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ON OVERCONVERGENT HILBERT MODULAR CUSP FORMS

by

Fabrizio Andreatta, Adrian Iovita & Vincent Pilloni

Abstract. — We p-adically interpolate modular invertible sheaves over a strict neighborhood of the ordinary locus of an Hilbert modular variety. We then prove the existence of finite slope families of cuspidal eigenforms.

Résumé **(À propos des formes modulaires surconvergentes cuspidales de Hilbert)**
Nous interpolons p-adiquement les faisceaux inversibles automorphes sur des voisinages stricts du lieu ordinaire d'une variété modulaire de HIlbert. Nous prouvons ensuite l'existence de familles de pente finie de formes propres et cuspidales.

1. Introduction

Let F be a totally real number field. The theme of this paper is the construction of eigenvarieties for Hilbert modular eigenforms defined over F. Several constructions already appeared in the literature:

- A construction by Buzzard in [**9**], where he interpolates automorphic functions on a quaternion algebra over F, ramified at infinity.
- A construction by Kisin-Lai in [**19**] of a 1-dimensional parallel weight eigenvariety. Their method is an extension of Coleman's original construction for $F = \mathbb{Q}$. It is based on twists by a lift of the Hasse invariant.
- A construction by Urban in [**25**], who interpolates the traces of the Hecke operators acting on the Betti cohomology of certain distribution sheaves defined over the Hilbert variety (Urban actually works in a much more general setting).
- A yet conjectural construction by Emerton in [**14**] who applies a Jaquet functor to the completed cohomology of the Hilbert variety.
- The work of Kassaei in [**18**] for unitary/quaternionic Shimura curves as a consequence of which a 1 dimensional eigenvariety for Hilbert modular forms was constructed (for weights of the form $(k, 2, 2, \ldots, 2)$, where k-varies.)
- A recent construction of Brasca in [**8**] who also constructs a 1-dimensional eigenvariety for Hilbert modular forms using Shimura curves.

In the present paper we work with the Hilbert variety and p-adically interpolate modular invertible sheaves over a strict neighborhood of the ordinary locus. We then prove the existence of finite slope families of cuspidal eigenforms. Thus our construction is geometric, in the sense that we work over (open subsets of) the Hilbert variety and interpolate sections of invertible sheaves. In the ordinary case, our construction boils down to the construction of Hida families using Katz's p-adic modular forms. In the parallel weight case, our construction is equivalent to Kisin-Lai's (and the cuspidal hypothesis is unnecessary).

This article is a natural continuation of both [**3**] and [**2**] and we would like to discuss what is new here. Let us first point out that the notion of "Hilbert modular form" is slightly ambiguous in the literature if $F \neq \mathbb{Q}$. More precisely, given the totally real number field F as above, there are two relevant algebraic groups associated to it: $G := \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ and $G^* := G \times_{\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m} \mathbb{G}_m$, where the morphism $G \longrightarrow \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ is the determinant morphism and the morphism $\mathbb{G}_m \longrightarrow \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ is the natural (diagonal) one. The Hilbert modular varieties considered in [**3**] and in section § 3 of this article are models for the Shimura varieties associated to the group G^*, therefore the classical, overconvergent and p-adic families of Hilbert modular forms considered there are all associated to the group G^*. Let us also point out that there are classical Hecke operators acting on those modular forms, but their definition depends on non-canonical choices and therefore these operators do not commute (although the U_p-operator is canonical.)

On the other hand there is a classical theory of automorphic forms for the group G, with a natural theory of eigenforms and to these eigenforms one can attach Galois representations. Moreover, recently D. Barrera in [**5**] showed that overconvergent modular symbols for the group G can be used in order to produce p-adic L-functions attached to classical automorphic eigenforms for this group. Therefore in view of various arithmetic applications it would be desirable to define overconvergent and p-adic families of modular forms for G. The main obstacle in doing this directly is that the moduli problem associated to G is not representable. If $F = \mathbb{Q}$, the ambiguity disappears as $G = G^\star$. As a result, we suppose that $F \neq \mathbb{Q}$ whenever we speak about the group G in the paper.

Let us now describe what is actually accomplished in this article.

a) In [**3**] we assumed that the prime p was unramified in F; in the present article we remove this assumption.

b) We start by constructing overconvergent and p-adic families of modular forms attached to the group G^*. As was mentioned before the construction is geometric but we do not follow the line of arguments started in [**3**]. Instead and for the sake of uniformity, we use the main ideas which appeared in [**2**].

c) We prove that the specialization map from finite slope p-adic families of cuspforms to finite slope overconvergent cuspforms of a given weight (still for the group G^*) is surjective, i.e., every overconvergent cuspform of finite slope can be deformed to a p-adic family. This was not proved in [**3**] even in the case when p was unramified in F.

d) We construct overconvergent and p-adic families of modular forms for the group G by descent, using the overconvergent and p-adic families of modular forms for G^* constructed at b).

e) We construct the cuspidal eigenvariety for the group G and study the associated Galois representations.

We shall now explain the main ideas of these constructions. For simplicity of the exposition, throughout the introduction we will only work with the open Hilbert modular varieties, but the reader should be aware that the shaves we construct extend to toroidal compactifications as will be explained in the main body of the article.

a) Let $N \geq 4$ be an integer prime to p, let $\mathfrak{c}$ denote a fractional ideal of F and $\mathfrak{c}^+$ its cone of positive elements. We fix K a finite extension of $\mathbb{Q}_p$ which splits F. We denote by $M(\mu_N, \mathfrak{c})$ the Hilbert modular scheme over $\mathrm{Spec}(\mathcal{O}_K)$ (for the group G^*) classifying abelian schemes of relative dimension $g := [F : \mathbb{Q}]$, level structure μ_N, $\mathcal{O}_F$-multiplication and polarization associated to $(\mathfrak{c}, \mathfrak{c}^+)$ (see section §3.1 for more details). Let $A \longrightarrow M(\mu_N, \mathfrak{c})$ be the universal abelian scheme with identity section $e\colon M(\mu_N, \mathfrak{c}) \longrightarrow A$ and let $\omega_A := e^*\big(\Omega^1_{A/M(\mu_N,\mathfrak{c})}\big)$ denote the co-normal sheaf of A. The sheaf ω_A is a locally free $\mathcal{O}_{M(\mu_N,\mathfrak{c})}$-module of rank g, with a natural action of $\mathcal{O}_F$ but if p ramifies in F it is not locally free as $\mathcal{O}_{M(\mu_N,\mathfrak{c})} \otimes \mathcal{O}_F$-module. There is a largest open subscheme $M^{\mathrm{R}}(\mu_N, \mathfrak{c})$ of $M(\mu_N, \mathfrak{c})$ such that the restriction of ω_A to it is an invertible $\mathcal{O}_{M^{\mathrm{R}}(\mu_N,\mathfrak{c})} \otimes \mathcal{O}_F$-module. Therefore the classical Hilbert modular forms are in this case defined as sections of the relevant sheaf over the open $M^{\mathrm{R}}(\mu_N, \mathfrak{c})$ (see section § 2.)

On the other hand the construction of the overconvergent Hilbert modular forms starts by fixing an integer $n > 0$, a multi-index $\underline{v} := (v_i)$ satisfying $0 < v_i < 1/p^n$ and a strict neighborhood $\mathcal{M}(\mu_n, \mathfrak{c})(\underline{v})$ of width $\underline{v}$ of the ordinary locus in the rigid analytic variety $\mathcal{M}(\mu_N, \mathfrak{c})$ associated to the generic fiber of $M(\mu_N, \mathfrak{c})$. Let us denote by $\mathfrak{M}(\mu_N, \mathfrak{c})(\underline{v})$ the natural formal model of $\mathcal{M}(\mu_N, \mathfrak{c})(\underline{v})$ described in section §3.2.1

It has the property that the universal abelian scheme $A_{\underline{v}}$ on it has a canonical subgroup H_n of order p^{ng}. We denote by $\mathcal{M}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v})$ the finite étale covering of $\mathcal{M}(\mu_N, \mathfrak{c})(\underline{v})$ on which H_n is trivialized and by $\mathfrak{M}\big(\Gamma_1(p^n), \mu_N, \mathfrak{c}\big)(\underline{v})$ the normalization of $\mathfrak{M}(\mu_N, \mathfrak{c})(\underline{v})$ in it. It turns out that over $\mathcal{M}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v})$, the canonical subgroup H_n is isomorphic to the constant group scheme $\mathcal{O}_F/p^n\mathcal{O}_F$. This isomorphism is compatible with the natural $\mathcal{O}_F$-actions on the two group schemes.

Therefore the sub-sheaf $\mathcal{F}$ of $\omega_A|_{\mathfrak{M}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})}$ defined in Proposition 5.1 is a locally free $\mathcal{O}_{\mathfrak{M}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})} \otimes \mathcal{O}_F$-module of rank one while the restriction of ω_A is not (if p is ramified in F).

It follows that the overconvergent modular sheaves are better behaved than the classical ones and the overconvergent modular forms are defined as sections of the relevant modular sheaves over opens of the form $\mathfrak{M}(\mu_N, \mathfrak{c})(\underline{v})$ wether p is ramified in F or not.

b) In **[3]** we defined modular sheaves by using certain universal torsors while in this article we content ourselves to only define their realization on the universal formal

Hilbert modular schemes. Instead of torsors we use overconvergent Igusa towers as in [**2**]. The present constructions are more restrictive but they suffice for the applications we have in mind.

c) The surjectiveness of the specialization map is proved as in [**2**]: we study the descent of our modular sheaves from a smooth toroidal compactification $\overline{\mathcal{M}}(\mu_N,\mathfrak{c})(\underline{v})$ of $\mathcal{M}(\mu_N,\mathfrak{c})(\underline{v})$ to the minimal compactification $\overline{\mathcal{M}}^*(\mu_N,\mathfrak{c})(\underline{v})$.

d) We shall first recall here the descent from the classical modular forms for the group G^* to modular forms for the group G; the latter are known to be identified, with their Hecke operators, to the classical automorphic forms for $G(\mathbb{A}_F)$, see for instance [**21**], section § 2.

Let us recall the notations at a) above: $N\geq 4$ is an integer prime to p, $\mathfrak{c}$ denotes a fractional ideal of F and $\mathfrak{c}^+$ its cone of positive elements. We denote by $M(\mu_N,\mathfrak{c})$ the Hilbert modular scheme over $\mathrm{Spec}(\mathcal{O}_K)$ (for the group G^*) classifying abelian schemes of relative dimension $g:=[F:\mathbb{Q}]$, level structure μ_N, $\mathcal{O}_F$-multiplication and polarization associated to $(\mathfrak{c},\mathfrak{c}^+)$ (see section §3.1 for more details). Let $A\longrightarrow M(\mu_N,\mathfrak{c})$ be the versal abelian scheme and let ω_A be the co-normal sheaf to the identity of A. We denoted by $M^{\mathrm{R}}(\mu_N,\mathfrak{c})$ the largest open subscheme of $M(\mu_n,\mathfrak{c})$ such that the restriction of ω_A to it is an invertible $\mathcal{O}_{M^{\mathrm{R}}(\mu_N,\mathfrak{c})}\otimes\mathcal{O}_F$-module.

We denote by $\mathbb{T}$ the algebraic group $\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathbb{G}_m$ over $\mathcal{O}_K$. This group is identified with the diagonal subgroup inside the derived group $G^{\star,der}=G^{der}$. Let $\mathbb{T}^\star$ be the diagonal subgroup in $G^\star$ and let $\mathbb{G}_m$ be the multiplicative group identified with the center of $G^\star$. The surjective map $\mathbb{T}\times\mathbb{G}_m\to\mathbb{T}^\star$ has kernel the group μ_2. Let I be the set of embeddings of F into K (recall that K was a finite extension of $\mathbb{Q}_p$ which splits F). Then $\mathbb{Z}[I]$ is the character group, over K, of the group $\mathbb{T}$. If we identify $\mathbb{Z}$ with the character group of $\mathbb{G}_m$, then the character group of $\mathbb{T}^\star$ is the subgroup of $\mathbb{Z}[I]\times\mathbb{Z}$ of elements $(\sum_{\sigma\in I}k_\sigma.\sigma,w)$ such that $\sum k_\sigma=w\mod 2$. According to [**20**], III.2, V.6., we can attach to each $(\kappa,w)\in\mathbb{Z}[I]\times\mathbb{Z}$ an invertible sheaf $\Omega_K^{(\kappa,w)}$ over $M(\mu_N,\mathfrak{c})_K$. Here we are using that the center of $G^\star$ is $\mathbb{G}_m$ and that $\mathbb{G}_m(\mathbb{Z})$ is a finite group. Actually, this sheaf does not depend on w (which must be considered like a Tate twist), so we suppress it from the notation as this is customary. This construction of the sheaf can be made explicit and extended integrally as follows. Let $\kappa:=\sum_{\sigma\in I}k_\sigma\cdot\sigma\in\mathbb{Z}[I]$ and let us set $\Omega^\kappa:=\bigoplus_{\sigma\in I}\omega_{A,\sigma}^{k_\sigma}$. The notation $\omega_{A,\sigma}$ means the sheaf $\omega_A\otimes_{(\mathcal{O}_K\otimes\mathcal{O}_F,1\otimes\sigma)}\mathcal{O}_K$ over $M^{\mathrm{R}}(\mu_N,\mathfrak{c})$. We refer to $\mathbb{Z}[I]$ as the set of *classical weights* for $G^\star$.

The module $\mathrm{M}(\mu_N,\mathfrak{c},\kappa):=\mathrm{H}^0\big(M^{\mathrm{R}}(\mu_N,\mathfrak{c})_K,\Omega^\kappa\big)$ is the K-vector space of tame level N, $\mathfrak{c}$-polarized, weight κ Hilbert modular forms (for the group G^*).

Let $\mathbb{T}^G$ be the diagonal subgroup in G. We have a map $p_1:\mathbb{T}\to\mathbb{T}^G$ given by the inclusion $G^\star\to G$. We can also identify $\mathbb{T}$ with the center of G. This provides a second map $p_2:\mathbb{T}\to\mathbb{T}^G$. The map $p_1\times p_2:\mathbb{T}\times\mathbb{T}\to\mathbb{T}_G$ is surjective, with kernel $\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mu_2$. As a result, the character group over K of $\mathbb{T}^G$ is the subgroup of $\mathbb{Z}[I]\times\mathbb{Z}[I]$ of pairs $(\sum_\sigma k_\sigma.\sigma,\sum_\sigma w_\sigma.\sigma)$ subject to the condition that $k_\sigma=w_\sigma\mod 2$. The map from characters of $\mathbb{T}^G$ to characters of $\mathbb{T}^\star$ is the one that sends $(\sum_\sigma k_\sigma.\sigma,\sum_\sigma w_\sigma.\sigma)$

to $(\sum_\sigma k_\sigma.\sigma, \sum_\sigma w_\sigma)$. There is an important subgroup of the character group of $\mathbb{T}^G$: they are the characters trivial on a finite index subgroup of the center $\mathbb{T}(\mathbb{Z})$. This conditions means, by Dirichlet unit theorem, that $w_\sigma = w$ for all $\sigma \in I$. We now define the group of *classical weights* for G: this is the subgroup of $\mathbb{Z}[I] \times \mathbb{Z} \subset \mathbb{Z}[I] \times \mathbb{Z}[I]$ of elements $(\sum_\sigma k_\sigma.\sigma, w)$ such that $k_\sigma = w \mod 2$ for all $\sigma \in I$. This group is isomorphic to $\mathbb{Z}[I] \times \mathbb{Z}$ via the map $(\sum_\sigma \nu_\sigma.\sigma, w) \mapsto (2\sum_\sigma \nu_\sigma.\sigma + w, w)$. According to **[20]**, one can attach to any classical weight (ν, w) for G an invertible sheaf on the tower of Shimura varieties for G, with global sections the space of classical modular forms for that given weight. We will now explain how this space can be obtained from the space $\mathrm{M}(\mu_N, \mathfrak{c}, \kappa = 2\nu + w)$ by application of a projector.

Let $\mathcal{O}_F^{\times,+}$ denote the group of totally positive units of $\mathcal{O}_F$. This group acts naturally on $M(\mu_N, \mathfrak{c})$ as follows. Let $y := (A, \iota, \psi, \lambda)$ be a point of $M(\mu_N, \mathfrak{c})$ and $\epsilon \in \mathcal{O}_F^{\times,+}$ (we recall that: A is an abelian scheme of relative dimension g, $\iota\colon \mathcal{O}_F \to \mathrm{End}(A)$ is a ring homomorphism, ψ is a μ_N-level structure and $\lambda\colon A \otimes_{\mathcal{O}_F} \mathfrak{c} \cong A^t$ is the polarization.)

