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DIFFERENTIAL EQUATIONS AND ALGEBRAIC TRANSCENDENTS: FRENCH EFFORTS AT THE CREATION OF A GALOIS THEORY OF DIFFERENTIAL EQUATIONS 1880–1910

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Abstract. — A "Galois theory" of differential equations was first proposed by Émile Picard in 1883. Picard, then a young mathematician in the course of making his name, sought an analogue to Galois's theory of polynomial equations for linear differential equations with rational coefficients. His main results were limited by unnecessary hypotheses, as was shown in 1892 by his student Ernest Vessiot, who both improved Picard's results and altered his approach, leading Picard to assert that his lay closest to the path of Galois. The subject became interesting to a number of French researchers in the next decade and more, most importantly Jules Drach, whose flawed 1898 doctoral thesis led to a further reworking of the subject by Vessiot. The present paper recounts these events, looking at the tools created and at the interpretation of the Galois legacy manifest in these different attempts.

Résumé (Équations différentielles et transcendants algébriques : les efforts français sur la création d'une théorie de Galois pour les équations différentielles 1880–1910)

Une « théorie de Galois » pour les équations différentielles a été créée pour la première fois par Émile Picard en 1883. Picard, à cette époque un jeune mathématicien qui cherchait faire une réputation, a façonné une théorie analogue à celle des équations algébriques de Galois pour les équations différentielles linéaires à coefficients rationnels. Ses résultats étaient limités par des hypothèses superflues, un fait démontré en 1892 par son élève Ernest Vessiot, qui a amélioré les résultats de Picard en modifiant son approche.

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Cette modification a mené Picard à affirmer que c'était son approche à lui qui restait plus fidèle au chemin tracé par Galois. Le sujet a intéressé plusieurs chercheurs en France dans les années qui suivirent, le plus important étant Jules Drach, dont la thèse erronée de 1898 a provoqué encore une intervention de Vessiot. Cet article relate ces évènements, en considérant les outils utilisés et l'interprétation du legs de Galois manifestée dans une série d'efforts divers.

1. INTRODUCTION

The reception of the work of Évariste Galois on the solution of polynomial equations, and the ways in which the name of Galois became emblematic for a certain kind of mathematical creativity, make a complicated story. In this paper we take on the question of what it meant in the context of the study of differential equations. As the pervasive presence of groups in mathematics dawned on at least some important researchers—Felix Klein, Sophus Lie, Henri Poincaré—the idea of employing an analogous theory for differential equations was enunciated by Poincaré's associate Émile Picard, whose first publication on the subject was in 1883. This was followed by further work of Picard, Ernest Vessiot, Jules Drach, and other French mathematicians, leading on the one hand to what has come to be called the Picard-Vessiot theory, an object of renewed research interest in recent years [Magid 1999]; and the "logical" integration theory of Drach. All three of these writers claimed their own approach as being the true heir to the essential ideas of Galois. In what follows we try to unpack what they meant by this, why there was some divergence, and what the claim means about values in mathematics and the relations between algebra and analysis in the late nineteenth century.

It was to become a commonplace of twentieth-century mathematics to pattern one theory on another, attempting to find analogous components and then exploiting similarities of "structure" in order to find results. Indeed, the structural turn has been described by Corry and others as characteristic of much pure mathematics of the twentieth century, though the idea of analogical building of theories is only one component of this approach [Corry 2004]. In fact, the notion of a mathematical theory was in transition in the last decades of the nineteenth century, when the term was used commonly in a non-technical way to denote a body of connected results on a single subject. Formal theories in the sense of Russell and others were a construction that was to come in the future. Indicative of the way in which the term was used are the following remarks of Francesco Brioschi, a senior observer describing what he sees as a modern tendency:

> The characteristic note of modern progress in mathematical studies can be recognized in the contribution that each special theory—that of functions, of substitutions, of forms, of transcendents, geometrical theories and so on—brings to the study of problems for which in other times only one seemed necessary … France, which, following the disaster of 1870, drew from it new and powerful scientific vitality, and has given proof of it in every realm of knowledge, has not remained outside this movement…[1] [Brioschi 1889, 72].

Despite the superficial resemblance between the problem of solving a polynomial equation and that of solving an ordinary or partial differential equation, the idea of creating a Galois theory for differential equations faced formidable obstacles. In the case of the original Galois theory, one starts with a polynomial equation. The theory relies on the correspondence between a splitting field that is an algebraic extension of $\mathbb{Q}$ and the automorphism group of the polynomial, that is, a subgroup of the permutation group of the roots. The main theorem of the subject states that if that group is solvable then the equation is solvable by radicals; this requires the notion of normal subgroup, a key construct of the theory. Yet the words with which we describe these objects easily now all emerged with their present meaning during and after the period we are discussing. In particular, the relationship between substitution groups (in the sense of Camille Jordan's 1870 treatise) and what Sophus Lie termed "transformation groups" was seen by many writers (including Jordan and Lie) as one of analogy rather than of identity of structure; and fundamental features of today's group concept (notably the presence of inverses) had a problematic status. Similarly the notion of an entity called a field, while adumbrated for example in Dedekind's work, existed alongside the notion of a slightly more fluid concept issuing from the work of Kronecker, the *domain of rationality*. The shifting meanings of these terms and a resulting

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1 "La nota caratteristica del progresso moderno degli studi matematici deve rintracciarsi nel contributo che ciascuna speciale teoria, quella delle funzioni, delle sostituzioni, delle forme, dei transcendenti, le geometriche e così via, porta nello studio di problemi pel quale in altri tempi forse una sola fra esse sembrava necessaria … La Francia, la quale dalle sciagure del 1870 seppe ritirarre nuova e potente vitalità scientifica, e ne ha dato ampie prove in ogni ramo dello scibile, non rimase estranea a quel movimento …".

vagueness in understanding the relationship between them pervades the work that we shall discuss in what follows.[2]

Thus when we move to the context of differential equations, we are immediately faced with a mass of difficulties. The “obvious” corresponding object to the rationals is the field of rational functions (in one variable) but since there is no result corresponding to the fundamental theorem of algebra there is not an evident analogy to the splitting field. Other complications include the fact that in the case of a differential equation of order p the set of solutions, far from finite, depends on up to p continuous parameters, and hence the groups involved would be continuous groups. This is where Lie’s theory comes in: the analogy to the symmetric group of finite permutations of roots is, in Lie’s work, the general linear group, and the appropriate subgroups are those that leave the equation fixed while transforming the solutions into each other (in which case the equation is usually described as *admitting* the transformation). Nothing here is as simple as in the algebraic context, and the search for the appropriate analogous structures was a large part of the struggle faced by the researcher.

