ISSN 0012-9593

quatrième série - tome 47 *fascicule 3* *mai-juin 2014*

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Frédéric BERNICOT & Sahbi KERAANI

On the global well-posedness of the 2D Euler equations for a large class of Yudovich type data

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4 e série, t. 47, 2014, p. 559 à 576

ON THE GLOBAL WELL-POSEDNESS OF THE 2D EULER EQUATIONS FOR A LARGE CLASS OF YUDOVICH TYPE DATA

BY FRÉDÉRIC BERNICOT AND SAHBI KERAANI

ABSTRACT. – The study of the 2D Euler equation with non Lipschitzian velocity was initiated by Yudovich in [20] where a result of global well-posedness for essentially bounded vorticity is proved. A lot of works have been since dedicated to the extension of this result to more general spaces. To the best of our knowledge all these contributions lack the proof of at least one of the following three fundamental properties: global existence, uniqueness and regularity persistence. In this paper we introduce a Banach space containing unbounded functions for which all these properties are shown to be satisfied.

RÉSUMÉ. – L'étude de l'équation d'Euler bidimensionnelle dans un cadre non lipschitzien a été initiée par Yudovich [20], qui a montré l'existence globale pour des tourbillons initiaux bornés. Depuis, de nombreux travaux ont été dédiés à l'extension de ce résultat à des espaces plus généraux. Au meilleur de notre connaissance aucun de ces travaux ne contient de résultat où les propriétés fondamentales suivantes soient vérifiées : existence globale, unicité et propagation de la régularité. Dans cet article, nous introduisons un nouvel espace de Banach contenant des fonctions non bornées et pour lequel ces trois propriétés sont vérifiées.

1. Introduction

We consider the Euler system related to an incompressible inviscid fluid with constant density, namely

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = 0, & x \in \mathbb{R}^d, t > 0, \\ \nabla . u = 0, \\ u_{|t=0} = u_0. \end{cases} \tag{1}$$

Here, the vector field $u = (u_2, u_1, \ldots, u_d)$ is a function of $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ denoting the velocity of the fluid and the scalar function P stands for the pressure. The second equation of the system $\nabla . u = 0$ is the condition of incompressibility. Mathematically, it guarantees

The two authors are supported by the ANR under the project AFoMEN no. 2011-JS01-001-01.

the preservation of Lebesgue measure by the particle-trajectory mapping (the classical flow associated to the velocity vector fields). It is worth noting that the pressure can be recovered from the velocity via an explicit Calderón-Zygmund type operator (see [5] for instance).

The question of local well-posedness of (1) with smooth data was resolved by many authors in different spaces (see for instance [5, 14]). In this context, the vorticity $\omega = \operatorname{curl} u$ plays a fundamental role. In fact, the well-known BKM criterion [2] ensures that the development of finite time singularities for these solutions is related to the blow-up of the L^∞ norm of the vorticity near the maximal time existence. A direct consequence of this result is the global well-posedness of the two-dimensional Euler solutions with smooth initial data, since the vorticity satisfies the transport equation

$$\partial_t \omega + (u \cdot \nabla)\omega = 0, \tag{2}$$

and then all its L^p norms are conserved.

Another class of solutions requiring lower regularity on the velocity can be considered: the weak solutions (see for instance [12, Chap 4]). They solve a weak form of the equation in the distribution sense, placing the equations in large spaces and using duality. The divergence form of Euler equations allows to put all the derivatives on the test functions and so to obtain

$$\int_0^\infty \int_{\mathbb{R}^d} (\partial_t \varphi + (u \cdot \nabla)\varphi).u \, dxdt + \int_{\mathbb{R}^d} \varphi(0,x) u_0(x) \, dx = 0,$$

for all $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}^d)$ with $\nabla.\varphi = 0$. In the two dimensional space and when the regularity is sufficient to give a sense to Biot-Savart law, then one can consider an alternative weak formulation: the vorticity-stream weak formulation. It consists in resolving the weak form of (2) supplemented with the Biot-Savart law:

$$u = K * \omega, \quad \text{with} \quad K(x) = \frac{x^\perp}{2\pi|x|^2}. \tag{3}$$

In this case, (v, ω) is a weak solution to the vorticity-stream formulation of the 2D Euler equation with initial data ω_0 if (3) is satisfied and

$$\int_0^\infty \int_{\mathbb{R}^2} (\partial_t \varphi + u.\nabla\varphi)\omega(t,x) dxdt + \int_{\mathbb{R}^2} \varphi(0,x)\omega_0(x) dx = 0,$$

for all $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^2, \mathbb{R})$.

The questions of existence/uniqueness of weak solutions have been extensively studied and a detailed account can be found in the books [5, 14, 12]. We emphasize that, unlike the fixed-point argument, the compactness method does not guarantee the uniqueness of the solutions and then the two issues (existence/uniqueness) are usually dealt with separately. These questions have been originally addressed by Yudovich in [20] where the existence and uniqueness of weak solution to 2D Euler systems (in bounded domain) are proved under the assumptions: $u_0 \in L^2$ and $\omega_0 \in L^\infty$. Serfati [15] proved the uniqueness and existence of a solution with initial velocity and vorticity which are only bounded (without any integrability condition). There is an extensive literature on the existence of weak solution to Euler system, possibly without uniqueness, with unbounded vorticity. DiPerna-Majda [7] proved the existence of weak solution for $\omega_0 \in L^1 \cap L^p$ with $2 < p < \infty$. The L^1 assumption in DiPerna-Majda's paper has been removed by Giga-Miyakawa-Osada [9]. Chae [4] proved an existence result for ω_0 in $L \ln^+ L$ with compact support. More recently, Taniuchi

[16] has proved the global existence (possibly without uniqueness nor regularity persistence) for $(u_0, \omega_0) \in L^\infty \times \mathrm{BMO}$. The papers [18] and [19] are concerned with the questions of existence and uniqueness of weak solutions for larger classes of vorticity. Both have intersections with the present paper and we will come back to them at the end of this section (Remark 2). A framework for measure-valued solutions can be found in [6] and [13] (see also [8] for more detailed references).

Roughly speaking, the proof of uniqueness of weak solutions requires a uniform, in time, bound of the log-Lipschitzian norm of the velocity. This "almost" Lipschitzian regularity of the velocity is enough to assure the existence and uniqueness of the associated flow (and then of the solution). Initial conditions of the type $\omega_0 \in L^\infty(\mathbb{R}^2)$ (or $\omega_0 \in \mathrm{BMO}, B^0_{\infty,\infty}, \ldots$) guarantee the log-Lipschitzian regularity of u_0. However, the persistence of such regularity when time varies requires an a priori bound of these quantities for the approximate-solution sequences. This is trivially done (via the conservation law) in the L^∞ case but is not at all clear for the other cases. The main issue in this context is the action of Lebesgue measure preserving homeomorphisms on these spaces. In fact, it is easy to prove that all these spaces are invariant under the action of such class of homeomorphisms, but the optimal form of the constants (depending on the homeomorphisms and important for the application) are not easy to find. It is worth mentioning, in this context, that the proof by Vishik [17] of the global existence for (1) in the borderline Besov spaces is based on a refined result on the action of Lebesgue measure preserving homeomorphisms on $B^0_{\infty,1}$.

