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Annette Huber & Stefan Kebekus & Shane Kelly

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Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
`bullsmf@ihp.fr` • `smf.emath.fr`

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DIFFERENTIAL FORMS IN POSITIVE CHARACTERISTIC AVOIDING RESOLUTION OF SINGULARITIES

BY ANNETTE HUBER, STEFAN KEBEKUS & SHANE KELLY

ABSTRACT. — This paper studies several notions of sheaves of differential forms that are better behaved on singular varieties than Kähler differentials. Our main focus lies on varieties that are defined over fields of positive characteristic. We identify two promising notions: the sheafification with respect to the cdh-topology, and right Kan extension from the subcategory of smooth varieties to the category of all varieties. Our main results are that both are cdh-sheaves and agree with Kähler differentials on smooth varieties. They agree on all varieties under weak resolution of singularities.

A number of examples highlight the difficulties that arise with torsion forms and with alternative candiates.

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ANNETTE HUBER, Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Eckerstraße 1, 79104 Freiburg im Breisgau, Germany • *E-mail :* annette.huber@math.uni-freiburg.de

STEFAN KEBEKUS, Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Eckerstraße 1, 79104 Freiburg im Breisgau, Germany and University of Strasbourg Institute for Advanced Study (USIAS), Strasbourg, France • *E-mail :* stefan.kebekus@math.uni-freiburg.de

SHANE KELLY, Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Eckerstraße 1, 79104 Freiburg im Breisgau, Germany • *E-mail :* shanekelly64@gmail.com

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1. Introduction

Sheaves of differential forms play a key role in many areas of algebraic and arithmetic geometry, including birational geometry and singularity theory. On singular schemes, however, their usefulness is limited by bad behavior such as the presence of torsion sections. There are a number of competing modifications of these sheaves, each generalizing one particular aspect. For a survey see the introduction of [14].

In this article we consider two modifications, Ω^n_{cdh} and Ω^n_{dvr}, to the presheaves Ω^n of relative k-differentials on the category $\mathsf{Sch}(k)$ of separated finite type k-schemes. By Ω^n_{cdh} we mean the sheafification of Ω^n with respect to the cdh-topology, cf. Definition 5.5, and by Ω^n_{dvr} we mean the right Kan extension along the inclusion $\mathsf{Sm}(k) \to \mathsf{Sch}(k)$ of the restriction of Ω^n to the category $\mathsf{Sm}(k)$ of smooth k-schemes, cf. Definition 5.2.

The following are three of our main results.

THEOREM 1.1. — *Let k be a perfect field and $n \geq 0$.*

1. *(Theorem 5.11). If X is a smooth k-variety then $\Omega^n(X) \cong \Omega^n_{\mathrm{cdh}}(X)$. The same is true in the rh- or eh-topology.*
2. *(Observation 5.3, Proposition 5.12). Ω^n_{dvr} is a* cdh-*sheaf and the canonical morphism*
$$\Omega^n_{\mathrm{cdh}} \to \Omega^n_{\mathrm{dvr}}$$
is a monomorphism.
3. *(Proposition 5.13). Under weak resolution of singularities, this canonical morphism is an isomorphism $\Omega^n_{\mathrm{cdh}} \cong \Omega^n_{\mathrm{dvr}}$.* □

Item 1 was already observed by Geisser, assuming a strong form of resolution of singularities, [8]. We are able to give a proof which does not assume any conjectures. The basic input into the proof is a fact about torsion forms (Theorem 5.8): given a torsion form on an integral variety, there is a blow-up such the pull-back of the form vanishes on the blow-up.

1.1. Comparison to known results in characteristic zero. — This paper aims to extend the results of [14] to positive characteristic, avoiding to assume resolution of singularities if possible. The following theorem summarizes the main results known in characteristic zero.

THEOREM 1.2 ([14]). — *Let k be a field of characteristic zero, X a separated finite type k-scheme, and $n \geq 0$.*

1. *The restriction of Ω^n_{h} to the small Zariski site of X is a torsion-free coherent sheaf of $\mathcal{O}_X$-modules.*
2. *If X is reduced we have*
$$\Omega^n(X)/\{\textit{torsion elements}\} \subseteq \Omega^n_{\mathrm{h}}(X)$$

and if X is Zariski-locally isomorphic to a normal crossings divisor in a smooth variety then

$$\Omega^n(X)/\{\textit{torsion elements}\} \cong \Omega^n_{\mathrm{h}}(X).$$

3. *If X is smooth, then $\Omega^n(X) \cong \Omega^n_{\mathrm{h}}(X)$ and $H^i_{Zar}(X, \Omega^n) \cong H^i_{\mathrm{h}}(X, \Omega^n_{\mathrm{h}})$ for all $i \geq 0$. The same is true using the* cdh-*or* eh-*topology in place of the* h-*topology.*
4. *We have $\Omega^n_{\mathrm{dvr}} \cong \Omega^n_h$, cf. Definition 5.2.* □

Failure of Items 1 and 2 in positive characteristic. — In positive characteristic, the first obstacle to this program one discovers is that $\Omega^n_{\mathrm{h}} = 0$ for $n \geq 1$, cf. Lemma 6.1. This is due to the fact that the geometric Frobenius is an h-cover, which induces the zero morphism on differentials. However, almost all of the results of [14] are already valid in the coarser cdh-topology, and remain valid in positive characteristic if one assumes that resolutions of singularities exist. So let us use the cdh-topology in place of the h-topology. But even then, Items 1 and 2 of Theorem 1.2 seem to be lost causes:

Corollary 1.3 (Corollary 5.16, Corollary 5.17, Example 3.6). — *For perfect fields of positive characteristic, there exist varieties X such that the restriction of Ω^1_{cdh} to the small Zariski site of X is not torsion-free.*

Moreover, there exist morphisms $Y \to X$ and torsion elements of $\Omega^1_{\mathrm{cdh}}(X)$ (resp. $\Omega^1(X)$) whose pull-back to $\Omega^1_{\mathrm{cdh}}(Y)$ (resp. $\Omega^1(Y)$) are not torsion. □

Note that functoriality of torsion forms over the complex numbers is true, cf. Theorem 3.3, [18, Corollary 2.7].

Positive results. — On the positive side, Item 1 in Theorem 1.1 can be seen as an analog of Item 3 in Theorem 1.2. In particular, we can give an unconditional statement of the case $i = 0$. In a similar vein, Items 2 and 3 of Theorem 1.1 relate to Item 4 in Theorem 1.2.

1.2. Other results. — Many of the properties of Ω^n_{dvr} hold for a more general class of presheaves, namely unramified presheaves, introduced by Morel, cf. Definition 4.5. The results mentioned above are based on the following very general result which should be of independent interest.

Proposition 1.4 (Proposition 4.18). — *Let S be a Noetherian scheme. If $\mathscr{F}$ is an unramified presheaf on $\mathsf{Sch}(S)$ then $\mathscr{F}_{\mathrm{dvr}}$ is an* rh-*sheaf. In particular, if $\mathscr{F}$ is an unramified Nisnevich (resp. étale) sheaf on $\mathsf{Sch}(S)$ then $\mathscr{F}_{\mathrm{dvr}}$ is a* cdh-*sheaf (resp.* eh-*sheaf).* □

In our effort to avoid assuming resolution of singularities, we investigated the possibility of a topology sitting between the cdh-and h-topologies which might allow the theorems of de Jong or Gabber on alterations to be used in place of resolution of singularities. An example of the successful application of such an idea is [19] where the $l$dh-topology is introduced and successfully used as a replacement to the cdh-topology. Section 6 proposes a number of new, initially promising sites, cf. Definitions 6.2 and 6.8, but then also shows in Example 6.5 that, somewhat surprisingly, the sheafification of Ω^1 on these sites does not preserve its values on regular schemes, cf. Proposition 6.6 and Lemma 6.12.

1.3. Outline of the paper. — After fixing notation in Section 2, the paper begins in Section 3 with a discussion of torsion- and torsion-free differentials. Section 4 contains a general discussion of the relevant properties of unramified presheaves, whereas properties that are specific to Ω^1 are collected in Section 5. Section 5 discusses our proposals for a good presheaf of differentials on singular schemes—Ω^1_{cdh} and Ω^1_{dvr}—and their properties. Appendix A gives the proof of the above mentioned result on killing torsion forms by blow-up. We also discuss a hyperplane section criterion for testing the vanishing of torsion forms.

1.4. Open problems. — What is missing from this paper is a full cdh-analog of Theorem 1.2, Item 3. Assuming resolutions of singularities, Geisser has shown [8] that the cdh-cohomology of Ω^n_{cdh} agrees with Zariski-cohomology of Ω^n on all smooth varieties X. It remains open if this can be extended unconditionally to Ω^n_{cdh} and Ω^n_{dvr}.

In a similar vein, we do not know if the assumption on resolutions of singularities can be removed from Item 3 of Theorem 1.1.

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2. Notation and Conventions

2.1. Global assumptions. — Throughout the present paper, all schemes are assumed to be separated. The letter S will always denote a fixed, separated, Noetherian base scheme.

2.2. Categories of schemes and presheaves. — Denote by $\mathsf{Sch}(S)$ the category of separated schemes of finite type over S, and let $\mathsf{Reg}(S)$ be the full subcategory of regular schemes in $\mathsf{Sch}(S)$. If S is the spectrum of a field k, we also write $\mathsf{Sch}(k)$ and $\mathsf{Reg}(k)$. If k is perfect, then $\mathsf{Reg}(k)$ is the category of smooth k-varieties, which need not necessarily be connected.

Notation 2.1 (Presheaf on $\mathsf{Sch}(S)$). — Given a Noetherian scheme S, a presheaf $\mathscr{F}$ of abelian groups on $\mathsf{Sch}(S)$ is simply a contravariant functor

$$\mathsf{Sch}(S) \to \{\text{abelian groups}\}.$$

A presheaf $\mathscr{F}$ is called a presheaf of $\mathscr{O}$-modules, if every $\mathscr{F}(X)$ has an $\mathscr{O}_X(X)$-module structure such that for every morphism $Y \to X$ in $\mathsf{Sch}(S)$, the induced maps $\mathscr{F}(X) \to \mathscr{F}(Y)$ are compatible with the morphisms of rings of functions $\mathscr{O}(X) \to \mathscr{O}(Y)$.

We are particularly interested in the presheaf of Kähler differentials.

Example 2.2 (Structure sheaf, Kähler differentials). — We denote by $\mathscr{O}$ the presheaf $X \mapsto \mathscr{O}_X(X)$. Given any $n \in \mathbb{N}$, we denote by Ω^n the presheaf $X \mapsto \Omega^n_{X/S}(X)$. Note that $\Omega^0 = \mathscr{O}$. We also abbreviate $\Omega = \Omega^1$.

The notation Ω^n_X means the usual Zariski-sheaf on X. That is, $\Omega^n_X = \Omega^n|_{X_{\mathrm{Zar}}}$ where X_{Zar} is the usual topological space associated to the scheme X.

Definition 2.3 (Torsion). — Let $\mathscr{F}$ be a presheaf on $\mathsf{Sch}(S)$ and $X \in \mathsf{Sch}(S)$. We write $\operatorname{tor}\mathscr{F}(X)$ for the set of those sections of $\mathscr{F}(X)$ which vanish on a dense open subscheme.

Warning 2.4 (Torsion groups might not form a presheaf). — In the setting of Definition 2.3 note that the groups $\operatorname{tor}\mathscr{F}(X)$ do not necessarily have the structure of a presheaf on $\mathsf{Sch}(S)$! For a morphism $Y \to X$ in $\mathsf{Sch}(S)$, the image of $\operatorname{tor}\mathscr{F}(X)$ under the morphism $\mathscr{F}(X) \to \mathscr{F}(Y)$ does not necessarily lie in $\operatorname{tor}\mathscr{F}(Y)$. A very simple example (pointed out to us by a referee) is the section x on $\operatorname{Spec} k[x,y]/(x^2,xy)$. It is torsion, but its pull-back to $\operatorname{Spec} k[x]/(x^2)$ is not. Example 3.6 on page 315 shows that the problem also happens for reduced varieties in positive characteristic in the case where $\mathscr{F} = \Omega$. On the positive side, note that the restriction of a torsion section to an open subscheme is again a torsion section.

Warning 2.5 (Possible conflict with commutative algebra). — Note that in the case that $\mathscr{F}$ is an $\mathscr{O}$-module, there is a competing notion of torsion element: an element $t \in \mathscr{F}(X)$ is torsion if Zariski locally on X there is a non-zero divisor $s \in \mathscr{O}(X)$ such that $s \cdot t = 0$. If X is reduced (and Noetherian) and $\mathscr{F}|_{X_{Zar}}$ is quasi-coherent then these two notions of "torsion" agree by [12, Proposition (8.4.6)], see also the Appendix of [18], but in general they may differ.

Indeed, if $\mathscr{F}|_{X_{Zar}}$ is the skyscraper sheaf at a point $x \in X$ with value $\mathscr{O}_{X,x}$ we get a counterexample in the non-quasi-coherent case. The characterization also fails when $\mathscr{F}|_{X_{Zar}}$ is quasi-coherent, and when X is non-reduced as the example $X = \operatorname{Spec}(k[x,y]/(xy,y^2))$ and the element $y \in \operatorname{tor}\mathscr{O}(X)$ shows.

Moreover, one can see that the "Zariski locally" part of the above definition is also necessary by considering the structure sheaf $i_*\mathscr{O}_x$ of a closed point $i : x \to X$ of any integral projective variety X of dimension > 0.

2.3. Schemes and morphisms of essentially finite type. — Parts of Section 4 use the notion of "S-schemes that are essentially of finite type". While this notion has been used at several places in the literature, we were not able to find a convenient reference for its definition. We have therefore chosen to include a definition and a brief discussion here.

DEFINITION 2.6. — We say that an S-scheme X' is *essentially of finite type* over S if there is a scheme X of finite type over S and a filtered inverse system $\{U_i\}_{i\in I}$ of open subschemes of X with affine transition maps such that $X' = \bigcap_{i\in I} U_i$.

LEMMA 2.7 (Morphisms of schemes essentially of finite type). — *Let X and Y be in $\mathsf{Sch}(S)$. Let $\{U_i\}_{i\in I}$ be a filtered inverse system of open subschemes of X with affine transitions maps with intersection X' and $\{V_j\}_{j\in J}$ a filtered inverse system of open subschemes of Y with affine transition maps with intersection Y'. Then*

$$\mathrm{Mor}_S(X', Y') = \varprojlim_j \varinjlim_i \mathrm{Mor}_S(U_i, V_j) \ .$$

REMARK 2.8. — Using the language of pro-categories, briefly recalled in Section 4.3, Lemma 2.7 asserts that the category of schemes essentially of finite type over S is a full subcategory of the pro-category of $\mathsf{Sch}(S)$.

