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SUBANALYTIC SHEAVES AND SOBOLEV SPACES

S. GUILLERMOU, G. LEBEAU, A. PARUSIŃSKI,
P. SCHAPIRA & J.-P. SCHNEIDERS

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Stéphane Guillermou
Institut Fourier, Université de Grenoble I, BP 74, F-38402 Saint-Martin d'Hères, France
`Stephane.Guillermou@ujf-grenoble.fr`

Gilles Lebeau
Département de Mathématiques, Université de Nice Sophia-Antipolis, Parc Valrose, F-06108 Nice Cedex 02, France.
`lebeau@unice.fr`

Adam Parusiński
Univ. Nice Sophia Antipolis, CNRS, LJAD, UMR 7351, F-06108 Nice, France
`adam.parusinski@unice.fr`

Pierre Schapira
Sorbonne Universités, UPMC Univ Paris 6 Institut de Mathématiques de Jussieu, F-75005 Paris France
`pierre.schapira@imj-prg.fr`

Jean-Pierre Schneiders
Institut de Mathématiques, Université de Liège, Belgique
`jpschneiders@ulg.ac.be`

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SUBANALYTIC SHEAVES AND SOBOLEV SPACES

Stéphane GUILLERMOU, Gilles LEBEAU, Adam PARUSIŃSKI,
Pierre SCHAPIRA and Jean-Pierre SCHNEIDERS

Abstract. — Sheaves on manifolds are perfectly suited to treat local problems, but many spaces one naturally encounters, especially in Analysis, are not of local nature. The subanalytic topology (in the sense of Grothendieck) on real analytic manifolds allows one to partially overcome this difficulty and to define for example sheaves of functions or distributions with temperate growth, but not to make the growth precise.

In this volume, one introduces the linear subanalytic topology, a refinement of the preceding one, and constructs various objects of the derived category of sheaves on the subanalytic site with the help of the Brown representability theorem.

In particular one constructs the Sobolev sheaves. These objects have the nice property that the complexes of their sections on open subsets with Lipschitz boundaries are concentrated in degree zero and coincide with the classical Sobolev spaces.

Another application of this topology is that it allows one to *functorially* endow regular holonomic D-modules with filtrations (in the derived sense).

In the course of the text, one also obtains some results on subanalytic geometry and one makes a detailed study of the derived category of filtered objects in symmetric monoidal categories.

Résumé **(Faisceaux sous-analytiques et espaces de Sobolev).** — Les faisceaux sur les variétés sont parfaitement adaptés à l'étude des problèmes locaux, mais de nombreux espaces que l'on rencontre naturellement, en particulier en Analyse, ne sont pas de nature locale. L'utilisation de la topologie sous-analytique (au sens de Grothendieck) sur les variétés analytiques réelles permet de surmonter partiellement cette difficulté et de définir par exemple des faisceaux de fonctions ou distributions à croissance tempérée, mais pas de préciser cette croissance.

Dans ce volume, on introduit la topologie sous-analytique linéaire, un raffinement de la précédente et l'on construit divers objets de la catégorie dérivée des faisceaux sur le site sous-analytique à l'aide du théorème de representabilité de Brown.

On construit en particulier les faisceaux de Sobolev. Ces objets ont la bonne propriété que les complexes de leurs sections sur les ouverts à frontière Lipschitz sont concentrés en degré zéro et coïncident avec les espaces de Sobolev classiques.

Une autre application de cette topologie est qu'elle permet de munir *fonctorielle-ment* les D-modules holonomes réguliers de filtrations (au sens dérivé).

Dans le cours du texte, on obtient aussi des résultats de géométrie analytique réelle et l'on fait une étude détaillée de la catégorie dérivée des objets filtrés dans les catégories monoidales symétriques.

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RÉSUMÉS DES ARTICLES

Sur une variété analytique réelle M nous construisons la topologie de Grothendieck linéaire M_{sal} et le morphisme naturel de sites ρ de M_{sa} vers M_{sal}, où M_{sa} est le site sous-analytique usuel. Notre premier résultat est que le foncteur dérivé de l'image directe par ρ admet un adjoint à droite, ce qui nous permet d'associer fonctoriellement un faisceau (au sens dérivé) sur M_{sa} à un préfaisceau sur M_{sa} satisfaisant certaines propriétés, ce faisceau ayant les mêmes sections que le préfaisceau sur tout ouvert à bord Lipschitz. Nous appliquons cette construction à divers préfaisceaux sur des variétés réelles, tels que le préfaisceau des fonctions à croissance tempérée d'un ordre donné le long du bord ou à croissance Gevrey le long du bord. (Dans un article séparé, Gilles Lebeau utilisera ces techniques pour construire les faisceaux de Sobolev.) Sur une variété complexe munie de la topologie sous-analytique, les complexes de Dolbeault associés à ces nouveaux faisceaux nous permettent d'obtenir divers faisceaux de fonctions holomorphes à croissance. Comme application, nous pouvons munir fonctoriellement les $\mathscr{D}$-modules holonomes réguliers d'une filtration, au sens dérivé.

Soit M une variété analytique réelle. Le site sous analytique M_{sa} est constitué des ouverts sous analytiques relativement compacts de M, les recouvrements étant finis à extraction près. Pour $s \in \mathbb{R}$, soit $H^s_{\mathrm{loc}}(M)$ l'espace de Sobolev usuel sur M. Pour tout $s \in \mathbb{R}, s \leq 0$ nous construisons un objet $\mathscr{H}^s$ de la catégorie dérivée $\mathsf{D}^+(\mathbf{C}_{M_{\mathrm{sa}}})$ des faisceau sur M_{sa}, qui vérifie la propriété suivante : pour tout ouvert $U \in M_{\mathrm{sa}}$ à frontière lipschitzienne, $\mathscr{H}^s(U) := \mathrm{R}\Gamma(U; \mathscr{H}^s)$ est concentré en degré 0 et coïncide avec l'espace de Sobolev usuel $H^s(U)$. Cette construction utilise les résultats de S. Guillermou et P. Schapira contenus dans ce volume.

Dans le cas où M est de dimension 2, nous explicitons le complexe $\mathscr{H}^s(U)$. Nous démontrons qu'il est toujours concentré en degré 0, mais ne s'identifie pas toujours à un sous espace de distributions sur U.

Soit U un ouvert sous-analytique relativement compact d'une vraiété analytique réelle M. Nous montrons qu'il existe un « recouvrement linéaire fini » (au sens de Guillermou-Schapira) de U par des ouverts sous-analytiques homéomorphes à une boule ouverte.

Nous montrons aussi que la fonction caractéristique de U peut s'écrire comme une combinaison linéaire finie de fonctions caractéristiques d'ouverts sous-analytiques relativement compacts de M homéomorphes, par des applications sous-analytiques et bi-lipschitz, à une boule ouverte.

Pour une catégorie abélienne $\mathscr{C}$ et un ensemble préordonné filtrant Λ, nous prouvons que la catégorie dérivée des objets filtrés de $\mathscr{C}$ indexés par Λ est équivalente à la catégorie dérivée de la catégorie abélienne des foncteurs de Λ dans $\mathscr{C}$. Nous appliquons ce résultat à l'étude de la catégorie des modules filtrés sur un anneau filtré d'une catégorie tensorielle.

ABSTRACTS

Construction of sheaves on the subanalytic site

On a real analytic manifold M, we construct the linear subanalytic Grothendieck topology M_{sal} together with the natural morphism of sites ρ from M_{sa} to M_{sal}, where M_{sa} is the usual subanalytic site. Our first result is that the derived direct image functor by ρ admits a right adjoint, allowing us to associate functorially a sheaf (in the derived sense) on M_{sa} to a presheaf on M_{sa} satisfying suitable properties, this sheaf having the same sections that the presheaf on any open set with Lipschitz boundary. We apply this construction to various presheaves on real manifolds, such as the presheaves of functions with temperate growth of a given order at the boundary or with Gevrey growth at the boundary. (In a separated paper, Gilles Lebeau will use these techniques to construct the Sobolev sheaves.) On a complex manifold endowed with the subanalytic topology, the Dolbeault complexes associated with these new sheaves allow us to obtain various sheaves of holomorphic functions with growth. As an application, we can endow functorially regular holonomic $\mathscr{D}$-modules with a filtration, in the derived sense.

Sobolev spaces and Sobolev sheaves

Sobolev spaces $H^s_{\mathrm{loc}}(M)$ on a real manifold M are classical objects of Analysis. In this paper, we assume that M is real analytic and denote by M_{sa} the associated subanalytic site, for which the open sets are the relatively compact subanalytic subsets and the coverings are, roughly speaking, the finite coverings. For $s \in \mathbb{R}, s \leq 0$, we construct an object $\mathscr{H}^s$ of the derived category $\mathsf{D}^+(\mathbf{C}_{M_{\mathrm{sa}}})$ of sheaves on M_{sa} with the property that if U is open in M_{sa} and has a Lipschitz boundary, then the object $\mathscr{H}^s(U) := \mathrm{R}\Gamma(U; \mathscr{H}^s)$ is concentrated in degree 0 and coincides with the classical Sobolev space $H^s(U)$. This construction is based on the results of S. Guillermou and P. Schapira in this volume.

Moreover, in the special case where the manifold M is of dimension 2, we will compute explicitly the complex $\mathscr{H}^s(U)$ and prove that it is always concentrated in degree 0, but is not necessarily a subspace of the space of distributions on U.

Let U be an open relatively compact subanalytic subset of a real analytic manifold M. We show that there exists a "finite linear covering" (in the sense of Guillermou-Schapira) of U by subanalytic open subsets of U homeomorphic to an open ball.

We also show that the characteristic function of U can be written as a finite linear combination of characteristic functions of open relatively compact subanalytic subsets of M homeomorphic, by subanalytic and bi-lipschitz maps, to an open ball.

For an abelian category $\mathscr{C}$ and a filtrant preordered set Λ, we prove that the derived category of the quasi-abelian category of filtered objects in $\mathscr{C}$ indexed by Λ is equivalent to the derived category of the abelian category of functors from Λ to $\mathscr{C}$. We apply this result to the study of the category of filtered modules over a filtered ring in a tensor category.

INTRODUCTION

(1) Sheaves on manifolds are perfectly suited to treat local problems, but many spaces one naturally encounters, especially in Analysis, are not of local nature. As noticed in [**5**], it is sometimes possible to overcome this difficulty by enlarging the category of sheaves to that of ind-sheaves, or to a special class of ind-sheaves which is sufficient for many applications, the subanalytic sheaves, and by extending to ind-sheaves or to subanalytic sheaves the machinery of sheaves (the six Grothendieck operations).

The idea of the subanalytic topology is extremely natural: the usual topology contains too many open sets, most of them being pathological (from the point of view of algebraic topology) and it is natural to look for families of reasonable open subsets. The family of open subanalytic subsets of a real analytic manifold M provides an excellent candidate. One then defines the subanalytic presite associated with M as the category $\mathrm{Op}_{M_{\mathrm{sa}}}$ of relatively compact subanalytic open subsets, the morphisms being the inclusions. Moreover, the usual coverings on M are too fine: if one wants for example to treat "properties at the boundary" such as functions with polynomial growth at the boundary or temperate distributions, it is natural to ask that if $\{U_i\}_{i\in I}$ is an open covering of U, then the boundary ∂U is contained in the union of the boundaries ∂U_i and this union is locally finite. The subanalytic site, denoted M_{sa}, is the presite $\mathrm{Op}_{M_{\mathrm{sa}}}$ endowed with the Grothendieck topology for which the coverings are, roughly speaking, the finite coverings.

Sheaves on M_{sa} have been intensely studied in [**5**], as particular cases of ind-sheaves, and allow one to define new sheaves which would not exist in the usual topology. For example, one has the sheaf $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$ of $\mathscr{C}^{\infty}$-functions with temperate growth, the sheaf $\mathscr{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ of temperate distributions or the sheaf $\mathscr{C}^{\infty,\mathrm{w}}_{M_{\mathrm{sa}}}$ of Whitney functions. On a complex manifold X, using the Dolbeault complexes of the first two ones, one obtains the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ (in the derived sense) of holomorphic functions with temperate growth. This last sheaf is implicitly used in the solution of the Riemann-Hilbert problem by Kashiwara [**3, 4**] and is also extremely important in the study of irregular holonomic $\mathscr{D}$-modules (see [**6**, § 7]).

However, the subanalytic topology is still too rough to treat more sophisticated spaces of analysis, such as the $\mathscr{C}^{\infty}$-functions of a given growth, the Gevrey functions of a given order, or, more important, the Sobolev spaces.

The aim of this volume is two-folds: to give the mathematical backgrounds to construct these new sheaves and then to apply this theory in order to obtain the Sobolev sheaves.

(2) The first task is performed in the paper "Construction of sheaves on the subanalytic site" by Stéphane Guillermou and Pierre Schapira (see **[1]**). In their paper, the authors modify the subanalytic topology by introducing what they call the linear subanalytic topology, denoted M_{sal}, as well as the natural morphism of sites $\rho_{\mathrm{sal}}\colon M_{\mathrm{sa}} \to M_{\mathrm{sal}}$. The presite underlying the site M_{sal} is the same as for M_{sa}, namely $\mathrm{Op}_{M_{\mathrm{sa}}}$, but the coverings are the linear coverings. Roughly speaking, a finite family $\{U_i\}_{i\in I}$ is a linear covering of their union U if there is a constant C such that the distance of any $x \in M$ to $M \setminus U$ is bounded by C-times the maximum of the distance of x to $M \setminus U_i$ $(i \in I)$.

Let $\mathbf{k}$ be a field. One easily shows that a presheaf F of $\mathbf{k}$-modules on M_{sal} is a sheaf as soon as, for any open sets U_1 and U_2 such that $\{U_1, U_2\}$ is a linear covering of $U_1 \cup U_2$, the Mayer-Vietoris sequence

$$0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2) \tag{0.1}$$

is exact. Moreover, if for any such a covering, the sequence

$$0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2) \to 0 \tag{0.2}$$

is exact, then the sheaf F is Γ-acyclic, that is, $\mathrm{R}\Gamma(U;F)$ is concentrated in degree 0 for all $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. Then the authors prove the two results below:

(A) the functor $\mathrm{R}\rho_{\mathrm{sal}*}\colon \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sa}}}) \to \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sal}}})$ admits a right adjoint $\rho^!_{\mathrm{sal}}$,

(B) if U has a Lipschitz boundary, then the object $\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_U$ is concentrated in degree 0.

The proof of (A) is based on the Brown representability theorem and the proof of (B) makes an essential use of the paper **[8]**.

It follows that to a presheaf F on M_{sa} satisfying a natural condition (namely the exactness of 0.2), one can now associate an object of the derived category of sheaves on M_{sa} which has the same sections as F on any Lipschitz open set. The authors give applications of their theory by defining and studying various sheaves: the sheaf $\mathscr{C}^{\infty,s}_{M_{\mathrm{sa}}}$ of $\mathscr{C}^\infty$-functions with polynomial growth of order $s \geq 0$ at the boundary and the sheaves $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sa}}}$ and $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sa}}}$ of $\mathscr{C}^\infty$-functions with Gevrey growth (*i.e.*, exponential growth) of type $s > 1$ at the boundary. By using a refined cut-off lemma (which follows from a refined partition of unity due to Hörmander **[2]**), they prove that these sheaves are Γ-acyclic. Then, on a complex manifold X, by considering the Dolbeault complexes of the sheaves of $\mathscr{C}^\infty$-functions considered above, they obtain new sheaves, namely the sheaf $\mathscr{O}^s_{X_{\mathrm{sa}}}$ of holomorphic functions with growth of order $s \geq 0$ and the sheaves $\mathscr{O}^{\mathrm{gev}(s)}_{X_{\mathrm{sa}}}$ and $\mathscr{O}^{\mathrm{gev}\{s\}}_{X_{\mathrm{sa}}}$ of holomorphic functions with Gevrey growth of type $s > 1$.

The sheaves $\mathscr{C}^{\infty,s}_{M_{\mathrm{sal}}}$ $(s \geq 0)$ on the linear subanalytic site define the filtered sheaf $\mathrm{F}_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$. Then, on a complex manifold X, by considering the Dolbeault complex of this filtered sheaf, they obtain the L^∞-filtration $\mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ on the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$. The Riemann-Hilbert problem, as solved in [**4**], is equivalent to saying that given a regular holonomic $\mathscr{D}_X$-module $\mathscr{M}$ and denoting by G the perverse sheaf of its holomorphic solution, the natural morphism $\mathscr{M} \to \rho_{\mathrm{sa}}^{-1}(G, \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})$ is an isomorphism. Replacing the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ with its filtered version $\mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$, one gets that any regular holonomic $\mathscr{D}_X$-module $\mathscr{M}$ can be *functorially* endowed with a filtration $\mathrm{F}_\infty\mathscr{M}$, in the derived sense on the manifold X.

Note that the category of filtered modules over a filtered ring is quasi-abelian in the sense of Schneiders [**10**] and one needs some tools to manipulate the derived categories of such modules on the sites M_{sa} or M_{sal}. This is achieved in [**9**].

(3) Using the tools elaborated in [**1**], the Sobolev sheaves on M_{sa} are constructed by Gilles Lebeau in the paper "Sobolev spaces and Sobolev sheaves" ([**7**]). More precisely, the author constructs for $s \leq 0$ a sheaf Y^s on M_{sal} such that the sequence (0.2) is exact as soon as $\{U_1, U_2\}$ is a linear covering of $U_1 \cup U_2$. Applying the functor $\rho^!_{\mathrm{sal}}$, he obtains the sheaf (in the derived sense) $\mathscr{H}^s_{M_{\mathrm{sa}}}$, an object of $\mathrm{D}^+(\mathbf{C}_{M_{\mathrm{sa}}})$. This sheaf has the property that if $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ has a Lipschitz boundary, then $\mathrm{R}\Gamma(U, \mathscr{H}_{M_{\mathrm{sa}}})$ is concentrated in degree 0 and coincides with the classical Sobolev space $H^s(U)$. One shall be aware that the definition of the Sobolev space $Y^s(U)$ may differ from the classical one when U is not Lipschitz. Finally, the author gives a detailed study of these sheaves in dimension 2.

The fact that Sobolev sheaves are objects of derived categories and are not concentrated in degree 0 shows that when dealing with spaces of functions or distributions defined on open subsets which are not regular (more precisely, which have not a Lipschitz boundary), it is natural to replace the notion of a space by that of a complex of spaces.

(4) In his paper "Regular subanalytic covers" (see [**8**]), Adam Parusinski proves that any open relatively compact subanalytic subset of a real analytic manifold M admits a finite covering for the linear subanalytic topology by open subsets which are topologically trivial. This is essential for proving that the open sets with Lipschitz boundary are acyclic for the functor $\rho_{\mathrm{sal}*}$ in [**1**]. He also obtains a decomposition of the characteristic functions of subanalytic open sets.

(5) The paper "Derived category of filtered objects" by Pierre Schapira and Jean-Pierre Schneiders (see [**9**]) is a complement to the classical paper [**10**] on quasi-abelian categories. It develops the necessary tools used in the construction of the object $\mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ of the derived category of filtered $\mathscr{D}$-modules on a complex manifold X_{sa} and of the functorial filtration on regular holonomic $\mathscr{D}$-modules on X.

These authors prove that for an abelian category $\mathscr{C}$ and a filtrant preordered set Λ, the derived category of the quasi-abelian category of filtered objects in $\mathscr{C}$ indexed by Λ is equivalent to the derived category of the abelian category of functors from Λ

to $\mathscr{C}$. They apply this result to the study of the category of filtered modules over a filtered ring in a tensor category.

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CONSTRUCTION OF SHEAVES ON THE SUBANALYTIC SITE

by

Stéphane Guillermou & Pierre Schapira

Abstract. — On a real analytic manifold M, we construct the linear subanalytic Grothendieck topology M_{sal} together with the natural morphism of sites ρ from M_{sa} to M_{sal}, where M_{sa} is the usual subanalytic site. Our first result is that the derived direct image functor by ρ admits a right adjoint, allowing us to associate functorially a sheaf (in the derived sense) on M_{sa} to a presheaf on M_{sa} satisfying suitable properties, this sheaf having the same sections that the presheaf on any open set with Lipschitz boundary. We apply this construction to various presheaves on real manifolds, such as the presheaves of functions with temperate growth of a given order at the boundary or with Gevrey growth at the boundary. (In a separated paper, Gilles Lebeau will use these techniques to construct the Sobolev sheaves.) On a complex manifold endowed with the subanalytic topology, the Dolbeault complexes associated with these new sheaves allow us to obtain various sheaves of holomorphic functions with growth. As an application, we can endow functorially regular holonomic $\mathscr{D}$-modules with a filtration, in the derived sense.

***Résumé* (Construction de faisceaux sur le site sous-analytique).** — Sur une variété analytique réelle M nous construisons la topologie de Grothendieck linéaire M_{sal} et le morphisme naturel de sites ρ de M_{sa} vers M_{sal}, où M_{sa} est le site sous-analytique usuel. Notre premier résultat est que le foncteur dérivé de l'image directe par ρ admet un adjoint à droite, ce qui nous permet d'associer fonctoriellement un faisceau (au sens dérivé) sur M_{sa} à un préfaisceau sur M_{sa} satisfaisant certaines propriétés, ce faisceau ayant les mêmes sections que le préfaisceau sur tout ouvert à bord Lipschitz. Nous appliquons cette construction à divers préfaisceaux sur des variétés réelles, tels que le préfaisceau des fonctions à croissance tempérée d'un ordre donné le long du bord ou à croissance Gevrey le long du bord. (Dans un article séparé, Gilles Lebeau utilisera ces techniques pour construire les faisceaux de Sobolev.) Sur une variété complexe munie de la topologie sous-analytique, les complexes de Dolbeault associés à ces nouveaux faisceaux nous permettent d'obtenir divers faisceaux de fonctions holomorphes à croissance. Comme application, nous pouvons munir fonctoriellement les $\mathscr{D}$-modules holonomes réguliers d'une filtration, au sens dérivé.

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Introduction

Let M be a real analytic manifold. The Grothendieck subanalytic topology on M, denoted M_{sa}, and the morphism of sites $\rho_{\mathrm{sa}}\colon M \to M_{\mathrm{sa}}$, were introduced in [**13**]. Recall that the objects of the site M_{sa} are the relatively compact subanalytic open subsets of M and the coverings are, roughly speaking, the finite coverings. In loc. cit. the authors use this topology to construct new sheaves which would have no meaning on the usual topology, such as the sheaf $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$ of $\mathscr{C}^{\infty}$-functions with temperate growth and the sheaf $\mathscr{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ of temperate distributions. On a complex manifold X, using the Dolbeault complexes, they constructed the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ (in the derived sense) of holomorphic functions with temperate growth. The last sheaf is implicitly used in the solution of the Riemann-Hilbert problem by Kashiwara [**8, 9**] and is also extremely important in the study of irregular holonomic $\mathscr{D}$-modules (see [**14**, § 7]).

In this paper, we shall modify the preceding construction in order to obtain sheaves of $\mathscr{C}^{\infty}$-functions with a given growth at the boundary. For example, functions whose growth at the boundary is bounded by a given power of the distance (temperate growth of order $s \geq 0$), or by an exponential of a given power of the distance (Gevrey growth of order $s > 1$), as well as their holomorphic counterparts. For that purpose, we have to refine the subanalytic topology and we introduce what we call the linear subanalytic topology, denoted M_{sal}.

Let us describe the contents of this paper with some details.

In Chapter 1 we construct the linear subanalytic topology on M. Denoting by $\mathrm{Op}_{M_{\mathrm{sa}}}$ the category of open relatively compact subanalytic subsets of M, the presite underlying the site M_{sal} is the same as for M_{sa}, namely $\mathrm{Op}_{M_{\mathrm{sa}}}$, but the coverings are the linear coverings. Roughly speaking, a finite family $\{U_i\}_{i\in I}$ is a linear covering of their union U if there is a constant C such that the distance of any $x \in M$ to $M \setminus U$ is bounded by C-times the maximum of the distance of x to $M \setminus U_i$ $(i \in I)$. (See Definition 1.1.) In this chapter, we also prove some technical results on linear coverings that we shall need in the course of the paper.

Chapter 2. Let $\mathbf{k}$ be a field. One easily shows that a presheaf F of $\mathbf{k}$-modules on M_{sal} is a sheaf as soon as, for any open sets U_1 and U_2 such that $\{U_1, U_2\}$ is a linear covering of $U_1 \cup U_2$, the Mayer-Vietoris sequence

$$0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2) \tag{0.1}$$

is exact. Moreover, if for any such a covering, the sequence

$$0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2) \to 0 \tag{0.2}$$

is exact, then the sheaf F is Γ-acyclic, that is, $\mathrm{R}\Gamma(U; F)$ is concentrated in degree 0 for all $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$.

There is a natural morphism of sites $\rho_{\mathrm{sal}}\colon M_{\mathrm{sa}} \to M_{\mathrm{sal}}$ and we shall prove the two results below (see Theorems 2.30 and 2.49):

(1) the functor $\mathrm{R}\rho_{\mathrm{sal}*}\colon \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sa}}}) \to \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sal}}})$ admits a right adjoint $\rho^!_{\mathrm{sal}}$,

(2) if U has a Lipschitz boundary, then the object $\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_U$ is concentrated in degree 0.

Therefore, if a presheaf F on M_{sa} has the property that the Mayer-Vietoris sequences (0.2) are exact, it follows that $\mathrm{R}\Gamma(U;\rho_{\mathrm{sal}}^{!}F)$ is concentrated in degree 0 and is isomorphic to $F(U)$ for any U with Lipschitz boundary. In other words, to a presheaf on M_{sa} satisfying a natural condition, we are able to associate an object of the derived category of sheaves on M_{sa} which has the same sections as F on any Lipschitz open set. This construction is in particular used by Gilles Lebeau [**21**] who obtains for $s \leq 0$ the "Sobolev sheaves $\mathscr{H}^s_{M_{\mathrm{sa}}}$," objects of $\mathrm{D}^+(\mathbb{C}_{M_{\mathrm{sa}}})$ with the property that if $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ has a Lipschitz boundary, then $\mathrm{R}\Gamma(U;\mathscr{H}^s_{M_{\mathrm{sa}}})$ is concentrated in degree 0 and coincides with the classical Sobolev space $H^s(U)$.

The fact that Sobolev sheaves are objects of derived categories and are not concentrated in degree 0 shows that when dealing with spaces of functions or distributions defined on open subsets which are not regular (more precisely, which have not a Lipschitz boundary), it is natural to replace the notion of a space by that of a complex of spaces.

In Chapter 3, we briefly study the natural operations on the linear subanalytic sites. The main difficulty is that a morphism $f\colon M \to N$ of real analytic manifolds does not induce a morphism of the linear subanalytic sites. This forces us to treat separately the direct or inverse images of sheaves for closed embeddings and for submersive maps.

In Chapter 4 we construct some sheaves on M_{sal}. We construct the sheaf $\mathscr{C}^{\infty,s}_{M_{\mathrm{sal}}}$ of $\mathscr{C}^\infty$-functions with growth of order $s \geq 0$ at the boundary and the sheaves $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}}$ and $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}$ of $\mathscr{C}^\infty$-functions with Gevrey growth of type $s > 1$ at the boundary. By using a refined cut-off lemma (which follows from a refined partition of unity due to Hörmander [**6**]), we prove that these sheaves are Γ-acyclic. Applying the functor $\rho_{\mathrm{sal}}^{!}$, we get new sheaves (in the derived sense) on M_{sa} whose sections on open sets with Lipschitz boundaries are concentrated in degree 0. Then, on a complex manifold X, by considering the Dolbeault complexes of the sheaves of $\mathscr{C}^\infty$-functions considered above, we obtain new sheaves of holomorphic functions with various growth.

As already mentioned, Sobolev sheaves are treated in a separate paper by G. Lebeau in [**21**].

Finally, in Chapter 5, we apply these results to endow the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ with a filtration (in the derived sense) that we call the L^∞-filtration.

Denote by $\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}$ the sheaf $\mathscr{D}_{M_{\mathrm{sa}}} := \rho_{\mathrm{sa}!}\mathscr{D}_M$ of differential operators on M_{sa}, endowed with its natural filtration and denote by $\mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}}$ the sheaf $\mathscr{D}_{M_{\mathrm{sal}}} := \rho_{\mathrm{sal}*}\mathscr{D}_{M_{\mathrm{sa}}}$ endowed with its natural filtration. For $\mathscr{T} = M, M_{\mathrm{sa}}, M_{\mathrm{sal}}$, the category $\mathrm{Mod}(\mathrm{F}\mathscr{D}_{\mathscr{T}})$ of filtered $\mathscr{D}$-modules on $\mathscr{T}$ is quasi-abelian in the sense of [**32**] and its derived category $\mathrm{D}^+(\mathrm{F}\mathscr{D}_{\mathscr{T}})$ is well-defined. We shall use here the recent results of [**31**] which give an easy description of these derived categories and we construct a right adjoint $\rho_{\mathrm{sal}}^{!}$ to the derived functor $\mathrm{R}\rho_{\mathrm{sal}*}\colon \mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}})$.

By considering the sheaves $\mathscr{C}^{\infty,s}_{M_{\mathrm{sal}}}$ $(s \geq 0)$ we obtain the filtered sheaf $\mathrm{F}_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$. Then, on a complex manifold X, by considering the Dolbeault complex of this filtered sheaf, we obtain the filtration $\mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ on the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$.

Recall now the Riemann-Hilbert correspondence. Let $\mathscr{M}$ be a regular holonomic $\mathscr{D}_X$-module and let $G := \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X)$ be the perverse sheaf of its holomorphic solutions. Kashiwara's theorem of [9] may be formulated by saying that the natural morphism $\mathscr{M} \to \rho_{\mathrm{sa}}^{-1}\mathrm{R}\mathscr{H}om(G,\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})$ is an isomorphism. Replacing the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ with its filtered version $\mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$, we define the filtered Riemann-Hilbert functors $\mathrm{RHF}_{\infty,\mathrm{sa}}$ and RHF_∞ by the formulas

$$\begin{aligned} \mathrm{RHF}_{\infty,\mathrm{sa}}\colon \mathrm{D}^+_{\mathrm{holreg}}(\mathscr{D}_X) &\to \mathrm{D}^+(\mathrm{F}\mathscr{D}_{X_{\mathrm{sa}}}), \\ \mathscr{M} &\mapsto \mathrm{FR}\mathscr{H}om(\mathrm{Sol}(\mathscr{M}),\mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}), \\ \mathrm{RHF}_\infty = \rho_{\mathrm{sa}}^{-1}\mathrm{RHF}_{\infty,\mathrm{sa}}\colon \mathrm{D}^+_{\mathrm{holreg}}(\mathscr{D}_X) &\to \mathrm{D}^+(\mathrm{F}\mathscr{D}_X) \end{aligned}$$

and we prove that the composition

$$\mathsf{D}^{\mathrm{b}}_{\mathrm{holreg}}(\mathscr{D}_X) \xrightarrow{\mathrm{RHF}_\infty} \mathrm{D}^+(\mathrm{F}\mathscr{D}_X) \xrightarrow{\mathrm{for}} \mathrm{D}^+(\mathscr{D}_X)$$

is isomorphic to the identity functor. In other words, any regular holonomic $\mathscr{D}_X$-module $\mathscr{M}$ can be *functorially* endowed with a filtration $\mathrm{F}_\infty\mathscr{M}$, in the derived sense.

We also briefly introduce an L^2-filtration better suited to apply Hörmander's theory (see [5]) and present some open problems.

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Finally Theorem 2.43 plays an essential role in the whole paper and we are extremely grateful to Adam Parusinski who has given a proof of this result.

1. Subanalytic topologies

1.1. Linear coverings

Notations and conventions. — We shall mainly follow the notations of [11, 13] and [15].

In this paper, unless otherwise specified, a manifold means a real analytic manifold. We shall freely use the theory of subanalytic sets, due to Gabrielov and Hironaka, after the pioneering work of Lojasiewicz. A short presentation of this theory may be found in [2].

For a subset A in a topological space X, $\overline{A}$ denotes its closure, $\operatorname{Int} A$ its interior and ∂A its boundary, $\partial A = \overline{A} \setminus \operatorname{Int} A$.

Recall that given two metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is Lipschitz if there exists a constant $C \geq 0$ such that $d_Y(f(x), f(x')) \leq C \cdot d_X(x, x')$ for all $x, x' \in X$.

$$\left\{\begin{array}{l}\text{All along this paper, if } M \text{ is a real analytic manifold, we}\\ \text{choose a distance } d_M \text{ on } M \text{ such that, for any } x \in M \text{ and}\\ \text{any local chart } (U, \varphi\colon U \hookrightarrow \mathbb{R}^n) \text{ around } x, \text{ there exists a}\\ \text{neighborhood of } x \text{ over which } d_M \text{ is Lipschitz equivalent to}\\ \text{the pull-back of the Euclidean distance by } \varphi. \text{ If there is no}\\ \text{risk of confusion, we write } d \text{ instead of } d_M.\end{array}\right. \tag{1.1}$$

In the following, we will adopt the convention

$$d(x, \varnothing) = D_M + 1, \qquad \text{for all } x \in M, \tag{1.2}$$

where $D_M = \sup\{d(y,z);\ y, z \in M\}$. In this way we avoid distinguishing the special case where $M = \bigcup_{i\in I} U_i$ in (1.4) below (which can happen if M is compact).

The site M_{sa}. — The subanalytic topology was introduced in [**13**].

Let M be a real analytic manifold and denote by $\mathrm{Op}_{M_{\mathrm{sa}}}$ the category of relatively compact subanalytic open subsets of M, the morphisms being the inclusion morphisms. Recall that one endows $\mathrm{Op}_{M_{\mathrm{sa}}}$ with a Grothendieck topology by saying that a family $\{U_i\}_{i\in I}$ of objects of $\mathrm{Op}_{M_{\mathrm{sa}}}$ is a covering of $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ if $U_i \subset U$ for all $i \in I$ and there exists a finite subset $J \subset I$ such that $\bigcup_{j\in J} U_j = U$. It follows from the theory of subanalytic sets that in this situation there exist a constant $C > 0$ and a positive integer N such that

$$d(x, M \setminus U)^N \leq C \cdot (\max_{j\in J} d(x, M \setminus U_j)). \tag{1.3}$$

One shall be aware that if U is an open subset of M, we may endow it with the subanalytic topology U_{sa}, but this topology does not coincide in general with the topology induced by M.

We denote by $\rho_{\mathrm{sa}}\colon M \to M_{\mathrm{sa}}$ (or simply ρ) the natural morphism of sites.

The site M_{sal}

Definition 1.1. — Let $\{U_i\}_{i\in I}$ be a finite family in $\mathrm{Op}_{M_{\mathrm{sa}}}$. We say that this family is 1-regularly situated if there is a constant C such that for any $x \in M$

$$d(x, M \setminus \bigcup_{i\in I} U_i) \leq C \cdot \max_{i\in I} d(x, M \setminus U_i). \tag{1.4}$$

Of course, this definition does not depend on the choice of the distance d.

When $M = \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ we have $d(x, M \setminus U) = d(x, \partial U)$, for all $x \in U$. In general we have the following comparison result.

Lemma 1.2. — *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ be such that ∂U is non empty (that is, U is not a union of connected components of M). Then there exists $C > 0$ such that for all $x \in U$ we have*

$$d(x, M \setminus U) \leq d(x, \partial U) \leq C\, d(x, M \setminus U).$$

Proof. — The first inequality is clear and we prove the second one. If it is false, there exist $x_n \in U$, $n \in \mathbb{N}$, such that $d(x_n, \partial U)/d(x_n, M \setminus U) \xrightarrow{n\to\infty} \infty$. Since $\overline{U}$ is compact, up to taking a subsequence we may assume that x_n converges to a point $x \in \overline{U}$. We see easily that $x \in \partial U$. We take a chart around x as in (1.1). Since $d_{\mathbb{R}^n}(y, \partial U) = d_{\mathbb{R}^n}(y, M \setminus U)$ for y in the chart near x, we can not have $d(x_n, \partial U)/d(x_n, M \setminus U) \xrightarrow{n\to\infty} \infty$, which proves the result. □

Example 1.3. — Let $U_1, U_2 \in \mathrm{Op}_{M_{\mathrm{sa}}}$ be two disjoint open sets. We prove that $\{U_1, U_2\}$ is 1-regularly situated. We set $U = U_1 \cup U_2$. We argue as in the proof of Lemma 1.2 and assume by contradiction that there exists a sequence $x_n \in U$, $n \in \mathbb{N}$, such that $d(x_n, M \setminus U)/\max_{i=1,2}\{d(x_n, M \setminus U_i)\}$ converges to ∞. We may as well assume $x_n \in U_1$ for all n. Up to taking a subsequence we may assume that x_n converges to a point $x \in \overline{U_1}$. We see that $x \in \partial U_1$. We take a chart around x as in (1.1). Then, for $n \gg 0$, $d_{\mathbb{R}^d}(x_n, M \setminus U_1)$ is realized by a point $y_n \in \partial U_1$. Since $U_2 \cap \partial U_1 = \varnothing$ we have in fact $y_n \in M \setminus U$. Hence $d_{\mathbb{R}^d}(x_n, M \setminus U_1) = d_{\mathbb{R}^d}(x_n, M \setminus U)$. Since d is Lipschitz equivalent to $d_{\mathbb{R}^d}$, the quotient $d(x_n, M \setminus U)/\max_{i=1,2}\{d(x_n, M \setminus U_i)\}$ remains bounded and we have a contradiction.

Example 1.4. — On $\mathbb{R}^2$ with coordinates (x_1, x_2) consider the open sets:

$$\begin{aligned}
U_1 &= \{(x_1, x_2);\ x_2 > -x_1^2,\ x_1 > 0\},\\
U_2 &= \{(x_1, x_2);\ x_2 < x_1^2,\ x_1 > 0\},\\
U_3 &= \{(x_1, x_2);\ x_1 > -x_2^2,\ x_2 > 0\}.
\end{aligned}$$

Then $\{U_1, U_2\}$ is not 1-regularly situated. Indeed, set $W := U_1 \cup U_2 = \{x_1 > 0\}$. Then, if $x = (x_1, 0), x_1 > 0$, $d(x, \mathbb{R}^2 \setminus W) = x_1$ and $d(x, \mathbb{R}^2 \setminus U_i)$ $(i = 1, 2)$ is less that x_1^2. On the other hand $\{U_1, U_3\}$ is 1-regularly situated. Indeed,

$$d(x, \mathbb{R}^2 \setminus (U_1 \cup U_3)) \leq \sqrt{2}\max(d(x, \mathbb{R}^2 \setminus U_1), d(x, \mathbb{R}^2 \setminus U_3)).$$

Definition 1.5. — A linear covering of U is a small family $\{U_i\}_{i\in I}$ of objects of $\mathrm{Op}_{M_{\mathrm{sa}}}$ such that $U_i \subset U$ for all $i \in I$ and

$$\begin{cases}\text{there exists a finite subset } I_0 \subset I \text{ such that the family}\\ \{U_i\}_{i\in I_0} \text{ is 1-regularly situated and } \bigcup_{i\in I_0} U_i = U.\end{cases} \tag{1.5}$$

Let $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ be two families of objects of $\mathrm{Op}_{M_{\mathrm{sa}}}$. Recall that one says that $\{U_i\}_{i\in I}$ is a refinement of $\{V_j\}_{j\in J}$ if for any $i \in I$, there exists $j \in J$ with $U_i \subset V_j$.

Proposition 1.6. — *The family of linear coverings satisfies the axioms of Grothendieck topologies below (see* [**15**, § 16.1]*).*

- **COV1:** *$\{U\}$ is a covering of U, for any $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$.*
- **COV2:** *If a covering $\{U_i\}_{i\in I}$ of U is a refinement of a family $\{V_j\}_{j\in J}$ in $\mathrm{Op}_{M_{\mathrm{sa}}}$ with $V_j \subset U$ for all $j \in J$, then $\{V_j\}_{j\in J}$ is a covering of U.*
- **COV3:** *If $V \subset U$ are in $\mathrm{Op}_{M_{\mathrm{sa}}}$ and $\{U_i\}_{i\in I}$ is a covering of U, then $\{V \cap U_i\}_{i\in I}$ is a covering of V.*

COV4: *If $\{U_i\}_{i\in I}$ is a covering of U and $\{V_j\}_{j\in J}$ is a small family in $\mathrm{Op}_{M_{\mathrm{sa}}}$ with $V_j \subset U$ such that $\{U_i \cap V_j\}_{j\in J}$ is a covering of U_i for all $i \in I$, then $\{V_j\}_{j\in J}$ is a covering of U.*

Proof. — We shall use the obvious fact stating that for two subsets $A \subset B$ in M, we have $d(x, M \setminus A) \leq d(x, M \setminus B)$.

COV1: Is trivial.

COV2: Let $I_0 \subset I$ be as in (1.5). Let $\sigma\colon I \to J$ be such that $U_i \subset V_{\sigma(i)}$, for all $i \in I$. Then, for all $x \in U_i$ we have $d(x, M \setminus U_i) \leq d(x, M \setminus V_{\sigma(i)})$. It follows that $\sigma(I_0)$ satisfies (1.5) with respect to $\{V_j\}_{j\in J}$.

COV3: Let $I_0 \subset I$ be as in (1.5) and let C be the constant in (1.4). Let x be a given point in $V \cap U$. We have $d(x, M \setminus (V \cap U)) \leq d(x, M \setminus U)$. We distinguish two cases.

(a) We assume that $d(x, M \setminus (V \cap U_i)) = d(x, M \setminus U_i)$, for all $i \in I_0$. Then we clearly have $d(x, M \setminus (V \cap U)) \leq C \max_{i\in I_0} d(x, M \setminus (V \cap U_i))$ and I_0 satisfies (1.5) with respect to $\{V \cap U_i\}_{i\in I}$.

(b) We assume $d(x, M \setminus (V \cap U_{i_0})) < d(x, M \setminus U_{i_0})$ for some $i_0 \in I_0$. We choose $y \in M \setminus (V \cap U_{i_0})$ such that $d(x, y) = d(x, M \setminus (V \cap U_{i_0}))$. Then we have $d(x, y) < d(x, M \setminus U_{i_0})$. We deduce that $y \in U_{i_0}$ and then that $y \in M \setminus V$. Hence $y \in M \setminus (V \cap U)$ and $d(x, M \setminus (V \cap U)) \leq d(x, y)$. Then

$$\begin{aligned} d(x, M \setminus (V \cap U)) &\leq d(x, M \setminus (V \cap U_{i_0})) \\ &\leq \max_{i\in I_0} d(x, M \setminus (V \cap U_i)). \end{aligned}$$

We obtain (1.4) for the family $\{V \cap U_i\}_{i\in I_0}$ with $C = 1$.

COV4: Let $I_0 \subset I$ be as in (1.5) and let C be the constant in (1.4). For each $i \in I_0$, let $J_i \subset J$ satisfying (1.5) with respect to U_i for the family $\{U_i \cap V_j\}_{j\in J}$ and let C_i be the corresponding constant. We set $J_0 = \bigcup_{i\in I_0} J_i$ and $B = \max\{C \cdot C_i; i \in I_0\}$. Then we have

$$\begin{aligned} d(x, M \setminus U) &\leq C \max_{i\in I_0} d(x, M \setminus U_i) \\ &\leq C \max_{i\in I_0}(C_i \max_{j\in J_i} d(x, M \setminus (U_i \cap V_j))) \\ &\leq B \max_{i\in I_0} \max_{j\in J_i} d(x, M \setminus V_j) \\ &\leq B \max_{j\in J_0} d(x, M \setminus V_j), \end{aligned}$$

which proves that J_0 satisfies (1.5) with respect to $\{V_j\}_{j\in J}$. □

As a particular case of COV4, we get

Corollary 1.7. — *If $\{U_i\}_{i\in I}$ is a linear covering of $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and $I = \bigsqcup_{\alpha\in A} I_\alpha$ is a partition of I, then setting $U_\alpha := \bigcup_{i\in I_\alpha} U_i$, $\{U_\alpha\}_{\alpha\in A}$ is a linear covering of U.*

The notion of a linear covering is of local nature (in the usual topology). More precisely, we have:

Proposition 1.8. — *Let $V \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $\{U_i\}_{i\in I}$ be a finite covering of $\overline{V}$ in M_{sa}. Then $\{V \cap U_i\}_{i\in I}$ is a linear covering of V.*

Proof. — Set $U = \bigcup_i U_i$ and let $W \in \mathrm{Op}_{M_{\mathrm{sa}}}$ be a neighborhood of the boundary ∂U such that $V \cap W = \varnothing$. Let us prove that the family $\{W, \{U_i\}_{i\in I}\}$ is a linear covering of $W \cup U$. We set $f(x) = \max\{d(x, M \setminus W), d(x, M \setminus U_i), i \in I\}$ and $Z = \{x \in M;\ d(x, M \setminus (W \cup U)) \geq d(x, U)\}$. Then Z is a compact subset of $W \cup U$. Hence there exists $\varepsilon > 0$ such that $f(x) > \varepsilon$ for all $x \in Z$.

We also see that $\overline{U} \subset Z$. Hence $f(x) = d(x, M \setminus W)$ for $x \notin Z$. Moreover, for a given $x \notin Z$ we have $d(x, M \setminus W) \leq d(x, M \setminus (W \cup U)) < d(x, U)$ by definition of Z. Hence a given $y \in M \setminus W$ realizing $d(x, M \setminus W)$ can not belong to U and we obtain $d(x, M \setminus (W \cup U)) = d(x, M \setminus W)$. Finally $d(x, M \setminus (W \cup U)) = f(x)$ for $x \notin Z$.

Now we deduce that $d(x, M \setminus (W \cup U)) \leq Cf(x)$ for some $C > 0$ and for all $x \in M$, that is, $\{W, \{U_i\}_{i\in I}\}$ is a linear covering of $W \cup U$.

Taking the intersection with V we obtain by COV3 that $\{V \cap U_i\}_{i\in I}$ is a linear covering of V. □

Corollary 1.9. — *Let $\{U_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ be two finite families in $\mathrm{Op}_{M_{\mathrm{sa}}}$. We set $U = \bigcup_i U_i$ and we assume that $\overline{U} \subset \bigcup_j B_j$. Then $\{U_i\}_{i\in I}$ is a linear covering of U if and only if $\{U_i \cap B_j\}_{i\in I}$ is a linear covering of $U \cap B_j$ for all $j \in J$.*

Proof. — (i) Assume that $\{U_i\}_i$ is a linear covering of U. Applying COV3 to $B_j \cap U \subset U$ we get that the family $\{U_i \cap B_j\}_{i\in I}$ is a linear covering of $U \cap B_j$ for all $j \in J$.
(ii) Assume that the family $\{U_i \cap B_j\}_{i\in I}$ is a linear covering of $U \cap B_j$ for all $j \in J$. By Proposition 1.8 the family $\{U \cap B_j\}_{j\in J}$ is a linear covering of U. Hence the result follows from COV4. □

Definition 1.10. — (a) The linear subanalytic site M_{sal} is the presite M_{sa} endowed with the Grothendieck topology for which the coverings are the linear coverings given by Definition 1.5.
(b) We denote by $\rho_{\mathrm{sal}}\colon M_{\mathrm{sa}} \to M_{\mathrm{sal}}$ and by $\rho_{\mathrm{sl}}\colon M \to M_{\mathrm{sal}}$ the natural morphisms of sites.

The morphisms of sites constructed above are summarized by the diagram

$$\begin{tikzcd} M \arrow[r, "\rho_{\mathrm{sa}}"] \arrow[dr, "\rho_{\mathrm{sl}}"'] & M_{\mathrm{sa}} \arrow[d, "\rho_{\mathrm{sal}}"] \\ & M_{\mathrm{sal}}. \end{tikzcd}$$

Remark 1.11. — Let M and N be two real analytic manifolds and let $f\colon M \to N$ be a topological isomorphism such that both f and f^{-1} are subanalytic Lipschitz maps. Then $f^{-1}\colon \mathrm{Op}_{M_{\mathrm{sa}}} \to \mathrm{Op}_{N_{\mathrm{sa}}}$ induces an isomorphism of sites $N_{\mathrm{sal}} \xrightarrow{\sim} M_{\mathrm{sal}}$.

1.2. Regular coverings. — We shall also use the following:

Definition 1.12. — Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. A regular covering of U is a sequence $\{U_i\}_{i\in[1,N]}$ with $1 \le N \in \mathbb{N}$ such that $U = \bigcup_{i\in[1,N]} U_i$ and, for all $1 \le k \le N$, $\{U_i\}_{i\in[1,k]}$ is a linear covering of $\bigcup_{1\le i\le k} U_i$.

We will use the following recipe to turn an arbitrary covering into a linear covering by a slight enlargement of the open subsets. For an open subset U of M, an arbitrary subset $V \subset U$ and $\varepsilon > 0$ we set

$$V^{\varepsilon,U} = \{x \in M;\ d(x,V) < \varepsilon\, d(x, M \setminus U)\}. \tag{1.6}$$

Then $V^{\varepsilon,U}$ is an open subset of U. If the distance d is a subanalytic function on $M\times M$, $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and V is a subanalytic subset, then $V^{\varepsilon,U}$ also belongs to $\mathrm{Op}_{M_{\mathrm{sa}}}$. We see easily that $(U \cap \overline{V}) \subset V^{\varepsilon,U} \subset U$.

Lemma 1.13. — *We assume that the distance d is a subanalytic function on $M \times M$. Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $V \subset U$ be a subanalytic subset. Let $0 < \varepsilon$ and $0 < \delta < 1$. We set $\varepsilon' = \frac{\varepsilon+\delta}{1-\delta}$. Then*

(i) *for any $x \in V^{\varepsilon,U}$ and $y \in M$ such that*

$$d(x,y) < \delta\, d(x, M\setminus U) \text{ or } d(x,y) < \delta\, d(y, M\setminus U),$$

we have $d(y,V) < \varepsilon' d(y, M\setminus U)$, that is, $y \in V^{\varepsilon',U}$,

(ii) *for any $x \in V^{\varepsilon,U}$ we have $d(x, M \setminus V^{\varepsilon',U}) \ge \delta\, d(x, M\setminus U)$,*

(iii) *$\{U \setminus \overline{V}, V^{\varepsilon',U}\}$ is a linear covering of U.*

We remark that any $\varepsilon' > 0$ can be written $\varepsilon' = \frac{\varepsilon+\delta}{1-\delta}$ with ε, δ as in the lemma.

Proof. — (i) The triangular inequality $d(x, M\setminus U) \le d(x,y) + d(y, M\setminus U)$ implies

$$\begin{cases} d(x, M\setminus U) < (1-\delta)^{-1} d(y, M\setminus U), & \text{if } d(x,y) < \delta\, d(x, M\setminus U), \\ d(x, M\setminus U) < (1+\delta) d(y, M\setminus U), & \text{if } d(x,y) < \delta\, d(y, M\setminus U). \end{cases}$$

Since $1+\delta < (1-\delta)^{-1}$ we obtain in both cases

$$d(x, M\setminus U) < (1-\delta)^{-1} d(y, M\setminus U). \tag{1.7}$$

In particular we have in both cases $d(x,y) < \delta(1-\delta)^{-1} d(y, M\setminus U)$. Now the definition of $V^{\varepsilon,U}$ implies

$$\begin{aligned} d(y,V) &\le d(x,y) + d(x,V) \\ &< \delta(1-\delta)^{-1} d(y, M\setminus U) + \varepsilon\, d(x, M\setminus U) \\ &< (\varepsilon+\delta)(1-\delta)^{-1} d(y, M\setminus U), \end{aligned}$$

where the last inequality follows from (1.7).

(ii) By (i), if a point $y \in M$ does not belong to $V^{\varepsilon',U}$, we have $d(x,y) \ge \delta\, d(x, M\setminus U)$. This gives (ii).

(iii) Since d is subanalytic, the open subset $V^{\varepsilon',U}$ is subanalytic. We also see easily that $U = (U \setminus \overline{V}) \cup V^{\varepsilon',U}$. Now let $x \in M$.
(a) If $x \notin V^{\varepsilon,U}$, then (1.6) gives $d(x,V) \geq \varepsilon\, d(x, M \setminus U)$. Since $d(x, M \setminus (U \setminus \overline{V})) = \min\{d(x, M \setminus U), d(x,V)\}$, we deduce $d(x, M \setminus (U \setminus \overline{V})) \geq \min\{\varepsilon, 1\} d(x, M \setminus U)$.
(b) If $x \in V^{\varepsilon,U}$, then (ii) gives $d(x, M \setminus V^{\varepsilon',U}) \geq \delta\, d(x, M \setminus U)$.
We obtain in both cases

$$\max\{d(x, M \setminus (U \setminus \overline{V})), d(x, M \setminus V^{\varepsilon',U})\} \geq C d(x, M \setminus U),$$

where $C = \min\{\delta, \varepsilon\}$. This proves (iii). □

Lemma 1.15 below will be used later to obtain subsets satisfying the hypothesis of Lemma 4.15. We will prove it by using Lemma 1.13 as follows. Let $U_1, U_2 \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $U = U_1 \cup U_2$. For $\varepsilon > 0$ we set, using Notation (1.6),

$$U_1^\varepsilon = (U_1 \setminus U_2)^{\varepsilon, U_1} = \{x \in U_1;\ d(x, U_1 \setminus U_2) < \varepsilon\, d(x, M \setminus U_1)\}, \tag{1.8}$$

$$U_2^\varepsilon = (U_2 \setminus U_1)^{\varepsilon, U_2} = \{x \in U_2;\ d(x, U_2 \setminus U_1) < \varepsilon\, d(x, M \setminus U_2)\}. \tag{1.9}$$

Lemma 1.14. — (i) *For $i = 1, 2$ and for any $\varepsilon > 0$, the pair $\{U_i^\varepsilon, U_1 \cap U_2\}$ is a linear covering of U_i.*
(ii) *For any $\varepsilon, \varepsilon' > 0$ such that $\varepsilon\varepsilon' < 1$, we have $\overline{U_1^\varepsilon} \cap \overline{U_2^{\varepsilon'}} \cap U = \varnothing$.*
(iii) *Let $\varepsilon > 0$, $0 < \delta < 1$ and set $\varepsilon' = \frac{\varepsilon+\delta}{1-\delta}$, $\varepsilon'' = \frac{\varepsilon'+\delta}{1-\delta}$. We assume $\varepsilon\varepsilon'' < 1$. Then, for any $x \in M$,*

$$\begin{cases} d(x, U_2^\varepsilon) \geq \delta\, d(x, M \setminus U_1) & \text{if } x \in U_1^{\varepsilon'}, \\ d(x, U_1^\varepsilon) \geq \delta\, d(x, M \setminus U_1) & \text{if } x \notin U_1^{\varepsilon'}. \end{cases}$$

Proof. — (i) By symmetry we can assume $i = 1$. By Lemma 1.13, the pair $\{U_1 \setminus \overline{(U_1 \setminus U_2)}, U_1^\varepsilon\}$ is a linear covering of U_1. Since U_2 is open we have $U_1 \setminus \overline{(U_1 \setminus U_2)} = U_1 \cap U_2$ and (i) follows.
(ii) We have

$$\overline{U_1^\varepsilon} \cap U \subset \{x \in U;\ d(x, U_1 \setminus U_2) \leq \varepsilon\, d(x, M \setminus U_1)\},$$
$$\overline{U_2^{\varepsilon'}} \cap U \subset \{x \in U;\ d(x, U_2 \setminus U_1) \leq \varepsilon'\, d(x, M \setminus U_2)\}.$$

We remark that $d(x, M \setminus U_2) \leq d(x, U_1 \setminus U_2)$ and $d(x, M \setminus U_1) \leq d(x, U_2 \setminus U_1)$ for any $x \in M$. Let $x \in \overline{U_1^\varepsilon} \cap \overline{U_2^{\varepsilon'}} \cap U$ and set $d_1 = d(x, U_2 \setminus U_1)$, $d_2 = d(x, U_1 \setminus U_2)$. We deduce $d_i \leq \varepsilon\varepsilon' d_i$, for $i = 1, 2$. Since $\varepsilon\varepsilon' < 1$ we obtain $d_1 = d_2 = 0$. Hence $x \notin U_1$ and $x \notin U_2$. Since $U = U_1 \cup U_2$, this proves (ii).

(iii) By Lemma 1.13 (ii), we have $d(x, M \setminus U_1^{\varepsilon''}) \geq \delta\, d(x, M \setminus U_1)$ for any $x \in U_1^{\varepsilon'}$. By (ii) we have $U_2^\varepsilon \subset M \setminus U_1^{\varepsilon''}$ and the first inequality follows.

By Lemma 1.13 (i), if $x \notin U_1^{\varepsilon'}$ and $z \in U_1^\varepsilon$, then $d(x,z) \geq \delta\, d(x, M \setminus U_1)$. This gives the second inequality. □

Lemma 1.15. — *Let $U_1, U_2 \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and set $U = U_1 \cup U_2$. We assume that $\{U_1, U_2\}$ is a linear covering of U. Then there exist $U_i' \subset U_i$, $i = 1, 2$, and $C > 0$ such that*

(i) $\{U'_i, U_1 \cap U_2\}$ *is a linear covering of* U_i $(i = 1, 2)$,
(ii) $\overline{U'_1} \cap \overline{U'_2} \cap U = \varnothing$,
(iii) *setting* $Z_i = (M \setminus U) \cup \overline{U'_i}$, *we have* $Z_1 \cap Z_2 = M \setminus U$ *and*

$$d(x, Z_1 \cap Z_2) \leq C(d(x, Z_1) + d(x, Z_2)), \qquad \textit{for any } x \in M.$$

Proof. — We set $\varepsilon = \delta = 1/3$, $\varepsilon' = \frac{\varepsilon+\delta}{1-\delta} = 1$ and $\varepsilon'' = \frac{\varepsilon'+\delta}{1-\delta} = 2$. Using the notations (1.8) and (1.9) we set $U'_i = U_i^\varepsilon$, $i = 1, 2$.

(i) and (ii) are given by Lemma 1.14 (i) and (ii).

(iii) The equality $Z_1 \cap Z_2 = M \setminus U$ follows from (ii). Let C' be the constant in (1.4) for the family $\{U_1, U_2\}$. We set $C_1 = \max\{1, \delta^{-1}C'\}$. Let $x \in M$ and let $x_i \in Z_i$ be such that $d(x, x_i) = d(x, Z_i)$. By the definition of Z_1, if $x_1 \notin \overline{U'_1}$, then $x_1 \in M \setminus U$. Hence $d(x, Z_1) = d(x, M \setminus U)$ and the inequality in (iii) is clear.

Hence we can assume $x_1 \in \overline{U'_1}$ and also $x_2 \in \overline{U'_2}$ by symmetry. Then we have $d(x, Z_1) + d(x, Z_2) = d(x, U_1^\varepsilon) + d(x, U_2^\varepsilon)$. Since $\varepsilon\varepsilon'' = 2/3 < 1$, Lemma 1.14 (iii) gives $d(x, U_1^\varepsilon) + d(x, U_2^\varepsilon) \geq \delta\, d(x, M \setminus U_1)$. The same holds with $M \setminus U_1$ replaced by $M \setminus U_2$ and (1.4) gives

$$d(x, U_1^\varepsilon) + d(x, U_2^\varepsilon) \geq \delta \max_{i=1,2}\{d(x, M \setminus U_i)\} \geq C_1^{-1} d(x, M \setminus U),$$

so that (iii) holds with $C = C_1$. □

Lemma 1.16. — *We assume that the distance d is a subanalytic function on $M \times M$. Let $\{U_i\}_{i=1}^N$ be a 1-regularly situated family in $\mathrm{Op}_{M_{\mathrm{sa}}}$ and let $C \geq 1$ be a constant satisfying* (1.4). *We choose $D > C$ and $1 > \varepsilon > 0$ such that $\varepsilon D < 1 - \varepsilon$. We define $U_i^0, V_i, U'_i \in \mathrm{Op}_{M_{\mathrm{sa}}}$ inductively on i by $U_1^0 = V_1 = U'_1 = U_1$ and*

$$\begin{aligned} U_i^0 &= \{x \in U_i;\ d(x, M \setminus (U_i \cup V_{i-1})) < D\, d(x, M \setminus U_i)\}, \\ V_i &= V_{i-1} \cup U_i^0, \\ U'_i &= (U_i^0)^{\varepsilon, V_i} \quad \textit{(using the notation (1.6))}. \end{aligned}$$

Then $V_N = \bigcup_{i=1}^N U_i$ and, for all $k = 1, \ldots, N$, we have $U'_k \subset U_k$, $V_k = \bigcup_{i=1}^k U'_i$ and $\{U'_i\}_{i=1}^k$ is a 1-regularly situated family in $\mathrm{Op}_{M_{\mathrm{sa}}}$.

