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A CASE OF MATHEMATICAL EPONYMY: THE VANDERMONDE DETERMINANT

Bernard Ycart

Abstract. — We study the historical process that led to the worldwide adoption, throughout mathematical research papers and textbooks, of the denomination "Vandermonde determinant". The mathematical object can be related to two passages in Vandermonde's writings, of which one inspired Cauchy's definition of determinants. Influential citations of Cauchy and Jacobi may have initiated the naming process. It started during the second half of the 19th century as a teaching practice in France. The spread in textbooks and research journals began during the first half of 20th century, and only reached full acceptance after the 1960's. The naming process is still ongoing, in the sense that the volume of publications using the denomination grows significantly faster than the overall volume of the field.

Résumé (Le déterminant de Vandemonde). — Nous étudions le processus historique qui a conduit à l'adoption dans le monde entier de la dénomination « déterminant de Vandermonde ». L'objet mathématique peut être relié à deux passages dans les écrits de Vandermonde, dont l'un a inspiré Cauchy pour sa définition des déterminants. Les citations de Cauchy et Jacobi ont pu déclencher le processus de dénomination. Celui-ci a démarré au cours de la seconde moitié du XIX^e^ siècle comme une pratique pédagogique. Cette pratique a précédé, plutôt que suivi, la diffusion dans les livres et les articles de recherche, qui a commencé pendant la première moitié du XX^e^ siècle, et n'a atteint un

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réel consensus qu'après les années 1960. Le processus de dénomination est encore en cours, au sens où l'usage du nom croît significativement plus vite que le volume global de publications du domaine.

1. INTRODUCTION

The Vandermonde determinant has become a standard example of Stigler's law of eponymy: "No scientific discovery is named after its original discoverer" (see [Stigler 1999, p. 277]). The source? An authority: Henri Lebesgue (1875–1941). On October 20, 1937, he gave a conference at Utrecht University, entitled "L'œuvre mathématique de Vandermonde". The text of that conference was published in 1939, reproduced in 1956, and again in a 1958 monography [Lebesgue 1958] to which we shall refer. In order to enhance Vandermonde's [1774] main achievement on the resolution of algebraic equations, Lebesgue [1958, p. 21] downplays his three other memoirs:[1]

> Thus the Vandermonde determinant is not due to Vandermonde; his theory of determinants is not very original, his notation of factorials is unimportant; his study of situation geometry is somewhat childish, what is left?[2]

Actually, the memoir on combinatorics Vandermonde [1775] contains more than just a notation for factorials: the identity

$$\binom{n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n-m}{k-i}$$

is still referred to as "Vandermonde's theorem" in probability and combinatorics textbooks (*e.g.* p. 315 of Santos [2011]). Though "childish", the memoir on situation geometry Vandermonde [1776b] made him regarded as a precursor of knot theory (see Przytycki [1992]).

The life of Alexandre Théophile Vandermonde (1735–1796), his engagement during the French revolution, his interests in music, mechanics, and political economy, and his short mathematical carrier, have all been amply documented: see Lebesgue [1958], Hecht [1971], Gillispie [1976],

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1 All translations are from the author.

2 Ainsi le déterminant de Vandermonde n'est pas de Vandermonde ; sa théorie des déterminants n'est pas très originale, sa notation des factorielles est sans importance ; son étude de géométrie de situation est un peu enfantine, que reste-t-il ?

Faccarello [1993], and Sullivan [1997]. We shall not attempt a new biography nor a mathematical assessment of Vandermonde's contribution. Neither shall we review here the early history of determinants. T. Muir's *Theory of determinants in their historical order of development* is the indispensable basis, and we shall often refer to the first two volumes: Muir [1906] and Muir [1911]. Our focus here is exclusively on the Vandermonde determinant, and more precisely on how that particular object came to be known under that name. We call *Vandermonde Determinant,* and denote by VD hereafter, the following determinant, depending on n variables $a_1, \ldots, a_n$:

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^n \\ 1 & a_2 & a_2^2 & \ldots & a_2^n \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & a_n & a_n^2 & \ldots & a_n^n \end{vmatrix}.$$

The VD has different mathematically equivalent interpretations, as a product of differences or an alternating polynomial, that will be developed in section 2.2.

Lebesgue makes the following assertion:

> What could have been personal, is the Vandermonde determinant? Yet it is not there, nor anywhere else in Vandermonde's work.[3] [Lebesgue 1958, p. 21]

Why then was Vandermonde's name given to that determinant? Lebesgue has a conjecture.[4]

> Vandermonde considers linear equations of which the unknowns are denoted by $\xi 1, \xi 2, \xi 3, \ldots$, and the coefficient of ξi in the k-th equation by $\overset{k}{i}$. The resolution of such a system, *e.g.* of
>
> $$\begin{aligned} \overset{1}{1}\xi 1 + \overset{1}{2}\xi 2 + \overset{1}{3}\xi 3 + \overset{1}{4} &= 0, \\ \overset{2}{1}\xi 1 + \overset{2}{2}\xi 2 + \overset{2}{3}\xi 3 + \overset{2}{4} &= 0, \\ \overset{3}{1}\xi 1 + \overset{3}{2}\xi 2 + \overset{3}{3}\xi 3 + \overset{3}{4} &= 0, \end{aligned}$$

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[3] Ce qui aurait pu être personnel, c'est le déterminant de Vandermonde ? Or il n'est pas là, ni nulle part ailleurs dans l'œuvre de Vandermonde!

[4] Lebesgue's notations have been reproduced.

will give determinants such as

$$\begin{vmatrix} \overset{1}{1} & \overset{1}{2} & \overset{1}{3} \\ \overset{2}{1} & \overset{2}{2} & \overset{2}{3} \\ \overset{3}{1} & \overset{3}{2} & \overset{3}{3} \end{vmatrix};$$

> Forgetting for a while the convention of notations that has been made, interpret the upper indices as exponents, you get a Vandermonde determinant. And perhaps, it is this mistake that saves Vandermonde's name from a more complete oblivion.[5]

As we shall see, no trace of such a mix-up can be found in the literature. Quite on the contrary, the mutation of exponents into indices in a VD is the very foundation of Cauchy's theory of determinants Cauchy [1815b]. Vandermonde [1776a] himself had made the observation that changing one of the indices of a general determinant into an exponent led to an alternating function. That remark did not escape either Cauchy nor Jacobi; this may have been the most solid argument in favor of the naming. On the other hand, it does not quite make the VD a counterexample to Stigler's law: linear systems with Vandermonde matrices had been written and solved long before Vandermonde, by Isaac Newton (1642–1727) and Abraham de Moivre (1667–1754).

Nevertheless, our purpose here is not to decide whether it is right or wrong to name that determinant after Vandermonde (the reader will be given enough elements to form his/her own opinion). Neither is it to enter the debate on mathematical eponymy (see Henwood & Rival [1980]; Smith [1980]). The naming of the VD is taken as a fact; and the *history of that fact*, we believe, is of independent interest. A mere attribution (citation: "a determinant introduced by Vandermonde") must be distinguished from an actual naming (eponymy: "a Vandermonde determinant"). R.K. Merton [1968] and his followers have long separated in the sociology of science, the respective roles of citation (that they consider as a moral norm) and eponymy (interpreted as a reward). We refer to Small [2004] for different

[5] Vandermonde considère des équations linéaires dont il désigne les inconnues par les notations, $\xi 1, \xi 2, \xi 3, \ldots$ et le coefficient de ξi dans la k-ième équation par $\overset{k}{i}$. La résolution d'un tel système, de [...] par exemple, donnera des déterminants tels que [...] or, oubliant un instant la convention des notations faites, interprétez les indices supérieurs comme des exposants, vous avez un déterminant de Vandermonde. Et peut-être est-ce cette méprise qui sauve le nom de Vandermonde d'un plus complet oubli.

theories of citation in science, and to Beaver [1976] for a historical perspective on eponymy. Eponymy has evolved together with successive sociological practices of science. In mathematics, it became a widespread habit essentially during the 19th century. Relatively few studies have been devoted to mathematical eponymy; among them Stigler's articles (see Stigler [1999] and references therein) stand out. The naming of a mathematical notion is in many cases a long term process that extends over several generations of mathematicians, and can be traced through historical accounts, textbooks, and research publications. By *naming process* we mean the penetration of the name as a function of time, "penetration" being taken in the statistical sense: the proportion of mathematicians knowing or using the name, measured as a proportion of texts where it can be found.