Proof of Lemma 2.7. — This is just a special case of [10, Corollaire 8.13.2]. The key point of the argument is that a morphism towards a finite type S-scheme with source Spec of the local ring of a variety always extends to an open neighborhood. □

EXAMPLE 2.9. — Let $X \in \mathsf{Sch}(S)$ and $\mathfrak{U} = \{U_i\}_{i\in I}$ be a filtered inverse system of open affine subschemes of X and with intersection X'. Let U be an open affine neighborhood of X' in X. By Lemma 2.7, we have

$$\mathrm{Mor}_X(X', U) = \varinjlim_i \mathrm{Mor}_X(U_i, U).$$

Hence the inclusion $X' \to U$ factors via some U_i. This means that $\mathfrak{U}$ is cofinal in the system of all affine open neighborhoods of X'.

2.4. Topologies. — We are going to use various topologies on $\mathsf{Sch}(S)$, which we want to introduce now. They are variants of the h-topology introduced by Voevodsky in [26]. Recall that a Grothendieck topology on $\mathsf{Sch}(S)$ is defined by specifying for each $X \in \mathsf{Sch}(S)$ which collections $\{U_i \to X\}_{i\in I}$ of S-morphisms should be considered as open covers. By definition, a presheaf $\mathscr{F}$ is a sheaf if for any such collection, $\mathscr{F}(X)$ is equal to the set of those elements $(s_i)_{i\in I}$ in $\prod_{i\in I} \mathscr{F}(U_i)$ for which $s_i|_{U_i\times_X U_j} = s_j|_{U_i\times_X U_j}$ for every $i, j \in I$.

We refer to the ordinary topology as the Zariski topology.

DEFINITION 2.10 (cdp-morphism). — A morphism $f : Y \to X$ is called a cdp-*morphism* if it is proper and completely decomposed, where by "completely decomposed" we mean that for every point $x \in X$ there is a point $y \in Y$ with $f(y) = x$ and $[k(y) : k(x)] = 1$.

These morphisms are also referred to as proper cdh-*covers*, or *envelopes* in the literature.

REMARK 2.11 (rh, cdh and eh-topologies). — Recall that the rh-topology on $\mathsf{Sch}(S)$ is generated by the Zariski topology and cdp-morphisms, [9]. In a similar vein, the cdh-topology is generated by the Nisnevich topology and cdp-morphisms, [25, § 5]. The eh-topology is generated by the étale topology and cdp-morphisms, [8].

We are going to need the following fact from algebraic geometry.

LEMMA 2.12. — *Let X be a regular, Noetherian scheme. Let $x \in X$ be a point of codimension n. Then there is point $y \in X$ of codimension $n-1$, a discrete valuation ring R essentially of finite type over X together with a map* Spec $R \to X$ *such that the special point of* Spec R *maps to x and the generic point to y, both inducing isomorphisms on their respective residue fields.*

Proof. — The local ring $\mathscr{O}_{X,x}$ is a regular local ring, and as such admits a regular sequence $f_1, \ldots, f_n$ generating its maximal ideal. The quotient ring

$$R := \mathscr{O}_{X,x} \big/ \langle f_1, \ldots, f_{n-1} \rangle$$

is then a regular local ring of dimension one, that is, a discrete valuation ring, [21, Theorems 36(3) and 17.G]. Let y be the image of the generic point of Spec R. By construction, this is a point of codimension $n-1$. □

PROPOSITION 2.13 (Birational– and cdp-morphisms). — *Suppose that X is a regular, Noetherian scheme. Then, every proper, birational morphism is a* cdp-*morphism, and every* cdp-*morphism is refinable by a proper, birational morphism.*

REMARK 2.14. — This fact is well-known over a field of characteristic zero and is usually proven using strong resolution of singularities. That is, by refining a proper birational morphism by a sequence of blow-ups with smooth centers. By contrast, the proof below works for *any* regular, Noetherian scheme X, without restriction on a potential base scheme, or structural morphism.

Proof of Proposition 2.13. — With X as in Proposition 2.13, we show the two statements separately.

Step 1: Birational morphisms are cdp. — Let $Y \to X$ be proper and birational. We must show that for every point $x \in X$ the canonical inclusion admits a factorization $x \to Y \to X$. We proceed by induction on the codimension. In codimension zero, the factorization is a consequence of birationality. Suppose that it is true up to codimension $n-1$ and let x be a point of codimension n. By Lemma 2.12 we can find a discrete valuation ring R and a diagram

$$\begin{tikzcd} y \arrow[dr] & & \\ & \operatorname{Spec} R \arrow[r] & X \\ x \arrow[ur] & & \end{tikzcd}$$

for some y of codimension $n-1$. By the inductive hypothesis, the inclusion of y into X admits a factorization through Y, so we have a commutative diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} R \arrow[r] & X \end{tikzcd}$$

and now the valuative criterion for properness implies that the inclusion of Spec R into X factors trough Y, and therefore so does the inclusion of x.

Step 2: cdp-*morphisms are refinable.* — If $Y \to X$ is a proper, completely decomposed morphism with X connected (hence irreducible), choose a factorization $\eta \to Y \to X$ of the inclusion of the generic point η of X. Then, the closure of the image of η in Y is birational and proper over X. □

Lemma 2.15. — *Let $Y \in \mathsf{Sch}(S)$ be an integral scheme and let $\{U_i \to Y\}_{i\in I}$ be an étale cover of Y by finitely many integral schemes. Assume further that for each for $i \in I$ we are given a proper, birational morphism $T_i \to U_i$. Then there exists a proper, birational morphism $Y' \to Y$, an étale cover $\{U'_i \to Y'\}_{i\in I}$ and for each $i \in I$ a commutative diagram of the following form,*

$$\begin{tikzcd} U'_i \arrow[rr, "\text{étale}"] \arrow[d, "\text{proper, biratl.}"'] & & Y' \arrow[d, "\text{proper, biratl.}"] \\ T_i \arrow[r, "\text{proper, biratl.}"'] & U_i \arrow[r, "\text{étale}"'] & Y \end{tikzcd}$$

If $\{U_i \to Y\}_{i\in I}$ is a Nisnevich– or a Zariski-cover, then so is the cover $\{U'_i \to Y'\}_{i\in I}$.

Proof. — It follows from flattening by blow-up, [23, Théorème 5.2.2], that there exists an integral scheme Y' and a proper birational morphism $Y' \to Y$ such that for any $i \in$, the strict transforms $T'_i \to Y'$ of the $T_i \to Y$ are flat. These morphisms factor via the pullbacks $U'_i = U_i \times_Y Y'$ of the U_i,

$$T'_i \to U'_i \to Y.$$

Since $\{U_i \to Y\}_{i\in I}$ is an étale (resp. Zariski, Nisnevich) cover, so is $\{U'_i \to Y'\}_{i\in I}$.

It remains to show that $T'_i \cong U'_i$. The scheme U'_i is integral because it is proper and birational over U_i. The scheme T'_i is integral because it is proper and birational over T_i. As $T'_i \to Y'$ is flat and $U'_i \to Y'$ is unramified, the morphism $T'_i \to U'_i$ is flat by [11, Proposition 17.7.10]. We now have a flat, proper and birational morphism between integral schemes, hence an isomorphism. In detail: by flatness, the morphism $T'_i \to U'_i$ has constant fiber dimension, which, by birationality, equals zero. This means that the morphism is quasi-finite. As it is also proper, this means that it is finite and still flat. It follows that $\mathscr{O}_{T'_i}$ is a locally free $\mathscr{O}_{U'_i}$-module of constant rank. The rank is one because again the morphism is birational. □

Corollary 2.16 (Normal form). — *Let $Y \in \mathsf{Sch}(S)$, and let $\{Y_i \to Y\}_{i\in I}$ be an* rh-, cdh-*or* eh-*cover, respectively. Then there exists a refinement of the following form,*

$$\{Y'_i \longrightarrow Y' \xrightarrow{\text{cdp-}covering} Y\}_{i\in I},$$

where $\{Y'_i \to Y'\}_{i\in I}$ is a Zariski-, Nisnevich- or étale cover, respectively, with I finite. If Y is regular, then we can even assume that $Y' \to Y$ is a proper birational morphism.

Proof. — In the cdh-case, this is precisely [25, Prop. 5.9]. In the two other cases, the same argument works using Noetherian induction, and using Lemma 2.15 in the appropriate place. The last claim uses Proposition 2.13. □

3. (Non-)Functoriality of torsion-forms

One very useful feature of differential forms on a smooth varieties is that they form a vector bundle, in particular, they are torsion-free. In characteristic zero, the different candidates for a good theory of differential forms share this behavior on all varieties. It is disappointing but true that this property *fails* in positive characteristic, as we are going to establish. The following notion will be used throughout.

Definition 3.1 (Torsion-differentials and torsion-free differentials, [18, Section 2.1]). — Let k be a field and $X \in \mathsf{Sch}(k)$. We define the sheaf $\check{\Omega}^n_X$ on X_{Zar} as the cokernel in the sequence

$$0 \longrightarrow \operatorname{tor}\Omega^n_X \xrightarrow{\alpha_X} \Omega^n_X \xrightarrow{\beta_X} \check{\Omega}^n_X \longrightarrow 0. \tag{3.1.1}$$

Sections in $\operatorname{tor}\Omega^n_X$ are called *torsion-differentials*. By slight abuse of language, we refer to sections in $\check{\Omega}^n_X$ as *torsion-free differentials*.

Remark 3.2 (Torsion-sheaves on reducible spaces). — Much of the literature discusses torsion-sheaves and torsion-free sheaves only in a setting where the underlying space is irreducible. We refer to [18, Appendix A and references there] for a brief discussion of torsion-sheaves on reduced, but possibly reducible spaces.

3.1. Torsion-free forms over the complex numbers. — Given a morphism between two reduced varieties that are defined over the complex numbers, the usual pull-back map of Kähler differentials induces pull-back maps for torsion-differentials and for torsion-free differentials, even if the image of the morphism is contained in the singular set of the target variety.

Theorem 3.3 (Pull-back for sheaves of torsion-free differentials, [18, Corollary 2.7]). — *Let $f : X \to Y$ be a morphism of reduced, quasi-projective schemes that are defined over the complex numbers. Then there exist unique morphisms $d_{\mathrm{tor}} f$ and $\check{d} f$ such that the following diagram, which has exact rows, becomes commutative*

$$\begin{tikzcd}
 & f^* \operatorname{tor} \Omega^n_Y \arrow[r, "f^*\alpha_Y"] \arrow[d, "d_{\mathrm{tor}} f"'] & f^*\Omega^n_Y \arrow[r, "f^*\beta_Y"] \arrow[d, "df"'] & f^*\check{\Omega}^n_Y \arrow[r] \arrow[d, "\check{d} f"] & 0 \\
0 \arrow[r] & \operatorname{tor} \Omega^n_X \arrow[r, "\alpha_X"'] & \Omega^n_X \arrow[r, "\beta_X"'] & \check{\Omega}^n_X \arrow[r] & 0.
\end{tikzcd}$$

In other words, $\operatorname{tor} \Omega^n$ *is a presheaf on* $\mathsf{Sch}(\mathbb{C})$. □

The same argument works for any field of characteristic zero.

Remark 3.4 (Earlier results). — For complex spaces, the existence of a map $\check{d}$ has been shown by Ferrari, [5, Proposition 1.1], although it is perhaps not obvious that the sheaf discussed in Ferrari's paper agrees with the sheaf of Kähler differentials modulo torsion.

Warning 3.5 (Theorem 3.3 is wrong in the relative setup). — One can easily define torsion-differentials and torsion-free differentials in the relative setting. The proof of Theorem 3.3, however, relies on the existence of a resolution of singularities for which no analog exists in the relative case. As a matter of fact, Theorem 3.3 becomes wrong when working with relative differentials, unless one makes rather strong additional assumptions. A simple example is given in [18, Warning 2.6].

3.2. Torsion-free forms in positive characteristic. — Now let $f : X \to Y$ be a morphism of reduced, quasi-projective schemes that are defined over a field of positive characteristic. We will see in this section that in stark contrast to the case of complex varieties, the pull-back map df of Kähler differential does generally *not* induce a pull-back map between the sheaves of torsion-free differential forms.

Indeed, if there exists a pull-back map $\check{d}f : f^*\check{\Omega}^n_Y \to \check{\Omega}^n_X$ which makes the following diagram commute,

$$\begin{tikzcd} f^*\Omega^n_Y \arrow[r, "f^*\beta_Y"] \arrow[d, "df"'] & f^*\check{\Omega}^n_Y \arrow[d, "\check{d}f"] \\ \Omega^n_X \arrow[r, "\beta_X"'] & \check{\Omega}^n_X, \end{tikzcd}$$

and if $\sigma \in \operatorname{tor}\Omega^1_Y$ is any torsion-differential, then $df(\sigma)$ is necessarily a torsion-differential on X, that is $df(\sigma) \in \operatorname{tor}\Omega^1_X$. The following example discusses a morphism between varieties for which this property does not hold.

Example 3.6 (Pull-back of torsion-form is generally not torsion). — Variants of this example work for any prime p, but we choose $p = 2$ for concreteness. Let k be an algebraically closed field of characteristic two and let $Y \subset \mathbb{A}^3{}_k$ be the Whitney umbrella. More precisely, consider the ring $R := k[x,y,z]\big/(y^2 - xz^2)$ and the schemes

$$X := \operatorname{Spec} k[x] \quad \text{and} \quad Y := \operatorname{Spec} R.$$

An elementary computation shows that the polynomial $y^2 - xz^2$ is irreducible. As a consequence, we see that Y is reduced and irreducible and that z is not a zerodivisor in R. Finally, let $f : X \to Y$ be the obvious inclusion map, which identifies X with the x-axis in $\mathbb{A}^3$, and which is given by the following map of rings,

$$f^\# : k[x,y,z]\big/(y^2 - xz^2) \to k[x] \qquad Q(x,y,z) \mapsto Q(x,0,0).$$

Note that X is nothing but the reduced singular locus of Y. We want to construct a torsion-differential σ on Y. To this end, notice that the differential form $dP \in \Gamma(\Omega^1_{\mathbb{A}^3})$, where $P = y^2 - xz^2$, induces the zero-form on Y,

$$0 = dP = -z^2 \cdot dx + 2y \cdot dy - 2xz \cdot dz = -z^2 \cdot dx \in \Gamma(\Omega^1_Y).$$

Since z^2 is not a zerodivisor, we see that the form $\sigma := dx$ is torsion, that is, $\sigma \in \Gamma(\operatorname{tor}\Omega^1_Y)$. On the other hand, the pull-back of σ to X is clearly given by $df(\sigma) = dx \in \Gamma(\Omega^1_X)$, which is not torsion.

Summary 3.7. — Example 3.6 shows that the assignments $X \mapsto \operatorname{tor}\Omega^n(X)$ and $X \mapsto \check{\Omega}^n_X(X)$ do not in general define presheaves on $\mathsf{Sch}(k)$.

4. The extension functor

4.1. Definition and first properties. — In analogy with the sheaves discussed in the introduction, we aim to define a "good" sheaf on $\mathsf{Sch}(S)$, which agrees on $\mathsf{Reg}(S)$ with S-relative[1] Kähler differentials and avoids the pathologies exposed by Kähler differentials on singular schemes. This section provides the technical framework for one construction in this direction: ignoring non-regular schemes, we define a sheaf on $\mathsf{Sch}(S)$ whose value group at one $X \in \mathsf{Sch}(S)$ is determined by differential forms on regular schemes over X. The following definition makes this idea precise.

