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## *Relaxation of the Incompressible Porous Media Equation*

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# RELAXATION OF THE INCOMPRESSIBLE POROUS MEDIA EQUATION

BY László SZÉKELYHIDI Jr.

ABSTRACT. – It was shown recently by Córdoba, Faraco and Gancedo in [1] that the 2D porous media equation admits weak solutions with compact support in time. The proof, based on the convex integration framework developed for the incompressible Euler equations in [4], uses ideas from the theory of laminates, in particular $T4$ configurations. In this note we calculate the explicit relaxation of IPM, thus avoiding $T4$ configurations. We then use this to construct weak solutions to the unstable interface problem (the Muskat problem), as a byproduct shedding new light on the gradient flow approach introduced by Otto in [12].

RÉSUMÉ. – Il a récemment été démontré par Córdoba, Faraco et Gancedo dans [1], que l'équation des milieux poreux en dimension 2 admet des solutions faibles avec support compact dans le temps. La démonstration, qui fait appel à la méthode par intégration convexe telle qu'elle a été développée dans [4], dans le contexte des équations d'Euler incompressibles, utilise certaines idées provenant de la théorie des « laminates », et en particulier les configurations dites $T4$. Dans cette note, nous calculons explicitement la relaxation du « IPM », évitant ainsi les configurations $T4$. Ceci nous permet ensuite de construire des solutions faibles au problème des interfaces instables (problème de Muskat) et a pour autre conséquence de clarifier l'approche par flot de gradient, introduite par Otto dans [12].

## 1. Introduction

We consider the incompressible porous media equation (IPM) in a 2-dimensional bounded domain $\Omega \subset \mathbb{R}^2$. The flow is described in Eulerian coordinates by a velocity

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field $v(x,t)$ and a pressure $p(x,t)$ obeying the conservation of mass and the conservation of momentum in the form of Darcy's law:

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \tag{IPM1}$$
$$\operatorname{div} v = 0, \tag{IPM2}$$
$$v + \nabla p = -(0, \rho). \tag{IPM3}$$

Here we chose $x_1$ as the horizontal and $x_2$ as the vertical direction, with the gravitational constant normalized to be 1. Equation (IPM2) amounts to the flow being incompressible, and this is coupled with the assumption that there is no flux across the boundary $\partial\Omega$, i.e., $v \cdot \nu = 0$ on $\partial\Omega$.

The system (IPM1)-(IPM3) can be used to model the flow of two immiscible fluids of different densities in a porous medium, or, equivalently, in a Hele-Shaw cell [15]. If initially the two fluids form a horizontal interface, with the heavier fluid on top, it is well known that the initial value problem, known as the Muskat problem, is ill-posed in classical function spaces [19, 17, 2]. Although some explicit solutions are known [7], there is no general existence theory, neither for the evolution problem for the interface, nor for weak solutions of IPM.

After a normalization we may assume that the density $\rho(x,t)$, indicating whether the pores at time $t$ near location $x \in \Omega$ are filled with the lighter or the heavier fluid, takes the values $\pm 1$. Hence, for the Muskat problem the Equations (IPM1)-(IPM3) should be complemented by

$$|\rho(x,t)| = 1 \quad \text{for a.e.}(x,t) \in \Omega \times (0,T). \tag{IPM4}$$

We remark in passing, that formally (IPM4) follows from (IPM1) if $|\rho(x,0)| = 1$ a.e., since the density $\rho$ is simply transported by the flow. However, for weak solutions this transport property need not hold, as shown for instance by Theorem 1.2 below.

As usual, a weak solution to the system (IPM1)-(IPM3) with initial data $\rho_0 \in L^\infty(\Omega)$ is defined as a pair $(\rho, v)$ with

$$\rho \in L^\infty(\Omega \times (0,T)), \quad v \in L^\infty(0,T; L^2(\Omega)),$$

such that for all $\phi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R})$ we have

$$\int_0^T \int_\Omega \rho(\partial_t \phi + v \cdot \nabla \phi)\, dxdt + \int_\Omega \rho_0(x)\phi(x,0)\, dx = 0, \tag{1}$$

$$\int_0^T \int_\Omega v \cdot \nabla \phi \, dxdt = 0, \tag{2}$$

and for all $\psi \in C_c^\infty(\Omega)$

$$\int_\Omega (v + (0,\rho)) \cdot \nabla^\perp \psi \, dx = 0. \tag{3}$$

We remark explicitly that (2) includes the no-flux boundary condition for $v$ whereas in (3) the pressure $p$ has been eliminated (observe also that there is no boundary condition on $p$).

Our main result can be stated as follows

THEOREM 1.1. – *Let $\Omega \subset \mathbb{R}^2$ be the unit square, and*

$$\rho_0(x) = \begin{cases} +1 & x_2 > 0, \\ -1 & x_2 < 0. \end{cases}$$

*For any $T > 0$ there exist infinitely many weak solutions $\rho \in L^\infty(\Omega \times (0,T))$ of* (IPM1)-(IPM4) *with initial data $\rho_0$.*

Recently, D. Córdoba, D. Faraco and F. Gancedo showed in [1], that on the 2-dimensional torus $\mathbb{T}^2$ the system (IPM1)-(IPM3) admits nontrivial weak solutions with compact support in time. More precisely

THEOREM 1.2 (Theorem 5.2, [1]). – *There exist infinitely many weak solutions to* (IPM1)-(IPM3) *with $(\rho, v) \in L^\infty(\mathbb{T}^2 \times \mathbb{R})$ such that*

$$|\rho(x,t)| = \begin{cases} 1 & \text{a.e. } (x,t) \in \mathbb{T}^2 \times (0,T), \\ 0 & \text{for } t < 0 \text{ or } t > T. \end{cases}$$

There is a subtle but quite important difference between the solutions in Theorem 1.1 and Theorem 1.2. In the latter the initial data (in the sense of Equation (1)) is $\rho_0 = 0$, so that although (IPM4) holds for $t > 0$, it is not satisfied by $\rho_0$. An interpretation of this is that the fluid is in an infinitely mixed state at time $t = 0$ (cf. discussion in Section 4). In contrast, for Theorem 1.1 the fluid is not mixed at the initial time. As a consequence the solutions are forced to have finite mixing speed, an effect that cannot be seen in the solutions from Theorem 1.2. More precisely, the solutions in Theorem 1.1 all satisfy

$$\rho(x,t) = \begin{cases} +1 & x_2 > 2t, \\ -1 & x_2 < -2t. \end{cases} \tag{4}$$

Moreover, the solutions in Theorem 1.1 are in good agreement and show interesting connections to predictions concerning the coarse-grained density and the growth of the mixing zone made in [12, 13]. In [12] F. Otto introduced a relaxation approach for (IPM1)-(IPM4), in particular for the Muskat problem, based on a gradient flow formulation of IPM and using ideas from mass transport. It was shown that under certain assumptions there exists a *unique* "relaxed" solution $\overline{\rho}$, representing a kind of coarse-grained density. Moreover, Otto showed in [13, Remark 2.1] that, in general, the mixing zone (where the coarse-grained density $\overline{\rho}$ is strictly between $\pm 1$) grows linearly in time as in (4), with the possible exception of a small set of volume fraction $O(t^{-1/2})$.

