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ROBUST TRANSITIVITY IN HAMILTONIAN DYNAMICS

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ABSTRACT. – A goal of this work is to study the dynamics in the complement of KAM tori with focus on non-local robust transitivity. We introduce C^r open sets ($r = 1, 2, \ldots, \infty$) of symplectic diffeomorphisms and Hamiltonian systems, exhibiting *large* robustly transitive sets. We show that the C^∞ closure of such open sets contains a variety of systems, including so-called *a priori* unstable integrable systems. In addition, the existence of ergodic measures with large support is obtained for all those systems. A main ingredient of the proof is a combination of studying minimal dynamics of symplectic iterated function systems and a new tool in Hamiltonian dynamics which we call "symplectic blender".

RÉSUMÉ. – Un objectif de ce travail est d'étudier la dynamique sur le complémentaire des tores KAM en mettant l'accent sur la transitivité robuste non locale. Nous introduisons les ensembles ouverts de difféomorphismes symplectiques et de systèmes hamiltoniens, présentant de grands ensembles robustement transitifs. L'adhérence de ces ensembles ouverts (en topologie C^r, $r = 1, 2, \ldots, \infty$) contient un grand nombre de systèmes, y compris les systèmes intégrables *a priori* instables. En outre, l'existence de mesures ergodiques avec un grand support est obtenue pour l'ensemble de ces systèmes. L'ingrédient principal des preuves est la combinaison de l'étude de systèmes itérés de fonctions de dynamique minimale et d'un nouvel outil de la dynamique hamiltonienne que nous appelons « mélangeurs symplectiques ».

1. Introduction and main results

The theory of Kolmogorov, Arnold and Moser (KAM) gives a precise description of the dynamics of a set of large measure of orbits for any small perturbation of a non-degenerate integrable Hamiltonian system. These orbits lie on the invariant KAM tori for which the dynamics are equivalent to irrational (Diophantine) rotations. In the case of autonomous systems in two degrees of freedom or time-periodic systems in one degree of freedom (i.e., 1.5 degree of freedom), the KAM Theorem proves the stability of *all* orbits, in the sense that the action variable does not vary much along the orbits. This, of course, is not the case if the degree of freedom is larger than two, where the KAM tori has codimension of at least two. A natural question arises: *Do generic perturbations of integrable systems in higher dimensions*

exhibit instabilities? The first example of instability is due to Arnold [4], who constructed a family of small perturbations of a non-degenerate integrable Hamiltonian system that exhibits instability in the sense that there are orbits with large action variation. This kind of topological instability is sometimes called the *Arnold diffusion*. Indeed, it was conjectured [3, p. 176] that instability is a common phenomenon in the complement of integrable systems. Aside the several deep contributions towards this conjecture, especially in recent years (see e.g. [12], [17], [14], [21], [22], [28], [27], [37], and references therein), it is still one of the central problems in Hamiltonian dynamics.

Here, we would like to suggest a different approach related to the instability problem. We propose to focus on the existence and abundance of a dynamical phenomenon, more sophisticated than instability, which is *"large" robustly transitive sets*. Roughly speaking, a set is transitive if it contains a dense orbit inside, and it is robustly transitive if the same holds for all nearby systems (see Definitions 1.6, 1.8).

The present paper is devoted to studying the non-local robust transitivity (global or non-global) in symplectic and Hamiltonian dynamics with the goal of better understanding the dynamics in the complement of KAM tori, and with application to the instability problem.

In the non-conservative context, there are many important recent contributions about robust transitivity. Note that a diffeomorphism of a manifold M is transitive if it has a dense orbit in the whole manifold. Such a diffeomorphism is called C^r robustly transitive if it belongs to the C^r interior of the set of transitive diffeomorphisms. It has been known since the 1960's that any (transitive) hyperbolic diffeomorphism is C^1 robustly transitive. The first examples of non-hyperbolic C^1 robustly transitive sets are credited to M. Shub [36] and R. Mañé [25]. For a long time their examples remained unique. Then, C. Bonatti and L. Díaz [7] introduced a semi-local source for transitivity, called *blender*, which is C^1 robust. Using this tool they constructed new examples of robustly transitive sets and diffeomorphisms. For recent results involving blenders, see [8]. For the recent surveys on this topic and robust transitivity on compact manifolds, see [10, Chapters 7,8], [33], [32].

In this paper, we develop the methods of robust transitivity within the context of symplectic and Hamiltonian systems. We apply them for the nearly integrable symplectic and Hamiltonian systems with more than two degrees of freedom. Following this approach, we introduce open sets of such Hamiltonian or symplectic diffeomorphisms exhibiting *large* robustly transitive sets and containing integrable systems in their closure. Then, the instability (Arnold diffusion) is obtained as a consequence of the existence of large robustly transitive sets. We want to point out that the results obtained also include systems not necessarily close to integrable ones.

We also obtain good information about the structure and dynamics of the robustly transitive sets that yield to topological mixing and even ergodicity. These are the scope of theorems stated in Sections 1.2 - 1.6.

We would like to compare the usual notion of instability (i.e. Arnold diffusion as treated in [14, 12, 37]) with robust transitivity (or topological mixing) obtained in the thesis of our theorems. Observe that the usual notion of instability is a C^0 robust property since it depends only on a finite number of iterations. However, there are no topologically mixing or transitive systems which are C^0 robust (see also Section 6.2).

Let us emphasize that in this paper we deal with the C^r-topology for any $r = 1, \ldots, \infty$. We also work with non-compact manifolds.

Section 1.1 introduces some definitions and notations; in Sections 1.2–1.6 the main theorems are stated. The two main ingredients used in the proofs are described informally in Sections 1.7 and 1.8. Finally, Section 1.9 provides a heuristic explanation of how these ingredients are combined and used.

1.1. Preliminaries and definitions

Some of the definitions below are standard in the literature so we only highlight the ones that are not common.

Let M be a boundaryless Riemannian manifold (not necessarily compact) and $f : M \longrightarrow M$ be a C^r diffeomorphism of a manifold M. From now on we assume that $r \in [1, \infty]$. We denote by $\mathrm{Diff}^r(M)$ the space of C^r diffeomorphisms of M endowed with the uniform C^r topology.

An f-invariant subset Λ is *partially hyperbolic* if its tangent bundle $T_\Lambda M$ splits as a Whitney sum of Df-invariant subbundles:

$$T_\Lambda M = E^u \oplus E^c \oplus E^s,$$

and there exist a Riemannian metric on M, a positive integer n_0 and constants $0 < \lambda < 1$ and $\mu > 1$ such that for every $p \in \Lambda$,

$$0 < \| D_p f^{n_0}|_{E^s} \| < \lambda < m(D_p f^{n_0}|_{E^c}) \leq \| D_p f^{n_0}|_{E^c} \| < \mu < m(D_p f^{n_0}|_{E^u}).$$

The co-norm $m(A)$ of a linear operator A between Banach spaces is defined by $m(A) := \inf\{\| A(v) \| : \| v \| = 1\}$. The subbundles E^u, E^c and E^s are referred to the unstable, center and stable bundles of f, respectively.

A partially hyperbolic set is called *hyperbolic* if its center bundle is trivial, i.e. $E^c = \{0\}$.

DEFINITION 1.1 (domination). – Let f and g be two diffeomorphisms on manifolds M and N respectively. Suppose that $\Lambda \subset M$ is an invariant hyperbolic set for f. We say that g is dominated by $f|_\Lambda$ if $\Lambda \times N$ is a partially hyperbolic set for $f \times g$, with $E^c = TN$.

The *homoclinic class of a hyperbolic set* is the closure of the transversal intersections of its stable and unstable manifolds. In the case of a hyperbolic periodic point P of a diffeomorphism F, we denote its homoclinic class by $H(P, F)$. Moreover, for any G nearby F, we denote by P_G the *analytic continuation* of P and by $H(P_G, G)$ its homoclinic class.

DEFINITION 1.2 (weak hyperbolic point). – Let p be a hyperbolic periodic point of g of period k; we say that p is δ-*weak hyperbolic* if

$$1 - \delta < m(D_p g^k|_{E^s_p}) < \| D_p g^k|_{E^s_p} \| < 1 < m(D_p g^k|_{E^u_p}) < \| D_p g^k|_{E^u_p} \| < \frac{1}{1-\delta}.$$

Let X be a metric space and $F : X \to X$ a continuous transformation. A set $Y \subset X$ (not necessarily compact) is *transitive* for F if for any U_1, U_2 open in X, such that $U_i \cap Y \neq \varnothing$, there is some n with $F^n(U_1) \cap U_2 \neq \varnothing$. If in addition, for any open sets $U_1, U_2 \subset Y$ (in the restricted topology), there is some n with $F^n(U_1) \cap U_2 \neq \varnothing$, then we say Y is *strictly transitive*. A stronger property is *topological mixing*, where $F^n(U_1) \cap U_2 \neq \varnothing$ holds for *any* sufficiently large n. Similarly we define *strictly topologically mixing*.

In the next definitions we denote by $\mathscr{D}^r$ a subspace of $\mathrm{Diff}^r(M)$ with the C^r topology.

DEFINITION 1.3 (continuation). – A set $X \subset M$ of f has *continuation* in $\mathscr{D}^r$ if there exist an open neighborhood $\mathscr{U}$ of f in $\mathscr{D}^r$ and a continuous map $\Phi : \mathscr{U} \to \mathscr{P}(M)$ such that $\Phi(f) = X$, where $\mathscr{P}(M)$ is the space of all subsets of M with the Hausdorff topology. Then, $\Phi(g)$ is called the *continuation* of X for g.

REMARK 1.4. – Note that it is not assumed that the continuation is neither homeomorphic to the initial set nor invariant. Compare with Definition 4.4.

DEFINITION 1.5 (exceptional set). – Let Λ be a partially hyperbolic set. We say that X is an *exceptional subset* of Λ if $X \subset \Lambda$ and for any central leaf L of Λ, the closure of $X \cap L$ in L has zero Lebesgue measure in L.

DEFINITION 1.6 (large set). – We say that a set X contained in Λ is large inside Λ if the Hausdorff distance of Λ and the interior of X in Λ is small.

DEFINITION 1.7 (compact robustly transitive set). – A compact set $Y \subset M$ is $\mathscr{D}^r$ *robustly (strictly) transitive* for $f \in \mathscr{D}^r$, if for any $g \in \mathscr{D}^r$ sufficiently close to f, the continuation of Y does exist and it is (strictly) transitive for g. In the same way one may define robustly (strictly) topologically mixing.

DEFINITION 1.8 (non-compact robustly transitive set). – If Y is not compact, then Y is called $\mathscr{D}^r$ *robustly (strictly) transitive* if it is the union of an increasing sequence of compact $\mathscr{D}^r$ robustly (strictly) transitive sets. In the same way one may define robustly (strictly) topologically mixing for non-compact sets.

A periodic point p of f of period n is called *quasi-elliptic* if $D_p f^n$ has a non-real eigenvalue of norm one, and all eigenvalues of norm one are non-real. If in addition all eigenvalues have norm one then it is called *elliptic*.

A point x is *non-wandering* for a diffeomorphism f if for any neighborhood U of x there is $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \varnothing$. By $\Omega(f)$ we denote the set of all non-wandering point of f. A point x is called (positively) *recurrent* for f if $\liminf_{n \to +\infty} \mathrm{dist}(x, f^n(x)) = 0$. A diffeomorphism is said *recurrent* if Lebesgue almost all points are recurrent.

Notations

Throughout this paper sometimes we use the following notation:

- $X \Subset Y$ means that $\overline{X}$ is a compact subset of the interior of Y;
- $\mathbb{U}^n := (0,1)^n$ and $\mathbb{D}^n$ is the unit open ball in $\mathbb{R}^n$.
- $\mathbb{T}^n$ is the n-dimensional torus (with the standard metric).

Let us recall some basic facts and definitions of symplectic topology. A symplectic manifold is a C^∞ smooth boundaryless manifold M together with a closed non-degenerate differential 2-form ω. We denote it by (M,ω), but sometimes we just write M. Examples of symplectic manifolds are orientable surfaces, even dimensional tori and cylinders, and the cotangent bundle T^*N of an arbitrary smooth manifold. A C^1 diffeomorphism f is *symplectic* if f preserves ω; i.e. $f^*\omega = \omega$. We denote by $\mathrm{Diff}^r_\omega(M)$ the space of C^r symplectic diffeomorphisms of M with the C^r topology, $1 \leq r \leq \infty$. If the symplectic form ω is exact, that is, $\omega = d\alpha$ for some 1-form α, and $f^*\alpha - \alpha = dS$ for some smooth function $S : M \to \mathbb{R}$, then we say that f is an exact symplectic diffeomorphism.

Let (M,ω) be a symplectic manifold and let $H : \mathbb{R} \times M \to \mathbb{R}$ be a C^r function called the (time dependent) Hamiltonian. For any $t \in \mathbb{R}$, the vector field X_{H_t} determined by the condition

$$\omega(X_{H_t}, Y) = dH_t(Y) \text{ or equivalently } i_{X_{H_t}}\omega = dH_t$$

is called the *Hamiltonian vector field* associated with $H_t := H(t,\cdot)$ or the symplectic gradient of H_t. The Hamiltonian H is called time periodic if $H_t = H_{t+T}$ for some $T > 0$. A diffeomorphism is called *Hamiltonian diffeomorphism* if it is the time-one map of some time periodic Hamiltonian flow.

An orbit is called quasi periodic if its closure, say $\mathcal{T}$, is diffeomorphic to a torus and the dynamics on $\mathcal{T}$ is conjugate to an irrational rotation on the torus.

A Hamiltonian on a $2n$-dimensional manifold is called *completely integrable* if it has n independent integrals in involution. Recall that an integral is a smooth real function on N (or $N \times \mathbb{R}$ in the case of time dependent Hamiltonian) which is constant along the orbits of the Hamiltonian flow.

A Hamiltonian is called *integrable* if it is locally completely integrable. A diffeomorphism is called integrable if it is the time-T map (for $T > 0$) of some integrable Hamiltonian flow. See Section 2.2 for more details and precise definitions.

Throughout this paper all diffeomorphisms and Hamiltonians on non compact manifolds are assumed to have finite norm in their corresponding C^r topology.

1.2. Robustly mixing sets for symplectic diffeomorphisms

We now state our main result concerning robust mixing in the context of symplectic diffeomorphisms. Roughly speaking, next theorem states that if the product of a hyperbolic basic set with any non-wandering dynamics is partially hyperbolic, then we can perturb the initial system in such a way that a *large* robustly topological mixing set is obtained.

THEOREM A. – *Let M and N be two symplectic manifolds so that N is compact, and $1 \leq r \leq \infty$. Let $F = f_1 \times f_2$ where $f_1 \in \mathrm{Diff}^r_\omega(M)$, $f_2 \in \mathrm{Diff}^r_\omega(N)$. Let $V \subset M$ be open and $\Gamma = \Lambda \times N$ such that*

(a) $\Lambda = \bigcap_{n\in\mathbb{Z}} f_1^n(V)$ *is a topologically mixing non-trivial hyperbolic set,*

(b) f_2 *is dominated by* $f_1|_\Lambda$,

(c) f_2 *has a δ-weak hyperbolic periodic point, and $\delta > 0$ is small enough.*

Then, there exist a connected open set $\mathcal{U} \subset \mathrm{Diff}^r_\omega(M \times N)$ and a periodic point $P \in \Lambda \times N$ of F such that

1. *F is contained in the C^r closure of $\mathcal{U}$.*
2. *For any $G \in \mathcal{U}$, the set $\Gamma_G := H(P_G; G) \bigcap_{n\in\mathbb{Z}} G^n(V \times N)$ verifies*
   (2.1) *Γ_G is a (partially hyperbolic) robustly topologically mixing set.*
   (2.2) *Γ_G tends to Γ in the Hausdorff topology as G tends to F.*

Roughly speaking, the item (2.2) shows that the robustly transitive set Γ_G is large along N, since it is close in the Hausdorff topology to $\Lambda \times N$. Observe that for the initial system F, the homoclinic class of P could be small along N. In fact, it could hold that $\Lambda \times N$ is non-transitive and $H(P, F) \cap (\Lambda \times N) = \Lambda \times \{q\}$, for some $q \in N$.

Related to the problem of instability, note that for the symplectic maps in the thesis of Theorem A, trajectories do not only "drift" along the N but belong to a large transitive set, and one obtains robustly transitive sets with "large variation of action" (see also Theorems C and D).

The proof of Theorem A follows essentially from Theorem 4.1 which could be considered as a parametric version of Theorem A. This parametric version gives a better description of the open set $\mathcal{U} \subset \mathrm{Diff}^r_\omega(M \times N)$ involved in the thesis of Theorem A. Moreover, Theorem 4.1 considers the case where the manifold N is not compact.

REMARK 1.9. – From a set of results proved in the C^1 topology ([9, 16, 20, 35]) and the theorem by Zehnder and Newhouse (about transversal homoclinic points in the presence of non hyperbolic periodic points, cf. Theorem 5.1), we conjecture that the hypothesis of Theorem A are optimal. For more details see Section 6.1.

DEFINITION 1.10. – A diffeomorphism satisfies the H.S. condition if it has a nontrivial transitive hyperbolic invariant set. A diffeomorphism f satisfies the A.H. condition if any neighborhood of f (in the corresponding space with the C^r topology) contains a diffeomorphism satisfying H.S. condition. In other words, the C^r closure of the set of diffeomorphisms satisfying the H.S. condition coincides with the set of diffeomorphisms satisfying the A.H. condition. Similar notions will be used for the Hamiltonian systems.

If the diffeomorphism f_2 in Theorem A is integrable and N is compact then we can prove that the (strong) continuation of Γ (which is a large subset) is in fact robustly transitive. So, we have the following.

THEOREM B. – *Let $r \geq 1$, $f_1 \in \mathrm{Diff}^r_\omega(M)$ satisfying the H.S. condition and $f_2 \in \mathrm{Diff}^r_\omega(N)$ be an integrable diffeomorphism on a compact boundaryless symplectic manifold N. Then $f_1 \times f_2$ is in the C^∞ closure of an open set $\mathcal{U} \in \mathrm{Diff}^r_\omega(M \times N)$ such that any $F \in \mathcal{U}$ has a robustly (strictly) transitive set whose projection on N is onto.*

Observe that in the thesis of Theorem A we do not state that the robustly transitive set projects onto N; actually, exceptional set outside the homoclinic class could appear. This is not the case in Theorem B. This stronger conclusion follows from the fact that the diffeomorphism f_2 in the hypothesis of Theorem B is integrable, which is not the case for Theorem A. To highlight the differences, compare Theorem 2.13 and Theorem 2.11. On the other hand, observe that in Theorem B the hypotheses on f_1 are weaker than the ones in Theorem A.

REMARK 1.11. – To highlight the broad range of application of Theorem B observe that the following systems satisfy the H.S. condition:

- any C^r generic perturbation of a symplectic diffeomorphism with an elliptic or quasi-elliptic periodic point,
- any C^r generic perturbation of an integrable system (i.e. generic nearly integrable diffeomorphism),
- any diffeomorphism in a C^1 open dense subset of $\mathrm{Diff}^1_\omega(M)$,
- any C^r surfaces diffeomorphism with positive entropy, if $r > 1$.

These are consequences of the results in [31] (cf. Theorem 5.1) and [23]. In fact, it is expected that the A.H. condition holds for any diffeomorphism in $\mathrm{Diff}^r_\omega(M)$. See also Section 6.1.

We would like to observe that in all our main results the robustly transitive sets are in fact robustly topological mixing for some power of the diffeomorphism.

1.3. Robust transitivity and instability of nearly integrable Hamiltonian systems

Theorems A and B could be stated in the context of exact symplectic and Hamiltonian diffeomorphisms, and also time-dependent Hamiltonians. The following theorem concerns the class of Hamiltonian systems that contains the so-called *a priori* unstable integrable Hamiltonian systems. This theorem can be considered as a Hamiltonian version of Theorem B.

THEOREM C. – *Let $H_0(p,q,x,y,t) = h_1(x,y,t) + h_2(p)$ be a time-periodic Hamiltonian, where $t \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ is the time, $(p,q) \in \mathbb{A}^{2m} = \mathbb{D}^m \times \mathbb{T}^m$, and $(x,y) \in \mathbb{A}^{2n} = \mathbb{D}^n \times \mathbb{T}^n$. Suppose that h_1 satisfies the A.H. condition and $h_2(p) = p \cdot {}^t p$ (where ${}^t p$ means transpose of p). Then for any compact set $K \subset \mathbb{A}^{2m}$, there exists a C^r $(r = 2, \dots, \infty)$ open set $\mathcal{H}$ of time periodic Hamiltonians that contains H_0 in its C^∞ closure and any Hamiltonian $H \in \mathcal{H}$ exhibits a robustly transitive invariant set such that its projection on $\mathbb{A}^{2m}$ contains K.*

This theorem is related to the results in [14], [12] and [37], where the case $m = n = 1$ and h_1 integrable (with a saddle loop) were considered. By quite different methods (and under extra conditions) it was shown that generic time-periodic perturbations of H_0 exhibit instability along the variable p (i.e. along the interval $\mathbb{D}^m$, $(m = 1)$).

