1}M_{T^\times} & \xrightarrow{h} & M_U \\ \downarrow & & \downarrow \\ f^{-1}\mathscr{O}_T & \longrightarrow & \mathscr{O}_U \end{array} \quad \text{and also} \quad \begin{array}{ccc} U & \xrightarrow{f} & T \\ \iota' \uparrow & & \uparrow \iota \\ U_0 & \xrightarrow{f_0} & T_0 \end{array}$$

Therefore, we have a commutative diagram

$$\begin{array}{ccccc} f_0^{-1}(\iota^{-1}M_{T^\times}) & = & (\iota')^{-1}f^{-1}M_{T^\times} & \longrightarrow & (\iota')^{-1}M_{U^\times} \\ \downarrow & & \downarrow & & \downarrow \\ f_0^{-1}(\iota^{-1}\mathscr{O}_T) & = & (\iota')^{-1}f^{-1}\mathscr{O}_T & \longrightarrow & (\iota')^{-1}\mathscr{O}_U \\ \downarrow & & & & \downarrow \\ f_0^{-1}(\mathscr{O}_{T_0}) & & \longrightarrow & & \mathscr{O}_{U_0} \end{array}$$

which defines a morphism $g_0 \colon U_0^\times \to T_0^\times$.

Definition 3.2. — *Let $X^\times, Y^\times$ be schemes or formal schemes with fine log structures and let $M \longrightarrow \mathscr{O}_X$ (respectively $N \longrightarrow \mathscr{O}_Y$) denote the morphisms of monoids on X (respectively on Y) giving the log structures. Let $f : X^\times \longrightarrow Y^\times$ be a morphism.*

i) We say that f is a closed immersion if the underlying morphism of schemes $X \longrightarrow Y$ is a closed immersion and the map $f^*N \longrightarrow M$ is surjective.

ii) We say that f is an exact closed immersion if f is a closed immersion and the map $f^*N \longrightarrow M$ is a bijection.

Definition 3.3. — *Let as above $X^\times, Y^\times$ be schemes or formal schemes with fine log structures given by the sheaves of monoids M respectively N and let $f : X^\times \longrightarrow Y^\times$ be a morphism. We say that f is smooth (respectively étale) if the underlying morphism of schemes $X \longrightarrow Y$ is locally of finite presentation and for any commutative diagram*

$$\begin{array}{ccc} T'^\times & \xrightarrow{s} & X^\times \\ \downarrow \iota & & \downarrow f \\ T^\times & \xrightarrow{t} & Y^\times \end{array}$$

where ι is an exact closed immersion such that the ideal of T' in T is nilpotent, there exists locally on T a morphism (respectively there exists a unique morphism) $g : T^\times \longrightarrow X^\times$ such that $g\iota = s$ and $fg = t$.

See **[23]** 2.9 for other equivalent formulations of Definition 3.3.

Moreover we have the following result from **[26]** 4.10:

Lemma 3.4. — *If $f : X^\times \longrightarrow Y^\times$ is a closed immersion, then there exists locally on X a factorization of f as: $X^\times \xrightarrow{\iota} T^\times \xrightarrow{g} Y^\times$ where $T^\times$ is a fine log scheme, ι is an exact closed immersion and g is an étale morphism.*

3.1.3. *Fibrations and rigid analytic Poincaré lemmas*

3.1.3.1. — Let us first consider a smooth affine scheme Z of finite type over k and let $\iota : Z \longrightarrow \mathscr{T}$ and $\iota : Z \longrightarrow \mathscr{T}'$ be closed immersions of Z into smooth p-adic formal affine schemes over W. Let us assume that we have a smooth morphism of formal schemes $u : \mathscr{T}' \longrightarrow \mathscr{T}$ such that $u \circ \iota' = \iota$. Let $\mathscr{T}'_{/Z}, \mathscr{T}_{/Z}$ denote the formal completions of $\mathscr{T}'$ respectively $\mathscr{T}$ along Z and let $T' := (\mathscr{T}'_{/Z})^{\mathrm{rig}}$ and $T := (\mathscr{T}_{/Z})^{\mathrm{rig}}$. Then locally on T' we have integers d and natural isomorphisms $T' \cong T \times_{K_0} S^d$, where let us recall that S is the open unit disk over K_0, such that the following diagram is commutative

$$\begin{array}{ccc} T' & \longrightarrow & T \times_{K_0} S^d \\ u\downarrow & & \downarrow \\ T & = & T \end{array}$$

In the above diagram the right vertical map is the natural projection. For a proof of the result see **[1]** Theorem 1.3.2. An easy consequence of this result on "fibrations" is the following

***Lemma 3.5* (Smooth Poincaré lemma).** — *Let the notations be as at the beginning of this section. Let $\mathscr{E}$ denote an isocrystal on Z/W (see Section 3.3) and let us consider the de Rham complexes of sheaves on T' and T denoted $DR(T', \mathscr{E})^\bullet$ and $DR(T, \mathscr{E})^\bullet$ obtained by evaluating $\mathscr{E}$ at the enlargements $\mathscr{T}_{/Z}$ and $\mathscr{T}_{/Z}$. The morphism $u : \mathscr{T}' \longrightarrow \mathscr{T}$ induces a morphism of complexes $DR(T, \mathscr{E})^\bullet \longrightarrow u_* DR(T', \mathscr{E})^\bullet$ which is a quasi-isomorphism.*

We'd like to recall the similar result in the relative situation and with log structures from **[34]**,**[35]** and **[36]**.

Let us now recall that we have denoted $\mathscr{S} = \mathrm{Spf}(W[[t]])$. Let us endow this formal scheme with the fine log structure given by the divisor $t = 0$ and denote this log formal scheme by $\mathscr{S}^\times$. The closed immersion $\mathrm{Spec}(k) \longrightarrow \mathscr{S}$ given by $t \to 0$ endows $\mathrm{Spec}(k)$ with the pull-back log structure. Let $Z^\times$ be a fine, smooth, affine log scheme over $\mathrm{Spec}(k)^\times$ and let $\iota : Z^\times \longrightarrow \mathscr{T}^\times$ and $\iota' : Z^\times \longrightarrow \mathscr{T}'^\times$ denote exact closed immersions over $\mathscr{S}^\times$ into smooth, affine log formal schemes (we assume that $\mathscr{T}, \mathscr{T}'$ are endowed with the (t, p)-topology). Suppose that $u : \mathscr{T}'^\times \longrightarrow \mathscr{T}^\times$ is a morphism of log formal schemes over $\mathscr{S}^\times$ such that $u \circ \iota' = \iota$. Let $\mathscr{T}'_{/Z}, \mathscr{T}_{/Z}$ denote the completions of $\mathscr{T}'$ respectively $\mathscr{T}$ along Z and let $]Z^\times[_{T'} := (\mathscr{T}'_{/Z})^{\mathrm{rig}},]Z^\times[_T := (\mathscr{T}_{/Z})^{\mathrm{rig}}$ denote the tubes of $Z^\times$ relative to $T'^\times$ and T respectively. We denote by $\omega^1_{]Z^\times[_{T'}}$ the sheaf on $]Z^\times[_{T'}$ given by: $\Omega^1_{(\mathscr{T}'_{/Z})^\times/\mathscr{S}^\times} \otimes_W K_0$ and similarly for $\omega^1_{]Z^\times[_T}$. Then we have the following log Poincaré lemma.

***Proposition 3.6* (Lemma 2.2.15, [34]).** — *Let $\mathscr{E}$ be an isocrystal (without log structures) on Z. If u is a smooth morphism of log formal schemes then the natural morphism of de Rham complexes*

$$DR(T, \mathscr{E})^\bullet := \mathscr{E}_{\mathscr{T}_{/Z}} \otimes_{\mathscr{O}_{\mathscr{T}_{/Z}}} \omega^\bullet_{]Z^\times[_T} \longrightarrow u_*\big(DR(T', \mathscr{E})^\bullet := \mathscr{E}_{\mathscr{T}_{/Z}} \otimes_{\mathscr{O}_{\mathscr{T}'_{/Z}}} \omega^\bullet_{]Z^\times[_{T'}}\big).$$

is a quasi-isomorphism.

3.1.4. *Weakly Complete Algebras*

3.1.4.1. *Weakly complete liftings.* — In this and the next sections we prove an important generalization of the "weak lifting theorem" (theorem A.1 of [**5**]) and give a geometric interpretation of it (in §3.1.5).

We start with some notations which will be used as such only in this section. Let R be a complete local ring of characteristic $(0,p)$ with maximal ideal $\mathfrak{p}$. If n is a non-negative integer set $R_n := R\langle T_1, T_2, \dots, T_n\rangle$. Fix now k a non-negative integer. For an R_k-algebra A, the *weak completion* $A^\dagger$ of A is the smallest sub-algebra of the $\mathfrak{p}$-adic completion of A which is $\mathfrak{p}$-adically saturated and contains the elements

$$\sum_{(I_1,\dots,i_n)\in\mathbb{N}^n} r_{i_1,\dots,i_n} a_1^{i_1}\cdots a_n^{i_n},$$

for any $a_j \in \mathfrak{p}A$, $1 \le j \le n$ and $r_{i_1,\dots,i_n} \in R_k$. (When R is discretely valued this is equivalent to the notion of weak completion of A over $(R,\mathfrak{p})$ in [**31**], §1.) The algebra A is weakly complete over R_k if $A = A^\dagger$. Let $A_m := A[x_1, x_2, \dots, x_m]$ and $R_{k,n} = (R_k)_n^\dagger$. A quotient of $R_{k,n}$ for some n by a finitely generated ideal is a *semi-dagger algebra* over R_k, [**10**]. Such algebras are weakly complete. Denote $\overline{A} := A/\mathfrak{p}A$. If $f : A \longrightarrow B$ is a homomorphism of semi-dagger R_k-algebras, we say B is formally smooth over A if $\overline{B}$ is smooth over $\overline{A}$ and

$$\mathrm{Ann}_B(\rho) = \mathrm{Ann}_A(\rho)B,$$

for all $\rho \in R$.

Theorem 3.7. — *Suppose A, B, C and D are flat semi-dagger algebras over R_k and we have a commutative diagram*

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array}$$

Suppose, in addition, $C \longrightarrow D$ is surjective, B is formally smooth over A and there exists an R_k-algebra homomorphism $\overline{s} : \overline{B} \longrightarrow \overline{C}$ which commutes with the reduction of the above diagram. Then there exists an R_k-algebra homomorphism $s : B \longrightarrow C$ which lifts $\overline{s}$ and commutes with this diagram.

Sketch of proof. The proof of the less general result Theorem A.1 of [**5**] translates easily. We first outline the proof.

There exists an integer n and $G_1, \dots, G_m \in A_n^\dagger$ so that we can take $B = A_n^\dagger/(G_1, \dots, G_m)$. Let g and $\overline{V}$ be the compositions $A_n^\dagger \longrightarrow B \longrightarrow D$ and $\overline{A}_n \longrightarrow \overline{B} \longrightarrow \overline{C}$ respectively. Let I be the kernel of $C \longrightarrow D$. Let $X := (x_1, \dots, x_n) \in A_n^n$ and $G = (G_1, \dots, G_m)$. First one shows there exists an R_k-algebra homomorphism $V_0 : A_n^\dagger \longrightarrow C$ over R_k which lifts $\overline{V}$ such that $V_0 = g(\mathrm{mod} I)$. Now one shows there exists an $n \times m$ matrix N an $m \times m$ matrix Q and an m-tuple of $m \times m$ matrices M with coefficients in $A_n^\dagger$ such that

$$G(X + GN) = GMG^t + GQ$$

where G^t is the transpose of G and the coordinates of Q are in $\mathfrak{p}A^\dagger$. Now for a non-negative integer s set

$$V_{s+1} = V_s(X) + G(V_s(X))N(V_s(X)).$$

The V_s converge to the required V as s goes to infinity. The proof of which we now explain:

Lemma 3.8. — *Suppose $f : A \longrightarrow B$ is a surjective map of R_k-semi-dagger algebras. The kernel of f is a finitely generated ideal.*

Proof. — Without loss of generality may suppose that $A = R_{k,a}$ and $B = R_{k,b}/J$, where J is a finitely generated ideal of $R_{k,b}$. Let us denote by $g : R_{k,b} \longrightarrow B$ the natural map (in particular J is the kernel of g) and call the "weak" variables in $R_{k,a}$ and $R_{k,b}$ by $x_1, \dots, x_a$ and respectively $y_1, \dots, y_b$. Let $h : R_{k,b} \longrightarrow R_{k,a}$ so that $f(h(x)) = g(x)$, $h(y_i) \in f^{-1}(g(y_i)), 1 \leq i \leq b$. Let $x'i \in g^{-1}(f(x_i))$. The kernel of f is generated by $h(J)$ and the finite set $\{x_i - h(x_i')\}_{i=1,a}$. □

In the notations of Theorem 3.7, because B is formally smooth over A, we may write $B = A_n^\dagger/(G_1, \dots, G_m)$. Let g and $\overline{V}$ be the compositions $A_n^\dagger \longrightarrow B \longrightarrow D$ and $\overline{A}_n \longrightarrow \overline{B} \xrightarrow{\overline{s}} \overline{C}$ respectively. Let I be the kernel of the homomorphism $C \longrightarrow D$ and let $X = (x_1, \dots, x_n) \in A_n^n$.

Lemma 3.9. — *There exists $V_0 : A_n^\dagger \longrightarrow C$ over R_k which lifts $\overline{V}$ such that $V_0 = g(\mathrm{mod} I)$.*

Proof. — Let $g'(X)$ be an element of C^n such that

$$g'(X) = g(X) \mathrm{mod} I$$

and define a homomorphism $V' : A_n^\dagger \longrightarrow C$ in the natural way. Similarly there is a homomorphism $V' : A_n^\dagger \longrightarrow C$ which lifts $\overline{V}$,

$$V' = g' \mathrm{mod}(\mathfrak{p}, I)C^n.$$

We can write

$$V'(X) - g'(X) = a - b,$$

where $a \in \mathfrak{p}C^n$ and $b \in IC^n$. Let $V_0 : A_n^\dagger \longrightarrow C$ such that $V_0(X) = V'(X) - a$. □

Let $G = (G_1, \dots, G_m)$ and $X = (x_1, \dots, x_n)$. Formal smoothness implies

Lemma 3.10. — *There exists a $n \times m$ matrix N an $m \times m$ matrix Q and an m-tuple of $m \times m$-matrices M over $A_n^\dagger$ such that*

$$G(X + GN) = GMG^t + GQ$$

where G^t is the transpose of G and the coordinates of Q are in $\mathfrak{p}A_n^\dagger$. Here we think of each G as a row vector of functions of X and by the notation $G(X+GN)$ we mean the composition of functions .

For an integer $s \geq 0$ set

$$V_{s+1}(X) := V_s(X) + G(V_s(X))N(V_s(X).$$

Suppose $Q, V_0(G) = 0 \mathrm{mod} q$, for $q \in \mathfrak{p}R$. Then for $s \geq 1$,

$$V_{s+1}(X) - V_s(X) = ((GMG^t + GQ)(V_{s-1}(X)))N(V_s(X)) = 0 \bmod q^{s+1}.$$

This is enough to show that the sequence V_s converges p-adically. We will now give some idea about why it "weakly converges".

If $r \in p^{\mathbb{Q}}$, $r > 1$, let $R_{k,n}(r)$ denote the sub-ring of $R_{k,n}$ consisting of series which converge on $B_k[1] \times B_n[r]$. If $f : R_{k,n} \longrightarrow A$ is a surjection and $r > 1$, let $A(f, r)$ denote the subring $f(R_{k,n}(r))$ and for $F \in A(f, r)$ set

$$||F||_{f,r} = \max\{||G||_r \mid G \in R_{k,n}(r), f(G) = F\}.$$

Choose once and for all surjective homomorphisms

$$R_{k,a} \longrightarrow A, \text{ and } R_{k,b} \longrightarrow C.$$

Let $R_{k,a+n} \longrightarrow A_n^\dagger$ be the induced surjection. If $e : R_{k,c} \longrightarrow E$ is one of these homomorphisms, let

$$E(r) = E(e, r) \text{ and } ||\ ||_r = ||\ ||_{e,r}.$$

We can show there exist real numbers $u > 1, d > 0$, and $L < 1$ such that for $1 \leq t \leq u$ the entries of N and G lie in $A_n^\dagger(u)$ and

(i) $V_s(A_n^\dagger(t^d)) \subset C(t)$,
(ii) $||V_s(X) - V_0(X)||_t < 1$,
(iii) $||G(V_s(X))||_t \leq L^s ||G(V_0(X))||_t$,
(iv) $L \geq ||N(V_s(X))||_t ||G(V_0(X))||_t$,
(v) $V_s = V_0(\mathrm{mod} I)$.

Now, (iii) and (iv) imply the sequences $V_s|_{A_n^\dagger(t^d)}$ converge to continuous homomorphisms $V_t : A_n^\dagger(t^d) \longrightarrow C(t)$, for $1 \leq t \leq u$, compatible with decreasing t. Let $V : A_n^\dagger \longrightarrow C$ be the direct limit of these V_t. Condition (ii) implies that V lifts $\overline{V}$, (iii) implies $G(V(X)) = 0$, so V factors through a a morphism $B \longrightarrow C$ which lifts $\overline{B} \longrightarrow \overline{C}$ and finally (v) implies this morphism commutes with the diagram.

Remark 3.11. — *A statement needed to prove (iv) which is analogous to a result used but not stated explicitly in* **[5]** *is, with notation as in the proof of lemma A-8 of* **[5]**,

$$||h(F)||_{g,t} \leq ||F||_{f,t^d}.$$

Corollary 3.12. — *Suppose R is discretely valued and B is a flat, formally smooth semi-dagger algebra over R_k. Then B is very smooth over $(R_k, \mathfrak{p}R_k)$ in the sense of* **[31]**, *Definition 2.5.*

Corollary 3.13. — *Suppose R is discretely valued and B and C are flat R_k semi-dagger algebras, formally smooth over R_k and there exists an R_k-algebra isomorphism $\overline{s} : \overline{B} \longrightarrow \overline{C}$. Then there exists an R_k-algebra isomorphism $s : B \longrightarrow C$ lifting $\overline{s}$.*

Proof. — This follows from the previous corollary and the proof of Theorem 3.3 of [**31**]. □

3.1.4.2. *Weak completions.* — Let the notations be as in §3.1.4.1. In this section, given a finitely generated R_k-algebra A, we give a geometric interpretation of the ring $A^\dagger \otimes_R K$, which will be used later in the article.

Suppose R is discretely valued.

Proposition 3.14. — *Let A be a finitely generated flat R_k-algebra. Set $\overline{A} = A/\mathfrak{p}A$, $\hat{A} = \lim_{\leftarrow, n} A/\mathfrak{p}^n A$, $U = \mathrm{Spec}(A)$, $\hat{U} = \mathrm{Spf}(\hat{A})$ and $\overline{U} = \mathrm{Spec}(\overline{A})$. Let $g : U \longrightarrow X$ be an open immersion of U into a scheme X proper and flat over R_k. Let $\hat{X}$ be the formal completion of X along its special fiber and $\hat{U}_K =]\overline{U}[_{\hat{X}}$. Then $A^\dagger \otimes_R K \cong \lim_{\rightarrow, V} A(V)$, where V ranges over all affinoid strict neighborhoods of $\hat{U}_K$ in $\hat{X}_K$ and $A(V)$ denotes the affinoid algebra of V.*

Proof. — Let Z be the complement of U in X with the reduced closed sub-scheme structure and let $\overline{Z}$ be its reduction modulo $\mathfrak{p}$. Let π be a uniformizer of R. Suppose $\{W_i\}_i$ is an affine cover of $\overline{X}$ and suppose that $f_{i1}, \dots, f_{in_i} \in \mathscr{O}_{\hat{X}_K}(]W_i[)$ are such that $\overline{f}_{i1}, \dots, \overline{f}_{in_i}$ generate the ideal in $\mathscr{O}_{W_i}$ defining $\overline{Z} \cap W_i$. For $\lambda \in p^{\mathbb{Q}}$, $|\lambda| \geq |\pi|$, let V_λ be the union over all i of

$$\{x \in]W_i[\mid \text{there exists } j, 1 \leq j \leq n_i \text{ such that } |f_j(x)| \geq \lambda\}.$$

As in [**1**] §1.2, the V_λ's are independent of the choices and form a co-final system of strict neighborhoods of $\hat{U}_K$ in X_K^{rig}. Then we see that V_λ is contained in $U_K^{\mathrm{rig}} (\subset X_K^{\mathrm{rig}})$. This implies that the inductive limit we consider does not depend on the choice of the embedding $U \longrightarrow X$. Choose a presentation $A = R_k[T_1, \dots, T_n]/I$, which gives a closed immersion $U \longrightarrow \mathbb{A}^n_{R_k}$ and let X be the closure of U in $\mathbb{P}^n_{R_k}$. Then we see that $A(V_\lambda)$ is isomorphic to $(R_k\langle T_1, \dots, T_n\rangle_\lambda / I) \otimes_R K$, where $R_k\langle T_1, \dots, T_n\rangle_\lambda$ denotes the ring of power series over R_k converging on the closed disk $\{(y, x) \in \overline{K}^{k+n} \mid |y| \leq 1, |x| \leq 1/\lambda\}$. Hence its inductive limit coincides with $(R_k[T_1, \dots, T_n]^\dagger / I) \otimes_R K \cong A^\dagger \otimes_R K$. □

Remark 3.15. — *It is possible to improve this result. If $Z \subset X$ are affinoids, set $|g|_Z = \sup\{|g(x)| \quad : \quad x \in Z\}$ and*

$$A_Z(X) = \{f \in A(X) \quad : |f|_Z \leq 1\}.$$

Then we can show, in the above notation, $A^\dagger \cong \lim_{\rightarrow, V} A_{\hat{U}_K}(V)$, where as before V ranges over all strict affinoid neighborhoods of $\hat{U}_K$ in $\hat{X}_K$ if $\overline{A}, A$ are normal, X is reduced and $\overline{U}$ is irreducible.

3.2. The geometry of the family. — Let us resume the notations of the introduction. We'll briefly recall from [**17**] how the family of curves $X \longrightarrow S$ in Section 2 is constructed. In this section we assume that P is empty.

As C is regular, $\overline{C}$ is a reduced divisor with simple normal crossings and each singular point is k-rational we may find a deformation of $\overline{C}$, $\mathfrak{X} \to \mathscr{S} := \mathrm{Spf}(W[[t]])$ with the following properties

- $\mathfrak{X}$ is defined over W
- the curve C is the base change of $\mathfrak{X}$ by the map $W[[t]] \to V$ sending t to π.
- Zariski locally $\mathfrak{X}$ is smooth over $W[[t]]$ or isomorphic to $W[[t]]\langle x, z\rangle/(xz - t)$.

Let $X := \mathfrak{X}^{\mathrm{rig}} \longrightarrow S := \mathscr{S}^{\mathrm{rig}}$ as defined in Section 3.1. In this particular case the general construction gives the following. Let $\mathscr{R}_0 := W[[t]]$ and for each integer $n \geq 1$ let $\mathscr{R}_n := W[[t]]\langle T\rangle/(t^n - pT)$; it turns out that R_n is the p-adic completion of $W[t,T]/(t^n - pT)$ and that we have natural maps

- $\mathscr{R}_n \to V$ defined by $t \to \pi, T \to \pi^n/p$ for all $n >[K : K_0]$

and

- $\mathscr{R}_{n+1} \to \mathscr{R}_n$ over $W[[t]]$ defined by $T \to tT$. Denote by $\mathfrak{X}_n$, $\mathfrak{X}_0 \times_{\mathrm{Spf}\mathscr{R}_0} \mathrm{Spf}\,\mathscr{R}_n$.

Let, for $n \geq 1$, X_n and S_n denote the generic fibers of the p-adic formal schemes $\mathfrak{X}_n$ and $\mathrm{Spf}(\mathscr{R}_n)$ and let

$$X := \lim_{\to,n} X_n \text{ and } S := \lim_{\to,n} S_n$$

The rest of this section will be devoted to understanding the rigid analytic structure of the family X/S. As $S_n := \mathrm{Spm}(\mathscr{R}_n \otimes K_0)$ is defined by $|t| \leq |p|^{1/n}$, it follows that S_n is the affinoid disk centered at 0 of radius $|p|^{1/n}$ and therefore S is isomorphic to the open disk of radius 1 centered at 0.

In [**7**] (see also [**9**]) a one-dimensional wide open was defined to be a rigid space which is isomorphic to the complement in a proper curve of a "discoid subdomain." We now define a wide open, in general, to be the rigid space associated to a complete, flat, topologically finitely generated, semi-local ring over W (or over V) (see §7 of [**25**]). Residue classes of affinoids are wide opens. One can show ([**11**]) that such spaces have a finite number of irreducible components. We suspect, when they are smooth, that they have finite dimensional de Rham cohomology.

First, as $\mathfrak{X}$ is a deformation of $\overline{C}$, the ideal $t\mathscr{O}_{\mathfrak{X}} + p\mathscr{O}_{\mathfrak{X}}$ of $\mathscr{O}_{\mathfrak{X}}$ is an ideal of definition for this formal scheme and the closed sub-scheme of $\mathfrak{X}$ defined by this ideal is isomorphic to $\overline{C}$ as schemes over k. Therefore, by Section 3.1 we have a reduction map $\mathrm{red}\colon X \to \overline{C}$, and we define the covering of X:

$$\mathscr{C} := \{\mathrm{red}^{-1}Z\colon Z \text{ is an irreducible component of } \overline{C}\}.$$

This is an admissible open cover of X. If v is an irreducible component of $\overline{C}$, we denote by $U_v \in \mathscr{C}$ the corresponding open and if e is a singular point of $\overline{C}$ we let $A_e = \mathrm{red}^{-1}(e)$. We'll see in Section 3.5 an interpretation of these notions in terms of graphs.

Moreover, if $s \in S^*$, then the restriction (i.e. base change) of $\mathscr{C}$ to the fiber X_s is an admissible covering $\mathscr{C}_s$ of X_s described in Section 2.2 for $s = \pi$. For every v irreducible component of $\overline{C}$ let us denote by

$$Z_v := U_v - \bigcup_{\substack{w \\ w \neq v}} U_w.$$

Then Z_v is a rigid space over S such that all of its fibers are affinoids for all v. Let e be a fixed singular point of $\overline{C}$. Then we have

Lemma 3.16. — *There are functions x_e and $x_{\tau(e)}$ on $A_e = A_{\tau(e)}$ such that $x_e x_{\tau(e)} = t$, $|x_e(u)| \to 1$ as u approaches $Z_{a(e)}$. Moreover, the map $\alpha \to (x_e(\alpha), x_{\tau(e)}(\alpha))$ maps A_e isomorphically to the open unit ball in $\mathbb{A}^2_{K_0}$, i.e. the rigid subspace of $\mathbb{A}^2_{K_0}$ defined by*

$$\{(x,z) \colon |x| < 1 \text{ and } |z| < 1\}.$$

Proof. — This follows easily from the fact that the singularities of X/S are given by local equations of the form $xz = t$. □

Let us recall that Y is the fiber of X/S above $0 \in S$. Let L be a finite, non-trivial, totally ramified extension of K_0 and π_L a uniformizer of L. Let also $\mathscr{B} := \mathrm{Spf}(\mathscr{O}_L\langle y\rangle)$ denote the formal scheme whose generic fiber is the closed disk centered at 0 of radius $|\pi_L|$. If $n > [L : K_0]$ we have a natural morphism $\phi : \mathscr{B} \longrightarrow \mathrm{Spf}(\mathscr{R}_n) \longrightarrow \mathscr{S}$ induced by the morphisms $\mathscr{R}_0 \longrightarrow \mathscr{R}_n \longrightarrow \mathscr{O}_L\langle y\rangle$ given by $t \to \pi_L y$ and $T \to (\pi_L^n/p)y^n$, whose generic fiber induces $B := B_L \subset S$. We denote by $\mathfrak{X}_{\mathscr{B}} := \mathfrak{X}_n \times_{\mathrm{Spf}(\mathscr{R}_n)} \mathscr{B}$, which is independent of $n > [L : K_0]$. Let us remark that by **[25]** 7.2.4, we have $(\mathfrak{X}_{\mathscr{B}})^{\mathrm{rig}} = X \times_S B$ which will be denoted X_B.

Lemma 3.17. — *In the notations above there is a natural isomorphism*

$$\xi_L : \overline{C} \times \mathbb{A}^1_k \longrightarrow (\mathfrak{X}_{\mathscr{B}})_1 \quad \text{as schemes over } \mathbb{A}^1_k$$

where let us recall, k is the residue field of K and if Z is a formal scheme over $\mathscr{O}_L$, Z_1 denotes the closed formal sub-scheme of Z of ideal $\pi_L\mathscr{O}_Z$.

Proof. — The special fiber of the map ϕ defined above, $\phi_1 \colon \mathscr{B}_1 = \mathbb{A}^1_k \longrightarrow \mathscr{S}_1 = \mathrm{Spf}(k[[t]])$ is the constant map, induced by the map sending t to 0.

Then $(\mathfrak{X}_{\mathscr{B}})_1 = (\mathscr{Y})_1 \times \mathbb{A}^1_k = \overline{C} \times \mathbb{A}^1_k$, where let us recall $\mathscr{Y}$ is the fiber at 0 of $\mathfrak{X} \longrightarrow \mathscr{S}$. □

Proposition 3.18. — *Let L, π_L, $\mathscr{B}$, B be as in Lemma 3.17. Then, for every vertex v of G there is an admissible wide-open strict neighborhood W_v of $Z_{v,B} := Z_v \times_S B$ in $U_{v,B} := U_v \times_S B$, and for every $s \in B$ an isomorphism*

$$\alpha_{v,s} := \alpha_{L,v,s} \colon W_{v,s} \times B \cong W_v \quad \text{over} \quad B,$$

lifting the isomorphism

$$\xi_L : \overline{C}^0_v \times \mathbb{A}^1_k \cong (Z_v)_1$$

given by Lemma 3.17. We have denoted by $W_{v,s}$ the fiber of W_v at s and by $\overline{C}_v^0$ the complement of singular points of $\overline{C}$ in the component $\overline{C}_v$ corresponding to v.

Proof. — Let $\mathscr{Z}_{\mathscr{B},v}$ denote the formal model of $Z_{B,v}$ in $\mathfrak{X}_{\mathscr{B}}$, which is the formal spectrum of the ring of integral valued rigid functions on $Z_{B,v}$. As the special fiber of $Z_{v,B}$ with respect to the ideal generated by (t, π_L) is the affine scheme $\overline{C}_v^0$ of finite type over k, $Z_{v,B}$ is an affinoid over B. By Lemma 3.17 we have $(\mathscr{Z}_{\mathscr{B},v})_1 \cong \overline{C}_v^0 \times \mathbb{A}_k^1$. We also have an isomorphism $\beta_{v,s} : (\mathscr{Z}_{v,s} \times \mathscr{B})_1 \cong \overline{C}_v^0 \times \mathbb{A}_k^1$, where $\mathscr{Z}_{v,s}$ is the fiber of $\mathscr{Z}_{\mathscr{B},v}$ at $s \in B$. Now using Theorem 3.7 the isomorphism between $(\mathscr{Z}_{\mathscr{B},v})_1$ and $(\mathscr{Z}_{v,s} \times \mathscr{B})_1$ lifts to an isomorphism over B of $\mathscr{Z}_{\mathscr{B},v}^\dagger$ and $(\mathscr{Z}_{v,s} \times \mathscr{B})^\dagger$. From Proposition 3.14 and Theorem 3.3 of **[31]** we deduce $\beta_{v,s}$ lifts to an isomorphism over B of strict affinoid neighborhoods T of $Z_{B,v}$ in $U_{B,v}$ and $T_s \times B$ of $Z_{v,s} \times B$ in $U_{v,s} \times B$, over B, where T_s denotes as usual the fiber of T at s. By Lemma 3.1, T_s has a canonical, p-adic formal model $\mathscr{T}_s$ over $\mathscr{O}_F$ (F being the residue field of s) with a morphism $\mathscr{T}_s \longrightarrow \mathfrak{X}_s$ which induces the inclusion $T_s \subset U_{v,s} \subset X_s$. This morphism induces a morphism between the special fiber $\overline{T}$ of $\mathscr{T}_s$ and $\overline{C}$. (In fact this morphism identifies $\overline{T}$ with a certain blow-up of the component $\overline{C}_v$ of $\overline{C}$ corresponding to v.) Let $\overline{T}_v$ denote the component of $\overline{T}$ isomorphic to $\overline{C}_v$ under this morphism.

Now, let $\mathscr{T} := \mathscr{T}_s \hat{\times} \mathscr{B}$, then $\mathscr{T}^{\mathrm{rig}} \cong T_s \times B \cong T$. We define W_v to be the inverse image under the reduction $T \xrightarrow{\mathrm{red}} \overline{T}$ of the component $\overline{T}_v$ of $\overline{T}$, i.e. $W_v :=]\overline{T}_v[_{\mathscr{T}}$. Similarly, let $W_{v,s}$ be the inverse image under the reduction $T_s \xrightarrow{\mathrm{red}} \overline{T}$ of $\overline{T}_v$, i.e. $W_{v,s} :=]\overline{T}_v[_{\mathscr{T}_s}$. Then both W_v and $W_{v,s} \times B$ are wide open spaces over B containing $Z_{v,B}$ and contained in $T \subset U_{v,B}$, respectively $T_s \times B \subset U_{v,s} \times B$, which are isomorphic under the restriction of the above isomorphism between T and $T_s \times B$. □

We have the following very easy consequence of the proof of Proposition 3.18, which we record for later use.

