

# RINGS OF MICRODIFFERENTIAL OPERATORS FOR ARITHMETIC *D*-MODULES

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# RINGS OF MICRODIFFERENTIAL OPERATORS FOR ARITHMETIC D-MODULES

# CONSTRUCTION AND AN APPLICATION TO THE CHARACTERISTIC VARIETIES FOR CURVES

by Tomoyuki Abe

ABSTRACT. — One aim of this paper is to develop a theory of microdifferential operators for arithmetic  $\mathscr{D}$ -modules. We first define the rings of microdifferential operators of arbitrary levels on arbitrary smooth formal schemes. A difficulty lies in the fact that there is no homomorphism between rings of microdifferential operators of different levels. To remedy this, we define the intermediate differential operators, and using these, we define the ring of microdifferential operators for  $\mathscr{D}^{\dagger}$ . We conjecture that the characteristic variety of a  $\mathscr{D}^{\dagger}$ -module is computed as the support of the microlocalization of a  $\mathscr{D}^{\dagger}$ -module, and prove it in the curve case.

#### Introduction

This paper is aimed to construct a theory of rings of microdifferential operators for arithmetic  $\mathscr{D}$ -modules. Let X be a smooth variety over  $\mathbb{C}$ . Then the sheaf of rings of microdifferential operators denoted by  $\mathscr{E}_X$  is defined on the cotangent bundle  $T^*X$  of X. This ring is one of basic tools to study  $\mathscr{D}$ -modules microlocally, and it is used in various contexts. One of the most important and fundamental properties is the equality

 $(\mathsf{V}) \qquad \qquad \operatorname{Char}(\mathscr{M}) = \operatorname{Supp}(\mathscr{E}_X \otimes_{\pi^{-1}\mathscr{D}_X} \mathscr{M})$ 

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for a coherent  $\mathscr{D}_X$ -module  $\mathscr{M}$ , where  $\pi : T^*X \to X$  is the projection. One goal of this study is to find an analogous equality in the theory of arithmetic  $\mathscr{D}$ -modules.

We should point out two attempts to construct rings of microdifferential operators. The first attempt was made by R. G. López in [13]. In there, he constructed the ring of microdifferential operators of *finite order on curves*. However, the relation between his construction and the theory of arithmetic  $\mathscr{D}$ -modules was not clear as he pointed out in the last remark of [13]. The second construction was carried out by A. Marmora in [23]. Our work can be seen as a generalization of this work, and we explain the relation with our construction in the following.

Now, let R be a complete discrete valuation ring of mixed characteristic (0, p). Let  $\mathscr{X}$  be a smooth formal scheme over  $\mathrm{Spf}(R)$ , and we denote the special fiber of  $\mathscr{X}$  by X. For an integer  $m \geq 0$ , P. Berthelot defined the ring of differential operators of level m denoted by  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ . He also defined the characteristic varieties for coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -modules in almost the same way we define the characteristic varieties for analytic (or algebraic)  $\mathcal{D}$ -modules. It is natural to hope that there exists a theory of microdifferential operators, and that we can define the ring of microdifferential operators  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)}$  of level m associated with  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$  satisfying an analog of  $(\vee)$ . When  $\mathscr{X}$  is a curve (and m = 0), this was done by Marmora in his study of Fourier transform. He fixed a system of local coordinates, constructed the ring of microdifferential operators using explicit descriptions as in [8, Chapter VIII], and proved that the construction does not depend on the choice of local coordinates. In this paper, we use a general technique of G. Laumon of formal microlocalization of certain filtered rings (cf. [22]) to define the ring of naive microdifferential operators of level mdenoted by  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)}$  (cf. 2.9). One advantage of this construction is that we do not need to choose coordinates. It follows also formally using the result of Laumon that for a coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -module  $\mathscr{M}$ , we get an analogous equality of  $(\mathcal{V})$ 

$$(\mathcal{Z}) \qquad \operatorname{Char}^{(m)}(\mathscr{M}) = \operatorname{Supp}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)} \otimes_{\pi^{-1}\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}} \pi^{-1}\mathscr{M})$$

in  $T^{(m)*}X := \operatorname{Spec}(\operatorname{gr}(\mathscr{D}_X^{(m)}))$ , where  $\pi \colon T^{(m)*}X \to X$  is the projection, and  $\operatorname{Char}^{(m)}$  denotes the characteristic variety (cf. 2.14).

Before explaining the construction of sheaves of microdifferential operators associated with  $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$ , let us review the theory of Berthelot, and see why we need to consider  $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$ -modules. Berthelot proved that many fundamental theorems in the theory of analytic  $\mathscr{D}$ -modules hold also for  $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$ -modules. For example, he defined pull-backs and push-forwards, and proved that push-forwards of coherent modules by proper morphisms remain coherent (cf. [7]). However,

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the analogue of Kashiwara's theorem, which states an equivalence between the category of coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -modules which are supported on a smooth closed formal subscheme  $\mathscr{Z}$  of  $\mathscr{X}$  and the category of coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -modules, does not hold. This failure makes it difficult to define a suitable subcategory of holonomic modules in the category of  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -modules. To remedy this, Berthelot took inductive limit on the levels to define the ring  $\mathscr{D}_{\mathscr{X},\mathbb{Q}}^{\dagger}$ , and proved an analogue of Kashiwara's theorem for coherent  $\mathscr{D}_{\mathscr{X},\mathbb{Q}}^{\dagger}$ -modules (cf. [7, 5.3.3]). As in the analytic  $\mathscr{D}$ -module theory, we need to consider holonomic modules to deal with push-forwards along open immersions, and we need to define characteristic varieties to define holonomic modules. When a coherent  $\mathscr{D}_{\mathscr{X},\mathbb{Q}}^{\dagger}$ -module possesses a *Frobenius structure* (*i.e.*, an isomorphism  $\mathscr{M} \xrightarrow{\sim} F^*\mathscr{M}$ ), Berthelot defined the characteristic variety. He reduced the definition to a finite level situation using a marvelous theorem of Frobenius descent, and proved Bernstein's inequality by using the analogue of Kashiwara's theorem. However, in the absence of Frobenius, the situation is mysterious.

In this paper, we propose a new formalism which allows us at least conjecturally to interpret this characteristic varieties by means of microlocalizations, and use them to define the characteristic varieties for general coherent  $\mathscr{D}_{\mathscr{X},\mathbb{Q}}^{\dagger}$ -modules which may not carry Frobenius structures. We also prove the conjecture in the case of curves (cf. Theorem 7.2). Let us describe a more precise statement and difficulties to carry this out.

One of the difficulties in defining microdifferential operators associated with  $\mathscr{D}^{\dagger}$  is that there are *no* transition homomorphism (cf. 4.1)

$$\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)} \to \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m+1)}$$

compatible with  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)} \to \widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m+1)}$ . This makes it hard to define the ring of microdifferential operators corresponding to  $\mathscr{D}_{\mathscr{X},\mathbb{Q}}^{\dagger}$  in a naive way. Let  $\pi: T^*\mathscr{X} \to \mathscr{X}$  be the projection. To remedy this, we define a  $\pi^{-1}\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -algebra  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$  for any integer  $m' \geq m$  called the "intermediate ring of microdifferential operators of level (m,m')" so that there exist homomorphisms of  $\pi^{-1}\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -algebras

$$\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m'+1)} \to \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}, \qquad \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')} \to \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m+1,m')},$$

and  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m)} = \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ . We define

$$\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m,\dagger)} := \varprojlim_{m'} \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}.$$

On this level, we have a transition homomorphism  $\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m,\dagger)} \to \mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m+1,\dagger)}$  compatible with  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)} \to \widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m+1)}$ . We define

$$\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{\dagger} := \varinjlim_{m} \mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m,\dagger)}.$$

Unfortunately, contrary to  $(\mathcal{Z})$ , we no longer have the equality

$$\operatorname{Char}^{(m)}(\mathscr{M}) = \operatorname{Supp}(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m,\dagger)} \otimes_{\pi^{-1}\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}} \pi^{-1}\mathscr{M})$$

for a coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -module  $\mathscr{M}$  in general (cf. 7.1). However, we conjecture the following.

**Conjecture**. — Let  $\mathscr{X}$  be a quasi-compact smooth formal scheme over R, and  $\mathscr{M}$  be a coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -module. Then there exists N > m such that for any  $m' \geq N$ ,

$$\operatorname{Char}^{(m')}(\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m')}\otimes_{\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}}\mathscr{M}) = \operatorname{Supp}(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m',\dagger)}\otimes_{\pi^{-1}\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}}\pi^{-1}\mathscr{M}).$$

This conjecture implies that  $\operatorname{Car}(\mathscr{M}) = \operatorname{Supp}(\mathscr{E}^{\dagger}_{\mathscr{X},\mathbb{Q}} \otimes \mathscr{M})$  for a coherent  $F \cdot \mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$ -module  $\mathscr{M}$  where Car denotes the characteristic variety defined by Berthelot. It is also worth noticing here that if this conjecture is true, the characteristic varieties for coherent  $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$ -modules stabilize when we raise the level m, and in particular we are able to define characteristic varieties for coherent  $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$ -modules structures. In the last part of this paper, we prove the following.

#### **Theorem 7.2.** — When $\mathscr{X}$ is a curve, the conjecture is true.

Finally, let us point out one of the most important applications of this theorem. The construction of the rings of microdifferential operators in this paper and Theorem 7.2 are crucial technical tools for the proof of the product formula for *p*-adic epsilon factors in [3]. Especially, the results of this paper is used to establish the theory of *p*-adic local Fourier transform and the "principle of stationary phase". While the present paper is on the review, this product formula was used as one of the most important ingredients to establish the "Langlands correspondence" of overconvergent *F*-isocrystals in [2] and [1]: In the celebrated paper of Lafforgue [20], Langlands correspondence for function field is established. This is a certain correspondence between  $\ell$ -adic Galois representations and cuspidal automorphic forms. A natural question is if there is any analogous correspondence for overconvergent *F*-isocrystals. In [1], a similar correspondence is established, and the conjecture of Deligne [12, 1.2.10 (vi)] (or more precisely [11, 4.13]) is proven for curves as an application.

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To conclude the introduction, let us see the structure of this paper. In § 1, we review the theory of formal microlocalization of certain filtered rings, and single out some basic cases where these rings are Noetherian (according to Definition 1.9). Using these results, we define the naive ring of microdifferential operators  $\widehat{\mathscr{C}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ , and prove some basic facts in the next section § 2. Before proceeding to the definition of the intermediate rings of microdifferential operators, we study some properties of  $\operatorname{gr}(\mathscr{C}_{\mathscr{X},\mathbb{Q}}^{(m)})$  in § 3. These are used to study the intermediate rings, which are defined in § 4. In § 5, we prove the flatness of transition homomorphisms and related results. One of the most important properties of  $\mathscr{C}_{\mathscr{X},\mathbb{Q}}^{(m,\dagger)}$  is that its sections over a strict affine open subscheme form a Fréchet-Stein algebra. In § 6, we prove a finiteness property of certain sheaves of modules, which may be useful to deal with sheaves on formal schemes. In the last section, we formulate the conjecture, and prove it in the case of curves.

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## Conventions

In this paper, all rings are assumed to be associative with unity. Filtered groups are assumed to be exhaustive (cf. 1.1.1), and modules are left modules unless otherwise stated. In principle, we use Roman fonts (e.g., X) for schemes, and script fonts (e.g.,  $\mathscr{X}$ ) for formal schemes.

If the reader is opening this paper in order to read [3], he might notice that the version of this paper used in it is not final. However, we notice that the numbering has not been changed since [3] was published.

# 1. Preliminaries on filtered rings

The aim of this section is to review the formal construction of the microlocalization of certain filtered rings due to O. Gabber and G. Laumon. To fix

notation and terminology, we begin by reviewing well-known definitions and properties of filtered modules.

1.1. — The reader can refer to  $[10, III, \S 2]$  and [18] for more details.

**1.1.1.** — An increasing sequence  $\{G_n\}_{n\in\mathbb{Z}}$  of subgroups of a group G is called an *increasing filtration on* G. The filtration is said to be *positive* if  $G_n = 0$  for all n < 0. We say that the filtration is *separated* if  $\bigcap_n G_n = \{e\}$  where e is the unit. If  $G_n$  are normal subgroups of G for all n, the filtration defines a canonical topology, which makes G a topological group (cf. [10, III, §2.5]). Unless otherwise stated, we always assume that filtrations are *exhaustive* (i.e.,  $\bigcup_i G_i = G$ ).

Let A be a ring (not necessary commutative), and  $\{A_i\}_{i\in\mathbb{Z}}$  be a filtration of the additive group A. We say that the couple  $(A, \{A_i\}_{i\in\mathbb{Z}})$  is a filtered ring if  $A_i \cdot A_j \subset A_{i+j}$ , and  $1 \in A_0$ . If there is no possible confusion, we abbreviate it by  $(A, A_i)$ . Let M be an A-module, and  $\{M_i\}_{i\in\mathbb{Z}}$  be a filtration of the additive group M such that  $A_i \cdot M_j \subset M_{i+j}$  for any  $i, j \in \mathbb{Z}$ . Then the couple  $(M, \{M_i\}_{i\in\mathbb{Z}})$  is said to be a filtered  $(A, A_i)$ -module. We often denote  $(M, \{M_i\}_{i\in\mathbb{Z}})$  by  $(M, M_i)$  for short.

**1.1.2.** — Let A be a ring, and I be a two-sided ideal. We put  $A_n := I^{-n}$  for  $n \leq 0$ , and  $A_n := A$  for n > 0. The couple  $(A, \{A_n\}_{n \in \mathbb{Z}})$  is a filtered ring, and the filtration is called the *I*-adic filtration.

Let  $(M, M_i)$  be a filtered  $(A, A_i)$ -module. We say that the filtration  $\{M_i\}_{i \in \mathbb{Z}}$ of M is good if there exist  $m_1, \ldots, m_s \in M$  and  $k_1, \ldots, k_s \in \mathbb{Z}$  such that  $M_n = \sum_{i=1}^s A_{n-k_i} \cdot m_i$  for any n.

**1.1.3.** — A filtered homomorphism  $f: (A, A_i) \to (B, B_i)$  is a ring homomorphism  $f: A \to B$  such that there exists an integer n satisfying  $f(A_i) \subset B_{i+n}$  for any integer i. Such a homomorphism is continuous with respect to the topology defined by the filtration on A and B. The filtered homomorphism f is said to be strict if  $f(A_i) = f(A) \cap B_i$  for any  $i \in \mathbb{Z}$ .

**1.1.4.** — For a filtered ring  $(A, A_i)$ , we put  $\operatorname{gr}_i(A) := A_i/A_{i-1}$ , and  $\operatorname{gr}(A) := \bigoplus_i \operatorname{gr}_i(A)$ . The module  $\operatorname{gr}(A)$  is naturally a graded ring, and it is called the *associated graded ring*. We define the *principal symbol* map  $\sigma: A \to \operatorname{gr}(A)$  in the following way: let  $x \in A$ . If  $x \in \bigcap_i A_i$ , then we put  $\sigma(x) = 0$ . Otherwise there exists an integer *i* such that  $x \in A_i$  and  $x \notin A_{i-1}$ . We define  $\sigma(x)$  to be the image of x in  $\operatorname{gr}_i(A) \subset \operatorname{gr}(A)$ .

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**1.1.5.** — We introduce the completion of a filtered ring. We refer to [18, Ch. I, § 3] for the details. Let  $(A, A_i)$  be a filtered ring. Let  $A[\nu, \nu^{-1}]$  be the ring of Laurent polynomials with one variable  $\nu$  over A, graded by the degree of  $\nu$ . Here, the element  $\nu$  is in the center by definition. We define the graded sub-algebra of  $A[\nu, \nu^{-1}]$  denoted by  $A_{\bullet}$ , called the *Rees ring* of  $(A, A_i)$ , by the formula

$$A_{\bullet} := \bigoplus_{i \in \mathbb{Z}} A_i \cdot \nu^i.$$

For an integer  $n \geq 1$ , we define a graded ring  $A_{\bullet,n} := A_{\bullet}/\nu^n A_{\bullet} \cong \bigoplus_{i \in \mathbb{Z}} A_i/A_{i-n} \cdot \nu^i$ . For  $i \in \mathbb{Z}$ , we put  $A_{i,n} := A_i/A_{i-n}$ , the part of degree i of  $A_{\bullet,n}$ . We get a projective system of graded rings

$$\to A_{\bullet,n+1} \to A_{\bullet,n} \to \cdots \to A_{\bullet,1} \cong \operatorname{gr}(A).$$

We define a module and a ring by

$$\widehat{A}_i := \lim_{n \to \infty} A_{i,n} \left( \cong \lim_{n \to \infty} A_i / A_{i-n} \right), \qquad \widehat{A} := \lim_{i \to \infty} \widehat{A}_i.$$

The couple  $(\widehat{A}, \{\widehat{A}_i\}_{i\in\mathbb{Z}})$  is a filtered ring, and is called the *completion of*  $(A, A_i)$ . This definition coincides with [18, Ch. I, 3.4.1] by [18, Ch. I, 3.5, (d) and (e)]. We note that the completion is separated, and the canonical homomorphism  $\operatorname{gr}(A) \to \operatorname{gr}(\widehat{A})$  is an isomorphism by [18, Ch. I, 4.2.2]. We say that the filtered ring  $(A, A_i)$  is *complete* if the canonical homomorphism  $A \to \widehat{A}$  is an isomorphism of filtered rings.

**1.1.6.** — We say that a filtered ring  $(A, A_i)$  is left (resp. right, two-sided) Noetherian filtered if the Rees ring  $A_{\bullet}$  is left (resp. right, two-sided) Noetherian. If A is a Noetherian filtered ring, the associated graded ring gr(A) is Noetherian since  $gr(A) \cong A_{\bullet}/\nu A_{\bullet}$ . If A is a complete filtered ring, A is Noetherian filtered if and only if gr(A) is a Noetherian ring (cf. [18, Ch. II, 1.2.3]). This shows that the completion of a Noetherian filtered ring is Noetherian filtered. Moreover, if  $(A, A_i)$  is a Noetherian filtered ring, the canonical homomorphism  $A \to \widehat{A}$  is flat by [18, Ch. I, 1.2.1]. We say that a Noetherian filtered ring is Zariskian if any good filtered module is separated. Any Noetherian filtered complete ring is known to be Zariskian (cf. [18, Ch. II, 2.2.1]).

**1.1.7.** — Let X be a topological space or, more generally, topos. The terminologies defined so far *except for those defined in* 1.1.6 *and principal symbol* in 1.1.4 can be defined also in the language of sheaves by replacing "ring" by "sheaf of rings on X" and so on. See [8, A.III.2] for more details.

**1.2.** — Let  $(A, A_i)$  be a filtered ring which is complete and the associated graded ring  $\operatorname{gr}(A)$  is *commutative*. Let  $S_1 \subset \operatorname{gr}(A)$  be a homogeneous multiplicative set (i.e., a multiplicative set consisting of homogeneous elements). Let  $c_n \colon A_{\bullet,n} \to A_{\bullet,1} \cong \operatorname{gr}(A)$  be the canonical homomorphism. We put

$$S_n := \{ x \in A_{\bullet,n} \mid c_n(x) \in S_1 \}.$$

By [22, A.2.1], the multiplicative set  $S_n$  satisfies the two-sided Ore condition (cf. [21, 4. § 10A]). We define a graded ring by

$$A'_{\bullet,n} := S_n^{-1} A_{\bullet,n} \cong A_{\bullet,n} S_n^{-1}.$$

This defines a projective system of graded rings  $\{A'_{\bullet,n}\}$ . Let us denote by  $A'_{i,n}$  the part of degree *i* of  $A'_{\bullet,n}$ . We define

$$A_i' := \varprojlim_{n \to \infty} A_{i,n}', \qquad A' := \varinjlim_{i \to \infty} A_i'$$

The filtered ring  $(A', A'_i)$  is complete. We denote this filtered ring by  $(A, A_i)_{S_1}$ , and we call it the *microlocalization of*  $(A, A_i)$  with respect to  $S_1$ . If gr(A)is Noetherian, then  $A'_0$  and A' are Noetherian and the canonical homomorphism  $A \to A'$  is flat by [22, Corollaire A.2.3.4].

Let us describe elements of A' concretely. Put  $S := \{a \in A \mid \sigma(a) \in S_1\}$ , and take  $s \in S$ . By definition, s is invertible in  $(A, A_i)_{S_1}$ . Given an element  $a \in A'_i$ , there exist  $a_k \in A_{l_k}$  and  $s_k \in S \cap A_{l_k-k}$  (thus  $a_k s_k^{-1} \in A'_k$ ) for each integer  $k \leq i$ , such that  $a = \sum_{k \leq i} a_k s_k^{-1}$ , where the sum is infinite and we consider the topology defined by the filtration of A (cf. 1.1.1). Moreover, assume that  $\sigma(s) \in \operatorname{gr}_N(A)$  and  $S_1 = \{\sigma(s)^n\}_{n\geq 0}$ . Then for any  $s' \in S$ , there exists an integer l such that  $\sigma(s') = \sigma(s)^l$ . Since  $a := s^l - s' \in A_{Nl-1}$ ,  $s'^{-1} = s^{-l} \cdot \sum_{k\geq 0} (as^{-l})^k$ . Thus for any  $a' \in A'_i$ , there exist an integer  $n_k \geq 0$ and  $a'_k \in A_{k+Nn_k}$  such that  $a' = \sum_{k \leq i} a'_k s^{-n_k}$ .

**1.3.** — Let  $(A, A_i)$  be a complete filtered ring whose associated graded ring is commutative. The constructions in the previous subsection can be carried out in almost the same way also for filtered  $(A, A_i)$ -modules. For the details see [22, A.2]. For example, for a filtered A-module  $(M, M_i)$  and a homogeneous multiplicative system  $S_1 \subset \operatorname{gr}(A)$ , we are able to define the microlocalization of  $(M, M_i)$  with respect to  $S_1$  denoted by  $(M, M_i)_{S_1}$ , which is complete.

**1.4.** — Let us sheafify the results. Let  $(A, \{A_i\}_{i \in \mathbb{Z}})$  be a positively filtered ring such that the associated graded ring gr(A) is commutative. Let R := gr(A) be the positively graded commutative ring. Note that  $A_0 = R_0$  is a commutative ring by assumption. We let  $X := \operatorname{Spec}(R_0), V := \operatorname{Spec}(R), P := \operatorname{Proj}(R)$ . Let  $s \colon X \to V$  be the morphism defined by the canonical projection  $R \to R_0$ .

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We put  $\mathring{V} := V \setminus s(X)$ . We have the following canonical commutative diagram (cf. [16, II, 8.3]).



We define a topological space V' in the following way: as a set, V' := V. The topology of V' is generated by the basis of open sets

 $\{D(f) \mid f \in R \text{ and } f \text{ is homogeneous}\}.$ 

We denote by  $\epsilon: V \to V'$  the identity map as sets, which is continuous. In the sequel, various sheaves are defined naturally on V'. A general strategy of [22] is to define sheaves on V, which are final outputs, by taking  $\epsilon^{-1}$ . Now, let us denote by  $\mathfrak{O}(T)$ , for a topological space T, the category of open sets of T. The canonical functor  $\epsilon^{-1}: \mathfrak{O}(V') \to \mathfrak{O}(V)$  admits a left adjoint denoted by  $\epsilon$ . For  $U \in \mathfrak{O}(V)$ , this functor can be described as  $\epsilon_{\cdot}(U) = \bigcup_{\lambda \in \mathbb{G}_{m,X}} \lambda \cdot U$  where  $\mathbb{G}_{m,X}$  acts on V naturally. Using this functor,  $\epsilon^{-1}$  can clearly be calculated: let  $\mathcal{F}'$  be a sheaf on V'. Then we have

(1.4.1) 
$$(\epsilon^{-1}\mathcal{F}')(U) = \mathcal{F}'(\epsilon_{\cdot}(U))$$

by [22, A.3.0.2]. Thanks to this equality, many properties of  $\mathcal{F}'$  also automatically be applied to that of  $\epsilon^{-1}\mathcal{F}'$ .

Having this strategy in mind, let us define first important sheaves on V. Let  $\mathcal{O}_{V'} := \epsilon_* \mathcal{O}_V$ . For  $n \in \mathbb{Z}$ , we denote by  $\mathcal{O}_{V'}(n)$ , the subsheaf of  $\mathcal{O}_{V'}$  consisting of the homogeneous sections of degree n. We put  $\mathcal{O}_V(n) := \epsilon^{-1}(\mathcal{O}_{V'}(n))$ , and  $\mathcal{O}_V(*) := \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_V(n)$ . We note that  $\mathcal{O}_V(*) \cong \epsilon^{-1} \epsilon_* \mathcal{O}_V$ . We get  $\mathcal{O}_V(n)|_{\dot{V}} \cong q^{-1} \mathcal{O}_P(n)$  for any integer n by [22, A.3.0.5]. By [22, A.3.0.5], we also have

(1.4.2) 
$$p_* \mathcal{O}_V(*) \cong s^{-1} \mathcal{O}_V(*) \cong R$$

where  $\tilde{\cdot}$  denotes the associated quasi-coherent  $\mathcal{O}_X$ -module.

**1.5.** — Let  $(\mathcal{A}, \mathcal{A}_i)$  be the filtered quasi-coherent  $\mathcal{O}_X$ -algebra associated to  $(A, A_i)$  on X. Let f be a homogeneous element of  $\operatorname{gr}(A)$ , and we put  $S_1(f) := \{f^m\}_{m\geq 0} \subset \operatorname{gr}(A)$ . Let  $S_n(f)$  be the multiplicative set of  $A_{\bullet,n}$  constructed from  $S_1(f)$  (see 1.2), and define  $A'_{\bullet,n}(f) := S_n(f)^{-1}A_{\bullet,n}$ . We define a sheaf  $\mathcal{B}'_{\bullet,n}$  on V' to be the sheaf associated to the presheaf  $D(f) \mapsto A'_{\bullet,n}(f)$ over the open basis of V' consisting of D(f) with a homogeneous element fin  $\operatorname{gr}(A)$ . By [22, A.3.1.1], we know that

(1.5.1) 
$$\Gamma(D(f), \mathcal{B}'_{\bullet,n}) = A'_{\bullet,n}(f).$$

We define

$$\mathcal{B}'_i := \lim_{n \to \infty} \mathcal{B}'_{i,n}, \qquad \mathcal{B}' := \varinjlim_{i \to \infty} \mathcal{B}'_i.$$

Then we have an isomorphism of complete filtered rings  $(A, A_i)_{S_1(f)} \xrightarrow{\sim} \Gamma(D(f), (\mathcal{B}', \mathcal{B}'_i))$  for a homogeneous element f of gr(A) by [22, (A.3.1.2)]. Now, let us use the general machinery to define a filtered sheaf of rings on V as follows:

$$(\mathcal{B}, \mathcal{B}_i) := \epsilon^{-1}(\mathcal{B}', \mathcal{B}'_i)$$

There is a canonical homomorphism of filtered rings  $\varphi: p^{-1}(\mathcal{A}, \mathcal{A}_i) \to (\mathcal{B}, \mathcal{B}_i)$ on V. The filtered ring  $(\mathcal{B}, \mathcal{B}_i)$  is called the *microlocalization of*  $(\mathcal{A}, \mathcal{A}_i)$ . By [22, A.3.1.6], we have canonical isomorphisms of graded rings

(1.5.2)  $\operatorname{gr}_n(\mathcal{B}) \cong \mathcal{O}_V(n), \quad \operatorname{gr}(\mathcal{B}) \cong \mathcal{O}_V(*).$ 

**Remark.** — Note that  $q_*(\mathcal{O}_{\check{V}}(n)) \cong \mathcal{O}_P(n)$  (resp.  $\mathcal{O}_{\check{V}}(n)$ ) is a quasi-coherent  $\mathcal{O}_P$ -module (resp.  $\mathcal{O}_{\check{V}}(0)$ -module). However, caution that, a priori,  $q_*(\mathscr{B}|_{\check{V}})$  and  $q_*(\mathscr{B}_i|_{\check{V}})$  do not have  $\mathcal{O}_P$ -module structure, nor do  $\mathscr{C}|_{\check{V}}$  and  $\mathscr{C}_i|_{\check{V}}$  have  $\mathcal{O}_{\check{V}}$ -module or  $\mathcal{O}_{\check{V}}(0)$ -module structure.

**1.6.** Let  $(M, M_i)$  be a filtered  $(A, A_i)$ -module such that  $M_i = 0$  for  $i \ll 0$  (and  $\bigcup_{i \in \mathbb{Z}} M_i = M$ ). Let  $(\mathcal{M}, \mathcal{M}_i)$  be the quasi-coherent  $\mathcal{O}_X$ -module associated to the filtered module  $(M, M_i)$ . This is a filtered  $(\mathcal{C}, \mathcal{C}_i)$ -module.

Using exactly the same construction (cf. [22, A.3.2]), we are able to define a  $(\mathcal{B}', \mathcal{B}'_i)$ -module  $(\mathcal{N}', \mathcal{N}'_i)$  on V' such that we have an isomorphism of complete filtered modules  $(M, M_i)_{S_1(f)} \xrightarrow{\sim} \Gamma(D(f), (\mathcal{N}', \mathcal{N}'_i))$  for a homogeneous element f of gr(A). We define a filtered  $(\mathcal{B}, \mathcal{B}_i)$ -module by

$$(\mathcal{N}, \mathcal{N}_i) := \epsilon^{-1}(\mathcal{N}', \mathcal{N}'_i).$$

There is a homomorphism  $\varphi_M : p^{-1}(\mathcal{M}, \mathcal{M}_i) \to (\mathcal{N}, \mathcal{N}_i)$  over  $\varphi$ .

**1.7. Lemma.** — Let  $\tilde{\operatorname{gr}}(\mathcal{M})$  be the quasi-coherent  $\mathcal{O}_V$ -module associated with the  $\operatorname{gr}(A) = R$ -module  $\operatorname{gr}(M)$ . Suppose  $\operatorname{gr}(M)$  is finitely presented over  $\operatorname{gr}(A)$ . Then we have the following equalities in V:

$$\operatorname{Supp}(\widetilde{\operatorname{gr}}(\mathcal{M})) = \operatorname{Supp}(\operatorname{gr}(\mathcal{N})) = \operatorname{Supp}(\mathcal{B} \otimes_{p^{-1}} \mathcal{B} p^{-1} \mathcal{M}).$$

*Proof.* — The first equality follows from [22, Proposition A.3.2.4 (i)]. Let us show the second one. Let U := D(f) with a homogeneous element f of gr(A). Since  $\Gamma(U, \mathcal{N})$  is complete, it is in particular separated with respect to the filtration. Thus,  $\Gamma(U, \mathcal{N}) = 0$  if and only if  $\Gamma(U, gr(\mathcal{N})) \cong gr(\Gamma(U, \mathcal{N})) = 0$  where the first isomorphism follows from [22, A.3.2]. Combining this with [22, Proposition A.3.2.4 (ii)], the lemma follows.

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**Definition**. — Assume  $\mathcal{M}$  is an  $\mathcal{A}$ -module of finite type. Then there exists a good filtration  $\{\mathcal{M}_i\}_{i\in\mathbb{Z}}$  (cf. [8, A.III 2.15]) of  $\mathcal{M}$ . Suppose  $\operatorname{gr}(A)$  is Noetherian. The above lemma implies that  $\operatorname{Supp}(\widetilde{\operatorname{gr}}(\mathcal{M}))$  does not depend on the choice of good filtrations. We call this the *characteristic variety of*  $\mathcal{M}$  and denote by  $\operatorname{Char}(\mathcal{M})$ .

**1.8. Remark.** — These construction localize and are functorial. In particular, we may globalize the definitions of microlocalizations on schemes not necessary affine (cf. [22, A.3.3]).

1.9. — Now, we collect some basic facts on Noetherian conditions.

**Definition**. — Let X be a topological space,  $\mathscr{C}$  be a sheaf of rings on X, and  $\mathfrak{B}$  be an open basis of the topology.

(i) The ring  $\mathscr{A}$  is said to be *left Noetherian with respect to*  $\mathfrak{B}$  if it satisfies the following conditions.

- 1. It is a left coherent ring (i.e., locally, any finitely generated left ideal of  $\mathscr{C}$  is finitely presented).
- 2. For any point  $x \in X$ , the stalk  $\mathscr{A}_x$  is a left Noetherian ring.
- 3. For any  $U \in \mathfrak{B}$ ,  $\Gamma(U, \mathfrak{A})$  is a left Noetherian ring.

In the same way, we define a right (resp. two-sided) Noetherian ring with respect to  $\mathfrak{B}$ . When there is no possible confusion, we abbreviate two-sided Noetherian sheaf of rings with respect to  $\mathfrak{B}$  as Noetherian ring.

(ii) A filtered ring  $(\mathcal{A}, \mathcal{A}_i)$  is said to be *pointwise left* (resp. *right, two-sided*) Zariskian if the stalk  $\mathcal{A}_x$  is left (resp. right, two-sided) Zariskian for any  $x \in X$ .

(iii) An  $\mathscr{A}$ -algebra  $\mathscr{B}$  is said to be of finite type over  $\mathscr{A}$  if for any  $x \in X$ , there exists an open neighborhood U of x and a surjection  $\mathscr{A}[T_1, \ldots, T_n]|_U \twoheadrightarrow \mathscr{B}|_U$ .

**Remark.** — This definition of Noetherian ring is slightly different from that of [19, Definition 1.1.1], who replaced 3 by Condition (c): for any open set U of X, a sum of left coherent  $\mathscr{C}|_U$ -ideals are also coherent. In §6, we show that a stronger property than Condition (c) holds for some of the Noetherian rings defined in this paper.

**Example.** — Let X be a Noetherian scheme. Let  $\mathfrak{B}$  be the open basis consisting of affine open subschemes of X. Then  $\mathcal{O}_X$  is a Noetherian ring with respect to  $\mathfrak{B}$ . More generally, let  $\mathscr{X}$  be a locally Noetherian adic formal scheme (cf. [16, I, 10.4.2]) and  $\mathfrak{C}$  be the open basis consisting of affine open formal subschemes of  $\mathscr{X}$ . Then  $\mathcal{O}_{\mathscr{X}}$  is Noetherian with respect to  $\mathfrak{C}$  by [16, I, 10.1.6].

**1.10.** — The following lemma is a generalization of [5, 3.3.6] to filtered rings.

**Lemma**. — Let  $(\mathcal{A}, \{\mathcal{A}_i\}_{i \in \mathbb{Z}})$  be a filtered ring on a topological space X. Let  $\mathfrak{B}$  be an open basis of the topological space X. Suppose that the following conditions hold:

- 1. For any  $U \in \mathfrak{B}$ , the filtered ring  $(\Gamma(U, \mathcal{C}), \Gamma(U, \mathcal{C}_i))$  is complete.
- 2. The graded ring  $gr(\mathscr{C})$  is left Noetherian with respect to  $\mathfrak{B}$ .
- 3. For  $V, U \in \mathfrak{B}$  such that  $V \subset U$ , the restriction homomorphism  $\Gamma(U, \operatorname{gr}(\mathscr{A})) \to \Gamma(V, \operatorname{gr}(\mathscr{A}))$  is right flat.
- 4. For any  $U \in \mathfrak{B}$ , the canonical homomorphism  $\operatorname{gr}(\Gamma(U, \mathcal{C})) \to \Gamma(U, \operatorname{gr}(\mathcal{C}))$ is an isomorphism.

Then, for any  $x \in X$ , the canonical homomorphism

$$(1.10.1) \qquad \qquad \mathcal{A}_x \to \mathcal{A}_x^{\wedge}$$

is right faithfully flat, where  $\wedge$  denotes the completion with respect to the filtration on  $\mathcal{A}_x$ . Moreover,  $(\mathcal{A}, \mathcal{A}_i)$  is pointwise left Zariskian, and  $\mathcal{A}$  is left Noetherian with respect to  $\mathfrak{B}$ . The statement is also valid if we replace left (resp. right) by right (resp. left).

