Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 138
Fascicule 4

2010

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 583-613

Bull. Soc. math. France
138 (4), 2010, p. 583–613

GLOBAL EXISTENCE OF SOLUTIONS TO SCHRÖDINGER EQUATIONS ON COMPACT RIEMANNIAN MANIFOLDS BELOW H^1

by Sijia Zhong

Abstract. — In this paper, we will study global well-posedness for the cubic defocusing nonlinear Schrödinger equations on the compact Riemannian manifold without boundary, below the energy space, i.e. $s < 1$, under some bilinear Strichartz assumption. We will find some $\tilde{s} < 1$, such that the solution is global for $s > \tilde{s}$.

Résumé (*Existence globale de solutions des équations de Schrödinger sur les variétés riemanniennes compactes en régularité plus faible que H^1*)

Nous nous intéressons dans cet article au caractère bien posé des équations de Schrödinger non-linéaires cubiques défocalisantes sur les variétés riemanniennes compactes sans bord, en régularité H^s, $s < 1$, sous certaines conditions bilinéaires de Strichartz. Nous trouvons un $\tilde{s} < 1$ tel que la solution est globale pour $s > \tilde{s}$.

Texte reçu le 7 octobre 2008, révisé le 14 juillet 2009, accepté le 23 juillet 2009

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2000 Mathematics Subject Classification. — 35Q55, 37K05, 37L50, 81Q20.

Key words and phrases. — Schrödinger equation, compact Riemannian manifold, global, I-method.

This work is supported by NSFC 10871175, NSFC 10926146, and NSFC 11001049.

1. Introduction

Suppose (M,g) is any compact Riemannian manifold of dimension 2, without boundary. In this paper, we will study the Cauchy problem for the cubic defocusing nonlinear Schrödinger equations posed on M,

$$\begin{cases} iu_t + \triangle u = |u|^2 u \\ u(0,x) = u_0(x) \in H^s(M), \end{cases} \tag{1.1}$$

where the solution u is a complex valued function on $\mathbb{R}\times M$, and $\triangle$ denotes the laplace operator associated to the metric g on M.

There are two conservation laws:

$$L^2\text{-mass} \quad \int_M |u(t,x)|^2 dx = \int_M |u_0(x)|^2 dx, \tag{1.2}$$

and

$$\text{energy } E(u(t)) = \frac{1}{2}\int_M |\nabla u(t)|_g^2 dx + \frac{1}{4}\int_M |u(t,x)|^4 dx = E_0. \tag{1.3}$$

From [10], N. Burq, P. Gérard, and N. Tzvetkov proved that, if bilinear Strichartz estimate $(\mathscr{P}_{s_0})$(see Definition 1.1) is satisfied for some $0 < s_0 < 1$, then the Cauchy problem (1.1) is locally well-posed on $H^s(M)$, $s > s_0$. Thus, as a corollary, if $s \geqslant 1$, combining with the conservation of energy and L^2-mass, the solution is global. The question we are interested in is whether they are global for $s_0 < s < 1$.

First of all, let us see the situation on the whole space $\mathbb{R}^2$. In this case, equation (1.1) is L^2-critical. From [12], the solution is locally well-posed on $H^s(\mathbb{R}^2)$, $s \geqslant 0$, and also by the conservation laws above, it is easy to get the global well-posedness for $s \geqslant 1$. Then, what about $s < 1$?

In 1998, J. Bourgain, by decomposing the initial data into high frequence part and low frequence part, proved that for $\frac{2}{3} < s < 1$, the solution is global. Then in 2002, the I-team (J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao), in [14] introduced the I-method, and improved it to be $\frac{4}{7} < s < 1$. After that, by combining with the refined Morawetz estimate, the result has been improved little by little, and the best result known is $\frac{2}{5} < s < 1$, (see [17], [13], and [15]). Meanwhile, in [15], the authors claimed that by the method available, $\frac{4}{13}$ could be achieved. Recently, in [19], R. Killip, T. Tao and M. Visan proved that, when the initial data is radial, the solution is global for $s \geqslant 0$, which is exactly the optimal one.

Now, let us see what happens on the compact Riemannian manifold without boundary. The first breakthrough was made by J. Bourgain in [5]. He proved that for $\mathbb{T}^2$, it is locally well posed for $s > 0$. Then in [7], he claimed that for $s > \frac{2}{3}$, the solution would be global and this result was proved by D. De Silva, N. Pavlović, G. Staffilani, and N. Tzirakis in [16]. Recently, Akahori in

[1] proved that for the compact manifold without boundary of dimension 2, if $s > \frac{11-7\alpha_1+4\alpha_2}{(1-\alpha_1)+(11-7\alpha_1+4\alpha_2)}$, the solution of (1.1) would be global exists in $H^s(M)$, here (α_1, α_2) is the pair of positive numbers satisfying

$$\sharp\{k \in \mathbb{N} : |\sqrt{\lambda_k} - \mu| \le A\} \le C\mu^{\alpha_1} A^{\alpha_2}.$$

For example for all the compact manifold without boundary, the above estimate holds at least with $(\alpha_1, \alpha_2) = (1, 1)$. Particularly, for $\mathbb{T}^2$ and $\mathbb{S}^2$, $(\alpha_1, \alpha_2) = (0, 1)$, and the s corresponding to them is $s > \frac{15}{16}$.

Here, we are also interested in obtaining an abstract result.

Let us give the main condition of this paper, which is the bilinear Strichartz estimates.

Definition 1.1. — *Let $0 \leqslant s_0 < 1$. We say that $S(t) = e^{it\triangle}$, the flow of the linear Schrödinger equation on M stated above, satisfies property $(\mathscr{P}_{s_0})$ if for all dyadic numbers N, L, and u_0, $v_0 \in L^2(M)$ localized on dyadic intervals of order N, L respectively, i.e.*

$$\mathbf{1}_{N\leqslant\sqrt{-\triangle}<2N}(u_0) = u_0, \text{ and } \mathbf{1}_{L\leqslant\sqrt{-\triangle}<2L}(v_0) = v_0, \tag{1.4}$$

the following estimate holds:

$$\|S(t)u_0 S(t)v_0\|_{L^2((0,1)_t\times M)} \leqslant C(\min(N, L))^{s_0}\|u_0\|_{L^2(M)}\|v_0\|_{L^2(M)}. \tag{1.5}$$

In fact, such kind of bilinear Strichartz estimates were established and used by several authors in the context of the wave equations and of the Schrödinger equations. For example, in [10], N. Burq, P. Gérard, N. Tzvetkov showed that for Zoll surface of dimension 2, especially for $\mathbb{S}^2$, $s_0 = \frac{1}{4}+$. Then in [11], they proved that $(\mathscr{P}_{s_0})$ holds for $s_0 = \frac{1}{2}+$ for $\mathbb{S}^3$ and $s_0 = \frac{3}{4}+$ for $\mathbb{S}^2 \times \mathbb{S}^1$. Also the results from J. Bourgain [5], [4], and [8] proved that $s_0 = 0+$ for $\mathbb{T}^2$, $s_0 = \frac{1}{2}+$ for $\mathbb{T}^3$, and $s_0 = \frac{2}{3}+$ for $\tilde{\mathbb{T}}^3 = \mathbb{R}^3/\prod_{j=1}^3(a_j\mathbb{Z})$, where a_j are pairwise irrational numbers. And R. Anton in [2] proved that $s_0 = \frac{3}{4}+$ for general manifolds with boundary and manifolds without boundary equipped with a Lipschitz metric g for dimension $d = 2$, and for dimension 3 for the nontrapping case. Recently, this result is improved by M. D. Blair, H. F. Smith and C. D. Sogge in [3] to be $\frac{2}{3}+$.

Then, the main result of the paper is,

Theorem 1.2. — *Assume that there exists some $s_0 < s < 1$, such that condition $(\mathscr{P}_{s_0})$ holds, then there must be some $s_0 < \tilde{s} < 1$, so that for $s > \tilde{s}$, the solution to the Cauchy problem (1.1) is global, here*

$$\tilde{s} = \frac{\sqrt{2}}{2} + (1 - \frac{\sqrt{2}}{2})s_0. \tag{1.6}$$

Moreover,

$$\|u(T)\|_{H^s} \lesssim T^{\frac{(s-s_0)(1-s)}{2s^2-4s_0s+s_0^2+2s_0-1}+}, \quad \textit{for } T \gg 1. \tag{1.7}$$

REMARK 1.3. — *1. $\tilde{s}$ is monotone increasing with respect to s_0, and for $s_0 \to 1$, $\tilde{s} \to 1$.*
2. For the special case $s_0 = 0+(\mathbb{T}^2)$, $\tilde{s} = \frac{\sqrt{2}}{2} \sim 0.707$. And for $s_0 = \frac{1}{4}+$(Zoll), $\tilde{s} = \frac{2+3\sqrt{2}}{8} \sim 0.78$.
3. When $s \to 1$, $\|u(T)\|_{H^s}$ is controlled by some constant.

Now, we will state the main idea for the proof briefly.

The aim is to imitate the H^1 argument with the energy. Hence we apply some smoothing operator to improve the regularity of the solution u, so that it makes sense for energy. However, the modified energy isn't conserved any more, so the crucial point is to estimate the growth of the energy. But, contrarily from the $\mathbb{R}^2$ case, the Fourier transformation couldn't be extended trivially. Although, we use eigenfunction expansion for M, there are still some obstacles, especially for the case high-high-low-low eigenvalues. Hence, we need to localized the function to some coordinate patch, and use some semiclassical analysis tools to deal with it.

The paper is organized as follows: in Section 2, we will give some notations and lemmas that will be used later, and in Section 3, we will prove the local well posedness for the modified equation. Then in Section 4, 5 and 6, the change of energy would be estimated, the results of which, will help to prove Theorem 1.2 in Section 7. Finally, in Section 8, which is also the appendix, we will prove two important lemmas that appear in the paper.

2. Notations

In this paper, we denote $s+$ for $s+\epsilon$, and $s-$ for $s-\epsilon$, with some constant $\epsilon > 0$ small enough, and by $<\xi>$, $(1+|\xi|^2)^{\frac{1}{2}}$.

$A \lesssim B$ means there is some constant C, such that $A \leqslant CB$, and $A \sim B$ means both $A \lesssim B$ and $B \lesssim A$.

As the spectrum of $\triangle$ is discrete, let $e_k \in L^2(M)$, $k \in \mathbb{N}$, be an orthonormal basis of eigenfunctions of $-\triangle$ associated to eigenvalues μ_k. Denote by P_k the orthogonal projector on e_k. The following space is called Bourgain spaces:

DEFINITION 2.1. — *The space* $X^{s,b}(\mathbb{R}\times M)$ *is the completion of* $C_0^\infty(\mathbb{R}_t;H^s(M))$ *for the norm*

$$\|u\|^2_{X^{s,b}(\mathbb{R}\times M)} = \sum_k \| <\tau+\mu_k>^b<\mu_k>^{\frac{s}{2}}\widehat{P_k u}(\tau)\|^2_{L^2(\mathbb{R}_t;L^2(M))}$$

$$= \|e^{-it\triangle}u(t,\cdot)\|^2_{H^b(\mathbb{R}_t;H^s)}, \tag{2.1}$$

where $\widehat{P_k u}(\tau)$ *denotes the Fourier transform of* $P_k u$ *with respect to the time variable. And we denote it as* $X^{s,b}$ *when there is no confusion.*

Then, for $1\geqslant T>0$, we denote by $X^{s,b}_T(M)$ the space of restrictions of elements of $X^{s,b}(\mathbb{R}\times M)$ endowed with the norm

$$\|u\|_{X^{s,b}_T} = \inf\{\|\tilde u\|_{X^{s,b}(\mathbb{R}\times M)},\ \tilde u|_{(-T,T)\times M}=u\}. \tag{2.2}$$

The following proposition, (see J. Ginibre [18], and N. Burq, P. Gérard, and N. Tzvetkov [10]), gathers basic properties of this space.