Lemma 3.19. — *There are canonical, isomorphic formal models $\mathscr{W}_v, \mathscr{W}_{v,s} \times \mathscr{B}$ of the wide opens W_v, $W_{v,s} \times B$ in Proposition 3.18, which are wide open enlargements of $\overline{C}_v$ (and so of $\overline{C}$). Moreover, there is a (non canonical) morphism of formal schemes $\mathscr{W}_v \longrightarrow \mathfrak{X}_{\mathscr{B}}$ over $\mathscr{B}$ whose generic fiber is the inclusion $W_v \subset X_B$ and whose special fiber is the morphism $\overline{C}_v \subset \overline{C}$.*

Proof. — Let us consider the formal scheme $\mathscr{W}_v := \mathscr{T}_{/\overline{T}_v}$ i.e. the formal completion of the formal scheme $\mathscr{T}$ defined in the proof of Proposition 3.18 along the closed sub-scheme $\overline{T}_v$. Then $\mathscr{W}_v^{\mathrm{rig}} \cong W_v$ as rigid spaces over B. Let us remark that $\mathscr{W}_v \cong \mathscr{W}_{v,s} \times \mathscr{B}$, where $\mathscr{W}_{v,s} := \mathscr{T}_{s/\overline{T}_v}$ is the formal completion of $\mathscr{T}_s$ along $\overline{T}_v$. The composition $\overline{T}_v \cong \overline{C}_v \longrightarrow \overline{C}$ makes the formal schemes $\mathscr{W}_v$ and $\mathscr{W}_{v,s}$ wide open enlargements of $\overline{C}_v$ and of $\overline{C}$ such that $\mathscr{W}_v \cong \mathscr{W}_{v,s} \times \mathscr{B}$ as formal schemes over $\mathscr{B}$. □

Remark 3.20. — *In the notations of Proposition 3.18 where now $s = 0$, the following diagram commutes*

$$\begin{array}{ccccc} W_v & \longrightarrow & X_B & \stackrel{\mathrm{mod}\pi_L}{\longrightarrow} & \overline{C} \times \mathbb{A}^1_k \\ \beta \downarrow & & & & \downarrow \\ W_{v,0} & \longrightarrow & Y_L & \stackrel{\mathrm{mod}\pi_L}{\longrightarrow} & \overline{C}. \end{array}$$

Proof. — The commutativity of the diagram follows from the fact that if we denote by $\iota_0 : Y_L \longrightarrow X_B$ the map induced by the the embedding of Y into X as its fiber at 0, the following diagram commutes

$$\begin{array}{ccc} X_B & \longrightarrow & \overline{C} \times \mathbb{A}^1_k \\ \iota_0 \uparrow & & \downarrow \\ Y_L & \longrightarrow & \overline{C}. \end{array} \qquad \square$$

Remark 3.21. — *Let B be as in Proposition 3.18. Then we have,*

$$\mathbb{H}_B \cong H^1_{dR}(X_B/B, (\mathscr{E}_{\mathfrak{X}^\times}|_{X_B})(\log Y)).$$

3.3. Isocrystals. — Our main references for F-isocrystals are [**32**], [**17**], [**16**], [**1**] and [**34**]. Let us briefly recall the definitions, in the cases in which we need them. Suppose that Z is a scheme over k and fix L a finite, totally ramified (possibly trivial) extension of K_0 and let $\mathscr{O}_L$ denote its ring of integers. Let us recall that if $L = K_0$, $\mathscr{O}_L = W$ and if $L = K$ then $\mathscr{O}_L = V$.

We begin by recalling the category of $\mathscr{O}_L$-enlargements of Z, on which the F-isocrystals take their values. First if $\mathscr{T}$ is a p-adic formal scheme over $\mathscr{O}_L$ we denote by $\mathscr{T}_0$ the reduced closed sub-scheme of the closed sub-scheme of $\mathscr{T}$ defined by the ideal $p\mathscr{O}_{\mathscr{T}}$.

Definition 3.22. — *A* $\mathscr{O}_L$-enlargement *of Z is a pair $(\mathscr{T}, z_{\mathscr{T}})$ consisting of a flat p-adic formal $\mathscr{O}_L$-scheme $\mathscr{T}$ (i.e., each open affine is isomorphic to $\mathrm{Spf} R$ where R is a quotient of $\mathscr{O}_L\langle X_1, \ldots, X_n\rangle$ for some n) together with a $\mathscr{O}_L$-morphism $z_{\mathscr{T}} \colon \mathscr{T}_0 \longrightarrow Z$. A morphism of $\mathscr{O}_L$-enlargements $(\mathscr{T}', z_{\mathscr{T}'}) \longrightarrow (\mathscr{T}, z_{\mathscr{T}})$ is an $\mathscr{O}_L$-morphism $g \colon \mathscr{T}' \longrightarrow \mathscr{T}$ such that $z_{\mathscr{T}} \circ g_0 = z_{\mathscr{T}'}$.*

Let, more generally, $\mathscr{T}$ be a locally noethering formal scheme over $\mathscr{O}_L$. We denote by $\mathscr{T}_0$ the reduced sub-scheme of the closed sub-scheme defined by an ideal of definition of $\mathscr{T}$. Let as above Z be a scheme over k.

Definition 3.23. — *By a* **wide open $\mathscr{O}_L$-enlargement** *of Z, we mean a pair $(\mathscr{T}, z_{\mathscr{T}})$ where $\mathscr{T}$ is a formal scheme such that the affine open sets are isomorphic to $\mathrm{Spf} R$ where R is a quotient of $\mathscr{O}_L\langle X_1, \ldots, X_m\rangle[[V_1, \ldots, V_n]]$ for some m and n and $z_Y : \mathscr{T}_0 \longrightarrow Z$ is a morphism of $\mathscr{O}_L$-schemes. The morphism of wide open enlargements is defined as in Definition 3.22.*

As in Section 3.1 one can attach a rigid analytic space over L, $\mathscr{T}^{\mathrm{rig}}$, to a formal $\mathscr{O}_L$-scheme as in the Definition 3.23. It satisfies the following universal property: if $\mathscr{T}$ is an affine formal scheme, say $\mathscr{T} = \operatorname{Spf} R$, there is a unique pair $(\iota_{\mathscr{T}}, \mathscr{T}^{\mathrm{rig}})$ which is the final element in the category of pairs (h, X) where X is rigid space over $\mathscr{O}_L$ and h is a continuous $\mathscr{O}_L$-homomorphism from R into $H^0(X, \mathscr{O}_X)$. A morphism in this category $(X, h) \to (Y, g)$ is a morphism $f\colon X \to Y$ such that $h = f^* \circ g$. See Proposition 0.2.3 of [**1**] for a discussion of this when $n = 0$. The tubes of Berthelot (see ibid.) are examples of these spaces.

Examples i) Let $\mathfrak{X}, \mathscr{S}, \mathfrak{X}_n$ be as in Section 3.2. Fix $n \geq 1$. As t generates the nilradical of $\mathscr{R}_n/p\mathscr{R}_n$, we have that $(\mathfrak{X}_n)_0$ is the closed sub-scheme of $\mathfrak{X}_n$ defined by the ideal generated by p and t. As a consequence we have a natural W-morphism $z_n\colon (\mathfrak{X}_n)_0 \longrightarrow \overline{C}$. Therefore the pairs $(\mathfrak{X}_n, z_n)$ are W-enlargements of $\overline{C}$ for all $n \geq 1$ and the morphisms $\mathfrak{X}_{n+1} \longrightarrow \mathfrak{X}_n$ induce morphisms of W-enlargements of $\overline{C}$.

ii) On the other hand $(\mathscr{S}, z_{\mathscr{S}})$ is a wide open enlargement of $\operatorname{Spec}(k)$, where $z_{\mathscr{S}} : \mathscr{S}_0 = \operatorname{Spec}(W[[t]]/tW[[t]]) \cong \operatorname{Spec}(k)$.

iii) As π generates the nilradical of V/pV, C_0 is the closed sub-scheme of C corresponding to the ideal $\pi\mathscr{O}_C$. As a consequence we have a natural isomorphism $z_C\colon C_0 \cong \overline{C}$, which makes (C, z_C) into a W-enlargement of $\overline{C}$.

iv) We can make the fibered product of two wide open enlargements $(\mathscr{S}, s)$ and $(\mathscr{T}, t)$ of Z, $\mathscr{S} \hat{\times} \mathscr{T}$. It equals (U, u) where U is the completion of $\mathscr{S} \times \mathscr{T}$ along $(s,t)^*\Delta(Z)$ and u is the composition

$$U_0 = (s,t)^*\Delta(Z) \to \mathscr{S}_0 \times \mathscr{T}_0 \xrightarrow{\pi_1} S_0 \xrightarrow{s} Z.$$

The existence of this fibered product is the main reason we consider wide open enlargements.

Definition 3.24. — *An* isocrystal $\mathscr{E}$ *on* $Z/\mathscr{O}_L$ *is the following set of data:*

(i) For every $\mathscr{O}_L$*-enlargement* $(\mathscr{T}, z_{\mathscr{T}})$ *of* Z *a coherent sheaf of* $L \otimes_{\mathscr{O}_L} \mathscr{O}_{\mathscr{T}}$*-modules* $\mathscr{E}_{(\mathscr{T}, z_{\mathscr{T}})}$. *In general and if there is no ambiguity this module will be denoted by* $\mathscr{E}_{\mathscr{T}}$.

(ii) For every $\mathscr{O}_L$*-morphism of enlargements of* Z, $g\colon (\mathscr{T}', z_{\mathscr{T}'}) \longrightarrow (\mathscr{T}, z_{\mathscr{T}})$ *an isomorphism of* $L \otimes_{\mathscr{O}_L} \mathscr{O}_{\mathscr{T}}$*-modules:* $\theta_g\colon g^*\mathscr{E}_{\mathscr{T}} \longrightarrow \mathscr{E}_{\mathscr{T}'}$. *The collection of isomorphisms* $\{\theta_g\}$ *is required to satisfy the cocycle condition.*

A morphism of isocrystals $\alpha\colon \mathscr{E}' \longrightarrow \mathscr{E}$ *is a collection of homomorphisms* $\alpha_{\mathscr{T}}\colon \mathscr{E}'_{\mathscr{T}} \longrightarrow \mathscr{E}_{\mathscr{T}}$ *compatible with the isomorphisms* θ_g*, for all* g.

For example, there is a natural isocrystal on Z/W denoted $\mathscr{O}_{Z/K_0}$ whose value on an enlargement $(\mathscr{T}, z_{\mathscr{T}})$ is $\mathscr{O}_{\mathscr{T}} \otimes_W K_0$. We call a direct sum of such isocrystals a *free isocrystal* on Z/W. Because every enlargement of $\operatorname{Spec} k$ factors through $\operatorname{Spf} W$, every isocrystal on a point is free.

Because the rigid space attached to a wide open enlargement may be admissibly covered by the rigid spaces attached to enlargements, the cocycle condition allows us to evaluate an isocrystal on a wide open enlargements $(\mathscr{T}, z_{\mathscr{T}})$ to get a coherent sheaf $\mathscr{E}_{(\mathscr{T},z_{\mathscr{T}})}$ on $\mathscr{T}^{\mathrm{rig}}$. (See Remark 2.3.4 of [**1**] for a discussion of this in the case of tubes.)

We'll now define F-isocrystals.

Definition 3.25. — *An* F-isocrystal *on Z/W is an isocrystal $\mathscr{E}$ on Z/W together with an isomorphism of isocrystals $F\colon \overline{F}^*\mathscr{E} \longrightarrow \mathscr{E}$.*

Let us recall what $\overline{F}^*$ means (see [**32**]). First we will recall a familiar notation, if $M \longrightarrow \mathrm{Spf}(W)$ is a formal scheme and $\tau\colon W \longrightarrow W$ is an automorphism we define $\alpha(\tau)\colon M^\tau \longrightarrow M$ by the Cartesian diagram

$$\begin{array}{ccc} M^\tau & \xrightarrow{\alpha(\tau)} & M \\ \downarrow & & \downarrow \\ \mathrm{Spf}(W) & \xrightarrow{\tau} & \mathrm{Spf}(W). \end{array}$$

where we also use τ to denote the corresponding endomorphism of $\mathrm{Spec}\, W$. If $f : M \longrightarrow M'$ is a morphism of formal schemes over $\mathrm{Spf}(W)$ we also define $f^\tau : M^\tau \longrightarrow (M')^\tau$ by functoriality.

Let now $\sigma : W \longrightarrow W$ be the Frobenius automorphism and $\overline{F} : Z \longrightarrow Z^\sigma$ be the absolute Frobenius. For every enlargement $(\mathscr{T}, z_{\mathscr{T}})$ of Z, $(\mathscr{T}, \overline{F}\circ z_{\mathscr{T}})$ is an enlargement of Z^σ and $(\mathscr{T}^{\sigma^{-1}}, (\overline{F}\circ z_{\mathscr{T}})^{\sigma^{-1}})$ is again an enlargement of Z. Then $\overline{F}^*(\mathscr{E})$ is the isocrystal on Z whose value on $(\mathscr{T}, z_{\mathscr{T}})$ is $\alpha(\sigma)_*\mathscr{E}_{(\mathscr{T}^{\sigma^{-1}},(\overline{F}\circ z_{\mathscr{T}})^{\sigma^{-1}})}$.

Remark 3.26. — (a) *Clearly the map of sections, $a\otimes\alpha \to a\alpha^\sigma$, defines an F-isocrystal structure on $\mathscr{O}_{Z/K_0}$.*

(b) *If $f\colon U \to Z$ is a morphism of schemes over k and $\mathscr{E}$ is an F-isocrystal on Z/W, there is a natural F-isocrystal on U/W, $f^*\mathscr{E}$, whose value on an enlargement $(\mathscr{T}, z_{\mathscr{T}})$ is $\mathscr{E}_{(\mathscr{T},f\circ z_{\mathscr{T}})}$.*

(c) *In* [**32**] *and* [**17**] *the object defined in Definition 5.4 is called "convergent isocrystal" and the object defined in Definition 3.25 is called "convergent F-isocrystal".*

(d) *In Section 2.1 we have used a filtered F-isocrystal $\mathscr{E}$ on Z. As we don't need to prove anything about the filtration in this paper we will not define this notion here. For the appropriate definition see* [**17**] *or* [**24**].

(e) *Let $\mathscr{E}$ be an F-isocrystal on $\overline{C}/W$. For each $n \geq 0$, $\mathscr{E}_{\mathfrak{X}_n}$ can be seen as a sheaf on the nilpotent site of $\mathfrak{X}_n$, or what is the same thing, as a $K_0\otimes_W \mathscr{O}_{\mathfrak{X}_n}$-module with an integrable, convergent connection D_n. The F-structure gives, for each open affine formal sub-scheme $\mathfrak{U}$ of $\mathfrak{X}_n$ with a lift of Frobenius $\phi_{\mathfrak{U}}$, a horizontal Frobenius $\Phi_n(\phi_{\mathfrak{U}})\colon \phi^*D_n \to D_n$ on $\mathfrak{U}^{\mathrm{rig}}$. Moreover the morphisms of W-enlargements $(\mathfrak{X}_{n+1}, z_{n+1}) \longrightarrow (\mathfrak{X}_n, z_n)$ induce isomorphisms $\theta_n\colon (\mathscr{E}_{\mathfrak{X}_{n+1}}, D_{n+1}) \cong (\mathscr{E}_n, D_n)$, therefore we obtain in the limit a coherent sheaf of $\mathscr{O}_X$-modules $\mathscr{E}_{\mathfrak{X}}$, together with an integrable connection $D_{X/K_0}\colon \mathscr{E}_{\mathfrak{X}} \longrightarrow \mathscr{E}_{\mathfrak{X}}\otimes\Omega^1_{X/K_0}$, which is compatible with Frobenii associated to local lifts of Frobenius. We will denote by the same symbol the composition*

$$\mathrm{D}_{X/K_0}\colon \mathscr{E}_X \longrightarrow \mathscr{E}_{\mathfrak{X}}\otimes\Omega^1_{X/K_0} \longrightarrow \mathscr{E}_{\mathfrak{X}}\otimes\Omega^1_{X/K_0}(\log Y).$$

We also get a relative connection by composing

$$\mathrm{D}_{X/S}\colon \mathscr{E}_{\mathfrak{X}} \overset{\mathrm{D}_{X/K_0}}{\longrightarrow} \mathscr{E}_{\mathfrak{X}} \otimes \Omega^1_{X/K_0}(\log Y) \longrightarrow \mathscr{E}_{\mathfrak{X}} \otimes \Omega^1_{X/S}(\log Y).$$

If $\mathscr{E} = \mathscr{O}_{Z/K_0}$*, we will denote* D_{X/K_0} *and* $\mathrm{D}_{X/S}$ *by* d_{X/K_0} *and* $\mathrm{d}_{X/S}$ *respectively.*

(f) $\mathscr{E}_C$*, by the same arguments as above can be thought of as a coherent sheaf of* $\mathscr{O}_{C_K}$*-modules with a convergent, in the sense of* **[32]***, integrable connection* D*. Moreover, the closed immersion* $g\colon C \longrightarrow \mathfrak{X}$ *identifying* C *with the fiber at* π *of* $\mathfrak{X}$ *and which is a morphism of enlargements, induces an isomorphism* $\theta_g\colon g^*\mathscr{E}_{\mathfrak{X}} \cong \mathscr{E}_C$*. 2.2.)*

Because every isocrystal on a point is free we have,

Proposition 3.27. — *Let* $\mathscr{E}$ *be an isocrystal on* $\overline{C}$*. Then* $(\mathscr{E}_X, \mathrm{D}_{X/K_0})$ *has the property that for every residue class* $M = \mathrm{red}_{\mathfrak{X}}^{-1}(x)$*, with* $x \in \overline{C}$*, of* X*, the* $\mathscr{O}_M$*-module with connection* $(\mathscr{E}_{\mathfrak{X}}|_M, \mathrm{D}_{X/K_0})$ *has a basis of horizontal sections.*

Lemma 2.2 of Section 2.2 follows.

3.4. Cohomology of an F-isocrystal. — We will recall here some constructions from **[1]** and **[34]**,**[35]** and **[36]** which will be used later.

3.4.1. — Let Z be a smooth, proper scheme of finite type over k and $\mathscr{E}$ an isocrystal on Z/W. We will recall the definition of $H^i_{\mathrm{cris}}(Z/W, \mathscr{E})$, for $i \geq 0$.

We choose an affine open covering $\{U_i\}_{1\leq i\leq s}$ of Z, and for each U_i a closed immersion into a smooth affine formal W-scheme T_i. For each subset J of $\{1, 2, \ldots, s\}$ we denote by T_J the completion of the fiber product of the T_j's for $j \in J$ along $\cap_{j\in J} U_j$. For each J consider the de Rham complex $H^0(T_J^{\mathrm{rig}}, \mathscr{E}_{T_J} \otimes \Omega^\bullet_{T_J^{\mathrm{rig}}/K_0})$ and connect them by the Čech differentials to make a double complex. We define $H^i_{\mathrm{cris}}(Z/W, \mathscr{E})$ to be the i-th cohomology group of this double complex. To show that this is independent of the choices of a covering $\{U_i\}_i$ and the formal schemes $\{T_i\}_i$, we take another pair of such $\{U'_k\}_{1\leq k\leq t}$ and closed immersions of the U'_k into smooth, affine formal W-schemes T'_k. To compare the constructions for the two choices consider the third, $\{U''_{i,k} := U_i \times_Z U'_k\}_{i,k}$ and $T''_{i,k} := T_i \times T'_k$. If, say $J \subset \{1, 2, \ldots, s\}$ and $K \subset \{1, 2, \ldots, t\}$ we have smooth morphisms of formal W-schemes $u : T''_{J\times K} \longrightarrow T_J$ and $v : T''_{J\times K} \longrightarrow T'_K$ and by the Poincaré lemma recorded in Section 3.1, the pairs of de Rham complexes of sheaves $DR(T_J, \mathscr{E})^\bullet := \mathscr{E}_{T_J} \otimes \Omega^\bullet_{T^{\mathrm{rig}}/K_0}$, and $u_*^{\mathrm{rig}} DR(T''_{J\times K}, \mathscr{E})^\bullet$ and $DR(T'_K, \mathscr{E})^\bullet := \mathscr{E}_{T'_K} \otimes \Omega^\bullet_{(T'_K)^{\mathrm{rig}}/K_0}$ and $v_*^{\mathrm{rig}} DR(T''_{J\times K}, \mathscr{E})^\bullet$ are quasi-isomorphic and so finally the cohomology of the double complexes constructed from them are all quasi-isomorphic.

3.4.2. — We will now recall the definition of log crystalline cohomology over a (certain) base. Let $\mathscr{S}^\times$ denote the formal scheme $\mathrm{Spf}(W[[t]])$ with the log structure given by the smooth divisor $t = 0$. Let $\mathrm{Spec}(k)^\times$ be the scheme $\mathrm{Spec}(k)$ with the inverse image log structure under the map induced by the natural morphism $W[[t]] \longrightarrow k$ sending t to 0. Let $Z^\times$ be a fine, log smooth, log proper scheme over $\mathrm{Spec}(k)^\times$, which we'll regard as a log smooth scheme over $\mathscr{S}^\times$. Let $\mathscr{E}$ be an F-isocrystal on Z/W (without log structure). We'll recall the definition of $H^i_{\mathrm{cris}}(Z^\times/\mathscr{S}^\times,\mathscr{E})$. It is a sheaf of $\mathscr{O}_S$-modules on S, where let us recall $S=\mathscr{S}^{\mathrm{rig}}$. In fact $H^i_{\mathrm{cris}}(Z^\times/\mathrm{Spec}(k)^\times,\mathscr{E})$ is an F-isocrystal on $\mathrm{Spec}(k)$ and $H^i_{\mathrm{cris}}(Z^\times/\mathscr{S}^\times,\mathscr{E})$ is its evaluation on the wide open enlargement $\mathscr{S}$ of $\mathrm{Spec}(k)$.

Let now $\{U_i\}_{1\le i\le s}$ be an affine covering of Z such that $U_i^\times$ is a log smooth, fine, log affine scheme over $\mathrm{Spec}(k)^\times$, where the log-structures are the induced ones. For each $1\le i\le s$ choose closed $\mathscr{S}^\times$-immersions $U_i^\times \longrightarrow T_i$ into log smooth, fine, log affine formal schemes over $\mathscr{S}^\times$. For each $J\subset\{1,2,\dots,s\}$ let T_J denote the log-formal scheme which is the log-completion along $U_J:=\cap_{j\in J}U_j^\times$ of the fibered product over $\mathscr{S}^\times$ of the $T_j^\times$'s, $j\in J$. For every admissible affinoid $B\subset S$, let $DR(T_J^{\mathrm{rig}}\times_S B,\mathscr{E})^\bullet$ denote the relative (to $S^\times$) log-de Rham complex of sheaves on $T_J^{\mathrm{rig}}\times_S B$ with coefficients in $\mathscr{E}_{T_J}$. We define the log rigid (or analytic) cohomology $H^i_{\mathrm{cris}}(Z^\times/\mathscr{S}^\times,\mathscr{E})$ to be the sheaf on S associated to the pre-sheaf $B\longrightarrow H^i((U_\bullet)_{\mathrm{Zar}},\mathrm{red}_*DR(T_\bullet^{\mathrm{rig}}\times_S B,\mathscr{E})^\bullet)$.

It is shown in [**34**] and [**35**] (using Proposition 3.6) that the definition is independent of choices.

Let us now assume that $Z^\times$ has a log smooth, exact global lifting $\mathfrak{X}^\times$ over $\mathscr{S}^\times$ and we write as usually $X:=\mathfrak{X}^{\mathrm{rig}}, S:=\mathscr{S}^{\mathrm{rig}}$.

Lemma 3.28. — *We have a natural isomorphism of sheaves on S, $H^i_{\mathrm{cris}}(Z^\times/\mathscr{S}^\times,\mathscr{E})\cong H^i_{\mathrm{dR}}(X^\times/S^\times,\mathscr{E}_{\mathfrak{X}})$. Here $\mathscr{E}_{\mathfrak{X}}$ is the evaluation of $\mathscr{E}$ at the enlargement $\mathfrak{X}$ of Z, seen as a coherent sheaf on $X:=\mathfrak{X}^{\mathrm{rig}}$ with an integrable connection.*

Proof. — Let $\{U_i\}_{1\le i\le s}$ be an affine open covering of Z, let T_i be the open log-formal sub-schemes of $\mathfrak{X}^\times$ whose underlying topological space is the same as U_i. For each $J\subset\{1,2,\dots,s\}$ define U_J and T_J as above. We also define T'_J to be the open log formal sub-scheme of $\mathfrak{X}^\times$ with underlying topological space U_J. The diagonal induces a log-smooth morphism $\Delta_J: T'_J\longrightarrow T_J$ compatible with the embeddings of U_J and for each admissible affinoid open $B\subset S$, we get quasi-isomorphisms for the relative, log de Rham complexes of sheaves

$$\mathrm{red}_*DR(T_J^{\mathrm{rig}}\times_S B,\mathscr{E})\longrightarrow \mathrm{red}_*DR((T'_J)^{\mathrm{rig}}\times_S B,\mathscr{E}).$$

The Čech complex of the latter complex computes $H^i_{\mathrm{dR}}(X^\times/\mathscr{S}^\times,\mathscr{E}_{\mathfrak{X}})(B)$, as $H^i_{\mathrm{dR}}(X/S,\mathscr{E}_{\mathfrak{X}})$ is a coherent sheaf and B is affinoid. Therefore the association

$$B\longrightarrow H^i((U_\bullet)_{\mathrm{Zar}},\mathrm{red}_*DR(T_J^{\mathrm{rig}}\times_S B,\mathscr{E}))$$

is already a coherent sheaf and we have an isomorphism $H^i_{\mathrm{dR}}(X^\times/S^\times,\mathscr{E}_{\mathfrak{X}})\cong H^i_{\mathrm{cris}}(Z^\times/\mathscr{S}^\times,\mathscr{E})$. □

3.4.3. — In the assumptions of Lemma 3.28 and for $i=1$ let us give an explicit description of the inverse of the isomorphism $\alpha: H^1_{\mathrm{cris}}(Z^\times/\mathscr{S}^\times,\mathscr{E})\cong H^1_{\mathrm{dR}}(X^\times/S^\times,\mathscr{E}_{\mathfrak{X}})$ in that lemma in terms of hyper-cocycles. Let, as in the proof of Lemma 3.28,

$\{U_i\}_{1\le i\le s}$ be an affine cover of Z and let $B\subset S$ be an admissible affinoid open. An element x of $H^1_{\mathrm{dR}}(X^\times/S^\times,\mathscr{E})(B)$ is then represented by a 1-hypercocycle (ω_i,f_{ij}) where $\omega_i\in H^0((T'_i)^{\mathrm{rig}}\times_S B,\mathscr{E}_{T'_i}\otimes\Omega^1_{(T'_i)^{\mathrm{rig}}/S^\times})$ for $1\le i\le s$ and $f_{ij}\in H^0((T'_{ij})^{\mathrm{rig}}\times_S B,\mathscr{E}_{\mathfrak{X}})$ for $1\le i<j\le s$ such that $\nabla(\omega_i)=0$ for all $1\le i\le s$, $\omega_i|_{(T'_{ij})^{\mathrm{rig}}}-\omega_j|_{(T'_{ij})^{\mathrm{rig}}}=\nabla(f_{ij})$ and for all $1\le i<j<k\le s$ we have $f_{ij}|_{(T'_{ijk})^{rig}}+f_{jk}|_{(T'_{ijk})^{rig}}-f_{ik}|_{(T'_{ijk})^{rig}}=0$.

Let as in the proof of Lemma 3.28, for every $1\le i\le s$, $T_i=T'_i$ and $T_{ij}:=(T'_i\times_{\mathscr{S}^\times}T'_j)_{/U_{ij}}$ i.e. T_{ij} is the formal completion of $T'_i\times_{S^\times}T'_j$ along U_{ij}.

We have a natural commutative diagram

$$\begin{array}{ccc} (T'_{ij})^{\mathrm{rig}} & \stackrel{\Delta}{\longrightarrow} & T_{ij}^{\mathrm{rig}} \\ \downarrow & & \pi_i\downarrow \\ (T'_i)^{\mathrm{rig}} & = & T_i^{\mathrm{rig}} \end{array}$$

and a similar one replacing i by j. Here π_i is induced by the natural projection $T'_i\times_{\mathscr{S}^\times}T'_j\longrightarrow T'_i=T_i$ which factors naturally through the formal completion of $T'_i\times_{\mathscr{S}^\times}T'_j$ along U_{ij}.

Lemma 3.29. — *In the notations above, for each $1\le i<j\le s$ there is a unique $h_{ij}\in H^0(T_{ij}^{\mathrm{rig}}\times_S B,\mathscr{E}_{T_{ij}})$ such that*

a) $\Delta^*(h_{ij})=0$

and

b) $\pi_i^*(\omega_i|_{(T'_{ij})^{\mathrm{rig}}})-\pi_j^*(\omega_i|_{(T'_{ij})^{\mathrm{rig}}})=\nabla_{ij}(h_{ij})$. *Here ∇_{ij} is the connection on $\mathscr{E}_{T_{ij}}$.*

Proof. — As Δ is log-smooth we may apply Proposition 3.6. Namely, let $\eta:=\pi_i^*(\omega_i|_{(T'_{ij})^{\mathrm{rig}}})-\pi_j^*(\omega_i|_{(T'_{ij})^{\mathrm{rig}}})$. Then $\nabla_{ij}(\eta)=0$ and moreover the above commutative diagram implies that $\Delta^*(\eta)=0$. Therefore, locally on T_{ij}^{rig}, there exist a_{ij}'s sections of $\mathscr{E}_{T_{ij}}$ such that $\nabla_{ij}(a_{ij})=\eta$. As $0=\Delta^*(\nabla_{ij}(a_{ij}))=\nabla(\Delta^*(a_{ij}))$, a_{ij} can be chosen such that $\Delta^*(a_{ij})=0$. For example replace a_{ij} by $a_{ij}-\pi_1^*(\Delta^*(a_{ij}))$. The conditions $\nabla_{ij}(a_{ij})=\eta$ and $\Delta^*(a_{ij})=0$ determine the a_{ij}'s uniquely, so they glue to give a section h_{ij} of $\mathscr{E}_{T_{ij}}$ over T_{ij}^{rig} satisfying the right properties. □

Now back to our original problem: to explicitly describe the isomorphism $H^1_{\mathrm{dR}}(X^\times/S^\times,\mathscr{E}_{\mathfrak{X}})\longrightarrow H^1_{\mathrm{cris}}(Z^\times/\mathscr{S}^\times,\mathscr{E})$. We have started with an element x of the first group represented by the 1-hyper-cocycle $(\omega_i,f_{ij})_{(i,i<j)}$. For each $1\le i<j\le s$ we determined the sections h_{ij} as in Lemma 3.29. Let us remark that for each $i<j$ we have the following calculation:

$$\pi_i^*(\omega_i)-\pi_j^*(\omega_j)=\pi_i^*(\omega_i)-\pi_j^*(\omega_i|_{(T'_{ij})^{\mathrm{rig}}})+\pi_j^*(\omega_i|_{(T'_{ij})^{\mathrm{rig}}})-\pi_j(\omega_j)=\nabla_{ij}(h_{ij})+\pi_j^*(\nabla(f_{ij}).$$

Moreover, for $1\le i<j<k\le s$ the section $h_{ijk}\in H^0(T_{ijk}^{\mathrm{rig}},\mathscr{E}_{T_{ijk}})$ defined by $h_{ijk}:=\pi_{ij}^*(h_{ij})+\pi_{jk}^*(h_{jk})-\pi_{ik}^*(h_{ik})$ satisfies: $\Delta^*(h_{ijk})=0$ and $\nabla_{ijk}(h_{ijk})=0$. Therefore $h_{ijk}=0$ and so finally $(\omega_i,h_{ij}+\pi_j(f_{ij}))_{(i,i<j)}$ is a 1-hyper-cocycle for the complex $DR(T_\bullet,\mathscr{E})^\bullet$ whose image in $H^1_{\mathrm{cris}}(Z^\times/\mathscr{S}^\times,\mathscr{E})$ is $\alpha^{-1}(x)$.

3.4.4. — In the notations and assumptions at §3.4.3 above let us assume that for each $1 \leq i \leq s$ we have a lifting of Frobenius on U_i, $F_i : T_i \longrightarrow T_i$ compatible with the lifting of Frobenius $F_{\mathscr{S}} : \mathscr{S} \longrightarrow \mathscr{S}$. $F_{\mathscr{S}}$ is defined as the arithmetic Frobenius σ on W and by $F_{\mathscr{S}}(t) = t^p$. Since T_i is affine and log smooth such liftings F_i always exist. Let us now assume that $\mathscr{E}$ is an F-isocrystal on Z/W. Then one defines a natural homomorphism, Frobenius,

$$\Phi : F_{\mathscr{S}}^* H^i_{\mathrm{cris}}(Z^\times/\mathscr{S}^\times, \mathscr{E}) \longrightarrow H^i_{\mathrm{cris}}(Z^\times/\mathscr{S}^\times, \mathscr{E}),$$

which is independent of all the choices. Let $i = 1$ and assume that $Z^\times$ has a log-smooth global lifting $\mathfrak{X}^\times/\mathscr{S}^\times$. We'll describe Φ on $H^1_{\mathrm{dR}}(X^\times/S^\times, \mathscr{E}_{\mathfrak{X}})$ under the identification $\alpha : H^1_{\mathrm{cris}}(Z^\times/\mathscr{S}^\times, \mathscr{E}) \cong H^1_{\mathrm{dR}}(X^\times/S^\times, \mathscr{E}_{\mathfrak{X}})$. Let $B \subset S$ be the affinoid disk centered at 0 of radius r and let $B' = F_{\mathscr{S}}(B) \subset S$ be the affinoid of radius r^p. $x \in H^1_{\mathrm{dR}}(X^\times/S^\times, \mathscr{E}_{\mathfrak{X}})(B')$, then we'd like to express $\Phi(x) := \alpha(\Phi(\alpha^{-1}(x))) \in H^1_{\mathrm{dR}}(X^\times/S^\times, \mathscr{E}_{\mathfrak{X}})(B)$. Suppose we fix an affine cover $\{U_i\}_{1 \leq i \leq s}$ of Z and use all the notations at b) above. If x is represented by the hypercocycle $(\omega_i, f_{ij})_{(i,i<j)}$ corresponding to B' let h_{ij} be as in Lemma 3.29. Then $\Phi(x)$ is represented by the hypercocycle

$$((F_i^{\mathrm{rig}})^*(\omega_i), (F_j^{\mathrm{rig}})^*(f_{ij}) + \Delta^*(F_{ij}^{\mathrm{rig}})^*(h_{ij}))$$

corresponding to B.

3.4.5. — Finally, let us recall the notations of Section 3.2. We have the morphism of formal schemes $f : \mathfrak{X} \longrightarrow \mathscr{S}$ and we denote by $\mathscr{Y} = \mathfrak{X} \times_{\mathscr{S}} \mathrm{Spf}(W)$, where the map $\mathrm{Spf}(W) \longrightarrow \mathscr{S}$ is induced by the W-algebra homomorphism $W[[t]] \longrightarrow W$ sending t to 0. In other words $\mathscr{Y}$ is the fiber of f at the point "0" of $\mathscr{S}$. Given the description of f in Section 3.2, $\mathscr{Y}$ is a divisor of $\mathfrak{X}$ with normal crossings (the irreducible components of $\mathscr{Y}$ are smooth and the singular points defined over W). Let us fix on $\mathfrak{X}$ the log structure corresponding to the divisor $\mathscr{Y}$ and denote this log formal W-scheme $\mathfrak{X}^\times$. Let us endow $\mathscr{Y}$ with the pull-back log structure and denote it $\mathscr{Y}^\times$. Let us remark that $\overline{C}$ is a divisor with normal crossings of C, endow C with the log structure defined by this divisor and by $\overline{C}^\times$ the log scheme $\overline{C}$ with the inverse image log structure.