*Proof.* — We only deal with the left case, and modules are always assumed to be left modules. Let  $x \in X$ , and take  $U \in \mathfrak{B}$  such that  $x \in U$ . Let us check that the restriction homomorphism

(1.10.2)  $r: \Gamma(U, \mathscr{C}) \to \mathscr{C}_r^{\wedge}$ 

is flat. Indeed, consider the following commutative diagram

$$\begin{array}{c|c} \operatorname{gr}(\Gamma(U,\,\mathscr{C})) & \stackrel{\sim}{\longrightarrow} \Gamma(U,\operatorname{gr}(\,\mathscr{C})) \\ & & & & \\ & & & & \\ \operatorname{gr}(\,\mathscr{C}_x^{\wedge}) & \stackrel{\sim}{\longleftarrow} \operatorname{gr}(\,\mathscr{C}_x) & \stackrel{\sim}{\longrightarrow} \operatorname{gr}(\,\mathscr{C})_x \end{array}$$

where the vertical homomorphisms are the restriction homomorphism of  $\mathscr{A}$ and  $\operatorname{gr}(\mathscr{A})$ . The upper horizontal homomorphism is an isomorphism by condition 4. The right vertical homomorphism is flat by condition 3, and thus  $\operatorname{gr}(r)$ is flat as well. Since the filtered rings  $\Gamma(U, \mathscr{A})$  and  $\mathscr{A}_x^{\wedge}$  are complete and their associated graded rings are Noetherian by conditions 2 and 4, these filtered rings are in fact Noetherian filtered (cf. 1.1.6). Since the source and the target of r are Noetherian filtered complete rings, r is flat by the flatness of  $\operatorname{gr}(r)$  and [18, Ch. II, 1.2.1]. By taking the inductive limit over U, (1.10.1) is flat.

We say that an  $\mathscr{A}_x$ -module M is monogenic of finite presentation if there exists a surjection  $\mathscr{A}_x \to M$  such that the kernel is a finitely generated ideal of  $\mathscr{A}_x$ . By [5, 3.3.5], to check that (1.10.1) is faithful, it suffices to show that for

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any  $\mathscr{A}_x$ -module M monogenic of finite presentation such that  $\mathscr{A}_x^{\wedge} \otimes M = 0$ , we get M = 0. Therefore, we assume M to be monogenic of finite presentation such that  $\mathscr{A}_x^{\wedge} \otimes M = 0$ . By this assumption, there exist  $U \in \mathfrak{B}$  and a  $\Gamma(U, \mathscr{A})$ -module  $M_U$  monogenic of finite presentation such that  $\mathscr{A}_x \otimes M_U \cong M$ . We fix a surjection  $\phi: \mathscr{A}_U := \Gamma(U, \mathscr{A}) \to M_U$ . This induces a good filtration on  $M_U$ . We define an  $\mathscr{A}_U$ -ideal K by the following short exact sequence

$$0 \to K \to \mathscr{C}_U \xrightarrow{\phi} M_U \to 0.$$

We consider the induced filtration from  $\mathscr{A}_U$  on K. Then we have the following exact sequence

$$0 \to \operatorname{gr}(K) \to \operatorname{gr}(\mathscr{A}_U) \xrightarrow{\operatorname{gr}(\phi)} \operatorname{gr}(M_U) \to 0.$$

Since  $\operatorname{gr}(\mathscr{C}^{\wedge}_{x})$  is flat over  $\operatorname{gr}(\mathscr{C}_{U}) \cong \Gamma(U, \operatorname{gr}(\mathscr{C}))$ , the sequence

$$(1.10.3) \ 0 \to \operatorname{gr}(\mathscr{A}_x) \otimes_{\operatorname{gr}(\mathscr{A}_U)} \operatorname{gr}(K) \to \operatorname{gr}(\mathscr{A}_x) \to \operatorname{gr}(\mathscr{A}_x) \otimes_{\operatorname{gr}(\mathscr{A}_U)} \operatorname{gr}(M_U) \to 0$$

is exact. The sequence

(1.10.4) 
$$0 \to \mathscr{Q}_{x}^{\wedge} \otimes_{\mathscr{Q}_{U}} K \to \mathscr{Q}_{x}^{\wedge} \otimes_{\mathscr{Q}_{U}} \mathscr{Q}_{U} \to 0$$

is also exact by the flatness of r in (1.10.2) and the hypothesis on M. We endow these two modules with the tensor filtrations (cf. [18, p.57]). Consider the following diagram:

$$\begin{array}{cccc} 0 & \longrightarrow \operatorname{gr}(\mathscr{A}_{x}^{\wedge}) \otimes \operatorname{gr}(K) & \stackrel{\beta}{\longrightarrow} \operatorname{gr}(\mathscr{A}_{x}^{\wedge}) & \longrightarrow \operatorname{gr}(\mathscr{A}_{x}^{\wedge}) \otimes \operatorname{gr}(M_{U}) & \longrightarrow 0 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & \\$$

where the vertical homomorphisms are canonical ones. The right vertical homomorphism is an isomorphism by [18, Ch. I, 6.15]. Since the upper row is exact and

$$\operatorname{gr}(\mathscr{C}_x^{\wedge}) \otimes \operatorname{gr}(K) \to \operatorname{gr}(\mathscr{C}_x^{\wedge} \otimes K)$$

is surjective by [18, p.58],  $\alpha$  is injective. This implies that the homomorphism (1.10.4) is strict by [18, Ch. I, 4.2.4 (2)], and  $\alpha$  is an isomorphism. Thus, by diagram chasing,  $\beta$  is also an isomorphism, and

$$\operatorname{gr}(\mathscr{A}_x) \otimes \operatorname{gr}(M_U) \cong \operatorname{gr}(\mathscr{A}_x^{\wedge}) \otimes \operatorname{gr}(M_U) = 0.$$

Since  $\operatorname{gr}(\mathscr{A}_x) \cong (\operatorname{gr}(\mathscr{A}))_x$ , this shows that there exists  $V \in \mathfrak{B}$  such that  $x \in V$ ,  $V \subset U$ , and

$$\Gamma(V, \operatorname{gr}(\mathcal{C})) \otimes_{\Gamma(U, \operatorname{gr}(\mathcal{C}))} \operatorname{gr}(M_U) = 0.$$

Let  $M_V := \Gamma(V, \mathscr{C}) \otimes_{\Gamma(U, \mathscr{C})} M_U$ , and equip it with the tensor filtration. Since the filtration on  $M_U$  is good, the filtration on  $M_V$  is also good by [18, Ch. I, 6.14]. Since the canonical homomorphism

$$\Gamma(V, \operatorname{gr}(\mathcal{A})) \otimes_{\Gamma(U, \operatorname{gr}(\mathcal{A}))} \operatorname{gr}(M_U) \cong \operatorname{gr}(\Gamma(V, \mathcal{A})) \otimes_{\operatorname{gr}(\Gamma(U, \mathcal{A}))} \operatorname{gr}(M_U) \to \operatorname{gr}(M_V)$$

is surjective, here the first isomorphism comes form condition 4, we have  $\operatorname{gr}(M_V) = 0$ . Since  $\Gamma(V, \mathscr{A})$  is complete and the filtration on  $M_V$  is good, we obtain that  $M_V = 0$  (cf. 1.1.6). Since  $M \cong \mathscr{A}_x \otimes_{\mathscr{A}_V} M_V = 0$ , the fully faithfulness follows.

Since  $\operatorname{gr}(\mathscr{A}_x)$  is Noetherian,  $\mathscr{A}_x$  is Zariskian by [18, Ch. II, 2.1.2 (4)], and in particular  $\mathscr{A}_x$  is Noetherian. To show that  $\mathscr{A}$  is Noetherian, it remains to prove that  $\mathscr{A}$  is coherent. For this, it suffices to check the conditions of [5, 3.1.1]. We have already checked (a). The flatness of the restriction  $\Gamma(U, \mathscr{A}) \to \Gamma(V, \mathscr{A})$  for open subsets  $V \subset U$  in  $\mathfrak{B}$  follows by [18, Ch. II, 1.2.1], thus (b) is satisfied, and the lemma follows.

**1.11. Lemma.** — We use the notation of 1.4 and 1.5, and we further assume that gr(A) is a Noetherian ring.

(i) The rings  $\Theta_V(0)$  and  $\Theta_V(*)$  are Noetherian. Moreover,  $\Theta_V(n)$  is a coherent  $\Theta_V(0)$ -module on  $\mathring{V}$  for any integer n.

(ii) The microlocalization  $\mathcal{B}$  is Noetherian and pointwise Zariskian on V. Moreover,  $\varphi$  is flat.

*Proof.* — Let us check (i). By [16, II, 2.1.5], the ring  $A_0$  is also Noetherian, and gr(A) is of finite type over  $A_0$ . Then by the same argument as [22, A.3.1.8],  $\mathcal{O}_V(0)$  and  $\mathcal{O}_V(*)$  are Noetherian. Now, since  $\mathcal{O}_P(n)$  is a coherent  $\mathcal{O}_P \cong \mathcal{O}_P(0)$ -module and  $\mathcal{O}_X(*)$  is of finite type over  $\mathcal{O}_X(0)$ , the second claim follows.

Let us check (ii). Applying Lemma 1.10 to the microlocalization  $\mathcal{B}$  on  $\mathring{V}$ ,  $\mathcal{B}$  is a Noetherian ring and pointwise Zariskian on  $\mathring{V}$  by (1.5.2). Let us show  $\mathcal{B}$  is in fact Noetherian and pointwise Zariskian on V. It is pointwise Zariskian on the zero section by (1.4.1) (or more directly by [22, A.3.1.5]). To check condition (i)-2 of Definition 1.9, we apply (1.4.2), and for (i)-3, we apply (1.4.1). It remains to show that  $\mathscr{B}$  is coherent, which follows directly from [5, 3.1.1]. The flatness follows by [22, A.3.1.7].

**1.12. Lemma.** — Let  $(A, \{A_i\}_{i \in \mathbb{Z}})$  be a filtered ring such that  $A_0$  is Noetherian filtered,  $\bigoplus_{i>0} \operatorname{gr}_i(A)$  is Noetherian. Then A is Noetherian filtered.

*Proof.* — By [16, II, 2.1.5, 2.1.6],  $\operatorname{gr}_i(A)$  is finitely generated over  $A_0$  for any  $i \in \mathbb{Z}$ . Then the statement is nothing but [19, Proposition 1.1.5] applying in the case where the topological space is just a point.

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**1.13. Lemma.** — Let  $(\mathcal{A}, \mathcal{A}_i)$  be a pointwise Zariskian filtered ring on a topological space X. Let  $(\mathcal{M}, \mathcal{M}_i)$  be a good filtered  $(\mathcal{A}, \mathcal{A}_i)$ -module. Then the filtration  $\{\mathcal{M}_i\}$  is separated (i.e.,  $\lim_{i \to i} \mathcal{M}_i = 0$ ).

*Proof.* — Since  $\varprojlim_i \mathcal{M}_i \hookrightarrow \mathcal{M}$ , we get the following commutative diagram for any  $x \in X$ .



Since  $\mathscr{A}$  is pointwise Zariskian,  $\lim_{i \to j} \mathscr{M}_{i,x} = 0$ , and thus,  $\lim_{i \to j} \mathscr{M} = 0$ .

#### 2. Microdifferential sheaves

We apply the results of the previous section to the theory of arithmetic  $\mathscr{D}$ -modules, and define the ring of naive microdifferential operators of finite level.

**2.1.** — Let *S* be a scheme over  $\mathbb{Z}_p$  (which may not be locally of finite type). Let *X* be a smooth scheme over *S*, and let *m* be a non-negative integer. Then we may consider the sheaf of *S*-linear differential operators of level *m* denoted by  $\mathscr{D}_{X/S}^{(m)}$  on *X*. We often abbreviate this as  $\mathscr{D}_X^{(m)}$ . For the details of this sheaf, we can refer to [5, 6, 7]. For  $i \in \mathbb{Z}$ , let  $\mathscr{D}_{X,i}^{(m)}$  be the sub- $\mathscr{O}_X$ -module consisting of operators whose orders are less than or equal to *i* in  $\mathscr{D}_X^{(m)}$  (cf. [5, 2.2.1]). By definition,  $\mathscr{D}_{X,i}^{(m)} = 0$  for i < 0. Then  $\{\mathscr{D}_{X,i}^{(m)}\}_{i\in\mathbb{Z}}$  is an increasing filtration of  $\mathscr{D}_X^{(m)}$ , which we call the *filtration by order*. By [5, 2.2.4], the ring  $\operatorname{gr}(\mathscr{D}_X^{(m)})$ is commutative. Let <sup>(1)</sup>

$$T^{(m)*}X := \operatorname{Spec}(\operatorname{gr}(\mathscr{D}_X^{(m)})), \qquad P^{(m)*}X := \operatorname{Proj}(\operatorname{gr}(\mathscr{D}_X^{(m)}))$$

We call these the pseudo cotangent bundles of level m. When we need to emphasize the base, we denote  $T^{(m)*}X$  by  $T^{(m)*}(X/S)$ . When m = 0, we denote  $T^{(m)*}X$  and  $P^{(m)*}X$  by  $T^*X$  and  $P^*X$  respectively, which are nothing but the usual cotangent bundles of X. Let  $\mathring{T}^{(m)*}X := T^{(m)*}X \setminus s(X)$  where

<sup>&</sup>lt;sup>(1)</sup> We warn the reader that the notation  $T^{(m)*}X$  is used in [5] for the associated *reduced* scheme  $\left(\operatorname{Spec}(\operatorname{gr}(\mathscr{D}_X^{(m)}))\right)_{\operatorname{red}}$ .

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 $s: X \to T^{(m)*}X$  denotes the zero section. Then there exist the canonical morphisms (cf. 1.4) as follows:



Recall the notation  $\mathcal{O}_{T^{(m)*}X}(n)$  for  $n \in \mathbb{Z}$  of 1.4 which is a subsheaf of  $\mathcal{O}_{T^{(m)*}X}(*)$  consisting of homogeneous elements of degree n. There is a canonical isomorphism  $q^{-1}\mathcal{O}_{P^{(m)*}X}(n) \cong \mathcal{O}_{T^{(m)*}X}(n)$  on  $\mathring{T}^*X$  for any integer n (cf. 1.4). We remind that  $\mathcal{O}_{T^{(m)*}X}$  does not coincide with  $\mathcal{O}_{T^{(m)*}X}(*)$ . The following lemma is immediate from Lemma 1.11.

**Lemma.** — The rings  $\mathcal{O}_{T^{(m)*}X}(0)$ ,  $\mathcal{O}_{T^{(m)*}X}(*)$  are Noetherian, and  $\mathcal{O}_{\mathring{T}^{(m)*}X}(n)$  is a coherent  $\mathcal{O}_{\mathring{T}^{(m)*}X}(0)$ -module for any integer n. Moreover,  $\mathcal{O}_{\mathring{T}^{(m)*}X}(*)$  is an  $\mathcal{O}_{\mathring{T}^{(m)*}X}(0)$ -algebra of finite type.

**2.2.** — We can consider the microlocalization of  $(\mathscr{D}_{X/S}^{(m)}, \mathscr{D}_{X/S,i}^{(m)})$  denoted by  $(\mathscr{E}_{X/S}^{(m)}, \mathscr{E}_{X/S,i}^{(m)})$  using the technique of 1.4. We often abbreviate this as  $(\mathscr{E}_{X}^{(m)}, \mathscr{E}_{X,i}^{(m)})$ . This is a filtered ring on  $T^{(m)*}X$ . Then there exists a canonical homomorphism of filtered rings

$$\varphi_m \colon \pi_m^{-1}(\mathscr{D}_X^{(m)}, \mathscr{D}_{X,i}^{(m)}) \to (\mathscr{E}_X^{(m)}, \mathscr{E}_{X,i}^{(m)}).$$

By (1.5.2), we have canonical isomorphisms

(2.2.1)  $\operatorname{gr}_{n}(\mathscr{E}_{X}^{(m)}) \cong \mathcal{O}_{T^{(m)}*X}(n), \quad \operatorname{gr}(\mathscr{E}_{X}^{(m)}) \cong \mathcal{O}_{T^{(m)}*X}(*).$ 

Since  $\operatorname{gr}(\mathscr{D}_X^{(m)})$  is a Noetherian ring by the proof of [5, 2.2.5],  $\mathscr{E}_X^{(m)}$  is pointwise Zariskian and Noetherian, and moreover  $\varphi_m$  is flat by Lemma 1.11. Since the canonical homomorphism  $\pi_m^{-1}\pi_{m*}\mathcal{O}_{T^{(m)*}X}(*) \to \mathcal{O}_{T^{(m)*}X}(*)$  is injective,  $\operatorname{gr}(\varphi_m)$  is injective as well, and thus  $\varphi_m$  is strictly injective by [18, Ch. I, 4.2.4 (2)].

**Remark.** — The  $\pi^{-1} \mathcal{O}_X$ -modules  $\mathscr{E}_X^{(m)}$  and  $\mathscr{E}_{X,i}^{(m)}$  do not possess  $\mathcal{O}_{T^{(m)*}X}$ -module structure (cf. Remark 1.5).

**2.3. Lemma.** — We assume that S and X are affine, and S = Spec(A). Let S' := Spec(B) be an affine scheme finite over S. We put  $X' := X \times_S S'$ , and we have the base change isomorphism  $T^{(m)*}(X'/S') \cong T^{(m)*}(X/S) \times_S S'$ (cf. [5, 2.2.2]). Let f be a homogeneous section of  $\Gamma(T^{(m)*}X, \Theta_{T^{(m)*}X})$ , and f'

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be the image in  $\Gamma(T^{(m)*}X', \mathcal{O}_{T^{(m)*}X'})$ . We put U := D(f) and U' := D(f'). Then there exists a canonical isomorphism of filtered rings

$$\Gamma\left(U, (\mathscr{E}_{X/S}^{(m)}, \mathscr{E}_{X/S,i}^{(m)})\right) \otimes_A B \xrightarrow{\sim} \Gamma\left(U', (\mathscr{E}_{X'/S'}^{(m)}, \mathscr{E}_{X'/S',i}^{(m)})\right).$$

*Proof.* — We may assume that  $\deg(f) > 0$ . By [5, 2.2.2], there exists an isomorphism

(2.3.1) 
$$\Gamma\left(X, (\mathscr{D}_{X/S}^{(m)}, \mathscr{D}_{X/S,i}^{(m)})\right) \otimes_A B \xrightarrow{\sim} \Gamma\left(X', (\mathscr{D}_{X'/S'}^{(m)}, \mathscr{D}_{X'/S',i}^{(m)})\right).$$

We denote  $\Gamma(U, (\mathscr{E}_X^{(m)}, \mathscr{E}_{X,i}^{(m)}))$  by  $(E_X, E_{X,i})$ , and  $\Gamma(X', (\mathscr{D}_{X'}^{(m)}, \mathscr{D}_{X',i}^{(m)}))$ by  $(D_{X'}, D_{X',i})$ . The isomorphism (2.3.1) induces a homomorphism of filtered rings  $(D_{X'}, D_{X',i}) \to (E_X, E_{X,i}) \otimes_A B$ . Since *B* is finite over *A*,  $(E_X, E_{X,i}) \otimes_A B$  is a complete filtered ring by [18, Ch. II, 1.2.10 (5)]. By the universality [22, Proposition A.2.3.3],  $(E_X, E_{X,i}) \otimes_A B$  is the microlocalization of  $(D_{X'}, D_{X',i})$ , and the lemma follows.

**Remark.** — Consider the general situation of 2.1, and let  $S' \to S$  be a finite morphism. Put  $X' := X \times_S S'$ . We have the base change isomorphism  $f': T^{(m)*}(X'/S') \xrightarrow{\sim} T^{(m)*}(X/S) \times_S S'$ , and using the lemma, we have the following isomorphism:

$$f'^{-1}\big((\mathscr{E}_{X/S}^{(m)}, \mathscr{E}_{X/S,i}^{(m)}) \otimes_S \mathcal{O}_{S'}\big) \xrightarrow{\sim} (\mathscr{E}_{X'/S'}^{(m)}, \mathscr{E}_{X'/S',i}^{(m)}).$$

Note, however, that since  $\mathscr{E}_X^{(m)}$  and  $\mathscr{E}_{X,i}^{(m)}$  are not quasi-coherent, a priori, the tensor product does not commute with global section functor over affine schemes. In this sense, the assertion of the lemma is slightly stronger than this global version.

**2.4.** — Now, we pass to the limit. Let R be a complete discrete valuation ring of mixed characteristic (0, p) whose residue field is denoted by k. We denote the field of fractions by K, and let  $\pi$  be a uniformizer of R. For a non-negative integer i, we put  $R_i := R/(\pi^{i+1})$ . From now on, we use these notation freely without referring to this subsection.

Let  $\mathscr{X}$  be a smooth formal scheme over R. We denote by  $X_i$  the reduction of  $\mathscr{X}$  over  $R_i$ . We define  $T^{(m)*}\mathscr{X}$  and  $P^{(m)*}\mathscr{X}$  by the limit of  $T^{(m)*}X_i$  and  $P^{(m)*}X_i$  over i respectively. We also define  $\mathcal{O}_{T^{(m)*}\mathscr{X}}(*)$  (resp.  $\mathcal{O}_{T^{(m)*}\mathscr{X}}(n)$ ) to be the limit of  $\mathcal{O}_{T^{(m)*}X_i}(*)$  (resp.  $\mathcal{O}_{T^{(m)*}X_i}(n)$ ) over i, and put  $\mathcal{O}_{T^{(m)*}\mathscr{X},\mathbb{Q}}(*)$ (resp.  $\mathcal{O}_{T^{(m)*}\mathscr{X},\mathbb{Q}}(n)$ ) to be  $\mathcal{O}_{T^{(m)*}\mathscr{X}}(*) \otimes \mathbb{Q}$  (resp.  $\mathcal{O}_{T^{(m)*}\mathscr{X}}(n) \otimes \mathbb{Q}$ ). Let  $\epsilon: \mathfrak{O}(V) \to \mathfrak{O}(V')$  be the functor in 1.4 where  $V = T^{(m)*}\mathscr{X}$ .

**Definition**. — (i) Let  $\mathfrak{B}'$  be the open basis of V' consisting of D(f) in  $T^{(m)*}\mathscr{X}$  over an affine open subscheme  $\mathscr{U}$  of  $\mathscr{X}$  where f is a homogeneous element of  $\Gamma(T^{(m)*}\mathscr{U}, \mathcal{O}_{T^{(m)*}\mathscr{U}})$ . An open subset  $\mathscr{U} \subset \mathring{T}^{(m)*}\mathscr{U}$  is said to be *strictly affine* if  $\mathscr{U} \in \mathfrak{B}'$ .

(ii) We define an open basis  $\mathfrak{B}$  of V to be the set consisting of  $U \in \mathfrak{O}(V)$  such that  $\epsilon_{\cdot}(U) \in \mathfrak{B}'$ .

**Lemma.** — The rings  $\mathcal{O}_{T^{(m)*}\mathscr{X}}(0)$  and  $\mathcal{O}_{T^{(m)*}\mathscr{X}}(*)$  are Noetherian with respect to  $\mathfrak{B}$ , and  $\mathcal{O}_{\mathring{T}^{(m)*}\mathscr{X}}(n)$  is a coherent  $\mathcal{O}_{\mathring{T}^{(m)*}\mathscr{X}}(0)$ -module for any integer n. Moreover,  $\mathcal{O}_{\mathring{T}^{(m)*}\mathscr{X}}(*)$  is an  $\mathcal{O}_{\mathring{T}^{(m)*}\mathscr{X}}(0)$ -algebra of finite type.

*Proof.* — The proof is the same as Lemma 1.11, so we only sketch here. We put  $\mathscr{V} := T^{(m)*}\mathscr{X}$  and  $\mathring{\mathscr{V}} := \mathring{T}^{(m)*}\mathscr{X}$ . To check that  $\mathcal{O}_{\mathring{\mathscr{V}}}(n)$  is a coherent  $\mathcal{O}_{\mathring{\mathscr{V}}}(0)$ -module, it suffices to point out that  $\mathcal{O}_{P^{(m)*}\mathscr{X}}(n)$  is a coherent  $\mathcal{O}_{P^{(m)*}\mathscr{X}} \cong \mathcal{O}_{P^{(m)*}\mathscr{X}}(0)$ -module. The proof that  $\mathcal{O}_{\mathring{\mathscr{V}}}(*)$  is of finite type is the same. It remains to show that  $\mathcal{O}_{\mathscr{V}}(0)$  is Noetherian. The only remaining thing we need to check is the coherence of  $\mathcal{O}_{\mathscr{V}}(0)$  and  $\mathcal{O}_{\mathscr{V}}(*)$  around the zero section, and for this, apply [5, 3.1.1] as Lemma 1.11.

We define a sheaf of rings on the topological space  $T^{(m)*}\mathscr{X} \approx T^{(m)*}X_0$ ( $\approx$  denotes the canonical homeomorphism of topological spaces) by

$$\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)} := \varprojlim_{i} \mathscr{E}_{X_{i}}^{(m)}$$

For  $j \in \mathbb{Z}$ , we also define

$$\mathscr{E}_{\mathscr{X},j}^{(m)} := \varprojlim_{i} \mathscr{E}_{X_{i},j}^{(m)}.$$

We remark that the "filtration"  $\mathscr{E}_{\mathscr{X},j}^{(m)}$  of  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$  is not exhaustive. We define a submodule (which is in fact a ring by Lemma 2.5 (iii) below) by

$$\mathscr{E}_{\mathscr{X}}^{(m)} := \varinjlim_{j \to \infty} \mathscr{E}_{\mathscr{X}, j}^{(m)} \subset \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}.$$

There is a canonical homomorphism of rings on  $T^{(m)*}\mathscr{X}$ 

(2.4.1) 
$$\widehat{\varphi}_m \colon \pi_m^{-1} \widehat{\mathscr{D}}_{\mathscr{X}}^{(m)} \to \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$$

This homomorphism is injective by the injectivity of  $\varphi_m$  in 2.2. Since  $\widehat{\varphi}_m(\pi_m^{-1}\mathscr{D}_{\mathscr{X},n}^{(m)}) \subset \mathscr{E}_{\mathscr{X},n}^{(m)}, \ \widehat{\varphi}_m|_{\pi_m^{-1}\mathscr{D}_{\mathscr{X}}^{(m)}}$  induces a homomorphism of modules  $\pi_m^{-1}\mathscr{D}_{\mathscr{X}}^{(m)} \to \mathscr{E}_{\mathscr{X}}^{(m)}$ . We abusively denote this homomorphism by  $\varphi_m$ . We see from the following Lemma 2.5 (iii) that this homomorphism is in fact a homomorphism of *rings*.

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**2.5. Lemma.** — Let  $\mathscr{X}$  be a smooth formal scheme over R. Let  $\mathscr{U}$  be an open formal subscheme of  $T^{(m)*}\mathscr{X}$  belonging to  $\mathfrak{B}$ . Let i be a non-negative integer, and we denote  $\mathscr{U} \otimes R_i$  by  $U_i$ .

(i) The ring  $\Gamma(\mathscr{U},\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)})$  is  $\pi$ -adically complete and flat over R. Moreover, the canonical homomorphisms  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)} \otimes R_i \to \mathscr{E}_{X_i}^{(m)}$  and  $\mathscr{E}_{\mathscr{X}}^{(m)} \otimes R_i \to \mathscr{E}_{X_i}^{(m)}$  are isomorphisms.

(ii) Let j be an integer, and k be a positive integer. Let  $\mathscr{E}$  be one of  $\mathscr{E}_{\mathscr{X},j+k}^{(m)}$ ,  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$ . We have

 $\Gamma(\mathscr{U}, \mathscr{E}/\mathscr{E}_{\mathscr{X}, j}^{(m)}) \cong \Gamma(\mathscr{U}, \mathscr{E})/\Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X}, j}^{(m)}), \quad \mathscr{E}_{\mathscr{X}, j+k}^{(m)}/\mathscr{E}_{\mathscr{X}, j}^{(m)} \cong \varprojlim_{i} \mathscr{E}_{X_{i}, j+k}^{(m)}/\mathscr{E}_{X_{i}, j}^{(m)}.$ 

(iii) Let j and k be integers. Then  $\mathscr{E}_{\mathscr{X},j}^{(m)} \cdot \mathscr{E}_{\mathscr{X},k}^{(m)} \subset \mathscr{E}_{\mathscr{X},j+k}^{(m)}$  in  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$ , and in particular,  $(\mathscr{E}_{\mathscr{X}}^{(m)}, \{\mathscr{E}_{\mathscr{X},j}^{(m)}\}_{j\in\mathbb{Z}})$  is a filtered ring. Moreover, the  $\pi$ -adic completion of  $\mathscr{E}_{\mathscr{X}}^{(m)}$  is isomorphic to  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$ .

(iv) The filtered rings  $\mathscr{E}_{X_i}^{(m)}$  and  $\mathscr{E}_{\mathscr{X}}^{(m)}$  are complete with respect to the filtration by order.

*Proof.* — For a projective system  $\{\mathcal{F}_i\}_{i\geq 0}$  on a topological space T and for an open subset U of T,

(2.5.1) 
$$\Gamma(U, \varprojlim_i \mathcal{F}_i) \xrightarrow{\sim} \varprojlim_i \Gamma(U, \mathcal{F}_i)$$

by [16, 0<sub>I</sub>, 3.2.6]. For an inductive system  $\{\mathcal{F}_i\}_{i\geq 0}$  on a Noetherian topological space T and for an open subset U of T,

(2.5.2) 
$$\varinjlim_{i} \Gamma(U, \mathcal{F}_{i}) \xrightarrow{\sim} \Gamma(U, \varinjlim_{i} \mathcal{F}_{i})$$

by [15, Ch. II, 3.10]. Since  $\mathscr{U}$  is an open subset of an affine formal scheme  $\epsilon_{\cdot}(\mathscr{U}), \mathscr{U}$  is a Noetherian space.

By (2.5.1) and the definition of  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$  in 2.4,  $\Gamma(\mathscr{U}, \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}) \cong \varprojlim_{i} \Gamma(U_{i}, \mathscr{E}_{X_{i}}^{(m)})$ . Since  $\Gamma(U_{i}, \mathscr{E}_{X_{i}}^{(m)})$  is flat over  $\Gamma(U_{i}, \mathscr{D}_{X_{i}}^{(m)})$  (cf. 2.2), the ring  $\Gamma(U_{i}, \mathscr{E}_{X_{i}}^{(m)})$  is flat over  $R_{i}$ . Thus we get first two claims of (i) by Lemma 2.3 and the following Lemma 2.6. For  $\mathscr{E}_{\mathscr{X}}^{(m)} \otimes R_{i} \xrightarrow{\sim} \mathscr{E}_{X_{i}}^{(m)}$ , we show  $\mathscr{E}_{\mathscr{X},j}^{(m)} \otimes R_{i} \xrightarrow{\sim} \mathscr{E}_{X_{i},j}^{(m)}$  for any  $j \in \mathbb{Z}$  by the same argument, and take the inductive limit over j.

Let us prove (ii) for the  $\mathscr{E} = \mathscr{E}_{\mathscr{X},j+k}^{(m)}$  case. By (1.5.1),

$$\Gamma(U_i, \mathscr{E}_{X_i, j+k}^{(m)} / \mathscr{E}_{X_i, j}^{(m)}) \cong \Gamma(U_i, \mathscr{E}_{X_i, j+k}^{(m)}) / \Gamma(U_i, \mathscr{E}_{X_i, j}^{(m)}).$$

Since the projective system  $\{\Gamma(U_i, \mathscr{E}_{X_i,j}^{(m)})\}_{i\geq 0}$  satisfies the Mittag-Leffler condition by (i), the sequence

$$0 \to \varprojlim_i \Gamma(U_i, \mathscr{E}_{X_i, j}^{(m)}) \to \varprojlim_i \Gamma(U_i, \mathscr{E}_{X_i, j+k}^{(m)}) \to \varprojlim_i \Gamma(U_i, \mathscr{E}_{X_i, j+k}^{(m)} / \mathscr{E}_{X_i, j}^{(m)}) \to 0$$

is exact. Considering (2.5.1), this shows that

(2.5.3) 
$$\Gamma(\mathscr{U}, \varprojlim_{i} \mathscr{E}_{X_{i}, j+k}^{(m)} / \mathscr{E}_{X_{i}, j}^{(m)}) \cong \Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X}, j+k}^{(m)}) / \Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X}, j}^{(m)})$$

Thus, since  $\mathfrak{B}$  is a basis of the topology, the canonical homomorphism  $\mathscr{E}_{\mathscr{X},j+k}^{(m)}/\mathscr{E}_{\mathscr{X},j}^{(m)} \to \varprojlim_i \mathscr{E}_{X_i,j+k}^{(m)}/\mathscr{E}_{X_i,j}^{(m)}$  is an isomorphism, and the second equality of (ii) follows. The first equality of (ii) follows by using (2.5.3) once again. To deal with the  $\mathscr{E} = \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$  case, just replace  $\mathscr{E}_{X_i,j+k}^{(m)}$  (resp.  $\mathscr{E}_{\mathscr{X},j+k}^{(m)}$ ) by  $\mathscr{E}_{X_i}^{(m)}$  (resp.  $\widehat{\mathscr{E}}_{\mathscr{X},j+k}^{(m)}$ ) in the argument above. For the  $\mathscr{E} = \mathscr{E}_{\mathscr{X}}^{(m)}$  case, use (2.5.2) and the  $\mathscr{E} = \mathscr{E}_{\mathscr{X},j+k}^{(m)}$  case.

The first claim of (iii) follows since  $\mathscr{E}_{X_i}^{(m)}$  is a filtered ring. By (i),  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$  is the  $\pi$ -adic completion of  $\mathscr{E}_{\mathscr{X}}^{(m)}$ .

Let us prove (iv). The completeness of  $\mathscr{E}_{X_i}^{(m)}$  follows by definition. Let us see the completeness of  $\mathscr{E}_{\mathscr{X}}^{(m)}$ . For an open affine subscheme  $\mathscr{U}$  in  $\mathfrak{B}$ , consider the following exact sequence

$$0 \to \Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X},k}^{(m)} / \mathscr{E}_{\mathscr{X},j}^{(m)}) \to \Gamma(\mathscr{U}, \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)} / \mathscr{E}_{\mathscr{X},j}^{(m)}) \to \Gamma(\mathscr{U}, \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)} / \mathscr{E}_{\mathscr{X},k}^{(m)}) \to 0$$

for integers  $k \geq j$ . The last surjection is deduced by using (ii). Since the projective system  $\{\Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X}}^{(m)} / \mathscr{E}_{\mathscr{X}, j}^{(m)})\}_{j}$  satisfies the Mittag-Leffler condition by (ii), the following sequence is exact:

where  $j \to -\infty$  and  $k \to \infty$ . The middle vertical homomorphism is an isomorphism as well by the commutativity (2.5.1) and the fact that two projective limits commute. Thus the lemma is proven.

**2.6. Lemma.** — Let  $\{E_i\}_{i\geq 0}$  be a projective system of *R*-modules such that for each *i*,  $E_i$  is a flat  $R_i$ -module. Assume that the homomorphism  $E_{i+1} \otimes R_i \to E_i$  induced by the transition homomorphism is an isomorphism for any non-negative integer *i*. Let  $E := \varprojlim_i E_i$ . Then the canonical homomorphism

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 $E \otimes R_i \to E_i$  is an isomorphism for any non-negative integer j. Moreover, E is  $\pi$ -adically complete and flat over R.

*Proof.* — We leave the proof to the reader.

