quatrième série - tome 48

fascicule 6 novembre-décembre 2015

ANNALES SCIENTIFIQUES des L'ÉCOLE NORMALE SUPÉRIEURE

Julien BARRAL

Inverse problems in multifractal analysis of measures

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Responsable du comité de rédaction / Editor-in-chief

Antoine CHAMBERT-LOIR

Publication fondée en 1864 par Louis Pasteur	Comité
Continuée de 1872 à 1882 par H. SAINTE-CLAIRE DEVILLE	N. Ana
de 1883 à 1888 par H. DEBRAY	E. Breu
de 1889 à 1900 par C. Негміте	R. Cerf
de 1901 à 1917 par G. DARBOUX	A. Chai
de 1918 à 1941 par É. PICARD	I. Gall
de 1942 à 1967 par P. MONTEL	

Comité de rédaction au 1 ^{er} janvier 2015		
N. Anantharaman	B. KLEINER	
E. Breuillard	E. Kowalski	
R. Cerf	P. LE CALVEZ	
A. Chambert-Loir	M. Mustață	
I. Gallagher	L. SALOFF-COSTE	

Rédaction / Editor

Annales Scientifiques de l'École Normale Supérieure, 45, rue d'Ulm, 75230 Paris Cedex 05, France. Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80. annales@ens.fr

Édition / Publication

Abonnements / Subscriptions

Société Mathématique de France Institut Henri Poincaré 11, rue Pierre et Marie Curie 75231 Paris Cedex 05 Tél. : (33) 01 44 27 67 99 Fax : (33) 01 40 46 90 96 Maison de la SMF Case 916 - Luminy 13288 Marseille Cedex 09 Fax : (33) 04 91 41 17 51 email : smf@smf.univ-mrs.fr

Tarifs

Europe : 515 €. Hors Europe : 545 €. Vente au numéro : 77 €.

© 2015 Société Mathématique de France, Paris

En application de la loi du 1^{er} juillet 1992, il est interdit de reproduire, même partiellement, la présente publication sans l'autorisation de l'éditeur ou du Centre français d'exploitation du droit de copie (20, rue des Grands-Augustins, 75006 Paris). *All rights reserved. No part of this publication may be translated, reproduced, stored in a retrieval system or transmitted in any form or by any other means, electronic, mechanical, photocopying, recording or otherwise, without prior permission of the publisher.*

ISSN 0012-9593

Directeur de la publication : Marc Peigné Périodicité : 6 nºs / an

INVERSE PROBLEMS IN MULTIFRACTAL ANALYSIS OF MEASURES

BY JULIEN BARRAL

ABSTRACT. – Multifractal formalism is designed to describe the distribution at small scales of the elements of $\mathcal{M}_c^+(\mathbb{R}^d)$, the set of positive, finite and compactly supported Borel measures on \mathbb{R}^d . It is valid for such a measure μ when its Hausdorff spectrum is the upper semi-continuous function given by the concave Legendre-Fenchel transform of the free energy function τ_{μ} associated with μ ; this is the case for fundamental classes of exactly dimensional measures.

For any function τ candidate to be the free energy function of some $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$, we construct such a measure, exactly dimensional, and obeying the multifractal formalism. This result is extended to a refined formalism considering jointly Hausdorff and packing spectra. Also, for any upper semicontinuous function candidate to be the lower Hausdorff spectrum of some exactly dimensional $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$, we construct such a measure.

RÉSUMÉ. – Le formalisme multifractal est un cadre adapté pour décrire la distribution aux petites échelles des mesures de Borel finies positives à support compact dans \mathbb{R}^d , dont l'ensemble est ici noté $\mathcal{M}_c^+(\mathbb{R}^d)$. Il est dit valide pour une mesure μ lorsque son spectre de Hausdorff est la fonction semi-continue supérieurement obtenue comme transformée de Legendre-Fenchel concave de sa fonction d'énergie libre τ_{μ} ; c'est le cas pour certaines classes fondamentales de mesures exactement dimensionnelles.

Pour toute fonction τ candidate à être la fonction d'énergie libre d'un élément μ de $\mathcal{M}_c^+(\mathbb{R}^d)$, nous construisons une telle mesure, exactement dimensionnelle, et validant le formalisme. Ce résultat s'étend à un formalisme plus fin considérant simultanément spectres de Hausdorff et de packing. D'autre part, pour toute fonction semi-continue supérieurement candidate à être le spectre de Hausdorff inférieur d'une mesure exactement dimensionnelle, nous construisons une telle mesure.

1. Introduction and main statements

1.1. Inverse problems in multifractal analysis of measures

Let $\mathcal{M}_{c}^{+}(\mathbb{R}^{d})$ stand for the set of compactly supported Borel positive and finite measures on \mathbb{R}^{d} $(d \geq 1)$, and for $\mu \in \mathcal{M}_{c}^{+}(\mathbb{R}^{d})$ denote by $\operatorname{supp}(\mu)$ the topological support of μ (i.e.,

The author is grateful to De-Jun Feng and Jacques Peyrière for their valuable comments on this work.

the compact set obtained as the complement of those points x for which $\mu(B(x,r)) = 0$ for some r > 0, where B(x, r) stands for the closed ball of radius r centered at x).

The upper and lower box dimensions of a bounded set $E \subset \mathbb{R}^d$ will be denoted $\overline{\dim}_B E$ and $\underline{\dim}_B E$ respectively, and its Hausdorff and packing dimensions will be denoted by $\dim_H E$ and $\dim_P E$ respectively (see [33, 60, 70, 81] for introductions to dimension theory).

Multifractal analysis is a natural framework to finely describe geometrically the heterogeneity in the distribution at small scales of the elements of $\mathcal{M}_c^+(\mathbb{R}^d)$. Specifically, if $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$, this heterogeneity can be described via the lower and upper local dimensions of μ , namely

$$\underline{d}(\mu, x) = \liminf_{r \to 0^+} \frac{\log(\mu(B(x, r)))}{\log(r)} \quad \text{and} \quad \overline{d}(\mu, x) = \limsup_{r \to 0^+} \frac{\log(\mu(B(x, r)))}{\log(r)},$$

and the level sets

$$E(\mu, \alpha, \beta) = \left\{ x \in \operatorname{supp}(\mu) : \underline{d}(\mu, x) = \alpha, \ \overline{d}(\mu, x) = \beta \right\} \quad (\alpha \le \beta \in \mathbb{R} \cup \{\infty\}),$$

which form a partition of supp(μ) (notice that $E(\mu, \alpha, \beta) = \emptyset$ whenever $\alpha < 0$). The sets

$$\underline{E}(\mu,\alpha) = \Big\{ x \in \operatorname{supp}(\mu) : \underline{d}(\mu,x) = \alpha \Big\}, \quad \overline{E}(\mu,\alpha) = \Big\{ x \in \operatorname{supp}(\mu) : \overline{d}(\mu,x) = \alpha \Big\},$$

and

$$E(\mu, \alpha) = \underline{E}(\mu, \alpha) \cap \overline{E}(\mu, \alpha) = E(\mu, \alpha, \alpha) \quad (\alpha \in \mathbb{R} \cup \{\infty\})$$

are also very natural, and the most studied in the literature (although the sets defined above are empty if $\alpha < 0$ because μ is a bounded function of Borel sets, it is convenient to include negative values of α in connection with the using along the paper of the Legendre-Fenchel transform of functions defined on \mathbb{R} or $\mathbb{R} \cup \{\infty\}$).

The *lower Hausdorff spectrum* of μ is the mapping defined as

$$\underline{f}^{H}_{\mu}: \alpha \in \mathbb{R} \cup \{\infty\} \mapsto \dim_{H} \underline{E}(\mu, \alpha),$$

with the convention that $\dim_H \emptyset = -\infty$, so that $\underline{f}_{\mu}^H(\alpha) = -\infty$ if $\alpha < 0$. This spectrum provides a geometric hierarchy between the sets $\underline{E}(\mu, \alpha)$, which partition the support of μ . Here, the lower local dimension is emphasized for it provides at any point the best pointwise Hölder control one can have on the measure μ at small scales. However, the upper local dimension is of course of interest, and much attention is paid in general to the sets $E(\mu, \alpha)$ of points at which one has an exact local dimension $\underline{d}(\mu, x) = \overline{d}(\mu, x)$, especially when studying ergodic measures in the context of hyperbolic and more generally non uniformly hyperbolic dynamical systems.

The *Hausdorff spectrum* of μ is the mapping defined as

$$f_{\mu}^{H}: \alpha \in \mathbb{R} \cup \{\infty\} \mapsto \dim_{H} E(\mu, \alpha).$$

Inspired by the observations made by physicists of turbulence and statistical mechanics [42, 40, 41], mathematicians derived, and in many situations justified the heuristic claiming that for a measure possessing a self-conformal like property, its Hausdorff spectrum should be obtained as the Legendre transform of a kind of free energy function, called L^q -spectrum. This gave birth to an abundant literature on the so-called multifractal formalisms [33, 21, 19, 63, 70, 52, 17, 71, 56], which aim at linking the asymptotic statistical properties of a given measure with its fine geometric properties.

To be more specific we need some definitions. Given $I \in \{\mathbb{R}, \mathbb{R} \cup \{\infty\}\}$ and a fonction $f: I \to \mathbb{R} \cup \{-\infty\}$, the domain of f is defined as $\operatorname{dom}(f) = \{x \in I : f(x) > -\infty\}$.

Let $\tau : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$. If dom $(\tau) \neq \emptyset$, the concave Legendre-Fenchel transform, or concave conjugate function, of τ is the upper-semi continuous concave function defined as $\tau^* : \alpha \in \mathbb{R} \mapsto \inf\{\alpha q - \tau(q) : q \in \operatorname{dom}(\tau)\}$ (see [77]). We will need a slight extension of this definition.

If $\tau : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, dom $(\tau) \neq \emptyset$, and $0 \in \text{dom}(\tau)$, we define its (extended) concave Legendre-Fenchel transform as

$$\tau^*: \alpha \in \mathbb{R} \cup \{\infty\} \mapsto \begin{cases} \inf\{\alpha q - \tau(q) : q \in \operatorname{dom}(\tau)\} & \text{if } \alpha \in \mathbb{R}, \\ \inf\{\alpha q - \tau(q) : q \in \operatorname{dom}(\tau) \cap \mathbb{R}_-\} & \text{if } \alpha = \infty, \end{cases}$$

with the conventions $\infty \times q = -\infty$ if q < 0 and $\infty \times 0 = 0$. Consequently, $\infty \in \text{dom}(\tau^*)$ if and only if $0 = \min(\text{dom}(\tau))$, and in this case $\tau^*(\infty) = -\tau(0) = \max(\tau^*)$. In any case, τ^* is upper semi-continuous over $\text{dom}(\tau^*)$, and concave over the interval $\text{dom}(\tau^*) \setminus \{\infty\}$ (here the notion of upper semi-continuous function is relative to $\mathbb{R} \cup \{\infty\}$ endowed with the topology generated by the open subsets of \mathbb{R} and the sets $(\alpha, \infty) \cup \{\infty\}$, $\alpha \in \mathbb{R}$).

Now, define the (lower) L^q -spectrum of $\mu \in \mathcal{M}^+_c(\mathbb{R}^d)$ as

$$\tau_{\mu}: q \in \mathbb{R} \mapsto \liminf_{r \to 0^+} \frac{\log \sup\left\{\sum_i \mu(B(x_i, r))^q\right\}}{\log(r)}$$

where the supremum is taken over all the centered packings of $supp(\mu)$ by closed balls of radius r.

By construction, τ_{μ} is concave and non decreasing, and

$$-d \le \tau_{\mu}(0) = -\overline{\dim}_B \operatorname{supp}(\mu) \le 0 = \tau_{\mu}(1),$$

so that one always has $\mathbb{R}_+ \subset \operatorname{dom}(\tau_\mu)$; also τ_μ^* takes values in $[0, d] \cup \{-\infty\}$, and $\operatorname{dom}(\tau_\mu^*)$ is a closed subinterval of $\mathbb{R}_+ \cup \{\infty\}$ (see Propositions 1.1 and 1.2).

For $\alpha \in \mathbb{R}$ we always have (see [63, Section 2.7] or [52, Section 3])

(1.1)
$$f_{\mu}^{H}(\alpha) \leq \underline{f}_{\mu}^{H}(\alpha) \leq \tau_{\mu}^{*}(\alpha) \leq \max(\alpha, -\tau_{\mu}(0)) \leq \max(\alpha, d);$$

we also have

$$f^H_\mu(\infty) \le \tau^*_\mu(\infty),$$

a dimension equal to $-\infty$ meaning that the set is empty (the second inequality is not standard, and will be proved in Section 5; the inequality $\tau_{\mu}^{*}(\alpha) \leq \max(\alpha, -\tau_{\mu}(0))$ is a direct consequence of the definition of τ_{μ}^{*} and the fact that $\tau_{\mu}(1) = 0$).

We notice that due to (1.1), if $f_{\mu}^{H}(\alpha) \geq \alpha$ at some α , then $0 \leq \alpha \leq d$ and $f_{\mu}^{H}(\alpha) = \tau_{\mu}^{*}(\alpha) = \alpha$, so that α is a fixed point of τ_{μ}^{*} . Moreover, since $\tau_{\mu}(1) = 0$ and τ_{μ} is concave, the set of fixed points of τ_{μ}^{*} is the interval $[\tau_{\mu}^{\prime}(1^{+}), \tau_{\mu}^{\prime}(1^{-})]$.

We will say that μ obeys the multifractal formalism at $\alpha \in \mathbb{R} \cup \{\infty\}$ if $\underline{f}_{\mu}^{H}(\alpha) = \tau_{\mu}^{*}(\alpha)$, and that the multifractal formalism holds (globally) for μ if it holds at any $\alpha \in \mathbb{R} \cup \{\infty\}$.

If $f_{\mu}^{H}(\alpha)$ can be replaced by $f_{\mu}^{H}(\alpha)$ in the previous definition, we will say that the multifractal formalism holds strongly, and it is in this form that this formalism has been introduced and studied the most. It turns out that in this case one has

$$\dim_H E(\mu, \alpha) = \dim_P E(\mu, \alpha) = \dim_H \underline{E}(\mu, \alpha) = \dim_H E(\mu, \alpha) = \tau^*_{\mu}(\alpha).$$

However, in general, nice families of discrete measures only obey the multifractal formalism associated with the lower Hausdorff spectrum as defined above.

The multifractal formalism turns out to hold globally, or on some non trivial subinterval of dom(τ_{μ}^{*}), for some important classes of continuous measures possessing (or close to have) self-conformal properties, namely some classes of self-conformal measures (among which some Bernoulli convolutions), Gibbs and weak Gibbs measures on conformal repellers (e.g., the harmonic measure on such a disconnected set) or attractors of Axiom A diffeomorphisms [25, 29, 73, 72, 59, 21, 54, 55, 76, 69, 70, 68, 58, 43, 50, 38, 34, 46, 79, 80, 35, 37, 49, 36], harmonic measure on the Brownian frontier [53], and scale invariant limits of certain multiplicative chaos [45, 30, 61, 1, 5, 8, 9, 74, 2]; in these cases it also holds strongly. It also holds for scale invariant discrete measures obtained as limits in law of Gibbs measures in the context of random directed polymers [48, 12, 11] (see also [3, 47, 32, 13, 67] for other classes of discrete measures on self-affine Sierpiński carpets [51, 64, 10, 6], or on almost all the attractors of IFS associated with certain families of $d \times d$ invertible matrices with small enough singular values [31, 7], as well as generic probability measures on a compact subset of \mathbb{R}^d [22, 23, 16].

The measures mentioned above share the geometric property to be exactly dimensional, i.e., for such a measure μ , there exists $D \in [0,d]$ such that $\lim_{r\to 0^+} \frac{\log(\mu(B(x,r)))}{\log(r)} = D$, μ -almost everywhere. This implies $f_{\mu}^{H}(D) \ge D$, hence $D \in [\tau'_{\mu}(1^+), \tau'_{\mu}(1^-)]$ and μ strongly obeys the multifractal formalism at D by a remark made above. In fact, for any $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$, for μ -almost every x one has $\tau'_{\mu}(1^+) \le \underline{d}(\mu, x) \le \overline{d}(\mu, x) \le \tau'_{\mu}(1^-)$ ([62]), and for most of the continuous measures in the previous references, $\tau'_{\mu}(1)$ exists, hence equals D; also, τ_{μ} is piecewise C^1 , and even analytic in certain cases, a typical example being Gibbs measures associated with Hölder potentials on repellers of $C^{1+\alpha}$ conformal mappings.

Another property of the previous measures is, when they obey globally the multifractal formalism, to be homogeneously multifractal (HM), this meaning that the lower Hausdorff spectrum of the restriction of μ to any closed ball whose interior intersects supp(μ) is equal to the lower Hausdorff spectrum of μ .

In this paper we solve the inverse problem consisting in constructing, for any concave function τ satisfying the necessary conditions to be the L^q -spectrum of an element of $\mathcal{M}_c^+(\mathbb{R}^d)$, an exactly dimensional and (HM) measure whose L^q -spectrum equals τ , and which strongly satisfies the multifractal formalism. More specifically:

THEOREM 1.1. – Let $\tau : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be a concave function satisfying the necessary properties (see Propositon 1.1) to be the L^q -spectrum of some element of $\mathcal{M}_c^+(\mathbb{R}^d)$. Let $D \in [\tau'(1^+), \tau'(1^-)]$. There exists an (HM) measure $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$, exactly dimensional with dimension D, and which strongly satisfies the multifractal formalism with $\tau_{\mu} = \tau$.

Theorem 1.1 will be obtained as a consequence of more general statements which also describe the Hausdorff and packing dimensions of the sets $E(\mu, \alpha, \beta)$ (Theorem 1.3 and Corollary 1.1 of Section 1.2). We will also study the inverse problem associated with a finer multifractal formalism designed to describe the more general situation where the Hausdorff spectrum f_{μ}^{H} and the packing spectrum $f_{\mu}^{P}: \alpha \mapsto \dim_{P} E(\mu, \alpha)$ differ (Theorem 1.4 and Corollary 1.2 of Section 1.2). As a by product of these results new multifractal behaviors are exhibited.

In general, $\operatorname{dom}(\underline{f}_{\mu}^{H}) = \{\alpha \in \mathbb{R} \cup \{\infty\} : \underline{E}(\mu, \alpha) \neq \emptyset\}$ is not necessarily a closed subinterval of $[0, \infty]$, and even when it is the case, the restriction of \underline{f}_{μ}^{H} to $\operatorname{dom}(\underline{f}_{\mu}^{H}) \cap \mathbb{R}_{+}$ is not necessarily concave. Consequently, we also study the inverse problem consisting in associating to a function $f : \mathbb{R} \cup \{\infty\} \to [0, d] \cup \{-\infty\}$ whose domain is a subset of $\mathbb{R}_{+} \cup \{\infty\}$ and such that $f(\alpha) \leq \alpha$ for all $\alpha \geq 0$, an (HM) measure whose lower Hausdorff spectrum is equal to f. We construct such a measure μ when dom(f) is a closed subset of $\mathbb{R}_{+} \cup \{\infty\}$, f is upper semi-continuous, and f has at least one fixed point, three properties shared with τ_{μ}^{*} . Moreover, the measure μ is exactly dimensional.

Thus, we will prescribe lower Hausdorff spectra in the family:

$$\mathcal{F}(d) = \left\{ f : \mathbb{R} \cup \{\infty\} \to [0, d] \cup \{-\infty\} : \begin{cases} \operatorname{Fix}(f) \neq \emptyset \\ \operatorname{dom}(f) \text{ is a closed subset of } [0, \infty] \\ f \text{ is upper semi-continuous} \\ f(\alpha) \le \alpha \text{ for all } \alpha \in \operatorname{dom}(f) \end{cases} \right\}$$

where $Fix(f) (\subset [0, d])$ stands for the set of fixed points of f.

THEOREM 1.2. – Let $f \in \mathcal{F}(d)$. For each $D \in \text{Fix}(f)$, there exists an (HM) measure $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$, exactly dimensional with dimension D, such that $f_{\mu}^H = f$.

This result will be strengthened in Theorem 1.5 of Section 1.2. It turns out that the approach used in this paper does not make it possible to replace $f_{\mu}^{H} = f$ by $f_{\mu}^{H} = f$ in the previous statement unless one of the following properties holds: dom(f) = Fix(f) (see Theorem 1.5), or dom(f) is an interval and f is concave over dom $(f) \cap \mathbb{R}_{+}$ (in this case we will get a measure obeying the strong multifractal formalism, see Theorem 1.3).

Before developing further results and comments, let us outline the main ideas leading to the construction of the measure μ provided by Theorem 1.2. To establish Theorem 1.1 one must improve this approach in order to control both the finer level sets $E(\mu, \alpha)$ and the upper large deviations spectrum of μ (to be defined in Section 1.2.1) when f is the concave function τ^* , and then use the duality property linking the L^q -spectrum and the upper large deviations spectrum to show that the multifractal formalism holds strongly.

For simplicity, we assume that dom(f) is a non trivial compact interval $[\alpha_{\min}, \alpha_{\max}] \subset \mathbb{R}_+$, f is continuous over $[\alpha_{\min}, \alpha_{\max}]$, $0 \leq f(\alpha) \leq \min(\alpha, d)$ over $[\alpha_{\min}, \alpha_{\max}]$, and f(D) = D for a unique point D in $[\alpha_{\min}, \alpha_{\max}]$. The homogeneity of the construction of the measure μ automatically implies that the measure is (HM).

At first one shows (independently of f) that for any $\gamma \in [0, d]$ and $\alpha \geq \gamma$, one can find two Borel probability measures $\mu_{\alpha,\gamma}$ and $\nu_{\alpha,\gamma}$ supported on $[0, 1]^d$ such that $\mu_{\gamma,\gamma} = \nu_{\gamma,\gamma}$, $\nu_{\alpha,\gamma}$ is exactly dimensional with dimension γ , and $\nu_{\alpha,\gamma}$ is concentrated on $E(\mu_{\alpha,\gamma}, \alpha)$, as well as on the set defined similarly but with $\alpha(\mu, x)$ replaced by $\lim_{n\to\infty} \frac{\log(\mu(I_n(x)))}{-n\log(2)}$, where $I_n(x)$ stands for the closure of a dyadic cube semi-open to the right containing x.

Set $A_1 = \{\alpha_1 = D\}$, and for each integer $m \ge 1$, define $A_{m+1} = A_m \cup \{\alpha_{m+1}\}$, where $\alpha_{m+1} \in [\alpha_{\min}, \alpha_{\max}] \setminus A_m$, in such a way that the set $\{\alpha_m : m \ge 1\}$ be dense in $[\alpha_{\min}, \alpha_{\max}]$. By using the previous property with $\gamma = f(\alpha)$, for all $m \ge 1$ one gets an integer n_m such that

for all $\alpha \in A_m$, for all $n \ge n_m$, there is a collection $G_{m,n}(\alpha)$ of about $2^{nf(\alpha)}$ dyadic subcubes of $[0,1]^d$ such that for all $I \in G_{m,n}(\alpha)$ one has $\mu_{\alpha,f(\alpha)}(I) \approx 2^{-n\alpha}$, $\nu_{\alpha,f(\alpha)}(I) \approx 2^{-nf(\alpha)}$, and $\sum_{I \in G_{m,n}(\alpha)} \nu_{\alpha,f(\alpha)}(I) \in [1/2,1]$.

For every integer $m \ge 2$, one considers m dyadic closed subcubes of $[0,1]^d$ of the same generation $n'_m, L_{\alpha_1}, \ldots, L_{\alpha_m}$, so that the $2^{-n'_m/5}$ neighborhood of each L_{α_i} does not intersect any of the other L_{α_i} .

The measure μ is constructed on a Cantor set $K = \bigcap_{m \ge 1} \bigcup_{I \in \mathbf{G}_m}$, where the \mathbf{G}_m are families of closed dyadic subcubes of $[0,1]^d$ of generation g_m tending to ∞ as $m \to \infty$, constructed recursively according to a scheme roughly as follows:

One obtains G_1 by considering the measure $\mu_{\alpha_1,f(\alpha_1)} = \mu_{D,D}$, an integer $N_1 \ge n_1$ much bigger than n'_2 and setting $G_1 = G_{1,N_1}(\alpha_1) = G_{1,N_1}(D)$. This yields the probability measure μ_1 defined on G_1 as

$$\mu_1(I) = \frac{\mu_{D,D}(I)}{\left(\sum_{I' \in \mathbf{G}_1} \mu_{D,D}(I')\right)}$$

This measure satisfies $\mu_1(I) \approx 2^{-N_1 D}$.

Suppose now that the set \mathbf{G}_m has been constructed, as well as a probability measure μ_m on its elements. One takes $N_{m+1} \ge n_{m+1}$ and integer much bigger than $\max(g_m, n'_{m+2})$, and for each $1 \le i \le m+1$, one considers the measure $\mu_{\alpha_i, f(\alpha_i)}$ and the associated set $G_{m+1}(\alpha_i) := G_{m+1,N_{m+1}}(\alpha_i)$. For each $1 \le i \le m+1$ and $I_m \in \mathbf{G}_m$, one defines the set of the elements of \mathbf{G}_{m+1} contained in I_m as $\bigcup_{i=1}^{m+1} \mathbf{G}_{m+1}(I_m, \alpha_i)$, where $\mathbf{G}_{m+1}(I_m, \alpha_i) =$ $\{I_m \cdot L_{\alpha_i} \cdot I : I \in G_{m+1}(\alpha_i)\}$, and the concatenation $J \cdot J'$ of two closed subcubes of $[0, 1]^d$ is obtained as the cube $f_J(J')$, where f_J is the natural contracting similitude mapping $[0, 1]^d$ onto J (this operation is associative). One gets a probability measure μ_{m+1} on \mathbf{G}_{m+1} by setting, for $I \in G_{m+1}(\alpha_i)$:

(1.2)
$$\mu_{m+1}(I_m \cdot L_{\alpha_i} \cdot I) = \mu_m(I_m) \frac{\mu_{\alpha_i, f(\alpha_i)}(I)}{\sum_{\alpha \in A_{m+1}} \sum_{I' \in G_{m+1}(\alpha)} \mu_{\alpha, f(\alpha)}(I')}$$

This makes it possible to define a Borel probability measure carried on K and coinciding with μ_m over \mathbf{G}_m for all $m \ge 1$.

Since $f(\alpha) < \alpha$ except for $\alpha = \alpha_1 = D$, if N_{m+1} is taken big enough, in (1.2) for each i > 1 the contribution of the elements of $G_{m+1}(\alpha_i)$ is roughly $2^{N_{m+1}(f(\alpha_i) - \alpha_i)}$ hence is negligible so that the denominator is equivalent to the single contribution of $\sum_{I' \in G_{m+1}(D)} \mu_{D,D}(I') \in [1/2, 1]$. Consequently, for $I_{m+1} \in \mathbf{G}_{m+1}$ of the form $I_m \cdot L_{\alpha_i} \cdot I$, $I \in G_{m+1}(\alpha_i)$, we have the following estimate:

(1.3)
$$\mu(I_{m+1}) \approx \mu_m(I_m) \mu_{\alpha_i, f(\alpha_i)}(I) \approx \mu_m(I_m) 2^{-\alpha_i N_{m+1}} \approx 2^{-\alpha_i g_{m+1}}$$

because $g_m \ll N_{m+1}$. Also, we have that $\#G_{m+1}(\alpha_i) \approx 2^{f(\alpha_i)N_{m+1}}$, hence

$$\#\{I \in \mathbf{G}_{m+1} : I \in \mathbf{G}_{m+1}(I_m, \alpha_i) \text{ for some } I_m \in \mathbf{G}_m\} = (\#\mathbf{G}_m)(\#G_{m+1}(\alpha_i)) \approx 2^{f(\alpha_i)g_{m+1}}, \mathbb{C}_m \in \mathbf{G}_m\}$$

again because $g_m \ll N_{m+1}$. The previous estimate and the continuity of f essentially yield that f is an upper bound for f^H . Combined with (1.3), it shows that at generation m+1, the

4° SÉRIE - TOME 48 - 2015 - Nº 6

mass of μ is essentially carried by the intervals $I_m \cdot L_D \cdot I$, $I \in G_{m+1}(D)$, since we have

$$1 = \|\mu\| \approx \sum_{i=1}^{m+1} 2^{f(\alpha_i)g_{m+1}} 2^{-\alpha_i g_{m+1}} = \sum_{i=1}^{m+1} 2^{(f(\alpha_i) - \alpha_i)g_{m+1}} \approx 2^{(f(\alpha_1) - \alpha_1)g_{m+1}} = 1$$

(recall that $\alpha_1 = f(\alpha_1) = D$). This can be strengthened to show that μ is exact *D*-dimensional.

Another important fact is the natural existence of a family of auxiliary measures used to find a sharp lower bound for \underline{f}^{H} : with each $\widehat{\beta} = (\beta_{m})_{m \ge 1} \in \prod_{m=1}^{\infty} A_{m}$ is associated the Cantor subset of K defined as

$$K_{\widehat{\beta}} = \bigcap_{m \ge 1} \bigcup_{I \in \mathbf{G}_{\widehat{\beta}, m}} I,$$

where $\mathbf{G}_{\widehat{\beta},m}$ is the subset of \mathbf{G}_m obtained by selecting only the intervals of the construction for which one considers the exponent $\beta_i \in A_i$ at step *i* for all $1 \leq i \leq m$. Using (1.3) and finer properties of the measures $\mu_{\alpha,\gamma}$ one can show that $K_{\widehat{\beta}} \subset \underline{E}(\mu,\beta)$, where $\beta = \liminf_{m\to\infty} \beta_m$. Moreover, the measures $\nu_{\beta_m,f(\beta_m)}$ can be used to construct a nice auxiliary probability measure $\nu_{\widehat{\beta}}$ carried by $K_{\widehat{\beta}}$. At first one defines recursively a sequence of measures $(\nu_{\widehat{\beta},m})_{m\geq 1}$ on the atoms of the sets $\mathbf{G}_{\widehat{\beta},m}, m \geq 1$, as follows: $\nu_{\widehat{\beta},1}$ is the restriction of $\nu_{D,D}$ to $\mathbf{G}_{\widehat{\beta},1}(=\mathbf{G}_1)$, and assuming that $\nu_{\widehat{\beta},m}$ is constructed on $\mathbf{G}_{\widehat{\beta},m}$, if $I_m \in \mathbf{G}_{\widehat{\beta},m}$, for $I \in G_{m+1}(\beta_{m+1})$ one sets

$$\nu_{\widehat{\beta},m+1}(I_m \cdot L_{\beta_{m+1}} \cdot I) = \nu_{\widehat{\beta},m}(I_m) \frac{\nu_{\beta_{m+1},f(\beta_{m+1})}(I)}{\sum_{I' \in G_{m+1}(\beta_{m+1})} \nu_{\beta_{m+1},f(\beta_{m+1})}(I')}$$

This yields a Borel probability measure $\nu_{\widehat{\beta}}$ supported on $K_{\widehat{\beta}}$ such that

$$\nu_{\widehat{\beta}}(I_m \cdot L_{\beta_{m+1}} \cdot I) = \nu_{\widehat{\beta}, m+1}(I_m \cdot L_{\beta_{m+1}} \cdot I) \approx \nu_{\widehat{\beta}, m}(I_m)\nu_{\beta_{m+1}, f(\beta_{m+1})}(I),$$

so that $\nu_{\widehat{\beta}}(I_m \cdot L_{\beta_{m+1}} \cdot I) \approx \nu_{\beta_{m+1},f(\beta_{m+1})}(I) \approx 2^{-f(\beta_{m+1})g_{m+1}}$ (again since $g_m \ll N_{m+1}$). This can be strengthened to $\dim_H(\nu_{\widehat{\beta}}) = \liminf_{m\to\infty} f(\beta_m)$, hence $\dim_H K_{\widehat{\beta}} \geq \liminf_{m\to\infty} f(\beta_m)$ by the mass distribution principle (see Section 6). Finally, if $\beta \in [\alpha_{\min}, \alpha_{\max}]$ and $\lim_{m\to\infty} \beta_m = \beta$, the continuity of f yields $\underline{f}^H(\beta) = \dim_H \underline{E}(\mu, \beta) \geq f(\beta)$.