Then: f is a log smooth morphism $\mathfrak{X}^\times \longrightarrow \mathscr{S}^\times$, which is a log smooth lifting of $\overline{C}^\times$ over $\mathscr{S}^\times$ as at 2) b) above. Finally $\mathscr{Y}^\times$ is a log smooth lifting of $\overline{C}^\times$ over $\mathrm{Spf}(W)^\times$ (this last log structure is given by the smooth divisor $p = 0$). Therefore, 1) and 2) above imply that if $\mathscr{E}$ is an F-isocrystal on Z then we have natural isomorphisms

$$H^1_{\mathrm{cris}}(Z^\times/\mathrm{Spec}(k)^\times, \mathscr{E}) \cong H^1_{\mathrm{cris}}(\mathscr{Y}^\times/\mathrm{Spf}(W)^\times, \mathscr{E}) \cong H^1_{\mathrm{dR}}(Y^\times/K_0, \mathscr{E}_{\mathscr{Y}}).$$

and

$$H^1_{\mathrm{cris}}(Z^\times/\mathscr{S}^\times, \mathscr{E}) \cong H^1_{\mathrm{dR}}(X^\times/S^\times, \mathscr{E}_{\mathfrak{X}}) = H^1_{dR}(X/S, \mathscr{E}_{\mathfrak{X}}(\log(Y)).$$

Moreover if we give ourselves local liftings of Frobenius as in 2) c) above all the isomorphisms are compatible with the Frobenii.

3.5. Hypercocycles and Mayer-Vietoris exact sequences. — In this section we collect a number of technical results showing how to relate Mayer-Vietoris exact sequences and representatives of de Rham cohomology classes for different admissible coverings.

3.5.1. *3.5.1Coverings and Graphs.* — Let T be a rigid analytic space over K and let $\mathscr{D} = \{U_\alpha\}_{\alpha\in I}$ be an admissible covering of T. We will suppose that all our coverings satisfy the assumption:

$$(*) \quad U_\alpha \cap U_\beta \cap U_\gamma \text{ is void for all } \alpha \neq \beta \neq \gamma \neq \alpha \in I.$$

We attach to $\mathscr{D}$ a graph $G = G(\mathscr{D})$ whose vertices $v(G)$ are the elements of $\mathscr{D}$ and whose oriented edges $\epsilon(G)$ correspond to triples $e = (U, V, W)$ where $U \neq V \in \mathscr{D}$ and $A_e := W$ is a connected component of $U \cap V$. If v is a vertex of G we denote U_v the element of $\mathscr{D}$ corresponding to it and also if $e = (U, V, W)$ is an edge then its origin $a(e)$ is U and its end $b(e)$ is V. If $U \cap V$ is connected we denote the edge e by $[a(e), b(e)]$.

We denote $\tau : \epsilon(G) \longrightarrow \epsilon(G)$ by $\tau(e = (U, V, W)) = (V, U, W)$ and we choose once for all a system of representatives $e(G)$ of the quotient set $\epsilon(G)/\tau$.

Let G be a graph. A local system F on G is the following collection of data:

a) for each vertex $v \in v(G)$, an abelian group F_v,

b) for each oriented edge $e \in e(G)$, an abelian group F_e,

c) if $e \in e(G)$, group homomorphisms $\varphi_{a(e)} : F_{a(e)} \longrightarrow F_e$ and $\varphi_{b(e)} : F_{b(e)} \longrightarrow F_e$.

To a local system F on the graph G we associate the complex of abelian groups

$$C^\bullet(G, F) : \quad C^0(G, F) = \oplus_{v\in v(G)} F_v \xrightarrow{d} C^1(G, F) = \oplus_{e\in e(G)} F_e,$$

where $(d(x_v)_{v\in v(G)})_e := \varphi_{a(e)}(x_{a(e)}) - \varphi_{b(e)}(x_{b(e)})$ for $e \in e(G)$. Let $H^i_{\mathrm{Betti}}(G, F) := H^i(C^\bullet(G, F))$ for $i \geq 0$.

Let us now suppose that the graph G is the graph associated to an admissible cover $\mathscr{D}$ of the rigid space T and that $(\mathscr{F}, \nabla)$ is a pair consisting of a coherent sheaf $\mathscr{F}$ of $\mathscr{O}_T$-modules with an integrable connection ∇, then we have a natural family of local systems F_j on G and Betti cohomology groups $H^{i,j}(\mathscr{D}, (\mathscr{F}, \nabla))$, for $i \geq 0, j \geq 0$, as follows:

a) for $v \in v(G)$ set $F_{j,v} := H^j_{dR}(U_v, \mathscr{F}|_{U_v})$,

b) for $e \in e(G)$ set $F_{j,e} := H^j_{dR}(A_e, \mathscr{F}|_{A_e})$,

c) for $e \in e(G)$ $\varphi_{a(e)}, \varphi_{b(e)}$ are pull-backs induced by the open immersions $A_e \subset U_{a(e)}$ and $A_e \subset U_{b(e)}$.

Then $H^{i,j}(\mathscr{D}, (\mathscr{F}, \nabla)) := H^i_{\mathrm{Betti}}(G, F_j)$.

Remark 3.30. — *We have the following variant of the definitions above. Suppose that $\mathscr{T}^\times := (\mathscr{T}, M)$ is a log formal scheme over* $\mathrm{Spf}(V)^\times$ *such that $\mathscr{T}^{\mathrm{rig}} \cong T$ as rigid spaces over K. Suppose that $(\mathscr{G}, \nabla_{\log})$ is a pair consisting of a coherent sheaf $\mathscr{G}$ of $\mathscr{O}_{\mathscr{T}}$-modules and a logarithmic integrable connection $\nabla_{\log}$ on it. Then one denotes $\mathscr{F} = \mathscr{G}^{\mathrm{rig}}, \nabla = (\nabla_{\log})^{\mathrm{rig}}$ and one has, for each $i \geq 0$ the local systems $F_{i,\log}$ obtained* *by taking the logarithmic de Rham cohomology with coefficients in* $(\mathscr{F}, \nabla)$ *and the Betti cohomology groups* $H^{i,j}(\mathscr{D}, \mathscr{F}) := H^i_{\mathrm{Betti}}(G, \mathscr{F}_{i,\log})$.

Remark 3.31. — *If the assumption* $(*)$ *is not satisfied by the covering* $\mathscr{D}$ *but the covering is finite (i.e. the index set* I *is finite) one may attach to it a finite dimensional simplex, local systems on the simplex and the corresponding Betti cohomology groups.*

3.5.2. *Hypercocycles and Mayer-Vietoris exact sequences attached to a covering.* — Let T be a rigid analytic space over K and $\mathscr{D} := \{U_\alpha\}_{\alpha\in I}$ an admissible covering of it which satisfies the assumption $(*)$ above. Let $(\mathscr{F}, \nabla)$ be a pair consisting of a coherent sheaf $\mathscr{F}$ of $\mathscr{O}_T$-modules which is locally free and an integrable connection ∇ on it.

Consider the diagram of rigid spaces and maps:

$$T_{v(G)} = \amalg_{v\in v(G)} U_v \xrightarrow{f} T \xleftarrow{g} T_{e(G)} := \amalg_{e\in e(G)} A_e.$$

We have then an exact sequence of sheaves on T:

$$0 \longrightarrow \mathscr{F} \longrightarrow f_*f^*\mathscr{F} \longrightarrow g_*g^*\mathscr{F} \longrightarrow 0.$$

If for $v \in v(G)$ and $e \in e(G)$ we denote by $\mathscr{F}^v := \mathscr{F}|_{U_v}$ respectively $\mathscr{F}^e := \mathscr{F}|_{A_e}$ then the exact sequence above becomes

$$0 \longrightarrow \mathscr{F} \longrightarrow f_*\big(\oplus_{v\in v(G)}\mathscr{F}^v\big) \longrightarrow g_*\big(\oplus_{e\in e(G)}\mathscr{F}^e\big) \longrightarrow 0.$$

This induces an exact sequence of de Rham complexes and therefore an exact sequence of cohomology groups (the Mayer-Vietoris exact sequence):

$$0 \longrightarrow H^0_{dR}(T,\mathscr{F}) \longrightarrow \oplus_{v\in v(G)} H^0_{dR}(U_v,\mathscr{F}) \longrightarrow \oplus_{e\in e(G)} H^0_{dR}(A_e,\mathscr{F}) \longrightarrow$$

$$\longrightarrow H^1_{dR}(T,\mathscr{F}) \longrightarrow \oplus_{v\in v(G)} H^1_{dR}(U_v,\mathscr{F}) \longrightarrow \oplus_{e\in e(G)} H^1_{dR}(A_e,\mathscr{F}) \longrightarrow \cdots$$

Using the graph and Betti cohomology notations in §3.5.1 we can re-write the Mayer-Vietoris exact sequence as the following short exact sequence

$$0 \longrightarrow H^{1,0}(\mathscr{D},\mathscr{F}) \longrightarrow H^1_{dR}(T,\mathscr{F}) \longrightarrow H^{0,1}(\mathscr{D},\mathscr{F}) \longrightarrow 0.$$

Let us keep the notations $T, \mathscr{D}, (\mathscr{F}, \nabla)$ as at the beginning of this section. In order to explicitly calculate the cohomology groups $H^i_{dR}(T,\mathscr{F})$ we use the following double complex:

$$C^{\bullet,\bullet}: \quad \begin{array}{ccccccc} \oplus_{e\in e(G)}\mathscr{F}_e & \xrightarrow{\nabla} & \oplus_{e\in e(G)}\mathscr{F}_e\otimes\Omega^1_{A_e} & \xrightarrow{\nabla} & \oplus_{e\in e(G)}\mathscr{F}_e\otimes\Omega^2_{A_e} & \xrightarrow{\nabla} \\ \uparrow\delta & & \uparrow\delta & & \uparrow\delta & \\ \oplus_{v\in v(G)}\mathscr{F}_v & \xrightarrow{\nabla} & \oplus_{v\in v(G)}\mathscr{F}_v\otimes\Omega^1_{U_v} & \xrightarrow{\nabla} & \oplus_{v\in v(G)}\mathscr{F}_v\otimes\Omega^2_{U_v} & \xrightarrow{\nabla} \end{array}$$

where $\mathscr{F}_e$, respectively $\mathscr{F}_v$ denote $H^0(A_e,\mathscr{F})$ respectively $H^0(U_v,\mathscr{F})$ for $e \in e(G)$ and $v \in v(G)$. Moreover the Čech differentials δ are defined by: $\delta((x_v)_{v\in v(G)})_e = x_{a(e)}|_{A_e} - x_{b(e)}|_{A_e}$, for $e \in e(G)$. The single complex

$$K^\bullet(T,(\mathscr{F},\nabla)) : K^0 \xrightarrow{D_0} K^1 \xrightarrow{D_1} K^2 \xrightarrow{D_2} \cdots$$

attached to the double complex $C^{\bullet,\bullet}$ is defined by: $K^0 := \oplus_{v\in v(G)}\mathscr{F}_v$, $K^1 := \big(\oplus_{v\in v(G)}\mathscr{F}_v\otimes\Omega^1_{U_v}\big)\oplus\big(\oplus_{e\in e(G)}\mathscr{F}_e\big)$ and $K^2 := \big(\oplus_{v\in v(G)}\mathscr{F}_v\otimes\Omega^2_{U_v}\big)\oplus\big(\oplus_{e\in e(G)}\mathscr{F}_e\otimes\Omega^1_{A_e}\big)$ etc. and

$$D_0((x_v)_{v\in v(G)} = \big((\nabla(x_v))_{v\in v(G)}, (x_{a(e)}|_{A_e} - x_{b(e)}|_{A_e})_{e\in e(G)}\big)$$

$$D_1\big((\omega_v)_{v\in v(G)}, (f_e)_{e\in e(G)}\big) = \big((\nabla(\omega_v))_{v\in v(G)}, (\omega_{a(e)}|_{A_e} - \omega_{b(e)}|_{A_e} - \nabla(f_e))_{e\in e(G)}\big)$$

$$D_2\big((\eta_v)_{v\in v(G)}, (\omega_e)_{e\in e(G)}\big) = \big((\nabla(\eta_v))_{v\in v(G)}, (\eta_{a(e)}|_{A_e} - \eta_{b(e)}|_{A_e} - \nabla(\omega_e))_{e\in e(G)}\big).$$

Then we have $H^i_{dR}(T,\mathscr{F}) = \mathrm{Ker}(D_i)/\mathrm{Im}(D_{i-1})$, for $i \geq 0$, where we set $K^{-1} = 0$, $D_{-1} = 0$. In particular, cohomology classes in $H^1_{dR}(T,\mathscr{F})$ are represented by 1-hypercocycles, i.e. families of elements $\big((\omega_v)_{v\in v(G)}, (f_e)_{e\in e(G)}\big)$ where $\omega_v \in \mathscr{F}_v\otimes\Omega^1_{U_v}$, $f_v \in \mathscr{F}_e$, for $v\in v(G), e\in e(G)$, which satisfy $\nabla(\omega_v) = 0$ for all v and $\omega_{a(e)}|_{A_e} - \omega_{b(e)}|_{A_e} = \nabla(f_e)$ for all e.

Remark 3.32. — *With the notations above, let us assume that the open sets U_α and A_e are acyclic for coherent sheaf cohomology. Then the maps $f : H^{1,0}(\mathscr{D},\mathscr{F}) \longrightarrow H^1_{dR}(Z,\mathscr{F})$ and $g : H^1_{dR}(Z,\mathscr{F}) \longrightarrow H^{0,1}(\mathscr{D},\mathscr{F})$ defining the Mayer-Vietoris sequence are given in terms of hypercocycles as follows.*

a) *If the cocycle $(x_e)_{e\in e(G)} \in \oplus_{e\in e(G)}H^0_{dR}(A_e,\mathscr{F})$ represents the cohomology class $x\in H^{1,0}(\mathscr{D},\mathscr{F})$, let us remark that by the assumptions above the $x_e\in\mathscr{F}_e$ such that $\nabla(x_e) = 0$. Therefore $f(x)$ is the class of the 1-hypercocycle $\big((0_v)_{v\in v(G)}, (x_e)_{e\in e(G)}\big)$.*

b) *If $\big((\omega_v)_{v\in v(G)}, (f_e)_{e\in e(G)}\big)$ is a 1-hypercocycle representing the class y in $H^1_{dR}(Z,\mathscr{F})$ then $g(y)$ is the image of $(\omega_v)_{v\in v(G)}$ in the group $\oplus_{v\in v(G)}H^1_{dR}(U_v,\mathscr{F})$, which is actually in $H^{0,1}(\mathscr{D},\mathscr{F})$.*

Remark 3.33. — *We have variants of these constructions for the logarithmic situation described in Remark 3.30. We need only replace the sheaves and modules of differentials $\Omega^i_{U_v}, \Omega^i_{A_e}$ by the sheaves and modules of logarithmic differentials.*

3.5.3. *Examples of coverings in our setting*

3.5.3.1. *First example.* — Let us now recall our geometric situation from §3.2. Let $\mathrm{red} : X \longrightarrow \overline{C}$ and for all $s\in S-\{0\}$, $\mathrm{red}_s\colon X_s = X\times_S s \longrightarrow \overline{C}$ denote the reduction maps. Let $\mathscr{C}$ (and for every $s\in S-\{0\}$, $\mathscr{C}_s$) denote the admissible covering of X (respectively of X_s) defined by $\mathscr{C} := \{\mathrm{red}^{-1}(Z)$ where Z is an irreducible component of $\overline{C}\}$ (respectively $\mathscr{C}_s := \{\mathrm{red}_s^{-1}(Z)$ where Z is an irreducible component of $\overline{C}\}$). Then we have $G := G(\mathscr{C}) = G(\mathscr{C}_s)$ for all $s\in S-\{0\}$. We fix once for all a choice of a system of representatives $e(G)$ of $\epsilon(G)/\tau$, see §3.5.1. Let us also remark that as $\overline{C}$ is a semi-stable curve $\mathscr{C}$ and $\mathscr{C}_s$ satisfy the condition $(*)$ of section §3.5.1. We use the following notations: for all $v\in v(G)$ we denote by $U_v\subset X$ the corresponding open set of $\mathscr{C}$ and for every s by $U_{v,s} = U_v\times_S s = U_v\cap X_s \subset X_s$ the respective open set of $\mathscr{C}_s$. Similarly, if $e\in\epsilon(G)$ we denote by $A_e = U_{a(e)}\cap U_{b(e)}$ and for every $s\in S-\{0\}$ we let $A_{e,s} := A_e\times_S s = A_e\cap X_s = U_{a(e),s}\cap U_{b(e),s}$. We'd like to recall that these coverings have already been defined in Section 3.2 and although the language of graphs was not used there, the definitions are the same.

3.5.3.2. *Second example.* — We keep the notations of section §3.5.3.1. For each $v \in v(G)$ let as in section §3.2,

$$Z_v := U_v - \bigcup_{\substack{w \\ w \neq v}} U_w.$$

Now, for each $v \in v(G)$ consider a strict neighborhood T_v of Z_v in U_v, which is wide open and such that $T_v \cap T_w = \phi$ if $v \neq w$. Let us recall that T_v is a "strict neighborhood" of Z_v in U_v means that the pair $\{T_v, U_v - Z_v\}$ is an admissible cover of U_v.

Such T's exist and let $\mathscr{C}' := \{T_v, A_e\}_{v,e}$ where v ranges over $v(G)$ and e over $e(G)$. Then $\mathscr{C}'$ is an admissible covering of X by wide open sets. This cover is a refinement of $\mathscr{C}$ and is appropriate for computing de Rham cohomology as the open sets are acyclic for coherent sheaf cohomology. We denote $G(\mathscr{C}')$ by G' and let us remark that: $v(G') = v(G) \amalg e(G)$ and $\epsilon(G') = \epsilon(G) \amalg \epsilon(G)$. We choose $e(G') = e(G) \amalg e(G)$ as follows. If $e \in e(G)$ then $(a(e), e)$ and $(e, b(e))$ belong to $e(G')$.

Moreover, as in section §3.5.3.1 if $s \in S$ (here s may be 0) we denote by $\mathscr{C}'_s := \{T_{v,s}, A_{e,s}\}_{v,e}$, where $T_{v,s} := T_v \times_S s = T_v \cap X_s$ for all $v \in v(G)$. Then $\mathscr{C}'_s$ is an admissible covering of X_s and $G(\mathscr{C}_s) = G(\mathscr{C}) = G'$.

3.5.3.3. *Third example.* — Let L be a totally ramified, non-trivial extension of K, as in section §3.2 and let $B = B_L \subset S$ denote the affinoid disk of centre 0 and radius $|\pi_L|$ as in Lemma 3.17. By Proposition 3.18, for every $v \in v(G)$ there exists a wide open neighborhood W_v of $Z_{v,B} := Z_v \times_S B$ in $U_{v,B} := U_v \times_S B$ and for all $s \in S$ an isomorphism over B:

$$\alpha_{v,s} : W_v \cong W_{v,s} \times B.$$

Set $\mathscr{C}''_B := \{W_v, A_{e,B}\}_{v,e}$, where v and e run over $v(G)$ and $e(G)$ respectively and $A_{e,B} := A_e \times_S B$. Then $\mathscr{C}''_B$ is an admissible covering of X_B and if $s \in S$, $\mathscr{C}''_s := \{W_{v,s}, A_{e,s}\}_{v,e}$ is an admissible covering of X_s. Then $G(\mathscr{C}''_B) = G(\mathscr{C}''_s) = G'$.

3.5.4. *Changing coverings.* — Let us fix $\mathscr{E}$ a W-isocrystal on $\overline{C}$. Let us also fix a closed point $s \in S - \{0\}$ defined over the finite extension F of K_0. Then one can see s as a W-algebra homomorphism $W[[t]] \longrightarrow \mathscr{O}_F$. If we denote by $X_s := X \times_S s$ and by $\mathfrak{X}_s := \mathfrak{X} \times_{\mathrm{Spf}(W[[t]]} s$, then X_s is the generic fiber of $\mathfrak{X}_s$. We denote by $(\mathscr{E}_s, D_s)$ the evaluation of $\mathscr{E}$ at the enlargement $\mathfrak{X}_s$ of $\overline{C}$, seen as a coherent sheaf $\mathscr{E}_s$ on X_s with an integrable connection D_s. Fix the coverings $\mathscr{C}_s := \{U_{v,s}\}_v$ as in section §3.5.3.1 and $\mathscr{C}'_s := \{T_{v,s}, A_{e,s}\}_{v,e}$ as in section §3.5.3.2 of graphs G and G' respectively. To simplify, for the next lemma we omit s from the notation i.e. we will use U_v, A_e, T_v to denote $U_{v,s}, A_{e,s}, T_{v,s}$. For $i \geq 0$, let $\mathscr{E}_i, \mathscr{E}'_i$ denote the local systems on G respectively G' associated as in section §3.5.1 to $(\mathscr{E}_s, D_s)$. We define the maps of abelian groups

$$\begin{array}{rccc} f_i^0 : & C^0(G, \mathscr{E}_i) & \longrightarrow & C^0(G', \mathscr{E}'_i) \\ f_i^1 : & C^1(G, \mathscr{E}_i) & \longrightarrow & C^1(G', \mathscr{E}'_i) \end{array}$$

by $f_i^0(((x_v)_v) = \big((x_v|_{T_v})_v, (\frac{x_{a(e)}|_{A_e} + x_{b(e)}|_{A_e}}{2})_e\big)$ and $f_i^1((y_e)_e) = \big(\frac{y_e|_{T_{a(e)}\cap A_e}}{2}, \frac{y_e|_{T_{b(e)}\cap A_e}}{2}\big)_e$, where everywhere v and e run over $v(G)$ and respectively $e(G)$.

Lemma 3.34. — a) *f_i^0, f_i^1 define morphisms of complexes $f_i^\bullet : C^\bullet(G, \mathscr{E}_i) \longrightarrow C^\bullet(G', \mathscr{E}'_i)$.*

b) *For $i = 0,1$ $f_i^\bullet$ induce isomorphisms $H^{1,0}(\mathscr{C}_s, \mathscr{E}_s) \cong H^{1,0}(\mathscr{C}'_s, \mathscr{E}_s)$ and $H^{0,1}(\mathscr{C}_s, \mathscr{E}_s) \cong H^{0,1}(\mathscr{C}'_s, \mathscr{E}_s)$ (the notations being as in section §3.5.1).*

c) *If $\big((\omega_v)_v, (f_e)_e\big)$ is a 1-hypercocycle for the complex $\mathscr{E}_s \otimes_{\mathscr{O}_{X_s}} \Omega^\bullet_{X_s/F}$ corresponding to the covering $\mathscr{C}_s$, then the co-chain $\big((\omega_v|_{T_v})_v, (\frac{\omega_{a(e)}|_{A_e} + \omega_{b(e)}|_{A_e}}{2})_e, (\frac{f_e|_{T_{a(e)}\cap A_e}}{2}, \frac{f_e|_{T_{b(e)}\cap A_e}}{2})_e\big)$ is a 1-hypercocycle for the same complex associated to the covering $\mathscr{C}'_s$, which represents the same cohomology class in $H^1_{dR}(X_s/F, \mathscr{E}_s)$.*

d) *The isomorphisms at* b) *make the following diagram of Mayer-Vietoris sequences commute.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{1,0}(\mathscr{C}_s, \mathscr{E}_s) & \longrightarrow & H^1_{dR}(X_s/F, \mathscr{E}_s) & \longrightarrow & H^{0,1}(\mathscr{C}_s, \mathscr{E}_s) & \longrightarrow & 0 \\ & & \downarrow & & || & & \downarrow & & \\ 0 & \longrightarrow & H^{1,0}(\mathscr{C}'_s, \mathscr{E}_s) & \longrightarrow & H^1_{dR}(X_s/F, \mathscr{E}_s) & \longrightarrow & H^{0,1}(\mathscr{C}'_s, \mathscr{E}_s) & \longrightarrow & 0 \end{array}$$

Proof. — We'll only sketch the prove of the fact that the morphism of complexes $f_1^\bullet$ induces an isomorphism $f : H^{0,1}(\mathscr{C}_s, \mathscr{E}_s) \cong H^{0,1}(\mathscr{C}'_s, \mathscr{E}_s)$. The main observation is that as U_v,T_v,A_e are wide opens, they are acyclic for coherent sheaf cohomology and so $H^i_{dR}(U_v, \mathscr{E}_s|_{U_v}), H^i_{dR}(T_v, \mathscr{E}_s|_{T_v}), H^i_{dR}(A_e, \mathscr{E}_s|_{A_e})$ can be calculated as hypercohomology of the de Rham complex relative to the admissible covering $\{U_v\}$ respectively $\{T_v\}$, respectively $\{A_e\}$. Moreover the first groups could also be calculated relative to the admissible covering $\{T_v, U_v - T_v = \amalg_{e\in e(G), v=a(e), v=b(e)} A_e\}$ of U_v.

Let us show the injectivity of f. Suppose that $(x_v)_v \in C^0(G, \mathscr{E}_1) = \oplus_v H^1_{dR}(U_v, \mathscr{E}_s|_{U_v})$ is such that

a) $d((x_v)_v) = 0$

and

b) $f((x_v)_v) = 0$ in $C^0(G', \mathscr{E}'_1)$.

Let $\omega_v \in H^0(U_v, \mathscr{E}_s \otimes \Omega^1_{U_v/F})$ be a representative of $x_v \in H^1_{dR}(U_v, \mathscr{E}_s|_{U_v})$. Condition a) implies that for all $e \in e(G)$ there is a section $u_e \in H^0(A_e, \mathscr{E}_s|_{A_e})$ such that $\omega_{a(e)}|_{A_e} - \omega_{b(e)}|_{A_e} = D(u_e)$. From condition b) we deduce there exist sections $u_v \in H^0(T_v, \mathscr{E}_s), w_e \in H^0(A_e, \mathscr{E}_s)$ such that $D_s(u_v) = \omega_v|_{T_v}, D_s(w_e) = \omega_{a(e)}|_{A_e} + \omega_{b(e)}|_{A_e}$, for all $v \in v(G), e \in e(G)$. This implies that the hypercochain

$$\big(D_s(u_v), D_s((w_e + u_e)/2), D_s((u_e - w_e)/2), (u_v|_{A_e\cap T_{a(e)}} - ((w_e + u_e)/2)|_{A_e\cap T_{a(e)}}),$$

$$(u_e - w_e)/2)|_{A_e\cap T_{a(e)}} - u_v|_{A_e\cap T_{a(e)}}\big)_{e\in e(G), e=a(e), e=b(e)}$$

is a hypercocycle for the covering $\{T_v, \amalg_{e\in e(G), v=a(e), v=b(e)} A_e\}$ of U_v representing the class x_v. Therefore $x_v = 0$ for all $v \in v(G)$.

For the surjectivity of f one makes similar calculations which we leave, together with the rest of the proof, to the reader. $\square$

Let us now fix L, B as in section §3.5.3.3. Let us also fix an isocrystal $\mathscr{E}$ on $\overline{C}$ and denote $\mathscr{E}_B$ its evaluation on the enlargement $\mathfrak{X}_{\mathscr{B}}$ (for notations see the section §3.2). Let us recall (see ibid.) that we have an absolute connection, D_B and a relative one $D_{X_B/B}$ on $\mathscr{E}_B$. For $i \geq 0$ let us denote by E^i_{abs} (respectively E^i_{rel}) the local system on G' defined by:

a) if $v \in v(G)$ then $E^i_{\mathrm{abs};v} := H^i_{dR}(W_v/L, \mathscr{E}_B|_{W_v}(\log(Y \cap W_v)))$ and if $e \in e(G)$ then $E^i_{\mathrm{abs};e} := H^i_{dR}(A_{e,B}/L, \mathscr{E}_B|_{A_{e,B}}(\log(Y \cap A_{e,B})))$,

b) if $e \in e(G)$ then $E^i_{\mathrm{abs};a(e),e} := H^i_{dR}(W_{a(e)} \cap A_{e,B}/L, \mathscr{E}_B(\log(Y \cap W_{a(e)} \cap A_{e,B})))$ and $E_{\mathrm{abs};e,b(e)} := H^i_{dR}(W_{b(e)} \cap A_{e,B}/L, \mathscr{E}_B(\log(Y \cap W_{b(e)} \cap A_{e,B})))$.

c) the maps are induced by the obvious restrictions.

We have similar definitions, using relative de Rham cohomology over B, for the local system E^i_{rel}.

We denote the the cohomology groups $H^{j,i}(\mathscr{C}''_B, E_*) := H^j_{\mathrm{Betti}}(G', E^i_*)$, for $* \in \{\mathrm{abs}, \mathrm{rel}\}$ and remark that $H^{i,j}(\mathscr{C}''_B, E_{\mathrm{rel}})$ are $\mathscr{O}_B$-modules.

Proposition 3.35. — a) *$H^{i,j}(\mathscr{C}''_B, E_{\mathrm{rel}})$ are free $\mathscr{O}_B$-modules of finite rank for all $0 \leq i, j \leq 1$, $i \neq j$. Moreover if $s \in B$ then we have $H^{i,j}(\mathscr{C}''_B, E_{\mathrm{rel}}) \cong H^{i,j}(\mathscr{C}''_s, \mathscr{E}_s) \otimes_L \mathscr{O}_B$ for i, j as above.*

b) *Let us denote by $\nabla^{i,j}$ the natural connection over K_0 of the modules $H^{i,j}(\mathscr{C}''_B, E_{\mathrm{rel}})$ whose space of horizontal sections is $H^{i,j}(\mathscr{C}''_0, \mathscr{E}_0)$ for $0 \leq i, j \leq 1$, $i \neq j$. Then for every $s \in B - \{0\}$ we have parallel transport isomorphisms $H^{i,j}(\mathscr{C}_s, \mathscr{E}_s) \cong H^{i,j}(\mathscr{C}''_s, \mathscr{E}_s) \cong H^{i,j}(\mathscr{C}''_0, \mathscr{E}_0) \otimes_{K_0} F_s$, where F_s is the residue field of s and i, j are as above.*

c) *The natural morphisms in the "relative Mayer-Vietoris" exact sequence*

$$0 \longrightarrow H^{1,0}(\mathscr{C}''_B, E_{\mathrm{rel}}) \longrightarrow H^1_{dR}(X_B/B, \mathscr{E}_B(\log(Y))) \longrightarrow H^{1,0}(\mathscr{C}''_B, E_{\mathrm{rel}}) \longrightarrow 0$$

are horizontal. Here the connection ∇_B on the $\mathbb{H}_B = H^1_{dR}(X_B/B, \mathscr{E}_B(\log(Y)))$ is the Gauss-Manin connection.

Proof. — a) Fix $s \in B$. Let us recall from Lemma 3.19 that the rigid spaces $W_v, W_{v,s}$ have canonical formal models $\mathscr{W}_v, \mathscr{W}_{v,s}$ with an isomorphism $\mathscr{W}_v \cong \mathscr{W}_{v,s} \times \mathscr{B}$ and natural morphisms

$$\begin{array}{ccccc} \overline{C}_v & \longrightarrow & \mathscr{W}_v & \longrightarrow & \mathfrak{X}_B \\ \| & & \cup & & \cup \\ \overline{C}_v & \longrightarrow & \mathscr{W}_{v,s} & \longrightarrow & \mathfrak{X}_s \end{array}$$

The first vertical maps are closed immersions and the last two vertical maps are the natural inclusions into $\mathscr{W}_v$ and $\mathfrak{X}_{\mathscr{B}}$ of their fibers at s. Thus $\mathscr{W}_v$ and $\mathscr{W}_{v,s}$ are wide open enlargements of $\overline{C}$. As $\mathscr{E}$ is a W-isocrystal on $\overline{C}$, we may evaluate it at $\mathscr{W}_v$ and $\mathscr{W}_{v,s}$ to obtain pairs $(\mathscr{E}_v, D_v)$ and $(\mathscr{E}_s, D_s)$ consisting of coherent sheaves of $\mathscr{O}_{W_v}$-modules, respectively $\mathscr{O}_{W_{v,s}}$-modules, with convergent integrable connections.

From the diagram above and its image under the functor “rig” we obtain: $(\mathscr{E}_v, D_v) \cong (\mathscr{E}_B, D_B)|_{W_v}$ and $(\mathscr{E}_s, D_s) \cong (\mathscr{E}_{\mathfrak{X}_s}, D_{\mathfrak{X}_s})|_{W_{v,s}}$.

Moreover, if we denote by $\beta : \mathscr{W}_v \longrightarrow \mathscr{W}_{v,s}$ the natural projection, the commutative diagram in Remark 3.20 implies that $\beta^*(\mathscr{E}_s, D_s) \cong (\mathscr{E}_v, D_v)$. Thus for all connected affinoid $B' \subset B$ we have $H^i_{dR}(W_v/B, \mathscr{E}_v)(B') \cong H^i_{dR}(W_{v,s}, \mathscr{E}_s) \otimes_L \mathscr{O}_{B'}$ for $i = 0, 1$. Since for all $e \in e(G)$ $A_{e,B}$ is contained in a residue class, $\mathscr{E}_e := \mathscr{E}_B|_{A_{e,B}}$ has a basis of horizontal sections for the absolute connection D_B. Hence similarly, for all connected affinoid $B' \subset B$ we have $H^i_{dR}(A_{e,B}/B, \mathscr{E}_e)(B') \cong H^i_{dR}(A_{e,s}, \mathscr{E}_s) \otimes \mathscr{O}_{B'}$, for $i = 0, 1$. Finally as $A_{e,B} \cap W_{a(e)}$ and $A_{e,B} \cap W_{b(e)}$ are contained in $A_{e,B}$ the same result holds for the cohomology of these spaces with values in $\mathscr{E}_e$. We deduce that $H^{i,j}(\mathscr{C}''_B, E_{\mathrm{rel}}) \cong H^{i,j}(\mathscr{C}''_s, \mathscr{E}_s) \otimes \mathscr{O}_B$ for $0 \le i, j \le 1$, $i \ne j$.

b) is now clear and in order to prove c) let us first recall the definition of the Gauss-Manin connection in our setting.

We have a natural exact sequence of de Rham complexes of sheaves on X_B

$$0 \longrightarrow f^*(\Omega^1_{B/L}(\log 0) \otimes \Omega_{X_B/B}(\log Y)^{\bullet-1} \otimes \mathscr{E}_B \longrightarrow \Omega^\bullet_{X_B/K_0}(\log Y) \otimes \mathscr{E}_B \longrightarrow \Omega^\bullet_{X_B/B}(\log Y) \otimes \mathscr{E}_B \longrightarrow 0$$

where we have denoted $f : X_B \longrightarrow B$ the structure morphism. Then the Gauss-Manin connection

$$\nabla_B : H^1_{dR}(X_B/B, \mathscr{E}_B(\log(Y))) \longrightarrow H^1_{dR}(X_B/B, \mathscr{E}_B(\log(Y))) \otimes \Omega^1_{B/L}(\log 0)$$

is the connecting homomorphism in the long exact sequence for hypercohomology.

