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COMPACT KÄHLER MANIFOLDS WITH COMPACTIFIABLE UNIVERSAL COVER

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Abstract. — Let X be a compact Kähler manifold such that the universal cover admits a compactification. We conjecture that the fundamental group is almost abelian and reduce this problem to a classical conjecture of Iitaka.

Résumé (*Variétés kähleriennes compactes à revêtement universel compactifiable*)
Nous étudions les variétés kählériennes compactes dont le revêtement universel se réalise comme un ouvert de Zariski d'une variété compacte. Nous formulons la conjecture selon laquelle le groupe fondamental d'une telle variété devrait être virtuellement abélien et nous ramenons ce problème à une conjecture classique d'Iitaka.

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1. Introduction

The aim of this paper is to study the following problem.

1.1. — **Conjecture.** *Let X be a compact Kähler manifold with infinite fundamental group $\pi_1(X)$. Suppose that the universal cover $\tilde{X}_{\rm univ}$ is a Zariski open subset $\tilde{X}_{\rm univ} \subset \overline{X}$ of some compact complex manifold $\overline{X}$. Then (after finite étale cover) there exists a locally trivial fibration $X \rightarrow A$ with simply connected fibre F onto a complex torus A. In particular we have $\tilde{X}_{\rm univ} \simeq F \times \mathbb{C}^{\dim A}$.*

This conjecture generalises Iitaka's classical conjecture claiming that a compact Kähler manifold X uniformised by $\mathbb{C}^{\dim X}$ is an étale quotient of a complex torus. In a recent paper with J. Kollár we studied this conjecture in the algebraic setting, i.e. under the additional hypothesis that X is projective and $\tilde{X}_{\rm univ}$ is quasi-projective. It turned out that the key issue is to show that the fundamental group is almost abelian and we established the following statement.

1.2. — **Proposition.** [14, Prop.1.3] *Let X have the smallest dimension among all normal, projective varieties that have an infinite, quasi-projective, étale Galois cover $\tilde{X} \rightarrow X$ whose Galois group is not almost abelian.*

Then X is smooth and its canonical bundle K_X is nef but not semiample. (That is, $(K_X \cdot C) \geq 0$ for every algebraic curve $C \subset X$ but $\mathcal{O}_X(mK_X)$ is not generated by global sections for any $m > 0$.)

By the *abundance conjecture* [31, Sec.2] the canonical bundle should always be semiample if it is nef. We then proved that in the algebraic case Conjecture 1.1 is implied by the abundance conjecture [14, Thm.1.1].

Since an infinite cover $\tilde{X} \rightarrow X$ is never an algebraic morphism, it is natural to look for an analogue of Proposition 1.2 in the analytic category. Note first that it is natural to impose that X is Kähler: as we know from Hodge theory the Kähler condition establishes a link between the complex and the differentiable (i.e. topological) structure of X. Moreover there exist plenty of non-Kähler compact manifolds covered by compactifiable complex spaces, the easiest examples being Hopf manifolds [14, 1.6]. Although the existence of a compactification $\tilde{X} \subset \overline{X}$ should already be quite restrictive we will see that the appropriate analytic analogue of the quasiprojectiveness is the existence of a Kähler compactification.

1.3. — **Theorem.** *Let X have the smallest dimension among all normal, compact Kähler spaces that have an infinite, étale Galois cover $\tilde{X} \rightarrow X$ whose*

Galois group Γ is not almost abelian and such that there exists a Kähler compactification $\tilde{X} \subset \overline{X}$. Then X is smooth, does not admit any Mori contraction[(1)], *and $\tilde{X}$ is not covered by positive-dimensional compact subspaces.*

In particular X has π_1-general type[(2)], *i.e. $\tilde{X}_{\mathrm{univ}}$ is not covered by positive-dimensional compact subspaces.*

Even in the algebraic case, this statement gives some new information: if X is projective, the absence of Mori contractions implies that K_X is nef. Thus the "minimal dimensional counterexample" in Proposition 1.2 is of π_1-general type. Note also that for a manifold of π_1-general type the Conjecture 1.1 simply claims that X is an étale quotient of a torus. Thus we are reduced to Iitaka's conjecture which has been studied by several authors [30, 29, 11, 20][(3)].

An important difference between the proof of Theorem 1.3 and the arguments in [14] is that the natural maps attached to compact Kähler manifolds (algebraic reduction, reduction maps for covering families of algebraic cycles) are in general not morphisms, as opposed to the classification theory of projective manifolds where we have Mori contractions and, assuming abundance, the Iitaka fibration at our disposal. Our key observation will be that for a general fibre of the Γ-reduction γ (cf. Definition 2.2) the aforementioned meromorphic maps are holomorphic. We then deduce a strong dichotomy: up to replacing γ by some factorisation the general fibre G is either projective or does not contain any positive-dimensional compact proper subspaces (cf. Theorem 2.13). In a similar spirit F. Campana shows in the Appendix (Theorem A.8) from a more general viewpoint that Iitaka's conjecture has only to be treated for projective manifolds and simple compact Kähler manifolds, i.e. those which are not covered by positive-dimensional compact proper subspaces.

If we try to avoid the Kähler assumption on $\overline{X}$ we still obtain some information of bimeromorphic nature:

1.4. — **Proposition.** *Let X have the smallest dimension among all normal, compact Kähler spaces that have an infinite, étale Galois cover $\tilde{X} \to X$ whose Galois group Γ is not almost abelian and such that there exists a compactification $\tilde{X} \subset \overline{X}$. Then X is smooth and special in the sense of Campana* [9].

(1) In the analytic setting we define a Mori contraction as a proper holomorphic morphism with connected fibres $\mu : X \to X'$ onto a normal complex space X' such that $-K_X$ is μ-ample.

(2) We follow the terminology of [7], this corresponds to the property of X having a generically large fundamental group in the sense of [24].

(3) Apart from [11] these papers do not really use that $\tilde{X}_{\mathrm{univ}} \simeq \mathbb{C}^{\dim X}$.

This proposition follows rather quickly from an orbifold version of the Kobayashi-Ochiai theorem (Theorem 3.1). By results of F. Campana and the first named author [9, Thm.3.33], [10, Thm.1.1] the fundamental group of a special manifold of dimension at most three is almost abelian, so our counterexample (if it exists) would have $\dim X \geq 4$.

Let us finally note that once we have understood the fundamental group, the geometric statement in Conjecture 1.1 is not far away.

1.5. — **Theorem.** *Let X be a compact Kähler manifold whose universal cover $\tilde{X}_{\mathrm{univ}}$ admits a Kähler compactification $\tilde{X}_{\mathrm{univ}} \subset \overline{X}$. If the fundamental group of X is almost abelian, the Albanese map of X is (up to finite étale cover) a locally trivial fibration whose fibre F is simply connected.*

Since the proof of the corresponding statement in the algebraic setting [14, Thm.1.4] relies on strong results of Hodge theory for birational morphisms which are unknown in the Kähler setting, our argument follows the lines of [25]. Indeed if X is of π_1-general type, [25, Thm. 16] implies that X is isomorphic to its Albanese torus (even without any further assumption on $\overline{X}$); see [16] and Remark 3.3 for a discussion around this general case.