1.2. Main statements, and comments

New definitions and properties are needed to state our main results.

1.2.1. Additional definitions and properties related to the multifractal formalism. – For $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$, recall that f_{μ}^H and f_{μ}^P stand for the Hausdorff spectrum $\alpha \in \mathbb{R} \cup \{\infty\} \mapsto \dim_H E(\mu, \alpha)$ and the packing spectrum $\alpha \in \mathbb{R} \cup \{\infty\} \mapsto \dim_P E(\mu, \alpha)$ respectively.

The functions defined below, as well as some variants, are well known in the literature ([33, 21, 19, 63, 52]). They naturally complete τ_{μ} and τ_{μ}^{*} to describe, in terms of large deviations, the asymptotic behavior of the distribution of the measure μ at small scales. They also yield a finer multifractal formalism, which connects geometric properties of the sets $E(\mu, \alpha)$ to large deviations properties associated with μ , both from the Hausdorff and packing dimensions point of views. In Remark 1.3 (Section 1.2.2) we will explain the connection with another multifractal formalism emphasized in [63], which is based on a purely geometric approach.

Define also the upper L^q -spectrum of μ as

$$q \in \mathbb{R} \mapsto \overline{\tau}_{\mu}(q) = \limsup_{r \to 0^+} \frac{\log \sup \left\{ \sum_i \mu(B(x_i, r))^q \right\}}{\log(r)}$$

(this function is not concave in general), as well as the lower and upper large deviations spectra f_{μ}^{LD} and $\overline{f}_{\mu}^{\text{LD}}$:

$$\begin{split} \alpha \in \mathbb{R} &\mapsto \underline{f}_{\mu}^{\mathrm{LD}}(\alpha) = \lim_{\epsilon \to 0^+} \liminf_{r \to 0^+} \frac{\log \sup \# \left\{ i : r^{\alpha + \epsilon} \le \mu(B(x_i, r)) \le r^{\alpha - \epsilon} \right\}}{-\log(r)}, \\ \alpha \in \mathbb{R} &\mapsto \overline{f}_{\mu}^{\mathrm{LD}}(\alpha) = \lim_{\epsilon \to 0^+} \limsup_{r \to 0^+} \frac{\log \sup \# \left\{ i : r^{\alpha + \epsilon} \le \mu(B(x_i, r)) \le r^{\alpha - \epsilon} \right\}}{-\log(r)}, \\ \underline{f}_{\mu}^{\mathrm{LD}}(\infty) &= \lim_{A \to \infty} \liminf_{r \to 0^+} \frac{\log \sup \# \left\{ i : \mu(B(x_i, r)) \le r^A \right\}}{-\log(r)}, \\ \overline{f}_{\mu}^{\mathrm{LD}}(\infty) &= \lim_{A \to \infty} \limsup_{r \to 0^+} \frac{\log \sup \# \left\{ i : \mu(B(x_i, r)) \le r^A \right\}}{-\log(r)}, \end{split}$$

where the suprema are taken over all the centered packings of $\operatorname{supp}(\mu)$ by closed balls of radius r. Notice that $0 \leq \underline{\dim}_B \operatorname{supp}(\mu) = -\overline{\tau}(0) \leq d$, and $\overline{\tau}_{\mu}(1) = 0$ (by the same arguments as for the equality $\tau_{\mu}(1) = 0$, see [63, Section 2.7] or [52, Section 3]).

One always has $\overline{\tau}_{\mu}^* \leq \tau_{\mu}^*$, and

(1.4)
$$\forall \alpha \in \mathbb{R} \cup \{\infty\}, \ f_{\mu}^{H}(\alpha) \leq \underline{f}_{\mu}^{\mathrm{LD}}(\alpha) \leq \overline{\tau}_{\mu}^{*}(\alpha) \leq \max(\alpha, -\overline{\tau}_{\mu}(0)) \leq \max(\alpha, d),$$

(1.5)
$$\forall \alpha \in \mathbb{R} \cup \{\infty\}, \ f_{\mu}^{P}(\alpha) \leq \overline{f}_{\mu}^{\text{LD}}(\alpha) \leq \tau_{\mu}^{*}(\alpha) \leq \max(\alpha, -\tau_{\mu}(0)) \leq \max(\alpha, d).$$

We will say that μ obeys the refined multifractal formalism at $\alpha \in \mathbb{R} \cup \{\infty\}$ if $f_{\mu}^{H}(\alpha) = \overline{\tau}_{\mu}^{*}(\alpha)$ and $f_{\mu}^{P}(\alpha) = \tau_{\mu}^{*}(\alpha)$. If $\alpha \in \operatorname{dom}(\tau_{\mu}^{*}) \setminus \operatorname{dom}(\overline{\tau}_{\mu}^{*})$, one necessarily has $E(\mu, \alpha) = \emptyset$, so that in (1.5), one can only expect the large deviations property $\overline{f}_{\mu}^{LD}(\alpha) = \tau_{\mu}^{*}(\alpha)$ to hold.

The inequalities $f_{\mu}^{H}(\alpha) \leq \overline{\tau}_{\mu}^{*}(\alpha)$ and $f_{\mu}^{P}(\alpha) \leq \tau_{\mu}^{*}(\alpha)$ are established for $\alpha < \infty$ when μ is doubling in [63, Section 2.7]. The inequalities $f_{\mu}^{H}(\alpha) \leq \overline{f}_{\mu}^{\text{LD}}(\alpha) \leq \tau_{\mu}^{*}(\alpha)$ are established in [52, Section 3] when $\alpha < \infty$. The other inequalities will be justified in Sections 5.2 and 5.3.

We notice that if μ is exactly dimensional with dimension D, then $D = f^H(D) = \underline{f}_{\mu}^{\text{LD}}(D) = \overline{\tau}_{\mu}^*(D) = \overline{f}_{\mu}^{\text{LD}}(D) = \tau_{\mu}^*(D)$, and in the case where $\overline{\tau}_{\mu}$ is concave, we have $D \in [\overline{\tau}'_{\mu}(1^+), \overline{\tau}'_{\mu}(1^-)](\subset [\tau'_{\mu}(1^+), \tau'_{\mu}(1^-)])$, since $\overline{\tau}_{\mu}(1) = 0$ implies that $\overline{\tau}_{\mu}^*(\alpha) = \alpha$ if and only if $\alpha \in [\overline{\tau}'_{\mu}(1^+), \overline{\tau}'_{\mu}(1^-)]$.

Let us now describe the possible behaviors of the L^q -spectrum and its Legendre transform. Before stating the corresponding propositions, we need to extend the notion of Legendre-Fentchel transform to functions $f : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{-\infty\}$.

If $f : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{-\infty\}$ and $\operatorname{dom}(f) \cap \mathbb{R} \neq \emptyset$, we define the concave Legendre-Fenchel transform of f as

 $f^*: q \in \mathbb{R} \mapsto \inf\{q\alpha - f(\alpha) : \alpha \in \operatorname{dom}(f)\},\$

with the conventions $q \times \infty = \frac{q}{|q|} \times \infty$ if $q \neq 0$ and $0 \times \infty = 0$.

4° SÉRIE - TOME 48 - 2015 - Nº 6

1464

Consequently, if $\infty \in \text{dom}(f)$ and f is bounded from above, then $0 = \min(\text{dom}(f^*))$ and $f^*(0) = -\max(\sup(f_{|\mathbb{R}}), f(\infty))$; moreover, f^* is concave over $\text{dom}(f^*)$, upper semicontinuous over $\text{dom}(f^*) \setminus \{0\}$, and upper semi-continuous at 0 if and only if $f(\infty) = \max(f)$.

PROPOSITION 1.1. – Let $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$.

- 1. τ_{μ} is concave, non-decreasing, $\tau_{\mu}(1) = 0$, and $-d \leq \tau_{\mu}(0) = -\overline{\dim}_B \operatorname{supp}(\mu) \leq 0$.
- 2. One has either dom $(\tau_{\mu}) = \mathbb{R}$, or dom $(\tau_{\mu}) = \mathbb{R}_+$, according to whether the exponent $\limsup_{r \to 0^+} \frac{\log(\inf\{\mu(B(x,r)) : x \in \operatorname{supp}(\mu)\})}{\log(r)}$ is finite or not. Moreover τ_{μ}^* is nonnegative on its domain.

PROPOSITION 1.2. – Suppose that $\tau : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ satisfies the properties of the L^q -spectrum described in Proposition 1.1.

- 1. Suppose that dom $(\tau) = \mathbb{R}$. Then dom (τ^*) is the compact interval $I = [\tau'(\infty), \tau'(-\infty)]$, τ^* is concave and continuous on its domain, and $(\tau^*)^* = \tau$ on \mathbb{R} .
- 2. Suppose that dom $(\tau) = \mathbb{R}_+$. Then $\infty \in dom(\tau^*)$ with $\tau^*(\infty) = -\tau(0)$ and:
 - (a) If $\tau(0) = 0$ then $\tau = 0$ over \mathbb{R}_+ , dom $(\tau^*) = \mathbb{R}_+ \cup \{\infty\}$ and $\tau^* = 0$ over $\mathbb{R}_+ \cup \{\infty\}$.
 - (b) If τ(0) < 0 and τ is continuous at 0⁺, then dom(τ^{*}) is the interval [τ'(∞),∞], τ^{*} is concave, continuous, and increasing over [τ'(∞), τ'(0⁺)), τ^{*}(α) = −τ(0) = τ^{*}(∞) = −τ(0) for all α ∈ [τ'(0⁺),∞) and τ^{*} is continuous at ∞; there are two distinct behaviors according to whether τ'(0⁺) < ∞ or not.</p>
 - (c) If $\tau(0) < 0$ and τ is discontinuous at 0^+ , then dom (τ^*) is the interval $I = [\tau'(\infty), \infty]$. Moreover, $\tau^*(\alpha) = -\tau(0^+) < \tau^*(\infty) = -\tau(0)$ for all $\alpha \in [\lim_{q \to 0^+} \tau'(q^-), \infty)$, so that τ^* is concave and continuous on $[\tau'(\infty), \infty)$ and discontinuous at ∞ (there are also two cases, according to whether $\lim_{q \to 0^+} \tau'(q^-) = \infty$ or not).
 - (d) In all the previous cases, $(\tau^*)^* = \tau$ on \mathbb{R} .

Proposition 1.1(1) is standard and proved for instance in [52] (Proposition 3.2). Propositions 1.1(2) and 1.2(1) are essentially restatements of Propositions 3.3-3.5 in [52]. However, for the reader's convenience we will provide a proof in Section 5, where the whole proofs of Propositions 1.1(2) and 1.2 are given.

1.2.2. Full illustration of the multifractal formalism. Complements to Theorem 1.1. – When μ is a Gibbs measure on a conformal repeller or a self-similar measure on an attractor satisfying suitable separation conditions, the Hausdorff and packing dimensions are also known for all the sets $E(\mu, \alpha, \beta)$:

(1.6)
$$\dim_{H} E(\mu, \alpha, \beta) = \min\{\tau_{\mu}^{*}(\gamma) : \gamma \in [\alpha, \beta]\},\\ \dim_{P} E(\mu, \alpha, \beta) = \max\{\tau_{\mu}^{*}(\gamma) : \gamma \in [\alpha, \beta]\},\\ \dim_{H} \underline{E}(\mu, \alpha) = \tau_{\mu}^{*}(\alpha) = \dim_{H} \overline{E}(\mu, \alpha),\\ \dim_{P} \underline{E}(\mu, \alpha) = \max\{\tau_{\mu}^{*}(\gamma) : \gamma \geq \alpha\}, \dim_{P} \overline{E}(\mu, \alpha) = \max\{\tau_{\mu}^{*}(\gamma) : \gamma \leq \alpha\}.$$

for all $\alpha \leq \beta \in \mathbb{R}_+ \cup \{\infty\}$ (see [66, 4, 83] and also [14, 39] for closely related questions). We notice that (1.6) implies that $\dim_H \operatorname{supp}(\mu) = \underline{\dim}_B \operatorname{supp}(\mu) = \dim_P \operatorname{supp}(\mu) = \overline{\dim}_B \operatorname{supp}(\mu) = -\tau_{\mu}(0)$.

It turns out that properties (1.6) enter in our exhaustive illustration of the multifractal formalism.

THEOREM 1.3. – Let $f \in \mathcal{F}(d)$. Suppose that dom(f) is a non empty closed subinterval of $[0, \infty]$ and f is concave over $\mathcal{I} \cap \mathbb{R}_+$.

For each fixed point D of f, there exists $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$, exactly dimensional with dimension D, such that $\tau_{\mu} = f^* = \overline{\tau}_{\mu}$, $\tau_{\mu}^* = f$, and (1.6) holds for all $\alpha \leq \beta \in \mathbb{R}_+ \cup \{\infty\}$. Moreover, the same properties hold if μ is replaced by its restriction to any closed ball whose interior intersects $\operatorname{supp}(\mu)$.

The following corollary, of which Theorem 1.1 is a consequence, then follows from the fact that if τ satisfies the properties of Proposition 1.1 (and so falls into the different situations described in Proposition 1.2(1) and (2)), then τ^* satisfies the assumptions of Theorem 1.3.

COROLLARY 1.1. – Let $\tau : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be a function satisfying properties (1) and (2) of Proposition 1.1. Let $D \in [\tau'(1^+), \tau'(1^-)]$.

There exists $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$, exactly dimensional with dimension D, satisfying (1.6) for all $\alpha \leq \beta \in \mathbb{R}_+ \cup \{\infty\}$ with $\tau_\mu = \tau = \overline{\tau}_\mu$. Moreover, the same properties hold if μ is replaced by its restriction to any closed ball whose interior intersects $\operatorname{supp}(\mu)$.

REMARK 1.1. – The behavior described in Proposition 1.2(1) is illustrated, for instance, by Gibbs and weak Gibbs measures on conformal repellers (see [63, 70, 38]). Such examples, which live on dynamical systems semi-conjugate to subshifts of finite type, cannot exhibit behaviors like those corresponding to Proposition 1.2(2). The behaviors described by Proposition 1.2(2)(b) are illustrated by some Gibbs measures on countable Markov shifts and their geometric realizations [46], which also obey the multifractal formalism, though in [46] the set $E(\mu, \infty)$ is not studied. The fact that the behaviors described in Proposition 1.2(2)(a) and (c) be illustrated by measures obeing the mutifractal formalism seems to be new. We notice that the extension of the Legendre transform including ∞ in the domain in this case yields Legendre transforms which are not necessarily upper semi-continuous, like τ at 0 in case (c).

REMARK 1.2. – Our results illustrate all the possible situations, in term of the function τ_{μ} , for which the measure μ is exactly dimensional though $\tau'_{\mu}(1)$ does not exist. In [44], when d = 1, for each $D \in (0,1)$ one finds an exactly dimensional measure μ with dimension D and L^q -spectrum equal to $\min(q - 1, 0)$ over \mathbb{R}_+ . It is also worth mentioning that in [15] one finds examples of inhomogeneous Bernoulli measures over [0, 1] with an L^q -spectrum presenting countably many points of non differentiability over $[1, +\infty)$.

In the previous results, due to (1.4) and (1.5) we have $f_{\mu}^{H} = f_{\mu}^{P} = \underline{f}_{\mu}^{LD} = \overline{f}_{\mu}^{LD}$, which reflects a strong homogeneity of the sets $E(\mu, \alpha)$. The purpose of the refined multifractal formalism is to describe situations irregular enough so that the Hausdorff and packing dimensions of $E(\mu, \alpha)$ differ for most of the α .

The next two results extend in a non trivial way the two previous ones , in particular by exhibiting a new formula for $\dim_H E(\mu, \alpha, \beta)$. They invoke an extension of (1.4) and (1.5),

which illustrates the following complement to the multifractal formalism: If $0 \le \alpha < \infty$ and $\alpha \leq \beta \leq \infty, 1 > r > 0$, and $\epsilon > 0$, set

$$f_{\mu}(\alpha,\beta,\epsilon,r) = \frac{\log \sup \# \left\{ i : r^{\beta+\epsilon} \le \mu(B(x_i,r)) \le r^{\alpha-\epsilon} \right\}}{-\log(r)},$$

where the suprema are taken over all the centered packing of $supp(\mu)$ by closed balls of radius r, and with the convention that $r^{\infty} = 0$. Then define

$$\underline{f}_{\mu}^{\mathrm{LD}}(\alpha,\beta) = \lim_{\epsilon \to 0} \liminf_{r \to 0+} f_{\mu}(\alpha,\beta,\epsilon,r).$$

Also, define $\underline{f}_{\mu}^{\mathrm{LD}}(\infty, \infty) = \underline{f}_{\mu}^{\mathrm{LD}}(\infty)$.

PROPOSITION 1.3. – Let $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$. For any $0 \le \alpha \le \beta \le \infty$, one has

1.
$$\dim_{H} E(\mu, \alpha, \beta) \leq f_{H}(\alpha, \beta) := \min(\overline{f}_{\mu}^{\text{LD}}(\alpha), \overline{f}_{\mu}^{\text{LD}}(\beta), \underline{f}_{\mu}^{\text{LD}}(\alpha, \beta))$$
$$\dim_{P} E(\mu, \alpha, \beta) \leq f_{P}(\alpha, \beta) := \max\{\overline{f}_{\mu}^{\text{LD}}(\alpha') : \alpha' \in [\alpha, \beta]\}$$
2.
$$\dim_{L} E(\mu, \alpha) \leq \sup\{f_{L}(\alpha, \beta) : \beta \geq \alpha\} \text{ for } L \in \{H, P\},$$

$$\dim_L \underline{E}(\mu, \alpha) \leq \sup\{f_L(\alpha, \beta) : \beta \geq \alpha\} \text{ for } L \in \{H, P\},$$
$$\dim_L \overline{E}(\mu, \alpha) \leq \sup\{f_L(\beta, \alpha) : \beta \leq \alpha\} \text{ for } L \in \{H, P\}.$$

THEOREM 1.4. – Let $d \in \mathbb{N}_+$. Let $\mathcal{J} \subset \mathcal{J}$ be two non empty closed subintervals of $[0, \infty]$. Let f and $g \in \mathcal{F}(d)$ such that dom $(f) = \mathcal{I}$, dom $(g) = \mathcal{J}$, and $f \leq g$. Suppose also that f and g are concave over $\mathcal{I} \cap \mathbb{R}_+$ and $\mathcal{J} \cap \mathbb{R}_+$ respectively.

For each $D \in Fix(f)$, there exists $\mu \in \mathcal{M}^+_c(\mathbb{R}^d)$, exactly dimensional with dimension D, such that

- 1. $\operatorname{dom}(f_{\mu}^{H}) = \operatorname{dom}(f_{\mu}^{P}) = \operatorname{dom}(\underline{f}_{\mu}^{\mathrm{LD}}) = \mathscr{I} \text{ and } \operatorname{dom}(\overline{f}_{\mu}^{\mathrm{LD}}) = \mathscr{J}.$ 2. For all $\alpha \in \mathscr{I}$, $f_{\mu}^{H}(\alpha) = \underline{f}_{\mu}^{\mathrm{LD}}(\alpha) = f(\alpha)$, $f_{\mu}^{P}(\alpha) = \overline{f}_{\mu}^{\mathrm{LD}}(\alpha) = g(\alpha)$ and $\overline{f}_{\mu}^{\mathrm{LD}}(\alpha) = g(\alpha)$ for $\alpha \in \mathcal{A} \setminus \mathcal{I}$.
- 3. *More generally, for all* $\alpha \leq \beta \in \mathbb{R}_+ \cup \{\infty\}$ *,*

$$\begin{split} & \underline{f}_{\mu}^{\mathrm{LD}}(\alpha,\beta) = f(\alpha,\beta) \coloneqq \max\{f(\alpha'):\alpha' \in [\alpha,\beta]\}, \\ & \dim_{H} E(\mu,\alpha,\beta) = \begin{cases} \min(g(\alpha),g(\beta),f(\alpha,\beta)) & \text{if } [\alpha,\beta] \subset \mathcal{J} \text{ and } [\alpha,\beta] \cap \mathcal{I} \neq \varnothing \\ -\infty & \text{otherwise}, \end{cases} \\ & \dim_{P} E(\mu,\alpha,\beta) = \begin{cases} \max\{g(\alpha'):\alpha' \in [\alpha,\beta]\} & \text{if } [\alpha,\beta] \subset \mathcal{J} \text{ and } [\alpha,\beta] \cap \mathcal{I} \neq \varnothing \\ -\infty & \text{otherwise}, \end{cases} \\ & \dim_{H} \underline{E}(\mu,\alpha) = \max\{\dim_{H} E(\mu,\alpha,\beta):\beta \geq \alpha\} = \min(g(\alpha),\max\{f(\beta):\beta \geq \alpha\}) \\ & \dim_{H} \overline{E}(\mu,\alpha) = \max\{\dim_{H} E(\mu,\beta,\alpha):\beta \leq \alpha\} = \min(g(\alpha),\max\{f(\beta):\beta \leq \alpha\}) \\ & \dim_{P} \underline{E}(\mu,\alpha) = \max\{\dim_{P} E(\mu,\alpha,\beta):\beta \geq \alpha\} \\ & = \begin{cases} \max\{g(\beta):\beta \geq \alpha\} & \text{if } \alpha \in [\min(\mathcal{J}),\max(\mathcal{I})] \\ -\infty & \text{otherwise}, \end{cases} \end{split}$$

$$\dim_{P} \overline{E}(\mu, \alpha) = \max\{\dim_{P} E(\mu, \beta, \alpha) : \beta \leq \alpha\}$$
$$= \begin{cases} \max\{g(\beta) : \beta \leq \alpha\} & \text{if } \alpha \in [\min(\mathscr{I}), \max(\mathscr{J})] \\ -\infty & \text{otherwise.} \end{cases}$$

4. $\bar{\tau}_{\mu} = f^*$, $\bar{\tau}_{\mu}^* = f$, $\tau_{\mu} = g^*$ and $\tau_{\mu}^* = g$.

Moreover, all the previous properties hold if μ is replaced by its restriction to any closed ball whose interior intersects supp(μ).

Notice that properties (2) and (4) of the previous statement imply $\dim_H \operatorname{supp}(\mu) = \underline{\dim}_B \operatorname{supp}(\mu) = -\overline{\tau}(0)$ and $\dim_P \operatorname{supp}(\mu) = \overline{\dim}_B \operatorname{supp}(\mu) = -\tau(0)$, because $\max\{\dim_H \underline{E}(\alpha) : \alpha \in \mathcal{J}\} = -\overline{\tau}(0)$ and $\max\{\dim_P \underline{E}(\alpha) : \alpha \in \mathcal{J}\} = -\tau(0)$.

COROLLARY 1.2. – Let $\tau, \overline{\tau} : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be two functions satisfying properties (1) and (2) of Proposition 1.1, and such that $\tau \leq \overline{\tau}$.

Let $D \in [\overline{\tau}'(1^+), \overline{\tau}'(1^-)] \subset [\tau'(1^+), \tau'(1^-)]$. There exists an exactly dimensional measure $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$ with dimension D such that :

- 1. $\tau_{\mu} = \tau$ and $\overline{\tau}_{\mu} = \overline{\tau}$;
- 2. $\dim_H \operatorname{supp}(\mu) = \underline{\dim}_B \operatorname{supp}(\mu) = -\overline{\tau}(0)$ and $\dim_P \operatorname{supp}(\mu) = \overline{\dim}_B \operatorname{supp}(\mu) = -\tau(0);$
- 3. properties (1)-(3) of Theorem 1.4 hold with $\mathcal{I} = \operatorname{dom}(\overline{\tau}^*)$, $\mathcal{J} = \operatorname{dom}(\tau^*)$, $f = \overline{\tau}^*$ and $g = \tau^*$.

Moreover, all the previous properties hold if μ is replaced by its restriction to any closed ball whose interior intersects supp(μ).

REMARK 1.3 (Link with Olsen's multifractal formalism). – In [63], Olsen introduces three "multifractal dimensions" functions $b_{\mu} \leq B_{\mu} \leq \Lambda_{\mu}$ derived from "multifractal" generalizations of Hausdorff and packing measures associated with μ (Λ_{μ} and B_{μ} are convex, while b_{μ} may be not), so that $f^{H}(\alpha) \leq (-b_{\mu})^{*}(\alpha)$ and $f^{P}(\alpha) \leq (-B_{\mu})^{*}(\alpha)$ ($\leq (-\Lambda_{\mu})^{*}(\alpha)$) for all $\alpha \in \mathbb{R}_{+}$; one can then say that Olsen's multifractal formalism holds at $\alpha \in \mathbb{R}_{+} \cup \{\infty\}$ if the previous inequalities are equalities (adding $\alpha = \infty$ in his formalism does not matter). The pair $\{b_{\mu}, B_{\mu}\}$ has a geometric meaning, while $\{\tau_{\mu}, \overline{\tau}_{\mu}\}$ relies on large deviations properties.

This formalism has recently found new illustrations by inhomogeneous Bernoulli measures on [0, 1] (cf. [18, 78]), and it is particularly well suited to describe $\dim_H E(\mu, \alpha)$ for self-affine measures or Gibbs measures on self-affine Sierpiński carpets and sponges [51, 64, 10, 6] (the packing dimension of the sets $E(\mu, \alpha)$ in these situations remains an open question in general).

Olsen pays a particular attention to compare the pairs of functions $\{b_{\mu}, \Lambda_{\mu}\}$ and $\{\tau_{\mu}, \overline{\tau}_{\mu}\}$. He proves $b_{\mu} \leq -\overline{\tau}_{\mu}$ and $B_{\mu} \leq \Lambda_{\mu} = -\tau_{\mu}$ when μ is doubling, which according to both multifractal formalisms inequalities, implies $(-b_{\mu})^*(\alpha) = \overline{\tau}^*_{\mu}(\alpha)$ and $(-B^*_{\mu})(\alpha) = (-\Lambda_{\mu})^*(\alpha)) = \tau^*_{\mu}(\alpha)$ when the refined multifractal formalism used in this paper holds at α , hence the validity of his formalism. It turns out that even if the measure μ we are going to construct to prove Corollary 1.2 are not doubling, they possess the weaker but close property that there exists a function $\epsilon(r)$ tending to 0^+ as $r \to 0^+$ such that $\mu(B(x, 2r)) \leq r^{-\epsilon(r)}\mu(B(x, r))$, uniformly in $x \in \text{supp}(\mu)$ (see Section 4.2). This is enough for $b_{\mu} \leq -\overline{\tau}_{\mu}$ and $B_{\mu} \leq \Lambda_{\mu} = -\tau_{\mu}$ to hold, hence for Olsen's multifractal formalism to be valid at each α of dom $(\overline{\tau}^*_{\mu})$. Moreover, using the equality $(-b_{\mu})^* = \overline{\tau}^*_{\mu}$ over dom $(\overline{\tau}^*_{\mu})$, taking the Legendre-Fenchel transforms of these functions, and

1468

using the inequality $b_{\mu} \leq -\overline{\tau}_{\mu} = -\overline{\tau}$, we can get $b_{\mu} = -\overline{\tau}_{\mu} = -\overline{\tau}$. Similarly $B_{\mu} = \Lambda_{\mu} = -\tau_{\mu} = -\tau$.

General upper bounds for $\dim_H \underline{E}(\mu, \alpha)$ and $\dim_H \overline{E}(\mu, \alpha)$ are given by [63, Theorem 2.17(ii)(iii)], namely $(-(b_{\mu} \Box B_{\mu}))^*(\alpha)$ and $(-(B_{\mu} \Box b_{\mu}))^*(\alpha)$ respectively, where $b \Box B$ equals b over $(-\infty, 0)$, $b(0) \wedge B(0)$ at 0, and B over $(0, \infty)$. The previous remarks and the formulas obtained in Theorem 1.4 for $\dim_H \underline{E}(\mu, \alpha)$ and $\dim_H \overline{E}(\mu, \alpha)$ show that for the measure μ we construct, the upper bounds estimates $(-(b_{\mu} \Box B_{\mu}))^*(\alpha)$ and $(-(B_{\mu} \Box b_{\mu}))^*(\alpha)$ do provide the correct values for the Hausdorff dimensions.

1.2.3. *Prescription of the lower Hausdorff spectrum.* – Theorem 1.2 can be refined as follows, according to the properties of the measure we construct:

THEOREM 1.5. – Let $f \in \mathcal{F}(d)$. For each $D \in \text{Fix}(f)$, there exists an (HM) measure $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$, exactly dimensional with dimension D, such that $f_{\mu}^H = f$.

Moreover, μ can be constructed so that one has: (1) if $\alpha \in \text{Fix}(f)$ then $f_{\mu}^{H}(\alpha) = \alpha$ and (2) if $\alpha \in \text{dom}(f) \setminus \text{Fix}(f) \neq \emptyset$ then $\underline{E}(\mu, \alpha) = E(\mu, \alpha, \infty)$, and properties (1) and (2) hold if μ is replaced by its restriction to any closed ball whose interior intersects $\text{supp}(\mu)$.

REMARK 1.4. – It can be shown for the measure we construct that $\dim_P \underline{E}(\mu, \alpha) = \max\{f(\alpha') : \alpha' \ge \alpha\}$ for all $\alpha \in \mathbb{R} \cup \{\infty\}$. The Hausdorff and packing dimensions of the sets $\overline{E}(\mu, \alpha)$ and $E(\mu, \alpha, \beta)$ can also be computed; we leave these calculations, based on Corollaries 3.1 and 3.2, to the reader.

REMARK 1.5. – (1) The prescription of the lower Hausdorff spectrum has also been studied in [24], where the authors work on \mathbb{R} and construct (HM) continuous measures, not exactly dimensional, but with upper Hausdorff dimension equal to 1, and whose support is equal to [0, 1], with a prescribed lower Hausdorff spectrum in the class \mathcal{F} of functions $f : \mathbb{R}_+ \to [0, 1] \cup \{-\infty\}$ which satisfy: f(1) = 1, dom(f) is a closed subinterval of [0, 1]of the form $[\alpha, 1]$ such that $\alpha > 0$, and $f_{|[\alpha,1]} = \max(g_{|[\alpha,1]}, 0)$, where (i) g is the supremum of a sequence of functions $(g_n)_{n\geq 1}$, such that each g_n is constant over its domain supposed to be a closed subinterval of [0, 1] and $g_n(\beta) \leq \beta$ for all $\beta \in [0, 1]$; (ii) $[\alpha, 1]$ is the smallest closed interval containing the support of g. It is also shown that for an (HM) measure to be supported by the whole interval [0, 1], it is necessary that the support of its lower Hausdorff spectrum contains an interval of the form $[\alpha, 1]$, $(0 \leq \alpha \leq 1)$.

The authors of [24] also study the case of non (HM) measures. In this case, they construct non exactly dimensional measures with upper Hausdorff dimension 1 whose support is equal to [0, 1], with a prescribed lower Hausdorff spectrum in the broader class $\tilde{\mathcal{F}}$ of functions fwhich satisfy that f(1) = 1, $0 < \inf(\operatorname{dom}(f))$, and $f_{|\operatorname{dom}(f)\setminus\{1\}|} = g_{|\operatorname{dom}(f)\setminus\{1\}|}$, where g satisfies property (i). This includes all such functions f for which g is lower semi-continuous. Simultaneously, they also construct a non (HM) measure with lower Hausdorff spectrum given by g.