Despite all this, in the years before 1880 many researchers had identified specific points of analogy between the theory of polynomial equations and those of linear differential equations, and this gave reason for optimism. Euler’s complete solution of homogeneous linear equations with constant coefficients through the very mechanism of looking at a closely analogous polynomial equation dated from 1750 [Euler 1753]. Euler begins with a linear ordinary differential equation of order n. Then “ante omnia ex ea formetur haec expressio Algebraica $P = A + Bz + Cz^2 + Dz^3 + Ez^4 +$etc. cuius quaeratur omnes factores simplices…”[3]. Here the algebraic character of the analogy is made explicit—the expression P is repeatedly referred to as an algebraic formula, with the word algebraic capitalized. The correspondence is established between the *order* of the differential equation and the *degree* of the associated polynomial employed as a tool in its solution. (Euler in fact uses the same term, *gradus*, for both.)

By 1881, Paul Appell took up the question of the analogy in a context remarkably close to that of Galois, seeking differential analogies to symmetric functions of the roots. In a two-page introduction, Appell gave an

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2 It is worth noting that the French term “structure” was explicitly introduced in a closely related context by one of the principal actors we discuss, Ernest Vessiot (1865-1952), who used this word as a translation for Sophus Lie’s *Zusammensetzung*. [Hawkins 2000, 168], [Vessiot & de Tannenberg 1889, 137].

3 “… before anything else one forms from it this algebraic expression … of which all the simple factors are sought.”

extended survey of earlier work that had explored other features of the analogy between the two mathematical contexts [Appell 1881, 391-392], aspects of which were very well known in the Paris mathematical world of that time. Appell mentions first Lagrange's result that a reduction of order of a linear differential equation is possible when a solution is known, analogous to using a linear factor to reduce the degree of a polynomial equation. He then rapidly lists work of Liouville, Libri, Frobenius and others, noting in particular the idea of irreducibility of a differential equation due to Frobenius. Much closer to home he notes recent work of Jules Tannery (from 1874) expounding and extending the work of Lazarus Fuchs on linear differential equations; and the 1879 thesis of Gustave Floquet exploiting an analogy with polynomials through the use of factorization of a differential equation. These works made familiar the notion of a fundamental system of n (linearly independent) solutions of a linear differential equation of order n, an idea due to Fuchs, and demonstrated some of its utility.[4] The same year, 1879, saw two papers by E. Laguerre who discussed the question of invariance of a linear differential equation under a transformation of the variables [Laguerre 1879a], [Laguerre 1879b]. Thus Appell (and Picard, soon to explore this path) entered into the study of the subject at a time when such analogies were being actively and widely explored both in France and elsewhere.

If $y_1, \ldots, y_n$ is a fundamental system of solutions Appell's own work identified the analogue of the symmetric functions of the roots:

> the functions in question are polynomials in $y_1, \ldots, y_n$ and their derivatives which reappear multiplied by a non-zero constant when we replace $y_1, \ldots, y_n$ by the elements $z_1, \ldots, z_N$ of another fundamental system, that is when we make a linear substitution of the form $y_i = C_{i1}z_1 + C_{i2}z_2 + \cdots + C_{in}z_n \ldots$ [5][Appell 1881, 392]

We see here the analogue of algebraic invariants in the presence of invariance up to a constant multiple. Appell's key theorem states that every such function for a monic linear differential equation of order n may be expressed as an algebraic function of the coefficients multiplied by $e^{-\int a_1 dx}$ where a_1 is the coefficient of the term of degree $n-1$. Without going into

[4] Jordan used this idea around the same time, in [Jordan 1873/74], on a suggestion by Hamburger. I thank F. Brechenmacher for this information.

[5] "Les fonctions en question sont des fonctions algébriques entières de $y_1, \ldots, y_n$ et de leurs dérivées qui se reproduisent multipliées par un facteur constant différent de zéro quand on remplace $y_1, \ldots, y_n$ par les éléments $z_1, \ldots, z_N$ d'un autre système fondamental, c'est-à-dire quand on fait une substitution linéaire de la forme $y_i = C_{i1}z_1 + C_{i2}z_2 + \cdots + C_{in}z_n \ldots$"

detail, we note that this continues Laguerre's approach, and thus rests on work of Hermite (on the algebraic invariant side) and Liouville, as the citations in Laguerre make clear. Appell provides several applications, including a necessary and sufficient condition for two n-th order linear differential equations to have a common solution.

Despite the proximity to the general approach of Galois theory, the name of Galois is not mentioned by Appell. The term *substitution* comes up, and the focus is on invariance (up to a constant multiple, as in the algebraic theory of invariants); the word group does not occur. And, indeed, there is no approach to the question of the existence of some "extension" of the "field" where the coefficient functions come from in which solutions would occur. It was this that was to be explored by Picard, who is likewise the first to claim explicitly his own role as an intellectual heir of Galois.

As [Brechenmacher 2007] illustrates well in a related context, the idea of what algebra was about was likewise very much in transition at this time. He draws attention to changes in algebraic practice and values, noting both the shift away from "generic" formulas (ones that admit exceptions, for example in the case of singularities) and the very capacity of algebra to attain generality. In this context, Drach's placing of Galois theory at the pinnacle of algebraic achievement, argued on both mathematical and broader "metaphysical" grounds, is of great interest, and gives a particularly privileged position to its transcendental analogue.

The efforts to create this analogue met with limited success, and we will not explore fully here the reasons that research in the area stalled for a long time. An account of later work related to this, centered on the activity of Ellis Kolchin, is presented in [Borel 2001], where the emphasis is on the history of the theory of Lie groups. In our period, the most active workers on the Galois theory of differential equations were principally French: Émile Picard, Ernest Vessiot, Jules Drach, Emile Cotton, Arthur Tresse.[6] The reasons for this French enthusiasm seem likely to stem in part from the image of Galois, which came to shine forth as a symbol of national brilliance, as is argued elsewhere in this volume. Further, the theories of ordinary and partial differential equations remained a large and important area of mathematics that had resisted efforts at systematization since the creation of the differential equation, though around 1880 the notion of transformation groups promised to provide an important conceptual tool

6 The main non-French exception is Ludwig Maurer, discussed in detail in [Borel 2001, 102 ff.], where the likelihood that Picard did not know of this work is expressed.

for this purpose, in the hands of Poincaré and Klein on the one hand; and due to the ideas of Lie on the other [Hawkins 2000], [Gray 2000]. This analogy is illusory: while an r-parameter Lie group leaving a differential equation invariant is to be the analogue of the Galois group in the sense that it permits a simplification in integrating the equation, most equations are not invariant under a Lie group of positive dimension. This is explicitly stated already by Drach:

> An ordinary differential equation .. of order higher than the first does not admit in general a group in the sense of M. Lie ... one can thus affirm that the *application indicated my M. Lie of his theory of groups to the integration of [differential] equations is not the true generalization of the method used by Galois for algebraic equations.*[7] (Emphasis in original) [Drach 1898, 247-248]

In the work we describe, there are several features that, if perhaps not surprising, have not received much attention in the usual accounts. One is the dominance of versions of Kronecker's "rationality domain" (*domaine de rationalité, Rationalitätsbereich*) as a central tool of the theory. Defined by Kronecker with this label in his 1882 *Grundzüge*, and used as early as 1853 with a slightly different label, these domains are rational functions of a collection (finite in Kronecker's way of thinking) of unspecified "quantities," with integer coefficients. For example, if the unspecified entity is the variable x, we have what we would now term the field of rational functions of a single variable [Kronecker 1882, 3-4]. However, these objects need not be restricted to variables that may take their value in some specified domain; they can, for example, include functions. And while Kronecker himself might have chosen to limit those functions (say, by requiring constructibility), the writers in France who worked with this tool had no such scruples, typically. The question of what happens when one includes transcendental functions in this list, for example, gave a specific direction to the research of Jules Drach, and in fact was a source of some of the problems with this work.