The proof of both Theorem 1.2 in [1] and Theorem 1.1 is based on the framework developed in [4] for the incompressible Euler equations, although there are several places where the authors in [1] had to modify the arguments. In technical terms, one of the crucial points in the general scheme of convex integration is to show that the relaxation with respect to the wave cone of a suitably defined constitutive set, the $\Lambda$-convex hull, contains the zero state in its interior. In [1] it was observed, that due to a lack of symmetry induced by the direction of gravity, this condition seems to fail for IPM; instead, a systematic method for obtaining a suitably modified constitutive set was introduced, based on so-called degenerate $T4$ configurations. The advantage of the method used in [1] is that it is rather robust, and can be used in situations where an explicit calculation of $K^\Lambda$ is out of reach due to the high complexity (see also [9, 11]). Indeed, the same technique has been recently applied successfully to a large class of active scalar equations by R. Shvydkoy [16].

On the other hand, there are certain advantages to obtaining an explicit formula for the $\Lambda$-convex hull $K^\Lambda$ rather than just showing that a fixed state is in the interior. The explicit formula allows one to identify "compatible boundary and initial conditions", for which the construction works. For the incompressible Euler equations, such initial conditions were called "wild initial data" in [5]. For IPM the explicit formula for the $\Lambda$-convex hull is necessary for studying the Muskat problem and leads to a concept of subsolution, analogously to Euler subsolutions in [4, 5]. In Section 4 we show that the relaxed solution $\overline{\rho}$ from [12] is very closely related to the concept of subsolution and in particular we construct weak solutions $\rho_k$ such that $\rho_k \overset{*}{\rightharpoonup} \overline{\rho}$. The interpretation is that $\overline{\rho}$ is the coarse-grained density obtained from $\rho_k$. It is interesting to note in this connection, that, although weak solutions are clearly not unique, there is a way to identify a selection criterion among subsolutions which leads to uniqueness.

The paper is organized as follows. In Section 2 we recall from [1] how to reformulate (IPM1)-(IPM3) as a differential inclusion, and then we calculate explicitly the relaxation, more precisely the $\Lambda$-convex hull of the constitutive set. These computations form the main contribution of this paper. If one is only interested in weak solutions as defined in this introduction (where $v$ can be *unbounded*), the "simpler" computations in Propositions 2.3 and 3.1 suffice. However, for completeness we include the computations that are required for *bounded* velocity $v$ in Propositions 2.4 and 3.3.

Then, in Section 3 we show how the explicit form of the $\Lambda$-convex hull can be used in conjunction with the Baire category method to obtain weak solutions. For the convenience of the reader we include the details of the Baire category method in the appendix.

Finally, in Section 4 we use the framework to construct weak solutions to the unstable interface problem. In this section, Theorem 1.1 is restated and proved as Theorem 4.2. Moreover, we show in Proposition 4.3 that if the coarse-grained density is independent of the horizontal direction, the linear growth estimate of [13] is sharp, in the sense that there is no exceptional set. As a consequence, we can interpret the uniqueness result of Otto as selecting the subsolution with "maximal mixing". In this light it is of interest to note that the analogous criterion for the incompressible Euler equations would be "maximally dissipating" [3, 6, 5].

## 2. The relaxation of IPM

We start by setting

$$u := 2v + (0, \rho). \tag{5}$$

Then (IPM1)-(IPM3) can be rewritten as

$$\begin{aligned} \partial_t \rho + \operatorname{div} m &= 0, \\ \operatorname{div}(u - (0,\rho)) &= 0, \\ \operatorname{curl}(u + (0,\rho)) &= 0, \end{aligned} \tag{6}$$

coupled with

$$\begin{aligned} m &= \tfrac{1}{2}(\rho u - (0,1)), \\ |\rho| &= 1 \end{aligned} \tag{7}$$

for almost every $(x,t)$. As in [1], we interpret (6)-(7) as a differential inclusion: the *state variable*

$$(\rho, u, m) : \Omega \times (0,T) \to \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$$

is subject to a linear system of conservation laws (6), and should take values in a *constitutive set* determined by (7).

The reason for introducing $u$ is purely cosmetic, it makes (6) symmetric. It also helps in simplifying some of the calculations below. Observe that (6) can be easily written as a genuine differential inclusion. Namely, we see that

$$\begin{pmatrix} \rho - u_2 & u_1 \\ u_1 & \rho + u_2 \end{pmatrix}$$

is curl-free, hence (locally) equal to $\nabla^2\phi$ for some function $\phi$. Then $\rho = \frac{1}{2}\Delta\phi$ and $\operatorname{div}\left(\frac{1}{2}\partial_t\nabla\phi + m\right) = 0$, hence $m = -\frac{1}{2}\partial_t\nabla\phi + \nabla^\perp\psi$ for some $\psi$. In particular we deduce that *for any* $(x,t)$*-periodic solution*

$$(\rho, u, m) : \mathbb{T}^3 \to \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$$

*of* (6)*, there exist periodic functions*

$$\phi, \psi : \mathbb{T}^3 \to \mathbb{R}$$

*such that*

$$\begin{aligned} \rho &= \tfrac{1}{2}\Delta\phi, \\ u &= \tfrac{1}{2}\left(2\partial_{12}\phi, \partial_{22}\phi - \partial_{11}\phi\right), \\ m &= -\tfrac{1}{2}\partial_t\nabla\phi + \nabla^\perp\psi. \end{aligned} \tag{8}$$

The periodic solutions of (6) are characterized by the *wave cone* $\Lambda$. This is easy to calculate, since it is clearly independent of $m$ and the other two equations have a div-curl structure. Hence

$$\Lambda = \{(\rho, u, m) : \; |\rho|^2 = |u|^2\}.$$

We conclude the existence of localized plane-waves, an observation which has already been made in [1]:

LEMMA 2.1. – *Let* $\bar{z} = (\bar{\rho}, \bar{u}, \bar{m}) \in \Lambda$. *There exists a sequence*

$$w_j = (\rho_j, u_j, m_j) \in C_c^\infty\left(B_1(0) \times (-1,1); \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2\right)$$

*solving* (6) *such that*

1. $\operatorname{dist}(w_j, [-\bar{z}, \bar{z}]) \to 0$ *uniformly,*
2. $w_j \rightharpoonup 0$ *weakly in* $L^2$*,*
3. $\iint |w_j|^2 \, dxdt \geq C|\bar{z}|^2$.

Next, we calculate the $\Lambda$-convex hull of the set in state-space defined by (7). Recall (e.g. from [9], see also [14, 8]) that for a closed set $K$ the $\Lambda$-convex hull is defined to be the largest closed set $K^\Lambda$ which cannot be separated from $K$: a state $z$ does not belong to $K^\Lambda$ if there exists a function $f$ which is $\Lambda$-convex in the sense that

$$s \mapsto f(w_0 + sw) \text{ is convex for all } w \in \Lambda$$

such that $f \leq 0$ on $K$ and $f(z) > 0$. It follows immediately from this definition that

$$z_1, z_2 \in K \text{ with } z_1 - z_2 \in \Lambda \implies [z_1, z_2] \subset K^\Lambda.$$

For further properties of $\Lambda$-convex hulls and functions, see [8].