Lebesgue addressed his 1937 audience as follows.

> [...] the name of Vandermonde would be ignored by the vast majority of mathematicians if it had not been attributed to the determinant that you know well, and which is not his![6]

The sentence seems to imply that the denomination "Vandermonde determinant" was familiar to any mathematics student or professor in 1937. We believe that the naming process started as a teaching practise during the second half of the 19th century in France. Initially, it was more like a rumor than an identified decision grounded on historical facts; actually, many mathematicians clearly resisted it. As Stigler expresses it:

> [...] resistance to eponymic recognition of close associates may in fact be the norm of scientific behavior, one which serves the role of protecting the practice from degenerating to a regional or factional basis, with the consequent fall in the reward's incentive power. [Stigler 1999, p. 283]

This raises the question of the differential penetration of the naming according to the countries, and the possible influence of nationalisms, which we did not try to assess; it may be the case that in 1937 the denomination was more familiar to Lebesgue than to his Dutch audience. The naming process of the VD slowly gained momentum during the first half of the 20th, but the denomination became universally used by mathematicians only after the 1960's. It may be considered that the naming process is still ongoing, in the sense that its growth rate remains higher than that of the field.

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6 [...] le nom de Vandermonde serait ignoré de l'immense majorité des mathématiciens si on ne lui avait attribué le déterminant que vous connaissez bien — et qui n'est pas de lui!

To support our assertions, we have examined a selection of influential textbooks, conducted a systematic search through available databases, and statistically studied numerical output data from MathSciNet. The first pedagogical publication we could find using the denomination, appeared in 1886; the first textbook in 1897; the first research paper in 1914. We have made a systematic query for the expressions "Vandermonde determinant" and "Vandermonde matrix", on the MathSciNet database. The occurrences start in 1929 and remain quite sporadic until 1960. After 1960, the numbers of occurrences grow exponentially. We have compared the growth rate with that of the (much larger) number of occurrences of "determinant" or "matrix". A statistical test has shown that the growth rate for "Vandermonde determinant" or matrix is significantly higher than the global rate of increase for determinant or matrix. With all necessary precautions on the use of quantitative methods (see Goldstein [1999]), our conclusion is that the naming process, far from being an immediate recognition of Vandermonde's achievements, is a rather recent, and still developing phenomenon. It appears to be posterior, and related, to the spread of matrix theory (see Brechenmacher [2010]).

The paper is organized as follows. Section 2 gives a historical sketch of the mathematical objects under consideration (difference-products and alternating functions). Vandermonde's notations will be briefly examined in 2.1, then Cauchy's definition of determinants, based on difference-products, will be exposed in 2.2. In 2.3, Newton's and de Moivre's anteriority on the Vandermonde matrix through the divided differences method will be reviewed. In 2.4, Vandermonde's actual contributions will be discussed. Section 3 is devoted to the naming process, that will be examined from three different points of view. Historical accounts will be described in 3.1, focusing on the credits explicitly given to Vandermonde. The appearance of the naming in textbooks is described in 3.2. The quantification of the naming process in research papers is treated in 3.3.

2. DIFFERENCE-PRODUCTS AND ALTERNATING FUNCTIONS

2.1. *Vandermonde's notation*

Before describing the mathematical objects under study, we shall briefly comment on Vandermonde's notations, of which Lebesgue thought they could have induced a mix-up between indices and exponents. Here is Vandermonde's [1776a, p. 517] definition of determinants:[7]

> I suppose that one represents by $\overset{1}{1}, \overset{2}{1}, \overset{3}{1}$, &c. $\overset{1}{2}, \overset{2}{2}, \overset{3}{2}$, &c. $\overset{1}{3}, \overset{2}{3}, \overset{3}{3}$, &c. as many different general quantities, of which any one be $\overset{\alpha}{a}$, another one be $\overset{\beta}{b}$, &c. & that the product of both be ordinarily denoted by $\overset{\alpha}{a} \cdot \overset{\beta}{b}$. Of the two ordinal numbers α & a, the first one, for instance, will designate from which equation the coefficient $\overset{\alpha}{a}$ is taken, and the second one will designate the rank that the coefficient has in the equation, as will be seen hereafter.
>
> I suppose moreover the following system of abbreviations, and that it be set[8]
>
> $$\frac{\alpha}{a}\Big|\frac{\beta}{b} = \overset{\alpha}{a} \cdot \overset{\beta}{b} - \overset{\alpha}{b} \cdot \overset{\beta}{a}$$
>
> $$\frac{\alpha}{a}\Big|\frac{\beta}{b}\Big|\frac{\gamma}{c} = \overset{\alpha}{a} \cdot \frac{\beta}{b}\Big|\frac{\gamma}{c} + \overset{\alpha}{b} \cdot \frac{\beta}{c}\Big|\frac{\gamma}{a} + \overset{\alpha}{c} \cdot \frac{\beta}{a}\Big|\frac{\gamma}{b} \qquad [\ldots].$$

Vandermonde's notations probably looked much less strange in the 19th century than they do nowadays. Referring to them, T. Muir said:

> [. . .] we observe first that Vandermonde proposes for coefficients a positional notation essentially the same as that of Leibnitz [*sic*], writing $\overset{1}{2}$ where Leibnitz wrote 12 or 1_2.
> [Muir 1906, p. 24]

Indeed, Vandermonde's notations were quite similar to some of the many systems tried by Leibniz (see Knobloch [2001]). During the first half of the 19th century, different ways of denoting the coefficients of an array or a linear system coexisted (see Muir [1906]):

$$^{i}j, \quad i_j, \quad (i,j), \quad {}^{i}a_j, \quad a_i^{(j)}, \quad \ldots$$

W. Spottiswoode used (i,j) in the first treatise ever published on determinants Spottiswoode [1851]. C.L. Dodgson (Lewis Carroll) was the only one

7 Vandermonde's notations have been reproduced.

8 Je suppose que l'on représente par $\overset{1}{1}, \overset{2}{1}, \overset{3}{1}$, &c. $\overset{1}{2}, \overset{2}{2}, \overset{3}{2}$, &c. $\overset{1}{3}, \overset{2}{3}, \overset{3}{3}$, &c. autant de différentes quantités générales, dont l'une quelconque soit $\overset{\alpha}{a}$, une autre quelconque soit $\overset{\beta}{b}$, &c. & que le produit des deux soit désigné à l'ordinaire par $\overset{\alpha}{a} \cdot \overset{\beta}{b}$. Des deux nombres ordinaux α & a, le premier, par exemple, désignera de quelle équation est pris le coëfficient $\overset{\alpha}{a}$, & le second désignera le rang que tient ce coëfficient dans l'équation, comme on le verra ci-après.

Je suppose encore le système suivant d'abréviations, & que l'on fasse [. . .].

who ever denoted coefficients by $i\big)j$ Dodgson [1867]. G. Dostor's classical treatise Dostor [1877] proposed two notations, "juxtaposed" and "superposed" indices. Suarez and Gascó describe and use six different notations Suarez & Gascó [1882]. The modern notation $a_{i,j}$ was already present in Cauchy's [1815b, p. 113] memoir. But Cauchy himself mostly preferred multiple letter notations such as $a_i, b_i, c_i, \ldots, e_i, f_i$ (*e.g.* p. 121).

