ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Ionut CHIFAN & Thomas SINCLAIR

On the structural theory of II_1 *factors of negatively curved groups*

ON THE STRUCTURAL THEORY OF II₁ FACTORS
OF NEGATIVELY CURVED GROUPS

BY Ionut CHIFAN and Thomas SINCLAIR

ABSTRACT. – Ozawa showed in [21] that for any i.c.c. hyperbolic group, the associated group factor $L\Gamma$ is solid. Developing a new approach that combines some methods of Peterson [29], Ozawa and Popa [27, 28], and Ozawa [25], we strengthen this result by showing that $L\Gamma$ is strongly solid. Using our methods in cooperation with a cocycle superrigidity result of Ioana [12], we show that profinite actions of lattices in $\mathrm{Sp}(n, 1)$, $n \geq 2$, are virtually W^*-superrigid.

RÉSUMÉ. – Ozawa a montré dans [21] que, pour un groupe c.c.i. hyperbolique, le facteur de type II₁ associé est solide. En dévéloppant une nouvelle approche, qui combine les méthodes de Peterson [29], d'Ozawa et Popa [27, 28], et d'Ozawa [25], nous renforçons ce résultat en montrant que ce facteur est fortement solide. En combinant nos méthodes avec un résultat d'Ioana de superrigidité des cocycles [12], nous prouvons que les actions des réseaux de $\mathrm{Sp}(n, 1)$, $n \geq 2$, sont virtuellement W*-superrigides.

Introduction

In a conceptual leap Ozawa established a broad property for group factors of Gromov hyperbolic groups—what he termed solidity—which essentially allowed him to reflect the "small cancellation" property such a group enjoys in terms of its associated von Neumann algebra.

OZAWA'S SOLIDITY THEOREM ([21]). – *If Γ is an i.c.c. Gromov hyperbolic group, then $L\Gamma$ is solid, i.e., $A' \cap L\Gamma$ is amenable for every diffuse von Neumann subalgebra $A \subset L\Gamma$.*

Notable for its generality, Ozawa's argument relies on a surprising interplay between C*-algebraic and von Neumann algebraic techniques [4].

Using his deformation/rigidity theory [32], Popa was able to offer an alternate, elementary proof of solidity for free group factors: more generally, for factors admitting a "free malleable deformation" [33]. Popa's approach exemplifies the use of spectral gap rigidity arguments that opened up many new directions in deformation/rigidity theory, cf. [32, 33, 34]. Of

ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE

particular importance, these techniques brought the necessary perspective for a remarkable new approach to the Cartan problem for free group factors in the work of Ozawa and Popa [27, 28]—an approach which this work directly builds upon.

A new von Neumann-algebraic approach to solidity was developed by Peterson in his important paper on L^2-rigidity [29]. Essentially, Peterson was able to exploit the "negative curvature" of the free group on two generators $\mathbb{F}_2$, in terms of a proper 1-cocycle into the left-regular representation, to rule out the existence of large relative commutants of diffuse subalgebras of $L\mathbb{F}_2$.

PETERSON'S SOLIDITY THEOREM ([29]). – *If Γ is an i.c.c. countable discrete group which admits a proper 1-cocycle $b : \Gamma \to \mathcal{H}_\pi$ for some unitary representation π which is weakly-ℓ^2 (i.e., weakly contained in the left-regular representation), then $L\Gamma$ is solid.*

It was later realized by the second author [39] that many of the explicit unbounded derivations (i.e., the ones constructed from 1-cocycles) that Peterson works with have natural dilations which are malleable deformations of their corresponding (group) von Neumann algebras.

However, the non-vanishing of 1-cohomology of Γ with coefficients in the left-regular representation does not reflect the full spectrum of negative curvature phenomena in geometric group theory as evidenced by the existence of non-elementary hyperbolic groups with Kazhdan's property (T), cf. [3]. In their fundamental works on the rigidity of group actions [17, 18], Monod and Shalom proposed a more inclusive cohomological definition of negative curvature in group theory which is given in terms of non-vanishing of the second-degree bounded cohomology for Γ with coefficients in the left-regular representation. Relying on Monod's work in bounded cohomology [16], we will make use of a related condition, which is the existence of a proper *quasi*-1-cocycle on Γ into the left-regular representation (more generally, into a representation weakly contained in the left-regular representation), cf. [16, 40]. By a result of Mineyev, Monod, and Shalom [15], this condition is satisfied for any hyperbolic group—the case of vanishing first ℓ^2-Betti number is due to Mineyev [14].

Statement of results

We now state the main results of the paper, in order to place them within the context of previous results in the structural theory of group von Neumann algebras. We begin with the motivating result of the paper, which unifies the solidity theorems of Ozawa and Peterson.

THEOREM A. – *Let Γ be an i.c.c. countable discrete group which is exact and admits a proper quasi-1-cocycle $q : \Gamma \to \mathcal{H}_\pi$ for some weakly-ℓ^2 unitary representation π (more generally, Γ is exact and belongs to the class $\mathcal{QH}_{\mathrm{reg}}$ of Definition 1.6). Then $L\Gamma$ is solid.*

In particular, all Gromov hyperbolic groups are exact, cf. [36], and admit a proper quasi-1-cocycle for the left-regular representation [15]. For the class of exact groups, belonging to the class $\mathcal{QH}_{\mathrm{reg}}$ is equivalent to bi-exactness (see Section 1), so the above result is equivalent to Ozawa's Solidity Theorem.

Following Ozawa's and Peterson's work on solidity, there was some hope that similar techniques could be used to approach to the Cartan subalgebra problem for group factors of hyperbolic groups, generalizing Voiculescu's celebrated theorem on the absence of

Cartan subalgebras for free group factors [43]. However, the Cartan problem for general hyperbolic groups would remain intractable until the breakthrough approach of Ozawa and Popa through Popa's deformation/rigidity theory resolved it in the positive for the group factor of any discrete group of isometries of the hyperbolic plane [28]. In fact, they were able to show that any such II$_1$ factor M is *strongly solid*, i.e., for every diffuse, amenable von Neumann subalgebra $A \subset M$, $\mathscr{N}_M(A)'' \subset M$ is an amenable von Neumann algebra, where $\mathscr{N}_M(A) = \{u \in \mathscr{U}(M) : uAu^* = A\}$.

Using the techniques developed by Ozawa and Popa [27, 28] and a recent result of Ozawa [25], we obtain the following strengthening of Theorem A.

THEOREM B. – *Let Γ be an i.c.c. countable discrete group which is weakly amenable (therefore, exact). If Γ admits a proper quasi-1-cocycle into a weakly-ℓ^2 representation, then $L\Gamma$ is strongly solid.*

Appealing to Ozawa's proof of the weak amenability of hyperbolic groups [23], Theorem B allows us to fully resolve in the positive the strong solidity problem—hence the Cartan problem—for i.c.c. hyperbolic groups and for lattices in connected rank one simple Lie groups. In particular, if Γ is an i.c.c. lattice in $\mathrm{Sp}(n, 1)$ or the exceptional group $\mathrm{F}_{4(-20)}$, then $L\Gamma$ is strongly solid. The strong solidity problem for the other rank one simple Lie groups— those locally isomorphic to $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$)—was resolved for $\mathrm{SO}(2, 1)$, $\mathrm{SO}(3, 1)$, and $\mathrm{SU}(1, 1)$ by the work of Ozawa and Popa [28] and, in the general case, by the work of the second author [39]. The results follow directly from Theorem B in the co-compact (i.e., uniform) case: in the non-uniform case, we must appeal to a result of Shalom (Theorem 3.7 in [38]) on the integrability of lattices in connected simple rank one Lie groups to produce a proper quasi-1-cocycle.

Building on Ioana's work on cocycle superrigidity [12], we are also able to obtain new examples of virtually W*-superrigid actions.

COROLLARY B.1. – *Let Γ be an i.c.c. countable discrete group which is weakly amenable and which admits a proper quasi-1-cocycle into a weakly-ℓ^2 representation. If $\Gamma \curvearrowright (X, \mu)$ is a profinite, free, ergodic measure-preserving action of Γ on a standard probability space (X, μ), then $L^\infty(X, \mu) \rtimes \Gamma$ has a unique Cartan subalgebra up to unitary conjugacy. If in addition Γ has Kazhdan's property (T) (e.g., Γ is a lattice in $\mathrm{Sp}(n, 1)$, $n \geq 2$), then any such action $\Gamma \curvearrowright (X, \mu)$ is virtually W*-superrigid.*

A natural question to ask is whether our techniques can be extended to demonstrate strong solidity of the group factor of any i.c.c. countable discrete group which is relatively hyperbolic [19] to a family of amenable subgroups.

The techniques used to prove Theorem B also allow us to deduce, by way of results of Cowling and Zimmer [8] and Ozawa [23], the following improvement of results of Adams (Corollary 6.2 in [1]) and of Monod and Shalom (Corollary 1.19 in [18]) on the structure of groups which are orbit equivalent to hyperbolic groups.

COROLLARY B.2. – *Let Γ be an i.c.c. countable discrete group which is weakly amenable and which admits a proper quasi-1-cocycle into a weakly-ℓ^2 representation. Let $\Gamma \curvearrowright (X, \mu)$ be a free, ergodic, measure-preserving action of Γ on a probability space and $\Lambda \curvearrowright (Y, \nu)$ be*

an arbitrary free, ergodic, measure-preserving action of some countable discrete group Λ on a probability space. If $\Gamma \curvearrowright (X, \mu)$ is orbit equivalent to $\Lambda \curvearrowright (Y, \nu)$, then Λ is not isomorphic to a non-trivial direct product of infinite groups and the normalizer of any infinite, amenable subgroup $\Sigma < \Lambda$ is amenable.

Beyond solidity results, we highlight that the techniques developed in this paper also enable us to reprove strong decomposition results for products of groups in the spirit of Popa's deformation/rigidity theory. Specifically, we are able to recover the following prime decomposition theorem of Ozawa and Popa.

THEOREM C (Ozawa and Popa [26]). – *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a non-trivial product of exact, i.c.c. countable discrete groups such that $\Gamma_i \in \mathcal{QH}_{\mathrm{reg}}, 1 \leq i \leq n$. If $N = N_1 \overline{\otimes} \cdots \overline{\otimes} N_m$ is a product of II_1 factors $N_j, 1 \leq j \leq m$, for some $m \geq n$, and $L\Gamma \cong N$, then $m = n$ and there exist $t_1, \ldots, t_n > 0$ with $t_1 \cdots t_n = 1$ so that, up to a permutation, $(L\Gamma_i)^{t_i} \cong N_i, 1 \leq i \leq n$.*

An advantage to our approach is that our proof naturally generalizes to unique measure-equivalence decomposition of products of bi-exact groups, first proven by Sako (Theorem 4 in [37]). This type of result was first achieved for products of groups of the class $\mathcal{C}_{\mathrm{reg}}$ by Monod and Shalom (Theorem 1.16 in [18]).

COROLLARY C (Sako [37]). – *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a non-trivial product of exact, i.c.c. countable discrete groups such that $\Gamma_i \in \mathcal{QH}_{\mathrm{reg}}, 1 \leq i \leq n$, and let $\Lambda = \Lambda_1 \times \cdots \times \Lambda_m$ be a product of arbitrary countably infinite discrete groups. Assume that $\Gamma \sim_{\mathrm{ME}} \Lambda$, i.e., there exist $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ free, ergodic, probability measure-preserving actions which are weakly orbit equivalent (Definition 2.2 in [9]). If $m \geq n$ then $m = n$ and, up to permuting indices, we have that $\Gamma_i \sim_{\mathrm{ME}} \Lambda_i, 1 \leq i \leq n$.*

On the method of proof

This paper began as an attempt to chart a "middle path" between the solidity theorems of Ozawa, Popa, and Peterson by recasting Ozawa's approach to solidity effectively as a deformation/rigidity argument. We did so by finding a "cohomological" characterization of Ozawa's notion of bi-exactness [22]. Interestingly, our reformulation of bi-exactness has many affinities with (strict) cohomological definitions of negative curvature proposed by Monod and Shalom [18] and Thom [40].

Working from the cohomological perspective, we were able to construct "deformations" of $L\Gamma$. Though these "deformations" no longer mapped $L\Gamma$ into itself, we were still able to control their convergence on a weakly dense C^*-subalgebra of $L\Gamma$ namely, the reduced group C^*-algebra $C^*_\lambda(\Gamma)$, then borrow Ozawa's insight of using local reflexivity to pass from $C^*_\lambda(\Gamma)$ to the entire von Neumann algebra.

After this initial undertaking had been completed, we turned our attention to applying these techniques to the foundational methods which Ozawa and Popa developed in proving strong solidity of free group factors. Our approach through deformation/rigidity-type arguments allowed us to exploit the "compactness" of deformations coming from quasi-cocycles to achieve a finer degree of control than is afforded by the use of bi-exactness. This extra control was crucial in our adaptation of Ozawa and Popa's fundamental techniques in the proof of Theorem B. This should be considered the main technical advance of the paper.

1. Cohomological-type properties and negative curvature

Let Γ be a countable discrete group. Recall that a *length function* $|\cdot| : \Gamma \to \mathbb{R}_{\geq 0}$ is a map satisfying: (1) $|\gamma| = 0$ if and only if $\gamma = e$ is the identity; (2) $|\gamma^{-1}| = |\gamma|$, for all $\gamma \in \Gamma$; and (3) $|\gamma\delta| \leq |\gamma| + |\delta|$, for all $\gamma, \delta \in \Gamma$.

1.1. Arrays

We introduce a general class of embeddings of a group Γ into Hilbert space that are compatible with some action of Γ by orthogonal transformations, which we refer to as arrays. These "arrays" distill the essential structural properties of proper affine isometric actions while adding a substantial amount of "geometric" flexibility. In fact, the simplest example of an array will be a length function, which can be thought of as taking values in the trivial orthogonal representation.

DEFINITION 1.1. – Let $\pi : \Gamma \to \mathcal{O}(\mathscr{H}_\pi)$ be an orthogonal representation of a countable discrete group Γ and let $\mathscr{G}$ be a family of subgroups of Γ. A map $q : \Gamma \to \mathscr{H}_\pi$ is called an *array* for every finite subset $F \subset \Gamma$ if there exists $K \geq 0$ such that

$$(1.1) \qquad \|\pi_\gamma(q(\delta)) - q(\gamma\delta)\| \leq K,$$

for all $\gamma \in F$, $\delta \in \Gamma$ (i.e., q is *boundedly equivariant*). It is an easy exercise to show that for any array q there exists a length function on Γ which bounds $\|q(\gamma)\|$ from above. An array $q : \Gamma \to \mathscr{H}_\pi$ is said to be:

- *proper with respect to* $\mathscr{G}$ if the map $\gamma \mapsto \|q(\gamma)\|$ is proper with respect to the family $\mathscr{G}$, i.e., for all $C > 0$ there exist finite subsets $G, H \subset \Gamma$, $\mathscr{K} \subset \mathscr{G}$ such that

$$\{\gamma \in \Gamma : \|q(\gamma)\| \leq C\} \subseteq G\mathscr{K}H.$$

  If $\mathscr{G} = \{\{e\}\}$, then this is the usual notion of metric properness, in which case the map q itself is referred to as *proper*;
- *symmetric* if $\pi_\gamma(q(\gamma^{-1})) = q(\gamma)$ for all $\gamma \in \Gamma$;
- *anti-symmetric* if $\pi_\gamma(q(\gamma^{-1})) = -q(\gamma)$ for all $\gamma \in \Gamma$; and
- *uniform* if there exist a proper length function $|\cdot|$ on Γ and an increasing function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\rho(t) \to \infty$ as $t \to \infty$ and such that

$$\rho(|\gamma^{-1}\delta|) \leq \|q(\gamma) - q(\delta)\|,$$

  for all $\gamma, \delta \in \Gamma$.

REMARK 1.2. – In the preceding definition we could as well have relaxed the condition of strict (anti-)symmetry to merely the condition that $\|\pi_\gamma(q(\gamma^{-1})) \pm q(\gamma)\|$ is bounded. However, it is easy to check that for any such function q, there exists an array $\tilde{q}$ which is a bounded distance from q; namely, $\tilde{q}(\gamma) = \frac{1}{2}(q(\gamma) \pm \pi_\gamma(q(\gamma^{-1})))$. This observation is essentially due to Andreas Thom [40].

It is easy to see that a length function on a group is a uniform, symmetric array for the trivial representation. Our primary examples of (uniform) anti-symmetric arrays will be quasi-1-cocycles.

