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# Julien BARRAL <br> Inverse problems in multifractal analysis of measures 

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#### Abstract

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# INVERSE PROBLEMS IN MULTIFRACTAL ANALYSIS OF MEASURES 

BY Julien BARRAL


#### Abstract

Multifractal formalism is designed to describe the distribution at small scales of the elements of $\mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$, the set of positive, finite and compactly supported Borel measures on $\mathbb{R}^{d}$. It is valid for such a measure $\mu$ when its Hausdorff spectrum is the upper semi-continuous function given by the concave Legendre-Fenchel transform of the free energy function $\tau_{\mu}$ associated with $\mu$; this is the case for fundamental classes of exactly dimensional measures.

For any function $\tau$ candidate to be the free energy function of some $\mu \in \mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$, we construct such a measure, exactly dimensional, and obeying the multifractal formalism. This result is extended to a refined formalism considering jointly Hausdorff and packing spectra. Also, for any upper semicontinuous function candidate to be the lower Hausdorff spectrum of some exactly dimensional $\mu \in \mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$, we construct such a measure.

Résumé. - Le formalisme multifractal est un cadre adapté pour décrire la distribution aux petites échelles des mesures de Borel finies positives à support compact dans $\mathbb{R}^{d}$, dont l'ensemble est ici noté $\mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$. Il est dit valide pour une mesure $\mu$ lorsque son spectre de Hausdorff est la fonction semi-continue supérieurement obtenue comme transformée de Legendre-Fenchel concave de sa fonction d'énergie libre $\tau_{\mu}$; c'est le cas pour certaines classes fondamentales de mesures exactement dimensionnelles.

Pour toute fonction $\tau$ candidate à être la fonction d'énergie libre d'un élément $\mu$ de $\mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$, nous construisons une telle mesure, exactement dimensionnelle, et validant le formalisme. Ce résultat s'étend à un formalisme plus fin considérant simultanément spectres de Hausdorff et de packing. D'autre part, pour toute fonction semi-continue supérieurement candidate à être le spectre de Hausdorff inférieur d'une mesure exactement dimensionnelle, nous construisons une telle mesure.


## 1. Introduction and main statements

### 1.1. Inverse problems in multifractal analysis of measures

Let $\mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$ stand for the set of compactly supported Borel positive and finite measures on $\mathbb{R}^{d}(d \geq 1)$, and for $\mu \in \mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$ denote by $\operatorname{supp}(\mu)$ the topological support of $\mu$ (i.e.,

[^0]the compact set obtained as the complement of those points $x$ for which $\mu(B(x, r))=0$ for some $r>0$, where $B(x, r)$ stands for the closed ball of radius $r$ centered at $x)$.

The upper and lower box dimensions of a bounded set $E \subset \mathbb{R}^{d}$ will be denoted $\overline{\operatorname{dim}}_{B} E$ and $\underline{\operatorname{dim}}_{B} E$ respectively, and its Hausdorff and packing dimensions will be denoted by $\operatorname{dim}_{H} E$ and $\operatorname{dim}_{P} E$ respectively (see [33, 60, 70, 81] for introductions to dimension theory).

Multifractal analysis is a natural framework to finely describe geometrically the heterogeneity in the distribution at small scales of the elements of $\mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$. Specifically, if $\mu \in \mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$, this heterogeneity can be described via the lower and upper local dimensions of $\mu$, namely

$$
\underline{d}(\mu, x)=\liminf _{r \rightarrow 0^{+}} \frac{\log (\mu(B(x, r)))}{\log (r)} \quad \text { and } \quad \bar{d}(\mu, x)=\limsup _{r \rightarrow 0^{+}} \frac{\log (\mu(B(x, r)))}{\log (r)}
$$

and the level sets

$$
E(\mu, \alpha, \beta)=\{x \in \operatorname{supp}(\mu): \underline{d}(\mu, x)=\alpha, \bar{d}(\mu, x)=\beta\} \quad(\alpha \leq \beta \in \mathbb{R} \cup\{\infty\})
$$

which form a partition of $\operatorname{supp}(\mu)($ notice that $E(\mu, \alpha, \beta)=\varnothing$ whenever $\alpha<0)$. The sets

$$
\underline{E}(\mu, \alpha)=\{x \in \operatorname{supp}(\mu): \underline{d}(\mu, x)=\alpha\}, \quad \bar{E}(\mu, \alpha)=\{x \in \operatorname{supp}(\mu): \bar{d}(\mu, x)=\alpha\}
$$

and

$$
E(\mu, \alpha)=\underline{E}(\mu, \alpha) \cap \bar{E}(\mu, \alpha)=E(\mu, \alpha, \alpha) \quad(\alpha \in \mathbb{R} \cup\{\infty\})
$$

are also very natural, and the most studied in the literature (although the sets defined above are empty if $\alpha<0$ because $\mu$ is a bounded function of Borel sets, it is convenient to include negative values of $\alpha$ in connection with the using along the paper of the Legendre-Fenchel transform of functions defined on $\mathbb{R}$ or $\mathbb{R} \cup\{\infty\}$ ).

The lower Hausdorff spectrum of $\mu$ is the mapping defined as

$$
\underline{f}_{\mu}^{H}: \alpha \in \mathbb{R} \cup\{\infty\} \mapsto \operatorname{dim}_{H} \underline{E}(\mu, \alpha)
$$

with the convention that $\operatorname{dim}_{H} \varnothing=-\infty$, so that $\underline{f}_{\mu}^{H}(\alpha)=-\infty$ if $\alpha<0$. This spectrum provides a geometric hierarchy between the sets $\underline{E}(\mu, \alpha)$, which partition the support of $\mu$. Here, the lower local dimension is emphasized for it provides at any point the best pointwise Hölder control one can have on the measure $\mu$ at small scales. However, the upper local dimension is of course of interest, and much attention is paid in general to the sets $E(\mu, \alpha)$ of points at which one has an exact local dimension $\underline{d}(\mu, x)=\bar{d}(\mu, x)$, especially when studying ergodic measures in the context of hyperbolic and more generally non uniformly hyperbolic dynamical systems.

The Hausdorff spectrum of $\mu$ is the mapping defined as

$$
f_{\mu}^{H}: \alpha \in \mathbb{R} \cup\{\infty\} \mapsto \operatorname{dim}_{H} E(\mu, \alpha)
$$

Inspired by the observations made by physicists of turbulence and statistical mechanics [42, 40, 41], mathematicians derived, and in many situations justified the heuristic claiming that for a measure possessing a self-conformal like property, its Hausdorff spectrum should be obtained as the Legendre transform of a kind of free energy function, called $L^{q}$-spectrum. This gave birth to an abundant literature on the so-called multifractal formalisms [33, 21, 19, $63,70,52,17,71,56]$, which aim at linking the asymptotic statistical properties of a given measure with its fine geometric properties.
$4^{\mathrm{e}}$ SÉRIE - TOME 48 - 2015 - $\mathrm{N}^{\mathrm{o}} 6$

To be more specific we need some definitions. Given $I \in\{\mathbb{R}, \mathbb{R} \cup\{\infty\}\}$ and a fonction $f: I \rightarrow \mathbb{R} \cup\{-\infty\}$, the domain of $f$ is defined as $\operatorname{dom}(f)=\{x \in I: f(x)>-\infty\}$.

Let $\tau: \mathbb{R} \rightarrow \mathbb{R} \cup\{-\infty\}$. If $\operatorname{dom}(\tau) \neq \varnothing$, the concave Legendre-Fenchel transform, or concave conjugate function, of $\tau$ is the upper-semi continuous concave function defined as $\tau^{*}: \alpha \in \mathbb{R} \mapsto \inf \{\alpha q-\tau(q): q \in \operatorname{dom}(\tau)\}$ (see [77]). We will need a slight extension of this definition.

If $\tau: \mathbb{R} \rightarrow \mathbb{R} \cup\{-\infty\}, \operatorname{dom}(\tau) \neq \varnothing$, and $0 \in \operatorname{dom}(\tau)$, we define its (extended) concave Legendre-Fenchel transform as

$$
\tau^{*}: \alpha \in \mathbb{R} \cup\{\infty\} \mapsto \begin{cases}\inf \{\alpha q-\tau(q): q \in \operatorname{dom}(\tau)\} & \text { if } \alpha \in \mathbb{R} \\ \inf \left\{\alpha q-\tau(q): q \in \operatorname{dom}(\tau) \cap \mathbb{R}_{-}\right\} & \text {if } \alpha=\infty\end{cases}
$$

with the conventions $\infty \times q=-\infty$ if $q<0$ and $\infty \times 0=0$. Consequently, $\infty \in \operatorname{dom}\left(\tau^{*}\right)$ if and only if $0=\min (\operatorname{dom}(\tau))$, and in this case $\tau^{*}(\infty)=-\tau(0)=\max \left(\tau^{*}\right)$. In any case, $\tau^{*}$ is upper semi-continuous over $\operatorname{dom}\left(\tau^{*}\right)$, and concave over the interval $\operatorname{dom}\left(\tau^{*}\right) \backslash\{\infty\}$ (here the notion of upper semi-continuous function is relative to $\mathbb{R} \cup\{\infty\}$ endowed with the topology generated by the open subsets of $\mathbb{R}$ and the sets $(\alpha, \infty) \cup\{\infty\}, \alpha \in \mathbb{R})$.

Now, define the (lower) $L^{q}$-spectrum of $\mu \in \mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$ as

$$
\tau_{\mu}: q \in \mathbb{R} \mapsto \liminf _{r \rightarrow 0^{+}} \frac{\log \sup \left\{\sum_{i} \mu\left(B\left(x_{i}, r\right)\right)^{q}\right\}}{\log (r)},
$$

where the supremum is taken over all the centered packings of $\operatorname{supp}(\mu)$ by closed balls of radius $r$.

By construction, $\tau_{\mu}$ is concave and non decreasing, and

$$
-d \leq \tau_{\mu}(0)=-\overline{\operatorname{dim}}_{B} \operatorname{supp}(\mu) \leq 0=\tau_{\mu}(1)
$$

so that one always has $\mathbb{R}_{+} \subset \operatorname{dom}\left(\tau_{\mu}\right)$; also $\tau_{\mu}^{*}$ takes values in $[0, d] \cup\{-\infty\}$, and $\operatorname{dom}\left(\tau_{\mu}^{*}\right)$ is a closed subinterval of $\mathbb{R}_{+} \cup\{\infty\}$ (see Propositions 1.1 and 1.2).

For $\alpha \in \mathbb{R}$ we always have (see [63, Section 2.7] or [52, Section 3])

$$
\begin{equation*}
f_{\mu}^{H}(\alpha) \leq \underline{f}_{\mu}^{H}(\alpha) \leq \tau_{\mu}^{*}(\alpha) \leq \max \left(\alpha,-\tau_{\mu}(0)\right) \leq \max (\alpha, d) ; \tag{1.1}
\end{equation*}
$$

we also have

$$
f_{\mu}^{H}(\infty) \leq \tau_{\mu}^{*}(\infty)
$$

a dimension equal to $-\infty$ meaning that the set is empty (the second inequality is not standard, and will be proved in Section 5; the inequality $\tau_{\mu}^{*}(\alpha) \leq \max \left(\alpha,-\tau_{\mu}(0)\right)$ is a direct consequence of the definition of $\tau_{\mu}^{*}$ and the fact that $\left.\tau_{\mu}(1)=0\right)$.

We notice that due to (1.1), if $f_{\mu}^{H}(\alpha) \geq \alpha$ at some $\alpha$, then $0 \leq \alpha \leq d$ and $f_{\mu}^{H}(\alpha)=$ $\tau_{\mu}^{*}(\alpha)=\alpha$, so that $\alpha$ is a fixed point of $\tau_{\mu}^{*}$. Moreover, since $\tau_{\mu}(1)=0$ and $\tau_{\mu}$ is concave, the set of fixed points of $\tau_{\mu}^{*}$ is the interval $\left[\tau_{\mu}^{\prime}\left(1^{+}\right), \tau_{\mu}^{\prime}\left(1^{-}\right)\right]$.

We will say that $\mu$ obeys the multifractal formalism at $\alpha \in \mathbb{R} \cup\{\infty\}$ if $\underline{f}_{\mu}^{H}(\alpha)=\tau_{\mu}^{*}(\alpha)$, and that the multifractal formalism holds (globally) for $\mu$ if it holds at any $\alpha \in \mathbb{R} \cup\{\infty\}$.

If $\underline{f}_{\mu}^{H}(\alpha)$ can be replaced by $f_{\mu}^{H}(\alpha)$ in the previous definition, we will say that the multifractal formalism holds strongly, and it is in this form that this formalism has been introduced and studied the most. It turns out that in this case one has

$$
\operatorname{dim}_{H} E(\mu, \alpha)=\operatorname{dim}_{P} E(\mu, \alpha)=\operatorname{dim}_{H} \underline{E}(\mu, \alpha)=\operatorname{dim}_{H} \bar{E}(\mu, \alpha)=\tau_{\mu}^{*}(\alpha) .
$$

However, in general, nice families of discrete measures only obey the multifractal formalism associated with the lower Hausdorff spectrum as defined above.

The multifractal formalism turns out to hold globally, or on some non trivial subinterval of $\operatorname{dom}\left(\tau_{\mu}^{*}\right)$, for some important classes of continuous measures possessing (or close to have) self-conformal properties, namely some classes of self-conformal measures (among which some Bernoulli convolutions), Gibbs and weak Gibbs measures on conformal repellers (e.g., the harmonic measure on such a disconnected set) or attractors of Axiom A diffeomorphisms $[25,29,73,72,59,21,54,55,76,69,70,68,58,43,50,38,34,46,79,80,35,37,49$, 36], harmonic measure on the Brownian frontier [53], and scale invariant limits of certain multiplicative chaos $[45,30,61,1,5,8,9,74,2]$; in these cases it also holds strongly. It also holds for scale invariant discrete measures obtained as limits in law of Gibbs measures in the context of random directed polymers [48, 12, 11] (see also [3, 47, 32, 13, 67] for other classes of discrete measures obeying the multifractal formalism). Other examples are special self-affine or Gibbs measures on self-affine Sierpiński carpets [51, 64, 10, 6], or on almost all the attractors of IFS associated with certain families of $d \times d$ invertible matrices with small enough singular values [31, 7], as well as generic probability measures on a compact subset of $\mathbb{R}^{d}[22,23,16]$.

The measures mentioned above share the geometric property to be exactly dimensional, i.e., for such a measure $\mu$, there exists $D \in[0, d]$ such that $\lim _{r \rightarrow 0^{+}} \frac{\log (\mu(B(x, r)))}{\log (r)}=D$, $\mu$-almost everywhere. This implies $f_{\mu}^{H}(D) \geq D$, hence $D \in\left[\tau_{\mu}^{\prime}\left(1^{+}\right), \tau_{\mu}^{\prime}\left(1^{-}\right)\right]$and $\mu$ strongly obeys the multifractal formalism at $D$ by a remark made above. In fact, for any $\mu \in \mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$, for $\mu$-almost every $x$ one has $\tau_{\mu}^{\prime}\left(1^{+}\right) \leq \underline{d}(\mu, x) \leq \bar{d}(\mu, x) \leq \tau_{\mu}^{\prime}\left(1^{-}\right)([62])$, and for most of the continuous measures in the previous references, $\tau_{\mu}^{\prime}(1)$ exists, hence equals $D$; also, $\tau_{\mu}$ is piecewise $C^{1}$, and even analytic in certain cases, a typical example being Gibbs measures associated with Hölder potentials on repellers of $C^{1+\alpha}$ conformal mappings.

Another property of the previous measures is, when they obey globally the multifractal formalism, to be homogeneously multifractal (HM), this meaning that the lower Hausdorff spectrum of the restriction of $\mu$ to any closed ball whose interior intersects $\operatorname{supp}(\mu)$ is equal to the lower Hausdorff spectrum of $\mu$.

In this paper we solve the inverse problem consisting in constructing, for any concave function $\tau$ satisfying the necessary conditions to be the $L^{q}$-spectrum of an element of $\mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$, an exactly dimensional and (HM) measure whose $L^{q}$-spectrum equals $\tau$, and which strongly satisfies the multifractal formalism. More specifically:

Theorem 1.1. - Let $\tau: \mathbb{R} \rightarrow \mathbb{R} \cup\{-\infty\}$ be a concave function satisfying the necessary properties (see Propositon 1.1) to be the $L^{q}$-spectrum of some element of $\mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$. Let $D \in$ $\left[\tau^{\prime}\left(1^{+}\right), \tau^{\prime}\left(1^{-}\right)\right]$. There exists an $(H M)$ measure $\mu \in \mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$, exactly dimensional with dimension $D$, and which strongly satisfies the multifractal formalism with $\tau_{\mu}=\tau$.

Theorem 1.1 will be obtained as a consequence of more general statements which also describe the Hausdorff and packing dimensions of the sets $E(\mu, \alpha, \beta)$ (Theorem 1.3 and Corollary 1.1 of Section 1.2 ). We will also study the inverse problem associated with a finer multifractal formalism designed to describe the more general situation where the Hausdorff spectrum $f_{\mu}^{H}$ and the packing spectrum $f_{\mu}^{P}: \alpha \mapsto \operatorname{dim}_{P} E(\mu, \alpha)$ differ (Theorem 1.4 and

[^1]Corollary 1.2 of Section 1.2). As a by product of these results new multifractal behaviors are exhibited.

In general, $\operatorname{dom}\left(\underline{f}_{\mu}^{H}\right)=\{\alpha \in \mathbb{R} \cup\{\infty\}: \underline{E}(\mu, \alpha) \neq \varnothing\}$ is not necessarily a closed subinterval of $[0, \infty]$, and even when it is the case, the restriction of $\underline{f}_{\mu}^{H}$ to $\operatorname{dom}\left(\underline{f}_{\mu}^{H}\right) \cap \mathbb{R}_{+}$ is not necessarily concave. Consequently, we also study the inverse problem consisting in associating to a function $f: \mathbb{R} \cup\{\infty\} \rightarrow[0, d] \cup\{-\infty\}$ whose domain is a subset of $\mathbb{R}_{+} \cup\{\infty\}$ and such that $f(\alpha) \leq \alpha$ for all $\alpha \geq 0$, an (HM) measure whose lower Hausdorff spectrum is equal to $f$. We construct such a measure $\mu$ when $\operatorname{dom}(f)$ is a closed subset of $\mathbb{R}_{+} \cup\{\infty\}, f$ is upper semi-continuous, and $f$ has at least one fixed point, three properties shared with $\tau_{\mu}^{*}$. Moreover, the measure $\mu$ is exactly dimensional.

Thus, we will prescribe lower Hausdorff spectra in the family:

$$
\mathcal{F}(d)=\left\{f: \mathbb{R} \cup\{\infty\} \rightarrow[0, d] \cup\{-\infty\}:\left\{\begin{array}{l}
\operatorname{Fix}(f) \neq \varnothing \\
\operatorname{dom}(f) \text { is a closed subset of }[0, \infty] \\
f \text { is upper semi-continuous } \\
f(\alpha) \leq \alpha \text { for all } \alpha \in \operatorname{dom}(f)
\end{array}\right\},\right.
$$

where $\operatorname{Fix}(f)(\subset[0, d])$ stands for the set of fixed points of $f$.
Theorem 1.2. - Let $f \in \mathscr{F}(d)$. For each $D \in \operatorname{Fix}(f)$, there exists an (HM) measure $\mu \in \mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$, exactly dimensional with dimension $D$, such that $\underline{f}_{\mu}^{H}=f$.

This result will be strengthened in Theorem 1.5 of Section 1.2. It turns out that the approach used in this paper does not make it possible to replace $\underline{f}_{\mu}^{H}=f$ by $f_{\mu}^{H}=f$ in the previous statement unless one of the following properties holds: $\operatorname{dom}(f)=\operatorname{Fix}(f)$ (see Theorem 1.5), or $\operatorname{dom}(f)$ is an interval and $f$ is concave over $\operatorname{dom}(f) \cap \mathbb{R}_{+}$(in this case we will get a measure obeying the strong multifractal formalism, see Theorem 1.3).

Before developing further results and comments, let us outline the main ideas leading to the construction of the measure $\mu$ provided by Theorem 1.2. To establish Theorem 1.1 one must improve this approach in order to control both the finer level sets $E(\mu, \alpha)$ and the upper large deviations spectrum of $\mu$ (to be defined in Section 1.2.1) when $f$ is the concave function $\tau^{*}$, and then use the duality property linking the $L^{q}$-spectrum and the upper large deviations spectrum to show that the multifractal formalism holds strongly.

For simplicity, we assume that $\operatorname{dom}(f)$ is a non trivial compact interval $\left[\alpha_{\min }, \alpha_{\max }\right] \subset \mathbb{R}_{+}$, $f$ is continuous over $\left[\alpha_{\min }, \alpha_{\max }\right], 0 \leq f(\alpha) \leq \min (\alpha, d)$ over $\left[\alpha_{\min }, \alpha_{\max }\right]$, and $f(D)=D$ for a unique point $D$ in $\left[\alpha_{\min }, \alpha_{\max }\right]$. The homogeneity of the construction of the measure $\mu$ automatically implies that the measure is (HM).

At first one shows (independently of $f$ ) that for any $\gamma \in[0, d]$ and $\alpha \geq \gamma$, one can find two Borel probability measures $\mu_{\alpha, \gamma}$ and $\nu_{\alpha, \gamma}$ supported on $[0,1]^{d}$ such that $\mu_{\gamma, \gamma}=\nu_{\gamma, \gamma}$, $\nu_{\alpha, \gamma}$ is exactly dimensional with dimension $\gamma$, and $\nu_{\alpha, \gamma}$ is concentrated on $E\left(\mu_{\alpha, \gamma}, \alpha\right)$, as well as on the set defined similarly but with $\alpha(\mu, x)$ replaced by $\lim _{n \rightarrow \infty} \frac{\log \left(\mu\left(I_{n}(x)\right)\right)}{-n \log (2)}$, where $I_{n}(x)$ stands for the closure of a dyadic cube semi-open to the right containing $x$.

Set $A_{1}=\left\{\alpha_{1}=D\right\}$, and for each integer $m \geq 1$, define $A_{m+1}=A_{m} \cup\left\{\alpha_{m+1}\right\}$, where $\alpha_{m+1} \in\left[\alpha_{\min }, \alpha_{\max }\right] \backslash A_{m}$, in such a way that the set $\left\{\alpha_{m}: m \geq 1\right\}$ be dense in $\left[\alpha_{\min }, \alpha_{\text {max }}\right]$. By using the previous property with $\gamma=f(\alpha)$, for all $m \geq 1$ one gets an integer $n_{m}$ such that
for all $\alpha \in A_{m}$, for all $n \geq n_{m}$, there is a collection $G_{m, n}(\alpha)$ of about $2^{n f(\alpha)}$ dyadic subcubes of $[0,1]^{d}$ such that for all $I \in G_{m, n}(\alpha)$ one has $\mu_{\alpha, f(\alpha)}(I) \approx 2^{-n \alpha}, \nu_{\alpha, f(\alpha)}(I) \approx 2^{-n f(\alpha)}$, and $\sum_{I \in G_{m, n}(\alpha)} \nu_{\alpha, f(\alpha)}(I) \in[1 / 2,1]$.

For every integer $m \geq 2$, one considers $m$ dyadic closed subcubes of $[0,1]^{d}$ of the same generation $n_{m}^{\prime}, L_{\alpha_{1}}, \ldots, L_{\alpha_{m}}$, so that the $2^{-n_{m}^{\prime} / 5}$ neighborhood of each $L_{\alpha_{i}}$ does not intersect any of the other $L_{\alpha_{j}}$.

The measure $\mu$ is constructed on a Cantor set $K=\bigcap_{m \geq 1} \bigcup_{I \in \mathbf{G}_{m}}$, where the $\mathbf{G}_{m}$ are families of closed dyadic subcubes of $[0,1]^{d}$ of generation $g_{m}$ tending to $\infty$ as $m \rightarrow \infty$, constructed recursively according to a scheme roughly as follows:

One obtains $\mathbf{G}_{1}$ by considering the measure $\mu_{\alpha_{1}, f\left(\alpha_{1}\right)}=\mu_{D, D}$, an integer $N_{1} \geq n_{1}$ much bigger than $n_{2}^{\prime}$ and setting $\mathbf{G}_{1}=G_{1, N_{1}}\left(\alpha_{1}\right)=G_{1, N_{1}}(D)$. This yields the probability measure $\mu_{1}$ defined on $\mathbf{G}_{1}$ as

$$
\mu_{1}(I)=\frac{\mu_{D, D}(I)}{\left(\sum_{I^{\prime} \in \mathbf{G}_{1}} \mu_{D, D}\left(I^{\prime}\right)\right)} .
$$

This measure satisfies $\mu_{1}(I) \approx 2^{-N_{1} D}$.
Suppose now that the set $\mathbf{G}_{m}$ has been constructed, as well as a probability measure $\mu_{m}$ on its elements. One takes $N_{m+1} \geq n_{m+1}$ and integer much bigger than $\max \left(g_{m}, n_{m+2}^{\prime}\right)$, and for each $1 \leq i \leq m+1$, one considers the measure $\mu_{\alpha_{i}, f\left(\alpha_{i}\right)}$ and the associated set $G_{m+1}\left(\alpha_{i}\right):=G_{m+1, N_{m+1}}\left(\alpha_{i}\right)$. For each $1 \leq i \leq m+1$ and $I_{m} \in \mathbf{G}_{m}$, one defines the set of the elements of $\mathbf{G}_{m+1}$ contained in $I_{m}$ as $\bigcup_{i=1}^{m+1} \mathbf{G}_{m+1}\left(I_{m}, \alpha_{i}\right)$, where $\mathbf{G}_{m+1}\left(I_{m}, \alpha_{i}\right)=$ $\left\{I_{m} \cdot L_{\alpha_{i}} \cdot I: I \in G_{m+1}\left(\alpha_{i}\right)\right\}$, and the concatenation $J \cdot J^{\prime}$ of two closed subcubes of $[0,1]^{d}$ is obtained as the cube $f_{J}\left(J^{\prime}\right)$, where $f_{J}$ is the natural contracting similitude mapping $[0,1]^{d}$ onto $J$ (this operation is associative). One gets a probability measure $\mu_{m+1}$ on $\mathbf{G}_{m+1}$ by setting, for $I \in G_{m+1}\left(\alpha_{i}\right)$ :

$$
\begin{equation*}
\mu_{m+1}\left(I_{m} \cdot L_{\alpha_{i}} \cdot I\right)=\mu_{m}\left(I_{m}\right) \frac{\mu_{\alpha_{i}, f\left(\alpha_{i}\right)}(I)}{\sum_{\alpha \in A_{m+1}} \sum_{I^{\prime} \in G_{m+1}(\alpha)} \mu_{\alpha, f(\alpha)}\left(I^{\prime}\right)} . \tag{1.2}
\end{equation*}
$$

This makes it possible to define a Borel probability measure carried on $K$ and coinciding with $\mu_{m}$ over $\mathbf{G}_{m}$ for all $m \geq 1$.

Since $f(\alpha)<\alpha$ except for $\alpha=\alpha_{1}=D$, if $N_{m+1}$ is taken big enough, in (1.2) for each $i>1$ the contribution of the elements of $G_{m+1}\left(\alpha_{i}\right)$ is roughly $2^{N_{m+1}\left(f\left(\alpha_{i}\right)-\alpha_{i}\right)}$ hence is negligible so that the denominator is equivalent to the single contribution of $\sum_{I^{\prime} \in G_{m+1}(D)} \mu_{D, D}\left(I^{\prime}\right) \in[1 / 2,1]$. Consequently, for $I_{m+1} \in \mathbf{G}_{m+1}$ of the form $I_{m} \cdot L_{\alpha_{i}} \cdot I$, $I \in G_{m+1}\left(\alpha_{i}\right)$, we have the following estimate:

$$
\begin{equation*}
\mu\left(I_{m+1}\right) \approx \mu_{m}\left(I_{m}\right) \mu_{\alpha_{i}, f\left(\alpha_{i}\right)}(I) \approx \mu_{m}\left(I_{m}\right) 2^{-\alpha_{i} N_{m+1}} \approx 2^{-\alpha_{i} g_{m+1}} \tag{1.3}
\end{equation*}
$$

because $g_{m} \ll N_{m+1}$. Also, we have that $\# G_{m+1}\left(\alpha_{i}\right) \approx 2^{f\left(\alpha_{i}\right) N_{m+1}}$, hence
$\#\left\{I \in \mathbf{G}_{m+1}: I \in \mathbf{G}_{m+1}\left(I_{m}, \alpha_{i}\right)\right.$ for some $\left.I_{m} \in \mathbf{G}_{m}\right\}=\left(\# \mathbf{G}_{m}\right)\left(\# G_{m+1}\left(\alpha_{i}\right)\right) \approx 2^{f\left(\alpha_{i}\right) g_{m+1}}$,
again because $g_{m} \ll N_{m+1}$. The previous estimate and the continuity of $f$ essentially yield that $f$ is an upper bound for $\underline{f}^{H}$. Combined with (1.3), it shows that at generation $m+1$, the
mass of $\mu$ is essentially carried by the intervals $I_{m} \cdot L_{D} \cdot I, I \in G_{m+1}(D)$, since we have

$$
1=\|\mu\| \approx \sum_{i=1}^{m+1} 2^{f\left(\alpha_{i}\right) g_{m+1}} 2^{-\alpha_{i} g_{m+1}}=\sum_{i=1}^{m+1} 2^{\left(f\left(\alpha_{i}\right)-\alpha_{i}\right) g_{m+1}} \approx 2^{\left(f\left(\alpha_{1}\right)-\alpha_{1}\right) g_{m+1}}=1
$$

(recall that $\alpha_{1}=f\left(\alpha_{1}\right)=D$ ). This can be strengthened to show that $\mu$ is exact $D$-dimensional.

Another important fact is the natural existence of a family of auxiliary measures used to find a sharp lower bound for $\underline{f}^{H}$ : with each $\widehat{\beta}=\left(\beta_{m}\right)_{m \geq 1} \in \prod_{m=1}^{\infty} A_{m}$ is associated the Cantor subset of $K$ defined as

$$
K_{\widehat{\beta}}=\bigcap_{m \geq 1} \bigcup_{I \in \mathbf{G}_{\widehat{\beta}, m}} I
$$

where $\mathbf{G}_{\widehat{\beta}, m}$ is the subset of $\mathbf{G}_{m}$ obtained by selecting only the intervals of the construction for which one considers the exponent $\beta_{i} \in A_{i}$ at step $i$ for all $1 \leq i \leq m$. Using (1.3) and finer properties of the measures $\mu_{\alpha, \gamma}$ one can show that $K_{\widehat{\beta}} \subset \underline{E}(\mu, \beta)$, where $\beta=\lim _{\inf }^{m \rightarrow \infty} \beta_{m}$. Moreover, the measures $\nu_{\beta_{m}, f\left(\beta_{m}\right)}$ can be used to construct a nice auxiliary probability measure $\nu_{\widehat{\beta}}$ carried by $K_{\widehat{\beta}}$. At first one defines recursively a sequence of measures $\left(\nu_{\widehat{\beta}, m}\right)_{m \geq 1}$ on the atoms of the sets $\mathbf{G}_{\widehat{\beta}, m}, m \geq 1$, as follows: $\nu_{\widehat{\beta}, 1}$ is the restriction of $\nu_{D, D}$ to $\mathbf{G}_{\widehat{\beta}, 1}\left(=\mathbf{G}_{1}\right)$, and assuming that $\nu_{\widehat{\beta}, m}$ is constructed on $\mathbf{G}_{\widehat{\beta}, m}$, if $I_{m} \in \mathbf{G}_{\widehat{\beta}, m}$, for $I \in G_{m+1}\left(\beta_{m+1}\right)$ one sets

$$
\nu_{\widehat{\beta}, m+1}\left(I_{m} \cdot L_{\beta_{m+1}} \cdot I\right)=\nu_{\widehat{\beta}, m}\left(I_{m}\right) \frac{\nu_{\beta_{m+1}, f\left(\beta_{m+1}\right)}(I)}{\sum_{I^{\prime} \in G_{m+1}\left(\beta_{m+1}\right)} \nu_{\beta_{m+1}, f\left(\beta_{m+1}\right)}\left(I^{\prime}\right)} .
$$

This yields a Borel probability measure $\nu_{\widehat{\beta}}$ supported on $K_{\widehat{\beta}}$ such that

$$
\nu_{\widehat{\beta}}\left(I_{m} \cdot L_{\beta_{m+1}} \cdot I\right)=\nu_{\widehat{\beta}, m+1}\left(I_{m} \cdot L_{\beta_{m+1}} \cdot I\right) \approx \nu_{\widehat{\beta}, m}\left(I_{m}\right) \nu_{\beta_{m+1}, f\left(\beta_{m+1}\right)}(I),
$$

so that $\nu_{\widehat{\beta}}\left(I_{m} \cdot L_{\beta_{m+1}} \cdot I\right) \approx \nu_{\beta_{m+1}, f\left(\beta_{m+1}\right)}(I) \approx 2^{-f\left(\beta_{m+1}\right) g_{m+1}}$ (again since $\left.g_{m} \ll N_{m+1}\right)$. This can be strengthened to $\operatorname{dim}_{H}\left(\nu_{\widehat{\beta}}\right)=\liminf _{m \rightarrow \infty} f\left(\beta_{m}\right)$, hence $\operatorname{dim}_{H} K_{\widehat{\beta}} \geq$ $\lim \inf _{m \rightarrow \infty} f\left(\beta_{m}\right)$ by the mass distribution principle (see Section 6). Finally, if $\beta \in\left[\alpha_{\min }, \alpha_{\max }\right]$ and $\lim _{m \rightarrow \infty} \beta_{m}=\beta$, the continuity of $f$ yields $\underline{f}^{H}(\beta)=\operatorname{dim}_{H} \underline{E}(\mu, \beta) \geq f(\beta)$.

### 1.2. Main statements, and comments

New definitions and properties are needed to state our main results.
1.2.1. Additional definitions and properties related to the multifractal formalism. - For $\mu \in \mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$, recall that $f_{\mu}^{H}$ and $f_{\mu}^{P}$ stand for the Hausdorff spectrum $\alpha \in \mathbb{R} \cup\{\infty\} \mapsto$ $\operatorname{dim}_{H} E(\mu, \alpha)$ and the packing spectrum $\alpha \in \mathbb{R} \cup\{\infty\} \mapsto \operatorname{dim}_{P} E(\mu, \alpha)$ respectively.

The functions defined below, as well as some variants, are well known in the literature ([33, $21,19,63,52]$ ). They naturally complete $\tau_{\mu}$ and $\tau_{\mu}^{*}$ to describe, in terms of large deviations, the asymptotic behavior of the distribution of the measure $\mu$ at small scales. They also yield a finer multifractal formalism, which connects geometric properties of the sets $E(\mu, \alpha)$ to large deviations properties associated with $\mu$, both from the Hausdorff and packing dimensions point of views. In Remark 1.3 (Section 1.2.2) we will explain the connection with another multifractal formalism emphasized in [63], which is based on a purely geometric approach.