PROPOSITION 2.2. — 1. $u\in X^{s,b}(\mathbb{R}\times M)\Longleftrightarrow e^{-it\triangle}u(t,\cdot)\in H^b(\mathbb{R},H^s(M))$.

2. *For* $b>\frac12$, $X^{s,b}(\mathbb{R}\times M)\hookrightarrow C(\mathbb{R},H^s(M))$, *and* $X^{s,b}_T(M)\hookrightarrow C((-T,T),H^s(M))$.

3. $X^{0,\frac14}(\mathbb{R}\times M)\hookrightarrow L^4(\mathbb{R},L^2(M))$.

4. *For* $s_1\leqslant s_2$, *and* $b_1\leqslant b_2$, $X^{s_2,b_2}(\mathbb{R}\times M)\hookrightarrow X^{s_1,b_1}(\mathbb{R}\times M)$.

5. *For* $0\leqslant b'<b<\frac12$, $\|u\|_{X^{s,b'}_T}\lesssim T^{b-b'}\|u\|_{X^{s,b}_T}$.

Then, from Lemma 2.3 of [10], the condition $(\mathscr{P}_{s_0})$ is equivalent to the following statement:

For any $b>\dfrac12$, and any $f,g\in X^{0,b}(\mathbb{R}\times M)$ satisfying
$\mathbf{1}_{N\leqslant\sqrt{-\triangle}<2N}(f)=f,\ \mathbf{1}_{L\leqslant\sqrt{-\triangle}<2L}(g)=g,$
one has

$$\|fg\|_{L^2(\mathbb{R}\times M)}\leqslant C(\min(N,L))^{s_0}\|f\|_{X^{0,b}(\mathbb{R}\times M)}\|g\|_{X^{0,b}(\mathbb{R}\times M)}. \tag{2.3}$$

By some interpolation, there would be

LEMMA 2.3. — *If condition* $(\mathscr{P}_{s_0})$ *holds, then for any* $1\geqslant s>s_0$, b, *which satisfies* $\frac14<b'(s)=\frac14(1+\frac{1-s}{1-s_0})+\leqslant b<\frac12$, *such that for any* f, $g\in X^{0,b}(\mathbb{R}\times M)$, *satisfying*

$$\mathbf{1}_{N\leqslant\sqrt{-\triangle}<2N}(f)=f,\ \textit{and}\ \mathbf{1}_{L\leqslant\sqrt{-\triangle}<2L}(g)=g,$$

one has

$$\|fg\|_{L^2(\mathbb{R}\times M)}\leqslant C(\min(N,L))^{s}\|f\|_{X^{0,b}(\mathbb{R}\times M)}\|g\|_{X^{0,b}(\mathbb{R}\times M)}. \tag{2.4}$$

The proof of this lemma could be referred to [10] and [21].

Now let us give the definition of I operator.

DEFINITION 2.4. — *For $N \gg 1$, define some smooth operator I_N, such that*

$$I_N u = m_N(\triangle)u = m(N^{-2}\triangle)u,$$

where

$$m(\xi) = \begin{cases} 1 & |\xi| \leqslant 1 \\ (\frac{1}{|\xi|})^{\frac{1-s}{2}} & |\xi| \geqslant 2, \end{cases} \tag{2.5}$$

is a smooth function. We denote I for I_N for simplicity when there is no confusion.

It is easy to check that

$$\|u\|_{X^{s,\frac{1}{2}+}} \lesssim \|I_N u\|_{X^{1,\frac{1}{2}+}} \lesssim N^{1-s}\|u\|_{X^{s,\frac{1}{2}+}}, \tag{2.6}$$

and

$$\|u\|_{H^s} \lesssim \|I_N u\|_{H^1} \lesssim N^{1-s}\|u\|_{H^s}. \tag{2.7}$$

3. Local well posedness

We impose the I_N operator to both sides of the equation (1.1), and denote v as $I_N u$, which satisfies the equation

$$\begin{cases} i\partial_t v + \triangle v = I_N(|I_N^{-1}v|^2 I_N^{-1}v) \\ v(0) = v_0. \end{cases} \tag{3.1}$$

So $v_0 \in H^1(M)$ ($\|v_0\|_{H^1} \lesssim N^{1-s}$ by (2.7)), and the following proposition ensures the local well posedness of (3.1).

PROPOSITION 3.1. — *Suppose $0 \leqslant s_0 < 1$, such that $(\mathscr{P}_{s_0})$ holds. For $1 > s > s_0$, and $v_0 \in H^1(M)$, there is some $\delta = C\|v_0\|_{H^1}^{-\frac{2(1-s_0)}{s-s_0}-} \gtrsim N^{-\frac{2(1-s_0)(1-s)}{s-s_0}-}$, such that (3.1) is locally well posed on $[0,\delta]$, and the solution $v \in C([0,\delta], H^1(M))$, satisfies*

$$\|I_N u\|_{X_\delta^{1,\frac{1}{2}+}} = \|v\|_{X_\delta^{1,\frac{1}{2}+}} \lesssim \|v_0\|_{H^1}, \tag{3.2}$$

where C is some constant that is independent on the time and initial data.

Proof. — To prove this proposition, we need the following lemma, the proof of which is in the appendix.

LEMMA 3.2. — *For* $s < 1$, $b'(s-) = \frac{1}{4}(1 + \frac{1-s}{1-s_0})+$, *and* $u \in X_\delta^{s,b'(s-)}$, *there exists*

$$\|I_N(|u|^2 u)\|_{X_\delta^{1,-b'(s-)}} \lesssim \|I_N u\|^3_{X_\delta^{1,b'(s-)}}, \quad s_0 < s < 1. \tag{3.3}$$

From this lemma, we can see that $\|I_N(|I_N^{-1}v|^2 I_N^{-1}v)\|_{X_\delta^{1,-b'(s-)}} \lesssim \|v\|^3_{X_\delta^{1,b'(s-)}}$. Then by Duhamel's formula, Proposition 2.11 in [10], and Proposition 2.2, we have

$$\begin{aligned}
\|v\|_{X_\delta^{1,\frac{1}{2}+}} &\lesssim \|v_0\|_{H^1} + \delta^{1-\frac{1}{2}-b'(s-)-}\|I_N(|I_N^{-1}v|^2 I_N^{-1}v)\|_{X_\delta^{1,-b'(s-)}} \\
&\lesssim \|v_0\|_{H^1} + \delta^{\frac{1}{2}-b'(s-)-}\|v\|^3_{X_\delta^{1,b'(s-)}} \\
&\lesssim \|v_0\|_{H^1} + \delta^{\frac{1}{2}-b'(s-)-}\delta^{3\times(\frac{1}{2}-b'(s-))-}\|v\|^3_{X_\delta^{1,\frac{1}{2}+}} \\
&\lesssim \|v_0\|_{H^1} + \delta^{\frac{s-s_0}{1-s_0}-}\|v\|^3_{X_\delta^{1,\frac{1}{2}+}}. \qquad (3.4)
\end{aligned}$$

Hence, by choosing $\delta \sim \|v_0\|_{H^1}^{-\frac{2(1-s_0)}{s-s_0}-} \gtrsim N^{-\frac{2(1-s_0)(1-s)}{s-s_0}-}$, and the standard contraction argument, we get the result of the proposition.

□

4. The change of energy

As $E(v(t))$ is not conserved any more, we have to calculate its variation on the time interval $[0,\delta]$.

PROPOSITION 4.1. — *For the solution* v *to the Cauchy problem (3.1), its variation of energy on the time interval* $[0,\delta]$ *is*

$$E(v(\delta)) \leqslant E(v_0) + C((N^{-(1-s_0)+}\delta^{\frac{1}{2}-} + N^{-2(1-s_0)+})\|v_0\|^4_{H^1} + N^{-2+s_0+}\delta^{\frac{1}{2}-}\|v_0\|^6_{H^1}). \tag{4.1}$$

Proof. — Since

$$\begin{aligned}
\partial_t E(v(t)) &= \operatorname{Re}\int_M \overline{\nabla v}\nabla v_t dx + \operatorname{Re}\int_M |v|^2\overline{v}v_t dx \\
&= -\operatorname{Re}\int_M \triangle\overline{v}v_t dx + \operatorname{Re}\int_M |v|^2\overline{v}v_t dx \\
&= \operatorname{Re}\int v_t(|v|^2\overline{v} - \triangle\overline{v})dx
\end{aligned}$$

$$= \text{Re}\int_M v_t(|v|^2\overline{v} - I_N(|I_N^{-1}v|^2\overline{I_N^{-1}v}))dx$$
$$= -\text{Im}\int_M (\triangle v - I_N(|I_N^{-1}v|^2 I_N^{-1}v))(|v|^2\overline{v} - I_N(|I_N^{-1}v|^2\overline{I_N^{-1}v}))dx$$
$$= -\text{Im}\int_M \triangle I_N u(|I_N u|^2\overline{I_N u} - I_N(|u|^2\bar{u}))dx$$
$$+ \text{Im}\int_M I_N(|u|^2 u)(|I_N u|^2\overline{I_N u} - I_N(|u|^2\bar{u}))dx,$$

Integral on $[0,\delta]$, then

$$E(v(\delta)) - E(v_0) = -\text{Im}\int_0^\delta\int_M \triangle I_N u(|I_N u|^2\overline{I_N u} - I_N(|u|^2\bar{u}))dxdt \tag{4.2}$$
$$+ \text{Im}\int_0^\delta\int_M I_N(|u|^2 u)(|I_N u|^2\overline{I_N u} - I_N(|u|^2\bar{u}))dxdt. \tag{4.3}$$

From now on, we will use the notation I instead of I_N.

The goal for Proposition 4.1 would be achieved if we have the following two estimates, which will be proved in Section 5 and 6 separately.