This is not the first time these questions have been approached by scholars. In particular [Pommaret 1988] discusses in some detail the relation of this work to contemporary efforts, particularly his own, with more attention to mathematical detail and to the relation to present theory. [Borel 2001] touches on these themes repeatedly with particular regard to the history of algebraic groups, again with a strongly mathematical

7 Une équation différentielle ordinaire ... d'ordre supérieur au premier, n'admet pas, en général, de groupe au sens de M. Lie ... on peut donc affirmer que *l'application, indiquée par M. Lie, de sa théorie de groupes à l'intégration des équations n'est pas la véritable généralisation de la méthode employée par Galois pour les équations algébriques.*

focus. [Bkouche forthcoming] has devoted a great deal of attention to the later reception of this work, particularly in the context of the work of Cartan. Finally, the thesis of Caroline Ehrhardt gives an account of these matters with particular historical focus on this work as part of the image of the heritage of Galois. Ehrhardt provides a nuanced account of the meaning of differential Galois theory for views about Galois's achievements, and we return to this in the conclusion.

2. PICARD'S FIRST EFFORTS OF THE 1880S

The first efforts to formulate and carry out this program were due to Émile Picard, who possessed several characteristics that conform with him being a key player. He was thoroughly familiar with the role of group theory in the study of linear differential equations as it was then being formulated by his close colleague Henri Poincaré. He was exposed to the ideas of Lie during Lie's visit to Paris in 1882. And, mentored by his father-in-law Charles Hermite, he was exposed to the notion that a French national revival in mathematics was of key importance at the time. Many years later he expressed himself repeatedly as a scientific nationalist, and while I have no evidence that he held these views as a young man, the later expressions are certainly consistent with this.

Sophus Lie came to Paris for a second visit in the Fall of 1882, where, famously, Poincaré told him that all mathematics was a story about groups and learned in return about Klein's Erlangen Programm. Hawkins has carefully discussed in some detail the ways in which Lie saw his theory as analogous to Galois theory in the 1870s and 1880s, though the exact character of Klein's and Lie's conceptions of Galois theory remains somewhat murky, particularly early in the period [Hawkins 2000, sections 1.3, 2.2 and passim].

Whatever Lie may have said exactly, by the Spring of 1883 Picard informed Lie that he hoped to take Lie's work—notably his 1880 *Annalen* paper [Lie 1880]—as the "starting point" for a memoir on linear differential equations—see the letter from Picard to Lie cited in [Hawkins 2000, p. 186, fn. 9]. Now [Lie 1880] is a lengthy exposition of his theory of groups of contact transformations, while Picard's work has a different focus and different methods, though the importance of the transformation group concept is common to both, and Picard makes a start at employing Lie's ideas in order to determine the relationship between Lie's infinitesimal transformations and his own algebraic groups. Lie himself points to the many "points of contact" (*Berührungspunkte* [Lie 1880, 525]) of his own work on transformation groups and Galois theory, but also with invariant theory, geometry and "the modern theory of manifolds" (in Riemann's sense, as he later clarifies) and with the theory of differential equations. However solvability of differential equations was not his primary concern. When describing analogies to his own theory he points instead to Jordan's theory of groups of substitutions, pointing out that Jordan's work is concerned with discrete rather than continuous collections.

In Picard's announcement to the *Académie* he was highly optimistic about the potential of his approach. As a relative newcomer in the world of French mathematics, the fact that he drew explicitly on Galois's name is evidence both of his perception of the broad reputation of Galois and of the potential power of that association:

> By using a method presenting the greatest analogy with that used by the illustrious geometer Galois, we arrive at a proposition that seems to correspond to Galois's fundamental theorem, and we are thus led to the notion of what I shall call the group of linear transformations corresponding to the differential equation. [8][Picard 1883, 1131].

What are the elements of analogical structure that are needed for a theory like that of Galois for algebraic equations? For clarity we describe this in today's terms. First of all there is the field of coefficients of the polynomial, originally the rationals. We then seek to identify the group of permutations of the roots of the polynomials such that any algebraic equation of the roots is still satisfied after the permutation. The coefficients are elementary symmetric polynomials in the roots, and we likewise need an analogue to this. For the polynomial case. this Galois group has a set of subgroups; we want normal subgroups which will correspond to the lattice of intermediate fields of a (normal, separable, i.e., Galois) extension of the base field that contains the elements needed to solve the equation. The fundamental theorem of this theory establishes a one-to-one correspondence between the lattice of subgroups of the Galois group and the intermediate fields, where the normal subgroups correspond to Galois extensions.

The announcement of 1883 was only followed up by a fuller version in 1887, in the *Annales de Toulouse*, [Picard 1887]. In order to see what Picard was doing, we follow the notation of the later paper, which is just a fuller

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[8] En employant une méthode présentant la plus grande analogie avec celle dont a fait usage l'illustre géomètre [Galois], on arrive à une proposition qui semble correspondre au théorème fondamental de Galois, et l'on est ainsi conduit à la notion de ce que j'appelerai le groupe de transformations linéaires correspondant à l'équation différentielle.

version of the former. Picard first considered a homogeneous linear differential equation of order n,

$$\frac{d^m y}{dx^m} + p_1 \frac{d^{m-1}y}{dx^{m-1}} + \cdots + p_m y = 0. \tag{2.1}$$

In order to get at the notion of transformations of the solutions leaving the equation fixed, he considers

$$V = A_{11}y_1 + \cdots + A_{1m}y_m + A_{21}\frac{dy_1}{dx} + \cdots + A_{2m}\frac{dy_m}{dx} + \cdots + A_{mm}\frac{d^{m-1}y}{dx^{m-1}},$$

a linear and homogeneous expression in the solutions and their derivatives up to order $m-1$. The coefficients here are arbitrary rational functions of x. This expression V serves Picard as a kind of differential equivalent to the Galois resolvent. By virtue of the fact that the y_i solve (2.1), V itself satisfies a linear homogeneous equation of order m^2

$$\frac{d^{m^2}V}{dx^{m^2}} + P_1 \frac{d^{m^2-1}V}{dx^{m^2-1}} + \cdots + P_{m^2}V = 0, \tag{2.2}$$

where the P are rational functions. By differentiating V sufficiently many times one obtains m^2 first-degree equations in V and its derivatives, that may be solved for y_i. Thus for any integral of (2.2) there is a system of solutions of (2.1). These can fail to be linearly independent, a situation Picard gave a determinantal criterion to avoid. Typically, Picard noted, the equation (2.2) will be irreducible, in the sense that it has no solutions in common with an equation of lower order than m^2. If not, let $f = 0$ be an equation of lowest order p that has this property. In this case, the solutions of this lowest-order—irreducible—equation constitute a fundamental set of solutions for (2.1).