Let us calculate the connection explicitly in terms of hypercocycles. For this let t denote a parameter of B at 0 and let $x \in H^1(dR)(X_B/B, \mathscr{E}_B(\log(Y)))(B)$. Let us suppose that x is represented by the following hypercocycle for the covering $\mathscr{C}''_B$: $\big((\omega_v)_v, (\omega_e)_e, (f_e, \overline{f}_e)_e\big)$, where v runs over $v(G)$ and e over $e(G)$. Here $\omega_v \in H^0(W_v, \Omega_{W_v/B}(\log W_{v,0}) \otimes \mathscr{E}_B)$, $\omega_e \in H^0(A_{e,B}, \Omega_{A_{e,B}/B}(\log A_{e,0}) \otimes \mathscr{E}_B)$, $f_e \in H^0(A_{e,B} \cap W_{a(e)}, \mathscr{E}_B)$ and $\overline{f}_e \in H^0(A_{e,B} \cap W_{b(e)}, \mathscr{E}_B)$ satisfying the relations:

a) $D_{X_B/B}(\omega_v) = D_{X_B/B}(\omega_e) = 0$ for all v, e.

b) $\omega_{a(e)}|_{W_{a(e)} \cap A_{e,B}} - \omega_e|_{W_{a(e)} \cap A_{e.B}} = D_{X_B/B}(f_e)$ and
$\omega_e|_{W_{b(e)} \cap A_{e,B}} - \omega_{b(e)}|_{W_{b(e)} \cap A_{e.B}} = D_{X_B/B}(\overline{f}_e)$ for all e.

Now we choose lifts of ω_v and ω_e to absolute forms, i.e. we choose $\tilde{\omega}_v \in H^0(W_v, \Omega^1_{W_v/K_0}(\log W_{v,0}) \otimes \mathscr{E}_B)$ and respectively $\tilde{\omega}_e \in H^0(A_{e,B}, \Omega^1_{A_e/K_0}(\log(A_{e,0}) \otimes \mathscr{E}_B)$ which project to ω_v and respectively ω_e and define the sections $\eta_v \in H^0(W_v, \Omega^1_{W_v/B}(\log W_{v,0}) \otimes \mathscr{E}_B), \eta_e \in H^0(A_{e,B}, \Omega^1_{A_{e,B}/B}(\log A_{e,0}) \otimes \mathscr{E}_B)$, $g_e \in H^0(W_{a(e)} \cap A_{e,B}, \mathscr{E}_B), \overline{g}_e \in H^0(W_{b(e)} \cap A_{e,B}, \mathscr{E}_B)$ by the relations.

i) $D_B(\tilde{\omega}_v) = \eta_v \wedge dy/y$, $D_B(\tilde{\omega}_e) = \eta_e \wedge dy/y$ for all v, e. Here y is a parameter at 0 on B.

ii) $\tilde{\omega}_{a(e)}|_{W_{a(e)} \cap A_{e,B}} - \tilde{\omega}_e|_{W_{a(e)} \cap A_{e,B}} - D_B(f_e) = g_e dy/y$ for all e.

iii) $\tilde{\omega}_e|_{W_{b(e)} \cap A_{e,B}} - \tilde{\omega}_{b(e)}|_{W_{b(e)} \cap A_{e,B}} - D_B(\overline{f}_e) = \overline{g}_e dy/y$ for all e.

Then the hyper-cochain $\big((\eta_v)_v, (\eta_e)_e, (g_e, \overline{g}_e)_e\big)$ is a hypercocycle and its cohomology class $\otimes dy/y$ represents $\nabla_B(x)$.

Using this the proof of c) is a simple calculation which we leave to the reader. □

We have the following easy consequence of Proposition 3.35.

Lemma 3.36. — *Suppose we have two choices $\{W_v\}_{v\in v(G)}$ and $\{W'_v\}_{v\in v(G)}$ as in Proposition 3.18. Let $\mathscr{C} := \{W_v, A_{e,B}\}_{v,e}$ and $\mathscr{C}' := \{W'_v, A_{e,B}\}_{v,e}$, where v, e run over $v(G)$ and respectively $e(G)$, be the corresponding admissible covers of X_B. Then we have natural isomorphisms of $\mathscr{O}_B$-modules:*

$$H^{i,j}(\mathscr{C}, E_{\rm rel}) \cong H^{i,j}(\mathscr{C}', E_{\rm rel}) \text{ for } 0 \le i, j \le 1, i \neq j.$$

Proof. — Let $0 \neq s \in B$. Then we have natural isomorphisms of $\mathscr{O}_B$-modules.

$$H^{i,j}(\mathscr{C}, E_{\rm rel}) \cong H^{i,j}(\mathscr{C}_s, \mathscr{E}_s) \otimes \mathscr{O}_B \text{ and } H^{i,j}(\mathscr{C}', E_{\rm rel}) \cong H^{i,j}(\mathscr{C}'_s, \mathscr{E}_s) \otimes \mathscr{O}_B,$$

for $0 \le i, j \le 1$, $i \neq j$.

Therefore it is enough to compare the groups $H^{i,j}(\mathscr{C}_s, \mathscr{E}_s)$ and $H^{i,j}(\mathscr{C}'_s, \mathscr{E}_s)$ and we may suppose that $W'_{v,s} \subset W_{v,s}$ for all v (if not take the intersections).

For the rest of the proof, in order to ease the notations we'll drop s from the notations everywhere, i.e. rename $\mathscr{E} = \mathscr{E}_s, W_v = W_{v,s}, W'_v = W'_{v,s}, A_e = A_{e,s}$, $\mathscr{C} = \mathscr{C}_s, \mathscr{C}' = \mathscr{C}'_s, D = D_s$ etc. The natural inclusions $W'_v \subset W_v$ induce by pull-back maps $H^{i,j}(\mathscr{C}, \mathscr{E}) \longrightarrow H^{i,j}(\mathscr{C}', \mathscr{E})$ which make the following diagram commutative.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{1,0}(\mathscr{C}, \mathscr{E}) & \longrightarrow & H^1_{dR}(X_s, \mathscr{E}) & \longrightarrow & H^{0,1}(\mathscr{C}, \mathscr{E}) & \longrightarrow & 0 \\ & & \alpha \downarrow & & || & & \downarrow \gamma & & \\ 0 & \longrightarrow & H^{1,0}(\mathscr{C}', \mathscr{E}) & \longrightarrow & H^1_{dR}(X_s, \mathscr{E}) & \longrightarrow & H^{0,1}(\mathscr{C}', \mathscr{E}) & \longrightarrow & 0 \end{array}$$

So it is enough to prove that α is an isomorphism. Let us remark that as W'_v is a strict neighborhood of Z_v in U_v (recall that we suppressed "s" from the notation), the set $\{W'_v, \amalg_{v=a(e),v=b(e)} A_e\}$ is an admissible covering of U_v. As W_v is an admissible open of U_v, the set $\{W'_v, \amalg_{v=a(e),v=b(e)} A_e \cap W_v\}$ is an admissible covering of W_v. But $\mathscr{E}$ has a basis of horizontal sections on $A_e \cap W_v$ for all $e \in e(G)$, therefore the restriction $H^0(W_v, \mathscr{E})^D \longrightarrow H^0(W'_v, \mathscr{E})^D$ is an isomorphism for all $v \in v(G)$. It follows that α is an isomorphism. □

Let us fix a collection $\{W_v\}_{v\in v(G)}$ as in Proposition 3.18 and let $s \in B$ (s may be 0). We consider again the admissible coverings $\mathscr{C}''_B$ of X_B and $\mathscr{C}''_s$ and the respective Mayer-Vietoris exact sequences. Pull back by the closed immersion $X_s \longrightarrow X_B$ provide vertical maps in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{1,0}(\mathscr{C}''_B/B, \mathscr{E}) & \longrightarrow & H^1_{dR}(X_B/B, \mathscr{E}_B(\log(Y))) & \longrightarrow & H^{0,1}(\mathscr{C}''_B/B, \mathscr{E}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^{1,0}(\mathscr{C}''_s, \mathscr{E}_s) & \longrightarrow & H^1_{dR}(X_s, \mathscr{E}_s(\log(Y \cap X_s))) & \longrightarrow & H^{0,1}(\mathscr{C}''_s, \mathscr{E}_s) & \longrightarrow & 0 \end{array}$$

If $s \neq 0$ the log structure on X_s is trivial.

Lemma 3.37. — *The above diagram of Mayer-Vietoris exact sequences is commutative.*

Proof. — The proof follows immediately from the definitions and we leave it to the reader. □

4. The Monodromy Operators

4.1. The global residue. — Let us fix the covering $\mathscr{C}' = \{T_v, A_e\}_{v\in v(G(X)), e\in e(G(X))}$ as in section §3.5.3.2, G' denote the graph of this cover and assume that $\mathscr{E}$ is an isocrystal on $\overline{C}$ i.e we assume that P and hence the log structure induced by it is trivial in this chapter (notations as in section §1.) We denote $(\mathscr{E}_{\mathfrak{X}}, D_{X/K_0})$ its evaluation on the wide open enlargement $\mathfrak{X}$ and by $D_{X/S}$ the associated relative connection. Let us also recall that we defined on $\mathfrak{X}$ the log structure given by the normal crossing divisor $\mathscr{Y} := \mathfrak{X}_0$, on $\mathscr{Y}$ itself the inverse image log structure defined by the closed immersion $\mathscr{Y} = \mathfrak{X}_0 \longrightarrow \mathfrak{X}$, and on $\mathscr{S}$ the log structure given by the divisor $t = 0$. The log schemes thus defined are denoted $\mathfrak{X}^{\times\times}, \mathscr{Y}^{\times\times}, \mathscr{S}^{\times}$. We denote $\Omega^i_{X^{\times\times}/S^{\times}} := (\Omega^i_{\mathfrak{X}^{\times\times}/\mathscr{S}^{\times}})^{\mathrm{rig}} = \Omega^i_{X/S}(\log(Y))$ and $\Omega^i_{Y^{\times\times}/K_0} := (\Omega^i_{\mathscr{Y}^{\times\times}/W^{\times}})^{\mathrm{rig}} = \Omega^i_{\mathscr{Y}^{\times\times}/W^{\times}} \otimes_W K_0$, for $i \geq 0$.

Let us first fix $e \in e(G)$ and recall that the sheaf $\mathscr{E}_{\mathfrak{X}}|_{A_e}$ has a basis of horizontal sections for $D_{X/S}$. We denote such a basis by $\{\epsilon_1, \ldots, \epsilon_\alpha\}$. Then using Lemma 3.16 every element $\omega \in H^0(A_e, \mathscr{E}_{\mathfrak{X}} \otimes \Omega^1_{X/S}(\log(Y)))$ can be written

$$\omega = \Big(\sum_{i=1}^{\alpha} \epsilon_i \otimes \sum_{n,m\geq 0} a_{i,n,m} x_e^n x_{\tau(e)}^m\Big) \frac{d_{\mathrm{X/S}} x_e}{x_e},$$

where $a_{i,n,m} \in K_0$ are such that the power series converge on A_e. We recall that the variables $x_e, x_{\tau(e)}$, defined in Lemma 3.16 satisfy $x_e x_{\tau(e)} = t$. Thus we define

$$\mathrm{Res}_e(\omega_e) := \Big(\frac{1}{2}\big(\sum_{i=1}^{\alpha} \epsilon_i|_{T_{a(e)}\cap A_e} \sum_{n\geq 0} a_{i,n,n} t^n\big), \frac{1}{2}\big(\sum_{i=1}^{\alpha} \epsilon_i|_{T_{b(e)}\cap A_e} \sum_{n\geq 0} a_{i,n,n} t^n\big)\Big)$$

$$\in H^0_{dR}((T_{a(e)} \cap A_e)/S, \mathscr{E}_{\mathfrak{X}}) \oplus H^0_{dR}((T_{b(e)} \cap A_e)/S, \mathscr{E}_{\mathfrak{X}}).$$

Therefore, for every $e \in e(G)$, Res_e can be seen as an $\mathscr{O}_S$-linear homomorphism

$$H^1_{dR}(A_e/S, \mathscr{E}_{\mathfrak{X}}(\log(Y))) \longrightarrow H^0_{dR}((A_e \cap T_{a(e)})/S, \mathscr{E}_{\mathfrak{X}}) \oplus H^0_{dR}(A_e \cap T_{b(e)}/S, \mathscr{E}_{\mathfrak{X}}).$$

Similarly, let $\mathscr{C}'_0 = \{T_{v,0}, A_{e,0}\}$ be the intersection of the covering $\mathscr{C}'$ with Y. It is an admissible cover of Y by acyclic wide opens. Let us fix $e \in e(G)$ and x, y be the restrictions of x_e and $x_{\tau(e)}$ to $A_{e,0}$ respectively. Denote by $\mathscr{E}_0$ the evaluation of $\mathscr{E}$ at $\mathscr{Y}$ and let $\omega \in H^0(A_{e,0}, \mathscr{E}_0 \otimes \Omega^1_{Y^{\times\times}/K_0})$. Then

$$\omega = \sum_{a=1}^{\alpha} \epsilon_a^0 \otimes \Big(\big(\sum_{n\geq 0} \alpha_{a,n} x^n\big)\frac{dx}{x} + \big(\sum_{n\geq 0} \beta_{a,n} y^n\big)\frac{dy}{y}\Big),$$

where $\{\epsilon_a^0\}_{1\leq a\leq s}$ is a basis of horizontal sections of $\mathscr{E}_0|_{A_{e,0}}$. As $xy = 0$ on $A_{e,0}$, $dx/x = -dy/y$ and we define

$$\mathrm{Res}_e(\omega) = \Big(\frac{1}{2}\sum_{a=1}^{s}\epsilon_a^0(\alpha_{a,0}-\beta_{a,0})|_{A_{e,0}\cap T_{a(e),0}}, \frac{1}{2}\sum_{a=1}^{s}\epsilon_a^0(\alpha_{a,0}-\beta_{a,0})|_{A_{e,0}\cap T_{b(e),0}}\Big)$$

$$\in H^0_{dR}(A_{e,0}\cap T_{a(e),0}/K_0,\mathscr{E}_0)\oplus H^0_{dR}(A_{e,0}\cap T_{b(e),0}/K_0,\mathscr{E}_0).$$

Thus we defined a K_0-linear homomorphism

$$\mathrm{Res}_e : H^1_{dR}(A_{e,0}^{\times\times}/K_0,\mathscr{E}_0) \longrightarrow H^0_{dR}(A_{e,0}\cap T_{a(e)}/K_0,\mathscr{E}_0)\oplus H^0_{dR}(A_{e,0}\cap T_{b(e),0}/K_0,\mathscr{E}_0)$$

for every $e\in e(G)$.

Now we define residue maps Res and respectively $\mathrm{Res}^{(0)}$ by the compositions:

$$\mathbb{H} = H^1_{dR}(X/S,\mathscr{E}_{\mathfrak{X}}(\log(Y))) \longrightarrow \oplus_{e\in e(G)}\big(H^1_{dR}(A_e/S,\mathscr{E}_{\mathfrak{X}}(\log(Y\cap A_e)))\big) \stackrel{\oplus_e\mathrm{Res}_e}{\longrightarrow} H^{1,0}(\mathscr{C}',E_{\mathrm{rel}}),$$

and

$$H^1(Y,\mathscr{E}) := H^1_{dR}(Y^{\times\times}/K_0,\mathscr{E}_0) \longrightarrow \oplus_{e\in e(G)}H^1_{dR}(A_{e,0}^{\times\times}/K_0,\mathscr{E}_0) \stackrel{\oplus_e\mathrm{Res}_e}{\longrightarrow} H^{1,0}(\mathscr{C}'_0,\mathscr{E}_0).$$

In the above sequences, the first arrows are restrictions.

Remark 4.1. — *Let L, B be as in section §3.2. Then we immediately obtain an $\mathscr{O}_B$-linear residue map* $\mathrm{Res}_B := \mathrm{Res}\otimes_{\mathscr{O}_S}\mathscr{O}_B : \mathbb{H}_B \longrightarrow H^{1,0}(\mathscr{C}''_B, E_{\mathrm{rel}})$.

Remark 4.2. — *Let*

$$\big((\omega_v)_v,(\omega_e)_e,(f_e,\overline{f}_e)_e\big) \tag{2}$$

be a hypercocycle for the complex of sheaves $\mathscr{E}_{\mathfrak{X}}\otimes\Omega^\bullet_{X/S}(\log(Y))$ with respect to the covering $\mathscr{C}'$, representing a cohomology class $x\in\mathbb{H}$. Here $\omega_v\in\mathscr{E}_X(T_v)\otimes\Omega^1_{T_v/S}$, $\omega_e\in\mathscr{E}_{\mathfrak{X}}(A_e)\otimes\Omega^1_{A_e/S}(\log Y)$, $f_e\in\mathscr{E}_{\mathfrak{X}}(T_{a(e)}\cap A_e)$ and $\overline{f}_e\in\mathscr{E}_{\mathfrak{X}}(T_{b(e)}\cap A_e)$ and they satisfy the cocycle conditions.

We may express Res *defined above explicitly in terms of cocycles as follows:* $\mathrm{Res}(x)$ *is the image in $H^{1,0}(\mathscr{C}',E_{\mathrm{rel}})$ of the cocycle* $(\mathrm{Res}_e(\omega_e))_{e\in e(G)}$.

Next we would like to describe the fibers of Res. Let $s\in S-\{0\}$ and $\mathscr{C}'_s$ the covering of the fiber X_s obtained by intersecting the open sets of $\mathscr{C}'$ with X_s. Let also $\mathscr{C}_s$ be the intersection of the covering $\mathscr{C}$ (defined in Section 3.5.3.1) with X_s. Both $\mathscr{C}'_s,\mathscr{C}_s$ are admissible covers of X_s by acyclic wide open subsets and $\mathscr{C}'_s$ is a refinement of $\mathscr{C}_s$. Let us consider the graphs associated to these covers, i.e., G' and G respectively. We have (see Remark 2.5)

Lemma 4.3. — *Let $s\in S-\{0\}$. Then under the identification between $H^{1,0}(\mathscr{C}_s,\mathscr{E}_s)$ and $H^{1,0}(\mathscr{C}'_s,\mathscr{E}_s)$ in Lemma 3.34* $(\mathrm{Res})_s = \mathrm{Res}^{(s)}$, *where* $(\mathrm{Res})_s$ *is the fiber of* Res *at s and for the notation* $\mathrm{Res}^{(s)}$ *see Remark 2.5.*

Proof. — This follows from the definitions and the explicit description of the isomorphism in Lemma 3.34 and we leave the details to the reader. □

Now let us concentrate on describing the fiber $(\mathrm{Res})_0$ of Res at $s=0$. Let us first remark that from the definition of an isocrystal and the definitions of the log structures on $\mathfrak{X}, \mathscr{Y}, \mathscr{S}$ we have natural isomorphisms

$$\left(\mathscr{E}_{\mathfrak{X}} \otimes_{\mathscr{O}_X} \Omega^i_{X^{\times\times}/S^\times}\right) \otimes_{\mathscr{O}_X} \mathscr{O}_Y \cong \mathscr{E}_0 \otimes_{\mathscr{O}_Y} \Omega^i_{Y^{\times\times}/K_0},$$

for $i \geq 0$. Let $j: Y \subset X$ be the natural inclusion.

Lemma 4.4. — $(\mathrm{Res})_0(x) = \mathrm{Res}^{(0)}(j^*x)$ *for all x section of* $\mathbb{H}$.

Proof. — The inclusion j induces an isomorphism $\mathbb{H}/t\mathbb{H} \overset{j^*}{\cong} H^1(Y, \mathscr{E})$ therefore it is enough to prove: if $x \in \mathbb{H}$ then we have $j^*(\mathrm{Res}(x)) = \mathrm{Res}^{(0)}(j^*x)$. Let x be represented by a hypercocycle as in formula (2) above. Then for each $e \in e(G)$ we have

$$\omega_e = \sum_{i=1}^{\alpha} \epsilon_i^{(e)} \otimes \Big(\sum_{n,m\geq 0} a^{(e)}_{i,n,m} x_e^n x_{\tau(e)}^m\Big) \frac{d_{\mathrm{X/S}}(x_e)}{x_e},$$

where $\{\epsilon_i^{(e)}\}$ is a basis of horizontal sections of $\mathscr{E}_{\mathfrak{X}}|_{A_e}$ for all e and $a^{(e)}_{i,n,m} \in K_0$ are such that the power series converge on A_e. With these notations we have $\mathrm{Res}_e(\omega_e) = (\frac{1}{2}\sum_{i=1}^{\alpha} \epsilon_i^{(e)}|_{T_{a(e)}\cap A_e} \sum_{n\geq 0} a_{i,n,n} t^n, \frac{1}{2}\sum_{i=1}^{\alpha} \epsilon_i^{(e)}|_{T_{b(e)}\cap A_e} \sum_{n\geq 0} a_{i,n,n} t^n)$. Now

$$j^*(\mathrm{Res}(\omega)) = \mathrm{Image}(\mathrm{Res}_e(\omega_e))_{e\in e(G(X))} (\mathrm{mod}\ tH^{1,0}(\mathscr{C}', E_{\mathrm{rel}}))$$

$$= \Big(\frac{1}{2}\Big(\sum_{i=1}^{\alpha} j^*(\epsilon_i^{(e)})|_{A_{e,0}\cap T_{a(e),0}} a^{(e)}_{i,0,0}, \frac{1}{2}\Big(\sum_{i=1}^{\alpha} j^*(\epsilon_i^{(e)})|_{A_{e,0}\cap T_{b(e),0}} a^{(e)}_{i,0,0}\Big)_e.$$

On the other hand, $j^*(x)$ is represented by the hypercocycle $\{(j^*(\omega_v))_v, (j^*(\omega_e))_e, (j^*(f_e), j^*(\overline{f}_e)_e\}$. In particular, for every $e \in e(G)$ let us denote by $y_e, y_{\tau(e)}$ the images $j^*(x_e)$ and respectively $j^*(x_{\tau(e)})$. With these notations $y_e y_{\tau(e)} = 0$ and we have

$$j^*(\omega_e) = \sum_{i=1}^{\alpha} j^*(\epsilon_i^{(e)}) \otimes \Big(a^{(e)}_{i,0,0} + \sum_{n\geq 1} a^{(e)}_{i,n,0} y_e^n + \sum_{m\geq 1} a^{(e)}_{i,0,m} y_{\tau(e)}^m\Big) \frac{d(y_e)}{y_e},$$

so

$$\mathrm{Res}_e^{(0)}(j^*(x)) = \Big(\frac{1}{2}\Big(\sum_{i=1}^{\alpha} j^*(\epsilon_i^{(e)})|_{A_{e,0}\cap T_{a(e),0}} a^{(e)}_{i,0,0}, \frac{1}{2}\Big(\sum_{i=1}^{\alpha} j^*(\epsilon_i^{(e)})|_{A_{e,0}\cap T_{b(e),0}} a^{(e)}_{i,0,0}\Big) = j^*(\mathrm{Res}_e(\omega_e)).$$

□

Let us define by $N_0 \colon H^1(Y, \mathscr{E}) \longrightarrow H^1(Y, \mathscr{E})$ the composition $(\mathrm{Res})_0 \circ \iota_0$ where

$$\iota_0 \colon H^{1,0}(\mathscr{C}_0', \mathscr{E}_0) \longrightarrow H^1(Y, \mathscr{E})$$

is the map induced from the Mayer-Vietoris exact sequence for Y and the covering $\mathscr{C}_0'$.

We have the following

Proposition 4.5. — *The $\mathscr{O}_S$-linear map* Res *is horizontal with respect to the connections, i.e.* $\mathrm{Res} \colon (\mathbb{H}, \nabla) \longrightarrow (H^{1,0}(\mathscr{C}', E_{\mathrm{rel}})), \nabla^{1,0})$ *satisfies* $\mathrm{Res} \circ \nabla^{1,0} = \nabla \circ \mathrm{Res}$.

Proof. — Let $x \in \mathbb{H}$ be represented by a hypercocycle as in formula (2). We have $\nabla(x) = y \otimes \mathrm{dlog}(t)$, where y is represented by a hypercocycle $\big((\eta_v)_v, (\eta_e)_e, (g_e, \overline{g}_e)_e\big)$ as in the proof of Proposition 3.35. To calculate $\mathrm{Res}(y)$ we only need to look at the η_e's. To start with, we may write

$$\omega_e = \sum_{i=1}^{\alpha} \epsilon_i \otimes r_i(t) \frac{d_{\mathrm{X/S}}(x_e)}{x_e} + D_{\mathrm{X/S}}(G_e),$$

where $\{\epsilon_i\}_{i=1,\alpha}$ is as before a basis of horizontal sections of $\mathscr{E}_{\mathfrak{X}}$ over A_e, $r_i(t) \in \mathscr{O}_S(S)$ and $G_e \in \mathscr{E}_X(A_e)$. Then, let us denote by

$$\tilde{\omega}_e :\ = \sum_{i=1}^{\alpha} \epsilon_i \otimes r_i(t) \frac{\mathrm{d}_{X/K_0}(x_e)}{x_e} + D_{\mathrm{X/K_0}}(G_e).$$

It is a lift of ω_e to "absolute differentials", i.e., to $\mathscr{E}_X(A_e) \otimes \Omega^1_{A_e/K_0}(\log Y)$. Then η_e may be chosen such that

$$\eta_e \wedge \mathrm{dlog}(t) = \mathrm{D}_{X/K_0}(\tilde{\omega}_e) = \sum_{i=1}^{\alpha} \epsilon_i \otimes t r_i'(t) \frac{\mathrm{d}_{X/K_0}(x_e)}{x_e} \wedge \mathrm{dlog}(t),$$

therefore

$$\mathrm{Res}_e(\eta_e) = (\frac{1}{2} \sum_{i=1}^{\alpha} \epsilon_i|_{A_e \cap T_{a(e)}} t r_i'(t), \frac{1}{2} \sum_{i=1}^{\alpha} \epsilon_i|_{A_e \cap T_{b(e)}} t r_i'(t)).$$

On the other hand

$$\nabla(\iota \circ \mathrm{Res}(\omega)) = \nabla[\big((0_v)_v, (0_e)_e, (\frac{1}{2} \sum_{i=1}^{\alpha} \epsilon_i|_{A_e \cap T_{a(e)}} \otimes r_i(t), \frac{1}{2} \sum_{i=1}^{\alpha} \epsilon_i|_{A_e \cap T_{b(e)}} \otimes r_i(t))_e\big)]$$

$$= [\big((0_v)_v, (0_e)_e, (\frac{1}{2} \sum_{1}^{\alpha} \epsilon_i|_{A_e \cap T_{a(e)}} \otimes t r_i'(t), \frac{1}{2} \sum_{i=1}^{\alpha} \epsilon_i|_{A_e \cap T_{b(e)}} \otimes t r_i'(t))_e\big)] \otimes \mathrm{dlog}(t).$$

This proves the proposition. □

Proposition 4.6. — *Under the parallel transport isomorphism of Theorem 2.6,* $N_0 \otimes \mathrm{id}_K$ *is identified with* N_{int}.

Proof. — Let $N : \mathbb{H} \longrightarrow \mathbb{H}$ be the composition $\mathbb{H} \xrightarrow{\mathrm{Res}} H^{1,0}(\mathscr{C}', E_{\mathrm{rel}}) \longrightarrow \mathbb{H}$ where the second morphism is the one coming from the Mayer-Vietoris sequence (see section §3.5.2). Then by Proposition 4.5 N is horizontal and hence it induces a homomorphism $N : (\mathbb{H}_{\log})^{\nabla} \longrightarrow (\mathbb{H}_{\log})^{\nabla}$. By Lemma 4.3 and Lemma 4.4 the following diagram is commutative

$$\begin{array}{ccccc} H^1(Y, \mathscr{E}) & \cong & (\mathbb{H}_{\log})^{\nabla} & \longrightarrow & H^1(C_K, \mathscr{E}_\pi) \\ N_0 \downarrow & & N \downarrow & & N_{\mathrm{int}} \downarrow \\ H^1(Y, \mathscr{E}) & \cong & (\mathbb{H}_{\log})^{\nabla} & \longrightarrow & H^1(C_K, \mathscr{E}_\pi). \end{array}$$ □

4.2. The proof of the equality of the monodromy operators. — The main result of this section is

Theorem 4.7. — *Under the notations of section §4.1 we have* $N_0 = N_{\deg}$.

Proof. — We will extend scalars to a finite, non-trivial, totally ramified extension L of K_0 and let $B = B_L \subset S$ be the affinoid disk as in Lemma 3.17. Recall Proposition 3.18 i.e., for all $v \in v(G)$ there is a wide open neighborhood W_v of $Z_{v,B}$ in $U_{v,B}$ and an isomorphism over B

$$\alpha_v = \alpha_{v,0} \colon W_v \cong B \times W_{v,0},$$

where $W_{v,0} = W_v \cap Y$. Let pr_i, $i = 1, 2$ be the i-th projection composed with α_v, i.e., $\mathrm{pr}_1 \colon W_v \to B$, $\mathrm{pr}_2 \colon W_v \to W_{v,0}$. As α_v is an isomorphism over B, pr_1 is the structure morphism of W_v over B.

Let us now fix v and let $U = \alpha_v^{-1}(U_0 \times B)$ where $U_0 \subset W_v \cap Y$ is any admissible open subset. We have

Lemma 4.8. — a) *The canonical isomorphism*

$$\Omega^1_{U^*/L} \cong \mathrm{pr}_1^* \Omega^1_{B^*/L} \oplus \mathrm{pr}_2^* \Omega^1_{U_0/L},$$

where $U^* = U - U_0$ *and* $B^* = B - 0$, *induces an isomorphism of sheaves on* U:

$$\Omega^1_{U/L}(\log Y) \cong \mathrm{pr}_1^* \Omega^1_{B/L}(\log 0) \oplus \mathrm{pr}_2^* \Omega^1_{U_0/L}.$$

b) *The isomorphism at a) induces an isomorphism of sheaves:*

$$\Omega^1_{U/B}(\log Y) \cong \mathrm{pr}_2^* \Omega^1_{U_0/L},$$

and an isomorphism of $\mathscr{O}_B(B)$*-modules*

$$\Omega^1_{U/B}(\log Y)(U) \cong \mathscr{O}_B(B) \hat{\otimes} \Omega^1_{U_0/L}(U_0)$$

where $\hat{\otimes}$ *denotes completed tensor product.*

Proof. — For a) it is enough to see that we have an isomorphism of "pairs"

$$(U, U_0) \cong (B, \{0\}) \times (U_0, \phi),$$

where ϕ is the void set, i.e., that $U \cong B \times U_0$ and under the above isomorphism $U_0 \cong (\{0\} \times U_0) \cup (B \times \phi)$.

For b) let us notice that we have an isomorphism of sheaves on U:

$$\Omega^1_{U/B}(\log Y) \cong \Omega^1_{U/L}(\log Y)/\mathrm{pr}_1^* \Omega^1_B(\log 0) \cong \mathrm{pr}_2^* \Omega_{U_0/L}(\log Y).$$

Now the lemma follows easily. □

Let us recall from section §3.5.3.3 that the set $\mathscr{C}''_B := \{W_v, A_{e,B}\}_{v \in v(G), e \in e(G)}$ is an admissible cover of $X_B := X \times_S B$. From Lemma 4.8 it follows that for all $v \in v(G)$ and $U \subset W_v$ as above, the canonical projection:

$$\Omega^1_{W_v/L}(\log Y)(U) \longrightarrow \Omega^1_{W_v/B}(\log Y)(U)$$

has a natural section, call it s_v with the property that its image is a submodule of $\Omega^1_{W_v/L}(U)$. Therefore for every section ω of $\Omega^1_{W_v/B}(\log Y)$ we have a lift of it $s_v(\omega)$ to absolute 1-forms, which is a regular absolute one-form by the remark above.

Moreover, if say $e \in e(G)$ then we also have a natural choice of a lift to absolute forms as follows. Let us recall that we have $\mathscr{O}_B(B) = L\langle y\rangle$ with the restriction $\mathscr{O}_S(S) \longrightarrow \mathscr{O}_B(B)$ given by: $t \longrightarrow \pi_L y$. Let $c := |\pi_L| < 1$.

Lemma 4.9. — *Let $\omega \in \Omega^1_{A_{e,B}/B}(\log Y)(A_{e,B})$, then we can write $\omega = r(y)\dfrac{d_{\mathrm{X/S}}(x_e)}{x_e} + d_{\mathrm{X/S}}(u_e)$ where $r(y)$ is a global section of $\mathscr{O}_B$ and $u_e \in \mathscr{O}_{X_B}(A_{e,B})$.*

Proof. — For this proof let us denote $U := A_{e,B}$ and $A(U) := \mathscr{O}_{X_B}(U)$, $x = x_e$ and $z = x_{\tau(e)}$. By Lemma 3.16, the natural functions $x, z \in A(U)$ satisfy $xz = \pi_L y$ and if $f \in A(U)$ then f may be written

$$f = \sum_{n=0}^{\infty} a_n x^n + \sum_{m=1}^{\infty} b_m z^m,$$

with $a_n, b_m \in \mathscr{O}_B(B)$ and such that, for every r such that $c < r < 1$ the sequences $|a_n|_B r^n \longrightarrow 0$ and $|b_n|_B (c/r)^n \longrightarrow 0$ as $n \longrightarrow \infty$.

Therefore $\omega = f d_{\mathrm{U/B}}(x)/x = d_{\mathrm{U/B}}(g) + a_0 d_{\mathrm{U/B}}(x)/x$, where

$$g = \sum_{n=1}^{\infty} \frac{a_n}{n} x^n + \sum_{m=1}^{\infty} \frac{b_m}{m} z^m \in A(U).$$

This proves the lemma. □

A lift to absolute 1-forms of ω as in Lemma 4.9 is then defined by:

$$\tilde{\omega}_e \colon = r(y)\frac{d_{\mathrm{X/K_0}}(x_e)}{x_e} + d_{\mathrm{X/K_0}}(u_e).$$

Proof of Theorem 4.7. Let $x \in \mathbb{H}_B$ be represented by the hypercocycle $\big((\omega_v)_v, (\omega_e)_e, (f_e, \overline{f})_e\big)$ with respect to $\mathscr{C}''_B$ (as in in Formula 3.3.2). Let us recall that v runs over $v(G)$ and e over $e(G)$. Then ω_e can be written as

$$\omega_e = \sum_{i=1}^{\alpha} \epsilon_i \otimes (r_{e,i}(y))\frac{d_{\mathrm{X/S}}(x_e)}{x_e} + D_{\mathrm{X/S}}(E_i)) = -\sum_{i=1}^{\alpha} \epsilon_i \otimes (r_{e,i}(y))\frac{d_{\mathrm{X/S}}(x_{\tau(e)}}{x_{\tau(e)}} + D_{\mathrm{X/S}}(E_i)),$$

where $\{\epsilon_i\}_{1\le i\le\alpha}$ is a horizontal basis of $\mathscr{E}_B|_{A_{e,B}}$, $E_i \in \mathscr{E}_B(A_{e,B})$ for all i and $r_{e,i}(y)$ are global sections of $\mathscr{O}_B$. The variables x_e and $x_{\tau(e)}$ have been defined in Lemma 3.16 and their restrictions to $A_{e,B}$ satisfy $x_e x_{\tau(e)} = \pi_L y$.