Proof. — (i) Let us prove that $U'_k \subset U_k$. Let $x \in U'_k$ and let us show that $x \in U_k$. By (1.6) we have $x \in V_k$ and there exists $y \in U_k^0$ such that $d(x, y) < \varepsilon\, d(x, M \setminus V_k)$. We deduce $d(x, y) < \varepsilon(d(x, y) + d(y, M \setminus V_k))$ and then

$$d(x, y) < (\varepsilon/(1-\varepsilon))\, d(y, M \setminus V_k). \tag{1.10}$$

On the other hand we have $U_k^0 \subset U_k$, hence $V_k \subset U_k \cup V_{k-1}$. Since $y \in U_k^0$ we deduce

$$d(y, M \setminus V_k) \leq d(y, M \setminus (U_k \cup V_{k-1})) < D\, d(y, M \setminus U_k). \tag{1.11}$$

The inequalities (1.10), (1.11) and the hypothesis on D and ε give $d(x, y) < d(y, M \setminus U_k)$. Hence $x \in U_k$.

(ii) We have $V_i = V_{i-1} \cup U_i^0$. Hence Lemma 1.13 implies that $\{V_{i-1}, U_i'\}$ is a covering of V_i in M_{sa}. Let us prove the last part of the lemma by induction on k. We immediately obtain that $V_k = \bigcup_{i=1}^k U_i'$. Moreover, $\{V_{k-1}, U_k'\}$ being a covering of V_k, we get by using COV4 that, for all $k = 1, \dots, N$, $\{U_i'\}_{i=1}^k$ is a 1-regularly situated family in $\mathrm{Op}_{M_{\mathrm{sa}}}$.

(iii) It remains to prove that $V_N = \bigcup_{i=1}^N U_i$. It is clear that $V_k \subset \bigcup_{i=1}^N U_i$, for all $k = 1, \dots, N$. Let $x \in \bigcup_{i=1}^N U_i$. Since $\{U_i\}_{i=1}^N$ is 1-regularly situated, there exists i_0 such that $d(x, M \setminus \bigcup_{i=1}^N U_i) \le C\, d(x, M \setminus U_{i_0})$. In particular $x \in U_{i_0}$ and moreover $d(x, M \setminus (U_{i_0} \cup V_{i_0-1})) \le C\, d(x, M \setminus U_{i_0}) < D\, d(x, M \setminus U_{i_0})$. Therefore $x \in U_{i_0}^0$. By definition $U_{i_0}^0 \subset V_{i_0} \subset V_N$. Hence $x \in V_N$ and we obtain $V_N = \bigcup_{i=1}^N U_i$. □

In particular, we have proved:

Proposition 1.17. — *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. Then for any linear covering $\{U_i\}_{i \in I}$ of U there exists a refinement which is a regular covering of U.*

2. Sheaves on subanalytic topologies

2.1. Sheaves

Usual notations. — We shall mainly follow the notations of [**11, 13**] and [**15**].

In this paper, we denote by $\mathbf{k}$ a field, although most of the results hold under the hypothesis that $\mathbf{k}$ is a commutative unital Noetherian ring with finite global dimension. Unless otherwise specified, a manifold means a real analytic manifold.

If $\mathscr{C}$ is an additive category, we denote by $\mathrm{C}(\mathscr{C})$ the additive category of complexes in $\mathscr{C}$. For $* = +, -, \mathrm{b}$ we also consider the full additive subcategory $\mathrm{C}^*(\mathscr{C})$ of $\mathrm{C}(\mathscr{C})$ consisting of complexes bounded from below (resp. from above, resp. bounded) and $\mathrm{C}^{\mathrm{ub}}(\mathscr{C})$ means $\mathrm{C}(\mathscr{C})$ ("ub" stands for "unbounded"). If $\mathscr{C}$ is an abelian category, we denote by $\mathrm{D}(\mathscr{C})$ its derived category and similarly with $\mathrm{D}^*(\mathscr{C})$ for $* = +, -, \mathrm{b}, \mathrm{ub}$.

For a site $\mathscr{T}$, we denote by $\mathrm{PSh}(\mathbf{k}_{\mathscr{T}})$ and $\mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ the abelian categories of presheaves and sheaves of $\mathbf{k}$-modules on $\mathscr{T}$. We denote by $\iota\colon \mathrm{Mod}(\mathbf{k}_{\mathscr{T}}) \to \mathrm{PSh}(\mathbf{k}_{\mathscr{T}})$ the forgetful functor and by $(\,\bullet\,)^a$ its left adjoint, the functor which associates a sheaf to a presheaf. Note that in practice we shall often not write ι. Recall that $\mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ is a Grothendieck category and, in particular, has enough injectives. We write $\mathrm{D}^*(\mathbf{k}_{\mathscr{T}})$ instead of $\mathrm{D}^*(\mathrm{Mod}(\mathbf{k}_{\mathscr{T}}))$ ($* = +, -, \mathrm{b}, \mathrm{ub}$).

For a site $\mathscr{T}$, we will often use the following well-known fact. For any $F \in \mathrm{D}(\mathbf{k}_{\mathscr{T}})$ and any $i \in \mathbb{Z}$, the cohomology sheaf $H^i(F)$ is the sheaf associated with the presheaf $U \mapsto H^i(U; F)$. In particular, if $H^i(U; F) = 0$ for all $U \in \mathscr{T}$, then $H^i(F) \simeq 0$.

For an object U of $\mathscr{T}$, recall that there is a sheaf naturally attached to U (see e.g., [**15**, § 17.6]). We shall denote it here by $\mathbf{k}_{U\mathscr{T}}$ or simply $\mathbf{k}_U$ if there is no risk of confusion. This is the sheaf associated with the presheaf (see loc. cit. Lemma 17.6.11):

$$V \mapsto \oplus_{V \to U} \mathbf{k}.$$

The functor "associated sheaf" is exact. If follows that, if $V \to U$ is a monomorphism in $\mathscr{T}$, then the natural morphism $\mathbf{k}_{V\mathscr{T}} \to \mathbf{k}_{U\mathscr{T}}$ also is a monomorphism.

Sheaves on M and M_{sa}. — We shall mainly use the subanalytic topology introduced in [**13**]. In loc. cit., sheaves on the subanalytic topology are studied in the more general framework of indsheaves. We refer to [**27**] for a direct and more elementary treatment of subanalytic sheaves.

Recall that $\rho_{\mathrm{sa}} \colon M \to M_{\mathrm{sa}}$ denotes the natural morphism of sites. The functor $\rho_{\mathrm{sa}*}$ is left exact and its left adjoint ρ_{sa}^{-1} is exact. Hence, we have the pairs of adjoint functors

$$\mathrm{Mod}(\mathbf{k}_M) \underset{\rho_{\mathrm{sa}}^{-1}}{\overset{\rho_{\mathrm{sa}*}}{\rightleftarrows}} \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}), \qquad \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \underset{\rho_{\mathrm{sa}}^{-1}}{\overset{\mathrm{R}\rho_{\mathrm{sa}*}}{\rightleftarrows}} \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{\mathrm{sa}}}). \tag{2.1}$$

The functor $\rho_{\mathrm{sa}*}$ is fully faithful and $\rho_{\mathrm{sa}}^{-1}\rho_{\mathrm{sa}*} \simeq \mathrm{id}$. Moreover, $\rho_{\mathrm{sa}}^{-1}\mathrm{R}\rho_{\mathrm{sa}*} \simeq \mathrm{id}$ and $\mathrm{R}\rho_{\mathrm{sa}*}$ in (2.1) is fully faithful.

The functor ρ_{sa}^{-1} also admits a left adjoint functor $\rho_{\mathrm{sa}!}$. For $F \in \mathrm{Mod}(\mathbf{k}_M)$, $\rho_{\mathrm{sa}!}F$ is the sheaf on M_{sa} associated with the presheaf $U \mapsto F(\overline{U})$. The functor $\rho_{\mathrm{sa}!}$ is exact, fully faithful and commutes with tensor products.

Proposition 2.1. — *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $F \in \mathrm{Mod}(\mathbf{k}_M)$. Then*

$$\mathrm{R}\Gamma(U; \mathrm{R}\rho_{\mathrm{sa}*}F) \simeq \mathrm{R}\Gamma(U; F).$$

Proof. — This follows from $\mathrm{R}\Gamma(U; G) \simeq \mathrm{RHom}\,(\mathbf{k}_U, G)$ for $G \in \mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ ($\mathscr{T} = M$ or $\mathscr{T} = M_{\mathrm{sa}}$) and by adjunction since $\rho_{\mathrm{sa}}^{-1}\mathbf{k}_{UM_{\mathrm{sa}}} \simeq \mathbf{k}_{UM}$. □

Also note that the functor $\rho_{\mathrm{sa}*}$ admitting an exact left adjoint functor, it sends injective objects of $\mathrm{Mod}(\mathbf{k}_M)$ to injective objects of $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$.

One denotes by $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ the category of $\mathbb{R}$-constructible sheaves on M. One denotes by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of objects with $\mathbb{R}$-constructible cohomologies.

Recall that $\rho_{\mathrm{sa}*}$ is exact when restricted to the subcategory $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$. Hence we shall consider this last category both as a full subcategory of $\mathrm{Mod}(\mathbf{k}_M)$ and a full subcategory of $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$.

For $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ we have the sheaf $\mathbf{k}_{UM_{\mathrm{sa}}} \simeq \rho_{\mathrm{sa}*}\mathbf{k}_{UM}$ on M_{sa} that we simply denote by $\mathbf{k}_U$.

Sheaves on M and M_{sal}. — Recall Definition 1.10. The functor $\rho_{\mathrm{sal}*}$ is left exact and its left adjoint ρ_{sal}^{-1} is exact since the presites underlying the sites M_{sa} and M_{sal} are the same (see [**15**, Th. 17.5.2]). Hence, we have the pairs of adjoint functors

$$\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \underset{\rho_{\mathrm{sal}}^{-1}}{\overset{\rho_{\mathrm{sal}*}}{\rightleftarrows}} \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sal}}}), \qquad \mathrm{D}^{+}(\mathbf{k}_{M_{\mathrm{sa}}}) \underset{\rho_{\mathrm{sal}}^{-1}}{\overset{\mathrm{R}\rho_{\mathrm{sal}*}}{\rightleftarrows}} \mathrm{D}^{+}(\mathbf{k}_{M_{\mathrm{sal}}}). \tag{2.2}$$

Lemma 2.2. — *The functor $\rho_{\mathrm{sal}*}$ in (2.2) is fully faithful and $\rho_{\mathrm{sal}}^{-1}\rho_{\mathrm{sal}*} \simeq \mathrm{id}$. Moreover, $\rho_{\mathrm{sal}}^{-1}\mathrm{R}\rho_{\mathrm{sal}*} \simeq \mathrm{id}$ and $\mathrm{R}\rho_{\mathrm{sal}*}$ in (2.2) is fully faithful.*

Proof. — (i) By its definition, $\rho_{\rm sal}^{-1}\rho_{{\rm sal}*}F$ is the sheaf associated with the presheaf $U \mapsto (\rho_{{\rm sal}*}F)(U) \simeq F(U)$ and this presheaf is already a sheaf.

(ii) Since $\rho_{\rm sal}^{-1}$ is exact, $\rho_{\rm sal}^{-1}\mathrm{R}\rho_{{\rm sal}*}$ is the derived functor of $\rho_{\rm sal}^{-1}\rho_{{\rm sal}*}$. □

In the sequel, if K is a compact subset of M, we set for a sheaf F on $M_{\rm sa}$ or $M_{\rm sal}$:

$$\Gamma(K;F) := \varinjlim_{K\subset U} \Gamma(U;F), \quad U \in \mathrm{Op}_{M_{\rm sa}}.$$

Lemma 2.3. — *Let $F \in \mathrm{Mod}(\mathbf{k}_{M_{\rm sal}})$. For K compact in M, we have the natural isomorphisms*

$$\Gamma(K;F) \xrightarrow{\sim} \Gamma(K;\rho_{\rm sal}^{-1}F) \xrightarrow{\sim} \Gamma(K;\rho_{\rm sl}^{-1}F).$$

Proof. — The first isomorphism follows from Proposition 1.8. The second one from [**13**, Prop. 6.6.2] since $\rho_{\rm sl}^{-1} \simeq \rho_{\rm sa}^{-1}\rho_{\rm sal}^{-1}$. □

The next result is analogue to [**13**, Prop. 6.6.2].

Proposition 2.4. — *Let $F \in \mathrm{Mod}(\mathbf{k}_{M_{\rm sal}})$. For U open in M, we have the natural isomorphism*

$$\Gamma(U;\rho_{\rm sl}^{-1}F) \simeq \varprojlim_{V\subset\subset U} \Gamma(V;F),\ V \in \mathrm{Op}_{M_{\rm sa}}.$$

Proof. — We have the chain of isomorphisms, the second one following from Lemma 2.3:

$$\Gamma(U;\rho_{\rm sl}^{-1}F) \simeq \varprojlim_{V\subset\subset U} \Gamma(\overline{V};\rho_{\rm sl}^{-1}F) \simeq \varprojlim_{V\subset\subset U} \Gamma(\overline{V};F) \simeq \varprojlim_{V\subset\subset U} \Gamma(V;F). \qquad \square$$

The next result is analogue to [**13**, Prop. 6.6.3, 6.6.4]. Since the proof of loc. cit. extends to our situation with the help of Proposition 2.4, we do not repeat it.

Proposition 2.5. — *The functor $\rho_{\rm sl}^{-1}$ admits a left adjoint that we denote by $\rho_{{\rm sl}!}$. For $F \in \mathrm{Mod}(\mathbf{k}_M)$, $\rho_{{\rm sl}!}F$ is the sheaf on $M_{\rm sal}$ associated with the presheaf $U \mapsto F(\overline{U})$. The functor $\rho_{{\rm sl}!}$ is exact and fully faithful.*

Sheaves on $M_{\rm sa}$ and $M_{\rm sal}$

Proposition 2.6. — *Let $U \in \mathrm{Op}_{M_{\rm sa}}$. Then we have $\rho_{{\rm sal}*}\mathbf{k}_{UM_{\rm sa}} \simeq \mathbf{k}_{UM_{\rm sal}}$ and $\rho_{\rm sal}^{-1}\mathbf{k}_{UM_{\rm sal}} \simeq \mathbf{k}_{UM_{\rm sa}}$.*

Proof. — The proof of [**13**, Prop. 6.3.1] gives the first isomorphism without any changes other than notational. The second isomorphism follows by Lemma 2.2. □

Proposition 2.7. — *Let $U \in \mathrm{Op}_{M_{\rm sa}}$ and let $F \in \mathrm{Mod}(\mathbf{k}_{M_{\rm sa}})$. Then*

$$\mathrm{R}\Gamma(U;\mathrm{R}\rho_{{\rm sal}*}F) \simeq \mathrm{R}\Gamma(U;F).$$

The proof goes as for Proposition 2.1.

In the sequel we shall simply denote by $\mathbf{k}_U$ the sheaf $\mathbf{k}_{U\mathscr{T}}$ for $\mathscr{T} = M_{\rm sa}$ or $\mathscr{T} = M_{\rm sal}$.

Proposition 2.8. — *Let $\mathscr{T}$ be either the site M_{sa} or the site M_{sal}. Then a presheaf F is a sheaf if and only if it satisfies:*

(i) $F(\varnothing) = 0$,

(ii) *for any $U_1, U_2 \in \mathrm{Op}_{M_{\mathrm{sa}}}$ such that $\{U_1, U_2\}$ is a covering of $U_1 \cup U_2$, the sequence $0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2)$ is exact.*

Of course, if $\mathscr{T} = M_{\mathrm{sa}}$, $\{U_1, U_2\}$ is always a covering of $U_1 \cup U_2$.

Proof. — In the case of the site M_{sa} this is Proposition 6.4.1 of [**13**]. Let F be a presheaf on M_{sal} such that (i) and (ii) are satisfied and let us prove that F is a sheaf. Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $\{U_i\}_{i\in I}$ be a linear covering of U. By Proposition 1.17 we can find a finite refinement $\{V_j\}_{j\in J}$ of $\{U_i\}_{i\in I}$ which is a regular covering of U. We choose $\sigma\colon J \to I$ such that $V_j \subset U_{\sigma(j)}$ for all $j \in J$ and we consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & F(U) \arrow[r, "u"] \arrow[d, equal] & \bigoplus_{i\in I} F(U_i) \arrow[r, "v"] \arrow[d, "a"] & \bigoplus_{i,j\in I} F(U_{ij}) \arrow[d, "b"] \\ 0 \arrow[r] & F(U) \arrow[r] & \bigoplus_{k\in J} F(V_k) \arrow[r] & \bigoplus_{k,l\in J} F(V_{kl}), \end{tikzcd} \tag{2.3}$$

where a and b are defined as follows. For $s = \{s_i\}_{i\in I} \in \bigoplus_{i\in I} F(U_i)$, we set $a(s) = \{t_k\}_{k\in J} \in \bigoplus_{k\in J} F(V_k)$ where $t_k = s_{\sigma(k)}|_{V_k}$. In the same way we set $b(\{s_{ij}\}_{i,j\in I}) = \{s_{\sigma(k)\sigma(l)}|_{V_{kl}}\}_{k,l\in J}$. The proof of [**13**, Prop. 6.4.1] applies to a regular covering in M_{sal} and we deduce that the bottom row of the diagram (2.3) is exact. It follows immediately that $\mathrm{Ker}\, u = 0$. This proves that F is a separated presheaf.

It remains to prove that $\mathrm{Ker}\, v = \mathrm{Im}\, u$. Let $s = \{s_i\}_{i\in I} \in \bigoplus_{i\in I} F(U_i)$ be such that $v(s) = 0$. By the exactness of the bottom row we can find $t \in F(U)$ such that $a(u(t) - s) = 0$. Let us check that $t|_{U_i} = s_i$ for any given $i \in I$. The family $\{U_i \cap V_k\}_{k\in J}$ is a covering of U_i in M_{sal}. Since F is separated it is enough to see that $t|_{U_i\cap V_k} = s_i|_{U_i\cap V_k}$ for all $k \in J$. Setting $W = U_i \cap V_k$, we have

$$t|_W = s_{\sigma(k)}|_W = (s_{\sigma(k)}|_{U_i\cap U_{\sigma(k)}})|_W = (s_i|_{U_i\cap U_{\sigma(k)}})|_W = s_i|_W,$$

where the first equality follows from $a(u(t) - s) = 0$ and the third one from $v(s) = 0$. $\square$

Lemma 2.9. — *Let $\mathscr{T}$ be either the site M_{sa} or the site M_{sal}. Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $\{F_i\}_{i\in I}$ be an inductive system in $\mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ indexed by a small filtrant category I. Then*

$$\varinjlim_i \Gamma(U; F_i) \xrightarrow{\sim} \Gamma(U; \varinjlim_i F_i). \tag{2.4}$$

This kind of results is well-known from the specialists (see e.g., [**13, 3**]) but for the reader's convenience, we give a proof.

Proof. — For a covering $\mathscr{S} = \{U_j\}_j$ of U set

$$\Gamma(\mathscr{S}; F) := \mathrm{Ker}\big(\prod_i F(U_i) \rightrightarrows \prod_{ij} F(U_i \cap U_j)\big).$$

Denote by "$\varinjlim$" the inductive limit in the category of presheaves and recall that $\varinjlim_i F_i$ is the sheaf associated with "$\varinjlim_i$" F_i. The presheaf "$\varinjlim_i$" F_i is separated. Denote by $\mathrm{Cov}(U)$ the family of coverings of U in $\mathscr{T}$ ordered as follows. For $\mathscr{S}_1$ and $\mathscr{S}_2$ in $\mathrm{Cov}(U)$, $\mathscr{S}_1 \preceq \mathscr{S}_2$ if $\mathscr{S}_1$ is a refinement of $\mathscr{S}_2$. Then $\mathrm{Cov}(U)^{\mathrm{op}}$ is filtrant and

$$\begin{aligned}\Gamma(U; \varinjlim_i F_i) &\simeq \varinjlim_{\mathscr{S} \in \mathrm{Cov}(U)} \Gamma(\mathscr{S}; \text{``}\varinjlim_i\text{''}\, F_i) \\ &\simeq \varinjlim_{\mathscr{S}} \varinjlim_i \Gamma(\mathscr{S}; F_i) \\ &\simeq \varinjlim_i \varinjlim_{\mathscr{S}} \Gamma(\mathscr{S}; F_i) \simeq \varinjlim_i \Gamma(U; F_i).\end{aligned}$$

Here, the second isomorphism follows from the fact that we may assume that the covering $\mathscr{S}$ is finite. □

Example 2.10. — Let $M = \mathbb{R}^2$ endowed with coordinates $x = (x_1, x_2)$. For $\varepsilon, A > 0$ we define the subanalytic open subset

$$U_{A,\varepsilon} = \{x;\ 0 < x_1 < \varepsilon,\ -Ax_1^2 < x_2 < Ax_1^2\}. \tag{2.5}$$

We define a presheaf F on M_{sal} by setting, for any $V \in \mathrm{Op}_{M_{\mathrm{sa}}}$,

$$F(V) = \begin{cases} \mathbf{k} & \text{if for any } A > 0, \text{ there exists } \varepsilon > 0 \text{ such that } U_{A,\varepsilon} \subset V, \\ 0 & \text{otherwise.} \end{cases}$$

The restriction map $F(V) \to F(V')$, for $V' \subset V$, is $\mathrm{id}_{\mathbf{k}}$ if $F(V') = \mathbf{k}$. We prove that F is a sheaf in (iii) below after the preliminary remarks (i) and (ii).

(i) For given $A, \varepsilon_0 > 0$ we have $d((\varepsilon, 0), M \setminus U_{A,\varepsilon_0}) \geq (A/4)\varepsilon^2$, for any $\varepsilon > 0$ small enough. In particular, if $F(V) = \mathbf{k}$, then

$$d((\varepsilon, 0), M \setminus V)/\varepsilon^2 \to +\infty \quad \text{when } \varepsilon \to 0. \tag{2.6}$$

(ii) Let us assume that there exist $A > 0$ and a sequence $\{\varepsilon_n\}$, $n \in \mathbb{N}$, such that $\varepsilon_n > 0$, $\varepsilon_n \to 0$ when $n \to \infty$ and V contains the closed balls $B((\varepsilon_n, 0), A\varepsilon_n^2)$ for all $n \in \mathbb{N}$. Then there exists $\varepsilon > 0$ such that V contains $\overline{U_{A,\varepsilon}} \setminus \{0\}$.

Before we prove this claim we translate the conclusion in terms of sheaf theory (in the usual site $\mathbb{R}^2$). Let $p\colon \mathbb{R}^2 \to \mathbb{R}$ be the projection $(x_1, x_2) \mapsto x_1$. Then, for $x_1 > 0$, the set $p^{-1}(x_1) \cap V \cap \overline{U_{A,\varepsilon}}$ is a finite disjoint union of intervals, say $I_1, \dots, I_N$. If $p^{-1}(x_1) \cap V$ contains $p^{-1}(x_1) \cap \overline{U_{A,\varepsilon}}$, then $N = 1$, I_1 is closed and $\mathrm{R}\Gamma(\mathbb{R}; \mathbf{k}_{I_1}) = \mathbf{k}$. In the other case none of these $I_1, \dots, I_N$ is closed and $H^0(\mathbb{R}; \mathbf{k}_{I_j}) = 0$, for all $j = 1, \dots, N$. By the base change formula we deduce that V contains $\overline{U_{A,\varepsilon}} \setminus \{0\}$ if and only if $\mathrm{R}p_*(\mathbf{k}_{V \cap \overline{U_{A,\varepsilon}}})|_{]0,\varepsilon]} \simeq \mathbf{k}_{]0,\varepsilon]}$.

We remark that, for $\varepsilon < 1$, we have $\mathrm{R}p_*(\mathbf{k}_{V\cap\overline{U_{A,\varepsilon}}})|_{]0,\varepsilon]} \simeq \mathrm{R}p_*(\mathbf{k}_{V\cap\overline{U_{A,1}}})|_{]0,\varepsilon]}$. The sheaf $\mathrm{R}p_*(\mathbf{k}_{V\cap\overline{U_{A,1}}})$ is constructible. Hence it is constant on $]0,\varepsilon]$ for $\varepsilon > 0$ small enough. Since $(\mathrm{R}p_*(\mathbf{k}_{V\cap\overline{U_{A,1}}}))_{\varepsilon_n} \simeq \mathbf{k}$ by hypothesis, the conclusion follows.

(iii) Now we check that F is a sheaf on M_{sal} with the criterion of Proposition 2.8. Let $U, U_1, U_2 \in \mathrm{Op}_{M_{\mathrm{sa}}}$ such that $\{U_1, U_2\}$ is a covering of U.

(iii-a) Let us prove that $F(U) \to F(U_1) \oplus F(U_2)$ is injective. So we assume that $F(U) = \mathbf{k}$ (otherwise this is obvious) and we prove that $F(U_1) = \mathbf{k}$ or $F(U_2) = \mathbf{k}$. Let $A > 0$. By (2.6) and (1.4) there exists $\varepsilon_0 > 0$ such that

$$\max\{d((\varepsilon,0), M \setminus U_1), d((\varepsilon,0), M \setminus U_2)\} \geq A\varepsilon^2, \quad \text{for all } \varepsilon \in]0,\varepsilon_0[.$$

Hence, for any integer $n \geq 1$, the ball $B((1/n,0), A/n^2)$ is included in U_1 or U_2. One of U_1 or U_2 must contain infinitely many such balls. By (ii) we deduce that it contains U_{A,ε_A}, for some $\varepsilon_A > 0$. When A runs over $\mathbb{N}$ we deduce that one of U_1 or U_2 contains infinitely many sets of the type U_{A,ε_A}, $A \in \mathbb{N}$. Hence $F(U_1) = \mathbf{k}$ or $F(U_2) = \mathbf{k}$.

(iii-b) Now we prove that the kernel of $F(U_1) \oplus F(U_2) \to F(U_{12})$ is $F(U)$. We see easily that the only case where this kernel could be bigger than $F(U)$ is $F(U_1) = F(U_2) = \mathbf{k}$ and $F(U_{12}) = 0$. In this case, for any $A > 0$, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $U_{A,\varepsilon_1} \subset U_1$ and $U_{A,\varepsilon_2} \subset U_2$. This gives $U_{A,\min\{\varepsilon_1,\varepsilon_2\}} \subset U_{12}$ which contradicts $F(U_{12}) = 0$.

(iv) By the definition of F we have a natural morphism $u\colon F \to \rho_{\mathrm{sal}*}\mathbf{k}_{\{0\}}$ which is surjective. We can see that $\rho_{\mathrm{sal}}^{-1}(u)$ is an isomorphism. We define $N \in \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sal}}})$ by the exact sequence

$$0 \to N \to F \to \rho_{\mathrm{sal}*}\mathbf{k}_{\{0\}} \to 0. \tag{2.7}$$

Then $\rho_{\mathrm{sal}}^{-1} N \simeq 0$ but $N \neq 0$. More precisely, for $V \in \mathrm{Op}_{M_{\mathrm{sa}}}$, we have $N(V) = 0$ if $0 \in V$ and $N(V) \xrightarrow{\sim} F(V)$ if $0 \notin V$.

2.2. Γ-acyclic sheaves

Čech complexes. — In this subsection, $\mathscr{T}$ denotes either the site M_{sa} or the site M_{sal}.

For a finite set I and a family of open subsets $\{U_i\}_{i\in I}$ we set for $\varnothing \neq J \subset I$,

$$U_J := \bigcap_{j\in J} U_j.$$

Lemma 2.11. — *Let $\mathscr{T}$ be either the site M_{sa} or the site M_{sal}. Let $\{U_1, U_2\}$ be a covering of $U_1 \cup U_2$. Then the sequence*

$$0 \to \mathbf{k}_{U_{12}} \to \mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2} \to \mathbf{k}_{U_1\cup U_2} \to 0 \tag{2.8}$$

is exact.

Proof. — The result is well-known for the site M_{sa} and the functor $\rho_{\mathrm{sal}*}$ being left exact, it remains to show that $\mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2} \to \mathbf{k}_{U_1\cup U_2}$ is an epimorphism. This follows from the fact that for any $F \in \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sal}}})$, the map $\mathrm{Hom}_{\mathbf{k}_{M_{\mathrm{sal}}}}(\mathbf{k}_{U_1\cup U_2}, F) \to \mathrm{Hom}_{\mathbf{k}_{M_{\mathrm{sal}}}}(\mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2}, F)$ is a monomorphism. □

Consider now a finite family $\{U_i\}_{i\in I}$ of objects of $\mathrm{Op}_{M_{\mathrm{sa}}}$ and let $N := |I|$. We choose a bijection $I = [1, N]$. Then we have the Čech complex in $\mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ in which the term corresponding to $|J| = 1$ is in degree 0.

$$\mathbf{k}^\bullet_{\mathscr{U}} := 0 \to \bigoplus_{J\subset I, |J|=N} \mathbf{k}_{U_J} \otimes e_J \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus_{J\subset I, |J|=1} \mathbf{k}_{U_J} \otimes e_J \to 0. \tag{2.9}$$

Recall that $\{e_J\}_{|J|=k}$ is a basis of $\bigwedge^k \mathbb{Z}^N$ and the differential is defined as usual by sending $\mathbf{k}_{U_J} \otimes e_J$ to $\bigoplus_{i\in I} \mathbf{k}_{U_{J\setminus\{i\}}} \otimes e_i \lfloor e_J$ using the natural morphism $\mathbf{k}_{U_J} \to \mathbf{k}_{U_{J\setminus\{i\}}}$.

Proposition 2.12. — *Let $\mathscr{T}$ be either the site M_{sa} or the site M_{sal}. Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $\mathscr{U} := \{U_i\}_i \in I$ be a finite covering of U in $\mathscr{T}$ (a regular covering in case $\mathscr{T} = M_{\mathrm{sal}}$). Then the natural morphism $\mathbf{k}^\bullet_{\mathscr{U}} \to \mathbf{k}_U$ is a quasi-isomorphism.*

Proof. — Recall that $N = |I|$. We may assume $I = [1, N]$. For $N = 2$ this is nothing but Lemma 2.11. We argue by induction and assume the result is proved for $N - 1$. Denote by $\mathscr{U}'$ the covering of $U' := \bigcup_{1\le i\le N-1} U_i$ by the family $\{U_i\}_{i\in[1,\dots,N-1]}$. Consider the subcomplex F_1 of $\mathbf{k}^\bullet_{\mathscr{U}}$ given by

$$F_1 := 0 \to \bigoplus_{N\in J\subset I, |J|=N} \mathbf{k}_{U_J} \otimes e_J \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus_{N\in J\subset I, |J|=1} \mathbf{k}_{U_J} \otimes e_J \to 0. \tag{2.10}$$

Note that F_1 is isomorphic to the complex $\mathbf{k}^\bullet_{\mathscr{U}'\cap U_N} \to \mathbf{k}_{U_N}$ where $\mathbf{k}_{U_N}$ is in degree 0 and we shall represent F_1 by this last complex. By [**15**, Th. 12.4.3], there is a natural morphism of complexes

$$u\colon \mathbf{k}^\bullet_{\mathscr{U}'}[-1] \to \big(\mathbf{k}^\bullet_{\mathscr{U}'\cap U_N} \to \mathbf{k}_{U_N}\big) \tag{2.11}$$

such that $\mathbf{k}^\bullet_{\mathscr{U}}$ is isomorphic to the mapping cone of u. Hence, writing the long exact sequence associated with the mapping cone of u, we are reduced, by the induction hypothesis, to prove that the morphism

$$\mathbf{k}_{U'\cap U_N} \to \mathbf{k}_{U'} \oplus \mathbf{k}_{U_N}$$

is a monomorphism and its cokernel is isomorphic to $\mathbf{k}_U$. Since $\{U', U_N\}$ is a covering of U, this follows from Lemma 2.11. □

Acyclic sheaves. — In this subsection, $\mathscr{T}$ denotes either the site M_{sa} or the site M_{sal}. In the literature, one often encounters sheaves which are $\Gamma(U;\bullet)$-acyclic for a given $U \in \mathscr{T}$ but the next definition does not seem to be frequently used.

Definition 2.13. — Let $F \in \mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$. We say that F is Γ-acyclic if we have $H^k(U;F) \simeq 0$ for all $k > 0$ and all $U \in \mathscr{T}$.

We shall give criteria in order that a sheaf F on the site $\mathscr{T}$ be Γ-acyclic.

Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $\mathscr{U} := \{U_i\}_i \in I$ be a finite covering of U in $\mathscr{T}$ (a regular covering in case $\mathscr{T} = M_{\mathrm{sal}}$). We denote by $C^\bullet(\mathscr{U};F)$ the associated Čech complex:

$$C^\bullet(\mathscr{U};F) := \mathrm{Hom}_{\mathbf{k}_{M_{\mathrm{sal}}}}(\mathbf{k}^\bullet_{\mathscr{U}}, F). \tag{2.12}$$

One can write more explicitly this complex as the complex:

$$0 \to \bigoplus_{J\subset I,|J|=1} F(U_J)\otimes e_J \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus_{J\subset I,|J|=N} F(U_J)\otimes e_J \to 0 \tag{2.13}$$

where the differential d is obtained by sending $F(U_J)\otimes e_J$ to $\bigoplus_{i\in I} F(U_J\cap U_i)\otimes e_i\wedge e_J$.

Proposition 2.14. — *Let $\mathscr{T}$ be either the site M_{sa} or the site M_{sal} and let $F\in\mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$. The conditions below are equivalent.*

(i) *For any $\{U_1,U_2\}$ which is a covering of $U_1\cup U_2$, the sequence $0\to F(U_1\cup U_2)\to F(U_1)\oplus F(U_2)\to F(U_1\cap U_2)\to 0$ is exact.*

(ii) *The sheaf F is Γ-acyclic.*

(iii) *For any exact sequence in $\mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$*

$$G^\bullet := 0 \to \bigoplus_{i_0\in A_0} \mathbf{k}_{U_{i_0}} \to \cdots \to \bigoplus_{i_N\in A_N} \mathbf{k}_{U_{i_N}} \to 0 \tag{2.14}$$

the sequence $\mathrm{Hom}_{\mathbf{k}_{\mathscr{T}}}(G^\bullet,F)$ is exact.

(iv) *For any finite covering $\mathscr{U}$ of U (regular covering in case $\mathscr{T}=M_{\mathrm{sal}}$), the morphism $F(U)\to C^\bullet(\mathscr{U};F)$ is a quasi-isomorphism.*

Proof. — (i)$\Rightarrow$(ii) (a) Let $U\in\mathrm{Op}_{M_{\mathrm{sa}}}$. Let us first show that for any exact sequence of sheaves $0\to F\xrightarrow{\varphi} F'\xrightarrow{\psi} F''\to 0$ and any $U\in\mathrm{Op}_{M_{\mathrm{sa}}}$, the sequence $0\to F(U)\to F'(U)\to F''(U)\to 0$ is exact. Let $s''\in F''(U)$. By the exactness of the sequence of sheaves, there exists a finite covering $U=\bigcup_{i=1}^N U_i$ and $s'_i\in F'(U_i)$ such that $\psi(s'_i)=s''|_{U_i}$. In case $\mathscr{T}=M_{\mathrm{sal}}$, we may assume that the covering is regular by Proposition 1.17. For $k=1,\dots,N$, we set $V_k=\bigcup_{i=1}^k U_i$. Let us prove by induction on k that there exists $t'_k\in F'(V_k)$ such that $\psi(t'_k)=s''|_{V_k}$. Starting with $t'_1=s'_1$ we assume that we have found t'_k. Since our covering is regular, $\{V_k,U_{k+1}\}$ is a covering of V_{k+1}. We set for short $W=V_k\cap U_{k+1}$. We have $\psi(t'_k|_W)=\psi(s'_{k+1}|_W)$. Hence there exists $s\in F(W)$ such that $\varphi(s)=t'_k|_W-s'_{k+1}|_W$. By hypothesis (i) there exists $s_V\in F(V_k)$ and $s_U\in F(U_{k+1})$ such that $s=s_V|_W-s_U|_W$. Setting $t'_V=t'_k-\varphi(s_V)$ and $s'_U=s'_{k+1}-\varphi(s_U)$ we obtain $t'_U|_W=s'_V|_W$ and we can glue $t'_U|_W$ and $s'_V|_W$ into $t'_{k+1}\in F'(V_{k+1})$. We check easily that $\psi(t'_{k+1})=s''|_{V_{k+1}}$ and the induction proceeds.

(i)$\Rightarrow$(ii) (b) Denote by $\mathscr{J}$ the full additive subcategory of $\mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ consisting of sheaves satisfying the condition (i). We shall show that the category $\mathscr{J}$ is $\Gamma(U;\bullet)$-injective for all $U\in\mathrm{Op}_{M_{\mathrm{sa}}}$. The category $\mathscr{J}$ contains the injective sheaves. By the first part of the proof, it thus remains to show that, for any short exact sequence of sheaves $F^\bullet:=0\to F'\to F\to F''\to 0$, if both F' and F belong to $\mathscr{J}$, then F'' belongs to $\mathscr{J}$.

Let U_1,U_2 as in (i) and denote by $\mathbf{k}^\bullet_{\mathscr{U}}$ the exact sequence $0\to\mathbf{k}_{U_1\cap U_2}\to\mathbf{k}_{U_1}\oplus\mathbf{k}_{U_2}\to\mathbf{k}_{U_1\cup U_2}\to 0$. Consider the double complex $\mathrm{Hom}_{\mathbf{k}_{\mathscr{T}}}(\mathbf{k}^\bullet_{\mathscr{U}},F^\bullet)$. By the preceding result all rows and columns except at most one (either one row or one column depending how one writes the double complex) are exact. It follows that the double complex is exact.

(ii)$\Rightarrow$(iii) Consider an injective resolution $I^\bullet$ of F, that is, a complex $I^\bullet$ of injective sheaves such that the sequence $I^{\bullet,+} := 0 \to F \to I^\bullet$ is exact. The hypothesis implies that $\Gamma(W; I^{\bullet,+})$ remains exact for all $W \in \mathrm{Op}_{M_{\mathrm{sa}}}$. Then the argument goes as in the proof of (i)$\Rightarrow$(ii) (b). Recall that $G^\bullet$ denotes the complex of (2.14) and consider the double complex $\mathrm{Hom}_{\mathbf{k}_{\mathscr{T}}}(G^\bullet, I^{\bullet,+})$. Then all its rows and columns except one (either one row or one column depending how one writes the double complex) will be exact. It follows that all rows and columns are exact.

(iii)$\Rightarrow$(iv) follows from Proposition 2.12.

(iv)$\Rightarrow$(i) is obvious. $\square$

Corollary 2.15. — *Let $\mathscr{T}$ be either the site M_{sa} or the site M_{sal}. A small filtrant inductive limit of Γ-acyclic sheaves is Γ-acyclic.*

Proof. — Since small filtrant inductive limits are exact in $\mathrm{Mod}(\mathbf{k})$, the family of sheaves satisfying condition (i) of Proposition 2.14 is stable by such limits by Lemma 2.9. $\square$

Definition 2.16. — Let $\mathscr{T}$ be either the site M_{sa} or the site M_{sal}. One says that $F \in \mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ is flabby if for any U and V in $\mathrm{Op}_{M_{\mathrm{sa}}}$ with $V \subset U$, the natural morphism $F(U) \to F(V)$ is surjective.

Lemma 2.17. — *Let $\mathscr{T}$ be either the site M_{sa} or the site M_{sal}.*

(i) *Injective sheaves are flabby.*
(ii) *Flabby sheaves are Γ-acyclic.*
(iii) *The category of flabby sheaves is stable by small filtrant inductive limits.*

Proof. — (i) Let F be an injective sheaf and let U and V in $\mathrm{Op}_{M_{\mathrm{sa}}}$ with $V \subset U$. Recall that the sequence $0 \to \mathbf{k}_V \to \mathbf{k}_U$ is exact. Applying the functor $\mathrm{Hom}_{\mathbf{k}_{\mathscr{T}}}(\,\bullet\,, F)$ we get the result.

(ii) If $F \in \mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ is flabby then it satisfies condition (i) of Proposition 2.14.

(iii) The proof of Corollary 2.15 also works in this case. $\square$

2.3. The functor $\rho^!_{\mathrm{sal}}$. — In this section we make an essential use of the Brown representability theorem (see for example [**15**, Th 14.3.1]).

Direct sums in derived categories. — In this subsection, we state and prove some elementary results that we shall need, some of them being well-known from the specialists.

Lemma 2.18. — *Let $\mathscr{C}$ be a Grothendieck category and let $d \in \mathbb{Z}$. Then the cohomology functor H^d and the truncation functors $\tau^{\leq d}$ and $\tau^{\geq d}$ commute with small direct sums in $\mathrm{D}(\mathscr{C})$. In other words, if $\{F_i\}_{i\in I}$ is a small family of objects of $\mathrm{D}(\mathscr{C})$, then*

$$\bigoplus_i \tau^{\leq d} F_i \xrightarrow{\sim} \tau^{\leq d}(\bigoplus_i F_i) \tag{2.15}$$

and similarly with $\tau^{\geq d}$ and H^d.

Proof. — (i) The case of H^d follows from [**15**, Prop. 10.2.8, Prop. 14.1.1].
(ii) The morphism in (2.15) is well-defined and it is enough to check that it induces an isomorphism on the cohomology. This follows from (i) since for any object $Y \in \mathrm{D}(\mathscr{C})$, $H^j(\tau^{\leq d} Y)$ is either 0 or $H^j(Y)$. □

Lemma 2.19. — *Let $\mathscr{C}$ and $\mathscr{C}'$ be two Grothendieck categories and let $\rho\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor. Let I be a small category. Assume*

(i) *I is either filtrant or discrete,*
(ii) *ρ commutes with inductive limits indexed by I,*
(iii) *inductive limits indexed by I of injective objects in $\mathscr{C}$ are acyclic for the functor ρ.*

Then for all $j \in \mathbb{Z}$, the functor $R^j\rho\colon \mathscr{C} \to \mathscr{C}'$ commutes with inductive limits indexed by I.

Proof. — Let $\alpha\colon I \to \mathscr{C}$ be a functor. Denote by $\mathscr{I}$ the full additive subcategory of $\mathscr{C}$ consisting of injective objects. It follows for example from [**15**, Cor. 9.6.6] that there exists a functor $\psi\colon I \to \mathscr{I}$ and a morphism of functors $\alpha \to \psi$ such that for each $i \in I$, $\alpha(i) \to \psi(i)$ is a monomorphism. Therefore one can construct a functor $\Psi\colon I \to \mathrm{C}^+(\mathscr{I})$ and a morphism of functor $\alpha \to \Psi$ such that for each $i \in I$, $\alpha(i) \to \Psi(i)$ is a quasi-isomorphism. Set $X_i = \alpha(i)$ and $G_i^\bullet = \Psi(i)$. We get a qis $X_i \to G_i^\bullet$, hence a qis

$$\varinjlim_i X_i \to \varinjlim_i G_i^\bullet .$$

On the other hand, we have

$$\begin{aligned}\varinjlim_i R^j\rho(X_i) &\simeq \varinjlim_i H^j(\rho(G_i^\bullet))\\ &\simeq H^j\rho(\varinjlim_i G_i^\bullet)\end{aligned}$$

where the second isomorphism follows from the fact that H^j commutes with direct sums and with filtrant inductive limits. Then the result follows from hypothesis (iii). □

Lemma 2.20. — *We make the same hypothesis as in Lemma 2.19. Let $-\infty < a \leq b < \infty$, let I be a small set and let $X_i \in \mathrm{D}^{[a,b]}(\mathscr{C})$. Then*

$$\bigoplus_i R\rho(X_i) \xrightarrow{\sim} R\rho(\bigoplus_i X_i). \tag{2.16}$$

Proof. — The morphism in (2.16) is well-defined and we have to prove it is an isomorphism. If $b = a$, the result follows from Lemma 2.19. The general case is deduced by induction on $b - a$ by considering the distinguished triangles

$$H^a(X_i)\,[-a] \to X_i \to \tau^{\geq a+1} X_i \xrightarrow{+1} . \qquad \square$$

Proposition 2.21. — *Let $\mathscr{C}$ and $\mathscr{C}'$ be two Grothendieck categories and let $\rho\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor. Assume that*

(a) *ρ has finite cohomological dimension,*
(b) *ρ commutes with small direct sums,*
(c) *small direct sums of injective objects in $\mathscr{C}$ are acyclic for the functor ρ.*

Then

(i) *the functor $R\rho\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}(\mathscr{C}')$ commutes with small direct sums,*
(ii) *the functor $R\rho\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}(\mathscr{C}')$ admits a right adjoint $\rho^!\colon \mathrm{D}(\mathscr{C}') \to \mathrm{D}(\mathscr{C})$,*
(iii) *the functor $\rho^!$ induces a functor $\rho^!\colon \mathrm{D}^+(\mathscr{C}') \to \mathrm{D}^+(\mathscr{C})$.*

Proof. — (i) Let $\{X_i\}_{i\in I}$ be a family of objects of $\mathrm{D}(\mathscr{C})$. It is enough to check that the natural morphism in $\mathrm{D}(\mathscr{C}')$

$$\bigoplus_{i\in I} R\rho(X_i) \to R\rho(\bigoplus_{i\in I} X_i) \tag{2.17}$$

induces an isomorphism on the cohomology groups. Assume that ρ has cohomological dimension $\leq d$. For $X \in \mathrm{D}(\mathscr{C})$ and for $j \in \mathbb{Z}$, we have

$$\tau^{\geq j} R\rho(X) \simeq \tau^{\geq j} R\rho(\tau^{\geq j-d-1} X).$$

The functor ρ being left exact we get for $k \geq j$:

$$H^k R\rho(X) \simeq H^k R\rho(\tau^{\leq k}\tau^{\geq j-d-1} X). \tag{2.18}$$

We have the sequence of isomorphisms:

$$\begin{aligned} H^k R\rho(\bigoplus_i X_i) &\simeq H^k R\rho(\tau^{\leq k}\tau^{\geq j-d-1} \bigoplus_i X_i) \\ &\simeq H^k R\rho(\bigoplus_i \tau^{\leq k}\tau^{\geq j-d-1} X_i) \\ &\simeq \bigoplus_i H^k R\rho(\tau^{\leq k}\tau^{\geq j-d-1} X_i) \\ &\simeq \bigoplus_i H^k R\rho(X_i). \end{aligned}$$

The first and last isomorphisms follow from (2.18).
The second isomorphism follows from Lemma 2.18.
The third isomorphism follows from Lemma 2.20.
(ii) follows from (i) and the Brown representability theorem (see for example [**15**, Th 14.3.1]).

(iii) This follows from hypothesis (a) and (the well-known) Lemma 2.22 below. $\square$

Lemma 2.22. — *Let $\rho\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor between two Grothendieck categories. Assume that $\rho\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}(\mathscr{C}')$ admits a right adjoint $\rho^!\colon \mathrm{D}(\mathscr{C}') \to \mathrm{D}(\mathscr{C})$ and assume moreover that ρ has finite cohomological dimension. Then the functor $\rho^!$ sends $\mathrm{D}^+(\mathscr{C}')$ to $\mathrm{D}^+(\mathscr{C})$.*

Proof. — By the hypothesis, we have for $X \in \mathrm{D}(\mathscr{C})$ and $Y \in \mathrm{D}(\mathscr{C}')$

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{C}')}(\rho(X), Y) \simeq \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, \rho^!(Y)).$$

Assume that the cohomological dimension of the functor ρ is $\leq r$. Let $Y \in \mathrm{D}^{\geq 0}(\mathscr{C}')$. Then $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, \rho^!(Y)) \simeq 0$ for all $X \in \mathrm{D}^{<-r}(\mathscr{C})$. This means that $\rho^!(Y)$ belongs to the right orthogonal to $\mathrm{D}^{<-r}(\mathscr{C})$ and this implies that $\rho^!(Y) \in \mathrm{D}^{\geq -r}(\mathscr{C}')$. □

The functor $\mathrm{R}\Gamma(U; \bullet\,)$

Lemma 2.23. — *Let $\mathscr{T}$ be either the site M_{sa} or the site M_{sal} and let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. Let I be a small filtrant category and $\alpha\colon I \to \mathrm{Mod}(\mathbf{k}_{\mathscr{T}})$ a functor. Set for short $F_i = \alpha(i)$. Then for any $j \in \mathbb{Z}$*

$$\varinjlim_i H^j\mathrm{R}\Gamma(U; F_i) \xrightarrow{\ \sim\ } H^j\mathrm{R}\Gamma(U; \varinjlim_i F_i). \tag{2.19}$$

Proof. — By Lemma 2.9, the functor $\Gamma(U; \bullet\,)$ commutes with small filtrant inductive limits and such limits of injective objects are $\Gamma(U; \bullet\,)$-acyclic by Lemma 2.17. Hence, we may apply Lemma 2.19. □

Proposition 2.24. — *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. The functor $\Gamma(U; \bullet\,)\colon \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \to \mathrm{Mod}(\mathbf{k})$ has cohomological dimension $\leq \dim M$.*

Proof. — We know that if $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$, then $H^j\mathrm{R}\Gamma(U; F) \simeq 0$ for $j > \dim M$. Since any $F \in \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$ is a small filtrant inductive limit of constructible sheaves, the result follows from Lemma 2.23. □

Corollary 2.25. — *Let $\mathscr{J}$ be the subcategory of $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$ consisting of sheaves which are Γ-acyclic. For any $F \in \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$, there exists an exact sequence $0 \to F \to F^0 \to \cdots \to F^n \to 0$ where $n = \dim M$ and the F^j's belong to $\mathscr{J}$.*

Proof. — Consider a resolution $0 \to F \to I^0 \xrightarrow{d^0} I^1 \to \cdots$ with the I^j's injective and define $F^j = I^j$ for $j \leq n-1$, $F^j = 0$ for $j > n$ and $F^n = \mathrm{Ker}\, d^n$. It follows from Proposition 2.24 that F^n is Γ-acyclic. □

Proposition 2.26. — *Let I be a small set and let $F_i \in \mathrm{D}(\mathbf{k}_{M_{\mathrm{sa}}})$ $(i \in I)$. For $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, we have the natural isomorphism*

$$\bigoplus_{i\in I} \mathrm{R}\Gamma(U; F_i) \xrightarrow{\ \sim\ } \mathrm{R}\Gamma(U; \bigoplus_{i\in I} F_i) \text{ in } \mathrm{D}(\mathbf{k}). \tag{2.20}$$

Proof. — The functor $\Gamma(U;\bullet)$ has finite cohomological dimension by Proposition 2.24, it commutes with small direct sums by Lemma 2.9 and inductive limits of injective objects are $\Gamma(U;\bullet)$-acyclic by Lemma 2.17. Hence, we may apply Proposition 2.21. $\square$

The functor $\mathrm{R}\rho_{\mathrm{sal}*}$

Lemma 2.27. — *Let $\mathscr{J}$ be the subcategory of* $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$ *consisting of sheaves which are Γ-acyclic. The category $\mathscr{J}$ is $\rho_{\mathrm{sal}*}$-injective* (*see* [**11**, Def. 1.8.2]).

Proof. — Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$.
(i) We see easily that if both F' and F belong to $\mathscr{J}$, then F'' belongs to $\mathscr{J}$.
(ii) It remains to prove that if $F' \in \mathscr{J}$, then the sequence $0 \to \rho_{\mathrm{sal}*}F' \to \rho_{\mathrm{sal}*}F \to \rho_{\mathrm{sal}*}F'' \to 0$ is exact. Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. By Proposition 2.7 and the hypothesis, the sequence $0 \to \rho_{\mathrm{sal}*}F'(U) \to \rho_{\mathrm{sal}*}F(U) \to \rho_{\mathrm{sal}*}F''(U) \to 0$ is exact. $\square$

Applying Corollary 2.25, we get:

Proposition 2.28. — *The functor $\rho_{\mathrm{sal}*}$ has cohomological dimension $\leq \dim M$.*

Proposition 2.29. — *Let I be a small set and let $F_i \in \mathrm{D}(\mathbf{k}_{M_{\mathrm{sa}}})$ $(i \in I)$. We have the natural isomorphism*

$$\bigoplus_{i\in I} \mathrm{R}\rho_{\mathrm{sal}*}F_i \xrightarrow{\sim} \mathrm{R}\rho_{\mathrm{sal}*}(\bigoplus_{i\in I} F_i) \ \textit{in}\ \mathrm{D}(\mathbf{k}_{M_{\mathrm{sal}}}). \tag{2.21}$$

Proof. — By Proposition 2.28, the functor $\rho_{\mathrm{sal}*}$ has finite cohomological dimension and by Lemma 2.9 it commutes with small direct sums. Moreover, inductive limits of injective objects are $\rho_{\mathrm{sal}*}$-acyclic by Lemmas 2.27 and 2.17. Hence, we may apply Proposition 2.21 (i). $\square$

Theorem 2.30. — (i) *The functor* $\mathrm{R}\rho_{\mathrm{sal}*}\colon \mathrm{D}(\mathbf{k}_{M_{\mathrm{sa}}}) \to \mathrm{D}(\mathbf{k}_{M_{\mathrm{sal}}})$ *admits a right adjoint* $\rho^!_{\mathrm{sal}}\colon \mathrm{D}(\mathbf{k}_{M_{\mathrm{sal}}}) \to \mathrm{D}(\mathbf{k}_{M_{\mathrm{sa}}})$.
(ii) *The functor $\rho^!_{\mathrm{sal}}$ induces a functor* $\rho^!_{\mathrm{sal}}\colon \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sal}}}) \to \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sa}}})$.

Proof. — These results follow from Propositions 2.21, 2.29 and 2.28. $\square$

Corollary 2.31. — *One has an isomorphism of functors on* $\mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sa}}})$:

$$\mathrm{id} \xrightarrow{\sim} \rho^!_{\mathrm{sal}}\mathrm{R}\rho_{\mathrm{sal}*}. \tag{2.22}$$

Proof. — This follows from the fact that $(\mathrm{R}\rho_{\mathrm{sal}*}, \rho^!_{\mathrm{sal}})$ is a pair of adjoint functors and that $\mathrm{R}\rho_{\mathrm{sal}*}$ is fully faithful by Lemma 2.2. $\square$

Remark 2.32. — (i) We don't know if the category M_{sal} has finite flabby dimension. We don't even know if for any $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{\mathrm{sal}}})$ and any $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, we have $\mathrm{R}\Gamma(U;F) \in \mathsf{D}^{\mathrm{b}}(\mathbf{k})$.
(ii) We don't know if the functor $\rho^!_{\mathrm{sal}}\colon \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sal}}}) \to \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sa}}})$ constructed in Theorem 2.30 induces a functor $\rho^!_{\mathrm{sal}}\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{\mathrm{sal}}}) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{\mathrm{sa}}})$.

2.4. Open sets with Lipschitz boundaries

Normal cones and Lipschitz boundaries. — In this paragraph $\mathbb{R}^n$ is equipped with coordinates (x', x_n), $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$.

Definition 2.33. — We say that $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ has Lipschitz boundary or simply that U is Lipschitz if, for any $x \in \partial U$, there exist an open neighborhood V of x and a bi-Lipschitz subanalytic homeomorphism $\psi\colon V \xrightarrow{\sim} W$ with W an open subset of $\mathbb{R}^n$ such that $\psi(V \cap U) = W \cap \{x_n > 0\}$.

Remark 2.34. — (i) The property of being Lipschitz is local and thus the preceding definition extends to subanalytic but not necessarily relatively compact open subsets of M.
(ii) If U_i is Lipschitz in M_i $(i = 1, 2)$ then $U_1 \times U_2$ is Lipschitz in $M_1 \times M_2$.
(iii) If U is Lipschitz and $x \in \partial U$, there exist a constant $C > 0$ and a sequence $\{y_n\}_{n \in \mathbb{N}}$, $y_n \in U$, such that $d(y_n, x) \to 0$ and $d(y_n, x) \leq C d(y_n, \partial U)$, for all $n \in \mathbb{N}$ (in the notations of the definition, assume $\psi(x) = (x', 0)$ and set $y_n = \psi^{-1}(x', 1/n)$).

Example 2.35. — (i) Lemma 2.37 below will provide many examples of Lipschitz open sets.
(ii) Let (x, y) denotes the coordinates on $\mathbb{R}^2$. Using (iii) of Remark 2.34 we see that the open set $U = \{(x, y); 0 < y < x^2\}$ is not Lipschitz.

Lemma 2.36. — *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. We assume that, for any $x \in \partial U$, there exist an open neighborhood V of x and a bi-analytic isomorphism $\psi\colon V \xrightarrow{\sim} W$ with W an open subset of $\mathbb{R}^n$ such that $\psi(V \cap U) = W \cap \{(x', x_n);\ x_n > \varphi(x')\}$ for a Lipschitz subanalytic function φ. Then U is Lipschitz.*

Proof. — We define $\psi_1\colon \mathbb{R}^n \to \mathbb{R}^n$, $(x', x_n) \mapsto (x', x_n - \varphi(x'))$. Then ψ_1 is a bi-Lipschitz subanalytic homeomorphism and we have $(\psi_1 \circ \psi)(V \cap U) = \psi_1(W) \cap \{x_n > 0\}$. Hence U is Lipschitz. □

Lemma 2.37. — *Let $\mathbb{V}$ be a vector space and let γ be a subanalytic proper closed convex cone with non empty interior. Let $U \in \mathrm{Op}_{\mathbb{V}_{\mathrm{sa}}}$. Then the open set $U + \gamma$ has Lipschitz boundary.*

Proof. — Let $p \in \partial(U + \gamma)$. We identify $\mathbb{V}$ with $\mathbb{R}^n$ so that p is the origin and γ contains the cone $\gamma_0 = \{(x', x_n);\ x_n > \|x'\|\}$. We have in particular

$$\gamma_0 \subset (U + \gamma) \subset (\mathbb{R}^n \setminus (-\gamma_0)). \tag{2.23}$$

For $x' \in \mathbb{R}^{n-1}$ we set $l_{x'} = (U + \gamma) \cap (\{x'\} \times \mathbb{R})$. Then $l_{x'} = l_{x'} + [0, +\infty[$. By (2.23) we also have $l_{x'} \neq \varnothing$ and $l_{x'} \neq \mathbb{R}$. Hence we can write $l_{x'} =]\varphi(x'), +\infty[$, for a well-defined function $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ whose graph is $\partial(U + \gamma)$.

Let us prove that φ is Lipschitz. Let $x' \in \mathbb{R}^{n-1}$ and let us set $q = (x', \varphi(x')) \in \partial(U + \gamma)$. We have the similar inclusion as (2.23), $(q + \gamma_0) \subset (U + \gamma) \subset (\mathbb{R}^n \setminus (q - \gamma_0))$. Hence $\partial(U + \gamma) \subset (\mathbb{R}^n \setminus ((q + \gamma_0) \cup (q - \gamma_0)))$. For any $y' \in \mathbb{R}^{n-1}$ we have $(y', \varphi(y')) \in \partial(U + \gamma)$

and the last inclusion translates into $|\varphi(y') - \varphi(x')| \leq \|y' - x'\|$. Hence φ is Lipschitz and $U + \gamma$ is Lipschitz by Lemma 2.36. $\square$

We refer to [**11**, Def 4.1.1] for the definition of the normal cone $C(A, B)$ associated with two subsets A and B of M.

Definition 2.38. — (See [**11**, § 5.3].) Let S be a subset of M. The strict normal cone $N_x(S)$ and the conormal cone $N_x^*(S)$ of S at $x \in M$ as well as the strict normal cone $N(S)$ and the conormal cone $N^*(S)$ of S are given by

$$\begin{aligned} N_x(S) &= T_xM \setminus C(M \setminus S, S), \text{ an open cone in } T_xM, \\ N_x^*(S) &= N_x(S)^\circ \text{ (where } ^\circ \text{ denotes the polar cone)}, \\ N(S) &= \bigcup_{x \in M} N_x(S), \text{ an open convex cone in } TM, \\ N^*(S) &= \bigcup_{x \in M} N_x^*(S). \end{aligned}$$

By loc. cit. Prop. 5.3.7, we have:

Lemma 2.39. — *Let U be an open subset of M and let $x \in \partial U$. Then the conditions below are equivalent:*

(i) *$N_x(U)$ is non empty,*
(ii) *$N_y(U)$ is non empty for all y in a neighborhood of x,*
(iii) *$N_x^*(U)$ is contained in a closed convex proper cone with non empty interior in T_x^*M,*
(iv) *there exists a local chart in a neighborhood of x such that identifying M with an open subset of $\mathbb{V}$, there exists a closed convex proper cone with non empty interior γ in $\mathbb{V}$ such that U is γ-open in an open neighborhood W of x, that is,*

$$W \cap ((U \cap W) + \gamma) \subset U.$$

Definition 2.40. — We shall say that an open subset U of M satisfies a cone condition if for any $x \in \partial U$, $N_x(U)$ is non empty.

