Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 142
Fascicule 4

2014

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 673-718

Bull. Soc. math. France
142 (4), 2014, p. 673–718

ESTIMATES OF THE LINEARIZATION OF CIRCLE DIFFEOMORPHISMS

by Mostapha Benhenda

Abstract. — A celebrated theorem by Herman and Yoccoz asserts that if the rotation number α of a C^{∞}-diffeomorphism of the circle f satisfies a Diophantine condition, then f is C^{∞}-conjugated to a rotation. In this paper, we establish explicit relationships between the C^k norms of this conjugacy and the Diophantine condition on α. To obtain these estimates, we follow a suitably modified version of Yoccoz's proof.

Résumé (*Estimées de la linéarisation de difféomorphismes du cercle*)
Un célèbre théorème de Herman et Yoccoz affirme que si le nombre de rotation α d'un C^{∞}-difféomorphisme du cercle f satisfait une condition diophantienne, alors f est C^{∞}-conjugué à une rotation. Dans cet article, nous établissons des relations explicites entre les C^k normes de cette conjuguée et la condition diophantienne sur α. Pour obtenir ces estimées, nous suivons une version convenablement modifiée de la preuve de Yoccoz.

Texte reçu le 12 janvier 2012, révisé le 13 octobre 2012 et le 1er juillet 2013, accepté le 2 septembre 2013.

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2010 Mathematics Subject Classification. — 37C05.

Key words and phrases. — Circle diffeomorphisms, rotation number, conjugacy, estimates.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2014/673/$ 5.00

1. Introduction

In his seminal work, M. Herman [5] shows the existence of a set A of Diophantine numbers of full Lebesgue measure such that for any circle diffeomorphism f of class C^ω (resp. C^∞) of rotation number $\alpha \in A$, there is a C^ω-diffeomorphism (resp. C^∞-diffeomorphism) h such that $hfh^{-1} = R_\alpha$. In the C^∞ case, J. C. Yoccoz [14] extended this result to all Diophantine rotation numbers. Results in analytic class and in finite differentiability class subsequently enriched the global theory of circle diffeomorphisms [9, 8, 7, 13, 6, 15, 4, 10]. In the perturbative theory, KAM theorems usually provide a bound on the norm of the conjugacy that involves the norm of the perturbation and the Diophantine constants of the number α (see [5, 12, 11] for example). We place ourselves in the global setting, we compute a bound on the norms of this conjugacy h in function of the class of differentiability k, of norms of f, and of the Diophantine parameters β and C_d of α (an irrational number $\alpha \in DC(C_d, \beta)$ satisfies a Diophantine condition of order $\beta \geq 0$ and constant $C_d > 0$ if for any rational number p/q, we have: $|\alpha - p/q| \geq C_d/q^{2+\beta}$). The dependency in C_d is particularly interesting to study, because for any fixed $\beta > 0$, the set of Diophantine numbers of parameter β has full Lebesgue measure. It follows that the control of the conjugacy for a typical diffeomorphism, with fixed norms, is approached as $C_d \to 0$.

To obtain these estimates, we follow a suitably modified version of Yoccoz's proof. Indeed, Yoccoz's proof needs to be modified because a priori, it does not exclude the fact that the following set could be unbounded for any fixed $X > 0$:

$$E_X = \{|Dh|_0 \,/ \exists f \in \mathrm{Diff}^k_+(\mathbb{T}^1),\, f = h^{-1}R_\alpha h,$$
$$\alpha \in DC(\beta, C_d), \max(k, \beta, C_d, |Df|_0, W(f), |Sf|_{k-3}) \leq X\}$$

where $\mathrm{Diff}^k_+(\mathbb{T}^1)$ denotes the group of orientation-preserving circle diffeomorphisms of class C^k, Df denotes the derivative of f, $W(f)$ the total variation of $\log Df$, and Sf the Schwarzian derivative of f.

These estimates have natural applications to the global study of circle diffeomorphisms with Liouville rotation number: in [2], they allow to show the following results: 1) there is a Baire-generic set $A_1 \subset \mathbb{R}$ such that for any $f \in D^\infty(\mathbb{T}^1)$ of rotation number $\alpha \in A_1$, there is a sequence $h_n \in D^\infty(\mathbb{T}^1)$ such that $h_n^{-1} f h_n \to R_\alpha$ in the C^∞-topology. 2) There is a Baire-generic set $A_2 \subset \mathbb{R}$ such that for any $f \in D^\infty(\mathbb{T}^1)$ of rotation number $\alpha \in A_2$ and any g of class C^∞ with $fg = gf$, f and g are accumulated in the C^∞-topology by commuting C^∞-diffeomorphisms that are C^∞-conjugated to rotations. Moreover, if β is the rotation number of g, R_α and R_β are accumulated in the C^∞-topology by commuting C^∞-diffeomorphisms that are C^∞-conjugated to f and g.

1.1. Notations. — We follow the notations of [14].

- The circle is denoted by $\mathbb{T}^1$. The group of $\mathbb{Z}$-periodic maps of class C^r of the real line is denoted by $C^r(\mathbb{T}^1)$. We work in $D^r(\mathbb{T}^1)$, which is the group of diffeomorphisms f of class C^r of the real line such that $f - Id \in C^r(\mathbb{T}^1)$. It is the universal cover of the group of orientation-preserving circle diffeomorphisms of class C^r. Note that if $f \in D^r(\mathbb{T}^1)$ and $r \geq 1$, then $Df \in C^{r-1}(\mathbb{T}^1)$.
- The Schwarzian derivative Sf of $f \in D^3(\mathbb{T}^1)$ is defined by:
$$Sf = D^2 \log Df - \frac{1}{2}(D \log Df)^2.$$
- The total variation of the logarithm of the first derivative of f is:
$$W(f) = \sup_{0 \leq a_0 \leq \cdots \leq a_n \leq 1} \sum_{i=0}^{n} |\log Df(a_{i+1}) - \log Df(a_i)|.$$
- For any continuous and $\mathbb{Z}$-periodic function ϕ, let:
$$|\phi|_0 = \|\phi\|_0 = \sup_{x \in \mathbb{R}} |\phi(x)|.$$
- Let $0 < \gamma' < 1$. The map $\phi \in C^0(\mathbb{T}^1)$ is Holder of order γ' if:
$$|\phi|_{\gamma'} = \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|^{\gamma'}} < +\infty.$$

  Let $\gamma \geq 1$ be a real number. All along the paper, we write $\gamma = r + \gamma'$ with $r \in \mathbb{N}$ and $0 \leq \gamma' < 1$.
- A function $\phi \in C^\gamma(\mathbb{T}^1)$ if $\phi \in C^r(\mathbb{T}^1)$ and if $D^r\phi \in C^{\gamma'}(\mathbb{T}^1)$. The set $C^\gamma(\mathbb{T}^1)$ is endowed with the norm:
$$\|\phi\|_\gamma = \max\left(\max_{0 \leq j \leq r} \|D^j\phi\|_0, |D^r\phi|_{\gamma'}\right).$$

  If $\gamma = 0$ or $\gamma \geq 1$, the C^γ-norm of ϕ is indifferently denoted $\|\phi\|_\gamma$ or $|\phi|_\gamma$. Thus, when possible, we favor the simpler notation $|\phi|_\gamma$.
- If $x \in \mathbb{T}^1$ and $\tilde{x}$ is a lift to $\mathbb{R}$, then:
$$|x| = \inf_{p \in \mathbb{Z}} |\tilde{x} + p|.$$
- For $x, y \in \mathbb{R}$, if $x \leq y$, $[x, y]$ denotes $\{t \in \mathbb{R}, x \leq t \leq y\}$ and if $x \geq y$, $[x, y]$ denotes $\{t \in \mathbb{R}, y \leq t \leq x\}$.
- For $\alpha \in \mathbb{R}$, we denote $R_\alpha \in D^\infty(\mathbb{T}^1)$ the map $x \mapsto x + \alpha$.
- An irrational number $\alpha \in DC(C_d, \beta)$ satisfies a Diophantine condition of order $\beta \geq 0$ and constant $C_d > 0$ if for any rational number p/q, we have:
$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C_d}{q^{2+\beta}}.$$

Moreover, if $\beta = 0$, then α is of constant type C_d.

- Let $\alpha_{-2} = \alpha$, $\alpha_{-1} = 1$. For $n \geq 0$, we define a real number α_n (the *Gauss sequence* of α) and an integer $\hat{a}_n$ by the relations $0 < \alpha_n < \alpha_{n-1}$ and

$$\alpha_{n-2} = \hat{a}_n \alpha_{n-1} + \alpha_n.$$

- In the following statements, $C_i[a, b, \ldots]$ denotes a positive numerical function of real variables $a, b, \ldots$, with an explicit formula that we compute.

  $C[a, b, \ldots]$ denotes a numerical function of $a, b, \ldots$, with an explicit formula that we do not compute.
- We use the notations $a \wedge b = a^b$, $e^{(n)} \wedge x$ the n^{th}-iterate of $x \mapsto \exp x$, $\lfloor x \rfloor$ for the largest integer such that $\lfloor x \rfloor \leq x$, and $\lceil x \rceil$ for the smallest integer such that $\lceil x \rceil \geq x$.

We recall Yoccoz's theorem [14]:

THEOREM 1.1. — *Let $k \geq 3$ be an integer and $f \in D^k(\mathbb{T}^1)$. We suppose that the rotation number α of f is Diophantine of order β. If $k > 2\beta + 1$, there exists a diffeomorphism $h \in D^1(\mathbb{T}^1)$ conjugating f to R_α. Moreover, for any $\eta > 0$, h is of class $C^{k-1-\beta-\eta}$.*

1.2. Statement of the results

1.2.1. C^1 *estimations*

THEOREM 1.2. — *Let $f \in D^3(\mathbb{T}^1)$ be of rotation number α, such that α is of constant type C_d. Then there exists a diffeomorphism $h \in D^1(\mathbb{T}^1)$ conjugating f to R_α, which satisfies the estimation:*

$$|Dh|_0 \leq e \wedge \left(\frac{C_1[W(f), |Sf|_0]}{C_d} \right).$$

The expression of $C_{1.2}$ is given in page 681.

More generally, for a Diophantine rotation number $\alpha \in DC(C_d, \beta)$, we have:

THEOREM 1.3. — *Let $k \geq 3$ be an integer and $f \in D^k(\mathbb{T}^1)$. Let $\alpha \in DC(C_d, \beta)$ be the rotation number of f. If $k > 2\beta + 1$, then there exists a diffeomorphism $h \in D^1(\mathbb{T}^1)$ conjugating f to R_α, which satisfies the estimation:*

$$|Dh|_0 \leq C_2[k, \beta, C_d, |Df|_0, W(f), |Sf|_{k-3}]. \tag{1}$$

The expression of C_2 is given in page 693.

Moreover, if $k \geq 3\beta + 9/2$, we have:

$$|Dh|_0 \leq e^{(3)} \wedge \left(C_3[\beta] C_4[C_d] C_5[|Df|_0, W(f), |Sf|_0] C_6[|Sf|_{\lceil 3\beta+3/2 \rceil}] \right). \tag{2}$$

The expressions of C_2, C_2, C_2, C_2 are given in page 695.

Let $\delta = k - 2\beta - 1$. When $\delta \to 0$, we have:

$$(3)\quad |Dh|_0 \leq e^{(3)} \wedge \big(\frac{1}{\delta^2} C_7[k, C_d, |Df|_0, W(f), |Sf|_0] + \frac{C[\delta]}{\delta^2} C[k, C_d, |Df|_0, W(f), |Sf|_0, |Sf|_{k-3}]\big)$$

where $C[\delta] \to_{\delta\to 0} 0$. The expression of C_3 is given in page 695.

REMARK 1.4. — Katznelson and Ornstein [8] showed that the assumption $k > 2\beta + 1$ in Yoccoz's theorem is not optimal (instead it is $k > \beta + 2$). Therefore, the divergence of the bound given by estimation (3) is because we compute the bound of the conjugacy following the Herman-Yoccoz method.

REMARK 1.5. — Let α_n be the Gauss sequence associated with α. Yoccoz's proof already gives the following result: if $k \geq 3\beta + 9/2$ and if, for any $n \geq 0$,

$$(4)\qquad \frac{\alpha_{n+1}}{\alpha_n} \geq C_8[n, k, W(f), |Sf|_{k-3}],$$

then:

$$|Dh|_0 \leq \exp\Big(C_9[k, W(f), |Sf|_{k-3}]^{C_{10}(\beta)}\Big)\, |Df|_0^2.$$

The expressions of $C_8, C_{1.5}, C_{1.5}$ are given in page 696.

1.2.2. *C^u estimations*

THEOREM 1.6. — *Let $k \geq 3$ be an integer, $\eta > 0$ and $f \in D^k(\mathbb{T}^1)$. Let $\alpha \in DC(C_d, \beta)$ be the rotation number of f. If $k > 2\beta + 1$, there exists a diffeomorphism $h \in D^{k-1-\beta-\eta}(\mathbb{T}^1)$ conjugating f to R_α, which satisfies the estimation:*

$$(5)\quad \|Dh\|_{k-2-\beta-\eta} \leq e^{(\lceil \log((k-2-\beta)/\eta)/\log(1+1/(2\beta+3))\rceil)} \wedge \left(C_{11}[\eta, k, \beta, C_d, |Df|_0, W(f), |Sf|_{k-3}]\right).$$

The expression of C_5 is given in page 712.

Moreover, if $k \geq 3\beta + 9/2$, we have:

$$(6)\quad \|Dh\|_{\frac{k}{2(\beta+2)}-\frac{1}{2}} \leq e \wedge \Big(C[k]e^{(2)} \wedge (2 + C_2[\beta]C_2[C_d]C_2[|Df|_0, W(f), |Sf|_0]C_2[|Sf|_{k-3}])\Big).$$

If α is of constant type, for any $k > 3$, we have:

$$(7)\qquad \|Dh\|_{\frac{k}{4}-\frac{1}{2}} \leq e \wedge \left(C[k]\left[C_{12}[W(f), |Sf|_{k-3}] + \frac{C_{1.2}[W(f), |Sf|_0]}{C_d}\right]^4\right)$$

The expression of C_7 is given in page 711.

REMARK 1.7. — In [3], we specify the dependency $C[k]$ in the parameter k of the C^u-estimates ($u = k - 1 - \beta - \eta$ or $\frac{k}{2(\beta+2)} + \frac{1}{2}$), a dependency that is not given in this paper.

2. Preliminaries

Let $f \in D^0(\mathbb{T}^1)$ be a homeomorphism and $x \in \mathbb{R}$. When n tends towards infinity, $(f^n(x)-x)/n$ admits a limit independent of x, which is denoted by $\rho(f)$. We call it the *translation number* of f. Two lifts f and f' of a given circle homeomorphism $\bar{f}$ only differ by a constant integer, so this is also the case for their translation numbers. We call the class of $\rho(f)$ mod $\mathbb{Z}$ the *rotation number* of f (and of $\bar{f}$). We still denote it $\rho(f)$. It is invariant by conjugacy.

Suppose, moreover, that $f \in D^2(\mathbb{T}^1)$. When $\alpha = \rho(f)$ is irrational, Denjoy showed that f is topologically conjugated to R_α. However, this conjugacy is not always differentiable (see [1, 5, 6, 15]). Its regularity depends on the Diophantine properties of the rotation number α and the regularity of f (see Yoccoz's Theorem 1.1).

Let α be an irrational number. Let $||\alpha||$ denote the distance from α to the closest integer, i.e.:

$$||\alpha|| = \inf_{p\in\mathbb{Z}} |\alpha - p|.$$

For $n \geq 1$, $\hat{a}_n \geq 1$. Let $\alpha = \hat{a}_0 + 1/(\hat{a}_1 + 1/(\hat{a}_2 + \cdots))$ be the development of α in continued fraction. We denote it $\alpha = [\hat{a}_0, \hat{a}_1, \hat{a}_2, \ldots]$. Let $p_{-2} = q_{-1} = 0$, $p_{-1} = q_{-2} = 1$. For $n \geq 0$, let p_n and q_n be:

$$p_n = \hat{a}_n p_{n-1} + p_{n-2}$$
$$q_n = \hat{a}_n q_{n-1} + q_{n-2}.$$

We have $q_0 = 1$, $q_n \geq 1$ for $n \geq 1$. The rationals p_n/q_n are called the convergents of α. They satisfy the following properties:

1. $\alpha_n = (-1)^n(q_n\alpha - p_n)$;
2. $\alpha_n = ||q_n\alpha||$, for $n \geq 1$;
3. $1/(q_{n+1} + q_n) < \alpha_n < 1/q_{n+1}$ for $n \geq 0$;
4. $\alpha_{n+2} < \frac{1}{2}\alpha_n$, $q_{n+2} \geq 2q_n$, for $n \geq -1$.

The set of Diophantine numbers of constants β and C_d is denoted by $DC(C_d, \beta)$. The elements of this set are characterized by any of the following relations:

1. $|\alpha - p_n/q_n| > C_d/q_n^{2+\beta}$ for any $n \geq 0$;
2. $\hat{a}_{n+1} < \frac{1}{C_d} q_n^\beta$ for any $n \geq 0$;
3. $q_{n+1} < \frac{1}{C_d} q_n^{1+\beta}$ for any $n \geq 0$;
4. $\alpha_{n+1} > C_d \alpha_n^{1+\beta}$ for any $n \geq 0$.

All along the paper, we denote $C'_d = 1/C_d$.

- For $x \in \mathbb{R}$, let $T(x) = x - 1$, $f_n(x) = f^{q_n}T^{p_n}(x)$, $m_n(x) = |f_n(x) - x|$, $n \geq 1$, let $M_n = \max_{x\in\mathbb{R}} m_n(x)$ and $m_n = \min_{x\in\mathbb{R}} m_n(x)$.