DEFINITION 1.3. – Let Γ be a countable discrete group and $\pi : \Gamma \to \mathcal{O}(\mathcal{H}_\pi)$ be an orthogonal representation of Γ on a real Hilbert space $\mathcal{H}_\pi$. A map $q : \Gamma \to \mathcal{H}_\pi$ is called a *quasi-1-cocycle* for the representation π if one can find a constant $K \geq 0$ such that for all $\gamma, \lambda \in \Gamma$ we have

$$(1.2) \qquad \|q(\gamma\lambda) - q(\gamma) - \pi_\gamma(q(\lambda))\| \leq K.$$

We denote by $D(q)$ the *defect* of the quasi-1-cocycle q, which is the infimum of all K satisfying Equation (1.2). Notice that when the defect is zero the quasi-1-cocycle q is actually a 1-cocycle for π [3]. In the sequel, we will drop the "1" and refer to (quasi-)1-cocycles as (quasi-)cocycles. Again, without (much) loss of generality we will require a quasi-cocycle q to be anti-symmetric, since every quasi-cocycle q is a bounded distance from some anti-symmetric quasi-cocycle $\tilde{q}$, cf. [40], which will suffice for our purposes.

A distinct advantage to working with arrays is that, unlike cocycles (or even quasi-cocycles), there is a well-defined notion of a tensor product.

PROPOSITION 1.4. – *Let Γ be a countable discrete group. Let $\pi_i : \Gamma \to \mathcal{O}(\mathcal{H}_i)$ be an orthogonal representation for $i = 1, 2$, and let $q_i : \Gamma \to \mathcal{H}_i$ be an array for π_i. Set*

$$\kappa(\gamma) = \max_{i=1,2} \|q_i(\gamma)\| + 1$$

for all $\gamma \in \Gamma$. Then the map $q_1 \wedge q_2 : \Gamma \to \mathcal{H}_1 \otimes \mathcal{H}_2$ defined by

$$(1.3) \qquad q_1 \wedge q_2(\gamma) = \kappa(\gamma)^{-1} q_1(\gamma) \otimes q_2(\gamma)$$

is an array into the tensor representation $\pi_1 \otimes \pi_2$. Moreover, if the arrays q_i are assumed to be symmetric, then $q_1 \wedge q_2$ is symmetric. If each of the arrays q_i is assumed to be proper relative to a given family $\mathcal{G}_i$ of subgroups of Γ then $q_1 \wedge q_2$ is proper with respect to the family $\mathcal{G}_1 \cup \mathcal{G}_2$.

Proof. – For brevity, we will denote $\pi_1 \otimes \pi_2$ as π and $q_1 \wedge q_2$ as q. First, we show that if q_1, q_2 are arrays then one can find a function $r : \Gamma \to \mathbb{R}_+$ such that for all $\gamma, \lambda \in \Gamma$, we have

$$(1.4) \qquad |\kappa(\gamma\lambda) - \kappa(\lambda)| \leq r(\gamma).$$

Indeed, by applying the triangle inequality, we see that

$$(1.5) \qquad \begin{aligned} \kappa(\gamma\lambda) &= \max\{\|q_1(\gamma\lambda)\|, \|q_2(\gamma\lambda)\|\} + 1 \\ &\leq \max\{\|q_1(\lambda)\|, \|q_2(\lambda)\|\} + 1 \\ &\quad + \max\{\|q_1(\gamma\lambda) - (\pi_1)_\gamma(q_1(\lambda))\|, \|q_2(\gamma\lambda) - (\pi_2)_\gamma(q_2(\lambda))\|\} \\ &\leq \kappa(\lambda) + \max\{\|q_1(\gamma\lambda) - (\pi_1)_\gamma(q_1(\lambda))\|, \|q_2(\gamma\lambda) - (\pi_2)_\gamma(q_2(\lambda))\|\}. \end{aligned}$$

Since q_1 and q_2 are arrays, there exists a function $\gamma \mapsto r'(\gamma)$ such that $\max\{\|q_1(\gamma\lambda) - (\pi_1)_\gamma(q_1(\lambda))\|, \|q_2(\gamma\lambda) - (\pi_2)_\gamma(q_2(\lambda))\|\} \leq r'(\gamma)$ for all $\lambda \in \Gamma$. Using this notation (1.5) can be rephrased as $\kappa(\gamma\lambda) \leq r'(\gamma) + \kappa(\lambda)$ for all $\gamma, \lambda \in \Gamma$. This implies that $\kappa(\lambda) = \kappa(\gamma^{-1}\gamma\lambda) \leq r'(\gamma^{-1}) + \kappa(\gamma\lambda)$ for all $\gamma, \lambda \in \Gamma$. Therefore, when combining the last two inequalities we conclude that

$$|\kappa(\gamma\lambda) - \kappa(\lambda)| \leq \max\{r'(\gamma), r'(\gamma^{-1})\},$$

for all $\gamma, \lambda \in \Gamma$. Letting $r(\gamma) = \max\{r'(\gamma), r'(\gamma^{-1})\}$ we obtain (1.4).

Next, we will show that q is an array; that is, that q is boundedly equivariant. Applying the triangle inequality, we have the following estimates:

$$\|q(\gamma\lambda) - \pi_\gamma(q(\lambda))\| = \|\kappa(\gamma\lambda)^{-1}q_1(\gamma\lambda) \otimes q_2(\gamma\lambda) - \kappa(\lambda)^{-1}(\pi_1)_\gamma(q_1(\lambda)) \otimes (\pi_2)_\gamma(q_2(\lambda))\|$$
$$\leq \kappa(\gamma\lambda)^{-1}\|(q_1(\gamma\lambda) - (\pi_1)_\gamma(q_1(\lambda))) \otimes q_2(\gamma\lambda)\|$$
$$+ |\kappa(\gamma\lambda)^{-1} - \kappa(\lambda)^{-1}|\,\|\pi_\gamma(q_1(\lambda)) \otimes q_2(\gamma\lambda)\|$$
$$+ \kappa(\lambda)^{-1}\|(\pi_1)_\gamma(q_1(\lambda)) \otimes (q_2(\gamma\lambda) - (\pi_2)_\gamma(q_2(\lambda)))\|$$
$$\leq \|q_1(\gamma\lambda) - (\pi_1)_\gamma(q_1(\lambda))\| + \|q_2(\gamma\lambda) - (\pi_2)_\gamma(q_2(\lambda))\| + |\kappa(\gamma\lambda) - \kappa(\lambda)|$$
$$\leq \|q_1(\gamma\lambda) - (\pi_1)_\gamma(q_1(\lambda))\| + \|q_2(\gamma\lambda) - (\pi_2)_\gamma(q_2(\lambda))\| + |r(\gamma)|.$$

Since q_i is boundedly equivariant, $i = 1, 2$, the previous inequality combined with (1.4) shows that q is boundedly equivariant.

From the definitions, one can easily see that if each q_i is symmetric, then q is again symmetric.

Finally, we verify the properness condition. Let $C > 0$ be a fixed arbitrary constant and set $K = \{\gamma \in \Gamma : \|q(\gamma)\| \leq C\}$. A straightforward computation shows that if $\gamma \in K$ then either $\|q_1(\gamma)\| \leq C$ or $\|q_2(\gamma)\| \leq C$. Since each q_i is proper relative to G_i, so there exist finite sets $G_1, G_2, H_1, H_2 \subset \Gamma$, $K_1 \subset \mathscr{G}_1$, and $K_2 \subset \mathscr{G}_2$ such that $K \subseteq G_1 K_1 H_1 \cup G_2 K_2 H_2$. Since this holds for all $C > 0$, we have obtained that q is proper relative to $\mathscr{G}_1 \cup \mathscr{G}_2$. $\qquad\square$

PROPOSITION 1.5. – *Let $\pi : \Gamma \to \mathcal{O}(\mathcal{H}_\pi)$ be an orthogonal representation. Assume that Γ admits a proper array $q : \Gamma \to \mathcal{H}_\pi$ for π which is boundedly bi-equivariant, i.e., $\|q(\gamma\delta\lambda) - \pi_\gamma q(\delta)\| < C(\gamma, \lambda)$, for all $\gamma, \delta, \lambda \in \Gamma$. Here, $C(\gamma, \lambda)$ denotes a constant only depending on γ and λ. Then there exists a symmetric proper array $\tilde{q} : \Gamma \to \mathcal{H}_\pi \otimes \mathcal{H}_\pi$ for the diagonal representation $\pi \otimes \pi$.*

Proof. – We begin by observing that if q is proper and boundedly bi-equivariant, then the map $q' : \Gamma \to \mathcal{H}_\pi$ defined by $q'(\gamma) = \pi_\gamma(q(\gamma^{-1}))$ is also boundedly equivariant and obviously proper.

Indeed, to see this we note that by definition we have

$$\|q'(\gamma\delta) - \pi_\gamma(q'(\delta))\| = \|q(\delta^{-1}\gamma^{-1}) - q(\delta^{-1})\| \leq C(e, \gamma^{-1}).$$

Notice that the above constant depends only on γ.

Now we consider $\tilde{q} : \Gamma \to \mathcal{H}_\pi \otimes \mathcal{H}_\pi$ to be the symmetric product of the boundedly equivariant maps $\tilde{q}(\gamma) = \frac{1}{2}(q \wedge q'(\gamma) + q' \wedge q(\gamma))$, which is also boundedly equivariant from the previous proposition. It is a straightforward exercise to check that $\tilde{q}$ is symmetric, i.e., $(\pi \otimes \pi)_\gamma(\tilde{q}(\gamma^{-1})) = \tilde{q}(\gamma)$.

Finally, since the square of the norm of the symmetric product of two vectors x and y is $\|x\|^2\|y\|^2 + |\langle x, y \rangle|^2$, we have that q is proper implies that $\tilde{q}$ is also proper. $\square$

1.2. The classes $\mathcal{QH}$ and $\mathcal{QH}_{\mathrm{reg}}$

We now proceed to describe some "cohomological" properties of countable discrete groups which capture many aspects of negative curvature from the perspective of representation theory.

DEFINITION 1.6. – We say that a countable discrete group Γ is in the class $\mathcal{QH}$ if it admits a proper, symmetric array $q : \Gamma \to \mathcal{H}_\pi$ for some non-amenable unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$. If the representation π can be chosen to be weakly-ℓ^2, then we say that Γ belongs to the class $\mathcal{QH}_{\mathrm{reg}}$.

By Proposition 1.5 we see that the class $\mathcal{QH}_{\mathrm{reg}}$ generalizes the class $\mathcal{D}_{\mathrm{reg}}$ of Thom [40] and that the class $\mathcal{QH}$ contains all groups having Ozawa and Popa's property (HH) [28].

PROPOSITION 1.7. – *The following statements are true.*

1. *If $\Gamma_1, \Gamma_2 \in \mathcal{QH}$, then so are $\Gamma_1 \times \Gamma_2$ and $\Gamma_1 * \Gamma_2$.*
2. *If $\Gamma_1, \Gamma_2 \in \mathcal{QH}_{\mathrm{reg}}$, then $\Gamma_1 * \Gamma_2 \in \mathcal{QH}_{\mathrm{reg}}$.*
3. *If $\Gamma_1, \Gamma_2 \in \mathcal{QH}_{\mathrm{reg}}$ are non-amenable, then $\Gamma_1 \times \Gamma_2 \notin \mathcal{QH}_{\mathrm{reg}}$.*
4. *If Γ is a lattice in a simple connected Lie group with real rank one, then $\Gamma \in \mathcal{QH}_{\mathrm{reg}}$.*
5. *If $\Gamma \in \mathcal{QH}$, then Γ is not inner amenable. If in addition Γ is weakly amenable, then Γ has no infinite normal amenable subgroups.*

Statement (5) is essentially Proposition 2.1 in [28] combined with Theorem A in [25].

Proof. – Statements (1) and (2) follow exactly as they do for groups which admit a proper cocycle into some non-amenable (respectively, weakly ℓ^2) unitary representation, cf. [40].

We prove statement (3) under the weaker assumption that $\Gamma \cong \Lambda \times \Sigma$, where Λ is non-amenable and Σ is an arbitrary infinite group. By contradiction, assume Γ admits a proper, symmetric, boundedly equivariant map $q : \Gamma \to \ell^2(\Gamma)$ (by inspection, the same argument will hold for $q : \Gamma \to \mathcal{H}_\pi$ for any weakly-ℓ^2 unitary representation π). Since the action of Λ on $\ell^2(\Gamma)$ has spectral gap and admits no non-zero invariant vectors, there exist a finite, symmetric subset $S \subset \Lambda$ and $K' > 0$ such that

$$\|\xi\| \leq K' \sum_{s \in S} \|\lambda_s(\xi) - \xi\|,$$

for all $\xi \in \ell^2(\Gamma)$. Let $K'' \geq 0$ be a constant so that inequality (1.1) is satisfied for $S \subset \Gamma$, and set $K = \max\{K', K''\}$. We then have for any $g \in \Sigma$ that

$$(1.6) \qquad \|q(g)\| \leq K \sum_{s \in S} \|\lambda_s(q(g)) - q(g)\|$$

$$\leq K \sum_{s \in S} \|q(sg) - q(g)\| + K^2 |S|$$

$$= K \sum_{s \in S} \|q(gs) - q(g)\| + K^2 |S|$$

$$= K \sum_{s \in S} \|\lambda_{gs}(q(s^{-1}g^{-1})) - \lambda_g(q(g^{-1}))\| + K^2|S|$$

$$= K \sum_{s \in S} \|\lambda_{s^{-1}}(q(g^{-1})) - q(s^{-1}g^{-1})\| + K^2|S| \leq 2K^2|S|.$$

Hence, $\|q(g)\|$ is bounded on Σ, which contradicts that q is proper.

For statement (4), it is well known that any co-compact lattice in a simple Lie group with real rank one is Gromov hyperbolic; hence, by [15] it admits a proper quasi-cocycle into the left-regular representation. A result of Shalom, Theorem 3.7 in [38], shows that any lattice in such a Lie group is integrable, and therefore ℓ^1-measure equivalent to any other lattice in the same Lie group. It is easy to check that having a proper quasi-cocycle into the left-regular representation is invariant under ℓ^1-measure equivalence, cf. Theorem 5.10 in [40].

In order to prove statement (5), we assume by contradiction that Γ is inner amenable, i.e., there exists a state φ on $\ell^\infty(\Gamma)$ such that $\varphi \perp \ell^1(\Gamma)$ and $\varphi \circ \mathrm{Ad}(\gamma) = \varphi$ for all $\gamma \in \Gamma$. Let $q : \Gamma \to \mathscr{H}_\pi$ be an array into a non-amenable representation π. Define a u.c.p. map $T : \mathfrak{B}(\mathscr{H}_\pi) \to \ell^\infty(\Gamma)$ by $T(x)(\gamma) = \frac{1}{\|q(\gamma)\|^2}\langle xq(\gamma), q(\gamma)\rangle$. Similarly to the proof of statement (3), by symmetry and bounded equivariance, for every $\gamma \in \Gamma$, there exists $K \geq 0$ such that

$$(1.7) \qquad \|q(\gamma^{-1}\delta\gamma) - \pi_{\gamma^{-1}}(q(\delta))\| \leq K,$$

for all $\delta \in \Gamma$. Since q is proper, this implies that the state $\Phi = \varphi \circ T$ on $\mathfrak{B}(\mathscr{H}_\pi)$ is $\mathrm{Ad}(\pi)$-invariant. However, this contradicts the fact that π is a non-amenable representation. The remaining assertion follows by Theorem A in [25]. □

The class $Q\mathscr{H}_{\mathrm{reg}}$ is intimately related with Ozawa's class of bi-exact groups (often denoted as the class $\mathcal{S}$ in the literature, e.g., [22, 37]). A reader unfamiliar with the theory of exact groups should consult Appendix A before proceeding further.

DEFINITION 1.8 (Ozawa [4, 22]). – A countable discrete group Γ is said to be *bi-exact* if it admits a sequence $\xi_n : \beta'\Gamma \to \ell^2(\Gamma)$ of continuous maps such that $\xi_n \geq 0$, $\|\xi_n(x)\|_2 = 1$, for all $x \in \beta'\Gamma$, $n \in \mathbb{N}$, which satisfy

$$(1.8) \qquad \sup_{x \in \beta'\Gamma} \|\lambda_\gamma(\xi_n(x)) - \xi_n(\gamma x \delta)\|_2 \to 0,$$

for all $\gamma, \delta \in \Gamma$. Here $\beta'\Gamma = \beta\Gamma \setminus \Gamma$ denotes the Stone–Čech boundary.

It is easy to see that if Γ is bi-exact in the sense of Definition 15.1.2 of [4] if and only if Γ is bi-exact in the sense of the above definition. By the same proof that "property A $\Rightarrow$ coarse embeddability into Hilbert space" (cf. [4, 36]), we have the following

PROPOSITION 1.9. – *If Γ is bi-exact, then it admits a uniform array into $\ell^2(\Gamma)^{\oplus\infty}$. In particular, Γ is exact and belongs to the class $Q\mathscr{H}_{\mathrm{reg}}$.*

Indeed, from the maps ξ_n, one may construct a proper, boundedly bi-equivariant map into $\ell^2(\Gamma)^{\oplus\infty}$, which may in turn be used to construct a proper, symmetric array by Proposition 1.5.