Here, H_0 is relaxed from such conditions and it is shown that H_0 is in the closure of some open set $\mathcal{H}$ of Hamiltonians, in which all systems exhibit robustly transitive (or topological mixing) sets, which contains arbitrary large regions of $\mathbb{A}^{2m}$. In particular, a very strong form of instability is present for all systems in $\mathcal{H}$.

THEOREM D. – *Let h_1 and h_2 be two time periodic Hamiltonians acting on manifolds M and N respectively, both with finite volume. Assume that h_1 satisfies the H.S. condition and h_2 has an elliptic periodic point. Let $H_\varepsilon = h_1 + \varepsilon h_2$.*

Then, for a sufficiently small $\varepsilon > 0$ and any $\alpha > 0$, there exists a C^r open set $\mathcal{V}$ of time periodic Hamiltonians on $M \times N$ that contains H_ε in its C^∞ closure and any Hamiltonian $H \in \mathcal{V}$ exhibits an invariant set Υ such that

1. *the projection of Υ on N has volume $> \mathrm{vol}(N) - \alpha$,*
2. *Υ is robustly transitive.*

The type of systems that appear in Theorem D resembles the sometimes called "slow-fast systems" (see e.g. [5]).

1.4. Ergodic measures with large support

The existence of ergodic measures with large supports is a very interesting phenomenon, specially for nearly integrable systems and for compact sets in the complement of KAM tori (recall that KAM tori are ergodic components). In this direction we have the following result.

COROLLARY E. – *The robustly transitive (or topologically mixing) sets obtained in Theorems A, B, C and D are contained in the support of an ergodic measure.*

This is a direct consequence of our main theorems together with the results in [1, Theorem 3.1], where a nice property of homoclinic classes was discovered: *any homoclinic class is the support of an ergodic measure*. See also Section 6.3 for more discussions about ergodicity.

1.5. Hausdorff dimension of the transitive sets and ergodic measures

It is clear from the statement of Theorem A (as well as from the other main theorems) that $\dim_H(\Gamma) = \dim_H(\Lambda) + 2n$, where $\dim_H$ is the Hausdorff dimension. As a matter of fact, the Hausdorff dimension of certain hyperbolic sets (e.g. some hyperbolic sets in the vicinity of elliptic points) may be arbitrarily close to the dimension of the ambient manifold. So, one may find examples for which $\dim_H(\Gamma)$ is close to $2n + 2m$. Consequently, in the context of Corollary E, there exist ergodic measures with large (non-local) supports and with Hausdorff dimension arbitrarily close to the (full) dimension of the ambient manifold.

1.6. A dichotomy: transitivity vs. escaping to infinity

Observe that once we remove the hypothesis on the compactness of N (and hence let it have infinite volume) in Theorem A then a diffeomorphism G near F may have wandering points in the continuation of Γ. However, the proof of Theorem A leads to the following dichotomy: *there exist either large robustly transitive sets or wandering orbits converging to infinity.* See Section 5.6 for the proof and more details.

In the next two subsections we explain the main tools involved in the proofs of the previous theorems. These tools, namely symplectic blenders and iterated functions systems, could be interesting in themselves.

1.7. Symplectic blender

A fundamental ingredient in the proofs is a new tool in symplectic dynamics called *symplectic blender*, a semi-local source of robust transitivity. It is based on the seminal work of Bonatti and Díaz [7]. The symplectic blender provides *robustness* of the density of the stable and unstable manifolds of a hyperbolic periodic point, in any compact region, which implies robustness of transitivity or even topological mixing. Theorem 3.16 is the main result in this direction. We believe that the construction of symplectic blenders presented in this paper could have more applications in symplectic dynamics. For more details, see the introduction of Section 3.

1.8. Iterated function systems

By an iterated function system we mean the action of a semi-group generated by a (finite) family of diffeomorphisms. Another main ingredient in the proofs is that we reduce the problem of transitivity to one about iterated function systems. That is, the problems related to instabilities and (robust) transitivity can be formulated for iterated function systems. This approach is very convenient to deal with the *whole* structure of homoclinic intersections associated to a normally hyperbolic submanifold. Moreover, iterated function systems are used to obtain symplectic blenders. Consequently, part of this work is devoted to the study of certain iterated function systems. The main novelties in this part—beyond the above mentioned reduction—are the results on global and local dynamics of iterated function systems (cf. Theorems 2.13 and 2.11). We believe this type of problems is in itself worth of study.

1.9. Heuristic model

As a basic model to understand the strategies of the proofs of Theorems A and B, one may consider perturbations of the product of a horseshoe and an integrable twist map, where applying results on the *iterated function systems* of recurrent diffeomorphisms lead to the *minimality* of (strong) stable and unstable foliations (giving instability as well). Then, using the *symplectic blender* one can show that transitivity (or even topological mixing) appears in a *robust* fashion. Observe that the hypotheses of Theorem B are in the framework of the described strategy. Similar arguments are applicable in the more general context of Theorem A that includes non-integrable systems.

Note specially that we do not use any KAM-type invariant sets in the proof. Instead, *recurrence* has an important role. Therefore, some difficulties such as appearance of large gaps between Diophantine tori (known as *the large gap problem*) are not present here.

The paper is organized as follows. In Section 2, we study transitivity of two different kinds of iterated function systems (IFS). Namely, the IFSs of expanding maps, and the IFSs of recurrent diffeomorphisms. In certain cases, the minimality of the IFSs is obtained, which is essential in the proof of Theorem B. In Section 3, we introduce the symplectic blenders and we use the expanding IFSs to build them. In Section 4 we prove a parametric version of Theorem A. We use this theorem and its proof in Section 5 where we prove our main theorems and also the dichotomy stated in Section 1.6. Finally, in Section 6 several remarks and open problems related to the main results are discussed. In particular, in Section 6.5, we

discuss the possibility of extending the methods and results introduced here to many fruitful contexts, including non-artificial mechanical and geometric problems.

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2. Iterated function system

In this section we study the transitivity of some iterated function systems (IFS). Loosely speaking, in the IFS, instead of taking iterations by only *one map*, all the possible compositions and iterations of *several maps* are considered. As a consequence, a point x may have an infinite number of orbit branches. The transitivity of an iterated function system of expanding maps has a fundamental role in the construction and properties of blenders (see Section 3). And the transitivity of an iterated function system of symplectic maps shall be used in the proof of the density of the (strong) stable and unstable manifolds of partially hyperbolic skew products (see Section 4).

Let $g_1, g_2, \ldots, g_n$ be maps defined on a metric space X. The iterated function system $\mathcal{G}(g_1, g_2, \ldots, g_n)$ is the action of the semi-group generated by $\{g_1, g_2, \ldots, g_n\}$ on X. We use the notion of multi-index $\sigma = (\sigma_1, \ldots, \sigma_k) \in \{1, 2, \ldots, n\}^k$ for $g_\sigma = g_{\sigma_k} \circ \cdots \circ g_{\sigma_1}$. We also denote the length of the multi-index σ by $|\sigma| = k$.

An orbit of $x \in X$ under the iterated function system $\mathcal{G} = \mathcal{G}(g_1, g_2, \ldots, g_n)$ is a sequence $\{g_{\Sigma_k}(x)\}_{k=1}^{\infty}$ where $\Sigma_k = (\sigma_1, \ldots, \sigma_k)$ and $\{\sigma_i\}_{i=1}^{\infty} \in \{1, 2, \ldots, n\}^{\mathbb{N}}$.

The $\mathcal{G}$-orbit of x denoted by $\mathcal{O}_{\mathcal{G}}^{+}(x)$ is the set of points lying on some orbit of $x \in X$ under the IFS $\mathcal{G}$. The $\mathcal{G}$-orbit of a subset $U \subset X$ is defined as the union of all its orbits, i.e. $\mathcal{O}_{\mathcal{G}}^{+}(U) = \bigcup_{x \in U} \mathcal{O}_{\mathcal{G}}^{+}(x)$.

Similarly, we denote $\mathcal{O}_{\mathcal{G}}^{-}(x)$ as the set of points such that x lies on (some of) their orbits.

Results similar to the one proved here can be found in [2], [29], [30], [18] and [24].

DEFINITION 2.1. – The IFS $\mathcal{G}(g_1, g_2, \ldots, g_n)$ is said *transitive* if the $\mathcal{G}$-orbit of any open set is dense. A set U is called transitive for $\mathcal{G}$ if the $\mathcal{G}$-orbit of any open subset of U is dense in U. This is equivalent to the existence of some point with a dense $\mathcal{G}$-orbit in U.

REMARK 2.2. – In similar way one defines the IFS of maps $g_i : U_i \subset X \to X$. In this case, the possible compositions of g_i's depend on each point: $g_i(U_i)$ is not necessarily a subset of U_j and so $g_j \circ g_i$ is only defined on $U_i \cap g_i^{-1}(U_j)$.

2.1. IFS of contracting and expanding maps

In this section we study the transitivity of the iterated function systems of contracting and expanding maps. In Section 2.1.1 we give the results for contracting ones and the theorem for expanding are sketched in Section 2.1.2 (the details are left to the reader).

2.1.1. *IFS of contracting maps.* – A map ϕ on a metric space (X, d) is called *contracting* if there is a constant $0 < K < 1$ such that $d(\phi(x), \phi(y)) < Kd(x, y)$, for all $x, y \in X$. The *contraction bound*, if it exists, is a number $\lambda > 0$ for which ϕ satisfies $\lambda d(x, y) < d(\phi(x), \phi(y))$ for all $x, y \in X$. This constant does not necessarily exist for any contracting map. For a generic smooth contracting map ϕ on $\mathbb{R}^n$, the contraction bound does exist if we consider its restriction on a compact domain U. In this case, the constant is equal to $\inf\{m(D\phi_z) : z \in U\}$.

PROPOSITION 2.3. – *Let $U \subset \mathbb{R}^n$ be an open disk containing 0 and $\phi : U \to U$ be a contracting map with the contraction bound λ and $\phi(0) = 0$. Then there exists $k \in \mathbb{N}$ such that for any $\varepsilon > 0$ small there exist vectors $c_1, \ldots, c_k \in B_\varepsilon(0)$ and a number $\delta > 0$ such that*

$$B_\delta(0) \subset \overline{\mathcal{O}^+_{\mathcal{G}}(0)},$$

where $\mathcal{G} = \mathcal{G}(\phi, \phi + c_1, \ldots, \phi + c_k)$.

Moreover, these properties are robust in the following sense:
Let $\phi_0 = \phi$ and $\phi_i = \phi + c_i$. Let $\mathcal{U}_i$ be a set of contracting maps C^0 close to ϕ_i such that their contraction bounds are also close to that of ϕ_i. Then the same is true if at each iteration in the $\mathcal{G}$-orbit of 0 one replaces the corresponding ϕ_i by any $\tilde{\phi}_i \in \mathcal{U}_i$.

Observe that if ϕ is smooth, all constants depend only on $m(D\phi(0))$.

In order to prove this proposition, we start with a non-perturbative version of it, which also clarifies the robustness of transitivity.

DEFINITION 2.4. – We say that an iterated function system $\mathcal{G}(\phi_1, \ldots, \phi_k)$ of contracting maps has the *covering property* if there is an open set $\mathcal{D}$ such that

$$\mathcal{D} \subset \bigcup_{i=1}^{k} \phi_i(\mathcal{D}).$$

We say that the set of (unique) fixed points z_i's of ϕ_i's is *well-distributed* if any open ball of diameter d and centered in $\mathcal{D}$ contains some z_i, where

$$d \geq \max\{r \mid \forall x \in \mathcal{D}, \exists i, B_r(x) \subset \phi_i(\mathcal{D})\}.$$

PROPOSITION 2.5. – *Let $\phi_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $i = 1, 2, \ldots, k$, be contracting maps, and let z_i be their unique fixed points. Suppose that the iterated function system $\mathcal{G} = \mathcal{G}(\phi_1, \ldots, \phi_k)$ has the covering property on $\mathcal{D}$. Then for any $x \in \mathcal{D}$ there exists a sequence $\{\sigma_j\}_{j=1}^{\infty}$ such that for all $j \in \mathbb{N}$, $\sigma_j \in \{1, 2, \ldots, k\}$, and*

$$\phi^{-1}_{\sigma_j} \circ \phi^{-1}_{\sigma_{j-1}} \circ \cdots \circ \phi^{-1}_{\sigma_1}(x) \in \mathcal{D}.$$

In addition, if the set of fixed points $\{z_i\}_{i=1}^k$ is well-distributed in $\mathcal{D}$ then

$$\mathcal{D} \subset \overline{\mathcal{O}^+_{\mathcal{G}}(0)}.$$

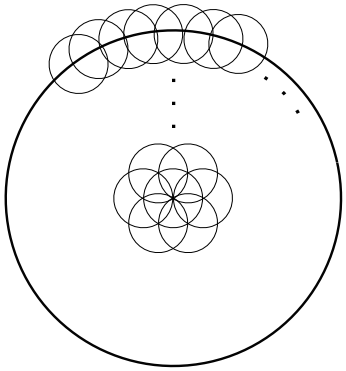

FIGURE 1. The covering and well-distributed properties. The disk $\mathcal{D}$ is the largest one and the other disks are its images under ϕ_i's.

Proof. – To prove the first part notice that given a point $x \in \mathcal{D}$, the covering property says that there is $\sigma_1 \in \{1, 2, \dots, k\}$ such that $\phi_{\sigma_1}^{-1}(x) \in \mathcal{D}$. Then, inductively, one constructs a sequence $\{\sigma_j\}_{j=1}^{\infty}$ such that $\phi_{\sigma_j}^{-1} \circ \phi_{\sigma_{j-1}}^{-1} \circ \cdots \circ \phi_{\sigma_1}^{-1}(x) \in \mathcal{D}$.

Now we prove the second part. The well-distributed property yields that for any small ball $B_r(x_0)$ in $\mathcal{D}$, either it belongs to some $\phi_i(\mathcal{D})$ or it contains the fixed point of some ϕ_i. Let $0 < K < 1$ such that $d(\phi_i(x), \phi_i(y)) < Kd(x, y)$ for any $i \in \{1, 2, \dots, k\}$. If the ball $B_r(x_0)$ is very small then it belongs to the domain of some ϕ_i, i.e. $B_r(x_0) \subset \phi_i(\mathcal{D})$, and so there is $x_1 \in \mathcal{D}$ such that $B_{K^{-1}.r}(x_1) \subset \phi_i^{-1}(B_r(x_0)) \subset \mathcal{D}$. We may continue this process inductively. Since the ratio of the balls is increasing exponentially at rate $K^{-1} > 1$, after some iteration, it would be large enough to contain the fixed point of some ϕ_i. This completes the proof. □

REMARK 2.6. – The well-distributed property yields that for any small ball $B_r(x_0)$ in $\mathcal{D}$, either it belongs to some $\phi_i(\mathcal{D})$ or it contains the fixed point of some ϕ_i.

Proof of Proposition 2.3. – It is enough to show that there exist a number k, and certain (small) translations of the map ϕ, such that the covering property and the well-distributed hypothesis hold in some open ball $B_\varepsilon(0)$. Then using Proposition 2.5 we obtain the density of the $\mathcal{G}$-orbit of 0. To show that, we consider a cover of the unit ball by k balls of radius λ. There is a number $C(n)$ such that for any $r < 1$, there is a cover of the unit ball in $\mathbb{R}^n$ by $C(n)r^{-n}$ number of balls of radius r. So there is a $k_1 = C(n)2^n \cdot \lambda^{-n}$ number of points $c_i \in \mathbb{D}^n$ (where $\mathbb{D}^n = B_1(0)$), such that

$$\overline{\mathbb{D}}^n \subset \bigcup_i B_{\lambda/2}(c_i),$$

and so

$$\overline{\mathbb{D}}^n \subset \bigcup_i (\phi(\mathbb{D}^n) + c_i).$$

Moreover, any ball of radius smaller than $\lambda/2$ is contained in some $\phi(\mathbb{D}^n) + c_i$.

We also choose a set of z_i, $i = 1, \dots, k_1$ which is λ-dense in the unit ball. So, any ball of radius greater than $\lambda/2$ in the unit ball contains some z_i. Let $c_{i+k_1} = (\mathrm{id} - \phi)(z_i)$. Indeed, z_i is the unique fixed point of the contraction $\phi + c_{i+k_1}$.

Now, let $k = 2k_1 = C(n)2^{n+1} \cdot \lambda^{-n}$, $\phi_0 = \phi$ and for $i = 1, \dots, k$, $\phi_i = \phi + c_i$. Again, robustness follows from the fact that the covering property and the well-distributed hypothesis are C^0 robust properties if the contraction bounds of the nearby maps are close to the initial ones. □

2.1.2. *IFS of expanding maps*. – A map ϕ on a metric space (X, d) is called *expanding* if and only if it is invertible and its inverse is a contracting map. Observe that in the expanding case, instead of the density of the forward orbit under the IFS, we look for the density of the backward orbits under the IFS. In particular, the covering property is also formulated for the inverse maps. Similar propositions to the one for contracting maps can be formulated for expanding maps, and although translation of a map is not a translation of its inverse, the proof of the equivalent propositions for expanding maps is done in the same way.

2.2. IFS of recurrent diffeomorphisms

In this section we study the transitivity of an iterated function system of recurrent diffeomorphisms. Here, by recurrent diffeomorphism we mean a diffeomorphism for which almost all points are recurrent. Recall that the Poincaré Recurrence Theorem asserts that any diffeomorphism preserving a finite volume form is recurrent.

The results of this section shall be used in the proof of Theorem A and other main theorems. We first prove several results for integrable systems and then we generalize them to general recurrent diffeomorphisms. In fact, in Proposition 2.8 we work with the IFS of integrable symplectic diffeomorphisms and in Theorem 2.13 we extend the previous proposition to the case of recurrent ones.

Let us first recall the statement of the Liouville-Arnold Theorem that allows us to re-define integrable systems as the following.

We say that $f \in \mathrm{Diff}^r_\omega(N)$ is *integrable* if there exist open sets N_i such that

- $N = \overline{\cup N_i}$,
- N_i's are mutually disjoint open sets,
- for any i, N_i is invariant and diffeomorphic to $\mathbb{U}^n \times \mathbb{T}^n$ by a (symplectic) diffeomorphism h_i,
- any torus $h_i^{-1}(\{x\} \times \mathbb{T}^n)$ is f-invariant and its dynamics is conjugate to a rotation.

Addendum

We also assume the following assumptions:

- bunching condition:

$$\limsup_{n\to\infty} \frac{1}{n} \log(||Df^n||/m(Df^n))$$

is sufficiently close to zero. (Observe that the limit is always zero on $\cup N_i$ and if there exists a periodic point in the boundary of any N_i, then the larger and smaller eigenvalues are close.)

– the family $\{N_i\}$ is locally finite in N, that is, any compact subset of N intersects with only finite number of $\{N_i\}$.
– the map $\mathbb{U}^n \ni x \mapsto \omega_x \in \mathbb{T}^n$ is a local diffeomorphism, where ω_x is the rotation vector of f on $h_i^{-1}(\{x\} \times \mathbb{T}^n)$.

LEMMA 2.7. – *Let f_1 be an integrable symplectic diffeomorphism on the symplectic manifold N. Then C^r-arbitrarily close to f_1 there exists another integrable symplectic diffeomorphism f_2 which is conjugate to f_1 by a smooth change of coordinates on N such that*

1. *any f_1-invariant torus intersects transversally some f_2-invariant torus, and vice versa,*
2. *given two open sets $U, V \subset N$, there is a chain of tori $\mathcal{T}_j$, $j = 1, 2, \ldots, s$, invariant for f_{σ_j}, $\sigma_j = 1$ or 2, such that, each $\mathcal{T}_j$ (for $j < s$) intersects transversally $\mathcal{T}_{j+1}$; $\mathcal{T}_1$ intersects U and $\mathcal{T}_s$ intersects V.*

Proof. – We construct a symplectic diffeomorphism $\phi \in \mathrm{Diff}^r_\omega(N)$ close to the identity such that $f_2 = \phi \circ f_1 \circ \phi^{-1}$ has the desired properties.

By the assumptions, $N = \cup \overline{N_i}$, where N_i is diffeomorphic to $\mathbb{U}^n \times \mathbb{T}^n$ by a symplectic diffeomorphism h_i. It is convenient to consider the polar coordinate system on $\mathbb{U}^n \times \mathbb{T}^n$, that is, any point is represented by

$$(r_1, \ldots, r_n, \theta_1, \ldots, \theta_n),$$

where $0 < r_i < 1$ and $\theta_i \in \mathbb{T}^1$.

The construction of ϕ has two steps.

Step 1. Let $\psi^\tau \in \mathrm{Diff}^r_\omega(\mathbb{U}^1 \times \mathbb{T}^1)$ be the time τ map of a completely integrable Hamiltonian flow such that in the polar coordinates we have

– $\psi^\tau(r, \theta) = (r, \theta)$, for any τ if $r \geq 1$,
– $\psi^1(\{r = c\}) \neq \{r = c\}$, if $0 \leq c < 1$.

In addition, given an open set W_0 in the annulus $\mathbb{U}^1 \times \mathbb{T}^1$ containing a curve that connects the two boundary components of the annulus we have:

– any two open sets in the unit disk $\{r < 1\}$ are connected by a finite chain of circles $\{r = c_j\}$ and $\psi^1(\{r = c_i\})$ such that the transition points are contained in W_0.