Define also the upper $L^{q}$-spectrum of $\mu$ as

$$
q \in \mathbb{R} \mapsto \bar{\tau}_{\mu}(q)=\limsup _{r \rightarrow 0^{+}} \frac{\log \sup \left\{\sum_{i} \mu\left(B\left(x_{i}, r\right)\right)^{q}\right\}}{\log (r)}
$$

(this function is not concave in general), as well as the lower and upper large deviations spectra ${\underset{f}{\mu}}_{\mathrm{LD}}$ and $\bar{f}_{\mu}^{\mathrm{LD}}$ :

$$
\left.\begin{array}{rl}
\alpha \in \mathbb{R} \mapsto \underline{f}_{\mu}^{\mathrm{LD}}(\alpha) & =\lim _{\epsilon \rightarrow 0^{+}} \liminf _{r \rightarrow 0^{+}} \frac{\log \sup \#\left\{i: r^{\alpha+\epsilon} \leq \mu\left(B\left(x_{i}, r\right)\right) \leq r^{\alpha-\epsilon}\right\}}{-\log (r)}, \\
\alpha \in \mathbb{R} \mapsto \bar{f}_{\mu}^{\mathrm{LD}}(\alpha) & =\lim _{\epsilon \rightarrow 0^{+}} \limsup _{r \rightarrow 0^{+}} \frac{\log \sup \#\left\{i: r^{\alpha+\epsilon} \leq \mu\left(B\left(x_{i}, r\right)\right) \leq r^{\alpha-\epsilon}\right\}}{-\log (r)}, \\
& \underline{f}_{\mu}^{\mathrm{LD}}(\infty)
\end{array}\right) \lim _{A \rightarrow \infty} \liminf _{r \rightarrow 0^{+}} \frac{\log \sup \#\left\{i: \mu\left(B\left(x_{i}, r\right)\right) \leq r^{A}\right\}}{-\log (r)}, ~=\lim _{A \rightarrow \infty} \limsup _{r \rightarrow 0^{+}} \frac{\log \sup \#\left\{i: \mu\left(B\left(x_{i}, r\right)\right) \leq r^{A}\right\}}{-\log (r)},
$$

where the suprema are taken over all the centered packings of $\operatorname{supp}(\mu)$ by closed balls of radius $r$. Notice that $0 \leq \operatorname{dim}_{B} \operatorname{supp}(\mu)=-\bar{\tau}(0) \leq d$, and $\bar{\tau}_{\mu}(1)=0$ (by the same arguments as for the equality $\tau_{\mu}(1)=0$, see [63, Section 2.7] or [52, Section 3]).

One always has $\bar{\tau}_{\mu}^{*} \leq \tau_{\mu}^{*}$, and

$$
\begin{align*}
& \forall \alpha \in \mathbb{R} \cup\{\infty\}, f_{\mu}^{H}(\alpha) \leq \underline{f}_{\mu}^{\mathrm{LD}}(\alpha) \leq \bar{\tau}_{\mu}^{*}(\alpha) \leq \max \left(\alpha,-\bar{\tau}_{\mu}(0)\right) \leq \max (\alpha, d)  \tag{1.4}\\
& \forall \alpha \in \mathbb{R} \cup\{\infty\}, f_{\mu}^{P}(\alpha) \leq \bar{f}_{\mu}^{\mathrm{LD}}(\alpha) \leq \tau_{\mu}^{*}(\alpha) \leq \max \left(\alpha,-\tau_{\mu}(0)\right) \leq \max (\alpha, d)
\end{align*}
$$

We will say that $\mu$ obeys the refined multifractal formalism at $\alpha \in \mathbb{R} \cup\{\infty\}$ if $f_{\mu}^{H}(\alpha)=\bar{\tau}_{\mu}^{*}(\alpha)$ and $f_{\mu}^{P}(\alpha)=\tau_{\mu}^{*}(\alpha)$. If $\alpha \in \operatorname{dom}\left(\tau_{\mu}^{*}\right) \backslash \operatorname{dom}\left(\bar{\tau}_{\mu}^{*}\right)$, one necessarily has $E(\mu, \alpha)=\varnothing$, so that in (1.5), one can only expect the large deviations property $\bar{f}_{\mu}^{\mathrm{LD}}(\alpha)=\tau_{\mu}^{*}(\alpha)$ to hold.

The inequalities $f_{\mu}^{H}(\alpha) \leq \bar{\tau}_{\mu}^{*}(\alpha)$ and $f_{\mu}^{P}(\alpha) \leq \tau_{\mu}^{*}(\alpha)$ are established for $\alpha<\infty$ when $\mu$ is doubling in [63, Section 2.7]. The inequalities $f_{\mu}^{H}(\alpha) \leq \bar{f}_{\mu}^{\mathrm{LD}}(\alpha) \leq \tau_{\mu}^{*}(\alpha)$ are established in [52, Section 3] when $\alpha<\infty$. The other inequalities will be justified in Sections 5.2 and 5.3.

We notice that if $\mu$ is exactly dimensional with dimension $D$, then $D=f^{H}(D)=$ $\underline{f}_{\mu}^{\mathrm{LD}}(D)=\bar{\tau}_{\mu}^{*}(D)=\bar{f}_{\mu}^{\mathrm{LD}}(D)=\tau_{\mu}^{*}(D)$, and in the case where $\bar{\tau}_{\mu}$ is concave, we have $D \in\left[\bar{\tau}_{\mu}^{\prime}\left(1^{+}\right), \bar{\tau}_{\mu}^{\prime}\left(1^{-}\right)\right]\left(\subset\left[\tau_{\mu}^{\prime}\left(1^{+}\right), \tau_{\mu}^{\prime}\left(1^{-}\right)\right]\right)$, since $\bar{\tau}_{\mu}(1)=0$ implies that $\bar{\tau}_{\mu}^{*}(\alpha)=\alpha$ if and only if $\alpha \in\left[\bar{\tau}_{\mu}^{\prime}\left(1^{+}\right), \bar{\tau}_{\mu}^{\prime}\left(1^{-}\right)\right]$.

Let us now describe the possible behaviors of the $L^{q}$-spectrum and its Legendre transform. Before stating the corresponding propositions, we need to extend the notion of LegendreFentchel transform to functions $f: \mathbb{R} \cup\{\infty\} \rightarrow \mathbb{R} \cup\{-\infty\}$.

If $f: \mathbb{R} \cup\{\infty\} \rightarrow \mathbb{R} \cup\{-\infty\}$ and $\operatorname{dom}(f) \cap \mathbb{R} \neq \varnothing$, we define the concave LegendreFenchel transform of $f$ as

$$
f^{*}: q \in \mathbb{R} \mapsto \inf \{q \alpha-f(\alpha): \alpha \in \operatorname{dom}(f)\}
$$

with the conventions $q \times \infty=\frac{q}{|q|} \times \infty$ if $q \neq 0$ and $0 \times \infty=0$.

Consequently, if $\infty \in \operatorname{dom}(f)$ and $f$ is bounded from above, then $0=\min \left(\operatorname{dom}\left(f^{*}\right)\right)$ and $f^{*}(0)=-\max \left(\sup \left(f_{\mid \mathbb{R}}\right), f(\infty)\right)$; moreover, $f^{*}$ is concave over $\operatorname{dom}\left(f^{*}\right)$, upper semicontinuous over $\operatorname{dom}\left(f^{*}\right) \backslash\{0\}$, and upper semi-continuous at 0 if and only if $f(\infty)=\max (f)$.

## Proposition 1.1. - Let $\mu \in \mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$.

1. $\tau_{\mu}$ is concave, non-decreasing, $\tau_{\mu}(1)=0$, and $-d \leq \tau_{\mu}(0)=-\overline{\operatorname{dim}}_{B} \operatorname{supp}(\mu) \leq 0$.
2. One has either $\operatorname{dom}\left(\tau_{\mu}\right)=\mathbb{R}$, or $\operatorname{dom}\left(\tau_{\mu}\right)=\mathbb{R}_{+}$, according to whether the exponent $\lim \sup _{r \rightarrow 0^{+}} \frac{\log (\inf \{\mu(B(x, r)): x \in \operatorname{supp}(\mu)\})}{\log (r)}$ is finite or not. Moreover $\tau_{\mu}^{*}$ is nonnegative on its domain.

Proposition 1.2. - Suppose that $\tau: \mathbb{R} \rightarrow \mathbb{R} \cup\{-\infty\}$ satisfies the properties of the $L^{q}$-spectrum described in Proposition 1.1.

1. Suppose that $\operatorname{dom}(\tau)=\mathbb{R}$. Then $\operatorname{dom}\left(\tau^{*}\right)$ is the compact interval $I=\left[\tau^{\prime}(\infty), \tau^{\prime}(-\infty)\right]$, $\tau^{*}$ is concave and continuous on its domain, and $\left(\tau^{*}\right)^{*}=\tau$ on $\mathbb{R}$.
2. Suppose that $\operatorname{dom}(\tau)=\mathbb{R}_{+}$. Then $\infty \in \operatorname{dom}\left(\tau^{*}\right)$ with $\tau^{*}(\infty)=-\tau(0)$ and:
(a) If $\tau(0)=0$ then $\tau=0$ over $\mathbb{R}_{+}, \operatorname{dom}\left(\tau^{*}\right)=\mathbb{R}_{+} \cup\{\infty\}$ and $\tau^{*}=0$ over $\mathbb{R}_{+} \cup\{\infty\}$.
(b) If $\tau(0)<0$ and $\tau$ is continuous at $0^{+}$, then $\operatorname{dom}\left(\tau^{*}\right)$ is the interval $\left[\tau^{\prime}(\infty), \infty\right]$, $\tau^{*}$ is concave, continuous, and increasing over $\left[\tau^{\prime}(\infty), \tau^{\prime}\left(0^{+}\right)\right), \tau^{*}(\alpha)=-\tau(0)=$ $\tau^{*}(\infty)=-\tau(0)$ for all $\alpha \in\left[\tau^{\prime}\left(0^{+}\right), \infty\right)$ and $\tau^{*}$ is continuous at $\infty$; there are two distinct behaviors according to whether $\tau^{\prime}\left(0^{+}\right)<\infty$ or not.
(c) If $\tau(0)<0$ and $\tau$ is discontinuous at $0^{+}$, then $\operatorname{dom}\left(\tau^{*}\right)$ is the interval $I=\left[\tau^{\prime}(\infty), \infty\right] . \quad$ Moreover, $\quad \tau^{*}(\alpha)=-\tau\left(0^{+}\right)<\tau^{*}(\infty)=-\tau(0) \quad$ for all $\alpha \in\left[\lim _{q \rightarrow 0^{+}} \tau^{\prime}\left(q^{-}\right), \infty\right)$, so that $\tau^{*}$ is concave and continuous on $\left[\tau^{\prime}(\infty), \infty\right)$ and discontinuous at $\infty$ (there are also two cases, according to whether $\lim _{q \rightarrow 0^{+}} \tau^{\prime}\left(q^{-}\right)=\infty$ or not $)$.
(d) In all the previous cases, $\left(\tau^{*}\right)^{*}=\tau$ on $\mathbb{R}$.

Proposition 1.1(1) is standard and proved for instance in [52] (Proposition 3.2). Propositions $1.1(2)$ and $1.2(1)$ are essentially restatements of Propositions 3.3-3.5 in [52]. However, for the reader's convenience we will provide a proof in Section 5, where the whole proofs of Propositions 1.1(2) and 1.2 are given.
1.2.2. Full illustration of the multifractal formalism. Complements to Theorem 1.1. - When $\mu$ is a Gibbs measure on a conformal repeller or a self-similar measure on an attractor satisfying suitable separation conditions, the Hausdorff and packing dimensions are also known for all the sets $E(\mu, \alpha, \beta)$ :

$$
\begin{align*}
& \operatorname{dim}_{H} E(\mu, \alpha, \beta)=\min \left\{\tau_{\mu}^{*}(\gamma): \gamma \in[\alpha, \beta]\right\} \\
& \operatorname{dim}_{P} E(\mu, \alpha, \beta)=\max \left\{\tau_{\mu}^{*}(\gamma): \gamma \in[\alpha, \beta]\right\} \\
& \operatorname{dim}_{H} \underline{E}(\mu, \alpha)=\tau_{\mu}^{*}(\alpha)=\operatorname{dim}_{H} \bar{E}(\mu, \alpha),  \tag{1.6}\\
& \operatorname{dim}_{P} \underline{E}(\mu, \alpha)=\max \left\{\tau_{\mu}^{*}(\gamma): \gamma \geq \alpha\right\}, \operatorname{dim}_{P} \bar{E}(\mu, \alpha)=\max \left\{\tau_{\mu}^{*}(\gamma): \gamma \leq \alpha\right\} .
\end{align*}
$$

for all $\alpha \leq \beta \in \mathbb{R}_{+} \cup\{\infty\}$ (see [66, 4, 83] and also [14, 39] for closely related questions). We notice that (1.6) implies that $\operatorname{dim}_{H} \operatorname{supp}(\mu)=\operatorname{dim}_{B} \operatorname{supp}(\mu)=\operatorname{dim}_{P} \operatorname{supp}(\mu)=$ $\overline{\operatorname{dim}}_{B} \operatorname{supp}(\mu)=-\tau_{\mu}(0)$.

It turns out that properties (1.6) enter in our exhaustive illustration of the multifractal formalism.

Theorem 1.3. - Let $f \in \mathscr{F}(d)$. Suppose that $\operatorname{dom}(f)$ is a non empty closed subinterval of $[0, \infty]$ and $f$ is concave over $\ddots \cap \mathbb{R}_{+}$.

For each fixed point $D$ of $f$, there exists $\mu \in \mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$, exactly dimensional with dimension $D$, such that $\tau_{\mu}=f^{*}=\bar{\tau}_{\mu}, \tau_{\mu}^{*}=f$, and (1.6) holds for all $\alpha \leq \beta \in \mathbb{R}_{+} \cup\{\infty\}$. Moreover, the same properties hold if $\mu$ is replaced by its restriction to any closed ball whose interior intersects $\operatorname{supp}(\mu)$.

The following corollary, of which Theorem 1.1 is a consequence, then follows from the fact that if $\tau$ satisfies the properties of Proposition 1.1 (and so falls into the different situations described in Proposition 1.2(1) and (2)), then $\tau^{*}$ satisfies the assumptions of Theorem 1.3.

Corollary 1.1. - Let $\tau: \mathbb{R} \rightarrow \mathbb{R} \cup\{-\infty\}$ be a function satisfying properties (1) and (2) of Proposition 1.1. Let $D \in\left[\tau^{\prime}\left(1^{+}\right), \tau^{\prime}\left(1^{-}\right)\right]$.

There exists $\mu \in \mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$, exactly dimensional with dimension $D$, satisfying (1.6) for all $\alpha \leq \beta \in \mathbb{R}_{+} \cup\{\infty\}$ with $\tau_{\mu}=\tau=\bar{\tau}_{\mu}$. Moreover, the same properties hold if $\mu$ is replaced by its restriction to any closed ball whose interior intersects $\operatorname{supp}(\mu)$.

Remark 1.1. - The behavior described in Proposition 1.2(1) is illustrated, for instance, by Gibbs and weak Gibbs measures on conformal repellers (see [63, 70, 38]). Such examples, which live on dynamical systems semi-conjugate to subshifts of finite type, cannot exhibit behaviors like those corresponding to Proposition 1.2(2). The behaviors described by Proposition $1.2(2)(b)$ are illustrated by some Gibbs measures on countable Markov shifts and their geometric realizations [46], which also obey the multifractal formalism, though in [46] the set $E(\mu, \infty)$ is not studied. The fact that the behaviors described in Proposition 1.2(2)(a) and (c) be illustrated by measures obeing the mutifractal formalism seems to be new. We notice that the extension of the Legendre transform including $\infty$ in the domain in this case yields Legendre transforms which are not necessarily upper semi-continuous, like $\tau$ at 0 in case (c).

Remark 1.2. - Our results illustrate all the possible situations, in term of the function $\tau_{\mu}$, for which the measure $\mu$ is exactly dimensional though $\tau_{\mu}^{\prime}(1)$ does not exist. In [44], when $d=1$, for each $D \in(0,1)$ one finds an exactly dimensional measure $\mu$ with dimension $D$ and $L^{q}$-spectrum equal to $\min (q-1,0)$ over $\mathbb{R}_{+}$. It is also worth mentioning that in [15] one finds examples of inhomogeneous Bernoulli measures over $[0,1]$ with an $L^{q}$-spectrum presenting countably many points of non differentiability over $[1,+\infty)$.

In the previous results, due to (1.4) and (1.5) we have $f_{\mu}^{H}=f_{\mu}^{P}=\underline{f}_{\mu}^{\mathrm{LD}}=\bar{f}_{\mu}^{\mathrm{LD}}$, which reflects a strong homogeneity of the sets $E(\mu, \alpha)$. The purpose of the refined multifractal formalism is to describe situations irregular enough so that the Hausdorff and packing dimensions of $E(\mu, \alpha)$ differ for most of the $\alpha$.

The next two results extend in a non trivial way the two previous ones, in particular by exhibiting a new formula for $\operatorname{dim}_{H} E(\mu, \alpha, \beta)$. They invoke an extension of (1.4) and (1.5),

[^2]which illustrates the following complement to the multifractal formalism: If $0 \leq \alpha<\infty$ and $\alpha \leq \beta \leq \infty, 1>r>0$, and $\epsilon>0$, set
$$
f_{\mu}(\alpha, \beta, \epsilon, r)=\frac{\log \sup \#\left\{i: r^{\beta+\epsilon} \leq \mu\left(B\left(x_{i}, r\right)\right) \leq r^{\alpha-\epsilon}\right\}}{-\log (r)},
$$
where the suprema are taken over all the centered packing of $\operatorname{supp}(\mu)$ by closed balls of radius $r$, and with the convention that $r^{\infty}=0$. Then define
$$
\underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)=\lim _{\epsilon \rightarrow 0} \liminf _{r \rightarrow 0+} f_{\mu}(\alpha, \beta, \epsilon, r) .
$$

Also, define $\underline{f}_{\mu}^{\mathrm{LD}}(\infty, \infty)=\underline{f}_{\mu}^{\mathrm{LD}}(\infty)$.
Proposition 1.3. - Let $\mu \in \mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$. For any $0 \leq \alpha \leq \beta \leq \infty$, one has

1. $\operatorname{dim}_{H} E(\mu, \alpha, \beta) \leq f_{H}(\alpha, \beta):=\min \left(\bar{f}_{\mu}^{\mathrm{LD}}(\alpha), \bar{f}_{\mu}^{\mathrm{LD}}(\beta), \underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)\right)$ $\operatorname{dim}_{P} E(\mu, \alpha, \beta) \leq f_{P}(\alpha, \beta):=\max \left\{\left\{_{\mu}^{\mathrm{LD}}\left(\alpha^{\prime}\right): \alpha^{\prime} \in[\alpha, \beta]\right\}\right.$
2. 

$$
\begin{aligned}
& \operatorname{dim}_{L} \underline{E}(\mu, \alpha) \leq \sup \left\{f_{L}(\alpha, \beta): \beta \geq \alpha\right\} \text { for } L \in\{H, P\} \\
& \operatorname{dim}_{L} \bar{E}(\mu, \alpha) \leq \sup \left\{f_{L}(\beta, \alpha): \beta \leq \alpha\right\} \text { for } L \in\{H, P\}
\end{aligned}
$$

Theorem 1.4. - Let $d \in \mathbb{N}_{+}$. Let $ป \subset \mathcal{O}$ be two non empty closed subintervals of $[0, \infty]$. Let $f$ and $g \in \mathcal{F}(d)$ such that $\operatorname{dom}(f)=\mathcal{I}, \operatorname{dom}(g)=\mathcal{I}$, and $f \leq g$. Suppose also that $f$ and $g$ are concave over $I \cap \mathbb{R}_{+}$and $\mathcal{I} \cap \mathbb{R}_{+}$respectively.

For each $D \in \operatorname{Fix}(f)$, there exists $\mu \in \mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$, exactly dimensional with dimension $D$, such that

1. $\operatorname{dom}\left(f_{\mu}^{H}\right)=\operatorname{dom}\left(f_{\mu}^{P}\right)=\operatorname{dom}\left(f_{\mu}^{\mathrm{LD}}\right)=才$ and $\operatorname{dom}\left(\bar{f}_{\mu}^{\mathrm{LD}}\right)=\mathscr{g}$.
2. For all $\alpha \in \mathcal{J}, f_{\mu}^{H}(\alpha)=\underline{f}_{\mu}^{\mathrm{LD}}(\alpha)=f(\alpha), f_{\mu}^{P}(\alpha)=\bar{f}_{\mu}^{\mathrm{LD}}(\alpha)=g(\alpha)$ and $\bar{f}_{\mu}^{\mathrm{LD}}(\alpha)=g(\alpha)$ for $\alpha \in \mathscr{J} \backslash 才$.
3. More generally, for all $\alpha \leq \beta \in \mathbb{R}_{+} \cup\{\infty\}$,

$$
\left.\begin{array}{rl}
\underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta) & =f(\alpha, \beta):=\max \left\{f\left(\alpha^{\prime}\right): \alpha^{\prime} \in[\alpha, \beta]\right\}, \\
\operatorname{dim}_{H} E(\mu, \alpha, \beta) & = \begin{cases}\min (g(\alpha), g(\beta), f(\alpha, \beta)) & \text { if }[\alpha, \beta] \subset \mathcal{J} \text { and }[\alpha, \beta] \cap \mathcal{I} \neq \varnothing \\
-\infty & \text { otherwise, }\end{cases} \\
\operatorname{dim}_{P} E(\mu, \alpha, \beta) & = \begin{cases}\max \left\{g\left(\alpha^{\prime}\right): \alpha^{\prime} \in[\alpha, \beta]\right\} & \text { if }[\alpha, \beta] \subset \mathcal{J} \text { and }[\alpha, \beta] \cap \mathcal{J} \neq \varnothing \\
-\infty & \text { otherwise, }\end{cases} \\
\operatorname{dim}_{H} \underline{E}(\mu, \alpha) & =\max \left\{\operatorname{dim}_{H} E(\mu, \alpha, \beta): \beta \geq \alpha\right\}=\min (g(\alpha), \max \{f(\beta): \beta \geq \alpha\}), \\
\operatorname{dim}_{H} \bar{E}(\mu, \alpha) & =\max \left\{\operatorname{dim}_{H} E(\mu, \beta, \alpha): \beta \leq \alpha\right\}=\min (g(\alpha), \max \{f(\beta): \beta \leq \alpha\}), \\
\operatorname{dim}_{P} \underline{E}(\mu, \alpha) & =\max \left\{\operatorname{dim}_{P} E(\mu, \alpha, \beta): \beta \geq \alpha\right\}
\end{array}\right] \begin{array}{ll}
\max \{g(\beta): \beta \geq \alpha\} & \text { if } \alpha \in[\min (\mathcal{J}), \max (\mathcal{J})] \\
-\infty & \text { otherwise, },
\end{array}
$$

$$
\begin{aligned}
\operatorname{dim}_{P} \bar{E}(\mu, \alpha) & =\max \left\{\operatorname{dim}_{P} E(\mu, \beta, \alpha): \beta \leq \alpha\right\} \\
& = \begin{cases}\max \{g(\beta): \beta \leq \alpha\} & \text { if } \alpha \in[\min (ף), \max (\nearrow)] \\
-\infty & \text { otherwise }\end{cases}
\end{aligned}
$$

4. $\bar{\tau}_{\mu}=f^{*}, \bar{\tau}_{\mu}^{*}=f, \tau_{\mu}=g^{*}$ and $\tau_{\mu}^{*}=g$.

Moreover, all the previous properties hold if $\mu$ is replaced by its restriction to any closed ball whose interior intersects $\operatorname{supp}(\mu)$.

Notice that properties (2) and (4) of the previous statement imply $\operatorname{dim}_{H} \operatorname{supp}(\mu)=$ $\operatorname{dim}_{B} \operatorname{supp}(\mu)=-\bar{\tau}(0) \quad$ and $\quad \operatorname{dim}_{P} \operatorname{supp}(\mu)=\overline{\operatorname{dim}}_{B} \operatorname{supp}(\mu)=-\tau(0)$, because $\max \left\{\operatorname{dim}_{H} \underline{E}(\alpha): \alpha \in \mathcal{J}\right\}=-\bar{\tau}(0)$ and $\max \left\{\operatorname{dim}_{P} \underline{E}(\alpha): \alpha \in \mathcal{J}\right\}=-\tau(0)$.

Corollary 1.2. - Let $\tau, \bar{\tau}: \mathbb{R} \rightarrow \mathbb{R} \cup\{-\infty\}$ be two functions satisfying properties (1) and (2) of Proposition 1.1, and such that $\tau \leq \bar{\tau}$.

Let $D \in\left[\bar{\tau}^{\prime}\left(1^{+}\right), \bar{\tau}^{\prime}\left(1^{-}\right)\right] \subset\left[\tau^{\prime}\left(1^{+}\right), \tau^{\prime}\left(1^{-}\right)\right]$. There exists an exactly dimensional measure $\mu \in \mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$ with dimension $D$ such that :

1. $\tau_{\mu}=\tau$ and $\bar{\tau}_{\mu}=\bar{\tau}$;
2. $\operatorname{dim}_{H} \operatorname{supp}(\mu)=\underline{\operatorname{dim}}_{B} \operatorname{supp}(\mu)=-\bar{\tau}(0)$ and $\operatorname{dim}_{P} \operatorname{supp}(\mu)=\overline{\operatorname{dim}}_{B} \operatorname{supp}(\mu)=$ $-\tau(0)$;
3. properties (1)-(3) of Theorem 1.4 hold with $\fallingdotseq=\operatorname{dom}\left(\bar{\tau}^{*}\right), \mathscr{g}=\operatorname{dom}\left(\tau^{*}\right), f=\bar{\tau}^{*}$ and $g=\tau^{*}$.
Moreover, all the previous properties hold if $\mu$ is replaced by its restriction to any closed ball whose interior intersects $\operatorname{supp}(\mu)$.

REMARK 1.3 (Link with Olsen's multifractal formalism). - In [63], Olsen introduces three "multifractal dimensions" functions $b_{\mu} \leq B_{\mu} \leq \Lambda_{\mu}$ derived from "multifractal" generalizations of Hausdorff and packing measures associated with $\mu\left(\Lambda_{\mu}\right.$ and $B_{\mu}$ are convex, while $b_{\mu}$ may be not), so that $f^{H}(\alpha) \leq\left(-b_{\mu}\right)^{*}(\alpha)$ and $f^{P}(\alpha) \leq\left(-B_{\mu}\right)^{*}(\alpha)\left(\leq\left(-\Lambda_{\mu}\right)^{*}(\alpha)\right)$ for all $\alpha \in \mathbb{R}_{+}$; one can then say that Olsen's multifractal formalism holds at $\alpha \in \mathbb{R}_{+} \cup\{\infty\}$ if the previous inequalities are equalities (adding $\alpha=\infty$ in his formalism does not matter). The pair $\left\{b_{\mu}, B_{\mu}\right\}$ has a geometric meaning, while $\left\{\tau_{\mu}, \bar{\tau}_{\mu}\right\}$ relies on large deviations properties.

This formalism has recently found new illustrations by inhomogeneous Bernoulli measures on $[0,1]$ (cf. $[18,78])$, and it is particularly well suited to describe $\operatorname{dim}_{H} E(\mu, \alpha)$ for self-affine measures or Gibbs measures on self-affine Sierpinski carpets and sponges $[51,64,10,6]$ (the packing dimension of the sets $E(\mu, \alpha)$ in these situations remains an open question in general).

Olsen pays a particular attention to compare the pairs offunctions $\left\{b_{\mu}, \Lambda_{\mu}\right\}$ and $\left\{\tau_{\mu}, \bar{\tau}_{\mu}\right\}$. He proves $b_{\mu} \leq-\bar{\tau}_{\mu}$ and $B_{\mu} \leq \Lambda_{\mu}=-\tau_{\mu}$ when $\mu$ is doubling, which according to both multifractal formalisms inequalities, implies $\left(-b_{\mu}\right)^{*}(\alpha)=\bar{\tau}_{\mu}^{*}(\alpha)$ and $\left.\left(-B_{\mu}^{*}\right)(\alpha)=\left(-\Lambda_{\mu}\right)^{*}(\alpha)\right)=\tau_{\mu}^{*}(\alpha)$ when the refined multifractal formalism used in this paper holds at $\alpha$, hence the validity of his formalism. It turns out that even if the measure $\mu$ we are going to construct to prove Corollary 1.2 are not doubling, they possess the weaker but close property that there exists a function $\epsilon(r)$ tending to $0^{+}$as $r \rightarrow 0^{+}$such that $\mu(B(x, 2 r)) \leq r^{-\epsilon(r)} \mu(B(x, r))$, uniformly in $x \in \operatorname{supp}(\mu)$ (see Section 4.2). This is enough for $b_{\mu} \leq-\bar{\tau}_{\mu}$ and $B_{\mu} \leq \Lambda_{\mu}=-\tau_{\mu}$ to hold, hence for Olsen's multifractal formalism to be valid at each $\alpha$ of $\operatorname{dom}\left(\bar{\tau}_{\mu}^{*}\right)$. Moreover, using the equality $\left(-b_{\mu}\right)^{*}=\bar{\tau}_{\mu}^{*}$ over $\operatorname{dom}\left(\bar{\tau}_{\mu}^{*}\right)$, taking the Legendre-Fenchel transforms of these functions, and
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using the inequality $b_{\mu} \leq-\bar{\tau}_{\mu}=-\bar{\tau}$, we can get $b_{\mu}=-\bar{\tau}_{\mu}=-\bar{\tau}$. Similarly $B_{\mu}=\Lambda_{\mu}=$ $-\tau_{\mu}=-\tau$.

General upper bounds for $\operatorname{dim}_{H} \underline{E}(\mu, \alpha)$ and $\operatorname{dim}_{H} \bar{E}(\mu, \alpha)$ are given by [63, Theorem 2.17(ii)(iii)], namely $\left(-\left(b_{\mu} \square B_{\mu}\right)\right)^{*}(\alpha)$ and $\left(-\left(B_{\mu} \square b_{\mu}\right)\right)^{*}(\alpha)$ respectively, where $b \square B$ equals b over $(-\infty, 0), b(0) \wedge B(0)$ at 0 , and $B$ over $(0, \infty)$. The previous remarks and the formulas obtained in Theorem 1.4 for $\operatorname{dim}_{H} \underline{E}(\mu, \alpha)$ and $\operatorname{dim}_{H} \bar{E}(\mu, \alpha)$ show that for the measure $\mu$ we construct, the upper bounds estimates $\left(-\left(b_{\mu} \square B_{\mu}\right)\right)^{*}(\alpha)$ and $\left(-\left(B_{\mu} \square b_{\mu}\right)\right)^{*}(\alpha)$ do provide the correct values for the Hausdorff dimensions.
1.2.3. Prescription of the lower Hausdorff spectrum. - Theorem 1.2 can be refined as follows, according to the properties of the measure we construct:

Theorem 1.5. - Let $f \in \mathscr{F}(d)$. For each $D \in \operatorname{Fix}(f)$, there exists an (HM) measure $\mu \in \mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$, exactly dimensional with dimension $D$, such that $\underline{f}_{\mu}^{H}=f$.

Moreover, $\mu$ can be constructed so that one has: (1) if $\alpha \in \operatorname{Fix}(f)$ then $f_{\mu}^{H}(\alpha)=\alpha$ and (2) if $\alpha \in \operatorname{dom}(f) \backslash \operatorname{Fix}(f) \neq \varnothing$ then $\underline{E}(\mu, \alpha)=E(\mu, \alpha, \infty)$, and properties (1) and (2) hold if $\mu$ is replaced by its restriction to any closed ball whose interior intersects $\operatorname{supp}(\mu)$.

Remark 1.4. - It can be shown for the measure we construct that $\operatorname{dim}_{P} \underline{E}(\mu, \alpha)=$ $\max \left\{f\left(\alpha^{\prime}\right): \alpha^{\prime} \geq \alpha\right\}$ for all $\alpha \in \mathbb{R} \cup\{\infty\}$. The Hausdorff and packing dimensions of the sets $\bar{E}(\mu, \alpha)$ and $E(\mu, \alpha, \beta)$ can also be computed; we leave these calculations, based on Corollaries 3.1 and 3.2, to the reader.

Remark 1.5. - (1) The prescription of the lower Hausdorff spectrum has also been studied in [24], where the authors work on $\mathbb{R}$ and construct (HM) continuous measures, not exactly dimensional, but with upper Hausdorff dimension equal to 1 , and whose support is equal to $[0,1]$, with a prescribed lower Hausdorff spectrum in the class $\mathscr{F}$ of functions $f: \mathbb{R}_{+} \rightarrow[0,1] \cup\{-\infty\}$ which satisfy: $f(1)=1, \operatorname{dom}(f)$ is a closed subinterval of $[0,1]$ of the form $[\alpha, 1]$ such that $\alpha>0$, and $f_{[[\alpha, 1)}=\max \left(g_{[[\alpha, 1)}, 0\right)$, where (i) $g$ is the supremum of a sequence of functions $\left(g_{n}\right)_{n \geq 1}$, such that each $g_{n}$ is constant over its domain supposed to be a closed subinterval of $[0,1]$ and $g_{n}(\beta) \leq \beta$ for all $\beta \in[0,1]$; (ii) $[\alpha, 1]$ is the smallest closed interval containing the support of $g$. It is also shown that for an (HM) measure to be supported by the whole interval $[0,1]$, it is necessary that the support of its lower Hausdorff spectrum contains an interval of the form $[\alpha, 1],(0 \leq \alpha \leq 1)$.

The authors of [24] also study the case of non (HM) measures. In this case, they construct non exactly dimensional measures with upper Hausdorff dimension 1 whose support is equal to $[0,1]$, with a prescribed lower Hausdorff spectrum in the broader class $\widetilde{\mathscr{G}}$ of functions $f$ which satisfy that $f(1)=1,0<\inf (\operatorname{dom}(f))$, and $f_{|\operatorname{dom}(f) \backslash\{1\}|}=g_{|\operatorname{dom}(f) \backslash\{1\}|}$, where $g$ satisfies property (i). This includes all such functions $f$ for which $g$ is lower semi-continuous. Simultaneously, they also construct a non (HM) measure with lower Hausdorff spectrum given by $g$.
(2) All the spectra defined in this paper make sense if measures are replaced by non negative functions of subsets of $\mathbb{R}^{d}$ to which a notion of support is associated. This is the case for instance of Choquet capacities. In [57], the prescription of the spectrum $\alpha \mapsto \operatorname{dim}_{H} E(C, \alpha)$ is studied, where $C$ is a Choquet capacity on subsets of $[0,1]$ but not a positive measure, which makes the situation easier to tract. The authors can find a capacity with spectrum given by $f$,
for any function $f=\sup _{i \geq 1} f_{i}$, where the functions $f_{i}: \mathbb{R} \rightarrow[0,1] \cup\{-\infty\}$ are such that $\operatorname{dom}\left(f_{i}\right)$ is a non empty closed subset of $\mathbb{R}_{+}$, and either $f_{i}=0$ over $\operatorname{dom}\left(f_{i}\right)$ or $f_{i}$ is invertible from $\operatorname{dom}\left(f_{i}\right)$ onto $f\left(\operatorname{dom}\left(f_{i}\right)\right)$, with a continuous inverse (this class of function contains $\widetilde{\mathscr{F}}$ ). Moreover the capacity is (HM).