We define $\epsilon \cdot (A, \iota, \psi, \lambda) := (A, \iota, \psi, \epsilon\lambda)$. In fact as $\epsilon\colon A \cong A$ defines an isomorphism such that $\epsilon^*\lambda = \epsilon^2\lambda$, it follows that if $\epsilon = \eta^2$ with $\eta \in \mathcal{O}_F^{\times,+}$ such that η is congruent to 1 modulo $N\mathcal{O}_F$, then we have for a point $y = (A, \iota, \psi, \lambda)$ of $M(\mu_N, \mathfrak{c})$:

$$\epsilon(A, \iota, \psi, \lambda) = (A, \iota, \psi, \epsilon\lambda) = \eta^*(A, \iota, \psi, \lambda) \cong (A, \iota, \psi, \lambda).$$

Therefore if we denote by U_N the subgroup of $\mathcal{O}_F^{\times,+}$ of units congruent to 1 modulo $N\mathcal{O}_F$, then the group U_N^2 of squares of elements of U_N acts trivially on the points of $M(\mu_N, \mathfrak{c})$ and thus the action of $\mathcal{O}_F^{\times,+}$ on $M(\mu_N, \mathfrak{c})$ factors through the finite group $\Delta := \mathcal{O}_F^{\times,+}/U_N^2$.

Now let $\kappa := \sum_{\sigma\in I} \in \mathbb{Z}[I]$ be a weight and let us suppose that the k_σ's have all the same parity. In that case we choose $\nu_\sigma \in \mathbb{Z}$ for all $\sigma \in I$ and $w \in \mathbb{Z}$ such that $k_\sigma = 2\nu_\sigma + w$ for all $\sigma \in I$. Denote by $\nu := \sum_{\sigma\in I} \nu_\sigma \cdot \sigma \in \mathbb{Z}[I]$ and $w := \sum_{\sigma\in I} w \cdot \sigma = wN_{F/\mathbb{Q}} \in \mathbb{Z}[I]$.

We define an action of Δ on Ω^κ as follows: let f be a (local) section of Ω^κ on $M^{\mathrm{R}}(\mu_N, \mathfrak{c})$ and let ω be a (local) generator of ω_A as $\mathcal{O}_{M^{\mathrm{R}}(\mu_N,\mathfrak{c})} \otimes \mathcal{O}_F$-module.

Let us remark that if $\epsilon = \eta^2$, with $\eta \in U_N$ we have

$$f\big(A, \iota, \psi, \lambda, \omega\big) = f\big(A, \iota, \psi, \epsilon\lambda, \eta\omega\big) = \kappa(\eta^{-1}) f\big(A, \iota, \psi, \epsilon\lambda, \omega\big) = \nu(\epsilon^{-1}) f\big(A, \iota, \psi, \epsilon\lambda, \omega\big).$$

In other words if we define the action of $\mathcal{O}_F^{\times,+}$ on $\mathrm{H}^0(M^{\mathrm{R}}(\mu_N, \mathfrak{c}), \Omega^\kappa)$ by

$$(\epsilon \cdot f)\big(A, \iota, \psi, \lambda, \omega\big) := \nu(\epsilon) f\big(A, \iota, \psi, \epsilon^{-1}\lambda, \omega\big),$$

then the action of U_N^2 is trivial and therefore the action factors through the finite group Δ.

We define the projector $e\colon \mathrm{M}(\mu_N, \mathfrak{c}, \kappa = 2\nu + w) \longrightarrow \mathrm{M}(\mu_N, \mathfrak{c}, \kappa = 2\nu + w)$ by

$$e := \frac{1}{\#\Delta} \sum_{\epsilon\in\Delta} \epsilon.$$

Explicitly this gives

$$(ef)(A,\iota,\psi,\lambda,\omega) := \frac{1}{\#\Delta}\sum_{\epsilon\in\Delta}\nu(\epsilon)f(A,\iota,\psi,\epsilon^{-1}\lambda,\omega).$$

Let us remark that this projector is compatible with the action of Δ on $M(\mu_N,\mathfrak{c})$ defined above and does not depend on the way we write κ as $2\nu+w$.

Definition 1.1. — We define the K-vector space of classical modular forms for G of tame level μ_N, polarization $\mathfrak{c}$ and weight (ν,w) by: $\mathrm{M}^G\big(\mu_N,\mathfrak{c},(\nu,w)\big) := \mathrm{Im}(e)$.

More precisely, $f\in \mathrm{M}(\mu_N,\mathfrak{c},\kappa=2\nu+w)$ is a modular form for G if and only if it satisfies the descent condition: for every $(A,\iota,\psi,\lambda,\omega)$ as above we have

$$f(A,\iota,\psi,\epsilon\lambda,\omega)=\nu(\epsilon)f(A,\iota,\psi,\lambda,\omega),\ \text{for every } \epsilon\in\Delta,\ \text{i.e., } f \text{ is invariant by } \Delta.$$

In other words, $\mathrm{M}^G\big(\mu_N,\mathfrak{c},(\nu,w)\big) := \Big(\mathrm{M}(\mu_N,\mathfrak{c},\kappa)\Big)^{\Delta}$.

In more geometric terms, if we denote by $M^G(\mu_N,\mathfrak{c})_K$ the quotient stack $M^{\mathrm{R}}(\mu_N,\mathfrak{c})_K/\Delta$, then the morphism $\pi\colon M^{\mathrm{R}}(\mu_N,\mathfrak{c})_K \longrightarrow M^G(\mu_N,\mathfrak{c})_K$ is finite Galois with Galois group Δ. We have an action of Δ on Ω^κ defined above.

$$\mathrm{M}^G\big(\mu_N,\mathfrak{c},(\nu,w)\big) := \mathrm{H}^0\Big(M^G\big(\mu_N,\mathfrak{c}\big)_K,\Omega^{(\nu,w)}\Big),\ \text{where } \Omega^{(\nu,w)} := \Big(\pi_*\big(\Omega^\kappa\big)\Big)^{\Delta}.$$

Let now $F^{\times,+}$ be a group of totally positive, non-zero elements of F. This group acts on the set of pairs $(\mathfrak{c},\mathfrak{c}^+)$ by: $x(\mathfrak{c},\mathfrak{c}^+) := (x\mathfrak{c},x\mathfrak{c}^+)$. If $\kappa=2\nu+wN_{F/\mathbb{Q}}$, for $\nu\in\mathbb{Z}[I]$ and $w\in\mathbb{Z}$, we have an isomorphism:

$$L_{(x\mathfrak{c},\mathfrak{c})}\colon M^G\big(\mu_N,\mathfrak{c},(\nu,w)\big) \longrightarrow M^G\big(\mu_N,x\mathfrak{c},(\nu,w)\big)$$

given by

$$L_{(x\mathfrak{c},\mathfrak{c})}(f)\big(A,\iota,\psi,\lambda,\omega\big) := \nu(x)f\big(A,\iota,\psi,x^{-1}\lambda,\omega\big).$$

We notice that if $x\in\mathcal{O}_F^{\times,+}$ then $L_{(x\mathfrak{c},\mathfrak{c})}(f)=f$ for all $f\in\mathrm{M}^G\big(\mu_N,\mathfrak{c},(\nu,w)\big)$.

Definition 1.2. — We define the K-vector space of modular forms for G of tame level μ_N and weight (ν,w), $\mathrm{M}^G\big(\mu_N,(\nu,w)\big)$ to be

$$\Big(\bigoplus_{(\mathfrak{c},\mathfrak{c}^+)}\mathrm{M}^G\big(\mu_N,\mathfrak{c},(\nu,w)\big)\Big)/\Big(f-L_{(x\mathfrak{c},\mathfrak{c})}(f)\Big)_{x\in F^{\times,+}/\mathcal{O}_F^{\times,+}}.$$

By choosing representatives of $Cl^+(F)$ we have a non-canonical isomorphism

$$\mathrm{M}^G\big(\mu_N,(\nu,w)\big)\cong\bigoplus_{\mathfrak{c}\in Cl^+(F)}\mathrm{M}^G\big(\mu_N,\mathfrak{c},(\nu,w)\big),$$

which implies that $\mathrm{M}^G\big(\mu_N,(\nu,w)\big)$ is a finite dimensional K-vector space.

This ends the discussion on the descent from G^* to G for classical modular forms. To construct overconvergent and p-adic families of modular forms for G we define the descent from G^* to G for overconvergent and p-adic families of forms for G^*.

2. The weight spaces

Let us recall (see Section 1) that to a totally real number field F we associate two algebraic groups G and G^* and each one of these algebraic groups have classical modular forms attached to them. In order to define overconvergent and p-adic families of modular forms for these groups we will first define the associated weight spaces.

We start by defining the weight space for G^*. Denote by $\mathbb{T} := \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathbb{G}_m$; it is a smooth commutative algebraic group over $\mathbb{Z}$ of dimension the degree $[F:\mathbb{Q}]$, it is a torus over the open subscheme of $\mathrm{Spec}(\mathbb{Z})$ where the discriminant of F is invertible and the fiber of $\mathbb{T}$ at a prime p dividing the discriminant of F has toric part of dimension equal to the sum of the degrees over $\mathbb{F}_p$ of the residue fields of $\mathcal{O}_F$ at the primes above p; see [**1**].

The space of weights for classical modular forms for G^* is the group of characters of the torus $\mathbb{T}_K$ (we recall from Section 1 that K was a finite extension of $\mathbb{Q}_p$ which splits F; in particular $\mathbb{T}_K$ is a split torus.) The yoga of weight spaces leads us to define the weight space for G^*, denoted $\mathcal{W}$, to be the rigid analytic space over K associated to the completed group algebra $\mathcal{O}_K[\![\mathbb{T}(\mathbb{Z}_p)]\!]$. There is a universal character:

$$\kappa^{\mathrm{un}}\colon \mathbb{T}(\mathbb{Z}_p) \to \mathcal{O}_K[\![\mathbb{T}(\mathbb{Z}_p)]\!]^\times$$

and in particular $\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cont}}\big(\mathbb{T}(\mathbb{Z}_p), \mathbb{C}_p^\times\big)$.

We have an isomorphism

$$\mathbb{T}(\mathbb{Z}_p) \simeq H \times \mathbb{Z}_p^g$$

where H is the torsion subgroup of $\mathbb{T}(\mathbb{Z}_p)$. Let $H^\vee$ be the character group of H and $B(1,1^-)$ be the open unit ball of center 1. Then $\mathcal{W} \simeq H^\vee \times B(1,1^-)^g$.

A p-adic character $\kappa \in \mathcal{W}(K)$ is called algebraic if it is in $\mathbb{Z}[I]$. It is called locally algebraic (or arithmetic, classical...) if it is the product $\kappa^{alg}\kappa^{fin}$ of an algebraic character by a finite character. The decomposition is unique.

Let $\widehat{\mathbb{T}}$ be the formal group obtained by completing $\mathbb{T}$ along its unit section. For any $w \in v(\mathcal{O}_K)$, we let $\mathbb{T}_w^0$ be the formal subgroup of $\widehat{\mathbb{T}}$, of elements congruent to 1 modulo p^w. We let $\mathbb{T}_w$ be the formal subgroup of $\widehat{\mathbb{T}}$ generated by $\mathbb{T}_w^0$ and $\mathbb{T}(\mathbb{Z}_p)$.

Let $\widehat{\mathbb{G}}_m$ be the formal completion along the identity of $\mathbb{G}_m$. Let $\mathcal{U}$ be a rigid analytic affinoid space and $\mathcal{U} \to \mathcal{W}$ be a morphism of rigid spaces. Let A be the algebra of power bounded functions on $\mathcal{U}$. Let $\kappa^{\mathcal{U}}\colon \mathbb{T}(\mathbb{Z}_p) \to A^\times$ be the restriction of the universal character to $\mathcal{U}$. We say that $\kappa^{\mathcal{U}}$ is n-analytic for some $n \in \mathbb{N}$ if the restriction of $\kappa^{\mathcal{U}}$ to $\mathbb{T}_n^0(\mathbb{Z}_p)$ factors as the composite $\exp\circ\psi\circ\log_F$ where $\log_F$ is the p-adic logarithm

$$\log_F\colon 1 + p^n\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \to p^n\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p,$$

ψ is some $\mathbb{Z}_p$-linear map:

$$\psi\colon p^n\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \to pA$$

and exp is the p-adic exponential $pA \to 1+pA$. It is easy to see that the character $\kappa^{\mathcal{U}}$ is n-analytic for n large enough depending on $\mathcal{U}$.

If $\kappa^{\mathcal{U}}$ is n-analytic then it extends to a character

$$\kappa^{\mathcal{U}} \colon \mathbb{T}_w \times \mathrm{Spf}\ A \to \widehat{\mathbb{G}}_m \times \mathrm{Spf}\ A$$

for all $w \geq n$.

We will denote by $\chi^{\mathcal{U}}$ the restriction of $\kappa^{\mathcal{U}}$ to $\mathbb{T}_w^0$.

The following lemma is obvious.

Lemma 2.1. — *Let $n \in \mathbb{N}$ such that $\kappa^{\mathcal{U}}$ is n-analytic. Let $w \geq n$. The character $\chi^{\mathcal{U}}$ of $\mathbb{T}_w^0$ takes values in the formal subgroup of $\widehat{\mathbb{G}}_m$ of elements congruent to* 1 *modulo p.*

Now we define the weight space for G which will be denoted $\mathcal{W}^G$. We recall from Section 1 that the space of classical weights for G is the space {characters of the torus $\mathbb{T}_K$} $\times\, \mathbb{Z}$ with the map to the classical weights for G^* given by: $(\nu, w) \to \kappa := 2\nu + wN_{F/\mathbb{Q}}$.

It is therefore natural to define the weight space for G to be the rigid analytic space over K associated to the completed group algebra $\mathcal{O}_K[\![\mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times]\!]$.

We also define the natural morphism of rigid spaces $\rho\colon \mathcal{W}^G \longrightarrow \mathcal{W}$ associated to the morphism of groups $\mathbb{T}(\mathbb{Z}_p) \longrightarrow \mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ given by $t \to \big(t^2, N_{F/\mathbb{Q}}(t)\big)$. This map can be described on points by: if $(\nu, w) \in \mathcal{W}^G(\mathbb{C}_p)$, where $\nu\colon \mathbb{T}(\mathbb{Z}_p) \longrightarrow \mathbb{C}_p^\times$ and $w\colon \mathbb{Z}_p^\times \longrightarrow \mathbb{C}_p^\times$ are continuous homomorphisms, then $\rho(\nu, w)(t) := \nu(t)^2 \cdot (w \circ N_{F/\mathbb{Q}})(t)$, for $t \in \mathbb{T}(\mathbb{Z}_p)$.

The universal character for $\mathcal{W}^G$ will be denoted $\kappa^{\mathrm{un,G}}\colon \mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times \longrightarrow \big(\mathcal{O}_K[\![\mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times]\!]\big)^\times$.

3. Overconvergent modular forms for the group G^*

3.1. Hilbert modular varieties. — Let F denote a totally real number field of degree g over $\mathbb{Q}$ with ring of integers $\mathcal{O}_F$ and different ideal $\mathcal{D}_F$. Fix an integer $N \geq 4$ and a prime p not dividing N. Let $\mathfrak{c}$ be a fractional ideal of F and let $\mathfrak{c}^+$ be the cone of totally positive elements. Denote by $\mathfrak{P}_1, \ldots, \mathfrak{P}_f$ the prime ideals of $\mathcal{O}_F$ over p. Let $e_1, \ldots, e_f \in \mathbb{N}$ be the ramification indexes defined by $p\mathcal{O}_F = \prod_j \mathfrak{P}_j^{e_j}$. Let $f_1, \ldots, f_f$ be the residual degrees.

We fix a finite extension $\mathbb{Q}_p \subset K$ such that $F \otimes K$ splits completely. Let $M(\mu_N, \mathfrak{c})$ be the Hilbert modular scheme over $\mathcal{O}_K$ classifying triples $(A, \iota, \Psi, \lambda)$ consisting of: (1) an abelian scheme $A \to S$ of relative dimension g over S, (2) an embedding $\iota\colon \mathcal{O}_F \subset \mathrm{End}_S(A)$, (3) a closed immersion $\Psi\colon \mu_N \otimes \mathcal{D}_F^{-1} \to A$ compatible with $\mathcal{O}_F$-actions, (4) if $P \subset \mathrm{Hom}_{\mathcal{O}_F}(A, A^\vee)$ is the sheaf for the étale topology on S of symmetric $\mathcal{O}_F$-linear homomorphisms from A to the dual abelian scheme $A^\vee$ and if $P^+ \subset P$ is the subset of polarizations, then λ is an isomorphism of étale sheaves $\lambda\colon (P, P^+) \cong (\mathfrak{c}, \mathfrak{c}^+)$, as invertible $\mathcal{O}_F$-modules with a notion of positivity. The triple is subject to the condition that the map $A \otimes_{\mathcal{O}_F} \mathfrak{c} \to A^\vee$ is an isomorphism of abelian schemes (the so called Deligne-Pappas condition).

We write $\overline{M}(\mu_N,\mathfrak{c})$ and $\overline{M}^*(\mu_N,\mathfrak{c})$ for a projective toroidal compactification, respectively the minimal or Satake compactification of $M(\mu_N,\mathfrak{c})$. They are Cohen-Macaulay flat and projective schemes over $\mathcal{O}_K$. Moreover $\overline{M}(\mu_N,\mathfrak{c})$ admits a versal semiabelian scheme A with real multiplication by $\mathcal{O}_F$. We let $D=\overline{M}(\mu_N,\mathfrak{c})\backslash M(\mu_N,\mathfrak{c})$ be the boundary divisor in the toroidal compactification. See [**11**] for details.