REMARK 1.10. – After a preliminary version of this manuscript was circulated, Narutaka Ozawa pointed out that the converse is also true. That is, if a countable discrete group Γ is exact and belongs to the class $\mathcal{QH}_{\mathrm{reg}}$, then Γ is bi-exact. A proof for the special case of the left-regular representation is contained in [6]: the general case may be found in [35].

The class $\mathcal{QH}_{\mathrm{reg}}$ is strictly larger than the class $\mathcal{D}_{\mathrm{reg}}$ of Thom [40]. This follows from Ozawa's proof that the group $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ is bi-exact [24], in conjunction with a theorem of Burger and Monod [5] demonstrating that $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ admits no proper quasi-cocycle for any representation. However, it is instructive to supply a direct proof without appealing to bi-exactness.

PROPOSITION 1.11. – *The group $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ is in the class $\mathcal{QH}_{\mathrm{reg}}$.*

The details of the construction are found in Appendix B.

2. Deformations of the uniform Roe algebra

2.1. Schur multipliers and the uniform Roe algebra

Using exponentiation, we now describe a canonical way to associate to an array $q : \Gamma \to \mathcal{H}_\pi$ a family of multipliers $\mathfrak{m}_t$ on $\mathfrak{B}(\ell^2(\Gamma))$. First notice that the kernel $(\gamma, \delta) \mapsto \|q(\gamma) - q(\delta)\|^2$ is conditionally negative definite (cf. Section 11.2 in [36] or Appendix D in [4]) and therefore by Schoenberg's theorem [36], for every $t \in \mathbb{R}$, the kernel $\kappa_t(\gamma, \delta) = \exp(-t^2\|q(\gamma) - q(\delta)\|^2)$ is positive definite. Hence for every t there is a unique unital, completely positive (u.c.p.) map $\mathfrak{m}_t : \mathfrak{B}(\ell^2(\Gamma)) \to \mathfrak{B}(\ell^2(\Gamma))$ called a *Schur multiplier*, such that

$$(2.1) \qquad\qquad \mathfrak{m}_t([x_{\gamma,\delta}]) = [\kappa_t(\gamma, \delta)x_{\gamma,\delta}],$$

for all $x \in \mathfrak{B}(\ell^2(\Gamma))$.

If Γ is a group then the *uniform Roe algebra* $C_u^*(\Gamma)$ is defined as the C^*-subalgebra of $\mathfrak{B}(\ell^2(\Gamma))$ generated by $C_\lambda^*(\Gamma)$ and $\ell^\infty(\Gamma)$. Notice that if one considers the action $\Gamma \curvearrowright^\lambda \ell^\infty(\Gamma)$ by left translation, then the uniform algebra $C_u^*(\Gamma)$ can be canonically identified with the reduced crossed product C^*-algebra $\ell^\infty(\Gamma) \rtimes_{\lambda,r} \Gamma$. Let F_0 denote the net of unital, symmetric, finite subsets of Γ. Given a finite subset $F \in F_0$, we define the operator space of *F-width operators* $X(F)$ to be the space of bounded operators $x \in \mathfrak{B}(\ell^2(\Gamma))$ such that $x_{\gamma,\delta} = 0$ whenever $\gamma^{-1}\delta \in \Gamma \setminus F$. Since it is easy to check that $X(F) = \ell^\infty(\Gamma)\mathbb{C}[F]\ell^\infty(\Gamma)$, we have $C_u^*(\Gamma) = \overline{\mathrm{span}\{X(F) : F \in F_0\}}^{\|\cdot\|}$. Our interest in the uniform Roe algebra stems from the fact that it is, in practical terms, the smallest C^*-algebra which contains $C_\lambda^*(\Gamma)$ and which is invariant under the class of Schur multipliers associated to Γ.

PROPOSITION 2.1. – *The algebra $C_u^*(\Gamma)$ is invariant under $\mathfrak{m}_t$.*

Proof. – Let $x \in C_u^*(\Gamma)$, then there exists a sequence (x_n) of elements of $\mathrm{span}\{X(F) : F \in F_0\}$ such that $\|x_n - x\|_\infty \to 0$. It is easy to see that if x_n is supported on the set $F_n \in F_0$, then $\mathfrak{m}_t(x_n) \in X(F_n)$. Since $\|\mathfrak{m}_t(x_n) - \mathfrak{m}_t(x)\|_\infty \to 0$, we have that $\mathfrak{m}_t(x) \in C_u^*(\Gamma)$. $\qquad\square$

We will also heavily use the following observation of Roe on the convergence of $\mathfrak{m}_t$ on $C_u^*(\Gamma)$, cf. Lemma 4.27 in [36].

PROPOSITION 2.2. – *If q is an array, then for all $x \in C_u^*(\Gamma)$, we have that $\|\mathfrak{m}_t(x) - x\|_\infty \to 0$ as $t \to 0$.*

2.2. Construction of the extended Roe algebra $C_u^*(\Gamma \curvearrowright^\rho Z)$.

Let $\Gamma \curvearrowright^\rho (Z, \eta)$ be a measure preserving action on a probability space Z. By abuse of notation, we still denote by ρ the Koopman representation of Γ on $L^2(Z, \eta)$ induced by the action ρ. Then consider the Hilbert space $L^2(Z, \eta) \otimes \ell^2(\Gamma)$ and for every $\gamma \in \Gamma$ define a unitary $u_\gamma \in \mathfrak{B}(L^2(Z) \otimes \ell^2(\Gamma))$ by the formula

$$u_\gamma(\xi \otimes \delta_h) = \rho_\gamma(\xi) \otimes \delta_{\gamma h},$$

where $\xi \in L^2(Z)$ and $h \in \Gamma$. Consider the algebra $L^\infty(Z \times \Gamma, \eta \times c) \subset \mathfrak{B}(L^2(Z) \otimes \ell^2(\Gamma))$, where c is the counting measure on Γ. Then the *extended Roe algebra $C_u^*(\Gamma \curvearrowright^\sigma Z)$* is defined as the C^*-algebra generated by $L^\infty(Z \times \Gamma)$ and the unitaries u_γ inside $\mathfrak{B}(L^2(Z) \otimes \ell^2(\Gamma))$. Notice that when X consists of a point, our definition recovers the regular uniform Roe algebra, i.e., $C_u^*(\Gamma \curvearrowright^\rho Z) = C_u^*(\Gamma)$.

As in the case of the uniform Roe algebra, we will see that $C_u^*(\Gamma \curvearrowright^\rho Z)$ can be realized as a reduced crossed product algebra. Specifically, we consider the action $\Gamma \curvearrowright L^\infty(Z \times \Gamma)$ given by $\lambda_\gamma^\rho(f)(x, h) = f(\gamma^{-1}x, \gamma^{-1}h)$, where $f \in L^\infty(Z \times \Gamma)$, $x \in Z$ and $\gamma, h \in \Gamma$. Then we show that $C_u^*(\Gamma \curvearrowright^\rho Z)$ is naturally identified with the reduced crossed product algebra corresponding to this action and the faithful representation $L^\infty(Z \times \Gamma) \subset \mathfrak{B}(L^2(Z) \otimes \ell^2(\Gamma))$.

PROPOSITION 2.3. – $C_u^*(\Gamma \curvearrowright^\rho Z) \cong L^\infty(Z \times \Gamma) \rtimes_{\lambda^\rho, r} \Gamma$.

Proof. – Consider the operator $U : L^2(Z) \otimes \ell^2(\Gamma) \otimes \ell^2(\Gamma) \to L^2(Z) \otimes \ell^2(\Gamma) \otimes \ell^2(\Gamma)$ defined by $U(\xi \otimes \delta_k \otimes \delta_h) = \sigma_h(\xi) \otimes \delta_k \otimes \delta_{hk}$, where $\xi \in L^2(Z)$ and $\gamma, h \in \Gamma$. One can easily check this is a unitary, and below we will show it implements a spatial isomorphism between the two algebras. For this purpose we will be seeing $C_u^*(\Gamma \curvearrowright Z)$ as the C^*-algebra generated by $L^\infty(Z \times \Gamma)$ and u_γ inside $\mathfrak{B}(L^2(Z) \otimes \ell^2(\Gamma) \otimes \ell^2(\Gamma))$, where

$$(2.2) \quad \begin{aligned} f(\xi \otimes \delta_k \otimes \delta_h) &= (f(\,\cdot\,, h)\xi) \otimes \delta_k \otimes \delta_h \\ u_\gamma(\xi \otimes \delta_k \otimes \delta_h) &= \rho_\gamma(\xi) \otimes \delta_k \otimes \delta_{\gamma h}, \end{aligned}$$

for all $f \in L^\infty(Z \times \Gamma)$, $\gamma, h, k \in \Gamma$, and $\xi \in L^2(Z)$.

Using the formula for U in combination with Equations (2.2), we have

$$\begin{aligned} U(1 \otimes \lambda_\gamma)(\xi \otimes \delta_k \otimes \delta_h) &= U(\xi \otimes \delta_k \otimes \delta_{\gamma h}) \\ &= \rho_{\gamma h}(\xi) \otimes \delta_k \otimes \delta_{\gamma h k} \\ &= u_\gamma(\rho_h(\xi) \otimes \delta_k \otimes \delta_{hk}) \\ &= u_\gamma U(\xi \otimes \delta_k \otimes \delta_h); \end{aligned}$$

hence, $U(1 \otimes \lambda_\gamma)U^* = u_\gamma$ for all $\gamma \in \Gamma$.

We also consider the representation of $L^\infty(Z \times \Gamma)$ on $\mathfrak{B}(L^2(Z) \otimes \ell^2(\Gamma) \otimes \ell^2(\Gamma))$ given by $\pi(f)(\xi \otimes \delta_k \otimes \delta_h) = \lambda_{h^{-1}}^\rho(f)(\xi \otimes \delta_k) \otimes \delta_h$, for every $f \in L^\infty(Z \times \Gamma)$.

Combining this with Equations (2.2) we see that

$$
\begin{aligned}
U\pi(f)(\xi \otimes \delta_k \otimes \delta_h) &= U((\lambda_{h^{-1}}^{\rho}(f)(\xi \otimes \delta_k) \otimes \delta_h) \\
&= U(\rho_{h^{-1}}(f(\cdot, hk))\xi \otimes \delta_k \otimes \delta_h) \\
&= f(\cdot, hk)\rho_h(\xi) \otimes \delta_k \otimes \delta_{h\gamma} \\
&= fU(\xi \otimes \delta_k \otimes \delta_h).
\end{aligned}
$$

Therefore, for all $f \in L^\infty(Z \times \Gamma)$ we have $U\pi(f)U^* = f$, and from the discussion above we conclude that $U(L^\infty(Z \times \Gamma) \rtimes_{\lambda^\rho, r} \Gamma)U^* = C_u^*(\Gamma \curvearrowright Z)$. $\qquad\square$

For further reference we keep in mind the following diagram of canonical inclusions:

$$
L^\infty(Z) \rtimes_{\rho, r} \Gamma \subset L^\infty(Z \times \Gamma) \rtimes_{\lambda^\rho, r} \Gamma = C_u^*(\Gamma \curvearrowright^\rho Z)
$$

(2.3)
$$
\cup \qquad\qquad \cup
$$
$$
C_r^*(\Gamma) \quad \subset \quad \ell^\infty(\Gamma) \rtimes_{\lambda, r} \Gamma \quad = \quad C_u^*(\Gamma).
$$

Note there exists a conditional expectation $E : L^\infty(Z \times \Gamma) \to \ell^\infty(\Gamma)$ defined by $E(f)(\gamma) = \int_Z f(x, \gamma)d\mu(x)$. This map is clearly Γ-equivariant and thus it extends to a conditional expectation $\tilde{E} : C_u^*(\Gamma \curvearrowright Z) \to C_u^*(\Gamma)$ by letting $\tilde{E}(\sum_\gamma x_\gamma u_\gamma) = \sum_\gamma E(x_\gamma)u_\gamma$ for any $\sum_\gamma x_\gamma u_\gamma \in C_u^*(\Gamma \curvearrowright Z)$ with $x_\gamma \in L^\infty(Z \times \Gamma)$.

2.3. A path of automorphisms of the extended Roe algebra associated with the Gaussian action

Let $\Gamma \curvearrowright^\sigma (X, \mu)$ be a measure preserving action of Γ on a probability space X. Following [30], any orthogonal group representation $\pi : \Gamma \to \mathcal{O}(\mathcal{H}_\pi)$ gives rise to a measure-preserving action which we still denote by $\Gamma \curvearrowright^\pi (Y^\pi, \nu^\pi)$, called the *Gaussian action*. We consider the diagonal action $\Gamma \curvearrowright^{\sigma \otimes \pi} (X \times Y^\pi, \mu \times \nu^\pi)$, which will be denoted by $\Gamma \curvearrowright^\rho (Z, \zeta)$. As in the previous subsection, to this action we associate the extended Roe algebra $C_u^*(\Gamma \curvearrowright^\rho Z)$. Below we indicate a procedure to construct a one-parameter family $(\alpha_t)_{t \in \mathbb{R}}$ of $*$-automorphisms of $C_u^*(\Gamma \curvearrowright^\rho Z)$. Specifically, α_t is obtained by exponentiating an array $q : \Gamma \to \mathcal{H}_\pi$ in a similar way to the construction of the malleable deformation of $L\Gamma$ from a cocycle b as carried out in §3 of [39]. Crucially, this family will be continuous with respect to the uniform norm as $t \to 0$ (Lemma 2.6).

Following the construction presented in §1.2 of [39], given an array $q : \Gamma \to \mathcal{H}_\pi$, there exists a one-parameter family of maps $\upsilon_t : \Gamma \to \mathcal{U}(L^\infty(Y^\pi, \nu^\pi))$ defined by $\upsilon_t(\gamma)(x) = \exp(itq(\gamma)(x))$, where $\gamma \in \Gamma$, $x \in Y^\pi$. Using similar computations as in [30, 39] one can verify that we have the following properties:

PROPOSITION 2.4. – *We have the following:*

(2.4)   *If π is weakly-ℓ^2 then the Koopman representation $\pi_{\sigma | L_0^2(Y^\pi, \nu^\pi)}$ is also weakly-ℓ^2;*

(2.5) $\int \upsilon_t(\gamma)(x)\upsilon_t(\delta)^*(x)d\mu(x) = \kappa_t(\gamma, \delta)$ *for all $\gamma, \delta \in \Gamma$.*

These maps give rise naturally to a path of operators $V_t \in \mathfrak{B}(L^2(Y^\pi) \otimes L^2(X) \otimes \ell^2(\Gamma))$ by letting $V_t(\xi \otimes \eta \otimes \delta_\gamma) = (\upsilon_t(\gamma)\xi) \otimes \eta \otimes \delta_\gamma$, for every $\xi \in L^2(Y^\pi)$, $\eta \in L^2(X)$ and $\gamma \in \Gamma$. For further reference we summarize below some basic properties of V_t.

PROPOSITION 2.5. – *For every $t, s \in \mathbb{R}$ we have the following properties:*

1. $V_t V_s = V_{t+s}$, $V_t V_t^* = V_t^* V_t = 1$;
2. *If the array is anti-symmetric we have* $JV_t J = V_t$ *and if it is symmetric we have* $JV_t J = V_{-t}$. *Here we denoted by* $J : L^2(L^\infty(Z) \rtimes \Gamma) \to L^2(L^\infty(Z) \rtimes \Gamma)$ *Tomita's conjugation.*

Proof. – The first part follows directly from the definitions, so we leave the details to the reader. To get the second part it suffices to verify that the two operators coincide on vectors of the form $\xi \otimes \eta \otimes \delta_\gamma \in L^2(Y^\pi) \otimes L^2(X) \otimes \ell^2(\Gamma)$. If we assume that q is an anti-symmetric array then employing the formulas for J, V_t, υ_t we see that

$$JV_t J(\xi \otimes \eta \otimes \delta_\gamma) = JV_t((\sigma_{\gamma^{-1}}(\xi^*)) \otimes \sigma_{\gamma^{-1}}(\eta^*) \otimes \delta_{\gamma^{-1}})$$
$$= J(\upsilon_t(\gamma^{-1})\sigma_{\gamma^{-1}}(\xi^*) \otimes \sigma_{\gamma^{-1}}(\eta^*) \otimes \delta_{\gamma^{-1}})$$
$$= (\sigma_\gamma(\upsilon_{-t}(\gamma^{-1}))\xi) \otimes \eta \otimes \delta_\gamma$$
$$= (\exp(-it\pi_\gamma(q(\gamma^{-1})))\xi) \otimes \eta \otimes \delta_\gamma$$
$$= (\exp(itq(\gamma))\xi) \otimes \eta \otimes \delta_\gamma$$
$$= V_t(\xi \otimes \eta \otimes \delta_\gamma),$$

which finishes the proof in this case.

