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HYPERBOLIC COMPONENTS OF McMULLEN MAPS

BY WEIYUAN QIU, PASCALE ROESCH, XIAOGUANG WANG
AND YONGCHENG YIN

ABSTRACT. – In this article, we completely settle a question raised by B. Devaney. We prove that all the hyperbolic components are Jordan domains in the family of rational maps of McMullen type. Moreover, we give a precise description of all the rational maps on the outer boundary. It follows that the cusps are dense on the outer boundary.

RÉSUMÉ. – Dans cet article nous résolvons complètement une question posée par B. Devaney. Nous montrons que toutes les composantes hyperboliques sont des domaines de Jordan dans la famille de fractions rationnelles de type McMullen. De plus nous donnons une description précise de toutes les fractions du bord de la composante non bornée. Il en découle que les cusps sont denses dans le bord de la composante non bornée.

1. Introduction

In his article [22], Curt McMullen presented a family of rational maps with the particularity that, viewed as a dynamical system, it exhibits very rich dynamical behavior. Nevertheless, this family has a very simple form. It consists in a singular perturbation of the monomial $z \mapsto z^n$ acting on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$:

$$\begin{aligned} \widehat{\mathbb{C}} &\rightarrow \widehat{\mathbb{C}} \\ z &\mapsto z^n + \lambda z^{-m} \end{aligned}$$

where λ varies in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $n, m \geq 1$.

The *Julia set*, which is the minimal totally invariant compact set of cardinality at least 3, appears in this family under several various classic fractals. Namely, Curt McMullen pointed out (in [22]) that when $(n, m) = (2, 3)$ and $\lambda \in \mathbb{C}^*$ is small, the Julia set is a Cantor set of circles. Moreover, in the hyperbolic case, the Julia set can also be homeomorphic to either

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a Cantor set, or a Sierpiński carpet as proven in [10]. A rational map is *hyperbolic* if every critical point converges under iterations to an attracting cycle.

The parameters $\lambda \in \mathbb{C}^*$ can then be divided into two classes, the hyperbolic ones and the others (see Figure 1, where hyperbolic parameters are in blue and yellow). Conjecturally the set of hyperbolic parameters (which is an open set) is dense in \mathbb{C}^* . In this article we study the boundaries of the hyperbolic components of the McMullen maps:

$$f_\lambda : z \mapsto z^n + \lambda z^{-n}, \quad \lambda \in \mathbb{C}^*, \quad n \geq 3.$$

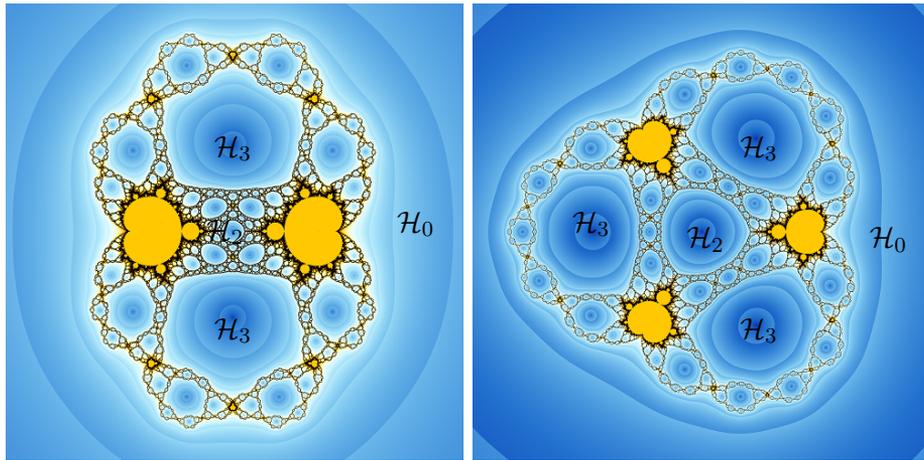


FIGURE 1. Parameter plane of McMullen maps, $n = 3, 4$.

Bob Devaney has proven in [6] that the boundary of the hyperbolic component containing the punctured neighborhood of the origin is a Jordan curve. He raised the question in 2004 at the Snowbird Conference (on the 25th Anniversary of the Mandelbrot set) whether all the other hyperbolic components of *escape type* (the free critical orbits escape to ∞) are Jordan domains.

The following Theorem 1.1 solves completely the question, it is the main result of the paper.

THEOREM 1.1. – *Fix any $n \geq 3$. The boundary of every hyperbolic component of the family $f_\lambda(z) = z^n + \lambda/z^n$, $\lambda \in \mathbb{C}^*$ is a Jordan curve.*

Moreover, we give a complete description of the dynamics of the McMullen maps lying at the boundary of the unbounded hyperbolic component \mathcal{H}_0 . In this component, the Julia sets of the maps are Cantor sets, but at the boundary the Julia set is connected (see Section 2). As a corollary we obtain our second main result :

THEOREM 1.2. – *Cusps are dense in $\partial\mathcal{H}_0$.*

Here, according to McMullen [25], a parameter $\lambda \in \mathbb{C}^*$ is called a *cuspidal* if the map f_λ has a parabolic cycle. Geometrically, a cusp is a point where the bifurcation locus is cusp-shaped.

Let us recall that McMullen proved that cusps are dense on the Bers' boundary of Teichmüller space in [23]. The analogue of this result in the world of rational maps, as a conjecture posed by McMullen [25], would be, in the space of degree d polynomials, the density of geometrically finite parabolics on the boundary of the hyperbolic component \mathcal{U}_d containing z^d . This conjecture is verified by P. Roesch [32] in the one-dimensional slice $z^d + cz^{d-1}$ ($d \geq 3$ and $c \in \mathbb{C}$). It remains open in full generality. The result we prove here is, in the spirit, related to this conjecture. But it has to be thought of as a phenomenon in some one-dimensional slices of rational maps.

We would like to explain why we concentrate on the case $m = n \geq 3$ of the general McMullen family $z \mapsto z^n + \lambda z^{-m}$, $\lambda \in \mathbb{C}^*$, $m, n \geq 1$. This is because our proof rests on the technical Yoccoz puzzle theory. When $m = n \geq 3$, we can succeed in applying this theory to study both the dynamical plane and the parameter plane. However when $m = n = 2$, it is impossible to find a non-degenerate critical annulus for the Yoccoz puzzle constructed in [28]. The existence of a non-degenerate critical annulus is technically necessary in this theory. In the general case when $m \neq n \geq 1$, 0 and ∞ have different folds of symmetries. To the best of our knowledge, whether the Yoccoz puzzle structure exists in this case is unknown.

1.1. Overview of the paper

Let us recall some definitions and give the basic notions to precise our results before to go to the proof.

For any $\lambda \in \mathbb{C}^*$, the map f_λ has a fixed point at ∞ which is superattracting since the derivative is 0. The immediate attracting basin of ∞ denoted by B_λ is the set of points converging under iteration to ∞ and lying in the connected component of ∞ . The set of critical points of f_λ is $\{0, \infty\} \cup C_\lambda$, where $C_\lambda = \{c \in \mathbb{C}; c^{2n} = \lambda\}$. Hence, there is only one free critical orbit (up to a sign) since besides ∞ , there are only two critical values: $v_\lambda^+ = 2\sqrt{\lambda}$ and $v_\lambda^- = -2\sqrt{\lambda}$ (here, when restricted to the fundamental domain, v_λ^+ and v_λ^- are well-defined, see Section 3).

A rational map is *hyperbolic* if all critical orbits are attracted by the attracting cycles (see [27, 24]). Hence, a McMullen map f_λ is hyperbolic if the free critical orbit is attracted either by ∞ or by an attracting cycle in \mathbb{C} . Every hyperbolic component is isomorphic to either the unit disk \mathbb{D} or $\mathbb{D}^* = \mathbb{D} - \{0\}$ (see Theorem 2.2). In particular, \mathcal{H}_0 is a topological punctured disk. Assuming Theorem 1.1, one gets a canonical parameterization $\nu : \mathbb{S} \rightarrow \partial\mathcal{H}_0$, where $\nu(\theta)$ is defined to be the landing point of the parameter ray $\mathcal{R}_0(\theta)$ (see Section 6) in \mathcal{H}_0 . The complete characterization of $\partial\mathcal{H}_0$ we give is the following :

- THEOREM 1.3** (Characterization of $\partial\mathcal{H}_0$ and cusps). – *We have*
1. $\lambda \in \partial\mathcal{H}_0$ if and only if ∂B_λ contains either C_λ or a parabolic cycle.
 2. $\nu(\theta)$ is a cusp if and only if $n^p\theta \equiv \theta \pmod{\mathbb{Z}}$ for some $p \geq 1$.

Theorem 1.2 is an immediate consequence of Theorem 1.3 since the set $\{\theta \mid n^p\theta \equiv \theta \pmod{\mathbb{Z}}, p \geq 1\}$ is a dense subset of the unit circle \mathbb{S} .

The main part of the paper is to prove Theorem 1.1. We briefly sketch the idea of the proof and the organization of the paper.

The proof rests on the dynamics and namely the fact that this family admits the “Yoccoz puzzle” structure (see [28]). The Yoccoz puzzle is induced by a kind of Jordan curve called “cut ray” that was first constructed by Bob Devaney [8].

To obtain results in the parameter plane, the idea is to use the rigidity of the maps. And for this we study deeply the dynamical planes. The idea is therefore different from the well known “*parapuzzle* techniques” (known to be a powerful tool to study the boundary of hyperbolic components, see [32, 30]). Until Section 7, we consider only the hyperbolic components of escape type, which are called *escape domains*, *i.e.*, for which the critical orbits tend to ∞ .

Precisely,

- In Section 2, we parameterize the escape domains.
- In Section 3, we recall (quickly) the construction of the cut rays, since it is necessary for the study of escape domains. The crucial fact regarding cut rays is that they move continuously in the Hausdorff topology with respect to the parameter.
- In Section 4, we give some characterizations of maps on $\partial\mathcal{H}_0$.
- In Section 5, we prove a rigidity result for the maps on $\partial\mathcal{H}_0$. We first construct a topological conjugacy between the maps with the same combinatorial information, using the Yoccoz puzzle techniques. Then we use the idea due to Kozlovski, Shen and van Strien [16], and a ‘zero measure argument’ following Lyubich, to get the rigidity result.
- In Section 6, we prove that $\partial\mathcal{H}_0$ is a Jordan curve and give some related consequences.
- In Section 7, we prove that the boundaries of all escape domains of level $k \geq 3$ are Jordan curves. These escape domains are called *Sierpiński holes*. The proof is based on three ingredients: the boundary regularity of \mathcal{H}_0 , holomorphic motion and continuity of cut rays. We remark that our approach also applies to the boundary of the hyperbolic component containing the punctured neighborhood of the origin. This yields a different proof from Devaney’s in [6].
- In Section 8, we show that every hyperbolic component is a Jordan domain by considering the ones which are not of “escape type”.

We refer the reader to [6, 7, 8, 9, 1, 10, 11, 15, 28, 31, 34] and the references therein for various related results on the dynamics of McMullen maps.

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2. Escape domains and parameterizations

There are two kinds of hyperbolic McMullen maps based on the behavior of the free critical orbit. If the orbit escapes to infinity, the corresponding hyperbolic component is called an *escape domain*. If the orbit tends to an attracting cycle other than ∞ , the corresponding hyperbolic component is of *renormalizable type*.

In this section, we present some known facts about escape domains. We refer the reader to [10] for more background materials. The hyperbolic components of renormalizable type will be discussed in Section 8.

For any $\lambda \in \mathbb{C}^*$, the *Julia set* $J(f_\lambda)$ of f_λ can be identified as the boundary of $\bigcup_{k \geq 0} f_\lambda^{-k}(B_\lambda)$. It satisfies $e^{\pi i/n} J(f_\lambda) = J(f_\lambda)$. The *Fatou set* $F(f_\lambda)$ of f_λ is defined by $F(f_\lambda) = \widehat{\mathbb{C}} - J(f_\lambda)$. We denote by T_λ the component of $f_\lambda^{-1}(B_\lambda)$ containing 0. It is possible that $B_\lambda = T_\lambda$. In that case, the critical set $C_\lambda \subset B_\lambda$ and $J(f_\lambda)$ is a Cantor set (see Theorem 2.1).

For any $k \geq 0$, we define a parameter set \mathcal{H}_k as follows:

$$\mathcal{H}_k = \{\lambda \in \mathbb{C}^*; k \text{ is the first integer such that } f_\lambda^k(C_\lambda) \subset B_\lambda\}.$$

A component of \mathcal{H}_k is called an *escape domain* of level k . One may verify that $\mathcal{H}_0 = \{\lambda \in \mathbb{C}^*; v_\lambda^+ \in B_\lambda\}$, $\mathcal{H}_1 = \emptyset$ and $\mathcal{H}_k = \{\lambda \in \mathbb{C}^*; f_\lambda^{k-2}(v_\lambda^+) \in T_\lambda \neq B_\lambda\}$ for $k \geq 2$. See Figure 1. The complement of the escape domains is called the *non-escape locus* \mathcal{M} . It can be written as

$$\mathcal{M} = \{\lambda \in \mathbb{C}^*; f_\lambda^k(v_\lambda^+) \text{ does not tend to infinity as } k \rightarrow \infty\}.$$

The set \mathcal{M} is invariant under the maps $z \mapsto \bar{z}$ and $z \mapsto e^{\frac{2\pi i}{n-1}} z$.

THEOREM 2.1 (Escape Trichotomy [10] and Connectivity [12]). – *We have*

1. *If $\lambda \in \mathcal{H}_0$, then $J(f_\lambda)$ is a Cantor set.*
2. *If $\lambda \in \mathcal{H}_2$, then $J(f_\lambda)$ is a Cantor set of circles.*
3. *If $\lambda \in \mathcal{H}_k$ for some $k \geq 3$, then $J(f_\lambda)$ is a Sierpiński curve.*
4. *If $\lambda \in \mathcal{M}$, then Julia set $J(f_\lambda)$ is connected.*

See Figure 2 for the four typical types of Julia sets.

Based on Theorem 2.1, we give some remarks on the escape domains. According to Devaney, \mathcal{H}_0 is called the *Cantor set locus*, \mathcal{H}_2 is called the *McMullen domain*, \mathcal{H}_k with $k \geq 3$ is called the *Sierpiński locus* and each of its component is called a *Sierpiński hole*. Devaney showed that the boundary $\partial\mathcal{H}_2$ is a Jordan curve [6] and \mathcal{H}_k with $k \geq 3$ consists of $(2n)^{k-3}(n-1)$ disk components [7].

The Böttcher map ϕ_λ of f_λ is defined in a neighborhood of ∞ by $\phi_\lambda(z) = \lim_{k \rightarrow \infty} (f_\lambda^k(z))^{n^{-k}}$. It is unique if we require $\phi'_\lambda(\infty) = 1$. The map ϕ_λ satisfies $\phi_\lambda(f_\lambda(z)) = \phi_\lambda(z)^n$ and $\phi_\lambda(e^{\pi i/n} z) = e^{\pi i/n} \phi_\lambda(z)$. One may verify that near infinity,

$$\phi_\lambda(z) = \sum_{k \geq 0} a_k(\lambda) z^{1-2kn}, \quad a_0(\lambda) = 1, a_1(\lambda) = \lambda/n, \dots$$

If $\lambda \in \mathbb{C}^* \setminus \mathcal{H}_0$, then both B_λ and T_λ are simply connected. In that case, there is a unique Riemann mapping $\psi_\lambda : T_\lambda \rightarrow \mathbb{D}$, such that $\psi_\lambda(w)^{-n} = \phi_\lambda(f_\lambda(w))$ for $w \in T_\lambda$ and $\psi'_\lambda(0) = \sqrt[n]{\lambda}$. The *external ray* $R_\lambda(t)$ of angle t in B_λ is defined by $R_\lambda(t) := \phi_\lambda^{-1}((1, +\infty)e^{2\pi it})$, the *internal ray* $R_{T_\lambda}(t)$ of angle t in T_λ is defined by $R_{T_\lambda}(t) := \psi_\lambda^{-1}((0, 1)e^{2\pi it})$.

THEOREM 2.2 (Parameterization of escape domains, [7, 31, 34]). – *We have*

1. *\mathcal{H}_0 is the unbounded component of $\mathbb{C}^* - \mathcal{M}$. The map $\Phi_0 : \mathcal{H}_0 \rightarrow \mathbb{C} - \overline{\mathbb{D}}$ defined by $\Phi_0(\lambda) = \phi_\lambda(v_\lambda^+)^2$ is a conformal isomorphism.*

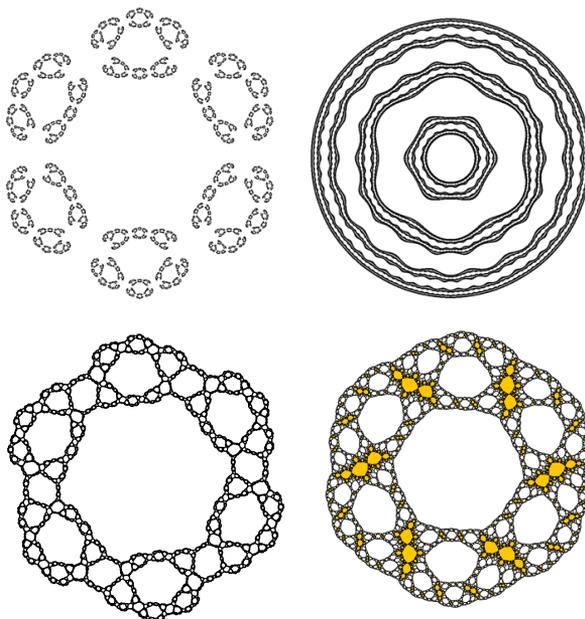


FIGURE 2. The Julia sets: a Cantor set (upper-left), a Cantor set of circles (upper-right), a Sierpiński curve (lower-left) and a connected set (lower-right).

