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COCYCLES OVER PARTIALLY HYPERBOLIC MAPS

by

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1. Partially hyperbolic diffeomorphisms

A diffeomorphism $f\colon M \to M$ on a compact manifold M is *partially hyperbolic* if there exists a continuous, nontrivial Df-invariant splitting

$$T_xM = E^s_x \oplus E^c_x \oplus E^u_x, \quad x \in M$$

of the tangent bundle such that the derivative is a contraction along E^s and an expansion along E^u, with uniform rates, and the behavior of Df along the *center bundle* E^c is in between its behaviors along E^s and E^u, again by a uniform factor. Partial hyperbolicity is a natural generalization of the notion of uniform hyperbolicity (Anosov or even Axiom A, see [**25**]), that includes many interesting additional examples, most notably: diffeomorphisms derived from Anosov through deformation by isotopy, many affine maps on homogeneous spaces, certain skew-products over hyperbolic maps, and time-1 maps of Anosov flows. Partial hyperbolicity is an open condition, so any C^1 small perturbation of these examples is partially hyperbolic as well.

The *stable* and *unstable bundles*, E^s and E^u, are uniquely integrable; that is, there exist unique f-invariant foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ tangent to E^s and E^u, respectively, at all points. The leaves of these foliations are C^k if the diffeomorphism is C^k, for any $1 \le k \le \infty$, but the foliations are usually not transversely smooth. On the other hand, if f is twice differentiable then each $\mathcal{W}^s$ and $\mathcal{W}^u$ is absolutely continuous, meaning that its holonomy maps preserve the class of zero Lebesgue measure sets. These facts go back to the pioneering work of Brin, Pesin [**6**] and Hirsch, Pugh, Shub [**15**] where partial hyperbolicity and the closely related notion of normally hyperbolic foliations were introduced.

In general, the center bundle E^c need not be integrable, and similarly for the center stable bundle $E^{cs} = E^c \oplus E^s$ and the center unstable bundle $E^{cu} = E^c \oplus E^u$. We call the diffeomorphism *dynamically coherent* if E^{cs} and E^{cu} are tangent to foliations

$\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ respectively. Then intersecting the leaves of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$, one obtains an integral foliation $\mathcal{W}^{c}$ for the center bundle as well. As it turns out, dynamical coherence does hold in many situations of interest.

Brin, Pesin **[6]** also introduced the notion of accessibility, which has played a central role in recent developments. A partially hyperbolic diffeomorphism is called *accessible* if any two points in the ambient manifold may be joined by an *su-path*, that is, a piecewise smooth path such that every smooth subpath is contained in a single leaf of $\mathcal{W}^{s}$ or a single leaf of $\mathcal{W}^{u}$. More generally, the diffeomorphism is *essentially accessible* if, given any two sets with positive volume, one can join some point of one to some point of the other by an *su*-path.

Interest in partially hyperbolic systems was greatly renewed in the mid-nineties, with two initial goals in mind. One goal was to characterize robust (or stable) transitivity, both in discrete time and continuous time. A dynamical system is *transitive* if it possesses orbits that are dense in the whole ambient space. The best known examples are all of the known constructions of Anosov diffeomorphisms (see **[25]**). Actually, since Anosov maps form an open subset of all C^1 diffeomorphisms, these are also examples of *robust* transitivity. On the other hand, early constructions by Shub **[24]** and Mañé **[17]** showed that diffeomorphisms can be robustly transitive without being Anosov. Many other examples were found by Bonatti, Díaz **[2]** and Bonatti, Viana **[5]**. A subsequent series of works started by Díaz, Pujals, Ures **[10]** for diffeomorphisms, and Morales, Pacifico, Pujals **[18]** for flows, established that in dimension three robustness implies partial hyperbolicity (where at least two of the bundles in the partially hyperbolic splitting are non-trivial). In higher dimensions one has to replace partial hyperbolicity by a related weaker condition called existence of a dominated splitting. See **[3, 5]** and also [**4**, Chapter 7] and references therein.

Another goal, initiated by Grayson, Pugh, Shub **[14]**, was to recover the original attempt by Brin, Pesin **[6]** to prove that most partially hyperbolic, volume preserving diffeomorphisms are actually ergodic. To this end, Pugh, Shub **[20]** proposed the following pair of conjectures:

Conjecture 1. — Accessibility holds for an open and dense subset of C^2 partially hyperbolic diffeomorphisms, volume preserving or not.

Conjecture 2. — A partially hyperbolic C^2 volume preserving diffeomorphism with the essential accessibility property is ergodic.

Concerning Conjecture 1, it was shown by Dolgopyat, Wilkinson **[12]** that accessibility holds for a C^1-open and dense subset of all partially hyperbolic diffeomorphisms, volume preserving or not. Moreover, Didier **[11]** proved that accessibility is C^1-open for systems with 1-dimensional center bundle. More recently, Rodriguez Hertz, Rodriguez Hertz, Ures **[23]** verified the complete conjecture for conservative systems whose center bundle is one-dimensional: accessibility is C^r-dense among C^r partially hyperbolic diffeomorphisms, for any $r \geq 1$. A version of this statement for non-conservative diffeomorphisms was obtained in [**7**]. It remains open whether C^r-density still holds when $\dim E^c > 1$.

Partial versions of Conjecture 2 were obtained by Pugh, Shub [**20, 21, 22**], assuming dynamical coherence and an additional technical condition they called center bunching. Roughly speaking, their notion of center bunching means that the diffeomorphism is close to being an *isometry* along center leaves. The best result to date on Conjecture 2 is due to Burns, Wilkinson [**8**] who proved ergodicity for any accessible, partially hyperbolic volume preserving diffeomorphism (not necessarily dynamically coherent) which is not too far from being *conformal* along center leaves. Although this property is also called center bunching, it is a lot milder than the one of Pugh, Shub. In particular, it is automatic when E^c has dimension one. Thus, the previous result contains as a corollary a complete proof of Conjecture 2 when the center bundle is one-dimensional. This corollary was also observed in [**23**].

2. Cocycles

The problems considered in this volume are situated in the following context. Let $f\colon M \to M$ be a diffeomorphism. We fix a (topological, Lie...) group H with identity element e and consider the set of all (continuous, Hölder continuous, smooth...) functions $\phi\colon M \to H$. Such a function is called a *cocycle*, for reasons that are explained in the sequel. Cocycles are objects that can be composed along orbits of f, and indeed, by the cocycle generated by ϕ we often mean the sequence ϕ_n defined by

$$\phi_n(x) = \begin{cases} \phi(f^{n-1}(x)) \cdots \phi(f(x)) \cdot \phi(x) & \text{if } n > 0,\\ \phi^{-1}(f^{-n}(x)) \cdots \phi^{-1}(f^{-2}(x)) \cdot \phi^{-1}(f^{-1}(x)) & \text{if } n < 0,\\ e & \text{if } n = 0.\end{cases}$$

An equivalent definition of a cocycle, and one that generalizes to actions of groups other than $\mathbb{Z}$, is the following. A 1-cocycle is a map $\alpha\colon \mathbb{Z} \times M \to H$ satisfying the *cocycle condition:*

$$\alpha(m+n, x) = \alpha(m, f^n(x)) \cdot \alpha(n, x), \qquad \forall n, m \in \mathbb{Z},\ x \in M. \tag{1}$$

Setting $\phi(x) = \alpha(1, x)$, we obtain from the cocycle condition that $\phi_n(x) = \alpha(n, x)$, thereby establishing the equivalence of the two notions.

There are several contexts in which cocycles arise immediately in smooth dynamics and related topics, which we now discuss.

Abelian cocycles. — The cocycle ϕ is called *abelian* when the group H is abelian. A fundamental example of an abelian cocycle is the Jacobian map $\operatorname{Jac} f\colon M \to \mathbb{R}_*$ that measures the volume distortion of a diffeomorphism $f\colon M \to M$ on a Riemannian manifold M:

$$\operatorname{Jac} f(x) = \frac{d(\mathrm{vol} \circ f)}{d\,\mathrm{vol}}(x).$$

The 1-cocycle generated by $\operatorname{Jac} f$ is $\alpha(n,x) = \operatorname{Jac} f^n(x)$; in this case the cocycle condition amounts to the composition law for Radon-Nikodym derivatives. Usually this cocycle is transformed to an additive cocycle by taking a logarithm: $\log \operatorname{Jac} f \colon M \to \mathbb{R}$.

Abelian cocycles appear more generally as potentials in thermodynamic formalism. In this setup, one associates to each cocycle $\phi\colon M \to \mathbb{R}$ over a dynamical system $f\colon M \to M$ one or more f-invariant probability measures μ_ϕ satisfying the variational equation

$$\int_M \phi \, d\mu_\phi + h(\mu_\phi) = \sup_\nu \left(\int_M \phi \, d\nu + h(\nu) \right),$$

where the supremum on the right is taken over all f-invariant probability measures ν, and $h(\nu)$ denotes the f-entropy of the measure ν. The functional

$$P(\phi) = \sup_\nu \left(\int_M \phi \, d\nu + h(\nu) \right),$$

called the *pressure* of ϕ, has the property that if

$$\phi - \psi = \Phi \circ f - \Phi, \tag{2}$$

for some function Φ, then $P(\phi) = P(\psi)$. Hence the measure μ_ϕ depends only on the equivalence equivalence class for the equivalence relation $\phi \sim \psi$ if and only if (2) holds. As we describe below, this equation can be viewed as a coboundary equation in the appropriate cohomology theory.

Another place in which abelian cocycles appear, this time in the context of $\mathbb{R}$-actions, is in time changes in flows. Suppose that φ_t is a flow. If $\gamma\colon M \to \mathbb{R}$, then the function $\alpha\colon \mathbb{R} \times M \to \mathbb{R}$ defined by

$$\alpha(t,x) = \int_0^t \gamma(\varphi_s(x)) \, ds$$

satisfies the cocycle condition:

$$\alpha(s+t,x) = \alpha(s, \varphi_t(x)) + \alpha(t,x), \tag{3}$$

which is the natural analogue of (1) for $\mathbb{R}$-actions. In general, if $\alpha\colon \mathbb{R} \times M \to \mathbb{R}$ is an arbitrary function, then the map $\psi^\alpha\colon \mathbb{R} \times M \to M$ given by

$$\psi^\alpha(t,x) = \varphi_{\alpha(t,x)}(x)$$

will define a flow on M if and only if α satisfies (3). Here too, one has a coboundary equation which corresponds to (2) for flows:

$$\alpha(t,x) - \beta(t,x) = \int_0^t \gamma(\varphi_s(x)) \, ds. \tag{4}$$

One can check that if Equation (2) is satisfied for cocycles α and β and some real-valued function γ, then the flows φ^α and φ^β are time changes of one another.

Linear cocycles. — By a *linear cocycle* we will mean a cocycle with values in a matrix group. Such non-abelian cocycles also arise naturally, most notably as derivative cocycles. Suppose that $f\colon M \to M$ is a diffeomorphism of an n-manifold M. To avoid technical issues, assume that the tangent bundle TM is trivial:

$$TM = M \times \mathbb{R}^d.$$

Then the derivative Df can be represented as a map $Df\colon M \to \mathrm{GL}(d, \mathbb{R})$ which, by the Chain Rule, satisfies the (non-abelian) cocycle condition:

$$D_x f^{n+m} = D_{f^m(x)} f^n \cdot D_x f^m.$$

(We remark that the case where TM is non-trivial can be handled with a slight generalization of the notion of cocycle, using sections of an appropriate bundle.) The group $\mathrm{GL}(d, \mathbb{R})$ can be replaced by other matrix groups, such as $\mathrm{SL}(d, \mathbb{R})$, $\mathrm{Sp}(d, \mathbb{R})$, $O(d)$, $U(d)$, etc. Such group-valued cocycles arise naturally as diffeomorphism cocycles that are volume preserving, symplectic, isometric, and so on, as well as in the study of frame flows on Riemannian manifolds.

Somewhat further afield, linear cocycles play a key role in analyzing the spectrum of the one-dimensional discrete Schrödinger operators. To any abelian cocycle ϕ over an ergodic system $f\colon M \to M$ and any $p \in M$ one can associate a one-dimensional discrete Schrödinger operator $H\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by

$$H(x)_n = x_n + x_{n-1} - \phi(f^n(p))\, x_n.$$

The properties of the $SL(2, \mathbb{R})$-valued cocycles defined by

$$A_E(p) = \begin{pmatrix} E - \phi & -1 \\ 1 & 0 \end{pmatrix}$$

for different choices of the parameter $E \in \mathbb{R}$ determine the spectral properties of the operator H. For example, if this cocycle is uniformly hyperbolic for some value of E, then E lies in the resolvent set of H.

3. The central problems

We briefly outline the main questions that are addressed in the two papers in this volume.

Cohomological equation. — The *cohomological (or coboundary) equation* is

$$\phi = \Phi^{-1} \cdot (\Phi \circ f). \tag{5}$$

For abelian cocycles this is usually written:

$$\phi = \Phi \circ f - \Phi. \tag{6}$$

If such a solution exists, then ϕ is called a *coboundary*. Coboundaries are in a natural sense orthogonal to f-invariant functions: they are the image of the linear operator $\phi \mapsto \phi \circ f - \phi$, whereas the f-invariant functions are the kernel. This orthogonality statement can be made precise. For example, if f preserves a probability measure μ, then in $L^2(\mu)$ the closed subspace

$$\mathcal{Z} = \{\phi \in L^2(\mu) \,|\, \phi \circ f = f\}$$

is the orthogonal complement of the L^2-closure of the space of coboundaries:

$$\mathcal{B} = \{\phi \circ f - \phi \,|\, \phi \in L^2(\mu)\}$$

This observation, which holds in some form in other function spaces as well, gives a method for proving ergodic theorems: establish the result for functions in $\mathcal{Z}$ and in $\mathcal{B}$ and then extend from the dense set $\mathcal{Z} \oplus \mathcal{B}$ using linear algebra, maximal inequalities, and so on.

An obvious obstruction to finding a continuous solution to (6) is obtained by integrating both sides against an f-invariant probability measure μ:

$$\int_M \phi \, d\mu = \int_M (\Phi \circ f - \Phi) \, d\mu = 0.$$

The natural question then arises whether this is the only obstruction; that is, if ϕ has average 0 with respect to every f-invariant probability measure μ, then does there exist a continuous solution to (6)? For transitive hyperbolic systems, the answer is "yes," as we explain below. For rigid rotations and other uniquely ergodic systems, the answer usually depends on finer arithmetic data.

For example, suppose that f is rotation on the circle by $\alpha \in \mathbb{R}/\mathbb{Z}$. A simple Fourier analysis of (6) shows that if α is Diophantine, then for any C^∞ function ϕ of average zero there exists a C^∞ solution to (6). On the other hand, if α is Liouvillean, then there exists a C^∞ function ϕ of average zero for which there is no measurable solution. For perturbations of rigid rotations, solving (6) is a key component of KAM theory, and the issue of small divisors presents obstructions to both solving the equation and establishing regularity of its solutions.

This a basic example of cohomological theory as applied to the so-called "elliptic systems." Related to these are the parabolic systems, which include flows on surfaces, polygonal billiard flows, interval exchange transformations, horocyclic flows and flows on nilmanifolds. In these systems, which are typically uniquely ergodic or possess finitely many invariant measures, solving the cohomological equation gives information about rates of convergence for ergodic averages. The relative paucity of invariant measures leads one to look at a broader class of functionals – the f-invariant distributions – as obstructions to solving the cohomological equation.

In contrast with the elliptic and parabolic systems, hyperbolic systems have a plethora of invariant measures, for example the Dirac measures supported on periodic orbits. The basic existence theory of Livšic shows that the invariant measures present a complete set of obstructions to finding a continuous solution to (6). What is more, for transitive hyperbolic systems (for which periodic orbits are dense), the periodic measures alone constitute a complete set of obstructions. Another feature of Anosov systems is that continuous solutions are always smooth.

Livšic theory for hyperbolic systems has several interesting applications. For example, applying this theory to the log Jac cocycle, it follows immediately that a transitive Anosov diffeomorphism f preserves a smooth invariant measure if and only if for every periodic point p of period n:

$$\operatorname{Jac} f^n(p) = 1.$$

Livšic theory for Anosov flows is also an ingredient in the proof of marked length spectrum rigidity for negatively curved surfaces, see **[19, 9]**.

In the second paper in this volume, this Livšic theory is extended to accessible partially hyperbolic diffeomorphisms.

The role of Lyapunov exponents. — If A is a linear cocycle over $f\colon M \to M$ with values in $\mathrm{GL}(k)$, then there is a well-defined notion of the *extremal Lyapunov exponents* of A at $p \in M$:

$$\lambda_+(A,p) := \limsup_{n\to\infty} \frac{1}{n} \log \|A_n(p)\| \quad \text{and} \quad \lambda_-(A,p) := -\limsup_{n\to\infty} \frac{1}{n} \log \|A_n(p)^{-1}\|$$

Kingman's ergodic theorem implies that if f preserves a finite measure μ, then for μ-almost every p, the limits exist and depend measurably on p; moreover, each limit is constant if μ is ergodic. More generally, Oseledec's theorem implies that μ-almost every $p \in M$, the limit

$$\lambda(A,p,v) := \lim_{n\to\infty} \frac{1}{n} \log \|A_n(p)v\|$$

exists for every $v \in \mathbb{R}^k$ and assumes finitely many values, called the *Lyapunov exponents at p*. The extremal Lyapunov exponents $\lambda_+(A,p)$ and $\lambda_-(A,p)$ coincide with the largest and smallest values of $\lambda(A,p,v)$ over all $v \in \mathbb{R}^k$.

The Lyapunov exponents carry important information about a linear cocycle. In the case of the derivative cocycle Df, non-vanishing of the Lyapunov exponents on a set of positive volume implies that f has various chaotic properties. For the Schrödinger cocycle, almost everywhere vanishing of the Lyapunov exponent (equivalently, vanishing of the extremal exponents) for a positive measure set of energies $E \in \mathbb{R}$ is equivalent to the existence of absolutely continuous spectrum for the associated operator. In the first paper in this volume, a criterion is developed to establish the *non-vanishing* of the extremal Lyapunov exponents for a linear cocycle over an accessible, volume preserving, partially hyperbolic diffeomorphism. Actually, as explained below, most of the theory extends to smooth (non-linear) cocycles.

4. The general theory

To place the preceding discussion into a larger context, we briefly describe the cohomology theory in which these cocycles fit. The abelian cohomological equations that arise in dynamical systems belong to a general cohomology theory developed to study groups. To be precise, the abelian cocycles considered above are 1-cocycles in the first cohomology group of $\mathbb{Z}$ with coefficients in a $\mathbb{Z}$-module of Hölder continuous functions on M. Let us explain what we mean by this.

Let G be a group. A *G-module* is an abelian group A together with an action of G by endomorphisms of A. In the simplest cases, A is an arbitrary abelian group and G acts trivially on A. The main example considered in dynamics arises as follows. We fix a group G acting by homeomorphisms on a space X (for example, the $\mathbb{Z}$-action generated by a single homeomorphism $f\colon X \to X$). We set A to be the space $C(X, \mathbb{R})$ of continuous, $\mathbb{R}$-valued functions on X, where the abelian group structure on A is given by pointwise addition. Then there is a natural G-action on A given by precomposition: $(g \cdot \phi)(x) = \phi(g(x))$, which makes A into a G-module. Clearly the target space $\mathbb{R}$ in this construction can be replaced by any abelian topological group. If we assume higher regularity, such as smoothness, for the G-action, then $C(X, \mathbb{R})$ can be replaced by other function spaces, such as the space of Hölder functions, or smooth functions. More generally, if V is a vector bundle over X to which the action of G extends, then we can take A to be the space of (continuous, smooth...) sections of V, such as the space of smooth vector fields on X, when X is a smooth manifold.

Now given a G-module A, we construct the cohomology groups $H^n(G, A)$ as follows. For $n \geq 0$, let $C^n(G, A)$ be the set of all functions from G^n to A, which forms an abelian group. The elements of $C^n(G, A)$ are called (inhomogeneous) n-cochains. The coboundary homomorphisms $d^n\colon C^n(G, A) \to C^{n+1}(G, A)$ are defined by

$$\begin{aligned}(d^n\psi)(g_1, \ldots, g_{n+1}) &= g_1 \cdot \psi(g_2, \ldots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i \psi(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) + (-1)^{n+1}\psi(g_1, \ldots, g_n).\end{aligned}$$

One can check that $d^{n+1} \circ d^n = 0$; thus, we have a cochain complex and we can compute cohomology in the standard way. The group of n-cocycles is defined by $Z^n(G, A) = \ker(d^n)$, and the group of n-coboundaries is defined by $B^0(G, A) = 0$, and

$$B^n(G, A) = d^{n-1}(C^{n-1}(G, A)), \quad n \geq 1.$$

Finally, we set $H^n(G, A) = Z^n(G, A)/B^n(G, A)$.

Going back to the dynamical setting, suppose that $f\colon X \to X$ is a homeomorphism, which generates an action of the integers $\mathbb{Z}$. Then the 0-cochains are just elements of the module $C(X, \mathbb{R})$, and any $\phi\colon X \to \mathbb{R}$ generates a 1-cochain $\alpha\colon \mathbb{Z} \to C(X, \mathbb{R})$ via the formula:

$$\alpha(n) = \phi \circ f^n.$$

It is easily checked that every such cochain is a 1-cocycle and, conversely, every 1-cocycle is generated by such a function ϕ. Indeed, the cocycle condition (1) in this setting reduces to $d^1\alpha = 0$. Moreover the abelian cohomological Equation (6) translates in this setting to:

$$\alpha = d^0\Phi.$$

This equation asks if the given cocycle α is trivial on cohomology. Higher order cohomology groups have been studied in the dynamical context, most notably for groups of diffeomorphisms of the circle. In this context, certain elements of H^2 generalize the notion of rotation number to non-amenable groups. See [**13**].

The non-abelian cocycles also fit into a similarly defined non-abelian cohomology theory. In this case, however, the cohomology spaces no longer carry a group structure.

5. Fibered systems

The unifying concept in this volume is that of cocycles over partially hyperbolic diffeomorphisms. Let us outline our basic approach to such systems. Linear cocycles can be studied through their induced action on the projective bundle associated to the underlying vector bundle. Similarly, the classical cohomological equation is associated to an action by translations on a trivial $\mathbb{R}$-bundle over the original space.

Both these constructions are special cases of a general notion of fibered dynamical system, acting on some bundle over the original space, possibly with low fiberwise regularity. Under suitable assumptions, the invariant (stable and unstable) foliations of the base partially hyperbolic diffeomorphism lift to invariant foliations of the fibered system. Solutions of the relevant cohomological equations correspond to sections of the fiber bundle that are saturated by the lifted foliations, a property that we call *holonomy invariance*. The rich structure of these foliations allow us to obtain strong properties for these sections, when they exist.

One main conclusion of the first paper in this volume applies when the diffeomorphism satisfies the assumptions of [**8**]: partial hyperbolicity, volume preserving and center bunching. According to Theorems D and E in this paper, in that case any measurable section which is essentially (i.e., almost everywhere) saturated under the lifted stable foliation and essentially saturated under the lifted unstable foliation coincides, almost everywhere, with some section that is saturated by both lifted foliations. Moreover, if the base diffeomorphism is accessible then such a *bi-saturated* section may be chosen to be continuous.

The goal in this first paper is to detect non-zero Lyapunov exponents for fibered systems that act smoothly on the fibers (smooth cocycles), including projective actions of linear cocycles as a special case. For this, it is convenient to consider yet another fibered system, namely the push-forward action on the space of probability measures on each fiber.

General methods going back to Ledrappier [**16**] in the linear case and extended by Avila, Viana [**1**] to the present setup, give that if the Lyapunov exponents vanish almost everywhere then there exist measurable sections that are essentially saturated by either one of the lifted foliations. In view of the previous observations, it follows (Theorems B and C in this paper) that if the Lyapunov exponents vanish almost everywhere then measurable bi-saturated sections do exist, and they may be chosen to be continuous if the base dynamics is accessible.

As it turns out, bi-saturated sections are very difficult to come by, at least in the accessible case. Indeed, given any point p in the base space, consider the group of su-loops, that is, su-paths from p to itself. Each su-loop is associated to a holonomy map on the fiber over p, and a bi-invariant section gives rise to a fixed point common to all those maps. When the base diffeomorphism is accessible, the loop group is very big, yielding a large set of obstructions to the existence of such a fixed point.

In this way one gets, in particular, that generic linear cocycles over an accessible, volume preserving, partially hyperbolic diffeomorphism have some non-vanishing extremal exponent (Theorem A of this paper).

These tools developed in the first paper to handle extremal Lyapunov exponents can be applied as well to abelian cocycles. This is the starting point for the second paper in this volume. Reinterpreting in the abelian context the results of the first paper, we obtain a reformulation in the partially hyperbolic context of two of the main conclusions of the Livšic theory: existence and measurable rigidity of solutions to the coboundary equation (Theorem A parts I and III of [**26**]). The second paper then completes the remaining task of establishing regularity of solutions to the coboundary equation (Theorem A parts II and IV of [**26**]). This gives a fairly complete extension of the main conclusions of the Livšic theory from the hyperbolic to the (accessible) partially hyperbolic context.

The task is simplified conceptually by the fibered system perspective. A solution to the coboundary equation is a bi-saturated section of the associated $\mathbb{R}$-bundle; the image of this section is invariant under the lifted stable and unstable holonomy maps. Accessibility implies that these local holonomy maps act transitively on the section, meaning that the section is homogeneous under a large groupoid of transformations. A condition on the diffeomorphism called *strong bunching* implies that the holonomy maps, while not smooth, are smooth along center directions in the base manifold. Under the strong bunching hypothesis, one can then invoke ideas from the study of transformation groups to show that the section is smooth along center directions. Smoothness of the leaves of the lifted foliation gives smoothness of the section along stable and unstable directions; combined with smoothness along center directions, this gives smoothness of the invariant section. As with the conclusions in this paper, the regularity results in [**26**] apply much more generally to saturated sections of smooth cocycles (Theorem C in [**26**]).

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HOLONOMY INVARIANCE: ROUGH REGULARITY AND APPLICATIONS TO LYAPUNOV EXPONENTS

by

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Abstract. — Un cocycle lisse est un produit gauche qui agit par des difféomorphismes dans les fibres. Si les exposants de Lyapounov extremaux du cocycle coincident alors les fibres possèdent certaines structures qui sont invariantes, à la fois, par la dynamique et par un pseudo-groupe canonique de transformations d'holonomie. Nous démontrons ce *principe d'invariance* pour les cocycles lisses au dessus des difféomorphismes conservatifs partiellement hyperboliques, et nous en donnons des applications aux cocycles linéaires et aux dynamiques partiellement hyperboliques.

Résumé. — Skew-products that act by diffeomorphisms on the fibers are called smooth cocycles. If the extremal Lyapunov exponents of a smooth cocycle coincide then the fibers carry quite a lot of structure that is invariant under the dynamics and under a canonical pseudo-group of holonomy maps. We state and prove this *invariance principle* for cocycles over partially hyperbolic volume preserving diffeomorphisms. It has several applications, e.g., to linear cocycles and to partially hyperbolic dynamics.

1. Introduction

Lyapunov exponents measure the asymptotic rates of contraction and expansion, in different directions, of smooth dynamical systems such as diffeomorphisms, cocycles, or their continuous-time counterparts. These numbers are well defined on a full measure subset of phase-space, relative to any finite invariant measure. Systems whose Lyapunov exponents are distinct/non-vanishing exhibit a wealth of geometric and dynamical structure (invariant laminations, entropy formula, abundance of periodic orbits, dimension of invariant measures) on which one can build to describe their evolution. The main theme we are interested in is that systems for which the Lyapunov exponents are *not* distinct are also special, in that they satisfy a very strong invariance principle. Thus, a detailed theory can be achieved also in this case, if only using very different ingredients.

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In the special case of linear systems, the invariance principle can be traced back to the classical results on random matrices by Furstenberg [**12**], Ledrappier [**19**], and others. Moreover, it has been refined in more recent works by Bonatti, Gomez-Mont, Viana [**7**], Bonatti, Viana [**8**], Viana [**25**] and Avila, Viana [**1, 2**]. An explicit and much more general formulation, that applies to smooth (possibly non-linear) systems, is proposed in Avila, Viana [**3**] and the present paper: while [**3**] deals with extensions of hyperbolic transformations, here we handle the case when the base dynamics is just partially hyperbolic and volume preserving. The two papers are contemporary and closely related: in particular, Theorem A of [**3**] relies on a version of the invariance principle proved in here, more precisely, Theorem B below.

As an illustration of the reach of our methods, let us state the following application in the realm of partially hyperbolic dynamics (for details, see Remark 2.9). Let $f : M \to M$ be a C^2 partially hyperbolic, dynamically coherent, volume preserving, accessible diffeomorphism satisfying a suitable center bunching condition. If the center bundle E^c has dimension 2 and the center Lyapunov exponents coincide almost everywhere then f admits

(a) either an invariant continuous field of directions $r \subset E^c$,
(b) or an invariant continuous field of pairs of directions $r_1 \cup r_2 \subset E^c$,
(c) or an invariant continuous conformal structure on E^c.

Sometimes, one can exclude all three alternatives a priori. That is the case, for instance, if f is known to have periodic points p and q that are, respectively, elliptic and hyperbolic along the center bundle E^c, in the following sense: the center eigenvalues of p are neither real nor pure imaginary, and the center eigenvalues of q are real and distinct. Then it follows that the center Lyapunov exponents are distinct and, in particular, at least one is non-zero. If f is symplectic then both center Lyapunov exponents are different from zero; compare Theorem A in [**3**].

Precise statements of our results, including the definitions of the objects involved, will appear in the next section. Right now, let us observe that important applications of the methods developed in here have been obtained by several authors: a Livšic theory of partially hyperbolic diffeomorphism, by Wilkinson [**27**]; existence and properties of physical measures, by Viana, Yang [**26**]; construction of measures of maximal entropy, by Hertz, Hertz, Tahzibi, Ures [**22**].

2. Preliminaries and statements

2.1. Partially hyperbolic diffeomorphisms. — Throughout the paper, unless stated otherwise, $f : M \to M$ is a partially hyperbolic diffeomorphism on a compact manifold M and μ is a probability measure in the Lebesgue class of M. In this section we define these and other related notions. See [**9, 15, 16, 24**] for more information.

A diffeomorphism $f : M \to M$ of a compact manifold M is *partially hyperbolic* if there exists a nontrivial splitting of the tangent bundle

$$TM = E^s \oplus E^c \oplus E^u \tag{2.1}$$

invariant under the derivative Df, a Riemannian metric $\|\cdot\|$ on M, and positive continuous functions ν, $\hat{\nu}$, γ, $\hat{\gamma}$ with ν, $\hat{\nu} < 1$ and $\nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1}$ such that, for any unit vector $v \in T_pM$,

$$\|Df(p)v\| < \nu(p) \qquad \text{if } v \in E^s(p), \tag{2.2}$$

$$\gamma(p) < \|Df(p)v\| < \hat{\gamma}(p)^{-1} \qquad \text{if } v \in E^c(p), \tag{2.3}$$

$$\hat{\nu}(p)^{-1} < \|Df(p)v\| \qquad \text{if } v \in E^u(p). \tag{2.4}$$

(Equivalently, one could ask these conditions for some iterate; see Gourmelon [**14**].) All three subbundles E^s, E^c, E^u are assumed to have positive dimension. However, in some cases (cf. Remarks 3.12 and 4.2) one may let either $\dim E^s = 0$ or $\dim E^u = 0$.

We take M to be endowed with the distance dist associated to such a Riemannian structure. The *Lebesgue class* is the measure class of the volume induced by this (or any other) Riemannian metric on M. These notions extend to any submanifold of M, just considering the restriction of the Riemannian metric to the submanifold. We say that f is *volume preserving* if it preserves some probability measure in the Lebesgue class of M.

Suppose that $f : M \to M$ is partially hyperbolic. The stable and unstable bundles E^s and E^u are uniquely integrable and their integral manifolds form two transverse continuous foliations $\mathcal{W}^s$ and $\mathcal{W}^u$, whose leaves are immersed submanifolds of the same class of differentiability as f. These foliations are referred to as the *strong-stable* and *strong-unstable* foliations. They are invariant under f, in the sense that

$$f(\mathcal{W}^s(x)) = \mathcal{W}^s(f(x)) \quad \text{and} \quad f(\mathcal{W}^u(x)) = \mathcal{W}^u(f(x)),$$

where $\mathcal{W}^s(x)$ and $\mathcal{W}^s(x)$ denote the leaves of $\mathcal{W}^s$ and $\mathcal{W}^u$, respectively, passing through any $x \in M$. These foliations are, usually, *not* transversely smooth: the holonomy maps between any pair of cross-sections are not even Lipschitz continuous, in general, although they are always γ-Hölder continuous for some $\gamma > 0$. Moreover, if f is C^2 then these foliations are absolutely continuous, meaning that the holonomy maps preserve the class of zero Lebesgue measure sets. Let us explain this key fact more precisely.

Let $d = \dim M$ and $\mathcal{F}$ be a continuous foliation of M with k-dimensional smooth leaves, $0 < k < d$. Let $\mathcal{F}(p)$ be the leaf through a point $p \in M$ and $\mathcal{F}(p, R) \subset \mathcal{F}(p)$ be the neighborhood of radius $R > 0$ around p, relative to the distance defined by the Riemannian metric restricted to $\mathcal{F}(p)$. A *foliation box* for $\mathcal{F}$ at p is the image of an embedding

$$\Phi : \mathcal{F}(p, R) \times \mathbb{R}^{d-k} \to M$$

such that $\Phi(\cdot, 0) = \mathrm{id}$, every $\Phi(\cdot, y)$ is a diffeomorphism from $\mathcal{F}(p, R)$ to some subset of a leaf of $\mathcal{F}$ (we call the image a *horizontal slice*), and these diffeomorphisms vary continuously with $y \in \mathbb{R}^{d-k}$. Foliation boxes exist at every $p \in M$, by definition of continuous foliation with smooth leaves. A *cross-section* to $\mathcal{F}$ is a smooth codimension-k disk inside a foliation box that intersects each horizontal slice exactly once, transversely and with angle uniformly bounded from zero.

Then, for any pair of cross-sections Σ and Σ', there is a well defined *holonomy map* $\Sigma \to \Sigma'$, assigning to each $x \in \Sigma$ the unique point of intersection of Σ' with the horizontal slice through x. The foliation is *absolutely continuous* if all these homeomorphisms map zero Lebesgue measure sets to zero Lebesgue measure sets. That holds, in particular, for the strong-stable and strong-unstable foliations of partially hyperbolic C^2 diffeomorphisms and, in fact, the Jacobians of all holonomy maps are bounded by a uniform constant.

A measurable subset of M is *s-saturated* (or $\mathcal{W}^s$*-saturated*) if it is a union of entire strong-stable leaves, *u-saturated* (or $\mathcal{W}^u$-saturated) if it is a union of entire strong-unstable leaves, and *bi-saturated* if it is both s-saturated and u-saturated. We say that f is *accessible* if $\varnothing$ and M are the only bi-saturated sets, and *essentially accessible* if every bi-saturated set has either zero or full measure, relative to any probability measure in the Lebesgue class. A measurable set $X \subset M$ is *essentially s-saturated* if there exists an s-saturated set $X^s \subset M$ such that $X \Delta X^s$ has measure zero, for any probability measure in the Lebesgue class. *Essentially u-saturated* sets are defined analogously. Moreover, X is *bi-essentially saturated* if it is both essentially s-saturated and essentially u-saturated.

Pugh, Shub conjectured in **[20]** that essential accessibility implies ergodicity, for a C^2 partially hyperbolic, volume preserving diffeomorphism. In **[21]** they showed that this does hold under a few additional assumptions, called dynamical coherence and center bunching. To date, the best result in this direction is due to Burns, Wilkinson **[10]**, who proved the Pugh-Shub conjecture assuming only the following mild form of center bunching:

Definition 2.1. — A C^2 partially hyperbolic diffeomorphism is *center bunched* if the functions ν, $\hat\nu$, γ, $\hat\gamma$ in (2.2)–(2.4) may be chosen to satisfy

$$\nu < \gamma\hat\gamma \quad \text{and} \quad \hat\nu < \gamma\hat\gamma. \tag{2.5}$$

When the diffeomorphism is just $C^{1+\alpha}$, for some $\alpha > 0$, the arguments of Burns, Wilkinson **[10]** can still be carried out, as long as one assumes what they call strong center bunching (see [**10**, Theorem 0.3]). All our results extend to this setting.

2.2. Fiber bundles. — In this paper we deal with a few different types of fiber bundles over the manifold M. The more general type we consider are *continuous fiber bundles* $\pi : \mathcal{E} \to M$ modeled on some topological space N. By this we mean that $\mathcal{E}$ is a topological space and there is a family of homeomorphisms (*local charts*)

$$\phi_U : U \times N \to \pi^{-1}(U), \tag{2.6}$$

indexed by the elements U of some finite open cover $\mathcal{U}$ of M, such that $\pi \circ \phi_U$ is the canonical projection $U \times N \to U$ for every $U \in \mathcal{U}$. Then each $\phi_{U,x} : \xi \mapsto \phi_U(x, \xi)$ is a homeomorphism between N and the *fiber* $\mathcal{E}_x = \pi^{-1}(x)$.

An important role will be played by the class of *fiber bundles with smooth fibers*, that is, continuous fiber bundles whose fibers are manifolds endowed with a continuous Riemannian metric. More precisely, take N to be a Riemannian manifold, not necessarily complete, and assume that all coordinate changes $\phi_V^{-1} \circ \phi_U$ have the form

$$\phi_V^{-1} \circ \phi_U : (U \cap V) \times N \to (U \cap V) \times N, \quad (x, \xi) \mapsto (x, g_x(\xi)) \tag{2.7}$$

where:

(i) $g_x : N \to N$ is a C^1 diffeomorphism and the map $x \mapsto g_x$ is continuous, relative to the uniform C^1 distance on $\mathrm{Diff}^1(N)$ (the uniform C^1 distance is defined by $\mathrm{dist}_{C^1}(g_x, g_y) = \sup\{|g_x(\xi) - g_y(\xi)|, \|Dg_x(\xi) - Dg_y(\xi)\| : \xi \in N\}$);

(ii) the derivatives $Dg_x(\xi)$ are $Dg_x^{-1}(\xi)$ are uniformly continuous and uniformly bounded in norm.

Endow each $\mathcal{E}_x$ with the manifold structure that makes $\phi_{U,x}$ a diffeomorphism. Condition (i) ensures that this does not depend on the choice of $U \in \mathcal{U}$ containing x. Moreover, consider on each $\mathcal{E}_x$ the Riemannian metric $\gamma_x = \sum_{U \in \mathcal{U}} \rho_U(x) \gamma_{U,x}$, where $\gamma_{U,x}$ is the Riemannian metric transported from N by the diffeomorphism $\phi_{U,x}$ and $\{\rho_U : U \in \mathcal{U}\}$ is a partition of unit subordinate to $\mathcal{U}$. It is clear that γ_x depends continuously on x. Condition (ii) ensures that different choices of the partition of unit give rise to Riemannian metrics γ_x that differ by a bounded factor only.

Restricting even further, we call $\pi : \mathcal{E} \to M$ a *continuous vector bundle* of dimension $d \geq 1$ if $N = \mathbb{K}^d$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and every g_x is a linear isomorphism, depending continuously on x and such that $\|g_x^{\pm 1}\|$ are uniformly bounded. Then each fiber $\mathcal{E}_x$ is isomorphic to $\mathbb{K}^d$ and is equipped with a scalar product (and, hence, a norm) which is canonical up to a bounded factor.

We also need to consider more regular vector bundles. Given $r \in \{0, 1, \dots, k, \dots\}$ and $\alpha \in [0, 1]$, we say that $\pi : \mathcal{E} \to M$ is a $C^{r,\alpha}$ vector bundle if, for any U, $V \in \mathcal{U}$ with non-empty intersection, the map

$$U \cap V \to \mathrm{GL}(d, \mathbb{K}), \quad x \mapsto g_x \tag{2.8}$$

is of class $C^{r,\alpha}$, that is, it is r times differentiable and the derivative of order r is α-Hölder continuous.

2.3. Linear cocycles. — Let $\pi : \mathcal{V} \to M$ be a continuous vector bundle of dimension $d \geq 1$. A *linear cocycle* over $f : M \to M$ is a continuous transformation $F : \mathcal{V} \to \mathcal{V}$ satisfying $\pi \circ F = f \circ \pi$ and acting by linear isomorphisms $F_x : \mathcal{V}_x \to \mathcal{V}_{f(x)}$ on the fibers. By Furstenberg, Kesten [**13**], the *extremal Lyapunov exponents*

$$\lambda_+(F, x) = \lim_{n \to \infty} \frac{1}{n} \log \|F_x^n\| \quad \text{and} \quad \lambda_-(F, x) = \lim_{n \to \infty} \frac{1}{n} \log \|(F_x^n)^{-1}\|^{-1}$$

exist at μ-almost every $x \in M$, relative to any f-invariant probability measure μ. If (f, μ) is ergodic then they are constant on a full μ-measure set. It is clear that $\lambda_-(F, x) \leq \lambda_+(F, x)$ whenever they are defined. We study conditions under which these two numbers coincide.

Suppose that $\pi : \mathcal{V} \to M$ is a $C^{r,\alpha}$ vector bundle, for some fixed r and α, and f is also of class $C^{r,\alpha}$ (this is contained in our standing assumptions if $r + \alpha \leq 2$). Then we call $F : \mathcal{V} \to \mathcal{V}$ a $C^{r,\alpha}$ linear cocycle if its expression in local coordinates

$$\phi_{U_1}^{-1} \circ F \circ \phi_{U_0} : (U_0 \cap f^{-1}(U_1)) \times \mathbb{K}^d \to U_1 \times \mathbb{K}^d, \quad (x, v) \mapsto (f(x), A(x)v) \tag{2.9}$$

is such that the function $x \mapsto A(x)$ is r times differentiable and the derivative of order r is bounded and α-Hölder continuous. The assumption on the vector bundle ensures that this condition does not depend on the choice of local charts.

The set $\mathcal{G}^{r,\alpha}(\mathcal{V}, f)$ of all $C^{r,\alpha}$ linear cocycles $F : \mathcal{V} \to \mathcal{V}$ over $f : M \to M$ is a $\mathbb{K}$-vector space and carries a natural $C^{r,\alpha}$ norm:

$$\|F\|_{r,\alpha} = \sup_{U,V\in\mathcal{U}} \left(\sup_{0\le i\le r} \sup_{x\in U\cap f^{-1}(V)} \|D^i A(x)\| + \sup_{x\neq y} \frac{\|D^r A(x) - D^r A(y)\|}{\operatorname{dist}(x,y)^\alpha} \right) \tag{2.10}$$

(for $\alpha = 0$ one may omit the last term). *We always assume that* $r + \alpha > 0$. Then every $F \in \mathcal{G}^{r,\alpha}(\mathcal{V}, f)$ is β-Hölder continuous, with

$$\beta = \begin{cases} \alpha & \text{if } r = 0 \\ 1 & \text{if } r \ge 1. \end{cases} \tag{2.11}$$

Definition 2.2. — We say that a cocycle $F \in \mathcal{G}^{r,\alpha}(\mathcal{V}, f)$ is *fiber bunched* if

$$\|F_x\| \, \|(F_x)^{-1}\| \, \nu(x)^\beta < 1 \quad \text{and} \quad \|F_x\| \, \|(F_x)^{-1}\| \, \hat\nu(x)^\beta < 1, \tag{2.12}$$

for every $x \in M$, where $\beta > 0$ is given by (2.11) and ν, $\hat\nu$ are functions as in (2.2)–(2.4), fixed once and for all.

Remark 2.3. — This notion appeared in **[7, 8, 25]**, where it was called *domination*. The present terminology seems preferable, on more than one account. To begin with, there is the analogy with the notion of center bunching in Definition 2.1. Perhaps more important, the natural notion of domination for smooth cocycles, that we are going to introduce in Definition 3.9, corresponds to a rather different condition. The relation between the two is explained in Remark 3.13: if a linear cocycle is fiber bunched then the associated *projective* cocycle is dominated. Finally, a notion of fiber bunching can be defined for smooth cocycles as well (see **[3]**), similar to (2.12) and stronger than domination.

Theorem A. — *Let $f : M \to M$ be a C^2 partially hyperbolic, volume preserving, center bunched, accessible diffeomorphism and let μ be an invariant probability in the Lebesgue class. Assume that $F \in \mathcal{G}^{r,\alpha}(\mathcal{V}, f)$ is fiber bunched.*

Then F is approximated, in the $C^{r,\alpha}$ norm, by open sets of cocycles $G \in \mathcal{G}^{r,\alpha}(\mathcal{V}, f)$ such that $\lambda_-(G, x) < \lambda_+(G, x)$ almost everywhere. Moreover, the set of $F \in \mathcal{G}^{r,\alpha}(\mathcal{V}, f)$ for which the extremal Lyapunov exponents do coincide has infinite codimension in the fiber bunched domain: locally, it is contained in finite unions of closed submanifolds with arbitrarily high codimension.

Notice that the Lyapunov exponents are constant on a full measure subset of M, because (cf. **[10]**) the hypothesis implies that f is ergodic.

There is an analogous statement in the space of $\mathrm{SL}(d,\mathbb{K})$-cocycles, that is, such that the functions $x \mapsto g_x$ and $x \mapsto A(x)$ in (2.8) and (2.9), respectively, take values in $\mathrm{SL}(d,\mathbb{K})$. In fact, our proof of Theorem A deals with the projectivization of the cocycle, and so it treats both cases, $\mathrm{GL}(d,\mathbb{K})$ and $\mathrm{SL}(d,\mathbb{K})$, on the same footing. It would be interesting to investigate the case of G-valued cocycles for more general subgroups of $\mathrm{GL}(d,\mathbb{K})$, for instance the symplectic group.

2.4. Smooth cocycles - invariant holonomies. — Let $\pi : \mathcal{E} \to M$ be a fiber bundle with smooth fibers modeled on some Riemannian manifold N. A *smooth cocycle* over $f : M \to M$ is a continuous transformation $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$ such that $\pi \circ \mathfrak{F} = f \circ \pi$, every $\mathfrak{F}_x : \mathcal{E}_x \to \mathcal{E}_{f(x)}$ is a C^1 diffeomorphism depending continuously on x, relative to the uniform C^1 distance in the space of C^1 diffeomorphisms on the fibers, and the norms of the derivative $D\mathfrak{F}_x(\xi)$ and its inverse are uniformly bounded. In particular, the functions

$$(x,\xi) \mapsto \log \|D\mathfrak{F}_x(\xi)\| \quad \text{and} \quad (x,\xi) \mapsto \log \|D\mathfrak{F}_x(\xi)^{-1}\|$$

are bounded. Then (Kingman [**18**]), given any $\mathfrak{F}$-invariant probability m on $\mathcal{E}$, the *extremal Lyapunov exponents* of $\mathfrak{F}$

$$\lambda_+(\mathfrak{F},x,\xi) = \lim_{n\to\infty} \frac{1}{n} \log \|D\mathfrak{F}_x^n(\xi)\| \quad \text{and} \quad \lambda_-(\mathfrak{F},x,\xi) = \lim_{n\to\infty} \frac{1}{n} \log \|D\mathfrak{F}_x^n(\xi)^{-1}\|^{-1}.$$

are well defined at m-almost every $(x,\xi) \in \mathcal{E}$. Clearly, $\lambda_-(\mathfrak{F},x,\xi) \le \lambda_+(\mathfrak{F},x,\xi)$. Notice that if m is $\mathfrak{F}$-invariant then its projection $\mu = \pi_* m$ is f-invariant. Most of the times we will be interested in measures m for which the projection is in the Lebesgue class of M.

Let $R > 0$ be fixed. The *local strong-stable leaf* $\mathcal{W}^s_{loc}(p)$ of a point $p \in M$ is the neighborhood of radius R around p inside $\mathcal{W}^s(p)$. The local strong-unstable leaf $\mathcal{W}^u_{loc}(p)$ is defined analogously. The choice of R is very much arbitrary, but in Section 5 we will be a bit more specific.

Definition 2.4. — We call *invariant stable holonomy* for $\mathfrak{F}$ a family H^s of homeomorphisms $H^s_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$, defined for all x and y in the same strong-stable leaf of f and satisfying

(a) $H^s_{y,z} \circ H^s_{x,y} = H^s_{x,z}$ and $H^s_{x,x} = \mathrm{id}$;
(b) $\mathfrak{F}_y \circ H^s_{x,y} = H^s_{f(x),f(y)} \circ \mathfrak{F}_x$;
(c) $(x,y,\xi) \mapsto H^s_{x,y}(\xi)$ is continuous when (x,y) varies in the set of pairs of points in the same local strong-stable leaf;
(d) there are $C > 0$ and $\gamma > 0$ such that $H^s_{x,y}$ is (C,γ)-Hölder continuous for every x and y in the same local strong-stable leaf.

Invariant unstable holonomy is defined analogously, for pairs of points in the same strong-unstable leaf.

Condition (c) in Definition 2.4 means that, given any $\varepsilon > 0$ and any (x,y,ξ) with $y \in \mathcal{W}^s_{loc}(x)$, there exists $\delta > 0$ such that $\mathrm{dist}(H^s_{x,y}(\xi), H^s_{x',y'}(\xi')) < \varepsilon$ for every

(x', y', ξ') with $y' \in \mathcal{W}^s_{loc}(x')$ and $\operatorname{dist}(x, x') < \delta$ and $\operatorname{dist}(y, y') < \delta$ and $\operatorname{dist}(\xi, \xi') < \delta$; for this to make sense, take the fiber bundle to be trivialized in the neighborhoods of $\mathcal{E}_x$ and $\mathcal{E}_y$. Condition (d), together with the invariance property (b), implies that $H^s_{x,y}$ is γ-Hölder continuous for every x and y in the same strong-stable leaf (the multiplicative Hölder constant C may not be uniform over global leaves).

Remark 2.5. — Uniformity of the multiplicative Hölder constant C on local strong-stable leaves is missing in the related definition in [**3**, Section 2.4], but is assumed in [**3**, Section 4.4] when arguing that the transformation $\tilde{G}$ is a deformation of G.

Example 2.6. — The *projective bundle* associated to a vector bundle $\mathcal{V} \to M$ is the continuous fiber bundle $\mathbb{P}(\mathcal{V}) \to M$ whose fibers are the projective quotients of the fibers of $\mathcal{V}$. Clearly, this is a fiber bundle with smooth leaves modeled on $N = \mathbb{P}(\mathbb{K}^d)$. The *projective cocycle* associated to a linear cocycle $F : \mathcal{V} \to \mathcal{V}$ is the smooth cocycle $\mathfrak{F} : \mathbb{P}(\mathcal{V}) \to \mathbb{P}(\mathcal{V})$ whose action $\mathfrak{F}_x : \mathbb{P}(\mathcal{V}_x) \to \mathbb{P}(\mathcal{V}_{f(x)})$ on the fibers is given by the projectivization of $F_x : \mathcal{V}_x \to \mathcal{V}_{f(x)}$:

$$\mathfrak{F}_x(\xi) = \frac{F_x(\xi)}{\|F_x(\xi)\|} \quad \text{for each } \xi \in \mathbb{P}(\mathcal{V}_x) \text{ and } x \in M$$

(on the right hand side of the equality, think of ξ as a unit vector in $\mathbb{K}^d$). Then $\mathfrak{F}^n_x(\xi) = F^n_x(\xi)/\|F^n_x(\xi)\|$ for every ξ, x and n. It follows that,

$$D\mathfrak{F}^n_x(\xi)\dot{\xi} = \frac{\operatorname{proj}_{F^n_x(\xi)}\left(F^n_x(\dot{\xi})\right)}{\|F^n_x(\xi)\|},$$

where $\operatorname{proj}_w v = v - w(w \cdot v)/(w \cdot w)$ is the projection of a vector v to the orthogonal complement of w. This implies that

$$\|D\mathfrak{F}^n_x(\xi)\| \le \|F^n_x\|/\|F^n_x(\xi)\| \le \|F^n_x\|\|(F^n_x)^{-1}\| \tag{2.13}$$

for every ξ, x and n. Analogously, replacing each F by its inverse,

$$\|D\mathfrak{F}^n_x(\xi)^{-1}\| \le \|(F^n_x)^{-1}\|\|F^n_x\| \tag{2.14}$$

for every ξ, x and n. These two inequalities imply

$$\lambda_+(\mathfrak{F}, x, \xi) \le \lambda_+(F, x) - \lambda_-(F, x) \quad \text{and} \quad \lambda_-(\mathfrak{F}, x, \xi) \ge \lambda_-(F, x) - \lambda_+(F, x)$$

whenever these exponents are defined. We will observe in Remark 3.13 that if F is fiber bunched then both F and $\mathfrak{F}$ admit invariant stable and unstable holonomies.

Example 2.7. — Suppose that the partially hyperbolic diffeomorphism $f : M \to M$ is *dynamically coherent*, that is, there exist invariant foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ with smooth leaves tangent to $E^c \oplus E^s$ and $E^c \oplus E^u$, respectively. Intersecting the leaves of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ one obtains a center foliation $\mathcal{W}^c$ whose leaves are tangent to the center subbundle E^c at every point. Let $\mathcal{E}$ be the disjoint union of the leaves of $\mathcal{W}^c$. In many cases (see Avila, Viana, Wilkinson [**4**]), the natural projection $\pi : \mathcal{E} \to M$ given by $\pi \mid \mathcal{W}^c(x) \equiv x$ is a fiber bundle with smooth fibers. Also, the map f induces a smooth cocycle $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$, mapping each $y \in \mathcal{W}^c(x)$ to $f(y) \in \mathcal{W}^c(f(x))$. Moreover, the cocycle $\mathfrak{F}$ admits invariant stable and unstable holonomies: for x close to y the image $H^s_{x,y}(\xi)$ is the point where the local strong-stable leaf through $\xi \in \mathcal{W}^c(x)$ intersects the center leaf $\mathcal{W}^c(y)$, and analogously for the unstable holonomy. This kind of construction, combined with Theorem 6.1 below, is used by Wilkinson **[27]** in her recent development of a Livšic theory for partially hyperbolic diffeomorphisms.

2.5. Lyapunov exponents and rigidity. — Theorem A will be deduced, in Section 8, from certain perturbation arguments together with an invariance principle for cocycles whose extremal Lyapunov exponents coincide. Here we state this invariance principle.

Let $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$ be a smooth cocycle that admits invariant stable holonomy. Let m be a probability measure on $\mathcal{E}$, let $\mu = \pi_* m$ be its projection, and let $\{m_x : x \in M\}$ be a *disintegration* of m into *conditional probabilities* along the fibers, that is, a measurable family of probability measures $\{m_x : x \in M\}$ such that $m_x(\mathcal{E}_x) = 1$ for μ-almost every $x \in M$ and

$$m(U) = \int m_x(\mathcal{E}_x \cap U)\, d\mu(x)$$

for every measurable set $U \subset \mathcal{E}$. Such a family exists and is essentially unique, meaning that any two coincide on a full measure subset. See Rokhlin **[23]**.

Definition 2.8. — A disintegration $\{m_x : x \in M\}$ is *s-invariant* if

$$(H^s_{x,y})_* m_x = m_y \quad \text{for every } x \text{ and } y \text{ in the same strong-stable leaf.} \tag{2.15}$$

One speaks of *essential s-invariance* if this holds for x and y in some full μ-measure subset of M. The definitions of *u-invariance* and *essential u-invariance* are analogous. The disintegration is *bi-invariant* if it is both s-invariant and u-invariant and we call it *bi-essentially invariant* if it is both essentially s-invariant and essentially u-invariant.

First, we state the invariance principle in the special case of linear cocycles:

Theorem B. — *Let $f : M \to M$ be a C^2 partially hyperbolic, volume preserving, center bunched diffeomorphism and μ be an invariant probability in the Lebesgue class. Let $F \in \mathcal{G}^{r,\alpha}(\mathcal{V}, f)$ be fiber bunched and suppose that $\lambda_-(F,x) = \lambda_+(F,x)$ at μ-almost every point.*

Then every $\mathbb{P}(F)$-invariant probability m on the projective fiber bundle $\mathbb{P}(\mathcal{V})$ with $\pi_* m = \mu$ admits a disintegration $\{\tilde{m}_x : x \in M\}$ along the fibers such that

(a) *the disintegration is bi-invariant over a full measure bi-saturated set $M_F \subset M$;*
(b) *if f is accessible then $M_F = M$ and the conditional probabilities $\tilde{m}_x$ depend continuously on the base point $x \in M$, relative to the weak* topology.*

Invariant probability measures m that project down to μ always exist in this setting, because $\mathbb{P}(F)$ is continuous and the domain $\mathbb{P}(\mathcal{V})$ is compact.

Remark 2.9. — If $f : M \to M$ is a C^2 partially hyperbolic diffeomorphism then (see [**9**, Corollary 2.1] and [**15**, Theorem 6.4]) the invariant vector bundles E^s, E^u and E^c are Hölder continuous. Indeed, if $\alpha > 0$ is close enough to zero that

$$(\nu/\gamma)\|Df^{-1}\|^\alpha < 1 \quad \text{and} \quad (\hat{\gamma}/\hat{\nu})\|Df\|^\alpha < 1 \tag{2.16}$$

then the center bundle E^c is α-Hölder continuous. The derivative of f induces a $C^{0,\alpha}$ linear cocycle $F : E^c \to E^c$ given by $F_x = Df \mid E^c_x$. Clearly, $\|F_x\| < \hat{\gamma}(x)^{-1}$ and $\|F_x^{-1}\| < \gamma(x)^{-1}$ for every x. Hence, F is fiber bunched whenever

$$\nu^\alpha < \gamma\hat{\gamma} \quad \text{and} \quad \hat{\nu}^\alpha < \gamma\hat{\gamma}. \tag{2.17}$$

Notice that this is compatible with (2.16). Moreover, (2.17) implies that f is center bunched, that is, $\nu < \gamma\hat{\gamma}$ and $\hat{\nu} < \gamma\hat{\gamma}$. Suppose that f is also dynamically coherent, volume preserving and accessible.

Now, assume that $\dim E^c = 2$ and the two center Lyapunov exponents of f coincide μ-almost everywhere. Let m be any $\mathfrak{F}$-invariant probability that projects down to Lebesgue measure μ. Then, as observed in Example 2.6, the Lyapunov exponents of $\mathfrak{F}$ vanish m-almost everywhere. By Theorem B, it follows that m admits a continuous, bi-invariant disintegration $\{m_x : x \in M\}$. Keep in mind that each m_x is a probability measure on the projective space $\mathbb{P}(E^c_x)$. Continuity, together with the assumption that m is invariant, implies that

$$m_{f(x)} = (\mathfrak{F}_x)_* m_x = Df(x)_* m_x \quad \text{for every } x \in M.$$

Suppose first that m_x admits some atom with mass $\geq 1/2$, for some $x \in M$. Since f is accessible, bi-invariance implies that the same holds for every $x \in M$. Clearly, either such an atom is unique or there exist exactly two of them. In the first case, we obtain a continuous map assigning to each point in M a point in $\mathbb{P}(E^c)$; moreover, this continuous field of directions is invariant under the derivative. The second case is analogous, except that one gets a continuous field of pairs of directions. Now, suppose that every m_x admits no atom with mass $\geq 1/2$. Then, by Douady, Earle [**11**, Section 2], the conditional measure m_x has a well defined conformal barycenter $\xi(x) \in \mathbb{D}$ and, consequently, it defines a conformal structure on E^c_x; moreover, this conformal structure depends continuously on x and is invariant under the derivative. This completes the proof of the alternative (a)-(c) in the Introduction.

Next, assume that f is known to have periodic points p and q that are, respectively, elliptic (eigenvalues neither real nor pure imaginary) and hyperbolic (eigenvalues real and distinct) along the center bundle E^c. On the one hand, the presence of p is an obstruction to f having an invariant field of directions or of pairs of directions. On the other hand, the presence of q ensures that there is no continuous invariant conformal structure. In this way we have excluded all three possibilities (a)-(c). This contradiction means that the center Lyapunov exponents of f must be distinct. In particular, at least one of them is non-zero. When f is symplectic, the center Lyapunov exponents are symmetric (see Bochi, Viana [**6**]); in this case, the previous conclusion means that all Lyapunov exponents of f are non-zero.

The statement of Theorem B extends to smooth cocycles:

Theorem C. — *Let $f : M \to M$ be a C^2 partially hyperbolic, volume preserving, center bunched diffeomorphism and μ be an invariant probability in the Lebesgue class. Let $\mathfrak{F}$ be a smooth cocycle over f admitting invariant stable and unstable holonomies. Let m be an $\mathfrak{F}$-invariant probability measure on $\mathcal{E}$ with $\pi_* m = \mu$, and suppose that $\lambda_-(\mathfrak{F}, x, \xi) = 0 = \lambda_+(\mathfrak{F}, x, \xi)$ at m-almost every point.*

Then m admits a disintegration $\{\tilde{m}_x : x \in M\}$ into conditional probabilities along the fibers such that

(a) *the disintegration is bi-invariant over a full measure bi-saturated set $M_{\mathfrak{F}} \subset M$;*
(b) *if f is accessible then $M_{\mathfrak{F}} = M$ and the conditional probabilities $\tilde{m}_x$ depend continuously on the base point $x \in M$, relative to the weak* topology.*

It is clear from the observations in Example 2.6 that Theorem B is contained in Theorem C. The proof of Theorem C is given in Sections 4 through 7. There are two main stages.

The first one, that will be stated as Theorem 4.1, is to show that every disintegration of m is essentially s-invariant and essentially u-invariant. This is based on a non-linear extension of an abstract criterion of Ledrappier [**19**] for linear cocycles, proposed in Avila, Viana [**3**] and quoted here as Theorem 4.4. At this stage we only need f to be a C^1 partially hyperbolic diffeomorphism (volume preserving, center bunching and accessibility are not needed) and μ can be any invariant probability, not necessarily in the Lebesgue class.

The second stage, that we state in Theorem D below, is to prove that any disintegration essentially s-invariant and essentially u-invariant is, in fact, fully invariant under both the stable holonomy and the unstable holonomy; moreover, it is continuous if f is accessible. This is a different kind of argument, that is more suitably presented in the following framework.

2.6. Sections of continuous fiber bundles. — Let $\pi : \mathcal{X} \to M$ be a continuous fiber bundle with fibers modeled on some topological space P. The next definition refers to the strong-stable and strong-unstable foliations of the partially hyperbolic diffeomorphism $f : M \to M$.

Definition 2.10. — A *stable holonomy* on $\mathcal{X}$ is a family $h^s_{x,y} : \mathcal{X}_x \to \mathcal{X}_y$ of γ-Hölder homeomorphisms, with uniform Hölder constant $\gamma > 0$, defined for all x, y in the same strong-stable leaf and satisfying

(α) $h^s_{y,z} \circ h^s_{x,y} = h^s_{x,z}$ and $h^s_{x,x} = \mathrm{id}$
(β) the map $(x, y, \xi) \mapsto h^s_{x,y}(\xi)$ is continuous when (x, y) varies in the set of pairs of points in the same local strong-stable leaf.

Unstable holonomy is defined analogously, for pairs of points in the same strong-unstable leaf.

The special case we have in mind are the *invariant* stable and unstable holonomies of smooth cocycles on fiber bundles with smooth leaves. Clearly, conditions (α) and (β) in Definition 2.10 correspond to conditions (a) and (c) in Definition 2.4. Notice, however, that there is no analogue to the invariance condition (b); indeed, cocycles are not mentioned at all in this section. We also have no analogue to condition (d) in Definition 2.4.

In what follows μ is a probability measure in the Lebesgue class of M, not necessarily invariant under f: here we do not assume f to be volume preserving. The next definition is a straightforward extension of Definition 2.8 to the present setting:

Definition 2.11. — Let $\pi : \mathcal{X} \to P$ be a continuous fiber bundle admitting stable holonomy. A measurable section $\Psi : M \to \mathcal{X}$ is *s-invariant* if

$$h^s_{x,y}(\Psi(x)) = \Psi(y) \quad \text{for every } x,\, y \text{ in the same strong-stable leaf}$$

and *essentially s-invariant* if this relation holds restricted to some full μ-measure subset. The definitions of *u-invariant* and *essentially u-invariant* functions are analogous, assuming that $\pi : \mathcal{X} \to M$ admits unstable holonomy and considering strong-unstable leaves instead. We call Ψ *bi-invariant* if it is both s-invariant and u-invariant, and we call it *bi-essentially invariant* if it is both essentially s-invariant and essentially u-invariant.

These notions extend, immediately, to measurable sections of $\mathcal{X}$ whose domain is just a bi-saturated subset of M. A measurable section Ψ is *essentially bi-invariant* if it coincides almost everywhere with a bi-invariant section defined on some full measure bi-saturated set.

Definition 2.12. — A (Hausdorff) topological space P is *refinable* if there exists an increasing sequence of finite or countable partitions $\mathcal{Q}_1 \prec \cdots \prec \mathcal{Q}_n \prec \cdots$ into Borel subsets such that any sequence $(Q_n)_n$ with $Q_n \in \mathcal{Q}_n$ for every n and $\cap_n Q_n \neq \varnothing$ converges to some point $\eta \in P$, in the sense that every neighborhood of η contains Q_n for all large n. (Then, clearly, η is unique and $\cap_n Q_n = \{\eta\}$.)

Notice that every Hausdorff space with a countable basis $\{U_n : n \in \mathbb{N}\}$ of open sets is refinable: just take $\mathcal{Q}_n$ to be the partition generated by $\{U_1, \dots, U_n\}$.

Theorem D. — *Let $f : M \to M$ be a C^2 partially hyperbolic, center bunched diffeomorphism and μ be any probability measure in the Lebesgue class. Let $\pi : \mathcal{X} \to M$ be a continuous fiber bundle with stable and unstable holonomies and assume that the fiber P is refinable. Then,*

(a) *every bi-essentially invariant section $\Psi : M \to \mathcal{X}$ coincides μ-almost everywhere with a bi-invariant section $\tilde{\Psi}$ defined on a full measure bi-saturated set $M_\Psi \subset M$;*
(b) *if f is accessible then $M_\Psi = M$ and $\tilde{\Psi}$ is continuous.*

The proof of part (a) is given in Section 6 (see Theorem 6.1), based on ideas of Burns, Wilkinson **[10]** that we recall in Section 5 (see Proposition 5.13). Concerning part (b), we should point out that the measure μ plays no role in it: if f is accessible then any non-empty bi-saturated set coincides with M and then one only has to check that bi-invariance implies continuity. That is done in Section 7 and uses neither center bunching nor refinability.

Actually, in Section 7 we prove a stronger fact: bi-continuity implies continuity, when f is accessible. The notion of bi-continuity is defined as follows:

Definition 2.13. — A measurable section $\Psi : M \to \mathcal{X}$ of the continuous fiber bundle $\pi : \mathcal{X} \to M$ is *s-continuous* if the map $(x, y, \Psi(x)) \mapsto \Psi(y)$ is continuous on the set of pairs of points (x, y) in the same local strong-stable leaf. The notion of *u-continuity* is analogous, considering strong-unstable leaves instead. Finally, Ψ is *bi-continuous* if it is both s-continuous and u-continuous.

More explicitly, a measurable section Ψ is s-continuous if for every $\varepsilon > 0$ and every (x, y) with $y \in \mathcal{W}^s_{loc}(x)$ there exists $\delta > 0$ such that $\operatorname{dist}(\Psi(y), \Psi(y')) < \varepsilon$ for every (x', y') with $y' \in \mathcal{W}^s_{loc}(x')$ and $\operatorname{dist}(x, x') < \delta$ and $\operatorname{dist}(y, y') < \delta$ and $\operatorname{dist}(\Psi(x), \Psi(x')) < \delta$; it is implicit in this formulation that the fiber bundle has been trivialized in the neighborhoods of the fibers $\mathcal{X}_x$ and $\mathcal{X}_y$.

Remark 2.14. — If a section $\Psi : M \to \mathcal{X}$ is s-invariant then it is s-continuous:

$$(x, y, \Psi(x)) \mapsto \Psi(y) = h^s_{x,y}(\Psi(x))$$

is continuous on the set of pairs of points in the same local strong-stable leaf. Moreover, s-continuity ensures that the section Ψ is continuous on every strong-stable leaf: taking $x = x' = y$ in the definition, we get that $\operatorname{dist}(\Psi(y), \Psi(y')) < \varepsilon$ for every $y' \in \mathcal{W}^s_{loc}(y)$ with $\operatorname{dist}(y, y') < \delta$. Analogously, u-invariance implies u-continuity and that implies continuity on every strong-unstable leaf.

Thus, part (b) of Theorem D is a direct consequence of the following result:

Theorem E. — *Let $f : M \to M$ be a C^1 partially hyperbolic, accessible diffeomorphism. Let $\pi : \mathcal{X} \to M$ be a continuous fiber bundle. Then every bi-continuous section $\Psi : M \to \mathcal{X}$ is continuous in M.*

The proof of this theorem is given in Section 7. Notice that we make no assumptions on the continuous fiber bundle: at this stage we do not need stable and unstable holonomies, and the fibers need not be refinable either.

The logical connections between our main results can be summarized as follows:

$$\begin{tikzcd}
\text{Prop. 8.2} \arrow[d] & & \text{Thm. C(a)} \arrow[dl] & \text{Thm. D(a)} \arrow[l] & \text{Thm. 6.1} \arrow[l] \\
\text{Thm. A} & \text{Thm. B} \arrow[l] \arrow[dl] & \text{Thm. 4.1} \arrow[u] \arrow[d] & \text{Thm. 4.4} \arrow[l] & \text{Prop. 5.13} \arrow[u] \\
\text{Rmk. 2.9} & & \text{Thm. C(b)} \arrow[ul] & \text{Thm. D(b)} \arrow[l] & \text{Thm. E} \arrow[l]
\end{tikzcd}$$

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3. Cocycles with holonomies

First, we explore the notions of domination and fiber bunching for linear cocycles. In Section 3.1 we prove that if a linear cocycle is fiber bunched then it admits invariant stable and unstable holonomies, and so does its projectivization. Moreover, in Section 3.2 we check that these invariant holonomies depend smoothly on the cocycle. Then, in Section 3.3, we discuss corresponding facts for smooth cocycles.

We will often use the following notational convention: given a continuous function $\tau : M \to \mathbb{R}^+$, we denote

$$\tau^n(p) = \tau(p)\tau(f(p)) \cdots \tau(f^{n-1}(p)) \quad \text{for any } n \geq 1.$$

3.1. Fiber bunched linear cocycles. — For simplicity of the presentation, we will focus on the case when the vector bundle $\pi : \mathcal{V} \to M$ is trivial, that is, $\mathcal{V} = M \times \mathbb{K}^d$ and $\pi : M \times \mathbb{K}^d \to M$ is the canonical projection. The general case is treated in the same way, using local charts (but the notations become rather cumbersome).

In the trivial bundle case, every linear cocycle $F : \mathcal{V} \to \mathcal{V}$ may be written in the form $F(x, v) = (f(x), A(x)v)$ for some continuous $A : M \to \mathrm{GL}(d, \mathbb{K})$. Notice that $F^n(x, v) = (f^n(x), A^n(x)v)$ for each $n \in \mathbb{Z}$, with

$$A^n(x) = A(f^{n-1}(x)) \cdots A(x) \quad \text{and} \quad A^{-n}(x) = A(f^{-1}(x))^{-1} \cdots A(f^n(x))^{-1}$$

for $n \neq 0$ and $A^0(x) = \mathrm{id}$. Notice also that $F \in \mathcal{G}^{r,\alpha}(\mathcal{V}, f)$ if, and only, if A belongs to the space $\mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ of $C^{r,\alpha}$ maps from M to $\mathrm{GL}(d, \mathbb{K})$. The $C^{r,\alpha}$ norm in $\mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ is defined by

$$\|A\|_{r,\alpha} = \sup_{0 \leq i \leq r} \sup_{x \in M} \|D^i A(x)\| + \sup_{x \neq y} \frac{\|D^r A(x) - D^r A(y)\|}{\mathrm{dist}(x, y)^\alpha}. \tag{3.1}$$

Recall that we assume that $r + \alpha > 0$ and take $\beta = \alpha$ if $r = 0$ and $\beta = 1$ if $r \geq 1$. Then every $A \in \mathcal{G}^{r,\alpha}(M, d K)$ is β-Hölder continuous. By the Definition (2.12), the cocycle F is fiber bunched if

$$\|A(x)\| \, \|A(x)^{-1}\| \, \nu(x)^\beta < 1 \quad \text{and} \quad \|A(x)\| \, \|A(x)^{-1}\| \, \hat{\nu}(x)^\beta < 1 \tag{3.2}$$

for every x in M. In this case we also say that the function A is fiber bunched. Up to suitable adjustments, all our arguments in the sequel hold under the weaker assumption that (3.2) holds for some power A^ℓ, $\ell \geq 1$.

Notice that fiber bunching is an open condition: if A is fiber bunched then so is every B in a neighborhood, just because M is compact. Even more, still by compactness, if A is fiber bunched then there exists $m < 1$ such that

$$\|B(x)\| \, \|B(x)^{-1}\| \nu(x)^{\beta m} < 1 \quad \text{and} \quad \|B(x)\| \, \|B(x)^{-1}\| \hat{\nu}(x)^{\beta m} < 1 \tag{3.3}$$

for every $x \in M$ and every B in a C^0 neighborhood of A. It is in this form that the definition will be used in the proofs.

Lemma 3.1. — *Suppose that $A \in \mathscr{G}^{r,\alpha}(M, d, \mathbb{K})$ is fiber bunched. Then there is $C > 0$ such that*

$$\|A^n(y)\| \, \|A^n(z)^{-1}\| \le C \nu^n(x)^{-\beta m}$$

for all y, $z \in \mathcal{W}^s_{\mathrm{loc}}(x)$, $x \in M$, and $n \ge 1$. Moreover, the constant C may be taken uniform on a neighborhood of A.

Proof. — Since $A \in \mathscr{G}^{r,\alpha}(M, d, \mathbb{K})$ is β-Hölder continuous, there exists $L_1 > 0$ such that

$$\begin{aligned} \|A(f^j(y))\| / \|A(f^j(x))\| &\le \exp(L_1 \operatorname{dist}(f^j(x), f^j(y))^\beta) \\ &\le \exp(L_1 \nu^j(x)^\beta \operatorname{dist}(x, y)^\beta) \end{aligned}$$

and similarly for $\|A(f^j(z))^{-1}\| / \|A(f^j(x))^{-1}\|$. By sub-multiplicativity of the norm

$$\|A^n(y)\| \, \|A^n(z)^{-1}\| \le \prod_{j=0}^{n-1} \|A(f^j(y))\| \, \|A(f^j(z))^{-1}\|.$$

In view of the previous observations, the right hand side is bounded by

$$\exp\Big[L_1 \sum_{j=0}^{n-1} \nu^j(x)^\beta (\operatorname{dist}(x,y)^\beta + \operatorname{dist}(x,z)^\beta)\Big] \prod_{j=0}^{n-1} \|A(f^j(x))\| \, \|A(f^j(x))^{-1}\|$$

Since $\nu(\cdot)$ is bounded away from 1, the first factor is bounded by some $C > 0$. By fiber bunching (3.3), the second factor is bounded by $\nu^n(x)^{-\beta m}$. It is clear from the construction that L_1 and C may be chosen uniform on a neighborhood. □

Proposition 3.2. — *Suppose that $A \in \mathscr{G}^{r,\alpha}(M, d, \mathbb{K})$ is fiber bunched. Then there is $L > 0$ such that for every pair of points x, y in the same leaf of the strong-stable foliation $\mathcal{W}^s$,*

(a) $H^s_{x,y} = \lim_{n\to\infty} A^n(y)^{-1} A^n(x)$ *exists (a linear isomorphism of $\mathbb{K}^d$)*
(b) $H^s_{f^j(x), f^j(y)} = A^j(y) \circ H^s_{x,y} \circ A^j(x)^{-1}$ *for every $j \ge 1$*
(c) $H^s_{x,x} = \mathrm{id}$ *and* $H^s_{x,y} = H^s_{z,y} \circ H^s_{x,z}$
(d) $\|H^s_{x,y} - \mathrm{id}\| \le L \operatorname{dist}(x,y)^\beta$ *whenever $y \in \mathcal{W}^s_{\mathrm{loc}}(x)$.*
(e) *Given $a > 0$ there is $\Gamma(a) > 0$ such that $\|H^s_{x,y}\| < \Gamma(a)$ for any x, $y \in M$ with $y \in \mathcal{W}^s(x)$ and $\operatorname{dist}_{\mathcal{W}^s}(x,y) < a$.*

Moreover, L and the function $\Gamma(\cdot)$ may be taken uniform on a neighborhood of A.

Proof. — In order to prove claim (a), it is sufficient to consider the case $y \in \mathcal{W}^s_{\mathrm{loc}}(x)$ because $A^{n+j}(y)^{-1}A^{n+j}(x) = A^j(y)^{-1}A^n(f^j(y))^{-1}A^n(f^j(x))A^j(x)$. Furthermore, once this is done, claim (2) follows immediately from this same relation. Each difference $\|A^{n+1}(y)^{-1}A^{n+1}(x) - A^n(y)^{-1}A^n(x)\|$ is bounded by

$$\|A^n(y)^{-1}\| \, \|A(f^n(y))^{-1}A(f^n(x)) - \mathrm{id}\,\| \, \|A^n(x)\|.$$

Since A is β-Hölder continuous, there is $L_2 > 0$ such that the middle factor in this expression is bounded by

$$L_2 \operatorname{dist}(f^n(x), f^n(y))^\beta \leq L_2 \big[\nu^n(x) \operatorname{dist}(x,y)\big]^\beta.$$

Using Lemma 3.1 to bound the product of the other factors, we obtain

$$\|A^{n+1}(y)^{-1}A^{n+1}(x) - A^n(y)^{-1}A^n(x)\| \leq CL_2\big[\nu^n(x)^{(1-m)} \operatorname{dist}(x,y)\big]^\beta. \tag{3.4}$$

The sequence $\nu^n(x)^{\beta(1-m)}$ is uniformly summable, since $\nu(\cdot)$ is bounded away from 1. Let $K > 0$ be an upper bound for the sum. It follows that $A^n(y)^{-1}A^n(x)$ is a Cauchy sequence, and so it does converge. This finishes the proof of claims (a) and (b). Claim (c) is a direct consequence.

