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HÖLDER CONTINUITY OF LYAPUNOV EXPONENT FOR QUASI-PERIODIC JACOBI OPERATORS

BY KAI TAO

ABSTRACT. — We consider the quasi-periodic Jacobi operator $H_{x,\omega}$ in $l^2(\mathbb{Z})$: $(H_{x,\omega}\phi)(n) = -b(x+(n+1)\omega)\phi(n+1) - b(x+n\omega)\phi(n-1) + a(x+n\omega)\phi(n) = E\phi(n)$, $n \in \mathbb{Z}$, where $a(x)$, $b(x)$ are analytic functions on $\mathbb{T}$, b is not identically zero, and ω obeys some strong Diophantine condition. We consider the corresponding unimodular cocycle. We prove that if the Lyapunov exponent $L(E)$ of the cocycle is positive for some $E = E_0$, then there exist $\rho_0 = \rho_0(a,b,\omega,E_0)$, $\beta = \beta(a,b,\omega)$ such that $|L(E) - L(E')| < |E - E'|^\beta$ for any $E, E' \in (E_0 - \rho_0, E_0 + \rho_0)$. If $L(E) > 0$ for all E in some compact interval I, then $L(E)$ is Hölder continuous on I with Hölder exponent $\beta = \beta(a,b,\omega,I)$. In our derivation we follow the refined version of the Goldstein-Schlag method [3] developed by Bourgain and Jitomirskaya [2].

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1. Introduction

We consider the quasi-periodic Jacobi operator $H_{x,\omega}$ in $l^2(\mathbb{Z})$:

$$(1.1)\quad (H_{x,\omega}\phi)(n) = -b(x+(n+1)\omega)\phi(n+1) - b(x+n\omega)\phi(n-1) + a(x+n\omega)\phi(n) = E\phi(n),\ n\in\mathbb{Z},$$

where $a(x),\ b(x)$ are real analytic functions on $\mathbb{T}$, b is not identically zero. Set

$$A(x,E,\omega) = \frac{1}{b(x+\omega)}\begin{pmatrix} a(x)-E & -b(x) \\ b(x+\omega) & 0 \end{pmatrix}.$$

$$M_N(x,E,\omega) = M_{[1,N]}(x,E,\omega) = A(x+(N-1)\omega,E,\omega)A(x+(N-2)\omega,E,\omega)\cdots A(x,E,\omega).$$

Define the unimodular matrix

$$\tilde{M}_N(x,E,\omega) = \tilde{M}_{[1,N]}(x,E,\omega) := \frac{M_{[1,N]}(x,E,\omega)}{|\det M_{[1,N]}(x,E,\omega)|^{\frac{1}{2}}}.$$

As

$$\det A(x,E,\omega) = \frac{b(x)}{b(x+\omega)},$$

then

$$(1.2)\qquad \det M_{[1,N]}(x,E,\omega) = \prod_{n=0}^{N-1}\frac{b(x+n\omega)}{b(x+(n+1)\omega)} = \frac{b(x)}{b(x+N\omega)},$$

and

$$(1.3)\quad \log\|\tilde{M}_{[1,N]}(x,E,\omega)\| = \log\|M_{[1,N]}(x,E,\omega)\| - \frac{1}{2}\log|\frac{b(x)}{b(x+N\omega)}|.$$

REMARK 1.1. — (1) Note that

$$\|A(x,E,\omega)\| \le \frac{C(a,b,E)}{|b(x+\omega)|},$$

where the constant $C(a,b,E)$ satisfies

$$C(a,b,E_0) = \sup_{|E|\le E_0} C(a,b,E) < +\infty.$$

Therefore,

$$\frac{1}{N}\log\|M_{[1,N]}(x,E,\omega)\| \le \log C(a,b,E) - \frac{1}{N}\sum_{n=1}^{N}\log|b(x+n\omega)|.$$

In this paper we always assume that $|E| \leq E_0$, where E_0 depends on a, b. For that matter we suppress E from the notation of some of the constants involved.

(2) $\log \|\tilde{M}_{[1,N]}(x, E, \omega)\| \geq 0$, since $\tilde{M}_{[1,N]}(x, E, \omega)$ is unimodular.

(3)

$$\begin{aligned} 0 &\leq \frac{1}{N} \log \|\tilde{M}_{[1,N]}(x, E, \omega)\| \\ &= \frac{1}{N} \log \|M_{[1,N]}(x, E, \omega)\| - \frac{1}{2N} \log |\frac{b(x)}{b(x+N\omega)}| \\ &\leq \log C(a, b, E) - \frac{1}{N} \sum_{n=1}^{N} \log |b(x+n\omega)| - \frac{1}{2N} \log |\frac{b(x)}{b(x+N\omega)}|. \end{aligned}$$

(4) It is a well-known fact that if b is an analytic function not identically zero, then $(\log |b|)^2$ is integrable. Set

$$D = \int_{\mathbb{T}} \log |b(\theta)| d\theta.$$

Therefore,

$$\begin{aligned} \int_{\mathbb{T}} \left| \frac{1}{N} \log \|\tilde{M}_{[1,N]}(x, E, \omega)\| \right| dx &= \int_{\mathbb{T}} \frac{1}{N} \log \|\tilde{M}_{[1,N]}(x, E, \omega)\| dx \\ &\leq C'(a, b) - D := C''(a, b). \end{aligned}$$

Similarly,

$$\int_{\mathbb{T}} (\frac{1}{N} \log \|\tilde{M}_N(x, E, \omega)\|)^2 dx \leq \tilde{C}(a, b).$$

(5) Combining (4) with (1.3), we conclude that $\frac{1}{N} \log \|M_{[1,N]}(x, E, \omega)\|$ is integrable, and

$$\frac{1}{N} \int_{\mathbb{T}} \log \|M_N(x, E, \omega)\| dx = \frac{1}{N} \int_{\mathbb{T}} \log \|\tilde{M}_N(x, E, \omega)\| dx.$$

Set

$$L_N(E, \omega) = \frac{1}{N} \int_{\mathbb{T}} \log \|M_N(x, E, \omega)\| dx = \frac{1}{N} \int_{\mathbb{T}} \log \|\tilde{M}_N(x, E, \omega)\| dx.$$

Note that $L_N(E, \omega) > 0$. Set

$$B(x, E, \omega) := \begin{pmatrix} a(x) - E & -b(x) \\ b(x+\omega) & 0 \end{pmatrix},$$

and

$$\begin{aligned} T_N(x, E, \omega) &= T_{[1,N]}(x, E, \omega) \\ &:= B(x+(N-1)\omega, E, \omega) B(x+(N-2)\omega, E, \omega) \cdots B(x, E, \omega). \end{aligned}$$

Then

$$M_{[1,N]}(x,E,\omega) = T_{[1,N]}(x,E,\omega)\prod_{n=N-1}^{0}\frac{1}{b(x+(n+1)\omega)},$$

$$\tag{1.4}\begin{aligned}\tilde{M}_{[1,N]}(x,E,\omega) &= \frac{|b(x+N\omega)|^{\frac12}}{|b(x)|^{\frac12}}M_{[1,N]}(x,E,\omega)\\ &= \frac{|b(x+N\omega)|^{\frac12}}{|b(x)|^{\frac12}}\prod_{n=0}^{N-1}\frac{1}{b(x+(n+1)\omega)}T_{[1,N]}(x,E,\omega),\end{aligned}$$

$$\tag{1.5}\|\tilde{M}_{[1,N]}(x,E,\omega)\| = \prod_{n=0}^{N-1}\frac{1}{|b(x+n\omega)b(x+(n+1)\omega)|^{\frac12}}\|T_{[1,N]}(x,E,\omega)\|,$$

and

$$\log\|\tilde{M}_{[1,N]}(x,E,\omega)\| = \log\|T_{[1,N]}(x,E,\omega)\| - \frac{1}{2}\sum_{n=0}^{N-1}\log|b(x+n\omega)b(x+(n+1)\omega)|.$$

Note also for future reference that

$$\tag{1.6}|\det T_{[1,N]}(x,E,\omega)| = \prod_{n=0}^{N-1}|b(x+n\omega)||b(x+(n+1)\omega)|.$$

Combining (1) with Remark 1.1, we conclude that $\frac{1}{N}\log\|T_{[1,N]}(x,E,\omega)\|$ is integrable,

$$\tag{1.7}J_N(E,\omega) := \frac{1}{N}\int_{\mathbb{T}}\log\|T_{[1,N]}(x,\omega)\|dx = L_N(E,\omega) + D.$$

Due to the subadditive property, the limits

$$\tag{1.8}\begin{aligned}L(E,\omega) &= \lim_{N\to\infty}\int_{\mathbb{T}}\frac{1}{N}\log\|M_N(x,E,\omega)\|dx = \lim_{N\to\infty}\int_{\mathbb{T}}\frac{1}{N}\log\|\tilde{M}_N(x,E,\omega)\|dx\\ &= \lim_{N\to\infty}L_N(E,\omega),\end{aligned}$$

$$\tag{1.9}J(E,\omega) = \lim_{N\to\infty}\int_{\mathbb{T}}\frac{1}{N}\log\|T_N(x,E,\omega)\|dx = \lim_{N\to\infty}J_N(E,\omega) = L(E,\omega) + D$$

exist. Moreover, $L(E,\omega) \geq 0$. Fix some $\alpha > 1$. Throughout this paper we assume that $\omega \in (0,1)$ satisfies the Diophantine condition

$$\tag{1.10}\|n\omega\| \geq \frac{C_\omega}{n(\log n)^\alpha} \quad \text{for all } n.$$

It is well known that for a fixed $\alpha > 1$ almost every ω satisfies (1.10).

The main theorem in this paper is

THEOREM 1.2. — *Assume $L(E_0) > 0$. Then there exists $\rho_0 > 0$ depending on $a(x), b(x), \omega$ and E_0 such that for any $E, E' \in (E_0 - \rho_0, E_0 + \rho_0)$, the following holds:*

$$|L(E) - L(E')| = |J(E) - J(E')| < |E - E'|^\beta,$$

where the constant β depends on $a(x), b(x), \omega$; i.e., $\beta = \beta(a, b, \omega)$, but does not depend on E_0.

For the Schrödinger equation:

$$(S_{x,\omega}\phi)(n) = \phi(n+1) + \phi(n-1) + V(x + n\omega)\phi(n) = E\phi(n), \ n \in \mathbb{Z},$$

Goldstein and Schlag [3] proved that if ω is Diophantine and $V(x)$ is analytic, then the Lyapunov exponent $L(E)$ is Hölder continuous. They developed two powerful tools, the Large Deviation Theorem and the Avalanche Principle, which was always used in the proof of the continuity of the Lyapunov exponent in the subsequent papers. Similar results were proved in [1] by Bourgain, Goldstein and Schlag for underlying dynamics being a shift or skew shift of a high dimensional torus. When ω is an irrational number, Bourgain and Jitomirakaya show that the Lyapunov exponent is jointly continuous in E and ω in [2]. Later, for the general analytic quasiperiodic cocycles with constant determinant, Jitomirskaya, Koslover and Schulteis ([5]) proved that the Lyapunov exponent is continuous. The author gave a proof of this result for the high dimensional torus in [8]. The jointly continuous dependence of the Lyapunov exponent on the cocycle with no restrictions on the determinant was established in [7].This improved an earlier,weaker version in [6]. Obviously, the Schrödinger equation is a special case of the Jacobi equation for $b(x) = -1$, but the result here is better than that in [3], as the Hölder constant here only depends on $a(x)$, $b(x)$ and ω, but not on E. The key to get a better result is Theorem 2.5 (Theorem 3.8 in [4]) and the uniform estimate from $L_N(E)$ to $L(E)$. In [4], Goldstein and Schlag proved that the Fourier coefficients of a subharmonic function only depend on the supremum of the subharmonic function, not on the infimum. In Section Three of this paper, there are several lemmas and theorems which emphasize the uniform estimate. Actually, if the potential function in Schrödinger equation is a small perturbation of a trigonometric polynomial function of degree k, Goldstein and Schlag proved that the Lyapunov exponent is Hölder $\frac{1}{2k_0} - \kappa$ continuous for any κ in [4].

2. Large Deviation Theorem

It is convenient to replace $a(x)$, $b(x)$ by $p(e(x))$ and $q(re(x))$ (with $e(x) = e^{2\pi i x}$), where $p(z), q(z)$ are analytic functions in the annulus

$A_\rho = \{z \in \mathbb{C} : 1-\rho < |z| < 1+\rho\}$ which assume only real values for $|z| = 1$. With this convention in place, we will use the notation $B(z)$, $T_{[1,N]}(z)$, etc.

LEMMA 2.1. — (1) $\sup_{z\in A_{\frac{\rho}{2}}} \|B(z)\| = \sup_{z\in A_{\frac{\rho}{2}}} \left\| \begin{pmatrix} p(z)-E & -q(z) \\ q(ze(\omega)) & 0 \end{pmatrix} \right\| \leq C,$ *where $C = C(p,q)$, and $C(p,q)$ is the same as in (1) of Remark 1.1.*

(2) $\sup_{z\in A_{\frac{\rho}{2}}} \|B^{-1}(z)\| \leq \frac{C(p,q)}{|q(ze(\omega))q(z))|}$

Proof. — (1) is obvious.