Now, let

$$K = \{(\rho, u, m) : |\rho| = 1,\, m = \tfrac{1}{2}\rho u\}.$$

Observe that (7) is equivalent to

$$\big(\rho, u, m + \tfrac{1}{2}(0,1)\big) \in K \quad \text{a.e.} \tag{9}$$

We note also that in the absence of spatial boundaries, any solution of (6) with $(\rho, u, m) \in K$ for a.e. $(x, t)$ is also a solution of (6)-(7). In particular, for solutions on the torus we can ignore the additional constant vector $\frac{1}{2}(0, 1)$.

LEMMA 2.2. – *The function*

$$f(\rho, u, m) := \big|m - \frac{1}{2}\rho u\big| + \frac{1}{4}(\rho^2 + |u|^2) \tag{10}$$

*is convex.*

*Proof.* – A short calculation and the triangle inequality shows that, for any $(\rho, u, m)$ and $(\bar\rho, \bar u, \bar m)$

$$f(\rho + \bar\rho t, u + t\bar u, m + t\bar m) \geq f(\rho, u, m) + ct + \frac{t^2}{4}(\bar\rho^2 + |\bar u|^2 - 2|\bar\rho \bar u|),$$

where

$$c = \frac{1}{2}(\rho\bar\rho + u \cdot \bar u) - |\bar m - \tfrac{1}{2}(\bar\rho u + \rho \bar u)|.$$

The convexity of

$$t \mapsto f(\rho + \bar\rho t, u + t\bar u, m + t\bar m),$$

and hence of $f : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ follows. □

PROPOSITION 2.3. – *We have*

$$K^\Lambda = \Big\{(\rho, u, m) : |\rho| \leq 1,\ \big|m - \tfrac{1}{2}\rho u\big| \leq \tfrac{1}{2}(1 - \rho^2)\Big\}.$$

*Proof.* – Let $|\rho_0| < 1$ and $u_0 \in \mathbb{R}^2$. Fix $e \in \mathbb{R}^2$ with $|e| = 1$ and define

$$u_1 = u_0 + (1-\rho_0)e \qquad \rho_1 = 1 \tag{11}$$

$$u_2 = u_0 - (1+\rho_0)e \qquad \rho_2 = -1. \tag{12}$$

Then it is easy to see that for any $m_1, m_2$

$$(\rho_1 - \rho_2, u_1 - u_2, m_1 - m_2) \in \Lambda,$$

hence in particular by setting $m_i = \frac{1}{2}\rho_i u_i$ we obtain that, with

$$\begin{aligned} m_0 :&= \tfrac{1}{2}(1+\rho_0)m_1 + \tfrac{1}{2}(1-\rho_0)m_2 \\ &= \tfrac{1}{2}\rho_0 u_0 + \tfrac{1}{2}(1-\rho_0^2)e, \end{aligned}$$

the state $(\rho_0, u_0, m_0)$ is contained in $K^\Lambda$. Thus any $(\rho, u, m)$ with

$$|\rho| \leq 1, \quad \left|m - \tfrac{1}{2}\rho u\right| = \tfrac{1}{2}(1-\rho^2)$$

is contained in $K^\Lambda$. Moreover, since for any $\bar{m} \in \mathbb{R}^2$ we have $(0, 0, \bar{m}) \in \Lambda$, we deduce that in fact any $(\rho, u, m)$ with

$$|\rho| \leq 1, \quad \left|m - \tfrac{1}{2}\rho u\right| \leq \tfrac{1}{2}(1-\rho^2)$$

is contained in $K^\Lambda$.

To see that this set is in fact the whole hull, we note that

$$K \subset \{z : g(z) \leq 1/2\},$$

where

$$g(\rho, u, m) := f(\rho, u, m) + \frac{1}{4}(\rho^2 - |u|^2) \tag{13}$$

and $f$ is the convex function in (10). On the other hand observe that $\rho^2 - |u|^2$ is $\Lambda$-convex (in fact $\Lambda$-affine). Therefore $g$ is $\Lambda$-convex, and therefore necessarily $K^\Lambda \subset \{z : g(z) \leq 1/2\}$. This completes the proof. □

*Remark 1.* – *Recalling* (9)*, we see that if* $\mathcal{K}$ *denotes the constitutive set given by* (7)*, then* $(0, 0, 0) \in \partial\mathcal{K}^\Lambda$. *This was the observation made in* [1, Remark 4.1].

Although working with the set $K$ would suffice to construct weak solutions to (IPM1)-(IPM4), there is a slight drawback in that $K$ is not bounded. More specifically, the solutions constructed will have $v \in L^2$ but not in $L^\infty$. For the details see the next section. To remedy this problem we next consider a compact subset of $K$. Here the calculation of the $\Lambda$-convex hull requires a bit more work.

Fix $\gamma > 1$ and let

$$K_\gamma = \{(\rho, u, m) : |\rho| = 1,\ m = \tfrac{1}{2}\rho u,\ |u| \leq \gamma\}. \tag{14}$$

PROPOSITION 2.4. – *For the set $K_\gamma$ above with $\gamma > 1$, $(K_\gamma)^\Lambda$ is given by the set of inequalities*

$$|\rho| \le 1, \tag{15}$$

$$|u|^2 \le \gamma^2 - (1-\rho^2), \tag{16}$$

$$|m - \tfrac{1}{2}\rho u| \le \tfrac{1}{2}(1-\rho^2), \tag{17}$$

$$|m - \tfrac{1}{2}u| \le \tfrac{\gamma}{2}(1-\rho), \tag{18}$$

$$|m + \tfrac{1}{2}u| \le \tfrac{\gamma}{2}(1+\rho). \tag{19}$$

*Proof*. – *Step 1.* — Observe that (15), (18) and (19) define convex sets whereas (16) and (17) define sublevel-sets of $\Lambda$-convex functions. Since states in $K_\gamma$ satisfy all 5 inequalities, it follows that $(K_\gamma)^\Lambda$ is certainly contained in the set defined by (15)-(19).

Conversely, let $U_\gamma$ be the set of all states $(\rho, u, m)$ with all inequalities (15)-(19) strict. We need to show that $\overline{U_\gamma} \subset (K_\gamma)^\Lambda$. To this end it suffices to show that

$$\begin{gathered}\text{for all } z \in \partial U_\gamma \setminus K_\gamma, \text{ there exists } \bar z \in \Lambda \setminus \{0\} \text{ such that}\\ z \pm \bar z \in \overline{U_\gamma}.\end{gathered} \tag{20}$$

Indeed, from this it follows that extr $\overline{U_\gamma} \subset K_\gamma$, so that $\overline{U_\gamma} \subset (K_\gamma)^\Lambda$ (cf. the Krein-Milman type theorem in the context of $\Lambda$-convexity [8, Lemma 4.16]). Thus, let $z = (\rho, u, m) \in \partial U_\gamma \setminus K_\gamma$.