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2. Notation and basic results

Manifolds and complex spaces will always be supposed to be irreducible.

If X is a normal complex space we denote by $\mathscr{C}(X)$ its cycle space [1]. We will use very often that if X is a compact Kähler space, then the irreducible components of $\mathscr{C}(X)$ are compact (Bishop's theorem, see [26]).

A fibration is a proper surjective map $\varphi : X \to Y$ with connected fibres between normal complex varieties. A meromorphic map $\varphi : X \to Y$ is almost holomorphic if there exists a Zariski open dense subset $X^0 \subset X$ such that the restriction $\varphi|_{X^0}$ is holomorphic and $\varphi|_{X^0} : X^0 \to Y$ is a proper map.

Recall that a fibration $\varphi : X \to Y$ from a manifold X onto a normal complex space Y is almost smooth if the reduction F_{red} of every fibre is smooth and has the expected dimension. In this case the complex space Y has at most quotient singularities, the local structure around $y \in Y$ being given by a finite representation of the fundamental group of $\pi_1(F_{\mathrm{red}})$ [28, Prop.3.7]. Thus there exists *locally* a finite base change $Y' \to Y$ such that the normalisation X' of $X \times_Y Y'$ is smooth over Y'.

2.1. — **Definition.** *We say that an almost smooth fibration $\varphi : X \to Y$ is almost locally trivial with fibre F if for every $y \in Y$ the fibration $X' \to Y'$ constructed above is locally trivial with fibre F.*

Note that while an almost locally trivial fibration is locally trivial in the neighbourhood of a generic point $y \in Y$, it is not true that the reduction $F_{0,\text{red}}$ of every fibre F_0 is isomorphic to F. For example if F is a K3 surface with a fixed point free involution $i : F \to F$ and $\Delta \subset \mathbb{C}$ the unit disc, then

$$X := (F \times \Delta)/ < i \times (z \mapsto -z) >$$

has an almost locally trivial fibration $X \to \Delta$ with fibre F and $F_{0,\text{red}}$ is isomorphic to the Enriques surface $F/ < i >$.

2.2. — **Definition.**[7, 24] *Let X be a compact Kähler manifold and Γ a quotient of the fundamental group $\pi_1(X)$. There exists a unique almost holomorphic fibration*[4]

$$\gamma : X \dashrightarrow \Gamma(X)$$

with the following property: let Z be a subspace with normalisation $Z' \to Z$ passing through a very general point $x \in X$. Then Z is contained in the fibre through x if and only if the natural map $\pi_1(Z') \to \pi_1(X) \to \Gamma$ has finite image. This fibration is called the Γ-reduction of X (Shafarevich map in the terminology of [24]*).*

By definition X is of π_1-general type (resp. the fundamental group $\pi_1(X)$ is generically large) if the $\pi_1(X)$-reduction is a bimeromorphic isomorphism [24, Defn.1.7] [7] (it corresponds to the case $\gamma d(X) = \dim(X)$ as defined in the Appendix).

2.A. Compactifiable subsets

2.3. — **Definition.** *Let $\tilde{X}$ be a normal complex space. We say that $\tilde{X}$ admits a Kähler compactification if there exists an embedding $\tilde{X} \hookrightarrow \overline{X}$ such that $\overline{X}$ is a normal, compact Kähler space and $\tilde{X}$ is Zariski open in $\overline{X}$.*

Let $\pi : \tilde{X} \to X$ be an infinite étale Galois cover with group Γ such that $\tilde{X}$ admits a compactification $\tilde{X} \subset \overline{X}$. In [14] we assumed that the compactification $\overline{X}$ is a projective variety and used the absence of algebraic Γ-invariant subsets to deduce important restrictions on the geometry of X. While algebraic subsets $Z \subset \tilde{X}$ are always Zariski open in some projective subset $\overline{Z} \subset \overline{X}$, this is no longer true if we consider analytic subspaces $Z \subset \tilde{X}$. We have to restrict our considerations to a smaller class:

[4] By unique we mean unique up to bimeromorphic equivalence of fibrations.

2.4. — **Definition.** *Let $\tilde{X}$ be a normal complex space such that we have a Kähler compactification $\tilde{X} \hookrightarrow \overline{X}$. A compactifiable subspace is a subset $Z \subset \tilde{X}$ such that there exists an analytic subspace $\bar{Z} \subsetneq \bar{X}$, an inclusion $Z \subset \bar{Z}$ and a subset $Z^* \subset Z$ such that such that $Z^* \subset \overline{Z}$ is dense and Zariski open. A compactifiable subset is a finite union of compactifiable subspaces.*

Remark. In general the compactification $\overline{Z}$ depends on the choice of $\overline{X}$. Moreover the property of being a compactifiable subset $Z \subset \tilde{X}$ might depend on the choice of the compactification since the natural map between two compactifications $\overline{X}$ and $\overline{X}'$ might have essential singularities. The subsets $Z \subset \tilde{X}$ which we will consider in this paper are typically defined by universal families over irreducible components of $\mathscr{C}(\tilde{X})$. The following lemma shows that these sets are always compactifiable if we assume that the compactification is Kähler.

2.5. — **Lemma.** *Let $\tilde{X}$ be a normal complex space such that we have a Kähler compactification $\tilde{X} \hookrightarrow \overline{X}$. Let $\tilde{\mathscr{H}}$ be an irreducible component of $\mathscr{C}(\tilde{X})$ and $\tilde{\mathscr{U}}$ be the universal family over $\tilde{\mathscr{H}}$. Let $\tilde{q} : \tilde{\mathscr{U}} \to \tilde{\mathscr{H}}$ and $\tilde{p} : \tilde{\mathscr{U}} \to \tilde{X}$ be the natural morphisms. Then $\tilde{p}(\tilde{\mathscr{U}})$ is a compactifiable subset.*

The idea of the proof is quite simple: $\tilde{\mathscr{H}}$ admits a natural compactification in $\mathscr{C}(\overline{X})$, the corresponding universal family compactifies $\tilde{p}(\tilde{\mathscr{U}})$. Since we use the statement several times we give the details of the proof.