2. \mathcal{H}_2 is the component of $\mathbb{C}^* - \mathcal{M}$ containing the punctured neighborhood of 0. The holomorphic map $\Phi_2 : \mathcal{H}_2 \rightarrow \mathbb{C} - \overline{\mathbb{D}}$ defined via $\Phi_2(\lambda)^{n-2} = \phi_\lambda(f_\lambda(v_\lambda^+))^2$ and $\lim_{\lambda \rightarrow 0} \lambda \Phi_2(\lambda) = 2^{\frac{2n}{2-n}}$, is a conformal isomorphism.
3. Let \mathcal{H} be an escape domain of level $k \geq 3$. The map $\Phi_{\mathcal{H}} : \mathcal{H} \rightarrow \mathbb{D}$ defined by $\Phi_{\mathcal{H}}(\lambda) = \psi_\lambda(f_\lambda^{k-2}(v_\lambda^+))$ is a conformal isomorphism.

Both Φ_0 and Φ_2 satisfy $\Phi_\epsilon(e^{\frac{2\pi i}{n-1}} \lambda) = e^{\frac{2\pi i}{n-1}} \Phi_\epsilon(\lambda)$ and $\Phi_\epsilon(\bar{\lambda}) = \overline{\Phi_\epsilon(\lambda)}$ for $\epsilon \in \{0, 2\}$ and $\lambda \in \mathcal{H}_\epsilon$. Thus they take the form $\Phi_\epsilon(\lambda) = \lambda \Psi_\epsilon(\lambda^{n-1})$, where Ψ_ϵ is a holomorphic function whose expansion has real coefficients.

THEOREM 2.3 (Connectivity of \mathcal{M}). – *The non-escape locus \mathcal{M} is connected and has logarithmic capacity $1/4$.*

Proof. – By Theorem 2.2, each component of $\widehat{\mathbb{C}} - \mathcal{M}$ is a topological disk. So \mathcal{M} is connected. The logarithmic capacity of \mathcal{M} follows from the expansion of Φ_0 near ∞ : $\Phi_0(\lambda) = 4\lambda + \mathcal{O}(\lambda^{2-n})$. \square

3. Cut rays in the dynamical plane

The topology of ∂B_λ is considered in [28], where the authors showed

THEOREM 3.1 ([28]). – *For any $n \geq 3$ and any $\lambda \in \mathbb{C}^*$,*

- ∂B_λ is either a Cantor set or a Jordan curve. In the latter case, all Fatou components eventually mapped to B_λ are Jordan domains.

- If ∂B_λ is a Jordan curve containing neither a parabolic point nor the recurrent critical set C_λ , then ∂B_λ is a quasi-circle.

Here, the critical set C_λ is called *recurrent* if $C_\lambda \subset J(f_\lambda)$ and the set $\bigcup_{k \geq 1} f_\lambda^k(C_\lambda)$ has an accumulation point in C_λ . The proof of Theorem 3.1 is based on the Yoccoz puzzle theory. To apply this theory, we need to construct a kind of Jordan curve which cuts the Julia set into two connected parts. These curves are called *cut rays*. They play a crucial role in our study of the boundaries of escape domains. For this, we briefly sketch their construction here.

To begin, we identify the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ with $(0, 1]$. We define a map $\tau : \mathbb{S} \rightarrow \mathbb{S}$ by $\tau(\theta) = n\theta \bmod 1$. Let $\Theta_k = (\frac{k}{2n}, \frac{k+1}{2n}]$ for $0 \leq k \leq n$ and $\Theta_{-k} = (\frac{k}{2n} + \frac{1}{2}, \frac{k+1}{2n} + \frac{1}{2}]$ for $1 \leq k \leq n-1$. Obviously, $(0, 1] = \bigcup_{-n < j \leq n} \Theta_j$.

Let Θ be the set of all angles $\theta \in (0, 1]$ whose orbits remain in $\bigcup_{k=1}^{n-1} (\Theta_k \cup \Theta_{-k})$ under all iterations of τ . One may verify that Θ is a Cantor set. Given an angle $\theta \in \Theta$, the *itinerary* of θ is a sequence of symbols $(s_0, s_1, s_2, \dots) \in \{\pm 1, \dots, \pm(n-1)\}^{\mathbb{N}}$ such that $\tau^k(\theta) \in \Theta_{s_k}$ for all $k \geq 0$. The angle $\theta \in \Theta$ and its itinerary (s_0, s_1, s_2, \dots) satisfy the identity [28, Lemma 3.1]:

$$\theta = \frac{1}{2} \left(\frac{\chi(s_0)}{n} + \sum_{k \geq 1} \frac{|s_k|}{n^{k+1}} \right),$$

where $\chi(s_0) = s_0$ if $0 \leq s_0 \leq n$ and $\chi(s_0) = n - s_0$ if $-(n-1) \leq s_0 \leq -1$.

Note that $e^{\pi i/(n-1)} f_\lambda(z) = (-1)^n f_{e^{2\pi i/(n-1)}\lambda}(e^{\pi i/(n-1)}z)$ for all $\lambda \in \mathbb{C}^*$. This implies that the fundamental domain of the parameter plane is

$$\mathcal{F}_0 = \{\lambda \in \mathbb{C}^*; 0 \leq \arg \lambda < 2\pi/(n-1)\}.$$

We denote the interior of \mathcal{F}_0 by

$$\mathcal{F} := \{\lambda \in \mathbb{C}^*; 0 < \arg \lambda < 2\pi/(n-1)\}.$$

In our discussion, we assume $\lambda \in \mathcal{F}$ and let $O_\lambda = \bigcup_{k \geq 0} f_\lambda^{-k}(\infty)$ be the grand orbit of ∞ . Let $c_0 = c_0(\lambda) = \sqrt[n]{\lambda}$ be the critical point that lies on $\mathbb{R}^+ := [0, +\infty)$ when $\lambda \in \mathbb{R}^+$ and varies analytically as λ ranges over \mathcal{F} . Let $c_k(\lambda) = c_0 e^{k\pi i/n}$ for $1 \leq k \leq 2n-1$. The critical points c_k with k even are mapped to v_λ^+ while the critical points c_k with k odd are mapped to v_λ^- .

Let $\ell_k = c_k[0, +\infty]$ be the closed straight line connecting 0 to ∞ and passing through c_k for $0 \leq k \leq 2n-1$. The closed sector bounded by ℓ_k and ℓ_{k+1} is denoted by S_k^λ for $0 \leq k \leq n$. Define $S_{-k}^\lambda = -S_k^\lambda$ for $1 \leq k \leq n-1$. These sectors are arranged counterclockwise about the origin as $S_0^\lambda, S_1^\lambda, \dots, S_n^\lambda, S_{-1}^\lambda, \dots, S_{-(n-1)}^\lambda$. See [28, Figure 2].

The critical value v_λ^+ always lies in S_0^λ because $\arg c_0 \leq \arg v_\lambda^+ < \arg c_1$ for all $\lambda \in \mathcal{F}_0$. Correspondingly, the critical value v_λ^- lies in S_n^λ . The image of ℓ_k under f_λ is a straight ray connecting one of the critical values to ∞ ; this ray is called a *critical value ray*. As a consequence, f_λ maps the interior of each of the sectors of $S_{\pm 1}^\lambda, \dots, S_{\pm(n-1)}^\lambda$ univalently onto a region Υ_λ , which can be identified as the complex sphere $\widehat{\mathbb{C}}$ minus two critical value rays. For any $\epsilon \in \{\pm 1, \dots, \pm(n-1)\}$, let $\text{int}(S_\epsilon^\lambda)$ be the interior of S_ϵ^λ , the inverse of $f_\lambda : \text{int}(S_\epsilon^\lambda) \rightarrow \Upsilon_\lambda$ is denoted by $h_\epsilon^\lambda : \Upsilon_\lambda \rightarrow \text{int}(S_\epsilon^\lambda)$.

THEOREM 3.2 (Cut ray, [8] [28]). – For any $\lambda \in \mathcal{F}$ and any angle $\theta \in \Theta$ with itinerary (s_0, s_1, s_2, \dots) , the set

$$\Omega_\lambda^\theta := \bigcap_{k \geq 0} f_\lambda^{-k}(S_{s_k}^\lambda \cup S_{-s_k}^\lambda)$$

is a Jordan curve intersecting the Julia set $J(f_\lambda)$ in a Cantor set.

Theorem 3.2 is originally proven for the parameters $\lambda \in \mathcal{F} \cap \mathcal{M}$ in [28]. The proof actually works for all $\lambda \in \mathcal{F}$ without any difference.

Here are some facts about the cut rays:

$$\Omega_\lambda^\theta = -\Omega_\lambda^\theta \text{ and } \Omega_\lambda^\theta = \Omega_\lambda^{\theta+1/2};$$

$$R_\lambda(\theta) \cup R_\lambda(\theta + \frac{1}{2}) \subset \Omega_\lambda^\theta \cap F(f_\lambda) \subset \bigcup_{k \geq 0} f_\lambda^{-k}(B_\lambda);$$

$0, \infty \in \Omega_\lambda^\theta$ and $\Omega_\lambda^\theta \setminus \{0, \infty\}$ is contained in the interior of $S_{s_0}^\lambda \cup S_{-s_0}^\lambda$; $f_\lambda(\Omega_\lambda^\theta) = \Omega_\lambda^{\tau(\theta)}$ and $f_\lambda : \Omega_\lambda^\theta \rightarrow \Omega_\lambda^{\tau(\theta)}$ is a two-to-one map. We refer the reader to [28] for more details of the cut rays.

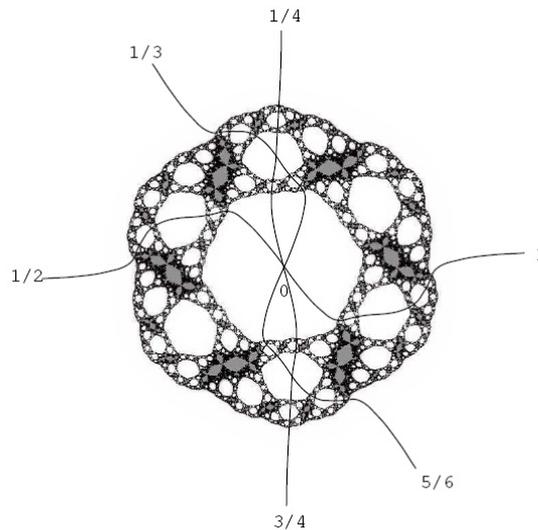


FIGURE 3. Some cut rays Ω_λ^θ with $\theta = 1/4, 1/3, 1/2$. ($n = 3$)

Now we give some new dynamical properties of the cut rays. These facts will be useful to study the parameter plane. We denote by $B(z, r)$ the open Euclidean disk centered at z with radius r . For any $\lambda \in \mathbb{C}^* \setminus \mathcal{H}_0$, set $B_\lambda^L := \{w \in B_\lambda; |\phi_\lambda(w)| > L\}$ for $L \geq 1$.

LEMMA 3.3 (Holomorphic motion of the cut rays). – Fix an angle $\theta \in \Theta$, the cut ray Ω_λ^θ moves holomorphically with respect to $\lambda \in \mathcal{F}$.

Proof. – Fix a parameter $\lambda_0 \in \mathcal{F}$. We will define a holomorphic motion $h : \mathcal{F} \times ((\Omega_{\lambda_0}^\theta \setminus O_{\lambda_0}) \cap F(f_{\lambda_0})) \rightarrow \widehat{\mathbb{C}}$ with base point λ_0 as follows. For any $\lambda \in \mathcal{F}$, there is a number $L \geq 1$ (depending on λ) such that the Böttcher map $\phi_u : B_u^L \rightarrow \{\zeta \in \widehat{\mathbb{C}}; |\zeta| > L\}$ is a conformal isomorphism, for $u \in \{\lambda, \lambda_0\}$.

If $z \in (\Omega_{\lambda_0}^\theta \setminus O_{\lambda_0}) \cap B_{\lambda_0}^L$, we define $h(\lambda, z) = \phi_\lambda^{-1} \circ \phi_{\lambda_0}(z)$. If $z \in (\Omega_{\lambda_0}^\theta \setminus O_{\lambda_0}) \cap (F(f_{\lambda_0}) \setminus B_{\lambda_0}^L)$, we consider the *itinerary* of z , which is the unique sequence of symbols $(\epsilon_0, \epsilon_1, \epsilon_2, \dots) \in \{\pm 1, \dots, \pm(n-1)\}^{\mathbb{N}}$ such that $f_{\lambda_0}^k(z) \in S_{\epsilon_k}^{\lambda_0}$ for all $k \geq 0$. Let $N \geq 1$ be the first integer such that $f_{\lambda_0}^N(z) \in B_{\lambda_0}^L$. We define $h(\lambda, z) = h_{\epsilon_0}^\lambda \circ \dots \circ h_{\epsilon_{N-1}}^\lambda \circ \phi_\lambda^{-1} \circ \phi_{\lambda_0}(f_{\lambda_0}^N(z))$. In this way, we get a well-defined map $h : \mathcal{F} \times ((\Omega_{\lambda_0}^\theta \setminus O_{\lambda_0}) \cap F(f_{\lambda_0})) \rightarrow \widehat{\mathbb{C}}$. Since both ϕ_λ and $h_{\epsilon_j}^\lambda$ are holomorphic with respect to $\lambda \in \mathcal{F}$, one may verify that the map h is a holomorphic motion parameterized by \mathcal{F} , with base point λ_0 (namely, $h(\lambda_0, z) \equiv z$). Moreover, for any $\lambda \in \mathcal{F}$, we have $h(\lambda, (\Omega_{\lambda_0}^\theta \setminus O_{\lambda_0}) \cap F(f_{\lambda_0})) = (\Omega_\lambda^\theta \setminus O_\lambda) \cap F(f_\lambda)$.

Note that for any $\lambda \in \mathcal{F}$, the closure of $(\Omega_\lambda^\theta \setminus O_\lambda) \cap F(f_\lambda)$ is Ω_λ^θ . By the λ -Lemma (see [21] or [24]), there is a holomorphic motion $H : \mathcal{F} \times \Omega_{\lambda_0}^\theta \rightarrow \widehat{\mathbb{C}}$ extending h and for any $\lambda \in \mathcal{F}$, one has $H(\lambda, \Omega_{\lambda_0}^\theta) = \Omega_\lambda^\theta$. That is to say, the cut ray Ω_λ^θ moves holomorphically when λ ranges over \mathcal{F} . □

Let Θ_{per} be a subset of $\Theta \setminus \{1, 1/2\}$, consisting of all periodic angles under the map τ . One may verify that Θ_{per} is a dense subset of Θ .

THEOREM 3.4 (Cut rays with real parameters). – *For any $\lambda \in (0, +\infty)$ and any angle $\theta \in \Theta_{per}$ with itinerary (s_0, s_1, s_2, \dots) , the set*

$$\Omega_\lambda^\theta := \bigcap_{k \geq 0} f_\lambda^{-k}((S_{s_k}^\lambda \cup S_{-s_k}^\lambda) \setminus \mathbb{R}^*)$$

is a Jordan curve intersecting the Julia set $J(f_\lambda)$ in a Cantor set, where $\mathbb{R}^ = \mathbb{R} \setminus \{0\}$. Moreover, if $\mathcal{F}_0 \ni \lambda_j \rightarrow \lambda \in (0, +\infty)$, then $\Omega_{\lambda_j}^\theta \rightarrow \Omega_\lambda^\theta$ in Hausdorff topology.*

Here is a remark. If $\lambda \in \mathcal{F}$, then $\bigcap_{k \geq 0} f_\lambda^{-k}(S_{s_k}^\lambda \cup S_{-s_k}^\lambda) = \bigcap_{k \geq 0} f_\lambda^{-k}((S_{s_k}^\lambda \cup S_{-s_k}^\lambda) \setminus \mathbb{R}^*)$, so the latter is also a reasonable definition of cut rays. However, if $\lambda \in (0, +\infty)$, the set $\bigcap_{k \geq 0} f_\lambda^{-k}(S_{s_k}^\lambda \cup S_{-s_k}^\lambda)$ is not a Jordan curve in general.

The proof of Theorem 3.4 is essentially the same as that of Proposition 3.9 in [28]. We would like to mention the idea of the proof here. Let $Y_\lambda = \widehat{\mathbb{C}} \setminus ([-\infty, v_\lambda^+] \cup [v_\lambda^+, +\infty] \cup \overline{B_\lambda^L})$ for some large $L > 1$ and p be the period of θ . The itinerary of θ satisfies $s_{p+k} = s_k$ for all $k \geq 0$. Since $\theta \neq 1, 1/2$, one of s_k will be in the set $\{\pm 1, \dots, \pm(n-2)\}$ and for any $k \geq 0$ and any $(\epsilon_1, \dots, \epsilon_p) = (\pm s_k, \dots, \pm s_{k+p-1})$, the set $h_{\epsilon_1}^\lambda \circ \dots \circ h_{\epsilon_p}^\lambda(Y_\lambda)$ is compactly contained in Y_λ (one should note that if $\theta = 1$ or $1/2$, then $h_1^\lambda \circ \dots \circ h_1^\lambda(Y_\lambda)$ is not compactly contained in Y_λ). Similar to the proof of Proposition 3.9 in [28], one can construct two sequences of Jordan curves converging to the boundaries of the two components of $\widehat{\mathbb{C}} - \Omega_\lambda^\theta$. In this way Ω_λ^θ is locally connected. One can show that its two complementary components share the same boundary, so Ω_λ^θ is a Jordan curve.

Using the same proof as Lemma 3.3, one can show that if $\theta \in \Theta_{per}$, then the cut ray Ω_λ^θ is a holomorphic motion in a neighborhood of the real and positive axis. This yields the continuity of cut rays. We omit the details.

PROPOSITION 3.5 (Preimages of cut ray, [28], Prop 3.5). – *For any $\lambda \in \mathcal{F}_0$ and any $\theta \in \Theta_{per}$, suppose that $(\Omega_\lambda^\theta - \{0, \infty\}) \cap (\bigcup_{1 \leq k \leq N} f_\lambda^k(C_\lambda)) = \emptyset$ for some $N \geq 1$. Then, for any $\alpha \in \bigcup_{0 \leq k \leq N} \tau^{-k}(\theta)$, there is a unique Jordan curve Ω_λ^α (or $\Omega_\lambda^{\alpha+1/2}$) containing 0 and ∞ , such that $f_\lambda(\Omega_\lambda^\alpha) = \Omega_\lambda^{\tau(\alpha)}$ and $R_\lambda(\alpha) \cup R_\lambda(\alpha + 1/2) \subset \Omega_\lambda^\alpha \cap B_\lambda$.*

The Jordan curve Ω_λ^α defined in Proposition 3.5 is also called a cut ray. We remark that the statement of Proposition 3.5 is slightly different from Prop. 3.5 in [28], but their proofs are the same.