**2.7. Lemma.** — Let  $\mathring{\mathscr{V}} := \mathring{T}^{(m)*} \mathscr{X}$ . Let  $\mathscr{A} = \bigoplus_{i \in \mathbb{Z}} \mathscr{A}_i$  be a graded  $\mathscr{O}_{\mathring{\mathscr{V}}}(0)$ -algebra of finite type on  $\mathring{\mathscr{V}}$  such that  $\mathscr{A}_i$  is a coherent  $\mathscr{O}_{\mathring{\mathscr{V}}}(0)$ -module for any  $i \in \mathbb{Z}$ . Then for any  $V \subset U$  in  $\mathfrak{B}$ , the restriction homomorphism  $\Gamma(U, \mathscr{A}) \to \Gamma(V, \mathscr{A})$  is flat, and  $\mathscr{A}$  is Noetherian with respect to  $\mathfrak{B}$ .

*Proof.* — We put  $\theta := \theta_{\hat{\gamma}}(0)$ . Let us check the conditions of Definition 1.9 (i). Condition 2 follows since  $\mathscr{A}$  is of finite type over  $\theta$ . Let U be an open subset of  $\hat{\mathscr{V}}$  in  $\mathfrak{B}$  such that there exists a surjection  $\phi : \theta[T_1, \ldots, T_n]|_U \to \mathscr{A}|_U$ . We claim that the homomorphism

$$\Gamma(U, \mathcal{O}[T_1, \ldots, T_n]) \to \Gamma(U, \mathcal{C})$$

is surjective. Indeed, since  $\mathscr{A}_i$  is a coherent  $\mathscr{O}$ -module for any i,  $\operatorname{Ker}(\phi)$  is an inductive limit of coherent  $\mathscr{O}|_U$ -modules. Since U is Noetherian and separated,  $H^1(U, -)$  commutes with inductive limit by [15, Ch. II, 4.12.1], and we have  $H^1(U, \operatorname{Ker}(\phi)) = 0$ , which implies the claim. Thus condition 3 is fulfilled. It remains to show that  $\mathscr{A}$  is a coherent ring. For this, it suffices to check the conditions of [5, 3.1.1]. For  $V \subset U$  in  $\mathfrak{B}$ , we have the restriction isomorphism  $\Gamma(V, \mathscr{O}) \otimes_{\Gamma(U, \mathscr{O})} \Gamma(U, \mathscr{A}_i) \xrightarrow{\sim} \Gamma(V, \mathscr{A}_i)$  for any i since  $\mathscr{A}_i$  is a coherent  $\mathscr{O}$ -module using (1.4.1). This induces an isomorphism

$$\Gamma(V, \mathcal{O}) \otimes_{\Gamma(U, \mathcal{O})} \Gamma(U, \mathcal{C}) \xrightarrow{\sim} \Gamma(V, \mathcal{C}).$$

Since the restriction homomorphism  $\Gamma(U, \emptyset) \to \Gamma(V, \emptyset)$  is flat, this isomorphism shows that  $\Gamma(U, \emptyset) \to \Gamma(V, \emptyset)$  is flat as well. Thus the claim follows.  $\Box$ 

**2.8. Proposition**. — Let  $\mathscr{X}$  be a smooth formal scheme over R.

- (i) The rings  $\mathscr{E}_{\mathscr{X}}^{(m)}$ ,  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$ ,  $\mathscr{E}_{\mathscr{X},0}^{(m)}$  are Noetherian with respect to  $\mathfrak{B}$ .
- (ii) The homomorphism  $\widehat{\varphi}_m$  of (2.4.1) is flat.

(iii) Let  $\mathscr{E}$  be either  $\mathscr{E}_{\mathscr{X},0}^{(m)}$  or  $\mathscr{E}_{\mathscr{X}}^{(m)}$  or  $\mathscr{E}_{\mathscr{X}}^{(m)}$ . For any open subsets  $\mathscr{U} \supset \mathscr{V}$  in  $\mathfrak{B}$ , the restriction homomorphism  $\Gamma(\mathscr{U},\mathscr{E}) \to \Gamma(\mathscr{V},\mathscr{E})$  is flat.

Proof. — Let us prove (i). First, we show the claim for  $\mathscr{E}_{\mathscr{X}}^{(m)}$  and  $\mathscr{E}_{\mathscr{X},0}^{(m)}$ . Let us check the conditions of Lemma 1.10 for  $\mathscr{E}_{\mathscr{X}}^{(m)}$  (resp.  $\mathscr{E}_{\mathscr{X},0}^{(m)}$ ) on  $\mathscr{V} := \mathring{T}^{(m)*}\mathscr{X}$ . Conditions 1 and 4 hold by Lemma 2.5. By Lemma 2.5 (ii),  $\operatorname{gr}(\mathscr{E}_{\mathscr{X}}^{(m)}) \cong \mathcal{O}_{\mathscr{V}}(*)$  as graded rings. By Lemma 2.4, this implies that  $\operatorname{gr}_i(\mathscr{E}_{\mathscr{X}}^{(m)})$  is a coherent  $\mathcal{O}_{\mathscr{V}}(0)$ -module on  $\mathscr{V}$  for any  $i \in \mathbb{Z}$ , and  $\operatorname{gr}(\mathscr{E}_{\mathscr{X}}^{(m)})$  (resp.  $\operatorname{gr}(\mathscr{E}_{\mathscr{X},0}^{(m)})$ ) is an  $\mathcal{O}_{\mathscr{V}}(0)$ -module of finite type on  $\mathscr{V}$ . Thus by Lemma 2.7, conditions 2 and 3

are fulfilled. This implies that  $\mathscr{E}_{\mathscr{X}}^{(m)}$  and  $\mathscr{E}_{\mathscr{X},0}^{(m)}$  are Noetherian with respect to  $\mathfrak{B}$  on  $\mathring{\mathscr{V}}$ . Using [18, Ch. II, 1.2.1],  $\widehat{\varphi}_m$  is flat, and (ii) follows. It remains to check that the rings are Noetherian around the zero section. By using (1.4.1), we only need to prove the coherence. This follows from [5, 3.1.1].

For  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$ , let us endow with the  $\pi$ -adic filtration  $\{\pi^{-i}\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}\}_{i\leq 0}$  (cf. 1.1.2). Since  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$  is  $\pi$ -torsion free by Lemma 2.5 (i), the homomorphism  $\mathscr{E}_{X_0}^{(m)}[T] \to \mathbb{C}_{X_0}^{(m)}$  $\operatorname{gr}(\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)})$  sending T to  $\pi \in \operatorname{gr}_1(\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)})$  is an isomorphism. It is straightforward to check the conditions of Lemma 1.10. We remind that the  $\pi$ -adic filtration can also be used when we apply Lemma 1.10 to show that  $\mathscr{E}_{\mathscr{X},0}^{(m)}$  is Noetherian. 

To prove (iii), it suffices to apply (i) and [18, Ch. II, 1.2.1].

**Remark**. — By the proof, we can moreover say that  $\mathscr{E}^{(m)}_{\mathscr{X}}$  and  $\mathscr{E}^{(m)}_{\mathscr{X},0}$  are pointwise Zariskian with respect to the filtration by order on  $\mathring{T}^{(m)*}\mathscr{X}$ , and  $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X}}$  and  $\mathscr{E}^{(m)}_{\mathscr{X},0}$  are pointwise Zariskian with respect to the  $\pi$ -adic filtration on  $\mathring{T}^{(m)*}\mathscr{X}$ .

2.9. - Now, we define

$$\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)} := \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)} \otimes \mathbb{Q}, \qquad \mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m)} := \mathscr{E}_{\mathscr{X}}^{(m)} \otimes \mathbb{Q}$$

Note that  $\otimes \mathbb{Q}$  commutes with global section functor over Noetherian space by [5, 3.4]. The homomorphism  $\hat{\varphi}_m$  of (2.4.1) induces a canonical injective homomorphism

$$\widehat{\varphi}_m \otimes \mathbb{Q} \colon \pi_m^{-1} \widehat{\mathscr{D}}_{\mathscr{X}, \mathbb{Q}}^{(m)} \to \widehat{\mathscr{E}}_{\mathscr{X}, \mathbb{Q}}^{(m)}.$$

If there is no risk of confusion, we sometimes denote  $\widehat{\varphi}_m \otimes \mathbb{Q}$  abusively by  $\widehat{\varphi}_m$ . We call the sheaves  $\mathscr{E}_{X_i}^{(m)}$ ,  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$ ,  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)}$  the rings of naive microdifferential operators of level m. Proposition 2.8 implies the following.

**Corollary.** — The rings  $\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m)}$  and  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)}$  are Noetherian with respect to  $\mathfrak{B}$ . Moreover,  $\widehat{\varphi}_m \otimes \mathbb{Q}$  and the restriction homomorphism  $\Gamma(\mathscr{U},\mathscr{E}) \to \Gamma(\mathscr{V},\mathscr{E})$  are flat for  $\mathscr{U} \supset \mathscr{V}$  in  $\mathfrak{B}$ , where  $\mathscr{E}$  is either  $\mathscr{E}_{\mathscr{X},\mathbb{O}}^{(m)}$  or  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{O}}^{(m)}$ .

2.10. — Let us describe sections of rings of microdifferential operators explicitly. We use the notation of 2.4. Suppose in addition that  $\mathscr X$  is affine, and possesses a system of local coordinates  $\{x_1, \ldots, x_d\}$ . Let  $\{\partial_1, \ldots, \partial_d\}$  be the corresponding differential operators, and  $\{\xi_1, \ldots, \xi_d\}$  be the corresponding basis of  $\Gamma(T^*\mathscr{X}, \mathcal{O}_{T^*\mathscr{X}})$ . Let k be a positive integer. We have a differential operator  $\partial_i^{\langle k \rangle_{(m)}}$  for any  $1 \leq i \leq d$  in  $\mathscr{D}_{X_l}^{(m)}$  for any integer  $l \geq 0$  or in  $\widehat{\mathscr{D}}_{\mathscr{X}}^{(m)}$  (cf. [5, 2.2.3]). Write  $k = p^m q + r$  with  $0 \leq r < p^m$ . Recall that there is a relation (cf. [5, (2.2.3.1)])

$$k! \,\partial_i^{\langle k \rangle_{(m)}} = q! \,\partial_i^k.$$

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Now, these operators define elements in  $\operatorname{gr}(\mathscr{D}_{X_{l}}^{(m)})$  by taking the principal symbol (cf. 1.1.4). We denote  $\sigma(\partial_{i}^{\langle k \rangle_{(m)}})$  by  $\xi_{i}^{\langle k \rangle_{(m)}}$  in  $\operatorname{gr}_{k}(\mathscr{D}_{X_{l}}^{(m)}) \subset \operatorname{gr}(\mathscr{D}_{X_{l}}^{(m)})$ . From now on, we use the multi-index notation. For example, for  $\underline{k} = (k_{1}, \ldots, k_{d}) \in \mathbb{N}^{d}$ , we denote by  $\underline{\xi}^{\langle \underline{k} \rangle_{(m)}} := \xi_{1}^{\langle k_{1} \rangle_{(m)}} \ldots \xi_{d}^{\langle k_{d} \rangle_{(m)}}, \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} := \partial_{1}^{\langle k_{1} \rangle_{(m)}} \ldots \partial_{d}^{\langle k_{d} \rangle_{(m)}}$ , and  $|\underline{k}| := k_{1} + \cdots + k_{d}$ . We denote by  $\underline{k} \geq \underline{k'}$  if  $k_{i} \geq k'_{i}$  for any  $1 \leq i \leq d$ . For  $n \in \mathbb{Z}$ , we sometimes denote  $(n, \ldots, n)$  by  $\underline{n}$  if there is no possible confusion.

Let

$$\Theta \in \Gamma(T^*\mathscr{X}, \mathscr{O}_{T^*\mathscr{X}})$$

be a non-zero homogeneous section of degree n. We consider  $\pi_m^{-1} \mathcal{O}_{\mathscr{X}}$  as a subring of  $\mathcal{O}_{T^{(m)*}\mathscr{X}}$ . This section induces a section  $\Theta^{(m)} \in \Gamma(T^{(m)*}\mathscr{X}, \mathcal{O}_{T^{(m)*}\mathscr{X}})$  for any integer  $m \geq 0$  as follows: we may write

(2.10.1) 
$$\Theta = \sum_{|\underline{k}|=n} a_{\underline{k}} \underline{\xi}^{\underline{k}}$$

where  $\underline{k} \in \mathbb{N}^d$  and  $a_k \in \Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$  in a unique way. We put

$$\Theta^{(m)} := \sum_{|\underline{k}|=n} a_{\underline{k}}^{p^m} (\underline{\xi}^{\langle p^m \rangle_{(m)}})^{\underline{k}}.$$

The homogeneous element  $\Theta$  induces also elements in  $\mathscr{D}_{\mathscr{X}}^{(m)}$  or  $\mathscr{D}_{X_{i}}^{(m)}$ . We put

$$\widetilde{\Theta}_{\mathrm{le}}^{(m)} := \sum_{|\underline{k}|=n} a_{\underline{k}}^{p^m} \big( \underline{\partial}^{\langle p^m \rangle_{(m)}} \big)^{\underline{k}}$$

where the subscript "le" stands for "left". Since  $\mathscr{D}_{\mathscr{X},\mathbb{Q}}^{(m)} \cong \mathscr{D}_{\mathscr{X},\mathbb{Q}}^{(m')}$  for any nonnegative integer m', we sometimes consider these operators as sections of  $\mathscr{D}_{\mathscr{X},\mathbb{Q}}^{(m')}$ .

Let  $\mathscr{U}$  be the open affine subset of  $T^{(m)*}\mathscr{X}$  defined by  $\Theta^{(m)}$ . We claim that the operator  $\widetilde{\Theta}_{le}^{(m)}$  is invertible is  $\Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X}}^{(m)})$ . Indeed, the inverse of  $\widetilde{\Theta}_{le}^{(m)}$ in  $\mathscr{E}_{X_l}^{(m)}$  has degree  $-np^m$  for any l. Since the inverse of  $\widetilde{\Theta}_{le}^{(m)}$  is unique in  $\mathscr{E}_{X_l}^{(m)}$ , these elements induce an element of  $\varprojlim_l \mathscr{E}_{X_l,-np^m}^{(m)} = \mathscr{E}_{\mathscr{X},-np^m}^{(m)} \subset \mathscr{E}_{\mathscr{X}}^{(m)}$ .

Let  $\{b_{\underline{k},i}\}$  be a sequence in  $\Gamma(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$  for  $\underline{k} \in \mathbb{N}^d$  and  $i \in \mathbb{N}$  such that the following holds: for each integer N, let  $\beta_{N,i} := \sup_{|\underline{k}| = inp^m + N} |b_{\underline{k},i}|$ , where  $|\cdot|$  denotes the spectral norm (cf. [5, 2.4.2]) on  $\Gamma(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ . Then

(2.10.2) 
$$\lim_{i \to \infty} \beta_{N,i} = 0, \qquad \lim_{N \to +\infty} \sup_{i} \{\beta_{N,i}\} = 0.$$

In the sequel, we consider the ring  $\pi_m^{-1}\mathscr{D}_{X_l}^{(m)}$  (resp.  $\pi_m^{-1}\widehat{\mathscr{D}}_{\mathscr{X}}^{(m)}$ ) as a subring of  $\mathscr{E}_{X_l}^{(m)}$  (resp.  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$ ) for any l by  $\varphi_m$  of 2.2 (resp.  $\widehat{\varphi}_m$  of (2.4.1)). The sum

(2.10.3) 
$$\sum_{N \in \mathbb{Z}} \sum_{|\underline{k}| - inp^m = N} b_{\underline{k},i} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \, (\widetilde{\Theta}_{le}^{(m)})^{-i}$$

converges in  $\Gamma(U_l, \mathscr{E}_{X_l}^{(m)})$  for any l. We note that the order of  $\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} (\widetilde{\Theta}_{le}^{(m)})^{-i}$ is  $N, \sum_{|\underline{k}|-inp^m = N} \dots$  is a finite sum by the first condition of (2.10.2), and  $\sum_{|\underline{k}|-inp^m = N} \dots = 0$  for  $N \gg 0$  by the second condition. Since these elements form a projective system over l, we have an element in  $\Gamma(\mathscr{U}, \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)})$ . Even though the sum of (2.10.3) does not converge in  $\Gamma(\mathscr{U}, \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)})$  with respect to the  $\pi$ -adic topology in general  $^{(2)}$ , we abusively denote by (2.10.3) the operator in  $\Gamma(\mathscr{U}, \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)})$ .

**Lemma**. — For any element  $P \in \Gamma(\mathscr{U}, \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)})$ , there exists a sequence  $\{b_{\underline{k},i}\}$ for  $\underline{k} \in \mathbb{N}^d$  and  $i \in \mathbb{N}$  satisfying (2.10.2) such that P can be written as (2.10.3). Moreover, if  $P \in \Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X},j}^{(m)})$  for some integer j, we can take  $b_{\underline{k},i} = 0$ for  $|\underline{k}| - inp^m > j$ .

This is called a left presentation of level m. We remind here that presentation is not unique.

*Proof.* — Since  $\Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X},j}^{(m)})$  and  $\Gamma(\mathscr{U}, \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)})$  are flat over R and  $\pi$ -adically complete by Lemma 2.5 (i), it suffices to show that any element of  $\Gamma(\mathscr{U}, \mathscr{E}_{X_0,j}^{(m)})$  can be written as

$$\sum_{N\in\mathbb{Z}}\sum_{|\underline{k}|-inp^m=N}c_{\underline{k},i}\,\underline{\partial}^{\langle\underline{k}\rangle_{(m)}}\,(\widetilde{\Theta}_{\mathrm{le}}^{(m)})^{-i},$$

with  $c_{\underline{k},i} \in \Gamma(X_0, \mathcal{O}_{X_0})$  such that the following holds: for each integer N,  $c_{\underline{k},i} = 0$  for almost all couples  $(\underline{k},i) \in \mathbb{N}^d \times \mathbb{N}$  such that  $|\underline{k}| - inp^m = N$ , and  $c_{\underline{k},i} = 0$  for any  $|\underline{k}| - inp^m > j$ . This follows from 1.2.

**Remark**. — Instead of using  $\widetilde{\Theta}_{le}^{(m)}$ , we can also use

$$\widetilde{\Theta}_{\mathrm{ri}}^{(m)} := \sum_{|\underline{k}|=n} (\underline{\partial}^{\langle p^m \rangle_{(m)}})^{\underline{k}} a_{\underline{k}}^{p^m},$$

to get *right presentations*. The construction is essentially the same, so we leave the details to the reader.

<sup>&</sup>lt;sup>(2)</sup> However, we are able to define a reasonable weaker topology on  $\Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X}}^{(m)})$  such that this sum converges. In the curve case, see [3, 1.2.2].

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**2.11.** — We used  $\widetilde{\Theta}_{le}^{(m)}$  to describe elements of  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$ . We may also use a variant of  $\widetilde{\Theta}_{le}^{(m+j)}$  for  $j \geq 0$  to describe them. Suppose  $\Theta$  is written as (2.10.1). Then we put

$$\Theta^{(m,m+j)} := \sum_{|\underline{k}|=n} a_{\underline{k}}^{p^{m+j}} \left(\underline{\xi}^{\langle p^m \rangle_{(m)}}\right)^{\underline{k} p^j}, \qquad \widetilde{\Theta}_{\mathrm{le}}^{(m,m+j)} := \sum_{|\underline{k}|=n} a_{\underline{k}}^{p^{m+j}} \left(\underline{\partial}^{\langle p^m \rangle_{(m)}}\right)^{\underline{k} p^j}.$$

If there is no risk of confusion, we sometimes abbreviate  $\widetilde{\Theta}_{le}^{(m)}$  (resp.  $\widetilde{\Theta}_{le}^{(m,m')}$ ) as  $\Theta^{(m)}$  (resp.  $\Theta^{(m,m')}$ ).

**Lemma.** — Let  $m' \ge m$  be an integer, and we put j := m' - m.

- (i) Let  $r_{m,m'} := (p^{m'}!) \cdot (p^m!)^{-p^j} \in \mathbb{Z}_p$ . Then  $\Theta^{(m,m')} = r_{m,m'}^n \cdot \Theta^{(m')}$ .
- (ii) The operator  $\widetilde{\Theta}_{le}^{(m,m')}$  is invertible in  $\Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X}}^{(m)})$ .

*Proof.* — We know that for any  $1 \le i \le d$ ,

$$\left(\xi_i^{\langle p^m \rangle_{(m)}}\right)^{p^j} = r_{m,m'} \cdot \xi_i^{\langle p^{m'} \rangle_{(m')}}$$

Since  $r_{m,m'}$  does not depend on *i*, we get (i) by definition. For the proof of (ii), just copy the proof of the invertibility of  $\widetilde{\Theta}_{le}^{(m)}$  in 2.10.

Let  $m' \ge m$  be an integer. We claim that any  $S \in \Gamma(\mathscr{U}, \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)})$  can also be written as

$$\sum_{N \in \mathbb{Z}} \sum_{|\underline{k}| - inp^{m'} = N} c_{\underline{k}, i} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \, (\widetilde{\Theta}_{\mathrm{le}}^{(m, m')})^{-i},$$

with a sequence  $\{c_{\underline{k},i}\}$  in  $\Gamma(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$  for  $\underline{k} \in \mathbb{N}^d$  and  $i \in \mathbb{N}$ , such that the following holds: for each integer N, let  $\gamma_{N,i} := \sup_{|k|=inp^{m'}+N} |c_{\underline{k},i}|$ . Then

$$\lim_{i \to \infty} \beta_{N,i} = 0, \qquad \lim_{N \to +\infty} \sup_{i} \{\gamma_{N,i}\} = 0.$$

The verification is left to the reader.

**2.12.** — Assume further that  $\Theta$  is of the form  $\xi^{\underline{k}_0} + \sum_{\underline{k}} a_{\underline{k}} \xi^{\underline{k}}$  where  $a_k \in \Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}}), |\underline{k}_0| > 0$ , and  $\underline{k}$  runs through  $\underline{k} \in \mathbb{N}^d$  such that  $|\underline{k}| = |\underline{k}_0|$  and  $\underline{k} \neq \underline{k}_0$ . Then we can check that  $\Gamma(D(\Theta), \mathscr{O}_{T^{(m)*}X_j}(*))$  is a free  $\Gamma(X_j, \mathscr{O}_{X_j})$ -module with basis  $\{\underline{\xi}^{\langle \underline{k} \rangle_{(m)}} \cdot (\Theta^{(m,m')})^{-i}\}_{(k,i) \in I}$  where

$$I := \left\{ (\underline{k}, i) \in \mathbb{N}^d \times \mathbb{Z} \, \big| \, \underline{k} - p^{m'} \underline{k}_0 \text{ is not } \ge \underline{0} \right\}.$$

Let  $m' \geq m$ . Using this basis, we define a continuous homomorphism of left  $\Gamma(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ -modules

$$\phi \colon \Gamma \big( D(\Theta), \mathcal{O}_{T^{(m)*} \mathscr{X}}(*) \big) \to \Gamma \big( D(\Theta), \mathscr{E}_{\mathscr{X}}^{(m)} \big)$$

by sending  $\underline{\xi}^{\langle \underline{k} \rangle_{(m)}} \cdot (\Theta^{(m,m')})^{-i}$  to  $\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \cdot (\widetilde{\Theta}_{le}^{(m,m')})^{-i}$ . Now, take  $P_k$  belonging to  $\Gamma(\mathscr{U}, \mathcal{O}_{T^{(m)*}\mathscr{X}}(k))$  for each  $k \in \mathbb{Z}$  such that  $\lim_{k \to \infty} |P_k| = 0$ . Then the infinite sum  $\sum_k \phi(P_k)$  makes sense in  $\Gamma(\mathscr{U}, \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)})$  as (2.10.3). Conversely, we have the following lemma whose verification is similar.

**Lemma.** — For any element  $P \in \Gamma(\mathscr{U}, \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)})$ , there exists a unique  $P_k \in \Gamma(\mathscr{U}, \mathcal{O}_{T^{(m)*}\mathscr{X}}(k))$  for each  $k \in \mathbb{Z}$  such that  $\lim_{k \to \infty} |P_k| = 0$ , and  $P = \sum_k \phi(P_k)$ .

**2.13. Example.** — Consider the case where  $\dim(\mathscr{X}) = 1$ , and  $\mathscr{X}$  possesses a local coordinate. Let  $\mathscr{U} := T^{(m)*}\mathscr{X} \setminus s(\mathscr{X})$  where s is the zero section. Take k > 0, and write  $k = q \cdot p^m - r$  where  $0 \le r < p^m$ . Put  $\partial^{\langle -k \rangle_{(m)}} := \partial^{\langle r \rangle_{(m)}} \cdot (\partial^{\langle p^m \rangle_{(m)}})^{-q}$ , which is defined in  $\Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X}}^{(m)})$ . Then any element of  $\Gamma(\mathscr{U}, \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)})$  can be written uniquely as  $\sum_{k \in \mathbb{Z}} a_k \cdot \partial^{\langle k \rangle_{(m)}}$  with  $a_k \in \Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$  such that  $\lim_{k \to \infty} a_k = 0$ .

**2.14.** — Let us clarify the relation between the characteristic varieties and the supports of microlocalizations. Let  $\mathscr{M}$  be a coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -module. Let us recall the definition of the characteristic variety of  $\mathscr{M}$  defined in [7, 5.2.4]. First, we take a  $\pi$ -torsion free coherent  $\widehat{\mathscr{D}}_{\mathscr{X}}^{(m)}$ -module  $\mathscr{M}'$  such that  $\mathscr{M}' \otimes \mathbb{Q} \cong \mathscr{M}$  using [5, 3.4.5]. Then,  $\mathscr{M}'/\pi$  is a coherent  $\mathscr{D}_{X_0}^{(m)}$ -module. Now, we can check that  $\operatorname{Char}(\mathscr{M}'/\pi) \subset T^{(m)*}X_0$  (cf. Definition 1.7) does not depend on the choice of  $\mathscr{M}'$ . This  $\operatorname{Char}(\mathscr{M}'/\pi)$  is called the *characteristic variety* of  $\mathscr{M}$  denoted <sup>(3)</sup> by  $\operatorname{Char}^{(m)}(\mathscr{M})$ . By using the canonical homeomorphism  $T^{(m)*}X_0 \approx T^{(m)*}\mathscr{X}$ , we consider that the characteristic varieties are in  $T^{(m)*}\mathscr{X}$ .

Let us define another subvariety of  $T^{(m)*}\mathscr{X}$  defined by  $\mathscr{M}$ . Consider the following coherent  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$ -module

$$\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)}(\mathscr{M}) := \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)} \otimes_{\pi_m^{-1}\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}} \pi_m^{-1} \mathscr{M},$$

which is called the *microlocalization* of  $\mathscr{M}$ . Note here that since  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)}(\mathscr{M})$  is an  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -module of finite type, the support  $\operatorname{Supp}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)}(\mathscr{M})) \subset T^{(m)*}\mathscr{X}$  is closed by [16, 0<sub>1</sub>, 5.2.2]<sup>(4)</sup>.

**2.15. Proposition**. — Let  $\mathscr{M}$  be a coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -module. Then, we have the following equality of closed subsets of  $T^{(m)*}\mathscr{X}$ :

$$\operatorname{Char}^{(m)}(\mathscr{M}) = \operatorname{Supp}(\widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}}(\mathscr{M})).$$

<sup>&</sup>lt;sup>(3)</sup> We warn that the characteristic variety  $\operatorname{Char}^{(m)}(\mathscr{M})$  is denoted by  $\operatorname{Car}^{(m)}(\mathscr{M})$  in [7].

 $<sup>^{(4)}</sup>$  In [16], only commutative case is treated, but the same argument can be used also for non-commutative case.

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*Proof.* — Take a coherent  $\widehat{\mathscr{D}}_{\mathscr{X}}^{(m)}$ -module  $\mathscr{M}'$  flat over R and  $\mathscr{M}' \otimes \mathbb{Q} \cong \mathscr{M}$ . Let us calculate the support of the microlocalization. Since  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$  is pointwise Zariskian with respect to the  $\pi$ -adic filtration by Remark 2.8, the  $\pi$ -adic filtration on  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)} \otimes \pi_m^{-1} \mathscr{M}'$  is separated by Lemma 1.13, and thus

$$\operatorname{Supp}(\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)} \otimes \pi_m^{-1} \mathscr{M}') = \operatorname{Supp}(\mathscr{E}_X^{(m)} \otimes \pi_m^{-1} \mathscr{M}') = \operatorname{Char}(\mathscr{M}' \otimes k) =: \operatorname{Char}^{(m)}(\mathscr{M}),$$

where the second equality holds by Lemma 1.7. Moreover, since  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$  is flat over  $\pi_m^{-1}\widehat{\mathscr{D}}_{\mathscr{X}}^{(m)}, \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)} \otimes \mathscr{M}'$  is  $\pi$ -torsion free. This implies that  $\operatorname{Supp}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)}(\mathscr{M})) = \operatorname{Supp}(\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)} \otimes \pi_m^{-1}\mathscr{M}')$ , and the proposition follows.

**2.16. Remark.** — P. Berthelot pointed out to the author another method to define  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m)}$ . Let  $\mathscr{X}$  be a smooth affine formal scheme over R. Let  $\Theta$  be a homogeneous section of  $\Gamma(T^*\mathscr{X}, \mathcal{O}_{T^*\mathscr{X}})$ . For each  $i \geq 0$ , there exists an integer  $m' \geq m$  such that  $\widetilde{\Theta}_{le}^{(m,m')}$  is contained in the center of  $\mathscr{D}_{X_i}^{(m)}$ . Note that in this case,  $\widetilde{\Theta}_{le}^{(m,m')} = \widetilde{\Theta}_{ri}^{(m,m')}$ . Let A be a ring, and S be a multiplicative system of A consisting of elements in the center of A. We can construct the ring of fractions  $S^{-1}A$  as the commutative case. (The details are left to the reader.) Using this, we define

$$\Gamma(D(\Theta^{(m)}), \mathscr{L}\mathscr{D}_{X_i}^{(m)}) := S_{\Theta^{(m,m')}}^{-1} \Gamma(D(\Theta^{(m)}), \pi^{-1}\mathscr{D}_{X_i}^{(m)}),$$

where  $S_{\Theta^{(m,m')}}$  denotes the multiplicative system generated by  $\widetilde{\Theta}_{le}^{(m,m')}$ . We can check easily that this does not depend on the choice of m' and defines a sheaf. By taking the completion with respect to the filtration by order, we get  $\mathscr{E}_{X_i}^{(m)}$ . By definition, the sheaf  $\mathscr{LD}_{X_i}^{(m)}$  is a Noetherian ring.

#### 3. Pseudo cotangent bundles and pseudo polynomials

**3.1.** — Recall the notation of 2.4. Let A be a commutative R-algebra, m be a non-negative integer, and d be a positive integer. We define

$$A[\xi_1, \dots, \xi_d]^{(m)} := A[\xi_j^{\langle p^i \rangle_{(m)}} \mid j = 1, \dots, d, \ i = 0, \dots, m]/I_m$$

where  $\xi_j^{\langle p^i \rangle_{(m)}}$  is an indeterminate for any *i* and *j*, and  $I_m$  is the ideal generated by the relations

$$(\xi_j^{\langle p^i \rangle_{(m)}})^p = \frac{(p^{i+1})!}{(p^i!)^p} \, \xi_j^{\langle p^{i+1} \rangle_{(m)}}$$

for  $1 \leq j \leq d$  and  $0 \leq i < m$ . We note that  $(p^{i+1}!) \cdot (p^i!)^{-p} \in \mathbb{Z}_p$ . We define  $\deg(\xi_j^{\langle p^i \rangle_{(m)}}) := p^i$ , which makes  $A[\xi_1, \ldots, \xi_d]^{(m)}$  a graded ring. We call this the ring of pseudo polynomials over A. We denote by  $A\{\xi_1, \ldots, \xi_d\}^{(m)}$  the  $\pi$ -adic

completion of  $A[\xi_1, \ldots, \xi_d]^{(m)}$ . We call this the *pseudo Tate algebra over A*. We note that for an *R*-algebra *A*,

$$A \otimes_R R[\xi_1, \dots, \xi_d]^{(m)} \cong A[\xi_1, \dots, \xi_d]^{(m)}$$

**Lemma.** — Let A be a commutative R-algebra. For any non-negative integers  $m' \ge m$ , there exists a unique isomorphism of graded rings

$$A[\xi_1, \dots, \xi_d]^{(m)} \otimes \mathbb{Q} \xrightarrow{\sim} A[\xi_1, \dots, \xi_d]^{(m')} \otimes \mathbb{Q}$$
  
sending  $\xi_i^{\langle 1 \rangle_{(m)}}$  to  $\xi_i^{\langle 1 \rangle_{(m')}}$  for  $1 \leq i \leq d$ .

Proof. — Left to the reader.

**3.2. Lemma.** — Let  $\mathscr{X} = \operatorname{Spf}(A)$  be an affine smooth formal scheme over R possessing a system of local coordinates  $\{x_1, \ldots, x_d\}$  on  $\mathscr{X}$ . Let  $A_i := A \otimes_R R_i$ . Then there exists a unique isomorphism of graded rings

 $A_i[\xi_1,\ldots,\xi_d]^{(m)} \xrightarrow{\sim} \Gamma(X_i,\operatorname{gr}(\mathscr{D}_{X_i}^{(m)}))$ 

sending  $\xi_j^{\langle p^i \rangle_{(m)}}$  to  $\sigma(\partial_j^{\langle p^i \rangle_{(m)}})$ , where  $\sigma$  denotes the principal symbol (cf. 1.1.4), for  $1 \leq k \leq d$ .

*Proof.* — To construct the homomorphism of graded rings, use [5, 2.2.4]. This homomorphism is surjective by [5, 2.2.5]. To check the injectivity, it suffices to show that, for any  $k \ge 0$ , the parts of degree k of both sides are free over  $A_i$  with the same ranks. The detail is left to the reader.

**3.3.** — Let X be a smooth scheme over k, and  $m \ge 0$  be an integer. Let  $X^{(m)} := X \otimes_{k, F_k^{m*}} k$  where  $F_k^{m*} : k \to k$  is the *m*-th absolute Frobenius homomorphism (i.e., the homomorphism sending x to  $x^{p^m}$ ). By [7, 5.2.2], we have a canonical isomorphism

$$(T^{(m)*}X)_{\mathrm{red}} \xrightarrow{\sim} X \times_{X^{(m)}} T^*X^{(m)}$$

where  $_{\rm red}$  denotes the associated reduced scheme. The scheme  $T^*X^{(m)}$  is deduced from  $T^*X$  by the base change  $X^{(m)} \to X$ . This induces the canonical morphism of schemes (which may not be a morphism over k)

$$(T^{(m)*}X)_{\mathrm{red}} \to T^*X$$

such that the underlying continuous map is a homeomorphism of topological spaces.

Now, let  $\mathscr{X}$  be a smooth formal scheme over R. Since the topological space of  $T^{(m)*}X_i$  is homeomorphic to that of  $T^{(m)*}\mathscr{X}$ , we also get a canonical homeomorphism  $T^{(m)*}\mathscr{X} \approx T^*\mathscr{X}$ . Consider the situation as in 2.10. The affine open subset of  $T^*\mathscr{X}$  defined by  $\Theta$  and that of  $T^{(m)*}\mathscr{X}$  defined by  $\Theta^{(m)}$  are

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homeomorphic under this canonical homeomorphism. From now on, we identify the spaces  $T^* \mathscr{X}$ ,  $T^{(m)*} \mathscr{X}$ ,  $T^* X_i$ , and  $T^{(m)*} X_i$  using these homeomorphisms. In particular, we consider  $\mathscr{E}_{\mathscr{X}}^{(m)}$  etc. as sheaves on  $T^* \mathscr{X}$  or  $T^* X_i$ . We denote the projection  $\pi: T^* \mathscr{X} \to \mathscr{X}$ . The notation " $\pi$ " is the same as the uniformizer of R, but we do not think there is any confusion. This identification also induces the identification of topological spaces

$$P^*\mathscr{X} \approx P^*X_i \approx P^{*(m)}X_i \approx P^{*(m)}\mathscr{X}.$$

**3.4. Lemma.** — Let  $\mathscr{X}$  be an affine smooth formal scheme over R of dimension d. We use the notation and the identifications in 3.3. We take non-negative integers  $m' \geq m$ .