*Step 2.* — If (17) is an equality, i.e.,

$$m = \frac{1}{2}\rho u + \frac{1}{2}(1-\rho^2)e$$

for some $e \in \mathcal{S}^1$, then $z$ lies on a $\Lambda$-segment connecting $z_1, z_2 \in K_\gamma$. More precisely, let

$$\bar z := \big(1, e, \tfrac{1}{2}(u - \rho e)\big). \tag{21}$$

Then $\bar z \in \Lambda$ and

$$z_1 := z + (1-\rho)\bar z, \quad z_2 := z - (1+\rho)\bar z$$

satisfy $z_1, z_2 \in K_\gamma$ and $z = \frac{1}{2}(1+\rho)z_1 + \frac{1}{2}(1-\rho)z_2$, just like in Proposition 2.3.

*Step 3.* — Assume now that (17) is a strict inequality, i.e.,

$$|m - \tfrac{1}{2}\rho u|^2 < \tfrac{1}{2}(1-\rho^2). \tag{22}$$

By elementary computations we verify the following identity:

$$\begin{aligned}\left[\frac{(1-\rho^2)^2}{4} - |m - \frac{1}{2}\rho u|^2\right] + \frac{(1-\rho^2)}{4}\left[\gamma^2 - |u|^2 - (1-\rho^2)\right] = \\ = \frac{1+\rho}{2}\left[\frac{\gamma^2}{4}(1-\rho)^2 - |m - \frac{1}{2}u|^2\right] + \frac{1-\rho}{2}\left[\frac{\gamma^2}{4}(1+\rho)^2 - |m + \frac{1}{2}u|^2\right].\end{aligned} \tag{23}$$

Observe that the terms in square brackets are all non-negative in $\overline{U_\gamma}$, and since we assumed (22), the left hand side is strictly positive. Therefore, by symmetry we may assume without loss of generality that also (19) is a strict inequality, i.e.,

$$|m + \tfrac{1}{2}u| < \tfrac{\gamma}{2}(1+\rho). \tag{24}$$

Next, we claim that there exists $\bar e \in S^1$ such that, with

$$\rho_t = \rho + t, \; u_t = u + t\bar e$$

the inequality (16) will remain valid for $|t| < \frac{1}{4}(\gamma^2 - 1)$. Indeed,

$$|u_t|^2 - \rho_t^2 = |u|^2 - \rho^2 + 2t(u \cdot \bar{e} - \rho).$$

Therefore, if $|u| < |\rho|$ this is true for any $\bar{e} \in S^1$, whereas if $|u| \geq |\rho|$, then choose $\bar{e} \in S^1$ so that $u \cdot \bar{e} = \rho$.

With this choice of $\bar{e}$ let

$$\bar{z} = \left(1, \bar{e}, \tfrac{1}{2}\bar{e} - \tfrac{1}{1-\rho}(m - \tfrac{1}{2}u)\right) \in \Lambda, \tag{25}$$

and consider $z_t = (\rho_t, u_t, m_t) := z + t\bar{z}$. Then

$$|m_t - \tfrac{1}{2}u_t| = \tfrac{1-\rho_t}{1-\rho}|m - \tfrac{1}{2}u| \leq \tfrac{\gamma}{2}(1 - \rho_t),$$

so that (18) is satisfied by $z_t$. Since we assumed (22) and (24), the inequalities (17) and (19) will continue to hold for $z_t$ provided $|t|$ is small. Thus, with this choice of $\bar{z} \in \Lambda$ we can ensure that

$$z + t\bar{z} \in \overline{U_\gamma}$$

for all $t$ with $|t|$ sufficiently small. This implies (20) and thus the proof is completed. □

*Remark 2. – The condition $\gamma > 1$ is sharp. Indeed, if $\gamma < 1$, then it is not difficult to see (using standard technology on gradient Young measures) that approximate solutions to the corresponding inclusion are compact. In fact, if $\Omega = \mathbb{T}^2$, by just looking at* (IPM2)-(IPM3) *we deduce easily that any weak solution $(\rho, u, m)$ of* (6)-(7) *with $\|u\|_{L^\infty} < 1$ is constant (see for instance* [10, 18]*).*

*Remark 3. – The computations in Proposition 2.3 and 2.4 do not depend on the vectors $u, m$ in state-space to be 2-dimensional. Therefore, the same formulae continue to hold for the relaxation of the IPM equation in $n$ space dimensions with any $n \geq 2$.*

## 3. Construction of weak solutions

There are several ways of constructing weak solutions to differential inclusions, depending on the particular problems at hand. For the system (6)-(7) the relaxation is sufficiently large so that a relatively simple iteration procedure, involving single localized plane-waves, suffices. As always, the crucial ingredient is to show that there exists an open set $U$ where *states are stable only near* $K$ (cf. [8, Definition 3.16]). This is expressed by conditions (H1)-(H2) in the appendix. The methodology of how to pass from this property to weak solutions via the Baire category theorem is well known [8, 1, 4, 5], but for the convenience of the reader we present the details in Theorem 5.1 in the Appendix.

Recall the definition of $K$ and $K_\gamma$ from Section 2, and let

$$U = \text{int } K^\Lambda, \quad U_\gamma = \text{int } (K_\gamma)^\Lambda, \tag{26}$$

so that

$$U = \left\{(\rho, u, m) : \ |\rho| < 1, \ \left|m - \tfrac{1}{2}\rho u\right| < \tfrac{1}{2}(1 - \rho^2)\right\}, \tag{27}$$

and $U_\gamma$ is analogously given by the inequalities (15)-(19) all being strict.

Proposition 3.1. – *There exists a constant $c > 0$ with the following property. For any $(\rho, u, m) \in U$ there exists $(\bar\rho, \bar u, \bar m) \in \Lambda$ with $|\bar\rho|^2 + |\bar m|^2 = 1$ such that*

$$(\rho, u, m) + t(\bar\rho, \bar u, \bar m) \in U \quad \textit{for } |t| < c(1-\rho^2).$$

*Proof.* – From the explicit Formulas (11)-(12) it follows that for any $(\rho, u, m)$ with $|\rho| < 1$ and $|m - \frac{1}{2}\rho u| = \frac{1}{2}(1-\rho^2)$ there exists $(\bar\rho, \bar u, \bar m) \in \Lambda$ with $\bar\rho = 1$ such that

$$(\rho, u, m) + t(\bar\rho, \bar u, \bar m) \in K^\Lambda \quad \text{for } |t| < c_0(1-\rho^2)$$

for some $c_0 > 0$. From this we deduce by continuity the claim (with $0 < c < c_0$) in the case where

$$\tfrac{1}{4}(1-\rho^2) < |m - \tfrac{1}{2}\rho u| < \tfrac{1}{2}(1-\rho^2).$$

In the remaining case, where $|m - \frac{1}{2}\rho u| < \frac{1}{4}(1-\rho^2)$, we can take $(\bar\rho, \bar u, \bar m) = (0, 0, \bar m)$ with $\bar m$ parallel to $m - \frac{1}{2}\rho u$. □

As a first application we obtain the following variant of [1, Theorem 5.2]:

Theorem 3.2. – *There exist infinitely many periodic weak solutions to (IPM1)-(IPM3) with*

$$\rho \in L^\infty(\mathbb{T}^2 \times \mathbb{R}),\ v \in L^\infty(\mathbb{R}; L^2(\mathbb{T}^2)),$$

*such that*

$$|\rho(x,t)| = \begin{cases} 1 & \textit{a.e. } (x,t) \in \mathbb{T}^2 \times (0,T), \\ 0 & \textit{for } t < 0 \textit{ or } t > T. \end{cases}$$

*Proof.* – We construct solutions $(\rho, u, m)$ of (6) such that

$$(\rho, u, m) \in K \ \text{ a.e. } (x,t).$$

For any such solution $|\rho| = 1$ and $m = \frac{1}{2}\rho u$ almost everywhere by definition of $K$ and, therefore, on the torus $\mathbb{T}^2$ satisfies

$$\partial_t \rho + \tfrac{1}{2}\mathrm{div}\,(\rho u - (0,1)).$$

Therefore, recalling that $v = \frac{1}{2}(u - (0,\rho))$ we deduce that $(\rho, v)$ is a weak solution of (IPM1)-(IPM3).