– For any $\phi, \psi \in C^\gamma(\mathbb{T}^1)$, we have:

$$|\phi\psi|_\gamma \leq \|\phi\|_0 |\psi|_\gamma + |\phi|_\gamma \|\psi\|_0 \tag{8}$$

$$\|\phi\psi\|_\gamma \leq \|\phi\|_0 |\psi|_\gamma + \|\phi\|_\gamma \|\psi\|_0. \tag{9}$$

In the rest of the paper, for any integer i, C_i^f denotes a constant depending only on $W(f)$ and $|Sf|_0$ (i.e., C_i^f is a numerical function of these variables). $C_i^{f,k}$ denotes a constant depending only on k, $W(f)$, $|Sf|_0$ and $|Sf|_{k-3}$. C_i denotes a constant that might depend on k, $W(f)$, $|Sf|_0$, $|Sf|_{k-3}$ and also β and C_d.

2.1. C^1 estimations: constant type. — The proof of Theorem 1.2 is divided in three steps. The first step is based on the improved Denjoy inequality, which estimates the C^0-norm of $\log Df^{q_l}$. In the second step, we extend this estimation to $\log Df^N$ for any integer N. To do this, following Denjoy and Herman, we write $N = \sum_{s=0}^S b_s q_s$, with b_s integers satisfying $0 \leq b_s \leq q_{s+1}/q_s$ and we apply the chain rule. In the third step, we derive a C^0-estimation of the derivative Dh of the conjugacy h.

The first step is based on the Denjoy inequality:

PROPOSITION 2.1. — *Let $f \in D^3(\mathbb{T}^1)$ and $x \in \mathbb{R}$. We have:*

$$|\log Df^{q_l}(x)| \leq W(f).$$

Proposition 2.1 is used to obtain an improved version of Denjoy inequality [14, p. 342]:

LEMMA 2.2. — *Let $f \in D^3(\mathbb{T}^1)$. We have:*

$$|\log Df^{q_l}|_0 \leq C_{13}^f M_l^{1/2} \tag{10}$$

$$|Df^{q_l} - 1|_0 \leq C_{14}^f M_l^{1/2}. \tag{11}$$

Moreover, we can take:

$$C_{10}^f = 2\sqrt{2}(2e^{W(f)} + 1)e^{W(f)}(|Sf|_0)^{1/2}$$

and

$$C_{11}^f = 6\sqrt{2}e^{3W(f)}|Sf|_0^{1/2}.$$

In the second step, we estimate $D \log Df^N$ independently of N. This step is based on the following lemma:

Lemma 2.3. — *Let $f \in D^3(\mathbb{T}^1)$. We have:*

$$\sum_{l\geq 0}\sqrt{M_l} \leq \frac{1}{\sqrt{C_{15}^f} - C_{2.3}^f}$$

with

$$C_{2.3}^f = \frac{1}{\sqrt{1+e^{-C_{16}^f}}} \tag{12}$$

and:

$$C_{16}^f = 6\sqrt{2}e^{2W(f)}\left(\max\left(|Sf|_0^{1/2}, 1\right)\right). \tag{13}$$

Proof. — To obtain this lemma, we need the claim:

Claim 2.4. — *Let $f \in D^2(\mathbb{T}^1)$ be of rotation number α, and let p_n/q_n be the convergents of α. Then for every $x \in \mathbb{R}$, we have:*

$$[x, f_{l+2}(x)] \subset [x, f_l(x)].$$

Proof. — By topological conjugation, it suffices to examine the case of a rotation of angle α. It is also sufficient to take $x = 0$.

Reasoning by contradiction, suppose that $0 < q_l\alpha - p_l < 2(q_{l+2}\alpha - p_{l+2})$. Then $-q_{l+2}\alpha + p_{l+2} < (q_l - q_{l+2})\alpha - (p_l - p_{l+2}) < q_{l+2}\alpha - p_{l+2}$. Therefore, $0 < |(q_{l+2} - q_l)\alpha - (p_{l+2} - p_l)| < |q_{l+2}\alpha - p_{l+2}|$. This contradicts the fact that

$$\|q_{l+2}\alpha\| = \inf\{\|q\alpha\|/0 < q \leq q_{l+2}\}. \qquad \square$$

Let I be an interval of length $|I|$. Lemma 2.2 implies the estimation:

$$\frac{|f^{q_{l+2}}(I)|}{|I|} \geq e^{-C_{16}^f M_{l+2}^{1/2}}.$$

Let $x \in \mathbb{R}$ be such that $M_{l+2} = |f^{q_{l+2}}(x) - x - p_{l+2}|$ and let $I = [x, f_{l+2}(x)]$. The former estimation implies:

$$|f^{2q_{l+2}}(x) - f^{q_{l+2}}(x)| \geq e^{-C_{16}^f M_{l+2}^{1/2}} M_{l+2}.$$

By applying claim 2.4, and since $M_n \leq 1$, we obtain:

$$M_{n+2} + e^{-C_{16}^f} M_{n+2} \leq M_{n+2} + e^{-C_{16}^f M_{n+2}^{1/2}} M_{n+2} \leq M_n.$$

Therefore, for any $l \geq 0$,

$$M_l \leq (C_{2.3}^f)^{l-1} \tag{14}$$

with

$$C_{2.3}^f = \frac{1}{\sqrt{1+e^{-C_{16}^f}}}.$$

Estimation (14) above gives:

$$\sum_{l\geq 0}\sqrt{M_l}\leq \frac{1}{\sqrt{C^f_{2.3}}}\frac{1}{1-\sqrt{C^f_{2.3}}}\leq \frac{1}{\sqrt{C^f_{2.3}}-C^f_{2.3}}.$$

Hence we get Lemma 2.3. □

Now, let N be an integer. Following Denjoy, since α is of constant type, we can write $N=\sum_{l=0}^{s} b_l q_l$, with b_l integers satisfying $0\leq b_l\leq q_{l+1}/q_l\leq C_d^{-1}$. By the chain rule and by Lemma 2.2, since for every $y\in\mathbb{R}$, $Df^N(y)>0$, then:

$$\begin{aligned}|\log D(f^N)(y)| &= |\log D(f^{\sum_{l=0}^{s} b_l q_l})(y)| = |\sum_{l=0}^{s}\sum_{i=0}^{b_s}\log Df^{q_l}\circ f^{iq_l}(y)|\\ &\leq \sup_{0\leq l\leq s} b_l\sum_{l=0}^{s}|\log|D(f^{q_l})|_0|\leq C_d^{-1}C^f_{16}\sum_{l\geq 0}M_l^{1/2}.\end{aligned}$$

By taking the upper bound on $y\in\mathbb{R}$ and $N\geq 0$, we obtain an estimation of $\sup_{N\geq 0}|\log D(f^N)|$.

We turn to the third step: we relate the norms of Dh and Df^N. By [14], h is C^1 and conjugates f to a rotation. Therefore, we have:

$$\log Dh-\log Dh\circ f=\log Df.$$

Hence, for any integer n:

$$\log Dh-\log Dh\circ f^n=\log D(f^n).$$

Since there is a point z such that $Dh(z)=1$, then we have:

$$|\log Dh\circ f^n(z)|=|\log D(f^n)(z)|\leq \sup_{i\geq 0}|\log D(f^i)|_0.$$

Moreover, since $(f^n(z))_{n\geq 0}$ mod 1 is dense in $\mathbb{T}$, and since Dh is continuous, then we obtain:

$$|\log Dh|_0\leq \sup_{i\geq 0}|\log D(f^i)|_0.$$

We conclude:

$$|Dh|_0\leq \exp\left(C_d^{-1}C^f_{16}\sqrt{e^{C^f_{16}\max(M_0^{1/2},M_1^{1/2})}+1}(\sqrt{M_0}+\sqrt{M_1})\right). \tag{15}$$

Finally, since $\max(M_0^{1/2},M_1^{1/2})\leq 1$, we obtain:

$$|Dh|_0\leq \exp\left(C^f_{1.2}/C_d\right)$$

where $C^f_{1.2}=2C^f_{16}\sqrt{e^{C^f_{16}}+1}$. We recall that:

$$C^f_{16}=6\sqrt{2}e^{2W(f)}\left(\max\left(|Sf|_0^{1/2},1\right)\right).$$

Corollary 2.5. — *Since* $\frac{1}{\min_{\mathbb{R}} Dh} \leq \exp\left(\sup_{i\geq 0} |\log D(f^i)|_0\right)$, *the proof above also provides an estimation on* $\frac{1}{\min_{\mathbb{R}} Dh}$*:*

$$\frac{1}{\min_{\mathbb{R}} Dh} \leq \exp\left(C^f_{1.2}/C_d\right).$$

3. C^1 estimations: non-constant type

We have $\max_{n\geq 0} |Df^n|_0 \leq \max_{n\geq 0} M_n/m_n$, by [14, p. 348]. Therefore, in order to prove Theorem 1.3, we can estimate M_n/m_n. To that end, we proceed in two steps: first, we establish some preliminary results. An important result is Corollary 3.6, which gives an estimation of M_{n+1}/M_n in function of M_n, α_{n+1}/α_n and a constant $C^{f,k}_{21}$. This estimation is already given in [14, p. 345], but we still recall the steps to reach it, because we need to estimate the constant $C^{f,k}_{21}$ in function of k, $W(f)$, $|Sf|_0$ and $|Sf|_{k-3}$.

In the second step, we establish an estimation of the C^1-conjugacy, based on a modification of the proof given in [14]. The main idea is to establish an alternative between two possible situations for the sequences M_n and α_n: the "favorable" situation (R_n) and the "unfavorable" situation (R'_n) (Proposition 3.10). The "unfavorable" situation only occurs a finite number of times, due to the Diophantine condition on α (Propositions 3.13 and 3.15).

In the "favorable" situation (R_n), we can estimate M_{n+1}/α_{n+1} in function of M_n/α_n (see estimation (31)) and likewise, we can estimate α_{n+1}/m_{n+1} in function of α_n/m_n. Therefore, we can estimate M_n/m_n in function of M_{n_4}/m_{n_4}, where n_4 is the integer such that for any $n \geq n_4$, only the favorable case occurs (see Proposition 3.20). We relate M_{n_4}/m_{n_4} to $Df|_0^{\frac{2}{\alpha_{n_4}}}$ (Proposition 3.18), and we compute a bound on α_{n_4} (Proposition 3.16). Yoccoz's proof needs to be modified because in its original version, it does not allow to compute a bound on α_{n_4}.

3.1. Preliminary results.

— We recall the following lemmas, which are in [14] (Lemmas 3,4 and 5):

Lemma 3.1. — *For $l \geq 1$ and $x \in \mathbb{R}$, we have:*

$$\sum_{i=0}^{q_{n+1}-1} \left(Df^i(x)\right)^l \leq C^f_{17} \frac{M_n^{l-1}}{m_n(x)^l}$$

with $C^f_{17}(l) = e^{lW(f)}$.

Remark 3.2. — This lemma is obtained by applying Denjoy inequality.

LEMMA 3.3. — *Let $f \in D^k(\mathbb{T}^1)$, $k \geq 3$. For any $x \in \mathbb{R}$, any $n \in \mathbb{N}$, any $0 \leq p \leq q_{n+1}$, we have:*

$$|Sf^p(x)| \leq C_{18}^f \frac{M_n}{m_n(x)^2} \tag{16}$$

$$|D \log Df^p(x)| \leq C_{19}^f \frac{M_n^{1/2}}{m_n(x)} \tag{17}$$

with $C_{16}^f = |Sf|_0 e^{2W(f)}$ and $C_{17}^f = 9\sqrt{2|Sf|_0} e^{4W(f)}$.

LEMMA 3.4. — *For $1 \leq r \leq k-1$, $n \geq 0$, $0 \leq p \leq q_{n+1}$, $x \in \mathbb{R}$, we have:*

$$|D^r \log Df^p(x)| \leq C_{20}^f(r) \left[\frac{M_n^{1/2}}{m_n(x)}\right]^r \tag{18}$$

with

$$C_{20}^f(1) = C_{17}^f,\ C_{20}^f(2) = 82|Sf|_0 e^{8W(f)}$$

and, for $r \geq 3$:

$$C_{20}^f(r) = \left[82(2r)^{2r} \left(\max\left(1, |Sf|_{r-2}\right)\right)^2 e^{(r+8)W(f)}\right]^{r!}.$$

In particular,

$$C_{20}^{f,k} := C_{20}^f(k-1) \leq \left[100(2k-2)^{2k-2} \left(\max\left(1, |Sf|_{k-3}\right)\right)^2 e^{(k+7)W(f)}\right]^{(k-1)!}.$$

Proof of Lemma 3.4. — The proof follows the line of [14], Lemma 5. See Appendix 5 for details. □

The important preliminary result, Corollary 3.6, is obtained from the following proposition. It is obtained by computing the constants in Proposition 2 of [14]:

PROPOSITION 3.5. — *Let*

$$C_{21}^{f,k} = (k+3)^{(k+3)!} e^{(k+2)!W(f)} (\max(1, |Sf|_{k-3}))^{k!}. \tag{19}$$

For any $x \in \mathbb{R}$, we have:

$$\left| m_{n+1}(x) - \frac{\alpha_{n+1}}{\alpha_n} m_n(x) \right| \leq C_{21}^{f,k} \left[M_n^{(k-1)/2} m_n(x) + M_n^{1/2} m_{n+1}(x) \right]. \tag{20}$$

COROLLARY 3.6. — *We have*

$$M_{n+1} \leq M_n \frac{\frac{\alpha_{n+1}}{\alpha_n} + C_{21}^{f,k} M_n^{(k-1)/2}}{1 - C_{21}^{f,k} M_n^{1/2}} \tag{21}$$

$$m_{n+1} \geq m_n \frac{\frac{\alpha_{n+1}}{\alpha_n} - C_{21}^{f,k} M_n^{(k-1)/2}}{1 + C_{21}^{f,k} M_n^{1/2}}. \tag{22}$$

The proof of Proposition 3.5 combines the following three lemmas [14, pp. 343-344] (Lemmas 6, 7 and 8):

LEMMA 3.7. — *For any $x \in \mathbb{R}$, there exists $y \in [x, f_n(x)], z \in [x, f_{n+1}(x)]$ such that*

$$m_{n+1}(y) = \frac{\alpha_{n+1}}{\alpha_n} m_n(z).$$

LEMMA 3.8. — *Suppose that m_{n+1} is monotonous on an interval $I_z = (z, f_n(z))$, $z \in \mathbb{R}$. Then, for any $x \in \mathbb{R}$, for any $y \in I_x$ ($I_x = (x, f_n(x))$), we have:*

$$\left|\frac{m_{n+1}(y)}{m_{n+1}(x)} - 1\right| \leq C_{22}^{f,k} M_n^{1/2}$$

with

$$C_{22}^{f,k} = 2^9(k+2)e^{(11+k/2)W(f)}(C_{11}^f)^2 C_{17}^f.$$

LEMMA 3.9. — *If m_{n+1} is not monotonous on any interval of the form $I_z = (z, f_n(z))$, $z \in \mathbb{R}$, then for any $x \in \mathbb{R}$, $y \in I_x$, we have:*

$$|m_{n+1}(y) - m_{n+1}(x)| \leq C_{23}^{f,k} M_n^{(k-1)/2} m_n(x)$$

with

$$C_{23}^{f,k} = (C_{20}^f(k-1))e^{W(f)} \left(e^{(k/2+2)W(f)}(1+e^{W(f)})^2 \frac{e^{(k/2+2)W(f)} - 1}{e^{W(f)} - 1}\right)^{k-1}.$$

Proof of Proposition 3.5. — The proof of Proposition 3.5 from these three lemmas is also found in [14, p. 344]. Let $x \in \mathbb{R}$ and $y \in I_x$, $z \in [x, f_{n+1}(x)]$ the points given by Lemma 3.7. By combining Lemmas 3.8 and 3.9, we obtain:

$$|m_{n+1}(y) - m_{n+1}(x)| \leq \left(\max\left(C_{22}^{f,k}, C_{23}^{f,k}\right)\right)\left(M_n^{1/2} m_{n+1}(x) + M_n^{(k-1)/2} m_n(x)\right).$$

Moreover, by Lemma 2.2, we have:

$$|m_n(z) - m_n(x)| \leq C_{11}^f M_n^{1/2} |z - x| \leq C_{11}^f M_n^{1/2} m_{n+1}(x).$$

By applying Lemma 3.7, and since $\alpha_{n+1}/\alpha_n \leq 1$, we get:

$$\left|m_{n+1}(x) - \frac{\alpha_{n+1}}{\alpha_n} m_n(x)\right| \leq \left|m_{n+1}(x) - \frac{\alpha_{n+1}}{\alpha_n} m_n(z)\right| + \frac{\alpha_{n+1}}{\alpha_n}|m_n(z) - m_n(x)|$$

$$\left|m_{n+1}(x) - \frac{\alpha_{n+1}}{\alpha_n} m_n(x)\right| \leq |m_{n+1}(y) - m_{n+1}(x)| + |m_n(z) - m_n(x)|.$$

Therefore, we have:

$$\left|m_{n+1}(x) - \frac{\alpha_{n+1}}{\alpha_n} m_n(x)\right| \leq C_{24}^{f,k}\left(M_n^{1/2} m_{n+1}(x) + M_n^{(k-1)/2} m_n(x)\right)$$

with $C_{24}^{f,k} = \max(C_{22}^{f,k}, C_{23}^{f,k}) + C_{11}^f$.