When the array is symmetric we get the conclusion by a similar computation. In this case the details are left to the reader. $\square$

Since V_t is a unitary on $L^2(Z) \otimes \ell^2(\Gamma)$, we may consider an inner automorphism α_t of $\mathfrak{B}(L^2(Z) \otimes \ell^2(\Gamma))$ by letting $\alpha_t(x) = V_t x V_t^*$ for all $x \in \mathfrak{B}(L^2(Z) \otimes \ell^2(\Gamma))$. Notice that this formula gives a family of inner automorphisms of the extended Roe algebra. Moreover, when restricting to the extended Roe algebra $C_u^*(\Gamma \curvearrowright X)$ one can recover from α_t the multipliers introduced above: $E \circ \alpha_t(x) = \mathrm{id}_X \otimes \mathfrak{m}_t(x)$ for all $x \in C_u^*(\Gamma \curvearrowright X)$.

However, one can see right away that these automorphisms do not move the group-measure space von Neumann algebra $L^\infty(X) \rtimes \Gamma$ into itself. Hence, applying the deformation/rigidity arguments at the level of von Neumann algebra $L^\infty(X) \rtimes \Gamma$ is rather inadequate. As we will see in the next section, this difficulty is overcome by working with the reduced crossed product C^*-algebra $L^\infty(X) \rtimes_{\sigma,r} \Gamma$ rather than $L^\infty(X) \rtimes \Gamma$. The following result underlines that the path α_t is a deformation at the C^*-algebraic level, i.e., with respect to the operatorial norm.

LEMMA 2.6. – *Let q be any symmetric or anti-symmetric array. Assuming the notations above, for every $x \in L^\infty(X) \rtimes_{\sigma,r} \Gamma$ we have*

$$(2.6) \qquad \|(\alpha_t(x) - x) \cdot e\|_\infty \to 0 \text{ as } t \to 0;$$

$$(2.7) \qquad \|(\alpha_t(JxJ) - JxJ) \cdot e\|_\infty \to 0 \text{ as } t \to 0,$$

where $\| \cdot \|_\infty$ denotes the operatorial norm in $\mathfrak{B}(L^2(Z) \otimes \ell^2(\Gamma))$. Here e denotes the orthogonal projection from $L^2(Z) \otimes \ell^2(\Gamma)$ onto $L^2(X) \otimes \ell^2(\Gamma)$.

Proof. – Since elements in $L^\infty(X) \rtimes_{\sigma,r} \Gamma$ can be approximated in the uniform norm by Γ-finitely supported elements, using the triangle inequality it suffices to show (2.6) only for $x = \sum_{g \in F} x_g u_g$, a finite sum where $x_g \in L^\infty(X)$. Fix an arbitrary vector

$\xi = \sum_\gamma \xi_\gamma \otimes \delta_\gamma \in L^2(X) \otimes \ell^2(\Gamma)$. Using the formula for α_t in combination with the Cauchy-Schwarz inequality, we have

$$\|(\alpha_t(x) - x)\xi\|^2 = \|\sum_{g \in F} \sum_{\gamma \in \Gamma} x_g (V_t u_g V_{-t} - u_g)(\xi_\gamma \otimes \delta_\gamma)\|^2$$

(2.8)

$$\leq \left(|F| \max_{g \in F} \|x_g\|_\infty^2\right) \left(\sum_{g \in F} \|\sum_{\gamma \in \Gamma} (V_t u_g V_{-t} - u_g)(\xi_\gamma \otimes \delta_\gamma)\|^2\right).$$

Applying the definitions and the formula for V_t we see that $u_g(\xi_\gamma \otimes \delta_\gamma) = \sigma_g(\xi_\gamma) \otimes \delta_{g\gamma}$ and $V_t u_g V_{-t}(\xi_\gamma \otimes \delta_\gamma) = v_t(g\gamma)\sigma_g(v_{-t}(\gamma)) \otimes \sigma_g(\xi_\gamma) \otimes \delta_{g\gamma}$. Therefore, continuing the estimate (2.8) we obtain

$$= \left(|F| \max_{g \in F} \|x_g\|_\infty^2\right) \sum_{g \in F} \|\sum_{\gamma \in \Gamma} (v_t(g\gamma)\sigma_g(v_{-t}(\gamma)) - 1) \otimes \sigma_g(\xi_\gamma) \otimes \delta_{g\gamma}\|^2$$

(2.9)

$$= \left(|F| \max_{g \in F} \|x_g\|_\infty^2\right) \sum_{g \in F} \sum_{\gamma \in \Gamma} \|\xi_\gamma\|^2 \|v_t(g\gamma)\sigma_g(v_{-t}(\gamma)) - 1\|^2$$

$$= 2\left(|F| \max_{g \in F} \|x_g\|_\infty^2\right) \sum_{g \in F} \sum_{\gamma \in \Gamma} \|\xi_\gamma\|^2 (1 - \tau(v_t(g\gamma)\sigma_g(v_{-t}(\gamma)))).$$

On the other hand, the same computations as in the proof of (2.5) together with inequality (1.1) imply that, there exist $K \geq 0$ such that for every $g \in F$ and $\gamma \in \Gamma$ we have

$$\tau(v_t(g\gamma)\sigma_g(v_{-t}(\gamma))) = \int_X \exp\left(it(q(g\gamma) - \pi_g(q(\gamma)))(x)\right) d\mu(x)$$

(2.10)

$$= \exp\left(-t^2 \|q(g\gamma) - \pi_g(q(\gamma))\|^2\right)$$

$$\geq \exp\left(-t^2 K\right).$$

Thus, combining (2.8), (2.9) and (2.10) we conclude that, for all $\xi \in L^2(X) \otimes \ell^2(\Gamma)$, we have

$$\|(\alpha_t(x) - x)\xi\|^2 \leq 2\left(\max_{g \in F} \|x_g\|_\infty\right) \|\xi\|^2 |F|^2 (1 - \exp(-t^2 K)),$$

which further implies

(2.11) $$\|(\alpha_t(x) - x) \cdot e\|_\infty \leq 2\left(\max_{g \in F} \|x_g\|_\infty\right) |F|^2 (1 - \exp(-t^2 K)).$$

Since F is finite, then $\exp(-t^2 K) \to 1$ as $t \to 0$, and (2.6) follows from (2.11).

It remains to show (2.7). Assume first that q is anti-symmetric. Since $[e, J] = 0$, by Proposition 2.5 we see that $(\alpha_t(JxJ) - JxJ) \cdot e = J((\alpha_t(x) - x) \cdot e)J$. Therefore, (2.7) follows from (2.6) because J is an anti-linear isometry. Similarly, when the array is symmetric, by the second part of Proposition 2.5, we have $(\alpha_t(JxJ) - JxJ) \cdot e = J((\alpha_{-t}(x) - x) \cdot e)J$ and the conclusion follows by the same reasoning. □

REMARK 2.7. – For future reference we make the following observation: in the proof of the previous proposition we used the symmetry or anti-symmetry of the array to show that the deformation converges pointwise in the operatorial norm to the identity on $JC^*_\lambda(\Gamma)J$. Except for establishing this fact, there is no other instance where symmetry or anti-symmetry will be used in the sequel.

Next we show that the path of unitaries V_t satisfies a "transversality" property very similar to Lemma 2.1 in [34]. Our proof follows closely the proof of Lemma 3.1 in [42]: we include it here only for the sake of completeness.

LEMMA 2.8. – *If V_t is the unitary defined above, then for all $\xi \in L^2(X) \otimes \ell^2(\Gamma)$ and all $t \in \mathbb{R}$ we have*

$$(2.12) \qquad\qquad 2\|V_t(\xi) - e \cdot V_t(\xi)\|^2 \geq \|\xi - V_t(\xi)\|^2.$$

Proof. – Fix $\xi \in L^2(X) \otimes \ell^2(\Gamma)$ and assume that it can be written as $\xi = \sum_\gamma \xi_\gamma \otimes \delta_\gamma$ with $\xi_\gamma \in L^2(X)$. Straightforward computations show that $V_t(\xi) = \sum_\gamma v_t(\gamma) \otimes \xi_\gamma \otimes \delta_\gamma$ and $e \cdot V_t(\xi) = \sum_\gamma \tau(v_t(\gamma))1 \otimes \xi_\gamma \otimes \delta_\gamma$; thus, the left side of (2.12) is equal to

$$(2.13) \qquad \begin{aligned} 2\|V_t(\xi) - e \cdot V_t(\xi)\|^2 &= 2\left(\|V_t(\xi)\|^2 - \|e \cdot V_t(\xi)\|^2\right) \\ &= 2\sum_\gamma \|\xi_\gamma\|^2 \|v_t(\gamma)\|^2 - \|\xi_\gamma\|^2|\tau(v_t(\gamma))|^2 \\ &= 2\sum_\gamma \|\xi_\gamma\|^2\left(1 - |\tau(v_t(\gamma))|^2\right). \end{aligned}$$

Applying the same formulas as above, we see that the right side of (2.12) is equal to

$$(2.14) \qquad \begin{aligned} \|\xi - V_t(\xi)\|^2 &= \|\xi\|^2 + \|V_t(\xi)\|^2 - 2Re\langle V_t(\xi), \xi\rangle \\ &= 2\left(\|\xi\|^2 - Re\langle V_t(\xi), \xi\rangle\right) \\ &= 2\sum_\gamma \|\xi_\gamma\|^2\left(1 - \tau(v_t(\gamma))\right). \end{aligned}$$

Since we have $\tau(v_t(\gamma)) = \exp(-t^2\|q(\gamma)\|^2) \geq \exp(-2t^2\|q(\gamma)\|^2) = |\tau(v_t(\gamma))|^2$, the conclusion follows from (2.13) and (2.14). $\qquad\square$

The multipliers $\mathfrak{m}_t$ arising from a proper quasi-cocycle behave in some sense as compact operators on $L\Gamma$, i.e., $\mathfrak{m}_t$ is continuous from the weak operator topology to the strong operator topology. Results of this type will be used crucially in the proof of Theorem 4.1.

PROPOSITION 2.9. – *Let $\mathfrak{m}_t$ be the Schur multiplier associated to some proper quasi-cocycle q on Γ. If $v_k \in M = L\Gamma$ is a bounded sequence of elements such that $v_k \to 0$ weakly, as $k \to \infty$, then for every $t > 0$ and every finite set $F \subset \Gamma$ we have that*

$$\lim_{k \to \infty}\left(\sup_{\|\xi\| \leq 1}\|(\mathfrak{m}_t(v_k) \otimes 1)(P_F \otimes 1)\xi\|\right) = 0,$$

where $\xi \in L^2(M) \otimes L^2(M)$. Here we denoted by P_F the orthogonal projection from $L^2(M)$ onto the linear span of the set $\{\delta_\alpha \,|\, \alpha \in F\}$.

Proof. – Let $\xi = \sum_s \xi_s \otimes \eta_s$ where $\{\eta_s\}_{s \in \mathbb{N}}$ is an orthonormal basis of $L^2(M)$ and $\xi_s = \sum_r \xi_r^s \delta_r$. Notice that $\|\xi\| \leq 1$ amounts to $\sum_s \sum_r |\xi_r^s|^2 \leq 1$. Without loss of generality we may assume that the set F consists of one element, i.e., $F = \{\gamma\}$. Let $v_k = \sum_h v_h^k u_h$ be the Fourier expansion, where $v_h^k \in \mathbb{C}$. Then applying the formula for $\mathfrak{m}_t$ and using $\sum_s |\xi_\gamma^s|^2 \leq 1$ we have that

$$\|(\mathfrak{m}_t(v_k) \otimes 1)(P_F \otimes 1)\xi\|^2 = \sum_s \|eV_t v_k V_{-t}(\xi_\gamma^s \delta_\gamma)\|^2$$

$$= \sum_s \| \sum_h eV_t v_h^k u_h V_{-t}(\xi_\gamma^s \delta_\gamma)\|^2$$

$$= \sum_s \| \sum_h (\tau(v_t(h\gamma)\sigma_h(v_{-t}(\gamma)))1)(v_h^k \xi_\gamma^s)\delta_{h\gamma}\|^2$$

(2.15)

$$= \sum_s \sum_h |v_h^k \xi_\gamma^s|^2 |\tau(v_t(h\gamma)\sigma_h(v_{-t}(\gamma)))|^2$$

$$= \sum_s \sum_h |v_h^k \xi_\gamma^s|^2 \exp\left(-t^2 \|q(h\gamma) - \pi_h(q(\gamma))\|^2\right)$$

$$\leq \sum_h |v_h^k|^2 \exp\left(-t^2 \|q(h\gamma) - \pi_h(q(\gamma))\|^2\right).$$

Furthermore, by the quasi-cocycle relation, the last term above is smaller than

(2.16)
$$\leq \sum_h |v_h^k|^2 \exp\left(-\frac{t^2}{2}\|q(h)\|^2 + t^2 D(q)\right).$$

Fix $\varepsilon > 0$. Since q is proper there exists a finite subset $F_\varepsilon \subset \Gamma$ such that $\frac{2}{t^2} \ln\left(\frac{2\exp(t^2 D(q))}{\varepsilon}\right) \leq \|q(h)\|^2$, for all $h \in \Gamma \setminus F_\varepsilon$. This obviously implies that

(2.17)
$$\exp\left(-\frac{t^2}{2}\|q(h)\|^2 + t^2 D(q)\right) \leq \frac{\varepsilon}{2\max_k \|v_k\|_\infty^2},$$

for all $h \in \Gamma \setminus F_\varepsilon$. Since the sequence v_k converges weakly to 0 as k approaches ∞ and F_ε is finite, one can find $k_\varepsilon \in \mathbb{N}$ such that, for all $k \geq k_\varepsilon$ and all $h \in F_\varepsilon$, we have

(2.18)
$$|v_h^k| \leq \left(\frac{\varepsilon}{2|F_\varepsilon|\max_{h\in F_\varepsilon} \exp\left(-\frac{t^2}{2}\|q(h)\|^2 + t^2 D(q)\right)}\right)^{\frac{1}{2}}.$$

Using (2.15), (2.16), (2.17), and (2.18) we obtain that for all $k \geq k_\varepsilon$ we have

$$\|(\mathfrak{m}_t(v_k) \otimes 1)(P_F \otimes 1)\xi\|^2$$

$$\leq \sum_h |v_h^k|^2 \exp\left(-\frac{t^2}{2}\|q(h)\|^2 + t^2 D(q)\right)$$

$$= \sum_{h\in F_\varepsilon} |v_h^k|^2 \exp\left(-\frac{t^2}{2}\|q(h)\|^2 + t^2 D(q)\right) + \sum_{h\in\Gamma\setminus F_\varepsilon} |v_h^k|^2 \exp\left(-\frac{t^2}{2}\|q(h)\|^2 + t^2 D(q)\right)$$

$$\leq \sum_{h\in F_\varepsilon} \frac{\varepsilon}{2|F_\varepsilon|\max_{h\in F_\varepsilon} \exp\left(-\frac{t^2}{2}\|q(h)\|^2 + t^2 D(q)\right)} \exp\left(-\frac{t^2}{2}\|q(h)\|^2 + t^2 D(q)\right)$$

$$+ \sum_{h\in\Gamma\setminus F_\varepsilon} |v_h^k|^2 \frac{\varepsilon}{2\max_k \|v_k\|_\infty^2}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which gives the desired conclusion. □

3. The proof of Theorem A

We begin by proving a general result (Theorem 3.2 below) describing the "position" of all commuting subalgebras in crossed products $L^\infty(X) \rtimes \Gamma$ arising from actions $\Gamma \curvearrowright X$ of exact groups Γ admitting proper arrays. The strategy of proof will essentially follow Theorem 4.3 in [29], formally replacing the family of almost derivations with the one-parameter group α_t constructed in §2.3. We note that unlike the proofs of solidity by Popa [33] and Vaes [42]—which also make use of one-parameter automorphism groups—we cannot directly appeal to spectral gap arguments and must, like Peterson, make fundamental use of Haagerup's criterion for amenability. We state Haagerup's criterion here for the convenience of the reader.

PROPOSITION 3.1 (Haagerup, Lemma 2.2 in [10]). – *Let M be a II$_1$ factor. A von Neumann subalgebra $N \subset M$ is amenable if and only if for every non-zero projection $p \in \mathcal{Z}(N)$ and every finite set of unitaries $F \subset \mathcal{U}(Np)$ we have*

$$(3.1) \qquad \| \sum_{u \in F} u \otimes \bar{u} \|_\infty = |F|.$$

Using this criterion, in cooperation with the technical results from the previous section, we show the following.

THEOREM 3.2. – *Let Γ be an exact group together with a finite family of subgroups $\mathcal{F}$. We assume that Γ admits an array into a weakly-ℓ^2 representation that is proper with respect to $\mathcal{F}$. Also, let $\Gamma \curvearrowright X$ be a free, ergodic, measure-preserving action of Γ on a probability space and denote by $M = L^\infty(X) \rtimes \Gamma$. Then for any diffuse von Neumann subalgebra $A \subset M$, either:*

1. *$A' \cap M$ is amenable; or,*
2. *$A \preceq_M L^\infty(X) \rtimes \Sigma$, for some $\Sigma \in \mathcal{F}$.*

Proof. – Denote by $N = A' \cap M$. Assuming that $A \npreceq_M L^\infty(X)$ we will show that N must be amenable. Let $p \in \mathcal{Z}(N)$ be a non-zero projection and $F \subset \mathcal{U}(Np)$ be a finite set of unitaries.