In this paper we place ourselves in some Banach space which is strictly imbricated between L^∞ and BMO. Although located beyond the reach of the conservation laws of the vorticity this space has many nice properties (namely with respect to the action of the group of Lebesgue measure preserving homeomorphisms) allowing to derive the above-mentioned a priori estimates for the approximate-solution sequences.

Before going any further, let us introduce this functional space (details about BMO spaces can be found in the book of Grafakos [10]).

DEFINITION 1. – *For a complex-valued locally integrable function on $\mathbb{R}^2$, set*

$$\|f\|_{\mathrm{LBMO}} := \|f\|_{\mathrm{BMO}} + \sup_{B_1, B_2} \frac{|\mathrm{Avg}_{B_2}(f) - \mathrm{Avg}_{B_1}(f)|}{1 + \ln\left(\frac{1-\ln r_2}{1-\ln r_1}\right)},$$

where the supremum is taken aver all pairs of balls $B_2 = B(x_2, r_2)$ and $B_1 = B(x_1, r_1)$ in $\mathbb{R}^2$ with $0 < r_1 \leq 1$ and $2B_2 \subset B_1$. Here and subsequently, we denote

$$\mathrm{Avg}_D(g) := \frac{1}{|D|} \int_D g(x)dx,$$

for every $g \in L^1_{loc}$ and every non negligible set $D \subset \mathbb{R}^2$. Also, for a ball B and $\lambda > 0$, λB denotes the ball that is concentric with B and whose radius is λ times the radius of B.

We recall that

$$\|f\|_{\mathrm{BMO}} := \sup_{\text{ball } B} \mathrm{Avg}_B |f - \mathrm{Avg}_B(f)|.$$

It is worth noting that if B_2 and B_1 are two balls such that $2B_2 \subset B_1$ then[1]

$$|\mathrm{Avg}_{B_2}(f) - \mathrm{Avg}_{B_1}(f)| \lesssim \ln(1+\frac{r_1}{r_2})\|f\|_{\mathrm{BMO}}. \tag{4}$$

In the definition of LBMO we replace the term $\ln(1+\frac{r_1}{r_2})$ by $\ln\left(\frac{1-\ln r_2}{1-\ln r_1}\right)$, which is smaller. This puts more constraints on the functions belonging to this space[2] and allows us to derive some crucial property on the composition of them with Lebesgue measure preserving homeomorphisms, which is the heart of our analysis.

The following statement is the main result of the paper.

THEOREM 1. – *Assume $\omega_0 \in L^p \cap \mathrm{LBMO}$ with $p \in [1,2[$. Then there exists a unique global weak solution (v,ω) to the vorticity-stream formulation of the 2D Euler equation. Besides, there exists a constant C_0 depending only on the $L^p \cap \mathrm{LBMO}$-norm of ω_0 such that*

$$\|\omega(t)\|_{L^p\cap\mathrm{LBMO}} \leq C_0 \exp(C_0 t), \qquad \forall\, t \in \mathbb{R}_+. \tag{5}$$

Some remarks are in order.

REMARK 1. – The proof gives more, namely $\omega \in \mathcal{C}(\mathbb{R}_+, L^q)$ for all $p \leq q < \infty$. Combined with the Biot-Savart law[3] this yields $u \in \mathcal{C}(\mathbb{R}_+, W^{1,r}) \cap \mathcal{C}(\mathbb{R}_+, L^\infty)$ for all $\frac{2p}{2-p} \leq r < \infty$.

REMARK 2. – The essential point of Theorem 1 is that it provides an initial space which is strictly larger than $L^p \cap L^\infty$ (it contains unbounded elements) which is a space of existence, uniqueness and persistence of regularity at once. We emphasize that the bound (5) is crucial since it implies that u is, uniformly in time, log-Lipschitzian which is the main ingredient for the uniqueness. Once this bound established the uniqueness follows from the work by Vishik [18]. In this paper Vishik also gave a result of existence (possibly without regularity persistence) in some large space characterized by growth of the partial sum of the L^∞-norm of its dyadic blocs. We should also mention the result of uniqueness by Yudovich [19] which establishes uniqueness (for bounded domain) for some space which contains unbounded functions. Note also that the example of unbounded function, given in [19], belongs actually to the space LBMO (see Proposition 3 below). Our approach is different from those in [18] and [19] and uses a classical harmonic analysis "à la Stein" without appealing to the Fourier analysis (para-differential calculus).

REMARK 3. – The main ingredient of the proof of (5) is a logarithmic estimate in the space $L^p \cap \mathrm{LBMO}$ (see Theorem 2 below). It would be desirable to prove this result for BMO instead of LBMO. Unfortunately, as it is proved in [3], the corresponding estimate with BMO is optimal (with the bi-Lipschitzian norm instead of the log-Lipschitzian norm of the homeomorphism) and so the argument presented here seems to be not extendable to BMO.

[1] Throughout this paper the notation $A \lesssim B$ means that there exists a positive universal constant C such that $A \leq CB$.

[2] Here, we identify all functions whose difference is a constant. In Section 2, we will prove that LBMO is complete and strictly imbricated between BMO and L^∞. The "L" in LBMO stands for "logarithmic".

[3] If $\omega_0 \in L^p$ with $p \in [1,2[$ then a classical Hardy-Littlewood-Sobolev inequality gives $u \in L^q$ with $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$.

The remainder of this paper is organized as follows. In the two next sections we introduce some functional spaces and prove a logarithmic estimate which is crucial to the proof of Theorem 1. The fourth and last section is dedicated to the proof of Theorem 1.

2. Functional spaces

Let us first recall that the set of log-Lipschitzian vector fields on $\mathbb{R}^2$, denoted by LL, is the set of bounded vector fields v such that

$$\|v\|_{LL} := \sup_{x\neq y} \frac{|v(x)-v(y)|}{|x-y|\big(1+\big|\ln|x-y|\big|\big)} < \infty.$$

The importance of this notion lies in the fact that if the vorticity belongs to the Yudovich type space (say $L^1 \cap L^\infty$) then the velocity is no longer Lipschitzian, but log-Lipschitzian. In this case we still have existence and uniqueness of flow but a loss of regularity may occur. Actually, this loss of regularity is unavoidable and its degree is related to the norm $L^1_t(LL)$ of the velocity. The reader is referred to Section 3.3 in [1] for more details about this issue.

To capture this behavior, and overcome the difficulty generated by it, we introduce the following definition.

DEFINITION 2. – *For every homeomorphism ψ, we set*

$$\|\psi\|_* := \sup_{x\neq y} \Phi\big(|\psi(x)-\psi(y)|, |x-y|\big),$$

where Φ is defined on $]0,+\infty[\times]0,+\infty[$ by

$$\Phi(r,s) = \begin{cases} \max\{\frac{1+|\ln(s)|}{1+|\ln r|}; \frac{1+|\ln r|}{1+|\ln(s)|}\}, & \text{if} \quad (1-s)(1-r)\geq 0, \\ (1+|\ln s|)(1+|\ln r|), & \text{if} \quad (1-s)(1-r)\leq 0. \end{cases}$$

Since Φ is symmetric then $\|\psi\|_* = \|\psi^{-1}\|_* \geq 1$. It is clear also that every homeomorphism ψ satisfying

$$\frac{1}{C}|x-y|^\alpha \leq |\psi(x)-\psi(y)| \leq C|x-y|^\beta$$

for some $\alpha, \beta, C > 0$, has its $\|\psi\|_*$ finite (see Proposition 2 for a reciprocal property).