The Hamiltonian flow ψ^τ can be defined explicitly by the Hamiltonian

$$h_\varepsilon(r, \theta) := \eta(r) \cdot \frac{r^2 \sin(\theta) + r^2 \cos(\theta)(1 + \varepsilon \cdot \eta(r))^2}{1 + \varepsilon \cdot \eta(r)},$$

where $\varepsilon > 0$ is small and η is a (suitable) non-negative C^∞ bump function on the real line such that $\eta(x) > 0$ if and only if $x \in (0, 1)$.

Now let

$$\psi_\tau = \overbrace{\psi^\tau \times \cdots \times \psi^\tau}^{n \text{ times}}.$$

Define $\varphi \in \mathrm{Diff}^r_\omega(N)$ by

$$\varphi = \begin{cases} h_i^{-1} \circ \psi_{\tau_i} \circ h_i & \text{on } N_i \\ \mathrm{id} & \text{on } N \setminus \cup N_i \end{cases}$$

where τ_i is small enough and depends on N_i.

The smoothness of φ on each N_i is obvious. The smoothness of φ on the boundary of N_i's follows from the following facts: (1) the time τ_i is small enough and depends on N_i; (2) the sets N_i are mutually disjoint and (3) ψ is equal to the identity on the boundary of $\mathbb{U}^n \times \mathbb{T}^n$. In order to get φ close to the identity, it is enough to take τ small enough.

The items (1) and (2) of the lemma hold for the pair of diffeomorphisms f_1 and $\varphi \circ f_1 \circ \varphi^{-1}$, at least on each N_i. Moreover, given any pair of open sets in N_i and any open set W_1 that intersects all the f_1-invariant tori in N_i, there exists a chain as in item (2) of the lemma such that the intersections between the tori are contained in W_1.

Step 2. Let $i > j$ such that $\partial N_i \cap \partial N_j$ contains a regular hypersurface (codimension one submanifolds) S_{ij}. Then for any such i, j we consider a small open neighborhood U_{ij} of some point of the hypersurface S_{ij}. The sets U_{ij} are pairwise disjoint. We choose U_{ij} small enough such that $W := N \setminus \cup U_{ij}$ intersects all the f_1-invariant tori.

Let us define $U_{ij}^+ = U_{ij} \cap N_i$ and $U_{ij}^- = U_{ij} \cap N_j$. Then consider a symplectic diffeomorphism φ_{ij} supported in U_{ij} and close to the identity such that

$$\varphi_{ij}(U_{ij}^-) \cap U_{ij}^+ \neq \varnothing \text{ and } \varphi_{ij}(U_{ij}^+) \cap U_{ij}^- \neq \varnothing.$$

Now we take the composition of all the above diffeomorphisms to define $\phi \in \mathrm{Diff}_\omega^r(N)$, that is

$$\phi := (\circ_{ij} \varphi_{ij}) \circ \varphi.$$

Now, we define

$$f_2 := \phi \circ f_1 \circ \phi^{-1}.$$

Observe that $\phi = \varphi$ on the open set W, and as mentioned before it follows that the items (1) and (2) hold on each N_i. So, given any two open sets U, V in N, if they intersect the same N_i, there is nothing left to prove. Otherwise, there exist j_1, j_2, such that $U \cap N_{j_1} \neq \varnothing$ and $U \cap N_{j_2} \neq \varnothing$. Then, one may find a chain of N_i from N_{j_1} to N_{j_2} by taking iteration by f_2. Consequently, f_2 has the desired properties. □

PROPOSITION 2.8. – *Let T_1 be an integrable symplectic diffeomorphism on the symplectic manifold N. Then, arbitrarily close to T_1 there exists an integrable symplectic diffeomorphism T_2 on N such that the iterated function system $\mathcal{G}(T_1^d, T_2^{d'})$ has a dense orbit, for any $d, d' \in \mathbb{Z}$. Moreover, almost all points have dense $\mathcal{G}$-orbits.*

Proof. – Let T_2 be an integrable diffeomorphism. Let $\mathcal{O}_0$ be the set of all quasi periodic points of T_1 and observe that the set is T_1-invariant. Similarly, let $\mathcal{O}_0'$ be the set of all quasi periodic points of T_2, which is T_2-invariant. The complement of $\mathcal{O}_0 \cap \mathcal{O}_0'$ has zero Lebesgue measure. Let $\mathcal{O}$ be the set of all points whose orbits under the iterated function system $\mathcal{G}(T_1, T_2)$ belong to $\mathcal{O}_0 \cap \mathcal{O}_0'$.

CLAIM. *The complement of $\mathcal{O}$ is contained in a countable union of codimension one submanifolds. So, $\mathcal{O}$ has total Lebesgue measure.*

Proof of Claim. – Let $\mathcal{B}_0 := N \backslash \mathcal{J}_0$ and $\mathcal{B} := N \backslash \mathcal{J}$. Then, $\mathcal{B}_0$ is contained in a countable union of codimension one submanifolds. On the other hand, $\mathcal{B}$ is the orbit of $\mathcal{B}_0$ under the IFS $\mathcal{G}(T_1, T_2)$. So, it is a *countable* union of the iterates of $(\mathcal{B}_0)$. This proves the claim.

Now we apply Lemma 2.7 for T_1 and we obtain $\phi \in \mathrm{Diff}^r_\omega(N)$ close to the identity. Now we set $T_2 := \phi \circ T_1 \circ \phi^{-1}$. Define $\mathcal{J}$ as above and observe that $\mathcal{J}$ has total Lebesgue measure. We want to show that the orbit of any point of $\mathcal{J}$ is dense.

Given two open sets U, V, there is a chain of tori $\mathcal{T}_j$, $j = 1, 2, \ldots, s$, invariant for T_{σ_j}, $\sigma_j = 1$ or 2, such that each $\mathcal{T}_j$ intersects (transversally) $\mathcal{T}_{j+1}$, $\mathcal{T}_1$ intersects U and $\mathcal{T}_s$ intersects V. It is not difficult to find an orbit of $\mathcal{G}$ which shadows this chain. For any $z \in \mathcal{J}$, there is n_z such that $T^{n_z}_{\sigma_j}(z)$ is close to $\mathcal{T}_{j+1}$ if z is sufficiently close to $\mathcal{T}_j$. The set $\mathcal{J}$ is $\mathcal{G}(T_1, T_2)$-invariant. So if $z \in \mathcal{J}$ is sufficiently close to $\mathcal{T}_1$, then it has a $\mathcal{G}$-orbit shadowing all $\mathcal{T}_j$, and therefore there is an orbit from U to V. Moreover, given any point $x \in \mathcal{J}$ and any open set U, there is a finite sequence of tori $\mathcal{T}_i$, $i = 1, \ldots, n$, invariant for T_1 or T_2 (alternatively), such that $x \in \mathcal{T}_1$, $\mathcal{T}_n \cap U \neq \varnothing$, and for any i, $\mathcal{T}_i$ intersects transversally $\mathcal{T}_{i+1}$. Then it follows that there exists $\Sigma = (\sigma_1, \ldots, \sigma_m)$ such that $T_\Sigma(x) \in U$. This completes the proof. □

REMARK 2.9. – If the set of quasi periodic points is residual then following the same argument in the proof of Proposition 2.8, we conclude that the set of all points with dense orbit for $\mathcal{G}(T_1, T_2)$ is also residual.

The following example helps to understand the above described techniques.

EXAMPLE 2.10. – Let $N = \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$ and

$$T_1 : (I, \theta) \longmapsto (I,\ \theta + h(I)).$$

In this case, we choose the change of coordinates

$$\phi : (I, \theta) \longmapsto (I + \epsilon \cos\theta,\ \theta).$$

Then we define $T_2 = \phi \circ T_1 \circ \phi^{-1}$.

THEOREM 2.11 (Minimal IFS). – *Let T_1 be an integrable diffeomorphism on the symplectic manifold N. Let $m = \dim(N) + 2$. Then, arbitrarily close to T_1 there exist integrable symplectic diffeomorphisms $T_2, \ldots, T_m$ on N such that the iterated function system $\mathcal{G}(T_1, \ldots, T_m)$ is minimal (i.e. every point has a dense orbit).*

Proof. – Let T_2 be the integrable diffeomorphism given by Proposition 2.8. It follows from the claim in the proof of Proposition 2.8 that the complement of $\mathcal{J}$ is contained in a countable union of codimension one submanifolds of N. Let $\mathcal{L}_1 := \{L_i\}_i$ be the family of those submanifolds and so $\mathcal{B} := N \setminus \mathcal{J} = \bigcup \mathcal{L}_1$. (Here we use the union $\bigcup$ as the unitary operator, as well.)

Since any point in $\mathcal{J}$ has a dense orbit, to prove minimality, it is enough to show that there exist integrable diffeomorphisms $T_3, \ldots, T_m$ close to T_1, such that any point $x \in N \setminus \mathcal{J}$ has an iterate in $\mathcal{J}$ for the iterated function system $\mathcal{G}(T_3, \ldots, T_m)$.

We give a proof of this statement using transversality theory. It is known that the set of diffeomorphisms transversal to a submanifold L is C^r residual. On the other hand, if

L and L' are two submanifolds of codimensions m and m', respectively, then the transversal intersection of them has codimension $m+m'$.

Therefore, for any i,j, there is a residual set $R_{1,i,j} \subset \mathrm{Diff}^r_\omega(N)$ such that for any $f \in R_{1,i,j}$ and $L_i, L_j \in \mathscr{L}_1$, the submanifolds L_i and $f(L_j)$ are transversal and so the intersection $L_i \cap f(L_j)$ is a countable union of codimension two submanifolds. Let $\phi_3 \in \mathscr{R}_1 := \bigcap_{i,j} R_{1,i,j}$ sufficiently close to the identity, then $\bigcup_{i,j}(L_i \cap \phi_3(L_j))$ is a countable union of submanifolds L'_j, $j \in \mathbb{N}$, of codimension two. Let $\mathscr{L}_2 := \{L'_i\}_i$ be the family of those submanifolds. We have $\mathscr{B} \cap \phi_3(\mathscr{B}) = \bigcup \mathscr{L}_2$.

We repeat this argument for any pair of $L_i \in \mathscr{L}_1$ and $L'_j \in \mathscr{L}_2$ to get the residual set of diffeomorphisms $R_{2,i,j}$. By choosing $\phi_4 \in \mathscr{R}_2 := \bigcap_{i,j} R_{2,i,j}$ sufficiently close to the identity, then $\bigcup_{i,j}(L_i \cap \phi_4(L_j))$ is a countable union of submanifolds L''_j, $j \in \mathbb{N}$, of codimension 3. We define $\mathscr{L}_3$ of such codimension 3 submanifolds. We have

$$\mathscr{B} \cap \phi_3(\mathscr{B}) \cap \phi_4(\mathscr{B} \cap \phi_3(\mathscr{B})) = \mathscr{B} \cap \phi_3(\mathscr{B}) \cap \phi_4\Big(\bigcup \mathscr{L}_2\Big) = \bigcup \mathscr{L}_3.$$

Repeating this argument inductively, we get $\phi_3, \dots, \phi_{m-1}$ and the countable set $\mathscr{L}_{m-2}$, (i.e., a countable set of codimension $\dim(N)$ submanifolds), such that

$$\mathscr{B} \bigcap_{j=3}^{m-1} \phi_j\Big(\bigcup \mathscr{L}_{j-2}\Big) = \bigcup \mathscr{L}_{m-2}.$$

Now, for a generic diffeomorphism ϕ_m which we assume to be sufficiently close to the identity, we have $\mathscr{B} \cap \phi_m(\bigcup \mathscr{L}_{m-2}) = \varnothing$. This means that for any $x \in \mathscr{B}$ there exist $j_1, \dots, j_i$ such that $\phi_{j_1} \circ \cdots \circ \phi_{j_i}(x) \in \mathscr{O}$. Now, we define $T_j := \phi_i \circ T_1 \circ \phi_i^{-1}$. It follows that any point $x \in N \setminus \mathscr{O}$ has an iterate in $\mathscr{O}$ for the iterated function system $\mathscr{G}(T_3, \dots, T_m)$. Clearly, all T_j are integrable (Hamiltonian) diffeomorphisms and close to T_1. This completes the proof. □

REMARK 2.12. – To apply Theorem 2.11 to the proof of the main theorems, it is essential that the number m of generators is finite and independent of T_1. However, one may ask about the optimal number m of generators needed to obtain minimality. In fact, one may prove that three generators are enough. To see this, just observe that given $k \in \mathbb{N}$ and a codimension one submanifold L in N, and for a C^r generic diffeomorphism $f \in \mathrm{Diff}^r_\omega(N)$, f^k is transversal to L, for any $k \in \mathbb{N}$. Then, one may easily prove that for a C^r generic f, the diffeomorphisms $T_i := f^{i-2}$, $i = 3, \dots, m$, satisfy the transversality conditions in the proof of Theorem 2.11. We choose $T_3 = f$ to be close to T_1. Notice that in this argument we do not assume T_3 being integrable.

Now, we establish a result about transitivity of the IFS of recurrent diffeomorphisms.

THEOREM 2.13. – *Let $T \in \mathrm{Diff}^r_\omega(N)$ be a recurrent diffeomorphism. Then for every $\epsilon > 0$,*

1. *there exist $T_1, T_2 \in B_\epsilon(T) \subset \mathrm{Diff}^r_\omega(N)$ such that $\mathscr{G}(T, T_1, T_2)$ is transitive;*
2. *for any open ball $V \subset N$ and any bounded domain $N_c \subset N$, there exist $k \in \mathbb{N}$ and $T_1, T_2, \dots, T_k \in B_\epsilon(T) \subset \mathrm{Diff}^r_\omega(N)$ such that $N_c \subset \mathscr{O}^-_{\mathscr{G}}(V)$, where $\mathscr{G} = \mathscr{G}(T, T_1, T_2, \dots, T_k)$.*

Proof. – If $T = \mathrm{id}$, then we choose ϕ_1 to be an integrable symplectic diffeomorphism on the manifold N such that almost all points are quasi periodic, and $d_{C^r}(\phi_1, \mathrm{id}) < \frac{1}{2}\epsilon$. Proposition 2.8 implies that for any open set V there exists ϕ_2 in $\mathrm{Diff}^r_\omega(N)$ and ϵ-close to the identity in the C^r topology, such that $\mathcal{O}^-_{\mathcal{G}_\phi}(V) \cap \mathcal{O}^+_{\mathcal{G}_\phi}(V)$ is open and dense in N, where $\mathcal{G}_\phi = \mathcal{G}(\phi_1, \phi_2)$. In other words, $\mathcal{G}_\phi$ is transitive. This completes the proof of (1) in the case where $T = \mathrm{id}$.

For an arbitrary recurrent T, let $\mathcal{R}$ be the set of recurrent points of T, which is also invariant by ϕ_1 and ϕ_2. This set is dense. In fact, following an argument similar to the claim in the proof of Proposition 2.8 this set is residual and of total Lebesgue measure.

Let V be an open set in N, and $z \in \mathcal{R} \cap \mathcal{O}^-_{\mathcal{G}_\phi}(V)$. This intersection is obviously non-empty. Then, there are $d \in \mathbb{N}$ and $\Sigma = (\sigma_1, \ldots, \sigma_d), \sigma_i = 1, 2$, such that

$$z \in (\phi_\Sigma)^{-1}(V).$$

Moreover, for any $i = 1, 2, \ldots, d$, and any $l_j \in \mathbb{Z}$, $j = 1, 2, \ldots, i$,

$$\tilde{z}_i := (T^{l_i} \circ \phi_{\sigma_i}) \circ \cdots \circ (T^{l_1} \circ \phi_{\sigma_1})(z) \in \mathcal{R}.$$

So, using recurrence, for some (large) $l_j \in \mathbb{N}$, the orbit $(\tilde{z}_i)_i$ shadows $(z_i)_i$, where $z_i = \phi_{\sigma_i} \circ \cdots \circ \phi_{\sigma_1}(z)$. This shows that for some $l_j \in \mathbb{N}$, the point $\tilde{z}_d$ belongs to V. But $(\tilde{z}_i)_i$ is an orbit of z under the iterated function system of

$$\mathcal{G}_2 = \mathcal{G}(T, T \circ \phi_1, T \circ \phi_2).$$

In other words, $\tilde{z}_d \in V \cap \mathcal{O}^+_{\mathcal{G}_2}(z)$. Recall that $\mathcal{R} \cap \mathcal{O}^-_{\mathcal{G}_\phi}(V)$ is dense in N. So, the $\mathcal{G}_2$-orbit of any point in a dense set intersects V. The same is true for backward $\mathcal{G}_2$-orbits. Thus, $\mathcal{O}^\pm_{\mathcal{G}_2}(V)$ is (open and) dense in N, and $\mathcal{G}_2$ is transitive. This completes the proof of (1).

Given $N_c \Subset N$ bounded, and $V \subset N$ open, we let $X = \overline{B_1(N_c)} \setminus \mathcal{O}^-_{\mathcal{G}_2}(V)$. Observe that X is a compact set with empty interior. So for any $x \in X$ there exists h_x in $\mathrm{Diff}^r_\omega(N)$ and ϵ-close to the identity in the C^r topology, such that $h_x^{-1}(x) \in V^- := \mathcal{O}^-_{\mathcal{G}_2}(V)$. Since V^- is an open set, there is a neighborhood U_x of x such that $h_x^{-1}(U_x) \subset V^-$. The family $\{U_x\}$ is an open cover of the compact set X. So there exist $k \in \mathbb{N}, x_1, x_2, \ldots, x_l \in X$ and $h_{x_1}, h_{x_2}, \ldots, h_{x_k} \in B_\epsilon(\mathrm{id}) \subset \mathrm{Diff}^r_\omega(N)$ such that

$$X \cap h_{x_1}^{-1}(X) \cap \cdots \cap h_{x_k}^{-1}(X) = \varnothing.$$

Thus

$$T^{-1}(X) \cap (h_{x_1} \circ T)^{-1}(X) \cap \cdots \cap (h_{x_k} \circ T)^{-1}(X) = \varnothing.$$

Therefore,

$$N_c \Subset T^{-1}(V^-) \cap (h_{x_1} \circ T)^{-1}(V^-) \cap \cdots \cap (h_{x_k} \circ T)^{-1}(V^-).$$

If we define

$$\mathcal{G} := \mathcal{G}(T, T \circ \phi_1, T \circ \phi_2, h_{x_1} \circ T, \ldots, h_{x_k} \circ T),$$

then we have that $N_c \Subset \mathcal{O}^-_{\mathcal{G}}(V)$. □

2.3. Skew products and IFS

In this section we explain the relation between iterated function systems and skew products over shifts. This relation has been used extensively in random dynamical systems (cf. [2]). Similar approach to the one we present here has been used by R. Moeckel in [29] where he studies the IFS of monotone twist maps on annulus, relating it to the instability (drift) for skew products of such maps over shift.

Here, we state a series of propositions (see Propositions 2.16 and 2.17) that describe the trajectories of the action of a semigroup of diffeomorphisms as the dynamics of the unstable sets of an appropriate skew product over shifts.

Let τ be the full shift with d symbols.

$$\tau : d^{\mathbb{Z}} \to d^{\mathbb{Z}}$$
$$x = (\dots, x_{-1}, x_0; x_1, \dots) \mapsto (\dots, x_0, x_1; x_2, \dots).$$

It is natural to define the local and global unstable manifolds of a point $x \in d^{\mathbb{Z}}$ for τ as the following

$$W^u_{\mathrm{loc}}(x;\tau) = \{(z_i) \mid \forall i \le 0, z_i = x_i\}$$
$$W^u(x;\tau) = \bigcup_{i\ge 0} \tau^i(W^u_{\mathrm{loc}}(\tau^{-i}(x);\tau)) = \{(z_i) \mid \exists i_0 \in \mathbb{Z}, \forall i \le i_0, z_i = x_i\}.$$

Let $\Phi : d^{\mathbb{Z}} \times Y \to d^{\mathbb{Z}} \times Y$ be a skew product such that

$$\Phi(x,y) = (\tau(x), \phi_x(y)),$$

where ϕ_x is a homeomorphism on Y, for any $x \in d^{\mathbb{Z}}$. Assume that the family of ϕ_x's is uniformly bi-Lipschitz, i.e., there exists $L > 1$ such that $\forall x \in d^{\mathbb{Z}}$, $\forall y, y' \in Y$,

$$\frac{1}{L}\mathrm{dist}_Y(y,y') \le \mathrm{dist}_Y(\phi_x(y), \phi_x(y')) \le L \cdot \mathrm{dist}_Y(y,y')$$

and let us assume that the Lipschitz constant varies continuously with respect to x.

We consider the following metric on $d^{\mathbb{Z}}$,

$$\mathrm{dist}(x,z) = \sum_{i\in\mathbb{Z}} e^{-|i|L}|x_i - z_i|.$$

One may define the strong unstable manifold as follows:

$$W^{uu}(x,y;\Phi) := \{(a,b) \mid \mathrm{dist}(\Phi^i(x,y), \Phi^i(a,b)) \sim \exp(iL) \text{ as } i \to -\infty\}.$$

Assume that ϕ_x depends only on $[x_{\le i_0}] := (\dots, x_{i_0-1}, x_{i_0})$, and denote it by $\phi_{[x_{\le i_0}]}$. To avoid complications we also assume that $i_0 = 0$.