Moreover, adding the last inequality over all n, we get $\|H^s_{x,y} - \mathrm{id}\,\| \leq L \operatorname{dist}(x,y)^\beta$ with $L = CL_2K$. This proves claim (d). As a consequence, we also get that there exists $\gamma > 0$ such that $\|H^s_{x,y}\| < \gamma$ for any points x, y in the same local strong-stable leaf. To deduce claim (e), notice that for any x, y in the same (global) strong-stable leaf there exist points $z_0, \ldots, z_n$, where n depends only on an upper bound for the distance between x and y along the leaf, such that $z_0 = x$, $z_n = y$, and each z_i belongs to the local strong-stable leaf of z_{i-1} for every $i = 1, \ldots, n$. Together with (c), this implies $\|H^s_{x,y}\| < \gamma^n$. It is clear from the construction that L_2 and $\Gamma(\cdot)$ may be taken uniform on a neighborhood. The proof of the proposition is complete. □

To show that the family of maps $H^s_{x,y}$ given by this proposition is an invariant stable holonomy for F (we also say that it is an invariant stable holonomy for A) we also need to check that these maps vary continuously with the base points. That is a consequence of the next proposition:

Proposition 3.3. — *Suppose that $A \in C^{r,\alpha}(M, d, \mathbb{K})$ is fiber bunched. Then the map*

$$(x,y) \mapsto H^s_{x,y}$$

is continuous on $W^s_N = \{(x,y) \in M \times M : f^N(y) \in \mathcal{W}^s_{\mathrm{loc}}(f^N(x))\}$, for every $N \geq 0$.

Proof. — Notice that $\operatorname{dist}(x,y) \leq 2R$ for all $(x,y) \in W^s_0$, by our definition of local strong-stable leaves. So, the Cauchy estimate in (3.4)

$$\begin{aligned} \|A^{n+1}(y)^{-1}A^{n+1}(x) - A^n(y)^{-1}A^n(x)\| &\leq CL_2\big[\nu^n(x)^{(1-m)} \operatorname{dist}(x,y)\big]^\beta. \\ &\leq CL_2(2R)^\beta \nu^n(x)^{\beta(1-m)} \end{aligned} \tag{3.5}$$

is uniform on W^s_0. This implies that the limit in part (a) of Proposition 3.2 is uniform on W^s_0. That implies case $N = 0$ of the present proposition. The general case follows immediately, using property (b) in Proposition 3.2. □

Remark 3.4. — Since the constants C and L_2 are uniform on some neighborhood of A, the Cauchy estimate (3.5) is also locally uniform on A. Thus, the limit in part (a) of Proposition 3.2 is locally uniform on A as well. Consequently, the stable holonomy also depends continuously on the cocycle, in the sense that

$$(A, x, y) \mapsto H^s_{A,x,y} \quad \text{is continuous on } \mathcal{G}^{r,\alpha}(M, d, \mathbb{K}) \times W^s_0.$$

Using property (b) in Proposition 3.2 we may even replace W^s_0 by any W^s_N.

Dually, one finds an *invariant unstable holonomy* $(x, y) \mapsto H^u_{x,y}$ for A (or the cocycle F), given by

$$H^u_{x,y} = \lim_{n\to-\infty} A^n(y)^{-1} A^n(x)$$

whenever x and y are on the same strong-unstable leaf, and it is continuous on $W^u_N = \{(x, y) \in M \times M : f^{-N}(y) \in \mathcal{W}^s_{\mathrm{loc}}(f^{-N}(x))\}$, for every $N \geq 0$. Even more,

$$(A, x, y) \mapsto H^u_{A,x,y} \quad \text{is continuous on every } \mathcal{G}^{r,\alpha}(M, d, \mathbb{K}) \times W^u_N.$$

3.2. Differentiability of holonomies. — Now we study the differentiability of stable holonomies $H^s_{A,x,y}$ as functions of $A \in \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$. Notice that $\mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ is an open subset of the Banach space of $C^{r,\alpha}$ maps from M to the space of all $d \times d$ matrices and so the tangent space at each point of $\mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ is naturally identified with that Banach space. The next proposition is similar to Lemma 2.9 in [**25**], but our proof is neater: the previous argument used a stronger fiber bunching condition.

Proposition 3.5. — *Suppose that $A \in \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ is fiber bunched. Then there exists a neighborhood $\mathcal{U} \subset \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ of A such that, for any $x \in M$ and any y, $z \in \mathcal{W}^s(x)$, the map $B \mapsto H^s_{B,y,z}$ is of class C^1 on $\mathcal{U}$, with derivative*

$$\begin{aligned}(3.6)\quad \partial_B H^s_{B,y,z} : \dot{B} \mapsto \sum_{i=0}^{\infty} B^i(z)^{-1} \Big[& H^s_{B,f^i(y),f^i(z)} B(f^i(y))^{-1} \dot{B}(f^i(y)) \\ & - B(f^i(z))^{-1} \dot{B}(f^i(z)) H^s_{B,f^i(y),f^i(z)} \Big] B^i(y).\end{aligned}$$

Proof. — There are three main steps. Recall that fiber bunching is an open condition and the constants in Lemma 3.1 and Proposition 3.2 may be taken uniform on some neighborhood $\mathcal{U}$ of A. First, we suppose that y, z are in the local strong-stable leaf of x, and prove that the expression $\partial_B H^s_{B,y,z} \dot{B}$ is well defined for every $B \in \mathcal{U}$ and every $\dot{B}$ in $T_B \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$. Next, still in the local case, we show that this expression indeed gives the derivative of our map with respect to the cocycle. Finally, we extend the conclusion to arbitrary points on the global strong-stable leaf of x.

Step 1. For each $i \geq 0$, write

$$(3.7)\qquad H^s_{B,f^i(y),f^i(z)} B(f^i(y))^{-1} \dot{B}(f^i(y)) - B(f^i(z))^{-1} \dot{B}(f^i(z)) H^s_{B,f^i(y),f^i(z)}$$

as the following sum

$$(H^s_{B,f^i(y),f^i(z)} - \mathrm{id})B(f^i(y))^{-1}\dot{B}(f^i(y)) + B(f^i(z))^{-1}\dot{B}(f^i(z))(\mathrm{id} - H^s_{B,f^i(y),f^i(z)}) \\ + [B(f^i(y))^{-1}\dot{B}(f^i(y)) - B(f^i(z))^{-1}\dot{B}(f^i(z))].$$

By property (d) in Proposition 3.2, the first term is bounded by

$$(3.8)\quad L\,\|B(f^i(y))^{-1}\|\,\|\dot{B}(f^i(y))\|\ \mathrm{dist}(f^i(y), f^i(z))^\beta \\ \leq L\,\|B^{-1}\|_{0,0}\,\|\dot{B}\|_{0,0}\left[\nu^i(x)\,\mathrm{dist}(y,z)\right]^\beta$$

and analogously for the second one. The third term may be written as

$$B(f^i(y))^{-1}[\dot{B}(f^i(y)) - \dot{B}(f^i(z))] + [B(f^i(y))^{-1} - B(f^i(z))^{-1}]\dot{B}(f^i(z)).$$

Using the triangle inequality, we conclude that this is bounded by

$$(3.9)\quad \Big(\|B(f^i(y))^{-1}\|\,H_\beta(\dot{B}) + H_\beta(B^{-1})\,\|\dot{B}(f^i(z))\|\Big)\,\mathrm{dist}(f^i(y), f^i(z))^\beta. \\ \leq \|B^{-1}\|_{0,\beta}\,\|\dot{B}\|_{0,\beta}\left[\nu^i(x)\,\mathrm{dist}(y,z)\right]^\beta,$$

where $H_\beta(\phi)$ means the smallest $C \geq 0$ such that $\|\phi(z) - \phi(w)\| \leq C\,\mathrm{dist}(z,w)^\beta$ for all z, $w \in M$. Notice, from the Definition (3.1), that

$$(3.10)\qquad \|\phi\|_{0,0} + H_\beta(\phi) = \|\phi\|_{0,\beta} \leq \|\phi\|_{r,\alpha} \quad \text{for any function } \phi.$$

Let $C_1 = \sup\left\{\|B^{-1}\|_{0,\beta} : B \in \mathcal{U}\right\}$. Replacing (3.8) and (3.9) in the expression preceding them, we find that the norm of (3.7) is bounded by

$$(2L+1)C_1\,\nu^i(x)^\beta\,\mathrm{dist}(y,z)^\beta\|\dot{B}\|_{0,\beta}$$

Hence, the norm of the ith term in the expression of $\partial_B H^s_{B,y,z}\,\dot{B}$ is bounded by

$$(3.11)\quad 2(L+1)C_1\ \nu^i(x)^\beta\|B^i(z)^{-1}\|\,\|B^i(y)\|\ \mathrm{dist}(y,z)^\beta\|\dot{B}\|_{0,\beta} \\ \leq C_2\,\nu^i(x)^{\beta(1-m)}\ \mathrm{dist}(y,z)^\beta\|\dot{B}\|_{0,\beta}$$

where $C_2 = 2C(L+1)C_1$ and C is the constant in Lemma 3.1. In this way we find,

$$(3.12)\qquad \|\partial_B H^s_{B,y,z}(\dot{B})\| \leq C_2\sum_{i=0}^{\infty}\nu^i(x)^{\beta(1-m)}\,\mathrm{dist}(y,z)^\beta\|\dot{B}\|_{0,\beta}$$

for any $x \in M$ and y, $z \in \mathcal{W}^s_{\mathrm{loc}}(x)$. This shows that the series defining $\partial_B H^s_{B,y,z}(\dot{B})$ does converge at such points.

Step 2. By part (a) of Proposition 3.2 together with Remark 3.4, the map $H^s_{B,y,z}$ is the uniform limit $H^n_{B,y,z} = B^n(z)^{-1}B^n(y)$ when $n \to \infty$. Clearly, every $H^n_{B,y,z}$ is a differentiable function of B, with derivative

$$\partial_B H^n_{B,y,z}(\dot B) = \sum_{i=0}^{n-1} B^i(z)^{-1}\big[H^{n-i}_{B,f^i(y),f^i(z)} B(f^i(y))^{-1}\dot B(f^i(y)) \\ - B(f^i(z))^{-1}\dot B(f^i(z)) H^{n-i}_{B,f^i(y),f^i(z)}\big] B^i(y).$$

So, to prove that $\partial_B H^s_{B,y,z}$ is indeed the derivative of the holonomy with respect to B, it suffices to show that $\partial H^n_{B,y,z}$ converges uniformly to $\partial H^s_{B,y,z}$ when $n \to \infty$.

Write $1 - m = 2\tau$. From (3.4) and the fact that $\nu(\cdot)$ is strictly smaller than 1,

$$\begin{aligned}\|H^n_{B,y,z} - H^s_{B,y,z}\| &\le CL_2 \sum_{j=n}^{\infty} \nu^j(x)^{\beta(1-m)} \operatorname{dist}(y,z)^\beta \\ &\le C_3 \nu^n(x)^{2\beta\tau} \operatorname{dist}(y,z)^\beta \le C_3 \nu^n(x)^{\beta\tau} \operatorname{dist}(y,z)^\beta\end{aligned}$$

for some uniform constant C_3 (the last inequality is trivial, but it will allow us to come out with a positive exponent for $\nu^i(x)$ in (3.13) below). More generally, and for the same reasons,

$$\begin{aligned}\|H^{n-i}_{B,f^i(y),f^i(z)} - H^s_{B,f^i(y),f^i(z)}\| &\le C_3 \nu^{n-i}(f^i(x))^{\beta\tau} \operatorname{dist}(f^i(y), f^i(z))^\beta \\ &\le C_3 \nu^{n-i}(f^i(x))^{\beta\tau} \nu^i(x)^\beta \operatorname{dist}(y,z)^\beta \\ &= C_3 \nu^n(x)^{\beta\tau} \nu^i(x)^{\beta(1-\tau)} \operatorname{dist}(y,z)^\beta\end{aligned}$$

for all $0 \le i \le n$, and all y, z in the same local strong-stable leaf. It follows, using also Lemma 3.1, that the norm of the difference between the ith terms in the expressions of $\partial_B H^n_{B,y,z}$ and $\partial_B H^s_{B,y,z}$ is bounded by

$$\begin{aligned}(3.13)\quad C_3 \nu^n(x)^{\beta\tau} \nu^i(x)^{\beta(1-\tau)} \operatorname{dist}(y,z)^\beta \|B^i(z)^{-1}\| \, \|B^i(y)\| \\ \le CC_3 \nu^n(x)^{\beta\tau} \nu^i(x)^{\beta\tau} \operatorname{dist}(y,z)^\beta.\end{aligned}$$

Combining this with (3.11), we find that $\|\partial_B H^n_{B,y,z} - \partial_B H^s_{B,y,z}\|$ is bounded by

$$CC_3 \sum_{i=0}^{n-1} \nu^i(x)^{\beta\tau} \nu^n(x)^{\beta\tau} \operatorname{dist}(y,z)^\beta + C_2 \sum_{i=n}^{\infty} \nu^i(x)^{2\beta\tau} \operatorname{dist}(y,z)^\beta.$$

Since $\nu^i(x)$ is bounded away from 1, the sum is bounded by $C_4 \nu^n(x)^{\beta\tau} \operatorname{dist}(y,z)^\beta$, for some uniform constant C_4. This latter expression tends to zero uniformly when $n \to \infty$, and so the argument is complete.

Step 3. From property (b) in Proposition 3.2, we find that if $H^s_{B,f(y),f(z)}$ is differentiable on B then so is $H^s_{B,y,z}$ and the derivative is determined by

$$(3.14)\quad \dot B(z)\, H^s_{B,y,z} + B(z) \cdot \partial_B H^s_{B,y,z}(\dot B) = H^s_{B,y,z} \cdot \dot B(y) + \partial_B H^s_{B,y,z}(\dot B) \cdot B(y).$$

Combining this observation with the previous two steps, we conclude that $H^s_{B,y,z}$ is differentiable on B for any pair of points y, z in the same (global) strong-stable leaf: just note that $f^n(y)$, $f^n(z)$ are in the same local strong-stable leaf for large n. Moreover, a straightforward calculation shows that the expression in (3.6) satisfies the relation (3.14). Therefore, (3.6) is the expression of the derivative for all points y, z in the same strong-stable leaf. The proof of the proposition is now complete. $\square$

Corollary 3.6. — *Suppose that $A \in \mathscr{G}^{r,\alpha}(M,d,\mathbb{K})$ is fiber bunched. Then there exists $\theta < 1$ and a neighborhood $\mathscr{U}$ of A and, for each $a > 0$, there exists $C_5(a) > 0$ such that*

$$(3.15)\quad \|\sum_{i=k}^{\infty} B^i(z)^{-1}\Big[H^s_{B,f^i(y),f^i(z)} B(f^i(y))^{-1}\dot{B}(f^i(y)) - B(f^i(z))^{-1}\dot{B}(f^i(z)) H^s_{B,f^i(y),f^i(z)}\Big] B^i(y)\| \le C_5(a)\,\theta^k\,\|\dot{B}\|_{0,\beta}$$

for any $B \in \mathscr{U}$, $k \ge 0$, $x \in M$, and y, $z \in \mathscr{W}^s(x)$ with $\operatorname{dist}_{\mathscr{W}^s}(y,z) < a$.

Proof. — Let $\theta < 1$ be an upper bound for $\nu(\cdot)^{\beta(1-m)}$. Begin by supposing that $\operatorname{dist}_{\mathscr{W}^s}(y,z) < R$. Then y, z are in the same local strong-stable leaf, and we may use (3.11) to get that the expression in (3.15) is bounded above by

$$C_2 \sum_{i=k}^{\infty} \nu^i(x)^{\beta(1-m)} \operatorname{dist}(y,z)^{\beta} \|\dot{B}\|_{0,\beta} \le C_5'\,\theta^k\,\|\dot{B}\|_{0,\beta}$$

for some uniform constant C_5'. This settles the case $a \le R$, with $C_5(a) = C_5'$.

In general, there is $l \ge 0$ such that $\operatorname{dist}_{\mathscr{W}^s}(y,z) < a$ implies $\operatorname{dist}_{\mathscr{W}^s}(f^l(y), f^l(z)) < R$. Suppose first that $k \ge l$. Clearly, the expression in (3.15) does not change if we replace y, z by $f^l(y)$, $f^l(z)$ and replace k by $k-l$. Then, by the previous special case, (3.15) is bounded above by

$$C_5'\,\theta^{k-l}\,\|\dot{B}\|_{0,\beta}$$

and so it suffices to choose $C_5(a) \ge C_5'\theta^{-l}$. If $k < l$ then begin by splitting (3.15) into two sums, respectively, over $k \le i < l$ and over $i \ge l$. The first sum is bounded by $C_5''(a)\|\dot{B}\|_{0,\beta}$ for some constant $C_5''(a) > 0$ that depends only on a (and l, which is itself a function of a). The second one is bounded by $C_5'\,\|\dot{B}\|_{0,\beta}$, as we have just seen. The conclusion follows, assuming we choose $C_5(a) \ge C_5'\theta^{-l} + C_5''(a)\theta^{-l}$. $\square$

For future reference, let us state the analogues of Proposition 3.5 and Corollary 3.6 for invariant unstable holonomies:

Proposition 3.7. — *Suppose that $A \in \mathscr{G}^{r,\alpha}(M,d,\mathbb{K})$ is fiber bunched. Then there exists a neighborhood $\mathscr{U} \subset \mathscr{G}^{r,\alpha}(M,d,\mathbb{K})$ of A such that, for any $x \in M$ and any $y, z \in \mathscr{W}^u(x)$, the map $B \mapsto H^u_{B,y,z}$ is of class C^1 on $\mathscr{U}$ with derivative*

$$(3.16)\quad \partial_B H^u_{B,y,z} : \dot{B} \mapsto -\sum_{i=1}^{\infty} B^{-i}(z)^{-1}\Big[H^u_{B,f^{-i}(y),f^{-i}(z)} B(f^{-i}(y))^{-1}\dot{B}(f^{-i}(y)) - B(f^{-i}(z))^{-1}\dot{B}(f^{-i}(z)) H^u_{B,f^{-i}(y),f^{-i}(z)}\Big] B^{-i}(y).$$

Corollary 3.8. — *In the same setting as Proposition 3.7,*

$$\|\sum_{i=k}^{\infty} B^{-i}(z)^{-1}\Big[H^u_{B,f^{-i}(y),f^{-i}(z)}B(f^{-i}(y))^{-1}\dot{B}(f^{-i}(y)) - B(f^{-i}(z))^{-1}\dot{B}(f^{-i}(z))H^u_{B,f^{-i}(y),f^{-i}(z)}\Big]B^{-i}(y)\| \le C_5(a)\,\theta^k\,\|\dot{B}\|_{0,\beta}. \tag{3.17}$$

for any $B \in \mathcal{U}$, $k \geq 0$, $x \in M$, and y, $z \in \mathcal{W}^u(x)$ with $\operatorname{dist}_{\mathcal{W}^u}(y,z) < a$.

3.3. Dominated smooth cocycles. — Now we introduce a concept of domination for smooth cocycles, related to the notion of fiber bunching in the linear setting. We observe that dominated smooth cocycles admit invariant stable and unstable holonomies, and these holonomies vary continuously with the cocycle. These facts are included to make the analogy to the linear case more apparent but, otherwise, they are not used in the present paper: whenever dealing with smooth cocycles we just assume that invariant stable and unstable holonomies do exist. In this section we do not consider any invariant measure.

Let $\beta > 0$ be fixed. A fiber bundle with smooth leaves $\pi : \mathcal{E} \to M$ is called β-Hölder if there exists $C > 0$ such that the coordinate changes (2.7) satisfy

$$\operatorname{dist}_{C^1}(g_x^{\pm 1}, g_y^{\pm 1}) \le C \operatorname{dist}(x,y)^\beta \quad \text{for every } x \text{ and } y. \tag{3.18}$$

Then we say that a smooth cocycle $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$ is β-Hölder if its local expressions $\phi_{U_1}^{-1} \circ \mathfrak{F} \circ \phi_{U_0} : (U_0 \cap f^{-1}(U_1)) \times N \to U_1 \times N$, $(x,\xi) \mapsto (f(x), \mathfrak{F}_x^U(\xi))$ satisfy

$$\operatorname{dist}_{C^1}(\mathfrak{F}_x^U, \mathfrak{F}_y^U) \le C_U \operatorname{dist}(x,y)^\beta \quad \text{for some } C_U > 0 \text{ and every } x \text{ and } y. \tag{3.19}$$

This does not depend on the choice of the local charts. Indeed, any other local expression has the form $\mathfrak{F}_x^V = g'_{f(x)} \circ \mathfrak{F}_x^U \circ g_x^{-1}$ on the intersection of the domains of definition. Then, a straightforward use of the triangle inequality gives

$$\operatorname{dist}_{C^1}(\mathfrak{F}_x^V, \mathfrak{F}_y^V) \le C_V \operatorname{dist}(x,y)^\beta \quad \text{for every } x \text{ and } y,$$

where C_V depends on β, C, C_U and upper bounds for the norms of $D\mathfrak{F}_x^U$, Dg'_y, Dg_x^{-1} and Df.

Definition 3.9. — Denote by $\mathcal{C}^\beta(f,\mathcal{E})$ the space of cocycles $\mathfrak{F}$ that are β-Hölder continuous. A cocycle $\mathfrak{F} \in \mathcal{C}^\beta(f,\mathcal{E})$ is *s-dominated* if there is $\theta < 1$ such that

$$\|D\mathfrak{F}_x(\xi)^{-1}\|\,\nu(x)^\beta \le \theta \quad \text{for all } (x,\xi) \in \mathcal{E} \tag{3.20}$$

and it is *u-dominated* if there is $\theta < 1$ such that

$$\|D\mathfrak{F}_x(\xi)\|\,\hat{\nu}(x)^\beta \le \theta \quad \text{for all } (x,\xi) \in \mathcal{E}. \tag{3.21}$$

We say that F is *dominated* if it is both s-dominated and u-dominated.

In geometric terms, (3.20) means that the contractions of $\mathfrak{F}$ along the fibers are strictly weaker than the contractions of f along strong-stable leaves and (3.21) expresses a similar property for the expansions of $\mathfrak{F}$. These conditions are designed so that the usual graph transform argument yields a "strong-stable" lamination and a "strong-unstable" lamination for the map $\mathfrak{F}$, as we are going to see. Then the holonomy maps for these laminations constitute invariant stable and unstable holonomies for the cocycle.

Observe that both conditions (3.20)–(3.21) become stronger as β decreases to zero; this may be seen as a sort of compensation for the decreasing regularity (Hölder continuity) of the cocycle. The observations that follow extend, up to straightforward adjustments, to the case when these conditions hold for some iterate $\mathfrak{F}^\ell$, $\ell \geq 1$.

Proposition 3.10. — *Let $\mathfrak{F} \in \mathscr{C}^\beta(f, \mathscr{E})$ be s-dominated. Then there exists a unique partition $\mathscr{W}^s = \{\mathscr{W}^s(x,\xi) : (x,\xi) \in \mathscr{E}\}$ of $\mathscr{E}$ and there exists $C > 0$ such that*

(a) *every $\mathscr{W}^s(x,\xi)$ is a (C,β)-Hölder continuous graph over $\mathscr{W}^s(x)$;*

(b) *the partition is invariant: $\mathfrak{F}(\mathscr{W}^s(x,\xi)) \subset \mathscr{W}^s(\mathfrak{F}(x,\xi))$ for all $(x,\xi) \in \mathscr{E}$.*

Consider the family of maps $H^s_{x,y} : \mathscr{E}_x \to \mathscr{E}_y$ defined by $(y, H^s_{x,y}(\xi)) \in \mathscr{W}^s(x,\xi)$ for each $y \in \mathscr{W}^s(x)$. Then, for every x, y and z in the same strong-stable leaf,

(c) $H^s_{y,z} \circ H^s_{x,y} = H^s_{x,z}$ *and* $H^s_{x,x} = \mathrm{id}$

(d) $\mathfrak{F}_y \circ H^s_{x,y} = H^s_{f(x),f(y)} \circ \mathfrak{F}_x$

(e) *$H^s_{x,y} : \mathscr{E}_x \to \mathscr{E}_y$ is the uniform limit of $(\mathfrak{F}^n_y)^{-1} \circ \mathfrak{F}^n_x$ as $n \to \infty$;*

(f) *$H^s_{x,y} : \mathscr{E}_x \to \mathscr{E}_y$ is γ-Hölder continuous, where $\gamma > 0$ depends only on $\mathfrak{F}$, and $H^s_{x,y}$ is (C,γ)-Hölder continuous if x and y are in the same strong-stable leaf;*

(g) *$(x,y,\xi) \mapsto H^s_{x,y}(\xi)$ is continuous when (x,y) varies in the set of pairs of points in the same local strong-stable leaf.*

Moreover, there are dual statements for strong-unstable leaves, assuming that $\mathfrak{F}$ is u-dominated.

Outline of the proof. — This follows from the same normal hyperbolicity methods (Hirsch, Pugh, Shub [**16**]) that were used in the previous section for linear cocycles. Existence (a) and invariance (b) of the family $\mathscr{W}^s$ follow from a standard application of the graph transform argument (see Chapter 5 of [**24**]). The pseudo-group property (c) is a direct consequence of the definition of $H^s_{x,y}$. The invariance property (d) is a restatement of (b). To prove (e), notice that

$$H^s_{x,y} = (\mathfrak{F}^n_y)^{-1} \circ H^s_{f^n(x),f^n(y)} \circ \mathfrak{F}^n_x,$$

because the lamination $\mathscr{W}^s$ is invariant under $\mathfrak{F}$. Also, by (a), the uniform C^0 distance from $H^s_{f^n(x),f^n(y)}$ to the identity is bounded by

$$C \operatorname{dist}(f^n(x), f^n(y))^\beta \leq C\big[\nu^n(x) \operatorname{dist}(x,y)\big]^\beta.$$

Putting these two observations together, we find that

$$\begin{aligned}\operatorname{dist}_{C^0}(H^s_{x,y}, (\mathfrak{F}^n_y)^{-1} \circ \mathfrak{F}^n_x) &\leq \operatorname{Lip}\left((\mathfrak{F}^n_y)^{-1}\right) \operatorname{dist}_{C^0}(H^s_{f^n(x),f^n(y)}, \mathrm{id}) \\ &\leq C \sup_\xi \|D\mathfrak{F}^n_y(\xi)^{-1}\| \, \nu^n(x)^\beta \operatorname{dist}(x,y)^\beta.\end{aligned}$$

So, by the domination condition (3.20),

$$\operatorname{dist}_{C^0}(H^s_{x,y}, (\mathfrak{F}^n_y)^{-1} \circ \mathfrak{F}^n_x) \le C\theta^n \operatorname{dist}(x,y)^\beta. \tag{3.22}$$

This proves (e). For pairs (x,y) in the same local strong-stable leaf, the right hand side of (3.22) is uniformly bounded by $CR^\beta\theta^n$. Since this converges to zero, we also get that the limit map $(x,y,\xi) \mapsto H^s_{x,y}(\xi)$ is continuous, as stated in (g).

The Hölder continuity property is another by-product of normal hyperbolicity theory. In this instance it can be derived as follows. In view of the invariance property (d), it suffices to consider the case when x and y are in the same local strong-stable leaf. Given nearby points $\xi, \eta \in \mathcal{E}_x$, let ξ', η' be their images under the holonomy map $H^s_{x,y}$. The domination Hypothesis (3.20) ensures that there exists $n \le -c_1 \log \operatorname{dist}(\xi', \eta')$ (where $c_1 > 0$ is a uniform constant) such that the distance $\operatorname{dist}(f^n(x), f^n(y))$ between the fibers is much smaller than the distance $\operatorname{dist}(\mathfrak{F}^n_x(\xi'), \mathfrak{F}_x(\eta'))$ along the fiber, in such a way that,

$$\operatorname{dist}(\mathfrak{F}^n_x(\xi), \mathfrak{F}^n_x(\eta)) \ge \frac{1}{2} \operatorname{dist}(\mathfrak{F}^n_y(\xi'), \mathfrak{F}^n_y(\eta')).$$

Let $c_2 > 0$ be an upper bound for $\log \|D\mathfrak{F}^{\pm 1}_w\|$ over all $w \in M$. Then

$$\frac{\operatorname{dist}(\xi', \eta')}{\operatorname{dist}(\xi, \eta)} \le e^{2c_2 n} \frac{\operatorname{dist}(\mathfrak{F}^n_y(\xi'), \mathfrak{F}^n_y(\eta'))}{\operatorname{dist}(\mathfrak{F}^n_x(\xi), \mathfrak{F}^n_x(\eta))} \le 2e^{2c_2 n} \le 2d(\xi', \eta')^{-2c_1 c_2}.$$

This gives $\operatorname{dist}(\xi', \eta') \le 2^\gamma \operatorname{dist}(\xi, \eta)^\gamma$ with $\gamma = 1/(1 + 2c_1 c_2)$. □

Next, let $\mathcal{D}^{s,\beta}(f, \mathcal{E}) \subset \mathcal{C}^\beta(f, \mathcal{E})$ be the subset of s-dominated cocycles. It is clear from the definition that $\mathcal{D}^{s,\beta}(f, \mathcal{E})$ is an open subset, relative to the uniform C^1 distance

$$\operatorname{dist}_{C^1}(\mathfrak{F}, \mathfrak{G}) = \sup\{\operatorname{dist}_{C^1}(\mathfrak{F}_x, \mathfrak{G}_x) : x \in M\}.$$

We are going to see that invariant stable holonomies vary continuously with the cocycle inside $\mathcal{D}^{s,\beta}(f, \mathcal{E})$, relative to this distance. Analogously, invariant unstable holonomies vary continuously with the cocycle inside the subset $\mathcal{D}^{u,\beta}(f, \mathcal{E}) \subset \mathcal{C}^\beta(f, \mathcal{E})$ of u-dominated cocycles. We also denote by $\mathcal{D}^\beta(f, \mathcal{E}) \subset \mathcal{C}^\beta(f, \mathcal{E})$ the (open) subset of dominated cocycles.

Let $\mathcal{W}^s(\mathfrak{G}) = \{\mathcal{W}^s(\mathfrak{G}, x, \xi) : (x, \xi) \in \mathcal{E}\}$ denote the strong-stable lamination of a dominated cocycle $\mathfrak{G}$, as in Proposition 3.10, and $H^s_{\mathfrak{G}} = H^s_{\mathfrak{G},x,y}$ be the corresponding stable holonomy:

$$(y, H^s_{\mathfrak{G},x,y}(\xi)) \in \mathcal{W}^s(\mathfrak{G}, x, \xi). \tag{3.23}$$

Recall that $\mathcal{W}^s(\mathfrak{G}, x, \xi)$ is a graph over $\mathcal{W}^s(x)$. We also denote by $\mathcal{W}^s_{\mathrm{loc}}(\mathfrak{G}, x, \xi)$ the subset of points $(y, H^s_{\mathfrak{G},x,y}(\xi))$ with $y \in \mathcal{W}^s_{\mathrm{loc}}(x)$.

Proposition 3.11. — *Let $(\mathfrak{F}_k)_k$ be a sequence of cocycles converging to $\mathfrak{F}$ in the space $\mathcal{D}^{s,\beta}(f, \mathcal{E})$. Then, for every $x \in M$, $y \in \mathcal{W}^s(x)$, and $\xi \in \mathcal{E}_x$,*

(a) *$\mathcal{W}^s(\mathfrak{F}_k, x, \xi)$ is a β-Hölder graph; restricted to local strong-stable leaves, the multiplicative Hölder constant is uniform on (k, x, ξ);*

(b) *the sequence* $(u_k)_k$ *of functions defined by* $\mathcal{W}^s_{\mathrm{loc}}(\mathfrak{F}_k, x, \xi) = \operatorname{graph} u_k$ *converges uniformly to the function* u *defined by* $\mathcal{W}^s_{\mathrm{loc}}(\mathfrak{F}, x, \xi) = \operatorname{graph} u$*; this convergence is uniform on* (x, ξ)*;*

(c) $H^s_{\mathfrak{F}_k,x,y}$ *converges uniformly to* $H^s_{\mathfrak{F},x,y}$*; this convergence is uniform on* (x, y)*, restricted to the set of pairs of points in the same local strong-stable leaf.*

Moreover, there are dual statements for invariant unstable holonomies, in the space of u-dominated cocycles.

Outline of the proof. — This is another standard consequence of the graph transform argument [**16**]. Indeed, the assumptions imply that the graph transform of $\mathfrak{F}_k$ converges to the graph transform of $\mathfrak{F}$ in an appropriate sense, so that the corresponding fixed points converge as well. This yields (a) and (b). When $y \in \mathcal{W}^s_{\mathrm{loc}}(x)$, claim (c) is a direct consequence of (b) and the Definition (3.23). The general statement follows, using the invariance property in Proposition 3.10:

$$H^s_{\mathfrak{F}_k,x,y} = (\mathfrak{F}^n_{k,y})^{-1} \circ H_{\mathfrak{F}_k,f^n(x),f^n(y)} \circ \mathfrak{F}^n_{k,x}.$$

Related facts were proved in [**25**, Section 4] for linear cocycles, along these lines. □

Remark 3.12. — The previous observations do not need the full strength of partial hyperbolicity. Indeed, the definition of s-dominated cocycle still makes sense if one allows the subbundle E^u in (2.1) to have dimension zero; moreover, all the statements about invariant stable holonomies in Propositions 3.10 and 3.11 remain valid in this case. Analogously, for defining u-domination and for the statements about invariant unstable holonomies one may allow E^s to have dimension zero.

Remark 3.13. — It follows from (2.13)-(2.14) that if a linear cocycle F is fiber bunched then the associated projective cocycle $\mathfrak{F} = \mathbb{P}(F)$ is dominated. Thus, we could use Proposition 3.10 to conclude that $\mathfrak{F}$ admits invariant stable and unstable holonomies. On the other hand, it is easy to exhibit these holonomies explicitly: if $H^s_{x,y}$ and $H^u_{x,y}$ are invariant holonomies for F then $\mathbb{P}(H^s_{x,y})$ and $\mathbb{P}(H^u_{x,y})$ are invariant holonomies for $\mathfrak{F}$.

4. Invariant measures of smooth cocycles

In this section we prove the following result and we use it to obtain Theorem C:

Theorem 4.1. — *Let* f *be a* C^1 *partially hyperbolic diffeomorphism,* $\mathfrak{F}$ *be a smooth cocycle over* f*,* μ *be an* f*-invariant probability, and* m *be an* $\mathfrak{F}$*-invariant probability on* $\mathcal{E}$ *such that* $\pi_* m = \mu$*.*

(a) *If* $\mathfrak{F}$ *admits invariant stable holonomies and* $\lambda_-(\mathfrak{F}, x, \xi) \geq 0$ *at* m*-almost every point* $(x, \xi) \in \mathcal{E}$ *then, for any disintegration* $\{m_x : x \in M\}$ *of* m *into conditional probabilities along the fibers, there exists a full* μ*-measure subset* M^s *such that* $m_z = (H^s_{y,z})_* m_y$ *for every* y*,* $z \in M^s$ *in the same strong-stable leaf.*

(b) *If $\mathfrak{F}$ admits invariant unstable holonomies and $\lambda_+(\mathfrak{F}, x, \xi) \leq 0$ at m-almost every point $(x, \xi) \in \mathcal{E}$ then, for any disintegration $\{m_x : x \in M\}$ of m into conditional probabilities along the fibers, there exists a full μ-measure subset M^u such that $m_z = (H^u_{y,z})_* m_y$ for every y, $z \in M^u$ in the same strong-unstable leaf.*

Remark 4.2. — Theorem 4.1 does not require full partial hyperbolicity. Indeed, the proof of part (a) that we will present in the sequel remains valid when $\dim E^u = 0$. Analogously, part (b) remains true when $\dim E^s = 0$.

Theorem C can be readily deduced from Theorem 4.1 and Theorem D, as follows. Given any disintegration $\{m_x : x \in M\}$ of the probability m, define $\Psi(x) = m_x$ at every point. According to Theorem 4.1, the function Ψ is essentially s-invariant and essentially u-invariant. By Theorem D, there exists a bi-invariant function $\tilde{\Psi}$ defined on some bi-saturated full measure set $\tilde{M}$ and coinciding with Ψ almost everywhere. Then we get a new disintegration $\{\tilde{m}_x : x \in M\}$ by setting $\tilde{m}_x = \tilde{\Psi}(x)$ when $x \in \tilde{M}$ and extending the definition arbitrarily to the complement. The conclusion of Theorem D means that this new disintegration is both s-invariant and u-invariant on $\tilde{M}$. Moreover, it is continuous if f is accessible.

The proof of Theorem 4.1 is given in Sections 4.1 through 4.4. Theorem D will be proved in Sections 6 and 7.

4.1. Abstract invariance principle. — Let $(M_*, \mathcal{M}_*, \mu_*)$ be a Lebesgue space, that is, a complete separable probability space. Every Lebesgue space is isomorphic mod 0 to the union of an interval, endowed with the Lebesgue measure, and a finite or countable set of atoms. See Rokhlin [**23**, § 2]. Let $T : M_* \to M_*$ be an invertible measurable transformation. A σ-algebra $\mathcal{B} \subset \mathcal{M}_*$ is *generating* if its iterates $T^n(\mathcal{B})$, $n \in \mathbb{Z}$ generate the whole $\mathcal{M}_*$ mod 0: for every $E \in \mathcal{M}_*$ there exists E' in the smallest σ-algebra that contains all the $T^n(\mathcal{B})$ such that $\mu_*(E \Delta E') = 0$.

***Theorem 4.3* (Ledrappier [19])**. — *Let $B : M_* \to \mathrm{GL}(d, \mathbb{K})$ be a measurable map such that the functions $x \mapsto \log \|B(x)^{\pm 1}\|$ are μ_*-integrable. Let $\mathcal{B} \subset \mathcal{M}_*$ be a generating σ-algebra such that both T and B are $\mathcal{B}$-measurable* mod 0.

If $\lambda_-(B, x) = \lambda_+(B, x)$ at μ_*-almost every $x \in M_*$ then, for any $\mathbb{P}(F_B)$-invariant probability m that projects down to μ_*, any disintegration $x \mapsto m_x$ of m along the fibers is $\mathcal{B}$-measurable mod 0.

The proof of Theorem 4.1 is based on an extension of this result to smooth cocycles that was recently proved by Avila, Viana [**3**]. For the statement one needs to introduce the following notion. A *deformation of a smooth cocycle* $\mathfrak{F}$ is a measurable transformation $\tilde{\mathfrak{F}} : \mathcal{E} \to \mathcal{E}$ which is conjugated to $\mathfrak{F}$,

$$\tilde{\mathfrak{F}} = \mathcal{H} \circ \mathfrak{F} \circ \mathcal{H}^{-1},$$

by some invertible measurable map $\mathcal{H} : \mathcal{E} \to \mathcal{E}$ of the form $\mathcal{H}(x, \xi) = (x, \mathcal{H}_x(\xi))$, such that all the $\mathcal{H}_x^{-1} : \mathcal{E}_x \to \mathcal{E}_x$ are Hölder continuous, with uniform Hölder constants:

there exist positive constants γ and Γ such that

$$\operatorname{dist}(\xi,\eta) \le \Gamma \operatorname{dist}(\mathcal{H}_x(\xi),\mathcal{H}_x(\eta))^\gamma \quad \text{for every } x \in M \text{ and } \xi,\eta \in \mathcal{E}_x.$$

To each $\mathfrak{F}$-invariant probability m corresponds an $\tilde{\mathfrak{F}}$-invariant probability $\tilde{m} = \mathcal{H}_* m$.

***Theorem 4.4* (Avila, Viana [3]).** — *Let $\tilde{\mathfrak{F}}$ be a deformation of a smooth cocycle $\mathfrak{F}$. Let $\mathcal{B} \subset \mathcal{M}_*$ be a generating σ-algebra such that both T and $x \mapsto \tilde{\mathfrak{F}}_x$ are $\mathcal{B}$-measurable* mod 0. *Let $\tilde{m}$ be an $\tilde{\mathfrak{F}}$-invariant probability that projects down to μ_*.*

If $\lambda_-(\mathfrak{F},x,\xi) \ge 0$ for m-almost every $(x,\xi) \in \mathcal{E}$ then any disintegration $x \mapsto \tilde{m}_x$ of $\tilde{m}$ along the fibers is $\mathcal{B}$-measurable mod 0.

4.2. Global essential invariance. — For proving Theorem 4.1 it suffices to consider the claim (a): then claim (b) is obtained just by reversing time. In this section we reduce the general case to a local version of the claim (Proposition 4.5 below), whose proof is postponed until Section 4.4.

For each symbol $* \in \{s,u\}$ and $r > 0$, denote by $\mathcal{W}^*(x,r)$ the neighborhood of radius r around x inside the leaf $\mathcal{W}^*(x)$. Recall that we write $\mathcal{W}^*_{\mathrm{loc}}(x) = \mathcal{W}^*(x,R)$.

Proposition 4.5. — *Consider the setting of Theorem 4.1(a). Let Σ be a cross-section to the strong-stable foliation $\mathcal{W}^s$ of f and let $\delta \in (0,R/2)$. Denote*

$$\mathcal{N}(\Sigma,\delta) = \bigcup_{z\in\Sigma} \mathcal{W}^s(z,\delta)$$

Then there exists a full μ-measure subset $\mathcal{N}^s$ of $\mathcal{N}(\Sigma,\delta)$ such that $m_y = (H^s_{x,y})_* m_x$ for every x, $y \in \mathcal{N}^s$ in the same $\mathcal{W}^s(z,\delta)$, $z \in \Sigma$.

Fix any $\delta \in (0,R/2)$. For each $p \in M$, consider a cross-section $\Sigma(p)$ such that $\mathcal{N}(\Sigma(p),\delta)$ contains p in its interior and let $\mathcal{N}^s(p) \subset \mathcal{N}(\Sigma(p),\delta)$ be a full measure subset as in Proposition 4.5. By compactness, we may find $\varepsilon \ll \delta$ and points $p_1,\dots,p_N$ such that the ball of radius ε around every point of M is contained in some $\mathcal{N}(\Sigma(p_j),\delta)$. Since the measure m is invariant under $\mathfrak{F}$, there exists an f-invariant set $M_m \subset M$ with full μ-measure such that $m_{f(x)} = (\mathfrak{F}_x)_* m_x$ for every $x \in M_m$. Take

$$M^s = \{x \in M_m : f^n(x) \notin \mathcal{N}(\Sigma(p_j),\delta) \setminus \mathcal{N}^s(p_j) \text{ for all } n \ge 0 \text{ and } j = 1,\dots,N.\}$$

Given any pair of points x, $y \in M^s$ in the same strong-stable leaf, take $n \ge 0$ large enough so that the distance from $f^n(x)$ to $f^n(y)$ along the corresponding strong-stable leaf is less than ε. Next, fix j such that $\mathcal{N}(\Sigma(p_j),\delta)$ contains the ball of radius ε around $f^n(x)$. Since x, $y \in M^s$, both points $f^n(x)$, $f^n(y)$ belong to $\mathcal{N}^s(p_j)$. So, by Proposition 4.5,

$$m_{f^n(y)} = (H^s_{f^n(x),f^n(y)})_* m_{f^n(x)}. \tag{4.1}$$

Since x, $y \in M_m$, we also have that $m_{f^n(x)} = (\mathfrak{F}^n_x)_* m_x$ and analogously for y. Then, using the invariance relation $H^s_{f^n(x),f^n(y)} \circ \mathfrak{F}^n_x = \mathfrak{F}^n_y \circ H^s_{x,y}$, the equality in (4.1) becomes $m_y = (H^s_{x,y})_* m_x$.

This proves claim (a) in Theorem 4.1. Claim (b) is analogous, up to time reversion. Thus, we have reduced the proof of Theorem 4.1 to proving Proposition 4.5.

4.3. A local Markov construction. — The proof of Proposition 4.5 can be outlined as follows. The assumption that the cocycle admits stable holonomy allows us to construct a special deformation $\tilde{\mathfrak{F}}$ of the smooth cocycle $\mathfrak{F}$ which is measurable mod 0 with respect to a certain σ-algebra $\mathcal{B}$. Applying Theorem 4.4 we get that the disintegration of $\tilde{m}$ is also $\mathcal{B}$-measurable mod 0, where $\tilde{m}$ is the $\tilde{\mathfrak{F}}$-invariant measure corresponding to m. When translated back to the original setting, this $\mathcal{B}$-measurability property means that the disintegration of m is essentially invariant on the domain $\mathcal{N}(\Sigma, \delta)$, as stated in Proposition 4.5.

In this section we construct $\tilde{\mathfrak{F}}$ and $\mathcal{B}$. The next proposition is the main tool. It is essentially taken from Proposition 3.3 in **[25]**, so here we just outline the construction.

Proposition 4.6. — *Let Σ be a cross-section to the strong-stable foliation $\mathcal{W}^s$ and $\delta \in (0, R/2)$. Then there exists $N \geq 1$ and a measurable family of sets $\{S(z) : z \in \Sigma\}$ such that*

(a) *$\mathcal{W}^s(z, \delta) \subset S(z) \subset \mathcal{W}^s_{\mathrm{loc}}(z)$ for all $z \in \Sigma$;*
(b) *for all $l \geq 1$ and $z, \zeta \in \Sigma$, if $f^{lN}(S(\zeta)) \cap S(z) \neq \varnothing$ then $f^{lN}(S(\zeta)) \subset S(z)$.*

Outline of the proof. — Fix N big enough so that $\nu^N(x) < 1/4$ for all $x \in M$, and denote $g = f^N$. For each $z \in \Sigma$ define $S_0 = \mathcal{W}^s(z, \delta)$ and

$$S_{n+1}(z) = S_0(z) \cup \bigcup_{(j,w) \in Z_n(z)} g^j(S_n(w)) \tag{4.2}$$

where $Z_n(z) = \{(j, w) \in \mathbb{N} \times \Sigma : g^j(S_n(w)) \cap S_0(z) \neq \varnothing\}$. Clearly, $S_0(z) \subset S_1(z)$ and $Z_0(z) \subset Z_1(z)$. Notice that if $S_{n-1}(z) \subset S_n(z)$ and $Z_{n-1}(z) \subset Z_n(z)$ for every $z \in \Sigma$, then,

$$\bigcup_{(j,w) \in Z_{n-1}(z)} g^j(S_{n-1}(w)) \subset \bigcup_{(j,w) \in Z_n(z)} g^j(S_n(w)).$$

Therefore, by induction, $S_n(z) \subset S_{n+1}(z)$ and $Z_n(z) \subset Z_{n+1}(z)$ for every $n \geq 0$. Define

$$S_\infty(z) = \bigcup_{n=0}^{\infty} S_n(z) \quad \text{and} \quad Z_\infty(z) = \bigcup_{n=0}^{\infty} Z_n(z).$$

Then $Z_\infty(z)$ is the set of $(j, w) \in \mathbb{N} \times \Sigma$ such that $g^j(S_\infty(w))$ intersects $S_0(z)$, and

$$S_\infty(z) = S_0(z) \cup \bigcup_{(j,w) \in Z_\infty(z)} g^j(S_\infty(w)).$$

The choice of N ensures that $S_\infty(z) \subset \mathcal{W}^s(z, 2\delta)$. Finally, define

$$S(z) = S_\infty(z) \setminus \bigcup_{(k,\xi) \in V(z)} g^k(S_\infty(\xi))$$

where $V(z) = \{(k, \xi) \in \mathbb{N} \times \Sigma : g^k(S_\infty(\xi)) \not\subset S_\infty(z)\}$. This family of sets satisfies the conclusion of the proposition. □

Since the conclusion of Proposition 4.5 is not affected when f and $\mathfrak{F}$ are replaced by its iterates f^N and $\mathfrak{F}^N$, we may take the integer N in Proposition 4.6 to be equal to 1. Let $M_* = M$ and $T = f$. Let $\mathscr{M}_*$ be the μ-completion of the Borel σ-algebra of M and μ_* be the canonical extension of μ to $\mathscr{M}_*$. Then $(M_*, \mathscr{M}_*, \mu_*)$ is a Lebesgue space and T is an automorphism in it.

For each $z \in \Sigma$, let $r(z) \geq 0$ be the largest integer (possibly infinite) such that $f^j(S(z))$ does not intersect any of the $S(w)$, $w \in \Sigma$ for all $0 < j \leq r(z)$. Let $\mathscr{B}$ be the σ-algebra of sets $E \in \mathscr{M}_*$ such that, for every z and j, either E contains $f^j(S(z))$ or is disjoint from it. Notice that an $\mathscr{M}$-measurable function on M is $\mathscr{B}$-measurable precisely if it is constant on every $f^j(S(z))$. Define $\tilde{\mathfrak{F}} : \mathscr{E} \to \mathscr{E}$ to be $\tilde{\mathfrak{F}} = \mathscr{H} \circ \mathfrak{F} \circ \mathscr{H}^{-1}$, where

$$\mathscr{H}_x = \begin{cases} H^s_{x,f^j(z)} & \text{if } x \in f^j(S(z)) \text{ for some } z \in \Sigma \text{ and } 0 \leq j \leq r(z) \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

Recall that $S(z) \subset \mathscr{W}^s_{\mathrm{loc}}(z)$ for every z, by construction. Reducing δ if necessary, we may assume that $f^j(S(z)) \subset \mathscr{W}^s_{\mathrm{loc}}(f^j(z))$ for every z and every $j \geq 0$. Then condition (d) in Definition 2.10 ensures that the family $\{\mathscr{H}_x : x \in M\}$ is uniformly Hölder continuous. The definition implies that

$$\tilde{\mathfrak{F}}_x = H^s_{f(x),f^{j+1}(z)} \circ \mathfrak{F}_x \circ H^s_{f^j(z),x} = \mathfrak{F}_{f^j(z)} \tag{4.3}$$

if $x \in f^j(S(z))$ for some $z \in \Sigma$ and $0 \leq j < r(z)$. Moreover,

$$\tilde{\mathfrak{F}}_x = H^s_{f(x),w} \circ \mathfrak{F}_x \circ H^s_{f^{r(z)}(z),x} \tag{4.4}$$

if $x \in f^{r(z)}(S(z))$ for some $z \in \Sigma$, where $w \in \Sigma$ is given by $f^{r(z)+1}(S(z)) \subset S(w)$. In all other cases, $\tilde{\mathfrak{F}}_x = \mathfrak{F}_x$.

Lemma 4.7. — *The following properties hold*

(a) $T = f$ *and* $x \mapsto \tilde{\mathfrak{F}}_x$ *are* $\mathscr{B}$*-measurable*
(b) $\mathrm{dist}_{C^0}(\mathscr{H}_x, \mathrm{id})$ *is uniformly bounded*
(c) $\{T^n(\mathscr{B}) : n \in \mathbb{N}\}$ *generates* $\mathscr{M}_*$ $\mathrm{mod}\, 0$.

Proof. — The relations (4.3) and (4.4) show that $\tilde{\mathfrak{F}}_x$ is constant on $f^j(S(z))$ for every $z \in \Sigma$ and $0 \leq j \leq r(z)$. Thus, $x \mapsto \tilde{\mathfrak{F}}_x$ is $\mathscr{B}$-measurable. $\mathscr{B}$-measurability of f is a simple consequence of the Markov property in Proposition 4.6. Indeed, let $E \in \mathscr{B}$ and let $z \in \Sigma$ and $0 \leq j \leq r(z)$ be such that $f^{-1}(E)$ intersects $f^j(S(z))$. Then E intersects $f^{j+1}(S(z))$. We claim that E contains $f^{j+1}(S(z))$. When $j+1 \leq r(z)$ this follows immediately from $E \in \mathscr{B}$. When $j = r(z)$, notice that $f^{j+1}(S(z)) \subset S(w)$ for some $w \in S(z)$, and $E \in \mathscr{B}$ must contain $S(w)$. So the claim holds in all cases. It follows that $f^{-1}(E)$ contains $f^j(S(z))$. This proves that $f^{-1}(E) \in \mathscr{B}$, and so the proof of part (a) is complete. To prove part (b), observe that

$$\mathrm{diam}\, f^j(S(z)) \leq \mathrm{diam}_{\mathscr{W}^s} S(z) \leq R,$$

for all $z \in \Sigma$ and $j \geq 0$, and so

$$\sup_{x\in M} \operatorname{dist}_{C^0}(\mathscr{H}_x, \mathrm{id}) \leq \sup_{\operatorname{dist}(a,b)\leq R} \operatorname{dist}_{C^0}(H^s_{a,b}, \mathrm{id}).$$

The right hand side is uniformly bounded, since the stable holonomy depends continuously on the base points, and the space of $(a, b) \in M \times M$ with $\operatorname{dist}(a, b) \leq R$ is compact. This proves part (b). To prove the last claim, observe that $f^n(\mathscr{B})$ is the σ-algebra of sets $E \in \mathscr{M}_*$ such that every $f^{j+n}(S(z))$ either is contained in E or is disjoint from E. Observe that the diameter of $f^{j+n}(S(z))$ goes to zero, uniformly, when n goes to ∞. It follows that every open set can be written as a union of sets $E_n \in f^n(\mathscr{B})$ and, hence, belongs to the σ-algebra generated by $\{f^n(\mathscr{B}) : n \in \mathbb{N}\}$. This proves that the latter σ-algebra coincides mod 0 with the completion $\mathscr{M}_*$ of the Borel σ-algebra, as stated in (c). □

4.4. Local essential invariance. — Next, we deduce Proposition 4.5. By assumption, $\lambda_-(\mathfrak{F}, x, \xi) \geq 0$ at m-almost every point. Lemma 4.7 ensures that all the other assumptions of Theorem 4.4 are fulfilled as well. We conclude from the theorem that the disintegration $\{\tilde{m}_x : x \in M\}$ of the measure $\tilde{m} = \mathscr{H}_* m$ is measurable mod 0 with respect to the σ-algebra $\mathscr{B}$. Then, there exists a full μ-measure set $X^s \subset M$ such that this restriction of the disintegration to X^s is constant on every $f^j(S(z))$ with $z \in \Sigma$ and $0 \leq j \leq r(z)$. The disintegrations of m and $\tilde{m}$ are related to one another by

$$\tilde{m}_x = \big(\mathscr{H}_x\big)_* m_x = \begin{cases} (H^s_{x,f^j(z)})_* m_x & \text{if } x \in f^j(S(z)) \text{ for } z \in \Sigma \text{ and } 0 \leq j \leq r(z) \\ m_x & \text{otherwise.} \end{cases}$$

Define $\mathscr{N}^s = X^s \cap \mathscr{N}(\Sigma, \delta)$. Recall that $\mathscr{W}(z, \delta) \subset S(z)$ for all $z \in \Sigma$. Then, for every x, $y \in \mathscr{N}^s$ in the same $\mathscr{W}(z, \delta)$,

$$(H^s_{x,z})_* m_x = \tilde{m}_x = \tilde{m}_y = (H^s_{y,z})_* m_y,$$

and so $m_y = (H^s_{y,z})_*^{-1} (H^s_{x,z})_* m_x = (H_{x,y})_* m_x$. This proves Proposition 4.5. The proof of Theorem 4.1 is now complete.

5. Density points

In this section we recall some ideas of Burns, Wilkinson **[10]** that will be important in Section 6. The conclusions that interest us more directly are collected in Proposition 5.13.

Let us start with a few preparatory remarks. Recall that we take M to carry a Riemannian metric adapted to $f : M \to M$, meaning that properties (2.2)-(2.4) hold. Clearly, these properties are not affected by rescaling. At a few steps in the course of the arguments that follow we do allow for the Riemannian metric to be multiplied by some large constant.

Recall that we write $\mathscr{W}^*_{\mathrm{loc}}(x) = \mathscr{W}^*(x, R)$ for every $x \in M$ and $* \in \{s, u\}$, where is R a fixed constant. In the sequel we suppose that $R > 1$. Up to rescaling the metric, we may assume that the Riemannian ball $B(p, R)$ is contained in foliation boxes for both $\mathcal{W}^s$ and $\mathcal{W}^u$, for every $p \in M$. By further rescaling the metric, we may ensure that, given any $p \in M$ and x, $y \in B(p, R)$,

$$y \in \mathcal{W}^s_{\mathrm{loc}}(x) \quad \text{implies} \quad \operatorname{dist}(f(x), f(y)) \leq \nu(p) \operatorname{dist}(x, y) \text{ and,}$$
$$y \in \mathcal{W}^u_{\mathrm{loc}}(x) \quad \text{implies} \quad \operatorname{dist}(f^{-1}(x), f^{-1}(y)) \leq \hat{\nu}(f^{-1}(p)) \operatorname{dist}(x, y).$$

As a consequence, given any p, x, $y \in M$,

(I) $f(\mathcal{W}^s_{\mathrm{loc}}(x)) \subset \mathcal{W}^s_{\mathrm{loc}}(f(x))$ and $f^{-1}(\mathcal{W}^u_{\mathrm{loc}}(x)) \subset \mathcal{W}^u_{\mathrm{loc}}(f^{-1}(x))$.

(II) If $f^j(x) \in B(f^j(p), R)$ for $0 \leq j < n$, and $y \in \mathcal{W}^s_{\mathrm{loc}}(x)$, then
$$\operatorname{dist}(f^n(x), f^n(y)) \leq \nu^n(p) \operatorname{dist}(x, y);$$

(III) If $f^{-j}(x) \in B(f^{-j}(p), R)$ for $0 \leq j < n$, and $y \in \mathcal{W}^u_{\mathrm{loc}}(x)$, then
$$\operatorname{dist}(f^{-n}(x), f^{-n}(y)) \leq \hat{\nu}^{-n}(p) \operatorname{dist}(x, y).$$

These properties of the strong-stable and strong-unstable foliations of f are useful guidelines to the notion of fake foliations, that we are going to recall in Section 5.2.

5.1. Density sequences. — Let λ be the volume associated to the (adapted) Riemannian metric on M. We denote by λ_S the volume of the Riemannian metric induced on any immersed submanifold S. Given a continuous foliation $\mathcal{F}$ of M with smooth leaves, we denote by $\lambda_{\mathcal{F}}(A)$ the volume of a measurable subset A of some leaf F, relative to the Riemannian metric λ_F induced on that leaf.

By definition, λ and the invariant volume μ have the same zero measure sets. More important for our proposes, they have the same Lebesgue density points. Recall that $x \in M$ is a *Lebesgue density point* of a set $X \subset M$ if

$$\lim_{\delta \to 0} \lambda(X : B(x, \delta)) = 1$$

where $\lambda(A : B) = \lambda(A \cap B)/\lambda(B)$ is defined for general subsets A, B with $\lambda(B) > 0$. The Lebesgue Density Theorem asserts that $\lambda(X \,\Delta\, \mathrm{DP}(X)) = 0$ for any measurable set X, where $\mathrm{DP}(X)$ is the set of Lebesgue density points of X.

Balls may be replaced in the definition by other, but not arbitrary, families of neighborhoods of the point.

Definition 5.1. — A sequence of measurable sets $(Y_n)_n$ is a *Lebesgue density sequence at $x \in M$* if

(a) $(Y_n)_n$ *nests* at the point x: $Y_n \supset Y_{n+1}$ for every n and $\cap_n Y_n = \{x\}$

(b) $(Y_n)_n$ is *regular*: there is $\delta > 0$ such that $\lambda(Y_{n+1}) \geq \delta \lambda(Y_n)$ for every n

(c) x is a Lebesgue density point of an arbitrary measurable set X if and only if $\lim_{n\to\infty} \lambda(X : Y_n) = 1$.

Some of the sequences we are going to mention satisfy these conditions for special classes of sets only. In particular, we say that $(Y_n)_n$ is a *Lebesgue density sequence at x for bi-essentially saturated sets* if (c) holds for every bi-essentially saturated set X (this notion was defined in Section 2.1).

Burns, Wilkinson [**10**] propose two main techniques for defining new Lebesgue density sequences: internested sequences and Cavalieri's principle. The first one is quite simple and applies to general measurable sets. Two sequences $(Y_n)_n$ and $(Z_n)_n$ that nest at x are said to be *internested* if there is $k \geq 1$ such that

$$Y_{n+k} \subseteq Z_n \quad \text{and} \quad Z_{n+k} \subseteq Y_n \quad \text{for all } n \geq 0.$$

***Lemma 5.2* (Lemma 2.1 in [10]).** — *If $(Y_n)_n$ and $(Z_n)_n$ are internested then one sequence is regular if and only if the other one is. Moreover,*

$$\lim_{n\to\infty} \lambda(X : Y_n) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \lambda(X : Z_n) = 1,$$

for any measurable set $X \subset M$.

Consequently, if two sequences are internested then one is a Lebesgue density sequence (respectively, a Lebesgue density sequence for bi-essentially saturated sets) if and only if the other is.

The second technique (Cavalieri's principle) is a lot more subtle and is specific to subsets essentially saturated by some absolutely continuous foliation $\mathcal{F}$ (with bounded Jacobians). Let U be a foliation box for $\mathcal{F}$ and Σ be a cross-section to $\mathcal{F}$ in U. The *fiber* of a set $Y \subset U$ over a point $q \in \Sigma$ is the intersection of Y with the local leaf of $\mathcal{F}$ in U containing q. The *base* of $Y \subset U$ is the set Σ_Y of points $q \in \Sigma$ whose fiber $Y(q)$ is a measurable set and has positive $\lambda_{\mathcal{F}}$-measure. The absolute continuity of $\mathcal{F}$ ensures that the base is a measurable set. We say that Y *fibers* over some set $Z \subset \Sigma$ if the basis Σ_Y equals Z. Given $c \geq 1$, a sequence of sets Y_n contained in U has *c-uniform fibers* if

$$c^{-1} \leq \frac{\lambda_{\mathcal{F}}(Y_n(q_1))}{\lambda_{\mathcal{F}}(Y_n(q_2))} \leq c \quad \text{for all } q_1,\, q_2 \in \Sigma_{Y_n} \text{ and every } n \geq 0. \tag{5.1}$$

***Proposition 5.3* (Proposition 2.7 in [10]).** — *Let $(Y_n)_n$ be a sequence of measurable sets in U with c-uniform fibers, for some c. Then, for any locally $\mathcal{F}$-saturated measurable set $X \subset U$,*

$$\lim_{n\to\infty} \lambda(X : Y_n) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \lambda_\Sigma(\Sigma_X : \Sigma_{Y_n}) = 1.$$

By *locally $\mathcal{F}$-saturated* we mean that the set is a union of local leaves of $\mathcal{F}$ in the foliation box U. Sets that differ from a locally $\mathcal{F}$-saturated one by zero Lebesgue measure subsets are called *essentially locally $\mathcal{F}$-saturated.*

***Proposition 5.4* (Proposition 2.5 in [10]).** — *Let $(Y_n)_n$ and $(Z_n)_n$ be two sequences of measurable subsets of U with c-uniform fibers, for some c, and $\Sigma_{Y_n} = \Sigma_{Z_n}$ for all n. Then, for any essentially locally $\mathcal{F}$-saturated set $X \subset U$,*

$$\lim_{n\to\infty} \lambda(X : Y_n) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \lambda(X : Z_n) = 1.$$

5.2. Fake foliations and juliennes. — Juliennes were initially proposed by Pugh, Shub [**20**] as density sequences particularly suited for partially hyperbolic dynamical systems. These are sets constructed by means of invariant foliations that are assumed to exist (dynamical coherence) tangent to the invariant subbundles E^s, E^u, $E^{cs} = E^c \oplus E^s$, $E^{cu} = E^c \oplus E^u$, and E^c, and they have nice properties of invariance under iteration and under the holonomy maps of the strong-stable and strong-unstable foliations. As mentioned before, strong-stable and strong-unstable foliations (tangent to the subbundles E^s and E^u, respectively) always exist in the partially hyperbolic setting. However, that is not always true about the center, center-stable, center-unstable subbundles E^c, E^{cs}, E^{cu}.

One main novelty in Burns, Wilkinson [**10**] was that, for the first time, the authors avoided the dynamical coherence assumption. A version of the julienne construction is still important in their approach, but now the definition is in terms of certain "approximations" to the (possibly nonexistent) invariant foliations, that they call fake foliations. We will not need to use fake foliations nor fake juliennes directly in this paper but, for the reader's convenience, we briefly describe their main features.

5.2.1. *Fake foliations.* — The central result about fake foliations is Proposition 3.1 in [**10**]: for any $\varepsilon > 0$ there exist constants $0 < \rho < r < R$ such that the ball of radius r around every point admits foliations

$$\widehat{\mathcal{W}}^u_p, \quad \widehat{\mathcal{W}}^s_p, \quad \widehat{\mathcal{W}}^c_p, \quad \widehat{\mathcal{W}}^{cu}_p, \quad \widehat{\mathcal{W}}^{cs}_p.$$

with the following properties, for any $* \in \{u, s, c, cs, cu\}$:

(i) For every $x \in B(p, \rho)$, the leaf $\widehat{\mathcal{W}}^*_p(x)$ is C^1 and the tangent space $T_x \widehat{\mathcal{W}}^*_p(x)$ is contained in the cone of radius ε around E^*_x.

(ii) For every $x \in B(p, \rho)$,

$$f(\widehat{\mathcal{W}}^*_p(x, \rho)) \subset \widehat{\mathcal{W}}^*_{f(p)}(f(x)) \quad \text{and} \quad f^{-1}(\widehat{\mathcal{W}}^*_p(x, \rho)) \subset \widehat{\mathcal{W}}^*_{f^{-1}(p)}(f^{-1}(x)).$$

(iii) Given $x \in B(p, \rho)$ and $n \geq 1$ such that $f^j(x) \in B(f^j(p), r)$ for $0 \leq j < n$, if $y \in \widehat{\mathcal{W}}^s_p(x, \rho)$ then $f^n(y) \in \widehat{\mathcal{W}}^s_{f^n(p)}(f^n(x), \rho)$ and

$$\operatorname{dist}(f^n(x), f^n(y)) \leq \nu^n(p) \operatorname{dist}(x, y).$$

Similarly for $\widehat{\mathcal{W}}^u$, with f replaced by its inverse.

(iv) Given $x \in B(p, \rho)$ and $n \geq 1$ such that $f^j(x) \in B(f^j(p), r)$ for $0 \leq j < n$, if $f^j(y) \in \widehat{\mathcal{W}}^{cs}_p(f^j(q), \rho)$ for $0 \leq j < n$ then $f^n(y) \in \widehat{\mathcal{W}}^{cs}_{f^n(p)}(f^n(x))$ and

$$\operatorname{dist}(f^n(x), f^n(y)) \leq \hat{\gamma}^n(p)^{-1} \operatorname{dist}(x, y).$$

Similarly for $\widehat{\mathcal{W}}^{cu}$, with f replaced by its inverse.

(v) $\widehat{\mathcal{W}}^u_p$ and $\widehat{\mathcal{W}}^c_p$ sub-foliate $\widehat{\mathcal{W}}^{cu}_p$, and $\widehat{\mathcal{W}}^s_p$ and $\widehat{\mathcal{W}}^c_p$ sub-foliate $\widehat{\mathcal{W}}^{cs}_p$.

(vi) $\widehat{\mathcal{W}}^s_p(p) = \mathcal{W}^s(p, r)$ and $\widehat{\mathcal{W}}^u_p(p) = \mathcal{W}^u(p, r)$.

(vii) All the fake foliations $\widehat{\mathcal{W}}^*$, $* \in \{u, s, c, cs, cu\}$ are Hölder continuous, and so are their tangent distributions.

(viii) Assuming f is center bunched, every leaf of $\widehat{\mathcal{W}}_p^{cs}$ is C^1 foliated by leaves of $\widehat{\mathcal{W}}_p^{s}$ and every leaf of $\widehat{\mathcal{W}}_p^{cu}$ is C^1 foliated by leaves of $\widehat{\mathcal{W}}_p^{u}$.

Properties (i) and (vi) are what we mean by “approximations”. Concerning the latter, let us emphasize that the fake strong-stable and strong-unstable foliations need not coincide with the genuine ones, $\mathcal{W}^s$ and $\mathcal{W}^u$, at points other than p. The local invariance property (ii) and the exponential bounds (iii) and (iv) should be compared to the corresponding properties (I), (II), (III) of, stated at the beginning of Section 5. The regularity properties (vi) and (vii) hold uniformly in $p \in M$.

5.2.2. *Juliennes.* — Another direct use of the center bunching condition, besides the smoothness property (viii) above, is in the definition of juliennes. In view of the first center bunching condition, $\nu < \gamma\hat{\gamma}$ (there is a dual construction starting from $\hat{\nu} < \gamma\hat{\gamma}$ instead), we may find continuous functions τ and σ such that

$$\nu < \tau < \sigma\gamma \quad \text{and} \quad \sigma < \min\{\hat{\gamma}, 1\}.$$

Let $p \in M$ be fixed. For any $x \in \mathcal{W}^s(p,1)$ and $n \geq 0$, define

$$\widehat{B}_n^c(x) = \widehat{\mathcal{W}}_p^c(x, \sigma^n(p)) \quad \text{and} \quad S_n(p) = \bigcup_{x \in \mathcal{W}^s(p,1)} \widehat{B}_n^c(x).$$

The (fake) *center-unstable julienne* of order $n \geq 0$ centered at $x \in \mathcal{W}^s(p,1)$ is defined by

$$\widehat{J}_n^{cu}(x) = \bigcup_{y \in \widehat{B}_n^c(x)} \widehat{J}_n^u(y), \quad \text{where} \quad \widehat{J}_n^u(y) = f^{-n}(\widehat{\mathcal{W}}_{f^n(p)}^u(f^n(y), \tau^n(p))).$$

The latter is the (fake) *unstable julienne* of order $n \geq 0$ centered at y, and is defined for every $y \in S_n(p)$. See Figure 1.

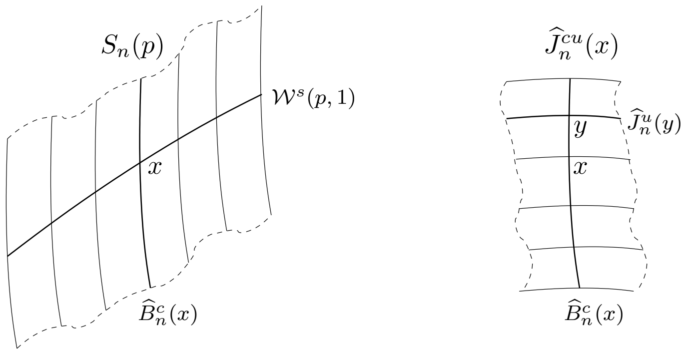

FIGURE 1.

Observe that $\widehat{\mathcal{J}}^{cu}_n(x)$ is contained in the smooth submanifold $\widehat{\mathcal{W}}^{cu}_p(x)$, by the coherence property (v) of fake foliations. Moreover, $\widehat{\mathcal{J}}^{cu}_n(x)$ has positive measure relative to the Riemannian volume $\lambda_{\widehat{cu}}$ defined by the restriction of the Riemannian metric to $\widehat{\mathcal{W}}^{cu}_p(x)$. Notice also that fake center-unstable leaves are transverse to the strong-stable foliation, as a consequence of property (i) of fake foliations. One key feature of center-unstable juliennes is that, unlike balls for instance, they are approximately preserved by the holonomy maps of the strong-stable foliation:

***Proposition 5.5* (Proposition 5.3 in [10]).** — *For any x, $x' \in \mathcal{W}^s(p,1)$, the sequences $h^s(\widehat{\mathcal{J}}^{cu}_n(x))$ and $\widehat{\mathcal{J}}^{cu}_n(x')$ are internested, where $h^s : \widehat{\mathcal{W}}^{cu}_p(x) \to \widehat{\mathcal{W}}^{cu}_p(x')$ is the holonomy map induced by the strong-stable foliation $\mathcal{W}^s$.*

5.3. Lebesgue and julienne density points. — Let S be a locally s-saturated set in a neighborhood of p. For notational simplicity, we write

$$\lambda_{\widehat{cu}}(S : \widehat{\mathcal{J}}^{cu}_n(x)) = \lambda_{\widehat{cu}}(S \cap \widehat{\mathcal{W}}^{cu}_p(x) : \widehat{\mathcal{J}}^{cu}_n(x)).$$

Notice that $S \cap \widehat{\mathcal{W}}^{cu}_p(x)$ coincides with the base of S over $\widehat{\mathcal{W}}^{cu}_p(x)$.

Definition 5.6. — We call $x \in \mathcal{W}^s(p,1)$ a *cu-julienne density point* of S if

$$\lim_{n\to\infty} \lambda_{\widehat{cu}}(S : \widehat{\mathcal{J}}^{cu}_n(x)) = 1.$$

Another crucial property of center-unstable juliennes is

***Proposition 5.7* (Proposition 5.5 in [10]).** — *Let X be a measurable set that is both s-saturated and essentially u-saturated. Then $x \in \mathcal{W}^s(p)$ is a Lebesgue density point of X if and only if x is a cu-julienne density point of X.*

We can not use this proposition directly, because the saturation hypotheses are not fully satisfied by the sets we deal with. However, we can rearrange the arguments in the proof of the proposition to obtain a statement that does suit our purposes. For this, let us recall the main steps in the proof of Proposition 5.7. They involve several nesting sequences $B_n(x)$, $C_n(x)$, $D_n(x)$, $G_n(x)$, that we introduce along the way.

By definition, $B_n(x)$ is just the Riemannian ball of radius $\sigma^n(p)$ centered at x:

$$B_n(x) = B(x, \sigma^n(p)).$$

Lemma 5.8. — *Let $S \subset M$ be any measurable set. Then, x is a Lebesgue density point of S if and only if $\lim_{n\to\infty} \lambda(S : B_n(x)) = 1$.*

Proof. — This follows from the fact that the ratio $\sigma^{n+1}(p)/\sigma^n(p) = \sigma(f^n(p))$ of successive radii is less than 1, and is uniformly bounded away from both 0 and 1. □

Next, for $x \in \mathcal{W}^s(p,1)$, let

$$C_n(x) = \bigcup_{q \in D^{cs}_n(x)} \mathcal{W}^u(q, \sigma^n(p)) \quad \text{and} \quad D_n(x) = \bigcup_{q \in D^{cs}_n(x)} f^{-n}(\mathcal{W}^u(f^n(q), \tau^n(p))).$$

Notice that these two nesting sequences fiber over the same sequence of bases

$$D_n^{cs}(x) = \bigcup_{y \in \widehat{\mathcal{W}}_p^s(x, \sigma^n(p))} \widehat{B}_n^c(y) = \bigcup_{y \in \widehat{\mathcal{W}}_p^s(x, \sigma^n(p))} \widehat{\mathcal{W}}_p^c(y, \sigma^n(p)).$$

Also, by the coherence property (v) of fake foliations, each set $D_n^{cs}(x)$ is contained in the submanifold $\widehat{\mathcal{W}}^{cs}(x)$.

Lemma 5.9. — *Let $S \subset M$ be any measurable set. Then,*

$$\lim_{n\to\infty} \lambda(S : B_n(x)) = 1 \iff \lim_{n\to\infty} \lambda(S : C_n(x)) = 1.$$

Proof. — Continuity and transversality of the fake foliations $\widehat{\mathcal{W}}_p^c$ and $\widehat{\mathcal{W}}_p^s$ imply that the sequences $D_n^{cs}(x)$ and $\widehat{\mathcal{W}}^{cs}(x, \sigma^n(p))$ are internested. Then, similarly, continuity and transversality of the foliations $\mathcal{W}^u$ and $\widehat{\mathcal{W}}_p^{cs}$ imply that the sequences $C_n(x)$ and $B_n(x)$ are internested. So, the claim follows from Lemma 5.2. □

Lemma 5.10. — *Let $S \subset M$ be locally essentially u-saturated. Then,*

$$\lim_{n\to\infty} \lambda(S : C_n(x)) = 1 \iff \lim_{n\to\infty} \lambda(S : D_n(x)) = 1.$$

Proof. — By definition, $C_n(x)$ and $D_n(x)$ both fiber over $D_n^{cs}(x)$, with fibers contained in strong-unstable leaves. The fibers of $C_n(x)$ are uniform, in the sense of (5.1), because they are all comparable to balls of fixed radius $\sigma^n(p)$ inside strong-unstable leaves. Proposition 5.4 in **[10]** gives that the fibers of $D_n(x)$ are uniform as well. Thus, the claim follows from Proposition 5.4 above. □

Finally, define

$$G_n(x) = \bigcup_{q \in \widehat{J}_n^{cu}(x)} \mathcal{W}^s(q, \sigma^n(p)).$$

Lemma 5.11. — *Let $S \subset M$ any measurable set. Then,*

$$\lim_{n\to\infty} \lambda(S : D_n(x)) = 1 \iff \lim_{n\to\infty} \lambda(S : G_n(x)) = 1.$$

Proof. — The sequences $D_n(x)$ and $G_n(x)$ are internested, according to Lemma 8.1 and Lemma 8.2 in **[10]**. So, the claim follows from Lemma 5.2. □

Lemma 5.12. — *Let $S \subset M$ be locally s-saturated. Then,*

$$\lim_{n\to\infty} \lambda(S : G_n(x)) = 1 \iff \lim_{n\to\infty} \lambda_{\widehat{cu}}(S : \widehat{J}_n^{cu}(x)) = 1.$$

Proof. — By definition, $G_n(x)$ fibers over $\widehat{J}_n^{cu}(x)$. The fibers are uniform, in the sense of (5.1), because they are all comparable to balls of fixed radius $\sigma^n(p)$ inside strong-stable leaves. Then the claim follows from Proposition 5.3 above. □

Proposition 5.7 was obtained in **[10]** by concatenating Lemmas 5.8 through 5.12. A variation of these arguments yields:

Proposition 5.13. — *Let $x \in \mathcal{W}^s(p,1)$ and $\delta > 0$.*

(a) *Let $X \subset M$ be a locally essentially u-saturated set in $B(x,\delta)$ and let Y be its local s-saturation inside $B(x,\delta)$. If x is a Lebesgue density point of X then x is a cu-julienne density point of Y.*
(b) *Let $X \subset M$ be a locally essentially s-saturated set in $B(x,\delta)$ and let Y be its local u-saturation inside $B(x,\delta)$. If x is a cu-julienne density point of X then x is a Lebesgue density point of Y.*
(c) *Let $S \subset M$ be any measurable set. If x is a cu-julienne density point of S then so is every $x' \in \mathcal{W}^s(p,1)$.*

Proof. — Applying Lemmas 5.8 through 5.11 to $S = X$, we get that

$$\lim_{n\to\infty} \lambda(X : G_n(x)) = 1$$

(Lemma 5.10 uses the assumption that X is essentially u-saturated). It follows that

$$\lim_{n\to\infty} \lambda(Y : G_n(x)) = 1,$$

because $Y \supset X$. Thus, applying Lemma 5.12 to $S = Y$, we get that x is a cu-julienne density point of Y, as claimed in part (a) of the proposition.

Next, we prove part (b). Given an essentially s-saturated set X in $B(x,\delta)$, we may use Lemmas 5.12 and 5.11 with $S = X$ to conclude that

$$\lim_{n\to\infty} \lambda(X : D_n(x)) = 1$$

(Lemma 5.12 uses the assumption that X is essentially s-saturated). It follows that

$$\lim_{n\to\infty} \lambda(Y : D_n(x)) = 1,$$

because $Y \supset X$. Then Lemmas 5.10 through 5.8, with $S = Y$, to conclude that x is a Lebesgue density point of Y, as claimed.