Definition 4.1 (Extension functor). — Given a presheaf $\mathscr{F}$ on $\mathsf{Reg}(S)$, define a presheaf $\mathscr{F}_{\text{dvr}}$ on $\mathsf{Sch}(S)$ by setting

$$\mathscr{F}_{\text{dvr}}(X) := \varprojlim_{Y \in \mathsf{Reg}(X)} \mathscr{F}(Y) \quad \text{for any } X \in \mathsf{Sch}(S).$$

The assignment

$$\begin{array}{ccc} \{\text{Presheaves on } \mathsf{Reg}(S)\} & \to & \{\text{Presheaves on } \mathsf{Sch}(S)\} \\ \mathscr{F} & \mapsto & \mathscr{F}_{\text{dvr}} \end{array}$$

is clearly functorial, and referred to as the *extension functor*.

Warning 4.2. —
1. Observe $\mathscr{F}_{\text{dvr}}(X)$ is defined via limits and not by colimits.
2. Given a morphism $X' \to X$, the induced morphism $\mathscr{F}_{\text{dvr}}(X) \to \mathscr{F}_{\text{dvr}}(X')$ is induced by the composition functor, $\mathsf{Reg}(X') \to \mathsf{Reg}(X)$, not $X' \times_X -$, which does not necessarily preserve regular schemes.

Remark 4.3 (Elementary properties of the extension functor). —
1. Explicitly, a section of $\mathscr{F}_{\text{dvr}}(X)$ is a sequence of compatible sections. To give a section, it is therefore equivalent to give an element $s_Y \in \mathscr{F}(Y)$ for every morphism $Y \to X$ with $Y \in \mathsf{Reg}(X)$, such that the following compatibility conditions hold: for every triangle $Y' \to Y \to X$ with $Y', Y \in \mathsf{Reg}(X)$, we have $s_Y|_{Y'} = s_{Y'}$.
2. The assignment $\mathscr{F} \mapsto \mathscr{F}_{\text{dvr}}$ could equivalently be defined as the right adjoint to the restriction functor from presheaves on $\mathsf{Sch}(S)$ to presheaves on $\mathsf{Reg}(S)$.
3. If X itself is regular, then $\mathscr{F}(X) = \mathscr{F}_{\text{dvr}}(X)$, since X is then a final object in $\mathsf{Reg}(X)$.
4. Let $S = \operatorname{Spec} k$ with k a field of characteristic zero. Consider the presheaf $\mathscr{F} = \Omega^n$. Under these assumptions, it has been shown in [14, Theorem 1] that $\Omega^n_{\text{dvr}} = \Omega^n_{\text{h}}$.

1. For applications to differential forms, S will be the spectrum of a perfect field, so a scheme is regular if and only if it is smooth. However it is the property of being regular that is used extensively in this section.

LEMMA 4.4 (Extension preserves sheaves). — *In the setting of Definition 4.1, suppose that τ is a topology on $\mathsf{Sch}(S)$ that is equal to or coarser than the étale topology. Observe that τ restricts to a topology τ_{dvr} on $\mathsf{Reg}(S)$. If $\mathscr{F}$ is a τ_{dvr}-sheaf (respectively, a τ_{dvr}-separated presheaf) on $\mathsf{Reg}(S)$, then $\mathscr{F}_{\mathrm{dvr}}$ is a τ-sheaf (respectively, a τ-separated presheaf) on $\mathsf{Sch}(S)$.*

Lemma 4.4 is in fact a consequence of $\mathsf{Reg}(S) \to \mathsf{Sch}(S)$ being a cocontinuous morphism of sites for such a topology, cf. [2, Definitions III.2.1 and II.1.2] and [1, Tags 00XF and 00XI]. For the reader who is not at ease with categorical constructions we give an explicit proof below. The key ingredient is the fact that a τ-cover of a regular scheme is regular when τ is coarser than or equal to the étale topology.

Proof of Lemma 4.4. — We prove the two assertions of Lemma 4.4 separately.

Step 1, $(-)_{\mathrm{dvr}}$ preserves separatedness: Assume that $\mathscr{F}$ is a τ_{dvr}-separated presheaf. To prove that $\mathscr{F}_{\mathrm{dvr}}$ is τ-separated, consider a scheme $X \in \mathsf{Sch}(S)$, a τ-cover $U \to X$ and two sections $s, t \in \mathscr{F}_{\mathrm{dvr}}(X)$. We would like to show that the assumption that $s|_U$ and $t|_U$ agree as elements of $\mathscr{F}_{\mathrm{dvr}}(U)$ implies that $s = t$. Notice that a consequence of the definition of $(-)_{\mathrm{dvr}}$ is that two sections $s, t \in \mathscr{F}_{\mathrm{dvr}}(X)$ are equal if and only if for every $Y \to X$ in $\mathsf{Reg}(X)$, their restrictions $s|_Y$ and $t|_Y$ are equal.

Consider the fiber product diagram,

$$\begin{tikzcd} U \times_X Y =: U_Y \arrow[r] \arrow[d, "\text{étale}"'] & U \arrow[d, "\text{étale}"] \\ Y \arrow[r] & X. \end{tikzcd}$$

Since $U \to X$ is étale by assumption, the fiber product $U_Y := U \times_X Y$ is étale over Y (which we are assuming is regular) and therefore U_Y is also regular. By functoriality, equality of $s|_U$ and $t|_U$ implies equality of the restrictions $s|_{U_Y}$ and $t|_{U_Y}$, and by Remark 3 above, we have $\mathscr{F}_{\mathrm{dvr}}(U_Y) = \mathscr{F}(U_Y)$ and $\mathscr{F}_{\mathrm{dvr}}(Y) = \mathscr{F}(Y)$. Since U_Y is a τ_{dvr}-cover of Y, separatedness of $\mathscr{F}$ therefore guarantees that the elements $s|_Y, t|_Y \in \mathscr{F}(Y)_{\mathrm{dvr}} = \mathscr{F}(Y)$ agree.

Step 2, $(-)_{\mathrm{dvr}}$ preserves sheaves: Assume that $\mathscr{F}$ is a τ_{dvr}-sheaf, that $X \in \mathsf{Sch}(S)$ and that $U \to X$ is a τ-cover, with connected components $(U_i)_{i \in I}$. Assume further that we are given one $s \in \mathscr{F}_{\mathrm{dvr}}(U)$ such that the associated restrictions satisfy the compatibility condition

$$s|_{U_i \times_X U_j} = s|_{U_i \times_X U_j} \quad \text{for every } i, j \in I. \tag{4.4.1}$$

To prove that $\mathscr{F}_{\mathrm{dvr}}$ is a τ-sheaf, we need to construct a section $t \in \mathscr{F}_{\mathrm{dvr}}(X)$ whose restriction $t|_U$ agrees with the $s|_U$. Once t is found, uniqueness follows from τ-separatedness of $\mathscr{F}_{\mathrm{dvr}}$ that was shown above.

To give $t \in \mathscr{F}_{\mathrm{dvr}}(X)$, it is equivalent to give compatible elements $t_Y \in \mathscr{F}(Y)$, for every $Y \in \mathsf{Reg}(X)$. Given one such Y, consider the base change diagram

$$\begin{tikzcd}
(U \times_X U)_Y \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & U_Y \arrow[r, "\text{étale}"] \arrow[d] & Y \arrow[d] \\
U \times_X U \arrow[r, shift left] \arrow[r, shift right] & U \arrow[r, "\text{étale}"'] & X.
\end{tikzcd}$$

As before, observe that $U_Y = U \times_X Y$ and $(U \times_X U)_Y = U \times_X U \times_X Y$ are regular, that $\mathscr{F}_{\mathrm{dvr}}(U_Y) = \mathscr{F}(U_Y)$ and $\mathscr{F}_{\mathrm{dvr}}((U\times_X U)_Y) = \mathscr{F}((U\times_X U)_Y)$, and that $s|_{U_Y} \in \mathscr{F}(U_Y)$ satisfies a compatibility condition, analogous to (4.4.1). Since $\mathscr{F}$ is a τ_{dvr}-sheaf, we obtain an element $t_Y \in \mathscr{F}(Y) = \mathscr{F}_{\mathrm{dvr}}(Y)$ such that $t_Y|_{U_Y} = s|_{U_Y}$. The elements t_Y so constructed clearly define an element $t \in \mathscr{F}_{\mathrm{dvr}}(X)$ where $t|_U = s|_U$. □

4.2. Unramified presheaves. — We will see in Section 4.4 that the extension functor admits a particularly simple description whenever the presheaf $\mathscr{F}$ is *unramified*. The notion is due to Morel. We refer the reader to [22, Definition 2.1 and Remarks 2.2, 2.4] for a detailed discussion of unramified presheaves, in the case where the base scheme S is the spectrum of a perfect field.

DEFINITION 4.5 (Unramified presheaf). — A presheaf $\mathscr{F}$ on $\mathsf{Reg}(S)$ is *unramified* if the following axioms are satisfied for all $X, Y \in \mathsf{Reg}(S)$.

(UNR1) The canonical morphism $\mathscr{F}(X \amalg Y) \to \mathscr{F}(X) \times \mathscr{F}(Y)$ is an isomorphism.

(UNR2) If $U \to X$ is a dense open immersion, then $\mathscr{F}(X) \to \mathscr{F}(U)$ is injective.

(UNR3) The presheaf $\mathscr{F}$ is a Zariski sheaf, and for every open immersion $U \to X$ which contains all points of codimension ≤ 1 the morphism $\mathscr{F}(X) \to \mathscr{F}(U)$ is an isomorphism.

We will say that a presheaf $\mathscr{F}$ on $\mathsf{Sch}(S)$ is unramified if its restriction to $\mathsf{Reg}(S)$ is unramified.

EXAMPLE 4.6. —
1. The sheaf $\mathscr{O}$ is unramified [21, Theorem 38, page 124].
2. If $\mathscr{F}$ is a presheaf on $\mathsf{Reg}(S)$ whose restrictions $\mathscr{F}|_{X_{\mathrm{Zar}}}$ to the small Zariski sites $X|_{\mathrm{Zar}}$ of each $X \in \mathsf{Reg}(S)$ are locally free, coherent $\mathscr{O}_X$-modules, then $\mathscr{F}$ is unramified.
3. If $S = \operatorname{Spec} k$ is the spectrum of a perfect field, then the presheaves Ω^n are unramified, for all $n \geq 0$.
4. There are other, important examples of unramified presheaves, which fall out of the scope of the article. These include the Zariski sheafifications of K-theory, étale cohomology with finite coefficients (prime to the characteristic), or homotopy invariant Nisnevich sheaves with transfers. More generally, all reciprocity sheaves in the sense of [17] which are

Zariski sheaves satisfy (UNR2) and satisfy (UNR3), by [17, Theorem 6 and Conjecture 1] and [24, Cor.0.3]..

4.3. Discrete valuation rings. — If $X \in \mathsf{Sch}(S)$ is any scheme over S and if $x \in X$ is any point, we consider Spec $\mathscr{O}_{X,x}$, which can be seen as the intersection of (usually infinitely many) open subschemes $U_i \subseteq X$. Hence it is an example of a scheme essentially of finite type over S.

Definition 4.7 (The category $\mathsf{Dvr}(S)$). — The category of schemes essentially of finite type over S will be denoted by $\mathsf{Sch}(S)^{\mathsf{ess}}$. It contains $\mathsf{Sch}(S)$ as a subcategory. Let $\mathsf{Dvr}(S)$ be the category of schemes essentially of finite type which are regular, local, and of dimension ≤ 1.

Remark 4.8 (A description of $\mathsf{Dvr}(S)$). — The schemes in $\mathsf{Dvr}(S)$ are of the form Spec R with R either a field or a discrete valuation ring. More precisely, $R = \mathscr{O}_{X,x}$ where $X \in \mathsf{Reg}(S)$ and $x \in X$ is a point of codimension one or zero. The latter amounts to a scheme of the form Spec K for a field extension $K/k(s)$ of finite transcendence degree of the residue field $k(s)$ of a point $s \in S$.

Any presheaf $\mathscr{F}$ on $\mathsf{Sch}(S)$ extends to a presheaf on the larger category $\mathsf{Sch}(S)^{\mathsf{ess}}$, in a canonical way[2].

Definition and Proposition 4.9 (Extension from $\mathsf{Sch}(S)$ to $\mathsf{Sch}(S)^{\mathsf{ess}}$). — Given a presheaf $\mathscr{F}$ on $\mathsf{Sch}(S)$, define a presheaf $\mathscr{F}^{\mathsf{ess}}$ on $\mathsf{Sch}(S)^{\mathsf{ess}}$ as follows. Given any $X' \in \mathsf{Sch}(S)^{\mathsf{ess}}$, choose $X \in \mathsf{Sch}(S)$ and any filtered inverse system $\{U_i\}_{i\in I}$ of open subschemes with affine transition maps with intersection $X' \cong \bigcap_{i\in I} U_i$, and write

$$\mathscr{F}^{\mathsf{ess}}(X') := \varinjlim_{i\in I} \mathscr{F}(U_i).$$

Then, $\mathscr{F}^{\mathsf{ess}}$ is well-defined and functorial, in particular independent of the choice of X and U_i. Moreover,

$$\mathscr{F}^{\mathsf{ess}}|_{\mathsf{Sch}(S)} = \mathscr{F}.$$

The proof of 4.9, given below on the next page, uses the pro-category of schemes. To prepare for the proof, we briefly recall the relevant definitions.

Reminder 4.10 (Cofiltered category). — A category is called *cofiltered* if the following conditions hold.

1. The category has at least one object.

2. In contrast to $(-)_{\mathrm{dvr}}$ however, $(-)^{\mathsf{ess}}$ is a *left* Kan extension as opposed to a right Kan extension and so the limits in the definition are *co*limits instead of (inverse) limits. That is, instead of a section being described as a coherent sequence of sections, it is given by an equivalence class of sections. Recall the difference ΠX_i vs $\amalg X_i$, or $\varprojlim_n \mathbb{Z}/p^n = \mathbb{Z}_p$ vs $\varinjlim_n \mathbb{Z}/p^n = \{e^{n\pi i/p^k} : n, k \in \mathbb{Z}\} \subset \mathbb{C}$.

2. For every pair of objects λ, λ' there is a third object λ'' with morphisms $\lambda''\to\lambda$, $\lambda''\to\lambda'$ towards both of them.
3. For every pair of parallel morphisms $\lambda' \rightrightarrows \lambda$ there is a third morphism $\lambda''\to\lambda'$ such that the two compositions are equal.