The definition above is motivated by this proposition (and by Theorem 2 below as well).

PROPOSITION 1. – *Let u be a smooth divergence-free vector field and let ψ be its flow:*

$$\partial_t \psi(t,x) = u(t,\psi(t,x)), \qquad \psi(0,x) = x.$$

Then, for every $t \geq 0$

$$\|\psi(t,\cdot)\|_* \leq \exp\Big(\int_0^t \|u(\tau)\|_{LL} d\tau\Big).$$

Proof. – It is well-known that for every $t \geq 0$ the mapping $x \mapsto \psi(t,x)$ is a Lebesgue measure preserving homeomorphism (see [5] for instance). We fix $t \geq 0$ and $x \neq y$ and set

$$z(t) = |\psi(t,x) - \psi(t,y)|.$$

Clearly the function Z is strictly positive and satisfies

$$|\dot{z}(t)| \leq \|u(t)\|_{LL}(1+|\ln z(t)|)z(t).$$

Accordingly, we infer

$$|g(z(t)) - g(z(0))| \leq \int_0^t \|u(\tau)\|_{LL} d\tau$$

where

$$g(\tau) := \begin{cases} \ln(1+\ln(\tau)), & \text{if} \quad \tau \geq 1, \\ -\ln(1-\ln(\tau)), & \text{if} \quad 0 < \tau < 1. \end{cases}$$

This yields in particular that $\frac{\exp(g(z(t)))}{\exp(g(z(0)))}$ and $\frac{\exp(g(z(0)))}{\exp(g(z(t)))}$ are both controlled by $\exp(\int_0^t \|u(\tau)\|_{LL} d\tau)$ leading to

$$\Phi(z(t), z(0)) \leq \exp(\int_0^t \|u(\tau)\|_{LL} d\tau),$$

as claimed. □

The following proposition follows directly from the definition by a straightforward computation.

PROPOSITION 2. – *Let ψ be a homeomorphism with $\|\psi\|_* < \infty$. Then for every $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ one has*

1. *If $|x-y| \geq 1$ and $|\psi(x) - \psi(y)| \geq 1$*

$$e^{-1}|x-y|^{\frac{1}{\|\psi\|_*}} \leq |\psi(x) - \psi(y)| \leq e^{\|\psi\|_*}|x-y|^{\|\psi\|_*}.$$

2. *If $|x-y| \leq 1$ and $|\psi(x) - \psi(y)| \leq 1$*

$$e^{-\|\psi\|_*}|x-y|^{\|\psi\|_*} \leq |\psi(x) - \psi(y)| \leq e|x-y|^{\frac{1}{\|\psi\|_*}}.$$

3. *In the other cases*

$$e^{-\|\psi\|_*}|x-y| \leq |\psi(x) - \psi(y)| \leq e^{\|\psi\|_*}|x-y|.$$

As an application we obtain the following useful lemma.

LEMMA 1. – *For every $r > 0$ and a homeomorphism ψ one has*

$$4\psi(B(x_0, r)) \subset B(\psi(x_0), g_\psi(r)),$$

where[4]*,*

$$g_\psi(r) := \begin{cases} 4e^{\|\psi\|_*} r^{\|\psi\|_*}, & \text{if} \quad r \geq 1, \\ 4\max\{er^{\frac{1}{\|\psi\|_*}}; e^{\|\psi\|_*} r\}, & \text{if} \quad 0 < r < 1. \end{cases}$$

In particular,

$$\left| \ln\left(\frac{1 + |\ln g_\psi(r)|}{1 + |\ln r|} \right) \right| \lesssim 1 + \ln\left(1 + \|\psi\|_*\right). \tag{6}$$

[4] This notation means that for every ball $B \subset \psi(B(x_0, r))$ we have $4B \subset B(\psi(x_0), g_\psi(r))$.

Proof. – The first inclusion follows from Proposition 2 and the definition of g_ψ. Let us check (6). This comes from an easy computation using the following trivial fact: if $\alpha, \beta, \gamma > 0$ then

$$\sup(\beta, \frac{1}{\beta}) \leq \alpha^\gamma \Longleftrightarrow |\ln(\beta)| \leq \gamma \ln(\alpha).$$

– If $r \geq 1$ then

$$1 \leq \frac{1 + |\ln g_\psi(r)|}{1 + |\ln r|} = \frac{1 + \ln 4 + \|\psi\|_* + \ln r}{1 + \ln r} \leq 3 + \|\psi\|_*.$$

– If $r < 1$ then we have to deal with two possible values of $g_\psi(r)$.

CASE 1. – If $g_\psi(r) = 4er^{\frac{1}{\|\psi\|_*}}$ then

$$|\ln g_\psi(r)| = |\ln 4 + 1 + \|\psi\|_*^{-1} \ln(r)|.$$

Since $\|\psi\|_* \geq 1$, we get

$$\frac{1 + |\ln g_\psi(r)|}{1 + |\ln r|} \leq \frac{3 + \frac{1}{\|\psi\|_*}|\ln r|}{1 + |\ln r|} \leq \frac{3 + |\ln r|}{1 + |\ln r|} \leq 3.$$

To estimate $\frac{1+|\ln r|}{1+|\ln g_\psi(r)|}$ we consider two possibilities.

– If $|\ln(r)| \leq 8\|\psi\|_*$ then

$$\frac{1 + |\ln r|}{1 + |\ln g_\psi(r)|} \leq 1 + |\ln r| \leq 1 + 8\|\psi\|_*.$$

– If $|\ln(r)| \geq 8\|\psi\|_*$ then

$$|\ln(4) + 1 + \|\psi\|_*^{-1} \ln(r)| \geq \frac{1}{2}\|\psi\|_*^{-1} |\ln(r)|,$$

and so

$$\frac{1 + |\ln r|}{1 + |\ln g_\psi(r)|} \leq \frac{1 + |\ln r|}{1 + \frac{1}{2}\|\psi\|_*^{-1}|\ln(r)|} \leq 2(1 + \|\psi\|_*).$$

CASE 2. – If $g_\psi(r) = 4e^{\|\psi\|_*} r$ then

$$|\ln g_\psi(r)| = |\ln 4 + \|\psi\|_* + \ln(r)|.$$

Thus,

$$\frac{1 + |\ln g_\psi(r)|}{1 + |\ln r|} \leq \frac{3 + \|\psi\|_* + |\ln r|}{1 + |\ln r|} \leq 3 + \|\psi\|_*.$$

As previously to estimate $\frac{1+|\ln r|}{1+|\ln g_\psi(r)|}$, we consider two possibilities.