Finally, absolute continuity (with bounded Jacobians) of the strong-stable foliation gives that

$$\lim_{n\to\infty} \lambda_{\widehat{cu}}(S : \widehat{J}^{cu}_n(x)) = 1 \quad \Rightarrow \quad \lim_{n\to\infty} \lambda_{\widehat{cu}}(S : h^s(\widehat{J}^{cu}_n(x))) = 1.$$

By Proposition 5.5, the sequences $h^s(\widehat{J}^{cu}_n(x))$ and $\widehat{J}^{cu}_n(x')$ are internested. Hence, by Lemma 5.2,

$$\lim_{n\to\infty} \lambda_{\widehat{cu}}(S : h^s(\widehat{J}^{cu}_n(x))) = 1 \quad \Rightarrow \quad \lim_{n\to\infty} \lambda_{\widehat{cu}}(S : \widehat{J}^{cu}_n(x')) = 1.$$

This proves part (c) of the theorem. $\square$

6. Bi-essential invariance implies essential bi-invariance

We call a continuous fiber bundle $\mathcal{X}$ *refinable* if the fibers $\mathcal{X}_x$, $x \in M$ are refinable.

Theorem 6.1. — *Let $f : M \to M$ be a C^2 partially hyperbolic center bunched diffeomorphism and $\mathcal{X}$ be a refinable fiber bundle with stable and unstable holonomies. Then, given any bi-essentially invariant section $\Psi : M \to \mathcal{X}$, there exists a bi-saturated set M_Ψ with full measure, and a bi-invariant section $\tilde{\Psi} : M_\Psi \to \mathcal{X}$ that coincides with Ψ at almost every point.*

Theorem D(a) is a particular case of this result, as we are going to explain. Indeed, let P be the space of probability measures on N, endowed with the weak* topology, that is, the smallest topology for which the integration operator

$$P \to \mathbb{R}, \quad \eta \mapsto \int \varphi \, d\eta$$

is continuous, for every bounded continuous function $\varphi : N \to \mathbb{R}$. It is well known (see [**5**, Section 6]) that this topology is separable and metrizable, because N is a separable metric space (if we were to assume that N is complete then the weak* topology would also be complete). In particular, P admits a countable basis of open sets and so it is refinable.

Associated to $\pi : \mathcal{E} \to M$, we have a new fiber bundle $\Pi : \mathcal{X} \to M$, whose fiber over a point $x \in M$ is the space of probability measures on the corresponding $\mathcal{E}_x$. It is easy to see that this is a continuous fiber bundle with leaves modeled on the space P we have just introduced: if $\pi^{-1}(U) \to U \times N$, $v \mapsto (\pi(v), \psi_{\pi(v)}(v))$ is a continuous local chart for $\mathcal{E}$ then

$$\Pi^{-1}(U) \to U \times P, \quad \eta \mapsto (\Pi(\eta), (\psi_{\Pi(\eta)})_*(\eta))$$

is a continuous local chart for $\mathcal{X}$. The cocycle $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$ induces a cocycle on $\mathcal{X}$, by push-forward, but this will not be needed here.

More important for our purposes, the stable and unstable holonomies of $\mathfrak{F}$ induce homeomorphisms

$$h^s_{x,y} = (H^s_{x,y})_* : \mathcal{X}_x \to \mathcal{X}_y \quad \text{and} \quad h^u_{x,y} = (H^u_{x,y})_* : \mathcal{X}_x \to \mathcal{X}_y$$

for points x, y in the same strong-stable leaf or the same strong-unstable leaf, respectively. These homeomorphisms form stable and unstable holonomies on $\mathcal{X}$. Indeed, the group property (α) in Definition 2.10 is an immediate consequence of property (a) in Definition 2.4, and the continuity property (β) can be verified as follows. Since the statement is local, we may pretend that the fiber bundle is trivial and the holonomies $H^s_{x,y}$ are homeomorphisms of N. Consider any sequence (x_k, y_k, ν_k) in $\mathcal{X}$ converging to $(x, y, \nu) \in \mathcal{X}$, with $y_k \in \mathcal{W}^s_{\mathrm{loc}}(x_k)$ and $y \in \mathcal{W}^s_{\mathrm{loc}}(x)$. Property (c) in Definition 2.4 implies that $H^s_{x_k,y_k}$ converges to $H^s_{x,y}$ uniformly on compact subsets. On its turn, this implies that $(H^s_{x_k,y_k})_*\nu_k$ converges to $(H^s_{x,y})_*\nu$ in the weak* topology.

Now it is clear that Theorem D(a) corresponds to the statement of Theorem 6.1 in the special case of the section $\Psi(x) = m_x$ of the fiber bundle $\mathcal{X}$ we have defined. In the remainder of this section we prove Theorem 6.1.

6.1. Lebesgue densities. — Let $\Psi : M \to P$ be a measurable function with values in a refinable space.

Definition 6.2. — We say that $x \in M$ is a *point of measurable continuity* of Ψ if there is $v \in P$ such that x is a Lebesgue density point of $\Psi^{-1}(V)$ for every neighborhood $V \subset P$ of v. Then v is called the *density value* of Ψ at x.

Clearly, the density value at x is unique, when it exists. Let $\mathrm{MC}(\Psi)$ denote the set of measurable continuity points of Ψ. The function $\tilde{\Psi} : \mathrm{MC}(\Psi) \to P$ assigning to each point x of measurable continuity its density value $\tilde{\Psi}(x)$ is called *Lebesgue density* of Ψ. Recall that $\mathrm{DP}(X)$ denotes the set of density points of a set X. The hypothesis that P is refinable is used in the next lemma:

Lemma 6.3. — *For any measurable function $\Psi : M \to P$, the set $\mathrm{MC}(\Psi)$ has full Lebesgue measure and $\Psi = \tilde{\Psi}$ almost everywhere.*

Proof. — Let $\mathcal{Q}_1 \prec \cdots \prec \mathcal{Q}_n \prec \cdots$ be a sequence of partitions of the space P as in Definition 2.12. Let

$$\tilde{M} = \bigcap_{n \geq 1} \bigcup_{Q \in \mathcal{Q}_n} \Psi^{-1}(Q) \cap \mathrm{DP}(\Psi^{-1}(Q)).$$

Since $\Psi^{-1}(Q) \cap \mathrm{DP}(\Psi^{-1}(Q))$ has full measure in $\Psi^{-1}(Q)$, and $\{\Psi^{-1}(Q) : Q \in \mathcal{Q}_n\}$ is a partition of M for every n, the set on the right hand side has full measure in M for every n. This proves that $\tilde{M}$ is a full measure subset of M. Next, we check that $\tilde{M}$ is contained in the set of points of measurable continuity of Ψ. Indeed, given any point $x \in \tilde{M}$, let $Q_n \in \mathcal{Q}_n$ be the sequence of atoms such that $x \in \Psi^{-1}(Q_n)$. Then x is a density point of $\Psi^{-1}(Q_n)$ for every $n \geq 1$, in view of the definition of $\tilde{M}$. Notice that $\cap_n Q_n$ is non-empty, since it contains $\Psi(x)$. Then, according to Definition 2.12, there exists $v \in \mathcal{X}$ such that every neighborhood V contains some Q_n. It follows that x is a density point of $\Psi^{-1}(V)$ for any neighborhood $V \subset \mathcal{X}$ of v, that is, v is the density value for Ψ at x. This shows that $x \in \mathrm{MC}(\Psi)$ with $\tilde{\Psi}(x) = v$. Moreover, v must coincide with $\Psi(x)$, since the intersection of all Q_n contains exactly one point. In other words, $\tilde{\Psi}(x) = \Psi(x)$ for every $x \in \tilde{M}$. □

More generally, let $\Psi : M \to \mathcal{X}$ be a measurable section of a refinable fiber bundle $\mathcal{X}$. Let $x \in M$ be fixed and U be a small neighborhood. Using a local chart, one may view $\Psi \mid U$ as a function with values in $\mathcal{X}_x$. Two such local expressions $\Psi_1 : U \to \mathcal{X}_x$ and $\Psi_2 : U \to \mathcal{X}_x$ of the section Ψ are related by

$$\Psi_1(z) = h_z(\Psi_2(z)),$$

where $(z, \xi) \mapsto (z, h_z(\xi))$ is a homeomorphism from $U \times \mathcal{X}_x$ to itself, with $h_x = \mathrm{id}$. So, a point $v \in \mathcal{X}_x$ is the density value of Ψ_1 at x if and only if it is the density value of Ψ_2 at x. More generally, given any point $y \in U$, the corresponding local expression $\Psi_3 : U \to \mathcal{X}_y$ of the section Ψ is related to $\Psi_1 : U \to \mathcal{X}_x$ by

$$\Psi_1(z) = g_z(\Psi_3(z)),$$

where $(z, \xi) \mapsto (z, g_z(\xi))$ is a homeomorphism from $U \times \mathcal{X}_y$ to $U \times \mathcal{X}_x$. So, a point z is a point of measurable continuity for Ψ_3 if and only if it is a point of measurable continuity for Ψ_1.

These observations allow us to extend Definition 6.2 to sections of refinable fiber bundles, as follows. We call $v \in \mathcal{X}_x$ a *density value* of the section $\Psi : M \to \mathcal{X}$ at the point x if it is the density value for some (and, hence, any) local expression $U \mapsto \mathcal{X}_x$ as before. We call x a *point of measurable continuity* of the section Ψ if it admits a density value or, equivalently, if it is a point of measurable continuity for some (and, hence, any) local expression of Ψ. The subset $\mathrm{MC}(\Psi)$ of points of measurable continuity has full Lebesgue measure in M, since it intersects every domain U of local chart on a full Lebesgue measure subset. Recall Lemma 6.3. Finally, the *Lebesgue density* of Ψ is the section $\mathrm{MC}(\Psi) \to \mathcal{X}$ assigning to each point x of measurable continuity its density value.

6.2. Proof of bi-invariance.

— Now Theorem 6.1 is a direct consequence of the next proposition: it suffices to take $M_\Psi = \mathrm{MC}(\Psi)$ and $\tilde{\Psi} =$ the Lebesgue density of Ψ, and apply the following proposition together with Lemma 6.3.

Proposition 6.4. — *Let $f : M \to M$ be a C^2 partially hyperbolic center bunched diffeomorphism and $\mathcal{X}$ be a refinable fiber bundle with stable and unstable holonomies. For any bi-essentially invariant section $\Psi : M \to \mathcal{X}$, the set* $\mathrm{MC}(\Psi)$ *is bi-saturated and the Lebesgue density $\tilde{\Psi} : \mathrm{MC}(\Psi) \to \mathcal{X}$ is bi-invariant on* $\mathrm{MC}(\Psi)$.

Proof. — For any $x \in \mathrm{MC}(\Psi)$ and $y \in \mathcal{W}^s(x, 1)$, we are going to prove $h^s_{x,y}(\tilde{\Psi}(x))$ is the density value of Ψ at y. It will follow that $y \in \mathrm{MC}(\Psi)$ and $\tilde{\Psi}(y) = h^s_{x,y}(\tilde{\Psi}(x))$. Analogously, one gets that if $x \in \mathrm{MC}(\Psi)$ and $y \in \mathcal{W}^u(x, 1)$ then $y \in \mathrm{MC}(\Psi)$ and $\tilde{\Psi}(y) = h^u_{x,y}(\tilde{\Psi}(x))$. The proposition is an immediate consequence of these facts.

It is convenient to think of $\pi : \mathcal{X} \to M$ as a trivial bundle on neighborhoods U_x of x and U_y of y, identifying $\pi^{-1}(U_x) \approx U_x \times P$ and $\pi^{-1}(U_y) \approx U_y \times P$ via local coordinates, and we do so in what follows. Let $V \subset P$ be a neighborhood of $h^s_{x,y}(\tilde{\Psi}(x))$. We are going to show that y is a density point of $\Psi^{-1}(V)$.

By the continuity property (β) in Definition 2.10, we can find $\varepsilon > 0$ and a neighborhood $W \subset V$ of $h^s_{x,y}(\tilde{\Psi}(x))$ such that

$$h^u_{w_1,w_2}(W) \subset V \quad \text{for all } w_1,\, w_2 \in B(y, \varepsilon) \text{ with } w_1 \in \mathcal{W}^u_{\mathrm{loc}}(w_2). \tag{6.1}$$

Similarly, up to reducing $\varepsilon > 0$, there exists a neighborhood $U \subset P$ of $\tilde{\Psi}(x)$ such that

$$h^s_{z,w}(U) \subset W \quad \text{for every } z \in B(x, \varepsilon) \text{ and } w \in B(y, \varepsilon) \text{ with } z \in \mathcal{W}^s_{\mathrm{loc}}(w). \tag{6.2}$$

The assumption that Ψ is bi-essentially invariant (Definition 2.11) implies that there exists a full measure set S^{su} such that

$$(6.3)\qquad \begin{aligned} h^s_{\xi,\eta}(\Psi(\xi)) = \Psi(\eta) \quad &\text{for any } \xi,\ \eta \in S^{su} \text{ in the same strong-stable leaf} \\ h^u_{\xi,\eta}(\Psi(\xi)) = \Psi(\eta) \quad &\text{for any } \xi,\ \eta \in S^{su} \text{ in the same strong-unstable leaf.} \end{aligned}$$

Lemma 6.5. — *Let $x \in \mathcal{W}^s(p,1)$ be a point of measurable continuity of Ψ. Then for any open neighborhood U of the point $\tilde{\Psi}(x) \in P$ there exist $\delta > 0$ and $L \subset B(x,\delta)$ such that*

(a) $\Psi(L \cap S^{su}) \subset U$.
(b) *L is a union of local strong-stable leaves inside $B(x,\delta)$.*
(c) *Each of these local leaves contains some point of S^{su}.*
(d) *x is a cu-julienne density point of L: $\lim_{n\to\infty} \lambda_{\widehat{cu}}(L : \widehat{J}^{cu}_n(x)) = 1$.*

Proof. — By the continuity property (β) in Definition 2.10, there exists $\delta_2 > 0$ and a neighborhood $U_2 \subset U$ of $\tilde{\Psi}(x)$ such that

$$(h^s_{z_1,z_2})(U_2) \subset U \quad \text{if } z_1,\ z_2 \in B(x,\delta_2) \text{ are in the same local strong-stable leaf.}$$

and there exists $\delta_1 > 0$ and a neighborhood $U_1 \subset U_2$ of $\tilde{\Psi}(x)$ such that

$$(h^u_{z_1,z_2})(U_1) \subset U_2 \quad \text{if } z_1,\ z_2 \in B(x,\delta_1) \text{ are in the same local strong-unstable leaf.}$$

Let $\delta = \min\{1, \delta_1, \delta_2\}$. Since x is a point of measurable continuity of Ψ, it is a Lebesgue density point of $\Psi^{-1}(U_1)$. Then, x is also a density point of $L_1 = \Psi^{-1}(U_1) \cap S^{su}$, because S^{su} has full Lebesgue measure. Let L_1^u be the local u-saturate of L_1 inside $B(x,\delta)$ and let $L_2 = L_1^u \cap S^{su}$. Then x is a Lebesgue density point of L_1^u, because $L_1^u \supset L_1$, and so it is also a density point of L_2, because S^{su} has full measure. Take L to be the local s-saturate of L_2 inside $B(x,\delta)$.

Consider any point $z \in L \cap S^{su}$. By definition, there exist $z_1 \in \Psi^{-1}(U_1) \cap S^{su}$ and $z_2 \in L_1^u \cap S^{su}$ such that z_1 is in the local strong-unstable leaf of z_2, and z_2 in the local strong-stable leaf of z. Consequently, in view of our choices of U_1 and U_2,

$$\Psi(z_2) = h^u_{z_1,z_2}(\Psi(z_1)) \in U_2 \quad \text{and then} \quad \Psi(z) = h^s_{z_2,z}(\Psi(z_2)) \in U.$$

This proves claim (a) in the lemma. Claims (b) and (c) are clear from the construction: L is a local s-saturate of a subset of S^{su}. Finally, applying Proposition 5.13(a) to $X = L_2$ we get that x is a cu-julienne density point of $Y = L$. This gives claim (d), and completes the proof of the lemma. □

Let L and δ be as in Lemma 6.5. Of course, we may suppose $\delta < \varepsilon$. We extend the local leaves in L along $\mathcal{W}^s_{\mathrm{loc}}(x)$, long enough so as to cross $B(y,\varepsilon)$. Let $\tilde{L}$ denote this extended set. See Figure 2. As we have seen in Proposition 5.13(c), cu-julienne density points of locally s-saturated sets are preserved by stable holonomy. Hence, Lemma 6.5(d) ensures that y is a cu-julienne density point of $\tilde{L}$. Then, clearly, y is also a cu-julienne density point of $X = \tilde{L} \cap S^{su} \cap B(y,\varepsilon)$. Let Y be the local u-saturation of X inside $B(y,\varepsilon)$. Since X is locally essentially s-saturated, we may use Proposition 5.13(b) to conclude that y is a Lebesgue density point of Y and, hence,

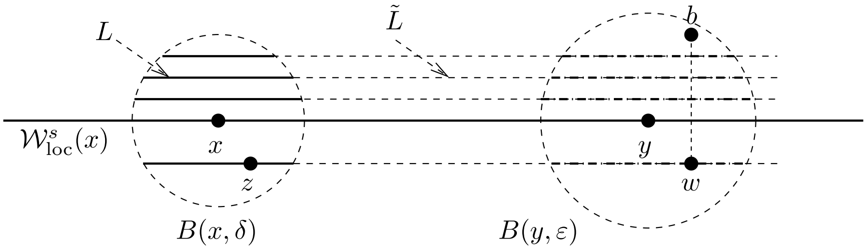

FIGURE 2.

also of $B = S^{su} \cap Y$. Thus, to prove that y is a Lebesgue density point of $\Psi^{-1}(V)$, as we claimed, it suffices to show that $\Psi(B) \subset V$.

Consider any point $b \in Y$. By definition, $b \in S^{su} \cap B(y,\varepsilon)$ and there exists some $w \in X$ such that b and w are in the same local strong-unstable leaf. By part (c) of Lemma 6.5, there exists $z \in L \cap S^{su}$ in the same local strong-stable leaf as w. By part (a) of Lemma 6.5, we have that $\Psi(z) \in U$. So, (6.3) and (6.2) imply that $\Psi(w) = h^s_{z,w}(\Psi(z)) \in W$. Then (6.3) and (6.1) imply that $\Psi(b) = h^u_{w,b}(\Psi(w)) \in V$, as we wanted to prove. This proves Proposition 6.4. □

Now the proof of Theorem 6.1 is complete.

Remark 6.6. — Let us say that a section $\Psi : M \to \mathcal{X}$ is *essentially s-continuous* if the s-continuity property (Definition 2.13) holds on some full measure subset M^s, uniformly on the neighborhood of every point. In formal terms: given any p, $q \in M$ and $\eta \in P$, there exists $\rho > 0$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that (trivialize the fiber bundle near p and q), given any x, $x' \in B(p,\rho) \cap M^s$ and y, $y' \in B(q,\rho) \cap M^s$ with $\Psi(x)$, $\Psi(x') \in B(\eta,\rho)$ and $y \in \mathcal{W}^s_{\mathrm{loc}}(x)$ and $y' \in \mathcal{W}^s_{\mathrm{loc}}(x')$,

$$\operatorname{dist}(x,x') < \delta, \quad \operatorname{dist}(y,y') < \delta, \quad \operatorname{dist}(\Psi(x),\Psi(x')) < \delta \Rightarrow \operatorname{dist}(\Psi(y),\Psi(y')) < \varepsilon.$$

Essential u-continuity is defined analogously. Moreover, Ψ is *bi-essentially continuous* if it is both essentially s-continuous and essentially u-continuous. A variation of the previous arguments yields the following statement (compare Proposition 6.4): If $f : M \to M$ is a C^2 partially hyperbolic center bunched diffeomorphism and $\mathcal{X}$ be a refinable fiber bundle then, for any bi-essentially continuous section $\Psi : M \to \mathcal{X}$, the set of points of measurable continuity is bi-saturated and the Lebesgue density $\tilde{\Psi} : \mathrm{MC}(\Psi) \to \mathcal{X}$ is bi-continuous.

7. Accessibility and continuity

Now we prove Theorem E. The main step is to show that small open sets can be reached by "nearby" su-paths starting from a fixed point in M. For the precise statement, to be given in Proposition 7.2, we need the following notion:

Definition 7.1. — Let z, $w \in M$. An *access sequence connecting z to w* is a finite sequence of points $[y_0, y_1, \dots, y_n]$ such that $y_0 = z$ and $y_j \in \mathcal{W}^*(y_{j-1})$ for $1 \le j \le n$, where each $* \in \{s, u\}$, and $y_n = w$.

Proposition 7.2. — *Given $x_0 \in M$, there is $w \in M$ and there is an access sequence $[y_0(w), \dots, y_N(w)]$ connecting x_0 to w and satisfying the following property: for any $\varepsilon > 0$ there exist $\delta > 0$ and $L > 0$ such that for every $z \in B(w, \delta)$ there exists an access sequence $[y_0(z), y_1(z), \dots, y_N(z)]$ connecting x_0 to z and such that*

$$\operatorname{dist}(y_j(z), y_j(w)) < \varepsilon \quad \textit{and} \quad \operatorname{dist}_{\mathcal{W}^*}(y_{j-1}(z), y_j(z)) < L \quad \textit{for } j = 1, \dots, N$$

where $\operatorname{dist}_{\mathcal{W}^*}$ denotes the distance along the strong (either stable or unstable) leaf common to the two points.

Let us deduce Theorem E from this proposition. Since the section Ψ is assumed to be bi-continuous, it suffices to prove it is continuous at *some* point in order to conclude that it is continuous everywhere. Fix $x_0 \in M$ and then let $w \in M$ and $[y_0(w), y_1(w), \dots, y_N(w)]$ be an access sequence connecting x_0 to w as in Proposition 7.2. We are going to prove that Ψ is continuous at w. Take the fiber bundle $\pi : \mathcal{X} \to M$ to be trivialized on the neighborhood of every node $y_j(w)$, via local coordinates. Let $V \subset P$ be any neighborhood of $\Psi(w) = \Psi(y_N(w))$. Since Ψ is bi-continuous, we may find numbers $\varepsilon_j > 0$ and neighborhoods V_j of $\Psi(y_j(w))$ such that $V_N = V$ and

$$\begin{aligned} &x \in B(y_{j-1}(w), \varepsilon_j), \quad y \in B(y_j(w), \varepsilon_j), \quad y \in \mathcal{W}^{*_j}(x), \\ &\qquad\qquad\qquad\qquad \text{and } \Psi(x) \in V_{j-1} \quad \Rightarrow \quad \Psi(y) \in V_j \end{aligned} \tag{7.1}$$

for every $j = 1, \dots, N$. Let $\varepsilon = \min\{\varepsilon_j : 1 \le j \le N\}$.

Using Proposition 7.2 we find $\delta > 0$ and, for each $z \in B(w, \delta)$, an access sequence $[y_0(z), y_1(z), \dots, y_N(z)]$ connecting x_0 to z, with

$$y_j(z) \in B(y_j(w), \varepsilon) \subset B(y_j(w), \varepsilon_j) \quad \text{for } j = 1, \dots, N. \tag{7.2}$$

It is no restriction to suppose that $\delta < \varepsilon$. Consider any $z \in B(w, \delta)$. Clearly, $\Psi(x) = \Psi(y_0(z)) \in V_0$. Hence, we may use (7.1)-(7.2) inductively to conclude that $\Psi(y_j(z)) \in V_j$ for every $j = 1, \dots, N$. The last case, $j = N$, gives $\Psi(z) \in V$. We have shown that $\Psi(B(w, \delta)) \subset V$. This proves that Ψ is continuous at w, as claimed.

In this way, we reduced the proof of Theorem E to proving Proposition 7.2.

7.1. Non-injective parametrizations. — In this section we prepare the proof of Proposition 7.2, that will be given in the next section. Roughly speaking, here we construct a kind of continuous parametrization of the space of *su*-paths with any given number of legs.

7.1.1. *Exhaustion of accessibility classes.* — Fix any point $x_0 \in M$. For each $r \in \mathbb{N}$, we consider the following sequence of sets $K_{r,n}$, $n \in \mathbb{N}$:

$$K_{r,1} = \left\{y \in \mathcal{W}^s(x_0) : \operatorname{dist}_{\mathcal{W}^s}(x_0, y) \le r\right\} \quad \text{and}$$
$$K_{r,n} = \bigcup_{x \in K_{r,n-1}} \left\{y \in \mathcal{W}^*(x) : \operatorname{dist}_{\mathcal{W}^*}(x, y) \le r\right\}, \quad \text{for } n \ge 2,$$

where $* = s$ when n is odd, and $* = u$ when n is even. That is, $K_{r,n}$ is the set of points that can be reached from x_0 using an access sequence with n legs whose lengths do not exceed r.

Lemma 7.3. — *Every $K_{r,n}$ is closed in M and, hence, compact.*

Proof. — It is clear from the definition that $K_{r,1}$ is closed. The general case follows by induction. Suppose $K_{r,n-1}$ is closed, and let z belong to the complement of $K_{r,n}$. Then, by definition,

$$Z = \left\{y \in \mathcal{W}^*(z) : \operatorname{dist}_{\mathcal{W}^*}(x, y) \le r\right\}$$

does not intersect the closed set $K_{r,n-1}$. It follows that $U \cap K_{r,n} = \varnothing$ for some neighborhood U of the set Z. By continuity of the strong-stable and strong-unstable foliations, and their induced Riemannian metrics, for every point w in a neighborhood of z,

$$\left\{y \in \mathcal{W}^*(z) : \operatorname{dist}_{\mathcal{W}^*}(x, y) \le r\right\} \subset U$$

and hence, the set on the left hand side is disjoint from $K_{r,n-1}$. This proves that points w in that neighborhood of z do not belong to $K_{r,n}$ either. Thus, $K_{r,n}$ is indeed closed. □

By definition, the union of $K_{r,n}$ over all (r, n) is the accessibility class of x_0. Since we are assuming that f is accessible, this union is the whole manifold:

$$M = \bigcup_{r,n \in \mathbb{N}} K_{r,n}.$$

Since M is a Baire space, it follows that $K_{r,n}$ has non-empty interior for some r and n, that we consider fixed from now on. Our immediate goal is to define a (non-injective) continuous "parametrization"

$$\Psi_n : \mathfrak{K}_{r,n} \to K_{r,n} \tag{7.3}$$

of the set $K_{r,n}$ by a convenient compact subspace $\mathfrak{K}_{r,n}$ of a Euclidean space, that we are going to introduce in the sequel. Let d_s and d_u denote the dimensions of the strong-stable leaves and the strong-unstable leaves, respectively. This Euclidean space will be the alternating product of $\mathbb{R}^{d_s}$ and $\mathbb{R}^{d_u}$, with n factors, each of which parametrizing one leg of the access sequence. The case $n = 2$ is described in Figure 3.

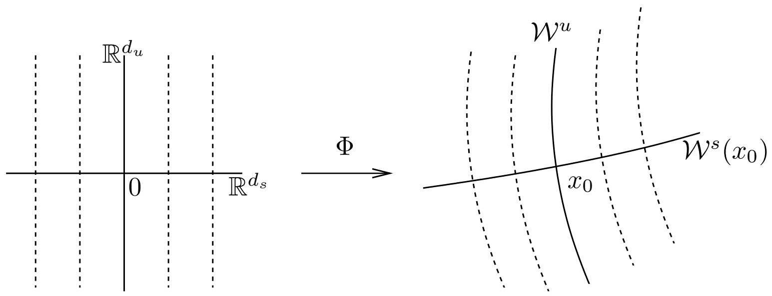

FIGURE 3.

7.1.2. *Fiber bundles induced by local strong leaves.* — The following lemma will be useful in the construction of (7.3). The whole point with the statement is that U does not need to be small. The diffeomorphisms in the statement are as regular as the partially hyperbolic diffeomorphism f itself.

Lemma 7.4. — *For any contractible space A, any continuous function $\Psi : A \to M$, and any symbol $* \in \{s, u\}$, there exists a homeomorphism*

$$\Theta : A \times \mathbb{R}^{d_*} \to \big\{(a, y) : a \in A \text{ and } y \in \mathcal{W}^*_{\mathrm{loc}}(\Psi(a))\big\}$$

mapping each $\{a\}\times\mathbb{R}^{d_*}$ diffeomorphically to $\{a\}\times\mathcal{W}^*_{\mathrm{loc}}(\Psi(a))$ with $\Theta(a,0) = (a, \Psi(a))$ for all $a \in A$.

Proof. — We consider the case $* = s$. Since $\mathcal{W}^s$ is a continuous foliation with smooth leaves, for each $p \in M$ we may find a neighborhood U_p and a continuous map

$$\Phi_p : U_p \times \mathbb{R}^{d_s} \to M$$

such that $\Phi_p(x, 0) = x$ and $\Phi_p(x, \cdot)$ maps $\mathbb{R}^{d_s}$ diffeomorphically to $\mathcal{W}^s_{\mathrm{loc}}(x)$, for every $x \in U_p$. Using these maps we may endow the set

$$F_s = \big\{(x, y) : x \in M \text{ and } y \in \mathcal{W}^s_{\mathrm{loc}}(x)\big\}$$

with the structure of a fiber bundle with smooth fibers, with local charts

$$U_p \times \mathbb{R}^{ds} \to \big\{(x, y) : x \in U_p \text{ and } y \in \mathcal{W}^s_{\mathrm{loc}}(x)\big\} \quad (x, v) \mapsto (x, \Phi_p(x, v)).$$

Then $F^s_\Psi = \big\{(a, y) : a \in A \text{ and } y \in \mathcal{W}^s_{\mathrm{loc}}(\Psi(a))\big\}$ also has a fiber bundle structure, with local coordinates

$$\Theta_p : \Psi^{-1}(U_p) \times \mathbb{R}^{d_s} \to \big\{(a, y) : \Psi(a) \in U_p \text{ and } y \in \mathcal{W}^s_{\mathrm{loc}}(\Psi(a))\big\}$$

given by $\Theta_p(a, v) = (a, \Phi_p(\Psi(a), v))$. This fiber bundle admits the space of diffeomorphisms of $\mathbb{R}^{d_s}$ that fix the origin as a structural group: all coordinate changes along the fibers belong to this group.

The core of the proof is the general fact (see [**17**, Chapter 4, Theorem 9.9]) that, for any topological group G, any fiber bundle over a contractible paracompact space that has G as a structural group is G-trivial. When applied to F^s_Ψ this result means that there exists a global chart

$$\Theta : A \times \mathbb{R}^{d_s} \to \big\{(a,y) : a \in A \text{ and } y \in \mathcal{W}^s_{\mathrm{loc}}(\Psi(a))\big\}, \quad \Theta(a,v) = (a, \Phi(a,v))$$

such that every $\Phi(a,\cdot)$ maps $\mathbb{R}^{d_s}$ to the strong-stable leaf through $\Psi(a)$, and every $\Phi(a,\cdot)^{-1} \circ \Phi_p(\Psi(a),\cdot)$ is a diffeomorphism that fixes the origin of $\mathbb{R}^{d_s}$. The latter gives that $\Phi(a,0) = \Phi_p(\Psi(a),0) = \Psi(a)$ for all $a \in A$. □

7.1.3. *Construction of non-injective parametrizations.* — We are ready to construct $\mathfrak{K}_{r,n}$ and Ψ as in (7.3). Let $l \geq 1$ be fixed such that, for any $x \in M$,

$$\begin{aligned}\big\{y \in \mathcal{W}^s(x) : \operatorname{dist}_{\mathcal{W}^s}(x,y) \leq 2r\big\} &\subset f^{-l}\big(\mathcal{W}^s_{\mathrm{loc}}(f^l(x))\big)\\ \big\{y \in \mathcal{W}^u(x) : \operatorname{dist}_{\mathcal{W}^u}(x,y) \leq 2r\big\} &\subset f^{l}\big(\mathcal{W}^s_{\mathrm{loc}}(f^{-l}(x))\big).\end{aligned} \tag{7.4}$$

Our argument is somewhat more transparent when $l = 0$, and so the reader should find it convenient to keep that case in mind throughout the construction.

Define $E_1 = \{y \in M : f^l(y) \in \mathcal{W}^s_{\mathrm{loc}}(f^l(x_0))\}$ and $\Phi_1 : E_1 \to M$ to be the inclusion. Notice that E_1 is contractible and $\Phi_1(E_1)$ contains $K_{r,1}$. Since E_1 is a smooth disc, there exists an diffeomorphism $\Theta_1 : \mathbb{R}^{d_s} \to E_1$ with $\Theta_1(0) = x_0$. Then

$$\Psi_1 = \Phi_1 \circ \Theta_1 : \mathbb{R}^{d_s} \to M$$

is a continuous function whose image contains $K_{r,1}$. Notice that the pre-image $\mathfrak{K}_{r,1} = \Psi_1^{-1}(K_{r,1})$ is compact: $K_{r,1} = \big\{y \in \mathcal{W}^s(x_0) : \operatorname{dist}_{\mathcal{W}^s}(x_0,y) \leq r\big\}$ and we have a factor 2 in (7.4). Next, define

$$E_2 = \big\{(a,y) : a \in \mathbb{R}^{d_s} \text{ and } f^{-l}(y) \in \mathcal{W}^u_{\mathrm{loc}}(f^{-l}(\Psi_1(a)))\big\}$$

and $\Phi_2 : E_2 \to M$, $\Phi_2(a,y) = y$. Notice that $\Phi_2(E_2)$ contains $K_{r,2}$. Using Lemma 7.4 with $A = \mathbb{R}^{d_s}$, $\Psi = f^{-l} \circ \Psi_1$, and $* = u$, we find a homeomorphism

$$\Theta_2 : \mathbb{R}^{d_s} \times \mathbb{R}^{d_u} \to \{(a,y) : a \in \mathbb{R}^{d_s} \text{ and } y \in \mathcal{W}^u_{\mathrm{loc}}(f^{-l}(\Psi_1(a)))\}$$

that maps each $\{a\} \times \mathbb{R}^{d_u}$ diffeomorphically to $\{a\} \times \mathcal{W}^u_{\mathrm{loc}}(f^{-l}(\Psi_1(a)))$ and satisfies $\Theta_2(a,0) = (a, f^{-l}(\Psi_1(a)))$. Clearly, the map

$$\Gamma_2 : \{(a,y) : a \in \mathbb{R}^{d_s} \text{ and } y \in \mathcal{W}^u_{\mathrm{loc}}(f^{-l}(\Psi_1(a)))\} \to E_2, \quad \Gamma_2(a,y) = (a, f^l(y))$$

is a homeomorphism, and $\Gamma_2(\Theta_2(a,0)) = (a, \Psi_1(a))$. Then

$$\Psi_2 = \Phi_2 \circ \Gamma_2 \circ \Theta_2 : \mathbb{R}^{d_s} \times \mathbb{R}^{d_u} \to M$$

is a continuous map whose image contains $K_{r,2}$. Moreover, Ψ_2 may be viewed as a continuous extension of Ψ_1, because

$$\Psi_2(a,0) = \Phi_2(\Gamma_2(\Theta_2(a,0))) = \Phi_2(a, \Psi_1(a)) = \Psi_1(a)$$

for all $a \in \mathbb{R}^{d_s}$. In general, $\Psi_2^{-1}(K_{r,2})$ needs not be compact. However,

$$\mathfrak{K}_{r,2} = \big\{(a,b) \in \mathbb{R}^{d_s} \times \mathbb{R}^{d_u} : a \in \mathfrak{K}_{r,1} \text{ and } \operatorname{dist}_{\mathcal{W}^u}(\Psi_2(a,0), \Psi_2(a,b)) \leq r\big\}$$

is compact and satisfies $\Psi_2(\mathfrak{K}_{r,2}) = K_{r,2}$. Repeating this procedure, we construct continuous maps

$$\Psi_j : \mathbb{R}^{d_s} \times \mathbb{R}^{d_u} \times \cdots \times \mathbb{R}^{d_*} \to M$$

(there are j factors, and so $* = u$ if j is even and $* = s$ if j is odd), contractible sets E_j, and compact sets $\mathfrak{K}_{r,j}$ such that each Ψ_j is a continuous extension of Ψ_{j-1}, in the previous sense, and $\Psi_j(\mathfrak{K}_{r,j}) = K_{r,j}$. We stop this procedure for $j = n$. The corresponding map Ψ_n is the parametrization announced in (7.3).

7.2. Selection of nearby access sequences. — Now we prove Proposition 7.2. We need the following general fact about regular values of continuous functions.

Definition 7.5. — Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a map between topological spaces $\mathcal{A}$ and $\mathcal{B}$. A point $x \in \mathcal{A}$ is *regular* for Φ, if for every neighborhood $\mathcal{V}$ of x we have $\Phi(x) \in \Phi(\mathcal{V})^\circ$. A point $y \in B$ is a *regular value* of Φ if every point of $\Phi^{-1}(y)$ is regular.

Proposition 7.6. — *Let $\mathcal{A}$ be a compact metrizable space and $\mathcal{B}$ a locally compact Hausdorff space. If $\Phi : \mathcal{A} \to \mathcal{B}$ is continuous then the set of regular values of Φ is residual.*

Proof. — We are going to prove that the image of the set of non-regular points is meager. The assumptions imply that $\mathcal{A}$ admits a countable base $\mathcal{T}$ of open sets, and the map Φ is closed. If x is a non-regular point of Φ, then there exists $\mathcal{V} \in \mathcal{T}$ such that $\Phi(x)$ does not belong to the interior of $\Phi(\overline{\mathcal{V}})$. Therefore, $\Phi(x)$ belongs to the closed set $\partial\Phi(\overline{\mathcal{V}})$, which has empty interior because $\Phi(\overline{\mathcal{V}})$ is closed. Then, the image of non-regular points is a subset of the meager set $\bigcup \left\{\partial\Phi(\overline{\mathcal{V}}) : \mathcal{V} \in \mathcal{T}\right\}$. $\square$

We apply this proposition to the continuous map $\Psi_n : \mathfrak{K}_{r,n} \to K_{r,n}$. Recall that, by construction, the image $K_{r,n}$ has non empty interior. Then, in particular, Ψ_n has some regular value $w \in K_{r,n}$. Let $(a_1, \ldots, a_n) \in \mathfrak{K}_{r,n}$ be any point in $\mathfrak{K}_{r,n}$ such that $\Psi_n(a_1, \ldots, a_n) = w$. Let $\varepsilon > 0$ be as in the statement of the proposition. Since the functions Ψ_1, Ψ_2, …, Ψ_n are continuous, there exists $\rho > 0$ such that if $|a_j - b_j| < \rho$, for $j = 1, \ldots, n$, then

$$\operatorname{dist}(\Psi_j(a_1, \ldots, a_j), \Psi_j(b_1, \ldots, b_j)) < \varepsilon \tag{7.5}$$

for all $j = 1, \ldots, n$. Using that the point $(a_1, \ldots, a_n)$ is regular (Definition 7.5), we get that the image $\Psi_n(V)$ of the neighborhood

$$V = \mathfrak{K}_{r,n} \cap \{(b_1, \ldots, b_n) : |a_j - b_j| < \rho, \text{ for } j = 1, \ldots, n\}$$

has w in its interior. In other words, there exists $\delta > 0$ such that $B(w, \delta) \subset \Psi_n(V)$. Consider any point $z \in B(w, \delta)$. Then there exists $(b_1(z), \ldots, b_n(z)) \in V$ such that $z = \Psi_n(b_1(z), \ldots, b_n(z))$. Define

$$y_j(z) = \Psi_j(b_1(z)), \ldots, y_j(z))$$

for $j = 1, \ldots, n$, and $y_0(z) = w$. Then $[y_1(z), \ldots, y_n(z)]$ is an access sequence connecting x_0 to z. The inequalities (7.5) mean that

$$\operatorname{dist}(y_j(z), y_j(w)) < \varepsilon \quad \text{for } j = 1, \ldots, n.$$

Moreover, since $\Psi_n(b_1(z), \ldots, b_n(z)) \in K_{r,n}$, the distance between every $y_{j-1}(z)$ and $y_j(z)$ along their common strong (stable or unstable) leaf does not exceed r. Proposition 7.2 follows taking $L = r$ and $N = n$.

8. Generic linear cocycles over partially hyperbolic maps

In this section we prove Theorem A. We will take the vector bundle $\pi : \mathcal{V} \to M$ to be trivial, that is, such that $\mathcal{V} = M \times \mathbb{K}^d$ and $\pi : M \times \mathbb{K}^d \to M$ is the canonical projection. This simplifies the presentation substantially, but is not really necessary for our arguments, which are local in nature: for obtaining the conclusion we consider modifications of the cocycle supported in a neighborhood of certain special points (the pivots, see Proposition 8.8), where triviality holds anyway, by definition.

Let us begin by giving an outline of the proof. Let $\mathbb{K}_x = \{x\} \times \mathbb{K}^d$ be the fiber of $\mathcal{V}$ and $\mathbb{P}(\mathbb{K}_x) = \{x\} \times \mathbb{P}(\mathbb{K})$ be the fiber of the projective bundle $\mathbb{P}(\mathcal{V})$ over the point x. We call *loop* of $f : M \to M$ at $x \in M$ any access sequence $\gamma = [y_0, \ldots, y_n]$ connecting a point $x \in M$ to itself, that is, such that $y_0 = y_n = x$. Then we denote

$$H_\gamma = H^{*_n}_{y_{n-1},y_n} \circ \cdots \circ H^{*_j}_{y_{j-1},y_j} \circ H^{*_1}_{y_0,y_1} : \mathbb{P}(\mathbb{K}_x) \to \mathbb{P}(\mathbb{K}_x)$$

where $*_j \in \{s, u\}$ is the symbol of the strong leaf common to the nodes y_{j-1} and y_j. Theorem B implies that if $\lambda_+(F) = \lambda_-(F)$ then any F-invariant probability measure m that projects down to μ admits a disintegration $\{m_z : z \in M\}$ such that

$$(H_\gamma)_* m_x = m_x \quad \text{for any loop } \gamma. \tag{8.1}$$

We consider *loops with slow recurrence*, for which some node y_r, that we call *pivot*, is slowly accumulated by the orbits of all the nodes including its own. Using perturbations of the cocycle supported on a small neighborhood of the pivot, we prove that the map $F \mapsto H_\gamma$ assigning to each cocycle the corresponding holonomy over the loop is a submersion. In fact, we are able to consider several independent loops with slow recurrence, $\gamma_1, \ldots, \gamma_m$, and prove that the map

$$F \mapsto (H_{\gamma_1}, \ldots, H_{\gamma_m})$$

is a submersion. Consequently, for typical cocycles, the matrices H_{γ_i} are in general position, and so they have no common invariant probability in the projective space. This shows that for typical cocycles the condition (8.1) fails and, hence, the extremal Lyapunov exponents are distinct.

8.1. Accessibility with slow recurrence. — An important step is to prove that loops with slow recurrence do exist. Beforehand, let us give the precise definition.

Definition 8.1. — A family $\{\gamma_1, \dots, \gamma_m\}$ of loops $\gamma_i = [y_0^i, \dots, y_{n(i)}^i]$ has *slow recurrence* if there exists $c > 0$ and for each $1 \le i \le m$ there exists $0 < r(i) < n(i)$ such that, for all $i, l = 1, \dots, m$, all $0 \le j \le n(i)$, and all $k \in \mathbb{Z}$,

$$\operatorname{dist}\left(f^k(y_j^i), y_{r(l)}^l\right) \ge c/(1 + k^2)$$

with the exception of $k = 0$ when $(i, j) = (l, r(l))$.

It is convenient to distinguish access sequences $[y_0, y_1, \dots, y_n]$ according to the nature of the last leg: we speak of accessibility *s-sequence* if y_{n-1} and y_n belong to the same strong-stable leaf, and we speak of accessibility *u-sequence* if y_{n-1} and y_n belong to the same strong-unstable leaf. Let d_s and d_u be the dimensions of the strong-stable leaves and strong-unstable leaves, respectively.

Proposition 8.2. — *For any $m \ge 1$ and any $(x_1, \dots, x_m) \in M^m$, there exists a family γ_i of loops with slow recurrence, where each γ_i is a loop at x_i.*

The proof of this proposition requires a number of preparatory results.

Lemma 8.3. — *Given any finite set $\{w_1, \dots, w_n\} \subset M$, any $y \in M$, and any symbol $* \in \{s, u\}$, there exists a full Lebesgue measure subset of points $w \in \mathcal{W}^*_{\mathrm{loc}}(y)$ such that*

$$\operatorname{dist}(f^k(w_j), w) \ge c/(1 + k^2) \tag{8.2}$$

for some $c > 0$ and for all $1 \le j \le n$ and all $k \in \mathbb{Z}$.

Proof. — Consider $* = s$: the case $* = u$ is analogous. Since local strong-stable leaves are a continuous family of C^2 embedded disks, there exists a constant $D_1 > 0$ such that

$$\lambda_{\mathcal{W}^s_{\mathrm{loc}}(y)}\left(\mathcal{W}^s_{\mathrm{loc}}(y) \cap B(z, c/(1 + k^2))\right) \le D_1 (c/(1 + k^2))^{d_s}$$

for any $z \in M$. Thus, the Lebesgue measure of the subset of points $w \in \mathcal{W}^s_{\mathrm{loc}}(y)$ not satisfying inequality (8.2) for some fixed $c > 0$ is bounded by

$$\sum_{j=1}^{n} \sum_{k \in \mathbb{Z}} D_1 c^{d_s} (1 + k^2)^{-d_s} \le D_2\, c^{d_s} \quad \text{with } D_2 = n D_1 \sum_{k \in \mathbb{Z}} (1 + k^2)^{-d_s} < \infty.$$

Making $c \to 0$, we conclude that the inequality (8.2) is indeed satisfied by Lebesgue almost every point in $\mathcal{W}^s_{\mathrm{loc}}(y)$. □

Corollary 8.4. — *Given any $m \ge 1$, any $(x_1, \dots, x_m) \in M^m$, and any $* \in \{s, u\}$, then for every $(z_1, \dots, z_m)$ in a full Lebesgue measure subset of M^m there exist $c > 0$ and accessibility $*$-sequences $[y_0^i, \dots, y_{n(i)}^i]$ connecting x_i to z_i such that*

$$\operatorname{dist}(f^k(y_j^i), z_l) \ge c/(1 + k^2)$$

for all $i, l = 1, \dots, m$, all $0 \le j < n(i)$, and all $k \in \mathbb{Z}$.

Proof. — Consider $* = s$: the case $* = u$ is analogous. Since the strong-stable foliation is absolutely continuous, it suffices to prove that, given any points $y_i \in M$, $1 \le i \le m$, the conclusion holds on a full Lebesgue measure subset of points $z_i \in \mathcal{W}^s_{\mathrm{loc}}(y_i)$, $1 \le i \le m$. Now, by the accessibility assumption, there exist accessibility sequences $[y^i_0, \dots, y^i_{r(i)}]$ connecting x_i to y_i. Consider each z_i in the full Lebesgue measure subset of $\mathcal{W}^s(y_i)$ given by Lemma 8.3, applied to the finite set

$$\left\{y^i_j : 1 \le i \le m \text{ and } 0 \le j \le r(i)\right\}.$$

and the point $y = y_i$. Then the accessibility s-sequences $[y^i_0, \dots, y^i_{k(i)}, z_i]$ satisfy the conditions in the conclusion. In view of the observation at the beginning, this proves the corollary. □

Lemma 8.5. — *For any $m \ge 1$ and any $(y_1, \dots, y_m) \in M^m$, there exists a full Lebesgue measure subset of $(z_1, \dots, z_m) \in \mathcal{W}^s_{\mathrm{loc}}(y_1) \times \cdots \times \mathcal{W}^s_{\mathrm{loc}}(y_m)$ such that*

$$\operatorname{dist}(f^k(z_i), z_l) \ge c/(1 + k^2)$$

for some $c > 0$ and for all $i, l = 1, \dots, m$ and all $k \ge 0$, except $k = 0$ when $i = l$. The statement remains true if one replaces $\mathcal{W}^s_{\mathrm{loc}}$ by $\mathcal{W}^u_{\mathrm{loc}}$ and $k \ge 0$ by $k \le 0$.

Proof. — It is clear that each strong-stable leaf contains at most one periodic point. As an easy consequence we get that, that given any $\kappa \ge 1$, there exists a full Lebesgue measure subset of $(z_1, \dots, z_m) \in \mathcal{W}^s_{\mathrm{loc}}(y_1) \times \cdots \times \mathcal{W}^s_{\mathrm{loc}}(y_m)$ such that $f^k(z_i) \ne z_l$ for all $i, l = 1, \dots, m$ and all $0 \le k < \kappa$, except $k = 0$ when $i = l$. Then the condition in the statement holds, for some $c > 0$, restricted to iterates $0 \le k < \kappa$. Let us focus on $k \ge \kappa$. For each $i, l = 1, \dots, m$, define

$$E^k_{i,l} = \left\{z_l \in \mathcal{W}^s_{\mathrm{loc}}(y_l) : \operatorname{dist}(f^k(z_i), z_l) < 1/(1 + k^2) \text{ for some } z_i \in \mathcal{W}^s_{\mathrm{loc}}(y_i)\right\}.$$

The diameter of $f^k(\mathcal{W}^s_{\mathrm{loc}}(y_i))$ is bounded by $C_1\theta^k$, where $C_1 > 0$ is some uniform constant and $\theta < 1$ is an upper bound for the contraction function $\nu(x)$ in (2.2). Consequently,

$$\operatorname{diam}(E^k_{i,l}) \le C_1\theta^k + 2/(1 + k^2) \le C_2/(1 + k^2)$$

for another uniform constant $C_2 > 0$. It follows that

$$\lambda_{\mathcal{W}^s_{\mathrm{loc}}(y_l)}\Big(\bigcup_{i=1}^{m}\bigcup_{k=\kappa}^{\infty} E^k_{i,l}\Big) \le m \sum_{k=\kappa}^{\infty} C_2(1 + k^2)^{-d_s}.$$

On the one hand, the right hand side of this expression goes to 0 when κ goes to infinity. On the other hand, in view of our previous observations, for any $\kappa \ge 1$, Lebesgue almost every $(z_1, \dots, z_m) \in \mathcal{W}^s_{\mathrm{loc}}(y_1) \times \cdots \times \mathcal{W}^s_{\mathrm{loc}}(y_m)$ with

$$z_l \notin \bigcup_{i=1}^{m}\bigcup_{k=\kappa}^{\infty} E^k_{i,l}$$

satisfies the conclusion of the lemma for some $c \in (0, 1)$. This proves that the subset of $(z_1, \dots, z_m)$ for which the conclusion of the lemma does not hold has zero Lebesgue measure, as claimed. □

Corollary 8.6. — *For any $m \geq 1$, and every $(z_1, \dots, z_m)$ in a full Lebesgue measure subset of M^m, there exists $c > 0$ such that*

$$\operatorname{dist}(f^k(z_i), z_l) \geq c/(1+k^2)$$

for all $i, l = 1, \dots, m$ and all $k \in \mathbb{Z}$, except $k = 0$ when $i = l$.

Proof. — It suffices to prove that the conditions obtained replacing $k \in \mathbb{Z}$ by either $k \geq 0$ or $k \leq 0$ are satisfied on full Lebesgue measure subsets of M^m, and then take the intersection of these two subsets. We consider the case $k \geq 0$, as the other one is analogous. Suppose there is a positive Lebesgue measure subset of $(z_1, \dots, z_m) \in M^m$ for which the condition is *not* satisfied: the forward orbit of some z_i accumulates some z_l faster than $c/(1+k^2)$ for any $c > 0$. Then, since M is covered by the foliation boxes of the strong-stable foliation, there exist foliation boxes U_i, $1 \leq i \leq m$ such that this exceptional subset intersects $U = U_1 \times \cdots \times U_m$ on a positive Lebesgue measure subset. The domain U is foliated by the products $\mathcal{W}^s_{\mathrm{loc}}(y_1) \times \cdots \times \mathcal{W}^s(y_m)$ of local strong-stable leaves. We denote this foliation as $\mathcal{W}^{s,m}$. Given any holonomy maps $h_i : \Sigma_i^1 \to \Sigma_i^2$ between cross-sections to the strong-stable foliation $\mathcal{W}^s$ inside U_i, the products $\Sigma^j = \Sigma_1^j \times \cdots \times \Sigma_m^j$ are cross-sections to $\mathcal{W}^{s,m}$, and the holonomy map of $\mathcal{W}^{s,m}$ is

$$h : \Sigma^1 \to \Sigma^2, \quad h(z_1, \dots, z_m) = (h_1(z_1), \dots, h_m(z_m)).$$

Since all the h_i are absolutely continuous, so is h: the Jacobians are related by $Jh(z_1, \dots, z_m) = Jh_1(z_1) \cdots Jh_m(z_m)$. This absolute continuity property implies that every positive Lebesgue measure subset of U intersects $\mathcal{W}^s_{\mathrm{loc}}(y_1) \times \cdots \mathcal{W}^s_{\mathrm{loc}}(y_m)$ on a positive Lebesgue measure subset, for a subset of $(y_1, \dots, y_m)$ with positive Lebesgue measure. In particular, the exceptional set intersects some leaf of $\mathcal{W}^{s,m}$ on a positive Lebesgue measure subset. This contradicts Lemma 8.5, and this contradiction proves the corollary. □

Corollary 8.7. — *For any $m \geq 1$, any $(x_1, \dots, x_m) \in M^m$, and any $* \in \{s, u\}$, and a full Lebesgue measure set D_* of $(z_1, \dots, z_m) \in M^m$, there exists $c > 0$ such that*

$$\operatorname{dist}(f^k(z_i), z_l) \geq c/(1+k^2) \tag{8.3}$$

for all $i, l = 1, \dots, m$ and all $k \in \mathbb{Z}$, except $k = 0$ when $i = l$, and there exist accessibility $*$-sequences $[y_0^i, \dots, y_{n(i)}^i]$ connecting x_i to z_i, for $1 \leq i \leq m$ such that

$$\operatorname{dist}(f^k(y_j^i), z_l) \geq c/(1+k^2) \tag{8.4}$$

for all $i, l = 1, \dots, m$, all $0 \leq j < n(i)$, and all $k \in \mathbb{Z}$.

Proof. — Just take the intersections of the full Lebesgue measure subsets given in Corollary 8.4, for $* \in \{s, u\}$, and in Corollary 8.6. □

Proof of Proposition 8.2. — Given $m \geq 1$ and $(x_1, \dots, x_m) \in M^m$, let D_s and D_u be the full Lebesgue measure sets given by Corollary 8.7, and then consider

$$(z_1, \dots, z_m) \in D_s \cap D_u\,.$$

The corollary yields, for each $1 \le i \le m$, an accessibility s-sequence $[y_0^i, \dots, y_{r(i)}^i]$ and an accessibility u-sequence $[w_0^i, \dots, w_{t(i)}^i]$ connecting x_i to z_i. Then

$$\gamma_i = [y_0^i, \dots, y_{r(i)}^i = w_{t(i)}^i, \dots, w_0^i]$$

is a loop at x_i, and properties (8.3)-(8.4) mean that the family $\{\gamma_1, \dots, \gamma_m\}$ of loops has slow recurrence. □

8.2. Holonomies on loops with slow recurrence. — As we pointed out before, the tangent space at each point $B \in \mathscr{G}^{r,\alpha}(M,d,\mathbb{K})$ is naturally identified with the Banach space of $C^{r,\alpha}$ maps from M to the space of linear maps in $\mathbb{K}^d$. This means that we may view the tangent vectors $\dot{B}$ as $C^{r,\alpha}$ functions assigning to each $z \in M$ a linear map $\dot{B}(z) : \mathbb{K}_z \to \mathbb{K}_{f(z)}$.

Let $A \in \mathscr{G}^{r,\alpha}(M,d,\mathbb{K})$ be fiber bunched. As we have seen in Section 3.2, there exists a neighborhood $\mathcal{U} \subset \mathscr{G}^{r,\alpha}(M,d,\mathbb{K})$ of A such that every $B \in \mathcal{U}$ is fiber bunched. Then, for any loop $\gamma = [y_0, \dots, y_n]$ at a point $x \in M$, and any $0 \le k < l \le n$, we have linear holonomy maps

$$H_{B,\gamma,k,l} = H^{*_l}_{B,y_{l-1},y_l} \circ \cdots \circ H^{*_{k+1}}_{B,y_k,y_{k+1}} : \mathbb{K}_{y_k} \to \mathbb{K}_{y_l}.$$

Furthermore, all the maps $B \mapsto H_{B,\gamma,k,l}$ are C^1 on $\mathcal{U}$. In particular, the derivative of $B \mapsto H_{B,\gamma} = H_{B,\gamma,0,n}$ is given by

$$\partial_B H_{B,\gamma} : \dot{B} \mapsto \sum_{l=1}^{n} H_{B,\gamma,l,n} \big[\partial_B H_{B,\gamma,l-1,l}(\dot{B})\big] H_{B,\gamma,0,l-1}. \tag{8.5}$$

The main result in this section is

Proposition 8.8. — *Let $A \in \mathscr{G}^{r,\alpha}(M,d,\mathbb{K})$ be fiber bunched and $\mathcal{U}$ be a neighborhood as above. For each $x \in M$ and $m \ge 1$, let $\gamma_i = [y_0^i, y_1^i, \dots, y_{n(i)}^i]$, $1 \le i \le m$ be a family of loops at x with slow recurrence. Then*

$$\mathcal{U} \ni B \mapsto (H_{B,\gamma_1}, \dots, H_{B,\gamma_m}) \in \mathrm{GL}(d, \mathbb{K}_x)^m$$

is a submersion: the derivative is surjective at every point, even restricted to the subspace of tangent vectors $\dot{B}$ supported on a small neighborhood of the pivots.

In the proof we use (8.5) together with the expressions for the $\partial_B H_{B,\gamma,l-1,l}(\dot{B})$ given in Propositions 3.5 and 3.7. The idea is quite simple. Perturbations in the neighborhood of the pivots affect the holonomies over all the loop legs, of course. However, Corollaries 3.6 and 3.8 show that the effect decreases exponentially fast with time, and slow recurrence means that the first iterates need not be considered. Combining these two ideas one shows (Corollary 8.12) that the derivative is a small perturbation of its term of order zero. The latter is easily seen to be surjective (Lemma 8.13), and then the same is true for any small perturbation.

Remark 8.9. — Essentially the same arguments yield an $\mathrm{SL}(d,\mathbb{K})$-version of this proposition: the map $\mathcal{U} \cap \mathscr{S}^{r,\alpha}(M,d,\mathbb{K}) \ni B \mapsto (H_{B,\gamma_1}, \dots, H_{B,\gamma_m}) \in \mathrm{SL}(d,\mathbb{K}_x)^m$ is a submersion. Clearly, it remains true that the derivative is a small perturbation of its term of order zero. Then the main point is to observe that the restriction of the operator S in Lemma 8.13 maps $T_B\mathcal{S}^{r,\alpha}(M,d,\mathbb{K})$ surjectively to $T_{H_{B,\gamma}}\operatorname{SL}(d,\mathbb{K}_x)$.

Before getting into the details, let us make an easy observation that allows for some simplification of our notations. If $\gamma=[y_0,\dots,y_n]$ is a loop with slow recurrence then so is $\bar\gamma=[y_n,\dots,y_0]$, and $H_{B,\bar\gamma}$ is the inverse of $H_{B,\gamma}$. Hence, the statement of the proposition is not affected if one reverses the orientation of any γ_i as described. So, it is no restriction to suppose that every loop γ has the orientation for which the pivot y_r satisfies

$$y_r\in\mathcal{W}^s(y_{r-1})\cap\mathcal{W}^u(y_{r+1}), \tag{8.6}$$

and we do so in all that follows.

Lemma 8.10. — *Let $\gamma=[y_0,\dots,y_n]$ be a loop with slow recurrence and y_r be the corresponding pivot. Then, there is $\tau>0$ such that for any small $\varepsilon>0$ and any tangent vector $\dot B$ supported on $B(y_r,\varepsilon)$,*

$$\|\partial_B H_{B,\gamma,l-1,l}(\dot B)\|\le\theta^{\sqrt{\tau/\varepsilon}}\,\|\dot B\|_{0,\beta}\quad\text{for any } l\neq r\text{, and}$$
$$\|\partial_B H_{B,\gamma,r-1,r}(\dot B)+B(y_r)^{-1}\dot B(y_r)H^s_{B,y_{r-1},y_r}\|\le\theta^{\sqrt{\tau/\varepsilon}}\,\|\dot B\|_{0,\beta}.$$

Proof. — By Definition 8.1, there exists $c>0$ such that

$$\operatorname{dist}(f^k(y_l),y_r)\ge c/(1+k^2)\quad\text{for all } (l,k)\in\{0,\dots,n\}\times\mathbb{Z},\ (l,k)\neq(r,0).$$

Consider $\varepsilon<c/2$. Then $B(y_r,\varepsilon)$ contains no other node of the loop. Moreover, for any $0\le l\le n$ and any $k\ge 1$,

$$f^k(y_l)\in B(y_r,\varepsilon)\Longrightarrow|k|\ge t(\varepsilon),\quad\text{where } t(\varepsilon)=\sqrt{c/\varepsilon-1}.$$

Let us denote by $\partial_B H_{B,\gamma,l-1,l,t(\varepsilon)}(\dot B)$ the t-tail of the derivative, that is, the sum over $i\ge t$ in Proposition 3.5 (case $*_l=s$) or Proposition 3.7 (case $*_l=u$). Then, for any $\dot B\in T_B\mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ supported in $B(y_r,\varepsilon)$, the expression in Proposition 3.5 becomes

$$\partial_B H_{B,\gamma,l-1,l}(\dot B)=\partial_B H_{B,\gamma,l-1,l,t(\varepsilon)}(\dot B) \tag{8.7}$$

for all $l\neq r$, and

$$\partial_B H_{B,\gamma,r-1,r}(\dot B)=-B(y_r)^{-1}\dot B(y_r)H^s_{B,y_{r-1},y_r}+\partial_B H_{B,\gamma,l-1,l,t(\varepsilon)}(\dot B) \tag{8.8}$$

for $l=r$. This applies to the loop legs with symbol $*_l=s$. Observing that the sum in Proposition 3.7 does not include the term $i=0$, we conclude that (8.7) extends to all loop legs with symbol $*_l=u$. Next, by Corollaries 3.6 and 3.8,

$$\|\partial_B H_{B,\gamma,l-1,l,t}(\dot B)\|\le C_5(a)\,\theta^t\,\|\dot B\|_{0,\beta}, \tag{8.9}$$

for every $1\le l\le n$ and any $t\ge 0$, where a is an upper bound for the distances between consecutive loop nodes. Choose any $\tau<c/2$. The lemma follows directly from (8.7), (8.8), (8.9) with $t=t(\varepsilon)$, because $\theta<1$ and the choices of ε and τ ensure $t(\varepsilon)>\sqrt{\tau/\varepsilon}$. □

Corollary 8.11. — *Let* $\gamma_i = [y_0^i, y_1^i, \dots, y_{n(i)}^i]$, $1 \le i \le m$ *be a family of loops at* x *with slow recurrence and* $y_{r(i)}$, $1 \le i \le m$ *be the corresponding pivots. Then there exists* $\tau > 0$ *such that, for any small* $\varepsilon > 0$, *any* $1 \le j \le m$, *and any tangent vector* $\dot{B}$ *supported on* $B(y_r^j, \varepsilon)$, $r = r(j)$

$$\|\partial_B H_{B,\gamma_i,l-1,l}(\dot{B})\| \le \theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta} \quad \textit{for all } (i,l) \neq (j,r), \textit{ and}$$
$$\|\partial_B H_{B,\gamma_j,r-1,r}(\dot{B}) + B(y_r^j)^{-1}\dot{B}(y_r^j)H^s_{B,y_{r-1}^j,y_r^j}\| \le \theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta}.$$

Proof. — The case $i = j$ is contained in Lemma 8.10. The cases $i \neq j$ follow from the same arguments, observing that

$$\operatorname{dist}(f^k(y_l^i), y_r^j) \ge c/(1+k^2) \quad \text{for every } k \in \mathbb{Z}$$

and so $f^k(y_l^i) \in B(y_r^j, \varepsilon)$ implies $|k| \ge t(\varepsilon)$, for every $0 \le l \le n(i)$. □

Corollary 8.12. — *Let* $\gamma_i = [y_0^i, y_1^i, \dots, y_{n(i)}^i]$, $1 \le i \le m$ *be a family of loops at* x *with slow recurrence, and* $y_{r(i)}$, $1 \le i \le m$ *be the corresponding pivots. Then, there exists* $K_1 > 0$ *such that, for any small* $\varepsilon > 0$, *any* $1 \le j \le m$, *and any tangent vector* $\dot{B}$ *supported on* $B(y_r^j, \varepsilon)$, $r = r(j)$

$$\|\partial_B H_{B,\gamma_i}(\dot{B})\| \le K_1\theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta} \quad \textit{for all } i \neq j, \textit{ and}$$
$$\|\partial_B H_{B,\gamma_j}(\dot{B}) + H_{B,\gamma_j,r,n(j)}B(y_r^j)^{-1}\dot{B}(y_r^j)H_{B,\gamma_j,0,r}\| \le K_1\theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta}$$

Proof. — This follows from replacing in (8.5) the estimates in Corollary 8.11. By part (e) of Proposition 3.2, the factors $H_{B,\gamma_i,0,l-1}$ and $H_{B,\gamma_i,l,n(i)}$ are bounded by some uniform constant K_2 that depends only on the loops. Then, for every $i \neq j$, Corollary 8.11 and the relation (8.5) gives

$$\|\partial_B H_{B,\gamma_i}(\dot{B})\| \le \sum_{l=1}^{n(i)} K_2^2 \|\partial_B H_{B,\gamma,l-1,l}(\dot{B})\| \le K_1\theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta},$$

as long as we choose $K_1 \ge K_2^2 \max_i n(i)$. This gives the first part of the corollary. Now we consider $i = j$. For the same reasons as before, all but one term in the expression (8.5) are bounded by $K_2^2\theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta}$. The possible exception is

$$H_{B,\gamma_j,r,n(j)}\big[\partial_B H_{B,\gamma_j,r-1,r}(\dot{B})\big]H_{B,\gamma_j,0,r-1},$$

corresponding to $l = r$. By Corollary 8.11, this last expression differs from

$$-H_{B,\gamma_j,r,n(j)}B(y_r^j)^{-1}\dot{B}(y_r^j)H^s_{B,y_{r-1}^j,y_r^j}H_{B,\gamma_j,0,r-1} = \\ -H_{B,\gamma_j,r,n(j)}B(y_r^j)^{-1}\dot{B}(y_r^j)H_{B,\gamma_j,0,r}$$

by a term bounded by $K_2^2\theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta}$. This completes the proof. □

Lemma 8.13. — *Let $\gamma = [y_0, \dots, y_n]$ be a loop at $x \in M$ and $0 < r < n$ be fixed. Then the linear map*

$$\begin{array}{rlcl} S: & T_B\mathscr{G}^{r,\alpha}(M,d,\mathbb{K}) & \to & T_{H_{B,\gamma}}\operatorname{GL}(d,\mathbb{K}_x) \simeq \mathscr{L}(\mathbb{K}_x^d,\mathbb{K}_x^d) \\ & \dot{B} & \mapsto & -H_{B,\gamma,r,n}B(y_r)^{-1}\dot{B}(y_r)H_{B,\gamma,0,r} \end{array}$$

is surjective, even restricted to the subspace of tangent vectors $\dot{B}$ vanishing outside some neighborhood of y_r. More precisely, there exists $K_3 > 0$ such that for $0 < \varepsilon < 1$ and $\Theta \in \mathscr{L}(\mathbb{K}^d, \mathbb{K}^d)$ there exists $\dot{B}_\Theta \in T_B\mathscr{G}^{r,\alpha}(M,d,\mathbb{K})$ vanishing outside $B(y_r,\varepsilon)$ and such that $S(\dot{B}_\Theta) = \Theta$ and $\|\dot{B}_\Theta\|_{0,\beta} \le K_3\,\varepsilon^{-\beta}\,\|\Theta\|$.

Proof. — Let $\tau : M \to [0,1]$ be a $C^{r,\alpha}$ function vanishing outside $B(y_r,\varepsilon)$ and such that $\tau(y_r) = 1$ and the Hölder constant $H_\beta(\tau) \le 2\varepsilon^{-\beta}$. For $\Theta \in \mathscr{L}(\mathbb{K}^d,\mathbb{K}^d)$, define $\dot{B}_\Theta \in T_B\mathscr{G}^{r,\alpha}(M,d,\mathbb{K})$ by

$$\dot{B}_\Theta(w) = B(y_r)\,H^{-1}_{B,\gamma,r,n}\,\Theta\,B(y_r)^{-1}\,\tau(w)\,B(w)\,H^{-1}_{B,\gamma,0,r}.$$

Notice that $\dot{B}_\Theta(y_r) = B(y_r)\,H^{-1}_{B,\gamma,r,n}\,\Theta\,H^{-1}_{B,\gamma,0,r}$ and so $S(\dot{B}_\Theta) = \Theta$. Moreover,

$$\|\dot{B}_\Theta\|_{0,0} \le \|H^{-1}_{B,\gamma,r,n}\|\,\|H^{-1}_{B,\gamma,0,r}\|\,\|B(y_r)\|\,\|B(y_r)^{-1}\|\,\|B\|_{0,0}\,\|\Theta\|. \tag{8.10}$$

For any $w_1, w_2 \in M$ the norm of $\dot{B}_\Theta(w_1) - \dot{B}_\Theta(w_2)$ is bounded by

$$\|H^{-1}_{B,\gamma,r,n}\|\,\|H^{-1}_{B,\gamma,0,r}\|\,\|B(y_r)\|\,\|B(y_r)^{-1}\| \\ \Big(\|\tau(w_1)-\tau(w_2)\|\|B(w_1)\| + |\tau(w_2)|\|B(w_1)-B(w_2)\|\Big)\,\|\Theta\|.$$

Consequently, the Hölder constant $H_\beta(\dot{B}_\Theta)$ of $\dot{B}_\Theta$ is bounded above by

$$\|H^{-1}_{B,\gamma,r,n}\|\,\|H^{-1}_{B,\gamma,0,r}\|\,\|B(y_r)\|\,\|B(y_r)^{-1}\|\,\big(2\varepsilon^{-\beta}\|B\|_{0,0} + H_\beta(B)\big)\,\|\Theta\|. \tag{8.11}$$

Adding the inequalities (8.10) and (8.11), and taking

$$K_3 = \|H^{-1}_{B,\gamma,r,n}\|\,\|H^{-1}_{B,\gamma,0,r}\|\,\|B(y_r)\|\,\|B(y_r)^{-1}\|\,\|B\|_{0,\beta},$$

one obtains $\|\dot{B}_\Theta\|_{0,\beta} \le K_3\varepsilon^{-\beta}\|\Theta\|$. □

Proof of Proposition 8.8. — For each $1 \le j \le m$, let S_j be the operator associated to $\gamma = \gamma_j$ as in Lemma 8.13. Let Θ_j be any element of the unit sphere in $\mathscr{L}(\mathbb{K}_x,\mathbb{K}_x)$. By Lemma 8.13, for any small $\varepsilon > 0$ there exists a tangent vector $\dot{B}(j,\Theta_j)$ supported in $B(y^j_{r(j)},\varepsilon)$ such that

$$S_j\big(\dot{B}(j,\Theta_j)\big) = \Theta_j \quad \text{and} \quad \|\dot{B}(j,\Theta_j)\| \le K_3\varepsilon^{-\beta}.$$

By Corollary 8.12, the norm of

$$(\partial_B H_{B,\gamma_1},\dots,\partial_B H_{B,\gamma_j},\dots,\partial_B H_{B,\gamma_m})(\dot{B}) - (0,\dots,0,S_j(\dot{B}),0,\dots,0)$$

is bounded above by $K_3\theta^{\sqrt{\tau/\varepsilon}}\|\dot{B}\|$, for any tangent vector supported in $B(y^j_{r(j)},\varepsilon)$. For $\dot{B} = \dot{B}(j,\Theta_j)$ this gives that

$$\|(\partial_B H_{B,\gamma_1},\dots,\partial_B H_{B,\gamma_j},\dots,\partial_B H_{B,\gamma_m})(\dot{B}(j,\Theta_j)) - (0,\dots,0,\Theta_j,0,\dots,0)\|$$

is bounded by $K_1K_3\theta^{\sqrt{\tau/\varepsilon}}\varepsilon^{-\beta}$. Assume $\varepsilon > 0$ is small enough so that

$$K_1K_3\theta^{\sqrt{\tau/\varepsilon}}\varepsilon^{-\beta} < 1/(2m).$$

Then for any $\Theta = (\Theta_1, \dots, \Theta_m)$ with Θ_j in the unit sphere of $\mathscr{L}(\mathbb{K}_x, \mathbb{K}_x)$ we find a tangent vector $\dot{B}(\Theta) = \sum_{j=1}^m \dot{B}(j, \Theta_j)$ supported on the ε-neighborhood of the pivots and such that

$$\|\big(\partial H_{B,\gamma_1}, \dots, \partial H_{B,\gamma_m}\big)\big(\dot{B}(\Theta)\big) - \Theta\| < 1/2.$$

This implies that the image of the derivative $(\partial H_{B,\gamma_1}, \dots, \partial H_{B,\gamma_m})$ is the whole target space $\mathscr{L}(\mathbb{K}_x^d, \mathbb{K}_x^d)^m$, as claimed. □

8.3. Invariant measures of generic matrices. — Finally, we prove Theorem A. The only missing ingredient is

Proposition 8.14. — *Given $\ell \geq 1$, let $G_{2\ell}$ be the set of $(\mathtt{A}_1, \dots, \mathtt{A}_{2\ell}) \in \mathrm{GL}(d, \mathbb{K})^{2\ell}$ such that there exists some probability η in $\mathbb{P}(\mathbb{C})$ invariant under the action of $\mathtt{A}_i$ for every $1 \leq i \leq 2\ell$. Then $G_{2\ell}$ is closed and nowhere dense, and it is contained in a finite union of closed submanifolds of codimension $\geq \ell$.*

Remark 8.15. — The arguments that we are going to present remain valid if one replaces $\mathrm{GL}(d, \mathbb{K})$ by the subgroup $\mathrm{SL}(d, \mathbb{K})$ of matrices with determinant 1: just note that the curves $\mathtt{B}(t)$ defined in (8.13) and (8.17) lie in $\mathrm{SL}(d, \mathbb{K})$ if the initial matrix $\mathtt{A}$ does. Thus, the proposition holds for $\mathrm{SL}(d, \mathbb{K})$ as well.

Let us assume this proposition for a while, and use it to conclude the proof of the theorem in the complex case. Let $A \in \mathscr{G}^{r,\alpha}(M, d, \mathbb{K})$ be fiber bunched. Fix any $\ell \geq 1$ and $x \in M$. By Proposition 8.2 there is a family γ_i, $1 \leq i \leq 2\ell$, of loops at x with slow recurrence. By Proposition 8.8, the map

$$\mathscr{U} \ni B \mapsto (H_{B,\gamma_1}, \dots, H_{B,\gamma_{2\ell}}) \in \mathrm{GL}(d, \mathbb{K}_x)^{2\ell}$$

is a submersion, where $\mathscr{U}$ is a neighborhood of A independent of ℓ. Let $\mathscr{Z}$ be the pre-image of $G_{2\ell}$ under this map. Then $\mathscr{Z}$ is closed and nowhere dense, and it is contained in a finite union of closed submanifolds of codimension $\geq \ell$.

We claim that $\lambda_-(B, \mu) < \lambda_+(B, \mu)$ for all $B \in \mathscr{U} \setminus \mathscr{Z}$. Indeed, suppose the equality holds, and let m be any $\mathbb{P}(F_B)$-invariant probability that projects down to μ. By Theorem B, the measure m admits a disintegration $\{m_z : z \in M\}$ which is invariant under strong-stable holonomies $h^s = \mathbb{P}(H^s)$ and strong-unstable holonomies $h^u = \mathbb{P}(H^u)$, on the whole manifold M. In particular,

$$\mathbb{P}(H_{B,\gamma_i})_* m_x = m_x \quad \text{for every } 1 \leq i \leq 2\ell. \tag{8.12}$$

This contradicts the definition of $G_{2\ell}$, and this contradiction proves our claim. Let $\mathscr{Z}_0$ be the set of fiber bunched $B \in \mathscr{G}^{r,\alpha}(M, d, \mathbb{K})$ for which $\lambda_-(B, \mu) = \lambda_+(B, \mu)$. We have shown that any fiber bunched $A \in \mathscr{G}^{r,\alpha}(M, d, \mathbb{K})$ admits a neighborhood $\mathscr{U}$ such that, for any $\ell \geq 1$, there exists a nowhere dense subset $\mathscr{Z}$ of $\mathscr{U}$ contained in a finite union of closed submanifolds of codimension $\geq \ell$ and such that $\mathscr{Z}_0 \cap \mathscr{U} \subset \mathscr{Z}$. Thus, the closure of $\mathscr{Z}_0$ has infinite codimension and, in particular, is nowhere dense.

The proof of Theorem A has been reduced to proving Proposition 8.14. The proof of the proposition is presented in the next two sections.