For convenience we recall that in Section 2.3 we considered a path of unitaries $V_t \in \mathfrak{B}(L^2(Y^\pi) \otimes L^2(X) \otimes \ell^2(\Gamma))$ defined as $V_t(\xi \otimes \eta \otimes \delta_\gamma) = v_t(\gamma)\xi \otimes \eta \otimes \delta_\gamma$, for every $\xi \in L^2(Y^\pi), \eta \in L^2(X)$, and $\gamma \in \Gamma$. We claim that, since $A \npreceq_M L^\infty(X) \rtimes \Sigma$ for all $\Sigma \in \mathcal{F}$, V_t cannot converge uniformly to the identity on $(Ap)_1$. Indeed, if $A \npreceq_M L^\infty(X) \rtimes \Sigma$ for all $\Sigma \in \mathcal{F}$ then, by Popa's intertwining techniques, there exists a sequence of unitaries $v_k \in \mathcal{U}(Ap)$ such that for all $s, t \in \Gamma$ we have that

$$(3.2) \qquad \|E_{L^\infty(X) \rtimes \Sigma}(u_t v_k u_s)\|_2 \to 0 \text{ as } k \to \infty.$$

Then letting $v_k = \sum_\gamma v_\gamma^k u_\gamma \in L^\infty(X) \rtimes \Gamma$ it is easy to see that

$$\|e \cdot V_t(v_k)\|^2 = \|e \cdot V_t(\sum_\gamma v_\gamma^k \delta_\gamma)\|^2$$

$$(3.3) \qquad = \sum_\gamma \|v_\gamma^k\|_2^2 \exp\left(-2t^2\|q(\gamma)\|^2\right)$$

$$= \sum_\gamma \|E_{L^\infty(X)}(v_k u_\gamma^*)\|_2^2 \exp\left(-2t^2\|q(\gamma)\|^2\right).$$

Since q is proper relative to the family $\mathcal{F}$, an easy computation shows that (3.2) together with (3.3) imply that $\|e \cdot V_t(v_k)\|$ converges to 0, as $k \to \infty$. Hence our claim follows because we have $\|V_t(x) - x\| \geq \|e \cdot V_t(x) - x\|$, for all $x \in Ap$.

In conclusion, one can find a constant $c > 0$ such that for every $t > 0$ there exists $x_t \in (Ap)_1$ such that $\|V_t(x_t) - x_t\| \geq c$. Let us denote $V_t(x_t)$ by ζ_t and define ξ_t to be $\zeta_t - e(\zeta_t)$. By Lemma 2.8, we have that $\|\xi_t\| \geq \dfrac{c}{2}$.

Let $E \subset L^\infty(X) \rtimes \Gamma$ be the operator system spanned by $\{p\} \cup F \cup F^*$. Since Γ is exact and $L^\infty(X)$ is abelian (hence, nuclear as a C^*-algebra), for every Γ-invariant, separable C^*-algebra A the reduced crossed product $A \rtimes_{\sigma,r} \Gamma$ is exact, which implies that $L^\infty(X) \rtimes_{\sigma,r} \Gamma$ is locally reflexive. Therefore, one can find a net $(\varphi_i)_{i \in I}$ of contractive completely positive maps $\varphi_i : E \to L^\infty(X) \rtimes_{\sigma,r} \Gamma$ such that $\varphi_i \to \mathrm{id}_E$ pointwise-ultraweakly. In fact, by passing to convex combinations of the φ_i's, we may assume that $\varphi_i(u) \to u$ in the strong* topology for all $u \in F$. Fixing $i \in I$, we have that for all $u \in F$

$$\lim_{t \to 0} \|\varphi_i(u)\xi_t\varphi_i(u^*) - \xi_t\|_2$$
$$= \lim_{t \to 0} \|(1-e)(\varphi_i(u)\zeta_t\varphi_i(u^*) - \zeta_t)\|_2$$
$$\leq \lim_{t \to 0} \|\varphi_i(u)J\varphi_i(u)J(\zeta_t) - \zeta_t\|_2$$
$$\leq \lim_{t \to 0} \|\varphi_i(u)V_t\left(\alpha_{-t}(J\varphi_i(u)J) - J\varphi_i(u)J\right)x_t\|_2$$
$$\quad + \lim_{t \to 0}\|V_t\left(\alpha_{-t}(\varphi_i(u)) - \varphi_i(u)\right)J\varphi_i(u)Jx_t\|_2 + \lim_{t \to 0}\|V_t\left(\varphi_i(u)J\varphi_i(u)Jx_t - x_t\right)\|_2$$
$$(3.4) \quad \leq \lim_{t \to 0}\|\left(\alpha_{-t}(J\varphi_i(u)J) - J\varphi_i(u)J\right)x_t\|_2 + \lim_{t \to 0}\|\left(\alpha_{-t}(\varphi_i(u)) - \varphi_i(u)\right)J\varphi_i(u)Jx_t\|_2$$
$$\quad + \lim_{t \to 0}\|\varphi_i(u))J\varphi_i(u)Jx_t - x_t\|_2$$
$$\leq 2\lim_{t \to 0}\|\left(\alpha_t(\varphi_i(u)) - \varphi_i(u)\right) \cdot e\|_\infty + \lim_{t \to 0}\|\varphi_i(u)x_t\varphi_i(u^*) - x_t\|_2$$
$$= \lim_{t \to 0}\|\varphi_i(u)x_t\varphi_i(u^*) - x_t\|_2$$
$$\leq \lim_{t \to 0}\|ux_tu^* - x_t\|_2 + 2\lim_{t \to 0}\|x_t\|_\infty\|\varphi_i(u) - u\|_2$$
$$\leq 2\|\varphi_i(u) - u\|_2.$$

Given $\varepsilon > 0$, let us choose $i \in I$ such that $\sum_{u \in F}\|\varphi_i(u) - u\|_2 \leq \frac{c\varepsilon}{4}$. Viewing $\mathcal{H} = L_0^2(Y^\pi) \otimes L^2(X) \otimes \ell^2(\Gamma)$ in the natural way as a Hilbert M-bimodule, we have that $\mathcal{H}$ is weakly contained in the coarse bimodule over M (cf. Lemma 5.1 in [28]). Hence, the induced representation $\pi' : M \otimes_{\mathrm{alg}} M^o \to \mathfrak{B}(\mathcal{H})$ given by $\pi'(x \otimes y^o)\xi = x\xi y$ extends in the minimal tensor norm. Thus, by the calculations above there exists $t > 0$ such that

$$\|\sum_{u \in F} u \otimes \bar{u}\|_\infty \geq \|\sum_{u \in F} \varphi_i(u) \otimes \overline{\varphi_i(u)}\|_\infty$$
$$\geq \frac{\|\sum_{u \in F}\varphi_i(u)\xi_t\varphi_i(u^*)\|_2}{\|\xi_t\|_2} \geq |F| - \varepsilon.$$

Hence, by Haagerup's criterion we have that N is amenable, a contradiction. $\qquad\square$

Proof of Theorem A. – Let Γ be an exact group belonging to the class $\mathcal{QH}_{\mathrm{reg}}$ and let $A \subset L\Gamma$ be a diffuse subalgebra. Applying the previous theorem in the case when X consists of a point, we have either $A' \cap M$ is amenable or $A \preceq_M \mathbb{C}$. However, the second case is impossible since A is diffuse; hence, it follows that $L\Gamma$ is solid. $\qquad\square$

4. The proof of Theorem B and corollaries

The above techniques for proving solidity can be upgraded to more general situations. Specifically, we obtain a result describing all weakly compact embeddings in the crossed product von Neumann algebras arising from actions of hyperbolic groups (Theorem 4.1). Though our approach follows the general outline of the proof of Theorem B in [28] there are substantial technical issues which arise in working with deformations from quasi-cocycles. For instance, one has to confront a lack of traciality in the spectral gap arguments. However, we believe that, by dealing with these obstacles, the techniques developed bring new insight in proving structural results for these factors.

To state the main theorem we need to recall the notion of weak compactness introduced by Ozawa and Popa in [27]. Briefly, a given inclusion of finite von Neumann algebras $B \subseteq Q$ is said to be a weakly compact embedding if the natural action by conjugation of the normalizer $\mathscr{N}_Q(B) \curvearrowright B$ is *weakly compact*. This means that there exists a net of positive unit vectors $(\eta_n)_{n \in \mathbf{N}}$ in $L^2(Q) \otimes L^2(\bar{Q})$ which simultaneously satisfies the following relations:

1. $\|\eta_n - (v \otimes \bar{v})\eta_n\| \to 0$, for all $v \in \mathscr{U}(B)$;
2. $\|[u \otimes \bar{u}, \eta_n]\| \to 0$, for all $u \in \mathscr{N}_Q(B)$; and
3. $\langle (x \otimes 1)\eta_n, \eta_n \rangle = \tau(x) = \langle (1 \otimes \bar{x})\eta_n, \eta_n \rangle$, for all $x \in Q$.

THEOREM 4.1. – *Let Γ be an exact group which admits a proper quasi-cocycle into a weakly-ℓ^2 representation. Let $\Gamma \curvearrowright X$ be a measure-preserving action on a probability space and set $M = L^\infty(X) \rtimes \Gamma$. If $P \subset M$ is a weakly compact embedding, then one of the following possibilities must hold:*

1. $P \preceq_M L^\infty(X)$;
2. $\mathscr{N}_M(P)''$ is amenable.

Proof. – We will denote by $N = \mathscr{N}_M(P)''$ and fix $p \in \mathscr{Z}(N' \cap M)$ a projection. The general strategy of the proof to show that the assumption $P \npreceq_M L^\infty(X)$ implies that Np is amenable. By assumption $P \subset M$ is weakly compact, so there exists a net of unit vectors $(\eta_n)_{n \in \mathbf{N}}$ in $L^2(M) \otimes L^2(\bar{M})$ as above.

Let $\mathscr{H} = L_0^2(Y^\pi) \otimes L^2(X) \otimes \ell^2(\Gamma)$ which as we remarked in the proof of Theorem 3.2 is weakly contained as an M-bimodule in the coarse bimodule. Fixing $t > 0$ we consider the unitary V_t associated with the quasi-cocycle q as defined in the previous section. Next set $\tilde{\eta}_{n,t} = (V_t \otimes 1)(p \otimes 1)\eta_n$, $\zeta_{n,t} = (e \otimes 1)\tilde{\eta}_{n,t} = (e \cdot V_t \otimes 1)(p \otimes 1)\eta_n$, and $\xi_{n,t} = \tilde{\eta}_{n,t} - \zeta_{n,t} = (e^\perp \otimes 1)\tilde{\eta}_{n,t} \in \mathscr{H} \otimes L^2(M)$. Using these notations we first prove the following result which is a technical adaptation of Proposition 2.9.

LEMMA 4.2. – *Let $\mathfrak{m}_t$ be the Schur multiplier associated to the proper quasi-cocycle q on Γ and let $\tilde{\mathfrak{m}}_t = \mathrm{id}_X \otimes \mathfrak{m}_t$. Let $v_k \in M$ be a bounded sequence of elements such that for all $x, y \in M$ we have $\|E_{L^\infty(X)}(xv_ky)\|_2 \to 0$ as $k \to \infty$. Then for every $t > 0$ and every finite set $F \subset \Gamma$ we have that*

$$\lim_{k \to \infty} \left(\sup_n \|(\tilde{\mathfrak{m}}_t(v_k) \otimes 1)(P_F \otimes 1)\zeta_{n,t}\| \right) = 0.$$

Here we denoted by P_F the orthogonal projection from $L^2(M)$ onto the linear span of the set $\{L^2(X) \otimes \delta_\alpha \mid \alpha \in F\}$.

Proof. – Let $(p \otimes 1)\eta_n = \sum_s a_s^n \otimes b_s$ where $\{b_s\}_{s\in\mathbb{N}}$ is an orthonormal basis of $L^2(M)$ and $a_s^n = \sum_r a_r^{s,n} \otimes \delta_r$. Without loss of generality we may assume that the set F consists of one element, i.e., $F = \{\gamma\}$. Let $v_k = \sum_h v_h^k u_h$ be the Fourier expansion, where $v_h^k \in L^\infty(X)$. Then applying the formula for $\mathfrak{m}_t$ together with $(P_F \otimes 1)(\zeta_{n,t}) = (P_F \otimes 1)(e \cdot V_t \otimes 1)((p \otimes 1)\eta_n) = \sum_s \left((\exp(-t^2\|q(\gamma)\|^2)a_\gamma^{s,n}) \otimes \delta_\gamma \right) \otimes b_s$ and $\exp(-2t^2\|q(\gamma)\|^2) \le 1$, we obtain the following formulas

$$\|(\tilde{\mathfrak{m}}_t(v_k) \otimes 1)(P_F \otimes 1)\zeta_{n,t}\|^2$$

$$= \sum_s \|eV_t v_k V_{-t}((\exp(-t^2\|q(\gamma)\|^2)a_\gamma^{s,n}) \otimes \delta_\gamma)\|^2$$

$$= \exp(-2t^2\|q(\gamma)\|^2) \sum_s \| \sum_h eV_t v_h^k u_h V_{-t}(a_\gamma^{s,n} \otimes \delta_\gamma)\|^2$$

(4.1)
$$\le \sum_s \| \sum_h (\tau(v_t(h\gamma)\sigma_h(v_{-t}(\gamma)))1) \otimes (v_h^k \sigma_h(a_\gamma^{s,n})) \otimes \delta_{h\gamma}\|^2$$

$$= \sum_s \sum_h \|v_h^k \sigma_h(a_\gamma^{s,n})\|^2 |\tau(v_t(h\gamma)\sigma_h(v_{-t}(\gamma)))|^2$$

$$= \sum_s \sum_h \|v_h^k \sigma_h(a_\gamma^{s,n})\|^2 \exp\left(-t^2\|q(h\gamma) - \pi_h(q(\gamma))\|^2\right).$$

Furthermore, by the quasi-cocycle relation, the last term in the equation above is smaller than

(4.2)
$$\le \sum_s \sum_h \|v_h^k \sigma_h(a_\gamma^{s,n})\|^2 \exp\left(-\frac{t^2}{2}\|q(h)\|^2 + t^2 D(q)\right).$$

Applying the identity $\|(x \otimes 1)(p \otimes 1)\eta_n\| = \|xp\|_2$ for all elements of the form $x = v_h^k u_h$, $h \in \Gamma$, a basic calculation shows that $\sum_s \sum_r \|v_h^k \sigma_h(a_r^{s,n})\|^2 = \|v_h^k u_h p\|_2^2$ for all $k \in \mathbb{N}$, $h \in \Gamma$, and n. In particular, this implies that, for all $k \in \mathbb{N}$, $h \in \Gamma$, and n, we have

$$\sum_s \|v_h^k \sigma_h(a_\gamma^{s,n})\|^2 \le \|v_h^k\|_2^2.$$

Using these estimates we see that the expression (4.2) is smaller than

(4.3)
$$\le \sum_h \|v_h^k\|_2^2 \exp\left(-\frac{t^2}{2}\|q(h)\|^2 + t^2 D(q)\right).$$

Fix $\varepsilon > 0$. Since the map q is proper, there exists a finite subset $F_\varepsilon \subset \Gamma$ such that $\frac{2}{t^2} \ln\left(\frac{2\exp(t^2 D(q))}{\varepsilon}\right) \le \|q(h)\|^2$ for all $h \in \Gamma \setminus F_\varepsilon$. This obviously implies that

(4.4)
$$\exp\left(-\frac{t^2}{2}\|q(h)\|^2 + t^2 D(q)\right) \le \frac{\varepsilon}{2\max_k \|v_k\|_\infty^2},$$

for all $h \in \Gamma \setminus F_\varepsilon$. Since for all $x, y \in M$, the sequence $\|E_{L^\infty(X)}(xv_k y)\|_2$ converges to 0 as k approaches ∞ and F_ε is finite (depending only on ε and q), then making a suitable choice for x and y one can find $k_\varepsilon \in \mathbb{N}$ such that, for all $k \ge k_\varepsilon$ and all $h \in F_\varepsilon$, we have

(4.5)
$$\|v_h^k\|_2 \le \left(\frac{\varepsilon}{2|F_\varepsilon| \max_{h\in F_\varepsilon} \exp\left(-\frac{t^2}{2}\|q(h)\|^2 + t^2 D(q)\right)}\right)^{\frac{1}{2}}.$$

Altogether, relations (4.1), (4.3), (4.4), and (4.5) show that, for all $k \geq k_\varepsilon$, we have

$$\sup_n \|(\tilde{\mathfrak{m}}_t(v_k) \otimes 1)(P_F \otimes 1)\zeta_{n,t}\|^2$$

$$\leq \sum_{h \in F_\varepsilon} \|v_h^k\|_2^2 \exp\left(-\frac{t^2}{2}\|q(h)\|^2 + t^2 D(q)\right) + \sum_{h \in \Gamma \setminus F_\varepsilon} \|v_h^k\|_2^2 \exp\left(-\frac{t^2}{2}\|q(h)\|^2 + t^2 D(q)\right)$$

$$\leq \sum_{h \in F_\varepsilon} \frac{\varepsilon}{2|F_\varepsilon| \max_{h \in F_\varepsilon} \exp\left(-\frac{t^2}{2}\|q(h)\|^2 + t^2 D(q)\right)} \exp\left(-\frac{t^2}{2}\|q(h)\|^2 + t^2 D(q)\right)$$

$$+ \sum_{h \in \Gamma \setminus F_\varepsilon} \|v_h^k\|_2^2 \left(\frac{\varepsilon}{2 \max_k \|v_k\|_\infty}\right)$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which gives the desired conclusion. □

Using the notations introduced at the beginning of the proof we show next the following inequality:

LEMMA 4.3. – *We have*

$$\operatorname*{Lim}_n \|\xi_{n,t}\| \geq \frac{5}{12}\|p\|_2,$$

where "Lim" is a limit along a non-principal ultrafilter.