Next, define the space of *subsolutions* as follows. Let

$$\mathscr{D} = \mathbb{T}^2 \times \mathbb{R},$$
$$\mathscr{U} = \mathbb{T}^2 \times (0,T),$$

and

$$X_0 = \Big\{ z = (\rho, u, m) \in C_c^\infty(\mathscr{U}) : \text{(6) holds and } z(x,t) \in U \quad \forall (x,t) \in \mathscr{U} \Big\}.$$

Note that $(0,0,0) \in U$, hence $X_0$ is nonempty. Any $(\rho, u, m) \in X_0$ satisfies $|\rho| \le 1$. Therefore, whenever $(\rho, u, m) \in X_0$, we have

$$\|\rho\|_{L^\infty_t L^2_x(\mathscr{D})} \le 1,$$

hence, using standard elliptic estimates and (6),

$$\|u\|_{L^\infty_t L^2_x(\mathscr{D})} \le C$$

for some constant $C$. Together with the definition of $U$ this implies that $X_0$ is a bounded subset of $L^2_{x,t}(\mathscr{D})$ and in particular $X_0$ satisfies condition (H3) in the appendix. Moreover, Lemma 2.1 and Proposition 3.1 imply that (H1)-(H2) in the appendix are satisfied, and consequently Theorem 3.2 follows from applying Theorem 5.1 to $X_0$. □

PROPOSITION 3.3. – *Let $\gamma > 1$. If $K_\gamma$ and $U_\gamma$ are as defined in (14) and (26), then for any $z \in U_\gamma$ with* $\operatorname{dist}(z, K_\gamma) \geq \alpha > 0$ *there exists $\bar{z} \in \Lambda$ with $|\bar{z}| = 1$ such that*

$$z + t\bar{z} \in U_\gamma \text{ whenever } |t| < \beta,$$

*where $\beta > 0$ depends only on $\alpha$ and $\gamma$.*

*Proof.* – Since $\partial U_\gamma$ is compact and locally the graph of a Lipschitz function, it suffices to prove the following quantitative version of (20):

$$\begin{aligned} &\text{for all } z_0 \in \partial U_\gamma \setminus K \text{ there exists } \varepsilon > 0, r > 0 \text{ so that} \\ &\text{for all } z \in B_\varepsilon(z_0) \cap \partial U_\gamma \text{ there exists } \bar{z} \in \Lambda \text{ with} \\ &|\bar{z}| = r \text{ and } z \pm \bar{z} \in \overline{U_\gamma}. \end{aligned} \tag{28}$$

In fact, since the wave-cone $\Lambda$ only restricts $\rho, u$ but not $m$, in the statement (28) it suffices to restrict to $z \in B_\varepsilon(z_0) \cap \partial U_\gamma$ with at least one of the inequalities (17)-(19) being an equality.

Let $z_0 = (\rho_0, u_0, m_0) \in \partial U_\gamma \setminus K_\gamma$. If $|m_0 - \frac{1}{2}\rho_0 u_0| < \frac{1}{2}(1-\rho_0^2)$, then using (23) as in Step 3 of the proof of Proposition 2.4 we may assume that $|m_0 + \frac{1}{2}u_0| < \frac{\gamma}{2}(1+\rho_0)$. But then the $\Lambda$-direction given in (25) works uniformly for a whole neighborhood of $(\rho_0, u_0, m_0)$.

Conversely, assume that $|m_0 - \frac{1}{2}\rho_0 u_0| = \frac{1}{2}(1-\rho_0^2)$, so that

$$m_0 = \tfrac{1}{2}\rho_0 u_0 + \tfrac{1}{2}(1-\rho_0^2)e_0 \text{ with } |e_0| = 1.$$

We know from Step 2 of the proof of Proposition 2.4 that there is a $\Lambda$-segment in $\partial U_\gamma$ of length $\min(1-\rho_0, 1+\rho_0)$ centered at $(\rho_0, u_0, m_0)$, but now we need to show that the length of the segment can be chosen uniformly large for a whole neighborhood. In other words we need to show that this $\Lambda$-segment is *stable*.

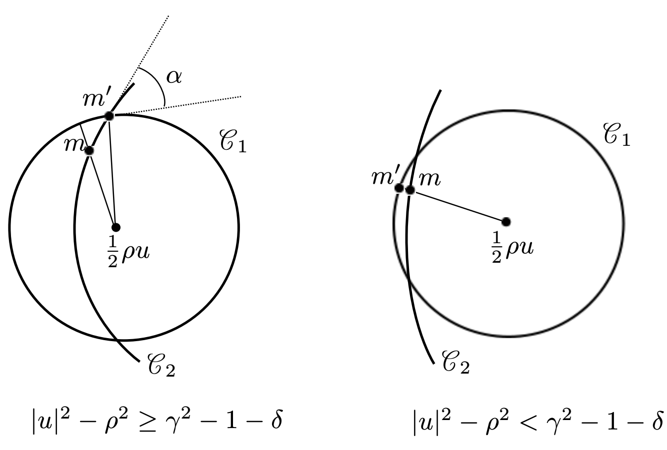


FIGURE 1. The choice of $m'$

To this end let $z = (\rho, u, m) \in \partial U_\gamma$ nearby and consider the three circles

$$\mathscr{C}_1 := \{m \in \mathbb{R}^2 : |m - \tfrac{1}{2}\rho u| = \tfrac{1}{2}(1-\rho^2)\},$$
$$\mathscr{C}_2 := \{m \in \mathbb{R}^2 : |m - \tfrac{1}{2}u| = \tfrac{\gamma}{2}(1-\rho)\},$$
$$\mathscr{C}_3 := \{m \in \mathbb{R}^2 : |m + \tfrac{1}{2}u| = \tfrac{\gamma}{2}(1+\rho)\}.$$

If $m$ lies on $\mathscr{C}_1$, then we are done as in Step 2 of the proof of Proposition 2.4 by choosing $\bar{z} \in \Lambda$ as in (21). Otherwise, we may assume without loss of generality that $m$ lies on $\mathscr{C}_2$. In this case the $\Lambda$-segment through $z$ necessarily has the form

$$\bar{z} = \Big(1, \bar{e}, \tfrac{1}{2}\bar{e} - \tfrac{1}{1-\rho}(m - \tfrac{1}{2}u)\Big)$$

for some $\bar{e} \in S^1$ still to be chosen, as in Step 3 of the proof of Proposition 2.4. Let $z_t = (\rho_t, u_t, m_t) := z + t\bar{z}$. We need to choose $\bar{e} \in S^1$ so that the inequalities (16)-(19) remain valid for an interval of $t$ whose size is independent of $|z - z_0|$. Observe, however, that (18) will remain an equality by our choice of $\bar{z}$, therefore, in light of the identity (23), (19) will remain valid as long as (16) and (17) are valid.