Proof. — We set $D := \overline{X} \setminus \tilde{X}$. We have a natural inclusion $\mathscr{C}(\tilde{X}) \hookrightarrow \mathscr{C}(\overline{X})$ and we choose an irreducible component $\overline{\mathscr{H}}$ that contains the image of $\tilde{\mathscr{H}}$. Denote by $\overline{\mathscr{U}}$ the universal family over $\overline{\mathscr{H}}$, endowed with the reduced structure, and by $\bar{q} : \overline{\mathscr{U}} \to \overline{\mathscr{H}}$ resp. $\bar{p} : \overline{\mathscr{U}} \to \overline{X}$ the natural morphisms. We summarise the construction in a commutative diagram:

(2.1)
$$\begin{tikzcd}
\overline{\mathscr{U}} \arrow[rr, "\bar{p}"] \arrow[dd, "\bar{q}"'] & & \overline{X} = \tilde{X} \sqcup D \\
& \mathscr{U} \arrow[ul, hook'] \arrow[r, "\tilde{p}"] \arrow[d, "\tilde{q}"] & \tilde{X} \arrow[u, hook] \\
\overline{\mathscr{H}} & \tilde{\mathscr{H}} \arrow[l, hook'] &
\end{tikzcd}$$

The complex space $\overline{X}$ being compact Kähler, the spaces $\overline{\mathscr{H}}$ and $\overline{\mathscr{U}}$ are compact, hence by Remmert's proper mapping theorem $\bar{p}(\overline{\mathscr{U}})$ is a finite union of analytic subspaces of $\overline{X}$. Moreover $\bar{q}(\bar{p}^{-1}(D))$ is a finite union of analytic spaces and the inclusion $\bar{q}(\bar{p}^{-1}(D)) \subset \overline{\mathscr{H}}$ is strict since $\bar{q}(\bar{p}^{-1}(D))$ is disjoint from $\tilde{\mathscr{H}}$. Since $\tilde{\mathscr{H}}$ is an irreducible component of $\mathscr{C}(\tilde{X})$ this actually shows that $\overline{\mathscr{H}} = \tilde{\mathscr{H}} \sqcup \bar{q}(\bar{p}^{-1}(D))$. In particular $\overline{\mathscr{H}}$ is unique and the Zariski closure of

$\tilde{\mathscr{H}} \subset \mathscr{C}(\overline{X})$. Note now that $\bar{p}(\bar{q}^{-1}(\bar{q}(\bar{p}^{-1}(D))))$ is a finite union of analytic subspaces of $\bar{p}(\overline{\mathscr{U}})$ which for reasons of dimension does not contain any irreducible component of $\bar{p}(\overline{\mathscr{U}})$. Thus

$$Z^* := \bar{p}(\overline{\mathscr{U}}) \setminus \bar{p}(\bar{q}^{-1}(\bar{q}(\bar{p}^{-1}(D))))$$

is dense and Zariski open in $\bar{p}(\overline{\mathscr{U}})$, moreover we have an inclusion

$$Z^* \subset \bar{p}(\tilde{\mathscr{U}}) \subset \bar{p}(\overline{\mathscr{U}}).$$

Since $\bar{p}(\tilde{\mathscr{U}}) = \tilde{p}(\tilde{\mathscr{U}})$ this proves the statement. □

For certain irreducible components of $\mathscr{C}(\tilde{X})$ we can say more:

2.6. — **Corollary.** *In the situation of the lemma above suppose moreover that $\tilde{\mathscr{U}}$ is irreducible and $\tilde{p}$ is onto and generically finite. Set*

$$\tilde{B} = \{\tilde{x} \in \tilde{X} \mid \dim \tilde{p}^{-1}(\tilde{x}) > 0\}$$

Then $\tilde{B}$ is a compactifiable subset.

Proof. — As in the proof of the preceding lemma we consider the compactification $\tilde{\mathscr{H}} \subset \overline{\mathscr{H}}$ and the corresponding compactification of universal families $\tilde{\mathscr{U}} \subset \overline{\mathscr{U}}$.

1st case. $\tilde{p}$ is bimeromorphic. The morphism $\bar{p}$ is onto and bimeromorphic and we denote by $\bar{B}$ the image of its exceptional locus, which is of course a finite union of analytic subspaces. We have an inclusion $\tilde{B} \subset \bar{B}$ and we are done if we show that $\tilde{B} = \bar{B} \cap \tilde{X}$. To see this take $\tilde{x} \in \tilde{X}$ such that $\tilde{x} \notin \tilde{B}$, then $\bar{q}(\bar{p}^{-1}(\tilde{x}))$ is the union of $\tilde{q}(\tilde{p}^{-1}(\tilde{x}))$ and the cycles parametrised by $\bar{q}(\bar{p}^{-1}(D))$ passing through $\tilde{x}$. Yet $\tilde{q}(\tilde{p}^{-1}(\tilde{x}))$ is a singleton in $\tilde{\mathscr{H}}$, hence disjoint from $\bar{q}(\bar{p}^{-1}(D))$. Moreover $\bar{q}(\bar{p}^{-1}(\tilde{x}))$ is connected by Zariski's main theorem, so $\bar{p}^{-1}(\tilde{x})$ is a unique point. This proves the claim.

2nd case. $\tilde{p}$ generically finite. The morphism $\bar{p}$ is onto and generically finite, and we denote by $\bar{p}_{St} : \overline{\mathscr{U}} \to \overline{X}_{St}$ and $\mu : \overline{X}_{St} \to \overline{X}$ the Stein factorisation. Note that $\overline{X}_{St}$ contains a Zariski open dense subset $\tilde{X}_{St} := \mu^{-1}(\tilde{X})$ and

$$\mu^{-1}(B) = \{\tilde{x} \in \tilde{X}_{St} \mid \dim \tilde{p}_{St}^{-1}(\tilde{x}) > 0\}.$$

The first case shows that the right hand side is compactifiable, hence B is compactifiable. □

2.B. The Γ-reduction. — Let Y be a normal Kähler space such that we have a Kähler compactification $Y \hookrightarrow \overline{Y}$, and let $U \to V$ be a flat, proper Kähler fibration. In [14, Section 2] we introduced the moduli spaces $\mathrm{FinMor}(U/V, Y, d)$ of finite morphisms $\phi : U_v \to Y$ where U_v is a fibre of $U \to V$ and d is the degree of the graph of ϕ with respect to some fixed Kähler forms on U and $\overline{Y}$. These spaces of morphisms are Zariski open sets in the relative cycle spaces $\mathscr{C}(U \times \bar{Y}/V)$, so we know by Bishop's theorem that for bounded degree d there are only finitely many irreducible components. Moreover we have seen in Lemma 2.5 that the images of universal families are compactifiable subsets of Y. We can now argue as in [14, Lemma 2.4] to prove the following:

2.7. — **Lemma.** *Let X be a normal, compact Kähler space and $\pi : \tilde{X} \to X$ an infinite étale Galois cover with group Γ such that $\tilde{X}$ admits a Kähler compactification $\tilde{X} \subset \overline{X}$. Let $X^0 \subset X$ be a dense, Zariski open subset and $g^0 : X^0 \to Z^0$ a flat, proper fibration with general fiber F such that π induces a finite covering $\tilde{F} \to F$. Let $\tilde{g}^0 : \tilde{X}^0 \to \tilde{Z}^0$ be the corresponding flat, proper fibration with general fiber $\tilde{F}$. Then (at least) one of the following holds:*

1.) $\tilde{g}^0$ extends to a locally trivial, Γ-equivariant fibration $\tilde{g} : \tilde{X} \to \tilde{Z}$ with fibre $\tilde{F}$, or

2.) $\tilde{X}$ contains a compactifiable Γ-invariant subspace that is disjoint from a general fiber of $\tilde{g}^0$.