So, any fundamental system of solutions $Y_1, \ldots, Y_m$ may be expressed in terms of a given system of solutions $y_1, \ldots, y_m$ corresponding to a given V in an $m \times m$ system

$$Y_i = \sum_j a_{ij}y_j, \quad i = 1, \ldots, m$$

where the coefficients depend on p parameters. Picard then argues that the coefficients must be algebraic functions of those parameters. This is a key point, since it implies that the transformation respects the algebraic relations between the roots.[9]Picard noted that we have here a group, due to the fact that the product of two members of this set of "substitutions" is again such a substitution. Here he cited [Lie 1880], introducing the term

9 This is equivalent to the idea that the group is an analogue to the group of automorphisms of the splitting field of a polynomial equation.

"groupe continu de transformations." In passing, we note that such transformations were termed finite by Lie, since they depend on a finite number of parameters. This usage made it also into the French literature; the group, denoted G, is termed a "groupe continu et algébrique" by Picard.

The centerpiece of both treatises is the following result:

> Every rational function of x, and of $y_1, y_2, \ldots, y_m$ as well as their derivatives, being expressed as rational functions of x, remains invariant when one performs the substitutions of the group G on $y_1, y_2, \ldots, y_m$.[10] [Picard 1887, A.5]

Picard adds to this the observation that the group of the equation in the older sense, that is, the group that takes one set of fundamental solutions to another as the independent (complex) variable traces its various paths in the plane, is contained in G—that is, corresponds to certain values of the parameters—and hence that rational functions of the y_i and their derivatives that are fixed by G are uniform as functions of x.

Why the delay in the publication of the extended version? Picard does not tell us, but [Picard 1883] adds to the result just cited its converse, "Every rational function of x, and of $y_1, y_2, \ldots, y_m$ as well as their derivatives, which remains invariant under the substitutions of the group G is a rational function of x."[11] [Picard 1883, 1133]. However, in the years in between, Picard realized it was necessary to add a hypothesis in order to have this converse. This matter had been taken up already by Fuchs in a number of papers in the 1850s and 1860s, and rendered clearer both notationally and as regards theoretical clarity by Frobenius in the early 1870s [Gray 2000]. In these papers, though Picard does not mention this, it had been shown necessary to add regularity conditions on the coefficients: the singularities of the coefficients of the terms of order i must be poles of order no more than i. This is corrected in the later paper. A further advance may be seen on a matter that Picard already formulated in [Picard 1883], namely the problem of actually identifying algebraic groups of this kind.

Just to assist the reader, we reproduce a pretty example from [Picard 1883, 1134]. Consider the equation

$$(x - x^2)\frac{d^2y}{dx^2} - \frac{x}{2}\frac{dy}{dx} + \alpha^2 y = 0,$$

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10 Toute fonction rationelle de x, de $y_1, y_2, \ldots, y_m$ ainsi que de leurs dérivées, s'exprimant rationellement en fonction de x, reste invariable quand on effectue sur $y_1, y_2, \ldots, y_m$ les substitutions du groupe G.

11 "Que nous ne ferons qu'énoncer: Toute fonction rationnelle de x et de $y_1, y_2, \ldots, y_m$, ainsi que de leurs dérivées, qui reste invariable par les substitutions du groupe G est une fonction rationnelle de x.

where α is a constant. The group of transformations depends on a single parameter and is of the form

$$Y_1 = \lambda y_1 + \sqrt{1-\lambda^2}\,y_2$$
$$Y_2 = \sqrt{1-\lambda^2}\,y_1 - \lambda y_2.$$

For any fundamental system of solutions, $y_1^2 + y_2^2 = 1$.

The use of the resolvent function V constitutes Picard's focus at this time. While he does not provide the details of the calculations, the resolvent V is found by finding the system of equations it satisfies and using the given equation to progressively reduce its order. We also note that, in restricting himself to rational functions as coefficients, he remains in close parallel to Galois's theory of equations with rational coefficients. A kind of side effect is that the linear differential equations that he treats are handled in a way that remains closely linked to the theory of Fuchs and Frobenius. Both of these features were to be altered by the thesis of Ernest Vessiot.

3. VESSIOT'S 1892 THESIS AND PICARD'S TRAITÉ

Picard was to continue his interest in this theory, which found a further augmentation in his *Traité d'analyse*, the third volume of which appeared in 1896. In the meantime the subject had been taken up and substantially augmented by Ernest Vessiot (1865–1952). Vessiot entered the École normale supérieure in 1884, second in the entrance examination after Hadamard. He obtained a position as a lycée instructor in Lyon, but in 1888 he was one of two students, the other being Wladimir de Tannenberg, to be selected to go for a term to Germany, in this case to Leipzig to study with Lie [Hawkins 2000, 196]. This program of sending *normaliens* (and others) to Germany had been instituted during the Ferry ministry in order to improve the knowledge of German intellectual and scientific trends among leading French students [Digeon 1959, 375–383]. Others included Arthur Tresse, who was to study under Lie in 1891–92, and Jules Molk, who studied with Kronecker in Berlin. Vessiot thus attained a considerable knowledge of the details of the current state of Lie's ideas, which he and de Tannenberg described at length in a 36-page *compte rendu* in the *Bulletin des sciences mathématiques* in 1889 [Vessiot & de Tannenberg 1889]. These studies and the work of Picard were the immediate background to Vessiot's thesis of 1892.

In the thesis and subsequent work Vessiot takes a different point of view from Picard with regard to the coefficients, generalizing them beyond the rational functions to an arbitrary domain of rationality. This notion, originating with Kronecker, and developed most extensively in [Kronecker 1882], includes in Vessiot's case the (real or complex) constants, the independent variable x and all rational functions of x and the coefficient functions. This marks a departure from anything Kronecker did, or likely could have done, since it undermines the idea of restricting the domain to constructible entities and hence the whole reason for using the generalized congruence approach that is fundamental to Kronecker's work.

Vessiot does not mention Kronecker and seems to come upon the idea, which he does not expressly label, in a natural way by beginning, not with a given equation and a fundamental system of solutions, but with a set of functions that will then satisfy a linear differential equation whose coefficients are given as the analogues of the elementary symmetric functions of the roots, that is, by Cramer's rule applied to the same system of linear equations of the functions and their derivatives that Picard derived starting from the equation. Vessiot refers to these as "rational functions of the integrals" [Vessiot 1892, 213]; the denomination "rationality domain" appears in his later work.