By Lemmas 2.37 and 2.39 we have:

Proposition 2.41. — *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. If U satisfies a cone condition, then U is Lipschitz.*

Remark 2.42. — One shall be aware that our definition of being Lipschitz differs from that of Lebeau in [**21**]. By Lemma 2.36, if U is Lipschitz in Lebeau's sense, then it is Lipschitz in our sense.

A vanishing theorem. — The next theorem is a key result for this paper and its proof is due to A. Parusinski [**26**].

Theorem 2.43. — (A. Parusinski) *Let $V \in \mathrm{Op}_{M_{\mathrm{sa}}}$. Then there exists a finite covering $V = \bigcup_{j\in J} V_j$ with $V_j \in \mathrm{Op}_{M_{\mathrm{sa}}}$ such that the family $\{V_j\}_{j\in J}$ is a covering of V in M_{sal} and moreover $H^k(V_j;\mathbf{k}_M) \simeq 0$ for all $k > 0$ and all $j \in J$.*

Recall that one denotes by $\rho_{\mathrm{sal}}\colon M_{\mathrm{sa}} \to M_{\mathrm{sal}}$ the natural morphism of sites.

Lemma 2.44. — *We have $\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_{M_{\mathrm{sa}}} \simeq \mathbf{k}_{M_{\mathrm{sal}}}$.*

Proof. — The sheaf $H^k(\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_{M_{\mathrm{sa}}})$ is the sheaf associated with the presheaf $U \mapsto H^k(U;\mathbf{k}_{M_{\mathrm{sa}}})$. This sheaf if zero for $k > 0$ by Theorem 2.43. □

Lemma 2.45. — *Let $M = \mathbb{R}^n$ and set $U =]0,+\infty[\, \times \mathbb{R}^{n-1}$. Then we have $\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_U \simeq \mathbf{k}_U$.*

Proof. — (i) The sheaf $H^k(\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_U)$ is the sheaf associated with the presheaf $V \mapsto H^k(V;\mathbf{k}_U)$. Hence it is enough to show that any $V \in \mathrm{Op}_{M_{\mathrm{sa}}}$ admits a finite covering $V = \bigcup_{j\in J} V_j$ in M_{sal} such that $H^k(V_j;\mathbf{k}_U) \simeq 0$ for all $k > 0$. We assume that the distance d is a subanalytic function. Let us set $V_{<0} = V \cap (]-\infty,0[\, \times \mathbb{R}^{n-1})$ and $V' = V_{<0}^{1,V}$, where we use the notation (1.6) with $\varepsilon = 1$. In our case we can write (1.6) as follows

$$V' = \{x \in V;\ d(x, V \setminus U) < d(x, M \setminus V)\}.$$

This is a subanalytic open subset of V. By Lemma 1.13 we have

$$\{V', V \cap U\} \text{ is a covering of } V \text{ in } M_{\mathrm{sal}}. \tag{2.24}$$

(ii) Let us prove that $\mathrm{R}\Gamma(V';\mathbf{k}_U) \simeq 0$. We denote by (x_1, x') the coordinates on $M = \mathbb{R}^n$. For $x = (x_1, x')$ with $x_1 \geq 0$, we have $d(x, V \setminus U) \geq d(x, M \setminus U) = x_1$. If $(x_1, x') \in V'$ we obtain $d(x, M \setminus V) > x_1$, hence $\overline{B(x, x_1)} \subset V$, where $B(x, x_1)$ is the ball with center x and radius x_1. This proves that $V' \cap \overline{U}$ is contained in the right hand side of the following equality

$$V' \cap \overline{U} = \{x = (x_1, x') \in V;\ x_1 \geq 0 \text{ and } \overline{B(x, x_1)} \subset V\} \tag{2.25}$$

and the reverse inclusion is easily checked. It follows that, if $(x_1, x') \in V' \cap \overline{U}$, then $(y_1, x') \in V' \cap \overline{U}$, for all $y_1 \in [0, x_1]$. Let $q\colon \mathbb{R}^n \to \{0\} \times \mathbb{R}^{n-1}$ be the projection. We deduce:

(a) q maps $V' \cap \overline{U}$ onto $V \cap \partial U$,

(b) $q^{-1}(x) \cap V' \cap U$ is an open interval, for any $x = (0, x') \in V \cap \partial U$.

For any $c < 0 < d$ we have $\mathrm{R}\Gamma(]c,d[;\mathbf{k}_{]0,d[}) \simeq 0$. Hence (a) and (b) give $\mathrm{R}q_*\mathrm{R}\Gamma_{V'}\mathbf{k}_U \simeq 0$, by the base change formula, and we obtain $\mathrm{R}\Gamma(V';\mathbf{k}_U) \simeq \mathrm{R}\Gamma(\mathbb{R}^{n-1};\mathrm{R}q_*\mathrm{R}\Gamma_{V'}\mathbf{k}_U) \simeq 0$.

(iii) By Theorem 2.43 we can choose a finite covering of $V \cap U$ in M_{sal}, say $\{W_j\}_{j\in J}$, such that $H^k(W_j;\mathbf{k}_U) \simeq 0$ for all $k > 0$. By (2.24) the family $\{V', \{W_j\}_{j\in J}\}$ is a covering of V in M_{sal}. By (ii) this covering satisfies the required condition in (i), which proves the result. □

We need to extend Definition 2.33.

Definition 2.46. — We say that $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ is weakly Lipschitz if for each $x \in M$ there exists a neighborhood $V \in \mathrm{Op}_{M_{\mathrm{sa}}}$ of x, a finite set I and $U_i \in \mathrm{Op}_{M_{\mathrm{sa}}}$, $i \in I$, such that $U \cap V = \bigcup_{i\in I} U_i$ and

$$\begin{cases}\text{for all } \varnothing \neq J \subset I, \text{ the set } U_J = \bigcap_{j\in J} U_j \text{ is a disjoint union} \\ \text{of Lipschitz open sets.}\end{cases} \tag{2.26}$$

By its definition, the property of being weakly Lipschitz is local on M.

Example 2.47. — The open subset $U = \mathbb{R}^2 \setminus \{0\}$ of $\mathbb{R}^2$ is not Lipschitz but it is weakly Lipschitz: setting $U_\pm = \{(x,y) \in \mathbb{R}^2;\ \pm y > -|x|\}$ we have $U = U_+ \cup U_-$ and U_+, U_-, $U_+ \cap U_-$ are disjoint unions of Lipschitz open subsets.

Proposition 2.48. — *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and consider a finite family of smooth submanifolds $\{Z_i\}_{i\in I}$, closed in a neighborhood of $\overline{U}$. Set $Z = \bigcup_{i\in I} Z_i$. Assume that*

(a) *U is Lipschitz,*
(b) *$Z_i \cap Z_j \cap \partial U = \varnothing$ for $i \neq j$, ∂U is smooth in a neighborhood of $Z \cap \partial U$ and the intersection is transversal,*
(c) *in a neighborhood of each point of $Z \cap U$ there exist a local coordinate system $(x_1, \dots, x_n)$ and for each $i \in I$, a subset I_i of $\{1, \dots, n\}$ such that $Z_i = \bigcap_{j\in I_i} \{x; x_j = 0\}$.*

Then $U \setminus Z$ is weakly Lipschitz.

Proof. — Since the property of being weakly Lipschitz is local on M, it is enough to prove the result in a neighborhood of each point $p \in \overline{U}$.
(i) Assume $p \in \partial U$. We choose a local coordinate system $(x_1, \dots, x_n)$ centered at p such that $U = \{x; x_n > 0\}$ and $Z = \{x; x_1 = \cdots = x_r = 0\}$ (with $r < n$). For $1 \leq i \leq r$, define $U_i = \{x; x_n > 0, x_i \neq 0\}$. Then the family $\{U_i\}_{i=1,\dots,r}$ satisfies (2.26).
(ii) Assume $p \in U$. We choose a local coordinate system $(x_1, \dots, x_n)$ such that $Z_i = \bigcap_{j_i \in I_i} \{x; x_{j_i} = 0\}$. For each $j_i \in I_i$ and $\varepsilon_i = \pm 1$, define $U_i^{\varepsilon_i, j_i} = \{x;\ \varepsilon_i\, x_{j_i} > 0\}$. Set $A = \prod_{i\in I} (\{\pm 1\} \times I_i)$ and for $\alpha \in A$ set $U_\alpha = \bigcap_{i\in I} U_i^{\alpha_i}$. Then the family $\{U_\alpha\}_{\alpha\in A}$ satisfies (2.26). □

Theorem 2.49. — *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and assume that U is weakly Lipschitz. Then*

(i) *$\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_{UM_{\mathrm{sa}}} \simeq \rho_{\mathrm{sal}*}\mathbf{k}_{UM_{\mathrm{sa}}} \simeq \mathbf{k}_{UM_{\mathrm{sal}}}$ is concentrated in degree zero.*
(ii) *For $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{\mathrm{sal}}})$, one has $\mathrm{R}\Gamma(U; \rho_{\mathrm{sal}}^! F) \simeq \mathrm{R}\Gamma(U; F)$.*
(iii) *Let $F \in \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sal}}})$ and assume that F is Γ-acyclic. Then $\mathrm{R}\Gamma(U; \rho_{\mathrm{sal}}^! F)$ is concentrated in degree 0 and is isomorphic to $F(U)$.*

Note that the result in (i) is local and it is not necessary to assume here that U is relatively compact.

Proof. — (i)–(a) First we assume that U is Lipschitz. The first isomorphism is a local problem. Hence, by Remark 1.11 and by the definition of "Lipschitz boundary" the first isomorphism follows from Lemma 2.45. The second isomorphism is given in Proposition 2.6.

(i)–(b) The first isomorphism is a local problem and we may assume that U has a covering by open sets U_i as in Definition 2.46. By using the Čech resolution associated with this covering, we find an exact sequence of sheaves in $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$:

$$0 \to L_r \to \cdots \to L_0 \to \mathbf{k}_U \to 0$$

where each L_i is a finite sum of sheaves isomorphic to $\mathbf{k}_V$ for some $V \in \mathrm{Op}_{M_{\mathrm{sa}}}$ with V Lipschitz. Therefore, $\mathrm{R}\rho_{\mathrm{sal}*} L_i$ is concentrated in degree 0 by (i)–(a) and the result follows.

(ii) follows from (i) and the adjunction between $\mathrm{R}\rho_{\mathrm{sal}*}$ and $\rho_{\mathrm{sal}}^!$.

(iii) follows from (ii). □

Example 2.50. — Let $M = \mathbb{R}^2$ endowed with coordinates $x = (x_1, x_2)$. Let $R > 0$ and denote by B_R the open Euclidean ball with center 0 and radius R. Consider the subanalytic sets:

$$U_1 = \{x \in B_R; x_1 > 0, x_2 < x_1^2\}, \quad U_2 = \{x \in B_R; x_1 > 0, x_2 > -x_1^2\},$$
$$U_{12} = U_1 \cap U_2, \quad U = U_1 \cup U_2 = \{x \in B_R; x_1 > 0\}.$$

Note that $\{U_1, U_2\}$ is a covering of U in M_{sa} but not in M_{sal}. Denote for short by $\rho\colon M_{\mathrm{sa}} \to M_{\mathrm{sal}}$ the morphism ρ_{sal}. We have the distinguished triangle in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{\mathrm{sal}}})$:

$$\mathrm{R}\rho_*\mathbf{k}_{U_{12}} \to \mathrm{R}\rho_*\mathbf{k}_{U_1} \oplus \mathrm{R}\rho_*\mathbf{k}_{U_2} \to \mathrm{R}\rho_*\mathbf{k}_U \xrightarrow{+1} . \tag{2.27}$$

Since U_1, U_2 and U are Lipschitz, $\mathrm{R}\rho_*\mathbf{k}_V$ is concentrated in degree 0 for $V = U_1, U_2, U$. It follows that $\mathrm{R}\rho_*\mathbf{k}_{U_{12}}$ is concentrated in degrees 0 and 1. Hence, we have the distinguished triangle

$$\rho_*\mathbf{k}_{U_{12}} \to \mathrm{R}\rho_*\mathbf{k}_{U_{12}} \to R^1\rho_*\mathbf{k}_{U_{12}}[-1] \xrightarrow{+1} . \tag{2.28}$$

Let us prove that $R^1\rho_*\mathbf{k}_{U_{12}}$ is isomorphic to the sheaf N introduced in (2.7). We easily see that there exists a natural morphism $\mathbf{k}_U \to N$ which is surjective. Hence we have to prove that the sequence

$$\mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2} \to \mathbf{k}_U \to N$$

is exact. This reduces to the following assertion: if $V \in \mathrm{Op}_{M_{\mathrm{sa}}}$ satisfies $V \subset U$ and $N(V) = 0$, then $\{V \cap U_1, V \cap U_2\}$ is a linear covering of V. We prove this claim now.

Let $V \subset U$ be such that $N(V) = 0$. By the definition of N, there exists $A > 0$ such that $U_{A,\varepsilon} \not\subset V$ for all $\varepsilon > 0$, where $U_{A,\varepsilon}$ is defined in (2.5). Hence there exists a sequence $\{(x_{1,n}, x_{2,n})\}_{n\in\mathbb{N}}$ such that $x_{1,n} > 0$, $x_{1,n} \to 0$ when $n \to \infty$, $|x_{2,n}| < Ax_{1,n}^2$ and $(x_{1,n}, x_{2,n}) \notin V$, for all $n \in \mathbb{N}$. We define $f(x) = d((x,0), M \setminus V)$, for $x \in \mathbb{R}$. Then f is a continuous subanalytic function and $f(x_{1,n}) < Ax_{1,n}^2$, for all $n \in \mathbb{N}$. The set $\{x \in \,]0,1[; f(x) < Ax^2\}$ is subanalytic and relatively compact, hence it is a finite disjoint union of points (but it is open) and intervals. Since it contains a sequence converging to 0, it must contain some interval $]0, x_0[$. We have then $f(x) \leq Ax^2$ for all $x \in]0, x_0[$. We deduce, for any $(x_1, x_2) \in \mathbb{R}^2$ with $x_1 \in]0, x_0[$,

$$d((x_1, x_2), M \setminus V) \leq |x_2| + d((x_1, 0), M \setminus V) \leq |x_2| + Ax_1^2. \tag{2.29}$$

On the other hand we can find $B > 0$ such that, for any $(x_1, x_2) \in U$,

$$\max\{d((x_1, x_2), M \setminus U_1), d((x_1, x_2), M \setminus U_2)\} \geq |x_2| + Bx_1^2. \tag{2.30}$$

We deduce easily from (2.29) and (2.30) that $\{V \cap U_1, V \cap U_2\}$ is a linear covering of V.

3. Operations on sheaves

All along this chapter, we follow the convention (1.1).

In this chapter we study the natural operations on sheaves for the linear subanalytic topology. In particular, given a morphism of real analytic manifolds, our aim is to define inverse and direct images for sheaves on the linear subanalytic topology. We are not able to do it in general (see Remark 3.20) and we shall distinguish the case of a closed embedding and the case of a submersion.

3.1. Tensor product and internal hom. — Since M_{sal} is a site, the category $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sal}}})$ admits a tensor product, denoted $\bullet \otimes \bullet$ and an internal hom, denoted $\mathscr{H}om$. The functor $\otimes$ is exact and the functor $\mathscr{H}om$ admits a right derived functor. For more details, we refer to [**15**, § 18.2].

3.2. Operations for closed embeddings

f-regular open sets. — In this section, $f\colon M \hookrightarrow N$ will be a closed embedding. We identify M with a subset of N. We assume for simplicity that d_M is the restriction of d_N to M and we write d for d_M or d_N. We also keep the convention (1.2) for $d(x, \varnothing)$.

Definition 3.1. — Let $V \in \mathrm{Op}_{N_{\mathrm{sa}}}$. We say that V is f-regular if there exists $C > 0$ such that

$$d(x, M \setminus M \cap V) \leq C\, d(x, N \setminus V) \quad \text{for all } x \in M. \tag{3.1}$$

- The property of being f-regular is local on M. More precisely, if $M = \bigcup_{i \in I} U_i$ is an open covering and $V \in \mathrm{Op}_{N_{\mathrm{sa}}}$ is $f|_{U_i}$-regular for each $i \in I$, then V is f-regular.
- If V and W belong to $\mathrm{Op}_{N_{\mathrm{sa}}}$ with $f^{-1}(V) = f^{-1}(W)$, $V \subset W$ and V is f-regular, then W is f regular.

Lemma 3.2. — *Let $f\colon M \hookrightarrow N$ be a closed embedding. The family $\{V \in \mathrm{Op}_{N_{\mathrm{sa}}}; V$ is f-regular$\}$ is stable by finite intersections.*

Proof. — We shall use the obvious fact which asserts that for two closed sets F_1 and F_2 in a metric space,

$$d(x, F_1 \cup F_2) = \inf(d(x, F_1), d(x, F_2)).$$

Let V_1 and V_2 be two f-regular objects of $\mathrm{Op}_{N_{\mathrm{sa}}}$ and let C_1 and C_2 be the corresponding constants as in (3.1). Let $x \in M$. We have

$$\begin{aligned} d(x, M \setminus (M \cap V_1 \cap V_2)) &= \inf_i d(x, M \setminus (M \cap V_i)) \\ &\leq \inf_i (C_i \cdot d(x, N \setminus V_i)) \\ &\leq (\max_i C_i) \cdot (\inf_i d(x, N \setminus V_i)) \\ &= (\max_i C_i) \cdot d(x, N \setminus (V_1 \cap V_2)). \end{aligned}$$

□

Lemma 3.3. — *Let $f\colon M \hookrightarrow N$ be a closed embedding and let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. Then there exists $V \in \mathrm{Op}_{N_{\mathrm{sa}}}$ such that V is f-regular and $M \cap V = U$.*

Proof. — We choose $V_0 \in \mathrm{Op}_{N_{\mathrm{sa}}}$ such that $\overline{U} \subset V_0$. We set

$$\delta = \inf\{d(x, N \setminus V_0);\ x \in U\}$$

and $V = (V_0 \setminus (V_0 \cap M)) \cup U$. We have $\delta > 0$. Let $x \in M$ and $y \in N$ be such that $d(x, N \setminus V) = d(x, y)$. If $y \in M$, then $d(x, N \setminus V) = d(x, M \setminus U)$. If $y \notin M$, then $d(x, N \setminus V) = d(x, N \setminus V_0) \geq \delta$. In any case we have $d(x, N \setminus V) \geq \min\{d(x, M \setminus U), \delta\}$. Hence (3.1) is satisfied with $C = \max\{1, D/\delta\}$, where $D = \max\{d(x, M \setminus U);\ x \in M\} < \infty$.

□

Lemma 3.4. — *Let $f\colon M \hookrightarrow N$ be a closed embedding. Let $V \in \mathrm{Op}_{N_{\mathrm{sal}}}$ be an f-regular open set and let $\{V_i\}_{i \in I}$ be a linear covering of V, that is, a covering in $\mathrm{Op}_{N_{\mathrm{sal}}}$. Then there exists a refinement $\{W_j\}_{j \in J}$ of $\{V_i\}_{i \in I}$ such that $\{W_j\}_{j \in J}$ is a linear covering of V and W_j is f-regular for all $j \in J$. We can even choose $J = I$ and $W_i \subset V_i$, for all $i \in I$.*

Proof. — Let C be a constant as in (3.1). Let $I_0 \subset I$ be a finite subset and let $C' > 0$ be such that

$$d(x, N \setminus V) \leq C' \cdot \max_{i \in I_0} d(x, N \setminus V_i) \quad \text{for all } x \in N. \tag{3.2}$$

Then, for any $x \in M$ we have

$$\begin{aligned} d(x, M \setminus (M \cap V)) &\leq C \cdot d(x, N \setminus V) \\ &\leq CC' \cdot \max_{i \in I_0} d(x, N \setminus V_i). \end{aligned} \tag{3.3}$$

We set $D = 2CC'$. For $i \in I_0$ we define $W_i \in \mathrm{Op}_{N_{\mathrm{sal}}}$ by

$$W_i = (V_i \setminus M) \cup \{x \in M \cap V_i;\ d(x, M \setminus (M \cap V)) < D\, d(x, N \setminus V_i)\}$$

and for $i \in I \setminus I_0$ we set $W_i = \varnothing$.

(i) Since $D \geq CC'$, the inequality (3.3) gives $V = \bigcup_{i\in I_0} W_i$. Let us prove that $\{W_i\}_{i\in I_0}$ is a linear covering of V. We first prove the following claim, for given $\varepsilon > 0$, $i \in I_0$ and $x \in N$:

$$\begin{aligned} &\text{if } d(x, N\setminus W_i) \leq \varepsilon d(x, N\setminus V), \\ &\text{then } d(x, N\setminus V_i) \leq (\varepsilon(1+\tfrac{C}{D}) + \tfrac{C}{D}) d(x, N\setminus V). \end{aligned} \tag{3.4}$$

If $d(x, N\setminus W_i) = d(x, N\setminus V_i)$, the claim is obvious. In the other case we choose $y \in N$ such that $d(x, N\setminus W_i) = d(x, y)$. Then we have $y \in V_i \setminus W_i$. Hence $y \in M$ and the definition of W_i gives $d(y, N\setminus V_i) \leq D^{-1} d(y, M\setminus (M\cap V))$. We deduce

$$\begin{aligned} d(x, N\setminus V_i) &\leq d(x,y) + d(y, N\setminus V_i) \\ &\leq d(x,y) + D^{-1} d(y, M\setminus(M\cap V)) \\ &\leq d(x,y) + CD^{-1} d(y, N\setminus V) \\ &\leq (1 + CD^{-1}) d(x,y) + CD^{-1} d(x, N\setminus V) \\ &\leq (\varepsilon(1+CD^{-1}) + CD^{-1}) d(x, N\setminus V), \end{aligned}$$

which proves (3.4).

Now we prove that $\{W_i\}_{i\in I_0}$ is a linear covering of V. We choose ε small enough so that $(\varepsilon(1+\frac{C}{D}) + \frac{C}{D}) < \frac{1}{C'}$ (recall that $D = 2CC'$) and we prove, for all $x \in N$,

$$d(x, N\setminus V) \leq \varepsilon^{-1} \cdot \max_{i\in I_0} d(x, N\setminus W_i). \tag{3.5}$$

Indeed, if (3.5) is false, then (3.4) implies $d(x, N\setminus V_i) < \frac{1}{C'} d(x, N\setminus V)$ for some $x \in V$ and all $i \in I_0$. But this contradicts (3.2).

(ii) Let us prove that W_i is f-regular, for any $i \in I_0$. We remark that $W_i \setminus M = V_i \setminus M$. Hence $d(x, N\setminus W_i) = d(x, N\setminus V_i)$ or $d(x, N\setminus W_i) = d(x, M\setminus(M\cap W_i))$, for all $x \in M$. In the first case we have, assuming $x \in M\cap W_i$ (the case $x \notin M\cap W_i$ being trivial),

$$d(x, M\setminus(M\cap W_i)) \leq d(x, M\setminus(M\cap V)) \leq D\, d(x, N\setminus V_i) = D\, d(x, N\setminus W_i).$$

In the second case we have obviously $d(x, M\setminus(M\cap W_i)) \leq d(x, N\setminus W_i)$. Hence (3.1) holds for W_i with the constant $\max\{D, 1\}$. □

Thanks to Lemma 3.2, to f we can associate a new site.

Definition 3.5. — Let $f\colon M \to N$ be a closed embedding.

(i) The presite N^f is given by $\mathrm{Op}_{N^f} = \{V \in N_{\mathrm{sa}}; V \text{ is } f\text{-regular}\}$.
(ii) The site N^f_{sal} is the presite N^f endowed with the topology such that a family $\{V_i\}_{i\in I}$ of objects Op_{N^f} is a covering of V in N^f if it is a covering in N_{sal}.

One denotes by $\rho_f\colon N_{\mathrm{sal}} \to N^f_{\mathrm{sal}}$ the natural morphism of sites.

Proposition 3.6. — *The functor $f^t\colon \mathrm{Op}_{N^f_{\mathrm{sal}}} \to \mathrm{Op}_{M_{\mathrm{sa}}}$, $V \mapsto f^{-1}(V)$, induces a morphism of sites $\widetilde{f}\colon M_{\mathrm{sal}} \to N^f_{\mathrm{sal}}$. Moreover, this functor of sites is left exact in the sense of* [**15**, Def. 17.2.4].

Proof. — (i) Let C be a constant as in (3.1). Let $\{V_i\}_{i\in I}$ be a covering of V in N_{sal} and let $I_0 \subset I$ be a finite subset and $C' > 0$ be such that

$$d(y, N \setminus V) \leq C' \cdot \max_{i\in I_0} d(y, N \setminus V_i) \quad \text{for all } y \in N.$$

We deduce, for $x \in M$,

$$\begin{aligned} d(x, M \setminus M \cap V) &\leq C \cdot d(x, N \setminus V) \\ &\leq CC' \cdot \max_{i\in I_0} d(x, N \setminus V_i) \\ &\leq CC' \cdot \max_{i\in I_0} d(x, M \setminus M \cap V_i). \end{aligned}$$

(ii) We have to prove that the functor $f^t\colon \mathrm{Op}_{N^f_{\mathrm{sal}}} \to \mathrm{Op}_{M_{\mathrm{sa}}}$ is left exact in the sense of [**15**, Def. 3.3.1], that is, for each $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, the category whose objects are the inclusions $U \to f^{-1}(V)$ ($V \in \mathrm{Op}_{N^f_{\mathrm{sal}}}$) is cofiltrant.

This category is non empty by Lemma 3.3 and then it is cofiltrant by Lemma 3.2. □

Hence, we have the morphisms of sites

$$\begin{tikzcd} & N_{\mathrm{sal}} \arrow[d, "\rho_f"] \\ M_{\mathrm{sal}} \arrow[r, "\widetilde{f}"'] & N^f_{\mathrm{sal}}. \end{tikzcd} \tag{3.6}$$

Now we consider two closed embeddings $f\colon M \to N$ and $g\colon N \to L$ of real analytic manifolds and we set $h := g \circ f$. We get the diagram of presites:

$$\begin{tikzcd} N_{\mathrm{sa}} \arrow[r, "\widetilde{g}"] \arrow[d, "\rho_f"'] & L^g \arrow[ddr, "\lambda_h"] & L_{\mathrm{sa}} \arrow[l, "\rho_g"'] \arrow[dd, "\rho_h"] \\ N^f \arrow[drr, "\overline{g}"] & & \\ M_{\mathrm{sa}} \arrow[u, "\widetilde{f}"] \arrow[rr, "\widetilde{h}"] & & L^g \cap L^h, \end{tikzcd} \tag{3.7}$$

where the objects of $L^g \cap L^h$ are the open sets $U \in \mathrm{Op}_{L_{\mathrm{sa}}}$ which are both g-regular and h-regular, $\overline{g}$ is induced by $\widetilde{g}$ and λ_h is the obvious inclusion. We will use the following lemma to prove that the direct images defined in the next section are compatible with the composition.

Lemma 3.7. — (i) *Let $W \in \mathrm{Op}_{L^h}$. Then $W \cap N \in \mathrm{Op}_{N^f}$.*

(ii) *Let $W \in \mathrm{Op}_{L^g}$ be such that $N \cap W \in \mathrm{Op}_{N^f}$. Then $W \in \mathrm{Op}_{L^h}$.*

(iii) *Let $W \in \mathrm{Op}_{L^g}$ and $V \in \mathrm{Op}_{N^f}$ be such that $V \subset N \cap W$. Then there exists $U \in \mathrm{Op}_{L^g} \cap \mathrm{Op}_{L^h}$ such that $U \subset W$ and $V \subset N \cap U$.*

Proof. — (i) By hypothesis there exists $C > 0$ such that $d(x, M \setminus M \cap W) \leq C\, d(x, L \setminus W)$, for any $x \in M$. Since $d(x, L \setminus W) \leq d(x, N \setminus N \cap W)$ we deduce (i).

(ii) By hypothesis there exist $C_1, C_2 > 0$ such that, for any $x \in M$,

$$d(x, M \setminus M \cap W) \leq C_1 d(x, N \setminus N \cap W) \leq C_1 C_2 d(x, L \setminus W),$$

which proves the result.

(iii) By Lemma 3.3 there exists $U_0 \in \mathrm{Op}_{L^g}$ such that $N \cap U_0 = V$. Then $U = U_0 \cap W$ is g-regular by Lemma 3.2 and $N \cap U = V$. Hence U is also h-regular by (ii). $\square$

Inverse and direct images by closed embeddings. — Let us first recall the inverse and direct images of presheaves.

Notation 3.8. — (i) For a morphism $f\colon \mathscr{T}_1 \to \mathscr{T}_2$ of presites, we denote by f_* and $f^\dagger$ the direct and inverse image functors for presheaves.
(ii) We recall that the direct image functor f_* has a right adjoint $f^\ddagger\colon \mathrm{PSh}(\mathbf{k}_{\mathscr{T}_2}) \to \mathrm{PSh}(\mathbf{k}_{\mathscr{T}_1})$ defined as follows (see [**15**, (17.1.4)]). For $P \in \mathrm{PSh}(\mathbf{k}_{\mathscr{T}_2})$ and $U \in \mathrm{Op}_{\mathscr{T}_1}$ we have $(f^\ddagger P)(U) = \varprojlim_{f^t(V) \to U} P(V)$.

Lemma 3.9. — *Let $f\colon M \to N$ be a closed embedding and let $G \in \mathrm{Mod}(\mathbf{k}_{N^f_{\mathrm{sal}}})$. Then, using the notations of* (3.6)*, we have $\rho_f^\ddagger G \in \mathrm{Mod}(\mathbf{k}_{N_{\mathrm{sal}}})$.*

Proof. — We have to prove that, for any $V \in \mathrm{Op}_{N_{\mathrm{sa}}}$ and any covering of V in N_{sal}, say $\{V_i\}_{i\in I}$, the following sequence is exact

$$0 \to \varprojlim_{W \subset V} G(W) \to \prod_{i\in I} \varprojlim_{W_i \subset V_i} G(W_i) \to \prod_{i,j\in I} \varprojlim_{W_{ij} \subset V_i \cap V_j} G(W_{ij}), \tag{3.8}$$

where W, W_i, W_{ij} run respectively over the f-regular open subsets of V, V_i, $V_i \cap V_j$. The limit in the second term of (3.8) can be replaced by the limit over the pairs (W, W_i) of f-regular open subsets with $W \subset V$, $W_i \subset W \cap V_i$. Then the family $\{W \cap V_i\}_{i\in I}$ is a covering of W in N_{sal}. By Lemma 3.4 it admits a refinement $\{W_i'\}_{i\in I}$ where the W_i''s are f-regular and $W_i' \subset V_i$. We may as well assume that W_i contains W_i', for any $i \in I$. Then $\{W_i\}_{i\in I}$ is a covering of W in N^f_{sal}. Hence the second term of (3.8) can be replaced by

$$\varprojlim_{W \subset V} \varprojlim_{\{W_i\}_{i\in I}} \prod_{i\in I} G(W_i),$$

where W runs over the f-regular open subsets of V and the family $\{W_i\}_{i\in I}$ runs over the coverings of W in N^f_{sal} such that $W_i \subset W \cap V_i$.

Now in the third term of (3.8) we may assume that W_{ij} contains $W_i \cap W_j$ and the exactness of the sequence follows from the hypothesis that $G \in \mathrm{Mod}(\mathbf{k}_{N^f_{\mathrm{sal}}})$. $\square$

Definition 3.10. — Let $f\colon M \to N$ be a closed embedding. We use the notations of (3.6).

(i) We denote by $f_{\mathrm{sal}*}\colon \mathrm{Mod}(M_{\mathrm{sal}}) \to \mathrm{Mod}(N_{\mathrm{sal}})$ the functor $\rho_f^{\ddagger} \circ \widetilde{f}_*$ and we call $f_{\mathrm{sal}*}$ the direct image functor.

(ii) We denote by $f_{\mathrm{sal}}^{-1}\colon \mathrm{Mod}(N_{\mathrm{sal}}) \to \mathrm{Mod}(M_{\mathrm{sal}})$ the functor $\widetilde{f}^{-1} \circ \rho_{f*}$ and we call f_{sal}^{-1} the inverse image functor.

For $F \in \mathrm{Mod}(M_{\mathrm{sal}})$, $G \in \mathrm{Mod}(N_{\mathrm{sal}})$, $U \in \mathrm{Op}_{M_{\mathrm{sal}}}$ and $V \in \mathrm{Op}_{N_{\mathrm{sal}}}$, we obtain

$$\Gamma(V; f_{\mathrm{sal}*}F) \simeq \varprojlim_{W \in \mathrm{Op}_{N^f},\, W \subset V} F(M \cap W), \tag{3.9}$$

$$\Gamma(U; f_{\mathrm{sal}}^{-1}G) \simeq \varinjlim_{W \in \mathrm{Op}_{N^f},\, W \cap M = U} G(W). \tag{3.10}$$

Lemma 3.11. — *Let $f\colon M \to N$ and $g\colon N \to L$ be closed embeddings and let $h = g \circ f$. We use the notations of the diagram (3.7). There is a natural isomorphism of functors*

$$\widetilde{g}_* \circ \rho_f^{\ddagger} \xrightarrow{\sim} \lambda_h^{\ddagger} \circ \overline{g}_*. \tag{3.11}$$

Proof. — The morphisms of functors $\lambda_{h*} \circ \widetilde{g}_* \circ \rho_f^{\ddagger} \simeq \overline{g}_* \circ \rho_{f*} \circ \rho_f^{\ddagger} \to \overline{g}_*$ gives by adjunction the morphism in (3.11). To prove that this morphism is an isomorphism, let us choose $G \in \mathrm{PSh}(\mathbf{k}_{N^f})$ and $W \in \mathrm{Op}_{L^g}$. We get the morphism

$$\Gamma(W; (\widetilde{g}_* \circ \rho_f^{\ddagger})(G)) \to \Gamma(W; (\lambda_h^{\ddagger} \circ \overline{g}_*)(G)), \tag{3.12}$$

where $\Gamma(W; (\widetilde{g}_* \circ \rho_f^{\ddagger})(G)) \simeq \varprojlim_{V \in \mathrm{Op}_{N^f},\, V \subset N \cap W} G(V)$ and $\Gamma(W; (\lambda_h^{\ddagger} \circ \overline{g}_*)(G)) \simeq \varprojlim_{U \in \mathrm{Op}_{L^h},\, U \subset W} G(N \cap U)$. Then the result follows from Lemma 3.7. ☐

Proposition 3.12. — *Let $f\colon M \to N$ and $g\colon N \to L$ be closed embeddings and let $h = g \circ f$. There is a natural isomorphism of functors $g_{\mathrm{sal}*} \circ f_{\mathrm{sal}*} \xrightarrow{\sim} h_{\mathrm{sal}*}$.*

Proof. — Applying Lemma 3.11, we define the isomorphism as the composition $\rho_g^{\ddagger} \circ \widetilde{g}_* \circ \rho_f^{\ddagger} \circ \widetilde{f}_* \xrightarrow{\sim} \rho_g^{\ddagger} \circ \lambda_h^{\ddagger} \circ \overline{g}_* \circ \widetilde{f}_* \simeq \rho_h^{\ddagger} \circ \widetilde{h}_*$. ☐

Theorem 3.13. — *Let $f\colon M \to N$ be a closed embedding.*

(i) *The functor $f_{\mathrm{sal}*}$ is right adjoint to the functor f_{sal}^{-1}.*

(ii) *The functor $f_{\mathrm{sal}*}$ is left exact and the functor f_{sal}^{-1} is exact.*

(iii) *If $g\colon N \to L$ is another closed embedding, we have $(g \circ f)_{\mathrm{sal}*} \simeq g_{\mathrm{sal}*} \circ f_{\mathrm{sal}*}$ and $(g \circ f)_{\mathrm{sal}}^{-1} \simeq f_{\mathrm{sal}}^{-1} \circ g_{\mathrm{sal}}^{-1}$.*

Proof. — (i) We have $f_{\mathrm{sal}*} = \rho_f^{\ddagger} \circ \widetilde{f}_*$ and $f_{\mathrm{sal}}^{-1} = \widetilde{f}^{-1} \circ \rho_{f*}$. Since $(\widetilde{f}^{-1}, \widetilde{f}_*)$ and $(\rho_{f*}, \rho_f^{\ddagger})$ are pairs of adjoint functors between categories of presheaves and since the category of sheaves is a fully faithful subcategory of the category of presheaves, the result follows.

(ii) By the adjunction property, it remains to show that functor f_{sal}^{-1} is left exact, hence that functor $\widetilde{f}^{-1}$ is exact. By Proposition 3.6, the morphism of sites $\widetilde{f}\colon M_{\mathrm{sal}} \to N_{\mathrm{sal}}^f$ is left exact in the sense of [**15**, Def. 17.2.4]. Then the result follows from [**15**, Th. 17.5.2].

(iii) The functoriality of direct images follows from Proposition 3.12 and that of inverse images results by adjunction. □

3.3. Operations for submersions. — Let $f\colon M \to N$ denote a morphism of real analytic manifolds. In this section we assume that f is a submersion. If f is proper, it induces a morphism of sites $M_{\mathrm{sal}} \to N_{\mathrm{sal}}$, but otherwise, it does not even give a morphism of presites. Following **[13]** we shall introduce other sites M_{sb} (denoted M_{sa} in loc. cit.), similar to M_{sa} but containing all open subanalytic subsets of M, and M_{sbl}, similar to M_{sal}. Then M_{sbl} has the same category of sheaves as M_{sal} and any submersion $f\colon M \to N$ induces a morphism of sites $f_{\mathrm{sbl}}\colon M_{\mathrm{sbl}} \to N_{\mathrm{sbl}}$.

Another subanalytic topology. — One denotes by $\mathrm{Op}_{M_{\mathrm{sb}}}$ the category of open subanalytic subsets of M and says that a family $\{U_i\}_{i\in I}$ of objects of $\mathrm{Op}_{M_{\mathrm{sb}}}$ is a covering of $U \in \mathrm{Op}_{M_{\mathrm{sb}}}$ if $U_i \subset U$ for all $i \in I$ and, for each compact subset K of M, there exists a finite subset $J \subset I$ such that $\bigcup_{j\in J} U_j \cap K = U \cap K$. We denote by M_{sb} the site so-defined. The next result is obvious (and already mentioned in **[13]**).

Proposition 3.14. — *The morphism of sites $M_{\mathrm{sb}} \to M_{\mathrm{sa}}$ induces an equivalence of categories* $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sb}}}) \simeq \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$.

Similarly, we introduce another linear subanalytic topology M_{sbl} as follows. The objects of the presite M_{sbl} are those of M_{sb}, namely the open subanalytic subsets of M. In order to define the topology, we have to generalize Definitions 1.1 and 1.5.

Definition 3.15. — Let $\{U_i\}_{i\in I}$ be a finite family in $\mathrm{Op}_{M_{\mathrm{sb}}}$. We say that this family is 1-regularly situated if for any compact subset $K \subset M$, there is a constant C such that for any $x \in K$

$$d(x, M \setminus \bigcup_{i\in I} U_i) \leq C \cdot \max_{i\in I} d(x, M \setminus U_i). \tag{3.13}$$

Definition 3.16. — A linear covering of $U \in \mathrm{Op}_{M_{\mathrm{sb}}}$ is a small family $\{U_i\}_{i\in I}$ of objects of $\mathrm{Op}_{M_{\mathrm{sb}}}$ such that $U_i \subset U$ for all $i \in I$ and

$$\begin{cases} \text{for each relatively compact subanalytic open subset } W \subset M \text{ there} \\ \text{exists a finite subset } I_0 \subset I \text{ such that the family } \{W \cap U_i\}_{i\in I_0} \text{ is} \\ \text{1-regularly situated in } W \text{ and } \bigcup_{i\in I_0}(U_i \cap W) = U \cap W. \end{cases} \tag{3.14}$$

Proposition 3.17. — (i) *The family of linear coverings satisfies the axioms of Grothendieck topologies.*

(ii) *The functor ρ_* associated with the morphism of sites $\rho\colon M_{\mathrm{sbl}} \to M_{\mathrm{sal}}$ defines an equivalence of categories* $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sbl}}}) \simeq \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sal}}})$.

The verification is left to the reader.

Inverse and direct images

Proposition 3.18. — *Let $f\colon M \to N$ be a morphism of real analytic manifolds. We assume that f is a submersion. Then f induces a morphism of sites $f_{\mathrm{sbl}}\colon M_{\mathrm{sbl}} \to N_{\mathrm{sbl}}$.*

Proof. — Let $V \in \mathrm{Op}_{N_{\mathrm{sb}}}$ and let $\{V_i\}_{i\in I}$ be a linear covering of V. We have to prove that $\{f^{-1}V_i\}_{i\in I}$ is a linear covering of $f^{-1}V$. As in the case of M_{sa}, the definition of the linear coverings is local (see Corollary 1.9). Hence we can assume that $M = N\times L$. We can also assume that $d_M((x,y),(x',y')) = \max\{d_N(x,x'), d_L(y,y')\}$, for $x,x' \in N$ and $y,y' \in L$. Then for any $(x,y)\in M$ we have $d_M((x,y), N\setminus f^{-1}V) = d_N(x, N\setminus V)$ and the result follows easily. □

By Propositions 3.17 and 3.18, any submersion $f\colon M \to N$ between real analytic manifolds induces a pair of adjoint functors $(f^{-1}_{\mathrm{sal}}, f_{\mathrm{sal}*})$ between $\mathrm{Mod}(M_{\mathrm{sal}})$ and $\mathrm{Mod}(N_{\mathrm{sal}})$ and we get the analogue of Theorem 3.13:

Theorem 3.19. — *Let $f\colon M \to N$ be a submersion.*

(i) *The functor $f_{\mathrm{sal}*}$ is right adjoint to the functor f^{-1}_{sal}.*
(ii) *The functor $f_{\mathrm{sal}*}$ is left exact and the functor f^{-1}_{sal} is exact.*
(iii) *If $g\colon N \to L$ is another submersion, we have $(g\circ f)_{\mathrm{sal}*} \simeq g_{\mathrm{sal}*}\circ f_{\mathrm{sal}*}$ and $(g\circ f)^{-1}_{\mathrm{sal}} \simeq f^{-1}_{\mathrm{sal}} \circ g^{-1}_{\mathrm{sal}}$.*

Remark 3.20. — Our two definitions of $f_{\mathrm{sal}*}$ for closed embeddings and submersions do not give a definition for a general f by composition. For example let us consider the following commutative diagram

$$\begin{tikzcd} M = \mathbb{R}^2 \arrow[r, "i"] \arrow[d, "p"'] \arrow[dr, "f"] & \mathbb{R}^3 \arrow[d, "q"] \\ \mathbb{R} \arrow[r, "j"] & N = \mathbb{R}^2, \end{tikzcd}$$

where $i(x,y) = (x,y,0)$, $p(x,y) = x$, $q(x,y,z) = (x,z)$ and $j(x) = (x,0)$. For $V \in \mathrm{Op}_{N_{\mathrm{sb}}}$ we define two families of open subsets of $f^{-1}(V)$:

$$I_1 = \{M\cap W;\ W \in \mathrm{Op}_{\mathbb{R}^3_{\mathrm{sb}}},\ W \subset q^{-1}V,\ W \text{ is } i\text{-regular}\},$$
$$I_2 = \{p^{-1}(\mathbb{R}\cap V');\ V' \in \mathrm{Op}_{N_{\mathrm{sb}}},\ V' \subset V,\ V' \text{ is } j\text{-regular}\}.$$

Then, for any $F \in \mathrm{Mod}(M_{\mathrm{sbl}})$ we have

$$\Gamma(V; q_{\mathrm{sal}*} i_{\mathrm{sal}*} F) \simeq \Gamma(q^{-1}V; i_{\mathrm{sal}*}F) \simeq \varprojlim_{U\in I_1} F(U), \tag{3.15}$$

$$\Gamma(V; j_{\mathrm{sal}*} p_{\mathrm{sal}*} F) \simeq \varprojlim_{V'\subset V,\, V' \ j\text{-regular}} \Gamma(\mathbb{R}\cap V'; p_{\mathrm{sal}*}F) \simeq \varprojlim_{U\in I_2} F(U). \tag{3.16}$$

Let us take for V the open set $V = \{(x,z);\ x^3 > z^2\}$. Then the two families I_1 and I_2 of open subsets of $f^{-1}(V) = \{(x,y);\ x>0\}$ are not cofinal. Indeed the set $W_0 \subset \mathbb{R}^3$ given by $W_0 = \{(x,y,z);\ x^3 > y^2+z^2\}$ is i-regular. Hence $M\cap W_0 = \{(x,y);\ x^3 > y^2\}$ belongs to I_1. On the other hand we see easily that, if V' is j-regular and $V' \subset V$, then $\mathbb{R}\cap V' \subset]\varepsilon,+\infty[$, for some $\varepsilon>0$. Hence $M\cap W_0$ is not contained in any set of the family I_2.

Let us define $F = \varinjlim_{\varepsilon>0} \mathbf{k}_{[0,\varepsilon]\times\{0\}} \in \mathrm{Mod}(M_{\mathrm{sbl}})$. Taking $U = M\cap W_0$ in (3.15) we can see that $\Gamma(V; q_{\mathrm{sal}*} i_{\mathrm{sal}*} F) \simeq \mathbf{k}$. On the other hand (3.16) implies $\Gamma(V; j_{\mathrm{sal}*} p_{\mathrm{sal}*} F) \simeq 0$. Hence $q_{\mathrm{sal}*} i_{\mathrm{sal}*} \not\simeq j_{\mathrm{sal}*} p_{\mathrm{sal}*}$.

4. Construction of sheaves

On the site M_{sa}, the sheaves $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$ and $\mathscr{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ below have been constructed in [**12, 13**]. By using the linear topology we shall construct sheaves on M_{sal} associated with more precise growth conditions.

All along this chapter, we follow the convention 1.1.

4.1. Sheaves on the subanalytic site

Temperate growth. — For the reader's convenience, let us recall first some definitions of [**12, 13**]. As usual, we denote by $\mathscr{C}^\infty_M$ (resp. $\mathscr{A}_M$) the sheaf of complex valued functions of class $\mathscr{C}^\infty$ (resp. real analytic), by $\mathscr{D}b_M$ (resp. $\mathscr{B}_M$) the sheaf of Schwartz's distributions (resp. Sato's hyperfunctions) and by $\mathscr{D}_M$ the sheaf of finite-order differential operators with coefficients in $\mathscr{A}_M$.

Definition 4.1. — Let $U\in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $f\in\mathscr{C}^\infty_M(U)$. One says that f has *polynomial growth* at $p\in M\backslash U$ if it satisfies the following condition. For a local coordinate system $(x_1,\dots,x_n)$ around p, there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$\sup_{x\in K\cap U} \big(d(x, K\setminus U)\big)^N |f(x)| < \infty. \tag{4.1}$$

We say that f is temperate at p if all its derivatives have polynomial growth at p. We say that f is temperate if it is temperate at any point $p\in M\setminus U$.

For $U\in\mathrm{Op}_{M_{\mathrm{sa}}}$, we shall denote by $\mathscr{C}^{\infty,\mathrm{tp}}_M(U)$ the subspace of $\mathscr{C}^\infty_M(U)$ consisting of temperate functions.

For $U\in\mathrm{Op}_{M_{\mathrm{sa}}}$, we shall denote by $\mathscr{D}b^{\mathrm{tp}}_M(U)$ the space of temperate distributions on U, defined by the exact sequence

$$0\to \Gamma_{M\backslash U}(M;\mathscr{D}b_M)\to\Gamma(M;\mathscr{D}b_M)\to\mathscr{D}b^{\mathrm{tp}}_M(U)\to 0.$$

It follows from (1.3) that $U\mapsto\mathscr{C}^{\infty,\mathrm{tp}}_M(U)$ is a sheaf and it follows from the work of Lojasiewicz [**22**] that $U\mapsto\mathscr{D}b^{\mathrm{tp}}_M(U)$ is also a sheaf. We denote by $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$ and $\mathscr{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ these sheaves on M_{sa}. The first one is called the sheaf of $\mathscr{C}^\infty$-functions with temperate growth and the second the sheaf of temperate distributions. Note that both sheaves are Γ-acyclic (see [**13**, Lem 7.2.4] or Proposition 4.4 below) and the sheaf $\mathscr{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ is flabby (see Definition 2.16).

We also introduce the sheaf $\mathscr{C}^\infty_{M_{\mathrm{sa}}}$ of $\mathscr{C}^\infty$-functions on M_{sa} as

$$\mathscr{C}^\infty_{M_{\mathrm{sa}}} := \rho_{\mathrm{sa}*}\mathscr{C}^\infty_M.$$

We denote as usual by $\mathscr{D}_M$ the sheaf of rings of finite order differential operators on the real analytic manifold M. If $\iota_M \colon M \hookrightarrow X$ is a complexification of M, then $\mathscr{D}_M \simeq \iota_M^{-1}\mathscr{D}_X$. We set, following [**13**]:

$$\mathscr{D}_{M_{\mathrm{sa}}} := \rho_{\mathrm{sa}!}\mathscr{D}_M. \tag{4.2}$$

The sheaves $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$, $\mathscr{C}^\infty_{M_{\mathrm{sa}}}$ and $\mathscr{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ are $\mathscr{D}_{M_{\mathrm{sa}}}$-modules.

Remark 4.2. — The sheaves $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$ and $\mathscr{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ are respectively denoted by $\mathscr{C}^{\infty,t}_M$ and $\mathscr{D}b^t_M$ in [**13**].

A cutoff lemma on M_{sa}. — Lemma 4.3 below is an immediate corollary of a result of Hörmander [**6**, Cor.1.4.11] and was already used in [**12**, Prop. 10.2].

Lemma 4.3. — *Let Z_1 and Z_2 be two closed subanalytic subsets of M. Then there exists $\psi \in \mathscr{C}^{\infty,\mathrm{tp}}_M(M \setminus (Z_1 \cap Z_2))$ such that $\psi = 0$ on a neighborhood of $Z_1 \setminus Z_2$ and $\psi = 1$ on a neighborhood of $Z_2 \setminus Z_1$.*

Proposition 4.4. — *Let $\mathscr{F}$ be a sheaf of $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$-modules on M_{sa}. Then $\mathscr{F}$ is Γ-acyclic.*

Proof. — By Proposition 2.14, it is enough to prove that for U_1, U_2 in $\mathrm{Op}_{M_{\mathrm{sa}}}$, the sequence $0 \to \mathscr{F}(U_1 \cup U_2) \to \mathscr{F}(U_1) \oplus \mathscr{F}(U_2) \to \mathscr{F}(U_1 \cap U_2) \to 0$ is exact. This follows from Lemma 4.3 (see [**12**, Prop. 10.2] or Proposition 4.18 below). □

Gevrey growth. — The definition below of the sheaf $\mathscr{C}^{\infty,\mathrm{gev}}_{M_{\mathrm{sa}}}$ is inspired by the definition of the sheaves of $\mathscr{C}^\infty$-functions of Gevrey classes, but is completely different from the classical one. Here we are interested in the growth of functions at the boundary contrarily to the classical setting where one is interested in the Taylor expansion of the function. As usual, there are two kinds of regularity which can be interesting: regularity at the interior or at the boundary. Since we shall soon consider the Dolbeault complexes of our new sheaves, the interior regularity is irrelevant and we are only interested in the growth at the boundary.

We refer to [**20**] for an exposition on classical Gevrey functions or distributions and their link with Sato's theory of boundary values of holomorphic functions. Note that there is also a recent study by [**4**] of these sheaves using the tools of subanalytic geometry.

In § 4.2 we shall define more refined sheaves by using the linear subanalytic topology.

Definition 4.5. — Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $f \in \mathscr{C}^\infty_M(U)$. We say that f has 0-*Gevrey growth* at $p \in M \setminus U$ if it satisfies the following condition. For a local coordinate system $(x_1, \dots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p, $h > 0$ and $s > 1$ such that

$$\sup_{x \in K \cap U} \big(\exp(-h \cdot d(x, K \setminus U)^{1-s})\big)|f(x)| < \infty. \tag{4.3}$$

We say that f has Gevrey growth at p if all its derivatives have 0-Gevrey growth at p. We say that f has Gevrey growth if it has such a growth at any point $p \in M \setminus U$.

We denote by $G_M(U)$ the subspace of $\mathscr{C}_M^\infty(U)$ consisting of functions with Gevrey growth and by $\mathscr{C}_{M_{\mathrm{sa}}}^{\infty,\mathrm{gev}}$ the presheaf $U \mapsto G_M(U)$ on M_{sa}.

The next result is clear in view of (1.3) and Proposition 4.4.

Proposition 4.6. — (a) *The presheaf* $\mathscr{C}_{M_{\mathrm{sa}}}^{\infty,\mathrm{gev}}$ *is a sheaf on* M_{sa}*,*
(b) *the sheaf* $\mathscr{C}_{M_{\mathrm{sa}}}^{\infty,\mathrm{gev}}$ *is a* $\mathscr{D}_{M_{\mathrm{sa}}}$*-module,*
(c) *the sheaf* $\mathscr{C}_{M_{\mathrm{sa}}}^{\infty,\mathrm{gev}}$ *is a* $\mathscr{C}_{M_{\mathrm{sa}}}^{\infty,\mathrm{tp}}$*-module,*
(d) *the sheaf* $\mathscr{C}_{M_{\mathrm{sa}}}^{\infty,\mathrm{gev}}$ *is* Γ*-acyclic.*

4.2. Sheaves on the linear subanalytic site. — By Lemma 2.27, if a sheaf $\mathscr{F}$ on M_{sa} is Γ-acyclic, then $\mathrm{R}\rho_{\mathrm{sal}*}\mathscr{F}$ is concentrated in degree 0. This applies in particular to the sheaves $\mathscr{C}_{M_{\mathrm{sa}}}^{\infty,\mathrm{tp}}$, $\mathscr{D}b_{M_{\mathrm{sa}}}^{\mathrm{tp}}$ and $\mathscr{C}_{M_{\mathrm{sa}}}^{\infty,\mathrm{gev}}$.

In the sequel, for $\mathscr{G}_{M_{\mathrm{sa}}}$ one of the sheaves $\mathscr{C}_{M_{\mathrm{sal}}}^{\infty,\mathrm{tp}}$, $\mathscr{D}b_{M_{\mathrm{sal}}}^{\mathrm{tp}}$, $\mathscr{C}_{M_{\mathrm{sal}}}^{\infty,\mathrm{gev}}$, $\mathscr{C}_{M_{\mathrm{sa}}}^{\infty}$, we set $\mathscr{G}_{M_{\mathrm{sal}}} := \rho_{\mathrm{sal}*}\mathscr{G}_{M_{\mathrm{sa}}}$.

Temperate growth of a given order

Definition 4.7. — Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, let $f \in \mathscr{C}_M^\infty(U)$ and let $t \in \mathbb{R}_{\geq 0}$. We say that f has *polynomial growth of order* $\leq t$ at $p \in M \setminus U$ if it satisfies the following condition. For a local coordinate system $(x_1, \dots, x_n)$ around p, there exists a sufficiently small compact neighborhood K of p such that

$$\sup_{x \in K \cap U} \big(d(x, K \setminus U)\big)^t |f(x)| < \infty. \tag{4.4}$$

We say that f is temperate of order t at p if, for each $m \in \mathbb{N}$, all its derivatives of order $\leq m$ have polynomial growth of order $\leq t+m$ at p. We say that f is temperate of order t if it is temperate of order t at any point $p \in M \setminus U$.

For $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, we denote by $\mathscr{C}_M^{\infty,t}(U)$ the subspace of $\mathscr{C}_M^\infty(U)$ consisting of functions temperate of order t and we denote by $\mathscr{C}_{M_{\mathrm{sal}}}^{\infty,t}$ the presheaf on M_{sal} so obtained.

The next result is clear by Proposition 2.8.

Proposition 4.8. — (i) *The presheaves* $\mathscr{C}_{M_{\mathrm{sal}}}^{\infty,t}$ $(t \geq 0)$ *are sheaves on* M_{sal}*,*
(ii) *the sheaf* $\mathscr{C}_{M_{\mathrm{sal}}}^{\infty,0}$ *is a sheaf of rings,*
(iii) *for* $t \geq 0$, $\mathscr{C}_{M_{\mathrm{sal}}}^{\infty,t}$ *is a* $\mathscr{C}_{M_{\mathrm{sal}}}^{\infty,0}$*-module and there are natural morphisms* $\mathscr{C}_{M_{\mathrm{sal}}}^{\infty,t} \otimes_{\mathscr{C}_{M_{\mathrm{sal}}}^{\infty,0}} \mathscr{C}_{M_{\mathrm{sal}}}^{\infty,t'} \to \mathscr{C}_{M_{\mathrm{sal}}}^{\infty,t+t'}$.

We also introduce the sheaf

$$\mathscr{C}_{M_{\mathrm{sal}}}^{\infty,\mathrm{tp}\,st} := \varinjlim_t \mathscr{C}_{M_{\mathrm{sal}}}^{\infty,t}.$$

(Of course, the limit is taken in the category of sheaves on M_{sal}.) Then, for $0 \leq t \leq t'$, there are natural monomorphisms of sheaves on M_{sal} :

$$\mathscr{C}_{M_{\mathrm{sal}}}^{\infty,0} \hookrightarrow \mathscr{C}_{M_{\mathrm{sal}}}^{\infty,t} \hookrightarrow \mathscr{C}_{M_{\mathrm{sal}}}^{\infty,t'} \hookrightarrow \mathscr{C}_{M_{\mathrm{sal}}}^{\infty,\mathrm{tp}\,st} \hookrightarrow \mathscr{C}_{M_{\mathrm{sal}}}^{\infty,\mathrm{tp}}. \tag{4.5}$$

Note that the inclusion $\mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$ is strict since there exists a function f (say on an open subset U of $\mathbb{R}$) with polynomial growth of order $\le t$ and such that its derivative does not have polynomial growth of order $\le t+1$.

Gevrey growth of a given order

Definition 4.9. — Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, let $s \in]1,+\infty[$ and let $f \in \mathscr{C}^{\infty}_{M}(U)$. We say that f has 0-*Gevrey growth of type* (s) at $p \in M \setminus U$ if it satisfies the following condition. For a local coordinate system $(x_1,\dots,x_n)$ around p, there exists a sufficiently small compact neighborhood K of p such that

$$\sup_{x\in K\cap U} \big(\exp(-h\cdot d(x,K\setminus U)^{1-s})\big)|f(x)| < \infty \tag{4.6}$$

for all $h \in]0,+\infty[$. We say that f has Gevrey growth of type (s) if all its derivatives have 0-Gevrey growth of type (s) at p. We say that f has Gevrey growth of type (s) if it has such a growth at any point $p \in M \setminus U$.

Similarly, one defines f of Gevrey growth of type $\{s\}$ when replacing (4.6) for all $h \in]0,+\infty[$ with the same condition for some $h \in]0,+\infty[$.

Definition 4.10. — For $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and $s \in]1,+\infty[$, we denote by $G^{(s)}_{M}(U)$ and $G^{\{s\}}_{M}(U)$ the spaces of functions of Gevrey growth of type (s) and $\{s\}$, respectively.

We denote by $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}}$ and $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}$ the presheaves on M_{sal} so obtained.

Clearly, the presheaves $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}}$ and $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}$ do not depend on the choice of the distance.

Proposition 4.11. — (i) *The presheaves $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}}$ and $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}$ are sheaves on M_{sal},*

(ii) *the sheaves $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}}$ and $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}$ are $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$-modules,*

(iii) *the presheaves $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}}$ and $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}$ are Γ-acyclic,*

(iv) *we have natural monomorphisms of sheaves on M_{sal} for $1 < s < s'$*

$$\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{gev}(s')}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{gev}\{s'\}}_{M_{\mathrm{sal}}}.$$

Proof. — (i), (ii) and (iv) are obvious and (iii) will follow from (ii) and Proposition 4.18 below (see Corollary 4.19). □

We set

$$\mathscr{C}^{\infty,\mathrm{gev\,st}}_{M_{\mathrm{sal}}} := \varinjlim_{s>1} \mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}.$$

Hence, we have monomorphisms of sheaves on M_{sal} for $0 \le t$ and $1 < s$

$$\begin{aligned}\mathscr{C}^{\infty,0}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}& \\ \hookrightarrow \mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{gev\,st}}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty,\mathrm{gev}}_{M_{\mathrm{sal}}} \hookrightarrow \mathscr{C}^{\infty}_{M_{\mathrm{sal}}}&.\end{aligned}$$

Definition 4.12. — If $\mathscr{F}_{M_{\mathrm{sal}}}$ is one of the sheaves $\mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}}$, $\mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}}$, $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}}$, $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}$ or $\mathscr{C}^{\infty,\mathrm{gev\,st}}_{M_{\mathrm{sal}}}$, we set $\mathscr{F}_{M_{\mathrm{sa}}} := \rho^{!}_{\mathrm{sal}}\,\mathscr{F}$.