Therefore, Φ is the product $\tau \times \phi_x$ on the set $\{z \in d^{\mathbb{Z}} \mid z_i = x_i, i \le 0\} \times Y$. So, the local unstable manifold of (x,y) for Φ contains $W^u_{\mathrm{loc}}(x;\tau) \times \{y\}$. Then we have proved the following proposition.

PROPOSITION 2.14. – *For any* $(x,y) \in d^{\mathbb{Z}} \times Y$ *and* $n \in \mathbb{N}$,

$$\begin{aligned}
\Phi^n(x,y) &= (\tau^n(x), \phi_{[x_{\le n-1}]} \circ \cdots \circ \phi_{[x_{\le 0}]}(y)),\\
\Phi^{-n}(x,y) &= (\tau^{-n}(x), \phi^{-1}_{[x_{\le -n}]} \circ \cdots \circ \phi^{-1}_{[x_{\le -1}]}(y)).\\
W^{uu}_{\mathrm{loc}}(x,y;\Phi) &= W^u_{\mathrm{loc}}(x;\tau) \times \{y\} = \{(z_i) \mid \forall i \le 0, z_i = x_i\} \times \{y\},\\
W^{uu}(x,y;\Phi) &= \bigcup_{i\ge 0} \Phi^i(W^{uu}_{\mathrm{loc}}(\Phi^{-i}(x,y);\Phi)).
\end{aligned}$$

Since $W^{uu}_{\mathrm{loc}}(x,y;\Phi)$ is a product set and Φ is a product on it, so $\Phi^i(W^{uu}_{\mathrm{loc}}(x,y;\Phi))$ is a finite union of some local strong unstable manifolds. Therefore, we have proved the following.

PROPOSITION 2.15. – *For any* $(x,y) \in d^{\mathbb{Z}} \times Y$, *the global strong unstable manifold* $W^{uu}_{\mathrm{loc}}(x,y;\Phi)$ *is a countable union of some local unstable manifolds* $W^{uu}_{\mathrm{loc}}((x^i,y^i);\Phi)$.

Locally constant skew products. – From now on we assume that ϕ_x depends only on x_0, where $x = (\dots, x_{-1}, x_0; x_1, \dots)$. Then, $\Phi = \tau \times \phi_j$ on the set $\{z \in d^{\mathbb{Z}} \mid z_0 = j\} \times Y$, for any $j \in \{1,2,\dots,d\}$. The next propositions shed some light on the relation between (locally constant) skew products over shifts and iterated function systems.

Let $\mathcal{G} = \mathcal{G}(\phi_1, \phi_2, \dots, \phi_d)$. Proposition 2.14 implies that for any $(x,y) \in d^{\mathbb{Z}} \times Y$ and $n \in \mathbb{N}$,

$$\Phi^n(x,y) = (\tau^n(x), \phi_{x_{n-1}} \circ \cdots \circ \phi_{x_0}(y)).$$

This shows that by taking different base points x, one can realize the orbit of y under the IFS $\mathcal{G}$. Since the skew product Φ does not depend on x_i, $i > 0$, so we get the entire positive $\mathcal{G}$-orbit of y by taking all points on $W^u_{\mathrm{loc}}(x;\tau)$. So, we have obtained the following proposition.

PROPOSITION 2.16. – *For any* $(x,y) \in d^{\mathbb{Z}} \times Y$, *the projection of* $\bigcup_{n>0} \Phi^n(W^{uu}_{\mathrm{loc}}(x,y;\Phi))$ *on* Y *is equal to* $\mathcal{O}^+_{\mathcal{G}}(\phi_{x_0}(y))$. *In particular, if* (x,y) *is a fixed point of* Φ, *then the projection of* $W^{uu}(x,y;\Phi)$ *on* Y *is equal to* $\mathcal{O}^+_{\mathcal{G}}(y)$.

As we said, this proposition turns out to be very useful in the study of dynamical properties of strong stable/unstable manifolds of certain partially hyperbolic skew product systems. For instance, to obtain the density of the strong unstable manifolds (see Section 4.2.3). More precisely let us mention the geometric meaning of this fact: by each iteration the length of $W^{uu}_{\mathrm{loc}}(x,y;\Phi)$ grows exponentially and it intersects the domain of all ϕ_i's. Therefore, all possible compositions of ϕ_i's do appear in the positive orbit of $W^{uu}_{\mathrm{loc}}(x,y;\Phi)$. Indeed, we have the following proposition which gives a precise description of the global strong unstable manifolds for Φ.

PROPOSITION 2.17. – *For any* $(x,y) \in d^{\mathbb{Z}} \times Y$,

$$W^{uu}(x,y;\Phi) = \bigcup_{\sigma \in \Sigma} W^{uu}_{\mathrm{loc}}(x^\sigma, \phi^{x,\sigma}(y);\Phi),$$

where

$$\begin{gathered}
\Sigma = \{\sigma = (\sigma_1, \dots, \sigma_n) \mid n \in \mathbb{N}, 1 \le \sigma_i \le d\},\\
\phi^{x,\sigma} = \phi_{\sigma_{n-1}} \circ \cdots \circ \phi_{\sigma_1} \circ \phi^{-1}_{x_{-n+1}} \circ \cdots \circ \phi^{-1}_{x_{-1}} \text{ and } \phi^{x,(\sigma_1)} = \mathrm{id},\\
x^\sigma = (\dots, x_{-n}, \sigma_1, \dots, \sigma_n; x_1, \dots).
\end{gathered}$$

Proof. – It is easy to see that for any $\sigma, \sigma' \in \Sigma$, $W^u_{\mathrm{loc}}(x^\sigma;\tau) = W^u_{\mathrm{loc}}(x^{\sigma'};\tau)$ if and only if $\sigma = \sigma'$. Moreover, $\tau^n(x^\sigma) \in W^u_{\mathrm{loc}}(x;\tau)$ if $\sigma = (\sigma_1, \dots, \sigma_n)$. Therefore,

$$W^u(x;\tau) = \bigcup_{\sigma\in\Sigma} W^u_{\mathrm{loc}}(x^\sigma;\tau).$$

On the other hand, the projection of $W^{uu}(x,y;\Phi)$ on $d^{\mathbb{Z}}$ is equal to $W^u(x;\tau)$, since Φ is a skew product.

$$\begin{aligned}
W^{uu}(x,y;\Phi) &= \bigcup_{n\geq 0} \Phi^n(W^{uu}_{\mathrm{loc}}(\Phi^{-n}(x,y);\Phi)).\\
\Phi(W^{uu}_{\mathrm{loc}}(\Phi^{-1}(x,y);\Phi)) &= \Phi(W^u_{\mathrm{loc}}(\tau^{-1}(x);\tau)\times\{\phi^{-1}_{x_{-1}}(y)\})\\
&= (\tau\times\phi_{x_{-1}})(W^u_{\mathrm{loc}}(\tau^{-1}(x);\tau)\times\{\phi^{-1}_{x_{-1}}(y)\})\\
&= \tau(W^u_{\mathrm{loc}}(\tau^{-1}(x);\tau))\times\{y\}\\
&= \bigcup_{|\sigma|=1} W^{uu}_{\mathrm{loc}}(x^\sigma,y;\Phi).
\end{aligned}$$

From the definition of global unstable manifolds and Proposition 2.14 it follows that for any $n \in \mathbb{N}$,

$$\begin{aligned}
\Phi^n(W^{uu}_{\mathrm{loc}}(p,q;\Phi)) &= \bigcup_{a\in W^u_{\mathrm{loc}}(p;\tau)} \{(\tau^n(a), \phi_{a_{n-1}}\circ\cdots\circ\phi_{a_0}(q))\}\\
&= \bigcup_{\substack{\eta=(p_0,a_1,\dots,a_{n-1})\\ a=(\dots,p_0;a_1,a_2,\dots)}} \{(\tau^n(a),\phi_\eta(q))\}.
\end{aligned}$$

Now let $(p,q) = \Phi^{-n}(x,y)$, then $p_i = x_{i-n}$, $(\forall i \in \mathbb{Z})$, and $q = \phi^{-1}_{x_{-n}}\circ\cdots\circ\phi^{-1}_{x_{-1}}(y)$. Thus, $\tau^n(a) = (\dots,p_0,a_1,\dots,a_n;a_{>n}) = (\dots,x_{-n},a_1,\dots,a_n;a_{>n})$ and $\eta = (x_{-n},a_1,\dots,a_{n-1})$. Therefore,

$$\begin{aligned}
\phi_\eta(q) &= \phi_{\eta_n}\circ\cdots\circ\phi_{\eta_1}\circ\phi^{-1}_{x_{-n}}\circ\cdots\circ\phi^{-1}_{x_{-1}}(y)\\
&= \phi_{a_{n-1}}\circ\cdots\circ\phi_{a_1}\circ\phi_{x_{-n}}\circ\phi^{-1}_{x_{-n}}\circ\phi^{-1}_{x_{-n+1}}\circ\cdots\circ\phi^{-1}_{x_{-1}}(y)\\
&= \phi_{a_{n-1}}\circ\cdots\circ\phi_{a_1}\circ\phi^{-1}_{x_{-n+1}}\circ\cdots\circ\phi^{-1}_{x_{-1}}(y).
\end{aligned}$$

It yields that,

$$\begin{aligned}
\Phi^n(W^{uu}_{\mathrm{loc}}(\Phi^{-n}(x,y);\Phi)) &= \bigcup_{\substack{\eta=(p_0,a_1,\dots,a_{n-1})\\ a=(\dots,p_0;a_1,a_2,\dots)}} \{(\tau^n(a),\phi_\eta(q))\}\\
&= \bigcup_{|\sigma|=n} W^{uu}_{\mathrm{loc}}(x^\sigma,\phi^{x,\sigma}(y);\Phi).
\end{aligned}$$

This completes the proof. □

The stable manifolds of these maps are defined as the unstable manifolds of corresponding inverse maps. Similar results hold for stable manifolds.

3. Blenders, double blenders and symplectic blenders

The definition, existence and properties of blenders, double blenders and symplectic blenders are discussed in this section.

Bonatti and Díaz in [7] introduced the notion of blender, which is a geometric model for certain hyperbolic sets that play an important role as a mechanism for creation of cycles and a semi-local source of transitivity. Although their methods may be easily modified for the conservative case, the symplectic one is more delicate.

In [7] a cu-blender, roughly speaking, is a hyperbolic (locally maximal) invariant set with a splitting of the form $E^{ss} \oplus E^u \oplus E^{uu}$, $\dim E^u = 1$, such that a convenient projection of its stable set has larger topological dimension than the dimension of the subbundle (cf. Lemma 3.7). This phenomenon is robust in the C^1 topology. Similarly, one may define a cs-blender.

Their constructions essentially use a hyperbolic set with a one-dimensional weakly hyperbolic subbundle. On the other hand, to apply this local tool for systems with higher dimensional central bundles they use a chain of blenders with one-dimensional central bundles with different indices (i.e. dimension of the stable bundle) connected to each other. This allows them to use such blenders in more general situations. This approach, of course, is impossible in the symplectic case, since all eigenvalues are pairwise conjugate and so all hyperbolic periodic points have the same index. *So, to build blenders in the symplectic case, we need to involve higher central dimensions.* We construct a new class of such blenders in the symplectic (or Hamiltonian) systems that works like a chain of cs-blenders and a chain of cu-blenders simultaneously.

In Section 3.1.1, regardless of the symplectic case, we give the definition of cs-blenders and we study an abstract model for blenders with arbitrary central dimensions. Using the inverse map, we can define the cu-blenders (see Section 3.2). For their construction we use the results in Section 2.3 and the ones about contracting (expanding) IFS stated in Section 2.1. Then, in Section 3.1.2 we consider a geometric model of cs-blender and prove in Section 3.1.4 that their main feature of being a blender is a robust property. To do that, first we recast the hypothesis of Proposition 2.5 for IFS in terms of properties of the geometric model of cs-blender (see Section 3.1.3).

In Section 3.3, we consider the case where the central bundle splits into two stable and unstable subbundles, that is, the maximal invariant set is hyperbolic of the form $E^{ss} \oplus E^s \oplus E^u \oplus E^{uu}$. We call it *double-blender*. Then we introduce an abstract model which exhibits the feature of both cu-and cs-blenders (see Section 3.3.1). Note that this case is very compatible with the symplectic case where the eigenvalues of periodic points are pairwise conjugate.

In Section 3.4, we introduce the above phenomenon in the context of symplectic diffeomorphisms. That is what we call *symplectic blender*. Moreover, in Theorem 3.16 we show how the symplectic blenders appear in the context of Theorem A . In Section 3.5 we give a series of results about the role of blenders inside partially hyperbolic sets. These results are essential in the proof of our mains theorems.

3.1. The cs-blenders

In this section we introduce the definition of cs-blender.

DEFINITION 3.1 (s-strip). – Let F be a diffeomorphism on the manifold M. Let $\mathcal{B}$ be an open embedded ball with three cone-fields $\mathcal{C}^{ss}$, $\mathcal{C}^{s}$, $\mathcal{C}^{u}$, invariant under the derivative DF defined in a compact neighborhood of $\mathcal{B}$. Observe that the cone fields induce three invariant subbundles E^{ss}, E^{s}, E^{u}. A horizontal strip (or s-strip) is an embedded $(s+ss)$-dimensional disk in $\mathcal{B}$ tangent to $E^{ss} \oplus E^{s}$ and which contains the ss-leaves of each of its points.

DEFINITION 3.2 (cs-blender). – The pair $(P, \mathcal{B})$ is a cs-blender for the diffeomorphism F if it satisfies the following features:

- B-1 P is a hyperbolic saddle periodic point of F contained in $\mathcal{B}$;
- B-2 $\mathcal{B}$ is an open embedded ball on which there are three hyperbolic cone fields $\mathcal{C}^{ss}$, $\mathcal{C}^{s}$ and $\mathcal{C}^{u}$ invariant under the derivative DF;
- B-3 any G sufficiently close to F in the C^1 topology verifies that any s-strip in $\mathcal{B}$ intersects the unstable manifold of P_G whose backward orbit is in $\mathcal{B}$. Here P_G is the continuation of P.

From the definition of cs-blender follows immediately the next lemma (the proof is left to the reader).

LEMMA 3.3. – *Let $(P, \mathcal{B})$ be a cs-blender. Let $B_G = \bigcap_{n\in\mathbb{Z}} G^n(closure(\mathcal{B}))$ for G close to F. Then it follows that B_G is a hyperbolic set such that for any s-strip S in $\mathcal{B}$ there is $x \in B_G$ such that $W^u_{\mathrm{loc}}(x) \cap \mathcal{B} \neq \varnothing$, where $W^u_{\mathrm{loc}}(x)$ denotes the local unstable manifold of x.*

REMARK 3.4. – Using the inverse map, and defining properly a u-strip (see Definition 3.9), we can define the notion of cu-blender (see details in Section 3.2). In this case, the three hyperbolic cone fields are $\mathcal{C}^{s}$, $\mathcal{C}^{u}$, $\mathcal{C}^{uu}$ and it is required that the stable manifold of the saddle periodic point intersects any u-strip.

3.1.1. *Symbolic cs-blender*. – In this section we use the language of Section 2.3 to introduce a symbolic interpretation and construction of cs-blenders.

Let $\mathfrak{S}$ be the space of all skew products $\Phi : k^{\mathbb{Z}} \times \mathbb{R}^n \to k^{\mathbb{Z}} \times \mathbb{R}^n$ such that

$$\Phi(x, y) = (\tau(x), \phi_x(y)),$$

where $\tau : k^{\mathbb{Z}} \to k^{\mathbb{Z}}$ is the full shift with k symbols and for any $x \in k^{\mathbb{Z}}$, $\phi_x = \phi_{W^u_{\mathrm{loc}}(x)}$ is a contracting map on $\mathbb{R}^n$ with positive contraction bound.

Let $\Phi = (\tau, \phi_x), \Psi = (\tau, \psi_x)$ in $\mathfrak{S}$. We say that Φ is close to Ψ if for any $x \in k^{\mathbb{Z}}$, ϕ_x and its contraction bound are close to ψ_x and its contraction bound, respectively.

By a s-strip we mean the set $W^s_{\mathrm{loc}}(x; \tau) \times U$ for some $x \in k^{\mathbb{Z}}$ and some open set $U \subset \mathbb{R}^n$.

DEFINITION 3.5. – Let D be an open set in $\mathbb{R}^n$. The set $\mathcal{B} = k^{\mathbb{Z}} \times D$ is a symbolic cs-blender of $\Phi \in \mathfrak{S}$ if there exists a fixed point (p, q) of Φ such that for any $\tilde{\Phi} \in \mathfrak{S}$ close to Φ the following hold:

1. (p, q) has a unique continuation $(p, \tilde{q})$ for $\tilde{\Phi}$,
2. the unstable manifold of $(p, \tilde{q})$ for $\tilde{\Phi}$ intersects any s-strip in $\mathcal{B}$.

A symbolic cu-blender is defined as a symbolic cs-blender for Φ^{-1}.

Observe that the existence of a symbolic *cs*-blender is not trivial, specially its robustness property. The next proposition guarantees the existence of a symbolic *cs*-blender.

PROPOSITION 3.6. – *Let $\Phi(x,y) = (\tau(x), \phi_{x_0}(y))$ be a locally constant skew product, where $x_0 \in \{1,\dots,k\}$ and $x = (\dots, x_{-1}, x_0; x_1, \dots)$. Assume that $\phi_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $i = 1,2,\dots,k$, are contracting maps, and $\phi_i(z_i) = z_i$ are their unique fixed points. Suppose that the iterated function system $\mathcal{G} = \mathcal{G}(\phi_1,\dots,\phi_k)$ has the covering property on $D \subset \mathbb{R}^n$ and the set $\{z_i\}_{i=1}^k$ is well-distributed in D. Then the set $\mathcal{B} = k^{\mathbb{Z}} \times D$ is a symbolic cs-blender of Φ.*

Proof. – First we are going to prove the properties of the *cs*-blender for the initial system and later we deal with the perturbations.

Let (p,q) be a fixed point of Φ, for instance, $p = (\dots,1,1;1,1,\dots)$ and $q = z_1$ is the unique fixed point of $\phi_p = \phi_1$.

Recall that the unstable manifold of (p,q) is defined as the set of all points (x,y) such that $\operatorname{dist}(\Phi^i(x,y), \Phi^i(p,q)) \to 0$ as $i \to -\infty$. Since the dynamic on $\mathbb{R}^n$ is contracting, then the unstable manifold of (p,q) coincides with $W^{uu}(p,q;\Phi)$. By Proposition 2.16, the projection of $W^{uu}(x,y;\Phi)$ on $\mathbb{R}^n$ is equal to $\overline{\mathcal{O}^+_{\mathcal{G}}(y)}$. Proposition 2.5 yields that $D \subset \overline{\mathcal{O}^+_{\mathcal{G}}(q)}$. In other words, the unstable manifold of (p,q) for Φ intersects any s-strip in $\mathcal{B}$.

Now we show the robustness of this property. Let $\tilde{\Phi} \in \mathfrak{S}$ be close to Φ. Since $\tilde{\phi}_p$ is a contracting map close to $\phi_p = \phi_1$, it follows that it has a unique fixed point $\tilde{q}$ close to q. We want to show that $W^{uu}(p,\tilde{q};\tilde{\Phi})$ is dense. Observe that each $\tilde{\phi}_x$ is close to ϕ_{x_0}. Now, given $x_0 \in \{1,\dots,k\}$ we take $\bar{x}_0 = (\dots,x_0,x_0;x_0\dots)$ and we consider the IFS $\tilde{\mathcal{G}} = \mathcal{G}(\{\tilde{\phi}_{\bar{x}_0}\}_{x_0\in\{1,\dots,k\}})$. The proof of the density of the unstable manifold of $(p,\tilde{q})$ follows from the density of the $\tilde{\mathcal{G}}$-orbit of $\tilde{q}$ which holds from the robustness of $\mathcal{G}$ (see Propositions 2.3 and 2.16). This yields that the unstable manifold of $(p,\tilde{q})$ for $\tilde{\Phi}$ intersects any s-strip in $\mathcal{B}$. □

3.1.2. *Geometric model of a symbolic cs-blender*. – Using the skew product construction it is easy to build a geometric model of the symbolic blender.

Let $\phi_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $i = 0,1,\dots,k$, be an affine contracting map as in Proposition 2.3. Let D be an open disk in $\mathbb{R}^n$ such that $\phi_i(D) \subset D$.

Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a diffeomorphism with a horseshoe type hyperbolic set $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$, and with the splitting of the form $E^{ss} \oplus E^u$. Then it has a Markov partition with $k+1$ symbols by rectangles $R_0,\dots,R_k$ such that for any $i \neq j$, $\overline{R_i} \cap \overline{R_j} = \varnothing$. We also assume that the contraction rate on E^{ss} is stronger than the contraction of ϕ_i's.

We take a diffeomorphism $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ such that

$$F|_{R_i\times\mathbb{R}^n} = f \times \phi_i.$$

Observe that $H := \bigcap_{n\in\mathbb{Z}} F^n(U \times D)$ is a hyperbolic basic set with splitting $E^{ss} \oplus E^s \oplus E^u$.

Now we apply Proposition 3.6 for F. It yields that the set $\Lambda \times D$ is a symbolic blender with a fixed point (p,q) of F.

The following lemma is an immediate consequence of the definition of blender. It highlights the distinguished property of a blender.