Proof. – Using the triangle inequality multiple times, we have that

$$\|\tilde{\eta}_{n,t} - (e \circ \alpha_t(v) \otimes \bar{v})\zeta_{n,t}\| \leq \|\tilde{\eta}_{n,t} - (e \cdot \alpha_t(v) \otimes \bar{v})\tilde{\eta}_{n,t}\| + \|\xi_{n,t}\|$$
$$\leq \|\zeta_{n,t} - (e \cdot \alpha_t(v) \otimes \bar{v})\tilde{\eta}_{n,t}\| + 2\|\xi_{n,t}\|$$
$$\leq \|\tilde{\eta}_{n,t} - (\alpha_t(v) \otimes \bar{v})\tilde{\eta}_{n,t}\| + 2\|\xi_{n,t}\|$$
$$\leq \|\eta_n - (v \otimes \bar{v})\eta_n\| + 2\|\xi_{n,t}\|,$$

for all $v \in \mathcal{U}(P)$ and all $n \in \mathbf{N}$.

Consequently, since by (3) we have $\|\tilde{\eta}_{n,t}\| = \|p\|_2$, using the triangle inequality again we get

$$(4.6) \qquad \|(e \circ \alpha_t(v) \otimes \bar{v})\zeta_{n,t}\| \geq \|p\|_2 - 2\|\xi_{n,t}\| - \|\eta_n - (v \otimes \bar{v})\eta_n\|.$$

Next we consider the operator $e \cdot V_t \otimes 1$ from $L^2(M) \overline{\otimes} L^2(M)$ to $L^2(M) \overline{\otimes} L^2(M)$. Since q is proper one can check that there exists a finite subset $F \subset \Gamma$ such that $\|(P_F \otimes 1)(e \cdot V_t \otimes 1) - e \cdot V_t \otimes 1\|_\infty \leq \frac{1}{6}\|p\|_2$: here P_F denotes the orthogonal projection on the linear span of $L^\infty(X)F$. Hence using the formula $e \circ \alpha_t \circ e = e \circ \tilde{\mathfrak{m}}_t$ together with relation (4.6) and the triangle inequality, we obtain

$$(4.7) \qquad \|(\tilde{\mathfrak{m}}_t(v) \otimes 1)(P_F \otimes 1)\zeta_{n,t}\| \geq \frac{5}{6}\|p\|_2 - 2\|\xi_{n,t}\| - \|\eta_n - (v \otimes \bar{v})\eta_n\|,$$

for all $v \in \mathcal{U}(P)$ and all $n \in \mathbf{N}$.

This further implies that

$$(4.8) \qquad \sup_s \|(\tilde{\mathfrak{m}}_t(v) \otimes 1)(P_F \otimes 1)\zeta_{s,t}\| \geq \frac{5}{6}\|p\|_2 - 2\|\xi_{n,t}\| - \|\eta_n - (v \otimes \bar{v})\eta_n\|,$$

for all $v \in \mathcal{U}(P)$ and all $n \in \mathbf{N}$.

Taking "Lim", an arbitrary limit along a non-principal ultrafilter, above and applying relation (1) we obtain

$$
(4.9) \qquad \sup_s \|(\tilde{\mathfrak{m}}_t(v) \otimes 1)(P_F \otimes 1)\zeta_{s,t}\| \geq \frac{5}{6}\|p\|_2 - 2\operatorname{Lim}_n\|\xi_{n,t}\|,
$$

for all $v \in \mathcal{U}(P)$. This shows that the limit $\operatorname{Lim}_n \|\xi_{n,t}\| \geq \frac{5}{12}\|p\|_2$. Indeed, since $P \npreceq L^\infty(X)$, by Popa's intertwining techniques there exists a sequence of unitaries $v_s \in \mathcal{U}(P)$ such that for all $x, y \in M$ we have $\|E_{L^\infty(X)}(xv_ky)\|_2 \to 0$, as $k \to \infty$. Applying inequality (4.9) for each v_k and taking the limit as $k \to \infty$ then Lemma 4.2 implies that the left side of (4.9) is 0 and we get the desired conclusion. $\qquad\square$

Following the same argument as in Theorem B of [28], we define a state ψ_t on $\mathcal{N} = \mathfrak{B}(\mathcal{H}) \cap \rho(M^{op})'$. Explicitly, $\psi_t(x) = \operatorname{Lim}_n \frac{1}{\|\xi_{n,t}\|^2}\langle(x \otimes 1)\xi_{n,t}, \xi_{n,t}\rangle$ for every $x \in \mathcal{N}$. Next we prove the following technical result

LEMMA 4.4. – *For every $\varepsilon > 0$ and every finite self-adjoint set $K \subset L^\infty(X) \rtimes_{\sigma,r} \Gamma$ with $\operatorname{dist}_{\|\cdot\|_2}(y, (N)_1) \leq \varepsilon$ for all $y \in K$ one can find $t_\varepsilon > 0$ and a finite set $L_{K,\varepsilon} \subset \mathcal{N}_M(P)$ such that*

$$
(4.10) \qquad |\langle((yx - xy) \otimes 1)\xi_{n,t}, \xi_{n,t}\rangle| \leq 10\varepsilon + 2\sum_{v \in L_{K,\varepsilon}} \|[v \otimes \bar{v}, \eta_n]\|,
$$

for all $y \in K$, $\|x\|_\infty \leq 1$, $t_\varepsilon > t > 0$, and n.

Proof. – Fix $\varepsilon > 0$ and $y \in K$. Since $N = \mathcal{N}_M(P)''$ by the Kaplansky density theorem there exists a finite set $F_y = \{v_i\} \subset \mathcal{N}_M(P)$ and scalars μ_i such that $\|\sum_i \mu_i v_i\|_\infty \leq 1$ and

$$
(4.11) \qquad \|y - \sum_i \mu_i v_i\|_2 \leq \varepsilon.
$$

Also using Proposition 2.6 one can find a positive number $t_\varepsilon > 0$ such that, for all $t_\varepsilon > t > 0$, we have

$$
(4.12) \qquad \begin{aligned} \|(y - \alpha_{-t}(y)) \cdot e\|_\infty &\leq \varepsilon; \\ \|(JyJ - \alpha_{-t}(JyJ)) \cdot e\|_\infty &\leq \varepsilon. \end{aligned}
$$

Next we will proceed in several steps to show inequality (4.10). First we fix $t_\varepsilon > t > 0$. Then, using the triangle inequality in combination with $\|x\|_\infty \leq 1$, (4.12), and the M-bimodularity of $1 - e = e^\perp$, we see that

$$
\begin{aligned}
&|\langle(x \otimes 1)\xi_{n,t}, (y^* \otimes 1)\xi_{n,t}\rangle - \langle(xy \otimes 1)\xi_{n,t}, \xi_{n,t}\rangle| \\
&\quad \leq \|(\alpha_{-t}(y^*) - y^*) \cdot e\|_\infty + |\langle(x \otimes 1)\xi_{n,t_\varepsilon}, (e^\perp V_t y^* p \otimes 1)\eta_n\rangle - \langle(xy \otimes 1)\xi_{n,t}, \xi_{n,t}\rangle| \\
&\quad \leq \varepsilon + |\langle(x \otimes 1)\xi_{n,t}, (e^\perp V_t y^* p \otimes 1)\eta_n\rangle - \langle(xy \otimes 1)\xi_{n,t}, \xi_{n,t}\rangle|.
\end{aligned}
$$

Furthermore, the Cauchy-Schwarz inequality together with (3) and (4.11) enable us to see that the last quantity above is smaller than

$$
\leq \varepsilon + \|((y^*p - \sum_i \bar{\mu}_i v_i^*)p \otimes 1)\eta_n\|
$$

$$
+ |\sum_i \mu_i \langle (x \otimes 1)\xi_{n,t}, (e^\perp V_t v_i^* p \otimes 1)\eta_n \rangle - \langle (xy \otimes 1)\xi_{n,t}, \xi_{n,t} \rangle|
$$

$$
\leq 2\varepsilon + |\sum_i \mu_i \langle (x \otimes \bar{v}_i^*)\xi_{n,t}, (e^\perp V_t p v_i^* \otimes \bar{v}_i^*)\eta_n \rangle - \langle (xy \otimes 1)\xi_{n,t}, \xi_{n,t} \rangle|.
$$

To this end we notice that, since η_n is a positive vector and J is an isometry then for all $z \in M$ we have

$$
(4.13) \qquad \|\eta_n(z \otimes 1)\| = \|J(z^* \otimes 1)J\eta_n\| = \|(z^* \otimes 1)\eta_n\| = \|z^*\|_2 = \|z\|_2.
$$

Using this identity in combination with (4.11) and v_i being a unitary, we see that the last quantity above is smaller than

$$
\leq 2\varepsilon + \sum_i \|[v_i^* \otimes \bar{v}_i^*, \eta_n]\|
$$

$$
+ |\sum_i \mu_i \langle (x \otimes \bar{v}_i^*)\xi_{n,t}, (e^\perp V_t p \otimes 1)(\eta_n v_i^* \otimes \bar{v}_i^*) \rangle - \langle (xy \otimes 1)\xi_{n,t}, \xi_{n,t} \rangle|
$$

$$
\leq 2\varepsilon + \sum_i \|[v_i^* \otimes \bar{v}_i^*, \eta_n]\|
$$

$$
(4.14) \qquad + |\langle (x \otimes 1)\xi_{n,t}, (e^\perp V_t p \otimes 1)(\eta_n(\sum_i \bar{\mu}_i v_i^*) \otimes 1) \rangle - \langle (xy \otimes 1)\xi_{n,t}, \xi_{n,t} \rangle|
$$

$$
\leq 3\varepsilon + \sum_i \|[v_i^* \otimes \bar{v}_i^*, \eta_n]\|
$$

$$
+ |\langle (x \otimes 1)\xi_{n,t}, (e^\perp V_t p \otimes 1)(\eta_n(y^* \otimes 1)) \rangle - \langle (xy \otimes 1)\xi_{n,t}, \xi_{n,t} \rangle|.
$$

Next we observe that using the second part of (4.12) and V_t being a unitary we have

$$
\|V_t \otimes 1((p \otimes 1)\eta_n(y^* \otimes 1)) - (V_t \otimes 1((p \otimes 1)\eta_n))(y^* \otimes 1)\|
$$

$$
= \|(V_t JyJ \otimes 1)(p \otimes 1)\eta_n - (JyJV_t \otimes 1)(p \otimes 1)\eta_n\|
$$

$$
\leq \|(V_t JyJ \otimes 1)(p \otimes 1)\eta_n - (JyJV_t \otimes 1)(p \otimes 1)\eta_n\|
$$

$$
\leq \|((JyJ - \alpha_{-t}(JyJ)) \otimes 1)(p \otimes 1)\eta_n\|
$$

$$
\leq \|(JyJ - \alpha_{-t}(JyJ)) \cdot e\|_\infty \leq \varepsilon.
$$

Therefore applying this estimate two times we see that the last expression in (4.14) is smaller that

$$\leq 4\varepsilon + \sum_i \|[v_i^* \otimes \bar{v}_i^*, \eta_n]\|$$
$$+ |\langle (x \otimes 1)\xi_{n,t}, ((e^\perp V_t p \otimes 1)\eta_n)(y^* \otimes 1)\rangle - \langle (xy \otimes 1)\xi_{n,t}, \xi_{n,t}\rangle|$$
$$= 4\varepsilon + \sum_i \|[v_i^* \otimes \bar{v}_i^*, \eta_n]\|$$
$$+ |\langle ((x \otimes 1)\xi_{n,t})(y \otimes 1), (e^\perp V_t p \otimes 1)\eta_n\rangle - \langle (xy \otimes 1)\xi_{n,t}, \xi_{n,t}\rangle|$$
$$\leq 5\varepsilon + \sum_i \|[v_i \otimes \bar{v}_i, \eta_n]\|$$
$$+ |\langle (xe^\perp V_t p \otimes 1)(\eta_n(y \otimes 1)), (e^\perp V_t p \otimes 1)\eta_n\rangle - \langle (xy \otimes 1)\xi_{n,t}, \xi_{n,t}\rangle|.$$

Using (4.11), (4.13), v_i being a unitary in combination with triangle inequality we see that the last quantity above is smaller than

$$\leq 6\varepsilon + \sum_i \|[v_i \otimes \bar{v}_i, \eta_n]\|$$
$$+ |\sum_i \mu_i \langle (xe^\perp V_t p \otimes 1)(\eta_n(v_i \otimes \bar{v}_i)), (e^\perp V_t p \otimes \bar{v}_i)\eta_n\rangle - \langle (xy \otimes 1)\xi_{n,t}, \xi_{n,t}\rangle|$$
$$\leq 6\varepsilon + 2\sum_i \|[v_i \otimes \bar{v}_i, \eta_n]\|$$
$$+ |\sum_i \mu_i \langle (xe^\perp V_t p \otimes 1)(v_i \otimes \bar{v}_i)\eta_n, (e^\perp V_t p \otimes \bar{v}_i)\eta_n\rangle - \langle (xy \otimes 1)\xi_{n,t}, \xi_{n,t}\rangle|$$
$$= 6\varepsilon + 2\sum_i \|[v_i \otimes \bar{v}_i, \eta_n]\|$$
$$+ \langle (xe^\perp V_t p(\sum_i \mu_i v_i) \otimes 1)\eta_n, (e^\perp V_t p \otimes 1)\eta_n\rangle - \langle (xy \otimes 1)\xi_{n,t}, \xi_{n,t}\rangle|.$$

Then (4.11) together with (3), the Cauchy-Schwarz inequality, and $\|x\|_\infty \leq 1$ show that the last quantity above is smaller than

$$\leq 7\varepsilon + 2\sum_i \|[v_i \otimes \bar{v}_i, \eta_n]\| + \langle (xe^\perp V_t py \otimes 1)\eta_n, (e^\perp V_t p \otimes 1)\eta_n\rangle - \langle (xy \otimes 1)\xi_{n,t}, \xi_{n,t}\rangle$$
$$\leq 7\varepsilon + 2\sum_i \|[v_i \otimes \bar{v}_i, \eta_n]\| + \|((V_t py - y V_t p) \otimes 1)\eta_n\|.$$

Finally, using (4.12) together with (3) and the initial assumption $\mathrm{dist}_{\|\cdot\|_2}(y, (N)_1) \leq \varepsilon$ we obtain that the last expression above is smaller than

$$\leq 7\epsilon + 2\sum_i \|[v_i \otimes \bar{v}_i, \eta_n]\| + \|((V_t(py - yp)) \otimes 1)\eta_n\| + \|((V_t y - y V_t) \otimes 1)(p \otimes 1)\eta_n)\|$$
$$\leq 7\varepsilon + 2\sum_i \|[v_i \otimes \bar{v}_i, \eta_n]\| + \|((py - yp) \otimes 1)\eta_n\| + \|(y - \alpha_{-t}(y)) \cdot e\|_\infty$$
$$\leq 8\varepsilon + 2\sum_i \|[v_i \otimes \bar{v}_i, \eta_n]\| + \|py - yp\|_2$$
$$\leq 10\varepsilon + 2\sum_i \|[v_i \otimes \bar{v}_i, \eta_n]\|.$$

In conclusion, (4.10) follows from the previous inequalities by taking $L_{K,\varepsilon} = \cup_{y \in K} F_y$.
□

LEMMA 4.5. – *For every $\varepsilon > 0$ and any finite set $F_0 \subset \mathcal{U}(N)$ there exist a finite set $F_0 \subset F \subset M$, a c.c.p. map $\varphi_{F,\varepsilon} : \mathrm{span}(F) \to L^\infty(X) \rtimes_{\sigma,r} \Gamma$, and $t_\varepsilon > 0$ such that*

$$(4.15) \qquad |\psi_{t_\varepsilon}(\varphi_{F,\varepsilon}(up)^* x \varphi_{F,\varepsilon}(up)) - \psi_{t_\varepsilon}(x)| \leq 116\varepsilon,$$

for all $u \in F_0$ and $\|x\|_\infty \leq 1$.