In [56], the authors construct non (HM) non negative functions $C$ of subsets of $[0,1]$, which are not measures, for which the spectrum

$$
\alpha \mapsto \lim _{\epsilon \rightarrow 0^{+}} \operatorname{dim}_{H} \bigcup_{s>0} \bigcap_{0<r<s}\left\{x \in \operatorname{supp}(C): r^{\alpha+\epsilon} \leq C(B(x, r)) \leq r^{\alpha-\epsilon}\right\}
$$

is prescribed in the class of upper semi-continuous functions $f: \mathbb{R}_{+} \mapsto[0,1] \cup\{-\infty\}$ with non empty compact domain. However, the spectrum which is prescribed is quite rough with respect to the Hausdorff spectrum.

Remark 1.6. - It is worth mentioning that in this paper our constructions provide continuous measures even when their dimension equals 0 , and are based on the properties of the simplest multifractal measures, namely Bernoulli products. These properties are combined in recursive concatenations (roughly described in Section 1 and more elaborated than those used for instance to lower bound the Hausdorff dimensions of the sets $E(\mu, \alpha)$ in the study of weak Gibbs measures) in order to converge asymptotically to a prescribed multifractal structure.

We will first prove Theorem 1.5 because its proof is shorter than that of Theorem 1.4, and it already contains some of the main ideas involved in the proof of Theorem 1.4. However, none of the two proofs can be reduced to the other one regarding the computation of $\underline{f}_{\mu}^{H}$.

The paper is organized as follows. Section 2 introduces preliminary information about Bernoulli measures. Section 3 is dedicated to the proof of Theorem 1.5, Section 4 to the proof of Theorem 1.4, and Section 5 contains the proofs of Propositions 1.1, 1.2 and 1.3, as well as some inequalities in (1.4) and (1.5). Section 6 gives a short account about the mass distribution principle.

## 2. Some notations, and preliminary facts about Bernoulli measures

### 2.1. Notations

For $n \geq 1$, define

$$
\mathcal{F}_{n}=\left\{\prod_{i=1}^{d}\left[k_{i} 2^{-n},\left(k_{i}+1\right) 2^{-n}\right]: 0 \leq k_{i}<2^{n}\right\} .
$$

If $x \in \mathbb{R}^{d}$ and $n \geq 0$ we denote by $I_{n}(x)$ the closure of the unique dyadic cube, semi-open to the right, of generation $n$, that contains $x$.

Given two elements $I=\prod_{i=1}^{d}\left[k_{i} 2^{-n},\left(k_{i}+1\right) 2^{-n}\right]$ and $I^{\prime}=\prod_{i=1}^{d}\left[k_{i}^{\prime} 2^{-n^{\prime}},\left(k_{i}^{\prime}+1\right) 2^{-n^{\prime}}\right]$ of $\bigcup_{p \geq 0} \mathscr{F}_{p}$, the concatenation $I \cdot I^{\prime}$ of $I$ and $I^{\prime}$ is defined as

$$
\begin{equation*}
I \cdot I^{\prime}=\prod_{i=1}^{d}\left[k_{i} 2^{-n}+k_{i}^{\prime} 2^{-n-n^{\prime}}, k_{i} 2^{-n}+\left(k_{i}^{\prime}+1\right) 2^{-n-n^{\prime}}\right] . \tag{2.1}
\end{equation*}
$$

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If $J$ is a closed dyadic cube of generation $j$, we denote by $\mathcal{N}_{1}(n, J)$ the set made of $J$ and the $3^{d}-1$ dyadic cubes of generation $n$ neighboring $J$, and denote by $\mathcal{N}_{2}(n, J)$ the union of $\mathcal{N}_{1}(n, J)$ and the $5^{d}-3^{d}$ closed dyadic cubes of generations $n$ neighboring $\mathcal{N}_{1}(n, J)$.

Fix $L_{0}$ a closed dyadic subcube of $[0,1]^{d}$ of generation 2 which does not touch $\partial[0,1]^{d}$. For each integer $k \geq 1$, we can fix a collection $\mathscr{L}(k)$ of $k$ closed dyadic cubes of generation $\ell(k)=\left\lfloor\frac{\log _{2}\left(6^{d} k\right)}{d}\right\rfloor+3$, all contained in $L_{0}$, such that the sets $\bigcup_{I \in \mathcal{N}_{2}(\ell(k), L)} I, L \in \mathcal{L}(k)$, are pairwise disjoint. This property will imply that the measure constructed in Section 4.1 is "weakly" doubling.

If $\nu$ is a positive Borel measure supported on $[0,1]^{d}$ and $x$ belongs to the support of $\nu$, we set

$$
d(\mu, x, n)=\frac{\log \nu\left(I_{n}(x)\right)}{-n \log (2)} .
$$

For the definitions of the $s$-dimensional Hausdorff and packing measures denoted respectively as $\mathscr{H}^{s}$ and $\mathscr{P}^{s}$ in this paper, the reader is referred to [33] or [60].

### 2.2. Some facts about Bernoulli measures

If $q \in[0,1]$, let $H(q)=-q \log _{2}(q)-(1-q) \log _{2}(1-q)$, with the convention $0 \times \infty=0$. Also, denote by $\nu_{q}$ the Bernoulli measure generated by $(q, 1-q)$ on $[0,1]$.

For each $0 \leq \gamma \leq d$ and $\alpha \geq \gamma$, we can fix $(p, q)=\left(p_{\alpha, \gamma}, q_{\alpha, \gamma}\right) \in[0,1]^{2}$ such that

$$
\left\{\begin{array}{l}
\alpha=-d \cdot\left(q \log _{2}(p)+(1-q) \log _{2}(1-p)\right) \\
\gamma=d \cdot H(q)
\end{array}\right.
$$

Indeed, since $\gamma / d \in[0,1]$, there are clearly two solutions to $H(q)=\gamma / d$ in $[0,1]$ if $\gamma<d$ and only one if $\gamma=d$, namely $1 / 2$. Fix $q$ one solution. Now we seek for $p \in[0,1]$ such that $\alpha(p)=-d\left(q \log _{2}(p)+(1-q) \log _{2}(1-p)\right)$ be equal to $\alpha$. If $q \in\{0,1\}$, this is immediate. Otherwise, the mapping $\alpha(p)$ decreases over $(0, q]$ from $\infty$ to $\alpha(q)=\gamma$, and it increases on $[q, \infty)$ from $\gamma$ to $\infty$. So in this case there is at least one and at most two solutions to $\alpha(p)=\alpha$ since we assumed that $\gamma \leq \alpha$.

We will use the following classical fact, which is just a consequence of the strong law of large numbers.

Proposition 2.1. - Suppose that $d=1$. Let $(p, q) \in[0,1]^{2}$. For $\nu_{q}$-almost every $x \in[0,1]$,

$$
\begin{aligned}
\lim _{n \rightarrow \infty} d\left(\nu_{p}, x, n\right) & =-q \log _{2}(p)-(1-q) \log _{2}(1-p) \\
\lim _{n \rightarrow \infty} d\left(\nu_{q}, x, n\right) & =H(q)
\end{aligned}
$$

Corollary 2.1. - For every $0 \leq \gamma \leq d$ and $\alpha \geq \gamma$, for $\nu_{q_{\alpha, \gamma}}^{\otimes d}$-almost every $x \in[0,1]^{d}$

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} d\left(\nu_{p_{\alpha, \gamma}}^{\otimes d}, x, n\right)=\alpha \\
& \lim _{n \rightarrow \infty} d\left(\nu_{q_{\alpha, \gamma}}^{\otimes d}, x, n\right)=\gamma .
\end{aligned}
$$

Notice that if $\gamma>0$, both the $\nu_{p_{\alpha, \gamma}}^{\otimes d}$ mass and the $\nu_{q_{\alpha, \gamma}}^{\otimes d}$ mass of the boundaries of closed dyadic subcubes of $[0,1]^{d}$ vanish.

Now for every $0 \leq \gamma \leq d, \alpha \geq \gamma, n \in \mathbb{N}$ and $\epsilon>0$ define

$$
E(\alpha, \gamma, n, \epsilon)=\left\{x \in[0,1)^{d}:\left\{\begin{array}{l}
d\left(\nu_{p_{\alpha, \gamma}}^{\otimes d}, x, n\right) \in[\alpha-\epsilon, \alpha+\epsilon] \\
d\left(\nu_{q_{\alpha, \gamma}}^{\otimes d}, x, n\right) \in[\gamma-\epsilon, \gamma+\epsilon]
\end{array}\right\}\right.
$$

Let $\left(\epsilon_{m}\right)_{m \geq 1} \in(0,1)^{\mathbb{N}_{+}}$be a decreasing sequence converging to 0 .
By Corollary 2.1, for each $m \in \mathbb{N}_{+}$we can fix an integer $n_{m}^{0}(\alpha, \gamma)$ such that

$$
\begin{equation*}
\nu_{q_{\alpha, \gamma}}^{\otimes d}\left(F_{m}(\alpha, \gamma)\right) \geq 1 / 2, \text { with } F_{m}(\alpha, \gamma)=\bigcap_{n \geq n_{m}^{0}(\alpha, \gamma)} E\left(\alpha, \gamma, n, \epsilon_{m} / 2\right) \tag{2.2}
\end{equation*}
$$

We notice that for all $n \geq n_{m}^{0}(\alpha, \gamma)$, since $\nu_{q_{\alpha, \gamma}}^{\otimes d}\left(F_{m}(\alpha, \gamma)\right) \leq 1$ and for each $I \in \mathcal{F}_{n}$ such that $I \cap F_{m}(\alpha, \gamma) \neq \varnothing$ we have $2^{-n\left(\gamma+\epsilon_{m} / 2\right)} \leq \nu_{q_{\alpha, \gamma}}^{\otimes d}(I)$, we have $\#\left\{I \in \mathcal{F}_{n}: I \cap F_{m}(\alpha, \gamma) \neq \varnothing\right\}$ $\leq 2^{n\left(\gamma+\epsilon_{m} / 2\right)}$.

Then, let

$$
\widetilde{E}(\alpha, \gamma, n, \epsilon)=\left\{x \in F_{m}(\alpha, \gamma): \frac{\log \nu_{q_{\alpha, \gamma}}^{\otimes d}\left(I_{n}(x) \cap F_{m}(\alpha, \gamma)\right)}{-n \log (2)} \in\left[\gamma-\epsilon_{m}, \gamma+\epsilon_{m}\right]\right\}
$$

We can find $n_{m}(\alpha, \gamma) \geq n_{m}^{0}(\alpha, \gamma)$ such that

$$
\begin{equation*}
\nu_{q_{\alpha, \gamma}}^{\otimes d}\left(\widetilde{F}_{m}(\alpha, \gamma)\right) \geq 1 / 2-1 / 2^{m}, \text { with } \widetilde{F}_{m}(\alpha, \gamma)=\bigcap_{n \geq n_{m}(\alpha, \gamma)} \widetilde{E}\left(\alpha, \gamma, n, \epsilon_{m}\right) \tag{2.3}
\end{equation*}
$$

Indeed, for $n \geq \widetilde{n}_{m}^{0}(\alpha, \gamma)$ we have

$$
\begin{align*}
\nu_{q_{\alpha, \gamma}}^{\otimes d}\left(F_{m}(\alpha, \gamma)\right. & \left.\backslash \widetilde{E}\left(\alpha, \gamma, n, \epsilon_{m}\right)\right) \\
& =\nu_{q_{\alpha, \gamma}}^{\otimes d}\left(\left\{x \in F_{m}(\alpha, \gamma): \nu_{q_{\alpha, \gamma}}^{\otimes d}\left(I_{n}(x) \cap F_{m}(\alpha, \gamma)\right)<2^{-n\left(\gamma+\epsilon_{m}\right)}\right\}\right) \\
& \leq \sum_{I \in \mathcal{F}_{n}: \nu_{q_{\alpha, \gamma}}^{\otimes d}\left(I \cap F_{m}(\alpha, \gamma)\right)<2^{-n\left(\gamma+\epsilon_{m}\right)}} \nu_{q_{\alpha, \gamma}}^{\otimes d}\left(I \cap F_{m}(\alpha, \gamma)\right) \\
& \leq\left(\#\left\{I \in \mathcal{F}_{n}: I \cap F_{m}(\alpha, \gamma) \neq \varnothing\right\}\right) 2^{-n\left(\gamma+\epsilon_{m}\right)} \\
& \leq 2^{n\left(\gamma+\epsilon_{m} / 2\right)} 2^{-n\left(\gamma+\epsilon_{m}\right)}=2^{-n \epsilon_{m} / 2} \tag{2.4}
\end{align*}
$$

so (2.3) follows if we choose $n_{m}(\alpha, \gamma)$ such that $\sum_{n \geq n_{m}(\alpha, \gamma)} 2^{-n \epsilon_{m}} \leq 2^{-m}$.
We define

$$
\begin{equation*}
\mu_{\alpha, \gamma}=\nu_{p_{\alpha, \gamma}}^{\otimes d} \text { and } \nu_{\alpha, \gamma}=\nu_{q_{\alpha, \gamma}}^{\otimes d}\left(\cdot \cap F_{m}(\alpha, \gamma)\right) \tag{2.5}
\end{equation*}
$$

We can now gather a series of properties which will be used in the proofs of our main results.

Properties 2.1. - Let $m \in \mathbb{N}_{+}, 0 \leq \gamma \leq d$ and $\alpha \geq \gamma$.
(1) If $N \geq n_{m}(\alpha, \gamma)$, by construction we have

$$
\begin{equation*}
1 / 2 \leq \nu_{\alpha, \gamma}\left(F_{m}(\alpha, \gamma)\right)=\sum_{I \in \mathscr{F}_{N}: I \cap F_{m}(\alpha, \gamma) \neq \varnothing} \nu_{\alpha, \gamma}(I) \leq 1 \tag{2.6}
\end{equation*}
$$

(notice that if $\alpha=\gamma$, by construction the above property also holds for $\mu_{\alpha, \alpha}$ ). Consequently, since for each $I \in \mathcal{F}_{N}$ such that $I \cap F_{m}(\alpha, \gamma) \neq \varnothing$ we have $2^{-N\left(\gamma+\epsilon_{m}\right)} \leq \nu_{\alpha, \gamma}(I) \leq$ $2^{-N\left(\gamma-\epsilon_{m}\right)}$, we get

$$
\begin{equation*}
2^{-1} 2^{N\left(\gamma-\epsilon_{m}\right)} \leq \#\left\{I \in \mathcal{F}_{N}: I \cap F_{m}(\alpha, \gamma) \neq \varnothing\right\} \leq 2^{N\left(\gamma+\epsilon_{m}\right)} . \tag{2.7}
\end{equation*}
$$

By construction of $\widetilde{F}_{m}(\alpha, \gamma)$, we also have

$$
\begin{equation*}
\sum_{I \in \mathcal{F}_{N}: I \cap \widetilde{F}_{m}(\alpha, \gamma)=\varnothing} \nu_{\alpha, \gamma}(I) \leq 2^{-m} \tag{2.8}
\end{equation*}
$$

(2) If $J$ is a dyadic cube of generation $n \geq n_{m}(\alpha, \gamma), J \cap F_{m}(\alpha, \gamma) \neq \varnothing$, and $N \geq n$, then we have by construction

$$
\begin{equation*}
\nu_{\alpha, \gamma}(J)=\sum_{\substack{J \supset I \in \mathcal{G}_{N}, I \cap F_{m}(\alpha, \gamma) \neq \varnothing}} \nu_{\alpha, \gamma}(I) \leq|J|^{\gamma-\epsilon_{m}}, \tag{2.9}
\end{equation*}
$$

and if, moreover, $J \cap \widetilde{F}_{m}(\alpha, \gamma) \neq \varnothing$,

$$
\begin{equation*}
|J|^{\gamma+\epsilon_{m}} \leq \nu_{\alpha, \gamma}(J)=\sum_{\substack{J \supset I \in \mathcal{F}_{N}, I \cap F_{m}(\alpha, \gamma) \neq \varnothing}} \nu_{\alpha, \gamma}(I) \leq|J|^{\gamma-\epsilon_{m}} . \tag{2.10}
\end{equation*}
$$

(3) If $J$ is a dyadic cube of generation $n \geq n_{m}(\alpha, \gamma), J \cap F_{m}(\alpha, \gamma) \neq \varnothing$, and $N \geq n$, we also have

$$
\begin{equation*}
\sum_{\substack{J \supset I \in \mathcal{F}_{N}, I \cap F_{m}(\alpha, \gamma) \neq \varnothing}} \mu_{\alpha, \gamma}(I) \leq \mu_{\alpha, \gamma}(J) \leq|J|^{\alpha-\epsilon_{m}} . \tag{2.11}
\end{equation*}
$$

Also, (2.9) implies

$$
\#\left\{J \supset I \in \mathcal{F}_{N}: I \cap F_{m}(\alpha, \gamma) \neq \varnothing\right\} \leq|J|^{\gamma-\epsilon_{m}} 2^{N\left(\gamma+\epsilon_{m}\right)}
$$

hence

$$
\begin{equation*}
\sum_{\substack{J \supset I \in \mathscr{G}_{N}, I \cap F_{m}(\alpha, \gamma) \neq \varnothing}} \mu_{\alpha, \gamma}(I) \leq 2^{-n\left(\gamma-\epsilon_{m}\right)} 2^{-N\left(\alpha-\gamma-2 \epsilon_{m}\right)} . \tag{2.12}
\end{equation*}
$$

## 3. Prescription of lower Hausdorff spectra: Proof of Theorem 1.5

### 3.1. Construction of $\mu$ and estimates of its local dimension

Setting apart some important details omitted in the outline provided in Section 1, the proof we present for the general case will be more sophisticated because: (1) $f$ is only upper semi-continuous; (2) $f$ may have more than one fixed point; (3) dom $(f)$ may contain $\infty$; (4) we will manage that all the sets $\underline{E}(\mu, \beta)$ are substantially big when they are not empty, this meaning that they contain a Cantor set. Referring to our sketch of proof, since we will use Bernoulli measures for $\mu_{\alpha_{i}, f\left(\alpha_{i}\right)}$ and $\nu_{\alpha_{i}, f\left(\alpha_{i}\right)}$, to getting such Cantor sets necessitates to avoid using couples $\left(\alpha_{i}, f\left(\alpha_{i}\right)\right)$ for which $f\left(\alpha_{i}\right)=0$ in the construction; indeed in this case we know that the Bernoulli measure $\nu_{\alpha_{i}, f\left(\alpha_{i}\right)}$ is a Dirac mass. These couples will be replaced by couples $\left(\alpha_{i}, \gamma_{m}\left(\alpha_{i}\right)\right)$, where $0<\gamma_{m}\left(\alpha_{i}\right)$ tends to 0 as $m \rightarrow \infty$. In particular, to illustrate
the case where $f(0)=0$ and $f(\alpha)=-\infty$ for all $\alpha \neq 0$, instead of taking $\mu=\delta_{x}$ for some $x \in \mathbb{R}^{d}$, we will construct a continuous measure on a Cantor set of dimension 0 .
3.1.1. Construction of the measure $\mu$. - We will denote $\operatorname{dom}(f)$ by $才$. Let $D \in \operatorname{Fix}(f)$. Due to our assumption requiring the upper semi-continuity of $f$, there exists a dense countable subset $\Delta$ of $\mathcal{I} \backslash\{\infty\}$ such that for all $\alpha \in \mathcal{J} \backslash\{\infty\}$, there exists a sequence $\left(\alpha_{n}\right)_{n \geq 1}$ in $\Delta^{\mathbb{N}_{+}}$ such that $\lim _{n \rightarrow \infty}\left(\alpha_{n}, f\left(\alpha_{n}\right)\right)=(\alpha, f(\alpha))$ (if $\infty \in \mathscr{f}$ and $f$ is continuous at $\infty$ the same holds at ( $\infty, f(\infty)$ ), but we will not need this property). This important fact is elementary (see for instance [56, Lemma 2] for a proof when $\operatorname{dom}(f)$ is a compact subset of $\mathbb{R}$; the general case considered here then follows immediately).

We fix once for all such a $\Delta$, and assume without loss of generality that it contains a dense subset of $\operatorname{Fix}(f)$.

Let $\alpha_{\text {min }}=\min (Л)$ and $\alpha_{\text {max }}=\max (Л) \in \mathbb{R}_{+} \cup\{\infty\}$.
If $\Delta \backslash\{0, D\} \neq \varnothing$, enumerate its elements in a sequence $\left(\alpha_{j}^{\Delta}\right)_{j \geq 1}$ (with necessary redundancies if $\Delta \backslash\{0, D\}$ is finite), and for each $m \in \mathbb{N}$ set

$$
\widetilde{A}_{m}=\left\{\alpha_{j}^{\Delta}: 1 \leq j \leq m, \alpha_{j}^{\Delta} \geq 4 \epsilon_{m}\right\}
$$

otherwise set $\widetilde{A}_{m}=\varnothing$. Also set

$$
\left\{\begin{array}{l}
D_{m}=2 \epsilon_{m} \text { if } D=0, \text { and } D_{m}=D \text { otherwise }, \\
\alpha_{m}(0)=D_{m} \text { if } D=0 \text { and } \alpha_{m}(0)=\min \left(2 \epsilon_{m}, D\right) / 2 \text { otherwise }, \\
\alpha_{m}(\infty)=\left(\max \left(d, m, \max \left(\alpha_{j}^{\Delta}: 1 \leq j \leq m\right)\right)^{2} \text { if } \alpha_{\max }=\infty .\right.
\end{array}\right.
$$

Then let

$$
A_{m}= \begin{cases}\widetilde{A}_{m} \cup\left\{D_{m}\right\} & \text { if } 0<\alpha_{\min } \leq \alpha_{\max }<\infty \\ \widetilde{A}_{m} \cup\left\{D_{m}\right\} \cup\left\{\alpha_{m}(\infty)\right\} & \text { if } 0<\alpha_{\min } \text { and } \alpha_{\max }=\infty, \\ \widetilde{A}_{m} \cup\left\{D_{m}\right\} \cup\left\{\alpha_{m}(0)\right\} & \text { if } \alpha_{\min }=0 \text { and } \alpha_{\max }<\infty, \\ \widetilde{A}_{m} \cup\left\{D_{m}\right\} \cup\left\{\alpha_{m}(0)\right\} \cup\left\{\alpha_{m}(\infty)\right\} & \text { if } \alpha_{\min }=0 \text { and } \alpha_{\max }=\infty .\end{cases}
$$

For $\alpha \in A_{m}$ let

$$
\gamma_{m}(\alpha)= \begin{cases}f(\alpha) & \text { if } \alpha \in \widetilde{A}_{m} \text { and } f(\alpha)>0 \\ f(\infty) & \text { if } \alpha=\alpha_{m}(\infty) \text { and } f(\infty)>0 \\ \alpha & \text { if } \alpha=D_{m} \\ \alpha_{m}(0) & \text { if } \alpha=\alpha_{m}(0) \\ \epsilon_{m} & \text { if } \alpha \in \widetilde{A}_{m} \text { and } f(\alpha)=0 \\ \epsilon_{m} & \text { if } \alpha=\alpha_{m}(\infty) \text { and } f(\infty)=0\end{cases}
$$

Notice that $\gamma_{m}(\alpha) \leq \alpha$ for all $\alpha \in A_{m}$. For the values of $\alpha \in A_{m} \backslash\{0, D\}$ such that $f(\alpha)=0$, we choose $\gamma_{m}(\alpha)>0$ so that the measure $\nu_{\alpha, \gamma_{m}(\alpha)}$ be continuous but of dimension tending to 0 as $m \rightarrow \infty$, and in our construction no level set $\underline{E}(\mu, \beta)$ be supported on a countable set; indeed, with this choice every non empty such set will contain a Cantor set when $f(\beta) \geq 0$. The choice of $\alpha_{m}(0)$ and $\gamma_{m}\left(\alpha_{m}(0)\right)$, and that of $D_{m}$ when $D=0$, correspond to the same goal.

Using the definitions of Section 2, for $\alpha \in A_{m}$, set

$$
\mu_{\alpha}=\mu_{\alpha, \gamma_{m}(\alpha)} \text { and } \nu_{\alpha}=\nu_{\alpha, \gamma_{m}(\alpha)} \text { for } \alpha \notin \operatorname{Fix}\left(\gamma_{m}\right) \text { and } \mu_{\alpha}=\nu_{\alpha}=\nu_{\alpha, \alpha} \text { for } \alpha \in \operatorname{Fix}\left(\gamma_{m}\right) .
$$

Strictly speaking, $\mu_{\alpha}$ and $\nu_{\alpha}$ should be written $\mu_{m, \alpha}$ and $\nu_{m, \alpha}$, but for the sake of readability we will omit the index $m$.

Also, let

$$
n_{m}=\max \left\{n_{m}\left(\alpha, \gamma_{m}(\alpha)\right): \alpha \in A_{m}\right\}
$$

Now let $\left(N_{m}\right)_{n \in \mathbb{N}}$ be an increasing sequence of integers defined recursively satisfying the following properties:

$$
\left\{\begin{array}{l}
\forall m \geq 1, N_{m} \geq n_{m},  \tag{3.1}\\
\max \left(\left(m+\# A_{m}+\max \left(A_{m}\right)\right)^{2}, n_{m}\right)=o\left(\sqrt{N_{m-1}}\right) \text { as } m \rightarrow \infty, \\
\left(\max \left(\{1\} \cup A_{m-1}\right)\right) \sum_{i=1}^{m-1} N_{i}=o\left(\min \left(\{1\} \cup \gamma_{m}\left(A_{m}\right)\right) \sqrt{N_{m}}\right) \text { as } m \rightarrow \infty
\end{array}\right.
$$

For each $m \geq 1$ and $\alpha \in A_{m}$ set

$$
G_{m}(\alpha)=\left\{I \in \mathscr{F}_{N_{m}}: I \cap F_{m}\left(\alpha, \gamma_{m}(\alpha)\right) \neq \varnothing\right\}
$$

and

$$
\rho_{m}(\alpha)= \begin{cases}1 & \text { if } \alpha=D_{m} \\ 2^{-m} / \# A_{m} & \text { otherwise }\end{cases}
$$

Due to (2.6) we have

$$
\begin{equation*}
2^{-1} \leq \sum_{I \in G_{m}(\alpha)} \nu_{\alpha}(I) \leq 1 \quad\left(\forall \alpha \in A_{m}\right), \tag{3.2}
\end{equation*}
$$

so

$$
\begin{equation*}
2^{-1} \leq \sum_{I \in G_{m}\left(D_{m}\right)} \rho_{m}\left(D_{m}\right)\left(\mu_{D_{m}}(I)=\nu_{D_{m}}(I)\right) \leq 1 \tag{3.3}
\end{equation*}
$$

Also,

$$
\begin{equation*}
\sum_{\alpha \in A_{m} \backslash\left\{D_{m}\right\}} \sum_{I \in G_{m}(\alpha)} \rho_{m}(\alpha) \mu_{\alpha}(I) \leq \sum_{\alpha \in A_{m} \backslash\left\{D_{m}\right\}} \frac{2^{-m}}{\# A_{m}}\left\|\mu_{\alpha}\right\| \leq 2^{-m} . \tag{3.4}
\end{equation*}
$$

For each $m \in \mathbb{N}_{+}$, we enumerate the elements of $A_{m}$ as $\alpha_{m, 1}, \ldots, \alpha_{m, \# A_{m}}$, we denote by $\mathcal{L}_{m}$ the set of disjoint closed dyadic cubes $\mathcal{L}\left(\# A_{m}\right)$ defined in Section 2.1, and denote its elements as $L_{m, \alpha_{m, 1}}, \ldots, L_{m, \alpha_{m, \# A_{m}}}$. We also denote $L_{m, \alpha_{m, i}}$ by $L_{m, i}$ and $\ell\left(\# A_{m}\right)$ by $\ell_{m}$.

We can start the construction of $\mu$. We will construct a Cantor set $K$ by defining recursively the sequence of families of cubes $\left(\mathbf{G}_{m}\right)_{m \in \mathbb{N}}$ such that $K=\bigcap_{m \geq 1} \bigcup_{J \in \mathbf{G}_{m}} J$, and simultaneously a consistent sequence of measures $\mu_{m}$ supported on $\bigcup_{J \in \mathbf{G}_{m}} J$ to get the desired measure $\mu$ on $K$.

Let

$$
\mathbf{G}_{1}=\bigcup_{\alpha_{1, i} \in A_{1}}\left\{L_{1, i} I_{1, i}: I_{1, i} \in G_{1}\left(\alpha_{1, i}\right)\right\}
$$

(the concatenation of intervals has been defined in (2.1)). By construction, the interiors of the elements of $\mathbf{G}_{1}$ are pairwise disjoint. Then the measure $\mu_{1}$ is defined on $\mathbf{G}_{1}$ by

$$
\begin{equation*}
\mu_{1}\left(L_{1, i} I_{1, i}\right)=\frac{\rho_{1}\left(\alpha_{1, i}\right) \mu_{\alpha_{1, i}}\left(I_{1, i}\right)}{\sum_{\alpha \in A_{1}} \sum_{I \in G_{1}(\alpha)} \rho_{1}(\alpha) \mu_{\alpha}(I)} . \tag{3.5}
\end{equation*}
$$

Combining (3.3) and (3.4) we have

$$
2^{-1} \leq \sum_{\alpha \in A_{1}} \sum_{I \in G_{1}(\alpha)} \rho_{1}(\alpha) \mu_{\alpha}(I) \leq 1+2^{-1} .
$$

Consequently, (3.5) yields

$$
\frac{2}{3} \rho_{1}\left(\alpha_{1, i}\right) \mu_{\alpha_{1, i}}\left(I_{1, i}\right) \leq \mu_{1}\left(L_{1, i} I_{1, i}\right) \leq 2 \rho_{1}\left(\alpha_{1, i}\right) \mu_{\alpha_{1, i}}\left(I_{1, i}\right) .
$$

Then, we define recursively for $m \geq 1$ :

$$
\mathbf{G}_{m+1}=\bigcup_{I_{m} \in \mathbf{G}_{m}} \bigcup_{\alpha_{m+1, i} \in A_{m+1}} \mathbf{G}_{m+1}\left(I_{m}, \alpha_{m+1, i}\right)
$$

where

$$
\mathbf{G}_{m+1}\left(I_{m}, \alpha_{m+1, i}\right)=\left\{I_{m} L_{m+1, i} I_{m+1, i}: I_{m+1, i} \in G_{m+1}\left(\alpha_{m+1, i}\right)\right\}
$$

and a measure $\mu_{m+1}$ on $\mathbf{G}_{m+1}$ by setting

$$
\mu_{m+1}\left(I_{m} L_{m+1, i} I_{m+1, i}\right)=\mu_{m}\left(I_{m}\right) \frac{\rho_{m+1}\left(\alpha_{m+1, i}\right) \mu_{\alpha_{m+1, i}}\left(I_{m+1, i}\right)}{\sum_{\alpha \in A_{m+1}} \sum_{I \in G_{m+1}(\alpha)} \rho_{m+1}(\alpha) \mu_{\alpha}(I)},
$$

which by construction satisfies

$$
\begin{equation*}
\frac{1}{1+2^{-m}} \leq \frac{\mu_{m+1}\left(I_{m} L_{m+1, i} I_{m+1, i}\right)}{\mu_{m}\left(I_{m}\right) \rho_{m+1}\left(\alpha_{m+1, i}\right) \mu_{\alpha_{m+1, i}}\left(I_{m+1, i}\right)} \leq 2 \tag{3.6}
\end{equation*}
$$

by (3.3) and (3.4).
Each measure $\mu_{m}$ can be trivially extended to a probability measure on $\mathcal{F}_{g_{m}}$, where $g_{m}=-\log _{2}\left|I_{m}\right|$, that we still denote by $\mu_{m}$. This measure yields an absolutely continuous Borel measure on $[0,1]^{d}$, denoted by $\mu_{m}$ again, whose density with respect to the Lebesgue measure is given by $2^{d g_{m}} \mu_{m}(I)$ over each cube $I \in \mathcal{F}_{g_{m}}$. By construction, the measures $\mu_{m}$ ( $m \in \mathbb{N}$ ) weakly converge to a Borel probability measure $\mu$ on $[0,1]^{d}$, supported on the Cantor set $K$ defined as

$$
K=\bigcap_{m \geq 1} \bigcup_{I \in \mathbf{G}_{m}} I
$$

and satisfying $\mu(I)=\mu_{m}(I)$ for all $m \geq 1$ and $I \in \mathscr{F}_{g_{m}}$. Moreover, since $K$ does not intersect the boundary of any dyadic cube due to the definition of the sets $\mathscr{L}_{m}, \mu$ vanishes on such a boundary.
3.1.2. Estimates of the local dimension of $\mu$. - Let $x \in K$ and $n>g_{1}=N_{1}+\log _{2}\left|L_{1, i_{1}}\right|^{-1}$. There exists a unique $m \in \mathbb{N}$ such that $g_{m}<n \leq g_{m+1}$. By construction, we have $I_{g_{m}}(x) \in \mathbf{G}_{m}$ and $I_{g_{m+1}}(x) \in \mathbf{G}_{m+1}$, and $I_{g_{m+1}}(x) \subset I_{n}(x) \subset I_{g_{m}}(x)$. Moreover, there exist a unique sequence of exponents $\alpha_{1}(x) \in A_{1}, \ldots, \alpha_{m}(x) \in A_{m}, \alpha_{m+1}(x) \in A_{m+1}$, and a unique sequence of pairs of intervals $\left\{L_{j}, I_{j}\right\}_{1 \leq j \leq m+1}$ such that

$$
\begin{equation*}
I_{g_{m}}(x)=L_{1} I_{1} \cdots L_{m} I_{m} \quad \text { and } \quad I_{g_{m+1}}(x)=L_{1} I_{1} \cdots L_{m} I_{m} L_{m+1} I_{m+1} \tag{3.7}
\end{equation*}
$$

with $I_{j} \in G_{j}\left(\alpha_{j}(x)\right)$ and $L_{j} \in \mathcal{L}_{j}$ for each $1 \leq j \leq m+1$. The intervals invoked in (3.7) will be used in the following statements.