(4.4)

$$(4.2) = -\text{Im}\int_0^\delta\int_M \Delta I_N u(|I_N u|^2\overline{I_N u} - I_N(|u|^2\bar{u}))dxdt \lesssim (N^{-(1-s_0)+}\delta^{\frac{1}{2}-} + N^{-2(1-s_0)+})\|v_0\|_{H^1}^4,$$

and

(4.5)

$$(4.3) = \text{Im}\int_0^\delta\int_M I_N(|u|^2 u)(|I_N u|^2\overline{I_N u} - I_N(|u|^2\bar{u}))dxdt \lesssim N^{-2+s_0+}\delta^{\frac{1}{2}-}\|v_0\|_{H^1}^6,$$

where $s > \max\{\frac{2}{3}, \frac{1+s_0}{2}\}$. □

5. Proof of estimate 4.4

Proof. — We have

$$(4.2) = -\text{Im}\int_0^\delta\int_M \nabla\overline{Iu}(\nabla(|Iu|^2 Iu) - I(\nabla(|u|^2 u)))$$
$$= -2\text{Im}\int_0^\delta\int_M \nabla\overline{Iu}(|Iu|^2\nabla Iu - I(|u|^2\nabla u))$$
$$- \text{Im}\int_0^\delta\int_M \nabla\overline{Iu}((Iu)^2\nabla\overline{Iu} - I(u^2\nabla\bar{u}))$$
$$= I + II. \tag{5.1}$$

5.1. Estimates for I. — The purpose of this subsection is to prove that

$$I \lesssim (N^{-(1-s_0)+}\delta^{\frac{1}{2}-} + N^{-2(1-s_0)+})\|v_0\|_{H^1}^4. \tag{5.2}$$

$$I = -2\mathrm{Im}\int_0^\delta\int_M \nabla\overline{Iu}(|Iu|^2\nabla Iu - I(|u|^2\nabla u))$$

$$= 2\mathrm{Im}\int_0^\delta\int_M \nabla\overline{Iu}I(|u|^2\nabla u)$$

$$= 2\mathrm{Im}\int_0^\delta\int_M \nabla\overline{Iu}I(|\chi_1(N^{-2+}\triangle)u|^2\nabla u) \tag{5.3}$$
$$+ \mathrm{III}$$

with some smooth cut off function χ_1, such that

$$\chi_1(\xi) = \begin{cases} 1 & |\xi| \leqslant 1 \\ 0 & |\xi| \geqslant 2. \end{cases} \tag{5.4}$$

And III is the sum of the terms:

$$2\mathrm{Im}\int_0^\delta\int \nabla\overline{Iu}I(\chi_1(N^{-2+}\triangle)u(1-\chi_1(N^{-2+}\triangle))\bar{u}\nabla u),$$

$$2\mathrm{Im}\int_0^\delta\int \nabla\overline{Iu}I(\chi_1(N^{-2+}\triangle)\bar{u}(1-\chi_1(N^{-2+}\triangle))u\nabla u),$$

and

$$2\mathrm{Im}\int_0^\delta\int_M \nabla\overline{Iu}I(|(1-\chi_1(N^{-2+}\triangle))u|^2\nabla u).$$

5.1.1. *Study of (5.3).* — We are going to show that

$$(5.3) = 2\mathrm{Im}\int_0^\delta\int \nabla\overline{Iu}I(|\chi_1(N^{-2+}\triangle)u|^2\nabla u) \lesssim N^{-1+s_0+}\delta^{\frac{1}{2}-}\|v_0\|_{H^1}^4. \tag{5.5}$$

Proof. — We have

$$(5.3) = 2\mathrm{Im}\int_0^\delta\int \nabla\overline{Iu}I(|\chi_1(N^{-2+}\triangle)u|^2\nabla u)$$
$$= 2\mathrm{Im}\int_0^\delta\int \nabla\overline{Iu}|\chi_1(N^{-2+}\triangle)u|^2\nabla Iu$$
$$+ 2\mathrm{Im}\int_0^\delta\int \nabla\overline{Iu}[I, |\chi_1(N^{-2+}\triangle)u|^2]\nabla u$$

$$= 2\mathrm{Im}\int_0^\delta\int \nabla\overline{Iu}[I,|\chi_1(N^{-2+}\triangle)u|^2]I^{-1}\nabla Iu$$

$$= 2\mathrm{Im}\int_0^\delta\int \nabla\overline{Iu}[I,g]I^{-1}\nabla Iu, \tag{5.6}$$

with $g(x) = |\chi_1(N^{-2+}\triangle)u|^2$.

By partition of unity, it is enough to estimate $2\mathrm{Im}\int_0^\delta\int \nabla\overline{Iu}\chi_2([I,g]I^{-1}\nabla Iu)$, where $\chi_2 \in C_0^\infty(M)$ is supported in a coordinate patch.

Then (5.5) follows from the following lemma,

Lemma 5.1. — *For the u, g, and χ_2 appearing above, we have,*

$$2\mathrm{Im}\int_0^\delta\int \nabla\overline{Iu}\chi_2([I,g]I^{-1}\nabla Iu) \lesssim N^{-1+s_0+}\delta^{\frac{1}{2}-}\|Iu\|^4_{X^{1,\frac{1}{2}+}} \lesssim N^{-1+s_0+}\delta^{\frac{1}{2}-}\|v_0\|^4_{H^1}. \tag{5.7}$$

We will postpone the proof of this lemma in the appendix (see section 8.2). □

5.1.2. *Estimate of III.* — For III, we are going to prove that

$$\mathrm{III} \lesssim (N^{-(1-s_0)+}\delta^{\frac{1}{2}-} + N^{-2(1-s_0)+})\|v_0\|^4_{H^1}. \tag{5.8}$$

Proof. — Because $\{e_k\}$ is an orthonomal basis of eigenfunction of $-\triangle$,

$$\mathrm{III} \sim 2\mathrm{Im}\sum_{N_0,N_1,N_2,N_3}\int_0^\delta\int \nabla\overline{Iu}^{N_0}\frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)}Iu^{N_1}\overline{Iu}^{N_2}\nabla Iu^{N_3}dxdt, \tag{5.9}$$

with

$$u^{N_j} = \sum_{N_j\leqslant<\mu_k>^{\frac{1}{2}}<2N_j} P_k u,\ j = 0,\ 1,\ 2,\ 3,$$

and N_1 or $N_2 \gtrsim N^{1-}$.

Let

$$I(N) = |\int_0^\delta\int \frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)}\nabla\overline{Iu}^{N_0}Iu^{N_1}I\overline{u}^{N_2}\nabla Iu^{N_3}dxdt|.$$

We will prove that part of the gradient of ∇Iu^{N_3} could be shared by Iu^{N_1} and Iu^{N_2}.

By symmetry argument and because the presence of complex conjugates will play no role here, we can assume $N_1 \geqslant N_2$. So $\max\{N_0,N_1,N_2,N_3\} \gtrsim N^{1-}$, and by Lemma 2.6 of [10], we can see that there exists $\tilde{C} > 0$ such that if $N_i \geqslant \tilde{C}\sum_{j\neq i}(N_j)$, then for every $p > 0$, there exists

$$\begin{aligned}
I(N) &\lesssim N_i^{-p}\max\{1,(\frac{N_i}{N})\}^{3(1-s)}\int_0^\delta \|\nabla I u^{N_0}\|_{L_x^2}\|Iu^{N_1}\|_{L_x^2}\|Iu^{N_2}\|_{L_x^2}\|\nabla Iu^{N_3}\|_{L_x^2}dt\\
&\lesssim N_i^{-p+3(1-s)}\prod_{j=0}^{3}\| <\nabla> Iu^{N_j}\|_{L_t^4L_x^2}\\
&\lesssim N_i^{-p+3(1-s)}\prod_{j=0}^{3}\| <\nabla> Iu^{N_j}\|^4_{X_\delta^{0,\frac14}}\\
&\lesssim N_i^{-p+3(1-s)}\delta^{1-}\|Iu\|^4_{X_\delta^{1,\frac12+}}\\
&\lesssim N_i^{0-}N^{-(p-3(1-s))+}\delta^{1-}\|Iu\|^4_{X_\delta^{1,\frac12+}}
\end{aligned}$$

(5.10)
$$\lesssim N_i^{0-}N^{-(p-3(1-s))+}\delta^{1-}\|v_0\|^4_{H^1}.$$

Hence, divide (5.9) into two parts, J_1 and J_2, where the summation is restricted to $N_i \geqslant \tilde{C}\sum_{j\neq i}N_j$ in J_1 and other possibilities are in J_2. Thus for J_1, as N_j $(j\neq i)$ could be controlled by N_i, we can choose p large enough such that J_1 could be controlled by (5.5), hence we just need to deal with J_2.

(a) $N_0, N_1, N_2, N_3 \ll N$.

Hence $\frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} = 1$.

By Lemma 2.3 and Proposition 2.2,

$$\begin{aligned}
I(N) &\lesssim \|\nabla Iu^{N_0}Iu^{N_2}\|_{L^2_{t,x}}\|Iu^{N_1}\nabla Iu^{N_3}\|_{L^2_{t,x}}\\
&\lesssim N_1^{s_0+}N_2^{1-}\|\nabla Iu^{N_0}\|_{X_\delta^{0,\frac14+}}\|Iu^{N_2}\|_{X_\delta^{0,\frac14+}}\|Iu^{N_1}\|_{X_\delta^{0,\frac12+}}\|\nabla Iu^{N_3}\|_{X_\delta^{0,\frac12+}}\\
&\lesssim N_1^{s_0-1+}N_2^{0-}\delta^{2\times(\frac12-\frac14)-}\prod_{i=0}^{3}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}}
\end{aligned}$$

$$\lesssim N_1^{-(1-s_0)+}N_2^{0-}\delta^{\frac12-}\prod_{i=0}^{3}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}}. \tag{5.11}$$

If $N_3 \lesssim N_1$, then $N_1 \gtrsim N^{1-}$, $N_0 \lesssim N_1$ and

(5.12)
$$\begin{aligned}
\sum_{N_0,N_1,N_2,N_3} I(N) &\lesssim \sum_{N_0,N_1,N_2,N_3} N_1^{-(1-s_0)+}N_0^{0-}N_1^{0-}N_2^{0-}N_3^{0-}\delta^{\frac12-}\prod_{i=0}^{3}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}}\\
&\lesssim N^{-(1-s_0)+}\delta^{\frac12-}\|Iu\|^4_{X_\delta^{1,\frac12+}}.
\end{aligned}$$

If $N_3 \gg N_1 \geqslant N_2$, then $N_1 \gtrsim N^{1-}$, $N_0 \sim N_3$,

$$\begin{aligned}\sum_{N_0,N_1,N_2,N_3} I(N) &\lesssim N^{-(1-s_0)+}\delta^{\frac12-}\|Iu\|^2_{X^{1,\frac12+}_\delta}\sum_{N_0\sim N_3}\|Iu^{N_0}\|_{X^{1,\frac12+}_\delta}\|Iu^{N_3}\|_{X^{1,\frac12+}_\delta}\\ &\lesssim N^{-(1-s_0)+}\delta^{\frac12-}\|Iu\|^2_{X^{1,\frac12+}_\delta}\sum_{N_0}\|Iu^{N_0}\|^2_{X^{1,\frac12+}_\delta}\\ &\lesssim N^{-(1-s_0)+}\delta^{\frac12-}\|Iu\|^4_{X^{1,\frac12+}_\delta}. \end{aligned}\tag{5.13}$$

Combining (5.12) with (5.13), we have

$$\sum_{N_0,N_1,N_2,N_3} I(N) \lesssim N^{-(1-s_0)+}\delta^{\frac12-}\|Iu\|^4_{X^{1,\frac12+}_\delta} \lesssim N^{-(1-s_0)+}\delta^{\frac12-}\|v_0\|^4_{H^1}.\tag{5.14}$$

(b) At least one of $N_i \gtrsim N$.

(b1) $N_1 \geqslant N_3$.

Hence, $N_0 \lesssim N_1$, and $N_1 \gtrsim N$.

(b11) $N_1 \gtrsim N \gg N_2, N_3$.

In this case, $N_1 \sim N_0 \gtrsim N$, which gives $\frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} = \frac{m_N(N_0^2)}{m_N(N_1^2)} \lesssim 1$, the left steps are the same as (a), and the result would even be better.

(b12) $N_1 \geqslant N_3 \gtrsim N \gg N_2$.

$$\frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} \lesssim N^{-2(1-s)}(N_1N_3)^{1-s}.$$

By Lemma 2.3 and Proposition 2.2,

$$\begin{aligned} I(N) &\lesssim N^{-2(1-s)}(N_1N_3)^{1-s}\|\nabla Iu^{N_0}Iu^{N_2}\|_{L^2_{t,x}}\|Iu^{N_1}\nabla Iu^{N_3}\|_{L^2_{t,x}}\\ &\lesssim N^{-2(1-s)}(N_1N_3)^{1-s}N_2^{1-}N_3^{s_0+}\\ &\quad\cdot\|\nabla Iu^{N_0}\|_{X^{0,\frac14+}_\delta}\|Iu^{N_2}\|_{X^{0,\frac14+}_\delta}\|Iu^{N_1}\|_{X^{0,\frac12+}_\delta}\|\nabla Iu^{N_3}\|_{X^{0,\frac12+}_\delta}\\ &\lesssim \delta^{\frac12-}N^{-2(1-s)}N_1^{-s}N_2^{0-}N_3^{1-s+s_0+}\prod_{i=0}^3\|Iu^{N_i}\|_{X^{1,\frac12+}_\delta}\\ &\lesssim \delta^{\frac12-}N^{-2(1-s)}N_1^{1-2s+s_0+}N_0^{0-}N_2^{0-}N_3^{0-}\prod_{i=0}^3\|Iu^{N_i}\|_{X^{1,\frac12+}_\delta}, \end{aligned}\tag{5.15}$$

then

$$\sum I(N) \lesssim \delta^{\frac12-}N^{-1+s_0+}\|Iu\|^4_{X^{1,\frac12+}_\delta} \lesssim \delta^{\frac12-}N^{-1+s_0+}\|v_0\|^4_{H^1},\tag{5.16}$$

with $s > \frac{1+s_0}{2}$.