Finally, we compute an estimation of $C_{24}^{f,k}$. Details can be found in [3]. □

3.2. Estimation of the C^1-conjugacy. Proof of Theorem 1.3. — We choose an integer n_1 such that for any $n \geq n_1$, we have:

$$C_{21}^{f,k} M_n^{1/2} \leq C_{21}^{f,k} (C_{2.3}^f)^{\frac{n-1}{2}} < 1/2. \tag{23}$$

We can take:

$$n_1 = \left\lceil \frac{-\log\left(2C_{21}^{f,k}/(C_{2.3}^f)^{1/2}\right)}{\log\left((C_{2.3}^f)^{1/2}\right)} \right\rceil .$$

We choose a parameter θ such that $(k+1)/2 - \theta > (1+\beta+\theta)(1+\theta)$ (for the interpretation of this parameter θ, see the remark after Proposition 3.10. For example, we cannot take $\theta = 0$ because we need that the infinite product $\prod_{n\geq 1}^{+\infty}\left(1 + M_n^\theta\right)$ converges). We can take:

$$\theta = \min\left(1/2, \left(\frac{3+\beta}{4}\right)\left(-1 + \left(1 + \frac{2(k-2\beta-1)}{(3+\beta)^2}\right)^{1/2}\right)\right) \tag{24}$$

(in the proof of estimation (2), we take $\theta = 1/2$ instead).

For $x \geq 0$, $1+x \leq e^x$ and for $0 \leq x \leq 1/2$, $\log\left(1/(1-x)\right) \leq x/(1-x) \leq 2x$. We apply estimation (14), we use the definition of n_1 and the fact that $\theta \leq 1/2$. We get:

$$\prod_{n=n_1}^{+\infty}\left(1 + M_n^\theta\right) \leq \exp\left(\sum_{n=n_1}^{+\infty} M_n^\theta\right) \leq \exp\left(\frac{1}{2C_{21}^{f,k}(1-(C_{2.3}^f)^\theta)}\right)$$

$$\prod_{n=n_1}^{+\infty}\left(\frac{1}{1 - C_{21}^{f,k} M_n^{1/2}}\right) \leq \exp\left(\sum_{n=n_1}^{+\infty} 2C_{21}^{f,k} M_n^{1/2}\right) \leq \exp\left(\frac{1}{1-(C_{2.3}^f)^{1/2}}\right).$$

Therefore,

$$\prod_{n=n_1}^{+\infty}\left(\frac{1 + M_n^\theta}{1 - C_{21}^{f,k} M_n^{1/2}}\right) \leq \exp\left(\frac{2}{1-(C_{2.3}^f)^\theta}\right). \tag{25}$$

Let:

$$C_{25} = \exp\left(\frac{2}{1-(C_{2.3}^f)^\theta}\right).$$

Let:

$$C_{26} = \max\left((4C_{21}^{f,k})^{\frac{1}{(1+\beta+\theta)(1+\theta)-1}}, C_{25}\right). \tag{26}$$

Let:

$$C_{27} = \frac{-\log\left(2(C_{26})^2\right)}{\log C_{2.3}^f} + 1.$$

For any

$$n \geq C_{27} \tag{27}$$

we have:

$$M_n \leq (C^f_{2.3})^{n-1} \leq \frac{1}{2C_{26}^2}. \tag{28}$$

We use this estimation in the second step of the proof, to which we come now. Let:

$$n_2 = \max(n_1, \tilde{n}_2) \tag{29}$$

where

$$\tilde{n}_2 = \left\lfloor C_{27} + \frac{4}{\log 2}\log(1/C_d) + 2 \right\rfloor. \tag{30}$$

Having defined the integer n_2, we can present the alternative between the "favorable" case (R_n) and the "unfavorable" case (R'_n).

Let $a_{n_2} = 1/((C_{26})^2)$. Let $1 \geq \eta_n \geq 0$ be a sequence such that $\alpha_n = \alpha_{n+1}^{1-\eta_n}$. For any $n \geq n_2$, we can define a sequence a_n by: if

$$(R_n)\quad C_{21}^{f,k} M_n^{(k+1)/2-\theta} \leq M_n \frac{\alpha_{n+1}}{\alpha_n} \quad \text{then} \quad a_{n+1} = a_n \frac{1+M_n^\theta}{1 - C_{21}^{f,k} M_n^{1/2}}$$

and if

$$(R'_n)\quad C_{21}^{f,k} M_n^{(k+1)/2-\theta} > M_n \frac{\alpha_{n+1}}{\alpha_n} \quad \text{then} \quad a_{n+1} = a_n.$$

We can also define a sequence ρ_n such that $M_n = a_n \alpha_n^{\rho_n}$.

Proposition 3.10. — *We have $1/((C_{26})^2) \leq a_n \leq 1/C_{26}$ and $\rho_n < 1$.*

Moreover, if (R_n) holds, then $\rho_{n+1} \geq \rho_n + \eta_n(1-\rho_n)$, and if (R'_n) holds, then $\rho_{n+1} \geq ((k+1)/2 - \theta)(1-\eta_n)\rho_n$. In particular, the sequence $(\rho_n)_{n \geq n_2}$ is increasing.

Remark 3.11. — The threshold between the alternatives (R_n) and (R'_n) is controlled with a parameter θ, which could be freely chosen such that $\theta > 0$ and $(k+1)/2 - \theta \geq (1+\beta+\theta)(1+\theta)$. When θ increases, the number n_3 of occurrences of (R'_n) increases. When n_3 increases, all other quantities being equal, the bound on the norm of the conjugacy increases. Moreover, if θ gets too large, we can no longer show that n_3 is finite (see Proposition 3.15), and therefore, we can no longer estimate the norm of the conjugacy.

On the other hand, when θ decreases, C_{25} increases. It increases the number n_2 above which we consider the alternatives (R_n) and (R'_n). C_{28} increases too (see Proposition 3.20). When C_{25} and C_{28} increase, all other quantities being equal, the bound on the norm of the conjugacy increases. Moreover, when $\theta \to 0$, $C_{25} \to +\infty$, which makes this bound on the conjugacy diverge.

Thus, the variation of θ has contradictory influences on the bound of the norm of the conjugacy, and there is a choice of θ that optimizes this bound. However, in this paper, we do not seek this optimal θ, since it would complicate further the expression of the final estimate. Instead, in estimation (2), we fix $\theta = 1/2$, which allows simplifying the expression of the estimate. In estimation (3), we take $\theta \to 0$, which also allows simplifying the estimate.

Proof of Proposition 3.10. — For any $n \geq n_2$, since $n_2 \geq n_1$,

$$a_{n_2} = \frac{1}{C_{26}^2} \leq a_n \leq a_{n_2} \prod_{n=n_1}^{+\infty} \left(\frac{1 + M_n^\theta}{1 - C_{21}^{f,k} M_n^{1/2}} \right) \leq \frac{C_{25}}{C_{26}^2} \leq \frac{1}{C_{26}}$$

and since

$$\alpha_n^{\rho_n} > a_n \alpha_n^{\rho_n} = M_n \geq \alpha_n$$

then $\rho_n < 1$.

Second, if (R_n) holds, then by applying Corollary 3.6, we have:

$$M_{n+1} \leq \frac{1 + M_n^\theta}{1 - C_{21}^{f,k} M_n^{1/2}} M_n \frac{\alpha_{n+1}}{\alpha_n}. \tag{31}$$

Therefore,

$$M_{n+1} = a_{n+1} \alpha_{n+1}^{\rho_{n+1}} \leq a_{n+1} \alpha_{n+1} \alpha_n^{\rho_n - 1} = a_{n+1} \alpha_{n+1} \alpha_{n+1}^{(1-\eta_n)(\rho_n - 1)}$$

and then:

$$\rho_{n+1} - 1 \geq (1 - \eta_n)(\rho_n - 1)$$

hence the estimation:

$$\rho_{n+1} \geq \rho_n + \eta_n (1 - \rho_n).$$

If (R'_n) holds, since $C_{21}^{f,k} M_n^{1/2} \leq 1/2$, then by applying Corollary 3.6, we obtain:

$$M_{n+1} \leq 4 C_{21}^{f,k} M_n^{(k+1)/2 - \theta}.$$

Moreover, since $a_n \leq 1/C_{26} < 1$, then:

$$a_n^{(k+1)/2-\theta} \leq a_n^{(1+\beta+\theta)(1+\theta)} = a_n a_n^{(1+\beta+\theta)(1+\theta)-1} \leq \frac{a_n}{C_{26}^{(1+\beta+\theta)(1+\theta)-1}} \leq \frac{a_n}{4C_{21}^{f,k}}.$$

Therefore, by combining these two estimations, we obtain:

$$\begin{aligned} a_{n+1} \alpha_{n+1}^{\rho_{n+1}} = M_{n+1} &\leq 4 C_{21}^{f,k} M_n^{(k+1)/2-\theta} \\ &\leq 4 C_{21}^{f,k} a_n^{(k+1)/2-\theta} \alpha_n^{\rho_n((k+1)/2-\theta)} \leq a_n \alpha_n^{\rho_n((k+1)/2-\theta)}. \end{aligned}$$

Moreover, since $a_{n+1} = a_n$, then

$$1 \leq \alpha_{n+1}^{(\rho_n((k+1)/2-\theta))(1-\eta_n) - \rho_{n+1}}$$

hence the estimation:

$$\rho_{n+1} \geq (\rho_n((k+1)/2 - \theta))(1 - \eta_n). \qquad \square$$

The reader can notice that until now, we have not used the Diophantine condition on α yet. Now, we introduce this condition in order to estimate ρ_{n_2} from below (Proposition 3.12), and in order to determine a bound ρ above which (R_n) always occurs (Proposition 3.13).

PROPOSITION 3.12. — *If $\beta > 0$, we have the estimation:*

$$\rho_{n_2} \geq \frac{\log 2}{((1+\beta)^{n_2+1}-1)\log(1/C_d)/\beta}.$$

If $\beta = 0$, we have the estimation:

$$\rho_{n_2} \geq \frac{\log 2}{(n_2+1)\log(1/C_d)}.$$

Proof. — Since α is Diophantine, we have: $\alpha_{n+1} \geq C_d \alpha_n^{1+\beta}$. Therefore, for $\beta > 0$,

$$\log\left(\frac{1}{\alpha_{n+1}}\right) + \frac{\log(1/C_d)}{\beta} \leq (1+\beta)\left(\log\left(1/\alpha_n\right) + \frac{\log(1/C_d)}{\beta}\right)$$

and since $\alpha_{-1} = 1$, then by iteration, for any $n \geq 0$,

$$\log\left(1/\alpha_n\right) \leq \left((1+\beta)^{n+1}-1\right)\frac{\log(1/C_d)}{\beta}.$$

If $\beta = 0$, we have:

$$\log\left(1/\alpha_n\right) \leq (n+1)\log(1/C_d).$$

Moreover, since $\rho_{n_2} = -\log(M_{n_2}/a_{n_2})/\log(1/\alpha_{n_2})$ and $M_{n_2}/a_{n_2} \leq 1/2$, then we get Proposition 3.12. □

PROPOSITION 3.13. — *Let $\beta_1 = \beta + \frac{2\log(1/C_d)}{(n_2-1)\log 2}$. If*

$$\rho_n \geq \frac{\beta_1}{(k-1)/2-\theta} = \rho \tag{32}$$

then (R_n) occurs.

REMARK 3.14. — Note that $\rho < 1$, because $(k+1)/2 - \theta \geq (1+\beta+\theta)(1+\theta)$ and $\beta_1 \leq \beta + 1/2$.

Proof. — Since $\alpha_n \leq (1/2)^{\frac{n-1}{2}}$, then

$$0 < \frac{\log C_d}{\log \alpha_n} \leq \frac{-\log C_d}{\frac{n-1}{2}\log 2}. \tag{33}$$

Furthermore, since $\alpha_{n+1} = \alpha_n^{\frac{1}{1-\eta_n}} \geq C_d \alpha_n^{1+\beta}$, then

$$\frac{1}{1-\eta_n}\log\alpha_n \geq \log C_d + (1+\beta)\log\alpha_n$$

and since $\log \alpha_n < 1$ for $n \geq 0$, then by (33),

$$\frac{1}{1-\eta_n} - 1 \leq \beta + \frac{\log C_d}{\log \alpha_n} \leq \beta + \frac{\log(1/C_d)}{\frac{n-1}{2}\log 2}.$$

Therefore, if estimation (32) holds, then

$$\left(\frac{k-1}{2} - \theta\right)\rho_n + 1 - \frac{1}{1-\eta_n} \geq 0$$

and therefore,

$$\left(\frac{1}{\alpha_n}\right)^{\left(\frac{k-1}{2}-\theta\right)\rho_n + 1 - \frac{1}{1-\eta_n}} \geq 1.$$

Hence

$$\begin{aligned} M_n \frac{\alpha_{n+1}}{\alpha_n} &= a_n \alpha_n^{\rho_n} \frac{\alpha_{n+1}}{\alpha_n} \geq a_n \alpha_n^{(\frac{k+1}{2}-\theta)\rho_n} = M_n^{\frac{k+1}{2}-\theta} a_n^{1-(\frac{k+1}{2}-\theta)} \\ &\geq M_n^{\frac{k+1}{2}-\theta} C_{26}^{\frac{k+1}{2}-\theta-1} \geq M_n^{\frac{k+1}{2}-\theta} C_{26}^{(1+\beta+\theta)(1+\theta)-1}. \end{aligned}$$

Therefore,

$$M_n \frac{\alpha_{n+1}}{\alpha_n} \geq C_{21}^{f,k} M_n^{\frac{k+1}{2}-\theta}. \qquad \square$$

PROPOSITION 3.15. — *Let n_3 be the number of times the alternative (R'_n) occurs. We have:*

$$n_3 - n_2 \leq \max\left(0, \frac{\log(\rho/\rho_{n_2})}{\log\left(\frac{(k+1)/2-\theta}{1+\beta_1}\right)}\right). \tag{34}$$

Proof. — If $\rho_{n_2} \geq \rho$, then (R'_n) does not occur for any $n \geq n_2$. We suppose $\rho_{n_2} < \rho$. For any $n \geq n_2$, since

$$((k+1)/2 - \theta)(1-\eta_n) \geq \frac{(k+1)/2-\theta}{1+\beta_1}$$

then

$$\rho_n \geq \left(\frac{(k+1)/2-\theta}{1+\beta_1}\right)^{n-n_2} \rho_{n_2}.$$

Moreover,

$$\left(\frac{(k+1)/2-\theta}{1+\beta_1}\right)^{n-n_2} \rho_{n_2} \geq \rho$$

when

$$n \geq n_2 + \frac{\log(\rho/\rho_{n_2})}{\log\left(\frac{(k+1)/2-\theta}{1+\beta_1}\right)}. \qquad \square$$

The next proposition gives a lower bound on α_{n_4}, which allows computing a bound on the C^1-conjugacy.

Proposition 3.16. — *Let $n_4 \geq 0$ be the smallest integer such that for any $n \geq n_4$, (R_n) occurs. We have:*

$$\alpha_{n_4} \geq C_d^{\exp((n_3+1+\rho/(1-\rho))(1+\beta_1))}.$$

Proof. — First, we suppose $n_4 \geq n_2 + 1$ We need the lemma:

Lemma 3.17. — *The set*

$$\{p \geq n_2 \ / \ \sum_{n=n_2}^{p} \eta_n \geq n_3 - n_2 + \rho/(1-\rho)\}$$

is not empty. Let n_5 be its minimum. We have $\rho_{n_5+1} \geq \rho$. In particular, for this integer n_5, we have that for any $n \geq n_5+1$, (R_n) occurs. Moreover, $n_5 \geq n_4 - 1$.

Proof. — First, let us show the existence of n_5. By absurd, suppose that

$$\sum_{n=n_2}^{+\infty} \eta_n < n_3 - n_2 + \rho/(1-\rho).$$

For any $1 > x \geq 0$,

$$\log\left(\frac{1}{1-x}\right) \leq \frac{x}{1-x}.$$

Therefore, for any integer $p \geq n_2 + 1$,

$$\prod_{n=n_2}^{p-1}\left(\frac{1}{1-\eta_n}\right) \leq \exp\left(\sum_{n=n_2}^{p-1} \frac{\eta_n}{1-\eta_n}\right).$$

Moreover, $\frac{1}{1-\eta_n} \leq 1 + \beta_1$ for any $n \geq 1$. Therefore,

$$\sum_{n=n_2}^{p-1} \frac{\eta_n}{1-\eta_n} \leq (n_3 - n_2 + \rho/(1-\rho))(1+\beta_1).$$

Since $\eta_n \leq 1$, then $\sum_{n=0}^{n_2-1} \eta_n \leq n_2$. Therefore,

$$\sum_{n=0}^{p-1} \frac{\eta_n}{1-\eta_n} \leq (n_3 + \rho/(1-\rho))(1+\beta_1).$$

Moreover, since $\alpha_0 = \alpha \geq C_d$ then for any $p \geq n_2 + 1$:

$$\alpha_p = \alpha_0^{\prod_{n=0}^{p-1}\left(\frac{1}{1-\eta_n}\right)} \geq C_d^{\exp((n_3+\rho/(1-\rho))(1+\beta_1))}.$$

However, since $\alpha_p \geq 2\alpha_{p+2}$, then $\alpha_p \to 0$ when $p \to +\infty$. Hence the contradiction and the existence of n_5. Note that $n_5 + 1 \geq n_4$.

Second, let us show that $\rho_{n_5+1} \geq \rho$. If there is $n_6 \leq n_5$ such that $\rho_{n_6} \geq \rho$, then $\rho_{n_5+1} \geq \rho$ because the sequence ρ_n is increasing. Otherwise, for any $n \leq n_5$, we have: $\rho_n \leq \rho$.

Let $E_1 = \{n_5 \geq n \geq n_2 \,/\, (R_n) \text{ occurs}\}$ and $E_2 = \{n_5 \geq n \geq n_2 \,/\, (R'_n) \text{ occurs}\}$.