2.2. *Cauchy's definition*

Cauchy's [1815a; 1815b] two founding memoirs were read to the Institute on November 30, 1812, but were only published in 1815. After a thorough analysis of both, T. Muir concludes with a very lively description of the respective roles of Vandermonde and Cauchy :

> If one bears this in mind, and recalls the fact, temporarily set aside, that Cauchy, instead of being a compiler, presented the subject from a perfectly new point of view, added many results previously unthought of, and opened up a whole avenue of fresh investigation, one cannot but assign to him the place of honour among all the workers from 1693 to 1812. It is, no doubt, impossible to call him, as some have done, the formal founder of the theory. This honour is certainly due to Vandermonde, who, however, erected on the foundation comparatively little of a superstructure. Those who followed Vandermonde contributed, knowingly or unknowingly, only a stone or two, larger or smaller, to the building. Cauchy relaid the foundation, rebuilt the whole, and initiated new enlargements; the result being an edifice which the architects of to-day may still admire and find worthy of study. [Muir 1906, p. 131]

What was that "perfectly new point of view"? Previously, Bézout, Laplace, and Vandermonde had all defined determinants by induction using, explicitly or not, what is now known as Laplace's formula: the development of a determinant along one of its lines or columns. Cauchy's [1815b, p. 113] definition is radically different:

> Let $a_1, a_2, \ldots, a_n$ be several different quantities in number equal to n. It has been shown above, that by multiplying the product of these quantities, or
>
> $$a_1 a_2\, a_3 \cdots a_n$$
>
> by the product of their respective differences, or else by
>
> $$(a_2 - a_1)(a_3 - a_1)\cdots(a_n - a_1) \\ \times (a_3 - a_2)(a_n - a_2)\cdots(a_n - a_{n-1})$$
>
> one obtained as a result the alternating symmetric function
>
> $$S(\pm a_1 a_2^2 \cdots a_n^n)$$

which, as a consequence, happens to be always equal to the product

$$a_1 a_2 \cdots a_n (a_2 - a_1) \cdots (a_n - a_1) \times (a_3 - a_2) \cdots (a_n - a_2) \cdots (a_n - a_{n-1}).$$

Let us suppose now that one develops this later product and that, in each term of the development, one replaces the exponent of each letter by a second index equal to that exponent: by writing, for instance, $a_{r,i}$ instead of a_r^i and $a_{i,r}$ instead of a_i^r, one will obtain as a result a new alternating symmetric function which, instead of being represented by

$$S(\pm a_1^1 a_2^2 \cdots a_n^n),$$

will be represented by

$$S(\pm a_{1,1} a_{2,2} \cdots a_{n,n}),$$

the sign S being relative to the first indices of each letter. Such is the most general form of the functions that I shall designate in what follows under the name of *determinants*.[9]

In order to understand Cauchy's reasoning, one must keep in mind that his main focus was on functions of n variables: Cauchy [1815b] came as a sequel to Cauchy [1815a] where he discussed functions of n variables that take less than $n!$ different values when the variables are permuted. He called "symmetric alternating functions" (fonctions symétriques alternées) those functions taking only two opposite values (they will be referred to as

[9] Soient $a_1, a_2, \ldots, a_n$ plusieurs quantités différentes en nombre égal à n. On a fait voir ci-dessus que, en multipliant le produit de ces quantités ou

$$a_1 a_2 a_3 \cdots a_n$$

par le produit de leurs différences respectives, ou par

$$(a_2 - a_1)(a_3 - a_1) \cdots (a_n - a_1)(a_3 - a_2) \cdots (a_n - a_2) \cdots (a_n - a_{n-1})$$

on obtenait pour résultat la fonction symétrique alternée

$$S(\pm a_1 a_2^2 \cdots a_n^n)$$

qui, par conséquent, se trouve toujours être égale au produit

$$a_1 a_2 \cdots a_n (a_2 - a_1)(a_3 - a_1) \cdots (a_n - a_1)(a_3 - a_2) \cdots (a_n - a_2) \cdots (a_n - a_{n-1}).$$

Supposons maintenant que l'on développe ce dernier produit et que, dans chaque terme du développement, on remplace l'exposant de chaque lettre par un second indice égal à l'exposant dont il s'agit : en écrivant, par exemple, $a_{r,i}$ au lieu de a_r^i et $a_{i,r}$ au lieu de a_i^r, on obtiendra pour résultat une nouvelle fonction symétrique alternée qui, au lieu d'être représentée par

$$S(\pm a_1^1 a_2^2 \cdots a_n^n),$$

sera représentée par

$$S(\pm a_{1,1} a_{2,2} \cdots a_{n,n}),$$

le signe S étant relatif aux premiers indices de chaque lettre. Telle est la forme la plus générale des fonctions que je désignerai dans la suite sous le nom de *déterminants*.

"alternating functions"). Among them, the polynomials in n variables are multiples of the "product of differences", later called difference-product (see Muir [1906]). The difference-product develops into a sum of monomials with alternating signs. Those signs depend on the permutation of the variables and their exponents, and the "rule of signs" had been described by Cauchy before defining determinants. (On the discovery by Leibniz in 1683 of the rule of signs, see Knobloch [2001].)

Different expressions in n variables $a_1, a_2, \ldots, a_n$, may both be mathematically equivalent, and have different interpretations. We shall distinguish between:

- difference-product: $\prod_{1\leqslant i<j\leqslant n} (a_j - a_i)$,
- alternating polynomial: $\sum_{\sigma\in\mathfrak{S}_n} (-1)^{\varepsilon(\sigma)} \prod_{i=1}^{n} a_i^{\sigma(i)-1}$,
- Vandermonde determinant: $\det(a_i^j)_{\substack{1\leqslant i\leqslant n\\ 0\leqslant j\leqslant n-1}}$.

They are written in modern notations: $\mathfrak{S}_n$ is the group of permutations of $\{a_1, \ldots, a_n\}$ onto itself and $\varepsilon(\sigma)$ denotes the signature of the permutation σ. Needless to say, the group of permutations and the signature as a homomorphism are anachronistic. Cauchy had recognized in the development of the difference-product, the same rule of signs as that of a general determinant. Hence his idea of using

$$\prod_{i=1}^{n} a_i \prod_{1\leqslant i<j\leqslant n} (a_j - a_i) = \sum_{\sigma\in\mathfrak{S}_n} (-1)^{\varepsilon(\sigma)} \prod_{i=1}^{n} a_i^{\sigma(i)}$$

as a general definition, after mutating the exponent of each variable into a second index.

As pointed out by Muir [1906, p. 247], the year 1841 marked a new spurt for determinant theory, fueled by the publication in *Crelle's Journal* of Jacobi's monograph, divided into three papers. There Jacobi rebuilds the whole theory, taking Cauchy's approach upside down. Here is Muir's account:

> At the outset, there is a reversal of former orders of things; Cramer's rule of signs for a permutation and Cauchy's rule being led up by a series of propositions instead of one of them being made a convention or definition. This implies, of course, that a new definition of a signed permutation is adopted, and that conversely this definition must have appeared as a deduced theorem in any exposition having either of this rules as its starting point. [Muir 1906, p. 254]

In other words, when Cauchy's started from the difference-product, then defined a general determinant by mutating exponents into indices, Jacobi first defined positive and negative permutations, then defined the determinant as a polynomial, with coefficients ± 1 according to the sign of the permutation. Eventually, Jacobi's definition prevailed upon Cauchy's, which was forgotten. Cauchy undoubdtedly saw both pedagogical and mathematical advantages to his approach. When he writes his famous *Cours d'Analyse* in 1821, he follows exactly the same path as in his 1812 memoir. He recommends the difference-product as a general method for solving linear systems of equations, and applies it immediately to the Lagrange interpolation problem (pp. 71, 72, 426, 429 of Cauchy [1821]). The third of Jacobi's memoirs in *Crelle's Journal* Jacobi [1841] deals with alternating functions. Cauchy responds with Cauchy [1841] in which he treats quotients of alternating functions by difference-products. In particular, he calculates the determinant $\det\left(\frac{1}{a_i+b_j}\right)_{1\leqslant i,j\leqslant n}$ (formula (10), p. 154 of Cauchy [1841]) in a quite simple way. (Interestingly enough, the denomination "Cauchy determinant" for that example seems to be rarely used outside France, whereas the particular case $a_i = i$, $b_j = j - 1$ is universally known as "Hilbert matrix").