There exists a greatest open subscheme $\overline{M}^{\mathrm{R}}(\mu_N,\mathfrak{c})\subset\overline{M}(\mu_N,\mathfrak{c})$ such that ω_A, the conormal sheaf to the identity of A, is an invertible $\mathcal{O}_{\overline{M}^{\mathrm{R}}(\mu_N,\mathfrak{c})}\otimes_{\mathbb{Z}}\mathcal{O}_F$-module (the so called Rapoport condition). The complement is empty if p does not divide the discriminant of F and, in general, it is of codimension 2 in the fiber of $M(\mu_N,\mathfrak{c})$ over the closed point of $\mathrm{Spec}(\mathcal{O}_K)$. Let $\mathbb{T}$ be the algebraic group $\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}(\mathbb{G}_m)$ over $\mathcal{O}_K$. Let I be the set of all embeddings of F into K (recall that K is chosen to split F). Then $\mathbb{Z}[I]$ is the character group of the torus $\mathbb{T}|_K$. Let $\kappa=\sum_{\sigma\in I}k_\sigma.\sigma$. To such a κ on can associate an invertible modular sheaf Ω^κ over $\overline{M}^{\mathrm{R}}(\mu_N,\mathfrak{c})$ by setting

$$\Omega^\kappa=\bigotimes_{\sigma\in I}\omega_{G,\sigma}^{\otimes k_\sigma}$$

where $\omega_{G,\sigma}=\omega_G\otimes_{\mathcal{O}_K\otimes_{\mathbb{Z}}\mathcal{O}_F,1\otimes\sigma}\mathcal{O}_K$.

Its global sections $\mathrm{H}^0(\overline{M}^{\mathrm{R}}(\mu_N,\mathfrak{c})_K,\Omega^\kappa)=\mathrm{M}(\mu_N,\mathfrak{c},\kappa)$ is the K-vector space of $\mathfrak{c}$-polarized, tame level N, weight κ Hilbert modular forms (let us recall: for the group G^*). We also let $\mathrm{H}^0(\overline{M}(\mu_N,\mathfrak{c})_K,\Omega^\kappa(-D))=\mathrm{S}(\mu_N,\mathfrak{c},\kappa)$ be the submodule of cusp forms.

3.2. The canonical subgroup theory

3.2.1. *Strict neighborhoods of the ordinary locus.* — The Hasse invariant on the reduction $\overline{M}(\mu_N,\mathfrak{c})_{\mathbb{F}_p}$ of $\overline{M}(\mu_N,\mathfrak{c})$ modulo p is the product of f-partial Hasse invariants $h_{\mathfrak{P}_1},\dots,h_{\mathfrak{P}_f}$ according to the decomposition of the Hodge bundle induced by the idempotents of $\mathfrak{e}_{\mathfrak{P}_1},\dots,\mathfrak{e}_{\mathfrak{P}_f}\in\mathcal{O}_F\otimes\mathbb{Z}_p$ associated to the prime ideals $\mathfrak{P}_1,\dots,\mathfrak{P}_f$ over p. Indeed let $\overline{A}\to\overline{M}(\mu_N,\mathfrak{c})_{\mathbb{F}_p}$ be the versal semiabelian scheme in characteristic p. Then the pull back $\omega_{\overline{A}}$ of the differentials via the 0-section is a locally free $\mathcal{O}_{\overline{M}(\mu_N,\mathfrak{c})_{\mathbb{F}_p}}$-module of rank g which we can decompose into a direct sum of locally free $\mathcal{O}_{\overline{M}(\mu_N,\mathfrak{c})_{\mathbb{F}_p}}$-modules:

$$\omega_{\overline{A}}=\bigoplus_{i=1}^{f}\omega_{\overline{A},i},\qquad \omega_{\overline{A},i}:=\mathfrak{e}_{\mathfrak{P}_i}\cdot\omega_{\overline{A}}.$$

As the $\mathcal{O}_F$-action commutes with the Vershiebung isogeny $\mathrm{V}\colon\overline{A}^{(p)}\to\overline{A}$, such decomposition is preserved by V. For every $i=1,\dots,r$ we let $h_{\mathfrak{P}_i}$ be the global section of $\det\omega_{\overline{A},i}^{p-1}$ given by the determinant of the induced map $\mathrm{V}_i^*:\omega_{\overline{A},i}\to\omega_{\overline{A},i}^{(p)}$. In particular $\prod_{i=1}^{f}h_{\mathfrak{P}_i}$ is the Hasse invariant.

Remark 3.1. — Over the Rapoport locus $\overline{M}^{\mathrm{R}}(\mu_N,\mathfrak{c})_{\mathbb{F}_p}$ one could consider refinements of the Hasse invariants $h_{\mathfrak{P}_i}$ according to the emebddings of $\mathcal{O}_F/\mathfrak{P}_i$ in the residue field of $\mathcal{O}_K$; see [**1**, § 7.11 & Def. 7.12] for details. In particular if e_i is the ramification index of $\mathfrak{P}_i$ and f_i is its residue degree then $h_{\mathfrak{P}_i}$ is the e_i-th power of the product of f_i partial Hasse invariants constructed in loc. cit. The advantage of our approach is that we can work over the whole $\overline{M}(\mu_N,\mathfrak{c})_{\mathbb{F}_p}$ and not simply over the Rapoport locus.

We let $\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})$ and $\overline{\mathfrak{M}}^*(\mu_N,\mathfrak{c})$ be the formal completions of $\overline{M}(\mu_N,\mathfrak{c})$ and $\overline{M}^*(\mu_N,\mathfrak{c})$ along their special fibers. We let $\overline{\mathcal{M}}(\mu_N,\mathfrak{c})$ and $\overline{\mathcal{M}}^*(\mu_N,\mathfrak{c})$ be, respectively, their rigid analytic fibers and we let $\overline{\mathcal{M}}(\mu_N,\mathfrak{c})^{ord}$ and $\overline{\mathcal{M}}^*(\mu_N,\mathfrak{c})^{ord}$ be the ordinary open subsets. It is of crucial importance for us that $\overline{\mathcal{M}}^*(\mu_N,\mathfrak{c})^{ord}$ is affinoid.

Given a multi-index $\underline{v}=(v_1,\dots,v_f)\in\mathbb{Q}^f_{>0}$ with $0\le v_i<\frac{1}{p}$, let $\overline{\mathcal{M}}(\mu_N,\mathfrak{c})(\underline{v})$ and respectively $\overline{\mathcal{M}}^*(\mu_N,\mathfrak{c})(\underline{v})$ be the neighborhoods of the ordinary locus of $\overline{\mathcal{M}}(\mu_N,\mathfrak{c})$ and $\overline{\mathcal{M}}^*(\mu_N,\mathfrak{c})$ defined by the conditions $|h_{\mathfrak{P}_i}|\le p^{v_i}$. Away from the boundary, they classify abelian schemes B whose reduction B_1 modulo p satisfy $|h_{\mathfrak{P}_i}(B_1)|\le p^{v_i}$. We set $\mathrm{Hdg}_{\mathfrak{P}_i}B:=\log_p|h_{\mathfrak{P}_i}(B_1)|$.

Let $\overline{\mathfrak{M}}'(\mu_N,\mathfrak{c})(\underline{v})$ be the formal model of $\overline{\mathcal{M}}(\mu_N,\mathfrak{c})(\underline{v})$, obtained by taking iterated blow-ups along the ideals $(h_{\mathfrak{P}_i},p^{v_i})$ of $\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})$ and by removing all the divisors at infinity. We let $\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})(\underline{v})$ be the normalization of $\overline{\mathfrak{M}}'(\mu_N,\mathfrak{c})(\underline{v})$ in $\overline{\mathcal{M}}(\mu_N,\mathfrak{c})(\underline{v})$. We also define in a similar fashion $\overline{\mathfrak{M}}^*(\mu_N,\mathfrak{c})(\underline{v})$ which is a formal model of $\overline{\mathcal{M}}^*(\mu_N,\mathfrak{c})(\underline{v})$. Let $\mathfrak{M}(\mu_N,\mathfrak{c})(\underline{v})$ be the open formal subscheme where the semi-abelian scheme is an abelian scheme.

Lemma 3.2. — *For every normal $\mathcal{O}_K$-algebra, p-adically complete and separated and topologically of finite type R the R-valued points of $\mathfrak{M}(\mu_N,\mathfrak{c})(\underline{v})$ are in one to one correspondence with isomorphism classes of*

a) *abelian schemes A over R of relative dimension g with real multiplication by $\mathcal{O}_F$ such that* $\mathrm{Hdg}_{\mathfrak{P}_i}A\le v_i$ *for $i=1,\dots,f$;*
b) *an isomorphism $\lambda\colon(P,P^+)\cong(\mathfrak{c},\mathfrak{c}^+)$, as in §3.1, so that the Deligne-Pappas condition holds;*
c) *a closed immersion $\Psi\colon\mu_N\otimes\mathcal{D}_F^{-1}\to A$ compatible with $\mathcal{O}_F$-actions;*

3.2.2. *The canonical subgroup.* — Let $n\in\mathbb{N}$ and $\underline{v}=(v_i)$ be a multi-index with $v_i\le\frac{1}{p^n}$. Under this assumption, it follows from **[17]** that there is a canonical subgroup

$$H_n\hookrightarrow A[p^n]|_{\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})(\underline{v})}.$$

This group has rank p^{ng}. Over K, this is an étale group scheme, locally isomorphic to $\mathbb{Z}/p^{ng}\mathbb{Z}$.

Lemma 3.3. — *The group H_n is stable under the action of $\mathcal{O}_F$. Moreover, over* Spec K, *it is locally (étale) isomorphic as an $\mathcal{O}_F$-module to $\mathcal{O}_F/p^n\mathcal{O}_F$.*

Proof. — From the functoriality of the canonical subgroup, it follows that it is stable under $\mathcal{O}_F$. To check the remaining properties, let $\mathcal{O}_L$ be a valuation ring, extension of $\mathcal{O}_K$. Let $x\colon\mathrm{Spec}\ \mathcal{O}_L\to\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})(\underline{v})$ be an $\mathcal{O}_L$-point and let A_x be the pull-back of A to x.

For every finite and flat group scheme $\mathcal{G}$ over $\mathcal{O}_L$, the $\mathcal{O}_L$-module of invariant differentials $\omega_{\mathcal{G}}$ of $\mathcal{G}$ is torsion of finite presentation and, thus, isomorphic to $\bigoplus_{i=1}^r \mathcal{O}_L/a_i$. Define the degree $\deg \mathcal{G}$ of $\mathcal{G}$ to be $\sum_i v(a_i)$ where $v\colon L^\times \to \mathbb{R}$ is the valuation normalized by $v(p)$. The function degree is additive, see [**16**]. Write $D = \bigoplus_{i=1}^f D_i$ for the p-divisible group associated to A_x, decomposed according to the primes above p in $\mathcal{O}_F$. Each D_i is a p-divisible group of dimension $e_i f_i$. By functoriality of the canonical subgroup we have $H_n = \bigoplus_i H_{n,i}$ where each $H_{n,i}$ is the canonical subgroup of D_i of level n of D_i. The degree of $H_{n,i}$ is $\deg D_i[p^n] - \deg D_i[p^r]/H_{n,i}$ by additivity. Since D_i is a p-divisible group we have $\deg D_i[p^n] = ne_i f_i$ and $\deg D_i[p^n]/H_{n,i} = \frac{p^n-1}{p-1}\mathrm{Hdg}_{\mathfrak{P}_i} A_x \le \frac{p^n-1}{p-1} v_i$ by [**16**, Thm. 6]. Thus $\deg H_{n,i} \ge ne_i f_i - \frac{p^n-1}{p-1}v_i \ge ne_i f_i - 1$. On the other hand, D_i is a flat $\mathcal{O}_{F_{\mathfrak{P}_i}}$-module. In particular $\deg D_i[\mathfrak{P}_i^r] = rf_i$. It follows that $\deg H_{n,i} \ge (ne_i - 1)f_i = \deg D_i[\mathfrak{P}_i^{ne_i-1}]$. Therefore $H_{n,i}$, which is a subgroup of $D_i[p^n] = D_i[\mathfrak{P}_i^{ne_i}]$, is not contained in $D_i[\mathfrak{P}_i^{ne_i-1}]$. Since $D_i[\mathfrak{P}_i^{ne_i}]$ over an algebraic closure $\overline{K}$ of K is finite and free of rank 2 as a $\mathcal{O}_F/\mathfrak{P}_i^{ne_i}$-module and $H_{n,i,\overline{K}}$ is free of rank $e_i f_i$ as a $\mathbb{Z}/p^n\mathbb{Z}$-module, then $H_{n,i,\overline{K}}$ has to be free of rank 1 as an $\mathcal{O}_F/\mathfrak{P}_i^{ne_i}$-module as claimed. □

For every $n \in \mathbb{N}$ and every $\underline{v} = (v_i)$ with $v_i < \frac{1}{p^n}$, we define

$$\overline{\mathcal{M}}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v}) := \mathrm{Isom}_{\overline{\mathcal{M}}(\mu_N,\mathfrak{c})(\underline{v})}\big(H_n, \mu_{p^n} \otimes_{\mathbb{Z}} \mathcal{D}_F^{-1}\big).$$

It is a finite, étale and Galois covering of $\overline{\mathcal{M}}(\mu_N, \mathfrak{c})(\underline{v})$, with Galois group $\mathbb{T}(\mathbb{Z}/p^n\mathbb{Z})$. We denote by $\overline{\mathfrak{M}}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v})$ the normalization of $\overline{\mathfrak{M}}(\mu_N, \mathfrak{c})(\underline{v})$ in $\overline{\mathcal{M}}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v})$.

3.3. The sheaf $\mathcal{F}$.

— Let $n \in \mathbb{N}$ and $\underline{v} = (v_i)$ a multi-index satisfying $v_i < \frac{1}{p^n}$. We have the following result from [**2**], prop. 4.3.1 :

Proposition 3.4. — *There is a unique subsheaf $\mathcal{F}$ of ω_A which is locally free of rank 1 as $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})}$-module and contains $p^{\frac{\sup\{v_i\}}{p-1}}\omega_A$. Moreover, for all $w \in]0, n - \sup\{v_i\}\frac{p^n}{p-1}]$ we have $\mathcal{O}_F$-linear maps:*

$$\mathrm{HT}_w\colon H_n^D \to h_n^*(\mathcal{F})/p^w h_n^*(\mathcal{F})$$

which induce isomorphisms:

$$\mathrm{HT}_w \otimes 1\colon H_n^D \otimes_{\mathbb{Z}} \mathcal{O}_{\overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})}/(p^w) \xrightarrow{\sim} h_n^*(\mathcal{F})/p^w h_n^*(\mathcal{F}).$$

Proof. — All statements except the fact that $\mathcal{F}$ is locally free of rank 1 as $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})}$-module, are proven in [**2**]. The fact that $\mathcal{F} \subset \omega_G$ is stable under the action of $\mathcal{O}_F$ and HT_w is $\mathcal{O}_F$-equivariant follow from functoriality. As $H_n^D \cong \mathcal{O}_F/p^n\mathcal{O}_F$ by Lemma 3.3 we conclude that $\mathcal{F}/p^w\mathcal{F}$ is free of rank 1 as $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})}/(p^w)$-module. As $\mathcal{F}$ is a coherent sheaf, we conclude that it is locally free of rank 1 as $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})}$-module as claimed. □

3.4. The modular sheaves. — We keep $\underline{v}$ and n as in the previous section and we choose $w \in]n-1, n - \sup\{v_i\}\frac{p^n}{p-1}]$. Define $\gamma_w\colon \mathfrak{IW}_w^+ \longrightarrow \overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})$ to be the formal affine morphism defined as follows. For every normal p-adically complete and separated, flat $\mathcal{O}_K$-algebra R and for every morphism $\gamma\colon \mathrm{Spf}(R) \to \overline{\mathfrak{M}}(\mathcal{O}_K,\Gamma(p^n),\mu_N,\mathfrak{c})(\underline{v})$, its R-valued points over γ classify isomorphisms $\alpha\colon \gamma^*(\mathcal{F}) \cong \mathcal{O}_F \otimes_{\mathbb{Z}} R$ as $\mathcal{O}_F \otimes_{\mathbb{Z}} R$-modules such that α modulo p^w composed with HT_w sends $1 \in \mathcal{O}_F/p^n\mathcal{O}_F \cong H_n^D(R)$ (isomorphism defined by the level $\Gamma_1(p^n)$-structure and the fact that $H_n^D(R) = H_n^D(R_K)$ by the normality of R) to $1 \in \mathcal{O}_F \otimes_{\mathbb{Z}} R/p^wR$, i.e.,

$$\mathcal{O}_F/p^n\mathcal{O}_F \cong H_n^D(R) \xrightarrow{\mathrm{HT}_w} \gamma^*(\mathcal{F})/p^w\gamma^*(\mathcal{F}) \xrightarrow{\alpha} \mathcal{O}_F \otimes_{\mathbb{Z}} R/p^wR$$

sends $1 \mapsto 1$.