(2) $\det B(z) = q(ze(\omega))q(z)$, and $B^{-1}(z) = \frac{1}{q(z)q(ze(\omega))} \begin{pmatrix} 0 & q(z) \\ -q(ze(\omega)) & p(z)-E \end{pmatrix}$, that implies (2). ☐

So

LEMMA 2.2. — *For any $z \in A_{\frac{\rho}{2}}$,*

$$\begin{aligned} -C_1(p,q) + \log|q(z)q(ze(\omega))| &\leq \log\|T_{[1,N]}(z)\| - \log\|T_{[1,N]}(ze(\omega))\| \\ &\leq C_1(p,q) - \log|q(ze((N-1)\omega))q(ze(N\omega))|. \end{aligned} \tag{2.1}$$

Proof. — $T_{[1,N]}(ze(\omega)) = B(ze(N\omega))T_{[1,N]}(z)B^{-1}(z)$,so $\|T_{[1,N]}(ze(\omega))\| \leq C(p,q)\|T_{[1,N]}(z)\| \frac{C(p,q)}{|q(z)q(ze(\omega))|}$ implies

$$\log\|T_{[1,N]}(ze(\omega))\| \leq 2\log C(p,q) + \log\|T_{[1,N]}(z)\| - \log|q(z)q(ze(\omega))|.$$

Similarly, we have

$$\log\|T_{[1,N]}(z)\| \leq 2\log C(p,q) + \log\|T_{[1,N]}(ze(\omega))\| - \log|q(ze((N-1)\omega))q(ze(N\omega))|.$$

Thus

$$\begin{aligned} -C_1 + \log|q(z)q(ze(\omega))| &\leq \log\|T_{[1,N]}(z)\| - \log\|T_{[1,N]}(ze(\omega))\| \\ &\leq C_1 - \log|q(ze((N-1)\omega))q(ze(N\omega))|, \end{aligned}$$

where $C_1 = C_1(p,q)$. ☐

COROLLARY 2.3. — (1) We have

$$\begin{aligned} -kC_1 + \sum_{m=0}^{k-1} \log|q(ze((m+1)\omega))q(ze((m)\omega))| \\ \leq \log\|T_{[1,N]}(z)\| - \log\|T_{[1,N]}(ze(k\omega))\| \\ \leq kC_1 - \sum_{m=0}^{k-1} \log|q(ze((N+m-1)\omega))q(ze((N+m)\omega))|, \end{aligned}$$

(2)

$$
\begin{aligned}
-KC_1+\sum_{k=0}^{K-1}\frac{K-k}{K}\log|q(ze(k\omega))q(ze((k+1)\omega))| \\
\leq \log\|T_{[1,N]}(z)\| - \frac{1}{K}\sum_{k=1}^{K}\log\|T_{[1,N]}(ze(k\omega))\| \\
\leq KC_1 - \sum_{k=0}^{K-1}\frac{K-k}{K}\log|q(ze((k-1+N)\omega))q(ze((k+N)\omega))|.
\end{aligned}
$$

Proof. — (1) is obvious.

(2) For $1\leq k\leq K$,

$$
\begin{aligned}
-\frac{kC_1}{K}+\sum_{m=0}^{k-1}\frac{1}{K}\log|q(ze((m+1)\omega))q(ze((m)\omega))| \\
\leq \frac{1}{K}\log\|T_{[1,N]}(z)\| - \frac{1}{K}\log\|T_{[1,N]}(ze(k\omega))\| \\
\leq \frac{kC_1}{K} - \sum_{m=0}^{k-1}\frac{1}{K}\log|q(ze((N+m-1)\omega))q(ze((N+m)\omega))|.
\end{aligned}
$$

As

$$
\sum_{k=1}^{K}\sum_{m=0}^{k-1}\frac{1}{K}\log|q(ze((m+1)\omega))q(ze((m)\omega))| = \sum_{k=0}^{K-1}\frac{K-k}{K}\log|q(ze((k+1)\omega))q(ze((k)\omega))|
$$

and

$$
\sum_{k=1}^{K}\frac{kC_1}{K}\leq\sum_{k=1}^{K}C_1 = KC_1,
$$

then

$$
\log\|T_{[1,N]}(z)\| - \sum_{k=1}^{K}\frac{1}{K}\log\|T_{[1,N]}(ze(k\omega))\| \geq -KC_1+\sum_{k=0}^{K-1}\frac{K-k}{K}\log|q(ze((k+1)\omega))q(ze((k)\omega))|.
$$

Also

$$\log \|T_{[1,N]}(z)\| - \sum_{k=1}^{K} \frac{1}{K} \log \|T_{[1,N]}(ze(k\omega))\|$$
$$\leq KC_1 - \sum_{k=0}^{K-1} \frac{K-k}{K} \log |q(ze((k-1+N)\omega))q(ze((k+N)\omega))|. \quad \square$$

LEMMA 2.4. — *Let $u : \Omega \to \mathbb{R}$ be a subharmonic function on a domain $\Omega \subset \mathbb{C}$. Suppose that $\partial\Omega$ consists of finitely many piece-wise C^1 curves. There exists a positive measure μ on Ω such that for any $\Omega_1 \Subset \Omega$ (i.e., Ω_1 is a compactly contained subregion of Ω),*

$$u(z) = \int_{\Omega_1} \log |z-\zeta| \, d\mu(\zeta) + h(z), \tag{2.2}$$

where h is harmonic on Ω_1 and μ is unique with this property. Moreover, μ and h satisfy the bounds

$$\mu(\Omega_1) \leq C(\Omega, \Omega_1) \, (\sup_{\Omega} u - \sup_{\Omega_1} u), \tag{2.3}$$

$$\|h - \sup_{\Omega_1} u\|_{L^\infty(\Omega_2)} \leq C(\Omega, \Omega_1, \Omega_2) \, (\sup_{\Omega} u - \sup_{\Omega_1} u) \tag{2.4}$$

for any $\Omega_2 \Subset \Omega_1$.

For the proof see Lemma 2.2 in [4].

THEOREM 2.5. — *Let u be a subharmonic function defined in the annulus A_ρ. Suppose furthermore that $|u(z)| \leq 1$. Then for any $1 - \frac{\rho}{2} \leq r \leq 1 + \frac{\rho}{2}$,*

$$\mathrm{mes}\,(\{x : |\sum_{k=1}^{n} u(re(x-k\omega)) - n < u(re(\cdot)) > | > \delta n\}) < \exp(-c\delta n + s_n), \tag{2.5}$$

where $< u(re(\cdot)) > := \int_0^1 u(re(y))dy$, $s_n \leq C(\log n)^A$ for general n and $s_n \leq C \log n$ if $n = q_j$ for any j.

For the proof see Theorem 3.8 in [3].

REMARK 2.6. — (1) The constants c, C here do not depend on δ.

(2) Actually, the condition $|u(z)| \leq 1$ can be replaced by $u(z) = \int \log |z-\zeta| d\mu(\zeta) + h(z)$, with $\|\mu\| + \|h\| \leq C$, see the proof of Theorem 3.8 in [3]

In what follows we will use the following version

THEOREM 2.7. — *Let u be a subharmonic function defined in the annulus A_ρ. Suppose furthermore that $u(z) = \int \log|z-\zeta| d\mu(\zeta) + h(z)$, with $\mu(A_{\frac{\rho}{2}}) + \|h\|_{L^\infty(A_{\frac{\rho}{2}})} \le \hat{C}$. Then for any $1-\frac{\rho}{2} \le r \le 1+\frac{\rho}{2}$,*

$$\operatorname{mes}\big(\{x : |\sum_{k=1}^{n} u(re(x+k\omega)) - n{<}u(re(\cdot)){>}| > \delta n\}\big) < \exp(-c\delta n), \tag{2.6}$$

where $c = c(\hat{C}, \omega)$.

Set $u_N(z,E,\omega) = \frac{1}{N}\log\|T_N(z,E,\omega)\|$, $\tilde{u}_N(z,E,\omega) = \frac{1}{N}\log\|\tilde{M}_N(z,E,\omega)\|$. Sometimes we use $u_N(z)$ or u_N for short; and similarly for $\tilde{u}_N(z,E,\omega)$. Let $L_{N,r}(E) = {<}\tilde{u}_N(re(\cdot)){>}$, $D_r = {<}\log|q(re(\cdot))|{>}$, $J_{N,r}(E) = {<}u_N(re(\cdot)){>} = L_{N,r}(E) + D_r$. For $r=1$ we use the notation $L_N(E)$, D, $J_N(E)$.

LEMMA 2.8. — *Function $u_N(z)$ is subharmonic in A_ρ and obeys $u_N(z) \le C(p,q)$.*

Proof. — Since $B(z)$ is analytic in A_ρ, so is $T_N(z)$. Therefore, $u_N(z) = \frac{1}{N}\log\|T_N(z)\|$ is subharmonic in A_ρ. The estimate $u_N(z) \le C(p,q)$ follows from Lemma 2.1. □

LEMMA 2.9. — *There exists $C_2(q) < +\infty$ s.t.*

$$\sup_{x\in\mathbb{T}} u_N(e(x)) \ge -C_2(q).$$

Proof. — One has $\|A\|^2 \ge |\det A|$ for any 2×2 matrix. So

$$u_N(z) \ge \frac{1}{2N}\log|\det T_N(z)| = \frac{1}{2N}\sum_{n=0}^{N-1}\log|q(ze(n\omega))q(ze((n+1)\omega))|,$$

see (1.6). Recall that $\log|q(e(x))|$ is integrable. So, $\frac{1}{2}\log|q(e(x))q(e(x+\omega))|$ is integrable. So,

$$\int_{\mathbb{T}}\left|\frac{1}{2}\log|q(e(x))q(e(x+\omega))|\right| dx = C_2(q) < +\infty. \tag{2.7}$$

Therefore

$$\int_{\mathbb{T}}\left|\frac{1}{2N}\sum_{n=0}^{N-1}\log|q(e(x+n\omega))q(e(x+(n+1)\omega))|\right| dx \le C_2(q) < +\infty.$$

Hence,

$$\sup_{x\in\mathbb{T}}\frac{1}{2N}\sum_{n=0}^{N-1}\log|q(e(x+n\omega))q(e(x+(n+1)\omega))| \ge -C_2(q). \qquad \square$$

Lemma 2.10. — *One has*

$$u_N(z) = \int \log|z-\zeta| d\mu(\zeta) + h(z),$$

where

$$\mu(A_{\frac{\rho}{2}}) \leq C_3(p,q), \;\; \|h\|_{L^\infty(A_{\frac{\rho}{2}})} \leq C_3(p,q).$$

Proof. — The statement follows from Lemmas 2.8, 2.9, and (2.3), (2.4). □

Remark 2.11. — By (2.7), we have

$$\sup_{x\in\mathbb{T}} \frac{1}{2}\log|q(re(x))q(re(x+\omega))| \geq -C_2(q).$$

Then by Lemma 2.9, and (2.3), (2.4), we have

$$\frac{1}{2}\log|q(re(x))q(re(x+\omega))| = \int \log|z-\zeta| d\mu_q(\zeta) + h_q(\zeta),$$

where

$$\mu_q(A_{\frac{\rho}{2}}) \leq C_q, \;\; \|h\|_{L^\infty(A_{\frac{\rho}{2}})} \leq C_q.$$

By Theorem 2.7 and Lemma 2.10, we have

Lemma 2.12. — *For any $1-\frac{\rho}{2} \leq r \leq 1+\frac{\rho}{2}$, δ and K,*

(2.8)

$$\operatorname{mes}(\{x : |\sum_{k=1}^{K} u_N(re(x+k\omega)) - K < u_N(re(\cdot)) > | > \delta K\}) < \exp(-c\delta K),$$

where $c = c(p,q,\omega)$.

Remark 2.13. — (1) By Remark 2.11 and Theorem 2.7, for any $1-\frac{\rho}{2} < r < 1+\frac{\rho}{2}$, δ and K, we also have

$$\begin{aligned}\operatorname{mes}(\{x : |\sum_{k=1}^{K} \frac{1}{2}\log|q(re(x+k\omega))q(re(x+(k+1)\omega))| - K \\ < \frac{1}{2}\log|q(re(x))q(re(x+\omega))| > | > \delta K\}) < \exp(-c_q\delta K).\end{aligned}$$

(2) It is well-known that $\int_0^1 |\log|q(re(x))|| \, dx \leq C'_q$, for any $1-\frac{\rho}{2} \leq r \leq 1+\frac{\rho}{2}$, if $b(x)$ is analytic.

(3) Due to (1) we have

$$\tilde{u}_N(re(x)) = u_N(re(x)) - \frac{1}{2N}\sum_{k=1}^{N}\log|q(re(x+k\omega))q(re(x+(k+1)\omega))|.$$

Hence,

$$\operatorname{mes}\{x : |\sum_{k=1}^{K}\tilde{u}_N(re(x+k\omega)) - K<\tilde{u}_N(re(\cdot))>| > \delta K\} \le \exp(-c\delta K) + K\exp(-c_q\delta K).$$

Moreover, there exists $\check{K} = \check{K}(p,q,\delta)$, s.t. for any $K \ge \check{K}$,

$$\operatorname{mes}\{x : |\sum_{k=1}^{K}\tilde{u}_N(re(x+k\omega)) - K < \tilde{u}_N(re(\cdot)) > | > \delta K\} \le \exp(-\hat{c}\delta K).$$

Lemma 2.14. — *For any $1-\frac{\rho}{2} \le r \le 1+\frac{\rho}{2}$, δ and K,*

$$\operatorname{mes}(\{x : |\sum_{k=0}^{K-1}\frac{K-k}{K}\log|q(re(x+k\omega))q(re((x+k+1)\omega))| - (K+1)D_r| > \delta K\}) \le K\exp(-c_q\delta K),$$

where c_q is as in Remark 2.13.