(b13) $N_1 \geqslant N_2 \gtrsim N \gg N_3$.

$$\frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} \lesssim N^{-2(1-s)}(N_1N_2)^{1-s}.$$

Similarly,

$$\begin{aligned} I(N) &\lesssim N^{-2(1-s)}(N_1N_2)^{1-s}(N_2N_3)^{s_0+} \\ &\quad \cdot \|\nabla Iu^{N_0}\|_{X_\delta^{0,\frac12+}}\|Iu^{N_1}\|_{X_\delta^{0,\frac12+}}\|Iu^{N_2}\|_{X_\delta^{0,\frac12+}}\|\nabla Iu^{N_3}\|_{X_\delta^{0,\frac12+}} \\ &\lesssim N^{-2(1-s)}N_1^{-s}N_2^{-s+s_0+}N_3^{s_0+}\prod_{i=0}^{3}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}} \end{aligned}$$

$$\lesssim N^{-2(1-s)}N_1^{-s+s_0+}N_2^{-s+s_0+}N_0^{0-}N_3^{0-}\prod_{i=0}^{3}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}}, \tag{5.17}$$

so

$$\sum I(N) \lesssim N^{-2(1-s_0)+}\|Iu\|^4_{X_\delta^{1,\frac12+}} \lesssim N^{-2(1-s_0)+}\|v_0\|^4_{H^1}. \tag{5.18}$$

(b14) $N_1 \geqslant N_2,\ N_3 \gtrsim N$.

$$\frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} \lesssim N^{-3(1-s)}(N_1N_2N_3)^{1-s}.$$

Also

$$\begin{aligned} I(N) &\lesssim N^{-3(1-s)}(N_1N_2N_3)^{1-s}(N_2N_3)^{s_0+} \\ &\quad \cdot \|\nabla Iu^{N_0}\|_{X_\delta^{0,\frac12+}}\|Iu^{N_1}\|_{X_\delta^{0,\frac12+}}\|Iu^{N_2}\|_{X_\delta^{0,\frac12+}}\|\nabla Iu^{N_3}\|_{X_\delta^{0,\frac12+}} \\ &\lesssim N^{-3(1-s)}N_1^{-s}N_2^{-s+s_0+}N_3^{1-s+s_0+}\prod_{i=0}^{3}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}} \end{aligned}$$

$$\lesssim N^{-3(1-s)}N_1^{1-2s+s_0+}N_2^{-s+s_0+}N_0^{0-}N_3^{0-}\prod_{i=0}^{3}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}}, \tag{5.19}$$

and

$$\sum I(N) \lesssim N^{-2(1-s_0)+}\|Iu\|^4_{X_\delta^{1,\frac12+}} \lesssim N^{-2(1-s_0)+}\|v_0\|^4_{H^1}. \tag{5.20}$$

(b2) $N_3 \geqslant N_1 \geqslant N_2$.
In this case, $N_3 \gtrsim N$, and $N_0 \lesssim N_3$.
(b21) $N_3 \gtrsim N \gg N_1 \gtrsim N^{1-}$.
So $N_0 \sim N_3 \gtrsim N$, and

$$\frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} \sim 1,$$

So this case can be dealt by the same way as case (a), and the result is the same.

(b22) $N_3 \geqslant N_1 \gtrsim N \gg N_2$.

(b221) $N_3 \sim N_1$.

$$\frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} \lesssim N^{-2(1-s)}(N_1N_3)^{1-s},$$

and

$$\begin{aligned} I(N) &\lesssim N^{-2(1-s)}(N_1N_3)^{1-s}N_1^{s_0+}N_2^{1-} \\ &\quad \cdot \|\nabla Iu^{N_0}\|_{X_\delta^{0,\frac12+}}\|Iu^{N_1}\|_{X_\delta^{0,\frac12+}}\|Iu^{N_2}\|_{X_\delta^{0,\frac14+}}\|\nabla Iu^{N_3}\|_{X_\delta^{0,\frac14+}} \\ &\lesssim N^{-2(1-s)}N_1^{-s+s_0+}N_2^{0-}N_3^{1-s}\delta^{\frac12-}\prod_{i=0}^{3}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}} \\ &\sim N^{-2(1-s)}N_3^{1-2s+s_0+}N_2^{0-}\delta^{\frac12-}\prod_{i=0}^{3}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}}. \end{aligned}$$

Hence,

$$\sum I(N) \lesssim N^{-1+s_0+}\delta^{\frac12-}\|Iu\|^4_{X_\delta^{1,\frac12+}} \lesssim N^{-1+s_0+}\delta^{\frac12-}\|v_0\|^4_{H^1}. \tag{5.21}$$

(b222) $N_1 \ll N_3$.

So $N_0 \sim N_3$ and

$$\frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} \lesssim N^{-(1-s)}N_1^{1-s}.$$

Then

$$\begin{aligned} I(N) &\lesssim N^{-(1-s)}N_1^{1-s}N_1^{s_0+}N_2^{1-} \\ &\quad \cdot \|\nabla Iu^{N_0}\|_{X_\delta^{0,\frac12+}}\|Iu^{N_1}\|_{X_\delta^{0,\frac12+}}\|Iu^{N_2}\|_{X_\delta^{0,\frac14+}}\|\nabla Iu^{N_3}\|_{X_\delta^{0,\frac14+}} \\ &\lesssim N^{-(1-s)}N_1^{-s+s_0+}N_2^{0-}\delta^{\frac12-}\prod_{i=0}^{3}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}}. \end{aligned}$$

And

$$\begin{aligned} \sum_{N_0,N_1,N_2,N_3} I(N) &\lesssim N^{-1+s_0+}\delta^{\frac12-}\|Iu\|^2_{X_\delta^{1,\frac12+}}\sum_{N_0\sim N_3}\|Iu^{N_0}\|_{X_\delta^{1,\frac12+}}\|Iu^{N_3}\|_{X_\delta^{1,\frac12+}} \\ &\lesssim N^{-1+s_0+}\delta^{\frac12-}\|Iu\|^2_{X_\delta^{1,\frac12+}}\sum_{N_0}\|Iu^{N_0}\|^2_{X_\delta^{1,\frac12+}} \\ &\lesssim N^{-1+s_0+}\delta^{\frac12-}\|Iu\|^4_{X_\delta^{1,\frac12+}} \\ &\lesssim N^{-1+s_0+}\delta^{\frac12-}\|v_0\|^4_{H^1}. \end{aligned} \tag{5.22}$$

(b23) $N_3 \geqslant N_1 \geqslant N_2 \gtrsim N$.

Also divide into two cases, $N_1 \sim N_3$ and $N_1 \ll N_3$.

First, if $N_1 \sim N_3$,

$$\frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} \lesssim N^{-3(1-s)}(N_1N_2N_3)^{1-s},$$

and

$$\begin{aligned} I(N) &\lesssim N^{-3(1-s)}(N_1N_2N_3)^{1-s}(N_0N_2)^{s_0+} \\ &\quad \cdot \|\nabla Iu^{N_0}\|_{X_\delta^{0,\frac12+}}\|Iu^{N_1}\|_{X_\delta^{0,\frac12+}}\|Iu^{N_2}\|_{X_\delta^{0,\frac12+}}\|\nabla Iu^{N_3}\|_{X_\delta^{0,\frac12+}} \\ &\lesssim N^{-3(1-s)}N_1^{-s+s_0+}N_2^{-s+s_0+}N_3^{1-s}\prod_{i=0}^{3}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}} \\ &\lesssim N^{-3(1-s)}N_3^{1-2s+s_0+}N_2^{-s+s_0+}\prod_{i=0}^{3}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}}. \end{aligned}$$

So

$$\sum I(N) \lesssim N^{-2+2s_0+}\|Iu\|^4_{X_\delta^{1,\frac12+}} \lesssim N^{-2+2s_0+}\|v_0\|^4_{H^1}. \tag{5.23}$$

For the second case, we have $N_3 \sim N_0 \gtrsim N$,

$$\frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} \lesssim N^{-2(1-s)}(N_1N_2)^{1-s},$$

and

$$\begin{aligned} I(N) &\lesssim N^{-2(1-s)}(N_1N_2)^{1-s}(N_1N_2)^{s_0+} \\ &\quad \cdot \|\nabla Iu^{N_0}\|_{X_\delta^{0,\frac12+}}\|Iu^{N_1}\|_{X_\delta^{0,\frac12+}}\|Iu^{N_2}\|_{X_\delta^{0,\frac12+}}\|\nabla Iu^{N_3}\|_{X_\delta^{0,\frac12+}} \\ &\lesssim N_1^{-s+s_0+}N_2^{-s+s_0+}N^{-2(1-s)}\prod_{i=0}^{3}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}}, \end{aligned}$$

which gives

$$\sum_{N_0,N_1,N_2,N_3} I(N) \lesssim N^{-2+2s_0+}\|Iu\|^4_{X_\delta^{1,\frac12+}} \lesssim N^{-2+2s_0+}\|v_0\|^4_{H^1}, \tag{5.24}$$

by the same arguments as (b222).

Conclusively,

$$\mathrm{III} \lesssim (N^{-(1-s_0)+}\delta^{\frac12-} + N^{-2(1-s_0)+})\|v_0\|^4_{H^1}. \tag{5.25}$$ □

Combining with the result of section 5.1.1,

$$I \lesssim (N^{-(1-s_0)+}\delta^{\frac12-} + N^{-2(1-s_0)+})\|v_0\|^4_{H^1}. \tag{5.26}$$

5.2. Estimates for II. — Because

$$\mathrm{II} = -\mathrm{Im}\int_0^\delta\int_M \nabla\overline{Iu}((Iu)^2\nabla\overline{Iu} - I(u^2\nabla\bar{u}))$$
$$= -\mathrm{Im}\sum_{N_0,N_1,N_2,N_3}\int_0^\delta\int_M \nabla\overline{Iu}^{N_0}\Big(1-\frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)}\Big)Iu^{N_1}Iu^{N_2}\nabla\overline{Iu}^{N_3},$$

and $|1-\frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)}| \lesssim \frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)}$, all the other cases could be estimated the same as I, except for

$$-\mathrm{Im}\int_0^\delta\int_M \nabla\overline{Iu}((\chi_1(N^{-2+}\triangle)Iu)^2\nabla\overline{Iu} - I((\chi_1(N^{-2+}\triangle)u)^2\nabla\bar{u})). \tag{5.27}$$

As

$$(5.27) = -\mathrm{Im}\int_0^\delta\int_M \nabla\overline{Iu}((\chi_1(N^{-2+}\triangle)Iu)^2\nabla\overline{Iu} - I((\chi_1(N^{-2+}\triangle)u)^2\nabla\bar{u}))$$
$$= -\mathrm{Im}\int_0^\delta\int \nabla\overline{Iu}((\chi_1(N^{-2+}\triangle)u)^2\nabla\overline{Iu} - (\chi_1(N^{-2+}\triangle)u)^2\nabla\overline{Iu} - [I,\underline{g}]I^{-1}\nabla\overline{Iu})$$
$$= \mathrm{Im}\int_0^\delta\int \nabla\overline{Iu}[I,\underline{g}]I^{-1}\nabla\overline{Iu},$$

where $\underline{g} = (\chi_1(N^{-2+}\triangle)u)^2$.