REMINDER 4.11 (Pro-category of schemes). — The objects of the pro-category of schemes are pairs (Λ, X) with Λ a small, cofiltered category and $\Lambda \xrightarrow{X} \mathsf{Sch}(S)$ a contravariant functor, and the set of morphisms between two pro-schemes $(\Lambda, X), (M, Y)$ is *defined* as:

$$\hom_{\mathsf{Pro}(\mathsf{Sch}(S))}\big((\Lambda, X),\, (M, Y)\big) := \varprojlim_{\mu\in M} \varinjlim_{\lambda\in\Lambda} \hom_{\mathsf{Sch}(S)}(X_\lambda, Y_\mu).$$

Proof of well-definedness in Proposition 4.9. — We have seen in Lemma 2.7 and Remark 2.8 that $\mathsf{Sch}(S)^{\mathsf{ess}}$ is equivalent to a subcategory of $\mathsf{Pro}\big(\mathsf{Sch}(S)\big)$. To prove well-definedness and functoriality, it will therefore suffice to note that $\mathscr{F}$ has a well-defined functorial extension, say $\mathscr{F}^{\mathsf{Pro}}$ to $\mathsf{Pro}\big(\mathsf{Sch}(S)\big)$, and that $\mathscr{F}^{\mathsf{ess}}$ is just $\mathscr{F}^{\mathsf{Pro}}|_{\mathsf{Sch}(S)^{\mathsf{ess}}}$, considering $\mathsf{Sch}(S)^{\mathsf{ess}}$ as a subcategory of $\mathsf{Pro}\big(\mathsf{Sch}(S)\big)$ by abuse of notation. Indeed, for any pro-scheme (Λ, X), we can define

$$\mathscr{F}^{\mathsf{Pro}}(\Lambda, X) = \varinjlim_{\lambda\in\Lambda} \mathscr{F}(X_\lambda).$$

It follows from functoriality, that given any two pro-schemes (Λ, X), (M, Y), an element of $\hom_{\mathsf{Pro}(\mathsf{Sch}(S))}\big((\Lambda, X),\, (M, Y)\big)$ induces an element of

$$\varprojlim_{\mu\in M} \varinjlim_{\lambda\in\Lambda} \operatorname{Hom}\big(\mathscr{F}(Y_\mu),\, \mathscr{F}(X_\lambda)\big),$$

and from therefore an element in

$$\operatorname{Hom}\Big(\varinjlim_{\mu\in M} \mathscr{F}(Y_\mu),\, \varinjlim_{\lambda\in\Lambda} \mathscr{F}(X_\lambda)\Big).$$

One checks that all this is compatible with the composition in the various categories. $\square$

REMARK 4.12. — In the setting of Definition 4.9, if $X \in \mathsf{Sch}(S)$, if $x \in X$ and if $\{U_i\}_{i\in I}$ are the open affine subschemes of X containing x, then $\mathscr{F}^{\mathsf{ess}}(\bigcap_{i\in I} U_i)$ is just the usual (Zariski) stalk of $\mathscr{F}$ at x.

REMARK 4.13. — Note, however, that in general, a presheaf $\mathcal{G}$ on $\mathsf{Sch}(S)^{\mathsf{ess}}$ will not necessarily satisfy $\mathcal{G}(\bigcap_{i\in I} U_i) = \varinjlim_{i\in I} \mathcal{G}(U_i)$, but many sheaves of interest do. One prominent example of a presheaf which does satisfy $\mathcal{G} = \mathcal{G}^{\mathsf{ess}}$ is Ω^n, for all $n \geq 0$.

4.4. Description of the extension functor. — The following is the main result of this section. It asserts that, under good assumptions, the values of an extended sheaf can be reconstructed from is values on elements of Dvr.

In this section we assume that S is J-2, by which we mean that for any $X \in \mathsf{Sch}(S)$, the set of regular points of X is open, cf. [1, Tags 07P7, 07QT, 07QW]. This is true, for example, if S is the spectrum of a quasi-excellent ring, such as any ring of finite type over a field, or a Dedekind domain whose fraction field is characteristic zero. This hypothesis is used in the proof of Proposition 4.14.

PROPOSITION 4.14 (Reconstruction of unramified presheaves). — *Suppose that S is J-2, for example the spectrum of a field. Let $\mathscr{F}$ be an unramified presheaf on $\mathsf{Reg}(S)$. If $X \in \mathsf{Sch}(S)$ is any scheme, then*

$$\mathscr{F}_{\mathrm{dvr}}(X) = \varprojlim_{W \in \mathsf{Dvr}(X)} \mathscr{F}^{\mathsf{ess}}(W).$$

The proof of this Proposition 4.14 will take the rest of the present Section 4.4. Before beginning the proof on page 323, we note two lemmas that (help to) prove the proposition in special cases.

LEMMA 4.15 (cf. [22, Remark 1.4]). — *In the setting of Proposition 4.14, if $X \in \mathsf{Reg}(S)$, then*

$$\mathscr{F}(X) = \varprojlim_{x \in X^{(\leq 1)}} \mathscr{F}^{\mathsf{ess}}\big(\mathrm{Spec}\ \mathscr{O}_{X,x}\big)$$

where $X^{(\leq 1)}$ is the subcategory of $\mathsf{Dvr}(X)$ consisting of inclusions of localisations of X at points of codimension ≤ 1.

Proof. — To keep notation short, we abuse notation slightly and write $\mathscr{F}$ for the presheaf, as well as for its extension $\mathscr{F}^{\mathsf{ess}}$ to $\mathsf{Sch}(S)^{\mathsf{ess}}$. Since $\mathscr{F}^{\mathsf{ess}}(Y) = \mathscr{F}(Y)$ for all $Y \in \mathsf{Sch}(S)$, no confusion is likely to occur. Using the Axiom (UNR1), we can restrict ourselves to the case where X is a connected, regular scheme. Given X, we aim to show that the following canonical map is an isomorphism,

$$\mathscr{F}(X) \to \varprojlim_{x \in X^{(\leq 1)}} \mathscr{F}\big(\mathrm{Spec}\ \mathscr{O}_{X,x}\big).$$

Injectivity. — Axiom (UNR2) implies that $\mathscr{F}(X) \to \mathscr{F}(\eta)$ is injective, where η is the generic point of X. Since this factors as

$$\mathscr{F}(X) \to \varprojlim_{x \in X^{(\leq 1)}} \mathscr{F}\big(\mathrm{Spec}\ \mathscr{O}_{X,x}\big) \xrightarrow{\text{canon. projection}} \mathscr{F}(\eta),$$

we obtain that the first map is injective.

Surjectivity. — Assume we are given a section

$$(s_x)_{x\in X^{(\leq 1)}} \quad \text{of} \quad \varprojlim_{x\in X^{(\leq 1)}} \mathscr{F}\big(\operatorname{Spec} \mathscr{O}_{X,x}\big).$$

By definition of the groups $\mathscr{F}\big(\operatorname{Spec} \mathscr{O}_{X,x}\big)$, for every $x \in X^{(\leq 1)}$ there is some open $U_x \subset X$ containing x and an element $t_x \in \mathscr{F}(U_x)$ which represents s_x. Furthermore, by the required coherency, for every $x \in X$ of codimension one there is an open subscheme $U_{x\eta}$ of $U_x \cap U_\eta$ such that the restrictions of t_x and t_η to $U_{x\eta}$ agree. By (UNR2), this means that we actually have

$$t_x|_{U_x\cap U_y} = t_y|_{U_x\cap U_y} \quad \text{for each } x, y \in X \text{ of codimension one.}$$

Since $\mathscr{F}$ is a Zariski sheaf by (UNR3), the sections t_x therefore lift to a section t on $\bigcup_{x\in X^{(1)}} U_x$, but by (UNR3), we have $\mathscr{F}\big(\bigcup_{x\in X^{(1)}} U_x\big) = \mathscr{F}(X)$. So the map is surjective. □

LEMMA 4.16 (cf. [19, Proof of 3.6.12]). — *In the setting of Proposition 4.14, if X is connected, regular and Noetherian with generic point η, then the projection map*

$$\varprojlim_{W\in \mathsf{Dvr}(X)} \mathscr{F}^{\mathsf{ess}}(W) \to \mathscr{F}^{\mathsf{ess}}(\eta) \tag{4.16.1}$$

is injective. Consequently, Proposition 4.14 is true when X is regular.

Proof. — As before, write $\mathscr{F}$ as a shorthand for $\mathscr{F}^{\mathsf{ess}}$. Recall from Remark 4.3(3) that when X is regular, $\mathscr{F}(X) = \mathscr{F}_{\mathrm{dvr}}(X)$. The composition

$$\mathscr{F}(X) \to \varprojlim_{W\in \mathsf{Dvr}(X)} \mathscr{F}(W) \to \varprojlim_{x\in X^{(\leq 1)}} \mathscr{F}\big(\operatorname{Spec} \mathscr{O}_{X,x}\big)$$

is the map we have shown is an isomorphism in Lemma 4.15. So to show that the first map is an isomorphism, it suffices to show that the second map is a monomorphism. Since (4.16.1) factors through this, it suffices to show that (4.16.1) is injective.

Assume we are given two sections

$$(s_W)_{W\in\mathsf{Dvr}(X)}, (t_W)_{W\in\mathsf{Dvr}(X)} \quad \text{of} \quad \varprojlim_{W\in \mathsf{Dvr}(X)} \mathscr{F}(W)$$

such that $s_\eta = t_\eta$. We wish to show that $t_W = s_W$ for all $W \in \mathsf{Dvr}(X)$.

First, we consider W of the form a point $x \in X$. This is by induction on the codimension. We already know $s_\eta = t_\eta$ for the point η of codimension zero, so suppose it is true for points of codimension at most $n-1$ and let x be of codimension n. By Lemma 2.12 there is a point y of codimension at most $n-1$ and a discrete valuation ring R satisfying the diagram

$$\begin{tikzcd}[row sep=small] y \arrow[dr] & & \\ & \operatorname{Spec} R \arrow[r] & X. \\ x \arrow[ur] & & \end{tikzcd}$$

By the inductive hypothesis $s_y = t_y$ and then by (UNR2) we have $s_{\mathrm{Spec}\ R} = t_{\mathrm{Spec}\ R}$. Therefore $s_{\mathrm{Spec}\ R}|_x = t_{\mathrm{Spec}\ R}|_x$ but by the coherency requirement on s and t and we have $s_{\mathrm{Spec}\ R}|_x = s_x$ and $t_{\mathrm{Spec}\ R}|_x = t_x$ and therefore $t_x = s_x$.

Now for an arbitrary $W \to X$ in $\mathsf{Dvr}(X)$ with W of dimension zero, if x is the image of W we have $t_W = t_x|_W = s_x|_W = s_W$. For an arbitrary W of dimension one and generic point $\widetilde{\eta}$ we have $t_W|_{\widetilde{\eta}} = t_{\widetilde{\eta}} = s_{\widetilde{\eta}} = s_Y|_{\widetilde{\eta}}$ and so by (UNR2) we have $t_W = s_W$. □

Proof of Proposition 4.14. — Again we write $\mathscr{F}$ as a shorthand for $\mathscr{F}^{\mathsf{ess}}$. By Lemma 4.16 we have

$$\mathscr{F}_{\mathrm{dvr}}(X) = \varprojlim_{Y \in \mathsf{Reg}(X)} \mathscr{F}(Y) = \varprojlim_{Y \in \mathsf{Reg}(X)} \varprojlim_{W \in \mathsf{Dvr}(Y)} \mathscr{F}(W).$$

Investigating this last double limit carefully, we observe that it can be described as sequences $(s_{W \to Y \to X})$, where $s_{W \to Y \to X} \in \mathscr{F}(W)$, indexed by pairs of composable morphisms $W \to Y \to X$ with $W \in \mathsf{Dvr}(X)$ and $Y \in \mathsf{Reg}(X)$, and subject to the following two conditions:

1. For each $W' \to W \to Y \to X$ with $W', W \in \mathsf{Dvr}(X)$ and $Y \in \mathsf{Reg}(X)$ we have $s_{W' \to Y \to X} = s_{W \to Y \to X}|_{W'}$.
2. For each $W \to Y' \to Y \to X$ with $W \in \mathsf{Dvr}(X)$ and $Y, Y' \in \mathsf{Reg}(X)$ we have $s_{W \to Y \to X} = s_{W \to Y' \to X}$.

By forgetting Y we get a natural map

$$\alpha : \varprojlim_{W \to X} \mathscr{F}(W) \to \varprojlim_{W \to Y \to X} \mathscr{F}(W)$$

with W, Y as before. It is given by sending an element

$$(t_{W \to X})_{\mathsf{Dvr}(X)} \in \varprojlim_{W \in \mathsf{Dvr}(X)} \mathscr{F}(W)$$

to the element

$$s \in \varprojlim_{W \to Y \to X} \mathscr{F}(W) \quad \text{with } s_{W \to Y \to X} = t_{W \to X}.$$

In order to check that α is surjective, we only need to check that for any

$$s \in \varprojlim_{W \to Y \to X} \mathscr{F}(W),$$

the section $s_{W \to Y \to X}$ is independent of Y. Let $W \to Y_1 \to X$ and $W \to Y_2 \to X$ be in the index system. By Example 2.9, W is the intersection of its affine open neighborhoods in some $Y_3 \in \mathsf{Sch}(S)$. Since S is J-2, the regular points of a scheme in $\mathsf{Sch}(S)$ form an open set, and we can assume that $Y_3 \in \mathsf{Reg}(S)$. By the description of morphisms of schemes essentially of finite type in Lemma 2.7

there exists an open affine subscheme U of Y_3 containing W such that we can lift the morphisms $W \to Y_1, Y_2$ to morphisms $U \to Y_1, Y_2$. Hence

$$s_{W\to Y_1\to X} = s_{W\to U\to X} = s_{W\to Y_2\to X}.$$

To show that α is a monomorphism, it suffices to notice that each $W \in \mathsf{Dvr}(X)$ can be "thickened" to a $Y \in \mathsf{Reg}(X)$: By Example 2.9 is W is of the form $\bigcap U$ for U running through the open neighborhoods of W in some $Y' \in \mathsf{Sch}(X)$. Let Y be the open subscheme of regular points of Y'. We claim that $W \to Y'$ factors via Y. The generic point of W maps to a generic point of Y', hence Y is non-empty. If W is of dimension 0, we are done. If W is of dimension one, consider the image $y' \in Y'$ of the closed point of W. Then $W = \operatorname{Spec} \mathscr{O}_{Y',y'}$ and y' is a regular point because W is regular. In particular, W is contained in Y. Hence, the functor of indexing categories which sends $W \to Y \to X$ to $W \to X$ is essentially surjective, and therefore the induced morphism α of limits is injective. □

4.5. Descent properties. — Recall the notion of a cdp-cover from Definition 2.10 and the rh, cdh and eh-topologies generated by cdp-covers together with open covers, resp. Nisnevich covers, resp. étale covers.

Proposition 4.18. — *Let S be a J-2 Noetherian scheme, such as the spectrum of a field. If $\mathscr{F}$ is an unramified presheaf on $\mathsf{Sch}(S)$ then $\mathscr{F}_{\mathrm{dvr}}$ is an* rh*-sheaf. Consequently, if $\mathscr{F}$ is an unramified Nisnevich (resp. étale) sheaf on $\mathsf{Sch}(S)$ then $\mathscr{F}_{\mathrm{dvr}}$ is a* cdh*-sheaf (resp.* eh*-sheaf).*

Example 4.19. — Let $S = \operatorname{Spec} k$ with k perfect. Then Ω^n_{dvr} is an eh-sheaf for all n. In particular, Ω^n_{dvr} is a cdh-sheaf since this is a weaker topology.