Let us apply Theorem 2.49 and Corollary 4.19. We get that if $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ is weakly Lipschitz and if $\mathscr{F}_{M_{\mathrm{sal}}}$ denotes one of the sheaves above, then

$$\mathrm{R}\Gamma(U;\mathscr{F}_{M_{\mathrm{sa}}}) \simeq \Gamma(U;\mathscr{F}_{M_{\mathrm{sal}}}).$$

We call $\mathscr{C}^{\infty,t}_{M_{\mathrm{sa}}}$, $\mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sa}}}$, $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sa}}}$, $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sa}}}$ and $\mathscr{C}^{\infty,\mathrm{gev\,st}}_{M_{\mathrm{sa}}}$ the sheaves on M_{sa} of $\mathscr{C}^\infty$-functions of growth t, strictly temperate growth, Gevrey growth of type (s) and $\{s\}$ and strictly Gevrey growth, respectively. Recall that on M_{sa}, we also have the sheaf $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$ of $\mathscr{C}^\infty$-functions of temperate growth, the sheaf $\mathscr{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ of temperate distributions and the sheaf $\mathscr{C}^{\infty,\mathrm{gev}}_{M_{\mathrm{sa}}}$ of $\mathscr{C}^\infty$-functions of Gevrey growth.

Rings of differential operators. — Let M be a real analytic manifold. Recall that $\mathscr{D}_M$ denotes the sheaf of finite order analytic differential operators on M and that we have set in (4.2)

$$\mathscr{D}_{M_{\mathrm{sa}}} := \rho_{\mathrm{sa}!}\mathscr{D}_M. \tag{4.7}$$

Now we set

$$\mathscr{D}_{M_{\mathrm{sal}}} := \rho_{\mathrm{sal}*}\mathscr{D}_{M_{\mathrm{sa}}}. \tag{4.8}$$

Hence, $\mathscr{D}_{M_{\mathrm{sa}}}$ is the sheaf on M_{sa} associated with the presheaf $U \mapsto \mathscr{D}_M(\overline{U})$ and M_{sal} is its direct image on M_{sal}. We define similarly the sheaves $\mathscr{D}_{\mathscr{T}}(m)$ of differential operators of order $\le m$ on the site $\mathscr{T} = M, M_{\mathrm{sa}}, M_{\mathrm{sal}}$.

Lemma 4.13. — *There are natural morphisms $\mathscr{D}_{M_{\mathrm{sal}}}(m) \otimes \mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}} \to \mathscr{C}^{\infty,t+m}_{M_{\mathrm{sal}}}$ making $\mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}}$ and $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$ left $\mathscr{D}_{M_{\mathrm{sal}}}$-modules.*
The sheaves $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}}$ and $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}$ are naturally left $\mathscr{D}_{M_{\mathrm{sal}}}$-modules.

Proof. — This follows immediately from Definitions 4.7 and 4.9. □

By using the functor $\rho^!_{\mathrm{sal}}$, we will construct new sheaves (in the derived sense) on M_{sa} associated with the sheaves previously constructed on M_{sal}.

Theorem 4.14. — (i) *The functor $\rho_{\mathrm{sal}*}\colon \mathrm{Mod}(\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{Mod}(\mathscr{D}_{M_{\mathrm{sal}}})$ has finite cohomological dimension.*
(ii) *The functor $\mathrm{R}\rho_{\mathrm{sal}*}\colon \mathrm{D}(\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{D}(\mathscr{D}_{M_{\mathrm{sal}}})$ commutes with small direct sums.*
(iii) *The functor $\mathrm{R}\rho_{\mathrm{sal}*}$ in* (ii) *admits a right adjoint $\rho^!_{\mathrm{sal}}\colon \mathrm{D}(\mathscr{D}_{M_{\mathrm{sal}}}) \to \mathrm{D}(\mathscr{D}_{M_{\mathrm{sa}}})$.*
(iv) *The functor $\rho^!_{\mathrm{sal}}$ induces a functor $\rho^!_{\mathrm{sal}}\colon \mathrm{D}^+(\mathscr{D}_{M_{\mathrm{sal}}}) \to \mathrm{D}^+(\mathscr{D}_{M_{\mathrm{sa}}})$.*

Proof. — Consider the quasi-commutative diagram of categories

$$\begin{tikzcd}
\mathrm{Mod}(\mathscr{D}_{M_{\mathrm{sa}}}) \arrow[r, "\rho_{\mathrm{sal}*}"] \arrow[d, "\mathrm{for}"'] & \mathrm{Mod}(\mathscr{D}_{M_{\mathrm{sal}}}) \arrow[d, "\mathrm{for}"'] \\
\mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sa}}}) \arrow[r, "\rho_{\mathrm{sal}*}"] & \mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sal}}}).
\end{tikzcd}$$

The functor for: $\mathrm{Mod}(\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sa}}})$ is exact and sends injective objects to injective objects, and similarly with M_{sal} instead of M_{sa}. It follows that the diagram below commutes:

$$\begin{tikzcd} \mathrm{D}(\mathscr{D}_{M_{\mathrm{sa}}}) \arrow[r, "\mathrm{R}\rho_{\mathrm{sal}*}"] \arrow[d, "\mathrm{for}"'] & \mathrm{D}(\mathscr{D}_{M_{\mathrm{sal}}}) \arrow[d, "\mathrm{for}"'] \\ \mathrm{D}(\mathbb{C}_{M_{\mathrm{sa}}}) \arrow[r, "\mathrm{R}\rho_{\mathrm{sal}*}"] & \mathrm{D}(\mathbb{C}_{M_{\mathrm{sal}}}). \end{tikzcd}$$

Moreover, the two functors for in the last diagram above are conservative. Then
(i) and (ii) follow from the corresponding result for $\mathbb{C}_{M_{\mathrm{sa}}}$-modules.
(iii) and (iv) follow from the Brown representability theorem (see Proposition 2.21). □

4.3. A refined cutoff lemma. — Lemma 4.15 below will play an important role in this paper and is an immediate corollary of a result of Hörmander [**6**, Cor.1.4.11]. Note that Hörmander's result was already used in [**12**, Prop. 10.2] (see Lemma 4.3 above).

Hörmander's result is stated for $M = \mathbb{R}^n$ but we check in Lemma 4.16 that it can be extended to an arbitrary manifold.

Lemma 4.15. — *Let Z_1 and Z_2 be two closed subsets of $M := \mathbb{R}^n$. Assume that there exists $C > 0$ such that*

$$d(x, Z_1 \cap Z_2) \leq C(d(x, Z_1) + d(x, Z_2)) \text{ for any } x \in M. \tag{4.9}$$

Then there exists $\psi \in \mathscr{C}_M^{\infty,0}(M \setminus (Z_1 \cap Z_2))$ such that $\psi = 0$ on a neighborhood of $Z_1 \setminus Z_2$ and $\psi = 1$ on a neighborhood of $Z_2 \setminus Z_1$.

Lemma 4.16. — *Let M be a manifold. Let Z_1 and Z_2 be two closed subsets of M such that $M \setminus (Z_1 \cap Z_2)$ is relatively compact and such that* (4.9) *holds for some $C > 0$. Then the conclusion of Lemma 4.15 holds true.*

Proof. — We consider an embedding of M in some $\mathbb{R}^N$ and we denote by d_M, $d_{\mathbb{R}^N}$ the distance on M or $\mathbb{R}^N$. We have a constant $D \geq 1$ such that

$$D^{-1} d_{\mathbb{R}^N}(x, y) \leq d_M(x, y) \leq D\, d_{\mathbb{R}^N}(x, y), \quad \text{for all } x, y \in M \setminus (Z_1 \cap Z_2).$$

Let $x \in \mathbb{R}^N$ and let $x' \in M$ such that $d_{\mathbb{R}^N}(x, x') = d_{\mathbb{R}^N}(x, M)$. In particular $d_{\mathbb{R}^N}(x, x') \leq d_{\mathbb{R}^N}(x, Z_1)$. Then we have, assuming $x' \notin Z_1 \cap Z_2$,

$$\begin{aligned} d_{\mathbb{R}^N}(x, Z_1 \cap Z_2) &\leq d_{\mathbb{R}^N}(x, x') + D\, d_M(x', Z_1 \cap Z_2) \\ &\leq d_{\mathbb{R}^N}(x, x') + DC(d_M(x', Z_1) + d_M(x', Z_2)) \\ &\leq d_{\mathbb{R}^N}(x, x') + D^2 C(d_{\mathbb{R}^N}(x', Z_1) + d_{\mathbb{R}^N}(x', Z_2)) \\ &\leq (1 + 2D^2 C) d_{\mathbb{R}^N}(x, x') + D^2 C(d_{\mathbb{R}^N}(x, Z_1) + d_{\mathbb{R}^N}(x, Z_2)) \\ &\leq (1 + 3D^2 C)(d_{\mathbb{R}^N}(x, Z_1) + d_{\mathbb{R}^N}(x, Z_2)). \end{aligned}$$

If $x' \in Z_1 \cap Z_2$, then $d_{\mathbb{R}^N}(x, Z_1 \cap Z_2) = d_{\mathbb{R}^N}(x, M) \leq d_{\mathbb{R}^N}(x, Z_1)$ and the same inequality holds trivially. Hence we can apply Lemma 4.15 to $Z_1, Z_2 \subset \mathbb{R}^N$ and obtain a function $\psi \in \mathscr{C}^{\infty,0}_{\mathbb{R}^N}(\mathbb{R}^N \backslash (Z_1 \cap Z_2))$. Then $\psi|_{M\backslash(Z_1\cap Z_2)}$ belongs to $\mathscr{C}^{\infty,0}_M(M\backslash(Z_1\cap Z_2))$ and satisfies the required properties. $\square$

Lemma 4.17. — *Let $U_1, U_2 \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and set $U = U_1 \cup U_2$. We assume that $\{U_1, U_2\}$ is a linear covering of U. Then there exist $U_i' \subset U_i$, $i = 1, 2$, and $\psi \in \mathscr{C}^{\infty,0}_M(U)$ such that*

(i) *$\{U_i', U_1 \cap U_2\}$ is a linear covering of U_i,*
(ii) *$\psi|_{U_1'} = 0$ and $\psi|_{U_2'} = 1$.*

Proof. — We choose $U_i' \subset U_i$, $i = 1, 2$, as in Lemma 1.15 and we set $Z_i = (M\backslash U) \cup \overline{U_i'}$. Then the result follows from Lemmas 1.15 and 4.16. $\square$

Proposition 4.18. — *Let $\mathscr{F}$ be a sheaf of $\mathscr{C}^{\infty,0}_{M_{\mathrm{sal}}}$-modules on M_{sal}. Then $\mathscr{F}$ is Γ-acyclic.*

Proof. — By Proposition 2.14, it is enough to prove that for any $\{U_1, U_2\}$ which is a covering of $U_1 \cup U_2$, the sequence $0 \to \mathscr{F}(U_1 \cup U_2) \to \mathscr{F}(U_1) \oplus \mathscr{F}(U_2) \to \mathscr{F}(U_1 \cap U_2) \to 0$ is exact. This follows from Lemma 4.17, similarly as in the proof of [**12**, Prop. 10.2]. The only non trivial fact is the surjectivity at the last term, which we check now.

We choose $U_i' \subset U_i$, $i = 1, 2$, and $\psi \in \mathscr{C}^{\infty,0}_M(U)$ as in Lemma 4.17. Let $s \in \Gamma(U_1 \cap U_2; \mathscr{F})$. Since $\{U_i', U_1 \cap U_2\}$ is a linear covering of U_i, $i = 1, 2$, we can define $s_1 \in \Gamma(U_1; \mathscr{F})$ and $s_2 \in \Gamma(U_2; \mathscr{F})$ by

$$s_1|_{U_1\cap U_2} = \psi \cdot s,\ s_1|_{U_1'} = 0 \quad \text{and} \quad s_2|_{U_1\cap U_2} = (1-\psi)\cdot s,\ s_2|_{U_2'} = 0.$$

Then $s_1|_{U_1\cap U_2} + s_2|_{U_1\cap U_2} = s$, as required. $\square$

Corollary 4.19. — *The sheaves $\mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}}$, $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$, $\mathscr{D}b^{\mathrm{tp}}_{M_{\mathrm{sal}}}$, $\mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}}$ ($t \in \mathbb{R}_{\geq 0}$), $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}}$ and $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}$ ($s > 1$), $\mathscr{C}^{\infty,\mathrm{gev\,st}}_{M_{\mathrm{sal}}}$ and $\mathscr{C}^{\infty,\mathrm{gev}}_{M_{\mathrm{sal}}}$ are Γ-acyclic.*

Let $\mathscr{F}_{M_{\mathrm{sal}}}$ denote one of the sheaves appearing in Corollary 4.19 and let $\mathscr{F}_{M_{\mathrm{sa}}} := \rho^!_{\mathrm{sal}} \mathscr{F}_{M_{\mathrm{sal}}} \in \mathrm{D}^+(\mathscr{D}_{M_{\mathrm{sa}}})$. Then, if U is weakly Lipschitz, $\mathrm{R}\Gamma(U; \mathscr{F}_{M_{\mathrm{sa}}})$ is concentrated in degree 0 and coincides with $\mathscr{F}_{M_{\mathrm{sal}}}(U)$.

4.4. A comparison result. — In the next lemma, we set $M := \mathbb{R}^n$ and we denote by dx the Lebesgue measure. As usual, for $\alpha \in \mathbb{N}^n$ we denote by D^α_x the differential operator $(\partial/\partial_{x_1})^{\alpha_1} \cdots (\partial/\partial_{x_n})^{\alpha_n}$ and we denote by $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ the Laplace operator on M.

In all this section, we consider an open set $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. We set for short

$$d(x) = d(x, M \setminus U).$$

For a locally integrable function φ on U and $s \in \mathbb{R}_{\geq 0}$, we set

$$\|\varphi\|_\infty = \sup_{x\in U} |\varphi(x)|, \quad \|\varphi\|^s_\infty = \|d(x)^s \varphi(x)\|_\infty. \tag{4.10}$$

Proposition 4.20. — *There exists a constant C_α such that for any locally integrable function φ on U, one has the estimate for $s \geq 0$:*

$$\|D_x^\alpha \varphi\|_\infty^{s+|\alpha|} \leq C_\alpha\big(\|\varphi\|_\infty^s + \|\Delta D_x^\alpha \varphi\|_\infty^{s+|\alpha|+2}\big). \tag{4.11}$$

Proof. — We shall adapt the proof of [**12**, Prop. 10.1].

(i) Let us take a distribution $K(x)$ and a $\mathscr{C}^\infty$ function $R(x)$ such that

$$\delta(x) = \Delta K(x) + R(x)$$

(where $\delta(x)$ is the Dirac distribution at the origin) and the support of $K(x)$ and the support of $R(x)$ are contained in $\{x \in M; |x| \leq 1\}$. Then $K(x)$ is integrable. For $c > 0$ and for a function ψ set:

$$\psi_c(x) = \psi(c^{-1}x),\ \widetilde{K}_c = c^{2-n}K_c \text{ and } \widetilde{R}_c = c^{-n}R_c.$$

Then we have again

$$\delta(x) = \Delta \widetilde{K}_c(x) + \widetilde{R}_c(x)\,.$$

Hence we have for any distribution ψ

$$\psi(x) = \int \widetilde{K}_c(x-y)(\Delta\psi)(y)dy + \int \widetilde{R}_c(x-y)\psi(y)dy\,. \tag{4.12}$$

Now for $x \in U$, set $c(x) = d(x)/2$. We set

$$\begin{aligned} A_\alpha(x) &= |\int \widetilde{K}_{c(x)}(x-y)(\Delta D_y^\alpha \varphi)(y)dy|,\\ B_\alpha(x) &= |\int \widetilde{R}_{c(x)}(x-y) D_y^\alpha \varphi(y)dy|. \end{aligned}$$

Since $\int |\widetilde{K}_{c(x)}(x-y)|dy = c(x)^2 \int |K(\frac{x}{c(x)} - y)|dy$, we get

$$\int |\widetilde{K}_{c(x)}(x-y)|dy \leq C_1 d(x)^2$$

for some constant C_1.

(ii) We have

$$\begin{aligned} A_\alpha(x) &\leq \Big(\sup_{|x-y|\leq c(x)} |(D_y^\alpha \Delta\varphi)(y)|\Big) \int |\widetilde{K}_{c(x)}(x-y)|dy\\ &\leq C_1 \Big(\sup_{|x-y|\leq c(x)} |(D_y^\alpha \Delta\varphi)(y)|\Big) \cdot d(x)^2. \end{aligned}$$

Hence,

$$\begin{aligned} d(x)^{s+|\alpha|}A_\alpha(x) &\leq C_1 \Big(\sup_{|x-y|\leq c(x)} |(D_y^\alpha \Delta\varphi)(y)|\Big) \cdot d(x)^{s+|\alpha|+2}\\ &\leq 2^{s+|\alpha|+2} C_1 \Big(\sup_{|x-y|\leq c(x)} |d(y)^{s+|\alpha|+2}(D_y^\alpha \Delta\varphi)(y)|\Big)\\ &\leq 2^{s+|\alpha|+2} C_1 \|\Delta D_x^\alpha \varphi\|_\infty^{s+|\alpha|+2}. \end{aligned} \tag{4.13}$$

Here we have used the fact that on the ball centered at x and radius $c(x)$, we have $d(x) \leq 2d(y)$.
(iii) Since $\widetilde{R}_c(x-y)$ is supported by the ball of center x and radius $c(x)$, we have

$$\begin{aligned} B_\alpha(x) &= |\int_{B(x,c(x))} D_y^\alpha \widetilde{R}_{c(x)}(x-y)\varphi(y)dy| \\ &= c(x)^{-|\alpha|}|\int_{B(x,c(x))} c(x)^{-n}(D_y^\alpha R)_{c(x)}(x-y)\varphi(y)dy| \\ &\leq c(x)^{-|\alpha|} \sup_{|x-y|\leq c(x)} |\varphi(y)| \cdot \int |D_y^\alpha R(y)|dy. \end{aligned}$$

Here we have used the fact that $D_y^\alpha R_{c(x)}(y) = c(x)^{-|\alpha|}(D_y^\alpha R)_{c(x)}(y)$.
As in (ii), we deduce that

$$\begin{aligned} d(x)^{s+|\alpha|}B_\alpha(x) &\leq C_2 \sup_{|x-y|\leq c(x)} |d(y)^s\varphi(y)| \\ &\leq C_2\|\varphi\|_\infty^s. \end{aligned} \tag{4.14}$$

for some constant C_2.
(iv) By choosing $\psi = D_x^\alpha\varphi$ in (4.12) the estimate (4.11) follows from (4.13) and (4.14). □

4.5. Sheaves on complex manifolds. — Let X be a complex manifold of complex dimension d_X and denote by $X_\mathbb{R}$ the real analytic underlying manifold. Denote by $\overline{X}$ the complex manifold conjugate to X. (The holomorphic functions on $\overline{X}$ are the anti-holomorphic functions on X.) Then $X \times \overline{X}$ is a complexification of $X_\mathbb{R}$ and $\mathscr{O}_{\overline{X}}$ is a $\mathscr{D}_{X\times\overline{X}}$-module which plays the role of the Dolbeault complex. In the sequel, when there is no risk of confusion, we write for short X instead of $X_\mathbb{R}$.

Notation 4.21. — In the sequel, we will often have to consider the composition $\mathrm{R}\rho_{\mathrm{sal}*} \circ \rho_{\mathrm{sa}!}$. For convenience, we introduce a notation. We set

$$\rho_{\mathrm{sl}*!} := \rho_{\mathrm{sal}*} \circ \rho_{\mathrm{sa}!}. \tag{4.15}$$

Sheaves on complex manifolds. — By applying the Dolbeault functor

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_{\overline{X}_{\mathrm{sal}}}}(\rho_{\mathrm{sl}*!}\mathscr{O}_{\overline{X}}, \bullet)$$

to one of the sheaves

$$\mathscr{C}^{\infty,\mathrm{tp}\,st}_{X_{\mathrm{sal}}},\quad \mathscr{C}^{\infty,\mathrm{tp}}_{X_{\mathrm{sal}}},\quad \mathscr{C}^{\infty,\mathrm{gev}(s)}_{X_{\mathrm{sal}}},\quad \mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{X_{\mathrm{sal}}},\quad \mathscr{C}^{\infty,\mathrm{gev\,st}}_{X_{\mathrm{sal}}},\quad \mathscr{C}^{\infty,\mathrm{gev}}_{X_{\mathrm{sal}}},\quad \mathscr{C}^{\infty}_{X_{\mathrm{sal}}},$$

we obtain respectively the sheaves

$$\mathscr{O}^{\mathrm{tp}\,st}_{X_{\mathrm{sal}}},\quad \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}},\quad \mathscr{O}^{\mathrm{gev}(s)}_{X_{\mathrm{sal}}},\quad \mathscr{O}^{\mathrm{gev}\{s\}}_{X_{\mathrm{sal}}},\quad \mathscr{O}^{\mathrm{gev\,st}}_{X_{\mathrm{sal}}},\quad \mathscr{O}^{\mathrm{gev}}_{X_{\mathrm{sal}}},\quad \mathscr{O}_{X_{\mathrm{sal}}}.$$

All these objects belong to $\mathrm{D}^+(\mathscr{D}_{X_{\mathrm{sal}}})$. Then we can apply the functor $\rho^!_{\mathrm{sal}}$ and we obtain the sheaves

$$\mathscr{O}^{\mathrm{tp}\,st}_{X_{\mathrm{sa}}},\quad \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}},\quad \mathscr{O}^{\mathrm{gev}(s)}_{X_{\mathrm{sa}}},\quad \mathscr{O}^{\mathrm{gev}\{s\}}_{X_{\mathrm{sa}}},\quad \mathscr{O}^{\mathrm{gev\,st}}_{X_{\mathrm{sa}}},\quad \mathscr{O}^{\mathrm{gev}}_{X_{\mathrm{sa}}},\quad \mathscr{O}_{X_{\mathrm{sa}}}.$$

Note that the functor $\rho_{\mathrm{sal}}^!$ commutes with the Dolbeault functor. More precisely:

Lemma 4.22. — *Let $\mathscr{C}$ be an object of $\mathrm{D}^+(\mathscr{D}_{X^{\mathbb{R}}_{\mathrm{sal}}})$. There is a natural isomorphism*

$$\rho_{\mathrm{sal}}^! \mathrm{R}\mathscr{H}om_{\mathscr{D}_{\overline{X}_{\mathrm{sal}}}}(\rho_{\mathrm{sl}*!}\mathscr{O}_{\overline{X}}, \mathscr{C}_{X_{\mathrm{sal}}}) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_{\overline{X}_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{O}_{\overline{X}}, \rho_{\mathrm{sal}}^!\mathscr{C}_{X_{\mathrm{sal}}}). \tag{4.16}$$

Proof. — This follows from the fact that the $\mathscr{D}_X$-module $\mathscr{O}_{\overline{X}}$ admits a global locally finite free resolution. □

Recall the natural isomorphism [**12**, Th. 10.5]

$$\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}} \xrightarrow{\ \sim\ } \mathrm{R}\mathscr{H}om_{\mathscr{D}_{\overline{X}_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{O}_{\overline{X}}, \mathscr{D}b^{\mathrm{tp}}_{X_{\mathrm{sa}}}).$$

Proposition 4.23. — *The natural morphism*

$$\mathscr{O}^{\mathrm{tp}\,st}_{X_{\mathrm{sal}}} \to \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}}$$

is an isomorphism in $\mathrm{D}^+(\mathscr{D}_{X_{\mathrm{sal}}})$.

Proof. — Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. Consider the diagram (in which $M = \mathbb{R}^{2n}$)

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(U; \mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}}) \arrow[r, "\Delta"] \arrow[d] & \Gamma(U; \mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(U; \mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}) \arrow[r, "\Delta"] & \Gamma(U; \mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}) \arrow[r] & 0.
\end{tikzcd}$$

As in the proof of [**12**, Th. 10.5], we are reduced to prove that the vertical arrows induce a qis from the top line to the bottom line. We shall apply Proposition 4.20.
(i) Let $\varphi \in \Gamma(U; \mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}})$ with $\Delta\varphi = 0$. There exists some $s \geq 0$ such that $\|d(x)^s\varphi\|_\infty < \infty$. Then $\|d(x)^{s+|\alpha|}D_x^\alpha\varphi\|_\infty < \infty$ by (4.11).
(ii) It follows from [**12**, Prop.10.1] that the arrow in the bottom is surjective. Now let $\psi \in \Gamma(U; \mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}})$. There exists $\varphi \in \Gamma(U; \mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}})$ with $\Delta\varphi = \psi$. Then it follows from (4.11) that $\varphi \in \Gamma(U; \mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}})$. □

Remark 4.24. — It is natural to expect that the morphism

$$\mathscr{O}^{\mathrm{gev\,st}}_{X_{\mathrm{sal}}} \to \mathscr{O}^{\mathrm{gev}}_{X_{\mathrm{sal}}}$$

is an isomorphism in $\mathrm{D}^+(\mathscr{D}_{X_{\mathrm{sal}}})$. The proof of Proposition 4.23 can be adapted with the exception that one does not know if the map $\Delta\colon \mathscr{C}^{\infty,\mathrm{gev}}_{X_{\mathrm{sa}}}(U) \to \mathscr{C}^{\infty,\mathrm{gev}}_{X_{\mathrm{sa}}}(U)$ is surjective.

Solutions of holonomic $\mathscr{D}$-modules. — The next result is a reformulation of a theorem of Kashiwara [**9**].

Theorem 4.25. — *Let $\mathscr{M}$ be a regular holonomic $\mathscr{D}_X$-module. Then the natural morphism*

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{M}, \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}) \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{M}, \mathscr{O}_{X_{\mathrm{sa}}})$$

is an isomorphism.

The next result was a conjecture of [**14**] and has recently been proved by Morando [**25**] (see also [**16**] for a rather different proof) by using the deep results of Mochizuki [**24**] (completed by those of Kedlaya [**17, 18**] for the analytic case).

Theorem 4.26. — *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then for any $G \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_X)$,*

$$\rho_{\mathrm{sa}}^{-1}\mathrm{R}\mathscr{H}om\,(G, \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{M}, \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})) \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_X).$$

It is natural to conjecture that this theorem still holds when replacing the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ with one of the sheaves $\mathscr{O}^{\mathrm{gev}(s)}_{X_{\mathrm{sa}}}$ or $\mathscr{O}^{\mathrm{gev}\{s\}}_{X_{\mathrm{sa}}}$.

In [**14**], the object $\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{M}, \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})$ is explicitly calculated when $X = \mathbb{C}$ and, denoting by t a holomorphic coordinate on X, $\mathscr{M}$ is associated with the operator $t^2\partial_t + 1$, that is, $\mathscr{M} = \mathscr{D}_X \exp(1/t)$.

It is well-known, after [**28**] (see also [**19**]), that the holomorphic solutions of an ordinary linear differential equation singular at the origin have Gevrey growth, the growth being related to the slopes of the Newton polygon.

Conjecture 4.27. — Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then the natural morphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{M}, \mathscr{O}^{\mathrm{gev}}_{X_{\mathrm{sa}}}) \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{M}, \mathscr{O}_{X_{\mathrm{sa}}})$$

is an isomorphism, or, equivalently,

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{M}, \mathscr{O}^{\mathrm{gev}}_{X_{\mathrm{sa}}}) \xrightarrow{\sim} \mathrm{R}\rho_{\mathrm{sa}*}\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X).$$

Moreover, there exists a discrete set $Z \subset \mathbb{R}_{>1}$ such that the morphisms $\mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\mathscr{M}, \mathscr{O}^{\mathrm{gev}(s)}_{X_{\mathrm{sa}}}) \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\mathscr{M}, \mathscr{O}^{\mathrm{gev}(t)}_{X_{\mathrm{sa}}})$ are isomorphisms for $s \leq t$ in the same components of $\mathbb{R}_{>1} \setminus Z$.

5. Filtrations

5.1. Derived categories of filtered objects. — In this section, we shall recall results of [**32**] completed in [**31**].

Complements on abelian categories. — In this subsection we state and prove some elementary results (some of them being well-known) on abelian and derived categories that we shall need.

Let $\mathscr{C}$ be an abelian category and let Λ be a small category. As usual, one denotes by $\mathrm{Fct}(\Lambda, \mathscr{C})$ the abelian category of functors from Λ to $\mathscr{C}$. Recall that the kernel of a morphism $u\colon X \to Y$ is the functor $\lambda \mapsto \operatorname{Ker} u(\lambda)$ and similarly with the cokernel or more generally with limits and colimits.

Lemma 5.1. — *Assume that $\mathscr{C}$ is a Grothendieck category. Then*

(a) *the category* $\mathrm{Fct}(\Lambda, \mathscr{C})$ *is a Grothendieck category,*

(b) *if $F \in \mathrm{Fct}(\Lambda, \mathscr{C})$ is injective, then for $\lambda \in \Lambda$, $F(\lambda)$ is injective in $\mathscr{C}$.*

Proof. — The category $\mathrm{Fct}(\Lambda, \mathscr{C})$ is equivalent to the category $\mathrm{PSh}(\Lambda^{\mathrm{op}}, \mathscr{C})$ of presheaves on Λ^{op} with values in $\mathscr{C}$. It follows that, for any given $\lambda \in \Lambda$, the functor $\mathrm{Fct}(\Lambda, \mathscr{C}) \to \mathscr{C}$, $F \mapsto F(\lambda)$ has a left adjoint. We can define it as follows (see e.g., [**15**, Not. 17.6.13]). For $G \in \mathscr{C}$ we define $G_\lambda \in \mathrm{Fct}(\Lambda, \mathscr{C})$ by

$$G_\lambda(\mu) = \bigoplus_{\mathrm{Hom}_{\Lambda}(\lambda,\mu)} G.$$

Then we can check directly that

(5.1) the functor $\mathscr{C} \ni G \mapsto G_\lambda \in \mathrm{Fct}(\Lambda, \mathscr{C})$ is exact,

(5.2) $\mathrm{Hom}_{\mathrm{Fct}(\Lambda,\mathscr{C})}(G_\lambda, F) \simeq \mathrm{Hom}_{\mathscr{C}}(G, F(\lambda))$ for any $F \in \mathrm{Fct}(\Lambda, \mathscr{C})$.

(a) Applying e.g., Th. 17.4.9 of loc. cit., it remains to show that $\mathrm{Fct}(\Lambda, \mathscr{C})$ admits a small system of generators. Let G be a generator of $\mathscr{C}$. It follows from (5.2) that the family $\{G_\lambda\}_{\lambda \in \Lambda}$ is a small system of generators in $\mathrm{Fct}(\Lambda, \mathscr{C})$.
(b) Follows from (5.2) and (5.1). $\square$

We consider two abelian categories $\mathscr{C}$ and $\mathscr{C}'$ and a left exact functor $\rho\colon \mathscr{C} \to \mathscr{C}'$. The functor ρ induces a functor

(5.3) $$\widetilde{\rho}\colon \mathrm{Fct}(\Lambda, \mathscr{C}) \to \mathrm{Fct}(\Lambda, \mathscr{C}').$$

Lemma 5.2. — *Assume that $\mathscr{C}$ is a Grothendieck category.*
(a) *The functor $\widetilde{\rho}$ is left exact.*
(b) *Let I be a small category and assume that ρ commutes with colimits indexed by I. Then the functor $\widetilde{\rho}$ in* (5.3) *commutes with colimits indexed by I.*
(c) *Assume that ρ has cohomological dimension $\leq d$, that is, $R^j\rho = 0$ for $j > d$. Then $\widetilde{\rho}$ has cohomological dimension $\leq d$.*
(d) *Assume that ρ commutes with small direct sums and that small direct sums of injective objects in $\mathscr{C}$ are acyclic for the functor ρ. Then small direct sums of injective objects in* $\mathrm{Fct}(\Lambda, \mathscr{C})$ *are acyclic for the functor $\widetilde{\rho}$.*

Proof. — (a) is obvious.
(b) follows from the equivalence $\mathrm{Fct}(I, \mathrm{Fct}(\Lambda, \mathscr{C})) \simeq \mathrm{Fct}(\Lambda, \mathrm{Fct}(I, \mathscr{C}))$ and similarly with $\mathscr{C}'$.
(c) By Lemma 5.1 (a), the category $\mathrm{Fct}(\Lambda, \mathscr{C})$ admits enough injectives. Let $F \in \mathrm{Fct}(\Lambda, \mathscr{C})$ and let $F \to F^\bullet$ be an injective resolution of F, that is, $F^\bullet$ is a complex in degrees ≥ 0 of injective objects and $F \to F^\bullet$ is a qis. By Lemma 5.1 (b), for $\lambda \in \Lambda$, $F^\bullet(\lambda)$ is an injective resolution of $F(\lambda)$ and by the hypothesis, $H^j(\rho(F^\bullet(\lambda))) \simeq 0$ for $j > d$ and $\lambda \in \Lambda$. This implies that $R^j\rho(F) \simeq H^j(\rho(F^\bullet))$ is 0 for $j > d$.
(d) For a given $\lambda \in \Lambda$ we denote by $i^{\mathscr{C}}_\lambda$ the functor $\mathrm{Fct}(\Lambda, \mathscr{C}) \to \mathscr{C}$, $F \mapsto F(\lambda)$. Then $i^{\mathscr{C}}_\lambda$ is exact and, by Lemma 5.1 (b), we have $i^{\mathscr{C}'}_\lambda \circ R\widetilde{\rho} \simeq R\rho \circ i^{\mathscr{C}}_\lambda$. Let $F \in \mathrm{Fct}(\Lambda, \mathscr{C})$ be a small direct sum of injective objects. Since $i^{\mathscr{C}}_\lambda$ commutes with direct sums, it follows from Lemma 5.1 (b) again that $i^{\mathscr{C}}_\lambda(F)$ is a small direct sum of injective objects in $\mathscr{C}$. By the hypothesis we obtain $R^j\rho \circ i^{\mathscr{C}}_\lambda(F) \simeq 0$, for all $j > 0$. Hence $i^{\mathscr{C}'}_\lambda \circ R^j\widetilde{\rho}(F) \simeq 0$,

for all $j > 0$. Since this holds for all $\lambda \in \Lambda$ we deduce $R^j\widetilde{\rho}(F) \simeq 0$, for all $j > 0$, as required. □

Abelian tensor categories. — Recall (see e.g., [**15**, Ch. 5]) that a tensor Grothendieck category $\mathscr{C}$ is a Grothendieck category endowed with a biadditive functor $\otimes\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ satisfying functorial associativity isomorphisms. We do not recall here what is a tensor category with unit, a ring object A in $\mathscr{C}$, a ring object with unit and an A-module M. In the sequel, all tensor categories will be with unit and a ring object means a ring object with unit.

We shall consider

$$\text{(5.4)}\qquad \begin{cases} \text{a Grothendieck tensor category } \mathscr{C} \text{ (with unit) in which} \\ \text{small inductive limits commute with } \otimes. \end{cases}$$

Lemma 5.3. — *Let $\mathscr{C}$ be as in* (5.4) *and let A be a ring object* (*with unit*) *in $\mathscr{C}$. Then*

(a) *The category* $\mathrm{Mod}(A)$ *is a Grothendieck category,*
(b) *the forgetful functor* $\mathrm{for}\colon \mathrm{Mod}(A) \to \mathscr{C}$ *is exact and conservative,*
(c) *the natural functor* $\widetilde{\mathrm{for}}\colon \mathrm{D}(A) \to \mathrm{D}(\mathscr{C})$ *is conservative.*

Proof. — (a) and (b) are proved in [**31**, Prop. 4.4].
(c) Since $\mathrm{D}(A)$ and $\mathrm{D}(\mathscr{C})$ are triangulated, it is enough to check that if $X \in \mathrm{D}(A)$ verifies $\widetilde{\mathrm{for}}(X) \simeq 0$, then $X \simeq 0$. Let X be such an object and let $j \in \mathbb{Z}$. Since for is exact, $\mathrm{for} H^j(X) \simeq H^j(\widetilde{\mathrm{for}}(X)) \simeq 0$. Since for is conservative, we get $H^j(X) \simeq 0$. □

Derived categories of filtered objects. — We shall consider

$$\text{(5.5)}\qquad \begin{cases} \text{a filtrant preordered additive monoid } \Lambda \text{ (viewed as a tensor} \\ \text{category with unit),} \\ \text{a category } \mathscr{C} \text{ as in (5.4).} \end{cases}$$

Denote by $\mathrm{Fct}(\Lambda, \mathscr{C})$ the abelian category of functors from Λ to $\mathscr{C}$. It is naturally endowed with a structure of a tensor category with unit by setting for $M_1, M_2 \in \mathrm{Fct}(\Lambda, \mathscr{C})$,

$$(M_1 \otimes M_2)(\lambda) = \varinjlim_{\lambda_1+\lambda_2 \leq \lambda} M_1(\lambda_1) \otimes M_2(\lambda_2).$$

A Λ-ring A of $\mathscr{C}$ is a ring with unit of the tensor category $\mathrm{Fct}(\Lambda, \mathscr{C})$ and we denote by $\mathrm{Mod}(A)$ the abelian category of A-modules.

We denote by $\mathrm{F}_\Lambda \mathscr{C}$ the full subcategory of $\mathrm{Fct}(\Lambda, \mathscr{C})$ consisting of functors M such that for each morphism $\lambda \to \lambda'$ in Λ, the morphism $M(\lambda) \to M(\lambda')$ is a monomorphism. This is a quasi-abelian category. Let

$$\iota\colon\ \mathrm{F}_\Lambda \mathscr{C} \to \mathrm{Fct}(\Lambda, \mathscr{C})$$

denote the inclusion functor. This functor admits a left adjoint κ and the category $\mathrm{F}_\Lambda \mathscr{C}$ is again a tensor category by setting

$$M_1 \otimes_F M_2 = \kappa(\iota(M_1) \otimes \iota(M_2)).$$

A ring object in the tensor category $\mathrm{F}_\Lambda\mathscr{C}$ will be called a Λ-filtered ring in $\mathscr{C}$ and usually denoted FA. An FA-module FM is then simply a module over FA in $\mathrm{F}_\Lambda\mathscr{C}$ and we denote by $\mathrm{Mod}(FA)$ the quasi-abelian category of FA-modules.

It follows from Lemmas 5.1 and 5.3 that $\mathrm{Mod}(\iota FA)$ is a Grothendieck category.

Notation 5.4. — In the sequel, for a ring object B in a tensor category, we shall write $\mathrm{D}^*(B)$ instead of $\mathrm{D}^*(\mathrm{Mod}(B))$, $* = +, -, \mathrm{b}, \mathrm{ub}$.

The next theorem is due to [**31**] and generalizes previous results of [**32**].

Theorem 5.5. — *Assume* (5.5). *Let FA be a Λ-filtered ring in $\mathscr{C}$. Then the category* $\mathrm{Mod}(FA)$ *is quasi-abelian, the functor* $\iota\colon \mathrm{Mod}(FA) \to \mathrm{Mod}(\iota FA)$ *is strictly exact and induces an equivalence of categories for* $* = \mathrm{ub}, +, -, \mathrm{b}$*:*

$$\iota\colon \mathrm{D}^*(FA) \xrightarrow{\sim} \mathrm{D}^*(\iota FA). \tag{5.6}$$

Notation 5.6. — Let Λ and $\mathscr{C}$ be as in (5.5). The functor $\varinjlim\colon \mathrm{Fct}(\Lambda,\mathscr{C}) \to \mathscr{C}$ is exact. Let FA be a Λ-filtered ring in $F_\Lambda\mathscr{C}$) and set

$$A := \varinjlim_{\lambda} A(\lambda). \tag{5.7}$$

(For short, we write $A(\lambda)$ instead of $FA(\lambda)$.) The functor $\varinjlim$ induces an exact functor

$$\varinjlim\colon \mathrm{Mod}(FA) \to \mathrm{Mod}(A), \tag{5.8}$$

thus, using Theorem 5.5, for $* = \mathrm{ub}, +, -, \mathrm{b}$, a functor

$$\varinjlim\colon \mathrm{D}^*(FA) \to \mathrm{D}^*(A). \tag{5.9}$$

Since one often considers FA as a filtration on the ring A, we shall denote by for (forgetful) the functor $\varinjlim$:

$$\mathrm{for}\colon \mathrm{D}^*(FA) \to \mathrm{D}^*(A), \quad \mathrm{for} := \varinjlim. \tag{5.10}$$

Complements on filtered objects

Lemma 5.7. — *Let Λ and $\mathscr{C}$ be as in* (5.5) *and let $\mathscr{C}'$ be another Grothendieck tensor category satisfying the same hypotheses as $\mathscr{C}$. Let FB be a Λ-filtered ring in $\mathscr{C}'$.*

(a) *Let $\sigma\colon \mathscr{C}' \to \mathscr{C}$ be an exact functor of tensor categories* (*see Definition 4.2.2 in* [**15**]). *Denote by $\widetilde{\sigma}\colon \mathrm{Fct}(\Lambda,\mathscr{C}') \to \mathrm{Fct}(\Lambda,\mathscr{C})$ the natural functor associated with σ. Then*
   - (i) *$FA := \widetilde{\sigma}(FB)$ has a natural structure of a Λ-filtered ring with values in $\mathscr{C}$,*
   - (ii) *the functor $\widetilde{\sigma}$ induces an exact functor $\widetilde{\sigma}_\Lambda\colon \mathrm{Mod}(\iota FB) \to \mathrm{Mod}(\iota FA)$ hence a functor $\sigma_\Lambda\colon \mathrm{Mod}(FB) \to \mathrm{Mod}(FA)$.*

(b) *Assume moreover that the functor σ has a right adjoint ρ which is fully faithful* (*hence ρ is left exact and $\sigma\rho \simeq \mathrm{id}_{\mathscr{C}}$*). *Denote by $\widetilde{\rho}\colon \mathrm{Fct}(\Lambda,\mathscr{C}) \to \mathrm{Fct}(\Lambda,\mathscr{C}')$ the natural functor associated with ρ. Then*
   - (i) *$\widetilde{\rho}$ is fully faithful and right adjoint to $\widetilde{\sigma}$,*

(ii) $\widetilde{\rho}$ *induces a left exact fully faithful functor* $\widetilde{\rho}_\Lambda\colon \mathrm{Mod}(\iota FA) \to \mathrm{Mod}(\iota FB)$ *right adjoint to* $\widetilde{\sigma}_\Lambda$ *and a fully faithful functor* $\rho_\Lambda\colon \mathrm{Mod}(FA) \to \mathrm{Mod}(FB)$ *right adjoint to* σ_Λ.

(c) *The diagram below, in which the horizontal arrows are the forgetful functors, is commutative when composing horizontal and down vertical arrows, or when composing horizontal and up vertical arrows*

$$\begin{tikzcd} \mathrm{Mod}(FA) \arrow[r] \arrow[d, shift left, "\rho_\Lambda"] & \mathrm{Mod}(\iota FA) \arrow[r] \arrow[d, shift left, "\widetilde{\rho}_\Lambda"] & \mathrm{Fct}(\Lambda, \mathscr{C}) \arrow[d, shift left, "\widetilde{\rho}"] \\ \mathrm{Mod}(FB) \arrow[r] \arrow[u, shift left, "\sigma_\Lambda"] & \mathrm{Mod}(\iota FB) \arrow[r] \arrow[u, shift left, "\widetilde{\sigma}_\Lambda"] & \mathrm{Fct}(\Lambda, \mathscr{C}'). \arrow[u, shift left, "\widetilde{\sigma}"] \end{tikzcd}$$

Proof. — (a) We first recall that a Λ-ring A of a tensor category $\mathscr{C}$ is the data of $A(\lambda) \in \mathscr{C}$, for each $\lambda \in \Lambda$, morphisms $\mu_A^{\lambda,\lambda'}\colon A(\lambda)\otimes A(\lambda') \to A(\lambda+\lambda')$, for all $\lambda, \lambda' \in \Lambda$, and $\varepsilon_A\colon \mathbf{1}_{\mathscr{C}} \to A(0)$, where $\mathbf{1}_{\mathscr{C}}$ is the unit of $\mathscr{C}$ and 0 the unit of Λ. These morphisms satisfy three commutative diagrams (which we do not recall here) expressing the associativity of μ_A and the fact that ε_A is a unit. Similarly a module M over A is the data of $M(\lambda) \in \mathscr{C}$, for each $\lambda \in \Lambda$, and morphisms $\mu_M^{\lambda,\lambda'}\colon A(\lambda)\otimes M(\lambda') \to M(\lambda+\lambda')$, for all $\lambda,\lambda' \in \Lambda$, satisfying two commutative diagrams left to the reader.

Let us go back to the situation of the lemma. For a Λ-filtered ring FB of $\mathscr{C}'$ and an FB-module N, setting $FA = \widetilde{\sigma}(FB)$, the morphisms $\mu_N^{\lambda,\lambda'}$ induce

$$\mu_{\widetilde{\sigma}(N)}^{\lambda,\lambda'}\colon A(\lambda)\otimes \sigma(N(\lambda')) \simeq \sigma(B(\lambda)\otimes N(\lambda')) \to \sigma(N(\lambda+\lambda')).$$

For $N = \iota FB$ we obtain $\mu_A^{\lambda,\lambda'}$. We define $\varepsilon_A = \sigma(\varepsilon_B)$. We leave to the reader the verification that ε_A, $\mu_A^{\cdot,\cdot}$ and $\mu_{\widetilde{\sigma}(N)}^{\cdot,\cdot}$ satisfy the required commutative diagrams. This defines the functor $\widetilde{\sigma}_\Lambda$. We see easily that $\widetilde{\sigma}_\Lambda$ is exact. Since FB is Λ-filtered, the exactness of σ implies that FA is Λ-filtered and that $\widetilde{\sigma}_\Lambda$ induces the functor σ_Λ of the lemma.

(b) The statement (i) is straightforward. Let us define $\widetilde{\rho}_\Lambda$. For a ιFA-module M the data of

$$\mu_M^{\lambda,\lambda'}\colon \sigma(B(\lambda)\otimes\rho(M(\lambda'))) \simeq A(\lambda)\otimes M(\lambda') \to M(\lambda+\lambda')$$

give by adjunction $\mu_{\widetilde{\rho}(M)}^{\lambda,\lambda'}\colon B(\lambda)\otimes\rho(M(\lambda')) \to \rho(M(\lambda+\lambda'))$ and define a structure of ιFB-module on $\rho(M)$. Since ρ is left exact $\widetilde{\rho}_\Lambda$ induces ρ_Λ. The adjunction properties are clear, as well as $\widetilde{\sigma}_\Lambda\widetilde{\rho}_\Lambda \simeq \mathrm{id}$ and $\sigma_\Lambda\rho_\Lambda \simeq \mathrm{id}$. Hence $\widetilde{\rho}_\Lambda$ and ρ_Λ are fully faithful.

(c) is clear. □

Theorem 5.8. — (1) *We make the assumptions of* Lemma 5.7 (a)-(b) *and assume moreover that*

(i) ρ *has cohomological dimension* $\leq d$,

(ii) *for any* $M \in \mathrm{Mod}(\iota FA)$, *there exists a monomorphism* $M \to I$ *in* $\mathrm{Mod}(\iota FA)$ *such that* $I(\lambda)$ *is* ρ*-acyclic, for all* $\lambda \in \Lambda$.

Then the derived functor $R\rho_\Lambda\colon \mathrm{D}^*(FA) \to \mathrm{D}^*(FB)$ $(* = \mathrm{ub}, +)$ *exists. It is fully faithful and admits a left adjoint* $\rho_\Lambda^{-1}\colon \mathrm{D}^*(FB) \to \mathrm{D}^*(FA)$ $(* = \mathrm{ub}, +)$.

(2) *Assume moreover that*

(iii) ρ *commutes with small direct sums,*
(iv) *small direct sums of injective objects in* $\mathscr{C}$ *are acyclic for the functor* ρ.

Then the derived functor $R\rho_\Lambda\colon \mathrm{D}(FA) \to \mathrm{D}(FB)$ *commutes with small direct sums and admits a right adjoint* $\rho_\Lambda^!\colon \mathrm{D}(FB) \to \mathrm{D}(FA)$. *Moreover,* $\rho_\Lambda^!$ *induces a functor* $\mathrm{D}^+(FB) \to \mathrm{D}^+(FA)$.

(3) *We make the assumptions of* Lemma 5.7 (a) *and assume moreover that* σ *is fully faithful and has a right adjoint* ρ *which is exact. Then the derived functor* $\sigma_\Lambda\colon \mathrm{D}^*(FA) \to \mathrm{D}^*(FB)$ $(* = \mathrm{ub}, +, \mathrm{b})$ *is well defined, is fully faithful and admits a right adjoint* $\rho_\Lambda\colon \mathrm{D}^*(FB) \to \mathrm{D}^*(FA)$ $(* = \mathrm{ub}, +, \mathrm{b})$.

Proof. — By Theorem 5.5, it is enough to prove the statements when replacing FA and FB with ιFA and ιFB, respectively and ρ_Λ with $\widetilde{\rho}_\Lambda$.

(1) Let us first prove that $\widetilde{\rho}_\Lambda\colon \mathrm{Mod}(\iota FA) \to \mathrm{Mod}(\iota FB)$ admits a derived functor and has cohomological dimension $\leq d$.

We let $\mathscr{I}$ be the subcategory of $\mathrm{Mod}(\iota FA)$ which consists of the $I \in \mathrm{Mod}(\iota FA)$ such that $I(\lambda)$ is ρ-acyclic, for all $\lambda \in \Lambda$. Using the hypothesis (iv) and the relation $\mathrm{for} \circ \widetilde{\rho}_\Lambda \simeq \widetilde{\rho} \circ \mathrm{for}$ we see that the subcategory $\mathscr{I}$ is $\widetilde{\rho}_\Lambda$-injective. Hence $R\widetilde{\rho}_\Lambda$ exists. We also see that $\mathrm{for}(\mathscr{I})$ is a $\widetilde{\rho}$-injective family. Hence $\mathrm{for} \circ R\widetilde{\rho}_\Lambda \simeq R\widetilde{\rho} \circ \mathrm{for}$. Now the assertion on the cohomological dimension follows from Lemma 5.2-(c).

By Lemma 5.7, the functor $\widetilde{\rho}_\Lambda$ is right adjoint to $\widetilde{\sigma}_\Lambda$. This functor $\widetilde{\sigma}_\Lambda$ induces $\widetilde{\rho}_\Lambda^{-1}$ on the derived category which is left adjoint to $R\widetilde{\rho}_\Lambda$. The relation $\widetilde{\sigma}_\Lambda\widetilde{\rho}_\Lambda \simeq \mathrm{id}$ gives $\widetilde{\rho}_\Lambda^{-1}R\widetilde{\rho}_\Lambda \simeq \mathrm{id}$. Hence $R\widetilde{\rho}_\Lambda$ is fully faithful.

(2) By the Brown representability theorem, it is enough to prove that

$$R\widetilde{\rho}_\Lambda \text{ commutes with small direct sums.} \tag{5.11}$$

We consider the functor $\widetilde{\rho}$: $\mathrm{Fct}(\Lambda, \mathscr{C}) \to \mathrm{Fct}(\Lambda, \mathscr{C}')$. The hypotheses of Proposition 2.21 are satisfied by Lemma 5.2. Therefore the functor $\widetilde{\rho}$ has cohomological dimension $\leq d$ and the functor $R\widetilde{\rho}$: $\mathrm{D}(\mathrm{Fct}(\Lambda, \mathscr{C})) \to \mathrm{D}(\mathrm{Fct}(\Lambda, \mathscr{C}'))$ commutes with small direct sums.

Now we prove (5.11). Let $\{X_i\}_{i\in I}$ be a family of objects of $\mathrm{D}(\iota FA)$. There is a natural morphism $\bigoplus_{i\in I} R\widetilde{\rho}_\Lambda(X_i) \to R\widetilde{\rho}_\Lambda(\bigoplus_{i\in I} X_i)$ in $\mathrm{D}(\iota FB)$ and it follows from Lemma 5.3 that this morphism is an isomorphism.

(3) is obvious. □

5.2. Filtrations on $\mathscr{O}_{X_{\mathrm{sal}}}$. — In the sequel, if FM is a filtered object in $\mathscr{C}$ over the ordered additive monoid $\mathbb{R}$, we shall write F^sM instead of $(FM)(s)$ to denote the image of the functor FM at $s \in \mathbb{R}$. This induces a functor $\mathrm{D}(\mathrm{F}_\mathbb{R}\mathscr{C}) \to \mathrm{D}(\mathscr{C})$ denoted in the same way $FM \mapsto F^sM$.

The filtered ring of differential operators. — Recall that the sheaf $\mathscr{D}_M$ of finite order differential operators on M has a natural $\mathbb{N}$-filtration given by the order.

Recall that the rings $\mathscr{D}_{M_{\mathrm{sa}}}$ and $\mathscr{D}_{M_{\mathrm{sal}}}$ as well as the sheaves $\mathscr{D}_{M_{\mathrm{sa}}}(m)$ and $\mathscr{D}_{M_{\mathrm{sal}}}(m)$ are defined in (4.7) and (4.8). We remark that $\rho_{\mathrm{sa}}^{-1}(\mathscr{D}_{M_{\mathrm{sa}}}(m)) \simeq \mathscr{D}_M(m)$ and $\rho_{\mathrm{sal}}^{-1}(\mathscr{D}_{M_{\mathrm{sal}}}(m)) \simeq \mathscr{D}_{M_{\mathrm{sa}}}(m)$.

Definition 5.9. — Let $\mathscr{T}$ be the site M or M_{sa} or M_{sal}. We define the filtered sheaf $\mathrm{F}\mathscr{D}_{\mathscr{T}}$ over $\Lambda = \mathbb{R}$ by setting:

$$\mathrm{F}^s\mathscr{D}_{\mathscr{T}} = \mathscr{D}_{\mathscr{T}}([s])$$

where $[s]$ is the integral part of s and $\mathscr{D}_{\mathscr{T}}([s])$ is the sheaf of differential operators of order $\leq [s]$. In particular, $\mathrm{F}^s\mathscr{D}_{\mathscr{T}} = 0$ for $s < 0$. We denote by $\mathrm{Mod}(\mathrm{F}\mathscr{D}_{\mathscr{T}})$ the category of filtered modules over $\mathscr{D}_{\mathscr{T}}$.

Let $M_{\mathscr{T}}$ be either M, M_{sa} or M_{sal}. In the sequel, we look at $\mathrm{Mod}(\mathbb{C}_{M_{\mathscr{T}}})$ as an abelian Grothendieck tensor category with unit and at $\mathrm{F}\mathscr{D}_{M_{\mathscr{T}}}$ as a Λ-ring object in $F_\Lambda\mathscr{C}$ (with $\Lambda = \mathbb{R}$) and $\mathscr{C} = \mathrm{Mod}(\mathbb{C}_{M_{\mathscr{T}}})$. Note that Definition 5.9 is in accordance with Lemma 5.7 (a) (i).

Since $\rho_{\mathrm{sa}}^{-1}(\mathscr{D}_{M_{\mathrm{sa}}}(m)) \simeq \mathscr{D}_M(m)$ and $\rho_{\mathrm{sal}}^{-1}(\mathscr{D}_{M_{\mathrm{sal}}}(m)) \simeq \mathscr{D}_{M_{\mathrm{sa}}}(m)$ we can apply Lemma 5.7 (a) with the exact functors $\sigma = \rho_{\mathrm{sa}}^{-1}$ or $\sigma = \rho_{\mathrm{sal}}^{-1}$. We obtain the functors

$$\begin{aligned} &\rho_{\mathrm{sa}}^{-1}\colon \mathrm{Mod}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{Mod}(\mathrm{F}\mathscr{D}_M),\\ &\rho_{\mathrm{sal}}^{-1}\colon \mathrm{Mod}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}}) \to \mathrm{Mod}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}). \end{aligned} \tag{5.12}$$

We will also use the fully faithful right adjoint of ρ_{sal}^{-1} given by Lemma 5.7 (b)

$$\rho_{\mathrm{sal}*}\colon \mathrm{Mod}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{Mod}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}}). \tag{5.13}$$

Theorem 5.10. — (i) *The functor $\rho_{\mathrm{sal}*}$ in (5.13) admits a right derived functor $\mathrm{R}\rho_{\mathrm{sal}*}\colon \mathrm{D}^*(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{D}^*(\mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}})$ $(* = \mathrm{ub}, +)$ which is fully faithful and admits a left adjoint functor $\rho_{\mathrm{sal}}^{-1}\colon \mathrm{D}^*(\mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}}) \to \mathrm{D}^*(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$ $(* = \mathrm{ub}, +)$.*

(ii) *The functor $\mathrm{R}\rho_{\mathrm{sal}*}$ $(* = \mathrm{ub}, +)$ commutes with small direct sums and admits a right adjoint $\rho_{\mathrm{sal}}^{!}\colon \mathrm{D}^*(\mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}}) \to \mathrm{D}^*(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$ $(* = \mathrm{ub}, +)$.*

(iii) *The functor $\rho_{\mathrm{sa}}^{-1}\colon \mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{D}^+(\mathrm{F}\mathscr{D}_M)$ has a fully faithful right adjoint $\mathrm{R}\rho_{\mathrm{sa}!}\colon \mathrm{D}^+(\mathrm{F}\mathscr{D}_M) \to \mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$.*

Proof. — (i)–(ii) We shall apply Theorem 5.8 (1)-(2) with $\mathscr{C} = \mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sa}}})$, $\mathscr{C}' = \mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sal}}})$, $\rho = \rho_{\mathrm{sal}*}$, $\sigma = \rho_{\mathrm{sal}}^{-1}$, $\Lambda = \mathbb{R}$, $FA = \mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}$, $FB = \mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}}$. Let us check hypotheses (i)–(iv) of Theorem 5.8. Hypothesis (i) follows from Proposition 2.28. The hypotheses (iii) and (iv) follow from Lemma 2.9. By Lemma 5.3 we know that $\mathrm{Mod}(\iota\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$ has enough injectives. Hence to check the hypothesis (ii) it is enough to prove that if $I \in \mathrm{Mod}(\iota\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$ is injective, then $I(\lambda)$ is $\rho_{\mathrm{sal}*}$-acyclic for any $\lambda \in \Lambda$.

By Lemmas 2.27 and 2.17 it is enough to prove that $I(\lambda)$ is flabby. For any $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ we have

$$\Gamma(U; I(\lambda)) \simeq \mathrm{Hom}_{\mathrm{Mod}(\iota \mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})}((\mathscr{D}_{M_{\mathrm{sa}}}^{[-\lambda]})_U, I), \tag{5.14}$$

where $\mathscr{D}_{M_{\mathrm{sa}}}^{[-\lambda]}$ denotes the object $\iota \mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}$ with the filtration shifted by λ, that is, $F^s\mathscr{D}_{M_{\mathrm{sa}}}^{[-\lambda]} = F^{s-\lambda}\mathscr{D}_{M_{\mathrm{sa}}}$; this isomorphism sends a section s of $I(\lambda)$ to the morphism $1 \mapsto s$ (which is filtered because $1 \in F^\lambda\mathscr{D}_{M_{\mathrm{sa}}}^{[-\lambda]}$). Hence the flabbiness of $I(\lambda)$ follows from the injectivity of I and the exact sequence $0 \to (\mathscr{D}_{M_{\mathrm{sa}}}^{[\lambda]})_U \to (\mathscr{D}_{M_{\mathrm{sa}}}^{[\lambda]})_V$, for any inclusion $U \subset V$. This completes the proof of (i)–(ii).
(iii) We apply Theorem 5.8 (3) with $\rho = \rho_{\mathrm{sa}}^{-1}$, $\sigma = \rho_{\mathrm{sa}!}$. □

We define a functor

$$\mathrm{F}\mathscr{H}\!om : \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \times \mathrm{Mod}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{Mod}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$$

by setting for $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$ and $\mathrm{F}\mathscr{M} \in \mathrm{Mod}(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$

$$\mathscr{H}\!om\,(G, \mathrm{F}\mathscr{M})(\lambda) = \mathscr{H}\!om\,(G, \mathscr{M}(\lambda)).$$

Using Theorem 5.5, this functor admits a derived functor

$$\mathrm{FR}\mathscr{H}\!om : \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \times \mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}).$$

Recall the functor for in (5.10).