One year before 1841, the difference-product approach had been rediscovered by James Joseph Sylvester (1814–1897). Sylvester [1840] (without any reference to Cauchy) called "zeta-ic multiplication" Cauchy's operation of mutating exponents into indices in a polynomial. Muir's comment is somewhat ironic:

> This early paper, one cannot but observe, has all the characteristics afterwards so familiar to readers of Sylvester's writings, — fervid imagination, vigorous originality, bold exuberance of diction, hasty if not contemptuous disregard of historical research, the outstripping of demonstration by enunciation, and an infective enthousiasm as to the vistas opened by his own work. [Muir 1906, p. 235]

2.3. *Newton, de Moivre, and the interpolation problem*

The difference-product could hardly be considered an original notion in Cauchy's time. Apart from being a very natural way of combining n variables, it appears in the Lagrange interpolation problem. This other interesting case of mathematical eponymy is connected to ours, as we shall now see. For a history of interpolation, see Fraser [1919], and section 10.4 of Chabert & Barbin [1999]. If $(x_1, y_1), \ldots, (x_n, y_n)$ are the Cartesian coordinates of the points to be interpolated and $P = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$ the unknown polynomial, then its coefficients $a_0, \ldots a_{n-1}$ satisfy the following linear system.

$$\text{(LIS)} \qquad \begin{cases} a_0 + a_1 x_1 + \cdots + a_{n-1} x_1^{n-1} &= y_1, \\ a_0 + a_1 x_2 + \cdots + a_{n-1} x_2^{n-1} &= y_2, \\ &\vdots \\ a_0 + a_1 x_n + \cdots + a_{n-1} x_n^{n-1} &= y_n. \end{cases}$$

Assuming the x_i's are all different, the solution is the Lagrange interpolation polynomial:

$$\text{(LIP)} \qquad P(X) = \sum_{i=1}^{n} y_i \prod_{j \neq i} \frac{X - x_j}{x_i - x_j}.$$

It may seem fair that whoever first wrote the system of equations (LIS) should get the credit for discovering the Vandermonde matrix and whoever wrote (LIP) for computing its inverse (and implicitly the VD). The naming "Lagrange interpolation" comes from one of the lessons that Joseph Louis Lagrange (1736–1813) gave at the École Normale in Paris in 1795 Lagrange [1795]. There, Lagrange did not pretend to expose his own research:

> Newton is the first one who has posed that problem. Here is the solution he gives:[10] [...].

Indeed, in the *Principia Mathematica*, Isaac Newton (1642–1727) had described a method to determine "a curved line of parabolic type which passes through any number of points"[11] [Newton 1687, Lemma V, book III]: what is now known as Newton's divided differences method. In the *Principia*, Newton did not explicitly write (LIS). However, in a famous letter to Oldenburg dated October 24 1676, he mentions a manuscript, *Methodus differentialis*, that appeared in print only after the *Principia*, in 1711. There, the system (LIS) is explicitly written (see p. 10 of Fraser [1919], where the *Methodus Differentialis* is reproduced and translated), but the explicit solution (LIP) is not given. One may think that writing down (LIP) would have seemed useless and even misleading to Newton: he must have been aware that his method was both faster and numerically more stable than the direct application of (LIP). The first one to explicitly write (LIP) is Newton's friend Abraham de Moivre (1667–1754), in 1730 (on de Moivre's relationship with Newton, see Bellhouse & Genest [2007]). Instead of interpolation, de Moivre's motivation was to calculate

10 Newton est le premier qui se soit proposé ce Problème ; voici la solution qu'il en donne : [...].

11 *Invenire lineam curvam generis parabilici, quæ per data quotcunque puncta transibit.*

the coefficients in a linear combination of geometric series, when that linear combination is supposed equal to another series. The coefficients turn out to be the solution of a system equivalent to (LIS). In Theorem IV, pp. 33–35 of the *Miscellanea analytica* de Moivre [1730], de Moivre explicitly writes a general system with power coefficients, and gives its solution, thus being the first one to write the inverse of a Vandermonde matrix. Actually, de Moivre had already published particular cases of that result in the first edition of his *Doctrine of chances*. There he said:

> And if a general theorem were desired, it might easily be formed from the inspection of the foregoing.
>
> These theorems are very useful for summing up readily those series which express the probability of the plays being ended in a given number of games. [de Moivre 1718, p. 132]

Indeed, de Moivre's motivation came from probability problems arising from dice games: the theorem is used for the solution of problem IV, p. 77 of *Miscellanea analytica*, and in later editions (1738 and 1756) of the *Doctrine of chances*. De Moivre gives full credit to Newton both for the interpolation problem and the divided differences method. The following extract of his preface to the *Doctrine of chances* is worth quoting: its last sentence has a particular resounding with our subject.

> There are other sorts of series, which tho' not properly infinite, yet are called series, from the regularity of the terms whereof they are composed; those terms following one another with a certain uniformity, which is always to be defined. Of this nature is the Theorem given by Sir *Isaac Newton*, in the fifth *Lemma* of the third Book of his *Principles*, for drawing a curve through any given number of points: of which the demonstration, as well as other things belonging to the same subject, may be deduced from the first Proposition of his *Methodus Differentialis*, printed with some other of his tracts, by the care of my intimate friend, and very skilful mathematician, Mr. W. Jones. The abovementionned theorem being very useful in summing up any number of terms whose last differences are equal (such as the numbers called triangular, pyramidal, &c. the squares, the cubes, or other powers of numbers in arithmetic progression) I have shewn in many places of this book how it might be applicable to these cases. I hope it will not be taken amiss that I have ascribed the invention of it to its proper author, tho' 'tis possible some persons may have found something like it by their own sagacity. [de Moivre 1718, p. x]

De Moivre's anteriority on the difference-product has been pointed out on several occasions, in particular by Tee [1993]; but of course, de Moivre does not express difference-products as determinants. Actually, the difference-product, and the explicit expression of the inverse matrix have been rediscovered many times, until late in the 20th century: see *e.g.* Klinger [1967].

2.4. *Vandermonde's writings*

We shall now examine what in Vandermonde's work can be connected to the VD. About his *Memoir on elimination*, Vandermonde says:

> This memoir was read to the Academy for the first time on the 12th of January 1771. It contained different things that I have suppressed here because they have been published since by other Geometers.[12] [Vandermonde 1776a, footnote p. 516]

These "other Geometers" certainly include Laplace, whose memoir though posterior, was published in the same volume as Vandermonde's. Guessing what exactly did Vandermonde suppress cannot but remain conjectural.

Just like Cauchy in 1812, Vandermonde wrote about determinants as a byproduct of symmetric functions; his memoir on elimination is a sequel to the memoir on the resolution of equations. The publication dates, 1774 and 1776, are misleading: Vandermonde [1774] was read to the academy "sometime in November 1770", *i.e.*, only two months before Vandermonde [1776a]. Vandermonde undoubtedly had the first memoir in mind when he wrote the second, and both should be examined as a whole. Here are two quotations, numbered for later reference:

[V1] [Vandermonde 1774, p. 369]:

> And yet, $(a^2b+b^2c+c^2a-a^2c-b^2a-c^2b)$, which equals $(a-b)(a-c)(b-c)$, squares as[13]

$$\begin{gathered} a^4b^2+a^4c^2+b^4c^2+c^4a^2+c^4b^2 \\ -2(a^4bc+b^4ac+c^4ab)-2(a^3b^3+a^3c^3+b^3c^3) \\ +2(a^3b^2c+a^3c^2b+b^3a^2c+b^3c^2a+c^3a^2b+c^3b^2a)-6a^2b^2c^2. \end{gathered}$$

[V2] [Vandermonde 1776a, p. 522]:

> Those acquainted with the abbreviated symbols that I have named *partial types of combination*, in my *Memoir on the resolution of equations*, will recognize here the formation of the *partial type* depending on the second degree, for any number of letters; they will easily see that, by taking our $\alpha, \beta, \gamma, \delta$, &c. for instance, as exponents, all terms with equal signs in the development of one of

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[12] Ce mémoire a été lû pour la première fois à l'Académie le 12 Janvier 1771. Il contenoit différentes choses que j'ai supprimées ici, parce qu'elles ont été publiées depuis par d'autres Géomètres.