(i) For any integer k, there exist integers  $a_k, b_k \ge 0$  such that the following holds: let  $\Theta \in \Gamma(T^* \mathscr{X}, \mathcal{O}_{T^* \mathscr{X}})$  be a homogeneous section of degree n.

(a) The operator

$$p^{a_k} \underline{\partial}^{\langle \underline{l} \rangle_{(m)}} (\Theta^{(m,m')})^{-i}$$

which is a priori contained in  $\Gamma(D(\Theta), \mathscr{E}_{\mathscr{X}, \mathbb{Q}}^{(m')})$  by Lemma 2.11 (i), is contained in  $\Gamma(D(\Theta), \mathscr{E}_{\mathscr{X}}^{(m')})$  for any  $|\underline{l}| - inp^{m'} \geq k$ . If  $dp^{m'+1} < k$ , we may take  $a_k = 0$ .

(b) The operator

$$p^{b_k} \partial^{\langle \underline{l} \rangle_{(m')}} (\Theta^{(m')})^{-i}$$

is in  $\Gamma(D(\Theta), \mathscr{E}_{\mathscr{X}}^{(m)})$  for any  $|\underline{l}| - inp^{m'} \leq k$ . If  $k < p^{m+1}$ , we may take  $b_k = 0$ .

(ii) Let  $\Theta \in \Gamma(T^*\mathscr{X}, \mathcal{O}_{T^*\mathscr{X}})$  be a homogeneous section. Take an integer m''such that  $m \leq m'' \leq m'$ . Suppose  $P = \alpha \cdot \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} (\Theta^{(m,m')})^{-i}$  with  $\alpha \in R$  is contained in  $\Gamma(D(\Theta), \mathscr{E}_{\mathscr{X}}^{(m')})$ . Then it is also contained in  $\Gamma(D(\Theta), \mathscr{E}_{\mathscr{X}}^{(m'')})$ .

*Proof.* — First, let us show (i). Since the proof for (b) is essentially the same, we concentrate on proving (a). We show the following.

Claim. — Let  $m' \ge 0$  be an integer. For integers m, a, k such that  $m' \ge m \ge 0$ ,  $m' - m \ge a \ge 0$ , there exists an integer  $\alpha_{k,m,a} \ge 0$  such that, for any  $\Theta$  and  $|\underline{l}| - inp^{m'} \ge k, p^{\alpha_{k,m,a}} \underline{\partial}^{\langle \underline{l} \rangle_{(m)}} (\Theta^{(m,m')})^{-i}$  is equal to  $\alpha \cdot \underline{\partial}^{\langle \underline{l} \rangle_{(m+a)}} (\Theta^{(m+a,m')})^{-i}$ with some  $\alpha \in \mathbb{Z}_p$ . If  $k > d p^{m'+1}$ , we can take  $\alpha_{k,m,a} = 0$ .

Once this claim is proven, the lemma follows by taking a = m' - m.

Proof of the claim. — Let  $b := m' - m - 1 \ge -1$ . We show the claim using the induction on b. When b = -1 or more generally a = 0, we can take  $\alpha_{k,m,a} = 0$ . Since we can take  $\alpha_{k,m,a} = \alpha_{k,m+1,a-1} + \alpha_{k,m,1}$ , it suffices to show the existence of  $\alpha_{k,m,1}$  by the induction hypothesis.

There exists a number  $c \in \mathbb{Z}_p^*$  such that

$$\Theta^{(m,m')} = c \, p^{np^b} \cdot \Theta^{(m+1,m')}.$$

For  $\underline{l'} \in \mathbb{N}^d$ , we put

$$g(\underline{l'}) := \sum_{j=1}^d \left[ \frac{l'_j}{p^{m+1}} \right]$$

where  $[\alpha]$  denotes the maximum integer less than or equal to  $\alpha$ . Then

$$\underline{\partial}^{\langle \underline{l} \rangle_{(m)}} = c' \, p^{g(\underline{l})} \, \underline{\partial}^{\langle \underline{l} \rangle_{(m+1)}}$$

with  $c' \in \mathbb{Z}_p^*$ . Since

$$\frac{l_j}{p^{m+1}} - 1 < \left[\frac{l_j}{p^{m+1}}\right] \le \frac{l_j}{p^{m+1}},$$

we get the inequalities

$$inp^{b} + \frac{k}{p^{m+1}} - d \le \frac{|\underline{l}|}{p^{m+1}} - d < \sum_{j=1}^{d} \left[\frac{l_{j}}{p^{m+1}}\right] = g(\underline{l}).$$

Thus,

$$\underline{\partial}^{\langle \underline{l} \rangle_{(m)}} (\Theta^{(m,m')})^{-i} = c' c^{-i} p^{g(\underline{l}) - inp^b} \underline{\partial}^{\langle \underline{l} \rangle_{(m+1)}} (\Theta^{(m+1,m')})^{-i},$$

and we may take  $\alpha_{k,m,1} = \max\{0, [d - kp^{-(m+1)} + 1]\}$ . Thus, we conclude the proof of the claim.

Let us prove (ii) on  $D(\Theta)$ . We get

$$\Theta^{(m,m')} = u \ (p^{m'-m} !)^n \cdot \Theta^{(m',m')}$$

where n denotes the order of  $\Theta$ , and u denotes a number in  $\mathbb{Z}_p^*$ . Thus, for  $m \leq l \leq m'$ ,

(3.4.1) 
$$\Theta^{(m,m')} = u' \frac{(p^{m'-m} !)^n}{(p^{m'-l} !)^n} \cdot \Theta^{(l,m')} = u_l \ p^{a_l} \cdot \Theta^{(l,m')}$$

where u' and  $u_l$  denote numbers in  $\mathbb{Z}_p^*$ , and  $a_l$  is equal to  $n \cdot (p^{m'-l} + \cdots + p^{m'-m-1})$ . We also get

(3.4.2) 
$$\partial^{\langle \underline{k} \rangle_{(m)}} = u' p^{b_l} \partial^{\langle \underline{k} \rangle_{(l)}}, \qquad b_l = \sum_{j=1}^d \sum_{i=m+1}^l [p^{-i}k_j].$$

Now, we define two functions  $f, g: [m, m'] \to \mathbb{R}$ . The function f is the continuous function such that it is affine on the interval [l, l+1] for any integer l in [m, m'], and

$$f(l) := \operatorname{ord}_p(\alpha) + b_l = \operatorname{ord}_p(\alpha) + \sum_{j=1}^d \sum_{i=m+1}^l [p^{-i}k_j],$$

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where  $\operatorname{ord}_p$  denotes the *p*-adic order normalized so that  $\operatorname{ord}_p(p) = 1$ . The function *g* is the continuous function such that it is affine on the interval [l, l+1]for any integer *l* in [m, m'], and

$$g(l) := i \cdot a_l = ni \cdot (p^{m'-l} + p^{m'-l+1} + \dots + p^{m'-m-1}).$$

Since the operator  $\partial^{\langle \underline{k} \rangle_{(m)}} (\Theta^{(m,m')})^{-i}$  is a section of  $\mathscr{E}_{\mathscr{X}}^{(m)}$ , we have  $g(m) \leq f(m)$ . By (3.4.1) and (3.4.2), it suffices to show that if  $g(m') \leq f(m')$ , then  $g(l) \leq f(l)$  for any integer l in [m, m']. We put

$$Df(l) := f(l) - f(l-1) = \sum_{j=1}^{d} [p^{-l} k_j], \qquad Dg(l) := g(l) - g(l-1) = nip^{m'-l}.$$

For any  $a \in \mathbb{R}$ , we have  $p^{-1} \cdot [a] \ge [p^{-1}a]$ . Indeed,  $p^{-1}a \ge [p^{-1}a]$ , and  $a \ge p \cdot [p^{-1}a]$ . Since  $p \cdot [p^{-1}a]$  is an integer, we get what we want by the definition of [·]. This implies that  $p^{-1} \cdot [p^{-l}k_j] \ge [p^{-(l+1)}k_j]$ , and thus

$$p^{-1} \cdot Df(l) \ge Df(l+1).$$

In turn, we have  $p^{-1} \cdot Dg(l) = Dg(l+1)$ . Suppose there exists an integer l in [m, m'] such that f(l) < g(l) and  $f(a) \ge g(a)$  for any integer a in [m, l]. This shows that Df(l) < Dg(l). Thus, Df(b) < Dg(b) for any  $l \le b$ , which implies f(m') < g(m'). This contradicts with the assumption, and we conclude that  $g(l) \le f(l)$  for any  $m \le l \le m'$ .

**3.5.** — Let M and M' be  $\pi$ -torsion free R-modules. A p-isogeny  $\phi: M \dashrightarrow M'$  is an isomorphism

$$\phi_{\mathbb{Q}}: M \otimes \mathbb{Q} \xrightarrow{\sim} M' \otimes \mathbb{Q}$$

such that there exist positive integers n and n' satisfying

$$p^n \cdot \phi_{\mathbb{Q}}(M) \subset M' \subset p^{-n'} \cdot \phi_{\mathbb{Q}}(M).$$

Here  $\phi_{\mathbb{Q}}$  is called the *realization* of the *p*-isogeny. We say that the *p*-isogeny is a *homomorphism* if we can take *n* to be 0.

**Lemma.** — Let M and M' be  $\pi$ -torsion free R-modules, and let  $\phi: M \dashrightarrow M'$  be a p-isogeny. Then this induces a canonical p-isogeny

$$\widehat{\phi} \colon M^{\wedge} \dashrightarrow M'^{\wedge}$$

where  $\wedge$  denotes the  $\pi$ -adic completion. If the given p-isogeny is a homomorphism, the induced p-isogeny is also a homomorphism.

*Proof.* — Let  $\phi_{\mathbb{Q}}: M \otimes \mathbb{Q} \to M' \otimes \mathbb{Q}$  be the realization of the isogeny. By definition, there exists an integer n such that  $p^n \cdot \phi_{\mathbb{Q}}$  induces a homomorphism  $M \to M'$ . We denote this homomorphism by  $\phi_n: M \to M'$ .

Let  $C := \operatorname{Coker}(\phi_n)$ . Since  $\phi$  is a *p*-isogeny, C is a  $\pi^{n'}$ -torsion module for some integer  $n' \ge 0$ . We have an exact sequence of projective systems

$$0 \to \{C\}'_{i \ge n'} \to \{M \otimes R_i\}_{i \ge n'} \to \{M' \otimes R_i\}_{i \ge n'} \to \{C\}_{i \ge n'} \to 0$$

where  $\{C\}'_{i\geq n'}$  denotes the projective system  $\{\cdots \to C \xrightarrow{\pi} C \xrightarrow{\pi} C \to \cdots\}$ , and  $\{C\}_{i\geq n'}$  is the projective system  $\{\cdots \to C \xrightarrow{\text{id}} C \xrightarrow{\text{id}} C \to \cdots\}$ . Since any projective system appearing in the short exact sequence above satisfies the Mittag-Leffler condition, the exact sequence induces an exact sequence

$$0 \to \widehat{M} \xrightarrow{\widehat{\phi_n}} \widehat{M'} \to C \to 0$$

by taking the projective limit. Thus, we get a *p*-isogeny  $\widehat{\phi}_{\mathbb{Q}} := p^{-n} \cdot \widehat{\phi}_n : \widehat{M} \otimes \mathbb{Q} \to \widehat{M'} \otimes \mathbb{Q}$  as desired. By construction, the homomorphism  $\widehat{\phi}_{\mathbb{Q}}$  does not depend on the choice of the integer *n*.

Let  $\mathcal{M}, \mathcal{M}'$  be  $\pi$ -torsion free R-modules a topological space X. Then exactly in the same way, we can define p-isogeny  $\phi \colon \mathcal{M} \dashrightarrow \mathcal{M}'$ . Namely, it is a homomorphism of sheaves of modules  $\phi_{\mathbb{Q}} \colon \mathcal{M} \otimes \mathbb{Q} \to \mathcal{M}' \otimes \mathbb{Q}$  such that there exist positive integers n and n' satisfying  $p^n \cdot \phi_{\mathbb{Q}}(\mathcal{M}) \subset \mathcal{M}' \subset p^{-n'} \phi_{\mathbb{Q}}(\mathcal{M})$ . We say that the p-isogeny is a homomorphism if we can take n to be 0.

**3.6.** — Let  $\mathscr{X} = \operatorname{Spf}(A)$  be an affine smooth formal scheme over R, and assume that it possesses a system of local coordinates  $\{x_1, \ldots, x_d\}$ . We identify the ring of global sections of  $\mathscr{O}_{T^{(m)*}\mathscr{X}}$  with  $A\{\xi_1, \ldots, \xi_d\}^{(m)}$  using Lemma 3.2. Let  $\Theta$  be a homogeneous element of  $A[\xi_1, \ldots, \xi_d]$  whose degree is *strictly greater* than 0. For a commutative graded ring  $\Lambda$  and a homogeneous element  $f \in \Lambda$ , we denote the submodule of degree n of the graded ring  $\Lambda_f$  by  $\Lambda_{(f)}(n)$ . Then by construction of  $\mathscr{O}_{P^{*(m)}\mathscr{X}}$ ,

$$\Gamma(D_{+}(\Theta), \mathcal{O}_{P^{*(m)}\mathscr{X}}(n)) \cong (A[\xi_{1}, \dots, \xi_{d}]_{(\Theta^{(m)})}^{(m)}(n))^{\wedge}$$

where  $^{\wedge}$  denotes the  $\pi$ -adic completion, and we used the notation of [16, II, (2.3.3)]. For  $m' \geq m$ , we note that

(3.6.1) 
$$(A[\xi_1, \dots, \xi_d]^{(m)}_{(\Theta^{(m)})}(n))^{\wedge} \cong (A[\xi_1, \dots, \xi_d]^{(m)}_{(\Theta^{(m,m')})}(n))^{\wedge}$$

since there exists  $Q \in A[\xi_1, \ldots, \xi_d]^{(m)}$  such that

$$(\Theta^{(m)})^{p^{m'-m}} = \Theta^{(m,m')} + pQ.$$

Lemma 2.11 (i) and the isomorphism  $A[\xi_1, \ldots, \xi_d]^{(m)} \otimes \mathbb{Q} \cong A[\xi_1, \ldots, \xi_d]^{(m')} \otimes \mathbb{Q}$  of Lemma 3.1 induces the following homomorphism

$$A[\xi_1,\ldots,\xi_d]^{(m')}_{\Theta^{(m')}} \to A[\xi_1,\ldots,\xi_d]^{(m)}_{\Theta^{(m,m')}} \otimes \mathbb{Q}.$$

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Using Lemma 3.4 (i)-(b), this homomorphism defines a p-isogeny

$$A[\xi_1,\ldots,\xi_d]^{(m')}_{(\Theta^{(m')})}(n) \dashrightarrow A[\xi_1,\ldots,\xi_d]^{(m)}_{(\Theta^{(m,m')})}(n)$$

for any  $n \in \mathbb{Z}$ . For  $n < p^{m+1}$ , this *p*-isogeny is moreover a homomorphism by the same lemma. This defines a *p*-isogeny

$$(A[\xi_1,\ldots,\xi_d]^{(m')}_{(\Theta^{(m')})}(n))^{\wedge} \dashrightarrow (A[\xi_1,\ldots,\xi_d]^{(m)}_{(\Theta^{(m,m')})}(n))^{\wedge}$$

by Lemma 3.5. Composing this with (3.6.1), we get a canonical *p*-isogeny

$$(A[\xi_1,\ldots,\xi_d]^{(m')}_{(\Theta^{(m')})}(n))^{\wedge} \dashrightarrow (A[\xi_1,\ldots,\xi_d]^{(m)}_{(\Theta^{(m)})}(n))^{\wedge},$$

which is a homomorphism for  $n < p^{m+1}$ . By construction, this *p*-isogeny is compatible with restrictions. Moreover, since  $b_n$  of Lemma 3.4 does not depend on  $\Theta$ , this induces a *p*-isogeny of *sheaves*. Summing up, we obtain the following lemma.

**Lemma.** — Let  $m' \ge m$  be non-negative integers. For any  $n \in \mathbb{Z}$ , there exist canonical p-isogenies of sheaves of modules

$$\mathcal{O}_{P^{*(m')}\mathscr{X}}(n) \dashrightarrow \mathcal{O}_{P^{*(m)}\mathscr{X}}(n), \qquad \mathcal{O}_{T^{*(m')}\mathscr{X}}(n) \dashrightarrow \mathcal{O}_{T^{*(m)}\mathscr{X}}(n)$$

on the topological spaces  $P^* \mathscr{X}$  and  $\mathring{T}^* \mathscr{X}$  respectively. These are homomorphisms for  $n < p^{m+1}$ .

**3.7. Lemma.** — By using the homomorphism of Lemma 3.6,  $\mathcal{O}_{P^{*(m)}\mathscr{X}}$  can be seen as an  $\mathcal{O}_{P^{*(m')}\mathscr{X}}$ -algebra. Then  $\mathcal{O}_{P^{*(m)}\mathscr{X}}$  is a coherent  $\mathcal{O}_{P^{*(m')}\mathscr{X}}$ -algebra.

*Proof.* — Let  $\Theta$  be a homogeneous element of  $A[\xi_1, \ldots, \xi_d]$  whose degree is strictly greater than 0. First of all, let us show that the homomorphism of rings

$$(3.7.1) \qquad (A[\xi_1, \dots, \xi_d]^{(m')}_{(\Theta^{(m')})}(0))^{\wedge} \to (A[\xi_1, \dots, \xi_d]^{(m)}_{(\Theta^{(m)})}(0))^{\wedge}$$

is finite. By construction of (3.7.1), it suffices to show the finiteness of the homomorphism

$$A[\xi_1, \dots, \xi_d]_{(\Theta^{(m')})}^{(m')}(0) \to A[\xi_1, \dots, \xi_d]_{(\Theta^{(m,m')})}^{(m)}(0).$$

Let

$$S := \left\{ (\underline{k}, \underline{k}', i) \in \mathbb{N}^d \times \mathbb{N}^d \times \mathbb{N} \middle| \begin{array}{l} k_j < p^{m'} \text{ for any } j, \, |\underline{k}'| < \operatorname{ord}(\Theta), \text{ and} \\ |\underline{k}| + |\underline{k}'| \, p^{m'} = ip^{m'} \operatorname{ord}(\Theta) \end{array} \right\}.$$

The condition  $|\underline{k}| + |\underline{k}'| p^{m'} = i p^{m'} \operatorname{ord}(\Theta)$  means that the order of

$$\underline{\xi}^{\langle \underline{k} + \underline{k}' p^{m'} \rangle_{(m)}} (\Theta^{(m,m')})^{-i}$$

is equal to 0. Obviously,  $\#S < \infty$ . Let

$$T := \left\{ \underline{k} \in \mathbb{N}^d \, \big| \, \underline{k} \in p^{m'} \, \mathbb{N}^d, |\underline{k}| = i p^{m'} \operatorname{ord}(\Theta) \text{ for some integer } i \right\}.$$

The set T is a submonoid of the commutative monoid  $\mathbb{N}^d$ . For any  $\underline{k} \in T$ , there exists  $u \in \mathbb{Z}_p^*$  such that

(3.7.2) 
$$\underline{\xi}^{\langle \underline{k} \rangle_{(m)}} (\Theta^{(m,m')})^{-i} = u \cdot \underline{\xi}^{\langle \underline{k} \rangle_{(m')}} (\Theta^{(m')})^{-i}.$$

Let

$$U := \left\{ \underline{k} \in \mathbb{N}^d \mid |\underline{k}| = ip^{m'} \operatorname{ord}(\Theta) \text{ for some } i \right\}.$$

This is also a submonoid of  $\mathbb{N}^d$ . Let

 $S' := \left\{ \underline{l} \in \mathbb{N}^d \, \big| \, \text{there exists } (\underline{k}, \underline{k'}, i) \in S \text{ such that } \underline{l} = \underline{k} + p^{m'} \underline{k'} \right\}.$ 

The monoid T is a submonoid of U, and S' is a finite subset of U. We claim that U = T + S'. Indeed, take  $\underline{l} \in U$ . We can write  $\underline{l} = \underline{i} + p^{m'}\underline{i'}$  such that  $\underline{i}, \underline{i'} \in \mathbb{N}^d$  and  $i_j < p^{m'}$  for any j. Now, there exists  $\underline{k'}$  such that  $|\underline{k'}| < \operatorname{ord}(\Theta)$ ,  $i'_j \geq k'_j$  for any j, and

$$|\underline{i'}| - |\underline{k'}| = \left[|\underline{i'}| \cdot (\operatorname{ord}(\Theta))^{-1}\right] \cdot \operatorname{ord}(\Theta)$$

where  $[\alpha]$  denotes the maximum integer less than or equal to  $\alpha$ . We put  $\underline{k} := \underline{i}$ . Then there exists an integer i such that  $|\underline{i}| + p^{m'} |\underline{k'}| = ip^{m'} \operatorname{ord}(\Theta)$ . By construction  $(\underline{k}, \underline{k'}, i) \in S$ , and  $p^{m'} \cdot (\underline{i'} - \underline{k'}) \in T$ . Since  $\underline{l} = p^{m'} \cdot (\underline{i'} - \underline{k'}) + (\underline{k} + p^{m'} \underline{k'})$ , the claim follows. Considering (3.7.2), this implies that the homomorphism

$$\bigoplus_{\underline{l}\in S} A[\xi_1,\ldots,\xi_d]^{(m')}_{(\Theta^{(m')})}(0) \to A[\xi_1,\ldots,\xi_d]^{(m)}_{(\Theta^{(m,m')})}(0)$$

sending 1 sitting at the  $(\underline{k}, \underline{k'}, i) \in S$  component to  $\underline{\xi}^{\langle \underline{k} + \underline{k'}p^{m'} \rangle_{(m)}} (\Theta^{(m,m')})^{-i}$  is surjective. Thus the homomorphism (3.7.1) is finite.

Let us prove the coherence. Let  $\Xi$  be another homogeneous element of  $A[\xi_1, \ldots, \xi_d]$  whose degree is strictly greater than 0. Let  $\mathscr{U}$  be the affine open subset of  $P^*\mathscr{X}$  defined by  $\Theta$ , and  $\mathscr{U}'$  by that of  $\Xi \cdot \Theta$ . It suffices to show that the canonical homomorphism

$$(3.7.3) \ \Gamma(\mathscr{U}, \mathcal{O}_{P^{*(m)}\mathscr{X}}) \otimes_{\Gamma(\mathscr{U}, \mathcal{O}_{P^{*(m')}\mathscr{X}})} \Gamma(\mathscr{U}', \mathcal{O}_{P^{*(m')}\mathscr{X}}) \to \Gamma(\mathscr{U}', \mathcal{O}_{P^{*(m)}\mathscr{X}})$$

is an isomorphism. By changing  $\Theta$  and  $\Xi$  to some powers of  $\Theta$  and  $\Xi$  respectively, we may assume that  $\operatorname{ord}(\Xi) = \operatorname{ord}(\Theta)$ . We put

$$\begin{aligned} &\mathcal{A} := A[\xi_1, \dots, \xi_d]_{(\Theta^{(m')})}^{(m')}(0), \qquad \Psi := \frac{\Xi^{(m')}}{\Theta^{(m')}}, \\ &\mathcal{B} := A[\xi_1, \dots, \xi_d]_{(\Theta^{(m,m')})}^{(m)}(0), \qquad \Phi := \frac{\Xi^{(m)}}{\Theta^{(m)}}, \qquad \Phi' := \frac{\Xi^{(m,m')}}{\Theta^{(m,m')}}. \end{aligned}$$

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Let  $\phi: \mathscr{A} \to \mathscr{B}$  be the canonical homomorphism. Firstly,  $\Phi' = \phi(\Psi)$  in  $\mathscr{B}$  by Lemma 2.11 (i). Secondly,

$$(\mathscr{B}_{\Phi})^{\wedge} \cong (\mathscr{B}_{\Phi'})^{\wedge}$$

by the same reason as (3.6.1). Thirdly,

$$\mathscr{B}_{\phi(\Psi)} \cong \mathscr{B} \otimes_{\mathscr{C}} \mathscr{U}_{\Psi}.$$

Combining these,  $(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{A}_{\Psi})^{\wedge} \cong (\mathscr{B}_{\Phi})^{\wedge}$ . This is implies that the  $\pi$ -adic completion of the left hand side of (3.7.3) is isomorphic to the right hand side. However, by the finiteness of (3.7.1), the left hand side of (3.7.3) is already  $\pi$ -adically complete by [16, 0<sub>I</sub>, 7.3.6], and as a result, (3.7.3) is an isomorphism. Thus we obtain the lemma.

**3.8.** — Let  $\mathscr{X}$  be an affine smooth formal scheme over R possessing a system of local coordinates. For any  $n \in \mathbb{Z}$ , the module  $\mathcal{O}_{T^{*(m)}\mathscr{X}}(n)$  is an  $\mathcal{O}_{T^{*(m)}\mathscr{X}}(0)$ -module on  $\mathring{T}^{*}\mathscr{X}$ , and by using Lemma 3.6,  $\mathcal{O}_{\mathring{T}^{*(m)}\mathscr{X}}(n)$  can be seen as an  $\mathcal{O}_{\mathring{T}^{*(m')}\mathscr{X}}(0)$ -module.

**Corollary.** — The  $\mathcal{O}_{\mathring{T}^{*(m')}\mathscr{X}}(0)$ -module  $\mathcal{O}_{\mathring{T}^{*(m)}\mathscr{X}}(n)$  is coherent.

**Remark.** — We show in Lemma 4.3 that the  $\mathcal{O}_{\mathring{T}^*(m')\mathscr{X}}(0)$ -module structure on  $\mathcal{O}_{\mathring{T}^*(m)\mathscr{X}}(n)$  does not depend on the choice of local coordinates, and the corollary can be globalized.

## 4. Intermediate microdifferential sheaves

In Section 2, we defined the ring of naive microdifferential operators. However, we do not have any natural homomorphism  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)} \to \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m+1)}$  as shown in 4.1. To remedy this situation, we consider the intermediate ring of microdifferential operators denoted by  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$  for  $m' \geq m$ , which is the "intersection" of  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)}$  and  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m')}$ . In this section, we define these rings and prove some basic properties.

**4.1.** — To start with, let us show the non-existence of a homomorphism  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)} \to \widehat{\mathscr{C}}_{\mathscr{X},\mathbb{Q}}^{(m+1)}$  compatible with the canonical homomorphism  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)} \to \widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m+1)}$ . Suppose the homomorphism existed. Then, for any coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -module  $\mathscr{M}$ , we would get

$$\operatorname{Char}^{(m)}(\mathscr{M}) \supset \operatorname{Char}^{(m+1)}(\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m+1)} \otimes_{\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}} \mathscr{M})$$

by Proposition 2.15. However, this does not hold as the following example shows:

**Example.** — Let  $\mathscr{X} := \widehat{\mathbb{A}}_R^1$ , X be the special fiber, x be the canonical coordinate, and  $\partial$  be the corresponding differential operator. We put  $\mathscr{M} = \widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(0)} / \widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(0)} (\partial - x)$ . Then,

1.  $\operatorname{Char}^{(0)}(\mathscr{M}) = s(X)$ , where  $s \colon X \to T^*X$  is the zero-section. 2.  $\operatorname{Char}^{(1)}(\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(1)} \otimes_{\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(0)}} \mathscr{M}) \cap \mathring{T}^*X \neq \emptyset.$ 

*Proof.* — Since  $\mathscr{M}$  is a coherent  $\mathscr{O}_{\mathscr{X},\mathbb{Q}}$ -module, the first claim follows. Let us check the second claim. First, let us prove that  $\mathscr{D}_{\mathscr{X},\mathbb{Q}}^{\dagger} \otimes_{\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(0)}} \mathscr{M} \neq 0$ . Let  $f_n \in K\{x\}$  (i.e., the Tate algebra), and  $\sum_{n\geq 0} f_n \partial^{[n]} \in \Gamma(\mathscr{X}, \mathscr{D}_{\mathscr{X},\mathbb{Q}}^{\dagger})$ . We get

$$\sum_{n\geq 0} f_n \,\partial^{[n]} \cdot (\partial - x) = \sum_{n\geq 0} \left( f_n \,\partial^{[n]} \,\partial - x \,f_n \,\partial^{[n]} - f_n \,\partial^{[n-1]} \right)$$
$$= \sum_{n\geq 0} \left( n \,f_{n-1} - x \,f_n - f_{n+1} \right) \partial^{[n]}.$$

Assume  $\sum_{n\geq 0} f_n \partial^{[n]} \cdot (\partial - x) = 1$ . Then there exist  $g_n, h_n \in K[x]$ ,  $\deg(g_n) < n-1$  and  $\deg(h_n) < n$ , such that the equality

$$(-1)^n f_n = (x^{n-1} + g_n) + (x^n + h_n) \cdot f_0$$

should hold for any n > 0. However, there is no  $f_0 \in K\{x\}$  such that  $\sum_{n\geq 0} f_n \partial^{[n]} \in \Gamma(\mathscr{X}, \mathscr{D}^{\dagger}_{\mathscr{X}, \mathbb{Q}})$  (since  $|f_n| = \max\{1, |f_0|\}$  by the equality), and  $\mathscr{D}^{\dagger}_{\mathscr{X}, \mathbb{Q}} \otimes \mathscr{M} \neq 0$ .

Now, let e be the element of  $\Gamma(\mathscr{X}, \mathscr{M})$  defined by  $1 \in \Gamma(\mathscr{X}, \widehat{\mathscr{D}}^{(0)}_{\mathscr{X}, \mathbb{Q}})$ . As an  $\mathcal{O}_{\mathscr{X}, \mathbb{Q}}$ -module,  $\mathscr{M}$  is free of rank 1. Since

 $\partial^n \cdot e = (x^n + (\text{polynomial in } K[x] \text{ whose degree is less than } n)) \cdot e,$ 

the  $\widehat{\mathscr{D}}^{(0)}_{\mathscr{X},\mathbb{Q}}$ -module structure on  $\mathscr{M}$  does not extend continuously to a  $\widehat{\mathscr{D}}^{(1)}_{\mathscr{X},\mathbb{Q}}$ -module structure. This shows that the canonical homomorphism  $\mathscr{M} \to \mathscr{M}^{(1)} := \widehat{\mathscr{D}}^{(1)}_{\mathscr{X},\mathbb{Q}} \otimes \mathscr{M}$  is not an isomorphism.

Garnier shows in [14, 5.2.4] that for any coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(0)}$ -module  $\mathscr{M}$ , the characteristic variety  $\operatorname{Char}^{(0)}(\mathscr{M})$  satisfies the Bernstein inequality (*i.e.*, the dimension of the characteristic variety is greater than or equal to 1 unless  $\mathscr{M} = 0$ ). Using the relation of characteristic varieties of Frobenius descents (cf. [7, 5.2.4 (iii)]), the Bernstein inequality also holds for any coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -module. Thus there are three possibilities for the characteristic variety V of  $\mathscr{M}^{(1)}$ : either  $\varnothing$ or [X] or  $V \cap \mathring{T}^* \mathscr{X} \neq \varnothing$ . Since  $\mathscr{M}^{(1)}$  is not 0, V is not empty. If  $V = [X], \mathscr{M}^{(1)}$ would be a coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}$ -module, and since  $\mathscr{M}$  is a coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}$ -module

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of rank 1, we would get that  $\mathscr{M} \cong \mathscr{M}^{(1)}$ , which is a contradiction. Thus the second claim follows.

**4.2.** — Before going to the main theme of this section, let us fix some frequently used notation. Consider the situation where an open subscheme  $\mathscr{U}$  of  $T^*\mathscr{X}$  is given. For non-negative integers  $i' \geq i$ , we put

$$\begin{split} D^{(i)} &:= \Gamma(\mathscr{U}, \mathscr{D}^{(i)}_{\mathscr{X}}), \quad D^{(i)}_{\mathbb{Q}} &:= \Gamma(\mathscr{U}, \mathscr{D}^{(i)}_{\mathscr{X}, \mathbb{Q}}), \quad E^{(i)} &:= \Gamma(\mathscr{U}, \mathscr{E}^{(i)}_{\mathscr{X}}), \\ E^{(i)}_{\mathbb{Q}} &:= \Gamma(\mathscr{U}, \mathscr{E}^{(i)}_{\mathscr{X}, \mathbb{Q}}), \quad E^{(i,i')} &:= \Gamma(\mathscr{U}, \mathscr{E}^{(i,i')}_{\mathscr{X}}), \quad E^{(i,i')}_{\mathbb{Q}} &:= \Gamma(\mathscr{U}, \mathscr{E}^{(i,i')}_{\mathscr{X}, \mathbb{Q}}), \\ \widehat{E}^{(i,i')} &:= \Gamma(\mathscr{U}, \widehat{\mathscr{E}}^{(i,i')}_{\mathscr{X}}), \quad \widehat{E}^{(i,i')}_{\mathbb{Q}} &:= \Gamma(\mathscr{U}, \widehat{\mathscr{E}}^{(i,i')}_{\mathscr{X}, \mathbb{Q}}), \quad E^{(i,i)}_{\mathbb{Q}} &:= \Gamma(\mathscr{U}, \mathscr{E}^{(i,i)}_{\mathscr{X}, \mathbb{Q}}). \end{split}$$

The last 5 rings are defined in 4.11. The first 6 rings are considered to be filtered rings.

**4.3.** — Let  $\mathscr{X}$  be a smooth formal scheme over R. For a non-negative integer m, we defined the filtered ring  $(\mathscr{E}_{\mathscr{X}}^{(m)}, \mathscr{E}_{\mathscr{X},n}^{(m)})$ . By (2.2.1) and Lemma 2.5 (ii),

Let  $m' \geq m$  be an integer. Consider the canonical homomorphism

$$\phi_{m',m} \colon \mathscr{D}_{\mathscr{X}}^{(m)} \to \mathscr{D}_{\mathscr{X}}^{(m')}$$

This homomorphism becomes an isomorphism if we tensor with  $\mathbb{Q}$ .

**Lemma**. — There exists a unique strictly injective homomorphism of filtered rings

$$\psi_{m,m'} \colon \mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m')} \to \mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m)}$$

such that the following diagram is commutative:

$$\begin{array}{c|c} \pi^{-1}\mathscr{D}_{\mathscr{X},\mathbb{Q}}^{(m')} \xleftarrow{\phi_{m',m}\otimes\mathbb{Q}} & \pi^{-1}\mathscr{D}_{\mathscr{X},\mathbb{Q}}^{(m)} \\ & \varphi_{m'} & & & & \\ \varphi_{m'} & & & & & \\ & \mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m')} & & & & \\ & & & & & & \\ & & & & & & \\ \end{array}$$

where we refer to 2.4 for  $\varphi_m$ . For  $m'' \ge m' \ge m$ ,  $\psi_{m,m'} \circ \psi_{m',m''} = \psi_{m,m''}$ . By using (4.3.1),  $\operatorname{gr}_n(\psi_{m,m'})$  can be identified with the p-isogeny in Lemma 3.6 locally.

*Proof.* — Once the existence and the uniqueness is proven, the compatibility  $\psi_{m,m'} \circ \psi_{m',m''} = \psi_{m,m''}$  automatically holds by the compatibility of  $\phi_{m',m}$ .