(2) All the spectra defined in this paper make sense if measures are replaced by non negative functions of subsets of \mathbb{R}^d to which a notion of support is associated. This is the case for instance of Choquet capacities. In [57], the prescription of the spectrum $\alpha \mapsto \dim_H E(C, \alpha)$ is studied, where C is a Choquet capacity on subsets of [0, 1] but not a positive measure, which makes the situation easier to tract. The authors can find a capacity with spectrum given by f,

for any function $f = \sup_{i \ge 1} f_i$, where the functions $f_i : \mathbb{R} \to [0, 1] \cup \{-\infty\}$ are such that $\operatorname{dom}(f_i)$ is a non empty closed subset of \mathbb{R}_+ , and either $f_i = 0$ over $\operatorname{dom}(f_i)$ or f_i is invertible from $\operatorname{dom}(f_i)$ onto $f(\operatorname{dom}(f_i))$, with a continuous inverse (this class of function contains $\widetilde{\mathcal{F}}$). Moreover the capacity is (HM).

In [56], the authors construct non (HM) non negative functions C of subsets of [0, 1], which are not measures, for which the spectrum

$$\alpha \mapsto \lim_{\epsilon \to 0^+} \dim_H \bigcup_{s > 0} \bigcap_{0 < r < s} \{ x \in \operatorname{supp}(C) : r^{\alpha + \epsilon} \le C(B(x, r)) \le r^{\alpha - \epsilon} \}$$

is prescribed in the class of upper semi-continuous functions $f : \mathbb{R}_+ \mapsto [0, 1] \cup \{-\infty\}$ with non empty compact domain. However, the spectrum which is prescribed is quite rough with respect to the Hausdorff spectrum.

REMARK 1.6. – It is worth mentioning that in this paper our constructions provide continuous measures even when their dimension equals 0, and are based on the properties of the simplest multifractal measures, namely Bernoulli products. These properties are combined in recursive concatenations (roughly described in Section 1 and more elaborated than those used for instance to lower bound the Hausdorff dimensions of the sets $E(\mu, \alpha)$ in the study of weak Gibbs measures) in order to converge asymptotically to a prescribed multifractal structure.

We will first prove Theorem 1.5 because its proof is shorter than that of Theorem 1.4, and it already contains some of the main ideas involved in the proof of Theorem 1.4. However, none of the two proofs can be reduced to the other one regarding the computation of f_{μ}^{H} .

The paper is organized as follows. Section 2 introduces preliminary information about Bernoulli measures. Section 3 is dedicated to the proof of Theorem 1.5, Section 4 to the proof of Theorem 1.4, and Section 5 contains the proofs of Propositions 1.1, 1.2 and 1.3, as well as some inequalities in (1.4) and (1.5). Section 6 gives a short account about the mass distribution principle.

2. Some notations, and preliminary facts about Bernoulli measures

2.1. Notations

For $n \ge 1$, define

$$\mathcal{F}_n = \left\{ \prod_{i=1}^d [k_i 2^{-n}, (k_i + 1) 2^{-n}] : 0 \le k_i < 2^n \right\}.$$

If $x \in \mathbb{R}^d$ and $n \ge 0$ we denote by $I_n(x)$ the closure of the unique dyadic cube, semi-open to the right, of generation n, that contains x.

Given two elements $I = \prod_{i=1}^{d} [k_i 2^{-n}, (k_i + 1)2^{-n}]$ and $I' = \prod_{i=1}^{d} [k'_i 2^{-n'}, (k'_i + 1)2^{-n'}]$ of $\bigcup_{p \ge 0} \mathcal{F}_p$, the concatenation $I \cdot I'$ of I and I' is defined as

(2.1)
$$I \cdot I' = \prod_{i=1}^{d} [k_i 2^{-n} + k'_i 2^{-n-n'}, k_i 2^{-n} + (k'_i + 1) 2^{-n-n'}].$$

4° SÉRIE - TOME 48 - 2015 - Nº 6

1470

If J is a closed dyadic cube of generation j, we denote by $\mathcal{N}_1(n, J)$ the set made of J and the $3^d - 1$ dyadic cubes of generation n neighboring J, and denote by $\mathcal{N}_2(n, J)$ the union of $\mathcal{N}_1(n, J)$ and the $5^d - 3^d$ closed dyadic cubes of generations n neighboring $\mathcal{N}_1(n, J)$.

Fix L_0 a closed dyadic subcube of $[0, 1]^d$ of generation 2 which does not touch $\partial [0, 1]^d$. For each integer $k \ge 1$, we can fix a collection $\mathcal{L}(k)$ of k closed dyadic cubes of generation $\ell(k) = \left\lfloor \frac{\log_2(6^d k)}{d} \right\rfloor + 3$, all contained in L_0 , such that the sets $\bigcup_{I \in \mathcal{N}_2(\ell(k), L)} I$, $L \in \mathcal{L}(k)$, are pairwise disjoint. This property will imply that the measure constructed in Section 4.1 is "weakly" doubling.

If ν is a positive Borel measure supported on $[0, 1]^d$ and x belongs to the support of ν , we set

$$d(\mu, x, n) = \frac{\log \nu(I_n(x))}{-n \log(2)}.$$

For the definitions of the s-dimensional Hausdorff and packing measures denoted respectively as \mathcal{H}^s and \mathcal{P}^s in this paper, the reader is referred to [33] or [60].

2.2. Some facts about Bernoulli measures

If $q \in [0, 1]$, let $H(q) = -q \log_2(q) - (1 - q) \log_2(1 - q)$, with the convention $0 \times \infty = 0$. Also, denote by ν_q the Bernoulli measure generated by (q, 1 - q) on [0, 1].

For each $0 \leq \gamma \leq d$ and $\alpha \geq \gamma$, we can fix $(p,q) = (p_{\alpha,\gamma}, q_{\alpha,\gamma}) \in [0,1]^2$ such that

$$\begin{cases} \alpha = -d \cdot (q \log_2(p) + (1-q) \log_2(1-p)) \\ \gamma = d \cdot H(q). \end{cases}$$

Indeed, since $\gamma/d \in [0, 1]$, there are clearly two solutions to $H(q) = \gamma/d$ in [0, 1] if $\gamma < d$ and only one if $\gamma = d$, namely 1/2. Fix q one solution. Now we seek for $p \in [0, 1]$ such that $\alpha(p) = -d(q \log_2(p) + (1 - q) \log_2(1 - p))$ be equal to α . If $q \in \{0, 1\}$, this is immediate. Otherwise, the mapping $\alpha(p)$ decreases over (0, q] from ∞ to $\alpha(q) = \gamma$, and it increases on $[q, \infty)$ from γ to ∞ . So in this case there is at least one and at most two solutions to $\alpha(p) = \alpha$ since we assumed that $\gamma \leq \alpha$.

We will use the following classical fact, which is just a consequence of the strong law of large numbers.

PROPOSITION 2.1. – Suppose that d = 1. Let $(p,q) \in [0,1]^2$. For ν_q -almost every $x \in [0,1]$,

$$\lim_{n \to \infty} d(\nu_p, x, n) = -q \log_2(p) - (1-q) \log_2(1-p),$$
$$\lim_{n \to \infty} d(\nu_q, x, n) = H(q).$$

COROLLARY 2.1. – For every $0 \le \gamma \le d$ and $\alpha \ge \gamma$, for $\nu_{q_{\alpha,\gamma}}^{\otimes d}$ -almost every $x \in [0,1]^d$

$$\begin{split} &\lim_{n\to\infty} d(\nu_{p_{\alpha,\gamma}}^{\otimes d},x,n) = \alpha, \\ &\lim_{n\to\infty} d(\nu_{q_{\alpha,\gamma}}^{\otimes d},x,n) = \gamma. \end{split}$$

Notice that if $\gamma > 0$, both the $\nu_{p_{\alpha,\gamma}}^{\otimes d}$ mass and the $\nu_{q_{\alpha,\gamma}}^{\otimes d}$ mass of the boundaries of closed dyadic subcubes of $[0,1]^d$ vanish.

Now for every $0 \le \gamma \le d$, $\alpha \ge \gamma$, $n \in \mathbb{N}$ and $\epsilon > 0$ define

$$E(\alpha,\gamma,n,\epsilon) = \left\{ x \in [0,1)^d : \begin{cases} d(\nu_{p_{\alpha,\gamma}}^{\otimes d}, x, n) \in [\alpha - \epsilon, \alpha + \epsilon], \\ d(\nu_{q_{\alpha,\gamma}}^{\otimes d}, x, n) \in [\gamma - \epsilon, \gamma + \epsilon] \end{cases} \right\}$$

Let $(\epsilon_m)_{m\geq 1} \in (0,1)^{\mathbb{N}_+}$ be a decreasing sequence converging to 0.

By Corollary 2.1, for each $m \in \mathbb{N}_+$ we can fix an integer $n_m^0(\alpha, \gamma)$ such that

(2.2)
$$\nu_{q_{\alpha,\gamma}}^{\otimes d}(F_m(\alpha,\gamma)) \ge 1/2, \text{ with } F_m(\alpha,\gamma) = \bigcap_{n \ge n_m^0(\alpha,\gamma)} E(\alpha,\gamma,n,\epsilon_m/2).$$

We notice that for all $n \geq n_m^0(\alpha, \gamma)$, since $\nu_{q_{\alpha,\gamma}}^{\otimes d}(F_m(\alpha, \gamma)) \leq 1$ and for each $I \in \mathcal{F}_n$ such that $I \cap F_m(\alpha, \gamma) \neq \emptyset$ we have $2^{-n(\gamma+\epsilon_m/2)} \leq \nu_{q_{\alpha,\gamma}}^{\otimes d}(I)$, we have $\#\{I \in \mathcal{F}_n : I \cap F_m(\alpha, \gamma) \neq \emptyset\} \leq 2^{n(\gamma+\epsilon_m/2)}$.

Then, let

$$\widetilde{E}(\alpha,\gamma,n,\epsilon) = \left\{ x \in F_m(\alpha,\gamma) : \frac{\log \nu_{q_{\alpha,\gamma}}^{\otimes d} \left(I_n(x) \cap F_m(\alpha,\gamma) \right)}{-n\log(2)} \in [\gamma - \epsilon_m, \gamma + \epsilon_m] \right\}.$$

We can find $n_m(\alpha, \gamma) \ge n_m^0(\alpha, \gamma)$ such that

(2.3)
$$\nu_{q_{\alpha,\gamma}}^{\otimes d}(\widetilde{F}_m(\alpha,\gamma)) \ge 1/2 - 1/2^m$$
, with $\widetilde{F}_m(\alpha,\gamma) = \bigcap_{n \ge n_m(\alpha,\gamma)} \widetilde{E}(\alpha,\gamma,n,\epsilon_m)$.

Indeed, for $n \geq \widetilde{n}_m^0(\alpha, \gamma)$ we have

$$\nu_{q_{\alpha,\gamma}}^{\otimes d} (F_m(\alpha,\gamma) \setminus \widetilde{E}(\alpha,\gamma,n,\epsilon_m)) = \nu_{q_{\alpha,\gamma}}^{\otimes d} (\{x \in F_m(\alpha,\gamma) : \nu_{q_{\alpha,\gamma}}^{\otimes d} (I_n(x) \cap F_m(\alpha,\gamma)) < 2^{-n(\gamma+\epsilon_m)}\})$$

$$\leq \sum_{I \in \mathcal{T}_n : \ \nu_{q_{\alpha,\gamma}}^{\otimes d} (I \cap F_m(\alpha,\gamma)) < 2^{-n(\gamma+\epsilon_m)}} \nu_{q_{\alpha,\gamma}}^{\otimes d} (I \cap F_m(\alpha,\gamma))$$

$$\leq (\#\{I \in \mathcal{T}_n : I \cap F_m(\alpha,\gamma) \neq \emptyset\}) 2^{-n(\gamma+\epsilon_m)}$$

$$\leq 2^{n(\gamma+\epsilon_m/2)} 2^{-n(\gamma+\epsilon_m)} = 2^{-n\epsilon_m/2},$$

so (2.3) follows if we choose $n_m(\alpha, \gamma)$ such that $\sum_{n \ge n_m(\alpha, \gamma)} 2^{-n\epsilon_m} \le 2^{-m}$. We define

we define

(2.5)
$$\mu_{\alpha,\gamma} = \nu_{p_{\alpha,\gamma}}^{\otimes d} \text{ and } \nu_{\alpha,\gamma} = \nu_{q_{\alpha,\gamma}}^{\otimes d} (\cdot \cap F_m(\alpha,\gamma)).$$

We can now gather a series of properties which will be used in the proofs of our main results.

PROPERTIES 2.1. – Let $m \in \mathbb{N}_+$, $0 \le \gamma \le d$ and $\alpha \ge \gamma$. (1) If $N \ge n_m(\alpha, \gamma)$, by construction we have

(2.6)
$$1/2 \le \nu_{\alpha,\gamma}(F_m(\alpha,\gamma)) = \sum_{I \in \mathcal{F}_N: I \cap F_m(\alpha,\gamma) \neq \varnothing} \nu_{\alpha,\gamma}(I) \le 1$$

4° SÉRIE – TOME 48 – 2015 – Nº 6

(notice that if $\alpha = \gamma$, by construction the above property also holds for $\mu_{\alpha,\alpha}$). Consequently, since for each $I \in \mathcal{F}_N$ such that $I \cap F_m(\alpha, \gamma) \neq \emptyset$ we have $2^{-N(\gamma + \epsilon_m)} \leq \nu_{\alpha,\gamma}(I) \leq 2^{-N(\gamma - \epsilon_m)}$, we get

(2.7)
$$2^{-1}2^{N(\gamma-\epsilon_m)} \le \#\{I \in \mathcal{F}_N : I \cap F_m(\alpha,\gamma) \ne \emptyset\} \le 2^{N(\gamma+\epsilon_m)}.$$

By construction of $\widetilde{F}_m(\alpha, \gamma)$, we also have

(2.8)
$$\sum_{I \in \mathcal{T}_N: I \cap \widetilde{F}_m(\alpha, \gamma) = \varnothing} \nu_{\alpha, \gamma}(I) \le 2^{-m}.$$

(2) If J is a dyadic cube of generation $n \ge n_m(\alpha, \gamma)$, $J \cap F_m(\alpha, \gamma) \ne \emptyset$, and $N \ge n$, then we have by construction

(2.9)
$$\nu_{\alpha,\gamma}(J) = \sum_{\substack{J \supset I \in \mathcal{G}_N, \\ I \cap F_m(\alpha,\gamma) \neq \varnothing}} \nu_{\alpha,\gamma}(I) \le |J|^{\gamma - \epsilon_m},$$

and if, moreover, $J \cap \widetilde{F}_m(\alpha, \gamma) \neq \emptyset$,

(2.10)
$$|J|^{\gamma+\epsilon_m} \leq \nu_{\alpha,\gamma}(J) = \sum_{\substack{J \supset I \in \mathcal{F}_N, \\ I \cap F_m(\alpha,\gamma) \neq \emptyset}} \nu_{\alpha,\gamma}(I) \leq |J|^{\gamma-\epsilon_m}.$$

(3) If J is a dyadic cube of generation $n \ge n_m(\alpha, \gamma)$, $J \cap F_m(\alpha, \gamma) \ne \emptyset$, and $N \ge n$, we also have

(2.11)
$$\sum_{\substack{J\supset I\in\mathcal{G}_N,\\I\cap F_m(\alpha,\gamma)\neq\emptyset}}\mu_{\alpha,\gamma}(I)\leq \mu_{\alpha,\gamma}(J)\leq |J|^{\alpha-\epsilon_m}.$$

Also, (2.9) implies

$$\#\{J \supset I \in \mathcal{G}_N : I \cap F_m(\alpha, \gamma) \neq \emptyset\} \le |J|^{\gamma - \epsilon_m} 2^{N(\gamma + \epsilon_m)},$$

hence

(2.12)
$$\sum_{\substack{J\supset I\in\mathcal{F}_N,\\I\cap F_m(\alpha,\gamma)\neq\varnothing}}\mu_{\alpha,\gamma}(I)\leq 2^{-n(\gamma-\epsilon_m)}2^{-N(\alpha-\gamma-2\epsilon_m)}.$$

3. Prescription of lower Hausdorff spectra: Proof of Theorem 1.5

3.1. Construction of μ and estimates of its local dimension

Setting apart some important details omitted in the outline provided in Section 1, the proof we present for the general case will be more sophisticated because: (1) f is only upper semi-continuous; (2) f may have more than one fixed point; (3) dom(f) may contain ∞ ; (4) we will manage that all the sets $\underline{E}(\mu, \beta)$ are substantially big when they are not empty, this meaning that they contain a Cantor set. Referring to our sketch of proof, since we will use Bernoulli measures for $\mu_{\alpha_i, f(\alpha_i)}$ and $\nu_{\alpha_i, f(\alpha_i)}$, to getting such Cantor sets necessitates to avoid using couples $(\alpha_i, f(\alpha_i))$ for which $f(\alpha_i) = 0$ in the construction; indeed in this case we know that the Bernoulli measure $\nu_{\alpha_i, f(\alpha_i)}$ is a Dirac mass. These couples will be replaced by couples $(\alpha_i, \gamma_m(\alpha_i))$, where $0 < \gamma_m(\alpha_i)$ tends to 0 as $m \to \infty$. In particular, to illustrate

the case where f(0) = 0 and $f(\alpha) = -\infty$ for all $\alpha \neq 0$, instead of taking $\mu = \delta_x$ for some $x \in \mathbb{R}^d$, we will construct a continuous measure on a Cantor set of dimension 0.

3.1.1. Construction of the measure μ . – We will denote dom(f) by \mathscr{I} . Let $D \in Fix(f)$. Due to our assumption requiring the upper semi-continuity of f, there exists a dense countable subset Δ of $\mathscr{I} \setminus \{\infty\}$ such that for all $\alpha \in \mathscr{I} \setminus \{\infty\}$, there exists a sequence $(\alpha_n)_{n\geq 1}$ in $\Delta^{\mathbb{N}_+}$ such that $\lim_{n\to\infty} (\alpha_n, f(\alpha_n)) = (\alpha, f(\alpha))$ (if $\infty \in \mathscr{I}$ and f is continuous at ∞ the same holds at $(\infty, f(\infty))$, but we will not need this property). This important fact is elementary (see for instance [56, Lemma 2] for a proof when dom(f) is a compact subset of \mathbb{R} ; the general case considered here then follows immediately).

We fix once for all such a Δ , and assume without loss of generality that it contains a dense subset of Fix(f).

Let $\alpha_{\min} = \min(\mathscr{I})$ and $\alpha_{\max} = \max(\mathscr{I}) \in \mathbb{R}_+ \cup \{\infty\}$.

If $\Delta \setminus \{0, D\} \neq \emptyset$, enumerate its elements in a sequence $(\alpha_j^{\Delta})_{j\geq 1}$ (with necessary redundancies if $\Delta \setminus \{0, D\}$ is finite), and for each $m \in \mathbb{N}$ set

$$A_m = \left\{ \alpha_j^\Delta : 1 \le j \le m, \; \alpha_j^\Delta \ge 4\epsilon_m \right\};$$

otherwise set $\widetilde{A}_m = \emptyset$. Also set

$$D_m = 2\epsilon_m \text{ if } D = 0, \text{ and } D_m = D \text{ otherwise,}$$

$$\alpha_m(0) = D_m \text{ if } D = 0 \text{ and } \alpha_m(0) = \min(2\epsilon_m, D)/2 \text{ otherwise,}$$

$$\alpha_m(\infty) = \left(\max(d, m, \max(\alpha_j^{\Delta} : 1 \le j \le m))^2 \text{ if } \alpha_{\max} = \infty.\right)$$

Then let

$$A_{m} = \begin{cases} \widetilde{A}_{m} \cup \{D_{m}\} & \text{if } 0 < \alpha_{\min} \le \alpha_{\max} < \infty \\ \widetilde{A}_{m} \cup \{D_{m}\} \cup \{\alpha_{m}(\infty)\} & \text{if } 0 < \alpha_{\min} \text{ and } \alpha_{\max} = \infty, \\ \widetilde{A}_{m} \cup \{D_{m}\} \cup \{\alpha_{m}(0)\} & \text{if } \alpha_{\min} = 0 \text{ and } \alpha_{\max} < \infty, \\ \widetilde{A}_{m} \cup \{D_{m}\} \cup \{\alpha_{m}(0)\} \cup \{\alpha_{m}(\infty)\} & \text{if } \alpha_{\min} = 0 \text{ and } \alpha_{\max} = \infty. \end{cases}$$

For $\alpha \in A_m$ let

$$\gamma_m(\alpha) = \begin{cases} f(\alpha) & \text{if } \alpha \in \widetilde{A}_m \text{ and } f(\alpha) > 0\\ f(\infty) & \text{if } \alpha = \alpha_m(\infty) \text{ and } f(\infty) > 0,\\ \alpha & \text{if } \alpha = D_m,\\ \alpha_m(0) & \text{if } \alpha = \alpha_m(0),\\ \epsilon_m & \text{if } \alpha \in \widetilde{A}_m \text{ and } f(\alpha) = 0,\\ \epsilon_m & \text{if } \alpha = \alpha_m(\infty) \text{ and } f(\infty) = 0. \end{cases}$$

Notice that $\gamma_m(\alpha) \leq \alpha$ for all $\alpha \in A_m$. For the values of $\alpha \in A_m \setminus \{0, D\}$ such that $f(\alpha) = 0$, we choose $\gamma_m(\alpha) > 0$ so that the measure $\nu_{\alpha,\gamma_m(\alpha)}$ be continuous but of dimension tending to 0 as $m \to \infty$, and in our construction no level set $\underline{E}(\mu, \beta)$ be supported on a countable set; indeed, with this choice every non empty such set will contain a Cantor set when $f(\beta) \geq 0$. The choice of $\alpha_m(0)$ and $\gamma_m(\alpha_m(0))$, and that of D_m when D = 0, correspond to the same goal.

4° SÉRIE - TOME 48 - 2015 - Nº 6

Using the definitions of Section 2, for $\alpha \in A_m$, set

 $\mu_{\alpha} = \mu_{\alpha,\gamma_m(\alpha)}$ and $\nu_{\alpha} = \nu_{\alpha,\gamma_m(\alpha)}$ for $\alpha \notin \operatorname{Fix}(\gamma_m)$ and $\mu_{\alpha} = \nu_{\alpha} = \nu_{\alpha,\alpha}$ for $\alpha \in \operatorname{Fix}(\gamma_m)$.

Strictly speaking, μ_{α} and ν_{α} should be written $\mu_{m,\alpha}$ and $\nu_{m,\alpha}$, but for the sake of readability we will omit the index m.

Also, let

$$n_m = \max\{n_m(\alpha, \gamma_m(\alpha)) : \alpha \in A_m\}.$$

Now let $(N_m)_{n \in \mathbb{N}}$ be an increasing sequence of integers defined recursively satisfying the following properties:

(3.1)
$$\begin{cases} \forall m \ge 1, \ N_m \ge n_m, \\ \max((m + \#A_m + \max(A_m))^2, n_m) = o(\sqrt{N_{m-1}}) \text{ as } m \to \infty, \\ (\max(\{1\} \cup A_{m-1})) \sum_{i=1}^{m-1} N_i = o(\min(\{1\} \cup \gamma_m(A_m))\sqrt{N_m}) \text{ as } m \to \infty \end{cases}$$

For each $m \ge 1$ and $\alpha \in A_m$ set

$$G_m(\alpha) = \{ I \in \mathcal{F}_{N_m} : I \cap F_m(\alpha, \gamma_m(\alpha)) \neq \emptyset \}$$

and

$$\rho_m(\alpha) = \begin{cases} 1 & \text{if } \alpha = D_m \\ 2^{-m} / \# A_m & \text{otherwise.} \end{cases}$$

Due to (2.6) we have

(3.2)
$$2^{-1} \le \sum_{I \in G_m(\alpha)} \nu_{\alpha}(I) \le 1 \quad (\forall \ \alpha \in A_m),$$

so

(3.3)
$$2^{-1} \le \sum_{I \in G_m(D_m)} \rho_m(D_m) \left(\mu_{D_m}(I) = \nu_{D_m}(I) \right) \le 1.$$

Also,

(3.4)
$$\sum_{\alpha \in A_m \setminus \{D_m\}} \sum_{I \in G_m(\alpha)} \rho_m(\alpha) \mu_\alpha(I) \le \sum_{\alpha \in A_m \setminus \{D_m\}} \frac{2^{-m}}{\#A_m} \|\mu_\alpha\| \le 2^{-m}.$$

For each $m \in \mathbb{N}_+$, we enumerate the elements of A_m as $\alpha_{m,1}, \ldots, \alpha_{m,\#A_m}$, we denote by \mathcal{L}_m the set of disjoint closed dyadic cubes $\mathcal{L}(\#A_m)$ defined in Section 2.1, and denote its elements as $L_{m,\alpha_{m,1}}, \ldots, L_{m,\alpha_{m,\#A_m}}$. We also denote $L_{m,\alpha_{m,i}}$ by $L_{m,i}$ and $\ell(\#A_m)$ by ℓ_m .

We can start the construction of μ . We will construct a Cantor set K by defining recursively the sequence of families of cubes $(\mathbf{G}_m)_{m\in\mathbb{N}}$ such that $K = \bigcap_{m\geq 1} \bigcup_{J\in\mathbf{G}_m} J$, and simultaneously a consistent sequence of measures μ_m supported on $\bigcup_{J\in\mathbf{G}_m} J$ to get the desired measure μ on K.

Let

$$\mathbf{G}_{1} = \bigcup_{\alpha_{1,i} \in A_{1}} \{ L_{1,i} I_{1,i} : I_{1,i} \in G_{1}(\alpha_{1,i}) \}$$

(the concatenation of intervals has been defined in (2.1)). By construction, the interiors of the elements of G_1 are pairwise disjoint. Then the measure μ_1 is defined on G_1 by

(3.5)
$$\mu_1(L_{1,i}I_{1,i}) = \frac{\rho_1(\alpha_{1,i})\mu_{\alpha_{1,i}}(I_{1,i})}{\sum_{\alpha \in A_1} \sum_{I \in G_1(\alpha)} \rho_1(\alpha)\mu_\alpha(I)}$$

Combining (3.3) and (3.4) we have

$$2^{-1} \le \sum_{\alpha \in A_1} \sum_{I \in G_1(\alpha)} \rho_1(\alpha) \mu_\alpha(I) \le 1 + 2^{-1}.$$

Consequently, (3.5) yields

$$\frac{2}{3}\rho_1(\alpha_{1,i})\mu_{\alpha_{1,i}}(I_{1,i}) \le \mu_1(L_{1,i}I_{1,i}) \le 2\rho_1(\alpha_{1,i})\mu_{\alpha_{1,i}}(I_{1,i}).$$

Then, we define recursively for $m \ge 1$:

$$\mathbf{G}_{m+1} = \bigcup_{I_m \in \mathbf{G}_m} \bigcup_{\alpha_{m+1,i} \in A_{m+1}} \mathbf{G}_{m+1}(I_m, \alpha_{m+1,i}),$$

where

$$\mathbf{G}_{m+1}(I_m, \alpha_{m+1,i}) = \left\{ I_m L_{m+1,i} I_{m+1,i} : I_{m+1,i} \in G_{m+1}(\alpha_{m+1,i}) \right\}$$

and a measure μ_{m+1} on \mathbf{G}_{m+1} by setting

$$\mu_{m+1}(I_m L_{m+1,i} I_{m+1,i}) = \mu_m(I_m) \frac{\rho_{m+1}(\alpha_{m+1,i}) \mu_{\alpha_{m+1,i}}(I_{m+1,i})}{\sum_{\alpha \in A_{m+1}} \sum_{I \in G_{m+1}(\alpha)} \rho_{m+1}(\alpha) \mu_\alpha(I)},$$

which by construction satisfies

(3.6)
$$\frac{1}{1+2^{-m}} \le \frac{\mu_{m+1}(I_m L_{m+1,i} I_{m+1,i})}{\mu_m(I_m)\rho_{m+1}(\alpha_{m+1,i})\mu_{\alpha_{m+1,i}}(I_{m+1,i})} \le 2$$

by (3.3) and (3.4).

Each measure μ_m can be trivially extended to a probability measure on \mathcal{F}_{g_m} , where $g_m = -\log_2 |I_m|$, that we still denote by μ_m . This measure yields an absolutely continuous Borel measure on $[0, 1]^d$, denoted by μ_m again, whose density with respect to the Lebesgue measure is given by $2^{dg_m}\mu_m(I)$ over each cube $I \in \mathcal{F}_{g_m}$. By construction, the measures μ_m $(m \in \mathbb{N})$ weakly converge to a Borel probability measure μ on $[0, 1]^d$, supported on the Cantor set K defined as

$$K = \bigcap_{m \ge 1} \bigcup_{I \in \mathbf{G}_m} I,$$

and satisfying $\mu(I) = \mu_m(I)$ for all $m \ge 1$ and $I \in \mathcal{F}_{g_m}$. Moreover, since K does not intersect the boundary of any dyadic cube due to the definition of the sets \mathcal{L}_m , μ vanishes on such a boundary.

4° SÉRIE - TOME 48 - 2015 - Nº 6

3.1.2. Estimates of the local dimension of μ . – Let $x \in K$ and $n > g_1 = N_1 + \log_2 |L_{1,i_1}|^{-1}$. There exists a unique $m \in \mathbb{N}$ such that $g_m < n \leq g_{m+1}$. By construction, we have $I_{g_m}(x) \in \mathbf{G}_m$ and $I_{g_{m+1}}(x) \in \mathbf{G}_{m+1}$, and $I_{g_{m+1}}(x) \subset I_n(x) \subset I_{g_m}(x)$. Moreover, there exist a unique sequence of exponents $\alpha_1(x) \in A_1, \ldots, \alpha_m(x) \in A_m, \alpha_{m+1}(x) \in A_{m+1}$, and a unique sequence of pairs of intervals $\{L_j, I_j\}_{1 \leq j \leq m+1}$ such that

(3.7)
$$I_{g_m}(x) = L_1 I_1 \cdots L_m I_m$$
 and $I_{g_{m+1}}(x) = L_1 I_1 \cdots L_m I_m L_{m+1} I_{m+1}$,

with $I_j \in G_j(\alpha_j(x))$ and $L_j \in \mathcal{L}_j$ for each $1 \le j \le m + 1$. The intervals invoked in (3.7) will be used in the following statements.

PROPOSITION 3.1. – With the notations introduced above, there exists a positive sequence $(\delta_n)_{n\geq 1}$ converging to 0 as $n \to \infty$ such that, uniformly in $x \in K$, for $g_m \leq n \leq g_{m+1}$ one has 1.

(3.8)
$$\frac{\mu(I_n(x))}{2^{-g_m\alpha_m(x)-(n-g_m)\alpha_{m+1}(x)}} \le 2^{n\delta_n};$$

2. *if either* $n = g_m$, or $\alpha_{m+1}(x) \in \text{Fix}(\gamma_{m+1})$ and $I_{m+1} \cap \widetilde{F}_{m+1}(\alpha_{m+1}(x), \alpha_{m+1}(x)) \neq \emptyset$, *then*

(3.9)
$$2^{-n\delta_n} \le \frac{\mu(I_n(x))}{2^{-g_m\alpha_m(x)-(n-g_m)\alpha_{m+1}(x)}} \le 2^{n\delta_n},$$

3. if
$$g_m + \ell_{m+1} + n_{m+1} < n \le g_{m+1}$$
 then

(3.10)
$$\frac{\mu(I_n(x))}{2^{-g_m\alpha_m(x)-(n-g_m)\gamma_{m+1}(\alpha_{m+1}(x))-N_{m+1}(\alpha_{m+1}(x)-\gamma_{m+1}(\alpha_{m+1}(x)))}} \le 2^{n\delta_n+2N_{m+1}\epsilon_{m+1}}.$$

COROLLARY 3.1. – For all $x \in K$ we have

- 1. $\underline{d}(\mu, x) = \liminf_{m \to \infty} \alpha_m(x).$
- 2. If for m large enough we have $\alpha_m(x) \in \operatorname{Fix}(\gamma_m)$ and $I_m \cap \widetilde{F}_m(\alpha_m(x), \alpha_m(x)) \neq \emptyset$, then $\overline{d}(\mu, x) = \limsup_{m \to \infty} \alpha_m(x)$.
- 3. If $\liminf_{m\to\infty} \alpha_m(x) \in \mathcal{J} \setminus \operatorname{Fix}(f)$ then $\overline{d}(\mu, x) = \infty$. In particular, if $\alpha \in \mathcal{J} \setminus \operatorname{Fix}(f)$, then $\underline{E}(\mu, \alpha) = E(\mu, \alpha, \infty)$.