Proposition 3.1. - With the notations introduced above, there exists a positive sequence $\left(\delta_{n}\right)_{n \geq 1}$ converging to 0 as $n \rightarrow \infty$ such that, uniformly in $x \in K$, for $g_{m} \leq n \leq g_{m+1}$ one has 1.

$$
\begin{equation*}
\frac{\mu\left(I_{n}(x)\right)}{2^{-g_{m} \alpha_{m}(x)-\left(n-g_{m}\right) \alpha_{m+1}(x)}} \leq 2^{n \delta_{n}} ; \tag{3.8}
\end{equation*}
$$

2. if either $n=g_{m}$, or $\alpha_{m+1}(x) \in \operatorname{Fix}\left(\gamma_{m+1}\right)$ and $I_{m+1} \cap \widetilde{F}_{m+1}\left(\alpha_{m+1}(x), \alpha_{m+1}(x)\right) \neq \varnothing$, then

$$
\begin{equation*}
2^{-n \delta_{n}} \leq \frac{\mu\left(I_{n}(x)\right)}{2^{-g_{m} \alpha_{m}(x)-\left(n-g_{m}\right) \alpha_{m+1}(x)}} \leq 2^{n \delta_{n}} \tag{3.9}
\end{equation*}
$$

3. if $g_{m}+\ell_{m+1}+n_{m+1}<n \leq g_{m+1}$ then

$$
\begin{equation*}
\frac{\mu\left(I_{n}(x)\right)}{2^{-g_{m} \alpha_{m}(x)-\left(n-g_{m}\right) \gamma_{m+1}\left(\alpha_{m+1}(x)\right)-N_{m+1}\left(\alpha_{m+1}(x)-\gamma_{m+1}\left(\alpha_{m+1}(x)\right)\right.}} \leq 2^{n \delta_{n}+2 N_{m+1} \epsilon_{m+1}} \tag{3.10}
\end{equation*}
$$

Corollary 3.1. - For all $x \in K$ we have

1. $\underline{d}(\mu, x)=\liminf _{m \rightarrow \infty} \alpha_{m}(x)$.
2. If for $m$ large enough we have $\alpha_{m}(x) \in \operatorname{Fix}\left(\gamma_{m}\right)$ and $I_{m} \cap \widetilde{F}_{m}\left(\alpha_{m}(x), \alpha_{m}(x)\right) \neq \varnothing$, then $\bar{d}(\mu, x)=\lim \sup _{m \rightarrow \infty} \alpha_{m}(x)$.
3. If $\lim \inf _{m \rightarrow \infty} \alpha_{m}(x) \in \mathcal{I} \backslash \operatorname{Fix}(f)$ then $\bar{d}(\mu, x)=\infty$. In particular, if $\alpha \in \mathcal{I} \backslash \operatorname{Fix}(f)$, then $\underline{E}(\mu, \alpha)=E(\mu, \alpha, \infty)$.

Proof of Proposition 3.1. We will write $\alpha_{j}$ for $\alpha_{j}(x)$.
At first we notice that by construction, and due to (3.6), we have

$$
c_{m} \prod_{j=1}^{m} \mu_{\alpha_{j}}\left(I_{j}\right) \leq \mu\left(I_{g_{m}(x)}\right) \leq 2^{m} \prod_{j=1}^{m} \mu_{\alpha_{j}}\left(I_{j}\right)
$$

where $c_{m}=\prod_{j=1}^{m}\left(1+2^{-m}\right)^{-1} \prod_{j=1}^{m} \rho_{j}\left(\alpha_{j}\right) \geq e^{-1} \prod_{j=1}^{m} \rho_{j}\left(\alpha_{j}\right)$. Then, due to the definition of $G_{j}\left(\alpha_{j}\right)$

$$
\begin{equation*}
c_{m} \exp \left(-\sum_{j=1}^{m}\left(\alpha_{j}+\epsilon_{j}\right) N_{j} \log (2)\right) \leq \mu\left(I_{g_{m}}(x)\right) \leq 2^{m} \exp \left(-\sum_{j=1}^{m}\left(\alpha_{j}-\epsilon_{j}\right) N_{j} \log (2)\right) . \tag{3.11}
\end{equation*}
$$

Proof of (1) and (2): We distinguish two cases. Let $g_{m}^{\prime}=g_{m}+\ell_{m+1}$.
CASE 1: $g_{m}<n \leq g_{m}^{\prime}+n_{m+1}$. Write $I_{g_{m}^{\prime}+n_{m+1}}(x)=I_{g_{m}(x)} L_{m+1} J_{n_{m+1}}$, where $J_{n_{m+1}}=I_{n_{m+1}}\left(\left\{2^{g_{m}^{\prime}} x\right\}\right),\{t\}$ standing for the vector whose entries are the fractional parts
of the entries of $t$. We have $J_{n_{m+1}} \supset I_{m+1}, I_{m+1} \cap F_{m+1}\left(\alpha_{m+1}, \gamma_{m+1}\left(\alpha_{m+1}\right)\right) \neq \varnothing$, and the generation of $J_{n_{m+1}}$ is $n_{m+1} \geq n_{m+1}\left(\alpha_{m+1}, \alpha_{m+1}\right)$.

We obviously have $\mu\left(I_{n}(x)\right) \leq \mu\left(I_{g_{m}}(x)\right)$, and by construction if $\alpha_{m+1} \in \operatorname{Fix}\left(\gamma_{m+1}\right)$ (remembering that $\mu_{\alpha_{m+1}}=\nu_{\alpha_{m+1}}$ and using the equality in (2.9)),

$$
\begin{aligned}
\mu\left(I_{g_{m}}(x)\right) & \geq \mu\left(I_{n}(x)\right) \geq \mu\left(I_{g_{m}^{\prime}+n_{m+1}}(x)\right)=\sum_{J_{n_{m+1}} \supset I \in G_{m+1}\left(\alpha_{m+1}\right)} \mu\left(I_{g_{m}}(x) L_{m} I\right) \\
& =\mu\left(I_{g_{m}}(x)\right) \frac{\sum_{J_{n_{m+1}} \supset I \in G_{m+1}\left(\alpha_{m+1}\right)} \rho_{m+1}\left(\alpha_{m+1}\right) \mu_{\alpha_{m+1}}(I)}{\sum_{\alpha \in A_{m+1}} \sum_{I \in G_{m+1}(\alpha)} \rho_{m+1}(\alpha) \mu_{\alpha}(I)} \\
& =\frac{\rho_{m+1}\left(\alpha_{m+1}\right)}{\sum_{\alpha \in A_{m+1}} \sum_{I \in G_{m+1}(\alpha)} \rho_{m+1}(\alpha) \mu_{\alpha}(I)} \mu\left(I_{g_{m}}(x)\right) \nu_{\alpha_{m+1}}\left(J_{n_{m+1}}\right) \\
& \geq \frac{\rho_{m+1}\left(\alpha_{m+1}\right)}{1+2^{-(m+1)}} \mu\left(I_{g_{m}}(x)\right) \nu_{\alpha_{m+1}}\left(J_{n_{m+1}}\right) \quad \text { (we have used (3.3) and (3.4) again). }
\end{aligned}
$$

Combining this with (3.11), if $\alpha_{m+1} \in \operatorname{Fix}\left(\gamma_{m+1}\right)$, we thus get

$$
c_{m+1} \widetilde{c}_{m}^{-1} \nu_{\alpha_{m+1}}\left(J_{n_{m+1}}\right) \leq \frac{\mu\left(I_{n}(x)\right)}{\exp \left(-\sum_{j=1}^{m} \alpha_{j} N_{j} \log (2)\right)} \leq \widetilde{c}_{m} 2^{m}
$$

with $\widetilde{c}_{m}=\exp \left(\sum_{j=1}^{m} N_{j} \epsilon_{j} \log (2)\right)$. If, moreover, $I_{m+1} \cap \widetilde{F}_{m+1}\left(\alpha_{m+1}, \alpha_{m+1}\right) \neq \varnothing$, then $J_{n_{m+1}} \cap \widetilde{F}_{m+1}\left(\alpha_{m+1}, \alpha_{m+1}\right) \neq \varnothing$, so due to (2.10) we have

$$
c_{m+1} \widetilde{c}_{m}^{-1} 2^{-n_{m+1}\left(\alpha_{m+1}+\epsilon_{m+1}\right)} \leq \frac{\mu\left(I_{n}(x)\right)}{\exp \left(-\sum_{j=1}^{m} \alpha_{j} N_{j} \log (2)\right)} \leq \widetilde{c}_{m} 2^{m}
$$

which finally yields

$$
C_{m}^{-1} \leq \frac{\mu\left(I_{n}(x)\right)}{\exp \left(-\alpha_{m+1}\left(n-g_{m}\right) \log (2)-\sum_{j=1}^{m} \alpha_{j} N_{j} \log (2)\right)} \leq C_{m}
$$

with $C_{m}=c_{m+1}^{-1} \widetilde{c}_{m} 2^{m} 2^{2\left(\ell_{m+1}+n_{m+1}\right)\left(\max \left(A_{m+1}\right)+\epsilon_{m+1}\right)}$. Moreover, it is readily seen that the previous upper bound holds whatever $\alpha_{m+1}$ be.

Now, due to the conditions (3.1) we have imposed to the sequence $N_{m}$ and the definition of $\rho_{j}\left(\alpha_{j}\right)$, we have $\log \left(\widetilde{C}_{m}\right)=o\left(g_{m}\right)$ and

$$
\sup \left\{\sum_{j=1}^{m-1} \alpha_{j} N_{j}:\left(\alpha_{j}\right)_{1 \leq j \leq m-1} \in \prod_{j=1}^{m-1} A_{j}\right\}=o\left(\min \left(A_{m}\right) g_{m}\right)
$$

Consequently, there exists a sequence $\left(\delta_{n}\right)_{n \in \mathbb{N}}$ converging to 0 as $n \rightarrow \infty$, such that (3.8) and (3.9) hold uniformly in $x \in K$ and $g_{m}<n \leq g_{m}^{\prime}+n_{m+1}$.

CASE 2: $g_{m}^{\prime}+n_{m+1}<n \leq g_{m+1}$. Write $I_{n}(x)=I_{g_{m}}(x) L_{m+1} J_{n-g_{m}^{\prime}}$, where $J_{n-g_{m}^{\prime}}=$ $I_{n-g_{m}^{\prime}}\left(\left\{2^{g_{m}^{\prime}} x\right\}\right)$. We have $J_{n-g_{m}^{\prime}} \supset I_{m+1}, I_{m+1} \cap F_{m+1}\left(\alpha_{m+1}, \gamma_{m+1}(\alpha)\right) \neq \varnothing$, and the generation of $J_{n-g_{m}^{\prime}}$ is $n-g_{m}^{\prime} \geq n_{m+1} \geq n_{m+1}\left(\alpha_{m+1}, \alpha_{m+1}\right)$.

By construction:

$$
\begin{aligned}
\mu\left(I_{n}(x)\right) & =\sum_{J_{n-g_{m}^{\prime}} \supset I \in G_{m+1}\left(\alpha_{m+1}\right)} \mu\left(I_{g_{m}}(x) L_{m} I\right) \\
& =\mu\left(I_{g_{m}}(x)\right) \frac{\sum_{J_{n-g_{m}^{\prime}} \supset I \in G_{m+1}\left(\alpha_{m+1}\right)} \rho_{m+1}\left(\alpha_{m+1}\right) \mu_{\alpha_{m+1}}(I)}{\sum_{\alpha \in A_{m+1}} \sum_{I \in G_{m+1}(\alpha)} \rho_{m+1}(\alpha) \mu_{\alpha}(I)} \\
& \leq \frac{\rho_{m+1}\left(\alpha_{m+1}\right)}{\sum_{\alpha \in A_{m+1}} \sum_{I \in G_{m+1}(\alpha)} \rho_{m+1}(\alpha) \mu_{\alpha}(I)} \mu\left(I_{g_{m}}(x)\right) \mu_{\alpha_{m+1}}\left(J_{n-g_{m}^{\prime}}\right) \quad \text { using (2.11), }
\end{aligned}
$$

with equality if $\alpha_{m+1} \in \operatorname{Fix}\left(\gamma_{m+1}\right)$, remembering that in this case $\mu_{\alpha_{m+1}}=\nu_{\alpha_{m+1}}$, and using the equality in (2.9). Consequently,

$$
\mu\left(I_{n}(x)\right) \leq 2 \rho_{m+1}\left(\alpha_{m+1}\right) \mu\left(I_{g_{m}}(x)\right) \mu_{\alpha_{m+1}}\left(J_{n-g_{m}^{\prime}}\right),
$$

and

$$
\frac{\rho_{m+1}\left(\alpha_{m+1}\right)}{1+2^{-(m+1)}} \mu\left(I_{g_{m}}(x)\right) \nu_{\alpha_{m+1}}\left(J_{n-g_{m}^{\prime}}\right) \leq \mu\left(I_{n}(x)\right) \leq 2 \rho_{m+1}\left(\alpha_{m+1}\right) \mu\left(I_{g_{m}}(x)\right) \nu_{\alpha_{m+1}}\left(J_{n-g_{m}^{\prime}}\right)
$$

if $\alpha_{m+1} \in \operatorname{Fix}\left(\gamma_{m+1}\right)$ (we have used (3.3) and (3.4) again). Set $C_{m, n}=\widetilde{C}_{m} 2^{\left(n-g_{m}^{\prime}\right) \epsilon_{m+1}}$. The previous estimates combined with (3.11) and the estimates (2.11) and (2.10) of $\mu_{\alpha_{m+1}}\left(J_{n-g_{m}^{\prime}}\right)$ and $\nu_{\alpha_{m+1}}\left(J_{n-g_{m}^{\prime}}\right)$ respectively, yield:

$$
\frac{\mu\left(I_{n}(x)\right)}{\exp \left(-\alpha_{m+1}\left(n-g_{m}\right) \log (2)-\sum_{j=1}^{m} \alpha_{j} N_{j} \log (2)\right)} \leq C_{m, n}
$$

and

$$
C_{m, n}^{-1} \leq \frac{\mu\left(I_{n}(x)\right)}{\exp \left(-\alpha_{m+1}\left(n-g_{m}\right) \log (2)-\sum_{j=1}^{m} \alpha_{j} N_{j} \log (2)\right)} \leq C_{m, n}
$$

if $\alpha_{m+1} \in \operatorname{Fix}\left(\gamma_{m+1}\right)$ and $I_{m+1} \cap \widetilde{F}_{m+1}\left(\alpha_{m+1}, \alpha_{m+1}\right) \neq \varnothing$. Then, due to (3.1) again, the above sequence $\left(\delta_{n}\right)_{n \in \mathbb{N}}$ can be modified so that (3.8) also holds uniformly in $x \in K$, and $g_{m}^{\prime}+n_{m+1}<n \leq g_{m+1}$.

Proof of (3). - It suffices to use (2.12) instead of (2.11) to estimate $\mu_{\alpha_{m+1}}\left(J_{n-g_{m}^{\prime}}\right)$ in the previous upper bounds for $\mu\left(I_{n}(x)\right)$.

Proof of Corollary 3.1. - (1) and (2) follow readily from (3.8) and (3.9), and the fact that by construction the neighboring dyadic cubes of generation $n$ of $I_{n}(x)$ have a $\mu$-mass equal to 0 or for which the estimates of Proposition 3.1(1)(2) also hold.

For (3), suppose that $\alpha=\liminf _{m \rightarrow \infty} \alpha_{m}(x) \in I \backslash \operatorname{Fix}(f)$. Since $\operatorname{Fix}(f)$ is closed (because $f(\alpha) \leq \alpha$ and $f$ is upper semi-continuous), there exists a subsequence $\left(m_{k}\right)_{k \geq 1}$
such that $\alpha_{m_{k}+1}(x)$ converges to $\alpha$ and $\gamma_{m_{k}+1}\left(\alpha_{m_{k}+1}(x)\right)$ converges to $\left.f(\alpha)<\alpha\right)$. Take $n=n(k)=g_{m_{k}}+\ell_{m_{k}+1}+n_{m_{k}+1}+\sqrt{N_{m_{k}+1}}$. By construction $g_{m_{k}}=o(n)$ and $n-g_{m} \sim n \sim \sqrt{N_{m_{k}+1}}$, and using (3.10) we have

$$
\frac{\mu\left(I_{n}(x)\right)}{2^{o(n)-n \gamma_{m_{k}+1}\left(\alpha_{m_{k}+1}(x)\right)-n^{2}\left(\alpha_{m_{k}+1}(x)-\gamma_{m_{k}+1}\left(\alpha_{m_{k}+1}(x)\right)\right.}} \leq 2^{n \delta_{n}+2 n^{2} \epsilon_{m_{k}+1}}
$$

so

$$
\frac{\mu\left(I_{n}(x)\right)}{-n \log (2)} \geq \frac{\gamma_{m_{k}+1}\left(\alpha_{m_{k}+1}(x)\right)}{\log (2)}+n \frac{\alpha_{m_{k}+1}(x)-\gamma_{m_{k}+1}\left(\alpha_{m_{k}+1}(x)\right)}{\log (2)}+o(n) .
$$

Letting $k$ tend to $\infty$ yields the desired conclusion on $\bar{d}(\mu, x)$, again because the neighboring dyadic cubes of generation $n$ of $I_{n}(x)$ have a $\mu$-mass equal to 0 or with the same behavior as $\mu\left(I_{n}(x)\right)$. Now, if $\alpha \in I \backslash \operatorname{Fix}(f)$ and $x \in \underline{E}(\mu, \alpha)$, then $\left(\alpha_{m}(x)\right)_{m \geq 1}$ must take infinitely many values in $I \backslash \operatorname{Fix}(f)$ since $\operatorname{Fix}(f)$ is a closed set, and consequently $\underline{E}(\mu, \alpha) \subset E(\mu, \alpha, \infty)$.

### 3.2. Auxiliary measures and lower bound for the lower Hausdorff spectrum

### 3.2.1. Construction of auxiliary measures. - Let $\widehat{\alpha}=\left(\alpha_{m}\right)_{m \geq 1} \in \prod_{m=1}^{\infty} A_{m}$.

Now, we construct a measure $\nu_{\widehat{\alpha}}$ as follows: Let

$$
\mathbf{G}_{\widehat{\alpha}, 1}=\left\{L_{1, \alpha_{1}} I_{1}: I_{1} \in G_{1}\left(\alpha_{1}\right)\right\}
$$

and define on $\mathbf{G}_{\widehat{\alpha}, 1}$ the measure

$$
\begin{equation*}
\nu_{\widehat{\alpha}, 1}\left(L_{1, \alpha_{1}} I_{1}\right)=\frac{\nu_{\alpha_{1}}\left(I_{1}\right)}{\sum_{I \in G_{1}\left(\alpha_{1}\right)} \nu_{\alpha_{1}}(I)} . \tag{3.12}
\end{equation*}
$$

Due to (3.2), (3.12) yields

$$
\nu_{\alpha_{1}}\left(I_{1}\right) \leq \nu_{\widehat{\alpha}, 1}\left(L_{1, \alpha_{1}} I_{1}\right) \leq 2 \nu_{\alpha_{1}}\left(I_{1}\right) .
$$

Then, recursively we define for $m \geq 1$

$$
\mathbf{G}_{\widehat{\alpha}, m+1}=\bigcup_{I_{m} \in \mathbf{G}_{\widehat{\alpha}, m}} \mathbf{G}_{\widehat{\alpha}, m+1}\left(I_{m}, \alpha_{m+1}\right),
$$

where

$$
\mathbf{G}_{\widehat{\alpha}, m+1}\left(I_{m}, \alpha_{m+1}\right)=\left\{I_{m} L_{m+1, \alpha_{m+1}} I_{m+1}: I_{m+1} \in G_{m+1}\left(\alpha_{m+1}\right)\right\}
$$

and a measure $\nu_{\widehat{\alpha}, m+1}$ on $\mathbf{G}_{\widehat{\alpha}, m+1}$ by

$$
\nu_{\widehat{\alpha}, m+1}\left(I_{m} L_{m+1, \alpha_{m+1}} I_{m+1}\right)=\nu_{\widehat{\alpha}, m}\left(I_{m}\right) \frac{\nu_{\alpha_{m+1}}\left(I_{m+1}\right)}{\sum_{I \in G_{m+1}\left(\alpha_{m+1}\right)} \nu_{\alpha_{m+1}}(I)} .
$$

Due to (3.2) we have

$$
\begin{equation*}
1 \leq \frac{\nu_{\widehat{\alpha}, m+1}\left(I_{m} L_{m+1, \alpha_{m+1}} I_{m+1}\right)}{\nu_{\widehat{\alpha}, m}\left(I_{m}\right) \nu_{\alpha_{m+1}}\left(I_{m+1}\right)} \leq 2 . \tag{3.13}
\end{equation*}
$$

Each measure $\nu_{\widehat{\alpha}, m}$ can be trivially extended into a probability measure on $\mathcal{F}_{g_{m}}$; we still denote this measure by $\nu_{\widehat{\alpha}, m}$. This measure yields an absolutely continuous Borel measure on $[0,1]^{d}$, denoted by $\nu_{\widehat{\alpha}, m}$ again, whose density with respect to the Lebesgue measure is given by $2^{d g_{m}} \nu_{\widehat{\alpha}, m}(I)$ over each cube $I \in \mathcal{F}_{g_{m}}$. By construction, the measures $\nu_{\widehat{\alpha}, m}(m \in \mathbb{N})$
converge weakly to a Borel probability measure $\nu_{\widehat{\alpha}}$ on $[0,1]^{d}$, supported on the Cantor set $K_{\widehat{\alpha}}$ defined as

$$
K \supset K_{\widehat{\alpha}}=\bigcap_{m \geq 1} \bigcup_{J \in \mathbf{G}_{\widehat{\alpha}, m}} J
$$

and satisfying $\nu_{\widehat{\alpha}}(I)=\nu_{\widehat{\alpha}, m}(I)$ for all $m \geq 1$ and $I \in \mathcal{F}_{g_{m}}$.
Estimation of the local dimension of $\nu_{\widehat{\alpha}}$. We use the same notations as in Section 3.1.2. Let $x \in K_{\widehat{\alpha}}$ and $n>g_{1}=N_{1}+\ell_{1}$. There exists a unique $m \in \mathbb{N}$ such that $g_{m}<n \leq g_{m+1}$. By construction, we have $I_{g_{m}}(x) \in \mathbf{G}_{\widehat{\alpha}, m}$ and $I_{g_{m+1}}(x) \in \mathbf{G}_{\widehat{\alpha}, m+1}$, and $I_{g_{m+1}}(x) \subset I_{n}(x) \subset$ $I_{g_{m}}(x)$. Moreover, we have (3.7).

Proposition 3.2. - There exists a positive sequence $\left(\delta_{n}\right)_{n \geq 1}$ converging to 0 as $n \rightarrow \infty$ such that, for $\nu_{\widehat{\alpha}}$-almost every $x \in K_{\widehat{\alpha}}$, for $n$ large enough,

$$
\begin{equation*}
2^{-n \delta_{n}} \leq \frac{\nu_{\widehat{\alpha}}\left(I_{n}(x)\right)}{2^{-g_{m} \gamma_{m}\left(\alpha_{m}\right)-\left(n-g_{m}\right) \gamma_{m+1}\left(\alpha_{m+1}\right)}} \leq 2^{n \delta_{n}} \tag{3.14}
\end{equation*}
$$

The previous proposition, the fact that by construction the neighboring dyadic cubes of generation $n$ of $I_{n}(x)$ have a $\nu_{\widehat{\alpha}}$-mass equal to 0 or for which the estimates (3.14) hold, and the mass distribution principle (Section 6) yield the Hausdorff and packing dimensions of $\nu_{\widehat{\alpha}}$ :

COROLLARY 3.2. - We have $\quad \underline{d}\left(\nu_{\widehat{\alpha}}, x\right)=\liminf _{m \rightarrow \infty} \gamma_{m}\left(\alpha_{m}\right) \quad$ and $\quad \bar{d}\left(\nu_{\widehat{\alpha}}, x\right)=$ $\limsup _{m \rightarrow \infty} \gamma_{m}\left(\alpha_{m}\right)$ for $\nu_{\widehat{\alpha}}$-almost every $x$. Consequently, $\operatorname{dim}_{H}\left(\nu_{\widehat{\alpha}}\right)=\liminf _{m \rightarrow \infty} \gamma\left(\alpha_{m}\right)$ and $\operatorname{dim}_{P}\left(\nu_{\widehat{\alpha}}\right)=\limsup \operatorname{sum}_{m \rightarrow \infty} \gamma\left(\alpha_{m}\right)$.

Proof of Proposition 3.2. At first, we prove the following fact: there exists $\widetilde{K}_{\widehat{\alpha}} \subset K_{\widehat{\alpha}}$, of full $\nu_{\widehat{\alpha}}$-measure, such that for all $x \in \widetilde{K}_{\widehat{\alpha}}$, for $m$ large enough, we have $I_{m} \cap \widetilde{F}_{m}\left(\alpha_{m}, \gamma_{m}\left(\alpha_{m}\right)\right) \neq \varnothing$, with $I_{m}$ defined as in (3.7).

Indeed, due to the multiplicative structure of $\nu_{\widehat{\alpha}}$, by construction we have

$$
\begin{aligned}
\nu_{\widehat{\alpha}}\left(\left\{x \in K_{\widehat{\alpha}}:\right.\right. & \left.\left.I_{m} \cap \widetilde{F}_{m}\left(\alpha_{m}, \gamma_{m}\left(\alpha_{m}\right)\right)=\varnothing\right\}\right) \\
& =\sum_{I \in \mathbf{G}_{\widehat{\alpha}, m-1}} \nu_{\widehat{\alpha}, m-1}(I) \frac{\sum_{J \in \mathcal{F}_{N_{m}}, J \cap \widetilde{F}_{m}\left(\alpha_{m}, \gamma_{m}\left(\alpha_{m}\right)\right)=\varnothing} \nu_{\alpha_{m}}(J)}{\nu_{\alpha_{m}}(J)} \\
& \leq \sum_{I \in \mathcal{G}_{\widehat{\alpha}, m-1}, J \cap F_{m}\left(\alpha_{m}, \gamma_{m}\left(\alpha_{m}\right)\right) \neq \varnothing} \nu_{\widehat{\alpha}, m-1}(I) \frac{2^{-m}}{2^{-1}}=2^{-(m-1)},
\end{aligned}
$$

where we have used the left hand side of (3.2), and (2.8). Then, by the Borel-Cantelli lemma, we have the desired conclusion.

Now the proof of the proposition follows from lines similar to those used to prove Proposition 3.1:

Let $x \in \widetilde{K}_{\widehat{\alpha}}$. At first we notice that by construction, and due to (3.13) we have

$$
\begin{equation*}
\prod_{j=1}^{m} \nu_{\alpha_{j}}\left(I_{j}\right) \leq \nu_{\widehat{\alpha}}\left(I_{g_{m}(x)}\right) \leq 2^{m} \prod_{j=1}^{m} \nu_{\alpha_{j}}\left(I_{j}\right) \tag{3.15}
\end{equation*}
$$

Then, due to the definition of $G_{j}\left(\alpha_{j}\right)$,

$$
\widetilde{c}_{m}^{-1} \leq \frac{\nu_{\hat{\alpha}}\left(I_{g_{m}}(x)\right)}{\exp \left(-\sum_{j=1}^{m} \gamma_{j}\left(\alpha_{j}\right) N_{j} \log (2)\right)} \leq 2^{m} \widetilde{c}_{m}
$$

where $\widetilde{c}_{m}$ is defined as in the proof of Proposition 3.1.
We distinguish two cases.

CASE 1: $g_{m}<n \leq g_{m}^{\prime}+n_{m+1}$. Write $I_{g_{m}^{\prime}+n_{m+1}}(x)=I_{g_{m}(x)} L_{m+1} J_{n_{m+1}}$, where $J_{n_{m+1}}=I_{n_{m+1}}\left(\left\{2^{g_{m}^{\prime}} x\right\}\right)$. We have

$$
\begin{aligned}
\nu_{\widehat{\alpha}}\left(I_{g_{m}}(x)\right) & \geq \nu_{\widehat{\alpha}}\left(I_{n}(x)\right) \geq \nu_{\widehat{\alpha}}\left(I_{g_{m}^{\prime}+n_{m+1}}(x)\right)=\sum_{J_{n_{m+1}} \supset I \in G_{m+1}\left(\alpha_{m+1}\right)} \nu_{\widehat{\alpha}}\left(I_{g_{m}}(x) L_{m} I\right) \\
& =\nu_{\widehat{\alpha}}\left(I_{g_{m}}(x)\right) \frac{\sum_{J_{m+1}} \supset I \in G_{m+1}\left(\alpha_{m+1}\right)}{\nu_{\alpha_{m+1}}(I)} \nu_{I \in G_{m+1}\left(\alpha_{m+1}\right)} \nu_{\alpha_{m+1}}(I) \\
& =\frac{\nu_{\widehat{\alpha}}\left(I_{g_{m}}(x)\right) \nu_{\alpha_{m+1}}\left(J_{n_{m+1}}\right)}{\sum_{I \in G_{m+1}\left(\alpha_{m+1}\right)} \nu_{\alpha_{m+1}}(I)} \geq \nu_{\widehat{\alpha}}\left(I_{g_{m}}(x)\right) \nu_{\alpha_{m+1}}\left(J_{n_{m+1}}\right)
\end{aligned}
$$

where we have used the right hand side of (3.2). For $m$ large enough so that $I_{m+1} \cap \widetilde{F}_{m+1}\left(\alpha_{m+1}, \gamma_{m+1}\left(\alpha_{m+1}\right)\right) \neq \varnothing$ we have $J_{n_{m+1}} \cap \widetilde{F}_{m+1}\left(\alpha_{m+1}, \gamma_{m+1}\left(\alpha_{m+1}\right)\right) \neq \varnothing$ and the generation of $J_{n_{m+1}}$ is $n_{m+1} \geq n\left(\alpha_{m+1}, \gamma_{m+1}\left(\alpha_{m+1}\right)\right)$, so (2.10) holds for $J_{n_{m+1}}$, and this combined with (3.15) and (2.10) yields

$$
\widetilde{c}_{m}^{-1} 2^{-n_{m+1}\left(\gamma_{m+1}\left(\alpha_{m+1}\right)+\epsilon_{m+1}\right)} \leq \frac{\nu_{\widehat{\alpha}}\left(I_{n}(x)\right)}{\exp \left(-\sum_{j=1}^{m} \gamma_{j}\left(\alpha_{j}\right) N_{j} \log (2)\right)} \leq 2^{m} \widetilde{c}_{m}
$$

Consequently,

$$
\begin{equation*}
\widetilde{C}_{m}^{-1} \leq \frac{\nu_{\widehat{\alpha}}\left(I_{n}(x)\right)}{\exp \left(-\gamma_{m+1}\left(\alpha_{m+1}\right)\left(n-g_{m}\right) \log (2)-\sum_{j=1}^{m} \gamma_{j}\left(\alpha_{j}\right) N_{j} \log (2)\right)} \leq \widetilde{C}_{m} \tag{3.16}
\end{equation*}
$$

with $\widetilde{C}_{m}=\widetilde{c}_{m} 2^{m} 2^{2\left(\ell_{m+1}+n_{m+1}\right)\left(\max \left(\gamma_{m+1}\left(A_{m+1}\right)\right)+\epsilon_{m+1}\right)}$.
Due to the conditions (3.1) we have imposed to the sequence $\left(N_{m}\right)_{m \geq 1}$, we have $\log \left(\widetilde{C}_{m}\right)=o\left(g_{m}\right)$ and

$$
\sup \left\{\sum_{j=1}^{m-1} \gamma_{j}\left(\alpha_{j}\right) N_{j}:\left(\alpha_{j}\right)_{1 \leq j \leq m-1} \in \prod_{j=1}^{m-1} A_{j}\right\}=o\left(\min \left(\gamma_{m}\left(A_{m}\right)\right) g_{m}\right) .
$$

Consequently, there exists a sequence $\left(\delta_{n}\right)_{n \in \mathbb{N}}$ converging to 0 as $n \rightarrow \infty$, such that (3.14) holds for all $x \in \widetilde{K}_{\widehat{\alpha}}$, for $m$ large enough and $g_{m}<n \leq g_{m}^{\prime}+n_{m+1}$.

CASE 2: $g_{m}^{\prime}+n_{m+1}<n \leq g_{m+1}$. Write $I_{n}(x)=I_{g_{m}}(x) L_{m+1} J_{n-g_{m}^{\prime}}$, where $J_{n-g_{m}^{\prime}}=$ $I_{n-g_{m}^{\prime}}\left(\left\{2^{g_{m}^{\prime}} x\right\}\right)$. We have

$$
\begin{aligned}
\nu_{\widehat{\alpha}}\left(I_{n}(x)\right) & =\sum_{J_{n-g_{m}^{\prime}} \supset I \in G_{m+1}\left(\alpha_{m+1}\right)} \nu_{\widehat{\alpha}}\left(I_{g_{m}}(x) L_{m} I\right) \\
& =\nu_{\widehat{\alpha}}\left(I_{g_{m}}(x)\right) \frac{\sum_{J_{n-g_{m}^{\prime}} \supset I \in G_{m+1}\left(\alpha_{m+1}\right)} \nu_{\alpha_{m+1}}(I)}{\sum_{I \in G_{m+1}\left(\alpha_{m+1}\right)} \nu_{\alpha_{m+1}}(I)},
\end{aligned}
$$

hence

$$
\nu_{\widehat{\alpha}}\left(I_{g_{m}}(x)\right) \nu_{\alpha_{m+1}}\left(J_{n-g_{m}^{\prime}}\right) \leq \nu_{\widehat{\alpha}}\left(I_{n}(x)\right) \leq 2 \nu_{\widehat{\alpha}}\left(I_{g_{m}}(x)\right) \nu_{\alpha_{m+1}}\left(J_{n-g_{m}^{\prime}}\right)
$$

(we have used (3.2)). For $m$ large enough we have $I_{m+1} \cap \widetilde{F}_{m+1}\left(\alpha_{m+1}, \gamma_{m+1}\left(\alpha_{m+1}\right)\right) \neq \varnothing$. This implies $J_{n-g_{m}^{\prime}} \cap \widetilde{F}_{m+1}\left(\alpha_{m+1}, \alpha_{m+1}\right) \neq \varnothing$.

Since, moreover, $n-g_{m}^{\prime} \geq n\left(\alpha_{m+1}, \gamma_{m+1}\left(\alpha_{m+1}\right)\right)$, (2.10) holds for $J_{n-g_{m}^{\prime}}$ and the previous estimates combined with (3.15) yield

$$
\widetilde{C}_{m, n}^{-1} \leq \frac{\nu_{\widehat{\alpha}}\left(I_{n}(x)\right)}{\exp \left(-\gamma_{m+1}\left(\alpha_{m+1}\right)\left(n-g_{m}\right) \log (2)-\sum_{j=1}^{m} \gamma_{j}\left(\alpha_{j}\right) N_{j} \log (2)\right)} \leq \widetilde{C}_{m, n}
$$

where

$$
\widetilde{C}_{m, n}=\widetilde{C}_{m} 2^{\max \left(\gamma_{m+1}\left(A_{m+1}\right)\right)\left(g_{m}^{\prime}-g_{m}\right)} 2^{\left(n-g_{m}^{\prime}\right) \epsilon_{m+1}}
$$

Then, due to (3.1), the above sequence $\left(\delta_{n}\right)_{n \in \mathbb{N}}$ can be modified so that (3.8) also holds for all $x \in \widetilde{K}_{\widehat{\alpha}}$, for $m$ large enough and $g_{m}^{\prime}+n_{m}<n \leq g_{m+1}$.