We want to calculate $\nabla(x)$ and its residue. $\nabla(x)$ is represented by the hypercocycle $\big((\eta_v)_v, (\eta_e)_e, (g_e, \overline{g}_e)_e\big)$, where

$$D_{\mathrm{X/K_0}}(s_v(\omega)_v) = \eta_v \wedge \mathrm{dlog}(y) \quad \text{and} \quad D_{\mathrm{X/S}}(\tilde{\omega}_e) = \eta_e \wedge \mathrm{dlog}(y),$$

for $v \in v(G)$ and $e \in e(G)$. Also

$$s_{a(e)}(\omega_{a(e)})|_{A_{e,B}\cap W_{a(e)}} - \tilde{\omega}_e|_{A_{e,B}\cap W_{a(e)}} - D_{\mathrm{X/S}}(f_e) = g_e \mathrm{dlog}(y),$$

and

$$\tilde{\omega}_e|_{A_{e,B}\cap W_{b(e)}} - s_{b(e)}(\omega_{b(e)})|_{A_{e,B}\cap W_{b(e)}} - D_{\mathrm{X/S}}(\overline{f}_e) = \overline{g}_e \mathrm{dlog}(y).$$

Let us recall that $s_v(\omega_v)$ is always a regular 1-form. Also,

$$\tilde{\omega}_e|_{A_{e,B}\cap W_{a(e)}} := r(y)\frac{d_{\mathrm{X/K_0}}(x_e)}{x_e} + d_{\mathrm{X/K_0}}(u_e)$$

is also regular as x_e is invertible on $A_{e,B} \cap W_{a(e)}$. On the other hand we have

$$\tilde{\omega}_e|_{A_{e,B}\cap W_{b(e)}} = r(y)\frac{d_{\mathrm{X/K_0}}(x_e)}{x_e} + d_{\mathrm{X/K_0}}(u_e) = r(y)\frac{d(y)}{y} - r(y)\frac{d_{\mathrm{X/K_0}}(x_{\tau(e)})}{x_{\tau(e)}} + d_{\mathrm{X/K_0}}(u_e),$$

and the form $-r(y)\frac{d_{\mathrm{X/K_0}}(x_{\tau(e)})}{x_{\tau(e)}} + d_{\mathrm{X/K_0}}(u_e)$ is regular on $W_{b(e)} \cap A_{e,B}$ because the function $x_{\tau(e)}$ is invertible on this open set.

Therefore we have: $\mathrm{Res}_{y=0}(\eta_v) = \mathrm{Res}_{y=0}(\eta_e) = 0$ for all $v \in v(G), e \in e(G)$, $\mathrm{Res}_{y=0}(g_e) = 0$ and $\mathrm{Res}_{y=0}(\overline{g}_e) = \sum_{i=1}^{\alpha} r_{e,i}(0)\epsilon_i|_{A_{e,B}\cap W_{b(e)}}$ for $e \in e(G)$. Thus, we have that $\mathrm{Res}_{y=0}(\nabla(x))$ is represented by the hypercocycle

$$\big((0_v)_v, (0_e)_e, (0_e, \sum_{i=1}^{\alpha} r_{e,i}(0)\epsilon_i|_{A_{e,B}\cap W_{b(e)}})_e\big)$$

whose cohomology class in $H^1(Y, \mathscr{E}) \otimes_{K_0} L$ is the same as the class of

$$\big((0_v)_v, (0_e)_e, (\frac{1}{2}\sum_{i=1}^{\alpha} r_{e,i}(0)\epsilon_i|_{A_{e,B}\cap W_{a(e)}}, \frac{1}{2}\sum_{i=1}^{\alpha} r_{e,i}(0)\epsilon_i|_{A_{e,B}\cap W_{b(e)}})_e\big)$$

which is

$$\mathrm{Res}(x) \pmod{y\mathbb{H}_B}.$$

This proves that $N_{\mathrm{deg}} \otimes_{K_0} \mathrm{id}_L = N_0 \otimes_{K_0} \mathrm{id}_L$. As N_{deg} and N_0 are both endomorphisms over K_0 of the finite dimensional K_0 vector space $H^1(Y, \mathscr{E})$, and as they become equal after base change to the extension L of K_0, they are equal. This ends the proof of Theorem 4.7. □

5. Frobenii

5.1. Frobenius and K_0-structures on $H^{i,j}(\mathscr{C}_s, \mathscr{E}_s)$. — In this section we supply a number of details needed in section §2.2. Namely let us resume the notations of section §3.2. Let $X \longrightarrow S$ be our family of curves, $\mathscr{C} = \{U_v\}_{v\in v(G)}$ be the admissible covering of X defined there. Fix $s \in S$ a point such that $s \neq 0$ and for an object M over S M_s will be the fiber of M over s. Let $\mathscr{C}_s := \{U_{v,s}\}_{v\in v(G)}$ and if $e = [u, v] \in e(G)$ then $A_{e,s} = A_e \times_S s = U_{u,s} \cap U_{v,s}$. Let us also denote by $\overline{s}$ the image under $\mathrm{red} : S \longrightarrow \mathscr{S} = \mathrm{Spf}(W[[t]])$ of the point $s \in S$ and by $\mathfrak{X}_s := \mathfrak{X} \otimes_{\mathscr{S}} \overline{s}$. In particular if $s = \pi$, then $X_s = C_K$ and $\mathfrak{X}_s = C$ in section §2.2. Let $\mathscr{E}$ denote an F-isocrystal on $\overline{C}$ and let $\mathscr{E}_s$ denote the evaluation of $\mathscr{E}$ on the enlargement $\mathfrak{X}_s$.

We will define the canonical K_0-structures and Frobenii on $H^{1,0}(\mathscr{C}_s, \mathscr{E}_s)$ and $H^{1,0}(\mathscr{C}_s, \mathscr{E}_s)$ needed in section §2.2.

For the rest of this section we fix s and denote $U_{v,s}, A_{e,s}$ simply by U_v, A_e.

Lemma 5.1. — *Suppose that the residue field of s is L. For every $e \in e(G)$ we have a canonical isomorphism of L-vector spaces*

$$H^0_{\mathrm{cris}}(e/W, \mathscr{E}) \otimes_{K_0} L \cong H^0_{dR}(A_e, \mathscr{E}_s|_{A_e}),$$

where above e denotes the singular point of $\overline{C}$ corresponding to the edge e.

Proof. — As mentioned before, A_e is a wide open enlargement of $e \in \overline{C}$, i.e. let us consider the formal completion of $\mathfrak{X}_s$ along e, $(\mathfrak{X}_s)_{/e}$. It is a formal scheme such that $(\mathfrak{X}_s)^{\mathrm{rig}}_{/e} \cong A_e$. Therefore $\mathscr{E}_s|_{A_e} \cong \mathscr{E}_{(\mathfrak{X}_s)_{/e}}$ and $H^0_{\mathrm{cris}}(e/W, \mathscr{E}) \otimes_{K_0} L \cong H^0_{dR}(A_e, \mathscr{E}_s|_{A_e})$. □

Let us remark that the isomorphism of lemma 5.1 endows $H^0_{dR}(A_e, \mathscr{E}_s|_{A_e})$ with a canonical K_0-structure and a Frobenius, namely $H^0_{\mathrm{cris}}(e/W, \mathscr{E})$ with its Frobenius, ϕ^0_e.

Let us fix $v \in v(G)$ and $\overline{C}_v$ the component of $\overline{C}$ corresponding to v. Let is denote by $\overline{C}_v^{\times\times}$ the log scheme $\overline{C}_v$ with log structure given by the smooth divisor of the singular points in $\overline{C}$ belonging to $\overline{C}_v$.

Lemma 5.2. — *In this lemma s may be 0. For $i = 0, 1$ we have natural isomorphisms of L-vector spaces*

$$H^i_{\mathrm{cris}}(\overline{C}_v^{\times\times}/W, \mathscr{E}) \otimes_{K_0} L \cong H^i_{dR}(U_v, \mathscr{E}_s|_{U_v}).$$

Proof. — Let red : $X_s \longrightarrow \overline{C}$ denote the reduction map and let $Z_v = \mathrm{red}^{-1}(\overline{C}^0_v)$, where $\overline{C}^0_v$ is the complement in $\overline{C}_v$ of the singular points in $\overline{C}$. Then Z_v is an underlying affinoid of U_v with good reduction (its reduction is $\overline{C}^0_v$). Let us denote by $\mathrm{Sing}_v := \overline{C}_v - \overline{C}^0_v$. As $\overline{C}_v$ is a smooth proper curve over k, there exists a pair (C', Q) consisting of a smooth proper curve C' over $\mathscr{O}_L$ and an étale divisor Q on C' such the special fiber of (C', Q) is $(\overline{C}_v, \mathrm{Sing}_v)$. Let us denote $\widehat{C'} := C'_{/\overline{C}_v}$ the formal completion of C' along its special fiber, let $C'_L := (\widehat{C'})^{\mathrm{rig}}$ and red : $C'_L \longrightarrow \overline{C}_v$ be the reduction map. If we denote $Z'_v := \mathrm{red}^{-1}(\overline{C}^0_v)$ then $Z_v \cong Z'_v$ and we'll identify the two. We claim that we may choose the pair (C', Q) such that the isomorphism $Z_v \cong Z'_v$ extends to an open immersion $U_v \hookrightarrow C'_L$. This can be seen as follows: let us "add the affinoid disks to U_v to close the holes". We obtain a smooth proper rigid curve with a smooth proper formal model whose special fiber is $\overline{C}_v$. This formal model is algebrizable, i.e. it is the formal completion along reduction of a smooth proper curve over $\mathscr{O}_L$, which may be taken to be C'. In any case, the open immersion $U_v \hookrightarrow C'_L$ has the property that its complement is a disjoint union of affinoid disks, containing Q and each contained in the residue class of the points $e \in \mathrm{Sing}_v$.

We have the natural morphisms of formal schemes over $\mathscr{O}_L$:

$$\overline{C} \hookleftarrow \overline{C}_v \hookrightarrow \widehat{C'},$$

which make $\widehat{C'}$ an enlargement of $\overline{C}$. Let us denote by $\mathscr{E}_{C'}$ the evaluation of $\mathscr{E}$ on this enlargement. It is a coherent sheaf with connection on C'_L.

Claim 1. — $\mathscr{E}_{C'}|_{U_v}$ is isomorphic to $\mathscr{E}_s|_{U_v}$ as coherent sheaves with connections.

To see this let us first recall that we have open immersions $U_v \hookrightarrow X_s$ and $U_v \hookrightarrow C'_L$ and X_s, C'_L have formal models $\mathfrak{X}_s, \widehat{C}'$ respectively. Moreover, by the description of the embedding $U_v \hookrightarrow C'_L$ given above the following diagram commutes

$$\begin{array}{ccccc} U_v & \hookrightarrow & X_s & \xrightarrow{\text{red}} & \overline{C} \\ || & & & & \cup \\ U_v & \hookrightarrow & C'_L & \xrightarrow{\text{red}} & \overline{C}_v \end{array}$$

Let now $V \subset U_v$ be an admissible open. By applying lemma 3.1 we obtain canonical formal models $\mathscr{V}' \longrightarrow \widehat{C}'$ and $\mathscr{V} \longrightarrow \mathfrak{X}_s$ and by the diagram above and section 3.1.2 we obtain a natural morphism $\mathscr{V}' \longrightarrow \mathscr{V}$ inducing the identity on generic fibers and such that the following diagram of special fibers commutes

$$\begin{array}{ccc} \overline{\mathscr{V}}' & \longrightarrow & \overline{\mathscr{V}} \\ \downarrow & & \downarrow \\ \overline{C}_v & \hookrightarrow & \overline{C} \end{array}$$

Thus we obtain a diagram of enlargements

$$\begin{array}{ccc} (\overline{\mathscr{V}}' \hookrightarrow \mathscr{V}') & \longrightarrow & (\overline{\mathscr{V}} \hookrightarrow \mathscr{V}) \\ \downarrow & & \downarrow \\ (\overline{C}_v \hookrightarrow \widehat{C}') & & (\overline{C} \hookrightarrow \mathfrak{X}_s) \end{array}$$

which shows that $\mathscr{E}_{C'}$ and $\mathscr{E}_s$ coincide on V. This proves the claim.

Let $\overline{C}_v^{\times\times}$ and $\widehat{C}'^{\times\times}$ denote the scheme $\overline{C}_v$, respectively formal scheme $\widehat{C}'$ with log structures given by the divisor Sing_v, respectively by the divisor Q. Now let us see that we have natural morphisms

$$H^i_{\mathrm{cris}}(\overline{C}_v^{\times\times}/W, \mathscr{E}) \otimes_{K_0} L \cong H^i_{\mathrm{cris}}(\overline{C}_v^{\times\times}/\mathscr{O}_L, \mathscr{E}) \cong H^i_{dR}(C'_L, \mathscr{E}_{C'}(\log(Q)) \longrightarrow H^i_{dR}(U_v, \mathscr{E}_s|_{U_v}),$$

the first two being naturally isomorphisms.

In order to prove the lemma let us remark that we have natural isomorphisms of L-vector spaces $H^i_{dR}(C'_L - Q, \mathscr{E}_{C'}|_{C'_L - Q}) \cong H^i_{dR}(C'_L, \mathscr{E}_{C'}(\log(Q)))$ for $i = 0, 1$. We will prove

Claim 2. — Restrictions induce isomorphisms between $H^i_{dR}(C'_L - Q, \mathscr{E}_{C'}|_{C'_L - Q}) \cong H^i_{dR}(U_v, \mathscr{E}_s|_{U_v})$ for all $i \geq 0$.

For $i = 0$ the statement of the claim is clear. The proof of the claim for $i = 1$ is by an excision argument presented in theorem 4.2 of **[7]** for the case of trivial $\mathscr{E}$. The main idea is for a rigid analytic space M to find good definitions of "closed subsets" and their "admissible open neighbourhoods" and to use the Gysin long exact sequence as in **[22]**.

We say that a subset Z of M is closed if it is the complement in M of an admissible open subset. Given such a Z, we say that U is an admissible neighbourhood of Z if

U is a strict neighbourhood of Z in M. Let us recall that this means $Z \subset U$, U is an admissible open of M and the family $\{U, M - Z\}$ is an admissible covering of M.

Now if $\mathscr{F}$ is a sheaf of abelian groups on M we define $\Gamma_Z(M, \mathscr{F})$ to be the sections $s \in \mathscr{F}(U)$ supported in Z for any strict neighbourhood U of Z. The functor $\mathscr{F} \longrightarrow \Gamma_Z(M, \mathscr{F})$ is left exact and therefore if $\mathscr{F}^\bullet$ is a complex of sheaves on M we define the hypercohomology groups with supports, $\mathbb{H}^i_Z(M, \mathscr{F}^\bullet)$ to be the hyper-right derived functors of $\Gamma_Z(M, -)$. By corollary 1.9 of **[22]** if $\mathscr{F}^\bullet$ is a complex of sheaves on M we have a long exact sequence (the Gysin sequence):

$$0 \longrightarrow \mathbb{H}^0_Z(M, \mathscr{F}^\bullet) \longrightarrow \mathbb{H}^0(M, \mathscr{F}^\bullet) \longrightarrow \mathbb{H}^0(X - Z, \mathscr{F}^\bullet) \longrightarrow \mathbb{H}^1_Z(M, \mathscr{F}^\bullet) \longrightarrow \cdots$$

Moreover, if U is a strict neighbourhood of Z in M we have excision, i.e. canonical isomorphisms

$$\mathbb{H}^i_Z(M, \mathscr{F}^\bullet) \cong \mathbb{H}^i_Z(U, \mathscr{F}^\bullet) \text{ for all } i \geq 0.$$

Let us now apply this theory to: $M = C'_L - Q$, $Z = (C'_L - U_v) - Q$. Let us remark that $C'_L - U_v$ is a disjoint union of closed disks contained each in the residue class of one point of Sing_v and containing exactly one point of Q. So in fact $Z = M - U_v$ is closed in M. Let us denote by $(E, D) = (\mathscr{E}_{C'}|_M, D|_M)$ the restriction to M of the coherent sheaf with connection $(\mathscr{E}_{C'}, D)$ and let $\mathscr{F}^\bullet := E \otimes_{\mathscr{O}_M} \Omega^\bullet_{M/L}$. The interesting part of the Gysin sequence reads:

$$\mathbb{H}^1_Z(M, R \otimes_{\mathscr{O}_M} \Omega^\bullet_{M/L}) \longrightarrow H^1_{dR}(C'_L - Q, E) \longrightarrow H^1_{dR}(U_v, E|_{U_v}) \longrightarrow \mathbb{H}^2_Z(M, E \otimes_{\mathscr{O}_M} \Omega^\bullet_{M/L}).$$

Let us now explicitly calculate $\mathbb{H}^i_Z(M, E \otimes_{\mathscr{O}_M} \Omega^\bullet_{M/L})$. Let U' denote a disjoint union of wide open disks in C'_L containing $C'_L - U_v$ and contained in the union of the residue disks of the points of Sing_v. Then $U' - Q$ is a strict neighbourhood of Z in M and excision implies

$$\mathbb{H}^i_Z(M, E \otimes_{\mathscr{O}_M} \Omega^\bullet_{M/L}) \cong \mathbb{H}^i_Z(U' - Q, E|_{U'-Q} \otimes_{\mathscr{O}_{U'-Q}} \Omega^\bullet_{(U'-Q)/L}) \text{ for all } i \geq 0.$$

The Gysin sequence for the pair $(U' - Q, Z)$ and the restriction of E to $U' - Q$ which we denote by E' gives

$$0 \longrightarrow \mathbb{H}^0_Z(U' - Q, E' \otimes \Omega^\bullet_{(U'-Q)/L}) \longrightarrow H^0_{dR}(U' - Q, E') \longrightarrow H^0_{dR}(U' - Z, E') \longrightarrow$$

$$\longrightarrow \mathbb{H}^1_Z(U' - Q, E' \otimes \Omega^\bullet_{(U'-Q)/L}) \longrightarrow H^1_{dR}(U' - Q, E') \longrightarrow H^1_{dR}(U' - Z, E') \ldots$$

First let us remark that as U' is contained in a union of residue classes, $(E|_{U'}, D|_{U'})$ has a basis of horizontal sections. Let us denote by $E^D := H^0_{dR}(U', E_{U'})$. Second let is remark that $U' - Q$ is a disjoint union of punctured disks containing the disjoint union of wide open annuli $U' - Z$. Therefore we have the following commutative diagram where the horizontal arrows are induced by restrictions and the last vertical ones are residue maps.

$$\begin{array}{ccc} H^1_{dR}(U'-Q,E') & \longrightarrow & H^1_{dR}(U'-Z,E') \\ \downarrow\cong & & \downarrow\cong \\ H^1_{dR}(U'-Q)\otimes_L E^D & \longrightarrow & H^1_{dR}(U'-Z)\otimes_L E^D \\ \downarrow & & \downarrow \\ H^0_{dR}(U'-Q,E') & =E^D= & H^0_{dR}(U'-Z,E') \end{array}$$

As the residue maps for punctured disks and annuli are isomorphisms the first horizontal arrow is an isomorphism and the Gysin sequence for $(U'-Q,Z)$ above implies that $\mathbb{H}^i_Z(M,E\otimes_{\mathscr{O}_M}\Omega^\bullet_{M/L})=0$ for all $i\geq 0$. This proves the claim.

Claim 3. — We claim that for $i=0,1$ the composed isomorphism

$$H^i_{\mathrm{cris}}(\overline{C}_v^{\times\times}/\mathscr{O}_L,\mathscr{E})\cong H^1_{dR}(U_v,\mathscr{E}_s|_{U_v})$$

is independent of the choice of C' and the choice of embedding $U_v\hookrightarrow C'_L$.

The proof of this claim is standard: suppose (C'',Q'') is another such pair defined over $\mathscr{O}_L$, with an embedding $U_v\hookrightarrow C''_L$. We let $\widehat{C_1}$ to be the formal completion along $\overline{C}_v$ of the fiber product $C'\times C''$. By the Poincaré lemma we have isomorphisms

$$H^i_{dR}(C'_L,\mathscr{E}_{C'}\log(Q))\longrightarrow H^i_{dR}((C_1)^{\mathrm{rig}},\mathscr{E}_{C_1}(\log(Q\cup Q''))\longleftarrow H^i_{dR}(C''_L,\mathscr{E}_{C''}(\log(Q'')),$$

compatible with the homomorphisms from $H^i_{dR}(U_v,\mathscr{E}_s|_{U_v})$ induced by the immersions $U_v\hookrightarrow C'_L$, $U_v\hookrightarrow C''_L$ and the diagonal immersion $U_v\hookrightarrow (C_1)^{\mathrm{rig}}$. □

As before the isomorphisms in lemma 5.2 endow the L-vector spaces $H^i_{dR}(U_v,\mathscr{E}_s|_{U_v})$ with natural K_0-structures with Frobenii, namely $H^i_{\mathrm{cris}}(\overline{C}_v^{\times\times},\mathscr{E})$ for $i=0,1$ with their Frobenii.

For $e\in e(G)$ let us denote by $\mathscr{E}_e:=\mathscr{E}_s|_{A_e}$ and let us now concentrate on the L-vector space $H^1_{dR}(A_e,\mathscr{E}_e)$. These spaces do not have an interpretation as crystalline cohomology groups, nevertheless we have residue isomorphisms

$$\mathrm{Res}_e: H^1_{dR}(A_e,\mathscr{E}_e)\cong H^0(A_e,\mathscr{E}_e),$$

and may define the K_0-structure of the domain to be the inverse image of the K_0-structure of the target, i.e. to be $\mathrm{Res}^{-1}(H^0_{\mathrm{cris}}(e/W,\mathscr{E}))$. Moreover let us endow this K_0-structure with a Frobenius ϕ^1_e defined by $\phi^1_e=p\mathrm{Res}_e^{-1}\circ\phi^0_e\circ\mathrm{Res}_e$. We have

Lemma 5.3. — *Let $e\in e(G)$ and suppose the vertex $v\in v(G)$ is the origin or the end of e. Then, for $i=0,1$ the natural restriction maps: $H^i_{dR}(U_v,\mathscr{E}_s|_{U_v})\longrightarrow H^i_{dR}(A_e,\mathscr{E}_e)$ respect the K_0-structures and the Frobenii.*

Proof. — For $i=0$ this follows from the commutativity of the diagram

$$\begin{array}{ccc} H^0_{dR}(U_v,\mathscr{E}_s|_{U_v}) & \longrightarrow & H^0_{dR}(A_e,\mathscr{E}_e) \\ \downarrow\cong & & \downarrow\cong \\ H^0_{\mathrm{cris}}(\overline{C}_v^{\times\times}/W,\mathscr{E})\otimes_{K_0}L & \longrightarrow & H^0_{\mathrm{cris}}(e/W,\mathscr{E})\otimes_{K_0}L \end{array}$$

where the lower horizontal map is the restriction $H^0_{\rm cris}(\overline{C}_v^{\times\times}/W, \mathscr{E}) \longrightarrow H^0_{\rm cris}(e/W, \mathscr{E})$ tensored with L over K_0.

For $i = 1$ we'll use residues. First we have a natural residue map Res which makes the following sequence exact:

$$0 \longrightarrow H^1_{\rm cris}(\overline{C}_v/W, \mathscr{E}) \longrightarrow H^1_{\rm cris}(\overline{C}_v^{\times\times}/W, \mathscr{E}) \xrightarrow{\rm Res} \oplus_{\rm e\in Sing_v} H^0_{\rm cris}(e/W, \mathscr{E})(1).$$

Here the twist by 1 refers to a twist as filtered, Frobenius modules. Moreover, the following diagram of L-vector spaces with exact rows is commutative

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1_{\rm cris}(\overline{C}_v/\mathscr{O}_L, \mathscr{E}) & \longrightarrow & H^1_{\rm cris}(\overline{C}_v^{\times\times}/\mathscr{O}_L, \mathscr{E}) & \xrightarrow{\rm Res} & \oplus_{\rm e\in Sing_v} H^0_{\rm cris}(e/\mathscr{O}_L, \mathscr{E}) \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
0 & \longrightarrow & H^1_{dR}(C'_L, \mathscr{E}_{C'}) & \longrightarrow & H^1_{dR}(C'_L, \mathscr{E}_{C'}(\log(Q))) & \xrightarrow{\rm Res} & \oplus_{\rm P\in Q}(\mathscr{E}_{C'})_P \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
0 & \longrightarrow & H^1_{dR}(\overline{C}_v/\mathscr{O}_L, \mathscr{E}_{C'}) & \longrightarrow & H^1_{dR}(U_v, \mathscr{E}_s|_{U_v}) & \xrightarrow{\rm Res} & \oplus_{\rm e\in Sing_v} H^0_{dR}(A_e, \mathscr{E}_s|_{A_e})
\end{array}$$

where:

• The map Res : $H^1_{dR}(U_v, \mathscr{E}_s|_{U_v}) \longrightarrow \oplus_{e\in {\rm Sing}_v} H^0_{dR}(A_e, \mathscr{E}_e)$ in that diagram is the composition of the restriction $H^1_{dR}(U_v, \mathscr{E}_s|_{U_v}) \longrightarrow \oplus_{e\in {\rm Sing}_v} H^1_{dR}(A_e, \mathscr{E}_e)$ and the direct sum of the residue maps ${\rm Res}_e : H^1_{dR}(A_e, \mathscr{E}_e) \longrightarrow H^0_{dR}(A_e, \mathscr{E}_e)$.

and

• If we denote by ϕ^0, ϕ^1 the natural Frobenii on $H^0_{\rm cris}(e/W, \mathscr{E})$ and $H^1_{\rm cris}(\overline{C}_v^{\times\times}/W, \mathscr{E})$ respectively and by ${\rm Res}_e : H^1_{\rm cris}(\overline{C}_v^{\times\times}/W, \mathscr{E}) \longrightarrow H^0_{\rm cris}(e/W, \mathscr{E})$ then we have: ${\rm Res}_e\phi^1 = p\phi^0{\rm Res}_e$.

These facts prove the lemma for $i = 1$. □

5.2. F-isocrystals. — Let us go back to our notations of section 5.1: $X \longrightarrow S$ is our family of curves over the wide open unit disk, $s \in S - \{0\}$ is a point defined over L, X_s the fiber of X over s, $\mathfrak{X}_s$ the canonical formal model of X_s over $\mathscr{O}_L$ (defined in section 5.1) and $\overline{C}$ the special fiber of $\mathfrak{X}_s$. For $v \in v(G)$ let $\overline{C}_v$ denote the component of $\overline{C}$ corresponding to v and $\overline{C}_v^0$ the complement in $\overline{C}_v$ of the singular points of $\overline{C}$.

Then the composition $\overline{C}_v \hookrightarrow \overline{C} \hookrightarrow \mathfrak{X}_s$ is a closed immersion of formal schemes over $\mathscr{O}_L$ and $\overline{C}_v^0 \hookrightarrow \overline{C}_v$ is an affine open, therefore we denote $U = U_v = {\rm red}^{-1}(\overline{C}_v) = \big(\mathfrak{X}_s\big)_{/\overline{C}_v}^{\rm rig}$ and $Z = Z_v = {\rm red}^{-1}(\overline{C}_v^0)$. Then U is a one-dimensional wide open of X_s and $Z \subset U$ is a an underlying affinoid with good reduction.

Let $U \longrightarrow U \times_{\rm Spm(L)} U$ be the diagonal embedding. It is locally a closed immersion so let us denote by Δ_U the formal neighbourhood of the diagonal i.e. the completion of $U \times_{\rm Spm(L)} U$ along the diagonal morphism. Let $\pi_1, \pi_2 : U \times_{\rm Spm(L)} U \longrightarrow U$ denote the two projections.

If M is a locally free, coherent sheaf of $\mathscr{O}_U$-modules on U with an integrable connection D there is a unique horizontal isomorphism

$$h\colon \pi_1^* M|_{\Delta_U} \to \pi_2^* M_{\Delta_U}$$

which restricts to the identity on U. Locally on U we may assume that $\Omega^1_{U/L}$ is a free $\mathscr{O}_U$-module generated by dt, let ∂ denote the derivation dual to dt and also by $\partial = D_\partial : M \longrightarrow M$ the induced morphism. Let us denote by $u = \pi_1^*(t) - \pi_2^*(t)$ seen as a rigid function on Δ_U. With these notations, h is given (locally) by formulae

$$h(\pi_1^* m) = \sum_{n=0}^{\infty} \frac{u^n}{n!} \pi_2^*(\partial^n m),$$

for m (local) section of M.

Now let us look at the sequence of morphisms:

$$\overline{C}_v \xrightarrow{\Delta} \overline{C}_v \times_{\mathrm{Spec(k)}} \overline{C}_v \hookrightarrow \mathfrak{X}_s^2 := \mathfrak{X}_s \times_{\mathrm{Spf}(\mathscr{O}_\mathrm{L})} \mathfrak{X}_s.$$

The composition is a closed immersion so let us define

$$\widetilde{\Delta}_U :=]\overline{C}_v[_{\mathfrak{X}_s^2} = \big((\mathfrak{X}_s^2)_{/\overline{C}_v}\big)^{\mathrm{rig}}.$$

Let us remark that $\widetilde{\Delta}_U$ is a tubular neighbourhood of the image under diagonal of U in $X_s \times_{\mathrm{Spm(L)}} X_s$.

Definition 5.4. — *We say the pair (M, D) is a* **convergent isocrystal** *on (U, Z) if h extends to $\widetilde{\Delta}_U$ (the extension is unique if it exists).*

Here are a few easy but very useful consequences of the definition. Suppose that (M, D) is a convergent isocrystal on (U, Z). If $f, g \colon T \to U$ are two morphisms from a rigid space T into U such that $(f, g)(T \times T) \subseteq \widetilde{\Delta}_U$, let $\chi_{f,g} = (f, g)^* h \colon f^* M \to g^* M$. As h is an isomorphism $\chi_{f,g}$ is an isomorphism of sheaves.

Lemma 5.5. — *The restriction of (M, D) to any residue class of (W, X) is trivial.*

Proof. — Let U be a residue class of (W, X). If there exists a point $P \in U(K)$, let $f, g \colon U \to W$ be the morphisms, the identity and $x \to P$, respectively. Then $f^* M = M|_U$, $g^* M$ is trivial and $\chi_{f\,g}$ is an isomorphism.

In general, base change to a Galois extension L of K such that $U(L) \neq \varnothing$, proceed as above for each irreducible component of U_L and then take invariants. □

Let us recall that $\overline{C}_v^0$ is a smooth affine curve over k contained in the smooth projective curve $\overline{C}_v$; therefore there is a smooth affine scheme of finite type over $\mathscr{O}_L$, Spec(A) lifting $\overline{C}_v^0$. The π_L-adic completion of A is isomorphic (non-canonically) to the ring of rigid functions on Z bounded by 1. Fix such an isomorphism and identify the two. Via this identification, proposition 3.14 (where R_k is been replaced by $\mathscr{O}_L$) gives

$$\mathrm{Spm}(A^\dagger \otimes_{\mathscr{O}_L} L) = \lim_{\to, T} H^0(T, \mathscr{O}_U)$$

where T ranges over all strict affinoid neighbourhoods of Z in U. We have natural restriction maps $\mathscr{O}_U(U) \longrightarrow H^0(T, \mathscr{O}_U)$ which induce an $\mathscr{O}_L$-algebra homomorphism $\mathscr{O}_U(U) \longrightarrow A^\dagger \otimes_{\mathscr{O}_L} L$.

Therefore if (M, D) is a locally free coherent sheaf of $\mathscr{O}_U$-modules on U with an integrable connection we denote

$$M^\dagger := H^0(U, M) \otimes_{\mathscr{O}_U} (A^\dagger \otimes L).$$

It is a projective $A^\dagger \otimes L$-module with an integrable connection

$$D^\dagger : M^\dagger \longrightarrow M^\dagger \otimes_{A^\dagger \otimes L} \Omega^1_{(A^\dagger \otimes L)/L},$$

induced by D. We have a description of $\Omega^1_{(A^\dagger \otimes L)/L}$ as $\varinjlim_{T} H^0(T, \Omega^1_{T/L})$, where T runs over the strict affinoid neighbourhoods of Z in U (see [**1**], section §2.5.)

Let $u_0 : \overline{C}_v^0 \longrightarrow \overline{C}_v^0$ be a morphism of schemes over k, let A, A' be smooth $\mathscr{O}_L$-algebras of finite type such that $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A')$ lift $\overline{C}_v^0$ and let $u : A^\dagger \longrightarrow A'^\dagger$ be a $\mathscr{O}_L$-algebra homomorphism lifting the k-algebra homomorphism corresponding to u_0 (see for example theorem 3.7.)

Define the category $\mathrm{Mic}_{A^\dagger \otimes L}$ to be the category of finitely generated projective $A^\dagger \otimes L$-modules with integrable convergent connections. Then the $\mathscr{O}_L$-algebra morphism u defines a functor which preserves convergence $u^* : \mathrm{Mic}_{A'^\dagger \otimes L} \longrightarrow \mathrm{Mic}_{A^\dagger \otimes L}$ and which is an equivalence of categories if u_0 is a isomorphism.

In particular for $u_0 = id_{\overline{C}_v^0}$, we see that $(M^\dagger, D^\dagger)$ is independent of the choice of the lifting A.

Also, let us first fix $\sigma : \mathscr{O}_L \longrightarrow \mathscr{O}_L$ an automorphism which lifts Frobenius of k. Let $f := [k : \mathbb{F}_p]$ and denote by $\overline{F} = \mathrm{Frob}^f : \overline{C}_v \longrightarrow \overline{C}_v$. Then $\overline{F}(\overline{C}_v^0) \subset \overline{C}_v^0$ and let $\phi : A^\dagger \longrightarrow A^\dagger$ be a lift of $\overline{F}$ over σ.

Definition 5.6. — *A convergent F-isocrystal on (U, Z) is the following family of data*
- *A convergent isocrystal (M, D) on (U, Z)*

and
- *a horizontal isomorphism $F_\phi : \phi^*(M^\dagger, D^\dagger) \longrightarrow (M^\dagger, D^\dagger)$ for every morphism $\phi : A^\dagger \longrightarrow A^\dagger$ which is a lifting of $\overline{F}$.*

Let us remark that if ϕ_1, ϕ_2 are two liftings as in definition 5.6 we have $F_{\phi_2} = F_{\phi_1} \circ \chi_{\phi_1, \phi_2}$.