Proof of Proposition 4.18. — By Lemma 4.4, $\mathscr{F}_{\mathrm{dvr}}$ is a Zariski-sheaf (resp. étale or Nisnevich sheaf). It remains to establish the sheaf property for cdp-morphisms. As before, write $\mathscr{F}$ as a shorthand for $\mathscr{F}^{\mathsf{ess}}$, in order to simplify notation.

Using the description $\mathscr{F}_{\mathrm{dvr}}(X) = \varprojlim_{W\in\mathsf{Dvr}(X)} \mathscr{F}(W)$ from Proposition 4.14, in order to show that $\mathscr{F}_{\mathrm{dvr}}$ is separated for cdp-morphisms, it suffices to show that for every cdp-morphism $X' \to X$, every $W \to X$ in $\mathsf{Dvr}(X)$ factors through $X' \to X$. Since $X' \to X$ is completely decomposed it is true for W of dimension zero, and therefore also true for the generic points of those W of dimension one. To factor all of a W of dimension one, we now use the valuative criterion for properness.

Now consider a cdp-morphism $X' \to X$ and a cocycle

$$s = (s_W)_{W\in\mathsf{Dvr}(X')} \in \ker\Big(\mathscr{F}_{\mathrm{dvr}}(X') \to \mathscr{F}_{\mathrm{dvr}}(X' \times_X X')\Big).$$

We have just observed that every scheme in $\mathsf{Dvr}(X)$ factors through X' and so making a choice of factorization for each $W \to X$ in $\mathsf{Dvr}(X)$, and taking t_W to be the s_W of this factorization, we have a potential section $t = (t_W)_{W \in \mathsf{Dvr}(X)} \in \mathscr{F}_{\mathrm{dvr}}(X)$, which potentially maps to s.

Independence of the choice: Suppose that $f_0, f_1 : W \rightrightarrows X' \to X$ are two factorizations of some $W \in \mathsf{Dvr}(X)$. Then there is a unique morphism $W \to X' \times_X X'$ such that composition with the two projections recovers the two factorizations. Saying that s is a cocycle is to say precisely that in this situation, the two s_W corresponding to f_0 and f_1 are equal as elements of $\mathscr{F}(W)$.

This independence of the choice implies that t is actually a section of $\mathscr{F}_{\mathrm{dvr}}(X)$. In other words, that $s_W|_{W'} = s_{W'}$ for any morphism $W' \to W$ in $\mathsf{Dvr}(X)$. It also implies that t is mapped to s. $\square$

5. cdh-differentials

Throughout the present Section 5, the letter k will always denote a perfect field. To keep our statements self-contained, we will repeat this assumption at times.

Remark 5.1 (Perfect fields). — Perfect fields have a number of characterisations. A pertinent one for us is: the field k is perfect if and only if $\Omega^1_{L/k} = 0$ for every algebraic extension L/k.

Definition 5.2 (Dvr differentials). — Let k be a perfect field and $n \in \mathbb{N}$ be any number. Let Ω^n_{dvr} be the extension of the presheaf Ω^n on $\mathsf{Reg}(k)$ to $\mathsf{Sch}(k)$ in the sense of Definition 4.1. A section of Ω^n_{dvr} is called a *dvr differential.* The justification for this name is Proposition 4.14.

Observation 5.3. — Let k be perfect field and $n \in \mathbb{N}$ be any number. Then Ω^n_{dvr} is an eh-sheaf. If X is regular, then $\Omega^n_{\mathrm{dvr}}(X) = \Omega^n(X)$.

Proof. — Recall from Example 4.6 that Ω^n is an unramified presheaf. The observations are therefore special cases of Proposition 4.18 and Remark 3, respectively. $\square$

Remark 5.4. — If the characteristic of k is zero, the description as dvr differentials is one of the equivalent characterisations of h-differentials given in [14, Theorem 1], without the terminology being introduced.

5.1. cdh-differentials. — We introduce an alternative candidate for a good theory of differentials and show that it agrees with Ω^n_{dvr}, if we assume that weak resolutions of singularities exist.

Definition 5.5 (cdh-differentials). — Let k be a perfect field and $n \in \mathbb{N}$ be any number. Let Ω^n_{cdh} be the sheafification of Ω^n on $\mathsf{Sch}(k)$ in the cdh-topology. Sections of Ω^n_{cdh} are called cdh-differentials. Analogously, we define Ω^n_{rh} and Ω^n_{eh} via the rh-or eh-topology.

REMARK 5.6 (Sheafification in characteristic zero). — If k has characteristic zero, then Ω^n_{cdh}, Ω^n_{rh} and Ω^n_{eh} agree. In fact, Ω^n_{cdh} is even an h-sheaf, [14, Theorem 3.6].

REMARK 5.7 (Comparison map). — In the setting of Definition 5.5, recall from Observation 5.3 that Ω^n_{dvr} is an eh-sheaf. By the universal property of sheafification, there are canonical morphisms

$$\Omega^n_{\mathrm{rh}} \to \Omega^n_{\mathrm{cdh}} \to \Omega^n_{\mathrm{eh}} \to \Omega^n_{\mathrm{dvr}}.$$

We aim to compare these sheaves. As it will turn out, the comparison is a question about torsion. The decisive input is the following theorem:

THEOREM 5.8 (Killing torsion in Ω). — *Let k be a perfect field, $Y \in \mathsf{Sch}(k)$ integral, $n \in \mathbb{N}$ and $\omega \in \operatorname{tor} \Omega^n(Y)$. Then there is a birational proper morphism $\pi : \widetilde{Y} \to Y$ such that the image of ω in $\Omega^n(\widetilde{Y})$ vanishes.*

REMARK 5.9. — The above result is a consequence of weak resolution of singularities, but in fact weaker. We will give a direct proof in Appendix A.

COROLLARY 5.10. — *Let k be a perfect field, $Y \in \mathsf{Sch}(k)$, $n \in \mathbb{N}$ and $\omega \in \Omega^n(Y)$ an element such that $\omega|_y = 0$ for every point $y \in Y$. Then there exists a* cdp-*morphism (Definition 2.10) $Y' \to Y$ such that $\omega|_{Y'} = 0$.*

Proof. — The proof is by induction on the dimension of Y. If the dimension of Y is zero then $Y_{\mathrm{red}} \to Y$ is a cdp-morphism such that $\omega|_{Y_{\mathrm{red}}}$ vanishes, so suppose that Y is of dimension $n > 0$ and that the result is true for schemes of dimension $< n$. Replacing Y by its reduced irreducible components we can assume that Y is integral. Then Theorem 5.8 gives a proper birational morphism $Y' \to Y$ for which $\omega|_{Y'}$ vanishes. Let $Z \subset Y$ be a closed nowhere dense subscheme outside of which $Y' \to Y$ is an isomorphism, and $Z' \to Z$ a cdp-morphism provided by the inductive hypothesis. Then $Z' \amalg Y' \to Y$ is a cdp-morphism on which ω vanishes. □

THEOREM 5.11. — *Let k be a perfect field, $n \in \mathbb{N}$ be any number and $X \in \mathsf{Reg}(k)$ any regular scheme. Then the following canonical morphisms are all isomorphisms:*

$$\Omega^n(X) \xrightarrow{\sim} \Omega^n_{\mathrm{rh}}(X) \xrightarrow{\sim} \Omega^n_{\mathrm{cdh}}(X) \xrightarrow{\sim} \Omega^n_{\mathrm{eh}}(X) \xrightarrow{\sim} \Omega^n_{\mathrm{dvr}}(X). \tag{5.11.1}$$

The isomorphism $\Omega^n(X) \cong \Omega^n_{\mathrm{eh}}(X)$ has been shown previously by Geisser in [8, Theorem 4.7] assuming the existence *strong* resolutions of singularities exists, i.e., under the additional assumption that any birational proper morphism between smooth varieties can be refined by a series of blow-ups with smooth centers.

Proof of Theorem 5.11. — We formulate the proof in the case of the cdh-topology. The same arguments also apply, *mutatis mutandis*, in the other cases.

Step 1, proof of (5.11.1) *up to torsion.* — Since X is regular, the composition of natural maps,

$$\Omega^n(X) \to \Omega^n_{\mathrm{cdh}}(X) \to \underbrace{\Omega^n_{\mathrm{dvr}}(X)}_{=\Omega^n(X) \text{ by Rem. 4.3(3)}}, \tag{5.11.2}$$

is an isomorphism by Remark 4.3(3). We claim that the direct complement $T_{\mathrm{cdh}}(X)$ of $\Omega^n(X)$ in $\Omega^n_{\mathrm{cdh}}(X)$ is torsion. To this end, we may assume X is connected, say with function field K. Applying the functor $(\bullet)^{\mathsf{ess}}$ to Sequence (5.11.2), we obtain

$$\underbrace{(\Omega^n)^{\mathsf{ess}}(K)}_{=\Omega^n(K) \text{ by Rem. 4.12}} \longrightarrow (\Omega^n_{\mathrm{cdh}})^{\mathsf{ess}}(K) \longrightarrow \underbrace{(\Omega^n_{\mathrm{dvr}})^{\mathsf{ess}}(K)}_{=\Omega^n(K) \text{ by Rems. 4.3(3), 4.12}}. \tag{5.11.3}$$

We claim that both morphisms in (5.11.3) are isomorphisms. Equation (5.11.1) then follows from (UNR2). In order to justify the claim, it suffices to show that the first morphism is surjective. Suppose that $s \in (\Omega^n_{\mathrm{cdh}})^{\mathsf{ess}}(K)$. By the definition of $(-)^{\mathsf{ess}}$, there is some open affine $U \subseteq X$ such that s is represented by a section $s' \in (\Omega^n_{\mathrm{cdh}})(U)$. Then, there exists a cdh-cover $V \to U$ such that $s'|_V$ is in the image $\Omega^n(V) \to \Omega^n_{\mathrm{cdh}}(V)$. Replacing U by some smaller open affine, we can assume that $V = U$. That is, the representative $s' \in (\Omega^n_{\mathrm{cdh}})(U)$ is in the image of $(\Omega^n)(U) \to (\Omega^n_{\mathrm{cdh}})(U)$. Hence, the section $s \in (\Omega^n_{\mathrm{cdh}})^{\mathsf{ess}}(K)$ it represents is in the image of $(\Omega^n)^{\mathsf{ess}}(K) \to (\Omega^n_{\mathrm{cdh}})^{\mathsf{ess}}(K)$.

Step 2, vanishing of torsion. — We will now show that $T_{\mathrm{cdh}}(X) = 0$. Let $\omega \in T_{\mathrm{cdh}}(X)$ be any element—so ω is an element of $\Omega^n_{\mathrm{cdh}}(X)$ which vanishes in $(\Omega^n_{\mathrm{cdh}})^{\mathsf{ess}}(K)$. We aim to show that $\omega = 0$. To this end, we construct a number of covering spaces, associated groups and preimages, which lead to Diagrams (5.11.4) and (5.11.5) below. We may assume without loss of generality that X is reduced.

By definition of sheafification, there exists a cdh-cover $V \to X$ of X for which $\omega|_V$ is in the image of $\Omega^n(V) \to \Omega^n_{\mathrm{cdh}}(V)$. Again we may assume that V is reduced. We choose one element $\omega' \in \Omega^n(V)$ contained in the preimage of ω. As $V \to X$ is a cdh-cover, we have a factorization $\operatorname{Spec} K \to V \to X$. Recalling from Step 1 that $(\Omega^n)^{\mathsf{ess}}(K) \to (\Omega^n_{\mathrm{cdh}})^{\mathsf{ess}}(K)$ is an isomorphism, it follows that $\omega' \in \Omega^n(V)$ is a torsion element.

By Corollary 2.16 we may refine the covering map and assume without loss of generality that it factorizes as follows,

$$V \xrightarrow{\text{Nisnevich cover}} Y \xrightarrow{\text{proper birational}} X.$$

Since $\omega' \in \Omega(V)$ is torsion, Theorem 5.8 gives a proper birational morphism $V' \to V$ such that $\omega'|_{V'} = 0$. Finally, Lemma 2.15 allows one to find Y'', Y'

fitting into the following commutative diagram

(5.11.4)
$$\begin{tikzcd}
Y'' \arrow[rrr, "\text{Nisnevich cover}"] \arrow[d, "\text{proper biratl.}"'] & & & Y' \arrow[d, "\text{proper biratl.}"] & \\
V' \arrow[r, "\text{proper biratl.}"'] & V \arrow[rr, "\text{Nisnevich cover}"'] & & Y \arrow[r, "\text{proper biratl.}"'] & X.
\end{tikzcd}$$

Proposition 2.13 implies that $Y' \to X$ and $Y'' \to Y$ are cdh-covers. A diagram chase will now finish the argument:

(5.11.5)
$$\begin{tikzcd}
 & \substack{\Omega^n_{\mathrm{cdh}}(Y'') \\ \text{contains } \omega|_{Y''}=0} & & \Omega^n_{\mathrm{cdh}}(Y') \arrow[ll] & \\
\substack{\Omega^n(Y'') \\ \text{contains } \omega'|_{Y''}=0} \arrow[ur] & & & & \\
 & \Omega^n_{\mathrm{cdh}}(V') \arrow[uu] & \substack{\Omega^n_{\mathrm{cdh}}(V) \\ \text{contains } \omega|_{V}} \arrow[l] & \Omega^n_{\mathrm{cdh}}(Y) \arrow[l] \arrow[uu] & \substack{\Omega^n_{\mathrm{cdh}}(X) \\ \text{contains } \omega} \arrow[l] \\
\substack{\Omega^n(V') \\ \text{contains } \omega'|_{V'}=0} \arrow[uu] \arrow[ur] & \substack{\Omega^n(V) \\ \text{contains } \omega'} \arrow[l] \arrow[ur] & & &
\end{tikzcd}$$
□

PROPOSITION 5.12. — *Let k be a perfect field and $n \in \mathbb{N}$ be any number. The canonical map $\Omega^n_{\mathrm{cdh}} \to \Omega^n_{\mathrm{dvr}}$ is a monomorphism.*

Proof. — Let $X \in \mathsf{Sch}(k)$ and ω be in the kernel of $\Omega^n_{\mathrm{cdh}}(X) \to \Omega^n_{\mathrm{dvr}}(X)$. Choose a cdh-cover $X' \to X$ such that the restriction of ω to X' is in the image of $\Omega^n(X') \to \Omega^n_{\mathrm{cdh}}(X')$, so now we have a section ω' in the kernel of $\Omega^n(X') \to \Omega^n_{\mathrm{dvr}}(X')$ and we wish to show that it vanishes on a cdh-cover of X'. By Corollary 5.10 it suffices to show that ω' vanishes on every point of X'. Let $x \in X'$ be a point, $\overline{\{x\}}$ its closure in X' with the reduced scheme structure, and let $V = (\overline{\{x\}})_{\mathrm{reg}}$ be its regular locus. Since the section ω' vanishes in $\Omega^n_{\mathrm{dvr}}(X')$, it vanishes on every scheme in $\mathsf{Reg}(X')$, and in particular, on V. But the generic point of V is isomorphic to x, and therefore ω' vanishes on x. □

PROPOSITION 5.13. — *Let k be a perfect field and $n \in \mathbb{N}$ be any number. Assume that weak resolutions of singularities exist for schemes defined over k. Then the natural maps*

$$\Omega^n_{\mathrm{rh}} \to \Omega^n_{\mathrm{cdh}} \to \Omega^n_{\mathrm{eh}} \to \Omega^n_{\mathrm{dvr}}$$

of presheaves on $\mathsf{Sch}(k)$ are isomorphisms. As such, all four are unramified.