### 3.2.2. Lower bound for the lower Hausdorff spectrum

Proposition 3.3. - For any closed ball $B$ whose interior intersects $K=\operatorname{supp}(\mu)$, we have $\operatorname{dim}_{H}(B \cap \underline{E}(\mu, \alpha)) \geq f(\alpha)$ for all $\alpha \in \mathcal{J}$.

Proof. - Fix $B$, a closed ball whose interior intersects $K$. There exist $m_{0} \in \mathbb{N}$, a sequence $\left(\alpha_{j}\right)_{1 \leq j \leq m_{0}} \in \prod_{j=1}^{m_{0}} A_{j}$, as well as pairs of dyadic cubes ( $\left\{L_{j}, I_{\alpha_{j}}\right\}_{1 \leq j \leq m_{0}}$, with $L_{j} \in \mathscr{L}_{j}$ and $I_{\alpha_{j}} \in G_{j}\left(\alpha_{j}\right)$ such that

$$
L_{1} I_{\alpha_{1}} \cdots L_{m_{0}} I_{\alpha_{m_{0}}} \subset B
$$

Now fix $\alpha \in 才$.
If $\alpha \in \mathcal{I} \backslash\{0, D, \infty\}$ then for each $m>m_{0}$, fix $\alpha_{m} \in A_{m} \backslash\left\{D_{m}\right\}$, so that $\lim _{m \rightarrow \infty}\left(\gamma_{m}\left(\alpha_{m}\right)=f\left(\alpha_{m}\right)\right)=f(\alpha)$. If, moreover, $\alpha \in \operatorname{Fix}(f) \backslash D$, take $\alpha_{m} \in \operatorname{Fix}(f)$.

If $\alpha=0$ then for each $m>m_{0}$ let $\alpha_{m}=\alpha_{m}(0)$.
We have $\lim _{m \rightarrow \infty}\left(\gamma_{m}\left(\alpha_{m}\right)=\alpha_{m}\right)=0=f(\alpha)$.
If $\alpha=D$ then for each $m>m_{0}$ let $\alpha_{m}=D_{m}$.
We have $\lim _{m \rightarrow \infty}\left(\gamma_{m}\left(\alpha_{m}\right)=\alpha_{m}\right)=\alpha=f(\alpha)$.
If $\alpha=\infty$ then for each $m>m_{0}$ let $\alpha_{m}=\alpha_{m}(\infty)$. We have $\gamma_{m}\left(\alpha_{m}\right)=f(\infty)$ if $f(\infty)>0$ and $\lim _{m \rightarrow \infty}\left(\gamma_{m}\left(\alpha_{m}\right)=\epsilon_{m}\right)=0=f(\infty)$ otherwise.

Let $\widehat{\alpha}=\left(\alpha_{m}\right)_{m \geq 1}$, and consider the measure $\nu_{\widehat{\alpha}}$ constructed in the previous section. This measure is supported on the set $\widetilde{K}_{\widehat{\alpha}} \subset K$ exhibited at the beginning of the proof of Proposition 3.2, and by construction $\nu_{\widehat{\alpha}}\left(B \cap \widetilde{K}_{\widehat{\alpha}}\right)>0$, so due to Corollary 3.2 we have $\operatorname{dim}_{H} \widetilde{K}_{\widehat{\alpha}} \cap B \geq f(\alpha)$.

Moreover, due to Corollary 3.1(1), we have $\widetilde{K}_{\widehat{\alpha}} \subset \underline{E}(\mu, \alpha)$, so $\operatorname{dim}_{H}(B \cap \underline{E}(\mu, \alpha)) \geq f(\alpha)$. If, moreover, $\alpha \in \operatorname{Fix}(f)$, then by Corollary 3.1(2), our choice of $\left(\alpha_{m}\right)_{m \geq 1}$ implies that $\widetilde{K}_{\widehat{\alpha}} \subset E(\mu, \alpha)$, so $\operatorname{dim}_{H}(B \cap E(\mu, \alpha))=f(\alpha)$.

### 3.3. Upper bound for the lower Hausdorff spectrum

Recall that $f$ is upper semi-continuous. Also, its domain, denoted by $f$ is a closed subset of $[0, \infty]$, so due to Corollary 3.1(1), if $\alpha \notin J$ then $\underline{E}(\mu, \alpha)=\varnothing$, hence $\operatorname{dim}_{H} \underline{E}(\mu, \alpha)=$ $-\infty=f(\alpha)$.

Let $\alpha \in \mathcal{I} \backslash\{\infty\}$. Fix $\eta>0$. Then choose $\delta>0$ such that $f(\beta) \leq f(\alpha)+\eta$ if $|\beta-\alpha| \leq \delta$. Due to Corollary 3.1(1), $\underline{E}(\mu, \alpha)=\left\{x \in K: \liminf _{m \rightarrow \infty} \alpha_{m}(x)=\alpha\right\}$, so

$$
\underline{E}(\mu, \alpha) \subset \bigcap_{M \in \mathbb{N}} \bigcup_{m \geq M} \bigcup_{I_{m} \in \mathbf{G}_{m}} \bigcup_{\substack{\alpha^{\prime} \in A_{m+1}: \\ \alpha^{\prime} \in[\alpha-\delta, \alpha+\delta]}} \bigcup_{I \in G_{m+1}\left(\alpha^{\prime}\right)} I_{m} L_{m+1, \alpha^{\prime}} I .
$$

For a fixed $m \geq 1$, all the cubes $I_{m} L_{m+1, \alpha^{\prime}} I$ are of the same generation $g_{m+1}$. Moreover the set $\mathscr{C}_{m}$ of these cubes has a cardinality

$$
\# \mathscr{C}_{m}=\sum_{I_{m} \in \mathbf{G}_{m}} \sum_{\substack{\alpha^{\prime} \in A_{m+1}: \\ \alpha^{\prime} \in[\alpha-\delta, \alpha+\delta]}} \# G_{m+1}\left(\alpha^{\prime}\right) \leq\left(\# \mathbf{G}_{m}\right)\left(\# A_{m+1}\right) \max _{\substack{\alpha^{\prime} \in A_{m+1}: \\ \alpha^{\prime} \in[\alpha-\delta, \alpha+\delta]}} \# G_{m+1}\left(\alpha^{\prime}\right)
$$

We can deduce from (2.7) and the definition of $\gamma_{m+1}$ that

$$
\# G_{m+1}\left(\alpha^{\prime}\right) \leq 2^{N_{m+1}\left(f\left(\alpha^{\prime}\right)+3 \epsilon_{m+1}\right)} .
$$

Thus, by using the upper semi-continuity of $f$ we get

$$
\# \mathscr{C}_{m} \leq\left(\# \mathbf{G}_{m}\right)\left(\# A_{m+1}\right) 2^{N_{m+1}\left(f(\alpha)+\eta+3 \epsilon_{m+1}\right)} .
$$

Moreover,

$$
\# \mathbf{G}_{m}=\prod_{j=1}^{m}\left(\sum_{\beta \in A_{j}} \# G_{j}(\beta)\right)
$$

Now, we deduce from (3.1) that $\log \left(\# \mathbf{G}_{m}\right)+\log \left(\# A_{m+1}\right)=o\left(\left(f(\alpha+\eta) N_{m+1}\right)\right.$, and finally obtain

$$
\limsup _{m \rightarrow \infty} \frac{\log \left(\# \mathscr{C}_{m}\right)}{N_{m+1} \log (2)} \leq f(\alpha)+\eta
$$

Noting that $\lim _{m \rightarrow \infty} g_{m+1} / N_{m+1}=1$, this is enough to conclude that for all $\epsilon>0$ we have

$$
\sum_{M \in \mathbb{N}} \sum_{m \geq M} \sum_{\substack{ \\I_{m} \in \mathbf{G}_{m}}} \sum_{\substack{\alpha^{\prime} \in A_{m+1}: \\ \alpha^{\prime} \in[\alpha-\delta, \alpha+\delta]}} \sum_{I \in G_{m+1}\left(\alpha^{\prime}\right)}\left|I_{m} L_{m+1, \alpha^{\prime}}\right|^{f(\alpha)+\eta+\epsilon}<\infty
$$

so $\mathscr{H}^{f(\alpha)+\eta+\epsilon}(\underline{E}(\mu, \alpha))=\lim _{m \rightarrow \infty} \mathscr{H}_{2^{-g_{m+1}}}^{f(\alpha)+\eta+\epsilon}(\underline{E}(\mu, \alpha))=0$, and $\operatorname{dim}_{H} \underline{E}(\mu, \alpha) \leq$ $f(\alpha)+\eta+\epsilon$. Since this holds for all $\eta>0$ and $\epsilon>0$, we get the conclusion.

Now suppose that $\alpha=\infty \in \mathcal{I}$. Let $\eta>0$ and $A>0$ such that $f(\alpha) \leq f(\infty)+\eta$ for $\alpha \geq A$ ( $f$ is upper semi-continuous over $I$ ). We have

$$
E(\mu, \infty) \subset \bigcap_{M \in \mathbb{N}} \bigcup_{m \geq M} \bigcup_{I_{m} \in \mathbf{G}_{m}} \bigcup_{\alpha^{\prime} \in A_{m+1}: \alpha^{\prime} \geq A} \bigcup_{I \in G_{m+1}\left(\alpha^{\prime}\right)} I_{m} L_{m+1, \alpha^{\prime}} I
$$

and following the same lines as for the case $\alpha<\infty$ yields $\operatorname{dim}_{H} E(\mu, \infty) \leq f(\infty)$, since $f$ is upper semi-continuous at $\infty$.
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## 3.4. $\mu$ is exactly dimensional, with dimension $D$

By construction, for each $m \geq 2$, we have

$$
\begin{aligned}
\mu\left(\left\{x \in K: \alpha_{m}(x) \neq D_{m}\right\}\right) & =\sum_{I \in \mathbf{G}_{m-1}} \sum_{\alpha \in A_{m} \backslash\left\{D_{m}\right\}} \sum_{J \in G_{m}(\alpha)} \mu\left(I L_{m, \alpha} J\right) \\
& =\sum_{I \in \mathbf{G}_{m-1}} \mu(I) \frac{\sum_{\alpha \in A_{m} \backslash\left\{D_{m}\right\}} \sum_{J \in G_{m}(\alpha)} \rho_{m}(\alpha) \mu_{\alpha}(J)}{\sum_{\alpha \in A_{m}} \sum_{J \in G_{m}(\alpha)} \rho_{m}(\alpha) \mu_{\alpha}(J)} \\
& \leq \sum_{I \in \mathbf{G}_{m-1}} \mu\left(I_{m}\right) \frac{2^{-m}}{2^{-1}}=2^{-(m-1)},
\end{aligned}
$$

where we have used (3.3) and (3.4). By the Borel-Cantelli lemma we get that $\mu$-almost everywhere, $\alpha_{m}(x)=D_{m}$ for $m$ large enough. Moreover, denoting by $I_{m}$ the same interval as in (3.7), we have

$$
\begin{aligned}
\mu\left(\left\{x \in K: \alpha_{m}(x)=D_{m},\right.\right. & \left.\left.I_{m} \cap \widetilde{F}_{m}\left(D_{m}, D_{m}\right)=\varnothing\right\}\right) \\
& =\sum_{I \in \mathbf{G}_{\widehat{\alpha}, m-1}} \mu_{m-1}(I) \frac{\sum_{J \in \mathscr{F}_{N_{m}}, J \cap \widetilde{F}_{m}\left(D_{m}, D_{m}\right)=\varnothing} \nu_{D_{m}}(J)}{\sum_{\alpha \in A_{m}} \sum_{J \in G_{m}(\alpha)} \rho_{m}(\alpha) \mu_{\alpha}(J)} \\
& \leq \sum_{I \in \mathbf{G}_{\widehat{\alpha}, m-1}} \mu_{m-1}(I) \frac{2^{-m}}{2^{-1}}=2^{-(m-1)},
\end{aligned}
$$

where we have used (2.8) and (3.3). Then, a new application of the Borel-Cantelli lemma implies that for $\mu$-almost every $x$, for $m$ large enough, we have both $\alpha_{m}(x)=D_{m}$ and $I_{m} \cap \widetilde{F}_{m}\left(D_{m}, D_{m}\right) \neq \varnothing$. We can then conclude from (3.9) and Corollary 3.1(1)(2) that $\mu$ is exactly dimensional with dimension $D$, since $D_{m}$ converges to $D$ as $m \rightarrow \infty$.

## 4. Full illustration of the multifractal formalism: Proof of Theorem 1.4

### 4.1. Construction of $\mu$

We will modify the scheme used in the proof of Theorem 1.5 by repeating recursively, for all $m \geq 1$, for some integers $R_{m}^{f}$ anf $R_{m}^{g}$ to be specified, $R_{m}^{f}$ times the $m$ th step with $\gamma_{m}(\alpha)$ approximating $f(\alpha)$, followed by $R_{m}^{g}$ times the $m$ th step with $\gamma_{m}(\alpha)$ approximating $g(\alpha)$; this will make it possible to both guarantee the non emptyness of the sets $E(\mu, \alpha)$, $\alpha \in \mathcal{I}$, and the control of the difference between the associated Hausdorff and packing spectra. Additional conditions on $\gamma_{m}(\alpha)$ will be also needed to obtain an exactly dimensional measure.

Let $D$ be a fixed point of $f$ (it is automatically a fixed point of $g$ ). Due to our assumption on the upper semi-continuity of $f$, there exists a countable subset $\Delta_{f}$ of $\mathscr{I} \backslash\{\infty\}$ such that for all $\alpha \in \mathscr{I} \backslash\{\infty\}$, there exists a sequence $\left(\alpha_{n}\right)_{n \geq 1}$ in $\Delta_{f}^{\mathbb{N}_{+}}$such that $\lim _{n \rightarrow \infty}\left(\alpha_{n}, f\left(\alpha_{n}\right)\right)=(\alpha, f(\alpha))$. Similarly, there exists a countable subset $\Delta_{g}$ of $\mathscr{g} \backslash\{\infty\}$ such that for all $\alpha \in \mathscr{J} \backslash\{\infty\}$, there exists a sequence $\left(\alpha_{n}\right)_{n \geq 1}$ in $\Delta_{g}^{\mathbb{N}_{+}}$such that $\lim _{n \rightarrow \infty}\left(\alpha_{n}, g\left(\alpha_{n}\right)\right)=(\alpha, g(\alpha))$.

We fix once for all such $\Delta_{f}$ and $\Delta_{g}$ and we can assume that they both contain $D$. Also take

$$
\epsilon_{m}=(m+1)^{-2} .
$$

Set $\alpha_{\text {min }}^{f}=\min (\mathcal{J}), \alpha_{\text {max }}^{f}=\max (\mathcal{J}) \in \mathbb{R}_{+} \cup\{\infty\}, \alpha_{\text {min }}^{g}=\min (\mathcal{J})$, and $\alpha_{\text {max }}^{g}=$ $\max (\mathcal{g}) \in \mathbb{R}_{+} \cup\{\infty\}$. For $h \in\{f, g\}$, if $\Delta_{h} \backslash\{0, D\} \neq \varnothing$, enumerate the elements of $\Delta_{h} \backslash\{0, D\}$ in a sequence $\left(\alpha_{j}^{\Delta_{h}}\right)_{j \geq 1}$, and for each $m \in \mathbb{N}$ set

$$
\widetilde{A}_{m}^{h}=\left\{\alpha_{j}^{\Delta_{h}}: 1 \leq j \leq m, \alpha_{j}^{\Delta_{h}} \geq 4 \epsilon_{m}^{1 / 3}\right\} ;
$$

otherwise, set $\widetilde{A}_{m}^{h}=\varnothing$. Also set

$$
\begin{cases}D_{m}=2 \epsilon_{m}^{1 / 3} & \text { if } D=0, \\ D_{m}=D & \text { otherwise } \\ \alpha_{m}^{h}(0)=D_{m} & \text { if } D=0 \\ \alpha_{m}(0)=\min \left(2 \epsilon_{m}^{1 / 3}, D\right) / 2 & \text { otherwise } \\ \alpha_{m}^{h}(\infty)=\left(\max \left(d, m, \max \left(\alpha_{j}^{\Delta_{h}}: 1 \leq j \leq m\right)\right)^{2}\right. & \text { if } \alpha_{\max }^{h}=\infty .\end{cases}
$$

Then let

$$
A_{m}^{h}= \begin{cases}\widetilde{A}_{m}^{h} \cup\left\{D_{m}\right\} & \text { if } 0<\alpha_{\min }^{h} \leq \alpha_{\max }^{h}<\infty \\ \widetilde{A}_{m}^{h} \cup\left\{D_{m}\right\} \cup\left\{\alpha_{m}^{h}(\infty)\right\} & \text { if } 0<\alpha_{\min }^{h} \text { and } \alpha_{\max }^{h}=\infty, \\ \widetilde{A}_{m}^{h} \cup\left\{D_{m}\right\} \cup\left\{\alpha_{m}^{h}(0)\right\} & \text { if } \alpha_{\min }^{h} 0 \text { and } \alpha_{\max }^{h}<\infty, \\ \widetilde{A}_{m}^{h} \cup\left\{D_{m}\right\} \cup\left\{\alpha_{m}^{h}(0)\right\} \cup\left\{\alpha_{m}^{h}(\infty)\right\} & \text { if } \alpha_{\min }^{h}=0 \text { and } \alpha_{\max }^{h}=\infty .\end{cases}
$$

For $\alpha \in A_{m}^{h}$ let

$$
\widetilde{\gamma}_{m}^{h}(\alpha)= \begin{cases}h(\alpha) & \text { if } \alpha \in \widetilde{A}_{m}^{h} \text { and } h(\alpha)>0 \\ h(\infty) & \text { if } \alpha=\alpha_{m}^{h}(\infty) \text { and } h(\infty)>0, \\ \alpha & \text { if } \alpha=D_{m}, \\ \alpha_{m}(0) & \text { if } \alpha=\alpha_{m}^{h}(0) \\ \epsilon_{m}^{1 / 3} & \text { if } \alpha \in \widetilde{A}_{m}^{h} \text { and } h(\alpha)=0, \\ \epsilon_{m}^{1 / 3} & \text { if } \alpha=\alpha_{m}^{h}(\infty) \text { and } h(\infty)=0\end{cases}
$$

(notice that $f(\infty)=0($ resp. $g(\infty)=0$ ) only if $f=0($ resp. $g=0)$ over $\mathscr{g}$ (resp. $\mathcal{g}$ ) due to our assumptions on $f$ (resp. $g$ )). Then, set (this will be used to entail that $\mu$ is exactly dimensional)

$$
\gamma_{m}^{h}(\alpha)= \begin{cases}\alpha=\widetilde{\gamma}_{m}^{h}(\alpha) & \text { if } \alpha=D_{m} \\ \left(1-\theta_{m}^{h}\right) \widetilde{\gamma}_{m}^{h}(\alpha) & \text { otherwise },\end{cases}
$$

where $\theta_{m}^{h} \in(0,1)$ tends to 0 slowly enough so that

$$
\forall \alpha \in A_{m}^{h} \backslash\left\{D_{m}\right\},\left\{\begin{array}{l}
\alpha-\gamma_{m}^{h}(\alpha)=\alpha-\left(1-\theta_{m}^{h}\right) \widetilde{\gamma}_{m}^{h}(\alpha) \geq D_{m} \sqrt{\epsilon_{m}}  \tag{4.1}\\
\left(1-\sqrt{\epsilon_{m}}\right) \alpha-\gamma_{m}^{h}(\alpha) \geq 0 .
\end{array}\right.
$$

We then have in particular $\gamma_{m}^{h}(\alpha)<\alpha$ for all $\alpha \in A_{m}^{h} \backslash\left\{D_{m}\right\}$.
Using the definitions of Section 2, for $\alpha \in A_{m}^{h}$ set
$\mu_{\alpha}=\mu_{\alpha, \gamma_{m}^{h}(\alpha)}$ and $\nu_{\alpha}=\nu_{\alpha, \gamma_{m}^{h}(\alpha)}$ for $\alpha \in A_{m}^{h} \backslash\left\{D_{m}\right\}$ and $\mu_{D_{m}}=\nu_{D_{m}}=\nu_{D_{m}, D_{m}}$ for $\alpha=D_{m}$.

Strictly speaking, $\mu_{\alpha}$ and $\nu_{\alpha}$ should be written $\mu_{m, \alpha}^{h}$ and $\nu_{m, \alpha}^{h}$, but for the sake of readability we will omit the indices $m$ and $h$.

Also, set

$$
n_{m}^{h}=\max \left\{n_{m}\left(\alpha, \gamma_{m}^{h}(\alpha)\right): \alpha \in A_{m}^{h}\right\} .
$$

Now let $\left(R_{m}^{f}\right)_{m \geq 1},\left(R_{m}^{g}\right)_{m \geq 1}$ and $\left(N_{m}\right)_{m \geq 1}$ be three increasing sequences of positive integers defined recursively and satisfying the following properties:
(4.2)
(1) $\forall m \geq 1, e^{m+1} \leq R_{m}^{f} \leq R_{m}^{g} \leq R_{m+1}^{f}$;
(2) $\forall m \geq 1, N_{m} \geq \max \left(n_{m}^{h}, \exp \left(\# A_{m}^{h}\right), \exp (m)\right)$ for $h \in\{f, g\}$, and, as $m \rightarrow \infty$ :
$\left\{\begin{array}{l}(3)\left(\max \left(\{1\} \cup A_{m}^{f} \cup A_{m}^{g}\right)\right) N_{m}=o\left(\left(\min \left(\{1\} \cup A_{m-1}^{f} \cup A_{m-1}^{g}\right)\right) \sum_{i=1}^{m-1}\left(R_{i}^{f}+R_{i}^{g}\right) N_{i}\right) ; \\ (4)\left(\max \left(\{1\} \cup A_{m-1}^{f} \cup A_{m-1}^{g}\right)\right) \sum_{i=1}^{m-1}\left(R_{i}^{f}+R_{i}^{g}\right) N_{i}=o\left(\min \left(\{1\} \cup \gamma_{m}\left(A_{m}^{f}\right)\right) R_{m}^{f} N_{m}\right) ; \\ (5)\left(\max \left(\{1\} \cup A_{m}^{f} \cup A_{m}^{g}\right)\right)\left(R_{m}^{f} N_{m}+\sum_{i=1}^{m-1}\left(R_{i}^{f}+R_{i}^{g}\right) N_{i}\right)=o\left(\min \left(\{1\} \cup \gamma_{m}\left(A_{m}^{g}\right)\right) R_{m}^{g} N_{m}\right) .\end{array}\right.$
Then, for $\alpha \in A_{m}^{h}$ set

$$
G_{m}^{h}(\alpha)=\left\{I \in \mathcal{F}_{N_{m}}: I \cap F_{m}\left(\alpha, \gamma_{m}^{h}(\alpha)\right) \neq \varnothing\right\}
$$

and

$$
\rho_{m}^{h}(\alpha)= \begin{cases}1 & \text { if } \alpha=D_{m}  \tag{4.3}\\ \left(2^{-m} / \# A_{m}^{h}\right)^{2} & \text { otherwise }\end{cases}
$$

We enumerate the elements of $A_{m}^{h}$ as $\alpha_{m, i}^{h}, 1 \leq i \leq \# A_{m}^{h}$ and denote by $L_{m, \alpha_{i}}^{h}$, $1 \leq i \leq \# A_{m}^{h}$, the disjoint closed dyadic cubes of generation $\ell_{m}^{h}=\ell\left(\# A_{m}^{h}\right)$ of the set $\mathcal{L}_{m}=\mathscr{L}\left(\# A_{m}^{h}\right)$ defined in Section 2.1. We also denote $L_{m, \alpha_{i}}^{h}$ by $L_{m, i}^{h}$.

Let us start the construction of $\mu$. We consider the same measure $\mu_{1}$ on $\mathbf{G}_{1}$ as in Section 3.1, except that we take the sets $G_{1}^{f}(\alpha), \alpha \in A_{1}^{f}$, instead of the sets $G_{1}(\alpha), \alpha \in A_{1}$, and the collection $\mathscr{L}_{1}^{f}$ instead of $\mathscr{L}_{1}$. Then, for $1 \leq s \leq R_{1}^{f}-1$ we define recursively

$$
\mathbf{G}_{s+1}=\bigcup_{I_{s} \in \mathbf{G}_{s}} \bigcup_{\alpha_{1, i} \in A_{1}^{f}} \mathbf{G}_{1}^{f}\left(I_{s}, \alpha_{1, i}\right),
$$

where

$$
\mathbf{G}_{1}\left(I_{s}, \alpha_{1, i}\right)=\left\{I_{s} L_{1, i}^{f} I_{1, i}: I_{1, i} \in G_{1}^{f}\left(\alpha_{1, i}\right)\right\}
$$

and a measure $\mu_{s+1}$ on $\mathbf{G}_{s+1}$ as

$$
\mu_{s+1}\left(I_{s} L_{1, i}^{f} I_{1, i}\right)=\mu_{s}\left(I_{s}\right) \frac{\rho_{1}^{f}\left(\alpha_{1, i}\right) \mu_{\alpha_{1, i}}\left(I_{1, i}\right)}{\sum_{\alpha \in A_{1}} \sum_{I \in G_{1}^{f}(\alpha)} \rho_{1}^{f}(\alpha) \mu_{\alpha}(I)} .
$$

Then, we define recursively a sequence $\left(\mathbf{G}_{s}\right)_{s \geq 1}$ of sets of intervals of the same generation and a sequence of measures $\left(\mu_{s}\right)_{s \geq 1}$ as follows:

For all $m \geq 1$, and $R_{m}^{f}+\sum_{j=1}^{m-1} R_{j}^{f}+R_{j}^{g}<s+1 \leq \sum_{j=1}^{m} R_{j}^{f}+R_{j}^{g}$,

$$
\mathbf{G}_{s+1}=\bigcup_{I_{s} \in \mathbf{G}_{s}} \bigcup_{\alpha_{m, i} \in A_{m}^{g}} \mathbf{G}_{m}\left(I_{s}, \alpha_{m, i}\right),
$$

where

$$
\mathbf{G}_{m}\left(I_{s}, \alpha_{m, i}\right)=\left\{I_{s} L_{m, i}^{g} I_{m, i}: I_{m, i} \in G_{m}^{g}\left(\alpha_{m, i}\right)\right\}
$$

and the measure $\mu_{s+1}$ on $\mathbf{G}_{s+1}$ is defined as

$$
\mu_{s+1}\left(I_{s} L_{m, i}^{g} I_{m, i}\right)=\mu_{s}\left(I_{s}\right) \frac{\rho_{m}^{g}\left(\alpha_{m, i}\right) \mu_{\alpha_{m, i}}\left(I_{m, i}\right)}{\sum_{\alpha \in A_{m}^{g}} \sum_{I \in G_{m}^{g}(\alpha)} \rho_{m}^{f}(\alpha) \mu_{\alpha}(I)} .
$$

For all $m \geq 2$, and $\sum_{j=1}^{m-1} R_{j}^{f}+R_{j}^{g}<s+1 \leq R_{m}^{f}+\sum_{j=1}^{m} R_{j}^{f}+R_{j}^{g}$,

$$
\mathbf{G}_{s+1}=\bigcup_{I_{s} \in \mathbf{G}_{s}} \bigcup_{\alpha_{m, i} \in A_{m}^{f}} \mathbf{G}_{m}\left(I_{s}, \alpha_{m, i}\right),
$$

where

$$
\mathbf{G}_{m}\left(I_{s}, \alpha_{m, i}\right)=\left\{I_{s} L_{m, i}^{f} I_{m, i}: I_{m, i} \in G_{m}^{f}\left(\alpha_{m, i}\right)\right\}
$$

and the measure $\mu_{s+1}$ on $\mathbf{G}_{s+1}$ is defined as

$$
\mu_{s+1}\left(I_{s} L_{m, i}^{f} I_{m, i}\right)=\mu_{s}\left(I_{s}\right) \frac{\rho_{m}^{f}\left(\alpha_{m, i}\right) \mu_{\alpha_{m, i}}\left(I_{m, i}\right)}{\sum_{\alpha \in A_{m}^{f}} \sum_{I \in G_{m}^{f}(\alpha)} \rho_{m}^{f}(\alpha) \mu_{\alpha}(I)} .
$$

This yields (in the same way as in Section 3.1) a Borel probability measure $\mu$ supported on

$$
K=\bigcap_{s \geq 1} \bigcup_{I \in \mathbf{G}_{s}} I
$$

such that $\mu(I)=\mu_{s}(I)$ for all $s \geq 1$ and $I \in \mathbf{G}_{s}$.
For each $m \geq 1$, we define

$$
s_{m}=R_{m}^{f}+\sum_{i=1}^{m-1} R_{i}^{f}+R_{i}^{g} \quad \text { and } \quad s_{m}^{\prime}=\sum_{i=1}^{m} R_{i}^{f}+R_{i}^{g} .
$$

Then, for $s \geq 1$, we denote by $n(s)$ the generation of the cubes belonging to $\mathbf{G}_{s}$, i.e.,

$$
n(s)=\left(s-s_{m-1}^{\prime}\right)\left(N_{m}+\ell_{m}^{f}\right)+\sum_{i=1}^{m-1} R_{i}^{f}\left(N_{i}+\ell_{i}^{f}\right)+R_{i}^{g}\left(N_{i}+\ell_{i}^{g}\right)
$$

if $s_{m-1}^{\prime}<s \leq s_{m}$, and

$$
n(s)=\left(s-s_{m}\right)\left(N_{m}+\ell_{m}^{g}\right)+R_{m}^{f}\left(N_{m}+\ell_{m}^{f}\right)+\sum_{i=1}^{m-1} R_{i}^{f}\left(N_{i}+\ell_{i}^{f}\right)+R_{i}^{g}\left(N_{i}+\ell_{i}^{g}\right)
$$

if $s_{m}<s \leq s_{m}^{\prime}$.
The following property, which follows immediately from (4.2) will be useful.
If $s_{m-1}^{\prime}<s \leq s_{m}$, set

$$
\begin{equation*}
n^{\prime}(s)=\left(s-s_{m-1}^{\prime}\right) N_{m}+\sum_{i=1}^{m-1} R_{i}^{f} N_{i}+R_{i}^{g} N_{i}, \tag{4.4}
\end{equation*}
$$

and if $s_{m}<s \leq s_{m}^{\prime}$, set

$$
n^{\prime}(s)=\left(s-s_{m}\right) N_{m}+R_{m}^{f} N_{m}+\sum_{i=1}^{m-1} R_{i}^{f} N_{i}+R_{i}^{g} N_{i} .
$$

Proposition 4.1. - If $s_{m-1}<s \leq s_{m}$, then

$$
\max \left(1, \max \left(A_{m}^{f} \cup A_{m}^{g}\right) \sum_{i=1}^{m} R_{m}^{f} \ell_{m}^{f}+R_{m}^{g} \ell_{m}^{g}=o\left(n^{\prime}(s)\right)\right.
$$

In particular, $n^{\prime}(s) \sim n(s)$ as $s \rightarrow \infty$.

Remark 4.1. - By construction, due to our choice for the cubes $L_{m, i}^{h}$, if $J=I_{s-1} L_{m, i}^{h} I_{m, i} \in \mathbf{G}_{s}$ and $I$ is a dyadic cube in $\mathcal{N}_{2}(n(s), J)$ (see Section 2.1 for the definition), either $\mu(I)=0$ or I takes the form $I_{s-1} L_{m, i}^{h} I_{m, i}^{\prime}$ and $\mu(J) 2^{-\epsilon(s) n(s)} \leq \mu(I) \leq$ $\mu(J) 2^{\epsilon(s) n(s)}$, where $\epsilon(s)$ is independent of $J$ and the sequence $(\epsilon(s))_{s \geq 1}$ tends to 0 as $s \rightarrow \infty$.

### 4.2. Reduction of the problem

In this section we explain why the measure $\mu$ constructed in the previous section has the nice property that it is possible to replace centered balls by dyadic cubes in all the sets and quantities involved in the multifractal formalism without modifying them (it is of course impossible to do so for any measure in $\mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$ ).

Let us start with two properties which easily follow from (4.2)(1-3) by construction:

$$
\begin{aligned}
\lim _{s \rightarrow \infty} \frac{n(s+1)}{n(s)} & =1 \\
\text { and } \lim _{s \rightarrow \infty} \sup _{x \in K}\left|\frac{\log _{2}\left(\mu\left(I_{n(s)}(x)\right)\right.}{n(s)}-\frac{\log _{2}\left(\mu\left(I_{n(s+1)}(x)\right)\right.}{n(s+1)}\right| & =0
\end{aligned}
$$

Moreover, if $2^{-n(s+1)}<r \leq 2^{-n(s)}$ we have

$$
I_{n(s+1)}(x) \subset B(x, r) \subset B(x, 2 r) \subset \bigcup_{I \in \mathcal{N}_{2}\left(n(s), I_{n(s)}(x)\right)} I
$$

and if $I \in \mathcal{N}_{2}\left(n(s), I_{n(s)}(x)\right)$, either $\mu\left(I_{n(s)}\right) e^{-\epsilon(s) n(s)} \leq \mu(I) \leq \mu\left(I_{n(s)}\right) e^{\epsilon(s) n(s)}$ or $\mu(I)=0$, where $\epsilon(s)$ is independent of $x$ and $I$, by Remark 4.1.