Let us prove the uniqueness first. Since the problem is local, we may assume that  $\mathscr{X}$  possesses a system of local coordinates, and it suffices to show the uniqueness for the ring of sections over  $D(\Theta)$  where  $\Theta \in \Gamma(\mathscr{X}, \Theta_{T^*\mathscr{X}})$  is a homogeneous element. Suppose there are two homomorphisms  $\psi, \psi'$  satisfying the condition. By the commutativity of the diagram,  $\psi(\Theta^{(m')}) = \psi'(\Theta^{(m')}) =: \theta$ . By the uniqueness of the inverse of  $\theta$ ,  $\psi((\Theta^{(m')})^{-1}) = \psi'((\Theta^{(m')})^{-1})$ . Since  $\psi$ and  $\psi'$  are filtered homomorphisms, these homomorphisms are continuous with respect to the topology defined by the filtrations (cf. 1.1.1). Let E be the subring of  $\Gamma(D(\Theta), \mathscr{E}_{\mathscr{X}}^{(m')})$  generated by  $\Gamma(D(\Theta), \mathscr{D}_{\mathscr{X}}^{(m')})$ and  $(\Theta^{(m')})^{-1}$ . Then,  $\psi|_E = \psi'|_E$ . Since  $\Gamma(D(\Theta), \mathscr{E}_{\mathscr{X}, \mathbb{Q}}^{(m)})$  is separated and E is dense in  $\Gamma(D(\Theta), \mathscr{E}_{\mathscr{X}}^{(m')})$ , we get  $\psi = \psi'$ , and the uniqueness follows.

Now, let us check the existence. Since the problem is local by the uniqueness, we may suppose that  $\mathscr{X}$  is affine. Let  $\Theta$  be a homogeneous element of  $\Gamma(\mathscr{X}, \mathcal{O}_{T^*\mathscr{X}})$ , and let  $\mathscr{U} := D(\Theta)$ . It suffices to construct  $\psi_{m,m'}$  on  $\mathscr{U} := D(\Theta)$ . We use the notation of 4.2. Let  $D_S^{(m')}$  be the microlocalization of  $D^{(m')}$  by using the multiplicative set S of  $\operatorname{gr}(D^{(m')})$  generated by  $\Theta^{(m')} \in \operatorname{gr}(D^{(m')})$  (cf. 1.2), and let  $(E_{\mathbb{Q}}^{(m)})'$  be the completion of  $E_{\mathbb{Q}}^{(m)}$  with respect to the filtration by order. Since  $\Theta^{(m,m')}$  is invertible in  $(E_{\mathbb{Q}}^{(m)})'$ ,  $\Theta^{(m')}$  is also invertible by Lemma 2.11 (i). Thus, by the universal property of the microlocalization [22, A.2.3.3], there exists a unique homomorphism of filtered rings  $\alpha \colon D_S^{(m')} \to (E_{\mathbb{Q}}^{(m)})'$  factoring the canonical homomorphism  $D^{(m')} \to (E_{\mathbb{Q}}^{(m)})'$ . For any n, there exists an integer N such that the homomorphism  $p^N \cdot \alpha_n$  induces a homomorphism  $D_{S,n}^{(m')} \to E_n^{(m)}$  by the concrete description Lemma 2.10 and Lemma 3.4 (i)-(b). Let  $(D_{S,n}^{(m')})^{\wedge}$  be the  $\pi$ -adic completion of  $D_{S,n}^{(m')}$ . Since  $E_n^{(m)}$  is  $\pi$ -adically complete, this induces the homomorphism  $(p^N \cdot \alpha_n)^{\wedge} \colon (D_{S,n}^{(m')})^{\wedge} \to E_n^{(m)}$ . We define

$$\widehat{\alpha}_n := p^{-N} \cdot (p^N \cdot \alpha_n)^{\wedge} \colon (D_{S,n}^{(m')})^{\wedge} \to E_n^{(m)} \otimes \mathbb{Q}.$$

By construction, we have  $\widehat{\alpha}_{n+1}|_{D_n^{\wedge}} = \widehat{\alpha}_n$  where  $D_n^{\wedge} := (D_{S,n}^{(m')})^{\wedge}$ . On the other hand, we have  $\beta_n : (D_{S,n}^{(m')})^{\wedge} \xrightarrow{\sim} E_n^{(m')}$ . Indeed, by Lemma 2.3,  $D_{S,n}^{(m')} \otimes R_i \cong \Gamma(\mathscr{U}, \mathscr{E}_{X_i,n}^{(m')})$ , and since  $E_n^{(m')}$  is  $\pi$ -adically complete, the isomorphism follows. Thus, we obtain

$$\varinjlim_{n} (\alpha_n \circ \beta_n^{-1}) \colon E^{(m')} \to E_{\mathbb{Q}}^{(m)},$$

which is what we are looking for.

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Finally, let us check that  $\psi_{m,m'}$  is strictly injective. By construction, locally,  $\operatorname{gr}_n(\psi_{m,m'})$  coincides with the *p*-isogeny of Lemma 3.6 for any *n*. This implies that the canonical homomorphism  $\operatorname{gr}(\mathscr{E}_{\mathscr{X}}^{(m')}) \to \operatorname{gr}(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m)})$  is injective. Since  $\mathscr{E}_{\mathscr{X}}^{(m')}$  is separated with respect to the filtration by order, we get the strict injectivity by [18, Ch. I, 4.2.4 (5)].

**4.4.** — We preserve the notation. For non-negative integers  $m' \ge m$ , we define a sheaf of rings

$$\mathscr{E}_{\mathscr{X}}^{(m,m')} := \psi_{m,m'}^{-1}(\mathscr{E}_{\mathscr{X}}^{(m)}) \cap \mathscr{E}_{\mathscr{X}}^{(m')},$$

where the intersection is taken in  $\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m')}$ . By definition,  $\mathscr{E}_{\mathscr{X}}^{(m,m)} = \mathscr{E}_{\mathscr{X}}^{(m)}$ . We denote  $\mathscr{E}_{\mathscr{X}}^{(m,m')} \otimes R_i$  by  $\mathscr{E}_{X_i}^{(m,m')}$ . Let  $\mathscr{U}$  be an open subset of  $T^*\mathscr{X}$ . Then the left exactness of the functor  $\Gamma$  implies that

$$\Gamma(\mathscr{U},\mathscr{E}_{\mathscr{X}}^{(m,m')}) \cong \psi_{m,m'}^{-1}(E^{(m)}) \cap E^{(m')} \subset E_{\mathbb{Q}}^{(m')}$$

using the notation of 4.2.

Since  $\psi_{m,m'}(\mathscr{E}_{\mathscr{X}}^{(m')})$  and  $\mathscr{E}_{\mathscr{X}}^{(m)}$  are  $\operatorname{sub} \pi^{-1}\mathscr{D}_{\mathscr{X}}^{(m)}$ -algebras of  $\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m)}$ , the ring  $\mathscr{E}_{\mathscr{X}}^{(m,m')}$  is also a  $\pi^{-1}\mathscr{D}_{\mathscr{X}}^{(m)}$ -algebra on  $T^*\mathscr{X}$ . Moreover, by putting

$$\mathscr{E}_{\mathscr{X},n}^{(m,m')} := \psi_{m,m'}^{-1}(\mathscr{E}_{\mathscr{X},n}^{(m)}) \cap \mathscr{E}_{\mathscr{X},n}^{(m')},$$

we may equip  $\mathscr{E}_{\mathscr{X}}^{(m,m')}$  with a filtration, and we consider  $\mathscr{E}_{\mathscr{X}}^{(m,m')}$  as a filtered ring. By Lemma 4.3,  $\psi_{m,m'}$  is a strict homomorphism, and the canonical homomorphisms of filtered rings

$$(\mathscr{E}_{\mathscr{X}}^{(m,m')},\mathscr{E}_{\mathscr{X},n}^{(m,m')}) \to (\mathscr{E}_{\mathscr{X}}^{(m)},\mathscr{E}_{\mathscr{X},n}^{(m)}), \qquad (\mathscr{E}_{\mathscr{X}}^{(m,m')},\mathscr{E}_{\mathscr{X},n}^{(m,m')}) \to (\mathscr{E}_{\mathscr{X}}^{(m')},\mathscr{E}_{\mathscr{X},n}^{(m')})$$

are also strict injective homomorphisms. By the explicit presentation Lemma 2.10 and Lemma 3.4 (i)-(b),  $\psi_{m,m'}(\mathscr{E}_{\mathscr{X},n}^{(m')}) \subset \mathscr{E}_{\mathscr{X},n}^{(m)}$  for  $n < p^{m+1}$ , and in particular

(4.4.1) 
$$\mathscr{E}_{\mathscr{X},0}^{(m,m')} = \mathscr{E}_{\mathscr{X},0}^{(m')}.$$

**4.5. Lemma.** — Assume  $\mathscr{X}$  to be affine, and let  $\Theta$  be a homogeneous section of  $\mathcal{O}_{T^*\mathscr{X}}$ . We put  $\mathscr{U} := D(\Theta)$ .

- 1. For non-negative integers  $M' \geq M \geq m' \geq m$ ,  $(\Theta^{(M,M')})^{-1} \in \Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X}}^{(m,m')}).$
- 2. For non-negative integers  $M' \ge M$  and  $M' \ge m' \ge m$ ,  $(\Theta^{(M,M')})^{-1} \in \Gamma(\mathscr{U}, \mathscr{E}^{(m,m')}_{\mathscr{X}, \mathbb{Q}})$ .

*Proof.* — For any integer m'' such that  $m'' \ge m' \ge m$ ,  $(\Theta^{(m',m'')})^{-1} \in \Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X}}^{(m)})$ . Indeed, there exist a non-negative integer n and a unit u of R such that  $\Theta^{(m',m'')} = u p^{-n} \cdot \Theta^{(m,m'')}$  by Lemma 2.11, and thus

$$(\Theta^{(m',m'')})^{-1} = u^{-1} p^n \cdot (\Theta^{(m,m'')})^{-1} \in \Gamma(\mathscr{U},\mathscr{E}_{\mathscr{X}}^{(m)}).$$

This yields the first claim.

In turn, for any integer m'' such that  $M' \geq m''$ , we can check that  $(\Theta^{(M,M')})^{-1} \in \Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X}, \mathbb{Q}}^{(m'')})$ , which implies the second claim.

**4.6.** — Let  $\iota: \mathscr{E}_{\mathscr{X}}^{(m)} \to \mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m)}$  be the canonical inclusion. Take an integer *n*. Consider the following commutative diagram (4.6.1)

whose rows are exact sequences.

**Lemma**. — Let inc:  $(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m)})_{n-1} \to (\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m)})_n$  be the canonical inclusion. The following sequence is exact:

$$0 \to \operatorname{Im}(\alpha_{n-1}) \to (\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m)})_{n-1} \oplus \operatorname{Im}(\alpha_n) \to (\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m)})_n,$$

where the second (resp. last) homomorphism is induced by  $(\alpha_{n-1}, inc)$  (resp.  $inc - \alpha_n$ ).

*Proof.* — Since the statement is local, we may suppose that  $\mathscr{X}$  is affine and possesses a system of local coordinates. Moreover, it suffices to show the exactness for the modules of sections over  $\mathscr{U} := D(\Theta)$  where  $\Theta$  is a homogeneous section of  $\Gamma(T^*\mathscr{X}, \Theta_{T^*\mathscr{X}})$ . We use the notation of 4.2. We also denote  $\psi_{m,m'}$  by  $\psi$  and  $(\psi_{m,m'})_n$  by  $(\psi)_n$  for short.

An operator P of  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$  is said to be *homogeneous of degree* l if we can write  $P = \sum_{|\underline{k}|=l} a_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}$  with  $a_{\underline{k}} \in \mathscr{O}_{\mathscr{X},\mathbb{Q}}$ . Take  $S \in \mathrm{Im}((\psi)_n, \iota_n)$ . Using a left presentation, we can write

$$S := \psi \left( \sum_{\substack{k \le n \\ i \in \mathbb{Z}}} P_{k,i}(\Theta^{(m')})^i \right) + \sum_{\substack{k \le n \\ i \in \mathbb{Z}}} Q_{k,i}(\Theta^{(m,m')})^i$$

where the first sum is an element of  $E_n^{(m')}$  and the second sum is one of  $E_n^{(m)}$ , and  $P_{k,i}, Q_{k,i}$  are homogeneous operators of degree  $k - ip^{m'} \operatorname{ord}(\Theta)$ 

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in  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m')}$  and  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$  respectively with some convergence conditions. Suppose  $S \in (E_{\mathbb{Q}}^{(m)})_{n-1}$ . We need to show that this element is contained in  $\operatorname{Im}(\alpha_{n-1})$ . Since  $\lim_{i\to\pm\infty} P_{n,i} = 0$  in  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m')}$  and by Corollary 3.8, there exists a *finite* subset  $I \subset \mathbb{Z}$  such that

$$\psi\left(\sum_{i\notin I} P_{n,i}(\Theta^{(m')})^i\right) \in E_n^{(m)}.$$

This is in fact contained in  $E_n^{(m)} \cap \psi(E_n^{(m')})$ . Then for  $N \gg 0$ , there exists  $a_{\underline{k}} \in \Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$  for  $|\underline{k}| = n + Np^{m'} \operatorname{ord}(\Theta) =: M$  and  $\underline{k} \geq \underline{0}$  such that

$$\sum_{i \in I} P_{n,i}(\Theta^{(m')})^i \in \sum_{|\underline{k}|=M} a_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m')}} (\Theta^{(m')})^{-N} + E_{n-1}^{(m')} + \left( \psi^{-1}(E_n^{(m)}) \cap E_n^{(m')} \right)$$

and  $a_{\underline{k}}\underline{\partial}^{\langle \underline{k} \rangle_{(m')}}(\Theta^{(m')})^{-N} \notin \psi^{-1}(E^{(m)})$  for any  $|\underline{k}| = M$  and  $\underline{k} \geq \underline{0}$  such that  $a_{\underline{k}} \neq \overline{0}$ . By the same argument for  $E^{(m)}$  and increasing N if necessary, we may also suppose that there exists  $b_{\underline{k}} \in \Gamma(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$  for  $|\underline{k}| = M$  such that  $b_{\underline{k}}\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}(\Theta^{(m,m')})^{-N} \notin E^{(m')}$  for any  $b_{\underline{k}} \neq 0$ , and

$$\sum_{i} Q_{n,i}(\Theta^{(m,m')})^i \in \sum_{|\underline{k}|=M} b_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} (\Theta^{(m,m')})^{-N} + E_{n-1}^{(m)} + (\psi(E_n^{(m')}) \cap E_n^{(m)})$$

in 
$$E_{\mathbb{Q}}^{(m)}$$
. However, since  $S \in (E_{\mathbb{Q}}^{(m)})_{n-1}$ ,  
(4.6.2)  

$$\sum_{|\underline{k}|=M} a_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m')}} (\Theta^{(m')})^{-N} + b_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} (\Theta^{(m,m')})^{-N} \in (E_{\mathbb{Q}}^{(m)})_{n-1} + (\psi(E_{n}^{(m')}) \cap E_{n}^{(m)}).$$

The finite set  $\{\underline{\xi}^{\underline{k}} \cdot (\Theta^{(m')})^{-N}\}_{|\underline{k}|=M}$  in  $\mathcal{O}_{T^*\mathscr{X},\mathbb{Q}}(n)$  is linearly independent over  $\mathcal{O}_{\mathscr{X},\mathbb{Q}}$ . Thus, by the choice of  $a_{\underline{k}}$  and  $b_{\underline{k}}$ , (4.6.2) is possible only when  $a_{\underline{k}} = b_{\underline{k}} = 0$ , and the lemma is proven.

Corollary. — We have

$$\operatorname{gr}(\mathscr{E}_{\mathscr{X}}^{(m,m')}) = \operatorname{gr}(\psi_{m,m'})^{-1} \left( \operatorname{gr}(\mathscr{E}_{\mathscr{X}}^{(m)}) \right) \cap \operatorname{gr}(\mathscr{E}_{\mathscr{X}}^{(m')})$$

where the intersection is taken in the ring

$$\operatorname{gr}(\mathscr{E}^{(m)}_{\mathscr{X},\mathbb{Q}}) \cong \bigoplus_{i\in\mathbb{Z}} \mathcal{O}_{T^*\mathscr{X},\mathbb{Q}}(i) = \mathcal{O}_{T^*\mathscr{X},\mathbb{Q}}(*).$$

*Proof.* — The canonical homomorphism  $\operatorname{Im}(\alpha_n)/\operatorname{Im}(\alpha_{n-1}) \to (\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m)})_n/(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m)})_{n-1}$  is injective by the lemma above. Thus, the corollary follows by (4.6.1).

**4.7. Lemma.** — Let  $m' \geq m$  be non-negative integers. We have an inclusion  $\mathscr{E}_{\mathscr{X}}^{(m-1,m')} \subset \mathscr{E}_{\mathscr{X}}^{(m,m')}$  in  $\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m')}$ . Moreover, there exists a unique strict injective homomorphisms of filtered  $\pi^{-1}\mathscr{D}_{\mathscr{X}}^{(m-1)}$ -algebras  $\alpha \colon \mathscr{E}_{\mathscr{X}}^{(m,m'+1)} \to \mathscr{E}_{\mathscr{X}}^{(m,m')}$  such that  $\psi_{m,m'} \circ \alpha = \psi_{m,m'+1}$ .

 $\begin{array}{l} Proof. \label{eq:proof.} \mbox{Let us show the first claim. It suffices to show that $\psi_{m-1,m'}: $\mathscr{E}_{\mathscr{X}}^{(m-1,m')} \to \mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m-1)}$ factors through the composition $\mathscr{E}_{\mathscr{X}}^{(m,m')} \to \mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m)} \to \mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m-1)}$. We consider <math display="inline">\mathscr{E}_{\mathscr{X}}^{(m-1,m')}$ and <math display="inline">\mathscr{E}_{\mathscr{X}}^{(m,m')}$ as subrings of <math display="inline">\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m-1)}$ using these injections. We may assume $\mathscr{X}$ is affine, and let $\mathscr{U}:=D(\Theta)$ where $\Theta$ is a homogeneous section of $\Gamma(T^*\mathscr{X}, \Theta_{T^*}\mathscr{X})$. Let us use the notation of 4.2. It suffices to show that $E^{(m-1,m')}$ is contained in $E^{(m,m')}$. Take $P \in (E^{(m-1,m')})_N$ for some integer $N$. We inductively define $P_i \in (E^{(m-1,m')})_{N-i}$ and $Q_i \in (E^{(m,m')})_{N-i}$ such that $P_{i+1} = P_i - Q_i$ for $i \geq 0$. Put $P_0 := P$. Assume $P_i$ is constructed. We can write $P_i = 0$ and $P_i = 0$. The substitutes the proof of $P_i$ and $P$ 

$$\sigma(P_i) = \sum_{|\underline{k}|=N-i} a_{\underline{k}} \underline{\xi}^{\langle \underline{k} \rangle_{(m-1)}} (\Theta^{(m-1,m')})^{-n}$$

with  $a_{\underline{k}} \in \Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$  and  $n \in \mathbb{Z}$ . By Corollary 4.6, this is contained in both  $\operatorname{gr}(E^{(m-1)})$  and  $\operatorname{gr}(E^{(m')})$ . Thus, Lemma 3.4 (ii) is shows that

$$Q_i := \sum_{|\underline{k}|=N-i} a_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m-1)}} (\Theta^{(m-1,m')})^{-n}$$

is in  $E^{(m,m')}$ . By construction,  $P_{i+1} := P_i - Q_i$  is contained in  $(E^{(m-1,m')})_{N-i-1}$ . The filtered ring  $E^{(m,m')}$  is complete by (4.4.1) and Lemma 2.5 (iv). Thus,  $P = \sum_{i \ge 0} Q_i \in E^{(m,m')}$ . The second claim can be checked similarly, and left to the reader.

**4.8. Lemma.** — For any  $n \in \mathbb{Z}$ ,  $\operatorname{gr}_n(\mathscr{E}_{\mathscr{X}}^{(m,m')})$  is a coherent  $\mathcal{O}_{T^{(m')*}\mathscr{X}}(0)$ -module on  $\mathring{T}^*\mathscr{X}$ . Moreover, on  $\mathring{T}^*\mathscr{X}$ ,  $\bigoplus_{i\geq 0} \operatorname{gr}_i(\mathscr{E}_{\mathscr{X}}^{(m,m')})$  and  $\operatorname{gr}(\mathscr{E}_{\mathscr{X}}^{(m,m')})$  are  $\mathcal{O}_{T^{(m')*}\mathscr{X}}(0)$ -algebras of finite type, and they are Noetherian.

*Proof.* — The modules  $\operatorname{gr}_n(\mathscr{E}_{\mathscr{X}}^{(m)})$  and  $\operatorname{gr}_n(\mathscr{E}_{\mathscr{X}}^{(m')})$  are coherent  $\mathscr{O}_{T^{(m')*}\mathscr{X}}(0)$ -modules on  $\mathring{T}^*\mathscr{X}$  by Corollary 3.8 and (4.3.1). Since

$$\operatorname{gr}_n(\mathscr{E}^{(m)}_{\mathscr{X},\mathbb{Q}}) \cong \mathcal{O}_{T^{(m)*}\mathscr{X},\mathbb{Q}}(n) = \sum_{i\geq 0} p^{-i} \mathcal{O}_{T^{(m)*}\mathscr{X}}(n),$$

there exists an integer  $i_0$  such that  $\operatorname{gr}_n(\mathscr{E}_{\mathscr{X}}^{(m)}) + \operatorname{gr}_n(\mathscr{E}_{\mathscr{X}}^{(m')}) \subset p^{-i_0} \mathcal{O}_{T^{(m)*}\mathscr{X}}(n)$ . Since the intersection of two coherent modules in a coherent module is coherent,  $\operatorname{gr}_n(\mathscr{E}_{\mathscr{X}}^{(m)}) \cap \operatorname{gr}_n(\mathscr{E}_{\mathscr{X}}^{(m')})$  is also a coherent  $\mathcal{O}_{T^{(m')*}\mathscr{X}}(0)$ -module, and  $\operatorname{gr}_n(\mathscr{E}_{\mathscr{X}}^{(m,m')})$  is coherent as well by Corollary 4.6.

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**Claim.** — Let  $\mathscr{A} = \bigoplus_{i \in \mathbb{Z}} \mathscr{A}_i$  be a finitely generated  $\mathscr{O} := \mathscr{O}_{\mathring{T}^{(m)*}\mathscr{X}}(0)$ -algebra such that  $\mathscr{A}_i$  are coherent  $\mathscr{O}$ -modules for all i. Let  $\mathscr{B} := \bigoplus_{i \in \mathbb{Z}} \mathscr{B}_i$  be a sub- $\mathscr{O}$ -algebra of  $\mathscr{A}$  such that  $\mathscr{B}_i \subset \mathscr{A}_i$  and  $\mathscr{B}_i$  are coherent  $\mathscr{O}$ -modules for all i. Assume  $\mathscr{A}$  is finite over  $\mathscr{B}$ . Then  $\mathscr{B}$  is a Noetherian ring and finitely generated over  $\mathscr{O}$ .

A proof is similar to that of Lemma 2.7 using [4, 7.8]. For any integer  $m'' \ge 0$ ,  $\bigoplus_{i\ge 0} \operatorname{gr}_i(\mathscr{E}_{\mathscr{X}}^{(m'')})$  is finitely generated  $\mathcal{O}_{T^{(m'')}*\mathscr{X}}(0)$ -algebras by Lemma 2.4,  $\mathcal{O}_{T^{(m)}*\mathscr{X}}(0)$  is a coherent  $\mathcal{O}_{T^{(m')}\mathscr{X}}(0)$ -module on  $\mathring{T}^*\mathscr{X}$  by Corollary 3.8, and  $\bigoplus_{i\ge 0} \operatorname{gr}_i(\mathscr{E}_{\mathscr{X}}^{(m)})$  is finite over  $\bigoplus_{i\ge 0} \operatorname{gr}_i(\mathscr{E}_{\mathscr{X}}^{(m,m')})$  by Lemma 3.4 (i)-(a). Using the claim, this shows that  $\bigoplus_{i\ge 0} \operatorname{gr}_i(\mathscr{E}_{\mathscr{X}}^{(m,m')})$  is an  $\mathcal{O}_{T^{(m')}*\mathscr{X}}(0)$ -algebra of finite type and Noetherian. Now, since  $\bigoplus_{i\le 0} \operatorname{gr}_i(\mathscr{E}_{\mathscr{X}}^{(m,m')}) \cong \bigoplus_{i\le 0} \operatorname{gr}_i(\mathscr{E}_{\mathscr{X}}^{(m')})$  is an  $\mathcal{O}_{T^{(m')}*\mathscr{X}}(0)$ -algebra of finite type,  $\operatorname{gr}(\mathscr{E}_{\mathscr{X}}^{(m,m')})$  is finitely generated as well. Applying the claim again to both  $\mathscr{C}$  and  $\mathscr{B}$  being equal to  $\operatorname{gr}(\mathscr{E}_{\mathscr{X}}^{(m,m')})$ , we conclude that the ring is Noetherian.  $\Box$ 

**4.9. Proposition**. — (i) The filtered ring  $(\mathscr{E}_{\mathscr{X}}^{(m,m')}, \mathscr{E}_{\mathscr{X},n}^{(m,m')})$  is complete.

(ii) The rings  $\mathscr{E}_{\mathscr{X},0}^{(m,m')}$  and  $\mathscr{E}_{\mathscr{X}}^{(m,m')}$  are Noetherian on  $\mathring{T}^*\mathscr{X}$ .

(iii) For open subsets  $\mathring{T}^* \mathscr{X} \supset \mathscr{U} \supset \mathscr{V}$  in  $\mathfrak{B}$ , the restriction homomorphism  $\Gamma(\mathscr{U}, \mathscr{E}) \rightarrow \Gamma(\mathscr{V}, \mathscr{E})$  is flat. Here  $\mathscr{E}$  denotes  $\mathscr{E}_{\mathscr{X}, 0}^{(m, m')}$  or  $\mathscr{E}_{\mathscr{X}}^{(m, m')}$ .

*Proof.* — We have already used (i) but we rewrite the statement because of the importance. This follows from the fact that  $\mathscr{E}_{\mathscr{X},n}^{(m,m')} = \mathscr{E}_{\mathscr{X},n}^{(m')}$  for any  $n < p^{m+1}$  by (4.4.1) and Lemma 2.5 (iv). We also get (ii) and (iii) for  $\mathscr{E} = \mathscr{E}_{\mathscr{X},0}^{(m,m')}$  by Proposition 2.8.

Let us prove (ii) for  $\mathscr{E}_{\mathscr{X}}^{(m,m')}$ . Let us check the conditions of Lemma 1.10 for the filtered ring  $(\mathscr{E}_{\mathscr{X}}^{(m,m')}, \mathscr{E}_{\mathscr{X},n}^{(m,m')})$ . The first condition is nothing but (i). The second and the third conditions follow from Lemma 2.7 together with Lemma 4.8. The last condition follows from Lemma 2.5 (ii) and Corollary 4.6. Hence  $\mathscr{E}_{\mathscr{X}}^{(m,m')}$  is a Noetherian ring. Thus, for any open subscheme  $\mathscr{U}$  in  $\mathfrak{B}$ ,  $\Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X}}^{(m,m')})$  is Noetherian, and (iii) holds by using [18, Ch. II, 1.2.1].  $\Box$ 

**Remark.** — By the proof, we can moreover say that  $\mathscr{E}_{\mathscr{X}}^{(m,m')}$  is pointwise Zariskian with respect to the filtration by order on  $\mathring{T}^*\mathscr{X}$ . This implies that  $\mathscr{E}_{X_i}^{(m,m')}$  is also pointwise Zariskian with respect to the filtration by order for any integer  $i \geq 0$  on  $\mathring{T}^*\mathscr{X}$ .

**4.10. Lemma.** — Let  $\mathscr{X}$  be an affine smooth formal scheme. Let  $\mathscr{U} \subset \mathring{T}^*\mathscr{X}$  be an open set in  $\mathfrak{B}$ , and  $\mathfrak{U}$  be a finite  $\mathfrak{B}$ -covering (i.e., a covering consisting of subsets in  $\mathfrak{B}$ ) of  $\mathscr{U}$ . Let  $\mathscr{E}$  be either  $\mathscr{E}_{\mathscr{X}}^{(m,m')}$  or  $\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$ . Then  $\check{H}^i_{\mathrm{aug}}(\mathfrak{U},\mathscr{E}) = 0$ for  $i \in \mathbb{Z}$ . Here  $\check{H}^i_{\mathrm{aug}}$  denotes the augmented Čech cohomology (cf. [9, 8.1.3]). In particular,  $H^1(\mathscr{U}, \mathscr{E}) = 0$ .

*Proof.* — Let *V*, *W* ∈ 𝔅. Since *ε*.(*V* ∩ *W*) = *ε*.(*V*) ∩ *ε*.(*W*), we may assume that *U* is strictly affine and 𝔅 is a finite strictly affine covering (cf. Definition 2.4). Let us show that  $\check{H}^i_{aug}(\mathfrak{U}, \mathscr{E}^{(m,m')}_{\mathscr{X},k}) = 0$  for any *k*. By Lemma 4.8,  $\mathscr{E}^{(m,m')}_{\mathscr{X},k} / \mathscr{E}^{(m,m')}_{\mathscr{X},-n}$  is a coherent  $\mathcal{O}_{T^{(m')*}\mathscr{X}}(0)$ -module. Thus,

$$\check{H}^i_{\mathrm{aug}}(\mathfrak{U}, \mathscr{E}^{(m,m')}_{\mathscr{X},k}/\mathscr{E}^{(m,m')}_{\mathscr{X},-n})=0$$

for  $i \in \mathbb{Z}$ . By the coherence, the projective system  $\left\{ \Gamma(\mathscr{V}, \mathscr{E}_{\mathscr{X},k}^{(m,m')} / \mathscr{E}_{\mathscr{X},-n}^{(m,m')}) \right\}_{n \geq 0}$  satisfies the Mittag-Leffler condition for any strictly affine open subset  $\mathscr{V}$ . This shows that  $C_{\text{aug}}^q(\mathfrak{U}, \mathscr{E}_{\mathscr{X},k}^{(m,m')} / \mathscr{E}_{\mathscr{X},-n}^{(m,m')})$  satisfies the Mittag-Leffler condition for any  $q \in \mathbb{Z}$ . Thus

$$\check{H}^{i}_{\mathrm{aug}}(\mathfrak{U},\mathscr{E}^{(m,m')}_{\mathscr{X},k}) \cong \check{H}^{i}_{\mathrm{aug}}(\mathfrak{U},\varprojlim \mathscr{E}^{(m,m')}_{\mathscr{X},k}/\mathscr{E}^{(m,m')}_{\mathscr{X},-n}) \cong \varprojlim \check{H}^{i}_{\mathrm{aug}}(\mathfrak{U},\mathscr{E}^{(m,m')}_{\mathscr{X},k}/\mathscr{E}^{(m,m')}_{\mathscr{X},-n}) = 0$$

for  $i \in \mathbb{Z}$ , where the first equality holds since  $\mathscr{E}_{\mathscr{X},k}^{(m,m')}$  is complete by Proposition 4.9 (i). Thus, we get what we wanted.

Now, since  $\mathscr{E}_{\mathscr{X}}^{(m,m')} \cong \lim_{k} \mathscr{E}_{\mathscr{X},k}^{(m,m')}$ , we have  $\check{H}_{\mathrm{aug}}^{i}(\mathfrak{U},\mathscr{E}_{\mathscr{X}}^{(m,m')}) = 0$  by using (2.5.2). The vanishing for  $\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$  follows immediately from the  $\mathscr{E} = \mathscr{E}_{\mathscr{X}}^{(m,m')}$  case. Finally, let us prove  $H^{1}(\mathscr{U},\mathscr{E}) = 0$ . We know that  $H^{1}(\mathscr{U},\mathscr{E}) \cong \varinjlim_{\mathfrak{V}} \check{H}^{1}(\mathfrak{V},\mathscr{E})$  where  $\mathfrak{V}$  runs over open coverings of  $\mathscr{U}$ . Given an open covering  $\mathfrak{V}$  of  $\mathscr{U}$ , there exists a refinement  $\mathfrak{V}'$  which is a  $\mathfrak{B}$ -covering. Thus the statement follows from the first claim.

**Remark.** — We may also prove that  $H^i(\mathcal{U}, \mathcal{E}) = 0$  for i > 0. This can be proven in the same way by using [16,  $0_{\text{III}}$  13.3.1].

4.11. — We define

$$\widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')} := \varprojlim_{i} \mathscr{E}_{X_{i}}^{(m,m')}, \qquad \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')} := \widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')} \otimes \mathbb{Q}.$$

These are  $\pi^{-1}\widehat{\mathscr{D}}_{\mathscr{X}}^{(m)}$ -algebras and the latter is moreover a  $\pi^{-1}\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -algebra. We call  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')}$  and  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$  the intermediate rings of microdifferential operators

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of level (m, m'). Let  $\mathscr{U}$  be an open subset of  $\mathring{T}^*\mathscr{X}$  in  $\mathfrak{B}$ . Applying Lemma 4.10 to the exact sequence

$$0 \to \mathscr{E}_{\mathscr{X}}^{(m,m')} \xrightarrow{\pi^{i+1}} \mathscr{E}_{\mathscr{X}}^{(m,m')} \to \mathscr{E}_{X_i}^{(m,m')} \to 0,$$

we get an isomorphism

(4.11.1) 
$$\Gamma(\mathscr{U}, \mathscr{E}_{X_i}^{(m,m')}) \cong \Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X}}^{(m,m')}) \otimes R_i$$

and by taking the projective limit over i, we have  $\Gamma(\mathscr{U}, \widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')}) \cong \Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X}}^{(m,m')})^{\wedge}$ , where  $^{\wedge}$  denotes the  $\pi$ -adic completion. By Lemma 2.6,  $\Gamma(\mathscr{U}, \widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')}) \otimes R_i \cong \Gamma(\mathscr{U}, \mathscr{E}_{X_i}^{(m,m')})$ , and thus  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')} \otimes R_i \cong \mathscr{E}_{X_i}^{(m,m')}$ , in particular,  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')}$  is  $\pi$ -adically complete. We also define

$$\mathscr{E}^{(m,\dagger)}_{\mathscr{X},\mathbb{Q}} := \lim_{m' \to \infty} \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}.$$

This is a ring on  $T^*\mathscr{X}$ . Note that there exists a canonical homomorphism  $\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m,\dagger)} \to \mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m+1,\dagger)}$  of rings by Lemma 4.7. We define

$$\mathscr{E}^{\dagger}_{\mathscr{X},\mathbb{Q}} := \varinjlim_{m \to \infty} \mathscr{E}^{(m,\dagger)}_{\mathscr{X},\mathbb{Q}}$$

For a quasi-compact open subscheme  $\mathscr{U}$  of  $\mathring{T}^*\mathscr{X}$ , (2.5.2) shows

$$\Gamma(\mathscr{U}, \mathscr{E}^{\dagger}_{\mathscr{X}, \mathbb{Q}}) \cong \varinjlim_{m' \to \infty} \Gamma(\mathscr{U}, \mathscr{E}^{(m', \dagger)}_{\mathscr{X}, \mathbb{Q}}).$$

**4.12. Proposition.** — The rings  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')}$ ,  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$  are Noetherian on  $\mathring{T}^*\mathscr{X}$ , and Proposition 4.9 (iii) and Lemma 4.10 are also valid if we take  $\mathscr{E}$  to be either  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')}$  or  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$ .