Since $\tilde{g} : \tilde{X} \to \tilde{Z}$ is Γ-equivariant, the Γ-action on $\tilde{X}$ descends to a Γ-action on $\tilde{Z}$. If $\tilde{F}$ has no fixed point free automorphisms, the Γ-action on $\tilde{Z}$ is fixed point free, but in general it can have finite stabilizers. Thus g^0 only extends to a fibration $g : X \to Z$ that is almost locally trivial (cf. Definition 2.1).

2.8. — **Corollary.** *Let X be a compact Kähler manifold and $\pi : \tilde{X} \to X$ an infinite étale Galois cover with group Γ such that $\tilde{X}$ admits a Kähler compactification $\tilde{X} \subset \overline{X}$. Suppose that $\tilde{X}$ does not contain any Γ-invariant compactifiable subsets.*

Then the Γ-reduction is an almost locally trivial holomorphic fibration $\gamma : X \to \Gamma(X)$ and the corresponding fibration $\tilde{\gamma} : \tilde{X} \to \Gamma(\tilde{X})$ is Γ-equivariant and locally trivial.

Let F be a general γ-fibre. If there exists an almost holomorphic map $\varphi_F : F \dashrightarrow W$ with general fibre G, this map extends to an almost locally trivial holomorphic map $\varphi : X \to Y$ and the corresponding fibration $\tilde{\varphi} : \tilde{X} \to \tilde{Y}$ is Γ-equivariant and locally trivial. We call φ a factorisation of the Γ-reduction with fibre G.

2.9. — **Remark.** Let G be a general φ-fibre. By definition of the Γ-reduction the natural map $\pi_1(G) \to \pi_1(X) \to \Gamma$ has finite image G_Γ, so π induces a finite étale cover $\tilde{G} \to G$ where $\tilde{G}$ is a general $\tilde{\varphi}$-fibre. Up to replacing X by the finite étale cover $X' \to X$ with Galois group G_Γ and Γ by

$$\Gamma' := \operatorname{im}(\pi_1(X') \to \pi_1(X) \to \Gamma)$$

we can suppose that G_Γ is trivial, hence $\tilde{G} \simeq G$. Since $\Gamma' \subset \Gamma$ has finite index, Γ is almost abelian if and only if this holds for Γ'.

2.10. — **Corollary.** *In the situation of Corollary 2.8, let $\varphi : X \to Y$ be a factorisation of the Γ-reduction with fibre G. Then G does not contain any rigid subspaces, i.e. there is no subspace $Z \subset G$ such that for all $m \in \mathbb{N}$ the Chow space has pure dimension 0 in the point $[mZ]$.*

Proof. — We argue by contradiction and suppose that such a subspace Z exists. For $y \in Y$ general, the fibration φ is locally trivial near y, i.e. there exists an analytic neighbourhood $y \in U \subset Y$ such that $\varphi^{-1}(U) \simeq U \times G$. In particular the relative Chow space $\mathscr{C}(\varphi^{-1}(U)/U)$ is isomorphic to a product $U \times \mathscr{C}(G)$, so there exists a unique irreducible component of $\mathscr{C}(X)$ parametrising deformations of Z in X that dominates Y. The reduction of this irreducible component is isomorphic to Y.

Note now that $\tilde{Z} := \pi^{-1}(Z)$ is a finite union of subspaces in $\tilde{G} := \pi^{-1}(G)$ which are rigid: otherwise their deformations would induce a deformation of some multiple of the cycle $[Z]$. Thus the deformations of $\tilde{Z}$ in $\tilde{X}$ correspond to an irreducible component of $\mathscr{C}(\tilde{X})$ whose reduction is isomorphic to $\tilde{Y}$. By Lemma 2.5 the deformations of $\tilde{Z}$ cover a compactifiable subset of $\tilde{X}$. Moreover it is Γ-invariant since it is the π-preimage of the locus covered by deformations of Z. Thus we have constructed a Γ-invariant compactifiable subset, a contradiction. □

2.11. — **Remark.** If $\mu : G \to G'$ is a bimeromorphic morphism onto a Kähler space, an irreducible component $Z \subset G$ of the μ-exceptional locus is rigid. Indeed if Z has dimension d, then $Z \cdot \mu^*\omega^d = \mu(Z) \cdot \omega^d = 0$ where ω is a Kähler form on G'. Thus if Z' is a small deformation of the cycle $[mZ]$, then $0 = Z' \cdot \mu^*\omega^d = \mu(Z') \cdot \omega^d$, so $\mu(Z')$ has dimension strictly smaller than d. Thus Z' is contained in the μ-exceptional locus. The complex space Z being an irreducible component of this locus, we have $\operatorname{Supp}(Z) = \operatorname{Supp}(Z')$.

2.12. — **Corollary.** *In the situation of Corollary 2.8, let $\varphi : X \to Y$ be a factorisation of the Γ-reduction with fibre G. Then the algebraic reduction $G \dashrightarrow A(G)$ is holomorphic.*

Proof. — By a theorem of Campana [6, Cor.10.1], the general fibres of the algebraic reduction[(5)] define an irreducible component of $\mathscr{C}(G)$, i.e. there exists an irreducible component $\mathscr{H}_G$ of $\mathscr{C}(G)$ such that the general point corresponds to a general fibre of the algebraic reduction. In particular if $\mathscr{U}_G$ is the universal family over $\mathscr{H}_G$, the natural morphism $\mathscr{U}_G \to G$ is onto and bimeromorphic. Using the local triviality of φ as in the proof of Corollary 2.10 above, we obtain an irreducible component $\mathscr{H}$ of $\mathscr{C}(X)$ such that the natural map $p : \mathscr{U} \to X$ is onto and bimeromorphic. If the image B of the p-exceptional locus is empty we are obviously done.

Suppose now that this is not the case. Then we have an irreducible component $\tilde{\mathscr{H}}$ of $\mathscr{C}(\tilde{X})$ such that the natural map $\tilde{p} : \tilde{\mathscr{U}} \to \tilde{X}$ is onto and bimeromorphic, moreover the set

$$\tilde{B} := \pi^{-1}(B) = \{\tilde{x} \in \tilde{X} \mid \dim \tilde{p}^{-1}(\tilde{x}) > 0\}$$

is Γ-invariant and compactifiable by Corollary 2.6. Again a contradiction to our assumption. □

Remark. Note that our proof heavily relies on the property that the general fibres of the algebraic reduction define an *irreducible component* of the Chow space. This holds for any almost holomorphic fibration, but fails in general: if $g : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ is the projection from a point x, the fibres correspond to lines through x, so they define a proper subset of the irreducible component of $\mathscr{C}(\mathbb{P}^n)$ parametrising lines.