8.3.1. *Complex case.* — Let S be the subset of matrices $\mathtt{A} \in \mathrm{GL}(d, \mathbb{C})$ whose eigenvalues are all distinct in norm. Then, S is an open and dense subset of $\mathrm{GL}(d, \mathbb{C})$ whose complement is contained in a finite union of closed manifolds of positive codimension. We use the following fact about variation of eigenvectors inside S:

Lemma 8.16. — *Let $\mathtt{A} \in S$. Then there exist C^∞ functions $\lambda_i : S_{\mathtt{A}} \to \mathbb{C}$ and $v_i : S_{\mathtt{A}} \to \mathbb{P}(\mathbb{C}^d)$ defined on an open neighborhood $S_{\mathtt{A}}$ of $\mathtt{A}$, for each $1 \leq i \leq d$, such that $v_i(\mathtt{B})$ is the direction of an eigenvector of $\mathtt{B}$ associated to the eigenvalue $\lambda_i(\mathtt{B})$, for any $B \in S_{\mathtt{A}}$. Furthermore, the map $S_{\mathtt{A}} \to \mathbb{P}(\mathbb{C}^d)^d$, $\mathtt{B} \mapsto (v_1(\mathtt{B}), \ldots, v_d(\mathtt{B}))$ is a submersion.*

Proof. — Since each eigenvalue $\lambda_i(\mathtt{A})$ is a simple root of the polynomial $\det(\mathtt{A} - \lambda \,\mathrm{id})$, it has a C^∞ continuation $\lambda_i(\mathtt{B})$ for all nearby matrices, given by the implicit function theorem. Denote $L_i(B) = \mathtt{B} - \lambda_i(\mathtt{B}) \,\mathrm{id}$. It depends smoothly on $\mathtt{B} \in S_{\mathtt{A}}$ and, since $\lambda_i(B)$ remains a simple eigenvalue of $\mathtt{B}$, it has rank $d-1$. Since the entries of $\mathrm{adj}(L_i(\mathtt{B}))$ are cofactors of $L_i(\mathtt{B})$, the adjoint is a non-zero matrix that also varies in a C^∞ fashion with $\mathtt{B}$. Moreover,

$$L_i(\mathtt{B}) \cdot \mathrm{adj}(L_i(\mathtt{B})) = \det(L_i(\mathtt{B})) \,\mathrm{id} = 0.$$

This means that any non-zero column of $\mathrm{adj}(L_i(B))$ is an eigenvector for $L_i(\mathtt{B})$, depending in a C^∞ fashion on the matrix, and so we may use it to define a function $v_i(\mathtt{B})$ as in the statement. To check that the derivative of v at $\mathtt{A}$ is onto just consider any differentiable curve $(-\varepsilon, \varepsilon) \ni t \mapsto (\beta_1(t), \ldots, \beta_d(t))$ such that $\beta_i(0) = v_i(\mathtt{A})$ for all $i = 1, \ldots, d$. Define $P(t) = [\beta_1(t), \ldots, \beta_d(t)]$, that is, $P(t)$ is the matrix whose column vectors are the $\beta_i(t)$. Then define

$$\mathtt{B}(t) = P(t) \,\mathrm{diag}[\lambda_1(\mathtt{A}), \ldots, \lambda_d(\mathtt{A})] P(t)^{-1}. \tag{8.13}$$

Then, $\mathtt{B}(0) = \mathtt{A}$ and $v(\mathtt{B}(t)) = (\beta_1(t), \ldots, \beta_d(t))$ for all t. In particular, the derivative $Dv(A)$ maps $\mathtt{B}'(0)$ to $(\beta_1'(0), \ldots, \beta_d'(0))$. So, the derivative is indeed surjective. □

Let $\mathcal{Z}_1$ be the subset of $\underline{\mathtt{A}} = (\mathtt{A}_1, \ldots, \mathtt{A}_{2\ell})$ such that $\mathtt{A}_i \notin S$ for at least ℓ values of i. Then $\mathcal{Z}_1$ is closed and it is contained in a finite union of closed submanifolds of codimension $\geq \ell$. For every $\underline{\mathtt{A}} \notin \mathcal{Z}_1$ there are at least $\ell+1$ matrices $\mathtt{A}_i$ whose eigenvalues all have distinct norms. Restricting to some open subset $\mathcal{V}$ of the complement of $\mathcal{Z}_1$, and renumbering if necessary, we may suppose that these matrices are $\mathtt{A}_1, \ldots, \mathtt{A}_{\ell+1}$. By Lemma 8.16, reducing $\mathcal{V}$ if necessary, the map

$$\mathcal{V} \setminus \mathcal{Z}_1 \ni \underline{\mathtt{A}} \mapsto \big(v_j(\mathtt{A}_i)\big)_{1 \leq j \leq d,\ 1 \leq i \leq \ell+1} \in \mathbb{P}(\mathbb{C}^d)^{d(\ell+1)}$$

is a submersion. Consequently, there exists a closed subset $\mathcal{Z}_2$ of $\mathcal{V} \setminus \mathcal{Z}_1$ contained in a finite union of closed submanifolds of codimension $\geq \ell$ such that for every $\underline{\mathtt{A}} \in \mathcal{V} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$ there exists some $1 \leq i \leq \ell$ such that

$$v_a(\mathtt{A}_i) \neq v_b(\mathtt{A}_{\ell+1}) \quad \text{for every } a, b \in \{1, \ldots, d\}. \tag{8.14}$$

Now it suffices to prove that $G_{2\ell} \cap \mathcal{V}$ is contained in $\mathcal{Z}_1 \cup \mathcal{Z}_2$. Indeed, suppose there is $\underline{\mathtt{A}} \in G_{2\ell} \cap \mathcal{V} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$. By the definition of $G_{2\ell}$, there exists some probability measure η on $\mathbb{P}(\mathbb{C}^d)$ such that

$$(\mathtt{A}_l)_*\eta = \eta \quad \text{for every } 1 \le l \le 2\ell. \tag{8.15}$$

Consider $l = i$, as in (8.14), and also $l = \ell + 1$. Since all the eigenvalues of $\mathtt{A}_i$ have distinct norms, η must be a convex combination of Dirac masses supported on the eigenspaces of $\mathtt{A}_i$. For the same reason, η must be supported on the set of eigenspaces of $\mathtt{A}_{\ell+1}$. However, (8.14) means that these two sets are disjoint, and so we reached a contradiction. This contradiction proves Proposition 8.14 in the complex case.

8.3.2. *Real case.* — The proof for real matrices is a bit more complicated due to the possibility of complex conjugate eigenvalues. In particular, the set of matrices whose eigenvalues are all distinct in norm is not dense. This difficulty has been met before by Bonatti, Gomez-Mont, Viana [**7**], and we use a similar approach in dimensions $d \ge 3$. For $d = 2$ we use a different argument, based on the conformal barycenter construction of Douady, Earle [**11**].

For each $r, s \ge 0$ with $r + 2s = d$, let $S(r,s)$ be the subset of matrices $\mathtt{A} \in \mathrm{GL}(d,\mathbb{R})$ having r real eigenvalues, and s pairs of (strictly) complex conjugate eigenvalues, such that all the eigenvalues that do not belong to the same complex conjugate pair have distinct norms. Every $S(r,s)$ is open and their union $S = \cup_{r,s} S(r,s)$ is an open and dense subset of $\mathrm{GL}(d,\mathbb{R})$ whose complement is contained in a finite union of closed submanifolds with positive codimension. Let $\mathrm{Grass}(k,d)$ denote the k-dimensional Grassmannian of $\mathbb{R}^d$, for $1 \le k \le d$. In what follows we often think of elements of $\mathrm{Grass}(2,d)$ as subsets of $\mathrm{Grass}(1,d) = \mathbb{P}(\mathbb{R}^d)$.

Lemma 8.17. — *Let* $\mathcal{F} = \{[(r_1,\dots,r_d)e^{i\theta}] \in \mathbb{P}(\mathbb{C}^d) : \theta \in [0,2\pi], (r_1,\dots,r_d) \in \mathbb{R}^d\}$. *Then* $\mathcal{F}$ *is closed in* $\mathbb{P}(\mathbb{C}^d)$ *and the map* $\Psi : \mathbb{P}(\mathbb{C}^d)\backslash\mathcal{F} \to \mathrm{Grass}(2,d)$ *defined by* $\Psi(v) = \mathrm{Span}\,\{\mathrm{Re}(v), \mathrm{Im}(v)\}$ *is a submersion.*

Proof. — First, we recall the usual local charts in $\mathrm{Grass}(2,d)$. Let $e_1,\dots,e_d$ the canonical base of $\mathbb{R}^d$ and $1 \le i < j \le d$ be fixed. For any $d \times 2$ matrix $\mathtt{A}$ we denote by $\varphi(\mathtt{A})$ the 2×2 matrix formed by the ith and jth rows of $\mathtt{A}$ and by $\varphi^*(\mathtt{A})$ the $(d-2) \times 2$ matrix formed by the other rows of $\mathtt{A}$. Let $U_{i,j}$ be the open set of planes $L \in \mathrm{Grass}(2,d)$ such that the orthogonal projection of L to $\mathrm{Span}\,\{e_i, e_j\}$ is an isomorphism. This means that if $L \in U_{i,j}$ with $L = \mathrm{Span}\,\{v_1, v_2\}$ then $\varphi(\mathtt{A}_L)$ is invertible, where $\mathtt{A}_L = [v_1, v_2]$ is the matrix whose columns are the vectors v_1, v_2. Then the map $\phi : U_{i,j} \to \mathbb{R}^{2(d-2)}$ defined by $\phi(L) = \varphi^*(\mathtt{A}_L)\varphi(\mathtt{A}_L)^{-1}$, where we identify $(d-2) \times 2$ matrices with points in $\mathbb{R}^{2(d-2)}$, is a local chart in the Grassmannian.

Now, note that v, $\overline{v} \in \mathbb{C}^d$ are linearly independent if and only if $v \in \mathbb{P}(\mathbb{C}^d)\backslash\mathcal{F}$. Moreover, in that case $\mathrm{Re}(v), \mathrm{Im}(v)$ are $\mathbb{C}$-linearly independent and, in particular, $\Psi(v)$ is well defined. It is clear from its expression in local charts that Ψ is differentiable. Moreover, still in local charts, its derivative is given by

$$D\Psi(v)\dot{v} = \varphi^*(\dot{\mathtt{A}})\varphi(\mathtt{A})^{-1} - \varphi^*(\mathtt{A})\varphi(\mathtt{A})^{-1}\varphi(\dot{\mathtt{A}})\varphi(\mathtt{A})^{-1},$$

where $\dot{v} \in T_v\mathbb{P}(\mathbb{C}^d)$, $\mathtt{A} = [\mathrm{Re}(v), \mathrm{Im}(v)]$ and $\dot{\mathtt{A}} = [\mathrm{Re}(\dot{v}), \mathrm{Im}(\dot{v})]$. Let $\dot{\mathtt{B}}$ be in the tangent space $T_{\Psi(v)}\,\mathrm{Grass}(2,d)$. Then $\dot{\mathtt{B}}$ is a $(d-2)\times 2$ matrix with real entries. Let $\dot{\mathtt{A}}_{\dot{\mathtt{B}}}$ be the $d\times 2$ matrix defined by $\varphi^*(\dot{\mathtt{A}}_{\dot{\mathtt{B}}}) = \dot{\mathtt{B}}\varphi(\mathtt{A})$ and $\varphi(\dot{\mathtt{A}}_{\dot{\mathtt{B}}}) = 0$. Since, $\dot{\mathtt{A}}_{\dot{\mathtt{B}}} = [\dot{v}_1, \dot{v}_2]$, we have that $D\Psi(v)(\dot{v}_1 + i\dot{v}_2) = \dot{\mathtt{B}}$. This finishes the proof of the lemma. $\square$

Lemma 8.18. — *Let $\mathtt{A} \in S(r,s)$. Then there exists an open neighborhood $S_{\mathtt{A}}$ of $\mathtt{A}$ and there exist C^∞ functions*

$$\begin{aligned}&\lambda_j : S_{\mathtt{A}} \to \mathbb{R}, \quad \xi_j : S_{\mathtt{A}} \to \mathrm{Grass}(1,d), \quad \textit{for } 1\le j\le r, \textit{ and}\\ &\mu_k : S_{\mathtt{A}} \to \mathbb{C}\setminus\mathbb{R}, \quad \eta_k : S_{\mathtt{A}} \to \mathrm{Grass}(2,d), \quad \textit{for } 1\le k\le s,\end{aligned}$$

such that $\xi_j(\mathtt{B})$ is the eigenspace of $\mathtt{B}$ associated to the eigenvalue $\lambda_j(\mathtt{B})$, and $\eta_k(\mathtt{B})$ is the characteristic space associated to the conjugate pair of eigenvalues $\mu_k(\mathtt{B})$ and $\bar{\mu}_k(\mathtt{B})$. Furthermore, the map

$$S_{\mathtt{A}} \to \mathrm{Grass}(1,d)^r \times \mathrm{Grass}(2,d)^s, \quad \mathtt{B} \mapsto (\xi_j(\mathtt{B})_{1\le j\le r}, \eta_k(\mathtt{B})_{1\le k\le s})$$

is a submersion.

Proof. — Existence and regularity of the eigenvalues λ_j and μ_k follow from the implicit function theorem. Moreover, the arguments in Lemma 8.16 imply that if $v_j(\mathtt{B})$ is an eigenvector associated to the eigenvalue $\lambda_j(\mathtt{B})$, for $j = 1,\dots,r$, and $v_{r+2k-1}(\mathtt{B}), v_{r+2k}(\mathtt{B})$ are eigenvectors associated to $\mu_k(\mathtt{B})$, $\bar{\mu}_k(\mathtt{B})$, respectively, for $k = 1,\dots,s$, then the map Φ defined by

$$\Phi(\mathtt{B}) = (v_1(\mathtt{B}),\dots,v_r(\mathtt{B}), v_{r+1}(\mathtt{B}),\dots,v_{r+2s}(\mathtt{B})) \in \mathbb{P}(\mathbb{R}^d)^r \times \mathbb{P}(\mathbb{C}^d)^s \tag{8.16}$$

is C^∞. We are going to show that this map is a submersion on some open neighborhood $S_{\mathtt{A}}$ of $\mathtt{A}$. For this, it is sufficient to show that the derivative $D\Phi(A)$ is onto. Consider any differentiable curve $(-\varepsilon,\varepsilon) \ni t \mapsto (\beta_1(t),\dots,\beta_{r+s}(t))$ such that $\beta_j(0) = v_j(\mathtt{A})$ for $j = 1,\dots,r$ and $\beta_{r+k}(0) = v_{r+2k-1}(\mathtt{A})$ for $k = 1,\dots,s$. Define

$$\begin{aligned}&P(t) = [\beta_1(t),\dots,\beta_r(t),\beta_{r+1},\bar{\beta}_{r+1},\dots,\beta_{r+s},\bar{\beta}_{r+s}], \textit{and}\\ &\mathtt{B}(t) = P(t)\ \mathrm{diag}[\lambda_1(\mathtt{A}),\dots,\lambda_r(\mathtt{A}),\mu_1(\mathtt{A}),\bar{\mu}_1(\mathtt{A}),\dots,\mu_s(\mathtt{A}),\bar{\mu}_s(\mathtt{A})]\,P(t)^{-1}.\end{aligned} \tag{8.17}$$

Observe that $t \mapsto \mathtt{B}(t)$ is a curve in $\mathrm{GL}(d,\mathbb{R})$, with $\mathtt{B}(0) = \mathtt{A}$. Observe also that $\Phi(\mathtt{B}(t)) = (\beta_1(t),\dots,\beta_{r+s}(t)$ for all $t \in (-\varepsilon,\varepsilon)$, and so $D\Phi(A)$ maps $\mathtt{B}'(0)$ to the vector $(\beta_1'(0),\dots,\beta_{r+s}'(0))$. So, the derivative is indeed surjective. Finally, define

$$\begin{aligned}&\xi_j(\mathtt{B}) = v_j(\mathtt{B}) \quad \text{for } j = 1,\dots,r \text{ and}\\ &\eta_k(\mathtt{B}) = \mathrm{Span}\,\{\mathrm{Re}(v_{r+2k-1}), \mathrm{Im}(v_{r+2k-1})\} \ \text{ for } k = 1,\dots,s.\end{aligned}$$

Clearly these maps are C^∞. Moreover, since (8.16) is a submersion, Lemma 8.17 implies that $\mathtt{B} \mapsto (\xi_j(\mathtt{B})_{1\le j\le r}, \eta_k(\mathtt{B})_{1\le k\le s})$ is a submersion. $\square$

Let $\mathcal{Z}_1$ be the subset of $\underline{\mathtt{A}} = (\mathtt{A}_1,\dots,\mathtt{A}_{2\ell})$ such that $\mathtt{A}_i \notin S$ for at least ℓ values of i. Then $\mathcal{Z}_1$ is closed and it is contained in a finite union of closed submanifolds of codimension $\ge \ell$. For every $\underline{\mathtt{A}} \notin \mathcal{Z}_1$ there are at least $\ell+1$ values of i such that $\mathtt{A}_i \in S$, that is, $\mathtt{A}_i \in S(r_i,s_i)$ for r_i and s_i. Restricting to some open subset $\mathcal{V}$ of the complement of $\mathcal{Z}_1$, and renumbering if necessary, we may suppose that these matrices are $\mathtt{A}_1, \dots, \mathtt{A}_{\ell+1}$. By Lemma 8.18, reducing $\mathcal{V}$ if necessary, the map

$$\mathcal{V} \setminus \mathcal{Z}_1 \ni \underline{\mathtt{A}} \mapsto \Big(\xi_j(\mathtt{A}_i)_{1\leq j\leq r_i},\ \eta_k(\mathtt{A}_i)_{1\leq k\leq s_i} \Big)_{1\leq i\leq \ell+1} \tag{8.18}$$

is a submersion.

Assume first that $d \geq 4$, and so $\dim \mathbb{P}(\mathbb{R}^d) \geq 3$. Since the $\xi_j(\mathtt{A})$ are points and the $\eta_k(\mathtt{A})$ are lines in the projective space, it follows that there exists a closed subset $\mathcal{Z}_2$ of $\mathcal{V} \setminus \mathcal{Z}_1$ contained in a finite union of closed submanifolds of codimension $\geq \ell$ such that for every $\mathtt{A} \in \mathcal{V} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$ there exists some $1 \leq i \leq \ell$ such that

$$\xi_a(\mathtt{A}_i) \neq \xi_b(\mathtt{A}_{\ell+1}) \tag{8.19}$$

$$\xi_a(\mathtt{A}_i) \notin \eta_c(\mathtt{A}_{\ell+1}) \quad \text{and} \quad \xi_b(\mathtt{A}_i) \notin \eta_d(\mathtt{A}_{\ell+1}) \tag{8.20}$$

$$\eta_c(\mathtt{A}_i) \cap \eta_d(\mathtt{A}_{\ell+1}) = \varnothing \tag{8.21}$$

for every $1 \leq a \leq r(\mathtt{A}_i)$, $1 \leq b \leq r(\mathtt{A}_{\ell+1})$, $1 \leq c \leq s(\mathtt{A}_i)$, and $1 \leq d \leq s(\mathtt{A}_{\ell+1})$. Now it suffices to prove that $G_{2\ell} \cap \mathcal{V}$ is contained in $\mathcal{Z}_1 \cup \mathcal{Z}_2$. Indeed, suppose there is $\underline{\mathtt{A}} \in G_{2\ell} \cap \mathcal{V} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$. By the definition of $G_{2\ell}$, there exists some probability measure η on $\mathbb{P}(\mathbb{C}^d)$ such that

$$(\mathtt{A}_l)_*\eta = \eta \quad \text{for every } 1 \leq l \leq 2\ell. \tag{8.22}$$

Consider both $l = i$, as in (8.19)–(8.21), and $l = \ell + 1$. Since all the eigenvalues of $\mathtt{A}_i$ have distinct norms, apart from the complex conjugate pairs, the measure η must be supported on

$$\Sigma(\mathtt{A}_i) = \bigcup_{j=1}^{r} \{\xi_j(\mathtt{A}_i)\} \cup \bigcup_{k=1}^{s} \eta_k(\mathtt{A}_i).$$

Analogously, η must be supported on $\Sigma(\mathtt{A}_{\ell+1})$. However, conditions (8.19)–(8.21) mean that the two sets $\Sigma(\mathtt{A}_i)$ and $\Sigma(\mathtt{A}_{\ell+1})$ are disjoint. This contradiction proves the proposition in any dimension $d \geq 4$.

For $d = 3$ the projective space $\mathbb{P}(\mathbb{R}^3)$ is only 2-dimensional, and so one can not force a pair of 1-dimensional submanifolds $\eta_k(\mathtt{A})$ to be disjoint, as required in (8.21). However, the argument can easily be adapted to cover the 3-dimensional case as well. Firstly, one replaces (8.21) by

$$\eta_c(\mathtt{A}_i) \neq \eta_d(\mathtt{A}_{\ell+1}) \tag{8.23}$$

for every $1 \leq c \leq s(\mathtt{A}_i)$ and $1 \leq d \leq s(\mathtt{A}_{\ell+1})$. (Both (8.21) and (8.23) are void if either $s(\mathtt{A}_i) = 0$ or $s(\mathtt{A}_{\ell+1}) = 0$; the only other possibility is $s(\mathtt{A}_i) = s(\mathtt{A}_{\ell+1}) = 1$, with $c = d = 1$.) Then the argument proceeds as before, except that we may no longer have disjointness: when $s = 1$,

$$\Sigma(\mathtt{A}_i) \cap \Sigma(\mathtt{A}_{\ell+1}) = \eta_1(\mathtt{A}_i) \cap \eta_1(\mathtt{A}_{\ell+1})$$

consists of exactly one point in projective space. Then η must be a Dirac measure supported on this point. However, in view of (8.22), this would have to be a fixed point of $\mathtt{A}_i$ contained in $\eta_1(\mathtt{A}_i)$, which is impossible because the eigenspace $\eta_i(\mathtt{A}_i)$ contains no invariant line. Thus, we reach a contradiction also in this case.

Now we deal with the case $d = 2$. Let $\mathcal{Z}_1$ be as in the previous cases: for every $\underline{\mathtt{A}} \notin \mathcal{Z}_1$ there are at least $\ell + 1$ values of i such that $\mathtt{A}_i \in S = S(2,0) \cup S(0,1)$. As before, it is no restriction to assume that these matrices are $\mathtt{A}_1, \dots, \mathtt{A}_{\ell+1}$. There are three cases to consider:

First, suppose there exist $1 \le i, j \le \ell + 1$ such that $\mathtt{A}_i \in S(2,0)$, that is, it has two real (distinct) eigenvalues, and $\mathtt{A}_j \in S(0,1)$, that is, it has a pair of complex eigenvalues. We claim that in this case $\underline{\mathtt{A}}$ can not belong to $G_{2\ell}$. Indeed, on the one hand, any probability measure η on $\mathbb{P}(\mathbb{R}^2)$ which is invariant under $\mathtt{A}_i \in S(2,0)$ must be a convex combination of Dirac masses at the two eigenspaces. On the other hand, the action of $\mathtt{A}_j \in S(0,1)$ on the projective space is a rotation whose angle is *not* a multiple of π, and so it admits no such invariant measure.

Next, suppose all the matrices are hyperbolic: $\mathtt{A}_i \in S(2,0)$ for all $1 \le i \le \ell$. In this case one can use precisely the same argument as we did before in higher dimensions (conditions (8.20) and (8.21)-(8.23) become void). One finds a closed subset $\mathcal{Z}_2$ contained in a finite union of submanifolds with codimension $\ge \ell$ such that $G_{2\ell} \cap \mathcal{V}$ is contained in $\mathcal{Z}_1 \cup \mathcal{Z}_2$.

Finally, suppose all the matrices are elliptic: $\mathtt{A}_i \in S(0,1)$ for all $1 \le i \le \ell$. Recall that every matrix $\mathtt{A} \in \mathrm{GL}(2,\mathbb{R})$ with positive determinant induces an automorphism $h_{\mathtt{A}}$ of the Poincaré half plane $\mathbb{H}$:

$$\mathtt{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow h_{\mathtt{A}}(z) = \frac{az+b}{cz+d}. \tag{8.24}$$

The action of $\mathtt{A}$ on the projective plane may be identified with the action of $h_{\mathtt{A}}$ on the boundary of $\mathbb{H}$, via

$$\partial\mathbb{H} \to \mathbb{P}(\mathbb{R}^2), \qquad x \mapsto [(x,1)]$$

(including $x = \infty$) so that $\mathbb{P}(\mathtt{A})$-invariant measures on the projective plane may be seen as $h_{\mathtt{A}}$-invariant measures sitting on the real axis. It is also easy to check that $h_{\mathtt{A}}$ has a fixed point in the open disc $\mathbb{H}$ if and only if $\mathtt{A} \in S(0,1)$. Define $\phi(\mathtt{A})$ to be this (unique) fixed point. It is easy to see that the $\mathtt{A} \mapsto \phi(\mathtt{A})$ is a C^∞ submersion: just use the explicit expression for the fixed point extracted from (8.24). The key feature is the following consequence of a classical construction of Douady, Earle [**11**]:

Lemma 8.19. — *If* $\mathtt{A}$, $\mathtt{B} \in S(0,1)$ *have some common invariant probability measure* μ *on* $\partial\mathbb{H}$ *then* $\phi(\mathtt{A}) = \phi(\mathtt{B})$.

Proof. — It is clear that elliptic matrices have no invariant measures with atoms of mass larger than $1/3$: such atoms would correspond to periodic points of $\mathtt{A}$ in the projective plane with period 1 or 2, which would contradict the definition of $S(0,1)$. In Proposition 1 of [**11**] a map $\mu \mapsto B(\mu)$ is constructed that assigns to each probability measure μ with no atoms of mass $\ge 1/2$ (see Remark 2 in [**11**, page26]) a point $B(\mu)$

in the half plane $\mathbb{H}$, in such a way that

$$B(h_*\mu) = h(B(\mu)) \quad \text{for every automorphism } h : \mathbb{H} \to \mathbb{H}.$$

When μ is $\mathtt{A}$-invariant this implies $h_{\mathtt{A}}(B(\mu)) = B((h_{\mathtt{A}})_*\mu) = B(\mu)$, and so the *conformal barycenter* $B(\mu)$ must coincide with the fixed point $\phi(\mathtt{A})$ of the automorphism $h_{\mathtt{A}}$. Thus, if μ is a common invariant measure then $\phi(\mathtt{A}) = B(\mu) = \phi(\mathtt{B})$. □

It follows from the previous observations that the map

$$\mathcal{V} \setminus \mathcal{Z}_1 \ni \underline{\mathtt{A}} \mapsto \big(\phi(\mathtt{A}_i)\big)_{1 \le i \le \ell+1} \in \mathbb{H}^{\ell+1}.$$

is a submersion. Hence, there exists a closed subset $\mathcal{Z}_2$ of $\mathcal{V} \setminus \mathcal{Z}_1$ contained in a finite union of closed submanifolds of codimension $\geq \ell$ such that for every $\underline{\mathtt{A}} \in \mathcal{V} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$ there exists some $1 \le i \le \ell$ such that $\phi(\mathtt{A}_i) \neq \phi(\mathtt{A}_{\ell+1})$. Thus, we may apply Lemma 8.19 to conclude that if $\underline{\mathtt{A}} \in \mathcal{V} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$. In other words, $G_{2\ell} \cap \mathcal{V}$ is contained in $\mathcal{Z}_1 \cup \mathcal{Z}_2$.

The proofs of Proposition 8.14 and Theorem A are now complete.

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THE COHOMOLOGICAL EQUATION FOR PARTIALLY HYPERBOLIC DIFFEOMORPHISMS

by

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Abstract. — We develop criteria for the existence and regularity of solutions to the cohomological equation over an accessible, partially hyperbolic diffeomorphism.

Résumé. — Nous développons des critères pour l'existence et la régularité des solutions de l'équation cohomologique au dessus d'un difféomorphisme partiellement hyperbolique et accessible.

Introduction

Let $f\colon M \to M$ be a dynamical system and let $\phi\colon M \to \mathbb{R}$ be a function. Considerable energy has been devoted to describing the set of solutions to the *cohomological equation:*

$$\phi = \Phi \circ f - \Phi, \tag{1}$$

under varying hypotheses on the dynamics of f and the regularity of ϕ. When a solution $\Phi\colon M \to \mathbb{R}$ to this equation exists, then ϕ is a called *coboundary*, for in the appropriate cohomology theory we have $\phi = d\Phi$. For historical reasons, a solution Φ to (1) is called a *transfer function.* The study of the cohomological equation has seen application in a variety of problems, among them: smoothness of invariant measures and conjugacies; mixing properties of suspended flows; rigidity of group actions; and geometric rigidity questions such as the isospectral problem. This paper studies solutions to the cohomological equation when f is a partially hyperbolic diffeomorphism and ϕ is C^r, for some real number $r > 0$.

A *partially hyperbolic diffeomorphism* $f\colon M \to M$ of a compact manifold M is one for which there exists a nontrivial, Tf-invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$ and a Riemannian metric on M such that vectors in E^s are uniformly contracted by Tf in this metric, vectors in E^u are uniformly expanded, and

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the expansion and contraction rates of vectors in E^c is dominated by the corresponding rates in E^u and E^s, respectively. An *Anosov diffeomorphism* is one for which the bundle E^c is trivial.

In the case where f is an Anosov diffeomorphism, there is a wealth of classical results on this subject, going back to the seminal work of Livšic, which we summarize here in Theorem 0.1. Here and in the rest of the paper, the notation $C^{k,\alpha}$, for $k \in \mathbb{Z}_+$, $\alpha \in (0,1]$, means C^k, with α-Hölder continuous kth derivative (where $C^{0,\alpha}$, $\alpha \in (0,1]$ simply means α-Hölder continuous). For $\alpha \in (0,1)$, C^α means α-Hölder continuous. More generally, if $r > 0$ is not an integer, then we will also write C^r for $C^{\lfloor r \rfloor, r - \lfloor r \rfloor}$.

Theorem 0.1. — **[23, 24, 25, 15, 16, 28, 19, 26]** *Let $f\colon M \to M$ be an Anosov diffeomorphism and let $\phi : M \to \mathbb{R}$ be Hölder continuous.*

I. Existence of solutions. *If f is C^1 and transitive, then* (1) *has a continuous solution Φ if and only if $\sum_{x\in\mathcal{O}} \phi(x) = 0$, for every f-periodic orbit $\mathcal{O}$.*

II. Hölder regularity of solutions. *If f is C^1, then every continuous solution to* (1) *is Hölder continuous.*

III. Measurable rigidity. *Let f be C^2 and volume-preserving. If there exists a measurable solution Φ to* (1)*, then there is a continuous solution Ψ, with $\Psi = \Phi$ a.e.*

More generally, if f is C^r and topologically transitive, for $r > 1$, and μ is a Gibbs state for f with Hölder potential, then the same result holds: if there exists a measurable function Φ such that (1) *holds μ-a.e., then there is a continuous solution Ψ, with $\Psi = \Phi$, μ-a.e.*

IV. Higher regularity of solutions. *Suppose that $r > 1$ is not an integer, and suppose that f and ϕ are C^r. Then every continuous solution to* (1) *is C^r.*

If f and ϕ are C^1, then every continuous solution to (1) *is C^1.*

If f and ϕ are real analytic, then every continuous solution to (1) *is real analytic.*

There are several serious obstacles to overcome in generalizing these results to partially hyperbolic systems. For one, while a transitive Anosov diffeomorphism has a dense set of periodic orbits, a transitive partially hyperbolic diffeomorphism might have *no* periodic orbits (for an example, one can take the time-t map of a transitive Anosov flow, for an appropriate choice of t). Hence the hypothesis appearing in part I can be empty: the vanishing of $\sum_{x\in\mathcal{O}} \phi(x)$ for every periodic orbit of f cannot be a complete invariant for solving (1).

This first obstacle was addressed by Katok and Kononenko **[20]**, who defined a new obstruction to solving equation (1) when f is partially hyperbolic. To define this obstruction, we first define a relevant collection of paths in M, called *su*-paths, determined by a partially hyperbolic structure.

The stable and unstable bundles E^s and E^u of a partially hyperbolic diffeomorphism are tangent to foliations, which we denote by $\mathcal{W}^s$ and $\mathcal{W}^u$ respectively **[5]**. The leaves of $\mathcal{W}^s$ and $\mathcal{W}^u$ are contractible, since they are increasing unions of submanifolds diffeomorphic to Euclidean space. An *su-path* in M is a concatenation of finitely many subpaths, each of which lies entirely in a single leaf of $\mathcal{W}^s$ or a single leaf of $\mathcal{W}^u$. An *su-loop* is an *su*-path beginning and ending at the same point.

We say that a partially hyperbolic diffeomorphism $f : M \to M$ is *accessible* if any point in M can be reached from any other along an *su*-path. The *accessibility class of* $x \in M$ is the set of all $y \in M$ that can be reached from x along an *su*-path. Accessibility means that there is one accessibility class, which contains all points. Accessibility is a key hypothesis in most of the results that follow. We remark that Anosov diffeomorphisms are easily seen to be accessible, by the transversality of E^u and E^s and the connectedness of M.

Any finite tuple of points $(x_0, x_1, \dots, x_k)$ in M with the property that x_i and x_{i+1} lie in the same leaf of either $\mathcal{W}^s$ or $\mathcal{W}^u$, for $i = 0, \dots, k-1$, determines an *su*-path from x_0 to x_k; if in addition $x_k = x_0$, then the sequence determines an *su*-loop. Following [**1**], we call such a tuple $(x_0, x_1, \dots, x_k)$ an *accessible sequence* and if $x_0 = x_k$, an *accessible cycle* (the term *periodic cycle* is used in [**20**]).

For f a partially hyperbolic diffeomorphism, there is a naturally-defined *periodic cycles functional*

$$PCF\colon \{\text{accessible sequences}\} \times C^\alpha(M) \to \mathbb{R}.$$

which was introduced in [**20**] as an obstruction to solving (1). For $x \in M$ and $x' \in \mathcal{W}^u(x)$, we define:

$$PCF_{(x,x')}\phi = \sum_{i=1}^{\infty} \phi(f^{-i}(x)) - \phi(f^{-i}(x')),$$

and for $x' \in \mathcal{W}^s(x)$, we define:

$$PCF_{(x,x')}\phi = \sum_{i=0}^{\infty} \phi(f^{i}(x')) - \phi(f^{i}(x)).$$

The convergence of these series follows from the Hölder continuity of ϕ and the expansion/contraction properties of the bundles E^u and E^s. This definition then extends to accessible sequences by setting $PCF_{(x_0,\dots,x_k)}\phi = \sum_{i=0}^{k-1} PCF_{(x_i,x_{i+1})}(\phi)$.

Assuming a hypothesis on f called *local accessibility*(1), [**20**] proved that the closely related *relative cohomological equation:*

$$\phi = \Phi \circ f - \Phi + c, \tag{2}$$

has a solution $\Phi\colon M \to \mathbb{R}$ and $c \in \mathbb{R}$, with Φ continuous, if and only if $PCF_\gamma(\phi) = 0$, for every accessible cycle γ.

The local accessibility hypothesis in [**20**] has been verified only for very special classes of partially hyperbolic systems, and it is not known whether there exist

(1) A partially hyperbolic diffeomorphism $f\colon M \to M$ is *locally accessible* if for every compact subset $M_1 \subset M$ there exists $k \geq 1$ such that for any $\varepsilon > 0$, there exists $\delta > 0$ that for every $x, x' \in M$ with $x \in M_1$ and $d(x, x') < \delta$, there is an accessible sequence $(x = x_0, \dots, x_k = x')$ from x to x' satisfying

$$d(x_i, x) \leq \varepsilon, \quad \text{and} \quad d_{\mathcal{W}^*}(x_{i+1}, x_i) < 2\varepsilon, \quad \text{for} \quad i = 0, \dots, k-1$$

where $d_{\mathcal{W}^*}$ denotes the distance along the $\mathcal{W}^s$ or $\mathcal{W}^u$ leaf common to the two points.

C^1-open sets of locally accessible diffeomorphisms, or more generally, whether accessibility implies local accessibility (although this seems unlikely). Assuming the strong hypothesis that E^u and E^s are C^∞ bundles, [**20**] also showed that a continuous transfer function for a C^∞ coboundary is always C^∞.

The starting point of the results here, part I of Theorem A below, is the observation that the local accessibility hypothesis in [**20**] can be replaced simply by accessibility. Accessibility is known to hold for a C^1 open and dense subset of all partially hyperbolic systems [**14**], is C^r open and dense among partially hyperbolic systems with 1-dimensional center [**36, 7**], and is conjectured to hold for a C^r open and dense subset of all partially hyperbolic diffeomorphisms, for all $r \geq 1$ [**32**]. Thus, part I of Theorem A gives a robust counterpart of part I of Theorem 0.1 for partially hyperbolic diffeomorphisms.

Another of the aforementioned major obstacles to generalizing Theorem 0.1 to the partially hyperbolic setting is that the regularity results in part IV fail to hold for general partially hyperbolic systems. Veech [**37**] and Dolgopyat [**13**] both exhibited examples of partially hyperbolic diffeomorphisms (volume-preserving and ergodic) where there is a sharp drop in regularity from ϕ to a solution Φ. These examples are not accessible. Here we show in Theorem A, part IV, that assuming accessibility and a C^1-open property called *strong r-bunching* (which incidentally is satisfied by the nonaccessible examples in [**37, 13**]), there is no significant loss of regularity between ϕ and Φ.

Part III of Theorem 0.1 is the most resistant to generalization, primarily because a general notion of Gibbs state for a partially hyperbolic diffeomorphism remains poorly understood. In the conservative setting, the most general result to date concerning ergodicity of for partially hyperbolic diffeomorphisms is due to Burns and Wilkinson [**9**], who show that every C^2, volume-preserving partially hyperbolic diffeomorphism that is center-bunched and accessible is ergodic. Center bunching is a C^1-open property that roughly requires that the action of Tf on E^c be close to conformal, relative to the expansion and contraction rates in E^s and E^u (see Section 2). Adopting the same hypotheses as in [**9**], we recover here the analogue of Theorem 0.1 part III for volume-preserving partially hyperbolic diffeomorphisms.

We now state our main result.

Theorem A. — *Let $f : M \to M$ be partially hyperbolic and accessible, and let $\phi : M \to \mathbb{R}$ be Hölder continuous.*

I. Existence of solutions. *If f is C^1, then* (2) *has a continuous solution Φ for some $c \in \mathbb{R}$ if and only if $PCF_{\mathscr{C}}(\phi) = 0$, for every accessible cycle $\mathscr{C}$.*

II. Hölder regularity of solutions. *If f is C^1, then every continuous solution to* (2) *is Hölder continuous.*

III. Measurable rigidity. *Let f be C^2, center bunched, and volume-preserving. If there exists a measurable solution Φ to* (2)*, then there is a continuous solution Ψ, with $\Psi = \Phi$ a.e.*

IV. Higher regularity of solutions. *Let $k \geq 2$ be an integer. Suppose that f and ϕ are both C^k and that f is strongly r-bunched, for some $r < k-1$ or $r = 1$. If Φ is a continuous solution to* (2), *then Φ is C^r.*

The center bunching and strong r-bunching hypotheses in parts III and IV are C^1-open conditions and are defined in Section 2. Theorem A part IV generalizes all known C^∞ Livšic regularity results for accessible partially hyperbolic diffeomorphisms. In particular, it applies to all time-t maps of Anosov flows and compact group extensions of Anosov diffeomorphisms. Accessibility is a C^1 open and C^∞ dense condition in these classes [**8, 6**]. In dimension 3, for example, the time-1 map of any mixing Anosov flow is stably accessible [**6**], unless the flow is a constant-time suspension of an Anosov diffeomorphism.

We also recover the results of [**13**] in the context of compact group extensions of volume-preserving Anosov diffeomorphisms. Finally, Theorem A also applies to all accessible, partially hyperbolic affine transformations of homogeneous manifolds. A direct corollary that encompasses these cases is:

Corollary 0.2. — *Let f be C^∞, partially hyperbolic and accessible. Assume that $Tf|_{E^c}$ is isometric in some continuous Riemannian metric. Let $\phi\colon M \to \mathbb{R}$ be C^∞. Suppose there exists a continuous function $\Phi\colon M \to \mathbb{R}$ such that*

$$\phi = \Phi \circ f - \Phi.$$

Then Φ is C^∞. If, in addition, f preserves volume, then any measurable solution Φ extends to a C^∞ solution.

For any such f, and any integer $k \geq 2$, there is a C^1 open neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^k(M)$ such that, for any accessible $g \in \mathcal{U}$, and any C^k function $\phi\colon M \to \mathbb{R}$, if

$$\phi = \Phi \circ g - \Phi,$$

has a continuous solution Φ, then Φ is C^1 and also C^r, for all $r < k-1$. If g also preserves volume, then any measurable solution extends to a C^r solution.

The vanishing of the periodic cycles obstruction in Theorem A, part I turns out to be a practical method in many contexts for determining whether (2) has a solution. On the one hand, this method has already been used by Damjanović and Katok to establish rigidity of certain partially hyperbolic abelian group actions [**12**]; in this (locally accessible, algebraic) context, checking that the PCF obstruction vanishes reduces to questions in classical algebraic K-theory (see also [**11, 21**]). On the other hand, for a given accessible partially hyperbolic system, the PCF obstruction provides an infinite codimension obstruction to solving (2), and so the generic cocycle ϕ has no solutions to (2). This latter fact follows from recent work of Avila, Santamaria and Viana on the related question of vanishing of Lyapunov exponents for linear cocycles over partially hyperbolic systems (see [**1**], Section 9).

As part of proof of Theorem A, part II, we also prove that stable and unstable foliations of any C^1 partially hyperbolic diffeomorphism are transversely Hölder continuous (Corollary 5.3). This extends to the C^1 setting the well-known fact that the stable and unstable foliations for a $C^{1+\theta}$ partially hyperbolic diffeomorphism are transversely Hölder continuous [**33**]. As far as we know, no previous regularity results were known for C^1 systems, including Anosov diffeomorphisms.

In a forthcoming work [**3**] we will use some of the results here to prove rigidity theorems for partially hyperbolic diffeomorphisms and group actions.

We now summarize in more detail the previous results in this area:

- Veech [**37**] studied the case when f is a partially hyperbolic toral automorphism and established existence and regularity results for solutions to (1). In these examples, there is a definite loss of regularity between coboundary and transfer function. The examples studied by Veech differ from those treated here in that they do not have the property of accessibility (although they have the weaker property of essential accessibility).
- Dolgopyat [**13**] studied equations (1) and (2) for a special class of partially hyperbolic diffeomorphisms – the compact group extensions of Anosov diffeomorphisms – in the case where the base map preserves a Gibbs state μ with Hölder potential. Assuming rapid mixing of the group extension with respect to μ, [**13**] showed that if the coboundary ϕ is C^∞, then any transfer function $\Phi \in L^2(\mu \times \text{Haar})$ is also C^∞. Dolgopyat also gave an example of a partially hyperbolic diffeomorphism with a C^∞ coboundary whose transfer map is continuous, but not C^1. This example, like Veech's, is essentially accessible, but not accessible. We note that when the Gibbs measure μ is volume, then the rapid mixing assumption in [**13**] is equivalent to accessibility.
- De la Llave [**27**], extended the work of [**20**] to give some regularity results for the transfer function under strong (nongeneric) local accessibility/regularity hypotheses on bundles. De la Llave's approach focuses on bootstrapping the regularity of the transfer function from L^p to continuity and higher smoothness classes using the transverse regularity of the stable and unstable foliations in M. For this reason, he makes strong regularity hypotheses on this transverse regularity.

While there are superficial similaries between these previous results and Theorem A, the approach here, especially in parts II and IV, is fundamentally new and does not rely on these results. In particular, to establish regularity of a transfer function, we take advantage of a form of self-similarity of its graph in the central directions of M. This self-similarity, known as C^r *homogeneity* is discussed in more detail in the following section.

1. Techniques in the proof of Theorem A

The proof of parts I and III of Theorem A use recent work of Avila, Santamaria and Viana on sections of bundles with various saturation properties. In [**1**], they apply these results to show that under suitable conditions, matrix cocycles over partially hyperbolic systems have a nonvanishing Lyapunov exponent. Parts I and III of Theorem A are translations of some of the main results in [**1**] to the abelian cocycle setting.

The regularity results in Theorem A – parts II and IV – comprise the bulk of this paper.

To investigate the regularity of a solution Φ, we examine the graph of Φ in $M \times \mathbb{R}$. If ϕ is Hölder continuous, then the stable and unstable foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ for f lift to two "stable and unstable" foliations $\mathcal{W}^s_\phi$ and $\mathcal{W}^u_\phi$ of $M \times \mathbb{R}$, whose leaves are graphs of Hölder continuous functions into $\mathbb{R}$. These lifted foliations are invariant under the skew product $(x, t) \mapsto (f(x), t + \phi(x))$. The fact that Φ satisfies the equation $\phi = \Phi \circ f - \Phi + c$, for some $c \in \mathbb{R}$, implies that the graph of Φ is saturated by leaves of the lifted foliations. The leafwise and transverse regularity of these foliations determine the regularity of Φ. In the most general setting of Theorem A, part II, these foliations are both leafwise and transversely Hölder continuous, and this implies the Hölder regularity of Φ when f is accessible.

The proof of higher regularity in part IV has two main components. We first describe a simplified version of the proof under an additional assumption on f called dynamical coherence.

Definition 1.1. — *A partially hyperbolic diffeomorphism f is* dynamically coherent *if the distributions $E^c \oplus E^u$, and $E^c \oplus E^s$ are integrable, and everywhere tangent to foliations $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$.*

If f is dynamically coherent, then there is also a central foliation $\mathcal{W}^c$, tangent to E^c, whose leaves are obtained by intersecting the leaves of $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$. The normally hyperbolic theory [**18**] implies that the leaves of $\mathcal{W}^{cu}$ are then bifoliated by the leaves of $\mathcal{W}^c$ and $\mathcal{W}^u$, and the leaves of $\mathcal{W}^{cs}$ are bifoliated by the leaves of $\mathcal{W}^c$ and $\mathcal{W}^s$.

Suppose that f is dynamically coherent and that f and ϕ satisfy the hypotheses of part IV of Theorem A, for some $k \geq 2$ and $r < k - 1$ or $r = 1$. Under these assumptions, here are the two components of the proof. The first part of the proof is to show that Φ is uniformly C^r along individual leaves of $\mathcal{W}^s$, $\mathcal{W}^u$ and $\mathcal{W}^c$. The second part is to employ a result of Journé to show that smoothness of Φ along leaves of these three foliations implies smoothness of Φ.

To show that Φ is smooth along the leaves of $\mathcal{W}^s$ and $\mathcal{W}^u$, we examine again the lifted foliations for the associated skew product. The assumption that ϕ is C^k implies that the leaves of these lifted foliations are C^r (in fact, they are C^k). This part of the proof does not require dynamical coherence or accessibility.

To show that Φ is smooth along leaves of the central foliation, one can use accessibility and strong r-bunching to show that the graph of Φ over any central leaf $\mathcal{W}^c(x)$ of f is C^r homogeneous. More precisely, setting $N' = \mathcal{W}^c(x) \times \mathbb{R}$ and $N = \{(y, \Phi(y)) : y \in \mathcal{W}^c(x)\} \subset N'$, we show that the manifold N is C^r *homogeneous in* N': for any two points $p, q \in N$, there is a C^r local diffeomorphism of N' sending

p to q and preserving N. C^1-homogeneous subsets of a manifold have a remarkable property:

Theorem 1.2. — [**35**] *Any locally compact subset N of a C^1 manifold N' that is C^1 homogeneous in N' is a C^1 submanifold of N'*

If $r = 1$, we can apply this result to obtain that the graph of Φ is C^1 over any center manifold. Hence Φ is C^1 over center, stable, and unstable leaves, which implies that Φ is C^1. This completes the proof in the case $r = 1$ (assuming dynamical coherence).

In fact we do not use the results in [**35**] in the proof of Theorem A but employ a different technique to establish smoothness, which also works for $r > 1$ and in the non-dynamically coherent case. Our methods also show:

Theorem B. — *For any integer $k \geq 2$, any C^k homogeneous, C^1 submanifold of a C^k manifold is a C^k submanifold.*

Theorem B also follows from the results in [**35**] (thanks to Bruce Kleiner for pointing this out). We give a somewhat different proof in Section 7 as it motivates later results.

Returning to the proof of Theorem A, assuming dynamical coherence and using Theorem B, one can obtain under the hypotheses of part IV that the graph of the transfer function Φ over each center manifold is $C^{\lfloor r \rfloor}$. With some more work, one can obtain that the graph of the transfer function Φ over each center manifold is C^r. A result of Journé [**19**] implies that for any $r > 1$ that is not an integer, and any two transverse foliations with uniformly C^r leaves, if a function Φ is uniformly C^r along the leaves of both foliations, then it is uniformly C^r. Since f is assumed to be dynamically coherent, the $\mathcal{W}^c$ and $\mathcal{W}^s$ foliations transversely subfoliate the leaves of $\mathcal{W}^{cs}$. Applying Journé's result using $\mathcal{W}^c$ and $\mathcal{W}^s$, we obtain that Φ is C^r along the leaves of $\mathcal{W}^{cs}$. Applying Journé's theorem again, this time with $\mathcal{W}^{cs}$ and $\mathcal{W}^u$, we obtain that Φ is C^r.

We have just described a proof of part IV under the assumption that f is dynamically coherent. If we drop the assumption of dynamical coherence, the assertion that Φ is "C^r along center manifolds" no longer makes sense, as f might not have center manifolds. One can find locally invariant center manifolds that are "nearly" tangent to the center distribution (as in [**9**]), but the argument described above does not work for these manifolds. The analysis becomes considerably more delicate and is described in more detail in Section 8. As one of the components in our argument, we prove a strengthened version of Journé's theorem (Theorem 8.4) that works for plaque families as well as foliations, and replaces the assumption of smoothness along leaves with the existence of an "approximate r-jet" at the basepoint of each plaque.

The main result that lies behind the proof of Theorem A, part IV is a saturated section theorem for fibered partially hyperbolic systems (Theorem C). A fibered partially hyperbolic diffeomorphism is defined on a fiber bundle and is also a bundle isomorphism, covering a partially hyperbolic diffeomorphism (see Section 9). In this context, Theorem C states that under the additional hypotheses that the bundle diffeomorphism is suitably bunched, and the base diffeomorphism is accessible, then any continuous section of the bundle whose image is an accessibility class for the lifted map is in fact a smooth section. Using Theorem C it is also possible to extend in part the conclusions of Theorem A part IV to (suitably bunched) cocycles taking values in other Lie groups. The details are not carried out here, but the reader is referred to [**31, 8, 2**], where some of the relevant technical considerations are addressed (see also the remark after the statement of Theorem C in Section 9).

Theorem C would follow immediately if the following conjecture is correct.

Conjecture 1.3. — *Let $f\colon M \to M$ be C^r, partially hyperbolic and r-bunched. Then every accessibility class for f is an injectively immersed, C^r submanifold of M.*

For locally compact accessibility classes, it should be possible to prove Conjecture 1.3 using the techniques from [**35**] to show that the accessibility class is a submanifold and the methods developed in this paper to show that the submanifold is smooth.

2. Partial hyperbolicity and bunching conditions

We now define the bunching hypotheses in Theorem A; to do so, we give a more precise definition of partial hyperbolicity. Let $f : M \to M$ be a diffeomorphism of a compact manifold M. We say that f is *partially hyperbolic* if the following holds. First, there is a nontrivial splitting of the tangent bundle, $TM = E^s \oplus E^c \oplus E^u$, that is invariant under the derivative map Tf. Further, there is a Riemannian metric for which we can choose continuous positive functions ν, $\hat{\nu}$, γ and $\hat{\gamma}$ with

$$\nu, \hat{\nu} < 1 \quad \text{and} \quad \nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1} \tag{3}$$

such that, for any unit vector $v \in T_pM$,

$$\|Tfv\| < \nu(p), \qquad \text{if } v \in E^s(p), \tag{4}$$

$$\gamma(p) < \|Tfv\| < \hat{\gamma}(p)^{-1}, \qquad \text{if } v \in E^c(p), \tag{5}$$

$$\hat{\nu}(p)^{-1} < \|Tfv\|, \qquad \text{if } v \in E^u(p). \tag{6}$$

We say that f is *center bunched* if the functions $\nu, \hat{\nu}, \gamma$, and $\hat{\gamma}$ can be chosen so that:

$$\max\{\nu, \hat{\nu}\} < \gamma\hat{\gamma}. \tag{7}$$

Center bunching means that the hyperbolicity of f dominates the nonconformality of Tf on the center. Inequality (7) always holds when $Tf|_{E^c}$ is conformal. For then we have $\|T_pfv\| = \|T_pf|_{E^c(p)}\|$ for any unit vector $v \in E^c(p)$, and hence we can choose $\gamma(p)$ slightly smaller and $\hat{\gamma}(p)^{-1}$ slightly bigger than

$$\|T_pf|_{E^c(p)}\|.$$

By doing this we may make the ratio $\gamma(p)/\hat{\gamma}(p)^{-1} = \gamma(p)\hat{\gamma}(p)$ arbitrarily close to 1, and hence larger than both $\nu(p)$ and $\hat{\nu}(p)$. In particular, center bunching holds whenever E^c is one-dimensional. The center bunching hypothesis considered here is natural and appears in other contexts, e.g., [**5, 4, 2, 31, 29**].

For $r > 0$, we say that f is *r-bunched* if the functions $\nu, \hat{\nu}, \gamma$, and $\hat{\gamma}$ can be chosen so that:

$$\nu < \gamma^r, \qquad \hat{\nu} < \hat{\gamma}^r \tag{8}$$

$$\nu < \gamma\hat{\gamma}^r, \quad \text{and} \quad \hat{\nu} < \hat{\gamma}\gamma^r. \tag{9}$$

Note that every partially hyperbolic diffeomorphism is r-bunched, for some $r > 0$. The condition of 0-bunching is merely a restatement of partial hyperbolicity, and 1-bunching is center bunching. The first pair of inequalities in (8) are r-normal hyperbolicity conditions; when f is dynamically coherent, these inequalities ensure that the leaves of $\mathcal{W}^{cu}$, $\mathcal{W}^{cs}$, and $\mathcal{W}^c$ are C^r. Combined with the first group of inequalities, the second group of inequalities imply that E^u and E^s are "C^r in the direction of E^c." More precisely, in the case that f is dynamically coherent, the r-bunching inequalities imply that the restriction of E^u to $\mathcal{W}^{cu}$ leaves is a C^r bundle and the restriction of E^s to $\mathcal{W}^{cs}$ leaves is a C^r bundle.

For $r > 0$, we say that f is *strongly r-bunched* if the functions $\nu, \hat{\nu}, \gamma$, and $\hat{\gamma}$ can be chosen so that:

$$\max\{\nu, \hat{\nu}\} < \gamma^r, \qquad \max\{\nu, \hat{\nu}\} < \hat{\gamma}^r \tag{10}$$

$$\nu < \gamma\hat{\gamma}^r, \quad \text{and} \quad \hat{\nu} < \hat{\gamma}\gamma^r. \tag{11}$$

We remark that if f is partially hyperbolic and there exists a Riemannian metric in which $Tf|_{E^c}$ is isometric, then f is strongly r-bunched, for every $r > 0$; given a metric $\|\cdot\|$ for which f satisfies (4), and another metric $\|\cdot\|'$ in which $Tf|_{E^c}$ is isometric, it is a straightforward exercise to construct a Riemannian metric $\|\cdot\|''$ for which inequalities (10) hold, with $\gamma = \hat{\gamma} \equiv 1$.

The reason strong r-bunching appears as a hypothesis in Theorem A is the following. Suppose that f is partially hyperbolic and that $\phi\colon M \to \mathbb{R}$ is C^1. Then the skew product $f_\phi\colon M \times \mathbb{R}/\mathbb{Z} \to M \times \mathbb{R}/\mathbb{Z}$ given by

$$f_\phi(x, t) = (f(x), t + \phi(x))$$

is partially hyperbolic, and if f is strongly r-bunched then f_ϕ is r-bunched. This skew product and the corresponding lifted skew product on $M \times \mathbb{R}$ appears in a central way in our analysis, as we explain in the following section.

2.1. Notation. — Let a and b be real-valued functions, with $b \neq 0$. The notation $a = O(b)$ means that the ratio $|a/b|$ is bounded above, and $a = \Omega(b)$ means $|a/b|$ is bounded below; $a = \Theta(b)$ means that $|a/b|$ is bounded above and below. Finally, $a = o(b)$ means that $|a/b| \to 0$ as $b \to 0$. Usually a and b will depend on either an integer j or a real number t and on one or more points in M. The constant C

bounding the appropriate ratios must be independent of n or t and the choice of the points.

The notation $\alpha < \beta$, where α and β are continuous functions, means that the inequality holds pointwise. The function $\min\{\alpha, \beta\}$ takes the value $\min\{\alpha(p), \beta(p)\}$ at the point p.

We denote the Euclidean norm by $|\cdot|$. If X is a metric space and $r > 0$ and $x \in X$, the notation $B_X(x, r)$ denotes the open ball about x of radius r. If the subscript is omitted, then the ball is understood to be in M. Throughout the paper, r always denotes a real number and j, k, ℓ, m, n always denote integers. I denotes the interval $(-1, 1) \subset \mathbb{R}$, and $I^n \subset \mathbb{R}^n$ the n-fold product.

If γ_1 and γ_2 are paths in M, then $\gamma_1 \cdot \gamma_2$ denotes the concatenated path, and $\overline{\gamma}_1$ denotes the reverse path.

Suppose that $\mathcal{F}$ is a foliation of an m-manifold M with d-dimensional smooth leaves. For $r > 0$, we denote by $\mathcal{F}(x, r)$ the connected component of x in the intersection of $\mathcal{F}(x)$ with the ball $B(x, r)$.

A *foliation box for* $\mathcal{F}$ is the image U of $\mathbb{R}^{m-d} \times \mathbb{R}^d$ under a homeomorphism that sends each vertical $\mathbb{R}^d$-slice into a leaf of $\mathcal{F}$. The images of the vertical $\mathbb{R}^d$-slices will be called *local leaves of* $\mathcal{F}$ *in* U.

A *smooth transversal* to $\mathcal{F}$ in U is a smooth codimension-d disk in U that intersects each local leaf in U exactly once and whose tangent bundle is uniformly transverse to $T\mathcal{F}$. If Σ_1 and Σ_2 are two smooth transversals to $\mathcal{F}$ in U, we have the *holonomy map* $h_{\mathcal{F}} : \Sigma_1 \to \Sigma_2$, which takes a point in Σ_1 to the intersection of its local leaf in U with Σ_2.

Finally, for $r > 1$ a nonintegral real number, M, N smooth manifolds, the C^r metric on $C^r(M, N)$ is defined in local charts by:

$$d_{C^r}(f, g) = d_{C^{\lfloor r \rfloor}}(f, g) + d_{C^0}(D^{\lfloor r \rfloor} f, D^{\lfloor r \rfloor} g).$$

This metric generates the (weak) C^r topology on $C^r(M, N)$.

3. The partially hyperbolic skew product associated to a cocycle

Let $f : M \to M$ be C^k and partially hyperbolic and let $\phi : M \to \mathbb{R}$ be $C^{\ell,\alpha}$, for some integer $\ell \geq 0$ and $\alpha \in [0, 1]$, with $0 < \ell + \alpha \leq k$. Define the skew product $f_\phi : M \times \mathbb{R} \to M \times \mathbb{R}$ by

$$f_\phi(p, t) = (f(p), t + \phi(p)).$$

The following proposition is the starting point for our proof of Theorem A.

Proposition 3.1. — *There exist foliations $\mathcal{W}^u_\phi$, $\mathcal{W}^s_\phi$ of $M \times \mathbb{R}$ with the following properties.*

1. *The leaves of $\mathcal{W}^u_\phi$, $\mathcal{W}^s_\phi$ are $C^{\ell,\alpha}$.*

2. *The leaves of* $\mathcal{W}^u_\phi$ *project to leaves of* $\mathcal{W}^u$, *and the leaves of* $\mathcal{W}^s_\phi$ *project to leaves of* $\mathcal{W}^s$. *Moreover,* $(x',t') \in \mathcal{W}^s_\phi(x,t)$ *if and only if* $x' \in \mathcal{W}^s(x)$ *and*

$$\liminf_{n\to\infty} d(f^n_\phi(x,t), f^n_\phi(x',t')) = 0.$$

3. *Define* $T\colon M \times \mathbb{R} \to M \times \mathbb{R}$ *by* $T_t(z,s) = (z, s+t)$. *Then for all* $z \in M$ *and* $s,t \in \mathbb{R}$*:*

$$\mathcal{W}^s_\phi(z, s+t) = T_t \mathcal{W}^s_\phi(z,s).$$

4. *If* $(x,t) \in M \times \mathbb{R}$ *and* $(x',t') \in \mathcal{W}^s_\phi(x,t)$, *then*

$$t' - t = \sum_{i=0}^{\infty} \phi(f^i(x')) - \phi(f^i(x)) = PCF_{(x,x')}\phi,$$

and if $(x',t') \in \mathcal{W}^u_\phi(x,t)$, *then*

$$t' - t = \sum_{i=1}^{\infty} \phi(f^{-i}(x)) - \phi(f^{-i}(x')) = PCF_{(x,x')}\phi.$$

Proof. — The map f_ϕ covers the map $(x,t) \mapsto (f(x), t + \phi(x))$ on the compact manifold $M \times \mathbb{R}/\mathbb{Z}$, which we also denote by f_ϕ

In the case where $\ell \geq 1$, (1) and (2) follow directly from the fact that f_ϕ is $C^{\ell,\alpha}$ and partially hyperbolic. The invariant foliations on $M \times \mathbb{R}/\mathbb{Z}$ lift to invariant foliations on $M \times \mathbb{R}$.

For $\ell = 0$, (1) and (2) are the content of Proposition 5.1, which is proved in Section 5.

Since $T_t \circ f_\phi = f_\phi \circ T_t$ for all $t \in \mathbb{R}$, (3) follows easily from (2). Finally, (4) is an easy consequence of (3). □

Throughout the rest of the paper, we will mine extensively the properties of the foliations $\mathcal{W}^s_\phi$ and $\mathcal{W}^u_\phi$: the regularity of the leaves, their transverse regularity, and their accessibility properties.

This focus on the lifted foliations $\mathcal{W}^s_\phi$ and $\mathcal{W}^u_\phi$ is not entirely new. Notably, Niticǎ and Török **[31]** established the regularity of solutions to equation (2) when f is an Anosov diffeomorphism by examining these lifted foliations. The key observation in **[31]** is that *the smoothness of the leaves of* $\mathcal{W}^s_\phi$ *and* $\mathcal{W}^u_\phi$ *determines the smoothness of the transfer function along the leaves of* $\mathcal{W}^s$ *and* $\mathcal{W}^u$. The advantage of the approach in **[31]** is that it allowed them to prove a natural generalization of Theorem 0.1 to cocycles taking values in nonabelian lie groups; provided that the induced skew product for such a cocycle is partially hyperbolic, the smoothness of the lifted invariant foliations determines the smoothness of transfer functions when f is Anosov. This focus on the foliations for the skew product associated to the cocycle turns out to be crucial in our setting.

4. Saturated sections of admissible bundles

In this section, we define a key property called *saturation* and present some general results about saturated sections of bundles. In the next section, we apply these results in the setting of abelian cocycles to prove parts I and III of Theorem A. Throughout this section, $f\colon M \to M$ denotes a partially hyperbolic diffeomorphism.

Let N be a manifold, and let $\pi\colon \mathcal{B} \to M$ be a fiber bundle, with fiber N. We say that $\mathcal{B}$ is *admissible* if there exist foliations $\mathcal{W}^s_{\mathrm{lift}}$, $\mathcal{W}^u_{\mathrm{lift}}$ of $\mathcal{B}$ (not necessarily with smooth leaves) such that, for every $z \in \mathcal{B}$ and $* \in \{s, u\}$, the restriction of π to $\mathcal{W}^*_{\mathrm{lift}}(z)$ is a homeomorphism onto $\mathcal{W}^*(\pi(z))$.

A more general definition of admissibility for more general bundles in terms of holonomy maps is given in [**1**]; we remark that two definitions are equivalent in this context. If $\pi\colon \mathcal{B} \to M$ is an admissible bundle, then given any su-path $\gamma\colon [0,1] \to M$ and any point $z \in \pi^{-1}(\gamma(0))$, there is a unique path $\tilde{\gamma}_z\colon [0,1] \to \mathcal{B}$ such that:

- $\pi\tilde{\gamma}_z = \gamma$,
- $\tilde{\gamma}_z(0) = z$,
- $\tilde{\gamma}_z$ is a concatenation of finitely many subpaths, each of which lies entirely in a single leaf of $\mathcal{W}^s_{\mathrm{lift}}$, or $\mathcal{W}^u_{\mathrm{lift}}$.

We call $\tilde{\gamma}_z$ an *su-lift path* and say that $\tilde{\gamma}_z$ is an *su-lift loop* if $\tilde{\gamma}_z(0) = \tilde{\gamma}_z(1) = z$. For a fixed su-path γ, the map $H_\gamma\colon \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1))$ that sends $z \in \pi^{-1}(\gamma(0))$ to $\tilde{\gamma}_z(1) \in \pi^{-1}(\gamma(1))$ is a homeomorphism. It is easy to see that $H_{\gamma_1\cdot\gamma_2} = H_{\gamma_2} \circ H_{\gamma_1}$ and $H_{\overline{\gamma}} = H_\gamma^{-1}$.

Recall that any accessible sequence $\mathcal{S} = (x_1, \dots, x_k)$ determines an su-path $\gamma_{\mathcal{S}}$. We fix the convention that $\gamma_{\mathcal{S}}$ is a concatenation of leafwise distance-minimizing arcs, each lying in an alternating sequences of single leaves of $\mathcal{W}^s$ or $\mathcal{W}^u$. Using this identification, we define the holonomy $H_{\mathcal{S}}\colon \pi^{-1}(x_1) \to \pi^{-1}(x_k)$ by setting $H_{\mathcal{S}} = H_{\gamma_{\mathcal{S}}}$; since the leaves of $\mathcal{W}^u$, $\mathcal{W}^s$, $\mathcal{W}^u_{\mathrm{lift}}$, and $\mathcal{W}^s_{\mathrm{lift}}$ are all contractible, $H_{\mathcal{S}}$ is well-defined.

Definition 4.1. — *Let $\pi\colon \mathcal{B} \to M$ be an admissible bundle. A section $\sigma\colon M \to \mathcal{B}$ is:*

- u-saturated *if for every $z \in \sigma(M)$ we have $\mathcal{W}^u_{lift}(z) \subset \sigma(M)$,*
- s-saturated *if for every $z \in \sigma(M)$ we have $\mathcal{W}^s_{lift}(z) \subset \sigma(M)$,*
- bisaturated *if σ is both u-and s-saturated, and*
- bi essentially saturated *if there exist sections σ^u (u-saturated) and σ^s (s-saturated) such that*

$$\sigma^u = \sigma^s = \sigma \quad \text{a.e. (volume on } M)$$

It follows from the preceding discussion that if $\sigma\colon M \to \mathcal{B}$ is a bisaturated section, then for any $x \in M$, for any accessible sequence $\mathcal{S}$, from x to x', we have $H_{\mathcal{S}}(\sigma(x)) = \sigma(x')$.

Theorem 4.2. — [**1**] *Let $f\colon M \to M$ be C^1 and partially hyperbolic, let $\pi\colon \mathcal{B} \to M$ be an admissible bundle over M, and let $\sigma\colon M \to \mathcal{B}$ be a section.*

1. *If σ is bisaturated, and f is accessible, then σ is continuous.*

2. *If f is C^2 and center bunched, and σ is bi essentially saturated, then there exists a bisaturated section σ^{su} such that $\sigma = \sigma^{su}$ a.e. (with respect to volume on M)*

Since we will use a proposition from the proof of Theorem 4.2, (1) in our later arguments, we give a sketch of the proof here, including a statement of the key proposition (Proposition 4.3 below). We remark that the proof of (2) adapts techniques from [**9**], where it is shown that if f is C^2 and center bunched, then any bi essentially saturated *subset* of M is essentially bisaturated; in effect, this is just Theorem 4.2 for the bundle $\mathscr{B} = M \times \{0,1\}$, with $\mathscr{W}^*_{\mathrm{lift}}(x,j) = \mathscr{W}^*(x) \times \{j\}$, for $j \in \{0,1\}$.

Sketch of proof of Theorem 4.2, (1). — We give a slightly modified version of the proof in [**1**], as we will need the results here in later sections. The key proposition in the proof is:

Proposition 4.3 **([1], Proposition 8.3)**. — *Suppose that f is accessible. Then for every $x_0 \in M$, there exists $w \in M$ and an accessible sequence $(y_0(w), \dots, y_k(w))$ connecting x_0 to w and satisfying the following property: for any $\varepsilon > 0$, there exist $\delta > 0$ and $L > 0$ such that, for every $z \in B_M(w, \delta)$, there exists an accessible sequence $(y_0(z), \dots y_K(z))$ connecting x_0 to z and such that*

$$d_M(y_j(z), y_j(w)) < \varepsilon \quad \textit{and} \quad d_{\mathscr{W}^*}(y_{j-1}(z), y_j(z)) < L, \quad \textit{for} \quad j = 1, \dots, K,$$

where $d_{\mathscr{W}^*}$ denotes the distance along the stable or unstable leaf common to the two points.

For $K \in \mathbb{Z}_+$ and $L \geq 0$, we say that $\mathscr{S}$ is an (K, L)-*accessible sequence* if $\mathscr{S} = (x_0, \dots, x_K)$ and

$$d_{\mathscr{W}^*}(x_{j-1}, x_j) \leq L, \quad \text{for} \quad j = 1, \dots, K,$$

where $d_{\mathscr{W}^*}$ denotes the distance along the stable or unstable leaf common to the two points.

If $\{\mathscr{S}_y = (x_0(y), \dots, x_K(y))\}_{y \in U}$ is a family of (K, L) accessible sequences in U and $x \in U$, we say that $\lim_{y \to x} \mathscr{S}_y = \mathscr{S}_x$ if

$$\lim_{y \to x} x_j(y) = x_j(x), \quad \text{for} \quad j = 0, \dots K,$$

and we say that $y \mapsto \mathscr{S}_y$ is uniformly continuous on U if $y \mapsto x_j(y)$ is uniformly continuous, for $j = 0, \dots, K$. An accessible cycle $(x_0, \dots, x_{2k} = x_0)$ is *palindromic* if $x_i = x_{2k-i}$, for $i = 1, \dots, k$. Note that a palindromic accessible cycle determines an su-path of the form $\eta \cdot \overline{\eta}$; in particular, if $\mathscr{S}$ is a palindromic accessible cycle from x to x , then $H_{\mathscr{S}}$ is the identity map on $\pi^{-1}(x)$.

The following lemma is stronger than we need for the proof of part (1) of Theorem 4.2, but will be used in later sections.

Lemma 4.4. — *Let f be accessible. There exist $K \in \mathbb{Z}_+$, $L \geq 0$ and $\delta > 0$ such that for every $x \in M$ there is a family of (K, L)-accessible sequences $\{\mathscr{S}_{x,y}\}_{y \in B_M(x,\delta)}$ such that $\mathscr{S}_{x,y}$ connects x to y, $\mathscr{S}_{x,x}$ is a palindromic accessible cycle and $\lim_{y \to x} \mathscr{S}_{x,y} = \mathscr{S}_{x,x}$. The convergence $\mathscr{S}_{x,y} \to \mathscr{S}_{x,x}$ is uniform in x.*

Proof of Lemma 4.4. — Fix an arbitrary point $x_0 \in M$. Proposition 4.3 gives a point $w \in M$, a neighborhood U_w of w, and a family of (K_0, L_0) -accessible sequences $\{(y_0(w'), \dots, y_{K_0}(w'))\}_{w' \in U_w}$ such that $(y_0(w'), \dots, y_{K_0}(w'))$ connects x_0 to w', and $(y_0(w'), \dots, y_{K_0}(w')) \to (y_0(w), \dots, y_{K_0}(w))$ uniformly in $w' \in U_w$.

***Lemma 4.5* (Accessibility implies uniform accessibility).** — *Let f be accessible. There exist constants K_M, L_M such that any two points x, x' in M can be connected by an (K_M, L_M)-accessible sequence.*

Proof of Lemma 4.5. — First note that, since any point in U_w can be connected to x_0 by an (K_0, L_0)-accessible sequence, we can connect any two points in U_w by a $(2K_0, L_0)$-accessible sequence.

Consider an arbitrary point $p \in M$ and let $(p = q_0, q_1, \dots, q_{K_p} = w)$ be an (K_p, L_p)-accessible sequence connecting p and w. Continuity of $\mathcal{W}^s$ and $\mathcal{W}^u$ implies that there is a neighborhood V_p of p and a family of (K_p, L_p)-accessible sequences $\{(p' = q_0(p'), q_1(p'), \dots, q_{K_p}(p'))\}_{p' \in V_p}$ with the property that $p' \mapsto (q_0(p'), \dots, q_{K_p}(p'))$ is uniformly continuous on V_p, and the map $p' \mapsto q_{K_p}(p')$ sends V_p into U_w and p to w. It easily follows that any two points in V_p can be connected by an $(K_0 + 2K_y, L_0 + L_y)$-accessible sequence. Covering M by neighborhoods V_p, and extracting a finite subcover, we obtain by concatenating accessible sequences that there exist constants K_M, L_M such that any two points x, x' in M can be connected by an (K_M, L_M)-accessible sequence. □

Returning to the proof of Lemma 4.4, we now fix a point $x \in M$, and let $(x = z_0, z_1, \dots, z_{K_M} = w)$ be an (K_M, L_M)-accessible sequence connecting x to w. As above, there exists a neighborhood V_x of x and a family of (K_M, L_M)-accessible sequences $\{(x' = z_0(x'), z_1(x'), \dots, z_{K_M}(x'))\}_{x' \in V_x}$ with the property that the map

$$x' \mapsto (z_0(x'), \dots, z_{K_M}(x'))$$

is uniformly continuous on V_x, and the map $x' \mapsto z_{K_M}(x')$ sends V_x into U_w and x to w.

For $x' \in V_x$, we define $\mathcal{S}_{x,x'}$ by concatenating the accessible sequences $(x = z_0(x), z_1(x), \dots, z_{K_M}(x) = w)$, $(w = y_{K_0}(w), \dots, y_0(w) = x_0)$, $(x_0 = y_0(z_{K_M}(x')), \dots, y_{K_0}(z_{K_M}(x')) = z_{K_M}(x'))$ and $(z_{K_M}(x'), \dots, z_0(x') = x')$. Then $\{\mathcal{S}_{x,x'}\}_{x' \in V_x}$ is a family of (K, L)-accessible sequences with the property that $\mathcal{S}_{x,x'}$ connects x to x', where $K = 2K_0 + 2K_M$ and $L = L_0 + L_M$.

Since $x' \mapsto (z_0(x'), \dots, z_{K_M}(x'))$ is uniformly continuous on V_x, and

$$\lim_{w' \to w} (y_0(w'), \dots, y_{K_0}(w')) = (y_0(w), \dots, y_{K_0}(w)),$$

we obtain that $\lim_{x' \to x} \mathcal{S}_{x,x'} = \mathcal{S}_{x,x}$. By construction, $\mathcal{S}_{x,x}$ is palindromic.

Finally, observe that all of the steps in this construction are uniform over x, and so we can choose $\delta > 0$ such that $B_M(x, \delta) \subset V_x$, for all x, and further, $\lim_{x' \to x} \mathcal{S}_{x,x'} = \mathcal{S}_{x,x}$ uniformly in x. This completes the proof of Lemma 4.4. □

Returning to the proof of Theorem 4.2, part (1), fix a point $x \in M$, and let $\{\mathcal{J}_{x,x'}\}_{x' \in B_M(x,\delta)}$ be the family of accessible paths given by Lemma 4.4. Since $\lim_{x' \to x} \mathcal{J}_{x,x'} = \mathcal{J}_{x,x}$ and the lifted foliations are continuous, it follows that

$$\lim_{x' \to x} H_{\mathcal{J}_{x,x'}} = H_{\mathcal{J}_{x,x}},$$

uniformly on compact sets. Since $\mathcal{J}_{x,x}$ is palindromic, we have $H_{\mathcal{J}_{x,x}} = id|_{\pi^{-1}(x)}$.

Let $\sigma \colon M \to \mathcal{B}$ be a bisaturated section. Then for any accessible sequence $\mathcal{J}$ from x to x', we have $H_{\mathcal{J}}(\sigma(x)) = \sigma(x')$. But then

$$\lim_{x' \to x} \sigma(x') = \lim_{x' \to x} H_{\mathcal{J}_{x,x'}}(\sigma(x)) = H_{\mathcal{J}_{x,x}}(\sigma(x)) = \sigma(x),$$

which shows that σ is continuous at x. □

***Proposition 4.6* (Criterion for existence of bisaturated section).** — *Let f be C^1, partially hyperbolic and accessible, and let $\pi \colon \mathcal{B} \to M$ be admissible. Let $z \in \mathcal{B}$ and let $x = \pi(z)$. Then there exists a bisaturated section $\sigma \colon M \to \mathcal{B}$ with $\sigma(x) = z$ if and only if for every su-loop γ in M with $\gamma(0) = \gamma(1) = x$, the lift $\tilde{\gamma}_z$ is an su-lift loop (with $\tilde{\gamma}_z(0) = \tilde{\gamma}_z(1) = z$).*

Proof. — We first prove the "if" part of the proposition. Define $\sigma : M \to \mathcal{B}$ as follows. We first set $\sigma(x) = z$. For each $x' \in M$, fix an su-path $\gamma \colon [0,1] \to M$ from x to x'. Since $\mathcal{B}$ is an admissible bundle, γ lifts to a path $\tilde{\gamma}_z \colon [0,1] \to \mathcal{B}$ along the leaves of $\mathcal{W}^s_{\mathrm{lift}}$ and $\mathcal{W}^u_{\mathrm{lift}}$ with $\tilde{\gamma}_z(0) = z$. We set $\sigma(x') = \tilde{\gamma}_z(1)$. Clearly $\pi\sigma(x') = x'$.

We first check that σ is well-defined. Suppose that $\gamma' \colon [0,1] \to M$ is another su-path from x to x'. Concatenating γ with $\overline{\gamma}'$, we obtain an su-loop $\gamma\overline{\gamma}'$ from x to x. By the hypotheses, the lift of $\gamma\overline{\gamma}'$ through z is an su-lift loop in $\mathcal{B}$. But this implies that $\tilde{\gamma}_z(1) = \tilde{\gamma}'_z(1)$.

The same argument shows that σ is bisaturated. Fix $y \in M$ and let $y' \in \mathcal{W}^s(y)$. We claim that $\sigma(y') \in \mathcal{W}^s_{\mathrm{lift}}(\sigma(y))$. To see this, fix two su-paths in M, one from x to y, and one from x to y'. Concatenating these paths with a path from y to y' along $\mathcal{W}^s(y)$, we obtain an su-loop γ through x. By hypothesis, the lift $\tilde{\gamma}_z$ is a lifted su-loop. It is easy to see that this means that $\sigma(y') \in \mathcal{W}^s_{\mathrm{lift}}(\sigma(y))$. Hence σ is s-saturated. Similarly, σ is u-saturated, and so σ is bisaturated.

The "only if" part of the proposition is straightforward. □

Remark: Upon careful inspection of the proofs in this subsection, one sees that the existence of foliations $\mathcal{W}^s_{\mathrm{lift}}$ and $\mathcal{W}^u_{\mathrm{lift}}$ is not an essential component of the arguments. For example, instead of assuming the existence of these foliations, one might instead assume (in the context where $\mathcal{B}$ is a smooth fiber bundle) the existence of E^u and E^s *connections* on $\mathcal{B}$, that is, the existence of subbundles E^u_ϕ and E^s_ϕ of $T\mathcal{B}$, disjoint from $\ker T\pi$, that project to E^u and E^s under $T\pi$. In this context, at least when E^u_ϕ and E^s_ϕ are smooth, there is a natural notion of a bisaturated section. In particular, for every us-path γ in M and $z \in \pi^{-1}(\gamma(0))$, there is a unique lift $\tilde{\gamma}_z$ to a path in $\mathcal{B}$, projecting to γ and everywhere tangent to E^u_ϕ or E^s_ϕ. Bisaturation of σ in this context means that for every su-path γ from x to x', one has $\tilde{\gamma}_{\sigma(x)}(1) = \sigma(x')$. The same proof as above shows that a bisaturated section in this sense is also continuous.

For this reason, [**1**] introduce the notions of *bi-continuous* and *bi-essentially continuous* sections, which extract the essential properties of a bisaturated section used in the proof of Theorem 4.2. While we have no need for this more general notion here, it is worth observing that bi-continuity might have applications in closely related contexts.