From the proof of section 5.1.1 and Lemma 5.1, we can see that the presence of complex conjugates makes no difference there, hence we have

$$(5.27) \lesssim N^{-1+s_0+}\delta^{\frac{1}{2}-}\|Iu\|^4_{X_\delta^{1,\frac{1}{2}+}} \lesssim N^{-1+s_0+}\delta^{\frac{1}{2}-}\|v_0\|^4_{H^1}. \tag{5.28}$$

So

$$II \lesssim (N^{-(1-s_0)+}\delta^{\frac{1}{2}-} + N^{-2(1-s_0)+})\|v_0\|^4_{H^1}. \tag{5.29}$$

Taking section 5.1 into consideration, it has

$$(4.2) \lesssim (N^{-(1-s_0)+}\delta^{\frac{1}{2}-} + N^{-2(1-s_0)+})\|v_0\|^4_{H^1}. \tag{5.30}$$ □

6. Proof of estimate (4.5)

Proof. — The purpose of this section is to prove estimate (4.5), i.e.

$$(4.3) = \mathrm{Im}\int_0^\delta\int_M I(|u|^2u)(|Iu|^2\overline{Iu} - I(|u|^2\bar{u}))dxdt \lesssim N^{-2+s_0+}\delta^{\frac{1}{2}-}\|v_0\|^6_{H^1}.$$

$$
\begin{aligned}
(4.3) &= \operatorname{Im} \int_0^\delta \int_M I(|u|^2 u)(|Iu|^2 \overline{Iu} - I(|u|^2 \bar{u}))dxdt \\
&= \operatorname{Im} \sum_{N_0,\cdots,N_6} \int_0^\delta \int I(u^{N_4}\bar{u}^{N_5}u^{N_6})^{N_0} \\
&\quad \cdot \left(1 - \frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)}\right) Iu^{N_1}\overline{Iu}^{N_2} Iu^{N_3},
\end{aligned} \tag{6.1}
$$

where u^{N_i} has the same definition as the ones in Section 5. By symmetry argument and because the presence of complex conjugates will play no role here, we can assume $N_1 \geqslant N_2 \geqslant N_3$, and $N_4 \geqslant N_5 \geqslant N_6$. As stated in Section 4, we can assume that $N_i \lesssim \sum_{j\in\{0,1,2,3\}\neq i} N_j$, $i = 0,1,2,3$, and also $N_i \lesssim \sum_{j\in\{0,4,5,6\}\neq i} N_j$, $i = 0,4,5,6$.

And if N_0, N_1, N_2, $N_3 \ll N$, then the right side of (6.1) would be 0, which is trivial, so we will just deal with the case at least one of them $\gtrsim N$.

Since

$$
|1 - \frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)}| \lesssim \frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)}, \tag{6.2}
$$

$$
\begin{aligned}
I(N) &\lesssim \frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} \|I(u^{N_4}\bar{u}^{N_5}u^{N_6})^{N_0}\|_{L^2_{t,x}} \|Iu^{N_1}Iu^{N_2}Iu^{N_3}\|_{L^2_{t,x}} \\
&\lesssim \frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} \|Iu^{N_1}Iu^{N_2}Iu^{N_3}\|_{L^2_{t,x}} \\
&\quad \cdot \frac{m_N(N_0^2)}{m_N(N_4^2)m_N(N_5^2)m_N(N_6^2)} \|Iu^{N_4}Iu^{N_5}Iu^{N_6}\|_{L^2_{t,x}} \\
&= (i) \times (ii),
\end{aligned} \tag{6.3}
$$

where

$$
(\mathrm{i}) = \frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} \|Iu^{N_1}Iu^{N_2}Iu^{N_3}\|_{L^2_{t,x}}
$$

and

$$
(\mathrm{ii}) = \frac{m_N(N_0^2)}{m_N(N_4^2)m_N(N_5^2)m_N(N_6^2)} \|Iu^{N_4}Iu^{N_5}Iu^{N_6}\|_{L^2_{t,x}}.
$$

Let us estimate (ii) first.

Lemma 6.1. — *We have*

$$
(\mathrm{ii}) \lesssim \begin{cases} N_4^{0-}\delta^{\frac{1}{2}-} \prod_{i=4}^6 \|Iu^{N_i}\|_{X_\delta^{1,\frac{1}{2}+}} & N_4 \ll N \\ N^{-1+}N_4^{0-}\delta^{\frac{1}{2}-} \prod_{i=4}^6 \|Iu^{N_i}\|_{X_\delta^{1,\frac{1}{2}+}} & N_4 \gtrsim N. \end{cases} \tag{6.4}
$$

Proof. — By Lemma 2.3, Proposition 2.2 and Sobolev embedding theory,

$$\begin{aligned}
\text{(ii)} &\lesssim \frac{m_N(N_0^2)}{m_N(N_4^2)m_N(N_5^2)m_N(N_6^2)}\|Iu^{N_4}Iu^{N_5}\|_{L^2_{t,x}}\|Iu^{N_6}\|_{L^\infty_{t,x}}\\
&\lesssim \frac{m_N(N_0^2)}{m_N(N_4^2)m_N(N_5^2)m_N(N_6^2)}N_5\|Iu^{N_4}\|_{X_\delta^{0,\frac14+}}\|Iu^{N_5}\|_{X_\delta^{0,\frac14+}}\|Iu^{N_6}\|_{X_\delta^{0,\frac12+}}\\
&\lesssim \frac{m_N(N_0^2)}{m_N(N_4^2)m_N(N_5^2)m_N(N_6^2)}\frac{1}{N_4}\delta^{\frac12-}\prod_{i=4}^{6}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}}. \qquad (6.5)
\end{aligned}$$

(1) $N_6 \leqslant N_5 \leqslant N_4 \ll N$.

$$\frac{m_N(N_0^2)}{m_N(N_4^2)m_N(N_5^2)m_N(N_6^2)} \lesssim 1,$$

and

$$(6.5) \lesssim \frac{1}{N_4}\delta^{\frac12-}\prod_{i=4}^{6}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}} \lesssim N_4^{0-}\delta^{\frac12-}\prod_{i=4}^{6}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}}. \qquad (6.6)$$

(2) $N_6 \leqslant N_5 \ll N \lesssim N_4$.

$$\frac{m_N(N_0^2)}{m_N(N_4^2)m_N(N_5^2)m_N(N_6^2)} \lesssim N^{-(1-s)}N_4^{1-s}.$$

So

$$\begin{aligned}
(6.5) &\lesssim N^{-(1-s)}N_4^{1-s}N_4^{-1}\delta^{\frac12-}\prod_{i=4}^{6}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}}\\
&= N^{-(1-s)}N_4^{-s}\delta^{\frac12-}\prod_{i=4}^{6}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}}\\
&\lesssim N^{-1+}N_4^{0-}\delta^{\frac12-}\prod_{i=4}^{6}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}}. \qquad (6.7)
\end{aligned}$$

(3) $N_6 \ll N \lesssim N_5 \leqslant N_4$.

Then

$$\frac{m_N(N_0^2)}{m_N(N_4^2)m_N(N_5^2)m_N(N_6^2)} \lesssim N^{-2(1-s)}(N_4N_5)^{1-s},$$

and

$$(6.5) \lesssim N^{-2(1-s)}(N_4N_5)^{1-s}N_4^{-1}\delta^{\frac{1}{2}-}\prod_{i=4}^{6}\|Iu^{N_i}\|_{X_\delta^{1,\frac{1}{2}+}}$$
$$\lesssim N^{-2(1-s)}N_4^{-s}N_5^{1-s}\delta^{\frac{1}{2}-}\prod_{i=4}^{6}\|Iu^{N_i}\|_{X_\delta^{1,\frac{1}{2}+}}$$
$$\lesssim N^{-1+}N_4^{0-}\delta^{\frac{1}{2}-}\prod_{i=4}^{6}\|Iu^{N_i}\|_{X_\delta^{1,\frac{1}{2}+}}, \tag{6.8}$$

with $s > \frac{1}{2}$.

(4) $N \lesssim N_6 \leqslant N_5 \leqslant N_4$.

Then

$$\frac{m_N(N_0^2)}{m_N(N_4^2)m_N(N_5^2)m_N(N_6^2)} \lesssim N^{-3(1-s)}(N_4N_5N_6)^{1-s},$$

and do the same thing as (3), we have

$$(6.5) \lesssim N^{-1+}N_4^{0-}\delta^{\frac{1}{2}-}\prod_{i=4}^{6}\|Iu^{N_i}\|_{X_\delta^{1,\frac{1}{2}+}}, \tag{6.9}$$

with the assumption that $s > \frac{2}{3}$.

Hence, conclusively,

$$(\mathrm{ii}) \lesssim N_4^{0-}\delta^{\frac{1}{2}-}\prod_{i=4}^{6}\|Iu^{N_i}\|_{X_\delta^{1,\frac{1}{2}+}}.$$ □

Then for (i), we can deal with it in the same cases as (ii). As at least $N_1 \gtrsim N$, we just need to consider about the case (2), (3), (4), which gives

$$(\mathrm{i}) \lesssim N^{-1+}N_1^{0-}\delta^{\frac{1}{2}-}\prod_{i=1}^{3}\|Iu^{N_i}\|_{X_\delta^{1,\frac{1}{2}+}}. \tag{6.10}$$

Therefore, when $N_4 \gtrsim N$, from (6.3), (6.4) and (6.10),

$$I(N) \lesssim N_1^{0-}N_4^{0-}N^{-2+}\delta^{1-}\prod_{i=1}^{6}\|Iu^{N_i}\|_{X_\delta^{1,\frac{1}{2}+}}, \tag{6.11}$$

and

$$\sum_{N_0\cdots N_6} I(N) \lesssim N^{-2+}\delta^{1-}\|Iu\|^6_{X_\delta^{1,\frac{1}{2}+}} \lesssim N^{-2+}\delta^{1-}\|v_0\|^6_{X_\delta^{1,\frac{1}{2}+}} \lesssim N^{-2+s_0+}\delta^{\frac{1}{2}-}\|v_0\|^6_{H^1}. \tag{6.12}$$

For $N_4 \ll N$, which means that $N_0 \lesssim N_4 \ll N$, so $N_1 \geqslant N_2 \gtrsim N$, and

$$\frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} \lesssim N^{-2(1-s)}(N_1N_2)^{1-s}\frac{1}{m_N(N_3^2)}.$$

Hence

$$
\begin{aligned}
\text{(i)} &\lesssim N^{-2(1-s)}(N_1N_2)^{1-s}\frac{1}{m_N(N_3^2)}\|Iu^{N_1}Iu^{N_2}\|_{L^2_{t,x}}\|Iu^{N_3}\|_{L^\infty_t H^1_x} \\
&\lesssim N^{-2(1-s)}(N_1N_2)^{1-s}\frac{1}{m_N(N_3^2)}N_2^{s_0+}\|Iu^{N_1}\|_{X_\delta^{0,\frac12+}}\|Iu^{N_2}\|_{X_\delta^{0,\frac12+}}\|Iu^{N_3}\|_{X_\delta^{1,\frac12+}} \\
&\lesssim N^{-2(1-s)}\frac{1}{m_N(N_3^2)}N_1^{-s}N_2^{-s+s_0+}\prod_{i=1}^{3}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}}.
\end{aligned}
\tag{6.13}
$$