LEMMA 3.7. – *Let $\Lambda \times D$ be the geometric model of the symbolic cs-blender as above. Then, for any $(x_1, x_2) \in U \times D$, there exists $(y_1, y_2) \in H$ such that*

$$W^{ss}_{\mathrm{loc}}(x_1, x_2; F) \cap W^{u}_{\mathrm{loc}}(y_1, y_2; F) \neq \varnothing.$$

This lemma says that the topological dimension of $W^u_{\mathrm{loc}}(H)$ is larger than the topological dimension of the local unstable manifold of each point in H.

Proof. – Let $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of nested open neighborhood of x_2, such that $\bigcap_{n\in\mathbb{N}} U_n = \{x_2\}$.

By definition, for any n, $W^s_{\mathrm{loc}}(x_1; f) \times U_n$ is a s-strip containing $W^{ss}_{\mathrm{loc}}(x_1, x_2; F)$ and intersecting $W^u(p, q; F)$. Now, let $\mathbf{z}_n \in H \cap W^u(p, q; F)$ such that

$$(W^s_{\mathrm{loc}}(x_1; f) \times U_n) \cap W^u_{\mathrm{loc}}(\mathbf{z}_n; F) \neq \varnothing.$$

Let $\mathbf{z}$ be an accumulation point of the sequence $\{\mathbf{z}_n\}$. Then $\mathbf{z} \in H$. Now, let $n \to \infty$, then from the election of the s-strips (i.e. $\bigcap_{n\in\mathbb{N}} U_n = \{x_2\}$) it follows that

$$W^{ss}_{\mathrm{loc}}(x_1, x_2, F) \cap W^u_{\mathrm{loc}}(\mathbf{z}; F) \neq \varnothing. \qquad \square$$

The following remark allows to recast the essential property of a *cs*-blender.

REMARK 3.8. – Given a blender $(p, \mathcal{B})$ and its maximal invariant set B, from Lemma 3.11 it follows that for any $x \in \mathcal{B}$ we have that $W^{ss}_{\mathrm{loc}}(x) \cap W^u_{\mathrm{loc}}(B) \neq \varnothing$. Moreover, the same holds for any G close to F.

3.1.3. *Covering and well distributed properties of the geometric model of a cs-blender*. – Now we would like to recast the hypothesis of Proposition 2.5, i.e., the *covering and well distributed properties* provided in Definition 2.4, in terms of properties of the geometric model of a *cs*-blender. Observe that the Markov partition $R_0, \ldots, R_k$ for Λ, and given by rectangles in $\mathbb{R}^m$, provides a Markov partition $\hat{R}_0, \ldots, \hat{R}_k$ for $\Lambda \times D$ in $\mathbb{R}^m \times \mathbb{R}^n$ where $\hat{R}_i = R_i \times D$.

Let $\mathcal{D}$ be the disk that is covered by the union of $\phi_i(\mathcal{D})$. Let us also define $R_{\mathcal{D}} := \cup_i R_i \times \mathcal{D}$. Now, given a point x in $\hat{R}_i$ we define $W^{ss}_{\hat{R}_i}(x)$ as the connected component of $W^{ss}(x) \cap \hat{R}_i$ that contains x. Then, for any $x \in R_{\mathcal{D}}$ and taking j such that $x \in \hat{R}_j$ it follows that

$$W^{ss}_{\hat{R}_j}(x) \cap \cup_i F(R_{\mathcal{D}} \cap \hat{R}_i) \neq \varnothing, \tag{1}$$

Moreover, observe that the Markov property implies that $\mathcal{D} \subset \cup_i \phi_i(\mathcal{D})$; it follows that if $W^{ss}_{\hat{R}_j}(x)$ intersects $F(R_{\mathcal{D}} \cap \hat{R}_i)$ then

$$F^{-1}(W^{ss}_{\hat{R}_j}(x) \cap F(R_{\mathcal{D}} \cap \hat{R}_i)) \cap \cup_i F(R_{\mathcal{D}} \cap \hat{R}_i) \neq \varnothing. \tag{2}$$

Conditions (1) and (2) are the *covering property* for the geometric model.

The *well-distributed* property is recasted in the following way: let $z_0, \cdots, z_k$ be the set of a unique fixed point of F in each $\hat{R}_i$, if any open ball of diameter d and centered in $R_{\mathcal{D}}$ contains some z_i, where

$$d \geq \max\{r \mid \forall x \in R_{\mathcal{D}}, \exists i, B_r(x) \subset F(R_i \times \mathcal{D})\}.$$

3.1.4. *Geometric model of a symbolic cs-blender is a cs-blender*. – In this section we show that the property that the unstable manifold of any point intersects any s-strip also holds for any perturbation. It is natural to think that any small perturbation of a geometric model of a symbolic cs-blender is also a geometric model of a symbolic cs-blender, in particular, that it is a skew-product locally constant over contracting IFS. But this is not generically true. In fact, if (under a change of coordinates if necessary) we get a skew-product locally constant over contracting IFS, it would follow that the laminations associated to the strong stable and unstable subbundles are Lipschitz, which is generically false. So, to prove that perturbations of a geometric model verify condition B-3 in Definition 3.2 we can not reduce to skew-product locally constant. However, the proof of Proposition 2.5 is easily adapted to the context of perturbations of the geometric model of a symbolic cs-blender, proving in this way that they are cs-blenders.

The proof runs as follows: First, given G C^1-close to F in a neighborhood of $\Lambda \times D$, we consider the set B_G which is the maximal invariant set of G in that neighborhood. Observe that the covering and well-distributed property also holds for B_G. This is immediate from the fact that $G(R_{\mathscr{D}} \cap \hat{R}_i)$ is C^0-close to $F(R_{\mathscr{D}} \cap \hat{R}_i)$ and $W^{ss}_{\hat{R}_j,G}(x)$ is C^1-close to $W^{ss}_{\hat{R}_j,F}(x)$. So, given x in $R_{\mathscr{D}}$ we consider a s-strip S around $W^{ss}_{\hat{R}_i}(x,G)$ (where i is such that $x \in \hat{R}_i$) and we take $S \cap G(\hat{R}_j \cap R_{\mathscr{D}})$, where j verifies that $S \cap G(\hat{R}_j \cap R_{\mathscr{D}}) \neq \varnothing$. Observe that $G^{-1}(S \cap G(\hat{R}_j \cap R_{\mathscr{D}}))$ is again an $s+ss$-embedded disk tangent to $E^{ss} \oplus E^s$ that intersects $R_{\mathscr{D}}$ and therefore we can choose new indexes i, j as before; since the $DG^{-1}_{|E^{ss}\oplus E^s}$ is expanding, in the same way that in Proposition 2.5 it follows that a backward orbit of a subdisk inside S eventually intersects the local unstable manifold of some fixed point z_{iG} (where z_{iG} is the analytic continuation of some z_i). Since all the fixed points are homoclinic connected for G, we prove in that way that the unstable manifold of any fixed point intersects any s-strip. This finishes the proof.

3.2. The cu-blenders

Using the inverse map, we can introduce the notions of cu-blender and symbolic cu-blender, its geometric model and also the properties of covering and well distribution. In the same way we see that the geometric models of symbolic cu-blenders are in fact cu-blenders. The details are left to the reader.

3.3. Double-blender

In this section we introduce the definition of double blender.

DEFINITION 3.9 (u-strip). – Let F be a diffeomorphism on the manifold M. Let $\mathscr{B}$ be an open embedded ball with four cone-fields $\mathscr{C}^{ss}$, $\mathscr{C}^{s}$, $\mathscr{C}^{u}$, $\mathscr{C}^{uu}$, invariant under the derivative DF defined in a compact neighborhood of $\mathscr{B}$. A vertical strip (or u-strip) is an embedded $(u+uu)$-dimensional disk in $\mathscr{B}$, which contains the uu-leaves of each of its points.

DEFINITION 3.10 (double blender). – The pair $(P, \mathscr{B})$ is a double blender for the diffeomorphism F if it satisfies the following features:

B-1 P is a hyperbolic saddle periodic point of F contained in $\mathcal{B}$;

B-2 $\mathcal{B}$ is an open embedded ball on which there are four hyperbolic cone fields $\mathcal{C}^{ss}$, $\mathcal{C}^{s}$, $\mathcal{C}^{u}$ and $\mathcal{C}^{uu}$ invariant under the derivative DF defined in a compact neighborhood of $\mathcal{B}$.

B-3 Any G sufficiently close to F in the C^1 topology verifies the following:

1. any u-strip in $\mathcal{B}$ intersects some s-strip contained in the stable manifold of P_G whose forward orbit is in $\mathcal{B}$;
2. any s-strip in $\mathcal{B}$ intersects some u-strip contained in the unstable manifold of P_G whose backward orbit is in $\mathcal{B}$.

Here P_G is the continuation of P.

The next lemma follows immediately from Definition 3.10 and the proof of Lemma 3.7.

LEMMA 3.11. – *Let $(P, \mathcal{B})$ be a double blender. Let $B_G = \bigcap_{n\in\mathbb{Z}} G^n(\text{closure}(\mathcal{B}))$ for G close to F. Then it follows that B_G is a hyperbolic set such that for any s-strip S in $\mathcal{B}$ there is $x \in B_G$ such that $W^u_{\text{loc}}(x) \cap \mathcal{B} \neq \varnothing$, where $W^u_{\text{loc}}(x)$ denotes the local stable manifold of x. A similar statement holds for any u-strip.*

The following remark allows to recast the essential property of a blender.

REMARK 3.12. – Given a blender $(p, \mathcal{B})$ and its maximal invariant set B, from Lemma 3.11 it follows that for any $x \in \mathcal{B}$ holds:

– $W^{ss}_{\text{loc}}(x) \cap W^{u}_{\text{loc}}(B) \neq \varnothing$,
– $W^{uu}_{\text{loc}}(x) \cap W^{s}_{\text{loc}}(B) \neq \varnothing$.

Moreover, the same holds for any G close to F.

3.3.1. *Symbolic double-blender*. – Here we introduce an abstract model with the features of symbolic *cu*-and *cs*-blenders, simultaneously. We call it a symbolic double-blender.

Let $\mathfrak{A}$ be the space of all skew products of the form $\Phi : k^{\mathbb{Z}} \times \mathbb{R}^n \times \mathbb{R}^m \to k^{\mathbb{Z}} \times \mathbb{R}^n \times \mathbb{R}^m$ such that

$$\Phi(x, y, z) = (\tau(x), \phi_x(y), \psi_x(z)),$$

where $\tau : k^{\mathbb{Z}} \to k^{\mathbb{Z}}$ is the full shift with k symbols; and for any $x \in k^{\mathbb{Z}}$, $\phi_x = \phi_{W^u_{\text{loc}}(x)}$ is a contracting map on $\mathbb{R}^n$ with contraction bound larger than a fixed positive number, and $\psi_x = \psi_{W^s_{\text{loc}}(x)}$ is an expanding map on $\mathbb{R}^m$ with expansion bound smaller than a fixed number.

Let us fix $\Phi = (\tau, \phi_x, \psi_x), \tilde{\Phi} = (\tau, \tilde{\phi}_x, \tilde{\psi}_x)$ in $\mathfrak{A}$. We say that Φ is close to $\tilde{\Phi}$ if for any $x \in k^{\mathbb{Z}}$, ϕ_x and its contraction bound are close to $\tilde{\phi}_x$ and its contraction bound, respectively, and ψ_x and its expansion bound are close to $\tilde{\psi}_x$ and its expansion bound, respectively.

For $\Phi = (\tau, \phi_x, \psi_x)$ in $\mathfrak{A}$, we set $\Phi_{cs} = (\tau, \phi_x)$, and $\Phi_{cu} = (\tau, \psi_x)$.

DEFINITION 3.13. – Let U and V be open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. The set $\mathcal{B} = k^{\mathbb{Z}} \times U \times V$ is a symbolic double-blender of $\Phi \in \mathfrak{A}$ if $k^{\mathbb{Z}} \times U$ is a *cs*-blender of Φ_{cs} and $k^{\mathbb{Z}} \times V$ is a *cu*-blender of Φ_{cu}.

A similar proposition to the one formulated for a symbolic cs-blender (Proposition 3.6) can be formulated in the context of a symbolic double blender.

PROPOSITION 3.14. – *Let $\Phi \in \mathfrak{A}$ be a locally constant skew product such that $\Phi(x,y,z) = (\tau(x), \phi_{x_0}(y), \psi_{x_1}(z))$, where $x = (\dots, x_{-1}, x_0; x_1, \dots) \in k^{\mathbb{Z}}$. Assume that ϕ_i and ψ_i^{-1}, $i = 1, 2, \dots, k$, are contracting maps. Suppose that the iterated function systems $\mathcal{G} = \mathcal{G}(\phi_1, \dots, \phi_k)$ and $\mathcal{G} = \mathcal{G}(\psi_1^{-1}, \dots, \psi_k^{-1})$ have covering and well-distributed properties on $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}^m$, respectively. Then the set $\mathcal{B} = k^{\mathbb{Z}} \times D_1 \times D_2$ is a symbolic double-blender of Φ.*

The proof is similar to the proof of Proposition 3.6 and it is left to the reader.

3.3.2. *Geometric model of symbolic double-blender*. – Using the skew product construction it is easy to build a geometric model of a symbolic double-blender. The idea is the same exposed in Section 3.1.1 where the geometric model of a symbolic *cs*-blender was considered. But instead of considering a set of affine contracting maps we use a set of pairs of affine maps, one contracting and one expanding. In a few words, we consider the skew product F such that $F|_{R_i \times \mathbb{R}^n \times \mathbb{R}^n} = f \times \phi_i^s \times \phi_i^u$, where for any i, $\phi_i^s : \mathbb{R}^n \to \mathbb{R}^n$ is contracting and $\phi_i^u : \mathbb{R}^n \to \mathbb{R}^n$ is expanding.

The geometric idea behind this definition is the following. In the 3-dimensional *cs*-blenders of [7], if one projects the cube and its pre-image along a stable direction, a figure like a Smale horseshoe appears but the right and left rectangles overlap. With this in mind, consider a 4-dimensional horseshoe with the splitting of the form $E^{ss} \oplus E^s \oplus E^u \oplus E^{uu}$ where the projection along E^{ss} gives a figure like a 3-dimensional horseshoe such that its two wings overlap and the same feature holds for the inverse map. Let F be such a diffeomorphism on the open set $\mathcal{B}$. Then, the maximal invariant set in $\mathcal{B}$, i.e., $\Lambda = \bigcap_n F^n(\mathcal{B})$ is a *cs*-blender, if we consider $E^{ss} \oplus E^s$ as the stable direction, E^u as the central unstable direction and E^{uu} as the strong unstable direction. Similarly Λ is a *cu*-blender if we consider E^{ss} as the strong stable direction, E^s as the central stable direction and $E^u \oplus E^{uu}$ as the unstable direction. Therefore, Λ is a double-blender. Note that using the results in Section 3.1.1, we may consider multi-dimensional central bundles, i.e., both of center stable and center unstable bundles of arbitrary dimensions.

In a similar way as it was done in Section 3.1.2 the notion of covering and well distributed properties, can be formulated for double-blenders.

3.3.3. *Geometric model of a symbolic double-blender is a double-blender*. – The proof is essentially the same as the one provided for a *cs*-blender in Section 3.1.4 but working simultaneously along the center stable and center unstable directions.

3.4. Symplectic blender

DEFINITION 3.15. – A *symplectic blender* is a double-blender for a symplectic (or Hamiltonian) diffeomorphism.

Observe that if we want to consider geometric models of symplectic blenders as it was done for the double-blenders, it is necessary to chose the pairs of contracting and expanding affine maps, in such a way that a symplectic relation is satisfied. For instance, $\phi_i^u = (\phi_i^s)^{-1}$.

The following theorem introduces a construction which yields the existence of a symplectic blender in the context of Theorem A.

THEOREM 3.16. – *Let M and N be two symplectic manifolds (not necessarily compact). Let $r = 1, 2, \ldots, \infty$. Suppose that $f_1 \in \mathrm{Diff}^r_\omega(M)$ has a hyperbolic periodic point $\hat{p}$ with transversal homoclinic intersections and $f_2 \in \mathrm{Diff}^r_\omega(N)$ has a δ-weak hyperbolic periodic point $\hat{q}$ with $\delta = \delta(f_1, \dim(N)) > 0$ small enough.*

Then, there is a C^r arc $\{F_\mu\}_{\mu\in[0,1]}$ of C^r symplectic diffeomorphisms on $M \times N$ such that,

1. *$F_0 = f_1 \times f_2$.*
2. *There is a neighborhood $\mathcal{V}$ of $\{F_\mu\}_{\mu\in(0,1]}$ in $\mathrm{Diff}^1(M \times N)$ such that for any $G \in \mathcal{V}$, the pair $(P_G, \mathcal{B})$ is a double blender, where P_G is the continuation of hyperbolic $P_0 = (\hat{p}, \hat{q})$ and $\mathcal{B}$ is an embedded open disk in $M \times N$.*

Proof. – Let $V \subset M$ be a small open disk which contains $\hat{p}$ and a point of transversal homoclinic intersection associated to $\hat{p}$, such that for some $k \in \mathbb{N}$, $\Lambda := \bigcap_{n\in\mathbb{Z}} f_1^{kn}(V)$ is an invariant hyperbolic compact set of saddle type for f_1^k. By choosing V suitable and k large enough, we may suppose that $f_1^k \mid_\Lambda$ is conjugate to a shift of $d + 1$ symbols $\{0, 1, \ldots, d\}$. Moreover, we can assume that the elements of Markov partition of Λ are contained in open sets such that their closures are pairwise disjoint. Also observe that if k is large enough and V is suitably chosen, then $d = d(f_1, k)$ can be taken arbitrarily large. Therefore if k is large enough, by taking f_1^k, and f_2^k instead of f_1 and f_2, we may assume that $P_0 = (\hat{p}, \hat{q})$ is a fixed point of $F_0 = f_1 \times f_2$ and Λ is f_1-invariant and f_1 on $\Lambda := \bigcap_{n\in\mathbb{Z}} f_1^n(V)$ is conjugate to the shift of symbols $\{0, 1, \ldots, d\}$. Moreover, we may assume that f_2 is dominated by $f_1|_\Lambda$. Indeed, we may replace f_2 by a map which is identity in the complement of $\hat{U}$, where we choose $\tilde{U} \Subset \hat{U}$ small neighborhoods of the fixed point $\hat{q}$, and in such a way that $\tilde{U}$ is much smaller than $\hat{U}$.

For a (linear) contraction map of $\mathbb{R}^n$, ($2n = \dim(N)$) with contraction bound equal to $1/2$, Proposition 2.3 gives a number l as a required number of elements of the IFS to obtain transitivity in some small disk.

Now we choose k large enough such that $d \geq 2l$.

Once we substitute f_2 by f_2^k it follows that $m(Df^k|_{E^s}) > (1 - \delta)^k$. Therefore, $\delta = \delta(f_1, \dim(N))$ is chosen close to zero in such a way that $(1 - \delta)^k > 1/2$.

From now on, we assume that $\hat{q}$ of f_2 is a δ-weak hyperbolic fixed point of f_2 with $\delta < 1/2$, $\hat{p} \in \Lambda$ is a hyperbolic fixed point of f_1, and Λ is f_1-invariant and it is conjugate to a shift with $d + 1$ symbols as above.

We want to perturb F_0 to construct a family F_μ such that each one has a symbolic double blender as a sub-system, and we will show how this leads to the existence of a symplectic blender.

Perturbations. – Let $\zeta > 0$ be small enough (it will be chosen later) and $\varepsilon : [0, 1] \longrightarrow [0, \zeta]$ be a smooth simple curve such that $\varepsilon(0) = 0$.

Let us define $F_0 = f_1 \times f_2$.
For $\mu \in (0, 1]$,

$$F_\mu := \Psi^{\varepsilon(\mu)} \circ F_0 \circ \Phi^{-\varepsilon(\mu)},$$

where Ψ^t and Φ^t are the time t maps of the Hamiltonians $\tilde{h}_1$ and $\tilde{h}_2$, respectively, to be chosen later.

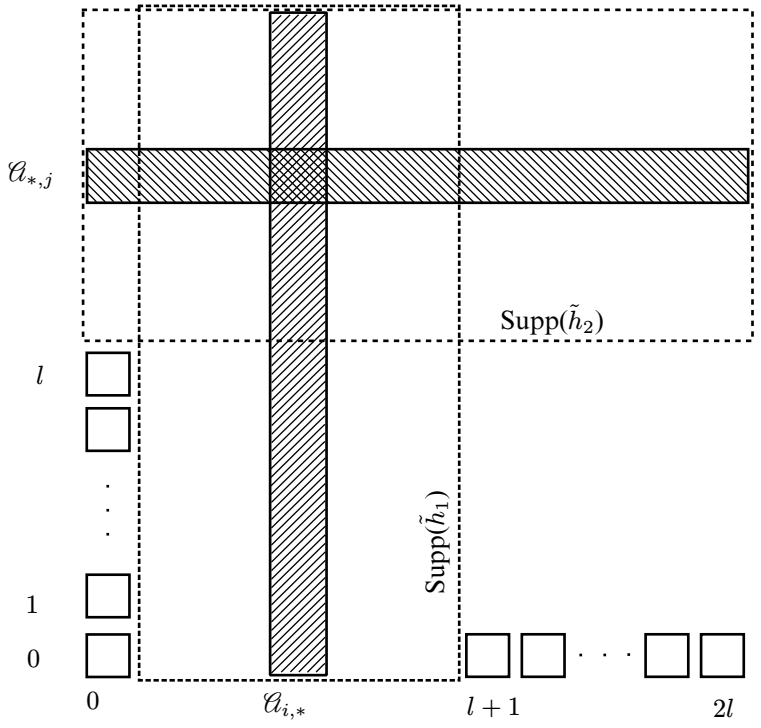

FIGURE 2. Support of local perturbations projected to Λ. No perturbation is made in white blocks.