Lemma 5.11. — *Let $G \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_M)$ and let $\mathrm{F}\mathscr{M} \in \mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$. Then*

$$\mathrm{F}^\lambda\mathrm{R}\mathscr{H}\!om\,(G, \mathrm{F}\mathscr{M}) \simeq \mathrm{R}\mathscr{H}\!om\,(G, \mathrm{F}^\lambda\mathscr{M}),$$
$$\mathrm{for}\mathrm{FR}\mathscr{H}\!om\,(G, \mathrm{F}\mathscr{M}) \simeq \mathrm{R}\mathscr{H}\!om\,(G, \mathrm{for}\mathrm{F}\mathscr{M}).$$

Proof. — The first isomorphism follows directly from Lemma 5.1 (b) and we only prove the second one.
(i) Since the problem is local on M, we may assume that G has compact support.
(ii) By standard arguments, we may then reduce to the case where $G = \mathbb{C}_U$, $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$.
(iii) Using Theorem 5.5, we may replace $\mathrm{F}\mathscr{M} \in \mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$ with an object $\widetilde{\mathscr{M}} \in \mathrm{D}^+(\mathrm{Fct}(\mathbb{R}, \mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sa}}})))$. Let us represent $\widetilde{\mathscr{M}}$ by a complex of injective objects $I^\bullet \in \mathrm{C}^+(\mathrm{Fct}(\mathbb{R}, \mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sa}}})))$. Then,

$$\begin{aligned}\mathrm{for}\mathrm{FR}\mathscr{H}\!om\,(\mathbb{C}_U, \mathrm{F}\mathscr{M}) &\simeq \varinjlim \mathrm{R}\Gamma(\mathbb{C}_U, \widetilde{\mathscr{M}}) \\ &\simeq \varinjlim \Gamma(U; I^\bullet) \\ &\underset{(\mathrm{a})}{\simeq} \Gamma(U; \varinjlim I^\bullet) \underset{(\mathrm{b})}{\simeq} \mathrm{R}\Gamma(U; \varinjlim I^\bullet) \\ &\simeq \mathrm{R}\Gamma(U; \varinjlim \widetilde{\mathscr{M}}) \simeq \mathrm{R}\Gamma(U; \mathrm{for}\mathrm{F}\mathscr{M}).\end{aligned}$$

Isomorphism (a) follows from Lemma 2.9 and isomorphism (b) follows from Lemma 5.1 (b) and Corollary 2.15. □

On a complex manifold X, we endow the $\mathscr{D}_X$-module $\mathscr{O}_X$ with the filtration $\mathrm{F}\mathscr{O}_X$ given by

$$\mathrm{F}^s\mathscr{O}_X = \begin{cases} 0 & \text{if } s < 0, \\ \mathscr{O}_X & \text{if } s \geq 0. \end{cases} \tag{5.15}$$

By applying the functors $\rho_{\mathrm{sa}!}$ and $\rho_{\mathrm{sal}*}$, we get the objects $\rho_{\mathrm{sa}!}\mathscr{O}_X$ and $\rho_{\mathrm{sl}*!}\mathscr{O}_X$ of $\mathrm{Mod}(F\mathscr{D}_{X_{\mathrm{sa}}})$ and $\mathrm{Mod}(F\mathscr{D}_{X_{\mathrm{sal}}})$, respectively. One shall be aware that these objects are in degree 0 contrarily to the sheaf $\mathscr{O}_{X_{\mathrm{sa}}}$ (when $d_X > 1$).

The L^∞-filtration on $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$. — Recall that on the site M_{sal}, the sheaf $\mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}}$ is endowed with a filtration, given by the sheaves $\mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}}$ ($t \in \mathbb{R}_{\geq 0}$). We also set

$$\mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}} = 0 \text{ for } t < 0.$$

Using Lemma 4.13 and Theorem 5.10, we set:

Definition 5.12. — (a) We denote by $\mathrm{F}_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$ the object of $\mathrm{Mod}(F\mathscr{D}_{M_{\mathrm{sal}}})$ given by the sheaves $\mathscr{C}^{\infty,t}_{M_{\mathrm{sal}}}$ ($t \in \mathbb{R}$).
(b) We set $\mathrm{F}_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}} := \rho^!_{\mathrm{sal}}\,\mathrm{F}_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$, an object of $\mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}})$.
We call these filtrations the L^∞-filtration on $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$ and $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$, respectively.

Hence,

- $\mathrm{F}^s_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}} = \mathscr{C}^{\infty,s}_{M_{\mathrm{sal}}}$ for $s \in \mathbb{R}$,
- we have morphisms $\mathrm{F}^r\mathscr{D}_{M_{\mathrm{sal}}} \otimes \mathrm{F}^s_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}} \to \mathrm{F}^{s+r}_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$,
- using Notation 5.6, $\mathrm{for}\mathrm{F}_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}} \simeq \mathscr{C}^{\infty,\mathrm{tp}\,st}_{M_{\mathrm{sal}}}$ and similarly with M_{sa} instead of M_{sal}.

If $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ is weakly Lipschitz, we thus have for $s \geq 0$:

$$\mathrm{R}\Gamma(U; \mathrm{F}^s_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}) \simeq \mathscr{C}^{\infty,s}_M(U). \tag{5.16}$$

Remark 5.13. — One could have also endowed $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$ with the L^2-filtration constructed similarly as the L^∞-filtration, when replacing the norm in (4.10) with the L^2-norm:

$$\|\varphi\|_2 = (\int_U |\varphi(x)|^2 dx)^{1/2}, \quad \|\varphi\|^s_2 = \|d(x)^s\varphi(x)\|_2. \tag{5.17}$$

One gets the filtered sheaves $\mathrm{F}_2\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$ and $\mathrm{F}_2\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$.

The L^∞-filtration on $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}}$. — On a complex manifold X, we set:

$$\mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}} := \mathrm{R}\mathscr{H}om_{\mathrm{F}\mathscr{D}_{\overline{X}_{\mathrm{sal}}}}(\rho_{\mathrm{sl}*!}\mathscr{O}_{\overline{X}}, \mathrm{F}_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{X_{\mathrm{sal}}}) \in \mathrm{D}^+(\mathrm{F}\mathscr{D}_{X_{\mathrm{sal}}}), \tag{5.18}$$

$$\begin{aligned} \mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}} &:= \mathrm{R}\mathscr{H}om_{\mathrm{F}\mathscr{D}_{\overline{X}_{\mathrm{sa}}}}(\rho_{\mathrm{sa}!}\mathscr{O}_{\overline{X}}, \mathrm{F}_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{X_{\mathrm{sa}}}) \\ &\simeq \rho^!_{\mathrm{sal}}\,\mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}} \in \mathrm{D}^+(\mathrm{F}\mathscr{D}_{X_{\mathrm{sa}}}). \end{aligned} \tag{5.19}$$

Proposition 5.14. — *The object* $\mathrm{F}^s_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}}$ *is represented by the complex of sheaves on* $X^{\mathbb{R}}_{\mathrm{sal}}$*:*

$$0 \to \mathrm{F}^s_\infty\mathscr{C}^{\infty,(0,0)}_{X_{\mathrm{sal}}} \xrightarrow{\overline{\partial}} \mathrm{F}^{s+1}_\infty\mathscr{C}^{\infty,(0,1)}_{X_{\mathrm{sal}}} \to \cdots \to \mathrm{F}^{s+d_X}_\infty\mathscr{C}^{\infty,(0,d_X)}_{X_{\mathrm{sal}}} \to 0. \tag{5.20}$$

Proof. — Recall that the Spencer complex $\mathrm{SP_X}(\mathscr{D}_\mathrm{X})$ is the complex of left $\mathscr{D}_X$-modules

$$\mathrm{SP_X}(\mathscr{D}_\mathrm{X}) := \quad 0 \to \mathscr{D}_\mathrm{X} \otimes_{\mathscr{O}} \bigwedge^{\mathrm{d_x}} \Theta_\mathrm{X} \xrightarrow{\mathrm{d}} \cdots \to \mathscr{D}_\mathrm{X} \otimes_{\mathscr{O}} \Theta_\mathrm{X} \to \mathscr{D}_\mathrm{X} \to 0. \tag{5.21}$$

Moreover, there is an isomorphism of complexes, in any local chart,

$$\mathrm{SP_X}(\mathscr{D}_\mathrm{X}) \simeq \mathrm{K}_\bullet(\mathscr{D}_\mathrm{X}; \cdot\partial_1, \dots, \cdot\partial_{\mathrm{d_X}}) \tag{5.22}$$

where the right hand side is the co-Koszul complex of the sequence $\cdot\partial_1, \dots, \cdot\partial_{d_X}$ acting on the right on $\mathscr{D}_X$. This implies that the left $\mathscr{D}$-linear morphism $\mathscr{D}_X \to \mathscr{O}_X$ induces an isomorphism $\mathrm{SP_X}(\mathscr{D}_\mathrm{X}) \xrightarrow{\sim} \mathscr{O}_\mathrm{X}$ in $\mathrm{D^b}(\mathscr{D}_X)$.

If we endow $\mathscr{D}_X \otimes_{\mathscr{O}} \bigwedge^k \Theta_X$, $k = 0, \dots, d_X$, with the filtration $\mathrm{F}^s(\mathscr{D}_X \otimes_{\mathscr{O}} \bigwedge^k \Theta_X) = \mathrm{F}^{s-k}(\mathscr{D}_X) \otimes_{\mathscr{O}} \bigwedge^k \Theta_X$, then $\mathrm{SP_X}(\mathscr{D}_\mathrm{X})$ gives a complex in $\mathrm{Mod}(\mathrm{F}\mathscr{D}_{\mathscr{T}})$ and we obtain $\mathrm{SP_X}(\mathscr{D}_\mathrm{X}) \xrightarrow{\sim} \mathscr{O}_\mathrm{X}$ in $\mathrm{D^b}(\mathrm{F}\mathscr{D}_X)$. Applying this to $\overline{X}$ and using the Definition (5.19) we obtain the result. ☐

Corollary 5.15. — *Let $U \subset X$ be an open relatively compact subanalytic subset. Assume that U is weakly Lipschitz. Then the object* $\mathrm{R}\Gamma(U; \mathrm{F}^s_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})$ *is represented by the complex*

$$0 \to \mathscr{C}^{\infty,s,(0,0)}_X(U) \xrightarrow{\overline{\partial}} \mathscr{C}^{\infty,s+1,(0,1)}_X(U) \to \cdots \to \mathscr{C}^{\infty,s+d_X,(0,d_X)}_X(U) \to 0. \tag{5.23}$$

Applying the functor ρ_{sa}^{-1}, one recovers the filtration introduced in (5.15):

$$\rho_{\mathrm{sa}}^{-1}\mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}} \simeq \mathrm{F}\mathscr{O}_X. \tag{5.24}$$

5.3. A functorial filtration on regular holonomic modules. — Good filtrations on holonomic modules already exist in the literature, in the regular case (see [**10, 1, 29, 30**]) and also in the irregular case (see [**23**]). But these filtrations are constructed on each holonomic module and are by no means functorial. Here we directly construct objects of $\mathrm{D}^+(\mathrm{F}\mathscr{D}_X)$, the derived category of filtered $\mathscr{D}$-modules.

Denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{holreg}}(\mathscr{D}_X)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ consisting of objects with regular holonomic cohomology. To $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{holreg}}(\mathscr{D}_X)$, one associates

$$\mathrm{Sol}(\mathscr{M}) := \mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{O}_X).$$

We know by [**7**] that $\mathrm{Sol}(\mathscr{M})$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$, that is, $\mathrm{Sol}(\mathscr{M})$ has $\mathbb{C}$-constructible cohomology. Moreover, one can recover $\mathscr{M}$ from $\mathrm{Sol}(\mathscr{M})$ by the formula:

$$\mathscr{M} \simeq \rho_{\mathrm{sa}}^{-1}\mathrm{R}\mathscr{H}om\,(\mathrm{Sol}(\mathscr{M}), \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}). \tag{5.25}$$

This is the Riemann-Hilbert correspondence obtained by Kashiwara in [**8, 9**].

Using the filtration $\mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ on $\mathscr{O}_{X_{\mathrm{sa}}}$ we can set:

Definition 5.16. — Let $\mathscr{M}$ be a regular holonomic module. We define the filtered Riemann-Hilbert functors $\mathrm{RHF}_{\infty,\mathrm{sa}}$ and RHF_∞ by the formulas

$$\begin{aligned}\mathrm{RHF}_{\infty,\mathrm{sa}}\colon \mathrm{D}^+_{\mathrm{holreg}}(\mathscr{D}_X) &\to \mathrm{D}^+(\mathrm{F}\mathscr{D}_{X_{\mathrm{sa}}}),\\ \mathscr{M} &\mapsto \mathrm{FR}\mathscr{H}om\,(\mathrm{Sol}(\mathscr{M}), \mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}),\\ \mathrm{RHF}_\infty = \rho_{\mathrm{sa}}^{-1}\mathrm{RHF}_{\infty,\mathrm{sa}}\colon \mathrm{D}^+_{\mathrm{holreg}}(\mathscr{D}_X) &\to \mathrm{D}^+(\mathrm{F}\mathscr{D}_X).\end{aligned}$$

Note that $\mathrm{RHF}_{\infty,\mathrm{sa}}$ and RHF_∞ are triangulated functors.

Recall Notation 5.6 and the functor for.

Proposition 5.17. — *In the diagram below*

$$\mathsf{D}^{\mathrm{b}}_{\mathrm{holreg}}(\mathscr{D}_X) \xrightarrow{\mathrm{RHF}_\infty} \mathrm{D}^+(\mathrm{F}\mathscr{D}_X) \xrightarrow{\mathrm{for}} \mathrm{D}^+(\mathscr{D}_X)$$

the composition is isomorphic to the identity functor.

Proof. — Since ρ_{sa}^{-1} commutes with inductive limits, the diagram below commutes:

$$\begin{tikzcd} \mathsf{D}^{\mathrm{b}}_{\mathrm{holreg}}(\mathscr{D}_X) \arrow[r, "\mathrm{RHF}_{\infty,\mathrm{sa}}"] & \mathrm{D}^+(\mathrm{F}\mathscr{D}_{X_{\mathrm{sa}}}) \arrow[r, "\mathrm{for}"] \arrow[d, "\rho_{\mathrm{sa}}^{-1}"'] & \mathrm{D}^+(\mathscr{D}_{X_{\mathrm{sa}}}) \arrow[d, "\rho_{\mathrm{sa}}^{-1}"'] \\ & \mathrm{D}^+(\mathrm{F}\mathscr{D}_X) \arrow[r, "\mathrm{for}"] & \mathrm{D}^+(\mathscr{D}_X). \end{tikzcd}$$

Now let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{holreg}}(\mathscr{D}_X)$ and set for short $G = \mathrm{Sol}_X(\mathscr{M})$. By using Lemma 5.11 we get

$$\begin{aligned}\mathrm{forFR}\mathscr{H}om\,(G, \mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}) &\simeq \mathrm{R}\mathscr{H}om\,(G, \mathrm{forF}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})\\ &\simeq \mathrm{R}\mathscr{H}om\,(G, \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})\end{aligned}$$

and we conclude with (5.25). □

Notation 5.18. — The module $\mathscr{M}$ endowed with the filtration obtained by applying the functor $\mathrm{RHF}_{\infty,\mathrm{sa}}$ or RHF_∞, will simply be denoted by $\mathrm{F}_{\infty,\mathrm{sa}}\mathscr{M}$ or $\mathrm{F}_\infty\mathscr{M}$, respectively.

Example 5.19. — Let D be a normal crossing divisor in X and let $\mathscr{M}$ be a regular holonomic module such that $\mathrm{Sol}(\mathscr{M}) \simeq \mathbb{C}_{X\setminus D}$. Let $W \in \mathrm{Op}_{X_{\mathrm{sa}}}$ with smooth boundary transversal to the strata of D so that $W \setminus D$ is weakly Lipschitz. Set $U := W \setminus D$. Then, by Lemma 5.11, $\mathrm{R}\Gamma(W; \mathrm{F}^s_{\infty,\mathrm{sa}}\mathscr{M}) \simeq \mathrm{R}\Gamma(U; \mathrm{F}^s_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})$ and therefore the object $\mathrm{R}\Gamma(W; \mathrm{F}^s_\infty\mathscr{M})$ is represented by the complex (5.23).

Remark 5.20. — By using the filtration F_2 on $\mathscr{C}^{\infty,\mathrm{tp}}_{X_{\mathrm{sal}}}$ (see Remark 5.13), one can also endow $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}}$ with an L^2-filtration and define similarly $\mathrm{F}_2\mathscr{O}^{\mathrm{tp}}_{\mathrm{sal}}$. Unfortunately, Hörmander's theory does not apply immediately to this situation. More precisely, for U open in $\mathbb{R}^n$, denote by $L^2(U;\mathrm{loc})$ the space of functions φ which are locally in L^2 for the Lebesgue measure and define

$$L^{2,s}(U) = \{\varphi \in L^2(U;\mathrm{loc});\, \|\varphi\|_2^s < \infty\}, \tag{5.26}$$

where $\|\varphi\|_2^s$ is defined in (5.26).

For U relatively compact and open in $\mathbb{C}^n$, denote by $W^{2,s,(p,q)}(U)$ the space of (p,q)-forms with coefficients in $L^{2,s}(U)$ and set

$$W_0^{2,s,(p,q)}(U) = \{\varphi \in W^{2,s,(p,q)}(U); \overline{\partial}\varphi \in W^{2,s,(p,q+1)}(U)\}.$$

Now we define $\widetilde{\mathrm{F}}_2\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}}$ as the Dolbeault complex

$$\widetilde{\mathrm{F}}^s_2\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}}(U) := 0 \to W_0^{2,s,(0,0)}(U) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} W_0^{2,s,(0,n)}(U) \to 0.$$

Then [**5**, Th 2.2.3] asserts that if U is pseudoconvex, $\widetilde{\mathrm{F}}^s_2\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}}(U)$ is concentrated in degree 0. However $\mathrm{F}^m\mathscr{D}_{X_{\mathrm{sal}}}$ does not send $W_0^{2,s,\cdot}$ to $W_0^{2,s+m,\cdot}$ and $\widetilde{\mathrm{F}}_2\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sal}}}$ is not defined as an object of $\mathrm{D}(\mathrm{F}\mathscr{D}_{X_{\mathrm{sal}}})$.

Given a regular holonomic $\mathscr{D}_X$-module $\mathscr{M}$, natural questions arise.

(i) Does there exist an integer r such that $H^j(\mathrm{F}^s_\infty\mathscr{L}) \to H^j(\mathrm{F}^{s+r}_\infty\mathscr{L})$ is the zero morphism for $s \gg 0$ and $j \neq 0$.

(ii) Is the filtration $H^0(\mathrm{F}_\infty\mathscr{M})$ a good filtration?

(iii) Does there exist a discrete set $Z \subset \mathbb{R}_{\geq 0}$ such that the morphisms $\mathrm{F}^s_\infty\mathscr{M} \to \mathrm{F}^t_\infty\mathscr{M}$ $(s \leq t)$ are isomorphisms for $[s,t]$ contained in a connected component of $\mathbb{R}_{\geq 0}\backslash Z$?

Note that it may be convenient to use better the L^2-filtration (see Remark 5.20).

One can also ask the question of comparing these filtrations with other filtrations already existing in the literature.

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S. Guillermou, Institut Fourier, Université de Grenoble I, BP 74, 38402 Saint-Martin d'Hères, France • *E-mail :* `Stephane.Guillermou@ujf-grenoble.fr`

P. Schapira, Sorbonne Universités, UPMC Univ Paris 6 Institut de Mathématiques de Jussieu, F-75005 Paris France • *E-mail :* `pierre.schapira@imj-prg.fr`

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SOBOLEV SPACES AND SOBOLEV SHEAVES

by

Gilles Lebeau

Abstract. — Sobolev spaces $H^s_{\mathrm{loc}}(M)$ on a real manifold M are classical objects of Analysis. In this paper, we assume that M is real analytic and denote by M_{sa} the associated subanalytic site, for which the open sets are the relatively compact subanalytic subsets and the coverings are, roughly speaking, the finite coverings. For $s \in \mathbb{R}, s \leq 0$, we construct an object $\mathscr{H}^s$ of the derived category $\mathsf{D}^+(\mathbb{C}_{M_{\mathrm{sa}}})$ of sheaves on M_{sa} with the property that if U is open in M_{sa} and has a Lipschitz boundary, then the object $\mathscr{H}^s(U) := \mathrm{R}\Gamma(U; \mathscr{H}^s)$ is concentrated in degree 0 and coincides with the classical Sobolev space $H^s(U)$. This construction is based on the results of S. Guillermou and P. Schapira in this volume.

Moreover, in the special case where the manifold M is of dimension 2, we will compute explicitly the complex $\mathscr{H}^s(U)$ and prove that it is always concentrated in degree 0, but is not necessarily a subspace of the space of distributions on U.

***Résumé* (Espaces de Sobolev et faisceaux de Sobolev).** — Soit M une variété analytique réelle. Le site sous analytique M_{sa} est constitué des ouverts sous analytiques relativement compacts de M, les recouvrements étant finis à extraction près. Pour $s \in \mathbb{R}$, soit $H^s_{\mathrm{loc}}(M)$ l'espace de Sobolev usuel sur M. Pour tout $s \in \mathbb{R}, s \leq 0$ nous construisons un objet $\mathscr{H}^s$ de la catégorie dérivée $\mathsf{D}^+(\mathbb{C}_{M_{\mathrm{sa}}})$ des faisceau sur M_{sa}, qui vérifie la propriété suivante: pour tout ouvert $U \in M_{\mathrm{sa}}$ à frontière lipschitzienne, $\mathscr{H}^s(U) := \mathrm{R}\Gamma(U; \mathscr{H}^s)$ est concentré en degré 0 et coïncide avec l'espace de Sobolev usuel $H^s(U)$. Cette construction utilise les résultats de S. Guillermou et P. Schapira contenus dans ce volume.

Dans le cas où M est de dimension 2, nous explicitons le complexe $\mathscr{H}^s(U)$. Nous démontrons qu'il est toujours concentré en degré 0, mais ne s'identifie pas toujours à un sous espace de distributions sur U.

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1. Introduction

Let M be a real analytic manifold. Let us recall that for $s \in \mathbb{R}$, and $x_0 \in M$ one says that a distribution $u \in \mathscr{D}'(M)$ belongs to the space $H^s_{x_0}(M)$ iff there exists a properly supported pseudodifferential operator P of degree s, elliptic at x_0, such that $Pu \in L^2_{\rm loc}(M)$. As usual, we denote by $H^s_{\rm loc}(M)$ the space of distributions u on M such that $u \in H^s_{x_0}(M)$ for all $x_0 \in M$. For U open and relatively compact in M, we define the space $H^s(U)$ by

$$H^s(U) = \{f \in \mathscr{D}'(U),\ \exists g \in H^s_{\rm loc}(M),\ g|_U = f\}.$$

Following [**3**], we endow the real analytic manifold M with the subanalytic topology and denote by $M_{\rm sa}$ the site so-obtained. Recall that the open sets of this Grothendieck topology are the relatively compact open subanalytic subsets of M and the coverings are the finite coverings. As usual, one denotes by $\mathsf{D}^+(\mathbb{C}_{M_{\rm sa}})$ the derived category of sheaves of $\mathbb{C}$-vector spaces on $M_{\rm sa}$ consisting of spaces bounded from below.

In this paper, we address the following question:

Let $s \in \mathbb{R}$ be given. Does there exists an object $\mathscr{H}^s$ of $\mathsf{D}^+(\mathbb{C}_{M_{\rm sa}})$, such that the following requirement holds true:

(1.1) If U is open, Lipschitz, and relatively compact, then the complex $\mathscr{H}^s(U)$ is concentrate in degree 0 and is equal to $H^s(U)$.

If 1.1 holds true, then we will say that the object $\mathscr{H}^s$ of $\mathsf{D}^+(\mathbb{C}_{M_{\rm sa}})$ is a "Sobolev sheaf". Clearly, this problem depends on the parameter $s \in \mathbb{R}$. It turns out that the answer to the above question is a straightforward byproduct of a theorem of A. Parusinski [**5**] for the values $s \in]-1/2, 1/2[$. More precisely, for $s \in]-1/2, 1/2[$, $U \mapsto H^s(U)$ is a sheaf on $M_{\rm sa}$, with cohomology concentrated in degree 0 (see Lemma 5.2 in Section 5).

In this paper, we will construct the Sobolev sheaf $\mathscr{H}^s$ for any $s \le 0$; this construction is based on the results of S. Guillermou and P. Schapira in [**1**]. Moreover, in the special case where the manifold M is of dimension 2, we will compute explicitly the complex $\mathscr{H}^s(U)$ for any bounded subanalytic open subset of M; it turns out that in dimension 2 and for $s \le 0$, $\mathscr{H}^s(U)$ is always concentrated in degree 0, but is not always a subspace of $\mathscr{D}'(U)$. We will address the existence of the Sobolev sheaf $\mathscr{H}^s$ for $s \ge 0$ in a forthcoming paper.

Let us recall that for any object $\mathscr{F}$ of $\mathsf{D}^+(\mathbb{C}_{M_{\rm sa}})$, if we denote by $\mathbb{H}^j(U, \mathscr{F})$ the jth cohomology space of the complex $\mathscr{F}(U)$, one has the exact long Mayer Vietoris sequence, where U, V are two open subanalytic relatively compact subsets of M

$$(1.2)\quad \to \mathbb{H}^j(U \cup V, \mathscr{F}) \to \mathbb{H}^j(U, \mathscr{F}) \oplus \mathbb{H}^j(V, \mathscr{F}) \to \mathbb{H}^j(U \cap V, \mathscr{F}) \to \mathbb{H}^{j+1}(U \cup V, \mathscr{F}) \to$$

The construction of a Sobolev sheaf $\mathscr{H}^s$ is a purely local problem near any point of M. In fact, all the Sobolev spaces introduced in this article are $C^\infty_0(M)$ modules. Hence we may and will assume $M = \mathbb{R}^n$ in all the paper.

The paper is organized as follows:

In Section 2, we recall some basic facts on Sobolev spaces on $\mathbb{R}^n$, and we introduce the spaces $H^s(U), H^s_0[U]$, $U \subset \mathbb{R}^n$ open, and the spaces H^s_F, $F \subset \mathbb{R}^n$ closed.

In Section 3, we study the Sobolev spaces $H^s(U)$ when U is an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary. The main result in this section is Proposition 3.6. From the requirement (1.1) and the exact long Mayer Vietoris sequence (1.2), the validity of Proposition 3.6 is a necessary condition for the existence of a Sobolev sheaf $\mathscr{H}^s$.

Section 4 is devoted to the study of the auxiliary spaces $X^t(U)$ and $Y^s(U)$. The main result in this section is Proposition 4.11 which implies that the sheaf $U \mapsto Y^s(U)$ is Γ-acyclic on the linear subanalytic site.

Section 5 is devoted to the construction of the Sobolev sheaf $\mathscr{H}^s$ in the case $s \leq 0$. In Subsection 5.1, using Proposition 4.11, the construction of the Sobolev sheaf $\mathscr{H}^s$ for $s \leq 0$ becomes a simple byproduct of the results of S. Guillermou and P. Schapira in [**1**]. In Subsection 5.2, we compute explicitly the cohomology of the complex $\mathscr{H}^s(U)$ on $\mathbb{R}^2$ for $s \leq 0$. In particular, we verify that this complex is in degree 0, but $\mathbb{H}^0(U, \mathscr{H}^s)$ is not always a subspace of $\mathscr{D}'(U)$.

Finally, in the appendix, we give in Section 6.1 some results about interpolation spaces, and we recall in Section 6.2 the "classical" definition of Sobolev spaces given in the book of Lions and Magenes [**4**], and their relations with our spaces.

In all the paper, we shall use the following notations:

$B(x,r) = \{y \in \mathbb{R}^n, |y-x| < r\}$ is the open Euclidean ball with center x and radius r.

For $s \in \mathbb{R}$, we denote by $[s]$ be the integer part of s and $\{s\} = s - [s] \in [0, 1[$.

We will denote by $\mathbb{H}^{j,s}(U)$ the $j$ème cohomology space of the complex $\mathscr{H}^s(U)$.

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2. Notations and basic results on Sobolev spaces

Let us first recall that for $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n)$ is the space of tempered distributions f such that the Fourier transform $\hat{f}$ is in L^2_{loc} and $(1+|\xi|^2)^{s/2}\hat{f}(\xi) \in L^2(\mathbb{R}^n)$. It is an Hilbert space with the norm

$$\|f\|^2_{H^s} = \int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi.$$

Let us recall (see [**2**], Section 7.9) that for $s \geq 0$, with $k = [s]$ and $r = \{s\}$, one has $f \in H^s(\mathbb{R}^n))$ if and only if $\partial^\alpha f \in L^2(\mathbb{R}^n)$ for all $\alpha, |\alpha| \leq k$, and (in the case $r > 0$), $\frac{\partial^\alpha f(x) - \partial^\alpha f(y)}{|x-y|^{n/2+r}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ for all $\alpha, |\alpha| = k$. Moreover, the square of the H^s norm is equivalent to

$$\sum_{|\alpha|\leq k}\int_{\mathbb{R}^n}|\partial^\alpha f(x)|^2\,dx+\mathbf{1}_{r>0}\sum_{|\alpha|=k}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|\partial^\alpha f(x)-\partial^\alpha f(y)|^2}{|x-y|^{n+2r}}\,dxdy. \tag{2.1}$$

If F is a closed subset of $\mathbb{R}^n$, we denote by H^s_F the closed subspace of $H^s(\mathbb{R}^n)$

$$H^s_F=\{f\in H^s(\mathbb{R}^n),\ \mathrm{support}(f)\subset F\}. \tag{2.2}$$

If U is an open subset of $\mathbb{R}^n$, we denote by $H^s_0[U]$ the closure of $C^\infty_0(U)$ for the topology of $H^s(\mathbb{R}^n)$. Obviously, $H^s_0[U]$ is a closed subspace of $H^s_{\overline{U}}$.

For U open in $\mathbb{R}^n$, we denote by $H^s(U)$ the subspace of $\mathscr{D}'(U)$

$$H^s(U)=\{f\in\mathscr{D}'(U),\exists g\in H^s(\mathbb{R}^n),g|_U=f\}. \tag{2.3}$$

We put on $H^s(U)$ the quotient topology:

$$\|f\|_{H^s(U)}=\inf(\|g\|_{H^s(\mathbb{R}^n)},\ g|_U=f). \tag{2.4}$$

Then one has the exact sequence

$$0\to H^s_{\mathbb{R}^n\setminus U}\to H^s(\mathbb{R}^n)\to H^s(U)\to 0 \tag{2.5}$$

which defines an Hilbert structure on $H^s(U)$.

Remark 2.1. — *The definition of $H^s(U)$ given by* (2.3) *is not the "usual" definition of the Sobolev space on U given in* [**4**]. *However, we will see in Section 6.2 that when U is Lipschitz and bounded,* (2.3) *coincides with the usual definition for all values of $s\in\mathbb{R}$, except for $s=-1/2-k,k\in\mathbb{N}$. Observe also that with the definition* (2.3), *it is obvious that for any s and α, the derivation ∂^α maps $H^s(U)$ into $H^{s-|\alpha|}(U)$. However, (see Section 6.2, Lemma 6.6) the map $f\mapsto\partial_x f$ does not map (!) $H^{1/2}(]0,\infty[)$ into $H^{-1/2}(]0,\infty[)$ with the usual definition of $H^{-1/2}(]0,\infty[)$ given in* [**4**].

Let U be an open subset of $\mathbb{R}^n$, $s\in\mathbb{R}$ and $t=-s$. There is a natural duality pairing between the spaces $H^s(U)$ and $H^t_0[U]$. It is defined for $f\in H^s(U)$ and $\psi\in H^t_0[U]$ by the formula

$$<f,\psi>=\lim_{n\to\infty}<g,\psi_n>,\quad g\in H^s(\mathbb{R}^n),\ g|_U=f \tag{2.6}$$

where $\psi_n\in C^\infty_0(U)$ is a sequence which converges to ψ in $H^t(\mathbb{R}^n)$. One has obviously from the above definitions

$$|<f,\psi>|\leq\|f\|_{H^s(U)}\|\psi\|_{H^t_0[U]}. \tag{2.7}$$

From (2.7), the canonical map j from $H^t_0[U]$ into the dual space of $H^s(U)$ defined by $j(\psi)(f)=<f,\psi>$ is continuous, and the map $\tilde{\jmath}$ from $H^s(U)$ into the dual space of $H^t_0[U]$ defined by $\tilde{\jmath}(f)(\psi)=<f,\psi>$ is continuous.

Lemma 2.2. — *The map j is an isomorphism of $H^t_0[U]$ onto the dual space of $H^s(U)$. The map $\tilde{\jmath}$ is an isomorphism of $H^s(U)$ onto the dual space of $H^t_0[U]$.*

Proof. — It is sufficient to prove that $\tilde{\jmath}$ is an isomorphism. One has $C_0^\infty(U) \subset H_0^t[U]$ and $H^s(U) \subset \mathscr{D}'(U)$. By (2.6), $\tilde{\jmath}(f) = 0$ implies $f = 0$ in $\mathscr{D}'(U)$, thus $\tilde{\jmath}$ is injective.

Let l be a continuous linear form on $H_0^t[U]$. Since $H_0^t[U]$ is a closed subspace of $H^t(\mathbb{R}^n)$, there exists a continuous linear form $\tilde{l}$ on $H^t(\mathbb{R}^n)$ such that $\tilde{l}(\psi) = l(\psi)$ for all $\psi \in H_0^t[U]$. Thus there exists $g \in H^s(\mathbb{R}^n)$ such that $l(\psi) = <g, \psi>$ for all $\psi \in H_0^t[U]$. Let $f = g|_U \in H^s(U)$. From (2.6) this means exactly $l = \tilde{\jmath}(f)$, thus $\tilde{\jmath}$ is surjective. □

Let $U \subset V$ be two open subsets of $\mathbb{R}^n$. Then using the identification of dual spaces given by Formula (2.6) and Lemma 2.2, one easily verifies that for any $t \in \mathbb{R}$, the dual map of the inclusion map

$$H_0^t[U] \to H_0^t[V] \tag{2.8}$$

is the restriction map r_U with $s = -t$

$$H^s(V) \to H^s(U). \tag{2.9}$$

Let us now describe what is the dual space of $H^t_{\overline{U}}$. Let $s = -t$ and let E be the closed subspace of $H^s(\mathbb{R}^n)$ defined by

$$E = \{g \in H^s(\mathbb{R}^n), \quad <g, \psi> = 0 \quad \forall \psi \in H^t_{\overline{U}}\} \subset H^s_{\mathbb{R}^n \setminus U}.$$

Let $\tilde{H}^s(U)$ be the Hilbert space defined by

$$0 \to E \to H^s(\mathbb{R}^n) \to \tilde{H}^s(U) \to 0. \tag{2.10}$$

The pairing $<f, \psi>$ is obviously defined for $f \in \tilde{H}^s(U)$ and $\psi \in H^t_{\overline{U}}$. It is easy to see like in the proof of Lemma 2.2 that this gives an isomorphism from $\tilde{H}^s(U)$ onto the dual space of $H^t_{\overline{U}}$. One has obviously a canonical surjection π

$$\tilde{H}^s(U) \xrightarrow{\pi} H^s(U) \to 0.$$

Observe that $\tilde{H}^s(U)$ is a subset of $\mathscr{D}'(U)$ if and only if π is injective, which is equivalent to $H_0^t[U] = H^t_{\overline{U}}$ or $E = H^s_{\mathbb{R}^n \setminus U}$. Therefore one gets the following corollary.

Corollary 2.3. — *The space $C_0^\infty(U)$ is dense in $H^t_{\overline{U}}$ (i.e one has the equality $H_0^t[U] = H^t_{\overline{U}}$) iff for any $g \in H^s_{\mathbb{R}^n \setminus U}$ and any $\psi \in H^t_{\overline{U}}$, with $t = -s$, one has $<g, \psi> = 0$.*

Example 2.4. — *For example, with $U = \mathbb{R} \setminus 0$, one has $H^1_{\overline{U}} = H^1(\mathbb{R})$, $H^{-1}_{\mathbb{R} \setminus U} = \{\mathbb{C}\delta_0\}$, and $H_0^1[U] = \{f \in H^1(\mathbb{R}), f(0) = 0\} \neq H^1_{\overline{U}}$.*

3. The case of Lipschitz U

Let us first recall the following definition.

Definition 3.1. — *A non void open set Ω in $\mathbb{R}^n$ is Lipschitz iff for any $q_0 \in \overline{\Omega}\backslash\Omega$, there exists an orthonormal system of coordinates $(x_1, \dots, x_{n-1}, x_n) = (x', x_n)$ centered at q_0, a constant $a > 0$, and a Lipschitz function $f(x')$ defined on $\{|x'| \leq a\}$, with Lipschitz constant K (i.e $|f(x') - f(y')| \leq K|x' - y'|$), such that $f(0) = 0$ and*

$$\Omega \cap C_{a,K} = \{(x', x_n) \in C_{a,K},\ x_n < f(x')\} \tag{3.1}$$

where $C_{a,K}$ is the cylinder

$$C_{a,K} = \{(x', x_n),\ |x'| < a,\ |x_n| < (1 + K)a\}. \tag{3.2}$$

It is obvious that if Ω is Lipschitz, its boundary $\partial\Omega = \overline{\Omega}\backslash\Omega$ has Lebesgue measure 0, and that $\mathbb{R}^n \setminus \overline{\Omega}$ is also Lipschitz (except in the trivial case $\Omega = \mathbb{R}^n$). There is a useful characterization of Lipschitz open sets by a cone condition. Let $S^{n-1} = \{\omega \in \mathbb{R}^n, |\omega| = 1\}$ be the unit sphere in $\mathbb{R}^n$. For $\omega \in S^{n-1}$, $b \in]0, \sqrt{2}[$ and $r > 0$, let $\Gamma_r^{\omega,b}$ be the open truncated cone

$$\Gamma_r^{\omega,b} = \{x = \rho v,\ \rho \in]0, r[,\ v \in S^{n-1}, |v - \omega| < b\}. \tag{3.3}$$

Then a non void open set Ω is Lipschitz iff for any $q \in \overline{\Omega}$, there exists $\epsilon > 0$, $\omega \in S^{n-1}$, $b \in]0, \sqrt{2}[$ and $r > 0$, such that

$$\forall x \in \overline{\Omega} \cap B(q, \epsilon), \quad x + \Gamma_r^{\omega,b} \subset \Omega. \tag{3.4}$$

The following lemma is elementary.

Lemma 3.2. — *Let Ω be a Lipschitz open subset of $\mathbb{R}^n$ and $q_0 \in \overline{\Omega} \setminus \Omega$. Then, with the notation of Definition 3.1, for any $r \in]0, a]$, $\Omega \cap (q_0 + C_{r,K})$ is Lipschitz and homeomorphic to the unit ball of $\mathbb{R}^n$.*

Proof. — One verifies easily that $\Omega_{r,K} = \Omega \cap C_{r,K}$ is homeomorphic to the unit ball of $\mathbb{R}^n$. In order to verify that $\Omega_{r,K}$ is Lipschitz, it is sufficient to prove the following: for any $p_0 = (x_0', f(x_0'))$ with $x_0' = r\omega_0, |\omega_0| = 1$, there exists $\alpha > 0, \epsilon > 0, t_0 > 0, \delta > 0$, such that for any $p = (x', x_n) \in \overline{\Omega_{r,K}}$ with $|x' - x_0'| \leq \delta$ and $|x_n - f(x_0')| \leq \delta$, one has

$$p + t\xi \in \Omega_{r,K}, \quad \forall (t, \xi) \in]0, t_0] \times \{\xi \in \mathbb{R}^n,\ |\xi - (\alpha\omega_0, -1)| < \epsilon\}$$

A simple calculation shows that this property holds true: choose first $\alpha > 0$ such that $\alpha(1 + 2K) < 1$. Then take $\epsilon \in]0, \alpha[$ and finally choose t_0 and δ small enough . We leave the details to the reader. □

Remark 3.3. — *The above lemma will be false if one replace $\Omega \cap (q_0 + C_{r,K})$ by $\Omega \cap B(q_0, r)$. Let us indicate how to construct a Lipschitz $\Omega \subset \mathbb{R}^2$ such that there exists a sequence $r_n \to 0$ such that $\Omega \cap B(0, r_n)$ is disconnected and not Lipschitz. Take $\alpha \in]0, \pi/4[$ and let $\gamma = \cos(\alpha)(\cos(\alpha) - \sin(\alpha))$. For $k \in \mathbb{N}$, set $P_k = (\gamma^k \cos(\alpha), \gamma^k \sin(\alpha))$ and $Q_k = (\gamma^k / \cos(\alpha), 0)$. Then define the function f on $\mathbb{R}$ by $f(x) = 0$ if $x \leq 0$ or $x \geq 1/\cos(\alpha)$, and on the interval $[0, 1/\cos(\alpha)]$, the graph of f is the union of the segments $[P_0, Q_0], [Q_0, P_1], [P_1, Q_1], \dots$ Then f is Lipschitz and the disk $B(0, \gamma^k)$ is tangent to $[P_k, Q_k]$ at P_k. The Lipschitz open set $\Omega = \{(x, y), y > f(x)\}$ does not give an example, but it is sufficient to modify slightly* *the function f above near each P_k to get an example. We leave the details to the reader.*

Lemma 3.4. — *Let U be a Lipschitz open and bounded subset of $\mathbb{R}^n$. For all $t \in \mathbb{R}$, one has the equality $H^t_{\overline{U}} = H^t_0[U]$. In particular, by Lemma 2.2, $H^t_{\overline{U}}$ is isomorphic to the dual space of $H^s(U)$ with $s = -t$.*

Proof. — Let $\psi \in H^t_{\overline{U}}$. Let $(\phi_j)_{1\leq j\leq N}$ be a smooth partition of unity with $\phi_j \in C^\infty_0(\mathbb{R}^n)$ and $\sum \phi_j = 1$ in a neighborhood of $\overline{U}$. It is clearly sufficient to prove $\phi_j\psi \in H^t_0[U]$ for all j. Thus with the notation of Definition 3.1 and $\Omega = U$, me may assume that ψ is supported in $\overline{\Omega \cap C_{r,K}}$ with $r < a/2$. Let $e = (e' = 0, e_n = -1) \in \mathbb{R}^n$. If T is a distribution on $\mathbb{R}^n$, let T_ε be the translation of T by εe (i.e $f_\varepsilon(x) = f(x-\varepsilon e)$ for a function f). With this notations one has $\lim_{\varepsilon\to 0} \psi_\varepsilon = \psi$ in the space $H^t(\mathbb{R}^n)$ and for $\varepsilon > 0$ small, support(ψ_ε) is a compact subset of $\Omega = U$. Thus by a classical regularization argument, one has $\psi_\varepsilon \in H^t_0[U]$, and since $H^t_0[U]$ is closed in $H^t(\mathbb{R}^n)$ we get $\psi \in H^t_0[U]$. □

The following lemma is a byproduct of Proposition 6.5, which gives equality of the space $H^s(U)$ and the "usual" Sobolev space on U defined in [4] in the case $s \geq 0$ and U Lipschitz and bounded.

Lemma 3.5. — *Let U be a Lipschitz open and bounded subset of $\mathbb{R}^n$. Let $s \geq 0$, $k = [s]$, $r = \{s\} = s - [s] \in [0, 1[$. Then one has $f \in H^s(U)$ iff for all $|\alpha| \leq k$ one has $\partial^\alpha f \in L^2(U)$ and (in the case $r \in]0, 1[$)*

$$\iint_{U\times U} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|^2}{|x-y|^{n+2r}}\, dxdy \; < \; \infty.$$

Proof. — If $f \in H^s(U)$, there exists $g \in H^s(\mathbb{R}^n)$ such that $g|_U = f$, and thus the integral condition follows from (2.1). On the other hand, by Proposition 6.5 and the usual definition of the Sobolev space (see Section 6.2), we may assume $s \in]0, 1[$. The problem is local near any point $x_0 \in \partial U$, and there exists $r > 0$ and a bi-Lipschitz diffeomorphism Φ defined on $B(x_0, 2r)$ such that $\Phi(U \cap B(x_0, r)) = \{x \in B(0, r), x_n > 0\}$. Since the spaces L^2 and H^1 are invariant by bi-Lipschitz diffeomorphisms , we may assume $U = \{x \in B(0, r), x_n > 0\}$ and f with support in $U \cap \{|x| \leq r/2\}$. Then if the integral condition on $U \times U$ is satisfied for f, it is satisfied on $\mathbb{R}^n \times \mathbb{R}^n$ for the function $\tilde{f} \in L^2(\mathbb{R}^n)$ defined by the even reflection of f across x_n=0, and therefore by (2.1) one has $f \in H^s(U)$. □

The validity of the following proposition, in the subanalytic case, is a necessary condition for the existence of a complex $\mathscr{H}^{s,\cdot}$ such that 1.1 and 1.2 hold true. The proof below is interesting since it does not use any subanalyticity assumption.

Proposition 3.6. — *Let U, V be two Lipschitz bounded subsets of $\mathbb{R}^n$ such that $U \cap V$ and $U \cup V$ are Lipschitz. Then for any $s \in \mathbb{R}$, the following sequence of Hilbert spaces* *is exact*

$$0 \to H^s(U \cup V) \xrightarrow{i} H^s(U) \oplus H^s(V) \xrightarrow{j} H^s(U \cap V) \to 0. \tag{3.5}$$

Proof. — Obviously, the map $f \mapsto i(f) = f|_U \oplus f|_V$ is injective . The fact that the map $f \oplus g \mapsto j(f \oplus g) = f|_{U\cap V} - g|_{U\cap V}$ is surjective is an obvious consequence of the Definition 2.5 of the spaces $H^s(U)$. Therefore, we just have to verify the following assertion

(3.6)

If $f \in \mathcal{D}'(U \cup V)$ is such that $f|_U \in H^s(U)$ and $f|_V \in H^s(V)$, one has $f \in H^s(U \cup V)$.

We will prove this result separately in the two cases $s \geq 0$ and $s \leq 0$.

The case $s \geq 0$

Let $(\phi_j)_{1\leq j\leq N}$ a smooth partition of unity, $\phi_j \in C_0^\infty(\mathbb{R}^n)$ and $\sum \phi_j = 1$ near $\overline{U \cup V}$. It is clearly sufficient to prove that (3.6) holds true for all $f_j = \phi_j f$. Therefore, one may assume that f is supported in any small neighborhood of a given point $q_0 \in \overline{U \cup V}$. The only non trivial case is the case $q_0 \in (\overline{U} \setminus U) \cap (\overline{V} \setminus V)$. With the notation of Definition 3.1 applied to $\Omega = U \cup V$, we will thus assume $f \in \mathcal{D}'(U \cup V)$, $f|_U \in H^s(U)$, $f|_V \in H^s(V)$ and support$(f) \subset (U \cup V) \cap C_{r,K}$ with $r < a/2$ (recall that support(f) is a closed subset of $U \cup V$).

When $s = k$ is an integer, the hypothesis $f|_U \in H^k(U)$, $f|_V \in H^k(V)$ is equivalent to (see Lemma 3.5) $\partial^\alpha f|_U \in L^2(U)$, $\partial^\alpha f|_V \in H^k(V)$ for all $|\alpha| \leq k$, which is equivalent to $\partial^\alpha f \in L^2(U \cup V)$ for all $|\alpha| \leq k$, which is equivalent to $f \in H^k(U \cup V)$ by Lemma 3.5 since $U \cup V$ is Lipschitz. By the same argument, and Lemma 3.5, we are thus reduced to the case $s \in]0, 1[$. Recall that for Ω Lipschitz and $s \in]0, 1[$, one has $f \in H^s(\Omega)$ iff

$$f \in L^2(\Omega) \text{ and } \iint_{\Omega\times\Omega} \frac{|f(x) - f(y)|^2}{|x - y|^{n+2s}} dxdy \ < \infty. \tag{3.7}$$

Since $f \in H^s(V)$, there exists $g \in H^s(\mathbb{R}^n)$ such that $g|_V = f|_V$. Since support$(f) \subset C_{r,K}$, we may assume support$(g) \subset C_{r',K}$ with $r < r' < a/2$. If one replaces r by r' and f by $f - g|_{U\cup V}$, we may thus assume $f|_V = 0$. One has clearly $f \in L^2(U \cup V)$, and it remains to verify

$$\iint_{(U\cup V)\times(U\cup V)} \frac{|f(x) - f(y)|^2}{|x - y|^{n+2s}} dxdy \ < \infty. \tag{3.8}$$

Since $f|_V = 0$, $f|_U \in H^s(U)$, and support$(f) \subset C_{r,K}$, dist$(C_{r,K}, \mathbb{R}^n \setminus C_{a,k}) > 0$, we just have to verify

$$\int_{x\in(U\setminus V)\cap C_{r,K}} |f(x)|^2 \int_{y\in(V\setminus U)\cap C_{a,K}} \frac{dy}{|x - y|^{n+2s}} dx \ < \infty. \tag{3.9}$$

Since V is Lipschitz, one has measure $(\overline{V} \setminus V) = 0)$, and since $f|_U \in H^s(U)$, we are reduced to prove that there exists $C_0 > 0$ such that

(3.10)

$$\forall x \in (U \setminus \overline{V}) \cap C_{r,K}, \quad I(x) = \int_{y\in(V\setminus U)\cap C_{a,K}} \frac{dy}{|x - y|^{n+2s}} \leq C_0 \int_{y\in U\cap V} \frac{dy}{|x - y|^{n+2s}}.$$

For an open set W, we will use the notation $W_{a,K} = W \cap C_{a,K}$. We set $\Omega = U \cup V$. Let us first prove that there exists $C_1 > 0$ such that

$$\forall x \in U_{a,K},\ \forall y \in V_{a,K}, \quad |x-y| \geq C_1 d(x, U \cap V). \tag{3.11}$$

For $x, y \in \Omega_{a,K}$, let $\mathrm{dist}_{\Omega_{a,K}}(x,y) = \inf_\gamma \int_0^1 |\gamma'(s)|ds$, where the infimum is over all C^1 map γ from $[0,1]$ into $\Omega_{a,K}$ such that $\gamma(0) = x, \gamma(1) = y$. Since by Lemma 3.2, $\Omega_{a,K}$ is Lipschitz connected and bounded, the function $\mathrm{dist}_{\Omega_{a,K}}(x,y)$ is bounded on $\Omega_{a,K} \times \Omega_{a,K}$ and there exists $c_0 \geq 1$ such that

$$\forall x, y \in \Omega_{a,K}, \quad c_0|x-y| \geq \mathrm{dist}_{\Omega_{a,K}}(x,y) \geq |x-y|. \tag{3.12}$$

Let $x \in U_{a,K}$ and $y \in V_{a,K}$ be given. Take $\varepsilon > 0$ small and $\gamma_\varepsilon : [0,1] \to \Omega_{a,K}$ such that $\gamma_\varepsilon(0) = x, \gamma_\varepsilon(1) = y$ and $\int_0^1 |\gamma'_\varepsilon(s)|ds \leq \mathrm{dist}_{\Omega_{a,K}}(x,y) + \varepsilon$. Since $[0,1]$ is connected, and $\{s, \gamma_\varepsilon(s) \in U\} \cup \{s, \gamma_\varepsilon(s) \in V\} = [0,1]$, there exists $s_0 \in [0,1]$ such that $z = \gamma_\varepsilon(s_0) \in U \cap V$. One has

$$d(x, U \cap V) \leq |x - z| \leq \int_0^{s_0} |\gamma'_\varepsilon(s)|ds \leq \mathrm{dist}_{\Omega_{a,K}}(x,y) + \varepsilon \leq c_0|x-y| + \varepsilon.$$

Therefore, (3.11) holds true with $C_1 = c_0^{-1}$. □

Let us now prove (3.10). Let $x \in (U \setminus \overline{V}) \cap C_{r,K}$ and $z \in \overline{U \cap V}$ such that $d(x, U \cap V) = |x - z| = \rho$. One has $\rho > 0$ since $x \notin \overline{V}$, and by (3.11) one gets

$$I(x) = \int_{y \in (V \setminus U) \cap C_{a,K}} \frac{dy}{|x-y|^{n+2s}} \leq \int_{C_1\rho \leq |x-y|} \frac{dy}{|x-y|^{n+2s}} = \rho^{-2s} \int_{|w| \geq C_1} \frac{dw}{|w|^{n+2s}}. \tag{3.13}$$

Since $U \cap V$ is Lipschitz and bounded, there exists $\delta > 0, b > 0$ and for any $p \in \overline{U \cap V}$, a unit vector $\omega = \omega(p) \in S^{n-1}$ such that $p + \Gamma_\delta^{\omega,b} \subset U \cap V$. Since $U \cap V$ is non void, one has $meas(U \cap V) > 0$, and therefore from (3.13), we get that (3.10) holds true for $\rho \geq \delta$. Let us now assume $\rho < \delta$. Let $\omega = \omega(z)$ and $b' = b/2$. Let θ be the unit vector such that $x - z = \rho\theta$. Since $\rho < \delta$, and $x \notin z + \Gamma_\delta^{\omega,b}$ one has $|\theta - \omega| \geq b$ and $\Gamma_1^{\omega,b'} \subset \Gamma_{\delta/\rho}^{\omega,b'}$. Therefore we get

$$\int_{y \in U \cap V} \frac{dy}{|x-y|^{n+2s}} \geq \int_{y \in z + \Gamma_\delta^{\omega,b'}} \frac{dy}{|x-y|^{n+2s}} \geq \rho^{-2s} \int_{w \in \Gamma_1^{\omega,b'}} \frac{dw}{|\theta - w|^{n+2s}}. \tag{3.14}$$

There exists constants $c_j(b,b') = c_j > 0$ such that for all $|\theta| = 1, |\theta - \omega| \geq b$, one has

$$c_1 \leq \int_{w \in \Gamma_1^{\omega,b'}} \frac{dw}{|\theta - w|^{n+2s}} \leq c_2. \tag{3.15}$$

Therefore, (3.10) is consequence of (3.13), (3.14), (3.15).

The case $s \leq 0$

Let $t = -s \geq 0$. By duality and Lemma 3.5, it is equivalent to prove the exactness of the sequence

$$0 \to H^t_{\overline{U \cap V}} \xrightarrow{\sigma} H^t_{\overline{U}} \oplus H^t_{\overline{V}} \xrightarrow{\tau} H^t_{\overline{U \cup V}} \to 0 \tag{3.16}$$

where $\sigma(f) = f \oplus (-f)$ and $\tau(f \oplus g) = f + g$. The map σ is obviously injective. Let $f \oplus g \in H^t_{\overline{U}} \oplus H^t_{\overline{V}}$ such that $\tau(f \oplus g) = 0$. Then there exists $h \in H^t_{\overline{U} \cap \overline{V}}$ such that $f = -g = h$, and it remains to verify $h \in H^t_{\overline{U \cap V}}$, which follows from the following lemma.

Lemma 3.7. — *Let U, V be two bounded and non void open sets in $\mathbb{R}^n$, such that $U \cup V$ is Lipschitz. Then $\overline{U} \cap \overline{V} = \overline{U \cap V}$.*

Proof. — Assume that there exists $q_0 \in \overline{U} \cap \overline{V}$, and $q_0 \notin \overline{U \cap V}$. Let $(x_n)_{n \geq 1} \in U$ and $(y_n)_{n \geq 1} \in V$ be two sequences such that $\lim x_n = \lim y_n = q_0$. Let $\Omega = U \cup V$. The open set Ω is Lipschitz and one has $q_0 \in \overline{\Omega} \setminus \Omega$. With the notations of Lemma 3.2, for all $r > 0$ small, there exists $N(r)$ such that for all $n \geq N(r)$ one has $x_n \in \Omega \cap C_{r,K}$ and $y_n \in \Omega \cap C_{r,K}$. By Lemma 3.2, $\Omega \cap C_{r,K}$ is connected. Thus there exists a continuous path $\gamma_r : [0,1] \to \Omega \cap C_{r,K}$ with $\gamma_r(0) = x_{N_r}, \gamma_r(1) = y_{N_r}$. One has $[0,1] = \gamma_r^{-1}(U) \cup \gamma_r^{-1}(V)$ and therefore, there exists $s_r \in \gamma_r^{-1}(U) \cap \gamma_r^{-1}(V)$. Then one has $z_r = \gamma_r(s_r) \in C_{r,K} \cap (U \cap V)$ and $\lim_{r \to 0} z_r = q_0$. This contradicts $q_0 \notin \overline{U \cap V}$. □

It remains to prove that τ is surjective. Let $h \in H^t_{\overline{U \cup V}}$. This means $h \in H^t(\mathbb{R}^n)$ and support$(h) \subset \overline{U \cup V}$. As before, we may assume that h is supported in an arbitrary small neighborhood of a given point $q_0 \in (\overline{U} \setminus U) \cap (\overline{V} \setminus V) \subset \overline{U} \cap \overline{V}$. Let $\Omega = \mathbb{R}^n \setminus (\overline{U \cap V})$ and let $C_{a,K}$ be the cylinder associated to q_0 and Ω as in Definition 3.1. Let $O = \Omega \cap C_{a,K} = C_{a,K} \setminus \overline{U \cap V}$. The open set O is Lipschitz and connected. We may assume support$(h) \subset C_{r,K}$ with $r < a/2$. Let θ be the function define on O by the formula

$$\begin{aligned} &\theta(x) = h(x) \quad \text{if} \quad x \notin V \cap O \\ &\theta(x) = 0 \quad \text{if} \quad x \in V \cap O. \end{aligned} \tag{3.17}$$

One has $\theta(x) = 0$ for all $x \in O \setminus \overline{U \cup V}$, thus $\theta \in L^2(O)$. Moreover, $\theta \in H^t_{\mathrm{loc}}(O)$ and for $|\alpha| \leq t$

$$\begin{aligned} &\partial^\alpha \theta(x) = \partial^\alpha h(x) \quad \text{if} \quad x \notin V \cap O \\ &\partial^\alpha \theta(x) = 0 \quad \text{if} \quad x \in V \cap O. \end{aligned} \tag{3.18}$$

If $t = k$ is an integer, this proves $\theta \in H^k(O)$. Thus there exists $f \in H^t(\mathbb{R}^n)$ such that $f|_O = \theta$. One has support$(\theta) \subset$ support$(h) \subset C_{r,K}$, thus we may assume support$(f) \subset C_{a,K}$. This implies support$(f) \subset \overline{U}$, and therefore $f \in H^t_{\overline{U}}$. Moreover, one has support$(h - f) \subset \overline{V}$, hence $g = h - f \in H^t_{\overline{V}}$. Therefore we get $h = \tau(f \oplus g)$. If t is not an integer, the same argument allows to reduce the problem to the case

$t \in]0,1[$, and it remains to prove $\theta \in H^t(O)$, i.e

$$\iint_{O\times O} \frac{|\theta(x)-\theta(y)|^2}{|x-y|^{n+2t}} dxdy < \infty. \tag{3.19}$$

Since $h \in H^t(\mathbb{R}^n)$, by the Definition (3.17) of θ, it is sufficient to verify

$$\int_{x\in U\cap O} |h(x)|^2 \int_{y\in V\cap O} \frac{dy}{|x-y|^{n+2t}} dx < \infty. \tag{3.20}$$

As in the case $s \geq 0$, let us denote by $\mathrm{dist}_O(x,y)$ the distance in the connected Lipschitz open set O. There exists $c_0 > 0$ such that

$$\forall x, y \in O, \quad |x-y| \geq c_0 \ \mathrm{dist}_O(x,y). \tag{3.21}$$

Let $x \in U\cap O$. One has $x \notin V$. Let $y \in V \cap O$ and let $\gamma : [0,1] \to O$ a path connecting x to y in O. Since there is no points of $U \cap V$ on γ, gamma is not contained in $U \cup V$. Let $W = \mathbb{R}^n \setminus \overline{(U\cup V)}$. Then there exists $s \in [0,1]$ such that $\gamma(s) = z \in \overline{W}$, and one has

$$|x-y| \geq c_0 \ d_O(x,y) \geq c_0 \ d_O(x,z) \geq c_0 \ \mathrm{dist}(x, \overline{W}). \tag{3.22}$$

Set $\rho(x) = \mathrm{dist}(x,\overline{W}) > 0$. From (3.22), we get $V\cap O \subset \mathbb{R}^n \setminus B(x, c_0\rho(x))$ and

$$\int_{y\in V\cap O} \frac{dy}{|x-y|^{n+2t}} \leq \int_{|z|\geq c_0\rho(x)} \frac{dz}{|z|^{n+2t}} \leq C\rho(x)^{-2t}. \tag{3.23}$$

Thus, it remains to verify

$$\int_{x\in U\cap O} |h(x)|^2 \rho(x)^{-2t} \ dx < \infty. \tag{3.24}$$

For $x \in U\cap O$, take $w = w(x) \in \overline{W}$ such that $\rho(x) = \mathrm{dist}(x,\overline{W}) = |x - w(x)|$. Then $w(x)$ varies in a compact set Q of $\overline{W}$. Since W is Lipschitz, there exists $r > 0, b > 0$ and for all $w \in Q$, a unit vector $\omega(w)$ such that $w + \Gamma_r^{\omega(w),b} \subset W$. Since $h \in H^t(\mathbb{R}^n)$ and $h|_W = 0$, one has

$$\int_{x\in U\cap O} |h(x)|^2 \int_{w\in W} \frac{dw}{|x-w|^{n+2t}} dx < \infty \tag{3.25}$$

and (3.24) follows from (3.25) and

$$\int_{w\in W} \frac{dw}{|x-w|^{n+2t}} \geq \int_{w\in w(x)+\Gamma_r^{\omega(w),b}} \frac{dw}{|x-w|^{n+2t}} \geq C_{b,r}|x-w(x)|^{-2t}. \tag{3.26}$$

The proof of Proposition 3.6 is complete.