Proof. — Set

$$\mathbb{X}_r := \{x : |\sum_{k=0}^{K-1}\frac{K-k}{K}\log|q(re(x+k\omega))q(re((x+k+1)\omega))| - (K+1)D_r| > \delta K\},$$

and

$$\begin{aligned} Q_r(x) &= \log|q(re(x))q(re(x+\omega))| - \int_{\mathbb{T}}\log|q(re(x))q(re(x+\omega))|dx \\ &= \log|q(re(x))q(re(x+\omega))| - 2D_r. \end{aligned}$$

Note also that $\sum_{k=0}^{K-1}\frac{K-k}{K} = \frac{K+1}{2}$, and

$$\sum_{k=0}^{K-1}\frac{K-k}{K}\log|q(re(x+k\omega))q(re((x+k+1)\omega))| - (K+1)D_r = \sum_{k=0}^{K-1}\frac{K-k}{K}Q_r(x+k\omega).$$

So

$$\left|\sum_{k=0}^{K-1}\frac{K-k}{K}\log|q(re(x+k\omega))q(re((x+k+1)\omega))| - (K+1)D_r\right| > \delta K$$

$$\Leftrightarrow |\sum_{k=0}^{K-1}\frac{K-k}{K}Q_r(x+k\omega)|>\delta K,\quad \text{and}\quad <Q_r>=0.$$

Define $S_{k,r}(x)=\sum_{j=0}^{k-1}Q_r(x+j\omega)$, then

$$\sum_{k=0}^{K-1}\frac{K-k}{K}Q_r(x+k\omega)=\frac{1}{K}\sum_{k=1}^{K}S_k(x).$$

Set $\mathbb{X}_{k,r}:=\{x:|S_{k,r}(x)|>\delta K\}$, then

$$\mathbb{X}_r\subseteq\bigcup_{k=1}^{K}\mathbb{X}_{k,r},\quad \operatorname{mes}(\mathbb{X}_r)\leq\sum_{k=1}^{K}\operatorname{mes}(\mathbb{X}_{k,r}).$$

Note that

$$\mathbb{X}_{k,r}=\{x:|\sum_{j=0}^{k-1}\log|q(re(x+j\omega))q(re(x+(j+1)\omega))| \\ -k\int_{\mathbb{T}}\log|q(re(x))q(re(x+\omega))|dx|>\delta K\}.$$

Thus, by Remark 2.13

$$\operatorname{mes}\mathbb{X}_{k,r}=\{x:|S_{k,r}(x)|>\delta K=\frac{\delta K}{k}k\})\leq\exp(-c_q\frac{\delta K}{k}k)=\exp(-c_q\delta K).$$

Thus

$$\operatorname{mes}\mathbb{X}_r\leq K\times\exp(-c_q\delta K).\qquad\square$$

THEOREM 2.15. — *There exists $\check{N}(p,q,\omega)$ such that for any $N\geq\check{N}$, any $1-\frac{\rho}{2}\leq r\leq 1+\frac{\rho}{2}$ and $\delta<1$,*

$$\operatorname{mes}(\{x:|\frac{1}{N}\log\|T_N(re(x))\|-J_{N,r}(E)|>\delta\})<\exp(-\check{c}\delta^2N),$$

where $\check{c}=\check{c}(p,q,\omega)$.

Proof. — Set

$$\mathbb{Y}_r:=\{x:|\frac{1}{N}\log\|T_N(re(x))\|-\frac{1}{N}\int\log\|T_N(re(x))\|dx|>\delta\}.$$

Then

$$\begin{aligned}\mathbb{Y}_r \subseteq \{x : &\left|\frac{1}{K}\sum_{k=1}^{K} u_N(re(x+k\omega)) - {<}u_N(re(\cdot)){>}\right| > \frac{\delta}{2}\}\\ &\bigcup\{x : u_N(re(x)) - \frac{1}{K}\sum_{k=1}^{K} u_N(re(x+k\omega)) > \frac{\delta}{2}\}\\ &\bigcup\{x : u_N(re(x)) - \frac{1}{K}\sum_{k=1}^{K} u_N(re(x+k\omega)) < -\frac{\delta}{2}\}\\ := \mathbb{Y}_{0,r}&\bigcup\mathbb{Y}_{+,r}\bigcup\mathbb{Y}_{-,r}.\end{aligned}$$

Set $K = C_4\delta N$. By Lemma 2.12,

$$\text{mes } \mathbb{Y}_{0,r} < \exp(-\frac{c}{2}\delta K) = \exp(-C_4 \times c\delta^2 N).$$

We need to estimate mes $\mathbb{Y}_{\pm,r}$. Due to part (2) of Corollary 2.3, we have

$$\mathbb{Y}_{+,r} \subseteq \{x : KC_1 - \sum_{k=0}^{K-1}\frac{K-k}{K}\log|q(ze((k-1+N)\omega))q(ze((k+N)\omega))| > \frac{N\delta}{2}\},$$

$$\mathbb{Y}_{-,r} \subseteq \{x : -KC_1 + \sum_{k=0}^{K-1}\frac{K-k}{K}\log|q(ze(k\omega))q(ze((k+1)\omega))| < -\frac{N\delta}{2}\}.$$

Set $C_4 < \frac{1}{4C_1}$, then

$$KC_1 - \sum_{k=0}^{K-1}\frac{K-k}{K}\log|q(ze((k-1+N)\omega))q(ze((k+N)\omega))| > \frac{N\delta}{2}$$

$$\Rightarrow \sum_{k=0}^{K-1}\frac{K-k}{K}\log|q(ze((k-1+N)\omega))q(ze((k+N)\omega))| < -\frac{\delta N}{4} = -\frac{K}{4C_4}.$$

(2.9)

$$\begin{aligned}&\sum_{k=0}^{K-1}\frac{K-k}{K}\log|q(ze((k-1+N)\omega))q(ze((k+N)\omega))|\\ &\qquad - \sum_{m=0}^{K-1} 2\frac{K-m}{K}\int_{\mathbb{T}}\log|q(re(x))|dx\\ &\quad = \sum_{k=0}^{K-1}\frac{K-k}{K}\log|q(ze((k-1+N)\omega))q(ze((k+N)\omega))| - (K+1)D_r\\ &\quad < -\frac{K}{4C_4} - (K+1)D_r.\end{aligned}$$

Recall that

$$\tilde{D} = \max_{1-\frac{\rho}{2} \le r \le 1+\frac{\rho}{2}} |D_r| < +\infty.$$

Let $C_4 < \frac{1}{8|\tilde{D}|}$ to make $\frac{1}{4C_4} + D_r = C_{5,r} > \tilde{D} > 0$. Note that $C_{5,r}$ is also continuous in r. Thus,

$$C_5 = \min_{1-\frac{\rho}{2} \le r \le 1+\frac{\rho}{2}} C_{5,r} > 0.$$

Then there exists $C_5' = C_5'(p,q) > 0$, s.t.

$$\frac{K}{4C_4} + (K+1)D_r = C_{5,r}K + D_r \ge C_5 K + D_r \ge C_5' K$$

for any K. Thus, Lemma 2.14 applies for $\delta = C_5'$,

$$\operatorname{mes}(\mathbb{Y}_{+,r}) \le K \exp(-c_q \times C_5' K) = C_4 \delta N \exp(-c_q \times C_5' \times C_4 \delta N).$$

As $y \exp(-\xi y) \le \xi^{-1}$ for any $y, \xi > 0$,

$$\begin{aligned}\operatorname{mes}(\mathbb{Y}_{+,r}) &\le C_4 \delta N \exp(-\frac{c_q \times C_5'}{2} \times C_4 \delta N) \times \exp(-\frac{c_q \times C_5'}{2} \times C_4 \delta N) \\ &< \frac{2}{c_q \times C_5'} \exp(-\frac{c_q \times C_5'}{2} \times C_4 \delta N) < \exp(-c_{p,q} \delta N),\end{aligned}$$

for $N \ge \check{N}$, where $\check{N}$ depends on $c_{p,q}$ and C_5', i.e., $\check{N} = \check{N}(p,q,\omega)$. Similarly,

$$\operatorname{mes}(\mathbb{Y}_{-,r}) < \exp(-c_{p,q} \delta N).$$

So for $N \ge \check{N}(p,q,\omega)$

$$\operatorname{mes} \mathbb{Y}_r < 2\exp(-c_{p,q}\delta N) + \exp(-C_4 \times c\delta^2 N) < \exp(-\check{c}\delta^2 N),$$

where $\check{c} = \check{c}(p,q,\omega)$. □

REMARK 2.16. — (1) Recall that

$$\begin{aligned}\tilde{u}(z) = u_N(z) - \frac{1}{2N} \sum_{n=0}^{N-1} \log |q(re(x+n\omega))q(re(x+(n+1)\omega))|, \\ <\tilde{u}_N(z)> = <u_N(z)> - D_r;\end{aligned}$$

see (1). By Remark 2.13, Theorem 2.15, for any $N \ge \check{N}$,

$$\begin{aligned}&\operatorname{mes}(\{x : |\tilde{u}_N(re(x)) - L_{N,r}(E)| > 2\delta\}) \\ &\qquad < \exp(-c_q \delta K) + \exp(-\check{c}\delta^2 N) < \exp(-4\tilde{c}\delta^2 N),\end{aligned}$$

where $L_{N,r}(E) = J_{N,r} - D_r$. Hence for any $N \ge \check{N}$,

$$\operatorname{mes}(\{x : |\tilde{u}_N(re(x)) - L_{N,r}(E)| > \delta\}) < \exp(-\tilde{c}\delta^2 N). \tag{2.10}$$

(2) Once again let us note that the constants c, $\tilde{c}$ here do not depend on δ. In particular, one can choose δ depending on N.

LEMMA 2.17. — *Let $1 > \rho > 0$ and suppose u is subharmonic on A_ρ such that $\sup_{z\in A_\rho} u(z) \le 1$ and $\int_{\mathbb{T}} u(re(x))\,dx \ge 0$. Then for any r_1, r_2 so that $1-\frac{\rho}{2} \le r_1, r_2 \le 1+\frac{\rho}{2}$ we have*

$$|\langle u(r_1 e(\cdot))\rangle - \langle u(r_2 e(\cdot))\rangle| \le C_\rho\,|r_1 - r_2|.$$

Proof. — See [4] Lemma 4.1. □

REMARK 2.18. — It is easy to see that this lemma also holds for $u_N(z)$ with $C_\rho = C_\rho(p,q,\omega)$. Thus,

$$\left|\int_0^1 u_N(re(\theta))d\theta - J_{N,1}(E)d\theta\right| \le C_\rho|r-1|$$

for any $1-\frac{\rho}{2} < r < 1+\frac{\rho}{2}$.

LEMMA 2.19. — *For any $N \ge \check{N}(p,q,\omega)$,*

$$\frac{1}{N}\log\|T_N(e(x),E)\| \le J_{N,1} + C_6\Big(\frac{\log N}{N}\Big)^{\frac{1}{2}}, \tag{2.11}$$

where $C_6 = C_6(p,q,\omega)$ and $\check{N}$ is as in Theorem 2.15.

Proof. — Let $0<\delta<\frac{\rho}{4}$ be arbitrary. Note that $e(x+iy) = e^{-2\pi y}e(x)$, $1-\frac{\rho}{4} < 1-\frac{\delta}{C'_\rho} \le e^{-2\pi y} \le 1+\frac{\delta}{C'_\rho} < 1+\frac{\rho}{4}$, if $|y| \le \frac{\delta}{4\pi e C'_\rho}$, where $C'_\rho = \max(1, C_\rho)$ and $e = \exp(1)$. By Remark 2.18 we have

$$\left|\int_0^1 u_N(re(\theta))d\theta - J_{N,1}(E)d\theta\right| \le \delta, \text{ if } |y| \le \frac{\delta}{4\pi e C'_\rho}. \tag{2.12}$$

Set

$$\mathbb{B}_y := \{x : |u_N(e(x+iy)) - J_{N,1}| > 2\delta\}.$$

It follows from (2.12) that for $|y| \le \frac{\delta}{4\pi e C'_\rho}$, the following holds:

$$\mathbb{B}_y \subseteq \{x : |u_N(e(x+iy)) - \int_0^1 u_N(e(\theta+iy))d\theta| > \delta\}.$$

Due to Theorem 2.15 we obtain $\operatorname{mes}\mathbb{B}_y \le \exp(-\check{c}\delta^2 N)$. The function $u_N(e(x+iy))$ is subharmonic, for $e(x+iy) \in A_\rho$. Let x_0 be arbitrary and $y_0 = 0$. Then $e(x_0) \in A_{\frac{\rho}{4}}$. Due to subharmonicity we have for any $t_0 < \frac{\rho}{4}$,

$$\begin{aligned} u_N(e(x_0)) - J_{N,1} &\le \frac{1}{\pi t_0^2}\iint_{|(x,y)-(x_0,0)|\le t_0} [u_N(e(x+iy)) - J_{N,1}]dxdy \\ &= \frac{1}{\pi t_0^2}\int_{|y|\le t_0}\int_{|x-x_0|\le\sqrt{t_0^2-|y|^2}} [u_N(e(x+iy)) - J_{N,1}]dxdy. \end{aligned}$$