Lemma 3.5. — *The formal scheme $\mathfrak{IW}_w^+$ is a formal torsor over $\overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})$ under the subgroup $\mathbb{T}_w^0$ of $\widehat{\mathrm{Res}}_{\mathcal{O}_F/\mathbb{Z}}(\mathbb{G}_m)$ of units congruent to 1 modulo p^w.*

Proof. — There is certainly an action of $\mathbb{T}_w^0$ on $\mathfrak{IW}_w^+$. Given a map of p-adic formal schemes $\gamma\colon \mathrm{Spf}(R) \to \overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})$ then $\gamma^*(\mathcal{F})$ is free of rank 1 as $\mathcal{O}_F \otimes_{\mathbb{Z}} R$-module with generator given by any element f reducing to $\gamma^*\big(\mathrm{HT}_w(1)\big)$ modulo p^w. Then, any other generator with these properties lies in $\mathbb{T}_w^0(R)f$ as claimed. □

Recall that $\mathbb{T}_w$ is the subgroup of the formal group $\widehat{\mathrm{Res}}_{\mathcal{O}_F/\mathbb{Z}}(\mathbb{G}_m)$ whose R-valued points are given by the inverse image of $\mathbb{T}(\mathbb{Z}/p^n\mathbb{Z}) = (\mathcal{O}_F/p^n\mathcal{O}_F)^\times$ via the projection

$$\widehat{\mathrm{Res}}_{\mathcal{O}_F/\mathbb{Z}}(\mathbb{G}_m)(R) = \big(\mathcal{O}_F \otimes_{\mathbb{Z}} R\big)^\times \longrightarrow \mathcal{O}_F \otimes_{\mathbb{Z}} R/p^wR.$$

As $\mathbb{T}(\mathbb{Z}/p^n\mathbb{Z})$ acts on $\overline{\mathfrak{M}}(\mathcal{O}_K,\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})$ via its action on the isomorphism $\mathcal{O}_F/p^n\mathcal{O}_F \cong H_n^D(R)$ the action of $\mathbb{T}_w^0$ extends to an action of $\mathbb{T}_w$ on

$$\pi_w\colon \mathfrak{IW}_w^+ \xrightarrow{\gamma_w} \overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v}) \xrightarrow{h_n} \overline{\mathfrak{M}}(\mathcal{O}_K,\mu_N,\mathfrak{c})(\underline{v}).$$

Definition 3.6. — For every n-analytic weight $\kappa \in \mathcal{W}(K)$ (see §2) define the sheaf of $\mathcal{O}_{\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})(\underline{v})}$-modules

$$\Omega_w^\kappa := \pi_{w,*}\big(\mathcal{O}_{\mathfrak{IW}_w^+}\big)[-\kappa].$$

More precisely, it is the subsheaf of $\pi_{w,*}\big(\mathcal{O}_{\mathfrak{IW}_w^+}\big)$ on which $\mathbb{T}_w$ acts via the character $-\kappa\colon \mathbb{T}_w \to \widehat{\mathbb{G}}_m$.

For technical reasons, it is also useful to introduce an intermediate sheaf.

Definition 3.7. — Define the sheaf of $\mathcal{O}_{\overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})}$-modules

$$\Omega_w^\chi := \gamma_{w,*}\big(\mathcal{O}_{\mathfrak{IW}_w^+}\big)[-\chi].$$

More precisely, it is the subsheaf of $\gamma_{w,*}\big(\mathcal{O}_{\mathfrak{IW}_w^+}\big)$ on which $\mathbb{T}_w^0$ acts via the character $-\chi\colon \mathbb{T}_w^0 \to \widehat{\mathbb{G}}_m$, where we recall that χ is the restriction of κ to $\mathbb{T}_w^0$.

Let us clarify the relation between these sheaves. There is an inclusion $\Omega_w^\kappa \hookrightarrow h_{n,*}\Omega_w^\chi$ of sheaves over $\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})(\underline{v})$. The group $\mathbb{T}_w$ acts on $h_{n,*}\Omega_w^\chi$. Let us twist the action by κ. Then the twisted action factors through the finite group $\mathbb{T}(\mathbb{Z}/p^n\mathbb{Z})$ and we obtain Ω_w^κ as the subsheaf of invariants of $h_{n,*}\Omega_w^\chi(\kappa)$ under $\mathbb{T}(\mathbb{Z}/p^n\mathbb{Z})$.

Proposition 3.8. — *The sheaf Ω_w^κ is a coherent sheaf of $\mathcal{O}_{\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})(\underline{v})}$-modules. Moreover, if $n-1 < w' \leq w < n - \sup\{v_i\}\frac{p^n}{p-1}$, there is a canonical isomorphism $\Omega_{w'}^\kappa \to \Omega_w^\kappa$. Its restriction to the rigid analytic generic fiber it is an invertible $\mathcal{O}_{\overline{\mathcal{M}}(\mu_N,\mathfrak{c})(\underline{v})}$-module.*

Proof. — The sheaf Ω_w^χ is an invertible sheaf of $\mathcal{O}_{\overline{\mathfrak{M}}(\mathcal{O}_K,\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})}$-modules, since the map γ_w is a $\mathbb{T}_w^0$-torsor.

As Ω_w^κ equals the subsheaf of invariants under $\mathbb{T}(\mathbb{Z}/p^n\mathbb{Z})$ of the sheaf with twisted action $h_{n,*}\Omega_w^\chi(\kappa)$, it follows that Ω_w^κ is coherent. Since h_n is finite étale with group $\mathbb{T}(\mathbb{Z}/p^n\mathbb{Z})$ after inverting p, we deduce that Ω_w^κ is invertible over $\overline{\mathcal{M}}(\mu_N,\mathfrak{c})(\underline{v})$.

Let $n-1 < w' \leq w < n - v\frac{p^n}{p-1}$. There is a canonical commutative diagram:

$$\begin{array}{ccc} \mathfrak{IW}_w^+ & \xrightarrow{\ i\ } & \mathfrak{IW}_{w'}^+ \\ \downarrow{\scriptstyle \gamma_w} & \swarrow{\scriptstyle \gamma_{w'}} & \\ \overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c}). & & \end{array}$$

Moreover, γ_w is a $\mathbb{T}_w^0$-torsor, $\gamma_{w'}$ is a $\mathbb{T}_{w'}^0$-torsor and the map i and the actions are compatible with the canonical map $\mathbb{T}_w^0 \to \mathbb{T}_{w'}^0$. It follows that the canonical map $\Omega_{w'}^\chi \to \Omega_w^\chi$ is an isomorphism and the proposition follows. □

Because the sheaf Ω_w^κ does not depend on w, but only on n we simply denote it Ω_n^κ. Similarly, Ω_w^χ does not depend on w and is denoted by Ω^χ (here the dependence on n is expressed in the fact that Ω^χ is a sheaf of $\overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})$. We now study the dependence on n of our sheaves.

Proposition 3.9. — *Let $n \leq n'$. Assume that $v_i \leq \frac{1}{p^{n'}}$. Assume that κ is n-analytic. There is a canonical map of sheaves:*

$$\Omega_{n'}^\kappa \to \Omega_n^\kappa$$

over $\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})$ which is an isomorphism of invertible sheaves over $\overline{\mathcal{M}}(\mu_N,\mathfrak{c})$.

Proof. — Let us take $n-1 < w < n - \sup\{v_i\}\frac{p^n}{p-1}$ and $n-1 < w' < n - \sup\{v_i\}\frac{p^n}{p-1}$. We have a commutative diagram:

$$\begin{array}{ccc} \mathfrak{IW}^+_{w'} & \longrightarrow & \mathfrak{IW}^+_w \\ \downarrow & & \downarrow \\ \overline{\mathfrak{M}}(\Gamma_1(p^{n'}), \mu_N, \mathfrak{c}) & \longrightarrow & \overline{\mathfrak{M}}(\Gamma_1(p^{n'}), \mu_N, \mathfrak{c}) \\ \downarrow & \swarrow & \\ \overline{\mathfrak{M}}(\mu_N, \mathfrak{c}). & & \end{array}$$

The diagram is equivariant under the actions of $\mathbb{T}_{w'}$ on $\mathfrak{IW}^+_{w'}$, of $\mathbb{T}_w$ on $\mathfrak{IW}^+_w$ with respect to the map $\mathbb{T}_{w'} \to \mathbb{T}_w$. We thus get a natural map:

$$\Omega^\kappa_{n'} \to \Omega^\kappa_n.$$

At the level of the rigid analytic generic fibers, $\mathfrak{IW}^+_{w'}$ and $\mathfrak{IW}^+_w$ become torsors under the rigid analytic groups associated to $\mathbb{T}_{w'}$ and $\mathbb{T}_w$. So the above map of sheaves becomes clearly an isomorphism. □

In the sequel of this paper we always write Ω^κ instead of Ω^κ_n but the reader should keep in mind that the integral structure of the sheaf Ω^κ depends on n.

Corollary 3.10. — *Let $\kappa \in \mathcal{W}(K)$ be a classical weight corresponding to an algebraic character $\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathbb{G}_m \to \mathbb{G}_m$. Then the sheaf Ω^κ constructed in this section on $\overline{\mathcal{M}}(\mu_N, \mathfrak{c})(\underline{v})$ is the classical modular sheaf of weight κ.*

Proof. — An algebraic weight is 0-analytic. Setting $n = 0$ above, one recovers the classical construction. □

Definition 3.11. — Define $\mathrm{M}(\mu_N, \mathfrak{c}, \kappa)(\underline{v}) := \mathrm{H}^0\big(\overline{\mathcal{M}}(\mu_N, \mathfrak{c})(\underline{v}), \Omega^\kappa\big)$ to be the K-Banach space of Hilbert overconvergent modular forms of tame level N, $\mathfrak{c}$-polarization, weight κ and degree of overconvergence $\underline{v}$. Define

$$\mathrm{M}(\mu_N, \mathfrak{c}, \kappa)^\dagger := \lim_{\underline{v} \to 0^+} \mathrm{M}(\mu_N, \mathfrak{c}, \kappa)(\underline{v})$$

to be the K-Frechet space of overconvergent modular forms of tame level N and weight κ.

We recall that the notion of "Hilbert overconvergent modular form" in Definition 3.11 refers to overconvergent modular forms for the group G^*.

3.5. Families of modular sheaves. — Let $\mathcal{U}$ be an affinoid rigid space together with a morphism of rigid spaces $\mathcal{U} \to \mathcal{W}$. Let $n \in \mathbb{N}$ such that the character $\kappa^{\mathcal{U}}$ is n-analytic. Let $\mathfrak{U} = \mathrm{Spf}\ A$ be a formal model of $\mathcal{U}$, where A consists of power bounded functions on $\mathcal{U}$. Let $v \leq \frac{1}{p^n}$ and $w \in]n-1, n - \sup\{v_i\}\frac{p^n}{p-1}[$. Recall that we have a universal character over $\mathfrak{U}$,

$$\kappa^{\mathcal{U}} \colon \mathbb{T}_w \times \mathfrak{U} \to \widehat{\mathbb{G}}_m \times \mathfrak{U}.$$

Definition 3.12. — Define the sheaf of $\mathscr{O}_{\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})(\underline{v})\times\mathfrak{U}}$-modules

$$\Omega^{\kappa^{\mathcal{U}}} := \pi_{w,*}\big(\mathscr{O}_{\mathfrak{IW}_w^+\times\mathfrak{U}}\big)[-\kappa^{\mathcal{U}}].$$

as the subsheaf of $\pi_{w,*}\big(\mathscr{O}_{\mathfrak{IW}_w^+\times\mathfrak{U}}\big)$ on which the formal group $\mathbb{T}_w$ over $\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})(\underline{v})\times\mathfrak{U}$ acts via the universal character $-\kappa^{\mathcal{U}}\colon \mathbb{T}_w\times\mathfrak{U}\to\widehat{\mathbb{G}}_m\times\mathfrak{U}$.

We also have an intermediate sheaf of $\mathscr{O}_{\overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})\times\mathfrak{U}}$-modules

$$\Omega^{\chi^{\mathcal{U}}} := \gamma_{w,*}\big(\mathscr{O}_{\mathfrak{IW}_w^+\times\mathfrak{U}}\big)[-\chi^{\mathcal{U}}].$$

We have an inclusion $\Omega^{\kappa^{\mathcal{U}}}\hookrightarrow h_{n,*}\Omega^{\chi^{\mathcal{U}}}$ of sheaves over $\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})(\underline{v})\times\mathfrak{U}$. We define $\Omega^{\kappa^{\mathcal{U}}}$ as the subsheaf of invariants of $h_{n,*}\Omega^{\chi^{\mathcal{U}}}(\kappa^{\mathcal{U}})$ under $\mathbb{T}(\mathbb{Z}/p^n\mathbb{Z})$.

The following proposition is a straightforward generalization of Propositions 3.8 and 3.9.

Proposition 3.13. — *The sheaf $\Omega^{\kappa^{\mathcal{U}}}$ is a coherent sheaf of $\mathscr{O}_{\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})(\underline{v})\times\mathfrak{U}}$-modules. Its definition does not depend on w. The restriction of $\Omega^{\kappa^{\mathcal{U}}}$ to the rigid analytic fiber $\overline{\mathcal{M}}(\mu_N,\mathfrak{c})(\underline{v})\times\mathcal{U}$ is invertible and independent of n.*

Define $\mathrm{M}(\mu_N,\mathfrak{c},\kappa^{\mathcal{U}})(\underline{v}) := \mathrm{H}^0\big(\overline{\mathcal{M}}(\mu_N,\mathfrak{c})(\underline{v})\times\mathcal{U},\Omega^{\kappa^{\mathcal{U}}}\big)$ to be the $A[1/p]$-module of Hilbert overconvergent modular forms of tame level N, weight parametrized by $\mathcal{U}$ and degree of overconvergence $\underline{v}$. Set $\mathrm{M}(\mu_N,\mathfrak{c},\kappa^{\mathcal{U}})^\dagger := \lim_w \mathrm{M}(\mu_N,\mathfrak{c},\kappa^{\mathcal{U}})(\underline{v})$.

Let $\mathcal{U}'$ be a second rigid space with a map $f\colon\mathcal{U}'\to\mathcal{U}$. Let $\mathfrak{U}'=\operatorname{Spf} A'$ be the formal model where A' is the algebra of power bounded functions on $\mathcal{U}'$. Let $\kappa^{\mathcal{U}'}$ be the universal character over $\mathfrak{U}'$.

Proposition 3.14. — *There is a canonical isomorphism*

$$f^*\big(\Omega^{\kappa^{\mathcal{U}}}\big)\to\Omega^{\kappa^{\mathcal{U}'}}$$

of invertible sheaves over $\overline{\mathcal{M}}(\mu_N,\mathfrak{c})(\underline{v})\times\mathcal{U}'$.

Proof. — Consider the following cartesian diagram:

$$\begin{array}{ccc}
\mathfrak{IW}_w^+\times\mathfrak{U}' & \xrightarrow{\quad 1\times f\quad} & \mathfrak{IW}_{w'}^+\times\mathfrak{U}\\
\big\downarrow{\scriptstyle \gamma_w\times 1} & & \big\downarrow{\scriptstyle \gamma_w\times 1}\\
\overline{\mathfrak{M}}(\mathscr{O}_K,\Gamma_1(p^n),\mu_N,\mathfrak{c})\times\mathfrak{U}' & \xrightarrow{1\times f} & \overline{\mathfrak{M}}(\mathscr{O}_K,\Gamma_1(p^n),\mu_N,\mathfrak{c})\times\mathfrak{U}
\end{array}$$

where the vertical maps are torsors under $\mathbb{T}_w^0$. We deduce immediately that we have an isomorphism

$$f^*\Omega^{\chi^{\mathcal{U}}}\to\Omega^{\chi^{\mathcal{U}'}}$$

of invertible sheaves over $\overline{\mathfrak{M}}(\mathscr{O}_K,\Gamma_1(p^n),\mu_N,\mathfrak{c})\times\mathfrak{U}'$.

Let $h_n\colon\overline{\mathfrak{M}}(\mathscr{O}_K,\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})\to\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})(\underline{v})$ be the finite, generically étale map. Pushing forward the previous isomorphism on $\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})(\underline{v})\times\mathfrak{U}'$ by $h_n\otimes 1$, passing to the generic fiber and decomposing according to the action of $\mathbb{T}(\mathbb{Z}/p^n\mathbb{Z})$ we finish the proof. □

Remark 3.15. — Note that this base change property may not hold at an integral level. The eventual obstruction lies in the cohomology of the finite group $\mathbb{T}(\mathbb{Z}/p^n\mathbb{Z})$.