Let now $\mathscr{E}$ be an F-isocrystal on $\overline{C}$. Let us recall that the formal completion of $\mathfrak{X}_s$ along the closed sub-scheme $\overline{C}_v$, $\mathfrak{U}_v := (\mathfrak{X}_s)_{/\overline{C}_v}$ is a smooth formal scheme over $\mathscr{O}_L$ such that $\mathfrak{U}_v^{\mathrm{rig}} = U_v = U$. Let us denote by $(\mathscr{E}_v, D_v)$ the evaluation of $\mathscr{E}$ on $\mathfrak{U}_v$, which is a wide open enlargement of $\overline{C}$. Here $(\mathscr{E}_v, D_v)$ is a pair consisting of a locally free, coherent $\mathscr{O}_U$-module with integral convergent connection D_v (convergence follows from [**1**] 2.2.2 and 2.3.4.) Moreover by definition 3.4 it follows that if $\phi_v : \mathfrak{U}_v \longrightarrow \mathfrak{U}_v$ is a lifting of $\overline{F}$ then we have an isomorphism $F_{\phi_v} : \phi_v^*(\mathscr{E}_v, D_v) \longrightarrow (\mathscr{E}_v, D_v)$.

We therefore clearly have

Lemma 5.7. — *The pair $(\mathscr{E}_v, D_v)$ is a convergent F-isocrystal on (U, Z).*

In fact by [**1**] corollary 2.5.8 the data of the F-isocrystal $(\mathscr{E}_v, D_v)$ is equivalent to the data: (M, D) where M is a finitely generated projective $A^\dagger \otimes L$-module, D :

$M \longrightarrow M \otimes_{A^\dagger \otimes L} \Omega^1_{(A^\dagger \otimes L)/L}$ is an integrable connection such that if $\phi : A^\dagger \longrightarrow A^\dagger$ is a lifting of $\overline{F}$, there is a horizontal isomorphism $\Phi : \phi^* M \longrightarrow M$. The convergence of the connection is a consequence of the existence of Φ.

We need to consider one example of a relative convergent isocrystal. Let as above Z be our affinoid over L and $f \in \mathscr{O}_Z(Z)^*$, $|f| < 1$. Let An be the rigid analytic space over L in $Z \times \mathbb{B}^1_L$ whose $\mathbb{C}_p$-points are $\{(z,b) \colon |f(z)| < |b| < 1\}$. This is a family of annuli over Z. Let T be the rigid function on An defined by $T(z,b) = b$ and $\tilde{\Delta}_{An/Z}$ be the neighbourhood of the relative diagonal $\Delta_{An/Z}$ in $An \times_Z An$ over Z whose points are

$$\{(x,y) \in An \times_Z An : |\frac{T(x)}{T(y)} - 1| < 1\}.$$

The diagonal morphism $An \longrightarrow An \times_Z An$ is a closed immersion. We denote by $\widehat{\Delta}_{An/Z}$ the formal completion of $An \times_Z An$ along its image. Let π_1, π_2 denote the natural projection from $An \times_Z An$ to An. Suppose M is a coherent sheaf of $\mathscr{O}_{An}$-modules, $D : M \longrightarrow M \otimes_{\mathscr{O}_{An}} \Omega^1_{An/Z}$ a (relative) integrable connection over Z and such that the formal horizontal isomorphism $h \colon \pi_1^* M|_{\widehat{\Delta}_{An/Z}} \to \pi_2^* M|_{\widehat{\Delta}_{An/Z}}$ which is the identity when restricted to $\Delta_{An/Z}$ extends to $\tilde{\Delta}_{An/Z}$ (i.e. (M,D) is a convergent isocrystal.)

Then we have

Lemma 5.8. — *Suppose that (M,D) is a locally free sheaf of $\mathscr{O}_{An}$-modules on An with a relative, integrable convergent connection D as above. We use h to identify $\pi_1^* M$ and $\pi_2^* M$ on $\tilde{\Delta}_{An/Z}$. Let ω be a section of $M \otimes_{\mathscr{O}_{An}} \Omega^1_{An/Z}$. Then there is a unique section ϵ of $\pi_1^*(M)$ on $\tilde{\Delta}_{An/Z}$ such that*

$$\pi_1^* D(\epsilon) = \pi_1^*(\omega)|_{\tilde{\Delta}_{An/Z}} - \pi_2^*(\omega)|_{\tilde{\Delta}_{An/Z}},$$

and such that $\epsilon|_{\Delta_{An/Z}} = 0$.

Proof. — For simplicity let us denote for this proof $U := \tilde{\Delta}_{An/Z}$. We claim that we have a natural isomorphism $\phi : U \cong An \times_{\mathrm{Sp(L)}} S_L$ as rigid spaces over Z, where let us recall S_L is the wide open unit disk over L. The isomorphism and its inverse $\psi : An \times_{\mathrm{Sp(L)}} S_L \longrightarrow U$ are defined as follows

$$\phi\big((z,b),(z,b')\big) := \big((z,b), b'b^{-1}\big) \text{ and } \psi\big((z,b),a\big) = \big(z,b),(z,(1+a)b\big).$$

This implies (see lemma 3.5 in section §3.1.3) that for any admissible affinoid open V of An the morphism of complexes

$$(M \otimes \Omega^\bullet_{An/Z})(V) \longrightarrow (\pi_1^*(M) \otimes \Omega^\bullet_{U/Z})(\pi_1^{-1}(V) \cap U)$$

is a quasi isomorphism and hence pull-back by the diagonal immersion

$$\Delta^* : (\pi_1^*(M) \otimes \Omega^\bullet_{U/Z})(\pi_1^{-1}(V) \cap U) \longrightarrow (M \otimes \Omega^1_{An/Z})(V)$$

is a quasi-isomorphism. In degree $0,1$ this implies that for any section $\eta \in (\pi_1^*(M) \otimes \Omega^1_{U/Z})(\pi_1^{-1}(V) \cap U)$ such that $D(\eta) = 0$ and $\Delta^*(\eta) = 0$, there exists a unique section

$\epsilon \in \pi_1^*(M)(\pi_1^{-1}(V)\cap U)$ such that $D(\epsilon) = \eta$ and $\Delta^*(\epsilon) = 0$. Now we apply this to the case $\pi_1^{-1}(V)\cap U = \pi_2^{-1}(V)\cap U$ and $\eta = \pi_1^*(\omega) - \pi_2^*(\omega)$ for a section $\omega \in (M \otimes \Omega^1_{An/Z})(V)$. □

Remark 5.9. — *In the notations of lemma 5.8 M has a basis of horizontal sections on An.*

Proof. — Let L' be a finite Galois extension of L such there exists a section $s : Z_{L'} \longrightarrow An_{L'}$ of the structure morphism $g : An_{L'} \longrightarrow Z_{L'}$ (the subscript L' denotes extension of scalars to L'). For example, suppose there is a $b_0 \in \mathbb{B}^1_L(L')$ such that $|f| < |b_0| < 1$. We may define s by $s(z) = (z, b_0)$ and thus we have a morphism $u =: (id_{An}, s) : An = An \times_Z Z \longrightarrow U$. Then u^*h gives a horizontal isomorphism of $M_{L'}$ to the module with trivial relative connection $g^*s^*M_{L'}$, defined over L'. Now take $\mathrm{Gal}(L'/L)$ invariants to get a basis of horizontal sections of M. □

Let us also notice that remark 5.9 implies that in lemma 5.8 one could reduced to the case where (M, D) is trivial and then prove the lemma by elementary calculations.

5.3. Lifts of Frobenius. — Recall $X \longrightarrow S$ is a family of curves over the wide open unit disk and $\mathscr{E}$ is an F-isocrystal on $\overline{C}$. We have defined a Frobenius $\varphi : S \longrightarrow S$ over the absolute Frobenius σ on $\mathrm{Spec}(K_0)$ in section 2.1 and $\mathscr{E}$ comes equipped with an isomorphism of isocrystals on $\overline{C}$

$$F : \overline{F}^*(\mathscr{E}) \longrightarrow \mathscr{E}$$

where $\overline{F}$ is the Frobenius on $\overline{C}$ over the absolute Frobenius σ on $\mathrm{Spf}(W)$.

Using F we have defined a Frobenius operator $\Phi_1 \colon \varphi^*\mathbb{H}^1 \longrightarrow \mathbb{H}^1$ in section 2.1. Let $f := [k : \mathbb{F}_p]$. We will give an explicit description of the “linearized Frobenius", Φ_1^f using “local lifts of Frobenius” to X.

Recall, from section 3.2, the admissible cover of X, $\mathscr{C}' = \{T_v, A_e\}_{v\in v(G), e\in e(G)}$. We intend to construct local lifts of F, so we will need to refine this cover in two ways. First let L be a finite, non-trivial, totally ramified extension of K_0 and $B^1 = B_L$ the affinoid disk around 0 of radius $|\pi_L|$, where π_L is some uniformizer of L. Let B^2 be the affinoid disk around 0 of radius $|\pi_L^{p^f}|$, where $f = [k : \mathbb{F}]$. Then $\psi = \varphi^f \otimes_{K_0} \mathrm{id}_L$, maps B^1 to B^2. Similarly, let $\overline{F}_k^*(\mathscr{E})$ denote the isocrystal on $\overline{C}$ defined by: $\overline{F}_k^*(\mathscr{E})_{(T,z_T)} = \mathscr{E}_{(T,\overline{F}_k\circ z_T)}$, where let us recall that $\overline{F}_k = \overline{F}^f$ is the Frobenius endomorphism over k of $\overline{C}$, and by $F_k : (\overline{F}_k)^*(\mathscr{E}) \longrightarrow \mathscr{E}$ the f-iterate of F.

For the rest of this chapter we use the following notations: for every $v \in v(G), i = 1, 2$ let $Z_v^i := Z_v \times_S B^i$, $U_{B^i,v} := U_v \times_S B^i$, $A_e^i := A_e \times_S B^i$.

We have

Proposition 5.10. — a) *For every $v \in v(G)$ there exist wide open strict neighbourhoods $U_v^i \subset U_{B^i,v}$ of Z_v^i over B^i and a rigid morphism $\phi_v : U_v^1 \longrightarrow U_v^2$ over ψ, i.e. such* *that the following diagram commutes*

$$\begin{array}{ccc} U_v^1 & \xrightarrow{\phi_v} & U_v^2 \\ \downarrow & & \downarrow \\ B^1 & \xrightarrow{\psi} & B^2 \end{array}$$

b) *The morphism* ϕ_v *at* a) *is a lift of Frobenius i.e. the following diagram commutes*

$$\begin{array}{ccccc} U_v^1 & \hookrightarrow & X & \xrightarrow{\text{red}} & \overline{C} \\ \phi_v \downarrow & & & & \overline{F}^f \downarrow \\ U_v^2 & \hookrightarrow & X & \xrightarrow{\text{red}} & \overline{C} \end{array}$$

Proof. — For $i = 1, 2$ let W_v^i denote wide open strict neighbourhoods of Z_v^i in $U_{B^i,v}$ such that there exist isomorphisms of rigid spaces over B^i (see proposition 3.18)

$$\alpha_v^i : W_v^i \cong W_{v,0}^i \times B^i,$$

where $W_{v,0}^i$ is the fiber at $s = 0 \in B^i$ of W_v^i. Then $W_{v,0}^i$ is a wide open strict neighbourhood of $Z_{v,0}$ in $U_{v,0}$. As $Z_{v,0} =]\overline{C}_v^0[_{\mathfrak{X}_0}$, as in the discussion after the proof of lemma 5.5 let A be a smooth $\mathscr{O}_L$-algebra of finite type such that $\mathrm{Spec}(A)$ is a lifting of $\overline{C}_v^0$. We identify $A^\dagger$ with a sub $\mathscr{O}_L$-algebra of the ring of rigid functions on $Z_{v,0}$ and let $\Phi_v : A^\dagger \longrightarrow A^\dagger$ be a lifting of $\overline{F}^f : \overline{C}_v \longrightarrow \overline{C}_v$. We may choose strict affinoid neighbourhoods T^i of $Z_{v,0}$ in $W_{v,0}^i$ such that $\Phi_v(T^1) \subset T_v^2$. As in the proof of proposition 3.18 define wide open neighbourhoods $U_{v,0}^i$ of $Z_{v,0}^i$ in $W_{v,0}^i$ over B^i such that $\Phi_v(U_{v,0}^1) \subset U_{v,0}^2$. For later use let us remark that we may choose $U_{v,0}^2$ such that $U_{v,0}^2 - Z_{v,0}$ is a disjoint union of wide open annuli. Let now $U_v^i := (\alpha_v^i)^{-1}(U_{v,0}^i \times B^i)$ and $\phi_v : U_v^1 \longrightarrow U_v^2$ the morphism $\phi_v = \alpha_v^2 \circ (\Phi_v, \psi) \circ (\alpha_v^1)^{-1}$. By definition we have the commutative diagram

$$\begin{array}{ccc} U_v^1 & \xrightarrow{\phi_v} & U_v^2 \\ \alpha_1 \downarrow & & \downarrow \alpha_2 \\ U_{v,0}^1 \times B^1 & \xrightarrow{(\Phi_v,\psi)} & U_{v,0}^2 \times B^2 \end{array}$$

compatible with the projections to B^1 respectively B^2. The conclusion follows. □

We now give a general definition of a "lifting of Frobenius" and some of its properties.

(1) For two admissible opens $U^i \subset X_{B^i}$, $i = 1, 2$, we say that an L-morphism $\phi : U^1 \longrightarrow U^2$ is a lifting of Frobenius over $\psi : B^1 \longrightarrow B^2$ if the following two natural diagrams commute

$$\begin{array}{ccc} U^1 & \xrightarrow{\phi} & U^2 \\ \downarrow & & \downarrow \\ B^1 & \xrightarrow{\psi} & B^2 \end{array}$$

and

$$\begin{array}{ccccccc} U^1 & \hookrightarrow & X_{B^1} & \xrightarrow{\text{red}} & (\mathfrak{X}_{\mathscr{B}^1})_1 & = & \overline{C}\times\mathbb{A}^1_k \\ \phi\downarrow & & & & & & F^f\downarrow \\ U^2 & \hookrightarrow & X_{B^2} & \xrightarrow{\text{red}} & (\mathfrak{X}_{\mathscr{B}^2})_1 & = & \overline{C}\times\mathbb{A}^1_k \end{array}$$

Let us recall that in the second diagram $\mathscr{B}^i$ denote the natural formal models of B^i defined in section 3.2 and $(\mathfrak{X}_{\mathscr{B}^i})_1$ the closed sub-schemes of $\mathfrak{X}_{\mathscr{B}^i}$ defined by the ideals $\pi_L\mathscr{O}_{\mathfrak{X}_{\mathscr{B}^i}}$, for $i=1,2$. F denotes the absolute Frobenius of $\overline{C}\times\mathbb{A}^1_k$.

The commutativity of the above two diagrams is equivalent to the commutativity of the diagram:

$$\begin{array}{ccccc} U^1 & \hookrightarrow & X & \xrightarrow{\text{red}} & \overline{C} \\ \phi\downarrow & & & & \overline{F}^f\downarrow \\ U^2 & \hookrightarrow & X & \xrightarrow{\text{red}} & \overline{C} \end{array}$$

(2) For any lifting of Frobenius $\phi : U^1 \longrightarrow U^2$, we have a canonical horizontal isomorphism $F_\phi : \phi^*(\mathscr{E}_{\mathfrak{X}}|_{U^1}) \cong \mathscr{E}_{\mathfrak{X}}|_{U^2}$. Here $\mathscr{E}_{\mathfrak{X}}$ denotes the evaluation of the F-isocrystal $\mathscr{E}$ on the (wide open) enlargement $\mathfrak{X}$ of $\overline{C}$.

Proof. — First let us assume that U^1, U^2 are affinoids. Let $\mathscr{U}^1, \mathscr{U}^2$ be the canonical formal models of of U^1, U^2 constructed as in lemma 3.1 using the p-adic formal models $\mathfrak{X}_{\mathscr{B}^1}, \mathfrak{X}_{\mathscr{B}^2}$ over $\mathscr{O}_L$. Moreover the commutative diagram in (1) above and the remarks after the proof of lemma 3.1 provide a morphism $\varphi : \mathscr{U}^1 \longrightarrow \mathscr{U}^2$ whose generic fiber is ϕ and which induces $\overline{F}^f$ in the special fiber. Now $\mathscr{E}_{\mathfrak{X}}|_{U^1}, \phi^*(\mathscr{E}_{\mathfrak{X}}|_{U^2})$ are in fact isomorphic to the evaluations of $\mathscr{E}$, respectively of $(\overline{F}^f)^*(\mathscr{E})$ on the enlargement $\mathscr{U}^1$. Now the definition of the F-isocrystal $\mathscr{E}$ provides the F_ϕ.

In general, choose an admissible affinoid covering of U^2 and an admissible affinoid covering of U^1 which refines the inverse image under ϕ of the covering of U^2. The functorially of the construction in lemma 3.1 imply that the local F_ϕ's glue. □

(3) If $\phi, \phi' : U^1 \longrightarrow U^2$ are two liftings of Frobenius there is a canonical horizontal isomorphism $\chi_{\phi,\phi'} : \phi^*(\mathscr{E}_{\mathfrak{X}}|_{U^2}) \longrightarrow \phi'^*(\mathscr{E}_{\mathfrak{X}}|_{U^2})$ compatible with $F_\phi, F_{\phi'}$. For three liftings, they satisfy the cocycle condition.

Proof. — This follows from the fact that $\phi^*(\mathscr{E}_{\mathfrak{X}}|_{U^2})$ is canonically isomorphic to the evaluation of $(\overline{F}^f)^*(\mathscr{E})$ on the enlargement $\mathscr{U}^1$ defined in the proof of (2) above and again from the properties of F-isocrystals. □

Let U^i_v, $i=1,2$, $v\in v(G)$ denote admissible open subsets of X_{B^i} over B^i whose properties were proved in proposition 5.10. In fact we will choose the U^i_v's as in the proof of proposition 5.10 i.e. such that for every $v\in v(G)$, $i=1,2$ there are isomorphisms of rigid spaces over B^i: $\alpha^i_v : U^i_v \cong U^i_{v,0}\times B^i$ where $U^i_{v,0}$ are the fibers of U^i_v at $s=0$ and they are wide open strict neighbourhoods of $Z_{v,0}$ in $W^i_{v,0}$. Moreover, $U^2_{v,0} - Z_{v,0}$ is a disjoint union of wide open annuli.

Let us note that $\mathscr{C}^i = \{U_v^i, A_e^i\}_{v\in v(G),e\in e(G)}$ where $i = 1, 2$ are admissible covers of X_{B^i} by acyclic, admissible open subsets. For every $e \in e(G)$ we have morphisms $\phi_e : A_e^1 \longrightarrow A_e^2$, over $\psi : B^1 \longrightarrow B^2$ defined by $\phi_e(x_e) = x_e^{p^f}$ and $\phi_e(x_{\tau(e)}) = x_{\tau(e)}^{p^f}$.

Let F_v, F_e denote the Frobenii provided by (2) above.

$$F_v : \phi_v^*(\mathscr{E}_X|_{U_v^2}) \longrightarrow \mathscr{E}_X|_{U_v^1} \quad \text{for all v}$$

and respectively

$$F_e : \phi_e^*(\mathscr{E}_X|_{A_e^2}) \longrightarrow \mathscr{E}_X|_{A_e^1} \quad \text{for all e.}$$

The description of the Frobenius $\Phi_1^f : \psi^*\mathbb{H}_{B^2} \longrightarrow \mathbb{H}_{B^1}$. — We can now give the description of the Frobenius operator. Let $\mathscr{C}^i = \{U_v^i, A_e^i\}_{v\in v(G),e\in e(G)}$ be the respective open covers of X_{B^i}.

Recall, $\mathscr{E}$ is an F-isocrystal on $\overline{C}$ and let F_v, F_e be as above. Let $\omega \in \mathbb{H}_{B^2} = H^1_{dR}(X_{B^2}/B^2, \mathscr{E}_{\mathfrak{X}}(\log(Y)))$ be represented by the hypercocycle with respect to $\mathscr{C}^2$:

$$\big((\omega_v)_{v\in v(G)}, (\omega_e)_{e\in e(G)}, (f_e)_{e\in e(G)}, (\overline{f}_e)_{e\in e(G)}\big).$$

Now we define a hypercocycle of the relative de Rham complex of $\mathscr{E}_{\mathfrak{X}}$ with respect to $\mathscr{C}^1$ whose cohomology class in $\mathbb{H}_{B^1}$ represents $\Phi_1^f(\psi^*\omega)$.

Let us remark that for $e \in e(G)$ we have (see the proof of proposition 5.10)

$$U^2_{a(e)} \cap A_e^2 = (U^2_{v,0} \cap A^2_{e,0}) \times B^2 = \{|a| < |x_{e,0}| < 1\} \times B^2,$$

where $x_{e,0}$ is the restriction of x_e to $A_{e,0}$ and $a \in L$ is such that $|\pi_L^{p^f}| < |a| < 1$. Thus the rigid space $An := U^2_{a(e)} \cap A^2_e$ is a family of annuli over the affinoid $Z = B^2$ and we may apply lemma 5.8 to the sheaf with relative connection $(\mathscr{E}_{\mathfrak{X}}|_{An}, D_{\mathrm{X}_{\mathrm{B}^2}/\mathrm{B}^2})$. We let $\tilde{\Delta}_{(U^2_{a(e)}\cap A^2_e)/B^2}$ denote the neighbourhood of the relative diagonal in $An \times_{B^2} An$ defined in that lemma. There exists a unique section $\epsilon_e \in \mathscr{E}_{\mathfrak{X}}(\tilde{\Delta}_{(U^2_{a(e)}\cap A^2_e)/B^2})$ such that

$$\pi_1^* D_{X_{B^2}/B^2}(\epsilon_e) = \pi_1^*(\omega_{a(e)}|_{\tilde{\Delta}_{U^2_{a(e)}\cap A^2_e/B^2}}) - \pi_2^*(\omega_{a(e)}|_{\tilde{\Delta}_{U^2_{a(e)}\cap A^2_e/B^2}}),$$

and whose restriction to the diagonal vanishes.

Let us define

$$\nu_v = F_v(\phi_v^*(\omega_v)), \quad \nu_e = F_e(\phi_e^*(\omega_e)) \quad h_e = \Delta^*(F_{a(e)} \circ \phi^*_{a(e)}, F_e \circ \phi_e^*)(\epsilon_e) + F_e(\phi_e^*(f_e)),$$

$$\overline{h}_e := \Delta^*(F_{b(e)} \circ \phi^*_{\underline{(}e)}, F_e \circ \phi_e^*)(\epsilon_e) + F_e(\phi_e^*(\overline{f}_e)).$$

Then the collection $\big((\nu_v)_{v\in v(G)}, (\nu_e)_{e\in e(G)}, (h_e)_{e\in e(G)}, (\overline{h}_e)_{e\in e(G)}\big)$ is a hypercocycle for the relative logarithmic de Rham complex of $\mathscr{E}_{\mathfrak{X}}$ on X_{B^1}/B^1 with respect to the covering $\mathscr{C}^1$. Its cohomology class depends only on ω and is equal to $\Phi_1^f(\omega)$.

To see this let us recall the notations and results of section 3.4.3. Namely let us recall that we denoted $\mathfrak{X}^{\times\times}$ the formal scheme $\mathfrak{X}$ with log structures on $\mathscr{Y}$, let $\mathscr{S}^\times$ denote the formal scheme $\mathscr{S} = \mathrm{Spf}(W[[t]])$ with log structures at $t = 0$ and let $\overline{C}^{\times\times}$

denote the scheme $\overline{C}$ with inverse image log structure from $\mathfrak{X}^{\times\times}$. If for $e \in e(G)$ we denote also by e the singular point of $\overline{C}$ corresponding to the edge e we have $\big((\mathfrak{X}^{\times\times})_{/e}\big)^{\mathrm{rig}} = A_e$ and

$$\big((\mathfrak{X}^{\times\times} \times_{\mathscr{S}^\times} \mathfrak{X}^{\times\times})_{/e}\big)^{\mathrm{rig}} \times_S B^2 \cong \tilde{\Delta}_{(U^2_{a(e)} \cap A^2_e)/B^2}.$$

Clearly, under the identification of

$$H^1_{dR}(X/S, \mathscr{E}_{\mathfrak{X}}(\log(Y)) \cong H^1_{\mathrm{cris}}(\overline{C}^{\times\times}/S^\times, \mathscr{E}),$$

in section 3.4.3, after restricting to B^1, B^2 respectively, the image of the linearized crystalline Frobenius Φ^f is exactly the one defined above in terms of hypercocycles.

Remark 5.11. — *Let us recall from section 2.1 that Φ induces Φ_{deg} on $H^1(Y, \mathscr{E})$ and that it is horizontal with respect to the connection, i.e. we have*

$$(\Phi \circ \varphi^*) \circ \nabla = \nabla \circ \Phi.$$

Here we have dropped the index (respectively upper index) 1 from the notation. Therefore we also have

$$(\Phi^f \circ \phi^*) \circ \nabla = \nabla \circ \Phi^f.$$

5.4. Integration. — The theory of p-adic integration of convergent F-isocrystals on curves is the generalization of that developed by the first author in [**8**] (see also [**6**].) For the convenience of the reader we will briefly review the theory in what follows and prove the necessary generalizations.

Let us go back to the notations of section §5.2, i.e. let $s \in S$, X_s is the fiber of X over s defined over L and let us fix $v \in v(G)$. Let us consider the pair (U, Z), where $U = U_{v,s}, Z = Z_{v,s}$. Let us recall that Z is an affinoid over L with good reduction and U is a wide open neighbourhood of Z in X_s such that $U - Z$ is a disjoint union of wide open annuli.

Let (M, D) be a convergent F-isocrystal on (U, Z). An admissible open subset T of U will be called a residue class of (U, Z) if T is a residue class of Z or a connected component of $U - Z$. Lemma 5.5 implies that the restriction of (M, D) to every residue class of (U, Z) is trivial. We now define the sheaf M^{flog} with connection D^{flog} on U, as follows: we choose a branch log of the p-adic logarithm on L^* and define for an admissible open V of U

$$M^{\mathrm{flog}}(V) = \prod_T M(V_T) \otimes_{\mathscr{O}_{V_T}} \mathscr{O}_U(V_T)[\log(f)]_{f \in \mathscr{O}_U(V_T)^\times}$$

where T runs over the residue classes of (U, Z) and $V_T = V \cap T$. Here, for every V and T as above $\mathscr{O}_U(V_T)[\log(f)]_{f \in \mathscr{O}_U(V_T)^\times}$ is the sub-ring of the ring of locally analytic functions on V_T generated by $\mathscr{O}_U(V_T)$ and the functions $\log(f)$ for $f \in \mathscr{O}_U(V)^\times$. The connection extends naturally to this sheaf. Let $\Omega^\bullet_U(M^\circ)$ be the naturally induced de Rham complex of sheaves on U, where $\circ =$ nothing or flog. Here we have denoted by $\Omega^i_U(M^\circ) := \Omega^i_U \otimes_{\mathscr{O}_U} M^\circ$ for $i = 0, 1$. Let $(M^\circ)^\dagger$ denote the pullback of M° to

$Z^\dagger$ and let $H^i(M^\circ, D) := \mathbb{H}^i(\Omega_U^\bullet((M^\circ)^\dagger))$. Suppose ϕ is a lifting of Frobenius to $Z^\dagger$ as in section 5.2. Then as explained in [C1, §7] ϕ induces endomorphisms $(\varphi^i)^\circ$ of $H^i(M^\circ, D)$ (morally, $(\phi^i)^\circ = F_\phi \circ \phi^*$).

Note that $H^1(M^{\text{flog}}, D^{\text{flog}}) = 0$. We have

Theorem 5.12. — *Let $\omega \in \Omega_U^1(M)(U)$. We denote by $[\omega]$ its image in $H^1(M, D)$. Suppose that there is a polynomial $G(t)$ with coefficients in L such that*
(a) $G(\phi^1)([\omega]) = 0$ *and* (b) $G((\phi^0)^{\text{flog}})$ *is an isomorphism.*
Then there exists a section u of $M^{\text{flog}}(U)$, unique up to a horizontal section of M on U such that
i) $D(u) = \omega$
ii) $G(F_\phi \circ \phi^*)(u|_{X^\dagger})$ *is overconvergent on X.*
Moreover, u does not depend on the choice of ϕ or $G(t)$.

The existence and uniqueness is, up to notation, Theorem 7.4 of [C1] (the notion of regular singular annuli is subsumed by Lemma 5.1). The independence follows from the fact that the map $(\phi^i)^\circ$ does not depend on the choice of ϕ and we may choose for $G(t)$ the minimal polynomial of ϕ^1 acting on the finite dimensional space spanned by the classes of the images of $\omega, F_\phi \circ \phi^*\omega, (F_\phi \circ \phi^*)^2\omega, \ldots$ in $H^1(M, D)$.

5.5. The Frobenius Operators

Definition 5.13. — *We say that the F-isocrystal $\mathscr{E}$ on $\overline{C}$ is* **regular** *if for every vertex $v \in v(G)$ the characteristic polynomials of Frobenius on $H^0_{\text{cris}}(x, \mathscr{E})$ and $H^1_{\text{cris}}(\overline{C}_v^{\times\times}, \mathscr{E})$ are relatively prime for all closed points $i_x : x \longrightarrow \overline{C}_v$. We have denoted, as in section §5.1 by $\overline{C}_v$ the irreducible component of $\overline{C}$ corresponding to v and by $\overline{C}_v^{\times\times}$ the log scheme $\overline{C}_v$ with log structures given by the divisor* Sing_v

We have

Lemma 5.14. — *Let C be the curve over V with semi-stable reduction introduced in the introduction, let $g : \mathscr{T} \longrightarrow C$ be a smooth proper morphism and let us consider the F-isocrystal on $\overline{C}$, $\mathscr{H}^i := R^i g_{*,cris}(\mathscr{O}_{\mathscr{T}})$. Then* $\text{Sym}^j(\mathscr{H}^i)$ *is a regular F-isocrystal for $i, j \geq 0$.*

Proof. — Let $\bar{\mathscr{T}}$ denote the special fiber of $\mathscr{T}$, $\bar{\mathscr{T}}_v$ the pull back of $\bar{\mathscr{T}} \longrightarrow \overline{C}$ by $\overline{C}_v \subset \overline{C}$

The Leray spectral sequence for log crystalline cohomology in the relative situation $g_v : \bar{\mathscr{T}}_v \longrightarrow \overline{C}_v$ for log structures on $\overline{C}_v$ given by Sing_v and on $\bar{\mathscr{T}}_v$ given by the fiber above Sing_v, reads

$$E_2^{i,j} = H^i(\overline{C}^{\times\times}, R^j g_{\text{v,cris},*}(\mathscr{O}_{\bar{\mathscr{T}}_v})) => H^{i+j}_{\text{cris}}(\bar{\mathscr{T}}_v^{\times\times}, \mathbb{Q}_p).$$

Let us first remark that $\mathscr{H}_v^j = R^j g_{\text{v,cris},*}(\mathscr{O}_{\bar{\mathscr{T}}_v})$ is the pull back of $\mathscr{H}^j$ by the inclusion $\overline{C}_v \hookrightarrow \overline{C}$.

As $\overline{C}_v$ is a smooth proper curve over k let us also remark that $E_2^{i,j} = 0$ unless $0 \leq i \leq 2$. This implies that the differential $d_2 : E_2^{1,j} \longrightarrow E_2^{3,j-1}$ vanishes as well as the differential d_2 whose target is $E_2^{1,j}$ for all $j \geq 0$. Therefore $E_3^{1,j} = E_2^{1,j}$ for all $j \geq 0$ and the spectral sequence collapses at E_3. Therefore, for $n \geq 0$ the K_0-vector space with Frobenius $H^{n+1} = H^{n+1}_{\text{cris}}(\overline{\mathcal{T}}_v^{\times\times}, \mathbb{Q}_p)$ has a filtration $0 \subset F^1 \subset F^2 \subset H^{n+1}$ where F_1, F^2 have the property that $F^2/F^1 \cong E_3^{1,n}$. By the comment above it follows that $H^1_{\text{cris}}(\overline{C}_v^{\times\times}, \mathscr{H}^n)$ is a quotient, as K_0-vector space with Frobenius, of a subspace, F_2 of H^{n+1}.

By the main result of **[28]** $H^{n+1}_{\text{cris}}(\overline{Z}_v^{\times\times}, \mathbb{Q}_p) \cong H^{n+1}_{\text{rig}}(\overline{Z}_v - \text{Sing}_v, \mathbb{Q}_p)$ and by **[3]** the weights of Frobenius on the last K_0-vector space are larger or equal to $(n+1)/2$. It follows that the weights of Frobenius on $H^1_{\text{cris}}(\overline{C}_v^{\times\times}, \mathscr{H}^n))$ are also larger or equal to $(n+1)/2$. On the other hand, since $\overline{Z}_x$ is smooth and $H^0(\mathscr{H}^n_x) \cong K \otimes H^i_{\text{cris}}(\overline{Z}_x)$ for any point x of $\overline{C}_v$, using the Riemann hypothesis, the weights of Frobenius on $H^0(\mathscr{H}^n_x) \cong H^0_{\text{cris}}(x, i_x^*\mathscr{H}^n)$ are all equal to $n/2$. Thus the characteristic polynomials of Frobenius on $H^1_{\text{cris}}(\overline{C}_v^{\times\times}, \mathscr{H}^n))$ and $H^0(x, \mathscr{H}^n)$ are relatively prime for all closed points x of $\overline{C}$. The statement for $\text{Sym}^j(\mathscr{H}^i)$ follows by the same type of arguments. □

For the rest of this chapter we assume $\mathscr{E}$ is regular. Let us now, as in the previous section, extend scalars to a finite, non-trivial, totally ramified extension L of K and let $B = B_L \subset S$ be the affinoid disk of lemma 3.17. Let us recall proposition 3.18 which asserts that for all $v \in v(G)$ there is a wide open neighbourhood W_v of $Z_{v,B}$ in $U_{v,B}$ over B and an isomorphism over B

$$\alpha_{v,0} : W_v \longrightarrow B \times W_{v,0},$$

where $W_{v,0}$ is the fiber of W_v at $0 \in B$. Let us denote by $f_B : X_B \longrightarrow B$ the restriction of our family of curves to B. Let us now fix v and denote $\alpha := \alpha_{v,0}$, $W_0 := W_{v,0}$. Let $\beta : W_v \longrightarrow W_0$ be $\pi_2 \circ \alpha$ and $j : W_0 \longrightarrow W_v$ be defined by $j(w) = \alpha^{-1}(0, w)$. Let $\mathscr{E}_{\mathfrak{X}}$ and $\mathscr{E}_{\mathscr{Y}}$,denote the evaluations of $\mathscr{E}$ on $\mathfrak{X}$ and $\mathscr{Y}$, where let us recall that $\mathscr{Y} := \mathfrak{X} \times_{\mathscr{S}} \text{Spf}(W)$ where the morphism $\text{Spf}(W) \longrightarrow \mathscr{S}$ is given by $t \longrightarrow 0$. $\mathscr{E}_{\mathfrak{X}}$ and $\mathscr{E}_{\mathscr{Y}}$ are coherent sheaves with connections on $X = \mathfrak{X}^{\text{rig}}$ and respectively $Y = \mathscr{Y}^{\text{rig}}$. Denote also by $(\mathscr{E}_v, D_v), (\mathscr{E}_0, D_0)$ the restrictions of the sheaves with connections $(\mathscr{E}_{\mathfrak{X}}, D_{X/S})$ and $(\mathscr{E}_{\mathscr{Y}}, D_Y)$ to W_v and respectively W_0. The isomorphism α induces the vertical isomorphisms in the following commutative diagram

$$\begin{array}{ccc} \mathscr{E}_v & \xrightarrow{D_v} & \mathscr{E}_v \otimes_{\mathscr{O}_{W_v}} \Omega^1_{W_v/B} \\ \downarrow\cong & & \downarrow\cong \\ \mathscr{E}_0 \otimes_L \mathscr{O}_B & \xrightarrow{D_0 \otimes \text{id}_B} & \mathscr{E}_0 \otimes_{\mathscr{O}_{W_0}} \Omega^1_{W_0/L} \end{array}$$

This implies

Lemma 5.15. — a) *The L-vector space $H^1_{dR}(W_0/L, \mathscr{E}_0)$ is finitely generated.*

b) *We have a natural isomorphism of sheaves on B induced by α: $H^1_{dR}(W_v/B, \mathscr{E}_v) \cong H^1_{dR}(W_0/L, \mathscr{E}_0) \otimes_L \mathscr{O}_B$.*

Proof. — a) is a consequence of lemma 5.2 and b) follows from the above commutative diagram. □

Let us fix $\omega_1, \omega_2, \dots, \omega_n$ global sections of $\mathscr{E}_0 \otimes_{\mathscr{O}_{W_0}} \Omega^1_{W_0/L}$ whose cohomology classes $[\omega_1], \dots, [\omega_n]$ form an L-basis of $H^1_{dR}(W_0/L, \mathscr{E}_0)$. Let now ω be a global section of $\mathscr{E}_v \otimes_{\mathscr{O}_{W_v}} \Omega^1_{W_v/B}$ and denote by $[\omega] \in H^1_{dR}(W_v/B, \mathscr{E}_v)(B)$ its cohomology class. Then $[\omega] = \sum_{i=1}^n a_i[\omega_i]$ for $a_i \in \mathscr{O}_B(B)$, $i = 1, n$ and therefore we have

$$\omega = \sum_{i=1}^n a_i\omega_i + D_v(f) \text{ for some } f \in \mathscr{E}_v(W_v).$$

Let us fix $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathscr{E}_0^{\mathrm{flog}}(W_0)$ p-adic integrals of $\omega_1, \dots, \omega_n$ (see section 5.4.)