Discussing for $N_3 \ll N$ or $N_3 \gtrsim N$, it is easy to get

$$
(6.13) \lesssim N^{-2+s_0+}N_1^{0-}\prod_{i=1}^{3}\|Iu^{N_i}\|_{X_\delta^{1,\frac12+}}. \tag{6.14}
$$

Therefore, for this case,

$$
\sum I(N) \lesssim N^{-2+s_0+}\delta^{\frac12-}\|Iu\|^6_{X_\delta^{1,\frac12+}} \lesssim N^{-2+s_0+}\delta^{\frac12-}\|v_0\|^6_{H^1}. \tag{6.15}
$$

□

7. Proof of Theorem 1.2

Proof. — By Proposition 4.1, (2.6) and (2.7), we have,

$$
\begin{aligned}
E(v(\delta)) - E(v_0) &\lesssim (N^{-(1-s_0)+}\delta^{\frac12-} + N^{-2(1-s_0)+})\|v_0\|^4_{H^1} + N^{-2+s_0+}\delta^{\frac12-}\|v_0\|^6_{H^1} \\
&\lesssim (N^{-(1-s_0)+}\delta^{\frac12-} + N^{-2(1-s_0)+})N^{4(1-s)} + N^{-2+s_0+}\delta^{\frac12-}N^{6(1-s)} \\
&\lesssim N^{3-4s+s_0+}\delta^{\frac12-} + N^{2-4s+2s_0+} + N^{4-6s+s_0+}\delta^{\frac12-}.
\end{aligned}
\tag{7.1}
$$

By the definition of energy, and Gagliardo-Nirenberg inequality, we have

$$
E(v_0) = \frac12\|\nabla v_0\|^2_{L^2_x} + \frac14\|v_0\|^4_{L^4_x} \leqslant \frac12\|\nabla v_0\|^2_{L^2_x} + C\|v_0\|^2_{L^2_x}\|\nabla v_0\|^2_{L^2_x} \leqslant C\|v_0\|^2_{H^1_x} \lesssim N^{2(1-s)}. \tag{7.2}
$$

Hence, for any given time $T \gg 1$, the number of iterate is $\frac{T}{\delta}$ with $\delta \gtrsim N^{-\frac{2(1-s_0)(1-s)}{s-s_0}-}$, and the total energy is

$$
\begin{aligned}
E(v(T)) &\lesssim E(v(0)) + \frac{T}{\delta}(N^{3-4s+s_0+}\delta^{\frac12-} + N^{2-4s+2s_0+} + N^{4-6s+s_0+}\delta^{\frac12-}) \\
&\lesssim N^{2(1-s)} + T(N^{3-4s+s_0+}\delta^{-\frac12-} + N^{2-4s+2s_0+}\delta^{-1} + N^{4-6s+s_0+}\delta^{-\frac12-}) \\
&\lesssim N^{2(1-s)} + T(N^{3-4s+s_0+\frac{(1-s)(1-s_0)}{s-s_0}+} \\
&\qquad + N^{2-4s+2s_0+\frac{2(1-s)(1-s_0)}{s-s_0}+} + N^{4-6s+s_0+\frac{(1-s)(1-s_0)}{s-s_0}+}).
\end{aligned}
\tag{7.3}
$$

Now we should take some proper $N = N(T)$, such that

$$
T(N^{3-4s+s_0+\frac{(1-s)(1-s_0)}{s-s_0}+} + N^{2-4s+2s_0+\frac{2(1-s)(1-s_0)}{s-s_0}+} + N^{4-6s+s_0+\frac{(1-s)(1-s_0)}{s-s_0}+}) \lesssim N^{2(1-s)},
$$

so that

(7.4)
$$\|u(T)\|_{H^s_x} \lesssim \|Iu(T)\|_{H^1_x} = \|v(T)\|_{H^1_x} \lesssim \sqrt{E(v(T))} + \|v(T)\|_{L^2_x} \lesssim N^{1-s} = T^\alpha,$$

with $\alpha > 0$. Here, we use the conservation of L^2-mass of $u(t)$, which gives

$$\|v(T)\|_{L^2_x} = \|I_N u(T)\|_{L^2_x} \leqslant \|u(T)\|_{L^2_x} = \|u_0\|_{L^2_x}.$$

Since

$$3 - 4s + s_0 + \frac{(1-s)(1-s_0)}{s-s_0} + > 2 - 4s + 2s_0 + \frac{2(1-s)(1-s_0)}{s-s_0} +$$

for $s > \frac{1+s_0}{2}$, and

$$3 - 4s + s_0 + \frac{(1-s)(1-s_0)}{s-s_0} + > 4 - 6s + s_0 + \frac{(1-s)(1-s_0)}{s-s_0} +$$

for $s > \frac{1}{2}$,

$$\text{(7.5)} \quad T(N^{3-4s+s_0+\frac{(1-s)(1-s_0)}{s-s_0}+} + N^{2-4s+2s_0+\frac{2(1-s)(1-s_0)}{s-s_0}+} + N^{4-6s+s_0+\frac{(1-s)(1-s_0)}{s-s_0}+}) \lesssim TN^{3-4s+s_0+\frac{(1-s)(1-s_0)}{s-s_0}+} \lesssim N^{2(1-s)},$$

by taking $N \sim T^{\frac{s-s_0}{2s^2-4s_0s+s_0^2+2s_0-1}+}$, for $s > \frac{\sqrt{2}}{2} + (1 - \frac{\sqrt{2}}{2})s_0 > \max\{\frac{1+s_0}{2}, \frac{2}{3}\}$.

Therefore, by (7.4)

$$\text{(7.6)} \qquad \|u(T)\|_{H^s} \lesssim T^{\frac{(s-s_0)(1-s)}{2s^2-4s_0s+s_0^2+2s_0-1}+}. \qquad \square$$

8. Appendix

In this section, we will prove Lemma 3.2 and Lemma 5.1.

8.1. Proof of Lemma 3.2

Proof. — It is just a little modification of Proposition 2.5 of [10].

Assume by density that $u \in C_0^\infty(\mathbb{R} \times M)$. By duality argument, we just need to prove

$$\text{(8.1)} \qquad \left| \int_{\mathbb{R}\times M} \bar{\varphi} I_N(|u|^2 u) dx dt \right| \leqslant C \|\varphi\|_{X^{-1,b'(s-)}} \|I_N u\|^3_{X^{1,b'(s-)}},$$

for $\varphi \in X^{-1,b'(s-)}$. Denote

$$\varphi^{N_0} = \sum_{N_0 \leqslant <\mu_k>^{\frac{1}{2}} < 2N_0} P_k \varphi,$$

$$u^{N_j} = \sum_{N_j \leqslant <\mu_k>^{\frac{1}{2}} < 2N_j} P_k u, \text{ for } j = 1, 2, 3,$$

and

$$I(N) = \left| \int_{\mathbb{R}\times M} \bar{\varphi}^{N_0} m(N_0) u^{N_1} \bar{u}^{N_2} u^{N_3} dxdt \right|,$$

then

$$\left| \int_{\mathbb{R}\times M} \bar{\varphi} I_N(|u|^2 u) dxdt \right| \lesssim \sum_{N_0,N_1,N_2,N_3} I(N). \tag{8.2}$$

By symmetry argument and because the presence of complex conjugates will play no role here, we can assume $N_1 \geqslant N_2 \geqslant N_3$. Also by Lemma 2.6 of [10], the condition that $s < 1$, which gives $\frac{1}{4} < b'(s-)$, we can manage the case $N_0 \geqslant C(N_1 + N_2 + N_3)$ by some similar argument as in section 5.1.2, so we just need to estimate the case $N_0 \lesssim N_1 + N_2 + N_3$, which implies $N_0 \lesssim N_1$.

Therefore,

$$\begin{aligned}
I(N) &\lesssim \frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} \int \bar{\varphi}^{N_0} I_N u^{N_1} \overline{I_N u}^{N_2} I_N u^{N_3} \\
&\lesssim \frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} \|\varphi^{N_0} I_N u^{N_2}\|_{L^2_{t,x}} \|I_N u^{N_1} I_N u^{N_3}\|_{L^2_{t,x}} \\
&\lesssim \frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} (N_2N_3)^{s-} \|\varphi^{N_0}\|_{X^{0,b'(s-)}} \prod_{i=1}^{3} \|I_N u\|_{X^{0,b'(s-)}} \\
&\lesssim \frac{m_N(N_0^2)}{m_N(N_1^2)m_N(N_2^2)m_N(N_3^2)} \left(\frac{N_0}{N_1}\right) (N_2N_3)^{s-1-} \\
&\quad \cdot \|\varphi^{N_0}\|_{X^{-1,b'(s-)}} \prod_{i=1}^{3} \|I_N u\|_{X^{1,b'(s-)}},
\end{aligned} \tag{8.3}$$

by Lemma 2.3.

If $N_i \ll N$,

$$\frac{N_i^{s-1-}}{m_N(N_i^2)} = N_i^{s-1-}.$$

If $N_i \gtrsim N$,

$$\frac{N_i^{s-1-}}{m_N(N_i^2)} = (\frac{N_i}{N})^{1-s} N_i^{s-1-} = N^{s-1} N_i^{0-},$$

with $s_0 < s < 1$.

Therefore, (8.3) turns to be

$$\begin{aligned}
&\sum_{N_0 \lesssim (N_1+N_2+N_3)} I(N) \\
&\lesssim \sum_{N_0 \lesssim N_1} \frac{m_N(N_0^2)}{m_N(N_1^2)} \left(\frac{N_0}{N_1}\right) \|\varphi^{N_0}\|_{X^{-1,b'(s-)}} \|I_N u^{N_1}\|_{X^{1,b'(s-)}} \|I_N u\|^2_{X^{1,b'(s-)}}.
\end{aligned} \tag{8.4}$$

Case 1. $N_0 \lesssim N_1 \ll N$: $\frac{m_N(N_0^2)}{m_N(N_1^2)}(\frac{N_0}{N_1}) \sim (\frac{N_0}{N_1}) \lesssim (\frac{N_0}{N_1})^s$.

Case 2. $N_0 \ll N \lesssim N_1$: $\frac{m_N(N_0^2)}{m_N(N_1^2)}(\frac{N_0}{N_1}) \lesssim (\frac{N_1}{N})^{1-s}(\frac{N_0}{N_1}) \lesssim (\frac{N_0}{N_1})^s$.

Case 3. $N \lesssim N_0 \lesssim N_1$: $\frac{m_N(N_0^2)}{m_N(N_1^2)}(\frac{N_0}{N_1}) \lesssim (\frac{N_1}{N_0})^{1-s}(\frac{N_0}{N_1}) = (\frac{N_0}{N_1})^s$.