In order to choose $\bar{e}$ we distinguish two cases. Let

$$\delta := \tfrac{1}{2}(1-\rho^2)(\gamma - 1) > 0.$$

If $|u|^2 - \rho^2 > \gamma^2 - 1 - \delta$, let $m' \in \mathscr{C}_1 \cap \mathscr{C}_2$ be the point of intersection of the two circles closer to $m$ (since $m$ is assumed to be close to $m_0$, this determines $m'$ uniquely). See Figure 1. Then $m'$ can be written as

$$m' = \tfrac{1}{2}\rho u + \tfrac{1}{2}(1-\rho^2)\bar{e}, \tag{29}$$

and this gives our choice of $\bar{e} \in S^1$. To check that with this choice of $\bar{e}$ the inequalities (16) and (17) remain valid for $z_t = z + t\bar{z}$, note that (16) remains valid for all $t \in (-(1+\rho), 1-\rho)$ by the same argument as in Step 2 of Proposition 2.4. Concerning (17), a short calculation shows that

$$m_t - \tfrac{1}{2}\rho_t u_t = \tfrac{1}{2}\bar{e}(1-\rho_t^2) + \frac{1-\rho_t}{1-\rho}\tilde{m},$$

where $\tilde{m} := m - m'$. In particular $|m_t - \frac{1}{2}\rho_t u_t| = \frac{1}{2}(1-\rho_t^2)$ if either $\rho_t = 1$ or $\rho_t$ satisfies

$$(1+\rho_t)(1-\rho)\bar{e}\cdot\tilde{m} = -|\tilde{m}|^2. \tag{30}$$

But now we claim that, with our choice of $\delta$, the angle between $\mathscr{C}_1$ and $\mathscr{C}_2$ at the point of intersection $m'$ is bounded away from zero, depending only on $\gamma$. Indeed, it can be easily verified that this angle $\alpha$ satisfies

$$\cos\alpha \le \tfrac{1}{\gamma}(1 + \tfrac{\delta}{2(1+\rho)}) \le \tfrac{1}{2}(\tfrac{1}{\gamma} + 1).$$

This implies

$$-\bar{e}\cdot\tilde{m} \ge \varepsilon|\tilde{m}|, \tag{31}$$

where $\varepsilon > 0$ depends only on $\gamma > 1$. Using (31) and (30) we deduce that whenever $|\rho - \rho_0|$, $|u - u_0|$, $|e - e_0|$, $|\tilde{m}|$ is sufficiently small, $z_t \in \partial U_\gamma$ for all $t$ with $|t| < 1/2\min(1-\rho_0, 1+\rho_0)$.

In the remaining case, when $|u|^2 - \rho^2 \le \gamma^2 - 1 - \delta$, the inequality (16) will obviously remain valid for $|t| < \frac{1}{2(\gamma+1)}$. Therefore, by choosing $m'$ to be the radial projection of $m$ onto $\mathscr{C}_1$ (see Figure 1), we choose $\bar{e}$ again according to (29). With this choice of $m'$ and $\bar{e}$ the inequality (31) is valid with $\varepsilon = 1$, therefore we can repeat the same argument as before. This concludes the proof. □

Using the same argument as in the proof of Theorem 3.2 but replacing $K, U$ by $K_\gamma, U_\gamma$ with $\gamma > 1$, we obtain another proof of Theorem 1.2 (Theorem 5.2 of [1]).

## 4. Evolution of microstructure

In this section we show how to construct solutions to (IPM1)-(IPM4) in the spatial domain

$$\Omega := (-1,1) \times (-1,1),$$

which exhibit the type of mixing behavior that is expected ([2, 12, 13]), when one starts with a horizontal interface with the heavier fluid on top. Thus, let

$$\rho_0(x) = \begin{cases} +1 & x_2 > 0 \\ -1 & x_2 < 0 \end{cases} \tag{32}$$

and define subsolutions as follows.

Let

$$\mathscr{D} = \Omega \times (0,T).$$

We consider triples $(\rho, v, m) \in L^\infty(\mathscr{D})$ such that for all $\phi \in C_c^\infty(\overline{\Omega} \times [0,T))$

$$\int_0^T \int_\Omega \partial_t \phi \rho + \nabla \phi \cdot m \, dx dt + \int_\Omega \phi(x,0)\rho_0(x)\, dx = 0, \tag{33}$$

$$\int_\Omega v \cdot \nabla \psi \, dx = 0 \qquad \forall \psi \in C^\infty(\overline{\Omega}), \tag{34}$$

$$\int_\Omega (v + (0,\rho)) \cdot \nabla^\perp \psi \, dx = 0 \qquad \forall \psi \in C_c^\infty(\Omega), \tag{35}$$

and

$$|m - \rho v + \tfrac{1}{2}(0, 1-\rho^2)| \leq \tfrac{1}{2}(1-\rho^2) \text{ in } \mathscr{D}. \tag{36}$$

We assume that there exists an open subset $\mathscr{U} \subset \mathscr{D}$ such that

$$|\rho| = 1 \text{ a.e. } \mathscr{D} \setminus \mathscr{U}, \tag{37}$$

$$(\rho, v, m) \text{ is continuous in } \mathscr{U}, \tag{38}$$

$$|m - \rho v + \tfrac{1}{2}(0, 1-\rho^2)| < \tfrac{1}{2}(1-\rho^2) \text{ in } \mathscr{U}. \tag{39}$$

DEFINITION 4.1. – *Let us call any $(\rho, v, m) \in L^\infty(\mathscr{D})$ satisfying (33)-(39) an admissible subsolution and the corresponding subset $\mathscr{U} \subset \mathscr{D}$ the mixing zone.*

Next, given an admissible subsolution $(\overline{\rho}, \overline{v}, \overline{m})$, let

$$X_0 = \Big\{ (\rho, u, m) \in L^\infty(\mathscr{D}) : (\rho, v, m) = (\overline{\rho}, \overline{v}, \overline{m}) \text{ a.e. } \mathscr{D} \setminus \mathscr{U} \\ \text{and satisfies (38)-(39) in } \mathscr{U} \Big\}.$$

As in the proof of Theorem 3.2, the space $X_0$ is bounded in $L^2(\mathscr{D})$, hence weak $L^2$ convergence is metrizable on $X_0$. Let $X$ be the closure of $X_0$ with respect to the induced metric.