*Proof.* — It suffices to show the proposition for  $\mathscr{E} = \widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')}$ . First, let us check Lemma 4.10 for this  $\mathscr{E}$ . Since  $\mathscr{E}_{\mathscr{X}}^{(m,m')}$  is  $\pi$ -torsion free, we have  $\check{H}_{\mathrm{aug}}^{i}(\mathfrak{U}, \mathscr{E}_{X_{i}}^{(m,m')}) = 0$  by (4.11.1) and Lemma 4.10. The projective system  $\{\Gamma(\mathscr{V}, \mathscr{E}_{X_{i}}^{(m,m')})\}_{i\geq 0}$  satisfies the Mittag-Leffler condition for any  $\mathscr{V} \in \mathfrak{B}$  again by (4.11.1), and we conclude that

$$\check{H}^{i}_{\mathrm{aug}}(\mathfrak{U},\widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')}) \cong \check{H}^{i}_{\mathrm{aug}}(\mathfrak{U},\varprojlim_{i} \mathscr{E}_{X_{i}}^{(m,m')}) \cong \varprojlim_{i} \check{H}^{i}_{\mathrm{aug}}(\mathfrak{U},\mathscr{E}_{X_{i}}^{(m,m')}) = 0.$$

To prove that  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')}$  is Noetherian, we check the conditions of Lemma 1.10 for the  $\pi$ -adic filtration. Conditions 2 and 3 follow from the fact that  $\mathscr{E}_{\mathscr{X}}^{(m,m')}$ 

is Noetherian by Proposition 4.9. Condition 4 follows from (4.11.1). Proposition 4.9 (iii) for  $\mathscr{E} = \widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')}$  follows directly from the  $\mathscr{E} = \mathscr{E}_{\mathscr{X}}^{(m,m')}$  case by using [5, 3.2.3 (vii)], and we finish the proof.

**Remark**. — By the proof we can moreover say that  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')}$  is pointwise Zariskian with respect to the  $\pi$ -adic filtration on  $\mathring{T}^*\mathscr{X}$ .

**4.13. Lemma.** — For non-negative integers  $m' \geq m$ , the canonical homomorphism  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')} \to \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)}$  is injective, and the canonical homomorphism  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')} / \mathscr{E}_{\mathscr{X},0}^{(m,m')} \to \widehat{\mathscr{E}}_{\mathscr{X}}^{(m)} / \mathscr{E}_{\mathscr{X},0}^{(m)}$  is a p-isogeny.

*Proof.* — We omit subscripts  $\mathscr{X}$  (e.g.,  $\mathscr{E}_0^{(m)}$  instead of  $\mathscr{E}_{\mathscr{X},0}^{(m)}$ ). Consider the following diagram whose rows are exact:

Since the injective homomorphism  $\mathscr{E}^{(m,m')} \to \mathscr{E}^{(m)}$  is strict,  $\gamma$  is injective as well. Moreover, by Lemma 3.4 (i)-(a), there exists an integer *a* such that  $\operatorname{Coker}(\gamma)$  is killed by  $p^a$ . Since  $\mathscr{E}^{(m,m')}/\mathscr{E}_0^{(m,m')}$  is  $\pi$ -torsion free, this implies that  $\gamma$  is a *p*-isogeny.

Thus we get the following commutative diagram, whose rows are exact



where  $\wedge$  denotes the  $\pi$ -adic completion. By 3.5,  $\widehat{\gamma}$  is also a *p*-isogeny, and in particular, it is an isomorphism after tensoring with  $\mathbb{Q}$ . Since  $\mathscr{E}_0^{(m)}$  and  $\mathscr{E}_0^{(m,m')}$  are already complete with respect to the  $\pi$ -adic topology by Lemma 2.5 (i) and  $\mathscr{E}_0^{(m,m')} \to \mathscr{E}_0^{(m)}$  is injective, the left vertical homomorphism is injective as well.

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**4.14.** — Let the situation be as in 2.10. Let  $m'' \ge m' \ge m$  be non-negative integers. We have the following concrete description:

$$\begin{split} \mathrm{Im} & \Big( \Gamma(\mathscr{U}, \widehat{\mathscr{E}}_{\mathscr{X}, \mathbb{Q}}^{(m, m')}) \to \Gamma(\mathscr{U}, \widehat{\mathscr{E}}_{\mathscr{X}, \mathbb{Q}}^{(m)}) \Big) \\ &= \Bigg\{ \sum_{|\underline{k}| - inp^{m''} < 0} a_{\underline{k}, i} \underline{\widehat{\varrho}}^{\langle \underline{k} \rangle_{(m')}} (\widetilde{\Theta}_{\mathrm{le}}^{(m', m'')})^{-i} + \sum_{|\underline{k}| - inp^{m''} \ge 0} a_{\underline{k}, i} \underline{\widehat{\varrho}}^{\langle \underline{k} \rangle_{(m)}} (\widetilde{\Theta}_{\mathrm{le}}^{(m, m'')})^{-i} \,\Big| \, (*) \Bigg\}. \end{split}$$

(\*) Let  $\underline{k} \in \mathbb{N}^d$ ,  $i \geq 0$ ,  $a_{\underline{k},i} \in \Gamma(\mathscr{X}, \mathcal{O}_{\mathscr{X},\mathbb{Q}})$ . For an integer N, put  $\alpha_{N,i} := \sup_{|\underline{k}|=inp^{m''}+N} |a_{\underline{k},i}|$ . Then  $\lim_{i\to\infty} \alpha_{N,i} = 0$  for any N,  $\lim_{N\to\infty} \sup_i \{\alpha_{N,i}\} = 0$ , and there exists a real number C > 0 such that  $C > \sup_i \{\alpha_{N,i}\}$  for any N < 0.

Moreover, consider the situation of 2.12. The homomorphism  $\phi$  induces  $\Gamma(D(\Theta), \mathcal{O}_{T^{(m)*}\mathscr{X}}(*) \cap \mathcal{O}_{T^{(m')*}\mathscr{X}}(*)) \to \Gamma(D(\Theta), \widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')})$ , where the intersection is taken in  $\mathcal{O}_{T^*\mathscr{X},\mathbb{Q}}(*)$ . This homomorphism is abusively denoted by  $\phi$ . Let us denote by  $|\cdot|_{(i)}$  the norm of  $\mathcal{O}_{T^*\mathscr{X},\mathbb{Q}}(*)$  defined by the *p*-adic norm on  $\mathcal{O}_{T^{(i)*}\mathscr{X}}(*)$ . Then any element of the above image can be written *uniquely* as

(4.14.1) 
$$\sum_{k\in\mathbb{Z}}\phi(P_k)$$

where  $P_k \in \Gamma(\mathscr{U}, \mathcal{O}_{T^*\mathscr{X}, \mathbb{Q}}(k))$  such that  $\lim_{k \to \infty} |P_k|_{(m)} = 0$  and there exists a real number C such that  $|P_k|_{(m')} < C$  for any k < 0.

**4.15. Lemma.** — Let m' > m be non-negative integers. The homomorphism  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')} \to \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m+1,m')}$  is injective, and induces an isomorphism

$$\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}/\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m'+1)} \xrightarrow{\sim} \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m+1,m')}/\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m+1,m'+1)}.$$

*Proof.* — Let us check the injectivity. Since the verification is local, we may assume that we are in the situation of 2.10. Since the restriction homomorphisms of  $\mathcal{O}_{T^*\mathscr{X}}(*)$  are injective, those of  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$  are injective as well, and the verification is generic. Thus, we may assume that we are in the situation of 2.12, and we can use the concrete description (4.14.1). Since the description is unique, the injectivity follows.

For the second claim, it suffices to show that the canonical homomorphism

$$(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m,m')})_0/(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m,m'+1)})_0 \to \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}/\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m'+1)}$$

is an isomorphism. This follows from Lemma 4.13.

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**4.16. Lemma**. — Let  $m' > m \ge 0$  be integers. Then the canonical injection  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m+1,m'+1)} \to \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m+1,m')}$  induces the isomorphism:

$$\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m+1,m'+1)}/\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m'+1)} \xrightarrow{\sim} \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m+1,m')}/\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}.$$

*Proof.* — We have the following diagram whose rows are exact.

The first two vertical homomorphisms are isomorphisms by Lemma 4.13. Thus the right vertical homomorphism is an isomorphism as well, and the corollary follows.  $\hfill \square$ 

By Lemma 4.13 and Lemma 4.15, we have the following big diagram of rings of microdifferential operators.

## 5. Flatness results

**5.1.** — Let  $\mathscr{X}$  be an affine smooth formal scheme, and  $\mathscr{U}$  be a strictly affine open subscheme of  $\mathring{T}^*\mathscr{X}$ . Let  $\mathscr{E}$  be one of the rings  $\mathscr{E}_{\mathscr{X}}^{(m,m')}$ ,  $\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$ ,  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')}$ ,  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$ . Let M be a finite  $\Gamma(\mathscr{U}, \mathscr{E})$ -module. We use the terminologies in [9, 9.1, 9.2] freely. Let  $\mathfrak{T}$  be the Grothendieck topology (in the sense of [9, 9.1.1/1]) on  $\mathscr{U}$  defined in the following way.

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- A subset is said to be *admissible open* if it is strictly affine open subset of  $\mathscr{U}$ .
- A covering is called an *admissible covering* if it is open covering in the usual sense.

We define a presheaf  $M^{\triangle}$  on  $(\mathscr{U}, \mathfrak{T})$  by associating  $\Gamma(\mathscr{V}, \mathscr{E}) \otimes_{\Gamma(\mathscr{U}, \mathscr{E})} M$  with an strictly affine open subscheme  $\mathscr{V}$ .

**5.2. Lemma.** — For any finite  $\mathfrak{B}$ -covering  $\mathfrak{U}$  of  $\mathscr{U}$ ,  $\check{H}^{i}_{aug}(\mathfrak{U}, M^{\triangle}) = 0$  for  $i \in \mathbb{Z}$ .

*Proof.* — We just copy the proof of [9, 8.2.1/5] using Lemma 4.10 and Lemma 4.12.

**5.3. Corollary.** — (i) For any finite  $\Gamma(\mathcal{U}, \mathcal{E})$ -module M, the presheaf  $M^{\triangle}$  defines a sheaf on  $(\mathcal{U}, \mathfrak{T})$ , and the functor  $\stackrel{\triangle}{}$  is exact.

(ii) Let  $\mathscr{U}$  be a strictly affine open subscheme of  $\mathring{T}^*\mathscr{X}$ , and suppose there exists a finite presentation on  $\mathscr{U}$ :

$$(\mathscr{E}|_{\mathscr{U}})^{\oplus a} \xrightarrow{\phi} (\mathscr{E}|_{\mathscr{U}})^{\oplus b} \to \mathscr{M} \to 0.$$

Then we have a canonical isomorphism  $\Gamma(\mathscr{U},\mathscr{M})^{\bigtriangleup} \xrightarrow{\sim} \mathscr{M}$ .

*Proof.* — Let us prove (i). Lemma 5.2 shows that the presheaf  $M^{\triangle}$  is a sheaf. The functor  $^{\triangle}$  is exact since the restriction homomorphism  $\Gamma(\mathscr{V}, \mathscr{E}) \to \Gamma(\mathscr{W}, \mathscr{E})$ is flat where  $\mathscr{W} \subset \mathscr{V} \subset \mathscr{U}$  are strictly affine by Proposition 4.9 and 4.12. Let us show (ii). We put  $M := \operatorname{Coker}(\Gamma(\mathscr{U}, \phi))$ . Let  $E := \Gamma(\mathscr{U}, \mathscr{E})$ . By the definition of M, we have the following exact sequence

$$E^{\oplus a} \xrightarrow{\Gamma(\mathscr{U}, \phi)} E^{\oplus b} \to M \to 0.$$

Taking the exact functor  $\triangle$ , we have an isomorphism  $\mathscr{M} \cong M^{\triangle}$ . Taking the global sections,  $\Gamma(\mathscr{U}, \mathscr{M}) \cong M$ , and the claim follows.

**Remark.** — We did not prove that any coherent  $\mathscr{E}$ -module on  $\mathscr{U}$  can be written as  $M^{\Delta}$  with a finite  $\Gamma(\mathscr{U}, \mathscr{E})$ -module M. We do not go into the problem further in this paper. We believe, however, that for any coherent  $\mathscr{E}$ -module  $\mathscr{M}$  on a strict affine open subscheme  $\mathscr{U}$ , the canonical homomorphism  $\Gamma(\mathscr{U}, \mathscr{M})^{\Delta} \to \mathscr{M}$  is an isomorphism.

Let us use the notation of 1.4. We consider the induced topology from  $(T^*\mathscr{X})'$  on the underlying set of  $\mathscr{U}$ , and denote the topological space by  $\mathscr{U}'$ . We denote by  $\epsilon \colon \mathscr{U} \to \mathscr{U}'$  the continuous map induced by the identity. The topology of  $\mathscr{U}'$  is slightly finer (cf. [9, 9.1.2/1]) than  $\mathfrak{T}$ . Thus by [9, 9.2.3/1], the sheaf  $M^{\triangle}$  extends uniquely to a sheaf on  $\mathscr{U}'$ , denoted by  $(M^{\triangle})'$ . Now, we get the sheaf  $\epsilon^{-1}((M^{\triangle})')$ . We also denote this sheaf on the topological space  $\mathscr{U}$ 

by  $M^{\triangle}$ , and from now on,  $M^{\triangle}$  indicates the associated sheaf on  $\mathscr U$  unless otherwise stated.

**5.4.** — Let  $\mathscr{X}$  be an affine smooth formal scheme over R possessing a system of local coordinates. Take a homogeneous element  $\Theta$  in  $\Gamma(T^*\mathscr{X}, \Theta_{T^*\mathscr{X}})$  such that  $\deg(\Theta) > 0$ . Let  $\mathscr{U} := D(\Theta)$ . In the rest of this section, we use the notation of 4.2 freely.

Recall that we have the canonical injection

$$\rho_{m,m'} \colon \widehat{E}^{(m,m'+1)}_{\mathbb{Q}} \to \widehat{E}^{(m,m')}_{\mathbb{Q}}.$$

We define  $E^{[m,m']} := \rho_{m,m'}^{-1}(\widehat{E}^{(m,m')})$ , and equip it with *non-exhaustive* filtration by order. Since  $\widehat{E}^{(m,m')} \otimes \mathbb{Q} = \widehat{E}_{\mathbb{Q}}^{(m,m')}$ , we get  $E^{[m,m']} \otimes \mathbb{Q} \cong \widehat{E}_{\mathbb{Q}}^{(m,m'+1)}$ . See 5.7 for an account why we need to introduce this ring.

**Lemma**. — There exists a subring  $\mathbb{E} \subset E^{[m,m']}$  such that the following holds. We equip  $\mathbb{E}$  with the induced non-exhaustive filtration from  $E^{[m,m']}$ .

- 1. The ring  $\mathbb{E}$  contains  $\widehat{E}^{(m,m'+1)}$ , and the inclusion  $\mathbb{E} \subset E^{[m,m']}$  is a *p*-isogeny.
- 2. The ring  $gr(\mathbb{E})$  is finitely generated over  $gr(E^{(m,m'+1)})$ .
- 3. The ring  $\mathbb{E}$  and the Rees ring  $(\mathbb{E}_0)_{\bullet}$  of  $\mathbb{E}_0$  are two-sided Noetherian.

*Proof.* — In this proof, we denote  $\widehat{E}^{(m,m'+1)}$  by E for simplicity and consider the non-exhaustive filtration by order. By Lemma 4.5, we have  $(\Theta^{(m',m'+1)})^{-1} \in \widehat{E}^{(m,m')}$  and  $(\Theta^{(m',m'+1)})^{-1} \in \widehat{E}^{(m,m'+1)}_{\mathbb{Q}}$ . This shows that  $(\Theta^{(m',m'+1)})^{-1} \in E^{[m,m']}$ . We put  $\mathbb{E}$  to be the subring of  $E^{[m,m']}$  generated by E and  $\theta := (\Theta^{(m',m'+1)})^{-1}$ .

Let us show that the inclusion  $\mathbb{E} \hookrightarrow E^{[m,m']}$  is a *p*-isogeny. Since  $E/E_0 \to \widehat{E}^{(m,m')}/E_0^{(m,m')}$  is a *p*-isogeny by Lemma 4.13, it suffices to check that  $\mathbb{E}_0 \hookrightarrow E_0^{[m,m']}$  is a *p*-isogeny. Let  $a := a_{\mathrm{ord}(\theta)}$  of Lemma 3.4 (i). Let  $\partial^{\langle \underline{k} \rangle_{(m')}} (\Theta^{(m',m'+1)})^{-i}$  for  $\underline{k} \ge 0$  and  $i \ge 0$  be an operator in  $E^{[m,m']}$  whose order is less than or equal to 0. Then there exists an integer j > 0 such that the order of  $\partial^{\langle \underline{k} \rangle_{(m')}} (\Theta^{(m',m'+1)})^{-i+j}$  is strictly greater than  $\mathrm{ord}(\theta)$  and less than or equal to 0. By the choice of a, the operator  $p^a \cdot \partial^{\langle \underline{k} \rangle_{(m')}} (\Theta^{(m',m'+1)})^{-i+j}$  is in E, and thus

$$p^{a} \cdot \partial^{\langle \underline{k} \rangle_{(m')}} (\Theta^{(m',m'+1)})^{-i} \in E \cdot \theta^{j} \subset \mathbb{E}.$$

Take any P in  $E_0^{[m,m']}$ . There exists an integer b such that  $p^b \cdot P \in \widehat{E}^{(m,m'+1)}$ . Take a left presentation (2.10.3) of level m' + 1 such that  $p^b \cdot b_{k,i} \in \Gamma(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ 

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for any  $\underline{k}$  and i. For an integer M' and  $? \in \{\leq, >\}$ , we put

$$P_{?M'} := \sum_{N?M'} \sum_{|\underline{k}| - inp^{m'+1} = N} b_{\underline{k},i} \underline{\partial}^{\langle \underline{k} \rangle_{(m'+1)}} (\Theta^{(m'+1)})^{-i}.$$

Since  $(\Theta^{(m'+1)})^{-1} \cdot \Theta^{(m',m'+1)} \in pE$ , the operator  $(P_{\leq M} \cdot (\Theta^{(m',m'+1)})^b)$  where  $M = b \cdot \operatorname{ord}(\theta) \ (< 0)$  is contained in E. Thus,

$$p^a \cdot P = p^a \cdot P_{>M} + p^a \cdot P_{\leq M} \in \mathbb{E} + E \cdot \theta^b \subset \mathbb{E},$$

which implies that  $p^a \cdot E_0^{[m,m']} \subset \mathbb{E}_0$ , and the claim follows.

Let us check condition 2 and show that this  $\mathbb{E}$  is left Noetherian. We can show  $\mathbb{E}$  to be right Noetherian similarly. We define a filtration  $G_i$  for  $i \geq 0$  on  $\mathbb{E}$  in the following way: we put  $G_0(\mathbb{E}) := E$ . For i > 0, we inductively define  $G_{i+1}(\mathbb{E}) := E + G_i(\mathbb{E}) \cdot \theta$ . Let  $P \in E_l$  for some integer l. Since  $p^n \theta = u \cdot (\Theta^{(m'+1)})^{-1}$  by Lemma 2.11 where  $u \in \mathbb{Z}^*$  and n denotes the order of  $\Theta$ , we can write  $(p^n \theta)P = P(p^n \theta) + \sum_{k>1} P_k(p^n \theta)^k$  with  $P_k \in E_{(k-1)np^{m'+1}+l-1}$ , thus

(5.4.1) 
$$\theta \cdot P \in P \cdot \theta + E_{l-1} \cdot \theta.$$

This implies that condition 2 holds. Moreover, the filtration G is compatible with ring structure and exhaustive. Let us show that  $\operatorname{gr}^{G}(\mathbb{E})$  is Noetherian. Once this is shown, since G is positive,  $\mathbb{E}$  is Noetherian as well.

We put  $\overline{E} := E/p^n E$ . Let  $A := E \oplus \bigoplus_{k>0} \overline{E} \cdot T^k$  be a graded ring, whose graduation is defined by the degree of the indeterminate T, and the multiplication is defined by

$$T \cdot P = (\theta P \theta^{-1}) \cdot T \quad \in (P + E_{l-1}) \cdot T$$

for  $P \in E_l$  where the membership relation holds by (5.4.1). It is straightforward to check that this gives us a ring structure. We denote by  $A_i$  the homogeneous part of degree *i*. Since  $p^n \theta \in E$ , there exist the surjection  $\overline{E} \twoheadrightarrow \operatorname{gr}_i^G(\mathbb{E})$  for  $i \geq 1$ sending 1 to  $\theta^i$ , which defines the surjection of rings

$$A \twoheadrightarrow \operatorname{gr}^{G}(\mathbb{E}).$$

It suffices to show that A is Noetherian. For  $Q \in \overline{E}$ , we denote by  $\sigma(Q)$  the principal symbol in  $\operatorname{gr}(\overline{E})$  where the filtration is taken with respect to the filtration by order, and for  $Q' \in E$ , we denote by  $\sigma(Q')$  the principal symbol of the image of Q' in  $\overline{E}$ . Let us define a "symbol map"  $\Sigma \colon A \to \operatorname{gr}(\overline{E})$ . Let  $P \in A$ . Then we may write in a unique way  $P = \sum P_i$  where  $P_i \in A_i$ . Let k be the largest integer such that  $P_k \neq 0$ . We define  $\Sigma(P)$  to be  $\sigma(P_k) \in \operatorname{gr}(\overline{E})$ . Let I be a left ideal of A. We define

$$S := \left\{ \Sigma(P) \mid P \in I \right\} \subset \operatorname{gr}(\overline{E}).$$

Since, for  $P \in E_l$ , we have  $T \cdot P \in P \cdot T + \overline{E}_{l-1} \cdot T$  in  $\operatorname{gr}_1^G(A)$ , the set S is closed under multiplication by homogeneous elements of  $\operatorname{gr}(\overline{E})$ . Moreover, S is also closed under addition of two homogeneous elements with the same degree. Let  $I_S$  in  $\operatorname{gr}(\overline{E})$  be the ideal generated by S. By the above properties, we get that  $S = I_S \cap \bigcup_{i \in \mathbb{Z}} \operatorname{gr}_i(\overline{E})$ . Since  $\operatorname{gr}(\overline{E})$  is Noetherian, we can take homogeneous generators  $P'_1, \ldots, P'_k \in S$  of  $I_S$ . There exist elements  $P_1, \ldots, P_k$  of A such that  $\Sigma(P_i) = P'_i$  and the degrees of  $P_i$  are the same d > 0 for all i. For any  $P \in I$ , the completeness of  $\overline{E}$  with respect to the filtration by order implies that there exists  $R_i \in A$  such that  $P - \sum R_i P_i$  is degree less than d, and thus contained in  $\bigoplus_{i < d} A_i$ . Since  $\bigoplus_{i < d} A_i$  is finite over the Noetherian ring E, there exists  $Q_1, \ldots, Q_{k'}$  generate I, and in particular, I is finitely generated.

It remains to show that  $(\mathbb{E}_0)_{\bullet}$  is Noetherian. Although the proof is slightly more complicated, the idea is essentially the same. We define a filtration F on  $\bigoplus_{j\leq 0} \mathbb{E}\nu^j$ :  $F_0$  is equal to  $\bigoplus_{j\leq 0} E\nu^j$ , and we inductively define  $F_{i+1} := F_0 + F_i \cdot \theta$ , namely  $F_i = \bigoplus_{j\leq 0} G_i(\mathbb{E})\nu^j$ . We define the induced filtration on  $(\mathbb{E}_0)_{\bullet}$  from  $\bigoplus_{j\leq 0} \mathbb{E}\nu^j$  also denoted by F. It suffices to show that  $\operatorname{gr}^F((\mathbb{E}_0)_{\bullet})$  is Noetherian. Let  $(\operatorname{gr}_i^G(\mathbb{E}))_j$  denotes the image of  $G_i(\mathbb{E}) \cap \mathbb{E}_j$ in  $\operatorname{gr}_i^G(\mathbb{E})$ , and we put  $N := np^{m'+1} = -\operatorname{ord}(\theta) > 0$ . Then, for i > 0, we get a surjection

$$\overline{E}_{Ni} \oplus \overline{E}_{Ni-1} \cdot \nu^{-1} \oplus \cdots \twoheadrightarrow \operatorname{gr}_{i}^{F}((\mathbb{E}_{0})_{\bullet}) \cong \bigoplus_{j \leq 0} (\operatorname{gr}_{i}^{G}(\mathbb{E}))_{j} \cdot \nu^{j}$$

sending  $P \cdot \nu^j$  with  $P \in \overline{E}_{Ni+j}$  to  $(P \cdot \theta^i) \cdot \nu^j$ . We define a ring graded both by the degree of T and  $\nu$  by

$$A' := \underbrace{(E_0 \oplus E_{-1} \cdot \nu^{-1} \oplus \dots)}_{=:A'_0} \oplus \bigoplus_{i>0} \underbrace{\left(\overline{E}_{Ni} \oplus \overline{E}_{Ni-1} \cdot \nu^{-1} \oplus \dots\right)}_{=:A'_i} T^i$$

and define the ring structure in the same way as before using (5.4.1). If we simply say degree, it means the degree of T. We denote by  $A'_i$  the part of degree *i*. Since there exists a surjection  $A' \to \operatorname{gr}^F((\mathbb{E}_0)_{\bullet})$ , it suffices to show that A' is Noetherian. Let  $\operatorname{gr}_{[j]}(\overline{E}) := \bigoplus_{i \leq j} \operatorname{gr}_i(\overline{E})$ . We define a *double graded* commutative ring

$$B := \bigoplus_{a,b \ge 0} \operatorname{gr}_{[Na-b]}(\overline{E}) \mu^a \nu^{-b},$$

whose ring structure is defined by the canonical homomorphism

$$\operatorname{gr}_{[Na-b]}(\overline{E}) \times \operatorname{gr}_{[Na'-b']}(\overline{E}) \to \operatorname{gr}_{[N(a+a')-(b+b')]}(\overline{E}).$$

We claim that this ring is Noetherian. For this, it suffices to show that B is finitely generated over  $\operatorname{gr}_0(\overline{E})$ . We know that  $\bigoplus_{i>0} \operatorname{gr}_i(\overline{E})$  and  $\bigoplus_{i<0} \operatorname{gr}_i(\overline{E})$  are

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finitely generated over  $\operatorname{gr}_0(\overline{E})$ . Then the following claim leads us to the desired conclusion.

**Claim.** — Let  $C = \bigoplus_{i \in \mathbb{Z}} C_i$  be a graded commutative ring. Assume that C is Noetherian, and that  $C_{\leq 0} := \bigoplus_{i < 0} C_i$  and  $C_{\geq 0} := \bigoplus_{i > 0} C_i$  are finitely generated over  $C_0$ . Let  $C_{[j]} := \bigoplus_{i>i} C_i$ . Then for any positive integer N, the ring  $D_N := \bigoplus_{i>0} C_{[N_j]} \nu^{N_j}$ , where  $\nu$  is an indeterminate, is also finitely generated over  $C_0$ . Moreover, the ring  $\bigoplus_{k,j>0} C_{[Nj-k]} \nu^j \mu^k$  is finitely generated over  $C_0$  where  $\nu$  and  $\mu$  are indeterminates.

*Proof.* — It suffices to check the N = 1 case. Indeed,  $D_1$  can be seen as a  $D_N$ -algebra, and  $D_1$  is integral over  $D_n$ . Thus if  $D_1$  is finitely generated over  $C_0$ ,  $D_N$  is also finitely generated over  $C_0$  by [4, 7.8].

It suffices to show that  $\bigoplus_{j\geq 0} \left( \bigoplus_{0\geq i} C_i \right) \nu^j \subset D_1$  and  $\bigoplus_{j\geq 0} \left( \bigoplus_{j\geq i\geq 0} C_i \right) \nu^j \subset D_1$ are finitely generated over  $C_0$ . Since the former one is isomorphic to  $C_{\leq 0}[\nu]$ , it is finitely generated. Let  $\{x_i\}_{i\in I}$  be a finite set of generators of  $C_{>0}\cong$  $\bigoplus_{i\geq 0} C_i \nu^i \subset D_1$  over  $C_0$ . Then the latter one is generated by  $\{x_i\}_{i\in I}$  and  $\nu$ , and the claim follows.  $\square$ 

For  $P \in A'$ , we can write  $P = \sum_i P_i$  with  $P_i \in A'_i$  in a unique way. Let s be the maximal integer such that  $P_s \neq 0$ . We denote  $P_s$  in  $A'_s$  by  $\tau(P)$ . Let  $\tau(P) = \sum_{0 \le i \le K} P_i \nu^{-i}$  with  $P_K \neq 0$ . We define  $\Sigma'(P) \in B$  to be  $\sigma(P_K)\mu^s \nu^{-K}$  where  $\sigma$  denotes the principal symbol with respect to the filtration by order of  $\overline{E}$ . Let I be an ideal of A', and we put  $S' := \{\Sigma'(P) \mid P \in I\} \subset B$ . This set is closed under addition of two elements with the same degree, and multiplication by homogeneous element. Take a finite set  $\{Q_i\}$  in I such that  $\{\Sigma'(Q_i)\}\$  is a set of generators of the ideal  $BS' \subset B$ . It is straightforward to check that the set  $\{Q_i\}$  generates I.

**5.5. Proposition**. — (i) The ring  $\widehat{E}_{\mathbb{Q}}^{[m,m']} := \widehat{E}^{[m,m']} \otimes \mathbb{Q}$  is Noetherian where  $\widehat{\cdot}$ indicates the  $\pi$ -adic completion.

- (ii) The canonical homomorphism α<sub>m,m'</sub>: Ê<sub>Q</sub><sup>(m,m'+1)</sup> → Ê<sub>Q</sub><sup>[m,m']</sup> is flat.
  (iii) The canonical homomorphism β<sub>m,m'</sub>: Ê<sub>Q</sub><sup>[m,m']</sup> → Ê<sub>Q</sub><sup>(m,m')</sup> is flat.

*Proof.* — We use the notation of Lemma 5.4. Since  $\mathbb{E}$  is Noetherian, the canonical homomorphism  $\mathbb{E} \to \widehat{\mathbb{E}}$  is flat and  $\widehat{\mathbb{E}}$  is Noetherian by [5, 3.2.3]. Since  $\mathbb{E}$ is p-isogeneous to  $E^{[m,m']}$ , they are also p-isogeneous even after taking  $\pi$ -adic completion by Lemma 3.5. Thus we get (i). Since  $E^{[m,m']} \otimes \mathbb{Q} \cong \widehat{E}^{(m,m'+1)}_{\mathbb{O}}$ , the flatness of  $\alpha_{m,m'}$  follows, which is (ii).

Let us prove (iii). We put  $\mathbb{E}_{\text{fin}} := \bigcup_n \mathbb{E}_n$ . By condition 2 of Lemma 5.4 and Lemma 4.8,  $\bigoplus_{i>0} \operatorname{gr}_i(\mathbb{E}_{\operatorname{fin}})$  is Noetherian. Since  $\mathbb{E}_0$  is a Noetherian filtered ring

with respect to the filtration by order by condition 3,  $\mathbb{E}_{\text{fin}}$  is also a Noetherian filtered ring by Lemma 1.12. Let  $\mathbb{E}'_{\text{fin}}$  be the completion of  $\mathbb{E}_{\text{fin}}$  with respect to the filtration. Then the canonical homomorphism  $\mathbb{E}_{\text{fin}} \to \mathbb{E}'_{\text{fin}}$  is flat and  $\mathbb{E}'_{\text{fin}}$  is Noetherian (cf. 1.1.6). Thus, by taking the  $\pi$ -adic completion, the canonical homomorphism  $\widehat{\mathbb{E}_{\text{fin}}} \to \widehat{\mathbb{E}'_{\text{fin}}}$  is flat by [5, 3.2.3 (vii)] where  $\wedge$  denote the  $\pi$ -adic completion. It suffices to show that

(5.5.1) 
$$\widehat{\mathbb{E}_{\mathrm{fin}}} \otimes \mathbb{Q} \cong \widehat{E}_{\mathbb{Q}}^{[m,m']}, \qquad \widehat{\mathbb{E}'_{\mathrm{fin}}} \otimes \mathbb{Q} \cong \widehat{E}_{\mathbb{Q}}^{(m,m')}.$$

Let  $E_{\text{fn}}^{[m,m']} := \bigcup_n E_n^{[m,m']}$ . Since  $\mathbb{E}_{\text{fn}} \subset E_{\text{fn}}^{[m,m']}$  is a *p*-isogeny and the  $\pi$ -adic completion of  $E_{\text{fn}}^{[m,m']}$  is  $\widehat{E}_{\mathbb{Q}}^{[m,m']}$ , we get the first isomorphism. Let us show the second one. Note that the completion of  $E_{\text{fn}}^{[m,m']}$  with respect to the filtration by order is  $E^{(m,m')}$ . There exists an integer *n* such that  $p^n E_{\text{fn}}^{[m,m']} \subset \mathbb{E}_{\text{fn}} \subset E_{\text{fn}}^{[m,m']}$ . Since these inclusions are strict homomorphisms, the inclusions are preserved even after taking the completion with respect to the filtration by order, and we get  $p^n E^{(m,m')} \subset \mathbb{E}'_{\text{fn}} \subset E^{(m,m')}$ . In particular, the inclusion  $\mathbb{E}'_{\text{fn}} \subset E^{(m,m')}$  is a *p*-isogeny, which implies the second isomorphism of (5.5.1), and the proposition follows.

**5.6. Lemma.** — Let m' > m. Put  $F^{(m')} := \iota(\widehat{E}_{\mathbb{Q}}^{(m,m')}) \cap \widehat{E}^{(m)}$  where  $\iota: \widehat{E}_{\mathbb{Q}}^{(m,m')} \to \widehat{E}_{\mathbb{Q}}^{(m)}$  is the canonical injection. Then, for any  $j \ge 0$ ,  $F_{X_j}^{(m')} := F^{(m')} \otimes R_j$  does not depend on m'.

 $\begin{array}{l} Proof. \ - \ \mbox{By definition, the canonical homomorphism $F_{X_j}^{(m')} \to E_{X_j}^{(m)} := Γ(𝔅, 𝔅_{X_j}^{(m)})$ is injective. There exists $m'' ≥ m'$ such that $\Theta^{(m,m'')}$ is in the center of $D_{X_j}^{(m)}$. Let $LD_{X_j}^{(m)}$ be the subring of $E_{X_j}^{(m)}$ generated by $D_{X_j}^{(m)}$ and $(\Theta^{(m,m'')})^{-1}$, which does not depend on the choice of $m''$. (In fact, $LD_{X_j}^{(m)} := Γ(𝔅, 𝔅𝔅_{X_j}^{(m)})$ using the notation of Remark 2.16.) It suffices to show that $LD_{X_j}^{(m)} = F_{X_j}^{(m')}$ in $E_{X_j}^{(m)}$. Since $(\Theta^{(m,m'')})^{-1} ∈ F^{(m')}$, we have $LD_{X_j}^{(m)} ⊂ F_{X_j}^{(m')}$. Let us show the opposite inclusion. By Remark 4.14, any element of $F^{(m')}$ can be written as$ 

$$\underbrace{\sum_{k<0} P_{k,i} \cdot (\Theta^{(m',m'')})^{-i}}_{\textcircled{1}} + \underbrace{\sum_{k\geq0} P_{k,i} \cdot (\Theta^{(m,m'')})^{-i}}_{\textcircled{2}}$$

where  $P_{k,i} \in (D_{\mathbb{Q}}^{(m)})_{k+np^{m''}i}$   $(n := \operatorname{ord}(\Theta))$  with some convergence conditions. The sum 2 becomes finite in  $E_{X_j}^{(m)}$  since  $\lim_{k\to\infty} P_{k,i} = 0$ . Since

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 $(\Theta^{(m',m'')})^{-1} \equiv 0 \mod p \widehat{E}^{(m)}, \text{ (1) is also a finite sum in } E_{X_j}^{(m)}, \text{ and we have } F_{X_j}^{(m')} \subset LD_{X_j}^{(m)}.$ 

**Corollary**. — The image of the homomorphism  $\widehat{E}_{\mathbb{Q}}^{(m,m'+2)} \to \widehat{E}_{\mathbb{Q}}^{[m,m']}$  is dense with respect to the  $\pi$ -adic topology on  $\widehat{E}_{\mathbb{Q}}^{[m,m']}$  for any  $m' \geq m$ .