Furthermore

$$\int_{|x-x_0|\leq\sqrt{t_0^2-|y|^2}}[u_N(e(x+iy))-J_{N,1}]dx$$
$$=\left(\int_{\{|x-x_0|\leq\sqrt{t_0^2-|y|^2}\}\bigcap\mathbb{B}_y}+\int_{\{|x-x_0|\leq\sqrt{t_0^2-|y|^2}\}\backslash\mathbb{B}_y}\right)[u_N(e(x+iy))-J_{N,1}]dx.$$

Note that

$$|u_N(e(x+iy))-J_{N,1}|\leq 2\delta,\ \text{if } x\notin\mathbb{B}_y \text{ and } y<\frac{\delta}{4e\pi C'_\rho}.$$

So

$$\left|\int_{\{|x-x_0|\leq\sqrt{t_0^2-|y|^2}\}\backslash\mathbb{B}_y}[u_N(e(x+iy))-J_{N,1}]dx\right|\leq 2\delta\times(2\sqrt{t_0^2-|y|^2}).$$

Due to Cauchy-Schwartz inequality,

$$\left|\int_{\{|x-x_0|\leq\sqrt{t_0^2-|y|^2}\}\bigcap\mathbb{B}_y}[u_N(e(x+iy))-J_{N,1}]dx\right|$$
$$\leq\left(\int_0^1|u_N(e(x+iy))-J_{N,1}|^2dx\right)^{\frac12}(\text{mes }\mathbb{B}_y)^{\frac12}\leq C_7\exp(-\frac{\check{c}}{2}\delta^2N).$$

Set $t_0=\frac{\delta}{4e\pi C'_\rho}$, then

$$\begin{aligned}u_N(e(x))-J_{N,1}&\leq\frac{1}{\pi t_0^2}\int_{|y|\leq t_0}[C_7\exp(-\frac{\check{c}}{2}\delta^2N)+2\delta\times(2\sqrt{t_0^2-|y|^2})]dy\\&\leq\frac{1}{\pi t_0^2}\times C_7\exp(-\frac{\check{c}}{2}\delta^2N)\times(2t_0)+2\delta\\&=\frac{8eC_7C'_\rho}{\delta}\exp(-\frac{\check{c}}{2}\delta^2N)+2\delta.\end{aligned}$$

Set $\delta=(\frac{C_8\log N}{N})^{\frac12}$, where $C_8>\frac{2}{\check{c}}$. Then $\exp(-\frac{\check{c}}{2}C_8\log N)<\frac1N$, and

$$u_N(e(x))\leq J_{N,1}+8eC_7C'_\rho\times(\frac{N}{C_8\log N})^{\frac12}\times\frac1N+2(\frac{C_8\log N}{N})^{\frac12}\leq J_{N,1}+C_6(\frac{\log N}{N})^{\frac12}.$$

□

LEMMA 2.20. — *For any $0\leq x\leq 1$ and any $N\geq\check{N}$, the following is true:*

$$\log\|\tilde{M}_N(e(x),E)\|\leq NL_N+C_6(N\log N)^{\frac12}-NF_N(x),$$

where

$$F_N(x)=\frac{1}{2N}\sum_{n=0}^{N-1}Q(x+n\omega),\qquad Q(x)=\log|q(e(x))q(e(x+\omega))|-2D.$$

Proof. — We have

(2.13)
$$\begin{aligned}
&\log\|\tilde{M}_{[1,N]}(e(x),E)\| \\
&\quad = \log\|T_{[1,N]}(e(x),E)\| - \frac{1}{2}\sum_{n=0}^{N-1}\log|q(e(x+n\omega))q(e(x+(n+1)\omega))| \\
&\quad \le NJ_N(E) + C_6(N\log N)^{\frac{1}{2}} - \frac{1}{2}\sum_{n=0}^{N-1}\log|q(e(x+n\omega))q(e(x+(n+1)\omega))|.
\end{aligned}$$

Recall that

$$J_N(E) = L_N(E) + D.$$

Then due to (2.13) we have

$$\log\|\tilde{M}_N(e(x),E)\| \le NL_N + C_6(N\log N)^{\frac{1}{2}} - NF_N(x). \qquad \square$$

Remark 2.21. — Note that Lemma 2.20 implies in particular that

$$NL_N + C_6(N\log N)^{\frac{1}{2}} - NF_N(x) \ge 0$$

for any x for large N.

Lemma 2.22. — *For any $0 \le x \le 1$ and any $k \ge \check{N}$, the following holds*

$$\begin{aligned}
&\Big|\log\|\tilde{M}_N(e(x+k\omega),E)\| - \log\|\tilde{M}_N(e(x))\|\Big| \\
&\qquad \le 2kL_k(E) + 2C_6(N\log N)^{\frac{1}{2}} - kF_k(x) - kF_k(x+N\omega).
\end{aligned}$$

Proof. — One has

$$\tilde{M}_N(e(x+k\omega),E)\tilde{M}_k(e(x),E) = \tilde{M}_k(e(x+N\omega),E)\tilde{M}_N(e(x),E).$$

Then

$$\begin{aligned}
&\Big|\log\|\tilde{M}_N(e(x+k\omega),E)\| - \log\|\tilde{M}_N(e(x))\|\Big| \\
&\qquad \le \log\|\tilde{M}_k(e(x),E)\| + \log\|\tilde{M}_k(x(x+N\omega),E)\|,
\end{aligned}$$

because $\|A^{-1}\| = \|A\| \ge 1$ if $\det A = 1$. Due to Lemma 2.20,

$$\begin{aligned}
&\log\|\tilde{M}_k(e(x),E)\| + \log\|\tilde{M}_k(e(x+N\omega),E)\| \\
&\qquad \le 2kL_k(E) + 2C_6(N\log N)^{\frac{1}{2}} - kF_k(x) - kF_k(x+N\omega). \quad \square
\end{aligned}$$

REMARK 2.23. — Due to Lemma 2.1,

$$u_N(e(x), E) \le \log C(p,q)$$

for any $x \in \mathbb{T}$, any N and any E. Similarly,

$$\Big|\log \|\tilde{M}_N(e(x+k\omega), E)\| - \log \|\tilde{M}_N(e(x))\|\Big| \\ \le 2k(\log C(p,q) - D) - kF_k(x) - kF_k(x+N\omega)$$

for any $x \in \mathbb{T}$, any N, any k and any E.

3. Using the avalanche principle

PROPOSITION 3.1. — *Let $A_1, \dots, A_n$ be a sequence of 2×2--matrices whose determinants satisfy*

$$\max_{1 \le j \le n} |\det A_j| \le 1. \tag{3.1}$$

Suppose that

$$\min_{1 \le j \le n} \|A_j\| \ge \mu > n \quad \textit{and} \tag{3.2}$$

$$\max_{1 \le j < n} [\log \|A_{j+1}\| + \log \|A_j\| - \log \|A_{j+1}A_j\|] < \frac{1}{2}\log \mu. \tag{3.3}$$

Then

$$\Big|\log \|A_n \cdot \ldots \cdot A_1\| + \sum_{j=2}^{n-1} \log \|A_j\| - \sum_{j=1}^{n-1} \log \|A_{j+1}A_j\|\Big| < C\frac{n}{\mu} \tag{3.4}$$

with some absolute constant C.

Proof. — See [3]. □

REMARK 3.2. — For the rest of the paper, we do not use $e(x+iy)$ with $y \ne 0$. For that reason we write x instead of $e(x)$ in all expressions. Moreover, unless specified otherwise, $N \ge \check{N}$ and $N \ge \check{K}$ from now on (δ in $\check{K}$ will be defined in Lemma 3.8).

LEMMA 3.3. — *Let $\tilde{c}$ be as in (2.10). Let $L_N(E) > 100\delta > 0$, where $\delta < 1$ is a constant not depending on N, and $L_{2N}(E) > \frac{9}{10}L_N(E)$. Let $N' = mN$, $m \in \mathbb{N}$ and $m \le \exp(\frac{\tilde{c}}{4}\delta^2 N)$. Then*

$$|L_{N'}(E) + L_N(E) - 2L_{2N}(E)| \le \exp(-\tilde{c}'\delta^2 N) + \frac{2}{9m}L_N(E),$$

where $\tilde{c}' = \tilde{c}'(p,q,\omega)$. If $\exp(\frac{\tilde{c}}{10}\delta^2 N) \le m \le \exp(\frac{\tilde{c}}{4}\delta^2 N)$, we have

$$|L_{N'}(E) + L_N(E) - 2L_{2N}(E)| \le \exp(-\hat{c}\delta^2 N), \tag{3.5}$$

where $\hat{c} = \hat{c}(p,q,\omega)$.

Proof. — By (2.10), we have, for $0 \leq j \leq m-1$,

$$|\tilde{u}_N(x+jN\omega, E) - L_N(E)| < \delta,$$
$$|\tilde{u}_{2N}(x+jN\omega, E) - L_{2N}(E)| < \delta$$

for $x \in \mathbb{G}_1$, with

$$\operatorname{mes}(\mathbb{T}\backslash\mathbb{G}_1) \leq 2m \times \exp(-\tilde{c}\delta^2 N) < \exp(-\frac{2\tilde{c}}{3}\delta^2 N).$$

Thus when $x \in \mathbb{G}_1$,

$$\|\tilde{M}_N(x+jN\omega, E)\| > \exp(N(L_N(E)-\delta)) > \exp(\frac{99}{100}NL_N(E)),$$

and

$$\begin{aligned}\Big|\log\|\tilde{M}_N(x+jN\omega, E)\| &+ \log\|\tilde{M}_N(x+(j+1)N\omega, E)\| \\ &- \log\|\tilde{M}_N(x+jN\omega, E)\tilde{M}_N(x+(j+1)N\omega, E)\|\Big| \\ &< 4N\delta + 2N|L_N(E) - L_{2N}(E)| < \frac{6}{25}NL_N(E).\end{aligned} \tag{3.6}$$

Since $0 \leq L_N(E) - L_{2N}(E) < \frac{1}{10}L_N(E)$, we have

$$\tilde{M}_{N'}(x, E) = \prod_{j=m}^{1} \tilde{M}_N(x+(j-1)N\omega, E).$$

The avalanche principle applies for $\mu = \exp(\frac{1}{2}NL_N(E))$. Integrating over $\mathbb{G}_1$, we obtain

$$\begin{aligned}\Big|\int_{\mathbb{G}_1} \tilde{u}_{N'}(x, E)dx &+ \int_{\mathbb{G}_1}\sum_{j=2}^{m-1} \tilde{u}_N(x+(j-1)N\omega, E)dx \\ &- \int_{\mathbb{G}_1}\sum_{j=1}^{m-1} \tilde{u}_{2N}(x+(j-1)N\omega, E)dx\Big| \leq C\frac{m}{N'}\exp(-\frac{1}{2}NL_N(E)),\end{aligned} \tag{3.7}$$

where $N' = m \times N$. We want to replace integration over $\mathbb{G}_1$ by integration over $\mathbb{T}$. Recall that, due to (4) in Remark 1.1,

$$\int_{\mathbb{T}} \tilde{u}_n^2(E)dx \leq \tilde{C}(p, q)$$

for any n and any E. Hence, by Cauchy-Schwartz inequality,

$$\Big|\int_{\mathbb{B}} \tilde{u}_K(E)dx\Big| \leq \tilde{C}(p, q)^{\frac{1}{2}}(\operatorname{mes}\mathbb{B})^{\frac{1}{2}} \tag{3.8}$$

for any K, any E and any $\mathbb{B} \subseteq \mathbb{T}$. Hence

$$\Big|\int_{\mathbb{T}\backslash\mathbb{G}_1} \tilde{u}_K(E)dx\Big| \leq \tilde{C}(p, q)^{\frac{1}{2}}\exp(-\frac{\tilde{c}}{3}\delta^2 N)$$

for any K and any E. Thus

$$\begin{aligned}(3.9)\quad \Big|\int_{\mathbb{T}\setminus\mathbb{G}_1} \tilde{u}_{N'}(x,E)dx + \frac{1}{m}\int_{\mathbb{T}\setminus\mathbb{G}_1} \sum_{j=2}^{m-1} \tilde{u}_N(x+(j-1)N\omega,E)dx \\ - \frac{2}{m}\int_{\mathbb{T}\setminus\mathbb{G}_1} \sum_{j=1}^{m-1} \tilde{u}_{2N}(x+(j-1)N\omega,E)dx\Big| \le 4\tilde{C}(p,q)^{\frac12}\exp(-\frac{\tilde{c}}{3}\delta^2 N).\end{aligned}$$

Combining (3.7) with (3.9), we have

$$\begin{aligned}&|L_{N'}(E) + \frac{m-2}{m}L_N(E) - \frac{2(m-1)}{m}L_{2N}(E)| \\ &\quad \le 4\tilde{C}(p,q)^{\frac12}\exp(-\frac{\tilde{c}}{3}\delta^2 N) + C\frac{m}{N'}\exp(-\frac12 N L_N(E)) \le \exp(-\tilde{c}'\delta^2 N).\end{aligned}$$

Thus

(3.10)

$$\begin{aligned}|L_{N'}(E) + L_N(E) - 2L_{2N}(E)| &\le \exp(-\tilde{c}'\delta^2 N) + \frac{2}{m}|L_N(E) - L_{2N}(E)| \\ &< \exp(-\tilde{c}'\delta^2 N) + \frac{2}{m}\times\frac{1}{10}L_N(E) \\ &\le \exp(-\tilde{c}'\delta^2 N) + \frac{1}{45m}C''(p,q),\end{aligned}$$

where $C''(p,q)$ is the same as in (4) of Remark 1.1. If $\exp(\frac{\tilde{c}}{10}\delta^2 N) \le m$, then

$$(3.11)\qquad |L_{N'}(E) + L_N(E) - 2L_{2N}(E)| \le \exp(-\hat{c}\delta^2 N). \qquad \square$$

Now, we can prove

LEMMA 3.4. — *Let $\tilde{c}$ be as in (2.10), $\hat{c}$ be as in Lemma 3.3. Assume that $L_{N_0}(E) > 100\delta > 0$ and $\exp(-\hat{c}\delta^2 N_0) \le \frac{\delta}{12}$, where $\delta < 1$ is a constant not depending on N_0, and $L_{2N_0}(E) > \frac{9}{10}L_{N_0}(E)$. There exists $\tilde{N}_0 = \tilde{N}_0(p,q,\delta,N_0) \le (\exp(\frac{\tilde{c}}{8}\delta^2 N_0)+1)N_0$ such that for any $N \ge \tilde{N}_0$, the following holds:*

$$|L_N(E) + L_{N_0}(E) - 2L_{2N_0}(E)| < \exp(-\bar{c}'\delta^2 N_0),$$

where $\bar{c}' = \bar{c}'(p,q,\omega)$. Furthermore,

$$(3.12)\qquad |L(E) + L_{N_0}(E) - 2L_{2N_0}(E)| < \exp(-\bar{c}\delta^2 N_0),$$

where $\bar{c} = \bar{c}(p,q,\omega)$.