4. The spaces $X^t(U)$ and $Y^s(U)$

4.1. The spaces $X^t(U)$. — Let U be an open subset of $\mathbb{R}^n$. Let d_U be the function on $\mathbb{R}^n$

$$d_U(x) = \mathrm{dist}(x, \mathbb{R}^n \setminus U) = \inf_{y \in \mathbb{R}^n \setminus U} |x - y|. \tag{4.1}$$

One has $d_U(x) = +\infty$ in the case $U = \mathbb{R}^n$, and for $U \neq \mathbb{R}^n$, d_U is finite, Lipschitz with Lipschitz constant 1. One has $U = \{x, d_U(x) > 0\}$. Let $x \mapsto g(x) \geq 0$ be a measurable function on $\mathbb{R}^n$ such that $g(x) < \infty$ for almost all x. For any $m > 0$, the measurable function gd_U^{-m} is defined on $\mathbb{R}^n$ by the rule: for $x \notin U$ $g(x)d_U^{-m}(x) = 0$ if $g(x) = 0$ and $g(x)d_U^{-m}(x) = +\infty$ if $g(x) > 0$. In particular

$$\int_{\mathbb{R}^n} g(x) d_U^{-m}(x)\, dx \in [0, \infty].$$

is well defined, and equal to 0 if $g(x) = 0$ for almost all $x \in \mathbb{R}^n$. Thus, the statement $gd_U^{-m} \in L^1(\mathbb{R}^n)$ makes sense. Moreover, $gd_U^{-m} \in L^1(\mathbb{R}^n)$ implies $g(x) = 0$ for almost all $x \in \mathbb{R}^n \setminus U$.

Definition 4.1. — *Let U be an open subset of $\mathbb{R}^n$, and $t \geq 0$. We define the space $X^t(U)$ as the subspace of $H^t(\mathbb{R}^n)$ of those $f \in H^t(\mathbb{R}^n)$ such that for all $\alpha, |\alpha| \leq [t]$, the following holds true.*

i) *Any measurable function $f_\alpha(x)$ equal to $\partial^\alpha f$ in the space $L^2(\mathbb{R}^n)$ satisfies $f_\alpha(x) = 0$ for almost all $x \in \mathbb{R}^n \setminus U$.*
ii) $|\partial^\alpha f| d_U^{|\alpha|-t} \in L^2(\mathbb{R}^n)$.

Observe that one has $X^t(\mathbb{R}^n) = H^t(\mathbb{R}^n)$. Observe also that the condition i) implies that $X^t(U)$ is a subspace of $H^t_{\overline{U}}$, but is stronger than $f \in H^t_{\overline{U}}$ in the case $meas(\overline{U} \setminus U) > 0$. An important fact for us will be that for $f \in X^t(U)$ one has the identities in $\mathscr{D}'(\mathbb{R}^n)$

$$\mathbf{1}_U \partial^\alpha f = \partial^\alpha f = \partial^\alpha(\mathbf{1}_U f), \quad \forall |\alpha| \leq [t]. \tag{4.2}$$

The canonical map $f \mapsto f|_U$ from $X^t(U)$ into $L^2(U, dx)$ is injective, and is a bijection for $t = 0$.

Lemma 4.2. — *$X^t(U)$ is an Hilbert space for the norm*

$$\|f\|^2_{X^t(U)} = \|f\|^2_{H^t(\mathbb{R}^n)} + \sum_{|\alpha| \leq [t]} \| |\partial^\alpha f| d_U^{|\alpha|-t} \|^2_{L^2(\mathbb{R}^n)}. \tag{4.3}$$

Proof. — Let f_m be a Cauchy sequence in $X^t(U)$ for the norm $\|.\|_{X^t(U)}$. We have to show that there exists a subsequence f_{m_k} and $f \in X^t(U)$ such that $\lim_{k\to\infty} \|f_{m_k} - f\|_{X^t(U)} = 0$. Since f_m is a Cauchy sequence for the norm $\|.\|_{H^t(\mathbb{R}^n)}$, there exists a unique $f \in H^t(\mathbb{R}^n)$ such that $\lim_{m\to\infty} \|f_m - f\|_{H^t(\mathbb{R}^n)} = 0$. Thus, there exists a subsequence f_{m_k} such that $\partial^\alpha f_{m_k}(x) \to \partial^\alpha f(x)$ for almost all $x \in \mathbb{R}^n$ and all $|\alpha| \leq [t]$. In particular, $\partial^\alpha f(x) = 0$ for almost all $x \in \mathbb{R}^n \setminus U$. By Fatou's lemma, one has

$$\int_{\mathbb{R}^n} |\partial^\alpha f_{m_l} - \partial^\alpha f|^2 d_U^{2|\alpha|-2t}\, dx \leq \liminf_{k\to\infty} \int_{\mathbb{R}^n} |\partial^\alpha f_{m_l} - \partial^\alpha f_{m_k}|^2 d_U^{2|\alpha|-2t}\, dx \ \to 0, \quad l \to \infty.$$

This shows that $|\partial^\alpha f| d_U^{|\alpha|-t} \in L^2(\mathbb{R}^n)$. Thus $f \in X^t(U)$ and $\lim_{l\to\infty} \|f_{m_l} - f\|_{X^t(U)} = 0$. The proof of Lemma 4.2 is complete. $\square$

Lemma 4.3. — *For any $t \geq 0$, and any $\alpha, |\alpha| \leq [t]$, the map $f \mapsto \partial^\alpha f$ is continuous from $X^t(U)$ into $X^{t-|\alpha|}(U)$.*

Proof. — This is an obvious consequence of the definition of the space $X^t(U)$. $\square$

By Definition 4.1, one has $C_0^\infty(U) \subset X^t(U)$, with continuous injection.

Lemma 4.4. — *For any $t \geq 0$, the space $C_0^\infty(U)$ is dense in $X^t(U)$.*

Proof. — We may assume $U \neq \mathbb{R}^n$. Let δ be a Whitney regularization of d_U. This means that $\delta(x) = 0$ for $x \notin U$, $\delta|_U \in C^\infty(U)$, for any β, there exists C_β such that for all $x \in U$ one has $|\partial^\beta \delta(x)| \leq C_\beta d_U^{1-|\beta|}(x)$, and there exists a constant $D \geq 1$ such that

$$D^{-1}\delta(x) \leq d_U(x) \leq D\delta(x), \quad \forall x \in \mathbb{R}^n. \tag{4.4}$$

Let $\chi \in C^\infty(\mathbb{R})$ such that $0 \leq \chi(u) \leq 1$, $\chi(u) = 0$ for $u \leq 1$, $\chi(u) = 1$ for $u \geq 2$, and $\chi'(u) > 0$ for $u \in]1, 2[$.

For $f \in X^t(U)$ and $\varepsilon \in]0, 1]$, let f_ε be the function on $\mathbb{R}^n$ defined by

$$f_\varepsilon(x) = \chi(\frac{\delta(x)}{\varepsilon}) f(x) \quad \text{if } x \in U, \quad f_\varepsilon(x) = 0 \quad \text{if } x \notin U.$$

For $a > 0$ set $U_a = \{x \in U,\ \delta(x) > a\}$; one has $\overline{U_a} \subset \{x \in U,\ \delta(x) \geq a\}$. Then $f_\varepsilon = f$ on $U_{2\varepsilon}$ and $f_\varepsilon = 0$ on $\mathbb{R}^n \setminus U_\varepsilon$. For any $\varepsilon \in]0, 1]$, one has $\chi(\frac{\delta}{\varepsilon}) \in C^\infty(\mathbb{R}^n)$ and all its derivatives are bounded on $\mathbb{R}^n$. Moreover, $\mathrm{support}(\chi(\frac{\delta}{\varepsilon})) \subset \overline{U_\varepsilon}$. Thus one has $f_\varepsilon \in H^t_{\overline{U_\varepsilon}}$. Since for any $a > 0$ $H^t_{\overline{U_a}} \subset X^t(U)$ with continuous embedding, one get $f_\varepsilon \in X^t(U)$. Let $\phi_\rho(x) = \rho^{-n}\phi(x/\rho)$ an approximation of identity with $\phi \in C_0^\infty(|x| < 1), \int \phi(x)dx = 1$ and $\psi \in C_0^\infty, \psi(x) = 1$ in $|x| \leq 1$. For $n \geq 1$ and $\rho < \varepsilon/2$, set $f_{n,\rho,\varepsilon} = \psi(nx)(\phi_\rho * f_\varepsilon)$. Then one has $f_{n,\rho,\varepsilon} \in C_0^\infty(U)$, $\mathrm{support}(f_{n,\rho,\varepsilon}) \subset \overline{U_{\varepsilon/2}}$ and $\lim_{\rho\to 0} \lim_{n\to\infty} \|f_{n,\rho,\varepsilon} - f_\varepsilon\|_{H^t(\mathbb{R}^n)} = 0$. Thus we are reduced to prove

$$\lim_{\varepsilon\to 0} \|f_\varepsilon - f\|_{X^t(U)} = 0. \tag{4.5}$$

Observe that for $|\nu| \geq 1$, there exists a constant C_ν such that

$$|\partial^\nu \chi(\frac{\delta}{\varepsilon})(x)| \leq C_\nu d_U^{-\nu}(x) \mathbf{1}_{\delta(x)\in[\varepsilon,2\varepsilon]}. \tag{4.6}$$

Set $g_{\alpha,\varepsilon} = (1-\chi(\frac{\delta}{\varepsilon}))\partial^\alpha f$ for $|\alpha| \le [t]$ and $h_{\alpha,\beta,\varepsilon} = d_U^{-|\alpha-\beta|}\mathbf{1}_{\delta\in[\varepsilon,2\varepsilon]}\partial^\beta f$ for $\beta < \alpha$ and $|\alpha| \le [t]$. Using Leibnitz formula for derivatives, (4.1), Definition 4.1, (4.3), and (4.6), we are reduce to verify the following statements, when $\varepsilon \to 0$:

$$g_{\alpha,\varepsilon} d_U^{|\alpha|-t} \to 0 \text{ in } L^2(\mathbb{R}^n), \quad \text{and} \quad h_{\alpha,\beta,\varepsilon} d_U^{|\alpha|-t} \to 0 \text{ in } L^2(\mathbb{R}^n) \tag{4.7}$$

and if $r = \{t\} > 0$,

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|g_{\alpha,\varepsilon}(x) - g_{\alpha,\varepsilon}(y)|^2}{|x-y|^{n+2r}}\, dxdy \to 0, \quad \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|h_{\alpha,\beta,\varepsilon}(x) - h_{\alpha,\beta,\varepsilon}(y)|^2}{|x-y|^{n+2r}}\, dxdy \to 0. \tag{4.8}$$

One has $|g_{\alpha,\varepsilon}|d_U^{|\alpha|-t} \le |\partial^\alpha f| d_U^{|\alpha|-t}$ and $\lim_{\varepsilon\to 0} g_{\alpha,\varepsilon} d_U^{|\alpha|-t}(x) = 0$ for almost all $x \in \mathbb{R}^n$. One has also $|h_{\alpha,\beta,\varepsilon}|d_U^{|\alpha|-t} \le |\partial^\beta f| d_U^{|\beta|-t}$ and $\lim_{\varepsilon\to 0} |h_{\alpha,\beta,\varepsilon}| d_U^{|\alpha|-t}(x) = 0$ for all $x \in \mathbb{R}^n$. Thus (4.7) is a consequence of Lebesgue dominated convergence theorem.

In the case $r = \{t\} > 0$, let us first prove the first item of (4.8). We have to verify

$$\int_{x\in U, y\notin U} \frac{|g_{\alpha,\varepsilon}(x)|^2}{|x-y|^{n+2r}}\, dxdy \to 0 \tag{4.9}$$

$$\int_{x\in U, y\in U, |x-y|\ge d_U(y)/4} \frac{|g_{\alpha,\varepsilon}(x) - g_{\alpha,\varepsilon}(y)|^2}{|x-y|^{n+2r}}\, dxdy \to 0 \tag{4.10}$$

$$\int_{x\in U, y\in U, |x-y|\le d_U(y)/4} \frac{|g_{\alpha,\varepsilon}(x) - g_{\alpha,\varepsilon}(y)|^2}{|x-y|^{n+2r}}\, dxdy \to 0. \tag{4.11}$$

Since $\partial^\alpha f \in H^r(\mathbb{R}^n)$, and $\partial^\alpha f(y) = 0$ for almost all $y \notin U$, one has by Fubini theorem

$$\int_{x\in U} |\partial^\alpha f(x)|^2 \Big(\int_{y\notin U} \frac{1}{|x-y|^{n+2r}}\, dy \Big)\, dx = \int_{x\in U, y\notin U} \frac{|\partial^\alpha f(x)|^2}{|x-y|^{n+2r}}\, dxdy < \infty \tag{4.12}$$

thus (4.9) is consequence of the Lebesgue theorem as above. Observe that, since

$$\int_{y\notin U} \frac{1}{|x-y|^{n+2r}}\, dy \le \int_{|y-x|\ge d_U(x)} \frac{1}{|x-y|^{n+2r}}\, dy \le C d_U(x)^{-2r}$$

the inequality (4.12) is in fact weaker that what we get from $\partial^\alpha f d_U^{-r} \in L^2(\mathbb{R}^n)$, since we have

$$\int_{x\in U} |\partial^\alpha f(x)|^2 d_U(x)^{-2r} < \infty. \tag{4.13}$$

To get (4.10), it is obviously sufficient to prove

$$\int_{x\in U, y\in U, |x-y|\ge d_U(y)/4} \frac{|g_{\alpha,\varepsilon}(x)|^2 + |g_{\alpha,\varepsilon}(y)|^2}{|x-y|^{n+2r}}\, dxdy \to 0. \tag{4.14}$$

Since $|x-y| \geq d_U(y)/4$ implies $|x-y| \geq d_U(x)/5$, the result follows as above from the Lebesgue theorem and

$$\begin{aligned}
&\int_{x\in U} |\partial^\alpha f(x)|^2 \Big(\int_{y\in U, |y-x|\geq d_U(x)/5} \frac{1}{|x-y|^{n+2r}}\ dy\Big)\ dx \\
(4.15) \qquad &\leq \int_{x\in U} |\partial^\alpha f(x)|^2 \Big(\int_{|y-x|\geq d_U(x)/5} \frac{1}{|x-y|^{n+2r}}\ dy\Big)\ dx \\
&\leq \mathrm{Cte} \int_{x\in U} |\partial^\alpha f(x)|^2 d_U(x)^{-2r} < \infty.
\end{aligned}$$

Let us now verify (4.11). One has $g_{\alpha,\varepsilon}(x) - g_{\alpha,\varepsilon}(y) = I_\varepsilon(x,y) + II_\varepsilon(x,y)$, $I_\varepsilon(x,y) = \theta_\varepsilon(x)(\partial^\alpha f(x) - \partial^\alpha f(y))$, $II_\varepsilon(x,y) = (\theta_\varepsilon(x) - \theta_\varepsilon(y))\partial^\alpha f(y)$ with $\theta_\varepsilon = (1-\chi(\frac{\delta}{\varepsilon}))$. By Lebesgue theorem, one has

$$\int_{x\in U, y\in U, |x-y|\leq d_U(y)/4} \frac{|I_\varepsilon(x,y)|^2}{|x-y|^{n+2r}}\ dxdy \to 0.$$

The ball $B(y, d_U(y)/4)$ is contained in U and for x in this ball, one has $3d_U(y)/4 \leq d_U(x) \leq 5d_U(y)/4$. From (4.6), we get that there exists $0 < c_0 < c_1$ such that for all $y \in U$ and all $x \in B(y, d_U(y)/4)$

$$|\theta_\varepsilon(x) - \theta_\varepsilon(y)| \leq \mathrm{Cte}\ \mathbf{1}_{\delta(y)\in[c_0\varepsilon, c_1\varepsilon]}\ |x-y|\ d_U(y)^{-1}.$$

Thus one get

(4.16)

$$\begin{aligned}
&\int_{x\in U, y\in U, |x-y|\leq d_U(y)/4} \frac{|II_\varepsilon(x,y)|^2}{|x-y|^{n+2r}}\ dxdy \\
&\leq \mathrm{Cte} \int_{y\in U} \mathbf{1}_{\delta(y)\in[c_0\varepsilon, c_1\varepsilon]} |\partial^\alpha f(y)|^2 d_U(y)^{-2} \Big(\int_{|x-y|\leq d_U(y)/4} |x-y|^{-n+2(1-r)}\ dx\Big)\ dy \\
&\leq \mathrm{Cte} \int_{y\in U, \delta(y)\in[c_0\varepsilon, c_1\varepsilon]} |\partial^\alpha f(y)|^2 d_U(y)^{-2r}\ dy
\end{aligned}$$

and we conclude by the Lebesgue theorem. The verification of the second item of (4.8) is similar. The proof of Lemma 4.4 is complete. □

Remark 4.5. — *By Lemma 4.4, we get that $X^t(U)$ is a subspace of $H^t_0[U]$, with continuous and dense injection. The equality $X^t(U) = H^t_0[U]$ is not always true, even when U is subanalytic. For example, in $\mathbb{R}^2$, one has $X^1(\mathbb{R}^2 \setminus \{0\}) = \{f \in H^1(\mathbb{R}^2)$, and $\frac{f}{|x|} \in L^2(\mathbb{R}^2)\}$, which is a strict subspace of $H^1_0[\mathbb{R}^2 \setminus \{0\}] = H^1(\mathbb{R}^2)$.*

Lemma 4.6. — *Let U be a Lipschitz open and bounded subset of $\mathbb{R}^n$. One has for all $t \geq 0$ the equality*

$$H^t_0[U] = X^t(U).$$

Proof. — Set $t = k + r, k \in \mathbb{N}, r \in [0,1[$. Since by Lemma 3.4, one has $H^t_0[U] = H^t_{\overline{U}}$, one has to verify that for any $f \in H^t_{\overline{U}}$, one has $\partial^\alpha f d_U^{|\alpha|-t} \in L^2$ for $|\alpha| \leq k$. Let us first assume $k=0$. The result is obvious for $r=0$ and follows from the characterization (2.1) of the H^r norm and $\int_{y\notin U}\frac{1}{|x-y|^{n+2r}}dy \simeq d_U^{-2r}(x)$ for $x\in U, r\in\,]0,1[$. For the general case $k\geq 1$, by induction on k, it remains to verify $fd_U^{-t}\in L^2$. We may assume that f is supported near a point of ∂U, where U is defined by an inequation of the form $x_n > F(x')$ with F Lipschitz. Since $f=0$ on the set $x_n < F(x')$, Taylor formula implies for $x_n > F(x')$

$$f(x',x_n)=\frac{(x_n-F(x'))^k}{(k-1)!}\int_0^1(1-s)^{k-1}(\partial_{x_n}^k f)(x',sx_n+(1-s)F(x'))ds.$$

Observe that $(x_n-F(x'))\simeq d_U(x)$. Thus the result follows from the case $k=0$ since the function $x\mapsto g_s(x)=1_U(\partial_{x_n}^k f)(x',sx_n+(1-s)F(x'))$ belongs to $H^r_{\overline{U}}$ and satisfy $\|g_s\|_{H^r}\leq s^{-1/2}\|\partial_{x_n}^k f\|_{H^r}$. The proof of Lemma 4.6 is complete. □

For $m\in\mathbb{R}, m\geq 0$, we denote by $C^{\infty,m}(U)$ the space of smooth functions ψ on U such that for any β, there exists C_β such that

$$|\partial^\beta\psi(x)|\leq C_\beta d_U^{-m-|\beta|}(x),\quad \forall x\in U.$$

For $\psi\in C^{\infty,m}(U)$ and $f\in X^t(U)$ with $0\leq m\leq t$, we define the function $\psi f\in L^2(\mathbb{R}^n)$ by the formula

$$\psi f(x)=\psi(x)f(x) \text{ if } x\in U,\quad \psi f(x)=0 \text{ if } x\notin U.$$

The following lemma is easy but fundamental. It will be untrue if one replace the space $X^t(U)$ by the space $H_0^t[U]$ or by the space $H^t_{\overline{U}}$. As an example with t=1, one has with $U=\mathbb{R}^2\setminus 0$, $H_0^1[U]=H^1_{\overline{U}}=H^1(\mathbb{R}^2)$; the function $f(x)=\sin(x/|x|)$ belongs to $C^{\infty,0}(U)$, and for $\phi\in C_0^\infty(\mathbb{R}^2)\subset H^1(\mathbb{R}^2)$, one has $f\phi\in H^1(\mathbb{R}^2)$ iff $\phi(0)=0$.

Lemma 4.7. — *For any $0\leq m\leq t$, and any $\psi\in C^{\infty,m}(U)$, the map $f\mapsto\psi f$ is continuous from $X^t(U)$ into $X^{t-m}(U)$.*

Proof. — We keep the notation of the proof of Lemma 4.4. By definition of the distribution $\psi f\in L^2(\mathbb{R}^n)$, one has by the Lebesgue theorem

$$\psi f=\lim_{L^2,\varepsilon\to 0}\chi(\delta/\varepsilon)\psi\mathbf{1}_U f.$$

Set $t-m=k+r, k=[t-m]$. By (4.2), one has $\partial^\nu(\mathbf{1}_U f)=\mathbf{1}_U\partial^\nu f$ for $|\nu|\leq[t]$. Using Leibnitz rule to compute derivatives, one gets for $|\alpha|\leq k$

$$\partial^\alpha(\psi f)=\lim_{\mathscr{D}',\varepsilon\to 0}\sum_{\beta\leq\alpha}C_\alpha^\beta\partial^{\alpha-\beta}(\chi(\delta/\varepsilon)\psi)\mathbf{1}_U\partial^\beta f.$$

By (4.6), there exists C independent of ε such that $|\partial^{\alpha-\beta}(\chi(\delta/\varepsilon)\psi)|\leq Cd_U^{-m-|\alpha-\beta|}$. Therefore $\partial^{\alpha-\beta}(\chi(\delta/\varepsilon)\psi)\mathbf{1}_U\partial^\beta f$ converges in $L^2(\mathbb{R}^n)$ to $\partial^{\alpha-\beta}(\psi)\mathbf{1}_U\partial^\beta f$ by Lebesgue theorem. Thus we get the equality in $\mathscr{D}'(\mathbb{R}^n)$

$$\partial^\alpha(\psi f)=\sum_{\beta\leq\alpha}C_\alpha^\beta\partial^{\alpha-\beta}(\psi)\mathbf{1}_U\partial^\beta f. \tag{4.17}$$

Since the right member of this identity belongs to $L^2(\mathbb{R}^n)$, this shows $\partial^\alpha(\psi f) \in L^2(\mathbb{R}^n)$, and any measurable representative of $\partial^\alpha(\psi f)$ is equal to 0 for almost all $x \notin U$. Then the point ii) of Definition 4.1 follows easily. Since $\partial^{\alpha-\beta}(\psi) \in C^{\infty,m+|\alpha-\beta|}$ and $\partial^\beta f \in X^{t-|\beta|}(U)$, it remains, in the case $r > 0$, to verify the following statement: for $\psi \in C^{\infty,m'}$, $f \in X^{t'}(U)$, $t' - m' = r \in]0,1[$, one has with $g = \mathbf{1}_U \psi f$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2r}}\, dxdy < \infty. \tag{4.18}$$

The proof of (4.18) is almost the same as the proof of Lemma 4.4. The only minor difference is in the estimation of the term

$$\int_{x \in U, y \in U, |x-y| \le d_U(y)/4} \frac{|(\psi(x) - \psi(y))f(y)|^2}{|x-y|^{n+2r}}\, dxdy.$$

Since the ball $B(y, d_U(y)/4)$ is contained in U, one has for $x \in B(y, d_U(y)/4)$,

$$\psi(x) - \psi(y) = (x-y).\int_0^1 \nabla\psi(y + s(x-y))ds.$$

Thus it is sufficient to prove

$$\sup_{s \in [0,1]} \int_{x \in U, y \in U, |x-y| \le d_U(y)/4} \frac{|(\nabla\psi(y + s(x-y)))f(y)|^2}{|x-y|^{n+2r-2}}\, dxdy < \infty. \tag{4.19}$$

Since for $|x-y| \le d_U(y)/4$ one has $d_U(y + s(x-y)) \ge d_U(y)/2$, (4.19) follows from

$$\int_{y \in U} |f(y)|^2 d_U(y)^{-2m'-2}\Big(\int_{|x-y| \le d_U(y)/4} \frac{1}{|x-y|^{n+2r-2}}\, dx\Big)dy \le \mathrm{Cte}\|f d_U^{-m'-r}\|^2_{L^2(U)} < \infty.$$

Thus we have proven $\psi f \in X^{t-m}(U)$. Since the map $f \mapsto \psi f$ from $X^t(U)$ into $L^2(\mathbb{R}^n)$ is obviously continuous, and since the injection $X^{t-m}(U) \subset L^2(\mathbb{R}^n)$ is continuous, the continuity of the map $f \mapsto \psi f$ from $X^t(U)$ into $X^{t-m}(U)$ follows from the closed graph theorem. The proof of Lemma 4.10 is complete. □

Let $U \subset V$ be two open sets in $\mathbb{R}^n$. One has for all $x \in \mathbb{R}^n$ $d_U(x) \le d_V(x)$, thus, for $f \in X^t(U)$, one has $f \in X^t(V)$, and the canonical injection $X^t(U) \to X^t(V)$ is continuous.

The following proposition is proven in **[1]** (Proposition 5.7) in the situation where $U \mapsto F(U)$ is a sheaf on the linear analytic site which is a $C^{\infty,0}$ module. For the convenience of the reader, we recall the proof here in the special case of the spaces $X^t(U)$. (observe that $U \mapsto X^t(U)$ is not a sheaf but is a cosheaf)

Proposition 4.8. — *Let U, V be two open sets in $\mathbb{R}^n$ and $t \ge 0$. Consider the complex of Hilbert spaces*

$$0 \to X^t(U \cap V) \xrightarrow{i} X^t(U) \oplus X^t(V) \xrightarrow{\tau} X^t(U \cup V) \to 0 \tag{4.20}$$

where the first arrow is $i(f) = f \oplus f$ and the second arrow is $\tau(f \oplus g) = f - g$. Then i is injective, $Ker(\tau) = \mathrm{Im}(i)$, and $\mathrm{Im}(\tau)$ is dense in $X^t(U \cup V)$.

If moreover, there exists a constant C *such that*

$$d_{U\cup V}(x) \leq C\max(d_U(x), d_V(x)), \quad \forall x \in \mathbb{R}^n \tag{4.21}$$

then τ *is surjective, i.e the sequence of Hilbert spaces* (4.20) *is exact.*

Proof. — The injectivity of i is obvious. Let $f \in X^t(U), g \in X^t(V)$ such that $f - g = 0$, then $f = g \in X^t(U) \cap X^t(V)$. Thus, for $|\alpha| \leq [t]$, and any measurable representative f_α of $\partial^\alpha f$, one has $f_\alpha(x)=0$ for almost all $x \notin U \cap V$. In order to prove $f \in X^t(U \cap V)$, it remains to show $|\partial^\alpha f| d_{U\cap V}^{|\alpha|-t} \in L^2(\mathbb{R}^n)$, for $|\alpha| \leq [t]$, but this is consequence of $f \in X^t(U) \cap X^t(V)$ and of the obvious identity

$$d_{U\cap V} = \min(d_U, d_V).$$

Thus one has $Ker(\tau) = \mathrm{Im}(i)$. By the classical partition of unity theorem, any function $h \in C_0^\infty(U \cup V)$ is of the form $h = f - g$, $f \in C_0^\infty(U) \subset X^t(U)$, $g \in C_0^\infty(V) \subset X^t(V)$. Therefore, by Lemma 4.4, $\mathrm{Im}(\tau)$ is dense in $X^t(U \cup V)$.

Let us now assume that (4.21) is satisfied. We may assume $U \neq \mathbb{R}^n$ and $V \neq \mathbb{R}^n$, since otherwise, the sequence (4.20) is trivially exact. Let G, F be the closed sets in $\mathbb{R}^n$

$$G = \{x \in \mathbb{R}^n, d_U(x) \leq d_V(x)/2\}, \quad F = \{x \in \mathbb{R}^n, d_V(x) \leq d_U(x)/2\}.$$

By a result of Whitney (see **[2]**, Corollary 1.4.11, for a proof and the notes therein), there exists a function $\psi \in C^\infty(\mathbb{R}^n \setminus (F \cap G))$, such that $\psi = 0$ near $G \setminus (F \cap G))$, $\psi = 1$ near $F \setminus (F \cap G))$, and such that for any α, there exists C_α such that, with $d(x) = \max(\mathrm{dist}(x, F), \mathrm{dist}(x, G))$, one has

$$\forall x \notin F \cap G, \quad |\partial^\alpha \psi(x)| \leq C_\alpha d(x)^{-|\alpha|}. \tag{4.22}$$

One has $F \cap G = \{x \in \mathbb{R}^n, d_V(x) = d_U(x) = 0\} = \mathbb{R}^n \setminus (U \cup V)$. Observe that $\mathbb{R}^n \setminus U \subset G$, and $\mathbb{R}^n \setminus V \subset F$, hence $\mathrm{dist}(x, G) \leq d_U(x)$ and $\mathrm{dist}(x, F) \leq d_V(x)$. Thus one has

$$d(x) \leq \max(d_U(x), d_V(x)), \quad \forall x \in \mathbb{R}^n .$$

Let $x \in U \setminus V$; one has $x \in F$. Let $z \in G$ such that $d(x) = \mathrm{dist}(x, G) = |x - z|$. Since $x \notin V$, one has $d_V(z) \leq |z - x|$. Thus we get

$$d_U(x) \leq |x - z| + d_U(z) \leq |x - z| + d_V(z)/2 \leq 3|x - z|/2 = 3d(x)/2.$$

Let $x \in U \cap V$ such that $d_V(x) \leq d_U(x)$. let $z \in G$ such that $\mathrm{dist}(x, G) = |x - z|$. One has

$$d_U(x) \leq |x - z| + d_U(z) \leq |x - z| + d_V(z)/2 \leq |x - z| + \frac{1}{2}(d_V(x) + |x - z|).$$

Therefore we get $d_U(x) \leq 3\mathrm{dist}(x, G) \leq 3d(x)$. Thus, we have proven

$$d(x) \leq \max(d_U(x), d_V(x)) \leq 3d(x), \quad \forall x \in \mathbb{R}^n .$$

Since (4.21) holds true, one has $\psi \in C^{\infty,0}(U \cup V)$. Let $f \in X^t(U \cup V)$. By Lemma 4.7, one has $\psi f \in X^t(U \cup V)$. By (4.17) and $(U \cup V) \setminus U \subset G$, any measurable representative of $\partial^\alpha(\psi f)$ vanishes for almost all $x \notin U$ and $|\alpha| \leq [t]$. Moreover, for all x such that

$\psi f(x) \neq 0$, one has $x \notin G$, hence $d_V(x)/2 < d_U(x)$ and this implies $2d_U(x) \geq \max(d_U(x), d_V(x))$. Thus one has $\psi f \in X^t(U)$. Similarly, $(1-\psi)f \in X^t(V)$. The proof of Proposition 4.8 is complete. □

4.2. The spaces $Y^s(U)$. — In this section, $s \in]-\infty, 0]$ is given and $t = -s \geq 0$.

Definition 4.9. — *Let U be an open set in $\mathbb{R}^n$. We define the space $Y^s(U) \subset \mathscr{D}'(U)$ by*

$$T \in Y^s(U) \quad \textit{iff} \quad \exists C,\ |\langle T, \phi\rangle| \leq C\|\phi\|_{X^t(U)}, \quad \forall \phi \in C_0^\infty(U). \tag{4.23}$$

For $T \in Y^s(U)$, we define $\|T\|_{Y^s(U)}$ as the infimum of C such that (4.23) *holds true.*

It is clear from the definition and from Lemma 4.4, that for $T \in Y^s(U)$ the duality $\langle T, \phi\rangle$ extends continuously and uniquely to a duality $\langle T, f\rangle$ with $f \in X^t(U)$, and one has

$$|\langle T, f\rangle| \leq \|T\|_{Y^s(U)}\|f\|_{X^t(U)}. \tag{4.24}$$

From Lemma 4.7, we get that for $0 \leq m \leq -s$, $\psi \in C^{\infty,m}(U)$, and $T \in Y^{s+m}(U)$, the distribution $\psi T \in \mathscr{D}'(U)$ defined by $\langle \psi T, \phi\rangle = \langle T, \psi\phi\rangle$, $\phi \in C_0^\infty(U)$, belongs to $Y^s(U)$, and the map $T \mapsto \psi T$ is continuous from $Y^{s+m}(U)$ into $Y^s(U)$.

Lemma 4.10. — *The duality $\langle T, f\rangle$ identifies $Y^s(U)$ to the dual space of $X^t(U)$. In particular, $Y^s(U)$ is an Hilbert space. Moreover, $H^s(U)$ is a dense subspace of $Y^s(U)$.*

Proof. — This is obvious. Let E be the dual space of $X^t(U)$. Since $X^t(U)$ is a dense subset of $H_0^t[U]$, by Lemma 2.2, $H^s(U)$ is a dense subset of E. The map $T \mapsto \langle T, .\rangle$ defines an isometric injection j from $Y^s(U)$ into E. Let $e \in E$. Then $\phi \mapsto e(\phi)$, $\phi \in C_0^\infty(U)$ defines a distribution $T \in \mathscr{D}'(U)$, which satisfies $|\langle T, \phi\rangle| \leq \|e\|_E\|\phi\|_{X^t(U)}$, thus one has $j(T) = e$ and therefore $\|e\|_E = \|T\|_{Y^s(U)}$. □

Let $U \subset V$ be two open sets in $\mathbb{R}^n$. Since we have a canonical imbedding $X^t(U) \to X^t(V)$, we deduce by duality and Lemma 4.10 a canonical continuous restriction map $r_{V,U}$

$$Y^s(V) \overset{r_{V,U}}{\to} Y^s(U) \tag{4.25}$$

and for $U \subset V \subset W$, one has $r_{W,U} = r_{W,V}r_{V,U}$. From Proposition 4.8, we deduce:

Proposition 4.11. — *Let U, V be two open sets in $\mathbb{R}^n$ and $s \leq 0$. Consider the complex of Hilbert spaces*

$$0 \to Y^s(U \cup V) \overset{\tilde{\tau}}{\to} Y^s(U) \oplus Y^s(V) \overset{\tilde{\imath}}{\to} Y^s(U \cap V) \to 0 \tag{4.26}$$

where the first arrow is $\tilde{\tau}(f) = r(f) \oplus -r(f)$ and the second arrow is $\tilde{\imath}(f \oplus g) = r(f) + r(g)$. Then $\tilde{\tau}$ is injective, $\tilde{\imath}$ is surjective, and $\mathrm{Im}(\tilde{\tau})$ is dense in $Ker(\tilde{\imath})$.

If moreover, there exists a constant C such that

$$d_{U\cup V}(x) \leq C\max(d_U(x), d_V(x)), \quad \forall x \in \mathbb{R}^n \tag{4.27}$$

then $\mathrm{Im}(\tilde{\tau}) = Ker(\tilde{\imath})$ i.e the sequence of Hilbert spaces (4.26) *is exact.*

Proof. — This is an obvious consequence of Proposition 4.8, since (4.26) is the dual sequence of (4.20). □

5. The sheaf $\mathscr{H}^s$

Let M be a real analytic manifold. In [**3**], M. Kashiwara and P. Schapira endow M with a Grothendieck topology, denoted M_{sa}, as follows. Denote by Op_M the category of open subsets of M (the morphisms are the inclusions) and denote by $\mathrm{Op}_{M_{\mathrm{sa}}}$ the full subcategory consisting of relatively compact subanalytic open subsets of M. These are the open sets for this new topology. Then a family $\{U_i\}_{i\in I}$ in $\mathrm{Op}_{M_{\mathrm{sa}}}$ is a covering of $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ if $U_i \subset U$ for all $i \in I$ and there exists a finite subset $J \subset I$ such that $\bigcup_{j\in J} U_j = U$. Roughly speaking, the coverings are the finite coverings.

One denotes by $\mathrm{Mod}(\mathbb{C}_M)$ the abelian category of sheaves of $\mathbb{C}$-vector spaces on M and one uses a similar notation for sheaves on M_{sa}. Note that a presheaf F on M_{sa} is a sheaf if and only if for any $\{U_1, U_2\}$ in $\mathrm{Op}_{M_{\mathrm{sa}}}$, the sequence $0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2)$ is exact. Denote by $\mathrm{D}^+(\mathbb{C}_M)$ the bounded from below derived category of sheaves of $\mathbb{C}$-vector spaces on M and by $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_M)$ the full subcategory consisting of bounded complexes. Define similarly the categories $\mathrm{D}^+(\mathbb{C}_{M_{\mathrm{sa}}})$ and $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{M_{\mathrm{sa}}})$. There is a natural morphism of sites

$$\rho_{\mathrm{sa}} : M \to M_{\mathrm{sa}},$$

which defines the pair of adjoint functors $(\rho_{\mathrm{sa}}^{-1}, \mathrm{R}\rho_{\mathrm{sa}*})$ (inverse and direct images):

$$\mathsf{D}^{\mathrm{b}}(\mathbb{C}_M) \underset{\rho_{\mathrm{sa}}^{-1}}{\overset{\mathrm{R}\rho_{\mathrm{sa}*}}{\rightleftarrows}} \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{M_{\mathrm{sa}}}).$$

and similarly with D^{b} replaced with D^+.

We start with the following lemma, which is a byproduct of a theorem of A. Parusinski in [**5**]. Let us recall from [**1**] the following definition of a Γ-acyclic sheaf on the analytic site M_{sa}.

Definition 5.1. — *A sheaf $U \mapsto F(U)$ on the analytic site M_{sa} is Γ-acyclic iff for any relatively compact open subanalytic subsets U, V of M, the following sequence is exact*

$$0 \to F(U \cup V) \to F(U) \oplus F(V) \to F(U \cap V) \to 0.$$

It is proven in [**1**] that if F is Γ-acyclic, then $\mathbb{H}^j(U, F) = 0$ for any $j \neq 0$.

Lemma 5.2. — *For any $s \in\,]-1/2, 1/2[$, $U \mapsto H^s(U)$ is a Γ-acyclic sheaf on the analytic site M_{sa}.*

Proof. — Let W be Lipschitz, relatively compact, open subset of M, and let 1_W be the characteristic function of W. It is well known that for any $f \in H^s_{\mathrm{loc}}(M)$, one has $1_W f \in H^s_{\mathrm{loc}}(M)$ for all $s \in\,]-1/2, 1/2[$. By Theorem 0.1 in [**5**], the set of relatively compact open subanalytic subsets of M is an algebra generated by the Lipschitz relatively compact open subanalytic subsets. Hence, if U is a relatively compact open subanalytic subset of M, the characteristic function 1_U is a linear combination of functions of the form $1_{W_1} \cdots 1_{W_m}$, where the W_j are Lipschitz relatively compact open subanalytic subsets of M. Thus, for any $f \in H^s_{\mathrm{loc}}(M)$, one has $1_U f \in H^s_{\mathrm{loc}}(M)$ for all $s \in \left]-1/2, 1/2\right[$. As an obvious consequence, if U, V are two relatively compact subanalytic subsets of M, the sequence

$$0 \to H^s(U \cup V)) \to H^s(U) \oplus H^s(V) \to H^s(U \cap V) \to 0$$

is exact for all $s \in \left]-1/2, 1/2\right[$. □

5.1. The sheaf $\mathscr{H}^s$ for $s \leq 0$. — Let d be a Riemannian distance on M. Let $(U_j)_j$ be a finite family of relatively compact open subanalytic subsets of M and $U = \cup_j U_j$. It follows from the theory of subanalytic sets that there exist a constant $C > 0$ and a positive integer N such that

$$d(x, M \setminus U)^N \leq C \cdot (\max_{j \in J} d(x, M \setminus U_j)). \tag{5.1}$$

Let $\{U_i\}_{i \in I}$ be a finite family in $\mathrm{Op}_{M_{\mathrm{sa}}}$. One says that this family is 1-regularly situated if there is a constant C such that for any $x \in M$

$$d(x, M \setminus \bigcup_{i \in I} U_i) \leq C \cdot \max_{i \in I} d(x, M \setminus U_i). \tag{5.2}$$

In [**1**], M. Guillermou and P. Schapira introduces a new Grothendieck topology on M, denoted M_{sal}, for which the open subsets are the same as for M_{sa} but the coverings are those for which one can extract a finite 1-regularly situated covering. There is a natural morphism of sites

$$\rho_{\mathrm{sal}} : M_{\mathrm{sa}} \to M_{\mathrm{sal}}.$$

and denoting as above by $\mathsf{D}^+(\mathbb{C}_{M_{\mathrm{sal}}})$ and $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{M_{\mathrm{sal}}})$ the derived categories of sheaves on M_{sal}, we have again a pair of adjoint functors $(\rho_{\mathrm{sal}}^{-1}, \mathrm{R}\rho_{\mathrm{sal}*})$:

$$\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{M_{\mathrm{sa}}}) \underset{\rho_{\mathrm{sal}}^{-1}}{\overset{\mathrm{R}\rho_{\mathrm{sal}*}}{\rightleftarrows}} \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{M_{\mathrm{sal}}}).$$

and similarly with D^{b} replaced with D^+.

Similarly as on the site M_{sa}, a presheaf on M_{sa} is a sheaf on M_{sal} if and only if, for any $\{U_1, U_2\}$ which is a covering of $U_1 \cup U_2$ in M_{sal}, the sequence $0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2)$ is exact. If moreover the sequence $0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2) \to 0$ is exact, one says that F is Γ-acyclic and this is equivalent ([**1**, Prop. 1.3.4]) to saying that $H^k(U; F) \simeq 0$ for all $k > 0$ and all $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. (Of course, the cohomology is calculated in the category of sheaves on M_{sal}.)

Let us recall the following theorems from [**1**].

Theorem 5.3. — [**1**, Th. 2.3.13] *The functor* $\mathrm{R}\rho_{\mathrm{sal}*}\colon \mathrm{D}^+(\mathbb{C}_{M_{\mathrm{sa}}}) \to \mathrm{D}^+(\mathbb{C}_{M_{\mathrm{sal}}})$ *admits a right adjoint* $\rho^!_{\mathrm{sal}}\colon \mathrm{D}^+(\mathbb{C}_{M_{\mathrm{sal}}}) \to \mathrm{D}^+(\mathbb{C}_{M_{\mathrm{sa}}})$.

In particular, we get that if $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, then

$$\mathrm{RHom}\,(\mathbb{C}_{UM_{\mathrm{sa}}}, \rho^!_{\mathrm{sal}}F) \simeq \mathrm{RHom}\,(\mathrm{R}\rho_{\mathrm{sal}*}\mathbb{C}_{UM_{\mathrm{sa}}}, F).$$

One says that $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ is weakly Lipschitz if for each $x \in M$ there exists a neighborhood $V \in \mathrm{Op}_{M_{\mathrm{sa}}}$ of x, a finite set I and $U_i \in \mathrm{Op}_{M_{\mathrm{sa}}}$ such that $U \cap V = \bigcup_i U_i$ and

$$\begin{cases} \text{for all } \varnothing \neq J \subset I, \text{ the set } U_J = \bigcap_{j\in J} U_j \text{ is a disjoint union} \\ \text{of Lipschitz open sets.} \end{cases} \tag{5.3}$$

Theorem 5.4. — [**1**, Th. 2.4.15] *Let* $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ *and assume that* U *is weakly Lipschitz. Then*

1. $\mathrm{R}\rho_{\mathrm{sal}*}\mathbb{C}_{UM_{\mathrm{sa}}} \simeq \rho_{\mathrm{sal}*}\mathbb{C}_{UM_{\mathrm{sa}}} \simeq \mathbb{C}_{UM_{\mathrm{sal}}}$ *is concentrated in degree zero.*
2. *For* $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{M_{\mathrm{sal}}})$, *one has* $\mathrm{R}\Gamma(U; \rho^!_{\mathrm{sal}}F) \simeq \mathrm{R}\Gamma(U;F)$.
3. *Let* $F \in \mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sal}}})$ *and assume that* F *is* Γ*-acyclic. Then* $\mathrm{R}\Gamma(U;\rho^!_{\mathrm{sal}}F)$ *is concentrated in degree* 0 *and is isomorphic to* $F(U)$.

Note that in this theorem, part (1) follows from a result of A. Parusinski [**5**].

As a corollary of Proposition 4.11, $U \mapsto Y^s(U)$ is a presheaf on the site M_{sa}, and a sheaf on the site M_{sal}. Moreover, on M_{sal}, this sheaf is Γ-acyclic. Thus the construction of the Sobolev sheaf $U \mapsto \mathscr{H}^s(U)$ follows now easily. Let us denote by Y^s the Γ-acyclic sheaf $U \mapsto Y^s(U)$ on M_{sal} .

Definition 5.5. — *For* $s \leq 0$, *we define the object* $\mathscr{H}^s$ *of* $\mathrm{D}^+(\mathbb{C}_{M_{\mathrm{sa}}})$ *by*

$$\mathscr{H}^s = \rho^! Y^s. \tag{5.4}$$

In particular, for any relatively compact subanalytic open set U, *the complex* $\mathscr{H}^s(U)$ *is defined by*

$$\mathscr{H}^s(U) = R\Gamma(U, \rho^! Y^s).$$

From Lemmas 2.2 and 4.6, one has $Y^s(U) = H^s(U)$ if U is Lipschitz. Hence, from Theorem 5.4, the requirement (1.1) holds true.

We end this subsection by the following lemma, which tell us that the above construction does not work for $s > 1$.

Lemma 5.6. — *Let* $n = dim(M) \geq 2$ *and* $s > 1$. *There is no* Γ*-acyclic sheaf* Y^s *on* M_{sal} *such that* $Y^s(U) = H^s(U)$ *when* U *is Lipschitz.*

Proof. — We may assume $M = \mathbb{R}^2$. Let Y^s be a Γ-acyclic sheaf on M_{sal} such that $Y^s(U) = H^s(U)$ when U is Lipschitz. We argue by contradiction, and we keep the notation of the case (o) in Proposition 5.7. Observe that $V_1 \cup V_2 = B^*_r$ is a linear covering, and $V_1 \cap V_2 = U_r(\gamma_1, \gamma_4) \cup U_r(\gamma_3, \gamma_2)$ with $U_r(\gamma_1, \gamma_4)$, $U_r(\gamma_3, \gamma_2)$ Lipschitz and $U_r(\gamma_1, \gamma_4) \cap U_r(\gamma_3, \gamma_2) = \varnothing$. Therefore the sequence

$$0 \to Y^s(V_1 \cup V_2) \to H^s(V_1) \oplus H^s(V_2) \to H^s(U_r(\gamma_1, \gamma_4)) \oplus H^s(U_r(\gamma_3, \gamma_2)) \to 0$$

is exact. This implies that any $f \oplus g \in H^s(U_r(\gamma_1, \gamma_4)) \oplus H^s(U_r(\gamma_3, \gamma_2))$ is the restriction to $U_r(\gamma_1, \gamma_4) \cup U_r(\gamma_3, \gamma_2)$ of some function $h \in H^s(\mathbb{R}^2)$, and this implies $f(0) = g(0)$, which is not always true: a contradiction. □

5.2. The sheaf $\mathscr{H}^s$ on $\mathbb{R}^2$, with $s \leq 0$. — Let us recall that a relatively compact open set $U \in \mathbb{R}^2$ is subanalytic iff U has a finite number of connected components U_j, and for all j, the boundary of U_j is a finite union of points and semi-analytic curves.

Let $p_0 = (0, 0)$ and $B_r = B(p_0, r)$ the open disk centered at the origin and of radius $r > 0$. If U is a subanalytic open set in $\mathbb{R}^2$ we set $U_r = U \cap B_r$. Then for $r > 0$ small enough, and if U is locally connected near p_0, U_r is of one of the following form (all semi-analytic curves γ below have p_0 as left extremity, $\gamma \setminus p_0$ is an analytic curve for r small enough, and $\dot\gamma$ denotes the unit vector tangent to γ at the left extremity p_0):

- Trivial cases: $U_r = B_r$ (iff $p_0 \in U$) or $U_r = \varnothing$ (iff $p_0 \notin \overline{U}$).
- Case (o): $U_r = B_r^* = B_r \setminus p_0$.
- Good case: There exists two semi-analytic curves γ_1, γ_2, such that $\dot\gamma_1 \neq \dot\gamma_2$ and $U_r = U_r(\gamma_1, \gamma_2)$ is the subset of B_r of points between γ_1 and γ_2 (one has $\dot\gamma_j \in S^1$ and we take the direct orientation on S^1). For example, if $\gamma_1 = (x \geq 0, y = 0)$ and $\gamma_2 = (x = 0, y \geq 0)$, one has $U_r(\gamma_1, \gamma_2) = B_r \cap \{x > 0 \text{ and } y > 0\}$ and $U_r(\gamma_2, \gamma_1) = B_r \cap \{x < 0 \text{ or } y < 0\}$.
- S case: There exists two distinct semi-analytic curves γ_1, γ_2 such that $\dot\gamma_1 = \dot\gamma_2$ and U_r is the subset of B_r of points between γ_1 and γ_2. For example, if $\gamma_1 = (x \geq 0, y = 0)$ and $\gamma_2 = (x \geq 0, y = x^2)$, one has $U_r(\gamma_1, \gamma_2) = B_r \cap \{x > 0 \text{ and } 0 < y < x^2\}$.
- B case: There exists two distinct semi-analytic curves γ_1, γ_2 such that $\dot\gamma_1 = \dot\gamma_2$ and $U_r = B_r \setminus \overline{U_r(\gamma_1, \gamma_2)}$.
- C case: There exists a semi-analytic curve γ such that $U_r = B_r \setminus \gamma$.

Let U be a bounded semi-analytic open subset of $\mathbb{R}^2$ such that $p_0 \in \partial U$. Since we are working in $\mathbb{R}^2$, the above classification of the geometry of U near p_0 is simple. As a consequence, we will get that the complex $\mathscr{H}^s(U)$ satisfying (1.1) is entirely determined by the long exact Mayer Vietoris sequence (1.2).

Proposition 5.7. — *Let $U \subset \mathbb{R}^2$ be a semi-analytic open set, $r > 0$ small, and $U_r = U \cap B_r$.*

i) *If the connected components of U near p_0 are the $U_j, 1 \leq j \leq N$, then*

$$\mathscr{H}^s(U_r) = \oplus_{j=1}^N \mathscr{H}^s(U_j \cap B_r). \tag{5.5}$$

ii) *If U is locally connected near p_0*

- *Trivial cases: If $U_r = B_r$ then $\mathscr{H}^s(B_r)$ is concentrate in degree 0 and isomorphic to $H^s(B_r)$. If $U_r = \varnothing$ then $\mathscr{H}^s(\varnothing) = 0$.*

- *Case (o): Let $\gamma_j, 1 \leq j \leq 4$ be algebraic curves with $\dot{\gamma_j} \neq \dot{\gamma_k}$ for $j \neq k$. Assume that $V_1 = U_r(\gamma_1, \gamma_2)$ and $V_2 = U_r(\gamma_3, \gamma_4)$ satisfy $V_1 \cup V_2 = B_r^*$ and $V_1 \cap V_2 = U_r(\gamma_1, \gamma_4) \cup U_r(\gamma_3, \gamma_2)$. Observe that $U_r(\gamma_1, \gamma_4) \cap U_r(\gamma_3, \gamma_2)$ is void. Then $\mathscr{H}^s(B_r^*)$ is the complex*

$$\cdots \to 0 \xrightarrow{d_{-1}} H^s(V_1) \oplus H^s(V_2) \xrightarrow{d_0} H^s(U_r(\gamma_1, \gamma_4)) \oplus H^s(U_r(\gamma_3, \gamma_2)) \xrightarrow{d_1} 0 \xrightarrow{d_2} \cdots. \tag{5.6}$$

- *Good case: If there exists two algebraic curves γ_1, γ_2 such that $\dot{\gamma_1} \neq \dot{\gamma_2}$ and $U_r = U_r(\gamma_1, \gamma_2)$, then U_r is Lipschitz and $\mathscr{H}^s(U_r(\gamma_1, \gamma_2))$ is concentrate in degree 0 and isomorphic to $H^s(U_r(\gamma_1, \gamma_2))$.*
- *S case: In that case, there exists two distinct algebraic curves γ_1, γ_2 such that $\dot{\gamma_1} = \dot{\gamma_2}$ and U_r is the subset of B_r of points between γ_1 and γ_2. Let γ_3 and γ_4 such that $\dot{\gamma_j} \neq \dot{\gamma_k}$ for $2 \leq j \neq k \leq 4$, and such that with $V_1 = U_r(\gamma_1, \gamma_3)$ and $V_2 = U_r(\gamma_4, \gamma_2)$ one has $V_1 \cap V_2 = U_r$ and $V_1 \cup V_2 = U_r(\gamma_4, \gamma_3)$. Then $\mathscr{H}^s(U_r(\gamma_1, \gamma_2))$ is the complex*

$$\cdots \to 0 \xrightarrow{d_{-2}} H^s(V_1 \cup V_2) \xrightarrow{d_{-1}} H^s(V_1) \oplus H^s(V_2) \xrightarrow{d_0} 0 \xrightarrow{d_1} \cdots. \tag{5.7}$$

- *B case: In that case, there exists two distinct algebraic curves γ_1, γ_2 such that $\dot{\gamma_1} = \dot{\gamma_2}$ and $U_r = B_r \setminus \overline{U_r(\gamma_1, \gamma_2)}$. Let γ_3 and γ_4 such that $\dot{\gamma_j} \neq \dot{\gamma_k}$ for $2 \leq j \neq k \leq 4$, and such that with $V_1 = U_r(\gamma_2, \gamma_3)$ and $V_2 = U_r(\gamma_4, \gamma_1)$ one has $V_1 \cup V_2 = U$ and $V_1 \cap V_2 = U_r(\gamma_4, \gamma_3)$. Then $\mathscr{H}^s(U_r(\gamma_1, \gamma_2))$ is the complex*

$$\cdots \to 0 \xrightarrow{d_{-1}} H^s(V_1) \oplus H^s(V_2) \xrightarrow{d_0} H^s(V_1 \cap V_2) \xrightarrow{d_1} 0 \xrightarrow{d_2} \cdots. \tag{5.8}$$

- *C case: In that case, there exists an algebraic curves γ_1 such that $U_r = B_r \setminus \gamma_1$. Take γ_2 and γ_3 such that $\dot{\gamma_j} \neq \dot{\gamma_k}$ for $j \neq k$, and such that with $V_1 = U_r(\gamma_1, \gamma_3)$ and $V_2 = U_r(\gamma_2, \gamma_1)$ one has $V_1 \cup V_2 = U$ and $V_1 \cap V_2 = U_r(\gamma_2, \gamma_3)$. Then $\mathscr{H}^s(U_r(\gamma_1, \gamma_2))$ is the complex*

$$\cdots \to 0 \xrightarrow{d_{-1}} H^s(V_1) \oplus H^s(V_2) \xrightarrow{d_0} H^s(V_1 \cap V_2) \xrightarrow{d_1} 0 \xrightarrow{d_2} \cdots. \tag{5.9}$$

Proof. — This proposition is an easy byproduct of the requirement (1.1) and of the long exact Mayer Vietoris sequence (1.2). Observe that (5.5) follows from $\mathscr{H}^s(\emptyset) = 0$ and (1.2). □

The following proposition shows that in $\mathbb{R}^2$ and for $s \leq 0$, the sheaf $\mathscr{H}^s$ is concentrated in degree 0.

Proposition 5.8. — *With the above notations, one has*

$$\mathbb{H}^{j,s}(U_r) = 0, \quad \forall j \neq 0 \tag{5.10}$$

and $\mathbb{H}^{0,s}(U_r) = Y^s(U_r)$ except in the S case.

Proof. — First observe that in all cases, except the S case, the open set U_r is weekly Lipschitz. Hence the proposition follows from Theorem 5.4, and in the S case from the fact that the map $H^s(V_1 \cup V_2) \to H^s(V_1) \oplus H^s(V_2)$ is always injective. □

Observe that from Proposition 5.8, the only non trivial cohomology space $\mathbb{H}^{0,s}(U_r)$ is a subspace of $\mathscr{D}'(U)$, except in the S case. Thus, we end this subsection by the following example which shows that $\mathbb{H}^{0,s}(U)$ is not always a subspace of $\mathscr{D}'(U)$.

Example 5.9. — *Let* $U = \{(x, y) \in \mathbb{R}^2,\ -x^2 < y < x^2,\ x^2 + y^2 < 1\}$. *Then for any* $s \in]-3/4, -1/2[$ *the canonical map*

$$\mathbb{H}^{0,s}(U) \to \mathscr{D}'(U)$$

is not injective.

Proof. — Let $V_\pm = \{(x, y),\ x > 0,\ \mp y < x^2,\ x^2 + y^2 < 1\}$. One has $V_+ \cap V_- = U$ and $V_+ \cup V_- = V = \{(x, y),\ x > 0,\ x^2 + y^2 < 1\}$. From (5.7), the canonical map $H^s(V_+) \oplus H^s(V_-)/\mathrm{Im}(d_{-1}) = K^{0,s}(U) \to \mathscr{D}'(U)$ is given by

$$T_+ \oplus T_- \mapsto T_+|_U - T_-|_U, \quad T_\pm \in H^s(V_\pm).$$

Thus it is sufficient to find a distribution $T \in \mathscr{D}'(V)$ such that $T|_{V_+} \in H^s(V_+)$, $T|_{V_-} = 0$, and $T \notin H^s(V)$. We define a distribution $T \in \mathscr{D}'(V)$ by the formula:

$$<T, \varphi> = \int_0^\infty x^{-\beta} p(x)\varphi(x, x^2)dx, \quad \varphi \in C_0^\infty(V).$$

Here, $\beta \in]1/2, 1[$ and p is a smooth non negative function such that $p(x) = 0$ for $x \geq 1/4$ and $p(x) = 1$ for $x \leq 1/8$. One has obviously $T|_{V_-} = 0$. Let us assume $T \in H^\sigma(V)$ with $\sigma < 0$. Then, there exists a constant C such that

$$|<T, \varphi>| \leq C\|\varphi\|_{H^{|\sigma|}(\mathbb{R}^2)}, \quad \forall \varphi \in C_0^\infty(V).$$

Take a test function in $C_0^\infty(V)$ of the form $\varphi(x, y) = \chi_1((x - 3\varepsilon)/\varepsilon)\chi_2(y/\varepsilon)$ with $\chi_{1,2} \in C_0^\infty(]-1, 1[)$ equal to 1 near 0. One has $\|\varphi\|_{H^{|\sigma|}} \simeq \varepsilon^{1-|\sigma|}$ and $|<T, \varphi>| \simeq \varepsilon^{1-\beta}$. Thus we get that $T \in H^\sigma(V)$ implies $-\sigma \geq \beta$. The proof will be complete if we show that $T|_{V_+} \in H^\sigma(V_+)$ as soon as $-\sigma > 1/4 + \beta/2$.

Let $S \in \mathscr{D}'(\mathbb{R}^2)$ be the distribution

$$<S, \varphi> = \int_0^\infty x^{-\beta} p(x)(\varphi(x, x^2) - \varphi(x, -x^2))dx, \quad \varphi \in C_0^\infty(V).$$

One has obviously $S|_{V_+} = T|_{V_+}$. It remains to verify $S \in H^\sigma(\mathbb{R}^2)$ for $-\sigma > 1/4 + \beta/2$. The Fourier transform of S is given by

$$\hat{S}(\xi, \eta) = -2i \int_0^\infty x^{-\beta} p(x) \sin(x^2\eta) e^{-ix\xi}\ dx\ .$$

This is a odd function of η and one has in $\eta > 0$

$$\hat{S}(\xi, \eta) = -2i\ \eta^{(\beta-1)/2} \int_0^\infty y^{-\beta} p(y/\sqrt{\eta}) \sin(y^2) e^{-iy\xi/\sqrt{\eta}}\ dy\ .$$

By 2 integrations by part in y, one find the estimates

$$|\hat{S}(\sqrt{\eta}\theta, \eta)| \leq C\eta^{(\beta-1)/2} \frac{1}{1 + \theta^2}, \quad \eta \geq 1$$

$$|\hat{S}(\sqrt{\eta}\theta,\eta)| \leq C\eta^{(\beta-1)/2}\frac{1}{1+\eta\theta^2}, \quad \eta \in]0,1[.$$

Then $S \in H^\sigma(\mathbb{R}^2)$ for $|\sigma| > 1/4 + \beta/2$ follows from the two above estimates and

$$\iint (1+\xi^2+\eta^2)^\sigma |\hat{S}(\xi,\eta)|^2 \; d\xi d\eta = 2\int_0^\infty (\int (1+\eta\theta^2+\eta^2)^\sigma |\hat{S}(\sqrt{\eta}\theta,\eta)|^2 \; d\theta)\sqrt{\eta} d\eta \; .$$

□

6. Appendix

6.1. Interpolation. — This section is devoted to the behavior with respect to interpolation of the spaces $H^s(U)$. Let us first recall the definition of the interpolation space $[X,Y]_\theta$ where $X \subset Y$ are two Hilbert spaces such that the inclusion map is continuous, and $\theta \in]0,1[$.