REMARK 3.6. – *The cut ray Ω_λ^α defined by Proposition 3.5 satisfies the following property:*

There is a neighborhood \mathcal{U} of λ , such that for all $u \in \mathcal{U} \cap \mathcal{F}_0$, $(\Omega_u^\theta - \{0, \infty\}) \cap (\bigcup_{1 \leq k \leq N} f_u^k(C_u)) = \emptyset$ (this implies the cut ray Ω_u^α exists). By Lemma 3.3 and Theorem 3.4, the cut ray Ω_u^α moves continuously with respect to $u \in \mathcal{U} \cap \mathcal{F}_0$.

LEMMA 3.7. – *For any $\lambda \in \mathcal{F}_0$ and any two different external rays $R_\lambda(t_1)$ and $R_\lambda(t_2)$, there is a cut ray Ω_λ^α with $\alpha \in \bigcup_{k \geq 0} \tau^{-k}(\Theta_{per})$ separating them.*

Proof. – Since Θ_{per} is an infinite set, we can find an angle $\theta \in \Theta_{per}$ such that $(\Omega_\lambda^\theta - \{0, \infty\}) \cap (\bigcup_{k \geq 1} f_\lambda^k(C_\lambda)) = \emptyset$. The preimages $\bigcup_{k \geq 0} \tau^{-k}(\theta)$ of θ are dense in the unit circle, so there is $\alpha \in \bigcup_{k \geq 0} \tau^{-k}(\theta)$ lying in between t_1 and t_2 . Then $R_\lambda(t_1)$ and $R_\lambda(t_2)$ are contained in different components of $\widehat{\mathbb{C}} - \Omega_\lambda^\alpha$. \square

4. Maps on $\partial\mathcal{H}_0$

In this section, we give some a priori characterizations of the maps on $\partial\mathcal{H}_0$. We begin with a dynamical result for our purpose. To prove Theorem 3.1 in [28], we reduce the situation to the following:

THEOREM 4.1 (Backward contraction on ∂B_λ , [28]). – *Suppose that $\lambda \in \mathbb{C}^* \setminus \mathcal{H}_0$ and ∂B_λ contains neither a parabolic point nor the recurrent critical set C_λ , then f_λ satisfies the following property on ∂B_λ : there exist three constants $\delta_0 > 0$, $C > 0$ and $0 < \rho < 1$ such that for any $0 < \delta < \delta_0$, any $z \in \partial B_\lambda$, any integer $k \geq 0$ and any component $U_k(z)$ of $f_\lambda^{-k}(B(z, \delta))$ that intersects with ∂B_λ , $U_k(z)$ is simply connected with Euclidean diameter $\text{diam}(U_k(z)) \leq C\delta\rho^k$.*

We refer the reader to [28] for a detailed proof based on the Yoccoz puzzle theory (to obtain Theorem 4.1, one should combine two results in [28]: Theorem 1.2 in Section 7.5 and Proposition 6.1 in Section 6). The proof of Theorem 4.1 rests on the crucial fact that the puzzle pieces around each point on ∂B_λ shrink to a single point.

LEMMA 4.2. – *Suppose that $J(f_\lambda)$ is not a Cantor set. If ∂B_λ contains neither a critical point nor a parabolic cycle, then there exist an integer $k \geq 1$ and two topological disks U_λ, V_λ with $\overline{B_\lambda} \subset V_\lambda \subset U_\lambda$, such that $f_\lambda^k : V_\lambda \rightarrow U_\lambda$ is a polynomial-like map of degree n^k with only one critical point ∞ .*

Proof. – The map f_λ satisfies the assumptions in Theorem 4.1. This guarantees the existence of three constants δ_0, C, ρ .

Let N_δ be δ -neighborhood of ∂B_λ , defined as the set of all points whose Euclidean distance to ∂B_λ is smaller than δ . We choose an integer $\ell > 0$ and a number $\delta < \delta_0$ such that $C\rho^\ell < 1$ and $(\bigcup_{0 \leq j < \ell} f_\lambda^{-j}(C_\lambda)) \cap N_\delta = \emptyset$.

Given a Jordan curve γ , we define its partial distance to ∂B_λ by $\varpi(\gamma) := \max_{z \in \gamma} d(z, \partial B_\lambda)$, where $d(\cdot, \cdot)$ is Euclidean distance. We choose a Jordan curve $\gamma_0 \subset \widehat{\mathbb{C}} \setminus \overline{B_\lambda}$ with $\varpi(\gamma_0) < \delta$. The annulus between γ_0 and ∂B_λ is denoted by A_0 . Since $(\bigcup_{0 \leq j < \ell} f_\lambda^{-j}(C_\lambda)) \cap N_\delta = \emptyset$, there is an annular component of $f_\lambda^{-\ell}(A_0)$, say A_1 , with ∂B_λ as one of its boundary components. The other boundary curve is denoted by γ_1 . Theorem 4.1 implies $\varpi(\gamma_1) \leq \varpi(\gamma_0)C\rho^\ell < \delta$. Continuing inductively, for any $k \geq 1$, there is an annular component of $f_\lambda^{-\ell}(A_{k-1})$, say A_k , whose boundary curves are ∂B_λ and γ_k . Then we have

$$\varpi(\gamma_k)/\varpi(\gamma_0) \leq C\rho^{k\ell}.$$

So we can choose $k_0 > 0$ such that $\varpi(\gamma_{k_0}) < \min_{z \in \gamma_0} d(z, \partial B_\lambda)$. Let V_λ be the unbounded component of $\widehat{\mathbb{C}} - \gamma_{k_0}$ and U_λ be the unbounded component of $\widehat{\mathbb{C}} - \gamma_0$. Then $f_\lambda^{k_0\ell} : V_\lambda \rightarrow U_\lambda$ is a polynomial-like map of degree $n^{k_0\ell}$, with only one critical point ∞ . It is actually quasiconformally conjugate to the power map $z \mapsto z^{n^{k_0\ell}}$. \square

We give here an alternative proof of Lemma 4.2 which uses Mañé’ Lemma. This proof does not need the precise estimate of Theorem 4.1.

Proof of Lemma 4.2. – From [28] it follows that ∂B_λ is a Jordan curve. Therefore, $\overline{B_\lambda}$ is full, i.e., $\widehat{\mathbb{C}} \setminus \overline{B_\lambda}$ is connected. Since there is no critical point on the boundary ∂B_λ , then $\overline{B_\lambda} \cap \overline{T_\lambda} = \emptyset$. Therefore there exists an open disk U containing $\overline{B_\lambda}$, avoiding $\overline{T_\lambda}$ and the critical set. Let ϕ be a Riemann map from $\widehat{\mathbb{C}} \setminus \overline{B_\lambda}$ to $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. The map $\phi \circ f_\lambda \circ \phi^{-1}$ defined on $\phi(U \setminus \overline{B_\lambda})$ extends by the Schwarz reflexion principle to a map noted by \tilde{f} on a neighborhood of the unit circle. Let g denote the restriction of \tilde{f} to the unit circle. It is a covering of degree n , the degree of the map f_λ on $U \setminus \overline{B_\lambda}$.

Let us prove by contradiction that g has no non-repelling periodic point. In the case of existence of an attracting or parabolic cycle of g on $\partial\mathbb{D}$, choose a point a in the cycle, let m be its period and let \mathcal{A} be its immediate basin. Let $\mathcal{B} = \phi^{-1}(\mathcal{A}) = \phi^{-1}(\mathcal{A} \cap (\mathbb{C} \setminus \overline{\mathbb{D}}))$. Then \mathcal{B} is bounded in \mathbb{C} and stable by f_λ^{mp} . It is therefore contained in a periodic component \mathcal{B}' of the Fatou set.

Since every point of \mathcal{A} tends to a under iteration of \tilde{f}^{mp} , every point of \mathcal{B} tends to ∂B_λ under iteration of f_λ^{mp} . Thus in particular the component \mathcal{B} is not part of a cycle of an attracting point nor a cycle of Siegel disks, because every point in a Siegel disk has an orbit that stays bounded away from the Julia set. It is therefore in a component of an immediate parabolic basin which contradicts the assumption that there is no parabolic point on ∂B_λ .

Let us cite Theorem A in [19] (see also [20]).

THEOREM 4.3 (Mañé). – *Let $N = S^1$ (the circle) or $N = [0, 1]$. If f is a C^2 map from N to N and $\Lambda \subset N$ is a compact invariant set that does not contain critical points, sinks or non-hyperbolic periodic points, then either $\Lambda = N = S^1$ and f is topologically equivalent to a rotation or Λ is a hyperbolic set.*

By the preceding discussion, we can apply this to g on $\partial\mathbb{D}$ with $\Lambda = N = \partial\mathbb{D}$. So g , and therefore \tilde{f} , is hyperbolic on $\partial\mathbb{D}$, which means that there is a continuous function $\rho(z) > 0$ on the unit circle such that \tilde{f} is uniformly expanding with respect to the metric $\rho(z)|dz|$, i.e., $\rho(\tilde{f}(z))|\tilde{f}'(z)| > \kappa\rho(z)$ for some $\kappa > 1$. Consider for some $\epsilon > 0$ the domain

$$V = \{z = re^{i\theta} \mid |\log r| \leq \epsilon\rho(e^{i\theta})\}.$$

Let U be the connected component of $\tilde{f}^{-1}(V)$ containing the unit circle. For ϵ small enough, U is compactly contained in V .

Let $U' = \overline{B_\lambda} \cup \phi^{-1}(U)$ and $V' = \overline{B_\lambda} \cup \phi^{-1}(V)$. Then U' and V' are open, connected, simply connected, U' is a connected component of $f_\lambda^{-p}(V')$ and U' is compactly contained in V' . Therefore the restriction of $f_\lambda^p : U' \rightarrow V'$ is a polynomial-like map. This restriction is also unicritical. \square

LEMMA 4.4. – *Suppose $\lambda \in \partial\mathcal{H}_0$, then ∂B_λ contains either the critical set C_λ or a parabolic cycle of f_λ .*

Proof. – If ∂B_λ contains neither the critical set C_λ nor a parabolic cycle, then it follows from Lemma 4.2 that there exist an integer $k \geq 1$ and two topological disks U_λ, V_λ with $\overline{B_\lambda} \subset V_\lambda \subset U_\lambda$, such that $f_\lambda^k : V_\lambda \rightarrow U_\lambda$ is a polynomial-like map of degree n^k with only one critical point ∞ . We may assume that $\overline{U_\lambda}$ has no intersection with $\bigcup_{0 \leq j < k} f_\lambda^{-j}(C_\lambda)$.

Then there is a neighborhood \mathcal{U} of λ , such that for all $u \in \mathcal{U}$, the set $\bigcup_{0 \leq j < k} f_u^{-j}(C_u)$ has no intersection with $\overline{U_\lambda}$, thus the component V_u of $f_u^{-k}(U_\lambda)$ that contains ∞ is a disk. Since ∂V_u moves holomorphically with respect to $u \in \mathcal{U}$, we may shrink \mathcal{U} a little bit so that for all $u \in \mathcal{U}$, ∂V_u is contained in U_λ . Set $U_u = U_\lambda$. In this way, we get a polynomial-like map $f_u^k : V_u \rightarrow U_u$ with only one critical point ∞ , for all $u \in \mathcal{U}$. As a consequence, the Julia set $J(f_u)$ is not a Cantor set for $u \in \mathcal{U}$.

But this is impossible since $\lambda \in \partial\mathcal{H}_0$. \square

Given a parameter $\lambda \in \mathcal{F}$, if $C_\lambda \subset \partial B_\lambda$, then there is a unique external ray $R_\lambda(t)$ landing at v_λ^+ . We define $\theta(\lambda) = t$. Note that $C_\lambda \subset \partial B_\lambda$ if and only if $v_\lambda^+ \in \partial B_\lambda$.

LEMMA 4.5. – *If $\lambda \in \mathcal{F}$ and $v_\lambda^+ \in \partial B_\lambda$, then $0 < \theta(\lambda) < \frac{1}{2(n-1)}$.*

Proof. – If $\lambda \in \mathcal{F}$, then v_λ^+ is contained in the interior of the closed sector S_0^λ . Note that $\Omega_\lambda^1 \subset S_{n-1}^\lambda \cup S_{-(n-1)}^\lambda$ and $\Omega_\lambda^{\frac{1}{2(n-1)}} \subset S_1^\lambda \cup S_{-1}^\lambda$, we have $0 < \theta(\lambda) < \frac{1}{2(n-1)}$. \square

LEMMA 4.6 ([28], Proposition 7.5). – *If ∂B_λ contains a parabolic cycle, then the following hold:*

1. *There is a symbol $\epsilon \in \{\pm 1\}$, an integer $p \geq 1$, a critical point $c \in C_\lambda$ and two topological disks U and V containing c , such that $\epsilon f_\lambda^p : U \rightarrow V$ is a quadratic-like map, hybrid equivalent to the polynomial $z \mapsto z^2 + 1/4$.*
2. *Let K be the filled Julia set of $\epsilon f_\lambda^p : U \rightarrow V$, then for any $j \geq 0$, the intersection $f_\lambda^j(K) \cap \partial B_\lambda$ is a singleton.*

Based on Lemma 4.6, let $K^+ \in \{f_\lambda(K), -f_\lambda(K)\}$ be the set containing v_λ^+ , and β_λ be the intersection point of K^+ and ∂B_λ .

REMARK 4.7. – If n is odd, since f_λ is an odd function, β_λ is necessarily a parabolic point; if n is even, either β_λ or $-\beta_\lambda$ is a parabolic point. Thus θ satisfies either $\tau^p(\theta) \equiv \theta$ or $\tau^p(\theta) \equiv \theta + \frac{1}{2}$ for some $p \geq 1$.

For any $t \in [0, 1)$, the parameter ray $\mathcal{R}_0(t)$ of angle t in \mathcal{H}_0 is defined by $\mathcal{R}_0(t) := \Phi_0^{-1}((1, +\infty)e^{2\pi it})$. Its impression \mathcal{X}_t is defined by

$$\mathcal{X}_t := \bigcap_{k \geq 1} \overline{\Phi_0^{-1}(\{re^{2\pi i t}; 1 < r < 1 + 1/k, |\theta - t| < 1/k\})}.$$

The set \mathcal{X}_t is a connected and compact subset of $\partial\mathcal{H}_0$. It satisfies

$$\mathcal{X}_{t+\frac{1}{n-1}} = e^{2\pi i/(n-1)}\mathcal{X}_t, \{\bar{\lambda}; \lambda \in \mathcal{X}_t\} = \mathcal{X}_{1-t}.$$

LEMMA 4.8. – Let $t \in [0, \frac{1}{n-1})$ and $\lambda \in \mathcal{X}_t \cap \mathcal{F}_0$.

1. If λ is not a cusp, then the external ray $R_\lambda(t/2)$ lands at v_λ^+ .
2. If λ is a cusp, then the external ray $R_\lambda(t/2)$ lands at β_λ .

Proof. – For any parameter $\lambda \in \mathcal{X}_t \cap \mathcal{F}_0$, it follows from Lemma 4.4 that either $C_\lambda \subset \partial B_\lambda$ or ∂B_λ contains a parabolic cycle. Since ∂B_λ is a Jordan curve (Theorem 3.1), there is an external ray $R_\lambda(t')$ landing at v_λ^+ (if λ is not a cusp) or β_λ (if λ is a cusp).

If $t' \notin \{t/2, (1+t)/2\}$, then there exist two cut rays Ω_λ^α and Ω_λ^β with $\alpha, \beta \in \bigcup_{k \geq 0} \tau^{-k}(\Theta_{per})$ (Lemma 3.7) such that the connected set $R_\lambda(t') \cup \{v_\lambda^+\}$ (if λ is not a cusp) or $R_\lambda(t') \cup K^+$ (if λ is a cusp), and the external rays $R_\lambda(t/2), R_\lambda((t+1)/2)$ are contained in three different components of $\widehat{\mathbb{C}} \setminus (\Omega_\lambda^\alpha \cup \Omega_\lambda^\beta)$. See Figure 4. Since the critical value $v_u^+ = 2\sqrt{u}$ and the

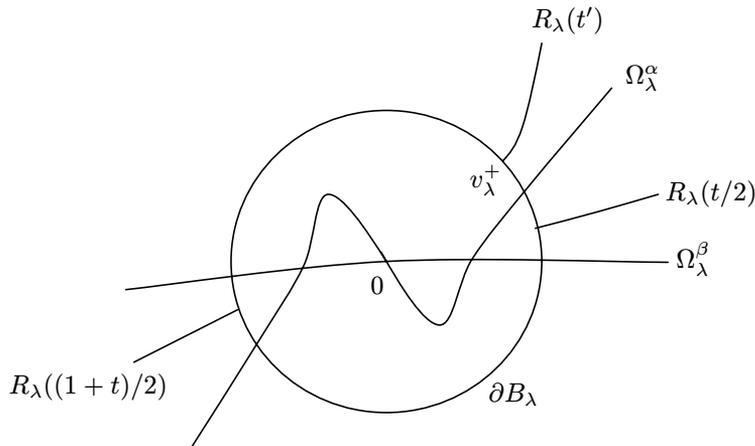


FIGURE 4. Two cut rays Ω_λ^α and Ω_λ^β separate the external rays $R_\lambda(t')$, $R_\lambda(t/2)$, $R_\lambda((t+1)/2)$ in case that λ is not a cusp.

cut rays $\Omega_u^\alpha, \Omega_u^\beta$ move continuously with respect to the parameter $u \in \mathcal{F}_0$ (Lemma 3.3 and Remark 3.6), there is a neighborhood \mathcal{V} of λ such that for all $u \in \mathcal{V} \cap \mathcal{F}_0$,

- $R_u(t')$ and v_u^+ are contained in the same component of $\widehat{\mathbb{C}} \setminus (\Omega_u^\alpha \cup \Omega_u^\beta)$.

• The external rays $R_u(t')$, $R_u(t/2)$, $R_u((t+1)/2)$ are contained in three different components of $\widehat{\mathbb{C}} \setminus (\Omega_u^\alpha \cup \Omega_u^\beta)$.