[13] Or, $(a^2b+b^2c+c^2a-a^2c-b^2a-c^2b)$, qui égale $(a-b)(a-c)(b-c)$, a pour carré [...].

our abbreviations, will also be the development of the *partial type* depending on the second degree, & formed with an equal number of letters.[14]

Actually, the difference-product of four variables appears in the following passage of [Vandermonde 1774, p. 386]:

> The first of these cubes is
>
> $$(A^3B^3) - \frac{3}{2}(A^3B^2C) + 6(A^3BCD) + 6(A^2B^2C^2) - 3(A^2B^2CD) + \frac{3}{2}(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)\sqrt{-3}\,;$$
>
> [...] as the square of the product of differences between the roots is a function of *types*, [...].[15]

However, the development is not explicitly written, and we have not found that sentence ever referred to.

Vandermonde [1774] details the resolution of second and third degree equations (hence [V1]), then states his general method, and illustrates it by the fourth degree equation. The rest of the paper is devoted to a discussion on the symmetric functions of the roots. Admittedly, the difference-product of three variables appears in [V1], and its development is given; but this does not establish that Vandermonde saw it as a determinant. [V2] certainly proves that he knew determinants were related to his "partial types depending on the second degree" (*i.e.*, alternating functions), through changing indices into exponents. He probably knew exactly to which "partial type" did the VD correspond, at least in dimension 3, and probably in dimension 4. There is no evidence he actually wrote a VD as a particular determinant, nor that he wrote difference-products of more than four variables. The impressive tables displayed on the three pages after p. 374 of Vandermonde [1774] show that he certainly had the capacity for much more difficult formal calculations. But they also prove that he did not have a general expression for symmetric nor alternating functions. The long footnote of pp. 374–375 seems to imply that he was on his way towards greater generality.

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14 Ceux qui ont connaissance des symboles abrégés que j'ai nommés *types partiels de combinaison*, dans mon *Mémoire sur la résolution des équations* ; reconnoîtront ici la formation du *type partiel* dépendant du second degré, pour un nombre quelconque de lettres ; ils verront sans peine qu'en prenant ici nos $\alpha, \beta, \gamma, \delta$, &c. par exemple, pour des exposans, tous les termes de même signe, dans le développement de l'une de nos abréviations, seront aussi le développement du *type partiel* dépendant du second degré, & formé d'un pareil nombre de lettres.

15 Le premier de ces cubes est [...] or comme le carré du produit des différences entre les racines est une fonction de *types*[...].

> […] By considering this formula as a multivariate finite difference equation, in which the difference of each variable is equal to unity, I can integrate & satisfy the conditions, by a particular procedure of which I propose to render an account in one of the future volumes.[16]

It is not very surprising that, by manipulating symmetric functions of three or four variables, Vandermonde had been led to write difference-products. Whether or not he viewed them as determinants may not be the most important. More interesting is the relation that he had seen in [V2]. He undoubtedly knew that by making an exponent of the second index in a determinant, an alternating function was obtained. But conversely, had he realized that *any* determinant could be obtained from a difference-product by the reverse operation? [V2] comes in Vandermonde [1776a], immediately after his four pages "proof" of the alternating property, before which he had announced:

> Instead of generally proving these two equations [the alternating property], which would demand an awkward rather than difficult calculation, I shall content myself with developing the simplest examples; this will suffice to grasp the spirit of the proof.[17]

The alternating property of the difference-product is trivial; and with Cauchy's definition, proving that a determinant changes sign when exchanging two columns becomes obvious. We do not think that Vandermonde would have written his four pages of "simplest examples" had he anticipated Cauchy's definition. Lebesgue appreciation on Vandermonde's contribution to the resolution of equations might still have some truth in it when applied to Vandermonde's determinants:

> Vandermonde never came back on his algebraic researches because at first he felt only imperfectly their importance, and if he did not understand it better afterwards, it is precisely because he had not reflected deeply on them; […].[18] [Lebesgue 1958, p. 38]

---

16 En considérant cette formule comme une équation aux différences finies à plusieurs variables, dans laquelle la différence de chaque variable soit égale à l'unité, je parviens à intégrer & à satisfaire aux conditions, par un procédé particulier dont je me propose de rendre compte dans l'un des volumes suivans.

17 Au lieu de démontrer généralement ces deux équations, ce qui exigeroit un calcul embarassant plutôt que difficile, je me contenterai de développer les exemples les plus simples ; cela suffira pour saisir l'esprit de la démonstration.

18 Or Vandermonde n'est jamais revenu sur ses recherches algébriques parce qu'il n'a tout d'abord senti qu'imparfaitement leur importance, et s'il ne l'a pas mieux comprise par la suite c'est précisément parce qu'il n'a pas réfléchi profondément sur elles ; […].

3. THE NAMING PROCESS

3.1. *Historical accounts*

We have searched historical notes in textbooks or research papers, for connections being made between Vandermonde and the VD. Many accounts have been given of Vandermonde's contribution to the resolution of equations: see Neuman [2007] or Stedall [2011] for recent references. Among the most famous, Nielsen [1929] and van der Waerden [1985] (as many others) do not mention the VD. Similarly Vandermonde's founding role is acknowledged in most historical accounts of determinant theory, but there again, his relation to the VD is seldom mentioned: throughout history, there seems to have been some embarrassment on the subject.

Muir's masterly treatise is quite significant, and it may have had some later influence on the naming. As many other authors, Muir calls "difference-product" the VD and "alternants" those determinants stemming from alternating functions or generalizing the VD; he has been quite an active contributor of the field in the last decades of the 19th century. In each volume, he devotes a chapter to alternants. Here are the first lines of that chapter in Volume 1:

> The first traces of the special functions now known as *alternating functions* are said by Cauchy to be discernible in certain work of Vandermonde's; and if we view the functions as originating in the study of the number of values which a function can assume through permutation of its variables, such an early date may in a certain sense be justifiable. To all intents and purposes, however, the theory is a creation of Cauchy's, and it is almost absolutely certain that its connection with determinants was never thought of until his time. [Muir 1906, p. 306]

In volume 2, Muir feels obliged to set some records straight:

> Further, as exagerated statements regarding Vandermonde's contribution to the subject have been widely accepted, it seems desirable to point out the exact foundation on which such statements rest. In a paper read in November 1770 Vandermonde says (p. 369), "Or $a^2b + b^2c + c^2a - a^2c - b^2a - c^2b$, qui égale $(a-b)(a-c)(b-c)$ a pour carré $a^4b^2 + \ldots$". This is the whole matter. [Muir 1911, p. 154]

As we have seen, there are essentially two ways to connect Vandermonde's writings to the VD:

[V1]: Vandermonde has written the difference-product of three variables and its development, hence a particular case of the VD.

[V2]: Vandermonde has anticipated Cauchy's definition by remarking that changing one of the indices into an exponent gives an alternating function.