Vessiot's thesis is a self-contained object that begins with a survey of the relevant points of Lie theory. He notes, in particular, Lie's idea of an integrable group, where by group he means transformation group with r parameters: a group G is integrable if it has a chain of invariant subgroups each of which has one parameter less than the previous one.[12]

This notion of integrable group is the tool that Vessiot employs to get at the analogue of solvability questions in the Galois case. He first requires analogues of the Picard theorems for the "domain of rationality" case, which requires a certain amount of machinery. The theorem he obtains states:

> To every linear equation corresponds a group Γ of homogeneous linear transformations, which possesses the two following properties: 1. Every rational function of the integrals that has a [singular] rational expression admits all the transformations of the group. 2. Every rational function of the integrals that is

[12] Vessiot cites Engel's 1887 result that a group is integrable if and only if it contains no 3-parameter subgroup with the structure of the general projective group in one variable.

invariant under all the transformations of the group has a rational expression.[13] [Vessiot 1892, 231]

Here the "rational expression" refers to the domain of rationality in question. This obscures the difference between Picard's theorem and Vessiot's; we note that in Vessiot's case the requirement for regularity of the coefficients in part 2 is eliminated. Vessiot notes that part 1 is just Picard's theorem.

With this in hand, Vessiot turns to the question that would assuredly have identified the theory as analogous to Galois's in the minds of many readers, namely, the relationship to solvability of differential equations. Vessiot's idea is to attempt a reduction of order by the adjunction of a particular "integral," that is, something that is expressed by a single integration of a function in the domain of rationality, that is, a solution of a first-order linear equation. In this situation the analogue of solvability of a polynomial by radicals becomes solvability by quadratures, in Vessiot's term. Because of the issue of what the domain of rationality may be, the idea of an integration by quadrature winds up feeling more than a little vague, in terms of what can actually be accomplished. The key result states that for a linear equation to be integrable by quadratures in this sense, it is necessary and sufficient that the transformation group of the equation be an integrable group. Necessity is easy; the proof of sufficiency requires a technical result of Lie about the structure of integrable groups of infinitesimal transformations, something that is discussed almost 600 dense pages into Lie's treatise on transformation groups. [Vessiot 1892, 241-245]. One observes that direct exposure to Lie would be, if not indispensable for this task, at least extremely useful. Even so, it was necessary for Vessiot to assume in the proof that the group of the equation have a normal decomposition consisting of algebraic groups in Picard's sense, something mentioned expressly by Vessiot as a limitation. [Vessiot 1892, 247] The last chapter of the thesis concerns second and third order linear equations, and enabled Vessiot to state that linear equations of these orders can present no "particularités intéressantes" beyond those already identified by Laguerre and Halphen by more traditional methods.

Whatever the limitations, the thesis contained very solid results. It was presented in detail in Picard's *Traité d'analyse*, as the culmination of the

[13] À toute équation linéaire correspond un groupe Γ de transformations linéaires homogènes, qui jouit des deux propriétés suivantes: 1. Toute fonction rationnelle des intégrales qui a une expression [singulière] rationnelle admet toutes les transformations de ce groupe; 2. Toute fonction rationnelle des intégrales invariantes par toutes les transformations de ce groupe a une expression rationnelle.

third volume. The last two chapters of that volume are in fact devoted to Galois theory and its differential counterpart, with the points of analogy spelled out in considerable detail [Picard 1896, Chs. XVI, XVII]. And Picard was not alone in finding the work seminal.

Yet Picard had already distanced himself in print from Vessiot's approach, claiming in [Picard 1894, 585] that:

> M. Vessiot in his work takes a point of view quite different from mine, and the path I followed to set up the bases of this theory, a path that is very close to that taken by Galois for algebraic equations, seems to me preferable in various ways.[14]

In this short paper Picard gives a separate proof of the converse portion of the main theorem, eliminating the regularity restriction. The argument revolves once again around the use of the resolvent V. The requirement is to show that if Φ is rational in x and the y_i and their derivatives, and invariant under actions of G, then there is a fundamental system of solutions Y_i in which Φ is rational in x. The insight of Picard was that the resolvent function could be used to achieve this: replacing the y_i and their derivatives by the corresponding values in terms of V, the invariance together with the original differential equation provide a straightforward argument based on the maximum degree of V that Φ must be rational in x alone.

The claim that this is closer to Galois than the work of Vessiot seems to be based largely, then, on the idea that it is the use of the resolvent that is the central feature of Galois's work. Picard was soon to make a similar claim made about the work of Jules Drach:

> M. Drach for his part is working on the application of the theory of groups to the theory of differential equations, however taking a point of view different from me. [15][Picard 1895, 792]

In this case, it is true that Drach takes a significantly different view of the essential features of Galois theory and its differential analogue. To this point we now turn.

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[14] "M. Vessiot se place dans son travail à un tout autre point de vue que moi, et la marche que j'ai suivie pour poser les bases de cette théorie, marche qui se rapproche beaucoup de celle de Galois pour les équations algébriques, me paraît à divers égards préférable."

[15] M. Drach s'occupe de son côté de l'application de la théorie des groupes à la théorie des équations différentielles, en se plaçant d'ailleurs à un autre point de vue que moi.

4. JULES DRACH AND THE CLASSIFICATION OF TRANSCENDENTS

4.1. *Drach and Galois Theory*

Jules Drach (1871-1949) came from an Alsatian family that fled the Prussian occupation to the non-annexed part of Lorraine. Jules entered the ENS in 1889, completing the agrégation in 1892; he was judged to have done rather poorly, and this lack of recognition may in part account for his enthusiasm for Galois. He nonetheless was encouraged by the influential Jules Tannery, then *directeur des études scientifiques* of the ENS and someone who had taught Drach, who ranked Drach with his contemporary, collaborator and friend Émile Borel. In the year of his *agrégation* Drach edited, with Borel, the lectures of Poincaré on elasticity; and they collaborated again in 1895 in reworking the 1891-1892 lectures of Tannery, published in 1895 as *Introduction à l'étude de la Théorie des nombres et de l'algèbre supérieure.* Borel and Drach appear as the authors of this text, Tannery explaining in the preface that he had only sketched some ideas, referring to these lectures as conversations. Of Drach's contribution he writes tellingly:

> ... he is one of those who concern themselves above all with the foundation of things, who remain discontent and anxious until they attain bedrock ... It is [Drach] who took on the most difficult task, and it is he who made it a personal *opus.*[16][Borel & Drach 1895, II-III]

In fact, Tannery notes further, he had scarcely raised the notion of algebraic numbers or Kronecker's basing of algebra on arithmetic, on the one hand, and of algebra as part of analysis on the other. It is relevant for our further discussion of Drach's work on *differential* systems to recall Tannery's further remark:

> The method of exposition to which he was led by the desire to reduce the construction of arithmetic and algebra to what is really essential consists essentially in viewing algebraic numbers, as well as positive or negative integers and rational numbers, as signs or symbols, *entirely defined* by a small number of properties posited *a priori* relative to two of their modes of composition. (Emphasis in original)[17] [Borel & Drach 1895, IV].

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[16] ... il est de ceux qui se préoccupent avant tout du fond des choses, qui restent mécontents et inquiets tant qu'ils n'ont pas atteint le roc ... c'est [Drach] qui avait assumé la plus lourde tâche, c'est lui surtout qui a fait œuvre personnelle.