By shrinking \mathcal{V} a little bit, we see that there is a small number $\varepsilon > 0$ such that $\arg \Phi_0(u) = 2 \arg \phi_u(v_u^+) \notin (t - \varepsilon, t + \varepsilon)$ for all $u \in \mathcal{V} \cap \mathcal{F}_0 \cap \mathcal{H}_0$. This is a contradiction since $\lambda \in \mathcal{X}_t$.

So either $t' = t/2$ or $t' = (1+t)/2$. To finish, we show that the latter is impossible. If $\lambda \in (0, +\infty)$, then ∂B_λ contains a cusp and $t' = 0$. If $\lambda \in \mathcal{F}$, then there is a component V of $\widehat{\mathbb{C}} \setminus (\Omega_\lambda^1 \cup \Omega_\lambda^{\frac{1}{2(n-1)}})$ such that $v_\lambda^+ \cup R_\lambda(t') \subset \bar{V}$. In this case, we have $0 \leq t \leq \frac{1}{2(n-1)}$.

So $t' = t/2$. □

5. A rigidity result

The main result of this section is the following:

THEOREM 5.1. – *Given two parameters $\lambda_1, \lambda_2 \in \mathcal{F}$, if $v_{\lambda_i}^+ \in \partial B_{\lambda_i}$ ($i=1,2$) and $\theta(\lambda_1) = \theta(\lambda_2)$, then $\lambda_1 = \lambda_2$.*

Recall that given a parameter $\lambda \in \mathcal{F}$ with $C_\lambda \subset \partial B_\lambda$, the angle $\theta(\lambda)$ is defined such that the external ray $R_\lambda(\theta(\lambda))$ lands at v_λ^+ . Theorem 5.1 is a crucial step in proving that $\partial \mathcal{H}_0$ is a Jordan curve in the next section.

When $\theta(\lambda_1)$ is a rational number, the proof is based on Thurston's Theorem [13], as follows:

Proof of Theorem 5.1 when $\theta(\lambda_1)$ is a rational number. – In this case, both f_{λ_1} and f_{λ_2} are postcritically finite. We define a homeomorphism $\psi : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that $\psi|_{B_{\lambda_1}} = \phi_{\lambda_2}^{-1} \circ \phi_{\lambda_1}$. Then there is a homeomorphism $\varphi : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ satisfying $\psi \circ f_{\lambda_1} = f_{\lambda_2} \circ \varphi$ and $\varphi|_{B_{\lambda_1}} = \psi|_{B_{\lambda_1}}$. (In fact, φ and ψ can be made quasiconformal because ∂B_{λ_1} and ∂B_{λ_2} are quasi-circles, see Theorem 3.1.) The condition $\theta(\lambda_1) = \theta(\lambda_2)$ implies that φ and ψ are isotopic relative to the postcritical set $P(f_{\lambda_1}) := \{\infty\} \cup \bigcup_{k \geq 1} f_{\lambda_1}^k(C_{\lambda_1})$. Thus f_{λ_1} and f_{λ_2} are combinatorially equivalent. It follows from Thurston's theorem (see [13]) that f_{λ_1} and f_{λ_2} are conjugate via a Möbius transformation. This Möbius map necessarily takes the form $\gamma(z) = az$ with $a^{n-1} = 1$ and $\lambda_2 = a^2 \lambda_1$. The condition $\lambda_1, \lambda_2 \in \mathcal{F}$ implies $\lambda_1 = \lambda_2$. □

When $\theta(\lambda_1)$ is an irrational number, The proof of Proposition 5.1 involves the Yoccoz puzzle theory. We first recall the Yoccoz puzzle construction in [28].

5.1. The Yoccoz puzzle

Given a parameter $\lambda \in \mathcal{M} \cap \mathcal{F}$, we define a graph $G_\lambda(\theta_1, \dots, \theta_N)$ by

$$G_\lambda(\theta_1, \dots, \theta_N) = \partial B_\lambda^L \cup \left((\widehat{\mathbb{C}} \setminus B_\lambda^L) \cap \bigcup_{k \geq 0} (\Omega_\lambda^{\tau^k(\theta_1)} \cup \dots \cup \Omega_\lambda^{\tau^k(\theta_N)}) \right),$$

where $L > 1$ is a fixed number and $\theta_1, \dots, \theta_N \in \Theta$ are τ -periodic angles. The angles $\theta_1, \dots, \theta_N$ are chosen so that the free critical orbit $\bigcup_{k \geq 1} f_\lambda^k(C_\lambda)$ avoids the graph. The puzzle pieces of depth $d \geq 0$ are defined to be all the connected components of $f_\lambda^{-d}((\widehat{\mathbb{C}} \setminus B_\lambda^L) \setminus G_\lambda(\theta_1, \dots, \theta_N))$. For any point $z \in J(f_\lambda)$ whose orbit avoids the graph, the puzzle piece of depth d containing z is denoted by $P_d^\lambda(z)$. We say the graph $G_\lambda(\theta_1, \dots, \theta_N)$ is *admissible* if there exists a non-degenerate critical annulus $P_d^\lambda(c) \setminus \overline{P_{d+1}^\lambda(c)}$ (or $P_0^\lambda(f_\lambda^d(c)) \setminus \overline{P_1^\lambda(f_\lambda^d(c))}$) for some $c \in C_\lambda$ and some $d \geq 1$.

LEMMA 5.2 ([28], Proposition 4.1). – *Suppose $\lambda \in \mathcal{M} \cap \mathcal{F}$ and the map f_λ is postcritically infinite, then there exists an admissible graph $G_\lambda(\theta_1, \dots, \theta_N)$.*

For $c \in C_\lambda$, the tableau $T(c)$ is defined as the two-dimensional array $(P_d^\lambda(f_\lambda^l(c)))_{d,l \geq 0}$. We say $T(c)$ is *periodic* if there is an integer $p \geq 1$ such that $P_d^\lambda(f_\lambda^p(c)) = P_d^\lambda(c)$ for all $d \geq 0$.

LEMMA 5.3 ([28], Lemma 5.2 and Propositions 7.2 and 7.3). – *Suppose $\lambda \in \mathcal{M} \cap \mathcal{F}$ and the graph $G_\lambda(\theta_1, \dots, \theta_N)$ is admissible.*

1. *If $T(c)$ is periodic for some $c \in C_\lambda$, then f_λ is either renormalizable or *-renormalizable. Let K be the small filled Julia set of this (*-)renormalization, then $K \cap \partial B_\lambda$ contains at most one point.*
2. *If none of $T(c)$ with $c \in C_\lambda$ is periodic, then for any sequence of shrinking puzzle pieces $P_0^\lambda \supset P_1^\lambda \supset P_2^\lambda \supset \dots$, the intersection $\bigcap_{d \geq 0} \overline{P_d^\lambda}$ is a singleton.*

See Section 8 for the definitions of renormalizable or *-renormalizable.

To prove Proposition 5.1, we need the following result:

THEOREM 5.4 (Lebesgue measure). – *If $f_\lambda^k(v_\lambda^+) \in \partial B_\lambda$ for some $k \geq 0$, then the Lebesgue measure of $J(f_\lambda)$ is zero.*

The proof of Theorem 5.4 is based on the Yoccoz puzzle theory following Lyubich [18]. For this, we put the proof in the appendix.

By Lemmas 5.2, 5.3 and assuming Theorem 5.4, the proof of Theorem 5.1 goes as follows: we first construct a topological conjugacy between f_{λ_1} and f_{λ_2} (Section 5.2), then we will show that this topological conjugacy is actually quasiconformal. To this end, we will need a shape bound result for puzzle pieces around every point on the Julia set (Section 5.4). Finally, we will apply a QC-criterion lemma by Kozlovski, Shen and van Strien [16, Lemma 12.1] (Sections 5.3) and Theorem 5.4 to get our result.

5.2. Constructing a topological conjugacy

We assume $\theta(\lambda_1)$ is an irrational number. In that case, both f_{λ_1} and f_{λ_2} are postcritically infinite. We will construct a topological conjugacy between f_{λ_1} and f_{λ_2} with the help of the Yoccoz puzzle.

First, by Lemma 5.2, we can find an admissible graph $G_{\lambda_1}(\theta_1, \dots, \theta_N)$ for f_{λ_1} . That is to say, there exist $c \in C_{\lambda_1}$ and $k \geq 1$ such that the annulus $P_0^{\lambda_1}(f_{\lambda_1}^k(c)) \setminus P_1^{\lambda_1}(f_{\lambda_1}^{k+1}(c))$ is non-degenerate.

Define a graph

$$\Gamma_0 = \bigcup_{k \geq 0} \bigcup_{1 \leq j \leq N} \Omega_{\lambda_1}^{\tau^k(\theta_j)}.$$

It is known from Theorem 3.1 that ∂B_{λ_1} and ∂B_{λ_2} are Jordan curves. This allows us to construct a homeomorphism ψ_0 from the dynamical plane of f_{λ_1} to that of f_{λ_2} satisfying

- $\psi_0|_{B_{\lambda_1}} = \phi_{\lambda_2}^{-1} \circ \phi_{\lambda_1}|_{B_{\lambda_1}}$.
- $\psi_0(\Omega_{\lambda_1}^\alpha) = \Omega_{\lambda_2}^\alpha, \forall \Omega_{\lambda_1}^\alpha \subset \Gamma_0$.

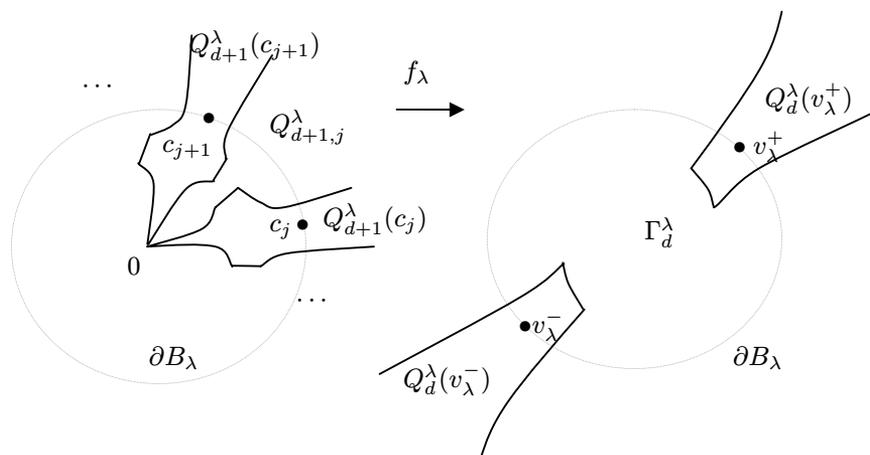


FIGURE 5. Partition and labeling.

We then construct a sequence of homeomorphisms ψ_j in the way that

- $f_{\lambda_2} \circ \psi_{j+1} = \psi_j \circ f_{\lambda_1}$ for all $j \geq 0$,
- $\psi_j(\Omega_{\lambda_1}^\alpha) = \Omega_{\lambda_2}^\alpha$ for any cut ray $\Omega_{\lambda_1}^\alpha \subset f_{\lambda_1}^{-j}(\Gamma_0)$.

The construction is inductively as follows. For $\lambda \in \{\lambda_1, \lambda_2\}$, any $d \geq 0$ and any $z \in C_\lambda \cup \{v_\lambda^+, v_\lambda^-\}$, let $Q_d^\lambda(z)$ be the component of $\overline{\mathbb{C}} \setminus f_\lambda^{-d}(\Gamma_0)$ containing z . The domain $\Gamma_d^\lambda := \overline{\mathbb{C}} \setminus (Q_d^\lambda(v_\lambda^+) \cup Q_d^\lambda(v_\lambda^-))$ either is empty or consists of one or two topological disks. Each component of $f_\lambda^{-1}(\overline{\mathbb{C}} \setminus \Gamma_d^\lambda)$ is a disk. Let $Q_{d+1,j}^\lambda$ be its component lying in between $Q_{d+1}^\lambda(c_j(\lambda))$ and $Q_{d+1}^\lambda(c_{j+1}(\lambda))$ for $0 \leq j < 2n$, $c_{2n}(\lambda) = c_0(\lambda)$. See Figure 5. Note that the map $f_\lambda|_{Q_{d+1,j}^\lambda} : Q_{d+1,j}^\lambda \rightarrow \Gamma_d^\lambda$ is a conformal isomorphism.

Suppose that $\psi_0, \psi_1, \dots, \psi_d$ are already defined and satisfy (a), (b). The assumption $\theta(\lambda_1) = \theta(\lambda_2)$ implies that the piece $Q_d^{\lambda_1}(v_{\lambda_1}^+)$ is bounded by $\Omega_{\lambda_1}^{\alpha_1}, \dots, \Omega_{\lambda_1}^{\alpha_s} \subset f_{\lambda_1}^{-d}(\Gamma_0)$ if and only if $Q_d^{\lambda_1}(v_{\lambda_1}^+)$ is bounded by $\Omega_{\lambda_2}^{\alpha_1} = \psi_d(\Omega_{\lambda_1}^{\alpha_1}), \dots, \Omega_{\lambda_2}^{\alpha_s} = \psi_d(\Omega_{\lambda_1}^{\alpha_s}) \subset f_{\lambda_2}^{-d}(\Gamma_0)$. This fact is very important. It enables us to get a lift ψ_{d+1} of ψ_d . Actually, we can define ψ_{d+1} piece by piece. Set $\psi_{d+1}|_{Q_{d+1,j}^{\lambda_1}} = (f_{\lambda_2}|_{Q_{d+1,j}^{\lambda_2}})^{-1} \circ \psi_d \circ (f_{\lambda_1}|_{Q_{d+1,j}^{\lambda_1}})$. We then define $\psi_{d+1}|_{\overline{Q_{d+1,j}^{\lambda_1}(c_j)}}$ so that it coincides with $\psi_{d+1}|_{Q_{d+1,j}^{\lambda_1}}$ in their common boundary and that the following diagram commutes

$$\begin{array}{ccc} \overline{Q_{d+1}^{\lambda_1}(c_j(\lambda_1))} & \xrightarrow{f_{\lambda_1}} & \overline{Q_d^{\lambda_1}(f_{\lambda_1}(c_j(\lambda_1)))} \\ \psi_{d+1} \downarrow & & \psi_d \downarrow \\ \overline{Q_{d+1}^{\lambda_2}(c_j(\lambda_2))} & \xrightarrow{f_{\lambda_2}} & \overline{Q_d^{\lambda_2}(f_{\lambda_2}(c_j(\lambda_2)))} \end{array}$$

One may verify that ψ_{d+1} is well defined and satisfies $f_{\lambda_2} \circ \psi_{d+1} = \psi_d \circ f_{\lambda_1}$. By induction assumption, ψ_d preserves the d -th preimages of Γ_0 . Then the condition $\theta(\lambda_1) = \theta(\lambda_2)$ and the construction of ψ_{d+1} implies that $\psi_{d+1}(\Omega_{\lambda_1}^\alpha) = \Omega_{\lambda_2}^\alpha$ for any cut ray $\Omega_{\lambda_1}^\alpha \subset f_{\lambda_1}^{-d-1}(\Gamma_0)$.

In this way, we get a sequence of homeomorphism $\psi_j, j \geq 0$. The construction implies that

1. For any $d \geq 0$,

$$\psi_{d+1}|_{f_{\lambda_1}^{-d}(B_{\lambda_1})} = \psi_d|_{f_{\lambda_1}^{-d}(B_{\lambda_1})}.$$

2. The graph $G_{\lambda_2}(\theta_1, \dots, \theta_N)$ is also admissible for f_{λ_2} and the annulus $P_0^{\lambda_2}(f_{\lambda_2}^k(c)) \setminus P_1^{\lambda_2}(f_{\lambda_2}^{k+1}(c))$ is non-degenerate.

By Lemma 5.3, the ψ_j converges to a continuous and one-to-one map $\psi_\infty : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$, holomorphic in the Fatou set $F(f_{\lambda_1}) = \cup_k f_{\lambda_1}^{-k}(B_{\lambda_1})$ and satisfies $f_{\lambda_2} \circ \psi_\infty = \psi_\infty \circ f_{\lambda_1}$. This is a topological conjugacy, as required.

5.3. Rigidity

In order to improve the quality of ψ_∞ constructed in the previous section, we first introduce the QC-Criterion result in [16].

For a homeomorphism $\phi : \Omega \rightarrow \tilde{\Omega}$ and for $x \in \Omega$, let

$$\underline{H}(\phi, x) = \liminf_{r \rightarrow 0} \frac{\sup_{|y-x|=r} |\phi(y) - \phi(x)|}{\inf_{|y-x|=r} |\phi(y) - \phi(x)|} \in [1, \infty].$$

LEMMA 5.5. – *Let $\phi : \Omega \rightarrow \tilde{\Omega}$ be a homeomorphism between two Jordan domains, $k \in (0, 1)$ be a constant. Let X be a subset of Ω such that both X and $\phi(X)$ have zero Lebesgue measures. Assume the following hold:*

1. $|\bar{\partial}\phi| \leq k|\partial\phi|$ a.e on $\Omega \setminus X$.
2. For each $x \in X$, $\underline{H}(\phi, x) < \infty$.

Then ϕ is a K -quasiconformal map, where K depends only on k .

Lemma 5.5 is a simplified version of [16, Lemma 12.1]. See therein for a detailed proof based on a standard extremal length argument.

Let $U \subsetneq \mathbb{C}$ be a simply connected planar domain and $z \in U$. The shape of U about z is defined by:

$$\text{Shape}(U, z) = \sup_{x \in \partial U} |x - z| / \inf_{x \in \partial U} |x - z|.$$

REMARK 5.6. – In Lemma 5.5, Condition 2 can be replaced by: There is a constant $M > 0$ such that for each $x \in X$, there exist a number $M_x > 0$ and a sequence of shrinking topological disks $D_1 \supset D_2 \supset \dots$ containing x , such that

- (a) $\bigcap \overline{D_j} = \{x\}$;
- (b) $\text{Shape}(D_j, x) \leq M$, $\text{Shape}(\phi(D_j), \phi(x)) \leq M_x$, $\forall j \geq 1$.

The original proof of [16, Lemma 12.1] goes through without essential differences.

By Lemma 5.5 and Remark 5.6, to show that ψ_∞ is a quasiconformal map it suffices to show that

LEMMA 5.7. – There is a constant $M > 0$ such that for any $z \in J(f_{\lambda_1})$, there exist a number $M_z > 0$ and a sequence of topological disks $D_1 \supset D_2 \supset \dots$ containing z , such that

- (a) $\bigcap \overline{D_j} = \{z\}$;
- (b) $\text{Shape}(D_j, z) \leq M$, $\text{Shape}(\psi_\infty(D_j), \psi_\infty(z)) \leq M_z$, $\forall j \geq 1$.