*Proof.* — Set (m, m') of Lemma 5.6 to be (m', m' + 1). Then  $F^{(m'+1)} = E^{[m',m']}$ , and the lemma implies that the image of the homomorphism  $\widehat{E}_{\mathbb{Q}}^{(m',m'+2)} \to \widehat{E}_{\mathbb{Q}}^{[m',m']}$  is dense. This shows that the image of the homomorphism  $(E_{\mathbb{Q}}^{(m',m'+2)})_0 \to (E_{\mathbb{Q}}^{[m',m']})_0$  is also dense, and so is the image of the composition

$$(E_{\mathbb{Q}}^{(m,m'+2)})_{0} \xrightarrow{\sim} (E_{\mathbb{Q}}^{(m',m'+2)})_{0} \to (\widehat{E}_{\mathbb{Q}}^{[m',m']})_{0} \xleftarrow{\sim} (\widehat{E}_{\mathbb{Q}}^{[m,m']})_{0}$$

which is nothing but the canonical homomorphism  $(E_{\mathbb{Q}}^{(m,m'+2)})_0 \to (\widehat{E}_{\mathbb{Q}}^{[m,m']})_0$ . Since

$$\widehat{E}_{\mathbb{Q}}^{(m,m'+2)}/(E_{\mathbb{Q}}^{(m,m'+2)})_{0} \xrightarrow{\sim} \widehat{E}_{\mathbb{Q}}^{[m,m']}/(\widehat{E}_{\mathbb{Q}}^{[m,m']})_{0},$$
  
e proof.  $\Box$ 

we conclude the proof.

5.7. — We recall the definition of Fréchet-Stein algebra. For more details, we refer to [25]. A K-algebra A together with a projective system of K-Banach algebras  $\{A_i\}_{i\geq 0}$  and a homomorphism of projective systems  $A \to \{A_i\}$  where A denotes the constant projective system is called a *Fréchet-Stein algebra* (cf. [25, § 3]) if the following hold.

- 1. For any  $i \ge 0$ , the ring  $A_i$  is Noetherian.
- 2. The transition homomorphism  $A_{i+1} \to A_i$  is flat and the image is dense in  $A_i$ .
- 3. The given homomorphism of projective systems induces an isomorphism of K-algebras  $A \to \varprojlim_i A_i$ .

In general, the image of the homomorphism  $\widehat{E}_{\mathbb{Q}}^{(m,m'+1)} \to \widehat{E}_{\mathbb{Q}}^{(m,m')}$  is not dense, and the projective system  $\{\widehat{E}_{\mathbb{Q}}^{(m,m')}\}_{m' \geq m}$  does not give a Fréchet-Stein structure on  $E_{\mathbb{Q}}^{(m,\dagger)}$ . We need to replace  $\widehat{E}_{\mathbb{Q}}^{(m,m')}$  by  $\widehat{E}_{\mathbb{Q}}^{[m,m']}$  to get such a structure as the following theorem shows.

**5.8. Theorem.** — (i) The ring  $E_{\mathbb{Q}}^{(m,\dagger)}$  is a Fréchet-Stein algebra with respect to the projective system  $\{\widehat{E}_{\mathbb{Q}}^{[m,m']}\}_{m' \geq m}$ .

(ii) For a finitely presented  $E_{\mathbb{Q}}^{(m,\dagger)}\mbox{-module }M,$  we have

$$R^{i} \underset{m'}{\varprojlim} (\widehat{E}_{\mathbb{Q}}^{(m,m')} \otimes_{E_{\mathbb{Q}}^{(m,\dagger)}} M) \xleftarrow{\sim} \begin{cases} M & \text{if } i = 0\\ 0 & \text{if } i \neq 0. \end{cases}$$

(iii) Let  $\mathscr{V} \subset \mathscr{U}$  be strictly affine open subschemes of  $\mathring{T}^*\mathscr{X}$ . Then the homomorphism  $\Gamma(\mathscr{U}, \mathscr{E}_{\mathscr{X}, \mathbb{Q}}^{(m, \dagger)}) \to \Gamma(\mathscr{V}, \mathscr{E}_{\mathscr{X}, \mathbb{Q}}^{(m, \dagger)})$  is flat.

*Proof.* — For (i), combine Proposition 5.5 and Corollary 5.4. To check (ii), the projective systems  $\{\widehat{E}_{\mathbb{Q}}^{[m,m']} \otimes M\}_{m' \geq m}$  and  $\{\widehat{E}_{\mathbb{Q}}^{(m,m')} \otimes M\}_{m' \geq m}$  are cofinal in the projective system

$$\cdots \to \widehat{E}_{\mathbb{Q}}^{[m,m'+1]} \otimes M \to \widehat{E}_{\mathbb{Q}}^{(m,m'+1)} \otimes M \to \widehat{E}_{\mathbb{Q}}^{[m,m']} \otimes M \to \widehat{E}_{\mathbb{Q}}^{(m,m')} \otimes M \to \cdots$$

and these three projective systems have the same  $R^i \varinjlim_{m'}$ . Thus, [25, §3 Theorem] leads us to (ii). Let us prove (iii). Put  $E_* := \Gamma(*, \mathscr{E}_{\mathscr{X}, \mathbb{Q}}^{(m, \dagger)})$  for  $* \in \{\mathscr{U}, \mathscr{V}\}$ . Let  $0 \to I \to E_{\mathscr{U}} \to M \to 0$  be an exact sequence of  $E_{\mathscr{U}}$ -modules such that Iis a finitely generated ideal. By [10, I, §4 Proposition 1], it suffices to show that  $\operatorname{Tor}_1^{E_{\mathscr{U}}}(E_{\mathscr{V}}, M) = 0$ . Using [25, Corollary 3.4-i], since M is finitely presented, M is coadmissible, and thus I is coadmissible as well. By [25, Remark 3.2, §3 Theorem] and Proposition 4.12, the sequence

$$0 \to E_{\mathscr{V}} \otimes_{E_{\mathscr{U}}} I \to E_{\mathscr{V}} \to E_{\mathscr{V}} \otimes_{E_{\mathscr{U}}} M \to 0$$

is exact, and we get the vanishing of Tor.

**Remark.** — In (ii) of the theorem, we can more generally take M to be a coadmissible  $E_{\mathbb{Q}}^{(m,\dagger)}$ -module (cf. [25, § 3]).

**5.9. Corollary.** — Let  $\mathscr{U}$  be a strictly affine open subscheme of  $\mathring{T}^*\mathscr{X}$ , and M be a finitely presented  $\Gamma(\mathscr{U}, \mathscr{E}^{(m,\dagger)}_{\mathscr{X},\mathbb{Q}})$ -module. We define the presheaf  $M^{\bigtriangleup}$  in the same way as 5.1. Then Lemma 4.10, Lemma 5.2, Corollary 5.3 are also valid for  $\mathscr{E} = \mathscr{E}^{(m,\dagger)}_{\mathscr{X},\mathbb{Q}}, \mathscr{E}^{\dagger}_{\mathscr{X},\mathbb{Q}}$ , and M.

*Proof.* — Let us check the claim for  $\mathscr{E} = \mathscr{E}^{(m,\dagger)}_{\mathscr{X},\mathbb{Q}}$ . For any strictly open subscheme  $\mathscr{V} \subset \mathscr{U}$ ,

$$R^{i} \varprojlim_{m'} \Bigl( \Gamma(\mathscr{V}, \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X}, \mathbb{Q}}) \otimes M \Bigr) = 0$$

for i > 0 by Theorem 5.8 (ii). Let us denote by  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')} \otimes M$  the coherent  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}|_{\mathscr{U}}$ -module associated with M. By Lemma 5.2, this shows that the sequence

$$\cdots \to \varprojlim_{m'} C^q_{\mathrm{aug}}(\mathfrak{U}, \widehat{\mathscr{E}}_{\mathscr{X}, \mathbb{Q}}^{(m, m')} \otimes M) \to \varprojlim_{m'} C^{q+1}_{\mathrm{aug}}(\mathfrak{U}, \widehat{\mathscr{E}}_{\mathscr{X}, \mathbb{Q}}^{(m, m')} \otimes M) \to \cdots$$

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is exact. Since

$$\varprojlim_{m'} C^q_{\mathrm{aug}}(\mathfrak{U}, \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X}, \mathbb{Q}} \otimes M) \cong C^q_{\mathrm{aug}}(\mathfrak{U}, \mathscr{E}^{(m,\dagger)}_{\mathscr{X}, \mathbb{Q}} \otimes M)$$

by Theorem 5.8 (ii), Lemma 4.10 and Lemma 5.2 for this  $\mathscr{E}$  follows. The verification of Corollary 5.3 is similar. For the claims on  $\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{\dagger}$ , we only note that the functor  $\varinjlim$  is exact.

**5.10. Corollary.** — Let m' > m be non-negative integers. Then the canonical injection  $\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m+1,\dagger)} \to \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m+1,m')}$  induces the isomorphism:

$$\mathscr{E}^{(m+1,\dagger)}_{\mathscr{X},\mathbb{Q}}/\mathscr{E}^{(m,\dagger)}_{\mathscr{X},\mathbb{Q}}\xrightarrow{\sim} \widehat{\mathscr{E}}^{(m+1,m')}_{\mathscr{X},\mathbb{Q}}/\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}.$$

*Proof.* — It suffices to show that  $E_{\mathbb{Q}}^{(m+1,\dagger)}/E_{\mathbb{Q}}^{(m,\dagger)} \xrightarrow{\sim} \widehat{E}_{\mathbb{Q}}^{(m+1,m')}/\widehat{E}_{\mathbb{Q}}^{(m,m')}$  for any strictly affine open subscheme  $\mathscr{U}$ . This follows from Lemma 4.16 and the fact that  $R^1 \varprojlim_{m'} \widehat{E}_{\mathbb{Q}}^{(m,m')} = 0$  by Theorem 5.8. □

**5.11.** — Now, we argue the flatness of  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')} \to \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m+1,m')}$ .

**Lemma**. — The canonical homomorphism  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')} \to \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m+1,m')}$  is flat for non-negative integers m' > m.

*Proof.* — Since the verification is local, we may assume that we are in the situation of 2.10. It suffices to check that  $\widehat{E}_{\mathbb{Q}}^{(m,m')} \to \widehat{E}_{\mathbb{Q}}^{(m+1,m')}$  is flat. The proof being similar to [5, 3.5.3], we only sketch. Let F be the subring of  $\widehat{E}_{\mathbb{Q}}^{(m,m')}$  generated over  $\widehat{E}^{(m,m')}$  by  $\{\partial_i^{\langle p^{m+1}\rangle_{(m+1)}}\}_{1\leq i\leq d}$ . Since  $[P,\partial_i^{\langle p^{m+1}\rangle_{(m+1)}}] \in D^{(m)}$  for  $P \in D^{(m)}$ ,

$$[(\Theta^{(m)})^{-1}, \partial_i^{(p^{m+1})_{(m+1)}}] = (\Theta^{(m)})^{-1} \cdot [\partial_i^{(p^{m+1})_{(m+1)}}, \Theta^{(m)}] \cdot (\Theta^{(m)})^{-1} \in \widehat{E}^{(m)}.$$

Thus,  $[Q, \partial_i^{(p^{(m+1)})}] \in \widehat{E}^{(m,m')}$  for  $Q \in \widehat{E}^{(m,m')}$ . This shows that

$$F = \sum_{\underline{k}} \widehat{E}^{(m,m')} \cdot \left(\underline{\partial}^{\langle p^{m+1} \rangle_{(m+1)}}\right)^{\underline{k}}.$$

Now, define the filtration on F by the order of  $\underline{\partial}^{\langle p^{m+1}\rangle_{(m+1)}}$ . Then we have a surjection  $\widehat{E}^{(m,m')}[T_1,\ldots,T_d] \to \operatorname{gr}(F)$  sending  $T_i$  to  $\sigma(\partial_i^{\langle p^{m+1}\rangle_{(m+1)}})$ . Thus, F is Noetherian since  $\widehat{E}^{(m,m')}$  is. This implies that the homomorphism  $F \to F^{\wedge}$  is flat. Using Lemma 3.4 (i), we can check that  $F^{\wedge}$  is *p*-isogeneous to  $\widehat{E}_{\mathbb{Q}}^{(m+1,m')}$ , and the lemma follows.

5.12. — We sum up the results we got in this section as the following theorem.

**Theorem**. — Let  $m' \ge m$  be non-negative integers.

- 1. The canonical injective homomorphism  $\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m,\dagger)} \to \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$  is flat. 2. The canonical injective homomorphism  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m'+1)} \to \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$  is flat.
- 3. Let  $\mathscr{M}$  be a finitely presented  $\mathscr{E}_{\mathscr{X},\mathbb{O}}^{(m,\dagger)}$ -module. Then we get

$$\mathscr{M} \xrightarrow{\sim} \varprojlim_{m'} \widehat{\mathscr{E}}_{\mathscr{X}, \mathbb{Q}}^{(m, m')} \otimes_{\mathscr{E}_{\mathscr{X}, \mathbb{Q}}^{(m, \dagger)}} \mathscr{M}.$$

4. The canonical injective homomorphism  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')} \to \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m+1,m')}$  is flat.

*Proof.* — We restate what we have proven for 2 and 4. To check 1, it suffices to apply [25, Remark 3.2]. Let us prove 3. Since  $\mathcal{M}$  is finitely presented, there exists a strictly affine open subscheme  $\mathscr U$  such that there exists a presentation on which  $\mathcal{M}$  possesses a finite presentation. Then we apply Corollary 5.9. 

**Remark**. — We do not know if  $\widehat{\mathscr{C}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$  is flat over  $\pi^{-1}\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ . However, in the curve case, this is flat by [3, 1.3.4].

## 6. On finiteness of sheaves of rings

In this section, we introduce a finiteness property for modules on certain topological spaces, and prove some stationary type theorem. This finiteness is especially useful when we consider modules on formal schemes.

- **6.1.** First, let us introduce conditions on topological spaces and on sheaves. A ringed space  $(X, \mathcal{O}_X)$  is said to satisfy condition (FT) if the following two conditions hold.
  - 1. The topological space X is sober  $^{(5)}$  (i.e., any irreducible closed subset has a unique generic point, see [17, Exp. IV, 4.2.1]) and Noetherian (cf.  $[16, 0_I, \S 2.2]$ ).
  - 2. The structure sheaf  $\mathcal{O}_X$  is a coherent ring, and  $\mathcal{O}_{X,x}$  is Noetherian for any  $x \in X$ .

Let  $(X, \mathcal{O}_X)$  be a ringed space satisfying (FT), and let  $\mathcal{M}$  be a coherent  $\mathcal{O}_X$ -module. Let  $\mathfrak{Z} := \{Z_i\}_{i \in I}$  be a *finite* family of irreducible closed subsets. The module  $\mathcal M$  is said to satisfy condition (SH) with respect to  $\mathfrak Z$  if the following holds.

<sup>&</sup>lt;sup>(5)</sup> In this paper, we do not use the uniqueness of generic points, and this assumption is a little stronger than what is really needed.

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For any section  $s \in \Gamma(U, \mathcal{M})$  over any open subset U, there exists a subset  $I' \subset I$  such that  $\operatorname{Supp}(s) = \bigcup_{i \in I'} Z_i \cap U$ .

We simply say that  $\mathcal{M}$  satisfies condition (SH) if there exists a finite family  $\mathfrak{Z}$  such that  $\mathcal{M}$  satisfies (SH) with respect to  $\mathfrak{Z}$ .

**6.2. Lemma.** — Let  $(X, \mathcal{O}_X)$  be a ringed space satisfying (FT). Let  $\mathcal{M}$  be a coherent  $\mathcal{O}_X$ -module satisfying (SH) with respect to  $\mathfrak{Z} = \{Z_i\}_{i \in I}$ . Then for any sub- $\mathcal{O}_X$ -module  $\mathcal{K}$  of  $\mathcal{M}$ , there exists an open subset  $Z'_i$  of  $Z_i$  for each  $i \in I$  such that  $\operatorname{Supp}(\mathcal{K}) = \bigcup_{i \in I} Z'_i$ .

*Proof.* — Let U be an open subset of X, and take  $0 \neq s \in \Gamma(U, \mathcal{K})$ . Let  $\varphi_U \colon \Gamma(U, \mathcal{K}) \hookrightarrow \Gamma(U, \mathcal{M})$  be the inclusion. Since  $\varphi$  is injective,  $\operatorname{Supp}(s) = \operatorname{Supp}(\varphi_U(s))$ . There exists a subset  $I_s \subset I$  such that

$$\operatorname{Supp}(s) = \operatorname{Supp}(\varphi_U(s)) = \bigcup_{i \in I_s} Z_i \cap U$$

by (SH) of  $\mathcal{M}$ . Note that this is an open subset of  $\bigcup_{i \in I_s} Z_i$ . Let  $\mathfrak{S} := \bigcup_{U \subset X} \Gamma(U, \mathcal{K})$  where U runs over open subsets of X, and  $\mathfrak{S}_i$  be the subset of  $\mathfrak{S}$  consisting of the elements s such that  $i \in I_s$ . Now, we get

$$\operatorname{Supp}(\mathcal{K}) = \bigcup_{s \in \mathfrak{S}} \operatorname{Supp}(s) = \bigcup_{i \in I} \left( \bigcup_{s \in \mathfrak{S}_i} \operatorname{Supp}(s) \cap Z_i \right).$$

Since  $\operatorname{Supp}(s) \cap Z_i$  is open in  $Z_i$ , the set  $Z'_i := \bigcup_{s \in \mathfrak{S}_i} \operatorname{Supp}(s) \cap Z_i$  is also open in  $Z_i$ .

**6.3. Proposition.** — Let  $(X, \mathcal{O}_X)$  be a ringed space satisfying (FT). Let  $\mathcal{M}$  be a coherent  $\mathcal{O}_X$ -module, and assume that for any open subset  $U \subset X$ , (SH) holds for any coherent subquotient of  $\mathcal{M}|_U$ . Now, let

$$\mathcal{K}_1 \subset \mathcal{K}_2 \subset \mathcal{K}_3 \subset \dots \subset \mathcal{M}$$

be an ascending chain of sub- $\mathcal{O}_X$ -modules (not necessarily coherent) of  $\mathcal{M}$ . Then the chain is stationary.

*Proof.* — Let  $n \in \mathbb{N}$  and Z be a closed subset of X. We say that the chain is stationary for (n, Z) if  $\mathcal{K}_n|_{X\setminus Z} = \mathcal{K}_i|_{X\setminus Z}$  for any  $i \geq n$ . We claim that if the chain is stationary for (n, Z) with  $Z \neq \emptyset$ , then there exists an integer n'and  $Z' \subsetneq Z$  such that the chain is stationary for (n', Z'). Once this is proven, (i) follows since X is a Noetherian space.

Let us show the claim. By Lemma 6.2, there exists an integer a such that  $\overline{\operatorname{Supp}(\mathcal{K}_i)} = \overline{\operatorname{Supp}(\mathcal{K}_a)}$  for any  $i \geq a$ . We may suppose that  $Z \subset \overline{\operatorname{Supp}(\mathcal{K}_a)}$ . Take a generic point  $\eta$  of Z. Since  $\mathcal{O}_{X,\eta}$  is Noetherian, there exists  $n' \geq \max\{a, n\}$  such that  $\mathcal{K}_{i,\eta} = \mathcal{K}_{n',\eta}$  for any  $i \geq n'$ . Fix a set of generators  $\{f_1, \ldots, f_{\alpha}\}$  of  $\mathcal{K}_{n',\eta}$ . There exists an open neighborhood U of  $\eta$  such

that  $\{f_1, \ldots, f_\alpha\}$  can be lifted on U and  $U \cap Z$  is irreducible. We fix a set of liftings  $\{\tilde{f}_1, \ldots, \tilde{f}_\alpha\}$  in  $\Gamma(U, \mathcal{K}_{n'})$ . Let  $\mathscr{S}$  be the sub- $\mathcal{O}_X|_U$ -module of  $\mathcal{M}|_U$  generated by  $\{\tilde{f}_1, \ldots, \tilde{f}_\alpha\}$ , which is coherent since  $\mathcal{O}_X$  is. Now, let  $\mathcal{M}'_U := \mathcal{M}|_U/\mathscr{S}$ be a coherent  $\mathcal{O}_X|_U$ -module, and denote by  $\mathcal{K}'_i$  the image of  $\mathcal{K}_i|_U$  in  $\mathcal{M}'_U$ . We know that  $\operatorname{Supp}(\mathcal{K}_i) \cap U \supset \operatorname{Supp}(\mathcal{K}'_i)$ . By construction,  $\eta \notin \operatorname{Supp}(\mathcal{K}'_i)$ for any  $i \ge n'$ . By assumption,  $\mathcal{M}'_U$  also satisfies (SH). Let  $\mathfrak{W} := \{W_j\}_{j \in J}$  be a finite family of irreducible closed subset of U such that  $\mathcal{M}'_U$  satisfies (SH) with respect to  $\mathfrak{W}$ . Let J' be the subset of J such that  $\eta \notin W_j$ , and we put  $W' := \bigcup_{i \in J'} W_j$ . We let

$$Z' := (Z \cap W') \cup (Z \setminus U).$$

Since  $\eta \notin Z'$ , we get  $Z' \subsetneq Z$ . For any  $i \ge n'$ , by Lemma 6.2, there exists an open subset  $W'_i$  of  $W_j$  for each  $j \in J$  such that

$$\operatorname{Supp}(\mathcal{K}'_i) = \bigcup_{j \in J} W'_j.$$

We claim that  $W'_j \cap Z = \emptyset$  for any  $j \notin J'$ . Indeed,  $j \notin J'$  implies  $\eta \in W_j$  and  $Z \cap U \subset W_j$ . If  $W'_j \cap Z \neq \emptyset$ , we would get  $\eta \in W'_j$  since  $Z \cap U$  is irreducible closed and  $W'_j$  is open in  $W_j$ . This contradicts with  $\eta \notin \text{Supp}(\mathcal{K}'_j)$ . Thus,

(6.3.1)  $\operatorname{Supp}(\mathcal{K}'_i) \cap (Z \setminus W') \cap U = \emptyset.$ 

Now, the chain is stationary for (n', Z'): it suffices to check  $\mathcal{K}_{i,z} = \mathcal{K}_{n',z}$  for any  $z \in Z \setminus Z' = (Z \setminus W') \cap U$ . However, we get  $\mathcal{K}'_{i,z} = 0$  for any  $i \ge n'$  by (6.3.1). Thus,  $\mathcal{K}_{i,z} = \mathcal{J}_z$  by the definition of  $\mathcal{K}'_i$ , which concludes the proof.  $\Box$ 

**6.4.** — We show that coherent modules over some Noetherian rings we have defined in this paper satisfy (SH). For this, we prepare some lemmas. In the following, let  $(X, \mathcal{O}_X)$  be a ringed space satisfying (FT).

**Lemma.** — Condition (SH) is closed under extensions: suppose there exists an exact sequence of coherent  $\mathcal{O}_X$ -modules  $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$  such that  $\mathcal{F}'$  and  $\mathcal{F}''$  satisfy condition (SH). Then  $\mathcal{F}$  also satisfies condition (SH).

*Proof.* — Left to the reader.

**6.5. Lemma.** — Let  $\mathcal{M}$  be a coherent  $\mathcal{O}_X$ -module. The module  $\mathcal{M}$  satisfies condition (SH) if and only if there exists a covering  $\{U_i\}_{i \in I}$  of X such that  $\mathcal{M}|_{U_i}$  satisfies the condition on  $U_i$  for any i.

*Proof.* — We only need the proof for the "if" part. Since X is quasi-compact, we may assume that the covering is finite. By assumption, for each  $i \in I$ , there exists a family  $\{Z_j\}_{j \in J_i}$  of closed subsets of  $U_i$  such that  $\mathcal{M}|_{U_i}$  satisfies (SH)

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with respect to this family. The module  $\mathcal{M}$  satisfies (SH) with respect to the family  $\bigcup_{i \in I} \{\overline{Z_j}\}_{j \in J_i}$ . Since the verification is straightforward, we leave the details to the reader.

**6.6. Lemma.** — Let  $(\mathcal{F}, \mathcal{F}_i)$  be a separated filtered sheaf. Suppose that  $gr(\mathcal{F})$  satisfies (SH). Then  $\mathcal{F}$  also satisfies (SH).

*Proof.* — Assume that gr(𝔅) satisfies (SH) with respect to  $\mathfrak{Z} = \{Z_i\}_{i \in I}$ . Let *U* be an open subset of *X*, and take a non-zero *s* ∈ Γ(*U*, 𝔅). There exists an integer *i*<sub>0</sub> such that *s* ∈ Γ(*U*, 𝔅<sub>*i*<sub>0</sub></sub>) and *s* ∉ Γ(*U*, 𝔅<sub>*i*<sub>0</sub>-1</sub>) since the filtration is exhaustive (see Convention) and separated. Let  $\sigma(s) \in \Gamma(U, \operatorname{gr}_{i_0}(\mathscr{F})) \subset \Gamma(U, \operatorname{gr}(\mathscr{F}))$  be the principal symbol of *s*. Then there exists *J*<sub>0</sub> ⊂ *I* such that  $\operatorname{Supp}(\sigma(s)) = \bigcup_{j \in J_0} Z_j \cap U$ . For  $k \ge 0$ , we inductively define an open subset *U*<sub>k</sub>, and a subset *J*<sub>k</sub> of *J* in the following way. We put *U*<sub>0</sub> := *U*. Now, let  $U_{k+1} := U_k \setminus \operatorname{Supp}(\sigma(s|_{U_k}))$ . Then there exists *J*<sub>k+1</sub> such that

$$\operatorname{Supp}(\sigma(s|_{U_{k+1}})) = \bigcup_{j \in J'_{k+1}} Z_j \cap U_{k+1}.$$

We define  $J_{k+1} := J_k \cup J'_{k+1}$ . Obviously,  $J_0 \subset J_1 \subset \cdots \subset I$ . Since I is a finite set, this sequence is stationary. Let  $J := \bigcup_i J_i \subset I$ . Then  $\operatorname{Supp}(s) = \bigcup_{j \in J} Z_j \cap U$ , and  $\mathcal{F}$  satisfies (SH) with respect to  $\mathfrak{Z}$  as well.

**6.7. Definition.** — Let  $(X, \mathcal{O}_X)$  be a ringed space satisfying (FT). The ringed space or  $\mathcal{O}_X$  is said to be (SH)-*Noetherian* if condition (SH) is satisfied for any coherent  $\mathcal{O}_X|_U$ -modules and open subset U of X. If, moreover,  $\mathcal{O}_X$  is Noetherian with respect to  $\mathfrak{B}$ , we say that  $\mathcal{O}_X$  is strictly Noetherian with respect to  $\mathfrak{B}$ .

**6.8. Lemma.** — Let  $(X, \mathcal{E})$  be a ringed space satisfying (FT), and let  $(\mathcal{E}, \mathcal{E}_i)$  be a filtration on  $\mathcal{E}$ . Suppose that the filtration is pointwise Zariskian (cf. Definition 1.9). If  $gr(\mathcal{E})$  is (SH)-Noetherian, then so is  $\mathcal{E}$ .

*Proof.* — Let  $\mathcal{M}$  be a coherent  $\mathcal{E}$ -module. Since the verification is local by Lemma 6.5, we may suppose that there exists a good filtration  $(\mathcal{M}, \mathcal{M}_i)$ . By Lemma 1.13, the filtration is separated. Now by Lemma 6.6, the corollary follows.

**6.9. Lemma.** — Let  $\mathscr{A}$  be a strictly Noetherian sheaf with respect to  $\mathfrak{B}$  (resp. (SH)-Noetherian sheaf) on a topological space X, then so is  $\mathscr{A}[T]$ .

*Proof.* — It is easy to check that  $\mathscr{C}$  is Noetherian if and only if  $\mathscr{C}[T]$  is a Noetherian ring since  $\Gamma(U, \mathscr{C}[T]) \cong \Gamma(U, \mathscr{C})[T]$  for any open set U. Assume  $\mathscr{C}$  to be (SH)-Noetherian, and let  $\mathscr{M}$  be a coherent  $\mathscr{C}[T]$ -module. It suffices to

show that  $\mathcal{M}$  satisfies (SH). Since the verification is local by Lemma 6.5, we may assume that there exist integers  $a, b \geq 0$  and a presentation

$$\mathscr{C}[T]^{\oplus a} \xrightarrow{\psi} \mathscr{C}[T]^{\oplus b} \xrightarrow{\phi} \mathscr{M} \to 0$$

Let  $\mathscr{G}_n := \bigoplus_{i \leq n} \mathscr{U} \cdot T^i$  and  $\mathscr{K}'_n := \psi(\mathscr{U}_n)$ . Let  $\mathscr{K}_{m,n} := \mathscr{K}'_m \cap \mathscr{U}_n^{\oplus b}$  in  $\mathscr{U}[T]^{\oplus b}$ , which is a coherent sub- $\mathscr{U}$ -module of  $\mathscr{U}_n^{\oplus b} \subset \mathscr{U}[T]^{\oplus b}$ . Since  $\mathscr{U}$  is strictly Noetherian,  $\mathscr{K}_n := \bigcup_{m \geq 0} \mathscr{K}_{m,n} = \operatorname{Ker}(\phi) \cap \mathscr{U}_n$  is a coherent  $\mathscr{U}$ -module by Proposition 6.3. We define a coherent  $\mathscr{U}$ -module  $\mathscr{M}_n$  by  $\mathscr{U}_n^{\oplus b}/\mathscr{K}_n \subset \mathscr{M}$ . By construction,  $\bigcup_n \mathscr{M}_n = \mathscr{M}$ , and  $(\mathscr{M}, \{\mathscr{M}_n\}_{n \in \mathbb{Z}})$  is a filtered  $(\mathscr{U}[T], \{\mathscr{U}_n\}_{n \in \mathbb{Z}})$ -module.

It suffices to show that  $\operatorname{gr}(\mathcal{M})$  satisfies (SH) by Lemma 6.6. We have checked  $\operatorname{gr}_i(\mathcal{M})$  is a coherent  $\mathcal{C}$ -module for any *i*. By construction, the homomorphism  $T : \operatorname{gr}_i(\mathcal{M}) \to \operatorname{gr}_{i+1}(\mathcal{M})$  is surjective for any *i*, and we have the following sequence of surjections.

$$\mathcal{M}_0 = \operatorname{gr}_0(\mathcal{M}) \twoheadrightarrow \cdots \twoheadrightarrow \operatorname{gr}_i(\mathcal{M}) \twoheadrightarrow \operatorname{gr}_{i+1}(\mathcal{M}) \twoheadrightarrow \cdots$$

Since  $\mathcal{M}_0$  is coherent and  $\mathcal{C}$  is strictly Noetherian, this sequence is stationary, and there exists an integer N such that  $T^i : \operatorname{gr}_N(\mathcal{M}) \to \operatorname{gr}_{N+i}(\mathcal{M})$  is an isomorphism for any  $i \geq 0$ . Since  $\bigoplus_{0 \leq i \leq N} \operatorname{gr}_i(\mathcal{M})$  is a coherent  $\mathcal{C}$ -module, it satisfies (SH) with respect to a family  $\mathfrak{Z}$ . Then  $\operatorname{gr}(\mathcal{M})$  satisfies (SH) with respect to  $\mathfrak{Z}$ .

**6.10. Corollary.** — Let X be a topological space satisfying (FT), and  $\mathscr{A}$  be a strictly Noetherian R-module on X with respect to  $\mathfrak{B}$  (resp. (SH)-Noetherian). Then so is  $\mathscr{A} \otimes \mathbb{Q}$ .

*Proof.* — We only prove the strictly Noetherian case. It is easily verified that  $\mathscr{A} \otimes \mathbb{Q}$  is Noetherian with respect to 𝔅. Let  $\mathscr{A}_n := \operatorname{Ker}(\mathscr{A} \xrightarrow{p^n} \mathscr{A})$ . Since  $\mathscr{A}$  is strictly Noetherian, the sequence  $\mathscr{A}_0 \subset \mathscr{A}_1 \subset \cdots \subset \mathscr{A}$  is stationary. Let  $\mathscr{A}_\infty := \lim_{m \to \infty} \mathscr{A}_n$ . Since  $\mathscr{A}/\mathscr{A}_\infty$  is a coherent  $\mathscr{A}$ -algebra, it is strictly Noetherian, and we may assume that  $\mathscr{A}$  is a flat *R*-module in the sequel. Let  $F_i(\mathscr{A} \otimes \mathbb{Q}) := \pi^{-i} \mathscr{A}$  for  $i \geq 0$  and  $F_i(\mathscr{A} \otimes \mathbb{Q}) = 0$  for i < 0. Then it suffices to show that  $\operatorname{gr}^F(\mathscr{A} \otimes \mathbb{Q})$  is strictly Noetherian by Lemma 6.8. Since  $\operatorname{gr}^F(\mathscr{A} \otimes \mathbb{Q})$  is a coherent  $\mathscr{A}[T]$ -algebra where the action of *T* is the multiplication by  $\pi^{-1} \in \operatorname{gr}_1^F(\mathscr{A} \otimes \mathbb{Q})$ , it is reduced to showing that  $\mathscr{A}[T]$  is strictly Noetherian, which follows from Lemma 6.9. □

**6.11. Lemma.** — Let X be a Noetherian scheme, and let  $\mathfrak{B}$  be the open basis consisting of open affine subschemes of X. Then  $\mathcal{O}_X$  is strictly Noetherian with respect to  $\mathfrak{B}$ .

*Proof.* — Condition (FT) on  $(X, \mathcal{O}_X)$  is a basic property of Noetherian schemes. Let  $\mathcal{M}$  be a coherent  $\mathcal{O}_X$ -module, and let us check (SH) for this  $\mathcal{M}$ . Since the statement is local by Lemma 6.5, we may suppose that X = Spec(A)

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for a Noetherian ring A. There exists a finite decreasing filtration  $\{\mathcal{M}_i\}_{0 \leq i \leq n}$ on  $\mathcal{M}$  such that  $\mathcal{M}_0 = \mathcal{M}, \mathcal{M}_n = 0$ , the quotient  $\mathcal{M}_i/\mathcal{M}_{i+1}$  is irredundant for any  $0 \leq i < n$ , and

$$\operatorname{Ass}(\mathcal{M}_i/\mathcal{M}_{i+1}) \subset \operatorname{Ass}(\mathcal{M})$$

by [16, IV 3.2.8]. By Lemma 6.4, it suffices to show the lemma for irredundant modules, but in this case, it follows by definition.  $\Box$ 

Using Lemma 6.9, we have the following corollary.

**Corollary.** — Let X be a Noetherian scheme and  $\mathcal{A}$  be a quasi-coherent  $\mathcal{O}_X$ -algebra of finite type. Then  $\mathcal{A}$  is strictly Noetherian.

**6.12. Lemma.** — Let  $(Y, \mathcal{A})$  be a ringed space satisfying (FT), and assume that  $\mathcal{A}$  is strictly Noetherian with respect to  $\mathfrak{B}$ . Let X be a sober and Noetherian topological space, and  $f: X \to Y$  be an open continuous map of topological spaces. Let  $\mathfrak{C}$  be an open basis of X consisting of V such that  $f(V) \in \mathfrak{B}$ . Then the ringed space  $(X, f^{-1}\mathcal{A})$  is strictly Noetherian with respect to  $\mathfrak{C}$ .