Proof of Proposition 3.1. We will write α_j for $\alpha_j(x)$.

At first we notice that by construction, and due to (3.6), we have

$$c_m \prod_{j=1}^m \mu_{\alpha_j}(I_j) \le \mu(I_{g_m(x)}) \le 2^m \prod_{j=1}^m \mu_{\alpha_j}(I_j)$$

where $c_m = \prod_{j=1}^m (1+2^{-m})^{-1} \prod_{j=1}^m \rho_j(\alpha_j) \ge e^{-1} \prod_{j=1}^m \rho_j(\alpha_j)$. Then, due to the definition of $G_j(\alpha_j)$ (3.11)

$$c_m \exp\Big(-\sum_{j=1}^m (\alpha_j + \epsilon_j)N_j \log(2)\Big) \le \mu(I_{g_m}(x)) \le 2^m \exp\Big(-\sum_{j=1}^m (\alpha_j - \epsilon_j)N_j \log(2)\Big).$$

Proof of (1) and (2): We distinguish two cases. Let $g'_m = g_m + \ell_{m+1}$.

CASE 1: $g_m < n \leq g'_m + n_{m+1}$. Write $I_{g'_m + n_{m+1}}(x) = I_{g_m(x)}L_{m+1}J_{n_{m+1}}$, where $J_{n_{m+1}} = I_{n_{m+1}}(\{2^{g'_m}x\}), \{t\}$ standing for the vector whose entries are the fractional parts

of the entries of t. We have $J_{n_{m+1}} \supset I_{m+1}, I_{m+1} \cap F_{m+1}(\alpha_{m+1}, \gamma_{m+1}(\alpha_{m+1})) \neq \emptyset$, and the generation of $J_{n_{m+1}}$ is $n_{m+1} \ge n_{m+1}(\alpha_{m+1}, \alpha_{m+1})$.

We obviously have $\mu(I_n(x)) \leq \mu(I_{g_m}(x))$, and by construction if $\alpha_{m+1} \in \text{Fix}(\gamma_{m+1})$ (remembering that $\mu_{\alpha_{m+1}} = \nu_{\alpha_{m+1}}$ and using the equality in (2.9)),

$$\begin{split} \mu(I_{g_m}(x)) &\geq \mu(I_n(x)) \geq \mu(I_{g'_m + n_{m+1}}(x)) = \sum_{\substack{J_{n_{m+1}} \supset I \in G_{m+1}(\alpha_{m+1})}} \mu(I_{g_m}(x)L_mI) \\ &= \mu(I_{g_m}(x)) \frac{\sum_{\substack{J_{n_{m+1}} \supset I \in G_{m+1}(\alpha_{m+1})}} \rho_{m+1}(\alpha_{m+1}) \mu_{\alpha_{m+1}}(I)}{\sum_{\alpha \in A_{m+1}} \sum_{\substack{I \in G_{m+1}(\alpha)}} \rho_{m+1}(\alpha) \mu_{\alpha}(I)} \\ &= \frac{\rho_{m+1}(\alpha_{m+1})}{\sum_{\alpha \in A_{m+1}} \sum_{\substack{I \in G_{m+1}(\alpha)}} \rho_{m+1}(\alpha) \mu_{\alpha}(I)} \mu(I_{g_m}(x)) \nu_{\alpha_{m+1}}(J_{n_{m+1}}) \\ &\geq \frac{\rho_{m+1}(\alpha_{m+1})}{1 + 2^{-(m+1)}} \mu(I_{g_m}(x)) \nu_{\alpha_{m+1}}(J_{n_{m+1}}) \quad \text{(we have used (3.3) and (3.4) again).} \end{split}$$

Combining this with (3.11), if $\alpha_{m+1} \in Fix(\gamma_{m+1})$, we thus get

$$c_{m+1}\widetilde{c}_m^{-1}\nu_{\alpha_{m+1}}(J_{n_{m+1}}) \le \frac{\mu(I_n(x))}{\exp\left(-\sum_{j=1}^m \alpha_j N_j \log(2)\right)} \le \widetilde{c}_m 2^m,$$

with $\widetilde{c}_m = \exp(\sum_{j=1}^m N_j \epsilon_j \log(2))$. If, moreover, $I_{m+1} \cap \widetilde{F}_{m+1}(\alpha_{m+1}, \alpha_{m+1}) \neq \emptyset$, then $J_{n_{m+1}} \cap \widetilde{F}_{m+1}(\alpha_{m+1}, \alpha_{m+1}) \neq \emptyset$, so due to (2.10) we have

$$c_{m+1}\tilde{c}_m^{-1}2^{-n_{m+1}(\alpha_{m+1}+\epsilon_{m+1})} \le \frac{\mu(I_n(x))}{\exp\left(-\sum_{j=1}^m \alpha_j N_j \log(2)\right)} \le \tilde{c}_m 2^m,$$

which finally yields

$$C_m^{-1} \le \frac{\mu(I_n(x))}{\exp\left(-\alpha_{m+1}(n-g_m)\log(2) - \sum_{j=1}^m \alpha_j N_j \log(2)\right)} \le C_m,$$

with $C_m = c_{m+1}^{-1} \tilde{c}_m 2^m 2^{2(\ell_{m+1}+n_{m+1})(\max(A_{m+1})+\epsilon_{m+1})}$. Moreover, it is readily seen that the previous upper bound holds whatever α_{m+1} be.

Now, due to the conditions (3.1) we have imposed to the sequence N_m and the definition of $\rho_j(\alpha_j)$, we have $\log(\tilde{C}_m) = o(g_m)$ and

$$\sup\left\{\sum_{j=1}^{m-1} \alpha_j N_j : (\alpha_j)_{1 \le j \le m-1} \in \prod_{j=1}^{m-1} A_j\right\} = o(\min(A_m)g_m).$$

Consequently, there exists a sequence $(\delta_n)_{n \in \mathbb{N}}$ converging to 0 as $n \to \infty$, such that (3.8) and (3.9) hold uniformly in $x \in K$ and $g_m < n \le g'_m + n_{m+1}$.

4° SÉRIE - TOME 48 - 2015 - Nº 6

1478

CASE 2: $g'_m + n_{m+1} < n \le g_{m+1}$. Write $I_n(x) = I_{g_m}(x)L_{m+1}J_{n-g'_m}$, where $J_{n-g'_m} = I_{n-g'_m}(\{2^{g'_m}x\})$. We have $J_{n-g'_m} \supset I_{m+1}, I_{m+1} \cap F_{m+1}(\alpha_{m+1}, \gamma_{m+1}(\alpha)) \ne \emptyset$, and the generation of $J_{n-g'_m}$ is $n - g'_m \ge n_{m+1} \ge n_{m+1}(\alpha_{m+1}, \alpha_{m+1})$.

By construction:

$$\begin{split} \mu(I_n(x)) &= \sum_{J_{n-g'_m} \supset I \in G_{m+1}(\alpha_{m+1})} \mu(I_{g_m}(x)L_m I) \\ &= \mu(I_{g_m}(x)) \frac{\sum_{J_{n-g'_m} \supset I \in G_{m+1}(\alpha_{m+1})} \rho_{m+1}(\alpha_{m+1}) \mu_{\alpha_{m+1}}(I)}{\sum_{\alpha \in A_{m+1}} \sum_{I \in G_{m+1}(\alpha)} \rho_{m+1}(\alpha) \mu_{\alpha}(I)} \\ &\leq \frac{\rho_{m+1}(\alpha_{m+1})}{\sum_{\alpha \in A_{m+1}} \sum_{I \in G_{m+1}(\alpha)} \rho_{m+1}(\alpha) \mu_{\alpha}(I)} \mu(I_{g_m}(x)) \mu_{\alpha_{m+1}}(J_{n-g'_m}) \quad \text{using (2.11)}, \end{split}$$

with equality if $\alpha_{m+1} \in \text{Fix}(\gamma_{m+1})$, remembering that in this case $\mu_{\alpha_{m+1}} = \nu_{\alpha_{m+1}}$, and using the equality in (2.9). Consequently,

$$\mu(I_n(x)) \le 2\rho_{m+1}(\alpha_{m+1})\mu(I_{g_m}(x))\mu_{\alpha_{m+1}}(J_{n-g'_m}),$$

and

$$\frac{\rho_{m+1}(\alpha_{m+1})}{1+2^{-(m+1)}}\mu(I_{g_m}(x))\nu_{\alpha_{m+1}}(J_{n-g'_m}) \le \mu(I_n(x)) \le 2\rho_{m+1}(\alpha_{m+1})\mu(I_{g_m}(x))\nu_{\alpha_{m+1}}(J_{n-g'_m})$$

if $\alpha_{m+1} \in \text{Fix}(\gamma_{m+1})$ (we have used (3.3) and (3.4) again). Set $C_{m,n} = \widetilde{C}_m 2^{(n-g'_m)\epsilon_{m+1}}$. The previous estimates combined with (3.11) and the estimates (2.11) and (2.10) of $\mu_{\alpha_{m+1}}(J_{n-g'_m})$ and $\nu_{\alpha_{m+1}}(J_{n-g'_m})$ respectively, yield:

$$\frac{\mu(I_n(x))}{\exp\left(-\alpha_{m+1}(n-g_m)\log(2)-\sum_{j=1}^m \alpha_j N_j \log(2)\right)} \le C_{m,n}$$

and

$$C_{m,n}^{-1} \le \frac{\mu(I_n(x))}{\exp\left(-\alpha_{m+1}(n-g_m)\log(2) - \sum_{j=1}^m \alpha_j N_j \log(2)\right)} \le C_{m,n}$$

if $\alpha_{m+1} \in \text{Fix}(\gamma_{m+1})$ and $I_{m+1} \cap \widetilde{F}_{m+1}(\alpha_{m+1}, \alpha_{m+1}) \neq \emptyset$. Then, due to (3.1) again, the above sequence $(\delta_n)_{n \in \mathbb{N}}$ can be modified so that (3.8) also holds uniformly in $x \in K$, and $g'_m + n_{m+1} < n \leq g_{m+1}$.

Proof of (3). – It suffices to use (2.12) instead of (2.11) to estimate $\mu_{\alpha_{m+1}}(J_{n-g'_m})$ in the previous upper bounds for $\mu(I_n(x))$.

Proof of Corollary 3.1. – (1) and (2) follow readily from (3.8) and (3.9), and the fact that by construction the neighboring dyadic cubes of generation n of $I_n(x)$ have a μ -mass equal to 0 or for which the estimates of Proposition 3.1(1)(2) also hold.

For (3), suppose that $\alpha = \liminf_{m \to \infty} \alpha_m(x) \in \mathcal{I} \setminus \operatorname{Fix}(f)$. Since $\operatorname{Fix}(f)$ is closed (because $f(\alpha) \leq \alpha$ and f is upper semi-continuous), there exists a subsequence $(m_k)_{k \geq 1}$

such that $\alpha_{m_k+1}(x)$ converges to α and $\gamma_{m_k+1}(\alpha_{m_k+1}(x))$ converges to $f(\alpha) < \alpha$). Take $n = n(k) = g_{m_k} + \ell_{m_k+1} + n_{m_k+1} + \sqrt{N_{m_k+1}}$. By construction $g_{m_k} = o(n)$ and $n - g_m \sim n \sim \sqrt{N_{m_k+1}}$, and using (3.10) we have

$$\frac{\mu(I_n(x))}{2^{o(n)-n\gamma_{m_k+1}(\alpha_{m_k+1}(x))-n^2(\alpha_{m_k+1}(x)-\gamma_{m_k+1}(\alpha_{m_k+1}(x)))}} \le 2^{n\delta_n+2n^2\epsilon_{m_k+1}},$$
$$\frac{\mu(I_n(x))}{-n\log(2)} \ge \frac{\gamma_{m_k+1}(\alpha_{m_k+1}(x))}{\log(2)} + n\frac{\alpha_{m_k+1}(x)-\gamma_{m_k+1}(\alpha_{m_k+1}(x))}{\log(2)} + o(n).$$

Letting k tend to ∞ yields the desired conclusion on $\overline{d}(\mu, x)$, again because the neighboring dyadic cubes of generation n of $I_n(x)$ have a μ -mass equal to 0 or with the same behavior as $\mu(I_n(x))$. Now, if $\alpha \in \mathcal{I} \setminus \operatorname{Fix}(f)$ and $x \in \underline{E}(\mu, \alpha)$, then $(\alpha_m(x))_{m \geq 1}$ must take infinitely many values in $\mathcal{I} \setminus \operatorname{Fix}(f)$ since $\operatorname{Fix}(f)$ is a closed set, and consequently $\underline{E}(\mu, \alpha) \subset E(\mu, \alpha, \infty)$.

3.2. Auxiliary measures and lower bound for the lower Hausdorff spectrum

3.2.1. Construction of auxiliary measures. – Let $\hat{\alpha} = (\alpha_m)_{m \ge 1} \in \prod_{m=1}^{\infty} A_m$. Now, we construct a measure $\nu_{\hat{\alpha}}$ as follows: Let

$$\mathbf{G}_{\widehat{\alpha},1} = \{ L_{1,\alpha_1} I_1 : I_1 \in G_1(\alpha_1) \},\$$

and define on $G_{\hat{\alpha},1}$ the measure

(3.12)
$$\nu_{\widehat{\alpha},1}(L_{1,\alpha_1}I_1) = \frac{\nu_{\alpha_1}(I_1)}{\sum_{I \in G_1(\alpha_1)} \nu_{\alpha_1}(I)}$$

Due to (3.2), (3.12) yields

$$\nu_{\alpha_1}(I_1) \le \nu_{\widehat{\alpha},1}(L_{1,\alpha_1}I_1) \le 2\nu_{\alpha_1}(I_1).$$

Then, recursively we define for $m \ge 1$

$$\mathbf{G}_{\widehat{\alpha},m+1} = \bigcup_{I_m \in \mathbf{G}_{\widehat{\alpha},m}} \mathbf{G}_{\widehat{\alpha},m+1}(I_m,\alpha_{m+1}),$$

where

$$\mathbf{G}_{\widehat{\alpha},m+1}(I_m,\alpha_{m+1}) = \{I_m L_{m+1,\alpha_{m+1}} I_{m+1} : I_{m+1} \in G_{m+1}(\alpha_{m+1})\}$$

and a measure $\nu_{\widehat{\alpha},m+1}$ on $\mathbf{G}_{\widehat{\alpha},m+1}$ by

$$\nu_{\widehat{\alpha},m+1}(I_m L_{m+1,\alpha_{m+1}} I_{m+1}) = \nu_{\widehat{\alpha},m}(I_m) \frac{\nu_{\alpha_{m+1}}(I_{m+1})}{\sum_{I \in G_{m+1}(\alpha_{m+1})} \nu_{\alpha_{m+1}}(I)}.$$

Due to (3.2) we have

(3.13)
$$1 \le \frac{\nu_{\widehat{\alpha},m+1}(I_m L_{m+1,\alpha_{m+1}} I_{m+1})}{\nu_{\widehat{\alpha},m}(I_m)\nu_{\alpha_{m+1}}(I_{m+1})} \le 2.$$

Each measure $\nu_{\widehat{\alpha},m}$ can be trivially extended into a probability measure on \mathcal{F}_{g_m} ; we still denote this measure by $\nu_{\widehat{\alpha},m}$. This measure yields an absolutely continuous Borel measure on $[0,1]^d$, denoted by $\nu_{\widehat{\alpha},m}$ again, whose density with respect to the Lebesgue measure is given by $2^{dg_m}\nu_{\widehat{\alpha},m}(I)$ over each cube $I \in \mathcal{F}_{g_m}$. By construction, the measures $\nu_{\widehat{\alpha},m}(m \in \mathbb{N})$

4° SÉRIE - TOME 48 - 2015 - Nº 6

so

converge weakly to a Borel probability measure $\nu_{\hat{\alpha}}$ on $[0, 1]^d$, supported on the Cantor set $K_{\hat{\alpha}}$ defined as

$$K \supset K_{\widehat{\alpha}} = \bigcap_{m \ge 1} \bigcup_{J \in \mathbf{G}_{\widehat{\alpha},m}} J,$$

and satisfying $\nu_{\widehat{\alpha}}(I) = \nu_{\widehat{\alpha},m}(I)$ for all $m \ge 1$ and $I \in \mathcal{F}_{g_m}$.

Estimation of the local dimension of $\nu_{\widehat{\alpha}}$. We use the same notations as in Section 3.1.2. Let $x \in K_{\widehat{\alpha}}$ and $n > g_1 = N_1 + \ell_1$. There exists a unique $m \in \mathbb{N}$ such that $g_m < n \leq g_{m+1}$. By construction, we have $I_{g_m}(x) \in \mathbf{G}_{\widehat{\alpha},m}$ and $I_{g_{m+1}}(x) \in \mathbf{G}_{\widehat{\alpha},m+1}$, and $I_{g_{m+1}}(x) \subset I_n(x) \subset I_{g_m}(x)$. Moreover, we have (3.7).

PROPOSITION 3.2. – There exists a positive sequence $(\delta_n)_{n\geq 1}$ converging to 0 as $n \to \infty$ such that, for $\nu_{\widehat{\alpha}}$ -almost every $x \in K_{\widehat{\alpha}}$, for n large enough,

(3.14)
$$2^{-n\delta_n} \le \frac{\nu_{\widehat{\alpha}}(I_n(x))}{2^{-g_m\gamma_m(\alpha_m) - (n - g_m)\gamma_{m+1}(\alpha_{m+1})}} \le 2^{n\delta_n}.$$

The previous proposition, the fact that by construction the neighboring dyadic cubes of generation n of $I_n(x)$ have a $\nu_{\hat{\alpha}}$ -mass equal to 0 or for which the estimates (3.14) hold, and the mass distribution principle (Section 6) yield the Hausdorff and packing dimensions of $\nu_{\hat{\alpha}}$:

COROLLARY 3.2. We have $\underline{d}(\nu_{\widehat{\alpha}}, x) = \liminf_{m \to \infty} \gamma_m(\alpha_m)$ and $\overline{d}(\nu_{\widehat{\alpha}}, x) = \limsup_{m \to \infty} \gamma_m(\alpha_m)$ for $\nu_{\widehat{\alpha}}$ -almost every x. Consequently, $\dim_H(\nu_{\widehat{\alpha}}) = \liminf_{m \to \infty} \gamma(\alpha_m)$ and $\dim_P(\nu_{\widehat{\alpha}}) = \limsup_{m \to \infty} \gamma(\alpha_m)$.

Proof of Proposition 3.2. At first, we prove the following fact: there exists $\widetilde{K}_{\widehat{\alpha}} \subset K_{\widehat{\alpha}}$, of full $\nu_{\widehat{\alpha}}$ -measure, such that for all $x \in \widetilde{K}_{\widehat{\alpha}}$, for *m* large enough, we have $I_m \cap \widetilde{F}_m(\alpha_m, \gamma_m(\alpha_m)) \neq \emptyset$, with I_m defined as in (3.7).

Indeed, due to the multiplicative structure of $\nu_{\hat{\alpha}}$, by construction we have

$$\nu_{\widehat{\alpha}}(\{x \in K_{\widehat{\alpha}} : I_m \cap F_m(\alpha_m, \gamma_m(\alpha_m)) = \varnothing\})$$

$$= \sum_{I \in \mathbf{G}_{\widehat{\alpha}, m-1}} \nu_{\widehat{\alpha}, m-1}(I) \frac{\sum_{J \in \mathcal{F}_{N_m}, J \cap \widehat{F}_m(\alpha_m, \gamma_m(\alpha_m)) = \varnothing}}{\sum_{J \in \mathcal{F}_{N_m}, J \cap F_m(\alpha_m, \gamma_m(\alpha_m)) \neq \varnothing}} \nu_{\alpha_m}(J)$$

$$\leq \sum_{I \in \mathbf{G}_{\widehat{\alpha}, m-1}} \nu_{\widehat{\alpha}, m-1}(I) \frac{2^{-m}}{2^{-1}} = 2^{-(m-1)},$$

where we have used the left hand side of (3.2), and (2.8). Then, by the Borel-Cantelli lemma, we have the desired conclusion.

Now the proof of the proposition follows from lines similar to those used to prove Proposition 3.1:

Let $x \in \widetilde{K}_{\widehat{\alpha}}$. At first we notice that by construction, and due to (3.13) we have

(3.15)
$$\prod_{j=1}^{m} \nu_{\alpha_j}(I_j) \le \nu_{\widehat{\alpha}}(I_{g_m(x)}) \le 2^m \prod_{j=1}^{m} \nu_{\alpha_j}(I_j).$$

Then, due to the definition of $G_j(\alpha_j)$,

$$\widetilde{c}_m^{-1} \le \frac{\nu_{\widehat{\alpha}}(I_{g_m}(x))}{\exp\left(-\sum_{j=1}^m \gamma_j(\alpha_j)N_j\log(2)\right)} \le 2^m \widetilde{c}_m$$

where \tilde{c}_m is defined as in the proof of Proposition 3.1.

We distinguish two cases.

CASE 1: $g_m < n \leq g'_m + n_{m+1}$. Write $I_{g'_m + n_{m+1}}(x) = I_{g_m(x)}L_{m+1}J_{n_{m+1}}$, where $J_{n_{m+1}} = I_{n_{m+1}}(\{2^{g'_m}x\})$. We have

$$\begin{split} \nu_{\widehat{\alpha}}(I_{g_{m}}(x)) &\geq \nu_{\widehat{\alpha}}(I_{n}(x)) \geq \nu_{\widehat{\alpha}}(I_{g'_{m}+n_{m+1}}(x)) = \sum_{J_{n_{m+1}} \supset I \in G_{m+1}(\alpha_{m+1})} \nu_{\widehat{\alpha}}(I_{g_{m}}(x)L_{m}I) \\ &= \nu_{\widehat{\alpha}}(I_{g_{m}}(x)) \frac{\sum_{J_{n_{m+1}} \supset I \in G_{m+1}(\alpha_{m+1})} \nu_{\alpha_{m+1}}(I)}{\sum_{I \in G_{m+1}(\alpha_{m+1})} \nu_{\alpha_{m+1}}(I)} \\ &= \frac{\nu_{\widehat{\alpha}}(I_{g_{m}}(x))\nu_{\alpha_{m+1}}(J_{n_{m+1}})}{\sum_{I \in G_{m+1}(\alpha_{m+1})} \nu_{\alpha_{m+1}}(I)} \geq \nu_{\widehat{\alpha}}(I_{g_{m}}(x))\nu_{\alpha_{m+1}}(J_{n_{m+1}}), \end{split}$$

where we have used the right hand side of (3.2). For *m* large enough so that $I_{m+1} \cap \widetilde{F}_{m+1}(\alpha_{m+1}, \gamma_{m+1}(\alpha_{m+1})) \neq \emptyset$ we have $J_{n_{m+1}} \cap \widetilde{F}_{m+1}(\alpha_{m+1}, \gamma_{m+1}(\alpha_{m+1})) \neq \emptyset$ and the generation of $J_{n_{m+1}}$ is $n_{m+1} \ge n(\alpha_{m+1}, \gamma_{m+1}(\alpha_{m+1}))$, so (2.10) holds for $J_{n_{m+1}}$, and this combined with (3.15) and (2.10) yields

$$\widetilde{c}_m^{-1} 2^{-n_{m+1}(\gamma_{m+1}(\alpha_{m+1})+\epsilon_{m+1})} \le \frac{\nu_{\widehat{\alpha}}(I_n(x))}{\exp\left(-\sum_{j=1}^m \gamma_j(\alpha_j)N_j\log(2)\right)} \le 2^m \widetilde{c}_m$$

Consequently,

$$(3.16) \quad \widetilde{C}_m^{-1} \le \frac{\nu_{\widehat{\alpha}}(I_n(x))}{\exp\left(-\gamma_{m+1}(\alpha_{m+1})(n-g_m)\log(2) - \sum_{j=1}^m \gamma_j(\alpha_j)N_j\log(2)\right)} \le \widetilde{C}_m,$$

with $\widetilde{C}_m = \widetilde{c}_m 2^m 2^{2(\ell_{m+1}+n_{m+1})(\max(\gamma_{m+1}(A_{m+1}))+\epsilon_{m+1})}$.

Due to the conditions (3.1) we have imposed to the sequence $(N_m)_{m\geq 1}$, we have $\log(\tilde{C}_m) = o(g_m)$ and

$$\sup\left\{\sum_{j=1}^{m-1} \gamma_j(\alpha_j) N_j : (\alpha_j)_{1 \le j \le m-1} \in \prod_{j=1}^{m-1} A_j\right\} = o(\min(\gamma_m(A_m))g_m).$$

Consequently, there exists a sequence $(\delta_n)_{n \in \mathbb{N}}$ converging to 0 as $n \to \infty$, such that (3.14) holds for all $x \in \widetilde{K}_{\widehat{\alpha}}$, for *m* large enough and $g_m < n \leq g'_m + n_{m+1}$.

4° SÉRIE – TOME 48 – 2015 – Nº 6

1482

CASE 2: $g'_m + n_{m+1} < n \le g_{m+1}$. Write $I_n(x) = I_{g_m}(x)L_{m+1}J_{n-g'_m}$, where $J_{n-g'_m} = I_{n-g'_m}(\{2^{g'_m}x\})$. We have

$$\begin{split} \nu_{\widehat{\alpha}}(I_n(x)) &= \sum_{J_{n-g'_m} \supset I \in G_{m+1}(\alpha_{m+1})} \nu_{\widehat{\alpha}}(I_{g_m}(x)L_mI) \\ &= \nu_{\widehat{\alpha}}(I_{g_m}(x)) \frac{\sum_{J_{n-g'_m} \supset I \in G_{m+1}(\alpha_{m+1})} \nu_{\alpha_{m+1}}(I)}{\sum_{I \in G_{m+1}(\alpha_{m+1})} \nu_{\alpha_{m+1}}(I)}, \end{split}$$

hence

$$\nu_{\widehat{\alpha}}(I_{g_m}(x))\nu_{\alpha_{m+1}}(J_{n-g'_m}) \le \nu_{\widehat{\alpha}}(I_n(x)) \le 2\nu_{\widehat{\alpha}}(I_{g_m}(x))\nu_{\alpha_{m+1}}(J_{n-g'_m})$$

(we have used (3.2)). For *m* large enough we have $I_{m+1} \cap \widetilde{F}_{m+1}(\alpha_{m+1}, \gamma_{m+1}(\alpha_{m+1})) \neq \emptyset$. This implies $J_{n-g'_m} \cap \widetilde{F}_{m+1}(\alpha_{m+1}, \alpha_{m+1}) \neq \emptyset$.

Since, moreover, $n - g'_m \ge n(\alpha_{m+1}, \gamma_{m+1}(\alpha_{m+1}))$, (2.10) holds for $J_{n-g'_m}$ and the previous estimates combined with (3.15) yield

$$\widetilde{C}_{m,n}^{-1} \le \frac{\nu_{\widehat{\alpha}}(I_n(x))}{\exp\left(-\gamma_{m+1}(\alpha_{m+1})(n-g_m)\log(2) - \sum_{j=1}^m \gamma_j(\alpha_j)N_j\log(2)\right)} \le \widetilde{C}_{m,n}$$

where

$$\widetilde{C}_{m,n} = \widetilde{C}_m 2^{\max(\gamma_{m+1}(A_{m+1}))(g'_m - g_m)} 2^{(n - g'_m)\epsilon_{m+1}}$$

Then, due to (3.1), the above sequence $(\delta_n)_{n \in \mathbb{N}}$ can be modified so that (3.8) also holds for all $x \in \widetilde{K}_{\widehat{\alpha}}$, for *m* large enough and $g'_m + n_m < n \leq g_{m+1}$.

3.2.2. Lower bound for the lower Hausdorff spectrum

PROPOSITION 3.3. – For any closed ball B whose interior intersects $K = \text{supp}(\mu)$, we have $\dim_H(B \cap \underline{E}(\mu, \alpha)) \ge f(\alpha)$ for all $\alpha \in \mathcal{I}$.

Proof. – Fix *B*, a closed ball whose interior intersects *K*. There exist $m_0 \in \mathbb{N}$, a sequence $(\alpha_j)_{1 \leq j \leq m_0} \in \prod_{j=1}^{m_0} A_j$, as well as pairs of dyadic cubes $(\{L_j, I_{\alpha_j}\}_{1 \leq j \leq m_0}, \text{ with } L_j \in \mathcal{L}_j$ and $I_{\alpha_j} \in G_j(\alpha_j)$ such that

$$L_1 I_{\alpha_1} \cdots L_{m_0} I_{\alpha_{m_0}} \subset B$$

Now fix $\alpha \in \mathscr{I}$.

If $\alpha \in \mathcal{I} \setminus \{0, D, \infty\}$ then for each $m > m_0$, fix $\alpha_m \in A_m \setminus \{D_m\}$, so that $\lim_{m\to\infty}(\gamma_m(\alpha_m) = f(\alpha_m)) = f(\alpha)$. If, moreover, $\alpha \in \operatorname{Fix}(f) \setminus D$, take $\alpha_m \in \operatorname{Fix}(f)$.

If $\alpha = 0$ then for each $m > m_0$ let $\alpha_m = \alpha_m(0)$.

We have $\lim_{m\to\infty} (\gamma_m(\alpha_m) = \alpha_m) = 0 = f(\alpha)$.

If $\alpha = D$ then for each $m > m_0$ let $\alpha_m = D_m$.

We have $\lim_{m\to\infty} (\gamma_m(\alpha_m) = \alpha_m) = \alpha = f(\alpha)$.

If $\alpha = \infty$ then for each $m > m_0$ let $\alpha_m = \alpha_m(\infty)$. We have $\gamma_m(\alpha_m) = f(\infty)$ if $f(\infty) > 0$ and $\lim_{m \to \infty} (\gamma_m(\alpha_m) = \epsilon_m) = 0 = f(\infty)$ otherwise.

Let $\widehat{\alpha} = (\alpha_m)_{m \geq 1}$, and consider the measure $\nu_{\widehat{\alpha}}$ constructed in the previous section. This measure is supported on the set $\widetilde{K}_{\widehat{\alpha}} \subset K$ exhibited at the beginning of the proof of Proposition 3.2, and by construction $\nu_{\widehat{\alpha}}(B \cap \widetilde{K}_{\widehat{\alpha}}) > 0$, so due to Corollary 3.2 we have $\dim_H \widetilde{K}_{\widehat{\alpha}} \cap B \geq f(\alpha)$.

ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE

1483

Moreover, due to Corollary 3.1(1), we have $\widetilde{K}_{\widehat{\alpha}} \subset \underline{E}(\mu, \alpha)$, so $\dim_H(B \cap \underline{E}(\mu, \alpha)) \geq f(\alpha)$. If, moreover, $\alpha \in \operatorname{Fix}(f)$, then by Corollary 3.1(2), our choice of $(\alpha_m)_{m\geq 1}$ implies that $\widetilde{K}_{\widehat{\alpha}} \subset E(\mu, \alpha)$, so $\dim_H(B \cap E(\mu, \alpha)) = f(\alpha)$.