Proof. — We argue by induction on the dimension. In dimension zero, there is nothing to show. Now, let $X \in \mathsf{Sch}(k)$ be of positive dimension. We may assume that X is reduced and hence generically regular. Let $\widetilde{X} \to X$ be a desingularisation, isomorphic outside a proper closed set $Z \subsetneq X$ and with exceptional locus $E \subsetneq \widetilde{X}$. It follows from cdp-descent that the sequences

$$0 \to \Omega^n_\bullet(X) \to \Omega^n_\bullet\big(\widetilde{X}\big) \oplus \Omega^n_\bullet(Z) \to \Omega^n_\bullet(E) \qquad \text{where } \bullet \in \{\mathrm{rh}, \mathrm{cdh}, \mathrm{eh}, \mathrm{dvr}\}$$

are all exact. By inductive hypothesis the comparison map is an isomorphism for Z and E. It is also an isomorphism for $\widetilde{X}$, by Proposition 5.11. Hence it is also an isomorphism for X. □

Remark 5.14. — It is currently unclear to us if Theorem 5.8 suffices to establish the conclusion of Proposition 5.13. For all we know, Theorem 5.8 only implies that the natural map $\Omega^n_{\mathrm{dvr}} \to (\Omega^n_\bullet)_{\mathrm{dvr}}$ for $\bullet \in \{\mathrm{rh}, \mathrm{cdh}, \mathrm{eh}\}$ is an isomorphism.

5.2. Torsion of cdh-sheaves. — Over fields of characteristic zero, the results [14, Theorem 1, Remark 3.11 and Proposition 4.2] show that $\Omega^n_{\mathrm{cdh}} = \Omega^n_{\mathrm{dvr}}$ is torsion free. We are going to show that this fails in positive characteristic.

Example 5.15 (Existence of cdh-torsion). — We maintain the setting and notation of Example 3.6: the field k is algebraically closed of characteristic two, $Y := \operatorname{Spec} k[x,y,z]\big/(y^2 - xz^2)$ is the Whitney umbrella, and $X := \operatorname{Spec} k[x]$ is its singular locus. Write $\widetilde{Y} := \operatorname{Spec} k[u,z] \cong \mathbb{A}^2$ and consider the (birational) desingularisation $\pi : \widetilde{Y} \to Y$, given by

$$\pi^{\#} : k[x,y,z]\big/(y^2 - xz^2) \to k[u,z], \qquad x \mapsto u^2,\ y \mapsto uz,\ z \mapsto z.$$

Let $E \subsetneq \widetilde{Y}$ be the exceptional locus of π. In other words, E equals the preimage of X and is hence given by $z = 0$. Note that the morphism $E \to X$ corresponds to the ring morphism $k[x] \to k[u]$, $x \mapsto u^2$, and therefore induces the zero morphism on Ω^1. We compute Ω^1_{dvr} of Y, $\widetilde{Y}$, X, and E. The last three are regular, hence $\Omega^1 \to \Omega^1_{\mathrm{dvr}}$ is an isomorphism on these varieties. Since Ω^1_{cdh} is a cdh-sheaf and $\widetilde{Y} \amalg X \to Y$ is a cdh-cover, we have the following exact sequence:

$$0 \to \Omega^1_{\mathrm{cdh}}(Y) \xrightarrow{a} \Omega^1\big(\widetilde{Y}\big) \oplus \Omega^1(X) \xrightarrow{b} \Omega^1(E).$$

We have seen above that $\Omega^1(X) \subseteq \ker b \cong \Omega^1_{\mathrm{cdh}}(Y)$. The associated sections of $\Omega^1_{\mathrm{cdh}}(Y)$ vanish on $Y \setminus X$ and are therefore torsion on Y. It follows that $\operatorname{tor} \Omega^1_{\mathrm{dvr}}(Y) \neq 0$, and that there are torsion elements whose restrictions to X do not vanish, and furthermore, are not torsion elements of $\Omega^1_{\mathrm{cdh}}(X)$.

Corollary 5.16. — *For perfect fields of positive characteristic, the sheaves $\Omega^\bullet_{\mathrm{cdh}}$ are not torsion-free in general.* □

COROLLARY 5.17. — *For perfect fields of positive characteristic, the pull-back maps of* $\Omega^1_{\mathrm{cdh}}(\cdot)$ *do not induce pull-back maps between the groups* $\operatorname{tor}\Omega^1_{\mathrm{cdh}}(\cdot)$. *In other words,* $\operatorname{tor}\Omega^1_{\mathrm{cdh}}$ *is generally not a presheaf on* $\mathsf{Sch}(k)$. □

REMARK 5.18. — The same computation works for any extension of Ω^1 to a sheaf on Sch which has cdp-descent and agrees with Kähler differentials on regular varieties. By Theorem 5.8 this includes dvr-, rh-, and eh-differentials.

6. Separably decomposed topologies

In many applications, de Jong's theorem on alterations [16] and Gabber's refinement [15] have proved a very good replacement for weak resolution of singularities, which is not (yet) available in positive characteristic. It is natural to ask if one can pass from the eh-topology to a suitable refinement that allows alterations as covers, but still preserves the notion of a differential in the smooth case. This turns out impossible.

6.1. h-topology. — We record the following for completeness.

LEMMA 6.1 (Sheafification of differentials in the h-topology). — *Assume that there is a positive integer* m *such that* $m = 0$ *in* $\mathscr{O}_S$ *(for example,* S *might be the spectrum of a field of positive characteristic). If* $n > 0$ *is any number, then the h-sheafification* Ω^n_h *of* Ω^n *on* $\mathsf{Sch}(S)$ *is zero. In fact, even the h-separated presheaf associated to* Ω^n *is zero. This is true regardless of whether* Ω^n *is the sheaf of absolute differentials, or the sheaf of* S*-relative differentials.*

Proof. — Since the h-topology is finer than the Zariski topology, it suffices to prove the statement for affine schemes. We claim that for any ring A, and any differential of the form $da \in \Omega(\operatorname{Spec} A)$, there exists an h-cover $Y \to \operatorname{Spec} A$ such that da is sent to zero in $\Omega(Y)$. Indeed, it suffices to consider the finite surjective morphism

$$\operatorname{Spec} A[T]\big/(T^m - a) \to \operatorname{Spec} A.$$

The image of da under this morphism is $da = d(T^m) = mT^{m-1} \cdot dT = 0$. □

6.2. sdh-topology. — To avoid the phenomenon encountered in Lemma 6.1, one could try to consider the following topology, which is coarser than the h-topology. We only allow proper maps that are generically separable. By making the notion stable under base change, we are led to the following notion.

DEFINITION 6.2 (sdh-topology). — We define the sdh-topology on $\mathsf{Sch}(S)$ as the topology generated by the étale topology, and by proper morphisms $f : Y \to X$ such that for every $x \in X$ there exists $y \in f^{-1}(x)$ with $[k(y) : k(x)]$ finite separable.

EXAMPLE 6.3. — Let $\pi : \widetilde{X} \to X$ be a proper morphism and let $Z \subset X$ be a closed subscheme such that π is finite and étale over $X \setminus Z$. The obvious map $\widetilde{X} \amalg Z \to X$ is then an sdh-cover.

REMARK 6.4. — In characteristic zero, the h-and sdh-topologies are the same, [26, Proof of Proposition 3.1.9].

EXAMPLE 6.5 (Failure of sdh-descent). — Let k be a perfect field of characteristic p, let n be a positive integer,

$$S := \frac{k[x,y,z]}{(z^p + zx^n - y)} \cong k[x,z], \qquad \text{and} \qquad R := k[x,y].$$

The obvious morphism $R \to S$ defines a covering map,

$$\pi : \underbrace{\operatorname{Spec} S}_{=:\widetilde{X}} \to \underbrace{\operatorname{Spec} R}_{=:X}.$$

Note that both X and $\widetilde{X}$ are regular. The covering map is finite of degree p. It is étale outside the exceptional set $Z = V(x) \subset X$. Indeed, the minimal polynomial of z over $k[x,y]$ is $T^p + x^n T - y$ with derivative x^n. Hence π is an alteration and generically separable. This also means that $\widetilde{X} \times_X \widetilde{X}$ is regular outside of $Z \times_X \widetilde{X} \times_X \widetilde{X}$.

We have observed in Example 6.3 that $\widetilde{X} \amalg Z \to X$ is an sdh-cover. We will now show that sdh-descent fails for Ω^1_{dvr} and this cover. Example 6.5 will also be used in Lemma 6.12 to show that Ω^1_{dvr} does not have descent for a local (in the birational geometry sense) version of the sdh-site. The reader who wishes to explore it in more detail can consult Appendix B where we make some explicit calculations.

PROPOSITION 6.6 (Failure of sdh-descent). — *In the setting of Example 6.5,* sdh-*descent fails for Ω^1_{cdh} and the cover $\widetilde{X} \amalg Z \to X$. In other words, $\Omega^1(X) \neq \Omega^1_{\mathrm{sdh}}(X)$ for this regular X.*

REMARK 6.7. — The argument shows more: no presheaf $\mathscr{F}$ with $\mathscr{F}|_{\mathsf{Reg}} \cong \Omega^1$ can ever be an sdh-sheaf.

Proof of Proposition 6.6. — We have to discuss exactness (or rather failure of exactness) of the sequence

(6.7.1)

$$0 \to \Omega^1_{\mathrm{dvr}}(X) \xrightarrow{\alpha} \Omega^1_{\mathrm{dvr}}(\widetilde{X}) \oplus \Omega^1_{\mathrm{dvr}}(Z)$$
$$\xrightarrow{\beta} \Omega^1_{\mathrm{dvr}}(\widetilde{X} \times_X \widetilde{X}) \oplus \Omega^1_{\mathrm{dvr}}(\widetilde{X} \times_X Z) \oplus \Omega^1_{\mathrm{dvr}}(Z \times_X \widetilde{X}) \oplus \Omega^1_{\mathrm{dvr}}(Z \times_X Z).$$

Notice that $Z \times_X Z = Z$, and that $\widetilde{Z} := \widetilde{X} \times_X Z$ is given as

$$\widetilde{Z} = \operatorname{Spec} \frac{k[x,y,z]}{(z^p + zx^n - y, x)} \cong \operatorname{Spec} \frac{k[y,z]}{(z^p - y)} \cong \operatorname{Spec} k[z]. \tag{6.7.2}$$

Using smoothness of $X, Z, \widetilde{X}$ and $\widetilde{Z}$, Sequence (6.7.1) simplifies to

$$0 \to \Omega^1(X) \xrightarrow{\alpha} \Omega^1(\widetilde{X}) \oplus \Omega^1(Z) \xrightarrow{\beta} \Omega^1_{\mathrm{dvr}}(\widetilde{X} \times_X \widetilde{X}) \oplus \Omega^1(\widetilde{Z}) \oplus \Omega^1(\widetilde{Z}) \oplus \Omega^1(Z).$$

We will work with this simplified description. Recalling that $\Omega^1(Z) = k[y] \cdot dy$, consider the element in the middle that is given by

$$0 \oplus dy \in \Omega^1(\widetilde{X}) \oplus \Omega^1(Z).$$

We claim that $\beta(0 \oplus dy) = 0$. This will be shown by considering the four components of $\beta(0 \oplus dy)$ one at a time. The component in $\Omega^1_{\mathrm{dvr}}(\widetilde{X} \times_X \widetilde{X})$ clearly vanishes because the first component of $0 \oplus dy$ does. The components in $\Omega^1(\widetilde{Z}) \oplus \Omega^1(\widetilde{Z})$ vanish because $d(\pi|_{\widetilde{Z}}) : \Omega^1(Z) \to \Omega^1(\widetilde{Z})$ is the map

$$k[y] \cdot dy \to k[z]dz; \qquad f(y) \cdot dy \mapsto f(z^p) \cdot d(z^p) = 0.$$

Finally, recall that we used the identity $Z \times_X Z = Z$ in the simplification. The two restriction maps $\Omega^1(Z) \to \Omega^1(Z \times_X Z) = \Omega^1(Z)$ agree, so that the last component vanishes as well. The claim is thus shown.

On the other hand, $0 \oplus dy$ cannot be in the image of α because the restriction map $\Omega^1(X) \to \Omega^1(\widetilde{X})$ is injective. In summary, we see that (6.7.1) can not possibly be exact. This concludes the proof. □

6.3. The site s-alt. — As the problem that arises in Example 6.5 seems to lie in the non-separable locus of $X' \to X$, one could try removing the need for Z, by considering the following version of [15, Exposé II, Section 1.2].

Definition 6.8 (Site s-alt(X)). — Let S be Noetherian, $X \in \mathsf{Sch}(S)$. We define the site s-alt(X) as follows. The objects are those morphisms $f : X' \to X$ in $\mathsf{Sch}(S)$ such that X' is reduced, and for every generic point $\xi \in X'$, the point $f(\xi)$ is a generic point of X and moreover, $k(\xi)/k\big(f(\xi)\big)$ is finite and separable. The topology is generated by the étale topology, and morphisms of s-alt(X), which are proper. Note that by virtue of being in s-alt(X), the latter are automatically generically separable.

Example 6.9. —
1. Let X be integral and $f : X' \to X$ a blowing-up. Then f is proper and birational, and hence an s-alt-cover.
2. Let X be reduced with irreducible components X_1, X_2. Then $\widetilde{X} = X_1 \amalg X_2 \to X$ is an s-alt-cover.
3. The morphism $\pi : \widetilde{X} \to X$ of Example 6.5 is an s-alt-cover.

The category s-alt(X) admits fiber products in the categorical sense, which can be calculated as follows: For morphisms $Y' \to Y$ and $Y'' \to Y$ in s-alt(X) let $Y' \times_Y^{\text{s-alt}} Y''$ be the union of the reduced irreducible components of the usual fiber product of schemes, $Y' \times_Y Y''$, that dominate an irreducible component of X.

EXAMPLE 6.10. — 1. Let X be reduced and connected with irreducible components X_1, X_2. Let $\widetilde{X} := X_1 \amalg X_2$. Then

$$\widetilde{X} \times_X \widetilde{X} = X_1 \amalg X_2 \amalg (X_1 \cap X_2) \amalg (X_2 \cap X_1).$$

In order to obtain the product in s-alt we have to drop the components which are not dominant over an irreducible component of X and get

$$\widetilde{X} \times_X^{\text{s-alt}} \widetilde{X} = \widetilde{X}.$$

2. Let k be a perfect field, C/k a nodal curve with normalization $\widetilde{C}$ — this means that locally for the étale topology we are in the situation of the previous example. Then $\widetilde{C} \times_C \widetilde{C}$ has one irreducible component isomorphic to $\widetilde{C}$ and two isolated points. Hence

$$\widetilde{C} \times_C^{\text{s-alt}} \widetilde{C} \cong \widetilde{C}.$$

3. In the setting of Example 6.5, the canonical inclusion $\widetilde{X} \times_X^{\text{s-alt}} \widetilde{X} \to \widetilde{X} \times_X \widetilde{X}$ is an isomorphism.