[17] "Le mode d'exposition auquel il a été amené par le désir de réduire à ce qui est indispensable la construction de l'Arithmétique et de l'Algèbre consiste essentiellement à regarder les nombres algébriques, aussi bien que les nombres entiers positifs ou négatifs et les nombres rationnels, comme des *signes* ou symboles, *entièrement définis*

This entails, in Drach's exposition, the use of Kronecker's conception that to calculate with algebraic numbers is identical to calculating with polynomials in one variable with integer coefficients, in which one neglects the multiples of a given polynomial: thus an extension of Gaussian modular arithmetic to this specific algebraic context. What is more, Tannery also notes that the motivation for this mode of proceeding rests on the fact that the real basis of algebra as Drach conceives of it is the theory of Galois, and that his treatment of this theory is the underlying justification of the method he employed.

This method is seen as "purely logical," "independent of any experimental notion … and in particular of the notion of magnitude [*grandeur*]." [Borel & Drach 1895, IV-V]. Tannery expresses mild unease with this symbolic, content-free approach, but in the end concludes that his misgivings are probably not really justified. Drach himself, at the conclusion of the work, places his formulation of the algebra in a broader context:

> A general study of the various kinds of symmetry that can present themselves—the theory of groups of substitutions (**)—shows the nature of the relations that tie different rational functions of n variables to each other. [18] [Borel & Drach 1895, 334]

and the footnote referred to by (**) states "nous avons naturellement adopté dans cette théorie les notations introduites dans la théorie générale des Groupes de Transformations par son créateur, le célèbre géomètre norvégien Sophus Lie."

We thus see that by 1895 Drach was sufficiently acquainted with Galois theory to give an account of higher algebra taking it as the pinnacle of the subject, and as the specific aim of his exposition. Furthermore, he did so with at least a passing knowledge of some version of the contributions of Lie. It seems likely that this acquaintance owed at least something to Picard's work. At any rate, this was the jumping off point for Drach's doctoral thesis, published in 1898, *Essai sur la théorie générale de l'intégration et sur la classification des transcendantes.*[Drach 1898], which discusses Picard and Vessiot in the following terms:

> It was reserved for M. Picard to establish in a few already classic pages that the transcendants that satisfy linear homogeneous differential equations with

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par un petit nombre de propriétés posées *a priori* relativement à deux de leurs modes de composition."

[18] "Une étude générale des divers genres de symétrie qui peuvent se présenter—théorie des groupes de substitutions (**)—montre de quelle nature sont les relations qui lient entre elles les diverses fonctions rationnelles de n indéterminées…"

> coefficients rational in x are also incompletely determined by the rational relations that link them to their derivatives and to the variable; the indeterminacy is defined in this case by a system of linear homogeneous transformations acting on the elements of a fundamental system of solutions. M. Vessiot, starting from there and employing the beautiful results obtained by M. Lie in the study of the structure of linear groups, has been able to obtain necessary and sufficient conditions for a linear equation to be integrable by quadratures.[19] [Drach 1898, 246]

The already classic pages of Picard refer to [Picard 1887], while Vessiot's account is his thesis, [Vessiot 1892].

4.2. *The Classification of Irreducible Transcendents: Drach's Thesis*

Drach's idea of the heritage and purpose of his own work, and the reasons for the importance of Galois', are made explicit in the opening pages of his thesis, which likewise depicts a set of ideas about the relation between different mathematical disciplines that diverges markedly from the traditional views embodied by the work of such writers as Picard. Drach begins with an observation by Lacroix that "ce qui peut le plus contribuer aux progrès du Calcul intégral, c'est la classification exacte des divers genres de transcendantes absolument irréductibles, et par là essentiellement distincts, et la recherche des propriétés particulières à chacun de ces genres." [Drach 1898, 243]

Galois is then portrayed as having carried out this classification program for algebraic numbers and algebraic functions. Drach immediately emphasizes that "les nombres algébriques ne sont jamais déterminés d'une manière unique par les relations algébriques entières à coéfficients rationnels qu'ils vérifient." [Drach 1898, 244] That is, they are determined only up to conjugation; there is a collection of substitutions that leaves the equation fixed. By this means, Drach puts Galois's work at the head of a line including Puiseux and Riemann, Klein and Poincaré.

Drach's approach is to start the subject from the ground up, beginning with the properties of operations on the integers (associativity and so forth). His aim would appear to be to produce a complete and

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19 Il était réservé á M. Picard d'établir en quelques pages, déjá classiques, que les transcendantes qui vérifient des équations différentielles linéaires et homogènes á coefficients rationnels en x sont aussi incomplètement déterminées par les relations rationnelles, qui les lient à leurs dérivées et à la variable; l'indétermination est définie cette fois par un système de transformations linéaires et homogènes portant sur les éléments d'un système fondamental de solutions. M. Vessiot, partant de là et utilisant les beaux résultats obtenus par M. Lie dans l'étude de la structure des groupes linéaires, a pu énoncer les conditions nécessaires et suffisantes pour qu'une équation linéaire soit intégrable par quadratures.

well-founded theory, an aim which he expresses in citing a remark of Weierstrass:

> The more I reflect on the principles of the theory of functions ... the firmer my conviction becomes that this must be built on the foundation of algebraic truths, and that is is therefore not the correct path if , in the other direction, the "Transcendant" is called into service for the grounding of simple and fundamental theorems of algebra.[20]" [Picard et al. 1981, 254]

Drach had already embarked on the research program described in his thesis by 1895 ([Drach 1895]), and his acquaintance with Lie theory had already led to a paper in 1893 ([Drach 1893]),. These papers introduce two notions that he was to harness in the thesis, the point transformation and what he terms "logical integration," a theme that continued through his work for many years.

The thesis begins with an exposition of Galois theory in a form that allows the analogy and key features of the theory that are useful for his generalization to emerge clearly. He quotes tellingly from Galois the concluding phrases of the letter to Chevalier:

> It had to do with seeing a priori, in a relation between transcendental quantities or functions, what exchanges one could make, which quantities one could substitute for given quantities, without the relation ceasing to hold. It makes one recognize right away the impossibility of many expression that one could look for. But I don't have time, and my ideas are not yet well developed on this terrain, which is immense.[21]

In fact in [Drach 1895, 76] Drach had already identified the "theory of ambiguity in analysis" that Galois did not have time to explore with his own "intégration logique," so we now explore what this meant to him in the context of the thesis. We follow in large measure the account of this very long paper given in abstract by Landberg, since it seemed hard to improve on for both accuracy and conciseness [Landsberg 1898].

After the fundamental introduction that leads to the work of Galois on algebraic equations, the second part of the paper introduces the concepts

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20 Je mehr ich über die Principien der Functionentheorie nachdenke ..., um so fester wird meine Ueberzeugung, dass diese auf dem Fundamente algebraischer Wahrheiten aufgebaut werden muss, und dass es deshalb nicht der richtige Weg ist, wenn umgekehrt zur Begründung einfacher und fundamentaler algebraischer Sätze, das "Transcendente" in Anspruch genommen ist...