We can now prove the main statement of this section:

2.13. — **Theorem.** *Let X be a compact Kähler manifold and $\pi : \tilde{X} \to X$ an infinite étale Galois cover with group Γ such that $\tilde{X}$ admits a Kähler compactification $\tilde{X} \subset \overline{X}$. Suppose that $\tilde{X}$ does not contain any Γ-invariant compactifiable subsets. Let $\varphi : X \to Y$ be a factorisation of the Γ-reduction such that the fibre G has minimal, but positive dimension. Then (up to replacing X by a finite étale cover) the manifold G is either projective or does not contain any positive-dimensional compact proper subspaces.*

Proof. — Suppose that G is not projective, i.e. $a(G) < \dim G$.

1st step. Suppose that $a(G) > 0$. Then by Corollary 2.12 the algebraic reduction $G \dashrightarrow A(G)$ is holomorphic. By Corollary 2.8 this induces a factorisation of the Γ-reduction whose fibres have strictly smaller dimension, a contradiction.

2nd step. Suppose that G is covered by positive-dimensional compact proper subspaces. Note first that a compact complex manifold G with $a(G) = 0$ contains only finitely many divisors [17]. Thus by Corollary 2.10 the manifold G

(5) Let $G' \subset G \times A(G)$ be the graph of $G \dashrightarrow A(G)$, and denote by $p_1 : G' \to G$ and $p_2 : G' \to A(G)$ the projections. Then the fibre over $a \in A(G)$ is defined as $p_1(p_2^{-1}(a))$.

contains no divisors since these would be rigid. Let now $\mathcal{H}_G \subset \mathcal{C}(G)$ be an irreducible component of the cycle space parametrising a covering family of positive-dimensional compact proper subspaces of maximal dimension. Then by Lemma 2.14 below the map from the universal family $p_G : \mathcal{U}_G \to G$ is generically finite. Arguing as in the proof of Corollary 2.12 we construct irreducible components $\mathcal{H} \subset \mathcal{C}(X)$ and $\tilde{\mathcal{H}} \subset \mathcal{C}(\tilde{X})$ such that the restriction to G is $\mathcal{H}_G$.

Thus the maps from the universal families $p : \mathcal{U} \to X$ and $\tilde{p} : \tilde{\mathcal{U}} \to \tilde{X}$ are onto and generically finite, hence the set

$$\tilde{B} = \{\tilde{x} \in \tilde{X} \mid \dim \tilde{p}^{-1}(\tilde{x}) > 0\}$$

is Γ-invariant and compactifiable by Corollary 2.6. By our hypothesis $\tilde{B}$ is empty, so the map p_G is finite. Since G is smooth and does not contain any divisors, we see by purity of the branch locus that (up to replacing $\mathcal{U}_G$ by its normalisation) the map p_G is étale. Hence p and $\tilde{p}$ are étale and $\tilde{\mathcal{U}}$ can be compactified by the universal family over the compactification $\tilde{\mathcal{H}} \subset \overline{\mathcal{H}}$. Thus up to replacing X by the finite étale cover $\mathcal{U} \to X$ we can suppose that p_G is an isomorphism. Yet then $G \simeq \mathcal{U}_G$ admits a natural fibration $q_G : \mathcal{U}_G \to \mathcal{H}_G$, so we get again a fibration with fibres of strictly smaller dimension, a contradiction.

3rd step. G has no positive-dimensional compact proper subspaces. Let $Z \subset G$ be a positive-dimensional compact proper subspace of maximal dimension and take $m \in \mathbb{N}$ arbitrary. Then the Chow scheme has dimension zero in the point $[mZ]$: indeed if $\mathcal{H}_G$ is an irreducible component passing through $[mZ]$, the map from the universal family $\mathcal{U}_G \to G$ is not onto by the 2nd step. By maximality of the dimension the image has dimension equal to $\dim Z$, so $\mathcal{H}_G$ has dimension zero. Thus Z is rigid, which is excluded by Corollary 2.10. □

2.14. — **Lemma.** *Let G be a compact Kähler manifold such that $a(G) = 0$. Let $\mathcal{H}_G \subset \mathcal{C}(G)$ be an irreducible component of the cycle space parametrising a covering family of positive-dimensional compact proper subspaces which have maximal dimension m, i.e. there is no covering family of proper subspaces with dimension strictly larger than m. Let $\mathcal{U}_G$ be the universal family over $\mathcal{H}_G$, and denote by $p_G : \mathcal{U}_G \to G$ and $q_G : \mathcal{U}_G \to \mathcal{H}_G$ the natural morphisms. Then p_G is generically finite.*

Proof. — Since G contains only finitely many divisors [17], we have $m < \dim G - 1$. We argue by contradiction, and suppose that the general p_G-fibre is positive-dimensional. Then for $g \in G$ general, the analytic set $q_G(p_G^{-1}(g))$ is positive-dimensional and Moishezon by [3, Cor.1]. In particular $q_G(p_G^{-1}(g))$ is covered by compact curves. Choose an irreducible curve $C_g \subset q_G(p_G^{-1}(g))$, then $p_G(q_G^{-1}(C_g))$ has dimension $m+1 < \dim G$. Since $g \in G$ is general we can construct in this way a covering family of strictly higher dimension, a contradiction. ☐

2.C. Fibre bundles. — Let us recall the following facts about the automorphism group of a compact Kähler manifold G [18, 26]. The identity component $\mathrm{Aut}^0(G)$ of the complex Lie group $\mathrm{Aut}(G)$ has a description in terms of the Albanese torus of G. Consider the natural map

$$\mathrm{Aut}^0(G) \longrightarrow \mathrm{Aut}^0(\mathrm{Alb}(G)) \simeq \mathrm{Alb}(G)$$

induced by the Albanese mapping. The kernel of this morphism is a linear algebraic group and the image is a subtorus of $\mathrm{Alb}(G)$; in particular, if G is not covered by rational curves, $\mathrm{Aut}^0(G)$ is a compact group, isogeneous to a subtorus of $\mathrm{Alb}(G)$. If $\mathrm{Aut}^0(G)$ is a point, fibre bundles with fibre G can be easily described.

2.15. — **Lemma.** [18, Cor.4.10] *Let $\varphi : X \to Y$ be a proper fibre bundle with fibre a manifold G. Suppose that the automorphism group $Aut(G)$ is discrete and that the fibration φ is Kähler, i.e. there exists a two-form ω on X such that the restriction $\omega|_G$ is a Kähler form. Then there exists a finite étale base change $Y' \to Y$ such that $X \times_Y Y' \simeq Y' \times G$.*

Sketch of proof.. — The Kähler assumption on the morphism implies that the structure group of the fibre bundle can be reduced to $\mathrm{Aut}(F, [\omega|_F])$, the group of automorphisms of F which preserves the cohomology class of $\omega|_F$. By Fujiki-Lieberman this group contains $\mathrm{Aut}^0(G) = \{1\}$ as a finite index subgroup and is thus finite. Thus the image of the monodromy presentation is finite, hence trivial after finite étale base change. ☐

3. Proofs of the main results

Proof of Theorem 1.3. — We claim that $\tilde{X}$ has no nontrivial, compactifiable subset $\tilde{W}$ invariant under a finite index subgroup $\Gamma' \subset \Gamma$. Note first that if such a $\tilde{W}$ exists, we can suppose it to be analytic: otherwise replace it by $\tilde{X} \cap \overline{W}$ where $\overline{W} \subset \overline{X}$ is a compactification. If we denote by $\tilde{W}_i$ the irreducible components of $\tilde{W}$, each of them is invariant under a finite index subgroup $\Gamma_i \subset \Gamma$. Taking the normalization $\tilde{W}_i^n$, we would get a smaller dimensional example $W_i^n := \tilde{W}_i^n / \Gamma_i$ as in Theorem 1.3; a contradiction.