In order to define the local perturbation, we first consider open sets $\mathcal{A}_{ij}$ and pairwise disjoint open sets $\widetilde{\mathcal{A}}_{ij}$ in a small neighborhood of Λ, such that,

$$\mathcal{A}_{ij} \cap \Lambda = \{(x_i)_{i\in\mathbb{Z}} \mid x_0 = i,\ x_1 = j\} \text{ and } \mathcal{A}_{ij} \Subset \widetilde{\mathcal{A}}_{ij} \tag{3}$$

where $(x_i)_{i\in\mathbb{Z}}$ is the itinerary of a point in the Markov partition.

In addition, we set

$$\mathcal{A}_{i,*} = \bigcup_{j=0}^{d} \mathcal{A}_{ij} \text{ and } \mathcal{A}_{*,j} = \bigcup_{i=0}^{d} \mathcal{A}_{ij}. \tag{4}$$

If $\mathcal{J}, \mathcal{I} \subset \{0, 1, \dots, d\}$ then we set

$$\mathcal{A}_{i,\mathcal{J}} = \bigcup_{j\in\mathcal{J}} \mathcal{A}_{ij}. \tag{5}$$

Similar notations shall be used for $\mathcal{A}_{\mathcal{I},j}$, $\mathcal{A}_{\mathcal{I},\mathcal{J}}$, $\widetilde{\mathcal{A}}_{i,*}$, $\widetilde{\mathcal{A}}_{*,j}$ and $\widetilde{\mathcal{A}}_{\mathcal{I},\mathcal{J}}$.

Using the Darboux Theorem we select local coordinates in a neighborhood $\tilde{U}$ of $\hat{q}$ in N, of the form $(a_1, \dots, a_n; b_1, \dots, b_n)$ such that on $\tilde{U}$ the symplectic form is $\Sigma_{i=1}^{n} da_i \wedge db_i$. These coordinates will be useful to define our local perturbation.

Let $U \Subset \tilde{U}$ be a small neighborhood of $\hat{q}$ such that $f_2(U) \Subset \tilde{U}$.

Given a sufficiently small vector $(u,v)=(u_1,\dots,u_n;v_1,\dots,v_n)\in\mathbb{R}^{2n}$, let the Hamiltonian $h_{(u,v)}$ on N be a bump function such that $h_{(u,v)}(y)=0$ if $y\in N\setminus\tilde{U}$ and if $y\in U$; then $h_{(u,v)}$ is expressed in the local coordinates as ($y=(a,b)\in U$) $h_{(u,v)}(a,b):=a\cdot v-b\cdot u$.

Note that the time t map of the Hamiltonian flow of $h_{(u,v)}$ in U is the translation

$$(a,b)\mapsto(a,b)+t(u,v).$$

We identify a neighborhood of zero in T_xN and N in the local coordinates on $\tilde{U}$, and let $\hat{q}=0$ in the local coordinates. So $Df_2(\hat{q})$ is very close to f_2 on the neighborhood U of $\hat{q}$, provided that U is small enough.

Let $\varphi^s=Df_2(\hat{q})|_{E^s_{\hat{q}}}$. By the assumption, the contraction bound of φ^s is $1-\delta>1/2$. Then, from Proposition 2.3 we get small vectors $c_1,\dots,c_l$ in $E^s_{\hat{q}}\subset T_{\hat{q}}N$ such that the linear maps $\varphi^s_0:=\varphi^s,\varphi^s_1:=\varphi^s+c_1,\dots,\varphi^s_l:=\varphi^s+c_l$ on $E^s_{\hat{q}}$ generate an iterated function system which is transitive in some small disk $D^s\subset E^s_{\hat{q}}$ (satisfying the covering and well-distributed properties).

We denote the vectors c_i in the local coordinates by $(u^i,v^i)\in\tilde{U}$. So we may define the Hamiltonians $h_{(u^i,v^i)}$ on N as above.

Now, we define the Hamiltonian perturbation $\tilde{h}_1:M\times N\to\mathbb{R}$, as a bump function such that

$$\tilde{h}_1(x,y)=h_{(u^i,v^i)}(y)\text{ if }x\in\mathcal{C}_{i,*},i=1,\dots,l,$$

$$\tilde{h}_1(x,y)=0\text{ if }x\notin\bigcup_{i=1}^{l}\widetilde{\mathcal{C}}_{i,*}.$$

Similarly, we get a family of contracting maps $\varphi^u_0:=Df_2^{-1}(\hat{q})|_{E^u_{\hat{q}}}$, $\varphi^u_1:=\varphi^u+c'_1$, $\dots,\varphi^u_l:=\varphi^u+c'_l$ on $E^u_{\hat{q}}$ that generate a transitive iterated function system in a small disk D^u. We denote the vectors c'_i in the local coordinates by $(u'^i,v'^i)\in\tilde{U}$.

The Hamiltonian perturbation $\tilde{h}_2:M\times N\to\mathbb{R}$ is a bump function such that

$$\tilde{h}_2(x,y)=h_{(u'^i,v'^i)}(y)\text{ if }x\in\mathcal{C}_{*,j},j=l+1,\dots,2l,$$

$$\tilde{h}_2(x,y)=0\text{ if }x\notin\bigcup_{j=l+1}^{2l}\widetilde{\mathcal{C}}_{*,j}.$$

The symplectic blender. – Now we explain how this construction creates a symplectic blender: First, observe that by the construction we get a geometric model of a symbolic double-blender and therefore its perturbations can be treated like the perturbations of a double-blender. Thus the proof reduces to the argument in Section 3.3.3. Since we preserve a symplectic structure, the double-blender becomes a symplectic blender.

Moreover, we have the following proposition which is a consequence of the first part of Proposition 2.5. The details of the proof are left to the reader.

PROPOSITION 3.17. – *Under the hypotheses of Theorem 3.16 it is possible to create a symplectic blender with the following additional property:*

B-4 *Any forward and backward iteration of a uu-leaf (ss-leaf) intersecting $\Lambda\times U$, intersects $\Lambda\times U$ in a uu-segment (ss-segment, respectively).*

Consequently, the set of all points whose strong (un)stable manifolds intersect $\Lambda\times U$, is an invariant set.

3.5. Blender inside partial hyperbolic sets

In the present section we give a series of lemmas about the role of blenders in obtaining robust transitivity. Roughly speaking, Lemmas 3.20 and 3.21 state that the points in a partially hyperbolic set connected to the blender through the strong leaves are contained in a homoclinic class. These lemmas are essential in the proof of Theorem 4.1.

DEFINITION 3.18. – Given a (not necessarily compact) partially hyperbolic invariant set Λ with splitting $E^{ss}\oplus E^{cs}\oplus E^{cu}\oplus E^{uu}$, for any $z\in\Lambda$ we define the strong stable (unstable) manifold of z with diameter L as the set

$$W^{ss}_L(z) := f^{-L}(W^{ss}_{\mathrm{loc}}(f^L(z))) \quad (W^{uu}_L(z) := f^L(W^{ss}_{\mathrm{loc}}(f^{-L}(z)))).$$

REMARK 3.19. – Recall that for a blender $(p,\mathcal{B})$, if $W^{ss(uu)}_L(x)\cap W^{u(s)}(B)\neq\varnothing$ then for any y sufficiently close to x it also holds that $W^{ss(uu)}_L(y)\cap W^{u(s)}(B)\neq\varnothing$, where B is the maximal invariant set generated by $\mathcal{B}$.

LEMMA 3.20. – *Let Λ be a (not necessarily compact) partially hyperbolic invariant set with splitting $E^{ss}\oplus E^{cs}\oplus E^{cu}\oplus E^{uu}$. Assume that*

1. *there exists a symplectic blender $(p,\mathcal{B})$, such that its maximal invariant hyperbolic set B is contained in Λ and has unstable dimension equal to $\dim(E^{cu}\oplus E^{uu})$;*
2. *there exists a non empty set $R\subset\Lambda$ such that if $x\in R$ then there are infinitely many forward and backward iterates of x in $\mathcal{B}$.*

Then, the homoclinic class of B contains the closure of R.

Proof. – Let us fix $x\in R$. From the fact that there are infinitely many backward iterates of x in $\mathcal{B}$ and from Remark 3.12, it follows that there is a sequence of points accumulating on x that belong to the intersection of the unstable manifold of B with the strong stable manifold of x. In the same way, and using now that there are infinitely many forward iterates of x in $\mathcal{B}$ it follows that there is a sequence of points accumulating on x that belong to the intersection of the stable manifold of B with the strong unstable manifold of x. Using Remark 3.19 we conclude that there is a sequence of points accumulating on x that belongs to the intersection of the stable and unstable manifold of B. □

LEMMA 3.21. – *Let Λ be a (not necessarily compact) partially hyperbolic invariant set with splitting $E^{ss}\oplus E^{cs}\oplus E^{cu}\oplus E^{uu}$. Assume that*

1. *there exists a symplectic blender $(p,\mathcal{B})$, such that its maximal invariant hyperbolic set B is contained in Λ and has unstable dimension equal to $\dim(E^{cu}\oplus E^{uu})$;*
2. *there exists a non empty set $R\subset\Lambda$ contained in the recurrent set of f such that if $x\in R$ then*
   (2.1) $W^{ss}(x)\cap W^{u}_{\mathrm{loc}}(B)\neq\varnothing$,
   (2.2) $W^{uu}(x)\cap W^{s}_{\mathrm{loc}}(B)\neq\varnothing$.

Then, the homoclinic class of the B contains the closure of R.

Proof. – If $W^{ss}(x) \cap W^u_{\mathrm{loc}}(B) \neq \varnothing$, there is an L sufficiently large such that $W^{ss}_L(x) \cap W^u_{\mathrm{loc}}(B) \neq \varnothing$. From Remark 3.19 and the fact that x is recurrent, it follows that there is a sequence $\{f^{-n_k}(x)\}$ such that $W^{ss}_L(f^{-n_k}(x)) \cap W^u(B) \neq \varnothing$ and therefore, there is a sequence of points accumulating on x that belong to the intersection of the unstable manifold of B with the strong stable manifold of x. Arguing in the same way, it follows that there is a sequence of points accumulating on x that belong to the intersection of the stable manifold of B with the strong unstable manifold of x. The lemma is now obtained as in the proof of Lemma 3.20. □

4. Parametric version of Theorem A

In this section, we prove a parametric version of Theorem A in a more general context. The proof of our main theorems is based on this theorem and its proof.

THEOREM 4.1. – *Let M and N be two symplectic manifolds (not necessarily compact), and $1 \leq r \leq \infty$. Let $f_1 \in \mathrm{Diff}^r_\omega(M)$ such that there exists an open set $V \subset M$ whose maximal invariant set Λ is a nontrivial topologically mixing hyperbolic compact set. Let $f_2 \in \mathrm{Diff}^r_\omega(N)$ such that:*

(a) *f_2 is dominated by $f_1|_\Lambda$, and f_2 has a δ-weak hyperbolic periodic point for some positive $\delta = \delta(f_1, \dim(N))$ close to zero.*
(b) *For any $\tilde{f}_2$ sufficiently C^r close to f_2, $\Omega(\tilde{f}_2) = N$.*

Then there is a C^r arc $\{F_\mu\}_{\mu\in[0,1]}$ of C^r symplectic diffeomorphisms on $M \times N$, such that $F_0 = f_1 \times f_2$, and there exists a periodic orbit P such that for all $\mu \in (0,1]$ the following hold,

1. *P is periodic for F_μ, $H(P; F_\mu) \bigcap_{n\in\mathbb{Z}} F^n_\mu(V \times N) = \Lambda \times N$ and it is robustly strictly topologically mixing (See Definition 1.8).*
2. *If G is close to F_μ, the set $\Gamma_G := H(P_G; G) \bigcap_{n\in\mathbb{Z}} G^n(V \times N)$ is robustly strictly topologically mixing.*
3. *For any compact neighborhood $N_c \subset N$ the map $G \in \mathcal{U} \mapsto \Gamma_G \bigcap (V \times N_c)$ defines a continuation of $\Lambda \times N_c$, for some neighborhood $\mathcal{U}$ of F_μ.*

REMARK 4.2. – We want to highlight that in Theorem 4.1 it is not assumed that the manifolds M and N are compact. However the hypotheses of Theorem 4.1 are contained in the hypotheses of Theorems A, B, C and D. In particular, observe that in Theorem 4.1 we assume that for any $\tilde{f}_2$ sufficiently C^r close to f_2 it holds that $\Omega(\tilde{f}_2) = N$, which is immediately satisfied whenever either M is compact or has finite volume.

REMARK 4.3. – First observe that for any diffeomorphism F_μ, with $\mu > 0$, it holds that the whole set $\Lambda \times N$ is transitive. In this sense, those maps satisfy a stronger property than the ones in the thesis of Theorem A. In fact, the ones in the thesis of Theorem A are obtained as perturbations of the family map $\{F_\mu\}_{\mu\in(0,1]}$. Also observe, that the non-wandering hypothesis (b) is obviously satisfied if the manifold N is compact or has a finite volume.

Before giving the proof of Theorem 4.1 we need some results stated in Section 4.1, about the persistence of normally hyperbolic laminations. This gives us the right framework to obtain the continuation of the set in the thesis of Theorem 4.1. The proof of Theorem 4.1 is postponed to Section 4.2.

4.1. Persistence of normally hyperbolic laminations

Now, we recall a technical version of the main results of Hirsch-Pugh-Shub [19] on persistence of normally hyperbolic laminations, extended to the non-compact embedded case. As we said before, the result provides the existence and precise characterization of the continuation of a partially hyperbolic set (for the definition of continuation used in the present context, see Section 4.4).

DEFINITION 4.4 (strong continuation). – An invariant set $X \subset M$ of f has *strong continuation* in $\mathcal{D}^r$, if there exist an open neighborhood $\mathcal{U}$ of f in $\mathcal{D}^r$, and a continuous map $\Phi : \mathcal{U} \to \mathcal{P}(M)$ such that, $\Phi(f) = X$, and for any $g \in \mathcal{U}$, the set $\Phi(g) \subset M$ is homeomorphic to X and invariant for g. Then, $\Phi(g)$ is called the *strong continuation* of X for g. Here, $\mathcal{P}(M)$ is the space of all subsets of M with the Hausdorff topology.

Compare this definition with Definition 1.3. In the present one, it is required that $\Phi(g)$ is homeomorphic to X.

THEOREM 4.5 (HPS). – *Let M and N be two boundaryless manifolds (not necessarily compact). Let $1 \leq r \leq \infty$ and let $f_1 \in \mathrm{Diff}^r_\omega(M)$ such that there exists an open set $V \subset M$ whose maximal invariant set Λ is a nontrivial topologically mixing hyperbolic compact set. Let $f_2 \in \mathrm{Diff}^r(N)$ be dominated by $f_1|_\Lambda$. Then the invariant set $\Lambda \times N$ of $F_0 = f_1 \times f_2$ has a unique strong continuation for any small perturbation of F_0 in $\mathrm{Diff}^r(M \times N)$.*

More precisely, the following hold,

- H-1 There is a neighborhood $\mathcal{U} \subset \mathrm{Diff}^1(M \times N)$ of F such that every $G \in \mathcal{U}$ has a (locally maximal) invariant Γ_G homeomorphic to $\Lambda \times N$ and is a continuation of Γ_{F_0}.
- H-2 There is a G-invariant lamination on Γ_G by manifolds diffeomorphic to N. So G induces a homeomorphism $\tilde{G}$ on the quotient of Γ_G such that $\tilde{G}$ is conjugate to $f_1|_\Lambda$.
- H-3 From previous item, G restricted to Γ_G is conjugate to a skew product $G^* : (x, w) \longmapsto (f_1(x), g_x(w))$ on $\Lambda \times N$, which depends continuously on G.

This result, in a more general setting, is proved in [19] in the compact case. However, as remarked there, the compactness is used only to obtain uniform estimates on the functions involved in the proof (i.e. the rate of contraction and expansion) and the results carry over as long as such uniform estimates hold. This is the case in our context since we consider the uniform C^r topology. See [11], [15, Appendix B] and [13, Appendix A] in which a similar situation is treated.

4.2. Proof of Theorem 4.1

The main idea of the proof is to find perturbations of $F_0 = f_1 \times f_2$ so that the following hold:

(i) the existence of a symplectic blender,

(ii) the "minimality" of (strong) stable and unstable foliations in the partially hyperbolic set $\Lambda \times N$.

These properties imply the transitivity (or even topological mixing) of the set $\Lambda \times N$ in a robust fashion. Following this approach, the proof is constructive and it is divided into four parts. First, in Section 4.2.1 we introduce the perturbations which are similar to the ones used in the construction of the family F_μ in the proof of Theorem 3.16. The perturbations are done using the result stated in Section 4.1. Then in the following subsections we prove that the perturbed systems satisfy the desired properties. In Section 4.2.2, applying Theorem 3.16 we show the existence of a symplectic blender. Then in Section 4.2.3 we use the results of iterated function systems of recurrent diffeomorphisms (see Section 2.2) to prove that the strong stable and unstable manifolds of almost all points in the central manifold intersect the constructed blender. In Section 4.2.4, using the lemmas in Section 3.5, we show that this property is robust under small perturbations.

Proof of Theorem 4.1. – Let $\hat{p}$ be a hyperbolic periodic point of f_1 with transversal homoclinic intersection, and $\hat{q}$ be a δ-weak hyperbolic periodic point of f_2. Let $P_0 = (\hat{p}, \hat{q})$. Let $V \subset M$ be a small open disk which contains $\hat{p}$ and a transversal homoclinic point associated to it, such that for some $k \in \mathbb{N}$, $\Lambda_0 := \bigcap_{n\in\mathbb{Z}} f_1^{kn}(V)$ is an invariant hyperbolic compact set of saddle type for f_1^k. By choosing V suitable and k large enough, we may suppose that $f_1^k|_{\Lambda_0}$ is conjugate to a shift of $d+1$ symbols $\{0, 1, \ldots, d\}$. Moreover, we can assume that the elements of the Markov partition of Λ_0 are contained in open sets such that their closures are pairwise disjoint. If k is large then d can be taken large. If k is large enough, by taking f_1^k, and f_2^k instead of f_1 and f_2, we may assume that $P_0 = (\hat{p}, \hat{q})$ is a fixed point of $F_0 = f_1 \times f_2$ and $\Lambda_0 \subset \Lambda$ are f_1-invariant. Moreover on $\Lambda_0 := \bigcap_{n\in\mathbb{Z}} f_1^n(V)$, f_1 is conjugate to the shift of symbols $\{0, 1, \ldots, d\}$.

As in the proof of Theorem 3.16 recall that for a (linear) contraction map of $\mathbb{R}^n$ ($2n = \dim(N)$), with contraction bound equal to $1/2$, Proposition 2.3 gives a number l as the required number of elements of the IFS needed to obtain transitivity in some small disk.

We choose k large enough so that $d \geq 2l + 4$ (observe that the number d here is larger than the one in Theorem 3.16).

Again, as in the proof of Theorem 3.16, once we substitute f_2 by f_2^k it follows that $m(Df^k|_{E^s}) > (1-\delta)^k$. Therefore, the constant $\delta = \delta(f_1, \dim(N))$ is chosen close to zero in such a way that $(1-\delta)^k > 1/2$.

From now on, we assume that $\hat{q}$ is a δ-weak hyperbolic fixed point of f_2 with $\delta < 1/2$, $\hat{p} \in \Lambda_0$ is a hyperbolic fixed point of f_1, and Λ_0 is f_1-invariant and it is conjugate to a shift with $d+1$ symbols as above.

In what follows, we are going to focus on the set $\Lambda_0 \times N$.

4.2.1. *The perturbations.* – Let $\zeta > 0$ be small enough (it will be chosen later) and $\varepsilon : [0,1] \longrightarrow [0,\zeta]$ such that $\varepsilon : x \mapsto x/\zeta$.

For $\mu \in [0,1]$,

$$F_\mu := \Psi^{\varepsilon(\mu)} \circ F_0 \circ \Phi^{-\varepsilon(\mu)},$$

where Ψ^t and Φ^t are the time t maps of the Hamiltonians h_1 and h_2, respectively, to be chosen later. It is clear that $F_0 = f_1 \times f_2$.

As in Theorem 3.16 using the Darboux Theorem we select local coordinates in a neighborhood $\tilde{U}$ of $\hat{q}$ in N, of the form $(a_1, \ldots, a_n; b_1, \ldots, b_n)$ such that on $\tilde{U}$ the symplectic form is $\Sigma_{i=1}^n da_i \wedge db_i$. This coordinates will be useful to define our local perturbation.

Let $U \Subset \tilde{U}$ be a small neighborhood of $\hat{q}$ such that $f_2(U) \Subset \tilde{U}$.