In particular, for every $\kappa \in \mathcal{U}(K)$ we get specialization maps

$$\mathrm{M}(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})^\dagger \longrightarrow \mathrm{M}(\mu_N, \mathfrak{c}, \kappa)^\dagger.$$

3.6. The specialization map for cusp forms. — Let $\mathcal{U}$ be an affinoid rigid space, with a morphism of rigid spaces $\mathcal{U} \to \mathcal{W}$. Let A be the algebra of power bounded functions on $\mathcal{U}$. Recall that for every multi-index $\underline{v}$ sufficiently small, we have defined a coherent sheaf $\Omega^{\kappa^{\mathcal{U}}}$ over $\overline{\mathfrak{M}}(\mu_N, \mathfrak{c})(\underline{v}) \times \mathrm{Spf}\ A$. We let D be the boundary divisor in $\overline{M}(\mu_N, \mathfrak{c})$. We let

$$\mathrm{S}(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})(\underline{v}) = \mathrm{H}^0(\overline{\mathcal{M}}(\mu_N, \mathfrak{c})(\underline{v}) \times \mathcal{U}, \Omega^{\kappa^{\mathcal{U}}}(-D))$$

be the $A[1/p]$-Banach module of cuspidal overconvergent modular forms of weights parametrized by $\mathcal{U}$, tame level N and $\mathrm{S}(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})^\dagger = \lim_{\underline{v}\to 0^+} \mathrm{S}(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})(\underline{v})$.

Theorem 3.16. — *The $A[1/p]$-module $\mathrm{S}(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})^\dagger$ is an inductive limit of projective Banach module and for any $\kappa \in \mathcal{U}(K)$, the specialization map*

$$\mathrm{S}(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})^\dagger \longrightarrow \mathrm{S}(\mu_N, \mathfrak{c}, \kappa)^\dagger$$

is surjective.

Recall that $\overline{M}^*(\mu_N, \mathfrak{c})$ is the minimal compactification of $M(\mu_N, \mathfrak{c})$. There is a projection $\overline{M}(\mu_N, \mathfrak{c}) \to \overline{M}^*(\mu_N, \mathfrak{c})$ from the the toroidal to the minimal compactification. Taking the formal p-adic completion and a certain open in a blow up, we get a map $\overline{\mathfrak{M}}(\mu_N, \mathfrak{c})(\underline{v}) \to \overline{\mathfrak{M}}^*(\mu_N, \mathfrak{c})(\underline{v})$. Passing to the generic fiber, there is a commutative diagram where all horizontal map are finite:

$$\begin{array}{ccc} \overline{\mathcal{M}}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v}) & \longrightarrow & \overline{\mathcal{M}}(\mu_N, \mathfrak{c})(\underline{v}) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}^\star(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v}) & \longrightarrow & \overline{\mathcal{M}}^\star(\mu_N, \mathfrak{c})(\underline{v}). \end{array}$$

Taking the normalization of $\overline{\mathfrak{M}}(\mu_N, \mathfrak{c})(\underline{v})$ and $\overline{\mathfrak{M}}^*(\mu_N, \mathfrak{c})(\underline{v})$ in $\overline{\mathcal{M}}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v})$ and $\overline{\mathcal{M}}^\star(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v})$ we obtain formal schemes $\overline{\mathfrak{M}}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v})$ and $\overline{\mathfrak{M}}^\star(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v})$ together with a map

$$\rho\colon \overline{\mathfrak{M}}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v}) \to \overline{\mathfrak{M}}^\star(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v}).$$

Theorem 3.17. — *Let D denote the boundary divisor in $\overline{\mathfrak{M}}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v})$. We have the vanishing:*

$$R^q\rho_*\mathcal{O}_{\overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})}(-D) = 0$$

for all $q \geq 1$.

Before proving this theorem, we need to describe the map ρ. Let C be a cusp on $\overline{\mathfrak{M}}^*(\Gamma_1(p^n), \mu_N, \mathfrak{c})$. Following [**11**, § 3.1], the cusp C is associated to:

- two fractional ideals of $\mathscr{O}_F$, $\mathscr{A}$ and $\mathscr{B}$,
- a positive polarization isomorphism $\beta\colon \mathscr{B}^{-1}\mathscr{A} \cong \mathfrak{c}$,
- a decomposition $N = N_1 N_2$ where N_1 and N_2 are prime to each other,
- an isomorphism $\mathscr{A}^{-1}/p^n N_1 \mathscr{A}^{-1} \simeq \mathscr{O}_F/p^n N_1 \mathscr{O}_F$ (the cusp is unramified at N_1 and p^n),
- an isomorphism $N_2^{-1}\mathscr{B}/\mathscr{B} \simeq \mu_{N_2} \otimes_{\mathbb{Z}} \delta^{-1}$.

We briefly recall how this data is used to construct 1-motives with level N and $\Gamma_1(p^n)$ structure. We let $M := \mathscr{A}\mathscr{B} \cong \mathfrak{c}\mathscr{B}^2$.

Over $\mathscr{O}_K[MN_2^{-1}]$ we have a commutative diagram:

$$\begin{array}{ccc} \mathscr{A} \otimes \mathscr{B}N_2^{-1} & \longrightarrow & \mathbb{G}_m \\ \uparrow & \nearrow & \\ p^n N_1 \mathscr{A} \otimes \mathscr{B}. & & \end{array}$$

This gives rise to two 1-motives and an isogeny:

$$\begin{array}{lccc} Mot(\mathscr{A}, \mathscr{B}): & \mathscr{B} & \longrightarrow & \mathbb{G}_m \otimes_{\mathbb{Z}} \mathscr{A}^{-1}\delta^{-1} \\ \quad\downarrow & \downarrow & & \downarrow \\ Mot(p^n N_1 \mathscr{A}, N_2^{-1}\mathscr{B}): & N_2^{-1}\mathscr{B} & \longrightarrow & \mathbb{G}_m \otimes_{\mathbb{Z}} p^{-n} N_1^{-1} \mathscr{A}^{-1}\delta^{-1}. \end{array}$$

The kernel of the isogeny $Mot(\mathscr{A}, \mathscr{B}) \to Mot(p^n N_1 \mathscr{A}, N_2^{-1}\mathscr{B})$ is the group $\mu_{p^n N_1} \otimes_{\mathbb{Z}} \mathscr{A}^{-1}\delta^{-1} \oplus N_2^{-1}\mathscr{B}/\mathscr{B}$ and the isomorphisms $\mathscr{A}^{-1}/p^n N_1 \mathscr{A}^{-1} \simeq \mathscr{O}_F/p^n N_1 \mathscr{O}_F$ and $N_2^{-1}\mathscr{B}/\mathscr{B} \simeq \mu_{N_2} \otimes_{\mathbb{Z}} \delta^{-1}$ provide the level N and $\Gamma_1(p^n)$ structure.

In the sequel, we shall suppose that the cusp is unramified (so $N_2 = 1$) and thus work with M. For ramified cusps, one simply has to work with $N_2^{-1}M$.

Let $M^* = \mathrm{Hom}(\mathscr{A} \otimes \mathscr{B}, \mathbb{R})$ and let $M^{*,+}$ be the cone of bilinear forms ϕ such that $\phi(\mathscr{A}^+, \mathscr{B}^+) \subset \mathbb{R}_{\geq 0}$.

The group $U_{p^n N} \subset \mathscr{O}_F^\times$ of units congruent to 1 modulo $p^n N$ acts on $M^{*,+}$ by $u\Phi = \Phi(u., u.)$. Let $\{\sigma_\alpha^C\}_{\alpha \in I_C}$ be the smooth projective rational polyhedral cone decomposition of $M^{*,+}$ defining the given toroidal compactification. It is invariant under the action of the group $U_{p^n N}$. Moreover, $\{\sigma_\alpha^C\}_{\alpha \in I_C}/U_{p^n N}$ is a finite set. The projectivity means that there exists a polarization function for $\{\sigma_\alpha^C\}_{\alpha \in I_C}$, see 6.1. The existence is shown in [**4**, Ch. 2]. Then, every σ_α^C defines an affine torus embedding

$$S^0 := \mathrm{Spec}\,\mathscr{O}_K[q^m]_{m \in M} \subset S\big(\sigma_\alpha^C\big) = \mathrm{Spec}\,\mathscr{O}_K[q^m]_{m \in M, m \geq 0 \text{ on } \sigma_\alpha^C}.$$

Let $\widehat{S}\big(\sigma_\alpha^C\big)$ be the completion of $S\big(\sigma_\alpha^C\big)$ along $S\big(\sigma_\alpha^C\big)^\infty := \mathrm{Spec}\,\mathscr{O}_K[q^m]_{m \in M, m = 0 \text{ on } \sigma_\alpha^C}$. We have

$$\widehat{S}\big(\sigma_\alpha^C\big) \cong \mathrm{Spf}\,\mathscr{O}_K[\![q^m]\!][q^s]_{m, s \in M, m > 0, s = 0 \text{ on } \sigma_\alpha^C}.$$

The schemes $S(\sigma_\alpha^C)$ and the formal schemes $\widehat{S}(\sigma_\alpha^C)$ glue for varying $\alpha \in I_C$ to a torus embedding $S^0 \subset S(\{\sigma_\alpha^C\})$ and, respectively, to a formal scheme $\widehat{S}(\{\sigma_\alpha^C\})$ which are endowed with an action of U_{p^nN}. The action on $\widehat{S}(\{\sigma_\alpha^C\})$ is free. Then, [**11**, Thm. 3.6 & Main Theorem § 4.3] state (for a slightly different level structure) that:

(1) the completion $\overline{\mathscr{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})^{\widehat{\rho^{-1}(C)}}$ of $\overline{\mathscr{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})$ along $\rho^{-1}(C)$ is isomorphic to the quotient $\widehat{S}(\{\sigma_\alpha^C\})/U_{p^nN}$;

(2) we have $\Gamma(\widehat{S}(\{\sigma_\alpha^C\}),\mathscr{O}_{\widehat{S}(\{\sigma_\alpha^C\})}) = \mathscr{O}_K[\![q^m]\!]_{m\in M^+\cup\{0\}}$ where U_{p^nN} acts via its action on M^+;

(3) the completion $\overline{\mathfrak{M}}^*(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})^{\widehat{C}}$ of $\overline{\mathfrak{M}}^*(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})$ at C is isomorphic to the formal spectrum of $\mathscr{O}_K[\![q^m]\!]_{m\in M^+}^{U_{p^nN}}$.

Summarizing we have

$$\widehat{S}(\{\sigma_\alpha^C\}) \xrightarrow{h_C} \widehat{S}(\{\sigma_\alpha^C\})/U_{p^nN} \cong \overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})^{\widehat{\rho^{-1}(C)}} \xrightarrow{\rho_C} \overline{\mathfrak{M}}^*(\mu_N,\mathfrak{c})(\underline{v})^{\widehat{C}},$$

where h_C is formally étale and ρ_C is projective due to the existence of a polarization function, see 6.2.

In order to show that

$$\mathrm{R}^q\rho_*\mathscr{O}_{\overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})}(-D) = 0$$

for $q > 0$, we simply need to show that the stalks of this sheaf are trivial. This is obvious away from the cusps because ρ is an isomorphism. By the theorem on formal functions, we are reduced to prove the vanishing of the cohomology on the formal fibers over the cusps. Let C be a cusp as above (which we take unramified for simplicity) and set $D_C := \rho_C^{-1}(C)$. We are thus left to prove:

$$\mathrm{R}^q\rho_{C,*}\mathscr{O}_{\widehat{S}(\{\sigma_\alpha^C\})/U_{p^nN}}(-D_C) = 0$$

for every $q \geq 1$. But this is Proposition 6.4 of the appendix. We now explain how we deduce Theorem 3.16 from Theorem 3.17.

Let $\rho'\colon \overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v}) \xrightarrow{\rho} \overline{\mathfrak{M}}^\star(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v}) \xrightarrow{h} \overline{\mathfrak{M}}^*(\mu_N,\mathfrak{c})(\underline{v})$.

Lemma 3.18. — *We have the vanishing:*

$$\mathrm{R}^q\rho'_*\mathscr{O}_{\overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})}(-D) = 0$$

for $q > 0$.

Proof. — The map h is finite. □

Recall the map $\gamma_w \times 1\colon \mathfrak{IW}_w^+ \times \mathfrak{U} \to \overline{\mathfrak{M}}(\Gamma(p^n),\mu_N,\mathfrak{c})(\underline{v}) \times \mathfrak{U}$.

We let $\chi^{\mathcal{U}}\colon \mathbb{T}_w^0 \times \mathfrak{U} \to \mathbb{G}_m \times \mathfrak{U}$ be the restriction of the character $\kappa^{\mathcal{U}}$ to $\mathbb{T}_w^0$. We let $\Omega^{\chi^{\mathcal{U}}} = (\gamma_w \times 1)_*\mathscr{O}_{\mathfrak{IW}_w^+\times\mathfrak{U}}[-\chi^{\mathcal{U}}]$.

Lemma 3.19. — *The sheaf* $\Omega^{\chi^{\mathcal{U}}}/p$ *is isomorphic to* $\mathscr{O}_{\overline{\mathfrak{M}}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})\times\mathfrak{U}}/p$.

Proof. — The character $\chi^{\mathcal{U}}$ mod p is constant by Lemma 2.1. □

Corollary 3.20. — *Let*

$$\rho' \times 1 \colon \overline{\mathfrak{M}}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v}) \times \mathfrak{U} \longrightarrow \overline{\mathfrak{M}}^*(\mu_N, \mathfrak{c})(\underline{v}) \times \mathfrak{U}$$

We have the vanishing:

$$\mathrm{R}^q(\rho' \times 1)_* \Omega^{\chi^{\mathcal{U}}}(-D) = 0$$

for $q > 0$.

Proof. — Apply the functor $\mathrm{R}(\rho' \times 1)_*$ to the short exact sequence $0 \to \Omega^{\chi^{\mathcal{U}}} \xrightarrow{p} \Omega^{\chi^{\mathcal{U}}} \to \Omega^{\chi^{\mathcal{U}}}/p \to 0$ and use Lemma 3.18 and 3.19. □

Let $(\mathfrak{V}_i)_{i \in I}$ be an affine covering of $\overline{\mathfrak{M}}^*(\mu_N, \mathfrak{c})(\underline{v})$.

Let $\underline{i} = (i_1, \ldots, i_r)$ be a multindex and $\mathfrak{V}_{\underline{i}}$ be the intersection of the $\mathfrak{V}_{i_j}$. Let

$$C_{\underline{i}} = \mathrm{H}^0(\mathfrak{V}_{\underline{i}} \times \mathfrak{U}, (\rho' \times 1)_* \Omega^{\chi^{\mathcal{U}}}(-D)).$$

Lemma 3.21. — *The A-module $C_{\underline{i}}$ is the p-adic completion of a free A-module.*

Proof. — This A-module is p-torsion free and p-adicaly complete. Moreover,

$$\begin{aligned} C_{\underline{i}}/p &= \mathrm{H}^0(\mathfrak{V}_{\underline{i}} \times \mathfrak{U}, (\rho' \times 1)_* \Omega^{\chi^{\mathcal{U}}}(-D)/p) \\ &\simeq H^0(\mathfrak{V}_{\underline{i}}, \rho'_* \mathcal{O}_{\overline{\mathfrak{M}}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v})(-D)}/p) \otimes_{\mathcal{O}_K/p} A/p \end{aligned}$$

by Corollary 3.20 and Lemma 3.19. It follows that $C_{\underline{i}}/p$ is a free A/p-module and that $C_{\underline{i}}$ is the completion of a free A-module. □

Proposition 3.22. — *The $A[1/p]$-module $\mathrm{S}(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})^\dagger$ is an inductive limit of projective Banach modules.*

Proof. — Consider the augmented Cech complex of A-modules $K_\bullet$ associated to the covering $(\mathfrak{V}_i)_{i \in I}$ and the sheaf $\Omega^{\chi^{\mathcal{U}}}(-D)$. The complex $K_\bullet[1/p]$ is exact since $\overline{\mathcal{M}}^*(\mu_N, \mathfrak{c})(\underline{v})$ is affinoid (1). It thus provides a resolution of

$$\mathrm{H}^0(\overline{\mathcal{M}}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v}) \times \mathcal{U}, \Omega^{\chi^{\mathcal{U}}}(-D))$$

by $A[1/p]$-orthonormalizable modules. It follows that $\mathrm{H}^0(\overline{\mathcal{M}}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v}) \times \mathcal{U}, \Omega^{\chi^{\mathcal{U}}}(-D))$ is projective. Since $\mathrm{S}(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})(\underline{v})$ is a direct factor (cut out by the action of $\mathbb{T}(\mathbb{Z}/p^n\mathbb{Z})$), it is also projective. □

Proposition 3.23. — *For any $\kappa \in \mathcal{U}(K)$, the specialization map*

$$\mathrm{S}(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})^\dagger \to \mathrm{S}(\mu_N, \mathfrak{c}, \kappa)^\dagger$$

is surjective.

---

(1) S. Hattori pointed out to us that the affinoid character of $\overline{\mathcal{M}}^*(\mu_N, \mathfrak{c})(\underline{v})$ is not clear because we are using the partial Hasse invariants to define the neighborhood instead of the global Hasse invariant. To finish the argument, we can restrict ourselves to the affinoid strict neighborhood $\{|\prod_{i=1}^f h_{\mathfrak{P}_i}| \geq |p^{-\inf v_i}|\} \subset \overline{\mathcal{M}}^*(\mu_N, \mathfrak{c})(\underline{v})$.