3.3. Upper bound for the lower Hausdorff spectrum

Recall that f is upper semi-continuous. Also, its domain, denoted by \mathscr{I} is a closed subset of $[0, \infty]$, so due to Corollary 3.1(1), if $\alpha \notin \mathscr{I}$ then $\underline{E}(\mu, \alpha) = \varnothing$, hence $\dim_H \underline{E}(\mu, \alpha) = -\infty = f(\alpha)$.

Let $\alpha \in \mathcal{J} \setminus \{\infty\}$. Fix $\eta > 0$. Then choose $\delta > 0$ such that $f(\beta) \leq f(\alpha) + \eta$ if $|\beta - \alpha| \leq \delta$. Due to Corollary 3.1(1), $\underline{E}(\mu, \alpha) = \{x \in K : \liminf_{m \to \infty} \alpha_m(x) = \alpha\}$, so

$$\underline{\underline{E}}(\mu,\alpha) \subset \bigcap_{M \in \mathbb{N}} \bigcup_{m \ge M} \bigcup_{I_m \in \mathbf{G}_m} \bigcup_{\substack{\alpha' \in A_{m+1}: \\ \alpha' \in [\alpha - \delta, \alpha + \delta]}} \bigcup_{I \in G_{m+1}(\alpha')} I_m L_{m+1,\alpha'} I.$$

For a fixed $m \ge 1$, all the cubes $I_m L_{m+1,\alpha'}I$ are of the same generation g_{m+1} . Moreover the set \mathcal{C}_m of these cubes has a cardinality

$$#\mathscr{C}_m = \sum_{I_m \in \mathbf{G}_m} \sum_{\substack{\alpha' \in A_{m+1}:\\ \alpha' \in [\alpha-\delta,\alpha+\delta]}} #G_{m+1}(\alpha') \le (\#\mathbf{G}_m)(\#A_{m+1}) \max_{\substack{\alpha' \in A_{m+1}:\\ \alpha' \in [\alpha-\delta,\alpha+\delta]}} #G_{m+1}(\alpha').$$

We can deduce from (2.7) and the definition of γ_{m+1} that

$$#G_{m+1}(\alpha') < 2^{N_{m+1}(f(\alpha')+3\epsilon_{m+1})}.$$

Thus, by using the upper semi-continuity of f we get

$$#\mathcal{C}_m \le (\#\mathbf{G}_m)(\#A_{m+1})2^{N_{m+1}(f(\alpha)+\eta+3\epsilon_{m+1})}.$$

Moreover,

$$#\mathbf{G}_m = \prod_{j=1}^m \Big(\sum_{\beta \in A_j} #G_j(\beta)\Big).$$

Now, we deduce from (3.1) that $\log(\#\mathbf{G}_m) + \log(\#A_{m+1}) = o((f(\alpha + \eta)N_{m+1}))$, and finally obtain

$$\limsup_{m \to \infty} \frac{\log(\#\mathcal{C}_m)}{N_{m+1}\log(2)} \le f(\alpha) + \eta.$$

Noting that $\lim_{m\to\infty} g_{m+1}/N_{m+1} = 1$, this is enough to conclude that for all $\epsilon > 0$ we have

$$\sum_{M \in \mathbb{N}} \sum_{m \ge M} \sum_{I_m \in \mathbf{G}_m} \sum_{\substack{\alpha' \in A_{m+1}: \\ \alpha' \in [\alpha - \delta, \alpha + \delta]}} \sum_{I \in G_{m+1}(\alpha')} |I_m L_{m+1, \alpha'} I|^{f(\alpha) + \eta + \epsilon} < \infty,$$

so $\mathcal{H}^{f(\alpha)+\eta+\epsilon}(\underline{E}(\mu,\alpha)) = \lim_{m\to\infty} \mathcal{H}^{f(\alpha)+\eta+\epsilon}_{2^{-g_{m+1}}}(\underline{E}(\mu,\alpha)) = 0$, and $\dim_H \underline{E}(\mu,\alpha) \leq f(\alpha) + \eta + \epsilon$. Since this holds for all $\eta > 0$ and $\epsilon > 0$, we get the conclusion.

Now suppose that $\alpha = \infty \in \mathcal{I}$. Let $\eta > 0$ and A > 0 such that $f(\alpha) \leq f(\infty) + \eta$ for $\alpha \geq A$ (*f* is upper semi-continuous over *I*). We have

and following the same lines as for the case $\alpha < \infty$ yields $\dim_H E(\mu, \infty) \leq f(\infty)$, since f is upper semi-continuous at ∞ .

4° SÉRIE - TOME 48 - 2015 - Nº 6

3.4. μ is exactly dimensional, with dimension D

By construction, for each $m \ge 2$, we have

$$\mu(\{x \in K : \alpha_m(x) \neq D_m\}) = \sum_{I \in \mathbf{G}_{m-1}} \sum_{\alpha \in A_m \setminus \{D_m\}} \sum_{J \in G_m(\alpha)} \mu(IL_{m,\alpha}J)$$
$$= \sum_{I \in \mathbf{G}_{m-1}} \mu(I) \frac{\sum_{\alpha \in A_m \setminus \{D_m\}} \sum_{J \in G_m(\alpha)} \rho_m(\alpha) \mu_\alpha(J)}{\sum_{\alpha \in A_m} \sum_{J \in G_m(\alpha)} \rho_m(\alpha) \mu_\alpha(J)}$$
$$\leq \sum_{I \in \mathbf{G}_{m-1}} \mu(I_m) \frac{2^{-m}}{2^{-1}} = 2^{-(m-1)},$$

where we have used (3.3) and (3.4). By the Borel-Cantelli lemma we get that μ -almost everywhere, $\alpha_m(x) = D_m$ for m large enough. Moreover, denoting by I_m the same interval as in (3.7), we have

$$\mu(\{x \in K : \alpha_m(x) = D_m, I_m \cap F_m(D_m, D_m) = \varnothing\})$$

$$= \sum_{I \in \mathbf{G}_{\hat{\alpha}, m-1}} \mu_{m-1}(I) \frac{\sum_{J \in \mathcal{F}_{N_m}, J \cap \tilde{F}_m(D_m, D_m) = \varnothing}}{\sum_{\alpha \in A_m} \sum_{J \in G_m(\alpha)} \rho_m(\alpha) \mu_\alpha(J)}$$

$$\leq \sum_{I \in \mathbf{G}_{\hat{\alpha}, m-1}} \mu_{m-1}(I) \frac{2^{-m}}{2^{-1}} = 2^{-(m-1)},$$

where we have used (2.8) and (3.3). Then, a new application of the Borel-Cantelli lemma implies that for μ -almost every x, for m large enough, we have both $\alpha_m(x) = D_m$ and $I_m \cap \tilde{F}_m(D_m, D_m) \neq \emptyset$. We can then conclude from (3.9) and Corollary 3.1(1)(2) that μ is exactly dimensional with dimension D, since D_m converges to D as $m \to \infty$.

4. Full illustration of the multifractal formalism: Proof of Theorem 1.4

4.1. Construction of μ

We will modify the scheme used in the proof of Theorem 1.5 by repeating recursively, for all $m \geq 1$, for some integers R_m^f and R_m^g to be specified, R_m^f times the *m*th step with $\gamma_m(\alpha)$ approximating $f(\alpha)$, followed by R_m^g times the *m*th step with $\gamma_m(\alpha)$ approximating $g(\alpha)$; this will make it possible to both guarantee the non emptyness of the sets $E(\mu, \alpha)$, $\alpha \in \mathcal{I}$, and the control of the difference between the associated Hausdorff and packing spectra. Additional conditions on $\gamma_m(\alpha)$ will be also needed to obtain an exactly dimensional measure.

Let *D* be a fixed point of *f* (it is automatically a fixed point of *g*). Due to our assumption on the upper semi-continuity of *f*, there exists a countable subset Δ_f of $\mathscr{I} \setminus \{\infty\}$ such that for all $\alpha \in \mathscr{I} \setminus \{\infty\}$, there exists a sequence $(\alpha_n)_{n\geq 1}$ in $\Delta_f^{\mathbb{N}_+}$ such that $\lim_{n\to\infty} (\alpha_n, f(\alpha_n)) = (\alpha, f(\alpha))$. Similarly, there exists a countable subset Δ_g of $\mathscr{J} \setminus \{\infty\}$ such that for all $\alpha \in \mathscr{J} \setminus \{\infty\}$, there exists a sequence $(\alpha_n)_{n\geq 1}$ in $\Delta_g^{\mathbb{N}_+}$ such that $\lim_{n\to\infty} (\alpha_n, g(\alpha_n)) = (\alpha, g(\alpha))$.

We fix once for all such Δ_f and Δ_g and we can assume that they both contain D. Also take

$$\epsilon_m = (m+1)^{-2}.$$

Set $\alpha_{\min}^f = \min(\mathscr{I}), \alpha_{\max}^f = \max(\mathscr{I}) \in \mathbb{R}_+ \cup \{\infty\}, \alpha_{\min}^g = \min(\mathscr{I}), \text{ and } \alpha_{\max}^g = \max(\mathscr{I}) \in \mathbb{R}_+ \cup \{\infty\}$. For $h \in \{f, g\}$, if $\Delta_h \setminus \{0, D\} \neq \emptyset$, enumerate the elements of $\Delta_h \setminus \{0, D\}$ in a sequence $(\alpha_i^{\Delta_h})_{j \geq 1}$, and for each $m \in \mathbb{N}$ set

$$\widetilde{A}_m^h = \left\{ \alpha_j^{\Delta_h} : 1 \le j \le m, \; \alpha_j^{\Delta_h} \ge 4\epsilon_m^{1/3} \right\};$$

otherwise, set $\widetilde{A}_m^h = \emptyset$. Also set

$$\begin{cases} D_m = 2\epsilon_m^{1/3} & \text{if } D = 0, \\ D_m = D & \text{otherwise,} \\ \alpha_m^h(0) = D_m & \text{if } D = 0 \\ \alpha_m(0) = \min(2\epsilon_m^{1/3}, D)/2 & \text{otherwise,} \\ \alpha_m^h(\infty) = \left(\max(d, m, \max(\alpha_j^{\Delta_h} : 1 \le j \le m))^2 & \text{if } \alpha_{\max}^h = \infty. \end{cases}$$

Then let

$$A_m^h = \begin{cases} \widetilde{A}_m^h \cup \{D_m\} & \text{if } 0 < \alpha_{\min}^h \le \alpha_{\max}^h < \infty \\ \widetilde{A}_m^h \cup \{D_m\} \cup \{\alpha_m^h(\infty)\} & \text{if } 0 < \alpha_{\min}^h \text{ and } \alpha_{\max}^h = \infty, \\ \widetilde{A}_m^h \cup \{D_m\} \cup \{\alpha_m^h(0)\} & \text{if } \alpha_{\min}^h = 0 \text{ and } \alpha_{\max}^h < \infty, \\ \widetilde{A}_m^h \cup \{D_m\} \cup \{\alpha_m^h(0)\} \cup \{\alpha_m^h(\infty)\} & \text{if } \alpha_{\min}^h = 0 \text{ and } \alpha_{\max}^h = \infty. \end{cases}$$

For $\alpha \in A_m^h$ let

$$\widetilde{\gamma}^h_m(\alpha) = \begin{cases} h(\alpha) & \text{if } \alpha \in \widetilde{A}^h_m \text{ and } h(\alpha) > 0\\ h(\infty) & \text{if } \alpha = \alpha^h_m(\infty) \text{ and } h(\infty) > 0, \\ \alpha & \text{if } \alpha = D_m, \\ \alpha_m(0) & \text{if } \alpha = \alpha^h_m(0), \\ \epsilon^{1/3}_m & \text{if } \alpha \in \widetilde{A}^h_m \text{ and } h(\alpha) = 0, \\ \epsilon^{1/3}_m & \text{if } \alpha = \alpha^h_m(\infty) \text{ and } h(\infty) = 0 \end{cases}$$

(notice that $f(\infty) = 0$ (resp. $g(\infty) = 0$) only if f = 0 (resp. g = 0) over \mathscr{I} (resp. \mathscr{J}) due to our assumptions on f (resp. g)). Then, set (this will be used to entail that μ is exactly dimensional)

$$\gamma_m^h(\alpha) = \begin{cases} \alpha = \widetilde{\gamma}_m^h(\alpha) & \text{if } \alpha = D_m \\ (1 - \theta_m^h) \widetilde{\gamma}_m^h(\alpha) & \text{otherwise,} \end{cases}$$

where $\theta^h_m \in (0,1)$ tends to 0 slowly enough so that

(4.1)
$$\forall \alpha \in A_m^h \setminus \{D_m\}, \begin{cases} \alpha - \gamma_m^h(\alpha) = \alpha - (1 - \theta_m^h) \widetilde{\gamma}_m^h(\alpha) \ge D_m \sqrt{\epsilon_m} \\ (1 - \sqrt{\epsilon_m}) \alpha - \gamma_m^h(\alpha) \ge 0. \end{cases}$$

We then have in particular $\gamma_m^h(\alpha) < \alpha$ for all $\alpha \in A_m^h \setminus \{D_m\}$.

Using the definitions of Section 2, for $\alpha \in A_m^h$ set

 $\mu_{\alpha} = \mu_{\alpha, \gamma_m^h(\alpha)} \text{ and } \nu_{\alpha} = \nu_{\alpha, \gamma_m^h(\alpha)} \text{ for } \alpha \in A_m^h \setminus \{D_m\} \text{ and } \mu_{D_m} = \nu_{D_m} = \nu_{D_m, D_m} \text{ for } \alpha = D_m.$

4° SÉRIE - TOME 48 - 2015 - Nº 6

1486

Strictly speaking, μ_{α} and ν_{α} should be written $\mu_{m,\alpha}^{h}$ and $\nu_{m,\alpha}^{h}$, but for the sake of readability we will omit the indices m and h.

Also, set

$$n_m^h = \max\{n_m(\alpha, \gamma_m^h(\alpha)) : \alpha \in A_m^h\}.$$

Now let $(R_m^f)_{m\geq 1}$, $(R_m^g)_{m\geq 1}$ and $(N_m)_{m\geq 1}$ be three increasing sequences of positive integers defined recursively and satisfying the following properties: (4.2)

$$\begin{cases} (1) \ \forall \ m \ge 1, \ e^{m+1} \le R_m^f \le R_m^g \le R_{m+1}^f; \\ (2) \ \forall \ m \ge 1, \ N_m \ge \max(n_m^h, \exp(\#A_m^h), \exp(m)) \ \text{for} \ h \in \{f, g\}, \ \text{and}, \ \text{as} \ m \to \infty: \\ (3) \ (\max(\{1\} \cup A_m^f \cup A_m^g))N_m = o\Big((\min(\{1\} \cup A_{m-1}^f \cup A_{m-1}^g))\sum_{i=1}^{m-1} (R_i^f + R_i^g)N_i\Big) \sum_{i=1}^{m-1} (R_i^f + R_i^g)N_i\Big); \\ (4) \ (\max(\{1\} \cup A_{m-1}^f \cup A_{m-1}^g))\sum_{i=1}^{m-1} (R_i^f + R_i^g)N_i = o\Big(\min(\{1\} \cup \gamma_m(A_m^f))R_m^fN_m\Big); \\ (5) \ (\max(\{1\} \cup A_m^f \cup A_m^g))(R_m^fN_m + \sum_{i=1}^{m-1} (R_i^f + R_i^g)N_i) = o\Big(\min(\{1\} \cup \gamma_m(A_m^g))R_m^gN_m\Big). \end{cases}$$

Then, for $\alpha \in A_m^h$ set

$$G_m^h(\alpha) = \{ I \in \mathcal{F}_{N_m} : I \cap F_m(\alpha, \gamma_m^h(\alpha)) \neq \emptyset \}$$

and

(4.3)
$$\rho_m^h(\alpha) = \begin{cases} 1 & \text{if } \alpha = D_m \\ (2^{-m}/\#A_m^h)^2 & \text{otherwise.} \end{cases}$$

We enumerate the elements of A_m^h as $\alpha_{m,i}^h$, $1 \leq i \leq \#A_m^h$ and denote by L_{m,α_i}^h , $1 \leq i \leq \#A_m^h$, the disjoint closed dyadic cubes of generation $\ell_m^h = \ell(\#A_m^h)$ of the set $\mathscr{L}_m = \mathscr{L}(\#A_m^h)$ defined in Section 2.1. We also denote L_{m,α_i}^h by $L_{m,i}^h$.

Let us start the construction of μ . We consider the same measure μ_1 on \mathbf{G}_1 as in Section 3.1, except that we take the sets $G_1^f(\alpha), \alpha \in A_1^f$, instead of the sets $G_1(\alpha), \alpha \in A_1$, and the collection \mathcal{L}_1^f instead of \mathcal{L}_1 . Then, for $1 \leq s \leq R_1^f - 1$ we define recursively

$$\mathbf{G}_{s+1} = \bigcup_{I_s \in \mathbf{G}_s} \bigcup_{\alpha_{1,i} \in A_1^f} \mathbf{G}_1^f(I_s, \alpha_{1,i}),$$

where

$$\mathbf{G}_{1}(I_{s},\alpha_{1,i}) = \left\{ I_{s}L_{1,i}^{f}I_{1,i}: I_{1,i} \in G_{1}^{f}(\alpha_{1,i}) \right\}$$

and a measure μ_{s+1} on \mathbf{G}_{s+1} as

$$\mu_{s+1}(I_s L_{1,i}^f I_{1,i}) = \mu_s(I_s) \frac{\rho_1^f(\alpha_{1,i}) \mu_{\alpha_{1,i}}(I_{1,i})}{\sum_{\alpha \in A_1} \sum_{I \in G_1^f(\alpha)} \rho_1^f(\alpha) \mu_\alpha(I)}$$

Then, we define recursively a sequence $(\mathbf{G}_s)_{s\geq 1}$ of sets of intervals of the same generation and a sequence of measures $(\mu_s)_{s\geq 1}$ as follows:

For all $m \ge 1$, and $R_m^f + \sum_{j=1}^{m-1} R_j^f + R_j^g < s + 1 \le \sum_{j=1}^m R_j^f + R_j^g$, $\mathbf{G}_{s+1} = \begin{bmatrix} 1 & \\ 1 & \end{bmatrix} \quad \mathbf{G}_m(I_s, \alpha_{m,i}),$

$$\mathbf{G}_{s+1} = \bigcup_{I_s \in \mathbf{G}_s} \bigcup_{\alpha_{m,i} \in A_m^g} \mathbf{G}_m(I_s, \alpha_{m,i})$$

where

$$\mathbf{G}_m(I_s, \alpha_{m,i}) = \left\{ I_s L^g_{m,i} I_{m,i} : I_{m,i} \in G^g_m(\alpha_{m,i}) \right\}$$

and the measure μ_{s+1} on \mathbf{G}_{s+1} is defined as

$$\begin{split} \mu_{s+1}(I_s L^g_{m,i} I_{m,i}) &= \mu_s(I_s) \frac{\rho^g_m(\alpha_{m,i}) \mu_{\alpha_{m,i}}(I_{m,i})}{\sum_{\alpha \in A^g_m} \sum_{I \in G^g_m(\alpha)} \rho^f_m(\alpha) \mu_\alpha(I)}. \end{split}$$

For all $m \geq 2$, and $\sum_{j=1}^{m-1} R^f_j + R^g_j < s+1 \leq R^f_m + \sum_{j=1}^m R^f_j + R^g_j,$
$$\mathbf{G}_{s+1} &= \bigcup_{I_s \in \mathbf{G}_s} \bigcup_{\alpha_{m,i} \in A^f_m} \mathbf{G}_m(I_s, \alpha_{m,i}), \end{split}$$

where

$$\mathbf{G}_m(I_s, \alpha_{m,i}) = \left\{ I_s L_{m,i}^f I_{m,i} : I_{m,i} \in G_m^f(\alpha_{m,i}) \right\}$$

and the measure μ_{s+1} on \mathbf{G}_{s+1} is defined as

$$\mu_{s+1}(I_s L_{m,i}^f I_{m,i}) = \mu_s(I_s) \frac{\rho_m^f(\alpha_{m,i}) \mu_{\alpha_{m,i}}(I_{m,i})}{\sum_{\alpha \in A_m^f} \sum_{I \in G_m^f(\alpha)} \rho_m^f(\alpha) \mu_\alpha(I)}.$$

This yields (in the same way as in Section 3.1) a Borel probability measure μ supported on

$$K = \bigcap_{s \ge 1} \bigcup_{I \in \mathbf{G}_s} I$$

such that $\mu(I) = \mu_s(I)$ for all $s \ge 1$ and $I \in \mathbf{G}_s$.

For each $m \ge 1$, we define

$$s_m = R_m^f + \sum_{i=1}^{m-1} R_i^f + R_i^g$$
 and $s'_m = \sum_{i=1}^m R_i^f + R_i^g$.

Then, for $s \ge 1$, we denote by n(s) the generation of the cubes belonging to \mathbf{G}_s , i.e.,

$$n(s) = (s - s'_{m-1})(N_m + \ell_m^f) + \sum_{i=1}^{m-1} R_i^f(N_i + \ell_i^f) + R_i^g(N_i + \ell_i^g)$$

if $s'_{m-1} < s \le s_m$, and

$$n(s) = (s - s_m)(N_m + \ell_m^g) + R_m^f(N_m + \ell_m^f) + \sum_{i=1}^{m-1} R_i^f(N_i + \ell_i^f) + R_i^g(N_i + \ell_i^g)$$

if $s_m < s \le s'_m$.

The following property, which follows immediately from (4.2) will be useful.

If $s'_{m-1} < s \le s_m$, set

(4.4)
$$n'(s) = (s - s'_{m-1})N_m + \sum_{i=1}^{m-1} R_i^f N_i + R_i^g N_i,$$

and if $s_m < s \le s'_m$, set

$$n'(s) = (s - s_m)N_m + R_m^f N_m + \sum_{i=1}^{m-1} R_i^f N_i + R_i^g N_i.$$

4° SÉRIE - TOME 48 - 2015 - Nº 6

1488

PROPOSITION 4.1. – If $s_{m-1} < s \leq s_m$, then

$$\max(1, \max(A_m^f \cup A_m^g) \sum_{i=1}^m R_m^f \ell_m^f + R_m^g \ell_m^g = o(n'(s)).$$

In particular, $n'(s) \sim n(s)$ as $s \to \infty$.

REMARK 4.1. – By construction, due to our choice for the cubes $L_{m,i}^h$, if $J = I_{s-1}L_{m,i}^hI_{m,i} \in \mathbf{G}_s$ and I is a dyadic cube in $\mathcal{N}_2(n(s), J)$ (see Section 2.1 for the definition), either $\mu(I) = 0$ or I takes the form $I_{s-1}L_{m,i}^hI'_{m,i}$ and $\mu(J)2^{-\epsilon(s)n(s)} \leq \mu(I) \leq \mu(J)2^{\epsilon(s)n(s)}$, where $\epsilon(s)$ is independent of J and the sequence $(\epsilon(s))_{s\geq 1}$ tends to 0 as $s \to \infty$.

4.2. Reduction of the problem

In this section we explain why the measure μ constructed in the previous section has the nice property that it is possible to replace centered balls by dyadic cubes in all the sets and quantities involved in the multifractal formalism without modifying them (it is of course impossible to do so for any measure in $\mathcal{M}_c^+(\mathbb{R}^d)$).

Let us start with two properties which easily follow from (4.2)(1-3) by construction:

and
$$\lim_{s \to \infty} \sup_{x \in K} \left| \frac{\log_2(\mu(I_{n(s)}(x)))}{n(s)} - \frac{\log_2(\mu(I_{n(s+1)}(x)))}{n(s+1)} \right| = 0.$$

Moreover, if $2^{-n(s+1)} < r \le 2^{-n(s)}$ we have

$$I_{n(s+1)}(x) \subset B(x,r) \subset B(x,2r) \subset \bigcup_{I \in \mathcal{N}_2(n(s),I_{n(s)}(x))} I_{x}(x)$$

and if $I \in \mathcal{N}_2(n(s), I_{n(s)}(x))$, either $\mu(I_{n(s)})e^{-\epsilon(s)n(s)} \leq \mu(I) \leq \mu(I_{n(s)})e^{\epsilon(s)n(s)}$ or $\mu(I) = 0$, where $\epsilon(s)$ is independent of x and I, by Remark 4.1.

It follows that for all $x \in K$ we have

(4.5)
$$\underline{d}(\mu, x) = \liminf_{s \to \infty} \frac{\log_2(\mu(I_{n(s)}(x)))}{-n(s)} \text{ and } \overline{d}(\mu, x) = \limsup_{s \to \infty} \frac{\log_2(\mu(I_{n(s)}(x)))}{-n(s)},$$

and μ is weakly doubling in the sense that there exists a function $\tilde{\epsilon}(r)$ tending to 0^+ as $r \to 0^+$ such that

$$\forall x \in K, \ \mu(B(x,2r)) \le r^{-\widetilde{\epsilon}(r)}\mu(B(x,r)).$$

Also, the above properties and standard covering arguments (see in particular [63, Section 4.6] where doubling measures are used) yield, for all $q \in \mathbb{R}$,

$$\tau_{\mu}(q) = \liminf_{s \to \infty} \frac{\log_2 \sum_{I_s \in \mathbf{G}_s} \mu(I_s)^q}{-n(s)} \text{ and } \overline{\tau}_{\mu}(q) = \limsup_{s \to \infty} \frac{\log_2 \sum_{I_s \in \mathbf{G}_s} \mu(I_s)^q}{-n(s)}.$$

Similarly, for $\alpha \in \mathbb{R}_+$ we have

$$\underline{f}_{\mu}^{\mathrm{LD}}(\alpha) = \lim_{\epsilon \to 0^+} \liminf_{s \to \infty} \frac{\log_2 \# \left\{ I \in \mathbf{G}_s : 2^{-n(s)(\alpha+\epsilon)} \le \mu(I) \le 2^{-n(s)(\alpha-\epsilon)} \right\}}{n(s)},$$
$$\overline{f}_{\mu}^{\mathrm{LD}}(\alpha) = \lim_{\epsilon \to 0^+} \limsup_{s \to \infty} \frac{\log_2 \# \left\{ I \in \mathbf{G}_s : 2^{-n(s)(\alpha+\epsilon)} \le \mu(I) \le 2^{-n(s)(\alpha-\epsilon)} \right\}}{n(s)},$$

and

$$\begin{split} & \underline{f}_{\mu}^{\mathrm{LD}}(\infty) = \lim_{A \to \infty} \liminf_{s \to \infty} \frac{\log_2 \# \left\{ I \in \mathbf{G}_s : \mu(I) \le 2^{-n(s)A} \right\}}{n(s)}, \\ & \overline{f}_{\mu}^{\mathrm{LD}}(\infty) = \lim_{A \to \infty} \limsup_{s \to \infty} \frac{\log_2 \# \left\{ I \in \mathbf{G}_s : \mu(I) \le 2^{-n(s)A} \right\}}{n(s)}, \end{split}$$

and for $0 \le \alpha \le \beta \le \infty$ such that $(\alpha, \beta) \ne (\infty, \infty)$,

$$\underline{f}_{\mu}^{\mathrm{LD}}(\alpha,\beta) = \lim_{\epsilon \to 0^+} \liminf_{s \to \infty} \frac{\log_2 \# \left\{ I \in \mathbf{G}_s : 2^{-n(s)(\beta+\epsilon)} \le \mu(I) \le 2^{-n(s)(\alpha-\epsilon)} \right\}}{n(s)}.$$

Finally, due to the multiplicative nature of the construction of $\mu,$ defining for each $m\geq 1$ and $q\in\mathbb{R}$

$$T_m^f(q) = \log_2 \frac{\sum_{\alpha \in A_m^f} \sum_{I \in G_m^f(\alpha)} \rho_m^f(\alpha)^q \mu_\alpha(I)^q}{\left(\sum_{\alpha \in A_m^f} \sum_{I \in G_m^f(\alpha)} \rho_m^f(\alpha) \mu_\alpha(I)\right)^q}$$

and

$$T_m^g(q) = \log_2 \frac{\sum_{\alpha \in A_m^g} \sum_{I \in G_m^g(\alpha)} \rho_m^g(\alpha)^q \mu_\alpha(I)^q}{\left(\sum_{\alpha \in A_m^g} \sum_{I \in G_m^g(\alpha)} \rho_m^g(\alpha) \mu_\alpha(I)\right)^q},$$

we have for $s\geq 1$ and $q\in \mathbb{R}$

$$\log_2 \sum_{I_s \in \mathbf{G}_s} \mu(I_s)^q = \sum_{i=1}^{m-1} R_i^f T_i^f(q) + R_i^g T_i^g(q) + (s - s'_{m-1}) T_m^f(q)$$

if $s'_{m-1} < s \leq s_m$ and

$$\log_2 \sum_{I_s \in \mathbf{G}_s} \mu(I_s)^q = \left(\sum_{i=1}^{m-1} R_i^f T_i^f(q) + R_i^g T_i^g(q)\right) + R_m^f T_m^f(q) + (s - s_m) T_m^g(q)$$

if $s_m < s \le s'_m$. It follows that

(4.6)
$$\tau_{\mu}(q) = \liminf_{S \ni s \to \infty} \frac{\log_2 \sum_{I_s \in \mathbf{G}_s} \mu(I_s)^q}{-n(s)} \text{ and } \overline{\tau}_{\mu}(q) = \limsup_{S \ni s \to \infty} \frac{\log_2 \sum_{I_s \in \mathbf{G}_s} \mu(I_s)^q}{-n(s)},$$

where $S = \{s_m, s'_m : m \ge 1\}.$

 $4^{\circ} \ S\acute{E}RIE - TOME \ 48 - 2015 - N^{o} \ 6$

4.3. Local dimension estimates for μ , auxiliary measures, and lower bounds for the different spectra

4.3.1. Local dimension estimates for the measure μ . – Let $x \in K$ and $s \ge 1$.

If $s'_{m-1} < s \le s_m$, for all $1 \le i \le m-1$, there are, uniquely determined, R_i^f elements $(\alpha_{i,j}^f(x))_{1\le j\le R_i^f} \in (A_i^f)^{R_i^f}$, R_i^g elements $(\alpha_{i,j}^g(x))_{1\le j\le R_i^g} \in (A_i^g)^{R_i^g}$, and $s' = s - s'_{m-1}$ elements $(\alpha_{m,j}^f(x))_{1\le j\le s'} \in (A_m^f)^{s'}$, and for each exponent $\alpha_{i,j}^h(x)$ of this collection a unique element $I_{\alpha_{i,j}^h(x)}$ of $G_i^h(\alpha_{i,j}^h(x))$ and a unique element $L_{i,\alpha_{i,j}^h(x)}^h$ of \mathcal{L}_i^h such that

$$I_{n(s)}(x) = \left(\odot_{i=1}^{m-1} \left(\odot_{j=1}^{R_i^f} L_{j,\alpha_{1,j}^f(x)}^f I_{\alpha_{1,j}^f(x)} \right) \cdot \left(\odot_{j=1}^{R_i^g} L_{j,\alpha_{1,j}^g(x)}^g I_{\alpha_{1,j}^g(x)} \right) \right) \cdot \left(\odot_{j=1}^{s'} L_{m,\alpha_{m,j}^f(x)}^f I_{\alpha_{m,j}^f(x)} \right)$$

(where the notation $\odot_{p=1}^{q} I_p$ stands for the cube obtained by concatenation $I_1 \cdot I_2 \cdots I_q$ of the cubes I_1, \ldots, I_q).