It follows that for all $x \in K$ we have

$$
\begin{equation*}
\underline{d}(\mu, x)=\liminf _{s \rightarrow \infty} \frac{\log _{2}\left(\mu\left(I_{n(s)}(x)\right)\right.}{-n(s)} \text { and } \bar{d}(\mu, x)=\limsup _{s \rightarrow \infty} \frac{\log _{2}\left(\mu\left(I_{n(s)}(x)\right)\right.}{-n(s)}, \tag{4.5}
\end{equation*}
$$

and $\mu$ is weakly doubling in the sense that there exists a function $\widetilde{\epsilon}(r)$ tending to $0^{+}$as $r \rightarrow 0^{+}$ such that

$$
\forall x \in K, \mu(B(x, 2 r)) \leq r^{-\widetilde{\epsilon}(r)} \mu(B(x, r)) .
$$

Also, the above properties and standard covering arguments (see in particular [63, Section 4.6] where doubling measures are used) yield, for all $q \in \mathbb{R}$,

$$
\tau_{\mu}(q)=\liminf _{s \rightarrow \infty} \frac{\log _{2} \sum_{I_{s} \in \mathbf{G}_{s}} \mu\left(I_{s}\right)^{q}}{-n(s)} \text { and } \bar{\tau}_{\mu}(q)=\limsup _{s \rightarrow \infty} \frac{\log _{2} \sum_{I_{s} \in \mathbf{G}_{s}} \mu\left(I_{s}\right)^{q}}{-n(s)} .
$$

Similarly, for $\alpha \in \mathbb{R}_{+}$we have

$$
\begin{aligned}
& \underline{f}_{\mu}^{\mathrm{LD}}(\alpha)=\lim _{\epsilon \rightarrow 0^{+}} \liminf _{s \rightarrow \infty} \frac{\log _{2} \#\left\{I \in \mathbf{G}_{s}: 2^{-n(s)(\alpha+\epsilon)} \leq \mu(I) \leq 2^{-n(s)(\alpha-\epsilon)}\right\}}{n(s)} \\
& \bar{f}_{\mu}^{\mathrm{LD}}(\alpha)=\lim _{\epsilon \rightarrow 0^{+}} \limsup _{s \rightarrow \infty} \frac{\log _{2} \#\left\{I \in \mathbf{G}_{s}: 2^{-n(s)(\alpha+\epsilon)} \leq \mu(I) \leq 2^{-n(s)(\alpha-\epsilon)}\right\}}{n(s)},
\end{aligned}
$$

and

$$
\begin{aligned}
& \underline{f}_{\mu}^{\mathrm{LD}}(\infty)=\lim _{A \rightarrow \infty} \liminf _{s \rightarrow \infty} \frac{\log _{2} \#\left\{I \in \mathbf{G}_{s}: \mu(I) \leq 2^{-n(s) A}\right\}}{n(s)}, \\
& \bar{f}_{\mu}^{\mathrm{LD}}(\infty)=\lim _{A \rightarrow \infty} \limsup _{s \rightarrow \infty} \frac{\log _{2} \#\left\{I \in \mathbf{G}_{s}: \mu(I) \leq 2^{-n(s) A}\right\}}{n(s)},
\end{aligned}
$$

and for $0 \leq \alpha \leq \beta \leq \infty$ such that $(\alpha, \beta) \neq(\infty, \infty)$,

$$
\underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)=\lim _{\epsilon \rightarrow 0^{+}} \lim _{s \rightarrow \infty} \frac{\log _{2} \#\left\{I \in \mathbf{G}_{s}: 2^{-n(s)(\beta+\epsilon)} \leq \mu(I) \leq 2^{-n(s)(\alpha-\epsilon)}\right\}}{n(s)} .
$$

Finally, due to the multiplicative nature of the construction of $\mu$, defining for each $m \geq 1$ and $q \in \mathbb{R}$

$$
T_{m}^{f}(q)=\log _{2} \frac{\sum_{\alpha \in A_{m}^{f}} \sum_{I \in G_{m}^{f}(\alpha)} \rho_{m}^{f}(\alpha)^{q} \mu_{\alpha}(I)^{q}}{\left(\sum_{\alpha \in A_{m}^{f}} \sum_{I \in G_{m}^{f}(\alpha)} \rho_{m}^{f}(\alpha) \mu_{\alpha}(I)\right)^{q}}
$$

and

$$
T_{m}^{g}(q)=\log _{2} \frac{\sum_{\alpha \in A_{m}^{g}} \sum_{I \in G_{m}^{g}(\alpha)} \rho_{m}^{g}(\alpha)^{q} \mu_{\alpha}(I)^{q}}{\left(\sum_{\alpha \in A_{m}^{g}} \sum_{I \in G_{m}^{g}(\alpha)} \rho_{m}^{g}(\alpha) \mu_{\alpha}(I)\right)^{q}},
$$

we have for $s \geq 1$ and $q \in \mathbb{R}$

$$
\log _{2} \sum_{I_{s} \in \mathbf{G}_{s}} \mu\left(I_{s}\right)^{q}=\sum_{i=1}^{m-1} R_{i}^{f} T_{i}^{f}(q)+R_{i}^{g} T_{i}^{g}(q)+\left(s-s_{m-1}^{\prime}\right) T_{m}^{f}(q)
$$

if $s_{m-1}^{\prime}<s \leq s_{m}$ and

$$
\log _{2} \sum_{I_{s} \in \mathbf{G}_{s}} \mu\left(I_{s}\right)^{q}=\left(\sum_{i=1}^{m-1} R_{i}^{f} T_{i}^{f}(q)+R_{i}^{g} T_{i}^{g}(q)\right)+R_{m}^{f} T_{m}^{f}(q)+\left(s-s_{m}\right) T_{m}^{g}(q)
$$

if $s_{m}<s \leq s_{m}^{\prime}$. It follows that

$$
\begin{equation*}
\tau_{\mu}(q)=\liminf _{S \ni s \rightarrow \infty} \frac{\log _{2} \sum_{I_{s} \in \mathbf{G}_{s}} \mu\left(I_{s}\right)^{q}}{-n(s)} \text { and } \bar{\tau}_{\mu}(q)=\limsup _{S \ni s \rightarrow \infty} \frac{\log _{2} \sum_{I_{s} \in \mathbf{G}_{s}} \mu\left(I_{s}\right)^{q}}{-n(s)}, \tag{4.6}
\end{equation*}
$$

where $S=\left\{s_{m}, s_{m}^{\prime}: m \geq 1\right\}$.

### 4.3. Local dimension estimates for $\mu$, auxiliary measures, and lower bounds for the different spectra

4.3.1. Local dimension estimates for the measure $\mu$. - Let $x \in K$ and $s \geq 1$.

If $s_{m-1}^{\prime}<s \leq s_{m}$, for all $1 \leq i \leq m-1$, there are, uniquely determined, $R_{i}^{f}$ elements $\left(\alpha_{i, j}^{f}(x)\right)_{1 \leq j \leq R_{i}^{f}} \in\left(A_{i}^{f}\right)^{R_{i}^{f}}, R_{i}^{g}$ elements $\left(\alpha_{i, j}^{g}(x)\right)_{1 \leq j \leq R_{i}^{g}} \in\left(A_{i}^{g}\right)^{R_{i}^{g}}$, and $s^{\prime}=s-s_{m-1}^{\prime}$ elements $\left(\alpha_{m, j}^{f}(x)\right)_{1 \leq j \leq s^{\prime}} \in\left(A_{m}^{f}\right)^{s^{\prime}}$, and for each exponent $\alpha_{i, j}^{h}(x)$ of this collection a unique element $I_{\alpha_{i, j}^{h}(x)}$ of $G_{i}^{h}\left(\alpha_{i, j}^{h}(x)\right)$ and a unique element $L_{i, \alpha_{i, j}^{h}(x)}^{h}$ of $\mathscr{L}_{i}^{h}$ such that $I_{n(s)}(x)=\left(\bigodot_{i=1}^{m-1}\left(\bigodot_{j=1}^{R_{i}^{f}} L_{j, \alpha_{1, j}^{f}(x)}^{f} I_{\alpha_{1, j}^{f}(x)}\right) \cdot\left(\bigodot_{j=1}^{R_{i}^{g}} L_{j, \alpha_{1, j}^{g}(x)}^{g} I_{\alpha_{1, j}^{g}(x)}\right)\right) \cdot\left(\bigodot_{j=1}^{s^{\prime}} L_{m, \alpha_{m, j}^{f}(x)}^{f} I_{\alpha_{m, j}^{f}(x)}\right)$ (where the notation $\odot_{p=1}^{q} I_{p}$ stands for the cube obtained by concatenation $I_{1} \cdot I_{2} \cdots I_{q}$ of the cubes $I_{1}, \ldots, I_{q}$ ).

By construction, using the analogue of (3.6) we get, writing $\alpha_{i, j}^{h}$ for $\alpha_{i, j}^{h}(x)$,

$$
\left.\begin{array}{rl}
\left(\frac{1}{1+2^{-m}}\right)^{s^{\prime}} & \left(\prod_{i=1}^{m-1}\left(\frac{1}{1+2^{-i}}\right)^{R_{i}^{f}+R_{i}^{g}}\right) \prod_{j=1}^{s^{\prime}} \rho_{m}^{f}\left(\alpha_{m, j}^{f}\right)\left(\prod_{i=1}^{m-1}\left(\prod_{j=1}^{R_{i}^{f}} \rho_{i}^{f}\left(\alpha_{i, j}^{f}\right)\right)\left(\prod_{j=1}^{R_{i}^{g}} \rho_{i}^{g}\left(\alpha_{i, j}^{g}\right)\right)\right. \\
& \leq \frac{\mu\left(I_{n(s)}\right)}{\prod_{j=1}^{s^{\prime}} \mu_{\alpha_{m, j}^{f}}\left(I_{\alpha_{m, j}^{f}}\right)}\left(\prod _ { i = 1 } ^ { m - 1 } ( \prod _ { j = 1 } ^ { R _ { i } ^ { f } } \mu _ { \alpha _ { i , j } ^ { f } } ( I _ { \alpha _ { i , j } ^ { f } } ) ) \left(\prod_{j=1}^{R_{i}^{g}} \mu_{\alpha_{i, j}^{g}}\left(I_{\alpha_{i, j}^{g}}\right)\right.\right.
\end{array}\right),
$$

Consequently, due to the fact that $I_{\alpha_{i, j}^{h}(x)} \in G_{i}^{h}\left(\alpha_{i, j}^{h}(x)\right)$ for $h \in\{f, g\}$, using (4.2)(1)(2) and Proposition 4.1 we get

$$
\begin{equation*}
2^{-\epsilon(s) n(s)} \leq \frac{\mu\left(I_{n(s)}(x)\right)}{2^{-n(s) \alpha_{s}(x)}} \leq 2^{\epsilon(s) n(s)} \tag{4.7}
\end{equation*}
$$

with

$$
\alpha_{s}(x)=\frac{\sum_{j=1}^{s^{\prime}} N_{m} \alpha_{m, j}^{f}(x)+\left(\sum_{i=1}^{m-1} \sum_{j=1}^{R_{i}^{f}} N_{i} \alpha_{i, j}^{f}(x)+\sum_{j=1}^{R_{i}^{g}} N_{i} \alpha_{i, j}^{g}(x)\right)}{s^{\prime} N_{m}+\sum_{i=1}^{m-1} R_{i}^{f} N_{i}+R_{i}^{g} N_{i}}
$$

and $\lim _{s \rightarrow \infty} \epsilon(s)=0$. Due to (4.2)(3-5),

$$
\begin{equation*}
\text { if } s=s_{m} \text {, we have } \quad \alpha_{s}(x)=\frac{\sum_{j=1}^{R_{m}^{f}} N_{m} \alpha_{m, j}^{f}(x)}{R_{m}^{f} N_{m}}+\epsilon^{\prime}(s), \tag{4.8}
\end{equation*}
$$

with $\lim _{s \rightarrow \infty} \epsilon^{\prime}(s)=0$ uniformly in $x$.
If now $s_{m}<s \leq s_{m}^{\prime}$ and $s^{\prime}=s-s_{m}^{\prime}$, with similar notations we get

$$
\left.\begin{array}{rl}
I_{n(s)}(x) & =\left(\odot_{i=1}^{m-1}\left(\odot_{j=1}^{R_{i}^{f}} L_{j, \alpha_{1, j}^{f}(x)}^{f} I_{\alpha_{1, j}^{f}(x)}\right) \cdot\left(\odot_{j=1}^{R_{i}^{g}} L_{j, \alpha_{1, j}^{g}(x)}^{g} I_{\alpha_{1, j}^{g}(x)}\right)\right)  \tag{4.9}\\
& \cdot\left(\odot_{j=1}^{R_{m}^{f}} L_{m, \alpha_{m, j}^{f}}^{f}(x)\right. \\
I_{\alpha_{m, j}^{f}}(x)
\end{array}\right) \cdot\left(\odot_{j=1}^{s^{\prime}} L_{m, \alpha_{m, j}^{g}(x)}^{g} I_{\alpha_{m, j}^{g}(x)}\right)
$$

and

$$
\begin{equation*}
2^{-\epsilon(s) n(s)} \leq \frac{\mu\left(I_{n(s)}(x)\right)}{2^{-n(s) \alpha_{s}(x)}} \leq 2^{\epsilon(s) n(s)} \tag{4.10}
\end{equation*}
$$

with
$\alpha_{s}(x)=\frac{\sum_{j=1}^{s^{\prime}} N_{m} \alpha_{m, j}^{g}(x)+\sum_{j=1}^{R_{m}^{f}} N_{m} \alpha_{m, j}^{f}(x)+\sum_{i=1}^{m-1} \sum_{j=1}^{R_{i}^{f}} N_{i} \alpha_{i, j}^{f}(x)+\sum_{j=1}^{R_{i}^{g}} N_{i} \alpha_{i, j}^{g}(x)}{s^{\prime} N_{m}+R_{m}^{f} N_{m}+\sum_{i=1}^{m-1} R_{i}^{f} N_{i}+R_{i}^{g} N_{i}}$ and $\lim _{s \rightarrow \infty} \epsilon(s)=0$. In addition,

$$
\begin{equation*}
\text { if } s=s_{m}^{\prime} \text {, we have } \alpha_{s}(x)=\frac{\sum_{j=1}^{R_{m}^{g}} N_{m} \alpha_{m, j}^{g}(x)}{R_{m}^{g} N_{m}}+\epsilon^{\prime}(s), \tag{4.11}
\end{equation*}
$$

with $\lim _{s \rightarrow \infty} \epsilon^{\prime}(s)=0$ uniformly in $x$.
4.3.2. Auxiliary measures. - Let $\widehat{\alpha}=\left(\alpha_{1}^{f}, \alpha_{1}^{g}, \ldots, \alpha_{m}^{f}, \alpha_{m}^{g}, \ldots\right) \in \prod_{m=1}^{\infty} A_{m}^{f} \times A_{m}^{g}$.

We construct a measure $\nu_{\widehat{\alpha}}$ as follows: Let

$$
\mathbf{G}_{\widehat{\alpha}, 1}=\left\{L_{1, \alpha_{1}^{f}}^{f} I_{1}: I_{1} \in G_{1}^{f}\left(\alpha_{1}^{f}\right)\right\},
$$

and define

$$
\nu_{\widehat{\alpha}, 1}\left(L_{1, \alpha_{1}^{f}}^{f} I_{1}\right)=\frac{\nu_{\alpha_{1}^{f}}\left(I_{1}\right)}{\sum_{I \in G_{1}^{f}\left(\alpha_{1}^{f}\right)} \nu_{\alpha_{1}^{f}}(I)} .
$$

Then, for $1 \leq s \leq R_{1}^{f}-1$ we define recursively

$$
\mathbf{G}_{\widehat{\alpha}, s+1}=\bigcup_{I_{s} \in \mathbf{G}_{\widehat{\alpha}, s}} \mathbf{G}_{\widehat{\alpha}, s+1}\left(I_{s}, \alpha_{1}^{f}\right),
$$

where

$$
\mathbf{G}_{\widehat{\alpha}, s+1}\left(I_{s}, \alpha_{1}^{f}\right)=\left\{I_{s} L_{1, \alpha_{1}^{f}}^{f} I_{1}: I_{1} \in G_{1}^{f}\left(\alpha_{1}^{f}\right)\right\}
$$

and a measure $\nu_{\widehat{\alpha}, s+1}$ on $\mathbf{G}_{\widehat{\alpha}, s+1}$ as

$$
\nu_{\widehat{\alpha}, s+1}\left(I_{s} L_{1, \alpha_{1}^{f}}^{f} I_{1}\right)=\nu_{\widehat{\alpha}, s}\left(I_{s}\right) \frac{\nu_{\alpha_{1}^{f}}\left(I_{1}\right)}{\sum_{I \in G_{1}^{f}\left(\alpha_{1}^{f}\right)} \nu_{\alpha_{1}^{f}}(I)} .
$$

Then, define recursively a sequence $\left(\mathbf{G}_{\widehat{\alpha}, s}\right)_{s \geq 1}$ of sets of intervals of the same generation and a sequence of measures $\left(\nu_{\widehat{\alpha}, s}\right)_{s \geq 1}$ as follows:

For all $m \geq 1$, and $s_{m}<s+1 \leq s_{m}^{\prime}$,

$$
\mathbf{G}_{\widehat{\alpha}, s+1}=\bigcup_{I_{s} \in \mathbf{G}_{\widehat{\alpha}, s}} \mathbf{G}_{\widehat{\alpha}, s+1}\left(I_{s}, \alpha_{m}^{g}\right),
$$

where

$$
\mathbf{G}_{\widehat{\alpha}, s+1}\left(I_{s}, \alpha_{m}^{g}\right)=\left\{I_{s} L_{m, \alpha_{m}^{g}}^{g} I_{m}: I_{m} \in G_{m}^{g}\left(\alpha_{m}^{g}\right)\right\}
$$

and the measure $\nu_{\widehat{\alpha}, s+1}$ on $\mathbf{G}_{\widehat{\alpha}, s+1}$ is defined as

$$
\nu_{\widehat{\alpha}, s+1}\left(I_{s} L_{m, \alpha_{m}^{g}}^{g} I_{m}\right)=\nu_{\widehat{\alpha}, s}\left(I_{s}\right) \frac{\nu_{\alpha_{m}^{g}}\left(I_{m}\right)}{\sum_{I \in G_{m}^{g}\left(\alpha_{m}^{g}\right)} \nu_{\alpha_{m}^{g}}(I)} .
$$

For all $m \geq 2$, and $s_{m-1}^{\prime}<s+1 \leq s_{m}$,

$$
\mathbf{G}_{\widehat{\alpha}, s+1}=\bigcup_{I_{s} \in \mathbf{G}_{\widehat{\alpha}, s}} \mathbf{G}_{\widehat{\alpha}, s+1}\left(I_{s}, \alpha_{m}^{f}\right),
$$

where

$$
\mathbf{G}_{\widehat{\alpha}, s+1}\left(I_{s}, \alpha_{m}^{f}\right)=\left\{I_{s} L_{m, \alpha_{m}^{f}}^{f} I_{m}: I_{m} \in G_{m}^{f}\left(\alpha_{m}^{f}\right)\right\}
$$

and the measure $\nu_{\widehat{\alpha}, s+1}$ on $\mathbf{G}_{\widehat{\alpha}, s+1}$ is defined as

$$
\nu_{\widehat{\alpha}, s+1}\left(I_{s} L_{m, \alpha_{m}^{f}}^{f} I_{m}\right)=\nu_{\widehat{\alpha}, s}\left(I_{s}\right) \frac{\nu_{\alpha_{m}^{f}}\left(I_{m}\right)}{\sum_{I \in G_{m}^{f}\left(\alpha_{m}^{f}\right)} \nu_{\alpha_{m}^{f}}(I)} .
$$

This yields a Borel probability measure $\nu_{\widehat{\alpha}}$ supported on

$$
K \supset K_{\widehat{\alpha}}=\bigcap_{s \geq 1} \bigcup_{I \in \mathbf{G}_{\hat{\alpha}, s}} I
$$

and such that $\nu_{\widehat{\alpha}}(I)=\nu_{\widehat{\alpha}, s}(I)$ for all $s \geq 1$ and $I \in \mathbf{G}_{\widehat{\alpha}, s}$. Moreover, estimates similar to those used to control the local dimension of $\mu$ show that there exists a positive sequence $(\epsilon(s))_{s \geq 1}$ such that $\lim _{s \rightarrow \infty} \epsilon(s)=0$ and for all $x \in K_{\widehat{\alpha}}$ and $s \geq 1$, if $s_{m-1}^{\prime}<s \leq s_{m}$ and $s^{\prime}=s-s_{m-1}^{\prime}$, we have

$$
\begin{equation*}
2^{-\epsilon(s) n(s)} \leq \frac{\nu_{\widehat{\alpha}}\left(I_{n(s)}(x)\right)}{2^{-n(s) \gamma_{s}(x)}} \leq 2^{\epsilon(s) n(s)} \tag{4.12}
\end{equation*}
$$

with

$$
\gamma_{s}(x)=\frac{\sum_{j=1}^{s^{\prime}} N_{m} \gamma_{m}^{f}\left(\alpha_{m, j}^{f}(x)\right)+\left(\sum_{i=1}^{m-1} \sum_{j=1}^{R_{i}^{f}} N_{i} \gamma_{i}^{f}\left(\alpha_{i, j}^{f}(x)\right)+\sum_{j=1}^{R_{i}^{g}} N_{i} \gamma_{i}^{g}\left(\alpha_{i, j}^{g}(x)\right)\right)}{s^{\prime} N_{m}+\sum_{i=1}^{m-1} R_{i}^{f} N_{i}+R_{i}^{g} N_{i}},
$$

and if $s_{m}<s \leq s_{m}^{\prime}$ and $s^{\prime}=s-s_{m}$, we have

$$
\begin{equation*}
2^{-\epsilon(s) n(s)} \leq \frac{\nu_{\widehat{\alpha}}\left(I_{n(s)}(x)\right)}{2^{-n(s) \gamma_{s}(x)}} \leq 2^{\epsilon(s) n(s)} \tag{4.13}
\end{equation*}
$$

with
$\gamma_{s}(x)=\frac{\sum_{j=1}^{s^{\prime}} N_{m} \gamma_{m}^{g}\left(\alpha_{m, j}^{g}(x)\right)+\sum_{j=1}^{R_{m}^{f}} N_{m} \gamma_{m}^{f}\left(\alpha_{m, j}^{f}(x)\right)+\sum_{i=1}^{m-1} \sum_{j=1}^{R_{i}^{f}} N_{i} \gamma_{i}^{f}\left(\alpha_{i, j}^{f}(x)\right)+\sum_{j=1}^{R_{i}^{g}} N_{i} \gamma_{i}^{g}\left(\alpha_{i, j}^{g}(x)\right)}{s^{\prime} N_{m}+R_{m}^{f} N_{m}+\sum_{i=1}^{m-1} R_{i}^{f} N_{i}+R_{i}^{g} N_{i}}$.
Moreover, since by construction for $x \in K_{\widehat{\alpha}}$, we have $\alpha_{i, j}^{f}(x)=\alpha_{i}^{f}$ and $\alpha_{i, j}^{g}(x)=\alpha_{i}^{g}$ for all $1 \leq j \leq R_{i}^{f}$ and $1 \leq j \leq R_{i}^{g}$ respectively, from (4.7) and (4.12) we get

$$
\begin{align*}
\alpha_{s}(x) & =\frac{s^{\prime} N_{m} \alpha_{m}^{f}+\sum_{i=1}^{m-1} R_{i}^{f} N_{i} \alpha_{i}^{f}+R_{i}^{g} N_{i} \alpha_{i}^{g}}{s^{\prime} N_{m}+\sum_{i=1}^{m-1} R_{i}^{f} N_{i}+R_{i}^{g} N_{i}}  \tag{4.14}\\
\gamma_{s}(x) & =\frac{s^{\prime} N_{m} \gamma_{m}^{f}\left(\alpha_{m}^{f}\right)+\sum_{i=1}^{m-1} R_{i}^{f} N_{i} \gamma_{i}^{f}\left(\alpha_{i}^{f}\right)+R_{i}^{g} N_{i} \gamma_{i}^{g}\left(\alpha_{i}^{g}\right)}{s^{\prime} N_{m}+\sum_{i=1}^{m-1} R_{i}^{f} N_{i}+R_{i}^{g} N_{i}}
\end{align*}
$$

if $s_{m-1}^{\prime}<s \leq s_{m}$, and from (4.10) and (4.13) we get

$$
\begin{align*}
\alpha_{s}(x) & =\frac{s^{\prime} N_{m} \alpha_{m}^{g}+R_{m}^{f} N_{m} \alpha_{m}^{f}+\sum_{i=1}^{m-1} R_{i}^{f} N_{i} \alpha_{i}^{f}+R_{i}^{g} N_{i} \alpha_{i}^{g}}{s^{\prime} N_{m}+R_{m}^{f} N_{m}+\sum_{i=1}^{m-1} R_{i}^{f} N_{i}+R_{i}^{g} N_{i}}  \tag{4.15}\\
\gamma_{s}(x) & =\frac{s^{\prime} N_{m} \gamma_{m}^{g}\left(\alpha_{m}^{g}\right)+R_{m}^{f} N_{m} \gamma_{m}^{f}\left(\alpha_{m}^{g}\right)+\sum_{i=1}^{m-1} R_{i}^{f} N_{i} \gamma_{i}^{f}\left(\alpha_{i}^{f}\right)+R_{i}^{g} N_{i} \gamma_{i}^{g}\left(\alpha_{i}^{g}\right)}{s^{\prime} N_{m}+R_{m}^{f} N_{m}+\sum_{i=1}^{m-1} R_{i}^{f} N_{i}+R_{i}^{g} N_{i}}
\end{align*}
$$

$$
\text { if } s_{m}<s \leq s_{m}^{\prime} \text {. }
$$

We also have

$$
\begin{equation*}
\underline{d}\left(\nu_{\widehat{\alpha}}, x\right)=\liminf _{s \rightarrow \infty} \frac{\log _{2}\left(\nu_{\widehat{\alpha}}\left(I_{n(s)}(x)\right)\right.}{-n(s)} \quad \text { and } \quad \bar{d}\left(\nu_{\widehat{\alpha}}, x\right)=\limsup _{s \rightarrow \infty} \frac{\log _{2}\left(\nu_{\widehat{\alpha}}\left(I_{n(s)}(x)\right)\right.}{-n(s)} \tag{4.16}
\end{equation*}
$$

for all $x \in K_{\widehat{\alpha}}$, for the same reasons as those leading to (4.5).
4.3.3. Lower bounds for the dimensions. - Suppose that we have proven that for all $0 \leq \alpha \leq \beta \leq \infty$ we have $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)=\max \left\{f\left(\alpha^{\prime}\right): \alpha^{\prime} \in[\alpha, \beta]\right\}$, a property which will be established in Section 4.4.3. Then, the lower bounds of Theorem 1.4(3) follow readily from the mass distribution principle (see Section 6), property (4.16), and the following proposition, which is a direct consequence of the estimates (4.14) and (4.15), and the assumptions (4.2)(4-5).

Proposition 4.2. - With the notations of the previous section, fix $0 \leq \alpha \leq \beta \leq \infty$ such that $[\alpha, \beta] \subset \mathcal{g}$ and $[\alpha, \beta] \cap \mathcal{I} \neq \varnothing$. Let $\alpha^{\prime}=\operatorname{argmax}\left(f_{[[\alpha, \beta]}\right)$ and $\beta^{\prime}=\operatorname{argmax}\left(g_{[[\alpha, \beta]}\right)$.

Fix a sequence $\widehat{\alpha}=\left(\alpha_{1}^{f}, \alpha_{1}^{g}, \ldots, \alpha_{m}^{f}, \alpha_{m}^{g}, \ldots\right) \in \prod_{m=1}^{\infty} A_{m}^{f} \times A_{m}^{g} \quad$ such that $\lim _{m \rightarrow \infty} \alpha_{m}^{f}=\alpha^{\prime}, \lim _{m \rightarrow \infty} \gamma_{m}^{f}\left(\alpha_{m}^{f}\right)=f\left(\alpha^{\prime}\right), \lim _{m \rightarrow \infty} \alpha_{3 m-2}^{g}=\alpha, \lim _{m \rightarrow \infty} \gamma_{3 m-2}^{g}\left(\alpha_{3 m-2}^{g}\right)=g(\alpha)$, $\lim _{m \rightarrow \infty} \alpha_{3 m-1}^{g}=\beta, \lim _{m \rightarrow \infty} \gamma_{3 m-1}^{g}\left(\alpha_{3 m-1}^{g}\right)=g(\beta), \lim _{m \rightarrow \infty} \alpha_{3 m}^{g}=\beta^{\prime}$, and $\lim _{m \rightarrow \infty} \gamma_{3 m}^{g}\left(\alpha_{3 m}^{g}\right)=g\left(\beta^{\prime}\right)$.
Then for all $x \in K_{\widehat{\alpha}}$, one has

$$
\underline{d}(\mu, x)=\alpha, \bar{d}(\mu, x)=\beta, \underline{d}\left(\nu_{\widehat{\alpha}}, x\right)=\min \left\{f\left(\alpha^{\prime}\right), g(\alpha), g(\beta)\right\}, \text { and } \bar{d}\left(\nu_{\widehat{\alpha}}, x\right)=g\left(\beta^{\prime}\right) .
$$

Consequently, $\nu_{\widehat{\alpha}}(E(\mu, \alpha, \beta))=1, \operatorname{dim}_{H} \nu_{\widehat{\alpha}}=\min \left\{f\left(\alpha^{\prime}\right), g(\alpha), g(\beta)\right\}$ and $\operatorname{dim}_{P} \nu_{\widehat{\alpha}}=g\left(\beta^{\prime}\right)$, so that $\operatorname{dim}_{H} E(\mu, \alpha, \beta) \geq \min \left\{f\left(\alpha^{\prime}\right), g(\alpha), g(\beta)\right\}$ and $\operatorname{dim}_{H} E(\mu, \alpha, \beta) \geq f\left(\beta^{\prime}\right)$.

### 4.4. Large deviations spectra and $L^{q}$-spectra

4.4.1. The large deviations spectra $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha)$ and $\bar{f}_{\mu}^{\mathrm{LD}}(\alpha)$. - It is clear from the construction of $\mu$, (4.7,4.10) and (4.2) that $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha)=-\infty$ if $\alpha \notin \mathscr{I}$ (take $s=s_{m}$ and use (4.8)) and $\bar{f}_{\mu}^{\mathrm{LD}}(\alpha)=-\infty$ if $\alpha \notin \mathscr{J}$ (use (4.7) and (4.10)). Moreover, by (1.4) we have $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha) \geq$ $\operatorname{dim}_{H} E(\mu, \alpha)$, hence $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha) \geq f(\alpha)$ for all $\alpha \in \mathcal{J}$ by Proposition 4.2. Similarly, $\bar{f}_{\mu}^{\mathrm{LD}}(\alpha) \geq$ $\operatorname{dim}_{P} E(\mu, \alpha)$, hence Proposition 4.2 yields $\bar{f}_{\mu}^{\mathrm{LD}}(\alpha) \geq g(\alpha)$ for all $\alpha \in \mathcal{J}$.

Let us show that $\bar{f}_{\mu}^{\mathrm{LD}}(\alpha) \leq g(\alpha)$ for $\alpha \in \mathcal{J}$.
Suppose first that $\alpha \in \mathcal{J} \backslash\{\infty\}$. Fix $\eta>0$ and $\epsilon_{\eta}>0$ such that $g\left(\alpha^{\prime}\right) \leq g(\alpha)+\eta$ if $\alpha^{\prime} \in\left[\alpha-2 \epsilon_{\eta}, \alpha+2 \epsilon_{\eta}\right]$. Fix $\epsilon \in\left(0, \epsilon_{\eta}\right)$. If $s_{m-1}^{\prime}<s \leq s_{m}$, due to (4.7), for $s$ large enough, if $I_{s} \in \mathbf{G}_{s}$ satisfies $2^{-n(s)(\alpha+\epsilon)} \leq \mu\left(I_{s}\right) \leq 2^{-n(s)(\alpha-\epsilon)}$, then for any $x \in I_{s}=I_{n(s)}(x)$ we have $\alpha-2 \epsilon \leq \alpha_{s}:=\alpha_{s}(x) \leq \alpha+2 \epsilon$, and the exponents $\alpha_{i, j}^{f}(x)$ and $\alpha_{i, j}^{g}(x)$ do not depend on $x \in I_{s}$.

Due to the multiplicative structure of the construction of $\mu$, for each such collection of exponents $\left\{\alpha_{i, j}^{f}, \alpha_{i, j}^{g}\right\}$, the set $\mathbf{G}_{s}\left(\left\{\alpha_{i, j}^{f}, \alpha_{i, j}^{g}\right\}\right)$ of those dyadic cubes $I_{s} \in \mathbf{G}_{s}$ such that
$\alpha_{i, j}^{h}(x)=\alpha_{i, j}^{h}$ for all $x \in I_{s}$ and $h \in\{f, g\}$ is such that (setting $s^{\prime}=s-s_{m-1}^{\prime}$ )

$$
\begin{aligned}
\# \mathbf{G}_{s}\left(\left\{\alpha_{i, j}^{f}, \alpha_{i, j}^{g}\right\}\right) & =\prod_{j=1}^{s^{\prime}} \# G_{m}^{f}\left(\alpha_{m, j}^{f}\right) \prod_{i=1}^{m-1}\left(\prod_{j=1}^{R_{i}^{f}} \# G_{i}^{f}\left(\alpha_{i, j}^{f}\right)\right) \prod_{j=1}^{R_{i}^{g}} \# G_{j}^{g}\left(\alpha_{i, j}^{g}\right) \\
& \leq \prod_{j=1}^{s^{\prime}} 2^{N_{m}\left(\gamma_{m}^{f}\left(\alpha_{m, j}^{f}\right)+\epsilon_{m}\right)} \prod_{i=1}^{m-1}\left(\prod_{j=1}^{R_{i}^{f}} 2^{N_{i}\left(\gamma_{i}^{f}\left(\alpha_{i, j}^{f}\right)+\epsilon_{i}\right)}\right) \prod_{j=1}^{R_{i}^{g}} 2^{N_{i}\left(\gamma_{i}^{g}\left(\alpha_{i, j}^{g}\right)+\epsilon_{i}\right)}
\end{aligned}
$$

Since for $\alpha \in A_{i}^{h}$ by construction we have $\gamma_{i}^{h}(\alpha) \leq h(\alpha)+2 \epsilon_{i}^{1 / 3}$, we get

$$
\begin{aligned}
\# \mathbf{G}_{s}\left(\left\{\alpha_{i, j}^{f}, \alpha_{i, j}^{g}\right\}\right) & \leq 2^{\eta(s) n^{\prime}(s)} \prod_{j=1}^{s^{\prime}} 2^{N_{m} f\left(\alpha_{m, j}^{f}\right)} \prod_{i=1}^{m-1}\left(\prod_{j=1}^{R_{i}^{f}} 2^{N_{i} f\left(\alpha_{i, j}^{f}\right)}\right) \prod_{j=1}^{R_{i}^{g}} 2^{N_{i} g\left(\alpha_{i, j}^{g}\right)} \\
& \leq 2^{\eta(s) n^{\prime}(s)} \prod_{j=1}^{s^{\prime}} 2^{N_{m} g\left(\alpha_{m, j}^{f}\right)} \prod_{i=1}^{m-1}\left(\prod_{j=1}^{R_{i}^{f}} 2^{N_{i} g\left(\alpha_{i, j}^{f}\right)}\right) \prod_{j=1}^{R_{i}^{g}} 2^{N_{i} g\left(\alpha_{i, j}^{g}\right)}
\end{aligned}
$$

with $\lim _{s \rightarrow \infty} \eta(s)=0$ and $n^{\prime}(s)$ defined like in (4.4). Now recall that $g$ is concave over $\mathscr{g} \backslash$ $\{\infty\}$. Thus

$$
\begin{aligned}
\sum_{j=1}^{s^{\prime}} N_{m} g\left(\alpha_{m, j}^{f}\right) & +\sum_{j=1}^{R_{i}^{f}} N_{i} g\left(\alpha_{i, j}^{f}\right)+\sum_{j=1}^{R_{i}^{g}} N_{i} g\left(\alpha_{i, j}^{g}\right) \\
& \leq n^{\prime}(s) g\left(\frac{\sum_{j=1}^{s^{\prime}} N_{m} \alpha_{m, j}+\sum_{i=1}^{m-1} \sum_{j=1}^{R_{i}^{f}} N_{i} \alpha_{i, j}^{f}+\sum_{j=1}^{R_{i}^{g}} N_{i} \alpha_{i, j}^{g}}{n^{\prime}(s)}\right) \\
& =n^{\prime}(s) g\left(\alpha_{s}\right) \leq n^{\prime}(s)(g(\alpha)+\eta)
\end{aligned}
$$

Consequently, due to Proposition 4.1 we get

$$
\# \mathbf{G}_{s}\left(\left\{\alpha_{i, j}^{f}, \alpha_{i, j}^{g}\right\}\right) \leq 2^{(g(\alpha)+\eta+\eta(s)) n(s)}
$$

Moreover, the number of such collections cannot exceed the total number of possible realizations of such a family when the condition $\alpha_{s} \in\left[\alpha-\epsilon_{\eta}, \alpha+\epsilon_{\eta}\right]$ is dropped, which by construction is equal to $\left(\# A_{m}^{f}\right)^{s^{\prime}} \prod_{i=1}^{m-1}\left(\# A_{i}^{f}\right)^{R_{i}^{f}}\left(\# A_{i}^{g}\right)^{R_{i}^{g}}=2^{\eta^{\prime}(s) n(s)}$, with $\lim _{s \rightarrow \infty} \eta^{\prime}(s)=0$, by $(4.2)(2)(3)$. We can conclude that

$$
\#\left\{I_{s} \in \mathbf{G}_{s}: 2^{-n(s)(\alpha+\epsilon)} \leq \mu\left(I_{s}\right) \leq 2^{-n(s)(\alpha-\epsilon)}\right\} \leq 2^{\left(g(\alpha)+\eta+\eta(s)+\eta^{\prime}(s)\right) n(s)}
$$

The same estimates hold if $s_{m}<s \leq s_{m}^{\prime}$, and this yields $\bar{f}_{\mu}^{\mathrm{LD}}(\alpha) \leq f(\alpha)+\eta$. Since this holds for all $\eta>0$, the have the desired conclusion.