– If $|\ln(r)| \leq 2(\ln(4) + \|\psi\|_*)$ then

$$\frac{1 + |\ln r|}{1 + |\ln g_\psi(r)|} \leq 1 + |\ln r| \leq 5 + 2\|\psi\|_*.$$

– If $|\ln(r)| \geq 2(\ln 4 + \|\psi\|_*)$ then $|\ln(4) + \|\psi\|_* + \ln r| \geq \frac{1}{2}|\ln(r)|$ and so

$$\frac{1 + |\ln r|}{1 + |\ln g_\psi(r)|} \leq \frac{1 + |\ln r|}{1 + \frac{1}{2}|\ln(r)|} \leq 2. \qquad \square$$

REMARK 4. – *The estimate* (6) *remains valid when we multiply* $g_\psi(r)$ *by any positive constant.*

3. The LBMO space

Let us now detail some properties of the space LBMO introduced in the first section of this paper.

PROPOSITION 3. – *The following properties hold true.*
(i) *The space* LBMO *is a Banach space included in* BMO *and strictly containing* $L^\infty(\mathbb{R}^2)$.
(ii) *For every* $g \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ *and* $f \in$ LBMO *one has*

$$\|g * f\|_{\mathrm{LBMO}} \leq \|g\|_{L^1} \|f\|_{\mathrm{LBMO}}. \tag{7}$$

Proof. – (i) Completeness of the space. Let $(f_n)_n$ be a Cauchy sequence in LBMO. Since BMO is complete then this sequence converges in BMO and then in L^1_{loc}. Using the definition and the convergence in L^1_{loc}, we get that the convergence holds in LBMO.

It remains to check that $L^\infty \subsetneq$ LBMO. Since L^∞ is obviously embedded into LBMO, we have just to build an unbounded function belonging to LBMO. Take

$$f(x) = \begin{cases} \ln(1 - \ln|x|) & \text{if} \quad |x| \leq 1 \\ 0, & \text{if} \quad |x| \geq 1. \end{cases}$$

It is clear that both f and ∇f belong to $L^2(\mathbb{R}^2)$ meaning that $f \in H^1(\mathbb{R}^2) \subset$ BMO.

Before going further three preliminary remarks are necessary.

– Since f is radially symmetric and decreasing then, for every $r > 0$, the mapping $x \mapsto \mathrm{Avg}_{B(x,r)} f$ is radial and decreasing.

– For the same reasons the mapping $r \mapsto \mathrm{Avg}_{B(0,r)}(f)$ is decreasing.

– Take $(r, \rho) \in]0, +\infty[^2$ and consider the problem of maximization of

$$\mathrm{Avg}_{B(x_1,r)}(f) - \mathrm{Avg}_{B(x_2,r)}(f)$$

when $|x_1 - x_2| = \rho$. The convexity of f implies that $x_1 = 0$ and $|x_2| = \rho$ are solutions of this problem.

We fix r_1 and r_2 such that $r_1 \leq 1$ and $2r_2 \leq r_1$. For every $x_1 \in \mathbb{R}^2$ one defines $\tilde{x}_1$ and $\hat{x}_1$ as follows:

$$\tilde{x}_1 = \begin{cases} x_1\left(1 - \frac{r_2 + r_1}{|x_1|}\right) & \text{if} \quad |x_1| \geq r_2 + r_1 \\ 0, & \text{if} \quad |x_1| \leq r_2 + r_1, \end{cases}$$

and

$$\hat{x}_1 = \begin{cases} x_1\left(1 + \frac{r_2 + r_1}{|x_1|}\right) & \text{if} \quad |x_1| \neq 0 \\ (r_2 + r_1, 0) & \text{if} \quad |x_1| = 0. \end{cases}$$

Let $A(x_1)$ be the set of admissible x_2: the set of x_2 such that $2B(x_2, r_2) \subset B(x_1, r_1)$. Using the two preliminary remarks above, we see that

$$\sup_{x_2 \in A(x_1)} |\mathrm{Avg}_{B(x_2,r_2)}(f) - \mathrm{Avg}_{B(x_1,r_1)}(f)| \leq \max\{J_1, J_2\},$$

with

$$J_1 = \mathrm{Avg}_{B(\tilde{x}_1,r_2)}(f) - \mathrm{Avg}_{B(x_1,r_1)}(f),$$
$$J_2 = \mathrm{Avg}_{B(x_1,r_1)}(f) - \mathrm{Avg}_{B(\hat{x}_1,r_2)}(f).$$

In fact, if $\mathrm{Avg}_{B(x_2,r_2)}(f) - \mathrm{Avg}_{B(x_1,r_1)}(f)$ is positive (resp. negative) then it is obviously dominated by J_1 (resp. J_2). Thus, we obtain

$$\sup_{x_2 \in A(x_1)} |\mathrm{Avg}_{B(x_2,r_2)}(f) - \mathrm{Avg}_{B(x_1,r_1)}(f)| \leq J_1 + J_2 = \mathrm{Avg}_{B(\tilde{x}_1,r_2)}(f) - \mathrm{Avg}_{B(\hat{x}_1,r_2)}(f).$$

The right hand side is maximal in the configuration when $\tilde{x}_1 = 0$ and $\hat{x}_1$ the farthest away from 0. This means when $|x_1| = r_1 + r_2$, $\tilde{x}_1 = 0$ and $|\hat{x}_1| = 2(r_1 + r_2)$.

Since f is increasing (going to the axe) then

$$\mathrm{Avg}_{B(\hat{x}_2,r_1)}(f) \geq f(4r_1).$$

Finally, we get for all $x_1 \in \mathbb{R}^2$ and $x_2 \in A(x_1)$

$$|\mathrm{Avg}_{B(x_2,r_2)}(f) - \mathrm{Avg}_{B(x_1,r_1)}(f)| \leq \mathrm{Avg}_{B(0,r_2)}(f) - f(4r_1).$$

Now it is easy to see that

$$f(4r_1) = \ln(1 - \ln(r_1)) + \mathcal{O}(1),$$

and (with an integration by parts)

$$\begin{aligned} \mathrm{Avg}_{B(0,r_2)}(f) &= \ln(1 - \ln(r_2)) + \frac{1}{r_2^2} \int_0^{r_1} \frac{1}{1 - \ln(\rho)} \rho d\rho \\ &= \ln(1 - \ln(r_2)) + \mathcal{O}(1). \end{aligned}$$

This yields

$$|\mathrm{Avg}_{B(x_2,r_2)}(f) - \mathrm{Avg}_{B(x_1,r_1)}(f)| \leq \ln\Big(\frac{1 - \ln(r_2)}{1 - \ln(r_1)}\Big) + \mathcal{O}(1),$$

as desired.

(ii) Stability by convolution. (7) follows from the fact that for all $r > 0$

$$x \mapsto \mathrm{Avg}_{B(x,r)}(g * f) = (g * \mathrm{Avg}_{B(\cdot,r)}(f))(x). \qquad \square$$

The advantage of using the space LBMO lies in the following logarithmic estimate which is the main ingredient for proving Theorem 1.

THEOREM 2. – *There exists a universal constant $C > 0$ such that*

$$\|f \mathrm{o} \psi\|_{\mathrm{LBMO} \cap L^p} \leq C \ln(1 + \|\psi\|_*) \|f\|_{\mathrm{LBMO} \cap L^p},$$

for any Lebesgue measure preserving the homeomorphism ψ.