Proof. — We first prove the second part. By Lemma 3.3 for $N_1' = mN_0$, $\exp(\frac{\tilde{c}}{8}\delta^2 N_0) \le m < \exp(\frac{\tilde{c}}{8}\delta^2 N_0)+1$, we have

$$(3.13)\qquad |L_{N_1'}(E) + L_{N_0}(E) - 2L_{2N_0}(E)| < \exp(-\hat{c}\delta^2 N_0)$$

and

$$|L_{2N_1'}(E) + L_{N_0}(E) - 2L_{2N_0}(E)| < \exp(-\hat{c}\delta^2 N_0).$$

In particular

$$|L_{N_1'}(E) - L_{2N_1'}(E)| < 2\exp(-\hat{c}\delta^2 N_0).$$

Since $0 \leq L_{N_0}(E) - L_{2N_0}(E) < \frac{1}{10}L_{N_0}(E)$, we obtain using (3.13) that

$$\begin{aligned} L_{N_1'}(E) &> L_{N_0}(E) - 2(L_{N_0}(E) - L_{2N_0}(E)) - \exp(-\hat{c}\delta^2 N_0) \\ &> \frac{4}{5}L_{N_0}(E) - \exp(-\hat{c}\delta^2 N_0) > 79\delta, \end{aligned}$$

and

$$|L_{N_1'}(E) - L_{2N_1'}(E)| \leq 2\exp(-\hat{c}\delta^2 N_0) < 2\delta < \frac{2}{79}L_{N_1'}(E) < \frac{1}{10}L_{N_1'}(E).$$

Set $\delta' = \frac{1}{2}\delta$, then $L_{N_1'}(E) > 100\delta'$, and Lemma 3.3 applies for $N_2' = m_1 N_1'$, $\exp(\frac{\tilde{c}}{8}\delta'^2 N_1') \leq m_1 < \exp(\frac{\tilde{c}}{8}\delta'^2 N_1') + 1$,

$$|L_{N_2'}(E) + L_{N_1'}(E) - 2L_{2N_1'}(E)| \leq \exp(-\hat{c}\delta'^2 N_1').$$

Also

$$\begin{aligned} L_{N_2'}(E) &> L_{N_1'}(E) - 2|L_{N_1'}(E) - L_{2N_1'}(E)| - \exp(-\hat{c}\delta'^2 N_1) \\ &\qquad > \frac{4}{5}L_{N_0}(E) - 6\exp(-\hat{c}\delta^2 N_0) > 79\delta > 100\delta', \end{aligned}$$

$$|L_{2N_2'}(E) + L_{N_1'}(E) - 2L_{2N_1'}(E)| \leq \exp(-\hat{c}\delta'^2 N_1'),$$

$$|L_{N_2'}(E) - L_{2N_2'}(E)| < 2\exp(-\hat{c}\delta'^2 N_1').$$

Since $N_1' > 8N_0$ we have

$$\exp(-\hat{c}\delta'^2 N_1') = \exp(-\hat{c}\frac{\delta^2}{4}N_1') < (\exp(-\hat{c}\delta^2 N_0))^2 < (\frac{\delta}{12})^2.$$

This implies in particular that

$$|L_{N_2'}(E) - L_{2N_2'}(E)| < 2\exp(-\hat{c}\delta'^2 N_1') < 2\delta < \frac{1}{10}L_{N_2'}(E).$$

Then Lemma 3.3 applies for $N_3' = m_2 N_2'$, $\exp(\frac{\tilde{c}}{8}\delta'^2 N_2') \leq m_2 < \exp(\frac{\tilde{c}}{8}\delta'^2 N_2') + 1$. E.T.C. We obtain $N_{i+1}' = m_i N_i'$, $\exp(\frac{\tilde{c}}{8}\delta'^2 N_i') \leq m_i < \exp(\frac{\tilde{c}}{8}\delta'^2 N_i') + 1$ with the same δ'. Then

$$|L_{N_{i+1}'}(E) + L_{N_i'}(E) - 2L_{2N_i'}(E)| \leq \exp(-\hat{c}\delta'^2 N_i'),$$

$$\begin{aligned} L_{N_{i+1}'}(E) &> L_{N_i'}(E) - 2|L_{N_i'}(E) - L_{2N_i'}(E)| - \exp(-\hat{c}\delta'^2 N_i') \\ &\qquad > \frac{4}{5}L_{N_0}(E) - \sum_{j=1}^{i}(\frac{1}{2})^j\delta \geq 79\delta > 50\delta = 100\delta', \end{aligned}$$

$$|L_{2N_{i+1}'}(E) + L_{N_i'}(E) - 2L_{2N_i'}(E)| \leq \exp(-\hat{c}\delta'^2 N_i'),$$

$$|L_{N_{i+1}'}(E) - L_{2N_{i+1}'}(E)| < 2\exp(-\hat{c}\delta'^2 N_i'),$$

$$4\exp(-\hat{c}\delta'^2 N_i') < (\frac{1}{2})^{i+1}\delta,$$

and

$$|L_{N_{i+1}'}(E) - L_{2N_{i+1}'}(E)| < 2\delta < \frac{1}{10}L_{N_{i+1}'}(E).$$

Moreover,

$$\begin{aligned}(3.14)\quad & |L_{N_{i+1}'}(E) - L_{N_i'}(E)| \\ &\le |L_{N_{i+1}'}(E) + L_{N_i'}(E) - 2L_{2N_i'}(E)| + 2|L_{N_i'}(E) - L_{2N_i'}(E)| \\ &< \exp(-\hat{c}\delta'^2 N_i') + 4\exp(-\hat{c}\delta'^2 N_{i-1}') < 5\exp(-\hat{c}\delta'^2 N_{i-1}'), \quad i \ge 2\end{aligned}$$

$$|L_{N_2'}(E) - L_{N_1'}(E)| < 5\exp(-\hat{c}\delta^2 N_0).$$

Since $L_{N_i'} \to L(E)$ with $i \to \infty$, we have

$$\begin{aligned}(3.15)\quad & |L(E)+L_{N_0}(E) - 2L_{2N_0}(E)| \\ &= \Big|\sum_{i\ge1}(L_{N_{i+1}'}(E) - L_{N_i'}(E)) + L_{N_1'}(E) + L_{N_0}(E) - 2L_{2N_0}(E)\Big| \\ &\le \sum_{s\ge1}|L_{N_{s+1}'}(E) - L_{N_s'}(E)| + |L_{N_1'}(E) + L_{N_0}(E) - 2L_{2N_0}(E)| \\ &= \sum_{s\ge2}|L_{N_{s+1}'}(E) - L_{N_s'}(E)| + |L_{N_2'}(E) - L_{N_1'}(E)| \\ &\quad + |L_{N_1'}(E) + L_{N_0}(E) - 2L_{2N_0}(E)| \\ &< \sum_{s\ge2} 5\exp(-\hat{c}\delta'^2 N_{i-1}') + 5\exp(-\hat{c}\delta^2 N_0) + \exp(-\hat{c}\delta^2 N_0) \\ &< \exp(-\bar{c}\delta^2 N_0).\end{aligned}$$

This proves part two. We prove now the first part. Note that just as in (3.15), we obtain

$$(3.16)\qquad |L_{N_i'}(E) + L_{N_0}(E) - 2L_{N_0}(E)| \le \exp(-\bar{c}\delta^2 N_0)$$

for $i \ge 1$. Let $N \ge \tilde{N}_0 := N_1'$ be arbitrary. Find i such that $N_i' \le N < N_{i+1}'$. Recall that

$$N_i' = m_{i-1}N_{i-1}',\ \exp(\frac{\tilde{c}}{8}\delta'^2 N_{i-1}') \le m_{i-1} < \exp(\frac{\tilde{c}}{8}\delta'^2 N_{i-1}') + 1,$$

$$N_{i+1}' = m_i N_i',\ \exp(\frac{\tilde{c}}{8}\delta'^2 N_i') \le m_i < \exp(\frac{\tilde{c}}{8}\delta'^2 N_i') + 1,$$

where $N_0' := N_0$ for convenience. Consider two cases:

(1) $N \le \exp(\frac{\tilde{c}}{4}\delta'^2 N'_{i-1})N'_{i-1}$. In this case, $\frac{N'_{i-1}}{N} \le \frac{N'_{i-1}}{N'_i} \le \exp(-\frac{\tilde{c}}{8}\delta'^2 N'_{i-1})$. Then find $\tilde{m}$, $m_{i-1} \le \tilde{m} \le \exp(\frac{\tilde{c}}{4}\delta'^2 N'_{i-1})$, such that

$$\tilde{m}N'_{i-1} \le N < (\tilde{m}+1)N'_{i-1}.$$

Then by Lemma 3.3,

$$|L_{\tilde{m}N'_{i-1}}(E) + L_{N'_{i-1}}(E) - 2L_{2N'_{i-1}}(E)| < \exp(-\hat{c}\delta'^2 N'_{i-1}). \tag{3.17}$$

Note that

$$N - \tilde{m}N'_{i-1} \le N'_{i-1},$$

and by Remark 2.23,

$$\begin{aligned}\Big|\log\|\tilde{M}_N(x,E)\| - \log\|\tilde{M}_{\tilde{m}N'_{i-1}}(x,E)\|\Big| &\le \log\|\tilde{M}_{N-\tilde{m}N'_{i-1}}(x+\tilde{m}N'_{i-1}\omega,E)\| \\ &\le N'_{i-1}(\log C(p,q) - D) - (N-\tilde{m}N'_{i-1})F_{N-\tilde{m}N'_{i-1}}(x+\tilde{m}N'_{i-1}\omega).\end{aligned}$$

We know that

$$\operatorname{mes}(\{x : |kF_k(x) - k < F_k(x) > | > k\delta\}) < \exp(-c\delta k)$$

for any k. Since $< F_k >= 0$, we have

$$\begin{aligned}\Big|\log\|\tilde{M}_N(x,E)\| - \log\|\tilde{M}_{\tilde{m}N'_{i-1}}(x,E)\|\Big| &\le N'_{i-1}(\log C(p,q) - D) + N'_{i-1} \\ &< \hat{C}'(p,q)N'_{i-1},\end{aligned} \tag{3.18}$$

if $x \notin \hat{\mathbb{B}}$, $\operatorname{mes}\hat{\mathbb{B}} < \exp(-cN'_{i-1})$. Integrating (3.18) over $\mathbb{T}\backslash\hat{\mathbb{B}}$ and using (3.8), we obtain

$$\begin{aligned}|L_N(E) - \frac{\tilde{m}N'_{i-1}}{N}L_{\tilde{m}N'_{i-1}}(E)| & \\ &< \hat{C}'(p,q)\frac{N'_{i-1}}{N} + 2\tilde{C}(p,q)^{\frac{1}{2}} \times \exp(-\frac{c}{2}N'_{i-1}) \\ &\le \hat{C}'(p,q)\exp(-\frac{\tilde{c}}{8}\delta'^2 N'_{i-1}) + 2\tilde{C}(p,q)^{\frac{1}{2}} \times \exp(-\frac{c}{2}N'_{i-1}) \\ &\le \exp(-\tilde{c}_1\delta'^2 N'_{i-1}).\end{aligned} \tag{3.19}$$

Note that

$$1 - \frac{\tilde{m}N'_{i-1}}{N} = \frac{N - \tilde{m}N'_{i-1}}{N} \le \frac{N'_{i-1}}{N} \le \exp(-\frac{\tilde{c}}{8}\delta'^2 N'_{i-1}).$$

Thus,

$$|L_N(E) - L_{\tilde{m}N'_{i-1}}(E)| \le \exp(-\tilde{c}_2\delta'^2 N'_{i-1}). \tag{3.20}$$