Proof. – Fix $\varepsilon > 0$. Set $F = \{up, u^*p\} \cup F_0 \cup F_0^*$ and $E = \mathrm{span}(F)$. By local reflexivity, we may choose a c.c.p. map $\varphi_{F,\varepsilon} : E \to L^\infty(X) \rtimes_{\sigma,r} \Gamma$ such that for all $u \in F$

$$(4.16) \qquad \|\varphi_{F,\varepsilon}(up) - up\|_2 \leq \varepsilon.$$

This shows in particular that $\mathrm{dist}_{\|\cdot\|_2}(\varphi_{F,\varepsilon}(up), (N)_1) \leq \varepsilon$ for all $u \in F$. Therefore, applying the previous lemma for $K = \{\varphi_{F,\varepsilon}(up) : u \in F\} \subset L^\infty(X) \rtimes_{\sigma,r} \Gamma$, there exist a $t_\varepsilon > 0$ and a finite set $K' \subset \mathcal{N}_M(P)$ such that, for all $u \in F$, all $\|x\|_\infty \leq 1$, and all n, we have
$$(4.17)$$
$$|\langle((\varphi_{F,\varepsilon}(up)^* x \varphi_{F,\varepsilon}(up) - x\varphi_{F,\varepsilon}(up)\varphi_{F,\varepsilon}(up)^*) \otimes 1)\xi_{n,t_\varepsilon}, \xi_{n,t_\varepsilon}\rangle| \leq 10\varepsilon + 2\sum_{v \in K'} \|[v \otimes \bar{v}, \eta_n]\|.$$

Also using Proposition 2.6, after shrinking t_ε if necessary, we can assume in addition that for all $u \in F$ we have

$$(4.18) \qquad \|(\varphi_{F,\varepsilon}(up) - \alpha_{-t_\varepsilon}(\varphi_{F,\varepsilon}(up))) \cdot e\|_\infty \leq \varepsilon.$$

Hence, using triangle inequality together with (4.17) and the Cauchy-Schwarz inequality, we have that

$$|\langle(\varphi_{F,\varepsilon}(up)^* x \varphi_{F,\varepsilon}(up) \otimes 1)\xi_{n,t_\varepsilon}, \xi_{n,t_\varepsilon}\rangle - \langle(x \otimes 1)\xi_{n,t_\varepsilon}, \xi_{n,t_\varepsilon}\rangle|$$

$$\leq 10\varepsilon + 2\sum_i \|[v \otimes \bar{v}, \eta_n]\| + |\langle(x(\varphi_{F,\varepsilon}(up)\varphi_{F,\varepsilon}(up)^* - 1) \otimes 1)\xi_{n,t_\varepsilon}, \xi_{n,t_\varepsilon}\rangle|$$

$$\leq 10\varepsilon + 2\sum_v \|[v \otimes \bar{v}, \eta_n]\| + \|x\|_\infty \|(\varphi_{F,\varepsilon}(up)\varphi_{F,\varepsilon}(up)^* - 1) \otimes 1)\xi_{n,t_\varepsilon}\|$$

$$\leq 10\varepsilon + 2\sum_v \|[v \otimes \bar{v}, \eta_n]\| + 2\|(\alpha_{-t_\varepsilon}(\varphi_{F,\varepsilon}(up)) - \varphi_{F,\varepsilon}(up)) \cdot e\|_\infty$$

$$\qquad + \|(V_{t_\varepsilon}(\varphi_{F,\varepsilon}(up)\varphi_{F,\varepsilon}(up)^* - 1)p \otimes 1)\eta_n\|.$$

Furthermore, using (4.16) together with the Cauchy-Schwarz inequality, (3) and (4.18) we see that the last quantity above is smaller than

$$\leq 12\varepsilon + 2\sum_v \|[v \otimes \bar{v}, \eta_n]\| + \|((\varphi_{F,\varepsilon}(up)\varphi_{F,\varepsilon}(up)^* - 1)p \otimes 1)\eta_n\|$$

$$\leq 12\varepsilon + 2\sum_v \|[v \otimes \bar{v}, \eta_n]\| + \|\varphi_{F,\varepsilon}(up)\varphi_{F,\varepsilon}(up)^* - p\|_2$$

$$\leq 12\varepsilon + 2\sum_v \|[v \otimes \bar{v}, \eta_n]\| + 2\|\varphi_{F,\varepsilon}(up) - up\|_2$$

$$\leq 20\varepsilon + 2\sum_v \|[v \otimes \bar{v}, \eta_n]\|.$$

Altogether, the above sequence of inequalities shows that

$$|\langle(\varphi_{F,\varepsilon}(up)^*x\varphi_{F,\varepsilon}(up)\otimes 1)\xi_{n,t_\varepsilon},\xi_{n,t_\varepsilon}\rangle - \langle(x\otimes 1)\xi_{n,t_\varepsilon},\xi_{n,t_\varepsilon}\rangle| \leq 20\varepsilon + 2\sum_{v\in K'}\|[v\otimes\bar{v},\eta_n]\|,$$

and combining this with (2) and Lemma 4.3 we obtain

$$|\psi_{t_\varepsilon}(\varphi_{F,\varepsilon}(up)^*x\varphi_{F,\varepsilon}(up)) - \psi_{t_\varepsilon}(x)| \leq \operatorname*{Lim}_n\left(\frac{20\varepsilon + 2\sum_v\|[v\otimes\bar{v},\eta_n]\|}{\|\xi_{n,t_\varepsilon}\|^2}\right)$$
$$\leq \frac{20\varepsilon}{\left(\frac{5}{12}\right)^2} < 116\varepsilon,$$

which finishes the proof.   □

For the remaining part of the proof we mention that one can use Haagerup criterion to show that Np is amenable. In fact the reasoning in Theorem B in [28] applies verbatim in our case and we leave the details to the reader.   □

Proof of Theorem B. – Let Γ be an i.c.c. group which is weakly amenable and admits a proper quasi-cocycle into the left-regular representation, and consider $A \subset L\Gamma = M$ a diffuse amenable subalgebra. By Theorem B in [25] it follows that A is weakly compact in $L\Gamma$. Also, weak amenability implies that Γ is exact, cf. Theorem 12.4.4 in [4]. Hence, applying the previous theorem for the case when X consists of a point, we obtain that either $\mathcal{N}_M(A)''$ is amenable or $A \preceq_M \mathbb{C}$. Since A is diffuse the second case is impossible and therefore $L\Gamma$ is strongly solid.   □

Proof of Corollary B.1. – In the case that Γ is hyperbolic, a result of Ozawa shows that Γ is weakly amenable [23]. In the case that Γ is a lattice in $\mathrm{Sp}(n,1)$, choose a co-compact lattice $\Lambda < \mathrm{Sp}(n,1)$. We have that Λ is Gromov hyperbolic; hence, by [15] Λ admits a proper quasi-cocycle into the left-regular representation. A result of Shalom (Theorem 3.7 in [38]) shows that $\Gamma < \mathrm{Sp}(n,1)$ is integrable, thus ℓ^1-measure equivalent to Λ. As explained in the proof of Proposition 1.7, item (4), this implies that Γ also admits a proper quasi-cocycle into the left-regular representation. The work of Cowling and Haagerup [7] shows that $\mathrm{Sp}(n,1)$ is weakly amenable, which implies, by an unpublished result of Haagerup, that any lattice in $\mathrm{Sp}(n,1)$ is also weakly amenable (cf. [25]). Therefore, the hypotheses of Theorem B are satisfied.

If Γ is an exact, weakly amenable group which admits a proper quasi-cocycle into a weakly-ℓ^2 representation, then for any profinite, free, ergodic measure-preserving action $\Gamma \curvearrowright X$ on a standard probability space, $M = L^\infty(X) \rtimes \Gamma$ is a weakly amenable II_1 factor. If $A \subset M$ is a Cartan subalgebra, then the normalizing algebra $\mathcal{N}_M(A)''$ is obviously non-amenable and therefore, by Theorem 4.1, we must have that $A \preceq_M L^\infty(X)$. Hence, by Appendix A of [31], there exists $u \in \mathcal{U}(M)$ such that $uAu^* = L^\infty(X)$. Next, if $\Lambda \curvearrowright Y$ is a free, ergodic measure-preserving action of a countable discrete group Λ on a standard probability space Y such that $\theta : L^\infty(Y) \rtimes \Lambda \to L^\infty(X) \rtimes \Gamma$ is an isomorphism of von Neumann algebras, then we may assume that $\theta(L^\infty(Y)) = L^\infty(X)$. In particular, the actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are orbit equivalent and by Theorem A of [12] it follows that $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are virtually conjugate.   □

Proof of Corollary B.2. – Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be two orbit equivalent actions. Therefore one can find an isomorphism $\theta : M = L^\infty(Y) \rtimes \Lambda \to L^\infty(X) \rtimes \Gamma$ such that $\theta(L^\infty(Y)) = L^\infty(X)$. Let $\Sigma < \Lambda$ be an infinite amenable subgroup and we assume by contradiction that its normalizing group $N_\Lambda(\Sigma)$ is non-amenable. From the assumption it follows that Λ is weakly amenable and therefore the action by conjugation of $N_\Lambda(\Sigma)$ on $L\Sigma$ is weakly compact and so is the action by conjugation of $\theta(N_\Lambda(\Sigma))$ on $\theta(L\Sigma)$ [25]. Since $N_\Lambda(\Sigma)$ is non-amenable the previous theorem implies that $\theta(L\Sigma) \preceq_M L^\infty(X)$ and since $\theta(L^\infty(Y)) = L^\infty(X)$ this is equivalent to $L\Sigma \preceq L^\infty(Y)$. This however is impossible. Indeed, by intertwining techniques this implies that one can find finitely many elements $x_i, y_i \in M$ and $C > 0$ such that

$$(4.19) \qquad \sum_i \|E_{L^\infty(Y)}(x_i v y_i)\|_2^2 \geq C, \text{ for all } v \in \mathcal{U}(L\Sigma).$$

By shrinking c a little we can assume that for all i the elements x_i and y_i have finite supports in Γ. Therefore the union $F \subset \Gamma$ of all these supports is still a finite set and so is $F^{-1}F^{-1}$. Since Σ is infinite one can find $\gamma \in \Sigma \setminus F^{-1}F^{-1}$. A simple computation shows that all elements $x_i u_\gamma y_i$ are supported on elements different than the identity and hence $E_{L^\infty(Y)}(x_i v y_i) = 0$ which contradicts (4.19). $\qquad\square$

REMARK 4.6. – Note that the same proof above shows that all i.c.c. groups in the orbit equivalence class of an i.c.c. hyperbolic group give rise to strongly solid factors. This should be compared with an observation of the second author and Peterson (Remark 6.4 in [30]) on orbit-equivalence and strong solidity of group factors for weakly amenable groups with the "L^2-Haagerup property".

5. The proof of Theorem C and corollary

This last section in the main body of the paper contains the proof of Theorem C on the uniqueness of decompositions of group von Neumann algebras of products of groups in $\mathcal{QH}_{\text{reg}}$ into prime factors. Our proof is designed to circumvent a technical subtlety in the proof of Theorem 6.1 in [29] on the norm estimates of fusion products of certain vectors $\tilde{\delta}_\alpha^i(x_\alpha^i)$-specifically, whether these vectors are uniformly bounded from below, so that the estimate "$\|\xi_\alpha\|_2 \geq c^m$" is achieved. Essentially, we will be using Theorem 3.2 together with the fact that there is a well-defined way to take a tensor product of arrays as explained in Proposition 1.4.

Proof of Theorem C. – Notice that via a canonical isomorphism we can view $M = L\Gamma_1 \overline{\otimes} \cdots \overline{\otimes} L\Gamma_n = L\Gamma$ where $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$. By assumption, for each $1 \leq i \leq n$, there exists an array $q_i : \Gamma_i \to \mathcal{H}_i$ into some weakly-ℓ^2 unitary representation $\pi_i : \Gamma_i \to \mathcal{U}(\mathcal{H}_i)$. Consider the tensor product representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, where $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ and $\pi = \pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_n$, and notice that since $\mathcal{H}_i$ is weakly-ℓ^2 then $\mathcal{H}$ is weakly-ℓ^2 for Γ. Setting $\hat{\Gamma}_i$ to be the kernel of the canonical projection from Γ onto Γ_i, using Proposition 1.4 inductively, one can construct an array $q : \Gamma \to \mathcal{H}$ which is proper with respect to the family $\{\hat{\Gamma}_i : 1 \leq i \leq n\}$.

Now, suppose that $B \subset L\Gamma$ is a II$_1$ subfactor whose relative commutant $N = B' \cap L\Gamma$ is a non-amenable factor. Therefore by Theorem 3.2 there exists $1 \leq j \leq n$ such that $B \preceq_M L\hat{\Gamma}_j$. The result then follows by appealing to Proposition 12 of [26].   □

For the remaining corollary we fix the following notation. Given a subset $F \subset \{1, \ldots, n\}$, we denote by $\hat{\Gamma}_F$ the subgroup of $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ which consists of all elements with trivial i-th coordinate, for all $i \in F$.

Proof of Corollary C. – Suppose that $m \leq n$. Since $\Gamma_1 \times \cdots \times \Gamma_n \sim_{ME} \Lambda_1 \times \cdots \times \Lambda_m$, there exists an isomorphism $\psi : A \rtimes (\Gamma_1 \times \cdots \times \Gamma_n) \to (B \rtimes (\Lambda_1 \times \cdots \times \Lambda_m))^t$ such that $\psi(A) = B^t$. For simplicity, we will assume that $t = 1$, and we denote $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$, $\Lambda = \Lambda_1 \times \cdots \times \Lambda_m$, $M = A \rtimes \Gamma$, and $N = B \rtimes \Lambda$. Also, throughout the proof, for every subset $F \subset \{1, \ldots, n\}$ and $K \subset \{1, \ldots, m\}$, we define $\hat{M}_F = A \rtimes \hat{\Gamma}_F$ and $\hat{N}_K = B \rtimes \hat{\Lambda}_K$.

To begin, we prove that for every proper subset $F \subset \{1, \ldots, n\}$ there exists $K \subset \{1, \ldots, m\}$ such that $|F| = |K|$ and

(5.1) $\phi(L(\hat{\Gamma}_F)) \preceq_N \hat{N}_K$.

First, we notice that the same argument as in the proof of previous theorem shows that if $P \subset N$ is a diffuse subfactor such that there exists a non-amenable subfactor $Q \subset P' \cap N$, then one can find $1 \leq l \leq m$ such that $P \preceq_N \hat{N}_l$. In particular, this shows our claim when F consists of a single element. To get the general case we will proceed by induction on the cardinality of F. To explain the inductive step, fix a proper subset F of $\{1, \ldots, n\}$ together with an element $k \in F$. By assumption, there exists $K' \subset \{1, \ldots, m\}$ with $|K'| = |F| - 1$ such that

(5.2) $\phi(L(\hat{\Gamma}_{F \setminus \{k\}})) \preceq_N \hat{N}_{K'}$.

Therefore, since all Γ_i are i.c.c., one can find projections $p_1 \in \phi(L(\hat{\Gamma}_F))$, $p_2 \in \phi(L(\Gamma_k))$, $q \in \hat{N}_{K'}$, and an injective homomorphism

$$\theta : (p_1 \otimes p_2)\, \phi(L(\hat{\Gamma}_{F \setminus \{k\}}))\, (p_1 \otimes p_2) \to q\hat{N}_{K'}q.$$

Next, we notice that $\theta(p_1 \otimes p_2\, \phi(L(\hat{\Gamma}_F))\, p_1 \otimes p_2)$ and $\theta(p_1 \otimes p_2\, \phi(L(\Gamma_k))\, p_1 \otimes p_2)$ are non-amenable, commuting subfactors of $\theta(p_1 \otimes p_2)\, \hat{N}_{K'}\, \theta(p_1 \otimes p_2)$; thus, applying the same argument as before, there exists an element $j \in \{1, \ldots, m\} \setminus K'$ such that

(5.3) $\theta(p_1 \otimes p_2\, \phi(L(\hat{\Gamma}_F))\, p_1 \otimes p_2) \preceq_{\hat{N}_{K'}} \hat{N}_{K' \cup \{j\}}$.

Finally, by Remark 3.8 in [41], relations (5.2) and (5.3) imply that $\phi(L(\hat{\Gamma}_F)) \preceq_N \hat{N}_{K' \cup \{j\}}$, which concludes the inductive step and the proof of (5.1).