We have:

$$n_3 - n_2 + \frac{\rho}{1-\rho} \leq \sum_{n=n_2}^{n_5} \eta_n = \sum_{n\in E_1} \eta_n + \sum_{n\in E_2} \eta_n \leq \sum_{n\in E_1} \eta_n + n_3 - n_2.$$

Therefore,

$$\sum_{n\in E_1} \eta_n \geq \rho/(1-\rho).$$

Since ρ_n is increasing and $\rho_n \leq \rho$, we get:

$$\begin{aligned}
\rho_{n_5+1} &= \rho_{n_2} + \sum_{n=n_2}^{n_5} \rho_{n+1} - \rho_n \\
\rho_{n_5+1} &\geq \rho_{n_2} + \sum_{n\in E_1} \rho_{n+1} - \rho_n \geq \rho_{n_2} + \sum_{n\in E_1} (1-\rho_n)\eta_n \\
&\geq \rho_{n_2} + (1-\rho) \sum_{n\in E_1} \eta_n \geq \rho.
\end{aligned}$$

□

Now, let us show Proposition 3.16. Since $\eta_n \leq 1$ for any n, then we have:

$$n_3 - n_2 + 1 + \frac{\rho}{1-\rho} > \sum_{n=n_2}^{n_5} \eta_n \geq n_3 - n_2 + \frac{\rho}{1-\rho}.$$

Since

$$n_3 - n_2 + \frac{\rho}{1-\rho} + 1 \geq \sum_{n=n_2}^{n_5} \eta_n \geq \sum_{n=n_2}^{n_4-1} \eta_n$$

then by proceeding in the same way as in the first part of the proof of Lemma 3.17, we obtain:

$$\alpha_{n_4} \geq C_d^{\exp((n_3+1+\rho/(1-\rho))(1+\beta_1))}. \tag{35}$$

Finally, if $n_4 \leq n_2$, then as in the proof of Lemma 3.17,

$$\alpha_{n_2} = \alpha_0^{\prod_{n=0}^{n_2-1}\left(\frac{1}{1-\eta_n}\right)} \geq C_d^{\exp(n_2(1+\beta_1))}.$$

Therefore, the estimation given in Proposition 3.16 still holds. □

Having bounded α_{n_4} from below, we show how this bound is related to M_n/m_n (and therefore, how this is related to the conjugacy).

PROPOSITION 3.18. — *Let $n \geq 1$. For any $j \leq n$,*

$$\frac{M_j}{m_j} \leq 3|Df|_0^{\frac{2}{\alpha_n}}.$$

Proof. — We need the following lemma, which is in [15, p. 140]:

LEMMA 3.19. — *For any $x \in \mathbb{R}$, let $J_x = [f_n^{-1}(x), f_n(x)]$. The intervals $f^i T^k(J_x)$, $0 \leq i < q_{n+1}$, $k \in \mathbb{Z}$, cover $\mathbb{R}$.*

First, note that since $f(x+1) - f(x) = 1$, then $|Df|_0 \geq 1$. Let $x, y \in \mathbb{R}$ be such that $M_n = m_n(x)$ and $m_n = m_n(y)$. Let $0 \leq i < q_{n+1}$ and $k \in \mathbb{Z}$ be such that $x \in f^i T^k(J_y)$. We have:

$$f^{i-q_n}(y) - k + p_n \leq x \leq f^{i+q_n}(y) - k - p_n$$
$$f^i(y) - k \leq f_n(x) \leq f^{i+2q_n}(y) - k - 2p_n.$$

Therefore, $[x, f^{q_n}(x)] \subset [f^{i-q_n}(y) - k + p_n, f^{i+2q_n}(y) - k - 2p_n]$. This implies:

$$\begin{aligned}
M_n &\leq f^{i+2q_n}(y) - k - 2p_n - f^{i-q_n}(y) - k + p_n \\
M_n &\leq f^{i+2q_n}(y) - f^{i+q_n}(y) + f^{i+q_n}(y) - f^i(y) + f^i(y) - f^{i-q_n}(y) - 3p_n \\
M_n &\leq f^{i+q_n}(f^{q_n}(y)) - f^{i+q_n}(y+p_n) + f^i(f^{q_n}(y)) - f^i(y+p_n) \\
&\quad + f^{i-q_n}(f^{q_n}(y)) - f^{i-q_n}(y+p_n) \\
M_n &\leq \left(|Df^{i+q_n}|_0 + |Df^i|_0 + |Df^{i-q_n}|_0\right)\left(f^{q_n}(y) - y - p_n\right)
\end{aligned}$$

and therefore,

$$\frac{M_n}{m_n} \leq \left(|Df^{i+q_n}|_0 + |Df^i|_0 + |Df^{i-q_n}|_0\right) \leq 3|Df|_0^{q_n+q_{n+1}}.$$

Since $q_n + q_{n+1} \leq 2q_{n+1} \leq \frac{2}{\alpha_n}$, and since α_n is decreasing, we obtain Proposition 3.18. □

PROPOSITION 3.20. — *For any $n \geq 1$,*

$$\frac{M_n}{m_n} \leq C_{28} \frac{M_{n_4}}{m_{n_4}} \tag{36}$$

with:

$$C_{28} = \exp\left(\frac{2(2C_{26}^2)^\theta - 1}{(2C_{26}^2)^\theta - 1} \frac{(C_{2.3}^f)^{(n_2-1)\theta}}{1 - (C_{2.3}^f)^\theta} + 3C_{21}^{f,k} \frac{(C_{2.3}^f)^{(n_2-1)/2}}{1 - (C_{2.3}^f)^{1/2}} \right). \tag{37}$$

Proof. — Since for any $n \geq n_4$, (R_n) occurs, then by Corollary 3.6, we have:

$$\frac{M_{n+1}}{M_n} \leq \frac{1 + M_n^\theta}{1 - C_{21}^{f,k} M_n^{1/2}} \frac{\alpha_{n+1}}{\alpha_n}$$
$$\frac{m_{n+1}}{m_n} \geq \frac{1 - M_n^\theta}{1 + C_{21}^{f,k} M_n^{1/2}} \frac{\alpha_{n+1}}{\alpha_n}.$$

Therefore,

$$\frac{M_{n+1}/m_{n+1}}{M_n/m_n} \leq \frac{1 + M_n^\theta}{1 - M_n^\theta} \frac{1 + C_{21}^{f,k} M_n^{1/2}}{1 - C_{21}^{f,k} M_n^{1/2}}. \tag{38}$$

Therefore, for any $n \geq n_4$,

$$\frac{M_n}{m_n} \leq \frac{M_{n_4}}{m_{n_4}} \prod_{j=n_4}^{+\infty} \frac{1+M_j^\theta}{1-M_j^\theta} \frac{1+C_{21}^{f,k} M_j^{1/2}}{1-C_{21}^{f,k} M_j^{1/2}}.$$

We finish the proof of estimation (25), using that $n_4 \geq n_2$) and for $j \geq n_2$, $M_j \leq 1/(2C_{26}^2)$. Detailed computations are in [3]. □

Proof of estimation (2). — By combining Propositions 3.18 and 3.20, and since by [14, p. 348], $|Dh|_0 \leq \sup_{n\geq 0} M_n/m_n$, we get:

$$|Dh|_0 \leq C_{29} |Df|_0^{\frac{2}{\alpha_{n_4}}}, \tag{39}$$

with

$$C_{29} = 3C_{28}.$$

We estimate C_{28}: since $(2x-1)/(x-1) = 2+1/(x-1)$, since $(C_{2.3}^f)^{(n_2-1)\theta} \leq 1/(2(C_{26})^2)^\theta$ and since $\theta \leq 1/2$, then:

$$C_{28} \leq \exp\left(\left(2 + \frac{1}{(2(C_{26})^2)^\theta - 1} + 3C_{21}^{f,k}\right) \frac{1}{(2(C_{26})^2)^\theta (1-(C_{2.3}^f)^\theta)}\right).$$

Since $C_{2.3}^f \geq 1$, we get:

$$|Dh|_0 \leq C_{30} |Df|_0^{\frac{2}{\alpha_{n_4}}}, \tag{40}$$

with

$$C_{30} = 3e \wedge \left(\left(2 + \frac{1}{(2(C_{26})^2)^\theta - 1} + 3C_{21}^{f,k}\right) \frac{1}{(2(C_{26})^2)^\theta (1-(C_{2.3}^f)^\theta)}\right).$$

We estimate C_{30} using expressions of θ (see (24)), of $C_{2.3}^f$ (see Lemma 2.3) and of C_{26} (see (26) and Proposition 3.5).

We estimate α_{n_4} using Propositions 3.16, 3.15, 3.13, 3.12, and the expressions of n_2 (see (2)) and estimates of θ, $C_{2.3}^f$ and C_{26}. We get:

$$|Dh|_0 \leq C_2(k, \beta, C_d, |Df|_0, W(f), |Sf|_{k-3})$$

where C_2 is the combination of the following functions:

1. $C_{2.3}^f = \left(1 + e \wedge \left(-6\sqrt{2} e^{2W(f)} \left(\max\left(|Sf|_0^{1/2}, 1\right)\right)\right)\right)^{-1/2}$ (since $|Sf|_0 \leq |Sf|_{k-3}$, we can estimate $C_{2.3}^f$ in function of $W(f), |Sf|_{k-3}$);
2. $C_{21}^{f,k} = (k+3)^{(k+3)!} e^{(k+3)! W(f)} (\max(1, |Sf|_{k-3}))^{(k+1)!}$;
3. $\theta = \min\left(1/2, \left(\frac{3+\beta}{4}\right)\left(-1 + \sqrt{1 + \frac{2(k-(2\beta+1))}{(3+\beta)^2}}\right)\right)$;
4. $C_{26} = \max\left(e^{\frac{2}{1-(C_{2.3}^f)^\theta}}, (4C_{21}^{f,k})^{\frac{1}{(1+\beta+\theta)(1+\theta)-1}}\right)$;
5. $n_2 = \lfloor \max\left(-\frac{\log(2C_{26}^2)}{\log C_{2.3}^f} + \frac{2\log(1/C_d)}{\theta \log 2} + 2, 2 + \frac{(2C_{21}^{f,k})}{\log((C_{2.3}^f)^{1/2})}\right) \rfloor$;

6. $\beta_1 = \beta + \frac{2\log(1/C_d)}{(n_2-1)\log 2}$;
7. $n_3 = \lceil \frac{1}{\log\left(\frac{(k+1)/2-\theta}{1+\beta_1}\right)} \left(n_2(1+\log(1+\beta)) + \log\left(\frac{(n_2+1)\log(1/C_d)}{\log 2}\right)\right)\rceil$;
8. $\rho = \frac{\beta_1}{\frac{k-1}{2}-\theta}$;
9. $\alpha'_{n_4} = C_d \wedge \left(e \wedge \left(\left(n_3+1+\frac{\rho}{1-\rho}\right)(1+\beta_1)\right)\right)$;
10. $C_{30} = 3e \wedge \left(\left(2+\frac{1}{(2(C_{26})^2)^\theta-1}+3C_{21}^{f,k}\right)\frac{1}{(2(C_{26})^2)^\theta(1-(C_{2.3}^f)^\theta)}\right)$;
11. $|Dh|_0 \leq C_{30}|Df|_0^{\frac{2}{\alpha'_{n_4}}}$.

Note that we have a bound $\alpha'_{n_4} \leq \alpha_{n_4}$, but we do not know the value of α_{n_4}. □

In order to obtain relatively simple estimates, we can take the parameter θ (defined in (24)) either vanishingly close to 0 (estimation (3)), or fixed independently of the other parameters (estimation (2)).

3.3. Proof of estimation (2). — When θ is fixed independently of the other parameters, we need to assume that $k-2\beta-1$ is sufficiently large, in order to keep $(k+1)/2-\theta \geq (1+\beta+\theta)(1+\theta)$. To illustrate this case, we take $\theta = 1/2$, which requires $k \geq 3\beta+9/2$ (for any fixed θ, we cannot obtain an assumption of the form $k \geq 2\beta+u$ for some number u: we necessarily have $k \geq \lambda\beta+u$ with $\lambda > 2$).

To simplify the function C_2, we successively estimate C_{28}, α'_{n_4} and n_2. Details of the computations are in [3]. We have:

$$C_{28} \leq \exp\left(C_{26}^{\frac{3\beta+1}{2}}\right)$$

where C_{28} and C_{26} defined in Proposition 3.20 and (26) respectively. We also have:

(41)
$$\frac{1}{\alpha'_{n_4}} \leq \left(\frac{1}{C_d}\right) \wedge e \wedge \left((\beta+3/2)\left(2+\frac{n_2}{\log(3/2)}\left(2+\log(1+\beta)+\log\log(1/C_d)\right)\right)\right)$$

(42)
$$n_2 \leq C_{31}(W(f),|Sf|_0)(k+4)!(1+\log(\max(1,|Sf|_{k-3})))(1+\log(1/C_d))$$

with:

$$C_{42}(W(f),|Sf|_0) = e^{(2)} \wedge (3W(f)+2\log(\max(1,|Sf|_0))+4).$$

By combining these estimates, and after some computations (that can be found in [3]), we obtain:

$$|Dh|_0 \leq e^{(3)} \wedge \big((\beta+3/2)(\beta+3)(k+4)!C_2(1+\log(1/C_d))^2 \\ (1+\log(\max(1,|Sf|_{k-3})))\big) \tag{43}$$

with $C_2 = 10(1+\log^{(2)}(|Df|_0))e^{(2)} \wedge (3W(f)+2\log(\max(1,|Sf|_0)))+2)$.

This estimation of $|Dh|_0$ is increasing with k. Therefore, to obtain a bound as low as we can, we take $k = \lceil 3\beta + 9/2 \rceil$. We obtain:

$$|Dh|_0 \leq e^{(3)} \wedge \big(C_2[\beta]C_2[C_d]C_2[|Df|_0, W(f), |Sf|_0]C_2[|Sf|_{\lceil 3\beta+3/2\rceil}]\big)$$

with:

1. $C_2[\beta] = (\lceil 3\beta + 21/2 \rceil)!$;
2. $C_2[C_d] = (1+\log(1/C_d))^2$;
3. $C_2[|Df|_0, W(f), |Sf|_0]$
 $= 10(1+\log^{(2)}(|Df|_0))e^{(2)} \wedge (3W(f)+2\log(\max(1,|Sf|_0)))+4)$;
4. $C_2[|Sf|_{\lceil 3\beta+3/2\rceil}] = 1+\log(\max(1,|Sf|_{\lceil 3\beta+3/2\rceil}))$.

3.4. Proof of estimation (3). — Let $\delta = k - 2\beta - 1$ and $\beta > 0$. We make a Taylor expansion with $\delta \to 0$ (since $k \geq 3$, this implies automatically $\beta > 0$). To estimate $|Dh|_0$, we successively estimate n_2, n_3, $\rho/(1-\rho)$ and α'_{n_4}. Details of the computations can be found in [3].

Since $\beta > 0$, then for δ sufficiently small, $C_{26} = e^{\overline{\frac{2}{1-(C^f_{2.3})^\theta}}}$. This makes the dependence on k and $|Sf|_{k-3}$ disappear. We have:

$$n_2 = \left(\frac{4}{(\log C^f_{2.3})^2} + \frac{2\log(1/C_d)}{\log 2}\right)\frac{1}{\theta} + o\left(\frac{1}{\theta}\right).$$

We denote $C_{32} = \frac{4}{(\log C^f_{2.3})^2}$ and $C_{33} = \frac{2\log(1/C_d)}{\log 2}$. We also have:

$$n_3 \leq \frac{(1+\beta)^2(C_{33}+C_{32})}{\theta^2(4+2\beta)} + o\left(\frac{1}{\theta^2}\right). \tag{44}$$

Moreover, $\rho/(1-\rho) = o(1/\delta^2)$ (we recall that $1/\delta = o(1/\delta^2)$).

We have:

$$\alpha'_{n_4} \geq C_d \wedge \left(e \wedge \left(\frac{C_3}{\delta^2} + o\left(\frac{1}{\delta^2}\right)\right)\right)$$

with:

$$C_3[k, C_d, W(f), |Sf|_0] = \frac{(k+5)^2(k+1)^3}{2k\log 2}\left(\frac{2\log 2}{(\log C^f_{2.3})^2} + \log(1/C_d)\right).$$

We recall that:

$$C^f_{2.3} = \left(1 + e \wedge \left(-6\sqrt{2}e^{2W(f)}\max\left(|Sf|_0^{1/2}, 1\right)\right)\right)^{-1/2}.$$

We have:

$$|Dh|_0 \leq C_{30}|Df|_0 \wedge \left(e \wedge \left(e \wedge \left(\frac{1}{\delta^2}C_3 + o(1/\delta^2)\right)\right)\right).$$

Since $|\log\log|Df|_0| \leq e^{o(1/\delta^2)}$ and $|\log C_{30}| \leq e \wedge e \wedge (o(1/\delta^2))$, we conclude:

$$|Dh|_0 \leq e^{(3)} \wedge \left(\frac{1}{\delta^2}C_3[k, C_d, W(f), |Sf|_0] + o(1/\delta^2)\right).$$

In estimations (2) and (3), three iterations of the exponential appear. This calls for explanation. A first exponential comes from the estimation $|Df^n|_0 \leq C|Df|_0^{2/\alpha_{n_4}}$, where n_4 is the rank above which the "favorable" case always occurs. A second exponential comes from writing $\alpha_{n_4} = \alpha_0^{\prod_{n=0}^{n_4-1}\left(\frac{1}{1-\eta_n}\right)}$. We bound each $\frac{1}{1-\eta_n}$ using the Diophantine condition, and a third exponential comes from the estimation $\prod_{n\in E_2}\left(\frac{1}{1-\eta_n}\right) \leq C^{n_3-n_2}$, where E_2 is the set and $n_3 - n_2$ is the number of "unfavorable" cases.

This number is bounded logarithmically, by $C\log C_{26}$. However, C_{26} is bounded by an exponential of the parameters. Indeed, when δ is small, $C_{26} \sim e^{\frac{1}{\delta}}$, which gives estimation (3). Otherwise, $C_{26} \sim C_{21}^{f,k}$. In this case, $C_{21}^{f,k} \sim C^k$. Indeed, in Lemma 20, we need $k-1$ iterations to estimate $|D^{k-1}\log Df^p(x)|_0$ $(p \leq q_{n+1})$, an estimation that, in turn, gives an estimate of $C_{21}^{f,k}$. This gives estimation (2). Thus, we have explained the occurence of three exponentials in the estimates.