Proof of Theorem 2. – Of course we are concerned with ψ such that $\|\psi\|_*$ is finite (if not the inequality is empty). Without loss of generality one can assume that $\|f\|_{\mathrm{LBMO} \cap L^p} = 1$. Since ψ preserves Lebesgue measure then the L^p-part of the norm is conserved. For the other two parts of the norm, we will proceed in two steps. In the first step we consider the BMO term of the norm and in the second one we deal with the other term.

STEP 1. – Let $B = B(x_0, r)$ be a given ball of $\mathbb{R}^2$. By using the L^p-norm we need only to deal with balls whose radius is smaller than a universal constant δ_0 (we want r to be small with respect to the constants appearing in the Whitney covering lemma below). Since ψ is a Lebesgue measure preserving homeomorphism then $\psi(B)$ is an open connected[(5)] set with

[(5)] We have also that $\psi(B)^C = \psi(B^C)$ and $\psi(\partial B) = \partial(\psi(B))$.

$|\psi(B)| = |B|$. By the Whitney covering lemma, there exists a collection of balls $(O_j)_j$ such that:

– The collection of double ball is a bounded covering:

$$\psi(B) \subset \bigcup 2O_j.$$

– The collection is disjoint and, for all j,

$$O_j \subset \psi(B).$$

– The Whitney property is verified:

$$r_{O_j} \simeq d(O_j, \psi(B)^c).$$

CASE 1: $r \leq \frac{1}{4}e^{-\|\psi\|_*}$. In this case

$$g_\psi(r) \leq 1.$$

We set $\tilde{B} := B(\psi(x_0), g_\psi(r))$. Since ψ preserves the Lebesgue measure we get

$$\begin{aligned}\mathrm{Avg}_B|f\circ\psi - \mathrm{Avg}_B(f\circ\psi)| &= \mathrm{Avg}_{\psi(B)}|f - \mathrm{Avg}_{\psi(B)}(f)| \\ &\leq 2\mathrm{Avg}_{\psi(B)}|f - \mathrm{Avg}_{\tilde{B}}(f)|.\end{aligned}$$

Using the notations above

$$\mathrm{Avg}_{\psi(B)}|f - \mathrm{Avg}_{\tilde{B}}(f)| \lesssim \frac{1}{|B|}\sum_j |O_j|\mathrm{Avg}_{2O_j}\big|f - \mathrm{Avg}_{\tilde{B}}(f)\big| \lesssim I_1 + I_2,$$

with

$$I_1 = \frac{1}{|B|}\sum_j |O_j|\mathrm{Avg}_{2O_j}\big|f - \mathrm{Avg}_{2O_j}(f)\big|$$

$$I_2 = \frac{1}{|B|}\sum_j |O_j|\big|\mathrm{Avg}_{2O_j}(f) - \mathrm{Avg}_{\tilde{B}}(f)\big|.$$

On one hand, since $\sum |O_j| \leq |B|$ then

$$I_1 \leq \frac{1}{|B|}\sum_j |O_j|\|f\|_{\mathrm{BMO}} \leq \|f\|_{\mathrm{BMO}}.$$

On the other hand, since $4O_j \subset \tilde{B}$ (remember Lemma 1) and $r_{\tilde{B}} \leq 1$, it yields that

$$\begin{aligned}I_2 &\lesssim \frac{1}{|B|}\sum_j |O_j|\big(1 + \ln\Big(\frac{1 - \ln 2r_j}{1 - \ln g_\psi(r)}\Big)\big) \\ &\lesssim \frac{1}{|B|}\sum_j |O_j|\big(1 + \ln\Big(\frac{1 - \ln r_j}{1 - \ln g_\psi(r)}\Big)\big).\end{aligned}$$

Thanks to (6) we get

$$\begin{aligned}\ln\Big(\frac{1 - \ln r_j}{1 - \ln g_\psi(r)}\Big) &\leq \ln\Big(\frac{1 - \ln r_j}{1 - \ln r}\Big) + \ln\Big(\frac{1 - \ln r}{1 - \ln g_\psi(r)}\Big) \\ &\lesssim 1 + \ln\Big(\frac{1 - \ln r_j}{1 - \ln r}\Big) + \ln(1 + \|\psi\|_*).\end{aligned} \tag{8}$$

Thus it remains to prove that

$$II := \frac{1}{|B|}\sum_j |O_j|(1+\ln\big(\frac{1-\ln r_j}{1-\ln r}\big)) \lesssim 1+\ln(1+\|\psi\|_*). \tag{9}$$

For every $k \in \mathbb{N}$ we set

$$u_k := \sum_{e^{-(k+1)}r<r_j\leq e^{-k}r} |O_j|,$$

so that

$$II \leq \frac{1}{|B|}\sum_{k\geq 0} u_k\big(1+\ln\big(\frac{k+2-\ln r}{1-\ln r}\big)\big). \tag{10}$$

We need the following lemma.

LEMMA 2. – *There exists a universal constant $C>0$ such that*

$$u_k \leq Ce^{-\frac{k}{\|\psi\|_*}} r^{1+\frac{1}{\|\psi\|_*}},$$

for every $k \in \mathbb{N}$.

Proof of Lemma 2. – If we denote by $C \geq 1$ the implicit constant appearing in the Whitney Lemma, then

$$u_k \leq |\{y \in \psi(B) : d(y,\psi(B)^c) \leq Ce^{-k}r\}|.$$

The preservation of the Lebesgue measure by ψ yields

$$|\{y \in \psi(B) : d(y,\psi(B)^c) \leq Ce^{-k}r\}| = |\{x \in B : d(\psi(x),\psi(B)^c) \leq Ce^{-k}r\}|.$$

Since $\psi(B)^c = \psi(B^c)$ then

$$u_k \leq |\{x \in B : d(\psi(x),\psi(B^c)) \leq Ce^{-k}r\}|.$$

We set

$$D_k = \{x \in B : d(\psi(x),\psi(B^c)) \leq Ce^{-k}r\}.$$

Since $\psi(\partial B)$ is the frontier of $\psi(B)$ and $d(\psi(x),\psi(B^c)) = d(\psi(x),\partial\psi(B))$ then

$$D_k \subset \{x \in B : \exists y \in \partial B \text{ with } |\psi(x)-\psi(y)| \leq Ce^{-k}r\}.$$

The condition on δ_0 is just to assure that $Cr \leq 1$ for all $r \leq \delta_0$. In this case Proposition 2 gives

$$D_k \subset \{x \in B : \exists y \in \partial B : |x-y| \leq Ce^{1-\frac{k}{\|\psi\|_*}} r^{\frac{1}{\|\psi\|_*}}\}.$$