Combining (3.17) with (3.20), we obtain

$$|L_N(E) + L_{N'_{i-1}}(E) - 2L_{2N'_{i-1}}(E)| < \exp(-\tilde{c}_3\delta^2 N'_{i-1}). \tag{3.21}$$

(2) $N > \exp(\frac{\tilde{c}}{4}\delta'^2 N'_{i-1})N'_{i-1}$. Find $\tilde{m}'$ such that

$$\tilde{m}' N'_i \leq N < (\tilde{m}'+1)N'_i.$$

Thus,

$$\tilde{m}' > \frac{\exp(\frac{\tilde{c}}{4}\delta'^2 N'_{i-1})}{\exp(\frac{\tilde{c}}{8}\delta'^2 N'_{i-1})+1} < \frac{\exp(\frac{\tilde{c}}{4}\delta'^2 N'_{i-1})}{2\exp(\frac{\tilde{c}}{8}\delta'^2 N'_{i-1})} = \frac{1}{2}\exp(\frac{\tilde{c}}{8}\delta'^2 N'_{i-1}).$$

Since $N < N'_{i+1}$, then $\tilde{m}' < m_i$. It implies, due to Lemma 3.4, that

$$|L_{\tilde{m}'N'_i}(E) + L_{N'_i}(E) - 2L_{2N'_i}(E)| < \exp(-\tilde{c}'\delta'^2 N'_i) + \frac{2}{9\tilde{m}'}L_{N'_i}(E). \tag{3.22}$$

As in Case (1), we have

$$|L_N(E) - L_{\tilde{m}N'_{i-1}}(E)| < \frac{\hat{C}''(p,q)}{\tilde{m}'} \leq \exp(-\tilde{c}_4\delta'^2 N'_{i-1}). \tag{3.23}$$

Combining (3.22) with (3.23), we obtain

$$|L_N(E) + L_{N'_i}(E) - 2L_{2N'_i}(E)| < \exp(-\tilde{c}_5\delta'^2 N'_{i-1}). \tag{3.24}$$

Combining (3.16) with (3.21) or (3.24), as in (3.15), we obtain

$$|L_N(E) + L_{N_0}(E) - 2L_{N_0}(E)| < \exp(-\overline{c}'\delta^2 N_0),$$

where $\overline{c}' = \overline{c}'(p,q,\omega)$. $\square$

LEMMA 3.5. — *Assume $L(E_0) > 0$. There exists $\check{C}(p,q,E_0)$ such that with $\rho'_0(E_0,N) = \frac{L(E_0)}{200}\exp(-\check{C}(p,q,E_0)N)$, we have*

$$|L_N(E_0) - L_N(E)| < \frac{L(E_0)}{100},$$

for any $|E-E_0| < \rho'_0(E_0,N)$ and any N.

Proof. — Note that

(3.25)
$$\begin{aligned}
&\Big|\|T_N(x,E_0)\| - \|T_N(x,E)\|\Big| \leq \|T_N(x,E_0) - T_N(x,E)\| \\
&\leq \sum_{j=0}^{N-1}(\|B(x+(N-1)\omega,E_0)\times\cdots\times B(x+(j+1)\omega,E_0)\| \\
&\quad \times\|B(x+j\omega,E_0) - B(x+j\omega,E)\| \times \|B(x+(j-1)\omega,E)\times\cdots\times B(x,E)\|) \\
&\leq NC(p,q)^{N-1}|E_0 - E|;
\end{aligned}$$

see (1) in Lemma 2.1. By (1.5), we have

$$\begin{aligned}
&\Big|\|\tilde{M}_N(x,E_0)\| - \|\tilde{M}_N(x,E)\|\Big| \\
&= \prod_{n=0}^{N-1} \frac{1}{|q(x+n\omega)q(x+(n+1)\omega)|^{\frac{1}{2}}} \big|\|T_N(x,E_0)\| - \|T_N(x,E)\|\big| \\
&\leq \frac{NC(p,q)^{N-1}|E_0-E|}{\prod_{n=0}^{N-1}|q(x+n\omega)q(x+(n+1)\omega)|^{\frac{1}{2}}}.
\end{aligned} \tag{3.26}$$

Assume, for instance, that $\|\tilde{M}_N(x,E_0)\| \geq \|\tilde{M}_N(x,E)\|$. Then

$$\begin{aligned}
&\big|\log\|\tilde{M}_N(x,E_0)\| - \log\|\tilde{M}_N(x,E)\|\big| \\
&= \log\frac{\|\tilde{M}_N(x,E_0)\|}{\|\tilde{M}_N(x,E)\|} = \log\Big(1+\frac{\|\tilde{M}_N(x,E_0)\| - \|\tilde{M}_N(x,E)\|}{\|\tilde{M}_N(x,E)\|}\Big) \\
&\leq \frac{\|\tilde{M}_N(x,E_0)\| - \|\tilde{M}_N(x,E)\|}{\|\tilde{M}_N(x,E)\|} \leq \|\tilde{M}_N(x,E_0)\| - \|\tilde{M}_N(x,E)\| \\
&\leq \frac{NC(p,q)^{N-1}|E_0-E|}{\prod_{n=0}^{N-1}|q(x+n\omega)q(x+(n+1)\omega)|^{\frac{1}{2}}}.
\end{aligned} \tag{3.27}$$

Due to (1) in Remark 2.13, for any δ and any K,

$$\begin{aligned}
\operatorname{mes}\Big(\Big\{x : \Big|\sum_{k=1}^{K}\frac{1}{2}\log|q(x+k\omega)q(x+(k+1)\omega)| - K{<}\frac{1}{2}\log|q(x)q(x+\omega)|{>}\Big| \\
> \delta K\Big\}\Big) < \exp(-c_q\delta K).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Big|\sum_{k=0}^{N-1}\frac{1}{2}\log|q(x+k\omega)q(x+(k+1)\omega)|\Big| \\
&< N\Big|<\frac{1}{2}\log|q(x)q(x+\omega)|>\Big| + \frac{800\tilde{C}(p,q)^{\frac{1}{2}}}{L(E_0)c_q}N \\
&= |D|N + \frac{800\tilde{C}(p,q)^{\frac{1}{2}}}{L(E_0)c_q}N = \hat{C}(q,p,E_0)N,
\end{aligned} \tag{3.28}$$

if $x \notin \mathbb{B}_1$, $\operatorname{mes}\mathbb{B}_1 < \exp(-c_q \times \frac{800\tilde{C}(p,q)^{\frac{1}{2}}}{L(E_0)c_q}N) = \exp(-\frac{800\tilde{C}(p,q)^{\frac{1}{2}}}{L(E_0)}N)$, where the constant $\tilde{C}(p,q)$ comes from (4) in Remark 1.1. The same estimate holds if

$\|\tilde{M}_N(x,E_0)\| \le \|\tilde{M}_N(x,E)\|$. So

$$\begin{aligned} &\Big|\log\|\tilde{M}_N(x,E_0)\| - \log\|\tilde{M}_N(x,E)\|\Big| \\ &\qquad \le NC(p,q)^{N-1}|E_0-E|\exp(\hat{C}(p,q,E_0)N) \\ &\qquad \le \exp(\check{C}(p,q,E_0)N)|E_0-E|, \end{aligned} \tag{3.29}$$

if $x \notin \mathbb{B}_1$, mes $\mathbb{B}_1 < \exp(-\frac{800\tilde{C}(p,q)^{\frac{1}{2}}}{L(E_0)}N)$. Set $\rho_0' = \frac{L(E_0)}{200}\exp(-\check{C}(p,q,E_0)N)$. Then, if $|E-E_0| \le \rho_0'$,

$$\Big|\log\|\tilde{M}_N(x,E_0)\| - \log\|\tilde{M}_N(x,E)\|\Big| < \frac{L(E_0)}{200},$$

if $x \notin \mathbb{B}_1$, mes $\mathbb{B}_1 < \exp(-\frac{800\tilde{C}(p,q)^{\frac{1}{2}}}{L(E_0)}N)$,

$$\left|\int_{\mathbb{T}\setminus\mathbb{B}_1}\log\|\tilde{M}_N(x,E_0)\| - \int_{\mathbb{T}\setminus\mathbb{B}_1}\log\|\tilde{M}_N(x,E)\|\right| < \frac{L(E_0)}{200}. \tag{3.30}$$

Due to (3.8),

$$\Big|\int_{\mathbb{B}_1}\tilde{u}_N dx\Big| \le \tilde{C}(p,q)^{\frac{1}{2}}\exp(-\frac{400\tilde{C}(p,q)^{\frac{1}{2}}}{L(E_0)}N)$$

for E or E_0. As $y\exp(-\xi y) \le \xi^{-1}$ for any $y,\xi > 0$. Thus,

$$\Big|\int_{\mathbb{B}_1}\tilde{u}_N dx\Big| \le \frac{L(E_0)}{400N} \le \frac{L(E_0)}{400} \tag{3.31}$$

for E or E_0. Combining (3.30) with (3.31), we have

$$|L_N(E_0) - L_N(E)| < \frac{L(E_0)}{200N} + 2\frac{L(E_0)}{400} \le \frac{L(E_0)}{100}. \qquad \square$$

Lemma 3.6. — *Assume $L(E_0) > 0$. There exists $\rho_0' = \rho_0'(p,q,E_0,\omega) > 0$ and $\tilde{N}_0 = \tilde{N}_0(p,q,E_0,\omega) < +\infty$ such that for any $N \ge \tilde{N}_0$ and any $|E-E_0| < \rho_0'$*

$$|L_N(E) - L(E)| < \frac{1}{20}L(E), \quad \frac{11}{10}L(E_0) > L(E) > \frac{9}{10}L(E_0).$$

Proof. — One has $\lim_{n\to\infty} L(E_0) = L(E_0)$. Therefore, there exists $N_0 = N_0(p,q,\omega,E_0)$ s.t. $|L_n(E_0) - L(E_0)| < \frac{L(E_0)}{100}$ for $n \ge N_0(p,q,\omega,E_0)$. It implies that $L_{N_0}(E_0) - L_{2N_0}(E_0) < \frac{L(E_0)}{100}$, as $L(E_0) \le L_{2N_0}(E_0) \le L_{N_0}(E_0)$. Set $\delta = \min(\frac{1}{200}L(E_0), \frac{1}{2})$. We can assume also that $\exp(-\hat{c}\delta^2 N_0) \le \frac{\delta}{12}$, $\exp(-\bar{c}\delta^2 N_0) < \frac{1}{50}L(E_0)$ and $\exp(-\bar{c}'\delta^2 N_0) < \frac{1}{50}L(E_0)$, where $\hat{c}$ is as in

Lemma 3.3, $\bar{c}$ and $\bar{c}'$ are as in Lemma 3.4. Using Lemma 3.5 applied to N_0 and $2N_0$, we have for $|E-E_0|<\rho_0'(E_0,2N_0)$,

$$L_{N_0}(E) \geq L(E_0) - |L_{N_0}(E) - L_{N_0}(E_0)| - |L_{N_0}(E_0) - L(E_0)| > L(E_0) - \frac{L(E_0)}{100} - \frac{L(E_0)}{100} = \frac{49}{50}L(E_0), \tag{3.32}$$

and

$$\begin{aligned} |L_{N_0}(E) - L_{2N_0}(E)| &\leq |L_{N_0}(E) - L_{N_0}(E_0)| + |L_{N_0}(E_0) - L_{2N_0}(E_0)| \\ &\quad + |L_{2N_0}(E_0) - L_{2N_0}(E)| \\ &< \frac{L(E_0)}{100} + \frac{L(E_0)}{100} + \frac{L(E_0)}{100} = \frac{3}{100}L(E_0) < \frac{1}{10}L_{N_0}(E). \end{aligned} \tag{3.33}$$

Thus, Lemma 3.4 applies for $L_{N_0}(E)$, δ, N_0 and E, then there exists $\tilde{N}_0 = \tilde{N}_0(p,q,\delta,N_0) \leq (\exp(\frac{\tilde{c}}{8}\delta^2 N_0)+1)N_0$ such that for any $N \geq \tilde{N}_0$ the following is true:

$$|L_N(E) + L_{N_0}(E) - 2L_{2N_0}(E)| < \exp(-\bar{c}'\delta^2 N_0),$$

and

$$|L(E) + L_{N_0}(E) - 2L_{2N_0}(E)| < \exp(-\bar{c}\delta^2 N_0), \tag{3.34}$$

where $\bar{c}' = \bar{c}'(p,q,\omega)$ and $\bar{c} = \bar{c}(p,q,\omega)$ are as in Lemma 3.5. These imply

$$|L(E) - L_N(E)| \leq \exp(-\bar{c}'\delta^2 N_0) + \exp(-\bar{c}\delta^2 N_0) < \frac{1}{50}L(E_0) + \frac{1}{50}L(E_0) = \frac{1}{25}L(E_0). \tag{3.35}$$

Combining (3.32), (3.33) with (3.34), we obtain

$$\begin{aligned} |L(E_0) - L(E)| &\leq |L(E) + L_{\tilde{N}_0}(E) - 2L_{2\tilde{N}_0}(E)| \\ &\quad + |L(E_0) - L_{\tilde{N}_0}(E)| + 2|L_{\tilde{N}_0}(E) - L_{2\tilde{N}_0}(E)| \\ &< \frac{1}{50}L(E_0) + \frac{1}{50}L(E_0) + 2\frac{3}{100}L(E_0) = \frac{1}{10}L(E_0). \end{aligned} \tag{3.36}$$

It implies

$$\frac{11}{10}L(E_0) > L(E) > \frac{9}{10}L(E_0), \tag{3.37}$$

and

$$|L(E) - L_N(E)| < \frac{1}{25}L(E_0) < \frac{1}{25} \times \frac{10}{9}L(E) = \frac{2}{45}L(E) < \frac{1}{20}L(E). \qquad \square$$

Lemma 3.7. — *Assume $L(E_0) > 0$. Let ρ_0' be as in Lemma 3.6. Fix any $0 < \kappa < 1$. Let $K = \frac{\kappa}{20}N$. Then there exists N_1 s.t. $N \geq N_1$ and for any $E \in (E_0 - \rho_0', E_0 + \rho_0')$, the following holds:*

$$\mathrm{mes}\,\{x : |\tilde{u}_N(x,E) - \frac{1}{K}\sum_{k=1}^{K}\tilde{u}_N(x+k\omega,E)| > \kappa L(E)\} < \exp(-c''\kappa L(E)N),$$

where the constant c'' depends only on p,q,ω, but not on E_0 or E or κ. The number N_1 depends on p,q,ω,E_0 and κ.