Now suppose that $\alpha=\infty \in \mathcal{J}$. Since $g(\infty) \geq \sup \{g(\alpha): \alpha \in \mathcal{J} \backslash\{\infty\}\}$, with the same notations as above, the only change is that for any $A>0$ we must consider those intervals in $\mathbf{G}_{s}$ such that $\alpha_{s} \geq A$, and conditioning on the realization of $\left\{\alpha_{i, j}^{f}(x), \alpha_{i, j}^{g}(x)\right\}$, the same calculations as above yield, even without using the concavity of $g$,

$$
\#\left\{I_{s} \in \mathbf{G}_{s}: \mu\left(I_{s}\right) \leq 2^{-n(s) A}\right\} \leq 2^{\left(g(\infty)+\eta+\eta(s)+\eta^{\prime}(s)\right) n(s)}
$$

hence the result.
Let us prove that $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha) \leq f(\alpha)$ for $\alpha \in 才$.

Suppose first that $\alpha<\infty$. Fix $\eta>0$ and $\epsilon_{\eta}>0$ such that $f\left(\alpha^{\prime}\right) \leq f(\alpha)+\eta$ if $\alpha^{\prime} \in\left[\alpha-2 \epsilon_{\eta}, \alpha+2 \epsilon_{\eta}\right]$. Fix $\epsilon \in\left(0, \epsilon_{\eta}\right)$. Suppose that $s=s_{m}$. Due to (4.8), for $s$ large enough, if $I_{s} \in \mathbf{G}_{s}$ satisfies $2^{-n(s)(\alpha+\epsilon)} \leq \mu\left(I_{s}\right) \leq 2^{-n(s)(\alpha-\epsilon)}$, then for any $x \in I_{s}=I_{n(s)}(x)$ we have $\alpha-2 \epsilon \leq \widetilde{\alpha}_{s}:=\frac{\sum_{j=1}^{R_{m}^{f} N_{m} \alpha_{m, j}^{f}(x)}}{R_{m}^{f} N_{m}} \leq \alpha+2 \epsilon$. Also, given $\left\{\alpha_{i, j}^{f}, \alpha_{i, j}^{g}\right\}$, the set $\mathbf{G}_{s}\left(\left\{\alpha_{i, j}^{f}, \alpha_{i, j}^{g}\right\}\right)$ of those dyadic cubes $I_{s} \in \mathbf{G}_{s}$ such that $\alpha_{i, j}^{h}(x)=\alpha_{i, j}^{h}$ for all $x \in I_{s}$ and $h \in\{f, g\}$ is bounded as above by

$$
\prod_{j=1}^{R_{m}^{f}} 2^{N_{m}\left(\gamma_{m}^{f}\left(\alpha_{m, j}^{f}\right)+\epsilon_{m}\right)} \prod_{i=1}^{m-1}\left(\prod_{j=1}^{R_{i}^{f}} 2^{N_{i}\left(\gamma_{i}^{f}\left(\alpha_{i, j}^{f}\right)+\epsilon_{i}\right)}\right) \prod_{j=1}^{R_{i}^{g}} 2^{N_{i}\left(\gamma_{i}^{g}\left(\alpha_{i, j}^{g}\right)+\epsilon_{i}\right)},
$$

which yields

$$
\# \mathbf{G}_{s}\left(\left\{\alpha_{i, j}^{f}, \alpha_{i, j}^{g}\right\}\right) \leq 2^{o\left(R_{m}^{f} N_{m}\right)} 2^{\sum_{j=1}^{R} R_{n}^{f}} N_{m} \gamma_{m}^{f}\left(\alpha_{m, j}^{f}\right) \leq 2^{o\left(R_{m}^{f} N_{m}\right)} 2^{\sum_{j=1}^{R f} N_{m} f\left(\alpha_{m, j}^{f}\right)}
$$

due to (4.2)(4). Using the concavity of $f$ this implies

$$
\# \mathbf{G}_{s}\left(\left\{\alpha_{i, j}^{f}, \alpha_{i, j}^{g}\right\}\right) \leq 2^{R_{m}^{f} N_{m}\left(f\left(\widetilde{\alpha}_{s}\right)+o(1)\right)} \leq 2^{R_{m}^{f} N_{m}(f(\alpha)+2 \eta)}
$$

for $s$ large enough. Finally, counting the possible number of collections $\left\{\alpha_{i, j}^{f}, \alpha_{i, j}^{g}\right\}$ as above yields

$$
\begin{equation*}
\#\left\{I_{s} \in \mathbf{G}_{s}: 2^{-n(s)(\alpha+\epsilon)} \leq \mu\left(I_{s}\right) \leq 2^{-n(s)(\alpha-\epsilon)}\right\} \leq 2^{(f(\alpha)+3 \eta) n(s)} \tag{4.17}
\end{equation*}
$$

for $s=s_{m}$ large enough. Since $\eta$ is arbitrary, this is enough to conclude.
If $\alpha=\infty$, the previous estimates with $\gamma_{m}^{f}\left(\alpha_{m, j}\right)$ bounded by $f(\infty)+2 \epsilon_{m}$ yield the conclusion.

It remains to prove that $\bar{f}_{\mu}^{\mathrm{LD}}(\alpha) \geq g(\alpha)$ for $\alpha \in \mathscr{g} \backslash \mathcal{J}$. In fact the argument is valid for all $\alpha \in \mathcal{Z}$. Let $\alpha \in \mathcal{Z}$. Suppose first that $\alpha<\infty$. Choose a sequence $\widehat{\alpha}=\left(\alpha_{1}^{f}, \alpha_{1}^{g}, \ldots, \alpha_{m}^{f}, \alpha_{m}^{g}, \ldots\right) \in \prod_{m=1}^{\infty} A_{m}^{f} \times A_{m}^{g}$ such that $\lim _{m \rightarrow \infty} \alpha_{m}^{g}=\alpha$ and $\lim _{m \rightarrow \infty} \gamma_{m}^{g}\left(\alpha_{m}^{g}\right)=g(\alpha)$. We leave the reader check that there exists a sequence $\epsilon^{\prime \prime}(s)$ converging to 0 as $s=s_{m}^{\prime}$ tends to $\infty$ such that, for all $x \in K_{\widehat{\alpha}}$ (the Cantor set constructed in Section 4.3.2), we have $\alpha_{s}(x)=\alpha+\epsilon(s)$ (this uses (4.11)) and

$$
\#\left\{\mathbf{G}_{\widehat{\alpha}, s}=\prod_{i=1}^{m}\left(\# G^{f}\left(i, \alpha^{f}\right)\right)^{R_{i}^{f}}\left(\# G^{f}\left(i, \alpha^{g}\right)\right)^{R_{i}^{f}} \geq 2^{n(s)\left(g(\alpha)-\epsilon^{\prime \prime}(s)\right)}\right.
$$

(this uses the left hand side of (2.7), and (4.2)(5)). This implies that $\bar{f}_{\mu}^{\mathrm{LD}}(\alpha) \geq g(\alpha)$.
If $\alpha=\infty$, the same argument with $\alpha_{m}^{g}=\alpha_{m}^{g}(\infty)$ yields the desired lower bound.
We notice that a similar argument would give another proof of $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha) \geq f(\alpha)$ for $\alpha \in \mathcal{J}$.

Remark 4.2. - Infact, the arguments developed in this section provide us with the following precious information: for $\alpha \in \mathbb{R}_{+}$we have

$$
\begin{aligned}
f(\alpha) & =\lim _{\epsilon \rightarrow 0^{+}} \liminf _{m \rightarrow \infty} \frac{\log _{2} \#\left\{I \in \mathbf{G}_{s_{m}}: 2^{-n\left(s_{m}\right)(\alpha+\epsilon)} \leq \mu(I) \leq 2^{-n\left(s_{m}\right)(\alpha-\epsilon)}\right\}}{n\left(s_{m}\right)} \\
& =\lim _{\epsilon \rightarrow 0^{+}} \limsup _{m \rightarrow \infty} \frac{\log _{2} \#\left\{I \in \mathbf{G}_{s_{m}}: 2^{-n\left(s_{m}\right)(\alpha+\epsilon)} \leq \mu(I) \leq 2^{-n\left(s_{m}\right)(\alpha-\epsilon)}\right\}}{n\left(s_{m}\right)} ; \\
g(\alpha) & =\lim _{\epsilon \rightarrow 0^{+}} \liminf _{m \rightarrow \infty} \frac{\log _{2} \#\left\{I \in \mathbf{G}_{s_{m}^{\prime}}: 2^{-n\left(s_{m}^{\prime}\right)(\alpha+\epsilon)} \leq \mu(I) \leq 2^{-n\left(s_{m}^{\prime}\right)(\alpha-\epsilon)}\right\}}{n\left(s_{m}^{\prime}\right)} \\
& =\lim _{\epsilon \rightarrow 0^{+}} \limsup _{m \rightarrow \infty} \frac{\log _{2} \#\left\{I \in \mathbf{G}_{s_{m}^{\prime}}: 2^{-n\left(s_{m}^{\prime}\right)(\alpha+\epsilon)} \leq \mu(I) \leq 2^{-n\left(s_{m}^{\prime}\right)(\alpha-\epsilon)}\right\}}{n\left(s_{m}^{\prime}\right)}
\end{aligned}
$$

and we also have

$$
\begin{aligned}
f(\infty) & =\lim _{A \rightarrow \infty} \liminf _{m \rightarrow \infty} \frac{\log _{2} \#\left\{I \in \mathbf{G}_{s_{m}}: \mu(I) \leq 2^{-n\left(s_{m}\right) A}\right\}}{n\left(s_{m}\right)} \\
& =\lim _{A \rightarrow \infty} \limsup _{m \rightarrow \infty} \frac{\log _{2} \#\left\{I \in \mathbf{G}_{s_{m}}: \mu(I) \leq 2^{-n\left(s_{m}\right) A}\right\}}{n\left(s_{m}\right)}
\end{aligned}
$$

and

$$
\begin{aligned}
g(\infty) & =\lim _{A \rightarrow \infty} \liminf _{m \rightarrow \infty} \frac{\log _{2} \#\left\{I \in \mathbf{G}_{s_{m}^{\prime}}: \mu(I) \leq 2^{-n\left(s_{m}^{\prime}\right) A}\right\}}{n\left(s_{m}^{\prime}\right)} \\
& =\lim _{A \rightarrow \infty} \limsup _{m \rightarrow \infty} \frac{\log _{2} \#\left\{I \in \mathbf{G}_{s_{m}^{\prime}}: \mu(I) \leq 2^{-n\left(s_{m}^{\prime}\right) A}\right\}}{n\left(s_{m}^{\prime}\right)}
\end{aligned}
$$

4.4.2. The functions $\tau_{\mu}$ and $\bar{\tau}_{\mu}$. - It follows from Remark 4.2 and standard large deviations estimates similar to those used for instance in the proof of [75, Theorem 4.2], that for all $q \in \mathbb{R}$ we have

$$
\lim _{m \rightarrow \infty} \frac{\log _{2} \sum_{I \in \mathbf{G}_{s_{m}}} \mu(I)^{q}}{-n\left(s_{m}\right)}=f^{*}(q) \quad \text { and } \quad \lim _{m \rightarrow \infty} \frac{\log _{2} \sum_{I \in \mathbf{G}_{s_{m}^{\prime}}} \mu(I)^{q}}{-n\left(s_{m}^{\prime}\right)}=g^{*}(q)
$$

(in case the domains of $f$ and $g$ are compact, this is a direct consequence of LaplaceVaradhan's integral lemma [28, Theorem 4.3.1]). Consequently, due to (4.6) we get $\tau_{\mu}=g^{*}$ and $\bar{\tau}_{\mu}=f^{*}$. Moreover, it is direct from the duality between upper semi-continuous concave functions [77, Theorem 12.2, Corollary 12.2.2] that if $\infty \notin \operatorname{dom}(f)$, we have $\tau_{\mu}^{*}=g$ and $\bar{\tau}_{\mu}^{*}=f$, and if $\infty \in \operatorname{dom}(f)$, then $\operatorname{dom}\left(\tau_{\mu}=f^{*}\right)=\mathbb{R}_{+}$, and $\tau_{\mu}^{*}$ coincides with $f$ over $\mathbb{R}$. Then, by our definition of the extended concave conjugate function, we have $\tau_{\mu}^{*}(\infty)=$ $-\tau_{\mu}(0)=f(\infty)$.
 defined as $\max \left\{f\left(\alpha^{\prime}\right): \alpha^{\prime} \in[\alpha, \beta]\right\}$.

If $[\alpha, \beta] \subset \mathbb{R}_{+} \cup\{\infty\} \backslash ף$, due to (4.8), we have $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)=-\infty=f(\alpha, \beta)$.

Assume now that $[\alpha, \beta] \cap \leftrightharpoons \neq \varnothing$, and $(\alpha, \beta) \neq(\infty, \infty)$. Denote the interval $[\alpha, \beta] \cap$ g by $\left[\alpha_{1}, \beta_{1}\right]$, and notice that $f(\alpha, \beta)=f\left(\alpha_{1}, \beta_{1}\right)$.

Suppose first that $\beta_{1}<\infty$. Let $\eta>0$ and for each $\alpha^{\prime} \in\left[\alpha_{1}, \beta_{1}\right]$ let $\epsilon\left(\alpha^{\prime}\right)>0$ such that $f(\beta) \leq f\left(\alpha^{\prime}\right)+\eta$ for all $\beta \in\left[\alpha^{\prime}-\epsilon\left(\alpha^{\prime}\right), \alpha^{\prime}+\epsilon\left(\alpha^{\prime}\right)\right]$. There exists $\alpha_{1}^{\prime}, \ldots, \alpha_{N}^{\prime}$ in $\left[\alpha_{1}, \beta_{1}\right]$ such that $\left[\alpha_{1}, \beta_{1}\right] \subset \bigcup_{i=1}^{N}\left[\alpha_{i}^{\prime}-\epsilon\left(\alpha_{i}^{\prime}\right), \alpha_{i}^{\prime}-\epsilon\left(\alpha_{i}^{\prime}\right)\right]$. Let $\epsilon \leq \min \left\{\epsilon\left(\alpha_{i}^{\prime}\right): 1 \leq i \leq N\right\}$. Property (4.8) implies that for $m$ large enough, if $I \in \mathbf{G}_{s_{m}}$ and $2^{-n\left(s_{m}\right)(\beta+\epsilon)} \leq \mu(I) \leq 2^{-n\left(s_{m}\right)(\alpha-\epsilon)}$, since there exists $x \in K$ such that $I=I_{n\left(s_{m}\right)}(x)$, in fact there exists $1 \leq i \leq N$ such that $2^{-n\left(s_{m}\right)\left(\alpha_{i}^{\prime}+\epsilon\left(\alpha_{i}^{\prime}\right)\right)} \leq \mu(I) \leq 2^{-n\left(s_{m}\right)\left(\alpha_{i}^{\prime}-\epsilon\left(\alpha_{i}^{\prime}\right)\right)}$. Then, the estimate (4.17) achieved in Section 4.4.1 yields

$$
\#\left\{I \in \mathbf{G}_{s_{m}}: 2^{-n\left(s_{m}\right)\left(\alpha_{i}^{\prime}+\epsilon\left(\alpha_{i}^{\prime}\right)\right)} \leq \mu(I) \leq 2^{-n\left(s_{m}\right)\left(\alpha_{i}^{\prime}-\epsilon\left(\alpha_{i}^{\prime}\right)\right)}\right\} \leq 2^{\left(f\left(\alpha_{i}^{\prime}\right)+3 \eta\right) n\left(s_{m}\right)}
$$

for $m$ large enough. This implies that for $m$ large enough

$$
\begin{aligned}
\#\left\{I \in \mathbf{G}_{s_{m}}: 2^{-n\left(s_{m}\right)(\beta+\epsilon)} \leq \mu(I) \leq\right. & \left.2^{-n\left(s_{m}\right)(\alpha-\epsilon)}\right\} \\
& \leq \sum_{i=1}^{N} 2^{\left(f\left(\alpha_{i}^{\prime}\right)+3 \eta\right) n\left(s_{m}\right)} \leq N 2^{(f(\alpha, \beta)+3 \eta) n\left(s_{m}\right)}
\end{aligned}
$$

If follows that $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta) \leq f(\alpha, \beta)+3 \eta$ for all $\eta>0$, hence the desired upper bound $f(\alpha, \beta)$ for $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)$.

For the lower bound, we just use the fact that by Proposition 4.2 we know that if $\alpha^{\prime}=\operatorname{argmax}\left(f_{[\alpha, \beta]}\right)$ we have $f\left(\alpha^{\prime}\right) \leq \operatorname{dim}_{H} E\left(\mu, \alpha^{\prime}\right)$, and on the other hand $\operatorname{dim}_{H} E\left(\mu, \alpha^{\prime}\right) \leq \underline{f}_{\mu}^{\mathrm{LD}}\left(\alpha^{\prime}\right) \leq \underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)$, the last inequality being obvious.

If $\beta_{1}=\infty$, the upper bound for $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)$ just comes from the observation already done in Section 4.4.1 that if $s=s_{m}$, we have $\# \mathbf{G}_{s} \leq 2^{n(s)(f(\infty)+o(1))}$, and the lower bound comes from the lower bound $f(\infty)$ for $\operatorname{dim}_{H} E(\mu, \infty)$.

### 4.5. Upper bounds for the different spectra

It is a direct application of Proposition 1.3, using the fact that $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)=f(\alpha, \beta)$.

## 4.6. $\mu$ is exactly dimensional, with dimension $D$

Fix $\epsilon>0$. For each $s \geq 1$, if $s_{m-1}^{\prime}<s \leq s_{m}$ and $s^{\prime}=s-s_{m-1}^{\prime}$, an application of Markov's inequality yields, for any $\eta>0$ :

$$
\begin{aligned}
\mu\left(E_{s,+}=\{x\right. & \left.\left.\in K: \frac{\mu\left(I_{n(s)}(x)\right)}{\mu\left(I_{n\left(s_{m-1}^{\prime}\right)}(x)\right)} \geq 2^{-\left(n^{\prime}(s)-n^{\prime}\left(s_{m-1}^{\prime}\right)\right)\left(D_{m}-\epsilon\right)}\right\}\right) \\
& \leq \sum_{I_{n(s)} \in \mathbf{G}_{s}} \mu\left(I_{n(s)}\right)\left(\frac{\mu\left(I_{n(s)}\right)}{\mu\left(I_{n\left(s_{m-1}^{\prime}\right)}\right)}\right)^{\eta} 2^{\eta\left(n^{\prime}(s)-n^{\prime}\left(s_{m-1}^{\prime}\right)\right)\left(D_{m}-\epsilon\right)} \\
& =\sum_{I_{n(s)} \in \mathbf{G}_{s}} \mu\left(I_{n\left(s_{m-1}^{\prime}\right)}\right)\left(\frac{\mu\left(I_{n(s)}\right)}{\mu\left(I_{n\left(s_{m-1}^{\prime}\right)}^{\prime}\right)}\right)^{1+\eta} 2^{\eta\left(n^{\prime}(s)-n^{\prime}\left(s_{m-1}^{\prime}\right)\right)\left(D_{m}-\epsilon\right)},
\end{aligned}
$$

where $I_{n\left(s_{m-1}^{\prime}\right)}$ stands for the unique element of $\mathbf{G}_{s_{m-1}^{\prime}}$ containing $I_{n(s)}$. We notice that by construction, given $I_{n\left(s_{m-1}^{\prime}\right)}$ in $\mathbf{G}_{s_{m-1}^{\prime}}$, the distribution of the collection $\left\{\frac{\mu\left(I_{n(s)}\right)}{\mu\left(I_{n\left(s_{m-1}^{\prime}\right)}\right)}\right\}$, where $I_{n(s)} \in \mathbf{G}_{s}$ and $I_{n(s)} \subset I_{n\left(s_{m-1}^{\prime}\right)}$, is independent of $I_{n\left(s_{m-1}^{\prime}\right)}$, and taking into account
the fact that between the steps $s_{m-1}^{\prime}$ and $s$ one uses $s^{\prime}$ times the same motive in the recursion defining $\mu$, we can get

$$
\begin{aligned}
\mu\left(E_{s,+}\right) & \leq \frac{\left(\sum_{\alpha \in A_{m}^{f}} \sum_{I \in G^{f}(m, \alpha)} \rho_{m}^{f}(\alpha)^{1+\eta} \mu_{\alpha}(I)^{1+\eta}\right)^{s^{\prime}}}{\left(\sum_{\alpha \in A_{m}^{f}} \sum_{I \in G^{f}(m, \alpha)} \rho_{m}^{f}(\alpha) \mu_{\alpha}(I)\right)^{(1+\eta) s^{\prime}}} 2^{\eta\left(n^{\prime}(s)-n^{\prime}\left(s_{m-1}^{\prime}\right)\right)\left(D_{m}-\epsilon\right)} \\
& \leq 2^{(1+\eta) s^{\prime}}\left(\sum_{\alpha \in A_{m}^{f}} \sum_{I \in G^{f}(m, \alpha)} \rho_{m}^{f}(\alpha)^{1+\eta} \mu_{\alpha}(I)^{1+\eta}\right)^{s^{\prime}} 2^{\eta\left(n^{\prime}(s)-n^{\prime}\left(s_{m-1}^{\prime}\right)\right)\left(D_{m}-\epsilon\right)}
\end{aligned}
$$

We have

$$
\begin{aligned}
& \sum_{I \in G^{f}\left(m, D_{m}\right)} \rho_{m}^{f}\left(D_{m}\right)^{1+\eta} \mu_{D_{m}}(I)^{1+\eta}=\sum_{I \in G^{f}\left(m, D_{m}\right)} \mu_{D_{m}}(I)^{1+\eta} \\
& \leq\left(\# G^{f}\left(m, D_{m}\right)\right) 2^{-N_{m}\left(D_{m}-\epsilon_{m}\right)(1+\eta)} \leq 2^{N_{m}\left(D_{m}+\epsilon_{m}\right)} 2^{-N_{m}\left(D_{m}-\epsilon_{m}\right)(1+\eta)} \\
&=2^{-N_{m} D_{m} \eta} 2^{N_{m} \epsilon_{m}(2+\eta)}
\end{aligned}
$$

On the other hand, if $A_{m}^{f} \backslash\left\{D_{m}\right\} \neq \varnothing$, fixing one of its elements $\alpha_{0}$, we have

$$
\begin{aligned}
\sum_{\alpha \in A_{m}^{f} \backslash\left\{D_{m}\right\}} & \sum_{I \in G^{f}(m, \alpha)} \rho_{m}^{f}(\alpha)^{1+\eta} \mu_{\alpha}(I)^{1+\eta} \\
& \leq \sum_{\alpha \in A_{m}^{f} \backslash\left\{D_{m}\right\}} \rho_{m}^{f}(\alpha)^{1+\eta}\left(\# G^{f}(m, \alpha)\right) 2^{-N_{m}\left(\alpha-\epsilon_{m}\right)(1+\eta)} \\
& \leq \sum_{\alpha \in A_{m}^{f} \backslash\left\{D_{m}\right\}} \rho_{m}^{f}(\alpha)^{1+\eta} 2^{N_{m}\left(\gamma_{m}^{f}(\alpha)+\epsilon_{m}\right)} 2^{-N_{m}\left(\alpha-\epsilon_{m}\right)(1+\eta)} \\
& \leq \rho_{m}^{f}\left(\alpha_{0}\right)^{1+\eta}\left(\# A_{m}^{f}\right)\left(\sup _{\alpha \in A_{m}^{f} \backslash\left\{D_{m}\right\}} 2^{N_{m}\left(\gamma_{m}^{f}(\alpha)-\alpha\right)}\right) 2^{N_{m} \epsilon_{m}(2+\eta)} .
\end{aligned}
$$

Now, take $\eta=\eta_{m}=\sqrt{\epsilon_{m}}$. Due to the Definition (4.3) of $\rho_{m}^{f}$ we have $\rho_{m}^{f}\left(\alpha_{0}\right)^{1+\eta}\left(\# A_{m}^{f}\right) \leq 1$, and due to (4.1), we have $\sup _{\alpha \in A_{m}^{f} \backslash\left\{D_{m}\right\}} 2^{N_{m}\left(\gamma_{m}^{f}(\alpha)-\alpha\right)} \leq 2^{-N_{m} D_{m} \eta_{m}}$. Finally,

$$
\begin{aligned}
\mu\left(E_{s,+}\right) & \leq 2^{s^{\prime}} \cdot 2^{\left(1+\eta_{m}\right) s^{\prime}}\left(2^{-N_{m} D_{m} \eta_{m}} 2^{N_{m} \epsilon_{m}\left(2+\eta_{m}\right)}\right)^{s^{\prime}} 2^{\eta_{m}\left(n^{\prime}(s)-n^{\prime}\left(s_{m-1}^{\prime}\right)\right)\left(D_{m}-\epsilon\right)} \\
& =2^{s^{\prime}} \cdot 2^{\left(1+\eta_{m}\right) s^{\prime}}\left(2^{-N_{m} D_{m} \eta_{m}} 2^{N_{m} \epsilon_{m}\left(2+\eta_{m}\right)}\right)^{s^{\prime}} 2^{\eta_{m} N_{m} s^{\prime}\left(D_{m}-\epsilon\right)} \\
& \leq 2^{3 s^{\prime}} 2^{-N_{m} s^{\prime} \eta_{m}\left(\epsilon-3 \eta_{m}\right)} .
\end{aligned}
$$

Also, using a similar estimate as above, and with the same choice $\eta=\eta_{m}=\sqrt{\epsilon_{m}}$, we have

$$
\begin{aligned}
\mu(\{x \in K: & \left.\left.\frac{\mu\left(I_{n(s)}(x)\right)}{\mu\left(I_{n\left(s_{m-1}^{\prime}\right)}^{\prime}(x)\right)} \leq 2^{-\left(n^{\prime}(s)-n^{\prime}\left(s_{m-1}^{\prime}\right)\right)\left(D_{m}+\epsilon\right)}\right\}\right) \\
& \leq \frac{\left(\sum_{\alpha \in A_{m}^{f}} \sum_{I \in G^{f}(m, \alpha)} \rho_{m}^{f}(\alpha)^{1-\eta} \mu_{\alpha}(I)^{1-\eta}\right)^{s^{\prime}}}{\left(\sum_{\alpha \in A_{m}^{f}} \sum_{I \in G^{f}(m, \alpha)} \rho_{m}^{f}(\alpha) \mu_{\alpha}(I)\right)^{(1-\eta) s^{\prime}}} 2^{-\eta\left(n^{\prime}(s)-n^{\prime}\left(s_{m-1}^{\prime}\right)\right)\left(D_{m}+\epsilon\right)} \\
& \leq 2^{(1-\eta) s^{\prime}}\left(\sum_{\alpha \in A_{m}^{f}} \sum_{I \in G^{f}(m, \alpha)} \rho_{m}^{f}(\alpha)^{1-\eta} \mu_{\alpha}(I)^{1-\eta}\right)^{s^{\prime}} 2^{-\eta\left(n^{\prime}(s)-n^{\prime}\left(s_{m-1}^{\prime}\right)\right)\left(D_{m}+\epsilon\right)} .
\end{aligned}
$$

On the one hand, we have

$$
\begin{aligned}
& \sum_{I \in G^{f}\left(m, D_{m}\right)} \rho_{m}^{f}\left(D_{m}\right)^{1-\eta} \mu_{D_{m}}(I)^{1-\eta}=\sum_{\substack{I \in G^{f}\left(m, D_{m}\right)}} \mu_{D_{m}}(I)^{1-\eta} \\
& \leq 2^{N_{m}\left(D_{m}+\epsilon_{m}\right)} 2^{-N_{m}\left(D_{m}-\epsilon_{m}\right)(1-\eta)}=2^{N_{m} D_{m} \eta} 2^{2 N_{m} \epsilon_{m}} .
\end{aligned}
$$

On the other hand, if $A_{m}^{f} \backslash\left\{D_{m}\right\} \neq \varnothing$, fixing one of its elements $\alpha_{0}$, we have

$$
\begin{aligned}
\sum_{\alpha \in A_{m}^{f} \backslash\left\{D_{m}\right\}} & \sum_{I \in G^{f}(m, \alpha)} \rho_{m}^{f}(\alpha)^{1-\eta} \mu_{\alpha}(I)^{1-\eta} \\
& \leq \sum_{\alpha \in A_{m}^{f} \backslash\left\{D_{m}\right\}} \rho_{m}^{f}(\alpha)^{1-\eta} 2^{N_{m}\left(\gamma_{m}^{f}(\alpha)+\epsilon_{m}\right)} 2^{-N_{m}\left(\alpha-\epsilon_{m}\right)(1-\eta)} \\
& \leq \rho_{m}^{f}\left(\alpha_{0}\right)^{1-\eta}\left(\# A_{m}^{f}\right)\left(\sup _{\alpha \in A_{m}^{f} \backslash\left\{D_{m}\right\}} 2^{N_{m}\left(\gamma_{m}^{f}(\alpha)-(1-\eta) \alpha\right)}\right) 2^{2 N_{m} \epsilon_{m}} .
\end{aligned}
$$

Due to the Definition (4.3) of $\rho_{m}^{f}$, since $\sqrt{\epsilon_{m}}=(m+1)^{-1} \leq 1 / 2$, we have $\rho_{m}^{f}\left(\alpha_{0}\right)^{1-\eta_{m}}\left(\# A_{m}^{f}\right) \leq 1$, and due to (4.1), we have $\sup _{\alpha \in A_{m}^{f} \backslash\left\{D_{m}\right\}} 2^{N_{m}\left(\gamma_{m}^{f}(\alpha)-\left(1-\eta_{m} \alpha\right)\right.} \leq 1$. Consequently, the previous estimates yield
$\mu\left(E_{s,-}=\left\{x \in K: \frac{\mu\left(I_{n(s)}(x)\right)}{\mu\left(I_{n\left(s_{m-1}^{\prime}\right)}(x)\right)} \leq 2^{-\left(n^{\prime}(s)-n^{\prime}\left(s_{m-1}^{\prime}\right)\right)\left(D_{m}+\epsilon\right)}\right\}\right) \leq 2^{3 s^{\prime}} 2^{-N_{m} s^{\prime} \eta_{m}\left(\epsilon-3 \eta_{m}\right)}$.
Similarly, if $s_{m}<s \leq s_{m}^{\prime}$ and $s^{\prime}=s-s_{m}$ we can get

$$
\begin{aligned}
& \mu\left(E_{s,+}=\left\{x \in K: \frac{\mu\left(I_{n(s)}(x)\right)}{\mu\left(I_{n\left(s_{m}\right)}(x)\right)} \geq 2^{-\left(n^{\prime}(s)-n^{\prime}\left(s_{m}\right)\right)\left(D_{m}-\epsilon\right)}\right\}\right) \leq 2^{3 s^{\prime}} 2^{-N_{m} s^{\prime} \eta_{m}\left(\epsilon-3 \eta_{m}\right)} \\
& \mu\left(E_{s,-}=\left\{x \in K: \frac{\mu\left(I_{n(s)}(x)\right)}{\mu\left(I_{n\left(s_{m}\right)}(x)\right)} \leq 2^{-\left(n^{\prime}(s)-n^{\prime}\left(s_{m}\right)\right)\left(D_{m}+\epsilon\right)}\right\}\right) \leq 2^{3 s^{\prime}} 2^{-N_{m} s^{\prime} \eta_{m}\left(\epsilon-3 \eta_{m}\right)} .
\end{aligned}
$$

Finally, for $m_{0}$ big enough so that $3 \eta_{m} \leq \epsilon / 2$, and $N_{m} \eta_{m} \epsilon / 2>4$ (remember that $N_{m} \geq e^{m}$ and $\eta_{m}=(m+1)^{-1}$ ), we have

$$
\begin{aligned}
& \sum_{m \geq m_{0}} \sum_{s_{m-1}^{\prime}<s \leq s_{m}} \mu\left(E_{s,+} \cup E_{s,-)}+\sum_{s_{m}<s \leq s_{m}^{\prime}} \mu\left(E_{s,+} \cup E_{s,-)}\right.\right. \\
& \quad \leq 2 \sum_{m \geq m_{0}} \sum_{s^{\prime}=1}^{R_{m}^{f}} 2^{3 s^{\prime}} 2^{-N_{m} s^{\prime} \eta_{m} \epsilon / 2}+\sum_{s^{\prime}=1}^{R_{m}^{g}} 2^{3 s^{\prime}} 2^{-N_{m} s^{\prime} \eta_{m} \epsilon / 2} \\
& \quad \leq 2 \sum_{m \geq m_{0}} 16 \cdot \frac{2^{-N_{m} \eta_{m} \epsilon / 2}}{1-8 \cdot 2^{-N_{m} \eta_{m} \epsilon / 2}} \leq 64 \sum_{m \geq m_{0}} 2^{-N_{m} \eta_{m} \epsilon / 2}<\infty .
\end{aligned}
$$

By the Borel-Cantelli lemma, we deduce that for $\mu$-almost every $x$, there exists an integer $m_{x}$ such that for all $m \geq m_{x}$, for all $s_{m-1}^{\prime}<s \leq s_{m}$ one has

$$
2^{-\left(n^{\prime}(s)-n^{\prime}\left(s_{m-1}^{\prime}\right)\right)\left(D_{m}+\epsilon\right)} \leq \frac{\mu\left(I_{n(s)}(x)\right)}{\mu\left(I_{n\left(s_{m-1}^{\prime}\right)}^{\prime}(x)\right)} \leq 2^{-\left(n^{\prime}(s)-n^{\prime}\left(s_{m-1}^{\prime}\right)\right)\left(D_{m}-\epsilon\right)}
$$

and for all $s_{m}<s \leq s_{m}^{\prime}$ one has

$$
2^{-\left(n^{\prime}(s)-n^{\prime}\left(s_{m}\right)\right)\left(D_{m}+\epsilon\right)} \leq \frac{\mu\left(I_{n(s)}(x)\right)}{\mu\left(I_{n\left(s_{m}\right)}(x)\right)} \leq 2^{-\left(n^{\prime}(s)-n^{\prime}\left(s_{m}\right)\right)\left(D_{m}-\epsilon\right)} .
$$

Using these inequalities telescopically and noting that $D_{m}$ converges to $D$ as $m \rightarrow \infty$ and $n^{\prime}(s) \sim n(s)$ as $s \rightarrow \infty$ we get $D-\epsilon \leq \underline{d}(\mu, x) \leq \bar{d}(\mu, x) \leq D+\epsilon$ for $\mu$-almost every $x$. Since $\epsilon$ is arbitrary, we have the desired exactly dimensionality.