Proof. — It follows easily from Corollary 3.20 that we have a surjective map of sheaves:

$$(\rho' \times 1)_* \Omega^{\chi^{\mathcal{U}}}(-D) \to (\rho')_* \Omega^{\chi}(-D).$$

Taking the global sections over $\overline{\mathcal{M}}^*(\mu_N, \mathfrak{c})(\underline{v}) \times \mathcal{U}$ which is affinoid, we get a surjection:

$$\mathrm{H}^0(\overline{\mathcal{M}}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v}) \times \mathcal{U}, \Omega^{\chi^{\mathcal{U}}}(-D)) \to \mathrm{H}^0(\overline{\mathcal{M}}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v}), \Omega^{\chi}(-D)).$$

Decomposing both sides according to the action of $\mathbb{T}(\mathbb{Z}/p^n\mathbb{Z})$, we conclude. □

We can summarize the results of this section as follows. Let us define the quasi-coherent sheaf $S(\mu_N, \mathfrak{c})^\dagger$ on $\mathcal{W}$ by the rule $\mathcal{U} \mapsto \mathrm{S}(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})^\dagger$.

Corollary 3.24. — *For every affinoid open $\mathcal{U} \to \mathcal{W}$, $\mathrm{S}(\mu_N, \mathfrak{c})^\dagger(\mathcal{U})$ is an inductive limit of projective Banach $\mathrm{H}^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$-modules with compact transition maps. Moreover, for all $\kappa \in \mathcal{W}(K)$, the fiber of $\mathrm{S}(\mu_N, \mathfrak{c})^\dagger$ at κ is $\mathrm{S}(\mu_N, \mathfrak{c}, \kappa)^\dagger$.*

3.7. Hecke operators. — Let $\ell \subset \mathcal{O}_F$ be an ideal prime to N. We let

$$\mathcal{Y}_\ell(\underline{v}) \subset \mathcal{M}(\mu_N, \mathfrak{c})(\underline{v}) \times \mathcal{M}(\mu_N, \ell\mathfrak{c})(\underline{v})$$

be the subspace classifying pairs $(A, \iota, \Psi, \lambda) \times (A', \iota', \Psi', \lambda')$ and an isogeny $\pi_\ell \colon A \to A'$ compatible with $(\iota, \iota', \psi, \psi', \lambda, \lambda')$, such that $\mathrm{Ker}\pi_\ell$ is étale locally isomorphic to $\mathcal{O}_F/\ell\mathcal{O}_F$ and $\mathrm{Ker}\pi_\ell \cap H_1 = \{0\}$, where H_1 is the canonical subgroup of A of level 1.

Denote by

$$p_1 \colon \mathcal{Y}_\ell(\underline{v}) \to \mathcal{M}(\mu_N, \mathfrak{c})(\underline{v}) \quad \text{and} \quad p_2 \colon \mathcal{Y}_\ell(\underline{v}) \to \mathcal{M}(\mu_N, \ell\mathfrak{c})(\underline{v})$$

the two projections.

Lemma 3.25. — 1. *The universal isogeny $\pi_\ell \colon A \to A'$ induces an isomorphism from the canonical subgroup of A onto the canonical subgroup of A'.*

2. *For ℓ coprime to p the maps p_1 and p_2 are finite and étale of degree $q_\ell + 1$ where $q_\ell := |\mathcal{O}_F/\ell\mathcal{O}_F|$.*
3. *If ℓ divides some power of p, the map p_1 is finite étale of degree $q_\ell := |\mathcal{O}_F/\ell\mathcal{O}_F|$.*
4. *For $\ell = \mathfrak{P}_i^{e_i}$ for some $i = 1, \ldots, f$, define $\underline{v}'$ by requiring that $v'_j = v_j$ for $j \neq i$ and $v'_i = \frac{v_i}{p}$. Then, the map p_2 induces an isomorphism on $\mathcal{M}(\mu_N, \ell\mathfrak{c})(\underline{v}')$.*

Proof. — Let $(A, \iota, \Psi, \lambda) \in \mathcal{M}(\mu_N, \mathfrak{c})(\underline{v})$. Let $\pi \colon A \to A'$ be an isogeny whose kernel $\mathrm{Ker}\pi$ is étale locally isomorphic to $\mathcal{O}_F/\ell\mathcal{O}_F$ and $\mathrm{Ker}\pi \cap H_1 = \{0\}$. Let us denote by H'_1 the image of H_1 in A'. It follows from the basic properties of the degree function that $\deg H'_1 \geq \deg H_1$. By [2], prop. 3.1.2, it follows that A' has a canonical subgroup, H'_1, and that $\mathrm{Hdg}_{\mathfrak{P}_i} A' \leq v_i$ for all i.

For the last statement, we use the following result of Fargues [17, Prop. 16]. Let $\mathcal{G}$ be a p-divisible group defined over $\mathcal{O}_L$ where L is some complete valued field extension of K with $\mathrm{Hdg}\mathcal{G} < \frac{p-2}{2p-2}$ so that the canonical subgroup $H_{can} \subset \mathcal{G}[p]$ exists. Let $D \subset \mathcal{G}[p]$ be a subgroup scheme finite and flat over $\mathcal{O}_L$, such that $D_L \oplus H_{can,L} = \mathcal{G}_L[p]$. Then the Hodge height of $\mathcal{G}' := \mathcal{G}/D$ is $\mathrm{Hdg}\mathcal{G}/p$ and the image of H_{can} in $\mathcal{G}'$

is the canonical subgroup H'_{can} of $\mathscr{G}'$. Also, note that $\mathscr{G} = \mathscr{G}'/H'_{can}$. For an abelian scheme A with RM by $\mathscr{O}_F$ over $\mathscr{O}_L$ we apply these results to the p-divisible group $\mathscr{G} = A[\mathfrak{P}_i^\infty]$ remarking that $\mathrm{Hdg}_{\mathfrak{P}_i} A = \mathrm{Hdg}\mathscr{G}$. □

Let $\mathscr{U}$ be some affinoid and $\mathscr{U} \to \mathscr{W}$ be a map to the weight space. We assume that $\kappa^{\mathscr{U}}$ is n-analytic.

Corollary 3.26. — *The induced map $\pi_\ell^*\colon p_2^*(\omega_{A'}) \to p_1^*(\omega_A)$ on invariant differentials induces an isomorphism:*

$$\pi_\ell^*\colon p_1^*\big(\Omega^{\kappa^{\mathscr{U}}}\big) \xrightarrow{\sim} p_2^*\big(\Omega^{\kappa^{\mathscr{U}}}\big)$$

of invertible $\mathscr{O}_{\mathscr{Y}_\ell(\underline{v})\times\mathscr{U}}$-modules.

Proof. — This follows easily from the functorial properties of the construction of the sheaves. We refer the reader to [**2**], Section 6.1 and 6.2.1 for more details. □

We let $\pi_\ell^{*,-1}$ be the inverse of the above map.

We now define an Hecke operator:

$$T_\ell\colon \mathrm{M}(\mu_N, \ell\mathfrak{c}, \kappa^{\mathscr{U}})(\underline{v}) \to \mathrm{M}(\mu_N, \mathfrak{c}, \kappa^{\mathscr{U}})(\underline{v}).$$

First of all, by Koecher's principle, we have the identification $\mathrm{M}(\mu_N, \mathfrak{a}, \kappa^{\mathscr{U}})(\underline{v}) = \mathrm{H}^0\big(\mathscr{M}(\mu_N, \mathfrak{a})\big(\underline{v}\big) \times \mathscr{U}, \Omega^{\kappa^{\mathscr{U}}}\big)$ for all fractional ideals $\mathfrak{a}$. We obtain T_ℓ as the composite

$$\mathrm{H}^0\big(\mathscr{M}(\mu_N, \ell\mathfrak{c})\big(\underline{v}\big) \times \mathscr{U}, \Omega^{\kappa^{\mathscr{U}}}\big) \xrightarrow{p_2^*} \mathrm{H}^0\big(\mathscr{Y}_\ell(\underline{v}) \times \mathscr{U}, p_2^*(\Omega^{\kappa^{\mathscr{U}}})\big)$$
$$\xrightarrow{\pi_\ell^{*,-1}} \mathrm{H}^0\big(\mathscr{Y}_\ell(\underline{v}) \times \mathscr{U}, p_1^*(\Omega^{\kappa^{\mathscr{U}}})\big) \xrightarrow{\frac{1}{q_\ell}\mathrm{Tr} p_1} \mathrm{H}^0\big(\mathscr{M}(\mu_N, \mathfrak{c})\big(\underline{v}\big) \times \mathscr{U}, \Omega^{\kappa^{\mathscr{U}}}\big).$$

If ℓ divides some power of p, we use the more traditional notation U_ℓ instead of T_ℓ. Let us now prove a fundamental property of the Hecke operator U_p.

The multiplication by p is a positive isomorphism $\mathfrak{c} \to p\mathfrak{c}$ and allows us to identify the spaces $\mathrm{M}(\mu_N, p\mathfrak{c}, \kappa^{\mathscr{U}})(\underline{v})$ and $\mathrm{M}(\mu_N, \mathfrak{c}, \kappa^{\mathscr{U}})(\underline{v})$ and to view U_p as an endomorphism. Similarly if $m \in \mathbb{Z}$ is a positive integer prime to pN, multiplication by m is a positive isomorphism $\mathfrak{c} \to m\mathfrak{c}$ and allows us to get an endomorphism T_m of $\mathrm{M}(\mu_N, \mathfrak{c}, \kappa^{\mathscr{U}})(\underline{v})$.

Lemma 3.27. — *The operator U_p is a compact operator.*

Proof. — Since $p = \prod_i \mathfrak{P}_i^{e_i}$, it follows from Lemma 3.25 4), that we may write U_p as the composite of a morphism $\mathrm{H}^0\big(\overline{\mathscr{M}}(\mu_N, \mathfrak{c})\big(\underline{v}/p\big) \times \mathscr{U}, \Omega^{\kappa^{\mathscr{U}}}\big) \to \mathrm{H}^0\big(\overline{\mathscr{M}}(\mu_N, \mathfrak{c})\big(\underline{v}\big) \times \mathscr{U}, \Omega^{\kappa^{\mathscr{U}}}\big)$ with the restriction map

$$\mathrm{H}^0\big(\overline{\mathscr{M}}(\mu_N, \mathfrak{c})\big(\underline{v}\big) \times \mathscr{U}, \Omega^{\kappa^{\mathscr{U}}}\big) \longrightarrow \mathrm{H}^0\big(\overline{\mathscr{M}}(\mu_N, \mathfrak{c})\big(\underline{v}/p\big) \times \mathscr{U}, \Omega^{\kappa^{\mathscr{U}}}\big)$$

which is compact. Hence, U_p is compact. □

For ℓ an ideal of $\mathscr{O}_F$, prime to Np, we also define a map

$$S_\ell\colon \mathfrak{M}(\mu_N, \mathfrak{c})(\underline{v}) \to \mathfrak{M}(\mu_N, \ell^2\mathfrak{c})(\underline{v})$$
$$(A, \iota, \Psi, \lambda) \mapsto (A \otimes \ell^{-1}, \iota', \Psi', \ell^2\lambda),$$

where ι' and Ψ' are induced by ι and Ψ. As before, it is not hard to see that there is an isomorphism $S_\ell^*\Omega^{\kappa^{\mathcal{U}}} \to \Omega^{\kappa^{\mathcal{U}}}$ induced by the universal isogeny $A \to A \otimes \ell^{-1}$. We thus get an Hecke operator:

$$S_\ell \colon \mathrm{M}(\mu_N, \ell^{-2}\mathfrak{c}, \kappa^{\mathcal{U}})(\underline{v}) \to \mathrm{M}(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})(\underline{v})$$
$$f \mapsto [(A, \iota, \Psi, \lambda) \mapsto q_\ell^{-2} S_\ell^* f(A \otimes \ell^{-1}, \iota, \Psi, \ell^2\lambda)]$$

Remark 3.28. — The action of the Hecke operators is clearly functorial in $\mathcal{U}$ and it also respects cuspidality.

4. Overconvergent modular forms for the group G

4.1. Overconvergent descent from G^* to G. — Let us recall the notations from the introduction: $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, we let $\det\colon G \to \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ be the determinant morphism and let G^* be the subgroup of G consisting of elements with determinant in $\mathbb{G}_m$, i.e.,

$$G^* = G \times_{\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m} \mathbb{G}_m.$$

As explained in the introduction, the Hilbert variety $M(\mu_N, \mathfrak{c})$ is a model for the Shimura variety for the group G^*. As a result, for classical weights κ, the modules $\mathrm{M}(\mu_N, \mathfrak{c}, \kappa)$, are related to automorphic forms on the group G^*. Therefore also for all p-adic weight $\kappa \in \mathcal{W}(K)$, the overconvergent modular forms defined in the previous sections are overconvergent modular forms for the group G^*.

The goal of this chapter is to define overconvergent and p-adic families of modular forms for the group G, using a descent similar to the one used to define classical modular forms for G.

Let $\mathcal{O}_F^{\times,+}$ be the sub-group of totally positive units in $\mathcal{O}_F^\times$. Let U_N be its subgroup of units which are congruent to 1 modulo N. Let us recall that we have defined (in Section 1) an action of $\mathcal{O}_F^{\times,+}$ on $M(\mu_N, \mathfrak{c})$, by sending a quadruple $(A, \iota, \psi, \lambda)$ to $(A, \iota, \psi, \epsilon{\cdot}\lambda)$ for any $\epsilon \in \mathcal{O}_F^{\times,+}$. This action factors through the group $\Delta = \mathcal{O}_F^{\times,+}/U_N^2$. If $n > 0$ is an integer and $\underline{v} = (v_i)_{1\le i\le f}$ satisfying $0 < v_i < 1/p^n$ for all i, the action of the elements of $\mathcal{O}_F^{\times,+}$ on $\mathcal{M}(\mu_N, \mathfrak{c})$ induced from the one on $M(\mu_N, \mathfrak{c})$, preserves $\mathcal{M}(\mu_N, \mathfrak{c})(\underline{v})$, as the definition of the last rigid space does not depend on the polarization.

Let us denote by $\mathcal{W}^G$ the weight space for G whose $\mathbb{C}_p$-points, let us recall, are equal to $\mathrm{Hom}(\mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, \mathbb{C}_p^\times)$. We have a map $k\colon \mathcal{W}^G \to \mathcal{W}$ induced by the group homomorphism:

$$\mathbb{T}(\mathbb{Z}_p) \to \mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$$

given by $x \mapsto (x^2, \mathrm{N}_{F/\mathbb{Q}}(x))$.

Let $(\nu, w) \in \mathcal{W}^G(K)$ be a weight for G, its image in $\mathcal{W}(K)$ is the weight $\kappa = k(\nu, w) = \nu^2 \cdot (w \circ N_{F/\mathbb{Q}})$. Suppose (ν, w) is n-analytic (it follows that κ is n-analytic as well) and choose $\underline{v} = (v_i)_{1\le i\le f}$ with $0 < v_i < 1/p^n$ for all i. Let $f \in \mathrm{M}(\mu_N, \mathfrak{c}, \kappa)(\underline{v})$ be an overconvergent modular form (for G^*) of weight κ. By

Lemma 3.2, Section 3.2 and Koecher principle, it can be seen as a rule which associates to every sequence $(A_{/R}, \iota, \psi, \lambda, u, \alpha)$ an element $f(A_{/R}, \iota, \psi, \lambda, u, \alpha) \in R_K$ such that: the association depends only on the isomorphism class of $(A_{/R}, \iota, \psi, \lambda, u, \alpha)$, commutes with base change and satisfies

$$f(A_{/R}, \iota, \psi, \lambda, t^{-1}u, t^{-1}\alpha) = \kappa(t) f(A_{/R}, \iota, \psi, \lambda, u, \alpha), \text{ for all } t \in (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times.$$

Let us recall that in the sequence $(A_{/R}, \iota, \psi, \lambda, u, \alpha)$ the symbols denote:

1. R denotes a normal, p-adically complete and separated, flat $\mathcal{O}_K$-algebra, which is topologically of finite type;
2. $A \to \mathrm{Spf}(R)$ is an abelian scheme of relative dimension g such that $\mathrm{Hdg}_{\mathfrak{P}_i}(A) \leq v_i$ for all $1 \leq i \leq f$;
3. ι is an $\mathcal{O}_F$-multiplication on A;
4. ψ is a level μ_N-structure;
5. λ is a $(\mathfrak{c}, \mathfrak{c}^+)$-polarization;
6. u is an isomorphism $u\colon H_{n,K} \cong \mu_{n,K} \otimes \mathcal{D}_F^{-1}$ of group-schemes over R_K, where H_n is the level n-canonical subgroup of $A[p^n]$.
   The sub-sequence $(A_{/R}, \iota, \psi, \lambda, u)$ defines a unique morphism $\gamma\colon \mathrm{Spf}(R) \longrightarrow \mathfrak{M}\big(\Gamma_1(p^n), \mu_N, \mathfrak{c}\big)(\underline{v})$. We also get from the isomorphism induced by u on Cartier duals an isomorphism $H_n^D(R) = H_n^D(R_K) \cong \mathcal{O}_F/p^n\mathcal{O}_F$.
7. α is an isomorphism $\alpha\colon \gamma^*(\mathcal{F}) \cong \mathcal{O}_F \otimes R$ such that $\alpha \pmod{p^w}$ composed with HT_w (see Proposition 3.4) sends the section $1 \in \mathcal{O}_F/p^n\mathcal{O}_F \cong H_n^D(R)$ to $1 \in \mathcal{O}_F \otimes R/p^wR$, for all $w \in]0, \sup(v_i)p^n/(p-1)]$.