Proof. — Choose $\bar{N}_0$ s.t.

$$\bar{N}_0 > \max(\frac{800(\log C(p,q) - D)}{\kappa^2 L(E_0)}, \frac{20C_6}{\kappa^2 L(E_0)}, 20, \frac{1}{\kappa}, \tilde{N}_0), \tag{3.38}$$

where $(\log C(p,q) - D)$ is as in Remark 2.23, C_6 is as in Lemma 2.19, $\tilde{N}_0$ is as in Lemma 3.6. Thus,

$$L_K(E) < (1 + \frac{1}{20})L(E), \tag{3.39}$$

for any $K \geq \bar{N}_0$ and any $E \in (E_0 - \rho_0', E_0 + \rho_0')$. Finally, we assume that

$$\log K < K^{\frac{1}{6}} \tag{3.40}$$

if $K \geq \bar{N}_0$. Using Lemma 2.22 and Remark 2.23, we obtain

$$\begin{aligned}
&|\tilde{u}_N(x,E) - \frac{1}{K}\sum_{k=1}^{K}\tilde{u}_N(x+k\omega,E)| \\
&\leq \frac{1}{KN}\big[\sum_{k=1}^{\bar{N}_0} 2k(\log C(p,q) - D) + \sum_{k=\bar{N}_0+1}^{K} 2kL_k(E) + \sum_{k=\bar{N}_0+1}^{K} 2C_6(k\log k)^{\frac{1}{2}}\big] \\
&\quad - \frac{1}{KN}\sum_{k=1}^{K}(kF_k(x) + kF_k(x+N\omega)) \\
&= (I) + (II).
\end{aligned}$$

Take $N \geq \bar{N}_0^3$, $K = \frac{\kappa}{20}N$. Note that $N_0\kappa > 1$, so $K = \frac{\kappa}{20}N > \frac{\bar{N}_0^2}{20} \geq \bar{N}_0$. Thus, (3.39) and (3.40) hold. We have

$$\begin{aligned}
(I) &< \frac{\bar{N}_0^2(\log C(p,q) - D)}{KN} + 4\frac{21}{20}L(E)\frac{K}{N} + + \frac{C_6 K^{\frac{1}{2}}(\log K)^{\frac{1}{2}}}{N} \\
&< \frac{\kappa L(E_0)}{20} + \frac{\kappa}{4}L(E) + \frac{\kappa L(E_0)}{20} < \frac{1}{2}\kappa L(E);
\end{aligned} \tag{3.41}$$

see (3.38), (3.39), (3.40) and Lemma 3.6. If

$$\sum_{k=1}^{K} kF_k(x,E) < -\frac{KN}{4}\kappa L(E),$$

then

$$\exists k,\ s.t.\ kF_k(x,E) < -\frac{N}{4}\kappa L(E).$$

We know that

$$\mathrm{mes}\,(\{x : |kF_k(x) - k < F_k(x) > | > k\delta\}) < \exp(-c\delta k).$$

Since $<F_k> = 0$,

$$\begin{aligned} &\mathrm{mes}\,(\{x : kF_k(x) < -\frac{N}{4}\kappa L(E)\}) \\ &\le \mathrm{mes}\,(\{x : |kF_k(x)| > \frac{N}{4}\kappa L(E)\}) < \exp(-c\frac{N\kappa L(E)}{4k}k) = \exp(-c_2\kappa N L(E)). \end{aligned}$$

So

$$\begin{aligned} &\mathrm{mes}\,\{x : \sum_{k=1}^{K} kF_k(x) < -\frac{KN}{4}L(E)\} \\ &\qquad \le K\exp(-c_2\kappa N L(E)) = K\exp(-c_2 20 K L(E)). \end{aligned}$$

Since $y\exp(-\xi y) \le \xi^{-1}$ for any ξ, $y > 0$, we have

(3.42)

$$\begin{aligned} \mathrm{mes}\,\{x : \sum_{k=1}^{K} kF_k(x) < -\frac{KN}{4}L(E)\} &\le K\exp(-c_2 20 K L(E)) \\ &= K\exp(-c_2 10 K L(E))\exp(-c_2 10 K L(E)) \\ &\le \frac{1}{10c_2 L(E)}\exp(-c_2 10 K L(E)) = \frac{1}{10c_2 L(E)}\exp(-\frac{c_2}{2}\kappa L(E) N) \\ &\le \exp(-c_2'\kappa L(E) N), \end{aligned}$$

if N is large enough depending on $L(E_0)$ and κ (see Lemma 3.6 and (3.38)). Combining (3.41) with (3.42), we have

$$\begin{aligned} &\mathrm{mes}\,\{x : |\tilde{u}_N(x,E) - \frac{1}{K}\sum_{k=1}^{K}\tilde{u}_N(x + j\omega, E)| > \kappa L(E)\} \\ &\qquad \le 2\exp(-c_2'\kappa N L(E)) < \exp(-c''\kappa L(E) N), \end{aligned}$$

where the constant c'' depends only on p, q, ω. □

LEMMA 3.8. — *Assume $L(E_0) > 0$. Let ρ_0' be as in Lemma 3.6, N_1 be as in Lemma 3.7 with $\kappa = \frac{1}{40}$. Then for $N \geq N_1$ and any $E \in (E_0 - \rho_0', E_0 + \rho_0')$, the following holds:*

$$\mathrm{mes}\,\{x : |\tilde{u}_N(x,E) - L(E)| > \frac{L(E)}{10}\} < \exp(-cL(E)N),$$

where the constant c depends only on p, q, ω, but not on E or E_0.

Proof. — Due to Remark 2.13 for $K > \check{K}$, the following holds:

$$\mathrm{mes}\,\{x : |\sum_{k=1}^{K} \tilde{u}_N(x+k\omega, E) - K < \tilde{u}_N(\cdot, E) > | > \delta K\} \leq \exp(-\hat{c}\delta K),$$

where $\hat{c} = \hat{c}(p,q,\omega)$. Set $\delta = \frac{L(E)}{40}$. Thus $\check{K} = \check{K}(p,q,E_0,\omega)$. Due to Lemma 3.6, $L_N(E) < (1+\frac{1}{20})L(E)$ if $N \geq \tilde{N}_0$, where $\tilde{N}_0$ is as in Lemma 3.6. Note that if

$$|\frac{1}{K}\sum_{k=1}^{K} \tilde{u}_N(x+k\omega, E) - < \tilde{u}_N(\cdot, E) > | \leq \delta,$$

then

$$\begin{aligned}|\frac{1}{K}\sum_{j=1}^{K} \tilde{u}_N(x+j\omega, E) - L(E)| &\leq |\frac{1}{K}\sum_{j=1}^{K} \tilde{u}_N(x+j\omega, E) - L_N(E)| \\ &\quad + |L_N(E) - L(E)| < \delta + \frac{1}{20}L(E) \\ &= \frac{1}{40}L(E) + \frac{1}{20}L(E) = \frac{3}{40}L(E).\end{aligned} \tag{3.43}$$

Therefore,

$$\mathrm{mes}\,\{x : |\frac{1}{K}\sum_{j=1}^{K} \tilde{u}_N(x+j\omega, E) - L(E)| > \frac{3}{40}L(E)\} < \exp(-\frac{\hat{c}}{40}L(E)K).$$

Let $K = \frac{N}{800}$ as in Lemma 3.7, with $\kappa = \frac{1}{40}$. Then for $N \geq N_1$, the following holds:

$$\mathrm{mes}\,\{x : |\tilde{u}_N(x,E) - \frac{1}{K}\sum_{k=1}^{K} \tilde{u}_N(x+k\omega, E)| > \frac{L(E)}{40}\} \leq \exp(-\frac{c''}{40}L(E)N).$$

Recall that $N_1 > \tilde{N}_0$ with $K = \frac{N}{800}$(see (3.38)). Let $N \geq N_1$. Then

$$\begin{aligned}&\mathrm{mes}\,\{x : |\tilde{u}_N(x,E) - L(E)| > \frac{L(E)}{10}\} \\ &\qquad < \exp(-\frac{\hat{c}}{32000}L(E)N) + \exp(-\frac{c''}{40}L(E)N) < \exp(-c_4L(E)N),\end{aligned}$$

where c_4 depends only on p, q, ω. Here, we replace c_4 with c for notational convenience. □

LEMMA 3.9. — *Assume $L(E_0) > 0$. Let ρ_0' be as in Lemma 3.6, N_1 be as in Lemma 3.7 with $\kappa = \frac{1}{40}$, c be as in Lemma 3.8. Let $N \geq N_1$ and $E \in (E_0 - \rho_0', E_0 + \rho_0')$ be arbitrary. Let $N' = mN, m \in \mathbb{N}$ and $\exp(\frac{c}{10}L(E)N) \leq m \leq \exp(\frac{c}{4}L(E)N)$. Then*

$$|L_{N'}(E) + L_N(E) - 2L_{2N}(E)| \leq \exp(-c_6 L(E)N), \tag{3.44}$$

where $c_6 = c_6(p, q, \omega)$.

Proof. — Let

$$\mathbb{G}_j = \{x : |\tilde{u}_N(x + j\omega N) - L(E)| \leq \frac{1}{10}L(E)\} \\ \cap \{x : |\tilde{u}_{2N}(x + j\omega N) - L(E)| \leq \frac{1}{10}L(E)\}.$$

By Lemma 3.8, $\operatorname{mes}(\mathbb{T}\backslash\mathbb{G}_j) \leq 2\exp(-cL(E)N)$ for any j. Set $\mathbb{G} = \bigcap_{0\leq j\leq m-1}\mathbb{G}_j$. Then $\operatorname{mes}(\mathbb{T}\backslash\mathbb{G}) \leq \exp(-\frac{2c}{3}L(E)N)$. We have

$$\|\tilde{M}_N(x + jN\omega, E)\| > \exp(\frac{9}{10}NL(E)),$$

and

$$\begin{aligned} &|\log\|\tilde{M}_N(x + jN\omega, E)\| + \log\|\tilde{M}_N(x + (j+1)N\omega, E)\| \\ &\quad - \log\|\tilde{M}_N(x + jN\omega, E)\tilde{M}_N(x + (j+1)N\omega, E)\|| \\ &\qquad < \frac{1}{10}NL(E) + \frac{1}{10}NL(E) + \frac{1}{10}2NL(E) = \frac{2}{5}NL(E), \end{aligned} \tag{3.45}$$

for any $x \in \mathbb{G}$, $0 \leq j \leq m$. We have

$$\tilde{M}_{N'}(x, E) = \prod_{j=m}^{1} \tilde{M}_N(x + (j-1)N\omega, E).$$

If $x \in \mathbb{G}$, the avalanche principle applies with $\mu = \exp(\frac{9}{10}NL(E))$. So

$$\Big|\log\|\tilde{M}_{N'}(x, E)\| + \sum_{j=2}^{m-1}\log\|\tilde{M}_N(x + (j-1)N\omega, E)\| \\ - \sum_{j=1}^{m-1}\log\|\tilde{M}_{2N}(x + (j-1)N\omega, E)\|\Big| \leq C\frac{m}{\mu} = Cm\exp(-\frac{9}{10}NL(E)).$$

Dividing by N' and integrating over $\mathbb{G}$, we obtain

$$(3.46)\quad |\int_{\mathbb{G}} \tilde{u}_{N'}(x,E)dx + \frac{1}{m}\int_{\mathbb{G}} \sum_{j=2}^{m-1} \tilde{u}_N(x+(j-1)N\omega,E)dx - \frac{2}{m}\int_{\mathbb{G}} \sum_{j=1}^{m-2} \tilde{u}_{2N}(x+(j-1)N\omega,E)dx| \leq C\frac{m}{N'}\exp(-\frac{9}{10}NL_N(E)) \leq C\exp(-\frac{9}{10}NL_N(E)).$$

Due to (3.8),

$$\Big|\int_{\mathbb{T}\setminus\mathbb{G}} \tilde{u}_K dx\Big| \leq \tilde{C}(p,q)^{\frac{1}{2}}\exp(-\frac{c}{3}L(E)N)$$

for any K. Thus,

$$(3.47)\quad \Big|\int_{\mathbb{T}\setminus\mathbb{G}} \tilde{u}_{N'}(x,E)dx + \frac{1}{m}\int_{\mathbb{T}\setminus\mathbb{G}} \sum_{j=2}^{m-1} \tilde{u}_N(x+jN\omega,E)dx - \frac{2}{m}\int_{\mathbb{T}\setminus\mathbb{G}} \sum_{j=1}^{m-1} \tilde{u}_{2N}(x+jN\omega,E)dx\Big| \leq 4\tilde{C}(p,q)^{\frac{1}{2}}\exp(-\frac{c}{3}L(E)N).$$

Combining (3.47) with (3.46), we have

$$\Big|L_{N'}(E) + \frac{m-2}{m}L_N(E) - \frac{2(m-1)}{m}L_{2N}(E)\Big| \leq 4\tilde{C}(p,q)^{\frac{1}{2}}\exp(-\frac{c}{3}L(E)N) + C\exp(-\frac{9}{10}NL(E)) \leq \exp(-c_5L(E)N).$$

As $\exp(\frac{c}{10}L(E)N) \leq m$, then

$$(3.48)\quad |L_{N'}(E) + L_N(E) - 2L_{2N}(E)| \leq \exp(-c_6L(E)N),$$

where $c_6 = c_6(p,q,\omega)$. □

LEMMA 3.10. — *Assume $L(E_0) > 0$. Let ρ_0' be as in Lemma 3.6, N_1 be as in Lemma 3.7 with $\kappa = \frac{1}{40}$. Let $N \geq N_1$ and $E \in (E_0 - \rho_0', E_0 + \rho_0')$ be arbitrary. Then*

$$(3.49)\quad |L(E) + L_N(E) - 2L_{2N}(E)| < \exp(-cL(E)N),$$

where $c = c(p,q)$.