### 4.7. Restrictions of $\mu$

Let $B$ be a closed ball whose interior intersects $\operatorname{supp}(\mu)$ at some point $x$. Let $\nu=\mu_{\mid B}$. We naturally have the inequalities $\operatorname{dim}_{L}(\nu, \alpha, \beta) \leq \operatorname{dim}_{L}(\mu, \alpha, \beta)$ for $L \in\{H, P\}$ and $0 \leq \alpha \leq \beta \leq \infty$, as well as $\underline{f}_{\nu}^{\mathrm{LD}}(\alpha) \leq \underline{f}_{\mu}^{\mathrm{LD}}(\alpha), \bar{f}_{\nu}^{\mathrm{LD}}(\alpha) \leq \bar{f}_{\mu}^{\mathrm{LD}}(\alpha), \underline{f}_{\nu}^{\mathrm{LD}}(\alpha, \beta) \leq \underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)$, $\tau_{\nu} \geq \tau_{\mu}$ and $\bar{\tau}_{\nu} \geq \bar{\tau}_{\mu}$. On the other hand, denoting by $s_{0}$ the smallest integer such that $I_{n\left(s_{0}\right)}(x) \subset \operatorname{Int}(B)$, we can modify the first terms of the sequence $\widehat{\alpha}$ of Proposition 4.2 so that $I_{n\left(s_{0}\right)}(x) \in \mathscr{G}_{\widehat{\alpha}, s_{0}}$, hence $\nu_{\widehat{\alpha}}\left(B \cap K_{\widehat{\alpha}}\right)>0$, and we have $\operatorname{dim}_{H} E(\nu, \alpha, \beta) \geq f(\alpha, \beta)$ and $\operatorname{dim}_{P} E(\nu, \alpha, \beta) \geq g(\alpha, \beta)$. This is enough to reverse all the previous inequalities since we also have for $\nu$ the general inequalities provided by (1.4), (1.5) and Proposition 1.3.

## 5. Proofs of Propositions 1.1(2), 1.2 and 1.3, and some inequalities in (1.4) and (1.5)

### 5.1. Proofs of Propositions 1.1(2) and 1.2

5.1.1. Proposition 1.1(2). - The fact that $\mathbb{R}_{+} \subset \operatorname{dom}\left(\tau_{\mu}\right)$ was explained after we defined $\tau_{\mu}$ in the introduction. Now suppose at first that there exist $\alpha \in \mathbb{R}_{+}$and $r_{0}>0$ such that for all $r \in\left(0, r_{0}\right)$ and $x \in \operatorname{supp}(\mu)$ we have $\mu(B(x, r))>r^{\alpha}$. Then, the definition of the $L^{q}$-spectrum yields $\tau_{\mu}(q) \geq \tau_{\mu}(0)+q \alpha$ for all $q<0$, hence $\tau_{\mu}$ is finite over $\mathbb{R}$. If, on the contrary, for all $\alpha>0$, for all $r_{0}>0$, there exist $r \in\left(0, r_{0}\right)$ and $x \in \operatorname{supp}(\mu)$ such that $\mu(B(x, r)) \leq r^{\alpha}$, by using again the definition of the $L^{q}$-spectrum we have $\tau_{\mu}(q) \leq \alpha q$ for all $\alpha>0$ and $q<0$, hence $\tau_{\mu}(q)=-\infty$ for $q<0$, so $\operatorname{dom}\left(\tau_{\mu}\right)=\mathbb{R}_{+}$.

Now let $\alpha \in \operatorname{dom}\left(\tau_{\mu}^{*}\right)$ and suppose that $-\infty<\tau_{\mu}^{*}(\alpha)<0$. Necessarily $\alpha<\infty$. Indeed one always has $\tau_{\mu}^{*}(\infty) \in\left\{-\infty,-\tau_{\mu}(0)\right\}$. Then, suppose first that $\alpha<\tau_{\mu}^{\prime}\left(0^{+}\right)$. Let $\alpha_{0}=\inf \left\{\beta \in\left(\alpha, \tau_{\mu}^{\prime}\left(0^{+}\right)\right): \tau_{\mu}^{*}(\beta) \geq 0\right\}$. The continuity of $\tau_{\mu}^{*}$ over the interior of its domain implies $\tau_{\mu}^{*}\left(\alpha_{0}\right)=0$. Then for all $\beta<\alpha_{0}$ we have $\tau_{\mu}^{*}(\beta)<0$, hence $\bar{f}^{\mathrm{LD}}(\beta)<0$. Consequently, for all $\epsilon>0$ there exists $r_{0}>0$ such that for all $r \in\left(0, r_{0}\right)$ and $x \in \operatorname{supp}(\mu)$ we have $\mu(B(x, r)) \leq r^{\alpha_{0}-\epsilon}$. This implies that $\tau_{\mu}(q) \geq \tau_{\mu}(0)+\alpha_{0} q$ for all $q \geq 0$, and finally $\tau_{\mu}^{*}(\beta) \leq \inf \left\{\beta q-\alpha_{0} q+\tau_{\mu}(0): q \geq 0\right\}=-\infty$ for all $\beta<\alpha_{0}$, which contradicts the fact that $-\infty<\tau_{\mu}^{*}(\alpha)$. Next suppose $\alpha>\tau_{\mu}^{\prime}\left(0^{-}\right)$and $\operatorname{dom}\left(\tau_{\mu}\right)=\mathbb{R}$. The same lines as above also yield a contradiction. If now $\alpha \in\left[-\tau_{\mu}^{\prime}\left(0^{+}\right), \tau_{\mu}^{\prime}\left(0^{-}\right)\right]$and $\operatorname{dom}\left(\tau_{\mu}\right)=\mathbb{R}$, then $\tau_{\mu}^{*}(\alpha)=-\tau_{\mu}(0) \geq 0$; new contradiction. It remains the case $\operatorname{dom}\left(\tau_{\mu}\right)=\mathbb{R}_{+}$and $\alpha \geq \tau_{\mu}^{\prime}\left(0^{+}\right)$. In this case, we necessarily have $\tau_{\mu}^{*}(\alpha)=\lim _{q \rightarrow 0^{+}}-\tau_{\mu}(q) \geq 0$. Finally, $\tau_{\mu}^{*}$ is non-negative on its domain.
5.1.2. Proposition 1.2. - (1) If $\operatorname{dom}(\tau)=\mathbb{R}$, the property $\operatorname{dom}\left(\tau^{*}\right)=\left[\tau^{\prime}(\infty), \tau^{\prime}(-\infty)\right]$ follows from standard considerations in convex analysis. Then, the fact that this interval is bounded from above follows from the boundedness from below of $\tau^{*}$. Also, since $\operatorname{dom}\left(\tau^{*}\right) \subset \mathbb{R}$, the equality $\left(\tau^{*}\right)^{*}=\tau$ on $\operatorname{dom}(\tau)=\mathbb{R}$ is just the usual duality between $\tau$ and its conjugate function ([77, Theorem 12.2, Corollary 12.2.2]) when this one is only defined on $\mathbb{R}$ and not also on $\mathbb{R} \cup\{\infty\}$ as in the convention used in this paper.
(2)(a) Since $\tau(0)=\tau(1)=0$, by concavity of the non decreasing function $\tau$, we have $\tau=0$ over $\mathbb{R}_{+}$, and a simple verification shows that $\tau^{*}=\tau$ over $\mathbb{R} \cup\{\infty\}$.
(2)(b) We suppose that $\tau(0)<0$ and $\tau$ is continuous at $0^{+}$. Here again, standard considerations in convex analysis show that min $\operatorname{dom}\left(\tau^{*}\right)=\tau_{\mu}^{\prime}(\infty)$ and $\left[\tau^{\prime}(\infty), \lim _{q \rightarrow 0^{+}} \tau^{\prime}\left(q^{-}\right)\right] \subset$ $\operatorname{dom}\left(\tau^{*}\right)$, as well as the continuity and the concavity of $\tau^{*}$ over $\left[\tau^{\prime}(\infty), \lim _{q \rightarrow 0^{+}} \tau^{\prime}\left(q^{-}\right)\right]$. If $\lim _{q \rightarrow 0^{+}} \tau^{\prime}\left(q^{-}\right)<\infty$, by using the definition of $\tau^{*}$ one checks that $\tau^{*}(\alpha)=-\tau\left(0^{+}\right)=$ $-\tau(0)=\tau^{*}(\infty)$ for all $\alpha \in\left[\lim _{q \rightarrow 0^{+}} \tau^{\prime}\left(q^{-}\right), \infty\right]$. So $\operatorname{dom}\left(\tau^{*}\right)=\left[\tau^{\prime}(\infty), \infty\right]$. The continuity of $\tau^{*}$ over $\operatorname{dom}\left(\tau^{*}\right)$ comes from the fact that $\tau^{*}(\infty)=-\tau(0)$. The fact that $\left(\tau^{*}\right)^{*}=\tau$ over $\mathbb{R}_{+}$is a direct consequence of the usual duality between $\tau$ and the restriction of $\tau^{*}$ to $\mathbb{R}$, and the fact that $\tau$ is continuous at $0^{+}$. The equality $\left(\tau^{*}\right)^{*}=\tau$ over $\mathbb{R}_{-}^{*}$ is obvious.
(2)(c) If $\tau$ is discontinuous at $0^{+}$, clearly $\tau^{\prime}\left(0^{+}\right)=\infty$. It is standard from convex analysis that $\min \operatorname{dom}\left(\tau^{*}\right)=\tau_{\mu}^{\prime}(\infty)$, and $\left[\tau^{\prime}(\infty), \lim _{q \rightarrow 0^{+}} \tau^{\prime}\left(q^{-}\right)\right] \subset \operatorname{dom}\left(\tau^{*}\right)$. Moreover, if $\lim _{q \rightarrow 0^{+}} \tau^{\prime}\left(q^{-}\right)<\infty$, by using the definition of $\tau^{*}$ one checks that $\tau^{*}(\alpha)=-\tau\left(0^{+}\right)<$ $-\tau(0)=\tau^{*}(\infty)$ for all $\alpha \in\left[\lim _{q \rightarrow 0^{+}} \tau^{\prime}\left(q^{-}\right), \infty\right)$. Consequently, we have $\operatorname{dom}\left(\tau^{*}\right)=$ $\left[\tau^{\prime}(\infty), \infty\right]$, as well as the concavity and continuity of $\tau^{*}$ over $\left[\tau_{\mu}^{\prime}(\infty), \infty\right)$. By using the usual duality between $\tau$ and the restriction of $\tau^{*}$ to $\mathbb{R}$, we would find that $\left(\tau^{*}\right)^{*}$ is equal to $\tau$ over $\mathbb{R}_{+}^{*}$ and equal to $\tau\left(0^{+}\right)$at 0 . Here, taking into account that $\alpha=\infty \in \operatorname{dom}\left(\tau^{*}\right)$, we find that $\left(\tau^{*}\right)^{*}(0)=-\tau^{*}(\infty)=\tau(0)$. Finally, $\left(\tau^{*}\right)^{*}=\tau$ on $\mathbb{R}_{+}$. The equality $\left(\tau^{*}\right)^{*}=\tau$ over $\mathbb{R}_{-}$is obvious.
(2)(d) It has been proved in the previous lines.

### 5.2. Proof of Proposition 1.3

(1) Fix $0 \leq \alpha<\beta \leq \infty$ (the case $\alpha=\beta$ is covered by (1.4) and (1.5)). Without loss of generality we assume that $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)>-\infty$, for otherwise one clearly has $E(\mu, \alpha, \beta)=\varnothing$.

We first show that $\operatorname{dim}_{H} F(\alpha, \beta) \leq \underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)$, where

$$
F(\alpha, \beta)=\{x \in \operatorname{supp}(\mu): \alpha \leq \underline{d}(\mu, x) \leq \bar{d}(\mu, x) \leq \beta\})
$$

Since $E(\mu, \alpha, \beta) \subset F(\alpha, \beta)$, this yields $\operatorname{dim}_{H} E(\mu, \alpha, \beta) \leq \underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)$.
Fix $\eta>0$. There exists $\epsilon>0$ such that for infinitely many $r>0$, we have $f_{\mu}(\alpha, \beta, \epsilon, r) \leq$ $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)+\eta$. Let $\left(r_{j}\right)_{j \geq 1}$ be a sequence converging to 0 such that for all $j$ we have $f_{\mu}\left(\alpha, \beta, \epsilon, r_{j}\right) \leq \underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)+\eta$.

By definition, we have $F(\alpha, \beta) \subset \bigcup_{N \geq 1} F_{N}$, where

$$
F_{N}=\bigcap_{0<r \leq 2^{-N}}\left\{x \in \operatorname{supp}(\mu): r^{\beta+\epsilon} \leq \mu(B(x, r)) \leq r^{\alpha-\epsilon}\right\}
$$

Fix $N \geq 1$. It follows from the previous line that for any $n \geq N$, there exists $j \geq 1$ such that $r_{j} \leq 2^{-n}$ and we have

$$
F_{N} \subset\left\{x \in \operatorname{supp}(\mu): r_{j}^{\beta+\epsilon} \leq \mu\left(B\left(x, r_{j}\right)\right) \leq r_{j}^{\alpha-\epsilon}\right\}
$$

It follows from Besicovitch's covering theorem (see [60]) that there exists an integer $Q(d)$ such that, defining $F_{j}(\epsilon)=\left\{x \in \operatorname{supp}(\mu): r_{j}^{\beta+\epsilon} \leq \mu\left(B\left(x, r_{j}\right)\right) \leq r_{j}^{\alpha-\epsilon}\right\}$, we can extract from $\left\{B\left(x, r_{j}\right): x \in F_{j}(\epsilon)\right\}, Q(d)$ families $\mathscr{F}_{k}(1 \leq k \leq Q(d))$ of disjoint balls such that $F_{j}(\epsilon) \subset \bigcup_{k=1}^{Q(d)} \bigcup_{B \in \mathscr{F}_{k}} B$.

Now, setting $\gamma=\underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)+2 \eta$, and using the covering of $F_{N}$ by the balls in $\bigcup_{k=1}^{Q(d)} \mathscr{F}_{k}$, we see that for $j$ large enough we have

$$
\begin{aligned}
\mathscr{H}_{2^{-n+1}}^{\gamma}\left(F_{N}\right) & \leq \sum_{k=1}^{Q(d)} \sum_{B \in \mathcal{F}_{k}}|B|^{\gamma} \\
& \leq Q(d)\left(\# \mathcal{G}_{k}\right)\left(2 r_{j}\right)^{\gamma} \leq 2^{\gamma} Q(d) r_{j}^{-\left(\underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)+\eta\right)+\gamma} \leq 2^{\gamma} Q(d) 2^{-n \eta}
\end{aligned}
$$

Letting $n$ tend to $\infty$ yields $\mathscr{H}^{\gamma}\left(F_{N}\right)=0$, so $\operatorname{dim} F_{N} \leq \gamma$ for all $N \geq 1$, and finally $\operatorname{dim} F(\alpha, \beta) \leq \underline{f}_{\mu}^{\mathrm{LD}}(\alpha, \beta)+2 \eta$. Since $\eta$ was arbitrary, we are done.

Now let us prove that $\operatorname{dim}_{H} E(\mu, \alpha, \beta) \leq \min \left(\bar{f}_{\mu}^{\mathrm{LD}}(\alpha), \bar{f}_{\mu}^{\mathrm{LD}}(\beta)\right)$. For $\alpha^{\prime} \in \mathbb{R} \cup\{\infty\}$, define $G_{\alpha^{\prime}}=\left\{x: \exists\left(n_{j}\right) \nearrow \infty: \lim _{j \rightarrow \infty} \frac{\log \left(\mu\left(B\left(x, 2^{-n_{j}}\right)\right)\right)}{\log \left(2^{-n_{j}}\right)}=\alpha^{\prime}\right\}$. We have $E(\mu, \alpha, \beta) \subset G_{\alpha} \cap G_{\beta}$. Consequently, the conclusion follows from the fact that $\operatorname{dim} G_{\alpha^{\prime}} \leq \bar{f}_{\mu}^{\mathrm{LD}}\left(\alpha^{\prime}\right)$. Indeed, if $\alpha^{\prime}<\infty$, fix $\eta>0$. Then let $\epsilon>0$ and $r_{0} \in \mathbb{N}_{+}$such that for all $0<r \leq r_{0}$ we have $f\left(\alpha^{\prime}, \epsilon, r\right) \leq \bar{f}_{\mu}^{\mathrm{LD}}\left(\alpha^{\prime}\right)+\eta$, where

$$
f\left(\alpha^{\prime}, \epsilon, r\right)=\frac{\log \sup \#\left\{i: r^{\alpha+\epsilon} \leq \mu\left(B\left(x_{i}, r\right)\right) \leq r^{\alpha-\epsilon}\right\}}{-\log (r)}
$$

the supremum being taken over the packings of $\operatorname{supp}(\mu)$ by balls of radii equal to $r$. For each $n \geq 1$ such that $2^{-n} \leq r_{0}$, we have $G_{\alpha^{\prime}} \subset \bigcup_{p \geq n} G_{\alpha^{\prime}, p}$, where $G_{\alpha^{\prime}, p}=\{x \in \operatorname{supp}(\mu)$ : $2^{-p\left(\alpha^{\prime}+\epsilon\right)} \leq \mu\left(B\left(x, 2^{-p}\right) \leq 2^{-p\left(\alpha^{\prime}+\epsilon\right)}\right\}$. Setting $\gamma=\bar{f}_{\mu}^{\mathrm{LD}}\left(\alpha^{\prime}\right)+2 \eta$ and using Besicovitch's covering theorem as above, we get $\mathscr{H}_{2^{-n+1}}^{\gamma}\left(G_{\alpha^{\prime}}\right) \leq 2^{\gamma} Q(d) \sum_{p \geq n} 2^{p\left(\bar{f}_{\mu}^{\mathrm{LD}}\left(\alpha^{\prime}\right)+\eta\right)-p \gamma}$. Letting $n$ tend to $\infty$ yields $\mathscr{H}^{\gamma}\left(G_{\alpha^{\prime}}\right)=0$, hence $\operatorname{dim}_{H} G_{\alpha^{\prime}} \leq \bar{f}_{\mu}^{\mathrm{LD}}\left(\alpha^{\prime}\right)+2 \eta, \eta>0$ being arbitrary. The case $\alpha^{\prime}=\infty$ can be treated similarly.

Now we prove that $\operatorname{dim}_{P} F(\alpha, \beta) \leq f_{P}(\alpha, \beta)=\sup \left\{\bar{f}_{\mu}^{\mathrm{LD}}\left(\alpha^{\prime}\right): \alpha^{\prime} \in[\alpha, \beta]\right\}$. Suppose first that $\beta<\infty$.

Fix $\eta>0$, and for each $\alpha^{\prime} \in[\alpha, \beta]$ fix $\epsilon\left(\alpha^{\prime}\right)>0$ and $r\left(\alpha^{\prime}\right)>0$ such that for all $0<r<r\left(\alpha^{\prime}\right)$, one has $f\left(\alpha^{\prime}, \epsilon\left(\alpha^{\prime}\right), r\right) \leq \bar{f}_{\mu}^{\mathrm{LD}}\left(\alpha^{\prime}\right)+\eta$. Then let $\alpha_{1}^{\prime}, \ldots, \alpha_{k}^{\prime}$ such that $[\alpha, \beta] \subset \bigcup_{i=1}^{k} B\left(\alpha_{i}^{\prime}, \epsilon\left(\alpha_{i}^{\prime}\right)\right)$.

Set $\epsilon=\min \left\{\epsilon\left(\alpha_{i}^{\prime}\right): 1 \leq i \leq k\right\}$ and $r_{\eta}=\min \left\{r\left(\alpha_{i}^{\prime}\right): 1 \leq i \leq k\right\}$.
Fix $N \geq 1$ and define $F_{N}$ as above. We have

$$
F_{N} \subset \bigcap_{0<2-p<\min \left(2^{-N}, r_{\eta}\right)} \bigcup_{i=1}^{k}\left\{x \in \operatorname{supp}(\mu): 2^{-p\left(\alpha_{i}^{\prime}+\epsilon\left(\alpha_{i}^{\prime}\right)\right)} \leq \mu\left(B\left(x, 2^{-p}\right)\right) \leq 2^{-p\left(\alpha_{i}^{\prime}-\epsilon\left(\alpha_{i}^{\prime}\right)\right)}\right\}
$$

Let $A \subset[0,1]^{d}$. Let $n \in \mathbb{N}$ such that $2^{-n} \leq \min \left(2^{-N}, r_{\eta}\right)$ and $\left\{B\left(x_{i}, r_{i}\right)\right\}$ a $2^{-n}$-packing of $A \cap F_{N}$. For each integer $p \geq n+1$, set $S_{p}=\left\{i: 2^{-p}<r_{i} \leq 2^{-p+1}\right\}$. The balls in $\left\{B\left(x_{i}, 2^{-p}\right): i \in S_{p}, 2^{-p\left(\alpha_{i}^{\prime}+\epsilon\left(\alpha_{i}^{\prime}\right)\right)} \leq \mu\left(B\left(x, 2^{-p}\right)\right) \leq 2^{-p\left(\alpha_{i}^{\prime}-\epsilon\left(\alpha_{i}^{\prime}\right)\right)}\right\}$ form a $2^{-p}$-packing of $\operatorname{supp}(\mu)$ of cardinality less than $2^{p\left(\bar{f}_{\mu}^{\mathrm{LD}}\left(\alpha_{i}^{\prime}\right)+\eta\right)}$, so $\# S_{p} \leq k 2^{p\left(f_{P}(\alpha, \beta)+\eta\right)}$.

Let $\gamma>f_{P}(\alpha, \beta)+2 \eta$. We have

$$
\begin{aligned}
\sum_{i}\left(2 r_{i}\right)^{\gamma} & \leq \sum_{p \geq n} \sum_{i \in S_{p}}\left(2 \cdot 2^{-p+1}\right)^{\gamma} \leq 4^{\gamma} \sum_{p \geq n}\left(\# S_{p}\right) 2^{-p \gamma} \\
& \leq 4^{\gamma} k \sum_{p \geq n} 2^{p(f(\alpha, \beta)+\eta-\gamma)} \leq 4^{\gamma} k \sum_{p \geq n} 2^{-p \eta}
\end{aligned}
$$

the upper bound being independent of the $2^{-n}$-packing $\left\{B\left(x_{i}, r_{i}\right)\right\}$, and going to 0 as $n \rightarrow \infty$. It follows that the pre-packing measure $\bar{P}^{\gamma}\left(A \cap F_{N}\right)$ equals 0 for all $A$, hence $\mathscr{P}^{\gamma}\left(F_{N}\right)=0$, so $\operatorname{dim}_{P}\left(F_{N}\right) \leq \gamma$. Since $\eta$ is arbitrary this yields $\operatorname{dim}_{P} F_{N} \leq f_{P}(\alpha, \beta)$, hence $\operatorname{dim}_{P} F(\alpha, \beta) \leq f_{P}(\alpha, \beta)$, and finally $\operatorname{dim}_{P} E(\mu, \alpha, \beta) \leq f_{P}(\alpha, \beta)$.

If $\beta=\infty$, take $\beta_{1}>0$ and $r(\beta)$ such that for all $0<r \leq r(\infty), f\left(\beta_{1}, \infty, r\right) \leq \bar{f}_{\mu}^{\mathrm{LD}}(\infty)+\eta$, where

$$
f\left(\beta_{1}, \infty, r\right)=\frac{\log \sup \#\left\{i: \mu\left(B\left(x_{i}, r\right)\right) \leq r^{\beta_{1}}\right\}}{-\log (r)}
$$

the supremum being taken over the packings of $\operatorname{supp}(\mu)$ by balls of radii equal to $r$. Then use a covering of $\left[\alpha, \beta_{1}\right]$ by intervals $B\left(\alpha_{i}^{\prime}, \epsilon\left(\alpha_{i}^{\prime}\right)\right)$ as above. The argument to conclude is the same as above, except that we have to bound the cardinality of $\left\{B\left(x_{i}, 2^{-p}\right): i \in S_{p}\right.$, $\left.\mu\left(B\left(x, 2^{-p}\right)\right) \leq 2^{-p \beta_{1}}\right\}$ by $2^{p\left(f_{\mu}^{\mathrm{LD}}(\infty)+\eta\right)}$.
(2) The result for packing dimensions easily follows from the inclusions $\underline{E}(\mu, \alpha) \subset$ $\bigcup_{\beta \geq \alpha} \uparrow F(\alpha, \beta)$ and $\bar{E}(\mu, \alpha) \subset \bigcup_{0 \leq \beta \leq \alpha} \uparrow F(\beta, \alpha)$, and the previous estimates for $\operatorname{dim}_{P} F(\alpha, \beta)$.

The upper bound for $\operatorname{dim}_{H} \underline{E}(\mu, \alpha)$ is obtained by writing $\underline{E}(\mu, \alpha)=\bigcup_{\beta \geq \alpha} \uparrow\{x \in \operatorname{supp}(\mu)$ : $\underline{d}(\mu, x)=\alpha, \bar{d}(\mu, x) \leq \beta\}$. If $\alpha=\infty$, there is nothing to prove, for $\underline{E}(\mu, \infty)=E(\mu, \infty)$. Suppose $\alpha<\infty$. Then, for each $\beta<\infty$, due to the estimates achieved to find an upper bound for $\operatorname{dim}_{H} E(\mu, \alpha, \beta)$, given $\eta>0$, for each $\alpha^{\prime} \in[\alpha, \beta]$ one can fix $\epsilon\left(\alpha^{\prime}\right)>0$ such that $\operatorname{dim}_{H}\left\{x \in \operatorname{supp}(\mu): \underline{d}(\mu, x)=\alpha, \alpha^{\prime}-\epsilon\left(\alpha^{\prime}\right) \leq \bar{d}(\mu, x) \leq \alpha^{\prime}+\epsilon\left(\alpha^{\prime}\right)\right\} \leq$ $\min \left\{f_{\mu}^{\mathrm{LD}}(\alpha), \bar{f}_{\mu}^{\mathrm{LD}}\left(\alpha^{\prime}\right), \bar{f}_{\mu}^{\mathrm{LD}}\left(\alpha, \alpha^{\prime}\right)\right\}+\eta=f_{H}\left(\alpha, \alpha^{\prime}\right)+\eta$. Since we can cover $[\alpha, \beta]$ by finitely many intervals of the form $\left[\alpha^{\prime}-\epsilon\left(\alpha^{\prime}\right), \alpha^{\prime}+\epsilon\left(\alpha^{\prime}\right)\right], \alpha^{\prime} \in[\alpha, \beta]$, we get $\operatorname{dim}_{H}\{x \in \operatorname{supp}(\mu):$ $\underline{d}(\mu, x)=\alpha, \bar{d}(\mu, x) \leq \beta\} \leq \sup \left\{f_{H}\left(\alpha, \alpha^{\prime}\right): \alpha^{\prime} \in[\alpha, \beta]\right\}+\eta$ for any $\eta>0$, hence $\operatorname{dim}_{H}\{x \in \operatorname{supp}(\mu): \underline{d}(\mu, x)=\alpha, \bar{d}(\mu, x) \leq \beta\} \leq \sup \left\{f_{H}\left(\alpha, \alpha^{\prime}\right): \alpha^{\prime} \in[\alpha, \beta]\right\}$. Since we also know that $\operatorname{dim}_{H} E(\mu, \alpha, \infty) \leq f_{H}(\alpha, \infty)$, writing $\underline{E}(\mu, \alpha)=E(\mu, \alpha, \infty) \bigcup \bigcup_{\infty>\beta \geq \alpha} \uparrow$ $\{x \in \operatorname{supp}(\mu): \underline{d}(\mu, x)=\alpha, \bar{d}(\mu, x) \leq \beta\}$ and using the previous estimates for the Hausdorff dimensions yields $\operatorname{dim}_{H} \underline{E}(\mu, \alpha) \leq \sup \left\{f_{H}(\alpha, \beta): \beta \geq \alpha\right\}$.

The upper bound for $\operatorname{dim}_{H} \bar{E}(\mu, \alpha)$ is obtained by using similar arguments.

### 5.3. Proof of some inequalities in (1.4) and (1.5)

We justify the inequalities $f_{\mu}^{H}(\alpha) \leq \underline{f}_{\mu}^{\mathrm{LD}}(\alpha) \leq \bar{\tau}_{\mu}^{*}(\alpha)$ and $\operatorname{dim}_{P} E(\mu, \alpha) \leq \bar{f}_{\mu}^{\mathrm{LD}}(\alpha)$ for $\alpha \in \mathbb{R}_{+} \cup\{\infty\}$, and $\bar{f}_{\mu}^{\mathrm{LD}}(\infty) \leq \tau_{\mu}^{*}(\infty)$.

Let $\alpha \in \mathbb{R}_{+} \cup\{\infty\}$. Suppose that $E(\mu, \alpha) \neq \varnothing$. The inequality $f_{\mu}^{H}(\alpha) \leq \underline{f}_{\mu}^{\mathrm{LD}}(\alpha)$ is a special case of the upper bound established for $\operatorname{dim}_{H} F(\alpha, \beta)$ in Section 5.2.

Similarly, the inequality $\operatorname{dim}_{P} E(\mu, \alpha) \leq \bar{f}_{\mu}^{\mathrm{LD}}(\alpha)$ follows from lines similar to those used to bound $\operatorname{dim}_{P} E(\mu, \alpha, \beta)$ in Section 5.2. Also, by Proposition 1.1(2), if $\operatorname{dom}\left(\tau_{\mu}\right)=\mathbb{R}$, then
one has $\bar{f}_{\mu}^{\mathrm{LD}}(\infty)=\tau_{\mu}^{*}(\infty)=-\infty$; otherwise $\overline{\operatorname{dim}}_{B}(\operatorname{supp}(\mu))=-\tau_{\mu}(0)=\tau_{\mu}^{*}(\infty)$, and by definition we have $\bar{f}_{\mu}^{\mathrm{LD}}(\infty) \leq \overline{\operatorname{dim}}_{B}(\operatorname{supp}(\mu))$. In any case, $\operatorname{dim}_{P} E(\mu, \infty) \leq \bar{f}_{\mu}^{\mathrm{LD}}(\infty) \leq$ $\tau_{\mu}^{*}(\infty)$.

To prove that $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha) \leq \bar{\tau}_{\mu}^{*}(\alpha)$, we assume without loss of generality that $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha)>-\infty$, hence $\underline{f}_{\mu}^{\mathrm{LD}}(\alpha) \geq 0$. The case $\alpha=\infty$ then occurs only if $\bar{\tau}_{\mu}(q)=-\infty$ if $q<0$ (same proof as when $\tau_{\mu}(q)=-\infty$ for $q<0$ ); then it is direct that ${\underset{\mu}{f}}_{\mathrm{LD}}^{\mathrm{LD}}(\infty) \leq-\bar{\tau}_{\mu}(0)=\bar{\tau}_{\mu}^{*}(\infty)$. Suppose now that $\alpha<\infty$. It is enough to prove that $\bar{\tau}_{\mu}(q) \leq\left(\underline{f}_{\mu}^{\mathrm{LD}}\right)^{*}(q)$ for all $q \in \mathbb{R}$. Then the result follows by taking the Legendre-transform and using (with $h=f_{\mu}^{\mathrm{LD}}$ ) the general inequality $\left(h^{*}\right)^{*} \geq h$ valid for any function whose domain is not empty. Fix $q \in \mathbb{R}, \beta \in \mathbb{R}$ and $\epsilon>0$. If $\left\{B\left(x_{i}, r\right)\right\}$ is a packing of $\operatorname{supp}(\mu)$ by disjoint balls, we have

$$
\sum_{i} \mu\left(B\left(x_{i}, r\right)\right)^{q} \geq\left(\#\left\{i: r^{\beta+\epsilon} \leq \mu\left(B\left(x_{i}, r\right)\right) \leq r^{\beta-\epsilon}\right\}\right) \cdot \begin{cases}r^{q(\beta-\epsilon)} & \text { if } q \geq 0 \\ r^{q(\beta+\epsilon)} & \text { otherwise }\end{cases}
$$

Taking the supremum over the packings, dividing by $\log (r)$, and taking the limsup as $r \rightarrow 0^{+}$yields $\bar{\tau}_{\mu}(q) \leq q(\beta \mp \epsilon)-\lim \inf _{r \rightarrow 0^{+}} f(\beta, \epsilon, r)$, and taking the limit as $\epsilon \rightarrow 0^{+}$ gives $\bar{\tau}_{\mu}(q) \leq q \beta-\underline{f}_{\mu}^{\mathrm{LD}}(\beta)$ for all $\beta$, hence $\bar{\tau}_{\mu}(q) \leq\left(\underline{f}_{\mu}^{\mathrm{LD}}\right)^{*}(q)$.

## 6. Dimensions of measures and mass distribution principle

Given $\nu \in \mathcal{M}_{c}^{+}\left(\mathbb{R}^{d}\right)$, if $\underline{d}(\nu, x)($ resp. $\bar{d}(\nu, x))$ takes the same value $\underline{D}$ (resp. $\bar{D}$ ) at $\nu$-almost every $x$, then $\operatorname{dim}_{H} \nu\left(\right.$ resp. $\left.\operatorname{dim}_{P} \nu\right)$ stands for the Hausdorff (resp. packing) dimension of the measure $\nu$, defined as the number $\underline{D}$ (resp. $\bar{D}$ ). Then, $\nu(E)>0$ implies $\operatorname{dim}_{H} E \geq \underline{D}$ (resp. $\operatorname{dim}_{P} E \geq \bar{D}$ ) for any Borel set $E$. This is the mass distribution principle we use to get lower bounds for the Hausdorff and packing dimensions of the level sets studied in this paper (about mass distribution principle and dimensions of measures, see the accounts proposed in [33, Section 4.2], [70, Ch. 2] and [44] (other possible references being [20, Section 14], [82], [26, 27] and in connection with multifractal formalism [43], [62] and [65]).

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