21 Il s'agissait de voir *a priori*, dans une relation entre des quantités ou fonctions transcendantes, quels échanges on pouvait faire, quelles quantités on pouvait substituer aux quantités données, sans que la relation pût cesser d'avoir lieu. Cela fait reconnaître de suite l'impossibilité de beaucoup d'expressions que l'on pourrait chercher. Mais je n'ai pas le temps, et mes idées ne sont pas encore bien développées sur ce terrain qui est immense.

of completely integrable system and logical integration. This is based on a kind of differential algebra, interestingly, in which a d operation (satisfying the sum and product rules, etc.) is added to the usual algebraic operations. A system of differential equations is called completely integrable if they are compatible (in the sense of having at least one solution) and if it suffices to determine all algebraic relations that obtain between the solutions and their derivatives [Drach 1898, 294]. The idea would seem to be that with such a system, while explicitly reversing the differentiations to solve may be impossible, the integral is determined by the data. Logical integration, then, is the reduction of a given system to a system which is both completely integrable and irreducible.

In the third part, if $z_1, z_2, \ldots, z_n$ are a fundamental system of solutions for

$$\frac{\partial z}{\partial x} + A_1 \frac{\partial z}{\partial x_1} + \cdots + A_n \frac{\partial z}{\partial x_n} = 0,$$

then the general (linear) group of substitutions from this system to another is called Γ_n. But if there is some algebraic relation between the z_i, there may also be a proper subgroup Γ leaving that equation fixed, as in the Galois case. The 'idea is that this "rationality" group of the equation, then, has the property that all the rational differential invariants are expressed rationally is functions of the independent variables x_i, and conversely anything that is rationally expressible in x_i is a function of the differential invariants. By extending the domain of rationality, adjoining appropriate transcendents, the rationality group can be reduced to a group that is simple in the sense of possessing no invariant subgroups.

The thesis is prolix, almost like a textbook. The basic ideas are often tangled up in an attempt to present at the same time a kind of formal theory, one in which calculation is eschewed but which is transparently founded. The central ideas relevant to differential equations are long in emerging and couched in unfamiliar language.

Despite these features, the reports on Drach's thesis by Picard and Darboux were highly favourable, and Picard in particular waxed enthusiastic. Prior to Drach's work, Picard noted, one could easily envisage the exension of his own and Vessiot's ideas to ordinary differential equations that shared with linear equations the property of having a fundamental system of integrals. Drach, on the other hand, has "complètement élucidé" the case of first order systems of algebraic differential equations, in Picard's enthusiastic announcement. Darboux writes more briefly, but likewise speaks of the work as "an original work of the highest value."[22] [Picard et al. 1981, 38-39]. The successful *soutenance* took place on June 24, 1898.

Drach's work proved, however, to contain serious errors, revolving around the fact that the rationality group he employed as the core of the theory need not always be defined. Given a fundamental system of solutions, the complete set of solutions should be obtained from one another by point transformations between them, in Drach's approach. However, in a given domain of rationality, the passage from one solution to another does not necessarily happen by means of a transformation that is rational (in that domain); and consequently there are cases that Drach's theory purports to cover where the rationality group fails to exist, rendering the entire theory murkyand the value of the approach suspect. This appears to have been noticed first of all by Vessiot, who wrote to Drach on October 3, 1898 on the matter [Picard et al. 1981, 40 ff].[23]

Vessiot expressed several concerns, and noted also that he had discussed the matter with Élie Cartan and Tannery. He gave a specific example, due to Cartan, where a key property claimed by Drach to guarantee the existence of the rationality group does not hold. This alone vitiates the thesis, but Vessiot identified other points of contention. Drach at first felt that the problems were simply resolved, but Vessiot was less convinced. Ultimately Vessiot consulted Paul Painlevé, who replied on October 17:

> I have just read the thesis of Drach, and I am completely in agreement with you in the incorrectness of the two fundamental theorems and of their proofs. The error is so great that I can hardly conceive that it escaped the author and the jury.[24] [Picard et al. 1981, 53]

Painlevé also noted that he had drawn the attention of Picard to the matter who agreed that the problem was as Vessiot had identified.

Errors in theses, or for that matter in published papers, are hardly unknown. Rarer are cases in which two leading researchers reporting on a work miss a flaw that undermines the main results fatally. For there was no

22 "Un travail original de la plus haute valeur"

23 These letters and related material have been collected carefully by. J.-P. Pommaret, who made them available to Dugac for publication in the *Cahiers du séminaire d'histoire des mathématiques de l'IHP*. Pommaret has written at some length about these events in [Pommaret 1988] and elsewhere, noting that Drach's approach was correctly anticipatory of later versions of the theory despite its errors.

24 Je viens de lire la thèse de Drach, et je suis absolument d'accord avec toi sur l'inexactitude des deux théorèmes fondamentaux et de leur démonstration. L'erreur est même tellement grosse que j'ai peine à concevoir qu'elle ait échappé à l'auteur et au jury.

quick fix in immediate view. In this case, Picard in particular, as a specialist in this precise field, could have been expected to identify the problem. Painlevé's opinion was that no-one who looked at the matter for 5 minutes should have overlooked it, as he noted to Vessiot in the letter cited above. One may wonder how this could have occurred, and I suggest various contributing factors.

Drach had presented the basic ideas of the work in the *Comptes Rendus* years prior, though in the form of announcements, that is, without detailed proof. Presented by Picard and Poincaré respectively, the general line of procedure as described in those papers recalled (explicitly) the 1892 thesis of Vessiot, with which Picard at least was intimately familiar. Yet one key algebraic tool, the domain of rationality, had not been used by Picard (though he did give an account of its use by Vessiot in his *Traité*). Hence the specific issue of the existence of the group that Drach employed possibly failing to exist due to shifts in the rationality group resulting from transformations performed on the fundamental system of solutions was at best a murky one, the more so since we are alerted by Drach to *no* issue of this character. We drew attention earlier to the long and rather unconventional presentation of the thesis, which likewise might have insulated the reader from a more energetic analysis of the work. Finally Drach was a known mathematical commodity, highly regarded by Tannery and with two books to his credit. It is not so hard to see how such an oversight might occur.

Nevertheless, the result could only be awkward at best for Drach, then teaching in a lycée. "C'est fort triste pour ce pauvre Drach," as Painlevé remarked [Picard et al. 1981, 53]. The fact that the thesis was badly flawed necessarily raised the question of whether the results could be fixed; and if so, could they be turned to good use, a question likewise raised by Landsberg at the end of his review, where he noted: "The exposition of the author is very general throughout, and it would be very desirable to see an account of the applicability of the conceptual developments used to genuinely interesting individual problems."[25]

25 [Landsberg 1898]: "Die Ausführungen des Verfassers sind durchwegs ganz allgemein gehalten, und eine Darlegung der Anwendbarkeit der eingeführten Begriffsbildungen auf wirklich interessante Einzelprobleme wäre sehr wünschenswert."

5. THE COMPETITION FOR THE GRAND PRIX: VESSIOT'S 1902 THEORY

By 1900, the Académie des sciences was responsible for a large number of prizes, but the Grand Prix des Sciences Mathématiques was highly prestigious. Awarded roughly every two years, it carried a 3000 Franc prize; previous winners included Minkowski, Picard, Painlevé, and Borel. The question announced for 1902 was "Perfectionner, en un point important, l'application de la théorie des groupes continues à l'étude des équations aux dérivées partielles."