4.1. Saturated cocycles: proof of Theorem A, parts I and III. — We now translate the previous results into the context of abelian cocycles. Let $\phi : M \to \mathbb{R}$ be such a cocycle, and let $\mathcal{B} = M \times \mathbb{R}$ be the trivial bundle with fiber $\mathbb{R}$. Then $\mathcal{B}$ is an admissible bundle; we define the lifted foliations $\mathcal{W}^*_{\mathrm{lift}}$, $* \in \{s, u\}$ to be the f_ϕ-invariant foliations $\mathcal{W}^*_\phi$ given by Proposition 3.1. There is a natural identification between functions $\Phi\colon M \to \mathbb{R}$ and sections $\sigma_\Phi\colon M \to \mathcal{B}$ via $\sigma_\Phi(x) = (x, \Phi(x))$. Definition 4.1 then extends to functions $\Phi\colon M \to \mathbb{R}$ in the obvious way, where saturation is defined with respect to the $\mathcal{W}^*_\phi$-foliations.

Proposition 4.7. — *Suppose that f is partially hyperbolic and ϕ is Hölder continuous.*

1. *Assume that f is accessible, and let $\Phi\colon M \to \mathbb{R}$ be continuous. Then there exists $c \in \mathbb{R}$ such that*

$$\phi = \Phi \circ f - \Phi + c, \tag{12}$$

if and only if Φ bisaturated.

2. *If f is volume-preserving and ergodic, and $\Phi : M \to \mathbb{R}$ is a measurable function satisfying* (12) *(m-a.e.), for some $c \in \mathbb{R}$, then Φ is bi essentially saturated.*

Proof. — (1) Suppose that Φ is a continuous solution to (12). Then (12) implies that for all $x \in M$ and all n, we have:

$$f^n_\phi(x, \Phi(x)) = (f^n(x), \Phi(f^n(x)) + cn).$$

Let $x' \in \mathcal{W}^s(x)$. Then

$$\liminf_{n\to\infty} d(f^n_\phi(x, \Phi(x)), f^n_\phi(x', \Phi(x'))) =$$

$$\lim_{n\to\infty} d((f^n(x), \Phi(f^n(x))), (f^n(x'), \Phi(f^n(x')))) = 0,$$

and so $(x, \Phi(x)), (x', \Phi(x'))$ lie on the same W^s_ϕ leaf. This implies that Φ is s-saturated. Similarly Φ is u-saturated, and hence bisaturated.

Suppose on the other hand that Φ is continuous and bisaturated. Define a function $c\colon M \to \mathbb{R}$ by $c(x) = \phi(x) - \Phi(f(x)) + \Phi(x)$. We want to show that c is a constant function. Proposition 3.1, (3) implies that, for all $z \in M$ and $s, t \in \mathbb{R}$:

$$\mathcal{W}^s_\phi(z, s+t) = T_t \mathcal{W}^s_\phi(z, s). \tag{13}$$

Suppose that $y \in \mathcal{W}^s(x)$. Saturation of Φ and f_ϕ-invariance of $\mathcal{W}^s_\phi$, $\mathcal{W}^u_\phi$ imply that:

$$\mathcal{W}^s_\phi(f(x), \Phi(f(x))) = \mathcal{W}^s_\phi(f(y), \Phi(f(y))), \quad \text{and} \tag{14}$$

$$f_\phi(\mathcal{W}^s_\phi(x, \Phi(x))) = f_\phi(\mathcal{W}^s_\phi(y, \Phi(y))). \tag{15}$$

On the other hand, invariance of the $\mathcal{W}^s_\phi$-foliation under f_ϕ implies that, for all $z \in M$:

$$\begin{aligned} f_\phi(\mathcal{W}^s_\phi(z, \Phi(z))) &= \mathcal{W}^s_\phi(f(z), \Phi(z) + \phi(z)) \\ &= \mathcal{W}^s_\phi(f(z), \Phi(f(z)) + (\Phi(z) - \Phi(f(z)) + \phi(z))) \\ &= T_{\Phi(z)-\Phi(f(z))+\phi(z)}\left(\mathcal{W}^s_\phi(f(z), \Phi(f(z)))\right). \end{aligned}$$

Equations (14) and (13) now imply that

$$\Phi(x) - \Phi(f(x)) + \phi(x) = \Phi(y) - \Phi(f(y)) + \phi(y);$$

in other words, $c(x) = c(y)$. Hence the function c is constant along $\mathcal{W}^s$-leaves; similarly, c is constant along $\mathcal{W}^u$-leaves. Accessibility implies that c is constant. Hence Φ and c satisfy (2).

(2) Let Φ be a measurable solution to (12). We may assume that (12) holds on an f-invariant set of full volume; for points in this set, we have

$$f^n_\phi(x, \Phi(x)) = (f^n(x), \Phi(f^n(x)) + cn),$$

for all n.

Choose a compact set $C \subset M$ such that $\mathrm{vol}(C) > .5\mathrm{vol}(M)$, on which Φ is uniformly continuous. Ergodicity of f and absolute continuity of $\mathcal{W}^s$ implies that for almost every $x \in M$, and almost every $x' \in \mathcal{W}^s(x)$, the pair of points x and x' will visit C simultaneously for a positive density set of times. For such a pair of points x, x' we have

$$\liminf_{n\to\infty} d(f^n_\phi(x, \Phi(x)), f^n_\phi(x', \Phi(x'))) =$$
$$\liminf_{n\to\infty} d((f^n(x), \Phi(f^n(x))), (f^n(x'), \Phi(f^n(x')))) = 0,$$

and so $(x, \Phi(x)), (x', \Phi(x'))$ lie on the same W^s_ϕ leaf. This implies that Φ is essentially s-saturated: one defines the s-saturate Φ^s of Φ at (almost every) x to be equal to the almost everywhere constant value of Φ on $\mathcal{W}^s(x)$ (see **[29]** for a version of this argument when f is Anosov).

Similarly Φ is essentially u-saturated, and hence bi essentially saturated. □

Proof of Theorem A, part I. — Let f be C^1 and accessible and let $\phi : M \to \mathbb{R}$ be Hölder continuous. Part I of Theorem A asserts that there exists a continuous function $\Phi\colon M \to \mathbb{R}$ and $c \in \mathbb{R}$ satisfying (2) if and only if $PCF_{\mathcal{C}}(\phi) = 0$, for every accessible cycle $\mathcal{C}$.

We start with a lemma:

Lemma 4.8. — *Let γ be an su-loop corresponding to the accessible cycle $\mathcal{C}$. Then $PCF_{\mathcal{C}}(\phi) = 0$ if and only if* every *lift of γ to an su-lift path in $M \times \mathbb{R}$ is an su-lift loop.*

Proof of Lemma 4.8. — Let $x \in M$ Proposition 3.1, part (4) implies that if $\mathscr{C} = (x_0, \ldots, x_k = x_0)$ is an accessible cycle, then for any $t \in \mathbb{R}$

$$H_{\mathscr{C}}(t) - t = \sum_{i=0}^{k-1} PCF_{(x_i, x_{i+1})}(\phi) = PCF_{\mathscr{C}}(\phi)$$

Let γ be an su-loop corresponding to $\mathscr{C}$. Then for any $t \in \mathbb{R}$, $H_\gamma(t) - t = PCF_{\mathscr{C}}(\phi)$

Fix $t \in \mathbb{R}$, and let $\tilde{\gamma}_t = \tilde{\gamma}_{x_0,t} \colon [0,1] \to M \times \mathbb{R}$ be the su-lift path projecting to γ, with $\tilde{\gamma}_t(0) = (x_0, t)$. Then $\tilde{\gamma}_t(1) = (x_0, H_\gamma(t)) = (x_0, t + PCF_{\mathscr{C}}(\phi) = 0)$. Thus $PCF_{\mathscr{C}}(\phi) = 0$ if and only if $\tilde{\gamma}_t(1) = t$ if and only if $\tilde{\gamma}_t$ is an su-lift loop. Since t was arbitrary, we obtain that $PCF_{\mathscr{C}}(\phi) = 0$ if and only if every lift of γ to an su-lift path is an su-lift loop. □

By Proposition 4.6 and Lemma 4.8, if $PCF_{\mathscr{C}}(\phi) = 0$, for every accessible cycle $\mathscr{C}$, then there exists a bisaturated function $\Phi : M \to M \times R$. Theorem 4.2, part (1), plus accessibility of f implies that Φ is continuous. Proposition 4.7 implies that there exists a $c \in \mathbb{R}$ such that (12) holds.

On the other hand, if Φ is continuous and there exists a $c \in \mathbb{R}$ such that (12) holds, then Proposition 4.7, (part 1) implies that Φ is bisaturated. Proposition 4.6 and Lemma 4.8 imply that $PCF_{\mathscr{C}}(\phi) = 0$, for every accessible cycle $\mathscr{C}$. □

Proof of Part III of Theorem B. — Assume that f is C^2, volume-preserving, center bunched and accessible. Let Φ be a measurable solution to (2), for some $c \in \mathbb{R}$. We prove that there exists a continuous function $\hat{\Phi}$ satisfying $\Phi = \hat{\Phi}$ almost everywhere.

Since f is center bunched and accessible, it is ergodic, by ([**9**], Theorem 0.1). Proposition 4.7, part (2) implies that Φ is bi essentially saturated. Theorem 4.2, part (2) then implies that Φ is essentially bisaturated, which means there exists a bisaturated function $\hat{\Phi}$, with $\hat{\Phi} = \Phi$ a.e. Since f is accessible, Theorem 4.2, part (1) then implies that $\hat{\Phi}$ is continuous. □

5. Hölder regularity: proof of Theorem A, part II.

Let $f \colon M \to M$ be partially hyperbolic and let $\phi \colon M \to \mathbb{R}$ be α-Hölder continuous, for some $\alpha > 0$. As above, define the skew product $f_\phi \colon M \times \mathbb{R} \to M \times \mathbb{R}$ by

$$f_\phi(p, t) = (f(p), t + \phi(p)).$$

We start with a standard proposition showing that the stable and unstable foliations for f lift to invariant stable and unstable foliations for f_ϕ.

Proposition 5.1. — *There exist foliations $\mathscr{W}^u_\phi$, $\mathscr{W}^s_\phi$ of $M \times \mathbb{R}$ with the following properties.*

1. *The leaves of $\mathscr{W}^u_\phi$, $\mathscr{W}^s_\phi$ are α-Hölder continuous.*

2. *The leaves of* $\mathcal{W}^u_\phi$ *project to leaves of* $\mathcal{W}^u$*, and the leaves of* $\mathcal{W}^s_\phi$ *project to leaves of* $\mathcal{W}^s$*. Moreover,* $(x',t') \in \mathcal{W}^s_\phi(x,t)$ *if and only if* $x' \in \mathcal{W}^s(x)$ *and*

$$\liminf_{n\to\infty} d(f^n_\phi(x,t), f^n_\phi(x',t')) = 0.$$

Proof. — This result is by now standard (see **[31]**), although strictly speaking, the proof appears in the literature only under a stronger partial hyperbolicity assumption (in which the functions $\nu, \hat\nu, \gamma, \hat\gamma$ are assumed to be constant). We sketch the proof under the slightly weaker hypotheses stated here.

For $x \in M$, let $\mathcal{G}_x = \{g\colon \mathcal{W}^u(x,\delta) \to \mathbb{R} : g \in C^\alpha, g(x) = 0\}$. The number $\delta > 0$ is chosen so that for all $x \in M$, if $y \in \mathcal{W}^u(x,\delta)$, then $d(f(x), f(y)) \geq \hat\nu(x)^{-1} d(x,y)$. Notice that the function $\psi(y) = \phi(y) - \phi(x)$ belongs to $\mathcal{G}_x$. The α-norm of an element $g \in \mathcal{G}_x$ is defined:

$$\|g\|_\alpha = \sup_{y\in\mathcal{W}^u(x,\delta)} \frac{|g(y)|}{d(x,y)^\alpha}.$$

The bundle $\mathcal{G}$ over M with fiber $\mathcal{G}_x$ over $x \in M$ has the structure of a Banach bundle. The fiber is modelled on the Banach space $B = \{g\colon B_{\mathbb{R}^u}(0,\delta) \to \mathbb{R} : g \in C^\alpha, g(0) = 0\}$, with the norm

$$\|g\|_\alpha = \sup_{v\in B_{\mathbb{R}^u}(0,\delta)} \frac{|g(v)|}{|v|^\alpha}.$$

The restriction of f to $\mathcal{W}^u$-leaves sends $\mathcal{W}^u(x,\delta)$ onto $\mathcal{W}^u(f(x), \hat\nu(x)^{-1}\delta)$, which contains $\mathcal{W}^u(f(x),\delta)$. On $\mathcal{W}^u(x) \times \mathbb{R}$, the map f_ϕ takes the form $f_\phi(p,t) = (f(p), t + \phi(p))$, and the induced graph transform map $\mathcal{T}_x : \mathcal{G}_x \to \mathcal{G}_{f(x)}$ takes the form: $\mathcal{T}_x(g)(y) = g(f^{-1}(y)) + \phi(f^{-1}(y)) - \phi(f^{-1}(x))$.

Suppose that $\|g\|_\alpha \leq C$. Then

$$\begin{aligned}
|\mathcal{T}_x(g)|_\alpha &= \sup_{z\in\mathcal{W}^u(f(x),\delta)} \frac{|\mathcal{T}_x(g)(z)|}{d(f(x),z)^\alpha} \\
&\leq \sup_{y\in\mathcal{W}^u(x,\delta)} \frac{|g(y) + \phi(y) - \phi(x)|}{d(f(x),f(y))^\alpha} \\
&= \sup_{y\in\mathcal{W}^u(x,\delta)} \frac{|g(y)|}{d(f(x),f(y))^\alpha} + \frac{|\phi(x) - \phi(y))|}{d(f(x),f(y))^\alpha} \\
&\leq \hat\nu(x)^\alpha \left(\sup_{y\in\mathcal{W}^u_\delta(x)} \frac{|g(y)|}{d(x,y)^\alpha} + \frac{|\phi(x) - \phi(y))|}{d(x,y)^\alpha} \right) \\
&\leq \hat\nu(x)^\alpha \left(\|g\|_\alpha + |\phi - \phi(x)|_\alpha \right) \\
&\leq \hat\nu(x)^\alpha (C + K) \leq C,
\end{aligned}$$

provided that C is larger than $\sup_x K/(1 - \hat\nu(x))$.

Hence the closed sets $\mathcal{G}_x(C) = \{g \in \mathcal{G}_x : \|g\|_\alpha \leq C\}$ are preserved by the maps $\mathcal{T}_x$. Next we show that $\mathcal{T}_x$ is a contraction in the α-norm. To this end, let $g, g' \in \mathcal{G}_x(C)$.

Then

$$\begin{aligned}
\|\mathcal{T}_x(g) - \mathcal{T}_x(g')\|_\alpha &= \sup_{z\in\mathcal{W}^u(f(x),\delta)} \frac{|\mathcal{T}_x(g)(z) - \mathcal{T}_x(g')(z)|}{d(f(x),z)^\alpha} \\
&\leq \sup_{y\in\mathcal{W}^u(x,\delta)} \frac{|g(y)+\phi(y)-\phi(x)-(g'(y)+\phi(y)-\phi(x))|}{d(f(x),f(y))^\alpha} \\
&= \sup_{y\in\mathcal{W}^u(x,\delta)} \frac{|g(y)-g'(y)|}{d(f(x),f(y))^\alpha} \\
&\leq \hat{\nu}(x)^\alpha \|g-g'\|_\alpha.
\end{aligned}$$

The invariant section theorem ([**18**], Theorem 3.1) now implies that there is a unique $\mathcal{T}$-invariant section $\sigma : M \to \mathcal{G}_x(C)$. It is easy to check that the set $\widehat{\mathcal{W}}^u_\phi(p,t) = \{(y, t+\sigma_p(y)) : y \in \mathcal{W}^u(p,\delta)\}$ is a local unstable manifold for f_ϕ. The rest of the proof is standard. □

Fix a foliation box U for $\mathcal{W}^s$. For any two smooth transversals Σ, Σ' in U, there is the $\mathcal{W}^s$-holonomy map from Σ to Σ' that sends $x \in \Sigma$ to the unique point of intersection x' between $\mathcal{W}^s(x)$ and Σ'. For any such Σ, Σ' there is also a well-defined $\mathcal{W}^s_\phi$-holonomy between $\Sigma\times\mathbb{R}$ and $\Sigma'\times\mathbb{R}$, sending $(x,t)\in\Sigma\times\mathbb{R}$ to the unique point of intersection (x',t') between $\mathcal{W}^s_\phi(x,t)$ and $\Sigma'\times\mathbb{R}$. Since the $\mathcal{W}^s$ leaves lift to $\mathcal{W}^s_\phi$-leaves, the $\mathcal{W}^s_\phi$ holonomy covers the $\mathcal{W}^s$ holonomy under the natural projection.

Proposition 5.2. — *Suppose that f is C^1 and ϕ is α-Hölder continuous, for some $\alpha \in (0,1]$. Then the $\mathcal{W}^s_\phi$ and $\mathcal{W}^u_\phi$ holonomy maps are uniformly Hölder continuous. Any $\theta\in(0,\alpha]$ satisfying the pointwise inequalities:*

$$\nu < (\nu\hat{\mu})^{\theta/\alpha} \quad \textit{and} \quad \nu\gamma^{-1} < (\nu\hat{\mu})^{\theta/\alpha} \tag{16}$$

is a Hölder exponent for the $\mathcal{W}^s_\phi$ holonomy, where $\nu,\gamma,\hat{\mu} : M\to\mathbb{R}$ are any continuous functions satisfying, for every $p\in M$ and any unit vector $v\in T_pM$:

$$v\in E^s(p) \Rightarrow \|T_pfv\| < \nu(p), \quad v\in E^c(p) \Rightarrow \gamma(p) < \|T_pfv\|,$$

and

$$v\in E^u(p) \Rightarrow \|T_pfv\| \leq \hat{\mu}(p)^{-1},$$

for some Riemannian metric.

By considering the trivial (constant) cocycle, we also obtain:

Corollary 5.3. — *The stable holonomy maps for a C^1 partially hyperbolic diffeomorphism f are uniformly Hölder continuous. Any $\theta\in(0,1]$ satisfying*

$$\nu < \gamma(\nu\hat{\mu})^\theta$$

is a Hölder exponent for the stable holonomy, where $\nu,\gamma,\hat{\mu}$ are defined as in Proposition 5.2.

Remark: In ([**33**], Theorem A) it is shown that the holonomy maps for $\mathcal{W}^u$ and $\mathcal{W}^s$ are Hölder continuous if f is at least C^2 (or $C^{1+\alpha}$, for some $\alpha > 0$). The proof in [**33**] uses a graph transform argument and an invariant section theorem to show that the plaques of $\mathcal{W}^u$ and $\mathcal{W}^s$ form a Hölder continuous family. Here in the proof of Proposition 3.1, as in the first part of the proof in [**33**], we have exhibited the plaques of $\mathcal{W}^u_\phi$ as an invariant section of a fiber-contracting bundle map $\mathcal{T}$. It is not possible, however, to carry over the rest of the proof in [**33**] to this setting: the low regularity of $\mathcal{T}$ prevents one from using a Hölder section theorem to conclude that the invariant section is Hölder continuous.

Hence we employ a different approach to prove that the holonomy maps are Hölder continuous. The proof here has some similarities with the proof that stable foliations are absolutely continuous. We fix two transversals τ and τ' to $\mathcal{W}^s_\phi$ and a pair of points $x, y \in \tau$. We iterate the picture forward until $f^n_\phi(\tau)$ and $f^n_\phi(\tau')$ are very close and then push $f^n_\phi(x)$ and $f^n_\phi(y)$ across a short distance to points $f^n_\phi(x'), f^n_\phi(y') \in f^n_\phi(\tau')$. The points x', y' are the images of x, y under $\mathcal{W}^s_\phi$-holonomy; the iterate n is chosen carefully so that the distance between x and y can be compared to some power of the distance between x' and y'. Unlike the proof of absolute continuity of stable foliations, in which n is chosen arbitrarily large, the choice of n is delicate and depends on the distance between x and y. We will employ this type of argument again in later sections.

As a final remark, we note that for every partially hyperbolic diffeomorphism f and every Hölder continuous cocycle ϕ, there is a choice of $\theta > 0$ satisfying (16), for some Riemannian metric.

Proof of Proposition 5.2.. — In this proof, we will use the convention that if q is a point in M and j is an integer, then q_j denotes the point $f^j(q)$, with $q_0 = q$. If $\alpha : M \to \mathbf{R}$ is a positive function, and $j \geq 1$ is an integer, we set

$$\alpha_j(p) = \alpha(p)\alpha(p_1)\cdots\alpha(p_{j-1}),$$

and

$$\alpha_{-j}(p) = \alpha(p_{-j})^{-1}\alpha(p_{-j+1})^{-1}\cdots\alpha(p_{-1})^{-1}.$$

We set $\alpha_0(p) = 1$. Observe that α_j is a multiplicative cocycle; in particular, we have $\alpha_{-j}(p)^{-1} = \alpha_j(p_{-j})$. Note also that $(\alpha\beta)_j = \alpha_j\beta_j$, and if α is a constant function, then $\alpha_n = \alpha^n$.

Fix $\theta \in (0, \alpha]$ satisfying (16). Next, fix a continuous positive function $\rho\colon M \to \mathbb{R}_+$ satisfying:

- $\rho < \min\{1, \gamma\}$, and
- $\nu\rho^{-1} \leq (\nu\hat{\mu}^{-1})^{\theta/\alpha}$.

We say that a smooth transversal Σ to $\mathcal{W}^s$ is *admissible* if the angle between $T\Sigma$ and E^s is at least $\pi/4$.

The next lemma follows from an elementary inductive argument and continuity of the functions ν, $\hat{\mu}$ and ρ (cf. [**9**], Lemma 1.1).

Lemma 5.4. — *There exists $\delta_0 > 0$ such that for any $p \in M$, and for any $p' \in \mathcal{W}^s(p, \delta_0)$:*

1. *for any $i \geq 0$,*
$$d(p_i, p'_i) \leq \nu_i(p) d(p, p');$$

2. *for any admissible transversal Σ' to $\mathcal{W}^s$ at p', and any point $q' \in \Sigma'$, if $d(p'_i, q'_i) < \delta_0$, for $i = 1, \dots, n$, then*
$$\rho_i(p) d(p', q') \leq d(p'_i, q'_i) \leq \hat{\mu}_i(p)^{-1} d(p', q'),$$
for $i = 1, \dots, n$.

Let $\delta_0 > 0$ be given by this lemma; by rescaling the metric, we may assume that $\delta_0 = 1$. Fix $p \in M$ and $p' \in \mathcal{W}^s(p, 1)$. Let Σ and Σ' be admissible transversals to $\mathcal{W}^s$, with $p \in \Sigma$ and $p' \in \Sigma'$, so that the $\mathcal{W}^s$-holonomy $h^s : \Sigma \to \Sigma'$, with $h^s(p) = p'$ is well-defined. Let $\tau = \Sigma \times \mathbb{R}$, and let $\tau' = \Sigma' \times \mathbb{R}$. Fix $q \in \Sigma$ with $d(p, q) < 1$, and let $q' = h^s(q)$.

For $(z, t) \in M \times \mathbb{R}$ and $n \geq 0$, write (z_n, t_n) for $f_\phi^n(z, t)$. We introduce the notation $S_n\phi(z) = \sum_{i=0}^{n-1} \phi(z_i)$, and note that $S_1\phi(z) = \phi(z)$. With these notations, we have $(z_n, t_n) = (z_n, t + S_n\phi(z))$. Denote by $h_\phi^s \colon \Sigma \times \mathbb{R} \to \Sigma \times \mathbb{R}$ the $\mathcal{W}_\phi^s$-holonomy, which covers the map h^s. We first establish Hölder continuity of the base holonomy map $h^s \colon \Sigma \to \Sigma'$.

Since $\nu < \hat{\mu}^{-1}$, there exists an n so that $d(p, q) = \Theta(\nu_n(p)\hat{\mu}_n(p))$; fix such an n. Lemma 5.4 applied in the transversal Σ implies that $d(p_i, q_i) \leq \hat{\mu}_i(p)^{-1} d(p, q) \leq O(\nu_n(p))$, for $i = 1, \dots, n$.

On the other hand, since $p' \in \mathcal{W}^s(p, 1)$, we have $d(p_i, p'_i) \leq O(\nu_i)$, for all i; in particular, $d(p_n, p'_n) \leq O(\nu_n)$. Similarly, $d(q_n, q'_n) \leq O(\nu_n)$. By the triangle inequality, we have that

$$\begin{aligned} d(p'_n, q'_n) &\leq d(p_n, q_n) + d(p_n, p'_n) + d(q_n, q'_n) \\ &= O(\nu_n(p)). \end{aligned}$$

Now applying f^{-n} to the pair of points p'_n, q'_n we obtain the pair of points p', q', which lie in the admissible transversal Σ'. Lemma 5.4 then implies that $d(p', q') \leq \rho_n(p)^{-1} d(p'_n, q'_n) \leq O(\rho_n(p)^{-1}\nu_n(p))$. Since $\rho_n(p)^{-1}\nu_n(p) < (\nu_n(p)\hat{\mu}_n(p))^{\theta/\alpha} = O(d(p, q)^{\theta/\alpha})$, we obtain that $d(p', q') \leq O(d(p, q)^{\theta/\alpha}) \leq O(d(p, q)^\theta)$, and so h^s is θ-Hölder continuous.

We next turn to the Hölder continuity of h_ϕ^s. Since h_ϕ^s covers h^s, it suffices to establish Hölder continuity in the $\mathbb{R}$-fiber. Fix a point $(p, r) \in \Sigma \times \mathbb{R}$ and write $h_\phi^s(p, r) = (p', r')$ and $h_\phi^s(q, s) = (q', s')$.

Hölder continuity of ϕ with exponent α implies that

$$\begin{aligned} |S_n\phi(p) - S_n\phi(q)| &\leq \sum_{i=0}^{n-1} O(d(p_i, q_i)^\alpha) \\ &\leq \sum_{i=0}^{n-1} O((\nu_n(p)\hat{\mu}_n(p)\hat{\mu}_i(p)^{-1})^\alpha) \\ &= \nu_n(p)^\alpha \sum_{i=0}^{n-1} O(\hat{\mu}_{-i}(p_n)^{-\alpha}) \\ &\leq \nu_n(p)^\alpha \sum_{i=0}^{n-1} O(\overline{\mu}^{i\alpha}) = O(\nu_n(p)^\alpha) \end{aligned}$$

where $\overline{\mu} < 1$ is an upper bound for $\hat{\mu}$. This means that $|r_n - s_n| \leq |r - s| + O(\nu_n(p)^\alpha)$.

Note that $(p'_n, r'_n) \in \mathcal{W}^s_\phi(p_n, r_n)$. Proposition 3.1 implies that $\mathcal{W}^s_\phi(p_n, r_n)$ is the graph of an α-Hölder continuous function from $\mathcal{W}^s(p_n)$ to $\mathbb{R}$. Hence

$$|r_n - r'_n| \leq O(d(p_n, p'_n)^\alpha) = O(\nu_n(p)^\alpha),$$

and similarly, $|s_n - s'_n| = O(\nu_n(p)^\alpha)$. Now, by the triangle inequality,

$$\begin{aligned} |r'_n - s'_n| &\leq |r_n - s_n| + |r_n - r'_n| + |s_n - s'_n| && (17) \\ &\leq |r - s| + O(\nu_n(p)^\alpha); && (18) \end{aligned}$$

Since $d(p'_{n-i}, q'_{n-i}) \leq O(\nu_n(p)\rho_{-i}(p_n))$, for $i = 1, \dots n$, the α-Hölder continuity of ϕ implies that $|S_n\phi(p')) - S_n\phi(q')| \leq \sum_{i=1}^n O((\nu_n(p)\rho_{-i}(p_n))^\alpha) = O((\nu_n(p)\rho_n(p)^{-1})^\alpha)$, since $\rho < 1$. The inequality $(\nu\rho^{-1})^\alpha < (\nu\hat{\mu})^\theta$ now implies that

$$|S_n\phi(p')) - S_n\phi(q')| \leq O((\nu_n(p)\hat{\mu}_n(p))^\theta). \tag{19}$$

Combining (17) and (19), we obtain:

$$\begin{aligned} |r' - s'| &= |(r'_n - s'_n) - (S_n\phi(p')) - S_n\phi(q'))| \\ &\leq |r - s| + O(\nu_n(p)^\alpha) + O((\nu_n(p)\hat{\mu}_n(p))^\theta) \\ &\leq |r - s| + O((\nu_n(p)\hat{\mu}_n(p))^\theta), \end{aligned}$$

since $\nu^\alpha < (\nu\hat{\mu})^\theta$.

We would like to compare $|r' - s'|$ to $d((p, r), (q, s))^\theta$; the latter quantity is equal to $(|r - s| + d(p, q))^\theta = (|r - s| + \Theta((\nu_n(p)\hat{\mu}_n(p))^\theta)$; by the preceding calculation, $|r' - s'| \leq O(d((p, r), (q, s))^\theta)$. Hence h^s_ϕ is θ-Hölder continuous. □

Having completed this preliminary step, we turn to the proof of the main result in this section.

Proof of Theorem A, part II. — Suppose that f is accessible and $\phi\colon M \to \mathbb{R}$ is Hölder continuous. Let $\Phi\colon M \to \mathbb{R}$ be a continuous map satisfying $\phi = \Phi \circ f - \Phi + c$, for some $c \in \mathbb{R}$. We show that Φ is Hölder continuous. The key ingredient in the proof is the following lemma.

Lemma 5.5. — *There exist $C > 0$, $r_0 > 0$ and $\kappa \in (0,1)$ with the following properties.*

For any pair of points $p, q \in M$, there exist functions $\alpha\colon B_M(p, r_0) \to B_M(q,1)$ and $\beta\colon B_M(p,r_0) \to \mathbb{R}$ with the following properties:

1. *$\alpha(p) = q$*
2. *for all $z, z' \in B_M(p,r_0)$,*
$$d(\alpha(z), \alpha(z')) \leq Cd(z,z')^\kappa,$$
and
$$|\beta(z) - \beta(z')| \leq Cd(z,z')^\kappa,$$
3. *for all $z \in B_M(p,r_0)$, $\alpha(z)$ is the endpoint of an su-path in M originating at z,*
4. *for all $z \in B_M(p,r_0)$, and $t \in \mathbb{R}$, $\Delta(z,t)$ is the endpoint of an su-lift path in $M \times \mathbb{R}$ originating at (z,t), where $\Delta\colon B_M(p,r_0) \times \mathbb{R} \to B_M(q,1) \times \mathbb{R}$ is the map $\Delta(z,t) = (\alpha(z), t + \beta(z))$.*

Assuming this lemma, the proof proceeds as follows. Let C, r_0, κ be given by Lemma 5.5. Fix $x_0, x_1 \in M$ with $d(x_0,x_1) < r_0$. For $i \geq 1$, we construct a sequence of points x_i and maps $\alpha_i\colon B_M(x_0,r_0) \to B_M(x_i,1)$, $\beta_i\colon B_M(x_0,r_0) \to \mathbb{R}$ and $\Delta_i\colon B_M(x_0,r_0) \times \mathbb{R} \to B_M(x_i,1) \times \mathbb{R}$ inductively as follows. The point x_1 is already defined. Assume that x_i, for $i \geq 1$ has been defined. Let α_i and β_i be given by the lemma, setting $p = x_0$ and $q = x_i$ (so that $h(x_0) = x_i$). Define Δ_i, as in Lemma 5.5, by $\Delta_i(z,t) = (\alpha_i(z), t + \beta_i(z))$. We then set $x_{i+1} = \alpha_i(x_1)$.

We next argue that, for any $i \geq 1$, the map Δ_i has the property that, for all $z \in B_M(x_0, r_0)$,
$$\Delta_i(z, \Phi(z)) = (\alpha(z), \Phi(z) + \beta_i(z)) = (\alpha(z), \Phi(\alpha(z))).$$
Since Φ is a continuous solution to (2), Proposition 4.7 implies then the graph of Φ is bisaturated. That is, for any $p, q \in M$, if (q,t) is the endpoint of any su-lift path originating at $(p, \Phi(p))$, then $t = \Phi(q)$. But properties 3 and 4 of the maps Δ_i given by Lemma 5.5 imply that $\alpha_i(z)$ is the endpoint of an su-path originating at z, and $\Delta_i(z,\Phi(z))$ is the endpoint of an su-lift path originating at $(z, \Phi(z))$. Hence we obtain that $\Delta_i(z, \Phi(z)) = (\alpha_i(z), \Phi(\alpha_i(z)))$, as claimed.

It now follows from the properties of Δ_i and the definition of x_i that, for $i \geq 1$:
$$\Phi(x_0) + \beta_i(x_0) = \Phi(\alpha_i(x_0)) = \Phi(x_i),$$
and
$$\Phi(x_1) + \beta_i(x_1) = \Phi(\alpha_i(x_1)) = \Phi(x_{i+1}).$$
Thus:
$$\Phi(x_1) - \Phi(x_0) \quad = \quad (\Phi(x_{i+1}) - \Phi(x_i)) + (\beta_i(x_0) - \beta_i(x_1)). \tag{20}$$
Summing equation (20) over $i \in \{1, \dots, n\}$, we obtain:
$$n\,(\Phi(x_1) - \Phi(x_0)) \quad = \quad (\Phi(x_{n+1}) - \Phi(x_1)) + \sum_{i=1}^{n} (\beta_i(x_0) - \beta_i(x_1)),$$

and so:

$$\begin{aligned}|\Phi(x_1) - \Phi(x_0)| &\leq \frac{1}{n}\,|\Phi(x_{n+1}) - \Phi(x_1)| + \frac{1}{n}\sum_{i=1}^{n}|\beta_i(x_0) - \beta_i(x_1)| \\ &\leq \frac{2}{n}\|\Phi\|_\infty + \frac{1}{n}\sum_{i=1}^{n} Cd(x_0,x_1)^\kappa \\ &\leq \frac{2}{n}\|\Phi\|_\infty + Cd(x_0,x_1)^\kappa.\end{aligned}$$

Sending $n \to \infty$, we obtain that $|\Phi(x_1) - \Phi(x_0)| \leq Cd(x_0,x_1)^\kappa$; since x_0 and x_1 were arbitrary points within distance r_0 of each other, this implies that Φ is κ-Hölder continuous. This completes the proof of Proposition 5.2, assuming Lemma 5.5. □

Proof of Lemma 5.5. — Let θ be given by Proposition 5.2, and let N_M, L_M be given by Lemma 4.5.

We first describe how to construct the maps α and β in the case where $q \in \mathcal{W}^s(p, L_M)$. The analogous construction works for $q \in \mathcal{W}^u(p, L_M)$. Lemma 4.5 implies that any p and q can be connected by an (K_M, L_M)-accessible sequence. We can therefore construct α, β for a general pair of points p and q by composing at most K_M maps along stable and unstable segments.

Suppose then that $p' \in \mathcal{W}^s(p, L_M)$. We define $\alpha = \alpha_{p,p'}$ as follows. Fix a foliation box U of $\mathcal{W}^s$ containing $\mathcal{W}^s(p, L_M)$, and let $\{\Sigma_x\}_{x\in U}$ be a (uniformly chosen) smooth foliation by admissible transversals to $\mathcal{W}^s$ in U. For $z \in U$, we define $\alpha_{p,p'}(z)$ to be the unique point of intersection of $\mathcal{W}^s(z, L_M)$ with $\Sigma_{p'}$ in U. The map $\alpha_{p,p'} \colon U \to \Sigma_{p'}$ sends p to p' and is θ-Hölder continuous when restricted to any transversal Σ_x. Since $\{\Sigma_x\}_{x\in\mathcal{W}^s(p)}$ is a smooth foliation, it follows that $\alpha_{p,p'}$ is θ-Hölder continuous, uniformly in $p' \in U$.

Similarly, for $(z,t) \in U \times \mathbb{R}$, we define $\Delta_{p,p'}(z,t)$ to be the unique point of intersection of $\mathcal{W}^s_\phi(z)$ with $\Sigma_{p'} \times \mathbb{R}$ in $U \times \mathbb{R}$. Proposition 3.1 implies that $\Delta_{p,p'}$ takes the form

$$\Delta_{p,p'}(z,t) = (\alpha_{p,p'}(z), t + \beta_{p,p'}(z)),$$

for some function $\beta_{p,p'} \colon U \to \mathbb{R}$. Proposition 5.2 implies that $\Delta_{p,p'}$, and so $\beta_{p,p'}$, is θ-Hölder continuous, uniformly in $p' \in U$.

The same construction defines $\alpha_{p,p'}$ and $\beta_{p,p'}$ for $p' \in \mathcal{W}^u(p, K_M)$. Finally, for p, q in M, we fix an (K_M, L_M)-accessible sequence $(y_0, y_1, \ldots, y_{K_M})$ connecting p and q and define

$$\alpha_{p,q} = \alpha_{y_{K_M-1},y_{K_M}} \circ \alpha_{y_{K_M-2},y_{K_M-1}} \circ \cdots \circ \alpha_{y_0,y_1}.$$

By construction, $\alpha_{p,q}(p) = q$. Similarly define $\beta_{p,q}$.

Then there exists $r_0 > 0$ such that for every pair p, q, $\alpha_{p,q}$ and $\beta_{p,q}$ are defined in the neighborhood $B_M(p, r_0)$ and $\alpha_{p,q}$ takes values in $B_M(q, 1)$. Furthermore, there exists $C > 0$ such that (1) and (2) in the statement of the lemma hold, for $\kappa = \theta^{K_M}$. Finally, property (4) holds by construction. □

Remark: The Hölder exponent for Φ obtained in this proof can be considerably smaller than the exponent for ϕ. In particular, the largest possible exponent for the $\mathcal{W}^s_\phi$ or $\mathcal{W}^u_\phi$ holonomy given by Proposition 5.2 is $\frac{1}{2}$. Concatenating these holonomies along K steps of an accessible sequence reduces this exponent further to $\frac{1}{2^K}$. In contrast, the exponents for Φ and ϕ in Theorem 0.1 are the same. This is because the transverse Hölder continuity of $\mathcal{W}^s_\phi$ and $\mathcal{W}^u_\phi$ does not play a role in the proof when f is Anosov, and so only the Hölder exponent of the leaves, which is the same as for ϕ, determines the exponent for Φ.

6. Jets

In this section we review basic facts about jets and jet bundles that will be needed in subsequent sections. The reader is referred to **[17, 22]** for a more detailed account.

If N_1 and N_2 are C^k manifolds and $\ell \leq k$, we denote by $\Gamma^\ell(N_1, N_2)$ the set of local C^k maps from N_1, N_2; each element of $\Gamma^\ell(N_1, N_2)$ is a triple (p, ϕ, U), where ϕ is a C^ℓ map from a neighborhood U of p in N_1 to N_2. For $p \in N_1$, we denote by $\Gamma^\ell_p(N_1, N_2)$ the set of elements of $\Gamma^\ell(N_1, N_2)$ based at p. We denote by $J^\ell(N_1, N_2)$ the bundle of C^ℓ jets from N_1 into N_2: each element of $J^\ell(N_1, N_2)$ is an equivalence class of triples $(p, \phi, U) \in \Gamma^\ell_p(N_1, N_2)$, where two triples (p, ϕ, U) and (p', ϕ', U') are equivalent if $p = p'$, and the partials of ϕ and ϕ' at p up to order ℓ coincide.

We denote by $[p, \phi, U]_\ell$ the equivalence class containing (p, ϕ, U), which is called a *ℓ-jet at p*. Alternately, we use the notation $j^\ell_p\phi$. The point p is called the *source* of (p, ϕ, U) and $\phi(p)$ is the *target*. The source map σ gives $J^\ell(N_1, N_2)$ the structure of a $C^{k-\ell}$ bundle over N_1; we denote by $J^\ell_p(N_1, N_2)$ the ℓ-jets with source $p \in N_1$. We also denote by $J^\ell(N_1, N_2)_q$ the set of jets with target q.

More generally one has the ℓ-jet bundle associated to a fiber bundle. If $\pi\colon \mathcal{B} \to M$ is a C^k fiber bundle, and $\ell \leq k$, we denote by $\Gamma^\ell(\pi\colon \mathcal{B} \to M)$ the set of C^ℓ local sections of $\mathcal{B}$, and by $\Gamma^\ell_p(\pi\colon \mathcal{B} \to M)$ the set of C^ℓ local sections whose domain contains $p \in M$. We then define the ℓ-jet bundle $J^\ell(\pi\colon \mathcal{B} \to M)$ to be the set of pairs (p, ϕ), where $\phi \in \Gamma^r_p(\pi\colon \mathcal{B} \to M)$, and two pairs (p, ϕ) and (p', ϕ') are equivalent if $p = p'$, and the partials of ϕ and ϕ' at p up to order ℓ coincide. Then $J^\ell(\pi\colon \mathcal{B} \to M)$ is a $C^{k-\ell}$ bundle over M. Observe that $J^\ell(N_1, N_2) = J^\ell(\mathrm{proj}_{N_1} : N_1 \times N_2 \to N_1)$ under the natural identification of sections of $N_1 \times N_2$ with functions $\phi : N_1 \to N_2$.

For $\ell' \leq \ell$, there is a natural projection $\pi_{\ell,\ell'}$ from the ℓ-jet bundle to the ℓ'-jet bundle that sends $j^\ell_p\phi$ to $j^{\ell'}_p\phi$. Under this projection, J^ℓ has the structure of a $C^{k-\ell'}$ fiber bundle over $J^{\ell'}$. Moreover, $J^{\ell-\ell'}(J^{\ell'}) = J^\ell$.

The bundle $J^\ell(\mathbb{R}^m, \mathbb{R}^n)$ is a trivial bundle over $\mathbb{R}^m$. The fiber space $J^\ell_v(\mathbb{R}^m, \mathbb{R}^n)$ is the $\ell + 1$-fold product $P^\ell(m, n) = \Pi^\ell_{i=0} L^i_{sym}(\mathbb{R}^m, \mathbb{R}^n)$, where $L^i_{sym}(\mathbb{R}^m, \mathbb{R}^n)$ is the vector space of of symmetric, i-multilinear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. Each ℓ-jet $[v, \phi, U]_\ell$ in $J^\ell_v(\mathbb{R}^m, \mathbb{R}^n)$ has a canonical representative, which is the ℓth order Taylor polynomial of ϕ about v. To denote an element of $J^\ell(\mathbb{R}^m, \mathbb{R}^n)$, we sometimes use the notation $(v, \wp)$ with $v \in \mathbb{R}^m$ and $\wp$ a degree ℓ polynomial (suppressing the neighborhood U, since polynomials are globally defined). These give C^∞ global coordinates on $J^\ell(\mathbb{R}^m, \mathbb{R}^n)$; in this way we regard $J^\ell(\mathbb{R}^m, \mathbb{R}^n)$ as a finite dimensional vector space with a Euclidean structure $|\cdot|$.

6.1. Prolongations. — If $\phi : N_1 \to N_2$ is a C^ℓ function, then ϕ gives rise to a section of the bundle $J^\ell(N_1, N_2)$ over N_1 via the map $v \mapsto j^\ell_v\phi$. This section, denoted $j^\ell\phi$ is called the *ℓ-prolongation* of ϕ. In the case $\ell = 0$, the jet bundle $J^0(N_1, N_2)$ is just the product $N_1 \times N_2$, and the image of N_1 under the prolongation $j^0\phi$ is the just the graph of ϕ.

The function $\phi : M \to M$ is C^k if and only if the ℓ-prolongation of ϕ is $C^{k-\ell}$. Not every continuous section of $J^\ell(M, N)$ is the prolongation of a C^ℓ function; however, the set of prolongations of smooth functions is closed:

Proposition 6.1. — *If $f_n \in C^\ell(M, N)$ and $j^\ell f_n \to j^\ell f$ in the weak topology on $C^0(M, J^\ell(M, N))$, then $f \in C^\ell(M, N)$.*

More generally, if $\sigma\colon M \to \mathcal{B}$ is a section (resp. local section) of a C^k bundle $\pi\colon \mathcal{B} \to M$, then the ℓ-prolongation $j^\ell\sigma\colon M \to J^\ell(\pi\colon \mathcal{B} \to M)$ is a $C^{k-\ell}$ section (resp. local section). The analogue of Proposition 6.1 holds for prolongations of sections.

6.2. Isomorphism of jet bundles. — The next lemma is used extensively in various forms in this paper.

Lemma 6.2. — *Let N_1, N_2, and N_3 be C^k manifolds.*

1. *Let $g : N_2 \to N_3$ be a C^k map. Then for every $\ell \le k$, the map $j^\ell_x\phi \mapsto j^\ell_x(g \circ \phi)$ is a $C^{k-\ell}$ map from $J^\ell(N_1, N_2)$ to $J^\ell(N_1, N_3)$.*
2. *Let $h\colon N_1 \to N_2$ be a C^k diffeomorphism. Then for every $\ell \le k$, the map $j^\ell_x\phi \mapsto j^\ell_{h(x)}(\phi \circ h^{-1})$ is a $C^{k-\ell}$ diffeomorphism from $J^\ell(N_1, N_3)$ to $J^\ell(N_2, N_3)$.*

Remark: There is some subtlety in item 2. If $h\colon N \to N$ is a C^k diffeomorphism other than the identity, then neither of the following maps is even differentiable on $J^\ell(N, N)$:

$$j^\ell_x\phi \mapsto j^\ell_{h(x)}\phi \qquad \text{or} \quad j^\ell_x\phi \mapsto j^\ell_x(\phi \circ h^{-1}).$$

It is at first glance a fortuitous fact that the composition of these maps is $C^{k-\ell}$. What item 2 expresses is the fact that the ℓ-jet bundle is a $C^{k-\ell}$ invariant under C^k-diffeomorphisms. More generally:

Corollary 6.3. — *(see, e.g., [**22**], Chapter 14.4) If $\pi\colon \mathcal{B} \to M$ and $\pi'\colon \mathcal{B}' \to M'$ are C^k fiber bundles, and $H\colon \mathcal{B} \to \mathcal{B}'$ is a C^k isomorphism of fiber bundles, covering the C^k diffeomorphism $h\colon M \to M'$, then for every $\ell \le k$ there is a canonical $C^{k-\ell}$ isomorphism of fiber bundles*

$$H^\ell\colon J^\ell(\pi\colon \mathcal{B} \to M) \to J^\ell(\pi'\colon \mathcal{B}' \to M')$$

covering h. For $\ell' \le \ell$, the map H^ℓ covers $H^{\ell'}$ under the natural projection.

The map H^ℓ is defined by:

$$H^\ell(j^\ell_x\sigma) = j^\ell_{h(x)}(H\circ\sigma\circ h^{-1}).$$

6.3. The graph transform on jets. — In its local form, Corollary 6.3 tells us that for diffeomorphisms of $\mathbb{R}^m\times\mathbb{R}^n$ of the form $H(x,y)=(h(x),g(x,y))$, the induced *graph transform* on functions $\Phi\colon\mathbb{R}^m\to\mathbb{R}^n$ produces a map that is smooth on the level of jets. By graph transform, we mean the map $\mathscr{T}_H\colon\{\Phi\colon\mathbb{R}^m\to\mathbb{R}^n\}\to\{\Phi\colon\mathbb{R}^m\to\mathbb{R}^n\}$ defined by:

$$\mathscr{T}_H(\Phi)(x) = g(h^{-1}(x),\Phi(h^{-1}(x))).$$

It is easy to see that if H is C^k, then $\mathscr{T}_H(C^\ell(\mathbb{R}^m,\mathbb{R}^n)) = C^\ell(\mathbb{R}^m,\mathbb{R}^n)$, for all $\ell\le k$; nonetheless, the restriction of $\mathscr{T}_H$ to $C^\ell(\mathbb{R}^m,\mathbb{R}^n)$ is not smooth at all, even for $\ell=0$. What is smooth, however, is the induced map $H^\ell\colon J^\ell(\mathbb{R}^m,\mathbb{R}^n)\to J^\ell(\mathbb{R}^m,\mathbb{R}^n)$:

$$H^\ell(j^\ell_x\psi) = j^\ell_{h(x)}(\mathscr{T}_H(\psi)).$$

This map on ℓ-jets is $C^{k-\ell}$.

More generally, whenever a graph transform is well-defined, it induces a continuous map on jets, which we now describe. Suppose that $H(x,y)=(h(x,y),g(x,y))$ is a C^k local diffeomorphism of $\mathbb{R}^m\times\mathbb{R}^n$. Write

$$D_vH = \begin{pmatrix} A_v & B_v\\ C_v & K_v\end{pmatrix},$$

where $A_v\colon\mathbb{R}^m\to\mathbb{R}^m$, $B_v\colon\mathbb{R}^n\to\mathbb{R}^m$, $C_v\colon\mathbb{R}^m\to\mathbb{R}^n$ and $K_v\colon\mathbb{R}^n\to\mathbb{R}^n$. Suppose that there exists $\rho_0>0$ such that for all $v\in B_{\mathbb{R}^{m+n}}(0,\rho_0)$, the map A_v is invertible.

Then there exists $\rho_1>0$ such that, for every $\ell\le k$, there exists a $C^{k-\ell}$ local diffeomorphism

$$H^\ell\colon J^\ell(\mathbb{R}^m,\mathbb{R}^n)\to J^\ell(\mathbb{R}^m,\mathbb{R}^n),$$

defined in the ρ_1-neighborhood of the 0-section of $J^\ell_{B_{\mathbb{R}^m}(0,\rho_0)}(\mathbb{R}^m,\mathbb{R}^n)$, given by:

$$H^\ell(j^\ell_x\psi) = j^\ell_{h(x,\psi(x))}\left((g\circ(id,\psi))\circ(h\circ(id,\psi))^{-1}\right).$$

The map H^ℓ has the defining property that for every $\psi\in\Gamma^\ell(\mathbb{R}^m,\mathbb{R}^n)$, if $j^\ell_x\psi$ is in the domain of H^ℓ, and $\psi'\in\Gamma^\ell(\mathbb{R}^m,\mathbb{R}^n)$ satisfies:

$$\mathrm{graph}(\psi') = H(\mathrm{graph}(\psi))$$

in a neighborhood of $h(x,\psi(x))$, then $H^\ell(j^\ell_x\psi) = j^\ell_{h(x,\psi(x))}\psi'$. This fact motivates the term "graph transform."

We explore the properties of these maps in more detail; this will be used in subsequent sections. Writing $P^\ell(m,n)=\Pi_{i=0}^\ell L^i_{sym}(\mathbb{R}^m,\mathbb{R}^n)$, we have coordinates

$$(x,\wp)\mapsto(x,\wp_0,\dots,\wp_\ell)$$

on $\mathbb{R}^m\times P^\ell(m,n)$, where $\wp_i = D^i_x\wp\in L^i_{sym}(\mathbb{R}^m,\mathbb{R}^n)$. Denote by $H^\ell(x,\wp)_i$ the $L^i_{sym}(\mathbb{R}^m,\mathbb{R}^n)$-coordinate of $H^\ell(x,\wp)$, so that

$$H^\ell(x,\wp) = (h(x,\wp_0),H^\ell(x,\wp)_0,\dots,H^\ell(x,\wp)_\ell).$$

Clearly $H^0(x,\wp_0)_0 = g(x,\wp_0)$. Because jets are natural, for $\ell' \leq \ell$, we have

$$H^\ell(x,\wp_0,\ldots,\wp_\ell)_{\ell'} = H^{\ell'}(x,\wp_0,\ldots,\wp_{\ell'})_{\ell'}.$$

Furthermore,

$$H^1(x,\wp_0,\wp_1)_1 = \left(C_{(x,\wp_0)} + K_{(x,\wp_0)}\wp_1\right)\left(A_{(x,\wp_0)} + B_{(x,\wp_0)}\wp_1\right)^{-1}.$$

Differentiating this expression ℓ times (implicitly), we get, for $\ell \geq 1$:

$$\begin{aligned} H^\ell(x,\wp_0,\ldots,\wp_\ell)_\ell &= (K_{(x,\wp_0)}\wp_\ell - H^1(v,\wp_0,\wp_1)_1 B_{(x,\wp_0)}\wp_\ell \\ &\quad + S^\ell(x,\wp_0,\ldots,\wp_{\ell-1})) \circ (A_{(x,\wp_0)} + B_{(x,\wp_0)}\wp_1)^{-1}, \end{aligned}$$

where S^ℓ is a polynomial in $(x,\wp_0,\ldots,\wp_{\ell-1})$ and in the partial derivatives of H at $(x,\wp_0)$ up to order ℓ.

Notice that if $B_{(x,\wp_0)} = 0$, then these expressions reduce to:

$$H^\ell(x,\wp_0,\ldots,\wp_\ell)_\ell = (K_{(x,\wp_0)}\wp_\ell + S^\ell(x,\wp_0,\ldots,\wp_{\ell-1})) \circ A^{-1}_{(x,\wp_0)}.$$

In particular, if $B_{(x,\wp_0)} = 0$, then there exists $\rho_2 > 0$ such that for all $(x',\wp')$ lying in the ρ_2-neighborhood of $(x,\wp)$ in $J^\ell(\mathbb{R}^m,\mathbb{R}^n)$, we have:

$$|H^\ell(x,\wp)_\ell - H^\ell(x',\wp')_\ell| \tag{21}$$

$$\leq Q^\ell_{(x,\wp_0)}(\wp_\ell - \wp'_\ell) + O\left(|(x,\wp_0,\ldots,\wp_{\ell-1}) - (x',\wp'_0,\ldots,\wp'_{\ell-1})|\right), \tag{22}$$

where $Q^\ell_{(x,\wp_0)}\colon L^\ell_{sym}(\mathbb{R}^m,\mathbb{R}^n) \to L^\ell_{sym}(\mathbb{R}^m,\mathbb{R}^n)$ is the linear map:

$$Q^\ell_{(x,\wp_0)}(\overline{\wp}_\ell) = K_{(x,\wp_0)} \circ \overline{\wp}_\ell \circ A^{-1}_{(x,\wp_0)}.$$

Observe that, because $\overline{\wp}_\ell$ is a symmetric map of order ℓ, we have $\|Q^\ell_{(x,\wp_0)}\| \leq \|K_{(x,\wp_0)}\|/m(A_{(x,\wp_0)})^\ell$, where $m(X) = \|X^{-1}\|^{-1}$ denotes the conorm of an invertible matrix X.

For $\ell \geq 1$, we may regard $J^\ell(\mathbb{R}^m,\mathbb{R}^n)$ as a vector bundle over $J^0(\mathbb{R}^m,\mathbb{R}^n)$ ($= \mathbb{R}^m \times \mathbb{R}^n$) under the natural projection $\pi_{\ell,0}$; the fiber is $\Pi_{i=1}^\ell L^i_{sym}(\mathbb{R}^m,\mathbb{R}^n)$. In a variety of contexts (see Section 10.1 ff.) we will consider the case where the map H^ℓ is a fiberwise contraction on a neighborhood of the 0-section of this bundle. We assume that $\|K_{(x,\wp_0)}\| < m(A_{(x,\wp_0)})$ and $\|K_{(x,\wp_0)}\| < m(A_{(x,\wp_0)})^\ell$ (which together imply that $\|K_{(x,\wp_0)}\| < m(A_{(x,\wp_0)})^i$, for $1 \leq i \leq \ell$).

Continuing to assume that $B_{(x,\wp_0)} = 0$, we next construct in the standard way a norm $|\cdot|'$ on $\Pi_{i=1}^\ell L^i_{sym}(\mathbb{R}^m,\mathbb{R}^n)$ such that:

$$|H^\ell(x,\wp) - H^\ell(x,\wp')|' \tag{23}$$

$$\leq \max\left\{\frac{\|A_{(x,\wp_0)}\|}{m(K_{(x,\wp_0)})}, \frac{\|K_{(x,\wp_0)}\|}{m(A_{(x,\wp_0)})^\ell}\right\} \cdot |(x,\wp)_\ell - (x,\wp')_\ell|', \tag{24}$$

for $(x,\wp),(x,\wp')$ lying in the set $\{(x,\wp_0,\overline{\wp}_1,\ldots,\overline{\wp}_\ell)\colon |(\overline{\wp}_1,\ldots,\overline{\wp}_\ell)|' \leq 1\}$. To do this, fix $L > 0$ and for $(\overline{\wp}_1,\ldots,\overline{\wp}_\ell) \in \Pi_{i=1}^\ell L^i_{sym}(\mathbb{R}^m,\mathbb{R}^n)$, define:

$$|(\overline{\wp}_1,\ldots,\overline{\wp}_\ell)|_L = L^\ell|\overline{\wp}_1| + \cdots + L|\overline{\wp}_\ell|.$$

It is not difficult to verify using (21) that if $L > 0$ is sufficiently large, then (23) holds for $|\cdot|' = |\cdot|_L$ and all $(x,\wp), (x,\wp')$ lying in the set $\{(x, \wp_0, \overline{\wp}_1, \dots, \overline{\wp}_\ell) \colon |(\overline{\wp}_1, \dots, \overline{\wp}_\ell)|' \leq 1\}$.

The same holds true if $\|B_{(x,\wp_0)}\|$ is sufficiently small. Summarizing this discussion, we have:

Lemma 6.4. — *Fix $\ell \geq 1$. For every $R > 0$ and $\kappa \in (0,1)$ there exist $\varepsilon > 0$ and $L > 0$ with the following properties.*

Let $H \colon B_{\mathbb{R}^{m+n}}(0,1) \to \mathbb{R}^{m+n}$ be a C^ℓ local diffeomorphism such that:

– *$d_{C^\ell}(H, Id) \leq R$, and*

– *writing $D_vH = \begin{pmatrix} A_v & B_v \\ C_v & K_v \end{pmatrix}$, we have:*

$$\inf_{v \in B_{\mathbb{R}^{m+n}}(0,1)} m(A_v) > 0,$$

$$\kappa > \sup_{v \in B_{\mathbb{R}^{m+n}}(0,1)} \max\left\{\frac{\|K_v\|}{m(A_v)}, \frac{\|K_v\|}{m(A_v)^\ell}\right\},$$

and

$$\sup_{v \in B_{\mathbb{R}^{m+n}}(0,1)} \|B_v\| < \varepsilon.$$

Then for all $v = (v^m, v^n) \in \mathbb{R}^{m+n}$ and all $j^\ell_{v^m}\psi, j^\ell_{v^m}\psi' \in \pi^{-1}_{\ell,0}(v)$, with $|j^\ell_{v^m}\psi|, |j^\ell_{v^m}\psi'| \leq 1$, we have:

$$|H^\ell(j^\ell_{v^m}\psi) - H^\ell(j^\ell_{v^m}\psi')|_L \leq \kappa |j^\ell_{v^m}\psi - j^\ell_{v^m}\psi'|_L.$$

7. Proof of Theorem B

Before proving our main higher regularity result (part IV of Theorem A), we give a proof of Theorem B, as the proof conveys some of the basic techniques we will use later, but in a simpler setting.

Suppose that N is an embedded C^1 submanifold of $\mathbb{R}^{m+n}$ such that, for every x, y in N, there exist neighborhoods U of x and V of y and a C^k diffeomorphism $H : U \to V$ such that $H(U) = V$ and $H(U \cap N) = V \cap N$, where $k \geq 2$.

We prove that N is a C^ℓ submanifold of $\mathbb{R}^{m+n}$, for all $\ell \leq k$, by induction on ℓ. By assumption, N is a C^1 submanifold. Suppose that N is a C^ℓ submanifold, for some $\ell \leq k-1$. We prove that N is $C^{\ell+1}$ submanifold. As the problem is local, we may restrict attention to a small neighborhood in N.

Fix a point $x_0 \in N$ and a neighborhood V of x_0 in N. By a local C^k change of coordinates in N' sending x_0 to $0 \in \mathbb{R}^n \times \mathbb{R}^m$, we may assume that N is the graph of a C^ℓ function $\Phi : \overline{B_{\mathbb{R}^n}(0,1)} \to \mathbb{R}^m$ satisfying $j^k_0\Phi = 0$. The first main step in the proof of Theorem B is the following lemma.

Lemma 7.1. — *For every $u \in \overline{B_{\mathbb{R}^n}(0,1)}$ there exists $\rho = \rho(u) > 0$, and for every $i \in \{0, \dots, \ell\}$, a C^{k-i} local diffeomorphism*

$$H_u^i\colon B_{J^i(\mathbb{R}^n,\mathbb{R}^m)}(0,\rho) \to J^i(\mathbb{R}^n,\mathbb{R}^m)$$

with the following properties:

1. *H_u^i covers H_u^{i-1} under the projection $J^i(\mathbb{R}^n,\mathbb{R}^m) \to J^{i-1}(\mathbb{R}^n,\mathbb{R}^m)$, and*
2. *writing $H_u^0(v,w) = (h_u(v,w), g_u(v,w))$, we have $h_u(0,\Phi(0)) = u$, and:*

$$H_u^\ell(j_v^\ell\Phi) = j^\ell_{h_u(v,\Phi(v))}\Phi,$$

for every v such that $j_v^\ell\Phi \in B_{J^\ell(\mathbb{R}^n,\mathbb{R}^m)}(0,\rho)$.

Proof. — For $i = 0$, this follows immediately from C^k homogeneity. Given $u \in \overline{B_{\mathbb{R}^n}(0,1)}$, select a C^k local diffeomorphism

$$H_u = (h_u, g_u)\colon B_{\mathbb{R}^n\times\mathbb{R}^m}(0,\rho_0) \to \mathbb{R}^n \times \mathbb{R}^m$$

sending $(0,0) = (0,\Phi(0))$ to $(u,\Phi(u))$ and preserving the graph of Φ. Under the natural identification of $J^0(\mathbb{R}^n,\mathbb{R}^m)$ with $\mathbb{R}^n \times \mathbb{R}^m$, this defines the map H_u^0:

$$H_u^0(v,w) = (h_u(v,w), g_u(v,w)).$$

Suppose $i \geq 1$, and fix a point $v' \in \mathbb{R}^n$ near 0, and a function $\psi \in \Gamma^i_{v'}(\mathbb{R}^n,\mathbb{R}^m)$. Consider the local map $h_u \circ (id,\psi) \in \Gamma^i_{v'}(\mathbb{R}^n,\mathbb{R}^n)$ given by:

$$H_u \circ (id,\psi)(v) = h_u(v,\psi(v)).$$

Its derivative at v' is

$$D_{v'}\left(h_u \circ (id,\psi)\right) = \frac{\partial h_u}{\partial v}(v',\psi(v')) + \frac{\partial h_u}{\partial w}(v',\psi(v'))D_{v'}\psi. \tag{25}$$

Since DH_u^0 preserves the tangent space to the graph of Φ, it follows that the map $\partial H_u/\partial v(0,0)$ is a diffeomorphism onto a neighborhood of u. On the other hand, plugging in $v' = 0$, $D_{v'}\psi = 0$ into equation (25) we obtain that for any $\psi \in \Gamma_0^i(\mathbb{R}^n,\mathbb{R}^m)$ with $j_0^1\psi = 0$, $D_0\left(h_u \circ (id,\psi)\right) = \frac{\partial h_u}{\partial v}(0,0)$.

Since H^0 is C^1, from this it follows that for $|j^i_{v'}\psi|$ and $|v'|$ sufficiently small, the derivative $D_{v'}\left(h_u \circ (id,\psi)\right)$ is invertible. The inverse function theorem then implies that $h_u \circ (id,\psi)$ is a C^i local diffeomorphism in a neighborhood of $v \in \mathbb{R}^n$, provided $|j_v^i\psi|$ is sufficiently small; in particular, $(h_u \circ (id,\psi))^{-1}$ is defined.

For $i \geq 1$, we then set

$$H_u^i(j_v^i\psi) = j^i_{h_u(v,\psi(v))}\left((g_u \circ (id,\psi)) \circ (h_u \circ (id,\psi))^{-1}\right).$$

Lemma 6.2 implies that H_u^i is a C^{k-i} local diffeomorphism. By construction, the maps H_u^i satisfy properties (1) and (2). □

Remark: Notice that Lemma 7.1 implies that the image of $\overline{B_{\mathbb{R}^n}(0,1)}$ under $j^\ell\Phi$ is a C^1 homogeneous submanifold of $J^\ell(\mathbb{R}^n,\mathbb{R}^m)$. At this point, it is possible to appeal to Theorem 1.2 to finish the proof.

Returning to the proof of Theorem B, our next step is to show:

If Φ is C^ℓ and $j^\ell\Phi$ is a C^1-homogeneous function (in the sense of Lemma 7.1), then $j^\ell\Phi$ is C^1, and so Φ is $C^{\ell+1}$.

To this end, let $A\colon J^\ell(\mathbb{R}^n,\mathbb{R}^m)\to J^\ell(\mathbb{R}^n,\mathbb{R}^m)$ be an invertible linear transformation, and let $\rho>0$. We next define a subset $\mathscr{G}(A,\rho)\subset\overline{B_{\mathbb{R}^n}(0,1)}$ consisting of the set of all $u\in\overline{B_{\mathbb{R}^n}(0,1)}$ with the following properties:

– For each $i\in\{0,\dots\ell\}$, there exists a bilipschitz embedding

$$\tilde{H}^i_u\colon B_{J^i(\mathbb{R}^n,\mathbb{R}^m)}(0,\rho)\to J^i(\mathbb{R}^n,\mathbb{R}^m)$$

such that:

– $\tilde{H}^i_u$ covers $\tilde{H}^{i-1}_u$ under the projection $J^i(\mathbb{R}^n,\mathbb{R}^m)\to J^{i-1}(\mathbb{R}^n,\mathbb{R}^m)$,

– writing $\tilde{H}^0_u(v,w)=(\tilde{h}_u(v,w),\tilde{g}_u(v,w))$, we have $\tilde{h}_u(0,\Phi(0))=u$, and:

$$\tilde{H}^\ell_u(j^\ell_v\Phi)=j_{\tilde{h}_u(v,\Phi(v))}\Phi,$$

for every v such that $j^\ell_v\Phi\in B_{J^\ell(\mathbb{R}^n,\mathbb{R}^m)}(0,\rho)$, and

– $\mathrm{Lip}(A-\tilde{H}^\ell_u)\le\frac{m(A)}{5}$ on $B_{J^\ell(\mathbb{R}^n,\mathbb{R}^m)}(0,\rho)$, where $m(A)=\|A^{-1}\|^{-1}$ denotes the conorm of A.

Fix a countable dense subset $\{A_j\}_{j\in\mathbb{Z}_+}\subset GL(J^\ell(\mathbb{R}^n,\mathbb{R}^m))$ of invertible linear transformations.

Lemma 7.2. — *For each $A\in GL(J^\ell(\mathbb{R}^m,\mathbb{R}^n))$, and $\rho>0$, the set $\mathscr{G}(A,\rho)$ is compact in $\overline{B_{\mathbb{R}^n}(0,1)}$. Moreover:*

$$\overline{B_{\mathbb{R}^n}(0,1)}=\bigcup_{j_1,j_2\in\mathbb{Z}_+}\mathscr{G}(A_{j_1},j_2^{-1}).$$

Proof. — Suppose that $\mathscr{G}(A,\rho)$ is nonempty. Let u_j be a sequence in $\mathscr{G}(A,\rho)$, and for each $i\in\{0,\dots,\ell\}$, let $\tilde{H}^i_{u_j}$ be the associated sequence of bilipschitz embeddings. Since the space of bilipschitz embeddings is locally compact in the uniform topology, there exists a convergent subsequence $u_{j_\ell}\to u\in\overline{B_{\mathbb{R}^n}(0,1)}$ with $\tilde{H}^i_{u_{j_\ell}}\to\tilde{H}^i_u$ uniformly for all i. The maps $\tilde{H}^i_u$ are bilipschitz embeddings, with $\tilde{H}^i_u$ covering $\tilde{H}^{i-1}_u$, and $\mathrm{Lip}(H^i_u-A)\le\frac{m(A)}{5}$. Since the ℓ-jet $j^\ell\Phi$ is a closed subset of $J^\ell(\mathbb{R}^n,\mathbb{R}^m)$, the limiting map $\tilde{H}^\ell_u$ preserves $j^\ell\Phi$. Hence $u\in\mathscr{G}(A,\rho)$, and so $\mathscr{G}(A,\rho)$ is compact.

Lemma 7.1 implies that for each u, and each i there exists a C^{r-i} diffeomorphism H^i_u satisfying the first two properties. Let $\varepsilon=m(D_0H^\ell_u)/11$. Fix $A_{j_1}\in GL(J^\ell(\mathbb{R}^n,\mathbb{R}^m))$ such that $\|D_0H^\ell_u-A_{j_1}\|<\varepsilon$. A simple estimate shows that $\|D_0H^\ell_u-A_{j_1}\|<\frac{m(A_{j_1})}{10}$. Next, fix j_2 such that $\mathrm{Lip}(D_0H^\ell_u-H^\ell_u)<\frac{m(A_{j_1})}{10}$ on $B_{J^\ell(\mathbb{R}^m,\mathbb{R}^n)}(0,j_2^{-1})$. Then $\mathrm{Lip}(A_{j_1}-H^\ell_u)<\frac{m(A_{j_1})}{5}$ on $B_{j^\ell(\mathbb{R}^n,\mathbb{R}^m)}(0,j_2^{-1})$, which implies that $u\in\mathscr{G}(A_{j_1},j_2^{-1})$. Hence:

$$\overline{B_{\mathbb{R}^n}(0,1)}=\bigcup_{j_1,j_2\in\mathbb{Z}_+}\mathscr{G}(A_{j_1},j_2^{-1}),$$

completing the proof of the lemma. □

Since $\overline{B_{\mathbb{R}^n}(0,1)}$ is a Baire space, there exist integers j_1, j_2 such that $\mathscr{G}(A_{j_1}, {j_2}^{-1})$ has nonempty interior. Let U be an open ball contained in $\mathscr{G}(A_{j_1}, j_2^{-1})$. For each pair $u, u' \in U$ and $i \in \{0, \dots, \ell\}$, we set $H^i_{(u,u')} = \tilde{H^i_{u'}} \circ \tilde{H^i_u}^{-1}$, which is defined on a neighborhood of $j^i_u \Phi$ in $J^i(\mathbb{R}^n, \mathbb{R}^m)$. We thus obtain:

Lemma 7.3. — *There exists $\rho > 0$ such that, for every pair $z = (u, u') \in U \times U$, the following hold:*

- *for each $i \in \{0, \dots \ell\}$, H^i_z is a bilipschitz homeomorphism, defined on a ρ-neighborhood of $j^i_u \Phi$,*
- *H^i_z covers H^{i-1}_z under the projection $J^i(\mathbb{R}^n, \mathbb{R}^m) \to J^{i-1}(\mathbb{R}^n, \mathbb{R}^m)$,*
- *writing $H^0_z(v, w) = (h_z(v, w), g_z(v, w))$, we have $h_z(u, \Phi(u)) = u'$, and:*
$$H^\ell_z(j^\ell_v \Phi) = j_{h_z(v, \Phi(v))}\Phi,$$
for every v such that $j^\ell_v \Phi \in B_{J^\ell(\mathbb{R}^n, \mathbb{R}^m)}(j_u \Phi, \rho)$, and
- *$Lip(I - H^\ell_z) \leq \frac{1}{2}$ on $B_{J^\ell(\mathbb{R}^n, \mathbb{R}^m)}(j^\ell_u \Phi, \rho)$.*

Let $K = 3/2$, which is a bound, over all $z = (u, u') \in U \times U$, for the Lipschitz norm of H^ℓ_z on $B_{J^\ell(\mathbb{R}^n, \mathbb{R}^m)}(j^\ell_u \Phi, \rho)$. Since Φ is assumed to be at least C^1, there exists a constant $C > 0$ such that, for all $u, u' \in U$,
$$|j^0_u \Phi - j^0_{u'} \Phi| \leq C|u - u'|.$$
Fix a point $u_0 \in U$, and let $\alpha = d(u_0, \mathbb{R}^n \setminus U)$ (which depends uniformly on u_0). Since $j^\ell \Phi$ is continuous, if u is sufficiently close to u_0 (uniformly in u_0), we will have $j^\ell_u \Phi \in B_{J^\ell(\mathbb{R}^m, \mathbb{R}^n)}(j^\ell_{u_0} \Phi, \rho)$.

Let $u_1 \in U$ be such a point. Fix $N \in \mathbb{Z}_+$ such that:
$$\frac{\alpha}{CK(N+1)} \leq |u_1 - u_0| < \frac{\alpha}{CKN}.$$

We construct a sequence of points $u_0, u_1, u_2, \dots, u_N$ in U inductively as follows. The points u_0 and u_1 have already been defined. For $i \in \{1, \dots, n-1\}$, we set $z_i = (u_0, u_i) \in U \times U$ and $u_{i+1} = h_{z_i}(u_1, \Phi(u_1))$. We need to check that if u_i is contained in U, then u_{i+1} is also contained in U.

To see this, note that, for $i \leq N$, we have:
$$\begin{array}{rcl} |u_i - u_{i-1}| & = & |h_{z_i}(u_1, \Phi(u_1)) - h_{z_i}(u_0, \Phi(u_0))| \\ & \leq & K|j^0_{u_1}\Phi - j^0_{u_0}\Phi| \\ & \leq & KC|u_1 - u_0| \end{array}$$
Hence, for $i \leq N$, this implies that $|u_i - u_0| \leq KCi|u_1 - u_0| < \alpha$, so that $u_i \in U$, for all $i \in \{1, \dots, N\}$.

Then, for each i:
$$\begin{array}{rcl} j^\ell_{u_i}\Phi - j^\ell_{u_{i-1}}\Phi & = & H^\ell_{z_i}(j^\ell_{u_1}\Phi) - H^\ell_{z_i}(j^\ell_{u_0}\Phi) \\ & = & j^\ell_{u_1}\Phi - j^\ell_{u_0}\Phi + (H^\ell_{z_i} - Id)(j^\ell_{u_1}\Phi) - (H^\ell_{z_i} - Id)(j^\ell_{u_2}\Phi) \end{array}$$

Summing these equations from $i = 1, \dots, N$, and taking the norm, we obtain:

$$\begin{aligned}
|j^\ell_{u_N}\Phi - j^\ell_{u_0}\Phi| &\geq N|j^\ell_{u_1}\Phi - j^\ell_{u_0}\Phi| \\
&\quad - \sum_{i=1}^{N}\left|(H^\ell_{z_i} - Id)(j^\ell_{u_1}\Phi) - (H^\ell_{z_i} - Id)(j^\ell_{u_0}\Phi)\right| \\
&\geq \frac{N}{2}|j^\ell_{u_1}\Phi - j^\ell_{u_0}\Phi|,
\end{aligned}$$

since $\mathrm{Lip}(H^\ell_{z_i} - Id) < \frac{1}{2}$, for $i = 1, \dots, N$.

Since $j^\ell\Phi$ is continuous, by assumption, there exists a constant $M > 0$ such that $|j^\ell_v\Phi| \leq M$, for all $v \in U$. Then:

$$\begin{aligned}
|j^\ell_{u_1}\Phi - j^\ell_{u_0}\Phi| &\leq \frac{2}{N}|j^\ell_{u_N}\Phi - j^\ell_{u_0}\Phi| \\
&\leq \frac{4M}{N} \\
&= \frac{4MCK(N+1)}{n\alpha}\frac{\alpha}{CK(N+1)} \\
&\leq \frac{12MC}{\alpha}|u_1 - u_0|.
\end{aligned}$$

From this it follows that $u \mapsto j^\ell_u\Phi$ is Lipschitz at u_0; since u_0 was arbitrary, the map is locally Lipschitz on U. Hence $j^\ell\Phi$ is differentiable almost everywhere on $U \subset V$. $C^{\ell+1}$-homogeneity of V now implies that $j^\ell\Phi$ is differentiable everywhere on V. Taking a point of continuity for the derivative of $j^\ell\Phi$, and applying $C^{\ell+1}$-homogeneity one more time, we obtain that $j^\ell\Phi$ is C^1, and so V is a $C^{\ell+1}$ submanifold of $\mathbb{R}^n \times \mathbb{R}^m$. This completes the inductive step of our proof, and so completes the proof that N is a C^k submanifold of $\mathbb{R}^{m+n}$.

8. Journé's theorem, re(re)visited.

Journé's theorem **[19]** is widely used in rigidity theory to show that a continuous function is smooth. The theorem states that any function that is uniformly smooth along leaves of two transverse foliations with uniformly smooth leaves is smooth. This theorem is typically applied in the Anosov setting as follows: according to Proposition 4.7, the graph of a continuous transfer function Φ for a smooth coboundary ϕ is bisaturated, i.e., saturated by leaves of the unstable and stable foliation for the skew product f_ϕ. Since f_ϕ is smooth, the leaves of these foliations are smooth graphs over the corresponding foliations for f. This implies that the function Φ is smooth along leaves of the stable and unstable foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ for f. In the Anosov setting, these foliations are transverse, so applying Journé's theorem, we obtain that Φ is smooth (see **[31]**).

Here in the partially hyperbolic setting, we reproduce this argument in part. Indeed, by the same argument, any continuous transfer function Φ of a smooth coboundary

ϕ is smooth along leaves of $\mathcal{W}^s$ and $\mathcal{W}^u$. Since the stable and unstable foliations are not necessarily transverse, we cannot apply Journé's theorem at this point. The idea is to use accessibility and center bunching to show that the restriction of Φ to leaves of a center foliation is also smooth. One then applies Journé's theorem twice, first to the pair of foliations $\mathcal{W}^c$ and $\mathcal{W}^u$, and then to the pair $\mathcal{W}^{cu}$ and $\mathcal{W}^s$, to conclude that Φ is smooth.

If one assumes that f is dynamically coherent, then it is possible to turn this idea into a rigorous argument, as we outlined above in Section 1. Here are a few more details on how one can show that Φ is smooth along leaves of $\mathcal{W}^c$ in the dynamically coherent setting. Bisaturation of Φ implies that the graph of Φ when restricted to the $\mathcal{W}^c$-manifolds is invariant under the $\mathcal{W}^s_\phi$ and $\mathcal{W}^u_\phi$-holonomy maps between lifted $\mathcal{W}^c_\phi$-manifolds. The strong bunching hypothesis on f implies that these holonomy maps are smooth when restricted to center manifolds of f_ϕ. Each center manifold $\mathcal{W}^c_\phi(p,t)$ of f_ϕ is the product $\mathcal{W}^c(p) \times \mathbb{R}$ of a center manifold for f with $\mathbb{R}$, and the $\mathcal{W}^s_\phi$ and $\mathcal{W}^u_\phi$-holonomies between $\mathcal{W}^c_\phi$-manifolds covers the corresponding $\mathcal{W}^s$ and $\mathcal{W}^u$-holonomies between $\mathcal{W}^c$-manifolds. Since f is accessible and Φ is bisaturated, any two points on the graph of Φ can be connected by an su-lift path. Corresponding to any such su-lift path is a composition of $\mathcal{W}^s_\phi$ and $\mathcal{W}^u_\phi$-holonomy diffeomorphisms between $\mathcal{W}^c_\phi$-manifolds that preserves the graph of Φ. Putting all of this together, we get that the graph of Φ over any given center manifold $\mathcal{W}^c(p)$ is a smoothly homogeneous submanifold of $\mathcal{W}^c(p) \times \mathbb{R}$ and so by Theorem B is a smooth submanifold. Hence the restriction of Φ to $\mathcal{W}^c$ leaves is also uniformly smooth.