Thus, D_k is contained in the annulus $\mathcal{A} = \{x \in B : d(x,\partial B) \leq Ce^{1-\frac{k}{\|\psi\|_*}} r^{\frac{1}{\|\psi\|_*}}\}$ and so

$$u_k \leq |D_k| \lesssim e^{-\frac{k}{\|\psi\|_*}} r^{1+\frac{1}{\|\psi\|_*}},$$

as claimed. □

Let us come back to (10). Let N be a large integer to be chosen later. We split the sum in the right hand side of (10) into two parts

$$II \lesssim \sum_{k\leq N}(\cdots)+\sum_{k>N}(\cdots) := II_1 + II_2.$$

Since $\sum u_k \leq |B|$ then

$$II_1 \leq 1+\ln\Big(\frac{N+2-\ln r}{1-\ln r}\Big). \tag{11}$$

On the other hand

$$II_2 \leq \sum_{k>N} e^{-\frac{k}{\|\psi\|_*}} r^{\frac{1}{\|\psi\|_*}-1}\big(1+\ln\big(\frac{k+2-\ln r}{1-\ln r}\big)\big).$$

The parameter N will be taken bigger than $\|\psi\|_*$ so that the function in k inside the sum is decreasing and an easy comparison with integral yields

$$II_2 \lesssim e^{-\frac{N}{\|\psi\|_*}}\|\psi\|_*^2 r^{\frac{1}{\|\psi\|_*}-1}\big(1+\ln\Big(\frac{N+2-\ln r}{1-\ln r}\Big)\big). \tag{12}$$

Putting (11) and (12) together and taking $N=[\|\psi\|_*(\|\psi\|_*-\ln r)]+1$,

$$II \lesssim \big(1+e^{-\frac{N}{\|\psi\|_*}}\|\psi\|_*^2 r^{\frac{1}{\|\psi\|_*}-1}\big)\big(1+\ln\Big(\frac{N+2-\ln r}{1-\ln r}\Big)\big).$$

Taking $N=[\|\psi\|_*(\|\psi\|_*-\ln r)]+1$,

$$\begin{aligned} II &\lesssim \big(1+e^{-\|\psi\|_*}\|\psi\|_*^2 r^{\frac{1}{\|\psi\|_*}}\big)\big(1+\ln\Big(\frac{\|\psi\|_*(\|\psi\|_*-\ln r)+2-\ln r}{1-\ln r}\Big)\big)\\ &\lesssim 1+\ln(1+\|\psi\|_*), \end{aligned}$$

where we have used the fact that $r\leq 1$ and the obvious inequality

$$\frac{\|\psi\|_*(\|\psi\|_*-\ln r)+2-\ln r}{1-\ln r} \lesssim (1+\|\psi\|_*)^2.$$

This ends the proof of (9).

Case 2: $\delta_0 \geq r \geq \frac{1}{4}e^{-\|\psi\|_*}$. In this case

$$|\ln r| \lesssim \|\psi\|_*.$$

Since ψ preserves the Lebesgue measure, we get

$$I := \mathrm{Avg}_B|f\circ\psi - \mathrm{Avg}_B(f\circ\psi)| \leq 2\mathrm{Avg}_{\psi(B)}|f|.$$

Let $\tilde{O}_j$ denote the ball which is concentric to O_j and whose radius is equal to 1 (we use the same Whitney covering as above). Without loss of generality we can assume $\delta_0 \leq \frac{1}{4}$. This guarantees $4O_j \subset \tilde{O}_j$ and yields by definition

$$\begin{aligned} I &\lesssim \frac{1}{|B|}\sum_j |O_j|\mathrm{Avg}_{2O_j}|f-\mathrm{Avg}_{\tilde{O}_j}(f)| + \frac{1}{|B|}\sum_j |O_j||\mathrm{Avg}_{\tilde{O}_j}(f)|\\ &\lesssim \frac{1}{|B|}\sum_j |O_j|\Big(1+\ln\big(1-\ln 2r_j\big)\Big)\|f\|_{\mathrm{LBMO}} + \frac{1}{|B|}\sum_j |O_j|\|f\|_{L^p}\\ &\lesssim 1+\frac{1}{|B|}\sum_j |O_j|\big(1+\ln\big(1-\ln r_j\big)\big). \end{aligned}$$

As before one writes

$$I \lesssim \frac{1}{|B|}\sum_{k\geq 0} u_k\big(1+\ln\big(k+2-\ln r\big)\big)$$
$$\lesssim 1+\ln\big(N+2-\ln r\big)+e^{-\frac{N}{\|\psi\|_*}}\|\psi\|_*^2 r^{\frac{1}{\|\psi\|_*}-1}\big(1+\ln\big(N+2-\ln r\big)\big).$$

Taking $N=[\|\psi\|_*(\|\psi\|_*-\ln r)]+1$ and using the fact that $|\ln r|\lesssim\|\psi\|_*$ leads to the desired result.

The outcome of this first step of the proof is

$$\|f\mathrm{o}\psi\|_{\mathrm{BMO}\cap L^p}\lesssim \ln(1+\|\psi\|_*)\|f\|_{\mathrm{LBMO}\cap L^p}.$$

STEP 2. – This step of the proof deals with the second term in the LBMO-norm. It is shorter than the first step because it makes use of the arguments developed above. Take $B_2=B(x_2,r_2)$ and $B_1=B(x_1,r_1)$ in $\mathbb{R}^2$ with $r_1\leq 1$ and $2B_2\subset B_1$. There are three cases to consider.

CASE 1: $r_1\lesssim e^{-\|\psi\|_*}$ (so that $g_\psi(r_2)\leq g_\psi(r_1)\leq\frac{1}{2}$).

We set $\tilde{B}_i:=B(\psi(x_i),g_\psi(r_i)), i=1,2$ and

$$J:=\frac{|\mathrm{Avg}_{B_2}(f\mathrm{o}\psi)-\mathrm{Avg}_{B_1}(f\mathrm{o}\psi)|}{1+\ln\big(\frac{1-\ln r_2}{1-\ln r_1}\big)}.$$

Since the denominator is bigger than 1 one gets

$$J\leq J_1+J_2+J_3,$$

with

$$J_1=|\mathrm{Avg}_{\psi(B_2)}(f)-\mathrm{Avg}_{\tilde{B}_2}(f)|+|\mathrm{Avg}_{\psi(B_1)}(f)-\mathrm{Avg}_{\tilde{B}_1}(f)|$$
$$J_2=\frac{|\mathrm{Avg}_{\tilde{B}_2}(f)-\mathrm{Avg}_{2\tilde{B}_1}(f)|}{1+\ln\left(\frac{1-\ln r_2}{1-\ln r_1}\right)}$$
$$J_3=|\mathrm{Avg}_{\tilde{B}_1}(f)-\mathrm{Avg}_{2\tilde{B}_1}(f)|.$$

Since $2\tilde{B}_2\subset 2\tilde{B}_1$ and $r_{2\tilde{B}_1}\leq 1$ then

$$J_2\leq\frac{1+\ln\big(\frac{1-\ln g_\psi(r_2)}{1-\ln(2g_\psi(r_1))}\big)}{1+\ln\big(\frac{1-\ln r_2}{1-\ln r_1}\big)}\|f\|_{\mathrm{LBMO}}.$$

Using a similar argument than (8) (and remembering Remark 4) we infer

$$\ln\Big(\frac{1-\ln g_\psi(r_2)}{1-\ln(2g_\psi(r_1))}\Big)\lesssim 1+\ln(1+\|\psi\|_*)+\ln\Big(\frac{1-\ln r_2}{1-\ln r_1}\Big).$$

Thus,

$$J_2\lesssim 1+\ln(1+\|\psi\|_*).$$

The estimation (4) yields

$$J_3\lesssim\|f\|_{\mathrm{BMO}}.$$

The term J_1 can be handled exactly as in the analysis of case 1 of step 1.