Since $\tilde{X}_{\mathrm{Sing}} \subset \tilde{X}$ is compactifiable and Γ-invariant, we conclude that X is smooth. We claim that X is not uniruled, i.e. it is not covered by rational curves: otherwise we can consider the MRC-fibration $\varphi : X \dashrightarrow Z$. Since the general φ-fibre G is rationally connected, hence projective and simply connected [7] [15, Cor.4.18], the MRC-fibration is a factorisation of the Γ-reduction. By Corollary

2.8 it extends to an almost locally trivial holomorphic map $\varphi : X \to Z$ and the corresponding fibration $\tilde{\varphi} : \tilde{X} \to \tilde{Z}$ is Γ-equivariant and locally trivial with fibre G. Note that G does not admit fixed point free actions by any finite group: the étale quotient would also be rationally connected, so simply connected. Therefore the stabilizer $\mathrm{stab}_\Gamma(F_z)$ is trivial for every $\tilde{\varphi}$-fiber F_z. Hence the Γ-action descends to a free Γ-action on $\tilde{Z}$; a contradiction to the minimality of the dimension of X.

Arguing by contradiction we will now prove that the Γ-reduction γ is an isomorphism. If this is not the case we know by Theorem 2.13 that there exists a factorisation $\varphi : X \to Y$ of the Γ-reduction such that the fibre G is either projective or without positive-dimensional compact proper subspaces. Let us consider the corresponding locally trivial fibration $\tilde{\varphi} : \tilde{X} \to \tilde{Y}$ with fibre G (cf. Remark 2.9).

1st case. $Aut^0(G)$ is a point. The structure group of the fiber bundle $\tilde{\varphi} : \tilde{X} \to \tilde{Y}$ is discrete, so by Lemma 2.15 we can suppose (after finite étale cover) that $\tilde{X} \simeq \tilde{Y} \times G$. The Γ-action on $\tilde{X}$ commutes with the projection on $\tilde{Y}$ and the group $\mathrm{Aut}(G)$ is discrete, so Γ acts diagonally on the product $\tilde{Y} \times G$. Consider now the $\tilde{\varphi}$-section $\tilde{Y} \times g$ for some $g \in G$: its orbit $\cup_{f \in \Gamma} f(\tilde{Y} \times g)$ is a finite union of sections of the form $\tilde{Y} \times g'$. The complex space $\tilde{Y}$ has a natural compactification in $\mathscr{C}(\overline{X})$, since it parametrises the $\tilde{\varphi}$-fibres. Thus we have constructed a Γ-invariant compactifiable subset of $\tilde{X}$, a contradiction to the minimality of X.

2nd case. $Aut^0(G)$ has positive dimension. Since G is not uniruled we know by Section 2.C that the group $\mathrm{Aut}^0(G)$ is isogeneous to a subtorus of the Albanese torus of G, in particular we have $q(G) > 0$. We claim that in this case G is a torus. Assuming this for the time being, let us see how to conclude. Since φ is almost locally trivial we can apply [12, §6], [19, Prop.4.5]: after a finite étale cover $X' \to X$ we can suppose that $q(X) = q(Y) + \dim G$. Since every φ-fibre is irreducible, the Albanese map $\alpha_X : X \to \mathrm{Alb}(X)$ maps each φ-fibre isomorphically onto a fibre of the locally trivial fibration $\varphi_* : \mathrm{Alb}(X) \to \mathrm{Alb}(Y)$. By the universal property of the fibre product we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Alb}(X) \times_{\mathrm{Alb}(Y)} Y \arrow[drr, "\psi"'] & & X \arrow[ll] \arrow[r, "\alpha_X"] \arrow[d, "\varphi"] & \mathrm{Alb}(X) \arrow[d, "\varphi_*"] \\
& & Y \arrow[r, "\alpha_Y"] & \mathrm{Alb}(Y)
\end{tikzcd}$$

The map ψ is the pull-back of φ_* by the fibre product, so it is a locally trivial fibration. The base Y is normal, so the total space $\mathrm{Alb}(X) \times_{\mathrm{Alb}(Y)} Y$ is normal. By what precedes the morphism $X \to \mathrm{Alb}(X) \times_{\mathrm{Alb}(Y)} Y$ is bimeromorphic and finite, hence an isomorphism by Zariski's main theorem. In particular $\varphi = \psi$ is smooth and locally trivial. Hence the Γ-action on $\tilde{X}$ descends to a free action on $\tilde{Y}$, i.e. X is not a minimal counterexample.

Proof of the claim. This is clear if G has no positive-dimensional compact proper subspaces, so we can suppose that G is projective. Since $q(G) \neq 0$ the Albanese map is non-trivial. By minimality of the factorisation we see that the Albanese map $G \to A(G)$ is generically finite onto its image. Since G admits no bimeromorphic map by Remark 2.11, it is actually finite onto its image. Moreover G is not of general type since $\mathrm{Aut}^0(G)$ has positive dimension. If we have $\dim G > \kappa(G) > 0$ we know by [22] that G admits a fibration by positive-dimensional abelian varieties, a contradiction to the minimality of the factorisation. Thus we have $\kappa(G) = 0$ and G is an abelian variety. This proves the claim.

Let us finally show that X does not admit any Mori contraction, i.e. does not admit any morphism with connected fibres $\mu : X \to X'$ onto a normal complex space X' such that $-K_X$ is μ-ample. Since X is not uniruled, μ would necessarily be bimeromorphic. Moreover μ is a projective morphism since it is polarised by $-K_X$. In particular the Ionescu-Wiśniewski inequality [21, Thm.0.4], [32, Thm.1.1] applies and shows that if E is an irreducible component of the exceptional locus and F a general fibre of $E \to \mu(E)$, then one has $\dim E + \dim F \geq \dim X$. Arguing as in the [14, Lemma 2.5] we can now prove that $\pi^{-1}(E)$ is a Γ-invariant compactifiable subset of $\tilde{X}$, contradicting the minimality of X. □

The proof of Proposition 1.4 relies on the following generalisation of the Kobayashi-Ochiai theorem to fibrations of general type.