Clearly, Muir is on the [V1] side, as all historians have been since. It was not quite so in the 19th century. As Muir points out, Cauchy had studied Vandermonde's two memoirs on the resolution of equations and on elimination, and quotes them. In [Cauchy 1815b, p. 110], [V1] is explicitly cited:

> Thus, supposing for instance $n = 3$, it will be found
>
> $$\begin{aligned} S^2(\pm a_2 a_3^2) &= a_2 a_3^2 + a_3 a_1^2 + a_1 a_2^2 - a_3 a_2^2 - a_2 a_1^2 - a_1 a_3^2 \\ &= (a_2 - a_1)(a_3 - a_1)(a_3 - a_2). \end{aligned}$$
>
> This last equation has been given by Vandermonde in his memoir on the resolution of equations.[19]

Cauchy does not explicitly acknowledge that [V2] inspired his definition of determinants from difference-products, but the following quotation clearly alludes to [V2]:

> The smallest divisor of this product is equal to 2 and it is easy to make sure, that, in any order, it is possible to form functions having only two different values. Vandermonde has given ways to compose functions of that kind. In general, to form with quantities
>
> $$a_1, a_2, \ldots, a_n$$
>
> an order n function with index 2, it will suffice to consider the positive or the negative part of the product
>
> $$(a_1 - a_2)(a_1 - a_3) \cdots (a_1 - a_n)(a_2 - a_3) \cdots (a_2 - a_n) \times \cdots (a_{n-1} - a_n)$$
>
> whose factors are the differences of the quantities $a_1, a_2, \ldots, a_n$ taken two by two.[20] [Cauchy 1815a, p. 70]

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19 Ainsi, par exemple, si l'on suppose $n = 3$, on trouvera [...] Cette dernière équation a été donnée par Vandermonde dans son Mémoire sur la résolution des équations.

20 Le plus petit diviseur de ce produit est toujours égal à 2 et il est facile de s'assurer que, dans un ordre quelconque, on peut former des fonctions qui n'aient que deux valeurs différentes. Vandermonde a donné les moyens de composer des fonctions de cette espèce. En général, pour former avec les quantités

$$a_1, a_2, \ldots, a_n$$

une fonction de l'ordre n dont l'indice soit égal à 2, il suffira de considérer la partie positive ou la partie négative du produit [...] qui a pour facteurs les différences des quantités $a_1, a_2, \ldots, a_n$ prises deux à deux.

We could find in the literature only four other citations of [V2]. The earliest comes in the very first words of Jacobi [1841]; admittedly, it is worth many others.

> The famous Vandermonde once elegantly observed that the proposed determinant
> $$\sum \pm a_0^{(0)} a_1^{(1)} a_2^{(2)} \cdots a_n^{(n)},$$
> if indices are changed into exponents, comes from the product formed from the differences of all elements $a_0, a_1, \ldots, a_n$[21]
> $$\begin{aligned} P = (a_1 - a_0)(a_2 - a_0)(a_3 - a_0) & \cdots (a_n - a_0) \\ (a_2 - a_1)(a_3 - a_1) & \cdots (a_n - a_1) \\ (a_3 - a_2) & \cdots (a_n - a_2) \\ & \cdots (a_n - a_{n-1}). \end{aligned}$$

The next citation that we are aware of, appears in Terquem [1846].

> A very ingenious observation of the same geometer [Vandermonde], about indices considered as exponents, has given birth to Mr. Cauchy's beautiful theory of *alternating functions* and to his proof of Cramer's formulae[22].

Our third citation comes from the preface of Spottiswoode's treatise. There he comments Cauchy [1815b] as follows:

> The second part of this paper refers immediately to determinants, and contains a large number of very general theorems. Amongst them is noticed a property of a class of functions closely connected with determinants, first given, so far as I am aware, by Vandermonde; if in the development of the expression
> $$\begin{aligned} a_1 a_2 \cdots a_n (a_2 - a_1) \cdots (a_n - a_1)(a_3 - a_2) \cdots (a_n - a_2) \\ \times \cdots (a_n - a_{n-1}) \end{aligned}$$
> the indices be replaced by a second series of suffixes, the result will be the determinant
> $$S(\pm a_{1,1} a_{2,2} \ldots a_{n,n}).$$
> [Spottiswoode 1851, p. vi]

The last citation appears in Prouhet [1856] who, before writing the difference-product of n variables "according to a theorem due to Vandermonde" gives [V2] as a reference.[23]

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21 *Eleganter olim observavit Cl. Vandermonde, proposito Determinante* [...] *si mutentur indices in exponentes, provenire Productum conflatum ex omnibus elementorum differentiis* $a_0, a_1, \ldots, a_n$.

22 Une observation très-ingénieuse du même géomètre, sur les indices considérés comme des exposants, a donné naissance à la belle théorie des *fonctions alternées* de M. Cauchy et à sa démonstration des formules de Cramer.

23 D'après un théorème de Vandermonde (A.A., 1772, 2ᵉ partie, p. 522), on a [...].

It is likely that, since Cauchy's definition never prevailed and soon fell into oblivion, so went with it Vandermonde's "elegant observation". From then on, [V1] has been the commonly accepted source for the naming. The position usually adopted is clearly expressed by an anonymous contributor to the *Nouvelles annales de Mathématiques*:

> Vandermonde [...] decomposes into factors a polynomial that can be considered as a 3rd order determinant: but nothing indicates that he had the general theorem in mind, not even that he had considered that polynomial as a determinant.[24] [Un professeur 1860, p. 181]

The same view has been expressed many times, from R. Baltzer [1857, p. 50] to J. Stedall [2011, p. 190], through S. Günther [1875, p. 66] and G. Kowalewski [1942, p. 315]; it appears in the *Encyclopedia of Mathematics* [Remeslennikov 1993, p. 363]. Only two of the early authors were less carefull in their attribution: F. Brioschi speaks of an "important relation due to Vandermonde" [Brioschi 1854, p. 75], and G.A. Gohierre de Longchamps [1883, p. 82] devotes a section to "Vandermonde's theorem".

Since the publication of Lebesgue's [1958] conference, his mix-up conjecture has been cited by several authors: see *e.g.* [Edwards 1984, p. 18], [Blyth & Robertson 2002, p. 197]; it even appears in Gillispie's [1976, p. 571] *Dictionary of Scientific Biography* . It has probably fostered the widely accepted idea that the attribution of the VD to Vandermonde is a misnomer. J. Dieudonné states it quite clearly:

> This naming, due to Cauchy, is not historically justified, since Vandermonde never explicitly introduced such a determinant[25]. [Dieudonné 1978, p. 59]

Yet, Dieudonné was aware of Cauchy's use of the exchange between exponents and indices, that he presents as an "elegant trick" (élégant artifice)...

3.2. *Textbooks*

We have made a selection of 24 treatises and textbooks having appeared in the 19th and 20th centuries, partially or completely devoted to determinants, and where the VD appears as a mathematical object, if only as a simple example or exercise. All of them have had several editions or

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24 Vandermonde (A.S., 1771, p. 369) décompose en facteur un polynôme qui peut être considéré comme un déterminant du 3e ordre : mais rien n'indique qu'il ait eu en vue le théorème général, ni même qu'il ait considéré ce polynôme comme un déterminant.

25 Cette appellation, due à Cauchy, n'est pas historiquement justifiée, Vandermonde n'ayant jamais introduit explicitement un tel déterminant.

translations, which we regard as a criterion of (relatively) large diffusion. Our selection is arbitrary, and we have examined only a very small sample of the full textbook production of these times. We have not systematically searched outside the (modern) area of linear algebra, though we are aware that early occurrences of the naming can be found in other fields. For instance, in one of the earliest and most influential treatises on numerical analysis, when the authors expose Newton's divided difference method, they write the interpolation system, its determinant, and add [Whittacker & Robinson 1924, p. 23]:

> Now a difference-product may be expressed as a determinant of the kind known as Vandermonde's [...].

As another example, Pólya and Szegő's [1998, p. 43] famous textbook contains a "generalized Vandermonde determinant". Nevertheless, we consider our sample as representative, in the statistical sense: our conclusion being that the denomination remains sporadic until 1950, we believe it would be confirmed on a broader corpus. Table 1 gives the references, the publication country (including translations), and the name given to the VD for each book in our sample.