CASE 2: $e^{-\|\psi\|_*}\lesssim r_2$. In this case we write

$$J\leq\mathrm{Avg}_{\psi(B_2)}|f|+\mathrm{Avg}_{\psi(B_1)}|f|.$$

Both terms can be handled as in the analysis of case 2 of the proof of BMO-part in step 1.

CASE 3: $r_2 \lesssim e^{-\|\psi\|_*}$ and $r_1 \gtrsim e^{-\|\psi\|_*}$. Again since the denominator is bigger than 1 we get

$$J \leq \mathrm{Avg}_{\psi(B_2)}|f - \mathrm{Avg}_{\tilde{B}_2}(f)| + \frac{|\mathrm{Avg}_{\tilde{B}_2}(f)|}{1 + \ln\left(\frac{1-\ln r_2}{1-\ln r_1}\right)} + \mathrm{Avg}_{\psi(B_1)}|f| = J_1 + J_2 + J_3.$$

The terms J_1 and J_3 can be controlled as before. The second term is controlled as follows (we make the average on $B(\psi(x_2), 1)$ appear and use Lemma 1 with $\|f\|_{L^p} \leq 1$)

$$J_2 \leq \frac{1 + \ln(1 - \ln r_2)}{1 + \ln\left(\frac{1-\ln r_2}{1-\ln r_1}\right)} \leq 1 + \ln(1 + |\ln r_1|) \leq 1 + \ln(1 + \|\psi\|_*). \qquad \square$$

4. Proof of Theorem 1

The proof falls naturally into three parts.

4.1. A priori estimates

The following estimates follow directly from Proposition 1 and Theorem 2.

PROPOSITION 4. – *Let u be a smooth solution of* (1) *and ω its vorticity. Then, there exists a constant C_0 depending only on the norm $L^p \cap \mathrm{LBMO}$ of ω_0 such that*

$$\|u(t)\|_{LL} + \|\omega(t)\|_{\mathrm{LBMO}} \leq C_0 \exp(C_0 t),$$

for every $t \geq 0$.

Proof. – One has $\omega(t,x) = \omega_0(\psi_t^{-1}(x))$ where ψ_t is the flow associated to the velocity u. Since u is smooth then $\psi_t^{\pm 1}$ is Lipschitzian for every $t \geq 0$. This implies in particular that $\|\psi_t^{\pm 1}\|_*$ is finite for every $t \geq 0$. Theorem 2 and Proposition 1 yield together

$$\begin{aligned}\|\omega(t)\|_{\mathrm{LBMO}} &\leq C\|\omega_0\|_{\mathrm{LBMO}\cap L^p} \ln(1 + \|\psi_t^{-1}\|_*) \\ &\leq C\|\omega_0\|_{\mathrm{LBMO}\cap L^p} \ln(1 + \exp(\int_0^t \|u(\tau)\|_{LL} d\tau)) \\ &\leq C_0(1 + \int_0^t \|u(\tau)\|_{LL} d\tau).\end{aligned}$$

On the other hand, one has

$$\begin{aligned}\|u(t)\|_{LL} &\leq \|\omega(t)\|_{L^2} + \|\omega(t)\|_{B^0_{\infty,\infty}} \\ &\leq C(\|\omega_0\|_{L^2} + \|\omega(t)\|_{BMO}).\end{aligned}$$

The first estimate is classical (see [1] for instance) and the second one is just the conservation of the L^2-norm of the vorticity and the continuity of the embedding $\mathrm{BMO} \hookrightarrow B^0_{\infty,\infty}$.

Consequently, we deduce that

$$\|u(t)\|_{LL} \leq C_0(1 + \int_0^t \|u(\tau)\|_{LL} d\tau),$$

and by Gronwall's Lemma

$$\|u(t)\|_{LL} \leq C_0 \exp(C_0 t), \qquad \forall\, t \geq 0.$$

This yields in particular

$$\|\omega(t)\|_{\mathrm{LBMO}} \leq C_0 \exp(C_0 t), \qquad \forall\, t \geq 0,$$

as claimed. □

4.2. Existence

Let $\omega_0 \in L^p \cap \mathrm{LBMO}$ and $u_0 = k * \omega_0$, with $K(x) = \frac{x^\perp}{2\pi|x|^2}$. We take $\rho \in \mathcal{C}_0^\infty$, with $\rho \geq 0$ and $\int \rho(x)dx = 1$ and set

$$\omega_0^n = \rho_n * \omega_0, \qquad u_0^n = \rho_n * u_0,$$

where $\rho_n(x) = n^2\rho(nx)$. Obviously, ω_0^n is a C^∞ bounded function for every $n \in \mathbb{N}^*$. Furthermore, thanks to (7),

$$\|\omega_0^n\|_{L^p} \leq \|\omega_0\|_{L^p} \qquad \text{and} \qquad \|\omega_0^n\|_{\mathrm{LBMO}} \leq \|\omega_0\|_{\mathrm{LBMO}}.$$

The classical interpolation result between Lebesgue and BMO spaces (see [10] for more details) implies that

$$\|\omega_0^n\|_{L^q} \leq \|\omega_0^n\|_{L^p \cap \mathrm{BMO}} \leq \|\omega_0\|_{L^p \cap \mathrm{BMO}}, \qquad \forall\, q \in [p, +\infty[.$$

Since, $\omega_0^n \in L^p \cap L^\infty$ then there exists a unique weak solution u^n with

$$\omega_n \in L^\infty(\mathbb{R}_+, L^p \cap L^\infty),$$

according to the classical result of Yudovich [20]. According to Proposition 4 one has

$$\|u^n(t)\|_{LL} + \|\omega^n(t)\|_{L^p \cap \mathrm{LBMO}} \leq C_0 \exp(C_0 t), \qquad \forall\, t \in \mathbb{R}_+. \tag{13}$$

With this uniform estimate in hand, we can perform the same analysis as in the case $\omega_0 \in L^p \cap L^\infty$ (see Paragraph 8.2.2 in [14] for more explanations). For the convenience of the reader we briefly outline the main arguments of the proof.