We denote by $\lambda_\omega := \sum_{i=1}^n a_i\lambda_i + f \in (\mathscr{E}_0^{\mathrm{flog}} \otimes_L \mathscr{O}_B)(W_v)$ and call it a p-adic integral of ω. It is well defined up to an element of $\mathscr{E}_v(W_v)^{D_v}$.

We have the following,

Lemma 5.16. — a) *With the notations above, λ_ω is a family of p-adic integrals of ω, i.e.*

i) $D_v(\lambda_\omega) = \omega$

and

ii) *for every $s \in B$, $\lambda_\omega|_{W_{v,s}}$ is a p-adic integral of $\omega|_{W_{v,s}}$.*

b) *If $\overline{\omega}$ is the natural lift of ω to $\mathscr{E}_v \otimes_{\mathscr{O}_{W_v}} \Omega^1_{W_v/L}(\log(W_0))$ defined in section 4.2, and η is defined by the equality $D_{\mathrm{W_v/L}}(\overline{\omega}) = \eta \wedge dy$, then*

$$\overline{\omega} - D_{\mathrm{W_v/L}}(\lambda_\omega) = \lambda_\eta dy.$$

Proof. — a) is clear and for b) let us write

$$\omega = \sum_{i=1}^n a_i(y)\omega_i + D_v(f),$$

where $a_i(y) \in \mathscr{O}_B(B)$, $f \in \mathscr{E}_v(W_v)$ and the ω_i's have been defined above. Then we have

$$\overline{\omega} = \sum_{i=1}^n a_i(y)\omega_i + D_{\mathrm{W_v/L}}(f)$$

and therefore $\eta = -\sum_{i=1}^n a_i'(y)\omega_i$ and

$$\overline{\omega} - D_{\mathrm{W_v/L}}(\lambda_\omega) = -(\sum_{i=1}^n a_i'(y)\lambda_i)dy = \lambda_\eta dy. \qquad \square$$

Let us choose now for the rest of this section the branch of the logarithm on $\mathbb{C}_p^\times$ such that $\log(\pi_L) = 0$.

We will give a general definition: let Z be a rigid space over L and let $\alpha : M \longrightarrow \mathscr{O}_Z$ be an integral log structure, where M is a sheaf of monoids.

Then if $W \subset Z$ is a admissible open subspace which is Stein we define $\mathscr{O}_Z(W)_{\log}$ to be the polynomial ring $\mathscr{O}_Z(W)[\ell(m)]_{m \in M(W)}$, where $\ell(m)$ are independent variables, divided by the relations: $\ell(m_1m_2) = \ell(m_1) + \ell(m_2)$ and $\ell(m) = \log(\alpha(m))$ if $\alpha(m) \in \mathscr{O}_Z(W)^\times$.

The natural derivation $d : \mathscr{O}_Z(W) \longrightarrow \Omega^1_{W/L}$ extends canonically to a derivation $d : \mathscr{O}_Z(W)_{\log} \longrightarrow \Omega^1_{W/L}(\log(M))$ by defining $d(\ell(m)) = d(\alpha(m))/\alpha(m)$ for $m \in M(W)$.

In particular, let us consider the log structure on B given by the divisor $0 \in B$ and choose a parameter $y \in \mathscr{O}_B(B)$ at 0. Then it is easy to see that $\mathscr{O}_B(B)_{\log} = \mathscr{O}_B(B)[\ell(y)]$ and we have $d(\ell(y)) = dy/y$.

Let $e \in e(G)$ and we denote in this section by $A_e := A_{e,B}$ and by $A_0 := A_{e,0}$ the fiber of A_e at $0 \in B$. If we consider on A_e the log structure given by the divisor over B with normal crossings A_0, we see that $\mathscr{O}_{A_e}(A_e)_{\log} = \mathscr{O}_{A_e}(A_e)[\ell(x_e), \ell(x_{\tau(e)})]$ with unique relation $\ell(x_e) + \ell(x_{\tau(e)}) = \ell(y)$. We have $d_{A_e/B}(\ell(x_e)) = d_{A_e/B}(x_e)/x_e$ and $d_{A_e/B}(\ell(x_{\tau(e)})) = d_{A_e/B}(x_{\tau(e)})/x_{\tau(e)}$.

We also denote by $(\mathscr{E}_e, D_e)$ the restriction of the sheaf with connection $(\mathscr{E}_{\mathfrak{X}}, D_{\mathrm{X/S}})$ to A_e. Let ω be a global section of the sheaf $\mathscr{E}_e \otimes_{\mathscr{O}_{A_e}} \Omega^1_{A_e/B}(\log A_0))$ and denote by $\epsilon_1, \ldots, \epsilon_\alpha$ a basis of horizontal sections of $(\mathscr{E}_e, D_e)$. Then using lemma 4.8 we can write

$$(*) \quad \omega = \sum_{i=1}^{\alpha} \epsilon_i \otimes r_i(y) \frac{d_{\mathrm{A_e/B}}(x_e)}{x_e} + D_e(u_e),$$

where $r_i(y) \in \mathscr{O}_B(B)$ and $u_e \in \mathscr{E}_e(A_e)$.

We set

$$\lambda_\omega := \sum_{i=1}^{\alpha} \epsilon_i \otimes r_i(y)\ell(x_e) + u_e \in \mathscr{E}_{\mathrm{e,log}} := \mathscr{E}_{\mathrm{e}}(A_e) \otimes_{\mathscr{O}_{A_e}(A_e)} \mathscr{O}_{A_e}(A_e)_{\log}.$$

Lemma 5.17. — *We have,*

a) *With the notations above λ_ω is a family of p-adic integrals of ω in the sense that*

i) $D_e(\lambda_\omega) = \omega$

and

ii) *λ_ω is an element of $\mathscr{E}_{\mathrm{e,log}}$ well defined up to an element of of $\mathscr{E}_e(A_e)^{\mathrm{D_e}}[\ell(y)] := \mathscr{E}(A_e)^{\mathrm{D_e}} \otimes_{\mathscr{O}_B(B)} \mathscr{O}_B(B)[\ell(y)]$.*

b) *Let $\widetilde{\omega}$ denote the lift of ω to absolute one-forms as in section 4.2 and let η be defined by the equality $D_{\mathrm{A_e/L}}(\widetilde{\omega}) = \eta \wedge dy$. Then $\widetilde{\omega} - D_{\mathrm{A_e/L}}(\lambda_\omega) = \lambda_\eta dy$.*

Proof. — Part i) of a) is clear and for part ii) let us remark that $(\mathscr{E}_{\mathrm{e,log}})^{\mathrm{D_e}} = \mathscr{E}(A_e)^{\mathrm{D_e}}[\ell(y)]$. For b) let us notice that

$$D_{\mathrm{A_e/L}}(\widetilde{\omega}) = -\sum_{i=1}^{\alpha} \epsilon_i \otimes r_i'(y) \frac{d_{\mathrm{A_e/L}}(x_e)}{x_e} \wedge dy,$$

and clearly

$$\widetilde{\omega} - D_{\mathrm{A_e/L}}(\lambda_\omega) = -(\sum_{i=1}^{\alpha} \epsilon_i \otimes r'(y)\ell(x_e))dy = \lambda_\eta dy. \qquad \square$$

Now we will use the p-adic integration discussed above in order to describe the Frobenius operator on $\mathbb{H}_B$. Let us remark that the collection $\mathscr{C}''_B = \{W_v, A_e\}_{v\in v(G), e\in e(G)}$ is an admissible cover of X_B by admissible, acyclic, wide open subsets over B. We will define an $\mathscr{O}_B$-linear map,

$$s_B : \mathbb{H}_B \longrightarrow H^{1,0}(\mathscr{C}''_B, \mathscr{E})_{\log} := H^{1,0}(\mathscr{C}''_B, \mathscr{E}) \otimes_{\mathscr{O}_B(B)} \mathscr{O}_B(B)[\ell(y)]$$

as follows: let $\omega \in \mathbb{H}_B$ be represented by the hypercocycle with respect to the covering $\mathscr{C}''_B$

$$\big((\omega_v)_{v\in v(G)}, (\omega_e)_{e\in e(G)}, (f_e)_{e\in e(G)}, (\overline{f}_e)_{e\in e(G)}\big).$$

where let us recall: $\omega_v \in (\mathscr{E}_v \otimes_{\mathscr{O}_{W_v}} \Omega^1_{W_v/B})(W_v)$, $\omega_e \in (\mathscr{E}_e \otimes_{\mathscr{O}_{A_e}} \Omega^1_{A_e/B}(\log(A_0))(A_e)$, $f_e \in \mathscr{E}_e(W_{a(e)} \cap A_e)$ and $\overline{f}_e \in \mathscr{E}_e(W_{b(e)} \cap A_e)$ satisfying the usual cocycle conditions.

For every $e \in e(G)$, let $s_B(\omega)_e$ be the section:

$$f_e - (\lambda_{\omega_{a(e)}}|_{W_{a(e)}\cap A_e} - \lambda_{\omega_e}|_{W_{a(e)}\cap A_e}),$$

and similarly let $(\overline{s}_B(\omega))_e$ be the section

$$\overline{f}_e - (\lambda_{\omega_{b(e)}}|_{W_{b(e)}\cap A_e} - \lambda_{\omega_e}|_{W_{b(e)}\cap A_e}).$$

Lemma 5.18. — *For every $e \in e(G)$ and $\omega \in \mathbb{H}_B$, $\big(s_B(\omega)_e, (\overline{s}_B(\omega))_e\big) \in \mathscr{E}_e^{\mathrm{D_e}}(W_{a(e)} \cap A_e)[\ell(y)] \oplus \mathscr{E}_e^{\mathrm{D_e}}(W_{b(e)} \cap A_e)[\ell(y)]$.*

Proof. — We will only prove that $s_B(\omega) \in \mathscr{E}^{\mathrm{D_e}}(W_{a(e)} \cap A_e)[\ell(y)]$, and leave the remaining similar argument to the reader. The isomorphism $\alpha_{a(e),0}$ induces an isomorphism

$$\alpha : W_{a(e)} \cap A_e \cong B \times U_0,$$

where U_0 is the annulus $W_{a(e),0} \cap A_{e,0}$. Let π_i for $i = 1, 2$ be the projections of $B \times U_0$ composed with α and denote by $x_0 := \pi_2^*(x_e|_{W_{a(e)}\cap A_e})$. Then x_0 is a parameter of U_0 (see the beginning of section 4.) If we write ω_e as in formula (*) before lemma (5.17) and use the isomorphism α above, we may integrate $\omega_e|_{W_{a(e)}\cap A_e}$ by the recipe outlined in lemma 5.16. Let us denote this integral by λ. We have

$$s_B(\omega)_e = f_e - (\lambda_{\omega_{a(e)}}|_{W_{a(e)}\cap A_e} - \lambda + \lambda - \lambda_{\omega_e}|_{W_{a(e)}\cap A_e}).$$

First let us first remark that $x_0 \in \mathscr{O}_{U_0}(U_0)^\times$ therefore $\ell(x_0) = \log(x_0)$ and that $\mathscr{O}_{U_0}[\log(f)]_{f\in\mathscr{O}_{U_0}^\times} = \mathscr{O}_{U_0}[\log(x_0)]$. Indeed every element $f \in \mathscr{O}_{U_0}(U_0)^\times$ can be written $f = ax_0^n g$, with $a \in L^\times$, $n \in \mathbb{Z}$ and $g \in \mathscr{O}_{U_0}(U_0)$ is such that $|g - 1| < 1$. Therefore $\log(f) = \log(a) + n\log(x_0) + \log(g)$, where $\log(g) \in \mathscr{O}_{U_0}(U_0)$.

As $W_{a(e)} \cap A_e$ is contained in the residue class A_e of X_B, $(\mathscr{E}_e, D_e)$ has a basis of horizontal sections on $W_{a(e)} \cap A_e$ and so we have

$$\big(\mathscr{E}_e((W_{a(e)} \cap A_e))[\log(x_0)]\big)^{\mathrm{D_e}} = \mathscr{E}_e((W_{a(e)} \cap A_e))^{\mathrm{D_e}}.$$

This implies that $f_e - \lambda_{\omega_{a(e)}}|_{W_{a(e)}\cap A_e} + \lambda \in \mathscr{E}_{A_e}(W_{a(e)} \cap A_e)[\ell(y)]$.

Let us remark that $x_0 = ux_e$, where $u \in \mathscr{O}_{A_e}(W_{a(e)} \cap A_e)^*$ such that $\log(u)$ is an analytic function on $W_{a(e)} \cap A_e$. Therefore lemma 5.17 shows that $\lambda - \lambda_{\omega_e}|_{W_{a(e)}\cap A_e} \in \mathscr{E}_e(W_{a(e)} \cap A_e)[\ell(y)]$. Now the fact that $D_e(s_B(\omega)_e) = 0$ implies the lemma. □

For every $\omega \in \mathbb{H}_B$ denote by $s_B(\omega)$ the class of the cocycle $(s_B(\omega)_e, \overline{s}_B(\omega)_e)_{e\in e(G)}$ in $H^{1,0}(\mathscr{C}_B'', \mathscr{E})_{\log}$ and by $s_B : \mathbb{H}_B \longrightarrow H^{1,0}(\mathscr{C}_B'', \mathscr{E})_{\log}$ the respective $\mathscr{O}_B$-linear homomorphism. Composing s_B with the inclusion $H^{1,0}(\mathscr{C}_B'', \mathscr{E})_{\log} \longrightarrow \mathbb{H}_{\mathrm{B,log}}$ obtained from (2), we may think of s_B as an $\mathscr{O}_B$-linear map from $\mathbb{H}_B$ to $\mathbb{H}_{\mathrm{B,log}}$. We have,

Theorem 5.19. — a) $s_B : \mathbb{H}_B \longrightarrow H^{1,0}(\mathscr{C}_B'', \mathscr{E})_{\log}$ *is a section of the inclusion.*
b) *For every $u \in B^* = B - \{0\}$, the fiber $s_{B,u}$ of s_B at u coincides with the map s_u defined in section 2.2.*
c) *We have $(s_B \otimes 1) \circ \nabla = \nabla \circ s_B$.*
d) *Let B^1 and B^2 as in section 5.3. We have $\Phi^f \circ s_{B^1} = s_{B^2} \circ \Phi^f$.*

Proof. — a) Let $x \in H^{1,0}(\mathscr{C}_B'', \mathscr{E})$ be represented by the cocycle $\big((f_e), (\overline{f}_e)\big)_{e\in e(G)}$. Then the image of x in $\mathbb{H}_B$ is the class of the hypercocycle:
$\big((0_v)_{v\in v(G)}, (0_e)_{e\in e(G)}, (f_e)_{e\in e(G)}, (\overline{f}_e)_{e\in e(G)}\big)$ and clearly the image of this class under s_B is x.

For b) if $u \in B^*$ we denote $\mathscr{C}_u'' = \{W_{v,u}, A_{e,u}\}$ the intersection of the cover $\mathscr{C}_B''$ with the fiber X_u. Let $\mathscr{C}_u = \{U_{v,u}\}_{v\in v(G)}$ denote the wide open cover of X_u described in section 2.2. We denote by $\mathscr{E}_u$ the restriction of $\mathscr{E}_{\mathfrak{X}}$ to the fiber X_u. We have the following diagram

$$\begin{array}{ccc} H^1_{dR}(X_u, \mathscr{E}_u) & \stackrel{s_{B,u}}{\longrightarrow} & H^{1,0}(\mathscr{C}_u'', \mathscr{E}_u) \\ || & & \downarrow\cong \\ H^1_{dR}(X_u, \mathscr{E}_u) & \stackrel{s_u}{\longrightarrow} & H^{1,0}(\mathscr{C}_u, \mathscr{E}_u) \end{array}$$

where the right vertical isomorphism is the one defined in section §3.5.4. Lemma 3.34 implies that the diagram is commutative and this proves b).

Let us now prove c). Let $\omega \in \mathbb{H}_B$ and let

$$\big((\omega_v)_{v\in v(G)}, (\omega_e)_{e\in e(G)}, (f_e)_{e\in e(G)}, (\overline{f}_e)_{e\in e(G)}\big)$$

be a hypercocycle with respect to the covering $\mathscr{C}_B''$ representing the class ω. Let $\overline{\omega}_v$ and $\widetilde{\omega}_e$ be the lifts of ω_v and ω_e respectively to absolute one-forms defined in section §4.2. Let $D_{\mathrm{X_B/L}}\overline{\omega}_v = \eta_v \wedge dy$, $D_{\mathrm{X_B/L}}\widetilde{\omega}_e = \eta_e \wedge dy$, $\overline{\omega}_{a(e)}|_{W_{a(e)}\cap A_e} - \widetilde{\omega}_e|_{W_{a(e)}\cap A_e} - D_{\mathrm{X_B/L}}(f_e) = g_e dy$ and $\overline{\omega}_{b(e)}|_{W_{b(e)}\cap A_e} - \widetilde{\omega}_e|_{W_{b(e)}-A_e}) - D_{\mathrm{X_B/L}}(\overline{f}_e) = \overline{g}_e dy$ for η_v, η_e, g_e and $\overline{g}_e$ global sections of $\mathscr{E}_v \otimes \Omega^1_{W_v/B}(\log W_0)$, $\mathscr{E}_e \otimes \Omega^1_{A_e/B}(\log A_0)$, $\mathscr{E}_{a(e)}|_{W_{a(e)}\cap A_e}$ $\mathscr{E}_{b(e)}|_{W_{b(e)}\cap A_e}$ respectively. Then $(s_B \otimes 1)(\nabla\omega)$, as an element of $\mathbb{H}_{B,log} \otimes dy$, is represented by the hypercocycle

$$\big((0_v)_{v\in v(G)}, (0_e)_{e\in e(G)}, (g_e - (\lambda_{\eta_{a(e)}}|_{W_{a(e)}\cap A_e} - \lambda_{\eta_e}|_{W_{a(e)}\cap A_e}))_{e\in e(G)},$$
$$(\overline{g}_e - (\lambda_{\eta_{b(e)}}|_{W_{b(e)}\cap A_e} - \lambda_{\eta_e}|_{W_{b(e)}\cap A_e}))_{e\in e(G)}\big) \otimes dy.$$

On the other hand $\nabla(s_B(\omega))$ is represented by the hypercocycle

$$\big((0_v)_{v\in v(G)}, (0_e)_{e\in e(G)}, (-D_{\mathrm{X_B/L}}(f_e) + D_{\mathrm{X_B/L}}\lambda_{\omega_{a(e)}}|_{W_{a(e)}\cap A_e} - D_{\mathrm{X_B/L}}\lambda_{\omega_e}|_{W_{a(e)}\cap A_e})_{e\in e(G)},$$
$$(-D_{\mathrm{X_B/L}}(\overline{f}_e) + D_{\mathrm{X_B/L}}\lambda_{\omega_{b(e)}}|_{W_{b(e)}\cap A_e} - D_{\mathrm{X_B/L}}\lambda_{\omega_e}|_{W_{b(e)}\cap A_e})_{e\in e(G)}\big) \otimes dy.$$

A calculation using the lemmas 5.16 and 5.17 shows that the two hypercocycles are cohomologous.

Now we prove d). For this let us recall the notations B^1, B^2 and the expression of Φ^f at the end of section 5.3. Let U_v^i, $i = 1, 2$ and $v \in v(G)$ denote admissible wide open subsets of X_{B^i} satisfying the properties of proposition 5.10 and the additional property that there are isomorphisms $\alpha_{v,i} : U_v^i \cong U_{v,0}^i \times B^i$. As in section §5.3 we consider the admissible covers $\mathscr{C}^i = \{U_v^i, A_e^i\}$ of X_{B^i}. Let the class $\omega \in \mathbb{H}_{B^2}$ be represented by the hypercocycle for the covering $\mathscr{C}^2$

$$\big((\omega_v)_{v\in v(G)}, (\omega_e)_{e\in e(G)}, (f_e)_{e\in e(G)}, (\overline{f}_e)_{e\in e(G)}\big).$$

Then $s_{B^2}(\omega)$ is represented by the hypercocycle

$$\big((0_v)_{v\in v(G)}, (0_e)_{e\in e(G)}, (g_e)_{e\in e(G)}, (\overline{g}_e)_{e\in e(G)}\big)$$

where $g_e = f_e - (\lambda_{\omega_{a(e)}}|_{U^2_{a(e)}\cap A^2_e} - \lambda_{\omega_e}|_{U^2_{a(e)}\cap A^2_e})$ and $\overline{g}_e = \overline{f}_e - (\lambda_{\omega_{b(e)}}|_{U^2_{b(e)}\cap A^2_e} - \lambda_{\omega_e}|_{U^2_{b(e)}\cap A^2_e})$.

Then $\Phi^f(s_{B^2}(\omega))$ is represented by

$$\big((0_v)_{v\in v(G)}, (0_e)_{e\in e(G)}, (F_e(\phi_e^*(g_e)))_{e\in e(G)}, (F_e(\phi_e^*(\overline{g}_e)))_{e\in e(G)}\big).$$

Let us recall from the end of the section §5.3 that $\Phi^f(\omega)$ is represented by the hypercocycle

$$\big((\nu_v)_{v\in v(G)}, (\nu_e)_{e\in e(G)}, (h_e)_{e\in e(G)}, (\overline{h}_e)_{e\in e(G)}\big)$$

where $\nu_v, \nu_e, h_e, \overline{h}_e$ are defined there.

Therefore, $s_{B^1}(\Phi^f(\omega))$ is represented by

$$\big((0_v)_{v\in v(G)}, (0_e)_{e\in e(G)}, (x_e)_{e\in e(G)}, (\overline{x}_e)_{e\in e(G)}\big)$$

with (see the end of section §5.3)

$$x_e = h_e - (\lambda_{\nu_{a(e)}}|_{U^1_{a(e)}\cap A^1_e} - \lambda_{\nu_e}|_{U^1_{a(e)}\cap A^1_e}) =$$
$$= \Delta^*(F_{a(e)}\circ\phi^*_{a(e)}, F_e\circ\phi_e)(\epsilon_e) + F_e(\phi_e^*(f_e)) - (F_{a(e)}(\phi^*_{a(e)}(\lambda_{\omega_{a(e)}}))|_{U^1_{a(e)}\cap A^1_e} - F_e(\phi_e^*(\lambda_{\omega_e}))|_{U^1_{a(e)}\cap A^1_e}).$$

Now we use the fact that $\epsilon_e = \pi_1^*(\lambda_{\omega_{a(e)}}|_{U^2_{a(e)}\cap A^2_e}) - \pi_2^*(\lambda_{\omega_e}|_{U^2_{a(e)}\cap A_e})$ and obtain

$$x_e = F_e(\phi_e^*(f_e - \lambda_{\omega_{a(e)}}|_{U^2_{a(e)}\cap A^2_e} + \lambda_{\omega_e}|_{U^2_{a(e)}\cap A^2_e})).$$

Similarly

$$\overline{x}_e = \overline{g}_e - (\lambda_{\nu_{b(e)}}|_{U^1_{b(e)}\cap A^1_e} - \lambda_{\nu_e}|_{U^1_{b(e)}\cap A^1_e}) =$$
$$= F_e(\phi_e^*(\overline{f}_e - \lambda_{\omega_{b(e)}}|_{U^2_{b(e)}\cap A^2_e} + \lambda_{\omega_e}|_{U^2_{b(e)}\cap A^2_e})).$$

This ends the proof of Theorem 5.19. □

Now we can finish the **proof of Theorem 2.6**. To prove that Φ_{deg} and Φ_{int} get identified by parallel transport. We have exact sequences

$$0 \longrightarrow H^{1,0}(C)\otimes_{K_0} L \longrightarrow H^1(C,\mathscr{E})\otimes_K L \longrightarrow H^{0,1}(C)\otimes_{K_0} L \longrightarrow 0$$

and

$$0 \longrightarrow H^{1,0}(\mathscr{C}_0'', \mathscr{E}_0) \longrightarrow H^1(Y, \mathscr{E}_0) \longrightarrow H^{0,1}(\mathscr{C}_0'', \mathscr{E}_0) \longrightarrow 0.$$

Proposition 3.35 implies that under the parallel transport isomorphism $H^1(Y, \mathscr{E}_0) \otimes_{K_0} L \cong H^1(C, \mathscr{E}) \otimes_K L$, $H^{1,0}(C)$ gets identified with $H^{1,0}(\mathscr{C}_0'', \mathscr{E}_0)$ and $H^{0,1}(C)$ gets identified with $H^{0,1}(\mathscr{C}_0'', \mathscr{E}_0)$. Moreover these last two isomorphisms commute with the respective Frobenii. We'll first show that Φ^f_{deg} corresponds to Φ^f_{int}. Let us parallel transport Φ^f_{deg} to $H^1(C, \mathscr{E}) \otimes_{K_0} L$ and let us denote by Φ^π_{deg} this endomorphism, i.e., if $\omega \in (\mathbb{H}_{\log})^\nabla$, we have seen that $(\Phi^f(\omega))_0 = \Phi^f_{\mathrm{deg}}(\omega_0)$ and as $\Phi(\omega) \in (\mathbb{H}_{\log})^\nabla$ we set $\Phi^\pi_{\mathrm{deg}}(\omega_\pi) = (\Phi^f(\omega))_\pi$. We have to show that $\Phi^\pi_{\mathrm{deg}} = \Phi^f_{\mathrm{int}}$ and so far we know that Φ^f_{int} and Φ^π_{deg} coincide both on the image of $H^{1,0}(C)$ and on the quotient $H^{0,1}(C)$ and $s_\pi \circ \Phi^f_{\mathrm{int}} = F^f_{0,\mathrm{cris}} \circ s_\pi$. Using Theorem 5.19 we have

$$s_\pi \circ \Phi^\pi_{\mathrm{deg}} = (s_{B^2} \circ \Phi^f)_\pi = (\Phi^f \circ s_{B^1})_\pi = F^f_{0,\mathrm{cris}} \circ s_\pi.$$

This proves that $\Phi^\pi_{\mathrm{deg}} = \Phi^f_{\mathrm{int}}$. Moreover, since $\mathscr{E}$ is regular it follows that the characteristic polynomials of $F_{0,\mathrm{cris}}$ on $H^{0,1}(C)$ and of $F_{1,\mathrm{cris}}$ on $H^{1,0}(C)$ are relatively prime. Thus both exact sequences above have natural Frobenii equivariant splittings and as $\Phi^\pi_{\mathrm{deg}} = \Phi^f_{\mathrm{int}}$, the splittings coincide under parallel transport. But the splitting produced by Φ^f_{int} is s_π, therefore we immediately deduce that $H^1(C, \mathscr{E})_{\mathrm{int}}$ and $H^1(Y, \mathscr{E}_0)$ become identified by parallel transport and the same is true for Φ_{int} and Φ_{deg}. This completes the proof of Theorem 2.6.

6. Logarithmic F-isocrystals

We start by defining the main objects of this section, the log F-isocrystals.

Let C be our semi-stable curve over V, let P be a finite set of smooth sections of C and $C^\times$ the corresponding log scheme. Let $\overline{P}$ be the special fiber of P. Then $\overline{P}$ is a smooth divisor of $\overline{C}$ and we denote, to the end of this section, by $\overline{C}^\times$ the corresponding log scheme.

Definition 6.1. — *A logarithmic enlargement of $\overline{C}^\times$ is a pair $(T^\times, z_T)$ consisting of a formal log scheme $T^\times$ and a morphism of log schemes $z_T : T_0^\times \to \overline{C}^\times$. If $(U^\times, z_U)$ and $(T^\times, z_T)$ are two log enlargements of $\overline{C}^\times$ then a morphism of log enlargements $g : (U^\times, z_U) \to (T^\times, z_T)$ is a morphism of formal log schemes $g : U^\times \to T^\times$ such that $z_T \circ g_0 = z_U$.*

Definition 6.2. — *A log isocrystal $\mathscr{E}$ on $\overline{C}^\times$ is the following set of data*

i) *for every log enlargement $(T^\times, z_T)$ of $\overline{C}^\times$ a coherent $K_0 \otimes_W \mathscr{O}_T$-module $\mathscr{E}_{(T^\times, z_T)}$ (sometimes in what follows we will use the shorthand notation $\mathscr{E}_{T^\times}$.)*

ii) *for every morphism of enlargements $g = (f, h) : (U^\times, z_U) \longrightarrow (T^\times, z_T)$ an isomorphism of $K_0 \otimes_U \mathscr{O}_W$-modules $\theta_g : f^{-1}\mathscr{E}_T \longrightarrow \mathscr{E}_U$. The collection $\{\theta_g\}$ is required to satisfy the cocycle condition.*

Remark 6.3. — *If $\mathscr{E}$ is a log isocrystal on $\overline{C}^{\times}$ and $(T^{\times}, z_T)$ is a log enlargement of $\overline{C}^{\times}$ such that the formal scheme T is locally Noetherian then one may interpret $\mathscr{E}_{T^{\times}}$ as a coherent sheaf on T^{rig}, the rigid analytic space associated to T. Moreover, applying the results in §6 of* [**26**] *one sees that $\mathscr{E}_T$ is endowed with an integrable connection*

$$D_T : \mathscr{E}_{T^{\times}} \longrightarrow \mathscr{E}_{\times}T \otimes_{\mathscr{O}_T} \omega_{T^{\times}/W^{\times}},$$

where $T^{\times} = (T, M_{T^{\times}})$ and $W^{\times}$ is the formal scheme $\mathrm{Spf}(W)$ *with the trivial log structure.*

Let now $k^{\times}$ denote the scheme $\mathrm{Spec}(k)$ with trivial log-structure and let $W^{\times}$ be the formal log scheme $\mathrm{Spf}(W)$ with trivial log structure. We denote by σ be the absolute Frobenius on $k^{\times}$ and on $W^{\times}$, respectively. Let us recall that σ is the absolute Frobenius on the respective schemes and multiplication by p on the respective monoids. Let now $f : A^{\times} \longrightarrow B^{\times}$ be a morphism of fine log schemes (or fine formal log schemes), where $B^{\times}$ is either $k^{\times}$ or $W^{\times}$. We'll denote by $(A^{\times})^{\sigma}$ the fiber product in the category of log schemes of the diagram

$$\begin{array}{ccc} & & A^{\times} \\ & & \downarrow \\ B^{\times} & \xrightarrow{\sigma} & B^{\times}. \end{array}$$

Let now $B^{\times}$ be $k^{\times}$, then we denote by $\overline{F} = F_{(A^{\times},k^{\times})} : A^{\times} \longrightarrow (A^{\times})^{\sigma}$ the morphism induced by the pair of maps: $f : A^{\times} \longrightarrow k^{\times}$ and the map form $A^{\times}$ to itself which is the identity on the underlying topological space, is $s \rightarrow s^p$ on $\mathscr{O}_A$ and is multiplication by p on M_A. If now, $(T^{\times}, z_T)$ is a log enlargement of $\overline{C}^{\times}$ then $(T^{\times}, \overline{F} \circ z_T)$ is a log enlargement of $(\overline{C}^{\times})^{\sigma}$ and $((T^{\times})^{\sigma^{-1}}, (\overline{F} \circ z_T)^{\sigma^{-1}})$ is again a log enlargement of $\overline{C}^{\times}$. If $\mathscr{E}$ is a log isocrystal on $\overline{C}^{\times}$ then we will denote by $\overline{F}^{*}\mathscr{E}$ the log isocrystal on $\overline{C}^{\times}$ such that

$$\overline{F}^{*}\mathscr{E}_{(T^{\times},z_T)} = \mathscr{E}_{((T^{\times})^{\sigma^{-1}},(\overline{F}\circ z_T)^{\sigma^{-1}})}.$$

Definition 6.4. — *A log F-isocrystal on $\overline{C}^{\times}$ is a log isocrystal on $\overline{C}^{\times}$, $\mathscr{E}$, together with an isomorphism of log isocrystals*

$$F : \overline{F}^{*}\mathscr{E} \longrightarrow \mathscr{E}.$$

Let C be a curve over V as in Section 2.1 and let P denote a finite collection of smooth sections of C over V, such that their image in $\overline{C}$ is the collection $\overline{P}$. By deformation theory the pair (C, P) may be regarded as the fiber at the point π of the formal model of the open unit disk $\mathscr{S}$ over W, of a pair $(\mathfrak{X}, \mathscr{P})$ consisting of a family of curves $\mathfrak{X} \rightarrow \mathscr{S}$ as in Section 2.1 and a smooth divisor $\mathscr{P}$ of $\mathfrak{X}$. We have a natural morphism of log schemes $z_{\mathfrak{X}} : (\mathfrak{X}^{\times}_{\mathscr{P}})_0 \rightarrow (C^{\times}_P)_0 = \overline{C}^{\times}$ so may regard $(\mathfrak{X}^{\times}, z_{\mathfrak{X}})$ (and any of its fibers above points of $\mathscr{S}$) as a log enlargement of $\overline{C}^{\times}$. Let now $\mathscr{E}$ be a log F-isocrystal on $\overline{C}^{\times}$. Denote by $X = \mathfrak{X}^{\mathrm{rig}}$ the rigid analytic space attached to $\mathfrak{X}$ and by P_X the intersection of the divisor $\mathscr{P}$ with X. Let us denote by $\mathscr{E}_{\mathfrak{X}^{\times}}$ the evaluation of the log F-isocrystal $\mathscr{E}$ on $(\mathfrak{X}^{\times}_{\mathscr{P}_X}, z_{\mathfrak{X}})$. It is a coherent sheaf of $\mathscr{O}_X$-modules with an integrable connection

$$\mathrm{D}_{X/K_0} : \mathscr{E}_{\mathfrak{X}^\times} \longrightarrow \mathscr{E}_{\mathfrak{X}^\times} \otimes_{\mathscr{O}_X} \Omega^1_{X/K_0}(\log P_X).$$

Composing D_{X/K_0} with the natural projections

$$\mathscr{E}_{\mathfrak{X}^\times} \otimes_{\mathscr{O}_X} \Omega^1_{X/K_0}(\log P_X) \longrightarrow \mathscr{E}_{\mathfrak{X}^\times} \otimes_{\mathscr{O}_X} \Omega^1_{X/K_0}(\log(P_X \cup Y)) \longrightarrow \mathscr{E}_{\mathfrak{X}^\times} \otimes_{\mathscr{O}_X} \Omega^1_{X/S}(\log(P_X \cup Y))$$

we get a relative integrable connection over S

$$\mathrm{D}_{X/S} : \mathscr{E}_{\mathfrak{X}^\times} \longrightarrow \mathscr{E}_{\mathfrak{X}^\times} \otimes_{\mathscr{O}_X} \Omega^1_{X/S}(\log(P_X \cup Y)).$$

Remark 6.5. — *$P_X \cup Y$ is a divisor of X with normal crossings and $P_X \cap Y$ is a finite set of smooth points of Y.*

Let us consider now, as in Section 2.1, $\mathbb{H}^i_P = H^i_{dR}(X/S, \mathscr{E}_{\mathfrak{X}^\times}(\log(P_X \cup Y)))$, for $i = 0, 1, 2$ with its logarithmic connection

$$\nabla^i \colon \mathbb{H}^i_P \longrightarrow \mathbb{H}^i_P \otimes_{\mathscr{O}_S} \Omega^1_S(\log 0),$$

and its Frobenius $\Phi_i \colon \varphi^*\mathbb{H}^i_P \to \mathbb{H}^i_P$. For every point $s \in S$ let us denote by P_s the fiber of P_X above s and by $\mathscr{E}_s = \mathscr{E}_{\mathfrak{X}^\times}|_{X_s}$. Then we have

a) if $s \in S - \{0\}$ then $H^i(C_s, P_s, \mathscr{E}) := \mathbb{H}^i_{P,s} \cong H^i_{dR}(X_s, \mathscr{E}_s(\log(P_s)))$

b) if $s = 0$ then $H^i(Y, P_0, \mathscr{E}) := \mathbb{H}^i_{P,0} \cong H^i_{dR}(Y^{\times\times}/K_0, \mathscr{E}_0)$, where let us recall $Y^{\times\times}$ is the log rigid space Y with inverse image log structure from the one on X induced by the divisor $P_X \cup Y$.