By construction, using the analogue of (3.6) we get, writing $\alpha_{i,j}^h$ for $\alpha_{i,j}^h(x)$,

$$\begin{split} \Big(\frac{1}{1+2^{-m}}\Big)^{s'}\Big(\prod_{i=1}^{m-1}\Big(\frac{1}{1+2^{-i}}\Big)^{R_i^f+R_i^g}\Big)\prod_{j=1}^{s'}\rho_m^f(\alpha_{m,j}^f)\Big(\prod_{i=1}^{m-1}\Big(\prod_{j=1}^{R_i^f}\rho_i^f(\alpha_{i,j}^f)\Big)\Big(\prod_{j=1}^{R_i^g}\rho_i^g(\alpha_{i,j}^g)\Big)\\ &\leq \frac{\mu(I_{n(s)})}{\prod_{j=1}^{s'}\mu_{\alpha_{m,j}^f}(I_{\alpha_{m,j}^f})\Big)\Big(\prod_{i=1}^{m-1}\Big(\prod_{j=1}^{R_i^f}\mu_{\alpha_{i,j}^f}(I_{\alpha_{i,j}^f})\Big)\Big(\prod_{j=1}^{R_g^g}\mu_{\alpha_{i,j}^g}(I_{\alpha_{i,j}^g})\Big)\\ &\leq 2^{ms'+\sum_{i=1}^{m-1}i(R_i^f+R_i^g)}. \end{split}$$

Consequently, due to the fact that $I_{\alpha_{i,j}^h(x)} \in G_i^h(\alpha_{i,j}^h(x))$ for $h \in \{f, g\}$, using (4.2)(1)(2) and Proposition 4.1 we get

(4.7)
$$2^{-\epsilon(s)n(s)} \le \frac{\mu(I_{n(s)}(x))}{2^{-n(s)\alpha_s(x)}} \le 2^{\epsilon(s)n(s)}$$

with

$$\alpha_s(x) = \frac{\sum_{j=1}^{s'} N_m \alpha_{m,j}^f(x) + \left(\sum_{i=1}^{m-1} \sum_{j=1}^{R_i^f} N_i \alpha_{i,j}^f(x) + \sum_{j=1}^{R_i^g} N_i \alpha_{i,j}^g(x)\right)}{s' N_m + \sum_{i=1}^{m-1} R_i^f N_i + R_i^g N_i}$$

and $\lim_{s\to\infty} \epsilon(s) = 0$. Due to (4.2)(3-5),

with $\lim_{s\to\infty}\epsilon'(s)=0$ uniformly in x.

If now $s_m < s \le s'_m$ and $s' = s - s'_m$, with similar notations we get

(4.9)
$$I_{n(s)}(x) = \left(\odot_{i=1}^{m-1} \left(\odot_{j=1}^{R_{i}^{f}} L_{j,\alpha_{1,j}^{f}(x)}^{f} I_{\alpha_{1,j}^{f}(x)} \right) \cdot \left(\odot_{j=1}^{R_{i}^{g}} L_{j,\alpha_{1,j}^{g}(x)}^{g} I_{\alpha_{1,j}^{g}(x)} \right) \right) \\ \cdot \left(\odot_{j=1}^{R_{m}^{f}} L_{m,\alpha_{m,j}^{f}(x)}^{f} I_{\alpha_{m,j}^{f}(x)} \right) \cdot \left(\odot_{j=1}^{s'} L_{m,\alpha_{m,j}^{g}(x)}^{g} I_{\alpha_{m,j}^{g}(x)} \right)$$

and

(4.10)
$$2^{-\epsilon(s)n(s)} \le \frac{\mu(I_{n(s)}(x))}{2^{-n(s)\alpha_s(x)}} \le 2^{\epsilon(s)n(s)}$$

with

$$\alpha_s(x) = \frac{\sum_{j=1}^{s'} N_m \alpha_{m,j}^g(x) + \sum_{j=1}^{R_m^f} N_m \alpha_{m,j}^f(x) + \sum_{i=1}^{m-1} \sum_{j=1}^{R_i^f} N_i \alpha_{i,j}^f(x) + \sum_{j=1}^{R_i^g} N_i \alpha_{i,j}^g(x)}{s' N_m + R_m^f N_m + \sum_{i=1}^{m-1} R_i^f N_i + R_i^g N_i}$$

and $\lim_{s\to\infty} \epsilon(s) = 0$. In addition,

with $\lim_{s\to\infty} \epsilon'(s) = 0$ uniformly in x.

4.3.2. Auxiliary measures. – Let $\widehat{\alpha} = (\alpha_1^f, \alpha_1^g, \dots, \alpha_m^f, \alpha_m^g, \dots) \in \prod_{m=1}^{\infty} A_m^f \times A_m^g$. We construct a measure $\nu_{\widehat{\alpha}}$ as follows: Let

$$\mathbf{G}_{\widehat{\alpha},1} = \{ L_{1,\alpha_1^f}^f I_1 : I_1 \in G_1^f(\alpha_1^f) \},\$$

and define

$$\nu_{\widehat{\alpha},1}(L_{1,\alpha_{1}^{f}}^{f}I_{1}) = \frac{\nu_{\alpha_{1}^{f}}(I_{1})}{\sum_{I \in G_{1}^{f}(\alpha_{1}^{f})} \nu_{\alpha_{1}^{f}}(I)}.$$

Then, for $1 \le s \le R_1^f - 1$ we define recursively

$$\mathbf{G}_{\widehat{\alpha},s+1} = \bigcup_{I_s \in \mathbf{G}_{\widehat{\alpha},s}} \mathbf{G}_{\widehat{\alpha},s+1}(I_s, \alpha_1^f),$$

where

$$\mathbf{G}_{\widehat{\alpha},s+1}(I_s,\alpha_1^f) = \left\{ I_s L_{1,\alpha_1^f}^f I_1 : I_1 \in G_1^f(\alpha_1^f) \right\}$$

and a measure $\nu_{\widehat{\alpha},s+1}$ on $\mathbf{G}_{\widehat{\alpha},s+1}$ as

$$\nu_{\widehat{\alpha},s+1}(I_{s}L_{1,\alpha_{1}^{f}}^{f}I_{1}) = \nu_{\widehat{\alpha},s}(I_{s})\frac{\nu_{\alpha_{1}^{f}}(I_{1})}{\sum_{I \in G_{1}^{f}(\alpha_{1}^{f})}\nu_{\alpha_{1}^{f}}(I)}$$

Then, define recursively a sequence $(\mathbf{G}_{\widehat{\alpha},s})_{s\geq 1}$ of sets of intervals of the same generation and a sequence of measures $(\nu_{\widehat{\alpha},s})_{s\geq 1}$ as follows:

For all $m \ge 1$, and $s_m < s + 1 \le s'_m$,

$$\mathbf{G}_{\widehat{\alpha},s+1} = \bigcup_{I_s \in \mathbf{G}_{\widehat{\alpha},s}} \mathbf{G}_{\widehat{\alpha},s+1}(I_s, \alpha_m^g),$$

where

$$\mathbf{G}_{\widehat{\alpha},s+1}(I_s,\alpha_m^g) = \left\{ I_s L_{m,\alpha_m^g}^g I_m : I_m \in G_m^g(\alpha_m^g) \right\}$$

and the measure $\nu_{\widehat{\alpha},s+1}$ on $\mathbf{G}_{\widehat{\alpha},s+1}$ is defined as

$$\nu_{\widehat{\alpha},s+1}(I_s L^g_{m,\alpha^g_m} I_m) = \nu_{\widehat{\alpha},s}(I_s) \frac{\nu_{\alpha^g_m}(I_m)}{\sum_{I \in G^g_m(\alpha^g_m)} \nu_{\alpha^g_m}(I)}.$$

For all $m \ge 2$, and $s'_{m-1} < s+1 \le s_m$,

$$\mathbf{G}_{\widehat{\alpha},s+1} = \bigcup_{I_s \in \mathbf{G}_{\widehat{\alpha},s}} \mathbf{G}_{\widehat{\alpha},s+1}(I_s, \alpha_m^f),$$

4° SÉRIE - TOME 48 - 2015 - Nº 6

where

$$\mathbf{G}_{\widehat{\alpha},s+1}(I_s,\alpha_m^f) = \left\{ I_s L_{m,\alpha_m^f}^f I_m : I_m \in G_m^f(\alpha_m^f) \right\}$$

and the measure $\nu_{\widehat{\alpha},s+1}$ on $\mathbf{G}_{\widehat{\alpha},s+1}$ is defined as

$$\nu_{\widehat{\alpha},s+1}(I_s L^f_{m,\alpha^f_m} I_m) = \nu_{\widehat{\alpha},s}(I_s) \frac{\nu_{\alpha^f_m}(I_m)}{\sum_{I \in G^f_m(\alpha^f_m)} \nu_{\alpha^f_m}(I)}.$$

This yields a Borel probability measure $\nu_{\widehat{\alpha}}$ supported on

$$K \supset K_{\widehat{\alpha}} = \bigcap_{s \ge 1} \bigcup_{I \in \mathbf{G}_{\widehat{\alpha},s}} I$$

and such that $\nu_{\widehat{\alpha}}(I) = \nu_{\widehat{\alpha},s}(I)$ for all $s \ge 1$ and $I \in \mathbf{G}_{\widehat{\alpha},s}$. Moreover, estimates similar to those used to control the local dimension of μ show that there exists a positive sequence $(\epsilon(s))_{s\ge 1}$ such that $\lim_{s\to\infty} \epsilon(s) = 0$ and for all $x \in K_{\widehat{\alpha}}$ and $s \ge 1$, if $s'_{m-1} < s \le s_m$ and $s' = s - s'_{m-1}$, we have

(4.12)
$$2^{-\epsilon(s)n(s)} \le \frac{\nu_{\widehat{\alpha}}(I_{n(s)}(x))}{2^{-n(s)\gamma_s(x)}} \le 2^{\epsilon(s)n(s)}$$

with

$$\gamma_s(x) = \frac{\sum_{j=1}^{s'} N_m \gamma_m^f(\alpha_{m,j}^f(x)) + \left(\sum_{i=1}^{m-1} \sum_{j=1}^{R_i^f} N_i \gamma_i^f(\alpha_{i,j}^f(x)) + \sum_{j=1}^{R_i^g} N_i \gamma_i^g(\alpha_{i,j}^g(x))\right)}{s' N_m + \sum_{i=1}^{m-1} R_i^f N_i + R_i^g N_i},$$

and if $s_m < s \le s'_m$ and $s' = s - s_m$, we have

(4.13)
$$2^{-\epsilon(s)n(s)} \le \frac{\nu_{\widehat{\alpha}}(I_{n(s)}(x))}{2^{-n(s)\gamma_s(x)}} \le 2^{\epsilon(s)n(s)}$$

with

$$\gamma_s(x) = \frac{\sum_{j=1}^{s'} N_m \gamma_m^g(\alpha_{m,j}^g(x)) + \sum_{j=1}^{R_m^f} N_m \gamma_m^f(\alpha_{m,j}^f(x)) + \sum_{i=1}^{m-1} \sum_{j=1}^{R_i^f} N_i \gamma_i^f(\alpha_{i,j}^f(x)) + \sum_{j=1}^{R_i^g} N_i \gamma_i^g(\alpha_{i,j}^g(x))}{s' N_m + R_m^f N_m + \sum_{i=1}^{m-1} R_i^f N_i + R_i^g N_i}.$$

Moreover, since by construction for $x \in K_{\widehat{\alpha}}$, we have $\alpha_{i,j}^f(x) = \alpha_i^f$ and $\alpha_{i,j}^g(x) = \alpha_i^g$ for all $1 \le j \le R_i^f$ and $1 \le j \le R_i^g$ respectively, from (4.7) and (4.12) we get

(4.14)

$$\alpha_{s}(x) = \frac{s' N_{m} \alpha_{m}^{f} + \sum_{i=1}^{m-1} R_{i}^{f} N_{i} \alpha_{i}^{f} + R_{i}^{g} N_{i} \alpha_{i}^{g}}{s' N_{m} + \sum_{i=1}^{m-1} R_{i}^{f} N_{i} + R_{i}^{g} N_{i}}$$

$$\gamma_{s}(x) = \frac{s' N_{m} \gamma_{m}^{f} (\alpha_{m}^{f}) + \sum_{i=1}^{m-1} R_{i}^{f} N_{i} \gamma_{i}^{f} (\alpha_{i}^{f}) + R_{i}^{g} N_{i} \gamma_{i}^{g} (\alpha_{i}^{g})}{s' N_{m} + \sum_{i=1}^{m-1} R_{i}^{f} N_{i} + R_{i}^{g} N_{i}}$$

if $s'_{m-1} < s \le s_m$, and from (4.10) and (4.13) we get

(4.15)

$$\alpha_{s}(x) = \frac{s'N_{m}\alpha_{m}^{g} + R_{m}^{f}N_{m}\alpha_{m}^{f} + \sum_{i=1}^{m-1} R_{i}^{f}N_{i}\alpha_{i}^{f} + R_{i}^{g}N_{i}\alpha_{i}^{g}}{s'N_{m} + R_{m}^{f}N_{m} + \sum_{i=1}^{m-1} R_{i}^{f}N_{i} + R_{i}^{g}N_{i}}$$

$$\gamma_{s}(x) = \frac{s'N_{m}\gamma_{m}^{g}(\alpha_{m}^{g}) + R_{m}^{f}N_{m}\gamma_{m}^{g}(\alpha_{m}^{g}) + \sum_{i=1}^{m-1} R_{i}^{f}N_{i}\gamma_{i}^{f}(\alpha_{i}^{f}) + R_{i}^{g}N_{i}\gamma_{i}^{g}(\alpha_{i}^{g})}{s'N_{m} + R_{m}^{f}N_{m} + \sum_{i=1}^{m-1} R_{i}^{f}N_{i} + R_{i}^{g}N_{i}}$$

if $s_m < s \le s'_m$.

We also have

(4.16)
$$\underline{d}(\nu_{\widehat{\alpha}}, x) = \liminf_{s \to \infty} \frac{\log_2(\nu_{\widehat{\alpha}}(I_{n(s)}(x)))}{-n(s)} \quad \text{and} \quad \overline{d}(\nu_{\widehat{\alpha}}, x) = \limsup_{s \to \infty} \frac{\log_2(\nu_{\widehat{\alpha}}(I_{n(s)}(x)))}{-n(s)}$$

for all $x \in K_{\hat{\alpha}}$, for the same reasons as those leading to (4.5).

4.3.3. Lower bounds for the dimensions. – Suppose that we have proven that for all $0 \le \alpha \le \beta \le \infty$ we have $f_{-\mu}^{\text{LD}}(\alpha,\beta) = \max\{f(\alpha') : \alpha' \in [\alpha,\beta]\}$, a property which will be established in Section 4.4.3. Then, the lower bounds of Theorem 1.4(3) follow readily from the mass distribution principle (see Section 6), property (4.16), and the following proposition, which is a direct consequence of the estimates (4.14) and (4.15), and the assumptions (4.2)(4-5).

PROPOSITION 4.2. – With the notations of the previous section, fix $0 \le \alpha \le \beta \le \infty$ such that $[\alpha, \beta] \subset \mathcal{J}$ and $[\alpha, \beta] \cap \mathcal{J} \ne \emptyset$. Let $\alpha' = \operatorname{argmax}(f_{|[\alpha, \beta]})$ and $\beta' = \operatorname{argmax}(g_{|[\alpha, \beta]})$.

Fix a sequence $\widehat{\alpha} = (\alpha_1^f, \alpha_1^g, \dots, \alpha_m^f, \alpha_m^g, \dots) \in \prod_{m=1}^{\infty} A_m^f \times A_m^g$ such that $\lim_{m \to \infty} \alpha_m^f = \alpha', \lim_{m \to \infty} \gamma_m^f(\alpha_m^f) = f(\alpha'), \lim_{m \to \infty} \alpha_{3m-2}^g = \alpha, \lim_{m \to \infty} \gamma_{3m-2}^g(\alpha_{3m-2}^g) = g(\alpha), \lim_{m \to \infty} \alpha_{3m-1}^g = \beta, \lim_{m \to \infty} \gamma_{3m-1}^g(\alpha_{3m-1}^g) = g(\beta), \lim_{m \to \infty} \alpha_{3m}^g = \beta', \text{ and } \lim_{m \to \infty} \gamma_{3m}^g(\alpha_{3m}^g) = g(\beta').$ Then for all $x \in K_{\widehat{\alpha}}$, one has

$$\underline{d}(\mu, x) = \alpha, \ \overline{d}(\mu, x) = \beta, \ \underline{d}(\nu_{\widehat{\alpha}}, x) = \min\{f(\alpha'), g(\alpha), g(\beta)\}, \ and \ \overline{d}(\nu_{\widehat{\alpha}}, x) = g(\beta').$$

Consequently, $\nu_{\widehat{\alpha}}(E(\mu, \alpha, \beta)) = 1$, $\dim_H \nu_{\widehat{\alpha}} = \min\{f(\alpha'), g(\alpha), g(\beta)\}$ and $\dim_P \nu_{\widehat{\alpha}} = g(\beta')$, so that $\dim_H E(\mu, \alpha, \beta) \ge \min\{f(\alpha'), g(\alpha), g(\beta)\}$ and $\dim_H E(\mu, \alpha, \beta) \ge f(\beta')$.

4.4. Large deviations spectra and L^q-spectra

4.4.1. The large deviations spectra $\underline{f}_{\mu}^{\text{LD}}(\alpha)$ and $\overline{f}_{\mu}^{\text{LD}}(\alpha)$. – It is clear from the construction of μ , (4.7,4.10) and (4.2) that $\underline{f}_{\mu}^{\text{LD}}(\alpha) = -\infty$ if $\alpha \notin \mathcal{I}$ (take $s = s_m$ and use (4.8)) and $\overline{f}_{\mu}^{\text{LD}}(\alpha) = -\infty$ if $\alpha \notin \mathcal{I}$ (use (4.7) and (4.10)). Moreover, by (1.4) we have $\underline{f}_{\mu}^{\text{LD}}(\alpha) \geq \dim_H E(\mu, \alpha)$, hence $\underline{f}_{\mu}^{\text{LD}}(\alpha) \geq f(\alpha)$ for all $\alpha \in \mathcal{I}$ by Proposition 4.2. Similarly, $\overline{f}_{\mu}^{\text{LD}}(\alpha) \geq \dim_P E(\mu, \alpha)$, hence Proposition 4.2 yields $\overline{f}_{\mu}^{\text{LD}}(\alpha) \geq g(\alpha)$ for all $\alpha \in \mathcal{I}$.

Let us show that $\overline{f}_{\mu}^{\text{LD}}(\alpha) \leq g(\alpha)$ for $\alpha \in \mathcal{J}$.

Suppose first that $\alpha \in \mathcal{J} \setminus \{\infty\}$. Fix $\eta > 0$ and $\epsilon_{\eta} > 0$ such that $g(\alpha') \leq g(\alpha) + \eta$ if $\alpha' \in [\alpha - 2\epsilon_{\eta}, \alpha + 2\epsilon_{\eta}]$. Fix $\epsilon \in (0, \epsilon_{\eta})$. If $s'_{m-1} < s \leq s_m$, due to (4.7), for *s* large enough, if $I_s \in \mathbf{G}_s$ satisfies $2^{-n(s)(\alpha+\epsilon)} \leq \mu(I_s) \leq 2^{-n(s)(\alpha-\epsilon)}$, then for any $x \in I_s = I_{n(s)}(x)$ we have $\alpha - 2\epsilon \leq \alpha_s := \alpha_s(x) \leq \alpha + 2\epsilon$, and the exponents $\alpha_{i,j}^f(x)$ and $\alpha_{i,j}^g(x)$ do not depend on $x \in I_s$.

Due to the multiplicative structure of the construction of μ , for each such collection of exponents $\{\alpha_{i,j}^f, \alpha_{i,j}^g\}$, the set $\mathbf{G}_s(\{\alpha_{i,j}^f, \alpha_{i,j}^g\})$ of those dyadic cubes $I_s \in \mathbf{G}_s$ such that

4° SÉRIE - TOME 48 - 2015 - Nº 6

$$\begin{aligned} \alpha_{i,j}^{h}(x) &= \alpha_{i,j}^{h} \text{ for all } x \in I_{s} \text{ and } h \in \{f,g\} \text{ is such that (setting } s' = s - s'_{m-1}) \\ &\#\mathbf{G}_{s}(\{\alpha_{i,j}^{f}, \alpha_{i,j}^{g}\}) = \prod_{j=1}^{s'} \#G_{m}^{f}(\alpha_{m,j}^{f}) \prod_{i=1}^{m-1} \left(\prod_{j=1}^{R_{i}^{f}} \#G_{i}^{f}(\alpha_{i,j}^{f})\right) \prod_{j=1}^{R_{i}^{g}} \#G_{j}^{g}(\alpha_{i,j}^{g}) \\ &\leq \prod_{j=1}^{s'} 2^{N_{m}(\gamma_{m}^{f}(\alpha_{m,j}^{f}) + \epsilon_{m})} \prod_{i=1}^{m-1} \left(\prod_{j=1}^{R_{i}^{f}} 2^{N_{i}(\gamma_{i}^{f}(\alpha_{i,j}^{f}) + \epsilon_{i})}\right) \prod_{j=1}^{R_{i}^{g}} 2^{N_{i}(\gamma_{i}^{g}(\alpha_{i,j}^{g}) + \epsilon_{i})}.\end{aligned}$$

Since for $\alpha \in A_i^h$ by construction we have $\gamma_i^h(\alpha) \le h(\alpha) + 2\epsilon_i^{1/3}$, we get

$$\begin{aligned} \#\mathbf{G}_{s}(\{\alpha_{i,j}^{f}, \alpha_{i,j}^{g}\}) &\leq 2^{\eta(s)n'(s)} \prod_{j=1}^{s'} 2^{N_{m}f(\alpha_{m,j}^{f})} \prod_{i=1}^{m-1} \left(\prod_{j=1}^{R_{i}^{f}} 2^{N_{i}f(\alpha_{i,j}^{f})}\right) \prod_{j=1}^{R_{i}^{g}} 2^{N_{i}g(\alpha_{i,j}^{g})} \\ &\leq 2^{\eta(s)n'(s)} \prod_{j=1}^{s'} 2^{N_{m}g(\alpha_{m,j}^{f})} \prod_{i=1}^{m-1} \left(\prod_{j=1}^{R_{i}^{f}} 2^{N_{i}g(\alpha_{i,j}^{f})}\right) \prod_{j=1}^{R_{i}^{g}} 2^{N_{i}g(\alpha_{i,j}^{g})} \end{aligned}$$

with $\lim_{s\to\infty} \eta(s) = 0$ and n'(s) defined like in (4.4). Now recall that g is concave over $\mathcal{J} \setminus \{\infty\}$. Thus

$$\sum_{j=1}^{s'} N_m g(\alpha_{m,j}^f) + \sum_{j=1}^{R_i^f} N_i g(\alpha_{i,j}^f) + \sum_{j=1}^{R_i^g} N_i g(\alpha_{i,j}^g)$$

$$\leq n'(s) g\left(\frac{\sum_{j=1}^{s'} N_m \alpha_{m,j} + \sum_{i=1}^{m-1} \sum_{j=1}^{R_i^f} N_i \alpha_{i,j}^f + \sum_{j=1}^{R_i^g} N_i \alpha_{i,j}^g}{n'(s)}\right)$$

$$= n'(s) g(\alpha_s) \leq n'(s) (g(\alpha) + \eta).$$

Consequently, due to Proposition 4.1 we get

$$\#\mathbf{G}_s(\{\alpha_{i,j}^f, \alpha_{i,j}^g\}) \le 2^{(g(\alpha)+\eta+\eta(s))n(s)}.$$

Moreover, the number of such collections cannot exceed the total number of possible realizations of such a family when the condition $\alpha_s \in [\alpha - \epsilon_\eta, \alpha + \epsilon_\eta]$ is dropped, which by construction is equal to $(\#A_m^f)^{s'} \prod_{i=1}^{m-1} (\#A_i^f)^{R_i^f} (\#A_i^g)^{R_i^g} = 2^{\eta'(s)n(s)}$, with $\lim_{s\to\infty} \eta'(s) = 0$, by (4.2)(2)(3). We can conclude that

$$#\left\{I_s \in \mathbf{G}_s : 2^{-n(s)(\alpha+\epsilon)} \le \mu(I_s) \le 2^{-n(s)(\alpha-\epsilon)}\right\} \le 2^{(g(\alpha)+\eta+\eta(s)+\eta'(s))n(s)}.$$

The same estimates hold if $s_m < s \le s'_m$, and this yields $\overline{f}_{\mu}^{\text{LD}}(\alpha) \le f(\alpha) + \eta$. Since this holds for all $\eta > 0$, the have the desired conclusion.

Now suppose that $\alpha = \infty \in \mathcal{J}$. Since $g(\infty) \ge \sup\{g(\alpha) : \alpha \in \mathcal{J} \setminus \{\infty\}\}$, with the same notations as above, the only change is that for any A > 0 we must consider those intervals in \mathbf{G}_s such that $\alpha_s \ge A$, and conditioning on the realization of $\{\alpha_{i,j}^f(x), \alpha_{i,j}^g(x)\}$, the same calculations as above yield, even without using the concavity of g,

$$\#\left\{I_s \in \mathbf{G}_s : \mu(I_s) \le 2^{-n(s)A}\right\} \le 2^{(g(\infty)+\eta+\eta(s)+\eta'(s))n(s)},$$

hence the result.

Let us prove that $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha) \leq f(\alpha)$ for $\alpha \in \mathscr{I}$.

Suppose first that $\alpha < \infty$. Fix $\eta > 0$ and $\epsilon_{\eta} > 0$ such that $f(\alpha') \leq f(\alpha) + \eta$ if $\alpha' \in [\alpha - 2\epsilon_{\eta}, \alpha + 2\epsilon_{\eta}]$. Fix $\epsilon \in (0, \epsilon_{\eta})$. Suppose that $s = s_m$. Due to (4.8), for slarge enough, if $I_s \in \mathbf{G}_s$ satisfies $2^{-n(s)(\alpha+\epsilon)} \leq \mu(I_s) \leq 2^{-n(s)(\alpha-\epsilon)}$, then for any $x \in I_s = I_{n(s)}(x)$ we have $\alpha - 2\epsilon \leq \widetilde{\alpha}_s := \frac{\sum_{j=1}^{R_m^f} N_m \alpha_{m,j}^f(x)}{R_m^f N_m} \leq \alpha + 2\epsilon$. Also, given $\{\alpha_{i,j}^f, \alpha_{i,j}^g\}$, the set $\mathbf{G}_s(\{\alpha_{i,j}^f, \alpha_{i,j}^g\})$ of those dyadic cubes $I_s \in \mathbf{G}_s$ such that $\alpha_{i,j}^h(x) = \alpha_{i,j}^h$ for all $x \in I_s$ and $h \in \{f, g\}$ is bounded as above by

$$\prod_{j=1}^{R_m^f} 2^{N_m(\gamma_m^f(\alpha_{m,j}^f) + \epsilon_m)} \prod_{i=1}^{m-1} \left(\prod_{j=1}^{R_i^f} 2^{N_i(\gamma_i^f(\alpha_{i,j}^f) + \epsilon_i)} \right) \prod_{j=1}^{R_i^g} 2^{N_i(\gamma_i^g(\alpha_{i,j}^g) + \epsilon_i)},$$

which yields

$$\#\mathbf{G}_{s}(\{\alpha_{i,j}^{f}, \alpha_{i,j}^{g}\}) \leq 2^{o(R_{m}^{f}N_{m})} 2^{\sum_{j=1}^{R_{m}^{f}}N_{m}\gamma_{m}^{f}(\alpha_{m,j}^{f})} \leq 2^{o(R_{m}^{f}N_{m})} 2^{\sum_{j=1}^{R_{m}^{f}}N_{m}f(\alpha_{m,j}^{f})}$$

due to (4.2)(4). Using the concavity of f this implies

$$\#\mathbf{G}_{s}(\{\alpha_{i,j}^{f}, \alpha_{i,j}^{g}\}) \leq 2^{R_{m}^{f}N_{m}(f(\tilde{\alpha}_{s}) + o(1))} \leq 2^{R_{m}^{f}N_{m}(f(\alpha) + 2\eta)}$$

for s large enough. Finally, counting the possible number of collections $\{\alpha_{i,j}^f, \alpha_{i,j}^g\}$ as above yields

(4.17)
$$\# \left\{ I_s \in \mathbf{G}_s : 2^{-n(s)(\alpha+\epsilon)} \le \mu(I_s) \le 2^{-n(s)(\alpha-\epsilon)} \right\} \le 2^{(f(\alpha)+3\eta)n(s)}$$

for $s = s_m$ large enough. Since η is arbitrary, this is enough to conclude.

If $\alpha = \infty$, the previous estimates with $\gamma_m^f(\alpha_{m,j})$ bounded by $f(\infty) + 2\epsilon_m$ yield the conclusion.

It remains to prove that $\overline{f}_{\mu}^{\text{LD}}(\alpha) \geq g(\alpha)$ for $\alpha \in \mathcal{J} \setminus \mathcal{J}$. In fact the argument is valid for all $\alpha \in \mathcal{J}$. Let $\alpha \in \mathcal{J}$. Suppose first that $\alpha < \infty$. Choose a sequence $\widehat{\alpha} = (\alpha_1^f, \alpha_1^g, \dots, \alpha_m^f, \alpha_m^g, \dots) \in \prod_{m=1}^{\infty} A_m^f \times A_m^g$ such that $\lim_{m\to\infty} \alpha_m^g = \alpha$ and $\lim_{m\to\infty} \gamma_m^g(\alpha_m^g) = g(\alpha)$. We leave the reader check that there exists a sequence $\epsilon''(s)$ converging to 0 as $s = s'_m$ tends to ∞ such that, for all $x \in K_{\widehat{\alpha}}$ (the Cantor set constructed in Section 4.3.2), we have $\alpha_s(x) = \alpha + \epsilon(s)$ (this uses (4.11)) and

$$\#\{\mathbf{G}_{\widehat{\alpha},s} = \prod_{i=1}^{m} (\#G^{f}(i,\alpha^{f}))^{R_{i}^{f}} (\#G^{f}(i,\alpha^{g}))^{R_{i}^{f}} \ge 2^{n(s)(g(\alpha) - \epsilon^{\prime\prime}(s))}$$

(this uses the left hand side of (2.7), and (4.2)(5)). This implies that $\overline{f}_{\mu}^{\text{LD}}(\alpha) \ge g(\alpha)$.

If $\alpha = \infty$, the same argument with $\alpha_m^g = \alpha_m^g(\infty)$ yields the desired lower bound.

We notice that a similar argument would give another proof of $\underline{f}_{\mu}^{\text{LD}}(\alpha) \ge f(\alpha)$ for $\alpha \in \mathscr{I}$.