Since the number of "unfavorable" cases drives the dominant term of these estimates, they can be substantially improved when the "favorable" case always occurs. In Remark 1.5, we make this assumption, together with the assumption $k \geq 3\beta + 9/2$. Thus, we can take $\theta = 1/2$, and a sufficient condition for the occurrence of the "favorable" case is:

$$\frac{\alpha_{n+1}}{\alpha_n} \geq C_{21}^{f,k}(C_{2.3}^f)^{(n-1)\frac{k}{2}} = C_8(n, k, \beta, W(f), |Sf|_{k-3})$$

which decreases geometrically with n.

We recall that:

$$C_{2.3}^f = \left(1 + e \wedge \left(-6\sqrt{2}e^{2W(f)}\max\left(|Sf|_0^{1/2}, 1\right)\right)\right)^{-1/2}$$
$$C_{21}^{f,k} = (k+3)^{(k+3)!}e^{(k+3)!W(f)}(\max(1, |Sf|_{k-3}))^{(k+1)!}.$$

We obtain the following estimation:

$$|Dh|_0 \leq \exp\left(C_{1.5}[k, W(f), |Sf|_{k-3}]^{C_{1.5}(\beta)}\right)|Df|_0^2$$

with:

$$C_{1.5}[k, W(f), |Sf|_{k-3}] = \max\left(e^{\frac{2}{1-(C_{2.3}^f)^{1/2}}}, 4C_{21}^{f,k}\right), \quad C_{1.5}[\beta] = \frac{3\beta+1}{2}.$$

Note that numbers of constant type do not always satisfy (8) for any n (they only satisfy it above some rank). Moreover, there are numbers satisfying (8) that are not of constant type.

4. C^k estimations

In this section, we compute estimates of higher order derivatives of the conjugacy h in function of bounds on the first derivative of h. We compute the values of some of the constants appearing in Yoccoz's proof [14] (we do not compute the dependency in k). However, in order to obtain our result, we need to modify slightly the proof of Proposition 5 in the work of Yoccoz [14]. If we strictly followed Yoccoz's proof, we would find an estimate that depends on the C^1-norm of h, and on $k, \beta, C_d, W(f), |Sf|_{k-3}, |D^{k-1}\log Df|_0$, but this estimate would diverge as f gets closer to a rotation.

The proof has four steps. We let real numbers $0 \leq \gamma_0 < \gamma_1 < g(\gamma_0)$, with $g(\gamma_0) = ((1+\beta)\gamma_0 + k - (2+\beta)) \,/\, (2+\beta)$, and we let an integer N. In the first three steps, we compute $\| \log Df^N \|_{\gamma_1}$ in function of $\sup_{p\geq 0} \| \log Df^p \|_{\gamma_0}$ (estimation (45)). In the first step, using convexity estimations (Proposition 4.7) and a consequence of the Faa-di-Bruno formula (Lemma 4.6), we establish an estimation of $\| \log Df^{q_s} \|_{\gamma}$ for $0 \leq \gamma \leq k-1$ (Lemma 4.8).

In the second step, we obtain an estimation of $\| \log Df^{nq_s} \|_{\gamma}$, $0 \leq n \leq q_{s+1}/q_s$ for $0 \leq \gamma \leq \gamma_1$ (estimation (58)).

In the third step, we write $N = \sum_{s=0}^{S} b_s q_s$, with b_s integers satisfying $0 \leq b_s \leq q_{s+1}/q_s$, in order to get an estimation of $\| \log Df^N \|_{\gamma_1}$ in function of $\sup_{p\geq 0} \| \log Df^p \|_{\gamma_0}$. Thus, in these three steps, the aim is to establish the following proposition:

PROPOSITION 4.1. — *Let* $0 \leq \gamma_0 < \gamma_1 < g(\gamma_0) = \frac{(1+\beta)\gamma_0 + k - (2+\beta)}{2+\beta}$. *We have:*

$$\begin{aligned}(45)\quad & \| \log Df^N \|_{\gamma_1} \\ & \leq e \wedge \left(C_{65}(k,\beta) \left(\log(C_d^{-1}) + C_{42}^{f,k} + C(k) \left(1 + \sup_{p\geq 0} \| \log Df^p \|_{\gamma_0} \right) \right)^4 \right).\end{aligned}$$

The expressions of C_{65} and $C_{42}^{f,k}$ are given in page 710. $C(k)$ denotes a positive numerical function of the variable k, with an explicit formula that we do not compute, and this is why it does not have a sub-index.

In the fourth step, we iterate this reasoning: the inductive step is given by Proposition 4.1: if we have an estimate of $\sup_{N\geq 0} \| \log Df^N \|_{\gamma_i}$, then we can get an estimate of $\sup_{N\geq 0} \| \log Df^N \|_{\gamma_{i+1}}$ for $\gamma_i < \gamma_{i+1} < g(\gamma_i)$. We can initiate the induction with $\gamma_0 = 0$, because we have C^1 estimates. We take $\gamma_{i+1} = \frac{1}{2}(g(\gamma_i) + \gamma_i)$ and we have:

$\lim_{i\to+\infty}\gamma_i = k-2-\beta$. Thus, we can obtain an estimation of $\|Dh\|_{k-2-\beta-\eta}$.

In all the rest of the paper, we denote:

$$M' = \exp\left(\sup_{i\geq 0} |\log D(f^i)|_0\right)$$
$$M = \exp\left(\sup_{i\geq 0} \|\log D(f^i)\|_{\gamma_0}\right).$$

Note that $M \geq M' \geq 1$.

The constants $C(k)$ only depend on k. This dependency is not specified. C denotes a numerical constant (independent of all parameters) of unspecified value. Numbered constants C_i, i integer, can depend on constants $C(k)$ or C.

4.1. Estimation of $\|\log Df^{q_s}\|_\gamma$, $0 \leq \gamma \leq k-1$.— The following lemma is a converse of the implication used in [14, p. 348], according to which if M_n/m_n is bounded, then the conjugacy of f to a rotation is C^1:

Lemma 4.2. — *Let $M' = \exp\left(\sup_{i\geq 0}|\log D(f^i)|_0\right)$. Then we have the following estimation:*

$$\frac{M_n}{m_n} \leq M'.$$

Proof. — Let $\epsilon > 0$, x, y be such that $M_n = |f^{q_n}(x) - x - p_n|$ and $m_n = |f^{q_n}(y) - y - p_n|$.

Since $f^p(y)_{p\geq 0}$ mod 1 is dense in $\mathbb{T}^1$, then there is a positive integer l and an integer k such

that $|f^l(y) - x - k| \leq \min\left(\frac{\epsilon}{|Df^{q_n}|_0}, \epsilon\right)$. Moreover, $m_n = |f^{q_n}(y-k) - (y-k) - p_n|$.

Then we obtain:

$$\begin{aligned}|f^{q_n}(x) - x - p_n|\\ &\leq |f^{q_n}(x) - f^{q_n}(f^l(y)-k)| + |f^l(f^{q_n}(y-k)) - f^l(y-k)| + |f^l(y-k) - x|\\ &\leq |Df^l|_0|f^{q_n}(y-k) - (y-k) - p_n| + 2\epsilon \leq M'm_n + 2\epsilon\end{aligned}$$

for every $\epsilon > 0$. □

The C^γ-norms, when γ varies in $\mathbb{R}^+$, are related with each other by convexity inequalities (also called interpolation inequalities):

Proposition 4.3. — *Let $\gamma_2, \gamma_3 \in \mathbb{R}^+$ with $0 \leq \gamma_2 \leq \gamma_3$ and $\gamma_3 > 0$. For any $\phi \in C^{\gamma_3}(\mathbb{T}^1)$, we have:*

$$\|\phi\|_{\gamma_2} \leq C(\gamma_3)\,\|\phi\|_0^{\frac{\gamma_3-\gamma_2}{\gamma_3}}\,\|\phi\|_{\gamma_3}^{\frac{\gamma_2}{\gamma_3}}.$$

Using these convexity inequalities, we establish various relations, among which is the important Corollary 4.5, which relate the norms of $\log Df^n$ and those of $Df^n - 1$.

For $0 \leq \gamma_1 \leq \gamma_2 \leq k-1$, $\gamma_2 \neq 0$, and $n \in \mathbb{Z}$, Proposition 4.3 gives:

$$\|\log Df^n\|_{\gamma_1} \leq C(\gamma_2) M'^{C(\gamma_2)} \|\log Df^n\|_{\gamma_2}^{\gamma_1/\gamma_2} \tag{46}$$

$$\|Df^n - 1\|_{\gamma_1} \leq C(\gamma_2) M'^{C(\gamma_2)} \|Df^n - 1\|_{\gamma_2}^{\gamma_1/\gamma_2}. \tag{47}$$

For $n \geq 0$, $j \in \mathbb{Z}$, we have:

$$\|(Df^n)^j\|_0 \leq M'^{|j|} \tag{48}$$

and, for $0 < \gamma' < 1$, since $|Df^n - 1|_{\gamma'} = |Df^n|_{\gamma'}$:

$$|(Df^n)^j|_{\gamma'} \leq |j| M'^{|j|-1} |Df^n - 1|_{\gamma'}. \tag{49}$$

Therefore, for $0 \leq \gamma' < 1$, $\phi \in C^{\gamma'}(\mathbb{T}^1)$, we get, by the relations (8) and (9):

$$\|(Df^n)^j \phi\|_{\gamma'} \leq C(j) M^{C(j)} (\|\phi\|_{\gamma'} + \|Df^n - 1\|_{\gamma'} \|\phi\|_0) \tag{50}$$

where $C(j)$ denotes a positive numerical function of the variable k, with an explicit formula that we do not compute, and this is why it does not have a sub-index.

Let $\Delta = X_1^{j_1} \cdots X_l^{j_l}$ be a monomial of l variables, such that $l = \sum_{p=1}^{l} p j_p \geq 1$. Let $0 \leq \gamma' < 1$, $n \in \mathbb{Z}$. We estimate $\|\Delta\|_{\gamma'}$ when $X_i = D^i \log Df^n$ or when $X_i = D^{i+1} f^n$, supposing that $l + \gamma' \leq k-1$.

The relations (8) and (9) allow estimating $\|\Delta\|_{\gamma'}$ by a sum of less than 2^l terms of the form $\|X_p\|_{\gamma'} \|\Delta / X_p\|_0$, $1 \leq p \leq l$, $j_p \neq 0$. By relation (46), we have:

$$\begin{aligned}
\|D^p \log Df^n\|_{\gamma'} &\leq C(l) M^{C(l)} \|\log Df^n\|_{l+\gamma'}^{(p+\gamma')/(l+\gamma')} \\
\|D^{p+1} f^n\|_{\gamma'} &\leq C(l) M^{C(l)} \|Df^n - 1\|_{l+\gamma'}^{(p+\gamma')/(l+\gamma')} \\
\left\| \frac{\Delta(D \log Df^n, \ldots, D^l \log Df^n)}{D^p \log Df^n} \right\|_0 &\leq C(l) M^{C(l)} \|\log Df^n\|_{l+\gamma'}^{(l-p)/(l+\gamma')} \\
\left\| \frac{\Delta(D^2 f^n, \ldots, D^{l+1} f^n)}{D^{p+1} f^n} \right\|_0 &\leq C(l) M^{C(l)} \|Df^n - 1\|_{l+\gamma'}^{(l-p)/(l+\gamma')}.
\end{aligned}$$

Scheme of the proof. — The first two estimates are straightforward. For the third estimate, we write

$$\begin{aligned}
&\frac{\Delta(D \log Df^n, \ldots, D^l \log Df^n)}{D^p \log Df^n} \\
&\qquad = (D \log Df^n)^{j_1} \cdots (D^p \log Df^n)^{j_p - 1} \cdots \left(D^l \log Df^n\right)^{j_l}
\end{aligned}$$

we apply estimation (46) to each $D^i \log Df^n$ and we use that $\sum j_k \leq l$.

The proof of the fourth estimate is analogous, by noting that for $i \geq 1$, $D^i(Df^n - 1) = D^{i+1}f^n$. $\square$

Therefore, when $X_i = D^i \log Df^n$, we get:

$$\|\Delta\|_{\gamma'} \leq C(l)M^{C(l)}\|\log Df^n\|_{l+\gamma'} \tag{51}$$

and when $X_i = D^{i+1}f^n$,

$$\|\Delta\|_{\gamma'} \leq C(l)M^{C(l)}\|Df^n - 1\|_{l+\gamma'}. \tag{52}$$

Using Appendix 6, this allows obtaining the following lemma:

Lemma 4.4. — *Let P be one of the polynomials of Appendix 6. P is a polynomial of l variables $X_1, \ldots, X_l$, homogeneous of degree l if X_i has a degree of i. For every $n \in \mathbb{Z}$, every $0 \leq \gamma' < 1$, we have:*

$$\|P(D\log Df^n, \ldots, D^l \log Df^n)\|_{\gamma'} \leq C(l)M^{C(l)}\|\log Df^n\|_{l+\gamma'}$$
$$\left\| P\left(\frac{D^2f^n}{Df^n}, \ldots, \frac{D^{l+1}f^n}{Df^n}\right)\right\|_{\gamma'} \leq C(l)M^{C(l)}\|Df^n - 1\|_{l+\gamma'}.$$

Corollary 4.5. — *For $n \in \mathbb{Z}$, $0 \leq \gamma \leq k-1$, $\gamma = \lfloor\gamma\rfloor + \gamma'$, $0 \leq \gamma' < 1$. we have:*

$$(C(\gamma)M^{C(\gamma)})^{-1}\|Df^n - 1\|_\gamma \leq \|\log Df^n\|_\gamma \leq C(\gamma)M^{C(\gamma)}\|Df^n - 1\|_\gamma.$$

Scheme of the proof. — For $0 \leq \gamma < 1$, we prove the estimates directly, using that $\log x \leq x - 1$.

When $\gamma \geq 1$, for the right-hand side of the estimation, we use formula (77) in Appendix 6 and the second estimate of Lemma 4.4.

For the left-hand side, we apply formula (76) in Appendix 6, the first estimate of Lemma 4.4, relation (50) with $\phi = D^{\lfloor\gamma\rfloor+1}f^n/Df^n$ and $j = 1$, the left-hand side of this estimate of Corollary 4.5 with $\gamma < 1$, and relation (46) twice. $\square$

Using mainly the Faa-d-Bruno formula, we have the lemma [14, p. 350]:

Lemma 4.6. — *Let $\gamma_0 \geq \gamma \geq 0$, $\psi \in D^{\max(1,\gamma)}(\mathbb{T}^1)$, $\phi \in C^\gamma(\mathbb{T}^1)$. We have:*

$$\|\phi \circ \psi\|_\gamma \leq C(\gamma)\|D\psi\|^{C(\gamma)}_{\max(\gamma,1)-1}\|\phi\|_\gamma.$$

We have: $f^n = hR_{n\alpha}h^{-1}$. We apply Lemma 4.6 with $\psi = h^{-1}$ and $\phi = hR_{n\alpha} - h - n\alpha$. To estimate $\|hR_{n\alpha} - h - n\alpha\|_\gamma$, we use the C^γ-norm of Dh. We get:

Proposition 4.7. — *For $n \in \mathbb{Z}$, $0 \leq \gamma \leq \gamma_0$, we have:*

$$\|f^n - id - n\alpha\|_\gamma \leq C(\gamma)M^{C(\gamma)}\|n\alpha\|.$$

Let $\alpha_n = (-1)^n(q_n\alpha - p_n)$ and let $\Delta_s = \|D^{k-1}\log Df^{q_s}\|_0 + \alpha_s$ (the role of this additional α_s is explained at the end of the proof of Lemma 4.12). We could also have taken $\Delta_s = \max\left(\|D^{k-1}\log Df^{q_s}\|_0, \alpha_s\right)$. By applying Lemma 3.4, and since $M_{s-1} \leq M'\alpha_{s-1}$, $1/m_{s-1} \leq M'/\alpha_{s-1}$, and $\alpha_{s-1} \leq 1/q_s$, then we have:

$$\Delta_s \leq (C_{20}^{f,k} {M'}^{\frac{3}{2}(k-1)} + 1) q_s^{(k-1)/2}. \tag{53}$$

Using Proposition 4.7 when $\gamma \leq \gamma_0 - 1$, using Corollary 4.5, Proposition 4.3(with $\gamma_3 = k$, $\gamma_2 = \gamma + 1$, $\gamma_1 = \gamma_0$), Proposition 4.7, and Corollary 4.5 again when $\gamma > \gamma_0 - 1$, we obtain the following lemma:

LEMMA 4.8. — *Let $\gamma \in [0, k-1]$ and $s \geq 0$. We have:*

$$\|\log Df^{q_s}\|_\gamma \leq C(k) M^{C(k)} q_{s+1}^{-1} (q_{s+1}\Delta_s)^{\max(0, \frac{\gamma+1-\gamma_0}{k-\gamma_0})}.$$

We make a remark on the method and notation: in this Lemma 4.8, we estimate the C^γ-norm for $0 \leq \gamma \leq k-1$, instead of only estimating the C^{γ_1}-norm, because of two reasons: first, this lemma is used to obtain Lemma 4.9, in which we need an estimation of all the norms of order $\gamma \leq k-1$. Second, in the proof of Proposition 4.10, we need an estimate of $\|\log Df^{q_s}\|_{k-1}$.

4.2. Estimation of $\|\log Df^{nq_s}\|_\gamma$, $0 \leq n \leq q_{s+1}/q_s$, $0 \leq \gamma \leq k-1$. — We use Lemma 4.8 to estimate $\|\log Df^{nq_s}\|_\gamma$, $0 \leq \gamma \leq k-1$ (Lemma 4.9) and second, we bootstrap this estimate (Lemma 4.12). This bootstrapping allows getting a higher degree of differentiability γ_1 at the end (see estimation (59)).