Assuming Lemma 5.7, we can give the following

Proof of Theorem 5.1 when $\theta(\lambda_1)$ is an irrational number. – In this case, with the help of the Yoccoz puzzle, we have constructed a topological conjugacy ψ_∞ , holomorphic in the Fatou set of f_{λ_1} . By Lemmas 5.5 and 5.7, ψ_∞ is a quasiconformal map.

By Theorem 5.4, the Lebesgue measure of $J(f_{\lambda_1})$ is zero, so ψ_∞ is a Möbius map of the form $\psi_\infty(z) = az$. One may verify that $a^{n-1} = 1$ and $\lambda_2 = a^2 \lambda_1$. The condition $\lambda_1, \lambda_2 \in \mathcal{F}$ implies that $\lambda_1 = \lambda_2$. \square

5.4. Proof of Lemma 5.7

Recall that for $\lambda \in \{\lambda_1, \lambda_2\}$, we have found an admissible graph $G_\lambda(\theta_1, \dots, \theta_N)$ and had a Yoccoz puzzle construction induced by this graph.

We first define the Yoccoz τ_Y -function as follows. We choose some $c \in C_\lambda$. For each $d \geq 1$, we define $\tau_Y(d)$ to be the biggest integer $k \in [0, d-1]$ such that the puzzle piece $f_\lambda^{d-k}(P_d^\lambda(c))$ contains some critical point in C_λ , we set $\tau_Y(d) = -1$ if no such integer exists. Since $f_\lambda(e^{\pi i/n} z) = -f_\lambda(z)$, by the symmetry of puzzle pieces (namely, $P_d^\lambda(e^{\pi i/n} z) = e^{\pi i/n} P_d^\lambda(z)$ for all $d \geq 1$, see [28, Lemma 4.1]), we see that the Yoccoz τ_Y -function is well-defined (independent of the choice of $c \in C_\lambda$). Moreover, it satisfies $\tau_Y(d+1) \leq \tau_Y(d) + 1$.

We say that the critical set C_λ is *non-recurrent* if $\tau_Y(d)$ is uniformly bounded for all $d \geq 1$; *recurrent* if $\limsup \tau_Y(d) = \infty$ (this definition is in fact consistent with the definition in Section 3). There are two cases when C_λ is recurrent. We say C_λ is *reluctantly recurrent* if $\liminf \tau_Y(d) < \infty$; *persistently recurrent* if $\liminf \tau_Y(d) = \infty$.

For $\lambda \in \{\lambda_1, \lambda_2\}$ and $z \in J(f_\lambda)$, let $\omega(z)$ be the ω -limit set of z , defined as

$$\{w \in J(f_\lambda); \text{ there exist } n_k \rightarrow \infty \text{ such that } f_\lambda^{n_k}(z) \rightarrow w\}.$$

Recall that Γ_0^λ is defined as

$$\Gamma_0^\lambda = \bigcup_{k \geq 0} \bigcup_{1 \leq j \leq N} \Omega_\lambda^{\tau^k(\theta_j)}.$$

We then decompose $J(f_\lambda)$ into three disjoint sets $J_0^\lambda, J_1^\lambda, J_2^\lambda$, where

$$\begin{aligned} J_0^\lambda &= J(f_\lambda) \cap \left(\bigcup_{k \geq 0} f_\lambda^{-k}(\Gamma_0^\lambda) \right), \\ J_1^\lambda &= \{z \in J(f_\lambda) \setminus J_0^\lambda; C_\lambda \cap \omega(z) = \emptyset\}, \\ J_2^\lambda &= \{z \in J(f_\lambda) \setminus J_0^\lambda; C_\lambda \cap \omega(z) \neq \emptyset\}. \end{aligned}$$

Our task is to verify Lemma 5.7 in these three cases.

Case 1: $z \in J_0^\lambda$. – Let us first look at the points $z \in J(f_\lambda) \cap \Gamma_0^\lambda$. The idea of the proof is based on the expanding property of f_λ on $J(f_\lambda) \cap \Gamma_0^\lambda$. Recall that S_k^λ with $k = \pm 1, \dots, \pm(n-1)$ are the closed sectors defined in Section 3, let $\widehat{B}_\lambda = \{w \in B_\lambda; |\phi_\lambda(w)| > 2\}$ be a subset of B_λ and $\widehat{S}_k^\lambda = S_k^\lambda \setminus f_\lambda^{-1}(\widehat{B}_\lambda)$. For each $z \in J(f_\lambda) \cap \Gamma_0^\lambda$ and for all $k \geq 0$, let $U_0(f_\lambda^k(z))$ be the \widehat{S}_j^λ containing $f_\lambda^k(z)$ (note that $U_0(f_\lambda^k(z))$ is always well-defined because $f_\lambda^k(z)$ cannot sit on the boundary of two sectors), and then let $U_k(z)$ be the component of $f_\lambda^{-k}(U_0(f_\lambda^k(z)))$ containing z . The following facts are immediate:

1. For all $z \in J(f_\lambda) \cap \Gamma_0^\lambda$, the compact set $U_{k+1}(z)$ is contained in the interior of $U_k(z)$.
2. There are two constants $C_1, C_2 > 0$ such that for all $z \in J(f_\lambda) \cap \Gamma_0^\lambda$, we have $\text{Shape}(U_1(z), z) \leq C_1$ and $\text{mod}(U_0(z) \setminus U_1(z)) \geq C_2$.

Note that for all $k \geq 0$, the map $f_\lambda^{k-1} : U_k(z) \rightarrow U_1(f_\lambda^k(z))$ is a conformal map. By the shape distortion [28, Lemma 6.1], we have

$$\text{Shape}(U_k(z), z) \leq C_3 \text{Shape}(U_1(f_\lambda^k(z)), f_\lambda^k(z)) \leq C_3 C_1,$$

where C_3 depends on C_2 .

By pulling back via f_λ and the shape distortion, one sees that the above inequality holds for all $z \in J_0^\lambda$.

In particular, the above discussion holds for $\lambda = \lambda_1$. In order to verify Lemma 5.7, it remains to show that $\text{Shape}(\psi_\infty(U_k(z)), \psi_\infty(z))$ is uniformly bounded above for all k and all $z \in J_0^{\lambda_1}$. By the uniform continuity of ψ_∞ , there are two constants $C'_1, C'_2 > 0$ such that for any $z \in J(f_{\lambda_1}) \cap \Gamma_0^{\lambda_1}$, $\text{Shape}(\psi_\infty(U_1(z)), \psi_\infty(z)) \leq C'_1$ and $\text{mod}(\psi_\infty(U_0(z) \setminus U_1(z))) \geq C'_2$. Because $f_{\lambda_2}^{k-1} : \psi_\infty(U_k(z)) \rightarrow \psi_\infty(U_0(f_{\lambda_1}^k(z)))$ is a conformal map, by the shape distortion [28, Lemma 6.1], we have

$$\text{Shape}(\psi_\infty(U_k(z)), \psi_\infty(z)) \leq C'_3 \text{Shape}(\psi_\infty(U_1(f_{\lambda_1}^k(z))), \psi_\infty(f_{\lambda_1}^k(z))) \leq C'_3 C'_1,$$

where C'_3 depends on C'_2 . By pulling back via f_{λ_2} and the shape distortion, we see that the above inequality holds for all $z \in J_0^{\lambda_1}$.

Case 2: $z \in J_1^{\lambda_1}$. – In this case, by Lemma 5.3, $C_{\lambda_1} \cap \omega(z) = \emptyset$ implies that there exists an integer $n_0 = n_0(z)$ such that $f_{\lambda_1}^k(z) \notin \bigcup_{c \in C_{\lambda_1}} P_{n_0}^{\lambda_1}(c)$ for all $k \geq 1$. There also exists a sequence of integers $k_j \rightarrow \infty$ such that $f_{\lambda_1}^{k_j}(z) \rightarrow z^* \in J(f_{\lambda_1}) \setminus J_0^{\lambda_1}$. By passing to a subsequence, we assume $f_{\lambda_1}^{k_j}(z) \in P_{n_0}^{\lambda_1}(z^*)$ for all j . We have $\deg(f_{\lambda_1}^{k_j} : P_{n_0+k_j}^{\lambda_1}(z) \rightarrow P_{n_0}^{\lambda_1}(z^*)) \leq 2$. Take a small number $r > 0$ such that $\text{mod}(P_{n_0}^{\lambda_1}(z^*) \setminus \overline{D(z^*, r)}) \geq 1$, where $D(z^*, r)$ is the open Euclidean disk centered at z^* with radius r . For large j , we have $f_{\lambda_1}^{k_j}(z) \in D(z^*, r/2)$. Let $D_j(z)$ be the component of $f_{\lambda_1}^{-k_j}(D(z^*, r))$ containing z . Then by the shape distortion [28, Lemma 6.1], there is a universal constant C such that for all large j ,

$$\text{Shape}(D_j(z), z) \leq C \text{Shape}(D(z^*, r), f_{\lambda_1}^{k_j}(z)) \leq 3C.$$

Let $m = \text{mod}(\psi_\infty(P_{n_0}^{\lambda_1}(z^*) \setminus \overline{D(z^*, r)})) = \text{mod}(P_{n_0}^{\lambda_2}(\psi_\infty(z^*)) \setminus \overline{\psi_\infty(D(z^*, r))})$. Again by the shape distortion [28, Lemma 6.1], there is a constant C_1 depending on m such that for all large j ,

$$\text{Shape}(\psi_\infty(D_j(z)), \psi_\infty(z)) \leq C_1 \text{Shape}(\psi_\infty(D(z^*, r)), \psi_\infty(f_{\lambda_1}^{k_j}(z))) \leq C_1 C_2,$$

where C_2 depends on z but is independent of j .

Case 3: $z \in J_2^\lambda$ and $c \in C_\lambda \cap \omega(z)$, $\lambda \in \{\lambda_1, \lambda_2\}$. – There are three subcases:

Case 3.1: C_λ is non-recurrent. – First there are three puzzle pieces $P_{n_2}^\lambda \Subset P_{n_1}^\lambda \Subset P_{n_0}^\lambda$ of depths n_2, n_1, n_0 and a sequence of integers $\{k_j\}_j$ such that $f_\lambda^{k_j}(c) \in P_{n_2}^\lambda$ and $\deg(f_\lambda^{k_j} : P_{n_0+k_j}^\lambda(c) \rightarrow P_{n_0}^\lambda) = 2$ for all j . For each j , let $l_j \geq 0$ be the smallest integer such that $f_\lambda^{l_j}(z) \in P_{n_2+k_j}^\lambda(c)$. Then

$$\deg(f_\lambda^{l_j} : P_{n_0+k_j+l_j}^\lambda(z) \rightarrow P_{n_0+k_j}^\lambda(c)) \leq C 2^{n_2-n_0}$$

for all j , where $C = 2^{\text{Card}(C_\lambda)} = 2^{2n}$. We have that for all j ,

$$\deg(f_\lambda^{k_j+l_j} : P_{n_0+k_j+l_j}^\lambda(z) \rightarrow P_{n_0}^\lambda) \leq C 2^{n_2-n_0+1}$$

and $f_\lambda^{k_j+l_j}(z) \in P_{n_2}^\lambda$. By the shape distortion [28, Lemma 6.1],

$$\text{Shape}(P_{n_1+k_j+l_j}^\lambda(z), z) \leq M_2 < +\infty.$$

This constant M_2 is independent of the points $z \in J_2^\lambda$ and j .

Case 3.2: C_λ is reluctantly recurrent. – In this case, there exist $c' \in C_\lambda$ (c' may be c) and integers n_0 and $\{k_j\}_j$ such that $f_\lambda^{k_j}$ maps $P_{n_0+k_j}^\lambda(c)$ to $P_{n_0}^\lambda(c')$ of degree two. Take puzzle pieces $P_{n_2}^\lambda(c') \Subset P_{n_1}^\lambda(c') \Subset P_{n_0}^\lambda(c')$. For each j , let $l_j \geq 0$ be the smallest integer such that $f_\lambda^{l_j}(f_\lambda^{k_j}(c)) \in P_{n_2}^\lambda(c')$, then for all j ,

$$\deg(f_\lambda^{k_j+l_j} : P_{n_0+k_j+l_j}^\lambda(c) \rightarrow P_{n_0}^\lambda(c')) \leq 2^{n_2-n_0}.$$

For each j , let m_j be the smallest integer such that $f_\lambda^{m_j}(z) \in P_{n_2+k_j+l_j}^\lambda(c)$. Then

$$\deg(f_\lambda^{m_j} : P_{n_0+k_j+l_j+m_j}^\lambda(z) \rightarrow P_{n_0+k_j+l_j}^\lambda(c)) \leq C 2^{n_2-n_0},$$

where $C = 2^{\text{Card}(C_\lambda)} = 2^{2n}$. We have that for all j ,

$$\deg(f_\lambda^{k_j+l_j+m_j} : P_{n_0+k_j+l_j+m_j}^\lambda(z) \rightarrow P_{n_0}^\lambda(c')) \leq C 4^{n_2-n_0}.$$

Therefore, by the shape distortion [28, Lemma 6.1],

$$\text{Shape}(P_{n_1+k_j+l_j+m_j}^\lambda(z), z) \leq M_3 < +\infty.$$

This constant M_3 is independent of the points $z \in J_2^\lambda$ and j .

Case 3.3: C_λ is persistently recurrent. – In this case, the following crucial property holds:

There is a number $m > 0$ and a sequence of puzzle pieces containing c :

$$K'_1(c) \supset K_1(c) \supset \widetilde{K}_1(c) \supset K'_2(c) \supset K_2(c) \supset \widetilde{K}_2(c) \supset \dots$$

such that $\bigcap \overline{K_j(c)} = \{c\}$ and for all $j \geq 1$,

$$\text{mod}(K'_j(c) \setminus \overline{K_j(c)}), \text{mod}(K_j(c) \setminus \overline{\widetilde{K}_j(c)}) \geq m$$

and

$$(K'_j(c) \setminus \overline{K_j(c)}) \cap \left(\bigcup_{k \geq 0} f_\lambda^k(C_\lambda)\right) = (K_j(c) \setminus \overline{\widetilde{K}_j(c)}) \cap \left(\bigcup_{k \geq 0} f_\lambda^k(C_\lambda)\right) = \emptyset.$$

This construction of the sequence of puzzle pieces is due to Kozlovski, Shen and van Strien in [16, Section 8]. The proof of the complex bounds of the moduli is given by Kozlovski-van Strien [17] and Qiu-Yin [29] independently. In literature, this sequence of puzzle pieces is called *KSS nest* or *enhanced nest*. See [16] for its original construction.

In our case, there is essentially only one free critical orbit, the construction is simplified. Following [35], we will sketch the construction of a sequence of puzzle pieces containing c inductively:

$$I_1(c) \supseteq K'_1(c) \supseteq K_1(c) \supseteq \dots \supseteq I_j(c) \supseteq K'_j(c) \supseteq K_j(c) \supseteq I_{j+1}(c) \supseteq \dots$$

Let $I_1(c) = P_L^\lambda(c)$ be a critical puzzle piece, the number L is chosen so that the annulus $P_L^\lambda(c) \setminus \overline{P_{L+1}^\lambda(c)}$ is non-degenerate. Assume that $I_j(c) = P_{k_j}^\lambda(c)$ is defined. It follows from [28, Lemma 4.7] that the set

$$S_j = \{k \geq L : \tau_Y(k) = k_j, \tau_Y(k + 1) = k_j + 1\}$$

consists of at least two elements. Since the critical set C_λ is persistently recurrent, the set S_j consists of at most finite elements. Let $a_j = \min S_j$ and $b_j = \max S_j$. Set $K'_j(c) = P_{b_j}^\lambda(c)$ and $K_j(c) = P_{t_j}^\lambda(c)$ with $t_j = b_j + a_j - k_j$. Let $s_j > b_j - k_j$ be the largest integer such that $P_{a_j}^\lambda(f_\lambda^k(c)) \cap C_\lambda = \emptyset$ for all $b_j - k_j < k < s_j$, and set $\widetilde{K}_j(c) = P_{s_j}^\lambda(c)$. Now let

$$Z_j = \{k \geq L : \tau_Y(k) = b_j + a_j - k_j, \tau_Y(k + 1) = b_j + a_j - k_j + 1\}.$$

Again, the set Z_j is a finite set containing at least two elements. Let $k_{j+1} = \max Z_j$ and $I_{j+1}(c) = P_{k_{j+1}}^\lambda(c)$. The sequence of puzzle pieces is then constructed inductively.

In this case, by [36, Proposition 1], there is a constant $M > 0$ such that the $\text{Shape}(K_j(c), c) \leq M$ for all j . For each j , let m_j be the smallest integer such that $f_\lambda^{m_j}(z) \in \widetilde{K}_j(c) = P_{s_j}^\lambda(c)$. It follows that

$$\text{deg}(f_\lambda^{m_j} : P_{m_j+s_j}^\lambda(z) \rightarrow P_{s_j}^\lambda(c)) \leq 2^{\text{Card}(C_\lambda)}.$$

The property

$$(K'_j(c) \setminus \overline{K_j(c)}) \cap \left(\bigcup_{k \geq 0} f_\lambda^k(C_\lambda)\right) = (K_j(c) \setminus \overline{\widetilde{K}_j(c)}) \cap \left(\bigcup_{k \geq 0} f_\lambda^k(C_\lambda)\right) = \emptyset$$

implies that

$$\deg(f_\lambda^{m_j} : P_{m_j+b_j}^\lambda(z) \rightarrow P_{b_j}^\lambda(c)) = \deg(f_\lambda^{m_j} : P_{m_j+t_j}^\lambda(z) \rightarrow P_{t_j}^\lambda(c)) \leq 2^{\text{Card}(C_\lambda)}.$$

By the complex bounds for moduli and the shape distortion, we have

$$\text{Shape}(P_{t_j+m_j}^\lambda(z), z) \leq M_4 < +\infty.$$

This constant M_4 is independent of the points $z \in J_2^\lambda$ and j .

This completes the proof of Lemma 5.7 in all possible cases.