Notice that (5.1) automatically implies that $m = n$. Indeed, if $m \geq n+1$, then applying the statement for any subset $F \subset \{1, \ldots, n\}$ with $|F| = m - 1$, we get that $\phi(L(\hat{\Gamma}_F)) \preceq_N B \rtimes \Lambda_l$ for some $1 \leq l \leq m$ which obviously contradicts Theorem 3.2. Also, (5.1) implies that for every $1 \leq i \leq n$ there exists $1 \leq \pi(i) \leq n$ such that $\phi(L(\Gamma_i)) \preceq_N B \rtimes \Lambda_{\pi(i)}$. Notice that since $\phi(A) = B$, for all $1 \leq i \leq n$, we also have that

(5.4) $\phi(A \rtimes \Gamma_i) \preceq_N B \rtimes \Lambda_{\pi(i)}$.

Applying the same procedure for ϕ^{-1}, for every $1 \leq i \leq n$, one can find $1 \leq \rho(\pi(i)) \leq n$ such that $\phi^{-1}(B \rtimes \Lambda_{\pi(i)}) \preceq_M A \rtimes \Gamma_{\rho(\pi(i))}$; equivalently,

$$(5.5) \qquad B \rtimes \Lambda_{\pi(i)} \preceq_N \phi(A \rtimes \Gamma_{\rho(\pi(i))}),$$

for all $1 \leq i \leq n$. Combining this with (5.4) and using that $\phi(A \rtimes \Gamma_i)$ is an irreducible subfactor of N, we obtain $\phi(A \rtimes \Gamma_i) \preceq_N \phi(A \rtimes \Gamma_{\rho(\pi(i))})$. In particular, this implies that $\rho(\pi(i)) = i$, for all $1 \leq i \leq n$; hence, π is permutation of $\{1, \ldots, n\}$. Therefore, using (5.4) and (5.5) together with Proposition 8.4 in [13], one can find unitaries $u_i \in \mathcal{U}(N)$ such that

$$(5.6) \qquad u_i \phi(A \rtimes \Gamma_i) u_i^* = B \rtimes \Lambda_{\pi(i)}.$$

This further gives that $\phi_{u_i} = \mathrm{Ad}(u_i) \circ \phi$ is an isomorphism from $A \rtimes \Gamma_i$ onto $B \rtimes \Lambda_{\pi(i)}$ which satisfies

$$\phi_{u_i}(a) u_i = u_i \phi(a),$$

for all $a \in A$.

Next, for $N = B \rtimes (\Lambda_{\pi(i)} \times \hat{\Lambda}_{\pi(i)})$, we consider the Fourier decomposition $u = \sum_{\lambda \in \hat{\Lambda}_{\pi(i)}} y_\lambda v_\lambda$ with $y_\lambda \in B \rtimes \Lambda_{\pi(i)}$ and, using the above equation, there exists a nonzero element $y_\lambda \in B \rtimes \Lambda_{\pi(i)}$ such that for all $a \in A$ we have

$$(5.7) \qquad \phi_{u_i}(a) y_\lambda = y_\lambda \delta_\lambda(\phi(a)).$$

Note that since $B = \phi(A)$ is a maximal abelian subalgebra of N, (5.7) implies that $y_\lambda^* y_\lambda \in B$. Furthermore, taking the polar decomposition $y_\lambda = w_\lambda |y_\lambda|$ with w_λ a partial isometry, we conclude that

$$\phi_{u_i}(a) w_\lambda = w_\lambda \delta_\lambda(\phi(a)),$$

for all $a \in A$. This shows in particular $\phi_{u_i}(A) \prec_{B \rtimes \Lambda_{\pi(i)}} \phi(A)$, and since $B = \phi(A)$ and $\phi_{u_i}(A)$ are Cartan subalgebras of $B \rtimes \Lambda_{\pi(i)}$, then by Theorem A2 in [31] there exists a unitary $u_i' \in B \rtimes \Lambda_{\pi(i)}$ such that

$$u_i' \psi_u(A) u_i'^* = \phi(A) = B.$$

Finally, letting $x_i = u_i' u_i \in \mathcal{N}_N(B)$, by (5.6) the map $\mathrm{Ad}(x_i) \circ \phi$ implements an isomorphism between $A \rtimes \Gamma_i$ and $B \rtimes \Lambda_{\pi(i)}$, identifying the Cartan subalgebras A and B; thus, $\Gamma_i \sim_{ME} \Lambda_{\pi(i)}$ for all $1 \leq i \leq n$. □

Appendix A

Amenable actions, exactness, and local reflexivity

DEFINITION A.1 (Anantharaman-Delaroche and Renault [2], cf. [11])

Let Γ be a countable discrete group and $\Gamma \curvearrowright X$ be an action of Γ by homeomorphisms on a compact topological space X. The action $\Gamma \curvearrowright X$ is said to be *amenable* if there exists a sequence (ξ_n) of continuous maps $\xi_n : X \to \ell^2(\Gamma)$ such that $\xi_n \geq 0$, $\|\xi_n(x)\|_2 = 1$, for all $x \in X$, $n \in \mathbb{N}$, and

$$(A.1) \qquad \sup_{x \in X} \|\lambda_\gamma(\xi_n(x)) - \xi_n(\gamma x)\|_2 \to 0,$$

for all $\gamma \in \Gamma$.

PROPOSITION A.2 (Higson and Roe [11]). – *A countable discrete group Γ has Guoliang Yu's property A* [44] *if and only if Γ acts amenably on its Stone–Čech boundary $\beta'\Gamma = \beta\Gamma \setminus \Gamma$.*

Property A is equivalent, cf. [36], to the nuclearity of $C_u^*(\Gamma)$ which is, in turn, equivalent to the exactness of $C_\lambda^*(\Gamma)$ by a result of Ozawa [20].

DEFINITION A.3. – A C*-algebra A is said to be *locally reflexive* if for every finite-dimensional operator system $E \subset A^{**}$, there exists a net $(\varphi_i)_{i \in I}$ of contractive completely positive (c.c.p.) maps $\varphi_i : E \to A$ which converges to the identity in the pointwise-ultraweak topology.

For the purposes of this paper, the crucial property implied by exactness is that $C_\lambda^*(\Gamma)$ is a locally reflexive C*-algebra, cf. [4], Chapter 9.

Appendix B

A proof of Proposition 1.11

The aim of this appendix is to provide an elementary proof that $\Gamma = \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ belongs to the class $\mathcal{QH}_{\text{reg}}$. Appealing to Theorem A then furnishes an alternate proof of the solidity of $L\Gamma$, the main result of [24]. As in [24], our proof will make use of the amenability of the natural action of $SL(2, \mathbb{Z})$ on $SL(2, \mathbb{R})/T \cong \mathbb{R}P^1$, where T is the group of upper-triangular 2×2 real matrices.

To begin, note that $\Gamma_0 = SL(2, \mathbb{Z})$ admits a proper cocycle $b : \Gamma_0 \to \ell^2(\Gamma_0)$ with respect to the left-regular representation. By Proposition 1.5, we may replace b with a proper, symmetric array b' into the left-regular representation. Let π be the representation of Γ on $\ell^2(\Gamma_0)$ obtained by pulling the left-regular representation of Γ_0 back along the quotient $\Gamma \to \Gamma/\mathbb{Z}^2 \cong \Gamma_0$, so that π is weakly contained in the left-regular representation of Γ. Let $p : \mathbb{Z}^2 \setminus \{(0,0)\} \to \mathbb{R}P^1$ be the projection defined by $p((x,y)) = x/y$, and note that p is equivariant with respect to the natural actions of $SL(2, \mathbb{Z})$ on $\mathbb{Z}^2$ and $\mathbb{R}P^1$.

Given a sequence of continuous maps $\xi_n : \mathbb{R}P^1 \to \ell^2(\Gamma_0)$ satisfying Definition A.1, define the maps $\xi'_n : \mathbb{Z}^2 \to \ell^2(\Gamma_0)$ by

$$\xi'_n(z) = \xi_n(p(z)),$$

for $z = (z_1, z_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, and $\xi'_n(z) = 0$, otherwise. Note that for any $a \in \mathbb{Z}^2$ we have

(B.1) $$\limsup_{z \to \infty} \|\xi'_n(z) - \xi'_n(z + a)\|_2 = 0,$$

for all $n \in \mathbb{N}$.

Now, consider finite, symmetric generating subsets $S' \subset \Gamma_0$ and $S'' \subset \mathbb{Z}^2$. Define $S_1 = S' \cup S''$ and $S_{k+1} = S_k \cup (S_1)^{k+1}$ for all $k \in \mathbb{N}$. By Equations A.1 and B.1, there exist an increasing sequence of finite, symmetric subsets $F_1 \subset F_2 \subset \cdots \subset F_k \subset \cdots \subset \mathbb{Z}^2$ such that $\bigcup_{k=1}^\infty F_k = \mathbb{Z}^2$ and a subsequence (n_k) such that

(B.2) $$\sup_{s \in S_k} \sup_{g \in \mathbb{Z}^2 \setminus F_k} \|\pi_s(\xi'_{n_k}(g)) - \xi'_{n_k}(s \cdot g)\|_2 \leq \frac{1}{2^k},$$

where $s \cdot g$ is the natural Γ-action on $\mathbb{Z}^2$. Define a map $\partial : \mathbb{Z}^2 \to \ell^2(\mathbb{N}; \ell^2(\Gamma_0)) = \mathcal{H}$ by $\partial(z)(k) = \xi'_{n_k}(z)$, if $z \notin F_k$, and 0, otherwise. It is then straightforward to check that ∂ is proper, symmetric, and boundedly Γ-equivariant. For $(z, \gamma) \in \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ we define the map $q((z, \gamma)) = b'(\gamma) \oplus \partial(z) \in \ell^2(\Gamma_0) \oplus \mathcal{H}$. It is easy to see that q is a proper, symmetric

array into the weakly-ℓ^2 representation $\pi \oplus \pi^{\oplus \infty}$. Thus, $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z}) \in \mathcal{QH}_{\mathrm{reg}}$ and we are done.

QUESTION B.1. – Does $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ admit a proper, *anti-symmetric* array into a weakly-ℓ^2 representation?

REFERENCES

[1] S. ADAMS, Indecomposability of equivalence relations generated by word hyperbolic groups, *Topology* **33** (1994), 785–798.

[2] C. ANANTHARAMAN-DELAROCHE, J. RENAULT, *Amenable groupoids*, Monographies de l'enseignement mathématique **36**, L'Enseignement mathématique, 2000.

[3] B. BEKKA, P. DE LA HARPE, A. VALETTE, *Kazhdan's property (T)*, New Mathematical Monographs **11**, Cambridge Univ. Press, 2008.

[4] N. P. BROWN, N. OZAWA, *C*-algebras and finite-dimensional approximations*, Graduate Studies in Math. **88**, Amer. Math. Soc., 2008.

[5] M. BURGER, N. MONOD, Continuous bounded cohomology and applications to rigidity theory, *Geom. Funct. Anal.* **12** (2002), 219–280.

[6] I. CHIFAN, T. SINCLAIR, B. UDREA, On the structure of II$_1$ factors of negatively curved groups, II. Actions by product groups, preprint, 2011.

[7] M. COWLING, U. HAAGERUP, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one, *Invent. Math.* **96** (1989), 507–549.

[8] M. COWLING, R. J. ZIMMER, Actions of lattices in $\mathrm{Sp}(1, n)$, *Ergodic Theory Dynam. Systems* **9** (1989), 221–237.

[9] A. FURMAN, Orbit equivalence rigidity, *Ann. of Math.* **150** (1999), 1083–1108.

[10] U. HAAGERUP, Injectivity and decomposition of completely bounded maps, in *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*, Lecture Notes in Math. **1132**, Springer, 1985, 170–222.

[11] N. HIGSON, J. ROE, Amenable group actions and the Novikov conjecture, *J. reine angew. Math.* **519** (2000), 143–153.

[12] A. IOANA, Cocycle superrigidity for profinite actions of property (T) groups, *Duke Math. J.* **157** (2011), 337–367.

[13] A. IOANA, J. PETERSON, S. POPA, Amalgamated free products of weakly rigid factors and calculation of their symmetry groups, *Acta Math.* **200** (2008), 85–153.

[14] I. MINEYEV, Straightening and bounded cohomology of hyperbolic groups, *Geom. Funct. Anal.* **11** (2001), 807–839.

[15] I. MINEYEV, N. MONOD, Y. SHALOM, Ideal bicombings for hyperbolic groups and applications, *Topology* **43** (2004), 1319–1344.

[16] N. MONOD, *Continuous bounded cohomology of locally compact groups*, Lecture Notes in Math. **1758**, Springer, 2001.

[17] N. MONOD, Y. SHALOM, Cocycle superrigidity and bounded cohomology for negatively curved spaces, *J. Differential Geom.* **67** (2004), 395–455.

[18] N. MONOD, Y. SHALOM, Orbit equivalence rigidity and bounded cohomology, *Ann. of Math.* **164** (2006), 825–878.

[19] D. V. OSIN, Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems, *Mem. Amer. Math. Soc.* **179** (2006).

[20] N. OZAWA, Amenable actions and exactness for discrete groups, *C. R. Acad. Sci. Paris Sér. I Math.* **330** (2000), 691–695.

[21] N. OZAWA, Solid von Neumann algebras, *Acta Math.* **192** (2004), 111–117.

[22] N. OZAWA, A Kurosh-type theorem for type II_1 factors, *Int. Math. Res. Not.* **2006** (2006), Art. ID 97560, 21.

[23] N. OZAWA, Weak amenability of hyperbolic groups, *Groups Geom. Dyn.* **2** (2008), 271–280.

[24] N. OZAWA, An example of a solid von Neumann algebra, *Hokkaido Math. J.* **38** (2009), 557–561.

[25] N. OZAWA, Examples of groups which are not weakly amenable, *Kyoto J. Math.* **52** (2012), 333–344.

[26] N. OZAWA, S. POPA, Some prime factorization results for type II_1 factors, *Invent. Math.* **156** (2004), 223–234.

[27] N. OZAWA, S. POPA, On a class of II_1 factors with at most one Cartan subalgebra, *Ann. of Math.* **172** (2010), 713–749.

[28] N. OZAWA, S. POPA, On a class of II_1 factors with at most one Cartan subalgebra, II, *Amer. J. Math.* **132** (2010), 841–866.

[29] J. PETERSON, L^2-rigidity in von Neumann algebras, *Invent. Math.* **175** (2009), 417–433.

[30] J. PETERSON, T. SINCLAIR, On cocycle superrigidity for Gaussian actions, *Ergodic Theory Dynam. Systems* **32** (2012), 249–272.

[31] S. POPA, On a class of type II_1 factors with Betti numbers invariants, *Ann. of Math.* **163** (2006), 809–899.

[32] S. POPA, Deformation and rigidity for group actions and von Neumann algebras, in *International Congress of Mathematicians. Vol. I*, Eur. Math. Soc., Zürich, 2007, 445–477.

[33] S. POPA, On Ozawa's property for free group factors, *Int. Math. Res. Not.* **2007** (2007), doi:10.1093/imrn/rnm036.

[34] S. POPA, On the superrigidity of malleable actions with spectral gap, *J. Amer. Math. Soc.* **21** (2008), 981–1000.

[35] S. POPA, S. VAES, Unique Cartan decomposition for II_1 factors arising from arbitrary actions of hyperbolic groups, preprint arXiv:1201.2824.

[36] J. ROE, *Lectures on coarse geometry*, University Lecture Series **31**, Amer. Math. Soc., 2003.

[37] H. SAKO, Measure equivalence rigidity and bi-exactness of groups, *J. Funct. Anal.* **257** (2009), 3167–3202.

[38] Y. SHALOM, Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group, *Ann. of Math.* **152** (2000), 113–182.

[39] T. Sinclair, Strong solidity of group factors from lattices in SO$(n, 1)$ and SU$(n, 1)$, *J. Funct. Anal.* **260** (2011), 3209–3221.

[40] A. Thom, Low degree bounded cohomology and L^2-invariants for negatively curved groups, *Groups Geom. Dyn.* **3** (2009), 343–358.

[41] S. Vaes, Explicit computations of all finite index bimodules for a family of II$_1$ factors, *Ann. Sci. Éc. Norm. Supér.* **41** (2008), 743–788.

[42] S. Vaes, One-cohomology and the uniqueness of the group measure space decomposition of a II$_1$ factor, preprint arXiv:1012.5377.

[43] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory. III. The absence of Cartan subalgebras, *Geom. Funct. Anal.* **6** (1996), 172–199.

[44] G. Yu, The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, *Invent. Math.* **139** (2000), 201–240.

Ionut Chifan
Department of Mathematics
Vanderbilt University
1326 Stevenson Center
Nashville, TN 37240, USA and
IMAR, Bucharest, Romania

Current address:
Department of Mathematics
University of Iowa
14 MacLean Hall
Iowa City, IA 52242-1419, USA
E-mail: `ionut-chifan@uiowa.edu`

Thomas Sinclair
Department of Mathematics
Vanderbilt University
1326 Stevenson Center
Nashville, TN 37240, USA

Current address:
Department of Mathematics
University of California at Los Angeles
Box 951555
Los Angeles, CA, 90095-1555, USA
E-mail: `thomas.sinclair@math.ucla.edu`