Before the second half of the 20th century, the denomination "Vandermonde determinant" can hardly be found in textbooks. Among the early treatises on determinants, Brioschi [1854, p. 75] mentions "an important relation due to Vandermonde", and Gohierre de Longchamps [1883] devotes a section to "Vandermonde's theorem". These attributions may have had some influence on the naming practice, but they are not *actual namings* of the VD as a mathematical object. Ernesto Pascal (1865–1940) seems to be the first one to actually name the VD in a textbook. His hesitations are very revealing. The running head of [Pascal 1897, p. 166] is indeed "Vandermonde determinant". But the title of the section is "Vandermonde or Cauchy determinant". Pascal cites Jacobi [1841] and mentions:

> It is usually called also Cauchy determinant, this last author having considered it in general, whereas Vandermonde studied it in a particular case.[26]

Many authors, although quite aware of Vandermonde's contributions, remain very cautious regarding the naming. Siegmund Günther (1848–1923) devotes the first chapter of his treatise to a careful historical exposition, where Vandermonde's role is thoroughly analyzed. Yet later

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26 *Si suol chiamare anche determinante di Cauchy, il quale ultimo autore lo consideró piu generale, mentre Vandermonde lo avea studiato in un caso speciale.*

Reference	Countries	Page	Naming
Brioschi [1854]	Italy	75	none
Baltzer [1857]	Germany, France	50	none
Salmon [1859]	Great-Britain	13	none
Bertrand [1859]	France, Italy	333	none
Trudi [1862]	Italy	31	none
Günter [1875]	Germany	66	difference product
Dostor [1877]	France	142	none
Scott [1880]	Great-Britain	115	difference product
Mansion [1880]	Belgium	27	none
Suarez & Gascó [1882]	Spain	360	none
Gohierre de Longchamps [1883]	France	82	none
Hanus [1886]	USA	187	difference product
Chrystal [1886]	Great-Britain	53	none
Pascal [1897]	Italy, Germany	166	Vandermonde
Kronecker [1903]	Germany	304	none
Hawkes [1905]	USA	218	none
Weld [1906]	USA	169	alternant
Wedderburn [1934]	USA	26	none
Barnard & Child [1936]	Great-Britain, USA	126	none
Aitken [1939]	USA	42	alternant
Kowalewski [1942]	Germany	315	none
Gantmacher [1959]	Russia, USA	99	Vandermonde
Bourbaki [1989]	France, USA	532	Vandermonde
Lang [1970]	USA	155	Vandermonde

TABLE 1. Textbooks including the VD, and whether ot not it is given a name.

on, the VD is named “Differenzenprodukt” and attributed to Vandermonde for $n = 3$ and to Cauchy for the general case [Günter 1875, p. 66]. Leopold Kronecker (1823–1891) cannot be suspected of downplaying Vandermonde’s achievements (see Lebesgue [1958]). However, when he writes his *Lessons on the theory of determinants*, he attributes the VD to Cauchy

[Kronecker 1903, p. 304] and does not name it. In his *Lessons on number theory*, the VD is named "Differenzenprodukt" [Kronecker 1901, p. 396]. Joseph Bertrand (1822–1900) has known Cauchy, and he is among the rare authors to follow Cauchy's definition of determinants. His *Traité élémentaire d'algèbre* had several editions since 1851. The determinants appear in the 1859 Italian edition [Bertrand 1859, p. 333] but no name is given to the VD.

3.3. *Research papers*

In order to evaluate the penetration of the expression "Vandermonde determinant" in the mathematical literature, we have searched through several databases: Gallica, Google Books, Göttinger Digitalisierungszentrum, Internet Archive, Jstor, Mathematical Reviews (or "MathSciNet"), Numdam, and Zentralblatt Math.[27] The earliest traces of the attribution that we could find in articles are:

(1) "According to a theorem due to Vandermonde"[28] [Prouhet 1856, p. 87];

(2) "This theorem, ordinarily attributed to Vandermonde," [...].[29] [Un professeur 1860, p. 181];

(3) "The last determinant, by virtue of the theorem known as *Vandermonde's*, [...]."[30] [Neuberg 1866, p. 517].

We cannot be sure that earlier appearances do not exist elsewhere. However we find it significant that the earliest references were found in pedagogy rather than research journals. They come from professors at the undergraduate level, sharing their solutions to particular problems. In quotations 2 and 3, some hesitation can be felt in the expressions "ordinarily attributed to" or "known as". As we have already seen, Prouhet [1856] cites [V2] to support the attribution, whereas "Un professeur" [1860] clearly resists it; both implicitly admit that the attribution to Vandermonde is already a usual practice. After 1886, maybe under the influence of Gohierre de Longchamps [1883], the attributions become more assertive. The first two actual namings seem to be:

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27 `http://gallica.bnf.fr`, `http://books.google.com`, `http://gdz.sub.uni-goettingen.de`, `http://www.archive.org`, `http://www.jstor.org`, `http://www.ams.org/mathscinet`, `http://www.numdam.org/`, `http://www.zentralblatt-math.org/zbmath`.

28 D'après un théorème de Vandermonde.

29 Ce théorème, ordinairement attribué à Vandermonde, [...].

30 Le dernier déterminant, en vertu du théorème dit *de Vandermonde*, [...].

(1) “The numerator is a Vandermonde determinant”[31] [Marchand 1886, p. 164];

(2) “On a form of Vandermonde determinant”[32] (title of the paper) Weill [1888].

The first occurrence of the naming in a research journal was found through Jstor: Bennett [1914]. This indicates that the denomination was already in use both among researchers and outside France, before World War One.

For our quantitative study, we chose to focus on MathSciNet, that seemed to give more easily interpretable results. As an example of the difficulties encountered with other bases, Zentralblatt has references to which the keyword “Vandermonde determinant” is associated, whereas it does not appear in the article: an example is de Jonquières [1895] whose denomination for the VD is “déterminant potentiel”; these false detections were difficult to sort. However we believe that searching in another database would give similar results (compare Figure 1 below with those of Annex 1.2 in Brechenmacher [2010]). We are aware of the limits to our quantitative approach. The MathSciNet database does not contain all published articles; moreover, we could not check each reference to make sure it was relevant. Nevertheless, we consider that MathSciNet is a representative sample, in the statistical sense, of the total mathematical production: we believe that our estimation of exponential growth rates would not be significantly (again in the statistical sense) modified if computed on another database.

We first searched for the other historical denominations, “alternant”, “difference-product” and “power determinant”. No publication could be found for “power determinant”, which seems to have disappeared (maybe for ambiguity reasons). Similarly, only two non ambiguous occurrences were found for “difference-product”. The name “alternant” is also ambiguous: it appears in “alternant code” and “alternant group”. After disambiguation, here are the occurrences per decade:

dates	< 1940	40’s	50’s	60’s	70’s	80’s	90’s	> 2000
alternant occurrences	10	7	8	12	4	9	5	4

The occurrence of “alternant” (as a determinant) did not completely disappear, but it has remained sporadic, and has not increased with the total

31 Le numérateur est un déterminant de Vandermonde.

32 Sur une forme du déterminant de Vandermonde.

mathematical production. Let us now turn to the Vandermonde denomination. It can be found under different forms.

– Vandermonde determinant or matrix,
– Vandermonde's determinant or matrix,
– Vandermondian.

The second one has 16 occurrences before 2011, the third one only 7. The first occurrence of "Vandermondian" was found in Farrel [1959]; however, the term seems to be more current in the physical literature than in the mathematical one: see Vein & Dale [1999], section 4.1, p. 51. It may be the case that the use of the Vandermonde determinant in the modelling of the quantum Hall effect (see Scharf et al. [1994]) boosted its popularity among physicists. This would match the effect that quantum mechanics had on the development of matrix theory, as described by Brechenmacher [2010].