If we do not assume dynamical coherence, then this argument fails. One can attempt to use in place of a center foliation a local "fake" center foliation $\widehat{\mathcal{W}}^c_x$, as is done in **[9]** to prove ergodicity. However, the fake center foliation $\widehat{\mathcal{W}}^c_x$ available to us is not sufficiently canonical to allow a dynamical proof that the graph of Φ is smoothly homogeneous over $\widehat{\mathcal{W}}^c_x$ leaves. Another difficulty is that the fake center foliation and the unstable foliation $\mathcal{W}^u$ are not jointly integrable, and so we cannot apply Journé's theorem in the two steps outlined above. Fortunately, both problems can be overcome, and it is possible to employ the fake foliations of **[9]** to prove Theorem A. The key observations that allow is to do this are:

1. the fake center foliation $\widehat{\mathcal{W}}^c_x$ and the *fake* unstable foliation $\widehat{\mathcal{W}}^u_x$ are jointly integrable,
2. one can show that Φ has continuous "approximate jets" along leaves of $\widehat{\mathcal{W}}^u_x$ and $\widehat{\mathcal{W}}^c_x$, and
3. Journé's theorem has a stronger formulation in terms of "approximate jets".

We detail the argument in the next section. In this section, we describe the stronger formulation of Journé's theorem and what we mean by "approximate jets."

Definition 8.1. — *Let D be a domain in $\mathbb{R}^m$, $C \geq 1$, $\alpha > 0$ and $\ell \in \mathbb{Z}_+$. A function $\psi\colon D \to \mathbb{R}^n$* has an (ℓ, α, C)-expansion at *z if there exists a polynomial $\wp_z$ of degree*

$\leq \ell$ such that:

$$|\psi(z') - \wp_z(z')| \leq C|z - z'|^{\ell+\alpha},$$

for all $z' \in D$.

The following theorem was proved by Campanato (in a more general context):

Theorem 8.2. — **[10]** *For $\ell \in \mathbb{Z}_+$ and $\alpha \in (0,1]$, a function $\psi : \mathbb{R}^m \to \mathbb{R}^n$ is $C^{\ell,\alpha}$ if and only if, for every compact set $D \in \mathbb{R}^m$, there exists $C > 0$ such that ψ has an (ℓ, α, C)-expansion at every $z \in D$.*

Furthermore, ψ is a polynomial of degree $\leq \ell$ if and only if there exists $\alpha > 1$ such that, for every compact set $D \in \mathbb{R}^m$, there exists a $C > 0$ such that ψ has an (ℓ, α, C)-expansion at every $z \in D$.

Definition 8.3. — *A* parametrized $C^{\ell,\alpha}$ transverse pair of plaque families *is a pair of maps (ω^H, ω^V), with*

$$\omega^H : I^{m+n} \times I^m \to \mathbb{R}^{m+n}, \quad \text{and} \quad \omega^V : I^{m+n} \times I^n \to \mathbb{R}^{m+n},$$

of the form:

$$\omega_z^H(x) = z + (x, \beta_z^H(x)), \quad \text{and} \quad \omega_z^V(y) = z + (\beta_z^V(y), y),$$

for $z \in I^{m+n}$, where $\beta_z^H \in C^{\ell,\alpha}(I^m, \mathbb{R}^n)$ and $\beta_z^V \in C^{\ell,\alpha}(I^n, \mathbb{R}^m)$ have the following additional properties:

1. *$\beta_z^H(0) = 0$ and $\beta_z^V(0) = 0$, for all $z \in I^{m+n}$,*
2. *$\beta_{(0,0)}^H(x) = 0$ for every $x \in I^m$, and $\beta_{(0,0)}^V(y) = 0$, for every $y \in I^n$,*
3. *The maps $z \mapsto \beta_z^H \in C^{\ell,\alpha}(I^m, \mathbb{R}^n)$ and $z \mapsto \omega_z^V \in C^{\ell,\alpha}(I^n, \mathbb{R}^m)$ are continuous.*

If (ω^H, ω^V) is a parametrized $C^{\ell,\alpha}$ transverse pair of plaque families, we define the norm $\|(\omega^H, \omega^V)\|_{\ell,\alpha}$ as follows:

$$\|(\omega^H, \omega^V)\|_{\ell,\alpha} := \sup_{z \in I^{m+n}} \|\beta_z^H\|_{C^{\ell,\alpha}(I^m,\mathbb{R}^n)} + \|\beta_z^V\|_{C^{\ell,\alpha}(I^n,\mathbb{R}^m)}.$$

Remark: A pair of transverse foliations with uniformly $C^{\ell,\alpha}$ leaves, after a $C^{\ell,\alpha}$ local change of coordinates, becomes a parametrized transverse pair of plaque families. Similarly, a pair of continuous plaque families (where the plaques depend continuously on the their center point in the $C^{\ell,\alpha}$ topology) transverse at every point gives a transverse pair of plaque families.

Theorem 8.4. — *Fix $\ell \in \mathbb{Z}_+$ and $\alpha \in (0,1)$. Let (ω^H, ω^V) be a parametrized $C^{\ell,\alpha}$ transverse pair of plaque families in $I^{n+m} \subset \mathbb{R}^n \times \mathbb{R}^m$. For every $C > 0$ there exist $C' = C'(C, \|(\omega^H, \omega^V)\|_{\ell,\alpha})$ and $\rho = \rho(C, \|(\omega^H, \omega^V)\|_{\ell,\alpha})$ such that the following holds.*

Suppose that $\psi : I^{n+m} \to \mathbb{R}$ has the properties:

(1) *for every $z \in I^{m+n}$, there exists a polynomial $\wp_z^H : I^m \to \mathbb{R}$ of degree $\leq \ell$ such that, for all $x \in I^m$:*

$$|\psi(\omega_z^H(x)) - \wp_z^H(x)| \leq C(|x|^{\ell+\alpha} + |z|^{\ell+\alpha}),$$

(2) *for every $z \in I^{m+n}$, there exists a polynomial $\wp_z^V : I^n \to \mathbb{R}$ of degree $\le \ell$ such that, for all $y \in I^n$:*

$$|\psi(\omega_z^V(y)) - \wp_z^V(y)| \le C(|y|^{\ell+\alpha} + |z|^{\ell+\alpha}),$$

Then ψ has an (ℓ, α, C')-expansion at $(0,0)$ in $B_{\mathbb{R}^{m+n}}(0,\rho)$.

Remark: Note that the hypotheses of Theorem 8.4 are weaker than requiring that $\psi \circ \omega_z^H$ and $\psi \circ \omega_z^H$ be $C^{\ell,\alpha}$ for every $z \in I^{m+n}$. They are also weaker than requiring that $\psi \circ \omega_z^H$ and $\psi \circ \omega_z^H$ have (ℓ, α, C)-expansions about 0 for every z. This latter condition corresponds to the stronger conditions:

$$|\psi(\omega_z^H(x)) - \wp_z^H(x)| \le C|x|^{\ell+\alpha}, \quad \text{and} \quad |\psi(\omega_z^V(y)) - \wp_z^V(y)| \le C|y|^{\ell+\alpha},$$

for every (x, y). Note also that the conclusion of Theorem 8.4 is in some aspects very weak: it does not even imply that ψ is continuous (except at the origin).

One can recover Journé's original result from Theorems 8.4 and 8.2 as follows. Suppose that ψ is uniformly $C^{\ell,\alpha}$ along the leaves of two transverse foliations with uniformly $C^{\ell,\alpha}$ leaves. Fix an arbitrary point x; in local coordinates sending x to 0, the transverse foliations give a parametrized $C^{\ell,\alpha}$ transverse pair of plaque families. In the coordinates given by this parametrization, ψ has a Taylor expansion at every point with uniform remainder term on the order $\ell+\alpha$. This implies that conditions (1) and (2) in Theorem 8.4 hold, for some C that is uniform in the point x. Theorem 8.4 implies that ψ has an (ℓ, α, C') expansion (in these coordinates) at x, where C' is uniform in x. Since x was arbitrary, Theorem 8.2 then implies that ψ is $C^{\ell,\alpha}$.

We also remark that whereas Theorem 8.2 holds for $\alpha = 1$, Theorem 8.4 is false for $\alpha = 1$, if $\ell > 1$ (see **[33]** for an example with $\alpha = 1$, $\ell = 1$).

Proof of Theorem 8.4. — The proof amounts to a careful inspection of the main result in **[19]**. We follow the format in **[30]**, where the structure of the original treatment in **[19]** has been clarified. We retain as much as possible the notation from **[19, 30]**, though there are some small changes. The two differences in the way the result is stated here and the way it is stated in **[19]** are the following:

1. In **[19]**, the transverse plaque family arises from a transverse pair of local foliations $\mathcal{F}_s$ and $\mathcal{F}_u$; this is not assumed here. An extra lemma (Lemma 8.9) deals with this.
2. In **[19]**, it is assumed that ψ is $C^{\ell,\alpha}$ along leaves of the foliations $\mathcal{F}_s$ and $\mathcal{F}_u$. This is replaced by (1) and (2). A slight adaptation of the proof of Lemma 8.11, part 1, deals with this.

As in **[19]** and **[30]**, we give the proof for $m = n = 1$; the proof for general m, n is completely analogous. We first reduce Theorem 8.4 to the following lemma.

***Lemma 8.5* (cf. [30], Lemma 4.4).** — *Under the hypotheses of Theorem 8.4, there is a polynomial $\wp = \wp(\psi)$ of degree $\le \ell$ with the following property. Given $\kappa > 0$*

and the cone $\mathscr{K}_\kappa = \{(u,v) \in \mathbb{R}^2 : |v| \le \kappa |u|\}$, *there exist positive constants* $C_1 = C_1(\kappa, C, \|(\omega^H, \omega^V)\|_{\ell,\alpha})$ *and* $\rho_1 = \rho_1(\kappa, C, \|(\omega^H, \omega^V)\|_{\ell,\alpha})$ *such that:*

$$|\psi(z) - \wp(z)| \le C_1 |z|^{\ell+\alpha}, \quad \textit{for}\, z \in \mathscr{K} \cap B(0,\rho_1). \tag{26}$$

We first prove Theorem 8.4 using Lemma 8.5. Fix $\kappa > 2$. Applying Lemma 8.5 to the cones $\mathscr{K} = \{(u,v) \in \mathbb{R}^2 : |v| \le \kappa|u|\}$ and $\mathscr{K}' = \{(u,v) \in \mathbb{R}^2 : |u| \le \kappa|v|\}$ (with the roles of u and v switched), we obtain polynomials $\wp$ and $\wp'$ of degree $\le \ell$ and constants C', ρ such that

$$|\psi(z) - \wp(z)| \le C'|z|^{\ell+\alpha}, \quad \text{for } z \in \mathscr{K} \cap B(0,\rho),$$

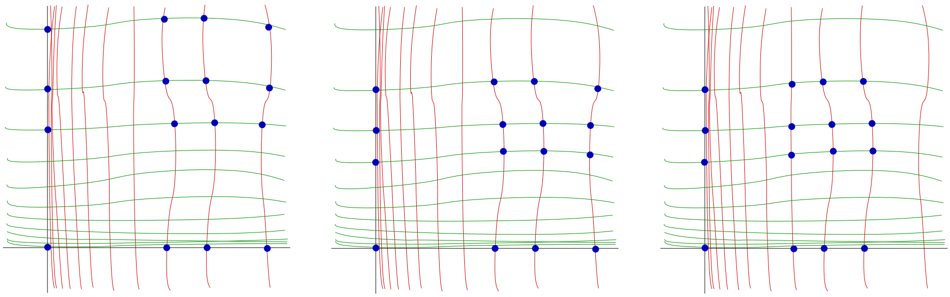
and

$$|\psi(z) - \wp'(z)| \le C'|z|^{\ell+\alpha}, \quad \text{for } z \in \mathscr{K}' \cap B(0,\rho).$$

Note that $V = B(0,\rho) \cap \mathscr{K} \cap \mathscr{K}'$ has nonempty interior. But then $\wp$ and $\wp'$ must agree because they have contact higher than ℓ on V. Hence ψ has an (ℓ, α, C') jet on $B_{\mathbb{R}^2}(0,\rho)$. This completes the proof of Theorem 8.4, assuming Lemma 8.5. □

Proof of Lemma 8.5. — Replacing ψ by $\psi(x,y) - \psi(x,0) - \psi(0,x) + \psi(0,0)$, we may assume that ψ vanishes along the x-and y-axes. For $z \in I^{m+n}$, let $\mathscr{F}^H(z) = \omega^H_z(I^m)$ and let $\mathscr{F}^V(z) = \omega^V_z(I^n)$.

The structure of the proof is as follows. We construct a sequence of degree $(\ell+1)^2$ polynomials $\wp_m$ on I^2 that interpolate the values of ψ on a carefully chosen collection S_m of $(\ell+1)^2$ points in $\mathbb{R}^2$. The terms of degree $\le \ell$ in $\wp_m$ converge to a degree ℓ polynomial $\wp$ that satisfies (26) on a cone $\mathscr{K}_\kappa$.



S_{2k} S_{2k+1} S_{2k+2}

FIGURE 1. The geometry of the sets S_m, when $\ell = 3$.

We say more about the selection of sets S_m. Each set S_m is the union of four subsets $S_m = \{(0,0)\} \cup (H_m \times \{0\}) \cup (\{0\} \times V_m) \cup J_m$, where H_m and V_m each contain ℓ distinct real positive numbers. The sets S_m are chosen with several properties:

- the minimum and maximum distance between any two points in S_m are comparable by a fixed factor $B \ge 1$ and are both $O(r^{m/2})$, for some fixed $r \in (0,1)$,

- J_m is approximately the cartesian product $H_m \times V_m$, with error $o(r^{m/2})$,
- any "vertical" collection of $\ell+1$ points in S_m lies on a vertical $\mathcal{F}^V$-plaque, and any "horizontal" collection of $\ell+1$ points in S_m lies on a horizontal $\mathcal{F}^H$-plaque,
- S_m and S_{m+1} agree on ℓ (horizontal or vertical, depending on the parity of m) collections of $\ell+1$ points.

These properties, combined with properties (1) and (2) of ψ ensure both that the degree $\leq \ell$ terms in the polynomials $\wp_m$ converge and that the limiting polynomial is a good approximation to ψ on any cone $\mathcal{K}_\kappa$. We will say more about the construction of S_m shortly; we note that it will be necessary to construct more than one such sequence, in order to prove that $\wp$ is a good approximation at all points in $\mathcal{K}$, and not just those points on which ψ was interpolated.

The starting point in Journé's argument is to prove a higher dimensional version of the following interpolation lemma.

***Lemma 8.6* (Basic interpolation lemma. [19]).** — *Fix $\ell \geq 1$. For each $B \geq 1$, there exists $C_0 = C_0(B) > 0$ with the following property. If the collection of points $\{z_0, z_1, \dots, z_\ell\} \subset \mathbb{R}$ satisfies $R/\eta < B$, where*

$$R = \sup_j |z_j| \quad \textit{and} \quad \eta = \inf_{j \neq j'} |z_j - z_{j'}|,$$

Then for any values $\{b_0, \dots, b_\ell\} \subset \mathbb{R}$, there exists a unique polynomial

$$\wp(x) = \sum_{p=0}^{\ell} c_p x^p$$

such that $\wp(z_j) = b_j$, for $0 \leq j \leq \ell$. Moreover,

$$\sum_p |c_p| R^p \leq C \sup_j |b_j|.$$

Journé's generalization of Lemma 8.6 allows one to interpolate values of a function on a collection of $(\ell+1)^2$ points in $\mathbb{R}^2$ that lie in a rectangle-like configuration – like the sets S_m described above – by a degree $(\ell+1)^2$ polynomial whose C^0 size is controlled on the scale of the grid:

***Lemma 8.7* (Rectangle interpolation lemma. [19], Lemma 1; cf. [30], Lemma 4.5)**

Fix $\ell \geq 1$. For each $B \geq 1$, there exist $\theta_0 = \theta_0(B) > 0$ and $C_0 = C_0(B) > 0$ with the following property. If the collections of points $\{z_{j,k} : 0 \leq j \leq \ell,\ 0 \leq k \leq \ell\} \subset \mathbb{R}^2$, $\{x_j : 0 \leq j \leq \ell\} \subset \mathbb{R}$ and $\{y_k : 0 \leq k \leq \ell\} \subset \mathbb{R}$ satisfy:

$$R/\eta < B, \quad \textit{and} \quad |z_{j,k} - (x_j, y_k)| \leq \theta_0 \eta,$$

where

$$R = \sup_{j,k} |z_{j,k}| \quad \textit{and} \quad \eta = \inf_{(j,k) \neq (j',k')} |z_{j,k} - z_{j',k'}|,$$

Then for any values $\{b_{j,k} : 0 \leq j \leq \ell, 0 \leq k \leq \ell\} \subset \mathbb{R}$, *there exists a unique polynomial*

$$\wp(x,y) = \sum_{0 \leq p,q \leq \ell} c_{pq} x^p y^q$$

such that $\wp(z_{j,k}) = b_{j,k}$, *for* $0 \leq j,k \leq \ell$. *Moreover,*

$$\sum_{p,q} |c_{p,q}| R^{p+q} \leq C_0 \sup_{j,k} |b_{j,k}|.$$

As mentioned above, to create the sets S_m, we will intersect plaques of our transverse plaque families. The next lemma gives control over the location of the intersection of two transverse plaques.

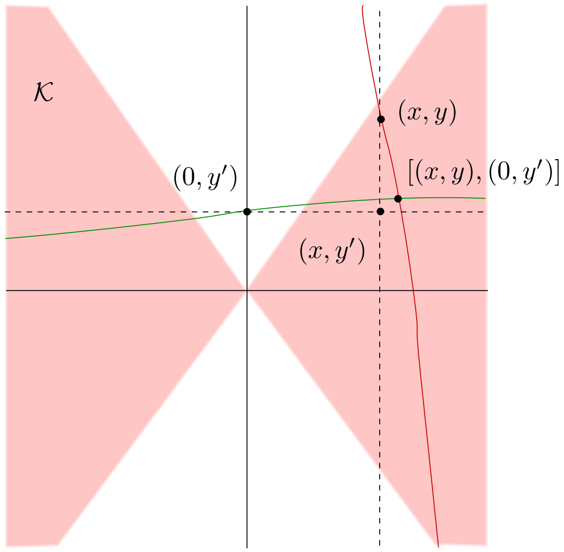


FIGURE 2. Lemma 8.8

***Lemma 8.8* (Local product structure).** — *For every* $K, \theta > 0$, *there exist* $\rho_0 = \rho_0(K) > 0$ *and* $\rho_1 = \rho_1(K,\theta) > 0$ *with* $\rho_1 < \rho_0$ *such that, for any parametrized* $C^{\ell,\alpha}$ *transverse pair of plaque families* (ω^H, ω^V) *with* $\|(\omega^H,\omega^V)\|_1 \leq K$, *and any* $z_1, z_2 \in B_{\mathbb{R}^{m+n}}(0,\rho_0)$, *the manifolds* $\omega^V_{z_1}(I^m)$ *and* $\omega^H_{z_2}(I^n)$ *intersect transversely in a single point* $[z_1,z_2] \in I^{m+n}$. *Moreover, if* $|(x,y)| < \rho_1$, *and* $|(x',y')| < \rho_1$ *then*

$$|[(x,y),(0,y')] - (x,y')| < \theta(|(x,y)| + |y'|),$$

and

$$|[(x',0),(x,y)] - (x',y)| < \theta(|(x,y) + |x'|).$$

Proof. — This is a simple consequence of the fact that the transverse plaque families are continuous in the C^1 topology. $\square$

Fix $K > 0$ and $\kappa \geq 1$ and let $\rho_0 = \rho_0(K)$. Fix (ω^H, ω^V) such that $\|(\omega^H, \omega^V)\|_{\ell,\alpha} \leq K$. We now define the *base grid:*

$$\mathcal{G}_0 = \mathcal{G}_0(\omega^H, \omega^V) = (\{\mathcal{F}_j^V\}_{j \in \mathbb{Z}_+ \cup \{\infty\}}, \{\mathcal{F}_k^H\}_{k \in \mathbb{Z}_+ \cup \{\infty\}})$$

of horizontal and vertical plaques from which we will eventually construct the sets S_m. We fix $r \in (0,1)$, and let $\mathcal{F}_\infty^H = \mathcal{F}^H(0,0)$ and $\mathcal{F}_\infty^V = \mathcal{F}^V(0,0)$, and for $j, k \geq 1$ set $\mathcal{F}_j^V = \mathcal{F}^V(r^j, 0)$ and $\mathcal{F}_k^V = \mathcal{F}^V(0, r^k)$.

For each (nonzero) $w \in B_{\mathbb{R}^{m+n}}(0, \rho_0)$, we also define a new grid $\mathcal{G}_w$ as follows. We choose $j = j(w) \in \mathbb{Z}_+$ such that the quantity

$$|[w, (0,0)] - r^j|$$

is minimized. The grid $\mathcal{G}_w$ is then the same as $\mathcal{G}_0$, except that the vertical leaf $\mathcal{F}_j^V$ in $\mathcal{G}_0$ is redefined: $\mathcal{F}_j^V = \mathcal{F}^V(w)$. This is illustrated in Figure 3.

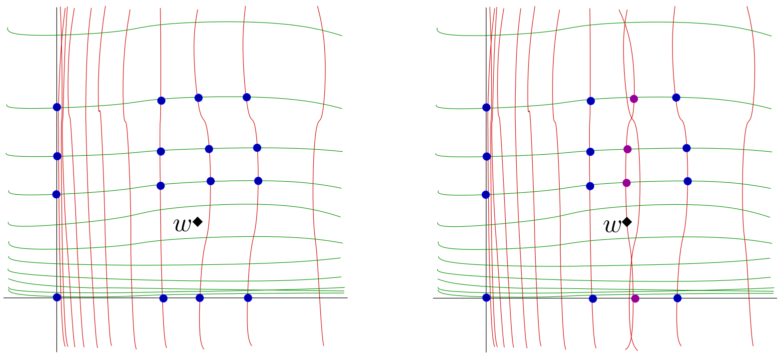


FIGURE 3. Grid substitution

Each grid $\mathcal{G} = (\{\mathcal{F}_j^V\}, \{\mathcal{F}_k^H\})$ defines sequences of points $\{z_{j,k}\}_{j,k \in \mathbb{Z}_+} \subset \mathbb{R}^2$, and $\{x_j\}, \{y_k\} \subset \mathbb{R}$ via: $\{z_{j,k}\} = \mathcal{F}_j^V \cap \mathcal{F}_k^H$, $\{(x_j, 0)\} = \mathcal{F}_j^V \cap \mathcal{F}_\infty^H$, and $\{(0, y_k)\} = \mathcal{F}_\infty^V \cap \mathcal{F}_k^H$. For each pair (j,k) with $|j - k| \leq 1$, we then define

$$H_{j,k} = H_{j,k}(\mathcal{G}) = \{x_{j'} : j \leq j' \leq j + \ell\}, V_{j,k} = V_{j,k}(\mathcal{G}) = \{y_{k'} : k \leq k' \leq k + \ell\}$$

and

$$J_{j,k} = J_{j,k}(\mathcal{G}) = \{z_{j',k'} : j \leq j' \leq j + \ell,\ k \leq k' \leq k + \ell\}.$$

***Lemma 8.9* (Grids are good).** — *For every $K > 0$ and $\kappa > 1$, there exists $\rho_2 = \rho_2(K, \kappa) > 0$ such that if $\|(\omega^H, \omega^V)\|_1 \leq K$, then for every $\theta > 0$, there exists an integer $k_0 = k_0(K, \kappa, \theta) > 0$ such that: for all $k \geq k_0$, for all j with $|j - k| \leq 1$, and for all $w \in B_{\mathbb{R}^{m+n}}(0, \rho_2) \cap \mathcal{K}_\kappa$, the grid $\mathcal{G}_w$ has the following properties.*

$$R_{j,k}/\eta_{j,k} \leq 6r^{\ell-2}, \quad \textit{and} \quad \sup_{z_{j',k'} \in J_{j,k}} |z_{j',k'} - (x_{j'}, y_{k'})| \leq \theta \eta_{j,k},$$

where

$$R_{j,k} = \sup_{z\in J_{j,k}} |z| \quad \textit{and} \quad \eta_{j,k} = \inf_{z,z'\in J_{j,k},\, z\neq z'} |z-z'|.$$

Moreover, $R_{j,k} \leq 3r^{k-1}$.

Proof of Lemma 8.9. — Let $K > 0$ and $\kappa > 1$ be given, and suppose that $\|(\omega^H, \omega^V)\|_1 \leq K$.

We choose ρ_2 such that for all $w \in B_{\mathbb{R}^2}(0,\rho_2) \cap \mathcal{K}_\kappa$, and for j sufficiently large (greater than some j_0), if j minimizes the quantity $|[w,(0,0)] - r^j|$, then $|w| \leq 2(1+\kappa)r^j$. This is possible, by Lemma 8.8.

Let $\theta > 0$ be given; we will describe below how to choose a constant $\theta_1 = \theta_1(K,\kappa,\theta)$. Assuming this choice has been made, let $\rho_1 = \rho_1(K,\theta_1)$ be given by Lemma 8.8. We choose $k_0 > j_0$ such that $\max\{2(1+\kappa)r^{k-1}, R_{j,k}\} < \rho_1$, for all $|j-k| \leq 1$ and $k \geq k_0$.

Let $w \in B_{\mathbb{R}^{m+n}}(0,\rho_2) \cap \mathcal{K}_\kappa$, and consider the grid $\mathcal{G}_w$. For $j,k \in \mathbb{Z}_+$ satisfying $|j-k| \leq 1$, and $k \geq k_0$, fix a point $z \in J_{j,k}$, which by definition is the point of intersection of $\mathcal{F}^V_{j'}$ and $\mathcal{F}^H_{k'}$, for some $k-1 \leq j', k' \leq k+\ell+1$. Write $z = (x,y)$ and $w = (x',y')$. There are two possibilities. Either $\mathcal{F}^V_{j'}$ is in the base grid $\mathcal{G}_0$, or $\mathcal{F}^V_{j'} = \mathcal{F}^V(w)$.

In the first case, since $z \in \mathcal{F}^V_{j'} \cap \mathcal{F}^H_{k'}$, we have $|z| < \rho_1$. Lemma 8.8 implies that

$$|[(0,0),(x,y)] - (0,y)| = |y_{k'} - y| = |r^{k'} - y| < \theta_1|(x,y)|$$

and

$$|[(x,y),(0,0)] - (x,0)| = |x_{j'} - x| = |r^{j'} - x| < \theta_1|(x,y)|.$$

and so $|z - (x_{j'}, y_{k'})| \leq |x_{j'} - x| + |y_{k'} - y| \leq 2\theta_1|z|$. Since $|(x_{j'}, y_{k'})| \leq 2r^{k-1}$, we therefore have, for θ_1 sufficiently small:

$$|z| \quad\leq\quad 3r^{k-1}, \tag{27}$$

and

$$|z - (x_{j'}, y_{k'})| \leq 6\theta_1 r^{k-1}. \tag{28}$$

Suppose, on the other hand, that $\mathcal{F}^V_{j'} = \mathcal{F}^V(w)$. Then the point $(x_{j'}, 0) = [w,(0,0)]$ has the property that

$$|x_{j'} - r^{j'}| \leq \frac{1}{2}|r^{j'} - r^{j'+1}| = \frac{(1-r)}{2} r^{j'} < \frac{r^{j'}}{2}.$$

Since $w \in B_{\mathbb{R}^2}(0,\rho_2) \cap \mathcal{K}_\kappa$, and $j' \geq k_0$, we have that $|w| < 2(1+\kappa)r^{j'-1} < \rho_1$. Hence Lemma 8.8 implies that $|x_{j'} - x'| = |[w,(0,0)] - (x',0)| \leq \theta_1(|w| + |x'|)$; This implies that $|x_{j'} - x'| =\leq \theta_1(|w| + |x'|) \leq 2\theta_1|w| \leq 4\theta_1(1+\kappa)r^{j'-1}$.

Now $z = [w,(0,r^{k'})]$ and $|[w,(0,r^{k'})] - (x', r^{k'})| \leq \theta_1(|w| + r^{k'}) \leq \theta_1(3+2\kappa)r^{k-1}$. Using the triangle inequality, we conclude that, for θ_1 sufficiently small, we have

$$|z| \quad\leq\quad 3r^{k-1} \tag{29}$$

and

$$|z-(x_{j'},y_{k'})| \leq |z-(x',r^{k'})| + |x_{j'}-x'| \leq \theta_1(7+6\kappa)r^{k-1}. \tag{30}$$

Hence, in either case, we conclude that

$$R_{j,k} \quad \leq \quad 3r^{k-1} \tag{31}$$

and

$$\sup_{z_{j',k'}\in J_{j,k}} |z_{j',k'}-(x_{j'},y_{k'})| \leq \theta_1(7+6\kappa)r^{k-1}. \tag{32}$$

On the other hand,

$$\eta_{j,k} \quad \geq \quad \inf_{j'\neq j''} |y_{j'}-y_{j''}| - \sup_{z_{j',k'}\in J_{j,k}} |z_{j',k'}-(x_{j'},y_{k'})| \tag{33}$$

$$\quad > \quad r^{\ell+k+1} - \theta_1(7+6\kappa)r^{k-1}, \tag{34}$$

and for θ_1 sufficiently small, we get $\eta_{j,k} \geq r^{\ell+k+1}/2$. Combining this with (31), we have $R_{j,k}/\eta_{j,k} \leq 6r^{\ell-2}$. Combining (33) with (32) we also get:

$$\sup_{z_{j',k'}\in J_{j,k}} |z_{j',k'}-(x_{j'},y_{k'})| \leq \eta_{j,k}\frac{\theta_1(7+6\kappa)r^{k-1}}{r^{\ell+k+1}-\theta_1(7+6\kappa)r^{k-1}}.$$

Choosing $\theta_1 = \theta_1(K,\kappa,\theta)$ small enough, we obtain that

$$\sup_{z_{j',k'}\in J_{j,k}} |z_{j',k'}-(x_{j'},y_{k'})| \leq \theta\eta_{j,k},$$

which finishes the proof. □

Let $B = 6r^{\ell-2}$ and let $\rho_2 = \rho_2(K,\kappa) > 0$ be given by Lemma 8.9. Let $\theta_0 = \theta_0(B) > 0$ and $C_0 = C_0(B) > 0$ be given by Lemma 8.7. Now let $k_0 = k_0(K,\kappa,\theta_0) > 0$ be given by Lemma 8.9.

Fix $w \in B_{\mathbb{R}^{m+n}}(0,\rho_2)$. We now define the sequence S_m of rectangles associated to the grid $\mathscr{G}_w$. For $|j-k| \leq 1$, we set:

$$S_{j,k} = \{0,0\} \cup (H_{j,k}\times\{0\}) \cup (\{0\}\times V_{j,k}) \cup J_{j,k}.$$

Now, let $S_{2k} = S_{k,k}$ and let $S_{2k+1} = S_{k,k+1}$. Define the sets H_m, V_m, and J_m analogously, for $m \in \mathbb{Z}_+$. Let $R_m = \sup_{z\in J_m} |z|$ and let $\eta_m = \inf_{z,z'\in J_m,\, z\neq z'} |z-z'|$. Lemma 8.9 implies that for $m \geq 2k_0$, we have $|R_m| \leq 3r^{(m-1)/2}$, and $R_m/\eta_m \leq B$.

By Lemma 8.7, there exists a constant $C_0 = C_0(B) > 0$ such that for each $m \geq 2k_0$, and any function ψ, there exists a unique (degree $(\ell+1)^2$) polynomial $\wp_m = \wp_m((\omega^H,\omega^V),w,\psi)$:

$$\wp_m(x,y) = \sum_{0\leq p,q\leq \ell} c^m_{p,q} x^p y^q$$

that interpolates ψ on the rectangle S_m. Furthermore:

$$\sum_{p,q} |c^m_{p,q}| R_m^{p+q} \leq C_0 \sup\{\psi(z) : z \in S_m\}, \tag{35}$$

where R_m is defined above.

Lemma 8.10. — *For every $K, C > 0$, there exist constants $C_1 = C_1(K,C) > 0$ and $\rho = \rho(K,C) > 0$, such that for all (ω^H, ω^V) with $\|(\omega^H,\omega^V)\|_{\ell,\alpha} \leq K$, for all $w \in B_{\mathbb{R}^2}(0,\rho_2) \cap \mathscr{K}$ and for all ψ satisfying hypotheses (1) and (2) of Theorem 8.4 for this value of C, the sequence $c^m_{p,q} = c^m_{p,q}(\mathscr{G}_w, \psi)$ has the following property.*

Let $\overline{\wp}_m(x,y) = \sum_{p+q\leq \ell} c^m_{p,q} x^p y^q$. Then there exists a polynomial $\overline{\wp}$ such that $\overline{\wp} = \lim_{m\to\infty} \overline{\wp}_m$ (uniformly on compact sets). Furthermore:

$$|\overline{\wp}(z) - \psi(z)| \leq C_1 |z|^{\ell+\alpha} \quad \textit{for} \quad z \in \mathscr{K} \cap \bigcup_{k\geq k_0} \mathscr{F}^V_k \cap B_{\mathbb{R}^{m+n}}(0,\rho).$$

We first finish the proof of Lemma 8.5, assuming Lemma 8.10. Let $C > 0$ and ψ be given satisfying hypotheses (1) and (2) for this value of C. Let $C_1 = C_1(K,C) > 0$ and $\rho = \rho(K,C) > 0$ be given by Lemma 8.10. Given $w \in B_{\mathbb{R}^2}(0,\rho) \cap \mathscr{K}$, let $\overline{\wp} = \overline{\wp}(\mathscr{G}_w, \psi)$ be given by Lemma 8.10. By construction of the grid $\mathscr{G}_w$, we have that $w \in \bigcup_{k\geq k_0} \mathscr{F}^V_k$. This implies in particular that

$$|\overline{\wp}(w) - \psi(w)| \leq C_1 |w|^{\ell+\alpha}.$$

Let $w' \in B_{\mathbb{R}^2}(0,\rho) \cap \mathscr{K}$ be another point, and let $\overline{\wp}' = \overline{\wp}(\mathscr{G}_{w'}, \psi)$. By the same reasoning,

$$|\overline{\wp}'(w') - \psi(w')| \leq C_1 |w'|^{\ell+\alpha}.$$

Note that the sequences $c^m_{p,q}(\mathscr{G}_w, \psi)$ and $c^m_{p,q}(\mathscr{G}_{w'}, \psi)$ differ in only finitely many places. This implies that $\overline{\wp}' = \overline{\wp}$. The polynomial $\wp = \overline{\wp}$ satisfies the conclusions of Lemma 8.5. This completes the proof of Lemma 8.5. □

Proof of Lemma 8.10. — The proof follows the proof of Lemma 4.4 in [**30**] very closely; the only slight change occurs in the proof of Lemma 8.11, part (1) below, which corresponds to Lemma 4.8 in [**30**]. We outline the proof and refer the reader to [**30**] or [**19**] for the details.

Fix k and let $\wp = \wp_{2k}$ and $\wp' = \wp'_{2k+1}$ be the interpolating polynomials on $S_{2k} = S_{k,k}$ and $S_{2k+1} = S_{k,k+1}$, respectively. Denote their coefficients by $c_{p,q}$ and $c'_{p,q}$ respectively. Let $T_k = 3r^{k-1}$. We will show that

$$|c_{p,q} - c'_{p,q}| = O(T_k^{\ell+\alpha-p-q}).$$

By Lemma 8.7, it is enough to consider the polynomial $\wp - \wp'$ and find an upper bound for $|\wp - \wp'|$ on $S_{k,k+1}$. Note that $\wp$ and $\wp'$ agree on $S_{k,k+1}$, except at the ℓ points $z_{j,k+\ell}$, $k \leq j \leq k+\ell$. On these points we have $\wp'(z_{j,k+\ell}) = \psi(z_{j,k+\ell})$. Hence we need only estimate $|\psi(z_{j,k+\ell}) - \wp(z_{j,k+\ell})|$, for $k \leq j \leq k+\ell$. For such a j, write $\mathscr{F}^V_j$ as a graph of a function of the second coordinate: $\mathscr{F}^V_j = \{(x_j(y), y) \colon y \in I\}$, and let $z_j(y) = (x_j(y), y)$. Notice that, in the case where $j = j(w)$, we have $z_j(y) = \omega^V_w(y - y_w)$, where $w = (x_w, y_w)$; otherwise, $z_j(y) = \omega^V_{(x_j,0)}(y)$. Note that in either case, $x_j(0) = x_j$, and the function $x_j(y)$ would be constant if the curve $\mathscr{F}^V_{j,k}$ were truly vertical. The following estimates would be trivial if x_j were a constant function. The hypothesis that (ω^H, ω^V) is uniformly $C^{\ell,\alpha}$ will be used as in [**19, 30**] to estimate the $C^{\ell,\alpha}$ size of $x_j(y)$.

Choose a constant $C_2 > 0$ so that $\{z_j(y) : y \in I_k\}$ contains all the points in $\omega^V_{(x_j,0)}(I) \cap S_{k,k} \cap \mathcal{K}$, for all $k \geq k_0$ and $k \leq j \leq k+\ell$, where I_k is the interval $I_k := [-C_2T_k, C_2T_k]$. We next show that $|\psi(z_j(y)) - \wp(z_j(y))| = O(T_k^{\ell+\alpha})$, for $k \leq j \leq k+\ell$ and any $y \in I_k$. Fix such a j. For $h\colon I^2 \to \mathbb{R}$, write $\tilde{h}(y)$ for $h(z_j(y))$. We will restrict attention to the domain I_k.

Lemma 8.11. — *There exists $C_3 > 0$ such that if $k \geq k_0$, $k \leq j \leq j+\ell$, and $y \in I_k$, then:*

1. $$|(\tilde{\psi} - \tilde{\wp})(y)| \leq C_3 \left\| \frac{d^\ell}{dy^\ell}(\tilde{\wp}) \right\|_\alpha T_k^{\ell+\alpha} + C_3 T_k^{\ell+\alpha},$$
2. *if $p, q \leq \ell$ and $p+q > \ell$, then*
$$\left\| \frac{d^\ell}{dy^\ell} x_j^p(y) y^q \right\|_\alpha \leq C_3 T_k^{p+q-\ell-\alpha} \|x_j\|_{C^{\ell,\alpha}(I_k)},$$
3. *if $p+q \leq \ell$, then*
$$\left\| \frac{d^\ell}{dy^\ell} x_j^p(y) y^q \right\|_\alpha \leq C_3,$$
4. *and therefore*
$$\left\| \frac{d^\ell}{dy^\ell} \tilde{\wp} \right\|_\alpha \leq C_3 \|x_j\|_{C^{\ell,\alpha}(I_k)} \sum_{p+q>\ell} |c_{p,q}| T_k^{p+q-\ell-\alpha} + C_3 \sum_{p+q\leq\ell} |c_{p,q}|.$$

Proof. — To prove (1), recall that $z_j(y) = \omega^V_w(y - y_w)$, if $j = j(w)$, and $z_j(y) = \omega^V_{(x_j,0)}(y)$ otherwise. The hypotheses of Theorem 8.4 imply that
$$\tilde{\psi}(z_j(y)) = \psi(\omega^V_{z_0}(y - y_0)) = \wp^V_z(y - y_0) + r^V_j(y - y_0),$$
where $z_0 \in \{w, (x_j, 0)\}$ and $y_0 \in \{0, y_w\}$, and $|r^V_j(y - y_0)| \leq C(|z|^{\ell+\alpha} + |y - y_0|^{\ell+\alpha})$. Now $|z_0| = O(T_k)$ and $|y_0| = O(T_k)$ (since $w \in \mathcal{K}$), which implies that $|r^V_j(y)| = O(T_k^{\ell+\alpha})$, for $y \in I_k$.

Writing the Taylor expansion of of the $C^{\ell,\alpha}$ function $\tilde{\wp}$ about 0, we have
$$\tilde{\wp}(y) = Q(y) + R_j(y),$$
where Q is a degree ℓ polynomial and $|R_j(y)| = O(|y|^{\ell+\alpha} \left\| \frac{d^\ell}{dy^\ell}\tilde{\wp} \right\|_\alpha) = O(T_k^{\ell+\alpha} \left\| \frac{d^\ell}{dy^\ell}\tilde{\wp} \right\|_\alpha)$, for $y \in I_k$. Recall that, since $k \leq j \leq k+\ell$, the polynomial $\wp$ interpolates ψ on the $\ell+1$ points in $S_{k,k+1} \cap \mathcal{F}^V(x_j, 0)$. Therefore the degree ℓ polynomial $\overline{Q}(y) = Q(y) - \wp^V_{z_0}(y - y_0)$ on I_k takes the value $r^V_j(t_i) + R_j(t_i)$ at the $\ell+1$ points
$$\{0 = t_0, t_1, \ldots, t_\ell\} = (\omega^V_{(x_j,0)})^{-1}\left(S_{k,k+1} \cap \mathcal{F}^V(x_j, 0)\right).$$
Lemma 8.8 implies the points $\{0, t_1, \ldots, t_\ell\}$ in I_k are spaced $\Theta(T_k)$ apart. Since $|\overline{Q}(t_i)| \leq |r^V_j(t_i) + R_j(t_i)| = O(T_k^{\ell+\alpha} + T_k^{\ell+\alpha} \left\| \frac{d^\ell}{dy^\ell}\tilde{\wp} \right\|_\alpha)$, for $i \in \{0, \ldots, \ell\}$, Lemma 8.6 then gives the desired inequality in (1).

The last three parts are proved in **[30]** (part (4) follows from (2) and (3)). □

Given $\delta > 0$, we may assume that $k_0 > 0$ was chosen sufficiently large so that $\|x_j\|_{C^{\ell,\alpha}(I_k)} < \delta$. Then we have

$$|(\psi - \wp)(z_j(y))| \leq C_3 T_k^{\ell+\alpha} + C_3 \delta \sum_{p+q>\ell} |c_{p,q}| T_k^{p+q} + C_3 \sum_{p+q\leq\ell} |c_{p,q}| T_k^{\ell+\alpha},$$

for all $y \in I_k$. Plugging $y = z_{j,k+\ell}$ into this equation (and recalling that $\wp'(z_{j,k+\ell}) = \psi(z_{j,k+\ell})$), and using (35) for $\wp - \wp'$ on $S_{k,k+1}$, we get:

$$\sum_{p,q} |c'_{p,q} - c_{p,q}| T_k^{p+q} \leq C_4 \left(T_k^{\ell+\alpha} + \delta \sum_{p+q>\ell} |c_{p,q}| T_k^{p+q} \sum_{p+q\leq\ell} |c_{p,q}| T_k^{\ell+\alpha} \right).$$

(cf. equation (4.11) in **[30]**).

Now the proof proceeds exactly as in **[30]**, and we obtain a polynomial $\overline{\wp}$ satisfying the conclusions of Lemma 8.10. □

9. Saturated sections of partially hyperbolic extensions

We recast Theorem A, part IV as a more general statement about saturated sections of partially hyperbolic extensions.

Definition 9.1. — *Let $f : M \to M$ be C^k and partially hyperbolic. A C^k* partially hyperbolic extension of f *is a tuple $(N, \mathcal{B}, \pi, F)$, where N is a C^∞ manifold, $\pi : \mathcal{B} \to M$ is a C^∞ fiber bundle over M with fiber N, and $F : \mathcal{B} \to \mathcal{B}$ is a C^k, partially hyperbolic diffeomorphism satisfying:*

1. *$\pi \circ F = f \circ \pi$, and*
2. *$E^c_F = T\pi^{-1}(E^c_f)$.*

We say that $(N, \mathcal{B}, \pi, F)$ is an r-bunched extension *if there exists a Riemannian metric $< \cdot, \cdot >$ on $\mathcal{B}$ and functions $\nu, \hat{\nu}, \gamma$, and $\hat{\gamma}$ on $\mathcal{B}$ satisfying (4)–(6) such that, for every $x \in M$:*

$$\sup_{z\in\pi^{-1}(x)} \nu(z) < \inf_{z\in\pi^{-1}(x)} \{\gamma(z), \gamma^r(z)\}, \quad \sup_{z\in\pi^{-1}(x)} \hat{\nu}(z) < \inf_{z\in\pi^{-1}(x)} \{\hat{\gamma}(z), \hat{\gamma}^r(z)\},$$

$$\frac{\sup_{z\in\pi^{-1}(x)} \nu(z)}{\inf_{z\in\pi^{-1}(x)} \gamma(z)} < \inf_{z\in\pi^{-1}(x)} \hat{\gamma}^r(z), \quad \textit{and} \quad \frac{\sup_{z\in\pi^{-1}(x)} \hat{\nu}(z)}{\inf_{z\in\pi^{-1}(x)} \hat{\gamma}(z)} < \inf_{z\in\pi^{-1}(x)} \gamma^r(z).$$

If $(N, \mathcal{B}, \pi, F)$ is an r-bunched extension of f, then f is r-bunched. To see this, we construct a Riemannian metric on M in which the inequalities in (8) and (9) hold. This is achieved by fixing a horizontal distribution $\mathrm{Hor} \subset T\mathcal{B}$, transverse to $\ker T\pi$ and containing $E^u_F \oplus E^s_F$, and defining, for $v \in T_xM$, the metric $< \cdot, \cdot >'$ by $< v_1, v_2 >'_x = \sup < w_1, w_2 >_z$, where the supremum is taken over all $w_i \in T\pi^{-1}(v_i) \cap \mathrm{Hor}(z)$, with $z \in \pi^{-1}(x)$. In this metric, the r-bunching inequalities hold for f, with $\nu(x) = \sup_{z\in\pi^{-1}(x)} \nu(z)$, $\hat{\nu}(x) = \sup_{z\in\pi^{-1}(x)} \hat{\nu}(z)$, $\gamma(x) = \inf_{z\in\pi^{-1}(x)} \gamma(z)$, and $\hat{\gamma}(x) = \inf_{z\in\pi^{-1}(x)} \hat{\gamma}(z)$.

If $(N, \mathcal{B}, \pi, F)$ is a partially hyperbolic extension of f, it follows that $\mathcal{B} \to M$ is an admissible bundle with $\mathcal{W}^s_{\mathrm{lift}} = \mathcal{W}^s_F$ and $\mathcal{W}^u_{\mathrm{lift}} = \mathcal{W}^u_F$. We say that a section $\sigma : M \to \mathcal{B}$ is *bisaturated* if it is bisaturated with respect to these lifted foliations (see Definition 4.1). We have the following theorem.

Theorem C. — *Let $f : M \to M$ be C^k, partially hyperbolic and accessible, for some integer $k \geq 2$. Let $(N, \mathcal{B}, \pi, F)$ be a C^k partially hyperbolic extension of f that is r-bunched, for some $r < k - 1$ or $r = 1$.*

Let $\sigma : M \to \mathcal{B}$ be a bisaturated section. Then σ is C^r.

Remark: One might ask whether the same conclusion holds if σ is instead assumed to be a continuous F-invariant section. The answer is no. De la Llave has constructed examples of an r-bunched extension of a linear Anosov diffeomorphism with a continuous F-invariant section that fails to be C^1. What is more, this section is $C^{(1/r)-\varepsilon}$, for all $\varepsilon > 0$, but fails to be $C^{1/r}$ (see [**31**], Theorem 4.1).

What is true is the following. Suppose that $(N, \mathcal{B}, \pi, F)$ is an r-bunched partially hyperbolic extension of f. Then there exists a critical Hölder exponent $\alpha_0 \geq 0$ such that, if σ is an F-invariant section of N that is Hölder continuous with exponent α_0, then σ is bisaturated, and hence C^r. The exponent α_0 is determined by $\nu, \hat{\nu}$ and the norm and conorm of the action of TF on fibers of N. When F is an isometric extension of f (as with abelian cocycles, or cocycles taking values in a compact Lie group), then $\alpha_0 = 0$, and any continuous invariant section is bisaturated. In general, if F is an r-bunched extension, then $\alpha_0 \leq 1/r$, but it can be smaller, as is the case with isometric extensions. The proof of these assertions is similar to the proof of Proposition 4.7; see also ([**31**], Theorem 2.2).

9.1. Proof of Theorem A, Part IV from Theorem C. — Suppose that f is C^k, accessible and strongly r-bunched and that ϕ is C^k, for some $k \geq 2$ and $r < k - 1$ or $r = 1$. Then the skew product $f_\phi \colon M \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a C^k, r-bunched, partially hyperbolic extension of f. If Φ is a continuous solution to (2), then Proposition 4.7 implies that Φ is bisaturated. Then the map $x \mapsto (x, \Phi(x) \,(\mathrm{mod} 1))$ is a bisaturated section of $M \times \mathbb{R}/\mathbb{Z}$. Theorem C implies that this section is C^r. This implies that Φ is C^r.

10. Tools for the proof of Theorem C

We finally delve into the details of the proof of Theorem C, which is the heart of this paper.

10.1. Fake invariant foliations. — Recall that to prove Theorem A, part IV, when f is dynamically coherent, one can make use of the stable and unstable holonomy maps for f and F between center manifolds; more generally this strategy can be used to prove Theorem C when f is dynamically coherent. Since we do not assume that

f is dynamically coherent, we use in place of the center foliation a locally-invariant family of center plaques (see [**18**], Theorem 5.5). The stable holonomy between center-manifolds is replaced by holonomy along locally-invariant, "fake" stable foliations, first introduced as a tool in [**9**]. These foliations are defined in the next proposition.

***Proposition 10.1* (cf. [9], Proposition 3.1).** — *Let $f : M \to M$ be a C^r partially hyperbolic diffeomorphism. For any $\varepsilon > 0$, there exist constants ρ and ρ_1 with $\rho > \rho_1 > 0$ such that, for every $p \in M$, the neighborhood $B_M(p, \rho)$ is foliated by foliations $\widehat{\mathcal{W}}^u_p$, $\widehat{\mathcal{W}}^s_p$, $\widehat{\mathcal{W}}^c_p$, $\widehat{\mathcal{W}}^{cu}_p$ and $\widehat{\mathcal{W}}^{cs}_p$ with the following properties:*

1. **Almost tangency to invariant distributions:** *For each $q \in B_M(p, \rho)$ and for each $* \in \{u, s, c, cu, cs\}$, the leaf $\widehat{\mathcal{W}}^*_p(q)$ is C^1 and the tangent space $T_q\widehat{\mathcal{W}}^*_p(q)$ lies in a cone of radius ε about $E^*(q)$.*
2. **Local invariance:** *for each $q \in B_M(p, \rho_1)$ and $* \in \{u, s, c, cu, cs\}$,*
$$f(\widehat{\mathcal{W}}^*_p(q, \rho_1)) \subset \widehat{\mathcal{W}}^*_{f(p)}(f(q)), \ \ \text{and } f^{-1}(\widehat{\mathcal{W}}^*_p(q, \rho_1)) \subset \widehat{\mathcal{W}}^*_{f^{-1}(p)}(f^{-1}(q)).$$
3. **Exponential growth bounds at local scales:** *The following hold for all $n \geq 0$.*
   (a) *Suppose that $q_j \in B_M(p_j, \rho_1)$ for $0 \leq j \leq n-1$.*
   If $q' \in \widehat{\mathcal{W}}^s_p(q, \rho_1)$, then $q'_n \in \widehat{\mathcal{W}}^s_p(q_n, \rho_1)$, and
   $$d(q_n, q'_n) \leq \nu_n(p) d(q, q').$$
   If $q'_j \in \widehat{\mathcal{W}}^{cs}_p(q_j, \rho_1)$ for $0 \leq j \leq n-1$, then $q'_n \in \widehat{\mathcal{W}}^{cs}_p(q_n)$, and
   $$d(q_n, q'_n) \leq \hat{\gamma}_n(p)^{-1} d(q, q').$$
   (b) *Suppose that $q_{-j} \in B_M(p_{-j}, \rho_1)$ for $0 \leq j \leq n-1$.*
   If $q' \in \widehat{\mathcal{W}}^u_p(q, \rho_1)$, then $q'_{-n} \in \widehat{\mathcal{W}}^u_p(q_{-n}, \rho_1)$, and
   $$d(q_{-n}, q'_{-n}) \leq \hat{\nu}_{-n}(p)^{-1} d(q, q').$$
   If $q'_{-j} \in \widehat{\mathcal{W}}^{cu}_p(q_{-j}, \rho_1)$ for $0 \leq j \leq n-1$, then $q'_{-n} \in \widehat{\mathcal{W}}^{cu}_p(q_{-n})$, and
   $$d(q_{-n}, q'_{-n}) \leq \gamma_{-n}(p) d(q, q').$$
4. **Coherence:** *$\widehat{\mathcal{W}}^s_p$ and $\widehat{\mathcal{W}}^c_p$ subfoliate $\widehat{\mathcal{W}}^{cs}_p$; $\widehat{\mathcal{W}}^u_p$ and $\widehat{\mathcal{W}}^c_p$ subfoliate $\widehat{\mathcal{W}}^{cu}_p$.*
5. **Uniqueness:** *$\widehat{\mathcal{W}}^s_p(p) = \mathcal{W}^s(p, \rho)$, and $\widehat{\mathcal{W}}^u_p(p) = \mathcal{W}^u(p, \rho)$.*
6. **Leafwise regularity:** *The following regularity statements hold:*
   (a) *The leaves of $\widehat{\mathcal{W}}^u_p$ and $\widehat{\mathcal{W}}^s_p$ are uniformly C^r, and for $* \in \{u, s\}$, the leaf $\widehat{\mathcal{W}}^*_p(x)$ depends continuously in the C^r topology on the pair $(p, x) \in M \times B_M(p, \rho_1)$.*
   (b) *If f is r-bunched, then the leaves of $\widehat{\mathcal{W}}^{cu}_p$, $\widehat{\mathcal{W}}^{cs}_p$ and $\widehat{\mathcal{W}}^c_p$ are uniformly C^r, and for $* \in \{cu, cs, c\}$, the leaf $\widehat{\mathcal{W}}^*_p(x)$ depends continuously in the C^r topology on $(p, x) \in M \times B_M(p, \rho_1)$.*

7. **Regularity of the strong foliation inside weak leaves:** *If f is C^k and r-bunched, for some $r < k-1$ or $r = 1$, and $k \geq 2$, then each leaf of $\widehat{\mathcal{W}}_p^{cs}$ is C^r foliated by leaves of the foliation $\widehat{\mathcal{W}}_p^{s}$, and each leaf of $\widehat{\mathcal{W}}_p^{cu}$ is C^r foliated by leaves of the foliation $\widehat{\mathcal{W}}_p^{u}$.*

   Furthermore, the distribution $\widehat{E}_p^s(x) = T_x\widehat{\mathcal{W}}_p^{s}$ is C^r in $x \in \widehat{\mathcal{W}}_p^{cs}(p)$, and the map $x \mapsto \widehat{E}_p^s(x)$ on $\widehat{\mathcal{W}}_p^{cs}(p)$ depends continuously on $p \in M$ in the C^r topology. The distribution $\widehat{E}_p^u(x) = T_x\widehat{\mathcal{W}}_p^{u}$ is C^r in $x \in \widehat{\mathcal{W}}_p^{cu}(p)$, and the map $x \mapsto \widehat{E}_p^u(x)$ on $\widehat{\mathcal{W}}_p^{cu}(p)$ depends continuously on $p \in M$ in the C^r topology.

Proof. — The proof of parts (1)–(5) is contained in [**9**]. We review the proof there, as we will use the same method to prove parts (6) and (7). Some of the discussion below is taken from [**9**].

Suppose that f is C^r, for some $r \geq 1$. After possibly reducing ε, we can assume that inequalities (3)–(6) hold for unit vectors in the ε-cones around the spaces in the partially hyperbolic splitting.

The construction is performed in two steps. The first step is to construct foliations of each tangent space T_pM. In the second step, we use the exponential map $\exp_p$ to project these foliations from a neighborhood of the origin in T_pM to a neighborhood of p.

Step 1. In the first step of the proof, we choose $\rho_0 > 0$ such that $\exp_p^{-1}$ is defined on $B_M(p, 2\rho_0)$. For $\rho \in (0, \rho_0]$, we define, in the standard way, a continuous map $\mathbf{f}_\rho\colon TM \to TM$ covering f, which is uniformly C^r on fibers, satisfying:

1. $\mathbf{f}_\rho(p, v) = \exp_{f(p)}^{-1} \circ f \circ \exp_p(v)$, for $\|v\| \leq \rho$;
2. $\mathbf{f}_\rho(p, v) = T_pf(v)$, for $\|v\| \geq 2\rho$;
3. $\|\mathbf{f}_\rho(p, \cdot) - T_pf(\cdot)\|_{C^1} \to 0$ as $\rho \to 0$, uniformly in p;
4. $p \mapsto \mathbf{f}_\rho(p, \cdot)$ is continuous in the C^r topology.

Endowing M with the discrete topology, we regard TM as the disjoint union of its fibers. if ρ is small enough, then $\mathbf{f}_\rho$ is partially hyperbolic, and each bundle in the partially hyperbolic splitting for $\mathbf{f}_\rho$ at $v \in T_pM$ lies within the $\varepsilon/2$-cone about the corresponding subspace of T_pM in the partially hyperbolic splitting for f at p (we are making the usual identification of T_vT_pM with T_pM). If ρ is small enough, the equivalents of inequalities (3) will hold with Tf replaced by $T\mathbf{f}_\rho$. Further, if f is r-bunched, then $\mathbf{f}_\rho$ will also be r-bunched, for ρ sufficiently small.

If ρ is sufficiently small, standard graph transform arguments give stable, unstable, center-stable, and center-unstable foliations for $\mathbf{f}_\rho$ inside each T_pM. These foliations are uniquely determined by the extension $\mathbf{f}_\rho$. and the requirement that their leaves be graphs of functions with bounded derivative. We obtain a center foliation by intersecting the leaves of the center-unstable and center-stable foliations. Since the restriction of $\mathbf{f}_\rho$ to T_pM depends continuously in the C^r topology on p, the foliations of T_pM depend continuously on p.

The uniqueness of the stable and unstable foliations implies, via a standard argument (see, e.g., [**18**], Theorem 6.1 (e)), that the stable foliation subfoliates the center-stable, and the unstable subfoliates the center-unstable.

We now discuss the regularity properties of these foliations of TM. Recall the standard method for determining the regularity of invariant bundles and foliations.

***Theorem 10.2* (cf. C^r Section Theorem ([18], Theorem 3.2)).** — *Let X be a C^r manifold, let $\pi\colon E \to X$ be a C^r Finslered Banach bundle, and let $g\colon E \to E$ be a C^r bundle map covering the C^r diffeomorphism $h : X \to X$. Assume that the image of the 0-section under g is bounded.*

Assume that for every $x \in X$ there exists a constant κ_x such that

$$\sup_{x\in X} |\kappa_x| < 1,$$

and for every $y, y' \in \pi^{-1}(x)$, $\|g(y) - g(y')\|_{\pi^{-1}(h(x))} \leq \kappa_x \|y - y'\|_{\pi^{-1}(x)}$. Then there is a unique bounded section $\sigma\colon X \to M$ such that $g(\sigma(X)) = \sigma(X)$, and σ is continuous.

Moreover, if

$$\sup_{x\in X} \frac{\kappa_x}{\lambda_x^r} < 1, \qquad \textit{where} \quad \lambda_x = m(T_x h)$$

then σ is C^r.

This theorem is used to prove the C^r regularity of the stable and unstable foliations for a C^r partially hyperbolic diffeomorphism f, once the C^1 regularity has been established (via Lipschitz jets, or some similar method). We review this argument, as it is prototypical.

Assume that the leaves of $\mathcal{W}^u$ are C^1. Note that since the leaves of $\mathcal{W}^u$ are tangent to the continuous distribution E^u, this automatically implies that the map $x \mapsto \mathcal{W}^u(x)$ is continuous in the C^1 topology.

To prove that the leaves of $\mathcal{W}^u$ are uniformly C^r for $r > 1$, one fixes a C^∞ approximation $TM = \widetilde{E}^s \oplus \widetilde{E}^c \oplus \widetilde{E}^u$ to the partially hyperbolic splitting. One then takes the C^1 manifold X to be the disjoint union of the leaves of the unstable foliation and the fiber of the bundle E over x to be the space $L_x(\widetilde{E}^u, \widetilde{E}^{cs})$ of linear maps from $\widetilde{E}^u(x)$ to $\widetilde{E}^{cs}(x)$. The linear graph transform on the bundle E covers the original partially hyperbolic diffeomorphism $f|_X$, contracts the fiber over x by $\kappa_x = \|T_x f|_{E^{cs}}\|/m(T_x f|_{E^u}) < 1$, and expands X at x by at least $\lambda_x = m(T_x f|_{E^u}) > 1$.

Since the ratio

$$\frac{\kappa_x}{\lambda_x} = \frac{\|T_x f|_{E^{cs}}\|/m(T_x f|_{E^u})}{m(T_x f|_{E^u})}$$

is bounded away from 1, Theorem 10.2 implies that the unique invariant bounded section of $\sigma\colon X \to E$ is C^1. But at the point $x \in X$, the graph of the map $\sigma(x)\colon \widetilde{E}^u(x) \to \widetilde{E}^{cs}(x)$ is precisely the bundle $E^u(x)$. Since E^u is C^1 along X, the manifold X is C^2.

Repeating the argument, using 2-jets of maps from $\widetilde{E}^u$ to $\widetilde{E}^{cs}$ instead of 1-jets, shows that X is C^3. An inductive argument using the $\ell - 1$ jet bundle shows that X is C^ℓ, for every integer $\ell \leq r$ To obtain that X is C^r, one applies Theorem 10.2 in its

Hölder formulation to show that the $\lfloor r\rfloor$ jet bundle is $C^{r-\lfloor r\rfloor}$. The leaves of $\mathcal{W}^u$ vary continuously in the C^r topology because the jets of E^u along $\mathcal{W}^u(x)$ are found as the fixed point of a fiberwise contraction that depends continuously on x. This fiberwise contraction preserves sections that depend continuously on x, and so the invariant section depends continuously on x as well.

Returning to the map $\mathbf{f}_\rho$, we see that the stable and unstable foliations for this map have uniformly C^r leaves, and for each $p \in M$ the leaves vary continuously inside of T_pM in the C^r topology. Moreover, since $p \mapsto \mathbf{f}_\rho(p, \cdot)$ is continuous in the C^r topology the leaves of unstable foliation for $\mathbf{f}_\rho$ also depend continuously on p in the C^r topology.

When f is r-bunched, a similar argument shows that the center-stable, center-unstable and center leaves for $\mathbf{f}_\rho$ are uniformly C^r. The condition $\hat{\nu} < \hat{\gamma}^r$ is an r-normal hyperbolicity condition for the center-unstable foliation, which implies that the leaves of this foliation are uniformly C^r (see Corollary 6.6 in [**18**]). In this application of Theorem 10.2, the base manifold X is the disjoint union of center-unstable manifolds, and the bundle E consists of jets of maps between the approximate center-unstable and approximate stable bundles. The fiber contraction on $\ell-1$-jets is $\kappa = \hat{\nu}/\hat{\gamma}^{\ell-1}$ and the base conorm of the bundle map on X is $\lambda = \hat{\gamma}$. The condition $\kappa/\lambda = \hat{\nu}/\hat{\gamma}^{\ell} < 1$ implies that the invariant section on $\ell - 1$ jets is C^1, and so the center unstable leaves are C^ℓ, for all $\ell < r$. As above, one obtains that the center-unstable leaves are uniformly C^r.

Similarly the condition $\nu < \gamma^r$ implies that the leaves of the center-stable foliation are uniformly C^r; intersecting center-unstable with center-stable leaves, one obtains that the leaves of the center foliation for $\mathbf{f}_\rho$ are uniformly C^r. The leaves of the center, center-stable and center-unstable foliations for $\mathbf{f}_\rho$ along T_pM also depend continuously on $p \in M$ in the C^r topology.

When $k \geq 2$, and f is r-bunched, for $r < k - 1$ or $r = 1$, another argument using Theorem 10.2 proves the C^r regularity of the unstable bundle along the leaves of the center-unstable foliation. The manifold X is the disjoint union of the leaves of the center-unstable foliation for $\mathbf{f}_\rho$, and the bundle E consists of linear maps from the approximate unstable into the approximate center-stable bundles. Note that X is uniformly C^r by the previous arguments, and the first $\lfloor r\rfloor$ derivatives of $\mathbf{f}_\rho$ vary $(r - \lfloor r\rfloor)$-Hölder continuously from leaf to leaf. Since X and E are C^r, we may apply the C^r section theorem directly (without inductive arguments).

In this case, the graph transform bundle map has fiber constant $\kappa = \hat{\nu}/\hat{\gamma}$ and the base conorm λ of $\mathbf{f}_\rho$ restricted to center-unstable leaves is bounded by γ. The r-bunching hypothesis $\hat{\nu} < \hat{\gamma}\gamma^r$ implies that $\kappa/\lambda^r < 1$, and so the unstable bundle for $\mathbf{f}_\rho$ is C^r when restricted to X. Moreover the jets of the unstable bundle along the center-unstable leaf vary $(r - \lfloor r\rfloor)$-Hölder continuously. Notice that we need $k-1 \geq r$ to carry out this argument, because the bundle map we consider is only C^{k-1} (in the fiber it is a linear graph transform determined by the derivative of $\mathbf{f}_\rho$, and we lose a derivative in this argument).

Similarly, this argument shows that the bunching hypothesis $\nu < \gamma\hat{\gamma}^r$ implies that the stable bundle for $\mathbf{f}_\rho$ is a C^r bundle over the leaves of the center-stable foliation, and we have (Hölder) continuous dependence of the appropriate jets on the basepoint. The details are described in **[33, 34]** in the case $r = 1$ and $k = 2$. The argument for general r, k is completely analogous.

Step 2. We now have foliations of T_pM, for each $p \in M$. We obtain the foliations $\widehat{\mathcal{W}}^u_p$, $\widehat{\mathcal{W}}^c_p$, $\widehat{\mathcal{W}}^s_p$, $\widehat{\mathcal{W}}^{cu}_p$, and $\widehat{\mathcal{W}}^{cs}_p$ by applying the exponential map $\exp_p$ to the corresponding foliations of T_pM inside the ball around the origin of radius ρ.

If ρ is sufficiently small, then the distribution $E^*(q)$ lies within the angular $\varepsilon/2$-cone about the parallel translate of $E^*(p)$, for every $* \in \{u, s, c, cu, cs\}$ and all p, q with $d(p, q) \le \rho$. Combining this fact with the preceding discussion, we obtain that property 1. holds if ρ is sufficiently small.

Property 2. — local invariance — follows from invariance under $\mathbf{f}_\rho$ of the foliations of TM and the fact that $\exp_{f(p)}(\mathbf{f}_\rho(p, v)) = f(\exp_p(p, v))$ provided $\|v\| \le \rho$.

Having chosen ρ, we now choose ρ_1 small enough so that $f(B_M(p, 2\rho_1)) \subset B_M(f(p), \rho)$ and $f^{-1}(B_M(p, 2\rho_1)) \subset B_M(f^{-1}(p), \rho)$, and so that, for all $q \in B_M(p, \rho_1)$,

$$\begin{aligned}
q' \in \widehat{\mathcal{W}}^s_p(q, \rho_1) &\implies d(f(q), f(q')) \le \nu(p)\, d(q, q'),\\
q' \in \widehat{\mathcal{W}}^u_p(q, \rho_1) &\implies d(f^{-1}(q), f^{-1}(q')) \le \hat{\nu}(f^{-1}(p))\, d(q, q'),\\
q' \in \widehat{\mathcal{W}}^{cs}_p(q, \rho_1) &\implies d(f(q), f(q')) \le \hat{\gamma}(p)^{-1}\, d(q, q'), \qquad \textit{and}\\
q' \in \widehat{\mathcal{W}}^{cu}_p(q, \rho_1) &\implies d(f^{-1}(q), f^{-1}(q')) \le \gamma(f^{-1}(p))^{-1}\, d(q, q').
\end{aligned}$$

Property 3. — exponential growth bounds at local scales — is now proved by an inductive argument.

Properties 4.– 7. — coherence, uniqueness, leafwise regularity and regularity of the strong foliation inside weak leaves — follow immediately from the corresponding properties of the foliations of TM discussed above. $\square$

Since there is no ambiguity in doing so, we write $\widehat{\mathcal{W}}^{cs}(x)$, $\widehat{\mathcal{W}}^{cu}(x)$, and $\widehat{\mathcal{W}}^{c}(x)$ for the corresponding manifolds $\widehat{\mathcal{W}}^{cs}_x(x)$, $\widehat{\mathcal{W}}^{cu}_x(x)$, and $\widehat{\mathcal{W}}^{c}_x(x)$. If f is C^k and r-bunched, for $k \ge 2$ and $r < k - 1$ or $r = 1$, then the collection of all $\widehat{\mathcal{W}}^{*}(x)$-manifolds forms a uniformly continuous C^r plaque family in M, but not in general a foliation.

Henceforth *we shall assume that $\mathcal{B}$ is the trivial bundle $\mathcal{B} = M \times N$.* All of the definitions and arguments that follow can be made for a general bundle $\mathcal{B}$ by fixing a connection on $\mathcal{B}$, at the expense of more cumbersome notation and the need to localize some of the objects, such as the fake foliations for F in the following lemma. Since Theorem C concerns the local property of smoothness, this simplifying assumption is benign.

Lemma 10.3. — *Let $k \geq 2$ and $r = 1$ or $r < k - 1$. If F is a C^k, r-bunched extension of f, then we can construct the fake foliations $\widehat{\mathcal{W}}^s_{F,z}$, $\widehat{\mathcal{W}}^u_{F,z}$, $\widehat{\mathcal{W}}^{cs}_{F,z}$, $\widehat{\mathcal{W}}^{cu}_{F,z}$ and $\widehat{\mathcal{W}}^c_{F,z}$ for F and $\widehat{\mathcal{W}}^s_p$, $\widehat{\mathcal{W}}^u_p$, $\widehat{\mathcal{W}}^{cs}_p$, $\widehat{\mathcal{W}}^{cu}_p$ and $\widehat{\mathcal{W}}^c_p$ for f so that:*

- *for each $p \in M$ and $z \in \pi^{-1}(p)$, the fake foliations $\widehat{\mathcal{W}}^*_{F,z}$ for F are defined in the entire neighborhood $\pi^{-1}(B_M(p,\rho))$ of $\pi^{-1}(p)$ and are independent of $z \in \pi^{-1}(p)$;*
- *for $* \in \{cs, cu, c\}$, we have:*
$$\widehat{\mathcal{W}}^*_{F,z}(w) = \pi^{-1}\left(\widehat{\mathcal{W}}^*_p(\pi(w))\right),$$
for all $p \in M$, all $z \in \pi^{-1}(p)$, and all $w \in \pi^{-1}(B_M(p,\rho))$;
- *for $* \in \{s, u\}$, we have:*
$$\pi\left(\widehat{\mathcal{W}}^*_{F,z}(w)\right) = \widehat{\mathcal{W}}^*_p(\pi(w)),$$
for all $p \in M$, all $z \in \pi^{-1}(p)$, and all $w \in \pi^{-1}(B_M(p,\rho))$; and
- *the conclusions of Proposition 10.1 hold for the fake foliations of F and f.*

Proof. — Let N be the fiber of $\mathcal{B}$. Fix $\rho_0 > 0$ such that the exponential map $\exp_p$ is a diffeomorphism from $B_{T_pM}(0,\rho_0)$ to $B_M(p,\rho_0)$, for every $p \in M$. Note that $\pi^{-1}(B_M(p,\rho_0))$ is a trivial bundle over $B_M(p,\rho_0)$, for each $p \in M$. Denote by $B_{TM}(0,\rho_0)$ the ρ_0-neighborhood of the 0-section of TM. The bundle $\mathcal{B}$ pulls back via the exponential map $\exp\colon B_{TM}(0,\rho_0) \to M$ to a C^r bundle $\tilde{\pi}_0\colon \widetilde{\mathcal{B}}_0 \to B_{TM}(0,\rho_0)$ with fiber N. The bundle $\widetilde{\mathcal{B}}_0$ is trivial over each fiber $\mathcal{B}_{T_pM}(0,\rho_0)$ of $B_{TM}(0,\rho_0)$ and pulls back to the original bundle $\mathcal{B}$ under the inclusion $M \hookrightarrow B_{TM}(0,\rho_0)$ of M into the 0-section of TM. Elements of $\widetilde{\mathcal{B}}_0$ are of the form $(p,v,z) \subset B_{TM}(0,\rho_0) \times \mathcal{B}$ such that $\pi(z) = \exp_p(v)$, and the projection $\tilde{\pi}_0$ sends (p,v,z) to (p,v). Extend $\widetilde{\mathcal{B}}_0$ to a C^r bundle $\tilde{\pi}\colon \widetilde{\mathcal{B}} \to TM$ over TM in such a way that $\widetilde{\mathcal{B}}$ is also a C^r bundle over M (with fiber $\mathbb{R}^m \times N$), and the restriction of $\widetilde{\mathcal{B}}$ to T_pM is a trivial bundle, for every $p \in M$.

In the proof of Proposition 10.1, we define $\mathbf{F}_r$ slightly differently, using the bundle $\widetilde{\mathcal{B}}$, rather than $T\mathcal{B}$. Fix $\rho_1 < \rho_0$ such that $\overline{f(B_M(p,\rho_1))} \subset B_M(f(p),\rho_0)$, for all $p \in M$. Let $\mathbf{f}\colon B_{TM}(0,\rho_1) \to B_{TM}(0,\rho_0)$ be the map:
$$\mathbf{f}(p,v) = \exp^{-1}_{f(p)} \circ f \circ \exp_p(v).$$
The map $F : \mathcal{B} \to \mathcal{B}$ induces a map $\mathbf{F}\colon \tilde{\pi}^{-1}(B_{TM}(0,\rho_1)) \to \tilde{\pi}^{-1}(B_{TM}(0,\rho_0))$, covering $\mathbf{f}$, defined by:
$$\mathbf{F}(p,v,z) = (\mathbf{f}(p,v), F(z)).$$

Since $\widetilde{\mathcal{B}}|_{TM}$ is a trivial bundle, we can write elements of $\tilde{\pi}^{-1}(T_pM)$ as triples (p,v,y), where $v \in T_pM$ and $y \in \pi^{-1}(p) \cong N$; we can choose this trivialization to depend smoothly on p. We also metrically trivialize the fibers $\widetilde{\mathcal{B}}|_{T_pM}$ of this bundle, using the product of the sup metric $< \cdot, \cdot >'_p$ on T_pM defined at the beginning of this section with the induced metric $< \cdot, \cdot >$ on the fiber $\pi^{-1}(p)$. If F is an r-bunched extension of f, then the r-bunching inequalities hold for this family of metrics on $\widetilde{\mathscr{B}}|_{B_{TM}(0,\rho)}$, if ρ is sufficiently small.

Then for each $\rho > 0$ there exists a C^r bundle isomorphism

$$\mathbf{F}_\rho \colon \widetilde{\mathscr{B}} \to \widetilde{\mathscr{B}},$$

covering the map $\mathbf{f}_\rho \colon TM \to TM$ constructed in the proof of Proposition 10.1, with the following properties:

- $\mathbf{F}_\rho(p,v,y) = \mathbf{F}(p,v,y)$ if $\|v\| \le \rho$; in particular, we have $\mathbf{F}_\rho(p,0,y) = (f(p),0,F(y))$,
- $\mathbf{F}_\rho(p,v,y) = (f(p),T_pf(v),\mathbf{F}(p,\rho v/\|v\|,y))$ if $\|v\| \ge 2\rho$,
- $\sup_{v\in T_pM} d_{C^r}(\mathbf{F}_\rho(p,v,\cdot),\mathbf{F}(p,0,\cdot)) \to 0$ as $\rho \to 0$,
- the C^r diffeomorphism $\mathbf{F}_\rho(p,\cdot,\cdot)$ depends continuously on p in the C^r topology.

The construction of $\mathbf{F}_\rho$ is straightforward, once one has proven the following lemma, and we omit the details.

Lemma 10.4. — *Let N be a compact manifold and let $\{F_v \colon N \to N\}_{v\in B_{\mathbb{R}^n}(0,2)}$ be a family of diffeomorphisms of N such that $(v,y)\mapsto F_v(y)$ is C^r.*

Then for every $\rho \in (0,1)$, there exists a family $\{F_{\rho,v} \colon N \to N\}_{v\in B_{\mathbb{R}^m}(0,\rho)}$ of diffeomorphisms with the following properties:

- *$(v,y)\mapsto F_{\rho,v}(y)$ is C^r;*
- *$F_{\rho,v} = F_v$, if $\|v\| \le \rho$;*
- *$F_{\rho,v} = F_{\rho v/\|v\|}$, if $\|v\| \ge 2\rho$; and*
- *$\sup_{v\in\mathbb{R}^n} d_{C^r}(F_{\rho,v},F_0) \to 0$ as $\rho \to 0$.*

Proof of Lemma 10.4. — We construct $F_{\rho,v}$ as follows. Consider the family of vector fields $\{X_v\}_{v\in B_{\mathbb{R}^m}(0,2)}$ on N defined by

$$X_v(y) = \frac{d}{dt}|_{t=0}F_{v+tv}(y),$$

and let $\varphi_{v,t}$ be the flow generated by X_v. For $v\in\mathbb{R}^n$, let $v_\rho = \rho v/\|v\|$.

For $\rho \in (0,1)$, let $\beta_\rho \colon \mathbb{R}^m \to [0,1]$ be a smooth radial bump function vanishing outside of $B_{\mathbb{R}^m}(0,2\rho)$ and identically 1 on $\overline{B_{\mathbb{R}}(0,\rho)}$ with derivative $|D\beta_\rho|$ bounded by 3ρ. We then define:

$$F_{\rho,v} = \begin{cases} F_v & \text{if} \quad \|v\| \le \rho \\ \varphi_{v_\rho,\beta(v)(\|v\|-\rho)} \circ F_{v_\rho} & \text{if} \quad \|v\| > \rho. \end{cases}$$

Then the family $\{F_{\rho,v}\}_{v\in\mathbb{R}^m}$ has the desired properties. □

Having constructed $\mathbf{F}_\rho$, the proof then proceeds exactly as in Proposition 10.1, except to construct the fake foliations for F, we consider the bundle $\widetilde{\mathscr{B}}$ over M (rather than TM over M) and take the disjoint union of its fibers. For ρ sufficiently small, $\mathbf{F}_\rho$ is partially hyperbolic and r-bunched, if F is an r-bunched extension of f. The fake foliations for F are constructed by first finding invariant foliations for $\mathbf{F}_\rho$ on $\widetilde{\mathscr{B}}$. One verifies as in Proposition 10.1 that these foliations have the required regularity properties. To construct the fake foliations for F, we first restrict these foliations to the bundle $\tilde{\pi}^{-1}(B_{TM}(0,\rho)) \subset \widetilde{\mathcal{B}}$. Fix $p \in M$. On $\tilde{\pi}^{-1}(B_{T_pM}(0,\rho))$, the projection $(p,v,z) \mapsto z$ is a diffeomorphism onto $\pi^{-1}(B_M(p,\rho))$; the image of the invariant foliations for $\mathbf{F}_\rho$ under this projection gives the fake invariant foliations for F on $\pi^{-1}(B_M(p,\rho))$.