If one denotes by $\psi_n(t,x)$ the associated flow to u^n then

$$\|\psi_n^{\pm 1}(t)\|_* \leq C_0 \exp(C_0 t), \qquad \forall\, t \in \mathbb{R}_+. \tag{14}$$

This yields the existence of explicit time continuous functions $\beta(t) > 0$ and $C(t)$ such that

$$|\psi_n^{\pm 1}(t, x_2) - \psi_n^{\pm 1}(t, x_1)| \leq C(t)|x_2 - x_1|^{\beta(t)}, \qquad \forall\, (x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

Moreover,

$$|\psi_n^{\pm 1}(t_2, x) - \psi_n^{\pm 1}(t_1, x)| \leq |t_2 - t_1| \|u^n\|_{L^\infty} \leq C_0 |t_2 - t_1|, \qquad \forall\, (t_1, t_2) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

Here, we have used the Biot-Savart law to get

$$\|u^n(t)\|_{L^\infty} \lesssim \|\omega^n(t)\|_{L^p \cap L^3} \leq \|\omega_0\|_{L^p \cap L^3}.$$

The family $\{\psi_n, n \in \mathbb{N}\}$ is bounded and equicontinuous on every compact $[0,T] \times \bar{B}(0,R) \subset \mathbb{R}_+ \times \mathbb{R}^2$. The Arzela-Ascoli theorem implies the existence of a limiting particle trajectory $\psi(t,x)$. Performing the same analysis for $\{\psi_n^{-1}, n \in \mathbb{N}\}$ we figure out that $\psi(t,x)$ is a Lebesgue measure preserving homeomorphism . Also, passing to the limit[6] in (14) leads to

$$\|\psi_t\|_* = \|\psi_t^{-1}\|_* \leq C_0 \exp(C_0 t), \qquad \forall\, t \in \mathbb{R}_+.$$

[6] We take the pointwise limit in the definition formula and then take the supremum.

One defines

$$\omega(t,x)=\omega_0(\psi_t^{-1}(x)),\qquad u(t,x)=(k*_x\omega(t,.))(x).$$

We easily check that for every $q\in[p,+\infty[$ one has

$$\omega^n(.,t)\longrightarrow\omega(.,t)\quad\text{in } L^q.$$
$$u^n(.,t)\longrightarrow_x u(.,t)\quad\text{uniformly}.$$

The last claim follows from the fact that

$$\|u^n(t)-u(t)\|_{L^\infty}\lesssim\|\omega^n(t)-\omega(t)\|_{L^p\cap L^3}.$$

All this allows us to pass to the limit in the integral equation on ω^n and then to prove that (u,ω) is a weak solution to the vorticity-stream formulation of the 2D Euler system. Furthermore, the convergence of $\{\omega^n(t)\}$ in $L^1_{\rm loc}$ and (13) imply that

$$\|\omega(t)\|_{L^p\cap\mathrm{LBMO}}\leq C_0\exp(C_0t),\qquad\forall\,t\in\mathbb{R}_+.$$

as claimed.

The continuity of ψ and the preservation of Lebesgue measure imply that $t\mapsto\omega(t)$ is continuous[(7)] with values in L^q for all $q\in[p,+\infty[$. This implies in particular that $u\in\mathscr{C}([0,+\infty[,L^r(\mathbb{R}^d))$ for every $r\in[\frac{2p}{2-p},+\infty]$.

4.3. Uniqueness

Since the vorticity remains bounded in BMO space then the uniqueness of the solutions follows from Theorem 7.1 in [18]. Another way to prove that is to add the information $u\in\mathscr{C}([0,+\infty,L^\infty(\mathbb{R}^d))$ (which is satisfied for the solution constructed above) to the theorem and in this case the uniqueness follows from Theorem 7.17 in [1].

(7) By approximation we are reduced to the following situation: $g_n(x)\to g(x)$ pointwise and

$$\|g_n\|_{L^q}=\|g\|_{L^q}.$$

This is enough to deduce that $g_n\to g$ in L^q (see Theorem 1.9 in [11] for instance).

BIBLIOGRAPHY

[1] H. Bahouri, J.-Y. Chemin, R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundl. Math. Wiss. **343**, Springer, Heidelberg, 2011.

[2] J. T. Beale, T. Kato, A. J. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, *Comm. Math. Phys.* **94** (1984), 61–66.

[3] F. Bernicot, S. Keraani, Sharp constants for composition with a bi-Lipschitz measure-preserving map, preprint arXiv:1204.6008.

[4] D. Chae, Weak solutions of 2-D Euler equations with initial vorticity in $L(\log L)$, *J. Differential Equations* **103** (1993), 323–337.

[5] J.-Y. Chemin, Fluides parfaits incompressibles, *Astérisque* **230** (1995), transl. by I. Gallagher and D. Iftimie, Oxford Lecture Series in Mathematics and Its Applications, Vol. **14**, Clarendon Press-Oxford University Press, New York (1998).

[6] J.-M. Delort, Existence de nappes de tourbillon en dimension deux, *J. Amer. Math. Soc.* **4** (1991), 553–586.

[7] R. J. DiPerna, A. J. Majda, Concentrations in regularizations for 2-D incompressible flow, *Comm. Pure Appl. Math.* **40** (1987), 301–345.

[8] P. Gérard, Résultats récents sur les fluides parfaits incompressibles bidimensionnels (d'après J.-Y. Chemin et J.-M. Delort), Séminaire Bourbaki, vol. 1991/92, exposé n° 757, *Astérisque* **206** (1992), 411–444.

[9] Y. Giga, T. Miyakawa, H. Osada, Two-dimensional Navier-Stokes flow with measures as initial vorticity, *Arch. Rational Mech. Anal.* **104** (1988), 223–250.

[10] L. Grafakos, *Classical and modern Fourier analysis*, Pearson Education, Inc., Upper Saddle River, NJ, 2004.

[11] E. H. Lieb, M. Loss, *Analysis*, Graduate Studies in Math. **14**, Amer. Math. Soc., Providence, RI, 1997.

[12] P.-L. Lions, *Mathematical topics in fluid mechanics. Vol. 1*, Oxford Lecture Series in Mathematics and its Applications **3**, The Clarendon Press, Oxford Univ. Press, New York, 1996.

[13] M. C. Lopes Filho, H. J. Nussenzveig Lopes, Z. Xin, Existence of vortex sheets with reflection symmetry in two space dimensions, *Arch. Ration. Mech. Anal.* **158** (2001), 235–257.

[14] A. J. Majda, A. L. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics **27**, Cambridge Univ. Press, Cambridge, 2002.

[15] P. Serfati, Structures holomorphes à faible régularité spatiale en mécanique des fluides, *J. Math. Pures Appl.* **74** (1995), 95–104.

[16] Y. Taniuchi, Uniformly local L^p estimate for 2-D vorticity equation and its application to Euler equations with initial vorticity in BMO, *Comm. Math. Phys.* **248** (2004), 169–186.

[17] M. Vishik, Hydrodynamics in Besov spaces, *Arch. Ration. Mech. Anal.* **145** (1998), 197–214.

[18] M. Vishik, Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type, *Ann. Sci. École Norm. Sup.* **32** (1999), 769–812.

[19] V. I. YUDOVICH, Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid, *Math. Res. Lett.* **2** (1995), 27–38.

[20] Y. YUDOVICH, Nonstationary flow of an ideal incompressible liquid, *USSR Comp. Math. and Math. Phys.* **3** (1963), 1407–1456.

(Manuscrit reçu le 13 septembre 2012 ;
accepté, après révision, le 12 avril 2013.)

Frédéric BERNICOT
CNRS et Université de Nantes
Laboratoire de Mathématiques Jean Leray
2, rue de la Houssinière
44322 Nantes Cedex 03, France
E-mail: frederic.bernicot@univ-nantes.fr

Sahbi KERAANI
Laboratoire Paul Painlevé
Université de Lille 1
59655 Villeneuve d'Ascq Cedex, France
E-mail: sahbi.keraani@univ-lille1.fr