Lemma 6.6. — *Let $\mathscr{E}$ be a log isocrystal on $\overline{C}^\times$. Then $(\mathscr{E}_{\mathfrak{X}^\times}, \mathrm{D}_{X/K_0})$ has the property that for every residue class $M = red^{-1}(x)$, with $x \in \overline{C} - \overline{P}$, of X, the $\mathscr{O}_M$-module with connection $(\mathscr{E}_{\mathfrak{X}^\times}|_M, \mathrm{D}_{X/K_0})$ has a basis of horizontal sections.*

Definition 6.7. — *Let $\mathscr{E}$ be a log F-isocrystal on $\overline{C}^\times$, and $\overline{P}$ a smooth divisor on $\overline{C}$. We say $\mathscr{E}$ is* regular outside of $\overline{P}$ *if for every vertex $v \in v(G)$ and for every closed point $x \in \overline{C}_v - \overline{P}$ the characteristic polynomials of Frobenii on $H^0_{\mathrm{cris}}(x, \mathscr{E})$ and $H^1_{\mathrm{cris}}(\overline{C}_v^{\times\times}, \mathscr{E})$ are relatively prime. Here $\overline{C}_v$ is the irreducible component of $\overline{C}$ corresponding to v and the log structure on $\overline{C}_v^{\times\times}$ is the one induced by the divisor $(\overline{P} \cap \overline{C}_v) \cup \mathrm{Sing}_v$.*

We have, similarly to Lemma 5.14,

Lemma 6.8. — *Let $g : Z^\times \longrightarrow C^\times$ be a log smooth, flat and proper morphism, where the log structure on $Z^\times$ is given by the fibers of g at the points in P. If $\mathscr{H}^i := R^i g_{*,\log-\mathrm{cris}}(\mathscr{O}_{Z^\times})$, the log F-isocrystal $\mathrm{Sym}^j(\mathscr{H}^i)$ is regular outside of $\overline{P}$, for $i, j \geq 0$.*

Proof. — The proof is very similar to the proof of Lemma 5.14. □

6.1. Convergent log F-isocrystals. — Fix a smooth divisor $\overline{P}$ of $\overline{C}$. Suppose from now on that the log F-isocrystal $\mathscr{E}$ on $\overline{C}^{\times}$ is regular outside of $\overline{P}$. We define FFM-modules $H^i_{\rm deg}(\mathscr{E})$ via degeneration, as in Section 2.1 and $H^i_{\rm int}(\mathscr{E})$ via integration as in Section 2.2, for $i = 0, 1, 2$. We only need to explain how the "integration splitting" $s : H^1(C, P, \mathscr{E}) \longrightarrow H^1(C, P, \mathscr{E})$ is defined. Recall that this splitting is defined in Section 2.2 in the case $\overline{P}$ is the void set.

We first need the notion of a convergent log F-isocrystal on a pair (U, Z) consisting of a one dimensional wide open rigid space and an underlying affinoid with good reduction. We fix $s \in S - \{0\}$ with residue field L as in section §5.1 and 5.2, and let $U = U_{v,s}, Z = Z_{v,s}$ be the admissible open subsets of X_s defined in those sections for some $v \in v(G)$. Let $U^{\times}, Z^{\times}$ denote the log rigid spaces with log structures induced by $\mathscr{P}_s \cap U$ and respectively $\mathscr{P}_s \cap Z$. Let us denote by $\Delta_{U^{\times}} = U^{\times} \times_{\rm Spm(L)} U^{\times}$ the product in the category of log spaces and let $\pi_i : \Delta_{U^{\times}} \longrightarrow U^{\times}$, $i = 1, 2$ be the natural projections. Let (M, D) be a pair consisting of a coherent sheaf of $\mathscr{O}_U$-modules M and an integrable connection $D : M \longrightarrow M \otimes_{\mathscr{O}_U} \Omega^1_{U^{\times}/L}$.

We say that (M, D) is a *convergent log isocrystal* on $U^{\times}$ if the natural isomorphism $\pi_1^*(M) \cong \pi_2^*(M)$ over the diagonal of $U^{\times}$ extends to an isomorphism over a tube of the diagonal of the reduction of $U^{\times}$ in $\Delta_{U^{\times}}$ (see Definition 5.4 for the case when $\mathscr{P}$ is void.)

A *convergent log F isocrystal* on $(U^{\times}, Z^{\times})$ is a convergent log isocrystal (M, D) on $U^{\times}$ with the assignment of a horizontal isomorphism $F_{\phi} : \phi^*(M|_{Z^{\dagger}}) \longrightarrow M|_{Z^{\dagger}}$ for every morphism of log spaces $\phi : Z^{\times,\dagger} \longrightarrow X^{\times,\dagger}$ which is a lift of Frobenius over k (see also Definition 5.6 for the case when $\mathscr{P}$ is void.) For two such lifts the respective isomorphisms should satisfy the cocycle relation.

Lemma 6.9. — *Let v be a vertex of G and $(U^{\times}, Z^{\times})$ be the pair fixed above. Then $\mathscr{E}_s|_U$ is a convergent log F-isocrystal on $(U^{\times}, Z^{\times})$.*

Proof. — The proof is similar to the proof of Lemma 5.7. □

Let us denote by $R = {\rm red_s}^{-1}(\overline{P}) \cap U$.

Lemma 6.10. — *Let the notations be as in Lemma 6.9 and denote by (E, D) the convergent log F-isocrystal on $(U^{\times}, Z^{\times})$ defined there. Then the restriction of (E, D) to $(U - R, Z - R)$ is a convergent F-isocrystal in the usual sense.*

Proof. — Let us first notice that $U - R$ and $Z - R$ are admissible open subsets of U and Z respectively. $Z - R$ is actually an affinoid. We may endow both $Z - R$ and $U - R$ with the induced log structures from $U^{\times}$ and denote by $(Z - R)^{\times}$, $(U - R)^{\times}$ the respective log spaces. Then we have

1) The restriction of (E, D) to $((U - R)^{\times}, (Z - R)^{\times})$ is a convergent log F-isocrystal Let us remark that $U - R$ is not a wide-open subset of X_s, but the pair $(U - R, Z - R)$ functions as a wide open and an underlying affinoid, i.e. $(U - R) - (Z - R)$ is a disjoint union of annuli, each contained in a residue class of X_s. Therefore the definition of a

convergent log F-isocrystal given above can be extended to the notion of a convergent log F-isocrystal on $((U-R)^{\times},(Z-R)^{\times})$.

2) The log structures on $U-R$ and $Z-R$ induced by $U^{\times}$ are trivial.

3) A convergent log F-isocrystal on a pair $(U^{\times},Z^{\times})$, where the log structures on $U^{\times}$ and $Z^{\times}$ are trivial is a (usual) convergent F-isocrystal on (U,Z).

The combination of 1), 2) and 3) above proves the lemma. □

Let (E,D) be the convergent log F-isocrystal on the pair $(U^{\times},Z^{\times})$ as in the Lemma 6.10, then the Theorem 5.12 of Section 5.4 applies to the convergent F-isocrystal (E,D) on $(U-R,Z-R)$ (here, as we have mentioned above, $U-R$ is not a wide-open anymore but the theorem works the same way.) More precisely, let $\omega\in\Omega^1_{U^{\times}/L}(E)(U)$ and denote by $[\omega]$ its image in $H^1(E,D)$. Using the notations of Theorem 5.12 we have:

There exists a section α of $E^{\mathrm{flog}}(U-R)$, unique up to a global section of $(E|_{U-R})^D$, such that

i) $D(\alpha)=\omega$

ii) $G(\varphi)(\alpha)\in E(U-R)$.

Having said this let us go back to the splitting $s: H^1(C,P,\mathscr{E})\longrightarrow H^1(C,P,\mathscr{E})$ and let us recall how it is defined: we take a cohomology class in $H^1(C,P,\mathscr{E})$ and a hypercocycle representing it $((\omega_v)_v,(f_e)_e)$ as in Section 2.2. Then the image of this class under s is obtained by integrating the differential forms ω_v on U_v-R_v, for every $v\in v(G)$, and taking differences on their restrictions to A_e's for $e\in e(G)$. Such integrals by the above are defined *a priori* up to horizontal sections of $\mathscr{E}_\pi$ on U_v-R_v (recall that C is the fiber of the family $\mathfrak{X}\longrightarrow\mathscr{S}$ at the point $s=\pi$ and $\mathscr{E}_\pi=\mathscr{E}_{C^{\times}}=\mathscr{E}_{\mathfrak{X}^{\times}}|_{C_K}$.) According to the definition in Section 2.2 we need to show that such a section extends to a horizontal section of $\mathscr{E}_\pi$ on U_v. In other words, we need

Proposition 6.11. — *Let $\mathscr{E}$ be a log F-isocrystal on $\overline{C}^{\times}$ and fix a vertex $v\in v(G)$. Then the natural map (restriction) $H^0_{\mathrm{cris}}(\overline{C}_v,\mathscr{E})\longrightarrow H^0_{\mathrm{cris}}(\overline{C}_v-\overline{P},\mathscr{E})$ is surjective.*

Proof. — Now let again for this proof denote $U=U_v$ and $Z=Z_v$ and let $(E^{\dagger},D^{\dagger})$ be the overconvergent F-isocrystal on $U-R$ defined by $\mathscr{E}_\pi|_U$. Let (E,D) be the underlying convergent F-isocrystal. It follows that E^D is finite dimensional and preserved by F_ϕ for any lifting ϕ of Frobenius. Let

$$M=(E^D\otimes_L\mathscr{O}_{U-R},1\otimes d)\text{ and }M^{\dagger}=(E^D\otimes_L\mathscr{O}^{\dagger}_{U-R},1\otimes d).$$

Then $M^{\dagger}$ has a natural structure of an overconvergent F-isocrystal on $U-R$ and M is its associated convergent F-isocrystal. It follows from the main theorem of [**27**] that the natural map $\mathrm{Hom}_{\mathrm{F-iso}}(M^{\dagger},E^{\dagger})\longrightarrow\mathrm{Hom}_{\mathrm{F-iso}}(M,E)$ is a bijection. Therefore the natural inclusion $M\hookrightarrow E$ extends uniquely to a morphism $M^{\dagger}\longrightarrow E^{\dagger}$, i.e. every section of E^D is overconvergent.

Suppose $\overline{Q}$ is an absolutely irreducible point of $\overline{P}$. Let T be the corresponding residue disk and $Q=T\cap P$. Then Q is a regular singular point for the connection

D and is the unique singular point for D in T. In fact, the log-monodromy matrix for $(E|_T, D)$ at Q is nilpotent. Moreover $(E|_T, D)$ has a Frobenius structure. Let t be a parameter on T which vanishes at Q. The main result of [**4**] implies that $(E|_T, D)$ has a basis B_T of horizontal sections over $\mathscr{O}_U(T)_{\log} = \mathscr{O}_U(T)[\ell(t)]$ (for the notations see section §5.5, the discussion after the proof of Lemma 5.16.)

Lemma 6.12. — *Let W be any annulus in T centered at Q. As the restriction of t to W is a unit of $\mathscr{O}_U(W)$, the restriction of $\ell(t)$ to W is $\log(t|_W)$. Then $\log(t|_W)$ is transcendental over $\mathscr{O}_U(W)$.*

Proof. — Let $u = t|_W$. Suppose $F(X) = \sum_{i=1}^n a_i(u)X^i$ is a polynomial of minimal degree over $\mathscr{O}_U(W)$ so that $F(\log(u)) = 0$. We may suppose $n > 0$ and $(a_0, a_1, \ldots, a_n) = 1$. We use the equation $F(\log(u)) = 0$ and

$$\sum_{i=1}^{n} a_i'(u)\log(u)^i + \sum_{i=1^n} ia_i(u)\log(u)^{i-1}/u = 0$$

and cancel the terms containing $\log(u)^n$. We must have

$$a_i a_n' - (i+1)a_{i+1}a_n/u - a_i' a_n = 0.$$

It follows that a_n is a unit which may be supposed to be 1. Thus $a_{n-1}' = -n/u$ which is impossible. □

Lemma 6.13. — *Let W be any annulus in T centered at Q. Then if $f(X) \in \mathscr{O}_U(W)[X]$, $f(\log(t|_W))$ does not vanish on any non-empty open set of W unless $f = 0$.*

Corollary 6.14. — *With notations as above $(B_T)|_W$ is a basis for the horizontal sections of $(E|_W, D)$ over $\mathscr{O}_U(W)_{\log}$.*

We can now finish the proof of Proposition 6.11. Suppose g is a horizontal section of (E, D) over $U - R$. We know that g is overconvergent i.e. it extends into U by the above. Thus it restricts to a horizontal section of D on W for an annulus W in T close to the boundary. By the above corollary it must be a linear combination of $B_T|_W$. Since it is analytic on W the above lemma implies it extends to a horizontal section across T. We can base extend and assume that P is a union of such points and see that g extends across U. □

Now we need to compare the FFM-modules $H^i_{\deg}(\mathscr{E})$ and $H^i_{\mathrm{int}}(\mathscr{E})$ for $i = 0, 1, 2$. Let us remark that the same arguments as in Section 2.1 show that ∇^i is the trivial connection on $\mathbb{H}^i_P$, for $i = 0, 2$. For $i = 1$, as $\mathbb{H}^1_P$ is a locally free coherent sheaf of $\mathscr{O}_S$-modules (see [**16**]), with a connection, whose only singularity (at 0) is regular, and a Frobenius endomorphism $\Phi_1^{\deg}$, the main result of [**4**] referred to above applies. This, combined with arguments similar to those used in Section 2.1, implies that the connection ∇^1 extended to $(\mathbb{H}^1_P)_{\log}$ is trivial.

Theorem 6.15. — *Suppose the filtered, log F-isocrystal $\mathscr{E}$ on $\overline{C}^{\times}$ is regular then the parallel transport isomorphism between $(\mathbb{H}^i_P)_0 \otimes_{K_0} K$ and $(\mathbb{H}^i_P)_\pi$ yields an isomorphism of FFM-modules*

$$H^i(\mathscr{E})_{\deg} \cong H^i(\mathscr{E})_{\mathrm{int}} \text{ for } i = 0, 1, 2.$$

The proof follows using arguments similar to those in the proof of Theorem 2.6.

7. Applications

7.1. The proof of Theorem 1.1. — We will apply the results of the previous sections to the following situation: Let K, V, k, π, K_0, W be as in Section 1. Let C be a proper curve over V with smooth generic fiber C_K and semi-stable special fiber $\overline{C}$ over k. Let $g : Z \longrightarrow C$ be a flat proper morphism and P a reduced flat sub-scheme of C of dimension 0 over V such that $\bar{P} \cap \mathrm{Sing} = \varnothing$. Let $C^{\times}$ be the log formal scheme over V associated to the pair (C, P) (i.e., the formal completion of C along the special fiber together with the log structure associated to P as in Section 6.) Let $\overline{C}^{\times}$ be the log scheme over k which is the special fiber of $C^{\times}$ and denote by $D_P := g^{-1}(P)$. Then D_P is a divisor of Z and we will suppose from now on that it is a reduced divisor with simple normal crossings and that the restriction of g induces a smooth proper map $(Z - D_P) \longrightarrow (C - P)$. Let $Z^{\times}$ denote the log formal scheme over V associated to the pair (Z, D_P) and we'll denote by $g : Z^{\times} \longrightarrow C^{\times}$ the morphism of log formal schemes induced by g and also by $\overline{g} : \overline{Z}^{\times} \longrightarrow \overline{C}^{\times}$ its special fiber. From the assumptions made it follows that g and $\overline{g}$ are log smooth maps of fine formal log schemes over V (with trivial log structure.)

Some important examples to keep in mind are:

0) $Z = C$, g the identity and $P = \varnothing$.

1) C is the complete modular curve classifying semi-stable elliptic curves with suitable level structure as in Section 1, P is the set of cusps, Z is the generalized universal elliptic curve.

2) C is the Shimura curve classifying abelian surfaces with quaternionic multiplication and full level structure, P is any finite set of sections which reduce to distinct, smooth points of $\overline{C}$ (P may be void), and Z is the universal abelian scheme.

We have the following,

Theorem 7.1. — *For $i \geq 0$ there exists a log F-isocrystal $\mathscr{E}^i := K_0 \otimes_W R^i g_{\mathrm{cris},*} \mathscr{O}_{\overline{Z}^{\times}/\overline{C}^{\times}}$ on $\overline{C}^{\times}$ whose evaluation on $(C^{\times}, z_{can})$, $\mathscr{E}^i_{C^{\times}}$, is*

$$K \otimes_V \mathbb{R}^i g_* \Omega^{\bullet}_{Z^{\times}/C^{\times}} = H^i_{dR}(Z_K/C_K, \Omega^{\bullet}_{Z_K/C_K}(\log D_P)),$$

and the connection is the Gauss-Manin connection. Here z_{can} is the canonical morphism $(C^{\times})_0 \longrightarrow \overline{C}^{\times}$.

In case (0) above, $\mathscr{E}^0_{C^{\times}} \cong \mathscr{O}_C$.

Proof. — The log crystalline site on $\overline{C}^{\times}$, log crystals and the higher direct images of g_{cris} are defined in [**26**], Section 6. These objects satisfy enough of the formal properties of the corresponding classical objects (i.e., without log structures) so that the proof follows the proof in [**32**], Section 3, formally. We will content ourselves to point out the main steps. In order to simplify the notations for the rest of this proof we'll drop the $\times$ from the symbols denoting log schemes.

1) If T is a log formal scheme over $\text{Spf}(W)$ and let us denote by T_1 the closed log sub-schemes of T of ideal $p\mathscr{O}_T$. Let $z'_T : T_1 \longrightarrow \overline{C}$ be a morphism of log schemes then we have the following Cartesian diagram

$$\begin{array}{ccc} \overline{Z}_{T_1} & \longrightarrow & \overline{Z} \\ g_T \downarrow & & \overline{g} \downarrow \\ T_1 & \xrightarrow{z'_T} & \overline{C} \end{array}$$

As T_1 and $\overline{C}$ are log schemes in characteristic p and the ideal $p\mathscr{O}_T$ has natural divided powers, we define

$$\mathscr{E}_T := K_0 \otimes_W R^1 g_{\text{T,cris},*} \mathscr{O}_{\overline{Z}_{T_1}/T_1}.$$

2) Now we'll define Frobenius. Let $\overline{F}$ denote the absolute Frobenius of the log-scheme $\overline{C}$ over the absolute Frobenius σ of k, as in Section 6. Consider the Cartesian diagram

$$\begin{array}{ccc} \overline{Z}' & \longrightarrow & \overline{Z} \\ g' \downarrow & & \overline{g} \downarrow \\ \overline{C} & \xrightarrow{F_C} & \overline{C} \end{array}$$

and one can see that the evaluation of the pullback by Frobenius $\overline{F}^* \mathscr{E}$ on (T, z'_T) is given by

$$(\overline{F}^* \mathscr{E})_{(T, z'_T)} := \mathscr{E}_{(T, F_C \circ z'_T)} \cong K_0 \otimes \overline{g}'_{\text{T,cris},*} \mathscr{O}_{\overline{Z}'_{T_1}/T_1}.$$

The relative Frobenius $F_{\overline{Z}/T_1} : \overline{Z} \longrightarrow \overline{Z}'$ induces an isomorphism

$$F_{\overline{Z}/T_1} : (\overline{F}^* \mathscr{E})_{(T, z'_T)} = K_0 \otimes_W R^i g'_{\text{T,cris},*} \mathscr{O}_{\overline{Z}'_{T_1}/T_1} \cong K_0 \otimes_W R^i g_{\text{T,cris},*} \mathscr{O}_{\overline{Z}_{T_1}} = \mathscr{E}_{(T, z'_T)}.$$

3) Now we will use 1) and 2) above to define the evaluation of $\mathscr{E}$ on log enlargements. Let (T, z_T) be a log enlargement of $\overline{C}$, i.e., T is a log formal scheme and $z_T : T_0 \longrightarrow \overline{C}$, where T_0 is the closed reduced sub-scheme of T_1. Let $\iota_T : T_0 \longrightarrow T_1$ be the canonical morphism. For $n >> 0$ we have a natural morphism $\rho^{(n)} : T_1 \longrightarrow T_0$ such that $\iota_T \circ \rho^{(n)} = F^n_{T_1}$ and $\rho^{(n)} \circ \iota_T = F^n_{T_0}$. Then we define

$$\mathscr{E}_{(T, z_T)} := \mathscr{E}_{(T, z_T \circ \rho^{(n)})},$$

where the right-hand side was defined at 1). If $n' > n$, say $n' = n + d$ we have

$$\mathscr{E}_{(T, z_T \circ \rho^{(n')})} = ((F^d_{\overline{Z}/T_1})^* \mathscr{E})_{(T, z_T \circ \rho^{(n)})} \cong \mathscr{E}_{(T, z_T \circ \rho^{(n)})},$$

so the definition is independent of n.

4) Now, if we consider (C, z_{can}) as a log enlargement of $\overline{C}$ as $g : Z \longrightarrow C$ is a lift of $g : \overline{Z} \longrightarrow \overline{C}$, the evaluation of $\mathscr{E}$ on it is the relative de Rham cohomology of Z_K/C_K, with its Gauss-Manin connection.

We will leave it to the reader to check the various compatibilities required in the definition of a log F-isocrystal. □

Now, let $j \geq 0$ be an integer and let $\mathscr{E}_j$:=Sym$^j\mathscr{E}$, where $\mathscr{E}$ is the log-F-isocrystal defined in the above mentioned theorem. Let $\mathbb{L}_j := \mathrm{Sym}^j(R^i g_* \mathbb{Q}_p)(j+1)$ be the p-adic étale local system on $C - P$ associated by the theory in **[16]** to $\mathscr{E}_j$.

Then Theorem 3.2 of **[17]** and Theorem 6.15 of the present article imply:

Theorem 7.2. — *Let C, $\mathscr{E}_j$ be as at the beginning of the section. Then we have that the FFM-modules $D_{\mathrm{st}}(H^1_{\mathrm{et}}((C-P)_{\overline{K}}, \mathbb{L}_j))$ and $H^1_{\mathrm{int}}(C, \mathscr{E}_j)$ are naturally isomorphic.*

Applying this to example (0) above gives a new proof of the main result in [CI] and applying it to the example in the introduction (i.e. $C = X(N,p)$ etc.) we get,

Corollary 7.3. — *If f is a weight $j+2$, where $j \geq 0$ is an even integer, cuspidal eigenform for $X(N,p)$ with $(N,p)=1$ (see Section 1) which is split multiplicative at p then all the $\mathscr{L}$-invariants attached to f are equal whenever they are defined. (See Section 1 for a brief discussion of these $\mathscr{L}$-invariants.)*

Corollary 7.4. — *Let $C = X(N,p)$, with $(N,p) = 1$ and for every $j \geq 0$ let $\mathscr{E}_j$ be the log F-isocrystal on $\overline{C}^{\times}$ as in the introduction. The the rank of $N_1^{\deg}$ acting on $H^1_{\mathrm{cris}}(\overline{C}^{\times}, \mathscr{E}_j)^{p-new}$ equals $\frac{1}{2}\dim_{K_0} H^1_{\mathrm{cris}}(\overline{C}^{\times}, \mathscr{E}_j)^{p-new}$.*

Proof. — It is enough to calculate the rank over K of $N_1^{\mathrm{int}} \otimes 1_K$ on $H^1_{\mathrm{cris}}(\overline{C}^{\times}, \mathscr{E}_j)^{p-new}$ and this follows from the study of the residue map on $H^1_{dR}(C_K, \mathscr{E}_j)^{p-new}$ in [C1]. □

As $H_{\mathrm{int}}(C, \mathscr{E}_j)$ has an explicit description, Theorem 7.2 gives an explicit description of $H^1_{\mathrm{et}}((C-P)_{\overline{K}}, \mathbb{L}_j)$ as a Galois representation. In particular if C is a modular curve or Shimura curve, we get explicit descriptions of the restriction of the Galois representation attached to a weight $j+2$ eigenforms F to a decomposition group at p. Corollary 7.4 implies

Corollary 7.5. — *If f is a cuspidal eigenform of weight $j+2 \geq 2$ on $X(N,p)$ which is p-new, the p-adic local Galois representation V_f attached to it is semi-stable but* not *crystalline.*

7.2. Gysin sequences. — Finally, we have another application to our theory, namely the compatibility of the comparison maps with respect to the p-adic étale, respectively crystalline Gysin sequences. More precisely, let the notations be as at the beginning of this section with the difference that $K = K_0$ is unramified over $\mathbb{Q}_p$. Moreover let $\mathbb{L}$ be an étale local system and $\mathscr{E}$ a regular filtered, F-isocrystal on C, which are associated as in **[16]**. Then we have

Proposition 7.6. — *The comparison isomorphisms determine a commutative diagram of FFM-modules with G_K-action*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1_{\mathrm{et}}(C_{\overline{K}}, \mathbb{L}) \otimes B_{\mathrm{st}} & \longrightarrow & H^1_{\mathrm{et}}((C-P)_{\overline{K}}, \mathbb{L}) \otimes B_{\mathrm{st}} & \longrightarrow & \oplus_{x \in P} \mathbb{L}_{\overline{x}}(-1) \otimes_{\mathbb{Q}_p} B_{\mathrm{st}} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1_{\mathrm{int}}(\mathscr{E}) \otimes B_{\mathrm{st}} & \longrightarrow & H^1_{\mathrm{int}}(P, \mathscr{E}) \otimes B_{\mathrm{st}} & \longrightarrow & \oplus_{x \in P_K} \mathscr{E}_{C,x}[1] \otimes_K B_{\mathrm{st}} \end{array}$$

Proof. — Let us first notice that we have an exact sequence of FFM-modules

$$0 \longrightarrow H^1_{\mathrm{int}}(\mathscr{E}) \longrightarrow H^1_{\mathrm{int}}(P, \mathscr{E}) \xrightarrow{\mathrm{Res}_P} \oplus_{x \in P_K} \mathscr{E}_{C,x}[1],$$

where Res_P is the residue map with respect to the points in P_K (let us recall from the Section 2.2 that $H^1_{\mathrm{int}}(P, \mathscr{E}) = H^1_{dR}(C_K, \mathscr{E}_C(\log(P_K))$ as K-vector spaces.) This follows from the fact that the following diagram commutes

$$\begin{array}{ccccccc} & & H^{1,0}(G, \mathscr{E}) & = & H^{1,0}(G, \mathscr{E}) & & \\ & & u \uparrow & & v \uparrow & & \\ 0 & \longrightarrow & H^1_{dR}(C_K, \mathscr{E}_C) & \longrightarrow & H^1_{dR}(C_K, \mathscr{E}_C(\log(P_K)) & \xrightarrow{\mathrm{Res}_P} & \oplus_{x \in P_K} \mathscr{E}_{C,x}[1] \end{array}$$

where u, v are either the residues with respect to the family of annuli $\{A_e\}_{e \in e(G)}$ or the integration splittings.

The proposition will follow from the following two facts:

a) We have a commutative diagram of FFM-modules with exact rows (notations as in Section 2)

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1_{\mathrm{deg}}(\mathscr{E}) & \longrightarrow & H^1_{\mathrm{deg}}(P, \mathscr{E}) & \longrightarrow & \oplus_{y \in P_0} \mathscr{E}_{Y,y}[1] \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1_{\mathrm{int}}(\mathscr{E}) & \longrightarrow & H^1_{\mathrm{int}}(P, \mathscr{E}) & \longrightarrow & \oplus_{x \in P} \mathscr{E}_{C,x}[1] \end{array}$$

and

b) We have a commutative diagram of FFM-modules with G_K-action

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1_{\mathrm{et}}(C_{\overline{K}}, \mathbb{L}) \otimes B_{\mathrm{st}} & \longrightarrow & H^1_{\mathrm{et}}((C-P)_{\overline{K}}, \mathbb{L}) \otimes B_{\mathrm{st}} & \longrightarrow & \oplus_{x \in P} \mathbb{L}_{\overline{x}}(-1) \otimes_{\mathbb{Q}_p} B_{\mathrm{st}} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1_{\mathrm{deg}}(\mathscr{E}) \otimes B_{\mathrm{st}} & \longrightarrow & H^1_{\mathrm{deg}}(P, \mathscr{E}) \otimes B_{\mathrm{st}} & \longrightarrow & \oplus_{y \in P_0} \mathscr{E}_{Y,y}[1] \otimes_K B_{\mathrm{st}} \end{array}$$

To prove a) above let us recall the notations of Section 2, i.e. let X be our family of curves over S, $\mathscr{P}_X$ the divisor corresponding to $\overline{P}$ and $\mathbb{H}^1, \mathbb{H}^1_P$ the respective cohomology sheaves. Then we have a horizontal exact sequence of $\mathscr{O}_S$-modules which is Frobenius equivariant:

$$(1) \quad 0 \longrightarrow \mathbb{H}^1 \longrightarrow \mathbb{H}^1_P \xrightarrow{\mathrm{Res}_{\mathscr{P}_X}} \mathscr{E}_{(\mathscr{P}_X, z_{can})}[1],$$

where let us recall z_{can} is the map identifying the reduction of $\mathscr{P}_X$ with $\overline{P}$. As $(\mathscr{P}_X, z_{can})$ is a log-enlargement of $\overline{P}$, the crystal $\mathscr{E}_{(\mathscr{P}_X, z_{can})}$ is trivial. Therefore after adjoining $\ell(t)$, we get parallel isomorphisms between the fibers at 0 and π of the exact sequence (1) (let's recall that $\mathbb{H}^1$ is free over $\mathscr{O}_S$) i.e. we get a).

For b) let us first notice that the left square is commutative as it arises from the embedding $U := C - P \subset C$. Let us prove that the right square is commutative (this is more or less explicitly contained in Faltings' papers [**17**], [**16**], [**15**]). $U = C - P$ is an affine curve over V. Let us fix a geometric generic point $\overline{\eta}$ of C and let $\mathscr{G}$ denote the quotient of the Galois group of the maximal cover of C étale over U_K, for which the inertia at the points in P is p-adic. Let $\Delta \subset \mathscr{G}$ denote the geometric Galois group. Then $H^1_{\mathrm{et}}(U_{\overline{K}}, \mathbb{L}) \cong H^1(\Delta, \mathbb{L}_{\overline{\eta}})$ and the Gysin map $H^1_{\mathrm{et}}(U_{\overline{K}}, \mathbb{L}) \longrightarrow \oplus_{x\in P}\mathbb{L}_{\overline{x}}(-1)$ is the specialization map:

$$H^1(\Delta, \mathbb{L}_{\overline{\eta}}) \longrightarrow \oplus_{x\in P} H^1(I_x, \mathbb{L}_{\overline{x}}) \cong \oplus_{x\in P}\mathbb{L}_{\overline{x}}(-1),$$

where $I_x \cong \mathbb{Z}_p(1)$ is the inertia at x. Now under the comparison map relating the étale cohomology of $U_{\overline{K}}$ with values in $\mathbb{L}$ to the de Rham cohomology of U_K with values in $\mathscr{E}$, the specialization to inertia at the points in P corresponds to the residue of the logarithmic differentials at the points with the same reduction in P_0 (see [**15**]). □

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