While no account of the reason for selection of the question is known to me at this writing, the motivation for the selection of this question can hardly have been other than the problems with Drach's thesis combined with the imagined promise of the result. Such competitions were frequently designed to allow one or more writers the possibility of displaying scientific prowess in a way that would be highly visible and hence potentially useful for their careers. Not infrequently protégés of one or the other of the academicians were among the expected competitors. The competitions were anonymous (the famed *pli cacheté* was only opened if the memoir was chosen). In this case, Vessiot was awarded the prize, the memoir [Vessiot 1904] appearing two years later. Vessiot's title, "Sur la théorie de Galois et ses diverses généralisations," seems both a reply to Picard's earlier claim that Vessiot had not taken the true Galoisian path, and a claim of innovation beyond Drach's more broadly framed idea of the reach and import of Galois's ideas for the mathematics of the time.

Vessiot's claim, on the opening page, to have abandoned the method of proof of Galois thus has dramatic effect. Noting Picard's earlier success in the use this method, Vessiot points out the problem with using this approach in the case of the first-order partial differential equations to which Drach applied it, namely, the fact that the passage from one solution to another is not accomplished by the use of rational transformations.

Vessiot's strategy, and his view of this alternative generalization of Galois's theory, is presented first in the algebraic context, presumably to enhance its claim to be a genuine generalization of the original. First replacing the algebraic equation by the system (S) of relations between its roots x_i and the coefficients, he asks, what advantage can we take of the knowledge of certain relations (A) between the roots when only employing rational operations? His answer is that from the system $[S, A]$ one can deduce an analogous *automorphic* system whose solutions still solve the original system. By an automorphic system, he means one in which the solutions may all be deduced from a single solution by means of a the substitutions that belong to a group G—the group of the system. Hence the rational relations between the roots may be studied by looking at associated rational automorphic systems, with the central theorem becoming: there exists a system $[S, A]$, automorphic and rational, such that all its solutions are shared by any rational system if that system shares one of its solutions. The group of this automorphic rational system is the rationality group (Galois group) of the equation.

Vessiot's description of the position of this method with respect to historical efforts in this direction reveals his understanding of what his own heritage consists of:

> This method shows the link that unites, in this theory, the point of view of the numerical invariance of the functions of the roots, to which Galois attached himself, and, after him, M. Jordan, with the point of view of formal invariance, which seems to have been that of Kronecker. [26] [Vessiot 1904, 10-11]

Hence the link to Kronecker, already implicit in the use of rationality domains in Vessiot's earlier work, is here made explicit along another axis, with *formal* invariance and numerical fixity being expressly tied together. Galois's point of view is thus presented as consisting of examining "les simplifications que peut présenter la résolution d'une équation donnée." Vessiot links this with the point of view of Abel and (more recently) Lie, which consists in seeing how one can take advantage of "certaines circonstances particulières données" for the solution of an arbitrary equation.[Vessiot 1904, 11]. The paper thus rephrases algebraic Galois theory in this form, and in a second chapter reprises the thesis work of Vessiot in this language; he claims an improved rigour.

Turning to the case of equations of the form

$$\frac{\partial x}{\partial t} + \sum_{i=1}^{n} p_i(t, t_1, \ldots, t_n)\frac{\partial x}{\partial t_i} = 0,$$

Vessiot follows the analogous method of replacing the equation by differential relations between a fundamental system of n linearly independent solutions of the system $x_1, \ldots, x_n$ and the independent variables. This part redoes Drach's basic setup in more conventional language, identifying automorphic systems in the differential case which are in general not rational. In the concluding chapter Vessiot devises a means for characterizing

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26 Cette méthode montre le lien qui unit, dans cette théorie, le point de vue de l'invariance numérique des fonctions des racines, auquel s'est attaché Galois, et, après lui, M. Jordan, au point de vue de l'invariance formelle, qui semble avoir été celui de Kronecker.

when they will be rational, and applying this new criterion to the class of first-order equations under study to obtain a corrected version of the main results of Drach. The statement is long and rather technical, and is thus omitted here [Vessiot 1904, 74].[27]

6. CONCLUDING REMARKS

The paper of Vessiot restored this differential cousin of Galois theory to its earlier promise, and indeed may have done something to lessen any tarnish that might have been associated with Drach's errors, since aspects of the initial insights of Drach remained, though heavily reworked. However, while it provided a theoretical framework and a certain amount of insight as to the nature of the analogies between problems involving algebraic and differential equations, it was not particularly rich in affording actual solution methods (beyond the method of successive reduction of Vessiot in his thesis). The group-theoretic viewpoint it espoused thus remained one in which potential was seen, but where neither extension (for example to higher-order cases), nor application to obtaining a concrete solution, seemed to issue forth with ease. Vessiot, Drach, and others continued to explore the field intermittently, but new fruit was hard to pick.

Vessiot took the opportunity to present the theory to a larger public in a more expository form in the *Encyklopädie der mathematischen Wissenschaften* where he authored the chapter on ordinary differential equations [Vessiot 1900]. The last section of this concerns "rational theories of integration," where the term rational is expressly linked to the idea of domains of rationality. Here the rational integration theories of Picard, Vessiot, and (somewhat elliptically) Drach are presented as formed in the image of Galois's theory of equations.[Vessiot 1900, 288]. This exposition received a considerable extension in the French version edited by Molk, with a much fuller bibliography and, of course 10 intervening years that included Vessiot's rehabilitation of Drach's work[Vessiot 1910].

Despite this, the success of the area in producing concrete results was quite limited in the period to 1910. As Vessiot put it, "la détermination du groupe de rationalité d'un système donné est un problème qui est loin d'être résolu." [Vessiot 1910, 170]. Nor was it clear that existing tools in the theory could usefully be re-interpreted in group-theoretic terms. The

[27] Vessiot also supplied a briefer version, but the brevity is obtained by introducing new concepts, notably the idea of a principal solution, and packing more into the definitions. The idea of added simplicity is thus a bit illusory.

analogy with Galois theory in this sense remained illusory. Despite ongoing efforts in these areas—Vessiot himself continued to publish on this until 1940, for example—the area did not revive until the work of Ritt and Kolchin, a set of developments detailed in [Borel 2001], and has come into its own much more recently.

The interesting features of these works in the assessment of the legacy of Galois have already been pointed to quite clearly by [Ehrhardt 2007]: what aspects of Galois's achievement are taken as fundamental varies with the author, as we have seen with Picard and Vessiot. There was likewise a tendency to assimilate one's own achievements in research to those of Galois, by taking his broadest statements and pointing to one's own work as exemplifying them, as we saw in the case of Drach. It would be, I think, anachronistic to say that these are mere rhetorical ploys by interested actors seeking to aggrandize the images of their achievements. It seems more sensible to take these statements as sincere expressions, of course at times perhaps self-serving, but nonetheless manifesting the inspirational power of the image of Galois's achievements, not only in the romantic fact of his tragically short life and career, but in the mathematical brilliance of specific features of the tools he created and the insights he brought to bear on basic, yet somehow model, problems.

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