The query "Vandermonde determinant" includes "Vandermonde's determinant" (and determinants); applied with the option "Anywhere", it returns 273 occurrences. The query "Vandermonde matrix" (including plural) returns 363 occurrences. Our query was the disjunction of these two, and it returned 623 occurrences (less than the sum of the previous two because "determinant" and "matrix" together are found in 13 references). The first occurrence appears in 1929. We have made the same query for each year from 1929 to 2010. The corresponding numbers will be referred to as "Vandermonde data". They remain quite sporadic during the first half of the 20th century (0, 1, 2, or 3 occurrences per year before 1958); then they gradually increase. Of course that increase was expected, since the total mathematical production grows exponentially: the increase in the output of any given query should be considered only relatively to the increase of the total production in the field. For the same years (1929–2010), we have made the query "determinant or matrix". The corresponding series will be referred to as "global data". The total number was 202 219. In order to compare both series, we have plotted on the same graphic (Figure 1), the Vandermonde and the global data, after dividing each by its sum. Of course the Vandermonde data are more irregular; however, both curves seem to grow exponentially, with a higher rate for the Vandermonde data.

In order to provide a statistical justification to the previous assertions, our treatment was the following. Firstly, the last two years (2009 and 2010) were truncated: they show a decrease that we do not consider as significant; it is probably due to the delay in entering new publications in the base. Then the data were binned over periods of 5 years (to account for

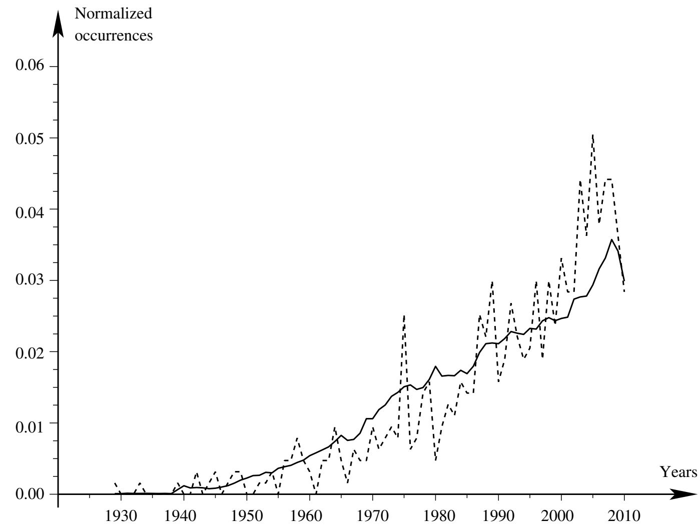

FIGURE 1. Occurrences of "Vandermonde determinant" or "Vandermonde matrix" (dashed) compared to "determinant" or "matrix" (solid) in the MathSciNet database. For each curve, the data per year have been divided by their sum.

sporadicity at the beginning of the Vandermonde series). Saying that the data grow exponentially means that they can be adjusted by a function of the type $y = \exp(ax+b)$ where x is a year, y a number of publication, and a is the exponential growth rate. Equivalently, the logarithm of the data can be adjusted by a linear function of the years: $ax+b$. The parameters a and b were estimated by a least-squares linear regression of the log-data over the years (see *e.g.* chap. 14 of Utts & Heckard [2004] as a general reference). Figure 2 displays the graphical results of the two linear regressions. Both regressions were found to be significant, with respective p-values of $3.6\ 10^{-12}$ and $3.1\ 10^{-7}$. The exponential growth rate (*i.e.*, the slope of the regression line) was found to be 0.0079 for the global data, and 0.0131 for the Vandermonde data. In other words, the global number of publications is multiplied by $e^a \simeq 1.0079$, or else increases by 0.79% per year on average, whereas the Vandermonde data increase by 1.31%. To test whether the 0.52% observed difference between growth rates was significant, we used another linear regression, that time on the logarithm of the *ratios*, *i.e.*, on the difference of the two previous sets. The new slope is of course the difference of the two previous ones, and it was found to be significantly positive, with a p-value of $6.9\ 10^{-4}$.

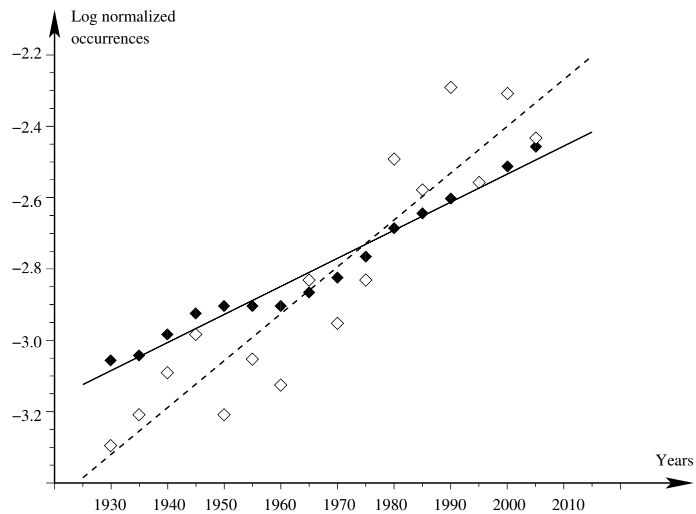

FIGURE 2. Linear regressions for the logarithms of occurrences of "Vandermonde determinant" or "Vandermonde matrix" (dashed line, empty diamonds) and "determinant" or "matrix" (solid line and diamonds) in the MathSciNet database. The data are binned by 5-year periods over the 80 years 1929–2008.

Having shown that the denomination "Vandermonde determinant or matrix" has a higher growth rate than "determinant or matrix" alone, the question of the interpretation arises. Comparing exponential growth rates may be a way of measuring the scientific dynamism of a research field. A field with a faster growth than the global production could be considered as booming; on the contrary a field with a lower growth rate would be seen as slowing down; among two fields, the more dynamic would be the one with a significantly higher growth rate. Here, the problem is different. The hypothesis of a higher dynamics of research on the VD compared to the rest of linear algebra can be ruled out: the VD has long been an undergraduate-level basic tool rather than a subject of research of its own. There remains two possible explanations.

(1) The fields of research using the Vandermonde determinant or matrix as a tool, are more fertile than those using other determinants or matrices.

(2) Mathematicians using a Vandermonde determinant or matrix tend more and more to give it its usual name.

We could not find any evidence supporting the first hypothesis, and we believe that the occurrence of the VD as an object is no more frequent in today's mathematical research than it was some decades ago. The only explanation we find plausible is that when mathematicians encounter a VD, they tend more and more to use the standard denomination, which has become a universally accepted shortcut.

4. CONCLUSION

In our study of the historical process that led to the worldwide adoption, throughout mathematical research papers and textbooks, of the denomination "Vandermonde determinant", we have established the following points. Although Vandermonde is not the first discoverer of the object, although he never expressed it in full generality, there still exist two connections between his writings and the VD: he has written down and developed the difference-product of three variables, and he has observed that changing indices into exponents in a general determinant gave an alternating function. Even if Vandermonde's calculation of the three variables difference-product was the only one eventually retained by historians, his second observation about changing exponents into indices probably inspired Cauchy's definition of determinants, and was quoted by Jacobi. Both may have sparked off the naming process. It started during the second half of the 19th century, essentially as a teaching practice. For quite a long time, textbook and research paper authors resisted the naming, for which no sufficient justification existed in their view. The naming process eventually gained momentum during the second half of the 20th century and from then on, its penetration of the mathematical community has been increasing. This was proved by a statistical treatment of numerical data from the MathSciNet database, that consisted in comparing the exponential growth rates of the naming to that of the global production.

Thus we believe that we have brought answers to the questions where?, when?, and how? The most important question may be the one we did not address: why? The sociological explanation of eponymy as a reward, may not be the only one. We believe that the pedagogical function of eponymy, which has been overlooked until now, should be taken into account. Here are some of the questions that would deserve an investigation. As the computation of the VD became a classical exercise or example, did the pressure to name it increase? More generally, do students prefer a mathematician's name rather than an impersonal one? Is a theorem easier to memorize when given a person's name? Does a mathematician necessarily transmit as a researcher the denominations he has learned as a student? Many questions remain to be asked, but we do not think that they are proper to mathematics, nor that can be answered by mathematicians alone: maybe the time has come for a collaboration between specialists of mathematics, pedagogy, and onomastics (see *e.g.* Nuessel [2011])...

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