6. $\partial\mathcal{H}_0$ is a Jordan curve

In this section, we prove that $\partial\mathcal{H}_0$ is a Jordan curve and give some consequences.

THEOREM 6.1. – *$\partial\mathcal{H}_0$ is a Jordan curve.*

Proof. – We first show that \mathcal{X}_0 is a singleton. To do this, first note that the parameter ray $\mathcal{R}_0(0)$ is contained in the real and positive axis. So \mathcal{X}_0 contains at least one positive number. We define $g_\lambda(z) = z^n(f_\lambda(z) - z) = z^{2n} - z^{n+1} + \lambda$ for $\lambda, z > 0$. The positive critical point of g_λ is $z_* = (\frac{n+1}{2n})^{\frac{1}{n-1}}$ and for all $z > z_*$, we have $g'_\lambda(z) > 0$.

Let λ_* solve $g_{\lambda_*}(z_*) = 0$, then $\lambda_* = \frac{n-1}{2n}(\frac{n+1}{2n})^{\frac{n+1}{n-1}}$. For any $\lambda > \lambda_*$, we have $g_\lambda > 0$. In this case, let us look at the graph of the real function $f_\lambda(z)$ with $\lambda, z > 0$, we see that for any $\lambda > \lambda_*, z > 0$, we have $f_\lambda^k(z) \rightarrow \infty$ as $k \rightarrow \infty$. This implies $(0, +\infty) \subset B_\lambda$. In particular, $v_\lambda^+ \in B_\lambda$. Thus $(\lambda_*, +\infty) \subset \mathcal{R}_0(0)$. On the other hand, we have $f_{\lambda_*}(z_*) = z_*$ and $f'_{\lambda_*}(z_*) = 1$. So λ_* is a cusp and $\lambda_* \in \mathcal{X}_0$. Moreover, by elementary properties of real functions, there is a small number $\epsilon > 0$ such that for all $\lambda \in (\lambda_* - \epsilon, \lambda_*)$, the map f_λ has an attracting cycle. So $(\lambda_* - \epsilon, \lambda_*)$ is contained in a hyperbolic component (of renormalizable type, see Section 8) and $(\lambda_* - \epsilon, \lambda_*) \cap \mathcal{X}_0 = \emptyset$. If $\mathcal{X}_0 \setminus \{\lambda_*\} \neq \emptyset$, then there is $\lambda \in \mathcal{X}_0 \cap \mathcal{F}$ which is not a cusp. By Lemma 4.5, we have $0 < \theta(\lambda) < \frac{1}{2(n-1)}$. However by Lemma 4.8, we have $\theta(\lambda) = 0$. This leads to a contradiction.

In the following, we assume $t \in (0, \frac{1}{n-1})$. Take two parameters $\lambda_1, \lambda_2 \in \mathcal{X}_t \cap \mathcal{F}$ which are not cusps, it follows from Lemma 4.8 that $\theta(\lambda_1) = \theta(\lambda_2) = t/2$. By Theorem 5.1 we have $\lambda_1 = \lambda_2$. Since there are countably many cusps, the impression \mathcal{X}_t is necessarily a singleton. So $\partial\mathcal{H}_0$ is locally connected.

If there are two different angles $t_1, t_2 \in [0, \frac{1}{n-1})$ with $\mathcal{X}_{t_1} = \mathcal{X}_{t_2} = \{\lambda\}$, then by Lemma 4.8, the external rays $R_\lambda(t_1/2)$ and $R_\lambda(t_2/2)$ land at the same point on ∂B_λ . But this is a contradiction since ∂B_λ is a Jordan curve (Theorem 3.1). \square

Theorem 6.1 has several consequences. First, one gets a canonical parameterization $\nu : \mathbb{S} \rightarrow \partial\mathcal{H}_0$, where $\nu(\theta)$ is defined to be the landing point of the parameter ray $\mathcal{R}_0(\theta)$ (namely, $\nu(\theta) := \lim_{r \rightarrow 1^+} \Phi_0^{-1}(r e^{2\pi i \theta})$).

THEOREM 6.2. – *$\nu(\theta)$ is a cusp if and only if θ is τ -periodic.*

Proof. – By Theorem 6.1, we see that $\nu(0) = \frac{n-1}{2n} \left(\frac{n+1}{2n}\right)^{\frac{n+1}{n-1}}$ is a cusp.

Note that $\nu(\theta + \frac{1}{n-1}) = e^{2\pi i/(n-1)}\nu(\theta)$ and $(-1)^n f_{\nu(\theta + \frac{1}{n-1})}(e^{\pi i/(n-1)}z) = e^{\pi i/(n-1)}f_{\nu(\theta)}(z)$, thus $\nu(\theta)$ is a cusp if and only if $\nu(\theta + \frac{1}{n-1})$ is a cusp. For this, we assume $\theta \in (0, \frac{1}{n-1})$.

If $\nu(\theta)$ is a cusp, then by Lemma 4.8, the external ray $R_{\nu(\theta)}(\theta/2)$ lands at $\beta_{\nu(\theta)}$. By Remark 4.7, $\frac{\theta}{2}$ satisfies either $\tau^p(\frac{\theta}{2}) \equiv \frac{\theta}{2}$ or $\tau^p(\frac{\theta}{2}) \equiv \frac{\theta}{2} + \frac{1}{2}$ for some $p \geq 1$. In either case, θ is τ -periodic.

Conversely, we assume θ is τ -periodic. If $\nu(\theta)$ is not a cusp, then by Lemma 4.8, the external ray $R_{\nu(\theta)}(\frac{\theta}{2})$ lands at $v_{\nu(\theta)}^+$. Note that $\frac{\theta}{2}$ satisfies either $\tau^p(\frac{\theta}{2}) = \frac{\theta}{2}$ or $\tau^p(\frac{\theta}{2}) = \frac{\theta}{2} + \frac{1}{2}$ for some $p \geq 1$. We have that either $f_{\nu(\theta)}^p(v_{\nu(\theta)}^+) = v_{\nu(\theta)}^+$ or $f_{\nu(\theta)}^p(v_{\nu(\theta)}^+) = v_{\nu(\theta)}^-$. In the former case, we get a periodic critical point $c \in f_{\nu(\theta)}^{-1}(v_{\nu(\theta)}^+)$; in the latter case, we get a periodic critical point $c \in f_{\nu(\theta)}^{-1}(v_{\nu(\theta)}^-)$. These critical points will be in the Fatou set. But this contradicts $v_{\nu(\theta)}^+ \in \partial B_{\nu(\theta)}$. □

REMARK 6.3. – *As a consequence of Lemma 4.8 and Theorem 6.2,*

1. *if θ is τ -periodic, then $\nu(\theta)$ is a cusp;*
2. *if θ is rational but not τ -periodic, then $f_{\nu(\theta)}$ is postcritically finite;*
3. *if θ is irrational, then $f_{\nu(\theta)}$ is postcritically infinite.*

In the last two cases, one has $C_{\nu(\theta)} \subset \partial B_{\nu(\theta)}$. Moreover, by Borel’s Normal Number Theorem [4], for almost all $\theta \in (0, 1]$, we have $\bigcup_{k \geq 1} f_{\nu(\theta)}^k(C_{\nu(\theta)}) = \partial B_{\nu(\theta)}$.

PROPOSITION 6.4. – *Set $\partial B_0 = \mathbb{S}$ and $\mathcal{V} = \mathbb{C} \setminus \overline{\mathcal{H}_0}$, then there is a holomorphic motion $H : \mathcal{V} \times \mathbb{S} \rightarrow \mathbb{C}$ parameterized by \mathcal{V} and with base point 0 such that $H(\lambda, \mathbb{S}) = \partial B_\lambda$ for all $\lambda \in \mathcal{V}$.*

Proof. – We first prove that every repelling periodic point of $f_0(z) = z^n$ moves holomorphically in $\mathcal{H}_2 \cup \{0\}$. Let $z_0 \in \mathbb{S} = J(f_0)$ be such a point with period k . For small λ , the map f_λ is a perturbation of f_0 . By Implicit Function Theorem, there is a neighborhood \mathcal{U}_0 of 0 such that z_0 becomes a repelling point z_λ of f_λ with the same period k , for all $\lambda \in \mathcal{U}_0$. On the other hand, for all $\lambda \in \mathcal{H}_2$, each repelling cycle of f_λ moves holomorphically throughout \mathcal{H}_2 (see [24], Theorem 4.2).

Since $\mathcal{H}_2 \cup \{0\}$ is simply connected, by the Monodromy Theorem [2], there is a holomorphic map $Z_{z_0} : \mathcal{H}_2 \cup \{0\} \rightarrow \mathbb{C}$ such that $Z_{z_0}(\lambda) = z_\lambda$ for $\lambda \in \mathcal{U}_0$. Let $\text{Per}(f_0)$ be all repelling periodic points of f_0 . One may verify that the map $y : \mathcal{H}_2 \cup \{0\} \times \text{Per}(f_0) \rightarrow \mathbb{C}$ defined by $y(\lambda, z) = Z_z(\lambda)$ is a holomorphic motion. Note that $\mathbb{S} = \overline{\text{Per}(f_0)}$, by λ -Lemma (see [21] or [24]), there is an extension of y , say $Y : \mathcal{H}_2 \cup \{0\} \times \mathbb{S} \rightarrow \mathbb{C}$. It is obvious that $Y(\lambda, \mathbb{S})$ is a connected component of $J(f_\lambda)$.

Now, we show $Y(\lambda, \mathbb{S}) = \partial B_\lambda$ for all $\lambda \in \mathcal{H}_2 \cup \{0\}$. By the uniqueness of the holomorphic motion of hyperbolic Julia sets, it suffices to show $Y(\lambda, \mathbb{S}) = \partial B_\lambda$ for a small and real parameter $\lambda \in (0, \epsilon)$, where $\epsilon > 0$. To see this, note that when $\lambda \in (0, \epsilon)$ the fixed point $p_0 = 1$ of f_0 becomes the repelling fixed point p_λ of f_λ , which is real and close to 1. The map f_λ has exactly two real and positive fixed points. One is p_λ and the other is p_λ^* , which is near 0. It is obvious that p_λ is the landing point of the zero external ray of f_λ . So $Y(\lambda, 1) = p_\lambda \in \partial B_\lambda$. This implies $Y(\lambda, \mathbb{S}) = \partial B_\lambda$ for all $\lambda \in (0, \epsilon)$.

By the above argument and Carathéodory convergence theorem, the following map defines a holomorphic motion of $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ parameterized by \mathcal{V} :

$$h : \mathcal{V} \times (\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}) \rightarrow \widehat{\mathbb{C}} \quad h(u, z) := \phi_u^{-1}(z) \text{ if } u \in \mathcal{U} \setminus \{0\} \text{ and } h(0, z) := z.$$

By Slodkowski's Theorem (see [14] or [33]), there is a holomorphic motion $H : \mathcal{V} \times \widehat{\mathbb{C}}$ extending h and for any $v \in \mathcal{V}$, we have $H(v, \mathbb{S}) = \partial B_v$. \square

THEOREM 6.5. – $\lambda \in \partial \mathcal{H}_0$ if and only if ∂B_λ contains either the critical set C_λ or a parabolic cycle of f_λ .

Proof. – By Lemma 4.4, it suffices to prove the ‘if’ part.

We first assume that f_λ has a parabolic cycle on ∂B_λ . By Lemma 4.6, the Julia set $J(f_\lambda)$ contains a quasiconformal copy of the Julia set of $z \mapsto z^2 + 1/4$. So the boundary ∂B_λ is not a quasi-circle. It follows from Proposition 6.4 that for all $v \in \mathbb{C} \setminus \overline{\mathcal{H}_0}$, ∂B_v is a quasi-circle. Thus $\lambda \in \partial \mathcal{H}_0$.

Now assume $\lambda \in \mathcal{F}$ and $C_\lambda \subset \partial B_\lambda$. Recall that $\theta(\lambda)$ is defined such that $R_\lambda(\theta(\lambda))$ lands at v_λ^+ . By Lemma 4.5, we have $0 < \theta(\lambda) < \frac{1}{2(n-1)}$.

Similar to the proof of Theorem 6.2, we conclude that $2\theta(\lambda)$ is not τ -periodic. Then $\lambda' = \nu(2\theta(\lambda)) \in \mathcal{F}$ is not a cusp (Theorem 6.2). It satisfies $\theta(\lambda') = \theta(\lambda)$. It follows from Proposition 5.1 that $\lambda = \lambda' \in \partial \mathcal{H}_0$. \square

7. Sierpiński holes are Jordan domains

Besides \mathcal{H}_0 , there are two kinds of escape domains: the McMullen domain \mathcal{H}_2 and the Sierpiński locus $\mathcal{H}_k, k \geq 3$. In [6], Devaney showed that the boundary $\partial \mathcal{H}_2$ is a Jordan curve by constructing a sequence of analytic curves converging to it. In this section, we will show that the boundary of every Sierpiński hole is a Jordan curve. We remark that our approach also applies to $\partial \mathcal{H}_2$. This will yield a different proof from Devaney's. An interesting fact is that our proof relies on the boundary regularity of \mathcal{H}_0 .

Let \mathcal{H} be an escape domain of level $k \geq 3$. It has no intersection with $\mathbb{R}^+ := (0, +\infty)$. (In fact, by elementary properties of real functions, one may verify that there is a positive parameter $\lambda^* \in (0, \nu(0))$ such that $(0, \lambda^*) \subset \mathcal{H}_2$ and for all $\lambda \in [\lambda^*, \nu(0)]$, the critical orbit of f_λ remains bounded, in that case, f_λ is renormalizable, see [28] Lemma 7.5.)

The relation $e^{\pi i/(n-1)} f_\lambda(z) = (-1)^n f_{e^{2\pi i/(n-1)} \lambda}(e^{\pi i/(n-1)} z)$ implies that $e^{2\pi i/(n-1)} \mathcal{H}_k = \mathcal{H}_k$. So we may assume $\mathcal{H} \subset \mathcal{F}$. The relation $\overline{f_\lambda(z)} = f_{\overline{\lambda}}(\overline{z})$ implies that \mathcal{H}_k is symmetric about the real axis. We may assume further: either \mathcal{H} is symmetric about $\{\lambda \in \mathbb{C}^*; \arg \lambda = \frac{\pi}{n-1}\}$ or $\mathcal{H} \subset \{\lambda \in \mathbb{C}^*; 0 < \arg \lambda < \frac{\pi}{n-1}\}$.

The parameter ray $\mathcal{R}_{\mathcal{H}}(t)$ of angle $t \in (0, 1]$ in \mathcal{H} is defined by $\mathcal{R}_{\mathcal{H}}(t) := \Phi_{\mathcal{H}}^{-1}((0, 1)e^{2\pi i t})$, its impression $\mathcal{X}_{\mathcal{H}}(t)$ is defined by

$$\mathcal{X}_{\mathcal{H}}(t) := \bigcap_{j \geq 1} \overline{\Phi_{\mathcal{H}}^{-1}(\{re^{2\pi i \theta}; 1 - 1/j < r < 1, |\theta - t| < 1/j\})}.$$

When λ ranges over $\overline{\mathcal{H}}$, the preimages $f_\lambda^{2-k}(0)$ move continuously and f_λ^{k-2} maps each component of $f_\lambda^{2-k}(T_\lambda)$ conformally onto T_λ . Let U_λ be the component of $f_\lambda^{2-k}(T_\lambda)$ containing v_λ^+ and g_λ be the inverse of $f_\lambda^{k-2}|_{U_\lambda}$. Both $g_\lambda(0)$ and U_λ move continuously

for $\lambda \in \overline{\mathcal{H}}$ (and holomorphically in \mathcal{H}). The internal ray $R_{U_\lambda}(t)$ of angle t in U_λ is defined by $R_{U_\lambda}(t) := g_\lambda(R_{T_\lambda}(t))$.

LEMMA 7.1. – For any integer $p \geq 0$, the set $f_\lambda^{-p}(\overline{B}_\lambda)$ moves continuously (in the Hausdorff topology) with respect to $\lambda \in \mathbb{C}^* \setminus \overline{\mathcal{H}_0}$.

Proof. – It is an immediate consequence of Proposition 6.4. □

LEMMA 7.2. – For any $t \in [0, 1)$ and any $\lambda \in \mathcal{X}_{\mathcal{H}}(t) \setminus \partial\mathcal{H}_0$, we have $v_\lambda^+ \in \partial U_\lambda$ and the internal ray $R_{U_\lambda}(t)$ lands at v_λ^+ .

Proof. – It follows from Lemma 7.1 that the closure of the external ray $R_\lambda(t)$ moves continuously (in the Hausdorff topology) for $\lambda \in \overline{\mathcal{H}} \setminus \partial\mathcal{H}_0$. Note that pulling back $R_\lambda(t)$ via f_λ^p preserves the continuity. □

PROPOSITION 7.3. – For any $t \in [0, 1)$, the impression $\mathcal{X}_{\mathcal{H}}(t)$ is either a singleton or contained in $\partial\mathcal{H}_0$.

Proof. – If not, there exist $t \in [0, 1)$ and a connected and compact subset \mathcal{E} of $\mathcal{X}_{\mathcal{H}}(t) \setminus \partial\mathcal{H}_0$ containing at least two points. In fact, the set \mathcal{E} can be chosen as follows: we first take a point $\lambda_0 \in \mathcal{X}_{\mathcal{H}}(t) \setminus \partial\mathcal{H}_0$, then let \mathcal{E} be a connected component of $\{\lambda : |\lambda - \lambda_0| \leq r\} \cap (\mathcal{X}_{\mathcal{H}}(t) \setminus \partial\mathcal{H}_0)$ where $r > 0$ is a very small number. By Lemma 7.2, the internal ray $R_{U_\lambda}(t)$ lands at v_λ^+ for $\lambda \in \mathcal{E}$. One may verify that for any $\lambda \in \mathcal{E}$, we have $f_\lambda^{k-2}(v_\lambda^+) \notin \partial B_\lambda$ and $f_\lambda^{k-1}(v_\lambda^+) \in \partial B_\lambda$. There is disk neighborhood $\mathcal{D} \subset \mathbb{C}^* \setminus \partial\mathcal{H}_0$ of \mathcal{E} such that for all $\lambda \in \mathcal{D}$, $f_\lambda^{k-2}(v_\lambda^+) \notin \overline{B}_\lambda$.