THEOREM 4.2. – *There exists a residual set in $X$ consisting of weak solutions $(\rho, v)$ to* (IPM1)-(IPM4) *in $\Omega \times (0,T)$ with initial condition $\rho_0$.*

*Proof.* – Recall from (5) that $v = \frac{1}{2}(u - (0,\rho))$, so that (39) is equivalent to

$$\big(\rho, u, m + (0, \tfrac{1}{2})\big) \in U \quad \text{for all } (x,t) \in \mathscr{U},$$

where $U$ is defined in (26). Therefore $X_0$ satisfies condition (H3) in the appendix. Moreover, (H1) and (H2) are satisfied by Lemma 2.1 and Proposition 3.1. Consequently Theorem 4.2 follows from Theorem 5.1. □

*Remark 4.* – *If we replace* (36) *and* (39) *by the full set of inequalities* (15)-(19) *in Proposition 2.4 for some $\gamma > 1$, and replace $U$ by $U_\gamma$ in the proof above, then Theorem 4.2 remains true with the added information that the weak solutions satisfy $v \in L^\infty(\mathscr{D})$.*

*Remark 5.* – *As a consequence of the residuality in $X$ we obtain, given an admissible subsolution $(\overline{\rho}, \overline{v}, \overline{m})$, the existence of a sequence of weak solutions $(\rho_k, v_k)$ of* (IPM1)-(IPM4) *such that*

$$\rho_k = \overline{\rho} \text{ a.e. in } \mathscr{D} \setminus \mathscr{U} \quad \text{and} \quad \rho_k \overset{*}{\rightharpoonup} \overline{\rho} \text{ in } L^\infty(\mathscr{U})$$

*as $k \to \infty$. In other words $\overline{\rho}$ represents a kind of coarse-grained density. This justifies calling $\mathscr{U}$ the mixing zone.*

To conclude this section we exhibit examples of (nontrivial) admissible subsolutions. In particular we set

$$\overline{v} \equiv 0, \quad \overline{m} = (0, \overline{m}_2),$$

and assume that $(\overline{\rho}, \overline{v}, \overline{m})$ are just functions of $t$ and the "height" $x_2$, i.e.,

$$\overline{\rho} = \overline{\rho}(x_2, t), \quad \overline{m}_2 = \overline{m}_2(x_2, t).$$

Then (34) and (35) are automatically satisfied, whereas (33) can be written as

$$\partial_t \overline{\rho} + \partial_2 \overline{m}_2 = 0 \quad \text{in } (-1,1) \times (0,T), \tag{40}$$

$$\overline{m}_2 = 0 \quad \text{for } x_2 = \pm 1, \tag{41}$$

$$\overline{\rho} = \begin{cases} -1 & x_2 < 0 \\ +1 & x_2 > 0 \end{cases} \quad \text{for } t = 0, \tag{42}$$

interpreted in the weak formulation. Furthermore, (36) becomes

$$-\left(1 - \overline{\rho}^2\right) \le \overline{m}_2 \le 0 \quad \text{in } (-1,1) \times (0,T). \tag{43}$$

Note also in connection with Remark 4 that in this case (16), (18) and (19) follow automatically from (43) provided $\gamma > 3$.

An obvious way to construct admissible subsolutions is then to prescribe

$$\overline{m}_2 = -\alpha(1 - \overline{\rho}^2)$$

for $\alpha \in (0,1)$. The resulting equation from (40) is essentially the inviscid Burgers' equation, with the (unique) entropy solution given by

$$\overline{\rho}(x_2,t) = \begin{cases} -1 & x_2 < -2\alpha t, \\ \frac{x_2}{2\alpha t} & |x_2| < 2\alpha t, \\ +1 & x_2 > 2\alpha t \end{cases} \tag{44}$$

for times $t < 1/(2\alpha)$. The corresponding mixing zone is

$$\mathscr{U} = \{(x,t) \in \mathscr{D} : \, |x_2| < 2\alpha t\}.$$

Since at $t = 1/(2\alpha)$ we have $|\overline{\rho}| < 1$ for all $x_2 \in (-1,1)$, the functions $\overline{\rho}, \overline{m}$ can easily be extended continuously to later times so that (40)-(43) continue to hold.

It is interesting to note that in the borderline case $\alpha = 1$ the subsolution $\overline{\rho}$ is precisely the unique solution of the relaxation approach to the problem (IPM1)-(IPM4) introduced by F. Otto in [12]. Thus, although weak solutions are clearly not unique, there seems to be a way to recover uniqueness at least for subsolutions. We plan to explore further this connection elsewhere. Here we contend ourselves with showing that among all subsolutions, for which $\overline{\rho}$ is a function of $t$ and the vertical direction $x_2$ only, the case $\alpha = 1$ in (44) corresponds to the one with "maximal mixing". This gives a new interpretation to the results in [12].

Proposition 4.3. – *Let $(\overline{\rho}, \overline{v}, \overline{m})$ be an admissible subsolution to the unstable initial condition (32) such that $\overline{\rho} = \overline{\rho}(x_2,t)$. Then the mixing zone $\mathscr{U}$ is contained in $\{(x,t) : |x_2| < 2t\}$.*

*Proof.* – We start by observing that since $\partial_1 \overline{\rho} = 0$, (34)-(35) imply that $\overline{v} = 0$. Next, let

$$\chi(s) = \begin{cases} 0 & s < 0, \\ s & s > 0, \end{cases}$$

and consider the test function $\phi(x,t) = \chi(x_2 - 2t)$ in (33) (it is easy to see by approximation that this is a valid test function). We obtain

$$\iint_{\{(x,t)\in\mathscr{D}:\, x_2 > 2t\}} \overline{m}_2 - 2\overline{\rho}\, dx dt + \int_{\{x \in \Omega:\, x_2 > 0\}} x_2 \rho_0(x) dx = 0.$$

Therefore

$$\iint_{\{(x,t)\in\mathscr{D}:\, x_2 > 2t\}} (\overline{m}_2 - 2\overline{\rho} + 2)\, dx dt = 0.$$

On the other hand (36) implies

$$\overline{m}_2 \geq -(1 - \overline{\rho}^2). \tag{45}$$

We deduce that

$$\overline{\rho} = 1 \text{ a.e. } (x,t) \in \{x_2 > 2t\}.$$

Analogously we also find

$$\overline{\rho} = -1 \text{ a.e. } (x,t) \in \{x_2 < -2t\}. \qquad \square$$

## 5. Appendix

We consider general systems in a domain $\mathscr{D} \subset \mathbb{R}^d$ of the form

$$\sum_{i=1}^{d} A_i \partial_i z = 0 \text{ in } \mathscr{D} \tag{46}$$

$$z(y) \in K \text{ a.e. } y \in \mathscr{D} \tag{47}$$

where

$$z : \mathscr{D} \subset \mathbb{R}^d \to \mathbb{R}^N$$

is the unknown state variable, $A_i$ are constant $m \times N$ matrices, and $K \subset \mathbb{R}^N$ is a closed set. We make the following assumptions.

(H1) The Wave Cone: There exist a closed cone $\Lambda \subset \mathbb{R}^N$ and a constant $C$ such that for all $\bar{z} \in \Lambda$ there exists a sequence $w_j \in C_c^\infty(B_1(0); \mathbb{R}^N)$ solving (46) such that

1. $\operatorname{dist}(w_j, [-\bar{z}, \bar{z}]) \to 0$ uniformly,
2. $w_j \rightharpoonup 0$ weakly in $L^2$,
3. $\int |w_j|^2 dy > C|\bar{z}|^2$.