We define an action of Δ on $\mathrm{M}(\mu_N, \mathfrak{c}, \kappa)(\underline{v})$ as follows. First, if $\epsilon = \eta^2$ with $\eta \in U_N$, then the isomorphism $\eta\colon A \cong A$ gives:

$$\begin{aligned} f(A/R, \iota, \psi, \lambda, u, \alpha) &:= f(A/R, \iota, \psi, \epsilon\lambda, \eta u, \eta\alpha) \\ &= \kappa(\eta^{-1}) f(A/R, \iota, \psi, \epsilon\lambda, u, \alpha) = \nu(\epsilon^{-1}) f(A/R, \iota, \psi, \epsilon\lambda, u, \alpha). \end{aligned}$$

Therefore if we define the action of $\mathcal{O}_F^{\times,+}$ on $\mathrm{M}(\mu_N, \mathfrak{c}, \kappa)(\underline{v})$ by:

$$(\epsilon \cdot f)(A_{/R}, \iota, \psi, \lambda, u, \alpha) := \nu(\epsilon) f(A_{/R}, \iota, \psi, \epsilon^{-1}\lambda, u, \alpha),$$

then the action of U_N^2 is trivial and so the action factors through Δ.

This allows us to define the projector (i.e., the idempotent endomorphism) $\mathfrak{e}\colon \mathrm{M}(\mu_N, \mathfrak{c}, \kappa)(\underline{v}) \longrightarrow \mathrm{M}(\mu_N, \mathfrak{c}, \kappa)(\underline{v})$ by:

$$(\mathfrak{e} \cdot f)(A_{/R}, \iota, \psi, \lambda, u, \alpha) = \frac{1}{\sharp\Delta} \sum_{\epsilon \in \Delta} \nu(\epsilon) f(A_{/R}, \iota, \psi, \epsilon^{-1}\lambda, u, \alpha).$$

Given the definition of classical modular forms for G in section §1, the following definition is now quite natural.

Definition 4.1. — The K-vector space of overconvergent modular forms for G of tame level μ_N, polarization $(\mathfrak{c}, \mathfrak{c}^+)$ and weight (ν, w), $\mathrm{M}^G\big(\mu_N, \mathfrak{c}, (\nu, w)\big)(\underline{v})$ is defined to be the image of the projector $\mathfrak{e}$.

In other words, an overconvergent modular form for G, of tame level μ_N, polarization $(\mathfrak{c},\mathfrak{c}^+)$, weight (ν,w) and degree of overconvergence $\underline{v}$ is an element $f\in \mathrm{M}(\mu_N,\mathfrak{c},\kappa)(\underline{v})$ such that $f(A_{/R},\iota,\psi,\epsilon\lambda,u,\alpha)=\nu(\epsilon)f(A_{/R},\iota,\psi,\lambda,u,\alpha)$, for all $\epsilon\in\Delta$ (where of course $\kappa:=k(\nu,w)$). In other words, as for classical forms, we have

$$\mathrm{M}^G\big(\mu_N,\mathfrak{c},(\nu,w)\big)(\underline{v})=\Big(\mathrm{M}(\mu_N,\mathfrak{c},\kappa)(\underline{v})\Big)^{\Delta}.$$

Moreover, we remark that $\mathrm{M}^G\big(\mu_N,\mathfrak{c},(\nu,w)\big)(\underline{v})$ is a K-Banach space.

We will now define p-adic families of modular forms for the group G. Let $\mathscr{U}$ be an affinoid with a morphism of rigid spaces $\mathscr{U}\longrightarrow\mathscr{W}^G$. By composing this morphism with k we obtain a morphism $\mathscr{U}\longrightarrow\mathscr{W}$. Let us denote by $(\nu^{\mathscr{U}},w^{\mathscr{U}})$ the restriction of the universal character to $\mathscr{U}$ and we set $\kappa^{\mathscr{U}}=2\nu^{\mathscr{U}}+w^{\mathscr{U}}$. Let $n\geq 1$ be such that $(\nu^{\mathscr{U}},w^{\mathscr{U}})$ is n-analytic (it follows that $\kappa^{\mathscr{U}}$ is n-analytic as well). We recall that in this case $\kappa^{\mathscr{U}}$ extends to a formal character

$$\kappa^{\mathscr{U}}\colon \mathbb{T}_w\times\mathfrak{U}\longrightarrow\widehat{\mathbb{G}}_m\times\mathfrak{U},$$

where let us recall $\mathfrak{U}=\mathrm{Spf}(B)$, with B the algebra of power bounded elements on $\mathscr{U}$, for all $w\in]0,\sup(v_i)p^n/(p-1)]$.

Let $f\in \mathrm{M}(\mu_N,\mathfrak{c},\kappa^{\mathscr{U}})(\underline{v})$. As before, the multiplication by an element $\eta\in U_N$ on $A_{/R}$, where this time R is a normal, p-adically complete and separated B-algebra, topologically of finite type, gives us the invariance formula:

$$f(A_{/R},\iota,\psi,\eta^2\lambda,u,\alpha)=\nu^{\mathscr{U}}(\eta^2)f(A_{/R},\iota,\psi,\lambda,u,\alpha).$$

This allows us to define an action of Δ on $\mathrm{M}(\mu_N,\mathfrak{c},\kappa^{\mathscr{U}})(\underline{v})$ by:

$$(\epsilon\cdot f)(A_{/R},\iota,\psi,\lambda,u,\alpha):=\nu^{\mathscr{U}}(\epsilon)f(A_{/R},\iota,\psi,\epsilon^{-1}\lambda,u,\alpha),\ \text{for all } \epsilon\in\Delta.$$

As a consequence we have the projector $\mathfrak{e}^{\mathscr{U}}$ on the space of p-adic families of modular forms (for G^*) with weights parameterized by $\mathscr{U}$, $\mathrm{M}(\mu_N,\mathfrak{c},\kappa^{\mathscr{U}})(\underline{v})$ defined by:

$$(\mathfrak{e}^{\mathscr{U}}\cdot f)(A_{/R},\iota,\psi,\lambda,u,\alpha)=\frac{1}{\sharp\Delta}\sum_{\epsilon\in\Delta}\nu^{\mathscr{U}}(\epsilon)f(A_{/R},\iota,\psi,\epsilon^{-1}\lambda,u,\alpha)$$

where f is a section over $\mathscr{M}(\mu_N,\mathfrak{c})(\underline{v})\times\mathscr{U}$ of the sheaf $\Omega^{\kappa^{\mathscr{U}}}$. To justify that the formula makes sense, we remark that the fibers of the sheaf $\Omega^{\kappa^{\mathscr{U}}}$ at $(A_{/R},\iota,\psi,\lambda)$ or $(A_{/R},\iota,\psi,\epsilon\lambda)$ are canonically isomorphic, as the construction of the sheaf $\Omega^{\kappa^{\mathscr{U}}}$ does not depend on the polarization, but only on the abelian variety $A_{/R}$ with action by $\mathscr{O}_F$.

Definition 4.2. — Let $\mathscr{U},\kappa^{\mathscr{U}},(\nu^{\mathscr{U}},w^{\mathscr{U}})$ as before and $\underline{v}$ a small enough multi-index. We define $\mathrm{M}^G\big(\mu_N,\mathfrak{c},\kappa^{\mathscr{U}}\big)(\underline{v})$ to be the image of the projector $\mathfrak{e}^{\mathscr{U}}$ defined above.

In particular if $\mathscr{U}\hookrightarrow\mathscr{W}^G$ is an open immersion then $\mathrm{M}^G\big(\mu_N,\mathfrak{c},(\nu^{\mathscr{U}},w^{\mathscr{U}})\big)(\underline{v})$ is the $(B\otimes K)$-module of p-adic families of modular forms for the group G with weights parameterized by $\mathscr{U}$.

As above, we have the alterative description:

$$\mathrm{M}^G\big(\mu_N, \mathfrak{c}, (\nu^{\mathcal{U}}, w^{\mathcal{U}})\big)(\underline{v}) = \Big(\mathrm{M}(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})(\underline{v})\Big)^{\Delta}.$$

Remark 4.3. — The attentive reader might have noticed that the construction of the space $\mathrm{M}^G\big(\mu_N, \mathfrak{c}, (\nu, w)\big)(\underline{v})$ of overconvergent modular forms for G is a particular case of the construction of $\mathrm{M}^G\big(\mu_N, \mathfrak{c}, (\nu^{\mathcal{U}}, w^{\mathcal{U}})\big)(\underline{v})$, namely it is enough to take $\mathcal{U} := \mathrm{Spm}(K)$ with the map $\mathcal{U} \longrightarrow \mathcal{W}^G$ sending the unique point of $\mathcal{U}$ to (ν, w). We preferred to present the two constructions here in order to make them clearer.

The projector $\mathfrak{e}^{\mathcal{U}}$ is compatible with the restriction to a smaller radius of overconvergence, is functorial in $\mathcal{U}$ and it respects cuspidality. We let $\mathrm{S}^G\big(\mu_N, \mathfrak{c}, (\nu^{\mathcal{U}}, w^{\mathcal{U}})\big)(\underline{v})$ and $\mathrm{S}^G\big(\mu_N, \mathfrak{c}, (\nu, w)\big)(\underline{v})$ be the cuspidal submodules of $\mathrm{M}^G\big(\mu_N, \mathfrak{c}, (\nu^{\mathcal{U}}, w^{\mathcal{U}})\big)(\underline{v})$ and $\mathrm{M}^G\big(\mu_N, \mathfrak{c}, (\nu, w)\big)(\underline{v})$. We also have:

Theorem 4.4. — *Suppose that $\mathcal{U}$ is an admissible open affinoid of $\mathcal{W}^G$ and let $(\nu^{\mathcal{U}}, w^{\mathcal{U}})$, $\kappa^{\mathcal{U}}$, $n, \underline{v}$ be as above. Then we have*

a) $\mathrm{S}^G\big(\mu_N, \mathfrak{c}, (\nu^{\mathcal{U}}, w^{\mathcal{U}})\big)(\underline{v})$ *is a projective, Banach-$(B \otimes_{\mathcal{O}_K} K)$-module.*

b) *Let $(\nu, w) \in \mathcal{U}(K)$ be a weight. Then the natural specialization map*

$$\mathrm{S}^G\big(\mu_N, \mathfrak{c}, (\nu^{\mathcal{U}}, w^{\mathcal{U}})\big)(\underline{v}) \longrightarrow \mathrm{S}^G\big(\mu_N, \mathfrak{c}, (\nu, w)\big)(\underline{v})$$

is surjective.

Proof. — The theorem follows immediately from Theorem 3.16 and the fact that the specialization of the idempotent $\mathfrak{e}^{\mathcal{U}}$ at (ν, w) is the idempotent $\mathfrak{e}$ defined by the action of Δ on $\mathrm{M}(\mu_N, \mathfrak{c}, \kappa)(\underline{v})$, where $\kappa = 2\nu + w$. $\square$

Let $x \in F^{\times}$ be a totally positive element. The multiplication by x is a positive isomorphism from $\mathfrak{c}$ to $x\mathfrak{c}$.

Lemma 4.5. — *Assume that x is also a p-adic unit. The map:*

$$\begin{aligned} L_x \colon \mathrm{M}^G(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})(\underline{v}) &\to \mathrm{M}^G(\mu_N, x\mathfrak{c}, \kappa^{\mathcal{U}})(\underline{v}) \\ f &\mapsto [(A_{/R}, \iota, \psi, \lambda, u, \alpha) \mapsto \nu^{\mathcal{U}}(x) f(A_{/R}, \iota, \psi, x^{-1}\lambda, u, \alpha)] \end{aligned}$$

does not depend on the element x, but only on the principal ideal with positivity (x) and thus defines a canonical isomorphism.

Proof. — For any $f \in \mathrm{M}^G(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})(\underline{v})$ and any positive unit ϵ, we have

$$\nu^{\mathcal{U}}(\epsilon) f(A_{/R}, \iota, \psi, \lambda, u, \alpha) = f(A_{/R}, \iota, \psi, \epsilon\lambda, u, \alpha). \qquad \square$$

The above isomorphism is functorial in $\mathcal{U}$, compatible with change of the radius of overconvergence and preserves cuspidality. The reason we impose that x be a p-adic unit is to be able to evaluate $\nu^{\mathcal{U}}$ at x. If ν is algebraic, this condition is unnecessary.

4.2. Arithmetic Hilbert modular forms. — Let $\mathrm{Frac}(F)^{(p)}$ be the group of fractional ideals prime to p. Let $\mathrm{Princ}(F)^{+,(p)}$ be the group of positive elements which are p-adic units. The quotient $\mathrm{Frac}(F)^{(p)}/\mathrm{Princ}(F)^{+,(p)} = \mathrm{Cl}^+(F)$ is the strict class group of F.

Definition 4.6. — The Banach module of $\underline{v}$-overconvergent, tame level N, weights parametrized by $\mathcal{U}$ arithmetic Hilbert modular forms (i.e., p-adic families of overconvergent modular forms for the group G over $\mathcal{U}$) is

$$\mathrm{M}^G\big(\mu_N, (\nu^{\mathcal{U}}, w^{\mathcal{U}})\big)(\underline{v}) = \Big(\bigoplus_{\mathfrak{c}\in\mathrm{Frac}(F)^{(p)}} \mathrm{M}^G(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})(\underline{v})\Big)/\Big(L_x(f) - f\Big)_{x\in\mathrm{Princ}(F)^{+,(p)}}.$$

One defines similarly:

- The sub-module $\mathrm{S}^G(\mu_N, (\nu^{\mathcal{U}}, w^{\mathcal{U}}))(\underline{v})$ of cuspidal forms for G,
- The associated Frechet spaces $\mathrm{M}^G(\mu_N, (\nu^{\mathcal{U}}, w^{\mathcal{U}}))^\dagger$ and $\mathrm{S}^G(\mu_N, (\nu^{\mathcal{U}}, w^{\mathcal{U}}))^\dagger$ for G, by passing to the limit over $\underline{v}$,
- The corresponding quasi-coherent sheaf of overconvergent cuspidal arithmetic Hilbert modular forms $\mathrm{S}^G(\mu_N)^\dagger$ over $\mathcal{W}^G$, whose value on an open affinoid $\mathcal{U} \subset \mathcal{W}^G$ is $\mathrm{S}^G(\mu_N, (\nu^{\mathcal{U}}, w^{\mathcal{U}}))^\dagger$.

4.3. Hecke operators. — For all ℓ prime to Np, we have defined in §3.7 Hecke operators T_ℓ and S_ℓ, which are homomorphisms from $S(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})^\dagger$ to $S(\mu_N, \ell\mathfrak{c}, \kappa^{\mathcal{U}})^\dagger$ and $S(\mu_N, \ell^2\mathfrak{c}, \kappa^{\mathcal{U}})^\dagger$ respectively. They thus induce operators on $\mathrm{S}(\mu_N)^\dagger$. We have also defined an action of U_p on each $S(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})^\dagger$ (see the paragraph before Lemma 3.27). This induces an action on $\mathrm{S}(\mu_N)^\dagger$, as U_p commutes with the projector.

Let $\mathcal{H}^{Np}$ be the commutative K-algebra, generated by the operators T_ℓ and S_ℓ for ℓ prime to Np.

In the same vein we now define operators $U_{\mathfrak{P}_i}$ for $1 \leq i \leq f$. Let us choose $x_i \in F^{\times>0}$ such that $v_{\mathfrak{P}_i}(x_i) = 1$ and $v_{\mathfrak{P}_j}(x_i) = 0$ if $j \neq i$. For all fractional ideal $\mathfrak{c} \in \mathrm{Frac}(F)^{(p)}$, the ideal $\mathfrak{P}_i x_i^{-1}\mathfrak{c} \in \mathrm{Frac}(F)^{(p)}$. Moreover, the multiplication by $x_i\colon \mathfrak{P}_i x_i^{-1}\mathfrak{c} \to \mathfrak{P}_i\mathfrak{c}$ is a positive isomorphism. Using such isomorphism we can identify $S(\mu_N, \mathfrak{P}_i x_i^{-1}\mathfrak{c}, \kappa^{\mathcal{U}})^\dagger$ with $S(\mu_N, \mathfrak{P}_i\mathfrak{c}, \kappa^{\mathcal{U}})^\dagger$. Thus we can view the operator $U_{\mathfrak{P}_i}$, defined in §3.7, as an homomorphism from $S(\mu_N, \mathfrak{c}, \kappa^{\mathcal{U}})^\dagger$ to $S(\mu_N, \mathfrak{P}_i x_i^{-1}\mathfrak{c}, \kappa^{\mathcal{U}})^\dagger$. This induces an action on $\mathrm{S}(\mu_N)^\dagger$. This action is non-canonical because it depends on the choice of the x_i's (for the U_p-operator we used the canonical choice p).