Take two different parameters $\lambda_1, \lambda_2 \in \mathcal{E}$ with $|\arg \lambda_1 - \arg \lambda_2| < \frac{2\pi}{n-1}$ and let $J = \{f_{\lambda_1}^j(v_{\lambda_1}^\varepsilon); 0 \leq j \leq k-2, \varepsilon = \pm\}$ $\cup \overline{B}_{\lambda_1}$. We define a continuous map $h : \mathcal{D} \times J \rightarrow \widehat{\mathbb{C}}$ in the following way:

1. $h(\lambda_1, z) = z$ for all $z \in J$;
2. $h(\lambda, z) = \phi_\lambda^{-1} \circ \phi_{\lambda_1}(z)$ for all $(\lambda, z) \in \mathcal{D} \times \overline{B}_{\lambda_1}$;
3. for any $\lambda \in \mathcal{D}$, we define $h(\lambda, f_{\lambda_1}^j(v_{\lambda_1}^\varepsilon)) = f_\lambda^j(v_\lambda^\varepsilon)$ for $0 \leq j \leq k-2$ and $\varepsilon \in \{\pm\}$.

The map h is a holomorphic motion parameterized by \mathcal{D} , with base point λ_1 . By Slodkowski's theorem [33], there is a holomorphic motion $H : \mathcal{D} \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ extending h . We consider the restriction $H_0 = H|_{\mathcal{E} \times \widehat{\mathbb{C}}}$ of H . Note that for any $\lambda \in \mathcal{E}$, the map $H_0(\lambda, \cdot)$ preserves the postcritical relation. So there is unique continuous map $H_1 : \mathcal{E} \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that $H_1(\lambda_1, \cdot) \equiv \text{id}$ and the following diagram commutes:

$$\begin{array}{ccc}
 \widehat{\mathbb{C}} & \xrightarrow{f_{\lambda_1}} & \widehat{\mathbb{C}} \\
 H_1(\lambda, \cdot) \downarrow & & \downarrow H_0(\lambda, \cdot) \\
 \widehat{\mathbb{C}} & \xrightarrow{f_\lambda} & \widehat{\mathbb{C}}
 \end{array}$$

Set $\psi_0 = H_0(\lambda_2, \cdot)$ and $\psi_1 = H_1(\lambda_2, \cdot)$. Both ψ_0 and ψ_1 are quasiconformal maps satisfying $f_{\lambda_2} \circ \psi_1 = \psi_0 \circ f_{\lambda_1}$. One may verify that ψ_0 and ψ_1 are homotopic relative to $P(f_{\lambda_1}) \cup \overline{B}_{\lambda_1}$. To see this, note that $H_1(\lambda, \cdot)^{-1} \circ H_0(\lambda, \cdot)$ is continuous in $\lambda \in \mathcal{E}$ with $H_1(\lambda_1, \cdot)^{-1} \circ H_0(\lambda_1, \cdot) = \text{id}$, this is to say the map $\psi_1^{-1} \circ \psi_0 = H_1(\lambda_2, \cdot)^{-1} \circ H_0(\lambda_2, \cdot)$ is homotopic to the identity map relative to $P(f_{\lambda_1}) \cup \overline{B}_{\lambda_1}$.

Then there is a sequence of quasi-conformal maps ψ_j such that

- (a) $f_{\lambda_2} \circ \psi_{j+1} = \psi_j \circ f_{\lambda_1}$ for all $j \geq 0$,
- (b) ψ_{j+1} and ψ_j are homotopic relative to $f_{\lambda_1}^{-j}(P(f_{\lambda_1}) \cup \overline{B_{\lambda_1}})$.

The maps ψ_j form a normal family since their dilations are uniformly bounded above. Let ψ_∞ be the limit map of ψ_j . It is holomorphic in the Fatou set $F(f_{\lambda_1}) = \cup_k f_{\lambda_1}^{-k}(B_{\lambda_1})$ and satisfies $f_{\lambda_2} \circ \psi_\infty = \psi_\infty \circ f_{\lambda_1}$ in $F(f_{\lambda_1})$. By continuity, $f_{\lambda_2} \circ \psi_\infty = \psi_\infty \circ f_{\lambda_1}$ in $\widehat{\mathbb{C}}$.

By Theorem 5.4, the Lebesgue measures of $J(f_{\lambda_1})$ and $J(f_{\lambda_2})$ are zero. Thus ψ_∞ is a Möbius map. It takes the form $\psi_\infty(z) = az$ where $a^{n-1} = 1$ and $\lambda_2 = a^2 \lambda_1$. The condition $|\arg \lambda_1 - \arg \lambda_2| < \frac{2\pi}{n-1}$ implies $\lambda_1 = \lambda_2$. But this is a contradiction.

So the set $\mathcal{X}_{\mathcal{H}}(t) \setminus \partial\mathcal{H}_0$ is either empty or a singleton. This implies that the impression $\mathcal{X}_{\mathcal{H}}(t)$ is either a singleton or contained in $\partial\mathcal{H}_0$. □

PROPOSITION 7.4. – *The boundary $\partial\mathcal{H}$ is locally connected.*

Proof. – It follows from Lemma 7.3 that for any t , the impression $\mathcal{X}_{\mathcal{H}}(t)$ is either a singleton or contained in $\partial\mathcal{H}_0$. In the latter case, for any $\lambda \in \mathcal{X}_{\mathcal{H}}(t) \cap \mathcal{F}$ which is not a cusp, it follows from Lemma 4.8 that there is an external ray $R_\lambda(\alpha)$ landing at v_λ^+ .

We claim that $nt = n^{k-1}\alpha \pmod 1$. If not, then by Lemma 3.7, there is a cut ray Ω_λ^β separating $R_\lambda(nt)$ and $R_\lambda(n^{k-1}\alpha)$. By stability of cut rays, there exists a neighborhood \mathcal{U} of λ and $\varepsilon > 0$ such that $Z_{nt,\varepsilon}^u$ and $Z_{n^{k-1}\alpha,\varepsilon}^u$ are contained in different components of $\overline{\mathbb{C}} - \Omega_u^\beta$ for all $u \in \mathcal{U} \cap (\overline{\mathbb{C}} - \mathcal{H}_0)$, where

$$Z_{t,\varepsilon}^u := \phi_u^{-1}(\{re^{2\pi i\theta}; r > 1, |\theta - t| < \varepsilon\}).$$

Moreover, by shrinking \mathcal{U} a little bit, we see that $f_u^{k-1}(v_u^+) \in Z_{n^{k-1}\alpha,\varepsilon}^u$ for all $u \in \mathcal{H} \cap \mathcal{U}$. Then there is a cut ray $\Omega_u^\eta \subset f_u^{1-k}(\Omega_u^\beta)$, separating v_u^+ and $\bigcup_{|\theta-t|<\varepsilon/n} R_{U_u}(\theta)$ for all $u \in \mathcal{H} \cap \mathcal{U}$. However, by the definition of $\mathcal{X}_{\mathcal{H}}(t)$, when k is large so that $1/k < \varepsilon/n$, there is $\lambda_k \in \mathcal{U} \cap \Phi_{\mathcal{H}}^{-1}(\{re^{2\pi i\theta}; 1 - 1/k < r < 1, |\theta - t| < 1/k\})$. So we have $v_{\lambda_k}^+ \subset \bigcup_{|\theta-t|<1/k} R_{U_{\lambda_k}}(\theta) \subset \bigcup_{|\theta-t|<\varepsilon/n} R_{U_{\lambda_k}}(\theta)$. But this is a contradiction. This completes the proof of the claim.

Thus each $\lambda \in \mathcal{X}_{\mathcal{H}}(t)$ is either a cusp or contained in $\{\nu(\alpha); nt = n^{k-1}\alpha\}$ (a finite set). The connectivity of $\mathcal{X}_{\mathcal{H}}(t)$ implies that it is a singleton. □

THEOREM 7.5. – *The boundary $\partial\mathcal{H}$ is a Jordan curve.*

Proof. – If not, then there exist a parameter $\lambda \in \partial\mathcal{H}$ with $0 \leq \arg \lambda \leq \frac{\pi}{n-1}$ (by assumption of \mathcal{H}) and two different angles t_1, t_2 such that $\mathcal{X}_{\mathcal{H}}(t_1) = \mathcal{X}_{\mathcal{H}}(t_2) = \{\lambda\}$.

By Lemma 3.7, there is a cut ray Ω_λ^α separating the internal rays $R_{U_\lambda}(t_1)$ and $R_{U_\lambda}(t_2)$. Suppose that v_λ^+ and $R_{U_\lambda}(t_1)$ are contained in the same component of $\widehat{\mathbb{C}} - \Omega_\lambda^\alpha$. By the stability of cut rays, there is a neighborhood \mathcal{U} of λ such that for any $u \in \mathcal{U} \cap \mathcal{H}$, the set $\{v_u^+\} \cup R_{U_u}(t_1)$ and the internal ray $R_{U_u}(t_2)$ are contained in different components of $\widehat{\mathbb{C}} - \Omega_u^\alpha$. But this contradicts the assumption that $\mathcal{X}_{\mathcal{H}}(t_2) = \{\lambda\}$. □

8. Hyperbolic components of renormalizable type

In this section, we study the hyperbolic components of renormalizable type.

We begin with a definition. We say a McMullen map f_λ is *renormalizable* (resp. **-renormalizable*) at $c \in C_\lambda$ if there exist an integer $p \geq 1$ and two disks U and V containing c , such that $f_\lambda^p : U \rightarrow V$ (resp. $-f_\lambda^p : U \rightarrow V$) is a quadratic-like map whose Julia set is connected. The triple (f_λ^p, U, V) (resp. $(-f_\lambda^p, U, V)$) is called the *renormalization* (resp. **-renormalization*) of f_λ .

Let \mathcal{B} be a hyperbolic component of renormalizable type. For any $\lambda \in \mathcal{B}$, the map f_λ has an attracting cycle in \mathbb{C} , say $z_\lambda \mapsto f_\lambda(z_\lambda) \mapsto \dots \mapsto f_\lambda^p(z_\lambda) = z_\lambda$, where p is the period. We may assume that the attracting cycle is suitably chosen and labeled so that z_λ is holomorphic with respect to $\lambda \in \mathcal{B}$.

LEMMA 8.1 ([28], Prop 5.4). – *If $\lambda \in \mathcal{B}$, then f_λ is either renormalizable or *-renormalizable. Moreover,*

1. *if f_λ is renormalizable and n is odd, then f_λ has exactly two attracting cycles in \mathbb{C} ;*
2. *if f_λ is *-renormalizable and n is odd, then p is even, $f_\lambda^{p/2}(z_\lambda) = -z_\lambda$ and f_λ has exactly one attracting cycle in \mathbb{C} ;*
3. *if n is even, then f_λ has exactly one attracting cycle in \mathbb{C} and there is a unique $c \in C_\lambda$, such that f_λ is renormalizable at c .*

The terminology ‘hyperbolic component of renormalizable type’ comes from Lemma 8.1.

Let $\rho(\lambda) = (f_\lambda^p)'(z_\lambda)$ be the multiplier of the attracting cycle of f_λ for $\lambda \in \mathcal{B}$. Based on Lemma 8.1, we set $(\epsilon, k) = (-1, p/2)$ if n is odd and f_λ is *-renormalizable, and $(\epsilon, k) = (1, p)$ in the other cases. We define a map $\kappa : \mathcal{B} \rightarrow \mathbb{D}$ by $\kappa(\lambda) = (\epsilon f_\lambda^k)'(z_\lambda)$. Note that either $\rho = \kappa^2$ or $\rho = \kappa$.

The main result of this section is:

THEOREM 8.2. – *The map $\kappa : \mathcal{B} \rightarrow \mathbb{D}$ is a conformal map. It can be extended continuously to a homeomorphism from $\overline{\mathcal{B}}$ to $\overline{\mathbb{D}}$.*

Proof. – Note that $\kappa(\lambda)$ is the multiplier of the map $g_\lambda = \epsilon f_\lambda^k$ at its fixed point z_λ . By the Implicit Function Theorem, if $\mathcal{B} \ni \lambda_n \rightarrow \partial\mathcal{B}$, then $|\kappa(\lambda_n)| \rightarrow 1$, so the map $\kappa : \mathcal{B} \rightarrow \mathbb{D}$ is proper.

In the following, we will show that κ is actually a covering map. To this end, we will construct a local inverse map of κ by means of quasiconformal surgery. The idea is similar to the quadratic case [5].

Fix $\lambda_0 \in \mathcal{B}$ and set $\kappa_0 = \kappa(\lambda_0)$. We may relabel z_{λ_0} so that the immediate attracting basin A_0 of z_{λ_0} contains a critical point $c \in C_{\lambda_0}$. Note that $\epsilon f_\lambda^k(A_0) = A_0$ and there is a conformal map $\phi : A_0 \rightarrow \mathbb{D}$ such that $\phi(z_{\lambda_0}) = 0$ and the following diagram commutes:

$$\begin{array}{ccc}
 A_0 & \xrightarrow{\epsilon f_{\lambda_1}^k} & A_0 \\
 \phi \downarrow & & \downarrow \phi \\
 \mathbb{D} & \xrightarrow{B_{\kappa_0}} & \mathbb{D}
 \end{array}$$

where B_ζ is the Blaschke product defined by $B_\zeta(z) = z \frac{z+\zeta}{1+\zeta z}$. Obviously $z = 0$ is an attracting fixed point of B_{κ_0} with multiplier $B'_\zeta(0) = \zeta$. Then there is a neighborhood \mathcal{U} of κ_0 and a continuous family of quasiregular maps $\tilde{B} : \mathcal{U} \times \mathbb{D} \rightarrow \mathbb{D}$ such that $\tilde{B}(\kappa_0, \cdot) = B_{\kappa_0}(\cdot)$ and $\tilde{B}(\zeta, z) = B_{\kappa_0}(z)$ for $\varepsilon < |z| < 1$ ($\varepsilon > 0$ is a small number), $\tilde{B}(\zeta, z) = B_\zeta(z)$ for $|z| < \varepsilon/2$ and $\tilde{B}(\zeta, \cdot)$ is quasi-regular elsewhere.

Then we get a continuous family $\{G_\zeta\}_{\zeta \in \mathcal{U}}$ of quasiregular maps:

$$G_\zeta(z) = \begin{cases} (-1)^q (f_{\lambda_0}^{k-1}|_{f_{\lambda_0}(A_0)})^{-1}(\varepsilon \phi^{-1} \tilde{B}(\zeta, \phi(e^{-q\pi i/n} z))), & z \in e^{q\pi i/n} A_0, 0 \leq q < 2n, \\ f_{\lambda_0}(z), & z \in \widehat{\mathbb{C}} \setminus \bigcup_{0 \leq q < 2n} e^{q\pi i/n} A_0. \end{cases}$$

We can construct a G_ζ -invariant complex structure σ_ζ such that

- σ_{κ_0} is the standard complex structure σ_0 on $\widehat{\mathbb{C}}$.
- σ_ζ is continuous with respect to $\zeta \in \mathcal{U}$.
- σ_ζ is invariant under the maps $z \mapsto e^{2\pi i/n} z$ and $z \mapsto -z$.
- σ_ζ is the standard complex structure near the attracting cycle and outside $\bigcup_{k \geq 0} f_{\lambda_0}^{-k}(\bigcup_{0 \leq q < 2n} e^{q\pi i/n} A_0)$.

The Beltrami coefficient μ_ζ of σ_ζ satisfies $\|\mu_\zeta\| < 1$. By the Measurable Riemann Mapping Theorem [3], there is a continuous family of quasiconformal maps ψ_ζ fixing $0, \infty$ and normalized so that $\psi'_\zeta(\infty) = 1$. The map ψ_ζ satisfies $\psi_\zeta(e^{2\pi i/n} z) = e^{2\pi i/n} \psi_\zeta(z)$ and $\psi_\zeta(-z) = -\psi_\zeta(z)$. Then $F_\zeta = \psi_\zeta \circ G_\zeta \circ \psi_\zeta^{-1}$ is a rational map of the form $z^{-n}(z^{2n} + \sum_{0 \leq k < 2n} b_k(\zeta) z^k)$. The symmetry $F_\zeta(e^{2\pi i/n} z) = F_\zeta(z)$ implies $F_\zeta(z) = z^n + b_0(\zeta) z^{-n} + b_n(\zeta)$. Since the two free critical values of F_ζ satisfy $\psi_\zeta(v_{\lambda_0}^+) + \psi_\zeta(v_{\lambda_0}^-) = 0$, we have $b_n(\zeta) = 0$. So $F_\zeta = f_{b_0(\zeta)}$. The coefficient $b_0 : \mathcal{U} \rightarrow \mathcal{B}$ is continuous with $\kappa(b_0(\zeta)) = \zeta$. So b_0 is the local inverse of κ . This implies κ is a covering map. Since \mathbb{D} is simply connected, κ is actually a conformal map.

The map κ has a continuation to the boundary $\partial\mathcal{B}$. By the Implicit Function Theorem, the boundary $\partial\mathcal{B}$ is an analytic curve except at $\kappa^{-1}(1)$. So $\partial\mathcal{B}$ is locally connected. Since for any $\lambda \in \partial\mathcal{B}$, the multiplier $e^{2\pi i t}$ of the non-repelling cycle of f_λ is uniquely determined by its angle $t \in \mathbb{S}$, the boundary $\partial\mathcal{B}$ is a Jordan curve. \square

REMARK 8.3. – *By Theorem 8.2, the multiplier map $\rho : \mathcal{B} \rightarrow \mathbb{D}$ is a double cover if and only if n is odd and f_λ is *-renormalizable. For example, when $n = 3$, let \mathcal{B}_+ (resp. \mathcal{B}_-) be the cardioid of the ‘largest’ baby Mandelbrot set intersecting the positive (resp. negative) real axis, then*

1. $\rho : \mathcal{B}_+ \rightarrow \mathbb{D}$ is a conformal map and near the center $\frac{1}{8}$ of \mathcal{B}_+ ,

$$\rho(\lambda) = 24\left(\lambda - \frac{1}{8}\right) + (216 + 156\sqrt{2})\left(\lambda - \frac{1}{8}\right)^2 + \mathcal{O}\left(\left(\lambda - \frac{1}{8}\right)^3\right).$$

2. $\rho : \mathcal{B}_- \rightarrow \mathbb{D}$ is a double covering and near the center $-\frac{1}{8}$ of \mathcal{B}_- ,

$$\rho(\lambda) = 576\left(\lambda + \frac{1}{8}\right)^2 + \mathcal{O}\left(\left(\lambda + \frac{1}{8}\right)^3\right).$$

As a concluding remark, we would like to mention that and using a result of McMullen [26], we can conclude that the parameter plane of f_λ contains many quasiconformal copies of the Mandelbrot set. In fact, every hyperbolic component \mathcal{B} of renormalizable type is a quasiconformal image of a hyperbolic component of the Mandelbrot set, as we can see in Figure 1.