Hence,

$$\sum_{N_0,N_1,N_2,N_3} I(N) \lesssim \sum_{N_0 \lesssim N_1} \left(\frac{N_0}{N_1}\right)^s \|\varphi^{N_0}\|_{X^{-1,b'(s-)}} \|I_N u^{N_1}\|_{X^{1,b'(s-)}} \|I_N u\|^2_{X^{1,b'(s-)}}. \tag{8.5}$$

Then, because for $N_0 \lesssim N_1$, there is some fixed positive integer l_0 such that $N_1 = 2^l N_0$, where $l \geqslant -l_0$, it has

$$\begin{aligned}
&\sum_{N_0 \lesssim N_1} \left(\frac{N_0}{N_1}\right)^s \|\varphi^{N_0}\|_{X^{-1,b'(s-)}} \|I_N u^{N_1}\|_{X^{1,b'(s-)}} \\
&\qquad = \sum_{l \geqslant -l_0} \sum_{N_0} 2^{-sl} \|\varphi^{N_0}\|_{X^{-1,b'(s-)}} \|I_N u^{2^l N_0}\|_{X^{1,b'(s-)}} \\
&\qquad \lesssim \sum_{l \geqslant -l_0} 2^{-sl} (\sum_{N_0} \|\varphi^{N_0}\|^2_{X^{-1,b'(s-)}})^{\frac{1}{2}} (\sum_{N_0} \|I_N u^{2^l N_0}\|^2_{X^{1,b'(s-)}})^{\frac{1}{2}} \\
&\qquad \lesssim \|\varphi\|_{X^{-1,b'(s-)}} \|I_N u\|_{X^{1,b'(s-)}},
\end{aligned} \tag{8.6}$$

which ends the proof for this lemma. □

8.2. Proof of Lemma 5.1

Proof. — Let

$$1 = \theta_0(\xi) + \sum_{j=1}^{\infty} \theta(2^{-j}\xi)$$

be a dyadic partition of unity in $\mathbb{R}^2$, $\theta_0 \in C_0^\infty(\mathbb{R}^2)$, $\theta \in C_0^\infty(\mathbb{R}^2 \backslash \{0\})$, and $\mathrm{supp}(\theta) \subset (1, +\infty)$.

So

$$\begin{aligned}
m(\xi) &= \theta_0(\xi) m(\xi) + \sum_{j=1}^{\infty} \theta(2^{-j}\xi) |\xi|^{-\frac{1-s}{2}} \\
&= \theta_0(\xi) m(\xi) + \sum_{j=1}^{\infty} 2^{-\frac{j}{2}(1-s)} \theta(2^{-j}\xi) |2^{-j}\xi|^{-\frac{1-s}{2}},
\end{aligned} \tag{8.7}$$

which gives

$$m(N^{-2}\triangle) = \theta_0(N^{-2}\triangle)m(N^{-2}\triangle) + \sum_{j=1}^{\infty} 2^{-\frac{j}{2}(1-s)}\theta(2^{-j}N^{-2}\triangle)|2^{-j}N^{-2}\triangle|^{-\frac{1-s}{2}}$$

$$= \sum_{j=0}^{\infty} 2^{-\frac{j}{2}(1-s)}\varphi_{1j}(h_j^2\triangle), \tag{8.8}$$

here $h_j = 2^{-\frac{j}{2}}N^{-1}$ $(j \geqslant 0)$, and

$$\varphi_{1j}(\xi) = \begin{cases} \theta_0(\xi)m(\xi) & j=0 \\ \theta(\xi)|\xi|^{-\frac{1-s}{2}} & j \geqslant 1 \end{cases} \in C_0^\infty(\mathbb{R}^2).$$

By Proposition 2.1 of [9], there is a sequence $\{\psi_{1ij}(x,\xi)\}$ of $C_0^\infty(M\times\mathbb{R}^d)$, and χ_3, which is equal to 1 near the support of χ_2 and $\mathrm{supp}(\chi_3)$ is a slightly larger than $\mathrm{supp}(\chi_2)$, such that for every $K\in\mathbb{N}$, $f\in C^\infty(M)$

$$\chi_2\varphi_{1j}(h_j^2\triangle)f - \sum_{i=0}^{K-1} h_j^i\psi_{1ij}(x,h_jD)\chi_3 f = R_{1j}(x,h_jD,h_j)f, \tag{8.9}$$

here $\psi_{10} = \chi_2(x)\varphi_{1j}(p_2(x,\xi))$, and for $i \geqslant 1$, $\psi_{1ij} = \sum_{k\geqslant 2}\frac{\varphi_1^{(k-1)}(p_2)}{(k-1)!}q_{1ijk}$, $q_{1ijk}(x,\xi)$ are polynomials in ξ of degree less or equal to $2(k-1)$ and supported in the set $\{x\in\mathrm{supp}(\chi_3)\}$, p_2 is the principal symbol of $\triangle_g$, $a|\xi|^2 \leqslant p_2(x,\xi) \leqslant b|\xi|^2$, and

$$\|R_{1j}f\|_{L_x^2} \lesssim h_j^K\|f\|_{L_x^2}. \tag{8.10}$$

Let

$$m_1(x,N^{-1}D) = \sum_{j=0}^{\infty}\sum_{i=0}^{K-1} 2^{-\frac{j}{2}(1-s)}h_j^i\psi_{1ij}(x,h_jD). \tag{8.11}$$

Then, similarly, for $m^{-1}(N^{-2}\triangle)$,

$$m^{-1}(N^{-2}\triangle) = \sum_{j=0}^{\infty} 2^{\frac{j}{2}(1-s)}\varphi_{2j}(h_j^2\triangle), \tag{8.12}$$

here

$$\varphi_{2j}(\xi) = \begin{cases} \theta_0(\xi)m^{-1}(\xi) & j=0 \\ \theta(\xi)|\xi|^{\frac{1-s}{2}} & j \geqslant 1 \end{cases} \in C_0^\infty(\mathbb{R}^2).$$

Also there are $\{\psi_{2ij}(x,\xi)\}$ and $R_{2j}(x,h_jD,h_j)$ such that

$$\chi_2\varphi_{2j}(h_j^2\triangle)f - \sum_{i=0}^{K-1} h_j^i\psi_{2ij}(x,h_jD)\chi_3 f = R_{2j}(x,h_jD,h_j)f, \tag{8.13}$$

and

$$\|R_{2j}f\|_{L_x^2} \lesssim h_j^K \|f\|_{L_x^2}. \tag{8.14}$$

Denote

$$m_2(x, N^{-1}D) = \sum_{j=0}^{\infty}\sum_{i=0}^{K-1} 2^{\frac{j}{2}(1-s)} h_j^i \psi_{2ij}(x, h_j D). \tag{8.15}$$

Now, let us deal with

$$\left|\int_0^\delta \int \nabla\overline{Iu}\chi_2[I,g]\chi_2 I^{-1}\nabla Iu\right| = \left|\int_0^\delta \int \bar{h}\chi_2[m(N^{-2}\triangle), g]m^{-1}(N^{-2}\triangle)h\right|,$$

here $h = \nabla Iu$.

By the properties of χ_3, we obtain

$$\begin{aligned}
&\int_0^\delta \int \bar{h}\chi_2([m(N^{-2}\triangle), g]m^{-1}(N^{-2}\triangle))h \\
&\quad = \int_0^\delta \int \bar{h}[\chi_2 m(N^{-2}\triangle) - m_1(x, N^{-1}D)\chi_3, g]\tilde{\chi}_2 m^{-1}(N^{-2}\triangle)h \\
&\qquad + \int_0^\delta \int \bar{h}[m_1(x, N^{-1}D)\chi_3, g]\tilde{\chi}_2 m^{-1}(N^{-2}\triangle)h \\
&\quad = \int_0^\delta \int \bar{h}[\chi_2 m(N^{-2}\triangle) - m_1(x, N^{-1}D)\chi_3, g](\tilde{\chi}_2 m^{-1}(N^{-2}\triangle) - m_2(x, N^{-1}D)\chi_3)h \\
&\qquad + \int_0^\delta \int \bar{h}[m_1(x, N^{-1}D)\chi_3, g](\tilde{\chi}_2 m^{-1}(N^{-2}\triangle) - m_2(x, N^{-1}D)\chi_3)h \\
&\qquad + \int_0^\delta \int \bar{h}[\chi_2 m(N^{-2}\triangle) - m_1(x, N^{-1}D)\chi_3, g]m_2(x, N^{-1}D)\chi_3 h \\
&\qquad + \int_0^\delta \int \bar{h}[m_1(x, N^{-1}D)\chi_3, g]m_2(x, N^{-1}D)\chi_3 h \\
&\quad = \text{(i)} + \text{(ii)} + \text{(iii)} + \text{(iv)},
\end{aligned} \tag{8.16}$$

with $\tilde{\chi}_2$ equal to 1 near the support of χ_2, and $\operatorname{supp}(\tilde{\chi}_2)$ is a little larger than $\operatorname{supp}(\chi_2)$.

8.2.1. *Study of* (i), (ii) *and* (iii). — For (i), (ii), (iii), we will use the same method, and take (iii) for example.

$$
\begin{aligned}
\text{(iii)} &\lesssim \|h\|_{L^2_{t,x}} \|[\chi_2 m(N^{-2}\triangle) - m_1(x,N^{-1}D)\chi_3, g] m_2(x,N^{-1}D)\chi_3 h\|_{L^2_{t,x}} \\
&\lesssim \sum_{j,k\geqslant 0}\sum_{i=0}^{K-1} 2^{\frac{k-j}{2}(1-s)} h_k^i \|h\|_{L^2_{t,x}} \|[R_{1j}(x,h_jD,h_j),g]\psi_{2ik}(x,h_kD)h\|_{L^2_{t,x}} \\
&\lesssim \sum_{j,k\geqslant 0}\sum_{i=0}^{K-1} 2^{\frac{k-j}{2}(1-s)} h_k^i \|h\|_{L^2_{t,x}} (\|R_{1j}(g\psi_{2ik}h)\|_{L^2_{t,x}} + \|g(R_{1j}\psi_{2ik}h)\|_{L^2_{t.x}}).
\end{aligned}
\tag{8.17}
$$

$$
\begin{aligned}
\|\partial_x^\alpha g\|_{L^\infty_x} &= \|\partial_x^\alpha (\chi_1(N^{-2+}\triangle)u\overline{\chi_1(N^{-2+}\triangle)u})\|_{L^\infty_x} \\
&\lesssim N^{|\alpha|-}\|\chi_1(N^{-2+}\triangle)u\|^2_{H^1_x} \lesssim N^{|\alpha|-}\|Iu\|^2_{H^1_x},\ \textit{for}\ |\alpha|\geqslant 1,
\end{aligned}
\tag{8.18}
$$

$$
\|g\|_{L^\infty_x} \lesssim N^{0+}\|Iu\|^2_{H^1}, \tag{8.19}
$$

$$
|\partial_x^\alpha \psi_{2ik}(x,h_k\xi)| \lesssim h_k^{1-s}|\xi|^{1-s} \lesssim 1, \tag{8.20}
$$

and

$$
h_k^{|\alpha|}N^{|\alpha|-} = 2^{-\frac{k}{2}|\alpha|}N^{0-},
$$

so by the classical pseudodifferential estimates, we just need to consider $|k-j|\leqslant \nu$, here ν is a fixed positive constant.

$$
\begin{aligned}
&\sum_j \sum_{|k-j|\leqslant\nu}\sum_{i=0}^{K-1} 2^{\frac{k-j}{2}(1-s)} h_k^i \|h\|_{L^2_{t,x}} (\|R_{1j}(g\psi_{2ik}h)\|_{L^2_{t,x}} + \|g(R_{1j}\psi_{2ik}h)\|_{L^2_{t.x}}) \\
&\quad\lesssim \sum_j \sum_{|k-j|\leqslant\nu}\sum_{i=0}^{K-1} 2^{\frac{k-j}{2}(1-s)} h_k^i \|h\|_{L^2_{t,x}} h_j^K \|g\|_{L^\infty_{t,x}} \|h\|_{L^2_{t,x}} \\
&\quad\lesssim \sum_j \sum_{|k-j|\leqslant\nu}\sum_{i=0}^{K-1} 2^{\frac{k-j}{2}(1-s)} h_k^i h_j^K \|h\|^2_{X^{0,0}_\delta} N^{0+}\|Iu\|^2_{L^\infty_t H^1_x} \\
&\quad\lesssim \sum_j \sum_{|k-j|\leqslant\nu}\sum_{i=0}^{K-1} 2^{\frac{k-j}{2}(1-s)} h_k^i h_j^K N^{0+}\delta^{1-}\|Iu\|^4_{X_\delta^{1,\frac12+}} \\
&\quad\lesssim N^{-K+}\delta^{1-}\|Iu\|^4_{X_\delta^{1,\frac12+}}.
\end{aligned}
\tag{8.21}
$$

Hence by taking $K=2$, the estimate for (iii) meets the requirement of the lemma . And for (i) and (ii), the results are the same or even better.