Since we have access to fiber products, a presheaf $\mathscr{F}$ on s-alt(X) is an s-alt-*sheaf* if the following sequence is exact for all covers $X_2 \to X_1$,

$$0 \to \mathscr{F}(X_1) \to \mathscr{F}(X_2) \to \mathscr{F}(X_2 \times_{X_1}^{\text{s-alt}} X_2).$$

By de Jong's theorem on alterations [16], the system of covers $Y \to X$ with Y regular is cofinal.

LEMMA 6.11. — *Let k be perfect, $X \in \mathsf{Sch}(k)$. For general X, the presheaf Ω^1_{dvr} restricted to* s-alt(X) *is not an* s-alt-*sheaf. In fact, it is not separated.*

Proof. — Assume that Ω^1_{dvr} is separated. In other words, assume that the natural map $\Omega^1_{\text{dvr}}(Y) \to \Omega^1_{\text{dvr}}(Y')$ is injective for all separable alterations $Y' \to Y$ in s-alt(X). Let X be irreducible. Choose a separable alteration $Y \to X$ with Y regular. Then, we have injective maps

$$\Omega^1_{\text{dvr}}(X) \to \underbrace{\Omega^1_{\text{dvr}}(Y)}_{=\Omega^1(Y)} \to \Omega^1\big(k(Y)\big).$$

The composition factors via $\Omega^1\big(k(X)\big) \to \Omega^1\big(k(Y)\big)$. This map is also injective, because $k(Y)/k(X)$ is separable. In total, we obtain that the map

$$\Omega^1_{\text{dvr}}(X) \to \Omega^1\big(k(X)\big)$$

is injective. In particular, we obtain that $\Omega^1_{\mathrm{dvr}}(X)$ is torsion-free, contradicting Corollary 5.16. $\square$

Torsion in $\Omega^1_{\mathrm{dvr}}(X)$, which played a role in the proof of Lemma 6.11, occurs only for singular X. The following example shows, however, that s-alt-descent also fails for regular X.

LEMMA 6.12. — *Let k be perfect. Then, Ω^1_{dvr} restricted to* s-alt(X) *does not have* s-alt-*descent for the morphism $\pi : \widetilde{X} \to X$ of Example 6.5 if $n \geq 2$. In particular, $\Omega^1_{\text{s-alt}}(X) \neq \Omega^1(X)$ for this particular X.*

The reader interested in following the computations here might also wish to look at Appendix B first, where many of the relevant rings and morphisms are explicitly computed.

Proof. — We consider the following commutative diagram,

$$\begin{tikzcd} 0 \arrow[r] & \Omega^1(X) \arrow[r] \arrow[d] & \Omega^1(\widetilde{X}) \arrow[r, "\alpha"] \arrow[d] \arrow[dr, "\beta"] & \Omega^1_{\mathrm{dvr}}(\widetilde{X}\times_X\widetilde{X}) \arrow[d] \\ 0 \arrow[r] & \Omega^1\big(k(X)\big) \arrow[r] & \Omega^1\big(k(\widetilde{X})\big) \arrow[r] & \Omega^1\big(k(\widetilde{X}\times_X\widetilde{X})\big). \end{tikzcd}$$

We wish to show that the top row is not exact. As X and $\widetilde{X}$ are smooth, the left two vertical morphisms are monomorphisms. Moreover, since $\widetilde{X} \to X$ is generically étale and Ω^1 is an étale sheaf, the lower row is exact. Consequently, the top row is exact at $\Omega^1(X)$. Now, for the moment, we ask the reader to admit the following claim.

Claim: The kernels of α and β are equal.

Application of the claim: If this claim holds, then from a diagram chase, exactness of the top row at $\Omega^1(\widetilde{X})$ would imply that $\Omega^1(X)$ is the intersection of $\Omega^1(k(X))$ and $\Omega^1(\widetilde{X})$ inside $\Omega^1(k(\widetilde{X}))$. That is, exactness at $\Omega^1(\widetilde{X})$ would imply the that inclusion

$$\underbrace{k[x,y]\cdot dx \oplus k[x,y]\cdot dy}_{=\Omega^1(X)} \subseteq \underbrace{k(x,y)\cdot dx \oplus k(x,y)\cdot dy}_{=\Omega^1\left(k(X)\right)} \cap \underbrace{k[x,z]\cdot dx \oplus k[x,z]\cdot dz}_{=\Omega^1(\widetilde{X})}$$

is in fact an equality inside $\Omega^1\big(k(\widetilde{X})\big) = k(x,z)\cdot dx \oplus k(x,z)\cdot dz$, where $y = z^p + zx^n$.

However, the element

$$x^{-1}\cdot dy = nzx^{n-2}\cdot dx + x^{n-1}\cdot dz \in \Omega^1\big(k(\widetilde{X})\big)$$

is in the intersection on the right, but cannot come from an element on the left, since for any element coming from the left the coefficient of dz is divisible by x^n. Hence, the inclusion is strict, as long as our claim that $\ker\alpha = \ker\beta$ holds.

Proof of the Claim: To see that $\ker\alpha = \ker\beta$, let $X^{(2)}$ be the normalization of $\widetilde{X} \times_X \widetilde{X}$. A direct computation, given in Section B.0.5 below, shows that $X^{(2)}$ is in fact smooth over $\operatorname{Spec} k$. The canonical map $\rho : X^{(2)} \to \widetilde{X} \times_X \widetilde{X}$ is proper, birational, and an isomorphism outside $Z \times_X \widetilde{X} \times_X \widetilde{X}$, cf. Section B.0.4 for that. Setting $Z^{(2)} = (Z \times_X \widetilde{X} \times_X \widetilde{X})_{\mathrm{red}}$, we now have a proper cdh-cover $Z^{(2)} \amalg X^{(2)}$ of $\widetilde{X} \times_X \widetilde{X}$. This is useful because we have seen in Example 4.19 that Ω^1_{dvr} is a cdh-sheaf and so

$$\Omega^1_{\mathrm{dvr}}\big(\widetilde{X} \times_X \widetilde{X}\big) \to \Omega^1_{\mathrm{dvr}}\big(X^{(2)} \amalg Z^{(2)}\big)$$

is injective. Now since $Z \times_X \widetilde{X} \times_X \widetilde{X} = (Z \times_X \widetilde{X}) \times_Z (Z \times_X \widetilde{X}) = \widetilde{Z} \times_Z \widetilde{Z}$, one calculates $Z^{(2)}$ as in (6.7.2) as

$$\big(\operatorname{Spec}\, k[z] \otimes_{k[y]} k[z]\big)_{\mathrm{red}} = \big(\operatorname{Spec}\, k[z] \otimes_{k[z^p]} k[z]\big)_{\mathrm{red}} = \operatorname{Spec}\, k[z].$$

From this calculation we glean two important pieces of information. Firstly $Z^{(2)}$ is smooth, and so

$$\Omega^1_{\mathrm{dvr}}\big(X^{(2)} \amalg Z^{(2)}\big) = \Omega^1\big(X^{(2)} \amalg Z^{(2)}\big)$$

and since Ω is torsion-free on regular schemes, the morphism

$$\Omega^1\big(X^{(2)}\big) \oplus \Omega^1\big(Z^{(2)}\big) \to \Omega^1\big(k(X^{(2)})\big) \oplus \Omega^1\big(k(Z^{(2)})\big)$$

is injective. Since $X^{(2)} \to \widetilde{X} \times_X \widetilde{X}$ was birational, all this implies that the morphism

$$\Omega^1_{\mathrm{dvr}}\big(\widetilde{X} \times_X \widetilde{X}\big) \to \Omega^1\big(k(\widetilde{X} \times_X \widetilde{X})\big) \oplus \Omega^1\big(k(Z^{(2)})\big)$$

is injective. So, to finish the proof of the claim, it suffices to show that the induced morphism $\Omega^1\big(\widetilde{X}\big) \to \Omega^1\big(k(Z^{(2)})\big)$ is zero. This is the second piece of information we obtain from the description of $Z^{(2)}$ above. Since $Z^{(2)} = \widetilde{Z}$, the two compositions

$$Z^{(2)} \to \widetilde{X} \times_X \widetilde{X} \rightrightarrows \widetilde{X}$$

induced by the projections are equal, and so $\Omega^1\big(\widetilde{X}\big) \to \Omega^1\big(k(Z^{(2)})\big)$ is indeed zero. $\square$

Appendix A. Theorem 5.8

We have seen in the main text that understanding the torsion in Ω^n is crucial in order to understand Ω^n_{cdh}. The main purpose of this appendix is to give a proof of Theorem 5.8. We are highly indepted to an anonymous referee who provided the crucial reference to the result of Gabber-Ramero [7, Corollary 6.5.21], which appeared as a hypothesis in an earlier version of this article. We also give criteria for testing the existence of torsion in special cases. These considerations are independent of the main text.

Before going into the direct proof, we explain how Theorem 5.8 follows easily from resolution of singularities.

LEMMA A.1. — *Let k be perfect. Assume weak resolution of singularities holds over k, that is, assume that for every reduced Y there is a proper birational morphism $X \to Y$ with X smooth. Then, Theorem 5.8 holds true.*

Proof. — Let $\omega \in \operatorname{tor}\Omega^n(Y)$. By definition there is a dense open subset $U \subset Y$ such that $\omega|_U$ vanishes. Let $\pi : X \to Y$ a desingularisation. Then $\pi^*\omega \in \Omega^n(X)$ is a torsion form because it vanishes on $\pi^{-1}U$. As X is smooth, this implies that $\pi^*\omega = 0$. □

A.1. Valuation rings. — We give a reformulation of Theorem 5.8 in terms of vanishing of differential forms on (not necessarily discrete) valuation rings. In this section, let k be a perfect field.

Let A be an integral k-algebra of finite type. Recall that the Riemann-Zariski space $\mathsf{RZ}(A)$, called the "Riemann surface" in [28, § 17, p. 110], as a set, is the set of (not necessarily discrete) valuation rings of $\mathsf{Frac}(A)$ which contain A. To a finitely generated sub-A-algebra A' is associated the set $E(A') = \{R \in \mathsf{RZ}(A) : A' \subseteq R\}$ and one defines a topology on $\mathsf{RZ}(A)$ taking the $E(A')$ as a basis. This topological space is quasi-compact, in the sense that every open cover admits a finite subcover [28, Theorem 40].

Consider the following hypothesis.

HYPOTHESIS V. — For every finitely generated extension K/k and every k-valuation ring R of K the map $\Omega^n(R) \to \Omega^n(K)$ is injective for all $n \geq 0$.

REMARK A.2 (Hypothesis V true). — The statement is true for $n = 0$ because valuation rings are torsion free. [7, Corollary 6.5.21] states that Hypothesis V is true for $n = 1$, and then Lemma A.4 says that this implies it is true for all $n \geq 0$.

We repeat Theorem 5.8 here as well for ease of reference.

HYPOTHESIS H (Theorem 5.8). — Given a perfect field k, assume that for every integral $Y \in \mathsf{Sch}(k)$, every number $n \in \mathbb{N}$ and every $\omega \in \operatorname{tor}\Omega^n(Y)$, there is a birational proper morphism $\pi : \widetilde{Y} \to Y$ such that the image of ω in $\Omega^n(\widetilde{Y})$ vanishes.

PROPOSITION A.3. — *Let k be perfect. Then Hypothesis V and Hypothesis H for k are equivalent. In particular, Hyothesis H is true.*

Proof. — For the reader's convenience, the proof is subdivided into steps.

Step 1: Proof V$\Rightarrow$H in the affine case. — Let $X = \operatorname{Spec} A \in \mathsf{Sch}(k)$ be integral and $\omega \in \Omega^n(X)$ an element which vanishes on a dense open, that is, the image of ω in $\Omega^n(\mathsf{Frac}(A))$ is zero. We wish to find a proper birational morphism $Y \to X$ such that $\omega|_Y = 0$.

Hypothesis V implies then that the image of ω in $\Omega^n(R)$ is zero for every valuation ring R of $\mathsf{Frac}(A)$ which contains A. As each R is the union of its finitely generated sub-A-algebras, for each such R there is a finitely generated sub-A-algebra, say A_R, for which ω vanishes in $\Omega^n(A_R)$. The $E(A_R)$ then form an open cover of $\mathsf{RZ}(A)$ and so since it is quasi-compact, there exists a finite subcover. That is, there is a finite set $\{A_i\}_{i=1}^m$ of finite generated sub-A-algebras of $\mathsf{Frac}(A)$ such that ω is zero in each $\Omega^n(A_i)$, and every valuation ring of $\mathsf{Frac}(A)$ which contains A, also contains one of the A_i's.

Zariski's Main Theorem in the form of Grothendieck, [10, Théorème 8.12.6], allows us, for each i, to choose a factorization $\operatorname{Spec} A_i \to Y_i \to X$ as a dense open immersion followed by a proper morphism and to define $Y := Y_1 \times_X \dots \times_X Y_m$ (or better, define Y to be the closure of the image of the generic point of Spec $\mathsf{Frac}(A)$ in this product) so that $Y \to X$ is now a proper birational morphism. Since $\omega|_{A_i} = 0$ for each i, it suffices now to show that the set of open immersions $\{(\operatorname{Spec} A_i) \times_{Y_i} Y \to Y\}_{i=1}^m$ is an open cover of Y to conclude that $\omega|_Y = 0$. But for every point $y \in Y$, there exists a valuation ring R_y of $\mathsf{Frac}(A)$ such that $\operatorname{Spec} R_y \to Y$ sends the closed point of $\operatorname{Spec} R_y$ to y, and since R_y contains some A_i, there is a factorization $\operatorname{Spec} R_y \to \operatorname{Spec} A_i \to Y$, and we see that $y \in (\operatorname{Spec} A_i) \times_{Y_i} Y$.

Step 2: Proof V$\Rightarrow$H in general. — For the case of a general integral $X \in \mathsf{Sch}(k)$ we use the same trick. Take an affine cover $\{U_i\}_{i=1}^m$ of X. We have just seen that there exist proper birational morphisms $V_i \to U_i$ such that $\omega|_{V_i} = 0$ for each i. Zariski's Main Theorem in the form of Grothendieck gives compactifications $V_i \to Y_i \to X$. We set $Y := Y_1 \times_X \dots \times_X Y_m$ so that $Y \to X$ is proper birational, and the same argument as above shows that $\{V_i \times_{Y_i} Y \to Y\}_{i=1}^m$ is an open cover. Since $\omega|_{V_i} = 0$ for each i, this implies that $\omega|_Y = 0$.