The Diophantine condition on α implies $q_{s+1} \leq C_d^{-1} q_s^{1+\beta}$. Therefore, by applying estimation (53), we get:

$$(\Delta_s q_{s+1})^{1/k} q_s^{-1} \leq C_{34}(0) q_s^{-\epsilon}. \tag{54}$$

With $\epsilon = \frac{1}{2} - \frac{1+2\beta}{2k} > 0$ and $C_{34}(0) = \left[(C_{20}^{f,k} M^{\frac{3}{2}(k-1)} + 1) C_d^{-1}\right]^{1/k}$.

The preceding estimates give the lemma:

LEMMA 4.9. — *Let $\gamma \in [0, k-1]$. For $s \geq 0$, $0 \leq n \leq q_{s+1}/q_s$, we have:*

$$\|\log Df^{nq_s}\|_\gamma \leq C(k) M^{C(k)} C_{34}(0)^{\lfloor\gamma\rfloor} q_s^{-1} (q_{s+1}\Delta_s)^{(\gamma+1)/k}.$$

Scheme of the proof. — This lemma is shown by induction on $r = \lfloor\gamma\rfloor$. If $r = 0$, we write $\log Df^{nq_s} = \sum_{i=0}^{n-1} \log Df^{q_s} \circ f^i$ and we apply Lemma 4.8.

Suppose the lemma holds for $r - 1 + \gamma'$, with $0 \leq \gamma' < 1$. We have, using the expression (80) in Appendix 6, and using estimations (8) and (9):

$$\|D^r \log Df^{nq_s}\|_{\gamma'} \leq \sum_{l=0}^{r-1}\sum_{i=0}^{n-1} (A_{i,l} + B_{i,l} + C_{i,l}) \tag{55}$$

with:

$$\begin{aligned}
A_{i,l} &= \|D^{r-l}\log Df^{q_s}\circ f^{iq_s}\|_{\gamma'}\|(Df^{iq_s})^{r-l}\|_0\|E_l^r\|_0\\
B_{i,l} &= \|D^{r-l}\log Df^{q_s}\circ f^{iq_s}\|_0|(Df^{iq_s})^{r-l}|_{\gamma'}\|E_l^r\|_0\\
C_{i,l} &= \|D^{r-l}\log Df^{q_s}\circ f^{iq_s}\|_0\|(Df^{iq_s})^{r-l}\|_0\|E_l^r\|_{\gamma'}\\
E_l^r &= E_l^r(D\log Df^{iq_s},\dots,D^l\log Df^{iq_s}).
\end{aligned}$$

We estimate E_l^r with Lemma 4.4 (with the polynomial $P=E_l^r$), with (46) (for $B_{i,l}$) and with the induction assumption. We estimate $\|D^{r-l}\log Df^{q_s}\circ f^{iq_s}\|_{\tilde\gamma}$, $\tilde\gamma=0$ or γ', by applying Lemma 4.6 with $\phi=D^{r-l}\log Df^{q_s}$ and $\psi=f^{iq_s}$, and by applying Lemma 4.8. We estimate $|(Df^{iq_s})^{r-l}|_0$ with (48). For $|(Df^{iq_s})^{r-l}|_{\gamma'}$, we apply (49), Corollary 4.5, (46) and the induction assumption. We get:

$$\begin{aligned}
A_{i,l} &\le C(k)M^{C(k)}C_{34}(0)^l q_{s+1}^{-1}q_s^{-1}(\Delta_s q_{s+1})^{\frac{r-l+\gamma'+1}{k}+\frac{l+1}{k}}\\
B_{i,l} &\le C(k)M^{C(k)}C_{34}(0)^l q_{s+1}^{-1}q_s^{-1}(\Delta_s q_{s+1})^{\frac{r-l+1}{k}+\frac{l+\gamma'+1}{k}}\\
C_{i,l} &\le C(k)M^{C(k)}C_{34}(0)^l q_{s+1}^{-1}q_s^{-1}(\Delta_s q_{s+1})^{\frac{r-l+1}{k}+\frac{l+\gamma'+1}{k}}.
\end{aligned}$$

Thus, we have:

$$A_{i,l}+B_{i,l}+C_{i,l}\le C(k)M^{C(k)}C_{34}(0)^l q_{s+1}^{-1}q_s^{-1}(\Delta_s q_{s+1})^{\frac{\gamma+1}{k}+1/k}.$$

We conclude using estimation (34), and using the fact that the sum (55) has $rn\le\lfloor\gamma\rfloor q_{s+1}/q_s$ terms. □

By applying this Lemma 4.9, together with estimate (34), Lemma 4.8 and Lemma 4.4, we get the proposition [14, p. 355]:

Proposition 4.10. — *The sequence $(\Delta_s/q_s)_{s\ge0}$ is bounded by C_{35}.*

C_{35} is defined by the following:

$$\begin{aligned}
C_{36} &= C(k)M^{C(k)}C_{34}(0)^{k-1}\\
C_{35} &= C_{20}^{f,k}M^{C(k)}\prod_{s=0}^{\infty}\left(1+\frac{C_{36}}{q_s^{\epsilon}}\right).
\end{aligned}$$

Proof. — We slightly modify Yoccoz's proof. Let $\Delta'_{-1}=0$ and, for $s\ge0$:

$$\Delta'_s=\sup\{|D^{k-1}\log Df^{q_t}\circ f^m(Df^m)^{k-1}|_0, 0\le t\le s, m\ge0\}.$$

For $s\ge0$, we have: $\Delta_s\le\Delta'_s+\alpha_s$ (this implies $\Delta_s\le\tilde C\Delta'_s$ when f is not a rotation, but contrary to Yoccoz's proof, we do not use this estimate, because the constant $\tilde C$ is of the form $\tilde C=1+\frac{M^{k-1}}{|D^{k-1}\log Df|_0}$, which diverges as f gets closer to a rotation). We compute a bound on $(\Delta'_s+\alpha_s)/q_s$.

Let $s \geq 0$ (this is another difference with Yoccoz's proof, which only considers $s \geq 1$). We have: $q_{s+1} = \hat{a}_{s+1} q_s + q_{s-1}$ (we recall that $q_{-1} = 0$). Using formula (79) in Appendix 6 with $g = f^{q_{s-1}}$ and $h = f^{\hat{a}_{s+1} q_s}$, we can write:

$$(D^{k-1} \log Df^{q_{s+1}} \circ f^m)(Df^m)^{k-1} = X' + Y' + Z'$$

with

$$\begin{aligned}
X' &= (D^{k-1} \log Df^{q_{s-1}} \circ f^{\hat{a}_{s+1} q_s + m})(Df^{\hat{a}_{s+1} q_s} \circ f^m)^{k-1}(Df^m)^{k-1} \\
Y' &= D^{k-1} \log Df^{\hat{a}_{s+1} q_s} \circ f^m (Df^m)^{k-1} \\
Z' &= \sum_{l=1}^{k-2} (D^{k-1-l} \log Df^{q_{s-1}} \circ f^{\hat{a}_{s+1} q_s + m})(Df^{\hat{a}_{s+1} q_s} \circ f^m)^{k-1-l} \\
&\quad \cdot (Df^m)^{k-1} G_l^{k-1}(D \log Df^{\hat{a}_{s+1} q_s} \circ f^m, \ldots, D^l \log Df^{\hat{a}_{s+1} q_s} \circ f^m).
\end{aligned}$$

We have:

$$|X'|_0 \leq \Delta'_{s-1}.$$

Using formula (80) in Appendix 6 with $g = f^{q_s}$, we have:

$$\begin{aligned}
Y' = \sum_{l=0}^{k-2} \sum_{n=0}^{\hat{a}_{s+1}-1} & (D^{k-1-l} \log Df^{q_s} \circ f^{n q_s + m})(Df^{n q_s + m})^{k-1-l} \\
& \cdot E_l^{k-1}(D \log Df^{n q_s}, \ldots, D^l \log Df^{n q_s}) \circ f^m (Df^m)^l = \sum_{l=0}^{k-2} Y'_l
\end{aligned}$$

(with the convention $E_0^{k-1} = 1$). We have: $|Y'_0|_0 \leq \hat{a}_{s+1} \Delta'_s$.

For $l \geq 1$, we estimate $E_l^{k-1}(D \log Df^{n q_s}, \ldots, D^l \log Df^{n q_s}) \circ f^m (Df^m)^l$ using Lemma 4.6 (with $\psi = f^m$ and $\gamma = 0$), Lemma 4.4 (with $P = E_l^{k-1}$), Lemma 4.9 (with $\gamma = l$) and estimation (34). We get:

$$\begin{aligned}
|E_l^{k-1}&(D \log Df^{n q_s}, \ldots, D^l \log Df^{n q_s}) \circ f^m (Df^m)^l|_0 \\
&\leq C(k) M^{C(k)} (C_{34}(0))^{l+1} (\Delta_s q_{s+1})^{l/k} q_s^{-\epsilon}.
\end{aligned}$$

By applying Lemma 4.8 (with $\gamma = k - 1 - l$ and $\gamma_0 = 0$), and using that $\Delta_s \leq \Delta'_s + \alpha_s$, we get:

$$|Y'_l|_0 \leq \hat{a}_{s+1}(\Delta'_s + \alpha_s) C(k) M^{C(k)} (C_{34}(0))^{l+1} q_s^{-\epsilon}.$$

Therefore,

$$|Y'|_0 \leq \hat{a}_{s+1} \Delta'_s + \hat{a}_{s+1}(\Delta'_s + \alpha_s) C_{36} q_s^{-\epsilon}.$$

Likewise, we can show that, for $s \geq 1$:

$$|Z'|_0 \leq C_{36} q_s^{-\epsilon} q_s^{-1} (q_s \Delta_{s-1})^{\frac{k-l}{k}} (q_{s+1} \Delta_s)^{l/k}.$$

(Yoccoz concludes the estimation of $|Z'|_0$ here, using the fact that $q_s^{1-l/k} \leq q_{s+1}^{1-l/k}$ and using the fact that $\Delta_t \leq \tilde{C}\Delta'_t, t = s-1, s$. We don't use these facts.)

Since $\Delta_t \leq \Delta'_t + \alpha_t$, $t = s-1, s$, and since $\Delta'_{s-1} \leq \Delta'_s$, we get:

$$|Z'|_0 \leq C_{36} q_s^{-\epsilon} \left(\frac{q_{s+1}}{q_s}\right)^{l/k} \left(\Delta'_{s-1} + \alpha_{s-1}\right)^{1-l/k} \left(\Delta'_s + \alpha_s\right)^{l/k}$$

$$|Z'|_0 \leq C_{36} q_s^{-\epsilon} \frac{q_{s+1}}{q_s} \left(\Delta'_s + \alpha_s\right) \left(\left(1 + \frac{\alpha_{s-1} - \alpha_s}{\Delta'_s + \alpha_s}\right)\left(\frac{q_s}{q_{s+1}}\right)\right)^{1-l/k}.$$

Since $\Delta'_s \geq 0$, and since $\hat{a}_{s+1} \leq q_{s+1}/q_s \leq 2\hat{a}_{s+1}$ and $\alpha_{s-1} \leq 2\hat{a}_{s+1}\alpha_s$, we get:

$$|Z'|_0 \leq 4C_{36} q_s^{-\epsilon} \hat{a}_{s+1} \left(\Delta'_s + \alpha_s\right).$$

If $s = 0$, $Z' = 0$. This estimate still holds.

Therefore, for $s \geq 0$,

$$\begin{aligned} \alpha_{s+1} &+ |(D^{k-1} \log Df^{q_{s+1}} \circ f^m)(Df^m)^{k-1}|_0 \\ &\leq \alpha_{s+1} + \Delta'_{s-1} + \hat{a}_{s+1}\Delta'_s + \hat{a}_{s+1}(\Delta'_s + \alpha_s) 5C_{36} q_s^{-\epsilon} \\ \alpha_{s+1} &+ |(D^{k-1} \log Df^{q_{s+1}} \circ f^m)(Df^m)^{k-1}|_0 \\ &\leq \alpha_{s+1} - \hat{a}_{s+1}\alpha_s + \Delta'_{s-1} + \hat{a}_{s+1}(\Delta'_s + \alpha_s)\left(1 + 5C_{36} q_s^{-\epsilon}\right). \end{aligned} \tag{56}$$

Moreover, we have: $\alpha_{s-1} = \hat{a}_{s+1}\alpha_s + \alpha_{s+1}$. Therefore, for $s \geq 1$, since $\alpha_{s+1} < \frac{1}{2}\alpha_{s-1}$, then

$$\alpha_{s+1} - a_{s+1}\alpha_s = 2\alpha_{s+1} - \alpha_{s-1} < 0 \leq \alpha_{s-1}.$$

Therefore,

$$\begin{aligned} \alpha_{s+1} &+ |(D^{k-1} \log Df^{q_{s+1}} \circ f^m)(Df^m)^{k-1}|_0 \\ &\leq \max_{t=s-1,s} \frac{\alpha_t + \Delta'_t}{q_t} (q_{s-1} + \hat{a}_{s+1} q_s)\left(1 + 5C_{36} q_s^{-\epsilon}\right). \end{aligned}$$

Since $q_{s-1} + \hat{a}_{s+1} q_s = q_{s+1}$, we get:

$$\frac{\alpha_{s+1} + |(D^{k-1} \log Df^{q_{s+1}} \circ f^m)(Df^m)^{k-1}|_0}{q_{s+1}} \leq \max_{t=s-1,s} \frac{\alpha_t + \Delta'_t}{q_t} \left(1 + 5C_{36} q_s^{-\epsilon}\right).$$

If $s = 0$, we have:

$$\frac{\alpha_1 + \Delta'_1}{q_1} \leq \frac{\alpha_0 + \Delta'_0}{q_0} \left(1 + 5C_{36}\right).$$

Let $\theta_s = \max_{0 \leq t \leq s} \frac{\alpha_t + \Delta'_t}{q_t}$. The preceding estimates give:

$$\theta_{s+1} \leq \theta_s \left(1 + 5C_{36} q_s^{-\epsilon}\right).$$

Moreover,

$$\left(\frac{\Delta_0' + \alpha_0}{q_0}\right) \leq 1 + M^{k-1}|D^{k-1}\log Df|_0.$$

Therefore, for any $s \geq 0$,

$$\frac{\Delta_s}{q_s} \leq \left(1 + M^{k-1}|D^{k-1}\log Df|_0\right) \prod_{s=0}^{+\infty} \left(1 + 5C_{36}q_s^{-\epsilon}\right).$$

To conclude, we apply the claim:

Claim 4.11. — *Let $C_{20}^{f,k}$ be defined in Lemma 3.4. For any $k \geq 3$, we have:*

$$|D^{k-1}\log Df|_0 \leq \tilde{C}_{20}(k, |Sf|_{k-3}) \leq C_{20}^{f,k}.$$

Proof. — First, we recall the observation (see e.g., [14]) that if x_0 is a point where $(D\log Df)^2$ is maximal, then we have:

$$\begin{aligned}|Sf|_0 \geq |Sf(x_0)| &= \left|D^2\log Df(x_0) - \frac{1}{2}(D\log Df(x_0))^2\right| \\ &= \left|\frac{1}{2}(D\log Df(x_0))^2\right| = \left|\frac{1}{2}(D\log Df)^2\right|_0.\end{aligned}$$

To prove the claim, we proceed by induction on k, using the fact that $|D\log Df|_0 \leq \sqrt{2|Sf|_0}$ and by applying formula (70) in the appendix.
If $k = 3$,

$$|D^2\log Df|_0 \leq |Sf|_0 + \frac{1}{2}|(D\log Df)^2|_0 \leq 2|Sf|_0.$$

Suppose the estimate holds for every $r \leq k-1$. By formula (70), we have:

$$D^k\log Df = D^{k-2}Sf - G_k(D\log Df, \ldots, D^{k-1}\log Df).$$

As in the proof of Lemma 3.4 (see [3] for details), and by applying the induction assumption, we have:

$$\left|G_k(D\log Df, \ldots, D^{k-1}\log Df)\right| \leq \frac{(2(k-1))!}{2k}\left(\tilde{C}_{20}(k, |Sf|_{k-3})\right)^k.$$

We conclude as in the proof of Lemma 3.4. □□

With Proposition 4.10, and by using the Diophantine condition $q_{s+1} \leq C_d^{-1}q_s^{1+\beta}$, we can generalize estimation (34) and Lemma 4.9, for $\gamma_0 > 0$. The generalization of estimation (34) is:

$$(\Delta_s q_{s+1})^{1/(k-\gamma_0)} q_s^{-1} \leq C_{34}(\gamma_0) q_s^{\frac{\beta+2}{k-\gamma_0}-1} \tag{57}$$

with $C_{34}(\gamma_0) = C_{35}^{\frac{1}{k-\gamma_0}} C_d^{\frac{-1}{k-\gamma_0}}$.

The generalization of Lemma 4.9 is:

Lemma 4.12. — *Let $\gamma \in [0, k-1]$. For $s \geq 0$, $0 \leq n \leq q_{s+1}/q_s$, we have:*

$$\|\log Df^{nq_s}\|_\gamma \leq C(k)M^{C(k)}(C_{34}(\gamma_0))^{\lfloor\gamma\rfloor} q_s^{-1}(q_{s+1}\Delta_s)^{\max\left(\frac{\gamma+1-\gamma_0}{k-\gamma_0},0\right)}.$$

Scheme of the proof. — We give the scheme of the proof in order to explain the additional α_s in the definition of Δ_s (this additional α_s makes necessary our modification of Yoccoz's proof of Proposition 4.10).