8.2.2. *Deal with* (iv). — Now, let us estimate (iv).

$$
\begin{aligned}
\text{(iv)} &= \iint \bar{h}[m_1(x,N^{-1}D)\chi_3, g]m_2(x,N^{-1}D)\chi_3 h \\
&= \sum_{j,k\geqslant 0}\sum_{i,l=0}^{K-1} 2^{\frac{k-j}{2}(1-s)} \iint \bar{h}[\psi_{1ij}(x,h_jD),g]\psi_{2lk}(x,h_kD)\tilde{\chi}_3 h \\
&= \sum_{N_0,N_1,N_2,N_3}\sum_{j,k\geqslant 0}\sum_{i,l=0}^{K-1} 2^{\frac{k-j}{2}(1-s)} \\
&\quad \cdot \iint \bar{h}^{N_0}[\psi_{1ij}(x,h_jD),\chi_1(N^{-2+}\triangle)u^{N_1}\overline{\chi_1(N^{-2+}\triangle)u^{N_2}}] \\
&\quad \psi_{2lk}(x,h_kD)\tilde{\chi}_3 h^{N_3},
\end{aligned}
\tag{8.22}
$$

where $\tilde{\chi}_3 \in C_0^\infty$, which equals 1 near support χ_3, h^{N_i}, u^{N_i} is defined as above. Denote $\tilde{g} = \chi_1(N^{-2+}\triangle)u^{N_1}\overline{\chi_1(N^{-2+}\triangle)u^{N_2}}$.

First, for the same reason as above, we just need to estimate the case $|k-j| \leqslant \nu$, N_1, $N_2 \lesssim N^{1-}$, and $N_0 \sim N_3 \gtrsim N$, otherwise, $[\psi_{1ij}(x,h_jD),\chi_1(N^{-2+}\triangle)u^{N_1}\overline{\chi_1(N^{-2+}\triangle)u^{N_2}}] = 0$.

By (8.18), the following lemma is from Theorem 2.6.5 and Remark 2.6.7 of [20].

Lemma 8.1. — *We have*

$$
[\psi_{1ij},\tilde{g}] = \sum_{|\alpha|=1}^{K'-1} \frac{h_j^{|\alpha|}}{i^{|\alpha|}|\alpha|!}\partial_\xi^\alpha \psi_{1ij}\partial_x^\alpha \tilde{g} + R_{ij}(x,h_j\xi,h_j),
\tag{8.23}
$$

such that

$$
\|R_{ij}\|_{L^2_x\to L^2_x} \lesssim h_j^{K'} N^{K'(1-\epsilon)}(N_1N_2)^{0-}\|Iu\|^2_{H^1},
$$

where $\epsilon > 0$ depends on the eigenvalue scale of $\chi_1(N^{-2+}\triangle)u$.

So take K' large enough such that $K'\epsilon > 1-s_0$, and do the same thing as (iii), then we can get the result what we want for R_i. Therefore, the left terms are $\sum_{|\alpha|=1}^{K'-1}\frac{h_j^{|\alpha|}}{i^{|\alpha|}|\alpha|!}\partial_\xi^\alpha\psi_{1ij}\partial_x^\alpha\tilde{g}$, and what we will estimate is

$$
\begin{aligned}
&\sum_{N_0,N_1,N_2,N_3}\sum_{j}\sum_{|k-j|\leqslant\nu}\sum_{i,l=0}^{K-1}\sum_{|\alpha|=1}^{K'-1} 2^{\frac{k-j}{2}(1-s)}\frac{h_j^{|\alpha|}}{i^{|\alpha|}\alpha!} \\
&\qquad \cdot\iint h^{N_0}\partial_x^\alpha\tilde{g}\partial_\xi^\alpha\psi_{1ij}(x,D)\psi_{2lk}(x,h_kD)\tilde{\chi}_3 h^{N_3},
\end{aligned}
\tag{8.24}
$$

with $\partial_x^\alpha\tilde{g} = \sum_{|\alpha_1|+|\alpha_2|=|\alpha|} C\partial_x^{\alpha_1}(\chi_1(N^{-2+}\triangle)u^{N_1})\overline{\partial_x^{\alpha_2}\chi_1(N^{-2+}\triangle)u^{N_2}}$.

First, we will estimate the case of $|\alpha| = 1$, and for the term

$$\partial_x(\chi_1(N^{-2+}\triangle)u^{N_1})\overline{\chi_1(N^{-2+}\triangle)u^{N_2}}.$$

For this case, the kernel of $\partial_\xi\psi_{1ij}(x,D)\psi_{2lk}(x,h_kD)$ is

$$K(x,z) = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^6} e^{ix\xi}e^{-iz\eta}e^{i(\eta-\xi)y}\partial_\xi\psi_{1ij}(x,\xi)\psi_{2lk}(y,h_k\eta)d\xi dyd\eta.$$

We will calculate the principle part, i.e. $\partial_\xi\psi_{10j}(x,h_j\xi)\psi_{20k}(y,h_k\eta)$, other terms would be no worse than it.

First for $\psi_{20k}(y,h_k\eta)$,

$$\psi_{20k}(y,h_k\eta) = \begin{cases} \chi_2(y)\theta_0(p_2(y,N^{-1}\eta))m(p_2(y,N^{-1}\eta)) & |\eta| \lesssim N \quad k=0 \\ \chi_2(y)\theta(p_2(y,h_j\eta))|p_2(y,h_k\eta)|^{\frac{1-s}{2}} & |\eta| \sim h_k^{-1} \ k \geqslant 1, \end{cases}$$

so

$$|\psi_{20k}(y,h_k\eta)| \lesssim 1. \tag{8.25}$$

Then for $\partial_\xi\psi_{10j}(x,\xi)$,

$$\partial_\xi\psi_{10j}(x,\xi) = \begin{cases} \chi_2(x)(\partial_\xi\theta_0(p_2(x,\xi))m(p_2(x,\xi)) \\ \quad +\theta_0(p_2(x,\xi))\partial_\xi m(p_2(x,\xi))) & j=0 \\ \chi_2(x)(\partial_\xi\theta(p_2(x,\xi))m(p_2(x,\xi)) \\ \quad +\theta(p_2(x,\xi))\partial_\xi(|p_2(x,\xi)|^{-\frac{1-s}{2}})) & j \geqslant 1. \end{cases}$$

Hence,

$$|\partial_\xi\psi_{10j}(x,\xi)| \lesssim 1.$$

By these estimates,

$$\int |K(x,z)|dx \lesssim 1, \int |K(x,z)|dz \lesssim 1,$$

then by Schur's Lemma

$$\|\partial_\xi\psi_{1ij}(x,D)\psi_{2lk}(x,h_kD)\tilde{\chi}_3h^{N_3}\|_{X_\delta^{0,\frac{1}{2}+}} \lesssim \|h^{N_3}\|_{X_\delta^{0,\frac{1}{2}+}}.$$

Therefore,

$$\Big|\int_0^\delta\int \bar{h}^{N_0}\partial_x(\chi_1(N^{-2+}\triangle)u^{N_1})\overline{\chi_1(N^{-2+}\triangle)u^{N_2}} \cdot \partial_\xi\psi_{1ij}(x,D)\psi_{2lk}(x,h_kD)\tilde{\chi}_3h^{N_3}\Big|$$

$$
\begin{aligned}
&\lesssim \|h^{N_0}\partial_x(\chi_1(N^{-2+}\triangle)u^{N_1})\\
&\quad\cdot \|_{L^2_{t,x}}\|\chi_1(N^{-2+}\triangle)u^{N_2}\partial_\xi\psi_{1ij}(x,D)\psi_{2lk}(x,h_kD)\tilde{\chi}_3h^{N_3}\|_{L^2_{t,x}}\\
&\lesssim N_1^{s_0+}\|h^{N_0}\|_{X_\delta^{0,\frac12+}}\|\partial_x(\chi_1(N^{-2+}\triangle)u^{N_1})\|_{X_\delta^{0,\frac12+}}\\
&\quad N_2^{1-}\|\chi_1(N^{-2+}\triangle)u^{N_2}\|_{X_\delta^{0,\frac14+}}\|\partial_\xi\psi_{1ij}(x,D)\psi_{2lk}(x,h_kD)\tilde{\chi}_3h^{N_3}\|_{X_\delta^{0,\frac14+}}\\
&\lesssim N^{s_0+}N_1^{0-}\|h^{N_0}\|_{X_\delta^{0,\frac12+}}\|Iu\|^2_{X_\delta^{1,\frac12+}}\\
&\quad\cdot\delta^{2\times(\frac12-\frac14)-}N_2^{0-}\|\partial_\xi\psi_{1ij}(x,D)\psi_{2lk}(y,h_kD)\tilde{\chi}_3h^{N_3}\|_{X_\delta^{0,\frac12+}}
\end{aligned}
$$

$$
\lesssim N^{s_0+}(N_1N_2)^{0-}\delta^{\frac12-}\|Iu\|^2_{X_\delta^{1,\frac12+}}\|h^{N_0}\|_{X_\delta^{0,\frac12+}}\|h^{N_3}\|_{X_\delta^{0,\frac12+}}. \tag{8.26}
$$

Then, for $|\alpha|>1$, we could estimate similarly, since

$$
\|\partial_x^{\alpha_1}\chi_1(N^{-2+}\triangle)Iu^{N_i}\|_{X_\delta^{1,\frac12+}}\lesssim N^{|\alpha_1|-}\|\chi_1(N^{-2+}\triangle)Iu^{N_i}\|_{X_\delta^{1,\frac12+}},\ |\alpha_1|\geqslant 1,
$$

and $h_j^{|\alpha|}N^{|\alpha|-}=2^{-\frac{j}{2}|\alpha|}N^{0-}$.

Conclusively,

$$
\begin{aligned}
|(8.24)|&\lesssim\delta^{\frac12-}\sum_j\sum_{|k-j|\leqslant\nu}\sum_{i,l=0}^{K-1}\sum_{|\alpha|=1}^{K'-1}2^{\frac{k-j}{2}(1-s)-\frac{j}{2}|\alpha|}N^{-1+s_0+}\\
&\qquad\cdot\|Iu\|^2_{X_\delta^{1,\frac12+}}\sum_{N_0\sim N_3}\|h^{N_0}\|_{X_\delta^{0,\frac12+}}\|h^{N_3}\|_{X_\delta^{0,\frac12+}}\\
&\lesssim N^{-1+s_0+}\delta^{\frac12-}\|Iu\|^2_{X_\delta^{1,\frac12+}}\sum_{N_0}\|h^{N_0}\|^2_{X_\delta^{0,\frac12+}}\\
&\lesssim N^{-1+s_0+}\delta^{\frac12-}\|Iu\|^4_{X_\delta^{1,\frac12+}}\lesssim N^{-1+s_0+}\delta^{\frac12-}\|v_0\|^4_{H^1}.
\end{aligned}\tag{8.27}
$$

Therefore, combining the results of section 8.2.1 and section 8.2.2, i.e. (8.21) and (8.27), gives the result of Lemma 5.1. □

Acknowledgment. — The author would like to thank her advisor Prof. D. Fang's guidance, and also her co-advisor Prof. N. Burq for his proposal to this problem and lots of discussion.

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