Step 3: Proof H$\Rightarrow$ V. — Let K be a finitely generated extension of k, let R be a k-valuation ring of K, and let ω be in the kernel of $\Omega^n(R) \to \Omega^n(K)$. There is some finitely generated sub-k-algebra A of R and $\omega' \in \Omega^n(A)$ such that $\operatorname{Spec} R \to \operatorname{Spec} A$ is birational, and $\omega'|_R = \omega$. Now by Hypothesis H there is a proper birational morphism $\widetilde{Y} \to \operatorname{Spec} A$ such that $\omega'|_{\widetilde{Y}}$ is zero. But by the valuative criterion for properness, there is a factorization $\operatorname{Spec} R \to \widetilde{Y} \to \operatorname{Spec} A$, and so $\omega = 0$. $\square$

The following lemma says that if Hypothesis V is true for $n = 1$ then it is true for all $n \geq 0$.

LEMMA A.4. — *Let R be a (possibly non-discrete) valuation ring. Let M be a torsion-free R-module. Then $\bigwedge^n M$ is torsion-free for all $n \geq 0$.*

Proof. — First note that for valuation rings, being a torsion-free module is equivalent to being flat, [3, Ch.VI, §3, n.6, Lemma 1] or [1, Tag 0539]. In general (even for noncommutative rings), a module M is flat if and only if it is possible to write M as $M = \varinjlim_{i\in I} M_i$, a colimit of *free* modules of finite type indexed by a filtered partially orderd set I, [20, Théorème 1.2]. So it suffices to show that if M is of this form, then so is $\bigwedge^n M$. Now it follows from the fact that tensor products commute with filtered colimits: We have

$$M^{\otimes n} = \varinjlim_{i\in I} M_i^{\otimes n}$$

because tensor product commutes with direct limits. By definition

$$\bigwedge^n M = M^{\otimes n}/N$$

where N is generated by elementary tensors of the form $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ with $x_j = x_{j'}$ for some $j \neq j'$. Hence $N = \varinjlim_{i\in I} N_i$ where N_i is generated by elementary tensors as above with all $x_j \in M_i$. The direct limit of the sequences

$$0 \to N_i \to M_i^{\otimes n} \to \bigwedge^n M_i \to 0$$

is

$$0 \to N \to M^{\otimes n} \to \varinjlim_{i\in I} \bigwedge^n M_i \to 0$$

and hence

$$\bigwedge^n M = \varinjlim_{i\in I} \bigwedge^n M_i.$$

Since the M_i are free modules of finite type, the same is true of the $\bigwedge^n M_i$. Hence, as mentioned at the beginning, these are flat, and therefore torsion-free. □

A.2. Hyperplane section criterion. — We give a criterion for testing the vanishing of torsion-forms.

LEMMA A.5 (Hyperplane section criterion). — *Let X be an integral, quasi-projective variety, defined over an algebraically closed field k. Let $H \subset X$ be any integral hyperplane that is not contained in the singular locus of X. Then, the natural map $(\operatorname{tor}\Omega^1_X)|_H \to \Omega^1_H$ is injective.*

In particular, if $U \subseteq X$ is open and $\sigma \in (\operatorname{tor}\Omega^1_X)(U)$ is any torsion-form with induced form $\sigma_H \in \Omega^1_H(U \cap H)$, then $\operatorname{supp}\sigma \cap H \subseteq \operatorname{supp}\sigma_H$.

Proof. — Consider the restriction $(\operatorname{tor}\Omega^1_X)|_H$. Since H is not contained in the singular locus of X, this is a torsion-sheaf on H. Recalling the exact sequence that defines torsion- and torsion-free forms, Sequence (3.1.1) on page 313, and using that $\check{\Omega}^1_X$ is torsion-free and $\operatorname{Tor}_1^{\mathscr{O}_X}(\mathscr{O}_H, \check{\Omega}^1_X)$ hence zero, observe that this sequence restricts to an exact sequence of sheaves on H,

$$0 \longrightarrow (\operatorname{tor}\Omega^1_X)|_H \xrightarrow{\alpha} \Omega^1_X|_H \xrightarrow{\beta} \check{\Omega}^1_X|_H \longrightarrow 0,$$

where $\bullet|_H$ denotes pull-back of in the sense of quasi-coherent modules. The sequence shows that α injects the torsion-sheaf $(\operatorname{tor}\Omega^1_X)|_H$ into the middle term of the second fundamental sequence for differentials, [13, II. Proposition 8.12],

$$\begin{tikzcd} & (\operatorname{tor}\Omega^1_X)|_H \arrow[d, "\alpha"', "\text{injection}"] & & \\ \mathscr{I}_H/\mathscr{I}^2{}_H \arrow[r, "a"'] & \Omega^1_X|_H \arrow[r, "b"'] & \Omega^1_H \arrow[r] & 0. \end{tikzcd}$$

We claim that the morphism a is also injective. To this end, recall that $\mathscr{I}_H$ is locally principal. In particular, $\mathscr{I}_H/\mathscr{I}^2{}_H$ is a locally free sheaf of $\mathscr{O}_H$-modules and that the morphism a is injective at the generic point of H, where X is smooth, [4, Exercise 17.12]. It follows that $\ker a$ is a torsion-subsheaf of the torsion-free sheaf $\mathscr{I}_H/\mathscr{I}^2{}_H$, hence zero. The image $\operatorname{img} a$ is hence isomorphic to $\mathscr{I}_H/\mathscr{I}^2{}_H$, and in particular torsion-free. As a consequence, note that the $\operatorname{img}\alpha$, which is the image of a torsion-sheaf and hence itself torsion, intersects $\operatorname{img} a = \ker b$ trivially. The composed morphism $b \circ \alpha$ is thus injective. This proves the main assertion of Lemma A.5.

To prove the "In particular…"-clause, let $U \subseteq X$ be any open set, $\sigma \in (\operatorname{tor}\Omega^1_X)(U)$ be any torsion-form and $x \in \operatorname{supp}\sigma$ be any closed point. It follows from Nakayama's lemma that $x \in \operatorname{supp}(\sigma|_{H\cap U}) \subseteq \operatorname{supp}\big((\operatorname{tor}\Omega^1_X)|_{H\cap U}\big)$. Since H is not contained in the singular locus, the sheaf Ω^1_X is locally free at the generic point of H, and σ_H is thus a torsion-form on $H \cap U$. Since $b \circ \alpha$ is injective, its support contains x as claimed. This finishes the proof of Lemma A.5. □

This lemma and Flenner's Bertini-type theorems, [6], imply the following two theorems.

THEOREM A.6 (Existence of good hyperplanes through normal points). — *Let X be a integral, quasi-projective variety of dimension $\dim X \geq 3$, defined over an algebraically closed field k, and let $x \in X$ be a closed, normal point. Then, there exists a hyperplane section H through x such that H is irreducible and reduced at x and such that the following holds: if $U \subseteq X$ is an open neighborhood of x and if $\sigma \in \operatorname{tor}\Omega^1_X(U)$ is any torsion-form whose induced form $\sigma_H \in$*

$\Omega^1_H(U \cap H)$ vanishes at x, then σ vanishes at x; in other words, its image in the fiber $\Omega^1_X \otimes k(x)$ is zero.

In particular, Ω^1_X is torsion-free at x if Ω^1_H is torsion-free at x.

Proof. — It follows from normality of $x \in X$ that the local ring $\mathscr{O}_{X,x}$ satisfies Serre's condition (R_1), has depth $\mathscr{O}_{X,x} \geq 2$ and that it is analytically irreducible, [27, Theorem on page 352]. We can thus apply [6, Korollar 3.6] and find a hyperplane section H through x that is irreducible and reduced at x. Shrinking X if need be, we can assume without loss of generality that H is irreducible and reduced. Lemma A.5 then yields the claim. □

Theorem A.7 (Existence of good hyperplanes in bpf linear systems). — *Let X be a integral, quasi-projective variety of dimension $\dim X \geq 3$, smooth in codimension one and defined over an algebraically closed field. Let $\mathbb{H}$ be a finite-dimensional, basepoint-free linear system. Then, there exists a dense, open subset $\mathbb{H}^\circ \subseteq \mathbb{H}$ such that any hyperplane section $H \subset X$ which corresponds to a closed point of $\mathbb{H}^\circ$ is irreducible, reduced, and satisfies the following additional property: if $U \subseteq X$ is open and $\sigma \in \operatorname{tor} \Omega^1_X(U)$ is any torsion-form with induced form $\sigma_H \in \Omega^1_H(U \cap H)$, then $\operatorname{supp} \sigma \cap H \subseteq \operatorname{supp} \sigma_H$.*

In particular, if Ω^1_H is torsion-free, then $\operatorname{supp} \operatorname{tor} \Omega^1_X$ is finite and disjoint from H.

Proof. — Recall from Flenner's version of Bertini's first theorem, [6, Satz 5.2], that any general hyperplane H is irreducible and reduced. The main assertion of Theorem A.7 thus follows from Lemma A.5.

If Ω^1_H is torsion-free, the support of $\operatorname{tor} \Omega^1_X$ necessarily avoids H. Since general hyperplanes can be made to intersect any positive-dimensional subvariety, we obtain the finiteness of $\operatorname{supp} \operatorname{tor} \Omega^1_X$. □

Appendix B. Explicit computations

Here we make some explicit calculations around Example 6.5. Recall that k is a perfect field of characteristic p, we let n be a positive integer,

$$S := \frac{k[x,y,z]}{(z^p + zx^n - y)} \cong k[x,z], \qquad \text{and} \qquad R := k[x,y].$$

We consider the morphism

$$\pi : \underbrace{\operatorname{Spec} S}_{=:\widetilde{X}} \to \underbrace{\operatorname{Spec} R.}_{=:X}$$

associated to the obvious morphism $R \to S$.

B.0.1. *Differentials of X and $\widetilde{X}$, and the pull-back map.* — The modules of k-differentials are given as $\Omega^1(X) = R \cdot dx \oplus R \cdot dy$ and $\Omega^1(\widetilde{X}) = S \cdot dx \oplus S \cdot dz$. In terms of these generators, the pull-back map $d\pi$ is given by

$$dx \mapsto dx \quad \text{and} \quad dy \mapsto d(z^p + zx^n) = x^n \cdot dz + nzx^{n-1} \cdot dx.$$

B.0.2. *The preimage of Z.* — We have seen above that $Z = V(x)$ is a regular subvariety of X. Its preimage $\widetilde{Z} := \pi^{-1}(Z)$ is then given as

$$\widetilde{Z} = \operatorname{Spec} \frac{k[x,y,z]}{(z^p + zx^n - y, x)} \cong \operatorname{Spec} \frac{k[y,z]}{(z^p - y)} \cong \operatorname{Spec} k[z].$$

In particular, we see that $\widetilde{Z}$ is likewise regular.

B.0.3. *Differentials of Z and $\widetilde{Z}$, and the pull-back map.* — The modules of k-differentials are given as $\Omega^1(Z) = k[y] \cdot dy$ and $\Omega^1(\widetilde{Z}) = k[z] \cdot dz$. In terms of these generators, the pull-back map $d(\pi|_{\widetilde{Z}})$ is given by $dy \mapsto d(z^p) = 0$. The map $d(\pi|_{\widetilde{Z}})$ is thus the zero map.

B.0.4. *Fibred products.* — Finally, $\widetilde{X} \times_X \widetilde{X}$ is the spectrum of the ring

$$\begin{aligned} \frac{k[x,y,z_1]}{(z_1^p + z_1x^n - y)} \otimes_{k[x,y]} \frac{k[x,y,z_2]}{(z_2^p + z_2x^n - y)} &= \frac{k[x,y,z_1,z_2]}{(z_1^p + z_1x^n - y, z_2^p + z_2x^n - y)} \\ &\cong \frac{k[x,z_1,z_2]}{(z_2^p + z_2x^n - z_1^p - z_1x^n)} \cong \frac{k[x,z_1,u]}{(u^p + x^n u)}, \end{aligned}$$

with $u = z_2 - z_1$. Under this identification, the two projections $\widetilde{X} \times_X \widetilde{X} \rightrightarrows \widetilde{X}$ correspond to the ring maps $z \mapsto z_1$ and $z \mapsto u + z_1$—notice that $y = z_1^p + z_1x^n = z_2^p + z_2x^n$ in this ring. The scheme $\widetilde{X} \times_X \widetilde{X}$ is regular outside of

$$Z^{(2)} := V(x,u) = \operatorname{Spec} \frac{k[x,z_1,u]}{(u^p + x^n u, x, u)} \cong \operatorname{Spec} k[z_1] \subset \widetilde{X} \times_X \widetilde{X}.$$

B.0.5. *The normalization of $\widetilde{X} \times_X \widetilde{X}$.* — The factorization $u^p + x^n u = u \cdot (u^{p-1} + x^n)$ decomposes $\widetilde{X} \times_X \widetilde{X}$ as the union of two closed subschemes,

$$\widetilde{X} \times_X \widetilde{X} = \underbrace{\operatorname{Spec} k[x,z_1]}_{\cong \mathbb{A}^2} \cup \underbrace{\operatorname{Spec} \frac{k[x,z_1,u]}{(u^{p-1} + x^n)}}_{\cong \mathbb{A}^1 \times (\text{curve } C)}$$

The k-curve C need not be smooth, but since k is perfect by assumption, its normalization will be smooth over $\operatorname{Spec} k$. It follows that normalization of $\widetilde{X} \times_X \widetilde{X}$ is likewise smooth over $\operatorname{Spec} k$.

B.0.6. *Dvr-differentials on $\widetilde{X} \times_X \widetilde{X}$.* — As we have seen in Subsections B.0.4 and B.0.5, the normalization of $\widetilde{X} \times_X \widetilde{X}$ provides a proper birational map $\rho : X^{(2)} \to \widetilde{X} \times_X \widetilde{X}$, which is an isomorphism outside $Z^{(2)}$. Recall from Remark 2.11 that the obvious morphism $X^{(2)} \amalg Z^{(2)} \to \widetilde{X} \times_X \widetilde{X}$ is a cover in the cdh-topology. We have seen in Example 4.19 that Ω^1_{dvr} is a cdh-sheaf. We obtain an injection

$$\Omega^1_{\mathrm{dvr}}(\widetilde{X} \times_X \widetilde{X}) \to \Omega^1_{\mathrm{dvr}}\big(X^{(2)}\big) \oplus \Omega^1_{\mathrm{dvr}}(Z^{(2)}).$$

Recall that $X^{(2)}$ and $Z^{(2)}$ are both smooth. This has two consequences. First, we have observed in Remark 4.3(3) that reg-differentials and Kähler differentials agree,

$$\Omega^1_{\mathrm{dvr}}\big(X^{(2)}\big) \oplus \Omega^1_{\mathrm{dvr}}(Z^{(2)}) \cong \Omega^1\big(X^{(2)}\big) \oplus \Omega^1(Z^{(2)}).$$

Secondly, the sheaves of Kähler-differentials are torsion-free, and inject into rational differentials. Summing up, we obtain an inclusion

$$\Omega^1_{\mathrm{dvr}}\big(\widetilde{X} \times_X \widetilde{X}\big) \to \Omega^1\left(\frac{k(x,z_1)[u]}{(u^p + x^n u)}\right) \oplus \Omega^1\big(k(z_1)\big).$$

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