Proof. — By Lemma 3.9 for $N' = mN$, with $m \in \mathbb{N}$, $\exp(\frac{c}{8}NL(E)) \leq m < \exp(\frac{c}{8}NL(E)) + 1$, we have

$$|L_{N'}(E) + L_N(E) - 2L_{2N}(E)| < \exp(-c_6L(E)N),$$

and

$$|L_{2N'}(E) + L_N(E) - 2L_{2N}(E)| < \exp(-c_6 L(E)N).$$

In particular,

$$|L_{N'}(E) - L_{2N'}(E)| < 2\exp(-c_6 L(E)N).$$

Pick $\exp(\frac{c}{8}NL(E)) \leq m_1 < \exp(\frac{c}{8}NL(E))+1$. Set $N_1' = m_1 N$. Similarly, define $N_2' = m_2 N_1'$, i.e., with N_1' in the role of N, etc. We obtain N_s' such that

$$\begin{aligned}|L_{N'_{s+1}}(E) + L_{N'_s}(E) - 2L_{2N'_s}(E)| &< \exp(-c_6 L(E)N'_s),\\ |L_{2N'_{s+1}}(E) + L_{N'_s}(E) - 2L_{2N'_s}(E)| &< \exp(-c_6 L(E)N'_s),\\ |L_{N'_{s+1}}(E) - L_{2N'_{s+1}}(E)| &< 2\exp(-c_6 L(E)N'_s).\end{aligned}$$

We have $L_{N'_s}(E) \to L(E)$ with $s \to \infty$. So

(3.50)

$$\begin{aligned}&|L(E) + L_N(E) - 2L_{2N}(E)|\\ &\quad= \left|\sum_{s\geq 1}(L_{N'_{s+1}}(E) - L_{N'_s}(E)) + L_{N'_1}(E) + L_N(E) - 2L_{2N}(E)\right|\\ &\quad\leq \sum_{s\geq 1}|L_{N'_{s+1}}(E) - L_{N'_s}(E)| + |L_{N'_1}(E) + L_N(E) - 2L_{2N}(E)|\\ &\quad< \sum_{s\geq 2} 5\exp(-c_6 L(E)N'_{s-1}) + 5\exp(-c_6 L(E)N) + \exp(-c_6 L(E)N)\\ &\quad< \exp(-c_7 L(E)N).\end{aligned}$$

Here again we change c_7 with c for notational convenience. □

4. Proof of the main theorem

Proof of Theorem 1.2. — Assume $\|\tilde{M}_N(x,E)\| \geq \|\tilde{M}_N(x,E')\|$, then

(4.1)

$$\begin{aligned}&\left|\log\|\tilde{M}_N(x,E)\| - \log\|\tilde{M}_N(x,E')\|\right| = \log\frac{\|\tilde{M}_N(x,E,\omega)\|}{\|\tilde{M}_N(x,E',\omega)\|}\\ &\quad= \log\left(1 + \frac{\|\tilde{M}_N(x,E)\| - \|\tilde{M}_N(x,E')\|}{\|\tilde{M}_N(x,E)\|}\right) \leq \frac{\|\tilde{M}_N(x,E)\| - \|\tilde{M}_N(x,E')\|}{\|\tilde{M}_N(x,E)\|}\\ &\quad\leq \|\tilde{M}_N(x,E)\| - \|\tilde{M}_N(x,E')\| = \frac{\|T_N(x,E)\| - \|T_N(x,E')\|}{\prod_{n=0}^{N-1}|q(x+n\omega)q(x+(n+1)\omega)|^{\frac{1}{2}}}\\ &\quad\leq \frac{\|T_N(x,E) - T_N(x,E')\|}{\prod_{n=0}^{N-1}|q(x+n\omega)q(x+(n+1)\omega)|^{\frac{1}{2}}}.\end{aligned}$$

We obtain

$$\begin{aligned}
(4.2)\quad & \|T_N(x,E)-T_N(x,E')\| \\
&\leq \sum_{j=0}^{N-1}(\|B(x+(N-1)\omega,E)\times\cdots\times B(x+(j+1)\omega,E)\| \\
&\quad\times\|B(x+j\omega,E)-B(x+j\omega,E')\| \\
&\quad\times\|B(x+(j-1)\omega,E')\times\cdots\times B(x,E')\|) \\
&= \sum_{j=0}^{N-1}\|\prod_{m=1}^{N-j}B(x+(N-m)\omega,E)\| \\
&\quad\times\|\prod_{m=j-1}^{0}B(x+m\omega,E')\|\times|E-E'|.
\end{aligned}$$

Combining (4.1), (4.2) and Lemma 2.20, we get

$$\begin{aligned}
(4.1) &\leq \sum_{j=0}^{N-1}\|\prod_{m=1}^{N-j}B(x+(N-m)\omega,E)\|\times\|\prod_{m=j-1}^{0}B(x+m\omega,E')\| \\
&\quad\times\exp(-ND)\times|E-E'| \\
&\quad\times\exp\left(-\sum_{n=o}^{N-1}\frac{1}{2}\log|q(x+n\omega)q(x+(n+1)\omega)|+ND\right) \qquad (4.3)\\
&= \sum_{j=0}^{N-1}\|\prod_{m=1}^{N-j}B(x+(N-m)\omega,E)\|\times\|\prod_{m=j-1}^{0}B(x+m\omega,E')\| \\
&\quad\times\exp(-ND)\times\exp(-NF_N(x))\times|E-E'|.
\end{aligned}$$

Combining (3.35) with (3.37), we have

$$L_N(E)\leq L(E)+\frac{1}{25}L(E_0)<\frac{11}{10}L(E_0)+\frac{1}{25}L(E_0)=\frac{57}{50}L(E_0) \qquad (4.4)$$

for any $N\geq\tilde{N}_0$ and any $E\in(E_0-\rho_0',E_0+\rho_0')$, where $\tilde{N}_0$ and ρ_0' are as in Lemma 3.6. Let $N>N_2:=2^{\frac{\log C(p,q)-D}{L(E_0)}}\tilde{N}_0\geq 2\tilde{N}_0$ and $E,E'\in(E_0-\rho_0',E_0+\rho_0')$, where $(\log(p,q)-D)$ is as in Remark 2.23,

$\tilde{N}_0$ and ρ_0' are as in Lemma 3.6. Due to Lemma 2.20 and Remark 2.23, we have

(4.5)

$$
\begin{aligned}
(4.1) \leq & \left(\sum_{j=1}^{\tilde{N}_0} + \sum_{j=\tilde{N}_0+1}^{N-\tilde{N}_0} + \sum_{j=N-\tilde{N}_0+1}^{N}\right) \| \prod_{m=1}^{N-j} B(x+(N-m)\omega, E)\| \\
& \times \| \prod_{m=j-1}^{0} B(x+m\omega, E')\| \times \exp(-ND) \times \exp(-NF_N(x)) \times |E-E'| \\
\leq & \left(2\sum_{j=1}^{\tilde{N}_0} \exp\left((\log C(p,q) - D)\tilde{N}_0 + \frac{57}{50}L(E_0)N + C_6(\frac{\log N}{N})^{\frac{1}{2}} - D\right)\right. \\
& \left. + \sum_{j=\tilde{N}_0+1}^{N-\tilde{N}_0} \exp\left(\frac{57}{50}L(E_0)N + 2C_6(\frac{\log N}{N})^{\frac{1}{2}} - D\right)\right) \\
& \times \exp(-NF_N(x)) \times |E-E'| \\
\leq & \sum_{j=1}^{N} \exp\left(\frac{1}{2}L(E_0)N + \frac{57}{50}L(E_0)N + 2C_6(\frac{\log N}{N})^{\frac{1}{2}} - D\right) \\
& \times \exp(-NF_N(x)) \times |E-E'|.
\end{aligned}
$$

There exists $N_3 = N_3(p,q,E_0,\omega)$ s.t. for any $N \geq N_3$, the following holds:

$$\sum_{j=1}^{N} \exp\left(\frac{82}{50}L(E_0)N + 2C_6(\frac{\log N}{N})^{\frac{1}{2}} - D\right) \leq \exp(2L(E_0)N).$$

It implies that for any $N \geq \max(N_2, N_3)$, the following holds:

$$(4.1) \leq \exp(2L(E_0)N) \times \exp(-NF_N(x)) \times |E-E'|.$$

It is obvious that

$$(4.1) \leq \exp(2L(E_0)N) \times \exp(-NF_N(x)) \times |E-E'|,$$

when $\|\tilde{M}_N(x,E,\omega)\| \geq \|\tilde{M}_N(x,E'\omega)\|$. Set

$$\mathbb{B} := \{x : NF_N(x) < -NL(E_0)\},$$

then

$$\operatorname{mes}(\mathbb{B}) \leq \operatorname{mes}(\{x : |NF_N(x) - N < F_N > | > NL(E_0)\}) < \exp(-cL(E_0)N),$$

since $< F_N >= 0$. It implies

$$(4.1) < \exp(3L(E_0)N)|E-E'|, \text{ if } x \in \mathbb{B}.$$

Due to (3.8), we have

$$
\begin{aligned}
|L_N(E) - L_N(E')| &= \int_{\mathbb{T}\setminus\mathbb{B}} |\tilde{u}_N(x,E) - \tilde{u}_N(x,E')|dx \\
&\quad + \int_{\mathbb{B}} |\tilde{u}_N(x,E) - \tilde{u}_N(x,E')|dx \\
&< \exp(3L(E_0)N)|E-E'| + 2\tilde{C}(p,q)^{\frac{1}{2}} \exp(-\frac{c}{2}L(E_0)N).
\end{aligned} \tag{4.6}
$$

Let $N \geq N_4 := \max(N_1, N_2, N_3)$, where N_1 is as in Lemma 3.7. Due to Lemma 3.10,

$$
\begin{aligned}
|L(E) - L(E')| &\leq |L(E) + L_N(E) - 2L_{2N}(E)| \\
&\quad + |L(E') + L_N(E') - 2L_{2N}(E')| + |L_N(E) - L_N(E')| \\
&\quad + 2|L_{2N}(E) - L_{2N}(E')| \\
&< 2\exp(-c_6 L(E_0)N) + 3\exp(6L(E_0)N)|E-E'| \\
&\quad + 3\tilde{C}(p,q)^{\frac{1}{2}} \exp(-\frac{c}{2}L(E_0)N) \\
&< \exp(-c_7 L(E_0)N) + 3\exp(6L(E_0)N)|E-E'|,
\end{aligned} \tag{4.7}
$$

where $c_7 = c_7(p,q,\omega)$. Set $\rho_0'' = \exp(-(6+c_7)L(E_0)N_4)$, and $\rho_0 = \min(\rho_0', \frac{\rho_0''}{2})$. Then for $E, E' \in (E_0 - \rho_0, E_0 + \rho_0)$, there exists $N \geq N_4$ such that

$$
\exp\big(-(6+c_7)L(E_0)(N+1)\big) \leq |E-E'| \leq \exp\big(-(6+c_7)L(E_0)N\big).
$$

It implies

$$
\begin{aligned}
|L(E) - L(E')| &< 4\exp(-c_7 L(E_0)N) \\
&= 4\exp\big(-\frac{N}{N+1} c_7 L(E_0)(N+1)\big) \\
&< 4\exp\big(-\frac{2c_7}{3} c_7 L(E_0)(N+1)\big) \\
&< \exp(-\frac{c_7}{2} L(E_0)N) < |E-E'|^{\beta},
\end{aligned} \tag{4.8}
$$

where $\beta = \frac{c_7}{12+2c_7}$, only depending on p,q and ω. By (1.9), we also have

$$
|J(E) - J(E')| = |L(E) - L(E')| < |E-E'|^{\beta}. \qquad \square
$$

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