If $\gamma_0 - 1 \leq \gamma < 1$, we proceed as in Lemma 4.9. If $\gamma \leq \gamma_0 - 1$ and $\gamma < 1$, we apply Corollary 4.5 and Proposition 4.7. The induction step is analogous to the proof of Lemma 4.9, except the end: indeed, by proceeding as in Lemma 4.9, we have:

$$A_{i,l} \leq C(k)M^{C(k)}C_{34}(\gamma_0)^l q_{s+1}^{-1}q_s^{-1}(\Delta_s q_{s+1})^{\max(0,\frac{r-l+\gamma'+1-\gamma_0}{k-\gamma_0})+\max(0,\frac{l+1-\gamma_0}{k-\gamma_0})}$$

$$B_{i,l} \leq C(k)M^{C(k)}C_{34}(\gamma_0)^l q_{s+1}^{-1}q_s^{-1}(\Delta_s q_{s+1})^{\max(0,\frac{r-l+1-\gamma_0}{k-\gamma_0})+\max(0,\frac{l+\gamma'+1-\gamma_0}{k-\gamma_0})}$$

$$C_{i,l} \leq C(k)M^{C(k)}C_{34}(\gamma_0)^l q_{s+1}^{-1}q_s^{-1}(\Delta_s q_{s+1})^{\max(0,\frac{r-l+1-\gamma_0}{k-\gamma_0})+\max(0,\frac{l+\gamma'+1-\gamma_0}{k-\gamma_0})}.$$

We have:

$$\begin{aligned}\max\Bigg(\max\left(0, \frac{r-l+1-\gamma_0}{k-\gamma_0}\right) + \max\left(0, \frac{l+\gamma'+1-\gamma_0}{k-\gamma_0}\right),\\ \max\left(0, \frac{r-l+\gamma'+1-\gamma_0}{k-\gamma_0}\right) + \max\left(0, \frac{l+1-\gamma_0}{k-\gamma_0}\right)\Bigg)\\ \leq \max\left(0, \frac{\gamma+1-\gamma_0}{k-\gamma_0}\right) + \frac{1}{k-\gamma_0}.\end{aligned}$$

Moreover, since $2q_{s+1}\Delta_s \geq 2q_{s+1}\alpha_s \geq 1$, then

$$A_{i,l} + B_{i,l} + C_{i,l} \leq C(k)M^{C(k)}C_{34}(\gamma_0)^l q_{s+1}^{-1}q_s^{-1}(\Delta_s q_{s+1})^{\max(0,\frac{\gamma+1-\gamma_0}{k-\gamma_0})+\frac{1}{k-\gamma_0}}.$$

This is why we define $\Delta_s = |D^{k-1}\log Df^{q_s}|_0 + \alpha_s$: if we defined $\Delta_s = |D^{k-1}\log Df^{q_s}|_0$ and if $|D^{k-1}\log Df^{q_s}|_0$ was too small, we could not do this last estimate. □

By using estimation (57) and Lemma 4.12, we obtain, for $0 \leq n \leq (q_{s+1})/q_s$, and $0 \leq \gamma \leq \gamma_1$ [14, p. 357]:

(58) $$\|\log Df^{nq_s}\|_\gamma \leq C(k)M^{C(k)}(C_{34}(\gamma_0))^{\lfloor\gamma\rfloor+1} q_s^{\rho(\gamma,\gamma_0)}$$

with

$$\rho(\gamma, \gamma_0) = \frac{(2+\beta)\max(0, (\gamma+1-\gamma_0))}{k-\gamma_0} - 1.$$

Notice that for any

(59) $$\gamma_1 < g(\gamma_0) = \frac{(1+\beta)\gamma_0 + k - (2+\beta)}{2+\beta}$$

we have $\rho(\gamma_1, \gamma_0) < 0$ (we will take $\gamma_1 = \frac{1}{2}(g(\gamma_0)+\gamma_0)$).

This implies $\sum_{s\geq 0} q_s^{\rho(\gamma_1,\gamma_0)} < +\infty$, which will allow estimating $\|\log Df^N\|_{\gamma_1}$, as we will see in the next subsection.

A remark on the method and notation: we establish estimate (58) for any $0 \leq \gamma \leq \gamma_1$ (and not just for γ_1) because we need it for the estimate of the quantity Z defined below.

4.3. Estimation of $\|\log Df^N\|_{\gamma_1}$

PROPOSITION 4.13. — *Let N be an integer and let us write $\gamma_1 = r + \gamma_1'$, with $0 \leq \gamma_1' < 1$ and r integer. We have:*

$$(60)\quad \|\log Df^N\|_{\gamma_1} \leq C(k)M^{C(k)}(C_{34}(\gamma_0))^{\lfloor\gamma_1\rfloor+1}\prod_{s=1}^{\infty}\left(1+\frac{C(k)M^{C(k)}(C_{34}(\gamma_0))^{\lfloor\gamma_1\rfloor+1}}{q_s^{-\rho(\gamma_1,\gamma_0)}}\right) = C_{??}.$$

Scheme of the proof. — We write $N = \sum_{s=0}^{S} b_s q_s$ with $0 \leq b_s \leq \frac{q_{s+1}}{q_s}$ and b_s integer. Let $N_s = \sum_{t=0}^{s} b_t q_t$ for $0 \leq s \leq S$. Moreover, let us write $\gamma_1 = r + \gamma_1'$, with $0 \leq \gamma_1' < 1$ and r integer. By formula (79) in Appendix 6, we can write $D^r \log Df^{N_s} = X + Y + Z$ with:

$$\begin{aligned}
X &= (D^r \log Df^{b_s q_s} \circ f^{N_{s-1}})(Df^{N_{s-1}})^r;\\
Y &= D^r \log Df^{N_{s-1}};\\
Z &= \sum_{l=1}^{r-1}(D^{r-l}\log Df^{b_s q_s}\circ f^{N_{s-1}})(Df^{N_{s-1}})^{r-l}G_l^r;\\
G_l^r &= G_l^r\left(D\log Df^{N_{s-1}},\ldots,D^l\log Df^{N_{s-1}}\right).
\end{aligned}$$

We successively estimate X and Z. For X, we use estimate (50), Corollary 4.5 and Lemma 4.6 with $\phi = D^r \log Df^{b_s q_s}$ and $\psi = f^{N_{s-1}}$. We also use estimate (58), and the fact that $q_s^{\rho(r,\gamma_0)} \leq q_s^{\rho(\gamma_1,\gamma_0)}$. We get:

$$\|X\|_{\gamma_1'} \leq C(k)M^{C(k)}(C_{34}(\gamma_0))^{r+1}q_s^{\rho(\gamma_1,\gamma_0)}(1+\|\log Df^{N_{s-1}}\|_{\gamma_1'}).$$

We estimate Z. By applying estimation (9), we have:

$$\begin{aligned}
\|Z\|_{\gamma_1'} \leq \sum_{l=1}^{r-1} &|(D^{r-l}\log Df^{b_s q_s}\circ f^{N_{s-1}})(Df^{N_{s-1}})^{r-l}|_0 |G_l^r|_{\gamma_1'}\\
&+ \|(D^{r-l}\log Df^{b_s q_s}\circ f^{N_{s-1}})(Df^{N_{s-1}})^{r-l}\|_{\gamma_1'}|G_l^r|_0.
\end{aligned}$$

As with X, we have:

$$\begin{aligned}
&\|(D^{r-l}\log Df^{b_s q_s}\circ f^{N_{s-1}})(Df^{N_{s-1}})^{r-l}\|_{\gamma_1'}\\
&\qquad\leq C(k)M^{C(k)}(C_{34}(\gamma_0))^{r+1}q_s^{\rho(\gamma_1,\gamma_0)}(1+\|\log Df^{N_{s-1}}\|_{\gamma_1'}).
\end{aligned}$$

Moreover, by estimate (58), we also have:

$$\begin{aligned}|(D^{r-l}\log Df^{b_sq_s}\circ f^{N_{s-1}})(Df^{N_{s-1}})^{r-l}|_0 &\leq C(k)M^{C(k)}(C_{34}(\gamma_0))^{r+1}q_s^{\rho(r-l,\gamma_0)}\\ &\leq C(k)M^{C(k)}(C_{34}(\gamma_0))^{\lfloor\gamma_1\rfloor+1}q_s^{\rho(\gamma_1,\gamma_0)}.\end{aligned}$$

For G_l^r, we use Lemma 4.4 with the polynomial $P=G_l^r$ (see Appendix 6). We estimate $\|Z\|_{\gamma_1'}$ by applying estimation (46) twice. We get:

$$\|Z\|_{\gamma_1'}\leq C(k)M^{C(k)}(C_{34}(\gamma_0))^{r+1}q_s^{\rho(\gamma_1,\gamma_0)}\|\log Df^{N_{s-1}}\|_{\gamma_1}.$$

Therefore, since $\|Y\|_{\gamma_1'}=\|D^r\log Df^{N_{s-1}}\|_{\gamma_1'}$, we get, for $s\geq 1$:

$$\begin{aligned}\|D^r\log Df^{N_s}\|_{\gamma_1'} &\leq \|\log Df^{N_s}\|_{\gamma_1}\\ &\leq \left(1+\frac{C(k)M^{C(k)}(C_{34}(\gamma_0))^{r+1}}{q_s^{-\rho(\gamma_1,\gamma_0)}}\right)\|D^r\log Df^{N_{s-1}}\|_{\gamma_1'}.\end{aligned}$$

Moreover, by estimate (58), since $N_0=b_0$, we also have:

$$\|D^r\log Df^{N_0}\|_{\gamma_1'}\leq C(k)M^{C(k)}(C_{34}(\gamma_0))^{r+1}.$$

We conclude that:

$$\begin{aligned}\|\log Df^N\|_{\gamma_1} &\leq C(k)M^{C(k)}(C_{34}(\gamma_0))^{r+1}\prod_{s=1}^{\infty}\left(1+\frac{C(k)M^{C(k)}(C_{34}(\gamma_0))^{r+1}}{q_s^{-\rho(\gamma_1,\gamma_0)}}\right)\\ &= C_{37}.\end{aligned}$$

□

In order to complete the proof of Proposition 4.1, we need to estimate $C_{??}$. This is done in the next paragraph.

4.3.1. *Computation of the estimations: proof of Proposition 4.1.*

Proof of Proposition 4.1. — We estimate $C_{??}$. We have:

$$C_{??}\leq\prod_{s=0}^{\infty}\left(1+\frac{C(k)M^{C(k)}(C_{34}(\gamma_0))^{\gamma_1+1}}{q_s^{-\rho(\gamma_1,\gamma_0)}}\right)\leq\prod_{s=0}^{\infty}\left(1+\frac{C(k)M^{C(k)}(C_{34}(\gamma_0))^{k}}{q_s^{-\rho(\gamma_1,\gamma_0)}}\right).$$

Moreover,

$$\begin{aligned}[C_{34}(\gamma_0)]^{k-\gamma_0} &= C_d^{-1}C_{35}\leq C(k)M^{C(k)}C_d^{-1}C_{20}^{f,k}\\ &\quad\cdot\prod_{s=0}^{\infty}\left(1+\frac{C(k)M^{C(k)}(C_{34}(0))^{k-1}}{q_s^{\epsilon}}\right)\\ [C_{34}(\gamma_0)]^{k-\gamma_0} &\leq\prod_{s=0}^{\infty}\left(1+\frac{C(k)M^{C(k)}C_{20}^{f,k}C_d^{-1}(C_{34}(0))^{k-1}}{q_s^{\epsilon}}\right)\\ (C_{34}(0))^{k-1} &\leq C(k)M^{C(k)}C_d^{-1}C_{20}^{f,k}.\end{aligned}$$

Therefore,

$$C_{??} \leq \prod_{s=0}^{\infty} \left(1 + q_s^{\rho(\gamma_1,\gamma_0)} \prod_{s=0}^{\infty} \left[1 + \frac{C(k)M^{C(k)} \left(C_{20}^{f,k} C_d^{-1} \right)^2}{q_s^{\epsilon}} \right]^{\frac{k}{k-\gamma_0}} \right).$$

Let

$$\tau_1 = \frac{k}{\beta + 2 + \eta}, \tag{61}$$

let $C_{38} = C(k)M^{C(k)} \left(C_{20}^{f,k} C_d^{-1} \right)^2$. Let also

$$\epsilon_1 = \min \left(\epsilon, \frac{\eta}{2(\beta + 2 + \eta)} \right). \tag{62}$$

We have: $\epsilon_1 \leq \min(\epsilon, -\rho(\gamma_1, \gamma_0))$ and for any $\gamma_0 \leq k - 2 - \beta - \eta$, we have $\tau_1 \geq \frac{k}{k-\gamma_0}$.

We have:

$$C_{??} \leq \prod_{s=0}^{\infty} \left(1 + \frac{\prod_{s=0}^{\infty} \left(1 + \frac{C_{38}}{q_s^{\epsilon_1}} \right)^{\tau_1}}{q_s^{\epsilon_1}} \right).$$

Since $q_s \geq (\sqrt{2})^{s-1}$, we get:

$$C_{??} \leq \prod_{s \geq 0} \left(1 + \frac{\sqrt{2} \prod_{s \geq 0} \left(1 + \frac{\sqrt{2} C_{38}}{2^{s \frac{\epsilon_1}{2}}} \right)^{\tau_1}}{2^{s \frac{\epsilon_1}{2}}} \right).$$

In order to obtain the final estimation, we need the claim:

Claim 4.14. — *Let $T \geq 10$. For any $2 \geq u > 1$, we have:*

$$\prod_{n=0}^{\infty} \left(1 + \frac{T}{u^n} \right) \leq e^{\frac{2^{2/3}}{\log u} (\log T)^2}.$$

Proof. — For $T \geq 10$.

$$\begin{aligned} \sum_{n \geq 0} \log\left(1 + T/u^n\right) &= \sum_{n \leq \frac{\log T}{\log u} - 1} \log\left(1 + T/u^n\right) + \sum_{n > \frac{\log T}{\log u} - 1} \log\left(1 + T/u^n\right) \\ &\leq \frac{\log T}{\log u} \log(1+T) + \sum_{n > \frac{\log T}{\log u} - 1} T/u^n \leq \frac{\log T}{\log u} \left(\log(1+T) + 1\right) \\ &\leq \frac{2^{2/3}}{\log u} (\log T)^2. \end{aligned}$$

□

By applying this proposition twice, we get the claim:

Claim 4.15. — *Let $T \geq 10$, $2 \geq u > 1$, $\tau \geq 1$. We have:*

$$\prod_{n\geq 0}\left(1+\frac{\prod_{n\geq 0}\left(1+\frac{T}{u^n}\right)^{\tau}}{u^n}\right) \leq e \wedge \left(\frac{C\tau^2}{(\log u)^3}(\log T)^4\right).$$

Let $C_{39} = \sqrt{2}C_{38}$. We apply claim 4.15 with $T = C_{39}$, $u = 2^{\frac{\epsilon_1}{2}}$, $\tau = \tau_1$. We obtain:

$$C_{??} \leq e \wedge \left(\frac{C\tau_1^2}{\epsilon_1^3}(\log C_{39})^4\right). \tag{63}$$

Moreover, let

$$C_{40} = \frac{C\tau_1^2}{\epsilon_1^3}. \tag{64}$$

By using the definitions of ϵ_1 (see (62)) and τ_1 (see (61)), since $\epsilon = \frac{k-(1+2\beta)}{2k}$ and since $\eta \leq k-2-\beta$, we have:

$$\begin{aligned} C_{40} &\leq C\frac{k^2}{\left(\min\left(\frac{k-(2\beta+1)}{2k},\frac{\eta}{2(\beta+2+\eta)}\right)\right)^3} \\ &\leq C\frac{k^5}{(\min(k-(2\beta+1),k-(\beta+2)))^3} = C_{41}(k,\beta). \end{aligned} \tag{65}$$

Therefore, we get:

(66)
$$\|\log Df^N\|_{\gamma_1} \leq e \wedge \left(C_{65}(k,\beta)\left(\log(C_d^{-1}) + C_{42}^{f,k} + C(k)\left(1+\sup_{p\geq 0}\|\log Df^p\|_{\gamma_0}\right)\right)^4\right)$$

with

$$C_{42}^{f,k} = W(f) + \log\left(\max(1,|Sf|_{k-3})\right).$$

Hence Proposition 4.1. $\square$

4.3.2. *Proof of Theorem 1.6: estimation* (6). — By Corollary 4.5, we have:

$$\|Df^N\|_{\frac{k}{2(\beta+2)}-\frac{1}{2}} \leq C(k)M^{C(k)}(1+\|\log Df^N\|_{\frac{k}{2(\beta+2)}-\frac{1}{2}}).$$

Moreover, $\|Dh\|_{\frac{k}{2(\beta+2)}-\frac{1}{2}} \leq \sup_{N\geq 0}\|Df^N\|_{\frac{k}{2(\beta+2)}-\frac{1}{2}}$. We get:

(67)
$$\|Dh\|_{\frac{k}{2(\beta+2)}-\frac{1}{2}} \leq e \wedge \left(C_{65}(k,\beta)\left(\log(C_d^{-1}) + C_{42}^{f,k} + C(k)\left(1+\log M'\right)\right)^4\right).$$

Proof of estimation (6). — We suppose $k \geq 3\beta + 9/2$. Let:

$$C_{43}(\beta, C_d, |Df|_0, W(f), |Sf|_0, |Sf|_{k-3}) = \\ e^{(3)} \wedge (C_2(\beta)C_2(C_d)C_2(|Df|_0, W(f), |Sf|_0)C_2(|Sf|_{k-3}))$$

i.e., we consider the bound given by estimation (2), except that we replace $|Sf|_{\lceil 3\beta+3/2 \rceil}$ with $|Sf|_{k-3}$.

$C_{20}^{f,k}$ depends on k, $|Sf|_{k-3}$ and $W(f)$. We have:

$\log C_{20}(k, |Sf|_{k-3}, W(f)) \leq C(k) \log C_{20}(3\beta + 9/2, |Sf|_{k-3}, W(f)) \leq C(k) \log C_{??}$.