As in the proof of Theorem 3.16, we set

$$\mathcal{C}_{i,*} = \bigcup_{j=0}^{d} \mathcal{C}_{ij} \text{ and } \mathcal{C}_{i,\mathcal{J}} = \bigcup_{j\in\mathcal{J}} \mathcal{C}_{ij}$$

where $\mathcal{J} \subset \{0, 1, \ldots, d\}$. Similar notations shall be used for $\mathcal{C}_{*,j}$, $\mathcal{C}_{\mathcal{J},j}$, $\mathcal{C}_{\mathcal{J},\mathcal{J}}$, $\widetilde{\mathcal{C}}_{i,*}$, $\widetilde{\mathcal{C}}_{*,j}$ and $\widetilde{\mathcal{C}}_{\mathcal{J},\mathcal{J}}$.

Now, we separate the support of perturbations in two groups:

1. one composed by $\mathcal{C}_{i,*}, i = 1, \ldots, l$, and $\mathcal{C}_{*,j}, j = l+1, \ldots, 2l$;
2. the other composed by $\mathcal{C}_{\mathcal{J}_1,\mathcal{J}_1} \cup \mathcal{C}_{\mathcal{J}_2,\mathcal{J}_2}$, where $\mathcal{J}_1 = \{0, 2l+1, 2l+2\}$ and $\mathcal{J}_2 = \{0, 2l+3, 2l+4\}$.

In the first group of sets, the same perturbations done in the proof of Theorem 3.16 are performed along the center direction. This shall provide us with a blender (see Section 4.2.2). In the second ones, other perturbations (explained below) are introduced such that they provide the minimality of the unstable and stable foliations (see Section 4.2.3).

So, using the same definitions as those given in the proof of Theorem 3.16 we define the Hamiltonian perturbation $\tilde{h}_1 : M \times N \to \mathbb{R}$, as a bump function such that

$$\tilde{h}_1(x,y) = h_{(u^i,v^i)}(y) \text{ if } x \in \mathcal{C}_{i,*}, i = 1, \ldots, l,$$

$$\tilde{h}_1(x,y) = 0 \text{ if } x \notin \bigcup_{i=1}^{l} \widetilde{\mathcal{C}}_{i,*}.$$

Similarly, the Hamiltonian perturbation $\tilde{h}_2 : M \times N \to \mathbb{R}$ is a bump function such that

$$\tilde{h}_2(x,y) = h_{(u'^i,v'^i)}(y) \text{ if } x \in \mathcal{C}_{*,j}, j = l+1, \ldots, 2l,$$

$$\tilde{h}_2(x,y) = 0 \text{ if } x \notin \bigcup_{j=l+1}^{2l} \widetilde{\mathcal{C}}_{*,j}.$$

Now we explain the perturbation performed in the set

$$\mathcal{C}_{\mathcal{J}_1,\mathcal{J}_1} \cup \mathcal{C}_{\mathcal{J}_2,\mathcal{J}_2} \quad (\mathcal{J}_1 = \{0, 2l+1, 2l+2\}, \mathcal{J}_2 = \{0, 2l+3, 2l+4\}).$$

Let $\hat{h}_1$ and $\hat{h}_2$ be two integrable Hamiltonians on N, close to the identity, such that the time one maps of their corresponding Hamiltonian flows satisfy the properties (1) and (2) in Lemma 2.7.

We define the Hamiltonian $\tilde{h}_3 : M \times N \to \mathbb{R}$ as a bump function such that

$$\tilde{h}_3(x,y) = \hat{h}_1(y) \text{ if } x \in \mathcal{C}_{2l+1,j},\ j \in \{0, 2l+1, 2l+2\},$$
$$\tilde{h}_3(x,y) = \hat{h}_2(y) \text{ if } x \in \mathcal{C}_{2l+2,j}, j \in \{0, 2l+1, 2l+2\},$$
$$\tilde{h}_3 = 0 \text{ if } x \notin \widetilde{\mathcal{C}}_{i,j}, i \in \{2l+1, 2l+2\}, j \in \{0, 2l+1, 2l+2\}.$$

Similarly, we define the Hamiltonian $\tilde{h}_4 : M \times N \to \mathbb{R}$ as a bump function such that

$$\tilde{h}_4(x,y) = -\hat{h}_1(y) \text{ if } x \in \mathcal{C}_{i,2l+3},\ i \in \{0, 2l+3, 2l+4\},$$
$$\tilde{h}_4(x,y) = -\hat{h}_2(y) \text{ if } x \in \mathcal{C}_{i,2l+4},\ i \in \{0, 2l+3, 2l+4\},$$
$$\tilde{h}_4 = 0 \text{ if } x \notin \widetilde{\mathcal{C}}_{i,j},\ i \in \{0, 2l+3, 2l+4\}, j \in \{2l+3, 2l+4\}.$$

Observe that the support of these Hamiltonians are disjoint from the supports of $\tilde{h}_1$ and $\tilde{h}_2$. Now we are able to define the Hamiltonians h_1 and h_2. Let $\epsilon > 0$ be small enough; we set

$$h_1 = \tilde{h}_1 + \epsilon \tilde{h}_3,$$
$$h_2 = \tilde{h}_2 + \epsilon \tilde{h}_4.$$

In the next two sections, we show how the above perturbation maps exhibit symplectic blenders and "almost" minimality of the strong foliations.

4.2.2. *The symplectic blender*. – We may repeat the proof of Theorem 3.16 for the family F_μ constructed in Section 4.2.1 to obtain a symplectic blender $\mathcal{B}$. In addition, Proposition 3.17 also holds for this family.

4.2.3. *Almost minimality of stable and unstable foliations*. – In this section it is shown that the strong stable and unstable manifolds of an open and dense set of points in the central manifold $N_0 := \{\hat{p}\} \times N$ intersect the constructed blender. From that, we obtain the existence of an open and dense set of points in $\Lambda \times N$ such that their strong (stable and unstable) manifolds are connected to the blender. We refer to this property by the almost minimality of the strong stable and strong unstable foliations.

PROPOSITION 4.6. – *Let us fix $\mu > 0$. Then there is an open and dense set $\mathcal{R}_\mu \subset N$ such that for every $q \in \mathcal{R}_\mu$ and for any $n \in \mathbb{Z}$ it follows that $W^{uu}(F_\mu^n(\hat{p}, q)) \cap \mathcal{B} \neq \varnothing$ and $W^{ss}(F_\mu^n(\hat{p}, q)) \cap \mathcal{B} \neq \varnothing$.*

Proof. – The key elements in the proof are the symbolic dynamics, the results of Section 2.2 and Proposition 3.17.

Here again we consider restriction of f_1 to Λ_0. For any $p = (p_i)_{i\in\mathbb{Z}} \in \Lambda_0 = \{0, 1, 2, \ldots, d\}^{\mathbb{Z}}$, the local and global unstable manifolds of p for f_1 are

$$W^u_{\mathrm{loc}}(p\,;\, f_1|_{\Lambda_0}) = \{(x_i) \mid \forall n \leq 0, x_i = p_i\},$$
$$W^u(p\,;\, f_1|_{\Lambda_0}) = \{(x_i) \mid \exists n_0 \in \mathbb{Z}, \forall n \leq n_0, x_i = p_i\}.$$

So, the local and global strong unstable manifolds of (p, q) for F_μ are

$$W^{uu}_{\mathrm{loc}}(p, q; F_\mu|_\Gamma) = W^u_{\mathrm{loc}}(p; f_1|_{\Lambda_0}) \times \{q\} = \{(x_i) \mid \forall n \leq 0, x_i = p_i\} \times \{q\},$$
$$W^{uu}(p, q; F_\mu|_\Gamma) = \bigcup_{n\geq 0} F^n_\mu(W^{uu}_{\mathrm{loc}}(F^{-n}_\mu(p, q); F_\mu|_\Gamma)).$$

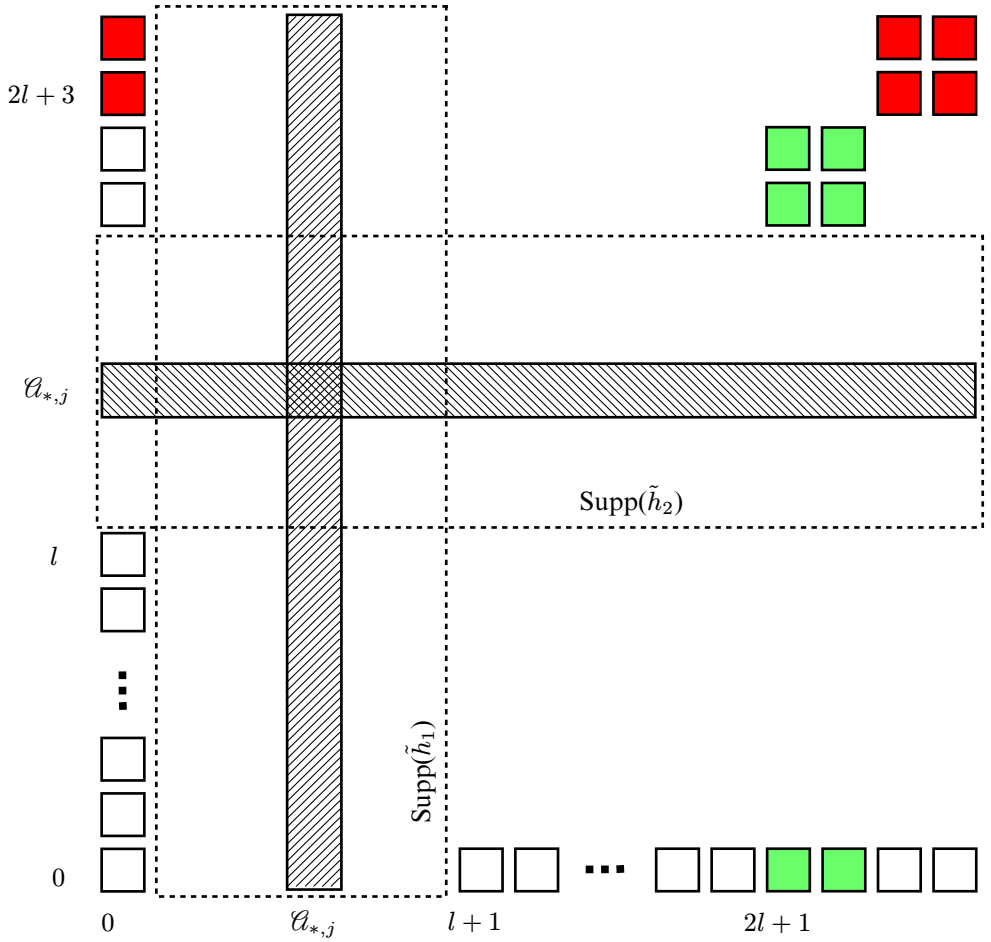

FIGURE 3. Support of local perturbations projected to Λ_0. The blocks with the same color are in the support of the same Hamiltonians. No perturbation is made in white blocks.

Let $T_1 = f_2$, $T_2 = \phi_1 \circ f_2$ and $T_3 = \phi_2 \circ f_2$, where ϕ_1 and ϕ_2 are the time $t = \varepsilon(\mu)$ maps of the Hamiltonian flows of $\epsilon\hat{h}_1$ and $\epsilon\hat{h}_2$, respectively. Then the proof of Theorem 2.13 yields the transitivity of the IFS $\mathcal{G}(T_1, T_2, T_3)$.

Let $q \in \mathrm{Rec}(f_2) \subset N$ such that there is a finite sequence $(\sigma_i)_{i=1}^n$ ($\sigma_i \in \{1,2,3\}$) and

$$T_{\sigma_n} \circ T_{\sigma_{n-1}} \circ \cdots \circ T_{\sigma_1}(q) \in T_1^{-2}(D).$$

We denote the set of all such points by R_μ.

Now, we consider

$$x = (x_i) = (\overbrace{\ldots, 0, 0, 0}^{W^u_{\mathrm{loc}}(\hat{p})}\,;\, \overbrace{a_1, a_2, \ldots, a_{n_0}}^{\text{IFS}}, 0, 0, \overbrace{x_{n_0+3}, \ldots}^{\text{arbitrary}}),$$

where for $i = 1, 2, \ldots, n_0$,

$$a_i = \begin{cases} 0, & \text{if } \sigma_i = 1; \\ 2l+1, & \text{if } \sigma_i = 2; \\ 2l+2, & \text{if } \sigma_i = 3. \end{cases}$$

The elements of the sequence (x_i) are chosen in such a way that $f_1^{n_0+1}(x) \in \mathcal{C}_{0,0}$, $x \in W^u_{\mathrm{loc}}(\hat{p}, f_1|_{\Lambda_0})$, and so $(x, q) \in W^{uu}(\hat{p}, q; F_\mu|_\Gamma)$. Moreover, the choice of $a_1, a_2, \ldots, a_{n_0}$ permits us to realize the IFS $\mathcal{G}(T_1, T_2, T_3)$.

We now take the iterations of the point (x, q) under F_μ. Since F_μ restricted to $\mathcal{C}_{a_i, \mathcal{J}_1} \times N$ ($\mathcal{J}_1 = \{0, 2l+1, 2l+2\}$) is equal to $f_1 \times T_{\sigma_i}$, inductively we have:

$$(f_1^i(x), T_{\sigma_i} \circ T_{\sigma_{i-1}} \circ \cdots \circ T_{\sigma_1}(q)) = F_\mu^i(x, q) \in W^{uu}(F_\mu^i(\hat{p}, q); F_\mu).$$

In particular, for $i = n_0 + 1$, $F_\mu^{n_0+1}(x, q) \in \mathcal{B}$. So,

$$W^{uu}(F_\mu^{n_0+1}(\hat{p}, q); F_\mu) \cap \mathcal{B} \neq \varnothing.$$

Now we apply Proposition 3.17 which implies that for all $n \in \mathbb{Z}$,

$$W^{uu}(F_\mu^n(\hat{p}, q); F_\mu) \cap \mathcal{B} \neq \varnothing.$$

Let $\mathcal{R}_\mu$ be the set of all points $q \in N$ such that the above intersection holds. Observe that we just proved that $R_\mu \subset \mathcal{R}_\mu$. The set $\mathcal{R}_\mu$ is open, because $\mathcal{B}$ is open and the strong stable and unstable manifolds depend continuously on the points. This completes the proof. □

REMARK 4.7. – Observe that the set R_μ has total Lebesgue measure. This follows from the results proved in Section 2.2.

COROLLARY 4.8. – *Let $\mu > 0$. Then there is an open and dense set $\hat{\mathcal{R}}_\mu \subset \Lambda \times N$ such that for every $x \in \hat{\mathcal{R}}_\mu$ and for any $n \in \mathbb{Z}$ it follows that $W^{uu}(F_\mu^n(x)) \cap \mathcal{B} \neq \varnothing$ and $W^{ss}(F_\mu^n(x)) \cap \mathcal{B} \neq \varnothing$.*

Proof. – First observe that Theorem 4.5 and the minimality of the stable and unstable foliations in the hyperbolic set Λ imply that $W^{ss}(\{\hat{p}\} \times N; F_\mu)$ and $W^{uu}(\{\hat{p}\} \times N; F_\mu)$ are dense in $\Lambda \times N$. On the other hand, the strong unstable foliations (with leaves $W^{uu}(\hat{p}, q; F_\mu)$) inside $W^{uu}(\{\hat{p}\} \times N; F_\mu)$ are continuous. Moreover, it follows from Proposition 4.6 that a dense open subset of uu-leaves in $W^{uu}(\{\hat{p}\} \times N; F_\mu)$ intersects $\mathcal{B}$. The same statements hold for the strong stable foliation.

Now, the density of $W^{uu}(\{\hat{p}\} \times N; F_\mu)$ and $W^{ss}(\{\hat{p}\} \times N; F_\mu)$ in $\Lambda \times N$ and Proposition 3.17 imply the corollary. □

Observe that Corollary 4.8 and Lemma 3.20 imply that $H(P; F_\mu) \bigcap_{n \in \mathbb{Z}} F_\mu^n(V \times N) = \Lambda \times N$, for any $\mu > 0$. So it remains to show that the set $H(P; F_\mu) \bigcap_{n \in \mathbb{Z}} F_\mu^n(V \times N)$ is robustly topologically mixing.

4.2.4. *Robustness of* $H(P;F_\mu)\bigcap_{n\in\mathbb{Z}}F_\mu^n(V\times N)$. – By the definition, we have to construct an increasing sequence of compact invariant robust transitive sets, such that its union gives $\Lambda\times N$. For any positive integer L large enough, we consider the set $\hat{\mathcal{R}}_{\mu,L}$ given by the points in $\hat{\mathcal{R}}$ such that for any iterate, it holds that the strong stable and strong unstable manifolds of diameter L are connected to the blender. Let us denote this set by $\Gamma_{F_\mu,L}$. It follows immediately that the set is compact invariant and is contained in the homoclinic class of p. By the definition it also holds that $\bigcup_{L>0}\hat{\mathcal{R}}_{\mu,L}=\hat{\mathcal{R}}$. In the same way, and following the details of the proof of Lemma 3.21 it holds that $\bigcup_{L>0}\Gamma_{F_\mu,L}=H(P;F_\mu)\bigcap_{n\in\mathbb{Z}}F_\mu^n(V\times N)$. To show that $\Gamma_{F_\mu,L}$ is robustly transitive for any L, observe that for any G nearby to F_μ it holds that $\Gamma_{G,L}$, defined as the set of trajectories with strong stable and strong unstable manifolds of diameter L connected to the continuation of the blender for G, is in fact a continuation of $\Gamma_{F_\mu,L}$.

4.2.5. *Continuation of compact parts*. – Following the argument described in the previous subsection, to conclude the last item of Theorem 4.1 it is enough to show that for any compact subset $\mathcal{K}\subset\mathcal{R}_\mu$, there exists $L>0$ such that any point in $\mathcal{K}$ is connected to the blender through the strong (stable and unstable) manifolds with diameter less than L. In fact, the invariant compact parts are given by $\Gamma_{\mu,L}\bigcap_{n\in\mathbb{Z}}F_\mu^n(V\times N)$.

The proof of Theorem 4.1 is completed. □

5. Proof of main theorems

5.1. Proof of Theorem A

Here we complete the proof of Theorem A.

Proof of Theorem A. – Observe that if $F=f_1\times f_2$ satisfies the hypothesis of Theorem A then it satisfies the hypothesis of Theorem 4.1. So, from Theorem 4.1 there exists a family $F_\mu,\mu\geq 0$ such that $F_0=f_1\times f_2$. Moreover, for any $\mu>0$, there exists a neighborhood $\mathcal{U}_\mu$ of F_μ such that any $G\in\mathcal{U}_\mu$ satisfies the (1)-(3) of Theorem 4.1 and so (2) of Theorem A. Let $\mathcal{U}=\bigcup_{\mu>0}\mathcal{U}_\mu$. Then $\mathcal{U}$ verifies (1) and any $G\in\mathcal{U}$ verifies (2). The proof of Theorem A is completed. □

5.2. Theorem B: Robust transitivity in nearly integrable diffeomorphisms

The proof is similar to the proof of Theorem A. However, using strongly the fact that the map f_2 is integrable, it is obtained that *no exceptional set appears.* More precisely, the stronger conclusion follows from applying Theorem 2.11 instead of Theorem 2.13.

Proof of Theorem B. – By the assumption, f_1 satisfies the H.S. hypothesis. So there exists a neighborhood $V\subset M$ such that its maximal invariant set Λ is a transitive hyperbolic basic set. Let $\tilde{f}_2$ be an integrable diffeomorphism sufficiently close to f_2 with a δ-weak hyperbolic saddle point (with small $\delta>0$). Then, the diffeomorphisms $f_1,\tilde{f}_2$ satisfy the hypothesis of Theorem 4.1. Here observe that we are assuming that $\tilde{f}_2$ is integrable.

So, in order to prove Theorem B we adapt the proof of Theorem 4.1 (and Theorem A). In fact, we do repeat the proof of Theorem 4.1 word by word, except in a few points to get a slightly stronger conclusion. In the following we point out where the proof of Theorem 4.1

needs changes. Beyond that part, the rest of the proof of Theorem 4.1 is repeated without any change.

First, we choose an iterate k large enough such that $f_1^k|\Lambda$ is conjugate to a shift with d symbols and $d \geq 2l + 2\dim(N) + 2$.

Second, in the definition of perturbations (cf. Section 4.2.1), we modify the second group of perturbations in order to apply Theorem 2.11 instead of Theorem 2.13 in the proof of Proposition 4.6.

Since $T_0 := \tilde{f}_2$ is integrable, it follows from Theorem 2.11 that there are $T_1, \ldots, T_m$, $m := \dim(N) + 1$, arbitrarily close to $\tilde{f}_2$ such that the IFS $\mathcal{G}(T_0, T_1, \ldots, T_m)$ is minimal. Similarly, we define $\hat{T}_1, \ldots, \hat{T}_m$ such that the IFS $\mathcal{G}(T_0^{-1}, \hat{T}_1^{-1}, \ldots, \hat{T}_m^{-1})$ is minimal.

We set $\mathcal{J}_1 = \{0, 2l+1, 2l+2, \ldots, 2l+m\}$ and $\mathcal{J}_2 = \{0, 2l+m+1, \ldots, 2l+2m\}$.