Proof. — Let  $\mathcal{N}$  be a coherent  $\mathcal{A}$ -module satisfying (SH) with respect to  $\mathfrak{Z}$ . Then  $f^{-1}\mathcal{N}$  satisfies (SH) with respect to  $f^{-1}(\mathfrak{Z})$ . We claim that for any coherent  $f^{-1}\mathcal{A}$ -module  $\mathcal{M}$ , there exists a coherent  $\mathcal{A}$ -module  $\mathcal{N}$  such that  $\mathcal{M} \cong f^{-1}(\mathcal{N})$ . Now, the functor  $f_*$  is exact since f is open. Thus, for a coherent  $f^{-1}\mathcal{A}$ -module  $\mathcal{M}$ , the canonical homomorphism  $f^{-1}f_*\mathcal{M} \to \mathcal{M}$  is an isomorphism, and  $f_*\mathcal{M}|_{f(X)}$  is a coherent  $\mathcal{A}|_{f(X)}$ -module. Thus  $f^{-1}\mathcal{A}$  is (SH)-Noetherian. We leave the details to check that it is Noetherian with respect to  $\mathfrak{C}$ .

**6.13. Theorem.** — Let  $\mathscr{X}$  be a smooth formal scheme of finite type over  $\operatorname{Spf}(R)$ . Then  $\mathcal{O}_{X_i}$ ,  $\mathcal{O}_{\mathscr{X}}$ ,  $\mathcal{O}_{\mathscr{X},\mathbb{Q}}$ ,  $\mathscr{D}_{X_i}^{(m)}$ ,  $\mathscr{D}_{\mathscr{X},\mathbb{Q}}^{(m)}$ ,  $\mathscr{D}_{\mathscr{X},\mathbb{Q}}^{(m)}$ ,  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$  are strictly Noetherian sheaves on  $\mathscr{X}$ . Moreover,  $\mathscr{E}_{X_i}^{(m,m')}$ ,  $\mathscr{E}_{\mathscr{X}}^{(m,m')}$ ,  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$ ,  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$  are strictly Noetherian on  $\mathring{T}^*\mathscr{X}$ .

Proof. — Note first that  $\mathscr{X}$  and  $\mathring{T}^*\mathscr{X}$  are Noetherian spaces. The ring  $\mathscr{O}_{X_i}$  is strictly Noetherian by Lemma 6.11. To check that  $\mathscr{O}_{\mathscr{X}}$  is strictly Noetherian, we consider the  $\pi$ -adic filtration. Since  $\mathscr{O}_{\mathscr{X}}$  is pointwise Zariskian with respect to the  $\pi$ -adic filtration by [5, 3.3.6] and [18, Ch. II, 2.2 (4)] (or we can use Lemma 1.10), it suffices to show that  $\operatorname{gr}_{\pi}(\mathscr{O}_{\mathscr{X}}) \cong \mathscr{O}_{X_0}[T]$  is strictly Noetherian by Lemma 6.8 where  $\operatorname{gr}_{\pi}$  denotes the gr with respect to the  $\pi$ -adic filtration and T denotes the class of  $\pi$ . This follows from Lemma 6.9. For  $\mathscr{O}_{\mathscr{X},\mathbb{Q}}$  use Corollary 6.10.

Let X be either  $\mathscr{X}$  or  $X_i$  for some  $i \geq 0$ . Let us prove that  $\mathscr{D}_X^{(m)}$  is strictly Noetherian. We consider the filtration by order. Since the filtration is positive, it suffices to show that  $\operatorname{gr}(\mathscr{D}_X^{(m)})$  is strictly Noetherian by Lemma 6.8. Since  $\operatorname{gr}(\mathscr{D}_X^{(m)})$  is of finite type over  $\operatorname{gr}_0(\mathscr{D}_X^{(m)})$ , and  $\operatorname{gr}_i(\mathscr{D}_X^{(m)})$ is coherent  $\operatorname{gr}_0(\mathscr{D}_X^{(m)})$ -module for any i,  $\operatorname{gr}(\mathscr{D}_X^{(m)})$  can be seen as a coherent  $\mathscr{O}_X[T_1,\ldots,T_n]$ -algebra for some n, and the claim follows by using Corollary 6.9.

Let us prove that  $\widehat{\mathscr{D}}_{\mathscr{X}}^{(m)}$  is strictly Noetherian. Consider the  $\pi$ -adic filtration. Then  $\widehat{\mathscr{D}}_{\mathscr{X}}^{(m)}$  is pointwise Zariskian filtered by [5, 3.3.6] and [18, Ch. II, 2.2 (4)]. By Lemma 6.8, it suffices to show that  $\operatorname{gr}_{\pi}(\widehat{\mathscr{D}}_{\mathscr{X}}^{(m)}) \cong (\mathscr{D}_{X_0}^{(m)})[T]$  is strictly Noetherian where  $\operatorname{gr}_{\pi}$  denotes the gr with respect to the  $\pi$ -adic filtration and T denotes the class of  $\pi$ . Since we checked that  $\mathscr{D}_{X_0}^{(m)}$  is strictly Noetherian,  $(\mathscr{D}_{X_0}^{(m)})[T]$  is strictly Noetherian by Lemma 6.9, and thus  $\widehat{\mathscr{D}}_{\mathscr{X}}^{(m)}$  is strictly Noetherian.

Let X be either  $\mathscr{X}$  or  $X_i$  for some *i*. Let us check that  $\mathscr{E} := \mathscr{E}_X^{(m,m')}$  is strictly Noetherian on  $\mathring{T}^*\mathscr{X}$ . Consider the filtration by order. It suffices to show that  $\operatorname{gr}(\mathscr{E})$  is strictly Noetherian by Lemma 6.8 and Remark 4.9. Let  $q : \mathring{T}^*X \to P^*X$  be the canonical surjection. Then, it suffices to show that  $q_*(\operatorname{gr}(\mathscr{E}))$  is strictly Noetherian by (1.4.1). By Lemma 4.8, we know that  $q_*(\operatorname{gr}(\mathscr{E}))$  is an  $\mathscr{O}_{P^{(m)}*X}$ -algebra of finite type and  $q_*(\operatorname{gr}_i(\mathscr{E}))$  is a coherent  $\mathscr{O}_{P^{(m)}*X}$ -module for any *i*, and we get the claim by using Lemma 6.9.

For  $\widehat{\mathscr{E}}_{\mathscr{X}}^{(m,m')}$  and  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$  the verifications are the same as those of  $\widehat{\mathscr{D}}_{\mathscr{X}}^{(m)}$  and  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ , and we leave the details to the reader.

## 7. Application: Stability theorem for curves

In this section, we focus on the relation between the support of the microlocalization and the characteristic variety. We formulate a conjecture on the relation, and prove the conjecture in the curve case.

7.1. — Recall the setting 2.4, and let  $\mathscr{X}$  be a *quasi-compact* smooth formal scheme over R. One might expect that, for a coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{O}}^{(m)}$ -module  $\mathscr{M}$ ,

$$\operatorname{Char}^{(m)}(\mathscr{M}) = \operatorname{Supp}(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m,\dagger)} \otimes_{\pi^{-1}\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}} \pi^{-1}\mathscr{M})$$

For the definition of the characteristic varieties, see 2.14. However, this does not hold in general. Indeed, suppose this were true. Then since  $\operatorname{Supp}(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m,\dagger)}\otimes\mathscr{M}) \supset$  $\operatorname{Supp}(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m+1,\dagger)}\otimes\mathscr{M})$ , we would get  $\operatorname{Char}^{(m)}(\mathscr{M}) \supset \operatorname{Char}^{(m+1)}(\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m+1)}\otimes\mathscr{M})$ . However, this does not hold by Example 4.1. Considering these, we conjecture the following.

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**Conjecture**. — Let  $\mathscr{X}$  be a quasi-compact smooth formal scheme over R, and  $\mathscr{M}$  be a coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -module. Then there exists an integer  $N \geq m$  such that for any  $m' \geq N$ ,

$$\operatorname{Char}^{(m')}(\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m')}\otimes_{\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}}\mathscr{M}) = \operatorname{Supp}(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m',\dagger)}\otimes_{\pi^{-1}\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}}\pi^{-1}\mathscr{M}).$$

We prove this conjecture in the case where  $\mathscr{X}$  is a formal curve (i.e., dimension 1 connected smooth formal scheme of finite type over R). Namely,

**7.2. Theorem.** — Let  $\mathscr{X}$  be a smooth formal curve over R. Let  $\mathscr{M}$  be a coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -module. Then there exists an integer  $N \geq m$  such that we have

$$\operatorname{Char}^{(m')}(\widehat{\mathscr{D}}^{(m')}_{\mathscr{X},\mathbb{Q}}\otimes\mathscr{M}) = \operatorname{Supp}(\mathscr{E}^{\dagger}_{\mathscr{X},\mathbb{Q}}\otimes\mathscr{M})$$

for  $m' \geq N$ .

This theorem is proven in the last part of this section.

**7.3. Remark.** — The conjecture and Theorem 7.2 may seem to be different since we used  $\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m',\dagger)}$  in the conjecture and  $\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{\dagger}$  in the theorem. However, these are equivalent. Indeed, since there exists a homomorphism  $\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m',\dagger)} \to \mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m'+1,\dagger)}$  and the topological space  $T^*\mathscr{X}$  is Noetherian, there exists an integer a such that for any  $m' \geq a$ ,

$$\operatorname{Supp}(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{\dagger}\otimes\mathscr{M})\subset\operatorname{Supp}(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m',\dagger)}\otimes\mathscr{M})=\operatorname{Supp}(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(a,\dagger)}\otimes\mathscr{M}).$$

We remind that these supports are closed by [16, 0<sub>I</sub>, 5.2.2]. Let us show that this inclusion is in fact an equality. Since the problem is local, we may assume that  $\mathscr{X}$  is affine, and take global generators  $m_1, \ldots, m_n \in \Gamma(\mathscr{X}, \mathscr{M})$  of  $\mathscr{M}$ over  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ . Suppose that the inclusion is not an equality, and take a point  $x \in \operatorname{Supp}(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(a,\dagger)} \otimes \mathscr{M})$  which is not contained in  $\operatorname{Supp}(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{\dagger} \otimes \mathscr{M})$ . This means that  $(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{\dagger} \otimes \mathscr{M})_x = 0$ . Now, we know that

$$(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{\dagger}\otimes\mathscr{M})_{x}\cong\varinjlim_{m'}(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m',\dagger)}\otimes\mathscr{M})_{x}$$

by [15, Ch. II, 1.11]. Thus there exists an integer  $m' \geq a$  such that the images of  $m_1, \ldots, m_n$  in  $(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m',\dagger)} \otimes \mathscr{M})_x$  are 0. Since the latter module is generated by these elements over  $(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m',\dagger)})_x$ , we would have  $(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m',\dagger)} \otimes \mathscr{M})_x = 0$ , which contradicts with the assumption. Summing up, we obtain

$$\operatorname{Supp}(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{\dagger}\otimes\mathscr{M})=\bigcap_{m'\geq m}\operatorname{Supp}(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m',\dagger)}\otimes\mathscr{M}).$$

7.4. — Before we start proving the theorem:

**Definition**. — Let  $\mathscr{X}$  be a smooth formal scheme of dimension 1 over R (not necessarily quasi-compact), and let  $\mathscr{M}$  be a coherent  $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$ -module. We define

$$\operatorname{Char}(\mathscr{M}) := \operatorname{Supp}(\mathscr{E}_{\mathscr{K},\mathbb{O}}^{\dagger} \otimes \mathscr{M}).$$

**7.5. Corollary.** — Suppose that k is perfect and there exists a lifting  $\sigma : R \xrightarrow{\sim} R$  of the absolute Frobenius automorphism of k, and fix one. Let  $\mathscr{X}$  be a smooth formal scheme of dimension 1, and let  $\mathscr{M}$  be a coherent  $F - \mathscr{D}^{\dagger}_{\mathscr{X}, \mathbb{Q}}$ -module. Then

$$\operatorname{Car}(\mathscr{M}) = \operatorname{Char}(\mathscr{M}),$$

where Car denotes the characteristic variety defined by Berthelot (cf. [7, 5.2.7]).

*Proof.* — The problem being local, we may assume that  $\mathscr{X}$  is quasi-compact, and thus a curve. Since this follows immediately from the definition of Car, we recall it briefly. For a large enough integer m, we can take the Frobenius descent of level m denoted by  $\mathscr{M}^{(m)}$  by [6, 4.5.4], which is a coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -module with an isomorphism  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m+1)} \otimes \mathscr{M}^{(m)} \xrightarrow{\sim} F^* \mathscr{M}^{(m)}$ . The characteristic variety of  $\mathscr{M}$ is by definition  $\operatorname{Char}^{(m)}(\mathscr{M}^{(m)})$ . Since we have  $\mathscr{M}^{(m+k)} := \widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m+k)} \otimes \mathscr{M}^{(m)} \cong$  $F^{k*}\mathscr{M}^{(m)}$ ,  $\operatorname{Char}^{(m)}(\mathscr{M}^{(m)}) = \operatorname{Char}^{(m+k)}(\mathscr{M}^{(m+k)})$  by [7, 5.2.4 (iii)]. Thus the corollary follows by applying Theorem 7.2 to  $\mathscr{M}^{(m)}$ . □

**7.6.** — Let us prove the theorem. To do it, we show the following proposition first.

**Proposition.** — Let  $\mathscr{X}$  be a smooth formal curve over R, and  $\mathscr{M}$  be a coherent  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{O}}^{(m)}$ -module. Suppose that there exists an integer  $N' \geq m$  such that

$$\operatorname{Char}^{(m')}(\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m')}\otimes\mathscr{M})=\operatorname{Char}^{(N')}(\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(N')}\otimes\mathscr{M})$$

for any  $m' \geq N'$ . Then the conclusion of Theorem 7.2 holds.

The proof is given in 7.9. For an interval  $I \subset \mathbb{R}$ , we denote  $I \cap \mathbb{Z}$  by  $I_{\mathbb{Z}}$  in the following.

**7.7. Lemma.** — Let  $\mathscr{X}$  be a smooth formal scheme. (We do not need to assume that  $\mathscr{X}$  is a curve in this lemma.) Let  $\mathscr{M}$  be a  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ -module and  $x \in T^*\mathscr{X}$ . Suppose there exist integers  $b \geq a \geq m$  such that

$$(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m')}\otimes\mathscr{M})_x=0$$

for any integer  $b \ge m' \ge a$ . Then the canonical homomorphism

(7.7.1) 
$$(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(l,\bullet)}\otimes\mathscr{M})_x\to(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m',\bullet)}\otimes\mathscr{M})_x$$

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is split surjective for any integers  $b \geq m' \geq a$ ,  $m' \geq l \geq m$ , and  $\bullet \in \{\dagger, [m', \infty[_{\mathbb{Z}}]\}$ . Here  $\widehat{\mathscr{E}}_{\mathscr{X}, \mathbb{Q}}^{(l, \dagger)}$  means  $\mathscr{E}_{\mathscr{X}, \mathbb{Q}}^{(l, \dagger)}$  by abuse of language. Moreover, we get

$$\operatorname{Ker}\left((\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(l,\bullet)}\otimes\mathscr{M})_{x}\to(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m',\bullet)}\otimes\mathscr{M})_{x}\right)\cong\operatorname{Tor}_{1}^{\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m',\bullet)}/\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(l,\bullet)},\mathscr{M})_{x}$$
$$\cong\begin{cases}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(l,a)}\otimes\mathscr{M})_{x}\oplus\bigoplus_{i=a}^{m'-1}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(i,i+1)}\otimes\mathscr{M})_{x}\quad \text{if }l< a\\\bigoplus_{i=l}^{m'-1}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(i,i+1)}\otimes\mathscr{M})_{x}\quad \text{if }l\geq a.\end{cases}$$

Proof. — For  $\bullet \in \{\dagger, [m', \infty[_{\mathbb{Z}}]\},$ 

$$\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m',\bullet)}/\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(l,\bullet)} \xrightarrow{\sim} \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m')}/\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(l,m')}$$

by Corollary 5.10. We denote this quotient by  $\mathscr{Q}$ . We get the following diagram whose rows are exact:

where the  $\otimes$  and Tor<sub>1</sub> are taken over  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ , and we omit the pullback of sheaves  $\pi^{-1}$  since it is obvious where to put them. Now, since  $\operatorname{Tor}_{1}^{\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m')},\mathscr{M}) = 0$  by the flatness of  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m')}$  over  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m')}$  and  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m')}$ over  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$  (cf. Proposition 2.8 (ii) and [5, 3.5.3]), the homomorphism  $\alpha$  is injective. Moreover, since  $(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m')} \otimes \mathscr{M})_x = 0$  by the hypothesis, the homomorphism  $\alpha_x$  is an isomorphism, and  $(\mathscr{Q} \otimes \mathscr{M})_x = 0$ . Since  $\alpha$  is injective,  $\beta$  is injective as well. Thus, the homomorphism (7.7.1) is split surjective and

$$\operatorname{Ker}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(l,\bullet)}\otimes\mathscr{M}\to\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m',\bullet)}\otimes\mathscr{M})_x\cong\operatorname{Tor}_1(\mathscr{Q},\mathscr{M})_x\xrightarrow{\sim}_{\alpha_x}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(l,m')}\otimes\mathscr{M})_x.$$

Let us calculate  $\operatorname{Tor}_1(\mathcal{Q}, \mathcal{M})_x$ . We only treat the case where l < a, and since the proof is similar, the other case is left to the reader. To calculate this, it suffices to show

$$(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(l,m')}\otimes\mathscr{M})_{x}\cong(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(l,a)}\otimes\mathscr{M})_{x}\oplus\bigoplus_{i=a}^{m'-1}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(i,i+1)}\otimes\mathscr{M})_{x}$$

We use the induction on k := m'-a. For k = 0, the claim is redundant. Suppose that the statement holds to be true for  $k = k_0 - 1 \ge 0$ . Then it suffices to show the following isomorphism for  $m'' = a + k_0$ :

(7.7.2) 
$$(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(l,m'')} \otimes \mathscr{M})_x \cong (\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(l,m''-1)} \otimes \mathscr{M})_x \oplus (\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m''-1,m'')} \otimes \mathscr{M})_x.$$

Indeed, we just apply the induction hypothesis to  $(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(l,m'-1)} \otimes \mathscr{M})_x$  to get the conclusion. Let us show (7.7.2). Note that  $m'' - 1 \geq a$ . This isomorphism can be shown using exactly the same method as before using the following diagram instead:

where

$$\mathscr{Q}' = \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(l,m''-1)} / \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(l,m'')} \cong \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m''-1)} / \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m''-1,m'')}$$

using Lemma 4.15.

**7.8. Lemma.** — Let  $\mathscr{X}$  be a curve. Then the canonical homomorphism

$$\pi^{-1}\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m+k)}/\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)} \to \mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m+k,\dagger)}/\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m,\dagger)}$$

is an isomorphism.

*Proof.* — It suffices to prove that the canonical homomorphism of sheaf of abelian groups  $\pi^{-1}\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)} \to \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)}/(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m)})_{-1}$  is an isomorphism. To show this, it suffices to check that  $\pi^{-1}\mathscr{D}_{X_i}^{(m)} \to \mathscr{E}_{X_i}^{(m)}/\mathscr{E}_{X_i,-1}^{(m)}$  is an isomorphism, whose verification is straightforward.

7.9. Proof of Proposition 7.6. — Consider the following diagram of sheaves on  $T^* \mathscr{X}$  for any integer  $m' \geq m$ .

Here  $\otimes$  and Tor are taken over  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ , and we omit the pull-back of sheaves  $\pi^{-1}$ . Since  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m')}$  and  $\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m')}$  are flat over  $\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$ , the left vertical arrow of the diagram is just  $0 \to 0$ . Thus the homomorphism  $\alpha$  is injective. Consider the following

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commutative diagram

$$\begin{array}{c|c} \operatorname{Tor}_{1}(\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m',\dagger)}/\mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m,\dagger)},\mathscr{M}) \xrightarrow{\beta} \mathscr{E}_{\mathscr{X},\mathbb{Q}}^{(m,\dagger)} \otimes \mathscr{M} \\ & & \\ & & \\ & & & \\ & & & \\ & & & \\ & & & \\ \operatorname{Tor}_{1}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m')}/\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')},\mathscr{M}) \xrightarrow{\alpha} \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')} \otimes \mathscr{M} \end{array}$$

where the isomorphism is by Lemma 5.10. Since  $\alpha$  is injective,  $\beta$  is also injective. This implies that

$$\begin{aligned} (*) \\ \mathscr{K}_{m'} &:= \operatorname{Ker} \left( \varphi_{m'} \colon \mathscr{E}_{\mathscr{X}, \mathbb{Q}}^{(m, \dagger)} \otimes \pi^{-1} \mathscr{M} \to \mathscr{E}_{\mathscr{X}, \mathbb{Q}}^{(m', \dagger)} \otimes \pi^{-1} \mathscr{M} \right) \\ &\cong \operatorname{Tor}_1 \left( \mathscr{E}_{\mathscr{X}, \mathbb{Q}}^{(m', \dagger)} / \mathscr{E}_{\mathscr{X}, \mathbb{Q}}^{(m, \dagger)}, \pi^{-1} \mathscr{M} \right) \\ &\cong \pi^{-1} \operatorname{Tor}_1 \left( \widehat{\mathscr{D}}_{\mathscr{X}, \mathbb{Q}}^{(m')} / \widehat{\mathscr{D}}_{\mathscr{X}, \mathbb{Q}}^{(m)}, \mathscr{M} \right) \hookrightarrow \pi^{-1} \mathscr{M}. \end{aligned}$$

Since  $\pi^{-1}\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m)}$  is strictly Noetherian by Lemma 6.12 and Theorem 6.13, there exists an integer N such that  $\mathscr{K}_k = \mathscr{K}_N$  for  $k \geq N$ . So far we have *not* used the assumption on the characteristic varieties.

By changing m if necessarily, we may assume that m = N'. Now, by this assumption,

$$Z := \operatorname{Supp}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m)} \otimes \mathscr{M}) = \operatorname{Supp}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m')} \otimes \mathscr{M})$$

for any  $m' \ge m$ . Take  $x \in \mathring{T}^* \mathscr{X} \setminus Z$ . Then Lemma 7.7 shows that  $\varphi_{m',x}$  (see (\*)) is split surjective. Thus, the homomorphism

$$\psi_{m',x} \colon (\mathscr{E}^{(N,\dagger)}_{\mathscr{X},\mathbb{Q}} \otimes \mathscr{M})_x \to (\mathscr{E}^{(m',\dagger)}_{\mathscr{X},\mathbb{Q}} \otimes \mathscr{M})_x$$

is also surjective for any  $m' \geq N$ . Since the kernel is isomorphic to  $(\mathscr{K}_{m'}/\mathscr{K}_N)_x$ , the homomorphism  $\psi_{m',x}$  is an isomorphism by the choice of N. Thus using Lemma 7.7 again,

$$(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m',m'')}\otimes\mathscr{M})_x=0$$

for any integer  $m' \ge N$  and  $m'' \ge m'$ .

Let  $\mathscr{U}$  be the complement of Z. Let  $\mathscr{V} \subset \mathscr{U}$  be a strictly affine open subscheme. By the above observation, we get  $\Gamma(\mathscr{V}, \widehat{\mathscr{E}}_{\mathscr{X}, \mathbb{Q}}^{(m', m'')} \otimes \mathscr{M}) = 0$ . Since  $\Gamma(\mathscr{V}, \mathscr{E}_{\mathscr{X}, \mathbb{Q}}^{(m', \dagger)})$  is a Fréchet-Stein algebra, we have  $\Gamma(\mathscr{V}, \mathscr{E}_{\mathscr{X}, \mathbb{Q}}^{(m', \dagger)} \otimes \mathscr{M}) = 0$ . Thus the proposition follows.

7.10. Proof of Theorem 7.2:— We use the notation in the proof of Proposition 7.6. There exists an integer M such that  $\mathscr{K}_k = \mathscr{K}_M$  for  $k \geq M$ . Note that to prove the existence of this M, we did not use the assumption of Proposition 7.6. Let  $x \in \mathring{T}^* \mathscr{X}$ . Suppose there is an integer m' > M such that  $(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{O}}^{(m')} \otimes \mathscr{M})_x = 0$ . Then by using Lemma 7.7,

$$0 = (\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m')} \otimes \mathscr{M})_x \cong (\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')} \otimes \mathscr{M})_x / \mathscr{K}_{m',x}$$
$$= (\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m')} \otimes \mathscr{M})_x / \mathscr{K}_{m'',x} \subset (\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m'',m')} \otimes \mathscr{M})_x$$

for an integer  $m' \geq m'' \geq M$ . However, since  $(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m'',m')} \otimes \mathscr{M})_x$  is generated by the image of  $\mathscr{M}$ , we have  $(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m'',m')} \otimes \mathscr{M})_x = 0$ . This implies that  $(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m'')} \otimes \mathscr{M})_x = 0$  for  $m' \geq m'' \geq M$ . Thus,

$$\operatorname{Supp}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m')}\otimes\mathscr{M})\subset\operatorname{Supp}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m'+1)}\otimes\mathscr{M})$$

for  $m' \geq M$ .

Now suppose there exists  $M' \geq M$  such that  $\operatorname{Supp}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(M')} \otimes \mathscr{M}) = T^*\mathscr{X}$ . Then there is nothing to prove. Using [14, 5.2.4] we may suppose that the dimension of  $\operatorname{Supp}(\widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m')} \otimes \mathscr{M})$  is equal to 1 for any  $m' \geq M$ .

In this case, for any  $m' \geq M$ , there exists an open formal subscheme  $\mathscr{U}_{m'}$ of  $\mathscr{X}$  such that

$$\operatorname{Char}^{(m')}(\widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m')}\otimes\mathscr{M}|_{\mathscr{U}_{m'}})\cap T^{*}\mathscr{U}_{m'}=\mathscr{U}_{m'}$$

and  $\mathscr{U}_{m'} \supset \mathscr{U}_{m'+1}$ . Let  $\mathscr{M}^{(m')} := \widehat{\mathscr{D}}_{\mathscr{X},\mathbb{Q}}^{(m')} \otimes \mathscr{M}$ . The module  $\mathscr{M}^{(m')}|_{\mathscr{U}_{m'}}$  is a coherent  $\mathcal{O}_{\mathscr{U}_{m'},\mathbb{Q}}$ -module. Let  $r_{m'}$  be the rank as a locally projective  $\mathcal{O}_{\mathscr{U}_{m'},\mathbb{Q}}$ -module. Then we know that  $r_{m'} \geq r_{m'+1}$  for any  $m' \geq M$ . There exists an integer  $N \geq M$  such that  $r_N = r_{m'}$  for any  $m' \geq N$ . Now,

**Claim.** — Let  $\mathscr{Y}$  be a smooth formal scheme, and  $\mathscr{N}$  be a coherent  $\widehat{\mathscr{D}}_{\mathscr{Y},\mathbb{Q}}^{(m)}$ -module which is also coherent as an  $\mathcal{O}_{\mathscr{Y},\mathbb{Q}}$ -module. For  $m' \geq m$ , the canonical homomorphism

$$\alpha \colon \mathscr{N} \to \mathscr{N}^{(m')} := \widehat{\mathscr{D}}_{\mathscr{Y}, \mathbb{Q}}^{(m')} \otimes_{\widehat{\mathscr{D}}_{\mathscr{Y}, \mathbb{Q}}^{(m)}} \mathscr{N}.$$

is an isomorphism in the following two cases:

- 1.  $\mathscr{N}^{(m')}$  is a coherent  $\mathscr{O}_{\mathscr{Y},\mathbb{Q}}\text{-module}$  with the same rank as  $\mathscr{N}.$
- 2. the  $\widehat{\mathscr{D}}_{\mathscr{Y},\mathbb{Q}}^{(m)}$ -module structure on  $\mathscr{N}$  extends continuously to a  $\widehat{\mathscr{D}}_{\mathscr{Y},\mathbb{Q}}^{(m')}$ -module structure.

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These claims seem more or less standard, but we provide a proof at the end for the sake of completeness. Let us finish the proof assuming the claims first. By the choice of N, the canonical homomorphism

$$\mathscr{M}^{(N)}|_{\mathscr{U}_{m'}} \to \mathscr{M}^{(m')}|_{\mathscr{U}_{m'}}$$

is an isomorphism for  $m' \geq N$  by Claim 1. The proof of [24, 2.16] is immediately translated to our situation to prove that for a smooth curve  $\mathscr{Y}$  and a  $\widehat{\mathscr{D}}_{\mathscr{Y},\mathbb{Q}}^{(m)}$ -module  $\mathscr{M}$  which is coherent as an  $\mathscr{O}_{\mathscr{Y},\mathbb{Q}}$ -module, if there exists an open formal subscheme  $\mathscr{V}$  such that  $\mathscr{M}|_{\mathscr{V}}$  is a  $\widehat{\mathscr{D}}_{\mathscr{Y},\mathbb{Q}}^{(m+1)}$ -module, then  $\mathscr{M}$  is a  $\widehat{\mathscr{D}}_{\mathscr{Y},\mathbb{Q}}^{(m+1)}$ -module. Thus  $\mathscr{M}^{(N)}|_{\mathscr{U}_N}$  is already a  $\widehat{\mathscr{D}}_{\mathscr{U}_N,\mathbb{Q}}^{(m')}$ -module, and thus  $\mathscr{M}^{(N)}|_{\mathscr{U}_N} \xrightarrow{\sim} \widehat{\mathscr{D}}_{\mathscr{U}_N,\mathbb{Q}}^{(m')} \otimes \mathscr{M}^{(N)}|_{\mathscr{U}_N}$  by Claim 2. This implies that the condition of Proposition 7.6 holds, and we obtain the theorem.

Finally, let us check the claims. Since the claims are local, we may assume  $\mathscr{Y}$  to be affine. In the following, we make no difference between sheaves and the modules of global sections. Consider the first situation. We denote by  $\mathscr{T}_{m'}$  (resp.  $\mathscr{T}'_{m'}$ ) the  $\mathscr{O}_{\mathscr{Y},\mathbb{Q}}$ -module (resp.  $\widehat{\mathscr{D}}^{(m')}_{\mathscr{Y},\mathbb{Q}}$ -module) topology on  $\mathscr{N}^{(m')}$ . Since  $\mathscr{N}^{(m')}$  is a finite  $\mathscr{O}_{\mathscr{Y},\mathbb{Q}}$ -module (resp.  $\widehat{\mathscr{D}}^{(m')}_{\mathscr{Y},\mathbb{Q}}$ -module) by assumption, it becomes a Banach space by [9, 3.7.3/1]. By construction,  $\operatorname{Im}(\alpha)$  is dense with respect to the topology  $\mathscr{T}'_{m'}$ . On the other hand,  $\operatorname{Im}(\alpha)$  is closed with respect to the topology  $\mathscr{T}'_{m'}$  is equivalent to  $\mathscr{T}_{m'}$ , which implies that  $\alpha$  is surjective. Since the ranks of the both sides are the same by assumption, the homomorphism is an isomorphism. Consider the second situation. The extended  $\widehat{\mathscr{D}}^{(m')}_{\mathscr{Y},\mathbb{Q}}$ -module structure yields the homomorphism  $\beta \colon \mathscr{N}^{(m')} \to \mathscr{N}$ . By definition, we have  $\beta \circ \alpha = \operatorname{id}$ . It remains to show that  $\alpha$  is surjective. To check this, it suffices to show that  $\alpha$  is in fact a homomorphism of  $\widehat{\mathscr{D}}^{(m')}_{\mathscr{Y},\mathbb{Q}}$ -modules. Since the  $\widehat{\mathscr{D}}^{(m)}_{\mathscr{Y},\mathbb{Q}}$ -module ule and  $\widehat{\mathscr{D}}^{(m')}_{\mathscr{Y},\mathbb{Q}}$ -module topology on  $\mathscr{N}$  coincide and  $\widehat{\mathscr{D}}^{(m)}_{\mathscr{Y},\mathbb{Q}}$  is dense in  $\widehat{\mathscr{D}}^{(m)}_{\mathscr{Y},\mathbb{Q}}$ , we get the desired assertion by a standard continuity argument.

## BIBLIOGRAPHY

- T. ABE "Langlands correspondence for isocrystals and existence of crystalline companion for curves", preprint arXiv:1310.0528.
- [2] \_\_\_\_\_, "Langlands program for p-adic coefficients and the petits camarades conjecture", preprint arXiv:1111.2479.
- [3] T. ABE & A. MARMORA "On p-adic product formula for epsilon factors", J. Inst. Math. Jussieu (2014), p. 1–103.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE

- [4] M. F. ATIYAH & I. G. MACDONALD Introduction to commutative algebra, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.
- [5] P. BERTHELOT "D-modules arithmétiques. I. Opérateurs différentiels de niveau fini", Ann. Sci. École Norm. Sup. 29 (1996), p. 185–272.
- [6] \_\_\_\_\_, "D-modules arithmétiques. II. Descente par Frobenius", Mém. Soc. Math. Fr. 81 (2000).
- [7] \_\_\_\_\_, "Introduction à la théorie arithmétique des  $\mathscr{D}$ -modules", Astérisque 279 (2002), p. 1–80.
- [8] J.-E. BJÖRK Analytic D-modules and applications, Mathematics and its Applications, vol. 247, Kluwer Academic Publishers Group, Dordrecht, 1993.
- [9] S. BOSCH, U. GÜNTZER & R. REMMERT Non-Archimedean analysis, Grundl. Math. Wiss., vol. 261, Springer, Berlin, 1984.
- [10] N. BOURBAKI Algèbre commutative, Hermann, 1985.
- [11] R. CREW "F-isocrystals and their monodromy groups", Ann. Sci. École Norm. Sup. 25 (1992), p. 429–464.
- [12] P. DELIGNE "La conjecture de Weil. II", Publ. Math. I.H.E.S. 52 (1980), p. 137–252.
- [13] R. GARCÍA LÓPEZ "Microlocalization and stationary phase", Asian J. Math. 8 (2004), p. 747–768.
- [14] L. GARNIER "Théorèmes de division sur  $\widehat{\mathscr{D}}_{\mathscr{X}\mathbf{Q}}^{(0)}$  et applications", Bull. Soc. Math. France 123 (1995), p. 547–589.
- [15] R. GODEMENT Topologie algébrique et théorie des faisceaux, Hermann, 1960.
- [16] A. GROTHENDIECK "Éléments de géométrie algébrique: I. Le langage des schémas", Publ. math. de l'IHÉS 4 (1960).
- [17] \_\_\_\_\_, Théorie des topos et cohomologie étale des schémas (SGA 4), Springer, 1972.
- [18] L. HUISHI & F. VAN OYSTAEYEN Zariskian filtrations, K-Monographs in math., vol. 2, Kluwer, 1996.
- [19] M. KASHIWARA & T. KAWAI "On holonomic systems of microdifferential equations. III. Systems with regular singularities", *Publ. Res. Inst. Math. Sci.* 17 (1981), p. 813–979.
- [20] L. LAFFORGUE "Chtoucas de Drinfeld et correspondance de Langlands", Invent. Math. 147 (2002), p. 1–241.
- [21] T. Y. LAM Lectures on modules and rings, Graduate Texts in Math., vol. 189, Springer, New York, 1999.
- [22] G. LAUMON "Transformations canoniques et spécialisation pour les Dmodules filtrés", Astérisque 130 (1985), p. 56–129.

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- [23] A. MARMORA "Microdifférentielles arithmétiques sur une courbe", personal notes.
- [24] A. OGUS "F-isocrystals and de Rham cohomology. II. Convergent isocrystals", Duke Math. J. 51 (1984), p. 765–850.
- [25] P. SCHNEIDER & J. TEITELBAUM "Algebras of *p*-adic distributions and admissible representations", *Invent. Math.* **153** (2003), p. 145–196.