To construct the fake invariant foliations for f, we take instead the image of the invariant foliations for $\mathbf{F}_\rho$ in $\tilde{\pi}^{-1}(B_{T_pM}(0,\rho))$ under the map $(p,v,z) \mapsto \exp_p(v)$. This construction ensures that the desired properties hold. $\square$

Fix $\varepsilon > 0$ small and let the fake foliations for f and F be defined by the preceding lemmas.

Since it does not depend on $z \in \pi^{-1}(p)$ we write $\widehat{\mathcal{W}}^*_{F,p}(w)$ for $\widehat{\mathcal{W}}^*_{F,z}(w)$, for $* \in \{s,u,cs,cu,c\}$. As with the fake foliations for f, for $* \in \{cs,cu,c\}$ and $p \in M$, we will denote by $\widehat{\mathcal{W}}^*_F(p)$ the plaque $\widehat{\mathcal{W}}^*_F(p) = \pi^{-1}(\widehat{\mathcal{W}}^*(p))$ in $\mathcal{B}$; it is the $\widehat{\mathcal{W}}^*_F$-leaf through any $z \in \pi^{-1}(p)$.

By rescaling the Riemannian metric on M, we may assume that $\rho_1 \gg 1$, so that all of the objects used in the sequel are well-defined on any ball of radius 1 in M.

10.2. Further consequences of r-bunching. — Here we explore in greater depth the properties of an r-bunched partially hyperbolic diffeomorphism. The goal is to bound the deviation between the fake foliations $\widehat{\mathcal{W}}^*_p$ and $\widehat{\mathcal{W}}^*_q$ for $q \in \widehat{\mathcal{W}}^*(p)$. In the dynamically coherent case, $\widehat{\mathcal{W}}^*_p(q)$ and $\widehat{\mathcal{W}}^*_q(q)$ coincide for $q \in \widehat{\mathcal{W}}^*(p)$. In a sense, the results in this section tell us that r-bunched systems are dynamically coherent "on the level of r-jets."

Throughout this and the following subsections, we continue to assume that F is a C^k, r-bunched extension of f, where $k \geq 2$ and $r < k-1$ or $r = 1$. In the statements of some of the lemmas, we will remind the reader of these hypotheses. We fix as above a choice of fake foliations and fake lifted foliations (we will not specify here the choice of $\varepsilon > 0$, but will indicate where it is relevant). Let $m = \dim(M)$, $s = \dim E^s$, $u = \dim E^u$, and $c = \dim E^c$, so that $m = s+u+c$.

Fix a point $p \in M$. We introduce C^r local $\mathbb{R}^u \times \mathbb{R}^s \times \mathbb{R}^c$ - coordinates (x^u, x^s, x^c) in the ρ-neighborhood of p, sending p to 0, $\widehat{\mathcal{W}}^{cs}(p)$ into the subspace $x^u = 0$, $\widehat{\mathcal{W}}^{cu}(p)$ into $x^s = 0$, $\mathcal{W}^s(p)$ to $x^u = x^c = 0$, $\mathcal{W}^s(p)$ to $x^u = x^c = 0$, and $\mathcal{W}^u(p)$ to $x^s = x^c = 0$. This is possible because all of the submanifolds in question are C^r. Since $\widehat{\mathcal{W}}^u_p$ is a C^r subfoliation of $\widehat{\mathcal{W}}^{cu}(p)$, and $\widehat{\mathcal{W}}^s_p$ is a C^r subfoliation of $\widehat{\mathcal{W}}^{cs}(p)$, we may also choose these coordinates so that each leaf $\widehat{\mathcal{W}}^u_p(q)$, for $q \in \widehat{\mathcal{W}}^{cu}(p)$ is sent into an affine space $x^s = 0, x^c \equiv x^c_0$ and each leaf $\widehat{\mathcal{W}}^s_p(q')$, for $q' \in \widehat{\mathcal{W}}^{cs}(p)$ is sent into an affine space $x^u = 0, x^c \equiv x^c_0{}'$.

We can choose these coordinates to depend uniformly on p. We call these coordinates *adapted coordinates at p*. Whenever we refer to adapted coordinates at a point p, we implicitly assume that they are chosen with a uniform bound on their C^r size.

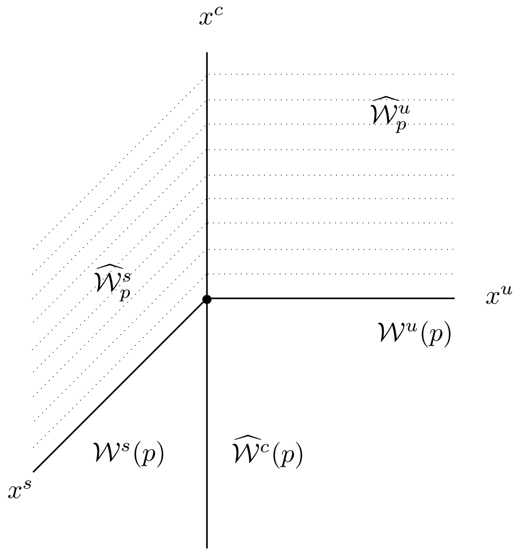


FIGURE 4. Coordinates adapted to the fake foliations at p.

According to Proposition 10.1 the leaves of the fake center, center-stable and center-unstable manifolds at each point z can be expressed using parametrized C^r plaque families:

$$\hat{\omega}^{cs}\colon I^m \times I^{c+s} \to \mathbb{R}^m, \qquad \hat{\omega}^{cu}\colon I^m \times I^{c+u} \to \mathbb{R}^m,$$

and

$$\hat{\omega}^c\colon I^m \times I^c \to \mathbb{R}^m,$$

where $\widehat{\mathcal{W}}^{cu}(z) = \hat{\omega}^{cu}_z(I^{c+u})$, $\widehat{\mathcal{W}}^{cs}(z) = \hat{\omega}^{cs}_z(I^{c+s})$ and $\widehat{\mathcal{W}}^{c}(z) = \hat{\omega}^{c}_z(I^{c})$. The map $\hat{\omega}^c$ is obtained from $\hat{\omega}^{cs}$ and $\hat{\omega}^{cu}$ using the implicit function theorem. We may assume these maps take the form:

$$\hat{\omega}^{cs}_z(x^c, x^s) = z + (\hat{\beta}^{cs}_z(x^c, x^s), x^s, x^c) \qquad \hat{\omega}^{cu}_z(x^c, x^u) = z + (\hat{\beta}^{cu}_z(x^u, x^c), x^u, x^c),$$

and

$$\hat{\omega}^{c}_z(x^c) = z + (\hat{\beta}^{c}(x^c), x^c),$$

where $\hat{\beta}^{cu}_z \in C^r(I^{c+u}, \mathbb{R}^s)$, $\hat{\beta}^{cs}_z \in C^r(I^{c+s}, \mathbb{R}^u)$, and $\hat{\beta}^{c}_z \in C^r(I^{c}, \mathbb{R}^{s+u})$, and $z \mapsto \hat{\beta}^*_z$ is continuous in the C^r topology. Moreover, we have $\hat{\beta}^*_z(0) = 0$ and $\hat{\omega}^*_0 \equiv 0$ for $* \in \{cs, cu, c\}$.

We now derive further consequences of the r-bunching hypothesis on f. The first concerns the behavior of the plaque families $\widehat{\mathcal{W}}^*(y)$ for $y \in \widehat{\mathcal{W}}^*(x)$, for $* \in \{cs, cu, c\}$.

Lemma 10.5. — *For each $v = (0, v^s, v^c) \in \widehat{\mathcal{W}}^{cs}(0)$, $w = (w^u, 0, w^c) \in \widehat{\mathcal{W}}^{cu}(0)$, and $z = (0, 0, z^c) \in \widehat{\mathcal{W}}^{c}(0)$, and for every positive integer $\ell \leq r$, we have:*

$$|j^\ell_0 \hat{\beta}^{cs}_v| = o(|v^c|^{r-\ell}), \quad |j^\ell_0 \hat{\beta}^{cu}_w| = o(|w^c|^{r-\ell}), \quad \text{and} \quad |j^\ell_0 \hat{\beta}^{c}_z| = o(|z^c|^{r-\ell}).$$

All of these statements hold uniformly in the coordinate system based at p.

Proof. — We prove the assertion for $\hat{\beta}^{cu}$; the argument for $\hat{\beta}^{cs}$ is the same but with f replaced by f^{-1}. The assertion for $\hat{\beta}^{c}$ follows from the first two.

As in the proof of Proposition 5.2, we will use the convention that if $q \in M$ and $j \in \mathbb{Z}$, then q_j denotes the point $f^j(q)$, with $q_0 = q$. For a positive function $\alpha\colon M \to \mathbb{R}_+$ we also use the cocycle notation described there.

Endow the disjoint union $\hat{M}_p = \bigsqcup_{n\geq 0} B(p_{-n}, \rho)$ with the C^r adapted coordinate system based at p_{-n} in the ball $B(p_{-n}, \rho)$. We thereby identify $\hat{M}_p$ with the disjoint union $\bigsqcup_{n\geq 0}(I^m)_{-n}$. This coordinate system is not invariant under f, but certain aspects of it are; in particular, the planes $x^u = 0$ and $x^s = 0$ are invariant, as are the families $x^u = 0, x^c \equiv x^c_0$ and $x^s = 0, x^c \equiv x^c_0$. Moreover, we may assume (having chosen $\varepsilon > 0$ small enough in the application of Proposition 10.1) that for any point of the form $(0, x^s, x^c) \in B(p_i, \rho)$, writing $f(0, x^s, x^c) = (0, x^s_1, x^c_1)$, we have that $|x^s_1| \leq \nu(p_i)|x^s|$ and $\gamma(p_i)|x^c| \leq |x^c_1| \leq \hat{\gamma}(p_i)^{-1}|x^c|$. Similarly for any point of the form $(x^u, 0, x^c) \in B(p_{i+1}, \rho)$, writing $f^{-1}(x^u, 0, x^c) = (x^u_{-1}, 0, x^c_{-1})$, we have that $|x^u_{-1}| \leq \hat{\nu}(p_i)|x^u|$ and $\hat{\gamma}(p_i)|x^c| \leq |x^c_{-1}| \leq \gamma(p_i)^{-1}|x^c|$.

Let $\hat{M}_p(1) = \bigsqcup_{n\geq 1} B(p_{-n}, 1)$, and note that $f(\hat{M}_p(1)) \subset \hat{M}_p$. Let φ be the change of coordinate $\varphi(x^u, x^s, x^c) = (x^c, x^u, x^s)$, and let $\tilde{f} = \varphi \circ f \circ \varphi^{-1}$. Now write, for $x \in \hat{M}_p(1)$:

$$D\tilde{f}(x) = \begin{pmatrix} A_x & B_x \\ C_x & K_x \end{pmatrix},$$

where $A_x\colon \mathbb{R}^{c+u} \to \mathbb{R}^{c+u}$, $B_x\colon \mathbb{R}^s \to \mathbb{R}^{c+u}$, $C_x\colon \mathbb{R}^{c+u} \to \mathbb{R}^s$ and $K_x\colon \mathbb{R}^s \to \mathbb{R}^s$. We may assume that $\varepsilon > 0$ was chosen small enough in the application of Proposition 10.1 that for every $x \in f^{-1}(B(p_{-n+1}, 1)) \cap B(p_{-n}, 1)$, we have that $m(A_x) \geq \gamma(p_{-n})$ and $\|K_x\| \leq \nu(p_{-n})$, and $\|B_x\|$ and $\|C_x\|$ are very small. The partial hyperbolicity and r-bunching hypotheses $\nu < \gamma$ and $\nu < \gamma^r$ then imply that, for all $\ell \leq r$:

$$\sup_{x\in\hat{M}_p} \max\left\{\frac{\|A_x\|}{m(K_x)}, \frac{\|K_x\|}{m(A_x)^\ell}\right\} < 1.$$

Fix $0 \leq \ell \leq r$, and let $\kappa = \max\{\nu\gamma^{-\ell}, \nu\gamma^{-1}\}$. Also fix a continuous function $\delta < \min\{1, \gamma\}$ such that $\kappa < \delta^{r-\ell}$; this is possible since f is r-bunched.

Consider the $C^{k-\ell}$ induced map

$$\mathcal{T}^\ell_f\colon \hat{M}_p(1) \times J^\ell_0(\mathbb{R}^{c+u}, \mathbb{R}^s)_0 \to \hat{M}_p \times J^\ell_0(\mathbb{R}^{c+u}, \mathbb{R}^s)_0$$

defined by:

$$\mathcal{T}^\ell_f(x, j^\ell_0\psi) = (f(x), j^\ell_0\psi'),$$

where $\psi' \in \Gamma^\ell_0(\mathbb{R}^{c+u}, \mathbb{R}^s)_0$ satisfies:

$$\tilde{f}(x + \operatorname{graph}(\psi)) = \tilde{f}(x) + \operatorname{graph}(\psi')$$

Lemma 6.4 implies that there is a metric $|\cdot|_L$ on $J^\ell_0(\mathbb{R}^{c+u}, \mathbb{R}^s)_0$ such that for all $n \geq 0$, all $x \in B(p_{-n-1}, 1) \subset \hat{M}_p(1)$ and all $j_0\psi, j_0\psi' \in J^\ell_0(I^{c+u}, \mathbb{R}^s)_0$, with

$|j_0\psi|_L, |j_0\psi'|_L \leq 1$, we have:

(36) $$|\mathcal{T}^\ell_f(x, j_0\psi) - \mathcal{T}^\ell_f(x, j_0\psi)|_L \leq \kappa(p_{-n})|j_0\psi - j_0\psi'|_L.$$

Given a point $w = (w^u, 0, w^c) \in \widehat{\mathcal{W}}^{cu}(p)$, we choose $n \in \mathbb{Z}_+$ such that $|w^c| = \Theta(\delta_{-n}(p)^{-1})$. This is possible, since $\delta < 1$ is a continuous function (remember that δ_{-n} is the product of reciprocal values of δ, and so $\delta_{-n}(p)^{-1}$ is less than 1). The planes $x^s = 0, x^c \equiv x^c_0$ lie in an ε-cone about the center-stable distribution for f. Hence under iteration by f^{-1}, the part of $x^s = 0, x^c \equiv x^c_0$ that remains inside of $\hat{M}_p(1)$ for n iterates is a smooth plane that remains in the ε-cone about the center-stable distribution. Write $w_{-n} = f^{-n}(w) = (w^u_{-n}, 0, w^c_{-n})$. Since $|w^c| = \Theta(\delta_{-n}(p)^{-1})$ and $|w^u| = O(1)$, and $\hat{\nu} < \delta\gamma < 1$, Proposition 10.1, parts (1)-(3) imply that $|w^u_{-n}| = O(\hat{\nu}_{-n}(p)^{-1}) = o(1)$ and $|w^c_{-n}| = O(\delta_{-n}(p)^{-1}\gamma_{-n}(p)) = o(1)$; in particular, we have that $w_{-i} \in B(p_{-i}, 1)$, for $i = 1, \ldots, n$.

Now consider the orbit of $(w_{-n}, j^\ell_0\hat{\beta}^{cu}_{w_{-n}}) \in \hat{M}_p(1) \times J^\ell_0(\mathbb{R}^{c+u}, \mathbb{R}^s)_0$ under $\mathcal{T}^\ell_f$. Local invariance of the $\widehat{\mathcal{W}}^{cu}_p$ plaque family implies that

$$\left(\mathcal{T}^\ell_f\right)^n(w_{-n}, j^\ell_0\hat{\beta}^{cu}_{w_{-n}}) = (w, j^\ell_0\hat{\beta}^{cu}_w).$$

On the other hand, since f leaves invariant the planes $x^s = 0$, we have that $\left(\mathcal{T}^\ell_f\right)^n(w_{-n}, 0) = (w, 0)$. But now (36) implies that

$$\begin{aligned} |j^\ell_0\hat{\beta}^{cu}_w|_L &\leq \kappa_{-n}(p)^{-1}|j^\ell_0\hat{\beta}^{cu}_{w_{-n}}|_L \\ &= O(\kappa_{-n}(p)^{-1}) \end{aligned}$$

On the other hand, $\kappa < \delta^{r-\ell}$, and $|w^c| = \Theta(\delta_{-n}(p)^{-1})$. This implies that $|j^\ell_0\hat{\beta}^{cu}_w| = o(|w^c|^{r-\ell})$, completing the proof of Lemma 10.5. □

The next consequence of r-bunching we derive concerns the discrepancy between the leaves of the real and fake stable (or unstable) foliation originating at a given point. To state these results, we introduce a parametrization of the fake stable and unstable foliations as follows. We are interested in the restriction of the fake stable foliation $\widehat{\mathcal{W}}^s_x$ to the center-stable leaf $\widehat{\mathcal{W}}^{cs}(x)$.

As above, fix an adapted coordinate system at p. Proposition 10.1 implies

that $\widehat{\mathcal{W}}^s_p$ is a C^r subfoliation when restricted to $\widehat{\mathcal{W}}^{cs}(p)$. We are going to give a different parametrization of $\widehat{\mathcal{W}}^{cs}(p)$ to reflect this fact. Recall our definition above: $\hat{\omega}^{cs}_z(x^c, x^s) = z + (\hat{\beta}^{cs}_z(x^c, x^s), x^s, x^c)$, and $\hat{\omega}^{cu}_z(x^c, x^u) = z + (x^u, \hat{\beta}^{cu}_z(x^c, x^u), x^c)$. Using the implicit function theorem, we can write instead:

$$\hat{\omega}^{cs}_z(x^c, x^s) = z + (\hat{\beta}^{s,u}_z(x^c, x^s), x^s, \hat{\beta}^{s,c}_z(x^c, x^s)),$$

and

$$\hat{\omega}^{cu}_z(x^u, x^c) = z + (x^u \hat{\beta}^{u,s}_z(x^c, x^u), \hat{\beta}^{u,c}_z(x^c, x^u)),$$

with the property that for fixed $x^c \in I^c$:

$$\hat{\omega}^{cs}_z(x^c, I^s) = \widehat{\mathcal{W}}^s_z(\hat{\omega}^{cs}(x^c, 0)), \quad \text{and} \quad \hat{\omega}^{cu}_z(x^c, I^u) = \widehat{\mathcal{W}}^u_z(\hat{\omega}^{cu}_z(x^c, 0)),$$

and such that $z \mapsto \hat\beta^s_z = (\hat\beta^{s,u}_z, \hat\beta^{s,c}_z) \in C^r(I^c \times I^s, \mathbb{R}^{u+c})$ and $z \mapsto \hat\beta^u_z = (\hat\beta^{u,s}_z, \hat\beta^{u,c}_z) \in C^r(I^c \times I^u, \mathbb{R}^{s+c})$ are all continuous in the C^r topologies. We may further assume that $\hat\beta^{s,c}_z(x^c, 0) = x^c = \hat\beta^{u,c}_z(x^c, 0)$. Our choice of coordinates also implies that $\hat\beta^s_0 \equiv 0$ and $\hat\beta^u_0 \equiv 0$. Finally, note that $\hat\omega^{cs}_z(0, I^s) = \widehat{\mathcal{W}}^s_z(z) = \mathcal{W}^s(z,\rho)$ and $\hat\omega^{cu}_z(0, I^u) = \widehat{\mathcal{W}}^u_z(z) = \mathcal{W}^u(z,\rho)$.

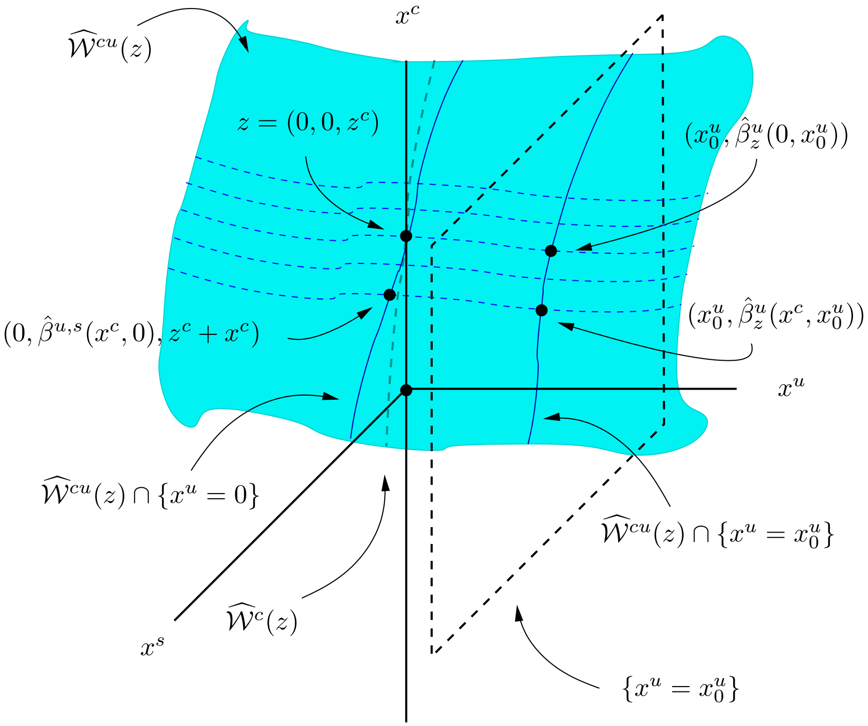


FIGURE 5. Parametrizing the fake unstable foliations at $(0, 0, z^c)$.

Fix $z^c \in I^c$. We are interested in the deviation between the true stable leaf $\hat\omega^{cs}_{(0,0,z^c)}(\{0\} \times I^s)$ and the fake stable leaf $\hat\omega^{cs}_0(\{z^c\} \times I^s)$; this is measured by the distance between the functions $\hat\beta^s_{(0,0,z^c)}(0, \cdot)$ and $\hat\beta^s_0(z^c, \cdot)$ at a point $x^s \in I^s$. We are interested not only in the C^0-distance between these functions, but in the distance between their transverse jets. By our choice of coordinate system, we have that $\hat\beta^s_0$ is identically 0; hence we will estimate just the jets of $\hat\beta^s_{(0,0,z^c)}$ in the x^c direction at $x^c = 0$ and a fixed value of x^s.

Lemma 10.6. — *For $z^c \in I^c$, $x^s \in I^s$ and $x^u \in I^u$ we have:*

$$\left| j^\ell_0 \left(x^c \mapsto \hat\beta^s_{(0,0,z^c)}(x^c, x^s) \right) \right| = |x^s| \cdot o(|z^c|^{r-\ell}),$$

and

$$\left|j_0^\ell\left(x^c \mapsto \hat\beta^u_{(0,0,z^c)}(x^c,x^u)\right)\right| = |x^u|\cdot o(|z^c|^{r-\ell}),$$

for every $\ell \leq r$.

Remark: Consider the transversals $x^u = 0$ and $x^u = x_0^u$ to the foliations $\widehat{\mathcal{W}}^u_0$ and $\widehat{\mathcal{W}}^u_{(0,0,z^c)}$. If we restrict to the space $x^u = x^s = 0$ inside the first transversal (which corresponds to the center manifold $\widehat{\mathcal{W}}^c(p)$), then the holonomy map for $\widehat{\mathcal{W}}^u_p|_{\widehat{\mathcal{W}}^{cu}(p)}$ to the second transversal is trivial in these coordinates, sending $(0,0,x^c)$ to $(x_0^u,0,x^c)$. If we consider instead the holonomy map for $\widehat{\mathcal{W}}^u_{(0,0,z^c)}|_{\widehat{\mathcal{W}}^{cu}(0,0,z^c)}$ between these transversals, then the point $(0,\hat\beta^{u,s}(x^c,0),z^c+x^c)$ is sent to $(x_0^u,\hat\beta^u_{(0,0,z^c)}(x^c,x_0^u))$ The ℓ-jet of this holonomy at $(0,0,z^c)$ (measured in the x^c coordinate) is precisely the quantity $j_0^\ell\left(x^c\mapsto \hat\beta^u_{(0,0,z^c)}(x^c,x_0^u)\right)$ estimated by Lemma 10.6.

Proof of Lemma 10.6. — We continue to adopt the conventions and notations in the proof of Lemma 10.5, we define $\hat M_p$ and $\hat M_p(1)$ as in that proof, and use the same coordinate system defined there. We prove the assertion for $\hat\beta^u$; the proof for $\hat\beta^s$ is the same, but with f replaced by f^{-1}.

Denote by f_0 the restriction of f to $\bigsqcup_{n\geq 1}\widehat{\mathcal{W}}^c(p_{-n}) \subset \hat M_p(1)$, which we regard locally as a map from I^c to I^c. We now focus attention on a single neighborhood $B(p_{-n},1)$, for some fixed $n\geq 1$, and regard $x^c\in I^c$ as coordinatizing $x^u=0, x^s=0$ and $(x^u,x^{s+c})\subset I^u\times I^{s+c} = I^m$ as coordinatizing points in this neighborhood.

In local coordinates respecting the decomposition $I^m = I^u\times I^{s+c}$, write:

$$f(x^u,x^{s+c}) = (f_u(x^u,x^{s+c}), f_{sc}(x^u,x^{s+c})).$$

In a neighborhood of each point, this map acts on graphs of C^1 functions from I^u to $\mathbb{R}^{s+c}$ by the usual graph transform, which is a contraction on the fibers of $\pi^{1,0}\colon J^1(I^u,\mathbb{R}^{s+c}) \to J^0(I^u,\mathbb{R}^{s+c}) = I^u\times\mathbb{R}^{s+c}$. Unstable manifolds for f are sent to unstable manifolds under this graph transform, and, locally, fake unstable manifolds are sent to fake unstable manifolds. For each point $(0,0,z^c)\in I^m$, we will consider a C^ℓ family of such 1-jets, expressed as a function of the coordinate x^c transverse to the fake unstable foliation in $\widehat{\mathcal{W}}^{cu}(p_{-n}) = \{x^s=0\}$; we study the variation of such graphs through points $(0,0,z^c+x^c)$ near $x^c=0$.

The space of all such ℓ-jets of 1-jets at the point $x^c = 0$ is the bundle $J_0^\ell(J^1_{I^u}(I^u,\mathbb{R}^{s+c}))$. Elements of this "mixed jet bundle" are of the form $j_0^\ell(j^1_{x^u}\beta)$, where $\beta(x^c,x^u)\colon I^c\times I^u\to\mathbb{R}^{s+c}$ is defined in a neighborhood of $\{0\}\times I^u$, the map $\beta(x^c,\cdot)$ is C^1, and the map $x^c\mapsto j^1_{x^u}\beta(x^c,\cdot)$ is C^ℓ. In particular, if β is $C^{\ell+1}$, then this property is satisfied. We denote this space $\Gamma_0^\ell(I^c,\Gamma^1_{I^u}(I^u,\mathbb{R}^{s+c}))$ of such local functions by $\Gamma^{\ell,1}_{\{0\}\times I^u}(I^c\times I^u,\mathbb{R}^{s+c})$. We also denote $j_0^\ell(j^1_{x^u}\beta)$ by $j^{\ell,1}_{0,x^u}\beta$ and the bundle $J_0^\ell(J^1_{I^u}(I^u,\mathbb{R}^{s+c}))$ by $J^{\ell,1}_{\{0\}\times I^u}(I^c\times I^u,\mathbb{R}^{s+c})$.

Note that in our parametrization $\hat{\beta}^u\colon I^m \times I^c \times I^u \to I^{s+c}$ of the fake unstable subfoliations, the set $\hat{\beta}^u_z(x^c, I^u)$ is the leaf of $\widehat{\mathcal{W}}^u_z$ through the point $\omega^{cu}_z(x^c, 0) = z + (0, \hat{\beta}^{cu}_z(x^c))$; if $z = (0, 0, z^c)$, then the unique point of $\widehat{\mathcal{W}}^u_z$ intersecting $x^u = 0$ is of the form $(0, x^s, z^c + x^c)$. Because the sets $\{x^u = 0, x^s = \mathrm{const}\}$ are invariant under f in our coordinate system, the image of the point $(0, x^s, z^c + x^c)$ is of the form $(0, x^{s\prime}, f_0(z^c + x^c))$. This is the unique point on the leaf of $\widehat{\mathcal{W}}^u_{f(z)}$ intersecting $x^u = 0$, which in turn lies in the set $\hat{\beta}^u_{f(z)}(\{f_0(z^c + x^c) - f_0(z^c)\} \times I^u)$. We will thus define the natural action of f on $I^c \times \Gamma^{\ell,1}_{\{0\}\times I^u}(I^c \times I^u, I^{s+c})$ so that it sends $(z_0, \hat{\beta}^u_{(0,0,z_0)}(\{x^c\} \times I^u))$ to $(f_0(z_0), \beta_{f(0,0,z^c)}(\{f_0(z^c + x^c) - f_0(z^c)\} \times I^u))$.

For $(z^c, \beta) \in I^c \times \Gamma^{\ell,1}_{\{0\}\times I^u}(I^c \times I^u, \mathbb{R}^{s+c})$, we would like to define the map $\mathcal{T}(z^c, \beta) \in \Gamma^{\ell,1}_{\{0\}\times I^u}(I^c \times I^u, \mathbb{R}^{s+c})$ implicitly by the equation

$$\mathcal{T}(z^c, \beta)\,(f_0(z^c + x^c) - f_0(z^c), f_u(x^u, \beta(x^c, x^u) + (0, z^c))) \tag{37}$$

$$= f_{sc}(\beta(x^c, x^u) + (0, z^c)) - (0, f_0(z^c)); \tag{38}$$

if such a map exists, then we will have:

$$\mathcal{T}(z^c, \hat{\beta}^u_{(0,0,z^c)}(x^c, I^u)) = \hat{\beta}^u_{(0,0,f_0(z^c))}(f_0(x^c + z^c) - f_0(x^c), I^u).$$

To check local invertibility, we must check that the map

$$g_{z^c}(x^c, x^u) = (f_0(z^c + x^c) - f_0(z^c), f_u(x^u, \beta(x^c, x^u) + (0, z^c)))$$

on $I^c \times I^u$ is invertible in a neighborhood of $(0, x^u)$. The derivative of this map at $(0, x^u)$ is

$$Dg_{z^c}(0, x^u) = \begin{pmatrix} Df_0(z^c) & 0 \\ C & K \end{pmatrix},$$

where

$$K = \frac{\partial f_u}{\partial x^u}(\beta(0, x^u) + (0, z^c)) + \frac{\partial f_u}{\partial x^{s+c}}(x^u, \beta(0, x^u) + (0, z^c)) \circ \frac{\partial \beta}{\partial x^u}(0, x^u)$$

and

$$C = \frac{\partial f_u}{\partial x^{s+c}}(x^u, \beta(0, x^u) + (0, z^c)) \circ \frac{\partial \beta}{\partial x^c}(0, x^u).$$

This map is invertible if $\frac{\partial \beta}{\partial x^u}(0, x^u)$ is sufficiently small. Let $\mathcal{T}(z^c, \beta)$ be defined by (37) on this subset.

Next, for $0 \le \ell \le k - 1$, consider the map

$$\mathcal{T}^{\ell,1}_f\colon I^c \times J^{\ell,1}_{\{0\}\times I^u}(I^c \times I^u, \mathbb{R}^{s+c}) \to \mathbb{R}^c \times J^{\ell,1}_{\{0\}\times I^u}(I^c \times I^u, \mathbb{R}^{s+c}),$$

defined (in a neighborhood of the 0-section) by

$$\mathcal{T}^{\ell,1}_f\left(z^c, j^\ell_0\left(j^1_{x^u}\beta\right)\right) = \left(f_0(z^c),\, j^\ell_0\left(j^1_{g_{z^c}(x^c,x^u)}\mathcal{T}(z^c, \beta)\right)\right).$$

Recall that we have been working in a single coordinate neighborhood $B(p_{-n},1)$. We combine these definitions of $\mathscr{T}^{\ell,1}_{f^{-1}}$ over all neighborhoods to define a global map

$$\mathscr{T}^{\ell,1}_{f}\colon \bigsqcup_{n\geq 1}\Big(I^c\times J^{\ell,1}_{\{0\}\times I^u}(I^c\times I^u,\mathbb{R}^{s+c})\Big)_{-n} \longrightarrow \bigsqcup_{n\geq 0}\Big(I^c\times J^{\ell,1}_{\{0\}\times I^u}(I^c\times I^u,\mathbb{R}^{s+c})\Big)_{-n}$$

(where the $-n$ subscript denotes the neighborhood $B(p_{-n},\rho)$ in the disjoint union). This map is fiberwise $C^{k-\ell-1}$ (in particular, it is C^1 if $\ell<k-1$) and has the property that $\mathscr{T}^{\ell,1}_{f}(z,j^{\ell,1}_{(0,x^u)}\hat\beta^u_z)=(f(z),j^{\ell,1}_{g_{z^c}(0,x^u)}\hat\beta^u_{f(z)})$.

A calculation very similar to the one in the proof of Lemma 6.4 shows that there is a norm $|\cdot|_L$ on $J^{\ell,1}_{\{0\}\times I^u}(I^c\times I^u,\mathbb{R}^{s+c})$ such that, for all $n\geq 0$, $z^c\in I^c_{-n-1}$, $x^s\in I^s_{-n-1}$, and all $j^{\ell,1}_{(0,x^u)}\beta$, $j^{\ell,1}_{(0,x^u)}\beta'\in J^{\ell,1}_{\{0\}\times I^u}(I^c\times I^u,\mathbb{R}^{s+c})_{-n-1}$ sufficiently close to the 0-section, we have:

$$\Big|\mathscr{T}^{\ell,1}_{f}(z^c,j^{\ell,1}_{(0,x^u)}\beta)-\mathscr{T}^{\ell,1}_{f}(z^c,j^{\ell,1}_{(0,x^u)}\beta')\Big|_L \tag{39}$$

$$\leq \kappa(p_{-n})\,\Big|j^{\ell,1}_{(0,x^u)}\beta-j^{\ell,1}_{(0,x^u)}\beta'\Big|_L, \tag{40}$$

where $\kappa=\max\{\nu/(\gamma\hat\gamma^{\ell}),\nu/(\gamma\hat\gamma)\}$. The r-bunching hypothesis implies that $\kappa<1$.

Having made these preliminary estimates, we finish the proof of Lemma 10.6. Fix $0\leq\ell\leq r$ and a continuous function $\delta<\min\{1,\gamma\}$ such that:

$$\kappa<\delta^{r-\ell}\quad\text{and}\quad \hat\nu\hat\gamma^{-1}<\delta^{r};$$

this is possible since f is partially hyperbolic and r-bunched. Fix a point $z^c\in I^c$ and an integer $n\geq 0$ such that $|z^c|=\Theta(\delta_{-n}(p)^{-1})$. Let $z=(0,0,z^c)\in I^m_0$. By our choice of n, we have that for $0\leq i\leq n$, $|f_0^{-i}(z^c)|\leq\gamma_{-i}(p)|z^c|\leq\gamma_{-i}(p)\Theta(\delta_{-n}(p)^{-1})\ll 1$, if $|z^c|$ sufficiently small (uniformly in p). Thus we may assume that $z_{-i}=f^{-i}(z)\in\hat M_p(1)$, for $0\leq i\leq n$.

Next, fix a point $x^u_0\in I^u$, and consider the point $w=\hat\omega^{cu}_z(0,x^u_0)=(x^u_0,\hat\beta^{u,s}_z(0,x^u_0),z^c+\hat\beta^{u,c}_z(0,x^u_0))$, which is the point of intersection of the unstable manifold $\mathscr{W}^u(z)$ with $x^u=x^u_0$. For $0\leq i\leq n$, write $w_{-i}=(w^s_{-i},w^u_{-i},w^c_{-i})$. Since w lies on the unstable manifold of z, which is uniformly contracted by f^{-1}, and since $z_{-i}\in\hat M_p(1)$ for $0\leq i\leq n$, we have that $w_{-i}\in I^m_{-i}$ for $0\leq i\leq n$.

We also will use a sequence of "twin points" in our calculations. The twin w' is defined $w'=(x^u_0,0,z^c)$; notice that $w'\in\widehat{\mathscr{W}}^u_p(z)$. We then set $w'_{-i}=f^{-i}(w')$, and write $w'_{-i}=(w^u_{-i}{}',0,w^c_{-i}{}')$, for $0\leq i\leq n-1$. Since $w\in\mathscr{W}^u(z)$, and $w'\in\widehat{\mathscr{W}}^u_p(z)$, it follows that

$$|w_{-n}-w'_{-n}|\leq|w_{-n}-f^{-n}(z)|+|w'_{-n}-f^{-n}(z)|\leq 2\hat\nu_{-n}(p)^{-1}|x^u_0|.$$

The vector $w-w'$ lies in a cone about the center-stable distribution for f at w'. Since this cone is mapped into itself by Tf^{-1}, acting as a strict contraction, it follows that $w_{-i}-w'_{-i}$ lies in this cone as well, for $0\leq i\leq n$. Recall that vectors in this cone are contracted/expanded under f by at most $\hat{\gamma}^{-1}$. Since $|w_{-n} - w'_{-n}| = O(\hat{\nu}_{-n}(p)^{-1})$, it follows from a simple inductive argument that $|w_{-i} - w'_{-i}| = O(\hat{\nu}_{-n}(p)^{-1}\hat{\gamma}_i(p_n)^{-1}|x^u_0|)$, for $i = 0, \ldots, n$. In particular, $|w - w'| = O(\hat{\nu}_{-n}(p)^{-1}\hat{\gamma}_n(p_n)^{-1}|x^u_0|) = O(\hat{\nu}_{-n}(p)^{-1}\hat{\gamma}_{-n}(p)|x^u|_0)$. Since $\hat{\nu}\hat{\gamma}^{-1} < \delta^r$, and $|z^c| = \Theta(\delta_{-n}(p)^{-1})$, we obtain that $|w - w'| \leq |x^u_0|o(|z^c|^r)$. But $w - w' = (\hat{\beta}^{u,s}_z(0, x^u), 0, \hat{\beta}^{u,c}_z(0, x^u))$, and so we have shown that $|\hat{\beta}^u_z(0, x^0)| \leq |x^u_0|o(|z^c|^r)$, proving the lemma for the case $\ell = 0$.

We next turn to the case $\ell > 1$. Consider the points $(z^c_{-n}, j^{\ell,1}_{(0,w^u_{-n})}\hat{\beta}^u_{z-n})$ and $(z^c_{-n}j^{\ell,1}_{(0,w^u_{-n}{}')}0)$ in $(I^c \times J^{\ell,1}_{\{0\}\times I^u}(I^c \times I^u, \mathbb{R}^{s+c}))_{-n}$.

To simply notation, we write "$\mathscr{T}$" for $\mathscr{T}^{\ell,1}_f$ and $j^{\ell,1}_{-i}\hat{\beta}^u$ for $j^{\ell,1}_{(0,w^u_{-i})}\hat{\beta}^u_{z-i}$. The notation $|\cdot|_L$ is the fiberwise norm on $I^c \times J^{\ell,1}_{\{0\}\times I^u}(I^c \times I^u, \mathbb{R}^{s+c})$ defined above (hence $|(x, j^{\ell,1}_y\beta)|_L = |j^{\ell,1}_y\beta|_L$). Having fixed this notation, we next estimate, for $0 \leq i \leq n$:

$$\begin{aligned}
|j^{\ell,1}_{-i+1}(\hat{\beta}^u)|_L &= |\mathscr{T}(z^c_{-i}, j^{\ell,1}_{-i}\hat{\beta}^u)|_L \\
&\leq |\mathscr{T}(z^c_{-i}, j^{\ell,1}_{-i}\hat{\beta}^u) - \mathscr{T}(z^c_{-i}, j^{\ell,1}_{(0,w^u_{-i})}0)|_L \\
&\qquad + |\mathscr{T}(z^c_{-i}, j^{\ell,1}_{(0,w^u_{-i})}0)|_L.
\end{aligned}$$

We estimate the first term in this latter sum using (39):

$$|(z^c_{-i}, j^{\ell,1}_{-i}\hat{\beta}^u) - \mathscr{T}(z^c_{-i}, j^{\ell,1}_{(0,w^u_{-i})}0)|_L \quad \leq \quad \kappa(p_{-i})|j^{\ell,1}_{-i}\hat{\beta}^u|_L.$$

The second term is estimated using two facts. First, we have that the map $\mathscr{T}$ is fiberwise C^1 (since $\ell \leq r < k - 1$), and so

$$|\mathscr{T}(z^c_{-i}, j^{\ell,1}_{(0,w^u_{-i})}0) - \mathscr{T}(z^c_{-i}, j^{\ell,1}_{(0,w^u_{-i}{}')}0)|_L = O(|w_{-i} - w'_{-i}|) = O(\hat{\nu}_{-n}(p)^{-1}\hat{\gamma}_i(p_{-n})^{-1}).$$

Second, we note that $\mathscr{T}\left(z^c_{-i}, j^{\ell,1}_{(0,w^u_{-i}{}'))}0\right) = (z^c_{-i+1}, j^{\ell,1}_{(0,w^u_{-i+1}{}')}0)$. Hence:

$$\begin{aligned}
|\mathscr{T}(z^c_{-i}, j^{\ell,1}_{(0,w^u_{-i})}0)|_L &\leq |\mathscr{T}(z^c_{-i}, j^{\ell,1}_{(0,w^u_{-i})}0) - \mathscr{T}(z^c_{-i}, j^{\ell,1}_{(0,w^u_{-i}{}'))}0)|_L \\
&= O(\hat{\nu}_{-n}(p)^{-1}\hat{\gamma}_i(p_{-n})^{-1}),
\end{aligned}$$

for $i = 0, \ldots, n$. Combining these calculations, we have, for $0 \leq i \leq n$:

$$|\, j^{\ell,1}_{-i+1}(\hat{\beta}^u)\,|_L = O(\kappa(p_{-i}))\,|\, j^{\ell,1}_{-i}\hat{\beta}^u\,|_L + O(\hat{\nu}_{-n}(p)^{-1}\hat{\gamma}_i(p_{-n})^{-1}).$$

By an inductive argument, we obtain:

$$\begin{aligned}
|\, j_0^{\ell,1}(\hat\beta^u)\,| &= O(\sum_{i=0}^{n} \kappa_{i-n}(p)^{-1}\hat\nu_{-n}(p)^{-1}\hat\gamma_i(p_{-n})^{-1})\\
&= o(\sum_{i=0}^{n} \delta_{i-n}(p)^{\ell-r}\hat\nu_{i-n}(p)^{-1}\hat\nu_i(p_{-n})\hat\gamma_i(p_{-n})^{-1})\\
&= o(\sum_{i=0}^{n} \delta_{i-n}(p)^{\ell-r}\hat\nu_{i-n}(p)^{-1}\delta_i(p_{-n})^{r})\\
&= o(\delta_{-n}(p)^{\ell-r}),
\end{aligned}$$

where we have used the facts that $\kappa < \delta^{r-\ell}$, and $\hat\nu/\hat\gamma < \delta^r$. Since $|z^c| = \Theta(\delta_{-n}(p)^{-1})$, and recalling our notation for $j^{\ell,1}_{0,x_0^u}\hat\beta^s_z$, we obtain that

$$|\, j_0^{\ell,1}(\hat\beta^u)\,| = |\, j^{\ell,1}_{0,x_0^u}\hat\beta^u_z\,| = o(|z^c|^{r-\ell}), \tag{41}$$

for all $x_0^u \in I^u$.

We are not quite done yet, as (41) is not exactly what is claimed in the statement of Lemma 10.6. To finish the proof, we note that if β is $C^{\ell+1}$, then by the equality of mixed partials, we have that $j^1_{x^u=x_0^u}(j^\ell_{x^c=0}\beta) = j^\ell_0(j^1_{x_0^u}\beta) = j^{\ell,1}_{0,x_0^u}\beta$. The quantity we want to estimate is

$$\left| j_0^\ell\left(x^c \mapsto \hat\beta^u_{(0,0,z^c)}(x^c,x^u)\right)\right|$$

Consider the function $\zeta\colon I^u \to J_0^\ell(\mathbb{R}^c,\mathbb{R}^{c+s})$ given by

$$\zeta(x^u) = j_0^\ell(x^c \mapsto \hat\beta^u_{(0,0,z^c)}(x^c,x^u)).$$

The value $\zeta(x_0^u)$ can be obtained by integrating its derivative along a smooth curve $\gamma(x^u)$, tangent to $\mathcal{W}^u_z(z)$, from 0 to x_0^u . But note that, since $\hat\beta^u_z$ is a $C^{\ell+1}$ function, we must have $j^1_{x^u}\zeta = j^{\ell,1}_{0,x^u}\beta$; (41) implies that $\zeta(x_0^u) \le |x_0^u|\cdot o(|z^c|^{r-\ell})$, for all $x_0^u \in I^u$. This completes the proof of Lemma 10.6. □

We remark that the same estimates hold for the lifted fake foliations $\widehat{\mathcal{W}}^*_F$ if F is C^k and r-bunched, for $k \ge 2$ and $r = 1$ or $r < k-1$.

10.3. Fake holonomy. — In the discussion that follows, we define holonomy maps for various fake foliations between fake center manifolds. Because we are interested in local properties, we will be deliberately careless in referring to the sizes of the domains of definition. For example, if x and x' lie within distance 1 on the same stable manifold, and τ and τ' are any smooth transversals to $\widehat{\mathcal{W}}^s_x$ inside $\widehat{\mathcal{W}}^{cs}(x)$, then there is a well-defined $\widehat{\mathcal{W}}^s_x$ holonomy map between a ρ'-ball $B_\tau(x,\rho')$ in τ and τ', if ρ' is sufficiently small. We will suppress this restriction of domain and just speak of the $\widehat{\mathcal{W}}^{cs}_x$-holonomy map between τ and τ'. This abuse of notation is justified because all of the holonomy maps we consider will be taken over paths of bounded length, and all foliations and fake foliations are continuous. Hence the restriction of domain can always be performed uniformly over the manifold. This will simplify greatly the notation in the sections that follow.

Let $x \in M$ and $x' \in \mathcal{W}^s(x,1)$. We define a C^r diffeomorphism

$$\hat{h}_{(x,x')} \colon \widehat{\mathcal{W}}^c(x) \to \widehat{\mathcal{W}}^c(x')$$

as the composition of two holonomy maps: first, $\widehat{\mathcal{W}}^s_x$ holonomy between the C^r manifolds $\widehat{\mathcal{W}}^c(x)$ and $\widehat{\mathcal{W}}^{cs}(x) \cap \widehat{\mathcal{W}}^{cu}(x')$, and second, the $\widehat{\mathcal{W}}^u_{x'}$ holonomy between $\widehat{\mathcal{W}}^{cs}(x) \cap \widehat{\mathcal{W}}^{cu}(x')$ and $\widehat{\mathcal{W}}^c(x')$.

We also define for $x' \in \mathcal{W}^s(x,1)$ the *lifted* fake holonomy map

$$\widehat{H}_{(x,x')} \colon \widehat{\mathcal{W}}^c_F(x) \to \widehat{\mathcal{W}}^c_F(x')$$

by composing $\widehat{\mathcal{W}}^s_{F,x}$ holonomy between $\widehat{\mathcal{W}}^c_F(x) = \pi^{-1}(\widehat{\mathcal{W}}^c(x))$ and $\widehat{\mathcal{W}}^{cs}_F(x) \cap \widehat{\mathcal{W}}^{cu}_F(x') = \pi^{-1}(\widehat{\mathcal{W}}^{cs}(x) \cap \widehat{\mathcal{W}}^{cu}(x'))$, and $\widehat{\mathcal{W}}^u_{F,x'}$ holonomy between $\widehat{\mathcal{W}}^{cs}_F(x) \cap \widehat{\mathcal{W}}^{cu}_F(x')$ and $\widehat{\mathcal{W}}^c_F(x') = \pi^{-1}(\widehat{\mathcal{W}}^c(x'))$. Lemma 10.3 implies that $\pi \circ \widehat{H}_{(x,x')} = \hat{h}_{(x,x')} \circ \pi$.

We similarly define, for $x \in M$ and $x' \in \mathcal{W}^u(x,1)$ a map

$$\hat{h}_{(x,x')} \colon \widehat{\mathcal{W}}^c(x) \to \widehat{\mathcal{W}}^c(x')$$

as the composition of $\widehat{\mathcal{W}}^u_x$ holonomy between $\widehat{\mathcal{W}}^c(x)$ and $\widehat{\mathcal{W}}^{cu}(x) \cap \widehat{\mathcal{W}}^{cs}(x')$, and $\widehat{\mathcal{W}}^s_{x'}$ holonomy between $\widehat{\mathcal{W}}^{cu}(x) \cap \widehat{\mathcal{W}}^{cs}(x')$ and $\widehat{\mathcal{W}}^c(x')$. Finally, we define, for $x \in M$ and $x' \in \mathcal{W}^y(x,1)$,

$$\widehat{H}_{(x,x')} \colon \widehat{\mathcal{W}}^c_F(x) \to \widehat{\mathcal{W}}^c_F(x')$$

to be the natural lift of $\hat{h}_{x,x')}$, as above.

Proposition 10.1, parts (6) and (7) and Lemma 10.3 immediately imply:

Lemma 10.7. — *Suppose f is C^k and r-bunched, for some $k \geq 2$ and $r < k-1$ or $r = 1$. Then for every $x \in M$ and $x' \in \mathcal{W}^*(x,1)$, for $* \in \{s,u\}$, the map $\hat{h}_{(x,x')}$ is a C^r diffeomorphism and depends continuously in the C^r topology on (x,x') .*

If F is a C^k, r-bunched extension of f, then $\widehat{H}_{(x,x')}$ is a C^r diffeomorphism for every $x \in M$, $x' \in \mathcal{W}^*(x,1)$, and $* \in \{s,u\}$and depends continuously in the C^r topology on (x,x'). Moreover, $\widehat{H}_{(x,x')}$ projects to $\hat{h}_{(x,x')}$ under π.

The definitions of $\hat{h}$ and $\widehat{H}$ readily extend to $(k,1)$-accessible sequences by composition (cf. Section 4 for the definition of accessible sequence). Note that any su-path corresponds to an $(k,1)$-accessible sequence if one uses sufficiently many successive points lying in the same stable or unstable leaf. Lemma 4.5 implies that if f is accessible, then there exists a $K_1 \in \mathbb{Z}_+$ such that any two points in M can be connected by a $(K_1,1)$-accessible sequence. For $\mathcal{S} = (y_0,\ldots,y_k)$ a $(k,1)$-accessible sequence, we define $\hat{h}_{\mathcal{S}} \colon \widehat{\mathcal{W}}^c(y_0) \to \widehat{\mathcal{W}}^c(y_k)$ by $\hat{h}_{\mathcal{S}} = \hat{h}_{(y_{k-1},y_k)} \circ \cdots \circ \hat{h}_{(y_0,y_1)}$ and $\widehat{H}_{\mathcal{S}} \colon \widehat{\mathcal{W}}^c_F(y_0) \to \widehat{\mathcal{W}}^c_F(y_k)$ by $\widehat{H}_{\mathcal{S}} = \widehat{H}_{(y_{k-1},y_k)} \circ \cdots \circ \hat{h}_{(y_0,y_1)}$.

Lemma 10.8. — *If F and f are C^k and r-bunched for $k \geq 2$ and $r = 1$ or $r < k-1$, then $\hat{h}_{\mathcal{S}}$ and $\widehat{H}_{\mathcal{S}}$ are C^r diffeomorphisms that depend continuously in the C^r topology on $\mathcal{S}$.*

We next define the notion of a *shadowing accessible sequence.* This concept will be crucial for proving that the C^r diffeomorphisms $\widehat{H}_{\mathcal{S}}$ can be well-approximated by homeomorphisms that preserve the image of any saturated section σ.

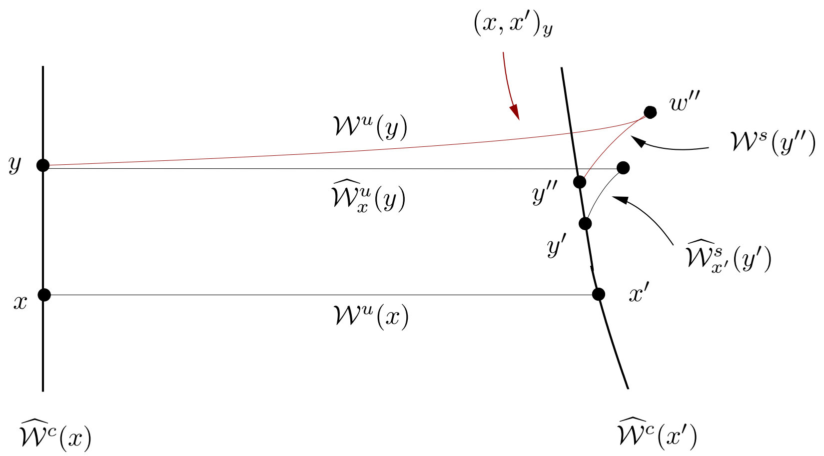


FIGURE 6. The shadowing accessible sequence $(x, x')_y$. The distance between $y' = \hat{h}_{(x,x')}(y)$ and $y'' = h_{(x,x')}(y)$ is $O(d(x,y)^r)$; the distance between x' and y' is $O(d(x,y))$ (see Lemma 10.9).

Let x be an arbitrary point in M, let $x' \in \mathcal{W}^u(x,1)$, and let $y \in \widehat{\mathcal{W}}^c(x)$. The *shadowing accessible sequence* $(x, x')_y$ is defined as follows. Let w'' be the unique point of intersection of $\mathcal{W}^u(y)$ with $\bigcup_{z \in \widehat{\mathcal{W}}^c(x')} \mathcal{W}^s_{\mathrm{loc}}(z)$, and let y'' be the unique point of intersection of $\mathcal{W}^s_{\mathrm{loc}}(w'')$ and $\widehat{\mathcal{W}}^c(x')$. We set $(x, x')_y = (y, w'', y'')$; it is an accessible sequence from y to a point $y'' \in \widehat{\mathcal{W}}^c(x')$. See Figure 6.

We have defined $(x, x')_y$ for $x' \in \mathcal{W}^u(x,1)$ and $y \in \widehat{\mathcal{W}}^c(x)$. Similarly, for $x' \in \mathcal{W}^s(x,1)$, and $y \in \widehat{\mathcal{W}}^c(x)$, define the shadowing accessible sequence $(x, x')_y = (x, w'', y'')$, where w'' is the unique point of intersection of $\mathcal{W}^s(y)$ with $\bigcup_{z \in \widehat{\mathcal{W}}^c(x')} \mathcal{W}^u_{\mathrm{loc}}(z)$, and y'' is the unique point of intersection of $\mathcal{W}^u_{\mathrm{loc}}(w'')$ and $\widehat{\mathcal{W}}^c(x')$. It is an accessible sequence from y to a point $y'' \in \widehat{\mathcal{W}}^c(x')$. Notice that $(x, x')_y$ is a $(2,1)$ accessible sequence, whereas (x, x') is a $(1,1)$-accessible sequence. We may regard (x, x') as a $(2,1)$ accessible sequence by expressing it as (x, x', x'). Then it is natural to say that $(x, x')_y \to (x, x')$ as $y \to x$.

We extend the definition of shadowing accessible sequences to all $(k,1)$-accessible sequences by concatenation. This defines, for each $(k,1)$-accessible sequence $\mathcal{S}$ connecting x and x', and for each $y \in \widehat{\mathcal{W}}^c(x)$, a $(2k,1)$-accessible sequence $\mathcal{S}_y$ connecting y to a point $y' \in \mathcal{S}^c(x')$. The $(k,1)$ accessible sequence may be regarded as a $(2k,1)$ accessible sequence by repeating the appropriate terms in the sequence. With this convention, we have that $\mathcal{S}_y \to \mathcal{S}$ as $y \to x$. Let $K = 2K_1$; henceforth we will restrict our attention to $(K,1)$-accessible sequences.

Now, for $x' \in \mathcal{W}^u(x,1)$ or $x' \in \mathcal{W}^s(x,1)$, we define the map:

$$h_{(x,x')}\colon \widehat{\mathcal{W}}^c(x) \to \widehat{\mathcal{W}}^c(x')$$

by $h_{(x,x')}(y) = \hat{h}_{(x,x')_y}(y)$; in other words, $h_{(x,x')}$ sends y to the endpoint of $(x,x')_y$. Notice that $h_{(x,x)'}$ is a local homeomorphism, but not a diffeomorphism. However, we will show that $h_{(x,x')}$ has "an r-jet at x" (Lemma 10.9); we will make this notion precise in the following subsections.

Similarly define $\mathcal{H}_{x,x'}\colon \widehat{\mathcal{W}}^c_F(x) \to \widehat{\mathcal{W}}^c_F(x')$ for $x' \in \mathcal{W}^u(x,1)$ or $x' \in \mathcal{W}^s(x,1)$ by $\mathcal{H}_{(x,x')}(z) = \widehat{\mathcal{H}}_{(x,x')_{\pi(z)}}(z)$. The definitions of h and $\mathcal{H}$ extend naturally to $(K,1)$-accessible sequences by composition; for $\mathcal{S}$ a $(K,1)$-accessible sequence from x to x', we denote by $h_{\mathcal{S}}\colon \widehat{\mathcal{W}}^c(x) \to \widehat{\mathcal{W}}^c(x')$ and $\mathcal{H}_{\mathcal{S}}\colon \widehat{\mathcal{W}}^c_F(x) \to \widehat{\mathcal{W}}^c_F(x')$ the corresponding maps.

Note the simple observation that if $\mathcal{S}$ is a $(K,1)$-accessible sequence from x to x', then $\hat{h}_{\mathcal{S}}(x) = x' = h_{\mathcal{S}}(x)$, and for every $z \in \pi^{-1}(x)$, $\widehat{\mathcal{H}}_{\mathcal{S}}(z) = \mathcal{H}_{\mathcal{S}}(z)$.

The next lemma is an important consequence of Lemmas 10.5 and 10.6. It tells us that the endpoint of the accessible sequence $(x,x')_y$ is a very good approximation to $\hat{h}_{(x,x')}(y)$, and this is true even on an infinitesimal level.

Lemma 10.9. — *If f is C^k and r-bunched, for $k \geq 2$ and $r = 1$ or $r < k-1$, then for every $(K,1)$ accessible sequence connecting x to x', every $y \in \widehat{\mathcal{W}}^c(x)$, and every integer $0 \leq \ell \leq r$:*

$$\|j^\ell_y \hat{h}_{\mathcal{S}} - j^\ell_y \hat{h}_{\mathcal{S}_y}\| = o(d(x,y)^{r-\ell}).$$

Moreover, if F is also C^k and r-bunched, then for any $z \in \pi^{-1}(x)$ and any $w \in B_{\mathcal{B}}(z,1) \cap \pi^{-1}(y)$:

$$\|j^\ell_w \widehat{\mathcal{H}}_{\mathcal{S}} - j^\ell_w \widehat{\mathcal{H}}_{\mathcal{S}_y}\| = o(d(z,w)^{r-\ell}),$$

where the distance is measured in a uniform coordinate system containing the su-path $\gamma_{\mathcal{S}}$.

Proof. — This is almost a direct consequence of Lemmas 10.5 and 10.6 in the previous subsection. We prove it for accessible sequences of the form $\mathcal{S} = (x,x')$ with $x' \in \mathcal{W}^u(x,1)$; the general case follows easily.

Fix x, $x' \in \mathcal{W}^u(x,1)$ and $y \in \widehat{\mathcal{W}}^c(x)$. Write $(x,x')_y = (y,w'',y'')$, as in the definition. Let v' be the unique point of intersection of $\widehat{\mathcal{W}}^u_x(y)$ and $\widehat{\mathcal{W}}^{cs}(x')$, and let v'' be the unique point of intersection of $\mathcal{W}^u(y)$ and $\widehat{\mathcal{W}}^{cs}(x')$. See Figure 7.

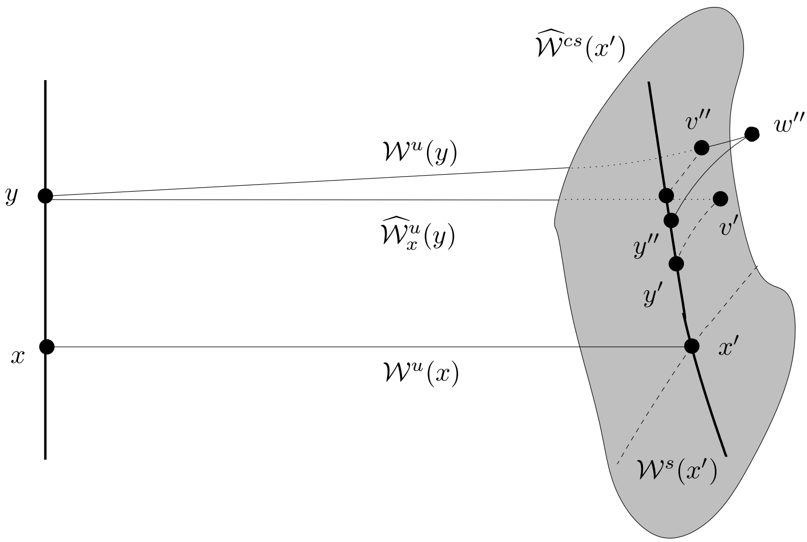


FIGURE 7. Points in the proof of Lemma 10.9

Fix a coordinate system adapted at x as in Subsection 10.2, sending x to the origin in I^m, $\widehat{\mathcal{W}}^{cu}(x)$ to $\{x^s = 0\}$, $\widehat{\mathcal{W}}^{cs}(x)$ to $\{x^u = 0\}$, $\widehat{\mathcal{W}}^{c}(x)$ to $\{x^s = 0\}$, $\{x^u = 0\}$, and sending the fake foliations $\widehat{\mathcal{W}}^{s}_x|_{\widehat{\mathcal{W}}^{cs}(x)}$ and $\widehat{\mathcal{W}}^{u}_x|_{\widehat{\mathcal{W}}^{cu}(x)}$ to the affine foliations $\{x^u = 0, x^s = \mathrm{const}\}$ and $\{x^s = 0, x^u = \mathrm{const}\}$, respectively. Suppose that y corresponds to the point $z = (0, 0, z^c)$ and y'' corresponds to the point z'' in the adapted coordinates at x.

In the coordinate system at x, we parametrize $\widehat{\mathcal{W}}^{c}(x)$ by $x^c \mapsto \hat\omega^c_0(x^c) = (0, 0, x^c)$ and $\widehat{\mathcal{W}}^{c}(y)$ by $x^c \mapsto \hat\omega_z(x^c)$. Similarly we parametrize $\widehat{\mathcal{W}}^{c}(x')$ by $x^c \mapsto (0, 0, x^c)$ and $\widehat{\mathcal{W}}^{c}(y'')$ by $x^c \mapsto \hat\omega_{z''}(x^c)$. We want to compare the ℓ-jets of $x^c \mapsto \hat h_{(x,x')}(0, 0, x^c)$ with $x^c \mapsto \hat h_{(x,x')_y} \circ \hat\omega_z(x^c)$ at the point $x^c = z^c$. We first observe that, by Lemma 10.5, we have that $j^\ell_{z^c}\hat\omega_z(x^c) = o(|z^c|^{r-\ell}) = o(d(x, y)^{r-\ell})$; hence we are left to compare the ℓ-jets of the holonomies $\hat h_{(x,x')}$ and $\hat h_{(x,x')_y}$ in the coordinates adapted at x, at the point z.

We write the maps $\hat h_{(x,x')}$ and $\hat h_{(x,x')_y}$ as compositions of several holonomy maps, and we compare the distance between the ℓ-jets of the corresponding terms in the compositions. First, we write

$$\hat h_{(x,x')} = h^s_{x'} \circ h^u_x,$$

where $h^u_x\colon \widehat{\mathcal{W}}^{c}(x) \to \widehat{\mathcal{W}}^{cu}(x) \cap \widehat{\mathcal{W}}^{cs}(x')$ is the $\widehat{\mathcal{W}}^{u}_x$-holonomy and $h^s_{x'}$ is the $\widehat{\mathcal{W}}^{s}_{x'}$ holonomy between $\widehat{\mathcal{W}}^{cu}(x) \cap \widehat{\mathcal{W}}^{cs}(x')$ and $\widehat{\mathcal{W}}^{c}(x')$. Next, we write:

$$\hat h_{(x,x')_y} = h^s_{y''} \circ h^u_{y,\flat} \circ h^u_y \circ h^u_{y,\sharp}$$

where $h^u_{y,\sharp}\colon \widehat{\mathcal{W}}^c(y) \to \widehat{\mathcal{W}}^{cs}(x)\cap\widehat{\mathcal{W}}^{cu}(y)$, $h^u_y\colon \widehat{\mathcal{W}}^{cs}(x)\cap\widehat{\mathcal{W}}^{cu}(y)\to\widehat{\mathcal{W}}^{cu}(y)\cap\widehat{\mathcal{W}}^{cs}(x')$ and $h^u_{y,\flat}\colon \widehat{\mathcal{W}}^{cu}(y)\cap\widehat{\mathcal{W}}^{cs}(x') \to \widehat{\mathcal{W}}^{cu}(y)\cap\widehat{\mathcal{W}}^{cs}(y'')$ are $\widehat{\mathcal{W}}^u_y$ holonomies, and $h^s_{y''}\colon \widehat{\mathcal{W}}^{cu}(y)\cap\widehat{\mathcal{W}}^{cs}(y'')\to\widehat{\mathcal{W}}^c(y'')$ is $\widehat{\mathcal{W}}^s_{y''}$-holonomy.

The term $h^u_{y,\sharp}$ in the second composition is expressed in the charts at x by the map $(\hat\omega^c_z(x^c),x^c)\mapsto(\overline{x}^s,0,\overline{x}^c)$, where $(\overline{x}^c,\overline{x}^s)$ are defined implicitly by the equation $\hat\beta^{c,s}_z(\overline{x}^c,\overline{x}^s)=0$. Lemma 10.5 implies that $|j_{z^c}\hat\omega^{cu}_z - j_{z^c}\hat\omega^{cu}_0|$ and $|j_{z^c}\hat\omega^c_z - j_{z^c}\hat\omega^c_0|$ are both $o(|z^c|)^{r-\ell}$, and so in these charts, $|j^\ell_z h^u_{y,\sharp} - j^\ell_z id| = o(|z^c|)^{r-\ell}$.

We may choose the coordinate system adapted at x so that x' is sent to the point $(x^u_0,0,0)$ and $\widehat{\mathcal{W}}^{cs}(x')$ is sent to $x^u=x^u_0$, and we may do this in a way that the C^r size of the chart is bounded independently of x,x'; this uses the fact that $p\mapsto\widehat{\mathcal{W}}^{cs}(p)$ is continuous in the C^r topology. Consider the $\widehat{\mathcal{W}}^u_x$ and $\widehat{\mathcal{W}}^u_y$ holonomies between $x^u=0$ and $x^c=x^u_0$, corresponding to the holonomies

$$h^u_x\colon \widehat{\mathcal{W}}^{cs}(x)\to\widehat{\mathcal{W}}^{cs}(x'),\quad\text{and}\quad h^u_y\colon \widehat{\mathcal{W}}^{cs}(x)\to\widehat{\mathcal{W}}^{cs}(x')$$

In the coordinates at x, these maps are expressed by the functions

$$(0,x^u,x^c)\mapsto\omega^{cs}_0(x^c,x^u),\quad\text{and}\quad(0,x^u,x^c)\mapsto\omega^{cs}_z(x^c,x^u)$$

Lemma 10.6 implies that $|j_{z^c}\hat\omega^{cs}_z(\cdot,x^u) - j_{z^c}\hat\omega^{cs}_0(\cdot,x^u) = o(|z^c|)^{r-\ell}$; in the charts at x we therefore have:

$$|j_z(h^u_x) - j_z(h^u_y)| = o(|z^c|)^{r-\ell} = o(d(x,r)^{r-\ell}).$$

Consider the image points $v'=h^u_x(y)$ and $v''=h^u_y(y)$ of these two holonomy maps in M. Since the distances $d(v',v'')$ and $d(v',y')$ are both $o(|z^c|^r)=o(d(x,y)^r)$, the transversality of the bundles in the partially hyperbolic splitting implies that $d(v'',w'')$ and $d(w'',y'')$ are also $o(d(x,y)^r)$ (see Figure 7). Hence the distance from y'' to x is $O(d(x',y')+d(y',y''))=O(d(x,y)+d(x,y)^r)=O(d(x,y))$, and similarly $d(x,v'')$ and $d(x,w'')$ are $O(d(x,y))$.

We are left to deal with the final terms in the compositions above: $h^s_{y''}\circ h^u_{y,\flat}$ and $h^s_{x'}$. All of these are C^r holonomy maps over very short distances, on the order of $o(d(x,y)^r)$. It follows that their ℓ-jets are close to the identity – within $o(d(x,y)^{r-\ell})$ – once we have shown that the transversals on which they are defined have ℓ-jets within $o(d(x,y)^{r-\ell})$ of the vertical foliation $\{(x^s,x^u)=\text{const}\}$.

Lemma 10.6 implies that the ℓ-jets of $\widehat{\mathcal{W}}^{cu}(x')$ and $\widehat{\mathcal{W}}^{cu}(x)$ coincide along $\mathcal{W}^u(x)$. In particular, in these coordinates, $\widehat{\mathcal{W}}^c(x')$ and the plane $\{x^s=0,x^u=x^u_0\}$ are tangent to order ℓ at x'. Furthermore, since $d(x',v'')$, $d(x',w'')$, $d(x',v'')$, $d(x',y')$ and $d(x',y'')$ are all $O(d(x,y))$, Lemma 10.6 implies that the manifolds $\widehat{\mathcal{W}}^{cu}(y)\cap\widehat{\mathcal{W}}^{cs}(x')$, $\widehat{\mathcal{W}}^{cu}(x)\cap\widehat{\mathcal{W}}^{cs}(x')$, $\widehat{\mathcal{W}}^{cu}(y)\cap\widehat{\mathcal{W}}^{cs}(y'')$, $\widehat{\mathcal{W}}^c(y')$ and $\widehat{\mathcal{W}}^c(y'')$ can all be expressed in the coordinates adapted at x as graphs of functions from $\{x^u=x^u_0,x^s=0\}$ to I^{s+u} whose ℓ-jets at v'', v', w'', y' and y'' respectively, are $o(d(x,y)^{r-\ell})$. Hence all of the the transversals for $h^u_{y,\flat}$, $h^s_{x'}$, and $h^s_{y''}$ have ℓ-jets within $o(d(x,y)^\ell)$ of the vertical foliation $\{(x^s,x^u)=\text{const}\}$ at their basepoints in the compositions. It follows that

$|j^\ell_{v''}(h^s_{y''}\circ h^u_{y,\flat}) - j^\ell_{v''}id| = o(d(x,y))^{r-\ell}$ and $|j^\ell_{v'}h^s_{x'} - j^\ell_{v'}id| = o(d(x,y))^{r-\ell}$, and so $|j^\ell_y\hat{h}_{(x,x')} - j^\ell_y\hat{h}_{(x,x')_y}| = o(d(x,y))^{r-\ell}$, as desired.

The proof for the maps $\widehat{\mathcal{H}}_{(x,x')}$ and $\widehat{\mathcal{H}}_{(x,x')_y}$ are completely analogous. □

10.4. Central jets. — Let $(N,\mathcal{B},\pi,F)$ be a C^k, r-bunched partially hyperbolic extension of f, for some $k\geq 2$, where $\mathcal{B}=M\times N$. We fix Riemannian metrics on M and N. Let $\exp\colon TM\to M$ be the exponential map for this metric (which we may assume to be C^∞), and fix $\rho_0>0$ such that $\exp_p$ is a diffeomorphism from $B_{T_pM}(0,\rho_0)$ to $B_M(p,\rho_0)$, for every $p\in M$. As in the proof of Lemma 10.3, the bundle $\mathcal{B}$ pulls back via $\exp\colon B_{TM}(0,\rho_0)\to M$ to a C^r bundle $\tilde\pi_0\colon\widetilde{\mathcal{B}}_0\to B_{TM}(0,\rho_0)$ with fiber N, where $B_{TM}(0,\rho_0)$ denotes the ρ_0-neighborhood of the 0-section of TM. As in the proof of Lemma 10.3, we fix, for each $p\in M$ a trivialization of $\widetilde{\mathcal{B}}_0|_{B_{T_pM}(0,\rho_0)}$, depending smoothly on $p\in M$. Any section $\sigma\colon M\to\mathcal{B}$ of $\mathcal{B}$ pulls back to a section $\tilde\sigma\colon B_{TM}(0,\rho_0)\to\widetilde{\mathcal{B}}_0$ via $\tilde\sigma(v)=(v,\sigma(\exp(v)))$.

Let $TM=\widetilde{E}^u\oplus\widetilde{E}^c\oplus\widetilde{E}^s$ be a C^∞ approximation to the partially hyperbolic splitting for f. Observe that TM is a C^∞ bundle over $\widetilde{E}^c$ under the map $\pi^c\colon TM\to\widetilde{E}^c$ that sends $v^u+v^c+v^s\in\widetilde{E}^u(p)\oplus\widetilde{E}^c(p)\oplus\widetilde{E}^s(p)$ to $v^c\in\widetilde{E}^c(p)$. This splitting will give us a global way to parametrize the fake center manifolds $\widehat{\mathcal{W}}^c(p)$.

If f is r-bunched, for $r=1$ or $r<k-1$, and the approximation $TM=\widetilde{E}^u\oplus\widetilde{E}^c\oplus\widetilde{E}^s$ to the hyperbolic splitting is sufficiently good, then Proposition 10.1 implies there exists a map $g^c\colon\widetilde{\mathcal{B}}_{\widetilde{E}^c}(0,\rho)\to B_{TM}(0,\rho_0)$ with the following properties:

1. g^c is a section of $\pi^c\colon B_{T_pM}(0,\rho)\to\widetilde{\mathcal{B}}_{\widetilde{E}^c}(0,\rho)$,
2. the restriction of g^c to $B_{\widetilde{E}^c(p)}(0,\rho)$ is a C^r embedding into T_pM, depending continuously in the C^r topology on $p\in M$;
3. for $p\in M$, the image $g^c(B_{\widetilde{E}^c(p)}(0,\rho))$ coincides with $\exp_p^{-1}(\widehat{\mathcal{W}}^c(p))$.

Let $\tilde\pi^c=\pi^c\circ\tilde\pi\colon\widetilde{\mathcal{B}}_0\to B_{\widetilde{E}^c}(0,\rho)$. The bundles and the relevant maps are summarized in the following commutative diagram.

$$\begin{tikzcd}
\widetilde{\mathcal{B}}_0 \arrow[r, "\mathrm{proj}_{\mathcal{B}}"] \arrow[d, shift right, "\tilde\pi"'] \arrow[dd, bend right=60, "\tilde\pi^c"'] & \mathcal{B} \arrow[d, shift right, "\pi"'] \\
B_{T_pM}(0,\rho) \arrow[u, shift right, "\tilde\sigma"'] \arrow[r, "\exp"] \arrow[d, shift right, "\pi^c"'] & M \arrow[u, shift right, "\sigma"'] \\
B_{\widetilde{E}^c}(0,\rho) \arrow[u, shift right, "g^c"']
\end{tikzcd}$$

Note that $\tilde{\pi}^c\colon \widetilde{\mathscr{B}}_0 \to B_{\widetilde{E^c}}(0,\rho)$ is a C^k bundle. A different choice of exponential map or approximation to the partially hyperbolic splitting gives an isomorphic bundle and a different section $g^{c\prime}$ related to the first by a uniform graph transform on fibers.

Consider the restriction $\widetilde{\mathscr{B}}_{0,p}$ of $\widetilde{\mathscr{B}}_0$ to any fiber $B_{\widetilde{E^c(p)}}(0,\rho)$ of $B_{\widetilde{E^c}}(0,\rho)$ over $p \in M$. For every positive integer $\ell \leq r$, we define a $C^{k-\ell}$ jet bundle $\mathscr{J}^\ell \to M$ whose fiber over $p \in M$ is the space $J_0^\ell(\tilde{\pi}^c\colon \widetilde{\mathscr{B}}_{0,p} \to B_{\widetilde{E^c(p)}}(0,\rho))$.

Suppose now that $\sigma\colon M \to \mathscr{B}$ is a section of $\mathscr{B}$, and that $\ell \leq r$. We say that σ *has a central ℓ-jet at p* if there exists a C^ℓ local section $s = s_{\sigma,p} \in \Gamma^\ell_{0_p}(\tilde{\pi}^c\colon \widetilde{\mathscr{B}}_{0,p} \to B_{\widetilde{E^c(p)}}(0,\rho))$ such that, for all $v \in B_{\widetilde{E^c(p)}}(0,\rho))$:

$$d_N(\mathrm{proj}_N \circ \tilde{\sigma} \circ g^c(v), \mathrm{proj}_N \circ s(v)) = o(|v|^\ell). \tag{42}$$

It is not hard to see that $\sigma\colon M \to \mathscr{B}$ has a central ℓ-jet at p if and only if the restriction of σ to $\widehat{\mathcal{W}}^c(p)$ is tangent to order ℓ at p to a C^ℓ local section $\sigma'\colon \widehat{\mathcal{W}}^c(p) \to \mathscr{B}$. If σ has a central ℓ-jet at p, for every $p \in M$ then σ induces a well-defined section $j^\ell\sigma^c : M \to \mathscr{J}^\ell$ that sends p to $j_0^\ell s_{\sigma,p}$. We call $j^\ell\sigma^c$ the *central ℓ-jet* of σ, and we write $j_p^\ell\sigma^c$ for the image of p under $j^\ell\sigma^c$. It is easy to see that the existence of a central ℓ-jet for σ is independent of the choice of smooth approximation to the partially hyperbolic splitting and independent of choice of exponential map. In general there is no reason to expect the central ℓ-jet $j^\ell\sigma^c$ to be a smooth section, even when σ itself is smooth, because g^c is not smooth.

Remark: If σ has a central ℓ-jet at p, then (in a fixed coordinate system about p), σ has an $(\ell - l, 1, C)$ expansion on $\widehat{\mathcal{W}}^c(p)$ at p. If $j^\ell\sigma^c$ is continuous, and the error term in (42) is uniform in p, then C can be chosen uniformly in a neighborhood of p.