Then, we define the perturbation performed in the set $\mathcal{C}_{\mathcal{J}_1,\mathcal{J}_1} \cup \mathcal{C}_{\mathcal{J}_2,\mathcal{J}_2}$ such that F_μ restricted on $\mathcal{C}_{2l+j,\mathcal{J}_1} \times N$ is equal to $f_1 \times T_j$, for any $j \in \{1, \ldots, m\}$; and F_μ restricted on $\mathcal{C}_{2l+m+j,\mathcal{J}_1} \times N$ is equal to $f_1 \times \hat{T}_j$, for any $j \in \{1, \ldots, m\}$.

Now, by adapting the proof of Proposition 4.6 we apply the minimality of the IFS $\mathcal{G}(T_0, T_1, \ldots, T_m)$ to prove the slightly stronger conclusion that $\mathcal{R}_\mu = N$. Consequently, in Corollary 4.8 we obtain $\hat{\mathcal{R}}_\mu = \Lambda \times N$ which is a compact set. On the other hand, the sets $\hat{\mathcal{R}}_{\mu,L}$ are open in $\hat{\mathcal{R}}_\mu$. From $\bigcup_{L>0} \hat{\mathcal{R}}_{\mu,L} = \hat{\mathcal{R}}$ and compactness, it follows that for some large $L_0 > 0$, $\hat{\mathcal{R}}_{\mu,L_0} = \hat{\mathcal{R}}_\mu = \Lambda \times N$. Therefore, $\Gamma_{F_\mu,L_0} = H(P; F_\mu) \bigcap_{n\in\mathbb{Z}} F_\mu^n(V \times N)$. Thus, for any G close to F_μ, the strong continuation $\Gamma_G = \Gamma_{G,L_0}$ is well defined and topologically mixing (from Theorem 4.1), and it is homeomorphic to Γ.

In other words, we obtain the family of diffeomorphisms F_μ that $F_0 = f_1 \times \tilde{f}_2$, and for $\mu > 0$ the set $\Gamma = \Lambda \times N$ is robustly topologically mixing. This implies that there is a neighborhood $\mathcal{U}_{\tilde{f}_2}$ of the arc $\{F_\mu | \mu > 0\}$ such that for any $G \in \mathcal{U}_{\tilde{f}_2}$ the (strong) continuation Γ_G of Γ is robustly topologically mixing and is a relative homoclinic class. It is clear that the projection of Γ_G to N is onto (it is homeomorphic to $\Lambda \times N$). Now let

$$\mathcal{U} = \bigcup_{\tilde{f}_2} \mathcal{U}_{\tilde{f}_2},$$

where the union is taken over all integrable diffeomorphisms $\tilde{f}_2$ sufficiently close to f_2 and with a saddle periodic point.

It is clear from definition that $f_1 \times f_2$ is in the closure of $\mathcal{U}$, and for any $F \in \mathcal{U}$, there exists a robustly topologically mixing set homeomorphic to the set $\Lambda \times N$. So its projection to N is surjective. This completes the proof. □

5.3. Theorem C: Instabilities in nearly integrable Hamiltonians.

To prove Theorem C, we use the fact that the time one map of H_0 satisfies the hypotheses of Theorem A.

Proof of Theorem C. – Let $M = \mathbb{D}^m \times \mathbb{T}^m$ and $N = \mathbb{D}^n \times \mathbb{T}^n$.

First we perturb the Hamiltonian h_1 on M to obtain a transversal homoclinic intersection. Since h_1 satisfies the A.H. condition, then there is a small perturbation $\tilde{h}_1$ of h_1 such that the time one map $\tilde{f}_1$ of its Hamiltonian flow has a nontrivial transitive hyperbolic invariant set.

Note that the time one map of the Hamiltonian flow of h_2 is dominated by the restriction of $\tilde{f}_1$ to its hyperbolic basic set.

Then, we take another small integrable perturbation $\tilde{h}_2$ of the integrable Hamiltonian h_2 on N to create a δ-weak hyperbolic periodic point, for $\delta > 0$ small enough.

Note that since the perturbations done in the proof of Theorem 4.1, A and B are performed by Hamiltonian diffeomorphism and by symplectic changes of coordinates, those theorems and proofs can easily be adapted for Hamiltonians.

Now we repeat the proof of Theorem A with the same changes we made in the proof of Theorem B. Note that here the manifold N is not compact so we obtain robust transitivity on (strong) continuation of any compact subset of Γ. This completes the proof of Theorem C. □

5.4. Theorem D: Slow-fast systems

Before giving the proof of Theorem D, let us recall a result due to Zehnder and Newhouse.

5.4.1. *Theorem of Zehnder and Newhouse.* – Recall that a periodic point p of f of period n is called *quasi-elliptic* if $T_p f^n$ has a non-real eigenvalue of norm one, and all eigenvalues of norm one are non-real. Indeed, C^r generically every periodic point is either hyperbolic or quasi-elliptic (cf. [34], [31]). Note also that if f is Anosov, then robustly there is no quasi-elliptic periodic point.

THEOREM 5.1 ([38, 31]). – *There is a residual set $\boldsymbol{R} \subset \mathrm{Diff}^r_\omega(M)$, $1 \leq r \leq \infty$, such that if $f \in \boldsymbol{R}$, then any quasi-elliptic periodic point of f is a limit of transversal homoclinic points of f.*

Moreover, C^r generically, any elliptic point is the limit of a sequence of hyperbolic orbits $\{p_n\}$ such that p_n is a δ_n-weak hyperbolic point where $\delta_n \to 0$ as $n \to \infty$ (cf. [31] and [6]). Thus, C^r generically, existence of an elliptic periodic point implies the existence of a δ-weak hyperbolic periodic point with arbitrarily small $\delta > 0$.

Proof of Theorem D. – In order to prove Theorem D, it is enough to show that for a sufficiently small $\varepsilon > 0$, H_ε is approximated by a sum of two Hamiltonians whose time one maps satisfy the hypotheses of Theorem A.

By assumption, h_1 satisfies the H.S. condition. So, the time one map f_1 of its Hamiltonian flow has a hyperbolic basic set Λ. Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the time one map of the Hamiltonian flow of εh_2, which we denote by f_2, is dominated by $f_1|_\Lambda$. Since, the domination is an open property, the same holds for any small perturbation of h_2.

By assumption, the time one map of h_2 has an elliptic periodic point, and so does time one map of any small perturbation of h_2. On the other hand, it follows from Section 5.4.1 that the time one map of the Hamiltonian flow of any generic perturbation $\tilde{h}_2$ of h_2, has an elliptic periodic point which is the limit of a sequence of δ-weak hyperbolic periodic points for arbitrarily small $\delta > 0$.

Now, we choose $\varepsilon = 1/k$, where $k \in \mathbb{N} \cap (1/\varepsilon_0, \infty)$. Then, $\tilde{f}_2$, the time one map of the Hamiltonian flow of $\varepsilon \tilde{h}_2$, has an elliptic periodic point which is the limit of a sequence of δ-weak hyperbolic periodic points, for arbitrarily small $\delta > 0$.

Therefore, f_1 and $\tilde{f}_2$ satisfy the hypothesis of Theorem 4.1. Thus, there exists a smooth arc of Hamiltonians beginning on $h_1 + \varepsilon\tilde{h}_2$ such that the corresponding time one maps verify the conclusion of Theorem 4.1.

Let $\mathcal{H}$ be the union of all such smooth arcs as $\tilde{h}_2$ tends to h_2. Clearly, H_ε belongs to the C^∞ closure of $\mathcal{H}$.

To finish the proof, we will show that for any $\alpha > 0$, there is an open neighborhood $\mathcal{V}$ of $\mathcal{H}$ that verifies (1) and (2) in thesis of the theorem.

Let $H \in \mathcal{H}$, then its corresponding time one map verifies all the properties of some F_μ in the proof of Theorem 4.1. Therefore, it is enough to show that the continuation of the set Γ (in the proof of Theorem 4.1) satisfies (1), whenever the neighborhood $\mathcal{V}$ is sufficiently small. This is a direct consequence of the following:

Let $\mathcal{R}_{\mu,L}$ be the set of points in $\mathcal{R}_\mu$ such that their strong stable and unstable manifolds of diameter L are connected to the blender. Then, the same holds for the continuation of this set for any nearby system.

Using Remark 4.7, the set $\mathcal{R}_\mu = \bigcup_{L>0} \mathcal{R}_{\mu,L} = \bigcup_{L>0} \Gamma_{F_\mu,L} \cap (\{\hat{p}\} \times N)$ is not open and dense but has total Lebesgue measure in $\{\hat{p}\} \times N$. So, for $L > 0$ large enough, $\mathrm{vol}(\mathcal{R}_{\mu,L}) > \mathrm{vol}(N) - \alpha/3$.

In addition, the normally hyperbolic manifold $\{\hat{p}\} \times N$ and its continuations are C^r close. Consequently, the volume of continuation of $\mathcal{R}_{\mu,L}$ is close to $\mathrm{vol}(\mathcal{R}_{\mu,L})$. So its projection to N has volume $> \mathrm{vol}(N) - \alpha$.

Now, let Υ be the continuation of $\Gamma_{F_\mu,L}$. So it is robustly topologically mixing and contains the continuation of $\mathcal{R}_{\mu,L} \subset \Gamma_{F_\mu,L}$. Therefore, the projection of Υ on N has volume $> \mathrm{vol}(N) - \alpha$. This completes the proof of Theorem D. □

5.5. Corollary E: Existence of ergodic measures

Proof of Corollary E. – In the proof of Theorem 4.1 we prove that the large transitive set Γ_G is in fact contained in the homoclinic class of some hyperbolic periodic point. In [1, Theorem 3.1] it is proved that any homoclinic class coincides with the support of some ergodic measure (with zero entropy). Consequently, the set Γ_G is contained in the support of some ergodic measure. □

It seems interesting that the support of ergodic measures varies (lower-semi) continuously in Hausdorff topology as the diffeomorphisms vary in C^r topology. This phenomenon suggests a notion of "stable ergodicity", weaker that the usual one (see also Section 6.3).

5.6. Dichotomy: wandering and instability versus recurrence and transitivity

In this section we are discussing the following dichotomy: *Suppose that the assumption (b) in Theorem 4.1 fails. Then, either there exist large robustly transitive sets as in Theorem 4.1 or there exist wandering orbits converging to infinity.*

Before proving it, we need to state some lemmas.

LEMMA 5.2. – *There is a residual subset $\mathcal{R}$ of* $\mathrm{int}(\Omega(f))$ *such that any point in $\mathcal{R}$ is a (positively and negatively) recurrent point.*

Proof. – Let $\mathfrak{B} = \{U_i : i \in \mathbb{N}\}$ be a countable topological base in $\operatorname{int}(\Omega(f))$. For every $i \in \mathbb{N}$, there is $n_i \in \mathbb{N}$ such that $f^{n_i}(U_i) \cap U_i = \varnothing$. Let $x_i \in V_i := f^{-n_i}(U_i) \cap U_i$. Since $\mathfrak{B}_k = \{U_i : i \geq k\}$ is also a topological base, the set $\{x_i\}_{r=k}^{\infty}$ is dense in $\operatorname{int}(\Omega(f))$. So $\bigcup_{i=k}^{\infty} V_i$ is an open and dense subset of $\operatorname{int}(\Omega(f))$. Then, $\mathscr{R}^+ := \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} V_i$ is residual. We claim that $\mathscr{R}^+ \subset \operatorname{Rec}^+(f)$. Since $\mathfrak{B}$ is a topological base, for any $\epsilon > 0$ there is a k_ϵ such that, if $i > k_\epsilon$ then $\operatorname{diam}(U_i) < \epsilon$. Now, for any $x \in \mathscr{R}^+$ and for $i > k_\epsilon$, $x \in V_i$. So there is $n_i \in \mathbb{N}$ such that, $d(f^{n_i}(x), x) < \operatorname{diam}(U_i) < \epsilon$. Since $\epsilon > 0$ was arbitrary, this implies that x is a positively recurrent point. We could do it for f^{-1} to obtain a residual subset $\mathscr{R}^-$ of negatively recurrent points. Any point in the residual set $\mathscr{R} = \mathscr{R}^- \cap \mathscr{R}^+$ is positively and negatively recurrent. □

REMARK 5.3. – It follows from this lemma that if the non-wandering set of a diffeomorphism has (large) non-empty interior, as in the case of Theorem 2.13, there is an iterated function system of its nearby systems exhibiting transitivity in the interior of the non-wandering set.

We say that a point x *converges to infinity* if for any compact set U there is a number n_0 such that for any $n > n_0$, $f^n(x) \notin U$.

The following lemma is a corollary of a variation of Poincaré Recurrence Theorem for unbounded measures (due to Hopf, cf. [26]) which yields that for conservative homeomorphisms on manifolds with unbounded measure, almost all points either are recurrent or converge to infinity.

LEMMA 5.4. – *Let f be a conservative homeomorphism on a non-compact manifold with unbounded volume. Then almost all points in $\Omega(f)^{\complement}$ converge to infinity, in the future and also past iterations.*

Proof of the dichotomy. – It follows from Lemma 5.4 that almost all points in $N \setminus \Omega(f_2)$ converge to infinity (for future and past iterations). Assume that $\Omega(f_2) = N$, but for some $\tilde{f}_2$ close to f_2, $\Omega(\tilde{f}_2) \subsetneq N$.

We show that the same statement about transitivity and topologically mixing as the one in Theorem 4.1 (or Theorem A) holds in the interior of $\Omega(G)$ for any G close to the constructed family F_μ. Indeed, we use the hypothesis $\Omega(\tilde{f}_2) = N$ only in the last step of the proof of Theorem 4.1, where we apply Lemma 3.21 which requires recurrence.

So we may repeat the proof of Theorem 4.1 to construct the family F_μ. Let us fix $L > 0$ large enough and define $\Gamma_{F_\mu,L}$ as in Section 4.2.4. For a G close to F_μ, we may apply Theorem 4.5, and so Γ_G, the continuation of $\Lambda \times N$, is well defined. As in Section 4.2.4, let $\Gamma_{G,L}$ be the set defined by the set of trajectories with strong stable and strong unstable manifolds of diameter L connected to the blender. By the continuity of the strong stable and unstable foliations we see that $\Gamma_{G,L}$ is close to $\Gamma_{F_\mu,L}$ in the Hausdorff topology.

Let $\tilde{\Gamma}_L$ be the interior of $\Gamma_{G,L} \cap \Omega(G)$ in Γ_G. We show that $\tilde{\Gamma}_L$ is (robustly) topologically mixing.

If $\tilde{\Gamma}_L = \Gamma_{G,L}$, then we may apply Lemma 5.2 for G restricted to the continuation of $\{\hat{p}\} \times N$, and the proof of Theorem 4.1 without any change can be repeated, obtaining therefore the transitivity (and topological mixing) of Γ_G.

Otherwise, assume that $\tilde{\Gamma}_L \subsetneq \Gamma_{G,L}$. Now, recall that the periodic center fibers are dense in Γ_G. For any periodic fiber $\tilde{N}_x$ of period j in Γ_G, almost all points in the interior of $\Omega(G^j|_{\tilde{N}_x})$ in $\tilde{N}_x$ are recurrent. So $\Gamma_{G,L} \bigcap \mathrm{int}(\Omega(G^j|_{\tilde{N}_x}))$ is contained in the homoclinic class of the periodic point P_G. This implies that $\tilde{\Gamma}_L$ is contained in the homoclinic class of the periodic point P_G and so it is topologically mixing. This means in particular that topologically mixing property holds in the interior of non-wandering set of G.

In fact, we have proved a stronger statement: it holds in the interior of $\Omega(G)$ in Γ_G. On the other hand, almost all points in the complement of these sets converge to infinity. This completes the proof of the dichotomy. □

6. Some remarks and open problems

Several natural questions arise from the main results of this paper. Here we just mention a few of them.

6.1. Are hypotheses of Theorem A optimal?

As we mentioned after the statement of Theorem A we wonder if its hypotheses are optimal. These speculations are based upon several results proved in the C^1 topology:

1. It has been shown that any robustly transitive invariant sets of symplectic diffeomorphisms on compact manifolds are partially hyperbolic [9, 16, 20, 35]. This means that, the hypothesis (b) is not avoidable to obtain robustness of transitivity (see also Section 6.2).
2. It has been conjectured that any symplectic diffeomorphism is C^r approximated either by an Anosov or by a diffeomorphism with a quasi-elliptic periodic point. The case $r = 1$ has been proved by Newhouse [31]. On the other hand, from Theorem 5.1 it follows that any diffeomorphism having a quasi-elliptic periodic point is C^r approximated by one exhibiting a transversal homoclinic point. Consequently, a C^1 generic diffeomorphism f_1 satisfies the hypothesis (a). Moreover, if $\dim(N) = 2$, a generic diffeomorphism f_2 either satisfies the hypothesis (c) or is an Anosov system and so the conclusions of Theorem A follow. This means that, up to the mentioned conjecture, for generic systems the only essential assumption in Theorem A is the hypothesis (b).

6.2. Transitivity and partial hyperbolicity

The first question concerns the genericity of robustly mixing partially hyperbolic sets. Theorem A suggests that the answer of the following problem would be positive.

PROBLEM 6.1. – *Does there exist a residual set* $\boldsymbol{R} \subset \mathrm{Diff}^r_\omega(M)$, $1 \leq r \leq \infty$, *such that if* $f \in \boldsymbol{R}$, *then any normally hyperbolic invariant submanifold* N *for* f *with transversal intersection between its stable and unstable manifolds is topologically mixing, provided that* $N \subset \mathrm{int}(\Omega(f))$*?*

As in the case of C^1 topology (see [16], [9] and [20]), we believe that the partial hyperbolicity condition is *necessary* for C^r robustness of mixing.

PROBLEM 6.2. – *Let* (M, ω) *be a symplectic manifold. Suppose that* Γ *is a robustly topologically mixing invariant set for* f *in* $\mathrm{Diff}^r_\omega(M)$. *Is it a partially hyperbolic set?*

The previous problem is also related to the C^r stability conjecture which is still open for $r \geq 2$.

6.3. Ergodicity and stable ergodicity

In the context of Theorem A, it follows that for any symplectic diffeomorphism G close to $f_1 \times f_2$, there is a natural invariant measure ρ_G supported on the continuation of $\Lambda \times N$, which has the form of skew product of the volume on the central fibers over the Bernoulli measure on shift. So, one may ask about the ergodicity of this measure:

PROBLEM 6.3. – *Is it possible to approximate the product $f_1 \times f_2$ of Theorem A in the C^∞ topology by a symplectic diffeomorphism G for which the invariant measure ρ_G supported on the continuation of $\Lambda \times N$ is ergodic or stably ergodic?*

In most works about stable ergodicity, a main step is proving (stable) accessibility (cf. [32], [11]). Accessibility, roughly speaking, means that any pair of points of a partially hyperbolic set are joined by a piecewise smooth arc that locally lies in the stable or unstable manifolds of its points. The "global" partial hyperbolicity is a fundamental assumption in all results involving "stable" accessibility. So, Problem 6.3 asks for new methods. More generally, it highlights the need of a theory for non-global stable ergodicity.

6.4. Iterated function systems

Motivated by results in Section 2.2, it is possible to formulate a series of problems for generic IFS.

PROBLEM 6.4. – *Is the iterated function system of any C^r-generic pair of symplectic diffeomorphisms robustly transitive (or topologically mixing, robustly topologically mixing, minimal)?*

Some progress on this problem has been done very recently in [24] for the case of surface diffeomorphisms.

PROBLEM 6.5. – *Is the iterated function system of any C^r-generic pair of conservative or symplectic diffeomorphisms ergodic (or stably ergodic, mixing, etc.)?*

Results in this direction would be very helpful to study the dynamics of certain partially hyperbolic sets.

6.5. Other contexts

We believe that the methods developed here could be applied in other contexts, somehow with mechanical meanings. We list a few ones.

Analytic symplectic and Hamiltonian systems. The perturbations involved in the present paper are typically C^r for any $r \geq 1$. However, we believe that some of them could be performed in the analytic context, specially the ones involving the creation of blenders.

The dynamics near the (quasi) elliptic periodic points in dimensions ≥ 4. The dynamic near elliptic periodic points could display transversal homoclinic intersections associated to a hyperbolic periodic point (with a hyperbolic part dominating a $\delta-$weaker hyperbolic one). This situation is similar to the one in Theorem A.

Perturbations of time independent Hamiltonian systems by periodic potentials. Observe that in Theorem C, the second Hamiltonian is time independent and it is perturbed by a time periodic one. It is natural to wonder if the same approach can be performed for the case that the perturbation is given by a time periodic potential.

Generic energy levels of time independent Hamiltonian systems. Observe that some of the Hamiltonians involved in Theorem C are time periodic. The goal, therefore, would be to extend some of the present results to the time independent context.

Specific mechanical problems. Of course, it would be essential to try to apply the present theorems to a concrete mechanical problem. We wonder if improvement of our results can be useful in these mechanical models with the goal to obtain robust transitivity. Of course, a major challenge would be to apply the present approach to the context of the restricted 3-body problem.

Perturbations of geodesic flows on surfaces by periodic potentials. It is natural to try to extend the results about robust transitivity to the case of a metric perturbed by a time periodic potential.

Geodesic flows on manifolds of dimensions larger than two. A natural context for this problem is to consider warped product (close to a direct product) between a metric of negative curvature with the spherical metric. Locally, the new metric resembles the product of a hyperbolic system with an integrable one.

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