9. Appendix: Lebesgue measure

In this appendix, we shall prove Theorem 5.4 based on the Yoccoz puzzle theory. See Sections 5.1 and 5.4 for basic introductions. The notations also follow there.

THEOREM 9.1 (Lebesgue measure). – *Suppose that $\lambda \in \mathcal{M} \cap \mathcal{F}$ and the graph $G_\lambda(\theta_1, \dots, \theta_N)$ is admissible. If none of $T(c)$ with $c \in C_\lambda$ is periodic, then the Lebesgue measure of $J(f_\lambda)$ is zero.*

Proof of Theorem 5.4 assuming Theorem 9.1. – We assume $\lambda \in \mathcal{F}_0$ (note that when λ is real and positive, the map f_λ is postcritically finite). It is known that if f_λ is postcritically finite, then the Lebesgue measure of $J(f_\lambda)$ is zero. So we assume further $\lambda \in \mathcal{F}$ and f_λ is postcritically infinite. By Lemma 5.2, there is an admissible graph $G_\lambda(\theta_1, \dots, \theta_N)$. By the assumption $f_\lambda^k(v_\lambda^+) \in \partial B_\lambda$, none of $T(c)$ with $c \in C_\lambda$ is periodic. It follows from Theorem 9.1 that the Lebesgue measure of $J(f_\lambda)$ is zero. \square

In this section, we will actually prove Theorem 9.1 following Lyubich [18].

For $k \geq 0$, let \mathcal{P}_k be the collection of all puzzle pieces of depth k . We first note that

LEMMA 9.2. – *We have $d_k = \max\{\text{diam}(P); P \in \mathcal{P}_k\} \rightarrow 0$ as $k \rightarrow \infty$.*

Proof. – If not, then there exist $\varepsilon > 0$ and a sequence of puzzle pieces $P_{n_k} \in \mathcal{P}_{n_k}$ with $n_1 < n_2 < \dots$ and $\text{diam}(P_{n_k}) \geq \varepsilon$. There is $P_{n_1}^* \in \mathcal{P}_{n_1}$ such that $I_1 = \{n_k; P_{n_k} \subset P_{n_1}^*\}$ is an infinite set. For $k > 1$, we define $P_{n_k}^*$ and I_k inductively as follows: $P_{n_{k-1}}^* \supset P_{n_k}^* \in \mathcal{P}_{n_k}$ and the set $I_k = \{j \in I_{k-1}; P_j \subset P_{n_k}^*\}$ is an infinite set. Then $P_{n_k}^*, k \geq 1$, is a sequence of shrinking puzzle pieces with $\text{diam}(P_{n_k}^*) \geq \varepsilon$. This contradicts the fact that $\bigcap_k \overline{P_{n_k}^*}$ consists of a single point (see Lemma 5.3). \square

LEMMA 9.3. – *Let U, V be two planar disks with $V \Subset U \neq \mathbb{C}$, $x \in V$. Suppose that $\text{Shape}(U, x) \leq C$, $\text{Shape}(V, x) \leq C$, $\text{mod}(U - \overline{V}) \leq m$, then there is a constant $\delta = \delta(C, m) \in (0, 1)$, such that $\text{area}(V) \geq \delta \text{area}(U)$.*

Here, the notation $V \Subset U$ means that V is compactly contained in U , i.e., $\overline{V} \subset U$. The proof of Lemma 9.3 is based on the Koebe distortion theorem. We leave it to the reader as an exercise.

LEMMA 9.4. – *Let f be a rational map with Julia set $J(f) \neq \widehat{\mathbb{C}}$. Let $z \in J(f)$, if there exist a number $\epsilon > 0$, a sequence of integers $0 \leq n_1 < n_2 < \dots$ and a constant $N > 0$ such that*

1. *for any $k \geq 0$, the component $U_k(z)$ of $f^{-n_k}(B(f^{n_k}(z), \epsilon))$ that contains z is a disk;*
2. *$\deg(f^{n_k}|_{U_k(z)}) \leq N$ for all $k \geq 1$.*

Then z is not a Lebesgue density point of $J(f)$.

Proof. – By passing to a subsequence, we assume $f^{n_k}(z) \rightarrow w \in J(f)$ as $k \rightarrow \infty$. We may assume further $z, w \neq \infty$ by a suitable change of coordinate. Choose $\epsilon_0 < \epsilon$, when k is large, we have $f^{n_k}(z) \in B(w, \epsilon_0/2) \subset B(w, \epsilon_0) \subset B(f^{n_k}(z), \epsilon)$. Let $V_k(z)$ be the component of $f^{-n_k}(B(w, \epsilon_0/2))$ that contains z . Then $V_k(z)$ is a disk and $\deg(f^{n_k}|_{V_k(z)}) \leq N$. By shape distortion (see [28], Lemma 6.1), the shape of $V_k(z)$ about z is bounded above by some constant depending on N . We then show that $\text{diam}(V_k(z)) \rightarrow 0$ as $k \rightarrow \infty$. In fact, if not, again by choosing a subsequence, we assume $V_k(z)$ contains a round disk $B(z, \rho)$ for some $\rho > 0$. Then for any large k , the image $f^{n_k}(B(z, \rho))$ is contained in $B(w, \epsilon_0/2)$. But this contradicts the fact that $J(f) \subset f^{n_k}(B(z, \rho))$ for large k (see [27]).

Since $J(f) \neq \widehat{\mathbb{C}}$, there is a round disk $B(\zeta, r) \Subset B(w, \epsilon_0/2) \cap F(f)$; here $F(f)$ is the Fatou set of f . Take a component D_k of $f^{-n_k}(B(\zeta, r))$ in $V_k(z)$ and $p \in f^{-n_k}(\zeta) \cap D_k$, then by shape distortion (see [28], Lemma 6.1), there is a constant $C > 0$ such that $\text{Shape}(D_k, p) \leq C \text{Shape}(B(\zeta, r), \zeta) = C \text{Shape}(V_k(z), p) \leq C \text{Shape}(B(w, \epsilon_0/2), \zeta) \leq C\epsilon_0/r$. Moreover, $\text{mod}(V_k(z) \setminus \overline{D_k}) \leq \text{mod}(B(w, \epsilon_0/2) \setminus \overline{B(\zeta, r)})$. It follows from Lemma 9.3 that there is a constant δ with $\text{area}(D_k) \geq \delta \text{area}(V_k(z))$. So

$$\text{area}(J(f) \cap V_k(z)) \leq \text{area}(V_k(z) - D_k) \leq (1 - \delta) \text{area}(V_k(z)).$$

This implies z is not a Lebesgue density point. \square

PROPOSITION 9.5. – *If C_λ is not persistently recurrent, then the Lebesgue measure of the Julia set $J(f_\lambda)$ is zero.*

Proof. – Let $P(f_\lambda) = \overline{\bigcup_{k \geq 1} f_\lambda^k(C_\lambda)}$. It is known ([24], Theorem 3.9) that for almost all $z \in J(f_\lambda)$, the spherical distance $d_{\widehat{\mathbb{C}}}(f_\lambda^n(z), P(f_\lambda)) \rightarrow 0$ as $n \rightarrow \infty$.

Suppose that the critical set C_λ is recurrent but not persistently recurrent; then there is a positive integer L such that the set $\{k; \tau_Y(k) \leq L\}$ is infinite. The recurrence of C_λ implies that there is $d \geq L$ such that the annulus $P_d^\lambda(c) \setminus \overline{P_{d+1}^\lambda(c)}$ for some (hence all) $c \in C_\lambda$ is non-degenerate. Since $\tau_Y(k+1) \leq \tau_Y(k) + 1$, the set $\Lambda = \{k; \tau_Y(k) = d, \tau_Y(k+1) = d+1\}$ is infinite. Moreover $\{f_\lambda^{k-d}(c); k \in \Lambda\} \subset \bigcup_{\zeta \in C_\lambda} P_{d+1}(\zeta) \Subset \bigcup_{\zeta \in C_\lambda} P_d(\zeta)$. So any critical point $c \in C_\lambda$ satisfies the conditions in Lemma 9.4, thus it is not a Lebesgue density point. We consider a point $z \in J(f_\lambda) \setminus C_\lambda$ with $\lim d_{\widehat{\mathbb{C}}}(f_\lambda^n(z), P(f_\lambda)) = 0$. We may assume that the forward orbit of z does not meet the graph $G_\lambda(\theta_1, \dots, \theta_N)$ (for else z is not a Lebesgue density point by Lemma 9.4). In that case, for each $k \in \Lambda$, there is $n_k > k$ and $c' \in C_\lambda$ such that $f_\lambda^{n_k - k - 1}(P_{n_k}^\lambda(z)) = P_{k+1}^\lambda(c')$ and $f_\lambda^j(P_{n_k}^\lambda(z)), 0 \leq j < n_k - k - 1$, meets no critical point. One can easily verify that z satisfies the conditions in Lemma 9.4, and is not a Lebesgue density point of $J(f_\lambda)$.

If the critical set C_λ is not recurrent, one can verify that each point $z \in J(f_\lambda)$ satisfies the condition in Lemma 9.4. Thus $J(f_\lambda)$ carries no Lebesgue density point. The proof is similar to, but easier than the previous argument. We omit the details. \square

We say a holomorphic map $g : \mathbf{U} \rightarrow \mathbf{V}$ is a *repelling system* if $\mathbf{U} \Subset \mathbf{V}$, the boundary $\partial \mathbf{U}$ avoids the critical orbit of g and both \mathbf{U} and \mathbf{V} consist of finitely many disk components. The filled Julia set of g is defined by $K(g) = \bigcap_{k \geq 1} g^{-k}(\mathbf{V})$, it can be an empty set.

THEOREM 9.6. – *If the critical set C_λ is persistently recurrent, then there is a repelling system $g : \mathbf{U} \rightarrow \mathbf{V}$ such that*

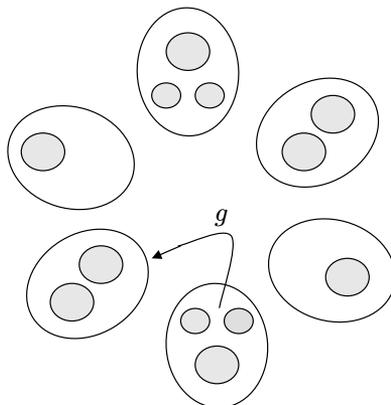


FIGURE 6. A repelling system $g : \mathbf{U} \rightarrow \mathbf{V}$, where \mathbf{U} is the union of all shadow disks and \mathbf{V} is the union of six larger disks.

1. each component of \mathbf{U} and \mathbf{V} is a puzzle piece;
2. for each component U_i of \mathbf{U} , $g|_{U_i} = f_\lambda^{l_i}$ for some l_i .
3. $C_\lambda \subset K(g)$.

Moreover, the Lebesgue measure of $J(f_\lambda) - \bigcup_{k \geq 0} f_\lambda^{-k}(K(g))$ is zero.

Proof. – Since the graph $G_\lambda(\theta_1, \dots, \theta_N)$ is admissible, we can find a non-degenerate critical annulus $P_d^\lambda(c) \setminus \overline{P_{d+1}^\lambda(c)}$ for some $d \geq 1$. Set $\mathbf{V} = \bigcup_{c \in C_\lambda} P_{d+1}^\lambda(c)$. Then $f_\lambda^j(\partial \mathbf{V}) \cap \overline{\mathbf{V}} = \emptyset$ for all $j \geq 1$. For any $j \geq 1$, either $f_\lambda^j(C_\lambda) \subset \mathbf{V}$ or $f_\lambda^j(C_\lambda) \cap \mathbf{V} = \emptyset$. Let $1 \leq n_1 < n_2 < \dots$ be all the integers such that $f_\lambda^{n_i}(C_\lambda) \subset \mathbf{V}$. Let $l_i = n_{i+1} - n_i$ (set $n_0 = 0$) for $i \geq 0$, we pull back \mathbf{V} along the orbit $\{f_\lambda^j(C_\lambda)\}_{j=n_i}^{n_{i+1}}$ and get \mathbf{V}_i . Namely, \mathbf{V}_i is the union of all components of $f_\lambda^{-l_i}(\mathbf{V})$ intersecting with $f_\lambda^{n_i}(C_\lambda)$. For any i , the intermediate pieces $f_\lambda^k(\mathbf{V}_i)$, $0 < k < l_i$ lie outside \mathbf{V} and for any component V of \mathbf{V}_i , the map $f_\lambda^{l_i}|_V$ is either univalent or a double covering.

Since C_λ is persistently recurrent, the set $\{k; \tau_Y(k) \leq d + 1\}$ is finite and there are only finitely many different \mathbf{V}_i 's. Moreover, if $\mathbf{V}_i \neq \mathbf{V}_j$, then $\mathbf{V}_i \cap \mathbf{V}_j = \emptyset$ (in fact, $\overline{\mathbf{V}_i} \cap \overline{\mathbf{V}_j} = \emptyset$).

Let $\mathbf{U} = \bigcup_i \mathbf{V}_i$ and define $g|_{\mathbf{V}_i} = f_\lambda^{l_i}$. Then $\mathbf{U} \Subset \mathbf{V}$ follows from the fact that $f_\lambda^{l_i}(\partial \mathbf{V}_i \cap \partial \mathbf{V}) \subset \partial \mathbf{V} \cap f_\lambda^{l_i}(\partial \mathbf{V}) = \emptyset$.

It follows from $\bigcup_{i \geq 0} f_\lambda^{n_i}(C_\lambda) = \bigcup_{k \geq 0} g^k(C_\lambda) \subset \mathbf{V}$ that $C_\lambda \subset K(g)$.

Similar to the proof of Proposition 9.5, we need only consider a point $z \in J(f_\lambda)$ with $d_{\mathbb{C}}(f_\lambda^n(z), P(f_\lambda)) \rightarrow 0$ as $n \rightarrow \infty$. For such point, there is an integer $N > 0$ such that for all $n \geq N$, $f_\lambda^n(z) \in \mathbf{V}$ implies $f_\lambda^n(z) \in \mathbf{U}$. Note that there is $p \geq N$ such that $f_\lambda^p(z) \in \mathbf{V}$. Then for all $j \geq 1$, we have $g^j(f_\lambda^p(z)) \in \mathbf{V}$. It turns out that $f_\lambda^p(z) \in K(g)$. This implies $J(f_\lambda) - \bigcup_{k \geq 0} f_\lambda^{-k}(K(g))$ has zero Lebesgue measure. \square

Let $D \subset \mathbb{C}$ be a topological disk containing a compact subset K (not necessarily connected), the modulus of $A = D - K$, denoted by $\mathbf{m}(A)$, is defined to be the extremal length of curves joining ∂D and ∂K . It is equal to the reciprocal of Dirichlet integral of the harmonic

measure u in A (namely, u is harmonic function in A which tends to 0 at regular points of ∂K and tends to 1 at regular points of ∂D):

$$\mathbf{m}(A) = \left(\int_A |\nabla u|^2 dx dy \right)^{-1}.$$

If we require further that K consists of finitely many components, then we have the following area-modulus inequality (see [18]):

$$\text{area}(D) \geq \text{area}(K)(1 + 4\pi\mathbf{m}(A)).$$

Now we consider the repelling system $g : \mathbf{U} \rightarrow \mathbf{V}$ defined in Theorem 9.6. Set $\mathbf{V}^0 = \mathbf{V}$ and consider the preimages $\mathbf{V}^d = g^{-d}(\mathbf{V})$ for $d \geq 1$. Note that $\mathbf{V}^{d+1} \Subset \mathbf{V}^d$. For any $z \in K(g)$ and $d \geq 0$, denote by $\mathbf{V}^d(z)$ the piece of level d containing z . Let $\mathbf{A}^d(z) = \mathbf{V}^d(z) - \overline{\mathbf{V}^{d+1}}$, it is a multiconnected domain. One can verify that for any $d \geq 1$, if $\mathbf{V}^d(z)$ contains no critical point in C_λ , then $\mathbf{m}(\mathbf{A}^d(z)) = \mathbf{m}(\mathbf{A}^{d-1}(f_\lambda(z)))$; if $\mathbf{V}^d(z)$ contains a critical point in C_λ , then $2\mathbf{m}(\mathbf{A}^d(z)) = \mathbf{m}(\mathbf{A}^{d-1}(f_\lambda(z)))$.

Using the same method as in [18], one can show that

LEMMA 9.7. – For any $z \in K(g)$, we have $\sum_{d \geq 1} \mathbf{m}(\mathbf{A}^d(z)) = \infty$. It turns out that $K(g)$ is a Cantor set.

Now we have

THEOREM 9.8. – Let $g : \mathbf{U} \rightarrow \mathbf{V}$ be the repelling system defined in Theorem 9.6, then the Lebesgue measure of $K(g)$ is zero.

Proof. – For any $d \geq 1$, let $\mathbf{V}^d(z_1), \dots, \mathbf{V}^d(z_{k_d})$ be all puzzle pieces of level d , where $z_1, \dots, z_{k_d} \in K(g)$. We define

$$M_d = \min_{1 \leq i \leq k_d} \sum_{0 \leq j < d} \mathbf{m}(\mathbf{A}^j(z_i)).$$

By Lemma 9.7, we have $M_d \rightarrow \infty$ as $d \rightarrow \infty$. By area-modulus inequality, we have

$$\text{area}(\mathbf{V}^d) \leq \frac{\text{area}(\mathbf{V})}{\min_{1 \leq i \leq k_d} \prod_{0 \leq j < d} (1 + 4\pi\mathbf{m}(\mathbf{A}^j(z_i)))} \leq \frac{\text{area}(\mathbf{V})}{1 + 4\pi M_d}.$$

This implies $\text{area}(\mathbf{V}^d) \rightarrow 0$ as $d \rightarrow \infty$. □

Theorem 9.1 then follows from Proposition 9.5 and Theorems 9.6 and 9.8.

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