3.1. — **Theorem.** [23, Thm.2][9, Thm.8.2] *Let X be a compact Kähler manifold, $\overline{X}$ be a complex manifold, and let $B \subset \overline{X}$ be a proper closed analytic subset. Let $\pi : \overline{X} \setminus B \dashrightarrow X$ be a nondegenerate meromorphic map,* i.e. *such that the tangent map $T_{\overline{X} \setminus B} \to T_X$ is surjective at least at one point $v \in \overline{X} \setminus B$. Let us finally consider $g : X \dashrightarrow Y$ a general type fibration*[(6)] *defined on X. Then $f = g \circ \pi$ extends to a meromorphic map $\overline{X} \dashrightarrow Y$.*

Proof of Proposition 1.4. — As in the proof of Theorem 1.3 the minimality condition implies that $\tilde{X}$ does not contain any Γ-invariant compactifiable subset. Since the singular locus $\tilde{X}_{\mathrm{Sing}}$ is Γ-invariant and naturally compactified by $\overline{X}_{\mathrm{Sing}}$, we see that X is smooth.

Let us now argue by contradiction and suppose that X is not special. Then the core fibration $c_X : X \dashrightarrow C(X)$ [9, section 3] is not trivial, i.e. the base has dimension at least one. Moreover it is a general type fibration, so by Theorem

(6) We refer to [9] for the basic definitions of the orbifold theory.

3.1 above the composed map $c_X \circ \pi : \tilde{X} \dashrightarrow C(X)$ extends to a meromorphic map $\bar{c}_X : \overline{X} \dashrightarrow C(X)$. Up to replacing $\overline{X}$ by a suitable bimeromorphic model, we can assume that $\bar{c}_X$ is holomorphic. Since $\overline{X}$ is proper, a general $\bar{c}_X$-fibre $\bar{F} := \bar{c}_X^{-1}(y)$ has finitely many irreducible components, each of them of dimension $\dim X - \dim C(X)$ and not contained in $\overline{X} \setminus \tilde{X}$. Thus if $F := c_X^{-1}(y)$ is the corresponding c_X-fibre, then $\pi^{-1}(F) = \bar{F} \cap \tilde{X}$. Thus $\pi^{-1}(F)$ is a Γ-invariant compactifiable subset of $\tilde{X}$, a contradiction. □

As mentioned in the introduction, the proof given in [14] of the local triviality of the Albanese map does not apply verbatim in this non algebraic setting. We need the following purely topological result.

3.2. — **Theorem.** [25, Thm. 14] *Let $f : Z \to A$ be a map between compact analytic spaces, the universal cover $\tilde{A}_{univ}$ being contractible. Let us denote by $\tilde{Z}$ the induced cover of Z. If $\tilde{Z}$ has the homotopy type of a compact metric space, then f is surjective.*

With this in mind, we can prove Theorem 1.5 in the spirit of [25].

Proof of Theorem 1.5. — To begin with, let us recall that classical arguments (see [14]) show that the Albanese map is always a fibration (*i.e.* a surjective map with connected fibres) when $\tilde{X}_{univ}$ is a Zariski open subset of a compact complex manifold. Since $\pi_1(X)$ is supposed to be almost abelian, we can also assume that

$$\alpha_X : X \to A := \mathrm{Alb}(X)$$

is a fibration which induces an isomorphism at the level of fundamental groups. Let us consider now F a smooth fibre of α_X and denote by f its homology class. Let us introduce $\mathcal{Z}_F(X)$ the unique irreducible component of $\mathcal{C}(F \times X)$ which contains the graph of the embeddings $j : F \hookrightarrow X$ whose homology class is fixed: $j_*[F] = f$ in $\mathrm{H}_*(X, \mathbb{Z})$. The homology class being fixed, there is a natural map:

$$\alpha_* : \mathcal{Z}_F(X) \to A$$

(the complex space $\mathcal{Z}_F(X)$ should be thought as the set of fibres of α_X which are isomorphic to F). Our aim is to apply Theorem 3.2 to show that α_* is surjective; we have to prove some topological finiteness of the fibre product:

$$\widetilde{\mathcal{Z}_F(X)} \longrightarrow \tilde{A}_{univ}.$$

Since α_X induces an isomorphism on the π_1, the induced map $\tilde{X}_{\mathrm{univ}} \to \tilde{A}_{\mathrm{univ}}$ between the universal covers is proper. We can then choose $\tilde{F}$ any lifting of F (it is a compact submanifold of $\tilde{X}_{univ} \subset \overline{X}$) and perform the same construction on $\overline{X}$: we denote by $\mathcal{Z}_{\tilde{F}}(\overline{X})$ the complex space of embeddings of $\tilde{F}$ into $\overline{X}$ whose homology class is given by $[\tilde{F}] \in \mathrm{H}_*(\overline{X}, \mathbb{Z})$. It is easily checked that there is a natural inclusion:

$$\widetilde{\mathcal{Z}_F(X)} \hookrightarrow \mathcal{Z}_{\tilde{F}}(\overline{X})$$

which realises $\widetilde{\mathcal{Z}_F(X)}$ as a Zariski open subset of $\mathcal{Z}_{\tilde{F}}(\overline{X})$. The compactification $\overline{X}$ being Kähler, the irreducible components of its cycle space are compact, furthermore there are only finitely many components since the homology class is fixed. Since $\widetilde{\mathcal{Z}_F(X)}$ is a Zariski open set in a compact complex space, it has the homotopy type of a finite CW complex. Thus $\alpha_* : \mathcal{Z}_F(X) \to A$ is surjective by Theorem 3.2. If the Albanese map α_X is equidimensional, this already shows that α_X is locally trivial: every φ-fibre contains the image of an embedding $F \hookrightarrow X$, since the cohomology class is fixed the manifold F is the whole fibre.

We will now prove by contradiction that α_X is equidimensional, and denote by $\varnothing \neq \Delta \subset A$ the locus where this is not the case. Set $Z := \alpha_X^{-1}(\Delta)$ and consider the map $f := \alpha_X|_Z : Z \to A$. The map f is not surjective since α_X is generically smooth. We claim that the induced cover $\tilde{Z} := Z \times_A \tilde{A}_{\mathrm{univ}}$ is Zariski open in a compact complex space, which as before leads to a contradiction to Theorem 3.2.

Proof of the claim. Let $\overline{\mathcal{U}}$ be the universal family over the unique component $\overline{\mathcal{H}}$ of $\mathcal{C}(\overline{X})$ such that the general point corresponds to a general fibre of the fibration $\tilde{X}_{\mathrm{univ}} \to \tilde{A}_{\mathrm{univ}}$. There is a natural bimeromorphic map $\overline{p} : \Gamma \to \overline{X}$ and $\tilde{Z} \subset \tilde{X}_{\mathrm{univ}}$ corresponds to the points $x \in \tilde{X}$ such that $\overline{p}^{-1}(x)$ has positive dimension. By Lemma 2.5 this set is compactifiable. □

3.3. — **Remark.** The arguments above are inspired by the proof of [25, Thm.20] and they rely heavily on the Kähler assumption on the compactification $\overline{X}$: in general the irreducible components of $\mathcal{C}(\overline{X})$ are not compact if $\overline{X}$ is merely a compact complex manifold.