For $\sigma = \frac{1}{2\theta} > 1/2$, let $W(\sigma,X,Y)$ be the Hilbert space

(6.1) $$W(\sigma,X,Y) = \{u(t), \quad u \in L^2([0,\infty[,X), \quad t^\sigma u \in L^2([0,\infty[,Y)\}.$$

For $u \in W(\sigma,X,Y)$, one has $(1+t^\sigma)u \in L^2([0,\infty[,Y)$. Since $\sigma > 1/2$, the map Θ from $W(\sigma,X,Y)$ into Y

$$u \mapsto \Theta(u) = \int_0^\infty u(t)dt$$

is well defined and continuous. Then $[X,Y]_\theta = \mathrm{Im}(\Theta) \subset Y$, and the Hilbert structure on $[X,Y]_\theta$ is the one of $W(\sigma,X,Y)/Ker(\Theta)$. Observe that, for this definition, we don't need X to be dense in Y. However, when X is the domain of an unbounded self-adjoint operator $A \geq Id$ on Y, using the spectral theorem for self-adjoint operators, it is easy to verify that $[X,Y]_\theta$ is the domain of $A^{1-\theta}$.

We will use below the notation $s = (1-\theta)s_1 + \theta s_2$ for all real $s_1 > s_2$ and all $\theta \in]0,1[$.

Let us recall that $H^s_{\mathbb{R}^n} = H^s(\mathbb{R}^n)$ and

(6.2) $$[H^{s_1}(\mathbb{R}^n), H^{s_2}(\mathbb{R}^n)]_\theta = H^s(\mathbb{R}^n).$$

Lemma 6.1. — *Let U be an open subset of $\mathbb{R}^n$, and F a closed subset of $\mathbb{R}^n$. Then one has the following continuous injections of Hilbert spaces.*

(6.3) $$\begin{gathered} 0 \to H^s(U) \to [H^{s_1}(U), H^{s_2}(U)]_\theta \\ 0 \to [H^{s_1}_F, H^{s_2}_F]_\theta \to H^s_F. \end{gathered}$$

Proof. — By definition of the spaces $H^s(U)$ and H^s_F, this is an obvious consequence of (6.1) and (6.2). □

We will now prove that for any open, Lipschitz and bounded subset U of $\mathbb{R}^n$, and $F = \overline{U}$, the right maps in (6.3) are onto.

Proposition 6.2. — *Let U be a Lipschitz open bounded subset of $\mathbb{R}^n$. Then the interpolation spaces satisfy*

$$[H^{s_1}(U), H^{s_2}(U)]_\theta = H^s(U)$$
$$[H^{s_1}_{\overline{U}}, H^{s_2}_{\overline{U}}]_\theta = H^s_{\overline{U}}. \tag{6.4}$$

Proof. — By the duality result of Lemma 3.4, it is sufficient to prove the results for the $H^s_{\overline{U}}$ spaces (observe that the inclusion of Hilbert spaces $H^{s_1}(U) \subset H^{s_2}(U)$ is continuous and dense). Set $F = \overline{U}$. By Lemma 6.1, we just have to prove that for a given $f \in H^s_F$, there exists $u \in W(\sigma, H^{s_1}_F, H^{s_2}_F)$ such that $\Theta(u) = f$.

This is a local problem near any point of $\overline{U}$, and since U is Lipschitz we may replace F by any closed set (still denoted by F) such that $F + \Gamma \subset F$, where $\Gamma = \Gamma_b$, $b > 0$, is the closed proper cone in $\mathbb{R}^n$

$$\Gamma = \{x = (x', x_n),\ x_n \leq 0,\ |x'| \leq -bx_n\}.$$

We will use the following lemma, the proof of which is postponed to the end of this section.

Lemma 6.3. — *Let $\Gamma^o = \{y \in \mathbb{R}^n,\ y.u > 0 \quad \forall u \in \Gamma \cap S^{n-1}\}$, and let W be the open tube in $\mathbb{C}^n$ defined by $W = \mathbb{R}^n - i\Gamma^o$. There exists an holomorphic function $E(z)$ defined on W, continues on $\overline{W}$, and $C > 0$ such that*

$$\begin{aligned}
&\frac{1}{C}(1+|z|^2)^{-1} \leq |E(z)| \leq C, \quad \forall z \in \overline{W}\\
&\mathrm{Im}(E(z)) < 0, \quad \forall z \in \overline{W}\\
&\frac{1}{C}(1+|x|)^{-1} \leq |E(x)| \leq C(1+|x|)^{-1} \quad \forall x \in \mathbb{R}^n.
\end{aligned} \tag{6.5}$$

Let $\Lambda(z) = E^{-1}(z)$. One has by (6.5) $\mathrm{Im}(\Lambda(z)) > 0$ for all $z \in \overline{W}$. Let $q > 0$ such that $2q\theta(s_1 - s_2) = 1$ and $p > 0$ such that $pq > 1$. Let $P(t, \lambda)$ be the function defined for $(t, \lambda) \in Q = [0, \infty[\times \{\lambda \in \mathbb{C},\ \mathrm{Im}(\lambda) > 0\}$ by the formula

$$P(t,\lambda) = \lambda^{p-(1+\theta)(s_1-s_2)}(it^q + \lambda)^{-p} \tag{6.6}$$

there exists a constant $c > 0$ such that for all $(t, \lambda) \in Q$ one has

$$|it^q + \lambda| \geq c(t^q + |\lambda|) \tag{6.7}$$

Let $P(t, \Lambda(D))$ be the operator defined by the Fourier multiplier $P(t, \Lambda(\xi))$. From (6.7) and (6.5) we get for some constant C and all $\xi \in \mathbb{R}^n$ (with $<\xi> = (1 + |\xi|)$)

$$|P(t,\Lambda(\xi))| \leq C \frac{<\xi>^{p-(1+\theta)(s_1-s_2)}}{(t^q + <\xi>)^p}. \tag{6.8}$$

Since P is holomorphic in λ, from the Paley-Wiener theorem and (6.5) one has

$$\mathrm{support}(P(t,\Lambda(D))(u)) \subset \Gamma + \mathrm{support}(u). \tag{6.9}$$

Let $\mu^{-1} = \int_0^\infty (1+it^q)^{-p} dt \neq 0$. One easily verifies that for any $\lambda \in \mathbb{C}$, $\mathrm{Im}(\lambda) > 0$, one has $\int_0^\infty P(t,\lambda)dt = \mu^{-1}\lambda^{(\theta-1)(s_1-s_2)}$. Let $f \in H_F^s$ be given. Set

$$g = \mu\Lambda(D)^{(1-\theta)(s_1-s_2)}(f) \in H_F^{s_2}$$

and define $u(t)$ by the formula

$$u(t) = P(t, \Lambda(D))(g). \tag{6.10}$$

Then one has by construction $\Theta(u) = f$, and from (6.8) and support$(u(t)) \subset F$, one gets easily $u \in W(\sigma, H_F^{s_1}, H_F^{s_2})$. The proof of Proposition 6.2 is complete. □

Let us now prove Lemma 6.3. Set $z = (z', z_n)$, and define $E(z)$ for $z \in W$ by

$$E(z) = \int_{|u'|<a} \frac{a^2 - |u'|^2}{z_n + z'.u' + i}\, du' \tag{6.11}$$

where $a > 0$ is small enough such that one has $v_n + v'.u' \geq 0$ for $v \in -\Gamma^o$, $u' \in \mathbb{R}^{n-1}$, $|u'| \leq 2a$. Then $H(z)$ is clearly holomorphic on W, and continuous on $\overline{W}$. One has for all $z \in \overline{W}$

$$\begin{aligned} |E(z)| &\leq \int_{|u'|<a} (a^2 - |u'|^2)\, du' \\ \mathrm{Im}(E(z)) &= -\int_{|u'|<a} \frac{(a^2 - |u'|^2)(y_n + y'.u' + 1)}{|z_n + z'.u' + i|^2} du' < 0 \\ (1+|z|^2)|\mathrm{Im}(E(z))| &\geq \int_{|u'|<a} (a^2 - |u'|^2) \frac{1+|z|^2}{|z_n + z'.u' + i|^2}\, du' \\ &\geq c \int_{|u'|<a} (a^2 - |u'|^2)\, du' \end{aligned} \tag{6.12}$$

with $c > 0$. Therefore it remains to show that the third line of (6.5) holds true. One has to study the behavior of $E(x)$ for large real x. By rotation in x', we may assume $x = (x_1, 0, \ldots, 0, x_n)$. Let $\lambda = |x|$ and set $x_1 = \lambda\cos(\theta)$, $x_n = \lambda\sin(\theta)$. One has for $x = (\lambda\cos(\theta), 0, \ldots, 0, \lambda\sin(\theta))$, with $\varepsilon = 1/\lambda$

(6.13)

$$\lambda E(x) = F(\theta, \varepsilon) = \int_{|u'|<a} \frac{a^2 - |u'|^2}{\sin(\theta) + u_1\cos(\theta) + i\varepsilon}\, du' = \gamma \int_{-a}^{+a} \frac{(a^2 - u_1^2)^{n/2}}{\sin(\theta) + u_1\cos(\theta) + i\varepsilon}\, du_1$$

with $\gamma = \int_{z\in\mathbb{R}^{n-2}, |z|\leq 1} (1 - z^2)dz > 0$. The proof will be complete if we show that the function $F(\theta, \varepsilon)$ is continuous on $S^1 \times [0, 1]$ and $F(\theta, 0)$ does not vanish. The continuity of F on $S^1 \times]0, 1]$ is obvious. The continuity up to the boundary is easy, since the only possible singularity of $F(\theta, \varepsilon)$ is for $\epsilon = 0$ and $-tg(\theta) = \pm a$, and comes from $u_1 = -tg(\theta) = \pm a$ in the integrand, but this is compensated by the fact that the numerator $(a^2 - u_1^2)^{n/2}$ vanishes at $u_1 = \pm a$. Thus it remains to show that the continuous function of $\theta \in S^1$

$$G(\theta) = \int_{-a}^{+a} \frac{(a^2 - u_1^2)^{n/2}}{\sin(\theta) + u_1\cos(\theta) + i0}\, du_1 \tag{6.14}$$

does not vanish. One has $G(-\theta) = -\overline{G(\theta)}$ and $G(\pi+\theta) = -\overline{G(\theta)}$, thus we may assume $\theta \in [0, \pi/2]$. Let $\theta_0 \in]0, \pi/2[$ be the unique solution of $tg(\theta_0) = a$, which is the unique analytic singularity of G on $[0, \pi/2]$. One has clearly $G > 0$ on $]\theta_0, \pi/2]$, and using the formula $\frac{1}{x+i0} = vp(\frac{1}{x}) - i\pi\delta_0$ we get $\mathrm{Im}(G(\theta)) < 0$ on $[0, \theta_0[$. It remains to show that $G(\theta_0) \neq 0$, which follows from

$$G(\theta_0) = \frac{a^n}{\cos(\theta_0)} \int_{-1}^{+1} \frac{(1-z^2)^{n/2}}{1+z}\, dz > 0. \tag{6.15}$$

The proof of Lemma 6.3 is complete.

6.2. The usual definition of Sobolev spaces. — Let $U \subset \mathbb{R}^n$ be a non void open set, and $s \in \mathbb{R}$. Let us recall the usual definition of the space of distribution on U with regularity H^s, for which we refer to the classical books of Lions and Magenes [**4**]. To prevent notational confusions, we will denote this "usual" space by $H^s_*(U)$. For $s = k \in \mathbb{N}$, $H^k_*(U)$ is defined by derivatives in L^2, i.e

$$H^k_*(U) = \{f \in L^2(U), \partial^\alpha f \in L^2(U)\ \forall \alpha, |\alpha| \leq k\}. \tag{6.16}$$

In particular, $H^0_*(U) = L^2(U)$. For $s \in]0,1[$, $H^s_*(U)$ is define by interpolation:

$$H^s_*(U) = [H^1_*(U), L^2(U)]_{1-s}. \tag{6.17}$$

For $k \geq 1$ integer and $s = k + \sigma, \sigma \in]0,1[$, $H^s_*(U)$ is defined by

$$H^s_*(U) = \{f \in L^2(U), \partial^\alpha f \in H^\sigma_*(U)\ \forall \alpha, |\alpha| \leq k\}. \tag{6.18}$$

All these spaces have a natural Hilbert structure, and one has $C^\infty_0(U) \subset H^s_*(U)$ for all $s \geq 0$. Then for all $s \geq 0$, one denotes by $H^s_{0,*}(U)$ the closed subspace of $H^s_*(U)$

$$H^s_{0,*}(U) = \text{ closure of } C^\infty_0(U) \text{ in } H^s_*(U). \tag{6.19}$$

Let us recall that for $t > 0$, one "usually" defines $H^{-t}_*(U)$ as the dual space of $H^t_{0,*}(U)$, i.e $H^{-t}_*(U)$ is the set of distribution $f \in \mathcal{D}'(U)$ such that there exists a constant C such that

$$|<f, \varphi>| \leq C\|\varphi\|_{H^t_*(U)}, \quad \forall \varphi \in C^\infty_0(U). \tag{6.20}$$

The subspace $H^{-t}_*(U)$ of $\mathcal{D}'(U)$, equipped with the norm $\|f\|_{H^{-t}_*(U)}$ = the best constant C in (6.20), is an Hilbert space.

The following lemma gives the rule of comparison between the spaces $H^s(U)$ and $H^s_*(U)$.

Lemma 6.4. — *Let U be an open subset of $\mathbb{R}^n$. One has the following continuous injections of Hilbert spaces.*

$$0 \to H^s(U) \to H^s_*(U) \quad \forall s \geq 0 \tag{6.21}$$

$$0 \to H^s_*(U) \to H^s(U) \quad \forall s \leq 0. \tag{6.22}$$

Proof. — For $s = k \in \mathbb{N}$, the continuous injection (6.21) is obvious. Then, by the first line of (6.3) and using interpolation we get that (6.21) holds true for $0 < s < 1$. For $s = \sigma + k$, $\sigma \in]0,1[$, $k \in \mathbb{N}$ and $f \in H^s(U)$, one has for $|\alpha| \le k$, $\partial^\alpha f \in H^\sigma(U) \subset H^\sigma_*(U)$. Thus (6.21) holds true for all $s \ge 0$.

Let $s < 0$ and $t = -s$. Since (6.21) holds true for t, we get that there exists C such that

$$\|\varphi\|_{H^t_*(U)} \le C\|\varphi\|_{H^t(U)} \le C\|\varphi\|_{H^t(\mathbb{R}^n)}, \quad \forall \varphi \in C_0^\infty(U). \tag{6.23}$$

Let $f \in H^s_*(U)$. Then by (6.20) and (6.23), f defines a continuous linear form l on $H^t_0[U]$ = closure of $C_0^\infty(U)$ in $H^t(\mathbb{R}^n)$. Let $\tilde{l}$ an extension of l to $H^t(\mathbb{R}^n)$. Let $g \in H^s(\mathbb{R}^n)$ such that $<g, \varphi> = \tilde{l}(\varphi)$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then $f = g|_U \in H^s(U)$. The proof of Lemma 6.4 is complete. □

The following proposition shows that the definition (2.3) of the H^s spaces for a Lipschitz and bounded open set U coincide with the usual definition for all s except $s = -1/2 - k, k \in \mathbb{N}$.

Proposition 6.5. — *Let U be a Lipschitz and bounded open subset of $\mathbb{R}^n$. For all $s \notin \{-1/2 - k, k \in \mathbb{N}\}$, one has $H^s(U) = H^s_*(U)$.*

Proof. — This proposition is well known if U has a smooth boundary (see [**4**]). For the convenience of the reader, we recall a proof for U Lipschitz. Since the problem is local near any point of $\overline{U}$, we may replace U by the Lipschitz open set (still denoted by U)

$$\begin{aligned} &U = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}, \quad x_n > H(x')\} \\ &|H(x') - H(y')| \le M|x' - y'|, \quad \forall x', y' \in \mathbb{R}^{n-1} \end{aligned} \tag{6.24}$$

with $H(0) = 0$. We may also assume that we work with distributions with support in

$$K = \{x = (x', x_n) \in \mathbb{R}^n, \ |x'| \le 1, \ |x_n| \le 1 + M\}.$$

We split the proof in the two cases $s \ge 0$ and $s < 0$.

The case $s \ge 0$

In order to prove our result, it is clearly sufficient to construct an extension map $\mathscr{E}$ from $L^2(U)$ into $L^2(\mathbb{R}^n)$ such that

$$\begin{aligned} &\forall f \in L^2(U), \text{ with support}(f) \subset K, \quad \mathscr{E}(f)|_U = f \\ &\forall s \ge 0, \forall f \in H^s_*(U) \text{ with support}(f) \subset K, \quad \mathscr{E}(f) \in H^s(\mathbb{R}^n). \end{aligned} \tag{6.25}$$

For the construction of $\mathscr{E}$, we refer to Stein [**6**], where (6.25) is proved for $s \in \mathbb{N}$ (see Theorem 5, p. 181). We recall here the Stein construction. Let $F = \overline{U}$ and let $\delta \in C^\infty(\mathbb{R}^n \setminus F)$ a Whitney regularization of the distance function $d(x, F)$. There exists positive constants A_1, A_2, c_α such that

$$\begin{aligned} &A_1 d(x, F) \le \delta(x) \le A_2 d(x, F), \quad \forall x \in \mathbb{R}^n \setminus F \\ &|\partial^\alpha \delta(x)| \le c_\alpha d(x, F)^{1-|\alpha|}, \quad \forall x \in \mathbb{R}^n \setminus F. \end{aligned} \tag{6.26}$$

Then δ is continuous on $\mathbb{R}^n \setminus U$ and $\delta(x) = 0$ for $x = (x', H(x'))$. Let $D \geq 2$ large enough such that the function $\delta^* = D\delta$ satisfies

$$\delta^*(x', x_n) \geq 2(H(x') - x_n), \quad \forall (x', x_n) \in \mathbb{R}^n \setminus U \tag{6.27}$$

and let $A_3 > 2$ such that $\delta^*(x', x_n) \leq A_3(H(x') - x_n)$ for all $(x', x_n) \in \mathbb{R}^n \setminus U$.

For $\lambda \geq 1$, and $x = (x', x_n) \in \mathbb{R}^n \setminus U$, set $\tau(x, \lambda) = (x', x_n + \lambda\delta^*(x', x_n))$. Then there exists $c > 0$ such that one has for all $x = (x', x_n) \in \mathbb{R}^n \setminus F$

$$\tau(x,\lambda) \in U \text{ and } c(H(x') - x_n) \leq d(\tau(x,\lambda), \overline{\mathbb{R}^n \setminus F}) \leq (A_3\lambda - 1)(H(x') - x_n). \tag{6.28}$$

Let $\psi(\lambda)$ be a Beurling function, i.e ψ is continuous on $[1, \infty[$, $\psi(\lambda) \in O(\lambda^{-k})$ for all $k \geq 1$, $\int_1^\infty \psi(\lambda)d\lambda = 1$ and $\int_1^\infty \lambda^k\psi(\lambda)d\lambda = 0$ for all $1 \leq k \in \mathbb{N}$. For $f \in L^2(U)$ with support$(f) \subset K$, set

$$\begin{aligned} &\mathscr{E}(f)(x) = f(x) \quad \forall x \in U, \\ &\mathscr{E}(f)(x) = \int_1^\infty f(x', x_n + \lambda\delta^*(x', x_n))\psi(\lambda)d\lambda \quad \forall x \notin F. \end{aligned} \tag{6.29}$$

Then the measurable function $\mathscr{E}(f)$ is defined for all $x \notin \partial U = \{(x', x_n),\ x_n = H(x')\}$, and the first line of (6.25) is obvious. It is proven in **[6]** that $\mathscr{E}(f) \in L^2(\mathbb{R}^n)$, and that the second line of (6.25) holds true for $s = k \in \mathbb{N}$. More precisely, it is proven in **[6]** that for any $k \in \mathbb{N}$, $C^\infty(F)$ is dense in $H^k_*(U)$, and there exists constants C_k such that

$$\|\mathscr{E}(f)\|_{H^k(\mathbb{R}^n)} \leq C_k \|f\|_{H^k_*(U)}, \quad \forall f \in C^\infty(F).$$

Here, $C^\infty(F)$ denotes the space of smooth functions on U with all partial derivatives continuous on $F = \overline{U}$. Therefore by interpolation, we get that the second line of (6.25) holds true for all $s \in [0,1]$. Let now $s = \sigma + k$ with $1 \leq k \in \mathbb{N}$ and $\sigma \in]0,1[$. Let $f \in H^s_*(U)$ with support$(f) \subset K$. One has to prove $\mathscr{E}(f) \in H^s(\mathbb{R}^n)$. For $|\alpha| \leq k$, one has $\partial^\alpha \mathscr{E}(f) \in L^2(\mathbb{R}^n)$ by the Stein result. By induction on k, it remains to show that for $|\alpha| = k$, one has $\partial^\alpha \mathscr{E}(f) \in H^\sigma(\mathbb{R}^n)$. We get from (6.29)

$$\partial^\alpha \mathscr{E}(f) = \mathscr{E}(\partial^\alpha f) + \sum_{p=1}^{|\alpha|} \sum_{(\beta, \mu_1, \dots, \mu_p) \in A_{\alpha,p}} T_{(\mu_1,\dots,\mu_p)}(\partial_x^\beta f) \tag{6.30}$$

with for all $g \in C^\infty(F)$

(6.31)

$$\begin{aligned} &T_{(\mu_1,\dots,\mu_p)}(g)(x) = 0 \quad \forall x \in F \\ &T_{(\mu_1,\dots,\mu_p)}(g)(x) = (\partial_x^{\mu_1}\delta^* \cdots \partial_x^{\mu_p}\delta^*) \int_1^\infty g(x', x_n + \lambda\delta^*(x', x_n))\lambda^p\psi(\lambda)d\lambda \quad \forall x \notin F. \end{aligned}$$

The precise definition of the set of multi-indices $A_{\alpha,p}$ is irrelevant, but one has $|\beta| \geq 1$, $|\mu_j| \geq 1$, and for all $p \geq 1$

$$|\beta| + \sum_{j=1}^p |\mu_j| = |\alpha| + p, \quad \forall (\beta, \mu_1, \dots, \mu_p) \in A_{\alpha,p}. \tag{6.32}$$

We already know that $\mathscr{E}(\partial^\alpha f) \in H^\sigma(\mathbb{R}^n)$. Thus it remains to prove that there exists a constant C such that for $|\alpha| = k$ and all $(\beta, \mu_1, \dots, \mu_p) \in A_{\alpha,p}$, one has with $m = -p + \sum_{j=1}^p |\mu_j|$ and all $\sigma \in [0,1]$.

$$\|T_{(\mu_1,\dots,\mu_p)}(g)\|_{H^\sigma(\mathbb{R}^n)} \leq C\|g\|_{H_*^{m+\sigma}(U)}, \quad \forall g \in C^\infty(F). \tag{6.33}$$

By the proof of Stein theorem, we know that (6.33) holds true for $\sigma = 0, 1$, but at this stage of the proof, we can't use interpolation since we do not know the equality $[H_*^{m+1}(U), H_*^m(U)]_{1-\sigma} = H_*^{m+\sigma}(U)$. The definition of the interpolation space given in Section 6.1, and the validity of this equality for $m = 0$, gives just the inclusion $[H_*^{m+1}(U), H_*^m(U)]_{1-\sigma} \subset H_*^{m+\sigma}(U)$. As in the proof of the Stein theorem, we use a Taylor expansion at order $m-1$ of $\lambda \mapsto g(x', x_n + \lambda\delta^*(x', x_n))$, and using the orthogonality condition $\int_1^\infty \lambda^j \psi(\lambda) d\lambda = 0$ for all $j \geq 1$, we get

$$T_{(\mu_1,\dots,\mu_p)}(g) = \frac{1}{(m-1)!} \int_0^1 (1-t)^{m-1} S_t(\partial_{x_n}^m g)\, dt \tag{6.34}$$

where the family of operators S_t from $C^\infty(F)$ into $L^2(\mathbb{R}^n)$ are defined by $S_t(h)(x) = 0$ for all $x \in F$ and by the formula
(6.35)

$$S_t(h)(x) = (\delta^*)^m (\partial_x^{\mu_1}\delta^* \cdots \partial_x^{\mu_p}\delta^*) \int_1^\infty h(x', x_n + (1 + t(\lambda - 1))\delta^*(x', x_n))(\lambda - 1)^m \lambda^p \psi(\lambda) d\lambda$$

for $x \notin F$. Therefore, it remains to prove that there exists a constant C such that for all $t \in [0,1]$ one has

$$\|S_t(h)\|_{H^\sigma(\mathbb{R}^n)} \leq C\|h\|_{H_*^\sigma(U)}, \quad \forall h \in C^\infty(F). \tag{6.36}$$

By the proof of Stein theorem, we know that (6.36) holds true for $\sigma = 0, 1$, and now we can use interpolation since by definition one has $H_*^\sigma(U) = [H_*^1(U), L^2(U)]_{1-\sigma}$. This complete the proof in the case $s \geq 0$.

The case $s < 0$

One has to verify that the canonical map $i : H_*^s(U) \to H^s(U)$ defined by (6.22) for $s < 0$ is an isomorphism if and only if s is not of the form $s = -1/2 - k$. Let $t = -s$. The dual space of $H_*^s(U)$ is $H_{0,*}^t(U)$ and since U is Lipschitz, the dual space of $H^s(U)$ is $H_{\overline{U}}^t$. The dual map of i is $j = r_U : H_{\overline{U}}^t \to H_{0,*}^t(U)$, and $j = Id$ on $C_0^\infty(U)$. One has to prove that j is an isomorphism iff $t \notin 1/2 + \mathbb{N}$.

Observe that j is injective since $t \geq 0$ and $mes(\overline{U} \setminus U) = 0$. Since C_0^∞ is dense both in $H^t_{\overline{U}}$ and $H^t_{0,*}(U)$, the fact that j is an isomorphism iff $t \notin 1/2 + \mathbb{N}$ is equivalent to the statement

$$\text{For } t \geq 0 \text{ the two norms } \|\varphi\|_{H^t_*(U)} \text{ and } \|\varphi\|_{H^t(\mathbb{R}^n)} \text{ on } C_0^\infty(U) \text{ are equivalent iff } t \notin 1/2 + \mathbb{N}. \tag{6.37}$$

By the first part of the proposition one has $H^t(U) = H^t_*(U)$, thus (6.37) is equivalent to

$$\text{For } t \geq 0 \text{ the two norms } \|\varphi\|_{H^t(U)} \leq \|\varphi\|_{H^t(\mathbb{R}^n)} \text{ on } C_0^\infty(U) \text{ are equivalent iff } t \notin 1/2 + \mathbb{N}. \tag{6.38}$$

For $t = k \in \mathbb{N}$, one has $\|\varphi\|_{H^k_*(U)} = \|\varphi\|_{H^k(\mathbb{R}^n)}$, thus $H^k_{\overline{U}} = H^k_{0,*}(U)$, and (6.37) holds true for $t = k$. By (6.24) $(x', x_n) \mapsto G(x', x_n) = (y' = x', y_n = x_n - H(x'))$ is a global bi-lipschitz diffeomorphism of $\mathbb{R}^n$ such that $G(U) = \mathbb{R}^n_+ = \{(y', y_n), y_n > 0\}$. It is well known (see [4]) that $[H^1_{0,*}(\mathbb{R}^n_+), L^2(\mathbb{R}^n_+)]_\theta = H^{1-\theta}_{0,*}(\mathbb{R}^n_+)$ for $\theta \neq 1/2$. Since the proof of Proposition 6.2, stated for bounded U, applies as well to $\mathbb{R}^n_+$, we get that (6.37) holds true for $t \in [0,1] \setminus \{1/2\}$. It is proven in [4] that one has $H^{1/2}_{0,*}(\mathbb{R}^n_+) = H^{1/2}_*(\mathbb{R}^n_+)$, and for $\theta = 1/2$, the interpolation space $H^{1/2}_{0,0,*}(\mathbb{R}^n_+) = [H^1_{0,*}(\mathbb{R}^n_+), L^2(\mathbb{R}^n_+)]_{1/2}$, is equal to

$$H^{1/2}_{0,0,*}(\mathbb{R}^n_+) = \{f \in H^{1/2}_*(\mathbb{R}^n_+), \int_{\mathbb{R}^{n-1}} \int_0^\infty \frac{|f|^2}{x_n} dx_n dx' < \infty\}. \tag{6.39}$$

In particular, $H^{1/2}_{0,0,*}(\mathbb{R}^n_+)$ is a strict subspace of $H^{1/2}_{0,*}(\mathbb{R}^n_+)$. Since, by the first part of the proposition, $H^{1/2}_*(\mathbb{R}^n_+) = H^{1/2}(\mathbb{R}^n_+)$, (6.39) implies that for $f \in H^{1/2}_{0,0,*}(\mathbb{R}^n_+)$, the function $\tilde{f}$ on $\mathbb{R}^n$ equal to f on $\mathbb{R}^n_+$ and equal to 0 elsewhere satisfies $\tilde{f} \in H^{1/2}(\mathbb{R}^n)$ and support$(\tilde{f}) \subset \{x_n \geq 0\}$. This shows the equality

$$H^{1/2}_{0,0,*}(\mathbb{R}^n_+) = H^{1/2}_{\overline{[\mathbb{R}^n_+]}}.$$

Thus, j is not an isomorphism for $t = 1/2$. For $t = \sigma + k, \sigma \in]0,1[, k \in \mathbb{N}$

$$\|\varphi\|^2_{H^t_*(U)} = \sum_{|\alpha| \leq k} \|\partial^\alpha \varphi\|^2_{H^\sigma_*(U)}.$$

Thus the validity of (6.37) for $t > 1$, and $t \notin \mathbb{N}$ follows from the case $t \in]0,1[$.

The proof of Proposition 6.5 is complete. □

We conclude this subsection by the following lemma which shows that for $s = -1/2 - k$, the space $H^s_*(]0,\infty[)$ does not enjoy good properties.

Lemma 6.6. — *Let $s = -1/2 - k$ with $k \in \mathbb{N}$.*

(i) *There exists $f \in H^s(\mathbb{R})$ such that $f|_{]0,\infty[} \notin H^s_*(]0,\infty[)$.*

(ii) *There exists $f \in H^{1/2}(]0,\infty[) = H^{1/2}_*(]0,\infty[)$ such that $\partial_x^{k+1} f \notin H^s_*(]0,\infty[)$.*

Proof. — We proceed by contradiction. Assume that (i) is not true. The imbedding of $H^s_*(]0,\infty[)$ into $H^s(]0,\infty[)$ is continuous by Lemma 6.4, and the restriction map $f \mapsto r(f) = f|_{]0,\infty[}$ is continuous from $H^s(\mathbb{R})$ into $H^s(]0,\infty[)$. Then by the closed graph theorem, if $r(H^s(\mathbb{R})) \subset H^s_*(]0,\infty[$, r is continuous from $H^s(\mathbb{R})$ into $H^s_*(]0,\infty[)$. Thus there exists a constant C such that

$$\|f|_{]0,\infty[}\|_{H^s_*(]0,\infty[} \leq C\|f\|_{H^s(\mathbb{R})}, \quad \forall f \in H^s(\mathbb{R}). \tag{6.40}$$

From (6.20) we thus get with $t = -s \geq 0$

$$|<f,\varphi>| \leq C\|f\|_{H^s(\mathbb{R})}\|\varphi\|_{H^t_*(]0,\infty[)}, \quad \forall \varphi \in C_0^\infty(]0,\infty[), \quad \forall f \in H^s(\mathbb{R}) \tag{6.41}$$

and this is equivalent to

$$\|\varphi\|_{H^t(\mathbb{R})} \leq C\|\varphi\|_{H^t_*(]0,\infty[)}, \quad \forall \varphi \in C_0^\infty(]0,\infty[) \tag{6.42}$$

which is untrue thanks to (6.37). For (ii) we just indicate the argument for $k = 0$: if (ii) is untrue, the map $f \mapsto f'$ is continuous from $H^{1/2}(]0,\infty[$ into $H_*^{-1/2}(]0,\infty[$, (since it is continuous from $H^{1/2}(]0,\infty[$ into $H^{-1/2}(]0,\infty[$ and its range is a subset of $H_*^{-1/2}(]0,\infty[)$. Thus for some constant C one has for all $f \in H^{1/2}(]0,\infty[)$ and all $\varphi \in C_0^\infty(]0,\infty[)$

$$|<f,\varphi'>| = |<f',\varphi>| \leq C\|f\|_{H^{1/2}(]0,\infty[)}\|\varphi\|_{H_*^{1/2}(]0,\infty[)}. \tag{6.43}$$

Take $f = g|_{]0,\infty[}$ with $g \in H^{1/2}(\mathbb{R})$. One has $\|f\|_{H^{1/2}(]0,\infty[)} \leq \|g\|_{H^{1/2}(\mathbb{R})}$. Thus (6.43) implies for all $g \in H^{1/2}(\mathbb{R})$ and all $\varphi \in C_0^\infty(]0,\infty[)$

$$|<g,\varphi'>| \leq C\|g\|_{H^{1/2}(\mathbb{R})}\|\varphi\|_{H_*^{1/2}(]0,\infty[)} \tag{6.44}$$

which is equivalent to

$$\|\varphi'\|_{H^{-1/2}(\mathbb{R})} \leq C\|\varphi\|_{H_*^{1/2}(]0,\infty[)}, \quad \forall \varphi \in C_0^\infty(]0,\infty[) \tag{6.45}$$

and this will imply that (6.42) holds true for $t = 1/2$, and all φ with support in $x \leq 1$, which is false. The proof of Lemma 6.6 is complete. □

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G. Lebeau, Département de Mathématiques, Université de Nice Sophia-Antipolis, Parc Valrose 06108 Nice Cedex 02, France. • *E-mail :* lebeau@unice.fr

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REGULAR SUBANALYTIC COVERS

by

Adam Parusiński

Abstract. — Let U be an open relatively compact subanalytic subset of a real analytic manifold M. We show that there exists a "finite linear covering" (in the sense of Guillermou-Schapira) of U by subanalytic open subsets of U homeomorphic to an open ball.

We also show that the characteristic function of U can be written as a finite linear combination of characteristic functions of open relatively compact subanalytic subsets of M homeomorphic, by subanalytic and bi-lipschitz maps, to an open ball.

Résumé. — Soit U un ouvert sous-analytique relativement compact d'une variété analytique réelle M. Nous montrons qu'il existe un « recouvrement linéaire fini » (au sens de Guillermou-Schapira) de U par des ouverts sous-analytiques homéomorphes à une boule ouverte.

Nous montrons aussi que la fonction caractéristique de U peut s'écrire comme une combinaison linéaire finie de fonctions caractéristiques d'ouverts sous-analytiques relativement compacts de M homéomorphes, par des applications sous-analytiques et bi-lipschitz, à une boule ouverte.

Let M be a real analytic manifold of dimension n. In this paper we study the subalgebra $\mathscr{S}(M)$ of integer valued functions on M generated by characteristic functions of relatively compact open subanalytic subsets of M (or equivalently by characteristic functions of compact subanalytic subsets of M). As we show this algebra is generated by characteristic functions of open subanalytic sets with Lipschitz regular boundaries. More precisely, we call a relatively compact open subanalytic subset $U \subset M$ *an open subanalytic Lipschitz ball* if its closure is subanalytically bi-Lipschitz homeomorphic to the unit ball of $\mathbb{R}^n$. Here we assume that M is equipped with a Riemannian metric. Any two such metrics are equivalent on relatively compact sets and hence the above definition is independent of the choice of a metric.

Theorem 0.1. — *The algebra $\mathscr{S}(M)$ is generated by characteristic functions of open subanalytic Lipschitz balls.*

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That is to say if U is a relatively compact open subanalytic subset of M then its characteristic function 1_U is a finite integral linear combination of characteristic functions $1_{W_1}, \dots, 1_{W_m}$, where the W_j are open subanalytic Lipschitz balls. Note that, in general, U cannot be covered by finitely many subanalytic Lipschitz balls, as it is easy to see for $\{(x,y) \in \mathbb{R}^2; y^2 < x^3, x < 1\}$, $M = \mathbb{R}^2$, due to the presence of cusps. Nevertheless we show the existence of a "regular" cover in the sense that we control the distance to the boundary.

Theorem 0.2. — *Let U be an open relatively compact subanalytic subset of M. Then there exist a finite cover $U = \bigcup_i U_i$ by open subanalytic sets such that :*

1. *every U_i is subanalytically homeomorphic to an open n-dimensional ball;*
2. *there is $C > 0$ such that for every $x \in U$, $\operatorname{dist}(x, M \setminus U) \le C \max_i \operatorname{dist}(x, M \setminus U_i)$.*

The proof of Theorem 0.1 is based on the classical cylindrical decomposition and the L-regular decomposition of subanalytic sets, cf. [**4, 9**], [**10**]. L-regular sets are natural multidimensional generalization of classical cusps. We recall them briefly in Subsection 1.6. For the proof of Theorem 0.2 we need also the regular projection theorem, cf. [**7, 8**], [**9**], that we recall in Subsection 1.4.

We also show the following strengthening of Theorem 0.2.

Theorem 0.3. — *In Theorem 0.2 we may require additionally that all U_i are open L-regular cells.*

For an open $U \subset M$ we denote $\partial U = \overline{U} \setminus U$.

1. Proofs

1.1. Reduction to the case $M = \mathbb{R}^n$.— Let U be an open relatively compact subanalytic subset of M. Choose a finite cover $\overline{U} \subset \bigcup_i V_i$ by open relatively compact sets such that for each V_i there is an open neighborhood of $\overline{V}_i$ analytically diffeomorphic to $\mathbb{R}^n$. Then there are finitely many open subanalytic U_{ij} such that $U_{ij} \subset V_i$ and 1_U is a combination of $1_{U_{ij}}$. Thus it suffices to show Theorem 0.1 for relatively compact open subanalytic subsets of $\mathbb{R}^n$.

Similarly, it suffices to show Theorems 0.2 and 0.3 for $M = \mathbb{R}^n$. Indeed, it follow from the observation that the function

$$x \to \max_i \operatorname{dist}(x, M \setminus V_i)$$

is continuous and nowhere zero on $\bigcup_i V_i$ and hence bounded from below by a nonzero constant $c > 0$ on $\overline{U}$. Then

$$\operatorname{dist}(x, M \setminus U) \le C_1 \le c^{-1} C_1 \max_i \operatorname{dist}(x, M \setminus V_i)$$

where C_1 is the diameter of $\overline{U}$ and hence, if $c^{-1}C_1 \ge 1$,

$$\operatorname{dist}(x, M \setminus U) \le c^{-1} C_1 \max_i (\min\{\operatorname{dist}(x, M \setminus U), \operatorname{dist}(x, M \setminus V_i)\}).$$

Now if for each $U \cap V_i$ we choose a cover $\bigcup_j U_{ij}$ satisfying the statement of Theorem 0.2 or 0.3 then for $x \in U$

$$\begin{aligned}\operatorname{dist}(x, M \setminus U) &\le c^{-1} C_1 \max_i (\min\{\operatorname{dist}(x, M \setminus U), \operatorname{dist}(x, M \setminus V_i)\} \\ &\le c^{-1} C_1 \max_i \operatorname{dist}(x, M \setminus U \cap V_i) \le C c^{-1} C_1 \max_{ij} \operatorname{dist}(x, M \setminus U_{ij}).\end{aligned}$$

Thus the cover $\bigcup_{i,j} U_{ij}$ satisfies the claim of Theorem 0.2, resp. of Theorem 0.3.

1.2. Regular projections. — We recall after **[8, 9]** the subanalytic version of the regular projection theorem of T. Mostowski introduced originally in **[7]** for complex analytic sets germs.

Let $X \subset \mathbb{R}^n$ be subanalytic. For $\xi \in \mathbb{R}^{n-1}$ we denote by $\pi_\xi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the linear projection parallel to $(\xi, 1) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Fix constants $C, \varepsilon > 0$. We say that $\pi = \pi_\xi$ is *(C,ε)-regular at $x_0 \in \mathbb{R}^n$ (with respect to X)* if

(a) $\pi|_X$ is finite;

(b) the intersection of X with the open cone

$$\mathscr{C}_\varepsilon(x_0, \xi) = \{x_0 + \lambda(\eta, 1); |\eta - \xi| < \varepsilon, \lambda \in \mathbb{R} \setminus 0\} \tag{1.1}$$

is empty or a finite disjoint union of sets of the form

$$\{x_0 + \lambda_i(\eta)(\eta, 1); |\eta - \xi| < \varepsilon\},$$

where λ_i are real analytic nowhere vanishing functions defined on $|\eta - \xi| < \varepsilon$.

(c) the functions λ_i from (b) satisfy for all $|\eta - \xi| < \varepsilon$

$$\|\operatorname{grad} \lambda_i(\eta)\| \le C |\lambda_i(\eta)|.$$

We say that $\mathscr{P} \subset \mathbb{R}^{n-1}$ defines *a set of regular projections for X* if there exists $C, \varepsilon > 0$ such that for every $x_0 \in \mathbb{R}^n$ there is $\xi \in \mathscr{P}$ such that π_ξ is (C,ε)-regular at x_0.

Theorem 1.1 **([8, 9])**. — *Let X be a compact subanalytic subset of $\mathbb{R}^n$ such that $\dim X < n$. Then the generic set of $n+1$ vectors $\xi_1, \dots, \xi_{n+1}$, $\xi_i \in \mathbb{R}^{n-1}$, defines a set of regular projections for X.*

Here by generic we mean in the complement of a subanalytic nowhere dense subset of $(\mathbb{R}^{n-1})^{n+1}$.

1.3. Cylindrical decomposition. — We recall the first step of a basic construction called the cylindrical algebraic decomposition in semialgebraic geometry or the cell decomposition in o-minimal geometry, for details see for instance **[2, 3]**.

Set $X = \overline{U} \setminus U$. Then X is a compact subanalytic subset of $\mathbb{R}^n$ of dimension $n-1$. We denote by $Z \subset X$ the set of singular points of X that is the complement in X of the set

$$Reg(X) := \{x \in X; (X, x) \text{ is the germ of a real analytic submanifold of dimension } n-1\}.$$

Then Z is closed in X, subanalytic and $\dim Z \le n-2$.

Assume that the standard projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ restricted to X is finite. Denote by $\Delta_\pi \subset \mathbb{R}^{n-1}$ the union of $\pi(Z)$ and the set of critical values of $\pi|_{Reg(X)}$. Then Δ_π, called *the discriminant set of* π, is compact and subanalytic. It is clear that $\overline{\pi(U)} = \pi(U) \cup \Delta_\pi$.

Proposition 1.2. — *Let $U' \subset \pi(U) \setminus \Delta_\pi$ be open and connected. Then there are finitely many bounded real analytic functions $\varphi_1 < \varphi_2 < \cdots < \varphi_k$ defined on U', such that $X \cap \pi^{-1}(U')$ is the union of graphs of φ_i's. In particular, $U \cap \pi^{-1}(U')$ is the union of the sets*

$$\{(x', x_n) \in \mathbb{R}^n; x' \in U', \varphi_i(x') < x_n < \varphi_{i+1}(x')\},$$

and moreover, if U' is subanalytically homeomorphic to an open $(n-1)$-dimensional ball, then each of these sets is subanalytically homeomorphic to an open n-dimensional ball.

1.4. The case of a regular projection. — Fix $x_0 \in U$ and suppose that $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is (C, ε)-regular at $x_0 \in \mathbb{R}^n$ with respect to X. Then the cone (1.1) contains no point of Z. By [9] Lemma 5.2, this cone contains no critical point of $\pi|_{Reg(X)}$, provided ε is chosen sufficiently small (for fixed C). In particular, $x'_0 = \pi(x_0) \notin \Delta_\pi$.

In what follows we fix $C, \varepsilon > 0$ and suppose ε small. We denote the cone (1.1) by $\mathscr{C}$ for short. Then for $\tilde{C}$ sufficiently large, that depends only on C and ε, we have

$$\operatorname{dist}(x_0, X \setminus \mathscr{C}) \le \tilde{C} \operatorname{dist}(x'_0, \pi(X \setminus \mathscr{C})) \le \tilde{C} \operatorname{dist}(x'_0, \Delta_\pi). \tag{1.2}$$

The first inequality is obvious, the second follow from the fact that the singular part of X and the critical points of $\pi|_{Reg(X)}$ are both outside the cone.

1.5. Proof of Theorem 0.2. — Induction on n. Set $X = \overline{U} \setminus U$ and let $\pi_{\xi_1}, \ldots, \pi_{\xi_{n+1}}$ be a set of (C, ε)-regular projections with respect to X. To each of these projections we apply the cylindrical decomposition. More precisely, let us fix one of these projections that for simplicity we suppose standard and denote it by π. Then we apply the inductive assumption to $\pi(U) \setminus \Delta_\pi$. Thus let $\pi(U) \setminus \Delta_\pi = \bigcup U'_i$ be a finite cover satisfying the statement of Theorem 0.2. Applying to each U'_i Proposition 1.2 we obtain a family of cylinders that covers $U \setminus \pi^{-1}(\Delta_\pi)$. In particular they cover the set of those points of U at which π is (C, ε)-regular.

Lemma 1.3. — *Suppose π is (C, ε)-regular at $x_0 \in U$. Let U' be an open subanalytic subset of $\pi(U) \setminus \Delta_\pi$ such that $x'_0 = \pi(x_0) \in U'$ and*

$$\operatorname{dist}(x'_0, \Delta_\pi) \le \tilde{C} \operatorname{dist}(x'_0, \partial U'), \tag{1.3}$$

with $\tilde{C} \ge 1$ for which (1.2) *holds. Then*

$$\operatorname{dist}(x_0, X) \le (\tilde{C})^2 \operatorname{dist}(x_0, \partial U_1), \tag{1.4}$$

where U_1 is the member of cylindrical decomposition of $U \cap \pi^{-1}(U')$ containing x_0.

Proof. — We decompose ∂U_1 into two parts. The first one is vertical, i.e., contained in $\pi^{-1}(\partial U')$, and the second part is contained in X. If $\operatorname{dist}(x_0, \partial U_1) < \operatorname{dist}(x_0, X)$ then the distance to the vertical part realizes the distance of x_0 to ∂U_1 and $\operatorname{dist}(x_0, \partial U_1) = \operatorname{dist}(x'_0, \partial U')$. Hence

$$\operatorname{dist}(x_0, \partial U_1) = \min\{\operatorname{dist}(x_0, X), \operatorname{dist}(x'_0, \partial U')\}. \tag{1.5}$$

If $\operatorname{dist}(x_0, \partial U_1) = \operatorname{dist}(x_0, X)$ then (1.4) holds with $\tilde{C} = 1$, otherwise by (1.2) and (1.3),

$$\begin{aligned} \operatorname{dist}(x_0, X \setminus \mathscr{C}) &\leq \tilde{C} \operatorname{dist}(x'_0, \Delta_\pi) \\ &\leq (\tilde{C})^2 \operatorname{dist}(x'_0, \partial U') = (\tilde{C})^2 \operatorname{dist}(x_0, \partial U_1). \quad \square \end{aligned} \tag{1.6}$$

Thus to complete the proof of Theorem 0.2 it suffices to show that (1.3) holds if U' is an element of the cover $\pi(U) \setminus \Delta_\pi = \bigcup U'_i$ for which $\operatorname{dist}(x'_0, \partial\pi(U)) \leq \tilde{C} \operatorname{dist}(x'_0, \partial U')$. This follows from the inclusion $\partial\pi(U) \subset \Delta_\pi$ that gives $\operatorname{dist}(x'_0, \Delta_\pi) \leq \operatorname{dist}(x'_0, \partial\pi(U))$. This ends the proof of Theorem 0.2.

1.6. L-regular sets. — Let $Y \subset \mathbb{R}^n$ be subanalytic, $\dim Y = n$. Then Y is called *L-regular (with respect to given system of coordinates)* if

1. if $n = 1$ then Y is a non-empty closed bounded interval;
2. if $n > 1$ then Y is of the form

$$Y = \{(x', x_n) \in \mathbb{R}^n; f(x') \leq x_n \leq g(x'), x' \in Y'\}, \tag{1.7}$$

   where $Y' \subset \mathbb{R}^{n-1}$ is L-regular, f and g are continuous subanalytic functions defined in Y'. It is also assumed that on the interior of Y', f and g are analytic, satisfy $f < g$, and have the first order partial derivatives bounded.

If $\dim Y = k < n$ then we say that Y is *L-regular (with respect to given system of coordinates)* if

$$Y = \{(y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k};\ z = h(y),\ y \in Y'\}, \tag{1.8}$$

where $Y' \subset \mathbb{R}^k$ is L-regular, $\dim Y' = k$, h is a continuous subanalytic map defined on Y', such that h is real analytic on the interior of Y', and has the first order partial derivatives bounded.

We say that Y is *L-regular* if it is L-regular with respect to a linear (or equivalently orthogonal) system of coordinates on $\mathbb{R}^n$.

We say that $A \subset \mathbb{R}^n$ is an *L-regular cell* if A is the relative interior of an L-regular set. That is, it is the interior of A if $\dim A = n$, and it is the graph of h restricted to $Int(Y')$ for an L-regular set of the form (1.8). By convention, every point is a zero-dimensional L-regular cell.

By **[4]**, see also Lemma 2.2 of **[9]** and Lemma 1.1 of **[5]**, L-regular sets and L-regular cells satisfy the following property, called in **[4]** quasi-convexity. We say that $Z \subset \mathbb{R}^n$ is *quasi-convex* if there is a constant $C > 0$ such that every two points x, y of Z can be connected in Z by a continuous subanalytic arc of length bounded by $C\|x - y\|$. It can be shown that for an L-regular set or cell Y of the form (1.7) or (1.8) the constant C depends only on n, the analogous constant for Y', and the bounds on first order partial derivatives of f and g, resp. h. By Lemma 2.2 of [9], an L-regular cell is subanalytically homeomorphic to the (open) unit ball.

Let Y be a subanalytic subset of a real analytic manifold M. We say that Y is *L-regular* if there exists its neighborhood V in M and an analytic diffeomorphism $\varphi : V \to \mathbb{R}^n$ such that $\varphi(Y)$ is L-regular. Similarly we define an L-regular cell in M.

1.7. Proof of Theorem 0.3. — Fix a constant C_1 sufficiently large and a projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ that is assumed, for simplicity, to be the standard one. We suppose that π restricted to $X = \partial U$ is finite. We say that $x' \in \pi(U) \setminus \Delta_\pi$ is *C_1-regularly covered* if there is a neighborhood $\tilde U'$ of x' in $\pi(U) \setminus \Delta_\pi$ such that $X \cap \pi^{-1}(\tilde U')$ is the union of graphs of analytic functions with the first order partial derivatives bounded (in the absolute value) by C_1. Denote by $U'(C_1)$ the set of all $x' \in \pi(U) \setminus \Delta_\pi$ that are C_1 regularly covered. Then $U'(C_1)$ is open (if we use strict inequalities while defining it) and subanalytic. By Lemma 5.2 of [9], if π is a (C, ε)-regular projection at x_0 then x_0' is C_1-regularly covered, for C_1 sufficiently big $C_1 \geq C_1(C, \varepsilon)$. Moreover we have the following result.

Lemma 1.4. — *Given positive constants C, ε. Suppose that the constants $\tilde C$ and C_1 are chosen sufficiently big, $C_1 \geq C_1(C, \varepsilon)$, $\tilde C \geq \tilde C(C, \varepsilon)$. Let π be (C, ε)-regular at $x_0 \notin X$ and let*

$$V' = \{x' \in \mathbb{R}^{n-1}; \operatorname{dist}(x', x_0') < (\tilde C)^{-1} dist(x_0, X \cap \mathscr{C})\}.$$

Then $\pi^{-1}(V') \cap X \cap \mathscr{C}$ is the union of graphs of φ_i with all first order partial derivatives bounded (in the absolute value) by C_1. Moreover, then either $\pi^{-1}(V') \cap (X \setminus \mathscr{C}) = \varnothing$ or

$$\operatorname{dist}(x_0', \Delta_\pi) = \operatorname{dist}(x_0', \pi(X \setminus \mathscr{C})) \leq \operatorname{dist}(x_0', \partial U'(C_1)).$$

Proof. — We only prove the second part of the statement since the first part follows from Lemma 5.2 of [9]. If $\pi^{-1}(V') \cap X \setminus \mathscr{C} \neq \varnothing$ then any point of $\pi(X \setminus \mathscr{C})$ realizing $\operatorname{dist}(x_0', \pi(X \setminus \mathscr{C}))$ must be in the discriminant set Δ_π. □

We now apply to $U'(C_1)$ the inductive hypothesis and thus assume that $U'(C_1) = \bigcup U_i'$ is a finite regular cover by open L-regular cells. Fix one of them U' and let U_1 be a member of the cylindrical decomposition of $U \cap \pi^{-1}(U')$. Then U_1 is an L-regular cell. Let $x_0 \in U_1$. We apply to x_0 Lemma 1.4.

If $\pi^{-1}(V') \cap (X \setminus \mathscr{C}) = \varnothing$ then

$$\operatorname{dist}(x_0, X) \leq \operatorname{dist}(x_0, X \cap \mathscr{C}) \leq \tilde C \operatorname{dist}(x_0', \partial U'(C_1)) \leq \tilde C^2 \operatorname{dist}(x_0', \partial U'),$$

where the second inequality follows from the first part of Lemma 1.4 and the last inequality by the induction hypothesis. Then $\operatorname{dist}(x_0, X) \leq \tilde C^2 \operatorname{dist}(x_0, \partial U_1)$ follows from (1.5).

Otherwise, $\operatorname{dist}(x_0', \Delta_\pi) \leq \operatorname{dist}(x_0', \partial U'(C_1)) \leq \tilde C \operatorname{dist}(x_0', \partial U')$ and the claim follows from Lemma 1.3. This ends the proof.

1.8. Proof of Theorem 0.1. — The proof is based on the following result.

Theorem 1.5. — [Theorem A of [**4**]] *Let $Z_i \subset \mathbb{R}^n$ be a finite family of bounded subanalytic sets. Then there is be a finite disjoint collection $\{A_j\}$ of L-regular cells such that each Z_i is the disjoint union of some of A_j.*

Similar results in the (more general) o-minimal set-up are proven in [**5**] and [**10**].

Let U be a relatively compact open subanalytic subset of $\mathbb{R}^n$. By Theorem 1.5, U is a disjoint union of L-regular cells and hence it suffices to show the statement of Theorem 0.1 for an L-regular cell. We consider first the case of an open L-regular cell. Thus suppose that

$$U = \{(x', x_n) \in \mathbb{R}^n; f(x') < x_n < g(x'), x' \in U'\}, \tag{1.9}$$

where U' is a relatively compact L-regular cell, f and g are subanalytic and analytic functions on U' with the first order partial derivatives bounded. Then, by the quasi-convexity of U', f and g are Lipschitz. By an extension formula of [**6**], see also [**11**] and [**1**], we may suppose that f and g are restrictions of Lipschitz subanalytic functions, that we denote later also by f and g, defined everywhere on $\mathbb{R}^{n-1}$ and satisfying $f \leq g$. Indeed, this extension of f is given by

$$\tilde{f}(p) = \sup_{q \in U'} f(q) - L\|p - q\|,$$

where L is the Lipschitz constant of f. Then $\tilde{f}$ is Lipschitz with the same constant as f and subanalytic. Therefore by the inductive assumption on dimension we may assume that U is given by (1.9) with U' a subanalytic Lipschitz ball. Denote U by $U_{f,g}$ to stress its dependence on f and g (with U' fixed). Then

$$1_{U_{f,g}} = 1_{U_{f-1,g}} + 1_{U_{f,g+1}} - 1_{U_{f-1,g+1}}$$

and $U_{f-1,g}$, $U_{f,g+1}$. and $U_{f-1,g+1}$ are open subanalytic Lipschitz balls.

Suppose now that

$$U = \{(y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k};\ z = h(y),\ y \in U'\}, \tag{1.10}$$

where U' is an open L-regular cell of $\mathbb{R}^k$, h is a subanalytic and analytic map defined on U' with the first order partial derivatives bounded. Hence h is Lipschitz. We may again assume that h is the restriction of a Lipschitz subanalytic map $h : \mathbb{R}^k \to \mathbb{R}^{n-k}$ and then, by the inductive hypothesis, that U' is a subanalytic Lipschitz ball. Let

$$U_\varnothing = \{(y, z) \in U' \times \mathbb{R}^{n-k};\ h_i(y) - 1 < z_i < h_i(y) + 1\,, i = 1, \dots, n-k\}.$$

For $I \subset \{1, \dots, n-k\}$ we denote

$$U_I = \{(y, z) \in U_\varnothing;\ z_i \neq h_i(y) \text{ for } i \in I\}.$$

Note that each U_I is the disjoint union of $2^{|I|}$ of open subanalytic Lipschitz balls and that

$$1_U = \sum_{I \subset \{1, \dots, n-k\}} (-1)^{|I|} 1_{U_I}.$$

This ends the proof.

2. Remarks on the o-minimal case

It would be interesting to know whether the main theorems of this paper, Theorems 0.1, 0.2, 0.3, hold true in an arbitrary o-minimal structure in the sense of [**3**], i.e., if we replace the word "subanalytic" by "definable in an o-minimal structure", and fix $M = \mathbb{R}^n$. This is the case for Proposition 1.2 and Theorem 1.5 by [**3**], resp. [**5**], [**10**], and therefore Theorem 0.1 holds true in the o-minimal set-up. But it is not clear whether the analog of Theorem 1.1 holds in an arbitrary o-minimal structure. Its proof in [**8**] uses Puiseux Theorem with parameters in an essential way. Thus we state the following questions.

Question 2.1. — *Does the regular projections theorem, Theorem 1.1, hold true in an arbitrary o-minimal strucure?*

Question 2.2. — *Do Theorems 0.2, 0.3, hold true in an arbitrary o-minimal strucure?*

One would expect the positive answers for the polynomially bounded o-minimal structures, though even this case in not entirely obvious.

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A. Parusiński, Univ. Nice Sophia Antipolis, CNRS, LJAD, UMR 7351, 06108 Nice, France
E-mail : `adam.parusinski@unice.fr`

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DERIVED CATEGORIES OF FILTERED OBJECTS

by

Pierre Schapira & Jean-Pierre Schneiders

Abstract. — For an abelian category $\mathscr{C}$ and a filtrant preordered set Λ, we prove that the derived category of the quasi-abelian category of filtered objects in $\mathscr{C}$ indexed by Λ is equivalent to the derived category of the abelian category of functors from Λ to $\mathscr{C}$. We apply this result to the study of the category of filtered modules over a filtered ring in a tensor category.

***Résumé* (Catégories dérivées d'objets filtrés).** — Pour une catégorie abélienne $\mathscr{C}$ et un ensemble préordonné filtrant Λ, nous prouvons que la catégorie dérivée des objets filtrés de $\mathscr{C}$ indexés par Λ est équivalente à la catégorie dérivée de la catégorie abélienne des foncteurs de Λ dans $\mathscr{C}$. Nous appliquons ce résultat à l'étude de la catégorie des modules filtrés sur un anneau filtré d'une catégorie tensorielle.

1. Introduction

Filtered modules over filtered sheaves of rings appear naturally in mathematics, such as for example when studying $\mathscr{D}_X$-modules on a complex manifold X, $\mathscr{D}_X$ denoting the filtered ring of differential operators (see [**3**]). As it is well-known, the category of filtered modules over a filtered ring is not abelian, only exact in the sense of Quillen [**7**] or quasi-abelian in the sense of [**8**], but this is enough to consider the derived category (see [**1, 6**]). However, quasi-abelian categories are not easy to manipulate, and we shall show in this paper how to substitute a very natural abelian category to this quasi-abelian category, giving the same derived category.

More precisely, consider an abelian category $\mathscr{C}$ admitting small exact filtrant (equivalently, "directed") colimits and a filtrant preordered set Λ. In this paper, we regard a filtered object in $\mathscr{C}$ as a functor $M\colon \Lambda \to \mathscr{C}$ with the property that all $M(\lambda)$ are sub-objects of $\varinjlim M$. We prove that the derived category of the quasi-abelian category of filtered objects in $\mathscr{C}$ indexed by Λ is equivalent to the derived category

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of the abelian category of functors from Λ to $\mathscr{C}$. Note that a particular case of this result, in which $\Lambda = \mathbb{Z}$ and $\mathscr{C}$ is the category of abelian groups, was already obtained in [**8**, § 3.1].