(H2) The $\Lambda$-convex hull: There exists an open set $U \subset \mathbb{R}^N$ with $U \cap K = \varnothing$, and such that for all $z \in U$ with $\operatorname{dist}(z, K) \geq \alpha > 0$ there exists $\bar{z} \in \Lambda \cap S^{N-1}$ such that

$$z + t\bar{z} \in U \text{ for all } |t| < \beta, \tag{48}$$

where $\beta = \beta(\alpha) > 0$.

(H3) Subsolutions: $X_0$ is a nonempty bounded subset of $L^2(\mathscr{D})$ consisting of functions which are "perturbable" in an open subdomain $\mathscr{U} \subset \mathscr{D}$. This means that any $z \in X_0$ is continuous on $\mathscr{U}$ with

$$z(y) \in U \quad \text{for } y \in \mathscr{U}, \tag{49}$$

and moreover, if $z \in X_0$ and $w \in C_c(\mathscr{U})$ such that $w$ solves (46) and $(z + w)(y) \in U$ for all $y \in \mathscr{U}$, then $z + w \in X_0$.

Finally, let $X$ be the closure of $X_0$ with respect to the weak $L^2$ topology. Since $X_0$ is bounded, the topology of weak $L^2$ convergence is metrizable on $X$, making it into a complete metric space. Denote its metric by $d_X(\cdot, \cdot)$.

THEOREM 5.1. – *Assuming (H1)-(H3), the set*

$$\{z \in X : z(y) \in K \text{ a.e. } y \in \mathscr{U}\}$$

*is residual in $X$.*

The proof relies on the following lemma, where, for notational convenience we set

$$F(z) := \min(1, \operatorname{dist}(z, K)).$$

LEMMA 5.2. – *Let $z \in X_0$ with $\int_{\mathscr{U}} F(z(y))dy \geq \varepsilon > 0$. For all $\eta > 0$ there exists $\tilde{z} \in X_0$ with $d_X(z, \tilde{z}) < \eta$ and $\int_{\mathscr{U}} |z - \tilde{z}|^2 dy \geq \delta$, where $\delta = \delta(\varepsilon) > 0$.*

*Proof of Lemma 5.2*. – Since $z \in X_0$ is continuous and $z(y) \notin K$ on $\mathscr{U}$, for any $y_0 \in \mathscr{U}$ there exists $r_0 = r_0(y_0) > 0$ such that

$$\frac{1}{2} F(z(y_0)) \leq F(z(y)) \leq 2\, F(z(y_0)) \tag{50}$$

for all $y \in B_{r_0}(y_0) \subset \mathscr{U}$. Then, by a simple domain exhaustion argument, we find disjoint balls $B_i := B_{r_i}(y_i) \subset \mathscr{U}$ for $i = 1 \dots k$ such that

$$m := \left| \bigcup_{i=1}^{k} B_i \right| \geq \frac{1}{2} |\mathscr{U}|, \tag{51}$$

$$\sum_i \int_{B_i} F(z(y))\, dy \geq \frac{1}{2} \int_{\mathscr{U}} F(z(y))\, dy. \tag{52}$$

Next, observe that (H2) implies the existence of a continuous function $\phi : [0,1] \to [0,1]$ such that $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$ and such that for any $z \in U$ there exists $\bar{z} \in \Lambda \cap S^{N-1}$ with

$$z + t\bar{z} \in U \qquad \text{whenever } |t| < \phi\big(F(z(y))\big). \tag{53}$$

By considering its convexification if necessary, we may assume without loss of generality that $\phi$ is convex and monotone increasing. Using (50), (52) and the convexity of $\phi$ we obtain

$$\begin{aligned} \phi\left(\frac{1}{4|\mathscr{U}|} \int_{\mathscr{U}} F(z(y))dy\right) &\leq \phi\left(\sum_i F(z(y_i)) \frac{|B_i|}{m}\right) \\ &\leq \sum_i \phi\big(F(z(y_i))\big) \frac{|B_i|}{m}. \end{aligned}$$

Moreover, using (53), (H1) and a simple rescaling, there exist $\tilde{z}_i \in C_c^\infty(B_i; \mathbb{R}^N)$ such that

1. $z(y) + \tilde{z}_i(y) \in U$ for all $y$,
2. $d(\tilde{z}_i, 0) < \eta/k$,
3. $\int_{B_i} |\tilde{z}_i|^2 dy > C\phi\big(F(z(y_i))\big)|B_i|$,

where $C$ is the constant in (H1). Therefore,

$$w := \sum_i \tilde{z}_i \in C_c(\mathscr{U})$$

and $z(y) + w(y) \in U$ for all $y \in \mathscr{U}$, hence $\tilde{z} := z + w \in X_0$ by (H3). Moreover, $\tilde{z} \in X_0$ satisfies

$$d_X(\tilde{z}, z) \leq \sum_i d_X(\tilde{z}_i, 0) < \eta$$

$$\begin{aligned} \int_{\mathscr{U}} |z - \tilde{z}|^2 dy &\geq C \sum_i \phi\big(F(z(y_i))\big)|B_i| \\ &\geq \frac{C}{2|\mathscr{U}|} \phi\left(\frac{1}{4|\mathscr{U}|} \int_{\mathscr{U}} F(z(y))dy\right). \end{aligned}$$

This concludes the proof. □

*Proof of Theorem 5.1*. – First of all the functional $I(z) = \int_{\mathscr{U}} |z(y)|^2\,dy$ is a Baire-1 function on $X$. Indeed, observe that

$$I_j(z) := \int_{\{y\in\mathscr{U}:\mathrm{dist}\,(y,\partial\mathscr{U})>1/j\}} |z*\rho_j(y)|^2 dy,$$

where $\rho_j \in C_c^\infty(B_{1/j}(0))$ is the usual mollifier sequence, is continuous as a map $X \to \mathbb{R}$, and moreover

$$I_j(z) \to I(z) \text{ as } j \to \infty.$$

Therefore, by the Baire category theorem the set

$$Y := \{z \in X : I \text{ is continuous at } z\}$$

is residual in $X$. We claim that $z \in Y$ implies $\int_{\mathscr{U}} F(z(y))dy = 0$.

If not, let $\varepsilon := \int_{\mathscr{U}} F(z(y))dy > 0$ for some $z \in Y$, and let $z_j \in X_0$ be a sequence such that $z_j \overset{d}{\to} z$ in $X$. Since $I$ is continuous at $z$, it follows that $z_j \to z$ strongly in $L^2(\mathscr{U})$, and in particular we may assume that $\int_{\mathscr{U}} F(z_j(y))dy > \varepsilon/2$.

Then, by applying Lemma 5.2 to each $z_j$, we obtain a new sequence $\tilde{z}_j \in X_0$ such that $\tilde{z}_j \overset{d}{\to} z$ in $X$ (and hence strongly in $L^2$), but $\int_{\mathscr{U}} |z_j - \tilde{z}_j|^2 dy \geq \delta > 0$, where $\delta$ only depends on $\varepsilon$. This contradicts the strong convergence of both sequences to $z$. □

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