Since $M' \leq C_{??}$, we get:

$$\|Dh\|_{\frac{k}{2(\beta+2)} - \frac{1}{2}} \leq e \wedge \left(C(k) \left(\log C_{??} \right)^4 \right).$$

Since $2 \geq 2 \log 2$, we conclude:

$$\|Dh\|_{\frac{k}{2(\beta+2)} - \frac{1}{2}} \leq e \wedge \big(C(k) e^{(2)} \\ \wedge (2 + C_2(\beta)C_2(C_d)C_2(|Df|_0, W(f), |Sf|_0)C_2(|Sf|_{k-3})) \big). \qquad \square$$

Proof of estimation (7). — If $\beta = 0$, we can use the C^1 estimate. We have: $\log M' \leq C_{1.2}/C_d$ and therefore, by using estimation (67), we obtain:

$$\|Dh\|_{\frac{k}{4} - \frac{1}{2}} \leq e \wedge \left(C[k] \left[C_7[W(f), |Sf|_{k-3}] + \frac{C_{1.2}[W(f), |Sf|_0]}{C_d} \right]^4 \right)$$

with:

$$C_7[W(f), |Sf|_{k-3}] = \log(\max(1, |Sf|_{k-3})) + W(f). \qquad \square$$

4.4. Iteration of the reasoning: proof of estimation (5) **of Theorem 1.6.** — To obtain an estimation of the $C^{k-1-\beta-\eta}$-norm of the conjugacy, we iterate estimation (45). We take $\gamma_0 = \gamma_i$ and $\gamma_1 = \gamma_{i+1} = \frac{1}{2}(g(\gamma_i) + \gamma_i)$. Thus, $\gamma_{i+1} < g(\gamma_i)$ and

$\lim_{i \to +\infty} \gamma_i = k - 2 - \beta$. We need to estimate the rank above which $\gamma_i \geq k - 2 - \beta - \eta$:

CLAIM 4.16. — *Let* $C_{44} = \frac{\beta+3/2}{\beta+2}, C_{45} = \frac{k-2-\beta}{2(\beta+2)}$. *If*

$$n \geq \log \left(\frac{C_{45}}{\eta(1 - C_{44})} \right) / \log \left(\frac{1}{C_{44}} \right) = C_{46},$$

we have $\gamma_n \geq k - 2 - \beta - \eta$.

Proof. — We have: $\gamma_{n+1} = C_{44}\gamma_n + C_{45}$. Therefore, $\gamma_n = C_{45} \frac{1 - C_{44}^n}{1 - C_{44}}$. Therefore, $|\gamma_n - (k - 2 - \beta)| = \left| C_{45} \frac{C_{44}^n}{1 - C_{44}} \right| \leq \eta$ if $n \geq C_{46}$. $\square$

By induction on n, we can also show the claim (see [3] for details):

Claim 4.17. — *Let $F(x) = e^{c(a+bx)^4}$. For any $x, c \geq 1$, $a, b \geq 4$, and integer $n \geq 1$, we have:*

$$F^n(x) \leq e^{(n)} \wedge \left((3+n)c(a+bx)^4\right).$$

We apply Proposition 4.1. In claim 4.17, we take $x = \log M'$, $a = C_{42}^{f,k} + \log(C_d^{-1})$, $b = C(k)$ (we have $C(k) \geq 4$), $c = C_{65}(k, \beta)$. Let n_7 be the integer such that: $C_{46} + 1 > n_7 \geq C_{46}$. We have:

$$\begin{aligned}\|\log Df^N\|_{k-2-\beta-\eta} \\ &\leq e^{(n_7)} \wedge \left((3+n_7)C_{65}(k,\beta)(C_{42}^{f,k} + \log(C_d^{-1}) + C(k)\log M')^4\right).\end{aligned}$$

Moreover, by Corollary 4.5, we have:

$$\|Df^N\|_{k-2-\beta-\eta} \leq C(k)M^{C(k)}(1 + \|\log Df^N\|_{k-2-\beta-\eta}).$$

Since $\|Dh\|_{k-2-\beta-\eta} \leq \|Df^N\|_{k-2-\beta-\eta}$, we get:

$$|Dh|_{k-2-\beta-\eta} \leq e^{(n_7)} \wedge \left((4+n_7)C_{65}(k,\beta)\left[C_{42}^{f,k} + \log(C_d^{-1}) + C(k)\log M'\right]^4\right).$$

Since $M' \leq C_2$, we let:

$$\begin{aligned}C_5[\eta, k, \beta, C_d, |Df|_0, W(f), |Sf|_{k-3}] \\ &= (4+n_7)C_{65}[k,\beta]\left[C_{42}^{f,k} + \log(C_d^{-1}) + C[k]\log C_2\right]^4.\end{aligned}$$

We recall that:

$$\begin{aligned}n_7 &= \left\lceil \frac{\log((k-2-\beta)/\eta)}{\log(1+1/(2\beta+3))} \right\rceil \\ C_{65}[k,\beta] &= C\frac{k^5}{(\min(k-(2\beta+1), k-(\beta+2)))^3} \\ C_{42}[W(f), |Sf|_{k-3}] &= W(f) + \log(\max(1, |Sf|_{k-3})).\end{aligned}$$

We have:

$$\begin{aligned}(68)\quad \|Dh\|_{k-2-\beta-\eta} &\leq e^{(\lceil \log((k-2-\beta)/\eta)/\log(1+1/(2\beta+3))\rceil)} \\ &\wedge (C_5[\eta, k, \beta, C_d, |Df|_0, W(f), |Sf|_{k-3}]).\end{aligned}$$

5. Appendix: proof of Lemma 3.4

We follow [14] but we give more details. Let $p \le q_{n+1}$. The case $r = 1$ stems from Lemma 3.3. For the case $r = 2$, we also use Lemma 3.3:

$$|D^2 \log Df^p(x)| \le |Sf^p(x)| + \frac{1}{2}|D \log Df^p(x)|^2 \le \left(C_{16}^f + \frac{1}{2}(C_{17}^f)^2\right) \frac{M_n}{m_n(x)^2}.$$

In particular, we can take

$$C_{20}^f(2) = 82|Sf|_0 e^{8W(f)}.$$

For $r > 2$, we prove Lemma 3.4 by induction. Suppose the lemma is proved up to $r \ge 2$. Since for any C^3-diffeomorphisms g and h,

$$S(g \circ h) = (Sg \circ h)(Dh)^2 + Sh$$

then for $p \ge 1$,

$$Sf^p = \sum_{i=0}^{p-1} (Sf \circ f^i)(Df^i)^2$$

and by differentiating this last equality, we get, for $r \ge 0$, $n \ge 1$,

$$D^r Sf^p = \sum_{l=0}^{r} \sum_{i=0}^{p-1} (D^{r-l} Sf \circ f^i)(Df^i)^{r-l+2} F_l^r(D \log Df^i, \dots, D^l \log Df^i) \tag{69}$$

where F_l^r is a polynomial in l variables $X_1, \dots, X_l$, homogenous of weight l if X_i is given the weight i. Moreover, since $Sf = D^2 \log Df - \frac{1}{2}(D \log Df)^2$, then for $r \ge 2$,

$$D^{r-2} Sf = D^r \log Df + G_r(D \log Df, \dots, D^{r-1} \log Df) \tag{70}$$

where G_r is a polynomial in $r - 1$ variables $X_1, \dots, X_{r-1}$, homogeneous of weight r if X_i is given the weight i. Therefore, in order to estimate $|D^r \log Df|_0$, it suffices to estimate $F_l^r(D \log Df^i, \dots, D^l \log Df^i)$ and $G_r(D \log Df, \dots, D^{r-1} \log Df)$. These estimations are given by Lemmas 5.1 and 5.2. They are used in [14] but we recall them here in order to compute the constants $C_{47}^f(r)$ in Lemma 5.1 and $C_{48}^f(r)$ in Lemma 5.2.

LEMMA 5.1. — *Under the induction assumption, for $0 \le l \le r$ and $0 \le p \le q_{n+1}$, we have:*

$$|F_l^r(D \log Df^p(x), \dots, D^l \log Df^p(x))| \le C_{47}^f(r) \left[\frac{M_n^{1/2}}{m_n(x)}\right]^l$$

with:

$$C_{47}^f(r) = (r)! \frac{(2r)!}{2} \left(C_{20}^f(r)\right)^r.$$

Proof. — We follow [14]. By derivating equation (69), we get:

(71)
$$D^{r+1}Sf^p = \sum_{i=0}^{n-1}\sum_{l=0}^{r}(D^{r+1-l}Sf\circ f^i)(Df^i)^{r+1-l+2}F_l^r(D\log Df^i,\dots,D^l\log Df^i)$$

$$+(D^{r-l}Sf\circ f^i)(Df^i)^{r-l+2}(r-l+2)D\log Df^i$$

(72)
$$\cdot F_l^r(D\log Df^i,\dots,D^l\log Df^i)$$

$$+\sum_{j=1}^{l}\frac{\partial F_l^r}{\partial X_j}(D\log Df^i,\dots,D^l\log Df^i)D^{j+1}\log Df^i(Df^i)^{r-l+2}$$

(73)
$$D^{r+1}Sf^p = \sum_{i=0}^{p-1}\sum_{l=0}^{r}(D^{r+1-l}Sf\circ f^i)(Df^i)^{r+1-l+2}F_l^r(D\log Df^i,\dots,D^l\log Df^i)$$

(74)
$$+\sum_{l=1}^{r+1}(D^{r+1-l}Sf\circ f^i)(Df^i)^{r-l+3}(r-l+3)D\log Df^i$$

$$\cdot F_{l-1}^r(D\log Df^i,\dots,D^{l-1}\log Df^i)$$

$$+\sum_{l=1}^{r+1}\sum_{j=2}^{l}\frac{\partial F_{l-1}^r}{\partial X_{j-1}}(D\log Df^i,\dots,D^{l-1}\log Df^i)D^j\log Df^i(Df^i)^{r+1-l+2}.$$

Therefore, for $1\le l\le r$,

$$F_l^{r+1} = F_l^r + (r-l+3)X_1F_{l-1}^r + \sum_{j=2}^{l}X_j\frac{\partial F_{l-1}^r}{\partial X_{j-1}} \tag{75}$$

for $l=0$,

$$F_l^{r+1} = F_l^r$$

and for $l=r+1$,

$$F_l^{r+1} = (r-l+3)X_1F_{l-1}^r + \sum_{j=2}^{l}X_j\frac{\partial F_{l-1}^r}{\partial X_{j-1}}.$$

Now, let us write

$$F_l^r = \sum_{i_1+2i_2+\cdots+li_l=l} a_{l,r}(i_1,\dots,i_l)X_1^{i_1}\cdots X_l^{i_l}.$$

We have $a_{l,r}(i_1,\dots,i_l)\ge 0$. Let

$$a_{l,r} = \max_{i_1+2i_2+\cdots+li_l=l} a_{l,r}(i_1,\dots,i_l)$$

and

$$\bar{a}_r = \max_{0 \leq j \leq r} a_{j,r}.$$

Consider $i_1, \ldots, i_l$ such that $a_{l,r}(i_1, \ldots, i_l) = a_{l,r}$. By applying equation (75), we have, for $1 \leq l \leq r$:

$$\begin{aligned} a_{l,r+1} &\leq a_{l,r} + (r+3-l)a_{l-1,r} + (l-1)(\max i_j)a_{l-1,r} \\ &\leq (r+3-l+l^2-l)\bar{a}_{l,r} \leq (r+1)^2\bar{a}_{l,r}. \end{aligned}$$

For $l = 0$ or $r+1$, this estimate still holds. Therefore, $\bar{a}_{r+1} \leq (r+1)^2\bar{a}_r$ and by iteration, we obtain:

$$\bar{a}_r \leq (r!)^2.$$

Moreover, since

$$\begin{aligned} &F_l^r(D\log Df^i, \ldots, D^l \log Df^i) \\ &\qquad = \sum_{i_1+2i_2+\cdots+li_l=l} a_{l,r}(i_1, \ldots, i_l)(D\log Df^i)^{i_1} \cdots (D^l \log Df^i)^{i_l} \end{aligned}$$

and since $\#\{(i_1, \ldots, i_l)/i_1+2i_2+\cdots+li_l = l\} \leq \#\{(i_1, \ldots, i_l)/i_1+i_2+\cdots+i_l = l\} = \frac{(2l-1)!}{l!(l-1)!}$ (this classical equality can be shown by induction) then by applying the induction assumption,

$$\begin{aligned} &|F_l^r(D\log Df^i(x), \ldots, D^l \log Df^i(x))| \\ &\qquad \leq (r!)^2 \frac{(2l-1)!}{l!(l-1)!} \max_{i_1+2i_2+\cdots+li_l=l} (C_{20}^f(1))^{i_1} \cdots (C_{20}^f(l))^{li_l} \left[\frac{M_n^{1/2}}{m_n(x)}\right]^l \end{aligned}$$

and since the $C_{20}^f(i)$ are increasing with i, we obtain:

$$\left|F_l^r(D\log Df^i(x), \ldots, D^l \log Df^i(x))\right| \leq C_{47}^f(r) \left[\frac{M_n^{1/2}}{m_n(x)}\right]^l. \qquad \square$$

Likewise, the estimation of $G_r(D\log Df^p, \ldots, D^{l-1}\log Df^p)$ is given by the lemma:

LEMMA 5.2. — *For any $x \in \mathbb{R}$, $0 \leq p \leq q_{n+1}$, $r \geq 2$,*

$$|G_r(D\log Df^p(x), \ldots, D^{l-1}\log Df^p(x))| \leq C_{48}^f(r) \left[\frac{M_n^{1/2}}{m_n(x)}\right]^r$$

with $C_{48}^f(r+1) = \frac{(2r)!}{2(r+1)}(C_{20}^f(r))^{r+1}$

Proof. — The polynomial G_r satisfies the following identity:

$$G_{r+1} = \sum_{j=2}^{r} X_j \frac{\partial G_r}{\partial X_{j-1}}.$$

We denote

$$G_r = \sum_{i_1+2i_2+\cdots+(r-1)i_{r-1}=r} b_r(i_1, \ldots, i_{r-1}) X_1^{i_1} \cdots X_{r-1}^{i_{r-1}}$$

(we have, for example, $G_2 = -\frac{1}{2}X_1^2$).

Let

$$b_r = \max_{i_1+2i_2+\cdots+(r-1)i_{r-1}=r} |b_r(i_1, \ldots, i_{r-1})|.$$

For $r \geq 2$, we have $b_{r+1} \leq r(\max_{1\leq j\leq r-1} i_j) b_r \leq r^2 b_r$ and therefore, $b_r \leq \frac{(r-1)!^2}{2}$.

Therefore,

$$\begin{aligned}
&|G_{r+1}(D\log Df^p(x), \ldots, D^r \log Df^p(x))| \\
&\quad \leq \frac{r!}{2} \frac{(2r)!}{r!(r+1)!} \max_{i_1+2i_2+\cdots+ri_r=r+1} (C_{20}^f(1))^{i_1} \cdots (C_{20}^f(r))^{i_r} \left[\frac{M_n^{1/2}}{m_n(x)}\right]^{r+1}.
\end{aligned}$$

Since the constants $C_{20}^f(r)$ are increasing with r, we can take:

$$C_{48}^f(r+1) = \frac{(2r)!}{2(r+1)} (C_{20}^f(r))^{r+1}. \qquad \square$$

We can now show estimation (20). By applying equation (70), we have, for $r \geq 2$:

$$D^{r+1} \log Df^p = D^{r-1} Sf^p - G_{r+1}(D\log Df^p, \ldots, D^r \log Df^p).$$

Therefore, by equation (69) and Lemma 3.1,

$$|D^{r+1} \log Df^p(x)| \leq \left(rC_{47}^f(r)|Sf|_{r-1} e^{(r+1)W(f)} + C_{48}^f(r+1)\right) \left(\frac{M_n^{1/2}}{m_n(x)}\right)^{r+1}$$

$$|D^{r+1} \log Df^p(x)| \leq (C_{20}^f(r))^r \frac{(2r)!}{2} \left(|Sf|_{r-1} e^{(r+1)W(f)} + C_{20}^f(r)\right) \left(\frac{M_n^{1/2}}{m_n(x)}\right)^{r+1}.$$

We can show by induction on r that we can take, for $r \geq 3$,

$$C_{20}^f(r) = \left[C_{20}^f(2)(2r)^{2r} (\max(1, |Sf|_{r-2})) e^{rW(f)}\right]^{r!}.$$

6. Appendix: some polynomials

Lemma 4.4 is used for some specific polynomials. There exist A_l, B_l, G_l, G_l^r, E_l^r, polynomials of l variables $X_1, \dots, X_l$ homogeneous of weight l if X_i has weight i, such that, for $l \geq 1$, and for any diffeomorphisms g and h sufficiently differentiable, we have [14, p. 337-338]:

$$D^{l+1} g = A_l \big(D \log Dg, \dots, D^l \log Dg \big) Dg \tag{76}$$

$$D^l \log Dg = B_l \left(\frac{D^2 g}{Dg}, \dots, \frac{D^{l+1} g}{Dg} \right) \tag{77}$$

$$D^{l-2} Sf = D^l \log Df + G_l (D \log Df, \dots, D^{l-1} \log Df). \tag{78}$$

For $r \geq 0$,

$$\begin{aligned} D^r \log D(g \circ h) &= (D^r \log Dg \circ h)(Dh)^r + D^r \log Dh \\ &+ \sum_{l=1}^{r-1} D^{r-l} \log Dg \circ h (Dh)^{r-l} G_l^r (D \log Dh, \dots, D^l \log Dh). \end{aligned} \tag{79}$$

For $r \geq 0$ and $n \geq 1$,

$$D^r \log Dg^n = \sum_{l=0}^{r-1} \sum_{i=0}^{n-1} (D^{r-l} \log Dg \circ g^i)(Dg^i)^{r-l} E_l^r (D \log Dg^i, \dots, D^l \log Dg^i). \tag{80}$$

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