Next, we assume that $\mathscr{C}$ is a tensor category and Λ is a preordered semigroup. In this case, we can define what is a filtered ring A indexed by Λ and a filtered A-module in $\mathscr{C}$ and we prove a similar result to the preceding one, namely that the derived category of the category of filtered A-modules is equivalent to the derived category of the abelian category of modules over the Λ-ring A.

Applications to the study of filtered $\mathscr{D}_X$-modules will be developed in the future. Indeed it is proved in [**2**] that, on a complex manifold X endowed with the subanalytic topology X_{sa}, the sheaf $\mathscr{O}_{X_{\mathrm{sa}}}$ (which is in fact an object of the derived category of sheaves, no more concentrated in degree zero) may be endowed with various filtrations and the results of this paper will be used when developing this point.

2. A review on quasi-abelian categories

In this section, we briefly review the main notions on quasi-abelian categories and their derived categories, after [**8**]. We refer to [**5**] for an exposition on abelian, triangulated and derived categories.

Let $\mathscr{C}$ be an additive category admitting kernels and cokernels. Recall that, for a morphism $f\colon X \to Y$ in $\mathscr{C}$, $\mathrm{Im}(f)$ is the kernel of $Y \to \mathrm{Coker}(f)$, and $\mathrm{Coim}(f)$ is the cokernel of $\mathrm{Ker}(f) \to X$. Then f decomposes as

$$X \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to Y.$$

One says that f is *strict* if $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism. Note that a monomorphism (resp. an epimorphism) $f\colon X \to Y$ is strict if and only if $X \to \mathrm{Im}(f)$ (resp. $\mathrm{Coim}(f) \to Y$) is an isomorphism. For any morphism $f\colon X \to Y$,

- $\mathrm{Ker}(f) \to X$ and $\mathrm{Im}(f) \to Y$ are strict monomorphisms,
- $X \to \mathrm{Coim}(f)$ and $Y \to \mathrm{Coker}(f)$ are strict epimorphisms.

Note also that a morphism f is strict if and only if it factors as $i \circ s$ with a strict epimorphism s and a strict monomorphism i.

Definition 2.1. — A *quasi-abelian* category is an additive category which admits kernels and cokernels and satisfies the following conditions:

(i) strict epimorphisms are stable by base changes,
(ii) strict monomorphisms are stable by co-base changes.

The condition (i) means that, for any strict epimorphism $u\colon X \to Y$ and any morphism $Y' \to Y$, setting $X' = X \times_Y Y' = \mathrm{Ker}(X \times Y' \to Y)$, the composition $X' \to X \times Y' \to Y'$ is a strict epimorphism. The condition (ii) is similar by reversing the arrows.

Note that, for any morphism $f\colon X \to Y$ in a quasi-abelian category, $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is both a monomorphism and an epimorphism.

Remark that if $\mathscr{C}$ is a quasi-abelian category, then its opposite category $\mathscr{C}^{\mathrm{op}}$ is also quasi-abelian.

Of course, an abelian category is a quasi-abelian category in which all morphisms are strict.

Definition 2.2. — Let $\mathscr{C}$ be a quasi-abelian category. A sequence $M' \xrightarrow{f} M \xrightarrow{f'} M''$ with $f' \circ f = 0$ is strictly exact if f is strict and the canonical morphism $\operatorname{Im} f \to \operatorname{Ker} f'$ is an isomorphism.

Equivalently such a sequence is strictly exact if the canonical morphism $\operatorname{Coim} f \to \operatorname{Ker} f'$ is an isomorphism.

One shall be aware that the notion of strict exactness is not auto-dual.

Consider a functor $F\colon \mathscr{C} \to \mathscr{C}'$ of quasi-abelian categories. Recall that F is

- strictly exact if it sends any strict exact sequence $X' \to X \to X''$ to a strict exact sequence,
- strictly left exact if it sends any strict exact sequence $0 \to X' \to X \to X''$ to a strict exact sequence $0 \to F(X') \to F(X) \to F(X'')$,
- left exact if it sends any strict exact sequence $0 \to X' \to X \to X'' \to 0$ to a strict exact sequence $0 \to F(X') \to F(X) \to F(X'')$.

Derived categories. — Let $\mathscr{C}$ be an additive category. One denotes as usual by $\mathrm{C}(\mathscr{C})$ the additive category consisting of complexes in $\mathscr{C}$. For $X \in \mathrm{C}(\mathscr{C})$, one denotes by X^n ($n \in \mathbb{Z}$) its n's component and by $d_X^n\colon X^n \to X^{n+1}$ the differential. For $k \in \mathbb{Z}$, one denotes by $X \mapsto X[k]$ the shift functor (also called translation functor). We denote by $\mathrm{C}^+(\mathscr{C})$ (resp. $\mathrm{C}^-(\mathscr{C})$, $\mathrm{C}^{\mathrm{b}}(\mathscr{C})$) the full subcategory of $\mathrm{C}(\mathscr{C})$ consisting of objects X such that $X^n = 0$ for $n \ll 0$ (resp. $n \gg 0$, $|n| \gg 0$). One also sets $\mathrm{C}^{\mathrm{ub}}(\mathscr{C}) := \mathrm{C}(\mathscr{C})$ (ub stands for unbounded).

We do not recall here neither the construction of the mapping cone $\mathrm{Mc}(f)$ of a morphism f in $\mathrm{C}(\mathscr{C})$ nor the construction of the triangulated categories $\mathrm{K}^*(\mathscr{C})$ ($* = \mathrm{ub}, +, -, \mathrm{b}$), called the homotopy categories of $\mathscr{C}$.

Recall that a null system $\mathscr{N}$ in a triangulated category $\mathscr{T}$ is a full triangulated saturated subcategory of $\mathscr{T}$, saturated meaning that an object X belongs to $\mathscr{N}$ whenever X is isomorphic to an object of $\mathscr{N}$. For a null system $\mathscr{N}$, the localization $\mathscr{T}/\mathscr{N}$ is a triangulated category. A distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathscr{T}/\mathscr{N}$ is a triangle isomorphic to the image of a distinguished triangle in $\mathscr{T}$.

Let $\mathscr{C}$ be quasi-abelian category. One says that a complex X is

- *strict* if all the differentials d_X^n are strict,
- *strictly exact in degree* n if the sequence $X^{n-1} \to X^n \to X^{n+1}$ is strictly exact.
- *strictly exact* if it is strictly exact in all degrees.

If X is strictly exact, then X is a strict complex and $0 \to \operatorname{Ker}(d_X^n) \to X^n \to \operatorname{Ker}(d_X^{n+1}) \to 0$ is strictly exact for all n.

Note that if two complexes X and Y are isomorphic in $\mathrm{K}(\mathscr{C})$, and if X is strictly exact, then so is Y. Let $\mathscr{E}$ be the full additive subcategory of $\mathrm{K}(\mathscr{C})$ consisting of strictly exact complexes. Then $\mathscr{E}$ is a null system in $\mathrm{K}(\mathscr{C})$.

Definition 2.3. — The derived category $\mathrm{D}(\mathscr{C})$ is the quotient category $\mathrm{K}(\mathscr{C})/\mathscr{E}$. where $\mathscr{E}$ is the null system in $\mathrm{K}(\mathscr{C})$ consisting of strictly exact complexes. One defines similarly the categories $\mathrm{D}^*(\mathscr{C})$ for $* = +, -, \mathrm{b}$.

A morphism $f\colon X \to Y$ in $\mathrm{K}(\mathscr{C})$ is called a *quasi-isomorphism* (a qis for short) if, after being embedded in a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$, Z belongs to $\mathscr{E}$. This is equivalent to saying that its image in $\mathrm{D}(\mathscr{C})$ is an isomorhism. It follows that given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathrm{K}(\mathscr{C})$, if two of f, g and $g \circ f$ are qis, then all the three are qis.

Note that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a sequence of morphisms in $\mathrm{C}(\mathscr{C})$ such that $0 \to X^n \to Y^n \to Z^n \to 0$ is strictly exact for all n, then the natural morphism $\mathrm{Mc}(f) \to Z$ is a qis, and we have a distinguished triangle

$$X \to Y \to Z \to X[1]$$

in $\mathrm{D}(\mathscr{C})$.

Left t-structure. — Let $\mathscr{C}$ be a quasi-abelian category. Recall that for $n \in \mathbb{Z}$, $\mathrm{D}^{\leq n}(\mathscr{C})$ (resp. $\mathrm{D}^{\geq n}(\mathscr{C})$) denotes the full subcategory of $\mathrm{D}(\mathscr{C})$ consisting of complexes X which are strictly exact in degrees $k > n$ (resp. $k < n$). Note that $\mathrm{D}^{+}(\mathscr{C})$ (resp. $\mathrm{D}^{-}(\mathscr{C})$) is the union of all the $\mathrm{D}^{\geq n}(\mathscr{C})$'s (resp. all the $\mathrm{D}^{\leq n}(\mathscr{C})$'s), and $\mathrm{D}^{\mathrm{b}}(\mathscr{C})$ is the intersection $\mathrm{D}^{+}(\mathscr{C}) \cap \mathrm{D}^{-}(\mathscr{C})$. The associated truncation functors are then given by:

$$\begin{array}{llllllllllll}
\tau^{\leq n}X\colon & \cdots & \to & X^{n-2} & \to & X^{n-1} & \to & \operatorname{Ker} d_X^n & \to & 0 & \to & \cdots \\
\tau^{\geq n}X\colon & \cdots & \to & 0 & \to & \operatorname{Coim} d_X^{n-1} & \to & X^n & \to & X^{n+1} & \to & \cdots .
\end{array}$$

The functor $\tau^{\leq n}\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}^{\leq n}(\mathscr{C})$ is a right adjoint to the inclusion functor $\mathrm{D}^{\leq n}(\mathscr{C}) \hookrightarrow \mathrm{D}(\mathscr{C})$, and $\tau^{\geq n}\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}^{\geq n}(\mathscr{C})$ is a left adjoint functor to the inclusion functor $\mathrm{D}^{\geq n}(\mathscr{C}) \hookrightarrow \mathrm{D}(\mathscr{C})$.

The pair $(\mathrm{D}^{\leq 0}(\mathscr{C}), \mathrm{D}^{\geq 0}(\mathscr{C}))$ defines a t-structure on $\mathrm{D}(\mathscr{C})$ by [**8**]. We refer to [**1**] for the general theory of t-structures (see also [**4**] for an exposition).

The heart $\mathrm{D}^{\leq 0}(\mathscr{C}) \cap \mathrm{D}^{\geq 0}(\mathscr{C})$ is an abelian category called the left heart of $\mathrm{D}(\mathscr{C})$ and denoted by $\mathrm{LH}(\mathscr{C})$ in [**8**]. The embedding $\mathscr{C} \hookrightarrow \mathrm{LH}(\mathscr{C})$ induces an equivalence

$$\mathrm{D}(\mathscr{C}) \xrightarrow{\sim} \mathrm{D}(\mathrm{LH}(\mathscr{C})).$$

By duality, one also defines the right t-structure and the right heart of $\mathrm{D}(\mathscr{C})$.

Derived functors. — Given an additive functor $F\colon \mathscr{C} \to \mathscr{C}'$ of quasi-abelian categories, its right or left derived functor is defined in [**8**, Def. 1.3.1] by the same procedure as for abelian categories.

Definition 2.4. — (See [**8**, Def. 1.3.2].) A full additive subcategory $\mathscr{P}$ of $\mathscr{C}$ is called F-projective if

(a) for any $X \in \mathscr{C}$, there exists a strict epimorphism $Y \to X$ with $Y \in \mathscr{P}$,
(b) for any strict exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$, if $X, X'' \in \mathscr{P}$, then $X' \in \mathscr{P}$,
(c) for any strict exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$, if $X', X, X'' \in \mathscr{P}$, then the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ in strictly exact in $\mathscr{C}'$.

If F admits an F-projective category, one says that F is explicitly left derivable. In this case, F admits a left derived functor LF and this functor is calculated as usual by the formula

$$LF(X) \simeq F(Y), \text{ where } Y \in \mathrm{K}^-(\mathscr{P}), Y \xrightarrow{\sim} X \text{ in } \mathrm{D}^-(\mathscr{C}).$$

We refer to [**8**, §1.3] for details.

If LF has bounded cohomological dimension, then it extends as a triangulated functor $LF\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}(\mathscr{C}')$.

3. Filtered objects

We shall assume

$$(1) \qquad \begin{cases} \Lambda \text{ is a small filtrant category,} \\ \mathscr{C} \text{ is an abelian category admitting small filtrant inductive} \\ \text{limits, such limits being exact.} \end{cases}$$

Denote by $\mathrm{Fct}(\Lambda, \mathscr{C})$ the abelian category of functors from Λ to $\mathscr{C}$, and denote as usual by $\Delta\colon \mathscr{C} \to \mathrm{Fct}(\Lambda, \mathscr{C})$ the functor which, to $X \in \mathscr{C}$, associates the constant functor $\lambda \mapsto X$ and by $\varinjlim\colon \mathrm{Fct}(\Lambda, \mathscr{C}) \to \mathscr{C}$ the inductive limit functor. Then $(\varinjlim, \Delta)$ is a pair of adjoint functors:

If $M \in \mathrm{Fct}(\Lambda, \mathscr{C})$, we set for short $M(\infty) := \varinjlim M$ and we denote by $j_M(\lambda)$ the morphism $M(\lambda) \to M(\infty)$. If $f\colon M \to M'$ is a morphism in $\mathrm{Fct}(\Lambda, \mathscr{C})$, we denote by $f(\infty)\colon M(\infty) \to M'(\infty)$ the associated morphism.

Definition 3.1. — (a) The category $\mathrm{F}_\Lambda \mathscr{C}$ of Λ-filtered objects in $\mathscr{C}$ is the full additive subcategory of $\mathrm{Fct}(\Lambda, \mathscr{C})$ formed by the functors which send morphisms to monomorphisms.
(b) We denote by $\iota\colon \mathrm{F}_\Lambda \mathscr{C} \hookrightarrow \mathrm{Fct}(\Lambda, \mathscr{C})$ the inclusion functor.

Remark 3.2. — (i) Inductive limits in $\mathscr{C}$ being exact, each morphisms $j_M(\lambda)$ is a monomorphism. As a matter of fact, such a morphism can be viewed as the inductive limit of the monomorphisms

$$M(s) : M(\lambda) \to M(\lambda')$$

over the category of arrows $s : \lambda \to \lambda'$ in Λ.
(ii) Let M be a Λ-filtered object of $\mathscr{C}$ and let $\lambda, \lambda' \in \Lambda$. Since $j_M(\lambda) \circ M(s) = j_M(\lambda')$ for any morphism $s : \lambda' \to \lambda$ of Λ and since $j_M(\lambda)$ is a monomorphism, it is clear that $M(s)$ does not depend on s. It follows that the category $\mathrm{F}_\Lambda(\mathscr{C})$ is equivalent to the

category $\mathrm{F}_{\Lambda_{\mathrm{pos}}}(\mathscr{C})$ where Λ_{pos} denotes the category corresponding to the preordered set associated with Λ, i.e., the category having the same objects as Λ but for which

$$\operatorname{Hom}_{\Lambda_{\mathrm{pos}}}(\lambda', \lambda) = \begin{cases} \{\mathrm{pt}\} & \text{if} \quad \operatorname{Hom}_{\Lambda}(\lambda', \lambda) \neq \varnothing, \\ \varnothing & \text{if} \quad \operatorname{Hom}_{\Lambda}(\lambda', \lambda) = \varnothing. \end{cases}$$

Therefore, when we study the properties of $\mathrm{F}_{\Lambda}(\mathscr{C})$ we can always assume that Λ is the category associated with a preordered set.

(iii) When Λ is a preordered set, M defines a increasing map from Λ to the poset of subobjects of $M(\infty)$. Moreover, $M(\infty)$ is the union of the $M(\lambda)$'s and we recover the usual notion of an exhaustive filtration.

(iv) Let Λ be a preordered set and let $\Lambda \sqcup \{\infty\}$ be the preordered set obtained by adding ∞ as a formal maximum to Λ. Thanks to (ii) it is clear that a $\Lambda \sqcup \{\infty\}$-filtered object of $\mathscr{C}$ can be identified with an object of $\mathscr{C}$ endowed with a non-exhaustive Λ-filtration.

Basic properties of $\mathrm{F}_{\Lambda}(\mathscr{C})$. — In this subsection, we shall prove that the category $\mathrm{F}_{\Lambda}(\mathscr{C})$ is quasi-abelian.

The next result is obvious.

Proposition 3.3. — *The subcategory* $\mathrm{F}_{\Lambda}(\mathscr{C})$ *of* $\mathrm{Fct}(\Lambda, \mathscr{C})$ *is stable by subobjects. In particular, the caterory* $\mathrm{F}_{\Lambda}(\mathscr{C})$ *admits kernels and the functor* ι *commutes with kernels.*

Definition 3.4. — Let $M \in \mathrm{Fct}(\Lambda, \mathscr{C})$. For $\lambda \in \Lambda$, we set $\kappa(M)(\lambda) = \operatorname{Im} j_M(\lambda)$ and for a morphism $s\colon \lambda' \to \lambda$ in Λ we define $\kappa(M)(s)$ as the morphism induced by the identity of $M(\infty)$.

These definitions turn obviously $\kappa(M)$ into an object of $\mathrm{F}_{\Lambda}(\mathscr{C})$ and give a functor

$$\kappa : \mathrm{Fct}(\Lambda, \mathscr{C}) \to \mathrm{F}_{\Lambda}(\mathscr{C}). \tag{2}$$

Proposition 3.5. — *The functor* κ *in* (2) *is left adjoint to the inclusion functor* ι *and* $\kappa \circ \iota \simeq \mathrm{id}_{\mathrm{F}_{\Lambda}(\mathscr{C})}$. *In particular the category* $\mathrm{F}_{\Lambda}(\mathscr{C})$ *admits cokernels and* κ *commutes with cokernels.*

Proof. — Let M be an object of $\mathrm{F}_{\Lambda}(\mathscr{C})$ and let M' be an object of $\mathrm{Fct}(\Lambda, \mathscr{C})$. Consider a morphism $f : M' \to \iota(M)$ in $\mathrm{Fct}(\Lambda, \mathscr{C})$. It induces a morphism $f(\infty) : M'(\infty) \to M(\infty)$ in $\mathscr{C}$. Since the diagram

$$\begin{tikzcd} M'(\lambda) \arrow[r, "f(\lambda)"'] \arrow[d, "j_{M'}(\lambda)"'] & M(\lambda) \arrow[d, "j_M(\lambda)"] \\ M'(\infty) \arrow[r, "f(\infty)"] & M(\infty) \end{tikzcd}$$

is commutative for every object λ of Λ and since $j_M(\lambda)$ is a monomorphism, the morphism $f(\lambda)$ induces a canonical morphism $f'(\lambda) : \operatorname{Im} j_{M'}(\lambda) \to M(\lambda)$ and these morphisms give us a morphism $f' : \kappa(M') \to M$. The preceding construction gives a morphism of abelian groups

$$\mathrm{Hom}_{\,\mathrm{Fct}(\Lambda,\mathscr{C})}(M', \iota(M)) \to \mathrm{Hom}_{\,\mathrm{F}_\Lambda\mathscr{C}}(\kappa(M'), M)$$

and it is clearly an isomorphism. Since ι is fully faithful, we have $\kappa \circ \iota \simeq \mathrm{id}_{\mathrm{F}_\Lambda \mathscr{C}}$ and the conclusion follows. □

By the preceding results, the category $\mathrm{F}_\Lambda(\mathscr{C})$ is additive and has kernels and cokernels and hence images and coimages. However, if $f\colon M' \to M$ is a morphism in $\mathrm{F}_\Lambda(\mathscr{C})$ the canonical morphism from $\mathrm{Coim}(f)$ to $\mathrm{Im}(f)$ is in general not an isomorphism, in other words, f is not in general a strict morphism and the inclusion functor ι does not commute with cokernels (see Example 3.7 below).

Corollary 3.6. — *Let $f\colon M' \to M$ be a morphism in $\mathrm{F}_\Lambda(\mathscr{C})$ and let* Ker, Coker, Im *and* Coim *be calculated in the category $\mathrm{F}_\Lambda(\mathscr{C})$. Then, one has the canonical isomorphisms for $\lambda \in \Lambda$:*
(a) $(\mathrm{Ker}\, f)(\lambda) \simeq \mathrm{Ker}\, f(\lambda)$,
(b) $(\mathrm{Coker}\, f)(\lambda) \simeq \mathrm{Im}[M(\lambda) \to \mathrm{Coker}\, f(\infty)]$,
(c) $(\mathrm{Im}\, f)(\lambda) \simeq \mathrm{Ker}[M(\lambda) \to \mathrm{Coker}\, f(\infty)]$,
(d) $(\mathrm{Coim}\, f)(\lambda) \simeq \mathrm{Im}\, f(\lambda)$.
In particular,
(i) *f is strict in $\mathrm{F}_\Lambda(\mathscr{C})$ if and only if the canonical square below is Cartesian*

$$\begin{tikzcd} \mathrm{Im}\, f(\lambda) \arrow[r] \arrow[d] & \mathrm{Im}\, f(\infty) \arrow[d] \\ M(\lambda) \arrow[r] & M(\infty), \end{tikzcd}$$

(ii) *a sequence $M' \xrightarrow{f} M \xrightarrow{f'} M''$ in $\mathrm{F}_\Lambda(\mathscr{C})$ with $f' \circ f = 0$ is strictly exact if and only if the canonical morphism $\mathrm{Im}\, f(\lambda) \to \mathrm{Ker}\, f'(\lambda)$ is an isomorphism for any $\lambda \in \Lambda$.*

Example 3.7. — Let $\Lambda = \mathbb{N}$, $\mathscr{C} = \mathrm{Mod}(\mathbb{C})$ and denote by $\mathbb{C}[X]^{\leq n}$ the space of polynomials in one variable X over $\mathbb{C}$ of degree $\leq n$. Consider the two objects M' and M of $\mathrm{F}_\Lambda \mathscr{C}$:

$$M' \colon n \mapsto \mathbb{C}[X]^{\leq n},$$
$$M \colon n \mapsto \mathbb{C}[X]^{\leq n+1}.$$

Denote by $f\colon M' \to M$ the natural morphism. Then $M'(\infty) \xrightarrow{\sim} M(\infty) \simeq \mathbb{C}[X]$ and $f(\infty)$ is an isomorphism. Therefore, $(\mathrm{Im}\, f)(n) \simeq M(n)$ and $(\mathrm{Coim}\, f)(n) \simeq \mathrm{Im}(f(n)) \simeq M'(n)$.

Proposition 3.8. — *Let $0 \to M' \xrightarrow{f} M \xrightarrow{f'} M'' \to 0$ be an exact sequence in $\mathrm{Fct}(\Lambda, \mathscr{C})$. Assume that M'' belongs to $\mathrm{F}_\Lambda(\mathscr{C})$. Then the sequence*

$$0 \to \kappa(M') \xrightarrow{\kappa(f)} \kappa(M) \xrightarrow{\kappa(f')} \kappa(M'') \to 0$$

is strictly exact in $\mathrm{F}_\Lambda(\mathscr{C})$ *(i.e.,* $\kappa(f)$ *is a kernel of* $\kappa(f')$ *and* $\kappa(f')$ *is a cokernel of* $\kappa(f)$*).*

Proof. — We know that the diagram

$$\begin{tikzcd}
0 \arrow[r] & M'(\lambda) \arrow[r, "f(\lambda)"] \arrow[d, "j_{M'}(\lambda)"'] & M(\lambda) \arrow[r, "f'(\lambda)"] \arrow[d, "j_M(\lambda)"'] & M''(\lambda) \arrow[r] \arrow[d, "j_{M''}(\lambda)"'] & 0 \\
0 \arrow[r] & M'(\infty) \arrow[r, "f(\infty)"] & M(\infty) \arrow[r, "f'(\infty)"] & M''(\infty) \arrow[r] & 0
\end{tikzcd}$$

is commutative and has exact rows. Since the last vertical arrow is a monomorphism it follows that we have a canonical isomorphism

$$\operatorname{Ker} j_{M'}(\lambda) \simeq \operatorname{Ker} j_M(\lambda).$$

Therefore, in the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Ker} j_{M'}(\lambda) \arrow[r] \arrow[d] & \operatorname{Ker} j_M(\lambda) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'(\lambda) \arrow[r] \arrow[d] & M(\lambda) \arrow[r] \arrow[d] & M''(\lambda) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \kappa(M')(\lambda) \arrow[r] \arrow[d] & \kappa(M)(\lambda) \arrow[r] \arrow[d] & \kappa(M'')(\lambda) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

the columns and the two lines in the top are exact. Therefore the last row is also exact and the conclusion follows from Corollary 3.6. □

Theorem 3.9. — *Assume* (1)*. The category* $\mathrm{F}_\Lambda(\mathscr{C})$ *is quasi-abelian.*

Proof. — Consider a Cartesian square in $\mathrm{F}_\Lambda(\mathscr{C})$

$$\begin{tikzcd}
N' \arrow[r, "g"] \arrow[d, "u'"] & N \arrow[d, "u"] \\
M' \arrow[r, "f"] & M
\end{tikzcd}$$

and assume that f is a strict epimorphism in $\mathrm{F}_\Lambda(\mathscr{C})$. It follows from Proposition 3.3 and Corollary 3.6 (ii) that this square is also Cartesian in $\mathrm{Fct}(\Lambda, \mathscr{C})$ and that f is an epimorphism in this category. Hence g is an epimorphism in $\mathrm{Fct}(\Lambda, \mathscr{C})$ and Corollary 3.6 (ii) shows that g is a strict epimorphism in $\mathrm{F}_\Lambda(\mathscr{C})$.

Consider now a co-Cartesian square in $\mathrm{F}_\Lambda(\mathscr{C})$

$$\begin{tikzcd} M' \arrow[r, "f"] \arrow[d, "u'"] & M \arrow[d, "u"] \\ N' \arrow[r, "g"] & N \end{tikzcd}$$

and assume that f is a strict monomorphism in $\mathrm{F}_\Lambda(\mathscr{C})$. We know from Proposition 3.3 and Proposition 3.5 that this square is the image by κ of the co-Cartesian square of $\mathrm{Fct}(\Lambda, \mathscr{C})$ with solid arrow

$$\begin{tikzcd} M' \arrow[r, "f"] \arrow[d, "u'"] & M \arrow[r, dotted, "q"] \arrow[d, "v"] & C \\ N' \arrow[r, "h"'] & P. \arrow[ur, dotted, "q'"'] & \end{tikzcd}$$

Denote by

$$q : M \to C$$

the canonical morphism from M to the cokernel of f in $\mathrm{F}_\Lambda(\mathscr{C})$. Since f is a strict monomorphism of $\mathrm{F}_\Lambda(\mathscr{C})$, C is also the cokernel of f in $\mathrm{Fct}(\Lambda, \mathscr{C})$ and there is a unique morphism $q' : P \to C$ such that $q' \circ v = q$ and $q' \circ h = 0$. Moreover, one checks easily that this morphism q' is a cokernel of h in $\mathrm{Fct}(\Lambda, \mathscr{C})$. It follows that the sequence

$$0 \to N' \xrightarrow{h} P \xrightarrow{q'} C \to 0$$

is exact in $\mathrm{Fct}(\Lambda, \mathscr{C})$. Applying κ to this sequence and using Proposition 3.8, we get a strictly exact sequence of the form:

$$0 \to N' \xrightarrow{g} N \to C \to 0.$$

This shows in particular that g is a strict monomorphism in $\mathrm{F}_\Lambda(\mathscr{C})$ and the conclusion follows. □

The Rees functor. — From now on we assume that Λ is a category associated with a preordered set. Thanks to Remark 3.2 (i), this assumption is not really restrictive.

In the sequel, given a direct sum $\bigoplus_{i \in I} X_i$ we denote by $\sigma_i \colon X_i \to \bigoplus_{i \in I} X_i$ the canonical morphism.

Definition 3.10. — For $M \in \mathrm{Fct}(\Lambda, \mathscr{C})$ we define $\Sigma(M) \in \mathrm{Fct}(\Lambda, \mathscr{C})$ as follows. For $\lambda_0 \in \Lambda$ and for $s \colon \lambda_0 \to \lambda_1$, we set

$$\Sigma(M)(\lambda_0) = \bigoplus_{s_0 \colon \lambda'_0 \to \lambda_0} M(\lambda'_0),$$

and define

$$\Sigma(M)(s) \colon \Sigma(M)(\lambda_0) \to \Sigma(M)(\lambda_1)$$

as the only morphism such that $\Sigma(M)(s) \circ \sigma_{s_0} = \sigma_{s \circ s_0}$ for any $s_0 \colon \lambda'_0 \to \lambda_0$.

Proposition 3.11. — *Let $M\colon \Lambda \to \mathscr{C}$ be a functor and let $s\colon \lambda_0 \to \lambda_1$ be a morphism of Λ. Then $\Sigma(M)(s)\colon\ \Sigma(M)(\lambda_0) \to \Sigma(M)(\lambda_1)$ is a split monomorphism. In particular $\Sigma(M)$ is an object of $\mathrm{F}_\Lambda\mathscr{C}$.*

Proof. — Let us define $\rho\colon\ \Sigma(M)(\lambda_1) \to \Sigma(M)(\lambda_0)$ as the unique morphism such that

$$\rho\circ\sigma_{s_1} = \begin{cases} \sigma_{s_0} & \text{if } s_1 = s\circ s_0 \text{ for some } s_0\colon \lambda_1' \to \lambda_0 \\ 0 & \text{otherwise.}\end{cases}$$

This definition makes sense since if $s_1 = s\circ s_0$ for some $s_0\colon \lambda_1' \to \lambda_0$ then such an s_0 is unique (recall that Λ is a poset). Since

$$\rho\circ\Sigma(M)(s)\circ\sigma_{s_0} = \rho\circ\sigma_{s\circ s_0} = \sigma_{s_0}$$

for any $s_0\colon \lambda_0' \to \lambda_0$ in Λ, the conclusion follows. □

Remark 3.12. — The preceding construction gives rise to a functor, that we call the Rees functor,

$$\Sigma\colon\ \mathrm{Fct}(\Lambda,\mathscr{C}) \to \mathrm{F}_\Lambda\mathscr{C}.$$

This functor sends exact sequences in $\mathrm{Fct}(\Lambda,\mathscr{C})$ to strict exact sequences in $\mathrm{F}_\Lambda\mathscr{C}$.

Definition 3.13. — For any $M \in \mathrm{Fct}(\Lambda,\mathscr{C})$ we define the morphism $\varepsilon_M\colon\ \Sigma(M) \to M$ by letting

$$\varepsilon_M(\lambda_0)\colon\ \Sigma(M)(\lambda_0) \to M(\lambda_0) \tag{3}$$

be the unique morphism such that $\varepsilon_M(\lambda_0)\circ\sigma_{s_0} = M(s_0)$ for any $s_0\colon \lambda_0' \to \lambda_0$ in Λ.

Proposition 3.14. — *For any $M \in \mathrm{Fct}(\Lambda,\mathscr{C})$ and $\lambda_0 \in \Lambda$, the morphism (3) is a split epimorphism of $\mathscr{C}$. In particular, the morphism $\varepsilon_M\colon\ \Sigma(M) \to M$ is an epimorphism in $\mathrm{Fct}(\Lambda,\mathscr{C})$.*

Proof. — This follows directly from

$$\varepsilon_M(\lambda_0)\circ\sigma_{\mathrm{id}_{\lambda_0}} = M(\mathrm{id}_{\lambda_0}) = \mathrm{id}_{M(\lambda_0)},\ \lambda_0 \in \Lambda.$$ □

Corollary 3.15. — *The category $\mathrm{F}_\Lambda\mathscr{C}$ is a κ-projective subcategory of the category $\mathrm{Fct}(\Lambda,\mathscr{C})$. In particular the functor*

$$\kappa\colon\ \mathrm{Fct}(\Lambda,\mathscr{C}) \to \mathrm{F}_\Lambda\mathscr{C}$$

is explicitly left derivable. Moreover, it has finite cohomological dimension.

Proof. — The properties (a), (b) and (c) of Definition 2.4 follow respectively from Proposition 3.14, Proposition 3.3 and Proposition 3.8. Hence the category $\mathrm{F}_\Lambda\mathscr{C}$ is κ-projective. Since it is also stable by subobjects it follows that any object of $\mathrm{Fct}(\Lambda,\mathscr{C})$ has a two terms resolution by objects of $\mathrm{F}_\Lambda\mathscr{C}$ and the conclusion follows. □

Theorem 3.16. — *Assume (1) and assume that Λ is a preordered set. The functor $\iota\colon \mathrm{F}_\Lambda\mathscr{C} \to \mathrm{Fct}(\Lambda,\mathscr{C})$ is strictly exact and induces an equivalence of categories for* $* = \mathrm{ub}, \mathrm{b}, +, -$

$$\iota\colon \mathrm{D}^*(\mathrm{F}_\Lambda\mathscr{C}) \to \mathrm{D}^*(\mathrm{Fct}(\Lambda,\mathscr{C}))$$

whose quasi-inverse is given by

$$L\kappa\colon \mathrm{D}^*(\mathrm{Fct}(\Lambda,\mathscr{C})) \to \mathrm{D}^*(\mathrm{F}_\Lambda\mathscr{C}).$$

Moreover, ι induces an equivalence of abelian categories

$$\mathrm{LH}(\mathrm{F}_\Lambda\mathscr{C}) \simeq \mathrm{Fct}(\Lambda,\mathscr{C}).$$

Proof. — Let $M^\bullet$ be an object of $\mathrm{D}^*(\mathrm{F}_\Lambda\mathscr{C})$. Since the category $\mathrm{F}_\Lambda\mathscr{C}$ is κ-projective, it is clear that

$$L\kappa(\iota(M^\bullet)) \simeq \kappa(\iota(M^\bullet) \simeq M^\bullet.$$

Let now $N^\bullet$ be an object of $\mathrm{D}^*(\mathrm{Fct}(\Lambda,\mathscr{C}))$. Since the category $\mathrm{F}_\Lambda\mathscr{C}$ is κ-projective and stable by subobjects, there is an object $M^\bullet$ of $\mathrm{D}^*(\mathrm{F}_\Lambda\mathscr{C})$ and an isomorphism

$$\iota(M^\bullet) \xrightarrow{\sim} N^\bullet$$

in $\mathrm{D}^*(\mathrm{Fct}(\Lambda,\mathscr{C}))$. It follows that

$$M^\bullet \simeq \kappa(\iota(M^\bullet)) \simeq L\kappa(N^\bullet)$$

in $\mathrm{D}^*(\mathrm{F}_\Lambda\mathscr{C})$ and that

$$N^\bullet \simeq \iota(M^\bullet) \simeq \iota(L\kappa(N^\bullet))$$

in $\mathrm{D}^*(\mathrm{Fct}(\Lambda,\mathscr{C}))$. Hence $L\kappa$ is a quasi-inverse of ι. To conclude, it is sufficient to note that a complex $M^\bullet$ of $C^*(\mathrm{F}_\Lambda\mathscr{C})$ is strictly exact in degree k if and only if $\iota(M^\bullet)$ is exact in degree k in $C^*(\mathrm{Fct}(\Lambda,\mathscr{C}))$. □

4. Filtered modules in an abelian tensor category

Abelian tensor categories. — In this subsection we recall a few facts about abelian tensor categories. References are made to [**5**, Ch. 5] for details.

Let $\mathscr{C}$ be an additive category. A biadditive tensor product on $\mathscr{C}$ is the data of a functor $\otimes\colon \mathscr{C}\times\mathscr{C} \to \mathscr{C}$ additive with respect to each argument together with functorial associativity isomorphisms

$$\alpha_{X,Y,Z}\colon (X\otimes Y)\otimes Z \xrightarrow{\sim} X\otimes(Y\otimes Z)$$

satisfying the natural compatibility conditions.

From now on, we assume that $\mathscr{C}$ is endowed with such a tensor product. A ring object of $\mathscr{C}$ (or, equivalently, "a ring in $\mathscr{C}$") is then an object A of $\mathscr{C}$ endowed with an associative multiplication $\mu_A\colon A\otimes A \to A$.

Let A be such a ring object. Then, an A-module of $\mathscr{C}$ is the data of an object M of $\mathscr{C}$ together with an associative action $\nu_M\colon A\otimes M \to M$.

The A-modules of $\mathscr{C}$ form a category denoted by $\mathrm{Mod}(A)$. A morphism $f\colon M \to N$ in this category is simply a morphism of $\mathscr{C}$ which is A-linear, i.e., which is compatible with the actions of A on M and N. Most of the properties of $\mathrm{Mod}(A)$ can be deduced from that of $\mathscr{C}$ thanks to following result, whose proof is left to the reader.

Lemma 4.1. — *The category* $\mathrm{Mod}(A)$ *is an additive category and the forgetful functor* $\mathrm{for}\colon \mathrm{Mod}(A) \to \mathscr{C}$ *is additive, faithful, conservative and reflects projective limits. This functor also reflects inductive limits which are preserved by* $A \otimes \bullet : \mathscr{C} \to \mathscr{C}$.

One easily deduces:

Proposition 4.2. — *Assume that* $\mathscr{C}$ *is abelian (resp. quasi-abelian) and the functor* $A \otimes \bullet$ *commutes with cokernels. Then* $\mathrm{Mod}(A)$ *is abelian (resp. quasi-abelian). Moreover, the forgetful functor* $\mathrm{for}\colon \mathrm{Mod}(A) \to \mathscr{C}$ *is additive, faithfull, conservative and commutes with kernels and cokernels. If one assumes moreover that* $\mathscr{C}$ *admits small inductive limits and that* $A \otimes \bullet$ *commutes with such limits, then* $\mathrm{Mod}(A)$ *admits small inductive limits and the forgetful functor* for *commutes with such limits.*

Remark 4.3. — Let A be a ring object in $\mathscr{C}$. We have defined an A-module by considering the left action of A. In other words, we have defined left A-modules. Clearly, one can define right A-modules similarly, which is equivalent to replacing the tensor product $\otimes$ with the opposite tensor product given by $X \otimes^{\mathrm{op}} Y = Y \otimes X$.

Assume that the tensor category $\mathscr{C}$ admits a unit, denoted by $\mathbf{1}$ and that the ring object A also admits a unit $e\colon \mathbf{1} \to A$. In this case, we consider the full subcategory $\mathrm{Mod}(A_e)$ of $\mathrm{Mod}(A)$ consisting of modules such that the action of A is unital, which is translated by saying that the diagram below commutes:

$$\begin{tikzcd} \mathbf{1} \otimes M \arrow[r, "e\otimes \mathrm{id}_M"] \arrow[dr, "\sim"'] & A \otimes M \arrow[d, "\nu_M"] \\ & M. \end{tikzcd}$$

All results concerning $\mathrm{Mod}(A)$ still hold for $\mathrm{Mod}(A_e)$.

Proposition 4.4. — *Let* $\mathscr{C}$ *be Grothendieck category with a generator* G *and assume that* $\mathscr{C}$ *is also a tensor category with unit* $\mathbf{1}$. *Let* A *be a ring in* $\mathscr{C}$ *with unit* e *and assume that the functor* $A \otimes \bullet$ *commutes with small inductive limits. Then* $\mathrm{Mod}(A_e)$ *is a Grothendieck category and* $A \otimes G$ *is a generator of this category.*

Proof. — For any $X \in \mathscr{C}$ the morphism

$$\bigoplus_{s \in \mathrm{Hom}_{\mathscr{C}}(G,X)} G \to X$$

is an epimorphism. It follows that for any $M \in \mathrm{Mod}(A_e)$ the morphism

$$\bigoplus_{s \in \mathrm{Hom}_{\mathscr{C}}(G,M)} A \otimes G \to A \otimes M$$

is an epimorphism. Since A has a unit, there is an epimorphism $A \otimes M \twoheadrightarrow M$. $\square$

Λ-rings and Λ-modules. — In this section, we shall assume

$$(4)\qquad \begin{cases} \Lambda \text{ is a filtrant preordered additive semigroup (viewed as a tensor category),} \\ \mathscr{C} \text{ is an abelian tensor category which admits small inductive limits which commute with } \otimes \text{ and small filtrant inductive limits are exact.} \end{cases}$$

Definition 4.5. — For $M_1, M_2 \in \mathrm{Fct}(\Lambda, \mathscr{C})$ we define $M_1 \otimes M_2 \in \mathrm{Fct}(\Lambda, \mathscr{C})$ as follows. For $\lambda, \lambda' \in \Lambda$ and $s\colon \lambda' \to \lambda$ we set

$$(M_1 \otimes M_2)(\lambda) = \varinjlim_{\lambda_1+\lambda_2 \le \lambda} M_1(\lambda_1) \otimes M_2(\lambda_2),$$

and we define $(M_1 \otimes M_2)(s)\colon (M_1 \otimes M_2)(\lambda') \to (M_1 \otimes M_2)(\lambda)$ to be the morphism induced by the inclusion

$$\{(\lambda_1', \lambda_2') \in \Lambda \times \Lambda\colon \lambda_1' + \lambda_2' \le \lambda'\} \subset \{(\lambda_1, \lambda_2) \in \Lambda \times \Lambda\colon \lambda_1 + \lambda_2 \le \lambda\}.$$

Proposition 4.6. — *The functor*

$$\otimes\colon \mathrm{Fct}(\Lambda, \mathscr{C}) \times \mathrm{Fct}(\Lambda, \mathscr{C}) \to \mathrm{Fct}(\Lambda, \mathscr{C})$$

defined above turns $\mathrm{Fct}(\Lambda, \mathscr{C})$ *into a tensor category. Moreover, it commutes with small inductive limits.*

Proof. — The fact that the functor commutes with small inductive limits follows from its definition. The associativity follows from the associativity of the tensor product in $\mathscr{C}$ and the canonical isomorphisms

$$((M_1 \otimes M_2) \otimes M_3)(\lambda) \simeq \varinjlim_{\lambda_1+\lambda_2+\lambda_3 \le \lambda} (M_1(\lambda_1) \otimes M_2(\lambda_2)) \otimes M_3(\lambda_3),$$
$$(M_1 \otimes (M_2 \otimes M_3))(\lambda) \simeq \varinjlim_{\lambda_1+\lambda_2+\lambda_3 \le \lambda} M_1(\lambda_1) \otimes (M_2(\lambda_2)) \otimes M_3(\lambda_3)). \qquad \square$$

Definition 4.7. — (a) A Λ-*ring of* $\mathscr{C}$ is a ring of the tensor category $\mathrm{Fct}(\Lambda, \mathscr{C})$ considered in Proposition 4.6.

(b) A Λ-*module of* $\mathscr{C}$ over a Λ-ring A of $\mathscr{C}$ is an A-module of the tensor category $\mathrm{Fct}(\Lambda, \mathscr{C})$.

(c) As usual, we denote by $\mathrm{Mod}(A)$ the category of A-modules, that is, Λ-modules in $\mathscr{C}$ over the Λ-ring A.

Remark 4.8. — It follows from the preceding definitions that a Λ-ring of $\mathscr{C}$ is the data of a functor $A\colon \Lambda \to \mathscr{C}$ together with a multiplication morphism

$$A(\lambda_1) \otimes A(\lambda_2) \to A(\lambda_1 + \lambda_2)$$

functorial in $\lambda_1, \lambda_2 \in \Lambda$ and associative in a natural way. Moreover, if A is such a ring, then a Λ-module of $\mathscr{C}$ over A is the data of a functor $M\colon \Lambda \to \mathscr{C}$ together with a functorial associative action morphism

$$A(\lambda_1) \otimes M(\lambda_2) \to M(\lambda_1 + \lambda_2).$$

Remark 4.9. — Assume that the semigroup Λ admits a unit, denoted by 0 (in which case, one says that Λ is a monoid), and the tensor category $\mathscr{C}$ admits a unit, denoted by $\mathbf{1}$. Then the tensor category $\mathrm{Fct}(\Lambda, \mathscr{C})$ admits a unit, still denoted by $\mathbf{1}_\Lambda$, defined as follows:

$$\mathbf{1}_\Lambda(\lambda) = \begin{cases} \mathbf{1} & \text{if } \lambda \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In such a case, the notion of a Λ-ring A with unit e makes sense as well as the notion of an A_e-module.

Proposition 4.10. — *Let A be a Λ-ring of $\mathscr{C}$. Then, the category* $\mathrm{Mod}(A)$ *is abelian and admits small inductive limits. Moreover, the forgetful functor* $\mathrm{for}\colon \mathrm{Mod}(A) \to \mathrm{Fct}(\Lambda, \mathscr{C})$ *is additive, faithfull, conservative and commutes with kernels and small inductive limits. In particular it is exact.*

Proof. — This follows directly from the preceding results and Proposition 4.2. □

Filtered rings and modules

Definition 4.11. — We define the functor

$$\otimes_F\colon \mathrm{F}_\Lambda\mathscr{C} \times \mathrm{F}_\Lambda\mathscr{C} \to \mathrm{F}_\Lambda\mathscr{C}$$

by the formula

$$M_1 \otimes_F M_2 = \kappa(\iota(M_1) \otimes \iota(M_2))$$

where the tensor product in the right-hand side is the tensor product of $\mathrm{Fct}(\Lambda, \mathscr{C})$.

Proposition 4.12. — *There is a canonical isomorphism*

$$\kappa(M_1 \otimes M_2) \simeq \kappa(M_1) \otimes_F \kappa(M_2)$$

functorial in $M_1, M_2 \in \mathrm{Fct}(\Lambda, \mathscr{C})$.

Proof. — We know that for $\lambda, \lambda_1, \lambda_2 \in \Lambda$, $\kappa(M_1 \otimes M_2)(\lambda)$ is the image of the canonical morphism $(M_1 \otimes M_2)(\lambda) \to (M_1 \otimes M_2)(\infty)$ and that $\kappa(M_1)(\lambda_1)$ and $\kappa(M_2)(\lambda_2)$ are respectively the images of the canonical morphisms $M_1(\lambda_1) \to M_1(\infty)$ and $M_2(\lambda_2) \to M_2(\infty)$. Since the morphisms

$$M_1(\lambda_1) \to \kappa(M_1)(\lambda_1) \quad \text{and} \quad M_2(\lambda_2) \to \kappa(M_2)(\lambda_2)$$

are epimorphisms, so is the morphism

$$M_1(\lambda_1) \otimes M_2(\lambda_2) \to \kappa(M_1)(\lambda_1) \otimes \kappa(M_2)(\lambda_2).$$

It follows that the canonical morphism

$$\varinjlim_{\lambda_1+\lambda_2 \leq \lambda} M_1(\lambda_1) \otimes M_2(\lambda_2) \to \varinjlim_{\lambda_1+\lambda_2 \leq \lambda} \kappa(M_1)(\lambda_1) \otimes \kappa(M_2)(\lambda_2).$$

is also an epimorphism. Since

$$\begin{aligned}(M_1 \otimes M_2)(\infty) \simeq M_1(\infty) \otimes M_2(\infty) &\simeq \kappa(M_1)(\infty) \otimes \kappa(M_2)(\infty) \\ &\simeq \kappa(M_1 \otimes M_2)(\infty)\end{aligned}$$

the conclusion follows. □

Proposition 4.13. — *The functor* $\otimes_F\colon \mathrm{F}_\Lambda\mathscr{C}\times\mathrm{F}_\Lambda\mathscr{C}\to\mathrm{F}_\Lambda\mathscr{C}$ *turns* $\mathrm{F}_\Lambda\mathscr{C}$ *into a tensor category. Moreover* $\otimes_F$ *commutes with small inductive limits.*

Proof. — By the preceding result we have

$$\begin{aligned}\kappa((\iota(M_1)\otimes\iota(M_2))\otimes\iota(M_3)) &\simeq \kappa(\iota(M_1)\otimes\iota(M_2))\otimes_F\kappa(\iota(M_3))\\ &\simeq (M_1\otimes_F M_2)\otimes_F M_3,\\ \kappa(\iota(M_1)\otimes(\iota(M_2)\otimes\iota(M_3))) &\simeq \kappa(\iota(M_1))\otimes_F\kappa(\iota(M_2)\otimes\iota(M_3))\\ &\simeq M_1\otimes_F(M_2\otimes_F M_3).\end{aligned}$$

Hence the associativity of $\otimes_F$ follows from that of the tensor product of $\mathrm{Fct}(\Lambda,\mathscr{C})$. Since κ commutes with small inductive limits, a similar argument shows that $\otimes_F$ has the same property. □

Definition 4.14. — (a) A Λ-*filtered ring of* $\mathscr{C}$ is a ring object in the tensor category $\mathrm{F}_\Lambda\mathscr{C}$.
(b) A Λ-*filtered module* FM over a Λ-filtered ring FA, or simply, an FA-module FM, is an FA-module in the tensor category $\mathrm{F}_\Lambda\mathscr{C}$.
(c) As usual, we denote by $\mathrm{Mod}(FA)$ the category of FA-modules.

Remark 4.15. — It follows from the preceding definitions that $\mathrm{Mod}(FA)$ is the full subcategory of $\mathrm{Mod}(A)$ formed by the functors which send morphisms of Λ to monomorphisms of $\mathscr{C}$. The multiplication on FA and the action of FA on a module FM may be described as in Remark 4.8.

Proposition 4.16. — *Let* FA *be a* Λ*-filtered ring of* $\mathscr{C}$*. The category* $\mathrm{Mod}(FA)$ *is quasi-abelian and admits small inductive limits. Moreover, the forgetful functor* $\mathrm{for}\colon\mathrm{Mod}(FA)\to\mathrm{F}_\Lambda\mathscr{C}$ *is additive, faithfull, conservative and commutes with kernels and inductive limits.*

Proof. — This follows directly from the preceding results and Proposition 4.2. □

Proposition 4.17. — *Let* FA *be a* Λ*-filtered ring of* $\mathscr{C}$*. Then* $A:=\iota FA$ *is a* Λ*-ring of* $\mathscr{C}$ *and the functors* $\iota\colon\mathrm{F}_\Lambda(\mathscr{C})\to\mathrm{Fct}(\Lambda,\mathscr{C})$ *and* $\kappa\colon\mathrm{Fct}(\Lambda,\mathscr{C})\to\mathrm{F}_\Lambda(\mathscr{C})$ *induce functors*

$$\iota_A\colon\mathrm{Mod}(FA)\to\mathrm{Mod}(A)\quad\textit{and}\quad\kappa_A\colon\mathrm{Mod}(A)\to\mathrm{Mod}(FA).$$

Moreover κ_A *is a left adjoint of* ι_A.

Proof. — This follows easily from Proposition 4.12 and the fact that κ is a left adjoint of ι. □

Proposition 4.18. — *Let* FA *be a* Λ*-filtered ring of* $\mathscr{C}$ *and set* $A:=\iota FA$*. Let* M *be an* A*-module. Then the functor* $\Sigma(M)$ *of Definition 3.10 has a canonical structure of an* A*-module and the morphism* $\varepsilon_M\colon\Sigma(M)\to M$ *is* A*-linear.*

Proof. — We define the action of FA on $\Sigma(M)$ as the composition of the morphisms

$$
\begin{aligned}
A(\lambda_1) \otimes \Sigma(M)(\lambda_2) &= A(\lambda_1) \otimes \bigoplus_{s_2\colon \lambda_2' \to \lambda_2} M(\lambda_2') \\
(*) \qquad &\simeq \bigoplus_{s_2\colon \lambda_2' \to \lambda_2} A(\lambda_1) \otimes M(\lambda_2') \\
(**) \qquad &\to \bigoplus_{s_2\colon \lambda_2' \to \lambda_2} M(\lambda_1 + \lambda_2') \\
(***) \qquad &\xrightarrow{v} \bigoplus_{s_3\colon \lambda_3' \to \lambda_1+\lambda_2} M(\lambda_3') \\
&= \Sigma(M)(\lambda_1 + \lambda_2)
\end{aligned}
$$

where (*) comes from the fact that $\otimes$ commutes with small inductive limits, (**) comes from the action of A on M and (***) is characterized by the fact that

$$v \circ \sigma_{s_2} = \sigma_{\mathrm{id}_{\lambda_1} + s_2}$$

where $\mathrm{id}_{\lambda_1} + s_2\colon \lambda_1 + \lambda_2' \to \lambda_1 + \lambda_2$ is the map induced by $s_2\colon \lambda_2' \to \lambda_2$. It is then easily verified that this action turns $\Sigma(M)$ into an A-module for which the morphism $\varepsilon_M\colon \Sigma(M) \to M$ becomes A-linear. $\square$

The following results can now be obtained by working as in Section 3.

Corollary 4.19. — *Let FA be a Λ-filtered ring of $\mathscr{C}$. Then the category* $\mathrm{Mod}(FA)$ *is a κ_A-projective subcategory of* $\mathrm{Mod}(A)$*. In particular the functor*

$$\kappa_A\colon \mathrm{Mod}(A) \to \mathrm{Mod}(FA)$$

is explicitly left derivable. Moreover, it has finite cohomological dimension.

Theorem 4.20. — *Assume Hypothesis 4. The functor*

$$\iota_A\colon \mathrm{Mod}(FA) \to \mathrm{Mod}(A)$$

is strictly exact and induces an equivalence of categories for $* = \mathrm{ub}, \mathrm{b}, +, -$:

$$\iota_A\colon \mathrm{D}^*(\mathrm{Mod}(FA)) \to \mathrm{D}^*(\mathrm{Mod}(A))$$

whose quasi-inverse is given by

$$L\kappa_A\colon \mathrm{D}^*(\mathrm{Mod}(A)) \to \mathrm{D}^*(\mathrm{Mod}(FA)).$$

Moreover, ι_A induces an equivalence of abelian categories

$$\mathrm{LH}(\mathrm{Mod}(FA)) \simeq \mathrm{Mod}(A).$$

Remark 4.21. — Assume that the semigroup Λ admits a unit, denoted by 0, and the tensor category $\mathscr{C}$ admits a unit, denoted by $\mathbf{1}$. Then the unit $\mathbf{1}_\Lambda$ of the category $\mathrm{Fct}(\Lambda, \mathscr{C})$ (see Remark 4.9) belongs to $\mathrm{F}_\Lambda\mathscr{C}$ and is a unit in this tensor category. In such a case, the notion of a Λ-filtered ring FA with unit e makes sense as well as the notion of FA_e-module.

Moreover, the results of Theorem 4.20 remain true with $\mathrm{Mod}(FA)$ and $\mathrm{Mod}(A)$ replaced with $\mathrm{Mod}(FA_e)$ and $\mathrm{Mod}(A_e)$, respectively.

Assume moreover that $\mathscr{C}$ is a Grothendieck category. In this case, $\mathrm{Mod}(A_e)$ is again a Grothendieck category by Proposition 4.4.

Example: modules over a filtered sheaf of rings. — Let X be a site and let $\mathbf{k}$ be a commutative unital algebra with finite global dimension. Consider the category $\mathscr{C} = \mathrm{Mod}(\mathbf{k}_X)$ of sheaves of $\mathbf{k}_X$-modules and its derived category, $\mathrm{D}(\mathbf{k}_X)$. Let Λ be as in (4). Applying the definitions and results obtained in the previous sections, we see that:

(a) A Λ-filtered sheaf $F\mathscr{F}$ is a sheaf $\mathscr{F} \in \mathrm{Mod}(\mathbf{k}_X)$ endowed with a family of subsheaves $\{F_\lambda \mathscr{F}\}_{\lambda\in\Lambda}$ such that $F_{\lambda'}\mathscr{F} \subset F_\lambda \mathscr{F}$ for any pair $\lambda' \leq \lambda$ and $\bigcup_j F_\lambda \mathscr{F} = \mathscr{F}$. (Of course, the union $\bigcup$ is taken in the category of sheaves.)
(b) A Λ-filtered sheaf of $\mathbf{k}_X$-algebras $F\mathscr{R}$ is a filtered sheaf such that the underlying sheaf $\mathscr{R}$ is a sheaf of $\mathbf{k}_X$-algebras and $F_\lambda\mathscr{R} \otimes F_{\lambda'}\mathscr{R} \subset F_{\lambda+\lambda'}\mathscr{R}$ for all $\lambda, \lambda' \in \Lambda$. (In particular, $F_0\mathscr{R}$ is a subring of $\mathscr{R}$.)
(c) Given $F\mathscr{R}$ as above, a left filtered module $F\mathscr{M}$ over $F\mathscr{R}$ is filtered sheaf such that the underlying sheaf $\mathscr{M}$ is a sheaf of modules over $\mathscr{R}$ and $F_\lambda\mathscr{R}\otimes F_{\lambda'}\mathscr{M} \subset F_{\lambda+\lambda'}\mathscr{M}$ for all $\lambda, \lambda' \in \Lambda$.
(d) If $\mathscr{R}$ is unital, we ask that the unit of $\mathscr{R}$ is a section of $F_0\mathscr{R}$ and acts as the identity on each $F_\lambda\mathscr{M}$.
(e) The category $\mathrm{Mod}(F\mathscr{R})$ of filtered modules over $F\mathscr{R}$ is quasi-abelian.

On the other-hand, an object $F\mathscr{N}$ of the abelian category $\mathrm{Mod}(\iota F\mathscr{R})$ is the data of a family of sheaves $\{F_\lambda\mathscr{N}\}_{\lambda\in\Lambda}$, morphisms $F_\lambda\mathscr{N} \to F_{\lambda'}\mathscr{N}$ for any pair $\lambda \leq \lambda'$ and morphisms $F_\lambda\mathscr{R} \otimes F_{\lambda'}\mathscr{N} \to F_{\lambda+\lambda'}\mathscr{N}$ for all $\lambda, \lambda' \in \Lambda$ satisfying the natural compatibility conditions but we do not ask any more that $F_{\lambda'}\mathscr{N}$ is a subsheaf of $F_\lambda\mathscr{N}$ for $\lambda' \leq \lambda$.

By Theorem 4.20, we have an equivalence of categories for $* = \mathrm{ub}, \mathrm{b}, +, -$:

$$\mathrm{D}^*(\mathrm{Mod}(F\mathscr{R})) \xrightarrow{\sim} \mathrm{D}^*(\mathrm{Mod}(\iota F\mathscr{R})).$$

Example 4.22. — Let $(X, \mathscr{O}_X)$ be a complex manifold and let $\mathscr{D}_X$ be the sheaf of finite order differential operators.

We apply the preceding construction to the tensor category $\mathrm{Mod}(\mathbb{C}_X)$. For $j \in \mathbb{Z}$, we denote by $F_j\mathscr{D}_X$ the subsheaf of $\mathscr{D}_X$ whose sections are differential operators of order $\leq j$, with $F_j\mathscr{D}_X = 0$ for $j < 0$, and we denote by $F\mathscr{D}_X$ the ring $\mathscr{D}_X$ endowed with this filtration. Recall that a filtered left $\mathscr{D}_X$-module $F\mathscr{M}$ is a left $\mathscr{D}_X$-module $\mathscr{M}$ endowed with a family of subsheaves $F_j\mathscr{M}$ ($j \in \mathbb{Z}$) and morphisms $F_i\mathscr{D}_X \otimes F_j\mathscr{M} \to F_{i+j}\mathscr{M}$ satisfying natural compatibility conditions (the $F_j\mathscr{M}$'s are thus $\mathscr{O}_X$-modules) and such that $\bigcup_j F_j\mathscr{M} = \mathscr{M}$. Therefore, $F\mathscr{D}_X$ is a $\mathbb{Z}$-ring in $\mathrm{Mod}(\mathbb{C}_X)$ and $F\mathscr{M}$ is an $F\mathscr{D}_X$-module, that is, an object of $\mathrm{Mod}(F\mathscr{D}_X)$.

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P. Schapira, Sorbonne Universités, UPMC Univ Paris 6, Institut de Mathématiques de Jussieu, F-75005 Paris France • *E-mail :* pierre.schapira@imj-prg.fr

J.-P. Schneiders, Institut de Mathématiques, Université de Liège, Belgique
E-mail : jpschneiders@ulg.ac.be

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Sheaves on manifolds are perfectly suited to treat local problems, but many spaces one naturally encounters, especially in Analysis, are not of local nature. The subanalytic topology (in the sense of Grothendieck) on real analytic manifolds allows one to partially overcome this difficulty and to define for example sheaves of functions or distributions with temperate growth, but not to make the growth precise.

In this volume, one introduces the linear subanalytic topology, a refinement of the preceding one, and constructs various objects of the derived category of sheaves on the subanalytic site with the help of the Brown representability theorem.

In particular one constructs the Sobolev sheaves. These objects have the nice property that the complexes of their sections on open subsets with Lipschitz boundaries are concentrated in degree zero and coincide with the classical Sobolev spaces.

Another application of this topology is that it allows one to *functorially* endow regular holonomic D-modules with filtrations (in the derived sense).

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