We let $\mathcal{U}_p$ be K-algebra generated by the operators $U_{\mathfrak{P}_i}$. In total, we have an action of the algebra $\mathcal{H}^{Np} \otimes_K \mathcal{U}_p$ on $\mathrm{S}(\mu_N)^\dagger$.

Remark 4.7. — It is worth remarking that our normalization of U_p does not coincide with the usual normalization in the classical case. Namely, if (ν, w) is an algebraic weight, let us set $(U_p)_{class} = \nu(p)U_p$. Then this operator coincides, on classical modular forms of weight $\kappa = 2\nu + w$, with the usual operator at p. The quantity $\nu(p)$ does not vary analytically with ν, this is the reason for this renormalisation.

In general if (ν, w) is an algebraic weight and we choose $x_i \in F^{\times>0}$ such that $v_{\mathfrak{P}_i}(x_i) = 1$ and $v_{\mathfrak{P}_j}(x_i) = 0$ if $j \neq i$, let us set $(U_{\mathfrak{P}_i})_{class} = \nu(x_i)U_{\mathfrak{P}_i}$. Then this operator is canonical and coincides, on classical modular forms of weight $\kappa = 2\nu + w$, with the usual operator at $\mathfrak{P}_i$. We warn the reader that, as endomorphisms of $\mathrm{S}(\mu_N)^\dagger$, we have (with our normalization) $\prod_i U_{\mathfrak{P}_i}^{e_i} = \nu(p \prod x_i^{-e_i}) U_p$. This makes sense as $p \prod x_i^{-e_i}$ is a p-adic unit.

For all $(\nu, w) \in \mathcal{W}^G$, the space $\mathrm{S}(\mu_N)^\dagger_{(\nu,w)}$ has a slope decomposition for the compact operator U_p. On the finite slope part, this slope decomposition can be refined by considering separately the slopes of the operators $U_{\mathfrak{P}_i}$. For a finite slope form $f \in \mathrm{S}(\mu_N)^\dagger_{(\nu,w)}$, we shall say that f has slope $\leq h$ for $\mathfrak{P}_i$ if f is in annihilated by a unitary polynomial in $U_{\mathfrak{P}_i}$ whose roots have valuation less than h.

5. The arithmetic eigenvariety

By the arithmetic eigenvariety in the title of this section we refer to the eigenvariety parameterizing overconvergent eigenforms for the group G. This is produced from the data of the action of the commutative algebra $\mathcal{H}^{Np} \otimes_K \mathcal{U}_p$ on $\mathrm{S}(\mu_N)^\dagger$ (see for example [**9**], sect. 1).

Let $\mathcal{Z} \to \mathcal{W}^G$ be the spectral variety associated to the compact operator U_p. Let $\mathrm{S}(\mu_N)^{U_p-fs}$ be the coherent sheaf "generalized eigenspace" over $\mathcal{Z}$.

We define a map $\Theta\colon \mathcal{H}^{Np} \otimes \mathcal{U}_p \to \mathrm{End}_{\mathcal{O}_\mathcal{Z}}(\mathrm{S}(\mu_N)^{U_p-fs})$ and let $\mathcal{O}_\mathcal{E}$ be the sheaf of finite torsion free $\mathcal{O}_\mathcal{Z}$-algebra generated by the image of Θ. We let $\mathcal{E} \to \mathcal{Z}$ be the finite map corresponding to the sheaf $\mathcal{O}_\mathcal{E}$ and $\mathrm{w}\colon \mathcal{E} \to \mathcal{W}^G$ be the projection to the weight space.

Theorem 5.1. — *The eigenvariety* $\mathrm{w}\colon \mathcal{E} \to \mathcal{W}^G$ *as the following properties:*

- *The eigenvariety is equidimensional of dimension* $g+1$*;*
- *The map* w *is, locally on* $\mathcal{E}$ *and* $\mathcal{W}^G$*, finite and surjective;*
- *We have a universal Hecke character* $\Theta\colon \mathcal{H}^{Np} \otimes \mathcal{U}_p \to \mathcal{O}_\mathcal{E}$*;*
- *For all* $(\nu, w) \in \mathcal{W}^G$*,* $\mathrm{w}^{-1}(\nu, w)$ *is in bijection with the finite slope eigensystems occuring in* $\mathrm{S}(\mu_N)^\dagger|_{(\nu,w)} = \mathrm{S}(\mu_N, (\nu, w))^\dagger$.
- *Let* (ν, w) *be locally algebraic characters. Set* $\nu^{alg} = \sum_{\sigma \in I} \nu_\sigma . \sigma$ *and* $w^{alg} = w_0(\sum_{\sigma\in I} \sigma)$ *where* $\nu_\sigma, w \in \mathbb{Z}$*. Set* $k_\sigma = 2\nu_\sigma + w_0$*. Let* $f \in \mathrm{S}(\mu_N, (\nu, w))^\dagger$ *be a finite slope overconvergent modular form whose* $\mathfrak{P}_i$*-slope is less than* s_i *for* $1 \leq i \leq f$*. Assume that for all* $\sigma \in I$ *and for all* $\mathfrak{P}_i$ *we have the small slope condition:*

  $$\textit{If } v(\sigma(\mathfrak{P}_i)) > 0,\ k_\sigma v(\sigma(\mathfrak{P}_i)) > s_i + f_i$$

  Then f *is a classical form.*
- *Let* $G_{F,Np}$ *be the Galois group of the maximal extension of* F *unramified outside* Np*. We have a universal pseudo-character* $T\colon G_{F,Np} \to \mathcal{O}_\mathcal{E}$ *satisfying* $T(Frob_\ell) = \theta(T_\ell)$ *for all prime ideal* ℓ *of* $\mathcal{O}_F$ *prime to* Np *(Here,* $Frob_\ell$ *is a geometric Frobenius).*
- *For all* $x \in \mathcal{E}$ *there is a semi-simple Galois representation* $\rho_x\colon G_{F,Np} \to \mathrm{GL}_2(\bar{k}(x))$ *characterized by* $\mathrm{Tr}(\rho_x) = T|_x$*,* $\det(\rho_x) = \Theta|_x(S_\ell) N_{F/\mathbb{Q}}(\ell)$.

Proof. — The first three points follow by construction. The fourth point is a consequence of Theorem 4.4. The fifth point is the main result of [**22**] in the unramified case, and [**7**] in the general case. Galois representations have been attached to classical eigenforms of cohomological weight by Carayol [**10**], completed by Taylor [**24**]. The technique of pseudo-representations gives the last two points of the theorem (see [**6**] sect. 7.5 for example). □

Remark 5.2. — For the convenience of the reader, let us add some precision on our normalizations. Let (ν, w) be an algebraic weight and $\kappa = \sum_\sigma k_\sigma.\sigma = 2\nu + w$ with $w = w_0(\sum_\sigma \sigma)$. The normalization of our Hecke operator T_ℓ acting on the space of classical arithmetic Hilbert modular forms of weight κ coincides with Shimura's operator $T'(\ell)$ (see [**23**], p. 650) if $w_0 = \sup_\sigma\{k_\sigma\}$. The Galois representations attached to a cuspidal eigenform of weight (ν, w) as above has determinant $\chi_f \chi_p^{1-w_0}$ where $w = w_0(\sum_\sigma \sigma)$, χ_f is a finite character and χ_p is the cyclotomic character. Its Hodge-Tate weights are the $(\frac{k_\sigma + w_0}{2}, \frac{-k_\sigma + w_0}{2} + 1)_{\sigma \in I}$.

6. An appendix: Some toric geometry

We put ourself in the following context. Let k be a field. Let T be a split torus over k with character group M and cocharacter group N and let $U \subset \mathrm{GL}(M)$ be a subgroup. Fix a rational polyhedral cone $C \subset N \otimes \mathbb{R}$ stable for the action of U and a smooth, U-admissible rational polyhedral cone decomposition $\Sigma = \{\sigma_\alpha\}$ of C in the sense of [**15**, Def. IV.2.2]. Let S_Σ be the associated torus embedding over k with action of U. Recall from [**15**, Def. IV.2.4 & Def. V.5.1] the following:

Definition 6.1. — A U-invariant polarization function for Σ is a continuous U-invariant function $h\colon C \to \mathbb{R}_{\geq 0}$ such that:

(a) it is $\mathbb{Z}$-valued on $N \cap C$ and it is linear on each $\sigma \in \Sigma$;

(b) it is upper convex, i. e., $h\big(tx + (1-t)y\big) \geq th(x) + (1-t)h(y)$ for every x and $y \in C$ and every $0 \leq t \leq 1$;

(c) it is *strictly upper convex* w. r. t Σ, i.e., Σ is the coarsest among the fans Σ' of $|\Sigma|$ for which h is Σ'-linear. Equivalently, the closure of the top dimensional cones of Σ are exactly the maximal polyhedral cones of C on which h is linear.

Associated to such a function we get a U-invariant Cartier divisor $D_h := -\sum_{\rho \in \Sigma(1)} \alpha_\rho D_\rho$ where $\Sigma(1)$ is the set of 1-dimensional faces of Σ and the positive integers α_ρ are defined as follows. Let $n(\rho)$ be the unique primitive element of $N \cap \rho$ such that $\rho = \mathbb{R}_{\geq 0} n(\rho)$. Then $\alpha_\rho := h\big(n(\rho)\big)$. Following [**15**, Def. 2.1, Appendix] we say that D_h is a very ample Cartier divisor if the global sections of $\mathcal{O}_{S_\Sigma}(D_h)$ form a basis of the topology of S_Σ. Then,

Proposition 6.2. — *The divisor D_h is a very ample Cartier divisor.*

Proof. — Conditions (b) and (c) in Definition 6.1 imply that the morphism

$$S_\Sigma \longrightarrow \mathbf{Proj}\left(\bigoplus_s \mathrm{H}^0\big(S_\Sigma, \mathscr{O}_{S_\Sigma}(D_h)\big)^{\otimes^s}\right)$$

of schemes over k is a closed immersion due to [**13**, Cor. IV.4.1(i) & Pf. Thm. IV.4.2]. The conclusion follows. □

Let $D' \subset S_\Sigma$ be a U-invariant toric $\mathbf{Q}$-Weil divisor given by $D' = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$. As the set $\Sigma(1)/U$ is finite there exists $\ell \in \mathbb{N}$ such that $a_\rho < \ell$ and ℓa_ρ is an integer for every $\rho \in \Sigma(1)$. In particular $\ell D' := \sum_{\rho\in\Sigma(1)} \ell a_\rho D_\rho$ is a Cartier divisor. For $a \in \mathbb{Q}$ define $\lfloor a\rfloor$ as the greatest integer $\leq a$. Denote by $\lfloor D'\rfloor$ the Cartier divisor (the "round down") associated to D' as

$$\lfloor D'\rfloor := \sum_\rho \lfloor a_\rho\rfloor D_\rho.$$

Multiplication by ℓ on M and N preserves Σ and induces a finite and flat, U-equivariant morphism $\Phi_\ell\colon S_\Sigma \to S_\Sigma$ which realizes S_Σ as the quotient of S_Σ by a group scheme G isomorphic to $(\mu_\ell)^s$ for some $s \in \mathbb{N}$. Moreover, $\Phi_\ell^*(\lfloor D'\rfloor) = \sum_\rho \ell\lfloor a_\rho\rfloor D_\rho$ so that $\ell D' - \Phi_\ell^*(\lfloor D'\rfloor) = \sum_\rho \ell\big(a_\rho - \lfloor a_\rho\rfloor\big) D_\rho$ is an effective Cartier divisor. By adjunction we get an injective map $\Phi_{\ell,*}\colon \mathscr{O}_{S_\Sigma}\big(\lfloor D'\rfloor\big) \to \Phi_{\ell,*}\big(\mathscr{O}_{S_\Sigma}(\ell D')\big)$.

Lemma 6.3. — *The map $\Phi_{\ell,*}$ admits a U-equivariant left inverse.*

Proof. — It is the content of [**12**, Lemma 9.3.4]. The splitting is constructed on the affine charts given by the σ_α's first and it is then proven to glue. It is easily verified to be U-equivariant. □

Take $\ell \in \mathbb{N}$ such that $\alpha_\rho < \ell$ for every ρ. Then, $\lfloor \ell^{-1} D_h\rfloor = \sum_{\rho\in\Sigma(1)} -D_\rho = -D$ defines the boundary of S_Σ. Due to Lemma 6.3 we get a U-equivariant map $\Phi_\ell\colon S_\Sigma \to S_\Sigma$ and a U-equivariant injective morphism $\Phi_{\ell,*}\colon \mathscr{O}_{S_\Sigma}(-D) \to \Phi_{\ell,*}\big(\mathscr{O}_{S_\Sigma}(D_h)\big)$ admitting a U-equivariant left inverse. Passing to completions with respect to the ideal $\mathscr{O}_{S_\Sigma}(-D)$ we get a finite and flat, U-equivariant map

$$\widehat{\Phi}_\ell\colon \widehat{S}_\Sigma \longrightarrow \widehat{S}_\Sigma,$$

inducing an injective U-equivariant morphism $\widehat{\Phi}_{\ell,*}\colon \mathscr{O}_{\widehat{S}_\Sigma}\big(-D\big) \to \widehat{\Phi}_{\ell,*}\big(\mathscr{O}_{\widehat{S}_\Sigma}(D_h)\big)$ which admits a U-equivariant left inverse.

Let $\widetilde{\widehat{S}}_\Sigma$ (resp. $\widetilde{D}$, resp. $\widetilde{D}_h$) be the quotient of $\widehat{S}_\Sigma$ for the action of U. We we get a finite and flat morphism

$$\widetilde{\widehat{\Phi}}_\ell\colon \widetilde{\widehat{S}}_\Sigma \longrightarrow \widetilde{\widehat{S}}_\Sigma,$$

inducing a split injective morphism $\widehat{\Phi}_{\ell,*}\colon \mathscr{O}_{\widetilde{\widehat{S}}_\Sigma}(-\widetilde{D}) \to \widehat{\Phi}_{\ell,*}\big(\mathscr{O}_{\widetilde{\widehat{S}}_\Sigma}(\widetilde{D}_h)\big)$. Taking cohomology, we deduce for ever $q \geq 0$ a split injective map

$$\mathrm{H}^q\big(\widetilde{\widehat{S}}_\Sigma, \mathscr{O}_{\widetilde{\widehat{S}}_\Sigma}(-\widetilde{D})\big) \longrightarrow \mathrm{H}^q\big(\widetilde{\widehat{S}}_\Sigma, \widehat{\Phi}_{\ell,*}\big(\mathscr{O}_{\widetilde{\widehat{S}}_\Sigma}(\widetilde{D}_h)\big)\big) \cong \mathrm{H}^q\big(\widetilde{\widehat{S}}_\Sigma, \mathscr{O}_{\widetilde{\widehat{S}}_\Sigma}(\widetilde{D}_h)\big).$$

The last isomorphism is induced by the finite map $\widetilde{\widetilde{\Phi}}_\ell$. The support of the divisor D is the union of irreducible components $Z := \bigcup_{i\in I} Z_i$ which are proper and smooth over k and the topological space underlying the formal scheme $\widehat{S}_\Sigma$ is the one defined by Z. Then, the support of $\widetilde{D}$ is the union of irreducible components $\widetilde{Z} = Z/U := \bigcup_{j=1}^n \widetilde{Z}_j$ which are proper over k and $\widetilde{Z}$ defines the topological space underlying $\widetilde{\widehat{S}}_\Sigma$. We claim that the sheaf $\mathcal{O}_{\widetilde{\widehat{S}}_\Sigma}(\widetilde{D}_h)$ is ample, i.e., by definition of ampleness of invertible sheaves on formal schemes, that its restriction $\widetilde{L}_h$ to the boundary $\widetilde{Z}$ is ample. Taking a finite subset $J \subset I$ large enough, the induced map $Z_J = \bigcup_{i\in J} Z_i \subset Z \to Z/U = \widetilde{Z}$ is finite and surjective and, hence, it suffices to prove that the pull–back of $\widetilde{L}$ to Z_J is ample. But this coincides with the restriction of $\mathcal{O}_{S_\Sigma}(D_h)$ to Z_J which is ample thanks to Lemma 6.2. Possibly replacing h by a polarization function defined by a positive multiple dh, which amounts to replace $\mathcal{O}_{\widetilde{\widehat{S}}_\Sigma}(D_h)$ with $\mathcal{O}_{\widetilde{\widehat{S}}_\Sigma}(D_{dh}) \cong \mathcal{O}_{\widetilde{\widehat{S}}_\Sigma}(D_h)^{\otimes^d}$, we may even assume that the invertible sheaf $\mathcal{O}_{\widetilde{\widehat{S}}_\Sigma}(D_h)$ is very ample. In particular, its cohomology groups in degrees $q \geq 1$ vanish. We conclude that

Proposition 6.4. — *We have* $\mathrm{H}^q\big(\widetilde{\widehat{S}}_\Sigma, \mathcal{O}_{\widetilde{\widehat{S}}_\Sigma}(-D)\big) = 0$ *for all* $q > 0$.

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