4° SÉRIE - TOME 48 - 2015 - Nº 6

REMARK 4.2. – In fact, the arguments developed in this section provide us with the following precious information: for $\alpha \in \mathbb{R}_+$ we have

$$f(\alpha) = \lim_{\epsilon \to 0^+} \liminf_{m \to \infty} \frac{\log_2 \# \left\{ I \in \mathbf{G}_{s_m} : 2^{-n(s_m)(\alpha+\epsilon)} \le \mu(I) \le 2^{-n(s_m)(\alpha-\epsilon)} \right\}}{n(s_m)}$$
$$= \lim_{\epsilon \to 0^+} \limsup_{m \to \infty} \frac{\log_2 \# \left\{ I \in \mathbf{G}_{s_m} : 2^{-n(s_m)(\alpha+\epsilon)} \le \mu(I) \le 2^{-n(s_m)(\alpha-\epsilon)} \right\}}{n(s_m)};$$
$$g(\alpha) = \lim_{\epsilon \to 0^+} \liminf_{m \to \infty} \frac{\log_2 \# \left\{ I \in \mathbf{G}_{s'_m} : 2^{-n(s'_m)(\alpha+\epsilon)} \le \mu(I) \le 2^{-n(s'_m)(\alpha-\epsilon)} \right\}}{n(s'_m)};$$

$$= \lim_{\epsilon \to 0^+} \limsup_{m \to \infty} \frac{\log_2 \# \left\{ I \in \mathbf{G}_{s'_m} : 2^{-n(s'_m)(\alpha+\epsilon)} \le \mu(I) \le 2^{-n(s'_m)(\alpha-\epsilon)} \right\}}{n(s'_m)}$$

and we also have

$$\begin{split} f(\infty) &= \lim_{A \to \infty} \liminf_{m \to \infty} \frac{\log_2 \# \left\{ I \in \mathbf{G}_{s_m} : \mu(I) \le 2^{-n(s_m)A} \right\}}{n(s_m)} \\ &= \lim_{A \to \infty} \limsup_{m \to \infty} \frac{\log_2 \# \left\{ I \in \mathbf{G}_{s_m} : \mu(I) \le 2^{-n(s_m)A} \right\}}{n(s_m)} \end{split}$$

and

$$g(\infty) = \lim_{A \to \infty} \liminf_{m \to \infty} \frac{\log_2 \# \left\{ I \in \mathbf{G}_{s'_m} : \mu(I) \le 2^{-n(s'_m)A} \right\}}{n(s'_m)}$$
$$= \lim_{A \to \infty} \limsup_{m \to \infty} \frac{\log_2 \# \left\{ I \in \mathbf{G}_{s'_m} : \mu(I) \le 2^{-n(s'_m)A} \right\}}{n(s'_m)}$$

4.4.2. The functions τ_{μ} and $\overline{\tau}_{\mu}$. – It follows from Remark 4.2 and standard large deviations estimates similar to those used for instance in the proof of [75, Theorem 4.2], that for all $q \in \mathbb{R}$ we have

$$\lim_{m \to \infty} \frac{\log_2 \sum_{I \in \mathbf{G}_{s_m}} \mu(I)^q}{-n(s_m)} = f^*(q) \quad \text{and} \quad \lim_{m \to \infty} \frac{\log_2 \sum_{I \in \mathbf{G}_{s'_m}} \mu(I)^q}{-n(s'_m)} = g^*(q)$$

(in case the domains of f and g are compact, this is a direct consequence of Laplace-Varadhan's integral lemma [28, Theorem 4.3.1]). Consequently, due to (4.6) we get $\tau_{\mu} = g^*$ and $\overline{\tau}_{\mu} = f^*$. Moreover, it is direct from the duality between upper semi-continuous concave functions [77, Theorem 12.2, Corollary 12.2.2] that if $\infty \notin \text{dom}(f)$, we have $\tau_{\mu}^* = g$ and $\overline{\tau}_{\mu}^* = f$, and if $\infty \in \text{dom}(f)$, then $\text{dom}(\tau_{\mu} = f^*) = \mathbb{R}_+$, and τ_{μ}^* coincides with fover \mathbb{R} . Then, by our definition of the extended concave conjugate function, we have $\tau_{\mu}^*(\infty) = -\tau_{\mu}(0) = f(\infty)$.

4.4.3. The large deviations spectrum $\underline{f}_{\mu}^{\text{LD}}(\alpha,\beta)$. – Recall that for $0 \le \alpha \le \beta \le \infty$, $f(\alpha,\beta)$ is defined as $\max\{f(\alpha') : \alpha' \in [\alpha,\beta]\}$.

If
$$[\alpha, \beta] \subset \mathbb{R}_+ \cup \{\infty\} \setminus \mathcal{I}$$
, due to (4.8), we have $\underline{f}_{\mu}^{\text{LD}}(\alpha, \beta) = -\infty = f(\alpha, \beta)$.

Assume now that $[\alpha, \beta] \cap \mathscr{I} \neq \varnothing$, and $(\alpha, \beta) \neq (\infty, \infty)$. Denote the interval $[\alpha, \beta] \cap \mathscr{I}$ by $[\alpha_1, \beta_1]$, and notice that $f(\alpha, \beta) = f(\alpha_1, \beta_1)$.

Suppose first that $\beta_1 < \infty$. Let $\eta > 0$ and for each $\alpha' \in [\alpha_1, \beta_1]$ let $\epsilon(\alpha') > 0$ such that $f(\beta) \leq f(\alpha') + \eta$ for all $\beta \in [\alpha' - \epsilon(\alpha'), \alpha' + \epsilon(\alpha')]$. There exists $\alpha'_1, \ldots, \alpha'_N$ in $[\alpha_1, \beta_1]$ such that $[\alpha_1, \beta_1] \subset \bigcup_{i=1}^N [\alpha'_i - \epsilon(\alpha'_i), \alpha'_i - \epsilon(\alpha'_i)]$. Let $\epsilon \leq \min\{\epsilon(\alpha'_i) : 1 \leq i \leq N\}$. Property (4.8) implies that for *m* large enough, if $I \in \mathbf{G}_{s_m}$ and $2^{-n(s_m)(\beta+\epsilon)} \leq \mu(I) \leq 2^{-n(s_m)(\alpha-\epsilon)}$, since there exists $x \in K$ such that $I = I_{n(s_m)}(x)$, in fact there exists $1 \leq i \leq N$ such that $2^{-n(s_m)(\alpha'_i + \epsilon(\alpha'_i))} \leq \mu(I) \leq 2^{-n(s_m)(\alpha'_i - \epsilon(\alpha'_i))}$. Then, the estimate (4.17) achieved in Section 4.4.1 yields

$$\#\Big\{I \in \mathbf{G}_{s_m} : 2^{-n(s_m)(\alpha'_i + \epsilon(\alpha'_i))} \le \mu(I) \le 2^{-n(s_m)(\alpha'_i - \epsilon(\alpha'_i))}\Big\} \le 2^{(f(\alpha'_i) + 3\eta)n(s_m)}$$

for m large enough. This implies that for m large enough

$$\begin{split} \#\Big\{I \in \mathbf{G}_{s_m} : 2^{-n(s_m)(\beta+\epsilon)} \leq \mu(I) \leq 2^{-n(s_m)(\alpha-\epsilon)}\Big\} \\ \leq \sum_{i=1}^N 2^{(f(\alpha'_i)+3\eta)n(s_m)} \leq N 2^{(f(\alpha,\beta)+3\eta)n(s_m)}. \end{split}$$

If follows that $\underline{f}_{\mu}^{\text{LD}}(\alpha,\beta) \leq f(\alpha,\beta) + 3\eta$ for all $\eta > 0$, hence the desired upper bound $f(\alpha,\beta)$ for $\underline{f}_{\mu}^{\text{LD}}(\alpha,\beta)$.

For the lower bound, we just use the fact that by Proposition 4.2 we know that if $\alpha' = \operatorname{argmax}(f_{|[\alpha,\beta]})$ we have $f(\alpha') \leq \dim_H E(\mu,\alpha')$, and on the other hand $\dim_H E(\mu,\alpha') \leq \underline{f}_{\mu}^{\text{LD}}(\alpha') \leq \underline{f}_{\mu}^{\text{LD}}(\alpha,\beta)$, the last inequality being obvious.

If $\beta_1 = \infty$, the upper bound for $\underline{f}_{\mu}^{\text{LD}}(\alpha, \beta)$ just comes from the observation already done in Section 4.4.1 that if $s = s_m$, we have $\#\mathbf{G}_s \leq 2^{n(s)(f(\infty)+o(1))}$, and the lower bound comes from the lower bound $f(\infty)$ for $\dim_H E(\mu, \infty)$.

4.5. Upper bounds for the different spectra

It is a direct application of Proposition 1.3, using the fact that $\underline{f}_{\mu}^{\text{LD}}(\alpha,\beta) = f(\alpha,\beta)$.

4.6. μ is exactly dimensional, with dimension D

Fix $\epsilon > 0$. For each $s \ge 1$, if $s'_{m-1} < s \le s_m$ and $s' = s - s'_{m-1}$, an application of Markov's inequality yields, for any $\eta > 0$:

$$\mu \Big(E_{s,+} = \Big\{ x \in K : \frac{\mu(I_{n(s)}(x))}{\mu(I_{n(s'_{m-1})}(x))} \ge 2^{-(n'(s)-n'(s'_{m-1}))(D_m-\epsilon)} \Big\} \Big)$$

$$\le \sum_{I_{n(s)} \in \mathbf{G}_s} \mu(I_{n(s)}) \Big(\frac{\mu(I_{n(s)})}{\mu(I_{n(s'_{m-1})})} \Big)^{\eta} 2^{\eta(n'(s)-n'(s'_{m-1}))(D_m-\epsilon)}$$

$$= \sum_{I_{n(s)} \in \mathbf{G}_s} \mu(I_{n(s'_{m-1})}) \Big(\frac{\mu(I_{n(s)})}{\mu(I_{n(s'_{m-1})})} \Big)^{1+\eta} 2^{\eta(n'(s)-n'(s'_{m-1}))(D_m-\epsilon)}$$

where $I_{n(s'_{m-1})}$ stands for the unique element of $\mathbf{G}_{s'_{m-1}}$ containing $I_{n(s)}$. We notice that by construction, given $I_{n(s'_{m-1})}$ in $\mathbf{G}_{s'_{m-1}}$, the distribution of the collection $\left\{\frac{\mu(I_{n(s)})}{\mu(I_{n(s'_{m-1})})}\right\}$, where $I_{n(s)} \in \mathbf{G}_s$ and $I_{n(s)} \subset I_{n(s'_{m-1})}$, is independent of $I_{n(s'_{m-1})}$, and taking into account

4° SÉRIE - TOME 48 - 2015 - Nº 6

1498

1499

the fact that between the steps s'_{m-1} and s one uses s' times the same motive in the recursion defining μ , we can get

$$\mu(E_{s,+}) \leq \frac{\left(\sum_{\alpha \in A_m^f} \sum_{I \in G^f(m,\alpha)} \rho_m^f(\alpha)^{1+\eta} \mu_{\alpha}(I)^{1+\eta}\right)^{s'}}{\left(\sum_{\alpha \in A_m^f} \sum_{I \in G^f(m,\alpha)} \rho_m^f(\alpha) \mu_{\alpha}(I)\right)^{(1+\eta)s'}} 2^{\eta(n'(s)-n'(s'_{m-1}))(D_m-\epsilon)}$$
$$\leq 2^{(1+\eta)s'} \left(\sum_{\alpha \in A_m^f} \sum_{I \in G^f(m,\alpha)} \rho_m^f(\alpha)^{1+\eta} \mu_{\alpha}(I)^{1+\eta}\right)^{s'} 2^{\eta(n'(s)-n'(s'_{m-1}))(D_m-\epsilon)}$$

We have

$$\sum_{I \in G^{f}(m,D_{m})} \rho_{m}^{f}(D_{m})^{1+\eta} \mu_{D_{m}}(I)^{1+\eta} = \sum_{I \in G^{f}(m,D_{m})} \mu_{D_{m}}(I)^{1+\eta}$$
$$\leq (\#G^{f}(m,D_{m}))2^{-N_{m}(D_{m}-\epsilon_{m})(1+\eta)} \leq 2^{N_{m}(D_{m}+\epsilon_{m})}2^{-N_{m}(D_{m}-\epsilon_{m})(1+\eta)}$$
$$= 2^{-N_{m}D_{m}\eta}2^{N_{m}\epsilon_{m}(2+\eta)}.$$

On the other hand, if $A_m^f \setminus \{D_m\} \neq \emptyset$, fixing one of its elements α_0 , we have

$$\sum_{\alpha \in A_m^f \setminus \{D_m\}} \sum_{I \in G^f(m,\alpha)} \rho_m^f(\alpha)^{1+\eta} \mu_\alpha(I)^{1+\eta}$$

$$\leq \sum_{\alpha \in A_m^f \setminus \{D_m\}} \rho_m^f(\alpha)^{1+\eta} (\#G^f(m,\alpha)) 2^{-N_m(\alpha-\epsilon_m)(1+\eta)}$$

$$\leq \sum_{\alpha \in A_m^f \setminus \{D_m\}} \rho_m^f(\alpha)^{1+\eta} 2^{N_m(\gamma_m^f(\alpha)+\epsilon_m)} 2^{-N_m(\alpha-\epsilon_m)(1+\eta)}$$

$$\leq \rho_m^f(\alpha_0)^{1+\eta} (\#A_m^f) (\sup_{\alpha \in A_m^f \setminus \{D_m\}} 2^{N_m(\gamma_m^f(\alpha)-\alpha)}) 2^{N_m\epsilon_m(2+\eta)}.$$

Now, take $\eta = \eta_m = \sqrt{\epsilon_m}$. Due to the Definition (4.3) of ρ_m^f we have $\rho_m^f(\alpha_0)^{1+\eta}(\#A_m^f) \leq 1$, and due to (4.1), we have $\sup_{\alpha \in A_m^f \setminus \{D_m\}} 2^{N_m(\gamma_m^f(\alpha) - \alpha)} \leq 2^{-N_m D_m \eta_m}$. Finally,

$$\mu(E_{s,+}) \leq 2^{s'} \cdot 2^{(1+\eta_m)s'} \left(2^{-N_m D_m \eta_m} 2^{N_m \epsilon_m (2+\eta_m)}\right)^{s'} 2^{\eta_m (n'(s)-n'(s'_{m-1}))(D_m-\epsilon)}$$

= $2^{s'} \cdot 2^{(1+\eta_m)s'} \left(2^{-N_m D_m \eta_m} 2^{N_m \epsilon_m (2+\eta_m)}\right)^{s'} 2^{\eta_m N_m s'(D_m-\epsilon)}$
 $\leq 2^{3s'} 2^{-N_m s' \eta_m (\epsilon-3\eta_m)}.$

Also, using a similar estimate as above, and with the same choice $\eta = \eta_m = \sqrt{\epsilon_m}$, we have

$$\begin{split} \mu\Big(\Big\{x\in K: \frac{\mu(I_{n(s)}(x))}{\mu(I_{n(s'_{m-1})}(x))} &\leq 2^{-(n'(s)-n'(s'_{m-1}))(D_{m}+\epsilon)}\Big\}\Big) \\ &\leq \frac{\Big(\sum_{\alpha\in A_{m}^{f}}\sum_{I\in G^{f}(m,\alpha)}\rho_{m}^{f}(\alpha)^{1-\eta}\mu_{\alpha}(I)^{1-\eta}\Big)^{s'}}{\Big(\sum_{\alpha\in A_{m}^{f}}\sum_{I\in G^{f}(m,\alpha)}\rho_{m}^{f}(\alpha)\mu_{\alpha}(I)\Big)^{(1-\eta)s'}}2^{-\eta(n'(s)-n'(s'_{m-1}))(D_{m}+\epsilon)} \\ &\leq 2^{(1-\eta)s'}\Big(\sum_{\alpha\in A_{m}^{f}}\sum_{I\in G^{f}(m,\alpha)}\rho_{m}^{f}(\alpha)^{1-\eta}\mu_{\alpha}(I)^{1-\eta}\Big)^{s'}2^{-\eta(n'(s)-n'(s'_{m-1}))(D_{m}+\epsilon)}. \end{split}$$

On the one hand, we have

$$\sum_{I \in G^{f}(m,D_{m})} \rho_{m}^{f}(D_{m})^{1-\eta} \mu_{D_{m}}(I)^{1-\eta} = \sum_{I \in G^{f}(m,D_{m})} \mu_{D_{m}}(I)^{1-\eta} \le 2^{N_{m}(D_{m}+\epsilon_{m})} 2^{-N_{m}(D_{m}-\epsilon_{m})(1-\eta)} = 2^{N_{m}D_{m}\eta} 2^{2N_{m}\epsilon_{m}}.$$

On the other hand, if $A_m^f \setminus \{D_m\} \neq \emptyset$, fixing one of its elements α_0 , we have

$$\sum_{\alpha \in A_m^f \setminus \{D_m\}} \sum_{I \in G^f(m,\alpha)} \rho_m^f(\alpha)^{1-\eta} \mu_\alpha(I)^{1-\eta}$$

$$\leq \sum_{\alpha \in A_m^f \setminus \{D_m\}} \rho_m^f(\alpha)^{1-\eta} 2^{N_m(\gamma_m^f(\alpha) + \epsilon_m)} 2^{-N_m(\alpha - \epsilon_m)(1-\eta)}$$

$$\leq \rho_m^f(\alpha_0)^{1-\eta} (\#A_m^f) (\sup_{\alpha \in A_m^f \setminus \{D_m\}} 2^{N_m(\gamma_m^f(\alpha) - (1-\eta)\alpha)}) 2^{2N_m \epsilon_m}.$$

Due to the Definition (4.3) of ρ_m^f , since $\sqrt{\epsilon_m} = (m+1)^{-1} \leq 1/2$, we have $\rho_m^f(\alpha_0)^{1-\eta_m}(\#A_m^f) \leq 1$, and due to (4.1), we have $\sup_{\alpha \in A_m^f \setminus \{D_m\}} 2^{N_m(\gamma_m^f(\alpha) - (1-\eta_m\alpha))} \leq 1$. Consequently, the previous estimates yield

$$\mu\Big(E_{s,-} = \Big\{x \in K : \frac{\mu(I_{n(s)}(x))}{\mu(I_{n(s'_{m-1})}(x))} \le 2^{-(n'(s) - n'(s'_{m-1}))(D_m + \epsilon)}\Big\}\Big) \le 2^{3s'} 2^{-N_m s' \eta_m(\epsilon - 3\eta_m)}.$$

Similarly, if $s_m < s \le s'_m$ and $s' = s - s_m$ we can get

$$\mu \Big(E_{s,+} = \Big\{ x \in K : \frac{\mu(I_{n(s)}(x))}{\mu(I_{n(s_m)}(x))} \ge 2^{-(n'(s)-n'(s_m))(D_m-\epsilon)} \Big\} \Big) \le 2^{3s'} 2^{-N_m s' \eta_m(\epsilon-3\eta_m)}$$

$$\mu \Big(E_{s,-} = \Big\{ x \in K : \frac{\mu(I_{n(s)}(x))}{\mu(I_{n(s_m)}(x))} \le 2^{-(n'(s)-n'(s_m))(D_m+\epsilon)} \Big\} \Big) \le 2^{3s'} 2^{-N_m s' \eta_m(\epsilon-3\eta_m)}.$$

Finally, for m_0 big enough so that $3\eta_m \leq \epsilon/2$, and $N_m\eta_m\epsilon/2 > 4$ (remember that $N_m \geq e^m$ and $\eta_m = (m+1)^{-1}$), we have

$$\sum_{m \ge m_0} \sum_{s'_{m-1} < s \le s_m} \mu(E_{s,+} \cup E_{s,-}) + \sum_{s_m < s \le s'_m} \mu(E_{s,+} \cup E_{s,-})$$

$$\leq 2 \sum_{m \ge m_0} \sum_{s'=1}^{R_m^f} 2^{3s'} 2^{-N_m s' \eta_m \epsilon/2} + \sum_{s'=1}^{R_m^g} 2^{3s'} 2^{-N_m s' \eta_m \epsilon/2}$$

$$\leq 2 \sum_{m \ge m_0} 16 \cdot \frac{2^{-N_m \eta_m \epsilon/2}}{1 - 8 \cdot 2^{-N_m \eta_m \epsilon/2}} \le 64 \sum_{m \ge m_0} 2^{-N_m \eta_m \epsilon/2} < \infty$$

By the Borel-Cantelli lemma, we deduce that for μ -almost every x, there exists an integer m_x such that for all $m \ge m_x$, for all $s'_{m-1} < s \le s_m$ one has

$$2^{-(n'(s)-n'(s'_{m-1}))(D_m+\epsilon)} \le \frac{\mu(I_{n(s)}(x))}{\mu(I_{n(s'_{m-1})}(x))} \le 2^{-(n'(s)-n'(s'_{m-1}))(D_m-\epsilon)},$$

and for all $s_m < s \leq s'_m$ one has

$$2^{-(n'(s)-n'(s_m))(D_m+\epsilon)} \le \frac{\mu(I_{n(s)}(x))}{\mu(I_{n(s_m)}(x))} \le 2^{-(n'(s)-n'(s_m))(D_m-\epsilon)}.$$

4° SÉRIE – TOME 48 – 2015 – Nº 6

Using these inequalities telescopically and noting that D_m converges to D as $m \to \infty$ and $n'(s) \sim n(s)$ as $s \to \infty$ we get $D - \epsilon \leq \underline{d}(\mu, x) \leq \overline{d}(\mu, x) \leq D + \epsilon$ for μ -almost every x. Since ϵ is arbitrary, we have the desired exactly dimensionality.

4.7. Restrictions of μ

Let *B* be a closed ball whose interior intersects $\operatorname{supp}(\mu)$ at some point *x*. Let $\nu = \mu_{|B}$. We naturally have the inequalities $\dim_L(\nu, \alpha, \beta) \leq \dim_L(\mu, \alpha, \beta)$ for $L \in \{H, P\}$ and $0 \leq \alpha \leq \beta \leq \infty$, as well as $\underline{f}_{\nu}^{\mathrm{LD}}(\alpha) \leq \underline{f}_{\mu}^{\mathrm{LD}}(\alpha)$, $\overline{f}_{\nu}^{\mathrm{LD}}(\alpha) \leq \overline{f}_{\mu}^{\mathrm{LD}}(\alpha)$, $\underline{f}_{\nu}^{\mathrm{LD}}(\alpha, \beta) \leq \underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)$, $\tau_{\nu} \geq \tau_{\mu}$ and $\overline{\tau}_{\nu} \geq \overline{\tau}_{\mu}$. On the other hand, denoting by s_0 the smallest integer such that $I_{n(s_0)}(x) \subset \operatorname{Int}(B)$, we can modify the first terms of the sequence $\widehat{\alpha}$ of Proposition 4.2 so that $I_{n(s_0)}(x) \in \mathcal{G}_{\widehat{\alpha},s_0}$, hence $\nu_{\widehat{\alpha}}(B \cap K_{\widehat{\alpha}}) > 0$, and we have $\dim_H E(\nu, \alpha, \beta) \geq f(\alpha, \beta)$ and $\dim_P E(\nu, \alpha, \beta) \geq g(\alpha, \beta)$. This is enough to reverse all the previous inequalities since we also have for ν the general inequalities provided by (1.4), (1.5) and Proposition 1.3.

5. Proofs of Propositions 1.1(2), 1.2 and 1.3, and some inequalities in (1.4) and (1.5)

5.1. Proofs of Propositions 1.1(2) and 1.2

5.1.1. Proposition 1.1(2). – The fact that $\mathbb{R}_+ \subset \operatorname{dom}(\tau_\mu)$ was explained after we defined τ_μ in the introduction. Now suppose at first that there exist $\alpha \in \mathbb{R}_+$ and $r_0 > 0$ such that for all $r \in (0, r_0)$ and $x \in \operatorname{supp}(\mu)$ we have $\mu(B(x, r)) > r^{\alpha}$. Then, the definition of the L^q -spectrum yields $\tau_{\mu}(q) \geq \tau_{\mu}(0) + q\alpha$ for all q < 0, hence τ_{μ} is finite over \mathbb{R} . If, on the contrary, for all $\alpha > 0$, for all $r_0 > 0$, there exist $r \in (0, r_0)$ and $x \in \operatorname{supp}(\mu)$ such that $\mu(B(x, r)) \leq r^{\alpha}$, by using again the definition of the L^q -spectrum we have $\tau_{\mu}(q) \leq \alpha q$ for all $\alpha > 0$ and q < 0, hence $\tau_{\mu}(q) = -\infty$ for q < 0, so dom $(\tau_{\mu}) = \mathbb{R}_+$.

Now let $\alpha \in \operatorname{dom}(\tau_{\mu}^{*})$ and suppose that $-\infty < \tau_{\mu}^{*}(\alpha) < 0$. Necessarily $\alpha < \infty$. Indeed one always has $\tau_{\mu}^{*}(\infty) \in \{-\infty, -\tau_{\mu}(0)\}$. Then, suppose first that $\alpha < \tau_{\mu}^{\prime}(0^{+})$. Let $\alpha_{0} = \inf\{\beta \in (\alpha, \tau_{\mu}^{\prime}(0^{+})) : \tau_{\mu}^{*}(\beta) \geq 0\}$. The continuity of τ_{μ}^{*} over the interior of its domain implies $\tau_{\mu}^{*}(\alpha_{0}) = 0$. Then for all $\beta < \alpha_{0}$ we have $\tau_{\mu}^{*}(\beta) < 0$, hence $\overline{f}^{\mathrm{LD}}(\beta) < 0$. Consequently, for all $\epsilon > 0$ there exists $r_{0} > 0$ such that for all $r \in (0, r_{0})$ and $x \in \operatorname{supp}(\mu)$ we have $\mu(B(x, r)) \leq r^{\alpha_{0}-\epsilon}$. This implies that $\tau_{\mu}(q) \geq \tau_{\mu}(0) + \alpha_{0}q$ for all $q \geq 0$, and finally $\tau_{\mu}^{*}(\beta) \leq \inf\{\beta q - \alpha_{0}q + \tau_{\mu}(0) : q \geq 0\} = -\infty$ for all $\beta < \alpha_{0}$, which contradicts the fact that $-\infty < \tau_{\mu}^{*}(\alpha)$. Next suppose $\alpha > \tau_{\mu}^{\prime}(0^{-})$ and $\operatorname{dom}(\tau_{\mu}) = \mathbb{R}$. The same lines as above also yield a contradiction. If now $\alpha \in [-\tau_{\mu}^{\prime}(0^{+}), \tau_{\mu}^{\prime}(0^{-})]$ and $\operatorname{dom}(\tau_{\mu}) = \mathbb{R}$, then $\tau_{\mu}^{*}(\alpha) = -\tau_{\mu}(0) \geq 0$; new contradiction. It remains the case $\operatorname{dom}(\tau_{\mu}) = \mathbb{R}_{+}$ and $\alpha \geq \tau_{\mu}^{\prime}(0^{+})$. In this case, we necessarily have $\tau_{\mu}^{*}(\alpha) = \lim_{q \to 0^{+}} -\tau_{\mu}(q) \geq 0$. Finally, τ_{μ}^{*} is non-negative on its domain.

5.1.2. Proposition 1.2. – (1) If dom(τ) = \mathbb{R} , the property dom(τ^*) = [$\tau'(\infty), \tau'(-\infty)$] follows from standard considerations in convex analysis. Then, the fact that this interval is bounded from above follows from the boundedness from below of τ^* . Also, since dom(τ^*) $\subset \mathbb{R}$, the equality (τ^*)^{*} = τ on dom(τ) = \mathbb{R} is just the usual duality between τ and its conjugate function ([77, Theorem 12.2, Corollary 12.2.2]) when this one is only defined on \mathbb{R} and not also on $\mathbb{R} \cup \{\infty\}$ as in the convention used in this paper.

(2)(a) Since $\tau(0) = \tau(1) = 0$, by concavity of the non decreasing function τ , we have $\tau = 0$ over \mathbb{R}_+ , and a simple verification shows that $\tau^* = \tau$ over $\mathbb{R} \cup \{\infty\}$.

(2)(b) We suppose that $\tau(0) < 0$ and τ is continuous at 0^+ . Here again, standard considerations in convex analysis show that min dom $(\tau^*) = \tau'_{\mu}(\infty)$ and $[\tau'(\infty), \lim_{q \to 0^+} \tau'(q^-)] \subset dom(\tau^*)$, as well as the continuity and the concavity of τ^* over $[\tau'(\infty), \lim_{q \to 0^+} \tau'(q^-)]$. If $\lim_{q \to 0^+} \tau'(q^-) < \infty$, by using the definition of τ^* one checks that $\tau^*(\alpha) = -\tau(0^+) = -\tau(0) = \tau^*(\infty)$ for all $\alpha \in [\lim_{q \to 0^+} \tau'(q^-), \infty]$. So dom $(\tau^*) = [\tau'(\infty), \infty]$. The continuity of τ^* over dom (τ^*) comes from the fact that $\tau^*(\infty) = -\tau(0)$. The fact that $(\tau^*)^* = \tau$ over \mathbb{R}_+ is a direct consequence of the usual duality between τ and the restriction of τ^* to \mathbb{R} , and the fact that τ is continuous at 0^+ . The equality $(\tau^*)^* = \tau$ over \mathbb{R}_+^* is obvious.

(2)(c) If τ is discontinuous at 0^+ , clearly $\tau'(0^+) = \infty$. It is standard from convex analysis that $\min \operatorname{dom}(\tau^*) = \tau'_{\mu}(\infty)$, and $[\tau'(\infty), \lim_{q \to 0^+} \tau'(q^-)] \subset \operatorname{dom}(\tau^*)$. Moreover, if $\lim_{q \to 0^+} \tau'(q^-) < \infty$, by using the definition of τ^* one checks that $\tau^*(\alpha) = -\tau(0^+) < -\tau(0) = \tau^*(\infty)$ for all $\alpha \in [\lim_{q \to 0^+} \tau'(q^-), \infty)$. Consequently, we have $\operatorname{dom}(\tau^*) = [\tau'(\infty), \infty]$, as well as the concavity and continuity of τ^* over $[\tau'_{\mu}(\infty), \infty)$. By using the usual duality between τ and the restriction of τ^* to \mathbb{R} , we would find that $(\tau^*)^*$ is equal to τ over \mathbb{R}^*_+ and equal to $\tau(0^+)$ at 0. Here, taking into account that $\alpha = \infty \in \operatorname{dom}(\tau^*)$, we find that $(\tau^*)^*(0) = -\tau^*(\infty) = \tau(0)$. Finally, $(\tau^*)^* = \tau$ on \mathbb{R}_+ . The equality $(\tau^*)^* = \tau$ over \mathbb{R}_- is obvious.

(2)(d) It has been proved in the previous lines.

5.2. Proof of Proposition 1.3

(1) Fix $0 \le \alpha < \beta \le \infty$ (the case $\alpha = \beta$ is covered by (1.4) and (1.5)). Without loss of generality we assume that $\underline{f}_{\mu}^{\text{LD}}(\alpha,\beta) > -\infty$, for otherwise one clearly has $E(\mu,\alpha,\beta) = \emptyset$.

We first show that $\dim_H F(\alpha, \beta) \leq \underline{f}_{\mu}^{\text{LD}}(\alpha, \beta)$, where

$$F(\alpha,\beta) = \{ x \in \operatorname{supp}(\mu) : \alpha \le \underline{d}(\mu,x) \le \overline{d}(\mu,x) \le \beta \} \}.$$

Since $E(\mu, \alpha, \beta) \subset F(\alpha, \beta)$, this yields $\dim_H E(\mu, \alpha, \beta) \leq \underline{f}_{\mu}^{\text{LD}}(\alpha, \beta)$.

Fix $\eta > 0$. There exists $\epsilon > 0$ such that for infinitely many r > 0, we have $f_{\mu}(\alpha, \beta, \epsilon, r) \leq \underline{f}_{\mu}^{\text{LD}}(\alpha, \beta) + \eta$. Let $(r_j)_{j\geq 1}$ be a sequence converging to 0 such that for all j we have $f_{\mu}(\alpha, \beta, \epsilon, r_j) \leq \underline{f}_{\mu}^{\text{LD}}(\alpha, \beta) + \eta$.

By definition, we have $F(\alpha, \beta) \subset \bigcup_{N>1} F_N$, where

$$F_N = \bigcap_{0 < r \le 2^{-N}} \Big\{ x \in \operatorname{supp}(\mu) : r^{\beta + \epsilon} \le \mu(B(x, r)) \le r^{\alpha - \epsilon} \Big\}.$$

Fix $N \ge 1$. It follows from the previous line that for any $n \ge N$, there exists $j \ge 1$ such that $r_j \le 2^{-n}$ and we have

$$F_N \subset \left\{ x \in \operatorname{supp}(\mu) : r_j^{\beta + \epsilon} \le \mu(B(x, r_j)) \le r_j^{\alpha - \epsilon} \right\}.$$

It follows from Besicovitch's covering theorem (see [60]) that there exists an integer Q(d) such that, defining $F_j(\epsilon) = \{x \in \operatorname{supp}(\mu) : r_j^{\beta+\epsilon} \leq \mu(B(x,r_j)) \leq r_j^{\alpha-\epsilon}\}$, we can extract from $\{B(x,r_j) : x \in F_j(\epsilon)\}$, Q(d) families \mathcal{F}_k $(1 \leq k \leq Q(d))$ of disjoint balls such that $F_j(\epsilon) \subset \bigcup_{k=1}^{Q(d)} \bigcup_{B \in \mathcal{F}_k} B$.