In the proof of Theorem C, we will focus attention on the pullbacks $\mathscr{J}^\ell|_{\widehat{\mathcal{W}}^c(x)}$ of $\mathscr{J}^\ell$ to various fake center manifolds over M. The central observation we will make use of is that, for each $x \in M$, there is an isomorphism I_x between the bundles $\mathscr{J}^\ell|_{\widehat{\mathcal{W}}^c(x)}$ and $J^\ell(\pi\colon \mathscr{B}_{\widehat{\mathcal{W}}^c(x)} \to \widehat{\mathcal{W}}^c(x))$. To compress notation, we will write $J^\ell(\widehat{\mathcal{W}}^c(x), N)$ for $J^\ell(\pi\colon \mathscr{B}_{\widehat{\mathcal{W}}^c(x)} \to \widehat{\mathcal{W}}^c(x))$. For $x \in M$, the isomorphism $I_x\colon \mathscr{J}^\ell|_{\widehat{\mathcal{W}}^c(x)} \to J^\ell(\widehat{\mathcal{W}}^c(x), N)$ is defined:

$$I_x(y, j_0^\ell\psi) = j_y^\ell(id_{\widehat{\mathcal{W}}^c(x)}, \mathrm{proj}_N \circ \psi \circ \pi^c \circ \exp_y^{-1}).$$

10.5. Coordinates on the central jet bundle. — Fix $\ell \leq r$. We describe here a natural system of $C^{r-\ell}$ coordinate charts on $\mathscr{J}^\ell$ based on adapted coordinates on M.

Let $\widetilde{E}^s \oplus \widetilde{E}^c \oplus \widetilde{E}^u$ be a C^∞ approximation to the hyperbolic splitting to M. Fix a point $p \in M$ and let (x^u, x^s, x^c) be a C^r adapted coordinate system on $B_M(p,\rho)$ based at p. Next fix C^r local trivializing coordinates $(x^m, v^c) \in \mathbb{R}^m \times \mathbb{R}^c$ for $\widetilde{E^c}$ over $B_M(p,\rho)$, covering the adapted charts at p and sending $B_{\widetilde{E^c}}(0,\rho_1)|_{B_M(p,\rho)}$ to $I^m \times I^c$.

Let $(x, v) \in I^m \times I^m$ be the corresponding charts on $B_{TM}(0, \rho_1)|_{B_M(p,\rho)}$. In these charts, the projection π^c sends (x^m, v^u, v^s, v^c) to (x^m, v^c).

We choose these charts such that the exponential map on $B_{TM}(0, \rho_1)$ over $B_M(p, \rho)$ in these coordinates sends (x^m, v) to $x^m + v \in I^m$ (these charts are not isometric, nor do they preserve the structure of TM as the tangent bundle to M, but they can be chosen to be uniformly C^r). Also fix C^r coordinates $(x^m, q) \in \mathbb{R}^m \times N$ for $\mathscr{B}$ over $B_M(p, \rho)$ sending $\pi^{-1}(B_M(p, \rho))$ to $I^m \times N$, with $\pi(x^m, q) = x^m$.

The induced coordinates on $\widetilde{\mathscr{B}}_0$ over $B_{\widetilde{E^c}}(0, \rho_0)|_{B_M(p,\rho)}$ take the form $(x^u, x^s, x^c + v^c, v^c, q) \in I^m \times N$. We may further choose these coordinates so that, $\tilde{\pi}$ and $\tilde{\pi}^c$ are the projections onto the $I^m \times I^c$ and I^m coordinates, respectively. These coordinates give a natural identification of $\mathscr{J}^\ell|_{B(p,\rho)}$ with $I^m \times J_0^\ell(I^c, N)$.

Finally, for each point $q \in N$, we fix C^r coordinates $z^n \in \mathbb{R}^n$, sending q to 0 and $B_N(q, \rho)$ to I^n. In this way, we define, for each $z \in \widetilde{\mathscr{B}}_0$, an adapted system of coordinates $(x^u, x^s, x^c + v^c, v^c, z^n) \in \mathbb{R}^m \times \mathbb{R}^c \times \mathbb{R}^n$ sending z to 0 and $B_{\widetilde{\mathscr{B}}_0}(z, \rho)$ to $I^m \times I^c \times I^n$.

In local coordinates, each element of $\mathscr{J}^\ell$ can thus be uniquely represented as a tuple $(x^m, \wp)$, where $x^m \in I^m$ and $\wp \in P^\ell(c, n)$. If σ has an ℓ-jet at p for every p, we can thus represent locally the section $j^\ell \sigma^c$ as a function from I^m to $P^\ell(c, n)$, using the adapted charts in a neighborhood of $\sigma(p)$.

Consider the set $I^c \times J_0^\ell(I^c, N)$. We may regard this as a natural object associated to $p \in M$ in either of two ways. First, $I^c \times J_0^\ell(I^c, N)$ embeds as the subset $\{x^u = 0, x^s = 0\} \times J_0(I^c, N)$ in an adapted coordinate system for $\mathscr{J}^\ell|_{B(p,\rho)}$, which gives an identification of $I^c \times J_0^\ell(I^c, N)$ with $\mathscr{J}^\ell|_{\widehat{\mathscr{W}}^c(p)}$. Second, in the same adapted coordinate system, we have the identification of $I^c \times J_0^\ell(I^c, N)$ with $J^\ell(\widehat{\mathscr{W}}^c(p), N)$. We will use both identifications in what follows. We can further put local coordinates on $I^c \times J_0^\ell(I^c, N)$, as follows. Given a point $z \in \pi^{-1}(x)$, we fix an adapted coordinate system $(x^c, z^n) \in I^c \times I^n$ for $\widehat{\mathscr{W}}^c_F(z)$, sending z to 0. This gives local coordinates $(x^c, \wp) \in I^c \times P^\ell(c, n)$ on $I^c \times J_0^\ell(I^c, N)$ sending z (regarded as an element of $J_0^0(I^c, N) \hookrightarrow J_0^\ell(I^c, N)$) to $(0, 0)$.

Let us give a name to these adapted coordinates and define them more precisely. For $z \in \mathscr{B}$, fix an adapted chart $\hat{\varphi}_z \colon I^m \times I^c \to B_{\mathscr{B}}(z, \rho)$ at z, sending $(0, 0)$ to z, sending $\{x^u = 0, x^s = 0\}$ to $\widehat{\mathscr{W}}^c_F(z)$, and so on. We may further assume that the projection $I^m \times I^c \to I^m$ is conjugate to π under $\hat{\varphi}$. The maps $\hat{\varphi}_z$ induce adapted coordinates $\varphi_z = \pi \circ \hat{\varphi}_z \circ \iota \colon I^m \to B_M(\pi(z), \rho)$ at $\pi(z)$, where ι is the inclusion $x^m \to (x^m, 0)$. We will denote by $\hat{\omega}^c$ the parametrization of $\widehat{\mathscr{W}}^c$ manifolds in the φ_z coordinates. Let $\theta_z \colon I^c \to B_{\widetilde{E}^c(\pi(z))}(0, \rho)$ be defined by:

$$\theta_z(x^c) = \pi^c \circ \exp^{-1}_{\pi(z)}(\varphi_z(0, 0, x^c)).$$

We now define the parametrizations η_z and ν_z of the bundles $\mathscr{J}^\ell|_{\widehat{\mathcal{W}}^c(\pi(z))}$ and $J^\ell(\widehat{\mathcal{W}}^c(\pi(z)), N)$ discussed above. Let $\eta_z\colon I^c \times P^\ell(c,n) \to \mathscr{J}^\ell_{\widehat{\mathcal{W}}^c(\pi(z))}$ be defined by

$$\eta_z(x^c, \wp) = (\varphi_z(0,0,x^c), j_0^\ell\,(id_{\widetilde{E}^c(\varphi_z(0,0,x^c))}, \hat{\varphi}_z(0,0,\theta_z^{-1}, \wp(\theta_z^{-1} - x^c)),$$

(recall here that elements of $\widetilde{\mathscr{B}}_{0,p}$ are of the form $(v,z) \in B_{\widetilde{E}^c(p)}(0,\rho) \times \mathscr{B}$ with $\exp_p(v) = \pi(z)$). Finally, let $\nu_z\colon I^c \times P^\ell(c,n) \to J^\ell(\widehat{\mathcal{W}}^c(\pi(z)), N)$ be the map:

$$\nu_z(x^c, \wp) = j^\ell_{\varphi_z(0,0,x^c)}(\hat{\varphi}_z \circ (\varphi_z{}^{-1},\ \wp\left(\mathrm{proj}_{I^c} \circ \varphi_z^{-1} - x^c\right))).$$

We make all of these choices uniformly in z. Strictly speaking, all of these parametrizations are defined only on a neighborhood of the zero-section in $P^\ell(c,n)$, but as with the holonomy maps, we will ignore restriction of domain issues to simplify notation.

Recall the isomorphism $I_x\colon \mathscr{J}^\ell|_{\widehat{\mathcal{W}}^c(x)} \to J^\ell(\widehat{\mathcal{W}}^c(x), N)$ constructed in the previous subsection. For $w \in \widehat{\mathcal{W}}^c_F(z,\rho)$, consider the map $I_{z,w}\colon I^c \times P^\ell(c,n) \to I^c \times P^\ell(c,n)$ given by $I_{z,w} = \nu_w^{-1} \circ I_{\pi(z)} \circ \eta_z$. We have constructed these coordinates so that $I_{z,z} = id_{I^c \times P^\ell(c,n)}$. The following lemma is a direct consequence of Lemmas 10.5 and 10.5.

Lemma 10.10. — *For every $z \in \mathscr{B}$ and $w \in \widehat{\mathcal{W}}^c_F(z,\rho)$, and $\ell \leq r$, we have:*

$$|j_0^\ell I_{z,w} - j_0^\ell id_{I^c \times P^\ell(c,n)}| = o(d(z,w)^{r-\ell}).$$

10.6. Holonomy on central jets. — Let $\mathscr{S}$ be a $(K,1)$-accessible sequence from x to x'. In this subsection, we will define, for each $0 \leq \ell \leq r$, and each $(K,1)$ accessible sequence from x to x', two bundle maps

$$\widehat{\mathscr{H}}^\ell_{\mathscr{S}}\colon J^\ell(\widehat{\mathcal{W}}^c(x), N) \to J^\ell(\widehat{\mathcal{W}}^c(x'), N)$$

and

$$\mathscr{H}^\ell_{\mathscr{S}}\colon \mathscr{J}^\ell|_{\widehat{\mathcal{W}}^c(x)} \to \mathscr{J}^\ell|_{\widehat{\mathcal{W}}^c(x')};$$

we will make use of the identification I_x between $J^\ell(\widehat{\mathcal{W}}^c(x), N)$ and $\mathscr{J}^\ell|_{\widehat{\mathcal{W}}^c(x)}$ to compare these maps. (Recall that "$J^\ell(\widehat{\mathcal{W}}^c(x), N)$" is shorthand notation for the jet bundle $J^\ell(\pi\colon \mathscr{B}_{\widehat{\mathcal{W}}^c(x)} \to \widehat{\mathcal{W}}^c(x)))$.

The map $\widehat{\mathscr{H}}^\ell_{\mathscr{S}}$ is just the action on ℓ-jets induced by the diffeomorphism $\widehat{\mathscr{H}}_{\mathscr{S}}$, defined by:

$$\widehat{\mathscr{H}}^\ell_{\mathscr{S}}(j^\ell_y\psi) = j^\ell_{\hat{h}_{\mathscr{S}}(y)}\widehat{\mathscr{H}}_{\mathscr{S}} \circ \psi \circ \hat{h}_{\mathscr{S}}^{-1};$$

Then $\widehat{\mathscr{H}}^\ell_{\mathscr{S}}$ is a $C^{r-\ell}$ bundle map, covering $\hat{h}_{\mathscr{S}}$ (see Section 6.3). Lemma 10.8 implies:

Lemma 10.11. — *If F and f are C^k and r-bunched for $k \geq 2$ and $r = 1$ or $r < k-1$, then $\widehat{H}^\ell_{\mathcal{S}}$ is a $C^{r-\ell}$ diffeomorphism that depends continuously in the $C^{r-\ell}$ topology on the $(K,1)$-accessible sequence $\mathcal{S}$.*

Fix a point $z \in \pi^{-1}(x)$ and let $z' = \mathcal{H}_{\mathcal{S}}(z)$. In coordinates on $J^\ell(\widehat{\mathcal{W}}^c(x), N)$ and $J^\ell(\widehat{\mathcal{W}}^c(x'), N)$ induced by the adapted coordinates at z and z', we have a map

$$\widehat{\mathcal{H}}^\ell_{\mathcal{S},z} = \nu_{z'}^{-1} \circ \widehat{\mathcal{H}}^\ell_{\mathcal{S}} \circ \nu_z \colon I^c \times P^\ell(c,n) \to I^c \times P^\ell(c,n).$$

Similarly, if $\mathcal{S}$ connects x and x', we set $\hat{h}_{\mathcal{S},x}(x^c) = \varphi_{x'}^{-1} \hat{h}_{\mathcal{S}} \circ \varphi_x \colon I^c \to I^c$.

Writing $P^\ell(c,n) = \Pi_{i=0}^\ell L^i_{sym}(\mathbb{R}^c, \mathbb{R}^n)$, we have coordinates

$$(x^c, \wp) \mapsto (x^c, \wp_0, \dots, \wp_\ell)$$

on $I^c \times P^\ell(c,n)$, where $\wp_i = D^i_{x^c}\wp$. Denote by $\widehat{\mathcal{H}}^\ell_{\mathcal{S},z}(x^c,\wp)_i$ the $L^i_{sym}(\mathbb{R}^c,\mathbb{R}^n)$-coordinate of $\widehat{\mathcal{H}}^\ell_{\mathcal{S},z}(x^c,\wp)$, so that

$$\widehat{\mathcal{H}}^\ell_{\mathcal{S},z}(x^c,\wp) = (\hat{h}_{\mathcal{S},z}(x^c), \widehat{\mathcal{H}}^\ell_{\mathcal{S},z}(x^c,\wp)_0, \dots, \widehat{\mathcal{H}}^\ell_{\mathcal{S},z}(x^c,\wp)_\ell),$$

where $\widehat{\mathcal{H}}^\ell_{\mathcal{S},z}(x^c,\wp)_0 = \widehat{\mathcal{H}}_{\mathcal{S},z}(x^c,\wp_0)$.

The following is an immediate consequence of the discussion in Section 6.3.

Lemma 10.12. — *For every $\ell \leq r$, there exists a $C^{r-\ell}$ map*

$$R^\ell \colon \mathbb{R}^c \times P^{\ell-1}(c,n) \to L^\ell_{sym}(\mathbb{R}^c, \mathbb{R}^n)$$

such that, for every $(x^c,\wp) \in \mathbb{R}^c \times P^\ell(c,n)$, we have:

$$\widehat{\mathcal{H}}^\ell_{\mathcal{S},z}(x^c,\wp)_\ell = R^\ell(x^c,\wp_0,\dots,\wp_{\ell-1}) + \frac{\partial \mathcal{H}_{\mathcal{S},z}}{\partial \wp_0}(x^c,\wp_0)\cdot \wp_\ell \circ (D_{x^c}\hat{h}_{\mathcal{S},z})^{-1}.$$

We have now defined, for each $(K,1)$-accessible sequence $\mathcal{S}$ connecting x and x', a natural lift of the C^r diffeomorphism $\widehat{\mathcal{H}}_{\mathcal{S}} \colon \widehat{\mathcal{W}}^c_F(x) \to \widehat{\mathcal{W}}^c_F(x')$ to a $C^{r-\ell}$ diffeomorphism $\widehat{\mathcal{H}}^\ell_{\mathcal{S}} \colon J^\ell(\widehat{\mathcal{W}}^c(x), N) \to J^\ell(\widehat{\mathcal{W}}^c(x'), N)$ on the corresponding central ℓ-jet bundles. We have also derived in Lemma 10.12 the important fact that $\widehat{\mathcal{H}}^\ell_{\mathcal{S}}$ has an upper triangular form with respect to the natural local adapted coordinate systems on $J^\ell(\widehat{\mathcal{W}}^c(x), N)$ and $J^\ell(\widehat{\mathcal{W}}^c(x'), N)$.

Our next task is to define, for each $(K,1)$-accessible sequence $\mathcal{S}$ from x to x', a lift of the homeomorphism $\mathcal{H}_{\mathcal{S}} \colon \widehat{\mathcal{W}}^c_F(x) \to \widehat{\mathcal{W}}^c_F(x')$ to a map $\mathcal{H}^\ell_{\mathcal{S}} \colon \mathcal{J}^\ell|_{\widehat{\mathcal{W}}^c(x)} \to \mathcal{J}^\ell|_{\widehat{\mathcal{W}}^c(x')}$ with two essential properties:

- $\mathcal{H}^\ell_{\mathcal{S}}$ and $\widehat{\mathcal{H}}^\ell_{\mathcal{S}}$ are tangent to order $r-\ell$ at x, under the natural identification of $J^\ell(\widehat{\mathcal{W}}^c(x), N)$ and $\mathcal{J}^\ell|_{\widehat{\mathcal{W}}^c(x)}$;
- $\mathcal{H}^\ell_{\mathcal{S}}$ preserves central ℓ-jets of bisaturated sections of $\mathcal{B}$.

Recall that for $x' \in \mathcal{W}^s(x,1)$ or $x' \in \mathcal{W}^u(x,1)$, we defined $h_{(x,x')}(y) = \hat{h}_{(x,x')_y}(y)$ and $\mathcal{H}_{(x,x')}(z) = \widehat{\mathcal{H}}_{(x,x')_{\pi(z)}}(z)$; we then extended this definition to $(K,1)$-accessible sequences via composition. We further extend this definition to central ℓ-jets. If $\mathcal{S}$ is a $(K,1)$-accessible sequence from x to x', we set:

$$\mathcal{H}^\ell_{\mathcal{S}}(y, j^\ell_0\psi) = I^{-1}_{h_{\mathcal{S}}(y)} \circ \widehat{\mathcal{H}}^\ell_{\mathcal{S}_y}(I_x \circ (y, j^\ell_0\psi)),$$

where $I_x\colon \mathcal{J}^\ell|_{\widehat{\mathcal{W}}^c(x)} \to J^\ell(\widehat{\mathcal{W}}^c(x), N)$ is the previously constructed isomorphism. Clearly we have that $\mathcal{H}^\ell_{\mathcal{S}}\colon \mathcal{J}^\ell|_{\widehat{\mathcal{W}}^c(x)} \to \mathcal{J}^\ell|_{\widehat{\mathcal{W}}^c(x')}$ is a map covering $\mathcal{H}_{\mathcal{S}}$, under the projection $\mathcal{J}^\ell|_{\widehat{\mathcal{W}}^c(x)} \to \pi^{-1}(\widehat{\mathcal{W}}^c(x)) = \widehat{\mathcal{W}}^c_F(x)$.

We now address the first important property of $\mathcal{H}^\ell_{\mathcal{S}}$: order $r-\ell$ tangency to $\widehat{\mathcal{H}}^\ell_{\mathcal{S}}$. For $\mathcal{S}$ connecting x and x', we set $h_{\mathcal{S},x}(x^c) = \varphi^{-1}_{x'} \circ h_{\mathcal{S}} \circ \varphi_x\colon I^c \to I^c$, and for $z \in \pi^{-1}(x)$, we define

$$\mathcal{H}^\ell_{\mathcal{S},z} = \eta^{-1}_{z'} \circ \mathcal{H}^\ell_{\mathcal{S}} \circ \eta_z\colon I^c \times P^\ell(c,n) \to I^c \times P^\ell(c,n),$$

where $z' = \widehat{\mathcal{H}}_{\mathcal{S}}(z) = \mathcal{H}_{\mathcal{S}}(z)$. Chasing down the definitions, we see that in $I^c \times P^\ell(c,n)$-coordinates, the map $\mathcal{H}^\ell_{\mathcal{S},z}$ takes the form

$$\mathcal{H}^\ell_{\mathcal{S},z}(x^c, \wp) = I^{-1}_{\mathcal{H}_{\mathcal{S}}(z(x^c,\wp_0)),z'} \circ \widehat{\mathcal{H}}^\ell_{\mathcal{S}_{y(x^c,\wp_0)}} \circ I_{z(x^c,\wp_0),z}(x^c,\wp)$$

where $y(x^c) = \varphi_z(0,0,x^c)$, $z(x^c,\wp_0) = \hat{\varphi}_z(0,0,x^c,\wp_0)$, and the maps $I_{z,w}$ are defined in the previous subsection.

Hence, by the definition of $\widehat{\mathcal{H}}^\ell$, the difference $|\widehat{\mathcal{H}}^\ell_{\mathcal{S},z}(x^c,\wp) - \mathcal{H}^\ell_{\mathcal{S},z}(x^c,\wp)|$ can by estimated by bounding:

- $|j^\ell_z\widehat{\mathcal{H}}_{\mathcal{S}} - j^\ell_{y(x^c,\wp_0)}\widehat{\mathcal{H}}_{\mathcal{S}_{z(x^c,\wp_0)}}|$ and $|j^\ell_z\hat{h}^{-1}_{\mathcal{S}} - j^\ell_{y(x^c,\wp_0)}\hat{h}^{-1}_{\mathcal{S}_{y(x^c,\wp_0)}}|$ which are both $o(|(x^c,\wp_0)|^{r-\ell})$, by Lemmas 10.5 and 10.9; and
- $|j^\ell_0 I^{-1}_{\mathcal{H}_{\mathcal{S}}(z(x^c,\wp_0)),z'} - j^\ell_0 id_{I^c\times P^\ell(c,n)}|$ and $|j^\ell_0(I_{z(x^c,\wp_0),z}(x^c,\wp)) - j^\ell_0 id_{I^c\times P^\ell(c,n)}|$, which are both $o|(x^c,\wp_0)|$, by Lemma 10.10.

We thereby obtain:

Lemma 10.13. — *Let $\mathcal{S}$ be a $(K,1)$-accessible sequence from x to x', and let $z \in \pi^{-1}(x)$.*

For each $x^c \in I^c$, $\wp \in P^\ell(c,n)$ with $|\wp|$ bounded, and for every $0 \le \ell \le r$ we have:

$$|\widehat{\mathcal{H}}^\ell_{\mathcal{S},z}(x^c,\wp) - \mathcal{H}^\ell_{\mathcal{S},z}(x^c,\wp)| = o(|(x^c,\wp_0)|^{r-\ell}).$$

In this sense, the maps $\mathcal{H}^\ell_{\mathcal{S}}$ and $\widehat{\mathcal{H}}^\ell_{\mathcal{S}}$ are tangent to order $r-\ell$ at x.

As mentioned above, another important property of $\mathcal{H}^\ell$ is that it preserves central ℓ-jets of saturated sections.

Lemma 10.14. — *Let $\sigma\colon M \to \mathscr{B}$ be a bisaturated section. Then for every $(K,1)$-accessible sequence from x to x', and any $y \in \widehat{\mathscr{W}}^c(x)$, we have $\mathscr{H}_{\mathscr{S}}(\sigma(y)) = \sigma(h_{\mathscr{S}}(y))$.*

If, in addition $\sigma : M \to \mathscr{B}$ is Lipschitz and has a central ℓ-jet $j^\ell_y\sigma^c$ at y for some $1 \le \ell < r$, then σ has a central ℓ-jet $j^\ell_{h_{\mathscr{S}}(y)}\sigma^c$ at $h_{\mathscr{S}}(y)$, and:

$$j^\ell_{h_{\mathscr{S}}(y)}\sigma^c = \mathscr{H}^\ell_{\mathscr{S}}(j^\ell_y\sigma^c).$$

Proof. — Fix $x \in M$ and $\mathscr{S}$ connecting x to x'. Let $\sigma\colon M \to \mathscr{B}$ be a bisaturated section. It suffices to prove the lemma in the case where $x' \in \mathscr{W}^u(x,1)$ and $\mathscr{S} = (x,x')$.

Let $y \in \widehat{\mathscr{W}}^c(x)$. By definition of $\widehat{\mathscr{H}}_{\mathscr{S}}$, the value $\widehat{\mathscr{H}}_{\mathscr{S}}(\sigma(y))$ is the endpoint of an su-lift path for the foliations $\mathscr{W}^s_F$ and $\mathscr{W}^u_F$, covering the path $(x,x')_y$. The endpoint of $(x,x')_y$ is $h_{\mathscr{S}}(y)$. It follows immediately from saturation of σ that $\mathscr{H}_{\mathscr{S}}(\sigma(y)) = \sigma(h_{\mathscr{S}}(y))$.

Next assume that σ is Lipschitz and has a central ℓ-jet $j^\ell_y\sigma^c$ at y, for some $1 \le \ell < r$. This means that the restriction of σ to $\widehat{\mathscr{W}}^c(y)$ is tangent to order ℓ at y to a C^ℓ local section $\sigma'\colon \widehat{\mathscr{W}}^c(y) \to \mathscr{B}$. Let $y' = \hat h_{(x,x')_y}(y) = h_{(x,x')}(y)$. Consider the images of σ and σ' under $\widehat{\mathscr{H}}_{(x,x')_y}$. Since $\widehat{\mathscr{H}}_{(x,x')_y}$ is a C^ℓ diffeomorphism and covers the C^ℓ diffeomorphism $\hat h_{(x,x')_y}$, the local sections $\widehat{\mathscr{H}}_{(x,x')_y}\circ\sigma\circ\hat h^{-1}_{(x,x')_y}$ and $\widehat{\mathscr{H}}_{(x,x')_y}\circ\sigma'\circ\hat h^{-1}_{(x,x')_y}$ over $\widehat{\mathscr{W}}^c(y')$ are tangent to order ℓ at y'.

Since $\mathscr{H}^\ell_{(x,y)}$ is defined by the induced action of $\mathscr{H}^\ell_{(x,y)_y}$ on $\widehat{\mathscr{W}}^c(y)$, it suffices to show that the local sections $\widehat{\mathscr{H}}_{(x,x')_y}\circ\sigma\circ\hat h^{-1}_{(x,x')_y}$ and $\sigma|_{\widehat{\mathscr{W}}^c(y')}$ are tangent to order ℓ at y'. If this is the case, then $\sigma|_{\widehat{\mathscr{W}}^c(h_{(x,x')}(y))}$ and $\widehat{\mathscr{H}}_{(x,x')_y}\circ\sigma'\circ\hat h^{-1}_{(x,x')_y}$ are also tangent to order ℓ at y'; since the latter section is C^ℓ, this implies that σ has a central ℓ-jet at y', and moreover that $j^\ell_{y'}\sigma^c = \mathscr{H}^\ell_{(x,x')}(j^\ell_y\sigma^c)$.

Lemma 10.9 implies that for all $z \in \widehat{\mathscr{W}}^c(x)$,

$$d_{\mathscr{B}}(\mathscr{H}_{(x,x')}(\sigma(z)), \widehat{\mathscr{H}}_{(x,x')_y}(\sigma(z))) = o(d(\sigma(y),\sigma(z))^r);$$

since σ is Lipschitz, we obtain that

$$d_{\mathscr{B}}(\mathscr{H}_{(x,x')}(\sigma(z)), \widehat{\mathscr{H}}_{(x,x')_y}(\sigma(z))) = o(d(y,z)^r).$$

We have already shown that for all $z \in \widehat{\mathscr{W}}^c(x)$, $\mathscr{H}_{(x,x')}(\sigma(z)) = \sigma(h_{(x,x')}(z))$. Hence $d_{\mathscr{B}}(\sigma(h_{(x,x')}(z)), \widehat{\mathscr{H}}_{(x,x')_y}(\sigma(z))) = o(d(y,z)^r)$, and so $\widehat{\mathscr{H}}_{(x,x')_y}\circ\sigma\circ\hat h^{-1}_{(x,x')_y}$ and $\sigma|_{\widehat{\mathscr{W}}^c(y')}$ are tangent to order r at y'. Since $\ell < r$, this completes the proof. □

10.7. E^c curves. — The final tool that we will need in our proof of Theorem C is the concept of an E^c-curve. As in the proof of Theorem B, we will use an inductive argument to prove that a bisaturated section has central ℓ-jets. In the inductive step of the proof of Theorem B, we prove that the ℓ-jets are Lipschitz continuous, and using Rademacher's theorem, we obtain $\ell+1$ jets. The analogue of that argument in this context would be to show that $j^\ell\sigma^c$ is Lipschitz and then apply Rademacher's theorem.

As mentioned before, this is not possible, since the function g^c is not Lipschitz, even along $\widehat{\mathcal{W}}^c$-manifolds. What we have shown in Lemma 10.5 is that g^c and its jets are Lipschitz along $\widehat{\mathcal{W}}^c(x)$ *at* x, and what we will show in our inductive step here is that $j^\ell\sigma^c$ is Lipschitz along $\widehat{\mathcal{W}}^c(x)$ at x, for every $x \in M$. This leaves the question of how to apply Rademacher's theorem to obtain anything at all, let alone $\ell+1$ central jets. The answer is E^c curves.

An E^c curve is simply a curve in M that is everywhere tangent to E^c. Such C^1 curves always exist by Peano's existence theorem, but we ask a little more: that they be C^r. Rather gratifyingly, there is a simple way to construct such curves, and when f is r-bunched, Campanato's theorem (Theorem 8.2) implies that they C^r. If a function s is Lipschitz along $\widehat{\mathcal{W}}^c(x)$ at x, for every $x \in M$, then for any E^c curve ζ, it is not hard to see that s must be Lipschitz along ζ, and so differentiable almost everywhere. What is more, if a section σ has a central ℓ-jet $j^\ell\sigma^c$, then restricting $j^\ell\sigma^c$ to an E^c curve ζ gives the *actual ℓ-jet for σ restricted to ζ* if $\sigma|_\zeta$ is C^ℓ. We will use both of these properties of E^c curves in our proof of Theorem C.

Lemma 10.15. — *Let f be C^k and r-bunched, where $k \geq 2$ and $r = 1$ or $r < k-1$. Let V be a coordinate neighborhood of p, and let $p^{su}_p : V \to \widehat{\mathcal{W}}^c(p)$ be a C^r submersion. For any C^r curve $\hat\zeta\colon (-1,1) \to \widehat{\mathcal{W}}^c(p)$ with $\hat\zeta(0) = p$, there exists a C^r (or $C^{r-1,1}$ if $r > 1$ is an integer) curve $\zeta\colon (-1,1) \to M$ such that, for all $t \in (-1,1)$:*

1. *$\hat\zeta(t) = p^{su}(\zeta(t))$,*
2. *$\zeta'(0) = \hat\zeta'(0)$,*
3. *$\zeta'(t) \in E^c(\zeta(t))$,*
4. *$d(\zeta(t), \hat\zeta(t)) \leq O(|t|^r)$, and*
5. *$|\zeta^{(\ell)}(t) - \hat\zeta^{(\ell)}(t))| \leq o(|t|^{r-\ell})$, for all $1 \leq \ell \leq r$; what is more, the distance between the ℓ-jets of $\widehat{\mathcal{W}}^c(\hat\zeta(t))$ at $\hat\zeta(t)$ and the ℓ-jets of $\widehat{\mathcal{W}}^c(\zeta(t))$ at $\zeta(t)$ is $o(|t|^{r-\ell})$, for all $1 \leq \ell \leq r$.*

Moreover, for each $y \in V$ there is a C^r submersion $p^{su}_y : V \to \widehat{\mathcal{W}}^c(y)$ with the following property. For each $s, t \in (-1,1)$, there exists a point $x_s \in \widehat{\mathcal{W}}^c(\zeta(t))$ such that x_s is connected to $p^{su}_{\zeta(t)}(\zeta(t+s))$ by an su-path whose length is $o(|s|^r)$, and such that:

(6) *properties (1)-(5) hold for the curves $\zeta_t(s) = \zeta(t+s)$ and $\hat\zeta_t(s) = p^{su}_{\zeta(t)}(\zeta(t+s))$, and*

(7) *$d(x_s, \zeta_t(s)) = o(|s|^r)$.*

All of these statements hold uniformly in $x \in M$.

Proof. — Let $\hat\zeta$ be given and assume without loss of generality that $\hat\zeta$ is unit speed. We may also assume that we are working in C^r local coordinates and that p^{su}_p is projection along an affine plane field E^{su} transverse to E^c. This planefield then defines for each $y \in M$ a smooth projection $p^{su}_y\colon V \to \widehat{\mathcal{W}}^c(y)$.

The curve $\hat{\zeta}$ induces a vector field on $(p^{su})^{-1}(\hat{\zeta})$ by intersecting E^c with $(Dp^{su})^{-1}(\dot{\hat{\zeta}})$, (note that the two distributions meet transversely in a linefield). Integrating this vector field, we get the E^c-curve ζ. Clearly ζ satisfies properties (1)-(3).

To prove (4), we show first that for every s and t, the distance between $\zeta(t+s)$ and the $p^{su}_{\zeta(t)}$-projection of $\zeta(t+s)$ onto $\widehat{\mathcal{W}}^c(\zeta(t))$ is $o(|s|^r)$. The proof of this fact is very similar to the proof of Lemma 10.5.

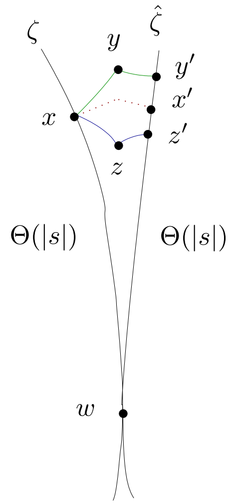


FIGURE 8. An E^c-curve ζ and its shadow $\hat{\zeta}$

Let $w = \zeta(t)$, let $x = \zeta(s+t)$, and let $x' = p^{su}_w(x)$. Let y be the unique point of intersection of $\mathcal{W}^u(x)$ with $\bigcup_{z\in\widehat{\mathcal{W}}^c(x)} \mathcal{W}^s_{\mathrm{loc}}(z)$, and let $y' \in \widehat{\mathcal{W}}^c(x)$ be the unique point of intersection of $\mathcal{W}^u_{\mathrm{loc}}(y)$ and $\widehat{\mathcal{W}}^c(x)$ Similarly, let z be the unique point of intersection of $\mathcal{W}^s(x)$ with $\bigcup_{z\in\widehat{\mathcal{W}}^c(x)} \mathcal{W}^u_{\mathrm{loc}}(z)$, and let $z' \in \widehat{\mathcal{W}}^c(x)$ be the unique point of intersection of $\mathcal{W}^s_{\mathrm{loc}}(z)$ and $\widehat{\mathcal{W}}^c(x)$ (note that y' and z' do not necessarily lie on $\hat{\zeta}$, but this is not important). Note that, because p^{su}_w is smooth, the distance between x' and x is $O(|s|)$. Continuity of the partially hyperbolic splitting and transversality of E^{su} to E^c then imply that $d(y', w)$ and $d(z', w)$ are also $O(|s|)$. We are going to show that $d(x, y)$ and $d(x, z)$ are both $o(|s|^r)$; continuity of the partially hyperbolic splitting and transversality of E^{su} to E^c then imply that $d(x, x') = o(|s|^{r+\varepsilon})$.

Assume that we have fixed a continuous function $\delta < \{\hat{\gamma}, 1\}$ satisfying $\delta\hat{\nu}\hat{\gamma}^{-1} < \gamma^r$; this is possible because f is r-bunched. Choose $n \geq 1$ such that $|s| = \Theta(\delta_n(w))$. Apply f^i to the picture, for $i = 1, \ldots, n$. Since x is connected to x_0 by a curve everywhere tangent to E^c, the distance between x_i and w_i is $O(\delta_n(w)\hat{\gamma}_i(w)^{-1})$. Since y' lies on $\widehat{\mathcal{W}}^c(w)$, the distance between x_i and y_i' is also $O(\delta_n(w)\hat{\gamma}_i(w)^{-1})$; these numbers are less than 1 for all $i = 1, \dots, n$. So the distance between x_n and y_n' is less than $d(x_n, w) + d(y_n', w) = O(\delta_n(w)\hat{\gamma}_n(w)^{-1})$.

Since $y \in \mathcal{W}^s(y')$, the distance between y_n and y_n' is $O(\nu_n(w))$. But 1-bunching implies that $\nu_n(w) = o(\delta_n(w)\hat{\gamma}_n(w)^{-1})$, and so the distance between y_n and x_n is $O(\delta_n(w)\hat{\gamma}_n(w)^{-1})$. Now apply f^{-n} to this picture. Since x_n and y_n lie on the same unstable manifold, the distance between their inverse iterates is contracted by $\hat{\nu}$ at each step. Thus $d(x, y) = O(\hat{\nu}_n(w)\delta_n(w)\hat{\gamma}_n(w)^{-1})$. But we chose δ so that $\delta\hat{\nu}\hat{\gamma}^{-1} < \gamma^r$. Hence $d(x, y) = o(\hat{\gamma}_n(w)^{-r}) = o(|s|^r)$. A similar argument replacing f by f^{-1} shows that $d(x, z) = o(|s|^r)$. Setting $t = 0$ we obtain conclusion (4).

To show that ζ is C^r we use Theorem 8.2. Note that for each $t \in (-1, 1)$, the projection $p^{su}_{\zeta(t)}\zeta$ onto $\widehat{\mathcal{W}}^c(\zeta(t))$ is the same as $p^{su}_{\zeta(t)}\hat{\zeta}$; in particular, $p^{su}_{\zeta(t)}\zeta$ is uniformly C^r, since $\hat{\zeta}$ and p^{su} are C^r, and $\widehat{\mathcal{W}}^c(\zeta(t))$ is uniformly C^r, by r-bunching of f. But the previous calculation now implies that there exists a constant $C > 0$, and for every $t \in (-1, 1)$, a C^r function $p^{su}_{\zeta(t)}\zeta \colon (-1, 1) \to M$ such that:

$$d(p^{su}_{\zeta(t)}\zeta(t+s), \zeta(t+s)) \leq C|s|^r,$$

for every $s \in (-1, 1)$. Theorem 8.2 implies that ζ is C^r (or $C^{r-1,1}$, if $r > 1$ and r is an integer).

The proof of item (5) is very similar to the proof of Lemma 10.5 and is left as an exercise.

Conclusion (6) of the lemma is immediate from the previous calculations. The proof of conclusion (7) is very similar to the calculation above, and is also left to the reader. □

Remark: In fact E^{cs}, E^{cu} and E^c are all C^r along E^c-curves. The proof uses Campanato's theorem again. This time the smooth approximating functions are parametrizations of the manifolds $\widehat{\mathcal{W}}^{cs}$ and $\widehat{\mathcal{W}}^{cu}$.

11. Proof of Theorem C

Suppose F is a C^k and r-bunched extension of f where $k \geq 2$ and $r = 1$ or $r < k-1$, and let $\sigma \colon M \to \mathcal{B}$ be a bisaturated section. The first step of the proof is to show:

Lemma 11.1. — *σ has a central $\lfloor r \rfloor$-jet at every point in M, and $j^{\lfloor r \rfloor}\sigma^c$ is continuous.*

Proof. — We prove the following inductive statements, for $\ell \in [0, \lfloor r \rfloor]$:

I$_\ell$. σ has a central ℓ-jet at every point.

II$_\ell$. The central $\ell - 1$-jets of σ along $\widehat{\mathcal{W}}^c(x)$ are Lipschitz at x, uniformly in $x \in M$, for $\ell \geq 1$.

III$_\ell$. The restriction of σ to E^c curves is uniformly C^ℓ.

We first verify I_0--III_0. Statement II_0 is empty. Since σ is bisaturated, Theorem 4.2 implies that σ is continuous. This implies I_0--III_0. Now assume that statements I_ℓ--III_ℓ hold, for some $\ell \in \{0, \dots, \lfloor r \rfloor - 1\}$.

The central ℓ-jets are continuous. We note that $\mathscr{J}^\ell$ is an admissible bundle; the holonomy map for the accessible sequence $\mathscr{S}$ for x to x' is just the restriction of the map $\mathscr{H}^\ell_{\mathscr{S}}$ to the fibers $\mathscr{J}^\ell|_{\{x\}}$ and $\mathscr{J}^\ell|_{\{x'\}}$. Lemma 10.14 implies that if σ has a central ℓ-jet $j^\ell\sigma^c$, then $j^\ell\sigma^c$ is a bisaturated section of $\mathscr{J}^\ell$. Continuity follows from Theorem 4.2.

The central ℓ-jets of σ along $\widehat{\mathcal{W}}^c(x)$ are Lipschitz at x. We first show that for every x, the restriction of $j^\ell\sigma^c$ to $\widehat{\mathcal{W}}^c(x)$ is Lipschitz at x (where the Lipschitz constant is uniform in x).

By Lemma 4.4 each point $x \in M$ has a uniformly large neighborhood U_x and a family of $(K, 1)$-accessible sequences $\{\mathscr{S}_{x,y}\}_{y \in U_x}$ such that $\mathscr{S}_{x,y}$ connects x to y, $\mathscr{S}_{x,x}$ is a palindromic accessible cycle and $\lim_{y \to x} \mathscr{S}_{x,y} = \mathscr{S}_{x,x}$, uniformly in x. We may assume that $\widehat{\mathcal{W}}^c(x)$ is contained in the neighborhood U_x.

We fix $x = x_0$ and $x_1 \in \widehat{\mathcal{W}}^c(x_0)$ and choose a sequence of points $x_i \in U_{x_0}$ as follows. Let U_{x_0} and $\{\mathscr{S}_{x,y}\}_{y \in U_{x_0}}$ be given by Lemma 4.4. For each $i \geq 1$, given $x_i \in U_{x_0}$, the accessible sequence $\mathscr{S}_i = \mathscr{S}_{x_0,x_i}$ determines a map $h_i := h_{\mathscr{S}_i} : \widehat{\mathcal{W}}^c(x_0) \to \widehat{\mathcal{W}}^c(x_i)$, satisfying $x_i = h_i(x_0)$. We set $x_{i+1} = h_i(x_1) \in \widehat{\mathcal{W}}^c(x_i)$.

We now write things in adapted coordinates. Let $\wp^\ell_\sigma : U_{x_0} \to P^\ell(c, n)$ be the function satisfying $j^\ell_y\sigma^c = \nu_{\sigma(y)}(\wp^\ell_\sigma(y))$. Then $\wp^\ell_\sigma$ assigns in adapted coordinates the appropriate central ℓ-jet of σ to each point in U_{x_0}. We are going to show that the restriction $\wp^\ell_\sigma : \widehat{\mathcal{W}}^c(x) \to P^\ell(c, n)$ is Lipschitz at x.

Let $\mathscr{H}^\ell_{\mathscr{S}_i} : \mathscr{J}^\ell_{\widehat{\mathcal{W}}^c_{x_0}} \to \mathscr{J}^\ell_{\widehat{\mathcal{W}}^c_{x_i}}$ be the lifted "true holonomy on jets," which covers $h_{\mathscr{S}_i}$ and let $\widehat{\mathscr{H}}^\ell_{\mathscr{S}_i} : J^\ell(\widehat{\mathcal{W}}^c(x_0), N) \to J^\ell(\widehat{\mathcal{W}}^c(x_i), N)$ be the lifted "fake holonomy on jets," which covers $\hat{h}_{\mathscr{S}_i}$. This defines maps $\mathscr{H}^\ell_i = \mathscr{H}^\ell_{\mathscr{S}_i, \sigma(x_0)}$ and $\widehat{\mathscr{H}}^\ell_i = \widehat{\mathscr{H}}^\ell_{\mathscr{S}_i, \sigma(x_0)}$ on $I^c \times P^\ell(c, n)$. Write $\mathscr{H}^\ell_i(v, \wp) = (h_i(v), H^\ell_i(v, \wp))$ and $\widehat{\mathscr{H}}^\ell_i(v, \wp) = (\hat{h}_i(v), \widehat{H}^\ell_i(v, \wp))$. Observe that $\varphi_{\sigma(x_i)}(0, 0, 0) = 0$ for all $i \geq 0$; let $v_{i+1} \in I^c$ be the point satisfying $\varphi_{\sigma(x_i)}(0, 0, v_{i+1}) = x_{i+1}$. Note that $|v_1| = O(|x_1 - x_0|)$, $|v_{i+1}| = O(|x_{i+1} - x_i|)$, and $v_{i+1} = \hat{h}_i(v_1)$, for all $i \geq 0$.

Then, since $j^\ell\sigma^c$ is bisaturated and continuous (and hence bounded) Lemma 10.14 implies:

$$\mathscr{H}^\ell_i(0, \wp^\ell_\sigma(x_0)) = (0, \wp^\ell_\sigma(x_i)), \quad \text{and} \quad \mathscr{H}^\ell_i(v_1, \wp^\ell_\sigma(x_1)) = (v_{i+1}, \wp^\ell_\sigma(x_{i+1})).$$

By definition of $\mathscr{H}_i^\ell$ and $\mathscr{H}_i^\ell$, we have $\widehat{\mathscr{H}}_i^\ell(0, \wp_\sigma^\ell(x_0)) = \mathscr{H}_i^\ell(0, \wp_\sigma^\ell(x_0))$; furthermore, Lemma 10.13 implies

$$
\begin{aligned}
&|\mathscr{H}_i^\ell(v_1, \wp_\sigma^\ell(x_1)) - \widehat{\mathscr{H}}_i^\ell(v_1, \wp_\sigma^\ell(x_1))| && (43)\\
&\qquad \leq o(|x_1 - x_0|^{r-\ell} + |\wp_\sigma^{\ell-1}(x_1) - \wp_\sigma^{\ell-1}(x_0)|^{r-\ell}). && (44)
\end{aligned}
$$

Now Lemma 10.11 implies that $\widehat{\mathscr{H}}_i^\ell$ is $C^{r-\ell}$, and uniformly close to the identity map, since $\mathscr{I}_{x_0,x_0}$ is palindromic and $\mathscr{I}_{x_0,y} \to \mathscr{I}_{x_0,x_0}$ as $y \to x_0$, uniformly in x_0.

Lemma 10.12 then implies that for every i with $|x_i - x_0| = O(1)$, there exist linear maps , $A_i = D\hat{h}_i(0)\colon \mathbb{R}^c \to \mathbb{R}^c$, $B_i = D_v\widehat{H}_i^\ell(0, \wp_\sigma^\ell(x_0))\colon \mathbb{R}^c \to P^\ell(c,n)$ and $C_i = D_{\wp_\ell}\widehat{H}_i^\ell(0, \wp_\sigma^\ell(x_0))\colon P^\ell(c,n) \to P^\ell(c,n)$, such that

$$v_{i+1} = \hat{h}_i(v_1) = A_i(v_1) + o(|v_1|), \tag{45}$$

and

$$
\begin{aligned}
\widehat{H}_i(v_1, \wp_\sigma^\ell(x_1)) - \widehat{H}_i(0, \wp_\sigma^\ell(x_0)) \quad = \quad & B_i(v_1) + C_i(\wp_\sigma^\ell(x_1) - \wp_\sigma^\ell(x_0)) \\
& + o(|v_1| + |\wp_\sigma^{\ell-1}(x_1) - \wp_\sigma^{\ell-1}(x_0)|)
\end{aligned}
$$

Moreover, we may assume that, for all i with $|x_i - x_0| = O(1)$:

$$\|A_i - Id_{\mathbb{R}^c}\| < \frac{1}{4}, \quad \|C_i - Id_{P^\ell(c,n)}\| < \frac{1}{4}, \quad \text{and} \quad \|B_i\| < \frac{1}{4}. \tag{46}$$

By the inductive hypothesis II_ℓ, the central $(\ell-1)$-jets of σ along $\widehat{\mathscr{W}}^c(x)$ are Lipschitz at x. Hence $|\wp_\sigma^{\ell-1}(x_1) - \wp_\sigma^{\ell-1}(x_0)| = O(|x_1 - x_0|)$, and so combining (43) and (45) we obtain

$$
\begin{aligned}
&\widehat{H}_i(v_1, \wp_\sigma^\ell(x_1)) - \widehat{H}_i(0, \wp_\sigma^\ell(x_0)) && (47)\\
&\qquad = B_i(v_1) + C_i(\wp_\sigma^\ell(x_1) - \wp_\sigma^\ell(x_0)) + o(|x_1 - x_0|). && (48)
\end{aligned}
$$

(Notice that when $\ell = 0$ the $|\wp_\sigma^{\ell-1}(x_1) - \wp_\sigma^{\ell-1}(x_0)|$ terms do not appear in these expressions, and so Lipschitz regularity of σ is not an issue. This is due to upper triangularity of $\widehat{\mathscr{H}}$.)

The proof now proceeds as the proof of Theorem B. Notice here that we do not need to assume a priori that σ is C^1; the reason is that the derivatives of $\widehat{\mathscr{H}}_i^\ell$ are upper triangular, (unlike the maps H_u^ℓ in the Proof of Theorem B) which allows for more precise estimates. We choose $N = \Theta(|x_1 - x_0|^{-1})$. By (45) and (46), this choice of N ensures that $|x_N - x_0| = O(1)$. Summing (47) from $i = 0$ to $N - 1$, we obtain:

$$
\begin{aligned}
\sum_{i=0}^{N-1} \widehat{H}_i(v_1, \wp_\sigma^\ell(x_1)) - \widehat{H}_i(0, \wp_\sigma^\ell(x_0)) \quad = \quad & (\sum_{i=0}^{N-1} B_i)(v_1) \\
& + (\sum_{i=1}^{N} C_i)(\wp_\sigma^\ell(x_1) - \wp_\sigma^\ell(x_0)) \\
& + No(|x_1 - x_0|).
\end{aligned}
$$

Equation (43) implies that $\sum_{i=0}^{N-1} \widehat{H}_i(v_1, \wp_\sigma^\ell(x_1)) - \widehat{H}_i(0, \wp_\sigma^\ell(x_0)) =$

$$\begin{aligned}
&= \sum_{i=0}^{N-1} \big(H_i(v_1, \wp_\sigma^\ell(x_1)) - H_i(0, \wp_\sigma^\ell(x_0))\big) \\
&\qquad\qquad + No(|x_1 - x_0|^{r-\ell}) \\
&= \sum_{i=0}^{N-1} \wp_\sigma^\ell(x_{i+1}) - \wp_\sigma^\ell(x_i) + No(|x_1 - x_0|^{r-\ell}) \\
&= \wp_\sigma^\ell(x_N) - \wp_\sigma^\ell(x_1) + No(|x_1 - x_0|^{r-\ell}).
\end{aligned}$$

Hence, since $r - \ell \geq 1$:

$$\begin{aligned}
\frac{1}{N}(\wp_\sigma^\ell(x_N) - \wp_\sigma^\ell(x_1)) &= \left(\frac{1}{N}\sum_{i=0}^{N-1} B_i\right)(v_1) \\
&\quad + \left(\frac{1}{N}\sum_{i=1}^{N} C_i\right)(\wp_\sigma^\ell(x_1) - \wp_\sigma^\ell(x_0)) + o(|x_1 - x_0|).
\end{aligned}$$

Rearranging terms and taking norms, we get

$$\begin{aligned}
|\frac{1}{N}(\sum_{i=1}^{N} C_i)(\wp_\sigma^\ell(x_1) - \wp_\sigma^\ell(x_0))| &\leq |\frac{1}{N}(\wp_\sigma^\ell(x_N) - \wp_\sigma^\ell(x_1))| \\
&\quad + |\frac{1}{N}(\sum_{i=0}^{N-1} B_i)(v_1)| + o(|x_1 - x_0|) \\
&\leq O(\frac{1}{N}) + \frac{1}{4}|(x_1 - x_0)| + o(|x_1 - x_0|),
\end{aligned}$$

using (46) and the fact that $\wp_\sigma^\ell$ is continuous, and hence bounded. Again using (46) we have that

$$\left|\left(\frac{1}{N}\sum_{i=1}^{N} C_i\right)(\wp_\sigma^\ell(x_1) - \wp_\sigma^\ell(x_0))\right| \geq \frac{3}{4}|\wp_\sigma^\ell(x_1) - \wp_\sigma^\ell(x_0)|.$$

Combining the previous two estimates, we get:

$$|\wp_\sigma^\ell(x_1) - \wp_\sigma^\ell(x_0)| \leq \frac{4}{3}\left(O(\frac{1}{N}) + \frac{1}{4}|(x_1 - x_0)| + o(|x_1 - x_0|)\right).$$

Finally, since $\frac{1}{N} = \Theta(|x_1 - x_0|)$, we obtain that

$$|\wp_\sigma^\ell(x_1) - \wp_\sigma^\ell(x_0)| = O(|x_1 - x_0|),$$

which is the desired estimate. This verifies $\mathrm{II}_{\ell+1}$.

σ **is Lipschitz.** If $\ell = 0$, we know that σ is Lipschitz at x along $\widehat{\mathcal{W}}^c(x)$ leaves, for every x, and differentiable along $\mathcal{W}^u$ leaves, and $\mathcal{W}^s$ leaves, with the partial derivatives continuous. This readily implies that σ is Lipschitz.

σ **has a central** $(\ell+1)$**-jet at every point.** We fix a uniform system of C^r submersions $p^{su}_x \colon V_x \to \widehat{\mathcal{W}}^c(x)$ defined in coordinate neighborhoods in M. We define E^c curves using these submersions.

Lemma 11.2. — *$j^\ell\sigma^c$ is uniformly Lipschitz along E^c curves.*

Proof. — This is a straightforward consequence of Lemma 10.15 and the fact that $j^\ell\sigma^c$ is Lipschitz along $\widehat{\mathcal{W}}^c(x)$ at x, for every $x \in M$.

□

Fix an E^c curve ζ^1 inside of a coordinate neighborhood V. Since $j^\ell\sigma^c$ is Lipschitz along ζ^1, it is differentiable almost everywhere. Fix a point $x_1 = \zeta^1(t)$ of differentiability. Then $j^\ell\sigma^c$ has a partial derivative along ζ^1 *at* x_1. Let $\{p^{su}_y \colon V \to \widehat{\mathcal{W}}^c(y)\}_{y\in V}$ be the system of submersions in the neighborhood V given by Lemma 10.15. Consider the C^r curve $\hat\zeta^1_{x_1}(s) := p^{su}_{x_1} \circ \zeta^1(t+s)$ in $\widehat{\mathcal{W}}^c(x_1)$. Lemma 10.15 implies that for each s, there is a point $x_s \in \widehat{\mathcal{W}}^c(\zeta(t+s))$ that is connected to $\hat\zeta^1_{x_1}(s)$ by a su-path $\mathcal{S}$ whose length is $o(|s|^r)$. Since $j^\ell\sigma^c$ is bisaturated, we have that $j^\ell_{x_s}\sigma^c = \mathcal{H}^\ell_{\mathcal{S}}(j^\ell_{\hat\zeta^1_{x_1}(s)}\sigma^c)$. Lemma 10.6 implies that

$$d(j^\ell_{x_s}\sigma^c, j^\ell_{\hat\zeta^1_{x_1}(s)}\sigma^c) = O(\mathrm{length}(\mathcal{S})) + O(d(j^\ell_{x_s}\widehat{\mathcal{W}}^c(x_s), j^\ell_{\hat\zeta^1_{x_1}(s)}\widehat{\mathcal{W}}^c(\hat\zeta^1_{x_1}(s)))).$$

Lemmas 10.15 (5), implies that $d(j^\ell_{x_s}\widehat{\mathcal{W}}^c(x_s), j^\ell_{\hat\zeta^1_{x_1}(s)}\widehat{\mathcal{W}}^c(\hat\zeta^1_{x_1}(s))) = o(|s|^{r-\ell})$. Hence:

$$d(j^\ell_{\hat\zeta^1_{x_1}(s)}\sigma^c, j^\ell_{x_s}\sigma^c) = o(|s|^r) + o(|s|^{r-\ell}) = o(|s|^{r-\ell}).$$

Since $j^\ell\sigma^c$ is Lipschitz along $\widehat{\mathcal{W}}^c(\zeta(t+s))$ at $\zeta(t+s)$, we also obtain that $d(j^\ell_{x_s}\sigma^c, j^\ell_{\zeta(t+s)}\sigma^c) = O(d(x_s, \zeta(t+s))) = o(|s|^r)$. Thus, in local coordinates, we have:

$$j^\ell_{\hat\zeta^1_{x_1}(s)}\sigma^c - j^\ell_{x_1}\sigma^c = j^\ell_{\zeta(t+s)}\sigma^c - j^\ell_{x_1}\sigma^c + o(|s|^{r-\ell});$$

since $\ell \le r-1$ and $j^\ell\sigma^c \circ \zeta$ is differentiable at $x_1 = \zeta(t)$, this implies that $j^\ell\sigma^c$ is differentiable at x_1 along the C^r curve $\hat\zeta^1_{x_1}$ in $\widehat{\mathcal{W}}^c(x_1)$.

Let U_{x_1} and $\{\mathcal{S}^1_y\}_{y\in U_{x_1}}$ be the family of accessible sequences given by Lemma 4.4. Since $j^\ell\sigma^c$ is bisaturated, Lemmas 10.13 and 10.14 imply that the image of $\hat\zeta^1_{x_1}$ under $\widehat{\mathcal{H}}_{\mathcal{S}^1_y}$ is a C^r path $\hat\zeta^1_y$ in $\widehat{\mathcal{W}}^c(y)$ along which $j^\ell\sigma^c$ is differentiable at y. Furthermore, $y \mapsto \hat\zeta^1_y$ is continuous at x_1 in the C^r topology, and and the derivative of $j^\ell\sigma^c$ along ζ^1_y at y is continuous at x_1.

Now choose another E^c curve ζ^2 through x_1, quasi-transverse to ζ^1 (that is, such that the tangent spaces to ζ^1 and ζ^2 at x_1 are linearly independent). Again $j^\ell\sigma^c$ is Lipschitz along ζ^2, and we choose a point of differentiability x_2. Since x_1 is a point of continuity of the curves $\{\hat\zeta^1_y\}_{y\in U_{x_1}}$, we may assume (by choosing x_2 close to x_1) that

ζ^2 and $\hat{\zeta}^1_{x_2}$ are quasi-transverse at x_2; hence $\hat{\zeta}^1_{x_2}$ and $\hat{\zeta}^2_{x_2} = p^{su}_{x_2}\zeta^2$ are quasi-transverse curves in $\widehat{\mathcal{W}}^c(x_2)$ along which $j^\ell\sigma^c$ has partial derivatives at x_2.

Let U_{x_2} and $\{\mathcal{S}^2_y\}_{y\in U_{x_2}}$ be given by Lemma 4.4 for the point x_2. Applying the fake holonomy $\widehat{\mathcal{H}}_{\gamma^2_y}$ to the transverse pair of curves $\hat{\zeta}^1_{x_2}$ and $\hat{\zeta}^2_{x_2}$, and reusing the label $\hat{\zeta}^1_y$ now to denote the curve $\widehat{\mathcal{H}}_{\gamma^2_y} \circ \hat{\zeta}^1_{x_2}$, we obtain a family of pairs $\{(\hat{\zeta}^1_y, \hat{\zeta}^2_y)\}_{y\in U_{x_2}}$ of quasi-transverse curves along which $j^\ell\sigma^c$ is differentiable at their intersection and such that $y \mapsto (\hat{\zeta}^1_y, \hat{\zeta}^2_y)$ is continuous at x_2 in the C^r topology.

Repeating this procedure $c = \dim(E^c)$ times, we obtain a point x_c, a neighborhood U_{x_c} of x_c, and a family of c-tuples of curves $\{(\hat{\zeta}^1_y, \ldots, \hat{\zeta}^c_y)\}_{y\in U_{x_c}}$ such that, for each $y \in U_{x_c}$:

1. the curves $(\hat{\zeta}^1_y, \ldots, \hat{\zeta}^c_y)$ contain y and lie in $\widehat{\mathcal{W}}^c(y)$;
2. the tangent lines to $(\hat{\zeta}^1_y, \ldots, \hat{\zeta}^c_y)$ at y span E^c_y;
3. $j^\ell\sigma^c$ is differentiable at y along $\hat{\zeta}^c_y$,
4. the map $z \mapsto (\hat{\zeta}^1_z, \ldots, \hat{\zeta}^c_z)$ is continuous at x_c in the C^r topology; and
5. for each i, the partial derivative of $j^\ell\sigma^c$ along ζ^i_z at z is continuous at $z = x_c$.

We claim that this implies that $j^\ell\sigma^c$ is differentiable along $\widehat{\mathcal{W}}^c(x_c)$ at x_c.

Lemma 11.3. — *Let x_c be given as above. Then for every $z \in \widehat{\mathcal{W}}^c(x_c)$, there exists a path η from x^c to a point w in M with the following properties. The path η is a concatenation of $\hat{\zeta}^i$ paths $\eta = \hat{\zeta}^1_1\hat{\zeta}^2_2\cdots\hat{\zeta}^c_c$, with $d(w, p^{su}_{x_c}(w)) = o(d(z, x_c)^r)$ and $d(p^{su}_{x_c}(w), z) = o(d(z, x_c))$.*

Proof. — Denote by ζ^i_y the ζ^i curve anchored at y (so that $\zeta^i_y(0) = y$). Starting with x_c, we take the union $\mathcal{P}_1 := \bigcup_{q\in\hat{\zeta}^1_{x_c}} \hat{\zeta}^2_q$. Similarly, for $i \geq 1$, we define $\mathcal{P}_{i+1} := \bigcup_{q\in\mathcal{P}_i} \hat{\zeta}^{i+1}_q$. The quasi transversality of the curves $\zeta^1, \ldots, \zeta^c$ at every point and continuity of ζ^i_y at $y = x^c$ implies that there exists a point $w' \in p^{su}_{x_c}(\mathcal{P}_c)$ with $d(w', z) = o(d(x_c, z))$. Fix a point $w \in (p^{su}_{x_c})^{-1}(w') \cap \mathcal{P}_c$. Tracing the $\hat{\zeta}^i$-curves in $\mathcal{P}_c$ back from w to x_c produces the desired path η from x_c to w. An inductive argument using Lemma 10.15 shows that $d(w', w) = o(d(x_c, z)^r)$. □

Let us see how this implies that $j^\ell\sigma^c$ is differentiable along $\widehat{\mathcal{W}}^c(x_c)$ at x_c. This is essentially the same as the proof that a function with continuous partial derivatives is C^1. We will use:

Lemma 11.4. — *For every $y \in V$ and every pair of points $z_1, z_2 \in \widehat{\mathcal{W}}^c(y)$:*

$$d(j^\ell_{z_1}\sigma^c, j^\ell_{z_2}\sigma^c) = O(d(z_1, z_2) + d(z_1, y)^{r-\ell} + d(z_2, y)^{r-\ell}).$$

Proof. — This follows from the facts that $j^\ell\sigma^c$ is saturated and Lipschitz along E^c curves, and that p^{su}_y has the properties given in Lemma 10.15. □

Working in local charts on $\widehat{\mathcal{W}}^c(x^c)$ sending x^c to 0, we may assume that the curves $\hat{\zeta}^i_{x^c}$ are unit speed and correspond to the axes $\cap_{i\neq j}\{x^j = 0\}$. Define constants $a^i = a^i(x_c) \in P^\ell_0(c,n)$, for $i = 1\dots,c$ by

$$a_i = \lim_{y\to x^c}(j^\ell\sigma^c \circ \hat{\zeta}^i_y)'(0).$$

We now define a linear map $A\colon \mathbb{R}^c \to P^\ell_0(c,n)$ by

$$A(t_1,\dots,t_c) = \sum_{i=1}^{c} a_i t_i.$$

We claim that this map is the derivative of $j^\ell\sigma^c$ along $\widehat{\mathcal{W}}^c(x_c)$ at x_c. Let $z \in \widehat{\mathcal{W}}^c(x_c)$ be given, and consider the path η from x_c to w given by Lemma 11.3. Let $v_1 = 0$, and write $\eta = \hat{\zeta}^1_{v_1}\cdot\hat{\zeta}^2_{v_2}\cdots\hat{\zeta}^c_{v_c}$; for $i = 1,\dots c-1$, let t_i satisfy $\hat{\zeta}^i_{v_i}(t_i) = v_{i+1} = \hat{\zeta}^{i+1}_{v_{i+1}}(0)$, and let t_c satisfy $\hat{\zeta}^c_{v^c}(t_c) = w$. The length of the curve η is $\Theta(\sum_{i=1}^c |t_i|) = \Theta(d(x_c,z))$. Lemma 10.5 readily implies that the distance between the ℓ-jets of $\widehat{\mathcal{W}}^c(w)$ at w and $\widehat{\mathcal{W}}^c(p^{su}_{x_c}(w))$ at $p^{su}_{x_c}(w)$ is $o(\mathrm{length}(\eta)^{r-\ell}) = o(d(x_c,z)^{r-\ell})$. Since $j^\ell\sigma^c$ is bisaturated and Lipschitz, we obtain from Lemma 10.6 that

$$\begin{aligned} d(j^\ell_w\sigma^c, j^\ell_{p^{su}_{x_c}(w)}\sigma^c) &= O(d(w,p^{su}_{x_c}(w))) + o(d(x_c,z))^{r-\ell}) \\ &= O(d(x_c,z)^r) + o(d(x_c,z))^{r-\ell}) \\ &= o(d(x_c,z)), \end{aligned}$$

where we have used the facts that $d(w,p^{su}_{x_c}(w)) = o(d(z,x_c)^r)$ and $\ell \le r-1$. Also, since $d(z,p^{su}(w)) = o(d(z,x_c))$, Lemma 11.4 implies that

$$d(j^\ell_z\sigma^c, j^\ell_{p^{su}_{x_c}(w)}\sigma^c) = o(d(z,x_c)),$$

and so

$$d(j^\ell_z\sigma^c, j^\ell_w\sigma^c) = o(d(z,x_c)).$$

Using the fact that $j^\ell\sigma^c$ has a directional derivative along each $\hat{\zeta}^i$ subpath of η at its anchor point $v_i = \hat{\zeta}^i_{v_i}(0)$, and writing things in local coordinates sending x^c to 0, we obtain that:

$$\begin{aligned} j^\ell_z\sigma^c - j^\ell_0\sigma^c &= \sum_{i=1}^{c}(j^\ell_{\hat{\zeta}^i_i(t_i)}\sigma^c - j^\ell_{\hat{\zeta}^i_i(0)}\sigma^c) + (j^\ell_z\sigma^c - j^\ell_w\sigma^c) \\ &= \sum_{i=1}^{c}(j^\ell\sigma^c\circ\hat{\zeta}^i_i)'(0)\cdot t_i + o(|z|) \\ &= A(z) + o(|z|). \end{aligned}$$

Hence $j^\ell\sigma^c$ is differentiable along $\widehat{\mathcal{W}}^c(x_c)$ at x_c, with derivative A.

Now we have that $j^\ell\sigma^c$ is differentiable at x_c along $\widehat{\mathcal{W}}^c(x_c)$, we can spread this derivative around using $\widehat{\mathcal{H}}^\ell$, and we get that the derivative of $j^\ell\sigma^c$ along $\widehat{\mathcal{W}}^c(x)$ at x

exists for every x and is a continuous function on M. We still need to show that σ has central $\ell+1$ jets, with uniform error term.

The derivative of $j^\ell\sigma^c$ at x gives a candidate $j^{\ell+1}_x\sigma^c$ for a central $\ell+1$ jet at x; the $\ell+1$st coordinate in $j^{\ell+1}_x\sigma^c$ is just the derivative at x along $\widehat{\mathcal{W}}^c_x$ of the ℓth coordinate of $j^\ell\sigma^c$. To show that σ has a central $\ell+1$-jet at x, we must show that for every $v\in B_{\widetilde{E^c(x)}}(0,\rho)$:

$$d_N(\mathrm{proj}_N\circ\tilde\sigma\circ g^c(v),\mathrm{proj}_N\circ j^{\ell+1}_x\sigma^c(v)) = o(|v|^{\ell+1}). \tag{49}$$

We first note that $j^\ell\sigma^c$ is differentiable along E^c curves. To see this, let ζ be an E^c curve in M. For each $t\in I$, Lemma 10.15 implies there exists a C^r curve $\hat\zeta_t$ in $\widehat{\mathcal{W}}^c(\zeta(t))$ with $\hat\zeta_t(0)=\zeta(t)$ and such that $\hat\zeta_t$ and $\zeta(s+t)$ are tangent to order r at 0. Furthermore, the previous arguments using saturation of $j^\ell\sigma$ show that the distance between $j^\ell_{\zeta(s+t)}\sigma^c$ and $j^\ell_{\hat\zeta_t(s)}\sigma^c$ is $o(|s|^{r-\ell})$. Since $j^\ell\sigma^c$ is differentiable along $\hat\zeta_t$ at $s=0$, this implies that $j^\ell\sigma^c$ is differentiable along $\zeta(s+t)$ at $s=0$. Since t was arbitrary, we see that $j^\ell\sigma^c$ is differentiable, and in fact C^1, along ζ.

Our induction hypothesis implies that σ is C^ℓ along E^c curves. We next observe that, for any E^c curve ζ, the ℓ-jet of $\sigma\circ\zeta$ at $t\in I$ satisfies:

$$\mathrm{proj}_N\circ j^\ell_t(\sigma\circ\zeta) = \mathrm{proj}_N\circ j^\ell_{\zeta(t)}\sigma^c\circ j^\ell_{\zeta(t)}(\pi^c\circ\exp^{-1}_{\zeta(t)})\circ j^\ell_t\zeta. \tag{50}$$

To see this, let $\hat\zeta_t$ be given by Lemma 10.15. Since $\zeta(t+s)$ and $\hat\zeta_t(s)$ have the same $\lfloor r\rfloor$ jets at $s=0$, and σ is Lipschitz, the functions $\sigma\circ\zeta_t(s)$ and $\sigma\circ\zeta(s+t)$ have the same ℓ-jets at $s=0$. But the definition of central ℓ-jets implies that:

$$d_N(\mathrm{proj}_N\circ\sigma\circ\hat\zeta_t(s),\ \mathrm{proj}_N\circ j^\ell_{\hat\zeta_t(0)}\sigma^c\circ\pi^c\circ\exp^{-1}_{\hat\zeta_t(0)}\circ\hat\zeta_t(s)) = o(|s|^\ell);$$

from the naturality of jets under composition, (50) follows immediately.

Now, since both $j^\ell\sigma^c$ and $j^\ell(\pi^c\circ\exp^{-1})$ are differentiable along E^c curves, it follows that σ is $C^{\ell+1}$ along every E^c curve ζ, and by Taylor's theorem, the $\ell+1$ jets of $\sigma\circ\zeta$ are given by the formula

$$j^{\ell+1}_t(\sigma\circ\zeta) = j^{\ell+1}_{\zeta(t)}\sigma^c\circ j^{\ell+1}_{\zeta(t)}(\pi^c\circ\exp^{-1}_{\zeta(t)})\circ j^{\ell+1}_t\zeta. \tag{51}$$

Finally, let $v\in B_{\widetilde{E^c(x)}}(0,\rho)$ be given, and let $y=\exp_x g^c(v)\in\widehat{\mathcal{W}}^c(x)$. Fix a geodesic arc $\hat\zeta$ in $\widehat{\mathcal{W}}^c(x)$ from x to y, with $\hat\zeta(0)=x$ and $\hat\zeta(1)=y$. Let ζ be the E^c curve given by Lemma 10.15, tangent to order r to $\hat\zeta$ at $\hat\zeta(0)=x$. Equation (51) now implies that

$$d_N(\mathrm{proj}_N\circ\sigma\circ\hat\zeta(t),\ \mathrm{proj}_N\circ j^{\ell+1}_x\sigma^c(tv)) = o(|tv|^{\ell+1}).$$

Since $d(\hat\zeta(t),\zeta(t))=o(|tv|^r)$, and σ is Lipschitz, we obtain (49). Hence σ has a central $\ell+1$ jet at x, and it is given by $j^{\ell+1}_x\sigma^c$. We have verified both $\mathrm{I}_{\ell+1}$ and $\mathrm{III}_{\ell+1}$.

Proposition 11.5. — *σ is C^r.*

Proof. — If $r = 1$, then we have already shown that the 0-jet of σ is differentiable along $\mathcal{W}^c(x)$ at x, for every x, and this derivative varies continuously at M. Since σ is C^1 along the leaves of $\mathcal{W}^s$ and $\mathcal{W}^u$, this readily implies that σ is C^1.

Assume, then that $1 < r < k - 1$. Let $\overline{\ell} = \lfloor r \rfloor$, and let $\overline{\alpha} = r - \overline{\ell}$. We first show:

$j^{\overline{\ell}}\sigma^c$ is $C^{\overline{\alpha}}$ at x along $\widehat{\mathcal{W}}^c(x)$, for every $x \in M$. The proof is a slight adaptation of the proof that $j^{\overline{\ell}}\sigma^c$ is Lipschitz at x along $\widehat{\mathcal{W}}^c(x)$, for every $x \in M$, for $\ell < r$; the central observation that allows one to modify this proof is that $H^{\overline{\ell}}_{\mathcal{J}}(x, \wp)$ still covers the diffeomorphism $H_{\mathcal{J}}(x, \wp)$, and for $i \geq 1$, $H^{\overline{\ell}}_{\mathcal{J}}(x, \wp)_i$ is $\overline{\alpha}$-Hölder continuous in the $(x, \wp_0)$-variable, and C^∞ in the $(\wp_1, \cdots, \wp_{\overline{\ell}})$-variables. (See the proof of part II of Theorem A as well). We omit the details.

σ has an $(\overline{\ell}, \overline{\alpha}, C)$ expansion at x along $\widehat{\mathcal{W}}^c(x)$, uniformly in $x \in M$. This is essentially the same as the proof that σ has a central ℓ-jet at every point for $\ell < r$, except one sharpens the estimates on the remainder of the Taylor expansions along E^c curves, using the $\overline{\alpha}$-Hölder continuity of the central $\overline{\ell}$-jets.

The section σ is C^r. Since r-bunching is an open condition, as is the condition $r < k - 1$, by increasing r slightly, we may assume that r is not an integer.

We have shown that σ has central $\overline{\ell}$-jets, and that $j^{\overline{\ell}}\sigma^c$ is $\overline{\alpha}$-Hölder continuous. Fix a point $p \in M$. The fake center-stable manifolds $\widehat{\mathcal{W}}^{cs}(x)$, for x in a neighborhood U of p, form a continuous family of $C^r = C^{\overline{\ell}, \overline{\alpha}}$ embedded disks.

Fix x in this neighborhood U, and consider the foliation $\{\widehat{\mathcal{W}}^s_x(y)\}_{y \in \widehat{\mathcal{W}}^{cs}(x)}$ of the plaque $\widehat{\mathcal{W}}^{cs}(x)$ by fake stable manifolds. Since σ is $\mathcal{W}^s$ saturated, it is C^k along $\mathcal{W}^s(y)$, for any $y \in M$. In particular, it has a $(\overline{\ell}, \overline{\alpha}, C)$-expansion along $\mathcal{W}^s(y)$, for any y. For $y \in \widehat{\mathcal{W}}^c(x)$ corresponding to $(0, 0, x^c)$ in adapted coordinates at x, Lemma 10.5 implies that the distance between $\hat{\omega}^{cs}_{(0,0,x^c)}(0, x^s)$ and $\hat{\omega}^{cs}_0(x^c, x^s)$ is $o(d(x, y)^r)$. Since σ is Lipschitz, and σ has a $(\overline{\ell}, \overline{\alpha}, C)$-expansion along $\hat{\omega}^{cs}_{(0,0,x^c)}(0, x^s)$ (which corresponds to $\mathcal{W}^s(y)$), this implies that σ has a $(\overline{\ell}, \overline{\alpha}, C)$-expansion along $\widehat{\mathcal{W}}^s(y)$ (corresponding to $\hat{\omega}^{cs}_0(x^c, x^s)$) with an error term that is on the order of $d(x, y)^r$.

Next consider the family of plaques $\{\widetilde{\mathcal{W}}^c(y)\}_{y \in \widehat{\mathcal{W}}^{cs}(x)}$ defined by $\widetilde{\mathcal{W}}^c(y) = \widehat{\mathcal{W}}^{cs}(x) \cap \widehat{\mathcal{W}}^{cu}(y)$. This forms a continuous family of C^r-embedded disks. Paired with the the $\widehat{\mathcal{W}}^s_x$ foliation, the family of $\widetilde{\mathcal{W}}^c$ plaques gives a C^r transverse pair of plaque families in $\widehat{\mathcal{W}}^{cs}(x)$. Lemma 10.5 implies that for each $y \in \widehat{\mathcal{W}}^{cs}(x)$, the distance between the $\overline{\ell}$-jets of $\widehat{\mathcal{W}}^{cs}(x)$ at x and $\widehat{\mathcal{W}}^{cs}(y)$ at y is $o(d(x, y)^{\overline{\alpha}})$. Since $\widehat{\mathcal{W}}^c(y) = \widehat{\mathcal{W}}^{cs}(y) \cap \widehat{\mathcal{W}}^{cu}(y)$, it follows that the the distance between the $\overline{\ell}$-jets at y of $\widetilde{\mathcal{W}}^c(y)$ and $\widehat{\mathcal{W}}^c(y)$ is also $o(d(x, y)^{\overline{\alpha}})$. But σ is Lipschitz, and σ has an $(\overline{\ell}, \overline{\alpha}, C)$ expansion at y along $\widehat{\mathcal{W}}^c(y)$, for every y. This implies that in an adapted coordinate system at x , we can write the plaques $\widetilde{W}^c(y)$ as a parametrized family along which σ has an $(\overline{\ell}, \overline{\alpha}, C)$ expansion at y

along $\widetilde{W}^c(y)$, for every $y \in \widehat{\mathcal{W}}^{cs}(x)$, with an error term that is on the order of $d(x,y)^r$. Hence we can apply Theorem 8.4 to conclude that σ has an $(\bar{\ell}, \overline{\alpha}, C)$-expansion along $\widehat{\mathcal{W}}^{cs}(x)$ at x, for every x in U, where C is uniform in x.

Now the family $\{\widehat{\mathcal{W}}^{cs}(x)\}_{x\in U}$ is a uniformly continuous family of C^r plaques in U. Paired with the local $\mathcal{W}^u$ foliation, it gives a transverse $C^{\bar{\ell},\overline{\alpha}}$ pair of plaque families in U. Since σ is u-saturated, it is C^k along $\mathcal{W}^u$-leaves and in particular has an $(\bar{\ell}, \overline{\alpha}, C)$-expansion along $\mathcal{W}^u(x)$ at every $x \in U$. Applying Journé's theorem again, we obtain that σ has a $(\bar{\ell}, \overline{\alpha}, C')$-expansion expansion at every $x \in U$, where C' is uniform in $x \in U$. Theorem 8.2 implies that σ is C^r in U. As p was arbitrary, we obtain that σ is C^r. □

This completes the proof of Theorem C. □

12. Final remarks and further questions

The proofs here could admit several improvements and generalizations. Some are not difficult: for example, the compactness of the manifold M was not essential. The definition of partial hyperbolicity in the noncompact cases merely needs to be modified to ensure that the functions $\nu, \hat{\nu}, \nu/\gamma, \hat{\nu}/\hat{\gamma}$ are uniformly bounded away from 1, and the definition of r-bunching must be similarly adjusted. Other improvements on Theorem A are more challenging. For example, there is no counterpart in Theorem A to the analyticity conclusions in Theorem 0.1, part IV. Another question is whether the Hölder exponent in Theorem A, part II can be improved. Finally, we ask whether the loss of one derivative in Theorem A part IV (and Theorem C) is really necessary: is it true that if ϕ is C^r, f is C^r, accessible and r-bunched, where $r \geq 1$, then any continuous solution to (2) is C^r (or perhaps $C^{r-\varepsilon}$, for all $\varepsilon > 0$)?

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