However, Theorem 3.2 and [25, Thm.16] are valid in the category of compact analytic spaces. This leads us to raise the following question: is there an equivalent statement of [25, Thm.20] in the compact complex category? More precisely let us consider $X \to A$ a morphism between compact complex spaces and let us assume that $\tilde{A}_{univ}$ is contractible. If the induced cover $\tilde{X}$ is a Zariski open set in a compact complex manifold $\overline{X}$, is $X \to A$ a fibre bundle ? We do not know any example where this is not the case.

Appendix A
Abelianity, Iitaka and S conjectures

by Frédéric Campana

Abstract. — In this appendix, we observe that Iitaka's conjecture fits in the more general context of special manifolds, in which the relevant statements follow from the particular cases of projective and simple manifolds.

Recall from [9] (for which we refer to the notions involved in the following):

A.1. — **Conjecture.**(Abelianity Conjecture, [9, Conj.7.11]) *Let X be a special manifold. Then $\pi_1(X)$ is almost abelian.*

A.2. — **Remark.** 1. This conjecture is, at least, true for the linear representations of the fundamental group, which have almost abelian image [9, th.7.8].
2. The conjecture is also true up to dimension three [10].

Recall from [7] that for any compact Kähler manifold there exists a unique connected surjective almost holomorphic map: $\gamma_X : X \dashrightarrow \Gamma(X)$ such that its fibre X_a through the general point $a \in X$ is the largest subspace Y of X through a such that the image of $\pi_1(\hat{Y})$ in $\pi_1(X)$ is finite, $\hat{Y}$ being the normalisation of Y. Then $\gamma d(X) := \dim(\Gamma(X))$ is called the γ-dimension of X. Thus $\gamma d(X) = 0$ if and only if $\pi_1(X)$ is finite. When $\gamma d(X) = \dim(X)$, we say (as in [13]) that X is of π_1-general type.

A.3. — **Conjecture.**(Conjecture **S**) *Let X be a special manifold with $\gamma d(X) = \dim(X)$. Then some finite étale cover of X is bimeromorphic to a complex torus.*

A.4. — **Remarks.** *1. Conversely, if some finite étale cover of X is bimeromorphic to a complex torus, X is special with $\gamma d(X) = \dim(X)$.*

2. This conjecture **S** *implies the conjecture of Iitaka which claims the same conclusion as in S assuming that the universal cover of X is $\mathbb{C}^n$. The latter hypothesis is indeed weaker (by the orbifold version of the theorem of Kobayashi-Ochiai* [9, th.7.11 and 8.11]*).*

3. The Abelianity conjecture implies the conjecture **S** *(and so the conjecture of Iitaka). Indeed, if X is special of maximal γ-dimension, its fundamental group is torsionfree and abelian (by going to a suitable finite étale cover), if one assumes the Abelianity conjecture. Its Albanese map is then surjective with connected fibres (by* [9, th.5.3]*) and induces an isomorphism on the first homology groups. It has thus to be bimeromorphic, by the maximality of $\gamma d(X)$.*

A.5. — **Definition.** *We shall say that X is* primitively special *if it is special but not covered by special submanifolds of intermediate dimension $0 < d < \dim(X)$.*

A.6. — **Lemma.** *X is primitively special if and only if either:*

1.) X is a rational or elliptic curve, or
2.) X is projective with K_X pseudo-effective and with $\kappa(X)$ either 0, or $-\infty$ (according to the abundance conjecture, this last case does not exist), or
*3.) X is simple (*i.e.* not a curve and not covered by compact subspaces of intermediate dimensions).*

Proof. — We assume that X is primitively special with $n \geq 2$. We distinguish 3 cases.

Let us first assume X to be projective. Then $\kappa(X) < n$, since X is special. If $0 \leq \kappa(X) < n$, X is covered by submanifolds having $\kappa = 0$ and intermediate dimension $n - \kappa(X)$ if $\kappa(X) > 0$, which is impossible if X is primitively special. Thus $\kappa(X) = 0$ and K_X is pseudo-effective. If K_X is not pseudo-effective, then X is uniruled, by [2] and [27]. The only remaining case is thus when K_X is pseudo-effective and $\kappa(X) = -\infty$ (which Abundance conjecture claims not to exist). The projective case is thus established.

Assume now that $0 < a(X) < n$. Because the fibres of the algebraic reduction are special, by [9, th.2.39], X is not primitively special.

Assume finally that $a(X) = 0$, and that X is not simple, but primitively special. Let Z_t be a covering family of X by an analytic family of subspaces which are generically irreducible and of intermediate dimension $0 < d < n$, chosen to be minimal. The generic member of this family is thus either of general type, or special and primitively special. The second possibility is excluded, since X is primitively special. Thus Z_t is of general type. Let then $\varphi : X \dashrightarrow Y$ be the quotient by the equivalence relation generated by the $Z_t's$ [4]. Its fibres are projective, by [4]. Since $a(X) = 0$, we have: $a(Y) = 0$, and $\dim(Y) > 0$. From [5], we get that the fibres of φ are almost-homogeneous, hence special. Contradiction since X was assumed to be primitively special. Thus X is simple. □

A.7. — **Remark.** The above argument is partially inspired by [20].

A.8. — **Theorem.** *The conjecture* **S** *(and so the conjecture of Iitaka) is true if* **S** *is true whenever X is primitively special. In particular, the conjecture* **S** *is true if it is true in the projective and simple cases.*

Proof. — Assume thus that X is special, with $\gamma d(X) = n$. If X is primitively special, we assume that **S** is true. So assume that X is not primitively special and let Z_t be a covering family of subspaces which are special of intermediate dimension $0 < d < n$. We may assume that the generic member Z_t is smooth, after suitable blow-ups of X. We may thus assume that the conjecture holds true for the generic Z_t, by working inductively on n. Thus the fundamental group of the generic Z_t is, in particular, almost abelian.

Let again $\varphi : X \dashrightarrow Y$ be the quotient by the equivalence relation generated by the $Z_t's$. Its fibres are special [9, th.3.3] (since they are connected by chains of special subspaces) and have maximal γ-dimension, since this is the case for X.

There are 2 cases, according to $m := dim(Y)$.

(i) $m = 0$. In this case, the fundamental group of X is almost abelian, by [8], since X is generated by connected chains of $Z_t's$, which have almost abelian fundamental groups. Since X has maximal γ-dimension, the conjecture **S** holds for X, by remark A.4 above.

(ii) $n > m > 0$. In this case the generic fibres of ϕ have an étale cover bimeromorphic to a torus. From [30], we conclude that X has an étale cover bimeromorphic to some X' having a submersion $\psi : X' \to V$ on a manifold V of maximal γ-dimension, with fibres complex tori. Because V is special, since so is X' [9, th.5.12], the conjecture **S** is true for V, so that its fundamental group is almost abelian. From [8] again, we deduce that the fundamental group of X is almost abelian and that the conjecture **S** holds true for X, as claimed. □

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