4° SÉRIE – TOME 48 – 2015 – Nº 6

Now, setting $\gamma = \underline{f}_{\mu}^{\text{LD}}(\alpha, \beta) + 2\eta$, and using the covering of F_N by the balls in $\bigcup_{k=1}^{Q(d)} \mathcal{F}_k$, we see that for j large enough we have

$$\begin{aligned} \mathcal{H}_{2^{-n+1}}^{\gamma}(F_N) &\leq \sum_{k=1}^{Q(d)} \sum_{B \in \mathcal{T}_k} |B|^{\gamma} \\ &\leq Q(d) (\#\mathcal{T}_k) (2r_j)^{\gamma} \leq 2^{\gamma} Q(d) r_j^{-(f_{-\mu}^{\mathrm{LD}}(\alpha,\beta)+\eta)+\gamma} \leq 2^{\gamma} Q(d) 2^{-n\eta} \end{aligned}$$

Letting *n* tend to ∞ yields $\mathscr{H}^{\gamma}(F_N) = 0$, so dim $F_N \leq \gamma$ for all $N \geq 1$, and finally dim $F(\alpha, \beta) \leq \underline{f}_{\mu}^{\text{LD}}(\alpha, \beta) + 2\eta$. Since η was arbitrary, we are done.

Now let us prove that $\dim_H E(\mu, \alpha, \beta) \leq \min(\overline{f}_{\mu}^{\mathrm{LD}}(\alpha), \overline{f}_{\mu}^{\mathrm{LD}}(\beta))$. For $\alpha' \in \mathbb{R} \cup \{\infty\}$, define $G_{\alpha'} = \{x : \exists (n_j) \nearrow \infty : \lim_{j \to \infty} \frac{\log(\mu(B(x, 2^{-n_j})))}{\log(2^{-n_j})} = \alpha'\}$. We have $E(\mu, \alpha, \beta) \subset G_{\alpha} \cap G_{\beta}$. Consequently, the conclusion follows from the fact that $\dim G_{\alpha'} \leq \overline{f}_{\mu}^{\mathrm{LD}}(\alpha')$. Indeed, if $\alpha' < \infty$, fix $\eta > 0$. Then let $\epsilon > 0$ and $r_0 \in \mathbb{N}_+$ such that for all $0 < r \leq r_0$ we have $f(\alpha', \epsilon, r) \leq \overline{f}_{\mu}^{\mathrm{LD}}(\alpha') + \eta$, where

$$f(\alpha',\epsilon,r) = \frac{\log \sup \#\left\{i: r^{\alpha+\epsilon} \le \mu(B(x_i,r)) \le r^{\alpha-\epsilon}\right\}}{-\log(r)}$$

the supremum being taken over the packings of $\operatorname{supp}(\mu)$ by balls of radii equal to r. For each $n \geq 1$ such that $2^{-n} \leq r_0$, we have $G_{\alpha'} \subset \bigcup_{p \geq n} G_{\alpha',p}$, where $G_{\alpha',p} = \{x \in \operatorname{supp}(\mu) : 2^{-p(\alpha'+\epsilon)} \leq \mu(B(x, 2^{-p}) \leq 2^{-p(\alpha'+\epsilon)}\}$. Setting $\gamma = \overline{f}_{\mu}^{\mathrm{LD}}(\alpha') + 2\eta$ and using Besicovitch's covering theorem as above, we get $\mathcal{H}_{2^{-n+1}}^{\gamma}(G_{\alpha'}) \leq 2^{\gamma}Q(d)\sum_{p\geq n} 2^{p(\overline{f}_{\mu}^{\mathrm{LD}}(\alpha')+\eta)-p\gamma}$. Letting n tend to ∞ yields $\mathcal{H}^{\gamma}(G_{\alpha'}) = 0$, hence $\dim_H G_{\alpha'} \leq \overline{f}_{\mu}^{\mathrm{LD}}(\alpha') + 2\eta$, $\eta > 0$ being arbitrary. The case $\alpha' = \infty$ can be treated similarly.

Now we prove that $\dim_P F(\alpha, \beta) \leq f_P(\alpha, \beta) = \sup\{\overline{f}_{\mu}^{\text{LD}}(\alpha') : \alpha' \in [\alpha, \beta]\}$. Suppose first that $\beta < \infty$.

Fix $\eta > 0$, and for each $\alpha' \in [\alpha, \beta]$ fix $\epsilon(\alpha') > 0$ and $r(\alpha') > 0$ such that for all $0 < r < r(\alpha')$, one has $f(\alpha', \epsilon(\alpha'), r) \leq \overline{f}_{\mu}^{\text{LD}}(\alpha') + \eta$. Then let $\alpha'_1, \ldots, \alpha'_k$ such that $[\alpha, \beta] \subset \bigcup_{i=1}^k B(\alpha'_i, \epsilon(\alpha'_i))$.

Set $\epsilon = \min\{\epsilon(\alpha'_i) : 1 \le i \le k\}$ and $r_\eta = \min\{r(\alpha'_i) : 1 \le i \le k\}$.

Fix $N \ge 1$ and define F_N as above. We have

$$F_N \subset \bigcap_{0 < 2^{-p} < \min(2^{-N}, r_\eta)} \bigcup_{i=1}^k \Big\{ x \in \operatorname{supp}(\mu) : 2^{-p(\alpha'_i + \epsilon(\alpha'_i))} \le \mu(B(x, 2^{-p})) \le 2^{-p(\alpha'_i - \epsilon(\alpha'_i))} \Big\}.$$

Let $A \subset [0,1]^d$. Let $n \in \mathbb{N}$ such that $2^{-n} \leq \min(2^{-N}, r_\eta)$ and $\{B(x_i, r_i)\}$ a 2^{-n} -packing of $A \cap F_N$. For each integer $p \geq n+1$, set $S_p = \{i : 2^{-p} < r_i \leq 2^{-p+1}\}$. The balls in $\{B(x_i, 2^{-p}) : i \in S_p, 2^{-p(\alpha'_i + \epsilon(\alpha'_i))} \leq \mu(B(x, 2^{-p})) \leq 2^{-p(\alpha'_i - \epsilon(\alpha'_i))}\}$ form a 2^{-p} -packing of supp(μ) of cardinality less than $2^{p(\overline{f}_{\mu}^{\mathrm{LD}}(\alpha'_i) + \eta)}$, so $\#S_p \leq k2^{p(f_P(\alpha,\beta) + \eta)}$.

Let $\gamma > f_P(\alpha, \beta) + 2\eta$. We have

$$\sum_{i} (2r_i)^{\gamma} \leq \sum_{p \geq n} \sum_{i \in S_p} (2 \cdot 2^{-p+1})^{\gamma} \leq 4^{\gamma} \sum_{p \geq n} (\#S_p) 2^{-p\gamma}$$
$$\leq 4^{\gamma} k \sum_{p \geq n} 2^{p(f(\alpha,\beta)+\eta-\gamma)} \leq 4^{\gamma} k \sum_{p \geq n} 2^{-p\eta},$$

the upper bound being independent of the 2^{-n} -packing $\{B(x_i, r_i)\}$, and going to 0 as $n \to \infty$. It follows that the pre-packing measure $\overline{P}^{\gamma}(A \cap F_N)$ equals 0 for all A, hence $\mathscr{P}^{\gamma}(F_N)=0$, so $\dim_P(F_N) \leq \gamma$. Since η is arbitrary this yields $\dim_P F_N \leq f_P(\alpha, \beta)$, hence $\dim_P F(\alpha, \beta) \leq f_P(\alpha, \beta)$, and finally $\dim_P E(\mu, \alpha, \beta) \leq f_P(\alpha, \beta)$.

If $\beta = \infty$, take $\beta_1 > 0$ and $r(\beta)$ such that for all $0 < r \le r(\infty)$, $f(\beta_1, \infty, r) \le \overline{f}_{\mu}^{\text{LD}}(\infty) + \eta$, where

$$f(\beta_1, \infty, r) = \frac{\log \sup \#\left\{i : \mu(B(x_i, r)) \le r^{\beta_1}\right\}}{-\log(r)}$$

the supremum being taken over the packings of $\operatorname{supp}(\mu)$ by balls of radii equal to r. Then use a covering of $[\alpha, \beta_1]$ by intervals $B(\alpha'_i, \epsilon(\alpha'_i))$ as above. The argument to conclude is the same as above, except that we have to bound the cardinality of $\{B(x_i, 2^{-p}) : i \in S_p, \mu(B(x, 2^{-p})) \le 2^{-p\beta_1}\}$ by $2^{p(\overline{f}_{\mu}^{\mathrm{LD}}(\infty) + \eta)}$.

(2) The result for packing dimensions easily follows from the inclusions $\underline{E}(\mu, \alpha) \subset \bigcup_{\beta \geq \alpha} \uparrow F(\alpha, \beta)$ and $\overline{E}(\mu, \alpha) \subset \bigcup_{0 \leq \beta \leq \alpha} \uparrow F(\beta, \alpha)$, and the previous estimates for $\dim_P F(\alpha, \beta)$.

The upper bound for dim_H $\underline{E}(\mu, \alpha)$ is obtained by writing $\underline{E}(\mu, \alpha) = \bigcup_{\beta \geq \alpha} \uparrow \{x \in \operatorname{supp}(\mu) : \underline{d}(\mu, x) = \alpha, \overline{d}(\mu, x) \leq \beta\}$. If $\alpha = \infty$, there is nothing to prove, for $\underline{E}(\mu, \infty) = E(\mu, \infty)$. Suppose $\alpha < \infty$. Then, for each $\beta < \infty$, due to the estimates achieved to find an upper bound for dim_H $E(\mu, \alpha, \beta)$, given $\eta > 0$, for each $\alpha' \in [\alpha, \beta]$ one can fix $\epsilon(\alpha') > 0$ such that dim_H $\{x \in \operatorname{supp}(\mu) : \underline{d}(\mu, x) = \alpha, \alpha' - \epsilon(\alpha') \leq \overline{d}(\mu, x) \leq \alpha' + \epsilon(\alpha')\} \leq \min\{f_{\mu}^{\mathrm{LD}}(\alpha), \overline{f}_{\mu}^{\mathrm{LD}}(\alpha', \alpha')\} + \eta = f_{H}(\alpha, \alpha') + \eta$. Since we can cover $[\alpha, \beta]$ by finitely many intervals of the form $[\alpha' - \epsilon(\alpha'), \alpha' + \epsilon(\alpha')], \alpha' \in [\alpha, \beta]$, we get dim_H $\{x \in \operatorname{supp}(\mu) : \underline{d}(\mu, x) = \alpha, \overline{d}(\mu, x) \leq \beta\} \leq \sup\{f_{H}(\alpha, \alpha') : \alpha' \in [\alpha, \beta]\} + \eta$ for any $\eta > 0$, hence dim_H $\{x \in \operatorname{supp}(\mu) : \underline{d}(\mu, x) = \alpha, \overline{d}(\mu, x) \leq \beta\} \leq \sup\{f_{H}(\alpha, \infty), \operatorname{writing} \underline{E}(\mu, \alpha) = E(\mu, \alpha, \infty) \bigcup \bigcup_{\infty > \beta \geq \alpha} \uparrow \{x \in \operatorname{supp}(\mu) : \underline{d}(\mu, x) = \alpha, \overline{d}(\mu, x) \leq \beta\}$ and using the previous estimates for the Hausdorff dimensions yields dim_H $\underline{E}(\mu, \alpha) \leq \sup\{f_{H}(\alpha, \beta) : \beta \geq \alpha\}$.

The upper bound for dim_H $\overline{E}(\mu, \alpha)$ is obtained by using similar arguments.

5.3. Proof of some inequalities in (1.4) and (1.5)

We justify the inequalities $f^H_{\mu}(\alpha) \leq \underline{f}^{\text{LD}}_{\mu}(\alpha) \leq \overline{\tau}^*_{\mu}(\alpha)$ and $\dim_P E(\mu, \alpha) \leq \overline{f}^{\text{LD}}_{\mu}(\alpha)$ for $\alpha \in \mathbb{R}_+ \cup \{\infty\}$, and $\overline{f}^{\text{LD}}_{\mu}(\infty) \leq \tau^*_{\mu}(\infty)$.

Let $\alpha \in \mathbb{R}_+ \cup \{\infty\}$. Suppose that $E(\mu, \alpha) \neq \emptyset$. The inequality $f^H_{\mu}(\alpha) \leq \underline{f}^{\text{LD}}_{\mu}(\alpha)$ is a special case of the upper bound established for $\dim_H F(\alpha, \beta)$ in Section 5.2.

Similarly, the inequality $\dim_P E(\mu, \alpha) \leq \overline{f}_{\mu}^{\text{LD}}(\alpha)$ follows from lines similar to those used to bound $\dim_P E(\mu, \alpha, \beta)$ in Section 5.2. Also, by Proposition 1.1(2), if $\operatorname{dom}(\tau_{\mu}) = \mathbb{R}$, then

1504

one has $\overline{f}_{\mu}^{\text{LD}}(\infty) = \tau_{\mu}^{*}(\infty) = -\infty$; otherwise $\overline{\dim}_{B}(\operatorname{supp}(\mu)) = -\tau_{\mu}(0) = \tau_{\mu}^{*}(\infty)$, and by definition we have $\overline{f}_{\mu}^{\text{LD}}(\infty) \leq \overline{\dim}_{B}(\operatorname{supp}(\mu))$. In any case, $\dim_{P} E(\mu, \infty) \leq \overline{f}_{\mu}^{\text{LD}}(\infty) \leq \tau_{\mu}^{*}(\infty)$.

To prove that $\underline{f}_{\mu}^{\text{LD}}(\alpha) \leq \overline{\tau}_{\mu}^{*}(\alpha)$, we assume without loss of generality that $\underline{f}_{\mu}^{\text{LD}}(\alpha) > -\infty$, hence $\underline{f}_{\mu}^{\text{LD}}(\alpha) \geq 0$. The case $\alpha = \infty$ then occurs only if $\overline{\tau}_{\mu}(q) = -\infty$ if q < 0 (same proof as when $\tau_{\mu}(q) = -\infty$ for q < 0); then it is direct that $\underline{f}_{\mu}^{\text{LD}}(\infty) \leq -\overline{\tau}_{\mu}(0) = \overline{\tau}_{\mu}^{*}(\infty)$. Suppose now that $\alpha < \infty$. It is enough to prove that $\overline{\tau}_{\mu}(q) \leq (\underline{f}_{\mu}^{\text{LD}})^{*}(q)$ for all $q \in \mathbb{R}$. Then the result follows by taking the Legendre-transform and using (with $h = \underline{f}_{\mu}^{\text{LD}}$) the general inequality $(h^{*})^{*} \geq h$ valid for any function whose domain is not empty. Fix $q \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $\epsilon > 0$. If $\{B(x_{i}, r)\}$ is a packing of $\text{supp}(\mu)$ by disjoint balls, we have

$$\sum_{i} \mu(B(x_i, r))^q \ge (\#\{i : r^{\beta + \epsilon} \le \mu(B(x_i, r)) \le r^{\beta - \epsilon}\}) \cdot \begin{cases} r^{q(\beta - \epsilon)} & \text{if } q \ge 0\\ r^{q(\beta + \epsilon)} & \text{otherwise} \end{cases}$$

Taking the supremum over the packings, dividing by $\log(r)$, and taking the lim sup as $r \to 0^+$ yields $\overline{\tau}_{\mu}(q) \leq q(\beta \mp \epsilon) - \liminf_{r \to 0^+} f(\beta, \epsilon, r)$, and taking the limit as $\epsilon \to 0^+$ gives $\overline{\tau}_{\mu}(q) \leq q\beta - \underline{f}_{\mu}^{\text{LD}}(\beta)$ for all β , hence $\overline{\tau}_{\mu}(q) \leq (\underline{f}_{\mu}^{\text{LD}})^*(q)$.

6. Dimensions of measures and mass distribution principle

Given $\nu \in \mathcal{M}_c^+(\mathbb{R}^d)$, if $\underline{d}(\nu, x)$ (resp. $\overline{d}(\nu, x)$) takes the same value \underline{D} (resp. \overline{D}) at ν -almost every x, then $\dim_H \nu$ (resp. $\dim_P \nu$) stands for the Hausdorff (resp. packing) dimension of the measure ν , defined as the number \underline{D} (resp. \overline{D}). Then, $\nu(E) > 0$ implies $\dim_H E \ge \underline{D}$ (resp. $\dim_P E \ge \overline{D}$) for any Borel set E. This is the mass distribution principle we use to get lower bounds for the Hausdorff and packing dimensions of the level sets studied in this paper (about mass distribution principle and dimensions of measures, see the accounts proposed in [33, Section 4.2], [70, Ch. 2] and [44] (other possible references being [20, Section 14], [82], [26, 27] and in connection with multifractal formalism [43], [62] and [65]).

BIBLIOGRAPHY

- M. ARBEITER, N. PATZSCHKE, Random self-similar multifractals, Math. Nachr. 181 (1996), 5–42.
- [2] N. ATTIA, J. BARRAL, Hausdorff and packing spectra, large deviations, and free energy for branching random walks in \mathbb{R}^d , *Comm. Math. Phys.* **331** (2014), 139–187.
- [3] V. AVERSA, C. BANDT, The multifractal spectrum of discrete measures, *Acta Univ. Carolin. Math. Phys.* **31** (1990), 5–8.
- [4] I. S. BAEK, L. OLSEN, N. SNIGIREVA, Divergence points of self-similar measures and packing dimension, Adv. Math. 214 (2007), 267–287.
- [5] J. BARRAL, Continuity of the multifractal spectrum of a random statistically self-similar measure, J. Theoret. Probab. 13 (2000), 1027–1060.
- [6] J. BARRAL, D.-J. FENG, Weighted thermodynamic formalism on subshifts and applications, Asian J. Math. 16 (2012), 319–352.
- [7] J. BARRAL, D.-J. FENG, Multifractal formalism for almost all self-affine measures, Comm. Math. Phys. 318 (2013), 473–504.
- [8] J. BARRAL, B. B. MANDELBROT, Multifractal products of cylindrical pulses, Probab. Theory Related Fields 124 (2002), 409–430.
- [9] J. BARRAL, B. B. MANDELBROT, Non-degeneracy, moments, dimension, and multifractal analysis for random multiplicative measures (Random multiplicative multifractal measures. II), in *Fractal geometry and applications: a jubilee of Benoît Mandelbrot, Part 2*, Proc. Sympos. Pure Math. **72**, Amer. Math. Soc., Providence, RI, 2004, 17–52.
- [10] J. BARRAL, M. MENSI, Gibbs measures on self-affine Sierpiński carpets and their singularity spectrum, *Ergodic Theory Dynam. Systems* 27 (2007), 1419–1443.
- [11] J. BARRAL, R. RHODES, V. VARGAS, Limiting laws of supercritical branching random walks, C. R. Math. Acad. Sci. Paris 350 (2012), 535–538.
- [12] J. BARRAL, S. SEURET, The singularity spectrum of Lévy processes in multifractal time, Adv. Math. 214 (2007), 437–468.
- [13] J. BARRAL, S. SEURET, The singularity spectrum of the inverse of cookie-cutters, Ergodic Theory Dynam. Systems 29 (2009), 1075–1095.
- [14] L. BARREIRA, J. SCHMELING, Sets of "non-typical" points have full topological entropy and full Hausdorff dimension, *Israel J. Math.* **116** (2000), 29–70.
- [15] A. BATAKIS, B. TESTUD, Multifractal analysis of inhomogeneous Bernoulli products, J. Stat. Phys. 142 (2011), 1105–1120.
- [16] F. BAYART, Multifractal spectra of typical and prevalent measures, Nonlinearity 26 (2013), 353–367.
- [17] F. BEN NASR, I. BHOURI, Y. HEURTEAUX, The validity of the multifractal formalism: results and examples, *Adv. Math.* 165 (2002), 264–284.
- [18] F. BEN NASR, J. PEYRIÈRE, Revisiting the multifractal analysis of measures, *Rev. Mat. Iberoam.* 29 (2013), 315–328.
- [19] F. BEN NASR, Analyse multifractale de mesures, C. R. Acad. Sci. Paris Sér. I Math. 319 (1994), 807–810.

4° SÉRIE - TOME 48 - 2015 - Nº 6

- [20] P. BILLINGSLEY, Ergodic theory and information, John Wiley & Sons, Inc., 1965.
- [21] G. BROWN, G. MICHON, J. PEYRIÈRE, On the multifractal analysis of measures, J. Statist. Phys. 66 (1992), 775–790.
- [22] Z. BUCZOLICH, J. NAGY, Hölder spectrum of typical monotone continuous functions, *Real Anal. Exchange* 26 (2000/01), 133–156.
- [23] Z. BUCZOLICH, S. SEURET, Typical Borel measures on [0,1]^d satisfy a multifractal formalism, *Nonlinearity* 23 (2010), 2905–2918.
- [24] Z. BUCZOLICH, S. SEURET, Measures and functions with prescribed homogeneous multifractal spectrum, J. Fractal Geom. 1 (2014), 295–333.
- [25] P. COLLET, J. L. LEBOWITZ, A. PORZIO, The dimension spectrum of some dynamical systems, in *Proceedings of the symposium on statistical mechanics of phase* transitions—mathematical and physical aspects (Trebon, 1986), 47, 1987, 609–644.
- [26] C. D. CUTLER, The Hausdorff dimension distribution of finite measures in Euclidean space, *Canad. J. Math.* 38 (1986), 1459–1484.
- [27] C. D. CUTLER, Measure disintegrations with respect to σ-stable monotone indices and the pointwise representation of packing dimension, *Rend. Circ. Mat. Palermo (2) Suppl.* 28 (1992), 319–339.
- [28] A. DEMBO, O. ZEITOUNI, Large deviations techniques and applications, second ed., Applications of Mathematics 38, Springer, 1998.
- [29] G. A. EDGAR, R. D. MAULDIN, Multifractal decompositions of digraph recursive fractals, Proc. London Math. Soc. 65 (1992), 604–628.
- [30] K. FALCONER, The multifractal spectrum of statistically self-similar measures, J. Theoret. Probab. 7 (1994), 681–702.
- [31] K. FALCONER, Generalized dimensions of measures on self-affine sets, *Nonlinearity* 12 (1999), 877–891.
- [32] K. FALCONER, Representation of families of sets by measures, dimension spectra and Diophantine approximation, *Math. Proc. Cambridge Philos. Soc.* 128 (2000), 111– 121.
- [33] K. FALCONER, Fractal geometry, third ed., John Wiley & Sons, Ltd., Chichester, 2014.
- [34] D.-J. FENG, The limited Rademacher functions and Bernoulli convolutions associated with Pisot numbers, Adv. Math. 195 (2005), 24–101.
- [35] D.-J. FENG, Gibbs properties of self-conformal measures and the multifractal formalism, *Ergodic Theory Dynam. Systems* 27 (2007), 787–812.
- [36] D.-J. FENG, Multifractal analysis of Bernoulli convolutions associated with Salem numbers, Adv. Math. 229 (2012), 3052–3077.
- [37] D.-J. FENG, K.-S. LAU, Multifractal formalism for self-similar measures with weak separation condition, J. Math. Pures Appl. 92 (2009), 407–428.
- [38] D.-J. FENG, E. OLIVIER, Multifractal analysis of weak Gibbs measures and phase transition—application to some Bernoulli convolutions, *Ergodic Theory Dynam. Systems* 23 (2003), 1751–1784.
- [39] D.-J. FENG, J. WU, The Hausdorff dimension of recurrent sets in symbolic spaces, Nonlinearity 14 (2001), 81–85.

- [40] U. FRISCH, G. PARISI, Fully developped turbulence and intermittency in turbulence, and predictability in geophysical fluid dynamics and climate dynamics, in *International school of Physics "Enrico Fermi"* (M. Ghil, ed.), North Holland, 84–88.
- [41] T. C. HALSEY, M. H. JENSEN, L. P. KADANOFF, I. PROCACCIA, B. I. SHRAIMAN, Fractal measures and their singularities: the characterization of strange sets, *Phys. Rev. A* 33 (1986), 1141–1151.
- [42] H. G. E. HENTSCHEL, I. PROCACCIA, The infinite number of generalized dimensions of fractals and strange attractors, *Phys. D* 8 (1983), 435–444.
- [43] Y. HEURTEAUX, Estimations de la dimension inférieure et de la dimension supérieure des mesures, Ann. Inst. H. Poincaré Probab. Statist. 34 (1998), 309–338.
- [44] Y. HEURTEAUX, Dimension of measures: the probabilistic approach, Publ. Mat. 51 (2007), 243–290.
- [45] R. HOLLEY, E. C. WAYMIRE, Multifractal dimensions and scaling exponents for strongly bounded random cascades, Ann. Appl. Probab. 2 (1992), 819–845.
- [46] G. IOMMI, Multifractal analysis for countable Markov shifts, *Ergodic Theory Dynam. Systems* **25** (2005), 1881–1907.
- [47] S. JAFFARD, Old friends revisited: the multifractal nature of some classical functions, J. Fourier Anal. Appl. 3 (1997), 1–22.
- [48] S. JAFFARD, The multifractal nature of Lévy processes, *Probab. Theory Related Fields* 114 (1999), 207–227.
- [49] T. JORDAN, M. RAMS, Multifractal analysis of weak Gibbs measures for non-uniformly expanding C¹ maps, Ergodic Theory Dynam. Systems 31 (2011), 143–164.
- [50] M. KESSEBÖHMER, Large deviation for weak Gibbs measures and multifractal spectra, *Nonlinearity* 14 (2001), 395–409.
- [51] J. F. KING, The singularity spectrum for general Sierpiński carpets, Adv. Math. 116 (1995), 1–11.
- [52] K.-S. LAU, S.-M. NGAI, Multifractal measures and a weak separation condition, Adv. Math. 141 (1999), 45–96.
- [53] G. F. LAWLER, The frontier of a Brownian path is multifractal, preprint http://www. math.duke.edu/~jose/multifrac.ps, 1998.
- [54] F. LEDRAPPIER, A. PORZIO, On the multifractal analysis of Bernoulli convolutions. I. Large-deviation results, J. Statist. Phys. 82 (1996), 367–395.
- [55] F. LEDRAPPIER, A. PORZIO, On the multifractal analysis of Bernoulli convolutions. II. Dimensions, J. Statist. Phys. 82 (1996), 397–420.
- [56] J. LÉVY VÉHEL, C. TRICOT, On various multifractal spectra, in *Fractal geometry and stochastics III*, Progr. Probab. 57, Birkhäuser, 2004, 23–42.
- [57] J. LÉVY VÉHEL, R. VOJAK, Multifractal analysis of Choquet capacities, Adv. in Appl. Math. 20 (1998), 1–43.
- [58] N. G. MAKAROV, Fine structure of harmonic measure, Algebra i Analiz 10 (1998), 1–62.
- [59] B. B. MANDELBROT, C. J. G. EVERTSZ, Y. HAYAKAWA, Exactly self-similar left sided multifractal measures, *Phys. Rev. A* 42 (1990), 4528–4536.
- [60] P. MATTILA, Geometry of sets and measures in Euclidean spaces, Cambridge Studies in Advanced Math. 44, Cambridge Univ. Press, Cambridge, 1995.

 $4^{\,e}\,\,S\acute{E}RIE-TOME\,48-2015-N^o\,\,6$

- [61] G. M. MOLCHAN, Scaling exponents and multifractal dimensions for independent random cascades, *Comm. Math. Phys.* 179 (1996), 681–702.
- [62] S.-M. NGAI, A dimension result arising from the L^q-spectrum of a measure, Proc. Amer. Math. Soc. 125 (1997), 2943–2951.
- [63] L. OLSEN, A multifractal formalism, Adv. Math. 116 (1995), 82–196.
- [64] L. OLSEN, Self-affine multifractal Sierpinski sponges in \mathbb{R}^d , *Pacific J. Math.* 183 (1998), 143–199.
- [65] L. OLSEN, Dimension inequalities of multifractal Hausdorff measures and multifractal packing measures, *Math. Scand.* 86 (2000), 109–129.
- [66] L. OLSEN, Multifractal analysis of divergence points of deformed measure theoretical Birkhoff averages, J. Math. Pures Appl. 82 (2003), 1591–1649.
- [67] L. OLSEN, N. SNIGIREVA, Multifractal spectra of in-homogenous self-similar measures, *Indiana Univ. Math. J.* 57 (2008), 1789–1843.
- [68] N. PATZSCHKE, Self-conformal multifractal measures, Adv. in Appl. Math. 19 (1997), 486–513.
- [69] Y. PESIN, H. WEISS, A multifractal analysis of equilibrium measures for conformal expanding maps and Moran-like geometric constructions, J. Statist. Phys. 86 (1997), 233–275.
- [70] Y. B. PESIN, *Dimension theory in dynamical systems*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1997.
- [71] J. PEYRIÈRE, A vectorial multifractal formalism, in *Fractal geometry and applications: a jubilee of Benoît Mandelbrot, Part 2*, Proc. Sympos. Pure Math. 72, Amer. Math. Soc., Providence, RI, 2004, 217–230.
- [72] A. PORZIO, The dimension spectrum of Axiom A attractors, J. Statist. Phys. 58 (1990), 923–937.
- [73] D. A. RAND, The singularity spectrum $f(\alpha)$ for cookie-cutters, *Ergodic Theory Dynam*. Systems **9** (1989), 527–541.
- [74] R. RHODES, V. VARGAS, Gaussian multiplicative chaos and applications: a review, *Probab. Surv.* 11 (2014), 315–392.
- [75] R. H. RIEDI, Multifractal processes, in *Theory and applications of long-range depen*dence, Birkhäuser, 2003, 625–716.
- [76] R. H. RIEDI, B. B. MANDELBROT, Multifractal formalism for infinite multinomial measures, Adv. in Appl. Math. 16 (1995), 132–150.
- [77] R. T. ROCKAFELLAR, Convex analysis, Princeton Mathematical Series 28, Princeton Univ. Press, Princeton, N.J., 1970.
- [78] S. SHEN, Multifractal analysis of some inhomogeneous multinomial measures with distinct analytic Olsen's *b* and *B* functions, *J. Stat. Phys.* **159** (2015), 1216–1235.
- [79] P. SHMERKIN, A modified multifractal formalism for a class of self-similar measures with overlap, *Asian J. Math.* **9** (2005), 323–348.
- [80] B. TESTUD, Mesures quasi-Bernoulli au sens faible: résultats et exemples, Ann. Inst. H. Poincaré Probab. Statist. 42 (2006), 1–35.
- [81] C. J. TRICOT, Two definitions of fractional dimension, *Math. Proc. Cambridge Philos.* Soc. 91 (1982), 57–74.

- [82] L. S. YOUNG, Dimension, entropy and Lyapunov exponents, *Ergodic Theory Dynam*. Systems 2 (1982), 109–124.
- [83] X. ZHOU, E. CHEN, Packing dimensions of the divergence points of self-similar measures with OSC, *Monatsh. Math.* 172 (2013), 233–246.

(Manuscrit reçu le 18 novembre 2013; accepté, après révision, le 24 septembre 2014.)

Julien BARRAL LAGA (UMR 7539) Département de Mathématiques Institut Galilée Université Paris 13 et Paris-Sorbonne-Cité 99 avenue Jean-Baptiste Clément 93430 Villetaneuse, France E-mail: barral@math.univ-paris13.fr